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Abir Ghannoum

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THÈSE

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dans l'**École Doctorale MSTII** et l'**École Doctorale des Sciences et de Technologie**

EDS réfléchies en moyenne avec sauts et EDS rétrogrades de type McKean-Vlasov: étude théorique et numérique.

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Chapitre 1

Introduction

Dans cette thèse, nous nous concentrons sur deux thématiques principales : les EDS réfléchies en moyenne avec sauts et les EDSR de type McKean-Vlasov. Dans une première partie, nous établissons le résultat d'existence et d'unicité d'une solution de l'EDS réfléchie en moyenne avec sauts et nous traitons un schéma numérique basé sur le système de particules. Dans une deuxième partie, nous étudions l'existence et l'unicité d'une solution de l'EDSR de type McKean-Vlasov, et nous abordons la vitesse de convergence d'une simulation numérique basée sur la décomposition en chaos de Wiener. Enfin, dans une troisième partie, nous nous concentrons sur des marches aléatoires pour trouver une autre simulation pour l'EDSR de type McKean-Vlasov et établir sa vitesse de convergence. Cette thèse est formée des travaux [BGL18], [ARBGL19] et [BGL19].

Dans ce chapitre introductif, nous rappelons dans les deux premières parties des résultats connus au sujet des EDS et des EDSR. Tandis que dans les trois parties suivantes nous présentons le contenu des chapitres qui constituent cette thèse, en décrivant les méthodes spécifiques utilisées et en énonçant les nouveaux résultats obtenus.

1.1 Préliminaires sur les EDS

1.1.1 Notations et définitions

Dans cette thèse, on considère un espace de probabilité complet $(\Omega, \mathcal{F}, \mathbb{P})$ sur lequel on définit un mouvement Brownien standard B d -dimensionnel, tel que $\{\mathcal{F}_t\}_{t \in [0, T]}$ est la filtration naturelle de B .

Considérons l'équation différentielle stochastique (EDS en abrégé) suivante

$$X_t = X_0 + \int_0^t b(s, X_s) ds + \int_0^t \sigma(s, X_s) dB_s, \quad 0 \leq t \leq T, \quad (1.1.1)$$

où

- la condition initiale $X_0 : \Omega \rightarrow \mathbb{R}^n$ est une variable aléatoire \mathcal{F}_0 -mesurable.
- la dérive $b : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ est une fonction mesurable.

- la diffusion $\sigma : [0, T] \times \mathbb{R}^n \longrightarrow \mathbb{R}^{n \times d}$ est une fonction mesurable.

Définition 1.1.1. Une solution (forte) de l'EDS (1.1.1) est un processus $(X_t)_{t \in [0, T]}$ à valeurs dans \mathbb{R}^n tel que

- $\{X_t\}_{t \in [0, T]}$ est continu et adapté,

- \mathbb{P} -p.s.,

$$\int_0^T (|b(s, X_s)| + \|\sigma(s, X_s)\|^2) ds < \infty, \quad \|\sigma\| = \text{trace}(\sigma\sigma^*),$$

- \mathbb{P} -p.s.,

$$X_t = X_0 + \int_0^t b(s, X_s) ds + \int_0^t \sigma(s, X_s) dB_s, \quad 0 \leq t \leq T.$$

Notons $S_T^2(\mathbb{R}^n)$ l'espace formé par les processus continus et adaptés, $\phi : \Omega \times [0, T] \longrightarrow \mathbb{R}^n$, tel que $\|\phi\|_{S_T^2}^2 := \mathbb{E}[\sup_{t \in [0, T]} |\phi_t|^2] < \infty$.

1.1.2 Existence et unicité

Il existe deux types de solutions principales à une EDS : les solutions faibles et les solutions fortes. Dans ce travail, nous nous concentrons sur le second type et fournissons un théorème d'existence et d'unicité d'une solution forte à l'EDS (1.1.1) (voir [KS91], [Øks03], [KP13]).

Hypothèse 1.1.1. Il existe une constante k telle que \mathbb{P} -p.s.,

- condition de Lipschitz en x :

$$\forall s, x, x', \quad |b(s, x) - b(s, x')| + \|\sigma(s, x) - \sigma(s, x')\| \leq k|x - x'|.$$

- condition de croissance linéaire en x :

$$\forall s, x, \quad |b(s, x)| + \|\sigma(s, x)\| \leq k(1 + |x|).$$

- condition d'intégrabilité en X_0 :

$$\mathbb{E}[|X_0|^2] < \infty.$$

Théorème 1.1.1 (Existence et unicité). Supposons que b , σ et X_0 satisfont l'Hypothèse 1.1.1. Alors il existe une solution unique X_t à l'EDS (1.1.1). Cette solution appartient à S_T^2 .

1.1.3 Exemples

La solution d'une EDS peut être obtenue par la formule d'Itô. Considérons quelques exemples et solutions correspondantes pour clarifier la technique de la solution.

Premièrement, donnons un exemple applicable au monde de la finance. Notons S_t le prix d'une action qui change de manière aléatoire. La dynamique du prix d'une action est donnée par l'EDS suivante :

$$dS_t = S_t(\mu dt + \sigma dB_t),$$

avec S_0 donné et σ considéré comme la volatilité. En appliquant la formule d'Itô, on peut trouver facilement la forme explicite de la solution S_t :

$$S_t = S_0 \exp \left((\mu - \sigma^2/2)t + \sigma B_t \right).$$

Deuxièmement, considérons l'EDS suivante :

$$dX_t = -cX_t dt + \sigma dB_t,$$

où $X_0 = x_0$. Notons que cette équation est une sorte de processus d'Ornstein-Uhlenbeck. Une intégration par parties nous donne

$$\begin{aligned} d(X_t e^{ct}) &= e^{ct} dX_t + c e^{ct} X_t dt \\ &= e^{ct} \sigma dB_t, \end{aligned}$$

et par suite nous obtenons

$$X_t = e^{-ct} x_0 + \int_0^t e^{c(s-t)} \sigma dB_s.$$

Dans cet exemple on peut vérifier que la solution X est un processus gaussien.

1.1.4 EDS avec sauts

Dans cette partie, on va donner un aperçu sur les EDS avec sauts. Soient $F : \mathbb{R}^{n \times d} \rightarrow \mathbb{R}^n$ et $G : \mathbb{R}^{n \times d} \rightarrow \mathbb{R}^n$. Considérons l'EDS avec sauts suivante :

$$\begin{aligned} X_t &= X_0 + \int_0^t b(X_{s-}) ds + \int_0^t \sigma(X_{s-}) dB_s \\ &\quad + \int_0^t \int_{\|z\| \leq c} F(X_{s-}, z) \tilde{N}(ds, dz) \\ &\quad + \int_0^t \int_{\|z\| > c} G(X_{s-}, z) N(ds, dz), \quad t \geq 0, \end{aligned} \tag{1.1.2}$$

où $c > 0$, \tilde{N} une mesure de Poisson compensée $\tilde{N}(ds, dz) = N(ds, dz) - \lambda(dz)ds$, N un processus de Poisson d'intensité λ et B un processus Brownien indépendant de N .

Les coefficients b , σ et F sont soumis aux hypothèses suivantes afin de garantir l'existence et l'unicité d'une solution de l'EDS avec sauts (1.1.2).

Hypothèse 1.1.2. *Il existe une constante k telle que \mathbb{P} -p.s.,*
— *condition de Lipschitz en x :*

$$\begin{aligned} \forall x, x', z, \quad &|b(x) - b(x')|^2 + \|\sigma(x) - \sigma(x')\|^2 + \int_{\|z\| \leq c} |F(x, z) - F(x', z)|^2 \lambda(dz) \\ &\leq k|x - x'|^2. \end{aligned}$$

— condition de croissance linéaire en x :

$$\forall x, \quad |b(x)|^2 + \|\sigma(x)\|^2 + \int_{\|z\| \leq c} |F(x, z)|^2 \lambda(dz) \leq k(1 + |x|^2).$$

Théorème 1.1.2 (Existence et unicité, [IW90]). *Supposons que X_0 satisfait la condition d'intégrabilité et que b et σ satisfont l'Hypothèse 1.1.2. Alors il existe une solution unique $(X_t)_{t \geq 0}$ de l'EDS avec sauts (1.1.2).*

1.1.5 EDS réfléchies

Dans le cadre classique, des nombreux auteurs ont étudié de manière approfondie les EDS réfléchies conduites par le mouvement Brownien. Parmi eux, Skorokhod est le premier qui introduit des processus de diffusion avec des frontières réfléchies dans les années 1960 (voir [Sko61], [Sko62]).

Définition 1.1.2. *Une paire de processus continus adaptés $(X_t, K_t)_{t \geq 0}$ est une solution de l'EDS*

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t + dK_t, \quad p.s.,$$

avec une réflexion en 0 et une condition initiale X_0 , si

1. $X_t \geq 0, t \geq 0$;
2. K un processus croissant, $K_0 = 0$;
3. $\int_0^t \mathbf{1}_{X_s > 0} dK_s = 0, \quad t \geq 0$;
4. $X_t = X_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dB_s + K_t, \quad p.s.,$

et toutes les intégrales sont bien définies.

La troisième contrainte est une condition de type Skorokhod qui signifie que K n'intervient que lorsque X atteint le point 0.

Théorème 1.1.3 (Skorokhod 1961). *Si b et σ sont deux fonctions lipschitziennes en x , alors l'EDS réfléchie*

$$\begin{cases} X_t = X_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dB_s + K_t, & t \geq 0, \\ X_t \geq 0, \quad \int_0^t \mathbf{1}_{X_s > 0} dK_s = 0, & t \geq 0, \end{cases} \quad (1.1.3)$$

admet une solution unique (X, K) .

Plusieurs autres articles sur les EDS réfléchies ont déjà été proposés dans la littérature : les diffusions réfléchies dans un demi-espace (voir par exemple Watanabe et al. [Wat71], El Karoui [EK75], Yamada [Yam76], El Karoui et Chaleyat-Maurel [EKCM78], El Karoui et al. [CMEKM80]), les EDS avec une réflexion sur les faces d'un polyèdre convexe (voir par exemple [VW85], [Wil87], [DW96], [Del08]) et les EDS rétrogrades réfléchies sous contrainte de laisser le processus X au-dessus d'un seuil S (voir El Karoui, Kapoudjian, Pardoux, Peng et Quenez [EKKP⁺97]). Cet article nous montre un résultat d'existence

et d'unicité de solution grâce au point fixe de Picard et la méthode de pénalisation.

D'une façon similaire, la forme d'une EDS réfléchié dans un domaine multi-dimensionnel D est la suivante :

$$dX_t = b(t, X_t)dt + \sum_{i=1}^d \sigma^i(t, X_t)dB_t^i + c(t, X_t)dK_t, \quad p.s.,$$

où c est un champ de vecteur réfléchi, $X_t \in \bar{D}$, $t \geq 0$, K est un processus continu croissant tel que $\int_0^t \mathbf{1}_{X_s \notin \partial D} dK_s = 0$, $t \geq 0$. L'étude des EDS multi-dimensionnelles remonte à Stroock et Varadhan [SV71], pour lesquelles l'existence et l'unicité de solutions faibles ont été prouvées lorsque le domaine est régulier. Ensuite, les solutions de telles équations ont été construites sur un domaine convexe par Tanaka [Tan79] en utilisant une méthode directe, alors que Menaldi [Men83] et Lions et al. [LMS81] ont adapté une méthode de pénalisation pour les construire. Concernant le problème de réflexion dans un domaine non convexe mais admissible, Lions et Sznitman [LS84] ont d'abord résolu le problème déterministe de Skorokhod. Plus tard, les résultats de [LS84] ont été améliorés par Saisho [Sai87], ainsi que par Saisho et Tanaka [ST87] en supprimant la condition d'admissibilité du domaine.

D'autre part, les EDS réfléchiées conduites par un processus de Lévy ont été étudiées par Menaldi et Robin [MR85] et Kohatsu-Higa [KH92]. Plus récemment, les EDS rétrogrades réfléchiées avec des sauts ont été introduites et étudiées par plusieurs auteurs (voir [HO03], [EHO05], [HH06], [Ess08], [CM08] et [QS14]).

1.1.6 Méthodes numériques

Plusieurs méthodes numériques ont été introduites dans la littérature pour résoudre l'EDS. Nous allons ici mentionner deux schémas importants. Le premier est le schéma d'Euler-Maruyama [Mar55] qui donnera un ordre fort 1/2 et le second est le schéma de Milstein [Mil94] qui a un ordre 1 pour la convergence forte. Commençons par donner une définition de la convergence forte. Soit n un entier positif non nul, on appelle $h = T/n$ le pas de temps et $t_k = kh$ les temps intermédiaires, où $k = 0, \dots, n$.

Définition 1.1.3. *Soit \tilde{X} une approximation en temps discret de la solution X de l'EDS (1.1.1). On dit que \tilde{X} converge fortement vers X en temps T avec un ordre $\gamma > 0$ s'il existe une constante $C > 0$ telle que*

$$\mathbb{E}[|\tilde{X}_T - X_T|] \leq Ch^\gamma.$$

Schéma d'Euler

La méthode numérique la plus simple pour approcher la solution d'une EDS est le schéma stochastique d'Euler (appelé schéma d'Euler Maruyama), qui utilise seulement les deux premiers termes de la décomposition de Taylor. Ce schéma est défini par :

$$\tilde{X}_{t_{i+1}} = \tilde{X}_{t_i} + b(t_i, \tilde{X}_{t_i})(t_{i+1} - t_i) + \sigma(t_i, \tilde{X}_{t_i})(B_{t_{i+1}} - B_{t_i}), \quad 0 \leq i \leq n - 1$$

où la valeur initiale $\tilde{X}_{t_0} = X_0$. L'approximation d'Euler-Maruyama converge fortement avec un ordre $1/2$.

Schéma de Milstein

Un autre méthode d'approximation numérique pour l'EDS est la méthode de Milstein. Si on prend la série de Taylor stochastique jusqu'aux termes du second ordre, on obtient la méthode de Milstein comme suit :

$$\begin{aligned} \tilde{X}_{t_{i+1}} = & \tilde{X}_{t_i} + b(t_i, \tilde{X}_{t_i})(t_{i+1} - t_i) + \sigma(t_i, \tilde{X}_{t_i})(B_{t_{i+1}} - B_{t_i}) \\ & + \frac{1}{2}\sigma(t_i, \tilde{X}_{t_i})\sigma'(t_i, \tilde{X}_{t_i})[(B_{t_{i+1}} - B_{t_i})^2 - (t_{i+1} - t_i)], \quad 0 \leq i \leq n - 1 \end{aligned}$$

où la valeur initiale $\tilde{X}_{t_0} = X_0$ et g' est la dérivée de g . L'approximation de Milstein converge fortement avec un ordre 1.

Il existe également de nombreux autres schémas numériques pour résoudre numériquement l'EDS : le schéma de type Runge-Kutta et le schéma de développement Itô-Taylor (voir [KPS12], [KP13]), des méthodes basées sur l'approximation gaussienne (voir [May82], [Jaz07], [Sar07], [SS13]) et des méthodes basées sur le domaine de Fourier et la fonction de base (voir [VT68], [PP02]).

En se basant sur le schéma d'Euler, des approximations numériques pour les EDS réfléchies ont été largement étudiées (voir [Slo94], [Slo01], [Lep95], [Pet95], [Pet97]). Plus récemment, des approximations numériques pour les EDS rétrogrades réfléchies avec des sauts ont été encore traitées (voir [DL16a], [DL16b]).

1.2 Préliminaires sur les EDSR

1.2.1 Notations et définitions

On se place toujours sur un espace de probabilité complet $(\Omega, \mathcal{F}, \mathbb{P})$ sur lequel on définit un mouvement Brownien standard B d -dimensionnel, tel que $\{\mathcal{F}_t\}_{t \in [0, T]}$ est la filtration naturelle de B .

Nous allons utiliser les espaces suivants de variables aléatoires ou processus :

- $L_T^2(\mathbb{R}^d)$ est l'espace formé par les variables aléatoires \mathcal{F}_T -mesurables $X : \Omega \rightarrow \mathbb{R}^d$, tel que $\|X\|_{L_T^2}^2 := \mathbb{E}[|X|^2] < \infty$.
- $S_T^2(\mathbb{R}^d)$ est l'espace formé par les processus continus et adaptés, $\phi : \Omega \times [0, T] \rightarrow \mathbb{R}^d$, tel que $\|\phi\|_{S_T^2}^2 := \mathbb{E}[\sup_{t \in [0, T]} |\phi_t|^2] < \infty$.
- $H_T^2(\mathbb{R}^d)$ est l'espace formé par tous les processus prévisibles $\phi : \Omega \times [0, T] \rightarrow \mathbb{R}^d$, tel que $\|\phi\|_{H_T^2}^2 := \mathbb{E} \int_0^T |\phi_t|^2 dt < \infty$.
- $L^2(0, T)$ est l'espace des fonctions de carré intégrable sur $[0, T]$.

Notons que dans la suite on écrira $L_T^2 = L_T^2(\mathbb{R}^d)$, $S_T^2 = S_T^2(\mathbb{R}^d)$ et $H_T^2 = H_T^2(\mathbb{R}^d)$.

Considérons l'équation différentielle stochastique rétrograde (EDSR en abrégé) suivante

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds - \int_t^T Z_s dB_s, \quad 0 \leq t \leq T, \quad (1.2.1)$$

où

- la condition terminale $\xi : \Omega \rightarrow \mathbb{R}^n$ est une variable aléatoire \mathcal{F}_T -mesurable.
- le générateur $f : [0, T] \times \mathbb{R}^n \times \mathbb{R}^{n \times d} \rightarrow \mathbb{R}^n$ est $\mathcal{P} \otimes \mathcal{B}^n \otimes \mathcal{B}^{n \times d}$ -mesurable où \mathcal{P} est la tribu des événements prévisibles.

Définition 1.2.1. Une solution de l'EDSR (1.2.1) est un couple de processus $(Y_t, Z_t)_{t \in [0, T]}$ à valeurs dans $\mathbb{R}^n \times \mathbb{R}^{n \times d}$ tel que

- $\{Y_t\}_{t \in [0, T]}$ est continu et adapté, $\{Z_t\}_{t \in [0, T]}$ est prévisible,
- \mathbb{P} -p.s., $t \mapsto Z_t$ appartient à $L^2(0, T)$, $t \mapsto f(t, Y_t, Z_t)$ appartient à $L^1(0, T)$,
- \mathbb{P} -p.s.,

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds - \int_t^T Z_s dB_s, \quad 0 \leq t \leq T.$$

1.2.2 Motivation

Considérons l'équation différentielle suivante :

$$\begin{cases} dY_t = 0, & t \in [0, T], \\ Y_T = \xi. \end{cases}$$

Cette équation possède une solution unique $Y_t = \xi$, qui n'est pas adaptée à la filtration $\{\mathcal{F}_t\}_{t \in [0, T]}$ si ξ n'est pas déterministe. La meilleure façon de garantir l'adaptabilité de la solution est de prendre $Y_t = \mathbb{E}[\xi | \mathcal{F}_t]$. Notons que $Y_T = \xi$, puisque ξ est \mathcal{F}_T -adapté. Le

théorème de représentation des martingales nous assure l'existence d'un processus adapté $Z \in H_T^2$ tel que

$$Y_t = \mathbb{E}[\xi | \mathcal{F}_t] = \mathbb{E}[\xi] + \int_0^t Z_s dB_s,$$

ce qui conduit en définitive à l'équation

$$\begin{cases} dY_t = Z_t dB_t, & t \in [0, T], \\ Y_T = \xi, \end{cases} \quad (1.2.2)$$

dont la solution est donnée par le couple (Y, Z) . La seconde inconnue Z joue le rôle d'un contrôle, qui permet au processus Y d'être adapté.

1.2.3 Existence et unicité dans L^2 dans le cas de générateurs lipschitziens

L'existence et l'unicité d'une solution de l'EDSR a été présentée et étudiée dans plusieurs théorèmes et sous différentes conditions. Le premier résultat a été établi par Bismut [Bis73a], dans le cas où le générateur f est linéaire. Ensuite, ce résultat a été généralisé par Pardoux et Peng [PP90], dans le cas où le générateur f est non linéaire, mais vérifie la condition de Lipschitz par rapport aux deux variables y et z . Dans cette section, nous rappelons le théorème d'existence et d'unicité de Pardoux et Peng.

Hypothèse 1.2.1. *Il existe une constante k telle que \mathbb{P} -p.s.,*

— *condition de Lipschitz en (y, z) :*

$$\forall s, y, z, y', z', \quad |f(s, y, z) - f(s, y', z')| \leq k(|y - y'| + \|z - z'\|).$$

— *condition d'intégrabilité en (ξ, f) :*

$$\mathbb{E} \left[|\xi|^2 + \int_0^T |f(s, 0, 0)|^2 ds \right] < \infty.$$

Théorème 1.2.1 (Pardoux-Peng 1990). *Supposons que (ξ, f) vérifie la condition d'intégrabilité et $f : [0, T] \times \mathbb{R}^n \times \mathbb{R}^{n \times d} \rightarrow \mathbb{R}^n$ le générateur $\mathcal{P} \otimes \mathcal{B}^n \otimes \mathcal{B}^{n \times d}$ -mesurable vérifie la condition de Lipschitz en (Y, Z) . Alors, il existe un couple unique $(Y, Z) \in S_T^2(\mathbb{R}^n) \times H_T^2(\mathbb{R}^{n \times d})$ qui est la solution de l'EDSR*

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds - \int_t^T Z_s dB_s, \quad 0 \leq t \leq T.$$

En 1997, El Karoui, Peng et Quenez [EKPQ97] ont prouvé le théorème cité précédemment en se basant sur l'argument du point fixe. Pour utiliser cet argument, l'idée fondamentale est de démontrer la stabilité des solutions par rapport à leur générateur et leur condition terminale. En particulier, cette estimation a priori garantit l'unicité de la solution de l'EDSR.

Proposition 1.2.1 (Stabilité). *Supposons que (ξ, f) et $(\bar{\xi}, \bar{f})$ vérifient la condition d'intégrabilité. Soient (Y, Z) et (\bar{Y}, \bar{Z}) les solutions de deux EDSR associées à (ξ, f) et $(\bar{\xi}, \bar{f})$ respectivement. Supposons que f est Lipschitzienne. Alors, il existe une constante C telle que*

$$\mathbb{E} \left(\sup_{0 \leq s \leq T} |Y_s - \bar{Y}_s|^2 \right) + \mathbb{E} \left(\int_0^T \|Z_s - \bar{Z}_s\|^2 ds \right) \leq C \mathbb{E} \left[|\xi - \bar{\xi}|^2 + \int_0^T |f(s, \bar{Y}_s, \bar{Z}_s) - \bar{f}(s, \bar{Y}_s, \bar{Z}_s)|^2 ds \right].$$

1.2.4 Théorème de comparaison

EDSR linéaires

Dans le cas d'une EDSR linéaire, on peut donner une formule explicite de la composante Y de la solution (voir [Pen90]). Fixons $n = 1$.

Proposition 1.2.2 (Peng 1990). *Soit (β, μ) un couple de processus mesurables bornés à valeurs dans $\mathbb{R} \times \mathbb{R}^d$, $\varphi \in H_T^2$ et $\xi \in L_T^2$. Considérons l'EDSR linéaire suivante :*

$$Y_t = \xi + \int_t^T (\varphi_s + \beta_s Y_s + \mu_s Z_s) ds - \int_t^T Z_s dB_s, \quad 0 \leq t \leq T. \quad (1.2.3)$$

— *L'EDSR linéaire (1.2.3) admet une unique solution $(Y, Z) \in S_T^2(\mathbb{R}) \times H_T^2(\mathbb{R}^d)$, où Y est donné par la formule explicite suivante*

$$Y_t = \Gamma_t^{-1} \mathbb{E} \left[\xi \Gamma_T + \int_t^T \Gamma_s \varphi_s ds \middle| \mathcal{F}_t \right], \quad \mathbb{P} - p.s.,$$

où

$$\Gamma_t = \exp \left(\int_0^t \mu_s dB_s - \frac{1}{2} \int_0^t |\mu_s|^2 ds + \int_0^t \beta_s ds \right).$$

— *Si φ et ξ sont positifs, alors le processus $(Y_t)_{t \leq T}$ est encore positif. Si en plus $Y_0 = 0$ alors pour tout $s \geq t$, $Y_s = 0$, $\xi = 0$ et $\varphi_s = 0$, $d\mathbb{P} \otimes ds - p.s.$*

Théorème de comparaison

La proposition précédente nous permet d'obtenir le théorème de comparaison suivant (voir S. Peng [Pen92]). On considère le cas des EDSR uni-dimensionnelles ($n=1$), où on peut établir le résultat de comparaison, qui est un outil essentiel dans l'étude des EDSR quadratiques.

Théorème 1.2.2 (Théorème de comparaison). *Supposons que (ξ, f) et $(\bar{\xi}, \bar{f})$ vérifient la condition d'intégrabilité. Soient (Y, Z) et (\bar{Y}, \bar{Z}) les solutions de deux EDSR associées à (ξ, f) et $(\bar{\xi}, \bar{f})$ respectivement. Supposons que f est Lipschitzienne. Si*

$$\xi \leq \bar{\xi}, \quad \mathbb{P} - p.s.,$$

et

$$f(t, \bar{Y}_t, \bar{Z}_t) \leq \bar{f}(t, \bar{Y}_t, \bar{Z}_t), \quad dt \otimes d\mathbb{P} - p.s.,$$

alors pour tout $t \in [0, T]$,

$$Y_t \leq \bar{Y}_t, \quad \mathbb{P} - p.s..$$

En outre, cette comparaison est stricte : si en plus $Y_0 = \bar{Y}_0$, alors $\xi = \bar{\xi}$, $f(t, \bar{Y}_t, \bar{Z}_t) = \bar{f}(t, \bar{Y}_t, \bar{Z}_t)$ et $Y_t = \bar{Y}_t$, $\forall t \in [0, T]$ $\mathbb{P} - p.s.$.

En 2006, Y. Hu et S. Peng ont étendu ce résultat au cas multi-dimensionnel (voir [HP06]).

1.2.5 Différentes généralisations

Après les résultats d'existence et d'unicité du Théorème 1.2.1, beaucoup d'auteurs ont travaillé pour affaiblir les hypothèses de ce théorème.

Dans le cas unidimensionnel où Y est un processus réel, les hypothèses peuvent être affaiblies à l'aide du théorème de comparaison. D'une part, Lepeltier et San Martin [LSM97] ont étudié l'existence des solutions minimale et maximale (sans l'unicité) dans le cas où f est continu en (y, z) et à croissance linéaire. D'autre part, l'existence et l'unicité d'une solution de l'EDSR quadratique (c-à-d f à croissance quadratique par rapport à z) a été montrée par Kobylanski [Kob00] pour des conditions terminales bornées et par Briand et Hu ([BH06],[BH08]) pour des conditions terminales non bornées. En ce qui concerne le cas multi-dimensionnel où Y est à valeurs dans \mathbb{R}^n , Briand et al. [BDH⁺03] ont généralisé le résultat d'existence et d'unicité en remplaçant la condition de Lipschitz en z par une condition de monotonie en z et avec une condition terminale appartenant à L_T^p pour $p > 1$ (le cas où $p = 1$ est encore vérifié dans le même article). Dans cette section, on va citer ces travaux en détail.

Générateur continu

Fixons $n = 1$. Nous allons maintenant présenter le résultat de Lepeltier et San Martin [LSM97] lorsque le générateur f est seulement continu et à croissance linéaire.

Hypothèse 1.2.2. *Il existe une constante k telle que \mathbb{P} -p.s.,*
— *condition de croissance linéaire en (y, z) :*

$$\forall s, y, z, \quad |f(s, y, z)| \leq |f(s, 0, 0)| + k|y| + k\|z\|.$$

Théorème 1.2.3 (Existence). *Supposons que (ξ, f) vérifie la condition d'intégrabilité avec f une fonction continue en (y, z) et à croissance linéaire. Alors il existe une solution minimale $(\underline{Y}, \underline{Z})$ et une solution maximale (\bar{Y}, \bar{Z}) de l'EDSR (1.2.1) c-à-d pour tout autre solution (Y, Z) , on a*

$$\underline{Y} \leq Y \leq \bar{Y}, \quad \mathbb{P} - p.s..$$

Remarque 1.2.1. *L'unicité de la solution est traitée dans d'autres articles pour des conditions plus difficiles. C'est le cas où il existe une fonction croissante concave κ telle que $\kappa(0) = 0$, $\kappa(x) > 0$ si $x \neq 0$, $\int_{0^+}^{\infty} (x/\kappa(x))dx = \infty$ et*

$$|f(s, y, z) - f(s, y', z')|^2 \leq \kappa(|y - y'|^2 + \|z - z'\|^2),$$

(voir [Mao95]). Pour des conditions différentes, voir [LR02].

EDSR quadratique

Commençons par le cas où la condition terminale ξ est bornée et le générateur f est à croissance linéaire en y et quadratique en z . Le théorème suivant est dû à Kobylanski (voir [Kob00]).

Hypothèse 1.2.3. *Il existe une constante k telle que \mathbb{P} -p.s.,*

- *f est continue par rapport à (y, z) .*
- *f est à croissance linéaire en y et quadratique en z :*

$$\forall s, y, z, \quad |f(s, y, z)| \leq k(1 + |y| + \|z\|^2).$$

- *la condition terminale ξ est bornée.*

Théorème 1.2.4 (Kobylanski 2000). *Supposons que (ξ, f) vérifie la condition d'intégrabilité et satisfait l'Hypothèse 1.2.3. Alors il existe une solution maximale (Y, Z) de l'EDSR (1.2.1).*

En 1998, Lepeltier et San Martin [LSM98] ont étendu les résultats de Kobylanski à un cadre plus général. Cette généralisation concerne le cas où

$$|f(s, y, z)| \leq l(y) + \|z\|^2,$$

avec l une fonction strictement positive telle que

$$\int_0^\infty \frac{dx}{l(x)} = \int_{-\infty}^0 \frac{dx}{l(x)} = \infty.$$

Un peu plus tard, Briand et Hu ont généralisé le résultat d'existence (voir [BH06]) des EDSR quadratiques lorsque la condition terminale ξ est non bornée.

Hypothèse 1.2.4. *Il existe une constante k telle que \mathbb{P} -p.s.,*

- *f est continu par rapport à (y, z) .*
- *f est à croissance linéaire en y et quadratique en z : il existe $\beta \geq 0$, $\gamma > 0$ et $\alpha \geq \beta/\gamma$ tel que*

$$\forall s, y, z, \quad |f(s, y, z)| \leq \alpha + \beta|y| + \frac{\gamma}{2}\|z\|^2.$$

- *il existe $\lambda > \gamma e^{\beta T}$ tel que $\mathbb{E}[e^{\lambda|\xi|}] < \infty$*

Théorème 1.2.5 (Briand-Hu 2006). *Supposons que (ξ, f) vérifie la condition d'intégrabilité et satisfait l'Hypothèse 1.2.4. Alors il existe une solution (Y, Z) de l'EDSR (1.2.1). En plus,*

$$-\frac{1}{\gamma} \ln \mathbb{E}[\phi_t(-\xi)|\mathcal{F}_t] \leq Y_t \leq \frac{1}{\gamma} \ln \mathbb{E}[\phi_t(\xi)|\mathcal{F}_t],$$

où ϕ est donnée dans [BH06].

On se réfère à [BH08] pour l'unicité de la solution lorsque le générateur est convexe par rapport à z .

Générateur monotone

Pour étudier les EDSR multi-dimensionnelles, le théorème de comparaison n'existe pas, alors pour cette difficulté, on va remplacer l'hypothèse de Lipschitz par une condition de monotonie en y . Cette idée est due à Darling et Pardoux [DP97]. Ils ont trouvé l'existence et l'unicité d'une solution de l'EDSR sous une condition de monotonie en y et de croissance linéaire en (y, z) . En 1999, Pardoux [Par99] a généralisé un peu ce résultat en prenant une hypothèse plus faible sur la croissance de f en y :

Hypothèse 1.2.5. *Il existe une fonction $\phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ continue et croissante et une constante k telles que \mathbb{P} -p.s.,*

- *f est continue par rapport à y .*
- *condition de Lipschitz en z :*

$$\forall s, y, z, z', \quad |f(s, y, z) - f(s, y, z')| \leq k \|z - z'\|, \quad \mathbb{P} - p.s..$$

- *condition de monotonie en y :*

$$\forall s, y, z, y', \quad (y - y')(f(s, y, z) - f(s, y', z)) \leq k |y - y'|^2.$$

- $\forall s, y, \quad |f(s, y, 0)| \leq |f(s, 0, 0)| + \phi(|y|).$

Les résultats d'existence et d'unicité d'une solution de l'EDSR reposent généralement sur l'obtention des estimations a priori qui sont nécessaires à l'utilisation du lemme de Gronwall ou du théorème du point fixe.

Proposition 1.2.3 (Stabilité). *Supposons que (ξ, f) et $(\bar{\xi}, \bar{f})$ vérifient la condition d'intégrabilité et satisfont l'Hypothèse 1.2.5. Soient (Y, Z) et (\bar{Y}, \bar{Z}) les solutions de deux EDSR associées à (ξ, f) et $(\bar{\xi}, \bar{f})$ respectivement. Alors, il existe une constante C telle que*

$$\mathbb{E} \left(\sup_{0 \leq s \leq T} |Y_s - \bar{Y}_s|^2 \right) + \mathbb{E} \left(\int_0^T \|Z_s - \bar{Z}_s\|^2 ds \right) \leq C \mathbb{E} \left[|\xi - \bar{\xi}|^2 + \int_0^T |f(s, \bar{Y}_s, \bar{Z}_s) - \bar{f}(s, \bar{Y}_s, \bar{Z}_s)|^2 ds \right].$$

Théorème 1.2.6 (Pardoux 1999). *Supposons que (ξ, f) vérifie la condition d'intégrabilité et satisfait l'Hypothèse 1.2.5. Alors il existe une solution unique (Y, Z) de l'EDSR (1.2.1).*

Ce résultat a été prouvé par Briand et Carmona [BC00] pour une fonction ϕ polynômiale.

EDSR dans L^p

Remarquons que la condition d'intégrabilité au carré sur les coefficients ξ et $f(\cdot, 0, 0)$ n'est pas toujours satisfaite, pour cette raison, on va améliorer notre étude sur la solution de l'EDSR en s'appuyant sur une condition d'intégrabilité dans L^p où $p < 2$. En 1997, El Karoui, Peng et Quenez [EKPQ97] ont prouvé l'existence et l'unicité d'une solution lorsque le générateur f est Lipschitz et les coefficients ξ et $f(\cdot, 0, 0)$ vérifient la condition d'intégrabilité dans L^p où $p \in (1, 2)$ c-à-d

$$\mathbb{E} \left[|\xi|^p + \left(\int_0^T |f(s, 0, 0)|^2 ds \right)^{p/2} \right] < \infty.$$

Dans [BDH⁺03], Briand et al. ont étendu ce résultat dans le cas d'une monotonie de f en y pour $p \in [1, \infty)$.

Hypothèse 1.2.6. *Il existe une constante k telle que \mathbb{P} -p.s.,*

- *f est continu par rapport à (t, z) .*
- *condition de Lipschitz en z :*

$$\forall s, y, z, z', \quad |f(s, y, z) - f(s, y, z')| \leq k \|z - z'\|, \quad \mathbb{P} - p.s..$$

- *condition de monotonie en y :*

$$\forall s, y, z, y', \quad (y - y')(f(s, y, z) - f(s, y', z)) \leq k |y - y'|^2.$$

- $\forall r > 0, \phi_r(s) := \sup_{|y| \leq r} |f(t, y, 0) - f(s, 0, 0)| \in L_T^1.$

Théorème 1.2.7 (Briand et al. 2003). *Pour $p > 1$, supposons que (ξ, f) vérifie la condition d'intégrabilité dans L^p et satisfait l'Hypothèse 1.2.6. Alors il existe une solution unique (Y, Z) de l'EDSR (1.2.1).*

Le cas où $p = 1$ est traité seul dans [BDH⁺03] à cause de quelques difficultés supplémentaires.

1.2.6 Lien avec les EDPs

EDSR markoviennes

Une EDSR markovienne est un type d'EDSR dans lequel le générateur et la condition terminale dépendent d'un processus qui est solution d'une EDS. Considérons l'EDS suivante :

$$X_s^{t,x} = x + \int_t^s b(r, X_r^{t,x}) dr + \int_t^s \sigma(r, X_r^{t,x}) dB_r, \quad s \in [t, T], \quad (1.2.4)$$

où $b : [0, T] \times \mathbb{R}^k \rightarrow \mathbb{R}^k, \sigma : [0, T] \times \mathbb{R}^k \rightarrow \mathbb{R}^{k \times d}$ et $x \in \mathbb{R}^k$.

Supposons que b et σ satisfont l'Hypothèse 1.1.1. Alors d'après le Théorème 1.1.1, il existe une solution unique $X_s^{t,x}$ de l'EDS (1.2.4). L'EDSR markovienne s'écrit alors :

$$Y_s^{t,x} = g(X_T^{t,x}) + \int_s^T f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) dr + \int_s^T Z_r^{t,x} dB_r, \quad s \in [t, T], \quad (1.2.5)$$

où $g : \mathbb{R}^k \rightarrow \mathbb{R}^n$ et $f : [0, T] \times \mathbb{R}^k \times \mathbb{R}^n \times \mathbb{R}^{n \times d} \rightarrow \mathbb{R}^n$.

Hypothèse 1.2.7. *Il existe une constante k telle que \mathbb{P} -p.s.,*

- *condition de Lipschitz de f :*

$$\forall t, x, x', y, y', z, z', \quad |f(t, x, y, z) - f(t, x', y', z')| \leq k(|x - x'| + |y - y'| + \|z - z'\|), \quad \mathbb{P} - p.s..$$

- *croissance linéaire de g :*

$$\forall x, \quad |g(x)| \leq k(1 + |x|).$$

D'après le Théorème 1.2.1 et sous l'Hypothèse 1.2.7, il existe une solution unique $(Y_s^{t,x}, Z_s^{t,x})$ à l'EDSR (1.2.5).

Remarquons que les coefficients de l'EDS précédente b et σ ne dépendent pas de $(Y_s^{t,x}, Z_s^{t,x})_{s \in [t, T]}$. Dans ce cas le système formé par les équations (1.2.4) et (1.2.5) est appelé système découplé. Plus généralement, lorsque ces coefficients dépendent de $(Y_s^{t,x}, Z_s^{t,x})_{s \in [t, T]}$, nous parlons alors d'un système couplé.

Rappelons quelques propriétés de régularité pour la solution de l'EDSR markovienne trouvées par [EKPQ97].

Proposition 1.2.4 (Propriété de régularité). *Pour tout $t, t' \in [0, T]$ et $x, x' \in \mathbb{R}^k$, il existe une constante C telle que*

$$\mathbb{E} \left(\sup_{0 \leq s \leq T} |Y_s^{t,x}|^2 \right) + \mathbb{E} \left(\int_0^T \|Z_s^{t,x}\|^2 ds \right) \leq C(1 + |x|^2),$$

$$\mathbb{E} \left(\sup_{0 \leq s \leq T} |Y_s^{t,x} - Y_s^{t',x'}|^2 \right) + \mathbb{E} \left(\int_0^T \|Z_s^{t,x} - Z_s^{t',x'}\|^2 ds \right) \leq C(1 + |x|^2)(|x - x'|^2 + |t - t'|^2).$$

Propriété de Markov

La propriété de Markov de la solution de l'EDSR markovienne nous permet d'étudier la relation entre ce type d'EDSR et les équations différentielles partielles (EDPs en abrégé). Notons que $\mathcal{F}_s^t = \sigma(B_u - B_t, t \leq u \leq s)$.

Proposition 1.2.5. *Soit $(t, x) \in [0, T] \times \mathbb{R}^k$. $(Y_s^{t,x}, Z_s^{t,x})_{s \in [t, T]}$ est une solution adaptée à la filtration \mathcal{F}_s^t . En plus, $Y_t^{t,x}$ est déterministe, et par suite on peut définir une fonction*

$$u(t, x) := Y_t^{t,x}.$$

Proposition 1.2.6 (Propriété de régularité). *Pour tout $t, t' \in [0, T]$ et $x, x' \in \mathbb{R}^k$, il existe une constante C telle que :*

$$|u(t, x)| \leq C(1 + |x|),$$

$$|u(t, x) - u(t', x')| \leq C(|x - x'| + |t - t'|^{1/2}(1 + |x|)).$$

Proposition 1.2.7 (Propriété de Markov). *Soient $t \in [0, T]$ et $x \in L_T^2$ une variable aléatoire. Alors, pour tout $s \in [t, T]$,*

$$Y_s^{t,x} = u(s, X_s^{t,x}), \quad \mathbb{P} - p.s..$$

Formule de Feynman-Kac

Considérons l'EDP parabolique semi-linéaire suivante :

$$\begin{cases} \partial_t u(t, x) + \mathcal{L}u(t, x) + f(t, x, u(t, x), \nabla u(t, x)) = 0, & (t, x) \in [0, T[\times \mathbb{R}^k, \\ u(t, x) = g(x), \end{cases} \quad (1.2.6)$$

où $g : \mathbb{R}^k \rightarrow \mathbb{R}$, $f : [0, T] \times \mathbb{R}^k \times \mathbb{R} \times \mathbb{R}^{1 \times d} \rightarrow \mathbb{R}$ et \mathcal{L} est l'opérateur différentiel linéaire suivant :

$$\mathcal{L}u(t, x) = b(t, x)\nabla u(t, x) + \frac{1}{2}\text{trace}(\sigma\sigma^*(t, x)\nabla^2 u(t, x)).$$

En 1992, Pardoux et Peng [PP92] ont étudié la relation entre l'EDSR (1.2.5) et l'EDP (1.2.6).

Théorème 1.2.8 (Pardoux-Peng 1992).

— Si $u \in C^{1,2}([0, T] \times \mathbb{R}^k)$ résout l'équation (1.2.6), alors pour tout $(t, x) \in [0, T] \times \mathbb{R}^k$,

$$Y_t^{t,x} := u(t, x)$$

où $(Y_s^{t,x}, Z_s^{t,x})_{s \in [t, T]}$ est la solution unique de l'EDSR markovienne (1.2.5). En plus, $(Y_s^{t,x}, Z_s^{t,x}) := (u(s, X_s^{t,x}), (\nabla u \cdot \sigma)(s, X_s^{t,x}))$.

— Sous de bonnes hypothèses sur les coefficients de l'EDSR (1.2.5) (f et g sont assez réguliers), alors $u(t, x) := Y_t^{t,x}$ est une solution de l'EDP (1.2.6).

Si f est linéaire, alors l'EDSR (1.2.5) admet une solution explicite, et par suite on obtient l'expression explicite de $u(t, x) := Y_t^{t,x}$ designé par la formule de Feynman-Kac.

Corollaire 1.2.1 (Formule de Feynman-Kac). Soit $f(t, x, y, z) = c(t, x)y + h(t, x)$, où c et h sont deux fonctions continues et bornées. Si $u \in C^{1,2}([0, T] \times \mathbb{R}^k)$ est une solution de l'équation (1.2.6), alors pour tout $(t, x) \in [0, T] \times \mathbb{R}^k$,

$$u(t, x) = \mathbb{E} \left[g(X_T^{t,x}) \exp \left(\int_t^T c(r, X_r^{t,x}) dr \right) + \int_t^T h(r, X_r^{t,x}) \exp \left(\int_t^r c(s, X_s^{t,x}) ds \right) dr \right].$$

D'autre part, si f et g sont seulement lipschitziennes, alors l'EDP (1.2.6) admet une solution faible au sens viscosité qui n'est pas une solution classique. En fait, Pardoux et Peng ([Pen91], [PP92]) ont été les premiers à trouver une représentation de la solution de viscosité de l'équation (1.2.6). Plusieurs autres articles ont étudié cette interprétation probabiliste dans des cadres différents (voir [BBP97], [EKKP⁺97], [Par99], [Kob00], [HH06], etc.).

Definition 1.2.1. Soit u une fonction continue sur $[0, T] \times \mathbb{R}^k$ qui satisfait la condition $u(T, x) = g(x)$, $x \in \mathbb{R}^k$. u est appelée une sous-solution (resp. sur-solution) de viscosité de l'EDP (1.2.6), si pour toute fonction $\varphi \in C^{1,2}([0, T] \times \mathbb{R}^k)$, on a,

$$\partial_t \varphi(t, x) + \mathcal{L}\varphi(t, x) + f(t, x, \varphi(t, x), \nabla \varphi(t, x)) \geq 0, \quad (\text{resp. } \leq 0),$$

pour tout $(t, x) \in [0, T] \times \mathbb{R}^k$ point de maximum (resp. minimum) local de $u - \varphi$.

Par conséquent, u est appelée une solution de viscosité de l'EDP (1.2.6) si elle est à la fois sous-solution et sur-solution de viscosité.

Théorème 1.2.9 (Solution de viscosité). Sous des hypothèses plus faibles sur f et g (hypothèse de Lipschitz seulement), alors la fonction $u(t, x) := Y_t^{t,x}$ est une solution de viscosité l'EDP (1.2.6).

1.2.7 EDSR de type McKean-Vlasov

Dans le Chapitre 3, nous avons étudié les EDSR de type McKean-Vlasov qui sont définies comme suit :

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s, [Y_s], [Z_s]) ds - \int_t^T Z_s \cdot dB_s, \quad 0 \leq t \leq T, \quad (1.2.7)$$

où $[\theta]$ est la loi d'une variable aléatoire θ , $f : [0, T] \times \mathbb{R} \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}) \times \mathcal{P}_2(\mathbb{R}^d) \rightarrow \mathbb{R}$ et $\mathcal{P}_2(\mathbb{R}^d)$ est l'ensemble des mesures de probabilité sur \mathbb{R}^d , de second moment fini, muni de la métrique \mathcal{W}_2 définie par

$$\mathcal{W}_2(\mu, \nu) := \inf_{\pi \in \Pi(\mu, \nu)} \left(\int_{\mathbb{R}^d \times \mathbb{R}^d} |x - x'|^2 d\pi(x, x') \right)^{1/2},$$

où $\Pi(\mu, \nu)$ est l'ensemble des mesures de probabilité sur $\mathbb{R}^d \times \mathbb{R}^d$ ayant μ et ν comme marginales.

Ce type d'EDSR a été introduit par Buckdahn, Djehiche, Li et Peng (voir [BDLP09] et [BLP09]) dans un cadre plus particulier : dans [BDLP09], les auteurs étudient le problème du champ moyen dans un cadre markovien et prouvent l'existence et l'unicité de la solution lorsque la condition terminale est de type $\xi = \mathbb{E}[g(x, X_T)]|_{x=X_T}$ où X est un processus stochastique adapté, et le générateur est défini par $\mathbb{E}[f(s, \lambda, \Lambda_s)]|_{\lambda=\Lambda_s}$ où $\Lambda_s = (X_s, Y_s, Z_s)$. Dans [BLP09], les auteurs étendent le résultat d'existence et d'unicité à un cadre plus général et relie l'EDSR de type champ moyen à une équation différentielle partielle (EDP) non locale. Dans le Chapitre 3, nous avons étudié le résultat d'existence et d'unicité pour les EDSR de type McKean-Vlasov (1.2.7). En outre, Carmona et Delarue ont étudié les EDS progressives rétrogrades de type McKean-Vlasov qui sont très utiles pour l'étude des jeux à champ moyen (voir [CD18b] et [CD18c]).

1.2.8 Méthodes numériques

Depuis 1994, des nombreux travaux sur la simulation des EDSR ont été traités. Commençons par citer ce qui concerne le problème de la discrétisation temporelle des EDSR.

Dans [MPY94], Ma, Protter et Yong ont proposé un schéma numérique appelé schéma à quatre étapes, qui permet de calculer la solution d'une EDS progressive rétrograde couplée. En fait, ils résolvent l'EDP quasi linéaire par une approximation par différence finie. Cependant, un tel algorithme ne peut pas fonctionner correctement en grande dimension. De plus, ils supposent des hypothèses restrictives sur les coefficients. Bally et Chevance ont proposé une discrétisation temporelle de l'EDSR (1.2.1) qui évite la résolution d'une EDP.

En 1997, Chevance propose dans sa thèse [Che97c] et dans son article [Che97a] un schéma de discrétisation pour les EDSR markoviennes (1.2.5), dont le générateur est indépendant de Z . L'auteur introduit une discrétisation temporelle et spatiale de l'EDSR où la dernière est basée sur une méthode de quantification aléatoire. L'algorithme obtenu

dans cette approximation est implémentable. Sous des hypothèses de régularité forte sur g et f , l'auteur a trouvé une borne pour l'erreur sur Y , tandis que l'erreur sur Z n'est pas observée. Indépendamment du travail [Che97c], Coquet, Mackevičius et Mémin ont prouvé la convergence de l'approximation de Y en utilisant l'outil de convergence des filtrations (voir [CMM99]).

La méthode développée par Bally [Bal97], s'applique au cas où la fonction f dépend à la fois des variables y et z . La discrétisation temporelle est effectuée sur les temps de sauts d'un processus de Poisson. Dans cette méthode, une vitesse de convergence par rapport à l'intensité du processus de Poisson a été démontrée, sans la nécessité d'utiliser la régularité inconnue de Z . Néanmoins, pour obtenir un schéma numérique implémentable, il faut calculer des intégrales de grande dimension, ce qui n'est pas effectué dans cet article.

En 2001, Briand, Delyon et Mémin [BDM01] ont généralisé le schéma déjà proposé dans [Che97a] au cas où f dépend de z , en se basant sur le théorème de Donsker. Cette méthode consiste à remplacer le mouvement Brownien par une marche aléatoire et ainsi l'intégrale stochastique est également discrétisée. Cela conduit à une EDSR à temps discret. Les auteurs ont prouvé la convergence de leur schéma, mais aucune vitesse de convergence n'est établie. Ensuite, ils ont étendu ce résultat à un cadre plus général (voir [BDM02]), où la condition terminale dépend de la trajectoire, tandis que la vitesse de convergence est restée une question ouverte. Plus récemment, Geiss, Labart et Luoto ([GLL18a], [GLL18b]) ont donné une première réponse sur la vitesse de convergence de l'approximation déjà introduit dans [BDM01]. Ils ont trouvé l'ordre de l'erreur entre la solution approchée et la solution exacte dans l'espace L^2 . Un peu plus tard, Briand, Geiss C., Geiss S. et Labart [BGGL19] ont amélioré ce résultat en obtenant un ordre de convergence plus élevé lorsque l'erreur est mesurée avec la distance de Wasserstein.

Pour traiter cette question, une approche basée sur l'équation de programmation dynamique a été introduite par Bouchard et Touzi dans [BT04] et Zhang dans [Zha04]. Cette équation est définie sous une forme explicite :

$$\begin{cases} Y_{t_n}^n = \xi, \\ Z_{t_i}^n = (t_{i+1} - t_i)^{-1} \mathbb{E}[Y_{t_{i+1}}^n (W_{t_{i+1}} - W_{t_i}) | \mathcal{F}_{t_i}], \\ Y_{t_i}^n = \mathbb{E}[Y_{t_{i+1}}^n + (t_{i+1} - t_i) f(t_i, Y_{t_{i+1}}^n, Z_{t_i}^n) | \mathcal{F}_{t_i}], \quad 0 \leq i \leq n-1, \end{cases} \quad (1.2.8)$$

La principale difficulté de ce schéma repose sur l'évaluation des espérances conditionnelles pour l'obtention d'un algorithme implémentable. Dans les deux articles [BT04] et [Zha04], les auteurs ont trouvé la vitesse de convergence du schéma (1.2.8), dans le cas des EDSR markoviennes. La question de l'évaluation des espérances conditionnelles qui reste ouverte dans [Zha04], a été traitée dans [BT04]. Bouchard et Touzi ont utilisé le calcul de Malliavin et la méthode de Monte Carlo pour estimer ces espérances conditionnelles. Ils ont étudié l'erreur seulement sur Y par un schéma numérique qui semble difficile et coûteux.

Ce résultat est généralisé par Gobet et Labart [GL07], ainsi que par Gobet et Makhlof [GM10]. Pour des EDSR markoviennes, et sous des conditions fortes sur les coefficients des équations, Gobet et Labart [GL07] ont trouvé une erreur d'approximation en temps plus fin du schéma explicite (1.2.8). Dans des cas particuliers, où le processus progressif X

est donné explicitement (ex. mouvement Brownien), les auteurs ont trouvé que la vitesse de convergence est meilleure que celle trouvée dans [GLW06]. Sous une condition plus faible sur g , Gobet et Makhlouf [GM10] ont étudié la vitesse de convergence du schéma explicite (1.2.8).

Pour que les algorithmes soient implémentables, ils nécessitent une bonne approximation de leurs espérances conditionnelles associées. Diverses méthodes ont été développées dans un cadre markovien (voir [GLW05], [CMT10], [CT17a]). Dans [GLW05], Gobet, Lemor et Warin ont étudié la vitesse de convergence d'un schéma implicite et ont vérifié que ce schéma est implémentable. Cette méthode est basée sur des fonctions de régression itératives approchées par des projections sur un sous-espace vectoriel fonctionnel, avec des coefficients évalués à l'aide de simulations de Monte Carlo. Ces mêmes auteurs ont proposé un travail similaire en utilisant le schéma explicite (1.2.8) (voir [GLW06]). En 2010, Crisan, Manolarakis et Touzi [CMT10] ont proposé une amélioration de l'algorithme déjà étudié par Bouchard et Touzi [BT04]. En se basant sur le calcul de Malliavin, les auteurs ont prouvé la convergence de cet algorithme amélioré et en ont déduit une borne de l'erreur. Un peu plus tard, dans [CT17a], Chassagneux et Trillos ont obtenu un développement d'erreur explicite en utilisant la méthode de la cubature sur les espaces de Wiener. Ce résultat a été prouvé sous un léger renforcement des hypothèses nécessaires pour l'application de la méthode de la cubature. Le développement explicite peut ensuite être utilisé pour augmenter l'ordre de la vitesse de convergence en utilisant les techniques d'extrapolation de Richardson-Romberg.

Des méthodes progressives ont été introduites pour approcher l'EDSR (1.1). En 2007, Bender et Denk [BD07] ont proposé une évaluation des espérances conditionnelles d'une manière progressive. L'idée dans cet article est d'étudier les itérations de Picard avant l'évaluation des espérances, puis rapprocher les espérances en se basant sur la méthode de Monte Carlo des moindres carrés comme dans les articles [GLW05] et [GLW06].

Plus récemment, on cite comme méthodes progressives : les variables de contrôle adaptatives (voir [GL10]), les méthodes de branchement (voir [HLTT14]), l'approximation de Picard multi-niveaux (voir [WHJK17]) et le développement en chaos de Wiener (voir [BL14]). Dans la troisième section de l'introduction, nous nous concentrons à développer cette dernière méthode "Développement en chaos de Wiener" et puis nous l'appliquons aux cas d'EDSR de type McKean-Vlasov dans le Chapitre 3.

De nombreuses extensions de l'EDSR (1.2.1) ont aussi été considérées : schémas d'ordre élevé (voir [Cha14], [CC14]), schéma pour les EDSR réfléchies (voir [BP03], [MZ05], [MPX08], [MSMT11], [CR16]), pour les EDSR couplées (voir [DM06], [BZ08]), pour les EDSR progressives découplées (voir [BE08]), pour les EDSR quadratiques (voir [CR15]), pour les EDSR avec sauts (voir [LMT14], [GL16]) et pour les EDSR de type McKean-Vlasov (voir [Ala15], [CT17b], [CCD17]). Dans la suite, nous développons deux méthodes différentes pour les EDSR de type McKean-Vlasov : développement en chaos de Wiener (voir Section 1.4 de l'introduction et Chapitre 3) et la méthode des marches aléatoires (voir Section 1.5 de l'introduction et Chapitre 4).

1.3 Schéma numérique pour les EDS réfléchies en moyenne avec sauts

Dans la première partie de cette thèse, nous nous intéressons à un type spécial d'EDS réfléchies, appelées EDS réfléchies en moyenne, dans lequel la contrainte n'agit pas directement sur les trajectoires de la solution de l'EDS comme dans le cas habituel, mais sur la loi de cette solution. Ces types d'EDS réfléchies ont été introduites par Briand, Elie et Hu sous leur forme rétrograde dans [BEH18]. Dans cette partie, on va tout d'abord introduire ce type d'EDS réfléchies en moyenne et présenter la convergence d'un schéma numérique étudiée dans [BCdRGL16] en se basant sur la propagation de chaos. Ensuite, on va donner une application dans la finance pour ce type d'EDS. Et finalement, on va énoncer les nouveaux résultats obtenus dans le Chapitre 2 sur la convergence dans le cas des EDS réfléchies en moyenne avec sauts.

1.3.1 EDS réfléchies en moyenne

Considérons l'EDSR réfléchi en moyenne comme suit :

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds - \int_t^T Z_s dB_s + K_T - K_t, \quad 0 \leq t \leq T, \quad (1.3.1)$$

$$\mathbb{E}[h(t, Y_t)] \geq 0, \quad 0 \leq t \leq T, \quad (1.3.2)$$

où $\mathbb{E}[h(t, Y_t)] \geq 0$ est la contrainte qui agit non pas sur la trajectoire du processus mais sur sa loi. Dans [BEH18], les auteurs (dans le cas $n = 1$) ont étudié l'EDSR réfléchi en moyenne (1.3.1)-(1.3.2) qui satisfait la condition de Skorokhod

$$\int_t^T \mathbb{E}[h(s, Y_s)] dK_s = 0, \quad 0 \leq t \leq T, \quad (1.3.3)$$

sous les hypothèses suivantes :

(H_h) la fonction $h : \Omega \times [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ est une fonction mesurable par rapport aux tribus $\mathcal{F}_T \times \mathcal{B}([0, T]) \times \mathcal{B}(\mathbb{R})$, continue en temps, continue et strictement croissante en espace et vérifiant \mathbb{P} -presque sûrement

$$\mathbb{E}[h(\cdot, \infty)] > 0 \text{ et } \exists C \geq 0, \quad |h(t, y)| \leq C(1 + |y|) \quad \forall (t, y) \in [0, T] \times \mathbb{R}.$$

(H_f) le générateur $f : [0, T] \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$ est une fonction lipschitzienne par rapport à ses deux dernières variables et $\mathbb{E} \left[\int_0^T |f(s, 0, 0)|^2 ds \right] < \infty$.

(H_ξ) la condition terminale ξ est une variable aléatoire \mathcal{F}_T -mesurable et de carré intégrable telle que $\mathbb{E}[h(T, \xi)] \geq 0$.

Definition 1.3.1. Une solution de carré intégrable à l'EDSR (1.3.1) avec une réflexion moyenne (1.3.2) est un triplet de processus (Y, Z, K) qui satisfait l'équation (1.3.1) avec la

contrainte (1.3.2). Une solution est dite plate si, en plus, K n'augmente que si nécessaire, c-à-d. lorsque nous avons

$$\int_t^T \mathbb{E}[h(s, Y_s)] dK_s = 0.$$

En outre, une solution est dite déterministe si le processus K est déterministe.

Théorème 1.3.1 (Briand-Elie-Hu 2016). *Dans le cas d'une réflexion linéaire*

$$h : (t, y) \longmapsto y - u_t$$

avec u déterministe continu, l'EDSR réfléchie en moyenne (1.3.1)-(1.3.2) admet une unique solution plate déterministe de carré intégrable (Y, Z, K) .

Dans le cas général, on introduit pour tout $t \in [0, T]$ l'opérateur $L_t : X \in L^2(\mathcal{F}_t) \longmapsto \inf\{x \geq 0 : \mathbb{E}[h(t, X + x)] \geq 0\}$, ce qui correspond à $(\mathbb{E}(X) - u_t)^-$ dans le cas linéaire.

Théorème 1.3.2 (Briand-Elie-Hu 2016). *Supposons de plus l'hypothèse suivante*

(H_L) la famille $\{L_t\}_{t \in [0, T]}$ est Lipschitz pour la norme L^1 uniformément en temps,

alors il existe une unique solution plate déterministe de carré intégrable (Y, Z, K) à (1.3.1)-(1.3.2).

Remarque 1.3.1. *Si h est continu en temps, strictement croissante en espace et qu'il existe des constantes $0 < c_h \leq C_h$ telles que*

$$|h(t, y)| \leq C_h(1 + |y|), \quad c_h|x - y| \leq |h(t, x) - h(t, y)| \leq C_h|x - y|, \quad \forall (t, x, y) \in [0, T] \times \mathbb{R}^2,$$

alors (H_h) et (H_L) sont vérifiées.

Remarque 1.3.2. *Nous observons que permettre à K d'être aléatoire conduit à l'existence d'une infinité de solutions plates de l'EDSR réfléchie en moyenne (1.3.1)-(1.3.2) (voir [Remarque 6, [BEH18]]). En plus, cela peut induire la non-existence d'une solution minimale pour cette EDSR (voir l'exemple fourni à la fin de la Section 5 dans [BEH18]). C'est pourquoi les auteurs ont choisi de trouver les solutions dites déterministes, c-à-d. les solutions (Y, Z, K) avec compensateur déterministe K , qui à son tour assure l'unicité de la solution.*

Rappelons que pour une EDSR réfléchie classique, la condition de Skorokhod garantit la minimalité de la solution dans la classe de toutes les super-solutions de l'EDSR réfléchie. Par minimalité, nous parlons de minimalité en termes de composante Y de la solution. La condition de Skorokhod indique que le compensateur K ne croît que lorsque c'est vraiment nécessaire. Dans cet esprit, les auteurs ont choisi dans cet article de rechercher des solutions aux EDSR réfléchies en moyenne satisfaisant la condition de planéité correspondante (1.3.3). Par suite, sous une hypothèse supplémentaire sur la structure du générateur f , ils ont vérifié que cette propriété de minimalité est bien satisfaite.

Théorème 1.3.3 (Briand-Elie-Hu 2016). *Supposons que le générateur f est une fonction de la forme*

$$f : (t, y, z) \longmapsto a_t y + l(t, z),$$

où a est une fonction mesurable déterministe et bornée. Si h est strictement croissante, une solution plate déterministe (Y, Z, K) est minimale parmi toutes les solutions déterministes.

L'idée de notre travail est d'étendre le résultat de l'article [BCdRGL16] au cas des EDS réfléchies en moyenne avec sauts (voir Chapitre 2). Dans [BCdRGL16], les auteurs se sont concentrés sur l'étude des EDS réfléchies en moyenne sous leur forme progressive :

$$\begin{cases} X_t = X_0 + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dB_s + K_t, & t \geq 0, \\ \mathbb{E}[h(X_t)] \geq 0, & \int_0^t \mathbb{E}[h(X_s)] dK_s = 0, & t \geq 0, \end{cases} \quad (1.3.4)$$

où b, σ sont des fonctions lipschitziennes, $X_0 \in L^2$ et h une fonction qui vérifie la Remarque 1.3.1, ce qui nous donne l'existence et l'unicité d'une solution plate déterministe de carré intégrable (X, K) du système (1.3.4) (voir Théorème 1.3.2). En plus,

$$\begin{aligned} \forall t \geq 0, K_t &= \sup_{s \leq t} \inf \{x \geq 0 : \mathbb{E}[h(x + U_s)] \geq 0\} \\ &= \sup_{s \leq t} G_0(\mu_s), \end{aligned} \quad (1.3.5)$$

où $(U_t)_{0 \leq t \leq T}$ est un processus défini par :

$$U_t = X_0 + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dB_s, \quad (1.3.6)$$

$$G_0 : \mathcal{P}(\mathbb{R}) \ni \nu \mapsto \inf \left\{ y \geq 0 : \int h(y+z)\nu(dz) \geq 0 \right\}, \quad (1.3.7)$$

et $(\mu_t)_{0 \leq t \leq T}$ est la famille des lois marginales de $(U_t)_{0 \leq t \leq T}$.

L'objectif principal dans cet article est d'étudier la convergence d'une approximation du système de particules de (1.3.4) afin de pouvoir trouver des schémas numériques pour les solutions de (1.3.4).

1.3.2 Propagation du chaos

L'étude numérique du système (1.3.4) est liée à la théorie des jeux à champ moyens, introduite par Lasry et Lions (voir [LL07a], [LL07b], [LL06a], [LL06b]) et dont le point de vue probabiliste est étudié dans [CD18b] et [CD18c]. Etant donné que le processus de réflexion K dépend de la loi de la position, les auteurs de [BCdRGL16] se sont inspirés de la théorie des jeux à champs moyens, pour étudier la convergence d'un schéma numérique, basé sur le système de particules.

La propagation du chaos se réfère à la convergence du système à N particules en interaction, défini par l'EDS

$$X_t^i = X_0^i + \int_0^t b(X_s^i)ds + \int_0^t \sigma(X_s^i)dB_s^i + \sup_{s \leq t} G_0(\mu_s^N), \quad 1 \leq i \leq N, \quad (1.3.8)$$

vers la solution de l'EDS réfléchi en moyenne (1.3.4). Notons que pour tout $1 \leq i \leq N$, les B^i sont des mouvements Browniens indépendants, les X_0^i sont des copies indépendantes

de X^0 et μ_s^N est la loi empirique au temps s des particules

$$U_t^i = X_0^i + \int_0^t b(X_s^i) ds + \int_0^t \sigma(X_s^i) dB_s^i, \quad 1 \leq i \leq N,$$

c-à-d

$$\mu_s^N = \frac{1}{N} \sum_{i=1}^N \delta_{U_s^i}.$$

Plus précisément, la propagation du chaos signifie généralement que les mesures empiriques μ^N convergent faiblement en probabilité vers la mesure déterministe μ .

Remarque 1.3.3. *Remarquons que le système de particules en interaction précédent peut être considéré comme une EDS réfléchie multi-dimensionnelle avec une réflexion oblique. En effet, si h est concave, l'ensemble*

$$C = \{(x_1, \dots, x_N) \in \mathbb{R}^N : h(x_1) + \dots + h(x_N) \geq 0\}$$

est convexe et le système

$$\begin{cases} X_t^i = X_0^i + G_0(\mu_0^N) + \int_0^t b(X_s^i) ds + \int_0^t \sigma(X_s^i) dB_s^i + K_t^N, & 1 \leq i \leq N, \\ \frac{1}{N} \sum_{i=1}^N h(X_t^i) \geq 0, & \frac{1}{N} \sum_{i=1}^N \int_0^t h(X_s^i) dK_s^N = 0, \end{cases}$$

n'est autre qu'une EDS réfléchie dans le convexe C avec une réflexion oblique dans la direction $(1, \dots, 1)$ (voir [LS84]).

Théorème 1.3.4 (Briand et al. 2017). *Introduisons les N copies indépendantes de X*

$$\bar{X}_t^i = X_0^i + \int_0^t b(\bar{X}_s^i) ds + \int_0^t \sigma(\bar{X}_s^i) dB_s^i + \sup_{s \leq t} G_0(\mu_s), \quad 1 \leq i \leq N,$$

et couplées avec les particules précédentes en choisissant le même mouvement Brownien. Pour $T > 0$,

- (i) *si il existe $p > 4$ tel que $X_0 \in L^p$, alors il existe une constante C qui dépend de b et σ telle que, pour tout $j \in \{1, \dots, N\}$,*

$$\mathbb{E} \left[\sup_{s \leq T} |X_s^j - \bar{X}_s^j|^2 \right] \leq C \exp \left(C \left(1 + \frac{M^2}{m^2} \right) (1 + T^2) \right) \frac{M^2}{m^2} N^{-1/2}.$$

- (ii) *si $h \in C^2$ avec ses dérivées bornées, alors il existe une constante C qui dépend de b et σ telle que, pour tout $j \in \{1, \dots, N\}$,*

$$\mathbb{E} \left[\sup_{s \leq T} |X_s^j - \bar{X}_s^j|^2 \right] \leq C \exp \left(C \left(1 + \frac{M^2}{m^2} \right) (1 + T^2) \right) \frac{1 + T^2}{m^2} \left(1 + \mathbb{E} \left[\sup_{s \leq T} |X_T|^2 \right] \right) N^{-1}.$$

En application, ce résultat de propagation du chaos permet de définir un algorithme basé sur le système de particules en interaction avec un schéma d'Euler classique, qui permet de trouver une approximation forte de la solution de (1.3.4). Cela conduit à une

erreur d'approximation inversement proportionnelle au nombre de points de la grille de discrétisation de l'intervalle de temps $((\log n/n)^{1/2})$, où n est le nombre de points de la grille de discrétisation) et sur le nombre de particules ($N^{-1/4}$ si la fonction h est seulement bi-Lipschitz et $N^{-1/2}$ si la fonction h est plus régulière) (voir Section 4, [BCdRGL16]).

1.3.3 Motivation et application

Notre principale motivation pour étudier (1.3.4) provient de problèmes financiers soumis à des contraintes de mesure du risque. Dans [BEH18], Briand, Elie et Hu ont développé le fait que les EDSR réfléchies sont liées aux stratégies d'investissement.

Mesures de risque

Soit (Ω, \mathcal{F}) un espace mesurable. Soit X une fonction mesurable à valeurs réelles définie sur Ω . Pour une position X , $\rho(X)$ s'interprète comme le montant des fonds propres exigés associé à cette position.

Definition 1.3.2. *On dit que l'application $\rho : L^2(\Omega) \rightarrow \mathbb{R}$ est une mesure de risque monétaire si :*

- *ρ est croissante : $X \leq Y \implies \rho(X) \geq \rho(Y)$, pour $X, Y \in L^2(\Omega)$;*
- *invariante par translation : $\rho(X + m) = \rho(X) - m$, pour $X \in L^2(\Omega)$ et $m \in \mathbb{R}$.*

Cette mesure peut être caractérisée par l'ensemble de toutes les positions acceptables, noté par l'ensemble d'acceptation et défini comme suit

$$\mathcal{A}_\rho = \{X \in L^2(\Omega) : \rho(X) \leq 0\}.$$

De même, pour un ensemble acceptable \mathcal{A} , on peut définir une mesure de risque

$$\rho(X) = \inf\{m \in \mathbb{R} : m + X \in \mathcal{A}\},$$

de sorte que l'ensemble d'acceptation \mathcal{A} et la mesure de risque ρ partagent une correspondance l'un à l'autre.

Exemple 1.3.1. *On peut considérer :*

1. *La Value At Risk au niveau $\alpha \in (0, 1)$: $VAR_\alpha(X) = \inf\{m : \mathbb{P}(m + X < 0) \leq \alpha\}$, c'est le quantile d'ordre α de X pour une probabilité \mathbb{P} sur (Ω, \mathcal{F}) donnée, avec un ensemble d'acceptation $\mathcal{A} = \{X : \mathbb{P}(X < 0) \leq \alpha\}$. Cela signifie que la probabilité de perte doit être inférieure au niveau α .*
2. *$\rho(X) = \inf\{m : \mathbb{E}[u(m + X)] \geq p\}$ où u est une fonction d'utilité (concave et croissante) et p est un seuil donné. Dans ce cas, l'ensemble d'acceptation est $\mathcal{A} = \{X : \mathbb{E}[u(X)] \geq p\}$, ce qui signifie qu'un bénéfice minimal est requis.*

Plusieurs autres exemples ont déjà été considérés dans la littérature : les mesures de risque cohérent (voir [ADEH99]), les mesures de risque convexe (voire [FS02]), etc.

Application

Nous passons maintenant à une application dans un marché financier avec une obligation de taux d'intérêt déterministe r et un vecteur d'actions d sous une dynamique

$$dS_t = S_t(\mu_t dt + \sigma_t dB_t), \quad 0 \leq t \leq T,$$

où μ et σ sont deux processus prévisibles de carré intégrables.

Pour un capital initial x donné, on considère les portefeuilles $X^{x,\pi,K}$ conduits par une stratégie de consommation-investissement (π, K) , et dont la dynamique du portefeuille est donnée par

$$\begin{aligned} dX_t^{x,\pi,K} &= X_t^{x,\pi,K} r_t dt + (\mu_t - r_t \mathbf{1})' \pi_t \frac{dS_t}{S_t} - dK_t \\ &= r_t X_t^{x,\pi,K} dt + (\mu_t - r_t \mathbf{1})' \pi_t dt + \pi_t' \sigma_t dB_t - dK_t, \quad 0 \leq t \leq T. \end{aligned}$$

Un ingénieur financier est disposé à couvrir un but $\xi \in L_T^2$. Le service de la gestion des risques qui dépend de son institution financière lui impose des restrictions sur les stratégies d'investissement admissibles. Notons que le portefeuille $X^{x,\pi,K}$ est dit admissible si et seulement si il satisfait la contrainte suivante

$$\rho_\alpha^{AVaR}(t, X_t^{x,\pi,K}) \leq q_t, \quad 0 \leq t \leq T,$$

où (α, q) est une famille indexée par le temps de niveaux de référence et quantiles déterministes. Cette référence peut être choisie de manière à ce que la contrainte devienne plus stricte ou plus faible à l'approche de l'échéance T , et ρ_α^{AVaR} est la valeur moyenne du risque (Average Value at Risk) qui est donnée par

$$\rho_\alpha^{AVaR}(t, X) = \sup \left\{ \mathbb{E}^{\mathbb{Q}}[-X] : \frac{d\mathbb{Q}}{d\mathbb{P}} \leq \frac{1}{\alpha_t} \right\}.$$

Dans un tel cas, l'investisseur cherche le "superhedging price" c-à-d l'investissement initial minimal qui permet de couvrir la somme ξ en temps T ,

$$Y_0 = \inf \left\{ x \in \mathbb{R} : \exists (\pi, K) \in \mathcal{A}, \quad X_T^{x,\pi,K} \geq \xi, \quad \rho_\alpha^{AVaR}(t, X_t^{x,\pi,K}) \leq q_t, \quad \forall t \in [0, T] \right\},$$

et la stratégie de consommation-investissement associée. Finalement, en appliquant les résultats de la sous-section "Étude théorique", on déduit que Y_0 est bien définie comme étant la valeur initiale de la solution déterministe unique de l'EDSR suivante

$$\begin{aligned} Y_t &= \xi + \int_t^T \left(-r_s Y_s - (\mu_s - r_s \mathbf{1})' \sigma_s^{-1} Z_s \right) ds, \quad 0 \leq t \leq T, \\ \rho_\alpha^{AVaR}(t, Y_t) &\leq q_t, \quad 0 \leq t \leq T, \quad \int_0^T [q_t - \rho_\alpha^{AVaR}(t, Y_t)] dK_t = 0. \end{aligned}$$

1.3.4 Nouveaux résultats

Le but du Chapitre 2 est d'étendre le résultat de l'article [BCdRGL16] au cas des EDS réfléchies en moyenne avec sauts :

$$\begin{cases} X_t = X_0 + \int_0^t b(X_{s-})ds + \int_0^t \sigma(X_{s-})dB_s + \int_0^t \int_E F(X_{s-}, z)\tilde{N}(ds, dz) + K_t, & t \geq 0, \\ \mathbb{E}[h(X_t)] \geq 0, \quad \int_0^t \mathbb{E}[h(X_s)] dK_s = 0, & t \geq 0, \end{cases} \quad (1.3.9)$$

où $E = \mathbb{R}^*$, \tilde{N} est une mesure de Poisson compensée $\tilde{N}(ds, dz) = N(ds, dz) - \lambda(dz)ds$, et B est un processus Brownien indépendant de N .

Sous de bonnes hypothèses et en se basant sur [BEH18], le résultat d'existence et d'unicité d'une solution plate déterministe de carré intégrable (X, K) du système (1.3.9) est étudié et vérifié (voir Chapitre 2). En plus, nous obtenons le résultat de la propagation du chaos, avec le même ordre de convergence que celui obtenu dans le cas où il n'y a pas de sauts :

Théorème 1.3.5 (Chapitre 2). *Soient $(X^i)_{i \in \{1, \dots, N\}}$ les N particules en interaction et $(\bar{X}^i)_{i \in \{1, \dots, N\}}$ les N copies indépendantes de X . Pour $T > 0$, et sous de bonnes hypothèses,*

- (i) *si il existe $p > 4$ tel que $X_0 \in L^p$, alors il existe une constante C qui dépend de T , b , σ , F et h telle que, pour tout $j \in \{1, \dots, N\}$,*

$$\mathbb{E} \left[\sup_{s \leq T} |X_s^j - \bar{X}_s^j|^2 \right] \leq CN^{-1/2}.$$

- (ii) *si $h \in C^2$ avec ses dérivées bornées, alors il existe une constante C qui dépend de T , b , σ , F et h telle que, pour tout $j \in \{1, \dots, N\}$,*

$$\mathbb{E} \left[\sup_{s \leq T} |X_s^j - \bar{X}_s^j|^2 \right] \leq CN^{-1}.$$

Ce résultat permet de définir à la section 4 du Chapitre 2 un algorithme basé sur le système de particules en interaction avec un schéma classique d'Euler qui donne une forte approximation de la solution de (1.3.9). Le point fort de notre travail est l'amélioration de la vitesse de convergence obtenue dans [BCdRGL16]. En effet, lorsque h est bi-Lipschitz, cela conduit à une erreur d'approximation au sens L^2 proportionnelle à $n^{-1} + N^{-1/2}$, où n est le nombre de points de la grille de discrétisation. Lorsque h est régulière, on obtient une erreur d'approximation proportionnelle à $n^{-1} + N^{-1}$. Finalement, nous avons illustré notre travail par des exemples numériques (voir Section 2.5 dans Chapitre 2).

1.4 Simulation par chaos de Wiener des EDSR de type McKean-Vlasov

L'idée du travail présenté dans la deuxième partie de cette thèse est d'étendre au cas McKean-Vlasov le travail de Briand et Labart [BL14]. Ils y proposent une méthode d'approximation progressive de solutions aux EDSR classiques (1.2.1) basée sur la décomposition en chaos de Wiener. Cette méthode est divisée en quatre types d'approximation : les itérations de Picard, la décomposition en chaos de Wiener, la troncature d'une base de L_T^2 et l'approximation de Monte Carlo. Dans cette partie, on va présenter cette méthode d'approximation et sa convergence vers la solution exacte, puis on va citer les nouveaux résultats obtenus dans le Chapitre 3 dans le cas d'EDSR de type McKean-Vlasov.

1.4.1 Décomposition en chaos de Wiener

Commençons par développer le schéma numérique proposée dans [BL14] :

Schéma de Picard

Le point de départ est l'utilisation des itérations de Picard de l'EDSR (1.2.1), $(Y^0, Z^0) = (0, 0)$, et pour tout $q \in \mathbb{N}$,

$$Y_t^{q+1} = \xi + \int_t^T f(s, Y_s^q, Z_s^q) ds - \int_t^T Z_s^{q+1} dB_s, \quad 0 \leq t \leq T.$$

Il est bien connu que la famille (Y^q, Z^q) converge exponentiellement vite vers la solution (Y, Z) de l'EDSR (1.2.1). Ensuite, les auteurs ont réécrit ce schéma de Picard d'une manière progressive

$$\begin{aligned} Y_t^{q+1} &= \mathbb{E} \left(\xi + \int_0^T f(s, Y_s^q, Z_s^q) ds \mid \mathcal{F}_t \right) - \int_0^t f(s, Y_s^q, Z_s^q) ds, \\ Z_t^{q+1} &= D_t Y_t^{q+1} = D_t \mathbb{E} \left(\xi + \int_0^T f(s, Y_s^q, Z_s^q) ds \mid \mathcal{F}_t \right), \end{aligned}$$

où $D_t X$ représente la dérivée de Malliavin de la variable aléatoire X .

La principale difficulté de cette approximation réside dans le calcul des espérances conditionnelles. La décomposition en chaos de Wiener est l'une des méthodes qui résout ce problème.

Décomposition en chaos

On se réfère à Nualart [Nua06] pour plus de détails sur cette section. Posons

$$F^q = \xi + \int_0^T f(s, Y_s^q, Z_s^q) ds.$$

Tout d'abord, comme $F^q \in L_T^2$, alors en se basant sur les polynômes d'Hermite, cette variable aléatoire F^q admet la décomposition orthogonale suivante

$$F^q = d_0 + \sum_{k \geq 1} \sum_{|n|=k} d_k^n \prod_{i \geq 1} K_{n_i} \left(\int_0^T g_i(s) dB_s \right), \quad (1.4.1)$$

où $(g_i)_{i \geq 1}$ est une base orthonormale de L_T^2 , K_n est le polynôme d'Hermite d'ordre n défini par

$$e^{xt-t^2/2} = \sum_{n \geq 0} K_n(x) t^n$$

avec la convention $K_{-1} \equiv 0$, $n = (n_i)_{i \geq 1}$ est une famille d'entiers positifs avec $|n| = \sum_{i \geq 1} n_i$ et les coefficients d_0 et $(d_k^n)_{k \geq 1, |n|=k}$ sont définis comme suit

$$d_0 = \mathbb{E}(F^q), \quad d_k^n = n! \mathbb{E} \left(F^q \times \prod_{i \geq 1} K_{n_i} \left(\int_0^T g_i(s) dB_s \right) \right),$$

avec $n! = \prod_{i \geq 1} n_i!$. Ensuite, pour obtenir des formules traitables, deux étapes principales ont été considérées :

- on travaille sur un nombre fini de chaos p ;
- on choisit un nombre fini de fonctions (g_1, \dots, g_N) .

La première étape consiste à considérer la projection $C_p(F^q)$ de la variable aléatoire F^q sur les p premiers chaos,

$$C_p(F^q) = d_0 + \sum_{1 \leq k \leq p} \sum_{|n|=k} d_k^n \prod_{i \geq 1} K_{n_i} \left(\int_0^T g_i(s) dB_s \right).$$

La deuxième étape consiste à travailler uniquement avec les N premières fonctions g_1, \dots, g_N d'une base orthonormale de L_T^2 . En effet, considérons un maillage $\mathcal{T} = \{\tilde{t}_i = i \frac{T}{N}, i = 0, \dots, N\}$ avec un pas de taille T/N et les N fonctions :

$$g_i(t) = \mathbf{1}_{] \tilde{t}_{i-1}, \tilde{t}_i]}(t) / \sqrt{h}, \quad i = 1, \dots, N, \text{ avec } h := \frac{T}{N},$$

puis complétons ces N fonctions dans une base orthonormale de L_T^2 .

Notons que, pour tout $i = 1, \dots, N$,

$$\int_0^T g_i(s) dB_s = G_i \quad \text{avec } G_i = \frac{B_{\tilde{t}_i} - B_{\tilde{t}_{i-1}}}{\sqrt{h}}.$$

Finalement, en choisissant ces N fonctions, une nouvelle approximation a été obtenue,

$$C_p^N(F^q) = d_0 + \sum_{k=1}^p \sum_{|n|=k} d_k^n \prod_{1 \leq i \leq N} K_{n_i}(G_i).$$

Par conséquent, cette décomposition est le point clé qui permet à Briand et Labart de trouver l'expression de $\mathbb{E}_t(C_p^N F^q)$ et $D_t \mathbb{E}_t(C_p^N(F^q))$.

Proposition 1.4.1 (Briand-Labart 2014). *Soit r un entier appartenant à $\{1, \dots, N\}$.*

Pour tout $\tilde{t}_{r-1} < t \leq \tilde{t}_r$, on a

$$\mathbb{E}_t(C_p^N F^q) = d_0 + \sum_{k=1}^p \sum_{|n(r)|=k} d_k^n \prod_{i < r} K_{n_i}(G_i) \times \left(\frac{t - \tilde{t}_{r-1}}{h} \right)^{(n_r)/2} K_{n_r} \left(\frac{B_t - B_{\tilde{t}_{r-1}}}{\sqrt{t - \tilde{t}_{r-1}}} \right),$$

$$D_t \mathbb{E}_t(C_p^N(F^q)) = h^{-1/2} \sum_{k=1}^p \sum_{\substack{|n(r)|=k \\ n(r) > 0}} d_k^n \prod_{i < r} K_{n_i}(G_i) \times \left(\frac{t - \tilde{t}_{r-1}}{h} \right)^{(n_r-1)/2} K_{n_r-1} \left(\frac{B_t - B_{\tilde{t}_{r-1}}}{\sqrt{t - \tilde{t}_{r-1}}} \right),$$

où, si $r \leq N$ et $n = (n_1, \dots, n_N)$, $n(r)$ représente (n_1, \dots, n_r) .

Remarquons que les coefficients d_0 et d_k^n des expressions précédentes ne sont pas calculables exactement dans la plupart des cas. Pour cette raison, une simulation de Monte Carlo a été proposée pour approcher ces coefficients. Et par suite on obtient une approximation des espérances conditionnelles $\mathbb{E}_t(C_p^N F^q)$ et de leurs dérivées de Malliavin $D_t \mathbb{E}_t(C_p^N(F^q))$ avec M trajectoires utilisées pour approcher d_0 et d_k^n par Monte-Carlo.

Convergence

Soit $(Y^{q,p,N,M}, Z^{q,p,N,M})$ l'approximation de la solution (Y, Z) de l'EDSR (1.2.1) obtenue par le schéma de Picard, la décomposition en chaos, la troncature et l'approximation de Monte Carlo.

Théorème 1.4.1 (Briand-Labart 2014). *Sous de bonnes hypothèses sur ξ et f , on a*

$$\|(Y - Y^{q,p,N,M}, Z - Z^{q,p,N,M})\|_{L^2}^2 \leq \frac{A_0}{2^q} + \frac{A_1(q,p)}{(p+1)!} + A_2(q,p) \left(\frac{T}{N} \right)^{2\beta_\xi \wedge 1} + \frac{A_3(q,p,N)}{M},$$

où β_ξ , A_0 , A_1 , A_2 et A_3 sont donnés par [BL14].

Finalement, le principal avantage de cette méthode est qu'une seule décomposition doit être calculée par itération de Picard : c'est la décomposition de F^q . En fait, ici réside la principale différence entre cette approximation et l'approximation basée sur la technique de régression développée par Bender et Denk dans [BD07]. Dans leur article, pour une itération de picard q et pour chaque instant t_i de la grille du maillage, il faut calculer deux projections, l'une pour $Y_{t_i}^q$ et l'autre pour $Z_{t_i}^q$. La deuxième différence provient de la façon de calculer Z^q . Dans notre méthode, une fois que la décomposition de F^q est calculée, Z^q est explicitement indiqué comme la dérivée de Malliavin de Y^q .

1.4.2 Nouveaux résultats

Le but du Chapitre 3 est d'étendre les résultats de [BL14] au cas des EDSR de type McKean-Vlasov (1.2.7), c-à-d de fournir un algorithme basé sur la décomposition en chaos de Wiener pour résoudre ce type d'EDSR.

D'une part, une estimation à priori a été étudiée et vérifiée dans le Chapitre 3, afin de trouver le résultat d'existence et d'unicité d'une solution de l'EDSR de type

McKean-Vlasov (1.2.7) en appliquant l'argument du point fixe. Comme nous n'avons pas trouvé ce résultat explicitement dans la littérature, nous l'avons écrite sous la condition d'intégrabilité de (ξ, f) et la condition de Lipschitz de f en (y, z, μ, ν) .

D'autre part, une méthode numérique a été proposée et étudiée dans le Chapitre 3 pour résoudre l'EDSR de type McKean-Vlasov (1.2.7). Cette méthode est basée sur la décomposition en chaos de Wiener en interaction avec une approximation du système de particules. La méthode d'approximation par la décomposition en chaos de Wiener a été introduite dans [BL14] pour résoudre les EDSR classiques (1.2.1) (voir la section précédente). Dans le cas d'EDSR de type McKean-Vlasov, le générateur dépend des lois des processus. L'idée principale pour résoudre ce cas est d'utiliser M particules qui serviront à la fois à approcher la loi de (Y, Z) et à calculer les coefficients $(d_k^n)_{k,n}$ de Monte Carlo. En appliquant cette idée, on a réussi à atteindre le même coût de calcul que celui obtenu pour les EDSR classiques pour certaines configurations du générateur f . Cependant, cette méthode nous a coûté l'indépendance dans l'approximation de Monte-Carlo, ce qui rend la preuve de la convergence plus difficile et conduit à une vitesse de convergence plus lente en M .

Théorème 1.4.2 (Chapitre 3). *Soit $(Y^{q,p,N,M,m}, Z^{q,p,N,M,m})$ l'approximation de la solution (Y, Z) de l'EDSR de type McKean-Vlasov (1.2.7) obtenue par le schéma de Picard, la décomposition en chaos, la troncature, l'approximation de Monte Carlo et l'approximation du système de particules. Sous de bonnes hypothèses sur ξ et f , on a pour tout $m \in \{1, \dots, M\}$,*

$$\begin{aligned} & \|(Y - Y^{q,p,N,M,m}, Z - Z^{q,p,N,M,m})\|_{L^2}^2 \\ & \leq \frac{A_0}{2^q} + \frac{A_1(q, p)}{(p+1)!} + A_2(q, p) \left(\frac{T}{N}\right)^{2\beta_\xi \wedge 1} + \frac{A_3(q, p, N)}{M} + \frac{A_4(q, p, N)(\log M)^{\frac{\alpha}{2}}}{M^{(\frac{1}{4} \wedge \frac{\beta}{2})\frac{1}{2q}}}, \end{aligned}$$

où $\alpha, \beta, \beta_\xi, A_0, A_1, A_2, A_3$ et A_4 sont donnés dans le Chapitre 3.

Des exemples numériques ont été étudiés pour illustrer notre travail et montrer l'efficacité de notre approximation (voir Section 3.5 du Chapitre 3).

1.5 Simulation par marche aléatoire et taux de convergence pour les EDSR de type McKean-Vlasov

La marche aléatoire est l'une des idées utilisées dans la littérature pour simuler les EDSR (voir [BDM01], [BDM02], [MPSMT02], [MPX08], [MSMT11], [PX11]). Dans cette partie, on va tout d'abord présenter l'approximation implicite d'une EDSR classique introduite par Briand, Delyon et Mémin [BDM01] en se basant sur un théorème de type Donsker. Ensuite, on va développer la vitesse de convergence de cette approximation prouvée par Briand, Geiss C., Geiss S. et Labart [BGGL19] dans un cas markovien. Enfin, on va énoncer notre nouveau résultat sur la vitesse de convergence obtenue dans le Chapitre 4 dans le cas d'une EDSR de type McKean-Vlasov.

1.5.1 Approximation par marche aléatoire

Commençons par la discrétisation des EDSR classiques (1.2.1). Considérons la marche aléatoire B^n :

$$B_t^n = \sqrt{h} \sum_{k=1}^{\lfloor t/h \rfloor} \zeta_k, \quad 0 \leq t \leq T, \quad (1.5.1)$$

où $(\zeta_k)_{k \geq 1}$ est une suite i.i.d. de variables aléatoires de Bernoulli symétriques et $h = T/n$ le pas de discrétisation. Posons $\mathcal{F}_k := \sigma(\zeta_1, \dots, \zeta_k)$ et ξ^n une variable aléatoire de carré intégrable \mathcal{F}_n^n -mesurable qui converge vers ξ dans L^1 . Le schéma proposé est donc

$$\begin{cases} Y_{t_n}^n = \xi^n, \\ Z_{t_k}^n = h^{-1/2} \mathbb{E}[Y_{t_{k+1}}^n \zeta_{k+1} | \mathcal{F}_k], \\ Y_{t_k}^n = Y_{t_{k+1}}^n + hf(Y_{t_k}^n, Z_{t_k}^n) - \sqrt{h} Z_{t_k}^n \zeta_{k+1}, \quad k = n-1, \dots, 0. \end{cases} \quad (1.5.2)$$

Notons que, pour n suffisamment grand et f lipschitzienne par rapport à y , le schéma est bien défini et que $Y_{t_k}^n$ est \mathcal{F}_k -mesurable puisqu'elle est \mathcal{F}_{k+1} -mesurable et orthogonale à ζ_{k+1} . Dans [BDM01], Briand, Delyon et Mémin ont trouvé la convergence faible de la solution (Y^n, Z^n) du schéma (1.5.2) vers la solution (Y, Z) de l'EDSR (1.2.1).

Théorème 1.5.1 (Briand-Delyon-Mémin 2001). *Supposons que f est Lipschitz et considérons une marche aléatoire B^n telle que*

$$\sup_{0 \leq t \leq T} |B_t^n - B_t| \longrightarrow 0 \quad \text{lorsque } n \longrightarrow \infty \quad \text{en probabilité,}$$

alors, on obtient

$$\sup_{0 \leq t \leq T} |Y_t^n - Y_t|^2 + \int_0^T |Z_s^n - Z_s|^2 ds \longrightarrow 0 \quad \text{lorsque } n \longrightarrow \infty \quad \text{en probabilité.}$$

La preuve de ce théorème est basée sur le schéma de Picard et la convergence des filtrations obtenue par Coquet, Mémin et Slominski dans [CMS01]. En plus, d'après le théorème de Donsker et le théorème de représentation de Skorokhod, on sait que la marche aléatoire B^n définie par (1.5.1) converge uniformément sur l'intervalle $[0, T]$ vers le mouvement Brownien B en probabilité et dans L^2 . Et par suite, on obtient la convergence

faible de (Y^n, Z^n) vers (Y, Z) . Par contre, la vitesse de convergence reste un problème ouvert. Ce problème semble être raisonnable dans le cas Markovien, où $\xi = g(B_T)$ du fait du lien avec les EDPs. Considérons alors le cas Markovien où l'EDSR est définie comme suit : pour tout $x \in \mathbb{R}$,

$$Y_s^{t,x} = g(B_T^{t,x}) + \int_s^T f(r, B_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) dr - \int_s^T Z_r^{t,x} dB_r, \quad 0 \leq t \leq s \leq T, \quad (1.5.3)$$

où $B_r^{t,x} = x + B_r - B_t$. En plus, l'approximation de l'équation précédente est donnée par

$$Y_s^{n,t,x} = g(B_T^{n,t,x}) + \int_s^T f(B_{r^-}^{n,t,x}, Y_{r^-}^{n,t,x}, Z_r^{n,t,x}) d\langle B^n \rangle_r - \int_s^T Z_r^{n,t,x} dB_r^n, \quad 0 \leq t \leq s \leq T, \quad (1.5.4)$$

où $\underline{t} = h\lfloor t/h \rfloor$ et B^n est la marche aléatoire définie par (1.5.1). En ajoutant la condition que g soit Lipschitz, le résultat de la convergence faible reste vérifié (voir [BDM01]) et le problème de la vitesse de convergence reste ouvert. Plus récemment, ce problème a été traité par Geiss, Labart et Luoto dans [GLL18a] et [GLL18b], en se basant sur le lien entre l'EDSR et l'EDP. Les auteurs ont trouvé une vitesse de convergence dans L^2 d'ordre $\epsilon/4$, sous les conditions que g et $f(\cdot, y, z)$ soient ϵ -höldériennes. Cependant, cette ordre de convergence n'est pas optimal. Dans [BGGL19], Briand, Geiss C., Geiss S. et Labart ont trouvé une vitesse de convergence d'ordre plus élevé ($\epsilon/2$) pour la distance de Wasserstein.

Le point de départ dans [BGGL19] est le résultat de Rio [Rio09] qui a prouvé que, lorsque $T = 1$, pour tout $r \geq 1$, il existe une constante C_r telle que, pour tout $n \geq 1$,

$$W_r(B_1^n, G) \leq C_r n^{-1/2},$$

où W_r est la distance de L^r -Wasserstein et G une variable aléatoire normale standard. Tout d'abord, les auteurs ont généralisé ce résultat afin d'obtenir la vitesse de convergence de la marche aléatoire B^n vers B pour la distance W_r (voir Proposition 2 dans [BGGL19]) ;

$$W_r(B_s^{n,t,x}, B_s^{t,x}) \leq C_r n^{-1/2}.$$

Ensuite, en utilisant l'EDP et ses propriétés, en particulier les formules de représentation tirées de [MZ02], ils ont prouvé la vitesse de convergence sous les conditions que g et $f(\cdot, y, z)$ soient ϵ -höldériennes et lorsque l'erreur est mesurée avec la distance de Wasserstein.

Hypothèse 1.5.1. *Il existe une constante k telle que \mathbb{P} -p.s.,*

— *condition de ϵ -Hölder de g :*

$$\forall x, x', \quad |g(x) - g(x')| \leq k|x - x'|^\epsilon.$$

— *condition de α -Hölder en t , ϵ -Hölder en x et Lipschitz en (y, z) de $f : \forall (t, x, y, z)$ et (t', x', y', z') ,*

$$|f(t, x, y, z) - f(t', x', y', z')| \leq k \left(|t - t'|^\alpha + |x - x'|^\epsilon + |y - y'| + |z - z'| \right).$$

D'une part, considérons l'équation de la chaleur semi-linéaire suivante :

$$\begin{cases} \partial_t u(t, x) + \frac{1}{2} \Delta u(t, x) = -f\left(t, x, u(t, x), \nabla u(t, x)\right), & (t, x) \in [0, T] \times \mathbb{R}, \\ u(T, \cdot) = g. \end{cases} \quad (1.5.5)$$

Sous l'Hypothèse 1.5.1, il existe une solution unique $u \in C^{1,\infty}([0, T] \times \mathbb{R})$ de l'EDP (1.5.5).

En plus, en se référant à la Section 1.2.6 de l'introduction et à [BGGL19], et en se basant sur [Zha05], on obtient les propriétés suivantes sur u et ∇u :

$$\begin{aligned} & - Y_s^{t,x} = u(s, B_s^{t,x}) \text{ et } Z_s^{t,x} = \nabla u(s, B_s^{t,x}). \\ & - u(t, x) = \mathbb{E} \left[g(B_T^{t,x}) + \int_t^T f(s, B_s^{t,x}, Y_s^{t,x}, Z_s^{t,x}) ds \right] \text{ et} \\ & \quad \nabla u(t, x) = \mathbb{E} \left[g(B_T^{t,x}) \frac{B_T - B_t}{T - t} + \int_t^T f(s, B_s^{t,x}, Y_s^{t,x}, Z_s^{t,x}) \frac{B_s - B_t}{s - t} ds \right]. \\ & - |u(t, x)| \leq C(1 + |x|)^\epsilon \text{ et } |\nabla u(t, x)| \leq \frac{C}{(T - t)^{(1-\epsilon)/2}}. \end{aligned}$$

D'autre part, considérons l'EDP discrétisée par la méthode de différence finie :

$$\begin{cases} U^n(kh, x) - hf((k+1)h, x, U^n(kh, x), \nabla^n U^n((k+1)h, x)) = D_+^n U^n((k+1)h, x), \\ U^n(nh, x) = g(x), \end{cases} \quad (1.5.6)$$

où $\nabla^n u(x) = \frac{1}{2\sqrt{h}}(u(x + \sqrt{h}) - u(x - \sqrt{h}))$ et $D_+^n u(x) = \frac{1}{2}(u(x + \sqrt{h}) + u(x - \sqrt{h}))$.

En se référant à [BGGL19], on trouve les propriétés sur U^n et $\nabla^n U^n$:

$$\begin{aligned} & - Y_s^{n,t,x} = U^n(s, B_s^{n,t,x}) \text{ et } Z_s^{n,t,x} = \begin{cases} \nabla^n U^n(s, B_s^{n,t,x}), & \text{si } s \in \{kh, k = [t/h] + 1, \dots, n\}, \\ \Delta^n(s, B_s^{n,t,x}) := \nabla^n U^n(\underline{s} + h, B_s^{n,t,x}), & \text{si non.} \end{cases} \\ & - U^n(t, x) = \mathbb{E} \left[g(B_T^{n,t,x}) + \int_{\underline{t}}^T f(\bar{s}, B_s^{n,t,x}, Y_s^{n,t,x}, Z_s^{n,t,x}) ds \right] \text{ et} \\ & \quad \Delta^n(t, x) = \mathbb{E} \left[g(B_T^{n,t,x}) \frac{B_T^n - B_t^n}{T - \underline{t}} + \int_{\underline{t}+h}^T f(\bar{s}, B_s^{n,t,x}, Y_s^{n,t,x}, Z_s^{n,t,x}) \frac{B_s^n - B_t^n}{\underline{s} - \underline{t}} ds \right]. \\ & - |U^n(t, x)| \leq C(1 + |x|)^\epsilon \text{ et } |\Delta^n(t, x)| \leq \frac{C}{(T - t)^{(1-\epsilon)/2}}. \end{aligned}$$

Finalement, en se basant sur les propriétés précédentes sur u , U^n , ∇u et Δ^n , on déduit la vitesse de convergence de (U^n, Δ^n) vers $(u, \nabla u)$ qui nous ramène à la vitesse de convergence de $(Y_s^{n,t,x}, Z_s^{n,t,x})$ vers $(Y_s^{t,x}, Z_s^{t,x})$ (voir [BGGL19]).

Théorème 1.5.2 (Briand-Geiss C.-Geiss S.-Labart 2018). *Sous l'Hypothèse 1.5.1, pour tout $r \geq 1$, il existe une constante C_r telle que, pour tout $x \in \mathbb{R}$,*

$$W_r(Y_s^{n,t,x}, Y_s^{t,x}) \leq C_r(1 + |x|)^\epsilon n^{-(\alpha \wedge \frac{\epsilon}{2})}, \quad 0 \leq t \leq s \leq T,$$

$$W_r(Z_s^{n,t,x}, Z_s^{t,x}) \leq C_r \frac{(1 + |x|)^\epsilon}{\sqrt{T - s}} n^{-(\alpha \wedge \frac{\epsilon}{2})}, \quad 0 \leq t \leq s < T.$$

1.5.2 Nouveaux résultats

Le but du Chapitre 4 est d'étendre le résultat de l'article [BGGL19] au cas des EDSR de type McKean-Vlasov :

$$Y_s^{t,x} = g(B_T^{t,x}) + \int_s^T f(r, B_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}, \mathbb{E}[\varphi_1(Y_r^{t,x})], \mathbb{E}[\varphi_2(Z_r^{t,x})]) dr - \int_s^T Z_r^{t,x} dB_r, \quad 0 \leq t \leq s \leq T, \quad (1.5.7)$$

où $\varphi_1, \varphi_2 : \mathbb{R} \rightarrow \mathbb{R}$. Supposons que l'Hypothèse 1.5.1 et l'hypothèse suivante sont satisfaites

Hypothèse 1.5.2. *Il existe une constante k telle que \mathbb{P} -p.s.,*

— *condition de Lipschitz de φ_1 et φ_2 :*

$$\forall x, x', \quad i \in \{1, 2\}, \quad |\varphi_i(x) - \varphi_i(x')| \leq k|x - x'|.$$

— *condition de Lipschitz en (p, q) de $f : \forall t, x, y, z, p, q, p', q'$,*

$$|f(t, x, y, z, p, q) - f(t, x, y, z, p', q')| \leq k(|p - p'| + |q - q'|).$$

Alors, en se basant sur [ARBGL19], on obtient le résultat d'existence et d'unicité d'une solution de l'EDSR (1.5.7). En plus, grâce au cas Markovien, cette solution est liée à l'équation de la chaleur semi-linéaire suivante : pour tout $0 \leq t \leq s < T$ et $(x, y) \in \mathbb{R}^2$

$$\begin{cases} \partial_s u(s, y; t, x) + \frac{1}{2} \Delta_y u(s, y; t, x) = -f\left(s, y, u(s, y; t, x), \nabla_y u(s, y; t, x), p(s, t, x), q(s, t, x)\right), \\ u(T, y; t, x) = g(y), \end{cases} \quad (1.5.8)$$

où

$$p(s, t, x) = \mathbb{E}[\varphi_1(u(s, B_s^{t,x}; t, x))] = \int \varphi_1(u(s, z; t, x)) \mu_s^{t,x} dz,$$

$$q(s, t, x) = \mathbb{E}[\varphi_2(\nabla_y u(s, B_s^{t,x}; t, x))] = \int \varphi_2(\nabla_y u(s, z; t, x)) \mu_s^{t,x} dz,$$

et $\mu_s^{t,x}$ est la loi de $B_s^{t,x}$. Sous les Hypothèses 1.5.1 et 1.5.2, il existe une solution unique $u(\cdot; t, x) \in \mathcal{C}^{1,2}([t, T] \times \mathbb{R}) \cap \mathcal{C}^{0,0}([t, T] \times \mathbb{R})$ de l'EDP (1.5.8) pour tout $(t, x) \in [0, T] \times \mathbb{R}$ (voir Chapitre 4). En plus, en se basant sur la formule d'Itô, on obtient la relation suivante

$$Y_s^{t,x} = u(s, B_s^{t,x}; t, x), \quad 0 \leq t \leq s \leq T,$$

$$Z_s^{t,x} = \nabla_y u(s, B_s^{t,x}; t, x) \quad 0 \leq t \leq s < T.$$

Remarque 1.5.1. *La solution de l'EDP peut être obtenue en utilisant l'EDSR dans un cas plus général. Considérons*

$$\bar{Y}_r^{s,y,t,x} = g(B_T^{s,y}) + \int_r^T f(a, B_a^{s,y}, \bar{Y}_a^{s,y,t,x}, \bar{Z}_a^{s,y,t,x}, \mathbb{E}[\varphi_1(Y_a^{t,x})], \mathbb{E}[\varphi_2(Z_a^{t,x})]) da - \int_r^T \bar{Z}_a^{s,y,t,x} dB_a. \quad (1.5.9)$$

Remarquons que les termes de McKean-Vlasov $\mathbb{E}[\varphi_1(Y_a^{t,x})]$ et $\mathbb{E}[\varphi_2(Z_a^{t,x})]$ sont indépendants de s et y , alors on obtient :

— *L'existence et l'unicité d'une solution de l'EDSR (1.5.9) (voir [PP90]).*

- La relation entre l'EDSR (1.5.9) et l'EDP (1.5.8) (voir Proposition 1 dans Chapitre 4) :

$$\begin{aligned}\bar{Y}_r^{s,y,t,x} &= u(r, B_r^{s,y}; t, x), \quad 0 \leq t \leq s \leq r \leq T, \\ \bar{Z}_r^{s,y,t,x} &= \nabla_y u(r, B_r^{s,y}; t, x), \quad 0 \leq t \leq s \leq r < T.\end{aligned}$$

- La formule de représentation de $\nabla_y u(s, y; t, x)$ (voir [Zha05]) :

$$\begin{aligned}\nabla_y u(s, y; t, x) &= \mathbb{E} \left(g(B_T^{s,y}) \frac{B_T - B_s}{T - s} \right) \\ &+ \mathbb{E} \left(\int_s^T f(r, B_r^{s,y}, \bar{Y}_r^{s,y,t,x}, \bar{Z}_r^{s,y,t,x}, \mathbb{E}[\varphi_1(Y_r^{t,x})], \mathbb{E}[\varphi_2(Z_r^{t,x})]) \frac{B_r - B_s}{r - s} dr \right).\end{aligned}$$

Ensuite, de la même façon que l'article [BGGL19], on peut trouver les solutions discrètes $(Y^{n,t,x}, Z^{n,t,x})$ de l'EDSR (1.5.7), $(\bar{Y}^{n,s,y,t,x}, \bar{Z}^{n,s,y,t,x})$ de l'EDSR (1.5.9) et $U^n(s, y; t, x)$ de l'EDP (1.5.8) ainsi que les relations entre elles. Et finalement, on peut étudier les propriétés sur $u, U^n, \nabla u$ et Δ^n , afin d'obtenir la vitesse de convergence de $(Y_s^{n,t,x}, Z_s^{n,t,x})$ vers $(Y_s^{t,x}, Z_s^{t,x})$ (voir Chapitre 4). Cette vitesse de convergence est du même ordre que celle obtenue dans le cas d'une EDSR classique .

Théorème 1.5.3 (Chapitre 4). *Sous les Hypothèses 1.5.1 et 1.5.2, pour tout $r \geq 1$, il existe une constante C_r telle que, pour tout $x \in \mathbb{R}$,*

$$\begin{aligned}W_r(Y_s^{n,t,x}, Y_s^{t,x}) &\leq C_r (1 + |x|)^\epsilon n^{-(\alpha \wedge \frac{\epsilon}{2})}, \quad 0 \leq t \leq s \leq T, \\ W_r(Z_s^{n,t,x}, Z_s^{t,x}) &\leq C_r \frac{(1 + |x|)^\epsilon}{\sqrt{T - s}} n^{-(\alpha \wedge \frac{\epsilon}{2})}, \quad 0 \leq t \leq s < T.\end{aligned}$$

Chapitre 2

Mean Reflected Stochastic Differential Equations with jumps

Abstract

In this chapter, a reflected Stochastic Differential Equation with jumps is studied for the case where the constraint acts on the law of the solution rather than on its paths. These reflected SDEs have been approximated in [BCdRGL16] using a numerical scheme based on particle systems, when no jumps occur. The main contribution of this chapter is to prove the existence and the uniqueness of the solutions to this kind of reflected SDEs with jumps and to generalize the results obtained in [BCdRGL16] to this context.

2.1 Introduction

Reflected stochastic differential equations have been introduced in the pioneering work of Skorokhod (see [Sko61]), and their numerical approximations by Euler schemes have been widely studied (see [Slo94], [Slo01], [Lep95], [Pet95], [Pet97]). Reflected stochastic differential equations driven by a Lévy process have also been studied in the literature (see [MR85], [KH92]). More recently, reflected backward stochastic differential equations with jumps have been introduced and studied (see [HO03], [EHO05], [HH06], [Ess08], [CM08], [QS14]), as well as their numerical approximation (see [DL16a] and [DL16b]). The main particularity of our work comes from the fact that the constraint acts on the law of the process X rather than on its paths. The study of such equations is linked to the mean field games theory, which has been introduced by Lasry and Lions (see [LL07a], [LL07b], [LL06b], [LL06a]) and whose probabilistic point of view is studied in [CD18b] and [CD18c]. Stochastic differential equations with mean reflection have been introduced by Briand, Elie and Hu in their backward forms in [BEH18]. In that work, they show that mean reflected stochastic processes exist and are uniquely defined by the associated system of equations of the following form :

$$\begin{cases} X_t = X_0 + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dB_s + K_t, & t \geq 0, \\ \mathbb{E}[h(X_t)] \geq 0, \quad \int_0^t \mathbb{E}[h(X_s)] dK_s = 0, & t \geq 0. \end{cases} \quad (2.1.1)$$

Due to the fact that the reflection process K depends on the law of the position, the authors of [BCdRGL16], inspired by mean field games, study the convergence of a numerical scheme based on particle systems to compute numerically solutions to (2.1.1).

In this chapter, we extend previous results to the case of jumps, i.e. we study existence and uniqueness of solutions to the following mean reflected stochastic differential equation (MR-SDE in the sequel)

$$\begin{cases} X_t = X_0 + \int_0^t b(X_{s-})ds + \int_0^t \sigma(X_{s-})dB_s + \int_0^t \int_E F(X_{s-}, z)\tilde{N}(ds, dz) + K_t, & t \geq 0, \\ \mathbb{E}[h(X_t)] \geq 0, \quad \int_0^t \mathbb{E}[h(X_s)] dK_s = 0, & t \geq 0, \end{cases} \quad (2.1.2)$$

where $E = \mathbb{R}^*$, \tilde{N} is a compensated Poisson measure $\tilde{N}(ds, dz) = N(ds, dz) - \lambda(dz)ds$, and B is a Brownian process independent of N . We also propose a numerical scheme based on a particle system to compute numerically solutions to (3.1.2) and study the rate of convergence of this scheme.

Our main motivation for studying (3.1.2) comes from financial problems submitted to risk measure constraints. Given any position X , its risk measure $\rho(X)$ can be seen as the amount of own fund needed by the investor to hold the position. For example, we can consider the following risk measure : $\rho(X) = \inf\{m : \mathbb{E}[u(m + X)] \geq p\}$ where u is a utility function (concave and increasing) and p is a given threshold (we refer the reader to [ADEH99] and to [FS02] for more details on risk measures). Suppose that we are given a portfolio X of assets whose dynamic, when there is no constraint, follows the jump diffusion model

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t + \int_E F(X_{t-}, z)\tilde{N}(dt, dz), \quad t \geq 0.$$

Given a risk measure ρ , one can ask that X_t remains an acceptable position at each time t . The constraint rewrites $\mathbb{E}[h(X_t)] \geq 0$ for $t \geq 0$ where $h = u - p$.

In order to satisfy this constraint, the agent has to add some cash in the portfolio through the time and the dynamic of the wealth of the portfolio becomes

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t + \int_E F(X_{t-}, z)\tilde{N}(dt, dz) + dK_t, \quad t \geq 0,$$

where K_t is the amount of cash added up to time t in the portfolio to balance the "risk" associated to X_t . Of course, the agent wants to cover the risk in a minimal way, adding cash only when needed : this leads to the Skorokhod condition $\mathbb{E}[h(X_t)]dK_t = 0$. Putting together all conditions, we end up with a dynamic of the form (3.1.2) for the portfolio.

This chapter is organized as follows. In Section 2.2, we show that, under Lipschitz assumptions on b , σ and F and bi-Lipchitz assumptions on h , the system admits a unique strong solution, i.e. there exists a unique pair of process (X, K) satisfying system (3.1.2) almost surely, the process K being an increasing and deterministic process. Then, we show that, by adding some regularity on the function h , the Stieltjes measure dK is absolutely

continuous with respect to the Lebesgue measure and we obtain the explicit expression of its density. In Section 2.3 we show that the system (3.1.2) can be seen as the limit of an interacting particle system with oblique reflection of mean field type. This result allows to define in Section 2.4 an algorithm based on this interacting particle system together with a classical Euler scheme which gives a strong approximation of the solution of (3.1.2). When h is bi-Lipschitz, this leads to an approximation error in L^2 -sense proportional to $n^{-1} + N^{-\frac{1}{2}}$, where n is the number of points of the discretization grid and N is the number of particles. When h is smooth, we get an approximation error proportional to $n^{-1} + N^{-1}$. By the way, we improve the speed of convergence obtained in [BCdRGL16]. Finally, we illustrate these results numerically in Section 2.5.

2.2 Existence, uniqueness and properties of the solution.

In this chapter, $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space endowed with a standard Brownian motion $B = \{B_t\}_{0 \leq t \leq T}$. $\{\mathcal{F}_t\}_{0 \leq t \leq T}$ is the usual augmented filtration of B . Before moving on, we give the following assumptions needed in the sequel.

Assumption (A.1).

(i) *Lipschitz assumption* : there exists a constant $C_p > 0$, such that for all $x, x' \in \mathbb{R}$ and $p > 0$, we have

$$|b(x) - b(x')|^p + |\sigma(x) - \sigma(x')|^p + \int_E |F(x, z) - F(x', z)|^p \lambda(dz) \leq C_p |x - x'|^p.$$

(ii) *The random variable X_0 is square integrable independent of B_t and N_t .*

Assumption (A.2).

(i) *The function $h : \mathbb{R} \rightarrow \mathbb{R}$ is increasing and bi-Lipschitz* : there exist $0 < m \leq M$ such that

$$\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, m|x - y| \leq |h(x) - h(y)| \leq M|x - y|.$$

(ii) *The initial condition X_0 satisfies* : $\mathbb{E}[h(X_0)] \geq 0$.

Assumption (A.3). $\exists p > 4$ such that $X_0 \in L^p$ i.e. $\mathbb{E}[|X_0|^p] < \infty$.

Assumption (A.4). *The function h is twice continuously differentiable with bounded derivatives.*

2.2.1 Preliminary results

Consider the function

$$H : \mathbb{R} \times \mathcal{P}_1(\mathbb{R}) \ni (x, \nu) \mapsto \int h(x + z) \nu(dz), \quad (2.2.1)$$

where $\mathcal{P}_1(\mathbb{R})$ is the set of probability measures with a finite first-order moment. Let \bar{G}_0 be the inverse function in space of H evaluated at 0 :

$$\bar{G}_0 : \mathcal{P}_1(\mathbb{R}) \ni \nu \mapsto \inf\{x \in \mathbb{R} : H(x, \nu) \geq 0\}, \quad (2.2.2)$$

and G_0 is the positive part of \bar{G}_0 :

$$G_0 : \mathcal{P}_1(\mathbb{R}) \ni \nu \mapsto \inf\{x \geq 0 : H(x, \nu) \geq 0\}. \quad (2.2.3)$$

We start by studying some properties of H and G_0 .

Lemma 2.2.1. *Under Assumption (A.2), we have :*

(i) *For all ν in $\mathcal{P}_1(\mathbb{R})$, the function $H(\cdot, \nu) : \mathbb{R} \ni x \mapsto H(x, \nu)$ is bi-Lipschitz :*

$$\forall x, y \in \mathbb{R}, m|x - y| \leq |H(x, \nu) - H(y, \nu)| \leq M|x - y|. \quad (2.2.4)$$

(ii) *For all x in \mathbb{R} , the function $H(x, \cdot) : \mathcal{P}_1(\mathbb{R}) \ni \nu \mapsto H(x, \nu)$ satisfies the following Lipschitz inequality :*

$$\forall \nu, \nu' \in \mathcal{P}_1(\mathbb{R}), |H(x, \nu) - H(x, \nu')| \leq \left| \int h(x + \cdot)(d\nu - d\nu') \right|. \quad (2.2.5)$$

Proof. Lemma 2.2.1 ensues from the definition of H (see (2.2.1)). \square

Let ν and ν' be two probability measures. The Wasserstein-1 distance between ν and ν' is defined by :

$$W_1(\nu, \nu') = \sup_{\varphi \text{ 1-Lipschitz}} \left| \int \varphi(d\nu - d\nu') \right| = \inf_{X \sim \nu; Y \sim \nu'} \mathbb{E}[|X - Y|].$$

Thus

$$\forall \nu, \nu' \in \mathcal{P}_1(\mathbb{R}), |H(x, \nu) - H(x, \nu')| \leq MW_1(\nu, \nu'). \quad (2.2.6)$$

According to Monge-Kantorovitch Theorem, the assertion (2.2.5) implies that for all x in \mathbb{R} , the function $H(x, \cdot)$ is Lipschitz continuous w.r.t. the Wasserstein-1 distance. Then, the regularity of G_0 is given in the following Lemma :

Lemma 2.2.2. *Under Assumption (A.2), the function $G_0 : \mathcal{P}_1(\mathbb{R}) \ni \nu \mapsto G_0(\nu)$ is Lipschitz continuous :*

$$|G_0(\nu) - G_0(\nu')| \leq \frac{1}{m} \left| \int h(\bar{G}_0(\nu) + \cdot)(d\nu - d\nu') \right|, \quad (2.2.7)$$

where $\bar{G}_0(\nu)$ is the inverse of $H(\cdot, \nu)$ at point 0. Especially,

$$|G_0(\nu) - G_0(\nu')| \leq \frac{M}{m} W_1(\nu, \nu'). \quad (2.2.8)$$

Proof. The proof is given in ([BCdRGL16], Lemma 2.5). \square

2.2.2 Existence and uniqueness of the solution of (3.1.2)

The set of Assumptions (A.1)-(A.4) will be used as follows :

- The existence and uniqueness results are stated under the standard assumption for SDEs (A.1) and the assumption used in [BEH18] (A.2).

- The convergence of particle systems is proved under (A.3).
 - Some of the results will be improved under the smoothness assumption (A.4).
- Firstly, we recall the existence and uniqueness result of [BEH18] in the case of SDEs.

Definition 2.2.1. *A couple of processes (X, K) is said to be a flat deterministic solution to (3.1.2) if (X, K) satisfy (3.1.2) with K being a non-decreasing continuous deterministic function with $K_0 = 0$.*

Given this definition we have the following result.

Theorem 2.2.1. *Under Assumptions (A.1) and (A.2), the mean reflected SDE (3.1.2) has a unique deterministic flat solution (X, K) . Moreover,*

$$\forall t \geq 0, K_t = \sup_{s \leq t} \inf \{x \geq 0 : \mathbb{E}[h(x + U_s)] \geq 0\} = \sup_{s \leq t} G_0(\mu_s), \quad (2.2.9)$$

where $(U_t)_{0 \leq t \leq T}$ is the process defined by :

$$U_t = X_0 + \int_0^t b(X_{s-}) ds + \int_0^t \sigma(X_{s-}) dB_s + \int_0^t \int_E F(X_{s-}, z) \tilde{N}(ds, dz) \quad (2.2.10)$$

and $(\mu_t)_{0 \leq t \leq T}$ is the family of marginal laws of $(U_t)_{0 \leq t \leq T}$.

Proof. We refer to [BEH18], for the proof in the case of continuous backward SDEs. We present here the proof of the forward case with jumps.

Let us consider the set $\mathcal{C}^2 = \{X \text{ } \mathcal{F}\text{-adapted cdlg, } \mathbb{E}(\sup_{t \leq T} |X_t|^2) < \infty\}$, and let $\hat{X} \in \mathcal{C}^2$ be a given process. We define

$$\hat{U}_t = X_0 + \int_0^t b(\hat{X}_{s-}) ds + \int_0^t \sigma(\hat{X}_{s-}) dB_s + \int_0^t \int_E F(\hat{X}_{s-}, z) \tilde{N}(ds, dz),$$

and the function K

$$K_t = \sup_{s \leq t} \inf \{x \geq 0 : \mathbb{E}[h(x + \hat{U}_s)] \geq 0\} = \sup_{s \leq t} G_0(\hat{\mu}_s). \quad (2.2.11)$$

Let us introduce the process X :

$$X_t = X_0 + \int_0^t b(\hat{X}_{s-}) ds + \int_0^t \sigma(\hat{X}_{s-}) dB_s + \int_0^t \int_E F(\hat{X}_{s-}, z) \tilde{N}(ds, dz) + K_t,$$

where K is given by (2.2.11), and check that (X, K) is the solution to (3.1.2) with U replaced by \hat{U} . First, based on the definition of K , we have $\mathbb{E}[h(X_t)] \geq 0$, $K_t = G_0(\hat{\mu}_t)$ $dK_t - a.e.$ and $G_0(\hat{\mu}_t) > 0$ $dK_t - a.e.$. Then, we obtain

$$\begin{aligned} \int_0^t \mathbb{E}[h(X_s)] dK_s &= \int_0^t \mathbb{E}[h(\hat{U}_s + K_s)] dK_s \\ &= \int_0^t \mathbb{E}[h(\hat{U}_s + G_0(\hat{\mu}_s))] dK_s \\ &= \int_0^t \mathbb{E}[h(\hat{U}_s + G_0(\hat{\mu}_s))] \mathbf{1}_{G_0(\hat{\mu}_s) > 0} dK_s. \end{aligned}$$

Moreover, since h is continuous, we have $\mathbb{E}[h(\hat{U}_s + G_0(\hat{\mu}_s))] = 0$ as soon as $G_0(\hat{\mu}_s) > 0$, so that

$$\int_0^t \mathbb{E}[h(X_s)] dK_s = 0.$$

Second, choose the map $\Phi : \mathcal{C}^2 \rightarrow \mathcal{C}^2$ which associates to \hat{X} the process X , solution to (3.1.2). Let us prove that Φ is a contraction. Using the same Brownian motion and Poisson process, we consider \hat{X} and $\hat{X}' \in \mathcal{C}^2$ and K and K' defined by (2.2.11). From Assumption (A.1), and by using Cauchy-Schwartz and Doob inequality, we get

$$\begin{aligned} \mathbb{E} \left[\sup_{t \leq T} |X_t - X'_t|^2 \right] &\leq 4 \mathbb{E} \left[\sup_{t \leq T} \left\{ \left| \int_0^t \left(b(\hat{X}_{s-}) - b(\hat{X}'_{s-}) \right) ds \right|^2 + \left| \int_0^t \left(\sigma(\hat{X}_{s-}) - \sigma(\hat{X}'_{s-}) \right) dB_s \right|^2 \right. \right. \\ &\quad \left. \left. + \left| \int_0^t \int_E \left(F(\hat{X}_{s-}, z) - F(\hat{X}'_{s-}, z) \right) \tilde{N}(ds, dz) \right|^2 + |K_t - K'_t|^2 \right\} \right] \\ &\leq 4 \left\{ \mathbb{E} \left[\sup_{t \leq T} t \int_0^t |b(\hat{X}_{s-}) - b(\hat{X}'_{s-})|^2 ds \right] + \mathbb{E} \left[\sup_{t \leq T} \left| \int_0^t \left(\sigma(\hat{X}_{s-}) - \sigma(\hat{X}'_{s-}) \right) dB_s \right|^2 \right] \right. \\ &\quad \left. + \mathbb{E} \left[\sup_{t \leq T} \left| \int_0^t \int_E \left(F(\hat{X}_{s-}, z) - F(\hat{X}'_{s-}, z) \right) \tilde{N}(ds, dz) \right|^2 \right] + \sup_{t \leq T} |K_t - K'_t|^2 \right\} \\ &\leq C \left\{ T \mathbb{E} \left[\int_0^T |b(\hat{X}_{s-}) - b(\hat{X}'_{s-})|^2 ds \right] + \mathbb{E} \left[\int_0^T |\sigma(\hat{X}_{s-}) - \sigma(\hat{X}'_{s-})|^2 ds \right] \right. \\ &\quad \left. + \int_0^T \int_E \mathbb{E} \left[|F(\hat{X}_{s-}, z) - F(\hat{X}'_{s-}, z)|^2 \right] \lambda(dz) ds + \sup_{t \leq T} |K_t - K'_t|^2 \right\} \\ &\leq C \left\{ T^2 C_1 \mathbb{E} \left[\sup_{t \leq T} |\hat{X}_{t-} - \hat{X}'_{t-}|^2 \right] + T C_1 \mathbb{E} \left[\sup_{t \leq T} |\hat{X}_{t-} - \hat{X}'_{t-}|^2 \right] \right. \\ &\quad \left. + T C_1 \mathbb{E} \left[\sup_{t \leq T} |\hat{X}_{t-} - \hat{X}'_{t-}|^2 \right] + \sup_{t \leq T} |K_t - K'_t|^2 \right\} \\ &\leq C \left(T^2 C_1 + T C_2 \right) \mathbb{E} \left[\sup_{t \leq T} |\hat{X}_t - \hat{X}'_t|^2 \right] + C \sup_{t \leq T} |K_t - K'_t|^2. \end{aligned}$$

From the representation (2.2.11) of the process K and Lemma 2.2.2, we have that

$$\begin{aligned} \sup_{t \leq T} |K_t - K'_t|^2 &\leq \frac{M}{m} \mathbb{E} \left[\sup_{t \leq T} |\hat{U}_t - \hat{U}'_t|^2 \right] \\ &\leq C(T^2 C_1 + T C_2) \mathbb{E} \left[\sup_{t \leq T} |\hat{X}_t - \hat{X}'_t|^2 \right]. \end{aligned}$$

This leads to

$$\mathbb{E} \left[\sup_{t \leq T} |X_t - X'_t|^2 \right] \leq C(1 + T) T \mathbb{E} \left[\sup_{t \leq T} |\hat{X}_t - \hat{X}'_t|^2 \right].$$

Therefore, there exists a positive \mathcal{T} , depending on b , σ , F and h only, such that for all

$T < \mathcal{T}$, the map Φ is a contraction. Consequently, we get the existence and uniqueness of solution on $[0, \mathcal{T}]$ and by iterating the construction the result is extended on \mathbb{R}^+ . \square

2.2.3 Regularity results on K , X and U

Remark 2.2.1. *In view of this construction, we derive that for all $0 \leq s < t$:*

$$K_t - K_s = \sup_{s \leq r \leq t} \inf \left\{ x \geq 0 : \mathbb{E} \left[h \left(x + X_s + \int_s^r b(X_{u-}) du + \int_s^r \sigma(X_{u-}) dB_u + \int_s^r \int_E F(X_{u-}, z) \tilde{N}(du, dz) \right) \right] \geq 0 \right\}.$$

Proof. From the representation (2.2.9) of the process K , we have

$$\begin{aligned} K_t &= \sup_{r \leq t} G_0(U_r) = \max \left\{ \sup_{r \leq s} G_0(U_r), \sup_{s \leq r \leq t} G_0(U_r) \right\} \\ &= \max \left\{ K_s, \sup_{s \leq r \leq t} G_0(U_r) \right\} \\ &= \max \left\{ K_s, \sup_{s \leq r \leq t} G_0(X_s - K_s + U_r - U_s) \right\} \\ &= \max \left\{ K_s, \sup_{s \leq r \leq t} \left[\bar{G}_0(X_s - K_s + U_r - U_s)^+ \right] \right\}. \end{aligned}$$

By the definition of \bar{G}_0 , we observe that for all $y \in \mathbb{R}$, $\bar{G}_0(X + y) = \bar{G}_0(X) - y$, so we get

$$\begin{aligned} K_t &= \max \left\{ K_s, \sup_{s \leq r \leq t} \left[\left(K_s + \bar{G}_0(X_s + U_r - U_s) \right)^+ \right] \right\} \\ &= K_s + \max \left\{ 0, \sup_{s \leq r \leq t} \left[\left(K_s + \bar{G}_0(X_s + U_r - U_s) \right)^+ - K_s \right] \right\}. \end{aligned}$$

Note that $\sup_r (f(r)^+) = (\sup_r f(r))^+ = \max(0, \sup_r f(r))$ for all function f , and obviously

$$\begin{aligned} K_t &= K_s + \sup_{s \leq r \leq t} \left[\left\{ \left(K_s + \bar{G}_0(X_s + U_r - U_s) \right)^+ - K_s \right\}^+ \right] \\ &= K_s + \sup_{s \leq r \leq t} \left[\left(\bar{G}_0(X_s + U_r - U_s) \right)^+ \right] \\ &= K_s + \sup_{s \leq r \leq t} G_0(X_s + U_r - U_s), \end{aligned}$$

and so

$$K_t - K_s = \sup_{s \leq r \leq t} G_0(X_s + U_r - U_s).$$

\square

Proposition 2.2.1. *Suppose that Assumptions (A.1) and (A.2) hold. Then, for all $p \geq 2$, there exists a positive constant K_p , depending on T , b , σ , F and h such that*

- (i) $\mathbb{E}[\sup_{t \leq T} |X_t|^p] \leq K_p(1 + \mathbb{E}[|X_0|^p])$.
(ii) $\forall 0 \leq s \leq t \leq T, \quad \mathbb{E}[\sup_{s \leq u \leq t} |X_u|^p | \mathcal{F}_s] \leq C(1 + |X_s|^p)$.

Remark 2.2.2. Under the same conditions, we conclude that

$$\mathbb{E}[\sup_{t \leq T} |U_t|^p] \leq K_p(1 + \mathbb{E}[|X_0|^p]).$$

Proof of (i). We have

$$\begin{aligned} \mathbb{E} \left[\sup_{t \leq T} |X_t|^p \right] &\leq 5^{p-1} \left\{ \mathbb{E}|X_0|^p + \mathbb{E} \sup_{t \leq T} \left(\int_0^t |b(X_{s-})| ds \right)^p + \mathbb{E} \sup_{t \leq T} \left| \int_0^t \sigma(X_{s-}) dB_s \right|^p \right. \\ &\quad \left. + \mathbb{E} \sup_{t \leq T} \left| \int_0^t \int_E F(X_{s-}, z) \tilde{N}(ds, dz) \right|^p + K_T^p \right\}. \end{aligned}$$

The last term $K_T = \sup_{t \leq T} G_0(\mu_t)$ is firstly studied. By using the Lipschitz property of Lemma 2.2.2 of G_0 and the definition of the Wasserstein metric, we have

$$\forall t \geq 0, \quad |G_0(\mu_t)| \leq \frac{M}{m} \mathbb{E}[|U_t - U_0|],$$

since $G_0(\mu_0) = 0$ as $\mathbb{E}[h(X_0)] \geq 0$ and where U is defined by (2.4.3). Therefore

$$\begin{aligned} |K_T|^p &= \left| \sup_{t \leq T} G_0(\mu_t) \right|^p \leq 3^{p-1} \left(\frac{M}{m} \right)^p \left\{ \mathbb{E} \sup_{t \leq T} \left(\int_0^t |b(X_{s-})| ds \right)^p + \mathbb{E} \sup_{t \leq T} \left| \int_0^t \sigma(X_{s-}) dB_s \right|^p \right. \\ &\quad \left. + \mathbb{E} \sup_{t \leq T} \left| \int_0^t \int_E F(X_{s-}, z) \tilde{N}(ds, dz) \right|^p \right\}, \end{aligned}$$

and so

$$\begin{aligned} \mathbb{E} \left[\sup_{t \leq T} |X_t|^p \right] &\leq C(p, M, m) \mathbb{E} \left[|X_0|^p + \sup_{t \leq T} \left(\int_0^t |b(X_{s-})| ds \right)^p + \sup_{t \leq T} \left| \int_0^t \sigma(X_{s-}) dB_s \right|^p \right. \\ &\quad \left. + \sup_{t \leq T} \left| \int_0^t \int_E F(X_{s-}, z) \tilde{N}(ds, dz) \right|^p \right]. \end{aligned}$$

Hence, using Assumption (A.1), Cauchy-Schwartz, Doob and BDG inequalities yields

$$\begin{aligned} \mathbb{E} \left[\sup_{t \leq T} |X_t|^p \right] &\leq C \left\{ \mathbb{E} \left[|X_0|^p \right] + T^{p-1} \mathbb{E} \left[\int_0^T (1 + |X_{s-}|)^p ds \right] + C_1 \mathbb{E} \left[\int_0^T (1 + |X_{s-}|)^2 ds \right]^{\frac{p}{2}} \right. \\ &\quad \left. + C_2 \mathbb{E} \left[\int_0^T (1 + |X_{s-}|)^p ds \right] \right\} \\ &\leq C_1 \left(1 + \mathbb{E}|X_0|^p \right) + C_2 \int_0^T \mathbb{E} \left[\sup_{t \leq r} |X_t|^p \right] dr, \end{aligned}$$

and from Gronwall's Lemma, we can conclude that for all $p \geq 2$, there exists a positive

constant K_p , depending on T, b, σ, F and h such that

$$\mathbb{E}\left[\sup_{t \leq T} |X_t|^p\right] \leq K_p(1 + \mathbb{E}[|X_0|^p]).$$

Proof of (ii). For the first part, we have

$$\begin{aligned} X_u &= U_u + K_u \\ &= X_s + (U_u - U_s) + (K_u - K_s) \\ &= X_s + \int_s^u b(X_{r-})dr + \int_s^u \sigma(X_{r-})dB_r + \int_s^u \int_E F(X_{r-}, z)\tilde{N}(dr, dz) \\ &\quad + (K_u - K_s). \end{aligned}$$

Let us denote $\mathbb{E}_s[\cdot] = \mathbb{E}[\cdot | \mathcal{F}_s]$. Then we get

$$\begin{aligned} \mathbb{E}_s \left[\sup_{s \leq u \leq t} |X_u|^p \right] &\leq 5^{p-1} \left\{ \mathbb{E}_s \left[|X_s|^p \right] + \mathbb{E}_s \left[\sup_{s \leq u \leq t} \left| \int_s^u b(X_{r-})dr \right|^p \right] + \mathbb{E}_s \left[\sup_{s \leq u \leq t} \left| \int_s^u \sigma(X_{r-})dB_r \right|^p \right] \right. \\ &\quad \left. + \mathbb{E}_s \left[\sup_{s \leq u \leq t} \left| \int_s^u \int_E F(X_{r-}, z)\tilde{N}(dr, dz) \right|^p \right] + \left| K_t - K_s \right|^p \right\} \\ &\leq C \left\{ |X_s|^p + T^{p-1} \int_s^t \mathbb{E}_s \left[\left| b(X_{r-}) \right|^p \right] dr + \int_s^t \mathbb{E}_s \left[\left| \sigma(X_{r-}) \right|^p \right] dr \right. \\ &\quad \left. + \int_s^t \int_E \mathbb{E}_s \left[\left| F(X_{r-}, z) \right|^p \right] \lambda(dz)dr + 2 \left| K_T \right|^p \right\} \\ &\leq C(T) \left\{ |X_s|^p + C_1 \int_s^t \mathbb{E}_s \left[1 + |X_{r-}|^p \right] dr + 2 \left| K_T \right|^p \right\} \\ &\leq C_1(1 + |X_s|^p) + C_2 \int_s^t \mathbb{E}_s \left[\sup_{s \leq u \leq r} |X_u|^p \right] dr. \end{aligned}$$

Finally, from Gronwall's Lemma, we deduce that for all $0 \leq s \leq t \leq T$, there exists a constant C , depending on p, T, b, σ, F and h such that

$$\mathbb{E} \left[\sup_{s \leq u \leq t} |X_u|^p | \mathcal{F}_s \right] \leq C(1 + |X_s|^p).$$

□

Proposition 2.2.2. *Let $p \geq 2$ and let Assumptions (A.1), (A.2) and (A.3) hold. There exists a constant C depending on p, T, b, σ, F and h such that*

- (i) $\forall 0 \leq s < t \leq T, \quad |K_t - K_s| \leq C|t - s|^{(1/2)}.$
- (ii) $\forall 0 \leq s \leq t \leq T, \quad \mathbb{E}[|U_t - U_s|^p] \leq C|t - s|.$
- (iii) $\forall 0 \leq r < s < t \leq T, \quad \mathbb{E}[|U_s - U_r|^p |U_t - U_s|^p] \leq C|t - r|^2.$

Remark 2.2.3. *Under the same conditions, we conclude that*

$$\forall 0 \leq s \leq t \leq T, \quad \mathbb{E}[|X_t - X_s|^p] \leq C|t - s|.$$

Proof of (i). Let us recall that, for all process X ,

$$\bar{G}_0(X) = \inf\{x \in \mathbb{R} : \mathbb{E}[h(x + X)] \geq 0\},$$

$$G_0(X) = (\bar{G}_0(X))^+ = \inf\{x \geq 0 : \mathbb{E}[h(x + X)] \geq 0\}.$$

From Remark 2.2.1, we have

$$K_t - K_s = \sup_{s \leq r \leq t} G_0(X_s + U_r - U_s). \quad (2.2.12)$$

Hence, from the previous representation of $K_t - K_s$, we deduce the $\frac{1}{2}$ -Hlder property of the function $t \mapsto K_t$. Indeed, since by definition $G_0(X_s) = 0$, if $s < t$, by using Lemma 2.2.2,

$$\begin{aligned} |K_t - K_s| &= \sup_{s \leq r \leq t} G_0(X_s + U_r - U_s) \\ &= \sup_{s \leq r \leq t} [G_0(X_s + U_r - U_s) - G_0(X_s)] \\ &= \frac{M}{m} \sup_{s \leq r \leq t} \mathbb{E}[|U_r - U_s|], \end{aligned}$$

and so

$$\begin{aligned} |K_t - K_s| &\leq C \left\{ \mathbb{E} \left[\sup_{s \leq r \leq t} \left| \int_s^r b(X_{u-}) du \right| \right] + \left(\mathbb{E} \left[\sup_{s \leq r \leq t} \left| \int_s^r \sigma(X_{u-}) dB_u \right|^2 \right] \right)^{1/2} \right. \\ &\quad \left. + \left(\mathbb{E} \left[\sup_{s \leq r \leq t} \left| \int_s^r \int_E F(X_{u-}, z) \tilde{N}(du, dz) \right|^2 \right] \right)^{1/2} \right\} \\ &\leq C \left\{ \int_s^t \mathbb{E} \left[|b(X_{u-})| \right] du + \left(\mathbb{E} \left[\int_s^t |\sigma(X_{u-})|^2 du \right] \right)^{1/2} \right. \\ &\quad \left. + \left(\mathbb{E} \left[\int_s^t \int_E |F(X_{u-}, z)|^2 \lambda(dz) du \right] \right)^{1/2} \right\} \\ &\leq C \left\{ |t - s| \mathbb{E} \left[1 + \sup_{u \leq T} |X_u| \right] + |t - s|^{1/2} \left(\mathbb{E} \left[1 + \sup_{u \leq T} |X_u|^2 \right] \right)^{1/2} \right\}. \end{aligned}$$

Therefore, if $X_0 \in L^p$ for some $p \geq 2$, it follows from Proposition 2.2.1 that

$$|K_t - K_s| \leq C|t - s|^{1/2}.$$

Proof of (ii).

$$\begin{aligned}
\mathbb{E} \left[|U_t - U_s|^p \right] &\leq 4^{p-1} \mathbb{E} \left[\left(\int_s^t |b(X_{r-})| dr \right)^p + \left| \int_s^t \sigma(X_{r-}) dB_r \right|^p \right. \\
&\quad \left. + \left| \int_s^t \int_E F(X_{r-}, z) \tilde{N}(dr, dz) \right|^p \right] \\
&\leq C \sup_{0 \leq r \leq t} \mathbb{E} \left[\left(\int_s^r |b(X_{u-})| du \right)^p + \left| \int_s^r \sigma(X_{u-}) dB_u \right|^p \right. \\
&\quad \left. + \left| \int_s^r \int_E F(X_{u-}, z) \tilde{N}(du, dz) \right|^p \right] \\
&\leq C \left\{ |t-s|^{p-1} \mathbb{E} \left[\int_s^t (1 + |X_{u-}|)^p du \right] + C_1 \mathbb{E} \left[\left(\int_s^t (1 + |X_{u-}|)^2 du \right)^{p/2} \right] \right. \\
&\quad \left. + C_2 \mathbb{E} \left[\int_s^t (1 + |X_{u-}|)^p du \right] \right\} \\
&\leq C_1 \mathbb{E} \left[1 + \sup_{t \leq T} |X_t|^p \right] |t-s|^p + C_2 \mathbb{E} \left[\left(1 + \sup_{t \leq T} |X_t|^2 \right)^{p/2} \right] |t-s|^{p/2} \\
&\quad + C_3 \mathbb{E} \left[1 + \sup_{t \leq T} |X_t|^p \right] |t-s|.
\end{aligned}$$

Finally, if $X_0 \in L^p$ for some $p \geq 2$, we conclude that there exists a constant C , depending on p, T, b, σ, F and h such that

$$\forall 0 \leq s \leq t \leq T, \quad \mathbb{E} [|X_t - X_s|^p] \leq C|t-s|.$$

Proof of (iii). Let $0 \leq r < s < t \leq T$, we have

$$\begin{aligned}
\mathbb{E} \left[|U_s - U_r|^p |U_t - U_s|^p \right] &\leq \mathbb{E} \left[|U_s - U_r|^p \mathbb{E}_s [|U_t - U_s|^p] \right] \\
&\leq C \mathbb{E} \left[|U_s - U_r|^p \left\{ \mathbb{E}_s \left[\left| \int_s^t b(X_{s-}) ds \right|^p \right] + \mathbb{E}_s \left[\left| \int_s^t \sigma(X_{s-}) dB_s \right|^p \right] \right. \right. \\
&\quad \left. \left. + \mathbb{E}_s \left[\left| \int_s^t \int_E F(X_{s-}, z) d\tilde{N}(ds, dz) \right|^p \right] \right\} \right].
\end{aligned}$$

Then, from Burkholder-Davis-Gundy inequality, we get

$$\begin{aligned}
\mathbb{E} \left[|U_s - U_r|^p |U_t - U_s|^p \right] &\leq C \mathbb{E} \left[|U_s - U_r|^p \left\{ \mathbb{E}_s \left[\left| \int_s^t b(X_{s^-}) ds \right|^p \right] + \left(\mathbb{E}_s \left[\int_s^t \left| \sigma(X_{s^-}) \right|^2 ds \right] \right)^{p/2} \right. \right. \\
&\quad \left. \left. + \mathbb{E}_s \left[\int_s^t \int_E \left| F(X_{s^-}, z) \right|^p \lambda(dz) ds \right] \right\} \right] \\
&\leq C \mathbb{E} \left[|U_s - U_r|^p \left\{ |t - s|^p \left(1 + \mathbb{E}_s \left[\sup_{s \leq u \leq t} |X_u|^p \right] \right) \right. \right. \\
&\quad \left. \left. + |t - s|^{p/2} \left(1 + \mathbb{E}_s \left[\sup_{s \leq u \leq t} |X_u|^2 \right] \right)^{p/2} + |t - s| \left(1 + \mathbb{E}_s \left[\sup_{s \leq u \leq t} |X_u|^p \right] \right) \right\} \right] \\
&\leq C \mathbb{E} \left[|U_s - U_r|^p \left\{ |t - s| \left(1 + \mathbb{E}_s \left[\sup_{s \leq u \leq t} |X_u|^p \right] \right) \right\} \right],
\end{aligned}$$

thus, from (i) and Proposition 2.2.1, we obtain

$$\begin{aligned}
\mathbb{E} \left[|U_s - U_r|^p |U_t - U_s|^p \right] &\leq C_1 |t - s| \mathbb{E} \left[|U_s - U_r|^p \right] + C_2 |t - s| \mathbb{E} \left[|U_s - U_r|^p |X_s|^p \right] \\
&\leq C_1 |t - s| |s - r| + C_2 |t - s| \mathbb{E} \left[|U_s - U_r|^p \left(|X_s - X_r|^p + |X_r|^p \right) \right] \\
&\leq C_1 |t - r|^2 + C_2 |t - s| \mathbb{E} \left[2^{p-1} |U_s - U_r|^p \left(|U_s - U_r|^p + |K_s - K_r|^p \right) \right] \\
&\quad + C_3 |t - s| \mathbb{E} \left[|U_s - U_r|^p |X_r|^p \right] \\
&\leq C_1 |t - r|^2 + C_2 |t - s| \mathbb{E} \left[|U_s - U_r|^{2p} \right] + C_3 |t - s| |s - r|^{p/2} \mathbb{E} \left[|U_s - U_r|^p \right] \\
&\quad + C_4 |t - s| \mathbb{E} \left[|U_s - U_r|^p |X_r|^p \right] \\
&\leq C_1 |t - r|^2 + C_4 |t - s| \mathbb{E} \left[|X_r|^p \mathbb{E}_r \left[|U_s - U_r|^p \right] \right].
\end{aligned}$$

Following the proof of (ii), we can also get

$$\mathbb{E}_r \left[|U_s - U_r|^p \right] \leq C |s - r| \left(1 + \mathbb{E}_r \left[\sup_{r \leq u \leq s} |X_u|^p \right] \right).$$

Then

$$\begin{aligned} \mathbb{E} \left[|U_s - U_r|^p |U_t - U_s|^p \right] &\leq C_1 |t - r|^2 + C_2 |t - s| |s - r| \mathbb{E} \left[|X_r|^p \left(1 + \mathbb{E}_r \left[\sup_{r \leq u \leq s} |X_u|^p \right] \right) \right] \\ &\leq C_1 |t - r|^2 + C_2 |t - r|^2 \mathbb{E} \left[|X_r|^p \left(1 + \sup_{r \leq u \leq s} |X_u|^p \right) \right]. \end{aligned}$$

Under (A.3), we conclude that

$$\mathbb{E}[|U_s - U_r|^p |U_t - U_s|^p] \leq C |t - r|^2, \quad \forall 0 \leq r < s < t \leq T.$$

□

2.2.4 Density of K

Consider \mathcal{L} the linear partial operator of second order described by

$$\mathcal{L}f(x) := b(x) \frac{\partial}{\partial x} f(x) + \frac{1}{2} \sigma \sigma^*(x) \frac{\partial^2}{\partial x^2} f(x) + \int_E \left(f(x + F(x, z)) - f(x) - F(x, z) f'(x) \right) \lambda(dz), \quad (2.2.13)$$

for any twice continuously differentiable function f .

Proposition 2.2.3. *Assume (A.1), (A.2) and (A.4). Let (X, K) be the unique deterministic flat solution to (3.1.2). Then the process K is Lipschitz continuous and the Stieljes measure dK has the following density*

$$k : \mathbb{R}^+ \ni t \mapsto \frac{(\mathbb{E}[\mathcal{L}h(X_{t-})])^-}{\mathbb{E}[h'(X_{t-})]} \mathbf{1}_{\mathbb{E}[h(X_t)]=0}. \quad (2.2.14)$$

Let us admit for the moment the following result that will be useful for our proof.

Lemma 2.2.3. *The functions $t \mapsto \mathbb{E}[h(X_t)]$ and $t \mapsto \mathbb{E}[\mathcal{L}h(X_t)]$ are continuous.*

Lemma 2.2.4. *If φ is a continuous function such that, for some $C \geq 0$ and $p \geq 1$,*

$$\forall x \in \mathbb{R}, \quad |\varphi(x)| \leq C(1 + |x|^p),$$

then the function $t \mapsto \mathbb{E}[\varphi(X_t)]$ is continuous.

The proof of Lemma 2.2.4 is given in Appendix A.1. We may now proceed to the proof of Proposition 2.2.3.

Proof. Firstly, we prove that K is Lipschitz continuous. In order to do it, we first prove that $s \mapsto \tilde{G}_0(\mu_s)$ is Lipschitz continuous on $[0, T]$. From the definition of \tilde{G}_0 , we have

$H(\bar{G}_0(\mu_t), \mu_t) = 0$ and by using (2.2.4), if $s < t$, we get

$$\begin{aligned}
|\bar{G}_0(\mu_s) - \bar{G}_0(\mu_t)| &\leq \frac{1}{m} |H(\bar{G}_0(\mu_s), \mu_t) - H(\bar{G}_0(\mu_t), \mu_t)| \\
&= \frac{1}{m} |H(\bar{G}_0(\mu_s), \mu_t)|, \\
&= \frac{1}{m} |\mathbb{E}[h(\bar{G}_0(\mu_s) + U_t)]| \\
&= \frac{1}{m} \left| \mathbb{E} \left[h \left(\bar{G}_0(\mu_s) + U_s + \int_s^t b(X_{r-}) dr + \int_s^t \sigma(X_{r-}) dB_r \right. \right. \right. \\
&\quad \left. \left. \left. + \int_s^t \int_E F(X_{r-}, z) \tilde{N}(dr, dz) \right) \right] \right|.
\end{aligned}$$

From Itô's formula, we obtain

$$\begin{aligned}
h(\bar{G}_0(\mu_s) + U_t) &= h(\bar{G}_0(\mu_s) + U_s) + \int_s^t b(X_{r-}) h'(\bar{G}_0(\mu_s) + U_{r-}) dr + \int_s^t \sigma(X_{r-}) h'(\bar{G}_0(\mu_s) + U_{r-}) dB_r \\
&\quad + \int_s^t \int_E F(X_{r-}, z) h'(\bar{G}_0(\mu_s) + U_{r-}) \tilde{N}(dr, dz) + \frac{1}{2} \int_s^t \sigma^2(X_{r-}) h''(\bar{G}_0(\mu_s) + U_{r-}) dr \\
&\quad + \int_s^t \int_E m(r, z) \lambda(dz) dr + \int_s^t \int_E m(r, z) \tilde{N}(dr, dz),
\end{aligned}$$

with

$$m(r, z) = \left(h(\bar{G}_0(\mu_s) + U_{r-} + F(X_{r-}, z)) - h(\bar{G}_0(\mu_s) + U_{r-}) - F(X_{r-}, z) h'(\bar{G}_0(\mu_s) + U_{r-}) \right).$$

This yields

$$\begin{aligned}
h(\bar{G}_0(\mu_s) + U_t) &= h(\bar{G}_0(\mu_s) + U_s) + \int_s^t \bar{\mathcal{L}}_{X_{r-}} h(\bar{G}_0(\mu_s) + U_{r-}) dr + \int_s^t \sigma(X_{r-}) h'(\bar{G}_0(\mu_s) + U_{r-}) dB_r \\
&\quad + \int_s^t \int_E \left(h(\bar{G}_0(\mu_s) + U_{r-} + F(X_{r-}, z)) - h(\bar{G}_0(\mu_s) + U_{r-}) \right) \tilde{N}(dr, dz),
\end{aligned}$$

where

$$\bar{\mathcal{L}}_y f(x) := b(y) \frac{\partial}{\partial x} f(x) + \frac{1}{2} \sigma \sigma^*(y) \frac{\partial^2}{\partial x^2} f(x) + \int_E \left(f(x + F(y, z)) - f(x) - F(y, z) f'(x) \right) \lambda(dz).$$

Therefore,

$$\begin{aligned}
\mathbb{E}[h(\bar{G}_0(\mu_s) + U_t)] &= \mathbb{E}[h(\bar{G}_0(\mu_s) + U_s)] + \int_s^t \mathbb{E}[\bar{\mathcal{L}}_{X_{r-}} h(\bar{G}_0(\mu_s) + U_{r-})] dr \\
&= H(\bar{G}_0(\mu_s), \mu_s) + \int_s^t \mathbb{E}[\bar{\mathcal{L}}_{X_{r-}} h(\bar{G}_0(\mu_s) + U_{r-})] dr \\
&= \int_s^t \mathbb{E}[\bar{\mathcal{L}}_{X_{r-}} h(\bar{G}_0(\mu_s) + U_{r-})] dr.
\end{aligned}$$

Consequently, the result immediately follows from the fact that h has bounded derivatives

and $\sup_{s \leq T} |X_s|$ is a square integrable random variable for each $T > 0$ (see Proposition 2.2.1).

Finally, we deduce that K is Lipschitz continuous and so has a bounded density on $[0, T]$ for each $T > 0$ (see Proposition 2.7 in [BCdRGL16] for more details).

Secondly, let us find the density of the measure dK . For all $0 \leq s \leq t \leq T$, we have

$$X_t = X_s + \int_s^t \left(b(X_{r-}) - \int_E F(X_{r-}, z) \lambda(dz) \right) dr + \int_s^t \sigma(X_{r-}) dB_r + \int_s^t \int_E F(X_{r-}, z) N(dr, dz) + K_t - K_s.$$

Under (A.4) and thanks to Itô's formula we get

$$\begin{aligned} h(X_t) - h(X_s) &= \int_s^t b(X_{r-}) h'(X_{r-}) dr + \int_s^t \sigma(X_{r-}) h'(X_{r-}) dB_r + \int_s^t \int_E F(X_{r-}, z) h'(X_{r-}) \tilde{N}(dr, dz) \\ &\quad + \int_s^t h'(X_{r-}) dK_r + \frac{1}{2} \int_s^t \sigma^2(X_{r-}) h''(X_{r-}) dr \\ &\quad + \int_s^t \int_E \left(h(X_{r-} + F(X_{r-}, z)) - h(X_{r-}) - F(X_{r-}, z) h'(X_{r-}) \right) N(dr, dz) \\ &= \int_s^t b(X_{r-}) h'(X_{r-}) dr + \int_s^t \sigma(X_{r-}) h'(X_{r-}) dB_r + \int_s^t \int_E F(X_{r-}, z) h'(X_{r-}) \tilde{N}(dr, dz) \\ &\quad + \int_s^t h'(X_{r-}) dK_r + \frac{1}{2} \int_s^t \sigma^2(X_{r-}) h''(X_{r-}) dr \\ &\quad + \int_s^t \int_E \left(h(X_{r-} + F(X_{r-})) - h(X_{r-}) - F(X_{r-}) h'(X_{r-}) \right) \lambda(dz) dr \\ &\quad + \int_s^t \int_E \left(h(X_{r-} + F(X_{r-}, z)) - h(X_{r-}) - F(X_{r-}, z) h'(X_{r-}) \right) \tilde{N}(dr, dz) \\ &= \int_s^t \mathcal{L}h(X_{r-}) dr + \int_s^t h'(X_{r-}) dK_r + \int_s^t \sigma(X_{r-}) h'(X_{r-}) dB_r \\ &\quad + \int_s^t \int_E \left(h(X_{r-} + F(X_{r-}, z)) - h(X_{r-}) \right) \tilde{N}(dr, dz), \end{aligned}$$

where \mathcal{L} is given by (2.2.13). Thus, we obtain

$$\mathbb{E} \left(\int_s^t h'(X_{r-}) dK_r \right) = \mathbb{E}h(X_t) - \mathbb{E}h(X_s) - \int_s^t \mathbb{E} \mathcal{L}h(X_{r-}) dr. \quad (2.2.15)$$

As a conclusion, using (2.2.15), Lemma 2.2.3 and the proof of Proposition 2.7 in [BCdRGL16], we deduce that the measure dK has the following density

$$k_t = \frac{(\mathbb{E}[\mathcal{L}h(X_{t-})])^-}{\mathbb{E}[h'(X_{t-})]} \mathbf{1}_{\mathbb{E}[h(X_t)]=0}.$$

□

Proof of Lemma 2.2.3. Under Assumption (A.2), and by using Lemma 2.2.4, we obtain the continuity of the function $t \mapsto \mathbb{E}h(X_t)$.

Under the assumptions (A.1), (A.2) and (A.4), we observe that $x \mapsto \mathcal{L}h(X_t)$ is a continuous function such that, for all $x \in \mathbb{R}$, there exist constants C_1, C_2 and $C_3 > 0$,

$$|b(x)h'(x)| \leq C_1(1 + |x|),$$

$$|\sigma^2(x)h''(x)| \leq C_2(1 + |x|^2),$$

and

$$\begin{aligned} & \left| \int_E \left(h(x + F(x, z)) - h(x) - F(x, z)h'(x) \right) \lambda(dz) \right| \\ & \leq C_3 \int_E |F(x, z)| \lambda(dz) \\ & \leq C_3 \left(\int_E |F(x, z) - F(0, z)| \lambda(dz) + \int_E |F(0, z)| \lambda(dz) \right) \\ & \leq C_3 \int_E |x| \lambda(dz) + C'_3 \\ & \leq C_3(1 + |x|). \end{aligned}$$

Finally, by using Lemma 2.2.4, we conclude that $t \mapsto \mathbb{E}\mathcal{L}h(X_t)$ is continuous. \square

2.3 Approximation of mean reflected SDEs by an interacting reflected particle system.

By using the notations presented in the beginning of Section 2.2, in particular equation (2.2.9), the unique solution of the SDE (3.1.2) can be derived as :

$$X_t = X_0 + \int_0^t b(X_{s-}) ds + \int_0^t \sigma(X_{s-}) dB_s + \int_0^t \int_E F(X_{s-}, z) \tilde{N}(ds, dz) + \sup_{s \leq t} G_0(\mu_s), \quad (2.3.1)$$

where μ_t stands for the law of

$$U_t = X_0 + \int_0^t b(X_{s-}) ds + \int_0^t \sigma(X_{s-}) dB_s + \int_0^t \int_E F(X_{s-}, z) \tilde{N}(ds, dz).$$

Let us consider the particle approximation of the above system. In order to do this, let us introduce the particles : for $1 \leq i \leq N$,

$$X_t^i = \bar{X}_0^i + \int_0^t b(X_{s-}^i) ds + \int_0^t \sigma(X_{s-}^i) dB_s^i + \int_0^t \int_E F(X_{s-}^i, z) \tilde{N}^i(ds, dz) + \sup_{s \leq t} G_0(\mu_s^N), \quad (2.3.2)$$

where $(B^i)_{1 \leq i \leq N}$ are independent Brownian motions, $(\tilde{N}^i)_{1 \leq i \leq N}$ are independent compensated Poisson measures, $(\bar{X}_0^i)_{1 \leq i \leq N}$ are independent copies of X_0 and μ_s^N represents the empirical distribution at time s of the particles

$$U_t^i = \bar{X}_0^i + \int_0^t b(X_{s-}^i) ds + \int_0^t \sigma(X_{s-}^i) dB_s^i + \int_0^t \int_E F(X_{s-}^i, z) \tilde{N}^i(ds, dz), \quad 1 \leq i \leq N,$$

namely $\mu_s^N = \frac{1}{N} \sum_{i=1}^N \delta_{U_s^i}$. Note that

$$G_0(\mu_s^N) = \inf \left\{ x \geq 0 : \frac{1}{N} \sum_{i=1}^N h(x + U_s^i) \geq 0 \right\},$$

$$K_t^N = \sup_{s \leq t} G_0(\mu_s^N).$$

Now, we can prove the propagation of chaos effect. In order to do it, let us introduce the following independent copies of X

$$\bar{X}_t^i = \bar{X}_0^i + \int_0^t b(\bar{X}_{s-}^i) ds + \int_0^t \sigma(\bar{X}_{s-}^i) dB_s^i + \int_0^t \int_E F(\bar{X}_{s-}^i, z) \tilde{N}^i(ds, dz) + \sup_{s \leq t} G_0(\mu_s), \quad 1 \leq i \leq N,$$

where the Brownian motions and the Poisson processes are the ones used in (2.3.2). In addition, we introduce the decoupled particles \bar{U}^i , $1 \leq i \leq N$:

$$\bar{U}_t^i = \bar{X}_0^i + \int_0^t b(\bar{X}_{s-}^i) ds + \int_0^t \sigma(\bar{X}_{s-}^i) dB_s^i + \int_0^t \int_E F(\bar{X}_{s-}^i, z) \tilde{N}^i(ds, dz).$$

It is worth noting that the particles $(\bar{U}_t^i)_{1 \leq i \leq N}$ are i.i.d.. Furthermore, we introduce $\bar{\mu}^N$ as the empirical measure associated to this system of particles.

Remark 2.3.1. (i) Under our assumptions, we have $\mathbb{E}[h(\bar{X}_0^i)] = \mathbb{E}[h(X_0)] \geq 0$. However, there is no reason to have

$$\frac{1}{N} \sum_{i=1}^N h(\bar{X}_0^i) \geq 0,$$

even if N is large. As a consequence,

$$G_0(\mu_0^N) = \inf \left\{ x \geq 0 : \frac{1}{N} \sum_{i=1}^N h(x + \bar{X}_0^i) \geq 0 \right\}$$

is not necessarily equal to 0. As a byproduct, we have $X_0^i = \bar{X}_0^i + G_0(\mu_0^N)$ and the non decreasing process $\sup_{s \leq t} G_0(\mu_s^N)$ is not equal to 0 at time $t = 0$. Written in this way, the particles defined by (2.3.2) can not be interpreted as the solution of a reflected SDE. To view the particles as the solution of a reflected SDE, instead of (2.3.2) one has to solve

$$X_t^i = \bar{X}_0^i + G_0(\mu_0^N) + \int_0^t b(X_{s-}^i) ds + \int_0^t \sigma(X_{s-}^i) dB_s^i + \int_0^t \int_E F(X_{s-}^i, z) \tilde{N}^i(ds, dz) + K_t^N,$$

$$\frac{1}{N} \sum_{i=1}^N h(X_t^i) \geq 0, \quad \int_0^t \frac{1}{N} \sum_{i=1}^N h(X_t^i) dK_s^N = 0,$$

with K^N non decreasing and $K_0^N = 0$. Since, we do not use this point in the sequel, we will work with the form (2.3.2).

(ii) Following the proof of Theorem 2.2.1, it is easy to demonstrate existence and uniqueness of a solution for the particle approximated system (2.3.2).

We have the following result concerning the approximation (3.1.2) by interacting particle system.

Theorem 2.3.1. *Let $T > 0$ and assume that (A.1) and (A.2) hold.*

(i) *Under Assumption (A.3), there exists a constant C depending on b , σ and F such that, for each $j \in \{1, \dots, N\}$,*

$$\mathbb{E} \left[\sup_{s \leq T} |X_s^j - \bar{X}_s^j|^2 \right] \leq C \exp \left(C \left(1 + \frac{M^2}{m^2} \right) (1 + T^2) \right) \frac{M^2}{m^2} N^{-1/2}.$$

(ii) *Under Assumption (A.4), there exists a constant C depending on b , σ and F such that, for each $j \in \{1, \dots, N\}$,*

$$\mathbb{E} \left[\sup_{s \leq T} |X_s^j - \bar{X}_s^j|^2 \right] \leq C \exp \left(C \left(1 + \frac{M^2}{m^2} \right) (1 + T^2) \right) \frac{1 + T^2}{m^2} \left(1 + \mathbb{E} \left[\sup_{s \leq T} |X_T|^2 \right] \right) N^{-1}.$$

Proof. Let $t > 0$. We have, for $r \leq t$,

$$\begin{aligned} |X_r^j - \bar{X}_r^j| &\leq \left| \int_0^r b(X_{s-}^j) - b(\bar{X}_{s-}^j) ds \right| + \left| \int_0^r \left(\sigma(X_{s-}^j) - \sigma(\bar{X}_{s-}^j) \right) dB_s^i \right| \\ &\quad + \left| \int_0^r \int_E \left(F(X_{s-}^j, z) - F(\bar{X}_{s-}^j, z) \right) \tilde{N}^j(ds, dz) \right| + \left| \sup_{s \leq r} G_0(\mu_s^N) - \sup_{s \leq r} G_0(\mu_s) \right|. \end{aligned}$$

Due to the following inequality

$$\begin{aligned} \left| \sup_{s \leq r} G_0(\mu_s^N) - \sup_{s \leq r} G_0(\mu_s) \right| &\leq \sup_{s \leq r} |G_0(\mu_s^N) - G_0(\mu_s)| \leq \sup_{s \leq t} |G_0(\mu_s^N) - G_0(\mu_s)| \\ &\leq \sup_{s \leq t} |G_0(\mu_s^N) - G_0(\bar{\mu}_s^N)| + \sup_{s \leq t} |G_0(\bar{\mu}_s^N) - G_0(\mu_s)|, \end{aligned}$$

we obtain

$$\sup_{r \leq t} |X_r^j - \bar{X}_r^j| \leq I_1 + \sup_{s \leq t} |G_0(\mu_s^N) - G_0(\bar{\mu}_s^N)| + \sup_{s \leq t} |G_0(\bar{\mu}_s^N) - G_0(\mu_s)|, \quad (2.3.3)$$

where I_1 is defined by

$$\begin{aligned} I_1 &= \int_0^t |b(X_{s-}^j) - b(\bar{X}_{s-}^j)| ds + \sup_{r \leq t} \left| \int_0^r \left(\sigma(X_{s-}^j) - \sigma(\bar{X}_{s-}^j) \right) dB_s^i \right| \\ &\quad + \sup_{r \leq t} \left| \int_0^r \int_E \left(F(X_{s-}^j, z) - F(\bar{X}_{s-}^j, z) \right) \tilde{N}^j(ds, dz) \right|. \end{aligned}$$

Firstly, due to Assumption (A.1), Doob and Cauchy-Schwartz inequalities, we have

$$\begin{aligned}
\mathbb{E}[|I_1|^2] &\leq C \left\{ \mathbb{E} \left[t \int_0^t \left| b(X_{s^-}^j) - b(\bar{X}_{s^-}^j) \right|^2 ds \right] + \mathbb{E} \left[\int_0^t \left| \sigma(X_{s^-}^j) - \sigma(\bar{X}_{s^-}^j) \right|^2 ds \right] \right. \\
&\quad \left. + \mathbb{E} \left[\int_0^t \int_E \left| F(X_{s^-}^j, z) - F(\bar{X}_{s^-}^j, z) \right|^2 \lambda(dz) ds \right] \right\} \\
&\leq C \left\{ tC_1 \int_0^t \mathbb{E} \left[|X_s^j - \bar{X}_s^j|^2 \right] ds + C_1 \int_0^t \mathbb{E} \left[|X_s^j - \bar{X}_s^j|^2 \right] ds \right. \\
&\quad \left. + C_1 \int_0^t \mathbb{E} \left[|X_s^j - \bar{X}_s^j|^2 \right] ds \right\} \\
&\leq C(1+t) \int_0^t \mathbb{E} \left[|X_s^j - \bar{X}_s^j|^2 \right] ds.
\end{aligned}$$

where C is a constant that depends only on b , σ and F . Note that C can change from line to line.

Secondly, in view of Lemma 2.2.2,

$$\sup_{s \leq t} |G_0(\mu_s^N) - G_0(\bar{\mu}_s^N)| \leq \frac{M}{m} \sup_{s \leq t} \frac{1}{N} \sum_{i=1}^N |U_s^i - \bar{U}_s^i| \leq \frac{M}{m} \frac{1}{N} \sum_{i=1}^N \sup_{s \leq t} |U_s^i - \bar{U}_s^i|.$$

Moreover, taking into account that the variables are exchangeable, Cauchy-Schwartz inequality implies

$$\mathbb{E} \left[\sup_{s \leq t} |G_0(\mu_s^N) - G_0(\bar{\mu}_s^N)|^2 \right] \leq \frac{M^2}{m^2} \frac{1}{N} \sum_{i=1}^N \mathbb{E} \left[\sup_{s \leq t} |U_s^i - \bar{U}_s^i|^2 \right] = \frac{M^2}{m^2} \mathbb{E} \left[\sup_{s \leq t} |U_s^j - \bar{U}_s^j|^2 \right].$$

Since

$$U_s^j - \bar{U}_s^j = \int_0^s (b(X_{r^-}^j) - b(\bar{X}_{r^-}^j)) dr + \int_0^s (\sigma(X_{r^-}^j) - \sigma(\bar{X}_{r^-}^j)) dB_r^j + \int_0^s \int_E (F(X_{r^-}^j, z) - F(\bar{X}_{r^-}^j, z)) \tilde{N}^j(dr, dz),$$

and following the previous computations, we get

$$\mathbb{E} \left[\sup_{s \leq t} |G_0(\mu_s^N) - G_0(\bar{\mu}_s^N)|^2 \right] \leq C \frac{M^2}{m^2} (1+t) \int_0^t \mathbb{E} \left[|X_s^j - \bar{X}_s^j|^2 \right] ds.$$

Consequently, combining the previous estimations with equation (2.3.3) gives

$$\begin{aligned}
\mathbb{E} \left[\sup_{r \leq t} |X_r^j - \bar{X}_r^j|^2 \right] &\leq K \int_0^t \mathbb{E} \left[|X_s^j - \bar{X}_s^j|^2 \right] ds + 4 \mathbb{E} \left[\sup_{s \leq t} |G_0(\bar{\mu}_s^N) - G_0(\mu_s)|^2 \right] \\
&\leq K \int_0^t \mathbb{E} \left[\sup_{r \leq s} |X_r^j - \bar{X}_r^j|^2 \right] ds + 4 \mathbb{E} \left[\sup_{s \leq t} |G_0(\bar{\mu}_s^N) - G_0(\mu_s)|^2 \right],
\end{aligned}$$

where $K = C(1+t)(1+M^2/m^2)$. According to Gronwall's Lemma, we get

$$\mathbb{E} \left[\sup_{r \leq t} |X_r^j - \bar{X}_r^j|^2 \right] \leq C e^{Kt} \mathbb{E} \left[\sup_{s \leq t} |G_0(\bar{\mu}_s^N) - G_0(\mu_s)|^2 \right].$$

In view of Lemma 2.2.2, we have

$$\mathbb{E} \left[\sup_{s \leq t} |G_0(\bar{\mu}_s^N) - G_0(\mu_s)|^2 \right] \leq \frac{1}{m^2} \mathbb{E} \left[\sup_{s \leq t} \left| \int h(\bar{G}_0(\mu_s) + \cdot)(d\bar{\mu}_s^N - d\mu_s) \right|^2 \right],$$

which leads to

$$\mathbb{E} \left[\sup_{r \leq t} |X_r^j - \bar{X}_r^j|^2 \right] \leq C e^{Kt} \frac{1}{m^2} \mathbb{E} \left[\sup_{s \leq t} \left| \int h(\bar{G}_0(\mu_s) + \cdot)(d\bar{\mu}_s^N - d\mu_s) \right|^2 \right]. \quad (2.3.4)$$

Proof of (i). Since h is at least a Lipschitz function, the rate of convergence will be given by the convergence of empirical measure of i.i.d. diffusion processes. As we consider a uniform convergence in time, getting the usual rate of convergence is not straightforward. If we only suppose that (A.2) holds, we obtain that :

$$\frac{1}{m^2} \mathbb{E} \left[\sup_{s \leq t} \left| \int h(\bar{G}_0(\mu_s) + \cdot)(d\bar{\mu}_s^N - d\mu_s) \right|^2 \right] \leq \frac{M^2}{m^2} \mathbb{E} \left[\sup_{s \leq t} W_1^2(\bar{\mu}_s^N, \mu_s) \right].$$

According to the additional Assumption (A.3), and in view of ([BCdRGL16], Theorem 3.2, Proof of (i)), we have

$$\mathbb{E} \left[\sup_{s \leq 1} W_1^2(\bar{\mu}_s^N, \mu_s) \right] \leq CN^{-1/2}.$$

Proof of (ii). Under Assumption (A.4), we can get rid of the supremum in time by using the sharp estimate

$$\mathbb{E} \left[\sup_{s \leq t} \left| \int h(\bar{G}_0(\mu_s) + \cdot)(d\bar{\mu}_s^N - d\mu_s) \right|^2 \right]. \quad (2.3.5)$$

According to Proposition 2.2.3, let ψ be the Radon-Nikodym derivative of $\bar{G}_0(\mu)$. Since $(\bar{U}^i)_{1 \leq i \leq N}$ are independent copies of U , we have

$$\begin{aligned} R_N(s) &:= \int h(\bar{G}_0(\mu_s) + \cdot)(d\bar{\mu}_s^N - d\mu_s) = \frac{1}{N} \sum_{i=1}^N h(\bar{G}_0(\mu_s) + \bar{U}_s^i) - \mathbb{E}[h(\bar{G}_0(\mu_s) + U_s)] \\ &= \frac{1}{N} \sum_{i=1}^N \{h(\bar{G}_0(\mu_s) + \bar{U}_s^i) - \mathbb{E}[h(\bar{G}_0(\mu_s) + \bar{U}_s^i)]\} \\ &= \frac{1}{N} \sum_{i=1}^N \{h(V_s^i) - \mathbb{E}[h(V_s^i)]\}, \end{aligned}$$

where V^i is the semi-martingale $s \mapsto \bar{G}_0(\mu_s) + \bar{U}_s^i$.

It follows from Itô's formula

$$\begin{aligned}
h(V_s^i) &= h(V_0^i) + \int_0^s h'(V_{r-}^i) d\bar{G}_0(\mu_r) + \int_0^s b(\bar{X}_{r-}^i) h'(V_{r-}^i) dr + \int_0^s \sigma(\bar{X}_{r-}^i) h'(V_{r-}^i) dB_r^i \\
&\quad + \int_0^s \int_E F(\bar{X}_{r-}^i, z) h'(V_{r-}^i) \tilde{N}^i(dr, dz) + \frac{1}{2} \int_0^s \sigma^2(\bar{X}_{r-}^i) h''(V_{r-}^i) dr \\
&\quad + \int_0^s \int_E \left(h(V_{r-}^i + F(\bar{X}_{r-}^i, z)) - h(V_{r-}^i) - F(\bar{X}_{r-}^i, z) h'(V_{r-}^i) \right) \lambda(dz) dr \\
&\quad + \int_0^s \int_E \left(h(V_{r-}^i + F(\bar{X}_{r-}^i, z)) - h(V_{r-}^i) - F(\bar{X}_{r-}^i, z) h'(V_{r-}^i) \right) \tilde{N}^i(dr, dz) \\
&= h(V_0^i) + \int_0^s h'(V_{r-}^i) \psi_r dr + \int_0^s b(\bar{X}_{r-}^i) h'(V_{r-}^i) dr + \frac{1}{2} \int_0^s \sigma^2(\bar{X}_{r-}^i) h''(V_{r-}^i) dr \\
&\quad + \int_0^s \int_E \left(h(V_{r-}^i + F(\bar{X}_{r-}^i, z)) - h(V_{r-}^i) - F(\bar{X}_{r-}^i, z) h'(V_{r-}^i) \right) \lambda(dz) dr \\
&\quad + \int_0^s \sigma(\bar{X}_{r-}^i) h'(V_{r-}^i) dB_r^i + \int_0^s \int_E \left(h(V_{r-}^i + F(\bar{X}_{r-}^i, z)) - h(V_{r-}^i) \right) \tilde{N}^i(dr, dz) \\
&= h(V_0^i) + \int_0^s h'(V_{r-}^i) \psi_r dr + \int_0^s \bar{\mathcal{L}}_{\bar{X}_{r-}^i} h(V_{r-}^i) dr + \int_0^s h'(V_{r-}^i) \sigma(\bar{X}_{r-}^i) dB_r^i \\
&\quad + \int_0^s \int_E \left(h(V_{r-}^i + F(\bar{X}_{r-}^i, z)) - h(V_{r-}^i) \right) \tilde{N}^i(dr, dz) \\
&= h(V_0^i) + \int_0^s \{ h'(V_{r-}^i) \psi_r + \bar{\mathcal{L}}_{\bar{X}_{r-}^i} h(V_{r-}^i) \} dr + \int_0^s h'(V_{r-}^i) \sigma(\bar{X}_{r-}^i) dB_r^i \\
&\quad + \int_0^s \int_E \left(h(V_{r-}^i + F(\bar{X}_{r-}^i, z)) - h(V_{r-}^i) \right) \tilde{N}^i(dr, dz),
\end{aligned}$$

Taking expectation gives

$$\begin{aligned}
\mathbb{E} \left[h(V_s^i) \right] &= \mathbb{E} \left[h(V_0^i) \right] + \int_0^s \mathbb{E} [h'(V_{r-}^i) \psi_r + \bar{\mathcal{L}}_{\bar{X}_{r-}^i} h(V_{r-}^i)] dr \\
&= H(\bar{G}_0(\mu_0), \mu_0) + \int_0^s \mathbb{E} [h'(V_{r-}^i) \psi_r + \bar{\mathcal{L}}_{\bar{X}_{r-}^i} h(V_{r-}^i)] dr \\
&= 0 + \int_0^s \mathbb{E} [h'(V_{r-}^i) \psi_r + \bar{\mathcal{L}}_{\bar{X}_{r-}^i} h(V_{r-}^i)] dr.
\end{aligned}$$

Immediately, we deduce that

$$\begin{aligned}
R_N(s) &= \frac{1}{N} \sum_{i=1}^N h(V_0^i) + \frac{1}{N} \sum_{i=1}^N \int_0^s C^i(r) dr + M_N(s) + L_N(s) \\
&= \frac{1}{N} \sum_{i=1}^N h(V_0^i) + \int_0^s \left(\frac{1}{N} \sum_{i=1}^N C^i(r) \right) dr + M_N(s) + L_N(s),
\end{aligned}$$

where

$$\begin{aligned}
C^i(r) &= h'(V_{r-}^i) \psi_r + \bar{\mathcal{L}}_{\bar{X}_{r-}^i} h(V_{r-}^i) - \mathbb{E} [h'(V_{r-}^i) \psi_r + \bar{\mathcal{L}}_{\bar{X}_{r-}^i} h(V_{r-}^i)], \\
M_N(s) &= \frac{1}{N} \sum_{i=1}^N \int_0^s h'(V_{r-}^i) \sigma(\bar{X}_{r-}^i) dB_r^i,
\end{aligned}$$

$$L_N(s) = \frac{1}{N} \sum_{i=1}^N \int_0^s \int_E \left(h(V_{r-}^i + F(\bar{X}_{r-}^i, z)) - h(V_{r-}^i) \right) \tilde{N}^i(dr, dz).$$

Then,

$$\begin{aligned} \sup_{s \leq t} |R_N(s)| &\leq \left| \frac{1}{N} \sum_{i=1}^N h(V_0^i) \right| + \sup_{s \leq t} \int_0^s \left| \frac{1}{N} \sum_{i=1}^N C^i(r) \right| dr + \sup_{s \leq t} |M_N(s)| + \sup_{s \leq t} |L_N(s)| \\ &\leq \left| \frac{1}{N} \sum_{i=1}^N h(V_0^i) \right| + \int_0^t \left| \frac{1}{N} \sum_{i=1}^N C^i(r) \right| dr + \sup_{s \leq t} |M_N(s)| + \sup_{s \leq t} |L_N(s)|. \end{aligned}$$

Since $(U^i)_{1 \leq i \leq N}$ and $(\bar{X}^i)_{1 \leq i \leq N}$ are i.i.d and by using Cauchy-Schwartz inequality, we obtain

$$\begin{aligned} \mathbb{E} \left[\sup_{s \leq t} |R_N(s)|^2 \right] &\leq 4 \left\{ \mathbb{V} \left[\frac{1}{N} \sum_{i=1}^N h(V_0^i) \right] + \mathbb{E} \left[\left(\int_0^t \left| \frac{1}{N} \sum_{i=1}^N C^i(r) \right| dr \right)^2 \right] \right. \\ &\quad \left. + \mathbb{E} \left[\sup_{s \leq t} |M_N(s)|^2 \right] + \mathbb{E} \left[\sup_{s \leq t} |L_N(s)|^2 \right] \right\} \\ &\leq 4 \left\{ \mathbb{V} \left[\frac{1}{N} \sum_{i=1}^N h(V_0^i) \right] + t \mathbb{E} \left[\int_0^t \left| \frac{1}{N} \sum_{i=1}^N C^i(r) \right|^2 dr \right] \right. \\ &\quad \left. + \mathbb{E} \left[\sup_{s \leq t} |M_N(s)|^2 \right] + \mathbb{E} \left[\sup_{s \leq t} |L_N(s)|^2 \right] \right\} \\ &= 4 \left\{ \mathbb{V} \left[\frac{1}{N} \sum_{i=1}^N h(V_0^i) \right] + t \int_0^t \mathbb{V} \left(\frac{1}{N} \sum_{i=1}^N C^i(r) \right) dr \right. \\ &\quad \left. + \mathbb{E} \left[\sup_{s \leq t} |M_N(s)|^2 \right] + \mathbb{E} \left[\sup_{s \leq t} |L_N(s)|^2 \right] \right\}. \end{aligned}$$

Hence, we get

$$\begin{aligned} \mathbb{E} \left[\sup_{s \leq t} |R_N(s)|^2 \right] &\leq \frac{4}{N} \mathbb{V}[h(V_0)] + \frac{4t}{N} \int_0^t \mathbb{V}(C(r)) dr + 4\mathbb{E} \left[\sup_{s \leq t} |M_N(s)|^2 \right] + 4\mathbb{E} \left[\sup_{s \leq t} |L_N(s)|^2 \right] \\ &= \frac{4}{N} \mathbb{V}[h(V_0)] + \frac{4t}{N} \int_0^t \mathbb{V}(h'(V_{r-}) \psi_r + \bar{\mathcal{L}}_{X_{r-}} h(V_{r-})) dr \\ &\quad + 4\mathbb{E} \left[\sup_{s \leq t} |M_N(s)|^2 \right] + 4\mathbb{E} \left[\sup_{s \leq t} |L_N(s)|^2 \right]. \end{aligned}$$

Since M_N is a martingale with

$$\langle M_N \rangle_t = \frac{1}{N^2} \sum_{i=1}^N \int_0^t (h'(V_{r-}^i) \sigma(\bar{X}_{r-}^i))^2 dr,$$

Doob's inequality leads to

$$\begin{aligned}
\mathbb{E} \left[\sup_{s \leq t} |M_N(s)|^2 \right] &\leq 4\mathbb{E} [|M_N(t)|^2] \\
&= \frac{4}{N^2} \sum_{i=1}^N \int_0^t \mathbb{E} \left[(h'(V_{r^-}^i) \sigma(\bar{X}_{r^-}^i))^2 \right] dr \\
&= \frac{4}{N} \int_0^t \mathbb{E} \left[(h'(V_{r^-}) \sigma(X_{r^-}))^2 \right] dr.
\end{aligned}$$

Then, using Doob inequality for the martingale L_N , we obtain

$$\begin{aligned}
\mathbb{E} \left[\sup_{s \leq t} |L_N(s)|^2 \right] &\leq 4\mathbb{E} [|L_N(t)|^2] \\
&= \frac{4}{N^2} \sum_{i=1}^N \mathbb{E} \left[\left(\int_0^t \int_E \left(h(V_{r^-}^i + F(\bar{X}_{r^-}^i, z)) - h(V_{r^-}^i) \right) \tilde{N}^i(dr, dz) \right)^2 \right] \\
&\quad + \frac{8}{N^2} \sum_{1 \leq i < j \leq N} \mathbb{E} \left[\int_0^t \int_E \left(h(V_{r^-}^i + F(\bar{X}_{r^-}^i, z)) - h(V_{r^-}^i) \right) \tilde{N}^i(dr, dz) \right. \\
&\quad \left. \int_0^t \int_E \left(h(V_{r^-}^j + F(\bar{X}_{r^-}^j, z)) - h(V_{r^-}^j) \right) \tilde{N}^j(dr, dz) \right] \\
&= \frac{4}{N^2} \sum_{i=1}^N \int_0^t \int_E \mathbb{E} \left[\left(h(V_{r^-}^i + F(\bar{X}_{r^-}^i, z)) - h(V_{r^-}^i) \right)^2 \right] \lambda(dz) dr \\
&= \frac{4}{N} \int_0^t \int_E \mathbb{E} \left[\left(h(V_{r^-}^i + F(\bar{X}_{r^-}^i, z)) - h(V_{r^-}^i) \right)^2 \right] \lambda(dz) dr.
\end{aligned}$$

Finally, using the fact that h has bounded derivatives, b , σ and F are Lipschitz, we get

$$\mathbb{E} \left[\sup_{s \leq t} |R_N(s)|^2 \right] \leq C(1 + t^2) \left(1 + \mathbb{E} \left[\sup_{s \leq t} |X_s|^2 \right] \right) N^{-1}.$$

This gives the result coming back to (2.3.4). □

2.4 Numerical approximation and its performance for MRSDE.

In this section, the numerical approximation of the SDE (3.1.2) on $[0, T]$ is studied. Let $0 = T_0 < T_1 < \dots < T_n = T$ be a subdivision of $[0, T]$ and define "·" the mapping $s \mapsto \underline{s} = T_k$ if $s \in [T_k, T_{k+1})$, $k \in \{0, \dots, n-1\}$. Let us consider the case of regular subdivisions : for a given integer n , $T_k = kT/n$, $k = 0, \dots, n$.

According to the previous section, we have shown that the particle system, for $1 \leq i \leq N$,

$$X_t^i = \bar{X}_0^i + \int_0^t b(X_{s^-}^i) ds + \int_0^t \sigma(X_{s^-}^i) dB_s^i + \int_0^t \int_E F(X_{s^-}^i, z) \tilde{N}^i(ds, dz) + \sup_{s \leq t} G_0(\mu_s^N),$$

where

$$\mu_t^N = \frac{1}{N} \sum_{i=1}^N \delta_{U_t^i},$$

$$U_t^i = \bar{X}_0^i + \int_0^t b(X_{s-}^i) ds + \int_0^t \sigma(X_{s-}^i) dB_s^i + \int_0^t \int_E F(X_{s-}^i, z) \tilde{N}^i(ds, dz), \quad 1 \leq i \leq N,$$

B^i being independent Brownian motions, N^i being independent Poisson processes and \bar{X}_0^i being independent copies of X_0 , converges to the solution of (3.1.2). Hence, to determine the numerical approximation, we apply an Euler scheme to this particle system. The discrete version of the particle system is : for $1 \leq i \leq N$,

$$\tilde{X}_t^i = \bar{X}_0^i + \int_0^t b(\tilde{X}_{\underline{s}-}^i) ds + \int_0^t \sigma(\tilde{X}_{\underline{s}-}^i) dB_s^i + \int_0^t \int_E F(\tilde{X}_{\underline{s}-}^i, z) \tilde{N}^i(ds, dz) + \sup_{s \leq t} G_0(\tilde{\mu}_{\underline{s}}^N),$$

where

$$\tilde{\mu}_{\underline{t}}^N = \frac{1}{N} \sum_{i=1}^N \delta_{\tilde{U}_{\underline{t}}^i},$$

$$\tilde{U}_{\underline{t}}^i = \bar{X}_0^i + \int_0^{\underline{t}} b(\tilde{X}_{\underline{s}-}^i) ds + \int_0^{\underline{t}} \sigma(\tilde{X}_{\underline{s}-}^i) dB_s^i + \int_0^{\underline{t}} \int_E F(\tilde{X}_{\underline{s}-}^i, z) \tilde{N}^i(ds, dz), \quad 1 \leq i \leq N.$$

2.4.1 Scheme.

In view of the above notations and taking into account the result on the interacting system of mean reflected particles of the MR-SDE of Section 2.3 and Remark 2.2.1, we deduce the following algorithm for the numerical approximation of the MR-SDE.

Remark 2.4.1. *It should be pointed out that, at each step k of the algorithm, the increment of the reflection process K is approximated by the increment of the following approximation :*

$$\Delta_k \hat{K}^N := \sup_{l \leq k} G_0(\tilde{\mu}_{T_l}^N) - \sup_{l \leq k-1} G_0(\tilde{\mu}_{T_l}^N). \quad (2.4.1)$$

First, we consider the special case when the SDE is defined by

$$\begin{cases} X_t = X_0 + \int_0^t b(X_{s-}) ds + \int_0^t \sigma(X_{s-}) dB_s + \int_0^t F(X_{s-}) d\tilde{N}_s + K_t, & t \geq 0, \\ \mathbb{E}[h(X_t)] \geq 0, \quad \int_0^t \mathbb{E}[h(X_s)] dK_s = 0, & t \geq 0. \end{cases}$$

where N is a Poisson process with intensity λ , and $\tilde{N}_t = N_t - \lambda t$.

According to Remark 2.2.1, the increment (2.4.1) can be estimated by :

$$\widehat{\Delta_k K}^N :=$$

$$\inf \left\{ x \geq 0 : \frac{1}{N} \sum_{i=1}^N h \left(x + \left(\tilde{X}_{T_{k-1}}^{\tilde{\mu}^N} \right)^i + \frac{T}{n} \left(b \left(\left(\tilde{X}_{T_{k-1}}^{\tilde{\mu}^N} \right)^i \right) - \lambda F \left(\left(\tilde{X}_{T_{k-1}}^{\tilde{\mu}^N} \right)^j \right) + \frac{\sqrt{T}}{\sqrt{n}} \sigma \left(\left(\tilde{X}_{T_{k-1}}^{\tilde{\mu}^N} \right)^i \right) G^i + F \left(\left(\tilde{X}_{T_{k-1}}^{\tilde{\mu}^N} \right)^i \right) H^i \right) \geq 0 \right\},$$

where $G^j \sim \mathcal{N}(0, 1)$ and $H^j \sim \mathcal{P}(\lambda(T/n))$ and are i.i.d..

Algorithm 1 Particle approximation

- 1: **for** $1 \leq j \leq N$ **do**
 - 2: $\left(\left(\tilde{X}_0^{\tilde{\mu}^N} \right)^j, \left(\tilde{U}_0^{\tilde{\mu}^N} \right)^j, \tilde{\mu}_0^N \right) = (x, x, \delta_x)$
 - 3: **end for**
 - 4: **for** $1 \leq k \leq n$ **do**
 - 5: **for** $1 \leq j \leq N$ **do**
 - 6: $G^j \sim \mathcal{N}(0, 1)$
 - 7: $H^j \sim \mathcal{P}(\lambda(T/n))$
 - 8: $\left(\tilde{U}_{T_k}^{\tilde{\mu}^N} \right)^j = \left(\tilde{U}_{T_{k-1}}^{\tilde{\mu}^N} \right)^j + (T/n) \left(b \left(\left(\tilde{X}_{T_{k-1}}^{\tilde{\mu}^N} \right)^j \right) - \lambda F \left(\left(\tilde{X}_{T_{k-1}}^{\tilde{\mu}^N} \right)^j \right) \right)$
 - 9: $+ \sqrt{(T/n)} \sigma \left(\left(\tilde{X}_{T_{k-1}}^{\tilde{\mu}^N} \right)^j \right) G^j + F \left(\left(\tilde{X}_{T_{k-1}}^{\tilde{\mu}^N} \right)^j \right) H^j$
 - 10: **end for**
 - 11: $\tilde{\mu}_{T_k}^N = N^{-1} \sum_{j=1}^N \delta_{\left(\tilde{U}_{T_k}^{\tilde{\mu}^N} \right)^j}$
 - 12: $\Delta_k \hat{K}^N = \sup_{l \leq k} G_0(\tilde{\mu}_{T_l}^N) - \sup_{l \leq k-1} G_0(\tilde{\mu}_{T_l}^N)$
 - 13: **for** $1 \leq j \leq N$ **do**
 - 14: $\left(\tilde{X}_{T_k}^{\tilde{\mu}^N} \right)^j = \left(\tilde{X}_{T_{k-1}}^{\tilde{\mu}^N} \right)^j + (T/n) \left(b \left(\left(\tilde{X}_{T_{k-1}}^{\tilde{\mu}^N} \right)^j \right) - \lambda F \left(\left(\tilde{X}_{T_{k-1}}^{\tilde{\mu}^N} \right)^j \right) \right)$
 - 15: $+ \sqrt{(T/n)} \sigma \left(\left(\tilde{X}_{T_{k-1}}^{\tilde{\mu}^N} \right)^j \right) G^j + F \left(\left(\tilde{X}_{T_{k-1}}^{\tilde{\mu}^N} \right)^j \right) H^j + \Delta_k \hat{K}^N$
 - 16: **end for**
 - 17: **end for**
-

In addition, similar procedures as in the proof of Theorem 2.2.1 can be used to verify that the increments of the approximated reflection process are equal to the approximation of the increments :

$$\forall k \in \{1, \dots, n\} : \widehat{\Delta_k K}^N = \Delta_k \hat{K}^N.$$

Returning to the general case (3.1.2), we can see in [YS12], $N = \{N(t) := N(E \times [0, t])\}$ is a stochastic process with intensity λ that counts the number of jumps until some given time. The Poisson random measure $N(dz, dt)$ generates a sequence of pairs $\{(\iota_i, \xi_i), i \in \{1, 2, \dots, N(T)\}\}$ for a given finite positive constant T if $\lambda < \infty$. Here $\{\iota_i, i \in \{1, 2, \dots, N(T)\}\}$ is a sequence of increasing nonnegative random variables representing the jump times of a standard Poisson process with intensity λ , and $\{\xi_i, i \in \{1, 2, \dots, N(T)\}\}$ is a sequence of independent identically distributed random variables, where ξ_i is distributed according to $f(z)$, where $\lambda(dz)dt = \lambda f(z)dzdt$. Then, the numerical approximation can equivalently be written in the following form

$$\begin{aligned} \bar{X}_{T_k}^j &= \bar{X}_{T_{k-1}}^j + \frac{T}{n} \left(b(\bar{X}_{T_{k-1}}^j) - \int_E \lambda F(\bar{X}_{T_{k-1}}^j, z) f(z) dz \right) + \sqrt{\frac{T}{n}} \sigma(\bar{X}_{T_{k-1}}^j) G^j \\ &+ \sum_{i=H_{T_{k-1}}^j+1}^{H_{T_k}^j} F(\bar{X}_{T_{k-1}}^j, \xi_i) + \Delta_k \hat{K}^N, \end{aligned}$$

$$\Delta_k \hat{K}^N = \widehat{\Delta_k K}^N =$$

$$\inf \left\{ x \geq 0 : \frac{1}{N} \sum_{i=1}^N h \left(x + \bar{X}_{T_{k-1}}^j + \frac{T}{n} \left(b(\bar{X}_{T_{k-1}}^j) - \int_E \lambda F(\bar{X}_{T_{k-1}}^j, z) f(z) dz \right) + \sqrt{\frac{T}{n}} \sigma(\bar{X}_{T_{k-1}}^j) G^j \right. \right. \\ \left. \left. + \sum_{i=H_{T_{k-1}}^j+1}^{H_{T_k}^j} F(\bar{X}_{T_{k-1}}^j, \xi_i) \right) \geq 0 \right\},$$

where $G^j \sim \mathcal{N}(0, 1)$ and $H^j \sim \mathcal{P}(\lambda(T/n))$ and are i.i.d..

2.4.2 Scheme error.

Proposition 2.4.1. (i) Let $T > 0$, N and n be two non-negative integers and let Assumptions (A.1), (A.2) and (A.3) hold. There exists a constant C , depending on T , b , σ , F , h and X_0 but independent of N , such that : for all $i = 1, \dots, N$

$$\mathbb{E} \left[\sup_{s \leq t} |X_s^i - \tilde{X}_s^i|^2 \right] \leq C \left(n^{-1} + N^{-1/2} \right).$$

(ii) Moreover, if Assumption (A.4) is in force, there exists a constant C , depending on T , b , σ , F , h and X_0 but independent of N , such that : for all $i = 1, \dots, N$

$$\mathbb{E} \left[\sup_{s \leq t} |X_s^i - \tilde{X}_s^i|^2 \right] \leq C \left(n^{-1} + N^{-1} \right).$$

Proof. Let us fix $i \in 1, \dots, N$ and $T > 0$. We have, for $t \leq T$,

$$\begin{aligned} |X_s^i - \tilde{X}_s^i| &\leq \left| \int_0^s b(X_{r-}^i) - b(\tilde{X}_{r-}^i) dr \right| + \left| \int_0^s \left(\sigma(X_{r-}^i) - \sigma(\tilde{X}_{r-}^i) \right) dB_r^i \right| \\ &+ \left| \int_0^s \int_E \left(F(X_{r-}^i, z) - F(\tilde{X}_{r-}^i, z) \right) \tilde{N}^i(dr, dz) \right| + \sup_{r \leq s} |G_0(\mu_r^N) - G_0(\tilde{\mu}_r^N)|. \end{aligned}$$

Hence, using Assumption (A.1), Cauchy-Schwartz, Doob and BDG inequalities gives

$$\begin{aligned}
\mathbb{E} \left[\sup_{s \leq t} |X_s^i - \tilde{X}_s^i|^2 \right] &\leq 4\mathbb{E} \left[\sup_{s \leq t} \left\{ \left| \int_0^s \left(b(X_{r^-}^i) - b(\tilde{X}_{r^-}^i) \right) dr \right|^2 + \left| \int_0^s \left(\sigma(X_{r^-}^i) - \sigma(\tilde{X}_{r^-}^i) \right) dB_r^i \right|^2 \right. \right. \\
&\quad \left. \left. + \left| \int_0^s \int_E \left(F(X_{r^-}^i, z) - F(\tilde{X}_{r^-}^i, z) \right) \tilde{N}^i(dr, dz) \right|^2 + \sup_{r \leq s} |G_0(\mu_r^N) - G_0(\tilde{\mu}_r^N)|^2 \right\} \right] \\
&\leq C \left\{ \mathbb{E} \left[t \int_0^t \left| b(X_{s^-}^i) - b(\tilde{X}_{s^-}^i) \right|^2 ds \right] + \mathbb{E} \left[\int_0^t \left| \sigma(X_{s^-}^i) - \sigma(\tilde{X}_{s^-}^i) \right|^2 ds \right] \right. \\
&\quad \left. + \mathbb{E} \left[\int_0^t \int_E \left| F(X_{s^-}^i, z) - F(\tilde{X}_{s^-}^i, z) \right|^2 \lambda(dz) ds \right] + \mathbb{E} \left[\sup_{s \leq t} |G_0(\mu_s^N) - G_0(\tilde{\mu}_s^N)|^2 \right] \right\} \\
&\leq C \left\{ TC_1 \int_0^t \mathbb{E} \left[|X_s^i - \tilde{X}_s^i|^2 \right] ds + C_1 \int_0^t \mathbb{E} \left[|X_s^i - \tilde{X}_s^i|^2 \right] ds \right. \\
&\quad \left. + C_1 \int_0^t \mathbb{E} \left[|X_s^i - \tilde{X}_s^i|^2 \right] ds + \mathbb{E} \left[\sup_{s \leq t} |G_0(\mu_s^N) - G_0(\tilde{\mu}_s^N)|^2 \right] \right\} \\
&\leq C \int_0^t \mathbb{E} \left[|X_s^i - \tilde{X}_s^i|^2 \right] ds + 4\mathbb{E} \left[\sup_{s \leq t} |G_0(\mu_s^N) - G_0(\tilde{\mu}_s^N)|^2 \right].
\end{aligned} \tag{2.4.2}$$

Denoting by $(\mu_t^i)_{0 \leq t \leq T}$ the family of marginal laws of $(U_t^i)_{0 \leq t \leq T}$ and $(\tilde{\mu}_t^i)_{0 \leq t \leq T}$ the family of marginal laws of $(\tilde{U}_t^i)_{0 \leq t \leq T}$, we have

$$\begin{aligned}
\mathbb{E} \left[\sup_{s \leq t} |G_0(\mu_s^N) - G_0(\tilde{\mu}_s^N)|^2 \right] &\leq 3 \left\{ \mathbb{E} \left[\sup_{s \leq t} |G_0(\mu_s^N) - G_0(\mu_s^i)|^2 \right] + \sup_{s \leq t} |G_0(\mu_s^i) - G_0(\tilde{\mu}_s^i)|^2 \right. \\
&\quad \left. + \mathbb{E} \left[\sup_{s \leq t} |G_0(\tilde{\mu}_s^i) - G_0(\tilde{\mu}_s^N)|^2 \right] \right\},
\end{aligned}$$

and from Lemma 2.2.2,

$$\begin{aligned}
&\leq 3 \left\{ \frac{1}{m^2} \mathbb{E} \left[\sup_{s \leq t} \left| \int h(\bar{G}_0(\mu_s^i) + \cdot)(d\mu_s^N - d\mu_s^i) \right|^2 \right] + \left(\frac{M}{m} \right)^2 \sup_{s \leq t} W_1^2(\mu_s^i, \tilde{\mu}_s^i) \right. \\
&\quad \left. + \frac{1}{m^2} \mathbb{E} \left[\sup_{s \leq t} \left| \int h(\bar{G}_0(\tilde{\mu}_s^i) + \cdot)(d\tilde{\mu}_s^N - d\tilde{\mu}_s^i) \right|^2 \right] \right\} \\
&\leq C \left\{ \mathbb{E} \left[\sup_{s \leq t} \left| \int h(\bar{G}_0(\mu_s^i) + \cdot)(d\mu_s^N - d\mu_s^i) \right|^2 \right] + \sup_{s \leq t} \mathbb{E} \left[|U_s^i - \tilde{U}_s^i|^2 \right] \right. \\
&\quad \left. + \mathbb{E} \left[\sup_{s \leq t} \left| \int h(\bar{G}_0(\tilde{\mu}_s^i) + \cdot)(d\tilde{\mu}_s^N - d\tilde{\mu}_s^i) \right|^2 \right] \right\}.
\end{aligned}$$

Proof of (i). Following the Proof of (i) in Theorem 2.3.1, we obtain

$$\mathbb{E} \left[\sup_{s \leq t} \left| \int h(\bar{G}_0(\mu_s^i) + \cdot)(d\mu_s^N - d\mu_s^i) \right|^2 \right] \leq C \mathbb{E} \left[\sup_{s \leq t} W_1^2(\mu_s^N, \mu_s^i) \right] \leq CN^{-1/2},$$

$$\mathbb{E} \left[\sup_{s \leq t} \left| \int h(\bar{G}_0(\tilde{\mu}_s^i) + \cdot)(d\tilde{\mu}_s^N - d\tilde{\mu}_s^i) \right|^2 \right] \leq C \mathbb{E} \left[\sup_{s \leq t} W_1^2(\tilde{\mu}_s^i, \tilde{\mu}_s^N) \right] \leq CN^{-1/2}.$$

From which we can derive the inequality

$$\begin{aligned} \mathbb{E} \left[\sup_{s \leq t} |G_0(\mu_s^N) - G_0(\tilde{\mu}_s^N)|^2 \right] &\leq C_1 \sup_{s \leq t} \mathbb{E} \left[\left| U_s^i - \tilde{U}_s^i \right|^2 \right] + C_2 N^{-1/2} \\ &\leq C_1 \left\{ \sup_{s \leq t} \mathbb{E} \left[\left| U_s^i - \tilde{U}_s^i \right|^2 \right] + \sup_{s \leq t} \mathbb{E} \left[\left| \tilde{U}_s^i - \tilde{U}_s^i \right|^2 \right] \right\} + C_2 N^{-1/2}. \end{aligned}$$

For the first term of the right hand side, we can observe that

$$\begin{aligned} \sup_{s \leq t} \mathbb{E} \left[\left| U_s^i - \tilde{U}_s^i \right|^2 \right] &\leq \mathbb{E} \left[\sup_{s \leq t} \left| U_s^i - \tilde{U}_s^i \right|^2 \right] \\ &\leq 3 \mathbb{E} \left[\sup_{s \leq t} \left\{ \left| \int_0^s \left(b(X_{r-}^i) - b(\tilde{X}_{r-}^i) \right) dr \right|^2 + \left| \int_0^s \left(\sigma(X_{r-}^i) - \sigma(\tilde{X}_{r-}^i) \right) dB_r^i \right|^2 \right. \right. \\ &\quad \left. \left. + \left| \int_0^s \int_E \left(F(X_{r-}^i, z) - F(\tilde{X}_{r-}^i, z) \right) \tilde{N}^i(dr, dz) \right|^2 \right\} \right] \\ &\leq C \left\{ \mathbb{E} \left[t \int_0^t \left| b(X_{s-}^i) - b(\tilde{X}_{s-}^i) \right|^2 ds \right] + \mathbb{E} \left[\int_0^t \left| \sigma(X_{s-}^i) - \sigma(\tilde{X}_{s-}^i) \right|^2 ds \right] \right. \\ &\quad \left. + \mathbb{E} \left[\int_0^t \int_E \left| F(X_{s-}^i, z) - F(\tilde{X}_{s-}^i, z) \right|^2 \lambda(dz) ds \right] \right\} \\ &\leq C \left\{ TC_1 \int_0^t \mathbb{E} \left[|X_s^i - \tilde{X}_s^i|^2 \right] dr + 2C_1 \int_0^t \mathbb{E} \left[|X_s^i - \tilde{X}_s^i|^2 \right] dr \right\} \\ &\leq C \int_0^t \mathbb{E} \left[|X_s^i - \tilde{X}_s^i|^2 \right] ds. \end{aligned}$$

Using Assumption (A.1), the second term $\sup_{s \leq t} \mathbb{E} \left[\left| \tilde{U}_s^i - \tilde{U}_{\underline{s}}^i \right|^2 \right]$ becomes

$$\begin{aligned}
\sup_{s \leq t} \mathbb{E} \left[\left| \tilde{U}_s^i - \tilde{U}_{\underline{s}}^i \right|^2 \right] &\leq 3 \sup_{s \leq t} \left\{ \mathbb{E} \left[\left| \int_{\underline{s}}^s b(\tilde{X}_{r^-}^i) dr \right|^2 + \left| \int_{\underline{s}}^s \sigma(\tilde{X}_{r^-}^i) dB_r^i \right|^2 + \left| \int_{\underline{s}}^s \int_E F(\tilde{X}_{r^-}^i, z) \tilde{N}^i(dr, dz) \right|^2 \right] \right\} \\
&\leq 3 \sup_{s \leq t} \left\{ \mathbb{E} \left[\left| b(\tilde{X}_{\underline{s}}^i) \right|^2 |s - \underline{s}|^2 + \left| \sigma(\tilde{X}_{\underline{s}}^i) \right|^2 |B_s^i - B_{\underline{s}}^i|^2 + \int_{\underline{s}}^s \int_E \left| F(\tilde{X}_{r^-}^i, z) \right|^2 \lambda(dz) dr \right] \right\} \\
&\leq 3 \sup_{s \leq t} \left\{ \mathbb{E} \left[\left| b(\tilde{X}_{\underline{s}}^i) \right|^2 |s - \underline{s}|^2 + \left| \sigma(\tilde{X}_{\underline{s}}^i) \right|^2 |B_s^i - B_{\underline{s}}^i|^2 + C \int_{\underline{s}}^s (1 + |\tilde{X}_{r^-}^i|^2) dr \right] \right\} \\
&\leq 3 \sup_{s \leq t} \left\{ \left(\frac{T}{n} \right)^2 \mathbb{E} \left[\sup_{\underline{s} \leq r \leq s} |b(\tilde{X}_{r^-}^i)|^2 \right] + \mathbb{E} \left[|B_s^i - B_{\underline{s}}^i|^2 \right] \mathbb{E} \left[|\sigma(\tilde{X}_{\underline{s}}^i)|^2 \right] \right. \\
&\quad \left. + C \left(\frac{T}{n} \right) \mathbb{E} \left[\sup_{\underline{s} \leq r \leq s} (1 + |\tilde{X}_r^i|^2) \right] \right\} \\
&\leq C_1 \left(\frac{T}{n} \right)^2 \mathbb{E} \left[\sup_{s \leq T} |b(\tilde{X}_s^i)|^2 \right] + C_2 \sup_{s \leq t} \mathbb{E} \left[|B_s^i - B_{\underline{s}}^i|^2 \right] \mathbb{E} \left[\sup_{s \leq T} |\sigma(\tilde{X}_s^i)|^2 \right] \\
&\quad + C_3 \left(\frac{T}{n} \right) \mathbb{E} \left[\sup_{s \leq T} (1 + |\tilde{X}_s^i|^2) \right] \\
&\leq C_1 \left(\frac{T}{n} \right)^2 \left(1 + \mathbb{E} \left[\sup_{s \leq T} |\tilde{X}_s^i|^2 \right] \right) + C_2 \sup_{s \leq t} \mathbb{E} \left[|B_s^i - B_{\underline{s}}^i|^2 \right] \left(1 + \mathbb{E} \left[\sup_{s \leq T} |\tilde{X}_s^i|^2 \right] \right) \\
&\quad + C_3 \left(\frac{T}{n} \right) \left(1 + \mathbb{E} \left[\sup_{s \leq T} |\tilde{X}_s^i|^2 \right] \right),
\end{aligned}$$

and from Proposition 2.2.1, we get

$$\sup_{s \leq t} \mathbb{E} \left[\left| \tilde{U}_s^i - \tilde{U}_{\underline{s}}^i \right|^2 \right] \leq C_1 \left(\frac{T}{n} \right) + C_2 \sup_{s \leq t} \mathbb{E} \left[|B_s^i - B_{\underline{s}}^i|^2 \right].$$

Then, by using BDG inequality, we obtain

$$\sup_{s \leq t} \mathbb{E} \left[|B_s^i - B_{\underline{s}}^i|^2 \right] = \sup_{s \leq t} \mathbb{E} \left[\left(\int_{\underline{s}}^s dB_u^i \right)^2 \right] \leq \sup_{s \leq t} |s - \underline{s}| \leq \frac{T}{n}.$$

Therefore, we conclude

$$\begin{aligned}
\sup_{s \leq t} \mathbb{E} \left[\left| \tilde{U}_s^i - \tilde{U}_{\underline{s}}^i \right|^2 \right] &\leq C_1 n^{-1} + C_2 n^{-1} \\
&\leq C n^{-1},
\end{aligned} \tag{2.4.3}$$

from which we derive the inequality

$$\mathbb{E} \left[\sup_{s \leq t} |G_0(\mu_s^N) - G_0(\tilde{\mu}_{\underline{s}}^N)|^2 \right] \leq C \left\{ n^{-1} + N^{-1/2} + \int_0^t \mathbb{E} \left[|X_s^i - \tilde{X}_{\underline{s}}^i|^2 \right] ds \right\}, \tag{2.4.4}$$

and taking into account (2.4.2) we get

$$\mathbb{E} \left[\sup_{s \leq t} |X_s^i - \tilde{X}_s^i|^2 \right] \leq C \left\{ n^{-1} + N^{-1/2} + \int_0^t \mathbb{E} \left[|X_s^i - \tilde{X}_s^i|^2 \right] ds \right\}. \quad (2.4.5)$$

Since

$$\begin{aligned} \mathbb{E} \left[|X_s^i - \tilde{X}_s^i|^2 \right] &\leq 2\mathbb{E} \left[|X_s^i - \tilde{X}_s^i|^2 \right] + 2\mathbb{E} \left[|\tilde{X}_s^i - \tilde{X}_s^i|^2 \right] \\ &= 2\mathbb{E} \left[|X_s^i - \tilde{X}_s^i|^2 \right] + 2\mathbb{E} \left[|\tilde{U}_s^i - \tilde{U}_s^i|^2 \right], \end{aligned}$$

it follows from (2.4.3) and (2.4.5) that

$$\mathbb{E} \left[\sup_{s \leq t} |X_s^i - \tilde{X}_s^i|^2 \right] \leq C \left\{ n^{-1} + N^{-1/2} + \int_0^t \mathbb{E} \left[|X_s^i - \tilde{X}_s^i|^2 \right] ds \right\}.$$

and finally, we conclude the proof of (i) with Gronwall's Lemma.

Proof of (ii). Following the proof of (ii) in Theorem 2.3.1, we obtain

$$\begin{aligned} \mathbb{E} \left[\sup_{s \leq t} \left| \int h(\bar{G}_0(\mu_s^i) + \cdot)(d\mu_s^N - d\mu_s^i) \right|^2 \right] &\leq CN^{-1}, \\ \mathbb{E} \left[\sup_{s \leq t} \left| \int h(\bar{G}_0(\tilde{\mu}_s^i) + \cdot)(d\tilde{\mu}_s^N - d\tilde{\mu}_s^i) \right|^2 \right] &\leq CN^{-1}. \end{aligned}$$

By using the same strategy as the one applied in the proof of (i) in Theorem 2.4.1, the result follows easily :

$$\mathbb{E} \left[\sup_{s \leq t} |X_s^i - \tilde{X}_s^i|^2 \right] \leq C \left(n^{-1} + N^{-1} \right).$$

□

Theorem 2.4.1. *Let $T > 0$, N and n be two non-negative integers and assumptions (A.1), (A.2) and (A.3) hold.*

(i) *There exists a constant C , depending on T , b , σ , F , h and X_0 but independent of N , such that : for all $i = 1, \dots, N$,*

$$\mathbb{E} \left[\sup_{t \leq T} |\bar{X}_t^i - \tilde{X}_t^i|^2 \right] \leq C \left(n^{-1} + N^{-1/2} \right).$$

(ii) *If in addition (A.4) holds, there exists a positive constant C , depending on T , b , σ , F , h and X_0 but independent of N , such that : for all $i = 1, \dots, N$,*

$$\mathbb{E} \left[\sup_{t \leq T} |\bar{X}_t^i - \tilde{X}_t^i|^2 \right] \leq C \left(n^{-1} + N^{-1} \right).$$

Proof. The proof is straightforward writing

$$|\bar{X}_t^i - \tilde{X}_t^i| \leq |\bar{X}_t^i - X_t^i| + |X_t^i - \tilde{X}_t^i|,$$

and using Theorem 2.3.1 and Proposition 2.4.1. \square

2.5 Numerical examples.

In this section, let us study on $[0, T]$ the following sort of processes : for all $t \geq 0$,

$$\begin{cases} X_t = X_0 - \int_0^t (\beta_s + a_s X_{s-}) ds + \int_0^t (\sigma_s + \gamma_s X_{s-}) dB_s + \int_0^t \int_E c(z)(\eta_s + \theta_s X_{s-}) \tilde{N}(ds, dz) + K_t, \\ \mathbb{E}[h(X_t)] \geq 0, \quad \int_0^t \mathbb{E}[h(X_s)] dK_s = 0, \end{cases} \quad (2.5.1)$$

where $(\beta_t)_{t \geq 0}$, $(a_t)_{t \geq 0}$, $(\sigma_t)_{t \geq 0}$, $(\gamma_t)_{t \geq 0}$, $(\eta_t)_{t \geq 0}$ and $(\theta_t)_{t \geq 0}$ are bounded adapted processes. This sort of processes is chosen to make some explicit computations which allow the illustration of the algorithm. Different diffusions and functions h are considered in order to illustrate our results.

Linear constraint. Firstly, we consider the cases where $h : \mathbb{R} \ni x \mapsto x - p \in \mathbb{R}$.

Case (i) Drifted Brownian motion and compensated Poisson process : $\beta_t = \beta > 0$, $a_t = \gamma_t = \theta_t = 0$, $\sigma_t = \sigma > 0$, $\eta_t = \eta > 0$, $X_0 = x_0 \geq p$, $c(z) = z$ and

$$f(z) = \frac{1}{\sqrt{2\pi z}} \exp\left(-\frac{(\ln z)^2}{2}\right) \mathbf{1}_{\{0 < z\}}.$$

We have

$$K_t = (p + \beta t - x_0)^+,$$

and

$$X_t = X_0 - (\beta + \lambda\sqrt{e})t + \sigma B_t + \sum_{i=0}^{N_t} \eta \xi_i + K_t,$$

where $N_t \sim \mathcal{P}(\lambda t)$ and $\xi_i \sim \text{lognormal}(0, 1)$.

Case (ii) Black and Scholes process : $\beta_t = \sigma_t = \eta_t = 0$, $a_t = a > 0$, $\gamma_t = \gamma > 0$, $\theta_t = \theta > 0$, $c(z) = \delta_1(z)$. Then

$$K_t = ap(t - t^*) \mathbf{1}_{t \geq t^*}, \text{ where } t^* = \frac{1}{a}(\ln(x_0) - \ln(p)),$$

and

$$X_t = Y_t + Y_t \int_0^t Y_s^{-1} dK_s,$$

where Y is the process defined by :

$$Y_t = X_0 \exp\left(-\left(a + \gamma^2/2 + \lambda\theta\right)t + \gamma B_t\right) (1 + \theta)^{N_t}.$$

Nonlinear constraint. Secondly, we illustrate the case of non-linear function h :

$$h : \mathbb{R} \ni x \mapsto x + \alpha \sin(x) - p \in \mathbb{R}, \quad -1 < \alpha < 1,$$

and we illustrate this case with

Case (iii) Ornstein Uhlenbeck process : $\beta_t = \beta > 0$, $a_t = a > 0$, $\gamma_t = \theta_t = 0$, $\sigma_t = \sigma > 0$, $\eta_t = \eta > 0$, $X_0 = x_0$ with $x_0 > |\alpha| + p$, $c(z) = \delta_1(z)$. We obtain

$$dK_t = e^{-at} d \sup_{s \leq t} (F_s^{-1}(0))^+,$$

where for all t in $[0, T]$,

$$\begin{aligned} F_t : \mathbb{R} \ni x \mapsto & \\ & \left\{ e^{-at} \left(x_0 - \beta \left(\frac{e^{at} - 1}{a} \right) + x \right) + \alpha \exp \left(- e^{-at} \frac{\sigma^2}{2a} \sinh(at) \right) \right. \\ & \times \left[\frac{1}{2} \left(\exp(\lambda t (e^{i\eta} - 1)) + \exp(\lambda t (e^{-i\eta} - 1)) \right) \sin \left(e^{-at} \left(x_0 - (\beta + \lambda\eta) \left(\frac{e^{-at} - 1}{a} \right) + x \right) \right) \right. \\ & \left. \left. + \frac{1}{2i} \left(\exp(\lambda t (e^{i\eta} - 1)) - \exp(\lambda t (e^{-i\eta} - 1)) \right) \cos \left(e^{-at} \left(x_0 - (\beta + \lambda\eta) \left(\frac{e^{-at} - 1}{a} \right) + x \right) \right) \right] \right. \\ & \left. - p \right\} \end{aligned}$$

Remark 2.5.1. *We choose these examples in order to obtain an analytic form of the true reflecting process K which can be compared numerically with its empirical approximation \hat{K} . Having the exact simulation of the underlying process, we can verify the efficiency of our algorithm.*

2.5.1 Proofs of the numerical illustrations.

In order to have closed, or almost closed, expression for the compensator K we introduce the process Y solution to the non-reflected SDE

$$Y_t = X_0 - \int_0^t (\beta_s + a_s Y_{s-}) ds + \int_0^t (\sigma_s + \gamma_s Y_{s-}) dB_s + \int_0^t \int_E c(z) (\eta_s + \theta_s Y_{s-}) \tilde{N}(ds, dz).$$

By letting $A_s = \int_0^s a_s ds$ and applying Its formula on $e^{A_t} X_t$ and $e^{A_t} Y_t$, we get

$$\begin{aligned} e^{A_t} X_t &= X_0 + \int_0^t e^{A_s} X_s a_s ds + \int_0^t e^{A_s} (-\beta_s - a_s X_{s-}) ds + \int_0^t e^{A_s} (\sigma_s + \gamma_s X_{s-}) dB_s \\ &\quad + \int_0^t \int_E e^{A_s} c(z) (\eta_s + \theta_s X_{s-}) \tilde{N}(ds, dz) + \int_0^t e^{A_s} dK_s \\ &= X_0 - \int_0^t e^{A_s} \beta_s ds + \int_0^t e^{A_s} (\sigma_s + \gamma_s X_{s-}) dB_s + \int_0^t \int_E e^{A_s} c(z) (\eta_s + \theta_s X_{s-}) \tilde{N}(ds, dz) \\ &\quad + \int_0^t e^{A_s} dK_s. \end{aligned}$$

In the same way,

$$e^{A_t} Y_t = X_0 - \int_0^t e^{A_s} \beta_s ds + \int_0^t e^{A_s} (\sigma_s + \gamma_s Y_{s-}) dB_s + \int_0^t \int_E e^{A_s} c(z) (\eta_s + \theta_s Y_{s-}) \tilde{N}(ds, dz),$$

and so

$$\begin{aligned} X_t = & Y_t + e^{-At} \int_0^t e^{As} dK_s + e^{-At} \int_0^t e^{As} \gamma_s (X_{s-} + Y_{s-}) dB_s \\ & + e^{-At} \int_0^t \int_E e^{As} c(z) \theta_s (X_{s-} + Y_{s-}) \tilde{N}(ds, dz). \end{aligned}$$

Remark 2.5.2. In all cases, we have $a_t = a$ i.e. $A_t = at$, so we get

$$\begin{aligned} \mathbb{E}[Y_t] &= \mathbb{E} \left[e^{-at} \left(x_0 - \int_0^t e^{as} \beta ds + \int_0^t e^{as} (\sigma_s + \gamma_s Y_{s-}) dB_s + \int_0^t \int_E e^{as} c(z) (\eta_s + \theta_s Y_{s-}) \tilde{N}(ds, dz) \right) \right] \\ &= e^{-at} \left(x_0 - \int_0^t e^{as} \beta ds \right) \\ &= e^{-at} \left(x_0 - \beta \left(\frac{e^{at} - 1}{a} \right) \right). \end{aligned}$$

Proof of assertions (i). From Proposition 2.2.3 and Remark 2.5.2, we have

$$\begin{aligned} k_t &= \beta \mathbf{1}_{\mathbb{E}(\mathbf{X}_t) = \mathbf{p}} \\ &= \beta \mathbf{1}_{\mathbb{E}(\mathbf{Y}_t) + \mathbf{K}_t - \mathbf{p} = \mathbf{0}} \\ &= \beta \mathbf{1}_{\mathbf{x}_0 - \beta t + \mathbf{K}_t - \mathbf{p} = \mathbf{0}}, \end{aligned}$$

so, we obtain that

$$\begin{aligned} K_t &= \int_0^t k_s ds \\ &= \int_0^t \beta \mathbf{1}_{\mathbf{K}_s = \mathbf{p} + \beta s - \mathbf{x}_0} ds, \end{aligned}$$

and as $K_t \geq 0$, we conclude that

$$K_t = (p + \beta t - x_0)^+.$$

Next, we have

$$f(z) = \frac{1}{\sqrt{2\pi}z} \exp \left(-\frac{(\ln z)^2}{2} \right),$$

the density function of a lognormal random variable, so we can obtain

$$\int_E \eta z \lambda(dz) = \lambda \eta \int_E z f(z) dz = \lambda \eta \mathbb{E}(\xi)$$

where $\xi \sim \text{lognormal}(0, 1)$, and we conclude that

$$\int_E \eta z \lambda(dz) = \lambda \eta \sqrt{e}.$$

Finally, we deduce the exact solution

$$X_t = X_0 - (\beta + \lambda \sqrt{e})t + \sigma B_t + \sum_{i=0}^{N_t} \eta \xi_i + K_t,$$

where $N_t \sim \mathcal{P}(\lambda t)$ and $\xi_i \sim \text{lognormal}(0, 1)$. □

Proof of assertions (ii). In this case, and using the same Proposition and Remark, we have

$$k_t = (\mathbb{E}(-aX_t))^{-1} \mathbf{1}_{\mathbb{E}(X_t)=p},$$

which

$$\begin{aligned} \mathbb{E}(X_t) = p &\iff \mathbb{E}(Y_t) - p + e^{-at} \int_0^t e^{as} dK_s = 0 \\ &\iff -x_0 e^{-at} + p = e^{-at} \int_0^t e^{as} dK_s \\ &\iff K_s = ap, \end{aligned}$$

and

$$\begin{aligned} K_t \geq 0 &\iff -x_0 e^{-at} + p \geq 0 \\ &\iff e^{-at} \leq \frac{p}{x_0} \\ &\iff t \geq \frac{1}{a} (\ln(x_0) - \ln(p)) := t^*. \end{aligned}$$

So, we conclude that $K_t = ap(t - t^*) \mathbf{1}_{t \geq t^*}$, where $t^* = \frac{1}{a} (\ln(x_0) - \ln(p))$. Next, by the definition of the process Y_t :

$$dY_t = -aY_t dt + \gamma Y_t dB_t + \theta Y_t d\tilde{N}_t,$$

we have

$$Y_t = X_0 \exp \left(- (a + \gamma^2/2 + \lambda\theta)t + \gamma B_t \right) (1 + \theta)^{N_t}.$$

Thanks to Itô's formula we get

$$\begin{aligned} d\left(\frac{1}{Y_t}\right) &= -\frac{1}{Y_t^2} dY_t + \frac{1}{2} \left(\frac{2}{Y_t^3}\right) \gamma^2 Y_t^2 dt + d \sum_{s \leq t} \left(\frac{1}{Y_{s^-} + \Delta Y_s} - \frac{1}{Y_{s^-}} + \frac{1}{Y_{s^-}^2} \Delta Y_s \right) \\ &= \frac{a}{Y_t} dt - \frac{\gamma}{Y_t} dB_t - \frac{\theta}{Y_{t^-}} d\tilde{N}_t + \frac{\gamma^2}{Y_t} dt + d \sum_{s \leq t} \left(\frac{1}{(1 + \theta)Y_{s^-}} - \frac{1}{Y_{s^-}} + \frac{\theta}{Y_{s^-}} \right), \end{aligned}$$

and so

$$\begin{aligned} dY_t^{-1} &= (a + \gamma^2) Y_t^{-1} dt - \gamma Y_t^{-1} dB_t - \theta Y_t^{-1} d\tilde{N}_t + \left(\frac{\theta^2}{1 + \theta} \right) d \sum_{s \leq t} Y_{s^-}^{-1} \\ &= \left(a + \gamma^2 + \frac{\lambda\theta^2}{1 + \theta} \right) Y_t^{-1} dt - \gamma Y_t^{-1} dB_t - \left(\frac{\theta}{1 + \theta} \right) Y_t^{-1} d\tilde{N}_t. \end{aligned}$$

Then, using integration by parts formula, we obtain

$$\begin{aligned} d(X_t Y_t^{-1}) &= X_{t^-} dY_t^{-1} + Y_{t^-}^{-1} dX_t + d[X, Y^{-1}]_t \\ &= (a + \gamma^2) X_t Y_t^{-1} dt - \gamma X_t Y_t^{-1} dB_t - \theta X_{t^-} Y_{t^-}^{-1} d\tilde{N}_t + \left(\frac{\theta^2}{1 + \theta} \right) d \sum_{s \leq t} X_{s^-} Y_{s^-}^{-1} \\ &\quad - a X_t Y_t^{-1} dt + \gamma X_t Y_t^{-1} dB_t + \theta X_{t^-} Y_{t^-}^{-1} d\tilde{N}_t + Y_t^{-1} dK_t \\ &\quad - \gamma^2 X_t Y_t^{-1} dt - \left(\frac{\theta^2}{1 + \theta} \right) d \sum_{s \leq t} X_{s^-} Y_{s^-}^{-1} \\ &= Y_t^{-1} dK_t. \end{aligned}$$

Finally, we deduce that

$$X_t = Y_t + Y_t \int_0^t Y_s^{-1} dK_s.$$

□

Proof of assertions (iii). In that case, we have

$$\begin{aligned} Y_t &= e^{-at} \left(x_0 - \beta \left(\frac{e^{at} - 1}{a} \right) \right) + \sigma_s e^{-at} \int_0^t e^{as} dB_s + e^{-at} \int_0^t \eta_s e^{as} d\tilde{N}_s \\ &= e^{-at} \left(x_0 - (\beta + \lambda\eta) \left(\frac{e^{at} - 1}{a} \right) \right) + \sigma_s e^{-at} \int_0^t e^{as} dB_s + e^{-at} \int_0^t \eta_s e^{as} dN_s \\ &:= f_t + G_t + F_t, \end{aligned}$$

and

$$X_t = Y_t + e^{-at} \bar{K}_t, \quad \bar{K}_t = \int_0^t e^{as} dK_s.$$

Hence

$$\begin{aligned} h(X_t) &= Y_t + e^{-at} \bar{K}_t + \alpha \sin(Y_t + e^{-at} \bar{K}_t) - p \\ &= Y_t + e^{-at} \bar{K}_t + \alpha \left(\sin(Y_t) \cos(e^{-at} \bar{K}_t) + \cos(Y_t) \sin(e^{-at} \bar{K}_t) \right) - p \\ &= Y_t + e^{-at} \bar{K}_t + \alpha \left[\cos(e^{-at} \bar{K}_t) \left\{ \sin(f_t) \cos(G_t) \cos(F_t) + \cos(f_t) \sin(G_t) \cos(F_t) \right. \right. \\ &\quad \left. \left. + \cos(f_t) \cos(G_t) \sin(F_t) - \sin(f_t) \sin(G_t) \sin(F_t) \right\} + \sin(e^{-at} \bar{K}_t) \left\{ \cos(f_t) \cos(G_t) \cos(F_t) \right. \right. \\ &\quad \left. \left. - \sin(f_t) \sin(G_t) \sin(F_t) - \sin(f_t) \cos(G_t) \sin(F_t) - \cos(f_t) \sin(G_t) \sin(F_t) \right\} \right] - p. \end{aligned}$$

On one side, since G_t is a centered gaussian random variable with variance

$$V = \sigma^2 \frac{1 - e^{-2at}}{2a} = \sigma^2 e^{-at} \frac{\sinh(at)}{a}, \text{ we obtain that}$$

$$\mathbb{E}[e^{iG_t}] = e^{-V/2},$$

$$\mathbb{E}[\sin(G_t)] = \mathbb{E} \left[\frac{e^{iG_t} - e^{-iG_t}}{2i} \right] = 0,$$

and

$$\mathbb{E}[\cos(G_t)] = \mathbb{E} \left[\frac{e^{iG_t} + e^{-iG_t}}{2} \right] = \mathbb{E}(e^{iG_t}) = \exp \left(-e^{-at} \frac{\sigma^2}{2a} \sinh(at) \right) =: g(t).$$

On the other side,

$$\mathbb{E}[e^{iF_t}] = \mathbb{E} \left[\exp \left(i\eta e^{-at} \int_0^t e^{as} dN_s \right) \right],$$

by taking ‘ a ’ small, we get

$$\begin{aligned}\mathbb{E}[e^{iF_t}] &\approx \mathbb{E}\left[\exp\left(i\eta \int_0^t dN_s\right)\right] \\ &\approx \mathbb{E}\left[\exp\left(i\eta N_t\right)\right] \\ &\approx \exp\left(\lambda t(e^{i\eta} - 1)\right),\end{aligned}$$

and so

$$\begin{aligned}\mathbb{E}[\sin(F_t)] &\approx \frac{\exp\left(\lambda t(e^{i\eta} - 1)\right) - \exp\left(\lambda t(e^{-i\eta} - 1)\right)}{2i} =: m(t), \\ \mathbb{E}[\cos(F_t)] &\approx \frac{\exp\left(\lambda t(e^{i\eta} - 1)\right) + \exp\left(\lambda t(e^{-i\eta} - 1)\right)}{2} =: n(t).\end{aligned}$$

Using Remark 2.5.2, we conclude that, for small ‘ a ’,

$$\begin{aligned}\mathbb{E}[h(X_t)] &\approx \mathbb{E}[Y_t] + e^{-at}\bar{K}_t + \alpha\left(g(t)m(t)\cos(f_t + e^{-at}\bar{K}_t) + g(t)n(t)\sin(f_t + e^{-at}\bar{K}_t)\right) - p \\ &:= F_t(\bar{K}_t).\end{aligned}$$

Therefore,

$$\bar{K}_t = \sup_{s \leq t} \left(F_s^{-1}(0)\right)^+ \quad \text{and} \quad dK_t = e^{-at} d \sup_{s \leq t} \left(F_s^{-1}(0)\right)^+.$$

□

2.5.2 Illustrations.

This computation works as follows. Let $0 = T_0 < T_1 < \dots < T_n = T$ be a subdivision of $[0, T]$ of step size T/n , n being a positive integer, let X be the unique solution of the MRSDE (2.5.1) and let, for a given i , $(\tilde{X}_{T_k}^i)_{0 \leq k \leq n}$ be its numerical approximation given by Algorithm 1. For a given integer L , we draw $(\bar{X}^l)_{0 \leq l \leq L}$ and $(\tilde{X}^{i,l})_{0 \leq l \leq L}$, L independent copies of X and \tilde{X}^i . Then, we approximate the \mathbb{L}^2 -error of Theorem 2.4.1 by :

$$\hat{E} = \frac{1}{L} \sum_{l=1}^L \max_{0 \leq k \leq n} \left| \bar{X}_{T_k}^l - \tilde{X}_{T_k}^{i,l} \right|^2. \quad (2.5.2)$$

Figure 2.1 illustrates the evolution in time of the true K (full line) and the estimated K (dotted line for particle method) (dashed line for density method) in case (i). It is confirmed that the approximation of K is almost the same as the exact solution. The evolution of $\log(\hat{E})$ w.r.t. $\log(N)$ is depicted in Figure 2.2. It can be seen that the slope is equal to 0.9, which is consistent with the statement of Theorem 2.4.1.

Figure 2.3 illustrates the evolution in time of the true K (full line) and the estimated K (dotted line for particle method) (dashed line for density method) in case (ii). As in the previous example, the approximation of K is almost the same as the exact solution.

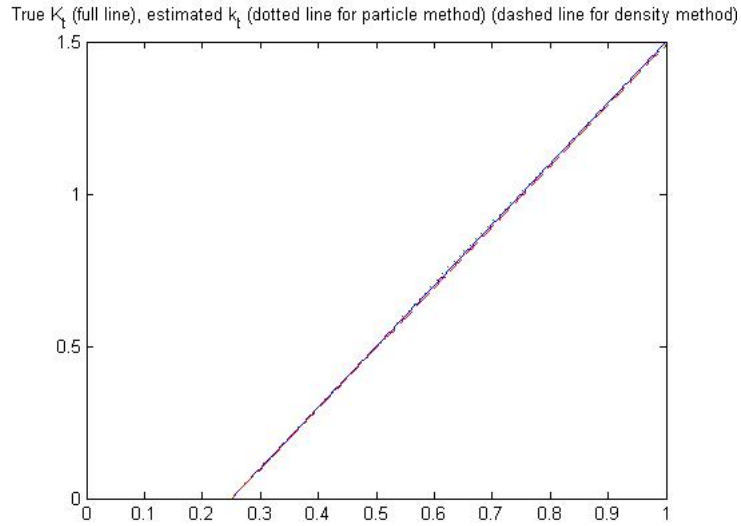


FIGURE 2.1 – Case (i). $n = 500$, $N = 100000$, $T = 1$, $\beta = 2$, $\sigma = 1$, $\lambda = 5$, $x_0 = 1$, $p = 1/2$.

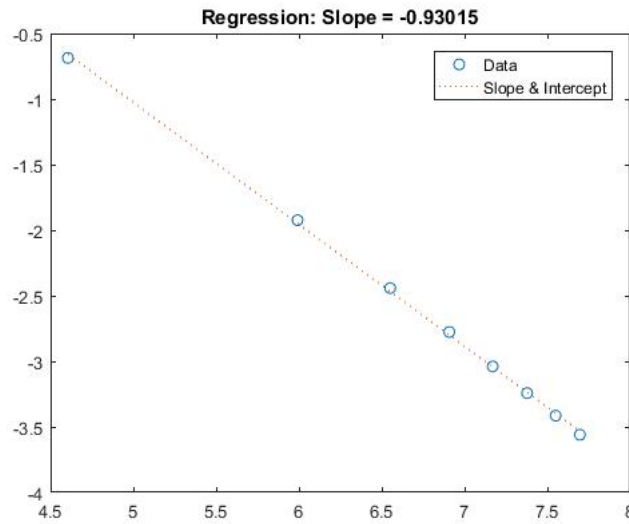


FIGURE 2.2 – Case (i). Regression of $\log(\hat{E})$ w.r.t. $\log(N)$. Data : \hat{E} when N varies from 100 to 2200 with step size 300. Parameters : $n = 100$, $T = 1$, $\beta = 2$, $\sigma = 1$, $\lambda = 5$, $x_0 = 1$, $p = 1/2$, $L = 1000$.

The evolution of $\log(\hat{E})$ w.r.t. $\log(N)$ is depicted in Figure 2.4. It can be seen that the slope is equal to 0.9, which is consistent with the statement of Theorem 2.4.1.

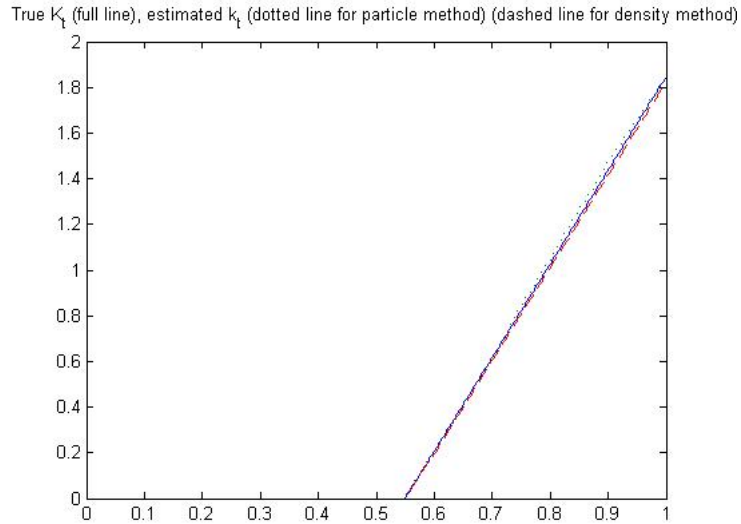


FIGURE 2.3 – Case (ii). Parameters : $n = 500$, $N = 10000$, $T = 1$, $\beta = 0$, $a = 3$, $\gamma = 1$, $\eta = 1$, $\lambda = 2$, $x_0 = 4$, $p = 1$.

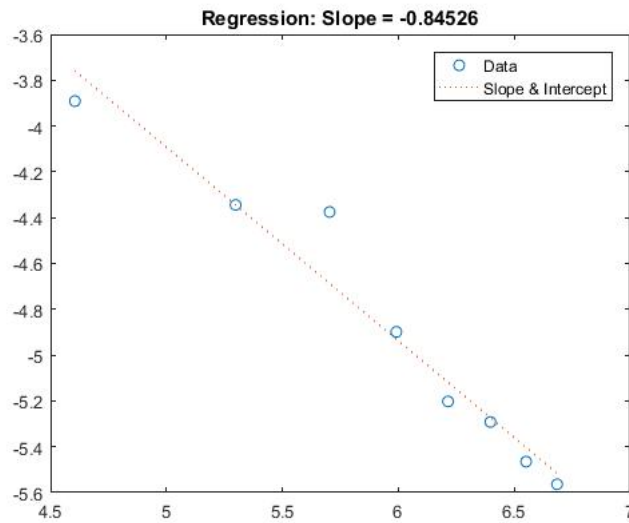


FIGURE 2.4 – Case (ii). Regression of $\log(\hat{E})$ w.r.t. $\log(N)$. Data : \hat{E} when N varies from 100 to 800 with step size 100. Parameters : $n = 1000$, $T = 1$, $\beta = 0$, $a = 3$, $\gamma = 1$, $\eta = 1$, $\lambda = 2$, $x_0 = 4$, $p = 1$, $L = 1000$.

Figure 2.5 illustrates the evolution in time of the true K (full line) and the estimated K (dotted line for particle method) (dashed line for density method) in case (iii). Moreover, we notice that the approximation of K with particle method is closer to the exact K than the one with density method.

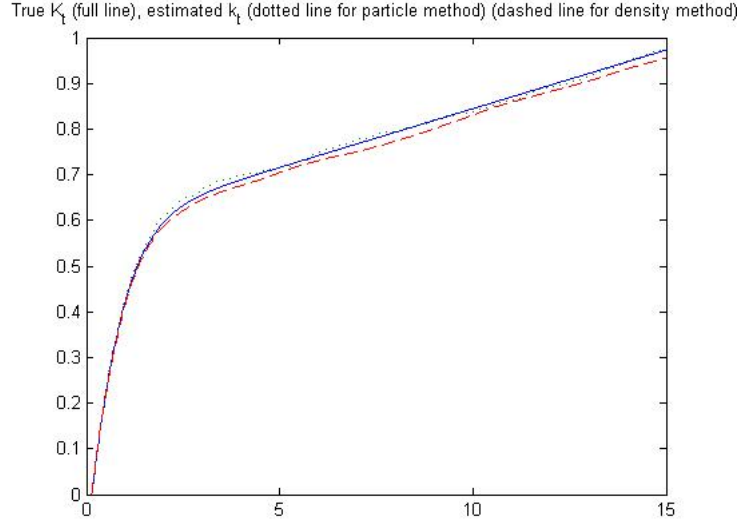


FIGURE 2.5 – Case (iii). Parameters : $n = 1000$, $N = 100000$, $T = 15$, $\beta = 10^{-2}$, $\sigma = 1$, $p = \pi/2$, $\alpha = 0.9$, $a = 10^{-2}$, x_0 is the unique solution of $x + \alpha \sin(x) - p = 0$ plus 10^{-1} .

2.6 Appendices

2.6.1 Proof of Lemma 2.2.4

Let s and t in $[0, T]$ be such that $s \leq t$.

Firstly, we suppose that φ is a continuous function with compact support. In this case, there exists a sequence of Lipschitz continuous functions φ_n with compact support which converges uniformly to φ . Therefore, by using Proposition 2.2.2, we get

$$\begin{aligned}
|\mathbb{E}[\varphi(X_t)] - \mathbb{E}[\varphi(X_s)]| &\leq |\mathbb{E}[\varphi(X_t)] - \mathbb{E}[\varphi_n(X_t)]| + |\mathbb{E}[\varphi_n(X_t)] - \mathbb{E}[\varphi_n(X_s)]| + |\mathbb{E}[\varphi_n(X_s)] - \mathbb{E}[\varphi(X_s)]| \\
&\leq \mathbb{E}[|(\varphi - \varphi_n)(X_t)|] + C_n \mathbb{E}[|X_t - X_s|] + \mathbb{E}[|(\varphi_n - \varphi)(X_s)|] \\
&\leq 2\mathbb{E}[\|\varphi_n - \varphi\|_\infty] + C_n (\mathbb{E}[|X_t - X_s|^2])^{1/2} \\
&\leq 2\mathbb{E}[\|\varphi_n - \varphi\|_\infty] + C_n |t - s|^{1/2}.
\end{aligned}$$

Thus, we obtain that

$$\limsup_{t \rightarrow s} |\mathbb{E}[\varphi(X_t)] - \mathbb{E}[\varphi(X_s)]| \leq 2\mathbb{E}[\|\varphi_n - \varphi\|_\infty].$$

This result is true for all $n \geq 1$, so we deduce that

$$\limsup_{t \rightarrow s} |\mathbb{E}[\varphi(X_t)] - \mathbb{E}[\varphi(X_s)]| = 0,$$

then we conclude the continuity of the function $t \mapsto \mathbb{E}[\varphi(X_t)]$.

Secondly, we consider the case where φ is a continuous function such that

$$\forall x \in \mathbb{R}, \exists C \in \mathbb{R}, \varphi(x) \leq C(1 + |x|^p).$$

We define a sequence of functions φ_n , such that for all $n \geq 1$ and $x \in \mathbb{R}$,

$$\varphi_n(x) = \varphi(x)\theta_n(x)$$

with

$$\theta_n(x) = \begin{cases} 1 & \text{if } |x| \leq n \\ 0 & \text{if } |x| > n \end{cases}$$

Based on this definition, φ_n is a continuous function with compact support. Then we get

$$\begin{aligned} & |\mathbb{E}[\varphi(X_t)] - \mathbb{E}[\varphi(X_s)]| \\ & \leq \left| \mathbb{E}[(\varphi - \varphi_n)(X_t)(\mathbf{1}_{|\mathbf{X}_t| \leq n} + \mathbf{1}_{|\mathbf{X}_t| > n})] \right| + \left| \mathbb{E}[\varphi_n(\mathbf{X}_t)] - \mathbb{E}[\varphi_n(\mathbf{X}_s)] \right| \\ & \quad + \left| \mathbb{E}[(\varphi_n - \varphi)(X_s)(\mathbf{1}_{|\mathbf{X}_s| \leq n} + \mathbf{1}_{|\mathbf{X}_s| > n})] \right| \\ & \leq \left| \mathbb{E}[(\varphi - \varphi_n)(X_t)\mathbf{1}_{|\mathbf{X}_t| > n}] \right| + \left| \mathbb{E}[\varphi_n(\mathbf{X}_t)] - \mathbb{E}[\varphi_n(\mathbf{X}_s)] \right| + \left| \mathbb{E}[(\varphi_n - \varphi)(\mathbf{X}_s)\mathbf{1}_{|\mathbf{X}_s| > n}] \right| \\ & \leq 2\mathbb{E}[|\varphi(X_t)|\mathbf{1}_{|\mathbf{X}_t| > n}] + \left| \mathbb{E}[\varphi_n(\mathbf{X}_t)] - \mathbb{E}[\varphi_n(\mathbf{X}_s)] \right| + 2\mathbb{E}[|\varphi(\mathbf{X}_s)|\mathbf{1}_{|\mathbf{X}_s| > n}] \\ & \leq C\mathbb{E}[(1 + |X_t|^p)\mathbf{1}_{|\mathbf{X}_t| > n}] + \left| \mathbb{E}[\varphi_n(\mathbf{X}_t)] - \mathbb{E}[\varphi_n(\mathbf{X}_s)] \right| + C\mathbb{E}[(1 + |\mathbf{X}_s|^p)\mathbf{1}_{|\mathbf{X}_s| > n}] \\ & \leq C\mathbb{E}\left[(1 + \sup_{t \leq T} |X_t|^p)\mathbf{1}_{\sup_{t \leq T} |\mathbf{X}_t| > n}\right] + \left| \mathbb{E}[\varphi_n(\mathbf{X}_t)] - \mathbb{E}[\varphi_n(\mathbf{X}_s)] \right|. \end{aligned}$$

Thus, by using the first part of this Lemma, we obtain that

$$\limsup_{t \rightarrow s} |\mathbb{E}[\varphi(X_t)] - \mathbb{E}[\varphi(X_s)]| \leq C\mathbb{E}\left[(1 + \sup_{t \leq T} |X_t|^p)\mathbf{1}_{\sup_{t \leq T} |\mathbf{X}_t| > n}\right].$$

This result is true for all $n \geq 1$, then by using the dominated convergence theorem, we deduce that

$$\limsup_{t \rightarrow s} |\mathbb{E}[\varphi(X_t)] - \mathbb{E}[\varphi(X_s)]| = 0,$$

and we conclude the continuity of the function $t \mapsto \mathbb{E}[\varphi(X_t)]$.

Chapitre 3

Simulation of McKean-Vlasov BSDEs by Wiener chaos expansion

Abstract

We present an algorithm to solve McKean-Vlasov BSDEs based on Wiener chaos expansion and Picard's iterations and study its convergence. This chapter extends the results obtained by Briand and Labart in [BL14] when standard BSDEs were considered. Here we are faced with the problem of the approximation of the law of (Y, Z) in the driver, that we solve by using a particle system. In order to avoid solving a system of BSDEs, which would not be feasible in practice, we use the same particles to approximate the law of (Y, Z) and to compute Monte Carlo approximations.

3.1 Introduction

Backward stochastic differential equations were introduced by Bismut in [Bis73b] for the linear case, and by Pardoux and Peng in [PP90] for the general case. These works consisted in finding a pair (Y_t, Z_t) of \mathcal{F}_t -adapted processes such that

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds - \int_t^T Z_s \cdot dB_s, \quad 0 \leq t \leq T, \quad (3.1.1)$$

where B is a d -dimensional standard Brownian motion, the terminal condition ξ is a real-valued \mathcal{F}_T -measurable random variable where $\{\mathcal{F}_t\}_{0 \leq t \leq T}$ stands for the augmented filtration of the Brownian motion B , the generator f is a map from $[0, T] \times \mathbb{R} \times \mathbb{R}^d$ into \mathbb{R} .

First results on the numerical approximation of (3.1.1) date from the end of the 90's. The case of a generator f independent of z has been studied in [Che97b] and in [CMM99]. The authors introduce a time and space discretization of the BSDE, which is somewhat reminiscent of the dynamic programming equation, introduced a couple of years later. The case of a generator dependent of z has first been done in [Bal97], where the

author introduces a random discretization. In [BDM01], the authors generalize the scheme proposed in [Che97b] to the case of f depending on z and prove the weak convergence of their scheme. In [BDM02], an approach for the case of path-dependent terminal condition ξ has been presented. The rate of the convergence of this method was left as an open problem. To deal with this question, an approach based on the dynamic programming equation has been introduced by Bouchard and Touzi in [BT04] and Zhang in [Zha04]. Both papers deal with the Markovian case, i.e. $\xi = g(X_T)$ where X is a solution of a stochastic differential equation. To be fully implementable, this algorithm requires to have a good approximation of its associated conditional expectation. Various methods have been developed (see [GLW05], [CMT10], [CT17b]). Forward methods have also been introduced to approximate (3.1.1) : branching diffusion method (see [HLTT14]), multilevel Picard approximation (see [WHJK17]) and Wiener chaos expansion (see [BL14]).

Many extensions of (3.1.1) have also been considered : high order schemes (see [Cha14], [CC14]), schemes for reflected BSDEs (see [BP03], [CR16]), for fully-coupled BSDEs (see [DM06], [BZ08]), for quadratic BSDEs (see [CR15]), for BSDEs with jumps (see [GL16]) and for McKean-Vlasov BSDEs (see [Ala15], [CdRT15], [CCD17]).

The aim of this chapter is to extend the results of [BL14] to the case of McKean-Vlasov BSDEs, i.e. to provide an algorithm based on Wiener chaos expansion to solve BSDEs of the following type

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s, [Y_s], [Z_s]) ds - \int_t^T Z_s \cdot dB_s, \quad 0 \leq t \leq T, \quad (3.1.2)$$

where $[\theta]$ is the notation for the law of a random variable θ and f is a map from $[0, T] \times \mathbb{R} \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}) \times \mathcal{P}_2(\mathbb{R}^d)$ into \mathbb{R} . The set $\mathcal{P}_2(\mathbb{R}^d)$ is the set of probability measures with a finite second-order moment, endowed with the Wasserstein distance i.e.

$$\mathcal{W}_2(\mu, \mu') := \inf_{\pi} \left(\int_{\mathbb{R}^d \times \mathbb{R}^d} |x - x'|^2 d\pi(x, x') \right)^{1/2},$$

for $(\mu, \mu') \in \mathcal{P}_2(\mathbb{R}^d) \times \mathcal{P}_2(\mathbb{R}^d)$, the infimum being taken over the probability distributions π on $\mathbb{R}^d \times \mathbb{R}^d$ whose marginals on \mathbb{R}^d are respectively μ and μ' . Notice that if X and X' are random variables of order 2 with values in \mathbb{R}^d , then by definition we have

$$\mathcal{W}_2([X], [X']) \leq \left[\mathbb{E}|X - X'|^2 \right]^{1/2}. \quad (3.1.3)$$

Such type of BSDEs have been introduced in [BDLP09] and [BLP09] in a more particular framework : in [BDLP09], the authors study the mean field problem in a Markovian setting and prove the existence and the uniqueness of the solution when the terminal condition is of type $\xi = \mathbb{E}[g(x, X_T)]|_{x=X_T}$ where X is a driving adapted stochastic process, and the generator is defined by $\mathbb{E}[f(s, \lambda, \Lambda_s)]|_{\lambda=\Lambda_s}$ where $\Lambda_s = (X_s, Y_s, Z_s)$. In [BLP09], the authors extend the result of existence and uniqueness to a more general framework and link the mean-field BSDE to non local partial differential equation.

The study of numerical methods for McKean-Vlasov BSDEs goes back to a few years (see [Ala15], [CdRT15], [CCD17]). Usually, forward McKean-Vlasov SDEs are solved by using particle algorithms (see [AKH02], [TV03], [Bos05]) in which the McKean term is

approximated by the empirical measure of a large number of interacting particles with independent noise. Adapting such algorithms to the backward problem is not obvious as the high dimension of the involved Brownian motion (given by the number of particles) induces, a priori, a high dimension backward problem with bad consequences for the numerical implementation. The above mentioned papers on numerical methods for McKean-Vlasov BSDEs do not use particle systems. In [CdRT15], the authors present a method based on cubature for decoupled McKean-Vlasov forward backward SDE. In [CCD17], the authors consider the case of strongly coupled forward-backward SDE of McKean-Vlasov type. They propose a scheme whose principle is to implement recursively Picard iterations on small time intervals, since Picard Theorem only applies in small time for fully coupled problems.

In this chapter we propose a method based on Wiener chaos expansion and particle system approximation which is neither more complex nor more costly than solving a standard BSDE of type (3.1.1) when $f(y, z, \mu, \nu)$ is a separated variable function in (y, z) and (μ, ν) . The method based on Wiener chaos expansion to solve standard BSDEs has been introduced in [BL14] and consists in writing the Picard scheme of (3.1.1) in a forward way

$$\begin{aligned} Y_t^{q+1} &= \mathbb{E} \left(\xi + \int_0^T f(s, Y_s^q, Z_s^q) ds \mid \mathcal{F}_t \right) - \int_0^t f(s, Y_s^q, Z_s^q) ds, \\ Z_t^{q+1} &= D_t Y_t^{q+1} = D_t \mathbb{E} \left(\xi + \int_0^T f(s, Y_s^q, Z_s^q) ds \mid \mathcal{F}_t \right), \end{aligned}$$

(where $D_t X$ stands for the Malliavin derivative of the random variable X) and to use Wiener chaos expansion to easily compute conditional expectations and their Malliavin derivatives. More precisely, all r.v. F in L^2 can be written

$$F = \mathbb{E}(F) + \sum_{k \geq 1} \sum_{|n|=k} d_k^n \prod_{i \geq 1} K_{n_i} \left(\int_0^T g_i(s) dB_s \right),$$

where K_l denotes the Hermite polynomial of degree l , $(g_i)_{i \geq 1}$ is an orthonormal basis of $L^2(0, T)$ and, if $n = (n_i)_{i \geq 1}$ is a sequence of integers, $|n| = \sum_{i \geq 1} n_i$. $(d_k^n)_{k \geq 1, |n|=k}$ is the sequence of coefficients ensuing from the decomposition of F . The numerical method consists in working with a finite number of chaos, a finite number of functions $(g_i)_i$ and in using Monte-Carlo approximation to compute the coefficients $(d_k^n)_{k, n}$. In case of McKean-Vlasov BSDE, the generator depends on the laws of the processes. The idea is to use M particles which will serve both to approximate the law of (Y, Z) and to compute the coefficients $(d_k^n)_{k, n}$ by Monte Carlo. By doing this, we manage to get a computational cost which is of the same order as the one obtained in case of standard BSDEs for many cases of the driver f . However, this pooling of particles costs the independence in the Monte Carlo approximation, making the proof of the convergence more difficult and leading to a slower speed of convergence in M .

The outline of this chapter is as follows. Section 3.2 state the notations and recall the main results of [BL14] in order to make this chapter as self-contained as possible. In Section 3.3 we generalize the existence and uniqueness results stated by Pardoux and Peng [PP90] to the case of BSDEs of type (3.1.2). Section 3.4 describes precisely the algorithm, Section 3.5 is devoted to the study of the convergence of the algorithm and finally Section

3.6 contains some numerical experiments.

3.2 Preliminaries.

3.2.1 Definitions and notations.

Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and an \mathbb{R}^d -valued Brownian motion B , we consider :

- $\{(\mathcal{F}_t); t \in [0, T]\}$, the filtration generated by the Brownian motion B and augmented.
- $L^p(\mathcal{F}_T) := L^p(\Omega, \mathcal{F}_T, \mathbb{P})$, $p \in \mathbb{N}^*$, the space of all \mathcal{F}_T -measurable random variables (r.v. in the following) $X : \Omega \rightarrow \mathbb{R}^d$ satisfying $\|X\|_p^p := \mathbb{E}(|X|^p) < \infty$.
- $\mathbb{E}_t(X) := \mathbb{E}(X|\mathcal{F}_t)$, the conditional expectation of X (in $L^1(\mathcal{F}_T)$) w.r.t. \mathcal{F}_t .
- $S_{\alpha, T}^p(\mathbb{R}^d)$, $p \in \mathbb{N}$, $p \geq 2$, $\alpha \geq 0$, the space of all càdlàg predictable processes $\phi : \Omega \times [0, T] \rightarrow \mathbb{R}^d$ such that $\|\phi\|_{S_{\alpha, T}^p}^p = \mathbb{E}(\sup_{t \in [0, T]} e^{\alpha t} |\phi_t|^p) < \infty$. Note that $S_T^p(\mathbb{R}^d) = S_{0, T}^p(\mathbb{R}^d)$.
- $H_{\alpha, T}^p(\mathbb{R}^d)$, $p \in \mathbb{N}$, $p \geq 2$, $\alpha \geq 0$, the space of all predictable processes $\phi : \Omega \times [0, T] \rightarrow \mathbb{R}^d$ such that $\|\phi\|_{H_{\alpha, T}^p}^p = \mathbb{E} \int_0^T e^{\alpha t} |\phi_t|^p dt < \infty$. Note that $H_T^p(\mathbb{R}^d) = H_{0, T}^p(\mathbb{R}^d)$.
- $L^2(0, T)$, the space of all square integrable functions in $[0, T]$.
- $C^{k, l}$, the set of continuously differentiable functions $\phi : (t, x) \in [0, T] \times \mathbb{R}^d$ with continuous derivatives w.r.t. t (resp. w.r.t. x) up to order k (resp. up to order l).
- $C_b^{k, l}$, the set of continuously differentiable functions $\phi : (t, x) \in [0, T] \times \mathbb{R}^d$ with continuous and uniformly bounded derivatives w.r.t. t (resp. w.r.t. x) up to order k (resp. up to order l). The function ϕ is also bounded.
- $\|\partial_{sp}^j f\|_{\infty}^2$, the norm of the derivatives of $f : ([0, T] \times \mathbb{R} \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}) \times \mathcal{P}(\mathbb{R}^d), \mathbb{R})$ w.r.t. the second and the third component which sum equals j : $\|\partial_{sp}^j f\|_{\infty}^2 := \sum_{|k|=j} \|\partial_y^{k_0} \partial_{z_1}^{k_1} \dots \partial_{z_d}^{k_d} f\|_{\infty}^2$, where $|k| = k_0 + \dots + k_d$.
- C_p^{∞} , the set of smooth functions $f : \mathbb{R}^n \rightarrow \mathbb{R}$ with partial derivatives of polynomial growth.
- $\|(\cdot, \cdot)\|_{L^p}^p$, $p \in \mathbb{N}$, $p \geq 2$, the norm on the space $S_T^p(\mathbb{R}) \times H_T^p(\mathbb{R}^d)$ defined by

$$\|(Y, Z)\|_{L^p}^p := \mathbb{E} \left(\sup_{t \in [0, T]} |Y_t|^p \right) + \int_0^T \mathbb{E}(|Z_t|^p) dt. \quad (3.2.1)$$

Note that this norm is different from the usual L^p norm for BSDE.

We also recall some useful definitions related to Malliavin calculus. We use the notations of [Nua06].

- \mathcal{S} denotes the class of random variables of the form $F = f(W(h_1), \dots, W(h_n))$, where $f \in C_p^{\infty}(\mathbb{R}^{n \times d}, \mathbb{R})$, for all $j \leq n$, $h_j = (h_j^1, \dots, h_j^d) \in L^2([0, T]; \mathbb{R}^d)$ and for all $i \leq d$, $W^i(h_j) = \int_0^T h_j^i(t) dW_t^i$.
- $\mathbb{D}^{r, 2}$ denotes the closure of \mathcal{S} w.r.t. the following norm on \mathcal{S}

$$\|F\|_{\mathbb{D}^{r, 2}}^2 := \mathbb{E}|F|^2 + \sum_{q=1}^r \sum_{|\alpha|=q} \mathbb{E} \left(\int_0^T \dots \int_0^T |D_{(t_1, \dots, t_q)}^{\alpha} F|^2 dt_1 \dots dt_q \right),$$

where α is a multi-index $(\alpha_1, \dots, \alpha_q) \in \{1, \dots, d\}^q$, $|\alpha|_1 := \sum_{i=1}^q \alpha_i = q$, and D^α represent the multi-index Malliavin derivative operator. We recall $\mathbb{D}^{\infty,2} = \bigcap_{r=1}^{\infty} \mathbb{D}^{r,2}$.

Remark 3.2.1. When $d = 1$, $\|F\|_{\mathbb{D}^{r,2}}^2 := \mathbb{E}|F|^2 + \sum_{q=1}^r \mathbb{E}(\int_0^T \dots \int_0^T |D_{(t_1, \dots, t_q)}^{(q)} F|^2 dt_1 \dots dt_q) = \mathbb{E}|F|^2 + \sum_{q=1}^r \|D^{(q)} F\|_{L^2(\Omega \times [0, T]^q)}^2$.

Let $m \in \mathbb{N}^*$ and $j \in \mathbb{N}$, $j \geq 2$. We also introduce the following notation :

— $\mathcal{D}^{m,j}$ denotes the space of all \mathcal{F}_T -measurable r.v. such that

$$\|F\|_{m,j}^j := \sum_{1 \leq l \leq m} \sum_{|\alpha|_1=l} \sup_{t_1 \leq \dots \leq t_l} \mathbb{E}(|D_{t_1, \dots, t_l}^\alpha F|^j) < \infty,$$

where $\sup_{t_1 \leq \dots \leq t_l}$ means $\sup_{(t_1, \dots, t_l): t_1 \leq \dots \leq t_l}$.

— $\mathcal{S}^{m,j}$ denotes the space of all couple of processes (Y, Z) belonging to $S_T^j(\mathbb{R}) \times H_T^j(\mathbb{R}^d)$ and such that

$$\|(Y, Z)\|_{m,j}^j := \sum_{1 \leq l \leq m} \sum_{|\alpha|_1=l} \sup_{t_1 \leq \dots \leq t_l} \|(D_{t_1, \dots, t_l}^\alpha Y, D_{t_1, \dots, t_l}^\alpha Z)\|_{L^j}^j < \infty,$$

i.e.

$$\|(Y, Z)\|_{m,j}^j = \sum_{1 \leq l \leq m} \sum_{|\alpha|_1=l} \sup_{t_1 \leq \dots \leq t_l} \left\{ \mathbb{E} \left(\sup_{t_1 \leq r \leq T} |D_{t_1, \dots, t_l}^\alpha Y_r|^j \right) + \int_{t_1}^T \mathbb{E}(|D_{t_1, \dots, t_l}^\alpha Z_r|^j) dr \right\}.$$

We also denote $\mathcal{S}^{m,\infty} := \bigcap_{j \geq 2} \mathcal{S}^{m,j}$.

3.2.2 Chaos decomposition formulas

We refer to the book [Nua06] for more details on this Section. The notations we use are the ones of [BL14]. Every square integrable random variable F , measurable w.r.t. \mathcal{F}_T , admits the following orthogonal decomposition

$$F = d_0 + \sum_{k \geq 1} \sum_{|n|=k} d_k^n \prod_{i \geq 1} K_{n_i} \left(\int_0^T g_i(s) dB_s \right), \quad (3.2.2)$$

where $(g_i)_{i \geq 1}$ is an orthonormal basis of $L^2(0, T)$, K_n is the Hermite polynomial of order n defined by the expansion

$$e^{xt-t^2/2} = \sum_{n \geq 0} K_n(x) t^n$$

with the convention $K_{-1} \equiv 0$, $n = (n_i)_{i \geq 1}$ is a sequence of positive integers and $|n|$ stands for $\sum_{i \geq 1} n_i$. Taking into account the normalization of the Hermite polynomials we use gives

$$d_0 = \mathbb{E}(F), \quad d_k^n = n! \mathbb{E} \left(F \times \prod_{i \geq 1} K_{n_i} \left(\int_0^T g_i(s) dB_s \right) \right),$$

where $n! = \prod_{i \geq 1} n_i!$.

To get tractable formulas, we consider a finite number of chaos and a finite number of functions (g_1, \dots, g_N) . The $(g_i)_{1 \leq i \leq N}$ functions are chosen such that we can quickly compute $\mathbb{E}(F|\mathcal{F}_t)$ and $D_t\mathbb{E}(F|\mathcal{F}_t)$ (see Section 3.4.1). We develop in this section the case $d = 1$, and we refer to [BL14, Section B.2] when $d > 1$.

The first step consists in considering a finite number of chaos. In order to approximate the random variable F , we consider its projection $C_p(F)$ onto the first p chaos, namely

$$C_p(F) = d_0 + \sum_{1 \leq k \leq p} \sum_{|n|=k} d_k^n \prod_{i \geq 1} K_{n_i} \left(\int_0^T g_i(s) dB_s \right). \quad (3.2.3)$$

The following two Lemmas give some useful properties of the operator C_p .

Lemma 3.2.1. *Let $1 \leq m \leq p + 1$ and $F \in \mathbb{D}^{m,2}$. We have*

$$\mathbb{E}[|F - C_p(F)|^2] \leq \frac{\|D^m F\|_{L^2(\Omega \times [0, T]^m)}^2}{(p + 2 - m) \cdots (p + 1)}.$$

We refer to [GL16, Lemma 2.4] for a proof.

Lemma 3.2.2.

- Let F be r.v. in $L^2(\mathcal{F}_T)$. $\forall p \geq 1$, we have $\mathbb{E}(|C_p(F)|^2) \leq \mathbb{E}(|F|^2)$. If F belongs to $L^j(\mathcal{F}_T)$, $\forall j > 2$, $\mathbb{E}(|C_p(F)|^j) \leq (1 + p(j - 1)^{p/2})^j \mathbb{E}(|F|^j)$.
- Let H be in $H_T^2(\mathbb{R})$. We have $C_p(\int_0^T H_s ds) = \int_0^T C_p(H_s) ds$.
- $\forall F \in \mathbb{D}^{1,2}$ and $\forall t \leq r$, $D_t \mathbb{E}_r[C_p(F)] = \mathbb{E}_r[C_{p-1}(D_t F)]$.

Of course, we still have an infinite number of terms in the sum in (3.2.3) and the second step consists in working with only the first N functions g_1, \dots, g_N of an orthonormal basis of $L^2(0, T)$.

Let us consider a regular mesh grid of N time steps $\mathcal{T} = \{\tilde{t}_i = i\frac{T}{N}, i = 0, \dots, N\}$ and the N step functions

$$g_i(t) = \mathbf{1}_{|\tilde{t}_{i-1}, \tilde{t}_i]}(t) / \sqrt{h}, \quad i = 1, \dots, N, \quad \text{where } h := \frac{T}{N}. \quad (3.2.4)$$

We complete these N functions g_1, \dots, g_N into an orthonormal basis of $L^2(0, T)$, $(g_i)_{i \geq 1}$. For instance, one can consider the Haar basis on each interval $(\tilde{t}_{i-1}, \tilde{t}_i)$, $i = 1, \dots, N$. We implicitly assume that $N \geq p$. This leads to the following approximation :

$$C_p^N(F) = d_0 + \sum_{1 \leq k \leq p} \sum_{|n|=k} d_k^n \prod_{1 \leq i \leq N} K_{n_i} \left(\int_0^T g_i(s) dB_s \right). \quad (3.2.5)$$

Due to the simplicity of the functions g_i , $i = 1, \dots, N$, we can compute explicitly

$$\int_0^T g_i(s) dB_s = G_i \quad \text{where } G_i = \frac{B_{\tilde{t}_i} - B_{\tilde{t}_{i-1}}}{\sqrt{h}}.$$

Roughly speaking this means that P_k , the k th chaos, is generated by

$$\{K_{n_1}(G_1) \cdots K_{n_N}(G_N) : n_1 + \cdots + n_N = k\}.$$

Thus the approximation we use for the random variable F is

$$\begin{aligned} C_p^N(F) &= d_0 + \sum_{k=1}^p \sum_{|n|=k} d_k^n K_{n_1}(G_1) \cdots K_{n_N}(G_N) \\ &= d_0 + \sum_{k=1}^p \sum_{|n|=k} d_k^n \prod_{1 \leq i \leq N} K_{n_i}(G_i), \end{aligned} \quad (3.2.6)$$

where the coefficients d_0 and d_k^n are given by

$$d_0 = \mathbb{E}(F), \quad d_k^n = n! \mathbb{E}(F K_{n_1}(G_1) \cdots K_{n_N}(G_N)). \quad (3.2.7)$$

The following Lemma, similar to Lemma 3.2.2, gives some useful properties of the operator C_p^N .

Lemma 3.2.3. *Let F be r.v. in $L^2(\mathcal{F}_T)$ and H be in $H_T^2(\mathbb{R})$. Then :*

- $\forall (p, N) \in (\mathbb{N}^*)^2$, $\mathbb{E}(|C_p^N(F)|^2) \leq \mathbb{E}(|C_p(F)|^2) \leq \mathbb{E}(|F|^2)$.
- $C_p^N(\int_0^T H_s ds) = \int_0^T C_p^N(H_s) ds$.
- $\forall t \leq r$, $D_t \mathbb{E}_r[C_p^N(F)] = \mathbb{E}_r[C_{p-1}^N(D_t F)]$.

From (3.2.6), we deduce the expressions of $\mathbb{E}_t(C_p^N F)$ and $D_t \mathbb{E}_t(C_p^N(F))$, useful for the approximation of (Y, Z) by the chaos decomposition (see Section 3.4.1).

Proposition 3.2.1 (Proposition 2.7, [BL14]). *Let F be a real random variable in $L^2(\mathcal{F}_T)$, and let r be an integer in $\{1, \dots, N\}$. For all $\tilde{t}_{r-1} < t \leq \tilde{t}_r$, we have*

$$\mathbb{E}_t(C_p^N F) = d_0 + \sum_{k=1}^p \sum_{|n(r)|=k} d_k^n \prod_{i < r} K_{n_i}(G_i) \times \left(\frac{t - \tilde{t}_{r-1}}{h} \right)^{(n_r)/2} K_{n_r} \left(\frac{B_t - B_{\tilde{t}_{r-1}}}{\sqrt{t - \tilde{t}_{r-1}}} \right),$$

$$D_t \mathbb{E}_t(C_p^N(F)) = h^{-1/2} \sum_{k=1}^p \sum_{\substack{|n(r)|=k \\ n(r) > 0}} d_k^n \prod_{i < r} K_{n_i}(G_i) \times \left(\frac{t - \tilde{t}_{r-1}}{h} \right)^{(n_r-1)/2} K_{n_r-1} \left(\frac{B_t - B_{\tilde{t}_{r-1}}}{\sqrt{t - \tilde{t}_{r-1}}} \right),$$

where, if $r \leq N$ and $n = (n_1, \dots, n_N)$, $n(r)$ stands for (n_1, \dots, n_r) .

Remark 3.2.2 (Remark 1, [BL14]). *For $t = \tilde{t}_r$ and $r \geq 1$, Proposition 3.2.1 leads to*

$$\mathbb{E}_{\tilde{t}_r}(C_p^N F) = d_0 + \sum_{k=1}^p \sum_{|n(r)|=k} d_k^n \prod_{i \leq r} K_{n_i}(G_i),$$

$$D_{\tilde{t}_r} \mathbb{E}_{\tilde{t}_r}(C_p^N F) = h^{-1/2} \sum_{k=1}^p \sum_{\substack{|n(r)|=k \\ n(r) > 0}} d_k^n \prod_{i < r} K_{n_i}(G_i) \times K_{n_r-1}(G_r),$$

When $r = 0$, we get $\mathbb{E}_{\tilde{t}_0}(C_p^N F) = d_0$, and we define $D_{\tilde{t}_0} \mathbb{E}_{\tilde{t}_0}(C_p^N F) = \frac{1}{\sqrt{h}} d_1^{e_1}$ (which is the limit of $D_t \mathbb{E}_t(C_p^N F)$ when t tends to 0).

Let us end this subsection by some examples.

Example 3.2.1 (Case $p = 2$). From (3.2.6)-(3.2.7), we have

$$C_2^N(F) = d_0 + \sum_{j=1}^N d_1^{e_j} K_1(G_j) + \sum_{j=1}^N \sum_{i=1}^{j-1} d_2^{e_{ij}} K_1(G_i) K_1(G_j) + \sum_{j=1}^N d_2^{2e_j} K_2(G_j),$$

where e_j denotes the unit vector whose j th component is one, and $e_{ij} = e_i + e_j$. For $j = 1, \dots, N$ and $i = 1, \dots, j-1$, it holds

$$d_1^{e_j} = \mathbb{E}(FK_1(G_j)), \quad d_2^{e_{ij}} = \mathbb{E}(FK_1(G_i)K_1(G_j)) \quad d_2^{2e_j} = 2\mathbb{E}(FK_2(G_j)).$$

Remark 3.2.2 leads to

$$\mathbb{E}_{\tilde{t}_r}(C_2^N F) = d_0 + \sum_{j=1}^r d_1^{e_j} K_1(G_j) + \sum_{j=1}^r \sum_{i=1}^{j-1} d_2^{e_{ij}} K_1(G_i) K_1(G_j) + \sum_{j=1}^r d_2^{2e_j} K_2(G_j),$$

$$D_{\tilde{t}_r} \mathbb{E}_{\tilde{t}_r}(C_2^N F) = h^{-1/2} \left(d_1^{e_r} + d_2^{2e_r} K_1(G_r) + \sum_{i=1}^{r-1} d_2^{e_{ir}} K_1(G_i) \right).$$

3.3 Existence, uniqueness and properties of the solution.

Note that the existence and the uniqueness of the solution of (3.1.2) have been proved in [BLP09] in the case $f(t, Y_t, Z_t, [Y_t], [Z_t]) = \mathbb{E}[g(t, \lambda, Y_t, Z_t)]_{|\lambda=(Y_t, Z_t)}$. We give a standard proof in our setting for the ease of the reader.

Hypothesis 3.3.1. We assume :

- the generator $f : \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}) \times \mathcal{P}_2(\mathbb{R}^d) \rightarrow \mathbb{R}$ is Lipschitz continuous : there exists a constant L_f such that for all $t \in \mathbb{R}^+$, $y_1, y_2 \in \mathbb{R}$, $z_1, z_2 \in \mathbb{R}^d$, $\mu_1, \mu_2 \in \mathcal{P}_2(\mathbb{R})$ and $\nu_1, \nu_2 \in \mathcal{P}_2(\mathbb{R}^d)$

$$|f(t, y_1, z_1, \mu_1, \nu_1) - f(t, y_2, z_2, \mu_2, \nu_2)| \leq L_f \left(|y_1 - y_2| + |z_1 - z_2| + \mathcal{W}_2(\mu_1, \mu_2) + \mathcal{W}_2(\nu_1, \nu_2) \right).$$

- $\mathbb{E}(|\xi|^2 + \int_0^T |f(s, 0, 0, [\delta_0], [\delta_0])|^2 ds) < \infty$.

Theorem 3.3.1. Given standard parameters (f, ξ) , there exists a unique pair $(Y, Z) \in S_T^2(\mathbb{R}) \times H_T^2(\mathbb{R}^d)$ which solves (3.1.2).

Let us start with a priori estimates that will be useful for our proof.

A Priori Estimates.

Proposition 3.3.1. Let $((f^i, \xi^i); i = 1, 2)$ be two standard parameters of the BSDE and $((Y^i, Z^i); i = 1, 2)$ be two square-integrable solutions. Let L_{f^1} be a Lipschitz constant for f^1 , and put $\delta Y_t = Y_t^1 - Y_t^2$ and $\delta_2 f_t = f^1(t, Y_t^2, Z_t^2, [Y_t^2], [Z_t^2]) - f^2(t, Y_t^2, Z_t^2, [Y_t^2], [Z_t^2])$. For any $\alpha, \lambda > 0$ such that $\alpha \geq 8L_{f^1}^2 + 4L_{f^1} + \lambda + \frac{1}{2}$, it follows that

$$\|\delta Y\|_{S_{\alpha, T}^2}^2 + \|\delta Z\|_{H_{\alpha, T}^2}^2 \leq (8L_{f^1} + 8C^2 + 5) \left[e^{\alpha T} \mathbb{E}(|\delta Y_T|^2) + \frac{1}{\lambda} \mathbb{E} \left(\int_0^T e^{\alpha s} |\delta_2 f_s|^2 ds \right) \right],$$

where C is a universal constant.

Démonstration. By applying Itô's formula from $s = t$ to $s = T$ on the semimartingale $e^{\alpha s}|\delta Y_s|^2$, we get

$$\begin{aligned}
& e^{\alpha t}|\delta Y_t|^2 + \alpha \int_t^T e^{\alpha s}|\delta Y_s|^2 ds + \int_t^T e^{\alpha s}|\delta Z_s|^2 ds \\
&= e^{\alpha T}|\delta Y_T|^2 + 2 \int_t^T e^{\alpha s} \delta Y_s \left(f^1(s, Y_s^1, Z_s^1, [Y_s^1], [Z_s^1]) - f^2(s, Y_s^2, Z_s^2, [Y_s^2], [Z_s^2]) \right) ds \\
& \quad - 2 \int_t^T e^{\alpha s} \delta Y_s \delta Z_s dB_s.
\end{aligned} \tag{3.3.1}$$

Moreover,

$$\begin{aligned}
& |f^1(s, Y_s^1, Z_s^1, [Y_s^1], [Z_s^1]) - f^2(s, Y_s^2, Z_s^2, [Y_s^2], [Z_s^2])| \\
& \leq |f^1(s, Y_s^1, Z_s^1, [Y_s^1], [Z_s^1]) - f^1(s, Y_s^2, Z_s^2, [Y_s^2], [Z_s^2])| + |\delta_2 f_s| \\
& \leq L_f^1 \left(|\delta Y_s| + |\delta Z_s| + \mathcal{W}_2([Y_s^1], [Y_s^2]) + \mathcal{W}_2([Z_s^1], [Z_s^2]) \right) + |\delta_2 f_s|,
\end{aligned}$$

where $L_{f^1} \geq 0$. By using (3.1.3), we obtain that

$$\begin{aligned}
& 2|\delta Y_s| \cdot |f^1(s, Y_s^1, Z_s^1, [Y_s^1], [Z_s^1]) - f^2(s, Y_s^2, Z_s^2, [Y_s^2], [Z_s^2])| \\
& \leq 2L_{f^1} \left(|\delta Y_s|^2 + |\delta Y_s| |\delta Z_s| + |\delta Y_s| (\mathbb{E}(|\delta Y_s|^2))^{1/2} + |\delta Y_s| (\mathbb{E}(|\delta Z_s|^2))^{1/2} \right) \\
& \quad + 2|\delta Y_s| |\delta_2 f_s|.
\end{aligned} \tag{3.3.2}$$

Therefore, by Young's inequality with $\lambda > 0$, we have

$$\begin{aligned}
& 2|\delta Y_s| \cdot |f^1(s, Y_s^1, Z_s^1, [Y_s^1], [Z_s^1]) - f^2(s, Y_s^2, Z_s^2, [Y_s^2], [Z_s^2])| \\
& \leq 2L_{f^1} |\delta Y_s|^2 + 4L_{f^1}^2 |\delta Y_s|^2 + \frac{1}{4} |\delta Z_s|^2 + 2L_{f^1} |\delta Y_s| (\mathbb{E}(|\delta Y_s|^2))^{1/2} \\
& \quad + 4L_{f^1}^2 |\delta Y_s|^2 + \frac{1}{4} \mathbb{E}(|\delta Z_s|^2) + \lambda |\delta Y_s|^2 + \frac{1}{\lambda} |\delta_2 f_s|^2 \\
& \leq (8L_{f^1}^2 + 2L_{f^1} + \lambda) |\delta Y_s|^2 + 2L_{f^1} |\delta Y_s| (\mathbb{E}(|\delta Y_s|^2))^{1/2} \\
& \quad + \frac{1}{4} |\delta Z_s|^2 + \frac{1}{4} \mathbb{E}(|\delta Z_s|^2) + \frac{1}{\lambda} |\delta_2 f_s|^2.
\end{aligned} \tag{3.3.3}$$

On the one hand, it follows from (3.3.1) and (3.3.3) that for $t = 0$,

$$\begin{aligned}
& \mathbb{E}(|\delta Y_0|^2) + \alpha \mathbb{E} \left(\int_0^T e^{\alpha s} |\delta Y_s|^2 ds \right) + \mathbb{E} \left(\int_0^T e^{\alpha s} |\delta Z_s|^2 ds \right) \\
& \leq e^{\alpha T} \mathbb{E}(|\delta Y_T|^2) + (8L_{f^1}^2 + 4L_{f^1} + \lambda) \mathbb{E} \left(\int_0^T e^{\alpha s} |\delta Y_s|^2 ds \right) + \frac{1}{2} \mathbb{E} \left(\int_0^T e^{\alpha s} |\delta Z_s|^2 ds \right) \\
& \quad + \frac{1}{\lambda} \mathbb{E} \left(\int_0^T e^{\alpha s} |\delta_2 f_s|^2 ds \right).
\end{aligned} \tag{3.3.4}$$

Choosing $\alpha \geq 8L_{f^1}^2 + 4L_{f^1} + \lambda + \frac{1}{2}$, this inequality implies

$$\mathbb{E}\left(\int_0^T e^{\alpha s} |\delta Y_s|^2 ds\right) + \mathbb{E}\left(\int_0^T e^{\alpha s} |\delta Z_s|^2 ds\right) \leq 2\left[e^{\alpha T} \mathbb{E}(|\delta Y_T|^2) + \frac{1}{\lambda} \mathbb{E}\left(\int_0^T e^{\alpha s} |\delta_2 f_s|^2 ds\right)\right]. \quad (3.3.5)$$

On the other hand, by combining equation (3.3.1) and inequality (3.3.3), and by using the fact that $\alpha \geq 8L_{f^1}^2 + 2L_{f^1} + \lambda$, we can also obtain

$$\begin{aligned} e^{\alpha t} |\delta Y_t|^2 &\leq e^{\alpha T} |\delta Y_T|^2 + \int_t^T e^{\alpha s} \left(2L_{f^1} |\delta Y_s| \mathbb{E}(|\delta Y_s|^2) + \frac{1}{4} \mathbb{E}(|\delta Z_s|^2) + \frac{1}{\lambda} |\delta_2 f_s|^2\right) ds \\ &\quad + 2 \left| \int_t^T e^{\alpha s} \delta Y_s \delta Z_s dB_s \right|, \end{aligned}$$

which leads to

$$\begin{aligned} \mathbb{E}\left(\sup_{0 \leq t \leq T} e^{\alpha t} |\delta Y_t|^2\right) &\leq e^{\alpha T} \mathbb{E}(|\delta Y_T|^2) + 2L_{f^1} \mathbb{E}\left(\int_0^T e^{\alpha s} |\delta Y_s|^2 ds\right) + \frac{1}{4} \mathbb{E}\left(\int_0^T e^{\alpha s} |\delta Z_s|^2 ds\right) \\ &\quad + \frac{1}{\lambda} \mathbb{E}\left(\int_0^T e^{\alpha s} |\delta_2 f_s|^2 ds\right) + 2\mathbb{E}\left(\sup_{0 \leq t \leq T} \left| \int_t^T e^{\alpha s} \delta Y_s \delta Z_s dB_s \right|\right). \end{aligned} \quad (3.3.6)$$

By the Burkholder-Davis-Gundy inequality, there exists a universal constant C such that

$$\begin{aligned} \mathbb{E}\left(\sup_{0 \leq t \leq T} \left| \int_t^T e^{\alpha s} \delta Y_s \delta Z_s dB_s \right|\right) &\leq C \mathbb{E}\left[\left(\int_0^T e^{2\alpha s} |\delta Y_s|^2 |\delta Z_s|^2 ds\right)^{1/2}\right] \\ &\leq C \mathbb{E}\left[\left(\sup_{0 \leq t \leq T} e^{\alpha t} |\delta Y_t|^2\right)^{1/2} \left(\int_0^T e^{\alpha s} |\delta Z_s|^2 ds\right)^{1/2}\right], \end{aligned}$$

and since $ab \leq a^2/2 + b^2/2$,

$$2\mathbb{E}\left(\sup_{0 \leq t \leq T} \left| \int_t^T e^{\alpha s} \delta Y_s \delta Z_s dB_s \right|\right) \leq \frac{1}{2} \mathbb{E}\left(\sup_{0 \leq t \leq T} e^{\alpha t} |\delta Y_t|^2\right) + 2C^2 \mathbb{E}\left(\int_0^T e^{\alpha s} |\delta Z_s|^2 ds\right). \quad (3.3.7)$$

Finally, by combining the inequalities (3.3.6)-(3.3.7) and by using (3.3.5), we derive that

$$\begin{aligned} \mathbb{E}\left(\sup_{0 \leq t \leq T} e^{\alpha t} |\delta Y_t|^2\right) &\leq 2e^{\alpha T} \mathbb{E}(|\delta Y_T|^2) + 4L_{f^1} \mathbb{E}\left(\int_0^T e^{\alpha s} |\delta Y_s|^2 ds\right) + \frac{2}{\lambda} \mathbb{E}\left(\int_0^T e^{\alpha s} |\delta_2 f_s|^2 ds\right) \\ &\quad + \frac{(8C^2 + 1)}{2} \mathbb{E}\left(\int_0^T e^{\alpha s} |\delta Z_s|^2 ds\right) \\ &\leq (8L_{f^1} + 8C^2 + 3) \left[e^{\alpha T} \mathbb{E}(|\delta Y_T|^2) + \frac{1}{\lambda} \mathbb{E}\left(\int_0^T e^{\alpha s} |\delta_2 f_s|^2 ds\right) \right], \end{aligned}$$

then, we can conclude that

$$\mathbb{E}\left(\sup_{0 \leq t \leq T} e^{\alpha t} |\delta Y_t|^2\right) + \mathbb{E}\left(\int_0^T e^{\alpha s} |\delta Z_s|^2 ds\right) \leq (8L_{f^1} + 8C^2 + 5) \left[e^{\alpha T} \mathbb{E}(|\delta Y_T|^2) + \frac{1}{\lambda} \mathbb{E}\left(\int_0^T e^{\alpha s} |\delta_2 f_s|^2 ds\right) \right]. \quad (3.3.8)$$

□

Proof of Theorem 3.3.1. We use a fixed-point theorem for the mapping ϕ from $S_{\alpha,T}^2(\mathbb{R}) \times H_{\alpha,T}^2(\mathbb{R}^d)$ into $S_{\alpha,T}^2(\mathbb{R}) \times H_{\alpha,T}^2(\mathbb{R}^d)$, which maps (y, z) onto the solution (Y, Z) of the BSDE with generator $f(t, y_t, z_t, [y_t], [z_t])$, i.e.,

$$Y_t = \xi + \int_t^T f(s, y_s, z_s, [y_s], [z_s]) ds - \int_t^T Z_s \cdot dB_s.$$

Let us remark that the solution $(Y, Z) \in S_T^2(\mathbb{R}) \times H_T^2(\mathbb{R}^d)$ is defined by [PP90], when $(y, z) \in S_T^2(\mathbb{R}) \times H_T^2(\mathbb{R}^d)$.

Let $(y^1, z^1), (y^2, z^2)$ be two elements of $S_{\alpha,T}^2(\mathbb{R}) \times H_{\alpha,T}^2(\mathbb{R}^d)$, and let (Y^1, Z^1) and (Y^2, Z^2) be the associated solutions. By applying Proposition 3.3.1 with $L_f = 0$ and $\alpha = \lambda + \frac{1}{2}$, we obtain

$$\|\delta Y\|_{S_{\alpha,T}^2}^2 + \|\delta Z\|_{H_{\alpha,T}^2}^2 \leq \frac{(8C^2 + 5)}{\lambda} \mathbb{E} \left(\int_0^T e^{\alpha t} |f(t, y_t^1, z_t^1, [y_t^1], [z_t^1]) - f(t, y_t^2, z_t^2, [y_t^2], [z_t^2])|^2 dt \right).$$

Now since f is Lipschitz with constant L_f , we have

$$\begin{aligned} \|\delta Y\|_{S_{\alpha,T}^2}^2 + \|\delta Z\|_{H_{\alpha,T}^2}^2 &\leq \frac{4(8C^2 + 5)L_f^2}{\lambda} \mathbb{E} \left(\int_0^T e^{\alpha t} (|\delta y_t|^2 + |\delta z_t|^2 + \mathcal{W}_2([y_t^1], [y_t^2])^2 + \mathcal{W}_2([z_t^1], [z_t^2])^2) dt \right) \\ &\leq \frac{4(8C^2 + 5)L_f^2}{\lambda} \mathbb{E} \left(\int_0^T e^{\alpha t} (|\delta y_t|^2 + |\delta z_t|^2 + \mathbb{E}(|\delta y_t|^2) + \mathbb{E}(|\delta z_t|^2)) dt \right) \\ &\leq \frac{8(8C^2 + 5)L_f^2}{\lambda} \int_0^T (\mathbb{E}(e^{\alpha t} |\delta y_t|^2) + \mathbb{E}(e^{\alpha t} |\delta z_t|^2)) dt \\ &\leq \frac{8(8C^2 + 5)(T + 1)L_f^2}{\lambda} \left(\mathbb{E}(\sup_{0 \leq t \leq T} e^{\alpha t} |\delta y_t|^2) + \mathbb{E} \int_0^T e^{\alpha t} |\delta z_t|^2 dt \right) \\ &\leq \frac{8(8C^2 + 5)(T + 1)L_f^2}{\lambda} \left(\|\delta y\|_{S_{\alpha,T}^2}^2 + \|\delta z\|_{H_{\alpha,T}^2}^2 \right). \end{aligned} \tag{3.3.9}$$

Choosing $\lambda \geq 16(8C^2 + 5)(T + 1)L_f^2$, we see that this mapping ϕ is a contraction from $S_{\alpha,T}^2(\mathbb{R}) \times H_{\alpha,T}^2(\mathbb{R}^d)$ onto itself and that there exists a fixed point, which is the unique continuous solution of the BSDE. □

From the proof of Proposition 3.3.1 (and more precisely from estimate (3.3.9)), we derive that the Picard iterative sequence converges almost surely to the solution of the BSDE.

Remark 3.3.1. Let α be such that $\alpha \geq 16(8C^2 + 5)(T + 1)L_f^2 + \frac{1}{2}$. Let (Y^q, Z^q) be the sequence defined recursively by $(Y^0 = 0, Z^0 = 0)$ and

$$Y_t^{q+1} = \xi + \int_t^T f(s, Y_s^q, Z_s^q, [Y_s^q], [Z_s^q]) ds - \int_t^T Z_s^{q+1} \cdot dB_s, \quad 0 \leq t \leq T, \tag{3.3.10}$$

Then the sequence (Y^q, Z^q) converges to (Y, Z) , $d\mathbb{P} \times dt$ a.s. and in $S_T^2(\mathbb{R}) \times H_T^2(\mathbb{R}^d)$ as q goes to $+\infty$.

Démonstration. Let (Y^q, Z^q) be the sequence defined recursively by (3.3.10). Then, by (3.3.9),

$$\|Y^{q+1} - Y^q\|_{S_T^2}^2 + \|Z^{q+1} - Z^q\|_{H_T^2}^2 \leq C_T 2^{-q},$$

and the result follows easily. \square

3.4 Description of the algorithm.

The algorithm is based on five types of approximations : Picards iterations, a Wiener chaos expansion up to a finite order, the truncation of an $L^2(0, T)$ basis in order to apply formulas of Proposition 3.2.1, a Monte Carlo method to approximate the coefficients d_0 and d_k^n defined in (3.2.7) and the particle system. We present these five steps of the approximation procedure in Section 3.4.1. The practical implementation is presented in Section 3.4.2.

3.4.1 Approximation procedure.

Picards iterations.

The first step consists in approximating (Y, Z) the solution to (3.1.2) by Picards sequence $(Y^q, Z^q)_q$, built as follows : $(Y^0 = 0, Z^0 = 0)$ and for all $q \geq 1$

$$Y_t^{q+1} = \xi + \int_t^T f(s, Y_s^q, Z_s^q, [Y_s^q], [Z_s^q]) ds - \int_t^T Z_s^{q+1} \cdot dB_s, \quad 0 \leq t \leq T. \quad (3.4.1)$$

From (3.4.1), under the assumptions that $\xi \in \mathbb{D}^{1,2}$ and $f \in C_b^{0,1,1,0,0}$, we express (Y^{q+1}, Z^{q+1}) as a function of the processes (Y^q, Z^q) ,

$$Y_t^{q+1} = \mathbb{E}_t \left(\xi + \int_t^T f(s, Y_s^q, Z_s^q, [Y_s^q], [Z_s^q]) ds \right), \quad Z_t^{q+1} = D_t Y_t^{q+1}, \quad (3.4.2)$$

which can also be written

$$\begin{aligned} Y_t^{q+1} &= \mathbb{E}_t \left(\xi + \int_0^T f(s, Y_s^q, Z_s^q, [Y_s^q], [Z_s^q]) ds \right) - \int_0^t f(s, Y_s^q, Z_s^q, [Y_s^q], [Z_s^q]) ds, \\ Z_t^{q+1} &= D_t Y_t^{q+1}, \end{aligned} \quad (3.4.3)$$

As recalled in the Introduction, the computation of the conditional expectation is the cornerstone in the numerical resolution of BSDEs. Chaos decomposition formulas enable us to circumvent this problem.

Wiener Chaos expansion

Computing the chaos decomposition of the r.v. $F = \xi + \int_t^T f(s, Y_s^q, Z_s^q, [Y_s^q], [Z_s^q]) ds$ (appearing in (3.4.2)) in order to compute Y_t^{q+1} is not judicious. F depends on t , and then the computation of Y^{q+1} on the grid $\mathcal{T} = \{\tilde{t}_i = i \frac{T}{N}, i = 0, \dots, N\}$ would require

$N + 1$ calls to the chaos decomposition function. To build an efficient algorithm, we need to call the chaos decomposition function as infrequently as possible, since each call is computationally demanding and brings an approximation error due to the truncation, the Monte Carlo approximation and to the particle approximation (see next sections). Then we look for a r.v. F^q independent of t such that Y_t^{q+1} and Z_t^{q+1} can be expressed as functions of $\mathbb{E}_t(F^q)$, $D_t\mathbb{E}_t(F^q)$ and of Y_q and Z_q . Equation (3.4.3) gives a more tractable expression of Y^{q+1} . Let F^q be defined by $F^q := \xi + \int_0^T f(s, Y_s^q, Z_s^q, [Y_s^q], [Z_s^q])ds$. Then

$$Y_t^{q+1} = \mathbb{E}_t(F^q) - \int_0^t f(s, Y_s^q, Z_s^q, [Y_s^q], [Z_s^q])ds, \quad Z_t^{q+1} = D_t\mathbb{E}_t(F^q). \quad (3.4.4)$$

The second type of approximation consists in computing the chaos decomposition of F^q up to order p . Since F^q does not depend on t , the chaos decomposition function \mathcal{C}_p is called only once per Picards iteration.

Let $(Y^{q,p}, Z^{q,p})$ denote the approximation of (Y^q, Z^q) built at step q using a chaos decomposition with order p : $(Y^{0,p}, Z^{0,p}) = (0, 0)$ and

$$\begin{aligned} Y_t^{q+1,p} &= \mathbb{E}_t(C_p(F^{q,p})) - \int_0^t f(s, Y_s^{q,p}, Z_s^{q,p}, [Y_s^{q,p}], [Z_s^{q,p}])ds, \\ Z_t^{q+1,p} &= D_t\mathbb{E}_t(C_p(F^{q,p})), \end{aligned} \quad (3.4.5)$$

where $F^{q,p} = \xi + \int_0^T f(s, Y_s^{q,p}, Z_s^{q,p}, [Y_s^{q,p}], [Z_s^{q,p}])ds$. In the sequel, we also use the following equality :

$$Z_t^{q+1,p} = \mathbb{E}_t(D_t C_p(F^{q,p})). \quad (3.4.6)$$

Truncation of the basis.

The third type of approximation comes from the truncation of the orthonormal $L^2(0, T)$ basis used in the definition of C_p (3.2.3). Instead of considering a basis of $L^2(0, T)$, we only keep the first N functions (g_1, \dots, g_N) defined by (3.2.4) to build the chaos decomposition function C_p^N (3.2.5). Proposition 3.2.1 gives us explicit formulas for $\mathbb{E}_t(C_p^N F)$ and $D_t\mathbb{E}_t(C_p^N F)$. From (3.4.5), we build $(Y^{q,p,N}, Z^{q,p,N})_q$ in the following way : $(Y^{0,p,N}, Z^{0,p,N}) = (0, 0)$ and

$$\begin{aligned} Y_t^{q+1,p,N} &= \mathbb{E}_t(C_p^N(F^{q,p,N})) - \int_0^t f(s, Y_s^{q,p,N}, Z_s^{q,p,N}, [Y_s^{q,p,N}], [Z_s^{q,p,N}])ds, \\ Z_t^{q+1,p,N} &= D_t\mathbb{E}_t(C_p^N(F^{q,p,N})), \end{aligned} \quad (3.4.7)$$

where $F^{q,p,N} = \xi + \int_0^T f(s, Y_s^{q,p,N}, Z_s^{q,p,N}, [Y_s^{q,p,N}], [Z_s^{q,p,N}])ds$.

Equation (3.4.7) is tractable as soon as we know closed formulas for the coefficients d_k^n of the chaos decomposition of $\mathbb{E}_t(C_p^N(F^{q,p,N}))$ and $D_t\mathbb{E}_t(C_p^N(F^{q,p,N}))$ (see Proposition 3.2.1). When it is not the case, we need to use a Monte Carlo method to approximate these coefficients. The next section is devoted to this method.

Monte Carlo simulations of the chaos decomposition

Let F denote a r.v. of $L^2(\mathcal{F}_T)$. Practically, when we are not able to compute exactly $d_0 = \mathbb{E}(F)$ and/or the coefficients $d_n^k = n! \mathbb{E}(F K_{n_1}(G_1) \cdots K_{n_N}(G_N))$ of the chaos decomposition (3.2.6)-(3.2.7) of F , we use Monte Carlo simulations to approximate them. Let $(F^m)_{1 \leq m \leq M}$ be a M i.i.d. sample of F and $(G_1^m, \dots, G_N^m)_{1 \leq m \leq M}$ be a M i.i.d. sample of (G_1, \dots, G_N) . We propose a method which consists in approximating the expectations $\mathbf{d} := (d_0, (d_k^n)_{1 \leq k \leq p, |n|=k})$ by empirical means $\widehat{\mathbf{d}}_M := (\hat{d}_0, (\hat{d}_k^n)_{1 \leq k \leq p, |n|=k})$ where

$$\hat{d}_0 := \frac{1}{M} \sum_{m=1}^M F^m, \quad \hat{d}_k^n := \frac{n!}{M} \sum_{m=1}^M F^m K_{n_1}(G_1^m) \cdots K_{n_N}(G_N^m). \quad (3.4.8)$$

Definition 3.4.1. Let F be a r.v. of $L^2(\mathcal{F}_T)$ and (F^1, \dots, F^M) be M identically distributed r.v. with the law of F . We denote $C_p^{N,M}((F^m)_{1 \leq m \leq M})$ the following approximation of $C_p^N(F)$

$$C_p^{N,M}((F^m)_{1 \leq m \leq M}) = \hat{d}_0 + \sum_{k=1}^p \sum_{|n|=k} \hat{d}_k^n \prod_{1 \leq i \leq N} K_{n_i}(G_i). \quad (3.4.9)$$

where $(\hat{d}_0, (\hat{d}_k^n)_{1 \leq k \leq p, |n|=k})$ are defined by (3.4.8). We also introduce the filtration $(\mathcal{G}_t)_{0 \leq t \leq T}$ defined by $\mathcal{G}_t = \sigma((B_s)_{s \leq t}, (G_1^m, \dots, G_N^m)_{1 \leq m \leq M})$.

Remark 3.4.1. When (F^1, \dots, F^M) is an independent and identically distributed sample of F , we adopt the short notation $C_p^{N,M}(F)$ to refer to $C_p^{N,M}((F^m)_{1 \leq m \leq M})$. The notation $C_p^{N,M}((F^m)_{1 \leq m \leq M})$ will only be used when the r.v. F^1, \dots, F^M are not independent. This will be the case in the next paragraph, when we introduce the particle system to approximate the law of (Y, Z) (see (3.4.15)).

Before introducing the processes $(Y^{q+1,p,N,M}, Z^{q+1,p,N,M})$, we define $\mathbb{E}_t(C_p^{N,M}((F^m)_m))$ the conditional expectation of $C_p^{N,M}((F^m)_m)$ w.r.t. \mathcal{G}_t which corresponds to the conditional expectation obtained in Proposition 3.2.1 when $(d_0, d_k^n)_{1 \leq k \leq p, |n|=k}$ are replaced by $(\hat{d}_0, \hat{d}_k^n)_{1 \leq k \leq p, |n|=k}$, i.e. for all $t \in]\tilde{t}_{r-1}, \tilde{t}_r]$,

$$\mathbb{E}_t(C_p^{N,M}((F^m)_m)) := \hat{d}_0 + \sum_{k=1}^p \sum_{|n(r)|=k} \hat{d}_k^n \prod_{i < r} K_{n_i}(G_i) \times \left(\frac{t - \tilde{t}_{r-1}}{h} \right)^{(n_r)/2} K_{n_r} \left(\frac{B_t - B_{\tilde{t}_{r-1}}}{\sqrt{t - \tilde{t}_{r-1}}} \right). \quad (3.4.10)$$

We also introduce

$$D_t \mathbb{E}_t(C_p^{N,M}((F^m)_m)) := h^{-1/2} \sum_{k=1}^p \sum_{\substack{|n(r)|=k \\ n(r) > 0}} \hat{d}_k^n \prod_{i < r} K_{n_i}(G_i) \times \left(\frac{t - \tilde{t}_{r-1}}{h} \right)^{(n_r-1)/2} K_{n_r-1} \left(\frac{B_t - B_{\tilde{t}_{r-1}}}{\sqrt{t - \tilde{t}_{r-1}}} \right). \quad (3.4.11)$$

which is linked to $\mathbb{E}_t(C_p^{N,M}((F^m)_m))$ by the following relation

$$\forall t \leq T, \hat{d}_0 + \int_0^t D_s \mathbb{E}_s(C_p^{N,M}((F^m)_m)) dB_s = \mathbb{E}_t(C_p^{N,M}((F^m)_m)). \quad (3.4.12)$$

We keep the notation ' D_t ' since the above equality is nothing else than the Clark-Ocone formula when considering B as a \mathcal{G} -Brownian motion.

Remark 3.4.2. When M samples of $(C_p^{N,M}((F^m)_m))_m$ are needed, we can use the same samples as the ones used to compute \widehat{d}_0 and \widehat{d}_k^n :

$(C_p^{N,M}((F^m)_m))_m = \widehat{d}_0 + \sum_{k=1}^p \sum_{|n|=k} \widehat{d}_k^n \prod_{1 \leq i \leq N} K_{n_i}(G_i^m)$, or use new ones. In the first case, we only require M samples (G_1, \dots, G_N) . We built F^1, \dots, F^M from these samples. The coefficients \widehat{d}_k^n and \widehat{d}_0 are not independent of $\prod_{1 \leq i \leq N} K_{n_i}(G_i^m)$. In this case the notation $\mathbb{E}_t(C_p^{N,M}((F^m)_m))$ introduced above is not equal to $\mathbb{E}(C_p^{N,M}((F^m)_m)|\mathcal{G}_t)$. In the second case, the coefficients \widehat{d}_k^n and \widehat{d}_0 are independent of $\prod_{1 \leq i \leq N} K_{n_i}((G'_i)^m)$ and we have $\mathbb{E}_t(C_p^{N,M}((F^m)_m)) = \mathbb{E}(C_p^{N,M}((F^m)_m)|\mathcal{G}_t)$. This second approach requires $2M$ samples of F and (G_1, \dots, G_N) and its variance increases with N . Practically, we use the first technique.

Let us now introduce the couple of processes $(Y^{q+1,p,N,M}, Z^{q+1,p,N,M})$, which corresponds to the approximation of $(Y^{q+1,p,N}, Z^{q+1,p,N})$ when we use $C_p^{N,M}$ instead of C_p^N , that is, when we use a Monte Carlo procedure to compute the coefficients d_k^n .

$$\begin{aligned} Y_t^{q+1,p,N,M} &:= \mathbb{E}_t(C_p^{N,M}(F^{q,p,N,M})) - \int_0^t f(\theta_s^{q,p,N,M}) ds, \\ Z_t^{q+1,p,N,M} &:= D_t \mathbb{E}_t(C_p^{N,M}(F^{q,p,N,M})), \end{aligned} \quad (3.4.13)$$

where $F^{q,p,N,M} := \xi + \int_0^T f(\theta_s^{q,p,N,M}) ds$ and $\theta_s^{q,p,N,M} := (s, Y_s^{q,p,N,M}, Z_s^{q,p,N,M}, [Y_s^{q,p,N,M}], [Z_s^{q,p,N,M}])$.

Particle system

In this section, we introduce an interacting particle system associated to (3.4.13) to approximate the law of $Y_s^{q,p,N,M}$ and $Z_s^{q,p,N,M}$. Indeed we replace a one single stochastic differential equation with unknown processes $Y_s^{q,p,N,M}$ and $Z_s^{q,p,N,M}$, with a system of M ordinary stochastic differential equations, whose solution consists in a system of particles $(Y_s^{q,p,N,M,m}, Z_s^{q,p,N,M,m})_{1 \leq m \leq M}$, replacing the law of the processes $Y_s^{q,p,N,M}$ and $Z_s^{q,p,N,M}$ by the empirical mean law

$$\begin{aligned} [Y_s^{q,p,N,M,m}]^M &= \frac{1}{M} \sum_{m=1}^M \delta_{Y_s^{q,p,N,M,m}}, \\ [Z_s^{q,p,N,M,m}]^M &= \frac{1}{M} \sum_{m=1}^M \delta_{Z_s^{q,p,N,M,m}}. \end{aligned}$$

Our candidates are the particles :

$$\begin{aligned} Y_t^{q+1,p,N,M,m} &:= \mathbb{E}_t(C_p^{N,M}((F^{q,p,N,M,m})_m)) \\ &\quad - \int_0^t f(s, Y_s^{q,p,N,M,m}, Z_s^{q,p,N,M,m}, [Y_s^{q,p,N,M,m}]^M, [Z_s^{q,p,N,M,m}]^M) ds, \\ Z_t^{q+1,p,N,M,m} &:= D_t \mathbb{E}_t(C_p^{N,M}((F^{q,p,N,M,m})_m)), \end{aligned} \quad (3.4.14)$$

where, for all $m \in \{1, \dots, M\}$,

$$F^{q,p,N,M,m} = \xi + \int_0^T f(s, Y_s^{q,p,N,M,m}, Z_s^{q,p,N,M,m}, [Y_s^{q,p,N,M,m}]^M, [Z_s^{q,p,N,M,m}]^M) ds. \quad (3.4.15)$$

Remark 3.4.3. Looking at (3.4.14) and (3.4.15), we notice that we use the M particles to compute $(\hat{d}_0, (\hat{d}_k^n)_{1 \leq k \leq p, |n|=k})$. Taking the same M drawings for the Monte Carlo simulations and the particle system has two impacts, one positive and one negative :

- the algorithm is not more costly than the one solving standard BSDEs when f is a separated variable function (see Remark 3.4.4),
- the samples $(F^{q,p,N,M,m})_{1 \leq m \leq M}$ being not independent, the speed of convergence of the algorithm will be badly impacted (see Remark (3.5.3)).
- As for (3.4.13), $\mathbb{E}_t(C_p^{N,M}((F^{q,p,N,M,m})_m))$ corresponds to $\mathbb{E}(C_p^{N,M}((F^{q,p,N,M,m})_m)|\mathcal{G}_t)$ when we consider the second approach of Remark 3.4.2.

Remark 3.4.4. The computational effort regarding the particle system of the algorithm depends on the form of the driver. To simplify we omit the z -terms in f .

- If $f(y, \mu) = f^1(y) + \int \phi(v)d\mu$ then $f(Y_t, [Y_t]) = f^1(Y_t) + \mathbb{E}(\phi(Y_t))$. Using a particle system $(Y_t^m)_{1 \leq m \leq M}$ we approximate $\mathbb{E}(\phi(Y_t))$ by $\frac{1}{M} \sum_{m=1}^M \phi(Y_t^m)$. This is done once and used for all the particles Y_t^m to evaluate $f(Y_t^m, [Y_t^m]^M)$. the complexity is of order M and then it is the same as the one we get when applying the algorithm for classical BSDEs.
- If the interaction is more complicated, say for example $f(y, \mu) = \int b(y, v)d\mu(v)$ the approximation of $f(Y_t, [Y_t])$ has to be recomputed for each particle and is approximated by

$$f(Y_t^m, [Y_t^m]^M) = \frac{1}{M} \sum_{m'=1}^M b(Y_t^m, [Y_t^{m'}]).$$

The computational cost is of $O(M^2)$ and thus different from the algorithm for classical BSDEs.

3.4.2 Pseudo-code of the algorithm

We aim at computing M trajectories of an approximation of (Y, Z) on the grid $\mathcal{T} = \{\tilde{t}_i = i\frac{T}{N}, i = 0, \dots, N\}$. Starting from $(Y^{0,p,N,M,m}, Z^{0,p,N,M,m}) = (0, 0)$, (3.4.14) enables to get $(Y^{q,p,N,M,m}, Z^{q,p,N,M,m})$ for each Picards iteration q on \mathcal{T} . Practically, we discretize the integral $\int_0^t f(\theta_s^{q,p,N,M,m})ds$ which leads to approximated values of $(Y^{q,p,N,M,m}, Z^{q,p,N,M,m})$ computed on a grid.

Let us introduce $(\tilde{Y}_{\tilde{t}_i}^{q+1,p,N,M,m}, \tilde{Z}_{\tilde{t}_i}^{q+1,p,N,M,m})_{1 \leq i \leq N}$, defined by $(\tilde{Y}^{0,p,N,M,m}, \tilde{Z}^{0,p,N,M,m}) = (0, 0)$ and for all $q \geq 0$

$$\begin{aligned} \tilde{Y}_{\tilde{t}_i}^{q+1,p,N,M,m} &= \mathbb{E}_{\tilde{t}_i}(C_p^{N,M}((\tilde{F}^{q,p,N,M,m})_m)) - h \sum_{j=1}^i f(\tilde{t}_j, \tilde{Y}_{\tilde{t}_j}^{q,p,N,M,m}, \tilde{Z}_{\tilde{t}_j}^{q,p,N,M,m}, [\tilde{Y}_{\tilde{t}_j}^{q,p,N,M,m}]^M, \\ &\quad [\tilde{Z}_{\tilde{t}_j}^{q,p,N,M,m}]^M), \\ \tilde{Z}_{\tilde{t}_i}^{q+1,p,N,M,m} &= D_{\tilde{t}_i} \mathbb{E}_{\tilde{t}_i}(C_p^{N,M}((\tilde{F}^{q,p,N,M,m})_m)), \end{aligned} \tag{3.4.16}$$

where $\tilde{F}^{q,p,N,M,m} := \xi + h \sum_{i=1}^N f(\tilde{t}_i, \tilde{Y}_{\tilde{t}_i}^{q,p,N,M,m}, \tilde{Z}_{\tilde{t}_i}^{q,p,N,M,m}, [\tilde{Y}_{\tilde{t}_i}^{q,p,N,M,m}]^M, [\tilde{Z}_{\tilde{t}_i}^{q,p,N,M,m}]^M)$. Here is the notation we use in the algorithm :

- d : dimension of the Brownian motion ;
- q : index of Picards iteration ;
- K_{it} : number of Picards iterations ;
- M : number of Monte Carlo samples ;
- N : number of time steps used for the discretization of Y and Z ;
- p : order of the chaos decomposition ;
- $\mathbf{Y}^q \in \mathcal{M}_{N+1,M}(\mathbb{R})$ represents M paths of $\tilde{Y}^{q,p,N,M,m}$ computed on the grid \mathcal{T} ;
- for all $l \in \{1, \dots, d\}$, $(\mathbf{Z}^q)_l \in \mathcal{M}_{N+1,M}(\mathbb{R})$ represents M paths of $(\tilde{Z}^{q,p,N,M,m})_l$ computed on the grid \mathcal{T} .

Since ξ in $L^2(\mathcal{F}_T)$, ξ can be written as a measurable function of the Brownian path. Then one gets one sample of ξ from one sample of (G_1, \dots, G_N) (where G_i represents $\frac{B_{\tilde{t}_i} - B_{\tilde{t}_{i-1}}}{\sqrt{h}}$).

For the sake of clarity, we detail the algorithm for $d = 1$ and in the first case presented in Remark 3.4.4.

Algorithm 2 Iterative algorithm

- 1: Pick at random $N \times M$ values of standard Gaussian r.v. stored in \mathbf{G} .
 - 2: Using \mathbf{G} , compute $(\xi^m)_{0 \leq m \leq M-1}$.
 - 3: $\mathbf{Y}^0 \equiv 0, \mathbf{Z}^0 \equiv 0$.
 - 4: **for** $q = 0 : K_{it} - 1$ **do**
 - 5: **for** $m = 0 : M - 1$ **do**
 - 6: Compute $F^{q,m} = \xi^m + h \sum_{i=1}^N f(\tilde{t}_i, (\mathbf{Y}^q)_{i,m}, (\mathbf{Z}^q)_{i,m}, [(\mathbf{Y}^q)_i]^M, [(\mathbf{Z}^q)_i]^M)$
 - 7: **end for**
 - 8: Compute the vector $\mathbf{d} := (\hat{d}_0, (\hat{d}_k^n)_{1 \leq k \leq p, |n|=k})$ of the chaos decomposition of $(F^{q,m})_m$
 - 9: $\hat{d}_0 := \frac{1}{M} \sum_{m=0}^{M-1} F^{q,m}, \hat{d}_k^n := \frac{n!}{M} \sum_{m=0}^{M-1} F^{q,m} K_{n_1}(G_1^m) \cdots K_{n_N}(G_N^m)$
 - 10: **for** $j = 1 : N$ **do**
 - 11: **for** $m = 0 : M - 1$ **do**
 - 12: Compute $(\mathbb{E}_{\tilde{t}_j}(C_p^{N,M}((F^{q,m})_m)))_m, (D_{\tilde{t}_j}(\mathbb{E}_{\tilde{t}_j}(C_p^{N,M}((F^{q,m})_m))))_m$
 - 13: $(\mathbf{Y}^{q+1})_{j,m} = (\mathbb{E}_{\tilde{t}_j}(C_p^{N,M}((F^{q,m})_m)))_m$ —
 - 14: $(\mathbf{Z}^{q+1})_{j,m} = (D_{\tilde{t}_j}(\mathbb{E}_{\tilde{t}_j}(C_p^{N,M}((F^{q,m})_m))))_m$
 - 15: **end for**
 - 16: Compute the empirical distribution of the particles $(\mathbf{Y}^{q+1})_{:,m}$ and $(\mathbf{Z}^{q+1})_{:,m}$
 - 17: $[(\mathbf{Y}^{q+1})_j]^M = M^{-1} \sum_{m=1}^M \delta_{(\mathbf{Y}^{q+1})_{j,m}}, [(\mathbf{Z}^{q+1})_j]^M = M^{-1} \sum_{m=1}^M \delta_{(\mathbf{Z}^{q+1})_{j,m}}$
 - 18: **end for**
 - 19: **end for**
 - 20: Return $(\mathbf{Y}^{K_{it}})_{0,:} = \hat{d}_0$ and $(\mathbf{Z}^{K_{it}})_{0,:} = \frac{1}{\sqrt{h}} \hat{d}_1^{e_1}$
-

3.5 Convergence results

We aim at bounding the error between (Y, Z) the solution of (3.1.2) and $(Y^{q,p,N,M,m}, Z^{q,p,N,M,m})$ defined by (3.4.14). Before stating the main result of this chapter, we recall some hypotheses introduced in [BL14].

In the following, (t_1, \dots, t_n) and (s_1, \dots, s_n) denote two vectors such that

$$0 \leq t_1 \leq \dots \leq t_n \leq T, \quad 0 \leq s_1 \leq \dots \leq s_n \leq T \quad \text{and} \quad \forall i, s_i \leq t_i.$$

Hypothesis 3.5.1. (Hypothesis \mathcal{H}_m). Let $m \in \mathbb{N}^*$. We say that F satisfies Hypothesis \mathcal{H}_m if F satisfies the two following hypotheses :

- \mathcal{H}_m^1 : $\forall j \geq 2$, $F \in \mathcal{D}^{m,j}$, that is $\|F\|_{m,j}^j < \infty$;
- \mathcal{H}_m^2 : $\forall j \geq 2$, $\forall i \in \{1, \dots, m\}$, $\forall l_0 \leq i - 1$, $\forall l_1 \leq m - i$, $\forall l \in \{1, \dots, d\}$ and for all multi-indices α_0 and α_1 such that $|\alpha_0| = l_0$ and $|\alpha_1| = l_1 + 1$, there exist two positive constants β_F and k_l^F such that

$$\sup_{t_1 \leq \dots \leq t_{l_0}} \sup_{s_{i+1} \leq \dots \leq s_{i+l_1}} \mathbb{E}(|D_{t_1, \dots, t_{l_0}}^{\alpha_0} (D_{t_i, s_{i+1}, \dots, s_{i+l_1}}^{\alpha_1} F - D_{s_i, \dots, s_{i+l_1}}^{\alpha_1} F)|^j) \leq k_l^F(j) (t_i - s_i)^{j\beta_F},$$

where $l = l_0 + l_1 + 1$. In the following, we denote $K_m^F(j) = \sup_{l \leq m} k_l^F(j)$.

Remark 3.5.1. If F satisfies \mathcal{H}_m^2 , for all multi-index α such that $|\alpha| = l$, we have

$$|\mathbb{E}(D_{t_1, \dots, t_l}^\alpha F) - \mathbb{E}(D_{s_1, \dots, s_l}^\alpha F)| \leq K_l^F((t_1 - s_1)^{\beta_F} + \dots + (t_l - s_l)^{\beta_F}), \quad (3.5.1)$$

where K_l^F is a constant.

Hypothesis 3.5.2. (Hypothesis $\mathcal{H}_{p,N}^3$). Let $(p, N) \in \mathbb{N}^2$. We say that an r.v. F satisfies $\mathcal{H}_{p,N}^3$ if

$$V_{p,N}(F) := \mathbb{V}(F) + \sum_{k=1}^p \sum_{|n|=k} n! \mathbb{V} \left(F \prod_{i=1}^N K_{n_i}(G_i) \right) < \infty.$$

Remark 3.5.2. If F is bounded by K , we get $V_{p,N}(F) \leq K^2 \sum_{k=0}^p \binom{k}{N}$. Then every bounded r.v. satisfies $\mathcal{H}_{p,N}^3$.

This remark ensues from $\mathbb{E}(\prod_{i=1}^N K_{n_i}^2(G_i)) = \frac{1}{n!}$.

Theorem 3.5.1. Assume that ξ satisfies \mathcal{H}_{p+q} and there exist reals $r > 4$ and $q' > 0$ s.t. $\xi \in \mathbb{L}^{q'r}$ and $f \in C_b^{0,p+q,p+q,0,0}$. We have

$$\begin{aligned} & \| (Y - Y^{q,p,N,M,m}, Z - Z^{q,p,N,M,m}) \|_{L^2}^2 \\ & \leq \frac{A_0}{2^q} + \frac{A_1(q,p)}{(p+1)!} + A_2(q,p) \left(\frac{T}{N} \right)^{2\beta_\xi \wedge 1} + \frac{A_3(q,p,N)}{M} + \frac{A_4(q,p,N)(\log M)^{\frac{\alpha}{2}}}{M^{\left(\frac{1}{4} \wedge \frac{\beta}{2}\right) \frac{1}{2q}}} \end{aligned}$$

where A_0 is given in Section 3.5.1, A_1 is given in Proposition 3.5.1, A_2 is given in Proposition 3.5.2, A_3 is given in Proposition 3.5.3, A_4 is given in Proposition 3.5.4 and (α, β) depends on the dimension d and is defined by (3.5.14).

If $f \in C_b^{0,\infty,\infty,0,0}$ and ξ satisfies \mathcal{H}_∞ and $\mathcal{H}_{\infty,\infty}^3$, we obtain that, for each $m \geq 1$,

$$\lim_{q \rightarrow \infty} \lim_{p \rightarrow \infty} \lim_{N \rightarrow \infty} \lim_{M \rightarrow \infty} \| (Y - Y^{q,p,N,M,m}, Z - Z^{q,p,N,M,m}) \|_{L^2}^2 = 0.$$

Remark 3.5.3. Compared to [BL14, Theorem 4.6] we notice that the additional term $\frac{A_4(q,p,N)(\log M)^{\frac{\alpha}{2}}}{M^{\left(\frac{1}{4} \wedge \frac{\beta}{2}\right) \frac{1}{2q}}}$ appears in the error bound. This term corresponds to the error approximation due to the particle system. It is clearly worse than $\frac{A_3(q,p,N)}{M}$ which corresponds to

the error due to the Monte Carlo approximation. As we will see in Lemma 3.5.4, introducing some dependency between M identically distributed r.v. gives a worse control of the error $C_p^N - C_p^{N,M}$ than the one obtained when considering i.i.d. r.v. (see Lemma 3.5.3).

Proof of Theorem 3.5.1. We split the error into 5 terms :

1. Picards iterations : $\varepsilon^q = \|(Y - Y^q, Z - Z^q)\|_{L^2}^2$, where (Y_q, Z_q) is defined by (3.4.1) ;
2. the truncation of the chaos decomposition : $\varepsilon^{q,p} = \|(Y^q - Y^{q,p}, Z^q - Z^{q,p})\|_{L^2}^2$, where $(Y_{q,p}, Z_{q,p})$ is defined by (3.4.5) ;
3. the truncation of the $L^2(0, T)$ basis : $\varepsilon^{q,p,N} = \|(Y^{q,p} - Y^{q,p,N}, Z^{q,p} - Z^{q,p,N})\|_{L^2}^2$, where $(Y_{q,p,N}, Z_{q,p,N})$ is defined by (3.4.7) ;
4. the Monte Carlo approximation to compute the expectations : $\varepsilon^{q,p,N,M} = \|(Y^{q,p,N} - Y^{q,p,N,M}, Z^{q,p,N} - Z^{q,p,N,M})\|_{L^2}^2$, where $(Y_{q,p,N,M}, Z_{q,p,N,M})$ is defined by (3.4.13).
5. the particle system : $\varepsilon^{q,p,N,M,m} = \|(Y^{q,p,N,M} - Y^{q,p,N,M,m}, Z^{q,p,N,M} - Z^{q,p,N,M,m})\|_{L^2}^2$, where $(Y_{q,p,N,M,m}, Z_{q,p,N,M,m})$ is defined by (3.4.14).

We have

$$\|(Y - Y^{q,p,N,M,m}, Z - Z^{q,p,N,M,m})\|_{L^2}^2 \leq 5(\varepsilon^q + \varepsilon^{q,p} + \varepsilon^{q,p,N} + \varepsilon^{q,p,N,M} + \varepsilon^{q,p,N,M,m}).$$

It remains to combine (3.5.2), Propositions 3.5.1, 3.5.2, 3.5.3 and 3.5.4 to get the first result. The second one is straightforward. \square

3.5.1 Picards iterations.

From Remark 3.3.1, we know that under Hypothesis 3.3.1, the sequence $(Y^q, Z^q)_q$ defined by (3.4.1) converges to $(Y, Z) d\mathbb{P} \times dt$ a.s. and in $S_T^2(\mathbb{R}) \times H_T^2(\mathbb{R}^d)$. Moreover, we have

$$\varepsilon^q := \|(Y - Y^q, Z - Z^q)\|_{L^2}^2 \leq \frac{A_0}{2^q}, \quad (3.5.2)$$

where A_0 depends on T , $\|\xi\|^2$ and on $\|f(\cdot, 0, 0, [0], [0])\|_{L^2(0,T)}^2$.

3.5.2 Error due to the truncation of the chaos decomposition.

We assume that the integrals are computed exactly, as well as expectations. The error is only due to the truncation of the chaos decomposition C_p introduced in [Equation(2.3), [BL14]]. For the sequel, we also need the following Lemma. We refer to [BL14, Appendix A.2] for a proof.

Lemma 3.5.1. *Assume that ξ satisfies \mathcal{H}_{m+q}^1 and $f \in C_b^{0,m+q,m+q,0,0}$. Then $\forall q' \leq q$, $\forall p \in \mathbb{N}$, $(Y^{q'}, Z^{q'})$ and $(Y^{q',p}, Z^{q',p})$ belong to $\mathcal{S}^{m,\infty}$. Moreover*

$$\|(Y^q, Z^q)\|_{m,j}^j + \|(Y^{q,p}, Z^{q,p})\|_{m,j}^j \leq C(\|\xi\|_{m+q,((m+q-1)!/m!)_j}, (\|\partial_{sp}^k f\|_\infty)_{k \leq m+q}),$$

where C is a constant depending on $\|\xi\|_{m+q,((m+q-1)!/m!)_j}$ and on $(\|\partial_{sp}^k f\|_\infty)_{k \leq m+q}$.

Proposition 3.5.1. *Let $m \leq p+1$. Assume that ξ satisfies \mathcal{H}_{m+q}^1 and $f \in C_b^{0,m+q,m+q,0,0}$. We recall $\varepsilon^{q,p} = \|(Y^q - Y^{q,p}, Z^q - Z^{q,p})\|_{L^2}^2$. We get*

$$\varepsilon^{q+1,p} \leq C_1 T(T+1) L_f^2 \varepsilon^{q,p} + \frac{K_1(q,m)}{(p+1) \cdots (p+2-m)}, \quad (3.5.3)$$

where C_1 is a scalar and $K_1(q,m)$ depends on T , m , $\|\xi\|_{m+q,2(m+q-1)!/(m-1)!}$ and on $(\|\partial_{sp}^k f\|_\infty)_{1 \leq k \leq m+q}$.

Since $\varepsilon^{0,p} = 0$, we deduce from (3.5.3) that $\varepsilon^{q,p} \leq \frac{A_1(q,m)}{(p+1) \cdots (p+2-m)}$ where $A_1(q,m) := \frac{(C_1 T(T+1) L_f^2)^{q-1}}{C_1 T(T+1) L_f^{2q-1}} K_1(q,m)$. Then, $(Y^{q,p}, Z^{q,p})$ converges to (Y^q, Z^q) when p tends to ∞ in $\|(\cdot, \cdot)\|_{L^2}$.

Remark 3.5.4. *We deduce from Proposition 3.5.1 that for all T and L_f , we have $\lim_{q \rightarrow \infty} \varepsilon^{q,p} = 0$. When $C_1 T(T+1) L_f^2 < 1$, that is, for T small enough, we also get $\lim_{p \rightarrow \infty} \lim_{q \rightarrow \infty} \varepsilon^{q,p} = 0$.*

Proof of Proposition 3.5.1. For the sake of clearness, we assume $d = 1$. In the following, one notes $\Delta Y_t^{q,p} := Y_t^{q,p} - Y_t^q$, $\Delta Z_t^{q,p} := Z_t^{q,p} - Z_t^q$ and $\Delta f_t^{q,p} := f(t, Y_t^{q,p}, Z_t^{q,p}, [Y_t^{q,p}], [Z_t^{q,p}]) - f(t, Y_t^q, Z_t^q, [Y_t^q], [Z_t^q])$. First, we deal with $\mathbb{E}(\sup_{0 \leq t \leq T} |\Delta Y_t^{q+1,p}|^2)$. From (3.4.4) and (3.4.5) we get

$$\begin{aligned} \Delta Y_t^{q+1,p} &= \mathbb{E}_t(C_p(F^{q,p}) - F^q) - \int_0^t \Delta f_s^{q,p} ds \\ &= \mathbb{E}_t(C_p(\xi) - \xi) \\ &\quad + \mathbb{E}_t\left(C_p\left(\int_0^T f(s, Y_s^{q,p}, Z_s^{q,p}, [Y_s^{q,p}], [Z_s^{q,p}]) ds\right) - \int_0^T f(s, Y_s^q, Z_s^q, [Y_s^q], [Z_s^q]) ds\right) \\ &\quad - \int_0^t \Delta f_s^{q,p} ds. \end{aligned}$$

We introduce $\pm C_p(\int_0^T f(s, Y_s^q, Z_s^q, [Y_s^q], [Z_s^q]) ds)$ in the second conditional expectation. This leads to

$$\begin{aligned} \Delta Y_t^{q+1,p} &= \mathbb{E}_t(C_p(\xi) - \xi) + \mathbb{E}_t\left(C_p\left(\int_0^T \Delta f_s^{q,p} ds\right)\right) \\ &\quad + \mathbb{E}_t\left(\int_0^T C_p(f(s, Y_s^q, Z_s^q, [Y_s^q], [Z_s^q])) - f(s, Y_s^q, Z_s^q, [Y_s^q], [Z_s^q]) ds\right) \\ &\quad - \int_0^t \Delta f_s^{q,p} ds. \end{aligned}$$

where we have used the second property of Lemma 3.2.2 to rewrite the third term.

From the previous equation, we bound $\mathbb{E}(\sup_{0 \leq t \leq T} |\Delta Y_t^{q+1,p}|^2)$ by using Doobs inequality,

the property of the Wasserstein-2 distance (see (3.1.3)) and the fact that f is Lipschitz :

$$\begin{aligned}
\mathbb{E}\left(\sup_{0 \leq t \leq T} |\Delta Y_t^{q+1,p}|^2\right) &\leq 16\mathbb{E}(|C_p(\xi) - \xi|^2) + 16\mathbb{E}\left(\left|C_p\left(\int_0^T \Delta f_s^{q,p} ds\right)\right|^2\right) \\
&\quad + 16T \int_0^T \mathbb{E}(|C_p(f(s, Y_s^q, Z_s^q, [Y_s^q], [Z_s^q])) - f(s, Y_s^q, Z_s^q, [Y_s^q], [Z_s^q])|^2) ds \\
&\quad + 16TL_f^2 \int_0^T \mathbb{E}(|\Delta Y_s^{q,p}|^2 + |\Delta Z_s^{q,p}|^2 + \mathcal{W}_2([Y_s^{q,p}], [Y_s^q])^2 + \mathcal{W}_2([Z_s^{q,p}], [Z_s^q])^2) ds \\
&\leq 16\mathbb{E}(|C_p(\xi) - \xi|^2) + 16\mathbb{E}\left(\left|C_p\left(\int_0^T \Delta f_s^{q,p} ds\right)\right|^2\right) \\
&\quad + 16T \int_0^T \mathbb{E}(|C_p(f(s, Y_s^q, Z_s^q, [Y_s^q], [Z_s^q])) - f(s, Y_s^q, Z_s^q, [Y_s^q], [Z_s^q])|^2) ds \\
&\quad + 32TL_f^2 \int_0^T \mathbb{E}(|\Delta Y_s^{q,p}|^2 + |\Delta Z_s^{q,p}|^2) ds.
\end{aligned}$$

To bound the second expectation of the previous inequality, we use the first property of Lemma 3.2.2 and the Lipschitz property of f . Then we bring together this term with the last one to get

$$\begin{aligned}
\mathbb{E}\left(\sup_{0 \leq t \leq T} |\Delta Y_t^{q+1,p}|^2\right) &\leq 16\mathbb{E}(|C_p(\xi) - \xi|^2) \\
&\quad + 16T \int_0^T \mathbb{E}(|C_p(f(s, Y_s^q, Z_s^q, [Y_s^q], [Z_s^q])) - f(s, Y_s^q, Z_s^q, [Y_s^q], [Z_s^q])|^2) ds \\
&\quad + 160TL_f^2 \int_0^T \mathbb{E}(|\Delta Y_s^{q,p}|^2 + |\Delta Z_s^{q,p}|^2) ds.
\end{aligned} \tag{3.5.4}$$

Let us now study the upper bound $\mathbb{E}(\int_0^T |\Delta Z_t^{q+1,p}|^2 ds)$. To do so, we use the Itô isometry $\mathbb{E}(\int_0^T |\Delta Z_t^{q+1,p}|^2 ds) = \mathbb{E}((\int_0^T \Delta Z_t^{q+1,p} dB_s)^2)$. Using the definitions (3.4.4)(3.4.6) and the ClarkOcone Theorem leads to

$$\begin{aligned}
\int_0^T \Delta Z_s^{q+1,p} dB_s &= F^q - \mathbb{E}(F^q) - (C_p(F^{q,p}) - \mathbb{E}(C_p(F^{q,p}))) \\
&= Y_T^{q+1} + \int_0^T f(s, Y_s^q, Z_s^q, [Y_s^q], [Z_s^q]) ds - Y_0^{q+1} \\
&\quad - \left(Y_T^{q+1,p} + \int_0^T f(s, Y_s^{q,p}, Z_s^{q,p}, [Y_s^{q,p}], [Z_s^{q,p}]) ds - Y_0^{q+1,p} \right).
\end{aligned}$$

Rearranging this summation makes $\Delta Y_T^{q+1,p} - (\Delta Y_0^{q+1,p})$ appear. We get

$$\begin{aligned}
\mathbb{E}\left(\int_0^T |\Delta Z_s^{q+1,p}|^2 ds\right) &\leq 6\mathbb{E}\left(\sup_{0 \leq t \leq T} |\Delta Y_t^{q+1,p}|^2\right) \\
&\quad + 24TL_f^2 \int_0^T \mathbb{E}(|\Delta Y_s^{q,p}|^2 + |\Delta Z_s^{q,p}|^2) ds.
\end{aligned} \tag{3.5.5}$$

Since $\int_0^T \mathbb{E}(|\Delta Y_s^{q,p}|^2 + |\Delta Z_s^{q,p}|^2) ds \leq (T+1)\varepsilon^{q,p}$, by computing $7 \times (3.5.4) + (3.5.5)$ we

obtain

$$\begin{aligned}\varepsilon^{q+1,p} &\leq 112\mathbb{E}(|C_p(\xi) - \xi|^2) \\ &\quad + 112T \int_0^T \mathbb{E}(|C_p(f(s, Y_s^q, Z_s^q, [Y_s^q], [Z_s^q])) - f(s, Y_s^q, Z_s^q, [Y_s^q], [Z_s^q])|^2) ds \\ &\quad + 1144T(T+1)L_f^2\varepsilon^{q,p}.\end{aligned}$$

Since ξ and $f(s, Y_s^q, Z_s^q, [Y_s^q], [Z_s^q])$ belong to $\mathbb{D}^{m,2}$ (ξ satisfies \mathcal{H}_{m+q}^1 , $f \in C_b^{0,m+q,m+q}$ and $(Y^q, Z^q) \in \mathcal{S}^{m,\infty}$ (see Lemma 3.5.1)), Lemma 3.2.1 gives

$$\begin{aligned}\varepsilon^{q+1,p} &\leq \frac{112}{(p+1)\cdots(p+2-m)} \|D^m \xi\|_{L^2(\Omega \times [0,T]^m)}^2 \\ &\quad + \frac{112T}{(p+1)\cdots(p+2-m)} \left(\int_0^T \|D^m f(s, Y_s^q, Z_s^q, [Y_s^q], [Z_s^q])\|_{L^2(\Omega \times [0,T]^m)}^2 ds \right) \\ &\quad + 1144T(T+1)L_f^2\varepsilon^{q,p}.\end{aligned}$$

Since $\int_0^T \|D^m f(s, Y_s^q, Z_s^q, [Y_s^q], [Z_s^q])\|_{L^2(\Omega \times [0,T]^m)}^2 ds$ is bounded by $C(T, m, (\|\partial_{sp}^k f\|_\infty)_{k \leq m}, \|(Y^q, Z^q)\|_{m,2m}^2)$, Lemma 3.5.1 gives the result. \square

3.5.3 Error due to the truncation of the basis.

We are now interested in bounding the error between $(Y^{q,p}, Z^{q,p})$ (defined by (3.4.5)) and $(Y^{q,p,N}, Z^{q,p,N})$ (defined by (3.4.7)).

Before giving an upper bound for the error, we measure the error between C_p and C_p^N for a r.v. satisfying (3.5.1) when $m = p$.

Remark 3.5.5 (Remark 4.13, [BL14]). *Let ξ satisfy \mathcal{H}_p and $f \in C_b^{0,p,p,0,0}$. Then, for all integer $q \geq 0$, $I_{q,p} := \int_0^T f(s, Y_s^{q,p}, Z_s^{q,p}, [Y_s^{q,p}], [Z_s^{q,p}]) ds$ satisfies (3.5.1); that is for all multi-index α such that $|\alpha| = r$, we have*

$$|\mathbb{E}(D_{t_1, \dots, t_r}^\alpha I_{q,p}) - \mathbb{E}(D_{s_1, \dots, s_r}^\alpha I_{q,p})| \leq K_r^{I_{q,p}} ((t_1 - s_1)^{\beta_{I_{q,p}}} + \dots + (t_r - s_r)^{\beta_{I_{q,p}}}),$$

where $\beta_{I_{q,p}} = \frac{1}{2} \wedge \beta_\xi$ and $K_r^{I_{q,p}}$ depends on K_r^ξ , $\|\xi\|_{p,1}$, T and on $(\|\partial_{sp}^k\|_\infty)_{1 \leq k \leq p}$.

Lemma 3.5.2 (Lemma 4.14, [BL14]). *Let F denote a r.v. in $L^2(\mathcal{F}_T)$ satisfying (3.5.1) for $m = p$. We have*

$$\mathbb{E}(|(C_p^N - C_p)(F)|^2) \leq (K_p^F)^2 \left(\frac{T}{N}\right)^{2\beta_F} \sum_{i=1}^p i^2 \frac{T^i}{i!} \leq (K_p^F)^2 \left(\frac{T}{N}\right)^{2\beta_F} T(1+T)e^T,$$

where K_p^F and β_F are defined in Hypothesis 3.5.1.

Proposition 3.5.2. *Assume that ξ satisfies \mathcal{H}_p and $f \in C_b^{0,p,p,0,0}$. We recall $\varepsilon^{q,p,N} := \|(Y^{q,p} - Y^{q,p,N}, Z^{q,p} - Z^{q,p,N})\|_{L^2}^2$. We get*

$$\varepsilon^{q+1,p,N} \leq C_2 T(T+1)L_f^2 \varepsilon^{q,p,N} + K_2(q,p)T(T+1)e^T \left(\frac{T}{N}\right)^{1 \wedge 2\beta_\xi}, \quad (3.5.6)$$

where C_2 is a scalar and $K_2(q, p)$ depends on K_p^ξ , T , $\|\xi\|_{p,1}$ and on $(\|\partial_{s_p}^k f\|_\infty)_{1 \leq k \leq p}$. Since $\varepsilon^{0,p,N} = 0$, we deduce from (3.5.6) that $\varepsilon^{q,p,N} \leq A_2(q, p) \left(\frac{T}{N}\right)^{1 \wedge 2\beta_\xi}$ where $A_2(q, p) := K_2(q, p)T(T+1)e^{T \frac{(C_2 T(T+1)L_f^2)^{q-1}}{C_2 T(T+1)L_f^2 - 1}}$. Then, $(Y^{q,p,N}, Z^{q,p,N})$ converges to $(Y^{q,p}, Z^{q,p})$ when N tends to ∞ in $\|(\cdot, \cdot)\|_{L^2}$.

Proof of Proposition 3.5.2. For the sake of clarity, we assume $d = 1$. In the following, one notes $\Delta Y_t^{q,p,N} := Y_t^{q,p,N} - Y_t^{q,p}$, $\Delta Z_t^{q,p,N} := Z_t^{q,p,N} - Z_t^{q,p}$ and $\Delta f_t^{q,p,N} := f(t, Y_t^{q,p,N}, Z_t^{q,p,N}, [Y_t^{q,p,N}], [Z_t^{q,p,N}]) - f(t, Y_t^{q,p}, Z_t^{q,p}, [Y_t^{q,p}], [Z_t^{q,p}])$. First, we deal with $\mathbb{E}(\sup_{0 \leq t \leq T} |\Delta Y_t^{q+1,p,N}|^2)$. From (3.4.5) and (3.4.7) we get

$$\Delta Y_t^{q+1,p,N} = \mathbb{E}_t(C_p(F^{q,p,N}) - F^{q,p}) - \int_0^t \Delta f_s^{q,p,N} ds.$$

Following the same steps as in the proof of Proposition 3.5.1, we get

$$\begin{aligned} \mathbb{E}\left(\sup_{0 \leq t \leq T} |\Delta Y_t^{q+1,p,N}|^2\right) &\leq 16\mathbb{E}(|C_p^N(\xi) - C_p(\xi)|^2) \\ &\quad + 16\mathbb{E}\left(\left|(C_p^N - C_p)\left(\int_0^T f(s, Y_s^{q,p}, Z_s^{q,p}, [Y_s^{q,p}], [Z_s^{q,p}]) ds\right)\right|^2\right) \\ &\quad + 160L_f^2 \int_0^T \mathbb{E}(|\Delta Y_s^{q,p,N}|^2 + |\Delta Z_s^{q,p,N}|^2) ds. \end{aligned} \tag{3.5.7}$$

Let us now study the upper bound $\mathbb{E}(\int_0^T |\Delta Z_s^{q+1,p,N}|^2 ds)$. Following the same steps as in the proof of Proposition 3.5.1, we get

$$\begin{aligned} \mathbb{E}\left(\int_0^T |\Delta Z_s^{q+1,p,N}|^2 ds\right) &\leq 6\mathbb{E}\left(\sup_{0 \leq t \leq T} |\Delta Y_t^{q+1,p,N}|^2\right) \\ &\quad + 24TL_f^2 \int_0^T \mathbb{E}(|\Delta Y_s^{q,p,N}|^2 + |\Delta Z_s^{q,p,N}|^2) ds. \end{aligned} \tag{3.5.8}$$

Adding $7 \times (3.5.7)$ and $(3.5.8)$ we obtain

$$\begin{aligned} \varepsilon^{q+1,p,N} &\leq 112\mathbb{E}(|(C_p^N - C_p)(\xi)|^2) \\ &\quad + 112\mathbb{E}\left(\left|(C_p^N - C_p)\left(\int_0^T f(s, Y_s^{q,p}, Z_s^{q,p}, [Y_s^{q,p}], [Z_s^{q,p}]) ds\right)\right|^2\right) \\ &\quad + 1144T(T+1)L_f^2 \varepsilon^{q,p,N}. \end{aligned}$$

Since ξ and $I_{q,p} := f(s, Y_s^{q,p}, Z_s^{q,p}, [Y_s^{q,p}], [Z_s^{q,p}])$ satisfy (3.5.1) (see Remarks 3.5.2 and 3.5.5), Lemma 3.5.2 gives

$$\begin{aligned} \varepsilon^{q+1,p,N} &\leq 112\left(\frac{T}{N}\right)^{1 \wedge 2\beta_\xi} T(T+1)e^T((K_p^\xi)^2 + (K_p^{I_{q,p}})^2) \\ &\quad + 1144T(T+1)L_f^2 \varepsilon^{q,p,N}. \end{aligned}$$

and (3.5.6) follows. \square

3.5.4 Error due to the Monte Carlo approximation.

We are now interested in bounding the error between $(Y^{q,p,N}, Z^{q,p,N})$ (defined by (3.4.7)) and $(Y^{q,p,N,M}, Z^{q,p,N,M})$ (defined by (3.4.13)). In this section, we assume that the coefficients \hat{d}_k^n are independent of the vector (G_1, \dots, G_N) , which corresponds to the second approach proposed in (Remark 3.2, [BL14]).

Before giving an upper bound for the error, we measure the error between C_p^N and $C_p^{N,M}$ for a r.v. satisfying $\mathcal{H}_{p,N}^3$ (see Hypothesis (3.5.2)). The following Lemma deals with $C_p^{N,M}(F)$, i.e. the case of i.i.d. samples of F (see Definition 3.4.1). The general case will be stated in Lemma 3.5.4.

Lemma 3.5.3. *Let F be a r.v. satisfying Hypothesis $\mathcal{H}_{p,N}^3$. We have*

$$\mathbb{E}(|(C_p^N - C_p^{N,M})(F)|^2) = \frac{1}{M} V_{p,N}(F),$$

and

$$\mathbb{E}(|C_p^{N,M}(F)|^2) \leq \mathbb{E}(|F|^2) + \frac{1}{M} V_{p,N}(F).$$

We refer to [BL14, Appendix A.5] for the proof of the lemma.

Proposition 3.5.3. *Let ξ satisfy Hypothesis $\mathcal{H}_{p,N}^3$ and f is a bounded function. Let $\varepsilon^{q,p,N,M} := \|(Y^{q,p,N} - Y^{q,p,N,M}, Z^{q,p,N} - Z^{q,p,N,M})\|_{L^2}^2$. We get*

$$\varepsilon^{q+1,p,N,M} \leq C_3 T(T+1) L_f^2 \varepsilon^{q,p,N,M} + \frac{K_3(p, N)}{M}, \quad (3.5.9)$$

where C_3 is a scalar and $K_3(p, N) := 724(V_{p,N}(\xi) + T^2 \|f\|_\infty^2 \sum_{k=0}^p \binom{k}{N})$.

Since $\varepsilon^{0,p,N,M} = 0$, we deduce from the previous inequality that $\varepsilon^{q,p,N,M} \leq \frac{A_3(q,p,N)}{M}$ where $A_3(q,p,N) := K_3(p, N) \frac{(C_3 T(T+1) L_f^2)^{q-1}}{C_3 T(T+1) L_f^2 - 1}$. Then, $(Y^{q,p,N,M}, Z^{q,p,N,M})$ converges to $(Y^{q,p,N}, Z^{q,p,N})$ when M tends to ∞ in $\|(\cdot, \cdot)\|_{L^2}$.

Proof of Proposition 3.5.3. For the sake of clarity, we assume $d = 1$. In the following, note that $\Delta Y_t^{q,p,N,M} := Y_t^{q,p,N,M} - Y_t^{q,p,N}$, $\Delta Z_t^{q,p,N,M} := Z_t^{q,p,N,M} - Z_t^{q,p,N}$ and $\Delta f_t^{q,p,N,M} := f(t, Y_t^{q,p,N,M}, Z_t^{q,p,N,M}, [Y_t^{q,p,N,M}], [Z_t^{q,p,N,M}]) - f(t, Y_t^{q,p,N}, Z_t^{q,p,N}, [Y_t^{q,p,N}], [Z_t^{q,p,N}])$. First, we deal with $\mathbb{E}(\sup_{0 \leq t \leq T} |\Delta Y_t^{q+1,p,N,M}|^2)$. From (3.4.7) and (3.4.13) we get

$$\Delta Y_t^{q+1,p,N,M} = \mathbb{E}_t(C_p^{N,M}(F^{q,p,N,M}) - C_p^N(F^{q,p,N})) - \int_0^t \Delta f_s^{q,p,N,M} ds.$$

By introducing $\pm C_p^N(F^{q,p,N,M})$ and by using Lemma 3.2.3, Doob's inequality, and the Lipschitz property of f , we obtain

$$\begin{aligned}
\mathbb{E}\left(\sup_{0 \leq t \leq T} |\Delta Y_t^{q+1,p,N,M}|^2\right) &\leq 12\mathbb{E}\left(|(C_p^{N,M} - C_p^N)(F^{q,p,N,M})|^2\right) \\
&\quad + 12\mathbb{E}\left(|C_p^N(F^{q,p,N,M} - F^{q,p,N})|^2\right) \\
&\quad + 12TL_f^2 \int_0^T \mathbb{E}(|\Delta Y_s^{q,p,N,M}|^2 + |\Delta Z_s^{q,p,N,M}|^2 \\
&\quad + \mathcal{W}_2([Y_s^{q,p,N,M}], [Y_s^{q,p,N}])^2 + \mathcal{W}_2([Z_s^{q,p,N,M}], [Z_s^{q,p,N}])^2) ds \\
&\leq 12\mathbb{E}\left(|(C_p^{N,M} - C_p^N)(F^{q,p,N,M})|^2\right) \\
&\quad + 12\mathbb{E}\left(|F^{q,p,N,M} - F^{q,p,N}|^2\right) \\
&\quad + 24TL_f^2 \int_0^T \mathbb{E}(|\Delta Y_s^{q,p,N,M}|^2 + |\Delta Z_s^{q,p,N,M}|^2) ds.
\end{aligned}$$

From Lemma 3.5.3, we get $\mathbb{E}(|(C_p^{N,M} - C_p^N)(F^{q,p,N,M})|^2) \leq \frac{2}{M}(V_{p,N}(\xi) + V_{p,N}(\int_0^t f(\theta_s^{q,p,N,M}) ds))$. Then, from Remark 3.5.2,

$$\begin{aligned}
\mathbb{E}\left(\sup_{0 \leq t \leq T} |\Delta Y_t^{q+1,p,N,M}|^2\right) &\leq \frac{24}{M}\left(V_{p,N}(\xi) + T^2\|f\|_\infty^2 \sum_{k=0}^p \binom{k}{N}\right) \\
&\quad + 120TL_f^2 \int_0^T \mathbb{E}(|\Delta Y_s^{q,p,N,M}|^2 + |\Delta Z_s^{q,p,N,M}|^2) ds.
\end{aligned} \tag{3.5.10}$$

Let us now bound $\mathbb{E}\left(\int_0^T |\Delta Z_s^{q+1,p,N,M}|^2 ds\right)$. From (3.4.13) and (3.4.12) we get

$$\hat{d}_0 + \int_0^T Z_s^{q+1,p,N,M} dB_s = Y_T^{q+1,p,N,M} + \int_0^T f(\theta_s^{q,p,N,M}) ds.$$

Since $\hat{d}_0 = Y_0^{q+1,p,N,M}$, we are back to the same proof as in Proposition 3.5.1.

$$\begin{aligned}
\mathbb{E}\left(\int_0^T |\Delta Z_s^{q+1,p,N,M}|^2 ds\right) &\leq 6\mathbb{E}\left(\sup_{0 \leq t \leq T} |\Delta Y_t^{q+1,p,N,M}|^2\right) \\
&\quad + 24TL_f^2 \int_0^T \mathbb{E}(|\Delta Y_s^{q,p,N,M}|^2 + |\Delta Z_s^{q,p,N,M}|^2) ds.
\end{aligned} \tag{3.5.11}$$

Adding $7 \times (3.5.10)$ and (3.5.11) gives the result. \square

3.5.5 Error due to the Particle approximation.

In order to prove the error between $(Y^{q,p,N,M}, Z^{q,p,N,M})$ (defined by (3.4.13)) and $(Y^{q,p,N,M,m}, Z^{q,p,N,M,m})$ (defined by (3.4.14)) for all $1 \leq m \leq M$, we introduce the follo-

wing independent copies of $(Y^{q,p,N,M}, Z^{q,p,N,M})$,

$$\begin{aligned}\bar{Y}_t^{q+1,p,N,M,m} &= \mathbb{E}_t(C_p^{N,M}((\bar{F}^{q,p,N,M,m})_m)) \\ &\quad - \int_0^t f(s, \bar{Y}_s^{q,p,N,M,m}, \bar{Z}_s^{q,p,N,M,m}, [\bar{Y}_s^{q,p,N,M,m}], [\bar{Z}_s^{q,p,N,M,m}]) ds, \quad (3.5.12) \\ \bar{Z}_t^{q+1,p,N,M,m} &= D_t \mathbb{E}_t(C_p^{N,M}((\bar{F}^{q,p,N,M,m})_m)),\end{aligned}$$

where $\bar{F}^{q,p,N,M,m} = \xi + \int_0^T f(s, \bar{Y}_s^{q,p,N,M,m}, \bar{Z}_s^{q,p,N,M,m}, [\bar{Y}_s^{q,p,N,M,m}], [\bar{Z}_s^{q,p,N,M,m}]) ds$.

Note that $[\bar{Y}_s^{q,p,N,M,m}]^M$ (resp. $[\bar{Z}_s^{q,p,N,M,m}]^M$) is the empirical distribution of the particles $\bar{Y}_s^{q,p,N,M,m}$ (resp. $\bar{Z}_s^{q,p,N,M,m}$).

We are now interested in bounding the error between $(\bar{Y}^{q,p,N,M,m}, \bar{Z}^{q,p,N,M,m})$ (defined by (3.5.12)) and $(Y^{q,p,N,M,m}, Z^{q,p,N,M,m})$ (defined by (3.4.14)), for all $1 \leq m \leq M$. Before giving an upper bound for the error, we measure the error between C_p^N and $C_p^{N,M}$ for a r.v. F satisfying $\mathcal{H}_{p,N}^3$ (see Hypothesis (3.5.2)) when $(F^m)_{1 \leq m \leq M}$ are identically distributed r.v., but not necessarily independent.

Lemma 3.5.4. *Let F be a r.v. satisfying Hypothesis $\mathcal{H}_{p,N}^3$ and let $(F^m)_{1 \leq m \leq M}$ be M identically distributed r.v. with law F . We get*

$$\mathbb{E}(|(C_p^N - C_p^{N,M})((F^m)_m)|^2) \leq \mathbb{E}(|F|^2) + \sum_{k=1}^p \sum_{|n|=k} \frac{n!}{2} \mathbb{E} \left(|F|^2 \prod_{i=1}^N K_{n_i}^2(G_i) \right),$$

and so, we obtain

$$\mathbb{E}(|C_p^{N,M}((F^m)_m)|^2) \leq 2\mathbb{E}(|F|^2) + \sum_{k=1}^p \sum_{|n|=k} \frac{n!}{2} \mathbb{E} \left(|F|^2 \prod_{i=1}^N K_{n_i}^2(G_i) \right).$$

Proof of Lemma 3.5.4. Using definitions (3.2.6) and (3.4.9), we have

$$(C_p^N - C_p^{N,M})((F^m)_m) = d_0 - \hat{d}_0 + \sum_{k=1}^p \sum_{|n|=k} (d_k^n - \hat{d}_k^n) \prod_{i=1}^N K_{n_i}(G_i).$$

Since \hat{d}_k^n is independent of $(G_i)_i$, it implies that

$$\mathbb{E}(|(C_p^N - C_p^{N,M})((F^m)_m)|^2) = \mathbb{E}(|d_0 - \hat{d}_0|^2) + \sum_{k=1}^p \sum_{|n|=k} \frac{1}{n!} \mathbb{E}(|d_k^n - \hat{d}_k^n|^2).$$

The definition of the coefficients d_0 and d_k^n given in (3.2.7) leads to

$$\begin{aligned}\mathbb{E}(|(C_p^N - C_p^{N,M})((F^m)_m)|^2) &= \mathbb{V}(\hat{d}_0) + \sum_{k=1}^p \sum_{|n|=k} \frac{1}{n!} \mathbb{V}(\hat{d}_k^n) \\ &\leq \mathbb{E}(|\hat{d}_0|^2) + \sum_{k=1}^p \sum_{|n|=k} \frac{1}{n!} \mathbb{E}(|\hat{d}_k^n|^2).\end{aligned}$$

From the representation (3.4.8) of the coefficients \hat{d}_0 and \hat{d}_k^n , we can prove for the first

term of the previous inequality that

$$\mathbb{E}(|\hat{d}_0|^2) = \mathbb{E}\left(\left|\frac{1}{M} \sum_{m=1}^M F^m\right|^2\right) = \frac{1}{M^2} \mathbb{E}\left(\left|\sum_{m=1}^M F^m\right|^2\right) \leq \frac{1}{M} \mathbb{E}\left(\sum_{m=1}^M |F^m|^2\right) = \mathbb{E}(|F|^2),$$

and for the second term $\mathbb{E}(|\hat{d}_k^n|^2)$, we obtain that

$$\begin{aligned} \mathbb{E}(|\hat{d}_k^n|^2) &= \mathbb{E}\left(\left|\frac{n!}{M} \sum_{m=1}^M F^m \prod_{i=1}^N K_{n_i}(G_i^m)\right|^2\right) \\ &= \frac{n!^2}{M^2} \mathbb{E}\left(\left|\sum_{m=1}^M F^m \prod_{i=1}^N K_{n_i}(G_i^m)\right|^2\right) \\ &= \frac{n!^2}{M^2} \left[\mathbb{E}\left(\sum_{m=1}^M |F^m|^2 \prod_{i=1}^N K_{n_i}^2(G_i^m)\right) + \frac{M(M-1)}{2} \mathbb{E}\left(\left|F^1 \prod_{i=1}^N K_{n_i}(G_i^1) \cdot F^2 \prod_{i=1}^N K_{n_i}(G_i^2)\right|^2\right) \right] \\ &\leq \frac{n!^2}{M} \mathbb{E}\left(|F|^2 \prod_{i=1}^N K_{n_i}^2(G_i)\right) + \frac{M(M-1)n!^2}{2M^2} \mathbb{E}\left(|F|^2 \prod_{i=1}^N K_{n_i}^2(G_i)\right) \\ &\leq n!^2 \left(\frac{1}{2} + \frac{1}{M}\right) \mathbb{E}\left(|F|^2 \prod_{i=1}^N K_{n_i}^2(G_i)\right). \end{aligned}$$

Then, the first result follows. To get the second result, we write $C_p^{N,M}((F^m)_m) = (C_p^{N,M} - C_p^N)((F^m)_m) + C_p^N(F)$. Since $\mathbb{E}((C_p^{N,M} - C_p^N)((F^m)_m)C_p^N(F)) = 0$, we get

$$\mathbb{E}(|C_p^{N,M}((F^m)_m)|^2) = \mathbb{E}(|(C_p^{N,M} - C_p^N)((F^m)_m)|^2) + \mathbb{E}(|C_p^N(F)|^2),$$

and Lemma 3.2.3 completes the proof. \square

Proposition 3.5.4. *Assume that there exist reals $r > 4$ and $q' > 0$ s.t. $\xi \in \mathbb{L}^{q'r}$ and f is a bounded function. Let $\varepsilon^{q,p,N,M,m} := \|(\bar{Y}^{q,p,N,M,m} - Y^{q,p,N,M,m}, \bar{Z}^{q,p,N,M,m} - Z^{q,p,N,M,m})\|_{L^2}^2$. We get*

$$\varepsilon^{q+1,p,N,M,m} \leq C_4(p, N) \left(\varepsilon^{q,p,N,M,m}\right)^{1/2} + \frac{K_4(p, N)(\log M)^{\frac{\alpha}{2}}}{M^{\frac{1}{4} \wedge \frac{\beta}{2}}}, \quad (3.5.13)$$

where $C_4(p, N)$ and $K_4(p, N)$ depends on p, N, T and $\|f\|_\infty$ and (α, β) is given by (3.5.14). Since $\varepsilon^{0,p,N,M,m} = 0$, we deduce from (3.5.13) that $\varepsilon^{q,p,N,M,m} \leq \frac{A_4(q,p,N)(\log M)^{\frac{\alpha}{2}}}{M^{\left(\frac{1}{4} \wedge \frac{\beta}{2}\right) \frac{1}{2q}}}$ where $A_4(q, p, N)$ depends on q, p, N, T and $\|f\|_\infty$. Then, $\forall 1 \leq m \leq M$, $(Y^{q,p,N,M,m}, Z^{q,p,N,M,m})$ converges to $(\bar{Y}^{q,p,N,M,m}, \bar{Z}^{q,p,N,M,m})$ when M tends to ∞ in $\|(\cdot, \cdot)\|_{L^2}$.

Lemma 3.5.5. *Let $r \geq 1$ and $q' > 0$. Assume that f is a bounded function and $\xi \in \mathbb{L}^{q'r}$. Then, for all $1 \leq m \leq M$, there exists a positive constant $K(p, N)$ depending on f, r, p, N and T such that*

$$\mathbb{E}(|\bar{Y}_t^{q,p,N,M,m}|^r) + \mathbb{E}(|\bar{Z}_t^{q,p,N,M,m}|^r) \leq K(p, N) \left(1 + \mathbb{E}(|\xi|^{q'r})\right)^{1/q'}.$$

Proof of Lemma 3.5.5. Let $r \geq 1, j \in \{0, \dots, N\}$ and $t \in]\tilde{t}_{j-1}, \tilde{t}_j]$. From (3.5.12) and by

using Cauchy-Schwartz inequality, we obtain

$$\begin{aligned}\mathbb{E}(|\bar{Y}_t^{q,p,N,M,m}|^r) &= \mathbb{E}\left(\left|\mathbb{E}_t(C_p^{N,M}(\bar{F}^{q,p,N,M,m}))\right.\right. \\ &\quad \left.\left.- \int_0^t f(s, \bar{Y}_s^{q,p,N,M,m}, \bar{Z}_s^{q,p,N,M,m}, [\bar{Y}_s^{q,p,N,M,m}], [\bar{Z}_s^{q,p,N,M,m}])ds\right|^r\right) \\ &\leq 2^{r-1}\mathbb{E}\left(\left|\mathbb{E}_t(C_p^{N,M}(\bar{F}^{q,p,N,M,m}))\right|^r\right) \\ &\quad + \left|\int_0^t f(s, \bar{Y}_s^{q,p,N,M,m}, \bar{Z}_s^{q,p,N,M,m}, [\bar{Y}_s^{q,p,N,M,m}], [\bar{Z}_s^{q,p,N,M,m}])ds\right|^r,\end{aligned}$$

where $\bar{F}^{q,p,N,M,m} = \xi^m + \int_0^T f(s, \bar{Y}_s^{q,p,N,M,m}, \bar{Z}_s^{q,p,N,M,m}, [\bar{Y}_s^{q,p,N,M,m}], [\bar{Z}_s^{q,p,N,M,m}])ds$. From (3.4.10) and by using Cauchy-Schwartz inequality, we have

$$\begin{aligned}\mathbb{E}\left(\left|\mathbb{E}_t(C_p^{N,M}(\bar{F}^{q,p,N,M,m}))\right|^r\right) \\ \leq \left(\sum_{k=0}^p \binom{N}{k}\right)^{r-1} \left[\mathbb{E}(|\hat{d}_0|^r) + \sum_{k=1}^p \sum_{|n(j)|=k} \mathbb{E}\left(\left|\hat{d}_k^n \prod_{i<j} K_{n_i}(G_i) \times \left(\frac{t - \tilde{t}_{j-1}}{h}\right)^{(n_j)/2} K_{n_j}\left(\frac{B_t - B_{\tilde{t}_{j-1}}}{\sqrt{t - \tilde{t}_{j-1}}}\right)\right|^r\right)\right],\end{aligned}$$

and, since \hat{d}_k^n is independent of $(G_i)_i$, it implies that

$$\mathbb{E}\left(\left|\mathbb{E}_t(C_p^{N,M}(\bar{F}^{q,p,N,M,m}))\right|^r\right) \leq K_1(p, N) \left(\mathbb{E}(|\hat{d}_0|^r) + \sum_{k=1}^p \sum_{|n|=k} \mathbb{E}(|\hat{d}_k^n|^r)\right),$$

where $K_1(p, N)$ depends on p , N and r .

Moreover, from the bounded property of the function f , there exist positive constants K_2 and K_3 depending on f , r and T such that

$$\begin{aligned}\mathbb{E}(|\hat{d}_0|^r) &= \mathbb{E}\left(\left|\frac{1}{M} \sum_{m=1}^M \bar{F}^{q,p,N,M,m}\right|^r\right) \leq K_2(1 + \mathbb{E}(|\xi^m|^r)), \\ \mathbb{E}(|\hat{d}_k^n|^r) &= \mathbb{E}\left(\left|\frac{n!}{M} \sum_{m=1}^M \bar{F}^{q,p,N,M,m} \prod_{i=1}^N K_{n_i}(G_i^m)\right|^r\right) \\ &\leq \frac{n!^r}{M^r} \mathbb{E}\left(\left|\sum_{m=1}^M K_3(1 + |\xi^m|) \prod_{i=1}^N K_{n_i}(G_i^m)\right|^r\right),\end{aligned}$$

and by using Cauchy-Schwartz inequality, there exist positive constants p' , $q' > 0$ such that $\frac{1}{p'} + \frac{1}{q'} = 1$ and $K_4(N)$ depending on f , r , p' , T and N such that

$$\begin{aligned}\mathbb{E}(|\hat{d}_k^n|^r) &\leq K_3 n!^r \mathbb{E}\left(\left|\prod_{i=1}^N K_{n_i}(G_i)\right|^{p'r}\right)^{1/p'} \left(1 + \mathbb{E}(|\xi^m|^{q'r})\right)^{1/q'} \\ &\leq K_4(N) \left(1 + \mathbb{E}(|\xi^m|^{q'r})\right)^{1/q'}.\end{aligned}$$

Then, since f is a bounded function, we can conclude that

$$\mathbb{E}(|\bar{Y}_t^{q,p,N,M,m}|^r) \leq K_5(p, N) \left(1 + \mathbb{E}(|\xi^m|^{q'r})\right)^{1/q'},$$

where $K_5(p, N)$ depends on p, N, r, f and T . In the same way and by using equation (3.4.11), we can obtain that

$$\mathbb{E}(|\bar{Z}_t^{q,p,N,M,m}|^r) \leq K_6(p, N) \left(1 + \mathbb{E}(|\xi^m|^{q'r})\right)^{1/q'},$$

and the result follows. \square

Proof of Proposition 3.5.4. For the sake of clarity, we assume $d = 1$. In the following, consider that $\Delta Y_t^{q,p,N,M,m} := Y_t^{q,p,N,M,m} - \bar{Y}_t^{q,p,N,M,m}$, $\Delta Z_t^{q,p,N,M,m} := Z_t^{q,p,N,M,m} - \bar{Z}_t^{q,p,N,M,m}$, $\Delta f_t^{q,p,N,M,m} := f(t, Y_t^{q,p,N,M,m}, Z_t^{q,p,N,M,m}, [Y_t^{q,p,N,M,m}]_M, [Z_t^{q,p,N,M,m}]_M) - f(t, \bar{Y}_t^{q,p,N,M,m}, \bar{Z}_t^{q,p,N,M,m}, [\bar{Y}_t^{q,p,N,M,m}], [\bar{Z}_t^{q,p,N,M,m}])$ and $\Delta F^{q,p,N,M,m} := F^{q,p,N,M,m} - \bar{F}^{q,p,N,M,m}$. First, we deal with $\mathbb{E}(\sup_{0 \leq t \leq T} |\Delta Y_t^{q+1,p,N,M,m}|^2)$. From (3.5.12) and (3.4.14) we get

$$\Delta Y_t^{q+1,p,N,M,m} = \mathbb{E}_t(C_p^{N,M}((\Delta F^{q,p,N,M,m})_m)) - \int_0^t \Delta f_s^{q,p,N,M,m} ds.$$

By using Doob and Cauchy-Schwartz inequalities, we obtain

$$\begin{aligned} & \mathbb{E}\left(\sup_{0 \leq t \leq T} |\Delta Y_t^{q+1,p,N,M,m}|^2\right) \\ & \leq 2\mathbb{E}\left(\sup_{0 \leq t \leq T} \left|\mathbb{E}_t(C_p^{N,M}((\Delta F^{q,p,N,M,m})_m))\right|^2\right) + 2\mathbb{E}\left(\sup_{0 \leq t \leq T} \left|\int_0^t \Delta f_s^{q,p,N,M,m} ds\right|^2\right) \\ & \leq 8\mathbb{E}\left(\left|C_p^{N,M}((\Delta F^{q,p,N,M,m})_m)\right|^2\right) + 2T\mathbb{E}\left(\int_0^T \left|\Delta f_s^{q,p,N,M,m}\right|^2 ds\right). \end{aligned}$$

From Lemma 3.5.4, we have

$$\mathbb{E}(|C_p^{N,M}((\Delta F^{q,p,N,M,m})_m)|^2) \leq 2\mathbb{E}(|\Delta F^{q,p,N,M,m}|^2) + \sum_{k=1}^p \sum_{|n|=k} \frac{n!}{2} \mathbb{E}\left(|\Delta F^{q,p,N,M,m}|^2 \prod_{i=1}^N K_{n_i}^2(G_i)\right),$$

and by using Cauchy-Schwartz inequality and the bounded property of the function f , we get

$$\begin{aligned} \mathbb{E}(|C_p^{N,M}((\Delta F^{q,p,N,M,m})_m)|^2) & \leq 2\mathbb{E}(|\Delta F^{q,p,N,M,m}|^2) \\ & \quad + \sum_{k=1}^p \sum_{|n|=k} \frac{n!}{2} \mathbb{E}\left(|\Delta F^{q,p,N,M,m}|^2\right)^{1/2} \mathbb{E}\left(|\Delta F^{q,p,N,M,m}|^2 \prod_{i=1}^N K_{n_i}^4(G_i)\right)^{1/2} \\ & \leq 2\mathbb{E}(|\Delta F^{q,p,N,M,m}|^2) + C(p, N) \mathbb{E}\left(|\Delta F^{q,p,N,M,m}|^2\right)^{1/2} \\ & \leq \left(C(p, N) + 4T\|f\|_\infty\right) \mathbb{E}\left(|\Delta F^{q,p,N,M,m}|^2\right)^{1/2}, \end{aligned}$$

where $C(p, N) = \sum_{k=1}^p \sum_{|n|=k} n! T \|f\|_\infty \mathbb{E} \left(\prod_{i=1}^N K_{n_i}^4(G_i) \right)^{1/2}$. Then, we can conclude that

$$\begin{aligned} \mathbb{E} \left(\sup_{0 \leq t \leq T} |\Delta Y_t^{q+1,p,N,M,m}|^2 \right) &\leq 8\sqrt{T} \left(C(p, N) + 4T \|f\|_\infty \right) \mathbb{E} \left(\int_0^T \left| \Delta f_s^{q,p,N,M,m} \right|^2 ds \right)^{1/2} \\ &\quad + 2T \mathbb{E} \left(\int_0^T \left| \Delta f_s^{q,p,N,M,m} \right|^2 ds \right) \\ &\leq \left(8\sqrt{T} (C(p, N) + 4T \|f\|_\infty) + 4T\sqrt{T} \|f\|_\infty \right) \mathbb{E} \left(\int_0^T \left| \Delta f_s^{q,p,N,M,m} \right|^2 ds \right)^{1/2} \\ &\leq C_1(p, N) \mathbb{E} \left(\int_0^T \left| \Delta f_s^{q,p,N,M,m} \right|^2 ds \right)^{1/2}, \end{aligned}$$

where $C_1(p, N) = 4\sqrt{T} (2C(p, N) + 9T \|f\|_\infty)$.

Moreover, the Lipschitz property of f gives

$$\begin{aligned} \mathbb{E} \left(\sup_{0 \leq t \leq T} |\Delta Y_t^{q+1,p,N,M,m}|^2 \right) &\leq 2C_1(p, N) L_f \left(\int_0^T \mathbb{E} (|\Delta Y_s^{q,p,N,M,m}|^2 + |\Delta Z_s^{q,p,N,M,m}|^2 \right. \\ &\quad \left. + \mathcal{W}_2([Y_s^{q,p,N,M,m}]^M, [\bar{Y}_s^{q,p,N,M,m}]^M)^2 \right. \\ &\quad \left. + \mathcal{W}_2([Z_s^{q,p,N,M,m}]^M, [\bar{Z}_s^{q,p,N,M,m}]^M)^2) ds \right)^{1/2}, \end{aligned}$$

where

$$\begin{aligned} \mathbb{E} (\mathcal{W}_2([Y_s^{q,p,N,M,m}]^M, [\bar{Y}_s^{q,p,N,M,m}]^M)^2) &\leq 2\mathbb{E} (\mathcal{W}_2([Y_s^{q,p,N,M,m}]^M, [\bar{Y}_s^{q,p,N,M,m}]^M)^2) \\ &\quad + 2\mathbb{E} (\mathcal{W}_2([\bar{Y}_s^{q,p,N,M,m}]^M, [\bar{Y}_s^{q,p,N,M,m}]^M)^2), \end{aligned}$$

and

$$\begin{aligned} \mathbb{E} (\mathcal{W}_2([Z_s^{q,p,N,M,m}]^M, [\bar{Z}_s^{q,p,N,M,m}]^M)^2) &\leq 2\mathbb{E} (\mathcal{W}_2([Z_s^{q,p,N,M,m}]^M, [\bar{Z}_s^{q,p,N,M,m}]^M)^2) \\ &\quad + 2\mathbb{E} (\mathcal{W}_2([\bar{Z}_s^{q,p,N,M,m}]^M, [\bar{Z}_s^{q,p,N,M,m}]^M)^2). \end{aligned}$$

On the one hand, by using the property of the Wasserstein distance \mathcal{W}_2 , we get

$$\begin{aligned} \mathbb{E} (\mathcal{W}_2([Y_s^{q,p,N,M,m}]^M, [\bar{Y}_s^{q,p,N,M,m}]^M)^2) &\leq \mathbb{E} \left(\mathcal{W}_2 \left(\frac{1}{M} \sum_{m=1}^M \delta_{Y_s^{q,p,N,M,m}}, \frac{1}{M} \sum_{m=1}^M \delta_{\bar{Y}_s^{q,p,N,M,m}} \right)^2 \right) \\ &\leq \frac{1}{M} \sum_{m=1}^M \mathbb{E} (|Y_s^{q,p,N,M,m} - \bar{Y}_s^{q,p,N,M,m}|^2) \\ &= \mathbb{E} (|\Delta Y_s^{q,p,N,M,m}|^2). \end{aligned}$$

On the other hand, thanks to Lemma 3.5.5 and to the assumption on ξ , we apply [FG15, Theorem 1] to obtain

$$\begin{aligned}\mathbb{E}(\mathcal{W}_2([\bar{Y}_s^{q,p,N,M,m}]^M, [\bar{Y}_s^{q,p,N,M,m}]^2)) &\leq \sup_{0 \leq s \leq T} \mathbb{E}(\mathcal{W}_2([\bar{Y}_s^{q,p,N,M,m}]^M, [\bar{Y}_s^{q,p,N,M,m}]^2)) \\ &\leq \frac{C_T}{\sqrt{M}},\end{aligned}$$

where C_T is a positive constant.

In the same way, we have

$$\mathbb{E}(\mathcal{W}_2([Z_s^{q,p,N,M,m}]^M, [\bar{Z}_s^{q,p,N,M,m}]^M)^2) \leq \mathbb{E}(|\Delta Z_s^{q,p,N,M,m}|^2),$$

and we apply [CD18a, Theorem 5.8] to get

$$\mathbb{E}(\mathcal{W}_2([\bar{Z}_s^{q,p,N,M,m}]^M, [\bar{Z}_s^{q,p,N,M,m}]^2)) \leq \frac{C'_T (\log M)^\alpha}{M^\beta}.$$

where

$$(\alpha, \beta) = \begin{cases} (0, \frac{1}{2}) & \text{if } d < 4, \\ (1, \frac{1}{2}) & \text{if } d = 4, \\ (0, \frac{2}{d}) & \text{if } d > 4. \end{cases} \quad (3.5.14)$$

and C'_T is a positive constant. Finally, we can derive the inequality

$$\begin{aligned}\mathbb{E}\left(\sup_{0 \leq t \leq T} |\Delta Y_t^{q+1,p,N,M,m}|^2\right) &\leq 2C_1(p, N)L_f \left[\int_0^T \mathbb{E}\left(3|\Delta Y_s^{q,p,N,M,m}|^2 + 3|\Delta Z_s^{q,p,N,M,m}|^2 + 2\frac{C_T}{\sqrt{M}} + 2\frac{C'_T (\log M)^\alpha}{M^\beta}\right) ds \right]^{1/2} \\ &\leq \frac{K(p, N)(\log M)^{\frac{\alpha}{2}}}{M^{\frac{1}{4} \wedge \frac{\beta}{2}}} + C_2(p, N) \left(\int_0^T \mathbb{E}(|\Delta Y_s^{q,p,N,M,m}|^2 + |\Delta Z_s^{q,p,N,M,m}|^2) ds \right)^{1/2},\end{aligned} \quad (3.5.15)$$

where $K(p, N)$ and $C_2(p, N) = 6C_1(p, N)L_f$ are two constants depending on p, N, T and $\|f\|_\infty$.

Let us now study the upper bound $\mathbb{E}(\int_0^T |\Delta Z_s^{q+1,p,N,M,m}|^2 ds)$. Following the same steps as in the proof of Proposition 3.5.1, we get

$$\mathbb{E}\left(\int_0^T |\Delta Z_s^{q+1,p,N,M,m}|^2 ds\right) \leq 6\mathbb{E}\left(\sup_{0 \leq t \leq T} |\Delta Y_t^{q+1,p,N,M,m}|^2\right) + 3T\mathbb{E}\left(\int_0^T |\Delta f_s^{q,p,N,M,m}|^2 ds\right),$$

and by using the same previous majoration for the second term of the right hand side, we deduce

$$\begin{aligned}\mathbb{E}\left(\int_0^T |\Delta Z_s^{q+1,p,N,M,m}|^2 ds\right) &\leq 6\mathbb{E}\left(\sup_{0 \leq t \leq T} |\Delta Y_t^{q+1,p,N,M,m}|^2\right) + \frac{K_1(p, N)(\log M)^{\frac{\alpha}{2}}}{M^{\frac{1}{4} \wedge \frac{\beta}{2}}} \\ &\quad + C_3(p, N) \left(\int_0^T \mathbb{E}(|\Delta Y_s^{q,p,N,M,m}|^2 + |\Delta Z_s^{q,p,N,M,m}|^2) ds \right)^{1/2}.\end{aligned} \quad (3.5.16)$$

By adding $7 \times (3.5.15)$ and $(3.5.16)$, the result can follow easily. \square

3.6 Numerical illustrations

In this section, we will illustrate the algorithm by presenting some explicit computations.

We consider on $[0, T]$ the following sort of processes

$$Y_t = \xi + \int_t^T \left(\alpha Y_s + \beta \mathbb{E}(Y_s) + \gamma \mathbb{E}(Z_s) \right) ds - \int_t^T Z_s \cdot dB_s, \quad 0 \leq t \leq T, \quad (3.6.1)$$

where α, β and $\gamma \in \mathbb{R}$. First, we study the solution of (3.6.1) when $\gamma = 0$. Then, we study the solution of (3.6.1) in the general case.

Case (1) $\gamma = 0$. We have

$$Y_t = e^{\alpha(T-t)} \left(\mathbb{E}_t(\xi) + \mathbb{E}(\xi)(e^{\beta(T-t)} - 1) \right),$$

and

$$Z_t = e^{\alpha(T-t)} \mathbb{E}_t(D_t \xi).$$

Now, if $\xi = B_T$, we obtain

$$(Y_t, Z_t) = \left(e^{\alpha(T-t)} B_t, e^{\alpha(T-t)} \right),$$

and if $\xi = B_T^2$, we get

$$(Y_t, Z_t) = \left(e^{\alpha(T-t)} (B_t^2 - t + T e^{\beta(T-t)}), 2e^{\alpha(T-t)} B_t \right).$$

Case (2) $\gamma \neq 0$. We consider two different values of ξ :

If $\xi = B_T$ and $\beta \neq 0$, we have

$$(Y_t, Z_t) = \left(e^{\alpha(T-t)} \left(B_t + \frac{\gamma}{\beta} (e^{\beta(T-t)} - 1) \right), e^{\alpha(T-t)} \right).$$

If $\xi = B_T^2$, we have

$$(Y_t, Z_t) = \left(e^{\alpha(T-t)} (B_t^2 - t + T e^{\beta(T-t)}), 2e^{\alpha(T-t)} B_t \right).$$

3.6.1 Proofs of the numerical illustrations

Proof of case (1). In this case, we have

$$\mathbb{E}(Y_t) = \mathbb{E}(\xi) + \int_t^T (\alpha + \beta) \mathbb{E}(Y_s) ds,$$

from which we derive

$$\mathbb{E}(Y_t) = \mathbb{E}(\xi) e^{(\alpha+\beta)(T-t)},$$

and taking into account (3.6.1), we get

$$Y_t = \xi + \int_t^T \left(\alpha Y_s + \beta \mathbb{E}(\xi) e^{(\alpha+\beta)(T-s)} \right) ds - \int_t^T Z_s dB_s.$$

Applying Itô's formula on $e^{\alpha t} Y_t$, we get

$$\begin{aligned} e^{\alpha t} Y_t &= e^{\alpha T} \xi + \beta \mathbb{E}(\xi) \int_t^T e^{\alpha s} e^{(\alpha+\beta)(T-s)} ds - \int_t^T e^{\alpha s} Z_s dB_s \\ &= e^{\alpha T} \xi + \mathbb{E}(\xi) e^{\alpha T} \left(e^{\beta(T-t)} - 1 \right) - \int_t^T e^{\alpha s} Z_s dB_s, \end{aligned}$$

which leads

$$Y_t = e^{\alpha(T-t)} \xi + \mathbb{E}(\xi) e^{\alpha(T-t)} \left(e^{\beta(T-t)} - 1 \right) - e^{-\alpha t} \int_t^T e^{\alpha s} Z_s dB_s.$$

Finally, by taking the conditional expectation of Y_t , we get the explicit form of (Y_t, Z_t) as follows

$$Y_t = e^{\alpha(T-t)} \left(\mathbb{E}_t(\xi) + \mathbb{E}(\xi) (e^{\beta(T-t)} - 1) \right),$$

and

$$Z_t = e^{\alpha(T-t)} \mathbb{E}_t(D_t \xi).$$

When the terminal condition is equal to B_T or B_T^2 , we easily get the results. \square

Proof of case (2). From (3.6.1), we have

$$Y_t = \mathbb{E}_t(\xi) + \mathbb{E}_t \left(\int_t^T (\alpha Y_s + \beta \mathbb{E}(Y_s) + \gamma \mathbb{E}(Z_s)) ds \right).$$

By applying the Malliavin derivative on Y_t , we obtain that for all $s \in [0, T]$,

$$\mathbb{E}(D_s Y_t) = \mathbb{E}(D_s \xi) + \alpha \int_t^T \mathbb{E}(D_s Y_s) ds,$$

and obviously when $s \rightarrow t$, we get

$$\begin{aligned} \mathbb{E}(Z_t) &= \mathbb{E}(D_t Y_t) \\ &= e^{\alpha(T-t)} \mathbb{E}(D_t \xi). \end{aligned}$$

Therefore,

$$\mathbb{E}(Z_t) = \begin{cases} e^{\alpha(T-t)} & \text{if } \xi = B_T, \\ 0 & \text{if } \xi = B_T^2. \end{cases}$$

When $\xi = B_T^2$, we obtain the same equation as in the case (1), and therefore, the same result.

Now, when $\xi = B_T$, we get the following form of the process Y_t

$$Y_t = \xi + \int_t^T \left(\alpha Y_s + \beta \mathbb{E}(Y_s) + \gamma e^{\alpha(T-s)} \right) ds - \int_t^T Z_s dB_s. \quad (3.6.2)$$

Since

$$\mathbb{E}(Y_t) = \int_t^T \left((\alpha + \beta)\mathbb{E}(Y_s) + \gamma e^{\alpha(T-s)} \right) ds,$$

it implies that

$$\begin{aligned} \mathbb{E}(Y_t) &= \gamma e^{(\alpha+\beta)(T-t)} \int_t^T e^{-(\alpha+\beta)(T-s)} e^{\alpha(T-s)} ds \\ &= \frac{\gamma}{\beta} e^{\alpha(T-t)} (e^{\beta(T-t)} - 1), \end{aligned}$$

and from (3.6.2), we deduce that

$$Y_t = \xi + \int_t^T \left(\alpha Y_s + \gamma e^{(\alpha+\beta)(T-s)} \right) ds - \int_t^T Z_s dB_s.$$

Finally, by applying Itô's formula and by taking the conditional expectation of $e^{\alpha t} Y_t$, we conclude that

$$(Y_t, Z_t) = \left(e^{\alpha(T-t)} \left(B_t + \frac{\gamma}{\beta} (e^{\beta(T-t)} - 1) \right), e^{\alpha(T-t)} \right).$$

□

3.6.2 Illustrations

The computations of this section have been done on the following computers

- Dell precision tower 3620 4 cores Intel(R) Xeon(R) CPU E3-1240 v6 @3.7 Ghz with 16 Go of memory for M varying from 10^4 to 10^7
- Dell precision T7920 with 2 Intel Xeon Gold 6128 with 6 cores @3.7 Ghz and 128 Go of memory for $M = 10^8$

Let $0 = t_0 < t_1 < \dots < t_N = T$ be a subdivision of $[0, T]$ of step size $h = T/N$, N being a positive integer, let (Y, Z) be the unique solution of McKean-Vlasov BSDE (3.1.2) and let, for a given q, p and m , $(\tilde{Y}_{t_k}^{q,p,m}, \tilde{Z}_{t_k}^{q,p,m})_{0 \leq k \leq N}$ be its numerical approximation given by Algorithm 2. For a given integer L , we draw $(\bar{Y}^l, \bar{Z}^l)_{0 \leq l \leq L}$ and $(\tilde{Y}^{q,p,m,l}, \tilde{Z}^{q,p,m,l})_{0 \leq l \leq L}$, L independent copies of (Y, Z) . Then we approximate the \mathbb{L}^2 -error of Theorem 3.5.1 by :

$$error = \frac{1}{L} \sum_{l=1}^L \left(\max_{0 \leq k \leq N} \left| \bar{Y}_{t_k}^l - \tilde{Y}_{t_k}^{q,p,m,l} \right|^2 + h \sum_{k=0}^N \left| \bar{Z}_{t_k}^l - \tilde{Z}_{t_k}^{q,p,m,l} \right|^2 \right). \quad (3.6.3)$$

- **Convergence in p .** Table 1 represents the evolution of $\tilde{Y}_0^{q,p,N,M}$ w.r.t q (Picards iteration index), when $p = 2$ and $p = 3$. We also give the CPU time needed to get $\tilde{Y}_0^{8,p,N,M}$ and $\tilde{Z}_0^{8,p,N,M}$. We fix $M = 10^7$, $N = 20$, $\xi = \sqrt{|B_T|}$ and $f(t, Y, Z) = Y + E(Z)$. The seed of the generator is also fixed. Note that the difference between the values of $\tilde{Y}_0^{8,2,N,M}$ and $\tilde{Y}_0^{8,3,N,M}$ does not exceed 0.12%. This is due to the fast convergence of the algorithm in p . CPU time is 5 times higher when $p = 3$ than when $p = 2$.

Iterations	1	2	3	4	5	6	7	8	Real time
p=2	0.822301	1.644047	2.033746	2.150164	2.174622	2.178412	2.178843	2.178874	49.596
p=3	0.822236	1.644453	2.035001	2.152181	2.177102	2.181102	2.181609	2.181662	284.947

TABLE 3.1 – Evolution of $\tilde{Y}_0^{q,p,N,M}$ ($p = 2$ and $p = 3$) w.r.t. Picard's iterations, $M = 10^7$, $N = 20$, $\xi = \sqrt{|B_T|}$, $f(t, Y, Z) = Y + E(Z)$ and the real time of calculation.

- **Convergence in M .** Figure 3.1 (resp. Figure 3.2) represents the evolution of $\tilde{Y}_0^{q,p,N,M}$ and $\tilde{Z}_0^{q,p,N,M}$ w.r.t. q when $p = 3$ (resp. when $p = 2$), $N = 20$, $f(t, Y, Z) = Y + E(Z)$ and $\xi = \sqrt{|B_T|}$ (resp. $\xi = B_T^2$) for different values of M . For this set of parameters, the exact solutions are $Y_0 = 2.2352$ (resp. $Y_0 = 2.7183$) and $Z_0 = 0$. The exact solution depicted in Figure 3.1 is obtained by applying a lot of drawings. Concerning Figure 3.1, we notice that $\tilde{Y}_0^{q,p,N,M}$ (resp. $\tilde{Z}_0^{q,p,N,M}$) converges to the exact solution when $M \geq 10^5$ (resp. $M \geq 10^6$). Concerning Figure 3.2, we notice that $\tilde{Y}_0^{q,p,N,M}$ (resp. $\tilde{Z}_0^{q,p,N,M}$) converges to the exact solution when $M \geq 10^5$ (resp. $M \geq 10^7$). In both cases the algorithm stabilizes after very few iterations.

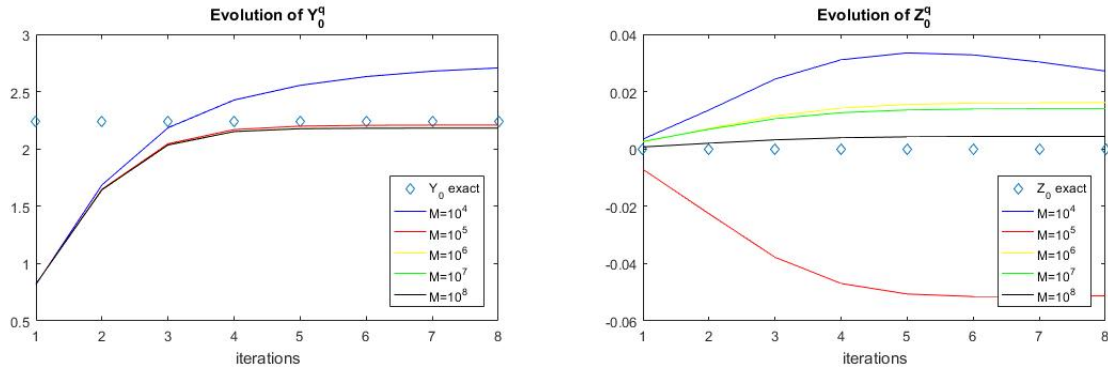


FIGURE 3.1 – Evolution of $\tilde{Y}_0^{q,p,N,M}$ and $\tilde{Z}_0^{q,p,N,M}$ w.r.t. q for different values of M when $N = 20$, $p = 3$, $\xi = \sqrt{|B_T|}$, $f(t, Y, Z) = Y + E(Z)$.

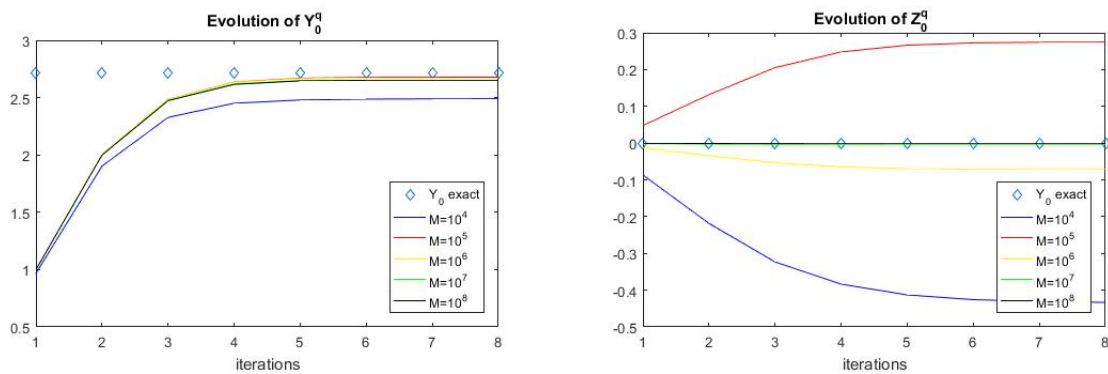


FIGURE 3.2 – Evolution of $\tilde{Y}_0^{q,p,N,M}$ and $\tilde{Z}_0^{q,p,N,M}$ w.r.t. q for different values of M when $N = 20$, $p = 2$, $\xi = B_T^2$, $f(t, Y, Z) = Y + E(Z)$.

- **Convergence in N .** Figure 3.3 represents the evolution of $\tilde{Y}_0^{q,p,N,M}$ and $\tilde{Z}_0^{q,p,N,M}$ w.r.t. q when $p = 2$, $M = 10^6$, $f(t, Y, Z) = Y + E(Y) + E(Z)$ and $\xi = B_T$ for different values of N . For this set of parameters, the exact solutions are $Y_0 = 4.6708$ and $Z_0 = 2.7183$. The algorithm converges even when $N = 10$, but $\tilde{Y}_0^{8,p,10,M}$ (resp.

$\tilde{Z}_0^{8,p,10,M}$) is quite below $\tilde{Y}_0^{8,p,40,M}$ (resp. $\tilde{Z}_0^{8,p,40,M}$). Notice that for $N = 40$ the approximation values are very close to the exact values.

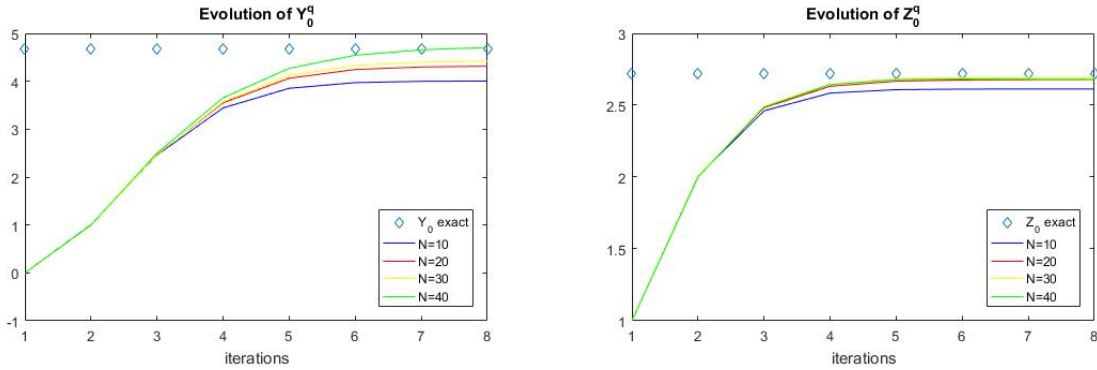


FIGURE 3.3 – Evolution of $\tilde{Y}_0^{q,p,N,M}$ and $\tilde{Z}_0^{q,p,N,M}$ w.r.t. q for different values of N when $M = 10^6$, $p = 3$, $\xi = B_T$, $f(t, Y, Z) = Y + E(Y) + E(Z)$.

- **Convergence in M of the error (3.6.3).** Figure 3.4 illustrates the error (3.6.3) (i.e. the error made on the whole path) for the case $p = 2$, $N = 20$, $q = 6$, $\xi = B_T^2$, $f(t, Y, Z) = Y + E(Z)$ for different values of M . We notice that the convergence in M is much faster than the theoretical one, which is $\frac{1}{M^{1/2q+2}}$.

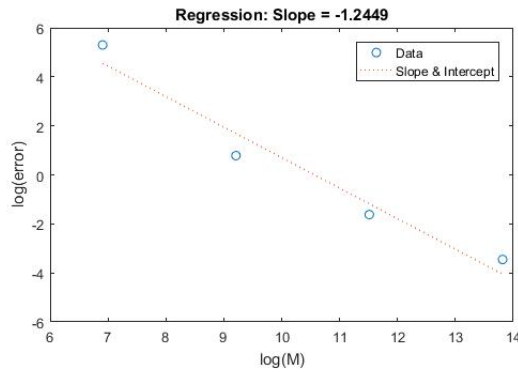


FIGURE 3.4 – Regression of $\log(\text{error})$ w.r.t. $\log(M)$. Data : error when $M = (10^3, 10^4, 10^5, 10^6)$. Parameters : $N = 20$, $p = 2$, $q = 6$, $\xi = B_T^2$, $f(t, Y, Z) = Y + E(Z)$.

Chapitre 4

Random walk approximation and its convergence rate for McKean-Vlasov BSDEs

Abstract

In this chapter, we provide a random walk approximation of the solution to a McKean-Vlasov BSDE in a Markovian case. Its convergence rate in the Wasserstein distance is derived using some properties of the associated PDE solution. This chapter can be viewed as a generalization of the results presented in [BGGL19] when standard BSDEs have been considered. The specific feature of this approximation is that it preserves the convergence rate as the one in [BGGL19] thanks to the regularity of solutions.

4.1 Introduction.

On a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, we consider the following backward stochastic differential equation (BSDE) in the Markovian case :

$$Y_s = g(B_T) + \int_s^T f(r, B_r, Y_r, Z_r) dr - \int_s^T Z_r dB_r, \quad 0 \leq s \leq T, \quad (4.1.1)$$

where $T > 0$ is a fixed positive real number, $\{B_t\}_{0 \leq t \leq T}$ is a standard real Brownian motion, $\{\mathcal{F}_t\}_{0 \leq t \leq T}$ stands for the augmented filtration of B , the terminal condition g is a map from \mathbb{R} into \mathbb{R} and the generator f is a map from $[0, T] \times \mathbb{R}^3$ into \mathbb{R} . This equation has been introduced by Bismut in [Bis73a] for the linear case, and by Pardoux and Peng in [PP90] for the general nonlinear case with Lipschitz continuous generators.

Since 1997, various numerical methods of (4.1.1) have been developed. In [Bal97], Bally was the first to present a random discretization based on the jump times of a Poisson process. Moreover, when the generator f is independent of z , Chevance in his PhD thesis (see [Che97c]) and Coquet, Mackevičius and Mémin (see [CMM99]) have proposed a time and space discretization of the solution of (4.1.1) and they have proved its convergence. In

the latter, the tool of convergence of filtrations has been used to obtain the convergence result. On the other hand, Briand, Delyon and Mémmin (see [BDM01]) have proposed another type of numerical method to approximate the BSDE when f is dependent of z . This method consists in replacing the Brownian motion B by a scaled random walks B^n defined as

$$B_t^n = \sqrt{T/n} \sum_{k=1}^{\lfloor nt/T \rfloor} \xi_k, \quad 0 \leq t \leq T,$$

where $(\xi_k)_{k \geq 1}$ is an i.i.d. sequence of symmetric Bernoulli random variables. Then, they have proved that the solution (Y^n, Z^n) of the following approximation :

$$Y_s^n = g(B_T^n) + \int_s^T f(r, B_{r-}^n, Y_{r-}^n, Z_r^n) d\langle B^n \rangle_s - \int_s^T Z_r^n dB_r^n, \quad 0 \leq s \leq T, \quad (4.1.2)$$

converges weakly to the solution (Y, Z) of the BSDE (4.1.1) by using the convergence of filtrations as in [CMM99], when the generator f is Lipschitz continuous and the terminal condition belongs to L^2 . However, the convergence rate was not given in this article. To address this issue, an approach based on the dynamic programming equation has been introduced by Bouchard and Touzi in [BT04] and by Zhang in [Zha04]. In these two articles, the authors have studied the rate of convergence in the case of the Markovian type BSDE. On the other hand, in the Markovian case where we have the Feynman-Kac formula, the solution of the BSDE can be seen as the viscosity solution of a semi-linear PDE. Based on this link between the BSDE and the PDE, Geiss, Labart and Luoto ([GLL18a], [GLL18b]) have recently answered to the first question on the convergence rate of the approximation already introduced in [BDM01], when g is ϵ -Hölder continuous and $f(t, \cdot)$ is Lipschitz continuous. They have found the rate of the L^2 -convergence of the approximated solution (Y^n, Z^n) of (4.1.2) to the exact one (Y, Z) of (4.1.1). Later, Briand, Geiss C., Geiss S. and Labart [BGGL19] have improved this result by obtaining a higher order of convergence when the error is measured with the Wasserstein distance and by assuming that g and $f(t, \cdot, y, z)$ are ϵ -Hölder continuous.

This chapter is concerned with the extension of the results presented in [BGGL19] to the following decoupled McKean-Vlasov BSDE :

$$Y_s^{t,x} = g(B_T^{t,x}) + \int_s^T f(r, B_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}, \mathbb{E}[\varphi_1(Y_r^{t,x})], \mathbb{E}[\varphi_2(Z_r^{t,x})]) dr - \int_s^T Z_r^{t,x} dB_r, \quad 0 \leq t \leq s \leq T, \quad (4.1.3)$$

where $x \in \mathbb{R}$, the functions φ_1 and φ_2 are maps from \mathbb{R} into \mathbb{R} and the generator f is a map from $[0, T] \times \mathbb{R}^5$ into \mathbb{R} . Moreover, for $0 \leq t \leq s \leq T$ and $x \in \mathbb{R}$, $B_s^{t,x} = x + B_s - B_t$.

A particular case of the above BSDE has been studied in [BDLP09] and [BLP09]. In [BDLP09], Buckdahn, Djehiche, Li and Peng have proved the existence and uniqueness of solutions of the mean-field BSDE driven by a forward McKean-Vlasov SDE, i.e. when the terminal condition of the BSDE is defined by $\mathbb{E}[g(x, X_T)]_{x=X_T}$ where X is the solution of the McKean-Vlasov SDE and the generator is presented by $\mathbb{E}[f(r, \lambda, \Lambda_r)]_{\lambda=\Lambda_r}$ where $\Lambda_r = (X_r, Y_r, Z_r)$. In [BLP09], the authors deepened the investigation of such mean-field BSDE in a more general framework. Indeed, they have proved the existence and uniqueness of solutions and they have presented the link between this mean-field BSDE and the

viscosity solution of a non local PDE. Recently, the authors of [ARBGL19] have studied the classical McKean-Vlasov BSDE which is a general case of (4.1.3), i.e. when ξ is the terminal condition and $f(t, Y_s, Z_s, [Y_s], [Z_s])$ is the generator where $[\theta]$ is the notation for the law of a random variable θ . They have proved the existence and uniqueness of solutions for this McKean-Vlasov BSDE with Lipschitz condition on f .

This work is not the first one to address the numerical methods of McKean-Vlasov BSDEs. Alanko in his PHD thesis (see [Ala15]) has presented a numerical method based on a Picard iteration for mean field games. In [CdRT15], the authors have developed a method based on cubatures for decoupled McKean-Vlasov forward-backward SDE. In [CCD17], the authors have considered the case of strongly coupled forward-backward SDE of McKean-Vlasov type. They have proposed a scheme whose principle is to implement recursively Picard iterations on small time intervals, since Picard Theorem only applies in small time for fully coupled problems. Recently, the authors of [ARBGL19] have presented an algorithm based on the Wiener chaos expansion and particle system to solve McKean-Vlasov BSDE and they have studied its convergence.

According to [ARBGL19] and by assuming that g and $f(t, \cdot, y, z, p, q)$ are ϵ -Hölder continuous and φ_1 and φ_2 are Lipschitz continuous, the BSDE (4.1.3) has a unique square integrable solution $(Y^{t,x}, Z^{t,x})$. Moreover, thanks to the Markovian framework, this solution is related to the following PDE : for $0 \leq t \leq s < T$ and $(x, y) \in \mathbb{R}^2$

$$\begin{cases} \partial_s u(s, y; t, x) + \frac{1}{2} \Delta_y u(s, y; t, x) = -f\left(s, y, u(s, y; t, x), \nabla_y u(s, y; t, x), p(s, t, x), q(s, t, x)\right), \\ u(T, y; t, x) = g(y), \end{cases} \quad (4.1.4)$$

where

$$p(s, t, x) = \mathbb{E}[\varphi_1(u(s, B_s^{t,x}; t, x))] = \int \varphi_1(u(s, z; t, x)) \mu_s^{t,x} dz,$$

$$q(s, t, x) = \mathbb{E}[\varphi_2(\nabla_y u(s, B_s^{t,x}; t, x))] = \int \varphi_2(\nabla_y u(s, z; t, x)) \mu_s^{t,x} dz,$$

and $\mu_s^{t,x}$ is the law of $B_s^{t,x}$. The aim of this chapter is to first propose a numerical approximation $(Y^{n,t,x}, Z^{n,t,x})$ to the solution of (4.1.3) based on the scaled random walk. Then, to prove that the rate of convergence is exactly the same as in [BGGL19] when the error is measured with the Wasserstein distance. In order to do this, we will follow the same methodology used in [BGGL19] which is based on the associated PDE and on the representation formulas that are introduced and studied in [MZ02] and [FT02] as a consequence of the Bismut-Elworthy formula. Then, based on the regularity of the solutions, we get the same rate of convergence as in [BGGL19] :

$$W_r(Y_s^{n,t,x}, Y_s^{t,x}) \leq C_r (1+|x|)^\epsilon n^{-\alpha \wedge \frac{\epsilon}{2}}, \quad W_r(Z_s^{n,t,x}, Z_s^{t,x}) \leq C_r \frac{(1+|x|)^\epsilon}{\sqrt{T-s}} n^{-\alpha \wedge \frac{\epsilon}{2}}, \quad 0 \leq t \leq s < T,$$

when g is ϵ -Hölder continuous, φ_1 and φ_2 are Lipschitz continuous and f is α -Hölder continuous in time, ϵ -Hölder continuous in space and Lipschitz continuous with respect to (y, z, p, q) .

This chapter is organized as follows : Section 4.2 presents the approximations of the

BSDE (4.1.3) and the PDE (4.1.4) and the relation between them. Then, it gives the existence and uniqueness of the solutions of these approximations. Section 4.3 is devoted to study the existence, the uniqueness and some regularity properties of the solution of the PDE (4.1.4) by using its relation with the BSDE (4.1.1). In Section 4.4, the main result of this chapter, which is the rate of convergence of the error on Y and Z for the Wasserstein distance, is provided. Finally, Section 4.5 contains some numerical experiments.

4.2 Discrete BSDE and PDE

Let us start by considering some assumptions for the terminal condition g , the functions φ_1 and φ_2 and the generator f .

Assumption (A1). *There exist $0 < \epsilon \leq 1$ and $0 < \alpha \leq 1$ such that*

(i) *The function $g : \mathbb{R} \rightarrow \mathbb{R}$ is ϵ -Hölder continuous :*

$$\exists C_g \geq 0, \quad \forall (x, x') \in \mathbb{R}^2, \quad |g(x) - g(x')| \leq C_g |x - x'|^\epsilon.$$

(ii) *The functions $\varphi_1, \varphi_2 : \mathbb{R} \rightarrow \mathbb{R}$ are Lipschitz continuous :*

$$\exists C_{\varphi_i} \geq 0, \quad i \in \{1, 2\}, \quad \forall (x, x') \in \mathbb{R}^2, \quad |\varphi_i(x) - \varphi_i(x')| \leq C_{\varphi_i} |x - x'|.$$

(iii) *The function $f : [0, T] \times \mathbb{R}^5 \rightarrow \mathbb{R}$ is α -Hölder continuous in time, ϵ -Hölder continuous in space and Lipschitz continuous with respect to (y, z, p, q) : there exists a constant $C_f \geq 0$ such that, for all (t, x, y, z, p, q) and (t', x', y', z', p', q') in $[0, T] \times \mathbb{R}^5$, we have :*

$$\begin{aligned} & |f(t, x, y, z, p, q) - f(t', x', y', z', p', q')| \\ & \leq C_f \left(|t - t'|^\alpha + |x - x'|^\epsilon + |y - y'| + |z - z'| + |p - p'| + |q - q'| \right). \end{aligned} \tag{4.2.1}$$

Note that $K_f := \sup_{t \in [0, T]} |f(t, 0, 0, 0, 0, 0)|$.

As already stated, this chapter studies the approximation of $(Y^{t,x}, Z^{t,x})$ - solution of (4.1.3) - by the solution of a McKean-Vlasov BSDE driven by a scaled random walk. To achieve this aim, we introduce $(\xi_k)_{k \geq 1}$ an i.i.d. sequence of symmetric Bernoulli random variables defined on some probability space, not necessary $(\Omega, \mathcal{F}, \mathbb{P})$. For $n \in \mathbb{N}^*$, we set $h = T/n$ and we consider the scaled random walk

$$B_t^n = \sqrt{h} \sum_{k=1}^{\lfloor t/h \rfloor} \xi_k, \quad 0 \leq t \leq T.$$

Notations :

- If x is a real number, $\lfloor x \rfloor = \max\{r \in \mathbb{Z} : r \leq x\}$ and $\lceil x \rceil = \min\{r \in \mathbb{Z} : r \geq x\}$.
For $0 \leq t \leq T$, $n_t = \lfloor t/h \rfloor$, $\underline{t} = h \lfloor t/h \rfloor = hn_t$ and $\bar{t} = h \lceil t/h \rceil$.
- As before, for $x \in \mathbb{R}$ and $0 \leq t \leq s \leq T$, $B_s^{n,t,x} = x + B_s^n - B_t^n$.

The McKean-Vlasov BSDE driven by B^n is described by

$$\begin{aligned} Y_s^{n,t,x} = & g(B_T^{n,t,x}) + \int_s^T f(r, B_{r^-}^{n,t,x}, Y_{r^-}^{n,t,x}, Z_r^{n,t,x}, \mathbb{E}[\varphi_1(Y_{r^-}^{n,t,x})], \mathbb{E}[\varphi_2(Z_r^{n,t,x})]) d\langle B^n \rangle_s \\ & - \int_s^T Z_r^{n,t,x} dB_r^n, \quad 0 \leq t \leq s \leq T. \end{aligned} \tag{4.2.2}$$

where $B^{n,t,x} = B^{n,t,x}$. Next, we set $Y_s^{n,t,x} = Y_s^{n,t,x}$ and $Z_s^{n,t,x} = Z_s^{n,t,x}$, for all $0 \leq t \leq s \leq T$.

Remark 4.2.1. *By construction, $Y^{n,t,x}$ is a stepwise constant càdlàg process : $Y_s^{n,t,x} = Y_{\bar{s}}^{n,t,x}$. Since the process $Z^{n,t,x} \in L^2(\Omega \times [0, T], d\mathbb{P} \otimes d\langle B \rangle)$ is defined only on the points $\{kh, k = n_t, \dots, n\}$, it will be extended on $]t, T]$ as a càglàd process by setting, for $t < s \leq T$, $Z_s^{n,t,x} = Z_{\bar{s}}^{n,t,x}$.*

The aforementioned BSDE is a discrete BSDE that can be solved by hand. Hence, for $k = n_t, \dots, n-1$, we have

$$\begin{cases} Y_{kh}^{n,t,x} = Y_{(k+1)h}^{n,t,x} + hf\left((k+1)h, B_{kh}^{n,t,x}, Y_{kh}^{n,t,x}, Z_{(k+1)h}^{n,t,x}, \mathbb{E}[\varphi_1(Y_{kh}^{n,t,x})], \mathbb{E}[\varphi_2(Z_{(k+1)h}^{n,t,x})]\right) \\ \quad - \sqrt{h} Z_{(k+1)h}^{n,t,x} \xi_{k+1}, \\ Y_{nh}^{n,t,x} = g(B_T^{n,t,x}). \end{cases} \tag{4.2.3}$$

Moreover, if $Y_{(k+1)h}^{n,t,x}$ is given,

$$Z_{(k+1)h}^{n,t,x} = h^{-1/2} \mathbb{E}\left(Y_{(k+1)h}^{n,t,x} \xi_{k+1} | \mathcal{F}_{kh}^{n,t,x}\right), \tag{4.2.4}$$

and thanks to the predictable representation property of B^n ,

$$Y_{kh}^{n,t,x} = \mathbb{E}\left(Y_{(k+1)h}^{n,t,x} | \mathcal{F}_{kh}^{n,t,x}\right) + hf\left((k+1)h, B_{kh}^{n,t,x}, Y_{kh}^{n,t,x}, Z_{(k+1)h}^{n,t,x}, \mathbb{E}[\varphi_1(Y_{kh}^{n,t,x})], \mathbb{E}[\varphi_2(Z_{(k+1)h}^{n,t,x})]\right). \tag{4.2.5}$$

In the following Proposition, we will prove that, if the condition $2hC_f \max(1, C_{\varphi_1}) < 1$ is satisfied, the BSDE (4.2.2) has a unique square integrable solution $(Y^{n,t,x}, Z^{n,t,x})$ with $Y^{n,t,x}$ being adapted and $Z^{n,t,x}$ being predictable with respect to the filtration generated by B^n .

Proposition 4.2.1. *(Existence and uniqueness)*

Under Assumption (A1) and if $2hC_f \max(1, C_{\varphi_1}) < 1$, then the BSDE (4.2.2) has a unique solution $(Y^{n,t,x}, Z^{n,t,x})$.

Démonstration. As pointed out before (see (4.2.3)), the existence of a unique solution $(Y^{n,t,x}, Z^{n,t,x})$ of (4.2.2) follows from the existence of a unique solution $Y_{kh}^{n,t,x}$ of (4.2.3).

In order to use a fixed-point theorem, we consider the map ϕ from $S_T^2(\mathbb{R})$ into $S_T^2(\mathbb{R})$, which associate to $y_{kh}^{n,t,x}$ the solution $Y_{kh}^{n,t,x}$ of

$$\begin{cases} Y_{kh}^{n,t,x} = Y_{(k+1)h}^{n,t,x} + hf \left((k+1)h, B_{kh}^{n,t,x}, y_{kh}^{n,t,x}, Z_{(k+1)h}^{n,t,x}, \mathbb{E}[\varphi_1(y_{kh}^{n,t,x})], \mathbb{E}[\varphi_2(Z_{(k+1)h}^{n,t,x})] \right) \\ \quad - \sqrt{h} Z_{(k+1)h}^{n,t,x} \xi_{k+1}, \\ Y_{nh}^{n,t,x} = g(B_T^{n,t,x}), \end{cases}$$

where $k = n_t, \dots, n-1$. Note that the space $S_T^2(\mathbb{R})$ is endowed with the norm $\|\varphi\|_{S_T^2(\mathbb{R})}^2 = \mathbb{E}[\sup_{t \in [0, T]} |\varphi_t|^2]$.

Let $y_{kh}^{n,t,x}, \tilde{y}_{kh}^{n,t,x}$ be two elements of $S_T^2(\mathbb{R})$, and let $Y_{kh}^{n,t,x}$ and $\tilde{Y}_{kh}^{n,t,x}$ be the associated solutions. We have

$$\begin{aligned} & \mathbb{E} \left[\sup_{0 \leq k \leq n-1} |Y_{kh}^{n,t,x} - \tilde{Y}_{kh}^{n,t,x}|^2 \right] \\ &= h^2 \mathbb{E} \left[\sup_{0 \leq k \leq n-1} \left| f \left((k+1)h, B_{kh}^{n,t,x}, y_{kh}^{n,t,x}, Z_{(k+1)h}^{n,t,x}, \mathbb{E}[\varphi_1(y_{kh}^{n,t,x})], \mathbb{E}[\varphi_2(Z_{(k+1)h}^{n,t,x})] \right) \right. \right. \\ & \quad \left. \left. - f \left((k+1)h, B_{kh}^{n,t,x}, \tilde{y}_{kh}^{n,t,x}, Z_{(k+1)h}^{n,t,x}, \mathbb{E}[\varphi_1(\tilde{y}_{kh}^{n,t,x})], \mathbb{E}[\varphi_2(Z_{(k+1)h}^{n,t,x})] \right) \right|^2 \right]. \end{aligned}$$

Since f and φ_1 are Lipschitz, we obtain

$$\begin{aligned} & \mathbb{E} \left[\sup_{0 \leq k \leq n-1} |Y_{kh}^{n,t,x} - \tilde{Y}_{kh}^{n,t,x}|^2 \right] \\ & \leq h^2 C_f^2 \mathbb{E} \left[\sup_{0 \leq k \leq n-1} \left(\left| y_{kh}^{n,t,x} - \tilde{y}_{kh}^{n,t,x} \right| + \left| \mathbb{E}[\varphi_1(y_{kh}^{n,t,x})] - \mathbb{E}[\varphi_1(\tilde{y}_{kh}^{n,t,x})] \right| \right)^2 \right] \\ & \leq 2h^2 C_f^2 \mathbb{E} \left[\sup_{0 \leq k \leq n-1} \left(\left| y_{kh}^{n,t,x} - \tilde{y}_{kh}^{n,t,x} \right|^2 + C_{\varphi_1}^2 \left| \mathbb{E}[y_{kh}^{n,t,x} - \tilde{y}_{kh}^{n,t,x}] \right|^2 \right) \right] \\ & \leq 4h^2 C_f^2 \max(1, C_{\varphi_1}^2) \mathbb{E} \left[\sup_{0 \leq k \leq n-1} \left(\left| y_{kh}^{n,t,x} - \tilde{y}_{kh}^{n,t,x} \right|^2 \right) \right]. \end{aligned}$$

Since $2hC_f \max(1, C_{\varphi_1}) < 1$, this implies that the mapping ϕ is a contraction from $S_T^2(\mathbb{R})$ onto itself. Hence, there exists a fixed point, which is the unique solution of (4.2.3). Finally, we conclude that there exists a unique solution of (4.2.2). \square

Thanks to the Markovian framework, we will prove in Proposition 4.2.2 that the solution of (4.2.2) is related to a discrete PDE of (4.1.4). We introduce

$$\begin{aligned} D_+^n u(x) &:= \frac{1}{2} \left(u(x + \sqrt{h}) + u(x - \sqrt{h}) \right), \\ D_-^n u(x) &:= \frac{1}{2} \left(u(x + \sqrt{h}) - u(x - \sqrt{h}) \right), \end{aligned}$$

and we set

$$\nabla^n u(x) := h^{-1/2} D_-^n u(x),$$

where u is a given function.

Next, we consider the finite difference equation which is a discrete PDE of (4.1.4) : for $k = n - 1, \dots, n_t$, $t \leq T$ and $(x, y) \in \mathbb{R}^2$,

$$\begin{cases} U^n(kh, y; t, x) - hf((k+1)h, y, U^n(kh, y; t, x), \nabla_y^n U^n((k+1)h, y; t, x), p^n(kh, t, x), q^n((k+1)h, t, x)) \\ \quad = D_+^n U^n((k+1)h, y; t, x), \\ U^n(nh, y; t, x) = g(y), \end{cases} \quad (4.2.6)$$

where

$$\begin{aligned} p^n(r, t, x) &= \mathbb{E}[\varphi_1(U^n(r, B_r^{n,t,x}; t, x))], \\ q^n(r, t, x) &= \mathbb{E}[\varphi_2(\nabla_y^n U^n(r, B_r^{n,t,x}; t, x))]. \end{aligned}$$

Furthermore, we consider the following BSDE driven by B^n in a general case,

$$\begin{cases} \bar{Y}_r^{n,s,y,t,x} = g(B_T^{n,s,y}) + \int_r^T f(a, B_{a^-}^{n,s,y,t,x}, \bar{Y}_{a^-}^{n,s,y,t,x}, \bar{Z}_a^{n,s,y,t,x}, \mathbb{E}[\varphi_1(Y_{a^-}^{n,t,x})], \mathbb{E}[\varphi_2(Z_a^{n,t,x})]) d\langle B^n \rangle_r \\ \quad - \int_r^T \bar{Z}_a^{n,s,y,t,x} dB_a^n, \quad \underline{t} \leq \underline{s} \leq r \leq T, \\ \bar{Y}_T^{n,T,y,t,x} = g(y). \end{cases} \quad (4.2.7)$$

Note that, $Y_r^{n,s,y,t,x} = Y_r^{n,s,y,t,x}$ and $Z_r^{n,s,y,t,x} = Z_r^{n,s,y,t,x}$, for all $0 \leq t \leq s \leq r \leq T$. Then, the existence of a unique solution $(\bar{Y}^{n,s,y,t,x}, \bar{Z}^{n,s,y,t,x})$ of the above system (4.2.7) is obtained from Proposition 4.2.1 and from the fact that the McKean terms $\mathbb{E}[\varphi_1(Y_{a^-}^{n,t,x})]$ and $\mathbb{E}[\varphi_2(Z_a^{n,t,x})]$ are independent of s and y . In the following, we assume that $n > 2TC_f \max(1, C_{\varphi_1})$; in particular $hC_f < 1$ then (4.2.7) has a unique solution (see [BDM01]).

Proposition 4.2.2. *Under Assumption (A1), there exists a unique solution U^n for the discrete PDE (4.2.6). Moreover, for $(x, y) \in \mathbb{R}^2$, we have*

$$\begin{aligned} \bar{Y}_{kh}^{n,s,y,t,x} &= U^n(kh, B_{kh}^{n,s,y}; t, x), \quad k = n_t, \dots, n, \\ \bar{Z}_{kh}^{n,s,y,t,x} &= \nabla_y^n U^n(kh, B_{(k-1)h}^{n,s,y}; t, x), \quad k = n_t + 1, \dots, n, \end{aligned}$$

where $(\bar{Y}^{n,s,y,t,x}, \bar{Z}^{n,s,y,t,x})$ is the unique solution of the discrete BSDE (4.2.7).

In particular,

$$\begin{aligned} Y_{kh}^{n,t,x} &= U^n(kh, B_{kh}^{n,t,x}; t, x) \quad k = n_t, \dots, n, \\ Z_{kh}^{n,t,x} &= \nabla_y^n U^n(kh, B_{(k-1)h}^{n,t,x}; t, x) \quad k = n_t + 1, \dots, n, \end{aligned}$$

where $(Y^{n,t,x}, Z^{n,t,x})$ is the unique solution of the discrete BSDE (4.2.2).

Démonstration. (i) Existence of solution.

According to Proposition 5.1 in [BDM01] and based on the fact that the McKean terms in (4.2.7) $\mathbb{E}[\varphi_1(Y_{a^-}^{n,t,x})]$ and $\mathbb{E}[\varphi_2(Z_a^{n,t,x})]$ do not depend on s and y , we have, for $(x, y) \in \mathbb{R}^2$

$$\begin{aligned} \bar{Y}_{kh}^{n,s,y,t,x} &= \bar{U}^n(kh, B_{kh}^{n,s,y}; t, x), \quad k = n_t, \dots, n, \\ \bar{Z}_{kh}^{n,s,y,t,x} &= \nabla_y^n \bar{U}^n(kh, B_{(k-1)h}^{n,s,y}; t, x), \quad k = n_t + 1, \dots, n, \end{aligned} \quad (4.2.8)$$

where \bar{U}^n solves the following discrete PDE :

$$\begin{cases} \bar{U}^n(kh, y; t, x) - hf((k+1)h, y, \bar{U}^n(kh, y; t, x), \nabla_y^n \bar{U}^n((k+1)h, y; t, x), \mathbb{E}[\varphi_1(Y_{kh}^{n,t,x})], \\ \quad \mathbb{E}[\varphi_2(Z_{(k+1)h}^{n,t,x})]) = D_+^n \bar{U}^n((k+1)h, y; t, x), \quad k = n_t, \dots, n-1, \\ \bar{U}^n(nh, y; t, x) = g(y). \end{cases} \quad (4.2.9)$$

Thanks to the uniqueness of the solution to (4.2.7), we have that

$$\begin{aligned} Y_{kh}^{n,t,x} &= \bar{Y}_{kh}^{n,t,x,t,x}, \quad k = n_t, \dots, n, \\ Z_{kh}^{n,t,x} &= \bar{Z}_{kh}^{n,t,x,t,x}, \quad k = n_t + 1, \dots, n. \end{aligned}$$

and from (4.2.8), we have

$$\begin{aligned} \bar{Y}_{kh}^{n,t,x,t,x} &= \bar{U}^n(kh, B_{kh}^{n,t,x}; t, x), \quad k = n_t, \dots, n, \\ \bar{Z}_{kh}^{n,t,x,t,x} &= \nabla_y^n \bar{U}^n(kh, B_{(k-1)h}^{n,t,x}; t, x), \quad k = n_t + 1, \dots, n, \end{aligned}$$

which leads to

$$\begin{aligned} \mathbb{E}[\varphi_1(Y_{kh}^{n,t,x})] &= \mathbb{E}[\varphi_1(\bar{U}^n(kh, B_{kh}^{n,t,x}; t, x))], \quad k = n_t, \dots, n, \\ \mathbb{E}[\varphi_2(Z_{kh}^{n,t,x})] &= \mathbb{E}[\varphi_2(\nabla_y^n \bar{U}^n(kh, B_{(k-1)h}^{n,t,x}; t, x))], \quad k = n_t + 1, \dots, n. \end{aligned}$$

Since (4.2.9) admits a unique solution, we get that there exists a solution to (4.2.6).

(ii) Uniqueness of solution.

Let $(t, x) \in [0, T[\times \mathbb{R}$. Suppose that for each $k = n-1, \dots, n_t$ and $y \in \mathbb{R}$, there exist two solutions $U_1^n(kh, y; t, x)$ and $U_2^n(kh, y; t, x)$ for the discrete PDE (4.2.6). On the one hand, suppose that $U_1^n((k+1)h, B_{(k+1)h}^{n,t,x}; t, x) = U_2^n((k+1)h, B_{(k+1)h}^{n,t,x}; t, x)$ for some $k \in \{n_t, \dots, n-1\}$. Then, we have

$$D_+^n U_1^n((k+1)h, B_{kh}^{n,t,x}; t, x) = D_+^n U_2^n((k+1)h, B_{kh}^{n,t,x}; t, x),$$

and

$$\nabla^n U_1^n((k+1)h, B_{kh}^{n,t,x}; t, x) = \nabla^n U_2^n((k+1)h, B_{kh}^{n,t,x}; t, x).$$

Next, by using (4.2.6), we get

$$\begin{aligned} &\mathbb{E}[|U_1^n(kh, B_{kh}^{n,t,x}; t, x) - U_2^n(kh, B_{kh}^{n,t,x}; t, x)|] \\ &= h \mathbb{E} \left[\left| f \left((k+1)h, B_{kh}^{n,t,x}, U_1^n(kh, B_{kh}^{n,t,x}; t, x), \nabla_y^n U_1^n((k+1)h, B_{kh}^{n,t,x}; t, x), \right. \right. \right. \\ &\quad \left. \left. \left. \mathbb{E}[\varphi_1(U_1^n(kh, B_{kh}^{n,t,x}; t, x))], \mathbb{E}[\varphi_2(\nabla_y^n U_1^n((k+1)h, B_{(k+1)h}^{n,t,x}; t, x))] \right) \right. \right. \\ &\quad \left. \left. - f \left((k+1)h, B_{kh}^{n,t,x}, U_2^n(kh, B_{kh}^{n,t,x}; t, x), \nabla_y^n U_1^n((k+1)h, B_{kh}^{n,t,x}; t, x), \right. \right. \right. \\ &\quad \left. \left. \left. \mathbb{E}[\varphi_1(U_2^n(kh, B_{kh}^{n,t,x}; t, x))], \mathbb{E}[\varphi_2(\nabla_y^n U_1^n((k+1)h, B_{(k+1)h}^{n,t,x}; t, x))] \right) \right| \right]. \end{aligned}$$

Moreover, from the fact that f and φ_1 are Lipschitz, we obtain that

$$\begin{aligned} \mathbb{E}[|U_1^n(kh, B_{kh}^{n,t,x}; t, x) - U_2^n(kh, B_{kh}^{n,t,x}; t, x)|] \\ \leq hC_f \mathbb{E}[|U_1^n(kh, B_{kh}^{n,t,x}; t, x) - U_2^n(kh, B_{kh}^{n,t,x}; t, x)|] \\ + hC_f C_{\varphi_1} \mathbb{E}[|U_1^n(kh, B_{kh}^{n,t,x}; t, x) - U_2^n(kh, B_{kh}^{n,t,x}; t, x)|]. \end{aligned}$$

Since $2hC_f \max(1, C_{\varphi_1}) < 1$, we get

$$\mathbb{E}[|U_1^n(kh, B_{kh}^{n,t,x}; t, x) - U_2^n(kh, B_{kh}^{n,t,x}; t, x)|] = 0.$$

Then, by induction and since $U_1^n(nh, B_{nh}^{n,t,x}; t, x) = U_2^n(nh, B_{nh}^{n,t,x}; t, x) = g(B_{nh}^{n,t,x})$, we conclude that

$$\mathbb{E}[|U_1^n(kh, B_{kh}^{n,t,x}; t, x) - U_2^n(kh, B_{kh}^{n,t,x}; t, x)|] = 0, \quad k = n_t, \dots, n.$$

Finally, from the fact that

$$\begin{aligned} \mathbb{E}[|U_1^n(kh, B_{kh}^{n,t,x}; t, x) - U_2^n(kh, B_{kh}^{n,t,x}; t, x)|] \\ = \sum_{j=0}^{(k-n_t)} \left[\mathbb{P}\left(B_{kh}^{n,t,x} = x + (k - n_t - 2j)\sqrt{h}\right) \right. \\ \left. \left| U_1^n(kh, x + (k - n_t - 2j)\sqrt{h}; t, x) - U_2^n(kh, x + (k - n_t - 2j)\sqrt{h}; t, x) \right| \right], \end{aligned}$$

we deduce that

$$U_1^n(kh, x + (k - n_t - 2j)\sqrt{h}; t, x) = U_2^n(kh, x + (k - n_t - 2j)\sqrt{h}; t, x), \quad j \in \{0, \dots, (k - n_t)\}.$$

Let us now prove that U_1^n and U_2^n are equal on \mathbb{R} . To do so, let us consider the map ϕ , which associate to $u^n(kh, y; t, x)$ the solution $U^n(kh, y; t, x)$ of

$$\begin{cases} U^n(kh, y; t, x) = D_+^n U^n((k+1)h, y; t, x) + hf \left((k+1)h, y, u^n(kh, y; t, x), \right. \\ \quad \left. \nabla_y^n U^n((k+1)h, y; t, x), \mathbb{E}[\varphi_1(U^n(kh, B_{kh}^{n,t,x}; t, x))] \right), \\ \quad \left. \mathbb{E}[\varphi_2(\nabla_y^n U^n((k+1)h, B_{(k+1)h}^{n,t,x}; t, x))] \right), \\ U^n(nh, y; t, x) = g(y), \end{cases} \quad (4.2.10)$$

where $k = n_t, \dots, n-1$. Let $u_1^n(kh, y; t, x)$ and $u_2^n(kh, y; t, x)$ be two elements, and

let $U_1^n(kh, y; t, x)$ and $U_2^n(kh, y; t, x)$ be the associated solutions. We have

$$\begin{aligned} & \sup_{n_t \leq k \leq n-1} |U_1^n(kh, y; t, x) - U_2^n(kh, y; t, x)|^2 \\ &= h^2 \sup_{n_t \leq k \leq n-1} \left| f \left((k+1)h, y, u_1^n(kh, y; t, x), \nabla_y^n U^n((k+1)h, y; t, x), \right. \right. \\ & \quad \left. \left. \mathbb{E}[\varphi_1(U_1^n(kh, B_{kh}^{n,t,x}; t, x))], \mathbb{E}[\varphi_2(\nabla_y^n U^n((k+1)h, B_{(k+1)h}^{n,t,x}; t, x))] \right) \right. \\ & \quad \left. - f \left((k+1)h, y, u_2^n(kh, y; t, x), \nabla_y^n U^n((k+1)h, y; t, x), \right. \right. \\ & \quad \left. \left. \mathbb{E}[\varphi_1(U_2^n(kh, B_{kh}^{n,t,x}; t, x))], \mathbb{E}[\varphi_2(\nabla_y^n U^n((k+1)h, B_{(k+1)h}^{n,t,x}; t, x))] \right) \right|^2. \end{aligned}$$

Since f is Lipschitz and from the fact that $U_1^n(kh, B_{kh}^{n,t,x}; t, x) = U_2^n(kh, B_{kh}^{n,t,x}; t, x)$, we get

$$\sup_{n_t \leq k \leq n-1} |U_1^n(kh, y; t, x) - U_2^n(kh, y; t, x)|^2 \leq h^2 C_f^2 \sup_{n_t \leq k \leq n-1} |u_1^n(kh, y; t, x) - u_2^n(kh, y; t, x)|^2.$$

Since $hC_f < 1$, this implies that the mapping ϕ is contraction. Then by using the fixed-point theorem, we conclude the uniqueness of solution to the discrete PDE (4.2.6). □

Next, if we define

$$U^n(s, y; t, x) = U^n(\underline{s}, y; t, x), \quad 0 \leq t \leq s \leq T,$$

we get

$$\begin{aligned} \bar{Y}_r^{n,s,y,t,x} &= \bar{Y}_r^{n,\underline{s},y,\underline{t},x} = U^n(r, B_r^{n,s,y}; t, x), \quad t \leq s \leq r \leq T, \\ \bar{Z}_r^{n,s,y,t,x} &= \bar{Z}_r^{n,\underline{s},y,\underline{t},x} = \nabla_y^n U^n(\bar{r}, B_{r^-}^{n,s,y}; t, x), \quad t \leq s \leq r \leq T. \end{aligned} \tag{4.2.11}$$

In particular,

$$\begin{aligned} Y_s^{n,t,x} &= Y_s^{n,\underline{t},x} = U^n(s, B_s^{n,t,x}; t, x), \quad 0 \leq t \leq s \leq T, \\ Z_s^{n,t,x} &= Z_s^{n,\underline{t},x} = \nabla_y^n U^n(\bar{s}, B_{s^-}^{n,t,x}; t, x), \quad 0 \leq t \leq s \leq T. \end{aligned}$$

Note that $\bar{Z}^{n,\underline{s},y,\underline{t},x}$ is defined at the points kh , $k = n_t + 1, \dots, n$. Furthermore, by using the same notations of [BGGL19], we have

$$\forall r \in]\underline{s}, T] \setminus \{kh, k = n_t + 1, \dots, n\}, \quad \bar{Z}_r^{n,s,y,t,x} = \Delta^n(r, B_{r^-}^{n,s,y}; t, x), \tag{4.2.12}$$

where $\Delta^n(s, y; t, x) = \nabla_y^n U^n(\underline{s} + h, y; t, x)$.

Now, we will prove some properties of U^n and Δ^n needed in the sequel.

Let us start by introducing some notations. We define

$$\begin{aligned}
F(s, y; t, x) &= f\left(s, y, u(s, y; t, x), \nabla_y u(s, y; t, x), p(s, t, x), q(s, t, x)\right), \\
F_n(s, y; t, x) &= f\left(s, y, U^n(s, y; t, x), \nabla_y U^n(s, y; t, x), p^n(s, t, x), q^n(s, t, x)\right), \\
\Theta_r^{n, s, y, t, x} &= \left(B_r^{n, s, y}, U^n(r, B_r^{n, s, y}; t, x), \nabla_y U^n(s, B_r^{n, s, y}; t, x), p^n(r, t, x), q^n(r, t, x)\right).
\end{aligned} \tag{4.2.13}$$

Then, we have $F_n(r, B_r^{n, s, y}; t, x) = f(r, \Theta_r^{n, s, y, t, x})$.

Remark 4.2.2. *Note that, under Assumption (A1), we have the following properties for F :*

(i) *For all $(t, s) \in [0, T]^2$ and for all $(x, y_1, y_2) \in \mathbb{R}^3$ such that $t \leq s \leq T$,*

$$\begin{aligned}
&|F(s, y_1; t, x) - F(s, y_2; t, x)| \\
&\leq C_f \left(|y_1 - y_2|^\epsilon + |u(s, y_1; t, x) - u(s, y_2; t, x)| + |\nabla_y u(s, y_1; t, x) - \nabla_y u(s, y_2; t, x)| \right).
\end{aligned}$$

(ii) *For all $(t, s_1, s_2) \in [0, T]^3$ and for all $(x, y) \in \mathbb{R}^2$ such that $t \leq s_2 < s_1 \leq T$,*

$$\begin{aligned}
&|F(s_1, y; t, x) - F(s_2, y; t, x)| \\
&\leq C_f \left(|s_1 - s_2|^\alpha + |u(s_1, y; t, x) - u(s_2, y; t, x)| + |\nabla_y u(s_1, y; t, x) - \nabla_y u(s_2, y; t, x)| \right. \\
&\quad \left. + |p(s_1, t, x) - p(s_2, t, x)| + |q(s_1, t, x) - q(s_2, t, x)| \right) \\
&\leq C_f \left(|s_1 - s_2|^\alpha + |u(s_1, y; t, x) - u(s_2, y; t, x)| + |\nabla_y u(s_1, y; t, x) - \nabla_y u(s_2, y; t, x)| \right. \\
&\quad \left. + C_{\varphi_1} \mathbb{E} \left[|u(s_1, B_{s_1}^{t, x}; t, x) - u(s_2, B_{s_2}^{t, x}; t, x)| \right] \right. \\
&\quad \left. + C_{\varphi_2} \mathbb{E} \left[|\nabla_y u(s_1, B_{s_1}^{t, x}; t, x) - \nabla_y u(s_2, B_{s_2}^{t, x}; t, x)| \right] \right).
\end{aligned}$$

Lemma 4.2.1. *(Properties of U^n and Δ^n)*

Under Assumption (A1), for all $(x, y) \in \mathbb{R}^2$,

(i) *for all $(t, s) \in [0, T]^2$, there exists a constant C depending on $T, \epsilon, f, \varphi_1, \varphi_2$ and g such that*

$$|U^n(s, y; t, x)| \leq C(1 + |x| + |y|)^\epsilon, \quad t \leq s \leq T,$$

(ii) *for all $(t, s) \in [0, T]^2$, there exists a constant C_n depending on T, ϵ, f, g and n such that*

$$|\Delta^n(s, y; t, x)| \leq \frac{C_n}{(T - s)^{(1-\epsilon)/2}}, \quad t \leq s < T.$$

Démonstration. (i) Firstly, let us observe that, from (4.2.2) and (4.2.7), we have

$$\begin{aligned}
Y_s^{n,t,x} &= g(B_T^{n,t,x}) + h \sum_{k=n_s+1}^n f\left(kh, B_{(k-1)h}^{n,t,x}, Y_{(k-1)h}^{n,t,x}, Z_{kh}^{n,t,x}, \mathbb{E}[\varphi_1(Y_{(k-1)h}^{n,t,x})], \mathbb{E}[\varphi_2(Z_{kh}^{n,t,x})]\right) \\
&\quad - \int_s^T Z_r^{n,t,x} dB_r^n \\
&= g(B_T^{n,t,x}) + \int_{\underline{s}}^T f\left(\bar{r}, B_r^{n,t,x}, Y_r^{n,t,x}, Z_r^{n,t,x}, \mathbb{E}[\varphi_1(Y_r^{n,t,x})], \mathbb{E}[\varphi_2(Z_r^{n,t,x})]\right) dr \\
&\quad - \int_s^T Z_r^{n,t,x} dB_r^n,
\end{aligned}$$

and

$$\begin{aligned}
\bar{Y}_r^{n,s,y,t,x} &= g(B_T^{n,s,y}) + h \sum_{k=n_r+1}^n f\left(kh, B_{(k-1)h}^{n,s,y}, \bar{Y}_{(k-1)h}^{n,s,y,t,x}, \bar{Z}_{kh}^{n,s,y,t,x}, \mathbb{E}[\varphi_1(Y_{(k-1)h}^{n,t,x})], \mathbb{E}[\varphi_2(Z_{kh}^{n,t,x})]\right) \\
&\quad - \int_r^T \bar{Z}_a^{n,s,y,t,x} dB_a^n \\
&= g(B_T^{n,s,y}) + \int_{\underline{r}}^T f\left(\bar{a}, B_a^{n,s,y}, \bar{Y}_a^{n,s,y,t,x}, \bar{Z}_a^{n,s,y,t,x}, \mathbb{E}[\varphi_1(Y_a^{n,t,x})], \mathbb{E}[\varphi_2(Z_a^{n,t,x})]\right) da \\
&\quad - \int_r^T \bar{Z}_a^{n,s,y,t,x} dB_a^n,
\end{aligned}$$

Then, according to the same strategy applied to the proof of Lemma 4.3.1-(i) and by using the prior estimates in the discret case (see the appendix in [BDM01]), we deduce that

$$\begin{aligned}
\mathbb{E} \left[\sup_{0 \leq r \leq T} |\bar{Y}_r^{n,s,y,t,x}|^2 + \int_0^T |\bar{Z}_r^{n,s,y,t,x}|^2 dr \right] \\
\leq C \left(1 + \mathbb{E} \left[\sup_{0 \leq r \leq T} |B_r^{n,s,y}|^{2\epsilon} \right] + \sup_{0 \leq r \leq T} \mathbb{E}[|Y_r^{n,t,x}|^2] + \int_0^T \mathbb{E}[|Z_r^{n,t,x}|^2] dr \right) \\
\leq C \left(1 + |y|^{2\epsilon} + \sup_{0 \leq r \leq T} \mathbb{E}[|Y_r^{n,t,x}|^2] + \int_0^T \mathbb{E}[|Z_r^{n,t,x}|^2] dr \right) \\
\leq C(1 + |x| + |y|)^{2\epsilon}.
\end{aligned}$$

Finally, since $U^n(s, y; t, x) = \bar{Y}_s^{n,s,y,t,x}$, we get the result.

(ii) Let us start by introducing the representation formula of Δ^n . From (4.2.7) and (4.2.11), we have,

$$\begin{aligned}
U^n(s, y; t, x) &= \mathbb{E} \left[g(B_T^{n,s,y}) + h \sum_{k=n_s+1}^n f\left(kh, B_{(k-1)h}^{n,s,y}, \bar{Y}_{(k-1)h}^{n,s,y,t,x}, \bar{Z}_{kh}^{n,s,y,t,x}, \mathbb{E}[\varphi_1(Y_{(k-1)h}^{n,t,x})], \mathbb{E}[\varphi_2(Z_{kh}^{n,t,x})]\right) \right] \\
&= \mathbb{E} \left[g(B_T^{n,s,y}) + \int_{\underline{s}}^T f\left(\bar{r}, B_r^{n,s,y}, \bar{Y}_r^{n,s,y,t,x}, \bar{Z}_r^{n,s,y,t,x}, \mathbb{E}[\varphi_1(Y_r^{n,t,x})], \mathbb{E}[\varphi_2(Z_r^{n,t,x})]\right) dr \right],
\end{aligned}$$

and taking into account (4.2.11) and (4.2.12),

$$\begin{aligned}
U^n(s, y; t, x) &= \mathbb{E} \left[g(B_T^{n,s,y}) + \int_{\underline{s}}^T f\left(\bar{r}, B_r^{n,s,y}, U^n(r, B_r^{n,s,y}; t, x), \Delta^n(r, B_r^{n,s,y}; t, x), p^n(r, t, x), q^n(r, t, x)\right) dr \right] \\
&= \mathbb{E} \left[g(B_T^{n,s,y}) + \int_{\underline{s}}^T f\left(\bar{r}, \Theta_r^{n,s,y,t,x}\right) dr \right].
\end{aligned} \tag{4.2.14}$$

Moreover, since $\Delta^n(s, y; t, x) = \nabla_y^n U^n(\underline{s} + h, y; t, x)$, we have (see the proof of Proposition 11 in [BGGL19] for more details)

$$\begin{aligned}
\Delta^n(s, y; t, x) &= \mathbb{E} \left[g(B_T^{n,s,y}) \frac{B_T^{n,s,y} - x}{T - \underline{s}} \right. \\
&\quad \left. + h \sum_{k=n_t+1}^{n-1} f\left((k+1)h, B_{kh}^{n,s,y}, \bar{Y}_{kh}^{n,s,y,t,x}, \bar{Z}_{(k+1)h}^{n,s,y,t,x}, \mathbb{E}[\varphi_1(Y_{kh}^{n,t,x})], \mathbb{E}[\varphi_2(Z_{(k+1)h}^{n,t,x})]\right) \frac{B_{kh}^{n,s,y} - x}{kh - \underline{s}} \right] \\
&= \mathbb{E} \left[g(B_T^{n,s,y}) \frac{B_T^{n,s,y} - x}{T - \underline{s}} \right. \\
&\quad \left. + \int_{\underline{s}+h}^T f\left(\bar{r}, B_r^{n,s,y}, \bar{Y}_r^{n,s,y,t,x}, \bar{Z}_r^{n,s,y,t,x}, \mathbb{E}[\varphi_1(Y_r^{n,t,x})], \mathbb{E}[\varphi_2(Z_r^{n,t,x})]\right) \frac{B_r^{n,s,y} - x}{r - \underline{s}} dr \right],
\end{aligned}$$

and in view of (4.2.11) and (4.2.12),

$$\Delta^n(s, y; t, x) = \mathbb{E} \left[g(B_T^{n,s,y}) \frac{B_T^{n,s,y} - x}{T - \underline{s}} + \int_{\underline{s}+h}^T f\left(\bar{r}, \Theta_r^{n,s,y,t,x}\right) \frac{B_r^{n,s,y} - x}{r - \underline{s}} dr \right]. \tag{4.2.15}$$

By applying the same steps as in the proof of [Lemma 8, [BGGL19]] where Δ^n is presented by (4.2.15), we deduce the result. \square

4.3 Continuous BSDE and PDE

In this Section, we firstly prove the existence and uniqueness of the solution u to the PDE (4.1.4). Then, thanks to the Markovian framework, we found the relation between u and the solutions to the BSDE (4.1.3). Finally, we study some properties and regularities of u and ∇u .

Let us consider the following general case of the BSDE (4.1.3) :

$$\bar{Y}_r^{s,y,t,x} = g(B_T^{s,y}) + \int_r^T f(a, B_a^{s,y}, \bar{Y}_a^{s,y,t,x}, \bar{Z}_a^{s,y,t,x}, \mathbb{E}[\varphi_1(Y_a^{t,x})], \mathbb{E}[\varphi_2(Z_a^{t,x})]) da - \int_r^T \bar{Z}_a^{s,y,t,x} dB_a, \tag{4.3.1}$$

where $0 \leq t \leq s \leq r \leq T$, $(x, y) \in \mathbb{R}^2$, $B_r^{s,y} = y + B_r - B_s$ and $(Y^{t,x}, Z^{t,x})$ the solution to (4.1.3).

In the above equation, the McKean terms $\mathbb{E}[\varphi_1(Y_a^{t,x})]$ and $\mathbb{E}[\varphi_2(Z_a^{t,x})]$ are independent

of s and y . Next, since the solution of (4.1.3) is known, the terms $\mathbb{E}[\varphi_1(Y_a^{t,x})]$ and $\mathbb{E}[\varphi_2(Z_a^{t,x})]$ are simply depending on time. Therefore, the existence of a unique solution $(\bar{Y}^{s,y,t,x}, \bar{Z}^{s,y,t,x})$ of the BSDE (4.3.1) can be obtained from [PP90]. Note that, the BSDE (4.2.7) defined in Section 4.2 is the discrete form of the above BSDE (4.3.1).

Proposition 4.3.1. *Under Assumption (A1), for each $(t, x) \in [0, T[\times \mathbb{R}$, there exists a unique classical solution $u(\cdot; t, x) \in \mathcal{C}^{1,2}([t, T[\times \mathbb{R}) \cap \mathcal{C}^{0,0}([t, T] \times \mathbb{R})$, to the PDE (4.1.4). Moreover, for $(x, y) \in \mathbb{R}^2$, we have*

$$\begin{aligned}\bar{Y}_r^{s,y,t,x} &= u(r, B_r^{s,y}; t, x), & 0 \leq t \leq s \leq r \leq T, \\ \bar{Z}_r^{s,y,t,x} &= \nabla_y u(r, B_r^{s,y}; t, x), & 0 \leq t \leq s \leq r < T,\end{aligned}\tag{4.3.2}$$

where $(\bar{Y}^{s,y,t,x}, \bar{Z}^{s,y,t,x})$ is the unique solution to the BSDE (4.3.1).

In particular,

$$\begin{aligned}Y_s^{t,x} &= u(s, B_s^{t,x}; t, x), & 0 \leq t \leq s \leq T, \\ Z_s^{t,x} &= \nabla_y u(s, B_s^{t,x}; t, x) & 0 \leq t \leq s < T,\end{aligned}$$

where $(Y^{t,x}, Z^{t,x})$ is the unique solution to the BSDE (4.1.3).

Démonstration. (i) Existence of solution.

According to [PP92] and based on the fact that the McKean terms in (4.3.1) $\mathbb{E}[\varphi_1(Y_a^{t,x})]$ and $\mathbb{E}[\varphi_2(Z_a^{t,x})]$ do not depend on s and y , there exists a deterministic $\bar{u}(s, y; t, x)$ such that $\bar{u}(\cdot; t, x) \in \mathcal{C}^{1,2}([t, T[\times \mathbb{R}) \cap \mathcal{C}^{0,0}([t, T] \times \mathbb{R})$ and for $s \in [0, T[$,

$$\bar{Y}_r^{s,y,t,x} = \bar{u}(r, B_r^{s,y}; t, x), \quad 0 \leq t \leq s \leq r \leq T, \quad \bar{Z}_r^{s,y,t,x} = \nabla_y \bar{u}(r, B_r^{s,y}; t, x), \quad 0 \leq t \leq s \leq r < T.$$

Moreover, \bar{u} satisfies the PDE :

$$\begin{cases} \partial_s \bar{u}(s, y; t, x) + \frac{1}{2} \Delta_y \bar{u}(s, y; t, x) \\ \quad = -f\left(s, y, \bar{u}(s, y; t, x), \nabla_y \bar{u}(s, y; t, x), \mathbb{E}[\varphi_1(Y_s^{t,x})], \mathbb{E}[\varphi_2(Z_s^{t,x})]\right), \\ \bar{u}(T, y; t, x) = g(y). \end{cases}$$

We refer to [[Zha01], Theorem 2.4.1] for the regularity of \bar{u} when f and g are Lipschitz continuous and to the approximation procedure used in the appendix of [BGGL19] for the Hölder case.

Thanks to the uniqueness of the solution to (4.3.1),

$$\bar{u}(s, B_s^{t,x}; t, x) = \bar{Y}_s^{s, B_s^{t,x}, t, x} = \bar{Y}_s^{t, x, t, x} = Y_s^{t, x},$$

$$\nabla_y \bar{u}(s, B_s^{t,x}; t, x) = \bar{Z}_s^{s, B_s^{t,x}, t, x} = \bar{Z}_s^{t, x, t, x} = Z_s^{t, x},$$

which imply that, for all $0 \leq t \leq s \leq T$

$$\mathbb{E}[\varphi_1(Y_s^{t,x})] = \mathbb{E}[\varphi_1(\bar{u}(s, B_s^{t,x}; t, x))],$$

$$\mathbb{E}[\varphi_2(Z_s^{t,x})] = \mathbb{E}[\varphi_2(\nabla_y \bar{u}(s, B_s^{t,x}; t, x))].$$

Then, u and \bar{u} satisfy the same equation. Consequently, we get the existence of a solution to (4.1.4).

(ii) Uniqueness of solution.

Let $(t, x) \in [0, T[\times \mathbb{R}$. Suppose that for each $s \geq t$ and $y \in \mathbb{R}$, there exist two solutions $u_1(s, y; t, x)$ and $u_2(s, y; t, x)$ for the PDE (4.1.4), such that $u_1(\cdot; t, x)$, $u_2(\cdot; t, x) \in \mathcal{C}^{1,2}([t, T[\times \mathbb{R}) \cap \mathcal{C}^{0,0}([t, T] \times \mathbb{R})$.

Let $i \in \{1, 2\}$. On the one hand, from the regularity of u_i and by applying Itô's formula, we obtain that

$$\begin{aligned} u_i(s, B_s^{t,x}; t, x) &= u_i(T, B_T^{t,x}; t, x) - \int_s^T \left(\partial_a u_i(a, B_a^{t,x}; t, x) + \frac{1}{2} \Delta_y u_i(a, B_a^{t,x}; t, x) \right) da \\ &\quad - \int_s^T \nabla_y u_i(a, B_a^{t,x}; t, x) dB_a. \end{aligned}$$

Since u_1 and u_2 are two solutions of the PDE (4.1.4), we get

$$\begin{aligned} u_i(s, B_s^{t,x}; t, x) &= g(B_T^{t,x}) + \int_s^T f\left(a, B_a^{t,x}, u_i(a, B_a^{t,x}; t, x), \nabla_y u_i(a, B_a^{t,x}; t, x), \right. \\ &\quad \left. \mathbb{E}[\varphi_1(u_i(a, B_a^{t,x}; t, x))], \mathbb{E}[\varphi_2(\nabla_y u_i(a, B_a^{t,x}; t, x))]\right) da \\ &\quad - \int_s^T \nabla_y u_i(a, B_a^{t,x}; t, x) dB_a. \end{aligned}$$

Then, we deduce that

$$Y_s^{i,t,x} = u_i(s, B_s^{t,x}; t, x), \quad Z_s^{i,t,x} = \nabla_y u_i(s, B_s^{t,x}; t, x),$$

where $(Y^{1,t,x}, Z^{1,t,x})$ and $(Y^{2,t,x}, Z^{2,t,x})$ are two solutions of the BSDE (4.1.3). And, thanks to the uniqueness of the solution to (4.1.3), we conclude that

$$u_1(s, B_s^{t,x}; t, x) = u_2(s, B_s^{t,x}; t, x). \quad (4.3.3)$$

On the other hand, let $s > t$ and let us define the set $N = \{y \in \mathbb{R}, |u_1 - u_2|(s, y; t, x) > 0\}$. Suppose that $\lambda(N) > 0$, where λ is the Lebesgue measure. Then, we obtain

$$\mathbb{P}(B_s^{t,x} \in N) > 0,$$

and obviously

$$\mathbb{P}\left(|u_1 - u_2|(s, B_s^{t,x}; t, x) > 0\right) > 0,$$

which contradicts the above result (4.3.3). Therefore, we conclude that $\lambda(N) = 0$, which implies that

$$u_1(s, \cdot; t, x) = u_2(s, \cdot; t, x), \quad \lambda - a.e..$$

Moreover, since $u_1(s, \cdot; t, x)$ and $u_2(s, \cdot; t, x)$ are continuous, we deduce that $u_1 = u_2$ and the uniqueness result has been verified. \square

Lemma 4.3.1. (Properties of u and ∇u)

Under Assumption (A1), for all $(x, y) \in \mathbb{R}^2$, there exists a constant C depending on $T, \epsilon, f, \varphi_1, \varphi_2$ and g such that

(i) for all $(t, s) \in [0, T]^2$ such that $t \leq s$, $|u(s, y; t, x)| \leq C(1 + |x| + |y|)^\epsilon$,

(ii) for all $(t, s) \in [0, T]^2$ such that $t \leq s$, $|\nabla_y u(s, y; t, x)| \leq \frac{C}{(T-s)^{(1-\epsilon)/2}}$.

Démonstration. (i) Let us start by writing equation (4.3.1) in the form of a classical BSDE. In order to do it, we define

$$\bar{\xi} = g(B_T^{s,y}), \quad \bar{f}(a, y_1, z_1) = f\left(a, B_a^{s,y}, y_1, z_1, \mathbb{E}[\varphi_1(Y_a^{t,x})], \mathbb{E}[\varphi_2(Z_a^{t,x})]\right),$$

where $(Y^{t,x}, Z^{t,x})$ is the unique solution of the BSDE (4.1.3). By using these notations, equation (4.3.1) is nothing else than the BSDE

$$\bar{Y}_r^{s,y,t,x} = \bar{\xi} + \int_r^T \bar{f}(a, \bar{Y}_a^{s,y,t,x}, \bar{Z}_a^{s,y,t,x}) da - \int_r^T \bar{Z}_a^{s,y,t,x} dB_a, \quad 0 \leq r \leq T.$$

Since g is ϵ -Hölder, φ_1 and φ_2 are Lipschitz continuous and f satisfies (4.2.1), we have

$$\begin{aligned} |\bar{\xi}|^2 + \int_0^T |\bar{f}(r, 0, 0)|^2 dr &\leq C \left(1 + \sup_{0 \leq r \leq T} |B_r^{s,y}|^{2\epsilon} + \sup_{0 \leq r \leq T} |\mathbb{E}[\varphi_1(Y_r^{t,x})]|^2 + \int_0^T |\mathbb{E}[\varphi_2(Z_r^{t,x})]|^2 dr \right) \\ &\leq C \left(1 + \sup_{0 \leq r \leq T} |B_r^{s,y}|^{2\epsilon} + \sup_{0 \leq r \leq T} \mathbb{E}[|Y_r^{t,x}|^2] + \int_0^T \mathbb{E}[|Z_r^{t,x}|^2] dr \right). \end{aligned} \quad (4.3.4)$$

To establish the majoration, we use the prior estimates of the solution $(\bar{Y}^{s,y,t,x}, \bar{Z}^{s,y,t,x})$ (see [EKPQ97]),

$$\mathbb{E} \left[\sup_{0 \leq r \leq T} |\bar{Y}_r^{s,y,t,x}|^2 + \int_0^T |\bar{Z}_r^{s,y,t,x}|^2 dr \right] \leq C \mathbb{E} \left[|\bar{\xi}|^2 + \int_0^T |\bar{f}(r, 0, 0)|^2 dr \right],$$

and taking into account (4.3.4), we get

$$\begin{aligned} \mathbb{E} \left[\sup_{0 \leq r \leq T} |\bar{Y}_r^{s,y,t,x}|^2 + \int_0^T |\bar{Z}_r^{s,y,t,x}|^2 dr \right] &\leq C \left(1 + \mathbb{E} \left[\sup_{0 \leq r \leq T} |B_r^{s,y}|^{2\epsilon} \right] + \sup_{0 \leq r \leq T} \mathbb{E}[|Y_r^{t,x}|^2] + \int_0^T \mathbb{E}[|Z_r^{t,x}|^2] dr \right) \\ &\leq C \left(1 + |y|^{2\epsilon} + \sup_{0 \leq r \leq T} \mathbb{E}[|Y_r^{t,x}|^2] + \int_0^T \mathbb{E}[|Z_r^{t,x}|^2] dr \right). \end{aligned}$$

By using the prior estimates of the solution $(Y^{t,x}, Z^{t,x})$ of (4.1.3) (see [ARBGL19]), we can deduce the upperbound for the last two terms of the previous inequality,

$$\sup_{0 \leq r \leq T} \mathbb{E} \left[|Y_r^{t,x}|^2 + \int_0^T |Z_r^{t,x}|^2 dr \right] \leq C(1 + |x|^{2\epsilon}).$$

Then, we deduce that

$$\mathbb{E} \left[\sup_{0 \leq r \leq T} |\bar{Y}_r^{s,y,t,x}|^2 + \int_0^T |\bar{Z}_r^{s,y,t,x}|^2 dr \right] \leq C \left(1 + |x|^{2\epsilon} + |y|^{2\epsilon} \right).$$

Finally, since $u(s, y; t, x) = \bar{Y}_s^{s,y,t,x}$, the results follows easily.

- (ii) The starting point is the representation formula of $\nabla_y u(s, y; t, x)$. From (4.3.1) and (4.3.2), we have

$$\begin{aligned} u(s, y; t, x) &= \mathbb{E} \left[g(B_T^{s,y}) + \int_s^T f \left(r, B_r^{s,y}, u(r, B_r^{s,y}; t, x), \nabla_y u(r, B_r^{s,y}; t, x), p(r, t, x), q(r, t, x) \right) dr \right] \\ &= \mathbb{E} \left[g(B_T^{s,y}) + \int_s^T F(r, B_r^{s,y}; t, x) dr \right]. \end{aligned} \quad (4.3.5)$$

According to [Zha05] and based on the fact that the McKean terms in the above equation, i.e. $p(r, t, x)$ and $q(r, t, x)$, are independent of s and y , we obtain

$$\begin{aligned} \nabla_y u(s, y; t, x) &= \mathbb{E} \left(g(B_T^{s,y}) \frac{B_T - B_s}{T - s} \right) + \mathbb{E} \left(\int_s^T F(r, B_r^{s,y}; t, x) \frac{B_r - B_s}{r - s} dr \right) \\ &= \mathbb{E} \left((g(B_T^{s,y}) - g(y)) \frac{B_T - B_s}{T - s} \right) + \mathbb{E} \left(\int_s^T (F(r, B_r^{s,y}; t, x) - F(r, y; t, x)) \frac{B_r - B_s}{r - s} dr \right). \end{aligned} \quad (4.3.6)$$

Under Assumption (A1), Remark 4.2.2-i, and since u satisfies Proposition 4.3.2-i and Proposition 4.3.3-i, we get

$$|\nabla_y u(s, y; t, x)| \leq \frac{C_g}{(T - s)^{(1-\epsilon)/2}} + \int_s^T C_f \left(\frac{1 + C}{(r - s)^{(1-\epsilon)/2}} + \frac{C}{\sqrt{T - r}(r - s)^{(1-\epsilon)/2}} \right) dr,$$

and the results can be immediately verified. \square

Proposition 4.3.2. *(Time and space regularity of u)*

Under Assumption (A1),

- (i) for all $(t, s) \in [0, T]^2$ and for all $(x, y_1, y_2) \in \mathbb{R}^3$, there exists a constant C depending on T, ϵ, f and g such that

$$|u(s, y_1; t, x) - u(s, y_2; t, x)| \leq C|y_1 - y_2|^\epsilon, \quad t \leq s < T,$$

- (ii) for all $(t, s_1, s_2) \in [0, T]^3$ and for all $(x, y) \in \mathbb{R}^2$ there exists a constant C depending on $T, \epsilon, f, \varphi_1, \varphi_2$ and g such that

$$|u(s_1, y; t, x) - u(s_2, y; t, x)| \leq C(1 + |x| + |y|)^\epsilon (s_1 - s_2)^{\epsilon/2}, \quad t \leq s_2 < s_1 < T.$$

Démonstration. (i) From classical result on BSDEs and based on the fact that the McKean terms in (4.3.1), $\mathbb{E}[\varphi_1(Y_a^{t,x})]$ and $\mathbb{E}[\varphi_2(Z_a^{t,x})]$ are independent of s and y , we obtain

$$\begin{aligned} &|\bar{Y}_s^{s,y_1,t,x} - \bar{Y}_s^{s,y_2,t,x}|^2 \\ &\leq \mathbb{E}|g(B_T^{s,y_1}) - g(B_T^{s,y_2})|^2 + \int_s^T \mathbb{E} \left| f \left(r, B_r^{s,y_1}, \bar{Y}_r^{s,y_1,t,x}, \bar{Z}_r^{s,y_1,t,x}, \mathbb{E}[\varphi_1(Y_r^{t,x})], \mathbb{E}[\varphi_2(Z_r^{t,x})] \right) \right. \\ &\quad \left. - f \left(r, B_r^{s,y_2}, \bar{Y}_r^{s,y_1,t,x}, \bar{Z}_r^{s,y_1,t,x}, \mathbb{E}[\varphi_1(Y_r^{t,x})], \mathbb{E}[\varphi_2(Z_r^{t,x})] \right) \right|^2 dr. \end{aligned}$$

Under Assumption (A1), the result follows easily.

(ii) From (4.3.5), we have

$$\begin{aligned}
& u(s_1, y; t, x) - u(s_2, y; t, x) \\
&= \mathbb{E} \left[g(B_T^{s_1, y}) - g(B_T^{s_2, y}) + \int_{s_1}^T F(r, B_r^{s_1, y}; t, x) dr - \int_{s_2}^T F(r, B_r^{s_2, y}; t, x) dr \right] \\
&= \mathbb{E} \left[g(B_T^{s_1, y}) - g(B_T^{s_2, y}) + \int_{s_1}^T \left(F(r, B_r^{s_1, y}; t, x) - F(r, B_r^{s_2, y}; t, x) \right) dr \right. \\
&\quad \left. - \int_{s_2}^{s_1} F(r, B_r^{s_2, y}; t, x) dr \right].
\end{aligned}$$

Under Assumption (A1) and Remark 4.2.2 and since u satisfies Proposition 4.3.2-(i) and $\nabla_y u$ satisfies Proposition 4.3.3-(i), we obtain

$$\begin{aligned}
& |u(s_1, y; t, x) - u(s_2, y; t, x)| \\
&\leq (C_g + TC_f(1 + C))|s_1 - s_2|^{\epsilon/2} + CC_f \int_{s_1}^T |s_1 - s_2|^{\epsilon/2} \frac{1}{\sqrt{T-r}} dr \\
&\quad + \mathbb{E} \left[\int_{s_2}^{s_1} \left(K_f + C_f(|B_r^{s_2, y}|^\epsilon + |u(r, B_r^{s_2, y}; t, x)| + |\nabla_y u(r, B_r^{s_2, y}; t, x)| \right. \right. \\
&\quad \left. \left. + |u(r, B_r^{t, x}; t, x)| + |\nabla_y u(r, B_r^{t, x}; t, x)| \right) dr \right].
\end{aligned}$$

Then, by using Lemma 4.3.1, we derive that

$$\begin{aligned}
& |u(s_1, y; t, x) - u(s_2, y; t, x)| \\
&\leq C|s_1 - s_2|^{\epsilon/2} + C_f \int_{s_2}^{s_1} \left((|y|^\epsilon + |r - s_2|^{\epsilon/2}) + C(1 + |x|^\epsilon + |y|^\epsilon + |r - s_2|^{\epsilon/2}) \right. \\
&\quad \left. + \frac{C}{(T-r)^{(1-\epsilon)/2}} + (1 + |x|^\epsilon + |r - t|^{\epsilon/2}) + \frac{C}{(T-r)^{(1-\epsilon)/2}} \right) dr.
\end{aligned}$$

Note that $x \mapsto x^{(\epsilon+1)/2}$ is concave, then the result is obvious. \square

Proposition 4.3.3. (Time and space regularity of $\nabla_y u$)

Under Assumption (A1),

(i) for all $(t, s) \in [0, T]^2$ and for all $(x, y_1, y_2) \in \mathbb{R}^3$, there exists a constant C depending on T, ϵ, f and g such that

$$|\nabla_y u(s, y_1; t, x) - \nabla_y u(s, y_2; t, x)| \leq C \frac{|y_1 - y_2|^\epsilon}{\sqrt{T-s}}, \quad t \leq s < T,$$

(ii) for all $(t, s_1, s_2) \in [0, T]^3$, and for all $(x, y) \in \mathbb{R}^2$ there exists a constant C depending on $T, \epsilon, f, \varphi_1, \varphi_2$ and g such that

$$|\nabla_y u(s_1, y; t, x) - \nabla_y u(s_2, y; t, x)| \leq C \frac{(s_1 - s_2)^{\epsilon/2}}{\sqrt{T-s_1}}, \quad t \leq s_2 < s_1 < T.$$

We refer to [Lemma 6 and Proposition 9, [BGGL19]] for the proof of Proposition 4.3.3.

4.4 Main results

In this section, we are interested in proving the rate of convergence for the Wasserstein distance of the solution of a McKean-Vlasov BSDE (4.1.3) towards its approximation which is given by the McKean-Vlasov BSDE driven by a scaled random walk (4.2.2).

Firstly, we start this section by recalling the definition of the Wasserstein distance. Let $X \rightsquigarrow \mu$ and $Y \rightsquigarrow \nu$ two random variables with their corresponding laws. The Wasserstein distance, $W_r(X, Y) = W_r(\mu, \nu)$ for two random variables is given by :

$$W_r(X, Y) = W_r(\mu, \nu) = \inf\{\mathbb{E}[|X - Y|^r]^{1/r} : X \rightsquigarrow \mu, Y \rightsquigarrow \nu\}.$$

Theorem 4.4.1. *Under Assumption (A1), for all $r \geq 1$, there exists a constant C_r such that, for all $x \in \mathbb{R}$,*

$$W_r(Y_s^{n,t,x}, Y_s^{t,x}) \leq C_r(1 + |x|)^\epsilon n^{-\alpha \wedge \frac{\epsilon}{2}}, \quad 0 \leq t \leq s \leq T,$$

$$W_r(Z_s^{n,t,x}, Z_s^{t,x}) \leq C_r \frac{(1 + |x|)^\epsilon}{\sqrt{T-s}} n^{-\alpha \wedge \frac{\epsilon}{2}}, \quad 0 \leq t \leq s < T.$$

Before giving these main results, it is necessary to find the rate of the point-wise convergence between the approximation U^n solution to (4.2.6) and the solution u to the semilinear heat equation (4.1.4).

Proposition 4.4.1. *Under Assumption (A1), for all $(x, y) \in \mathbb{R}^2$, there exists a constant C such that,*

$$|u(s, y; t, x) - U^n(s, y; t, x)| \leq C(1 + |x| + |y|)^\epsilon n^{-\alpha \wedge \frac{\epsilon}{2}}, \quad 0 \leq t \leq s \leq T,$$

$$|\nabla_y u(s, y; t, x) - \Delta^n(s, y; t, x)| \leq C \frac{(1 + |x| + |y|)^\epsilon}{\sqrt{T-s}} n^{-\alpha \wedge \frac{\epsilon}{2}}, \quad 0 \leq t \leq s < T.$$

Under Lemma 4.2.1-(i) and following the proof of [Proposition 11, [BGGL19]], we get the results.

Finally, the proof of Theorem 4.4.1 is deduced from [Theorem 10, [BGGL19]] and by taking into account Proposition 4.3.2-(ii) and Proposition 4.4.1.

4.5 Practical Implementation and Numerical examples

The implemented numerical scheme slightly differs from (4.2.4)-(4.2.5). Looking at (4.2.5), we notice that this equation is implicit in $Y_{kh}^{n,t,x}$. In order to circumvent this problem, we consider the following explicit scheme, where we have fixed $t = x = 0$ and $t_k := kh$ for all $k \in \{0, \dots, n\}$:

$$\begin{cases} Y_{t_n}^n = g(B_{t_n}^n), \text{ and for all } k \in \{0, \dots, n-1\} : \\ Z_{t_{k+1}}^n = h^{-1/2} \mathbb{E} \left(Y_{t_{k+1}}^n \xi_{k+1} | \mathcal{F}_{t_k}^n \right), \\ Y_{t_k}^n = \mathbb{E} \left(Y_{t_{k+1}}^n + hf \left(t_{k+1}, B_{t_{k+1}}^n, Y_{t_{k+1}}^n, Z_{t_{k+1}}^n, \mathbb{E}[\varphi_1(Y_{t_{k+1}}^n)], \mathbb{E}[\varphi_2(Z_{t_{k+1}}^n)] \right) | \mathcal{F}_{t_k}^n \right). \end{cases} \quad (4.5.1)$$

Since B^n is built by using a random walk, its law is known at each time $(t_k)_k$ of the grid. For all $k \in \{0, \dots, n\}$ $B_{t_k}^n$ is a discrete random variable taking values in $\{k\sqrt{h}, (k-2)\sqrt{h}, \dots, -k\sqrt{h}\}$ and we have

$$\forall j \in \{0, \dots, k\}, \quad \mathbb{P} \left(B_{t_k}^n = (k-2j)\sqrt{h} \right) = \frac{1}{2^k} \binom{k}{j}. \quad (4.5.2)$$

4.5.1 Backward computation of $(Y_{t_k}^n, Z_{t_k}^n)_k$

We detail one step of the backward computation of $(Y_{t_k}^n, Z_{t_k}^n)_k$. From the law of $B_{t_n}^n$ we get the law of $Y_{t_n}^n$. Let us compute the law of $Z_{t_n}^n$. From (4.5.1), we get

$$\begin{aligned} Z_{t_n}^n &= h^{-1/2} \mathbb{E} \left(Y_{t_n}^n \xi_n | \mathcal{F}_{t_{n-1}}^n \right) = h^{-1/2} \mathbb{E} \left(g(B_{t_{n-1}}^n + \sqrt{h} \xi_n) \xi_n | \mathcal{F}_{t_{n-1}}^n \right) \\ &= \frac{1}{2\sqrt{h}} \left(g(B_{t_{n-1}}^n + \sqrt{h}) - g(B_{t_{n-1}}^n - \sqrt{h}) \right). \end{aligned} \quad (4.5.3)$$

Combining the previous equality and (4.5.2) gives us the law of $Z_{t_n}^n$. Once obtained the law of $Y_{t_n}^n$ and $Z_{t_n}^n$, we are able to compute $\mathbb{E}[\varphi_1(Y_{t_n}^n)]$ and $\mathbb{E}[\varphi_2(Z_{t_n}^n)]$. (4.5.1) gives

$$Y_{t_{n-1}}^n = \mathbb{E} \left(Y_{t_n}^n + hf \left(t_n, B_{t_n}^n, Y_{t_n}^n, Z_{t_n}^n, \mathbb{E}[\varphi_1(Y_{t_n}^n)], \mathbb{E}[\varphi_2(Z_{t_n}^n)] \right) | \mathcal{F}_{t_{n-1}}^n \right).$$

As $Y_{t_n}^n = g(B_{t_{n-1}}^n + \xi_n)$ and $Z_{t_n}^n$ satisfies (4.5.3), we immediately see that $Y_{t_{n-1}}^n$ is an explicit function of $B_{t_{n-1}}^n$. The value of $Y_{t_{n-1}}^n$ is known at each node of the tree and (4.5.2) gives us the law of $Y_{t_{n-1}}^n$.

4.5.2 Numerical examples

In this Section we highlight the results obtained in Theorem 4.4.1 through numerical examples. According to this Theorem, we have

$$W_1(Y_t^n, Y_t) \leq \frac{C}{n^{\alpha \wedge \frac{\alpha}{2}}} \text{ and } W_1(Z_t^n, Z_t) \leq \frac{C}{\sqrt{T-t} n^{\alpha \wedge \frac{\alpha}{2}}}. \quad (4.5.4)$$

W_1 is the infimum over all couplings between the law of Y_t^n and the law of Y_t , which boils down to take the infimum over all couplings between the law of B_t^n and the law of B_t . Since numerically we can not compute this infimum, we consider one special case of coupling : for a fixed B_t , B_t^n corresponds to the value of the tree closest to B_t . We also use a Monte Carlo procedure to approximate $\mathbb{E}|Y_t^n - Y_t|$ (and the same approximation

holds for $\mathbb{E}|Z_t^n - Z_t|$)

$$\hat{E}_Y = \frac{1}{M} \sum_{i=1}^M |Y_t^{n,i} - Y_t^i|. \quad (4.5.5)$$

Asymptotically in M , \hat{E}_Y (resp. \hat{E}_Z) upper bounds $W_1(Y_t^n, Y_t)$ (resp. $W_1(Z_t^n, Z_t)$.) In the following examples, we consider the equation

$$Y_t = \xi + \int_t^T \alpha Y_s + \beta \mathbb{E}(Y_s) + \gamma \mathbb{E}(Z_s) ds - \int_t^T Z_s dB_s \quad (4.5.6)$$

First example : $\xi = B_T$, $\alpha = \beta = 1$ and $\gamma = 0$

The solution of (4.5.6) is given in [ARBGL19]. We have

$$(Y_t, Z_t) = (e^{T-t} B_t, e^{T-t}).$$

Figure 4.5.2 represents the evolution of $\log(\hat{E}_Y)$ and $\log(\hat{E}_Z)$ given in (4.5.5) w.r.t. $\log(n)$. In this case $\varepsilon = 1$, (4.5.4) indicates that we should get a slope of $-\frac{1}{2}$ for both errors. We get a slope of -0.5679408 for the error on Y and a slope of -1.2124148 for the error on Z . This last result is undoubtedly due to the fact in this case Z is deterministic.

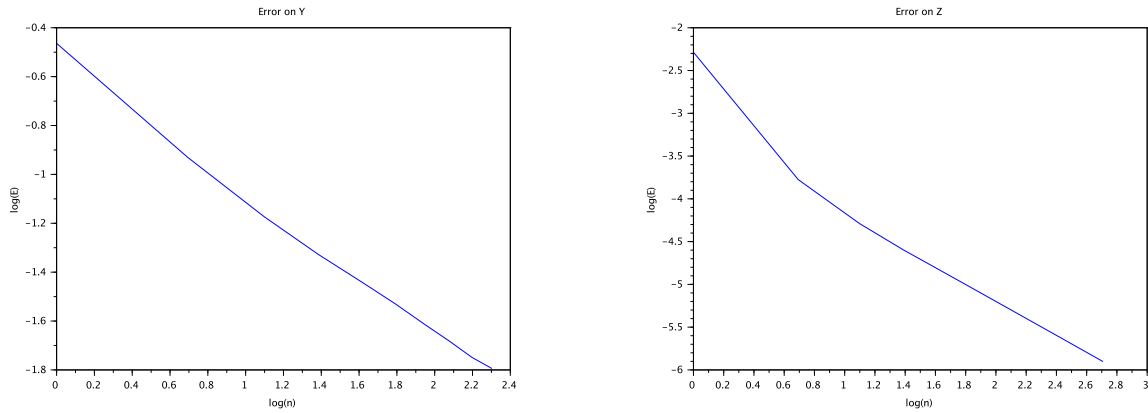


FIGURE 4.1 – Evolution of $\log(\hat{E}_Y)$ and $\log(\hat{E}_Z)$ w.r.t. $\log(n)$ when $T = 1$, $t = 0.5$ and $M = 10^4$.

Second example : $\xi = B_T$, $\alpha = \beta = \gamma = 1$

The solution of (4.5.6) is given in [ARBGL19]. We have

$$(Y_t, Z_t) = (e^{T-t}(B_t + e^{T-t} - 1), e^{T-t}).$$

Figure 4.5.2 represents the evolution of $\log(\hat{E}_Y)$ and $\log(\hat{E}_Z)$ given in (4.5.5) w.r.t. $\log(n)$. In this case $\varepsilon = 1$, (4.5.4) indicates that we should get a slope of $-\frac{1}{2}$ for both

errors. We get a slope of -0.2338094 for the error on Y (which is not inconsistent, given that we compute an upper bound of $W_1(Y_t^n, Y_t)$) and a slope of -1.2017365 for the error on Z .

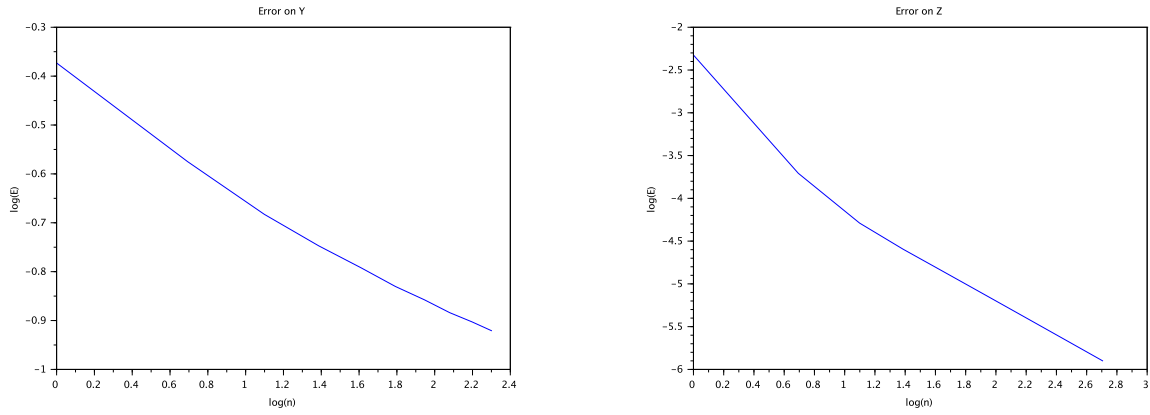


FIGURE 4.2 – Evolution of $\log(\hat{E}_Y)$ and $\log(\hat{E}_Z)$ w.r.t. $\log(n)$ when $T = 1$, $t = 0.5$ and $M = 10^4$.

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Résumé

Cette thèse est consacrée à l'étude théorique et numérique de deux principaux sujets de recherche : les équations différentielles stochastiques (EDS) réfléchies en moyenne avec sauts et les équations différentielles stochastiques rétrogrades (EDSR) de type McKean-Vlasov.

Le premier travail de ma thèse établit la propagation du chaos pour les EDS réfléchies en moyenne avec sauts. Nous avons étudié dans un premier temps l'existence et l'unicité d'une solution. Nous avons développé ensuite un schéma numérique via un système de particules. Enfin nous avons obtenu une vitesse de convergence pour ce schéma.

Le deuxième travail de ma thèse consiste à étudier les EDSR de type McKean-Vlasov. Nous avons prouvé l'existence et l'unicité de solutions de telles équations, et nous avons proposé une approximation numérique basée sur la décomposition en chaos de Wiener ainsi que sa vitesse de convergence.

Le troisième travail de ma thèse s'intéresse à une autre type de simulation pour les EDSR de type McKean-Vlasov. Nous avons proposé un schéma numérique basé sur l'approximation du mouvement brownien par une marche aléatoire et nous avons obtenu une vitesse de convergence pour ce schéma.

Par ailleurs, quelques exemples numériques dans ces trois travaux permettent de constater l'efficacité de nos schémas et les vitesses de convergences annoncées par les résultats théoriques.

Mots Clés : EDS réfléchies en moyenne avec sauts, système de particules, propagation du chaos, EDSR de type McKean-Vlasov, décomposition en chaos de Wiener, méthode de Monte-Carlo, marche aléatoire.

Abstract

This thesis is devoted to the theoretical and numerical study of two main subjects in the context of stochastic differential equations (SDEs) : mean reflected SDEs with jumps and McKean-Vlasov backward SDEs.

The first part of my thesis establishes the propagation of chaos for the mean reflected SDEs with jumps. First, we study the existence and uniqueness of a solution. Then, we develop a numerical scheme based on the particle system. Finally, we obtain the rate of convergence of this scheme.

The second part of my thesis studies the McKean-Vlasov backward SDEs. In this case, we prove the existence and uniqueness of a solution for such equations. Then, thanks to the Wiener chaos expansion, we provide a numerical approximation. Moreover, the convergence rate of this approximation is also determined.

The third part of my thesis proposes another type of simulation for the McKean-Vlasov backward SDEs. Due to the approximation of Brownian motion by a scaled random walk, we develop a numerical scheme and we get its convergence rate.

In addition, a few numerical examples in these three parts are given to illustrate the efficiency of our schemes and their convergence rates stated by the theoretical results.

Keywords : mean reflected SDEs with jumps, particle system, propagation of chaos, McKean-Vlasov backward SDEs, Wiener chaos expansion, Monte-Carlo method, random walk.