Contractible 3-manifold and Positive scalar curvature

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Abstract

The purposes of this thesis is to understand spaces which carry metrics of positive scalar curvature. There are several topological obstructions for a smooth manifold to have a complete metric of positive scalar curvature. Our goal is to find all obstructions for contractible 3-manifolds and closed 4-manifolds.

In dimension 3, we are concerned with the question whether a complete contractible 3-manifold of positive scalar curvature is homeomorphic to $\mathbb{R}^3$. The topological structure of contractible 3-manifolds could be complicated. For example, the Whitehead manifold is a contractible 3-manifold which is not homeomorphic to $\mathbb{R}^3$.

We first prove that the Whitehead manifold does not carry a complete metric of positive scalar curvature. This result can be generalised to the so-called genus one case. Precisely, we show that no contractible genus one 3-manifold admits a complete metric of positive scalar curvature.

We then study the fundamental group at infinity, π_1^∞, and its relationship with the existence of positive scalar curvature metric. The fundamental group at infinity of a manifold is the inverse limit of the fundamental groups of complements of compact subsets. In this thesis, we give a partial answer to the above question. We prove that a complete contractible 3-manifold with positive scalar curvature and trivial π_1^∞ is homeomorphic to $\mathbb{R}^3$.

Finally, we study closed aspherical 4-manifolds. Together with minimal surface theory and the geometrisation conjecture, we show that no closed aspherical 4-manifold with non-trivial first Betti number carries a metric of positive scalar curvature.

Résumé

Un des objectifs de ce mémoire est de comprendre les espaces munis de métriques complète de courbure scalaire positive. Il y a plusieurs obstructions topologiques à l'existence d'une métrique complète de courbure scalaire positive. Notre but est de trouver toutes les obstructions pour les variétés contractiles de dimension 3 et les variétés fermées de dimension 4.

En dimension 3, nous considérons la question de savoir si une variété contractile complète de courbure scalaire positive est homéomorphe à $\mathbb{R}^3$. La structure topologique des variétés contractiles de dimension 3 est assez compliquée. Par exemple, Whitehead a construit une variété dimension 3 contractile qui n'est pas homéomorphe à $\mathbb{R}^3$.

Nous prouvons, tout d'abord, que la variété de Whitehead n'a pas de métrique complète de courbure scalaire positive. Ce résultat peut être généralisé au cas dit de genre un. Précisément, nous montrons qu'aucune variété contractile de dimension 3 et de genre un ne possède de métrique complète de courbure scalaire positive.

Nous étudions ensuite le groupe fondamental à l'infini, π_1^∞, et son lien avec l'existence d'une métrique de courbure scalaire positive. Le groupe fondamental à l'infini d'une variété est la limite projective des groupes fondamentaux des complémentaires des sous-ensembles compacts. Dans ce mémoire, nous apportons une réponse partielle à la question évoquée plus haut. Nous prouvons qu'une variété complète de dimension 3 de courbure scalaire positive dont le groupe π_1^∞ est trivial est homéomorphe à $\mathbb{R}^3$.

Enfin, nous étudions les variétés fermées asphériques de dimension 4. En utilisant la théorie des surfaces minimales et la conjecture de géométrisation, nous montrons qu'aucune variété fermée asphérique de dimension 4 avec un premier nombre de Betti non trivial ne possède de métrique à courbure scalaire positive.

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Dedicated to my mother, for her 60th birthday.

Contents

Introduction

Riemannian geometry aims to study Riemannian manifolds which are smooth manifolds with metric structures. One of the fundamental questions is to understand the relationship between the curvature, which is locally defined, and the global properties of smooth manifolds. The earliest result is the classical theorem of Gauss and Bonnet, which links the curvature and the Euler number, a topological invariant. This theorem implies, for example, that any compact surface of genus $g > 0$ has no metric of positive curvature.

In higher dimensions, the existence of metrics of positive curvature becomes much more complicated, because there are several topological obstructions for a smooth manifold to have a complete metric of positive curvature.

We then take 3-manifolds to explain this fact.

(M^3, g)	The compact case	The non-compact case
$K > 0$	$\mathbb{S}^3/\Gamma$	$\mathbb{R}^3$
$Ric > 0$		$\mathbb{R}^3$
$Scal > 0$	$(\#_{i=1}^{k}\mathbb{S}^3/\Gamma_i)\#(\#_{j=1}^{l}\mathbb{S}^1 \times \mathbb{S}^2)$	?

TABLE 1.

For a compact Riemannian 3-manifold, there is a unique short-time solution to the so-called (normalized) Ricci flow, introduced by Hamilton [**Ham82**]. If the manifold has positive Ricci curvature, the short-time solution can be extended to be the long-time solution. The limit of this flow is a metric of constant sectional curvature. That is to say, if a compact 3-manifold has positive Ricci curvature, then it is homeomorphic to the quotient $\mathbb{S}^3/\Gamma$ of the sphere $\mathbb{S}^3$ by a finite subgroup $\Gamma \subset O(4)$. Such a quotient is called a spherical 3-manifold.

The next major contribution to the subject was made by Perelman [**Per02a**, **Per02b**, **Per03**] who developed the Ricci flow with surgery. One surprising and beautiful result of this study is a proof that a compact 3-manifold of positive scalar curvature is homeomorphic to a connected sum of some spherical 3-manifolds and some copies of $\mathbb{S}^1 \times \mathbb{S}^2$ (See also [**BBB$^+$10**] and [**MT07**]). Its generalization to the non-compact case is due to Bessières, Besson, and Maillot [**BBM11**].

For the non-compact case, the first result shown by Gromoll and Meyer [**GM69**] was that a complete non-compact 3-manifold of positive sectional curvature is homeomorphic to $\mathbb{R}^3$.

The next step in the development of this subject is due to Schoen and Yau [**SY82**] who used minimal surfaces theory and the splitting theorem [**CG71**] to study the topology of 3-manifolds. They proved that a complete non-compact 3-manifold of positive Ricci curvature is homeomorphic to $\mathbb{R}^3$.

Although all of these works are very impressive, they still left the open question (See Problem 27 in [**Yau82**]):

How to classify non-compact 3*-manifolds with positive scalar curvature, up to diffeomorphism?*

The goal now is to find all obstructions and to characterize all open 3-manifolds with positive scalar curvature. Although Gromov-Lawson [**GL83**] and Schoen-Yau [**SY82**] gave several topological obstructions, all those obstruction classes both vanish for contractible 3-manifolds.

Let us consider contractible 3-manifolds. For example, $\mathbb{R}^3$ admits a complete metric g_1 of positive scalar curvature, where

$$g_1 = \sum_{i=1}^{3}(dx_i)^2 + (\sum_{i=1}^{3} x_i dx_i)^2.$$

So far, it is the only known contractible 3-manifold which admits a complete metric of positive scalar curvature. This suggests the following question:

Is any complete contractible 3*-manifold of positive scalar curvature homeomorphic to* $\mathbb{R}^3$ *?*

A complete contractible 3-manifold of uniformly positive scalar curvature (i.e. its scalar curvature is bounded away from zero) is homeomorphic to $\mathbb{R}^3$. It was first proved by Gromov and Lawson [**GL83**]. Recently, it was generalized by Chang, Weinberger and Yu [**CWY10**], to contractible 3-manifolds whose scalar curvature is uniform positive outside a compact set. Using minimal surfaces theory, we further generalize it.

THEOREM *A*. (See Theorem 3.3.12 and Theorem 1.1 in [**Wan19c**]) Assume that (M^3, g) is a contractible complete 3-manifold. If there exists a number $\alpha \in (-\infty, 2)$ such that

$$\liminf_{r(x)\to\infty} r^{\alpha}(x)\kappa(x) > 0,$$

where $\kappa(x)$ is the scalar curvature of (M, g) and $r(x)$ is the distance function from some point $0 \in M$ to x, then M^3 is diffeomorphic to $\mathbb{R}^3$.

The proof follows the argument of Gromov and Lawson (See Corollary 10.9 in [**GL83**]).

0.1. Contractible 3-manifolds

Thurston's Geometrisation conjecture [**Per02a**, **Per02b**, **Per03**] (See also [**BBB$^+$10**] and [**MT07**]) shows that for a compact 3-manifold, its topology is fully determined by its

homotopy type. However, the topological structure of contractible 3-manifolds is much more complicated. For example, the Whitehead manifold (constructed in [**Whi35**]) is a contractible 3-manifold but not homeomorphic to $\mathbb{R}^3$.

In order to explain the construction of the Whitehead manifold, let us introduce the concept of a meridian curve. A meridian $\gamma \subset \partial N$ of a closed solid torus N is an embedded closed curve which is null-homotopic in N but not contractible in ∂N. A meridian disc $(D, \partial D) \subset (N, \partial N)$ of the solid torus N is an embedded disc whose boundary is a meridian of N. (See Definition 1.2.1)

The Whitehead manifold is constructed from the Whitehead link. Recall that the Whitehead link is a link with two components illustrated in the following figure:

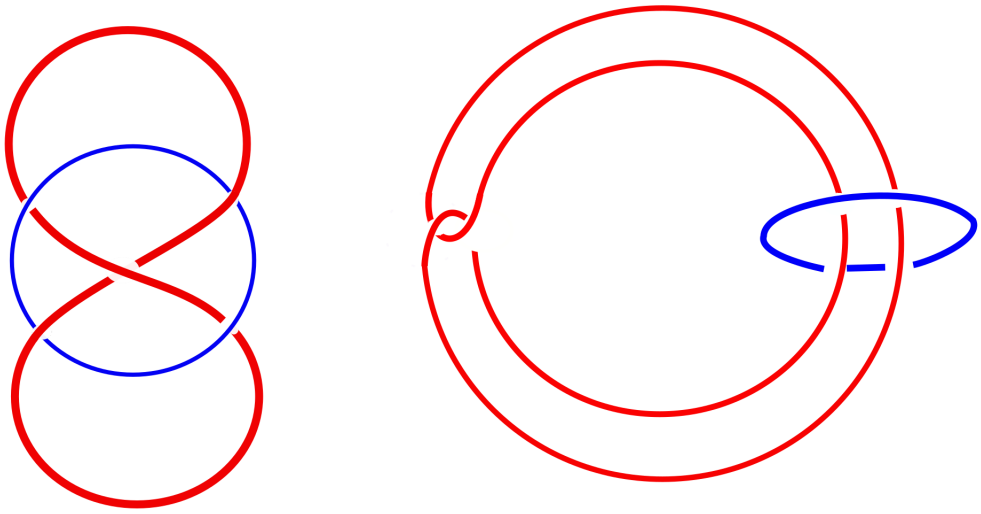


FIGURE 0.1.

Choose a closed unknotted solid torus T_1 in $\mathbb{S}^3$. Its complement inside $\mathbb{S}^3$ is another solid torus. Take a second solid torus T_2 inside T_1 so that the core K_2 of T_2 forms a Whitehead link with any meridian of T_1 as in the following figure.

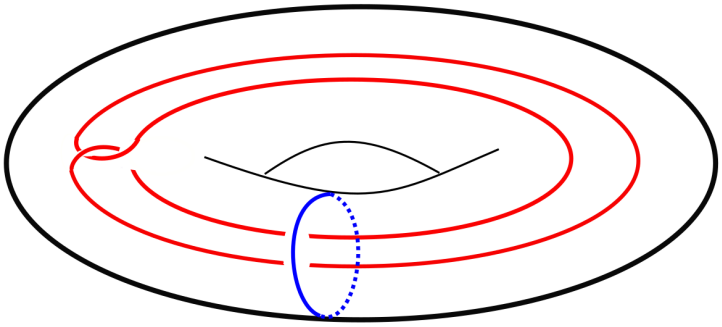


FIGURE 0.2.

The solid torus T_2 is unknotted in $\mathbb{S}^3$. Then, embed T_3 inside T_2 in the same way as T_2 lies into T_1 and so on infinitely many times. Define the set $T_\infty = \bigcap_{k=1}^{\infty} T_k$, called the Whitehead continuum.

The Whitehead manifold is defined as $Wh := \mathbb{S}^3 \backslash T_\infty$ which is a non-compact 3-manifold without boundary.

REMARK. Since each T_k is unknotted in $\mathbb{S}^3$, its complement N_k is a solid torus. Therefore, the Whitehead manifold is an increasing union of solid tori $\{N_k\}_k$. Each N_k is embedded inside N_{k+1} in the same way as T_2 lies in T_1. This follows from the symmetry of the Whitehead link.

Variation on the construction, like changing the knot at each step k, gives a family of so-called genus one 3-manifold, introduced in [**MJ62**]. Their construction is involved with the *geometric index*.

If $N' \subset N$ are solid tori, the *geometric index*, $I(N', N)$, of N' in N is equal to the minimal number of points of the intersection of the core of N' with a meridian disc of N. A genus one 3-manifold is the ascending union of solid tori $\{N_k\}$, so that for each k, $N_k \subset \mathrm{Int} N_{k+1}$ and the geometric index of N_k in N_{k+1} is not equal to zero. (See Definition 1.3.8, Definition 1.3.12 and [**GRW18**])

For example, Wh is a contractible genus one 3-manifold. The geometric index $I(N_k, N_{k+1})$ equals two for each k, where N_k is illustrated as above. Remark that $\mathbb{R}^3$ is not genus one but genus zero, since it is an increasing union of 0-handlebodies (i.e. 3-balls).

An interesting question is whether the Whitehead manifold admits a complete metric of positive scalar curvature. In this thesis, we answer negatively:

THEOREM B_1. (See Theorem 1.1 of [**Wan19a**]) The Whitehead manifold has no complete metric of positive scalar curvature.

This result can be generalized to the genus one case.

THEOREM B_2. (See Theorem 1.2 of [**Wan19a**]) No contractible genus one 3-manifold has a complete metric of positive scalar curvature.

Combining with Kazdan's work [**Kaz82**], we generalize these results to the nonnegative scalar curvature.

COROLLARY. (See Corollary 6.4.3) No contractible genus one 3-manifold admits a complete metric of non-negative scalar curvature.

The existence of complete metrics of positive scalar curvature is related with *the fundamental group at infinity*. The fundamental group π_1^∞ at infinity of a connected space is the inverse limit of the fundamental groups of complements of compact subsets (See Definition 1.1.9).

The triviality of the fundamental group at infinity is not equivalent to the simply-connectedness at infinity(See Definition 1.1.7). For example, the Whitehead manifold is not simply-connected at infinity but its fundamental group at infinity is trivial.

We prove the following:

THEOREM C. (See Theorem 1.1 of [**Wan19b**]) A contractible 3-manifold with positive scalar curvature and trivial π_1^∞ is homeomorphic to $\mathbb{R}^3$.

This result can also be generalised to the non-negative scalar curvature.

COROLLARY. (See Corollary 6.4.4) A contractible 3-manifold with non-negative scalar curvature and trivial π_1^∞ is homeomorphic to $\mathbb{R}^3$.

However, there are uncountably many mutually non-homeomorphic contractible 3-manifolds with non-trivial π_1^∞. In Chapter 1.3, we construct such a manifold and show that this manifold has no complete metric of positive scalar curvature (See Theorem 1.3.15).

0.2. The idea of the proof of Theorem B_2

It is classical that minimal surfaces theory gives topological information about 3-manifolds. This fact appeared in the articles of Schoen and Yau [**SY82**, **SY79b**, **SY79a**] as well as Gromov and Lawson's [**GL83**] and various other works.

For the proof of Theorem B_2, we argue by contradiction. Suppose that (M, g) is a complete Riemannian manifold of positive scalar curvature, where $M := \bigcup_k N_k$ is a contractible genus one 3-manifold and the family $\{N_k\}_k$ of solid tori is assumed as in Theorem 1.3.13.

0.2.1. Minimal surfaces and Limits. As in [**SY82**] and [**GL83**], our first step is to construct minimal surfaces. Choose $\gamma_k \subset \partial N_k$ as a meridian of N_k (See Definition 1.2.1). Roughly, it is spanned by an embedded stable minimal disc Ω_k. Its existence is ensured by the result of Meeks and Yau (See [**MIY80**, **MIY82**] or Theorem 3.1.8) when the boundary ∂N_k is mean convex.

Let us consider the simplest case when Ω_k converges to a connected stable minimal surface Σ.

On the one side, we show that the number of connected components of $\Omega_k \cap N_1$ intersecting N_0 goes to infinity as k goes to infinity (See Chapter 2.1). Therefore, there are infinitely many connected components of $\Sigma \cap N_1$ intersecting N_0. By a result of Meeks and Yau (See Theorem 3.2.7 anc [**MIY80**]), each of these components contains a definitive amount of area. Hence, $\Sigma \cap N_1$ has infinite area.

On the other side, since (M, g) has positive scalar curvature, not only Σ is conformally diffeomorphic to $\mathbb{R}^2$ (See Corollary 3.3.11), but also its geometry is constrained by the so-called extrinsic Cohn-Vossen inequality:

THEOREM D. (See Theorem 3.3.10 and [**Wan19a**]) Let $\Sigma^2 \subset (M^3, g)$ be a complete (non-compact) immersed stable minimal surface. If the complete manifold (M^3, g) has non-negative scalar curvature ($\kappa(x) \geqslant 0$), then

$$\int_\Sigma \kappa(x) + 1/2|A|^2 dv \leqslant 2\pi\chi(\Sigma)$$

where $|A|^2$ is the square norm of the second fundamental form of Σ. Moreover, if $\kappa > 0$ and Σ is embedded, then Σ is a properly embedded plane.

Since the scalar curvature is bounded away zero on N_1, this is in contradiction with the infinite area contained in $\Sigma \cap N_1$.

If ∂N_k is not mean convex, we modify the metric in a smaller tubular neighborhood of ∂N_k so that for the new metric, it becomes mean convex. Then Ω_k is stable minimal for the new metric and for the original away from the neighborhood ∂N_k, (for example, near N_k), which is sufficient for our proof.

0.2.2. Properties of the limit surface. Generally, Ω_k sub-converges to a minimal lamination $\mathscr{L} := \bigcup_{t\in\Gamma} L_t$ (that is, a disjoint union of some embedded minimal surfaces) instead of a single surface. It may have infinitely many components. However, each leaf L_t of $\mathscr{L}$ is a complete (non-compact) stable minimal surface (See Theorem 4.2.3). Since (M, g) has positive scalar curvature, it is homeomorphic to $\mathbb{R}^3$(See Corollary 3.3.11).

The geometry of each leaf is influenced by the extrinsic Cohn-Vossen inequality (See Theorem D) as well as by a topological property of M, called *Property P* (See Definition 2.1.3). These two aspects tell us that the lamination $\mathscr{L}$ has *the Vanishing property* for $\{N_k\}_k$. That is to say,

there is a positive integer k_0 such that for any $k \geqslant k_0$ and any $t \in \Gamma$, any circle in $L_t \cap \partial N_k$ is null-homotopic in ∂N_k.

The reason is as follows: Suppose that there exists a sequence $\{k_n\}$ of increasing integers and a sequence $\{L_{t_n}\}$ of leaves so that for each n, $L_{t_n} \cap \partial N_{k_n}$ has at least one non-nullhomotopic circle in ∂N_{k_n}. Similar to the property of the sequence $\{\Omega_k\}_k$ (Property P), we know that the number of connected components of $L_{t_n} \cap N_1$ intersecting N_0 goes to infinity as k_n goes to infinity (See Definition 2.1.3 and Theorem 2.1.6).

The sequence $\{L_{t_n}\}$ sub-converges to some leaf L_{t_∞} in the lamination $\mathscr{L}$ with finite multiplicity. The sub-convergence is ensured by a result of Schoen [**Sch83**] (See Lemma 4.1.5) and the extrinsic Cohn-Vossen inequality (See Theorem D). Therefore, $L_{t_\infty} \cap N_1$ has infinitely many components intersecting N_0. As in the above case, each component has a definite amount of area. The extrinsic Cohn-Vesson inequality gives a contradiction.

Let us explain how to deduce a contradiction from the Vanishing property. We show that for any $k \geqslant k_0$, $\Omega_k \cap \partial N_{k_0}$ contains a closed curve which is not null-homotopic in ∂N_{k_0} (See Lemma 1.3.11). Roughly speaking, these non-nullhomotopic circles will sub-converge to some closed curve in $\mathscr{L} \cap \partial N_{k_0}$ which is not contractible in ∂N_{k_0}. It follows the fact that Ω_k sub-converges to the lamination $\mathscr{L}$. Therefore, some leaf of $\mathscr{L}$ has a non-contractible circle in ∂N_{k_0}. This is in contradiction with the above Vanishing property of $\mathscr{L}$.

0.3. π_1^∞ and the Vanishing property

0.3.1. Handlebodies and Property H. Let (M, g) be a complete contractible 3-manifold of positive scalar curvature. It is an increasing union of closed handlebodies $\{N_k\}$ (See Theorem 1.1.12).

In the following, we consider that M is not homeomorphic to $\mathbb{R}^3$. We may assume that none of the N_k is contained in a 3-ball (i.e. homeomorphic to a unit ball in $\mathbb{R}^3$) in M (See Remark 1.1.8).

In the genus one case, the family $\{N_k\}$ has several good properties. For example, the maps $\pi_1(\partial N_k) \to \pi_1(\overline{M\backslash N_k})$ and $\pi_1(\partial N_k) \to \pi_1(\overline{N_k\backslash N_0})$ are both injective (See Lemma 1.3.10). These properties are crucial and necessary in the study of the existence of complete metrics of positive scalar curvature. In general, the family $\{N_k\}$ may not have the above properties.

For example, the map $\pi_1(\partial N_0) \to \pi_1(\overline{M\backslash N_0})$ may not be injective. To overcome it, we use topological surgeries on N_0 and find a new handlebody to replace it. Precisely, we use the loop lemma to find an embedded disc $(D, \partial D) \subset (\overline{M\backslash N_0}, \partial N_0)$ whose boundary is a non-contractible simple curve in ∂N_0. The new handlebody is obtained from N_0 by attaching a closed tubular neighborhood $N_\epsilon(D)$ of D in $\overline{M\backslash N_0}$.

We repeatedly use topological surgeries on each N_k to obtain a new family $\{R_k\}_k$ of closed handlebodies with the following properties, called *Property H* (See Definition 2.2.5):

(1) the map $\pi_1(\partial R_k) \to \pi_1(\overline{R_k\backslash R_0})$ is injective for $k > 0$;
(2) the map $\pi_1(\partial R_k) \to \pi_1(\overline{M\backslash R_k})$ is injective for $k \geqslant 0$;
(3) each R_k is contractible in R_{k+1} but not contained in a 3-ball in M ;
(4) there exists a sequence of increasing integers $\{j_k\}_k$, such that $\pi_1(\partial R_k \cap \partial N_{j_k}) \to \pi_1(\partial R_k)$ is surjective.

REMARK. If M is not homeomorphic to $\mathbb{R}^3$, the existence of such a family is ensured by Theorem 2.2.6. It is not unique. In addition, the union of such a family may be not equal to M.

For example, if $M := \bigcup_k N_k$ is a contractible genus one 3-manifold, the family $\{N_k\}$ (assumed as in Theorem 1.3.13) satisfies the above Property H (See Lemma 2.10 in [**Wan19a**] or Lemma 1.3.10).

0.3.2. The Vanishing property. In the genus one case, the geometry of a stable minimal surface is constrained by the geometric index (See *Property P* in [**Wan19a**] or Definition 2.1.3). In the higher genus case, the behavior of a stable minimal surface is related to the fundamental group at infinity.

In order to clarify their relationship, let us introduce a geometric property, called *the Vanishing property*. First, we consider a complete contractible 3-manifold (M, g) of positive scalar curvature which is not homeomorphic to $\mathbb{R}^3$. As indicated above, there is an increasing family $\{R_k\}_k$ of closed handlebodies with Property H.

A complete embedded stable minimal surface $\Sigma \subset (M, g)$ is called to satisfy *the Vanishing property* for the family $\{R_k\}_k$ if there is a positive integer $k(\Sigma)$ so that for $k \geqslant k(\Sigma)$, any circle in $\Sigma \cap \partial R_k$ is nullhomotopic in ∂R_k (See Chapter 5).

If a complete stable minimal surface does not satisfy the Vanishing property for $\{R_k\}_k$, it gives a non-trivial element in $\pi_1^\infty(M)$(See Lemma 5.2.1). As a consequence, if π_1^∞ is trivial, any complete stable minimal surface in M has the Vanishing property for $\{R_k\}_k$ (See Corollary 5.2.2).

0.3.3. The idea of the proof of Theorem C. We argue by contradiction. Suppose that a complete contractible 3-manifold (M, g) with positive scalar curvature and trivial $\pi_1^\infty(M)$ is not homeomorphic to $\mathbb{R}^3$.

Before constructing minimal surfaces, let us introduce a notation from 3-dimensional topology.

A *system of meridians* of a handlebody N is a collection of g distinct meridians $\{\gamma^l\}_{l=1}^g$ with the property that $\partial N \backslash \coprod_{l=1}^g \gamma^l$ is homeomorphic to an open disc with some closed subdiscs removed (See Lemma 1.2.7). Its existence is ensured by Lemma 1.2.7.

Let $\{N_k\}_k$ and $\{R_k\}_k$ be as above. Since N_0 is not contained in a 3-ball (See Remark 1.1.8), the genus of N_k is greater than zero. The handlebody N_k has a system of meridians $\{\gamma_k^l\}_{l=1}^{g(N_k)}$. Roughly , there are $g(N_k)$ disjoint area-minimizing discs $\{\Omega_k^l\}_l$ with $\partial\Omega_k^l = \gamma_k^l$. Their existence is ensured by the works of Meeks and Yau [**MIY80**,**MIY82**] (See Theorem 6.28 of [**CM11**]) when the boundary ∂N_k is mean convex.

Let us explain their existence. We construct these discs by induction on l.

When $l = 1$, there is an embedded area-minimizing disc $\Omega_k^1 \subset N_k$ with boundary γ_k^1 (See [**MIY80**, **MIY82**] or Theorem 6.28 of [**CM11**]).

Suppose that there are l disjointly embedded stable minimal discs $\{\Omega_k^i\}_{i=1}^l$ with $\partial\Omega_k^i = \gamma_k^i$. Our target is to construct a stable minimal surface Ω_k^{l+1} with boundary γ_k^{l+1}.

Let us consider the Riemannian manifold $(T_{k,l}, g|_{T_{k,l}})$, where $T_{k,l} := N_k \backslash \coprod_{i=1}^l \Omega_k^l$. It is a handlebody of genus $g(N_k) - l$. For example, see the following figure.

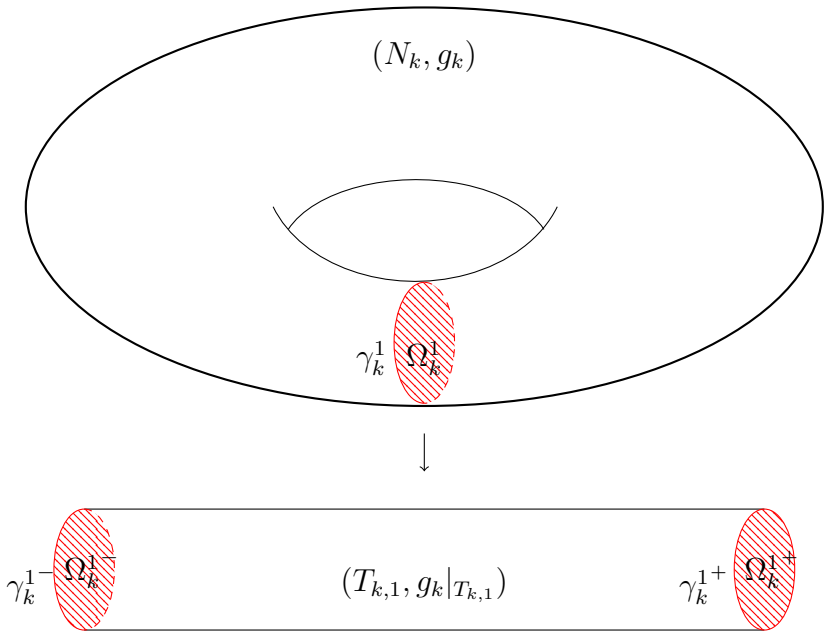


FIGURE 0.3.

The boundary of $(T_{k,l}, g|_{T_{k,l}})$ consists of $\partial N_k \backslash \coprod_{i=1}^l \gamma_k^i$ and some disjoint discs $\{\Omega_k^{i\,-}\}_{i=1}^l$ and $\{\Omega_k^{i\,+}\}_{i=1}^l$. The two discs $\Omega_k^{i\,-}$ and $\Omega_k^{i\,+}$ both come from the same minimal disc

Ω_k^i. Therefore, the mean curvature of the boundary of $(T_{k,l}, g|_{T_{k,l}})$ is non-negative. (See Chapter 4.2)

In addition, $\{\gamma_k^i\}_{i>l}$ is a system of meridians of the handlebody $(T_{k,l}, g|_{T_{k,l}})$. Then, we use the result of Meeks and Yau to find an embedded stable minimal surface $\Omega_k^{l+1} \subset T_{k,l}$ with boundary γ_k^{l+1}. These discs $\{\Omega_k^i\}_{i=1}^{l+1}$ are disjoint in N_k. This finishes the inductive construction.

As in the genus one case, if ∂N_k is not mean convex, we can deform the metric in a small neighborhood of it so that it becomes mean convex.

Define the lamination $\mathscr{L}_k := \coprod_l \Omega_k^l$ (i.e. a disjoint union of embedded surfaces). We show that each lamination $\mathscr{L}_k$ intersects the compact set R_0 (Corollary 1.2.8). According to Colding-Mincozzi's theory (See Appendix B of [**CM04**]), the sequence $\{\mathscr{L}_k\}_k$ subconverges to a lamination $\mathscr{L} := \bigcup_{t\in\Lambda} L_t$ in (M, g) (See Theorem 4.2.3). Note that each leaf L_t is a complete (non-compact) stable minimal surface.

As indicated above, since (M, g) has positive scalar curvature and $\pi_1^\infty(M)$ is trivial, each leaf L_t in $\mathscr{L}$ has the Vanishing property for $\{R_k\}_k$ (See Lemma 5.2.1 and Corollary 5.2.2). Furthermore, the lamination $\mathscr{L}$ also satisfies the Vanishing property (See Corollary 5.2.4). That is to say,

there exists a positive integer k_0 such that for any $k \geqslant k_0$ and any $t \in \Lambda$, any circle in $L_t \cap \partial R_k$ is nullhomotopic in ∂R_k.

The reason is described as follows.

We argue by contradiction. Suppose that there exists a sequence $\{k_n\}_n$ of increasing integers and a sequence $\{L_{t_n}\}$ of leaves in $\mathscr{L}$ satisfying that $L_{t_n} \cap \partial R_{k_n}$ has at least one non-nullhomotopic circle(s) in ∂R_{k_n} for each n.

The sequence $\{L_{t_n}\}$ smoothly subconverges to some leaf in $\mathscr{L}$. For our convenience, we may assume that the sequence $\{L_{t_n}\}$ converges to the leaf L_{t_∞}. The leaf L_{t_∞} satisfies the Vanishing property. That is to say, there is a positive integer $k(L_{t_\infty})$ such that for $k \geqslant k(L_{t_\infty})$, any circle $\partial R_k \cap L_{t_\infty}$ is nullhomotopic in ∂R_k.

However, since $L_{t_n} \cap \partial R_{k_n}$ has some non-null-homotopic circle in ∂R_{k_n}, we know that for $k_n > k(L_{t_\infty})$, $L_{t_n} \cap \partial R_{k(L_{t_\infty})}$ has a meridian of $R_{k(L_{t_\infty})}$ (See Remark 2.2.7 and Corollary 1.2.6). These meridians of $R_{k(L_{t_\infty})}$ will converge to a meridian of $R_{k(L_{t_\infty})}$ which is contained in $L_{t_\infty} \cap \partial R_{k(L_{t_\infty})}$. This is in contradiction with the last paragraph.

Let us explain how to deduce a contradiction from the Vanishing property of $\mathscr{L}$.

We show that if N_k contains R_{k_0} (for k large enough), then $\mathscr{L}_k \cap \partial R_{k_0}$ contains at least one meridian of R_{k_0} (See Corollary 1.2.8). As in the above case, these meridians of R_{k_0} will sub-converge to a non-contractible circle in $\mathscr{L} \cap \partial R_{k_0}$. The Vanishing property of $\mathscr{L}$ gives a contradiction.

0.4. Closed Aspherical 4-manifolds

A manifold M is called aspherical if it is path-connected and all its higher homotopy groups vanish (i.e. $\pi_k(M)$ is trivial for $k \geqslant 2$). The class of aspherical manifolds contains all hyperbolic manifolds and all manifolds with non-positive curvature.

An interesting question posed by Geroch is whether the torus $\mathbb{T}^n$, $n \geqslant 3$, carries a metric of positive scalar curvature. This question was settled by Gromov-Lawson [**GL83**] and Schoen-Yau [**SY79b**, **SY82**, **SY17**]. Generally, it is conjectured that

CONJECTURE. No compact aspherical manifold has a metric of positive scalar curvature.

This conjecture was proved for 3-manifolds by Gromov and Lawson [**GL83**]. In dimension four, it is confirmed for 4-manifolds which contains incompressible surfaces [**GL83**].

In this thesis, we prove that

THEOREM F. No closed aspherical 4-manifolds with non-zero first Betti number has a metric of positive scalar curvature.

Note that there is a closed aspherical 4-manifold whose first Betti number vanishes (See [**RT05**]).

We argue by contradiction. Suppose that there is a compact aspherical 4-manifold (M^4, g) of positive scalar curvature, where the first Betti number $b_1(M^4)$ is greater than zero.

Choose a circle $\gamma \subset M^4$ so that $[\gamma]$ has infinite order in $H_1(M^4, \mathbb{Z})$. We use the Poincaré duality to find a class $u \in H_3(M^4, \mathbb{Z})$ with $< u, [\gamma] >= 1$. A theorem of Fleming-Federer (See [**FF60**] or Chapter 7 of [**Sim83**]) tells us that there is a volume-minimizing hypersurface Σ^3 in this class. Therefore, the intersection number of γ and Σ^3 is equal to one.

Since (M, g) has positive scalar curvature, then Σ^3 admits a metric of positive scalar curvature (See Proposition 3.3.5). The manifold Σ^3 is homeomorphic to a connected sum of spherical 3-manifolds and some copies of $\mathbb{S}^1 \times \mathbb{S}^2$ (See [**Per02a**, **Per02b**, **Per03**], [**BBB$^+$10**] and [**MT07**]).

Because $\pi_2(M)$ and $\pi_3(M)$ are both trivial, then the spherical part of Σ and the 2-spheres in Σ are homotopic to a point in M^4. That is to say, Σ is homotopic to a wedge sum of some circles in M (these circles come from the $\mathbb{S}^2 \times \mathbb{S}^1$'s part of Σ^3). Because $\dim(M) = 4$, we see that the intersection number of Σ and γ equals zero, which is in contradiction with the intersection number of γ and Σ.

0.5. Organization of the thesis

The plan of this thesis is as follows:

For the first part, we discuss contractible 3-manifolds and related topological properties. In Chapter 1, we recall related background from 3-manifolds, such as simply-connectedness at infinity, the fundamental group at infinity and handlebodies. Subsequently, we discuss the topological structure of contractible 3-manifolds and derive some notations such as meridians of a handlebody, a system of meridians. Finally, we give some examples of 3-manifolds such as the Whitehead manifold and genus one 3-manifolds.

In Chapter 2, we start with the embedded discs in the Whitehead manifold. An interesting fact is that the behavior of these discs is influenced by the geometric index. Their relation is suggested by Theorem 2.1.2. Based on this relation, we introduce a

new property, called *Property P* and we show that any contractible genus one 3-manifold satisfies this property (See Theorem 2.1.6).

Generally, a contractible 3-manifold may not satisfy Property P. The reason is that it may be made up of some handlebodies of higher genus. Some handles in these handlebodies may make no contribution to its topology and yield technical difficulties. To overcome it, we introduce two types of surgeries. Using these surgeries, we find a new family of handlebodies with good properties, called *Property H* (See Definition 2.2.5).

In the second part of the thesis, we focus on minimal surfaces and related convergence theories. In Chapter 3, we recall some notations such as the so-called first and second variation formulas, Morse index and the stable condition (See Chapter 3.1.1). Then we discuss Plateau's problem(See Chapter 3.1.2).

Subsequently, we focus on the local properties of minimal surfaces, including the strong maximal principle (See Corollary 3.2.3) and the monotonicity formula (See Proposition 3.2.5).

Then, we study the topology of stable minimal hypersurfaces. These hypersurfaces are characterized by the first eigenvalue of the stable operator (See Lemma 3.3.1 and Theorem 3.3.4). In a manifold of positive scalar curvature, there are many topological constraint for stable minimal surfaces. For example, if a complete 3-manifold has nonnegative scalar curvature, a complete stable minimal surface in it satisfies the extrinsic Cohn-Vossen inequality (See Corollary 3.3.6 and Theorem 3.3.10). As a consequence, we give a new proof of the topological classification of stable minimal surfaces in a 3-manifold with nonnegative scalar curvature (See Corollary 3.3.11 and [**SY82**]). Finally, as an application of minimal surfaces theory, we give the proof of Theorem *A* (See Theorem 3.3.12).

In Chapter 4, we discuss the convergence theory of minimal surfaces. We begin with the convergence of minimal surface equations (See Lemma 4.1.1). It can be generalized to the Riemannian case. Therefore we get a compactness theorem for minimal surfaces (See Theorem 4.1.4).

Next, we discuss the convergence without area estimate. In this case, the limit is a minimal lamination (i.e. a disjoint union of some embedded discs) instead of a single surface. We recall the minimal lamination theory of Colding-Minicozzi (See Appendix B of [**CM04**]). Then we construct a required family of minimal laminations in a contractible 3-manifold. Their limit is a stable minimal lamination. Each leaf is a complete minimal surface. If the manifold has positive scalar curvature, it is a properly embedded plane (See Theorem *D*). As an application, we give a new proof of the topological classification of contractible 3-manifolds with uniformly positive scalar curvature (See Corollary 4.2.7 and [**GL83**]).

For the third part, we give the complete proofs of the main theorems. In Chapter 5, we introduce the *Vanishing property* and study its relationship with the fundamental group at infinity, which is suggested by Lemma 5.2.1. Note that, in the genus one case, Property P implies the Vanishing property (See Theorem 5.1.1).

In Chapter 6, we reduce the proof of the main theorems to a cover lemma (See Lemma 6.1.3). For the proof of this lemma, we use the Vanishing property of the lamination (constructed in Chapter 4.2) to define a set S (See Definition 6.3.1). Positivity of the scalar curvature implies the finiteness of S (See Lemma 6.3.4 and Lemma 6.3.5). We use

the finiteness to prove the cover lemma. Finally, we discuss several related questions and further research about 3-manifolds.

In the fourth part of the thesis, we discuss the existence of metrics of positive scalar curvature over a compact aspherical 4-manifold. Together with a result of Perelman [**Per02a**, **Per02b**, **Per03**], we give a proof of Theorem F. Finally, we talk about several further questions about 4-manifolds.

Introduction (français)

La géométrie riemannienne vise à étudier les variétés riemanniennes qui sont des variétés lisses à structures métriques. Une des questions fondamentales est de comprendre la relation entre la courbure, définie localement, et les propriétés globales des variétés lisses. Le résultat le plus ancien est le théorème classique de Gauss et Bonnet, qui relie la courbure au nombre d'Euler, un invariant topologique. Ce théorème implique, par exemple, que toute surface compacte du genre $g > 0$ n'a pas de métrique à courbure positive.

En dimension supérieure, l'existence de mtriques à courbure positive devient beaucoup plus compliquée parce qu'il existe plusieurs obstructions topologiques pour qu'une variété lisse ait une métrique complète à courbure positive.

Nous prenons ensuite 3-variétés pour expliquer ce fait.

(M^3, g)	cas compact	cas non compact
$K > 0$	$\mathbb{S}^3/\Gamma$	$\mathbb{R}^3$
$Ric > 0$		$\mathbb{R}^3$
$Scal > 0$	$(\#_{i=1}^k \mathbb{S}^3/\Gamma_i)\#(\#_{j=1}^l \mathbb{S}^1 \times \mathbb{S}^2)$	?

TABLE 2.

Pour une 3-variété riemannienne compacte, il existe une solution unique en temps court au flot de Ricci (normalisé), introduit par Hamilton [**Ham82**]. Si la variété a une courbure de Ricci positive, la solution en temps court peut être étendue en la solution en temps long. La limite de ce flot est une métrique à courbure sectionnelle constante. Autrement dit, si une 3-variété compacte a courbure de Ricci positive, elle est homéomorphe au quotient $\mathbb{S}^3/\Gamma$ de la sphère $\mathbb{S}^3$ par un sous-groupe fini $\Gamma \subset O(4)$. Un tel quotient est appelé une 3-variété sphérique.

La contribution majeure suivante au sujet a été apportée par Perelman [**Per02a**, **Per02b**, **Per03**], qui a développé le flot de Ricci avec chirurgie. Un résultat surprenant et magnifique de cette étude est la preuve qu'une 3-variété compacte á courbure scalaire positive est homéomorphe à une somme connexe de certaines 3-variétés sphériques et de copies de $\mathbb{S}^1 \times \mathbb{S}^2$ (Voir aussi [**BBB$^+$10**] et [**MT07**]). Sa généralisation au cas non compact est due à Bessières, Besson et Maillot [**BBM11**].

Pour le cas non compact, les premiers résultats montrés par Gromoll et Meyer [**GM69**] sont qu'une 3-variété complète non compacte à courbure sectionnelle positive est homéomorphe à $\mathbb{R}^3$ (Voir [**GM69**]).

L'étape suivante du développement de ce sujet est due à Schoen et Yau [**SY82**] qui ont utilisé la théorie des surfaces minimales et le théorème de splitting [**CG71**] pour étudier la topologie des 3-variétés . Ils ont prouvé qu'une 3-variété complète non compacte à courbure de Ricci positive est homéomorphe à $\mathbb{R}^3$.

Bien que toutes ces travauw soient trés impressionnants, ils ont toujours laissé la ouverte question (voir le problème 27 dans [**Yau82**]):

Comment classifier les 3*-variétés non compactes à courbure scalaire positive, à difféomorphisme près ?*

Le but est de trouver toutes les obstructions et de caractriser toutes les 3-variètés ouvertes à courbure scalaire positive. Bien que Gromov-Lawson [**GL83**] et Schoen-Yau [**SY82**] aient donné plusieurs obstructions topologiques, toutes ces classes d'obstruction disparaissent pour les 3-variétés contractiles.

Considérons des 3-variétés contractiles. Par exemple, $\mathbb{R}^3$ possède une métrique complète g_1 à courbure scalaire positive, où

$$g_1 = \sum_{i=1}^{3}(dx_i)^2 + (\sum_{i=1}^{3} x_i dx_i)^2.$$

Jusqu'à présent, c'est la seule 3-variété contractile connue qui admet une métrique complète à courbure scalaire positive. Ceci suggère la question suivante :

Est-ce qu'une 3*-variété complète contractile à courbure scalaire positive est homéomorphe à* $\mathbb{R}^3$ *?*

Une 3-variété complète contractile à courbure scalaire uniformément positive (c'est-à-dire que sa courbure scalaire est minorée par une constante strictement positive) est homéomorphe à $\mathbb{R}^3$. Cela a été prouvé pour la premire fois par Gromov et Lawson [**GL83**]. Récemment, ce résultat a été généralisé par Chang, Weinberger et Yu [**CWY10**] à des 3-variétés contractile dont la courbure scalaire est uniformément positive à l'extérieur d'un ensemble compact. En utilisant la théorie des surfaces minimales, nous généralisons davantage.

Théroème 1 (=Theorem A) Supposons que (M^3, g) est une 3-variété complète contractile. S'il existe un réel $\alpha \in (-\infty, 2)$ tel que

$$\liminf_{r(x)\to\infty} r^{\alpha}(x)\kappa(x) > 0,$$

o $\kappa(x)$ est la courbure scalaire de (M, g) et $r(x)$ est la fonction de distance d'un point $0 \in M$ à x, alors M^3 est difféomorphe à $\mathbb{R}^3$.

La preuve suit les arguments de Gromov et Lawson (voir le corollaire 10.9 dans [**GL83**]).

0.6. 3-variétés contractiles

La conjecture de géométrisation, formulée par William Thurston [**Per02a**, **Per02b**, **Per03**] (voir aussi [**BBB$^+$10**] et [**MT07**]) énonce que pour la topologie d'une 3-variété compacte est entièrement déterminée par son type d'homotopie. Cependant, la structure topologique des 3-variétés contractiles est trés complique. Par exemple, la variété Whitehead (construite dans [**Whi35**]) est une 3-variété contractile qui n'est pas homéomorphe à $\mathbb{R}^3$.

Pour expliquer la construction de la variété de Whitehead, introduisons le concept de méridien. Un méridien $\gamma \subset \partial N$ d'un tore solide fermé N est une courbe fermée plongée qui est homotopiquement triviale dans N mais non contractile dans ∂N. Un disque méridien $(D, \partial D) \subset (N, \partial N)$ d'un tore solide N est un disque plongé dont la frontière est un méridien de N. (Voir Définition 1.2.1).

La variété de Whitehead est construite à partir de l'entrelacs de Whitehead. Rappelons que l'entrelacs de Whitehead est un entrelacs á deux composantes comme illustr sur la figure suivante:

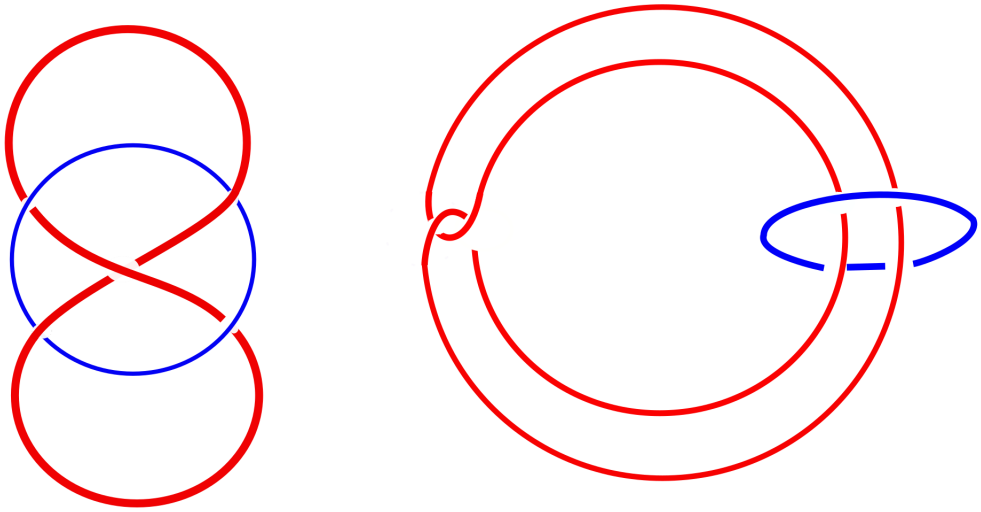


FIGURE 0.4.

Choisissons un tore solide fermé T_1 qui est non noué dans $\mathbb{S}^3$. L'intérieur du tore solide dans $\mathbb{S}^3$ est un autre tore solide. Prenons un deuxième tore solide T_2 à l'intérieur de T_1 de sorte que l'intérieur de K_2 forme un entrelacs de Whitehead avec un méridien de T_1 comme sur la figure suivante.

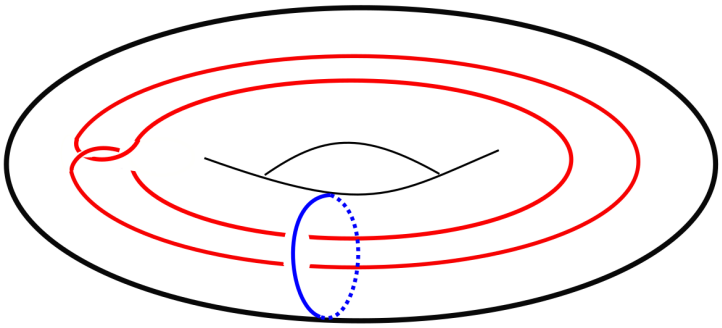


FIGURE 0.5.

Le tore solide T_2 est non nou dans $\mathbb{S}^3$. Ensuite, on plonge T_3 dans T_2 de la même manière que T_2 se trouve dans T_1 et ainsi de suite un nombre infini de fois. Définissons l'ensemble $T_\infty = \bigcap_{k=1}^{\infty} T_k$, appelé le continuum de Whitehead.

La variété de Whitehead est définie comme suit: $Wh := \mathbb{S}^3 \backslash T_\infty$, qui est une 3-variété non compacte.

Remarque Puisque chaque T_k est non noué dans $\mathbb{S}^3$, son complémentaire N_k est un tore solide. Par conséquent, la variété de Whitehead est une union croissante de tores solides $\{N_k\}_k$. Chaque N_k est plongé dans N_{k+1} de la même manière que T_2 dans T_1. Cela découle de la symétrie de l'entrelacs de Whitehead.

Une variation de la construction, comme changer le nœud à chaque étape k, donne une famille de ce qu'on appelle les 3-variétés de genre un, introduite dans [**MJ62**]. La construction est reliée à l'*indice géométrique*.

Si $N' \subset N$ est un tore solide, l'*indice géométrique*, $I(N', N)$, de N' dans N est égal au nombre minimal de points de l'intersection de l'intérieur de N' avec un disque mridien de N. Une 3-variété de genre un est une union croissante de tores solides $\{N_k\}$ telle que pour chaque k, $N_k \subset \text{Int } N_{k+1}$, et que l'indice géométrique de N_k dans N_{k+1} ne soit pas égal à zéro. (Voir Définition 1.3.8, Définition 1.3.12 et [**GRW18**]).

Par exemple, Wh est une 3-variété contractile de genre un. L'indice géométrique $I(N_k, N_{k+1})$ est égal à deux pour chaque k, où N_k est illustr comme ci-dessus. Remarquons que $\mathbb{R}^3$ n'est pas de genre un, mais de genre zéro, puisqu'il s'agit d'une union croissante de 0-corps à anses. (c'est-à-dire de 3-boules).

Une question intéressante est de savoir si la variété de Whitehead admet une métrique complète de courbure scalaire positive. Dans cette thèse, nous répondons négativement :

Théroème 2 (=Theorem B_1) La variété de Whitehead n'a pas de métrique complète á courbure scalaire positive.

Ce résultat peut être généralisé au cas du genre un.

Théroème 3 (=Theorem B_2) Une 3-variété contractile genre un n'a pas de métrique complète á courbure scalaire positive.

En combinant le travail de Kazdan [**Kaz82**], nous généralisons ces résultats à la courbure scalaire non négative.

Corollarie (voir Corollaire 6.4.3)Une 3-variété contractile genre un n'a pas de métrique complète á courbure scalaire non négative

L'existence de métriques complètes à courbure scalaire positive est liée au *groupe fondamental à l'infini*. Le groupe fondamental à l'infini π_1^∞ d'une 3-variété est la limite projective des groupes fondamentaux des complémentaires de sous-ensembles compacts (voir Définition 1.1.9).

La trivialité du groupe fondamental à l'infini n'est pas équivalente pas à la connexité simple à l'infini. Par exemple, la variété de Whitehead n'est pas simplement connexe à l'infini mais son groupe fondamental à l'infini est trivial.

Nous prouvons le résultat suivant :

Théroème 4 (=Theorem C) Une 3-variété contractile à courbure scalaire positive et π_1^∞ trivial est homéomorphe à $\mathbb{R}^3$.

Ce résultat peut également être généralisé à la courbure scalaire non négative.

Corollarie (voir Corollaire 6.4.4)Une 3-variété contractile à courbure scalaire non négative et π_1^∞ trivial est homéomorphe à $\mathbb{R}^3$.

Cependant, il existe une quantité indénombrable de 3-variétés contractiles deux à deux non homéomorphes dont le π_1^∞ n'est pas trivial. Dans le Chapitre 1.3, nous construisons une telle variété et montrons que cette variété n'a pas de métrique complète de courbure scalaire positive (voir Théorème 1.3.15).

0.7. L'idée de la preuve du Théorème 3

Il est classique que la théorie des surfaces minimales donne des informations topologiques sur les 3-variétés. Ce fait est apparu dans les articles de Schoen et Yau [**SY82**, **SY79b**, **SY79a**] ainsi que de Gromov et Lawson cite GL et divers autres travaux.

Pour la preuve du théorème 3, raisonnons par l'absurde. Supposons que (M, g) soit une 3-variété riemannienne complète à courbure scalaire positive, où $M := \bigcup_k N_k$ est une 3-variété contractile de genre un et la famille $\{N_k\}_k$ de tores solides est supposée comme dans Théorème 1.3.13.

0.7.1. Surfaces minimales et limites. Comme dans [**SY82**] et [**GL83**], notre première étape consiste à construire des surfaces minimales. Choisissons $\gamma_k \subset \partial N_k$ comme méridien de N_k (voir Définition 1.2.1). En gros, il borde un disque minimal stable plongé Ω_k. Son existence est assurée par le résultat de Meeks et Yau (voir [**MIY80**, **MIY82**] ou le théorème 3.1.8) lorsque la frontière ∂N_k est mean convex.

Considérons le cas le plus simple où Ω_k converge vers une surface minimale stable connexe Σ.

D'une part, nous montrons que le nombre de composantes connexes de $\Omega_k \cap N_1$ intersectant N_0 tend vers l'infini quand k tend vers l'infini (voir Chapitre 2.1). Par conséquent, il existe une infinité de composantes connexes de $\Sigma \cap N_1$ intersectant N_0. Par un résultat

de Meeks et Yau (voir Théorème 3.2.7 et [**MIY80**]), chacune de ces composantes contient une certaine quantité d'aire. Ainsi, $\Sigma \cap N_1$ a une aire infinie.

D'autre part, puisque (M, g) a courbure scalaire positive , non seulement Σ est conformément difféomorphe à $\mathbb{R}^2$ (voir Corollaire 3.3.11), mais sa géométrie est également contrainte par l'inégalité extrinsèque de Cohn-Vossen:

Théroème 5 (=Theorem *D*) Soit $\Sigma^2 \subset (M^3, g)$ une surface minimale complète immergée (non compacte). Si la variété complète (M^3, g) a courbure scalaire positive ou nulle $(\kappa(x) \geqslant 0)$, alors

$$\int_\Sigma \kappa(x) + 1/2|A|^2 dv \leqslant 2\pi\chi(\Sigma).$$

où $|A|^2$ est le carré de la norme de la seconde forme fondamentale de Σ. De plus, si $\kappa > 0$ et Σ est plongée, alors Σ est un plan proprement plongé.

Puisque la courbure scalaire est minorée par une constante strictement positive sur N_1, cela est en contradiction avec la surface infinie contenue dans $\Sigma \cap N_1$.

Si ∂N_k n'est pas mean convex, on modifies la métrique dans un plus petit voisinage tubulaire de ∂N_k afin que, pour la nouvelle métrique, elle devienne mean convex.. Alors Ω_k est minimal stable pour la nouvelle métrique, et pour la métrique originale il l'est loin du voisinage ∂N_k (par exemple, près de N_k), ce qui est suffisant pour notre preuve.

0.7.2. Propriétés de la surface limite. En général, Ω_k sous-converge vers une lamination minimale $\mathscr{L} := \bigcup_{t\ in\Gamma} L_t$ (c'est-à-dire une union disjointe de certaines surfaces minimales plongées). Elle peut avoir une infinité de composantes. Cependant, chaque feuille L_t de $\mathscr{L}$ est une surface minimale stable complète (non compacte) (voir Théorème 4.2.3). Puisque (M, g) a une métrique à courbure scalaire positive, elle est homéomorphe à $\mathbb{R}^3$ (voir Corollaire 3.3.11).

La géométrie de chaque feuille est influencée par l'inégalité extrinsèque de Cohn-Vossen (cf. Théorème 5) ainsi que par une propriété topologique de M, appelée *Propriété P* (voir Définition 2.1.3). Ces deux aspects nous indiquent que la lamination $\mathscr{L}$ a la *propriété d'annulation* pour $\{N_k\}_k$. C'est-à-dire,

il existe un entier positif k_0 tel que pour tout $k \geqslant k_0$ et tout $t \in \Gamma$, tout cercle dans $L_t \cap \partial N_k$ est homotopiquement trivial à ∂N_k.

La raison en est la suivante : supposons qu'il existe une suite décroissante d'entiers $\{k_n\}$ et une suite $\{L_{t_n}\}$ de feuilles telles que pour chaque n , $L_{t_n} \cap \partial N_{k_n}$ a au moins un cercle non homotopiquement trivial dans ∂N_{k_n}. Comme pour la propriété de la suite $\{\Omega_k\}_k$ (Propriété P), nous savons que le nombre de composantes connexes de $L_{t_n} \cap N_1$ intersectant N_0 tend vers l'infini quand k_n tend vers l'infini (Voir Définition 2.1.3 et Théorème 2.1.6).

La suite $\{L_{t_n}\}$ sous-converge vers une feuille $L_{t\ infty}$ dans la lamination $\mathscr{L}$ avec une multiplicit finie. La sous-convergence est assurée par un résultat de Schoen [**Sch83**] (voir Lemme 4.1.5) et l'inégalité extrinsèque de Cohn-Vossen (voir Théorème 5). Par conséquent, $L_{t_\infty} \cap N_1$ a une infinité de composantes intersectant N_0. Comme dans le cas ci-dessus, chaque composante a une aide donnée. L'inégalité extrinsèque de Cohn-Vesson donne une contradiction.

Expliquons comment déduire une contradiction avec la propriété d'annulation. Nous montrons que pour tout $k \geqslant k_0$, $\Omega_k \cap \partial N_{k_0}$ contient une courbe ferme qui n'est pas homotopiquement triviale dans ∂N_{k_0} (voir le Lemme 1.3.11). Grosso modo, ces cercles non homotopiquement triviaux vont sous-converger vers une courbe fermée dans $\mathscr{L} \cap \partial N_{k_0}$ qui n'est pas contractile dans ∂N_{k_0}. Il s'ensuit que Ω_k sous-converge vers la lamination $\mathscr{L}$. Par conséquent, certaines feuilles de $\mathscr{L}$ ont un cercle non contractile dans ∂N_{k_0}. Ceci est en contradiction avec la propriété d'annulation ci-dessus de $\mathscr{L}$.

0.8. π_1^∞ et la propriété d'annulation

0.8.1. Corps à anses et propriété H. Soit (M, g) une 3-variété complète contractile à courbure scalaire positive. Il s'agit d'une union croissante de corps à anses $\{N_k\}$ (voir Théorème 1.1.12).

Dans ce qui suit, nous considérons que M n'est pas homéomorphe à $\mathbb{R}^3$. Nous pouvons supposer qu'aucun des N_k n'est contenu dans une 3-boule (c'est-à-dire homéomorphe à une boule unitaire dans $\mathbb{R}^3$) dans M (voir Remarque 1.1.8). Cela joue un rôle crucial dans notre argument.

Dans le cas du genre un, la famille $\{N_k\}$ a plusieurs bonnes propriétés. Par exemple, les applications $\pi_1(\partial N_k) \to i_1(\overline{M \backslash N_k})$ et $\pi_1(\partial N_k) \to \pi_1(\overline{N_k \backslash N_0})$ sont toutes les deux injectives (voir le Lemme 1.3.10). Ces propriétés sont cruciales et nécessaires dans l'étude de l'existence de métriques complètes à courbure scalaire positive. En général, la famille $\{N_k\}$ peut ne pas avoir les propriétés ci-dessus.

Par exemple, l'application $\pi_1(\partial N_0) \to \pi_1(\overline{M \backslash N_0})$ peut ne pas être injective. Pour surmonter cette difficulté, nous utilisons des chirurgies topologiques sur N_0 et nous trouvons un nouveau corps à anses pour le remplacer. Précisément, nous utilisons le lemme de la boucle pour trouver un disque plong $(D, \partial D) \subset (\overline{M \backslash N_0}, \partial N_0)$ dont le bord est une courbe simple non contractile dans ∂N_0. Le nouveau corps à anses est obtenu à partir de N_0 en attachant un voisinage tubulaire fermé $N_\epsilon(D)$ de D dans $\overline{M \backslash N_0}$.

Nous utilisons à plusieurs reprises des chirurgies topologiques sur chaque N_k pour obtenir une nouvelle famille $\{R_k\}_k$ de corps à anses fermés avec les propriétés suivantes, appelée *Propriété H* (voir Définition 2.2.5):

(1) l'application $\pi_1(\partial R_k) \to \pi_1(\overline{R_k \backslash R_0})$ est injective pour $k > 0$;
(2) l'application $\pi_1(\partial R_k) \to \pi_1(\overline{M \backslash R_k})$ est injective pour $k \geqslant 0$;
(3) chaque R_k est contractile dans R_{k+1} mais n'est pas contenu dans une 3-boule de M;
(4) il existe une suite d'entiers croissante $\{j_k\}_k$, telle que $\pi_1(\partial R_k \cap \partial N_{j_k}) \to \pi_1(\partial R_k)$ soit surjective.

Remarque Si M n'est pas homéomorphe à $\mathbb{R}^3$, l'existence d'une telle famille est assure par le Thorme 2.2.6. Elle n'est pas unique. De plus, l'union d'une telle famille peut ne pas correspondre à M.

Par exemple, si $M := \bigcup_k N_k$ est une 3-variété contractile de genre un, la famille $\{N_k\}$ (supposée comme dans le Théorème 1.3.13) vérifie la propriété ci-dessus (propriété H) (voir le lemme 2.10 dans [**Wan19a**] ou Lemme 1.3.10).

0.8.2. La proriété d'annulation. Dans le cas du genre un, la géométrie d'une surface minimale stable est contrainte par l'indice géométrique (voir *Propriété P* dans [**Wan19a**] ou la Définition 2.1.3). Dans le cas du genre supérieur, le comportement d'une surface minimale stable est lié au groupe fondamental à l'infini.

Afin de clarifier leur relation, introduisons une propriété géométrique, appelée la *propriété d'annulation*. Premièrement, nous considérons une 3-variété complte contractile (M, g) qui n'est pas homéomorphe à $\mathbb{R}^3$. Comme indiqué ci-dessus, il existe une famille croissante $\{R_k\}_k$ de corps à anses avec la propriété H.

On dit qu'une surface minimale stable complète plongée $\Sigma \subset (M, g)$ satisfait la *propriété d'annulation* pour la famille $\{R_k\}_k$ s'il existe un entier positif $k(\Sigma)$ tel que pour $k \geqslant k(\Sigma)$, tout cercle dans $\Sigma \cap \partial R_k$ soit homotopiquement trivial dans ∂R_k (Voir Chapitre 5).

Si une surface minimale stable complète ne satisfait pas la propriété d'annulation pour $\{R_k\}_k$, elle donne un élément non trivial dans $\pi_1^\infty(M)$ (voir le Lemme 5.2.1). En conséquence, si π_1^∞ est trivial, toute surface minimale stable complète dans M a la propriété d'annulation pour $\{R_k\}_k$ (voir le Corollaire 5.2.2).

0.8.3. L'idée de la preuve du théorème 4. Raisonnons par l'absurde. Supposons qu'une 3-variété (M, g) complète contractile à courbure scalaire positive et $\pi_1^\infty(M)$ trivial, ne soit pas homéomorphe à $\mathbb{R}^3$.

Avant de construire des surfaces minimales, introduisons une notation de topologie en dimension 3.

Un *système de méridiens* d'un corps à anses N est une collection de g méridiens distincts $\{\gamma^l\}_{l=1}^g$ tels que $\partial N \backslash \amalg_{l=1}^g \gamma^l$ soit hoéomorphe à un disque ouvert privé de certains sous-disques fermés (voir le Lemme 1.2.7). Son existence est assurée par le Lemme 1.2.7.

Soit $\{N_k\}_k$ et $\{R_k\}_k$ comme ci-dessus. Puisque N_0 n'est pas contenu dans une 3-boule (voir la remarque 1.1.8), le genre de N_k est suprieur zro. Le corps à anses N_k a un système de méridiens $\{\gamma_k^l\}_{l=1}^{g(N_k)}$. En gros, il existe $g(N_k)$ disques disjoints d'aire minimale $\{\Omega_k^l\}_l$ avec $\partial\Omega_k^l = \gamma_k^l$. Leur existence est assurée par les travaux de Meeks et Yau [**MIY80**, **MIY82**] (voir le théorème 6.28 de [**CM11**]) lorsque la frontière ∂N_k est mean convex.

Expliquons leur existence. Nous construisons ces disques par récurrence sur l.

Lorsque $l = 1$, il existe un disque plongé d'aire minimale $\Omega_k^1 \subset N_k$ avec $\partial\Omega_k^1 = \gamma_k^1$ (voir [**MIY80**, **MIY82**] ou le théorème 6.28 de [**CM11**]).

Supposons qu'il existe l disques stables plongés disjoints d'aire minimale $\{\Omega_k^i\}_{i=1}^l$ avec $\partial\Omega_k^i = \gamma_k^i$. Notre objectif est de construire une surface minimale stable Ω_k^{l+1} avec $\partial\Omega_k^{l+1} = \gamma_k^{l+1}$.

Considérons la variété riemannienne $(T_{k,l}, g|_{T_{k,l}})$, où $T_{k,l} := N_k \backslash \coprod_{i=1}^l \Omega_k^l$. C'est un corps à anses du genre $g(N_k) - l$. Par exemple, voir la figure suivante.

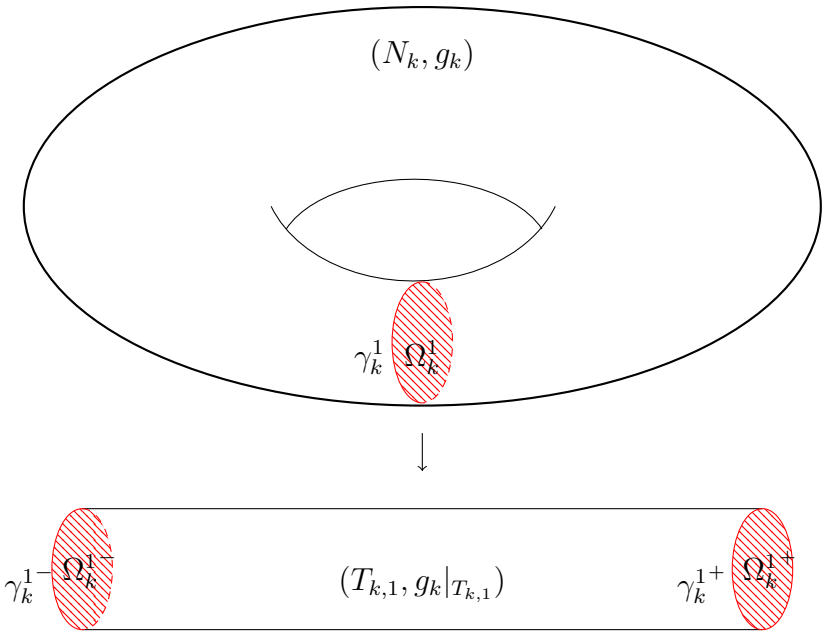


FIGURE 0.6.

La frontière de $(T_{k,l}, g|_{T_{k,l}})$ est constituée de $\partial N_k \backslash \coprod_{i=1}^{l} \gamma_k^i$ et certains disques disjoints $\{\Omega_k^{i\,-}\}_{i=1}^{l}$ et $\{\Omega_k^{i\,+}\}_{i=1}^{l}$. Les deux disques $\Omega_k^{i\,-}$ et $\Omega_k^{i\,+}$ proviennent tous du même disque minimal Ω_k^i. Par conséquent, la courbure moyenne de la frontière de $(T_{k,l}, g|_{T_{k,l}})$ est positive. (Voir Chapitre 4.2)

De plus, $\{\gamma_k^i\}_{i>l}$ est un système de méridiens du corps à anses $(T_{k,l}, g|_{T_{k,l}})$. Ensuite, nous utilisons le résultat de Meeks et Yau pour trouver une surface minimale stable plongée $\Omega_k^{l+1} \subset T_{k,l}$ dont le bord est γ_k^{l+1}. Ces disques $\{\Omega_k^i\}_{i=1}^{l+1}$ sont disjoints dans N_k. Ceci termine la construction par récurrence.

Comme dans le cas du genre un, si ∂N_k n'est pas mean convex, nous pouvons dformer la mtrique dans un petit voisinage de celle-ci pour qu'elle devienne mean convex.

Définissons la lamination $\mathscr{L}_k := \coprod_l \Omega_k^l$ (qui est une union disjointe de surfaces plongées). Nous montrons que chaque lamination $\mathscr{L}_k$ intersecte l'ensemble compact R_0 (Corollaire 1.2.8). Selon la théorie de Colding-Mincozzi (voir l'annexe B de [**CM04**]), la suite $\{\mathscr{L}_k\}_k$ sous-converge vers une laminiation $\mathscr{L} := \bigcup_{t\in\Lambda} L_t$ in (M, g) (voir le Théorème 4.2.3). Notons que chaque feuille L_t est une surface minimale stable complète (non compacte).

Comme indiqué ci-dessus, puisque (M, g) est a 3-variété complète à courbure scalaire positive et que $\pi_1^\infty(M)$ est trivial, chaque feuille L_t dans $\mathscr{L}$ a la propriété d'annulation pour $\{R_k\}_k$ (voir le Lemme 5.2.1 et le Corollaire 5.2.2). En outre, la lamination $\mathscr{L}$ satisfait la propriété d'annulation (voir le Corollaire 5.2.4). C'est-à-dire,

il existe un entier positif k_0 tel que pour tout $k \geqslant k_0$ et tout $t \in \Lambda$, n'importe quel cercle dans $L_t \cap \partial R_k$ soit homotopiquement trivial dans ∂R_k.

La raison est dcrite comme suit.

Raisonnons par l'absurde. Supposons qu'il existe une suite croissante $\{k_n\}_n$ d'entiers et une suite $\{L_{t_n}\}$ de feuilles dans $\mathscr{L}$ telles que $L_{t_n} \cap \partial R_{k_n}$ ait au moins un cercle non homotopiquement trivial dans ∂R_{k_n} pour chaque n.

La suite $\{L_{t_n}\}$ sous-converge vers une feuille de $\mathscr{L}$. Pour simplifier, nous pouvons supposer que la suite $\{L_{t_n}\}$ converge vers la feuille L_{t_∞}. La feuille $L_{t\ infty}$ satisfait la propriété d'annulation. C'est-à-dire qu'il existe un entier positif $k(L_{t_\infty})$ tel que pour $k \geqslant k(L_{t_\infty})$, tout cercle $\partial R_k \cap L_{t_\infty}$ soit homotopiquement trivial dans ∂R_k.

Cependant, comme $L_{t_n} \cap \partial R_{k_n}$ a un cercle non homotopiquement trivial dans $partial R_{k_n}$, nous savons que pour $k_n > k(L_{t_\infty})$, $L_{t_n} \cap \partial R_{k(L_{t_\infty})}$ a un méridien de $R_{k(L_{t_\infty})}$ (voir la remarque 2.2.7 et le corollaire 1.2.6). Ces méridiens de $R_{k(L_{t_\infty})}$ convergeront vers un méridien de $R_{k(L_{t_\infty})}$ contenu dans $L_{t_\infty} \cap \partial R_{k(L_{t_\infty})}$. Ceci est en contradiction avec le dernier paragraphe.

Expliquons comment déduire une contradiction de la propriété d'annulation de $\mathscr{L}$.

Nous montrons que si N_k contient R_{k_0} (pour k assez grand), alors $\mathscr{L}_k \cap \partial R_{k_0}$ contient au moins un méridien de R_{k_0} (voir le Corollaire 1.2.8). Comme dans le cas ci-dessus, ces méridiens de R_{k_0} convergeront vers un cercle non contractile dans $\mathscr{L} \cap \partial R_{k_0}$. La propriété d'annulation de $\mathscr{L}$ donne une contradiction.

0.9. 4-variétés fermées asphériques

Une variété M est dite asphérique s'il est connexe par arcs et si tous ses groupes d'homotopie suérieurs s'annulent (c'est-à-dire que $\pi_k(M)$ est trivial pour $k \geqslant 2$). La classe des variétés asphériques comprend toutes les variétés hyperboliques et toutes les variétés à courbure négative.

Une question intéressante est de savoir si le tore $\mathbb{T}^n$, $n \geqslant 3$, possède une métrique à courbure scalaire positive. Cette question a été traitée par Gromov-Lawson [**GL83**] et Schoen-Yau [**SY79b**, **SY82**, **SY17**]. En général, on conjecture que

Conjecture. Aucune variété compacte asphérique n'a de métrique à courbure scalaire positive.

Cette conjecture a été démontrée pour les 3-variétés par Gromov et Lawson [**GL83**]. En dimension quatre, elle est confirmée pour les 4-variété qui contiennent des surfaces incompressibles [**GL83**].

Dans cette thèse, nous prouvons que

Théroème 6 (=Theorem F) Aucune 4-variété asphérique fermée avec premier nombre de Betti non nul n'a de métrique à courbure scalaire positive.

Notons qu'il existe une 4-variété asphérique fermée dont le premier nombre de Betti est nul (Voir [**RT05**]).

Raisonnons par l'absurde. Supposons qu'il existe une 4-variété asphrique fermée (M^4, g) à courbure scalaire positive, dont le premier nombre de Betti $b_1(M^4)$ est strictement supérieur à zéro.

Choisissons un cercle $\gamma \subset M^4$ tel que $[\gamma]$ soit d'ordre infini dans $H_1(M^4, \mathbb{Z})$. Nous utilisons la dualité de Poincaré pour trouver une classe $u \in H_3(M^4, \mathbb{Z})$ avec $< u, [\ g] >= 1$. Un théorème de Fleming-Federer (voir [**FF60**] ou le chapitre 7 de [**Sim83**]) nous indique

qu'il existe une hypersurface Σ^3 minimisant le volume dans cette classe. Par conséquent, le nombre d'intersection de γ et Σ^3 est égal à un.

Puisque (M, g) a une courbure scalaire positive, alors Σ^3 admet une métrique de courbure scalaire positive (voir la proposition 3.3.5). La variété Σ^3 est homéomorphe à une somme connexe de 3-variété sphériques et de copies de $\mathbb{S}^1 \times \mathbb{S}^2$ (Voir [**Per02a**,**Per02b**, **Per03**], [**BBB$^+$10**] et [**MT07**]).

Comme $\pi_2(M)$ et $\pi_3(M)$ sont triviaux, la partie sphérique de Σ et les 2-sphères de Σ sont homotopes à un point dans M^4. C'est-à-dire que Σ est homotope à un bouquet de cercles dans M (ces cercles proviennent de la partie de $\mathbb{S}^2 \times \mathbb{S}^1$ dans Σ^3). Comme $\dim(M) = 4$, nous voyons que le nombre d'intersection de Σ et γ est égal à zéro, ce qui est en contradiction avec les nombres d'intersection de γ et Σ.

0.10. Organisation de la thèse

Le plan de cette thèse est le suivant:

Dans la première partie, nous discutons des 3-variétés contractile et des propriétés topologiques associées. Au Chapitre 1, nous formulons des rappels liés aux 3-variétés, comme la connexité simple à l'infini, le groupe fondamental à l'infini et les corps à anses. Par la suite, nous discutons de la structure topologique des 3-variétés contractiles et présentons quelques notions telles que les méridiens d'un corps à anses, ou encore les systèmes de méridiens. Enfin, nous donnons quelques exemples de 3-variétés, telles que les variétés de Whitehead et les 3-variétés de genre un.

Au chapitre 2, nous commenons par les disques plongés dans la variété de Whitehead. Un fait intéressant est que le comportement de ces disques est influencé par l'indice géométrique. Leur relation est suggérée par le Théoréme 2.1.2. Sur la base de cette relation, nous introduisons une nouvelle propriété, appelée *Propriété P*, et montrons que toute variété contractile de genre un satisfait cette propriété (voir Théorème 2.1.6).

En règle générale, une 3-variété contractible peut ne pas satisfaire la propriété P. La raison est que cette variété peut être composée de corps à anses de genre supérieur. Certaines anses peuvent ne pas contribuer à la topologie et engendrer des difficultés techniques. Pour surmonter cela, nous introduisons deux types de chirurgies. En utilisant ces chirurgies, nous trouvons une nouvelle famille de corps à anses avec de bonnes propriétés, appelée *Proprit H* (voir Définition 2.2.5).

Dans la deuxième partie de la thèse, nous nous concentrons sur les surfaces minimales et les théories de convergence associées. Au Chapitre 3, nous rappelons certaines notions telles que les formules dites de première et deuxième variation, l'indice de Morse et la condition de stabilité (Voir Chapitre 3.1.1). Ensuite, nous discutons du problème de Plateau (voir Chapitre 3.1.2).

Nous nous intéresserons ensuite aux propriétés locales des surfaces minimales, y compris le principe du maximum et la formule de monotonie (voir Proposition 3.2.5).

Ensuite, nous étudions la topologie des hypersurfaces minimales stables. Ces hypersurfaces sont caractérisées par la première valeur propre de l'opérateur stable (voir le Lemme 3.3.1 et le Théorème 3.3.4). Dans une variété à courbure scalaire positive, il existe de nombreuses contraintes topologiques sur les surfaces minimales stables. Par

exemple, si une 3-variété compléte a courbure scalaire positive ou nulle, une surface minimale stable complète vérifie l'inégalit extrinsèque de Cohn-Vossen (Voir Corollary 3.3.6 et Theorem 3.3.10). En conséquence, nous donnons une nouvelle preuve de la classification topologique des surfaces minimales stables dans une 3-variété à courbure scalaire positive ou nulle (voir le Corollaire 3.3.11 et [**SY82**]). Enfin, en tant qu'application de la théorie des surfaces minimales, nous donnons la preuve du Théorème 4.2.1 (voir Théorème 3.3.12).

Au Chapitre 4, nous discutons de la théorie de la convergence des surfaces minimales. Nous commenons par la convergence des équations de surfaces minimales (Voir Lemme 4.1.1). Cela peut être généralisé au cas riemannien. Par conséquent, nous obtenons un théorème de compacité pour les surfaces minimales (voir Théorème 4.1.4).

Ensuite, nous discutons de la convergence sans estimation d'aire. Dans ce cas, la limite est une lamination minimale (c'est-à-dire une union disjointe de disques plongés). Nous rappelons la thorie de la lamination minimale de Colding-Minicozzi (Voir l'Annexe B de [**CM04**]). Ensuite, nous construisons une famille requise de laminations minimales dans une 3-variété contractile. Leur limite est une lamination minimale stable. Chaque feuille est une surface minimale compléte. Si la variété a une courbure scalaire positive, il s'agit d'un plan proprement plongé (voir le Théorème 5). En guise d'application, nous donnons une nouvelle preuve de la classification topologique des 3-variétés contractiles à courbure scalaire uniformément positive (Voir Corollaire 4.2.7 et [**GL83**]).

Dans la troisième partie, nous donnons les preuves complètes des théorèmes principaux. Au Chapitre 5, nous introduisons la *propriété d'annulation* et étudions sa relation avec le groupe fondamental à l'infini. La relation est suggérée par le Lemme 5.2.1. Notez que, dans le cas du genre un, la propriété P implique la propriété d'annulation (voir Théorème 5.1.1).

Au Chapitre 6, nous réduisons la preuve des théorèmes principaux à un lemme de recouvrement (voir Lemme 6.1.3). Pour prouver ce lemme, nous utilisons la propriété d'annulation de la lamination (construit au Chapitre 4.2) pour définir un ensemble S (voir Définition 6.3.1). La positivité de la courbure scalaire implique la finitude de S (voir Lemme 6.3.4 et Lemme 6.3.5). Nous utilisons la finitude pour prouver le lemme de recouvrement. Enfin, nous discutons de plusieurs questions connexes et d'autres sujets de recherche sur les 3-variétés.

Dans la quatrième partie de la thèse, nous discutons de l'existence de métriques à courbure scalaire positive sur une 4-variété asphérique compacte. Avec un résultat de Perelman [**Per02a**, **Per02b**, **Per03**], nous donnons une preuve du Théorème 6. Enfin, nous abordons plusieurs autres questions sur les 4-variétés.

Part 1

3-Manifolds

CHAPTER 1

3-Manifolds

In this chapter, we review related background in geometric topology and algebraic topology.

We begin with several classical theorems in 3-manifolds, such as the loop lemma. Subsequently, we study the topological structures of contractible 3-manifolds. Then we introduce several notations (for example, effective meridians and a system of meridians) and discuss their topological properties.

Finally, we give several examples, such as genus one 3-manifolds.

1.1. Background

In this part, we recall several classical theorems for 3-manifolds and discuss the topological structure of contractible 3-manifolds.

1.1.1. Preliminary. A 3-manifold is irreducible if any embedded 2-sphere bounds a closed 3-ball (namely, it is homeomorphic to a closed unit ball in $\mathbb{R}^3$).

REMARK 1.1.1. We know from the so-called Alexander's theorem (See [Theorem 1.1, Page 1] of [**Hat00**]) that any embedded 2-sphere in $\mathbb{R}^3$ bounds an embedded 3-ball.

Further, the proof of Poincaré Conjecture [**Per02a**, **Per02b**, **Per03**] (See [**BBB**$^+$**10**] or [**MT07**]) tells that any contractible 3-manifold is irreducible.

It is well-known that there are many links between the geometric properties of 3-manifolds and homotopy theory, specially π_1. For example, the loop lemma.

LEMMA 1.1.2. *(See [Theorem 3.1, Page 54] of [**Hat00**])Let M be a 3-manifold with boundary ∂M, not necessarily compact or orientable. If there is a map $f : (D^2, \partial D^2) \to (M, \partial M)$ with the property that $f|_{\partial D^2}$ is not nullhomotopic in ∂M. Then there is an embedding h with the same property.*

REMARK 1.1.3. We may assume that $h(\text{Int } D^2) \subset \text{Int } M$. The reason is described below:

Let us consider a 1-sided open neighborhood $M_\epsilon \cong \partial M \times [0, \epsilon)$ of ∂M in M. Shrinking the image of f into $M(\epsilon) := M \backslash M_\epsilon$, we find a map $f_\epsilon : (D^2, \partial D^2) \to (M(\epsilon), \partial M(\epsilon))$ with the property that $f_\epsilon(\partial D^2)$ is not nullhomotopic in $\partial M(\epsilon)$. By Lemma 1.1.2 there is an embedding h_ϵ with the same property. Its image is contained in $(M(\epsilon), \partial M(\epsilon))$. Therefore, the image of h_ϵ is contained in Int M.

In addition, there is an embedded circle $\gamma \subset \partial M$ which is homotopic to $h_\epsilon(\partial D^2)$ in $\overline{M}_\epsilon$. There is an embedded annulus $A_\epsilon \subset \overline{M_\epsilon}$ joining γ and $h_\epsilon(\partial D^2)$. We have a map $h : (D^2, \partial D^2) \to (M, \partial M)$ so that its image is an embedded disc (i.e. the union of A_ϵ and the image of h_ϵ). It has the same property as f and $h(\text{Int } D^2) \subset \text{Int } M$.

One of tools for computing the fundamental group is the so-called Van-Kampen's theorem. It will be frequently used in the following.

THEOREM 1.1.4. *(Van-Kampen's Theorem, See [Theorem11.60, Page 396] of [**Rot12**]) Let K be a connected complex having connected subcomplexes L_1 and L_2 with $K = L_1 \cup L_2$. If $L_1 \cap L_2$ is connected, then $\pi_1(K)$ is the pushout of the data.*

$$\begin{array}{ccc} \pi_1(L_1) & \longrightarrow & \pi_1(K) \\ {\scriptstyle j_{1*}}\uparrow & & \uparrow \\ \pi_1(L_1 \cap L_2) & \xrightarrow{j_{2*}} & \pi_1(L_2) \end{array}$$

where $j_k : L_1 \cap L_2 \to L_k$ is the inclusion for $k = 1, 2$.

Moreover, $\pi_1(K)$ is isomorphic to $\pi_1(L_1) *_{\pi_1(L_1 \cap L_2)} \pi_1(L_2)$.

If the map $j_{k*} : \pi_1(L_1 \cap L_2) \to \pi_1(L_k)$ is injective for $k = 1, 2$, $\pi_1(K)$ can be written as a free product with amalgamation (See Chapter 11 of [**Rot12**]).

LEMMA 1.1.5. *(See [Theorem11.67, Page 404] of [**Rot12**]) Let B, A_1 and A_2 be groups. . Let $A_1 *_B A_2$ be the pushout of the following data.*

$$\begin{array}{ccc} A_1 & \xrightarrow{\lambda_1} & A_1 *_B A_2 \\ {\scriptstyle i_1}\uparrow & & {\scriptstyle \lambda_2}\uparrow \\ B & \xrightarrow{i_2} & A_2 \end{array}$$

If the map $i_k : B \to A_k$ is injective for $k = 1, 2$, one has

(1) the map λ_k is injective for $k = 1, 2$;

(2) if $A'_k = \lambda_k(A_k)$, then $< A'_1, A'_2 >= A_1 *_B A_2$ and $A'_1 \cap A'_2$ is isomorphic to B.

We now introduce several concepts about the disjoint closed curves in a disc.

DEFINITION 1.1.6. (See Definition 2.11 of [**Wan19a**]) Let $C := \{c_i\}_{i \in I}$ be a finite set of pairwise disjoint circles in the disc D^2 and $D_i \subset D^2$ the unique disc with boundary c_i. Consider the set $\{D_i\}_{i \in I}$ and define the partially ordered relation induced by the inclusion. For each maximal element D_j in $(\{D_i\}_{i \in I}, \subset)$, its boundary c_j is defined as a *maximal circle* in C. For each minimal element D_j, its boundary c_j is called a *minimal circle* in C.

1.1.2. Simply-connectedness at infinity and π_1^∞.

DEFINITION 1.1.7. A topological space M is *simply connected at infinity* if for any compact set $K \subset M$, there exists a compact set K' containing K so that the induced map $\pi_1(M \backslash K') \to \pi_1(M \backslash K)$ is trivial.

A result of Stallings [**Sta72**] and Remark 1.1.1 tell us that the only contractible and simple-connected at infinity 3-manifold is $\mathbb{R}^3$.

REMARK 1.1.8. If a contractible 3-manifold M is not homeomorphic to $\mathbb{R}^3$, it is not simply-connected at infinity. That is to say, there is a compact set $K \subset M$ so that for any compact set $K' \subset M$ containing K, the induced map $\pi_1(M \backslash K') \to \pi_1(M \backslash K)$ is not

trivial. We also have that the set K is not contained in a 3-ball in M. The reason is described below:

If a closed 3-ball B contains K, Theorem 1.1.4 shows that $\pi_1(M) \cong \pi_1(\overline{M\backslash B}) *_{\pi_1(\partial B)} \pi_1(B)$. In addition, $\pi_1(B)$ and $\pi_1(\partial B)$ are both trivial. Therefore, $\pi_1(\overline{M\backslash B}) \cong \pi_1(M)$ is trivial. That is to say, the map $\pi_1(M\backslash B) \to \pi_1(M\backslash K)$ is trivial. This is a contradiction.

Definition 1.1.9. The fundamental group at infinity π_1^∞ of a path-connected space is the inverse limit of the fundamental groups of complements of compact subsets.

For example, the fundamental group at infinity of any compact manifold is trivial. For any contractible n-manifold M^n, it is simply-connected at infinity if and only if $\pi_1^\infty(M^n)$ is trivial, when $n \geqslant 4$ (See [**CWY10**]). However, this result is not true in dimension 3. For example, the Whitehead manifold is not simply-connected at infinity but its fundamental group at infinity is trivial.

Remark 1.1.10. Let us consider a contractible 3-manifold M and an exhaustion $\{N_k\}_k$ of M. We have the following:

$$\pi_1^\infty(M) = \varprojlim_k \pi_1(M\backslash N_k) = \left\{ ([\gamma_k]) \in \prod_k \pi_1(M\backslash N_k) \text{ so that } (f_{k,j})_*([\gamma_k]) = [\gamma_j] \text{ for any } k \geqslant j \right\},$$

where $f_{k,j}$ is the inclusion from $M\backslash N_k$ to $M\backslash N_j$.

Therefore, $\pi_1^\infty(M)$ is non-trivial if and only if there exists a non-trivial element $([\gamma_k]) \in \prod_k \pi_1(M\backslash N_k)$ satisfying

1) for some k_0, the closed curve γ_{k_0} is non-contractible in $M\backslash N_{k_0}$;
2) for $k \geqslant k_0$, γ_k is homotopic to γ_k in $M\backslash N_{k_0}$.

That is to say, there is a compact set K and a family of closed curves $\{\alpha_n\}_n$ going to infinity with the following property: for each n

a) α_n is nullhomotopic in $M\backslash K$ for ;
b) α_n is homotopic to α_{n+1} in $M\backslash K$.

Note that this family of circles gives a non-trivial element in $\pi_1^\infty(M)$.

1.1.3. Handlebodies.

Definition 1.1.11. [Page 59, [**Rol03**]] A handlebody is any space obtained from the 3-ball D^3 (0-handle) by attaching g distinct copies of $D^2 \times [-1, 1]$ (1-handle) with the homeomorphisms identifying the $2g$ discs $D^2 \times \{\pm 1\}$ to $2g$ disjoint 2-disks on ∂D^3, all to be done in such a way that the resulting 3-manifold is orientable. The integer g is called the genus of the handlebody.

Remark that a handlebody of genus g is homeomorphic to a boundary connected sum of g solid tori. Therefore, its boundary is a compact surface of genus g. (See Page 59 in [**Rol03**])

From a result of McMillan [**MJ61**] and Remark 1.1.1, we know that:

Theorem 1.1.12. *[Page 511, Theorem 1] [**MJ61**] Any contractible 3-manifold can be written as an ascending union of handlebodies .*

REMARK 1.1.13. Let consider a contractible 3-manifold M. If it is not homeomorphic to $\mathbb{R}^3$, it can written as an increasing family of handlebodies $\{N_k\}$ satisfying that for each k,

- N_k is homotopically trivial in N_{k+1};
- none of the N_k is contained in a 3-ball (See Remark 1.1.8).

In the following, we consider a closed handlebody N in $\mathbb{S}^3$.

DEFINITION 1.1.14. A closed handlebody $N \subset \mathbb{S}^3$ of genus g is said to be unknotted in $\mathbb{S}^3$ if it complement in $\mathbb{S}^3$ is also a handlebody of genus g.

For example, an unknotted solid torus in $\mathbb{S}^3$.

1.2. Meridians

In the following, we consider a closed handlebody N.

DEFINITION 1.2.1. An embedded circle $\gamma \subset \partial N$ is called a *meridian* if γ is nullhomotpic in N, but not contractible in ∂N.

An embedded closed disc $(D, \partial D) \subset (N, \partial N)$ is called a *meridian disc* if its boundary is a meridian of N.

The disc D is a *splitting meridian disc*, if $N \backslash D$ is not connected. Its boundary is called a *splitting meridian*.

The disc D is a *non-splitting disc*, if $N \backslash D$ is connected. Its boundary is called a *non-splitting meridian*.

REMARK. Let γ be a meridian of N. If γ is a splitting meridian, it cuts ∂N into two components. The class $[\gamma]$ is equal to zero in $H_1(\partial N)$.

If γ is a non-splitting meridian, then $\partial N \backslash \gamma$ is connected. The class $[\gamma]$ is a non-trivial element in $H_1(\partial N)$.

LEMMA 1.2.2. *Let N' and N be two closed handlebodies with $N' \subset Int\ N$. If N' is homotopically trivial in N, then any non-splitting meridian of N' is non-trivial in $H_1(\overline{N \backslash N'})$ and any meridian of N is trivial in $H_1(\overline{N \backslash N'})$.*

PROOF. The Mayer-Vietoris sequence gives:

$$H_2(N) \to H_1(\partial N') \xrightarrow{i_1} H_1(N') \oplus H_1(\overline{N \backslash N'}) \xrightarrow{i_2} H_1(N) \to \hat{H}_0(\partial N'). \tag{1.2.1}$$

We know that $H_2(N)$ and $\hat{H}_0(\partial N')$ are both trivial. Then, the map $H_1(\partial N') \to H_1(N') \oplus H_1(\overline{N \backslash N'})$ is injective. As indicated above, any non-splitting meridian of N' is non-trivial in $H_1(\partial N')$. Therefore, it is not equal to zero in $H_1(\overline{N \backslash N'})$.

In the following, we will show that any meridian γ of N is trivial in $H_1(\overline{N \backslash N'})$.

Embed N into $\mathbb{S}^3$ as an unknotted handlebody. The set N' can be viewed as a handlebody in $\mathbb{S}^3$. The core $K' = \vee_{i=1}^{g'} \alpha_i$ of N' is a wedge sum of g' circles $\{\alpha_i\}_i$, where g' is the genus of N'.

Choose $\gamma_i' \subset \partial N'$ as a normal circle of α_i in $\mathbb{S}^3$. That is to say, the linking number of $\gamma_i' \amalg \alpha_j$ in $\mathbb{S}^3$ is equal to δ_{ij}. In addition, the kernel of the map $H_1(\partial N') \to H_1(N')$ is a linear subspace of dimension g', spanned by $\{[\gamma_i']\}_{i=1}^{g'}$.

Since N' is homotopically trivial in N, the map $H_1(N') \to H_1(N)$ is a zero map. Choose the element $(0, [\gamma]) \in H_1(N') \oplus H_1(\overline{N \backslash N'})$. Since γ is a meridian of N, the element is in the kernel of the map i_2. From the sequence (1.2.1), it is also contained in the image of i_1. Hence, $[\gamma]$ can be written as $\sum_i n_i[\gamma_i']$ in $H_1(\overline{N \backslash N'})$.

Claim: the coefficient n_i is equal to the linking number of $\alpha_i \amalg \gamma$.

From the Mayer-Vietoris sequence, the group $H_1(\mathbb{S}^3 \backslash K')$ is a free Abelian group of rank g' spanned by $\{[\gamma_i']\}_{i=1}^{g'}$. Hence, $[\gamma]$ is equal to $\sum_i n_i[\gamma_i']$ in $H_1(\mathbb{S}^3 \backslash K')$. Similarly, $H_1(\mathbb{S}^3 \backslash \alpha_i)$ is of rank one and generated by γ_i'. One has that

$$\begin{aligned} H_1(\overline{N \backslash N'}) &\to H_1(\mathbb{S}^3 \backslash K') \to H_1(\mathbb{S}^3 \backslash \alpha_i) \\ [\gamma] = \sum_i n_i[\gamma_i'] &\mapsto \quad \sum_i n_i[\gamma_i'] \mapsto n_i[\gamma_i'] \end{aligned}$$

That is to say, $[\gamma]$ is equal to $n_i[\gamma_i']$ in $H_1(\mathbb{S}^3 \backslash \alpha_i)$. From the definition of the linking number (See Page 132 in [**Rol03**]), n_i is the linking number of $\gamma \amalg \alpha_i$.

Since each $\alpha_i \subset N'$ is nullhomotopic in N, then the linking number of $\gamma \amalg \alpha_i$ is zero. Namely, n_i is equal to zero, for each i. Therefore, $[\gamma]$ is equal to zero in $H_1(\overline{N \backslash N'})$. □

1.2.1. The effective meridian. Consider two closed handlebodies N' and N with $N' \subset \text{Int } N$.

Definition 1.2.3. A meridian γ of N is called *an effective meridian relative to* N' if any meridian disc with boundary γ intersects the core of N'.

The handbody N is called *an effective handlebody relative to* N', if any meridian of N is an effective meridian relative to N'.

Note that if N' is contained in a 3-ball $B \subset \text{Int } N$, there is no effective meridian relative to N'.

Lemma 1.2.4. *Let N' and N be two closed handlebodies with $N' \subset$ Int N. The handlebody N is an effective handlebody relative to N' if and only if the map $\pi_1(\partial N) \to \pi_1(\overline{N \backslash N'})$ is injective.*

Proof. If N is not an effective handlebody relative to N', there is a meridian disc $(D, \partial D) \subset (N, \partial N)$ with $D \cap N' = \varnothing$. Therefore, the map $\pi_1(\partial N) \to \pi_1(\overline{N \backslash N'})$ is not injective.

If the map $\pi_1(\partial N) \to \pi_1(\overline{N \backslash N'})$ is not injective, we apply Lemma 1.1.2 to the 3-manifold $\overline{N \backslash N'}$. There is an embedded disc $(D', \partial D') \subset (\overline{N \backslash N'}, \partial N)$ whose boundary is not contractible in ∂N. As in Remark 1.1.3, we may assume that Int $D' \subset$ Int $(N \backslash N')$. We see that D' is a meridian disc with $D' \cap N' = \varnothing$. Therefore, N is not an effective handlebody relative to N'. □

Lemma 1.2.5. *Let N' and N be two closed handlebodies satisfying that 1) $N' \subset$ Int N and 2) $\pi_1(\partial N') \to \pi_1(\overline{N \backslash N'})$ is injective. If N is an effective handlebody relative to N', then any meridian disc $(D, \partial D) \subset (N, \partial N)$ contains a meridian of N'.*

The proof is the same as the proof of Lemma 2.12 in [**Wan19a**].

PROOF. Suppose that the closed meridian disc D intersects $\partial N'$ transversally where $\gamma := \partial D$ is a meridian of N. The intersection $D \cap \partial N'$ is a disjoint union of circles $\{c_i\}_{i\in I}$. Each c_i bounds a unique closed disc $D_i \subset \text{Int } D$.

Consider the set $C^{non} := \{c_i \mid c_i \text{ is not contractible in } \partial N'\}$ and the set $C^{max} = \{c_i | \ c_i$ is a maximal circle in $\{c_i\}_{i\in I}\}$.

We will show that C^{non} is nonempty and a minimal circle in C^{non} is a desired meridian.

Suppose the contrary that C^{non} is empty. Hence, each $c_i \in C^{max}$ is contractible in $\partial N'$ and bounds a disc $D_i' \subset \partial N'$. Consider the immersed disc

$$\hat{D} := (D \setminus \cup_{c_i\in C^{max}} D_i) \cup (\cup_{c_i\in C^{max}} D_i')$$

with boundary γ. Since $\hat{D} \cap \text{Int } N' = \varnothing$, we see that γ is contractible in $\overline{N\setminus N'}$.

However, Lemma 1.2.4 shows that the map $\pi_1(\partial N) \to \pi_1(\overline{N\setminus N'})$ is injective. That is to say, the circle γ is nullhomotopic in ∂N. This is in contradiction with our hypothesis that γ is non-trivial in $\pi_1(\partial N)$. We conclude that $C^{non} \neq \varnothing$.

In the following, we will prove that each minimal circle c_j in C^{non} is a required meridian. From Definition 1.2.1, it is sufficient to show that c_j is homotopically trivial in N'. Our strategy is to construct an immersed disc $\hat{D}_j \subset N'$ with boundary c_j.

Let $C_j := \{c_i \, | c_i \subset \text{Int } D_j \text{ for } i \in I\}$ and C_j^{max} be the set of maximal circles in C_j. We now have two cases: $C_j = \varnothing$ or $C_j \neq \varnothing$.

Case I: If C_j is empty, we consider the set $Z :=$ Int D_j and define the disc $\hat{D}_j$ as Int D_j.

Case II: If C_j is not empty, then C_j^{max} is also nonempty. From the minimality of c_j in C^{non}, each $c_i \in C_j^{max}$ is nullhomotopic in $\partial N'$ and bounds a disc $D_i'' \subset \partial N'$.

Define the set $Z :=$ Int $D_j \setminus \cup_{c_i\in C_j^{max}} D_i$ and the new disc $\hat{D}_j := Z \cup (\cup_{c_i\in C_j^{max}} D_i'')$ with boundary c_j.

Let us explain why $\hat{D}_j$ is contained in N'. In any case, $\partial N'$ cuts N into two connected components, $N\setminus N'$ and Int N'. The set Z is one of these components of Int $D_j\setminus\partial N'$. Therefore, it must be contained in Int N' or $N\setminus N'$.

If Z is in $N\setminus N'$, the disc $\hat{D}_j$ is contained in $\overline{N\setminus N'}$. Thus, c_j is contractible in $\overline{N\setminus N'}$. However, since the induced map $\pi_1(\partial N') \to \pi_1(\overline{N\setminus N'})$ is injective, then c_j is homotopically trivial in $\partial N'$. This contradicts the choice of $c_j \in C^{non}$. We conclude that Z is contained in Int N'.

Therefore, $\hat{D}_j$ is contained in N'. That is to say, c_j is null-homotopic in N'. However, $[c_j]$ is a non-trivial element in $\pi_1(\partial N')$. From Definition 1.2.1, we conclude that $c_j \subset D$ is a meridian of N'. This finishes the proof. □

As a consequence, we have

COROLLARY 1.2.6. *Let N' and N be two closed handlebodies in a contractible 3-manifold M satisfying that 1) $N' \subset$ Int N and 2) the map $\pi_1(\partial N') \to \pi_1(\overline{M\setminus N'})$ is injective. If an embedded circle $\gamma \subset \partial N$ is not nullhomotopic in $\overline{M\setminus N'}$, then any embedded disc $D \subset M$ with boundary γ contains a meridian of N'.*

The proof is the same as Lemma 1.2.5.

1.2.2. The system of meridians.

LEMMA 1.2.7. *For a closed handlebody N of genus g, there are g disjoint non-splitting meridians $\{\gamma^l\}_{l=1}^g$ so that $N\setminus \amalg_l N_{\epsilon_l}(D_l)$ is a closed 3-ball, where D_l is a closed meridian disc with boundary γ^l and $N_{\epsilon_l}(D_l)$ is an open neighborhood of D_l in N with small radius ϵ_l .*

The set of these meridians $\{\gamma^l\}_{l=1}^g$ is called *a system of the handlebody N of genus g*. In general, it is not unique.

PROOF. Pick any non-splitting meridian γ^1 of N. We use Lemma 1.1.2 to find an embedded disc $D_1 \subset N$.

As Remark 1.1.3, we may assume that Int $D_1 \subset$ Int N. The set $N_1 := N\setminus N_\epsilon(D_1)$ is a closed handlebody of genus $g-1$, where $N_{\epsilon_1}(D_1)$ is the open tubular neighborhood of D_1 in N with small radius ϵ_1. In particular, the map $\pi_1(\partial N \cap \partial N_1) \to \pi_1(\partial N_1)$ is surjective.

Choose a non-splitting meridian $\gamma^2 \subset \partial N \cap \partial N_1$ of N_1. By Lemma 1.1.2, there exists a meridian disc D_2 of $N_1 = N\setminus N_{\epsilon_1}(D_1)$. The set $N_2 := N\setminus N_{\epsilon_1}(D_1) \amalg N_{\epsilon_2}(D_2)$ is a closed handlebody of genus $g-2$, where $N_{\epsilon_2}(D_2)$ is an open tubular neighborhood of D_2 in N.

We repeat this process $g-2$ times and obtain g disjointly embedded discs $\{D_l\}$ so that $N\setminus \amalg_l N_{\epsilon_l}(D_l)$ is a handlebody of genus zero (a 3-ball). The boundaries $\{\gamma^l\}_{l=1}^g$ of these discs are g distinct meridians which are the required candidates in the assertion. □

COROLLARY 1.2.8. *Let $N \subset M$, $\{\gamma^l\}$ and $\{D_l\}$ be as in Lemma 1.2.7, where M is a 3-manifold without boundary. If $R \subset$ Int N is a closed handlebody satisfying that 1) it is not contained in a 3-ball in M; 2) $\pi_1(\partial R) \to \pi_1(\overline{M\setminus R})$ is injective, then $\partial R \cap \amalg_l D_l$ contains at least a meridian of R.*

The poof is also similar to the proof of Lemma 2.12 [**Wan19a**].

PROOF. We may assume that ∂R intersects $\amalg_l D_l$ transversally. The intersection $\partial R \cap \amalg D_l := \{\gamma\}_{\gamma\in C}$ has finitely many components. Let us consider the set $C^{non} := \{\gamma \in C$ is not contractible in $\partial R\}$.

Claim: C^{non} is nonempty.

We argue by contradiction. Suppose that C^{non} is empty. We have that any circle in $D_l \cap \partial R$ is contractible in ∂R. As in the proof of Lemma 1.2.5, we get a new disc in $\overline{N\setminus R}$ with boundary γ^l. Therefore, γ^l is null-homotopic in $\overline{N\setminus R}$.

We use Lemma 1.1.2 to find an embedded disc $D_1' \subset \overline{N\setminus R}$ with boundary γ^1. As in Remark 1.1.3, we may assume that Int $D_1' \subset$ Int $\overline{N\setminus R}$ (or $D_1' \subset N\setminus R$). Choose the open tubular neighborhood $N_{\epsilon_1'}(D_1')$ of D_1' in $N\setminus R$ with small radius ϵ_1'. The set $N_1' := N\setminus N_{\epsilon_1'}(D_1')$ is a closed handlebody of genus $g-1$ containing R.

In addition, for $l > 1$, γ^l is a non-splitting meridian of N_1' but contractible in $N\setminus(N_{\epsilon_1'}(D_1') \amalg R)$.

Repeating this process $g-1$ times, we obtain g embedded discs $\{D_l'\}_{l=1}^g$ so that

1) $R \cap \amalg_l N_{\epsilon_l'}(D_l') = \varnothing$;

2) The handlebody $N\setminus \amalg_l N_{\epsilon_l'}(D_l')$ is of genus zero (a closed 3-ball),

where $N_{\epsilon_l'}(D_l')$ is the open tubular neighborhood of D_l' in N with small radius ϵ_l'.

Therefore, R is contained in the 3-ball $N \backslash \amalg_l N_{\epsilon_l'}(D_l')$. This contradicts our hypothesis. The claim follows.

As in the proof of Lemma 1.2.5, we use the condition 2) to show that each minimal circle in C^{non} is a required meridian. □

1.3. Examples

In this part, we begin with Knot theory in a closed solid torus. Subsequently, we introduce several notations, such as the geometric index, the Whitehead manifold and contractible genus one 3-manifolds. In addition, we construct two non-homeomorphic contractible 3-manifolds whose fundamental group at infinity are both non-trivial.

1.3.1. Knots basic.

DEFINITION 1.3.1. A subset K of a 3-manifold X is a *knot* if K is homeomorphic with a circle $\mathbb{S}^1$. More generally, K is a *link* if K is homeomorphic with a disjoint union of one or some circle(s).

Two knots or links K and K' are *ambient isotopic* if there is a homeomorphism $h : X \to X$ such that (1) h is isotopic to the identity map; (2) $h(K) = K'$.

A knot K is called to be *trivial* (or *unknotted*) in X if there exists an embedded disc in X with boundary K.

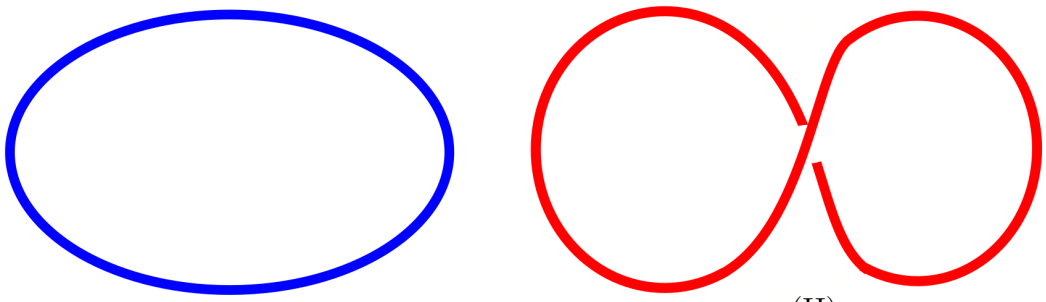


FIGURE 1.1.

For example, if X is $\mathbb{R}^3$ or $\mathbb{S}^3$, the knot (I) is ambient isotopic to the knot (II) in X (See Figure 1.1). These two knots are both trivial in $\mathbb{S}^3$.

If X is a torus $\mathbb{T}^2$, two knots K and K' are ambient isotopic if and only if $[K] = \pm[K']$ in $\pi_1(\mathbb{T}^2)$ (See [16.Theorem, Page 25] of [**Rol03**]).

We consider a closed solid torus N and a knot $K \subset N$. Embed N into $\mathbb{S}^3$ as an unknotted solid torus (See Definition 1.1.14). The knot K can be viewed as a knot in $\mathbb{S}^3$. The disjoint union $K \amalg \gamma$ is a link in $\mathbb{S}^3$, where γ is a meridian of N.

Recall that an n-component link $L \subset \mathbb{S}^3$ is unlinked if and only if there exist n disjointly embedded discs $D_i \subset \mathbb{S}^3$ so that $L = \amalg_i \partial D_i$.

REMARK 1.3.2. The knot K is trivial in N if and only if the link $K \amalg \gamma$ is unlinked in $\mathbb{S}^3$. The reason is as follows:

If K is trivial in N, there exists an embedded closed disc $D \subset N$ with boundary K. This disc is away from the knot γ in $\mathbb{S}^3$.

Since $N \subset \mathbb{S}^3$ is unknotted , the meridian γ is a trivial knot in $\mathbb{S}^3$. Note that $\mathbb{S}^3 \backslash D$ is homeomorphic to an open 3-ball. Then, γ is a trivial knot in $\mathbb{S}^3 \backslash D$. Hence, one finds an embedded disc $D_0 \subset \mathbb{S}^3 \backslash D$ with boundary γ. Therefore, $K \amalg \gamma$ is the boundary of $D \amalg D_0$. That is to say, the link $\gamma \amalg K \subset \mathbb{S}^3$ is unlinked.

If $K \amalg \gamma$ is unlinked in $\mathbb{S}^3$, there exists an embedded closed disc D in $S^3 \backslash \gamma$ with boundary K. Therefore, K is trivial in the complement of the knot γ.

A closed solid torus N is homeomorphic with $\mathbb{S}^1 \times D^2$, where D^2 is a closed unit disc in $\mathbb{R}^2$. A special homeomorphism $h : \mathbb{S}^1 \times D^2 \to N$ is called a *framing* of N.

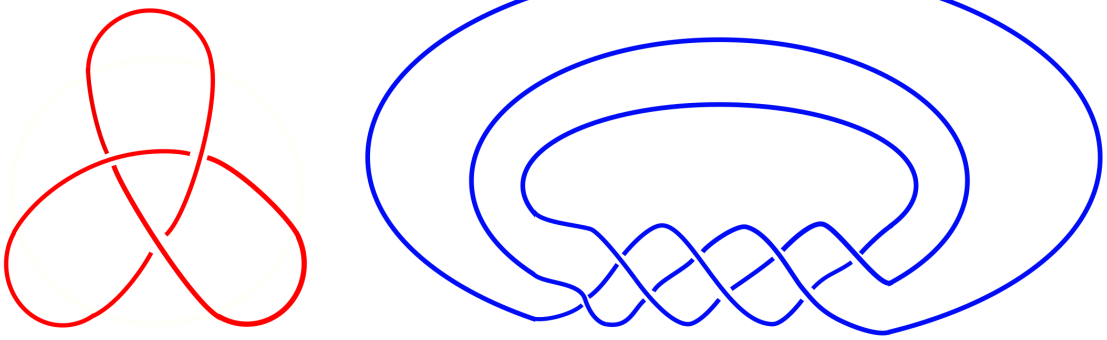
A *longitude* of N is any simple closed curve in ∂N of form $h(\mathbb{S}^1 \times x_0)$, for some framing h of N and some point x_0 in D^2.

REMARK 1.3.3. In a closed solid torus N, the kernel of the induced map $\pi_1(\partial N) \to \pi_1(N)$ is isomorphic to $\mathbb{Z}$. Each meridian γ of N belongs to the kernel. Since γ is an embedded curve, it is a generator of the kernel.

An embedded circle $\theta \subset N$ is a longitude if and only if $[\theta]$ and $[\gamma]$ generate $H_1(\partial N, \mathbb{Z})$, where γ is a meridian of N (See Page 29 of [**Rol03**]). In addition, any longitude of N is isotopic to the core of N in N.

DEFINITION 1.3.4. Assume that p, q are relatively prime and N is an unknotted solid torus in $\mathbb{S}^3$. The *torus knot* $T_{p,q} \subset \partial N \subset \mathbb{S}^3$ of type (p, q) is the knot which wraps around N in the longitudinal direction p times and in the meridional direction q times.

For example, the trefoil is $T_{2,3}$. Here are $T_{2,3}$ and $T_{3,4}$.



The $(2, 3)$ torus knot

The $(3, 4)$ torus knot

FIGURE 1.2.

REMARK 1.3.5. The knot $T_{p,q}$ is trivial in $\mathbb{S}^3$ if and only if $p = \pm 1$ or $q = \pm 1$ (See Page 53 of [**Rol03**]).

In the following, we consider two closed solid tori N' and N with $N' \subset \mathrm{Int}N$.

LEMMA 1.3.6. *If the closed solid torus $N' \subset$ Int N is homotopically trivial in the closed solid torus N, then $H_1(\overline{N \backslash N'}) = \mathbb{Z}^2$ and the kernel of the induced map $H_1(\partial N') \to H_1(\overline{N \backslash N'})$ is generated by a longitude of N'.*

PROOF. As in Lemma 1.2.2, we use the Mayer-Vietoris sequence to show that $H_1(\overline{N \backslash N'})$ is isomorphic to $\mathbb{Z}^2$. It is generated by a meridian of N' and a longitude of N.

The image of the map $H_1(\partial N') \to H_1(\overline{N \backslash N'})$ is a subgroup of rank one which is generated by the meridian γ' of N'. The kernel of $H_1(\partial N') \to H_1(\overline{N \backslash N'})$ is also of rank one and generated by $[\theta']$, where $\theta' \subset \partial N'$ is an embedded circle. Therefore, $H_1(\partial N')$ is generated by $[\gamma']$ and $[\theta']$. The circle θ' is a longitude of N' (See Page 29 of [**Rol03**] or Remark 1.3.3). That is to say, the longitude θ' is a generator of the kernel of $H_1(\partial N') \to H_1(\overline{N \backslash N'})$. □

1.3.2. The Whitehead manifold. The Whitehead manifold is constructed from the Whitehead link. Recall that the Whitehead link is a link with two components illustrated in Figure 1.3:

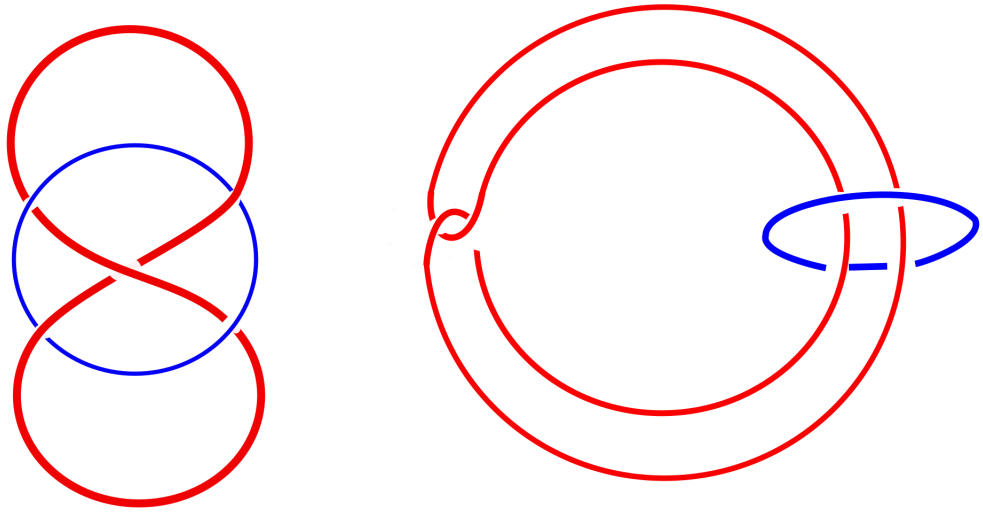


FIGURE 1.3.

Choose a closed unknotted solid torus T_1 in $\mathbb{S}^3$. Its complement inside $\mathbb{S}^3$ is another solid torus. Take a second solid torus T_2 inside T_1 so that the core of K_2 forms a Whitehead link with any meridian of T_1 as in Figure 1.4.

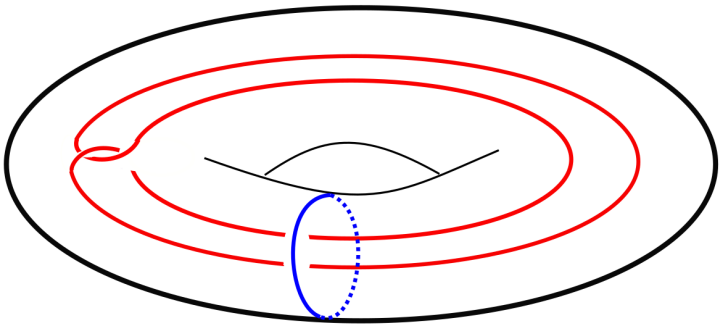


FIGURE 1.4.

The solid torus T_2 is unknotted in $\mathbb{S}^3$. Then, embed T_3 inside T_2 in the same way as T_2 lies in T_1 and so on infinitely many times. Define the set $T_\infty = \cap_{k=1}^{\infty} T_k$, called the Whitehead continuum.

The Whitehead manifold is defined as $Wh := \mathbb{S}^3 \backslash T_\infty$ which is an open 3-manifold.

REMARK 1.3.7. From the above construction, we know that

(1) Since each T_k is unknotted in $\mathbb{S}^3$, then its complement N_k is a solid torus. Therefore, the Whitehead manifold is an increasing union of solid tori $\{N_k\}_k$ as in Remark 1.1.13. In addition, each N_k is embedded inside N_{k+1} in the same way as T_2 lies in T_1. This follows from the symmetry of the Whitehead link.
(2) The core K_k of N_k is a non-trivial knot in the solid torus N_{k+1}. Furthermore, the link $K_k \amalg \gamma_{k+1}$ is a Whitehead link for each meridian γ_{k+1} of N_{k+1}. This is a consequence of the symmetry of the Whitehead link.
(3) Each K_k is unknotted in $\mathbb{S}^3$. For each $j > k$, K_k is nullhomotopic in N_j but a non-trivial knot in N_j.

Remark that the Whitehead manifold has no complete metric of positive scalar curvature (See Theorem B_1).

1.3.3. Geometric Index.

DEFINITION 1.3.8. [**Sch53**] If $N' \subset \text{Int } N$ are solid tori, the *geometric index* of N' in N, $I(N', N)$, is the minimal number of points of the intersection of the core of N' with a meridian disc of N.

REMARK 1.3.9. If the geometric index $I(N', N)$ is greater than zero, the solid torus N is an effective handlebody relative to N' (See Definition 1.2.3).

If the core K' of N' is a trivial knot in N, there is a meridian disc $(D, \partial D) \subset (N \backslash N', \partial N)$. Moreover, $I(N', N)$ is equal to zero. (See Corollary 2.9 of [**Wan19a**])

For example, in Wh, the geometric index $I(N_k, N_{k+1}) = 2$ for each k, where N_k is illustrated as in Chapter 1.3.2.

See [**Sch53**] for the following results about the geometric index.

1) Let N_0, N_1, and N_2 be solid tori so that $N_0 \subset \text{Int } N_1$ and $N_1 \subset \text{Int } N_2$. Then $I(N_0, N_2) = I(N_0, N_1)I(N_1, N_2)$.

2) If N_0 and N_1 are unknotted solid tori in $\mathbb{S}^3$ with $N_0 \subset \text{Int } N_1$, and if N_0 is homotopically trivial in N_1, then $I(N_0, N_1)$ is even.

LEMMA 1.3.10. *Suppose that the closed solid torus $N' \subset Int\ N$ is homotopically trivial in the closed solid torus N. If $I(N', N) > 0$, then the two induced maps $i_1 : \pi_1(\partial N) \to \pi_1(\overline{N \backslash N'})$ and $i_2 : \pi_1(\partial N') \to \pi_1(\overline{N \backslash N'})$ are both injective.*

PROOF. Since $I(N', N) > 0$, the solid torus N is an effective handlebody relative to N' (See Remark 1.3.9). We use Lemma 1.2.4 to see that the map $i_1 : \pi_1(\partial N) \to \pi_1(\overline{N \backslash N'})$ is injective.

Suppose that the map i_2 is not injective. We use Lemma 1.1.2 to find an embedded disc $(D_2, \partial D_2) \subset (\overline{N \backslash N'}, \partial N')$. The embedded circle $\theta := \partial D_2$ is not contractible in $\partial N'$.

Since θ bounds an embedded disc $D_2 \subset N \backslash N'$, it is a trivial knot in N. Furthermore, $[\theta]$ belongs to the kernel of the map $H_1(\partial N') \to H_1(\overline{N \backslash N'})$. From Lemma 1.3.6, the embedded circle θ is a longitude of N'.

Recall that as a knot, any longitude of N' is isotopic to the core K' of N' in N' (See Remark 1.3.3). Therefore, K' is isotopic to θ and a trivial knot in N. From Remark 1.3.9, $I(N', N) = 0$. This is a contradiction. □

LEMMA 1.3.11. *Suppose that the closed solid torus $N' \subset Int\ N$ is homotopically trivial in the closed solid torus N. If $I(N', N) > 0$, then any meridian disc D of N contains a meridian of N'.*

It follows from Lemma 1.3.10 and Lemma 1.2.5.

1.3.4. Genus one 3-manifold. (See [**GRW18**]) Let us describe McMillan's construction in [**MJ62**].

DEFINITION 1.3.12. (Genus one 3-manifold) A *genus one 3-manifold M* is the ascending union of solid tori $\{N_k\}_{k\in\mathbb{N}}$, so that for each k, $N_k \subset \text{Int } N_{k+1}$ and the geometric index of N_k in N_{k+1} is not equal to zero.

THEOREM 1.3.13. *(See [Theorem 2.8, Page 2042] of [**GRW18**])*

(1) A genus one 3-manifold defined with a sequence of open solid tori $\{N_k\}_{k\in\mathbb{N}}$ so that each N_k is contractible in N_{k+1}, is a contractible 3-manifold that is not homeomorphic to $\mathbb{R}^3$.

(2) Any contractible genus one 3-manifold can be written as an ascending union of solid tori $\{N_k\}_{k\in\mathbb{N}}$ so that 1) N_k is contractible in N_{k+1}; 2) $I(N_k, N_{k+1}) \geqslant 2$ for each k.

For example, the Whitehead manifold is a contractible genus one 3-manifold.

Any contractible genus one 3-manifold $M := \cup_{i=0}^{\infty} N_k$ satisfies the following:

(1) For each k, N_k is homotopically trivial in N_{k+1}. Moreover, $I(N_k, N_{k+1}) \geqslant 2$.

(2) For each $j > k$, the core K_k of N_k is null-homotopic in N_j but a nontrivial knot in N_j.

(3) If N_j is viewed as an unknotted solid torus in $\mathbb{S}^3$, then the link $K_k \amalg \gamma_j \subset \mathbb{S}^3$ is linked in $\mathbb{S}^3$, for each meridian γ_j of N_j for $j > k$. Moreover, its linking number is zero.
(4) However, the knot $K_k \subset \mathbb{S}^3$ may be knotted in $\mathbb{S}^3$.

Together these properties, we will show that no contractible genus one 3-manifold admits a complete metric of positive scalar curvature (See Theorem B_2).

1.3.5. More examples. In this part, we construct two different contractible 3-manifolds, M_1 and M_2. Their fundamental group at infinity are both non-trivial.

First, we construct the example, M_1.

Choose an unknotted handlebody $W_0 \subset \mathbb{S}^3$ of genus two (See Definition 1.1.14). Take a second handlebody $W_1 \subset \text{Int } W_0$ of genus two which is a tubular neighborhood of the curve in Figure 1.5. Then, embed another handlebody W_2 of genus two inside W_1 in the same way as W_1 lies in W_0 and so on infinitely many times. Therefore, we obtain a decreasing family $\{W_k\}_k$ of handlebodies of genus two.

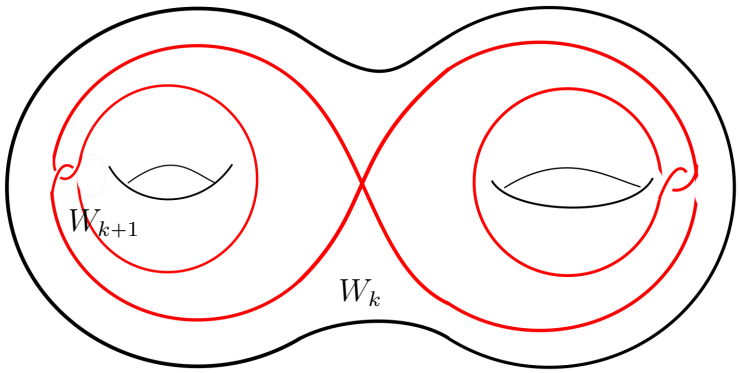


FIGURE 1.5.

The manifold M_1 is defined as $M_1 := \mathbb{S}^3 \setminus \cap_{k=0}^{\infty} W_k$. It is an open 3-manifold.

We have that each W_k is unknotted in $\mathbb{S}^3$ (See Definition 1.1.14). That is, the complement N_k of W_k in $\mathbb{S}^3$ is a handlebody of genus two. Therefore, M_1 can be written as the increasing union of handlebodies $\{N_k\}_k$ of genus two. In addition, each N_k lies in N_{k+1} as in Figure 1.6. (The set K_k is the core of N_k.) Since each N_k is homotopically trivial in N_{k+1}, M_1 is a contractible 3-manifold.

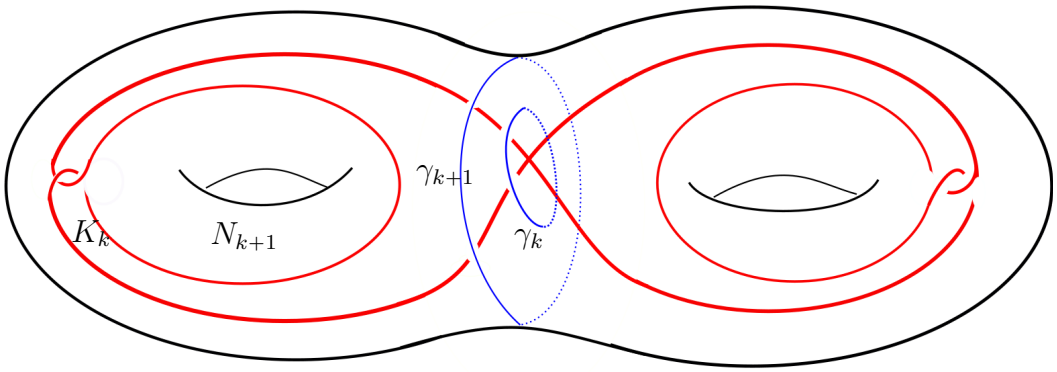


FIGURE 1.6.

Next, we construct a properly embedded plane $P_1 \subset M_1$ and show that $\pi_1^\infty(M_1)$ is non-trivial.

Choose the splitting meridian $\gamma_{k+1} \subset \partial N_{k+1}$ of N_{k+1} as in Figure 1.6. We have that γ_k is homotopic to γ_{k+1} in $\overline{N_{k+1}\backslash N_k}$. Choose an embedded annulus $A_k \subset \overline{N_{k+1}\backslash N_k}$ (namely, it is homeomorphic to $\mathbb{S}^1 \times [0,1]$) with boundary $\gamma_k \amalg \gamma_{k+1}$. We define the plane P_1 as

$$P_1 := \cup_{k\geqslant 0} A_k \cup_{\gamma_0} D_0$$

where $D_0 \subset N_0$ is a meridian disc with boundary γ_0.

PROPOSITION 1.3.14. *Let M_1 and P_1 be constructed as above. Then,*

- *the fundamental group at infinity $\pi_1^\infty(M_1)$ is non-trivial;*
- *the properly embedded plane P_1 cuts M_1 into two Whitehead manifolds.*

REMARK. Since $\pi_1^\infty(M_1)$ is non-trivial, M_1 is not simply-connected at infinity. That is, M_1 is not homeomorphic to $\mathbb{R}^3$.

The family $\{N_k\}$ of handlebodies satisfies Property H (See Definition 2.2.5).

PROOF. First, we will show that $\pi_1^\infty(M_1)$ is non-trivial. Since γ_k is homotopic to γ_{k+1} in $\overline{M_1\backslash N_0}$, it is sufficient to show that γ_0 is not contractible in $\overline{M_1\backslash N_0}$ (See Remark 1.1.10).

We see from Figure 1.6 that N_{k+1} is an effective handlebody relative to N_k. From Lemma 1.2.4, the map $\pi_1(\partial N_{k+1}) \to \pi_1(\overline{N_{k+1}\backslash N_k})$ is injective for $k \geqslant 0$.

From Figure 1.5, we have that W_k is an effective handlebody relative to W_{k+1}. By Lemma 1.2.4, the maps $\pi_1(\partial W_k) \to \pi_1(\overline{W_k\backslash W_{k+1}})$ is also injective. In addition, the space $\overline{W_k\backslash W_{k+1}}$ is equal to $\overline{N_{k+1}\backslash N_k}$. Then, we can conclude that the map $\pi_1(\partial N_k) \to \pi_1(\overline{N_{k+1}\backslash N_k})$ is injective for $k \geqslant 0$.

Claim: the map $\pi_1(\partial N_0) \to \pi_1(\overline{M_1\backslash N_0})$ is injective.

Theorem 1.1.4 gives an isomorphism between $\pi_1(\overline{N_2\backslash N_0})$ and $\pi_1(\overline{N_1\backslash N_0}) *_{\pi_1(\partial N_1)} \pi_1(\overline{N_1\backslash N_0})$. From the above fact, the maps $\pi_1(\partial N_1) \to \pi_1(\overline{N_1\backslash N_0})$ and $\pi_1(\partial N_1) \to \pi_1(\overline{N_2\backslash N_1})$ are both injective. We use Lemma 1.1.5 to show that the map $\pi_1(\overline{N_1\backslash N_0}) \to \pi_1(\overline{N_2\backslash N_0})$ is injective.

In addition, the map $\pi_1(\partial N_0) \to \pi_1(\overline{N_1 \backslash N_0})$ is injective. The composition of these two maps $\pi_1(\partial N_0) \to \pi_1(\overline{N_2 \backslash N_0})$ is injective.

Repeating the above argument several times, we obtain that the map $\pi_1(\partial N_0) \to \pi_1(\overline{N_j \backslash N_0})$ is injective for each $j > 0$. Hence, the map $\pi_1(\partial N_0) \to \pi_1(\overline{M_1 \backslash N_0})$ is injective. This finishes the proof of this claim.

Since γ_0 is not homotopically trivial in ∂N_0, it is also non-contractible in $\overline{M_1 \backslash N_0}$. Since each γ_k is homotopic to γ_{k+1} in $\overline{M_1 \backslash N_0}$, each γ_k is a non-trivial element in $\pi_1(\overline{M_1 \backslash N_0})$. We see from Remark 1.1.10 that $\pi_1^\infty(M_1)$ is non-trivial.

It remains to show that P_1 cuts M_1 into two Whitehead manifolds.

The plane P_1 cuts M_1 into two contractible 3-manifolds M_1' and M_1''. In addition, $P_1 \cap N_k$ is a splitting meridian disc of N_k with boundary γ_k.

From the sequence $\{N_k\}$, we obtain two increasing families, $\{N_k'\}$ and $\{N_k''\}$, of solid tori in M_1 satisfying that

- $M_1' = \cup_k N_k'$ and $M_1'' = \cup_k N_k''$;
- the set $N_k \backslash (N_k' \amalg N_k'')$ is a tubular neighborhood of the meridian disc $P_1 \cap N_k$.

Furthermore, each N_k' is embedded into N_{k+1}' as in Figure 1.7. From Chapter 1.3.2, we see that M_1' is homeomorphic to the Whitehead manifold. Similarly, the contractible 3-manifold M_1'' is also homeomorphic to the Whitehead manifold. Therefore, P_1 cuts M_1 into two Whitehead manifolds. □

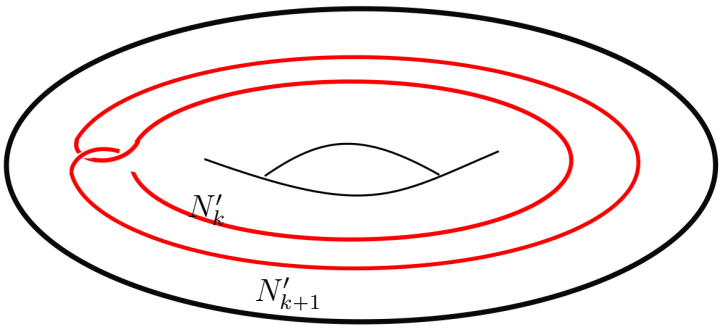


FIGURE 1.7.

Together with the proof of Theorem B_2, we have that

THEOREM 1.3.15. *The contractible 3-manifold, M_1, has no complete metric of positive scalar curvature.*

We will prove it in Chapter 6.

In the following, we construct the second example, M_2. The construction of M_2 is similar to M_1's.

Choose an unknotted handlebody $W_0 \subset \mathbb{S}^3$ of genus two. Take a second handlebody $W_1 \subset \text{Int } W_0$ of genus two which is a tubular neighborhood of the curve in Figure 1.8.

Then, embed another handlebody W_2 of genus two inside W_1 in the same way as W_1 lies in W_0 and so on infinitely many times. Therefore, we obtain a decreasing family $\{W_k\}$ of handlebodies.

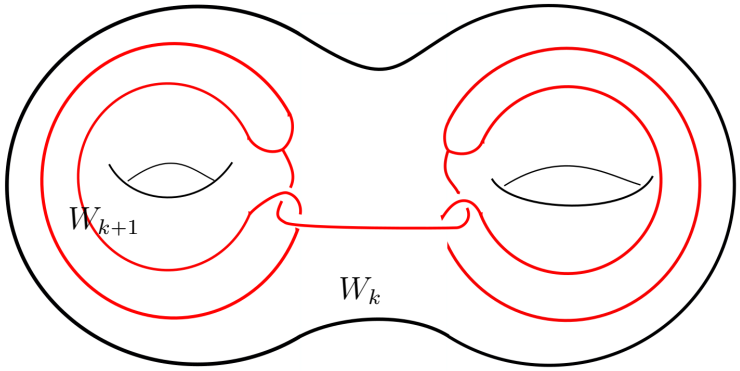


FIGURE 1.8.

The manifold M_2 is defined as $M_2 := \mathbb{S}^3 \setminus \cap_k^\infty W_k$. Note that each W_k is unknotted in $\mathbb{S}^3$

As above, M_2 can be written as an increasing union of handlebodies $\{N_k\}_k$ of genus two, where $N_k := \mathbb{S}^3 \setminus W_k$. Furthermore, each N_k lies in N_{k+1} as in Figure 1.9.

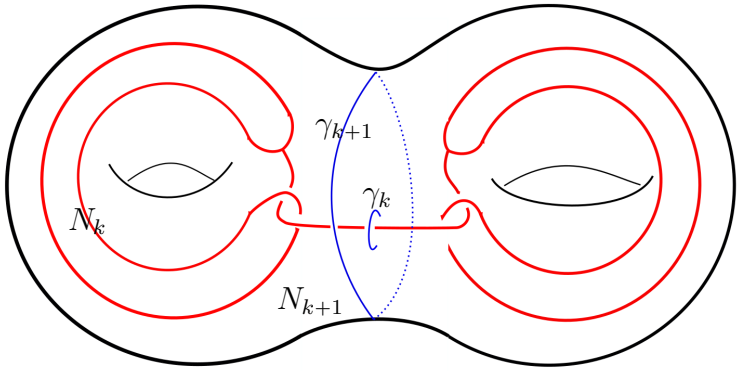


FIGURE 1.9.

Pick the splitting meridian $\gamma_k \subset N_k$ of N_k. Each γ_k is homotopic to γ_{k+1} in $\overline{N_{k+1} \setminus N_k}$. Choose an embedded annulus $A_k \subset \overline{N_{k+1} \setminus N_k}$ with boundary $\gamma_k \amalg \gamma_{k+1}$. We define the plane P_2 as

$$P_2 := \cup_{k \geqslant 0} A_k \cup_{\gamma_0} D_0$$

where $D_0 \subset N_0$ is a meridian disc with boundary γ_0.

PROPOSITION 1.3.16. *Let M_2 and P be constructed as above. Then,*

(1) $\pi_1^\infty(M_2)$ is non-trivial;

(2) the plane P_2 cuts M_2 into two contractible 3-manifolds. Each of them is homeomorphic to $\mathbb{R}^3$

PROOF. As in the proof of Proposition 1.3.14, we know that $\pi_1(M_2^\infty)$ is non-trivial. It remains to show that P_2 cuts M_2 into two $\mathbb{R}^3$s.

The plane P_2 cuts M_2 into two contractible 3-manifolds M_2' and M_2''. In addition, $P_2 \cap N_k$ is also a meridian disc of N_k with boundary γ_k.

As the first example, we obtain two increasing families, $\{N_k'\}$ and $\{N_k''\}$, of solid tori in M_2 satisfying that

- $M_2' = \cup_k N_k'$ and $M_2'' = \cup_k N_k''$;
- the set $N_k \backslash (N_k' \amalg N_k'')$ is a tubular neighborhood of the meridian disc $P_2 \cap N_k$.

Furthermore, each N_k' is embedded into N_{k+1}' as in Figure 1.10. There is a closed 3-ball $B_k \subset N_{k+1}'$ containing N_k'. Therefore, M_2' is equal to $\cup_k B_k$. Hence, M_1' is simply-connected at infinity (See Remark 1.1.8). Namely, it is homeomorphic to $\mathbb{R}^3$. Similarly, M_2'' is also homeomorphic to $\mathbb{R}^3$. □

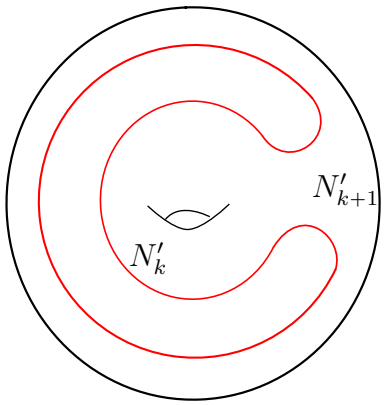


FIGURE 1.10.

REMARK 1.3.17. It is unknown whether M_2 admits a complete metric of positive scalar curvature.

CHAPTER 2

Topological Properties

In this chapter, we discuss several topological properties of contractible 3-manifolds.

We first study the behavior of embedded discs in the Whitehead manifold and their relationship with the geometric indexes. Their relation is clarified by Theorem 2.1.2. Based on their relation, we introduce the topological property, called Property P. Furthermore, we show that any contractible genus one 3-manifold satisfies this property.

Next, we consider contractible 3-manifolds. We introduce two types of surgeries on handlebodies. We use these surgeries to show the existence of effective handlebodies (See Theorem 2.2.3). Then we inductively find an increasing family of handlebodies with good properties, called Property (H).

2.1. Property P

2.1.1. The Whitehead case. As in Chapter 1.3.2, $\mathrm{Wh} \subset \mathbb{S}^3$ is an increasing union of closed solid tori $\{N_k\}_{k=0}^{\infty}$ so that the geometric index $I(N_k, N_{k+1}) = 2$, for each k. For any $j > k$, the core K_k of N_k is a non-trivial knot in N_j but unknotted in $\mathbb{S}^3$. In addition, the link $K_k \amalg \gamma_j$ is linked with zero linking number, for any meridian γ_j of N_j.

LEMMA 2.1.1. *Any embedded circle $\gamma \subset \partial N_k$ which is the boundary of a closed embedded disc D in Wh but not nullhomotopic in ∂N_k, is a meridian of N_k.*

PROOF. Since the disc D is compact, there is some $k_0 > k$ such that D is contained in N_{k_0}.

Let γ belong to the homology class $p[\gamma_k] + q[\theta_k]$ in $H_1(\partial N_k)$, where γ_k and θ_k are a meridian and a longitude of N_k. Since N_k is an unknotted solid torus in $\mathbb{S}^3$(See Remark 1.3.7), γ (as a knot in $\mathbb{S}^3$) is isotopic to the torus knot $T_{p,q}$ in $\mathbb{S}^3$.

Because the knot γ bounds an embedded disc D in N_{k_0}, it is a trivial knot in $N_{k_0} \subset \mathbb{S}^3$. Hence, γ is unknotted in $\mathbb{S}^3$. We see from Remark 1.3.5 that $p = \pm 1$ or $q = \pm 1$.

Since the knot γ is trivial in N_{k_0}, we use Remark 1.3.9 to find a meridian disc $(D_1, \partial D_1) \subset (N_{k_0}, \partial N_{k_0})$ with $D_1 \cap \gamma = \varnothing$. Because the geometric index $I(N_k, N_{k_0}) > 0$, the disc D_1 contains at least one meridian γ'_k of ∂N_k (Lemma 1.3.11 or Lemma 1.2.5).

Therefore, $\gamma'_k \cap \gamma$ is empty. Their intersection number on ∂N_k must be zero.

We knows that the intersection number of γ and γ'_k is q. Therefore, we knows that $p = \pm 1, q = 0$. That is to say, γ is homotopic to the meridian γ'_k on ∂N_k. This completes the proof. □

THEOREM 2.1.2. *Any $\gamma \subset \partial N_k$ bounding an embedded disc D in Wh satisfies one of the following:*

(1) $[\gamma]$ is trivial in $\pi_1(\partial N_k)$,

(2) $D \cap \mathrm{Int}\, N_l$ has at least $I(N_l, N_k)$ components intersecting N_0, for each $l < k$.

Note that the geometric index $I(N_l, N_k)$ is equal to 2^{k-l}.

PROOF. We argue by induction on k.

- When $k = 0$, it is trivial.
- We suppose that it holds for N_{k-1}.

We suppose that the closed curve γ is not contractible in ∂N_k. From Lemma 2.1.1, it is a meridian of N_k. In addition, the linking number of $\gamma \amalg K_{k-1}$ is zero, where K_{k-1} is the core of N_{k-1} (See Remark 1.3.7).

We may assume that D intersects ∂N_{k-1} transversally. The set $D \cap \partial N_{k-1}$ has finitely many components $C := \{\gamma_i\}_{i\in I}$. Each component γ_i is an embedded circle and bounds a unique closed disc $D_i \subset \text{Int } D$.

Let $\{\gamma_j\}_{j\in I_0}$ be the set of maximal circles in C where $I_0 \subset I$. Each γ_j is the boundary of the disc D_j, for $j \in I_0$.

Claim: There exist at least two elements in $\{\gamma_j\}_{j\in I_0}$, which are meridians of N_{k-1}.

By Lemma 1.3.10, the maps $\pi_1(\partial N_k) \to \pi_1(\overline{N_k \backslash N_0})$ and $\pi_1(\partial N_k) \to \pi_1(\overline{N_m \backslash N_k})$ are both injective for any $m > k$. Van-Kampen's Theorem (See Theorem 1.1.4) gives an isomorphism between $\pi_1(\overline{N_m \backslash N_0})$ and $\pi_1(\overline{N_k \backslash N_0}) *_{\pi_1(\partial N_k)} \pi_1(\overline{N_m \backslash N_k})$. We use Lemma 1.1.5 to see that the map $\pi_1(\partial N_k) \to \pi_1(\overline{N_m \backslash N_0})$ is also injective. Therefore, since γ is not contractible in ∂N_k, we can conclude that it is not contractible in $\overline{Wh \backslash N_0}$.

If γ_j is homotopically trivial in ∂N_{k-1} for each $j \in I_0$, then one finds a disc $D'_j \subset \partial N_{k-1}$. Consider a new disc $D' := (\Sigma \backslash \cup_{j\in I_0} D_j) \cup (\cup_{j\in I_0} D'_j)$ in $\overline{Wh \backslash N_0}$ with boundary γ. Therefore, γ is contractible in $\overline{Wh \backslash N_0}$. This contradicts the last paragraph. We see that one of $\{\gamma_j\}_{j\in I_0}$ is non-contractible in ∂N_{k-1}. Hence, by Lemma 2.1.1, there is at least one meridian of N_{k-1} in $\{\gamma_j\}_{j\in I_0}$.

In the following, we argue by contradiction.

Suppose that there is a unique meridian of N_{k-1} in the set $\{\gamma_j\}_{j\in I_0}$. That is to say, there is a unique $j_0 \in I_0$ such that γ_{j_0} is a meridian of N_{k-1}. Remark that each γ_j bounds a unique disc $D_j \subset D$.

If γ_j is not contractible ∂N_k for some $j \in I_0 \backslash \{j_0\}$, Lemma 2.1.1 shows that it is a meridian, which contradicts the uniqueness of j_0. We see that γ_j is nullhomotopic in ∂N_{k-1}, for each $j \in I_0 \backslash \{j_0\}$.

Consider a meridian disc $\hat{D}_{j_0}$ of N_{k-1} with boundary γ_{j_0}, which intersects the core K_{k-1} of N_{k-1} transversally at one point. For $j \in I_0 \backslash \{j_0\}$, there exists a disc $\hat{D}_j \subset \partial N_{k-1}$ with boundary γ_j.

Define a new disc $\hat{D} := (D \backslash \cup_{j\in I_0} D_j) \cup_{j\in I_0} (\cup_{\gamma_j} \hat{D}_j)$ with boundary γ. It intersects K_{k-1} transversally at one point, which implies that the intersection number of $\hat{D}$ and K_{k-1} is ± 1.

Therefore, the linking number of $\gamma \amalg K_{k-1}$ is ± 1. This is in contradiction with the fact that its linking number is zero.

This completes the proof of the claim.

From the above claim, there are at least two distinct meridians, γ_{j_0} and γ_{j_1}, of N_{k-1} in $\{\gamma_j\}_{j\in I_0}$. Applying our inductive assumption to D_{j_0} and D_{j_1} respectively, we know that

$D_{j_t} \cap \text{Int } N_l$ has at least 2^{k-1-l} components intersecting N_0 for $t = 0, 1$ for $l \leqslant k-1$. Therefore, $D \cap \text{Int } N_l$ has at least 2^{k-l} components intersecting N_0. □

Based on Theorem 2.1.2, we introduce a topological property.

DEFINITION 2.1.3. A contractible genus one 3-manifold M is called to satisfy *Property P* if for any properly embedded plane $\Sigma \subset M$, any $k > 0$ and any closed curve $\gamma \subset \partial N_k \cap \Sigma$, it holds one of the following:

(1) γ is contractible in ∂N_k;
(2) for $l < k$, $D \cap \text{Int } N_l$ has at least $I(N_l, N_k)$ components intersecting N_0,

where $D \subset \Sigma$ is a unique disc with boundary γ and $\{N_k\}_k$ is a sequence as described in Theorem 1.3.13.

We will show that all contractible genus one 3-manifolds satisfy Property P (Theorem 2.1.6).

2.1.2. The Genus one case. In this part, we show that any contractible genus one 3-manifold satisfies Property P.

First, recall some notations from Chapter 1.3.4. Any contractible genus one 3-manifold M is the ascending union of closed solid tori $\{N_k\}_{k=0}^{\infty}$ so that N_k is homotopically trivial in N_{k+1} and the geometric index $I(N_k, N_{k+1}) \geqslant 2$ (See Theorem 1.3.13).

In the genus one case, Lemma 2.1.1 can be generalized as follows:

LEMMA 2.1.4. *A circle $\gamma \subset \Sigma \cap \partial N_k$, which is not contractible in ∂N_k, is a meridian of N_k, where $\Sigma \subset M$ is a properly embedded plane. Moreover, the unique disc $D \subset \Sigma$ with boundary γ intersects the core K_0 of N_0.*

PROOF. We may assume that Σ intersects ∂N_k transversally. Since Σ is properly embedded, $\Sigma \cap \partial N_k := \{\gamma_i\}_{i=0}^{n}$ has finitely many components, where $\gamma_0 = \gamma$. Each γ_i bounds a unique closed disc $D_i \subset \Sigma$ (where $D_0 = D$).

Define the set $C := \{\gamma_i | \gamma_i \subset D_0 \text{ is not contractible in } \partial N_k\}$. It is not empty ($\gamma_0 \subset D_0$).

Since $\{\gamma_i\}_{i=0}^{n}$ is a family of disjoint circles, we see that the intersection number of γ and γ_i in ∂N_k is zero for each $i \neq 0$.

If $[\gamma_i]$ is not equal to $\pm[\gamma]$ in $\pi_1(\partial N_k)$ for some $\gamma_i \in C$, the intersection number of γ and γ_i is nonzero in ∂N_k. This contradicts the above fact. We can conclude that each $\gamma_i \in C$ is homotopic to γ in ∂N_k, up to orientation.

In the following, we will show that each minimal circle γ_j in C is a meridian. This is to say, γ is also a meridian of N_k.

The remaining proof is similar to the proof of Lemma 1.2.4. It is sufficient to show that γ_j is homotopically trivial in N_j. We begin by constructing an immersed disc $\hat{D}_j \subset N_k$ with boundary γ_j.

Let us consider the set $C_j := \{\gamma_i \,|\gamma_i \subset \text{Int } D_j\} \subset C$ and the set C_j^{max} of maximal circles in C_j. One has two cases: $C_j = \varnothing$ or $C_j \neq \varnothing$.

Case I: If C_j is empty, we consider the set $Z := \text{Int } D_j$ and define the disc $\hat{D}_j$ as Int D_j;

Case II: If C_j is not empty, then C_j^{max} is also non-empty. From the minimality of γ_j, each $\gamma_i \in C_j^{max}$ is contractible in ∂N_k and bounds a disc $D_i' \subset \partial N_k$.

Define the set $Z :=$ Int $D_j \setminus \cup_{\gamma_i \in C_j^{max}} D_i$ and the disc $\hat{D}_j := Z \cup (\cup_{\gamma_i \in C_j^{max}} D_i')$ with boundary γ_j.

Let us explain why $\hat{D}_j$ is contained in N_k. In any case, ∂N_k cuts M into two connected components, $M \setminus N_k$ and Int N_k. The set Z is one of these components of Int $D_j \setminus \partial N_k$.Therefore, it is in $M \setminus N_k$ or Int N_k.

If Z is in $M \setminus N_k$, then the disc $\hat{D}_j$ with boundary γ_j is contained in $\overline{M \setminus N_k}$. Therefore, we see that $[\gamma_j] = 0$ in $\pi_1(\overline{M \setminus N_k})$. However, the map $\pi_1(\partial N_k) \to \pi_1(\overline{M \setminus N_k})$ is injective (Lemma 1.3.10). That is to say, γ_j is null-homotopic in ∂N_k. This contradicts the fact that $[\gamma_j] \neq 0$ in $\pi_1(\partial N_k)$. We can conclude that Z is contained in Int N_k.

Therefore, $\hat{D}_j$ is contained in N_k. Its boundary γ_j is nullhomotopic in N_k. Since γ is homotopic to γ_j in ∂N_k, it is also contractible in N_k. By Definition 1.2.1, γ must be a meridian of N_k.

By Lemma 1.3.10, the two induced maps $\pi_1(\partial N_k) \to \pi_1(\overline{M \setminus N_k})$ and $\pi_1(\partial N_k) \to \pi_1(N_k \setminus K_0)$ are both injective. Van-Kampen's theorem (See Theorem 1.1.4) shows that $\pi_1(M \setminus K_0) \cong \pi_1(\overline{M \setminus N_k}) *_{\pi_1(\partial N_k)} \pi_1(N_k \setminus K_0)$. We see from Lemma 1.1.5 that the map $\pi_1(\partial N_k) \to \pi_1(M \setminus K_0)$ is also injective. Therefore, $[\gamma] \neq 0$ in $\pi_1(M \setminus K_0)$. We can conclude that the disc $D \subset \Sigma$ with boundary γ must intersect the core K_0 of N_0. □

Remark 2.1.5.

- In the proof, the set $\hat{D}_j \cap$ Int N_k is equal to the set Z and a subset of $D \cap$ Int N_k.
- The disc $\hat{D}_j$ may not be embedded, because D_i' may be contained in some $D_{i'}'$. When it is not an embedding, we can deform $\hat{D}_j$ in a small neighborhood of ∂N_k in N_k so that it becomes an embedded disc in N_k.

Theorem 2.1.6. *Any contractible genus one 3-manifold M satisfies Property P.*

Proof. Consider a properly embedded plane $\Sigma \subset M$. Suppose there is some closed curve $\gamma \subset \Sigma \cap \partial N_k$ which is not contractible in ∂N_k for some $k \in \mathbb{N}_{>0}$. By Lemma 2.1.4, γ is a meridian of N_k and the unique closed disc $D \subset \Sigma$ with boundary γ intersects N_0.

We may assume that Σ intersects ∂N_k transversally. The set $D \cap \partial N_k := \{\gamma_i\}_{i=0}^n$ has finitely many components where $\gamma_0 = \gamma$.

Define the set $C := \{\gamma_i \mid$ the circle $\gamma_i \subset D \cap \partial N_k$ is not contractible in $\partial N_k\}$. (It is not empty because $\gamma \subset D$). We use Lemma 2.1.4 to see that each minimal circle γ_j in C is a meridian of N_k. It bounds a unique closed disc $D_j \subset D$. As in the proof of Lemma 2.1.4, we construct a disc $\hat{D}_j \subset N_k$ with boundary γ_j. Remark that $\hat{D}_j \cap \text{Int} N_k$ is a subset of $D \cap \text{Int} N_k$ (See the above Remark).

As described in the above remark, the disc $\hat{D}_j$ may be not embedded. If necessary, we can deform it in a small neighborhood of ∂N_k in N_k so that it becomes an embedded disc. For $l < k$, $\hat{D}_j \cap$ Int N_l is still a subset of $D \cap$ Int $N_l \subset \Sigma$.

It is sufficient to show that $\hat{D}_j \cap$ Int N_l has at least $I(N_l, N_k)$ components intersecting N_0.

We may assume that $\hat{D}_j$ intersects ∂N_l transversally. The intersection $\hat{D}_j \cap \partial N_l := \{\gamma'_t\}_{t\in T}$ has finitely many components. Let us consider the set $\hat{C}^{max}$ of maximal circles in $\{\gamma'_t\}_{t\in T}$ and its subset $\hat{C}^{non} := \{\gamma'_t \in \hat{C}^{max} | \gamma'_t \text{ is not contractible in } \partial N_l\}$.

Claim: $|\hat{C}^{non}| \geqslant I(N_l, N_k)$.

We argue by contradiction. Suppose that $|\hat{C}^{non}| < I(N_i, N_k)$. Each $\gamma'_t \in \hat{C}^{max}$ bounds a unique disc $D'_t \subset \hat{D}_j$.

If γ'_t is in $\hat{C}^{non}$, it is a meridian of N_l (See Lemma 2.1.4). Therefore, we can find a meridian disc D''_t of N_l which intersects the core K_l of N_l transversally at one point. If $\gamma'_t \in \hat{C}^{max}\backslash\hat{C}^{non}$, γ'_t is contractible in ∂N_l and bounds a disc D''_t in ∂N_l.

Define a disc $\hat{D}'_j$ with boundary γ_j

$$\hat{D}'_j := (\hat{D}_j \backslash \cup_{\gamma'_t\in\hat{C}^{max}} D'_t) \cup (\cup_{\gamma'_t\in\hat{C}^{max}} D''_t).$$

The number $\#(\hat{D}'_j \cap K_l)$ of points of $\hat{D}'_j \cap K_l$ is less than $I(N_l, N_j)$.

As above, the disc $\hat{D}'_j$ may be not embedded (because D''_t may be contained in some $D''_{t'}$). If necessary, we modify the disc $\hat{D}'_j$ in a small neighborhood of ∂N_l so that it becomes an embedded disc in N_k.

Therefore, we may assume that $(\hat{D}'_j, \partial\hat{D}'_j) \subset (\overline{N}_k, \partial N_k)$ is an embedded disc with boundary γ_j. Since γ_j is a meridian of N_k(See Lemma 2.1.4), $\hat{D}'_j$ is a meridian disc of N_k with $\#(\hat{D}'_j \cap K_l) < I(N_l, N_k)$. However, the definition of the geometric index (See Definition 1.3.8) gives that $\#(\hat{D}'_j \cap K_l) \geqslant I(N_l, N_k)$, a contradiction. This finishes the proof of the claim.

In the following, we will finish the proof of the theorem.

Let $\{\gamma'_s\}_{s=1}^m$ be the circles in $\hat{C}^{non}$ and $D'_s \subset \hat{D}_j$ the unique disc with boundary γ'_s, where $m = |\hat{C}^{non}_j|$. From the maximality of γ'_s in $\{\gamma'_t\}_{t\in T}$, $\{D'_s\}_{s=1}^m$ is a family of pairwise disjoint discs in $\hat{D}_j$.

We use Lemma 2.1.4 to see that each $\gamma'_s \in \hat{C}^{non}$ is a meridian. Thus, D'_s intersects the core K_0 of N_0. The intersection $D'_s \cap \text{Int } N_l$ contains at least one component intersecting N_0.

We conclude that $\hat{D}_j \cap \text{Int } N_l$ has at least m components intersecting N_0. From the above claim, we know that $m \geqslant I(N_l, N_k)$. Therefore, $D \cap \text{Int } N_l$ has at least $I(N_l, N_k)$ components intersecting N_0. □

Remark 2.1.7.

- The proof of Lemma 2.1.4 and Theorem 2.1.6 just depend on the injectivity of the two maps $\pi_1(\partial N_k) \to \pi_1(\overline{N_k\backslash N_0})$ and $\pi_1(\partial N_k) \to \pi_1(\overline{M\backslash N_k})$.
- Let $\{R_k\}_k$ be an increasing family of solid tori in a contractible 3-manifold with the property that
  (1) $\pi_1(\partial R_k) \to \pi_1(\overline{R_k\backslash R_0})$ is injective ;
  (2) $\pi_1(\partial R_k) \to \pi_1(\overline{M\backslash R_k})$ is injective .
  However, the union $\cup_k R_k$ may not be equal to M.

  From the above fact, Lemma 2.1.4 holds for each R_k. Further, as in the proof of Theorem 2.1.6, the family $\{R_k\}$ satisfies Property P. That is to say, for any

properly embedded plane $\Sigma \subset M$, any $k > 0$ and any closed curve $\gamma \subset \partial R_k \cap \Sigma$, it holds one of the following:
(1) γ is contractible in ∂R_k;
(2) for $l < k$, $D \cap \text{Int } R_l$ has at least $I(R_l, R_k)$ components intersecting R_0,
where $D \subset \Sigma$ is a unique disc with boundary γ.

2.2. Property H

2.2.1. Surgeries. Consider two closed handlebodies N' and N in a 3-manifold M with $N' \subset \text{Int } N$. We introduce two types of surgeries on handlebodies:

Type I: If there exists a meridian disc $D \subset N \backslash N'$ of N, then we consider an open tubular neighborhood $N_\epsilon(D) \subset N \backslash N'$ of D. We then have two cases:

Case (1): If D is a splitting meridian disc, $N \backslash N_\epsilon(D)$ has two components. The closed handlebody W_1 is defined as the component containing N';

Case(2): If D is a non-splitting meridian disc, $N \backslash N_\epsilon(D)$ is connected. The closed handlebody W_1 is defined by $N \backslash N_\epsilon(D)$.

Type II: If there exists an embedded disc $D_1 \subset \overline{M \backslash N}$ satisfying that 1) Int $D_1 \subset M \backslash N$ and 2) its boundary $\gamma \subset \partial N$ is not contractible in ∂N, we consider a closed tubular neighborhood $\overline{N_{\epsilon_1}(D_1)}$ of D_1 in $\overline{M \backslash N}$. Define a new handlebody W_2 as $N \cup \overline{N_{\epsilon_1}(D_1)}$.

Remark 2.2.1. For $i = 1, 2$, the genus $g(\partial W_i)$ of ∂W_i is less than $g(\partial N)$. In addition, ∂W_i is a union of $\partial W_i \cap \partial N$ and some disjoint discs. It tells us that the map $\pi_1(\partial W_i \cap \partial N) \to \pi_1(\partial W_i)$ is surjective.

Lemma 2.2.2. *If N' is homotopically trivial in N, then N' is also homotopically trivial in W_i for each i, where W_i is obtained from the above surgeries.*

Proof. For the type II surgery, we see that N is contained in W_2. Therefore, N' is homotopically trivial in W_2.

For the type I surgery, it is sufficient to show that any circle $c \subset N'$ bounds some disc $\hat{D}' \subset W_1$.

The closed curve c bounds an immersed disc $D' \subset \text{Int } N$. We will construct the required disc $\hat{D}' \subset W_1$ from D'.

We may assume that D' intersects $D^- \amalg D^+ := \text{Int } N \cap \partial N_\epsilon(D')$ transversally. Each component c_i of $D' \cap (D^+ \amalg D^-)$ is a circle in D' and bounds a closed sub-disc $D'_i \subset D'$.

Since D^+ and D^- are two disjoint discs, each c_i is contractible in $D^+ \amalg D^-$. It also bounds a disc $D''_i \subset D^+ \amalg D^-$. Let C^{max} be the set of the maximal circles of $\{c_i\}_{i \in I}$ in D'. We construct a disc

$$\hat{D}' := D' \backslash \cup_{c_i \in C^{max}} D'_i \cup (\cup_{c_i \in C^{max}} D''_i)$$

with boundary c. It stays in $\overline{N \backslash N_\epsilon(D')}$. That is to say, c is contractible in W_1. Therefore, N' is homotopically trivial in W_1. □

2.2.2. Effective Handlebodies. In the following, let us consider a contractible 3-manifold M.

THEOREM 2.2.3. *Let N' and N be two closed handlebodies in M satisfying that 1) $N' \subset$ Int N and 2) N' is homotopically trivial in N. Then there exists a closed handlebody $R \subset M$ containing N' satisfying that*

(1) the map $\pi_1(\partial R) \to \pi_1(\overline{R \backslash N'})$ is injective;
(2) the map $\pi_1(\partial R) \to \pi_1(\overline{M \backslash R})$ is injective;
(3) N' is homotopically trivial in R;
(4) ∂R is a union of $\partial R \cap \partial N$ and some disjoint discs.

REMARK. From (1), R is an effective handlebody relative to N' (Lemma 1.2.4).

PROOF. Suppose that either the map $i_1 : \pi_1(\partial N) \to \pi_1(\overline{N \backslash N'})$ is not injective or the map $i_2 : \pi_1(\partial N) \to \pi_1(\overline{M \backslash N})$ is not injective. (If these two maps are both injective, R is defined as N.)

If i_1 is not injective, Lemma 1.1.2 shows that there exists a meridian disc D_1 of N with $D_1 \cap N' = \varnothing$. We do the type I surgery on N with the disc D_1 to obtain a new handlebody W.

If i_2 is not injective, we use Lemma 1.1.2 to find an embedded circle $\gamma \subset \partial N$ and an embedded disc $D_2 \subset \overline{M \backslash N}$ (Int $D_2 \subset M \backslash N$) where $\gamma = \partial D_2$ is not nullhomotopic in ∂N. We do the type II surgery with the disc D_2 to get a new handlebody W.

In any case, we have that $g(\partial W) < g(\partial N)$. The boundary ∂W is a union of $\partial W \cap \partial N$ and some disjoint discs $\{D_i'\}_i$. Therefore, $\pi_1(\partial W \cap \partial N) \to \pi_1(\partial W)$ is surjective. In addition, we see from Lemma 2.2.2 that N' is contractible in W.

When picking a circle $\gamma \subset \partial W$ which is not nullhomotopic in ∂W, we may assume that γ is an embedded circle in $\partial W \cap \partial N$. Therefore, when repeating these two types of surgeries, we may assume that the new surgeries are operated away from these disjoint discs $\{D_i'\}$.

Iterate this process until we find a handlebody R satisfying (1) and (2). At each step, the genus of the handlebody obtained from the surgery is less than the original one. Therefore, this process stops in no more than $g(N)$ steps.

As above, N' is contractible in R and ∂R is a union of $\partial R \cap \partial N$ and some disjoint discs. □

REMARK. If N' is not contained in a 3-ball in M, then the genus of R is greater than zero.

LEMMA 2.2.4. *Let $R \subset M$ be a closed effective handlebody relative to the closed handlebody $N' \subset$ Int R satisfying that $\pi_1(\partial R) \to \pi_1(\overline{M \backslash R})$ is injective. If a closed handlebody N is an effective handlebody relative to $R \subset$ Int N, then N is an effective handlebody relative to N'.*

PROOF. Based on Lemma 1.2.4, it is sufficient to show that the map $\pi_1(\partial N) \to \pi_1(\overline{N \backslash N'})$ is injective.

We use Lemma 1.2.4 to show that the induced map $\pi_1(\partial R) \to \pi_1(\overline{R \backslash N'})$ is injective. Since $\pi_1(\partial R) \to \pi_1(\overline{M \backslash R})$ is injective, then the map $\pi_1(\partial R) \to \pi_1(\overline{N \backslash R})$ is also injective.

Van Kampen's theorem (Theorem 1.1.4) gives an isomorphism between $\pi_1(\overline{N \backslash N'})$ and $\pi_1(\overline{N \backslash R}) *_{\pi_1(\partial R)} \pi_1(\overline{R \backslash N'})$. A classical result (See [Theorem 11.67, Page 404] of [**Rot12**] or Lemma 1.1.5) shows that the induced map $\pi_1(\overline{N \backslash R}) \to \pi_1(\overline{N \backslash N'})$ is injective.

Lemma 1.2.4 shows that the map $\pi_1(\partial N) \to \pi_1(\overline{N\backslash R})$ is injective. Therefore, the composition $\pi_1(\partial N) \to \pi_1(\overline{N\backslash R}) \to \pi_1(\overline{N\backslash N'})$ is also injective. This finishes the proof. □

2.2.3. Property H. In the following, let us consider a contractible 3-manifold M which is not homeomorphic to $\mathbb{R}^3$.

By Theorem 1.1.12, M can be written as an ascending union of handlebodies $\{N_k\}_{k=0}^{\infty}$. Each N_k is contractible in N_{k+1}. As in Remark 1.1.8, we can choose N_0 so that it is not contained in a 3-ball in M (because M is not homeomorphic to $\mathbb{R}^3$).

In the genus one case, the family $\{N_k\}$ has several good properties. For example, each N_k is an effective handlebody relative to N_0 and the map $\pi_1(\partial N_k) \to \pi_1(\overline{M\backslash N_k})$ is injective (See Lemma 1.3.10 or Lemma 2.10 of [**Wan19a**]). These properties are necessary and crucial in our proof. In general, the family $\{N_k\}$ may not have these properties. To overcome this difficulty, we introduce a topological property, called Property H.

Definition 2.2.5. A family $\{R_k\}_k$ of handlebodies in a contractible 3-manifold $M := \cup_k N_k$ is called to have *Property H* if it satisfies that

(1) the map $\pi_1(\partial R_k) \to \pi_1(\overline{R_k\backslash R_0})$ is injective for $k > 0$;
(2) the map $\pi_1(\partial R_k) \to \pi_1(\overline{M\backslash R_k})$ is injective for $k \geqslant 0$;
(3) each R_k is contractible in R_{k+1} but not contained in a 3-ball in M ;
(4) there exists a sequence of increasing integers $\{j_k\}_k$, such that $\pi_1(\partial R_k \cap \partial N_{j_k}) \to \pi_1(\partial R_k)$ is surjective.

where $\{N_k\}$ is assumed as in Remark 1.1.13.

For example, in a contractible genus one 3-manifold $M := \cup_k N_k$, the family $\{N_k\}_k$ satisfies Property H, where $\{N_k\}$ is assumed in Chapter 1.3.4 (See Lemma 1.3.10).

In the following, we will prove if a contractible 3-manifold M is not homeomorphic to $\mathbb{R}^3$, there is a family of handlebodies with Property H (See Theorem 2.2.6). However, such a family is not unique .

Theorem 2.2.6. *If a contractible 3-manifold $M := \cup_k N_k$ (as above) is not homeomorphic to $\mathbb{R}^3$, then there is an ascending family $\{R_k\}_k$ of closed handlebodies in M satisfying that*

(1) the map $\pi_1(\partial R_k) \to \pi_1(\overline{R_k\backslash R_0})$ is injective for $k > 0$;
(2) the map $\pi_1(\partial R_k) \to \pi_1(\overline{M\backslash R_k})$ is injective for $k \geqslant 0$;
(3) each R_k is contractible in R_{k+1} but not contained in a 3-ball in M ;
(4) there exists a sequence of increasing integers $\{j_k\}_k$, such that $\pi_1(\partial R_k \cap \partial N_{j_k}) \to \pi_1(\partial R_k)$ is surjective.

Remark 2.2.7.

- The union $\cup_k R_k$ may be not equal to M.
- For $k > 0$, Van-Kampen's Theorem gives an isomorphism between $\pi_1(\overline{M\backslash R_0})$ and $\pi_1(\overline{M\backslash R_k}) *_{\pi_1(\partial R_k)} \pi_1(\overline{R_k\backslash R_0})$. Based on (1) and (2) in Theorem 2.2.6, we use Lemma 1.1.5 to show that the map $\pi_1(\partial R_k) \to \pi_1(\overline{M\backslash R_0})$ is injective.
- As (4) in Theorem 2.2.3, ∂R_k is the union of $\partial R_k \cap \partial N_{j_k}$ and disjoint discs.

PROOF. First, we construct R_0. We repeatedly apply the Type II surgery to N_0, until we find a handlebody R_0 containing N_0 so that $\pi_1(\partial R_0) \to \pi_1(\overline{M \backslash R_0})$ is injective.

From Remark 2.2.1, we see that, at each step, the genus of the handlebody obtained from the surgery is less than the original one. Therefore, this process stops in no more than $g(N_0)$ steps.

In addition, since N_0 is not contained in a 3-ball in M, then R_0 has the same property.

It remains to construct the sequence $\{R_k\}_k$ inductively.

When k is equal to 1, we pick a handlebody N_{j_1} containing R_0 satisfying that R_0 is homotopically trivial in N_{j_1}. Its existence is ensured by the following fact:

Because R_0 is compact, there is some handlebody N_{j_1-1} containing R_0. Since N_{j_1-1} is homotopically trivial in N_{j_1}, R_0 is contained in N_{j_1} and contractible in N_{j_1}.

By Theorem 2.2.3, there exists a handlebody R_1 containing R_0 so that

- $\pi_1(\partial R_1) \to \pi_1(\overline{R_1 \backslash R_0})$ is injective;
- $\pi_1(\partial R_1) \to \pi_1(\overline{M \backslash R_1})$ is injective;
- R_0 is contractible in R_1;
- ∂R_1 is a union of $\partial R_1 \cap \partial N_{j_1}$ and some disjoint closed discs. Therefore, $\pi_1(\partial R_1 \cap \partial N_{j_1}) \to \pi_1(\partial R_1)$ is surjective.

In particular, since R_0 is not contained in a 3-ball in M, R_1 has the same property.

Suppose that there exists a handlebody R_{k-1} and a positive integer j_{k-1} satisfying (1), (2), (3) and (4) in Theorem 2.2.6.

As the existence of N_{j_1}, there exists a handlebody N_{j_k} containing R_{k-1} satisfying that R_{k-1} is homotopically trivial in N_{j_k}. We use Theorem 2.2.3 to find an effective handlebody R_k relative to R_{k-1} satisfying (2), (3) and (4).

Since the map $\pi_1(\partial R_{k-1}) \to \pi_1(\overline{R_{k-1} \backslash R_0})$ is injective, R_{k-1} is an effective handlebody relative to R_0 (Lemma 1.2.4). Lemma 2.2.4 shows that R_k is an effective handlebody relative to R_0. We apply Lemma 1.2.4 again and get that R_k also satisfies (1). This finishes the proof. □

Part 2

Minimal Surfaces

CHAPTER 3

Minimal Surfaces

A *minimal surface* is a submanifold in a Riemannian manifold whose mean curvature is identically zero. It is also the critical point of the area functional (See the first variation formula, Equation (3.1.12)).

In Section 3.1, we first introduce the so-called first and second variation formulas for the area functional. Subsequently, we derive some notations, such as minimal surfaces, Morse index and the stability condition. Finally, we focus on the Plateau Problem and related results.

In Section 3.2, we discuss some local properties of minimal surfaces, including the strong maximal principle (See Corollary 3.2.3) and the monotonicity formula (See Proposition 3.2.5) for the area. Particularly, the monotonicity formula gives a quantitative estimate for the area (See Corollary 3.2.6). This estimate can be generalized to the Riemannian case (See Theorem 3.2.7).

In Section 3.3, we study the topology of stable minimal surfaces. The stable minimal hypersurface is characterized by the first eigenvalue of the stable operator (See Lemma 3.3.1 and Theorem 3.3.4). In a manifold of positive scalar curvature, there are many topological constraints for stable minimals surfaces (See Proposition 3.3.5). For example, in the case of 3-manifolds with positive scalar curvature, the geometry of stable minimal surfaces is influenced by the extrinsic version of Cohn-Vossen inequality (See Corollary 3.3.6 and Theorem 3.3.10). As an application, we give a new proof of Theorem 2 of [**SY82**]. Finally, we use Theorem 3.3.8 to study contractible 3-manifolds whose scalar curvature has a decay at infinity.

3.1. Background

3.1.1. Mean Curvature. Let us consider a k-dimensional submanifold $\Sigma^k \subset (M^n, g)$ possibly with boundary.

In the following, if X is a vector field on $\Sigma \subset M$, then we let X^T and X^N denote the tangential and normal components, respectively. The covariant derivative ∇ on M induces a covariant derivative ∇^Σ on Σ and the second fundamental form A of Σ. More precisely, the induced covariant derivative ∇^Σ is given by

$$\nabla^\Sigma = (\nabla)^T \tag{3.1.1}$$

and the vector-valued bilinear form A on Σ is given for $X, Y \in T\Sigma$ by

$$A(X, Y) = (\nabla_X Y)^N. \tag{3.1.2}$$

Since the Lie bracket of X and Y is a tangential vector field in $T\Sigma$, it is easy to see that A is symmetric, i.e., $A(X, Y) = A(Y, X)$.

The *mean curvature vector* H at a point $x \in \Sigma$ is defined

$$H = \sum_{i=1}^{k} A(E_i, E_i),$$

where $\{E_i\}$ is an orthonormal basis for $T_x\Sigma$. Furthermore, the *squared norm of the second fundamental form* at x is given by

$$|A|^2 = \sum_{i,j=1}^{k} |A(E_i, E_j)|^2. \tag{3.1.3}$$

Recall also that the *Gauss equations* assert if $X, Y \in T_x\Sigma$, then

$$\begin{aligned} K_\Sigma(X,Y)|X \wedge Y|^2 &= K_M(X,Y)|X \wedge Y|^2 \\ &+ g(A(X,X), A(Y,Y)) - g(A(X,Y), A(X,Y)) \end{aligned} \tag{3.1.4}$$

where $|X \wedge Y|^2$ is given by

$$|X \wedge Y|^2 = g(X,X)g(Y,Y) - g(X,Y)^2$$

and $K_X(X,Y)$ and $K_\Sigma(X,Y)$ are the sectional curvatures of M and Σ , respectively, in the 2-plane spanned by X and Y.

For example, let $\Sigma^{n-1} \subset M^n$ be a hypersurface and N a unit normal vector field in a neighborhood of $x \in \Sigma$, then

$$\nabla_{(\cdot)} N : T_x\Sigma \to T_x\Sigma$$

is a symmetric map (often referred to as the *Weingarten map*) and its eigenvalues $\{\kappa_i\}_i^{n-1}$ are called the principle curvatures. Moreover,

$$g(H, N) = -\sum_{i=1}^{n-1} \kappa_i.$$

Finally, if X is a vector field over Σ, then the *divergence* of X at $x \in \Sigma$ is defined as

$$div_\Sigma X = \sum_{i=1}^{n-1} g(\nabla^T_{E_i} X, E_i) \tag{3.1.5}$$

where $\{E_i\}$ is an orthonormal basis for $T_x\Sigma$. Notice that div_Σ satisfies the Leibniz rule

$$div_\Sigma(fX) = g(\nabla_\Sigma f, X) + f\ div_\Sigma(X).$$

We can also use div_Σ to define the Laplace operator Δ_Σ on Σ by

$$\Delta_\Sigma f = div_\Sigma(\nabla^\Sigma f).$$

A function f is a *harmonic* function on Σ if $\Delta_\Sigma f = 0$.

Remark 3.1.1. Note that

$$\begin{aligned} div_\Sigma Y^N &= \sum_i g(E_i, \nabla_{E_i} Y^N) = -\sum_i g(Y^N, \nabla_{E_i} E_i) \\ &= -g(Y^N, H). \end{aligned} \tag{3.1.6}$$

3.1.2. First Variation Formula. Let $F : \Sigma^k \times (-\epsilon, \epsilon) \to M^n$ be a variation of a k-dimensional submanifold Σ^k with compact support and fixed boundary. That is, $F = \mathrm{Id}$ outside a compact set,

$$F(x, 0) = x$$

and for all $x \in \partial\Sigma$,

$$F(x, t) = x.$$

The vector F_t restricted to Σ is called the *variational vector field*. Now we want to compute the first variation of area for this one parameter family of surfaces. Let (x_i) be a local coordinate on Σ. Set

$$\begin{aligned} g_{ij}(t) &= g(F_{x_i}, F_{x_j}), \\ \nu(t) &= \sqrt{\det(g_{ij}(t))}\sqrt{\det(g^{ij}(0))}, \end{aligned} \tag{3.1.7}$$

where (a^{ij}) denotes the inverse of the metric (a_{ij}) and $1 \leqslant i, j \leqslant k$. Further, the area formula is

$$\mathrm{Vol}(F(\Sigma, t)) = \int_\Sigma \nu(t)\sqrt{\det\left(g_{ij}(0)\right)} \tag{3.1.8}$$

Differentiating it gives

$$\frac{d}{dt}\mathrm{Vol}(F(\Sigma, t))\Big|_{t=0} = \int_\Sigma \frac{d}{dt}\nu(t)|_{t=0}\sqrt{\det\left(g_{ij}(0)\right)}. \tag{3.1.9}$$

We may choose an orthonormal coordinate system, i.e. so that at the point x

$$g_{ij}(0) = \delta_{ij} = \begin{cases} 0, & i \neq j; \\ 1, & i = j. \end{cases}$$

Using that $[F_t, F_{x_i}] = 0$, under this coordinate, we get at x,

$$\begin{aligned} \frac{d}{dt}\det\left(g_{ij}(0)\right)(t)\Big|_{t=0} &= \sum_{ij}\frac{d}{dt}g_{ij}(t)\Big|_{t=0} g^{ij}(0) = \sum_{i=1}^k \frac{d}{dt}(g(F_{x_i}, F_{x_i}))\Big|_{t=0} \\ &= 2\sum_{i=1}^k g(\nabla_{F_t}F_{x_i}, F_{x_i}) = 2\sum_{i=1}^k g(\nabla_{F_{x_i}}F_t, F_{x_i}) \\ &= 2div_\Sigma(F_t). \end{aligned} \tag{3.1.10}$$

Therefore, from Equation (3.1.6) we have

$$\begin{aligned} \frac{d}{dt}\nu(t)\Big|_{t=0} &= div_\Sigma(F_t) = div_\Sigma(F_t^T) + div_\Sigma(F_t^N) \\ &= -g(H, F_t^N) + div_\Sigma(F_t^T). \end{aligned} \tag{3.1.11}$$

Integrating Equation (3.1.11) gives the so-called first variation formula:

$$\frac{d}{dt}\mathrm{Vol}(F(\Sigma, t)) = -\int_\Sigma g(F_t^N, H) = \int_\Sigma div_\Sigma(F_t). \tag{3.1.12}$$

Note that Stokes' Theorem was used to see that $\int_\Sigma div_\Sigma(F_t^T) = 0$. As a consequence of Equation (3.1.12), we see that Σ is a critical point for the area functional if and only if the mean curvature H vanishes identically.

DEFINITION 3.1.2. An immersed submanifold $\Sigma^k \subset (M^n, g)$ is said to be minimal if the mean curvature H vanishes identically.

For example, let Σ be the graph of a function $u : \mathbb{R}^n \to \mathbb{R}$. The hypersurface Σ is minimal in $\mathbb{R}^{n+1}$ if and only if u satisfies

$$div(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}) = 0. \tag{3.1.13}$$

It is the so-called minimal surface equation. Furthermore, in dimension three ($n = 2$), we know

$$\frac{|\mathrm{Hess}(u)|^2}{(1+|\nabla u|^2)^3} \leqslant |A|^2 \leqslant 2\frac{|\mathrm{Hess}(u)|^2}{1+|\nabla u|^2} \tag{3.1.14}$$

See Pages 28 and 29 in [**CM11**].

3.1.3. Second Variation Formula. Suppose now that $\Sigma^k \subset (M^n, g)$ is a minimal submanifold. We want to compute the second derivative of the area functional for a variation of Σ. Therefore, we consider a variation F of Σ with compact support. In fact, we assume that F is a normal variation, that is, on Σ we have

$$F_t^T(\cdot, 0) = 0.$$

As before, let $(x_i)_i$ be a local coordinate on Σ and set

$$\begin{aligned} g_{ij}(t) &= g(F_{x_i}, F_{x_j}), \\ \nu(t) &= \sqrt{\det(g_{ij}(t))}\sqrt{\det(g^{ij}(0))}. \end{aligned}$$

Differentiating the measure $\nu(t)$ gives

$$\frac{d^2}{dt^2}\mathrm{Vol}(F(\Sigma, t))\Big|_{t=0} = \int_\Sigma \frac{d^2}{dt^2}\nu(t)\Big|_{t=0}\sqrt{\det(g_{ij}(0))}. \tag{3.1.15}$$

Recall that the first derivative of the measure $\nu(t)$ can be written as

$$2\frac{d}{dt}\nu(t) = \sum_{ij} g'_{ij}(t)g^{ij}(t)\nu(t). \tag{3.1.16}$$

To evaluate $\frac{d^2}{dt^2}\nu(t)\Big|_{t=0}$ at some point $x \in \Sigma$, we may choose an orthonormal coordinate $(x_i)_i$ at x. Since the metric (g_{ij}) is the identity at x, the vectors F_{x_i} give an orthonormal basis for $T\Sigma$ at x. Differentiating Equation (3.1.16) then gives at x,

$$2\frac{d^2}{dt^2}\nu(t)\Big|_{t=0} = \sum_i g''_{ii}(0) - \sum_{ij} g'_{ij}(0)^2 + 1/2(\sum_i g'_{ii}(0))^2. \tag{3.1.17}$$

We have that $g'_{ii}(0) = g(\nabla_{F_t} F_{x_i}, F_{x_i}) = g(\nabla_{F_{x_i}} F_t, F_{x_i}) = -g(\nabla_{F_{x_i}} F_{x_i}, F_t)$. Since Σ is minimal and $F_t^T = 0$, we have $\sum_i g'_{ii}(0) = -g(\sum_i \nabla_{F_{x_i}} F_{x_i}, F_t) = -g(F_t, H) = 0$ as in Equation (3.1.6). Therefore, we get

$$2\frac{d^2}{dt^2}\nu(t)\Big|_{t=0} = \sum_i g''_{ii}(0) - \sum_{ij} g'_{ij}(0)^2. \tag{3.1.18}$$

LEMMA 3.1.3. *At the point x, we get*

$$\begin{aligned}\sum_{ij} g'_{ij}(0)^2 =& 4\sum_{ij} g(A(F_{x_i}, F_{x_j}), F_t)^2,\\ \sum_i g''_{ii}(0) =& 2\sum_{ij} g(A(F_{x_i}, F_{x_j}), F_t)^2 + 2|\nabla^N_\Sigma F_t|^2\\ &+ 2\sum_i g(R_M(F_{x_i}, F_t)F_t, F_{x_i}) + 2div_\Sigma(F_{tt}).\end{aligned}$$

PROOF. An easy computation gives that

$$g'_{ij}(0) = g(\nabla_{F_t} F_{x_i}, F_{x_j}) + g(F_{x_i}, \nabla_{F_t} F_{x_j}) = -2g(A(F_{x_i}, F_{x_j}), F_t).$$

This implies the first equation.

We compute that

$$g''_{ii}(0) = 2g(\nabla_{F_t}\nabla_{F_t} F_{x_i}, F_{x_i}) + 2g(\nabla_{F_t} F_{x_i}, \nabla_{F_t} F_{x_i}).$$

Next use the definition of the Riemann curvature tensor R_M of M to get

$$\begin{aligned}\sum_i g(\nabla_{F_t}\nabla_{F_t} F_{x_i}, F_{x_i}) &= \sum_i g(\nabla_{F_t}\nabla_{F_{x_i}} F_t, F_{x_i}) && \text{since}[F_t, F_{x_i}] = 0\\ &= \sum_i g(R(F_{x_i}, F_t)F_t, F_{x_i}) + \sum_i g(\nabla_{F_{x_i}}\nabla_{F_t} F_t, F_{x_i})\\ &= \sum_i g(R(F_{x_i}, F_{F_t})F_t, F_{x_i}) + div_\Sigma(F_{tt}).\end{aligned}$$

Therefore, we have

$$\begin{aligned}\sum_i g''_{ii}(0) &= 2\sum_i g(\nabla^T_{F_t} F_{x_i}, \nabla^T_{F_t} F_{x_i}) + 2\sum_i g(\nabla^N_{F_t} F_{x_i}, \nabla^N_{F_t} F_{x_i})\\ &\quad + 2\sum_i g(R(F_{x_i}, F_{F_t})F_t, F_{x_i}) + 2div_\Sigma(F_{tt})\\ &= 2\sum_{ij} g(A(F_{x_i}, F_{x_j}), F_t)^2 + 2|\nabla^N_\Sigma F_t|^2\\ &\quad + 2\sum_i g(R_M(F_{x_i}, F_t)F_t, F_{x_i}) + 2div_\Sigma(F_{tt}).\end{aligned}$$

□

The map $g(A(\cdot,\cdot), F_t) : T_x\Sigma \times T_x\Sigma \to \mathbb{R}$ is a symmetric bilinear map. Since $\{F_{x_i}\}$ is an orthonormal basis of $T\Sigma$ at x, the squared norm $|g(A(\cdot,\cdot), F_t)|^2(x)$ is equal to $\sum_{ij} g(A(F_{x_i}, F_{x_j}), F_t)^2$. Similarly, the trace $\mathrm{Tr}_\Sigma\, g(R_M(\cdot, F_t)F_t, \cdot)(x)$ equals $\sum_i g(R_M(F_{x_i}, F_t)F_t, F_{x_i})$.

Therefore, we get at x

$$\frac{d^2}{dt^2}\nu(t)\Big|_{t=0} = -|g(A(\cdot,\cdot),F_t)|^2 + |\nabla^N_\Sigma F_t|^2 - \mathrm{Tr}_\Sigma\, g(R_M(\cdot,F_t)\cdot,F_t) + div_\Sigma(F_{tt}). \tag{3.1.19}$$

Note that we used the skew symmetry of R_M to reverse the sign.

The vector $F_{tt}(\cdot,0)$ can be decomposed into two parts, the tangential part F^T_{tt} and the normal partial part F^N_{tt}. We use Stokes' theorem to see that $\int_\Sigma div_\Sigma(F^T_{tt}) = 0$. From the minimality of Σ and Equation (3.1.6), we have that $\int_\Sigma div(F^N_{tt}) = -\int_\Sigma g(F^N_{tt}, H) = 0$. Inserting Equation (3.1.19) into Equation (3.1.15), integrating and using the minimality of Σ, we get

$$\begin{aligned}\frac{d^2}{dt^2}\mathrm{Vol}(F(\Sigma,t)) &= -\int_\Sigma |g(A(\cdot,\cdot),F_t)|^2 \\ &\quad + \int_\Sigma |\nabla^N_\Sigma F_t|^2 - \int_\Sigma \mathrm{Tr}_\Sigma g(R_M(\cdot,F_t)\cdot,F_t) \\ &= -\int_\Sigma g(F_t, L(F_t)).\end{aligned} \tag{3.1.20}$$

The self-adjoint operator L is the so-called *stability operator* (or *Jacobi operator*) defined on a normal vector field X to Σ by

$$L(X) = \Delta^N_\Sigma X + \mathrm{Tr}_\Sigma g(R_M(\cdot,X)\cdot,X) + \hat{A}(X)$$

where $\hat{A}$ is *Simons' operator* defined by

$$\hat{A}(X) = \sum_{i,j=1}^k g(A(E_i,E_j),X)A(E_i,E_j)$$

and Δ^N_Σ is the *Laplacian on the normal bundle*, that is

$$\Delta^N_\Sigma X = \sum_{i=1}^k (\nabla_{E_i}\nabla_{E_i}X)^N - \sum_{i=1}^k (\nabla_{(\nabla_{E_i}E_i)^T}X)^N.$$

A normal vector field X with $L(X) = 0$ is said to be a *Jacobi field*.

We will adopt the convention that λ is a (Dirichlet) eigenvalue of L on $\Omega \subset \Sigma$ if there exists a non-trivial normal vector field X which vanishes on $\partial\Omega$ so that

$$L(X) + \lambda X = 0.$$

Definition 3.1.4. The *Morse index* of a compact minimal surface $\Sigma^k \subset (M^n, g)$ is the number of negative eigenvalues of the stability operator L (counting with multiplicity) acting on the space of smooth sections of the normal bundle which vanishes on the boundary.

A minimal surface submanifold $\Sigma^k \subset (M^n, g)$ is *stable* if for all variations F with fixed boundary and compact support

$$\frac{d^2}{dt^2}\mathrm{Vol}(F(\Sigma,t))\Big|_{t=0} = -\int_\Sigma g(F_t, L(F_t)) \geqslant 0.$$

A complete (possibly non-compact) minimal submanifold without boundary is said to be *stable* if all compact subdomains are stable.

For a minimal hypersurface $\Sigma^{n-1} \subset (M^n, g)$ with trivial normal bundle, the stability operator simplifies significantly since, in this case, it becomes an operator on functions. Namely, if we identify a normal vector field $X = \eta N$, then

$$L(\eta) = \Delta_\Sigma \eta + |A|^2\eta + Ric_M(N,N)\eta, \tag{3.1.21}$$

where Ric_M is the Ricci tensor of M.

Schoen and Yau [**SY79b**] pointed out that the stability operator (See Equation (3.1.21)) is linked with the scalar curvature of M(See Page 7 in [**SY79b**]).

PROPOSITION 3.1.5. *(See Page 7-8 of [**SY79b**])Let $\Sigma^n \subset (M^{n+1}, g)$ be a minimal surface with trivial normal bundle. Then the operator can be written as*

$$L = \Delta_\Sigma - \kappa_\Sigma + \kappa_M + 1/2|A|^2 \tag{3.1.22}$$

where κ_Σ and κ_M are the scalar curvature of Σ and M, respectively.

PROOF. Fixed a point $x \in \Sigma$, we may choose an orthonormal basis $\{E_i\}_{i=1}^{n+1}$ of T_xM. The unit vector E_{n+1} is equal to the unit normal vector of Σ. The Gauss Equations (See Equation (3.1.4)) assert

$$K_\Sigma(E_i, E_j) = K_M(E_i, E_j) + A_{ii}A_{jj} - A_{ij}^2 \tag{3.1.23}$$

where K_Σ and K_M are the sectional curvature of Σ and M respectively, $A_{ij} := A(E_i, E_j)$.

Summing Equation (3.1.23), we have

$$\sum_{1\leqslant i<j\leqslant n} K_\Sigma(E_i,E_j) = \sum_{1\leqslant i<j\leqslant n} K_M(E_i,E_j) + \sum_{1\leqslant i<j\leqslant n} A_{ii}A_{jj} - A_{ij}^2.$$

Therefore, by the minimality of Σ, the scalar curvature of M is

$$\begin{aligned}
\kappa_M &= \sum_{1\leqslant i<j\leqslant n} K_M(E_i,E_j) + Ric_M(E_{n+1},E_{n+1}) \\
&= \sum_{1\leqslant i<j\leqslant n} K_\Sigma(E_i,E_j) - \Big(\sum_{1\leqslant i<j\leqslant n} A_{ii}A_{jj} - A_{ij}^2\Big) \\
&+ Ric_M(E_{n+1},E_{n+1}) \\
&= \kappa_\Sigma + 1/2|A|^2 + Ric_M(N,N).
\end{aligned} \tag{3.1.24}$$

Therefore, putting Equation (3.1.24) into Equation (3.1.21), we get

$$L = \Delta_\Sigma - \kappa_\Sigma + 1/2|A|^2 + \kappa_M.$$

□

3.1.4. Existence of Minimal surfaces. The following fundamental existence problem for minimal surfaces is known as the Plateau problem:

Given a closed curve Γ*, find a minimal surface with boundary* Γ*.*

This problem was first formulated by Lagrange in 1760 and was studied extensively by Plateau in the 19th century. This question had led to many significant developments in partial differential equations and geometric measure theory, such as Morrey's works [**Mor48**, **Mor09**] and Fleming-Federer's works [**FF60**].

There are various solutions to this problem, depending on the exact definition of a surface (parametrized disc, integral current, $\mathbb{Z}_2$-current or varifold). In the following, we consider the version of the Plateau problem for parametrized discs.

In the case of $\mathbb{R}^3$, the solution was obtained in 1930 by J.Douglas [**Dou31**] and simultaneously by T.Radó [**Rad30**].

THEOREM 3.1.6. *(See [Theorem 4.1, Page 134] in [**CM11**]) Given a piecewise* C^1 *closed Jordan curve* $\Gamma \subset \mathbb{R}^3$*. there exists a map* $u : D \subset \mathbb{R}^2 \to \mathbb{R}^3$ *so that*

(1) $u : \partial D \to \Gamma$ *is monotone and onto;*
(2) $u \in C^0(\overline{D}) \cap W^{1,2}(D)$ *and is* C^∞ *in the interior of* D*;*
(3) The image of u *minimizes area among all maps from the discs with boundary* Γ*.*

The generalisation to the Riemannian manifold is due to C.B.Morrey [**Mor48**,**Mor09**].

For fixed boundary Γ, an area-minimizing disc is a solution to the Plateau Problem. Therefore, Morrey [**Mor48**, **Mor09**] used the variation method to find such a disc in a homegenous Riemannian 3-manifold. That is, take a sequence of mappings from the disc to the 3-manifold whose area are going to the infimum and attempt to extract a convergent subsequence. Morrey [**Mor48**, **Mor09**] pointed out that the limit of a subsequence of mappings is also the solution to the Dirichlet problem for the harmonic map.

For example, in $\mathbb{R}^3$, the existence of weak ($W^{1,2}$-)solution to the Dirichlet problem for the harmonic map is ensured by the Kondrachov compactness theorem for $W^{1,2}$(See [Theorem 7.22, Page 167] of [**GT15**]). Weyl's Lemma told us that the weak solution is smooth in the interior of domain. This result also follows from the standard regularity theory (See [Theorem 2.10, Page 23] of [**GT15**]). The above argument can be generalized to a homogenous Riemannian 3-manifold.

This solution is called Morrey's solution to the Plateau problem. In addition, the argument of Gulliver [**Gul73**] and Osserman [**Oss70**] pointed out that this solution has no interior branched point.

The remaining issue is the regularity of this solution up to the boundary. In the case of $\mathbb{R}^3$, J.J. Nitsche [**Nit69**] gave an answer:

THEOREM 3.1.7. *([Theorem 1, 315] in [**Nit69**])Let* u *be a function from* $D \subset \mathbb{R}^2$ *to* $\mathbb{R}$ *as in Theorem 3.1.6. If* Γ *is a regular Jordon curve of class* $C^{k,\alpha}$*, where* $k \geqslant 1$ *and* $0 < \alpha < 1$*, then* u *is* $C^{k,\alpha}$ *on* $\overline{D}$*.*

This result can be generalized to the Riemannian 3-manifolds in [**HH70**].

In various works, compact 3-manifolds with mean convex boundary were intensively studied. Let (M, g) be a compact 3-manifold with boundary. The boundary ∂M is mean convex if

- ∂M is a piecewise smooth 2-manifold consisting of smooth surfaces $\{H_i\}_i$
- for each i, the mean curvature of H_i is nonnegative.

THEOREM 3.1.8. *(See [**MIY80, MIY82**] or[Theorem 6.28, Page 224] of [**CM11**]) Let (M^3, g) be a compact Riemannian 3-manifold whose boundary is mean convex and γ a simple closed curve in ∂M which is null-homotopic in M. Then, γ bounds an area-minimizing disc and any such least area disc is properly embedded.*

This theorem will be repeatedly used in the following.

3.2. Local structures of Minimal Surfaces

It is classical that the minimal surface theory consists of two aspects: PDE and Geometry. These two aspects give various results about local structures of minimal surfaces, such as the maximum principle and the monotonicity formula.

3.2.1. Minimal surfaces are locally graphical. We begin with a minimal surface $\Sigma \subset \mathbb{R}^3$. The *Gauss map* is a continuous choice of a unit normal

$$N : \Sigma \to \mathbb{S}^2 \subset \mathbb{R}^3.$$

There are two choices of such a map N and $-N$ corresponding to a choice of orientation of Σ. Suppose that E_1,E_2 is an orthonormal frame on Σ. We know that

$$< \nabla_{E_i} N, E_j >= -A_\Sigma(E_i, E_j)$$

where A_Σ is the second fundamental form (See Equation (3.1.2)) of Σ. Therefore,

$$|dN| \leqslant |A|. \tag{3.2.1}$$

LEMMA 3.2.1. *(See [Lemma 2.4, Page 74] in [**CM11**])Let $\Sigma \subset \mathbb{R}^3$ be an immersed minimal surface with*

$$16s^2 \sup_\Sigma |A_\Sigma|^2 \leqslant 1. \tag{3.2.2}$$

If $x \in \Sigma$ and $d^\Sigma(x, \partial\Sigma) \geqslant 2s$, then the geodesic ball $B^\Sigma(x, 2s)$, centered at x with radius $2s$, can be written as a graph of a function u over $T_x\Sigma$ with $|\nabla u| \leqslant 1$ and $\sqrt{2}s|Hess(u)| \leqslant 1$.

PROOF. We define

$$d_{x,y} := d^{\mathbb{S}^2}(N(x), N(y)). \tag{3.2.3}$$

Therefore, we see that

$$< N(x), N(y) >= \cos d_{x,y}. \tag{3.2.4}$$

Recall that $|dN| \leqslant |A|$ (See Equation (3.2.1)). Therefore, given $y \in B^\Sigma(x, 2s)$, integrating (3.2.2) along a geodesic from x to y gives that

$$\sup_{y \in B^\Sigma(x,2s)} d_{x,y} \leqslant \frac{1}{2} < \frac{\pi}{4}. \tag{3.2.5}$$

Therefore, it follows that $B^\Sigma(x, 2s)$ is contained in the graph of a function u over a subset of $T_x\Sigma$.

We may choose a coordinate $(x_k)_{k=1}^3$ on $\mathbb{R}^3$ so that

$$x = (0,0,0),$$
$$T_x\Sigma = \{(x_1, x_2, 0)\}.$$

Therefore, $N(x) = (0,0,1)$ and $N(y) = \frac{(-u_{x_1}, -u_{x_2}, 1)}{\sqrt{1+|\nabla u|^2}}$. From Equation (3.2.3), we have

$$(1 + |\nabla u|^2) =< N_x, N_y >^{-2} = \cos^{-2}(d_{x,y})$$

where the last equality comes from Equation (3.2.1).

If $y \in B^\Sigma(x, 2s)$, Equation (3.2.5) implies that $|\nabla u(y)| \leqslant 1$. The Hessian estimate of u comes from the gradient and curvature estimate together with Equation (3.1.14):

$$|Hess(u)|^2 \leqslant (1 + |\nabla u|^2)^3 |A|^2 \leqslant \frac{1}{2} s^{-2}.$$

□

3.2.2. Strong Maximal Principle. First note that the difference of two solutions to the minimal surface equation (See Equation (3.1.13)) satisfies an elliptic divergent form equation (where the bound on the ellipticity depends on the gradient of the minimal graphs).

LEMMA 3.2.2. *If u_1 and u_2 are two solutions to the minimal surface equation (See Equation (3.1.13)) on a domain $\Omega \subset \mathbb{R}^n$, then $v := u_1 - u_2$ satisfies an equation of the form:*

$$div(A(x)\nabla v) = 0 \tag{3.2.6}$$

where each eigenvalue λ_k of the matrix $A(x) := (a_{i,j}(x))_{n\times n}$ satisfies $\mu < \lambda_k < 1/\mu$, where μ depends only on the upper bounds for the gradient of $|\nabla u_1|$ and $|\nabla u_2|$.

PROOF. Define the mapping $F : \mathbb{R}^n \to \mathbb{R}^n$ by

$$F(X) = \frac{X}{(1 + |X|^2)^{1/2}}.$$

Note that each u_k satisfies $div(F(\nabla u_k)) = 0$. We know that

$$\begin{aligned} F(\nabla u_1) - F(\nabla u_2) &= \int_0^1 \frac{d}{dt} F(t\nabla u_1 + (1-t)\nabla u_2) dt \\ &= \int_0^1 dF(t\nabla u_1 + (1-t)\nabla u_2) \cdot \nabla(u_1 - u_2) dt \\ &= (\int_0^1 dF(t\nabla u_1 + (1-t)\nabla u_2) dt) \cdot \nabla(u_1 - u_2). \end{aligned} \tag{3.2.7}$$

From this, we can conclude that $v = u_1 - u_2$ satisfies an equation of the form $div(A(x)\nabla v) = 0$, where the matrix $A(x)$ is given by Equation (3.2.7).

The remaining is to show that Equation (3.2.6) is a uniformly elliptic equation.

Given a unit vector $V \in \mathbb{S}^{n-1} \subset \mathbb{R}^n$ and $X \in \mathbb{R}^n$, we see that

$$dF(X)V = \frac{V}{(1 + |X|^2)^{1/2}} - \frac{< X, V >}{(1 + |X|^2)^{3/2}} X.$$

In particular, taking inner product with V gives

$$\begin{aligned}(1+|X|^2)^{3/2} < dF(X)V, V > &= (1+|X|^2) - < X, V >^2 \\ &\geqslant (1+|X|^2) - |X|^2 = 1.\end{aligned}$$

It follows that $A(x)$ is a weighted average of positive definite matrix and thus also a positive definite matrix. □

COROLLARY 3.2.3. *Let $\Omega \subset \mathbb{R}^n$ be an open connected neighborhood of the origin. If $u_1, u_2 : \Omega \to \mathbb{R}$ are two solutions to the minimal surface equation (See Equation* (3.1.13)*) with $u_1 \leqslant u_2$ and $u_1(0) = u_2(0)$, then $u_1 \equiv u_2$.*

PROOF. Lemma 3.2.2 tells us that the difference $v := u_1 - u_2$ satisfies $div(a_{i,j}\nabla v) = 0$, where the matrix $a_{i,j}$ is positive definite. We apply the strong maximum principle for the linear elliptic equation to v. (See [**HL11**] or [Theorem 3.5, Page 48] of [**GT15**]). □

As in Lemma 3.2.1, a minimal hypersurface can be locally written as the graph of a solution to the minimal surface equation. As a consequence of Corollary 3.2.3 we know that

COROLLARY 3.2.4. *If $\Sigma_1, \Sigma_2 \subset \mathbb{R}^n$ are two complete connected minimal hypersurfaces (without boundary), $\Sigma_1 \cap \Sigma_2 \neq \varnothing$ and Σ_2 lies on one side of Σ_1, then $\Sigma_1 = \Sigma_2$.*

3.2.3. Monotonicity Formula and Area estimates.

PROPOSITION 3.2.5. *(The Monotonicity Formula) Suppose that $\Sigma^k \subset \mathbb{R}^n$ is a minimal submanifold and $x_0 \in \Sigma$. Then for all $0 < s < t$,*

$$\frac{Vol(B(x_0,t)\cap\Sigma)}{t^k} - \frac{Vol(B(x_0,s)\cap\Sigma)}{s^k} = \int_{B(x_0,t)\setminus B(x_0,s)\cap\Sigma} \frac{|(x-x_0)^N|^2}{|x-x_0|^{k+2}}, \tag{3.2.8}$$

where $(x-x_0)^N$ is the projection of the normal part of Σ of the vector $(x-x_0)$.

See [Chapter 3.2, Page 24-26] of [**CM11**] for a proof.
As a consequence, we have

COROLLARY 3.2.6. *Suppose that $\Sigma^k \subset \mathbb{R}^n$ is a minimal submanifold and $x_0 \in \mathbb{R}^n$. Then the function*

$$\Theta_{x_0}(s) = \frac{Vol\, B(x_0,s)\cap\Sigma)}{Vol(B(0,s)\subset\mathbb{R}^k)}$$

is a non-decreasing function of s. Moreover, if $x_0 \in \Sigma$, then $\Theta_{x_0}(s) \geqslant 1$ and

$$Vol(\Sigma \cap B(x_0,s)) \geqslant \omega_k s^k, \tag{3.2.9}$$

where ω_k is the volume of the unit ball $B(0,1) \subset \mathbb{R}^n$.

The area estimate (3.2.9) can be generalized to the Riemannian manifolds. Before we state it, we will recall the coarea formula. This formula asserts (See, for instant, Chapter 3.2 of [**Fed14**] for a proof) that if (M, g) is a Riemannian manifold and the function

$$h : M \to \mathbb{R}$$

is a proper (i.e. $h^{-1}((\infty,t])$ is compact for all $t\in\mathbb{R}$) Lipschitz function, then for any locally integral function f on M and $t\in\mathbb{R}$,

$$\int_{h\leqslant t} f|\nabla_M h| = \int_\infty^t \int_{h=\tau} f d\tau. \tag{3.2.10}$$

THEOREM 3.2.7. *(See [Lemma 1, Page 445] of [**MIY80**]) Let (M^{n+1},g) be a Riemannian manifold whose sectional curvature is bounded by a positive constant K and Σ^n a minimal submanifold. If for some point $x_0\in\Sigma$, $d^M(x_0,\partial M)$ and $d^M(x_0,\partial\Sigma)$ are both greater than a constant $\epsilon>0$, then for any $\delta\leqslant\min\{\epsilon,i(M)\}$*

$$Vol(B(x_0,\delta)\cap\Sigma)\geqslant C_n K\int_0^\delta t^{-1}(sin(Kt))^n dt, \tag{3.2.11}$$

where $i(M)$ is the radius of injectivity of M and C_n depends only on n.

PROOF. Let $r(x)$ be the distance function of M from x_0 to $x\in\Sigma$. If $r(x)$ is smaller than $i(M)$, the Hessian comparison theorem (See [Theorem 27, Page 175] of [**PAR06**]) gives

$$\mathrm{Hess}(e_i,e_i)(r)(x) := \nabla_{e_i}\nabla_{e_i} r - \nabla_{\nabla_{e_i}e_i}(r)\geqslant K\cot(Kr) \tag{3.2.12}$$

where $x\in\Sigma$, $\{e_i\}_{i=1}^n$ is an orthonormal basis of $T_x\Sigma$ and N is a unit normal vector of Σ at x.

Summing Equation (3.2.12) and using the minimality of Σ, we have

$$\begin{aligned}
\Delta_\Sigma r^2(x) &\geqslant 2r(x)\Delta_\Sigma r(x) = 2r(x)(\sum_{i=1}^n \nabla_{e_i}\nabla_{e_i} r - \nabla_{\nabla^\Sigma_{e_i}(e_i)} r)\\
&= 2r(x)(\sum_{i=1}^n \mathrm{Hess}(e_i,e_i)(r) + \sum_{i=1}^n \nabla_{\nabla^N_{e_i}(e_i)} r),\\
&= 2r(x)(\sum_{i=1}^n \mathrm{Hess}(e_i,e_i)(r) + \nabla_{H(x)} r), \qquad \text{because } H(x)=\sum_{i=1}^n \nabla^N_{e_i} e_i\\
&= 2r(x)\sum_{i=1}^n \mathrm{Hess}(e_i,e_i)(r), \qquad \text{since } H(x)=0\\
&\geqslant 2nKr\cot(Kr),
\end{aligned} \tag{3.2.13}$$

where $x\in\Sigma$, ∇^Σ, ∇^N and $H(x)$ are defined as in Section 3.1, (See also the argument [Page 243] of [**SY77**]). Integrating Equation (3.2.13) over $B(x,t)$ and noting that $|\nabla r|\leqslant 1$, we have,

(3.2.14)

$$\begin{aligned} 2t\mathrm{Vol}(\partial(\Sigma\cap B(x_0,t)) &\geqslant \int_{\partial(\Sigma\cap B(x_0,t))} <\nabla^{\Sigma} r^2, \mathbf{n}>, \qquad \text{since } |\nabla_{\Sigma} r| \leqslant |\nabla r| \leqslant 1 \\ &= \int_{\Sigma\cap B(x_0,t)} \Delta_{\Sigma} r^2, \qquad \text{doing integration by parts,} \\ &\geqslant 2nK \int_{B(x_0,t)\cap\Sigma} r\cot(Kr) \qquad \text{from Equation(3.2.13).} \end{aligned}$$

where $\mathbf{n}$ is the outward unit normal vector of $\partial(\Sigma\cap B(x_0,t))$ in Σ.

Let $C(t)=\int_{B(x_0,t)\cap\Sigma} r\cot(Kr)$. Then by the coarea formula (See Equation (3.2.10)) and the fact that $|\nabla r|\leqslant 1$, we have that

$$\frac{\partial C(t)}{\partial t} = \int_{\{x \mid r(x)=t\}\cap\Sigma} \frac{r\cot(Kr)}{|\nabla r|} \geqslant t\cot(Kt)\mathrm{Vol}(\partial(\Sigma\cap B(x_0,t))). \tag{3.2.15}$$

Inserting Equation (3.2.14) into Equation (3.2.15), we find

$$\frac{\partial C(t)}{\partial t} \geqslant nK\cot(Kt)C(t). \tag{3.2.16}$$

It is easy to verify that

$$\lim_{t\to 0} C(t)\sin(Kt)^{-1} = K^{-n-1}C_n$$

where C_n is a positive constant depending only on n. It follows from Equation (3.2.16) that

$$C(t) \geqslant K^{-n-1}C_n(\sin(Kt))^n \tag{3.2.17}$$

for all $t\leqslant \min\{\epsilon, i(M)\}$. Therefore, Equation (3.2.14) shows that

$$\mathrm{Vol}(\partial(\Sigma\cap B(x_0,t))) \geqslant nC_nK^{-n}t^{-1}(\sin(Kt))^n. \tag{3.2.18}$$

By the coarea formula (See Equation (3.2.14)) again,

$$\begin{aligned} \mathrm{Vol}(\Sigma\cap B(x_0,\delta)) &\geqslant \int_0^{\delta} \mathrm{Vol}(\partial(\Sigma\cap B(x_0,\tau)))d\tau \\ &\geqslant nC_nK^{-n}\int_0^{\delta} \tau^{-1}(\sin(K\tau))^n d\tau. \end{aligned} \tag{3.2.19}$$

□

3.3. Stable minimal surfaces

In this section, let us consider a stable minimal surface $\Sigma^n\subset(M^{n+1},g)$ with trivial normal bundle. First, the stable condition is linked with the first eigenvalue of the operator L (See (3.1.21) or (3.1.22)). Second, if (M,g) has positive curvature, the stability condition gives several topological restrictions on Σ.

3.3.1. Eigenvalues and Stability. The stability of a minimal surface is characterized by the first eigenvalue of L (See (3.1.21) or (3.1.22)). A minimal hypersurface $\Sigma^n \subset (M, g)$ is stable if for any smooth function η with compact support,

$$\int_\Sigma -\eta L(\eta) = \int_\Sigma |\nabla_\Sigma \eta|^2 - Ric(N,N)\eta^2 - |A|^2\eta^2 \geqslant 0.$$

We see that the stable condition is equivalent to the first eigenvalue $\lambda_1(L, \Omega) \geqslant 0$ for each $\Omega \subset \Sigma$ where

$$\lambda_1(L,\Omega) := \inf\{\int_\Sigma -\eta L(\eta) | \eta \in W_0^{1,2}(\Omega) \text{ and } \int_\Sigma \eta^2 = 1\}. \tag{3.3.1}$$

By standard elliptic theory (see [**GT15**] or [**HL11**]), we get the following:

LEMMA 3.3.1. *If L and $\Omega \subset \Sigma$ are assumed as above and $\lambda_1 := \lambda_1(L, \Omega)$, then the eigenfunction $u \in W_0^{1,2}(\Omega)$ of the first eigenvalue λ_1 (i.e. $L(u) = -\lambda_1 u$) is smooth.*

It follows from the regularity theory for elliptic equations (See [Theorem 8.14, Page 188] of [**GT15**] or [**HL11**]).

Together with the Harnack inequality, we see that any eigenfunction of the first eigenvalue can not change sign.

LEMMA 3.3.2. *Assume that u is a smooth function on Ω that vanishes on $\partial\Omega$. If $L(u) = -\lambda_1 u$ where $\lambda_1 = \lambda_1(L, \Omega)$, then u can not change sign.*

PROOF. We may assume that u is not identically zero and $\int_\Sigma u^2 = 1$. Since u vanishes on $\partial\Omega$, so does $|u|$. In fact, $|u|$ also achieve the minimum in (3.3.1). By Lemma 3.3.1, $|u|$ is smooth and $L(|u|) = -\lambda_1|u|$. Since $|u| \geqslant 0$ and $|u|$ is not identically zero, the Harnack inequality (See [Theorem 8.20, Page 199] of [**GT15**]) implies $|u| > 0$ in Ω. Hence, u can not change sign. □

Next, let us consider the positive solution to the stable operator L (See Equation (3.1.21) or Equation (3.1.22)).

PROPOSITION 3.3.3. *Let $\Sigma^n \subset (M^{n+1}, g)$ be a minimal hypersurface with trivial normal bundle, L its stability operator (See Equation (3.1.21) or (3.1.22)), and $\Omega \subset \Sigma$ a bounded domain. If there exists a positive function u on Ω with $L(u) = 0$, then Ω is a stable minimal surface.*

PROOF. Set $q(x) = |A|^2 + Ric_M(N,N)$ so that $L = \Delta_\Sigma + q$. Since $u > 0$, the function $w := \log u$ is well-defined and satisfies

$$\Delta_\Sigma w = -q - |\nabla_\Sigma w|^2. \tag{3.3.2}$$

Let us consider any compactly supported function f on Ω. Multiplying both sides of Equation (3.3.2) with f^2 and using integrating by parts give

$$\begin{aligned} \int_\Omega f^2 q + f^2|\nabla_\Sigma w|^2 = -\int_\Omega f^2 \Delta_\Sigma w &\leqslant 2\int_\Sigma f|\nabla_\Sigma f||\nabla_\Sigma w| \\ &\leqslant \int_\Sigma f^2|\nabla_\Sigma w|^2 + |\nabla_\Sigma f|^2, \end{aligned} \tag{3.3.3}$$

where the second inequality follows from the Cauchy-Schwarz inequality. Canceling the $\int_\Sigma f^2|\nabla_\Sigma w|^2$ term, we have that

$$\int_S -fL(f) \geqslant 0$$

This finishes the proof. □

We will give a characterization for a complete (non-compact) stable minimal hypersurfaces with trivial normal bundle. For such a hypersurface, the stability is equivalent to the existence of positive solution to the stability operator.

THEOREM 3.3.4. *(See [Theorem 1, Page 201] [**FCS80**]) Let $\Sigma^n \subset (M, g)$ be a complete non-compact minimal hypersurface with trivial normal bundle, then the following are equivalent:*

(1) $\lambda_1(L, \Omega) \geqslant 0$ for any bounded domain $\Omega \subset \Sigma$;
(2) $\lambda_1(L, \Omega) > 0$ for any bounded domain $\Omega \subset \Sigma$;
(3) there is a positive function u over Σ with $L(u) = 0$.

PROOF. By Proposition 3.3.3, (3) implies (1).

Clearly, (2) implies (1). To see the equivalence of (1) and (2), we consider any bounded domain Ω_0 and choose a strictly larger bounded domain Ω_1. The variational characterization of the first eigenvalue (See (3.3.1)) implies that

$$\lambda_1(L, \Omega_0) \geqslant \lambda_1(L, \Omega_1) \geqslant 0,$$

where the second inequality follows from (1). Let u_0 be the first eigenfunction for L in Ω_0. We define u_1 on Ω_1 by

$$u_1(x) = \begin{cases} u_0(x), & if \quad x \in \Omega_0; \\ 0, & otherwise. \end{cases}$$

where $u_0 \geqslant 0$ and u_0 is not identically zero (See Lemma 3.3.2).

If we had that $\lambda_1(L, \Omega_0) = 0$, then the nonnegative function u_1 is an eigenfunction of $\lambda_1(L, \Omega_1)$. Lemma 3.3.1 tells that $L(u_1) = \lambda_1(L, \Omega)u_1$ and u_1 is smooth. Since $u_1 = 0$ on $\Omega_1 \backslash \Omega_0$, the Harnack inequality (See [Theorem 8.20, Page 199] of [**GT15**]) implies that u_1 is identically zero on Ω_1. This is not possible. We can conclude that $\lambda_1(L, \Omega_0) > 0$. The equivalence of (1) and (2) follows.

The remaining is to show that (2) implies (3). To do this, fixed a point $p \in \Sigma$ and any $r > 0$, let $B^\Sigma(p, r)$ be the geodesic ball in Σ with radius r and centered at p. Then the first eigenvalue $\lambda_1(L, B^\Sigma(p, r))$ is greater than zero. By Fredholm alternative (See [Theorem 6.15, Page 107] of [**GT15**]), there exists a function v_r satisfying:

$$L(v_r) = -|A|^2 - Ric(N, N) \text{ on } B^\Sigma(p, r) \text{ and } v_r = 0 \text{ on } \partial B^\Sigma(p, r). \tag{3.3.4}$$

Setting $u_r = v_r + 1$, Equation (3.3.4) gives

$$L(u_r) = 0 \text{ in } B^\Sigma(p, r) \text{ and } u_r = 1 \text{ on } \partial B^\Sigma(p, r). \tag{3.3.5}$$

We claim that

$$u_r > 0 \text{ on } B^{\Sigma}(p,r). \tag{3.3.6}$$

If u_r is not a non-negative function, we choose a nonempty connected component $\Omega \subset \Sigma$ of the set

$$\{x \in B^{\Sigma}(p,r) | u_r(x) < 0\}.$$

Then, $L(u_r) = 0$ on Ω and $u_r = 0$ on $\partial\Omega$. This is to say, the first eigenvalue $\lambda_1(L,\Omega) \leqslant 0$. This is in contradiction with (2). Therefore, we conclude that $u_r \geqslant 0$.

The claim follows from the Harnark inequality (See [Theorem 8.20, Page 199] of [**GT15**]).

For each r, we define a positive function by

$$w_r(x) := u_r(x)(u_r(p))^{-1} \text{ on } B^{\Sigma}(p,r)$$

and see that $L(w_r) = 0$ and $w_r(p) = 1$.

Now, let us consider a compact set $K \subset B^{\Sigma}(p,R_0)$. We use the Harnack inequality (See [Theorem 8.20, Page 199] of [**GT15**]) to have a positive constant $C(p,R_0)$, only depending on p and R_0, satisfying for any $r > 2R_0$,

$$|w_r(x)| \leqslant C(p,R_0) \text{ if } x \in B^{\Sigma}(p,R_0). \tag{3.3.7}$$

The interior Schauder estimate (See [Theorem 6.2, Page 90] of [**GT15**]) gives that

$$|w_r|_{C^{2,\alpha}_K} \leqslant C_K$$

where the constant C_K only depends on K and $0 < \alpha < 1$.

To sum up, we have a uniform $C^{2,\alpha}$ estimate for any w_r where $r > 2R_0$. We use the Arzela-Ascoli theorem to extract a subsequence of w_r that converges uniformly to w. This convergence ensures that $L(w) = 0$ and $w \geqslant 0$. The Harnack inequality (See [Theorem 8.20, Page 199] of [**GT15**]) tells us that w is a positive function. This finishes the proof of the theorem. □

3.3.2. Global Structure (I): the compact case.

PROPOSITION 3.3.5. *(See Page 8 of [**SY79b**])Let $\Sigma^n \subset (M^{n+1}, g)$ be a compact stable minimal surface with trivial normal bundle. If (M,g) has positive scalar curvature, then (Σ, g_Σ) is conformally equivalent to a metric of positive scalar curvature, where g_Σ is the induced metric and $n > 2$.*

PROOF. The stability of the minimal surface Σ and Equation (3.1.22) give

$$\int_\Sigma \kappa_M \eta^2 - \kappa_\Sigma \eta^2 + 1/2|A|^2\eta^2 \leqslant \int_\Sigma |\nabla_\Sigma \eta|^2 \tag{3.3.8}$$

for any smooth function η on Σ.

Since $\kappa_M > 0$ on Σ, we can conclude that

$$-\int_\Sigma \kappa_\Sigma \eta^2 < \int_\Sigma |\nabla_\Sigma \eta|^2 \tag{3.3.9}$$

for all smooth function η.

Let λ be the first eigenvalue of the operator $\Delta_\Sigma - \frac{n-2}{2(n-1)}\kappa_\Sigma$ and u an eigenfunction of λ. That is,

$$\Delta_\Sigma u - \frac{n-2}{4(n-1)}\kappa_\Sigma u = -\lambda u. \tag{3.3.10}$$

From (3.3.9), we can conclude that $\lambda > 0$.

Otherwise ($\lambda \leqslant 0$), multipling two sides of (3.3.10) by u and integrating, we see that

$$\int_\Sigma |\nabla_\Sigma u|^2 = -\frac{n-2}{4(n-1)}\int_\Sigma \kappa_\Sigma u^2 + \lambda\int_\Sigma u^2 \leqslant -\frac{n-2}{2(n-1)}\int_\Sigma \kappa_\Sigma u^2 \leqslant \frac{n-2}{2(n-1)}\int_\Sigma |\nabla_\Sigma u|^2$$

where the last inequality follows from (3.3.9). This is impossible.

As the argument in Lemma 3.3.2, we have that the eigenfunction u is smooth and a positive function. Multiplying the metric g_Σ by $u^{\frac{4}{n-2}}$, under the new metric, the scalar curvature of Σ is

$$u^{-\frac{n+2}{n-2}}(\kappa_\Sigma - \frac{4(n-1)}{n-2}\Delta_\Sigma u) = \frac{4(n-1)}{n-2}\lambda u^{-\frac{n+2}{n-2}} > 0.$$

□

Next, we consider a stable minimal surface $\Sigma \subset (M^3, g)$.

Corollary 3.3.6. *(See [Theorem 5.1, Page 139] of [**SY79a**])Let $\Sigma^2 \subset (M^3, g)$ be a closed stable minimal surface with trivial normal bundle. If the complete manifold (M^3, g) has positive scalar curvature, then Σ is $\mathbb{S}^2$ or $\mathbb{R}P^2$ and*

$$\int_\Sigma \kappa_M + 1/2|A|^2 \leqslant 4\pi.$$

Proof. Since Σ is compact and has no boundary, we choose the constant function $\eta = 1$ as a cut-off function. Equation (3.1.22) gives

$$\int_\Sigma \kappa_M + 1/2|A|^2 \leqslant \int_\Sigma K_\Sigma = 2\pi\chi(\Sigma), \tag{3.3.11}$$

where the last equity follows from the Gauss-Bonnet formula. As a consequence, we see that $\chi(\Sigma) > 0$. That is to say, Σ is $\mathbb{S}^2$ or $\mathbb{R}P^2$. Therefore, $\chi(\Sigma) \leqslant 2$. It gives the inequality in the assertion. □

In the next part, we will use Cohn-Vesson's inequality [**Coh35**] to generalize (3.3.11) to the non-compact case.

3.3.3. Global Structure (II): the non-compact case. By Theorem 3.3.4, for a complete (non-compact) stable minimal surface $\Sigma \subset (M^3, g)$, there is a positive function u over Σ with $L(u) = 0$.

Theorem 3.3.7. *([Theorem 2, Page 126] in [**Fis85**]) Let $\Sigma^2 \subset (M, g)$ be a complete (non-compact) stable minimal hypersurface. If the complete manifold (M^3, g) has non-negative scalar curvature ($\kappa(x) \geqslant 0$), then the new metric $u^2 d^2 s$ is a complete metric on Σ with non-negative sectional curvature, where $d^2 s$ is the induced metric and u is the positive function with $L(u) = 0$ (as in Theorem 3.3.4).*

PROOF. By Theorem 3.3.4, there is a positive function u with $L(u) = 0$. In addition, there exists a minimal geodesic ray $\gamma(t) : [0, \infty) \to \Sigma$ in the metric $\tilde{d}^2s := u^2 d^2s$, where t is arclength in the original metric d^2s. It is obtained as below:

Fixed $x \in M$ and for any $R > 0$, let us consider a geodesic ball $B^\Sigma(x, R)$ in the complete manifold (Σ, d^2s). Define $u_R := u + \eta_R$ where η_R is a smooth positive function satisfying

$$\eta_R = \begin{cases} 0, & |x| < R; \\ 1, & |x| > 2R. \end{cases}$$

Then, since u_R is bounded away from zero, the metric $u_R^2 d^2s$ is complete. Therefore, there is a shortest geodesic γ_R from x to $\partial B^\Sigma(x, R)$ in the metric $u_R^2 d^2s$. We can conclude that γ_R must stay in $B^\Sigma(x, R)$. (Otherwise, under the metric $u_R^2 d^2s$, there is another curve connecting x to $\partial B^\Sigma(x, R)$ whose length is shorter than γ_R's.)

Since $u = u_R$ on $B^\Sigma(x, R)$, γ_R is also a minimizing geodesic in $\tilde{d}^2s$. Each γ_R can be parametrized with respect to arclength in the metric d^2s. Let us consider the limit of these minimizing geodesics in $(\Sigma, \tilde{d}^2s)$. The sequence $\{\gamma_R\}$ sub-converges to a minimizing geodesic ray $\gamma(t)$ that is parametrized by arclength in the metric d^2s.

It remains to show the completeness of $\tilde{d}^2s$.

By the construction of γ, the completeness of $\tilde{d}^2s$ will follow if we can show that γ has infinite length under the metric $\tilde{d}^2s$, i.e. it is sufficient to show that

$$\int_0^\infty u(\gamma(t))ds = \infty. \tag{3.3.12}$$

Since γ is a minimizing geodesic in $\tilde{d}^2s = u^2 d^2s$, the second variation formula of acrlength gives

$$\int_0^\infty ((\frac{d\phi}{\tilde{ds}})^2 - \tilde{K}\phi^2)\tilde{ds} \geqslant 0, \tag{3.3.13}$$

where the smooth function ϕ has compact support in $(0, \infty)$, $\frac{d\phi}{\tilde{ds}} = u^{-1}\frac{d\phi}{ds}$, $\tilde{ds} = uds$ and the sectional curvature $\tilde{K}$ of $(\Sigma, \tilde{d}^2s)$ is

$$\tilde{K} = u^{-2}(K - \Delta_\Sigma \log u). \tag{3.3.14}$$

In addition, u is a positive function with $L(u) = 0$. Namely,

$$L(u) = \Delta_\Sigma u - Ku + (\kappa(x) + 1/2|A|^2)u = 0$$

where Δ_Σ is the Laplace-Beltrami operator respect to (Σ, d^2s) and K is the sectional curvature of (Σ, d^2s). Since $\kappa \geqslant 0$, we see that $\Delta_\Sigma u \leqslant Ku$. In addition,

$$\Delta_\Sigma \log u = u^{-1}\Delta_\Sigma u - \frac{|\nabla_\Sigma u|^2}{u^2}. \tag{3.3.15}$$

Note that together with (3.3.14) and (3.3.15), we can conclude that $\tilde{K} \geqslant 0$. Inserting (3.3.14),(3.3.15) into (3.3.13), we see that

$$\int_0^\infty \frac{(u')^2}{u^3}\phi^2 ds \leqslant \int_0^\infty u^{-1}(K - \Delta_\Sigma \log u)\phi^2 ds \leqslant \int_0^\infty u^{-1}(\phi'(s))^2 ds \tag{3.3.16}$$

where the first inequality follows from (3.3.14) and (3.3.15), the last inequality follows from (3.3.13) and $u(s) = u(\gamma(s))$.

We now show that $\int_0^\infty uds = \infty$. Take $\phi = u\psi$, where the smooth function ψ has compact support in $(0, \infty)$. Then $\phi' = u'\psi + u\psi'$ and

$$u^{-1}(\phi')^2 = u^{-1}(u')^2\psi^2 + u(\psi')^2 + 2u'\psi'\psi.$$

The inequality (3.3.16) shows that

$$\int_0^\infty \frac{(u')^2}{u^3}\phi^2 ds = \int_0^\infty \frac{(u')^2}{u}\psi^2 ds \leqslant \int_0^\infty u^{-1}(u')^2\psi^2 + u(\psi')^2 + 2u'\psi'\psi ds.$$

Therefore,

$$0 \leqslant \int_0^\infty u(\psi')^2 + 2u'\psi\psi' ds.$$

The integration by parts gives

$$0 \leqslant \int_0^\infty -u(\psi')^2 - 2u\psi''\psi ds. \tag{3.3.17}$$

Set $\psi(s) = s\xi(s)$, where the smooth function ξ has compact support in $[0, \infty)$. Then,

$$\psi'(s) = \xi(s) + s\xi'(s),$$
$$\psi''(s) = s\xi''(s) + 2\xi'(s).$$

Putting these equations into (3.3.17), we have

$$\int_0^\infty \xi^2 uds \leqslant \int_0^\infty (-6s\xi\xi' - 2s^2\xi\xi'' - s^2(\xi')^2)uds.$$

Choose a smooth decreasing function ξ so that

$$\xi(s) = 1, \text{for} \quad 0 \leqslant s \leqslant R,$$

$$\xi(s) = 0, \text{for} \quad s > 2R,$$

and $|\xi'|$ and $|\xi''|$ are bounded by $2R^{-1}$ and $4R^{-2}$ respectively, for $R \leqslant s \leqslant 2R$. Then $|s\xi'(s)| \leqslant 4$ and $|s\xi''(s)| \leqslant 16$. We see that

$$\int_0^R uds \leqslant \int_0^\infty u\xi^2 ds \leqslant \int_0^\infty (-6s\xi\xi' - 2s^2\xi\xi'' - s^2(\xi')^2)uds \leqslant 72\int_R^\infty uds.$$

This inequality implies that $\int_0^\infty uds = \infty$. That is to say, $\tilde{d}^2 s = u^2 d^2 s$ is a complete metric with nonnegative sectional curvature $\tilde{K} \geqslant 0$. □

We apply the above argument to stable minimal surfaces in a 3-manifolds of uniformly positive scalar curvature (i.e. its scalar curvature is bounded away from zero).

THEOREM 3.3.8. *(See [Theorem 10.2, Page 384] of [**GL83**] or [Theorem 1, Page 228] of [**Ros06**]) Let (M^3, g) be a Riemannian 3-manifold with the scalar curvature $\kappa \geqslant c$*

where c is a positive constant. If Σ is a stable minimal surface immersed in M, then for $x \in \Sigma$:

$$d^{\Sigma}(x, \partial\Sigma) \leqslant \frac{2\pi}{\sqrt{3c}}.$$

This result was firstly proved by Gromov-Lawson [**GL83**] and Schoen-Yau in [**SY83**]. The generalization to the stable H-surface was due to H.Rosenberg [**Ros06**].

PROOF. For any $x \in \Sigma$, let $R := d^{\Sigma}(x, \partial\Sigma)$, where d^{Σ} is the induced distance function in (Σ, d^2s). For any $\epsilon > 0$, the geodesic ball $B^{\Sigma}(x, R-\epsilon)$ in Σ, centered at x with radius $R-\epsilon$, is contained in Int Σ.

It is sufficient to show that for any $\epsilon > 0$, $R - \epsilon \leqslant \frac{2\pi}{\sqrt{3c}}$.

First, we solve the equation $L(u) = 0$ on the $B^{\Sigma}(x, R-\epsilon)$, where L is the stability operator (See Equation 3.1.22). As in the proof of Theorem 3.3.4, the first eigenvalue $\lambda_1(L, B^{\Sigma}(0, R-\epsilon)) > 0$.

Let us consider the Dirichlet problem:

$$\begin{cases} L(v) = K - (\kappa + 1/2|A|^2) & \text{in} \quad B^{\Sigma}(0, R-\epsilon) \\ \qquad v = 0 & \text{on} \quad \partial B^{\Sigma}(0, R-\epsilon). \end{cases} \tag{3.3.18}$$

Since $\lambda_1(L, B^{\Sigma}(x, R-\epsilon)) > 0$, we use Fredholm alternative (See [Theorem 6.15, Page 107] of [**GT15**]) to find a solution v. Setting $u = v + 1$. Equations (3.3.8) give that

$$L(u) = 0 \text{ in } B^{\Sigma}(x, R-\epsilon) \text{ and } u = 1 \text{ on } \partial B^{\Sigma}(x, R-\epsilon).$$

As in the proof of Theorem 3.3.4, the positivity of the operator L implies that u is a positive function.

Make a conformal change of the metric, $\tilde{d}^2s = u^2 d^2s$ on $B^{\Sigma}(x, R-\epsilon)$. Let consider the minimizing geodesic γ from x to $\partial B^{\Sigma}(x, R-\epsilon)$.

Let $\tilde{a}$ and a be the length of γ in the metrics $\tilde{d}^2s$ and d^2s, respectively. Note that $R - \epsilon \leqslant a$.

The second variation formula of arclength shows that

$$\int_0^{\tilde{a}} ((\frac{d\phi}{\tilde{ds}})^2 - \tilde{K}\phi^2)\tilde{ds} \geqslant 0, \tag{3.3.19}$$

where the smooth function ϕ has compact support in $(0, a)$, $\tilde{ds} = uds$ and $\frac{d\phi}{\tilde{ds}} = u^{-1}\frac{d\phi}{ds}$. We have that

$$\begin{aligned} L(u) &= \Delta_{\Sigma} u - Ku + (\kappa(x) + 1/2|A|^2)u = 0 \\ \tilde{K} &= u^{-2}(K - \Delta_{\Sigma} \log u). \end{aligned}$$

where Δ_{Σ} is the Laplace-Beltrami operator in (Σ, d^2s), $\tilde{K}$ and K are the sectional curvature of $\tilde{d}^2s$ and d^2s, respectively.

Therefore,

$$\phi^2\tilde{K}u = \frac{\phi^2}{u^2}(Ku - \Delta_{\Sigma} u + \frac{|\nabla_{\Sigma} u|^2}{u}) \geqslant \frac{\phi^2}{u^2}(cu + \frac{|\nabla_{\Sigma} u|^2}{u}) \tag{3.3.20}$$

where the second inequality follows from $L(u) = 0$ and $\kappa \geqslant c$.

Putting (3.3.20) into (3.3.19), we see that

$$\int_0^a u^{-1}\phi^2(c+u^{-2}|u'|^2)ds \leqslant \int_0^a u^{-1}(\frac{d\phi}{ds})^2 ds. \tag{3.3.21}$$

where $u(s) = u(\gamma(s))$. Setting $\phi := u^{1/2}\psi$, where ψ has compact support in $(0, a)$. We have

$$\begin{aligned}\int_0^a \psi^2(c+u^{-2}|u'|^2)ds &\leqslant \int_0^a u^{-1}(u^{1/2}\psi' + 1/2u^{-1/2}u'\psi)^2 ds\\ &= \int_0^a (\psi')^2 + 1/4(\frac{u'}{u})^2\psi^2 + u^{-1}u'\psi'\psi ds,\\ &\leqslant \int_0^a (\psi')^2 + (\frac{u'}{u})^2\psi^2 + 1/3(\psi')^2 ds.\end{aligned}$$

where the last inequality follows from the Cauchy-Schwarz inequality ($|u^{-1}u'\psi'\psi| \leqslant 3/4(\frac{u'}{u})^2\psi^2 + 1/3(\psi')^2$). Canceling the term $(\frac{u'}{u})^2\psi^2$ on the two sides of the inequality, we have

$$\int_0^a \psi^2 ds \leqslant \frac{4}{3c}\int_1^a (\psi')^2 ds.$$

Choosing $\psi(s) = \sin(\pi a^{-1}s)$, we know that $1 \leqslant \frac{4}{3c}(\frac{\pi}{a})^2$. That is to say, $a \leqslant \frac{2\pi}{\sqrt{3c}}$.

Therefore, we see that for each $\epsilon > 0$,

$$R - \epsilon \leqslant a \leqslant \frac{2\pi}{\sqrt{3c}}.$$

Namely, $d^{\Sigma}(0, \partial\Sigma) = R \leqslant \frac{2\pi}{\sqrt{3c}}$. □

As a consequence, we have

Corollary 3.3.9. *In a complete Riemannian 3-manifold of uniformly positive scalar curvature, any orientable complete stable minimal surface is compact and homeomorphic to $\mathbb{S}^2$.*

Theorem 3.3.8 implies that a stable surface in a 3-manifold of uniformly positive scalar curvature is compact. By Corollary 3.3.6, it is homeomorphic to $\mathbb{S}^2$.

Together with Cohn-Vesson's inequality [**Coh35**], we generalize Corollary 3.3.6 to the non-compact case and obtain the so-called extrinsic Cohn-Vesson inequality (See [Theorem 5.8, Page 18] of [**Wan19a**]).

Theorem 3.3.10. *(See Theorem 5.8 in [**Wan19a**]) Let $\Sigma^2 \subset (M^3, g)$ be a complete (non-compact) immersed stable minimal surface. If the complete manifold (M^3, g) has non-negative scalar curvature ($\kappa(x) \geqslant 0$), then*

$$\int_{\Sigma} \kappa(x) + 1/2|A|^2 dv \leqslant 2\pi\chi(\Sigma). \tag{3.3.22}$$

Moreover, if $\kappa > 0$ and Σ is embedded, then Σ is properly embedded.

PROOF. From Equation (3.1.22), the stability operator can be written as $L := \Delta_\Sigma - K + (\kappa(x) + 1/2|A|^2)$, where Δ_Σ is the Laplace-Beltrami operator of (Σ, d^2s). Since the non-compact surface Σ is stable minimal , we use Theorem 3.3.4 to find a positive fuction u with $L(u) = 0$.

Consider the metric $\tilde{d}^2s = u^2 d^2s$. Let $\tilde{K}$ and $\tilde{dv}$ be its sectional curvature and its volume form. We see that

$$\tilde{K} = u^{-2}(K - \Delta_\Sigma \log u) \text{ and } \tilde{dv} = u^2 dv. \tag{3.3.23}$$

Theorem 3.3.7 shows that $(\Sigma, \tilde{d}^2s)$ is a complete surface with nonnegative sectional curvature $\tilde{K} \geqslant 0$. We use the Cohn-Vossen inequality [**Coh35**] to have

$$\int_\Sigma \tilde{K}\tilde{dv} \leqslant 2\pi\chi(\Sigma). \tag{3.3.24}$$

Since $L(u) = 0$, then $\int_{B^\Sigma(x,R)} u^{-1}L(u)dv = 0$, where $B^\Sigma(x, R)$ is the geodesic ball in Σ centered at $x \in \Sigma$ with radius R. We deduce that

$$\begin{aligned}
\int_{B^\Sigma(0,R)} \kappa(x) + \frac{1}{2}|A|^2 dv &= \int_{B^\Sigma(0,R)} (K - u^{-1}\Delta_\Sigma\, u)dv \\
&= \int_{B^\Sigma(0,R)} K_\Sigma - (\Delta_\Sigma \log\ u + u^{-2}|\nabla_\Sigma u|)dv \\
&\leqslant \int_{B^\Sigma(0,R)} u^{-2}(K - \Delta_\Sigma log\ u)u^2 dv \qquad (3.3.25) \\
&= \int_{B^\Sigma(0,R))} \tilde{K}\tilde{dv} \\
&\leqslant \int_\Sigma \tilde{K}\tilde{dv}
\end{aligned}$$

Putting (3.3.24) into (3.3.25) and taking $R \to \infty$, we have that,

$$\int_\Sigma \kappa(x) + 1/2|A|^2 dv \leqslant 2\pi\chi(\Sigma).$$

Remark that since Σ admits a complete metric $\tilde{d}^2s$ of nonnegative sectional curvature, we see $\chi(\Sigma) \leqslant 1$ (See details in Corollary 3.3.11).

In the following, we consider the case that Σ is embedded and $\kappa(x) > 0$. We have that

$$\int_\Sigma \kappa(x)dv \leqslant 2\pi. \tag{3.3.26}$$

Suppose that Σ is not proper. There is an accumulation point p of Σ so that the set $B(p, r/2) \cap \Sigma$ is a non-compact closed set in Σ. Namely, it is unbounded in (Σ, d^2s). Hence, there is a sequence $\{p_k\}$ of points in $B(p, r/2) \cap \Sigma$ going to infinity in (Σ, d^2s).

Therefore, we may assume that the geodesic discs $B^\Sigma(p_k, r/2)$ in Σ are disjoint.

Define two constants $K_0 := \sup_{x\in B(p,r)} |K_M(x)|$ $r_0 := \frac{1}{2}\min\{r, i_0, \frac{\pi}{\sqrt{K}}\}$ where $i_0 := \inf_{x\in B(p,r)} \mathrm{Inj}_M(x)$ and K_M is the sectional curvature of (M, g). The geodesic ball $B^\Sigma(p_k, r_0/2)$

in Σ is contained in $B(p,r)$. Applying [Theorem 3, Appendix, Page 139] of [**Fre96**] to the geodesic disc $B^{\Sigma}(p_k, r_0/2) \subset B(p,r)$, we have

$$\mathrm{Area}(B^{\Sigma}(p_k, r_0/2)) \geqslant C(r_0, K_0).$$

This leads to a contradiction as follows:

$$\begin{aligned} 2\pi &\geqslant \int_{\Sigma} \kappa(x) dv \geqslant \int_{B(p,r)\cap\Sigma} \kappa(x) dv \\ &\geqslant \sum_k \int_{B^{\Sigma}(p_k, r_0/2)} \kappa(x) \\ &\geqslant \inf_{x\in B(p,r)} \kappa(x) \cdot \sum_k \mathrm{Area}(B^{\Sigma}(p_k, r_0/2)) \\ &\geqslant \inf_{x\in B(p,r)} \kappa(x) \cdot \sum_k C = \infty \end{aligned}$$

□

Combining Theorem 3.3.7 and Theorem 3.3.10, we will give a new proof of the result of Theorem 2 in [**SY82**].

COROLLARY 3.3.11. *(See [Theorem 2, Page 211] of [**SY82**]) Let $\Sigma \subset (M,g)$ be an oriented complete non-compact stable minimal surface. If the complete manifold (M^3, g) has nonnegative scalar curvature($\kappa(x) \geqslant 0$), then Σ is diffeomorphic to $\mathbb{R}^2$ or $\mathbb{S}^1 \times \mathbb{R}$. If the latter case occurs, then Σ is totally geodesic and the scalar curvature κ of M is zero along Σ.*

Moreover, if $\kappa(x) > 0$, then Σ is diffeomorphic to $\mathbb{R}^2$.

PROOF. Since Σ is stable minimal, we use Theorem 3.3.4 to find a positive function u with $L(u) = 0$, where L is the stability operator (See (3.1.21) and (3.1.22)). By Theorem 3.3.7, $(\Sigma, u^2 d^2 s)$ is a complete 2-manifold with nonnegative sectional curvature, where $d^2 s$ is the induced metric.

We apply the Soul theorem (See Theorem 1.11 and Theorem 2.1 in [**CG72**]) to $(\Sigma, u^2 d^2 s)$. This theorem asserts that if (X, g) is a connected complete manifold with nonnegative sectional curvature, there is a compact totally convex, totally geodesic submanifold (called a soul of (X, g)) such that X is diffeomorphic to the normal bundle of the submanifold.

Therefore, there is a submanifold $S \subset \Sigma$ (i.e. a soul) such that Σ is the normal bundle of S. In addition, since Σ is non-compact, we see that $\dim(S) < \dim(\Sigma)$ ($\dim(S) = 0$ or $\dim(S) = 1$).

Case (I) If $\dim(S) = 0$, S is a point. That is to say, Σ is diffeomorphic to $\mathbb{R}^2$.

Case (II) If $\dim(S) = 1$, S is homeomorphic to $\mathbb{S}^1$. Since Σ is oriented, the normal bundle is trivial. That is to say, Σ is homeomorphic to $\mathbb{S}^1 \times \mathbb{R}$. We use Theorem 3.3.10 to see that

$$\int_{\Sigma} \kappa(x) + 1/2|A|^2 dv \leqslant 2\pi\chi(\Sigma).$$

In this case, we see that $\chi(\Sigma) = 0$. Therefore, Σ is totally geodesic and $\kappa = 0$ on Σ.

If $\kappa > 0$, the latter case never occurs. We can conclude that Σ is diffeomorphic to $\mathbb{R}^2$. □

Finally, we give an application of Lemma 3.3.8 in a complete 3-manifold.

THEOREM 3.3.12 (See Theorem A and Theorem 1.1 in [**Wan19c**]). *Assume that (M^3, g) is a contractible complete 3-manifold. If there exists a number $\alpha \in (-\infty, 2)$ such that*

$$\liminf_{r(x)\to\infty} r^{\alpha}(x)\kappa(x) > 0,$$

where $\kappa(x)$ is the scalar curvature of (M, g) and $r(x)$ is the distance function from some point $0 \in M$ to x, then M^3 is diffeomorphic to $\mathbb{R}^3$.

PROOF. From our assumption, there are two positive constants C and R_0 such that if $r(x) > R_0$, then

$$\kappa(x) \geqslant \frac{C}{r^{\alpha}(x)}. \tag{3.3.27}$$

Claim: If $R > 2\max\left\{R_0, (\frac{4^{1+\alpha/2}\pi}{(3C)^{1/2}})^{\frac{2}{2-\alpha}}\right\}$, then the induced map $\pi_1(M\backslash B(0,4R)) \to \pi_1(M\backslash B(0,R))$ is trivial.

Suppose the contrary that there exists some $R > 2\max\left\{R_0, (\frac{4^{1+\alpha/2}\pi}{(3C)^{1/2}})^{\frac{2}{2-\alpha}}\right\}$ so that the induced map $\pi_1(M\backslash B(0,4R)) \to \pi_1(M\backslash B(0,R))$ is non-trivial. That is to say, there is a simple closed curve $\gamma \subset M\backslash B(0,4R)$ which is not contractible in $M\backslash B(0,R)$.

We use the work of Morrey [**Mor09**, **Mor48**] to find an area-minimizing disc Ω with boundary γ. The surface Ω intersects the set $B(0,R)$. Therefore, $\Omega \cap \partial B(0,2R)$ and $\Omega \cap \partial B(0,4R)$ are both nonempty.

Let us consider the set $\Sigma := \Omega \cap (B(0,4R)\backslash B(0,2R))$. It is a stable minimal surface in (M, g) whose boundary is contained in the disjoint union of $\partial B(0,4R)$ and $\partial B(0,2R)$. Since $\kappa(x) \geqslant \frac{C}{(4R)^{\alpha}}$ on $B(0,4R)\backslash B(0,2R)$, we use Lemma 3.3.8 to know that

$$\Sigma \text{ is contained in the } \frac{2(4R)^{\alpha/2}\pi}{(3C)^{1/2}}\text{-neighborhood of } \partial\Sigma.$$

Since $\partial\Sigma \cap \partial B(0,2R)$ and $\partial\Sigma \cap \partial B(0,4R)$ are both nonempty, we see that

$$2R = d(\partial B(0,2R), \partial B(0,4R)) \leqslant 2\frac{2(4R)^{\alpha/2}\pi}{(3C)^{1/2}}. \tag{3.3.28}$$

That is to say, $R < (\frac{4^{1+\alpha/2}\pi}{(3C)^{1/2}})^{\frac{2}{2-\alpha}}$. This is in contradiction of the choice of R. This finishes the proof of Claim.

From the above Claim, we see that M is simply-connected at infinity. From Remark 1.1.8, we know that M is diffeomorphic to $\mathbb{R}^3$. □

COROLLARY 3.3.13 (See Theorem 3.5 in [**Wan19c**]). *Assume that (M^3, g) is a simply-connected open 3-manifold with $\pi_2(M) = \mathbb{Z}$. Let $0 \in M$ be a point and $r(x)$ a distance function from x to 0. If there exists a real number $\alpha \in [0, 2)$, such that,*

$$\liminf_{r(x)\to\infty} r^{\alpha}(x)\kappa(x) > 0,$$

then M^3 is diffeomorphic to $\mathbb{R} \times \mathbb{S}^2$.

See the proof in [**Wan19c**]

CHAPTER 4

Convergence

In this chapter, we consider the convergence theory for minimal surfaces. First, we consider the convergence theory for minimal surface equations (See Equation (3.1.13)). Together with local properties of minimal surfaces, we discuss a classical theorem about convergence with finite multiplicity.

Then, we will introduce a new concept, the lamination. Our focus is the convergence theory for stable minimal laminations. Finally, we construct a family of stable minimal lamination in a complete contractible 3-manifold and discuss its limit.

4.1. The smooth convergence

Let u be a function from the unit disc $\mathbb{B}^2$ to $\mathbb{R}$. Its image in $\mathbb{R}^3$ is a minimal surface if and only if

$$div(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}) = 0.$$

LEMMA 4.1.1. *Let $\{f_i\}_i$ be a sequence of functions from the unit disc $\mathbb{B}^2$ to $\mathbb{R}$. Each graph of f_i in $\mathbb{R}^3$ is minimal. If $||f_i||_{C^2} \leqslant C < \infty$, then up to extracting a subsequence, f_i converges smoothly on compact sets of $\mathbb{B}^2$ to f and the graph of f is also a minimal surface.*

PROOF. By Arzelà-Ascoli Lemma, we may extract a subsequence so that f_k converges to f in $C^{1,\alpha}$-topology on a compact set for $0 < \alpha < 1$. According to the minimal surface equation (See Equation (3.1.13)) and the Schauder estimate for linear elliptic equation (See [Theorem 6.2, Page 90] of [**GT15**]), we see that this sequence converges in $C^{m,\alpha}$-topology on a compact set for each m. As a consequence, f also satisfies the minimal surface equation (see Equation (3.1.13)). That is to say, its graph is minimal. □

DEFINITION 4.1.2. In a complete Riemannian 3-manifold (M, g), a sequence $\{\Sigma_n\}$ of immersed minimal surfaces *converges smoothly with finite multiplicity* (at most m) to an immersed minimal surface Σ, if for each point p of Σ, there is a disc neighborhood D in Σ of p, an integer m and a neighborhood U of D in M (consisting of geodesics of M orthogonal to D and centered at the points of D) so that for n large enough, each Σ_n intersects U in at most m connected components. Each component is a graph over D in the geodesic coordinates. Moreover, each component converges to D in $C^{2,\alpha}$-topology as n goes to infinity.

Note that in the case that each Σ_n is embedded, the surface Σ is also embedded. The *multiplicity* at p is equal to the number of connected component of $\Sigma_n \cap U$ for n large enough. It remains constant on each component of Σ.

REMARK 4.1.3. Let us consider a family $\{\Sigma_n\}_n$ of properly embedded minimal surfaces converging to the minimal surface Σ with finite multiplicity. Fix a compact simply-connected subset $D \subset \Sigma$. Let U be the tubular neighborhood of D in M with radius ϵ and $\pi : U \to D$ the projection from U onto D. It follows that the restriction $\pi|_{\Sigma_n \cap U} : \Sigma_n \cap U \to D$ is a m-sheeted covering map for ϵ small enough and n large enough, where m is the multiplicity.

Therefore, the restriction of π to each component of $\Sigma_n \cap U$ is also a covering map. Hence, since D is simply-connected, it is bijective. Therefore, each component of $\Sigma_n \cap U$ is a normal graph over D.

Let us recall a classical theorem about convergence with finite multiplicity.

THEOREM 4.1.4. *(See [Theorem 4.37, Page 49] of [**MIRR02**] or [Compactness Theorem, Page 96] of [**And85**]) Let $\{\Sigma_k\}_{k\in\mathbb{N}}$ be a family of properly embedded minimal surfaces in a complete 3-manifold (M^3, g) satisfying (1) each Σ_k intersects a given compact set K_0; (2) for any compact set K in M, there are three constants $C_1 = C_1(K) > 0$, $C_2 = C_2(K) > 0$ and $k_0 = k_0(K) \in \mathbb{N}$ such that for each $k \geqslant k_0$, it holds that*

(1) $|A_{\Sigma_k}|^2 \leqslant C_1$ on $K \cap \Sigma_k$, where $|A_{\Sigma_k}|^2$ is the square length of the second fundamental form of Σ_k,
(2) Area$(\Sigma_k \cap K) \leqslant C_2$.

Then, after passing to a subsequence, Σ_k converges to a properly embedded minimal surface with finite multiplicity in the C^∞-topology.

PROOF. Choose a point $p_k \in \Sigma_k \cap K_0$. Extracting a subsequence, the sequence $\{p_k\}$ converges to some point $p \in M$. We may suppose that the unit normal vector $\vec{\mathbf{n}}_{\Sigma_k}|_{p_k}$ to the surface Σ_k at p_k converges to some unit vector in T_pM. Namely, the tangent space $T_{p_k}\Sigma_k \subset T_{p_k}M$ converges to some plane $T \subset T_pM$. From (1) and Lemma 3.2.1, we can express Σ_k locally (near p) as some graphs of functions over T. That is to say, there is an open geodesic ball U centered at p such that for k large enough,

- each component of $U \cap \Sigma_k$ is the graph of some function on T. The Hessian of this function is bounded by the bound of the second fundamental form (See Lemma 3.2.1 and (1));
- the number of components of $U \cap \Sigma_k$ is bounded by the area bound from Theorem 3.2.7 and (2);

We use Lemma 4.1.1 to extract a subsequence converging to a minimal surface. Then, a diagonal argument allows us find a subsequence converging smoothly to a minimal surface Σ. The area bound (See (2)) implies that this sequence converges with finite multiplicity.

From the strong maximal principle (See Corollary 3.2.1), the minimal surface Σ is embedded or self-intersects transversally. Since Σ_k is embedded, we can conclude that Σ is also embedded. □

In the following, we consider the convergence theory for stable minimal surfaces. For stable minimal surfaces in a 3-manifold, Schoen gave a uniform bound of the second fundamental form.

LEMMA 4.1.5. *(See [Theorem 3, Page 122] in [**Sch83**]) Let Σ be an immersed stable surface in a Riemannian manifold (M^3, g). Given $r_0 \in (0, 1]$, and a point $p_0 \in \Sigma$, if $\Sigma \cap B(p_0, r_0)$ has compact closure in Σ, there are two constants $0 < \epsilon_0 < 1$ and $C > 0$ depending only on the metric g near p_0 and the injective radius of (M, g) at p_0 so that*

$$|A_\Sigma|^2 \leqslant C r_0^{-2} \text{ on } B(p_0, \epsilon_0 r_0) \cap \Sigma.$$

As a consequence, we have

COROLLARY 4.1.6. *Let (M, g) be a complete Riemannian manifold and $\{\Sigma_k\}$ a sequence of complete embedded stable minimal surfaces. If for any compact set $K \subset M$, there is a constant C_1 depending on K, satisfying that for each k*

$$Area(\Sigma_k \cap K) \leqslant C_1,$$

then after extracting a subsequence, Σ_k converges smoothly to a complete stable minimal surface with finite multiplicity.

4.2. Minimal lamination

In the following,we assume that a complete contractible 3-manifold (M, g) is not homeomorphic to $\mathbb{R}^3$. As in Remark 1.1.13, M is an increasing union of closed handlebodies $\{N_k\}_k$ satisfying that for each k,

- N_k is homtopically trivial in N_{k+1};
- None of the N_k is contained in a 3-ball.

In addition, for each k, the genus of N_k is greater than zero. (If not, there is some handlebody N_k of genus zero, namely a 3-ball. That is to say, N_0 is contained in a 3-ball N_k which is in contradiction with the last paragraph.)

4.2.1. Construction of minimal laminations. From Lemma 1.2.7, each N_k has a system of meridians $\{\gamma_k^l\}_{l=1}^{g(N_k)}$, where $g(N_k)$ is the genus of N_k. Our target now is to construct a lamination $\mathscr{L}_k := \cup_l \Omega_k^l \subset N_k$ (i.e. a disjoint union of embedded surfaces) with $\partial\Omega_k^l = \gamma_k^l$ and "good" properties.

Let us recall a result of Meeks and Yau (See Theorem 3.1.8 or [Theorem 6.28 Page 224] of [**CM11**]). It provides us a geometric version of loop theorem to construct them.

THEOREM. (See [**MIY80**, **MIY82**], [Theorem 6.28 Page 224] of [**CM11**] or Theorem 3.1.8) Let (M^3, g) be a compact Riemannian 3-manifold whose boundary is mean convex and γ a simple closed curve in ∂M which is null-homotopic in M. Then, γ bounds an area-minimizing disc and any such least area disc is properly embedded.

Remark 4.2 The boundary ∂M is mean convex. That is, ∂M is a piecewise smooth 2-manifold consisting of smooth surfaces $\{H\}_i$. On each H_i, the mean curvature is non-negative.

Let Σ be an embedded area-minimizing disc with boundary γ. It intersects ∂M transversally. Therefore, Int Σ is contained in Int M.

Our strategy is to apply this theorem to $(N_k, g|_{N_k})$ for each k. However, the boundary of N_k may be not mean convex. To overcome it, we find a new metric g_k on N_k so that

1) (N_k, g_k) is a 3-manifold with mean convex boundary;

2) $g_k|_{N_{k-1}} = g|_{N_{k-1}}$.

The metric g_k is constructed as below:

Let $h(t)$ be a positive smooth function on $\mathbb{R}$ so that $h(t) = 1$, for any $t \in \mathbb{R}\backslash[-\epsilon, \epsilon]$. Consider the function $f(x) := h(d(x, \partial N_k))$ and the metric $g_k := f^2 g|_{N_k}$. Under (N_k, g_k), the mean curvature $\hat{H}(x)$ of ∂N_k is

$$\hat{H}(x) = h^{-1}(0)(H(x) + 2h'(0)h^{-1}(0))$$

Choosing $\epsilon > 0$ small enough and a function h with $h(0) = 2$ and $h'(0) > 2\max_{x\in\partial N_k} |H(x)|+2$, one gets the metric g_k which is the required candidate in the assertion.

In the following, we inductively construct the lamination $\mathscr{L}_k \subset (N_k, g_k)$.

When $l = 1$, there is an embedded area-minimizing disc $\Omega_k^1 \subset (N_k, g_k)$ with boundary γ_k^1 (See Theorem 3.1.8 or Theorem 6.28 of [**CM11**]). As in Remark 4.2, it intersects ∂N_k transversally. Then, Int $\Omega_k^1 \subset$ Int N_k.

Suppose that there are l disjointly embedded stable minimal discs $\{\Omega_k^i\}_{i=1}^l$ with $\partial\Omega_k^i = \gamma_k^i$.

Let us consider the Riemannian manifold $(T_{k,l}, g_k|_{T_{k,l}})$, where $T_{k,l} := N_k \backslash \amalg_{i=1}^l \Omega_k^l$. It is a handlebody of genus $g(N_k) - l$. For example, see the following figure.

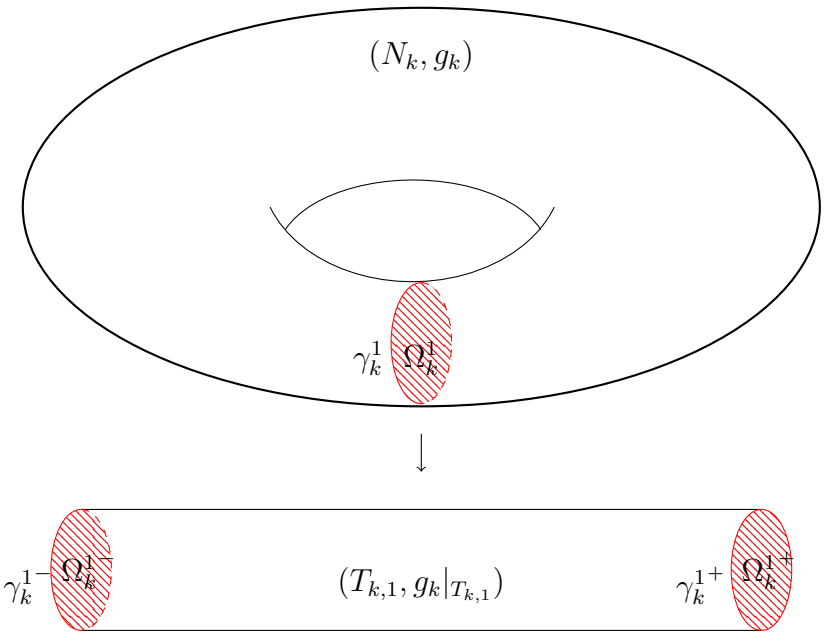


FIGURE 4.1.

The boundary of $(T_{k,l}, g_k|_{T_{k,l}})$ consists of two different parts. One is $\partial N_k \backslash \amalg_{i=1}^l \gamma_l^i$. The mean curvature is positive on this part. The other is $2l$ disjoint discs $\{\Omega_k^{i\,-}\}_{i=1}^l$ and $\{\Omega_k^{i\,+}\}_{i=1}^l$. The two discs $\Omega_k^{i\,-}$ and $\Omega_k^{i\,+}$ are two sides of the same minimal disc Ω_k^i. The mean curvature vanishes on these discs.

Therefore, the boundary of $(T_{k,l}, g_k|_{T_{k,l}})$ is mean convex (See Remark 4.2). In addition, $\{\gamma_k^i\}_{i>l}$ is a system of meridian of the handlebody $(T_{k,l}, g_k|_{T_{k,l}})$.

Then, we use Theorem 3.1.8 and the above remark to find an embedded stable minimal surface Ω_k^{l+1} in the closure of $(T_{k,l}, g_k|_{T_{k,l}})$ with boundary γ_k^{l+1}.

From Remark 4.2, Ω_k^{l+1} intersects the boundary of $(T_{k,l}, g_k|_{T_{k,l}})$ transversally. Hence, Int Ω_k^{l+1} is contained in Int $T_{k,l}$. That is to say, $\{\Omega_k^i\}_{i=1}^{l+1}$ are disjoint stable minimal surfaces for (N_k, g_k).

This finishes the inductive construction.

To sum up, there exist $g(N_k)$ disjointly embedded meridian discs $\{\Omega_k^l\}$. Define the lamination $\mathscr{L}_k$ by $\amalg_l \Omega_k^l$. It is a stable minimal lamination for the new metric g_k and for the original one away from ∂N_k (near N_{k-1}, for example).

The set $\mathscr{L}_k \cap N_{k-1}$ is a stable minimal lamination in (M, g). Each leaf has its boundary contained in ∂N_{k-1}. In addition, since N_0 is not contained in a 3-ball, We can conclude that each lamination $\mathscr{L}_k$ intersects N_0. The reason is below:

If the set $\mathscr{L}_k \cap N_0$ is empty, we choose a tubular neighborhood $N(\mathscr{L}_k)$ in N_k with small radius so that the set $N(\mathscr{L}_k) \cap N_0$ is also empty. That is to say, N_0 lies in the handlebody $N_k \backslash N(\mathscr{L}_k)$ of genus zero (i.e. a 3-ball). This is in contradiction with our assumption that N_0 is not contained in a 3-ball.

4.2.2. Limits of minimal laminations. Let us consider the sequence $\{\mathscr{L}_k\}_k$ and its limit. From Lemma 4.1.5, we know that the sequence $\{\mathscr{L}_k\}_k$ satisfies the condition (1) in Theorem 4.1.4. However, it may not hold the condition (2) in Theorem 4.1.4.

For example, in the Whitehead manifold, each N_k is of genus one. The lamination $\mathscr{L}_k$ is a meridian disc $\Omega_k^1 \subset N_k$. From Theorem 2.1.2, $\Omega_k^1 \cap \text{Int}N_1$ has at least 2^{k-1} components intersecting N_0. We know that for $k > 1$, each component $(\Sigma, \partial\Sigma) \subset (N_1, \partial N_1)$ of $\Omega_k^1 \cap \text{Int}N_1$ is a stable minimal surface in (M, g).

Choose $x_0 \in \Sigma \cap N_0$ and $r_0 = \frac{1}{2}\min\{r, i_0\}$, where $r := \text{dist}(\partial N_0, \partial N_1)$ and $i_0 := \inf_{x \in N_1} \text{Inj}_M(x)$. We see that the ball $B(x_0, r_0)$ is in N_1. We apply a result [Lemma 1, Page 445] in [**MIY80**] to $(N_1, \partial N_1)$. Hence, it follows that

$$\text{Area}(\Sigma) \geqslant \text{Area}(\Sigma \cap B(x_0, r_0)) \geqslant C(i_0, r_0, K)$$

where $K := \sup_{x \in N_1} |K_M(x)|$ and K_M is the sectional curvature.

Therefore, one has that $\text{Area}(N_1 \cap \Omega_k^1) \geqslant 2^{k-1}C$. The area of $\mathscr{L}_k \cap N_1$ goes to infinity as k goes to infinity. That is to say, the sequence $\{\mathscr{L}_k\}_k$ does not satisfy Condition (2).

Generally, the sequence $\{\mathscr{L}_k\}_k$ may not sub-converge with finite multiplicity. In the following, we consider the convergence toward a lamination.

Definition 4.2.1. A *codimension one lamination* in a 3-manifold M^3 is a collection $\mathscr{L}$ of smooth disjoint surfaces (called leaves) such that $\bigcup_{L \in \mathscr{L}} L$ is closed in M^3. Moreover, for each $x \in M$ there exists an open neighborhood U of x and a coordinate chart (U, Φ), with $\Phi(U) \subset \mathbb{R}^3$ so that in these coordinates the leaves in $\mathscr{L}$ pass through $\Phi(U)$ in slices form

$$\mathbb{R}^2 \times \{t\} \cap \Phi(U).$$

A *minimal lamination* is a lamination whose leaves are minimal. Finally, a sequence of laminations is said to converge if the corresponding coordinate maps converge in $C^{0,\alpha}$-topology.

For example, $\mathbb{R}^2 \times \Lambda$ is a lamination in $\mathbb{R}^3$, where Λ is a closed set in $\mathbb{R}$.

Note that any (compact) embedded surface (connected or not) is a lamination. In [Appendix B, Laminations] of [**CM04**], Coding and Minicozzi describ the limit of laminations with uniformly bounded curvatures.

PROPOSITION 4.2.2. *(See Proposition B.1, Page 610] in [**CM04**])Let M^3 be a fixed 3-manifold. If $\mathscr{L}_i \subset B(x, 2R) \subset M$ is a sequence of minimal laminations with uniformly bounded curvatures (where each leaf has boundary contained in $\partial B(0, 2R)$), then a subsequence, $\mathscr{L}_j$, converges in the C^α-topology for any $\alpha < 1$ to a (Lipschitz) lamination $\mathscr{L}$ in $B(x, R)$ with minimal leaves.*

We use Proposition 4.2.2 to show that

THEOREM 4.2.3. *The sequence $\{\mathscr{L}_k\}$of laminations sub-converges to a lamination $\mathscr{L}$. Moreover, Each leaf in $\mathscr{L}$ is a complete minimal surface.*

PROOF. As constructed above, the intersection $\mathscr{L}_j \cap N_k$ is a stable minimal lamination or any $j > k$. It may have many leaves (connected components). Each leaf has boundary contained in ∂N_k. In addition, $\mathscr{L}_j$ intersects N_0.

From Lemma 4.1.5, there is a constant $C(N_{k-1})$, depending on N_k and g, so that for any $j > k$,

$$|A_{\mathscr{L}_j}|^2 \leqslant C(N_{k-1}) \text{ on } N_{k-1}.$$

Therefore, for $j > 3$, $\{\mathscr{L}_j \cap N_2\}$ is a sequence of minimal laminations with uniformly bounded curvature where each leaf has boundary contained in ∂N_2. We use Proposition 4.2.2 to extract a subsequence converging to a minimal lamination in N_1. Each leaf has boundary contained in ∂N_1.

We repeat the argument on each N_k. A diagonal argument allows us to find a subsequence of $\{\mathscr{L}_k\}$ converging to a lamination $\mathscr{L}$. Each leaf is a complete minimal surface. □

For our convenience, we may assume that the sequence $\mathscr{L}_k$ converges to $\mathscr{L}$. In the following, we will show that if (M, g) has positive scalar curvature, then each leaf in $\mathscr{L}$ is a (non-compact) stable minimal surface.

LEMMA 4.2.4. *Let Σ be a compact minimal surface in a 3-manifold (X, g) (possibly with boundary) and $(\hat{X}, \hat{g})$ the double cover of (X, g). The lift $\hat{\Sigma}$ of Σ is a connected minimal surface in $(\hat{X}, \hat{g})$. Then Σ is stable minimal if an only if $\hat{\Sigma}$ is stable minimal.*

PROOF. Let L and $\hat{L}$ be the stable operators of Σ and $\hat{\Sigma}_k$ respectively. The operator $p^*(L)$ is equal to $\hat{L}$, where $p : \hat{\Sigma} \to \Sigma$ is the double cover. Let λ_1 and $\hat{\lambda}_1$ be the first eigenvalues of L and $\hat{L}$ respectively.

It is sufficient to show that $\hat{\lambda}_1 = \lambda_1$.

Let f be an eigenfunction for λ_1 (i.e. $L(f) = -\lambda_1 f$). The function $\hat{f} := p^*(f)$ satisfies that $\hat{L}(\hat{f}) = -\lambda_1 \hat{f}$. Hence, $\lambda_1 \geqslant \hat{\lambda}_1$.

Let τ be the desk transformation of the double cover p. The surface $\hat{\Sigma}$ is τ-invariant, (namely $\tau(\hat{\Sigma}) = \hat{\Sigma}$). The map τ is isometric on $\hat{\Sigma}$.

Let $\hat{h}$ be an eigenfunction for $\hat{\lambda}_1$ (i.e. $\hat{L}(\hat{h}) = -\hat{\lambda}_1\hat{h}$). We may assume that $\hat{h}$ is τ-invariant ($\tau(\hat{h}) = \hat{h}$). (If not, we replace it by $\tau(\hat{h}) + \hat{h}$.) The function $\hat{h}$ is equal to $p^*(h)$, where h is a smooth function on Σ. We have that $L(h) = -\hat{\lambda}_1 h$. Therefore, $\hat{\lambda}_1 \geqslant \lambda_1$. We can conclude that $\hat{\lambda}_1 = \lambda_1$. □

THEOREM 4.2.5. *Each leaf in $\mathscr{L}$ is stable minimal.*

PROOF. Let L_t be a leaf in the minimal lamination $\mathscr{L}$.

Case (I): If L_t is a limit leaf (that is to say, the closure of $\mathscr{L}\backslash L_t$ contains L_t), we use the result of Meek, Pérez and Rosenberg (See [Theorem 1, Page 4] of [**MIPR08**]) to have that it is stable minimal.

Case (II): If L_t is not a limit leaf, the intersection $L_t \cap \overline{\mathscr{L}\backslash L_t}$ is empty. There is a tubular neighborhood $N(L_t)$ of L_t such that the intersection $N(L_t) \cap \overline{\mathscr{L}\backslash L_t}$ is empty.

Let π be the projection from $N(L_t)$ to L_t. For any point p and $r > 0$, we consider the geodesic disc $B^{L_t}(p, r)$ in L_t and the set $N(p, r) := \pi^{-1}(B^{L_t}(p, r))$.

Choose one component Σ_k of $N(p, r) \cap \mathscr{L}_k$ for $k \geqslant 0$. Since $\mathscr{L}_k$ converges to $\mathscr{L}$, the sequence $\{\Sigma_k\}_k$ converges to some subset of $\mathscr{L}$. In addition, the intersection $\mathscr{L} \cap N(p, r)$ has the unique component, $B^{L_t}(p, r)$. Hence, Σ_k converges to $B^{L_t}(p, r)$.

We first consider the case when L_t is 2-sided. In the following, we show that $B^{L_t}(p, r)$ is stable minimal for any $r > 0$.

Step 1: Define the function $d_k : \Sigma_k \to \mathbb{R}$.

Let $\vec{\mathbf{n}}(x)$ be the unit normal vector to L_t at x and $\pi_k := \pi|_{\Sigma_k}$. The map $\pi_k : \Sigma_k \to B^{L_t}(p, r)$ is a covering map for k large enough.

We define the function $d_k : \Sigma_k \to \mathbb{R}$ as follows:

$$d_k(x) =< \exp^{-1}_{\pi_k(x)}(x), \vec{\mathbf{n}}(\pi_k(x)) > .$$

Step 2: π_k is injective.

Recall that the area of $\mathscr{L}_k$ is finite. The subset Σ_k has a finite area. Therefore, π_k is a finite cover for k large enough.

We argue it by contradiction. Suppose that π_k is a m-sheeted covering ($m > 1$).

Let us consider three sets in Σ_k as follows:

$$I^{Top} := \left\{x \in \Sigma_k | d_k(x) = \max\{d_k(x') | x' \in \pi_k^{-1}(\pi_k(x))\}\right\};$$

$$I^{Med} = \left\{x \in \Sigma_k | \min\{d_k(x') | x' \in \pi_k^{-1}(\pi_k(x))\} < d_k(x) < \max\{d_k(x') | x' \in \pi_k^{-1}(\pi_k(x))\}\right\};$$

$$I^{Bot} = \left\{x \in \Sigma_k | d_k(x) = \min\{d_k(x') | x' \in \pi_k^{-1}(\pi_k(x))\}\right\}.$$

From the homotopy lifting property of π_k and $m > 1$, these disjoint three set are open. In addition, $\Sigma = I^{Top} \amalg I^{Med} \amalg I^{Bot}$. Since Σ_k is connected, there are at least two empty sets in these three sets.

However, since $m > 1$, I^{Top} and I^{Bot} are two nonempty sets. This is in contradiction with the last paragraph. We conclude that π_k is injective.

Step 3: The stability of $B^{L_t}(x, r)$.

As in Step 2, Σ_k can be written as the graph of some function f_k over $B^{L_t}(p,r)$ for k large enough. The sequence $\{f_k\}$ converges in $C^{0,\alpha}$-topology. Since Σ_k is stable minimal, we use Lemma 4.1.5 and Lemma 4.1.1 to have that the sequence $\{f_k\}$ converges in $C^{2,\alpha}$-topology. Namely, Σ_k converges to $B^{L_t}(p,r)$ in $C^{2,\alpha}$-topology. Therefore, $B^{L_t}(p,r)$ is stable minimal for any $r>0$.

Therefore, we can conclude that L_t is stable minimal.

If L_t is 1-sided, we consider the double cover $\widehat{N(L_t)}$ of $N(L_t)$ and the lift $\hat{L}_t$ of L_t. We choose r large enough so that $B^{L_t}(p,r)\subset L_t$ is 1-sided. The lift $\widehat{B^{L_t}(p,r)}$ of $B^{L_t}(p,r)$ is connected and 2-sided in the lift $\widehat{N(p,r)}$ of $N(p,r)$.

Let $\hat{\Sigma}_k$ be one component of the pre-image of Σ_k. It is a stable minimal surface. The reason is as follows:

The map $\hat{\Sigma}_k\to\Sigma_k$ is a m'-sheeted cover map, where $m'\leqslant 2$.
If $m'=1$, the map $\hat{\Sigma}_k\to\Sigma_k$ is isometric. Therefore, $\hat{\Sigma}_k$ is stable minimal.
If $m'=2$, we use Lemma 4.2.4 to have that $\hat{\Sigma}_k$ is stable minimal.

As in Step 1, we define the projection $\hat{\pi}:\widehat{N(L_t)}\to\hat{L}_t$ and the function $\hat{d}_k:\hat{\Sigma}_k\to\mathbb{R}$.

The sequence $\{\hat{\Sigma}_k\}$ converges to $\widehat{B^{L_t}(p,r)}$. The map $\hat{\pi}_k:=\hat{\pi}|_{\hat{\Sigma}_k}$ is a cover map for k large enough. Since $\mathrm{Area}(\Sigma_k)\leqslant\mathrm{Area}(\mathscr{L}_k)<\infty$ and $\mathrm{Area}(\hat{\Sigma}_k)\leqslant 2\mathrm{Area}(\Sigma_k)$, $\hat{\Sigma}_k$ has a finite area. Therefore, $\hat{\pi}_k$ is a finite cover for k large enlarge.

As in the above case, $\widehat{B^{L_t}(p,r)}$ is stable minimal. From Lemma 4.2.4, $B^{L_t}(p,r)$ is stable minimal for r large enough. Namely, L_t is stable minimal. □

THEOREM 4.2.6. *If (M,g) has positive scalar curvature, each leaf in $\mathscr{L}$ is non-compact.*

Recall that a component Ω_k^l of $\mathscr{L}_k$ is an area-minimizing disc with boundary $\partial\Omega_k^l$ in the closure of $(T_{k,l-1},g_k)$, where $T_{k,l-1}=N_k\backslash\amalg_{j=1}^{l-1}\Omega_k^j$ and g_k is obtained by modifying the metric g. In addition, $g_k|_{N_{k-1}}$ is equal to $g|_{N_{k-1}}$.

PROOF. We argue by contradiction. Suppose that there exists a compact leaf L_t in $\mathscr{L}$.

Step 1: Topology of L_t

From the positivity of the scalar curvature, we use Corollary 3.3.6 to have that L_t is a 2-sphere or a projective plane.

If L_t is a projective plane, L_t is 1-sided. Hence, $M\backslash L_t$ is connected. There is a an embedded curve γ in M which intersects L_t transversally at one point. The intersection numberof L_t and γ is ±1.

However, from the contractibility of M, γ is homotopically trivial. Hence, the intersection number of γ and L_t is zero, a contradiction.

We conclude that L_t is a 2-sphere.

Step 2: Area Estimate.

Since M is irreducible (See Remark 1.1.1), there is a 3-ball $B \subset M$ with boundary L_t. Let $N_{2\epsilon}(B)$ be the tubular neighborhood of B with radius 2ϵ. The set $N_{2\epsilon}(L_t)$ is a subset of $N_{2\epsilon}(B)$.

Since $N_{2\epsilon}(B)$ is relatively compact, there is a positive integer k_0, such that $N_{2\epsilon}(B) \subset N_{k_0-1}$.

From now on, we fix the integer $k > k_0$. Let $\{\Sigma_k^j\}_j$ be the set of components of $\mathscr{L}_k \cap N_{2\epsilon}(L_t)$. The component Σ_k^j is contained in some component Ω_k^j of $\mathscr{L}_k := \amalg_l \Omega_k^l$.

In the following, we show that there is a constant C, independent of k and j, so that the area of Σ_k^j in (M, g) is less than C

We may assume that Ω_k^j intersects $\partial N_{2\epsilon}(B)$ transversally. The intersection $\Omega_k^j \cap \partial N_{2\epsilon}(B) := \{\gamma_i\}_i$ has finitely many components. Each component γ_i is a circle and bounds a unique closed disc D_i in Ω_k^j.

Since $\partial N_{2\epsilon}(B)$ is a 2-sphere, there is an embedded disc $D_i' \subset \partial N_{2\epsilon}(B)$ with boundary γ_i.

We claim that for any $D_i \subset \Omega_k^j$,

$$\mathrm{Area}(D_i, g_k) \leqslant \mathrm{Area}(D_i', g_k),$$

where $\mathrm{Area}(D_i, g_k)$ is the area of D_i in (N_k, g_k).

We prove it by induction on j.

When $j = 1$, Ω_k^1 is an area-minimizing disc in (N_k, g_k). If the claim does not hold for some $D_i \subset \Omega_k^1$, we consider the disc $(\Omega_k^1 \backslash D_i) \cup_{\gamma_i} D_i'$ with boundary $\partial\Omega_k^1$. Its area is less than the area of Ω_k^1 in (N_k, g_k), a contradiction.

Therefore, for any $D_i \subset \Omega_k^1$, we have $\mathrm{Area}(D_i, g_k) \leqslant \mathrm{Area}(D_i', g_k)$.

We suppose that it holds for any $l \leqslant j$ and any $D_i \subset \Omega_k^l$.

In the following, we consider that D_i is contained in Ω_k^{j+1}. If $D_i' \cap (\amalg_{l \leqslant j} \Omega_k^l)$ is empty, this claim follows from the above argument.

If not, we may assume that D_i' intersects $\amalg_{l \leqslant j} \Omega_k^l$ transversally. The intersection $D_i' \cap \cup_{l \leqslant j} \Omega_k^l := \{c_m\}_{c_m \in C_i}$ has finitely many components. Each component c_m bounds a disc $D_m' \subset D_i'$. In addition, it also bounds a disc $D_m \subset \amalg_{l \leqslant j} \Omega_k^j$.

Let C_i^{max} be the set of maximal circles of C_i in D_i'. These discs $\{D_m'\}_{c_m \in C_i^{max}}$ are disjoint. The set $D_i' \backslash (\cup_{c_m \in C_i^{max}} D_m')$ is contained in $N_k \backslash \amalg_{l \leqslant j} \Omega_k^l$.

We consider the disc

$$D_i'' := D_i' \backslash (\cup_{c_m \in C_i^{max}} D_m') \cup_{c_m \in C_i^{max}} (\cup_{c_m} D_m).$$

It is contained in the closure of $(T_{k,j}, g_k)$.

Since Ω_k^{j+1} is an area-minimizing disc in the closure of $(T_{k,j}, g_k)$, we have that

$$\mathrm{Area}(D_i, g_k) \leqslant \mathrm{Area}(D_i'', g_k).$$

If not, we consider the disc $(\Omega_k^{j+1} \backslash D_i) \cup_{\gamma_i} D_i''$. Its area is less than the area of Ω_k^{j+1}, a contradiction.

From the inductive hypothesis,

$$\text{Area}(D_m, g_k) \leqslant \text{Area}(D'_m g_k),$$

for any c_m. Hence,

$$\begin{aligned}\text{Area}(D'_i, g_k) &= \text{Area}(D'_i \backslash (\cup_{c_m \in C_i^{max}} D'_m), g_k) + \sum \text{Area}(D'_m g_k) \\ &\geqslant \text{Area}(D'_i \backslash (\cup_{c_m \in C_i^{max}} D'_m), g_k) + \sum \text{Area}(D_m g_k) \\ &= \text{Area}(D''_i, g_k) \geqslant \text{Area}(D_i, g_k).\end{aligned}$$

Therefore, we finish the proof of the claim.

We will show that the above claim implies an area estimate.

Let C^{max} be the set of maximal circles of $\{\gamma_i\}_i$ in Ω_k^j. We have that

$$\Omega_k^j \cap N_{2\epsilon}(L_t) \subset \Omega_k^j \cap N_{2\epsilon}(B) \subset \cup_{\gamma_i \in C^{max}} D_i.$$

Hence, Σ_j^k is a subset of some D_i.

Recall that $g_k|_{N_{2\epsilon}(B)} = g|_{N_{2\epsilon}(B)}$ for $k > k_0$. For each k and j, we have that

$$\begin{aligned}\text{Area}(\Sigma_k^j, g_k) &= \text{Area}(\Sigma_k^j, g) \\ \text{Area}(\partial N_{2\epsilon}(B), g_k) &= \text{Area}(\partial N_{2\epsilon}(B), g)\end{aligned}$$

We then have that

$$\begin{aligned}\text{Area}(\Sigma_k^j, g) &= \text{Area}(\Sigma_k^j, g_k) \leqslant \text{Area}(D_i, g_k) \\ &\leqslant \text{Area}(D'_i, g_k) \leqslant \text{Area}(\partial N_{2\epsilon}(B), g_k) \\ &= \text{Area}(\partial N_{2\epsilon}(B), g).\end{aligned}$$

We conclude that for each $k > k_0$ and j,

$$\text{Area}(\Sigma_k^j, g) \leqslant \text{Area}(\partial N_{2\epsilon}(B), g).$$

Step 3: Contradiction.

Choose a point $p \in L_t$ and a point $p_k \in \mathscr{L}_k \cap N_\epsilon(L_t)$ so that $\lim_{k\to\infty} p_k = p$.

Let $\Sigma_k^{j_k}$ be the component of $\mathscr{L}_k \cap N_{2\epsilon}(L_t)$ passing through p_k. As the proof in Step 2, we have that for $k \geqslant k_0$

$$\text{Area}(\Sigma_k^{j_k}, g) \leqslant \text{Area}(\partial N_{2\epsilon}(B), g).$$

From Lemma 4.1.5, the curvatures of these surfaces $\{\Sigma_k^{j_k}\}_k$ are uniformly bounded in $N_{2\epsilon}(L_t)$. By Theorem 4.1.4, the sequence $\{\Sigma_k^{j_k}\}$ sub-converges smoothly to a properly embedded surface Σ with finite multiplicity in $N_\epsilon(L_t)$.

For our convenience, we assume that $\{\Sigma_k^{j_k}\}$ converges smoothly to Σ in $N_\epsilon(L_t)$. The limit $\Sigma \subset \mathscr{L}$ is a disjoint union of connected embedded surfaces. Its boundary is contained in $\partial N_\epsilon(L_t)$. In addition, p lies in $\mathscr{L}$. Hence, L_t is one component of Σ.

Since Σ is properly embedded, the set $\Sigma' := \Sigma \backslash L_t$ is a closed set. The sets Σ' and L_t are two disjoint closed sets. Choose $\delta < \epsilon/2$ small enough such that

$$N_{2\delta}(L_t) \cap N_{2\delta}(\Sigma') = \varnothing.$$

Claim: For k large enough, $\Sigma_k^{j_k}$ is contained in $N_\delta(L_t) \amalg (N_{2\epsilon}(L_t) \backslash N_{2\delta}(L_t))$.

Since $\Sigma_k^{j_k}$ is a subset of $N_{2\epsilon}(L_t)$, $\Sigma_k^{j_k} \backslash N_{2\delta}(L_t)$ is contained in $N_{2\epsilon}(L_t) \backslash N_{2\delta}(L_t)$. It is sufficient to show that $\Sigma_k^{j_k} \cap N_{2\delta}(L_t)$ is contained in $N_\delta(L_t)$.

For k large enough, $\Sigma_k^{j_k} \cap N_\epsilon(L_t)$ is contained in $N_\delta(\Sigma)$, because $\Sigma_k^{j_k} \cap N_\epsilon(L_t)$ converges to Σ. Hence, $\Sigma_k^{j_k} \cap N_{2\delta}(L_t)$ is a subset of $N_{2\delta}(L_t) \cap N_\delta(\Sigma)$. From the choice of δ, we have that

a) $N_\delta(\Sigma)$ is equal to $N_\delta(L_t) \amalg N_\delta(\Sigma')$;
b) $N_\delta(\Sigma') \cap N_{2\delta}(L_t)$ is empty.

By a), $N_{2\delta}(L_t) \cap N_\delta(\Sigma)$ is equal to $N_\delta(L_t) \amalg (N_\delta(\Sigma') \cap N_{2\delta}(L_t))$. From b), it is equal to $N_\delta(L_t)$. Therefore, $\Sigma_{j_k}^k \cap N_{2\delta}(L_t)$ is contained in $N_\delta(L_t)$.
This finishes the proof of the claim.

For k large enough, p_k is located in $N_\delta(L_t)$. Namely, $\Sigma_k^{j_k} \cap N_\delta(L_t)$ is non-empty. In addition, since $\partial \Sigma_k^{j_k} \subset \partial N_{2\epsilon}(L_t)$ is non-empty, $\Sigma_k^{j_k} \cap (N_{2\epsilon}(L_t) \backslash N_{2\delta}(L_t))$ is also nonempty.

The sets $N_\delta(L_t)$ and $\overline{N_{2\epsilon}(L_t) \backslash N_{2\delta}(L_t)}$ are disjoint. Since $\Sigma_k^{j_k}$ is connected, we use the claim to have that one of these two sets $\Sigma_k^{j_k} \cap N_\delta(L_t)$ and $\Sigma_k^{j_k} \cap (N_{2\epsilon}(L_t) \backslash N_{2\delta}(L_t))$ is empty. This is in contradiction with the last paragraph.
We can conclude that each leaf L_t is non-compact. □

As an consequence, we give a new proof of [Corollary 10.8, Page 173] in [**GL83**].

COROLLARY 4.2.7. *A complete contractible 3-manifold with uniformly positive scalar curvature (i.e. that is, its scalar curvature is bounded away from zero) is homeomorphic to* $\mathbb{R}^3$

PROOF. Suppose that M is not diffeomorphic to $\mathbb{R}^3$. As described above, there exists a complete (non-compact) stable minimal surface Σ. By Corollary 3.3.11, Σ is conformally diffeomorphic to $\mathbb{R}^2$.

Since the scalar curvature $\kappa(x)$ of M is uniformly positive, $\inf_{x \in M} \kappa(x) > 0$. From Theorem 3.3.10, one has,

$$\begin{aligned} 2\pi &\geqslant \int_\Sigma \kappa(x) dv \\ &\geqslant \inf_{x \in M} \kappa(x) \cdot \int_\Sigma dv \\ &= \inf_{x \in M} \kappa(x) \cdot Area(\Sigma). \end{aligned}$$

Therefore, Σ is a surface of finite area.

However, we apply the theorem of Gromov and Lawson [Theorem 8.8] in [**GL83**]. This theorem asserts that if (X, g) is a Riemannian manifold of positive scalar curvature, then any complete stable minimal surface of finite area in X is homeomorphic to $\mathbb{S}^2$. Hence, Σ is homeomorphic to $\mathbb{S}^2$, which leads to a contradiction with the topological structure of Σ (Σ is homeomorphic to $\mathbb{R}^2$). □

Part 3

Proof of Main Theorems

CHAPTER 5

The Vanishing Property

In this chapter, we consider the geometry of a complete stable minimal surface and its relationship with the topological property of contractible 3-manifolds.

Let us consider a complete contractible Riemannian 3-manifold (M, g) of positive scalar curvature and a complete (non-compact) embedded stable minimal surface $\Sigma \subset (M, g)$. From Theorem 3.3.10 and Corollary 3.3.11, the surface Σ is a properly embedded plane (i.e. it is diffeomorphic to $\mathbb{R}^2$).

In the following, we assume that $M := \cup_k N_k$ is not homeomorphic to $\mathbb{R}^3$ where $\{N_k\}$ is assumed as in Remark 1.1.13. By Theorem 2.2.6, there is an increasing family $\{R_k\}_k$ of closed handlebodies with Property H.

Definition. A complete embedded stable minimal surface $\Sigma \subset (M, g)$ is called to satisfy the *Vanishing Property* for $\{R_k\}_k$, if there exists a positive integer $k(\Sigma)$ so that for any $k \geqslant k(\Sigma)$, any circle in $\Sigma \cap \partial R_k$ is contractible in ∂R_k.

Let us consider a stable minimal lamination $\mathscr{L} \subset (M, g)$, where each leaf is a complete (non-compact) stable minimal surface. It is called to have the *Vanishing Property* for $\{R_k\}_k$, if there is a positive integer k_0 so that for any $k \geqslant k_0$ and each leaf L_t in $\mathscr{L}$, then any circle in $L_t \cap \partial R_k$ is contractible in ∂R_k.

We will consider the Vanishing property and its relationship with Property P and the fundamental group at infinity.

5.1. The vanishing Property and Property P

In this section, we consider the case that a complete contractible genus one 3-manifold (M, g). In this case, we see from Lemma 1.3.10 that the family $\{N_k\}$ (as in Theorem 1.3.13) satisfies Property H. Namely, R_k is defined as N_k.

In addition, we see from Theorem 2.1.6 that the manifold M satisfies Property P (See Definition 2.1.3). If (M, g) has positive scalar curvature, the geometry of a stable minimal lamination is constrained by the extrinsic Cohn-Vesson inequality (See Theorem 3.3.10) as well as by Property P. Their relationship is clarified by the following theorem:

Theorem 5.1.1. *Let $\mathscr{L} := \cup_{t\in\Gamma} L_t$ be a stable minimal lamination in a complete contractible genus one 3-manifold (M, g). Each leaf L_t is a complete (non-compact) stable minimal lamination. If the manifold (M, g) has positive scalar curvature ($\kappa(x) > 0$), then $\mathscr{L}$ satisfies the Vanishing property for $\{N_k\}_k$, where $\{N_k\}_k$ is assumed as in Theorem 1.3.13.*

Precisely, there exists a positive integer $k_0 = k_0(M, g)$, such that for each $k \geqslant k_0$ and any $t \in \Lambda$, each embedded circle γ in $L_t \cap \partial N_k$ is contractible in ∂N_k.

PROOF. Since (M, g) has positive scalar curvature, we know from Corollary 3.3.11 that each L_t is diffeomorphic to $\mathbb{R}^2$.

We prove by contradiction. We suppose that there exists a sequence of increasing integers $\{k_n\}_n$ such that :

for each k_n, there exists a minimal surface L_{t_n} in $\mathscr{L}$ and an embedded circle $c_{k_n} \subset L_{t_n} \cap \partial N_{k_n}$ which is not contractible in ∂N_{k_n}.

Since $\lim_{n\to\infty} k_n = \infty$, we know that $\lim_{n\to\infty} I(N_1, N_{k_n}) = \infty$.

Because L_{t_n} is homeomorphic to $\mathbb{R}^2$, there exists a unique disc $D_n \subset L_{t_n}$ with boundary c_{k_n}. From Property P (Definition 2.1.3), we see that $D_n \cap \operatorname{Int} N_1$ has at least $I(N_1, N_{k_n})$ components intersecting N_0, denoted by $\{\Sigma_j\}_{j=1}^m$.

Define the constants $r := d^M(\partial N_0, \partial N_1)$, $C := \inf_{x\in N_1} \kappa(x)$, $K := \sup_{x\in N_1} |K_M|$ and $i_0 := \inf_{x\in N_1}(\operatorname{Inj}_M(x))$, where K_M is the sectional curvature of (M, g) and $\operatorname{Inj}_M(x)$ is the injective radius at x of (M, g).

Choose $r_0 = \frac{1}{2}\min\{i_0, r\}$ and $x_j \in \Sigma_j \cap N_0$, then $B(x_j, r_0)$ is in N_1. We apply Theorem 3.2.7 to the minimal surface $(\Sigma_j, \partial\Sigma_j) \subset (N_1, \partial N_1)$. Hence, one has that

$$\operatorname{Area}(\Sigma_j \cap B(x_j, r_0)) \geqslant C_1(K, i_0, r_0).$$

From Theorem 3.3.10, we have:

$$\begin{aligned} 2\pi &\geqslant \int_{L_{t_n}} \kappa(x)dv \geqslant \sum_{j=1}^m \int_{\Sigma_j} \kappa(x)dv \geqslant \sum_{j=1}^m \int_{\Sigma_j\cap B(x_j,r_0)} \kappa(x)dv \\ &\geqslant \sum_{j=1}^m C\operatorname{Area}(\Sigma_j \cap B(x_j, r_0)) \\ &\geqslant CC_1 m \geqslant CC_1 I(N_1, N_{k_n}) \end{aligned}$$

This contradicts the fact that $\lim_{n\to\infty} I(N_1, N_{k_n}) = \infty$ and completes the proof. □

REMARK 5.1.2. In the following, our proof requires that ∂N_k intersects some leaf L_t transversally. To overcome it , we will deform the solid torus N_k in a small tubular neighborhood of ∂N_k so that the boundary of the new solid torus intersects L_t transversally.

This new solid torus also holds for Theorem 5.1.1. The reason is as follows:

The proof of Theorem 5.1.1 only depends on the extrinsic Cohn-Vossen inequality (See Theorem 3.3.8) and the geometric indexes. If we replace N_k by a new solid torus obtained from deforming N_k, all geometric indexes remain unchanged. Therefore, N_k' also holds for Theorem 5.1.1.

REMARK 5.1.3. Let $\{R_k\}_k$ be a family of solid tori in a complete contractible 3-manifold (M, g) with the properties that

$$\lim_{k\to\infty} I(R_1, R_k) = \infty.$$

Now we consider the case that the maps $\pi_1(\partial R_k) \to \pi_1(\overline{M\backslash R_k})$ and $\pi_1(\partial R_k) \to \pi_1(\overline{R_k\backslash R_0})$ are both injective. From Remark 2.1.7, the family satisfies Property P (See the detail in Remark 2.1.7).

From the proof of Theorem 5.1.1, we have that if (M, g) has positive scalar curvature, any complete stable minimal lamination in (M, g) has the Vanishing Property for $\{R_k\}$.

5.2. The vanishing Property and π_1^∞

Generally, the geometry of a stable minimal surface is constrained by some topological properties of the 3-manifold. For example, the fundamental group at infinity. In the following, we will study a complete (non-compact) stable minimal surface $\Sigma \subset (M, g)$ and its relationship with the fundamental group at infinity.

LEMMA 5.2.1. *Let (M, g) be a complete contractible Riemannian 3-manifold with positive scalar curvature $\kappa(x) > 0$ and $\{R_k\}_k$ a family of handlebodies with Property H. If a complete embedded stable minimal surface Σ does not satisfy the Vanishing Property for $\{R_k\}_k$, then $\pi_1^\infty(M)$ is non-trivial.*

Roughly, there is a sequence of non-trivial circles in Σ going to infinity. This sequence gives a non-trivial element in $\pi_1^\infty(M)$.

PROOF. Since Σ does not satisfy the Vanishing property for $\{R_k\}$, there exists a sequence $\{k_n\}_n$ of increasing integers so that for each k_n, there is a circle $\gamma_n \subset \partial R_{k_n} \cap \Sigma$ which is not nullhomotopic in ∂R_{k_n}. By Corollary 3.3.11, Σ is conformally diffeomorphic to $\mathbb{R}^2$. Each γ_n bounds a unique closed disc $D_n \subset \Sigma$.

However, γ_n may not be a meridian. We will choose a meridian in D_n of R_{k_n} to replace it.

Since the map $\pi_1(\partial R_{k_n}) \to \pi_1(\overline{M \backslash R_{k_n}})$ is injective (See Definition 2.2.5), we use Corollary 1.2.6 to see that D_n contains at least one meridian of R_{k_n}. Without loss of generality, we may assume that γ_n is a meridian of R_{k_n} and Int D_n has no meridian of R_{k_n}. (If not, we can replace γ_n by the meridian in Int D_n).

Since $\{\gamma_n\}_n$ is a collection of disjointly embedded circles in Σ, one of the following holds: for each n' and n',

- $D_{n'} \subset D_n$;
- $D_n \subset D_{n'}$;
- $D_n \cap D_{n'} = \varnothing$.

We claim that:
$(*)$:*if any $n' > n$, then $D_n \subset D_{n'}$ or $D_n \cap D_{n'} = \varnothing$.*'

The reason is below: If not, $D_{n'}$ is a subset of D_n. Since $\gamma_{n'} \subset \partial R_{k_{n'}}$ is not contractible in $M \backslash R_0$ (See Remark 2.2.7) and the map $\pi_1(\partial R_{k_n}) \to \pi_1(\overline{M \backslash R_{k_n}})$ is injective, we use Corollary 1.2.6 to see that $D_{n'} \subset$ Int D_n contains at least one meridian of R_{k_n}. This is in contradiction with the above assumption.

We will show there is an increasing subsequence of $\{D_n\}$. Furthermore, the boundaries of these discs in the subsequence gives a non-trivial element in $\pi_1^\infty(M)$.

***Step 1**: the existence of the ascending subsequence of $\{D_n\}$.*

We argue by contradiction. Suppose these is no ascending subsequence in $\{D_n\}$. Consider the partially ordered set $(\{D_n\}_n, \subset)$ induced by the inclusion. Let C be the set of minimal elements in $(\{D_n\}_n, \subset)$. These discs in C are disjoint in Σ.

If the set C is finite, we consider the integer $n_0 := \max\{n|\ D_n \in C\}$. From the above fact $(*)$, the subsequence $\{D_n\}_{n>n_0}$ is an increasing subsequence, which contradicts our hypothesis. Therefore, we can conclude that the set C is infinite. That is to say, there is a subsequence $\{D_{n_s}\}_s$ of disjointly embedded discs.

From Remark 2.2.7, the map $\pi_1(\partial R_{k_{n_s}}) \to \pi_1(M\backslash R_0)$ is injective. Therefore, the disc D_{n_s} intersects N_0. Since N_0 is a subset of R_0, it also intersects R_0.

Choose $x_{n_s} \in R_0 \cap D_{n_s}$ and $r_0 = \frac{1}{2}\min\{i_0, r\}$, where $r := d^M(\partial R_0, \partial R_1)$ and $i_0 := \inf_{x\in R_1}(\mathrm{Inj}_M(x))$. Hence, the geodesic ball $B(x_{n_s}, r_0)$ in M lies in R_1.

Define the constants $C := \inf_{x\in R_1}\kappa(x)$, $K := \sup_{x\in R_1}|K_M|$ where K_M is the sectional curvature of (M, g).We apply Theorem 3.2.7 (See [Lemma 1, Page 445] of [**MIY80**]) to the minimal surface $D_{n_s} \cap R_1$ in $(R_1, \partial R_1)$ and obtain that

$$\mathrm{Area}(D_{n_s} \cap B(x_{n_s}, r_0)) \geqslant C_1(K, i_0, r_0).$$

This leads to a contradiction from Theorem 3.3.10 as follows:

$$\begin{aligned} 2\pi \geqslant \int_\Sigma \kappa dv &\geqslant \int_{R_1\cap\Sigma} \kappa dv \geqslant \sum_s \int_{D_{n_s}\cap B(x_{n_s}, r_0)} \kappa dv \\ &\geqslant \sum_s C\mathrm{Area}(D_{n_s} \cap B(x_{N_s}, r_0)) \\ &\geqslant \sum_s CC_1 = \infty \end{aligned}$$

Therefore, we can conclude that there is an ascending subsequence of $\{D_n\}_n$.

From now on, we abuse the notation and write $\{D_n\}$ for an ascending subsequence.

***Step 2**: $\pi_1^\infty(M)$ is non-trivial.*

Claim: There is an integer N so that for $n \geqslant N$, $(D_n\backslash D_{n-1}) \cap R_0$ is empty.

We argue by contradiction. Suppose that there exists a family $\{n_l\}$ of increasing integers such that $D_{n_l}\backslash D_{n_{l-1}}$ intersects R_0.

Choose $x_l \in D_{n_l}\backslash D_{n_{l-1}} \cap R_0$. Hence the geodesic ball $B(x_l, r_0)$ in M is contained in R_1, where r_0 is assumed as above. We apply Theorem 3.2.7 (See [Lemma 1, Page 445] in [**MIY80**]) to the minimal surface $D_{n_l}\backslash D_{n_{l-1}} \cap R_1$ in $(R_1, \partial R_1)$.

$$\mathrm{Area}((D_{n_l}\backslash D_{n_{l-1}}) \cap B(x_l, r_0)) \geqslant C_1(K, i_0, r_0).$$

From Theorem 3.3.10, one gets a contradiction as follows:

$$\begin{aligned} 2\pi \geqslant \int_\Sigma \kappa dv &\geqslant \int_{R_1\cap\Sigma} \kappa dv \\ &\geqslant \sum_l \int_{(D_{n_l}\backslash D_{n_{l-1}})\cap B(x_l, r_0)} \kappa dv \\ &\geqslant \sum_l C\mathrm{Area}(B(x_l, r_0) \cap D_{n_l}\backslash D_{n_{l-1}}) \\ &\geqslant C\sum_l C_1 = \infty \end{aligned}$$

This proves Claim 1.

Therefore, for $n > N$, γ_n is homotopic to γ_N in $M\backslash R_0$ and not nullhomotopic in $M\backslash R_0$.

Because $\cup_k R_k$ may not equal to M, the sequence $\{\gamma_n\}_{n>N}$ of circles may not go to infinity. For overcoming it, we choose a new family $\{\gamma'_n\}_{n>N}$ of circles going to infinity to replace it.

The map $\pi_1(\partial R_{k_n} \cap N_{j_{k_n}}) \to \pi_1(\partial R_{k_n})$ is surjective (See Theorem 2.2.6 and Definition 2.2.5). Hence, we can find a circle $\gamma'_n \subset \partial N_{j_{k_n}} \cap \partial R_{k_n}$ which is homotopic to γ_n in ∂R_{k_n}. The sequence of circles $\{\gamma'_n\}_{n\geqslant N}$ goes to infinity.

The sequence $\{\gamma'_n\}$ also have the property that for $n > N$,

- γ'_n is homotopic to γ'_{n+1} in $M\backslash R_0$;
- γ'_n is not nullhomotopic in $M\backslash R_0$.

From Remark 1.1.10, $\pi_1^\infty(M)$ is not trivial. □

As a corollary, we have

COROLLARY 5.2.2. *Let (M, g) be a Riemannian 3-manifold of positive scalar curvature and $\{R_k\}_k$ a family of handlebodies with Property (H). If $\pi_1^\infty(M)$ is trivial, then any complete stable minimal surface in (M, g) has the Vanishing property for $\{R_k\}_k$.*

THEOREM 5.2.3. *Let (M, g) be a Riemannian manifold of positive scalar curvature and a family of handlebodies $\{R_k\}_k$ with Property (H). If each leaf in a lamination $\mathscr{L}$ is a complete (non-compact) stable minimal surface satisfying the Vanishing Property for $\{R_k\}_k$, then the lamination $\mathscr{L}$ also has the Vanishing property for $\{R_k\}_k$.*

PROOF. We argue by contradiction. Suppose that there exists a sequence $\{L_{t_n}\}$ of leaves in $\mathscr{L}$ and a sequence of increasing integers $\{k_n\}_n$ so that some circle $\gamma_n \subset L_{t_n} \cap \partial R_{k_n}$ is not contractible in ∂R_{k_n} for each n.

The leaf L_{t_n} is a complete (non-compact) stable minimal surface. From Corollary 3.3.11 (See [Theorem 2, Page 211] of [**SY82**]), it is diffeomorphic to $\mathbb{R}^2$. The circle γ_n bounds a unique closed disc $D_n \subset L_{t_n}$. Since γ_n is not nullhomotopic in $\overline{M\backslash R_0}$ (See Remark 2.2.7), the disc D_n intersects R_0.

Step 1: *The sequence $\{L_{t_n}\}_n$ sub-converges with finite multiplicity.*

Since each L_{t_n} is a stable minimal surface, we use Lemma 4.1.5 (See [Theorem 3, Page 122] of [**Sch83**]) to show that, fixed a compact set $K \subset M$, there exists a constant $C_1 = C_1(K, M, g)$ satisfying that

$$|A_{L_{t_n}}|^2 \leqslant C_1 \text{ on } K \cap L_{t_n}$$

where $|A_{L_{t_n}}|^2$ is the squared norm of the second fundamental form of L_{t_n}.

From Theorem 3.3.10, $\int_{L_{t_n}} \kappa dv \leqslant 2\pi$. Hence,

$$\text{Area}(K \cap L_{t_n}) \leqslant 2\pi(\inf_{x\in K} \kappa(x))^{-1}.$$

From Theorem 4.1.4, the sequence $\{L_{t_n}\}_n$ sub-converges to a sublamination $\mathscr{L}'$ of $\mathscr{L}$ with finite multiplicity. In addition, $\mathscr{L}'$ is proper embedded.

The lamination $\mathscr{L}'$ may has infinitely many components. Let $\mathscr{L}'' \subset \mathscr{L}'$ be a set of leaves intersecting R_0. Since $\mathscr{L}''$ is properly embedded, $\mathscr{L}''$ has finitely many leaves.

Since each leaf L_t in $\mathscr{L}'$ is homeomorphic to $\mathbb{R}^2$ (See Corollary 3.3.11), any embedded circle $\gamma \subset \partial R_k \cap L_t$ bounds a unique closed disc $D \subset L_t$ for $k > 0$.

If L_t is in $\mathscr{L}'\backslash\mathscr{L}''$, the intersection $D \cap R_0$ is empty. Namely, γ is contractible in $\overline{M\backslash R_0}$. Since $\pi_1(\partial R_k) \to \pi_1(\overline{M\backslash R_0})$ is injective, then γ is nullhomotopic in ∂R_k.

Therefore, we conclude that for each $k > 0$ and any leaf $L_t \in \mathscr{L}'\backslash\mathscr{L}''$, any circle in $L_t \cap \partial R_k$ is homotopically trivial in ∂R_k.

Step 2: *The Vanishing property gives a contradiction.*

From now on, we abuse the notation and write $\{L_{t_n}\}$ for a convergent sequence. In addition, we assume that the lamination $\mathscr{L}'' := \amalg_s^m L_{t_s}$ ($\mathscr{L}''$ has finitely many leaves).

The Vanishing property gives an integer $k(L_{t_s})$ for L_{t_s}. For $k \geqslant \sum_{s=1}^m k(L_{t_s})$, any circle in $\partial R_k \cap \mathscr{L}''$ is contractible in ∂R_k. From the above fact, for $k > 0$, any closed curve in $\partial R_k \cap \mathscr{L}'\backslash\mathscr{L}''$ is also homotopically trivial in ∂R_k.

Therefore, for any $k \geqslant \sum_{s=1}^m k(L_{t_s})$, any circle in $\partial R_k \cap \mathscr{L}'$ is contractible in ∂R_k.

In the following, we fix the integer $k \geqslant \sum_{s=1}^m k(L_{t_s})$ and have the following:

Claim: For n large enough, any circle in $\partial R_k \cap L_{t_n}$ is homotopically trivial

we may assume that $\mathscr{L}'$ intersects ∂R_k transversally. Since $\mathscr{L}'$ is properly embedded, $\partial R_k \cap \mathscr{L}'$ has finitely many components. Each component of $\partial R_k \cap \mathscr{L}'$ is an embedded circle. From the above fact, it is homotopically trivial in ∂R_k. That is to say,

$$\pi_1(\partial R_k \cap \mathscr{L}') \to \pi_1(\partial R_k) \text{ is a trivial map.}$$

Choose an open tubular neighborhood U of $\mathscr{L}' \cap \partial R_k$ in ∂R_k. It is homotopic to $\mathscr{L}' \cap \partial R_k$ in ∂R_k. Therefore, $\pi_1(U) \to \pi_1(\partial R_k)$ is a trivial map.

Since $\{L_{t_n}\}$ converges to $\mathscr{L}'$, we see that $L_{t_n} \cap \partial R_k$ is contained in U for n large enough. Hence, the map $\pi_1(\partial R_k \cap L_{t_n}) \to \pi_1(\partial R_k)$ is trivial. Namely, any circle in $\partial R_k \cap L_{t_n}$ is contractible in ∂R_k. The claim follows.

The boundary $\gamma_n \subset \partial R_{k_n}$ of D_n is non-contractible in ∂R_{k_n}. From Remark 2.2.7, it is non-contractible in $\overline{M\backslash R_0}$. If $k_n > k$, we use Corollary 1.2.6 and Property H (See Definition 2.2.5) to find a meridian $\gamma' \subset D_n \subset L_{t_n}$ of R_k. This is in contradiction with the above claim. □

As a consequence, we have

COROLLARY 5.2.4. *Let (M, g) be a Riemannian manifold of positive scalar curvature and $\{R_k\}_k$ a family of handlebodies with Property (H). If $\pi_1^\infty(M)$ is trivial, then any complete stable minimal lamination in (M, g) has the Vanishing property for $\{R_k\}_k$.*

CHAPTER 6

Proof of Main Theorems

In this chapter, we will explain the proof of the main theorems. For a contractible 3-manifold, the existence of complete metrics of positive scalar curvature and its topological properties (for example, Property H) can be related through the limit of a sequence of lamination (constructed in Chapter 4). Combining all these, we will finish the proof of Theorem B_2 and Theorem C.

6.1. Proof of Main theorems

For the proof of the main theorems, we will argue by contradiction. In this chapter, we assume that (M, g) is a complete contractible 3-manifold of positive scalar curvature which is not homeomorphic to $\mathbb{R}^3$.

As in Remark 1.1.13, M is an increasing union of handlebodies $\{N_k\}_k$ with the property that for each k, (1) N_k is homotopically trivial in N_{k+1}; (2) none of the N_k is contained in a 3-ball. In addition, the genus of N_k is greater than zero for $k \geqslant 0$.

From Lemma 1.2.7, each N_k has a system of meridians $\{\gamma_k^l\}_{l=1}^{g(N_k)}$. As in Chapter 4.2, there is a lamination $\mathscr{L}_k := \amalg_l \Omega_k^l \subset N_k$. Each leaf Ω_k^l is a meridian disc with boundary γ_k^l. As described in Chapter 4.2, since N_0 is not contained in a 3-ball, the lamination $\mathscr{L}_k$ intersects N_0.

The intersection $\mathscr{L}_k \cap N_{k-1}$ is a stable minimal lamination in (M, g). From Chapter 4.2, the sequence $\{\mathscr{L}_k\}$ sub-converges to a stable minimal lamination $\mathscr{L} := \cup_{t\in\Gamma} L_t$ in (M, g). Each leaf L_t is a complete (non-compact) stable minimal surface in (M, g).

Since (M, g) has positive scalar curvature, each leaf in $\mathscr{L}$ is a properly embedded plane (See Theorem 3.3.10 and Corollary 3.3.11).

6.1.1. Properties of $\mathscr{L}$. In the following, we consider the lamination $\mathscr{L}$ and its properties.

If one of the following holds:

- $\pi_1^\infty(M)$ is trivial,
- M is a contractible genus one 3-manifold;
- M is homeomorphic to M_1 (constructed in Chapter 1.3.5)

there is a family of ascending handlebodies $\{R_k\}_k$ satisfying Property H, so that

a) the lamination $\mathscr{L}$ has the Vanishing property for $\{R_k\}_k$;

b) for each k and any N_j containing R_k, the intersection $\mathscr{L}_j \cap \partial R_k$ has at least one meridian of R_k.

Remark 6.1.1.

- The three different conditions for M correspond respectively to Theorem C, Theorem B_2 and Theorem 1.3.15.
- If M is a contractible genus one 3-manifold, then $\pi_1^\infty(M)$ is trivial.

In the following, let us explain the above properties, a) and b).

If $\pi_1^\infty(M)$ is trivial, we know from Theorem 2.2.6 that there is an increasing family $\{R_k\}_k$ of closed handlebodies with Property H (See Definition 2.2.5). Corollary 5.2.4 shows that the lamination $\mathscr{L}$ has the Vanishing Property for this family.

In addition, none of the R_k is contained in a 3-ball (See Definition 2.2.5). Together with Property H, we use Corollary 1.2.8 to know that if N_j contains R_k, the intersection $\mathscr{L}_j \cap \partial R_k$ has at least one meridian of R_k.

If M is a contractible genus one 3-manifold, the family $\{N_k\}$ can be assumed as in Theorem 1.3.13. That is to say, the geometric index $I(N_k, N_{k+1})$ is greater than zero.

From Lemma 1.3.10, Theorem 1.1.4 and Lemma 1.1.5, we know that the map $\pi_1(\partial N_k) \to \pi_1(\overline{M \backslash N_k})$ and $\pi_1(\partial N_k) \to \pi_1(\overline{N_k \backslash N_0})$ are also injective. That is to say, the family $\{N_k\}$ satisfies Property H.

From Theorem 2.1.6, M satisfies Property P. Theorem 5.1.1 implies that $\mathscr{L}$ satisfies the Vanishing property for $\{N_k\}_k$. In addition, since the geometric index $I(N_j, N_k)$ is greater than zero for $j < k$, $\mathscr{L}_j \cap \partial N_j$ has at least one meridian of N_j (See Lemma 1.3.11).

If M is homeomorphic to M_1 (constructed in Chapter 1.3.5), then there is a properly embedded plane P_1 as constructed in Chapter 1.3.4. It cuts M_1 into two contractible 3-manifolds, M_1' and M_1'' (See Proposition 1.3.14). In addition, M_1' is homeomorphic to the Whitehead manifold. As in the construction of the Whitehead manifold (in Chapter 1.3.2) there is a family $\{R_k\}$ of solid tori in M satisfying:

- the union $\cup_k R_k$ is equal to M_1';
- the geometric index $I(R_k, R_{k+1})$ is equal to two for each k.

As in the genus one case, we know that $\pi_1(\partial R_k) \to \pi_1(\overline{M \backslash R_k})$ and $\pi_1(\partial R_k) \to \pi_1(\overline{R_k \backslash R_0})$ are both injective (See Lemma 1.3.10). In addition, none of the R_k is contained in a 3-ball in M. That is to say, the family $\{R_k\}_k$ satisfies Property H.

From Remark 2.1.7, the family $\{R_k\}_k$ satisfies Property P. Since $\lim_{k\to\infty} I(R_0, R_k) = \lim_{k\to\infty} 2^{k-1} = \infty$, we see from Remark 5.1.3 that $\mathscr{L}$ satisfies the Vanishing property for $\{R_k\}$.

In addition, we know from Corollary 1.2.8, that for any N_k containing R_j, $\mathscr{L}_k$ has at least one meridian of R_j for $k \geqslant j$.

Remark 6.1.2. In the following, our proof requires that ∂R_k intersects some leaf L_t transversally. To overcome it , we will deform the handlebody R_k in a small tubular neighborhood of ∂R_k so that the boundary of the new handlebody intersects L_t transversally.

This new handlebody also satisfies a) and b). The reason is as follows:

For any handlebody R_k' obtained by deforming R_k, the maps $\pi_1(\partial R_k') \to \pi_1(\overline{R_k' \backslash R_0})$ and $\pi_1(\partial R_k') \to \pi_1(\overline{M \backslash R_k'})$ are both injective. The proof of a) and b) just depends on the injectivity of these two maps. Hence, the handlebody R_k' also holds a) and b).

REMARK. The positivity of the scalar curvature gives the property, a). The topological properties of M implies that $\mathscr{L}_k$ satisfies the property, b).

In the following, we just give the complete proof of Theorem C. The remaining proofs of Theorem B_2 and Theorem 1.3.15 are the same as Theorem C.

6.1.2. The proof of Theorem C. From the above property a), there is a family of handlebodies $\{R_k\}_k$ so that $\mathscr{L}$ has the Vanishing property for the family $\{R_k\}_k$. That is to say,

There is a integer $k_0 > 0$ so that for any $k \geqslant k_0$, any circle in $\mathscr{L} \cap \partial R_k$ is contractible in ∂R_k.

If $\mathscr{L} := \amalg_{t\in\Lambda} L_t$ has finitely many components, we may assume that each leaf L_t intersects ∂R_k transversally for $k \geqslant k_0$. Since L_t is properly embedded, $L_t \cap \partial R_k := \{\gamma_i^t\}_{i\in I_t}$ has finitely many components. Each component is a circle.

From the above fact, each γ_i^t is null-homotopic in ∂R_k. Consider the unique closed disc $D_i^t \subset \partial R_k$ with boundary γ_i^t and the partially ordered set $(\{D_i^t\}_{t\in\Lambda, i\in C_t}, \subset)$. Let C be the set of maximal elements. In particular, it is a finite set. The set $\mathscr{L} \cap \partial R_k$ is contained in the disjoint union of closed discs in C.

In the general case, we also have a similar result.

LEMMA 6.1.3. *For any $k \geqslant k_0$, $\partial R_k(\epsilon) \cap \mathscr{L}$ is contained in a disjoint union of finitely many closed discs in $\partial R_k(\epsilon)$, where $R_k(\epsilon) := R_k \backslash N_\epsilon(\partial R_k)$, $N_\epsilon(\partial R_k)$ is some tubular neighborhood of ∂R_k in R_k.*

In the general case when $\mathscr{L}$ has infinitely many components, we will prove it in Chapter 6.3.

We now finish the proof of Theorem C.

PROOF. Suppose that some complete contractible 3-manifold (M, g) with positive scalar curvature and trivial $\pi_1^\infty(M)$ is not homeomorphic to $\mathbb{R}^3$. As above, there is an ascending family $\{R_k\}_k$ of handlebodies with Property H, so that

a) the lamination $\mathscr{L}$ has the Vanishing property for $\{R_k\}_k$;

b) for each k and any N_j containing $R_k(\epsilon)$, the intersection $\mathscr{L}_j \cap \partial R_k(\epsilon)$ has at least one meridian of $R_k(\epsilon)$.

The Vanishing property implies Lemma 6.1.3 (We will prove it in Chapter 6.3). That is to say, the intersection $\mathscr{L} \cap \partial R_k(\epsilon)$ is in the union of disjoint closed discs $\{D_i\}_{i=1}^s$ for $k \geqslant k_0$.

Choose an open neighborhood U of the closed set $\mathscr{L} \cap R_{k+1}$ so that $U \cap \partial R_k(\epsilon)$ is contained in a disjoint union $\amalg_{i=1}^s D_i'$, where D_i' is an open tubular neighborhood of D_i in $\partial R_k(\epsilon)$ with small radius. Each D_i' is an open disc in $\partial R_k(\epsilon)$.

Since $\mathscr{L}_k$ subconverges to $\mathscr{L}$, there exists an integer j, large enough, satisfying

- $\mathscr{L}_j \cap R_{k+1} \subset U$;
- $R_k(\epsilon)$ is contained in N_j.

Therefore, $\mathscr{L}_j \cap \partial R_k(\epsilon)$ is contained in $U \cap \partial R_k(\epsilon) \subset \amalg D_i'$. The induced map $\pi_1(\mathscr{L}_j \cap \partial R_k(\epsilon)) \to \pi_1(\amalg_i D_i') \to \pi_1(\partial R_k(\epsilon))$ is a trivial map. We can conclude that any circle in $\mathscr{L}_j \cap \partial R_k(\epsilon)$ is contractible in $\partial R_k(\epsilon)$.

However, from b), there exists a meridian $\gamma \subset \mathscr{L}_j \cap \partial R_k(\epsilon)$ of $R_k(\epsilon)$. This contradicts the last paragraph and finishes the proof of Theorem C. □

6.2. Two topological lemmas

Before proving Lemma 6.1.3, we introduce two topological lemmas. These two lemmas play a crucial role in the proof of Lemma 6.1.3

LEMMA 6.2.1. *Let $(\Omega, \partial\Omega) \subset (N, \partial N)$ be a 2-sided embedded disc with some closed sub-discs removed, where N is a closed handlebody of genus $g > 0$. Each circle γ_i is contractible in ∂N, where $\partial\Omega = \amalg_i \gamma_i$. Then $N\backslash\Omega$ has two connected components. Moreover, there is a unique component B satisfying that the induced map $\pi_1(B) \to \pi_1(N)$ is trivial.*

PROOF. We argue by contradiction. Suppose that $N\backslash\Omega$ is path-connected. That is to say, there is an embedded circle $\sigma \subset N$ which intersects Ω transversally at one point.

Since each γ_i is contractible in ∂N, it bounds a unique disc $D_i \subset \partial N$. The surface $\hat{\Omega} := \Omega \bigcup_i \cup_{\gamma_i} D_i$ also intersects σ transversally at one point. The intersection number between $\hat{\Omega}$ and σ is ± 1.

However, $\hat{\Omega}$ is the image of a continuous map $g : \mathbb{S}^2 \to N$. It is contractible in N, since $\pi_2(N) = \{0\}$. The intersection number between $\hat{\Omega}$ and σ must be zero, which leads to a contradiction.

Therefore, $N\backslash\Omega$ is not connected. Since Ω is 2-sided and connected, $N\backslash\Omega$ just has two components B_0 and B_1.

Remark that the surface $\hat{\Omega} := \Omega \bigcup (\cup_{\gamma_i} D_i)$ is an immersed 2-sphere in N. This deduces that the map $\pi_1(\Omega) \to \pi_1(\hat{\Omega})$ is trivial map. Therefore, the map $\pi_1(\Omega) \to \pi_1(N)$ is trivial.

In the following, let us explain the existence of B.

Consider the partially ordered relationship over $\{D_i\}$ induced by inclusion. Therefore, $\cup_i D_i$ is equals to a disjoint union of maximal elements in $(\{D_i\}, \subset)$. The set $\partial N \backslash \cup_i D_i$ is a compact surface with some disjoint closed sub-discs removed.

Therefore, the induced map $\pi_1(\partial N \backslash \cup_i D_i) \to \pi_1(\partial N)$ is surjective. The induced map $\pi_1(\partial N) \to \pi_1(N)$ is also surjective. We can conclude that the composition of these two maps $\pi_1(\partial N \backslash \cup_i D_i) \to \pi_1(N)$ is also surjective.

The set $\partial N \backslash \cup_i D_i$ is contained in one of two components, B_1 and B_2, of $N\backslash\Omega$. Without loss of generality, we may assume that B_1 contains $\partial N \backslash \cup_i D_i$. Based on the last paragraph, the induced map $\pi_1(B_1) \to \pi_1(N)$ is surjective.

Let G_i be the image of the map $\pi_1(B_i) \to \pi_1(N)$, a subgroup of $\pi_1(N)$. Van-Kampen's Theorem (See Theorem 1.1.4) gives an isomorphism between $\pi_1(N)$ and $\pi_1(B_1) *_{\pi_1(\Omega)} \pi_1(B_2)$. Since the image of $\pi_1(\Omega) \to \pi_1(N)$ is trivial, $\pi_1(N)$ is isomorphic to $G_1 * G_2$. Grushko's Theorem [**Gru40**] shows that $\mathrm{rank}(G_1) + \mathrm{rank}(G_2) = \mathrm{rank}(\pi_1(N))$. (The rank of a group is the smallest cardinality of a generating set for the group.) From the last paragraph, the image, G_1, of the map $\pi_1(B_1) \to \pi_1(N)$ is isomorphic to $\pi_1(N)$. That is to say, $\mathrm{rank}(G_1) = \mathrm{rank}(\pi_1(N))$. Therefore, $\mathrm{rank}(G_2)$ is equal to zero. That is to say, G_2 is a trivial group. We know that $B := B_2$ is the required candidate in the assertion.

Finally, we prove the uniqueness. Suppose that the two induced maps are both trivial. Therefore, the map $H_1(B_i) \to H_1(N)$ is trivial for each i. Applying the Mayer-Vietoris

sequence to $N = B_0 \cup_\Omega B_1$, one has:

$$H_1(B_0) \oplus H_1(B_1) \to H_1(N) \to \hat{H}_0(\Omega).$$

Since Ω is connected, $\hat{H}_0(\Omega)$ is trivial. Therefore, $H_1(N)$ is also trivial. This contradicts the fact that $H_1(N)$ is isomorphic to $\mathbb{Z}^{g(N)}$. This completes the proof. □

Consider two disjoint surfaces $(\Omega_1, \partial\Omega_1)$ and $(\Omega_2, \partial\Omega_2)$ as assumed in Lemma 6.2.1. For $t = 1, 2$, $N\backslash\Omega_t$ has two components. Let B_t be the unique component of $N\backslash\Omega_t$ satisfying $\pi_1(B_t) \to \pi_1(N)$ is trivial. One has:

LEMMA 6.2.2. *Let $(\Omega_1, \partial\Omega_1)$ and $(\Omega_2, \partial\Omega_2)$ be two disjoint surfaces as assumed in Lemma 6.2.1. For each $t = 1, 2$, $N\backslash\Omega_t$ has a unique component B_t with the property that the map $\pi_1(B_t) \to \pi_1(N)$ is trivial. Then it holds one of the following:*

(1) $B_1 \cap B_2 = \varnothing$;
(2) $B_1 \subset B_2$;
(3) $B_2 \subset B_1$.

PROOF. Suppose $B_1 \cap B_2$ and $B_1\backslash B_2$ are both nonempty. Say, there are two points $p_1 \in B_1\backslash B_2$ and $p_2 \in B_1 \cap B_2$.

First, Ω_2 is contained in B_1. The reason follows as below: B_1 includes a curve γ joining p_1 and p_2 (since B_1 is connected). γ must intersect Ω_2 at some point(s). Hence, $\Omega_2 \cap B_1$ is not empty. Since $\Omega_1 \cap \Omega_2$ is empty, Ω_2 lies in one of component of $N\backslash\Omega_1$. Therefore, Ω_2 is contained in B_1.

Second, Ω_2 cuts B_1 into two components. Otherwise, there is a circle in B_1 which intersects Ω_2 at one point. As argued in Lemma 6.2.1, such a circle can not exist.

Finally, take the component B of $B_1\backslash\Omega_2$ satisfying that $\partial B \cap \Omega_1$ is empty. Then, B is also a component of $N\backslash\Omega_2$. In addition, the map $\pi_1(B) \to \pi_1(B_1) \to \pi_1(N)$ is trivial. From the uniqueness of B_2, one has that $B = B_2$. This implies $B_2 \subset B_1$. □

6.3. Proof of Lemma 6.1.3

In order to prove Lemma 6.1.3, we will introduce the set S and prove the finiteness of S. The finiteness of S will imply Lemma 6.1.3.

6.3.1. Definition of the set S. Let (M, g), $\mathscr{L}$, $\{\mathscr{L}_k\}$ and $\{N_k\}$ be assumed as in the proof of Theorem D. As in Chapter 6.2, there is an ascending family $\{R_k\}_k$ of handlebodies with Property H with the property that $\mathscr{L}$ satisfies the Vanishing property for $\{R_k\}_k$. That is to say,

there is a positive integer k_0 so that for each $k \geqslant k_0$ and each $t \in \Lambda$, each circle in $L_t \cap \partial R_k$ is contractible in ∂R_k.

In the following, we will work on the open handlebody Int R_k and construct the set S, for a fixed integer $k \geqslant k_0$.

6.3.1.1. *Elements in S.* Let $\{\Sigma_i^t\}_{i\in I_t}$ be the set of components of $L_t \cap \operatorname{Int} R_k$ for each $t \in \Lambda$. (It may be empty.) We will show that for each component Σ_i^t, $R_k \backslash \overline{\Sigma_i^t}$ has a unique component B_i^t satisfying that $\pi_1(B_i^t) \to \pi_1(R_k)$ is trivial.

If L_t intersects ∂R_k transversally, the boundary $\partial\Sigma_i^t \subset L_t \cap \partial R_k$ is the union of some disjointly embedded circles. From the Vanishing property, any circles in the boundary $\partial\Sigma_i^t \subset L_t \cap \partial R_k$ is contractible in ∂R_k.

In addition, since L_t is homeomorphic to $\mathbb{R}^2$ and Σ_i^t is relatively compact, Σ_i^t is homeomorphic to an open disc with some disjoint closed subdiscs removed. By Lemma 6.2.1, $R_k \backslash \overline{\Sigma_i^t}$ has a unique component B_i^t satisfying that $\pi_1(B_i^t) \to \pi_1(R_k)$ is trivial.

In general, L_t may not intersect ∂R_k transversally. To overcome it, we will deform the surface ∂R_k. Precisely, for the leaf L_t, there is a new handlebody $\tilde{R}_k(\epsilon_t)$ containing R_k so that L_t intersects $\partial\tilde{R}(\epsilon_t)$ transversally, where $\tilde{R}_k(\epsilon_t)$ is a closed tubular neighborhood of R_k in M.

We consider the component $\tilde{\Sigma}_i^t$ of $L_t \cap \operatorname{Int}\tilde{R}_k\ (\epsilon_t)$ containing Σ_i^t. As above, $\tilde{R}_k\ (\epsilon_t) \backslash \overline{\tilde{\Sigma}_i^t}$ has a unique component $\tilde{B}_i^t$ so that the map $\pi_1(\tilde{B}_i^t) \to \pi_1(\tilde{R}_k(\epsilon_t))$ is trivial.

Choose the component B_i^t of $\tilde{B}_i^t \cap R_k$ whose boundary contains Σ_i^t. It is a component of $R_k \backslash \overline{\Sigma_i^t}$. In addition, the map $\pi_1(B_i^t) \to \pi_1(\tilde{B}_i^t) \to \pi_1(\tilde{R}_k(\epsilon_t))$ is trivial. Since R_k and $\tilde{R}_k(\epsilon_t)$ are homotopy equivalent, the map $\pi_1(B_i^t) \to \pi_1(R_k)$ is also trivial. This finishes the construction of B_i^t.

6.3.1.2. *Properties of S.* From Lemma 6.2.2, for any B_i^t and $B_{i'}^{t'}$, it holds one of the following

(1) $B_i^t \cap B_{i'}^{t'} = \varnothing$;
(2) $B_i^t \subset B_{i'}^{t'}$;
(3) $B_{i'}^{t'} \subset B_i^t$,

where $t, t' \in \Lambda$, $i \in I_t$ and $i' \in I_{t'}$.

Therefore, $(\{B_i^t\}_{t\in\Lambda, i\in I_t}, \subset)$ is a partially ordered set. We consider the set $\{B_j\}_{j\in J}$ of maximal elements. However this set may be infinite.

DEFINITION 6.3.1. $S := \{B_j | B_j \cap R_k(\epsilon/2) \neq \varnothing$, for any $j \in J\}$, where $R_k(\epsilon/2)$ is $R_k \backslash N_{\epsilon/2}(\partial R_k)$ and $N_{\epsilon/2}(\partial R_k)$ is a 2-sided tubular neighborhood of ∂R_k with radius $\epsilon/2$.

PROPOSITION 6.3.2. *Let Σ_i^t be one component of $L_t \cap R_k$ and B_i^t assumed as above. If B_i^t is an element in S, then $\Sigma_i^t \cap R_k(\epsilon/2)$ is nonempty.*

PROOF. We argue by contradiction. Suppose that $\Sigma_i^t \cap R_k(\epsilon/2)$ is empty. As mentioned above, $\overline{\Sigma_i^t}$ cuts R_k into two components. Hence, $R_k(\epsilon/2)$ must be in one of these two components.

In addition, from the definition S, the component B_i^t of $R_k \backslash \overline{\Sigma_i^t}$ must intersect $R_k(\epsilon/2)$. One knows that $R_k(\epsilon/2)$ is contained in B_i^t.

However, the composition of maps $\pi_1(R_k(\epsilon)) \to \pi_1(B_i^t) \to \pi_1(R_k)$ is an isomorphism. Therefore, the map $\pi_1(B_i^t) \to \pi_1(R_k)$ is non-trivial and surjective, which contradicts the fact that the map $\pi_1(B_i^t) \to \pi_1(R_k)$ is trivial. This finishes the proof. □

PROPOSITION 6.3.3. $R_k(\epsilon) \cap \mathscr{L} \subset \bigcup_{B_j \in S} \overline{B}_j \cap R_k(\epsilon)$. *Moreover,* $\partial R_k(\epsilon) \cap \mathscr{L} \subset \bigcup_{B_j \in S} \overline{B}_j \cap \partial R_k(\epsilon)$.

PROOF. Each component Σ_i^t of $L_t \cap \text{Int } R_k$ is contained in $\overline{B_i^t}$. Hence, $L_t \cap R_k$ is in $\cup_{i \in I_t} \overline{B_i^t}$. We can conclude that $\mathscr{L} \cap R_k$ is contained in $\cup \overline{B_i^t}$.

The set $\cup B_i^t$ is equal to $\cup_{j \in J} B_j$, because $\{B_j\}_{j \in J}$ is the set of all maximal elements in the partially ordered set $(\{B_i^t\}, \subset)$. Therefore, $\mathscr{L} \cap R_k$ is in $\cup_{j \in J} \overline{B_j}$.

From the definition of S, $\cup_{j \in J} \overline{B_j} \cap R_k(\epsilon)$ equals $\cup_{B_j \in S} \overline{B_j} \cap R_k(\epsilon)$. Therefore, $R_k(\epsilon) \cap \mathscr{L} \subset \cup_{B_j \in S} \overline{B_j} \cap R_k(\epsilon)$.

Similarly, one has that $\cup_{j \in J} \overline{B_j} \cap \partial R_k(\epsilon)$ equals $\cup_{B_j \in S} \overline{B_j} \cap \partial R_k(\epsilon)$. Hence, $\partial R_k(\epsilon) \cap \mathscr{L} \subset \cup_{B_j \in S} \overline{B_j} \cap \partial R_k(\epsilon)$ □

6.3.2. The finiteness of the set S. The set $\partial B_j \cap \text{Int } R_k$ equals some $\Sigma_i^t \subset L_t$ for $t \in \Lambda$. Let us consider the set $S_t := \{B_j \in S | \partial B_j \cap \text{Int } R_k \subset L_t\}$. Then, $S = \amalg_{t \in \Lambda} S_t$. Note that each $B_j \in S_t$ is a B_i^t for some $i \in I_t$.

In this subsection, we first show that each S_t is finite. Then, we argue that $\{S_t\}_{t \in \Lambda}$ contains at most finitely many nonempty sets. These imply the finiteness of S.

LEMMA 6.3.4. *Each S_t is finite.*

PROOF. We argue by contradiction. Suppose that S_t is infinite for some t.

For each $B_j \in S_t$, there exists a $i \in I_t$ so that B_j is equal to B_i^t, where B_i^t is a component of $R_k \backslash \overline{\Sigma_i^t}$ and Σ_i^t is one component of $L_t \cap \text{Int } R_k$. By Proposition 6.3.2, $\Sigma_i^t \cap R_k(\epsilon/2)$ is nonempty.

Choose $x_j \in \Sigma_i^t \cap R_k(\epsilon/2)$ and $r_0 = \frac{1}{2} \min\{\epsilon/2, i_0\}$, where $i_0 := \inf_{x \in R_k} \text{Inj}_M(x)$. Then the geodesic ball $B(x_j, r_0)$ in M is contained in R_k.

We apply Theorem 3.2.7 to the minimal surface $(\Sigma_i^t, \partial \Sigma_i^t) \subset (R_k, \partial R_k)$. One knows that,

$$\text{Area}(\Sigma_i^t \cap B(x_j, r_0)) \geqslant C(r_0, i_0, K)$$

where $K = \sup_{x \in R_k} |K_M|$. This leads to a contradiction from Theorem 3.3.10 as below:

$$\begin{aligned} 2\pi \geqslant \int_{L_t} \kappa(x) dv &\geqslant \sum_{B_j \in S_t} \int_{\Sigma_i^t} \kappa(x) dv \geqslant \sum_{B_j \in S_t} \int_{\Sigma_i^t \cap B(x_j, r_0)} \kappa(x) dv \\ &\geqslant \inf_{x \in R_k} (\kappa(x)) \sum_{B_j \in S_t} \text{Area}(B(x_j, r_0) \cap \Sigma_i^t) \\ &\geqslant C \inf_{x \in R_k} (\kappa(x)) |S_t| = \infty \end{aligned}$$

This finishes the proof. □

LEMMA 6.3.5. $\{S_t\}_{t \in \Lambda}$ *contains at most finitely many nonempty sets.*

PROOF. We argue by contradiction. Suppose that there exists a sequence $\{S_{t_n}\}_{n \in \mathbb{N}}$ of nonempty sets. For an element $B_{j_{t_n}} \in S_{t_n}$, there is some $i_n \in I_{t_n}$ so that B_{j_n} equals $B_{i_n}^{t_n}$ where $B_{i_n}^{t_n}$ is one component of $R_k \backslash \overline{\Sigma_{i_n}^{t_n}}$ and $\Sigma_{i_n}^{t_n}$ is one of components of $L_{t_n} \cap \text{Int } R_k$. Note that $\pi_1(B_{i_n}^{t_n}) \to \pi_1(R_k)$ is trivial.

By Proposition 6.3.2, $\Sigma_{i_n}^{t_n} \cap R_k(\epsilon/2)$ is not empty. Pick a point p_{t_n} in $\Sigma_{i_n}^{t_n} \cap R_k(\epsilon/2)$.

Step1*: $\{L_{t_n}\}$ subconverges to a lamination $\mathscr{L}' \subset \mathscr{L}$ with finite multiplicity.*

Since L_{t_n} is a stable minimal surface, We know from Lemma 4.1.5 that for any compact set $K \subset M$, there is a constant $C_1 := C_1(K, M, g)$ such that

$$|A_{L_{t_n}}|^2 \leqslant C_1 \text{ on } K \cap L_{t_n}.$$

From Theorem 3.3.10, $\int_{L_{t_n}} \kappa(x) dv \leqslant 2\pi$. Hence,

$$\text{Area}(K \cap L_{t_n}) \leqslant 2\pi(\inf_{x \in K} \kappa(x))^{-1}.$$

We use Theorem 4.1.4 (See [Compactness Theorem, Page 96] in [**And85**]) to find a sub-sequence of $\{L_{t_n}\}$ subconverging to a properly embedded lamination $\mathscr{L}'$ with finite multiplicity. Since $\mathscr{L}$ is a closed set in M, $\mathscr{L}' \subset \mathscr{L}$ is a sublamination.

From now on, we abuse notation and write $\{L_{t_n}\}$ and $\{p_{t_n}\}$ for the convergent subsequence.

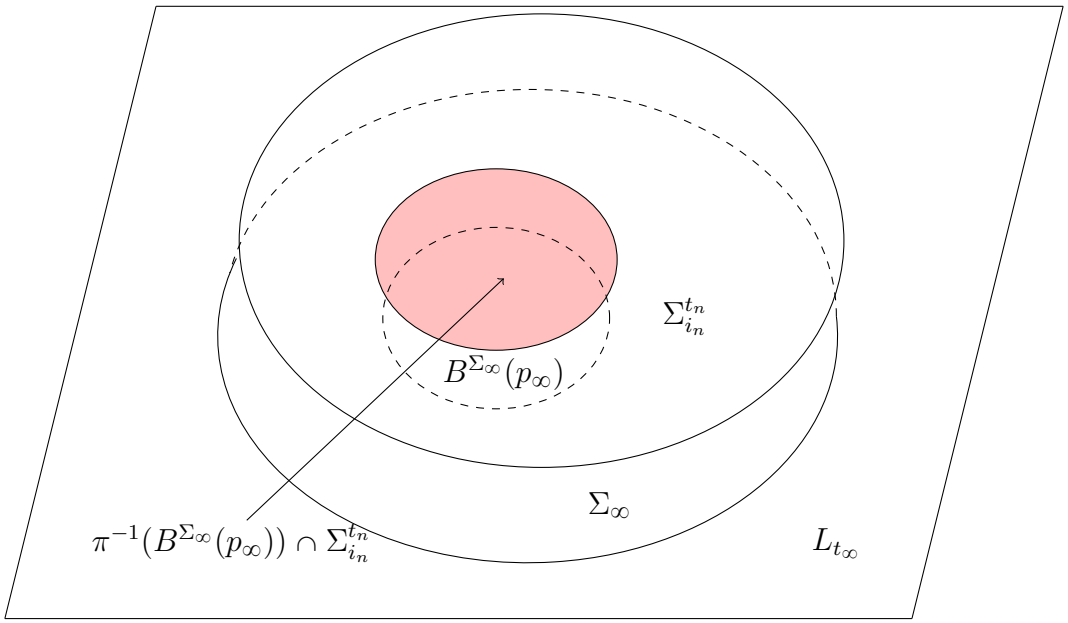


FIGURE 6.1.

Step 2*: $\{\Sigma_{i_n}^{t_n}\}$ converges with multiplicity one.*

Let L_{t_∞} be the unique component of $\mathscr{L}'$ passing through p_∞, where $p_\infty = \lim_{n\to\infty} p_{t_n}$. The limit of $\{\Sigma_{i_n}^{t_n}\}$ is the component Σ_∞ of $L_{t_\infty} \cap R_k$ passing through p_∞, where $\Sigma_{i_n}^{t_n}$ is the unique component of $R_k \cap L_{t_n}$ passing though p_{t_n}.

Let $D \subset L_{t_\infty}$ be a simply-connected subset satisfying $\Sigma_\infty \subset D$. Since $\{L_{t_n}\}$ converges smoothly to L_{t_∞}, there exists $\epsilon_1 > 0$ and an integer N such that

$$\Sigma_{i_n}^{t_n} \subset D(\epsilon_1), \text{ for } n > N,$$

where $D(\epsilon_1)$ is the tubular neighborhood of D with radius ϵ_1 in M. (See Definition 4.1.2 and Remark 4.1.3).

Let $\pi : D(\epsilon_1) \to D$ be the projection. For n large enough, the restriction of π to each component of $L_{t_n} \cap D(\epsilon_1)$ is injective (See Remark 4.1.3).

Hence, $\pi|_{\Sigma_{i_n}^{t_n}} : \Sigma_{i_n}^{t_n} \to D$ is injective. That is to say, $\Sigma_{i_n}^{t_n}$ is a normal graph over a subset of D. Therefore, $\{\Sigma_{i_n}^{t_n}\}$ converges to Σ_∞ with multiplicity one (See Definition 4.1.2). That is to say, there is a geodesic disc $B^{\Sigma_\infty}(p_\infty) \subset \Sigma_\infty$ centered at p_∞ with small raduis so that

($**$): *the set $\pi^{-1}(B^{\Sigma_\infty}(p_\infty)) \cap \Sigma_{i_n}^{t_n}$ is connected and a normal graph over $B^{\Sigma_\infty}(p_\infty)$, for large n.*

***Step 3**: Get a contradiction.*

There exists a neighborhood U of p_∞ and a coordinate map Φ, such that each component of $\Phi(\mathscr{L} \cap U)$ is $\mathbb{R}^2 \times \{x\} \cap \Phi(U)$ for some $x \in \mathbb{R}$. (See Definition 4.2.1 or Appendix B of [**CM11**].) Choose the disc $B^{\Sigma_\infty}(p_\infty)$ and ϵ_1 small enough such that $\pi^{-1}(B^{\Sigma_\infty}(p_\infty)) \subset U$. We may assume that $U = \pi^{-1}(B^{\Sigma_\infty}(p_\infty))$.

From ($**$), $\Sigma_{i_n}^{t_n} \cap U \subset L_{t_n}$ is connected and a graph over $B^{\Sigma_\infty}(p_\infty)$, for n large enough. Since $\partial B_{j_{t_n}} \cap U \subset L_{t_n}$ equals $\Sigma_{i_n}^{t_n} \cap U$, it is also connected. Therefore $\Phi(\partial B_{j_{t_n}} \cap U)$ is the set $\mathbb{R}^2 \times \{x_{t_n}\} \cap \Phi(U)$ for some $x_{t_n} \in \mathbb{R}$. In addition, $\Phi(\Sigma_\infty \cap U)$ equals $\mathbb{R}^2 \times \{x_\infty\} \cap \Phi(U)$ for some $x_\infty \in \mathbb{R}$. Since $\lim_{n\to\infty} p_{t_n} = p_\infty$, we have $\lim_{n\to\infty} x_{t_n} = x_\infty$.

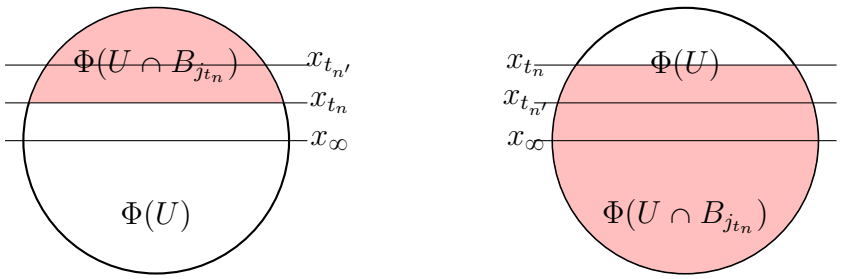


FIGURE 6.2.

The set $U \backslash \partial B_{j_{t_n}}$ has two components. Therefore, $\Phi(B_{j_{t_n}} \cap U)$ is $\Phi(U) \cap \{x | x_3 > x_{t_n}\}$ or $\Phi(U) \cap \{x | x_3 < x_{t_n}\}$. For n large enough, there exists some $n' \neq n$ such that $\mathbb{R}^2 \times \{x_{t_{n'}}\} \cap \Phi(U) \subset \Phi(B_{j_{t_n}} \cap U)$. This implies that $B_{j_{t_n}} \cap B_{j_{t_{n'}}}$ is non-empty.

Since S consists of maximal elements in $(\{B_i^t\}, \subset)$, the set $B_{j_{t_n}} \cap B_{j_{t_{n'}}}$ is empty which leads to a contradiction. This finishes the proof. □

6.3.3. The finiteness of S implies Lemma 6.1.3. We will explain how to deduce Lemma 6.1.3 from the finiteness of S.

PROOF. Since S is finite , we may assume that ∂B_j intersects $\partial R_k(\epsilon)$ transversally for each $B_j \in S$. Remark that each B_j is equal to some B_i^t and $\partial B_j \cap \partial R_k(\epsilon)$ equals $\Sigma_i^t \cap \partial R_k(\epsilon)$. Since each Σ_i^t is properly embedded, $\{c_i\}_{i\in I} := \partial R_k(\epsilon) \cap (\cup_{B_j \in S} \partial B_j)$ has finitely many components. Each component is an embedded circle.

The Vanishing property of $\mathscr{L}$ and Remark 6.1.2 show that each c_i is contractible in $\partial R_k(\epsilon)$ and bounds a unique closed disc $D_i \subset \partial R_k(\epsilon)$ (since $k \geqslant k_0$). The set $(D_i, \subset)$ is a partially ordered set. Let $\{D_{j'}\}_{j' \in J'}$ be the set of maximal elements. The set J' is finite .

Since the boundary of $\partial R_k(\epsilon) \cap \overline{B}_j$ is a subset of $\partial B_j \cap \partial R_k(\epsilon) \subset \amalg_{i \in I} c_i$, it is contained in $\amalg_{j' \in J'} D_{j'}$.

Next we show that for any $B_j \in S$, $\partial R_k(\epsilon) \cap \overline{B_j}$ is contained in $\amalg_{j' \in J'} D_{j'}$.

If not, $\partial R_k(\epsilon) \setminus \amalg_{j' \in J'} D_{j'}$ is contained in $\partial R_k(\epsilon) \cap \overline{B}_j$ for some $B_j \in S$. This implies that the composition of two maps $\pi_1(\partial R_k(\epsilon) \setminus (\amalg_{j' \in J'} D_{j'})) \to \pi_1(\overline{B}_j) \to \pi_1(R_k)$ is not a zero map. However, the induced map $\pi_1(\overline{B}_j) \to \pi_1(R_k)$ is trivial. This is impossible. We conclude that for each $B_j \in S$, $\partial R_k(\epsilon) \cap \overline{B}_j$ is contained in $\amalg_{j' \in J'} D_{j'}$.

Therefore, $\cup_{B_j \in S} \overline{B}_j \cap \partial R_k(\epsilon)$ is contained in $\amalg_{j' \in J'} D_{j'}$. From Proposition 6.3.3, $\mathscr{L} \cap \partial R_k(\epsilon)$ is contained in a disjoint union of finite discs $\{D_{j'}\}_{j' \in J'}$. This completes the proof. □

6.4. Deformation to Positive Scalar curvature

This section follows Kazdan's result [**Kaz82**]. In this section, we show that a complete non-Ricci-flat metric of nonnegative scalar curvature can deformed to be a complete metric of positive scalar curvature.

Let (M^n, g) be a complete n-manifold. We consider the operator

$$L(u) = -\Delta u + fu$$

where f is a smooth function on M.

For a bounded open set $\Omega \subset M$ with smooth boundary and outer normal derivative $\partial/\partial\nu$ on $\partial\Omega$, let $\mu_1(L, \Omega)$ be the lowest eigenvalue of L with Neumann boundary conditions, $\partial u/\partial\nu = 0$ on $\partial\Omega$. One has the well-known variational characterization of $\mu_1(\Omega)$

$$\mu_1(L, \Omega) = \inf_v \frac{\int_\Omega (|\nabla v|^2 + fv^2) dx}{\int_\Omega v^2 dx},$$

where dx is the volume form and the infimum is taken over all v in the Sobolev space $H^1(\Omega)$.

LEMMA 6.4.1. *(See [Theorem A, Page 228] in [**Kaz82**]) Assume there is a bounded open set $\Omega_0 \subset M$ such that $\mu_1(L, \Omega_0) > 0$ and $f \geqslant 0$ on $M \setminus \Omega_0$. Then there is a solution $u > 0$ on M of $L(u) > 0$; in fact one can find a solution of $L(u) > 0$ satisfying $0 < C_1 < u < C_2$, where C_1 and C_2 are two constants.*

In the following, we consider that the conformal Laplacian L_g,

$$L_g := -\frac{4(n-1)}{n-2}\Delta_g + \kappa_g.$$

THEOREM 6.4.2. *A complete non-Ricci-flat metric of nonnegative scalar curvature can be deformed to be a complete metric of positive scalar curvature.*

PROOF. Assume that (M, g_0) is a complete non-Ricci-flat manifold with non-negative ($\kappa_{g_0} \geqslant 0$). Let p be a point in (M, g_0) satisfying that

$$\mathrm{Ric}_{g_0}(p) \neq 0. \tag{6.4.1}$$

Choose the geodesic ball $B(p,r)$ centered at p with radius r, where r is less than the injective radius of (M,g) at p. Since $\kappa_{g_0} \geqslant 0$, then $\mu_1(L_{g_0}, B(p,r))$ is non-negative.

If $\mu_1(L_{g_0}, B(p,r)) > 0$, we use Lemma 6.4.1 to find a positive function u with the following properties:

- $L_{g_0}(u) > 0$;
- there is a positive constant C such that $u \geqslant C$.

Choose the new metric $g := u^{\frac{4}{n-2}} g_0$. It is a complete metric over M. The scalar curvature is

$$\kappa_g = L_{g_0}(u) u^{-\frac{n+2}{n-2}} > 0.$$

The metric g is the required candidate as in the statement.

If $\mu_1(L_{g_0}, B(p,r)) = 0$, we have that the scalar curvature $\kappa_{g_0} = 0$ on $B(p,r)$ and the eigenfunction ϕ for $\mu_1(L_{g_0}, B(p,r))$ is a constant function. The reason is as follows:

We have that

$$\begin{cases} -\Delta_{g_0}\phi + \kappa_{g_0}\phi = 0 \\ \partial\phi/\partial\nu = 0 \end{cases}$$

Dong integration by parts, we have that $\int_{B(p,r)} |\nabla\phi|^2 + \kappa_{g_0}\phi^2 = 0$. Thus, ϕ is a constant function and $\kappa_{g_0} = 0$ on $B(p,r)$.

From [Lemma 3.3, Page 232] in [**Kaz82**], there exists a new metric g_{t_0} so that

(1) $\mu_1(L_{g_{t_0}}, B(p,r)) > 0$;
(2) g_{t_0} is equal to g_0 outside $B(p,r)$.

As the above case, we could find a complete metric with positive scalar curvature.

The metric g_{t_0} is constructed as follows:

Pick a a function $\eta \in C_0^\infty(B(p,r), \mathbb{R}^{\geqslant 0})$ with $\eta(p) > 0$ and consider a family of metrics

$$g_t = g_0 - t \cdot \eta \cdot \mathrm{Ric}_{g_0}$$

Since $\kappa_{g_0} = 0$ on $B(x,r)$, the first variation formal (See [Page 233] of [**Kaz82**] or [**KW75**]) gives that

$$\frac{d}{dt}\mu_1(L_{g_t}, B(x,r))|_{t=0} = \frac{1}{\mathrm{Vol}(B(p,r))} \int_{B(x,r)} \eta |Ric|^2 \tag{6.4.2}$$

Since $\eta(p) > 0$, Equations (6.4.1) and (6.4.2) gives

$$\frac{d}{dt}\mu_1(L_{g_t}, B(p,r))|_{t=0} > 0.$$

Since $\mu_1(L_{g_0}, B(p,r)) = 0$, we find that $\mu_1(L_{g_{t_0}}, B(x,r)) > 0$ for some $t_0 > 0$.

The metric g_{t_0} is the required metric. □

As a consequence, we have that

COROLLARY 6.4.3. *No contractible genus one 3-manifold admits a complete metric of nonnegative scalar curvature.*

PROOF. We argue by contradiction. Suppose that (M^3, g) is a complete contractible genus one 3-manifold of nonnegative scalar curvature.

If g is Ricci-flat, it is a flat metric. Thus, M is homeomorphic to $\mathbb{R}^3$. This is in contradiction with Theorem 1.3.13(M is not homeomorphic to $\mathbb{R}^3$).

If not, g can be deformed to be a complete metric of positive scalar curvature. This contradicts Theorem B_2. □

As the above argument, we have that

COROLLARY 6.4.4. *A complete contractible 3-manifold with non-negative scalar curvature and trivial π_1^∞ is homeomorphic to $\mathbb{R}^3$.*

6.5. Further questions

6.5.1. The General Case. In [**Wan19a**, **Wan19b**], we verified Question 1 for 3-manifolds with trivial π_1^∞. The remaining case is contractible 3-manifolds with non-trivial π_1^∞.

In this case, we require more techniques combining minimal surfaces theory and topological surgeries on 3-manifolds.

The key point is to understand stable minimal surfaces (as constructed in Chapter 4.2) and its relationships with the fundamental group at infinity.

Based on this relationship, we attempt to devise a new topological surgery which can reduce Question 1 to Theorem B_2.

6.5.2. RCD metrics. By [**Liu13**], the Whitehead manifold does not admit any complete metric with positive Ricci curvature. In metric geometry, Riemannian manifolds with lower bounds on Ricci curvature correspond to the RCD spaces. It would be interesting to know whether the Whitehead manifold has a RCD(0, 3) metric.

6.5.3. Spherical Decomposition.

DEFINITION 6.5.1. An embedded 2-sphere S in a 3-manifold M is called *compressible* if S bounds a 3-ball in M.

A spherical decomposition $\mathcal{S}$ of a 3-manifold is a locally finite collection of (possibly non-separating) pairwise disjoint embedded 2-sphere in M such that the operation of cutting M along $\mathcal{S}$ and gluing a ball to each boundary component of the resulting manifold yields a collection of irreducible manifolds.

A 3 manifold M is irreducible if and only if all 2-spheres in M are compressible.

If $\mathcal{S}$ is a spherical decomposition, then the collection of sphere obtained by removing compressible spheres in $\mathcal{S}$ is still a spherical decomposition.

The prime decomposition theorem for 3-manifold (See [**Hat00**] or [**Kne29**]) is equivalent to the statement that every compact 3-manifold has a spherical decomposition. This result does not generalize to open manifolds (See [**Sco77**] and [**Mai07**, **Mai08**]). The first example was given by Scott [**Sco77**].

A question posed by Maillot is the following:

Question: *Does a complete 3-manifold of uniformly positive scalar curvature have a spherical decomposition ?*

For this question, a possible approach is to use the minimal surface theory to study the geometry of incompressible spheres.

This question is related to the open problem, introduced by Bessières, Besson and Maillot [**BBM11**]:

Question :(See [**BBM11**]) *Let M be a connected, orientable 3-manifold which admits a complete metric of uniformly positive scalar curvature. Is M a connected sum of spherical manifolds and copies of $\mathbb{S}^2 \times \mathbb{S}^1$?*

Generally, we attempt to use the spherical decomposition to study the following question:

Question: (See Problem 27 in [**Yau82**]) *Classify 3-manifolds admitting complete Riemannian metrics of positive (resp. nonnegative) scalar curvature up to diffeomorphism.*

Part 4

Closed aspherical 4-Manifolds

CHAPTER 7

Gromov-Lawson Conjecture

7.1. Aspherical 4-manifolds

DEFINITION 7.1.1. A space M is called aspherical if it is path-connected and all its higher homotopy groups vanish (i.e. $\pi_k(M)$ is trivial for $k \geqslant 2$).

For example, a CW complex M is aspherical if and only if its universal cover is contractible.

LEMMA 7.1.2. *If M^n is an aspherical manifold, then its fundamental group is torsion-free.*

However, if M^n is an aspherical manifold, the first homology group $H_1(M)$ may be not torsion-free. For example, the Klein bottle K^2 is aspherical (since its universal cover is $\mathbb{R}^2$). However, $H_1(K^2) \cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.

CONJECTURE 7.1.3. *(Gromov-Lawson Conjecture) No closed aspherical manifold has a metric of positive scalar curvature.*

For example, a n-dimensional torus $\mathbb{T}^n$ has no metric of positive scalar curvature for $n \geqslant 2$. It was proved by Gromov and Lawson [**GL83**] and by Schoen and Yau [**SY82**, **SY17**].

It is well-known from [**GL83**] that this conjecture holds for the 3-dimensional case. In dimension four, Gromov and Lawson [**GL83**] gave a partial solution requiring an additional hypothesis about incompressible surfaces.

We prove that

THEOREM. *F* No closed aspherical 4-manifolds with nonzero first Betti number has a metric of positive scalar curvature.

Remark that there is some closed aspherical 4-manifold whose first homology group vanishes (See [**RT05**]).

7.2. The non-existence result

7.2.1. Topological preliminary. For two topological spaces X and Y, let us consider the topological space $\mathbf{Map}(X, Y)$, consisting of all continuous maps from X to Y and the compact-open topology.

For a manifold M, the space $\mathbf{Map}(\mathbb{S}^1 \times \mathbb{S}^2, M)$ is homeomorphic to the space $\mathbf{Map}(\mathbb{S}^1, \mathbf{Map}(\mathbb{S}^2, M))$. The homeomorphism is given by

$$\begin{aligned} \mathbf{Map}(\mathbb{S}^1 \times \mathbb{S}^2, M) &\to \mathbf{Map}(\mathbb{S}^1, \mathbf{Map}(\mathbb{S}^2, M)) \\ f(\cdot, \cdot) &\mapsto \text{The map } \; x \to f(x, \cdot) \end{aligned}$$

For a point $x \in \mathbb{S}^1$, $f(x,\cdot)$ can be considered as an element in $\mathbf{Map}(\mathbb{S}^2, M)$).

The projection $p_0 : \mathbb{S}^1 \times \mathbb{S}^2 \to \mathbb{S}^1$ induces the map

$$\begin{aligned} p_0^* : \mathbf{Map}(\mathbb{S}^1, M) &\to \mathbf{Map}(\mathbb{S}^1 \times \mathbb{S}^2, M) \\ f \qquad &\mapsto \qquad p_0 \circ f \end{aligned}$$

LEMMA 7.2.1. *If $\pi_2(M)$ is trivial, the map p_0^* is a homotopy equivalence.*

PROOF. First, we show that M and $\mathbf{Map}(\mathbb{S}^2, M)$ are homotopic equivalent .

For a fixed point $x_0 \in \mathbb{S}^2$, we consider the map

$$\begin{aligned} p : \mathbf{Map}(\mathbb{S}^2, M) &\longrightarrow M \\ f \qquad &\longmapsto f(x_0) \end{aligned}$$

For any point $x \in M$, $p^{-1}(x)$ is a set of all continuous maps from $(\mathbb{S}^2, x_0)$ to (M, x). Since $\pi_2(M)$ is trivial, the fiber $p^{-1}(x)$ is contractible for each $x \in M$. Therefore, p satisfies the homotopy lifting property (i.e. p is a fibration). Applying the Puppe sequence (See [Theorem 4.41, Page 376] in [**Hat05**]), one has

$$\pi_k(p^{-1}(x)) \to \pi_k(\mathbf{Map}(\mathbb{S}^2, M)) \xrightarrow{p_*} \pi_k(M) \to \pi_{k-1}(p^{-1}(x))$$

Since $p^{-1}(x)$ is contractible, the map p_* is an isomorphism for each k. Whitehead's theorem (See [Theorem4.5, Page 346] in [**Hat05**]) shows that the map $p : \mathbf{Map}(\mathbb{S}^2, M) \to M$ is a homotopy equivalence.

As a consequence, the induced map $p_* : \mathbf{Map}(\mathbb{S}^1 \times \mathbb{S}^2, M) \to \mathbf{Map}(\mathbb{S}^1, M)$ is also a homotopy equivalence, since $\mathbf{Map}(\mathbb{S}^1 \times \mathbb{S}^2, M)$ is homeomorphic to $\mathbf{Map}(\mathbb{S}^1, \mathbf{Map}(\mathbb{S}^2, M))$.

We know that $p_0^* \circ p_* = Id_{\mathbf{Map}(\mathbb{S}^1, M)}$. Since p_* is a homotopy equivalence, then p_0^* is also a homotopy equivalence. The lemma follows. □

7.2.2. Proof of Theorem F. We begin by a compact manifold (M^4, g) with $b_1(M) > 0$. There is an embedded circle $\gamma \subset M$ with the property that $[\gamma]$ is a torsion-free element in $H_1(M)$. We use the Poincaré duality to find a class $u \in H_3(M^4) \cong H^1(M^4)$ satisfying that $([\gamma], u) = 1$.

We apply a theorem of Fleming-Federer (See [**FF60**]) and the regularity theory for area-minimizing currents (See Chapter 7 of [**Sim83**]). This result asserts that, in a Riemannian manifold (X^n, g), for a non-trivial class in $H_{n-1}(X, \mathbb{Z})$, there is a volume-minimizing hypersurface in the class satisfying that it is smooth outside a set of Hausdorff dimension $\leqslant n - 8$. Therefore, there is a stable minimal hypersurface $\Sigma^3 \subset (M^4, g)$ in the class u. In addition, the intersection number (γ, Σ) of Σ and γ is $+1$.

If (M^4, g) has positive scalar curvature, then Σ admits a metric of positive scalar curvature (See Proposition 3.3.5). In this case, Σ is a connected sum of some spherical manifolds and some copies of $\mathbb{S}^1 \times \mathbb{S}^2$ (See [**Per02a**,**Per02b**,**Per03**], [**MT07**], [**BBB$^+$10**]).

In the following, we prove Theorem F.

PROOF. We argue by contradiction. Suppose a closed aspherical manifold M^4 with $b_1(M^4) > 0$ has a metric g of positive scalar curvature.

Let γ and Σ be constructed as above. The intersection number (Σ, γ) of Σ and γ is not zero. In addition, Σ has a metric of positive scalar curvature. By [**Per02a**, **Per02b**, **Per03**], we see that

$$\Sigma \cong \mathbb{S}^3/\Gamma_1 \# \ldots \# \mathbb{S}^3/\Gamma_j \# (\mathbb{S}^1 \times \mathbb{S}^2) \# \ldots \# (\mathbb{S}^1 \times \mathbb{S}^2)$$

where each Γ_i is a finite subgroup of $O(4)$ for $1 \leqslant i \leqslant j$.

There exists a family of disjointly embedded splitting 2-spheres $\{S_i\}_{i=1}^{n-1}$ in Σ. They cut M into n-components, denoted by $\{X_i\}_{i=1}^{n}$. That is,

$$\Sigma \cong X_1 \cup_{S_1} X_2 \cup_{S_2} \cdots \cup_{S_{n-1}} X_n.$$

Set

$$X'_k = \begin{cases} X_1 \cup_{S_1} \mathbb{B}^3 & k = 1 \\ X_n \cup_{S_{n-1}} \mathbb{B}^3 & k = n \\ X_k \cup_{S_{k-1}} \mathbb{B}^3 \cup_{S_k} \mathbb{B}^3 & \text{otherwise,} \end{cases}$$

where $\mathbb{B}^3$ is a unit ball in $\mathbb{R}^3$. Therefore, $\Sigma \cong X'_1 \# \ldots \# X'_n$. Each X'_k is a spherical manifold or $\mathbb{S}^1 \times \mathbb{S}^2$. (Note that X'_k is not a subset of M^4.)

Since $\pi_2(M)$ is trivial, each 2-sphere S_k bounds an immersed 3-ball $B_k \subset M^4$. Let us consider a submanifold

$$X''_k = \begin{cases} X_1 \cup_{S_1} B_1 & k = 1 \\ X_n \cup_{S_{n-1}} B_{n-1} & k = n \\ X_k \cup_{S_{k-1}} B_{k-1} \cup_{S_k} B_k & \text{otherwise,} \end{cases}$$

Each $X''_k \subset M^4$ can be viewed as the image of some map f_k from X'_k to M^4. In addition, we know that $[\Sigma] = \sum_{k=1}^{n} [X''_k]$ in $H_3(M^4, \mathbb{Z})$.

Since the intersection number (γ, Σ) is not equal to zero, there is some k such that the intersect number (γ, X''_k) is not zero. As mentioned above, X'_k is a spherical 3-manifold or $\mathbb{S}^1 \times \mathbb{S}^2$.

Case I: If X'_k is a spherical manifold (that is, it is $\mathbb{S}^3/\Gamma_k$), X''_k can be considered as the image of the map from $\mathbb{S}^3$ to M. However, since $\pi_3(M^4)$ is trivial, we see that X''_k is contractible in M^4. Therefore, the intersection number (γ, X''_k) is zero. This is in contradiction with the last paragraph.

Case II: If X'_k is a $\mathbb{S}^1 \times \mathbb{S}^2$, then f_k is an element in $\mathbf{Map}(\mathbb{S}^1 \times \mathbb{S}^2, M^4)$. Since $\pi_2(M^4)$ is trivial, Lemma 7.2.1 shows that there is a map $f'_k \in \mathbf{Map}(\mathbb{S}^1, M^4)$ such that $p_0^*(f'_k)$ is homotopic to f_k in M^4.

We may assume that the image of $p_0^*(f'_k)$ is an embedded circle γ'. Therefore, X''_k is homotopic to γ' in M^4. Hence, the intersection number (γ, X''_k) is equal to the intersection number (γ, γ'). Since $\dim(M^4) = 4$, we see that the intersection number (γ, γ') is zero in M^4. This contradicts the above fact that $(\gamma, X''_k) \neq 0$.

This finishes the proof of Theorem F. $\square$

7.3. Further questions

7.3.1. Stable minimal hypersurfaces in $\mathbb{R}^4$. When using the minimal hypersurfaces to study 4-manifolds, understanding the geometry of minimal hypersurfaces is necessary and crucial. However, the geometry of stable minimal hypersurfaces in a 4-manifold is not known. R. Schoen has conjectured:

CONJECTURE. (See [Conjecture 2.12, Page 79] of [**CM11**]):If $\Sigma^3 \subset \mathbb{R}^4$ is a complete immersed stable minimal hypersurface with trivial bundle, then Σ is flat.

7.3.2. Closed Aspherical 4-manifolds. Generally, Gromov-Lawson conjecture is still unknown, particularly for 4-manifolds with zero first Betti number.

The proof of Theorem F involves stable minimal hypersurfaces and the geometrization conjecture. In the general case, the main issue is the existence of stable minimal hypersurfaces. In the proof of Theorem F, its existence is ensured by our hypothesis that the first Betti number is non-zero.

In order to overcome it, we attempt to find a covering space whose first betti number is nonzero. We next use topological conditions to construct a complete stable minimal surface. Then one is led to study stable minimal surface in a 4-manifold with uniformly positive scalar curvature.

Combining with the argument in [**GL83**] and some metric inequalities in [**Gro18**], we plan to argue by contradiction. We expect to show that such a minimal surface does not exist, which would lead to a contradiction.

7.3.3. Exotic $\mathbb{R}^4$. An exotic $\mathbb{R}^4$ is a differential manifold that is homeomorphic but not diffeomorphic to the Euclidean space $\mathbb{R}^4$. The first example were found by Freedman (See [**FQ14**]). Actually, there are infinitely many non-diffeomorphic differential structures of $\mathbb{R}^4$, as was shown first by Taubes [**Tau87**].

An interesting question is whether an exotic $\mathbb{R}^4$ admits a complete metric of positive scalar (or Ricci) curvature.

For this question, we might follow the following scheme. First, based on the construction of an exotic $\mathbb{R}^4$, we attempt to construct a complete stable minimal hypersurface. Then, one is led to understand the geometry of such a hypersurface and its relationship with the differential structures and the positivity of scalar (or Ricci) curvature.

We plan to argue by contradiction. Combining the geometry of the stable minimal hypersurface, we expect to show that such a hypersurface does not exist, which would lead to a contradiction.

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