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## To cite this version:

Jian Wang. Contractible 3-manifolds and Positive scalar curvature. Differential Geometry [math.DG].
Université Grenoble Alpes, 2019. English. NNT: . tel-02953229v1

HAL Id: tel-02953229
https://theses.hal.science/tel-02953229v1
Submitted on 2 Oct 2020 (v1), last revised 28 Sep 2020 (v2)

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## THÈSE

Pour obtenir le grade de

## DOCTEUR DE L’UNIVERSITE GRENOBLE ALPES

Spécialité : Mathématique
Arrêté ministériel : 25 mai 2016

Présentée par

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Thèse dirigée par Gérard Besson, Directeur de Recherche Class exceptionnelle, CNRS

Préparée au sein du Laboratoire Institut Fourier dans I'École Doctorale Mathématiques, Sciences et technologies de l'information, Informatique

## Les 3-variétés contractiles et Le courbure scalaire positive

## Contractible 3-manifolds and Positive Scalar curature

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# Contractible 3-manifolds and Positive scalar curvature 

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Key words and phrases. Thurston's Geometrisation Conjecture, Positive Scalar Curvature, Contractible 3-manifolds, Simply-connectedness at infinity, Fundamental group at infinity, Loop Lemma, Handlebody, Genus One 3-manifolds, Minimal Surface

Technique, Minimal Surface Equation, Plateau's Problem, Stable Minimal Surface, Lamination, Extrinsic Cohn-Vesson's Inequality, Aspherical manifolds, Gromov-Lawson Conjecture

The author is supported by ERC Advanced Grant 320939, GETOM.


#### Abstract

The purposes of this thesis is to understand spaces which carry metrics of positive scalar curvature. There are several topological obstructions for a smooth manifold to have a complete metric of positive scalar curvature. Our goal is to find all obstructions for contractible 3-manifolds and closed 4-manifolds.

In dimension 3, we are concerned with the question whether a complete contractible 3 -manifold of positive scalar curvature is homeomorphic to $\mathbb{R}^{3}$. The topological structure of contractible 3-manifolds could be complicated. For example, the Whitehead manifold is a contractible 3 -manifold which is not homeomorphic to $\mathbb{R}^{3}$.

We first prove that the Whitehead manifold does not carry a complete metric of positive scalar curvature. This result can be generalised to the so-called genus one case. Precisely, we show that no contractible genus one 3-manifold admits a complete metric of positive scalar curvature.

We then study the fundamental group at infinity, $\pi_{1}^{\infty}$, and its relationship with the existence of positive scalar curvature metric. The fundamental group at infinity of a manifold is the inverse limit of the fundamental groups of complements of compact subsets. In this thesis, we give a partial answer to the above question. We prove that a complete contractible 3-manifold with positive scalar curvature and trivial $\pi_{1}^{\infty}$ is homeomorphic to $\mathbb{R}^{3}$.

Finally, we study closed aspherical 4-manifolds. Together with minimal surface theory and the geometrisation conjecture, we show that no closed aspherical 4-manifold with nontrivial first Betti number carries a metric of positive scalar curvature.


## Résumé

Un des objectifs de ce mémoire est de comprendre les espaces munis de métriques complète de courbure scalaire positive. Il y a plusieurs obstructions topologiques à l'existence d'une métrique complète de courbure scalaire positive. Notre but est de trouver toutes les obstructions pour les variétés contractiles de dimension 3 et les variétés fermées de dimension 4.

En dimension 3, nous considérons la question de savoir si une variété contractile complète de courbure scalaire positive est homéomorphe à $\mathbb{R}^{3}$. La structure topologique des variétés contractiles de dimension 3 est assez compliquée. Par exemple, Whitehead a construit une variété dimension 3 contractile qui n'est pas homéomorphe à $\mathbb{R}^{3}$.

Nous prouvons, tout d'abord, que la variété de Whitehead n'a pas de métrique complète de courbure scalaire positive. Ce résultat peut être généralisé au cas dit de genre un. Précisément, nous montrons qu'aucune variété contractile de dimension 3 et de genre un ne possède de métrique complète de courbure scalaire positive.

Nous étudions ensuite le groupe fondamental à l'infini, $\pi_{1}^{\infty}$, et son lien avec l'existence d'une métrique de courbure scalaire positive. Le groupe fondamental à l'infini d'une variété est la limite projective des groupes fondamentaux des complémentaires des sousensembles compacts. Dans ce mémoire, nous apportons une réponse partielle à la question évoquée plus haut. Nous prouvons qu'une variété complète de dimension 3 de courbure scalaire positive dont le groupe $\pi_{1}^{\infty}$ est trivial est homéomorphe à $\mathbb{R}^{3}$.

Enfin, nous étudions les variétés fermées asphériques de dimension 4. En utilisant la théorie des surfaces minimales et la conjecture de géométrisation, nous montrons qu'aucune variété fermée asphérique de dimension 4 avec un premier nombre de Betti non trivial ne possède de métrique à courbure scalaire positive.

## Acknowledgement

I would like to thank my advisor Gérard Besson for all of his guidance throughout the time in Grenoble. He suggested this great project and generously shared many amazing ideas with me. Over the years, he has led my way from a student to a junior researcher though patient and dedicated suggestions. Without his constant encouragement, this dissertation may not be completed. Beside Mathematics, I respect him for his kindness and humbleness.

I also thank Laurent Mazet, Blaine Lawson for spending time and effort on reveiwing my thesis. They made many helpful and insightful suggestions that improved the exposition of my thesis. I am thankful to Yunhui Wu for pointing out an improvement of my results. In addition, I thank Louis Funar, Sylvain Maillot, David Gabai and Harold Rosenberg for their enlightening discussions and sharing their ideas .

I also thank Laurent Bessières, Christoph Böhm, Gilles Carron, Xiuxiong Chen, Jian Ge, Misha Gromov, Bernhard Hanke, Laurent Hauswirth, Ilaria Mondello, Yuguang Shi, Samuel Tapie, Richard Thom, Wilderich Tuschmann, Marina Ville, Burkhard Wilking for their interests in my thesis and for their enlightening conversations.

I also thank my former advisors Yuxiang Li and Pin Yu in Tsinghua University for the encouragement and inspiration to pursue research. Beside, I am grateful to Jean-Pierre

Demailly. I enjoyed the lunch time with him. I am thankful to my collaborator, Long Li, who led my way at the beginning stage of my other research project.

I had a good time at Fourier Institute. I thank many faculty members for their support. I also would like to thank deeply Christine Haccart and Géraldine Rahal for their patience and helps over the past three years. I also thank Loren Coquille, Sylvain Courte, François Dahmani, Pierre Dehornoy, Philippe Eyssidieux, Jean Fasel, Stéphane Guillermou, Vincent Lafforgue, Dietrich Häfner, Jean-Baptiste Meilhan, Hervé Pajot, Pierre Will for their lectures and discussion I enjoyed much. I am also thankful to many graduate students, including Clément Berat, Gabriella Clemente, Clément Debin, Bruno Laurent, David Leturcq, Louis-Clément Lefèvre, Alejandro Rivera, Deng Ya, Zhizhong Huang, Jian Xiao, Xiaojun Wu, Peng Du.

I would like to thank my friends, Nicola Cavallucci, Simone Calamai, Jiaming Chen, Jiao He, Si Chen, Yahui Leng, Yu Xie, Chenlei Yu, Honghao Gao, Bingyu Zhang, Chenmin Sun, Chao Qian, Yuexun Wang, Jun Wang, Jie Liu, Jie Shen, Sheng Rao.

I want to thank my parents for their care and support of my eduction in the past twenty years. I am grateful to their efforts that they made for me through these years.

Dedicated to my mother, for her 60th birthday.

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## Introduction

Riemannian geometry aims to study Riemannian manifolds which are smooth manifolds with metric structures. One of the fundamental questions is to understand the relationship between the curvature, which is locally defined, and the global properties of smooth manifolds. The earliest result is the classical theorem of Gauss and Bonnet, which links the curvature and the Euler number, a topological invariant. This theorem implies, for example, that any compact surface of genus $g>0$ has no metric of positive curvature.

In higher dimensions, the existence of metrics of positive curvature becomes much more complicated, because there are several topological obstructions for a smooth manifold to have a complete metric of positive curvature.

We then take 3-manifolds to explain this fact.

| $\left(M^{3}, g\right)$ | The compact case | The non-compact case |
| :---: | :---: | :---: |
| $K>0$ | $\mathbb{S}^{3} / \Gamma$ | $\mathbb{R}^{3}$ |
| Ric $>0$ |  | $\mathbb{R}^{3}$ |
| Scal $>0$ | $\left(\#_{i=1}^{k} \mathbb{S}^{3} / \Gamma_{i}\right) \#\left(\#_{j=1}^{l} \mathbb{S}^{1} \times \mathbb{S}^{2}\right)$ | $?$ |

Table 1.

For a compact Riemannian 3-manifold, there is a unique short-time solution to the so-called (normalized) Ricci flow, introduced by Hamilton Ham82. If the manifold has positive Ricci curvature, the short-time solution can be extended to be the long-time solution. The limit of this flow is a metric of constant sectional curvature. That is to say, if a compact 3-manifold has positive Ricci curvature, then it is homeomorphic to the quotient $\mathbb{S}^{3} / \Gamma$ of the sphere $\mathbb{S}^{3}$ by a finite subgroup $\Gamma \subset O(4)$. Such a quotient is called a spherical 3-manifold.

The next major contribution to the subject was made by Perelman Per02a, Per02b, Per03 who developed the Ricci flow with surgery. One surprising and beautiful result of this study is a proof that a compact 3-manifold of positive scalar curvature is homeomorphic to a connected sum of some spherical 3-manifolds and some copies of $\mathbb{S}^{1} \times \mathbb{S}^{2}$ (See also $\overline{\mathbf{B B B}^{+} \mathbf{1 0}}$ and $\left.\mathbf{M T 0 7} \mid\right)$. Its generalization to the non-compact case is due to Bessières, Besson, and Maillot [BBM11].

For the non-compact case, the first result shown by Gromoll and Meyer GM69 was that a complete non-compact 3-manifold of positive sectional curvature is homeomorphic to $\mathbb{R}^{3}$.

The next step in the development of this subject is due to Schoen and Yau [SY82] who used minimal surfaces theory and the splitting theorem CG71 to study the topology of 3 -manifolds. They proved that a complete non-compact 3-manifold of positive Ricci curvature is homeomorphic to $\mathbb{R}^{3}$.

Although all of these works are very impressive, they still left the open question (See Problem 27 in Yau82):

How to classify non-compact 3-manifolds with positive scalar curvature, up to diffeomorphism?

The goal now is to find all obstructions and to characterize all open 3-manifolds with positive scalar curvature. Although Gromov-Lawson [GL83] and Schoen-Yau [SY82 gave several topological obstructions, all those obstruction classes both vanish for contractible 3-manifolds.

Let us consider contractible 3-manifolds. For example, $\mathbb{R}^{3}$ admits a complete metric $g_{1}$ of positive scalar curvature, where

$$
g_{1}=\sum_{i=1}^{3}\left(d x_{i}\right)^{2}+\left(\sum_{i=1}^{3} x_{i} d x_{i}\right)^{2}
$$

So far, it is the only known contractible 3-manifold which admits a complete metric of positive scalar curvature. This suggests the following question:

Is any complete contractible 3-manifold of positive scalar curvature homeomorphic to $\mathbb{R}^{3}$ ?

A complete contractible 3-manifold of uniformly positive scalar curvature (i.e. its scalar curvature is bounded away from zero) is homeomorphic to $\mathbb{R}^{3}$. It was first proved by Gromov and Lawson GL83. Recently, it was generalized by Chang, Weinberger and Yu $\mathbf{C W Y 1 0}$, to contractible 3-manifolds whose scalar curvature is uniform positive outside a compact set. Using minimal surfaces theory, we further generalize it.

Theorem $A$. (See Theorem 3.3.12 and Theorem 1.1 in Wan19c ) Assume that $\left(M^{3}, g\right)$ is a contractible complete 3 -manifold. If there exists a number $\alpha \in(-\infty, 2)$ such that

$$
\liminf _{r(x) \rightarrow \infty} r^{\alpha}(x) \kappa(x)>0
$$

where $\kappa(x)$ is the scalar curvature of $(M, g)$ and $r(x)$ is the distance function from some point $0 \in M$ to $x$, then $M^{3}$ is diffeomorphic to $\mathbb{R}^{3}$.

The proof follows the argument of Gromov and Lawson (See Corollary 10.9 in $\widehat{\mathbf{G L 8 3} \mid) . ~}$

### 0.1. Contractible 3 -manifolds

Thurston's Geometrisation conjecture $\mathbf{P e r 0 2 a}$ Per02b, Per03 (See also $\mathbf{B B B}^{+} \mathbf{1 0}$ and MT07]) shows that for a compact 3-manifold, its topology is fully determined by its
homotopy type. However, the topological structure of contractible 3-manifolds is much more complicated. For example, the Whitehead manifold (constructed in [Whi35]) is a contractible 3-manifold but not homeomorphic to $\mathbb{R}^{3}$.

In order to explain the construction of the Whitehead manifold, let us introduce the concept of a meridian curve. A meridian $\gamma \subset \partial N$ of a closed solid torus $N$ is an embedded closed curve which is null-homotopic in $N$ but not contractible in $\partial N$. A meridian disc $(D, \partial D) \subset(N, \partial N)$ of the solid torus $N$ is an embedded disc whose boundary is a meridian of $N$. (See Definition 1.2.1)

The Whitehead manifold is constructed from the Whitehead link. Recall that the Whitehead link is a link with two components illustrated in the following figure:


Figure 0.1.

Choose a closed unknotted solid torus $T_{1}$ in $\mathbb{S}^{3}$. Its complement inside $\mathbb{S}^{3}$ is another solid torus. Take a second solid torus $T_{2}$ inside $T_{1}$ so that the core $K_{2}$ of $T_{2}$ forms a Whitehead link with any meridian of $T_{1}$ as in the following figure.


Figure 0.2.

The solid torus $T_{2}$ is unknotted in $\mathbb{S}^{3}$. Then, embed $T_{3}$ inside $T_{2}$ in the same way as $T_{2}$ lies into $T_{1}$ and so on infinitely many times. Define the set $T_{\infty}=\bigcap_{k=1}^{\infty} T_{k}$, called the Whitehead continuum.

The Whitehead manifold is defined as $W h:=\mathbb{S}^{3} \backslash T_{\infty}$ which is a non-compact 3manifold without boundary.

Remark. Since each $T_{k}$ is unknotted in $\mathbb{S}^{3}$, its complement $N_{k}$ is a solid torus. Therefore, the Whitehead manifold is an increasing union of solid tori $\left\{N_{k}\right\}_{k}$. Each $N_{k}$ is embedded inside $N_{k+1}$ in the same way as $T_{2}$ lies in $T_{1}$. This follows from the symmetry of the Whitehead link.

Variation on the construction, like changing the knot at each step $k$, gives a family of so-called genus one 3-manifold, introduced in McM62]. Their construction is involved with the geometric index.

If $N^{\prime} \subset N$ are solid tori, the geometric index, $I\left(N^{\prime}, N\right)$, of $N^{\prime}$ in $N$ is equal to the minimal number of points of the intersection of the core of $N^{\prime}$ with a meridian disc of $N$. A genus one 3 -manifold is the ascending union of solid tori $\left\{N_{k}\right\}$, so that for each $k$, $N_{k} \subset \operatorname{Int} N_{k+1}$ and the geometric index of $N_{k}$ in $N_{k+1}$ is not equal to zero. (See Definition 1.3.8. Definition 1.3 .12 and GRW18)

For example, $W h$ is a contractible genus one 3-manifold. The geometric index $I\left(N_{k}, N_{k+1}\right)$ equals two for each $k$, where $N_{k}$ is illustrated as above. Remark that $\mathbb{R}^{3}$ is not genus one but genus zero, since it is an increasing union of 0 -handlebodies (i.e. 3-balls).

An interesting question is whether the Whitehead manifold admits a complete metric of positive scalar curvature. In this thesis, we answer negatively:

Theorem $B_{1}$. (See Theorem 1.1 of Wan19a) The Whitehead manifold has no complete metric of positive scalar curvature.

This result can be generalized to the genus one case.
Theorem $B_{2}$. (See Theorem 1.2 of Wan19a) No contractible genus one 3-manifold has a complete metric of positive scalar curvature.

Combining with Kazdan's work Kaz82, we generalize these results to the nonnegative scalar curvature.

Corollary. (See Corollary 6.4.3) No contractible genus one 3-manifold admits a complete metric of non-negative scalar curvature.

The existence of complete metrics of positive scalar curvature is related with the fundamental group at infinity. The fundamental group $\pi_{1}^{\infty}$ at infinity of a connected space is the inverse limit of the fundamental groups of complements of compact subsets (See Definition 1.1.9).

The triviality of the fundamental group at infinity is not equivalent to the simplyconnectedness at infinity(See Definition 1.1.7). For example, the Whitehead manifold is not simply-connected at infinity but its fundamental group at infinity is trivial.

We prove the following:

Theorem C. (See Theorem 1.1 of Wan19b]) A contractible 3-manifold with positive scalar curvature and trivial $\pi_{1}^{\infty}$ is homeomorphic to $\mathbb{R}^{3}$.

This result can also be generalised to the non-negative scalar curvature.
Corollary. (See Corollary 6.4.4) A contractible 3-manifold with non-negative scalar curvature and trivial $\pi_{1}^{\infty}$ is homeomorphic to $\mathbb{R}^{3}$.

However, there are uncountably many mutually non-homeomorphic contractible 3manifolds with non-trivial $\pi_{1}^{\infty}$. In Chapter 1.3, we construct such a manifold and show that this manifold has no complete metric of positive scalar curvature (See Theorem 1.3.15.

### 0.2. The idea of the proof of Theorem $B_{2}$

It is classical that minimal surfaces theory gives topological information about 3manifolds. This fact appeared in the articles of Schoen and Yau [SY82,SY79b,SY79a as well as Gromov and Lawson's GL83 and various other works.

For the proof of Theorem $B_{2}$, we argue by contradiction. Suppose that $(M, g)$ is a complete Riemannian manifold of positive scalar curvature, where $M:=\bigcup_{k} N_{k}$ is a contractible genus one 3 -manifold and the family $\left\{N_{k}\right\}_{k}$ of solid tori is assumed as in Theorem 1.3.13.
0.2.1. Minimal surfaces and Limits. As in SY82 and GL83, our first step is to construct minimal surfaces. Choose $\gamma_{k} \subset \partial N_{k}$ as a meridian of $N_{k}$ (See Definition 1.2.1). Roughly, it is spanned by an embedded stable minimal disc $\Omega_{k}$. Its existence is ensured by the result of Meeks and Yau (See MY80, MY82 or Theorem 3.1.8) when the boundary $\partial N_{k}$ is mean convex.

Let us consider the simplest case when $\Omega_{k}$ converges to a connected stable minimal surface $\Sigma$.

On the one side, we show that the number of connected components of $\Omega_{k} \cap N_{1}$ intersecting $N_{0}$ goes to infinity as $k$ goes to infinity (See Chapter 2.1). Therefore, there are infinitely many connected components of $\Sigma \cap N_{1}$ intersecting $N_{0}$. By a result of Meeks and Yau (See Theorem 3.2 .7 anc $\mathbf{M Y 8 0}]$ ), each of these components contains a definitive amount of area. Hence, $\Sigma \cap N_{1}$ has infinite area.

On the other side, since $(M, g)$ has positive scalar curvature, not only $\Sigma$ is conformally diffeomorphic to $\mathbb{R}^{2}$ (See Corollary 3.3.11), but also its geometry is constrained by the so-called extrinsic Cohn-Vossen inequality:

Theorem $D$. (See Theorem 3.3 .10 and Wan19a) Let $\Sigma^{2} \subset\left(M^{3}, g\right)$ be a complete (non-compact) immersed stable minimal surface. If the complete manifold $\left(M^{3}, g\right)$ has non-negative scalar curvature $(\kappa(x) \geqslant 0)$, then

$$
\int_{\Sigma} \kappa(x)+1 / 2|A|^{2} d v \leqslant 2 \pi \chi(\Sigma)
$$

where $|A|^{2}$ is the square norm of the second fundamental form of $\Sigma$. Moreover, if $\kappa>0$ and $\Sigma$ is embedded, then $\Sigma$ is a properly embedded plane.

Since the scalar curvature is bounded away zero on $N_{1}$, this is in contradiction with the infinite area contained in $\Sigma \cap N_{1}$.

If $\partial N_{k}$ is not mean convex, we modify the metric in a smaller tubular neighborhood of $\partial N_{k}$ so that for the new metric, it becomes mean convex. Then $\Omega_{k}$ is stable minimal for the new metric and for the original away from the neighborhood $\partial N_{k}$, (for example, near $N_{k}$ ), which is sufficient for our proof.
0.2.2. Properties of the limit surface. Generally, $\Omega_{k}$ sub-converges to a minimal lamination $\mathscr{L}:=\bigcup_{t \in \Gamma} L_{t}$ (that is, a disjoint union of some embedded minimal surfaces) instead of a single surface. It may have infinitely many components. However, each leaf $L_{t}$ of $\mathscr{L}$ is a complete (non-compact) stable minimal surface (See Theorem 4.2.3). Since $(M, g)$ has positive scalar curvature, it is homeomorphic to $\mathbb{R}^{3}$ (See Corollary 3.3.11).

The geometry of each leaf is influenced by the extrinsic Cohn-Vossen inequality (See Theorem $D$ ) as well as by a topological property of $M$, called Property P (See Definition 2.1.3). These two aspects tell us that the lamination $\mathscr{L}$ has the Vanishing property for $\left\{N_{k}\right\}_{k}$. That is to say,
there is a positive integer $k_{0}$ such that for any $k \geqslant k_{0}$ and any $t \in \Gamma$, any circle in $L_{t} \cap \partial N_{k}$ is null-homotopic in $\partial N_{k}$.

The reason is as follows: Suppose that there exists a sequence $\left\{k_{n}\right\}$ of increasing integers and a sequence $\left\{L_{t_{n}}\right\}$ of leaves so that for each $n, L_{t_{n}} \cap \partial N_{k_{n}}$ has at least one non-nullhomotopic circle in $\partial N_{k_{n}}$. Similar to the property of the sequence $\left\{\Omega_{k}\right\}_{k}$ (Property $P$ ), we know that the number of connected components of $L_{t_{n}} \cap N_{1}$ intersecting $N_{0}$ goes to infinity as $k_{n}$ goes to infinity (See Definition 2.1.3 and Theorem 2.1.6).

The sequence $\left\{L_{t_{n}}\right\}$ sub-converges to some leaf $L_{t_{\infty}}$ in the lamination $\mathscr{L}$ with finite multiplicity. The sub-convergence is ensured by a result of Schoen Sch83 (See Lemma 4.1 .5 ) and the extrinsic Cohn-Vossen inequality (See Theorem D). Therefore, $L_{t_{\infty}} \cap N_{1}$ has infinitely many components intersecting $N_{0}$. As in the above case, each component has a definite amount of area. The extrinsic Cohn-Vesson inequality gives a contradiction.

Let us explain how to deduce a contradiction from the Vanishing property. We show that for any $k \geqslant k_{0}, \Omega_{k} \cap \partial N_{k_{0}}$ contains a closed curve which is not null-homotopic in $\partial N_{k_{0}}$ (See Lemma 1.3.11). Roughly speaking, these non-nullhomotopic circles will sub-converge to some closed curve in $\mathscr{L} \cap \partial N_{k_{0}}$ which is not contractible in $\partial N_{k_{0}}$. It follows the fact that $\Omega_{k}$ sub-converges to the lamination $\mathscr{L}$. Therefore, some leaf of $\mathscr{L}$ has a non-contractible circle in $\partial N_{k_{0}}$. This is in contradiction with the above Vanishing property of $\mathscr{L}$.

## 0.3. $\pi_{1}^{\infty}$ and the Vanishing property

0.3.1. Handlebodies and Property H. Let $(M, g)$ be a complete contractible 3manifold of positive scalar curvature. It is an increasing union of closed handlebodies $\left\{N_{k}\right\}$ (See Theorem 1.1.12).

In the following, we consider that $M$ is not homeomorphic to $\mathbb{R}^{3}$. We may assume that none of the $N_{k}$ is contained in a 3-ball (i.e. homeomorphic to a unit ball in $\mathbb{R}^{3}$ ) in $M$ (See Remark 1.1.8).

In the genus one case, the family $\left\{N_{k}\right\}$ has several good properties. For example, the maps $\pi_{1}\left(\partial N_{k}\right) \rightarrow \pi_{1}\left(\overline{M \backslash N_{k}}\right)$ and $\pi_{1}\left(\partial N_{k}\right) \rightarrow \pi_{1}\left(\overline{N_{k} \backslash N_{0}}\right)$ are both injective (See Lemma 1.3.10). These properties are crucial and necessary in the study of the existence of complete metrics of positive scalar curvature. In general, the family $\left\{N_{k}\right\}$ may not have the above properties.

For example, the map $\pi_{1}\left(\partial N_{0}\right) \rightarrow \pi_{1}\left(\overline{M \backslash N_{0}}\right)$ may not be injective. To overcome it, we use topological surgeries on $N_{0}$ and find a new handlebody to replace it. Precisely, we use the loop lemma to find an embedded disc $(D, \partial D) \subset\left(\overline{M \backslash N_{0}}, \partial N_{0}\right)$ whose boundary is a non-contractible simple curve in $\partial N_{0}$. The new handlebody is obtained from $N_{0}$ by attaching a closed tubular neighborhood $N_{\epsilon}(D)$ of $D$ in $\overline{M \backslash N_{0}}$.

We repeatedly use topological surgeries on each $N_{k}$ to obtain a new family $\left\{R_{k}\right\}_{k}$ of closed handlebodies with the following properties, called Property $H$ (See Definition 2.2.5):
(1) the map $\pi_{1}\left(\partial R_{k}\right) \rightarrow \pi_{1}\left(\overline{R_{k} \backslash R_{0}}\right)$ is injective for $k>0$;
(2) the map $\pi_{1}\left(\partial R_{k}\right) \rightarrow \pi_{1}\left(\overline{M \backslash R_{k}}\right)$ is injective for $k \geqslant 0$;
(3) each $R_{k}$ is contractible in $R_{k+1}$ but not contained in a 3 -ball in $M$;
(4) there exists a sequence of increasing integers $\left\{j_{k}\right\}_{k}$, such that $\pi_{1}\left(\partial R_{k} \cap \partial N_{j_{k}}\right) \rightarrow$ $\pi_{1}\left(\partial R_{k}\right)$ is surjective.

Remark. If $M$ is not homeomorphic to $\mathbb{R}^{3}$, the existence of such a family is ensured by Theorem 2.2.6. It is not unique. In addition, the union of such a family may be not equal to $M$.

For example, if $M:=\bigcup_{k} N_{k}$ is a contractible genus one 3-manifold, the family $\left\{N_{k}\right\}$ (assumed as in Theorem 1.3.13) satisfies the above Property $H$ (See Lemma 2.10 in Wan19a or Lemma 1.3.10).
0.3.2. The Vanishing property. In the genus one case, the geometry of a stable minimal surface is constrained by the geometric index (See Property $P$ in Wan19a or Definition 2.1.3). In the higher genus case, the behavior of a stable minimal surface is related to the fundamental group at infinity.

In order to clarify their relationship, let us introduce a geometric property, called the Vanishing property. First, we consider a complete contractible 3-manifold ( $M, g$ ) of positive scalar curvature which is not homeomorphic to $\mathbb{R}^{3}$. As indicated above, there is an increasing family $\left\{R_{k}\right\}_{k}$ of closed handlebodies with Property H .

A complete embedded stable minimal surface $\Sigma \subset(M, g)$ is called to satisfy the Vanishing property for the family $\left\{R_{k}\right\}_{k}$ if there is a positive integer $k(\Sigma)$ so that for $k \geqslant k(\Sigma)$, any circle in $\Sigma \cap \partial R_{k}$ is nullhomotopic in $\partial R_{k}$ (See Chapter 5).

If a complete stable minimal surface does not satisfy the Vanishing property for $\left\{R_{k}\right\}_{k}$, it gives a non-trivial element in $\pi_{1}^{\infty}(M)$ (See Lemma 5.2.1). As a consequence, if $\pi_{1}^{\infty}$ is trivial, any complete stable minimal surface in $M$ has the Vanishing property for $\left\{R_{k}\right\}_{k}$ (See Corollary 5.2.2).
0.3.3. The idea of the proof of Theorem C. We argue by contradiction. Suppose that a complete contractible 3-manifold $(M, g)$ with positive scalar curvature and trivial $\pi_{1}^{\infty}(M)$ is not homeomorphic to $\mathbb{R}^{3}$.

Before constructing minimal surfaces, let us introduce a notation from 3-dimensional topology.

A system of meridians of a handlebody $N$ is a collection of $g$ distinct meridians $\left\{\gamma^{l}\right\}_{l=1}^{g}$ with the property that $\partial N \backslash \coprod_{l=1}^{g} \gamma^{l}$ is homeomorphic to an open disc with some closed subdiscs removed (See Lemma 1.2.7). Its existence is ensured by Lemma 1.2.7.

Let $\left\{N_{k}\right\}_{k}$ and $\left\{R_{k}\right\}_{k}$ be as above. Since $N_{0}$ is not contained in a 3-ball (See Remark 1.1.8), the genus of $N_{k}$ is greater than zero. The handlebody $N_{k}$ has a system of meridians $\left\{\gamma_{k}^{l}\right\}_{l=1}^{g\left(N_{k}\right)}$. Roughly, there are $g\left(N_{k}\right)$ disjoint area-minimizing discs $\left\{\Omega_{k}^{l}\right\}_{l}$ with $\partial \Omega_{k}^{l}=\gamma_{k}^{l}$. Their existence is ensured by the works of Meeks and Yau MY80 MY82 (See Theorem 6.28 of $[\mathbf{C M 1 1}]$ ) when the boundary $\partial N_{k}$ is mean convex.

Let us explain their existence. We construct these discs by induction on $l$.
When $l=1$, there is an embedded area-minimizing disc $\Omega_{k}^{1} \subset N_{k}$ with boundary $\gamma_{k}^{1}$ (See MY80, MY82 or Theorem 6.28 of (CM11]).

Suppose that there are $l$ disjointly embedded stable minimal discs $\left\{\Omega_{k}^{i}\right\}_{i=1}^{l}$ with $\partial \Omega_{k}^{i}=$ $\gamma_{k}^{i}$. Our target is to construct a stable minimal surface $\Omega_{k}^{l+1}$ with boundary $\gamma_{k}^{l+1}$.

Let us consider the Riemannian manifold $\left(T_{k, l},\left.g\right|_{T_{k, l}}\right)$, where $T_{k, l}:=N_{k} \backslash \coprod_{i=1}^{l} \Omega_{k}^{l}$. It is a handlebody of genus $g\left(N_{k}\right)-l$. For example, see the following figure.


Figure 0.3.

The boundary of $\left(T_{k, l},\left.g\right|_{T_{k, l}}\right)$ consists of $\partial N_{k} \backslash \coprod_{i=1}^{l} \gamma_{k}^{i}$ and some disjoint discs $\left\{\Omega_{k}^{i-}\right\}_{i=1}^{l}$ and $\left\{\Omega_{k}^{i+}\right\}_{i=1}^{l}$. The two discs $\Omega_{k}^{i-}$ and $\Omega_{k}^{i+}$ both come from the same minimal disc
$\Omega_{k}^{i}$. Therefore, the mean curvature of the boundary of $\left(T_{k, l},\left.g\right|_{T_{k, l}}\right)$ is non-negative. (See Chapter 4.2)

In addition, $\left\{\gamma_{k}^{i}\right\}_{i>l}$ is a system of meridians of the handlebody $\left(T_{k, l},\left.g\right|_{T_{k, l}}\right)$. Then, we use the result of Meeks and Yau to find an embedded stable minimal surface $\Omega_{k}^{l+1} \subset T_{k, l}$ with boundary $\gamma_{k}^{l+1}$. These discs $\left\{\Omega_{k}^{i}\right\}_{i=1}^{l+1}$ are disjoint in $N_{k}$. This finishes the inductive construction.

As in the genus one case, if $\partial N_{k}$ is not mean convex, we can deform the metric in a small neighborhood of it so that it becomes mean convex.

Define the lamination $\mathscr{L}_{k}:=\coprod_{l} \Omega_{k}^{l}$ (i.e. a disjoint union of embedded surfaces). We show that each lamination $\mathscr{L}_{k}$ intersects the compact set $R_{0}$ (Corollary 1.2.8). According to Colding-Mincozzi's theory (See Appendix B of $\mid \overline{\mathrm{CM04}})$, the sequence $\left\{\mathscr{L}_{k}\right\}_{k}$ subconverges to a lamination $\mathscr{L}:=\bigcup_{t \in \Lambda} L_{t}$ in $(M, g)$ (See Theorem 4.2.3). Note that each leaf $L_{t}$ is a complete (non-compact) stable minimal surface.

As indicated above, since $(M, g)$ has positive scalar curvature and $\pi_{1}^{\infty}(M)$ is trivial, each leaf $L_{t}$ in $\mathscr{L}$ has the Vanishing property for $\left\{R_{k}\right\}_{k}$ (See Lemma 5.2.1 and Corollary 5.2 .2 ). Furthermore, the lamination $\mathscr{L}$ also satisfies the Vanishing property (See Corollary 5.2.4. That is to say,
there exists a positive integer $k_{0}$ such that for any $k \geqslant k_{0}$ and any $t \in \Lambda$, any circle in $L_{t} \cap \partial R_{k}$ is nullhomotopic in $\partial R_{k}$.

The reason is described as follows.
We argue by contradiction. Suppose that there exists a sequence $\left\{k_{n}\right\}_{n}$ of increasing integers and a sequence $\left\{L_{t_{n}}\right\}$ of leaves in $\mathscr{L}$ satisfying that $L_{t_{n}} \cap \partial R_{k_{n}}$ has at least one non-nullhomotopic circle(s) in $\partial R_{k_{n}}$ for each $n$.

The sequence $\left\{L_{t_{n}}\right\}$ smoothly subconverges to some leaf in $\mathscr{L}$. For our convenience, we may assume that the sequence $\left\{L_{t_{n}}\right\}$ converges to the leaf $L_{t_{\infty}}$. The leaf $L_{t_{\infty}}$ satisfies the Vanishing property. That is to say, there is a positive integer $k\left(L_{t_{\infty}}\right)$ such that for $k \geqslant k\left(L_{t_{\infty}}\right)$, any circle $\partial R_{k} \cap L_{t_{\infty}}$ is nullhomotopic in $\partial R_{k}$.

However, since $L_{t_{n}} \cap \partial R_{k_{n}}$ has some non-null-homotopic circle in $\partial R_{k_{n}}$, we know that for $k_{n}>k\left(L_{t_{\infty}}\right), L_{t_{n}} \cap \partial R_{k\left(L_{t_{\infty}}\right)}$ has a meridian of $R_{k\left(L_{t_{\infty}}\right)}$ (See Remark 2.2.7 and Corollary 1.2.6). These meridians of $R_{k\left(L_{\left.t_{\infty}\right)}\right)}$ will converge to a meridian of $R_{k\left(L_{\left.t_{\infty}\right)}\right)}$ which is contained in $L_{t_{\infty}} \cap \partial R_{k\left(L_{t_{\infty}}\right)}$. This is in contradiction with the last paragraph.

Let us explain how to deduce a contradiction from the Vanishing property of $\mathscr{L}$.
We show that if $N_{k}$ contains $R_{k_{0}}$ (for $k$ large enough), then $\mathscr{L}_{k} \cap \partial R_{k_{0}}$ contains at least one meridian of $R_{k_{0}}$ (See Corollary 1.2.8). As in the above case, these meridians of $R_{k_{0}}$ will sub-converge to a non-contractible circle in $\mathscr{L} \cap \partial R_{k_{0}}$. The Vanishing property of $\mathscr{L}$ gives a contradiction.

### 0.4. Closed Aspherical 4-manifolds

A manifold $M$ is called aspherical if it is path-connected and all its higher homotopy groups vanish (i.e. $\pi_{k}(M)$ is trivial for $k \geqslant 2$ ). The class of aspherical manifolds contains all hyperbolic manifolds and all manifolds with non-positive curvature.

An interesting question posed by Geroch is whether the torus $\mathbb{T}^{n}, n \geqslant 3$, carries a metric of positive scalar curvature. This question was settled by Gromov-Lawson GL83 and Schoen-Yau [SY79b, SY82, SY17]. Generally, it is conjectured that

Conjecture. No compact aspherical manifold has a metric of positive scalar curvature.

This conjecture was proved for 3-manifolds by Gromov and Lawson [GL83]. In dimension four, it is confirmed for 4-manifolds which contains incompressible surfaces [GL83].

In this thesis, we prove that
Theorem $F$. No closed aspherical 4-manifold with non-zero first Betti number has a metric of positive scalar curvature.

Note that there is a closed aspherical 4-manifold whose first Betti number vanishes (See RT05]).

We argue by contradiction. Suppose that there is a compact aspherical 4-manifold $\left(M^{4}, g\right)$ of positive scalar curvature, where the first Betti number $b_{1}\left(M^{4}\right)$ is greater than zero.

Choose a circle $\gamma \subset M^{4}$ so that $[\gamma]$ has infinite order in $H_{1}\left(M^{4}, \mathbb{Z}\right)$. We use the Poincaré duality to find a class $u \in H_{3}\left(M^{4}, \mathbb{Z}\right)$ with $<u,[\gamma]>=1$. A theorem of Fleming-Federer (See $[\mathbf{F F 6 0}]$ or Chapter 7 of $[\mathbf{S i m 8 3}$ ) tells us that there is a volumeminimizing hypersurface $\Sigma^{3}$ in this class. Therefore, the intersection number of $\gamma$ and $\Sigma^{3}$ is equal to one.

Since $(M, g)$ has positive scalar curvature, then $\Sigma^{3}$ admits a metric of positive scalar curvature (See Proposition 3.3.5). The manifold $\Sigma^{3}$ is homeomorphic to a connected sum of spherical 3-manifolds and some copies of $\mathbb{S}^{1} \times \mathbb{S}^{2}$ (See Per02a, Per02b, Per03, ( $\mathrm{BBB}^{+} \mathbf{1 0}$ and (MT07]).

Because $\pi_{2}(M)$ and $\pi_{3}(M)$ are both trivial, then the spherical part of $\Sigma$ and the 2spheres in $\Sigma$ are homotopic to a point in $M^{4}$. That is to say, $\Sigma$ is homotopic to a wedge sum of some circles in $M$ (these circles come from the $\mathbb{S}^{2} \times \mathbb{S}^{1}$ s part of $\Sigma^{3}$ ). Because $\operatorname{dim}(M)=4$, we see that the intersection number of $\Sigma$ and $\gamma$ equals zero, which is in contradiction with the intersection number of $\gamma$ and $\Sigma$.

### 0.5. Organization of the thesis

The plan of this thesis is as follows:
For the first part, we discuss contractible 3-manifolds and related topological properties. In Chapter 1, we recall related background from 3 -manifolds, such as simplyconnectedness at infinity, the fundamental group at infinity and handlebodies. Subsequently, we discuss the topological structure of contractible 3-manifolds and derive some notations such as meridians of a handlebody, a system of meridians. Finally, we give some examples of 3-manifolds such as the Whitehead manifold and genus one 3-manifolds.

In Chapter 2, we start with the embedded discs in the Whitehead manifold. An interesting fact is that the behavior of these discs is influenced by the geometric index. Their relation is suggested by Theorem 2.1.2. Based on this relation, we introduce a
new property, called Property $P$ and we show that any contractible genus one 3-manifold satisfies this property (See Theorem 2.1.6).

Generally, a contractible 3-manifold may not satisfy Property P. The reason is that it may be made up of some handlebodies of higher genus. Some handles in these handlebodies may make no contribution to its topology and yield technical difficulties. To overcome it, we introduce two types of surgeries. Using these surgeries, we find a new family of handlebodies with good properties, called Property H (See Definition 2.2.5).

In the second part of the thesis, we focus on minimal surfaces and related convergence theories. In Chapter 3, we recall some notations such as the so-called first and second variation formulas, Morse index and the stable condition (See Chapter 3.1.1). Then we discuss Plateau's problem(See Chapter 3.1.2).

Subsequently, we focus on the local properties of minimal surfaces, including the strong maximal principle (See Corollary 3.2.3) and the monotonicity formula (See Proposition 3.2.5).

Then, we study the topology of stable minimal hypersurfaces. These hypersurfaces are characterized by the first eigenvalue of the stable operator (See Lemma 3.3.1 and Theorem 3.3.4). In a manifold of positive scalar curvature, there are many topological constraint for stable minimal surfaces. For example, if a complete 3-manifold has nonnegative scalar curvature, a complete stable minimal surface in it satisfies the extrinsic Cohn-Vossen inequality (See Corollary 3.3.6 and Theorem 3.3.10). As a consequence, we give a new proof of the topological classification of stable minimal surfaces in a 3-manifold with nonnegative scalar curvature (See Corollary 3.3 .11 and $\mathbf{S Y 8 2}$ ). Finally, as an application of minimal surfaces theory, we give the proof of Theorem $A$ (See Theorem 3.3.12).

In Chapter 4, we discuss the convergence theory of minimal surfaces. We begin with the convergence of minimal surface equations (See Lemma 4.1.1). It can be generalized to the Riemannian case. Therefore we get a compactness theorem for minimal surfaces (See Theorem 4.1.4).

Next, we discuss the convergence without area estimate. In this case, the limit is a minimal lamination (i.e. a disjoint union of some embedded discs) instead of a single surface. We recall the minimal lamination theory of Colding-Minicozzi (See Appendix B of (CM04). Then we construct a required family of minimal laminations in a contractible 3 -manifold. Their limit is a stable minimal lamination. Each leaf is a complete minimal surface. If the manifold has positive scalar curvature, it is a properly embedded plane (See Theorem $D$ ). As an application, we give a new proof of the topological classification of contractible 3-manifolds with uniformly positive scalar curvature (See Corollary 4.2.7 and (GL83).

For the third part, we give the complete proofs of the main theorems. In Chapter 5, we introduce the Vanishing property and study its relationship with the fundamental group at infinity, which is suggested by Lemma 5.2.1. Note that, in the genus one case, Property P implies the Vanishing property (See Theorem 5.1.1).

In Chapter 6, we reduce the proof of the main theorems to a cover lemma (See Lemma 6.1.3). For the proof of this lemma, we use the Vanishing property of the lamination (constructed in Chapter 4.2) to define a set $S$ (See Definition 6.3.1). Positivity of the scalar curvature implies the finiteness of $S$ (See Lemma 6.3.4 and Lemma 6.3.5). We use
the finiteness to prove the cover lemma. Finally, we discuss several related questions and further research about 3-manifolds.

In the fourth part of the thesis, we discuss the existence of metrics of positive scalar curvature over a compact aspherical 4-manifold. Together with a result of Perelman Per02a, Per02b, Per03, we give a proof of Theorem F. Finally, we talk about several further questions about 4-manifolds.

## Introduction (français)

La géométrie riemannienne vise à étudier les variétés riemanniennes qui sont des variétés lisses à structures métriques. Une des questions fondamentales est de comprendre la relation entre la courbure, définie localement, et les propriétés globales des variétés lisses. Le résultat le plus ancien est le théorème classique de Gauss et Bonnet, qui relie la courbure au nombre d'Euler, un invariant topologique. Ce théorème implique, par exemple, que toute surface compacte du genre $g>0$ n'a pas de métrique à courbure positive.

En dimension supérieure, l'existence de métriques à courbure positive devient beaucoup plus compliquée parce qu'il existe plusieurs obstructions topologiques pour qu'une variété lisse ait une métrique complète à courbure positive.

Nous prenons ensuite 3 -variétés pour expliquer ce fait.

| $\left(M^{3}, g\right)$ | cas compact | cas non compact |
| :---: | :---: | :---: |
| $K>0$ | $\mathbb{S}^{3} / \Gamma$ | $\mathbb{R}^{3}$ |
| Ric>0 |  | $\mathbb{R}^{3}$ |
| Scal $>0$ | $\left(\#_{i=1}^{k} \mathbb{S}^{3} / \Gamma_{i}\right) \#\left(\#_{j=1}^{l} \mathbb{S}^{1} \times \mathbb{S}^{2}\right)$ | $?$ |

Table 2.

Pour une 3-variété riemannienne compacte, il existe une solution unique en temps court au flot de Ricci (normalisé), introduit par Hamilton Ham82. Si la variété a une courbure de Ricci positive, la solution en temps court peut être étendue en la solution en temps long. La limite de ce flot est une métrique à courbure sectionnelle constante. Autrement dit, si une 3 -variété compacte a courbure de Ricci positive, elle est homéomorphe au quotient $\mathbb{S}^{3} / \Gamma$ de la sphère $\mathbb{S}^{3}$ par un sous-groupe fini $\Gamma \subset O(4)$. Un tel quotient est appelé une 3 -variété sphérique.

La contribution majeure suivante au sujet a été apportée par Perelman $\mathbf{P e r 0 2 a}$, Per02b, Per03], qui a développé le flot de Ricci avec chirurgie. Un résultat surprenant et magnifique de cette étude est la preuve qu'une 3 -variété compacte á courbure scalaire positive est homéomorphe à une somme connexe de certaines 3 -variétés sphériques et de copies de $\mathbb{S}^{1} \times \mathbb{S}^{2}\left(\right.$ Voir aussi $\mathbf{B B B}^{+} \mathbf{1 0}$ et MT07]). Sa généralisation au cas non compact est due à Bessières, Besson et Maillot [BBM11].

Pour le cas non compact, les premiers résultats montrés par Gromoll et Meyer GM69 sont qu'une 3-variété complète non compacte à courbure sectionnelle positive est homéomorphe à $\mathbb{R}^{3}$ (Voir [GM69]).

L'étape suivante du développement de ce sujet est due à Schoen et Yau [SY82] qui ont utilisé la théorie des surfaces minimales et le théorème de splitting [CG71] pour étudier la topologie des 3 -variétés . Ils ont prouvé qu'une 3 -variété complète non compacte à courbure de Ricci positive est homéomorphe à $\mathbb{R}^{3}$.

Bien que toutes ces travauw soient trés impressionnants, ils ont toujours laissé la ouverte question (voir le problème 27 dans Yau82):

Comment classifier les 3-variétés non compactes à courbure scalaire positive, à difféomorphisme près ?

Le but est de trouver toutes les obstructions et de caractériser toutes les 3 -variètés ouvertes à courbure scalaire positive. Bien que Gromov-Lawson $\mathbf{G L 8 3}$ et Schoen-Yau [SY82] aient donné plusieurs obstructions topologiques, toutes ces classes d'obstruction disparaissent pour les 3 -variétés contractiles.

Considérons des 3 -variétés contractiles. Par exemple, $\mathbb{R}^{3}$ possède une métrique complète $g_{1}$ à courbure scalaire positive, où

$$
g_{1}=\sum_{i=1}^{3}\left(d x_{i}\right)^{2}+\left(\sum_{i=1}^{3} x_{i} d x_{i}\right)^{2} .
$$

Jusqu'à présent, c'est la seule 3 -variété contractile connue qui admet une métrique complète à courbure scalaire positive. Ceci suggère la question suivante :

Est-ce qu'une 3-variété complète contractile à courbure scalaire positive est homéomorphe $\grave{a} \mathbb{R}^{3}$ ?

Une 3-variété complète contractile à courbure scalaire uniformément positive (c'est-à-dire que sa courbure scalaire est minorée par une constante strictement positive) est homéomorphe à $\mathbb{R}^{3}$. Cela a été prouvé pour la première fois par Gromov et Lawson GL83]. Récemment, ce résultat a été généralisé par Chang, Weinberger et Yu CWY10 à des 3-variétés contractile dont la courbure scalaire est uniformément positive à l'extérieur d'un ensemble compact. En utilisant la théorie des surfaces minimales, nous généralisons davantage.
Théroème 1 (=Theorem (A) Supposons que $\left(M^{3}, g\right)$ est une 3 -variété complète contractile. S'il existe un réel $\alpha \in(-\infty, 2)$ tel que

$$
\liminf _{r(x) \rightarrow \infty} r^{\alpha}(x) \kappa(x)>0,
$$

où $\kappa(x)$ est la courbure scalaire de $(M, g)$ et $r(x)$ est la fonction de distance d'un point $0 \in M$ à $x$, alors $M^{3}$ est difféomorphe à $\mathbb{R}^{3}$.

La preuve suit les arguments de Gromov et Lawson (voir le corollaire 10.9 dans (GL83]).

### 0.6. 3-variétés contractiles

La conjecture de géométrisation, formulée par William Thurston Per02a, Per02b, Per03] (voir aussi $\left[\mathbf{B B B}^{+} \mathbf{1 0}\right]$ et $[\mathbf{M T 0 7 ]}]$ ) énonce que pour la topologie d'une 3-variété compacte est entièrement déterminée par son type d'homotopie. Cependant, la structure topologique des 3 -variétés contractiles est trés compliquée. Par exemple, la variété Whitehead (construite dans $\mathbf{W h i 3 5}$ ) est une 3 -variété contractile qui n'est pas homéomorphe à $\mathbb{R}^{3}$.

Pour expliquer la construction de la variété de Whitehead, introduisons le concept de méridien. Un méridien $\gamma \subset \partial N$ d'un tore solide fermé $N$ est une courbe fermée plongée qui est homotopiquement triviale dans $N$ mais non contractile dans $\partial N$. Un disque méridien $(D, \partial D) \subset(N, \partial N)$ d'un tore solide $N$ est un disque plongé dont la frontière est un méridien de $N$. (Voir Définition 1.2.1).

La variété de Whitehead est construite à partir de l'entrelacs de Whitehead. Rappelons que l'entrelacs de Whitehead est un entrelacs á deux composantes comme illustré sur la figure suivante:


Figure 0.4.

Choisissons un tore solide fermé $T_{1}$ qui est non noué dans $\mathbb{S}^{3}$. L'intérieur du tore solide dans $\mathbb{S}^{3}$ est un autre tore solide. Prenons un deuxième tore solide $T_{2}$ à l'intérieur de $T_{1}$ de sorte que l'intérieur de $K_{2}$ forme un entrelacs de Whitehead avec un méridien de $T_{1}$ comme sur la figure suivante.


Figure 0.5.
Le tore solide $T_{2}$ est non noué dans $\mathbb{S}^{3}$. Ensuite, on plonge $T_{3}$ dans $T_{2}$ de la même manière que $T_{2}$ se trouve dans $T_{1}$ et ainsi de suite un nombre infini de fois. Définissons l'ensemble $T_{\infty}=\bigcap_{k=1}^{\infty} T_{k}$, appelé le continuum de Whitehead.

La variété de Whitehead est définie comme suit: $W h:=\mathbb{S}^{3} \backslash T_{\infty}$, qui est une 3-variété non compacte.
Remarque Puisque chaque $T_{k}$ est non noué dans $\mathbb{S}^{3}$, son complémentaire $N_{k}$ est un tore solide. Par conséquent, la variété de Whitehead est une union croissante de tores solides $\left\{N_{k}\right\}_{k}$. Chaque $N_{k}$ est plongé dans $N_{k+1}$ de la même manière que $T_{2}$ dans $T_{1}$. Cela découle de la symétrie de l'entrelacs de Whitehead.

Une variation de la construction, comme changer le nœud à chaque étape $k$, donne une famille de ce qu'on appelle les 3 -variétés de genre un, introduite dans McM62. La construction est reliée à l'indice géométrique.

Si $N^{\prime} \subset N$ est un tore solide, l'indice géométrique, $I\left(N^{\prime}, N\right)$, de $N^{\prime}$ dans $N$ est égal au nombre minimal de points de l'intersection de l'intérieur de $N^{\prime}$ avec un disque méridien de $N$. Une 3 -variété de genre un est une union croissante de tores solides $\left\{N_{k}\right\}$ telle que pour chaque $k, N_{k} \subset \operatorname{Int} N_{k+1}$, et que l'indice géométrique de $N_{k}$ dans $N_{k+1}$ ne soit pas égal à zéro. (Voir Définition 1.3.8, Définition 1.3 .12 et [GRW18]).

Par exemple, Wh est une 3 -variété contractile de genre un. L'indice géométrique $I\left(N_{k}, N_{k+1}\right)$ est égal à deux pour chaque $k$, où $N_{k}$ est illustré comme ci-dessus. Remarquons que $\mathbb{R}^{3}$ n'est pas de genre un, mais de genre zéro, puisqu'il s'agit d'une union croissante de 0 -corps à anses. (c'est-à-dire de 3 -boules).

Une question intéressante est de savoir si la variété de Whitehead admet une métrique complète de courbure scalaire positive. Dans cette thèse, nous répondons négativement :

Théroème 2 (=Theorem $\left(B_{1}\right)$ La variété de Whitehead n'a pas de métrique complète á courbure scalaire positive.

Ce résultat peut être généralisé au cas du genre un.
Théroème 3 (=Theorem $B_{2}$ ) Une 3-variété contractile genre un n'a pas de métrique complète á courbure scalaire positive.

En combinant le travail de Kazdan Kaz82, nous généralisons ces résultats à la courbure scalaire non négative.

Corollarie (voir Corollaire 6.4.3) Une 3-variété contractile genre un n'a pas de métrique complète á courbure scalaire non négative

L'existence de métriques complètes à courbure scalaire positive est liée au groupe fondamental à l'infini. Le groupe fondamental à l'infini $\pi_{1}^{\infty}$ d'une 3 -variété est la limite projective des groupes fondamentaux des complémentaires de sous-ensembles compacts (voir Définition 1.1.9).

La trivialité du groupe fondamental à l'infini n'est pas équivalente pas à la connexité simple à l'infini. Par exemple, la variété de Whitehead n'est pas simplement connexe à l'infini mais son groupe fondamental à l'infini est trivial.

Nous prouvons le résultat suivant:
Théroème 4 (=Theorem $(C)$ Une 3 -variété contractile à courbure scalaire positive et $\pi_{1}^{\infty}$ trivial est homéomorphe à $\mathbb{R}^{3}$.

Ce résultat peut également être généralisé à la courbure scalaire non négative.
Corollarie (voir Corollaire 6.4.4) Une 3-variété contractile à courbure scalaire non négative et $\pi_{1}^{\infty}$ trivial est homéomorphe à $\mathbb{R}^{3}$.

Cependant, il existe une quantité indénombrable de 3 -variétés contractiles deux à deux non homéomorphes dont le $\pi_{1}^{\infty}$ n'est pas trivial. Dans le Chapitre 1.3, nous construisons une telle variété et montrons que cette variété n'a pas de métrique complète de courbure scalaire positive (voir Théorème 1.3.15).

### 0.7. L'idée de la preuve du Théorème 3

Il est classique que la théorie des surfaces minimales donne des informations topologiques sur les 3-variétés. Ce fait est apparu dans les articles de Schoen et Yau SY82,SY79b, SY79a ainsi que de Gromov et Lawson cite GL et divers autres travaux.

Pour la preuve du théorème 3, raisonnons par l'absurde. Supposons que $(M, g)$ soit une 3-variété riemannienne complète à courbure scalaire positive, où $M:=\bigcup_{k} N_{k}$ est une 3 -variété contractile de genre un et la famille $\left\{N_{k}\right\}_{k}$ de tores solides est supposée comme dans Théorème 1.3.13.
0.7.1. Surfaces minimales et limites. Comme dans [SY82] et GL83], notre première étape consiste à construire des surfaces minimales. Choisissons $\gamma_{k} \subset \partial N_{k}$ comme méridien de $N_{k}$ (voir Définition 1.2.1). En gros, il borde un disque minimal stable plongé $\Omega_{k}$. Son existence est assurée par le résultat de Meeks et Yau (voir MY80, MY82 ou le théorème 3.1.8 lorsque la frontière $\partial N_{k}$ est mean convex.

Considérons le cas le plus simple où $\Omega_{k}$ converge vers une surface minimale stable connexe $\Sigma$.

D'une part, nous montrons que le nombre de composantes connexes de $\Omega_{k} \cap N_{1}$ intersectant $N_{0}$ tend vers l'infini quand $k$ tend vers l'infini (voir Chapitre 2.1). Par conséquent, il existe une infinité de composantes connexes de $\Sigma \cap N_{1}$ intersectant $N_{0}$. Par un résultat
de Meeks et Yau (voir Théorème 3.2.7 et $\mathbf{M Y 8 0}]$ ), chacune de ces composantes contient une certaine quantité d'aire. Ainsi, $\Sigma \cap N_{1}$ a une aire infinie.

D'autre part, puisque $(M, g)$ a courbure scalaire positive, non seulement $\Sigma$ est conformément difféomorphe à $\mathbb{R}^{2}$ (voir Corollaire 3.3.11), mais sa géométrie est également contrainte par l'inégalité extrinsèque de Cohn-Vossen:

Théroème 5 (=Theorem $D$ ) Soit $\Sigma^{2} \subset\left(M^{3}, g\right)$ une surface minimale complète immergée (non compacte). Si la variété complète $\left(M^{3}, g\right)$ a courbure scalaire positive ou nulle $(\kappa(x) \geqslant 0)$, alors

$$
\int_{\Sigma} \kappa(x)+1 / 2|A|^{2} d v \leqslant 2 \pi \chi(\Sigma)
$$

où $|A|^{2}$ est le carré de la norme de la seconde forme fondamentale de $\Sigma$. De plus, si $\kappa>0$ et $\Sigma$ est plongée, alors $\Sigma$ est un plan proprement plongé.

Puisque la courbure scalaire est minorée par une constante strictement positive sur $N_{1}$, cela est en contradiction avec la surface infinie contenue dans $\Sigma \cap N_{1}$.

Si $\partial N_{k}$ n'est pas mean convex, on modifies la métrique dans un plus petit voisinage tubulaire de $\partial N_{k}$ afin que, pour la nouvelle métrique, elle devienne mean convex.. Alors $\Omega_{k}$ est minimal stable pour la nouvelle métrique, et pour la métrique originale il l'est loin du voisinage $\partial N_{k}$ (par exemple, près de $N_{k}$ ), ce qui est suffisant pour notre preuve.
0.7.2. Propriétés de la surface limite. En général, $\Omega_{k}$ sous-converge vers une lamination minimale $\mathscr{L}:=\bigcup_{t i n \Gamma} L_{t}$ (c'est-à-dire une union disjointe de certaines surfaces minimales plongées). Elle peut avoir une infinité de composantes. Cependant, chaque feuille $L_{t}$ de $\mathscr{L}$ est une surface minimale stable complète (non compacte) (voir Théorème 4.2.3). Puisque $(M, g)$ a une métrique à courbure scalaire positive, elle est homéomorphe à $\mathbb{R}^{3}$ (voir Corollaire 3.3.11).

La géométrie de chaque feuille est influencée par l'inégalité extrinsèque de Cohn-Vossen (cf. Théorème5) ainsi que par une propriété topologique de $M$, appelée Propriété $P$ (voir Définition 2.1.3). Ces deux aspects nous indiquent que la lamination $\mathscr{L}$ a la propriété d'annulation pour $\left\{N_{k}\right\}_{k}$. C'est-à-dire,
il existe un entier positif $k_{0}$ tel que pour tout $k \geqslant k_{0}$ et tout $t \in \Gamma$, tout cercle dans $L_{t} \cap \partial N_{k}$ est homotopiquement trivial à $\partial N_{k}$.

La raison en est la suivante : supposons qu'il existe une suite décroissante d'entiers $\left\{k_{n}\right\}$ et une suite $\left\{L_{t_{n}}\right\}$ de feuilles telles que pour chaque $n, L_{t_{n}} \cap \partial N_{k_{n}}$ a au moins un cercle non homotopiquement trivial dans $\partial N_{k_{n}}$. Comme pour la propriété de la suite $\left\{\Omega_{k}\right\}_{k}$ (Propriété $P$ ), nous savons que le nombre de composantes connexes de $L_{t_{n}} \cap N_{1}$ intersectant $N_{0}$ tend vers l'infini quand $k_{n}$ tend vers l'infini (Voir Définition 2.1.3 et Théorème 2.1.6).

La suite $\left\{L_{t_{n}}\right\}$ sous-converge vers une feuille $L_{t_{\text {infty }}}$ dans la lamination $\mathscr{L}$ avec une multiplicité finie. La sous-convergence est assurée par un résultat de Schoen [Sch83] (voir Lemme 4.1.5) et l'inégalité extrinsèque de Cohn-Vossen (voir Théorème 5). Par conséquent, $L_{t_{\infty}} \cap N_{1}$ a une infinité de composantes intersectant $N_{0}$. Comme dans le cas ci-dessus, chaque composante a une aide donnée. L'inégalité extrinsèque de Cohn-Vesson donne une contradiction.

Expliquons comment déduire une contradiction avec la propriété d'annulation. Nous montrons que pour tout $k \geqslant k_{0}, \Omega_{k} \cap \partial N_{k_{0}}$ contient une courbe fermée qui n'est pas homotopiquement triviale dans $\partial N_{k_{0}}$ (voir le Lemme 1.3.11). Grosso modo, ces cercles non homotopiquement triviaux vont sous-converger vers une courbe fermée dans $\mathscr{L} \cap \partial N_{k_{0}}$ qui n'est pas contractile dans $\partial N_{k_{0}}$. Il s'ensuit que $\Omega_{k}$ sous-converge vers la lamination $\mathscr{L}$. Par conséquent, certaines feuilles de $\mathscr{L}$ ont un cercle non contractile dans $\partial N_{k_{0}}$. Ceci est en contradiction avec la propriété d'annulation ci-dessus de $\mathscr{L}$.

## 0.8. $\pi_{1}^{\infty}$ et la propriété d'annulation

0.8.1. Corps à anses et propriété $\mathbf{H}$. Soit $(M, g)$ une 3 -variété complète contractile à courbure scalaire positive. Il s'agit d'une union croissante de corps à anses $\left\{N_{k}\right\}$ (voir Théorème 1.1.12).

Dans ce qui suit, nous considérons que $M$ n'est pas homéomorphe à $\mathbb{R}^{3}$. Nous pouvons supposer qu'aucun des $N_{k}$ n'est contenu dans une 3-boule (c'est-à-dire homéomorphe à une boule unitaire dans $\mathbb{R}^{3}$ ) dans $M$ (voir Remarque 1.1.8). Cela joue un rôle crucial dans notre argument.

Dans le cas du genre un, la famille $\left\{N_{k}\right\}$ a plusieurs bonnes propriétés. Par exemple, les applications $\pi_{1}\left(\partial N_{k}\right) \rightarrow i_{1}\left(\overline{M \backslash N_{k}}\right)$ et $\pi_{1}\left(\partial N_{k}\right) \rightarrow \pi_{1}\left(\overline{N_{k} \backslash N_{0}}\right)$ sont toutes les deux injectives (voir le Lemme 1.3.10). Ces propriétés sont cruciales et nécessaires dans l'étude de l'existence de métriques complètes à courbure scalaire positive. En général, la famille $\left\{N_{k}\right\}$ peut ne pas avoir les propriétés ci-dessus.

Par exemple, l'application $\pi_{1}\left(\partial N_{0}\right) \rightarrow \pi_{1}\left(\overline{M \backslash N_{0}}\right)$ peut ne pas être injective. Pour surmonter cette difficulté, nous utilisons des chirurgies topologiques sur $N_{0}$ et nous trouvons un nouveau corps à anses pour le remplacer. Précisément, nous utilisons le lemme de la boucle pour trouver un disque plongé $(D, \partial D) \subset\left(\overline{M \backslash N_{0}}, \partial N_{0}\right)$ dont le bord est une courbe simple non contractile dans $\partial N_{0}$. Le nouveau corps à anses est obtenu à partir de $N_{0}$ en attachant un voisinage tubulaire fermé $N_{\epsilon}(D)$ de $D$ dans $\overline{M \backslash N_{0}}$.

Nous utilisons à plusieurs reprises des chirurgies topologiques sur chaque $N_{k}$ pour obtenir une nouvelle famille $\left\{R_{k}\right\}_{k}$ de corps à anses fermés avec les propriétés suivantes, appelée Propriété $H$ (voir Définition 2.2.5):
(1) l'application $\pi_{1}\left(\partial R_{k}\right) \rightarrow \pi_{1}\left(\overline{R_{k} \backslash R_{0}}\right)$ est injective pour $k>0$;
(2) l'application $\pi_{1}\left(\partial R_{k}\right) \rightarrow \pi_{1}\left(\overline{M \backslash R_{k}}\right)$ est injective pour $k \geqslant 0$;
(3) chaque $R_{k}$ est contractile dans $R_{k+1}$ mais n'est pas contenu dans une 3-boule de $M$;
(4) il existe une suite d'entiers croissante $\left\{j_{k}\right\}_{k}$, telle que $\pi_{1}\left(\partial R_{k} \cap \partial N_{j_{k}}\right) \rightarrow \pi_{1}\left(\partial R_{k}\right)$ soit surjective.
Remarque Si $M$ n'est pas homéomorphe à $\mathbb{R}^{3}$, l'existence d'une telle famille est assurée par le Théorème 2.2.6. Elle n'est pas unique. De plus, l'union d'une telle famille peut ne pas correspondre à $M$.

Par exemple, si $M:=\bigcup_{k} N_{k}$ est une 3-variété contractile de genre un, la famille $\left\{N_{k}\right\}$ (supposée comme dans le Théorème 1.3.13) vérifie la propriété ci-dessus (propriété $H$ ) (voir le lemme 2.10 dans Wan19a ou Lemme 1.3.10).
0.8.2. La proriété d'annulation. Dans le cas du genre un, la géométrie d'une surface minimale stable est contrainte par l'indice géométrique (voir Propriété $P$ dans Wan19a ou la Définition 2.1.3). Dans le cas du genre supérieur, le comportement d'une surface minimale stable est lié au groupe fondamental à l'infini.

Afin de clarifier leur relation, introduisons une propriété géométrique, appelée la propriété d'annulation. Premièrement, nous considérons une 3 -variété complète contractile $(M, g)$ qui n'est pas homéomorphe à $\mathbb{R}^{3}$. Comme indiqué ci-dessus, il existe une famille croissante $\left\{R_{k}\right\}_{k}$ de corps à anses avec la propriété $H$.

On dit qu'une surface minimale stable complète plongée $\Sigma \subset(M, g)$ satisfait la propriété d'annulation pour la famille $\left\{R_{k}\right\}_{k}$ s'il existe un entier positif $k(\Sigma)$ tel que pour $k \geqslant k(\Sigma)$, tout cercle dans $\Sigma \cap \partial R_{k}$ soit homotopiquement trivial dans $\partial R_{k}$ (Voir Chapitre 5).

Si une surface minimale stable complète ne satisfait pas la propriété d'annulation pour $\left\{R_{k}\right\}_{k}$, elle donne un élément non trivial dans $\pi_{1}^{\infty}(M)$ (voir le Lemme 5.2.1). En conséquence, si $\pi_{1}^{\infty}$ est trivial, toute surface minimale stable complète dans $M$ a la propriété d'annulation pour $\left\{R_{k}\right\}_{k}$ (voir le Corollaire 5.2.2).
0.8.3. L'idée de la preuve du théorème 4. Raisonnons par l'absurde. Supposons qu'une 3 -variété $(M, g)$ complète contractile à courbure scalaire positive et $\pi_{1}^{\infty}(M)$ trivial, ne soit pas homéomorphe à $\mathbb{R}^{3}$.

Avant de construire des surfaces minimales, introduisons une notation de topologie en dimension 3.

Un système de méridiens d'un corps à anses $N$ est une collection de $g$ méridiens distincts $\left\{\gamma^{l}\right\}_{l=1}^{g}$ tels que $\partial N \backslash \amalg_{l=1}^{g} \gamma^{l}$ soit hoéomorphe à un disque ouvert privé de certains sous-disques fermés (voir le Lemme 1.2.7). Son existence est assurée par le Lemme 1.2.7.

Soit $\left\{N_{k}\right\}_{k}$ et $\left\{R_{k}\right\}_{k}$ comme ci-dessus. Puisque $N_{0}$ n'est pas contenu dans une 3boule (voir la remarque 1.1.8), le genre de $N_{k}$ est supérieur à zéro. Le corps à anses $N_{k}$ a un système de méridiens $\left\{\gamma_{k}^{l}\right\}_{l=1}^{g\left(N_{k}\right)}$. En gros, il existe $g\left(N_{k}\right)$ disques disjoints d'aire minimale $\left\{\Omega_{k}^{l}\right\}_{l}$ avec $\partial \Omega_{k}^{l}=\gamma_{k}^{l}$. Leur existence est assurée par les travaux de Meeks et Yau MY80 MY82] (voir le théorème 6.28 de $\mathbf{C M 1 1 ]}$ ) lorsque la frontière $\partial N_{k}$ est mean convex.

Expliquons leur existence. Nous construisons ces disques par récurrence sur $l$.
Lorsque $l=1$, il existe un disque plongé d'aire minimale $\Omega_{k}^{1} \subset N_{k}$ avec $\partial \Omega_{k}^{1}=\gamma_{k}^{1}$ (voir MY80, MY82 ou le théorème 6.28 de [CM11]).

Supposons qu'il existe $l$ disques stables plongés disjoints d'aire minimale $\left\{\Omega_{k}^{i}\right\}_{i=1}^{l}$ avec $\partial \Omega_{k}^{i}=\gamma_{k}^{i}$. Notre objectif est de construire une surface minimale stable $\Omega_{k}^{l+1}$ avec $\partial \Omega_{k}^{l+1}=$ $\gamma_{k}^{l+1}$.

Considérons la variété riemannienne $\left(T_{k, l},\left.g\right|_{T_{k}, l}\right)$, où $T_{k, l}:=N_{k} \backslash \coprod_{i=1}^{l} \Omega_{k}^{l}$. C'est un corps à anses du genre $g\left(N_{k}\right)-l$. Par exemple, voir la figure suivante.


Figure 0.6.

La frontière de $\left(T_{k, l},\left.g\right|_{T_{k, l}}\right)$ est constituée de $\partial N_{k} \backslash \coprod_{i=1}^{l} \gamma_{k}^{i}$ et certains disques disjoints $\left\{\Omega_{k}^{i-}\right\}_{i=1}^{l}$ et $\left\{\Omega_{k}^{i+}\right\}_{i=1}^{l}$. Les deux disques $\Omega_{k}^{i-}$ et $\Omega_{k}^{i+}$ proviennent tous du même disque minimal $\Omega_{k}^{i}$. Par conséquent, la courbure moyenne de la frontière de $\left(T_{k, l},\left.g\right|_{T_{k, l}}\right)$ est positive. (Voir Chapitre 4.2)

De plus, $\left\{\gamma_{k}^{i}\right\}_{i>l}$ est un système de méridiens du corps à anses $\left(T_{k, l},\left.g\right|_{T_{k, l}}\right)$. Ensuite, nous utilisons le résultat de Meeks et Yau pour trouver une surface minimale stable plongée $\Omega_{k}^{l+1} \subset T_{k, l}$ dont le bord est $\gamma_{k}^{l+1}$. Ces disques $\left\{\Omega_{k}^{i}\right\}_{i=1}^{l+1}$ sont disjoints dans $N_{k}$. Ceci termine la construction par récurrence.

Comme dans le cas du genre un, si $\partial N_{k}$ n'est pas mean convex, nous pouvons déformer la métrique dans un petit voisinage de celle-ci pour qu'elle devienne mean convex.

Définissons la lamination $\mathscr{L}_{k}:=\coprod_{l} \Omega_{k}^{l}$ (qui est une union disjointe de surfaces plongées). Nous montrons que chaque lamination $\mathscr{L}_{k}$ intersecte l'ensemble compact $R_{0}$ (Corollaire 1.2.8). Selon la théorie de Colding-Mincozzi (voir l'annexe B de [CM04]), la suite $\left\{\mathscr{L}_{k}\right\}_{k}$ sous-converge vers une laminiation $\mathscr{L}:=\bigcup_{t \in \Lambda} L_{t}$ in $(M, g)$ (voir le Théorème 4.2.3). Notons que chaque feuille $L_{t}$ est une surface minimale stable complète (non compacte).

Comme indiqué ci-dessus, puisque $(M, g)$ est a 3 -variété complète à courbure scalaire positive et que $\pi_{1}^{\infty}(M)$ est trivial, chaque feuille $L_{t}$ dans $\mathscr{L}$ a la propriété d'annulation pour $\left\{R_{k}\right\}_{k}$ (voir le Lemme 5.2 .1 et le Corollaire 5.2.2). En outre, la lamination $\mathscr{L}$ satisfait la propriété d'annulation (voir le Corollaire 5.2.4). C'est-à-dire,
il existe un entier positif $k_{0}$ tel que pour tout $k \geqslant k_{0}$ et tout $t \in \Lambda$, n'importe quel cercle dans $L_{t} \cap \partial R_{k}$ soit homotopiquement trivial dans $\partial R_{k}$.

La raison est décrite comme suit.

Raisonnons par l'absurde. Supposons qu'il existe une suite croissante $\left\{k_{n}\right\}_{n}$ d'entiers et une suite $\left\{L_{t_{n}}\right\}$ de feuilles dans $\mathscr{L}$ telles que $L_{t_{n}} \cap \partial R_{k_{n}}$ ait au moins un cercle non homotopiquement trivial dans $\partial R_{k_{n}}$ pour chaque $n$.

La suite $\left\{L_{t_{n}}\right\}$ sous-converge vers une feuille de $\mathscr{L}$. Pour simplifier, nous pouvons supposer que la suite $\left\{L_{t_{n}}\right\}$ converge vers la feuille $L_{t_{\infty}}$. La feuille $L_{t_{\text {infty }}}$ satisfait la propriété d'annulation. C'est-à-dire qu'il existe un entier positif $k\left(L_{t_{\infty}}\right)$ tel que pour $k \geqslant k\left(L_{t_{\infty}}\right)$, tout cercle $\partial R_{k} \cap L_{t_{\infty}}$ soit homotopiquement trivial dans $\partial R_{k}$.

Cependant, comme $L_{t_{n}} \cap \partial R_{k_{n}}$ a un cercle non homotopiquement trivial dans partial $R_{k_{n}}$, nous savons que pour $k_{n}>k\left(L_{t_{\infty}}\right), L_{t_{n}} \cap \partial R_{k\left(L_{\left.t_{\infty}\right)}\right)}$ a un méridien de $R_{k\left(L_{\left.t_{\infty}\right)}\right)}$ (voir la remarque 2.2 .7 et le corollaire 1.2.6). Ces méridiens de $R_{k\left(L_{\left.t_{\infty}\right)}\right)}$ convergeront vers un méridien de $R_{k\left(L_{\left.t_{\infty}\right)}\right)}$ contenu dans $L_{t_{\infty}} \cap \partial R_{k\left(L_{t_{\infty}}\right)}$. Ceci est en contradiction avec le dernier paragraphe.

Expliquons comment déduire une contradiction de la propriété d'annulation de $\mathscr{L}$.
Nous montrons que si $N_{k}$ contient $R_{k_{0}}$ (pour $k$ assez grand), alors $\mathscr{L}_{k} \cap \partial R_{k_{0}}$ contient au moins un méridien de $R_{k_{0}}$ (voir le Corollaire 1.2.8). Comme dans le cas ci-dessus, ces méridiens de $R_{k_{0}}$ convergeront vers un cercle non contractile dans $\mathscr{L} \cap \partial R_{k_{0}}$. La propriété d'annulation de $\mathscr{L}$ donne une contradiction.

### 0.9. 4-variétés fermées asphériques

Une variété $M$ est dite asphérique s'il est connexe par arcs et si tous ses groupes d'homotopie suérieurs s'annulent (c'est-à-dire que $\pi_{k}(M)$ est trivial pour $k \geqslant 2$ ). La classe des variétés asphériques comprend toutes les variétés hyperboliques et toutes les variétés à courbure négative.

Une question intéressante est de savoir si le tore $\mathbb{T}^{n}, n \geqslant 3$, possède une métrique à courbure scalaire positive. Cette question a été traitée par Gromov-Lawson [GL83] et Schoen-Yau SY79b,SY82,SY17. En général, on conjecture que

Conjecture. Aucune variété compacte asphérique n'a de métrique à courbure scalaire positive.

Cette conjecture a été démontrée pour les 3 -variétés par Gromov et Lawson $\mathbf{G L 8 3}$. En dimension quatre, elle est confirmée pour les 4 -variété qui contiennent des surfaces incompressibles GL83.

Dans cette thèse, nous prouvons que
Théroème 6 (=Theorem $\sqrt[F]{ }$ ) Aucune 4 -variété asphérique fermée avec premier nombre de Betti non nul n'a de métrique à courbure scalaire positive.

Notons qu'il existe une 4 -variété asphérique fermée dont le premier nombre de Betti est nul (Voir RT05).

Raisonnons par l'absurde. Supposons qu'il existe une 4-variété asphérique fermée $\left(M^{4}, g\right)$ à courbure scalaire positive, dont le premier nombre de Betti $b_{1}\left(M^{4}\right)$ est strictement supérieur à zéro.

Choisissons un cercle $\gamma \subset M^{4}$ tel que $[\gamma]$ soit d'ordre infini dans $H_{1}\left(M^{4}, \mathbb{Z}\right)$. Nous utilisons la dualité de Poincaré pour trouver une classe $u \in H_{3}\left(M^{4}, \mathbb{Z}\right)$ avec $<u,[g]>=1$. Un théorème de Fleming-Federer (voir $[\mathbf{F F 6 0}]$ ou le chapitre 7 de $[$ Sim83] $)$ nous indique
qu'il existe une hypersurface $\Sigma^{3}$ minimisant le volume dans cette classe. Par conséquent, le nombre d'intersection de $\gamma$ et $\Sigma^{3}$ est égal à un.

Puisque $(M, g)$ a une courbure scalaire positive, alors $\Sigma^{3}$ admet une métrique de courbure scalaire positive (voir la proposition 3.3.5). La variété $\Sigma^{3}$ est homéomorphe à une somme connexe de 3 -variété sphériques et de copies de $\mathbb{S}^{1} \times \mathbb{S}^{2}$ (Voir Per02a Per02b, Per03], $\overline{\mathrm{BBB}^{+} \mathbf{1 0}}$ et (MT07]).

Comme $\pi_{2}(M)$ et $\pi_{3}(M)$ sont triviaux, la partie sphérique de $\Sigma$ et les 2 -sphères de $\Sigma$ sont homotopes à un point dans $M^{4}$. C'est-à-dire que $\Sigma$ est homotope à un bouquet de cercles dans $M$ (ces cercles proviennent de la partie de $\mathbb{S}^{2} \times \mathbb{S}^{1}$ dans $\Sigma^{3}$ ). Comme $\operatorname{dim}(M)=4$, nous voyons que le nombre d'intersection de $\Sigma$ et $\gamma$ est égal à zéro, ce qui est en contradiction avec les nombres d'intersection de $\gamma$ et $\Sigma$.

### 0.10. Organisation de la thèse

Le plan de cette thèse est le suivant:
Dans la première partie, nous discutons des 3 -variétés contractile et des propriétés topologiques associées. Au Chapitre 1, nous formulons des rappels liés aux 3-variétés, comme la connexité simple à l'infini, le groupe fondamental à l'infini et les corps à anses. Par la suite, nous discutons de la structure topologique des 3 -variétés contractiles et présentons quelques notions telles que les méridiens d'un corps à anses, ou encore les systèmes de méridiens. Enfin, nous donnons quelques exemples de 3 -variétés, telles que les variétés de Whitehead et les 3 -variétés de genre un.

Au chapitre 2, nous commençons par les disques plongés dans la variété de Whitehead. Un fait intéressant est que le comportement de ces disques est influencé par l'indice géométrique. Leur relation est suggérée par le Théoréme 2.1.2. Sur la base de cette relation, nous introduisons une nouvelle propriété, appelée Propriété $P$, et montrons que toute variété contractile de genre un satisfait cette propriété (voir Théorème 2.1.6).

En règle générale, une 3 -variété contractible peut ne pas satisfaire la propriété P . La raison est que cette variété peut être composée de corps à anses de genre supérieur. Certaines anses peuvent ne pas contribuer à la topologie et engendrer des difficultés techniques. Pour surmonter cela, nous introduisons deux types de chirurgies. En utilisant ces chirurgies, nous trouvons une nouvelle famille de corps à anses avec de bonnes propriétés, appelée Propriété $H$ (voir Définition 2.2.5).

Dans la deuxième partie de la thèse, nous nous concentrons sur les surfaces minimales et les théories de convergence associées. Au Chapitre 3, nous rappelons certaines notions telles que les formules dites de première et deuxième variation, l'indice de Morse et la condition de stabilité (Voir Chapitre 3.1.1). Ensuite, nous discutons du problème de Plateau (voir Chapitre 3.1.2).

Nous nous intéresserons ensuite aux propriétés locales des surfaces minimales, y compris le principe du maximum et la formule de monotonie (voir Proposition 3.2.5).

Ensuite, nous étudions la topologie des hypersurfaces minimales stables. Ces hypersurfaces sont caractérisées par la première valeur propre de l'opérateur stable (voir le Lemme 3.3.1 et le Théorème 3.3.4). Dans une variété à courbure scalaire positive, il existe de nombreuses contraintes topologiques sur les surfaces minimales stables. Par
exemple, si une 3 -variété compléte a courbure scalaire positive ou nulle, une surface minimale stable complète vérifie l'inégalité extrinsèque de Cohn-Vossen (Voir Corollary 3.3.6 et Theorem 3.3.10). En conséquence, nous donnons une nouvelle preuve de la classification topologique des surfaces minimales stables dans une 3 -variété à courbure scalaire positive ou nulle (voir le Corollaire 3.3.11 et [SY82]). Enfin, en tant qu'application de la théorie des surfaces minimales, nous donnons la preuve du Théorème 4.2.1 (voir Théorème 3.3.12).

Au Chapitre 4, nous discutons de la théorie de la convergence des surfaces minimales. Nous commençons par la convergence des équations de surfaces minimales (Voir Lemme 4.1.1). Cela peut être généralisé au cas riemannien. Par conséquent, nous obtenons un théorème de compacité pour les surfaces minimales (voir Théorème 4.1.4).

Ensuite, nous discutons de la convergence sans estimation d'aire. Dans ce cas, la limite est une lamination minimale (c'est-à-dire une union disjointe de disques plongés). Nous rappelons la théorie de la lamination minimale de Colding-Minicozzi (Voir l'Annexe B de [CM04]). Ensuite, nous construisons une famille requise de laminations minimales dans une 3-variété contractile. Leur limite est une lamination minimale stable. Chaque feuille est une surface minimale compléte. Si la variété a une courbure scalaire positive, il s'agit d'un plan proprement plongé (voir le Théorème 5). En guise d'application, nous donnons une nouvelle preuve de la classification topologique des 3 -variétés contractiles à courbure scalaire uniformément positive (Voir Corollaire 4.2.7 et GL83]).

Dans la troisième partie, nous donnons les preuves complètes des théorèmes principaux. Au Chapitre 5, nous introduisons la propriété d'annulation et étudions sa relation avec le groupe fondamental à l'infini. La relation est suggérée par le Lemme 5.2.1. Notez que, dans le cas du genre un, la propriété P implique la propriété d'annulation (voir Théorème 5.1.1.

Au Chapitre 6, nous réduisons la preuve des théorèmes principaux à un lemme de recouvrement (voir Lemme 6.1.3). Pour prouver ce lemme, nous utilisons la propriété d'annulation de la lamination (construit au Chapitre 4.2) pour définir un ensemble $S$ (voir Définition 6.3.1). La positivité de la courbure scalaire implique la finitude de $S$ (voir Lemme 6.3.4 et Lemme 6.3.5). Nous utilisons la finitude pour prouver le lemme de recouvrement. Enfin, nous discutons de plusieurs questions connexes et d'autres sujets de recherche sur les 3 -variétés.

Dans la quatrième partie de la thèse, nous discutons de l'existence de métriques à courbure scalaire positive sur une 4 -variété asphérique compacte. Avec un résultat de Perelman Per02a, Per02b, Per03, nous donnons une preuve du Théorème 6. Enfin, nous abordons plusieurs autres questions sur les 4 -variétés.

## Part 1

3-Manifolds

## CHAPTER 1

## 3-Manifolds

In this chapter, we review related background in geometric topology and algebraic topology.

We begin with several classical theorems in 3-manifolds, such as the loop lemma. Subsequently, we study the topological structures of contractible 3-manifolds. Then we introduce several notations (for example, effective meridians and a system of meridians) and discuss their topological properties.

Finally, we give several examples, such as genus one 3-manifolds.

### 1.1. Background

In this part, we recall several classical theorems for 3-manifolds and discuss the topological structure of contractible 3-manifolds.
1.1.1. Preliminary. A 3-manifold is irreducible if any embedded 2 -sphere bounds a closed 3-ball (namely, it is homeomorphic to a closed unit ball in $\mathbb{R}^{3}$ ).

Remark 1.1.1. We know from the so-called Alexander's theorem (See [Theorem 1.1, Page 1] of $[\mathbf{H a t 0 0}]$ ) that any embedded 2 -sphere in $\mathbb{R}^{3}$ bounds an embedded 3-ball.

Further, the proof of Poincaré Conjecture $\overline{\mathrm{Per02a}, \mathrm{Per02b}, \mathrm{Per03}]}$ (See $\mathrm{BBB}^{+} \mathbf{1 0}$ or (MT07) tells that any contractible 3-manifold is irreducible.

It is well-known that there are many links between the geometric properties of 3manifolds and homotopy theory, specially $\pi_{1}$. For example, the loop lemma.

Lemma 1.1.2. (See [Theorem 3.1, Page 54] of [Hat00])Let $M$ be a 3-manifold with boundary $\partial M$, not necessarily compact or orientable. If there is a map $f:\left(D^{2}, \partial D^{2}\right) \rightarrow$ $(M, \partial M)$ with the property that $\left.f\right|_{\partial D^{2}}$ is not nullhomotopic in $\partial M$. Then there is an embedding $h$ with the same property.

Remark 1.1.3. We may assume that $h\left(\operatorname{Int} D^{2}\right) \subset \operatorname{Int} M$. The reason is described below:

Let us consider a 1 -sided open neighborhood $M_{\epsilon} \cong \partial M \times[0, \epsilon)$ of $\partial M$ in $M$. Shrinking the image of $f$ into $M(\epsilon):=M \backslash M_{\epsilon}$, we find a map $f_{\epsilon}:\left(D^{2}, \partial D^{2}\right) \rightarrow(M(\epsilon), \partial M(\epsilon))$ with the property that $f_{\epsilon}\left(\partial D^{2}\right)$ is not nullhomotopic in $\partial M(\epsilon)$. By Lemma 1.1.2 there is an embedding $h_{\epsilon}$ with the same property. Its image is contained in $(M(\epsilon), \partial M(\epsilon))$. Therefore, the image of $h_{\epsilon}$ is contained in Int $M$.

In addition, there is an embedded circle $\gamma \subset \partial M$ which is homotopic to $h_{\epsilon}\left(\partial D^{2}\right)$ in $\bar{M}_{\epsilon}$. There is an embedded annulus $A_{\epsilon} \subset \bar{M}_{\epsilon}$ joining $\gamma$ and $h_{\epsilon}\left(\partial D^{2}\right)$. We have a map $h:\left(D^{2}, \partial D^{2}\right) \rightarrow(M, \partial M)$ so that its image is an embedded disc (i.e. the union of $A_{\epsilon}$ and the image of $h_{\epsilon}$ ). It has the same property as $f$ and $h\left(\operatorname{Int} D^{2}\right) \subset \operatorname{Int} M$.

One of tools for computing the fundamental group is the so-called Van-Kampen's theorem. It will be frequently used in the following.

Theorem 1.1.4. (Van-Kampen's Theorem, See [Theorem11.60, Page 396] of Rot12]) Let $K$ be a connected complex having connected subcomplexes $L_{1}$ and $L_{2}$ with $K=L_{1} \cup L_{2}$. If $L_{1} \cap L_{2}$ is connected, then $\pi_{1}(K)$ is the pushout of the data.

where $j_{k}: L_{1} \cap L_{2} \rightarrow L_{k}$ is the inclusion for $k=1,2$.
Moreover, $\pi_{1}(K)$ is isomorphic to $\pi_{1}\left(L_{1}\right) *_{\pi_{1}\left(L_{1} \cap L_{2}\right)} \pi_{1}\left(L_{2}\right)$.
If the map $j_{k_{*}}: \pi_{1}\left(L_{1} \cap L_{2}\right) \rightarrow \pi_{1}\left(L_{k}\right)$ is injective for $k=1,2, \pi_{1}(K)$ can be written as a free product with amalgamation (See Chapter 11 of [Rot12]).

Lemma 1.1.5. (See [Theorem11.67, Page 404] of Rot12]) Let B, $A_{1}$ and $A_{2}$ be groups. . Let $A_{1} *_{B} A_{2}$ be the pushout of the following data.


If the map $i_{k}: B \rightarrow A_{k}$ is injective for $k=1,2$, one has
(1) the map $\lambda_{k}$ is injective for $k=1,2$;
(2) if $A_{k}^{\prime}=\lambda_{k}\left(A_{k}\right)$, then $<A_{1}^{\prime}, A_{2}^{\prime}>=A_{1} *_{B} A_{2}$ and $A_{1}^{\prime} \cap A_{2}^{\prime}$ is isomorphic to $B$.

We now introduce several concepts about the disjoint closed curves in a disc.
Definition 1.1.6. (See Definition 2.11 of $|\overline{W a n 19 a}|$ ) Let $C:=\left\{c_{i}\right\}_{i \in I}$ be a finite set of pairwise disjoint circles in the disc $D^{2}$ and $D_{i} \subset D^{2}$ the unique disc with boundary $c_{i}$. Consider the set $\left\{D_{i}\right\}_{i \in I}$ and define the partially ordered relation induced by the inclusion. For each maximal element $D_{j}$ in $\left(\left\{D_{i}\right\}_{i \in I}, \subset\right)$, its boundary $c_{j}$ is defined as a maximal circle in $C$. For each minimal element $D_{j}$, its boundary $c_{j}$ is called a minimal circle in $C$.

### 1.1.2. Simply-connectedness at infinity and $\pi_{1}^{\infty}$.

Definition 1.1.7. A topological space $M$ is simply connected at infinity if for any compact set $K \subset M$, there exists a compact set $K^{\prime}$ containing $K$ so that the induced map $\pi_{1}\left(M \backslash K^{\prime}\right) \rightarrow \pi_{1}(M \backslash K)$ is trivial.

A result of Stallings Sta72] and Remark 1.1.1 tell us that the only contractible and simple-connected at infinity 3-manifold is $\mathbb{R}^{3}$.

REmARK 1.1.8. If a contractible 3-manifold $M$ is not homeomorphic to $\mathbb{R}^{3}$, it is not simply-connected at infinity. That is to say, there is a compact set $K \subset M$ so that for any compact set $K^{\prime} \subset M$ containing $K$, the induced map $\pi_{1}\left(M \backslash K^{\prime}\right) \rightarrow \pi_{1}(M \backslash K)$ is not
trivial. We also have that the set $K$ is not contained in a 3 -ball in $M$. The reason is described below:

If a closed 3 -ball $B$ contains $K$, Theorem 1.1 .4 shows that $\pi_{1}(M) \cong \pi_{1}(\overline{M \backslash B}) *_{\pi_{1}(\partial B)}$ $\pi_{1}(B)$. In addition, $\pi_{1}(B)$ and $\pi_{1}(\partial B)$ are both trivial. Therefore, $\pi_{1}(\overline{M \backslash B}) \cong \pi_{1}(M)$ is trivial. That is to say, the map $\pi_{1}(M \backslash B) \rightarrow \pi_{1}(M \backslash K)$ is trivial. This is a contradiction.

Definition 1.1.9. The fundamental group at infinity $\pi_{1}^{\infty}$ of a path-connected space is the inverse limit of the fundamental groups of complements of compact subsets.

For example, the fundamental group at infinity of any compact manifold is trivial. For any contractible $n$-manifold $M^{n}$, it is simply-connected at infinity if and only if $\pi_{1}^{\infty}\left(M^{n}\right)$ is trivial, when $n \geqslant 4$ (See $[\mathbf{C W Y 1 0}]$ ). However, this result is not true in dimension 3. For example, the Whitehead manifold is not simply-connected at infinity but its fundamental group at infinity is trivial.

Remark 1.1.10. Let us consider a contractible 3 -manifold $M$ and an exhaustion $\left\{N_{k}\right\}_{k}$ of $M$. We have the following:
$\pi_{1}^{\infty}(M)=\underset{k}{\lim } \pi_{1}\left(M \backslash N_{k}\right)=\left\{\left(\left[\gamma_{k}\right]\right) \in \prod_{k} \pi_{1}\left(M \backslash N_{k}\right)\right.$ so that $\left(f_{k, j}\right)_{*}\left(\left[\gamma_{k}\right]\right)=\left[\gamma_{j}\right]$ for any $\left.k \geqslant j\right\}$, where $f_{k, j}$ is the inclusion from $M \backslash N_{k}$ to $M \backslash N_{j}$.

Therefore, $\pi_{1}^{\infty}(M)$ is non-trivial if and only if there exists a non-trivial element $\left(\left[\gamma_{k}\right]\right) \in$ $\prod_{k} \pi_{1}\left(M \backslash N_{k}\right)$ satisfying

1) for some $k_{0}$, the closed curve $\gamma_{k_{0}}$ is non-contractible in $M \backslash N_{k_{0}}$;
2) for $k \geqslant k_{0}, \gamma_{k}$ is homotopic to $\gamma_{k}$ in $M \backslash N_{k_{0}}$.

That is to say, there is a compact set $K$ and a family of closed curves $\left\{\alpha_{n}\right\}_{n}$ going to infinity with the following property: for each $n$
a) $\alpha_{n}$ is nullhomotopic in $M \backslash K$ for ;
b) $\alpha_{n}$ is homotopic to $\alpha_{n+1}$ in $M \backslash K$.

Note that this family of circles gives a non-trivial element in $\pi_{1}^{\infty}(M)$.

### 1.1.3. Handlebodies.

Definition 1.1.11. [Page 59, Rol03] A handlebody is any space obtained from the 3 -ball $D^{3}$ ( 0 -handle) by attaching $g$ distinct copies of $D^{2} \times[-1,1]$ (1-handle) with the homeomorphisms identifying the $2 g$ discs $D^{2} \times\{ \pm 1\}$ to $2 g$ disjoint 2-disks on $\partial D^{3}$, all to be done in such a way that the resulting 3 -manifold is orientable. The integer $g$ is called the genus of the handlebody.

Remark that a handlebody of genus $g$ is homeomorphic to a boundary connected sum of $g$ solid tori. Therefore, its boundary is a compact surface of genus $g$. (See Page 59 in (Rol03)

From a result of McMillan McM61 and Remark 1.1.1, we know that:
Theorem 1.1.12. [Page 511, Theorem 1] [McM61] Any contractible 3-manifold can be written as an ascending union of handlebodies .

Remark 1.1.13. Let consider a contractible 3-manifold $M$. If it is not homeomorphic to $\mathbb{R}^{3}$, it can written as an increasing family of handlebodies $\left\{N_{k}\right\}$ satisfying that for each k,

- $N_{k}$ is homotopically trivial in $N_{k+1}$;
- none of the $N_{k}$ is contained in a 3 -ball (See Remark 1.1.8).

In the following, we consider a closed handlebody $N$ in $\mathbb{S}^{3}$.
Definition 1.1.14. A closed handlebody $N \subset \mathbb{S}^{3}$ of genus $g$ is said to be unknotted in $\mathbb{S}^{3}$ if it complement in $\mathbb{S}^{3}$ is also a handlebody of genus $g$.

For example, an unknotted solid torus in $\mathbb{S}^{3}$.

### 1.2. Meridians

In the following, we consider a closed handlebody $N$.
Definition 1.2.1. An embedded circle $\gamma \subset \partial N$ is called a meridian if $\gamma$ is nullhomotpic in $N$, but not contractible in $\partial N$.

An embedded closed disc $(D, \partial D) \subset(N, \partial N)$ is called a meridian disc if its boundary is a meridian of $N$.

The disc $D$ is a splitting meridian disc, if $N \backslash D$ is not connected. Its boundary is called a splitting meridian.

The disc $D$ is a non-splitting disc, if $N \backslash D$ is connected. Its boundary is called a non-splitting meridian.

Remark. Let $\gamma$ be a meridian of $N$. If $\gamma$ is a splitting meridian, it cuts $\partial N$ into two components. The class $[\gamma]$ is equal to zero in $H_{1}(\partial N)$.

If $\gamma$ is a non-splitting meridian, then $\partial N \backslash \gamma$ is connected. The class $[\gamma]$ is a non-trivial element in $H_{1}(\partial N)$.

Lemma 1.2.2. Let $N^{\prime}$ and $N$ be two closed handlebodies with $N^{\prime} \subset$ Int $N$. If $N^{\prime}$ is homotopically trivial in $N$, then any non-splitting meridian of $N^{\prime}$ is non-trivial in $H_{1}\left(\overline{N \backslash N^{\prime}}\right)$ and any meridian of $N$ is trivial in $H_{1}\left(\overline{N \backslash N^{\prime}}\right)$.

Proof. The Mayer-Vietoris sequence gives:

$$
\begin{equation*}
H_{2}(N) \rightarrow H_{1}\left(\partial N^{\prime}\right) \xrightarrow{i_{1}} H_{1}\left(N^{\prime}\right) \oplus H_{1}\left(\overline{N \backslash N^{\prime}}\right) \xrightarrow{i_{2}} H_{1}(N) \rightarrow \hat{H}_{0}\left(\partial N^{\prime}\right) . \tag{1.2.1}
\end{equation*}
$$

We know that $H_{2}(N)$ and $\hat{H}_{0}\left(\partial N^{\prime}\right)$ are both trivial. Then, the map $H_{1}\left(\partial N^{\prime}\right) \rightarrow$ $H_{1}\left(N^{\prime}\right) \oplus H_{1}\left(\overline{N \backslash N^{\prime}}\right)$ is injective. As indicated above, any non-splitting meridian of $N^{\prime}$ is non-trivial in $H_{1}\left(\partial N^{\prime}\right)$. Therefore, it is not equal to zero in $H_{1}\left(\overline{N \backslash N^{\prime}}\right)$.

In the following, we will show that any meridian $\gamma$ of $N$ is trivial in $H_{1}\left(\overline{N \backslash N^{\prime}}\right)$.
Embed $N$ into $\mathbb{S}^{3}$ as an unknotted handlebody. The set $N^{\prime}$ can be viewed as a handlebody in $\mathbb{S}^{3}$. The core $K^{\prime}=\vee_{i=1}^{g^{\prime}} \alpha_{i}$ of $N^{\prime}$ is a wedge sum of $g^{\prime}$ circles $\left\{\alpha_{i}\right\}_{i}$, where $g^{\prime}$ is the genus of $N^{\prime}$.

Choose $\gamma_{i}^{\prime} \subset \partial N^{\prime}$ as a normal circle of $\alpha_{i}$ in $\mathbb{S}^{3}$. That is to say, the linking number of $\gamma_{i}^{\prime} \amalg \alpha_{j}$ in $\mathbb{S}^{3}$ is equal to $\delta_{i j}$. In addition, the kernel of the map $H_{1}\left(\partial N^{\prime}\right) \rightarrow H_{1}\left(N^{\prime}\right)$ is a linear subspace of dimension $g^{\prime}$, spanned by $\left\{\left[\gamma_{i}^{\prime}\right]\right\}_{i=1}^{g^{\prime}}$.

Since $N^{\prime}$ is homotopically trivial in $N$, the map $H_{1}\left(N^{\prime}\right) \rightarrow H_{1}(N)$ is a zero map. Choose the element $(0,[\gamma]) \in H_{1}\left(N^{\prime}\right) \oplus H_{1}\left(\overline{N \backslash N^{\prime}}\right)$. Since $\gamma$ is a meridian of $N$, the element is in the kernel of the map $i_{2}$. From the sequence 1.2 .1 , it is also contained in the image of $i_{1}$. Hence, $[\gamma]$ can be written as $\sum_{i} n_{i}\left[\gamma_{i}^{\prime}\right]$ in $H_{1}\left(\overline{N \backslash N^{\prime}}\right)$.

Claim: the coefficient $n_{i}$ is equal to the linking number of $\alpha_{i} \amalg \gamma$.
From the Mayer-Vietoris sequence, the group $H_{1}\left(\mathbb{S}^{3} \backslash K^{\prime}\right)$ is a free Abelian group of rank $g^{\prime}$ spanned by $\left\{\left[\gamma_{i}^{\prime}\right]\right\}_{i=1}^{g^{\prime}}$. Hence, $[\gamma]$ is equal to $\sum_{i} n_{i}\left[\gamma_{i}^{\prime}\right]$ in $H_{1}\left(\mathbb{S}^{3} \backslash K^{\prime}\right)$. Similarly, $H_{1}\left(\mathbb{S}^{3} \backslash \alpha_{i}\right)$ is of rank one and generated by $\gamma_{i}^{\prime}$. One has that

$$
\begin{aligned}
& H_{1}\left(\overline{N \backslash N^{\prime}}\right) \rightarrow H_{1}\left(\mathbb{S}^{3} \backslash K^{\prime}\right) \\
& {[\gamma]=H_{1}\left(\mathbb{S}^{3} \backslash \alpha_{i}\right) } \\
& {\left[\sum_{i} n_{i}\left[\gamma_{i}^{\prime}\right]\right.} \mapsto \quad \sum_{i} n_{i}\left[\gamma_{i}^{\prime}\right] \mapsto n_{i}\left[\gamma_{i}^{\prime}\right]
\end{aligned}
$$

That is to say, $[\gamma]$ is equal to $n_{i}\left[\gamma_{i}^{\prime}\right]$ in $H_{1}\left(\mathbb{S}^{3} \backslash \alpha_{i}\right)$. From the definition of the linking number (See Page 132 in Rol03), $n_{i}$ is the linking number of $\gamma \amalg \alpha_{i}$.

Since each $\alpha_{i} \subset N^{\prime}$ is nullhomotopic in $N$, then the linking number of $\gamma \amalg \alpha_{i}$ is zero. Namely, $n_{i}$ is equal to zero, for each $i$. Therefore, $[\gamma]$ is equal to zero in $H_{1}\left(\overline{N \backslash N^{\prime}}\right)$.
1.2.1. The effective meridian. Consider two closed handlebodies $N^{\prime}$ and $N$ with $N^{\prime} \subset \operatorname{Int} N$.

Definition 1.2.3. A meridian $\gamma$ of $N$ is called an effective meridian relative to $N^{\prime}$ if any meridian disc with boundary $\gamma$ intersects the core of $N^{\prime}$.

The handbody $N$ is called an effective handlebody relative to $N^{\prime}$, if any meridian of $N$ is an effective meridian relative to $N^{\prime}$.

Note that if $N^{\prime}$ is contained in a 3 -ball $B \subset \operatorname{Int} N$, there is no effective meridian relative to $N^{\prime}$.

Lemma 1.2.4. Let $N^{\prime}$ and $N$ be two closed handlebodies with $N^{\prime} \subset$ Int $N$. The handlebody $N$ is an effective handlebody relative to $N^{\prime}$ if and only if the map $\pi_{1}(\partial N) \rightarrow$ $\pi_{1}\left(\overline{N \backslash N^{\prime}}\right)$ is injective.

Proof. If $N$ is not an effective handlebody relative to $N^{\prime}$, there is a meridian disc $(D, \partial D) \subset(N, \partial N)$ with $D \cap N^{\prime}=\varnothing$. Therefore, the map $\pi_{1}(\partial N) \rightarrow \pi_{1}\left(\overline{N \backslash N^{\prime}}\right)$ is not injective.

If the map $\pi_{1}(\partial N) \rightarrow \pi_{1}\left(\overline{N \backslash N^{\prime}}\right)$ is not injective, we apply Lemma 1.1.2 to the 3manifold $\overline{N \backslash N^{\prime}}$. There is an embedded disc $\left(D^{\prime}, \partial D^{\prime}\right) \subset\left(\overline{N \backslash N^{\prime}}, \partial N\right)$ whose boundary is not contractible in $\partial N$. As in Remark 1.1.3, we may assume that Int $D^{\prime} \subset \operatorname{Int}\left(N \backslash N^{\prime}\right)$. We see that $D^{\prime}$ is a meridian disc with $D^{\prime} \cap N^{\prime}=\varnothing$. Therefore, $N$ is not an effective handlebody relative to $N^{\prime}$.

Lemma 1.2.5. Let $N^{\prime}$ and $N$ be two closed handlebodies satisfying that 1) $N^{\prime} \subset$ Int $N$ and 2) $\pi_{1}\left(\partial N^{\prime}\right) \rightarrow \pi_{1}\left(\overline{N \backslash N^{\prime}}\right)$ is injective. If $N$ is an effective handlebody relative to $N^{\prime}$, then any meridian disc $(D, \partial D) \subset(N, \partial N)$ contains a meridian of $N^{\prime}$.

The proof is the same as the proof of Lemma 2.12 in Wan19a.

Proof. Suppose that the closed meridian disc $D$ intersects $\partial N^{\prime}$ transversally where $\gamma:=\partial D$ is a meridian of $N$. The intersection $D \cap \partial N^{\prime}$ is a disjoint union of circles $\left\{c_{i}\right\}_{i \in I}$. Each $c_{i}$ bounds a unique closed disc $D_{i} \subset \operatorname{Int} D$.

Consider the set $C^{\text {non }}:=\left\{c_{i} \mid c_{i}\right.$ is not contractible in $\left.\partial N^{\prime}\right\}$ and the set $C^{\max }=\left\{c_{i} \mid c_{i}\right.$ is a maximal circle in $\left.\left\{c_{i}\right\}_{i \in I}\right\}$.

We will show that $C^{n o n}$ is nonempty and a minimal circle in $C^{\text {non }}$ is a desired meridian.
Suppose the contrary that $C^{n o n}$ is empty. Hence, each $c_{i} \in C^{\max }$ is contractible in $\partial N^{\prime}$ and bounds a disc $D_{i}^{\prime} \subset \partial N^{\prime}$. Consider the immersed disc

$$
\hat{D}:=\left(D \backslash \cup_{c_{i} \in C^{\max }} D_{i}\right) \cup\left(\cup_{c_{i} \in C^{\max }} D_{i}^{\prime}\right)
$$

with boundary $\gamma$. Since $\hat{D} \cap \operatorname{Int} N^{\prime}=\varnothing$, we see that $\gamma$ is contractible in $\overline{N \backslash N^{\prime}}$.
However, Lemma 1.2.4 shows that the map $\pi_{1}(\partial N) \rightarrow \pi_{1}\left(\overline{N \backslash N^{\prime}}\right)$ is injective. That is to say, the circle $\gamma$ is nullhomotopic in $\partial N$. This is in contradiction with our hypothesis that $\gamma$ is non-trivial in $\pi_{1}(\partial N)$. We conclude that $C^{\text {non }} \neq \varnothing$.

In the following, we will prove that each minimal circle $c_{j}$ in $C^{n o n}$ is a required meridian. From Definition 1.2.1, it is sufficient to show that $c_{j}$ is homotopically trivial in $N^{\prime}$. Our strategy is to construct an immersed disc $\hat{D}_{j} \subset N^{\prime}$ with boundary $c_{j}$.

Let $C_{j}:=\left\{c_{i} \mid c_{i} \subset \operatorname{Int} D_{j}\right.$ for $\left.i \in I\right\}$ and $C_{j}^{\max }$ be the set of maximal circles in $C_{j}$. We now have two cases: $C_{j}=\varnothing$ or $C_{j} \neq \varnothing$.

Case I: If $C_{j}$ is empty, we consider the set $Z:=\operatorname{Int} D_{j}$ and define the disc $\hat{D}_{j}$ as Int $D_{j}$.

Case II: If $C_{j}$ is not empty, then $C_{j}^{\max }$ is also nonempty. From the minimality of $c_{j}$ in $C^{n o n}$, each $c_{i} \in C_{j}^{\max }$ is nullhomotopic in $\partial N^{\prime}$ and bounds a disc $D_{i}^{\prime \prime} \subset \partial N^{\prime}$.

Define the set $Z:=\operatorname{Int} D_{j} \backslash \cup_{c_{i} \in C_{j}^{\max }} D_{i}$ and the new disc $\hat{D}_{j}:=Z \cup\left(\cup_{c_{i} \in C_{j}^{\max }} D_{i}^{\prime \prime}\right)$ with boundary $c_{j}$.

Let us explain why $\hat{D}_{j}$ is contained in $N^{\prime}$. In any case, $\partial N^{\prime}$ cuts $N$ into two connected components, $N \backslash N^{\prime}$ and Int $N^{\prime}$. The set $Z$ is one of these components of Int $D_{j} \backslash \partial N^{\prime}$. Therefore, it must be contained in Int $N^{\prime}$ or $N \backslash N^{\prime}$.

If $Z$ is in $N \backslash N^{\prime}$, the disc $\hat{D}_{j}$ is contained in $\overline{N \backslash N^{\prime}}$. Thus, $c_{j}$ is contractible in $\overline{N \backslash N^{\prime}}$. However, since the induced map $\pi_{1}\left(\partial N^{\prime}\right) \rightarrow \pi_{1}\left(\overline{N \backslash N^{\prime}}\right)$ is injective, then $c_{j}$ is homotopically trivial in $\partial N^{\prime}$. This contradicts the choice of $c_{j} \in C^{n o n}$. We conclude that $Z$ is contained in Int $N^{\prime}$.

Therefore, $\hat{D}_{j}$ is contained in $N^{\prime}$. That is to say, $c_{j}$ is null-homotopic in $N^{\prime}$. However, [ $c_{j}$ ] is a non-trivial element in $\pi_{1}\left(\partial N^{\prime}\right)$. From Definition 1.2.1, we conclude that $c_{j} \subset D$ is a meridian of $N^{\prime}$. This finishes the proof.

As a consequence, we have
Corollary 1.2.6. Let $N^{\prime}$ and $N$ be two closed handlebodies in a contractible 3manifold $M$ satisfying that 1) $N^{\prime} \subset$ Int $N$ and 2) the map $\pi_{1}\left(\partial N^{\prime}\right) \rightarrow \pi_{1}\left(\overline{M \backslash N^{\prime}}\right)$ is injective. If an embedded circle $\gamma \subset \partial N$ is not nullhomotopic in $\overline{M \backslash N^{\prime}}$, then any embedded disc $D \subset M$ with boundary $\gamma$ contains a meridian of $N^{\prime}$.

The proof is the same as Lemma 1.2.5.

### 1.2.2. The system of meridians.

Lemma 1.2.7. For a closed handlebody $N$ of genus $g$, there are $g$ disjoint non-splitting meridians $\left\{\gamma^{l}\right\}_{l=1}^{g}$ so that $N \backslash \amalg_{l} N_{\epsilon_{l}}\left(D_{l}\right)$ is a closed 3-ball, where $D_{l}$ is a closed meridian disc with boundary $\gamma^{l}$ and $N_{\epsilon_{l}}\left(D_{l}\right)$ is an open neighborhood of $D_{l}$ in $N$ with small radius $\epsilon_{l}$.

The set of these meridians $\left\{\gamma^{l}\right\}_{l=1}^{g}$ is called a system of the handlebody $N$ of genus $g$. In general, it is not unique.

Proof. Pick any non-splitting meridian $\gamma^{1}$ of $N$. We use Lemma 1.1.2 to find an embedded disc $D_{1} \subset N$.

As Remark 1.1.3, we may assume that Int $D_{1} \subset \operatorname{Int} N$. The set $N_{1}:=N \backslash N_{\epsilon}\left(D_{1}\right)$ is a closed handlebody of genus $g-1$, where $N_{\epsilon_{1}}\left(D_{1}\right)$ is the open tubular neighborhood of $D_{1}$ in $N$ with small radius $\epsilon_{1}$. In particular, the map $\pi_{1}\left(\partial N \cap \partial N_{1}\right) \rightarrow \pi_{1}\left(\partial N_{1}\right)$ is surjective.

Choose a non-splitting meridian $\gamma^{2} \subset \partial N \cap \partial N_{1}$ of $N_{1}$. By Lemma 1.1.2, there exists a meridian disc $D_{2}$ of $N_{1}=N \backslash N_{\epsilon_{1}}\left(D_{1}\right)$. The set $N_{2}:=N \backslash N_{\epsilon_{1}}\left(D_{1}\right) 山 N_{\epsilon_{2}}\left(D_{2}\right)$ is a closed handlebody of genus $g-2$, where $N_{\epsilon_{2}}\left(D_{2}\right)$ is an open tubular neighborhood of $D_{2}$ in $N$.

We repeat this process $g-2$ times and obtain $g$ disjointly embedded discs $\left\{D_{l}\right\}$ so that $N \backslash \amalg_{l} N_{\epsilon_{l}}\left(D_{l}\right)$ is a handlebody of genus zero (a 3-ball). The boundaries $\left\{\gamma^{l}\right\}_{l=1}^{g}$ of these discs are $g$ distinct meridians which are the required candidates in the assertion.

Corollary 1.2.8. Let $N \subset M,\left\{\gamma^{l}\right\}$ and $\left\{D_{l}\right\}$ be as in Lemma 1.2.7, where $M$ is a 3-manifold without boundary. If $R \subset$ Int $N$ is a closed handlebody satisfying that 1) it is not contained in a 3-ball in $M$; 2) $\pi_{1}(\partial R) \rightarrow \pi_{1}(\overline{M \backslash R})$ is injective, then $\partial R \cap \amalg_{l} D_{l}$ contains at least a meridian of $R$.

The poof is also similar to the proof of Lemma 2.12 Wan19a.
Proof. We may assume that $\partial R$ intersects $\amalg_{l} D_{l}$ transversally. The intersection $\partial R \cap$ $\amalg D_{l}:=\{\gamma\}_{\gamma \in C}$ has finitely many components. Let us consider the set $C^{n o n}:=\{\gamma \in C$ is not contractible in $\partial R\}$.
Claim: $C^{n o n}$ is nonempty.
We argue by contradiction. Suppose that $C^{\text {non }}$ is empty. We have that any circle in $D_{l} \cap \partial R$ is contractible in $\partial R$. As in the proof of Lemma 1.2.5, we get a new disc in $\overline{N \backslash R}$ with boundary $\gamma^{l}$. Therefore, $\gamma^{l}$ is null-homotopic in $\overline{N \backslash R}$.

We use Lemma 1.1.2 to find an embedded disc $D_{1}^{\prime} \subset \overline{N \backslash R}$ with boundary $\gamma^{1}$. As in Remark 1.1.3, we may assume that Int $D_{1}^{\prime} \subset$ Int $\overline{N \backslash R}$ (or $D_{1}^{\prime} \subset N \backslash R$ ). Choose the open tubular neighborhood $N_{\epsilon_{1}^{\prime}}\left(D_{1}^{\prime}\right)$ of $D_{1}^{\prime}$ in $N \backslash R$ with small radius $\epsilon_{1}^{\prime}$. The set $N_{1}^{\prime}:=N \backslash N_{\epsilon_{1}^{\prime}}\left(D_{1}^{\prime}\right)$ is a closed handlebody of genus $g-1$ containing $R$.

In addition, for $l>1, \gamma^{l}$ is a non-splitting meridian of $N_{1}^{\prime}$ but contractible in $N \backslash\left(N_{\epsilon_{1}^{\prime}}\left(D_{1}^{\prime}\right) \amalg R\right)$.

Repeating this process $g-1$ times, we obtain $g$ embedded discs $\left\{D_{l}^{\prime}\right\}_{l=1}^{g}$ so that

1) $R \cap \amalg_{l} N_{\epsilon_{l}^{\prime}}\left(D_{l}^{\prime}\right)=\varnothing$;
2) The handlebody $N \backslash \amalg_{l} N_{\epsilon_{l}^{\prime}}\left(D_{l}^{\prime}\right)$ is of genus zero (a closed 3-ball), where $N_{\epsilon_{l}^{\prime}}\left(D_{l}^{\prime}\right)$ is the open tubular neighborhood of $D_{l}^{\prime}$ in $N$ with small radius $\epsilon_{l}^{\prime}$.

Therefore, $R$ is contained in the 3-ball $N \backslash \amalg_{l} N_{\epsilon_{l}^{\prime}}\left(D_{l}^{\prime}\right)$. This contradicts our hypothesis. The claim follows.

As in the proof of Lemma 1.2.5, we use the condition 2) to show that each minimal circle in $C^{n o n}$ is a required meridian.

### 1.3. Examples

In this part, we begin with Knot theory in a closed solid torus. Subsequently, we introduce several notations, such as the geometric index, the Whitehead manifold and contractible genus one 3 -manifolds. In addition, we construct two non-homeomorphic contractible 3-manifolds whose fundamental group at infinity are both non-trivial.

### 1.3.1. Knots basic.

Definition 1.3.1. A subset $K$ of a 3 -manifold $X$ is a knot if $K$ is homeomorphic with a circle $\mathbb{S}^{1}$. More generally, $K$ is a link if $K$ is homeomorphic with a disjoint union of one or some circle(s).

Two knots or links $K$ and $K^{\prime}$ are ambient isotopic if there is a homeomorphism $h: X \rightarrow X$ such that (1) $h$ is isotopic to the identity map; (2) $h(K)=K^{\prime}$.

A knot $K$ is called to be trivial (or unknotted) in $X$ if there exists an embedded disc in $X$ with boundary $K$.


Figure 1.1.
For example, if $X$ is $\mathbb{R}^{3}$ or $\mathbb{S}^{3}$, the knot (I) is ambient isotopic to the knot (II) in $X$ (See Figure 1.1). These two knots are both trivial in $\mathbb{S}^{3}$.

If $X$ is a torus $\mathbb{T}^{2}$, two knots $K$ and $K^{\prime}$ are ambient isotopic if and only if $[K]= \pm\left[K^{\prime}\right]$ in $\pi_{1}\left(\mathbb{T}^{2}\right)$ (See [16.Theorem, Page 25] of Rol03).

We consider a closed solid torus $N$ and a knot $K \subset N$. Embed $N$ into $\mathbb{S}^{3}$ as an unknotted solid torus (See Definition 1.1.14). The knot $K$ can be viewed as a knot in $\mathbb{S}^{3}$. The disjoint union $K \amalg \gamma$ is a link in $\mathbb{S}^{3}$, where $\gamma$ is a meridian of $N$.

Recall that an $n$-component link $L \subset \mathbb{S}^{3}$ is unlinked if and only if there exist $n$ disjointly embedded discs $D_{i} \subset \mathbb{S}^{3}$ so that $L=\amalg_{i} \partial D_{i}$.

Remark 1.3.2. The knot $K$ is trivial in $N$ if and only if the link $K \amalg \gamma$ is unlinked in $\mathbb{S}^{3}$. The reason is as follows:

If $K$ is trivial in $N$, there exists an embedded closed disc $D \subset N$ with boundary $K$. This disc is away from the knot $\gamma$ in $\mathbb{S}^{3}$.

Since $N \subset \mathbb{S}^{3}$ is unknotted, the meridian $\gamma$ is a trivial knot in $\mathbb{S}^{3}$. Note that $\mathbb{S}^{3} \backslash D$ is homeomorphic to an open 3-ball. Then, $\gamma$ is a trivial knot in $\mathbb{S}^{3} \backslash D$. Hence, one finds an embedded disc $D_{0} \subset \mathbb{S}^{3} \backslash D$ with boundary $\gamma$. Therefore, $K \amalg \gamma$ is the boundary of $D \amalg D_{0}$. That is to say, the link $\gamma \amalg K \subset \mathbb{S}^{3}$ is unlinked.

If $K \amalg \gamma$ is unlinked in $\mathbb{S}^{3}$, there exists an embedded closed disc $D$ in $S^{3} \backslash \gamma$ with boundary $K$. Therefore, $K$ is trivial in the complement of the knot $\gamma$.

A closed solid torus $N$ is homeomorphic with $\mathbb{S}^{1} \times D^{2}$, where $D^{2}$ is a closed unit disc in $\mathbb{R}^{2}$. A special homeomorphism $h: \mathbb{S}^{1} \times D^{2} \rightarrow N$ is called a framing of $N$.

A longitude of $N$ is any simple closed curve in $\partial N$ of form $h\left(\mathbb{S}^{1} \times x_{0}\right)$, for some framing $h$ of $N$ and some point $x_{0}$ in $D^{2}$.

Remark 1.3.3. In a closed solid torus $N$, the kernel of the induced map $\pi_{1}(\partial N) \rightarrow$ $\pi_{1}(N)$ is isomorphic to $\mathbb{Z}$. Each meridian $\gamma$ of $N$ belongs to the kernel. Since $\gamma$ is an embedded curve, it is a generator of the kernel.

An embedded circle $\theta \subset N$ is a longitude if and only if $[\theta]$ and $[\gamma]$ generate $H_{1}(\partial N, \mathbb{Z})$, where $\gamma$ is a meridian of $N$ (See Page 29 of Rol03). In addition, any longitude of $N$ is isotopic to the core of $N$ in $N$.

Definition 1.3.4. Assume that $p, q$ are relatively prime and $N$ is an unknotted solid torus in $\mathbb{S}^{3}$. The torus knot $T_{p, q} \subset \partial N \subset \mathbb{S}^{3}$ of type $(p, q)$ is the knot which wraps around $N$ in the longitudinal direction $p$ times and in the meridional direction $q$ times.

For example, the trefoil is $T_{2,3}$. Here are $T_{2,3}$ and $T_{3,4}$.


The $(2,3)$ torus knot


The $(3,4)$ torus knot

Figure 1.2.
Remark 1.3.5. The knot $T_{p, q}$ is trivial in $\mathbb{S}^{3}$ if and only if $p= \pm 1$ or $q= \pm 1$ (See Page 53 of Rol03 ).

In the following, we consider two closed solid tori $N^{\prime}$ and $N$ with $N^{\prime} \subset \operatorname{Int} N$.
Lemma 1.3.6. If the closed solid torus $N^{\prime} \subset$ Int $N$ is homotopically trivial in the closed solid torus $N$, then $H_{1}\left(\overline{N \backslash N^{\prime}}\right)=\mathbb{Z}^{2}$ and the kernel of the induced map $H_{1}\left(\partial N^{\prime}\right) \rightarrow$ $H_{1}\left(\overline{N \backslash N^{\prime}}\right)$ is generated by a longitude of $N^{\prime}$.

Proof. As in Lemma 1.2.2, we use the Mayer-Vietoris sequence to show that $H_{1}\left(\overline{N \backslash N^{\prime}}\right)$ is isomorphic to $\mathbb{Z}^{2}$. It is generated by a meridian of $N^{\prime}$ and a longitude of $N$.

The image of the map $H_{1}\left(\partial N^{\prime}\right) \rightarrow H_{1}\left(\overline{N \backslash N^{\prime}}\right)$ is a subgroup of rank one which is generated by the meridian $\gamma^{\prime}$ of $N^{\prime}$. The kernel of $H_{1}\left(\partial N^{\prime}\right) \rightarrow H_{1}\left(\overline{N \backslash N^{\prime}}\right)$ is also of rank one and generated by $\left[\theta^{\prime}\right]$, where $\theta^{\prime} \subset \partial N^{\prime}$ is an embedded circle. Therefore, $H_{1}\left(\partial N^{\prime}\right)$ is generated by $\left[\gamma^{\prime}\right]$ and $\left[\theta^{\prime}\right]$. The circle $\theta^{\prime}$ is a longitude of $N^{\prime}$ (See Page 29 of Rol03] or Remark 1.3.3). That is to say, the longitude $\theta^{\prime}$ is a generator of the kernel of $\overline{H_{1}\left(\partial N^{\prime}\right)} \rightarrow$ $H_{1}\left(\overline{\left.N \backslash N^{\prime}\right)}\right.$.
1.3.2. The Whitehead manifold. The Whitehead manifold is constructed from the Whitehead link. Recall that the Whitehead link is a link with two components illustrated in Figure 1.3 .


Figure 1.3.

Choose a closed unknotted solid torus $T_{1}$ in $\mathbb{S}^{3}$. Its complement inside $\mathbb{S}^{3}$ is another solid torus. Take a second solid torus $T_{2}$ inside $T_{1}$ so that the core of $K_{2}$ forms a Whitehead link with any meridian of $T_{1}$ as in Figure 1.4 .


Figure 1.4.
The solid torus $T_{2}$ is unknotted in $\mathbb{S}^{3}$. Then, embed $T_{3}$ inside $T_{2}$ in the same way as $T_{2}$ lies in $T_{1}$ and so on infinitely many times. Define the set $T_{\infty}=\cap_{k=1}^{\infty} T_{k}$, called the Whitehead continuum.

The Whitehead manifold is defined as $W h:=\mathbb{S}^{3} \backslash T_{\infty}$ which is an open 3-manifold.
Remark 1.3.7. From the above construction, we know that
(1) Since each $T_{k}$ is unknotted in $\mathbb{S}^{3}$, then its complement $N_{k}$ is a solid torus. Therefore, the Whitehead manifold is an increasing union of solid tori $\left\{N_{k}\right\}_{k}$ as in Remark 1.1.13. In addition, each $N_{k}$ is embedded inside $N_{k+1}$ in the same way as $T_{2}$ lies in $T_{1}$. This follows from the symmetry of the Whitehead link.
(2) The core $K_{k}$ of $N_{k}$ is a non-trivial knot in the solid torus $N_{k+1}$. Furthermore, the link $K_{k} \amalg \gamma_{k+1}$ is a Whitehead link for each meridian $\gamma_{k+1}$ of $N_{k+1}$. This is a consequence of the symmetry of the Whitehead link.
(3) Each $K_{k}$ is unknotted in $\mathbb{S}^{3}$. For each $j>k, K_{k}$ is nullhomotopic in $N_{j}$ but a non-trivial knot in $N_{j}$.

Remark that the Whitehead manifold has no complete metric of positive scalar curvature (See Theorem $B_{1}$ ).

### 1.3.3. Geometric Index.

Definition 1.3.8. Sch53 If $N^{\prime} \subset$ Int $N$ are solid tori, the geometric index of $N^{\prime}$ in $N, I\left(N^{\prime}, N\right)$, is the minimal number of points of the intersection of the core of $N^{\prime}$ with a meridian disc of $N$.

Remark 1.3.9. If the geometric index $I\left(N^{\prime}, N\right)$ is greater than zero, the solid torus $N$ is an effective handlebody relative to $N^{\prime}$ (See Definition 1.2.3).

If the core $K^{\prime}$ of $N^{\prime}$ is a trivial knot in $N$, there is a meridian disc $(D, \partial D) \subset$ ( $\left.N \backslash N^{\prime}, \partial N\right)$. Moreover, $I\left(N^{\prime}, N\right)$ is equal to zero. (See Corollary 2.9 of Wan19a)

For example, in $W h$, the geometric index $I\left(N_{k}, N_{k+1}\right)=2$ for each $k$, where $N_{k}$ is illustrated as in Chapter 1.3.2.

See [Sch53] for the following results about the geometric index.

1) Let $N_{0}, N_{1}$, and $N_{2}$ be solid tori so that $N_{0} \subset \operatorname{Int} N_{1}$ and $N_{1} \subset \operatorname{Int} N_{2}$. Then $I\left(N_{0}, N_{2}\right)=I\left(N_{0}, N_{1}\right) I\left(N_{1}, N_{2}\right)$.
2) If $N_{0}$ and $N_{1}$ are unknotted solid tori in $\mathbb{S}^{3}$ with $N_{0} \subset \operatorname{Int} N_{1}$, and if $N_{0}$ is homotopically trivial in $N_{1}$, then $I\left(N_{0}, N_{1}\right)$ is even.

Lemma 1.3.10. Suppose that the closed solid torus $N^{\prime} \subset$ Int $N$ is homotopically trivial in the closed solid torus $N$. If $I\left(N^{\prime}, N\right)>0$, then the two induced maps $i_{1}: \pi_{1}(\partial N) \rightarrow$ $\pi_{1}\left(\overline{N \backslash N^{\prime}}\right)$ and $i_{2}: \pi_{1}\left(\partial N^{\prime}\right) \rightarrow \pi_{1}\left(\overline{N \backslash N^{\prime}}\right)$ are both injective.

Proof. Since $I\left(N^{\prime}, N\right)>0$, the solid torus $N$ is an effective handlebody relative to $N^{\prime}$ (See Remark 1.3.9). We use Lemma 1.2.4 to see that the map $i_{1}: \pi_{1}(\partial N) \rightarrow \pi_{1}\left(\overline{N \backslash N^{\prime}}\right)$ is injective.

Suppose that the map $i_{2}$ is not injective. We use Lemma 1.1.2 to find an embedded $\operatorname{disc}\left(D_{2}, \partial D_{2}\right) \subset\left(\overline{N \backslash N^{\prime}}, \partial N^{\prime}\right)$. The embedded circle $\theta:=\partial D_{2}$ is not contractible in $\partial N^{\prime}$.

Since $\theta$ bounds an embedded disc $D_{2} \subset N \backslash N^{\prime}$, it is a trivial knot in $N$. Furthermore, [ $\theta$ ] belongs to the kernel of the map $H_{1}\left(\partial N^{\prime}\right) \rightarrow H_{1}\left(\overline{N \backslash N^{\prime}}\right)$. From Lemma 1.3.6, the embedded circle $\theta$ is a longitude of $N^{\prime}$.

Recall that as a knot, any longitude of $N^{\prime}$ is isotopic to the core $K^{\prime}$ of $N^{\prime}$ in $N^{\prime}$ (See Remark 1.3.3). Therefore, $K^{\prime}$ is isotopic to $\theta$ and a trivial knot in $N$. From Remark 1.3.9, $I\left(N^{\prime}, N\right)=0$. This is a contradiction.

Lemma 1.3.11. Suppose that the closed solid torus $N^{\prime} \subset$ Int $N$ is homotopically trivial in the closed solid torus $N$. If $I\left(N^{\prime}, N\right)>0$, then any meridian disc $D$ of $N$ contains a meridian of $N^{\prime}$.

It follows from Lemma 1.3.10 and Lemma 1.2.5.
1.3.4. Genus one 3 -manifold. (See GRW18]) Let us describe McMillan's construction in McM62.

Definition 1.3.12. (Genus one 3-manifold) A genus one 3-manifold $M$ is the ascending union of solid tori $\left\{N_{k}\right\}_{k \in \mathbb{N}}$, so that for each $k, N_{k} \subset \operatorname{Int} N_{k+1}$ and the geometric index of $N_{k}$ in $N_{k+1}$ is not equal to zero.

Theorem 1.3.13. (See [Theorem 2.8, Page 2042] of [GRW18])
(1) A genus one 3-manifold defined with a sequence of open solid tori $\left\{N_{k}\right\}_{k \in \mathbb{N}}$ so that each $N_{k}$ is contractible in $N_{k+1}$, is a contractible 3-manifold that is not homeomorphic to $\mathbb{R}^{3}$.
(2) Any contractible genus one 3-manifold can be written as an ascending union of solid tori $\left\{N_{k}\right\}_{k \in \mathbb{N}}$ so that 1) $N_{k}$ is contractible in $N_{k+1}$; 2) $I\left(N_{k}, N_{k+1}\right) \geqslant 2$ for each $k$.

For example, the Whitehead manifold is a contractible genus one 3-manifold.
Any contractible genus one 3-manifold $M:=\cup_{i=0}^{\infty} N_{k}$ satisfies the following:
(1) For each $k, N_{k}$ is homotopically trivial in $N_{k+1}$. Moreover, $I\left(N_{k}, N_{k+1}\right) \geqslant 2$.
(2) For each $j>k$, the core $K_{k}$ of $N_{k}$ is null-homotopic in $N_{j}$ but a nontrivial knot in $N_{j}$.
(3) If $N_{j}$ is viewed as an unknotted solid torus in $\mathbb{S}^{3}$, then the link $K_{k} \amalg \gamma_{j} \subset \mathbb{S}^{3}$ is linked in $\mathbb{S}^{3}$, for each meridian $\gamma_{j}$ of $N_{j}$ for $j>k$. Moreover, its linking number is zero.
(4) However, the knot $K_{k} \subset \mathbb{S}^{3}$ may be knotted in $\mathbb{S}^{3}$.

Together these properties, we will show that no contractible genus one 3-manifold admits a complete metric of positive scalar curvature (See Theorem $B_{2}$.
1.3.5. More examples. In this part, we construct a contractible 3-manifold whose fundamental group at infinity is non-trivial.

Let us describe the construction of such a manifold.
Choose an unknotted handlebody $W_{0} \subset \mathbb{S}^{3}$ of genus two (See Definition 1.1.14). Take a second handlebody $W_{1} \subset$ Int $W_{0}$ of genus two which is a tubular neighborhood of the curve in Figure 1.5. Then, embed another handlebody $W_{2}$ of genus two inside $W_{1}$ in the same way as $W_{1}$ lies in $W_{0}$ and so on infinitely many times. Therefore, we obtain a decreasing family $\left\{W_{k}\right\}_{k}$ of handlebodies of genus two.


Figure 1.5.

The manifold $M_{1}$ is defined as $M_{1}:=\mathbb{S}^{3} \backslash \cap_{k=0}^{\infty} W_{k}$. It is an open 3-manifold.
We have that each $W_{k}$ is unknotted in $\mathbb{S}^{3}$ (See Definition 1.1.14). That is, the complement $N_{k}$ of $W_{k}$ in $\mathbb{S}^{3}$ is a handlebody of genus two. Therefore, $M_{1}$ can be written as the increasing union of handlebodies $\left\{N_{k}\right\}_{k}$ of genus two. In addition, each $N_{k}$ lies in $N_{k+1}$ as in Figure 1.6. (The set $K_{k}$ is the core of $N_{k}$.) Since each $N_{k}$ is homotopically trivial in $N_{k+1}, M_{1}$ is a contractible 3-manifold.


Figure 1.6.
Next, we construct a properly embedded plane $P_{1} \subset M_{1}$ and show that $\pi_{1}^{\infty}\left(M_{1}\right)$ is non-trivial.

Choose the splitting meridian $\gamma_{k+1} \subset \partial N_{k+1}$ of $N_{k+1}$ as in Figure 1.6. We have that $\gamma_{k}$ is homotopic to $\gamma_{k+1}$ in $\overline{N_{k+1} \backslash N_{k}}$. Choose an embedded annulus $A_{k} \subset \overline{N_{k+1} \backslash N_{k}}$ (namely, it is homeomorphic to $\left.\mathbb{S}^{1} \times[0,1]\right)$ with boundary $\gamma_{k} \amalg \gamma_{k+1}$. We define the plane $P_{1}$ as

$$
P_{1}:=\cup_{k \geqslant 0} A_{k} \cup_{\gamma_{0}} D_{0}
$$

where $D_{0} \subset N_{0}$ is a meridian disc with boundary $\gamma_{0}$.
Proposition 1.3.14. Let $M_{1}$ and $P_{1}$ be constructed as above. Then,

- the fundamental group at infinity $\pi_{1}^{\infty}\left(M_{1}\right)$ is non-trivial;
- the properly embedded plane $P_{1}$ cuts $M_{1}$ into two Whitehead manifolds.

Remark. Since $\pi_{1}^{\infty}\left(M_{1}\right)$ is non-trivial, $M_{1}$ is not simply-connected at infinity. That is, $M_{1}$ is not homeomorphic to $\mathbb{R}^{3}$.

The family $\left\{N_{k}\right\}$ of handlebodies satisfies Property H (See Definition 2.2.5).
Proof. First, we will show that $\pi_{1}^{\infty}\left(M_{1}\right)$ is non-trivial. Since $\gamma_{k}$ is homotopic to $\gamma_{k+1}$ in $\overline{M_{1} \backslash N_{0}}$, it is sufficient to show that $\gamma_{0}$ is not contractible in $\overline{M_{1} \backslash N_{0}}$ (See Remark 1.1.10).

We see from Figure 1.6 that $N_{k+1}$ is an effective handlebody relative to $N_{k}$. From Lemma 1.2.4, the map $\pi_{1}\left(\partial N_{k+1}\right) \rightarrow \pi_{1}\left(\overline{N_{k+1} \backslash N_{k}}\right)$ is injective for $k \geqslant 0$.

From Figure 1.5, we have that $W_{k}$ is an effective handlebody relative to $W_{k+1}$. By Lemma 1.2.4, the maps $\pi_{1}\left(\partial W_{k}\right) \rightarrow \pi_{1}\left(\overline{W_{k} \backslash W_{k+1}}\right)$ is also injective. In addition, the space $\overline{W_{k} \backslash W_{k+1}}$ is equal to $\overline{N_{k+1} \backslash N_{k}}$. Then, we can conclude that the map $\pi_{1}\left(\partial N_{k}\right) \rightarrow$ $\pi_{1}\left(\overline{N_{k+1} \backslash N_{k}}\right)$ is injective for $k \geqslant 0$.

Claim: the $\operatorname{map} \pi_{1}\left(\partial N_{0}\right) \rightarrow \pi_{1}\left(\overline{M_{1} \backslash N_{0}}\right)$ is injective.
Theorem 1.1.4 gives an isomorphism between $\pi_{1}\left(\overline{N_{2} \backslash N_{0}}\right)$ and $\pi_{1}\left(\overline{N_{1} \backslash N_{0}}\right) * \pi_{1}\left(\partial N_{1}\right) \pi_{1}\left(\overline{N_{1} \backslash N_{0}}\right)$. From the above fact, the maps $\pi_{1}\left(\partial N_{1}\right) \rightarrow \pi_{1}\left(\overline{N_{1} \backslash N_{0}}\right)$ and $\pi_{1}\left(\partial N_{1}\right) \rightarrow \pi_{1}\left(\overline{N_{2} \backslash N_{1}}\right)$ are both injective. We use Lemma 1.1 .5 to show that the map $\pi_{1}\left(\overline{N_{1} \backslash N_{0}}\right) \rightarrow \pi_{1}\left(\overline{N_{2} \backslash N_{0}}\right)$ is injective.

In addition, the map $\pi_{1}\left(\partial N_{0}\right) \rightarrow \pi_{1}\left(\overline{N_{1} \backslash N_{0}}\right)$ is injective. The composition of these two maps $\pi_{1}\left(\partial N_{0}\right) \rightarrow \pi_{1}\left(\overline{N_{2} \backslash N_{0}}\right)$ is injective.

Repeating the above argument several times, we obtain that the map $\pi_{1}\left(\partial N_{0}\right) \rightarrow$ $\pi_{1}\left(\overline{N_{j} \backslash N_{0}}\right)$ is injective for each $j>0$. Hence, the map $\pi_{1}\left(\partial N_{0}\right) \rightarrow \pi_{1}\left(\overline{M_{1} \backslash N_{0}}\right)$ is injective. This finishes the proof of this claim.

Since $\gamma_{0}$ is not homotopically trivial in $\partial N_{0}$, it is also non-contractible in $\overline{M_{1} \backslash N_{0}}$. Since each $\gamma_{k}$ is homotopic to $\gamma_{k+1}$ in $\overline{M_{1} \backslash N_{0}}$, each $\gamma_{k}$ is a non-trivial element in $\pi_{1}\left(\overline{M_{1} \backslash N_{0}}\right)$. We see from Remark 1.1.10 that $\pi_{1}^{\infty}\left(M_{1}\right)$ is non-trivial.

It remains to show that $P_{1}$ cuts $M_{1}$ into two Whitehead manifolds.
The plane $P_{1}$ cuts $M_{1}$ into two contractible 3-manifolds $M_{1}^{\prime}$ and $M_{1}^{\prime \prime}$. In addition, $P_{1} \cap N_{k}$ is a splitting meridian disc of $N_{k}$ with boundary $\gamma_{k}$.

From the sequence $\left\{N_{k}\right\}$, we obtain two increasing families, $\left\{N_{k}^{\prime}\right\}$ and $\left\{N_{k}^{\prime \prime}\right\}$, of solid tori in $M_{1}$ satisfying that

- $M_{1}^{\prime}=\cup_{k} N_{k}^{\prime}$ and $M_{1}^{\prime \prime}=\cup_{k} N_{k}^{\prime \prime}$;
- the set $N_{k} \backslash\left(N_{k}^{\prime} \amalg N_{k}^{\prime \prime}\right)$ is a tubular neighborhood of the meridian disc $P_{1} \cap N_{k}$.

Furthermore, each $N_{k}^{\prime}$ is embedded into $N_{k+1}^{\prime}$ as in Figure 1.7. From Chapter 1.3.2, we see that $M_{1}^{\prime}$ is homeomorphic to the Whitehead manifold. Similarly, the contractible 3-manifold $M_{1}^{\prime \prime}$ is also homeomorphic to the Whitehead manifold. Therefore, $P_{1}$ cuts $M_{1}$ into two Whitehead manifolds.


Figure 1.7.
Together with the proof of Theorem $B_{2}$, we have that
Theorem 1.3.15. The contractible 3 -manifold, $M_{1}$, has no complete metric of positive scalar curvature.

We will prove it in Chapter 6.

## CHAPTER 2

## Topological Properties

In this chapter, we discuss several topological properties of contractible 3-manifolds.
We first study the behavior of embedded discs in the Whitehead manifold and their relationship with the geometric indexes. Their relation is clarified by Theorem 2.1.2. Based on their relation, we introduce the topological property, called Property $P$. Furthermore, we show that any contractible genus one 3-manifold satisfies this property.

Next, we consider contractible 3-manifolds. We introduce two types of surgeries on handlebodies. We use these surgeries to show the existence of effective handlebodies (See Theorem 2.2.3). Then we inductively find an increasing family of handlebodies with good properties, called Property (H).

### 2.1. Property $P$

2.1.1. The Whitehead case. As in Chapter 1.3.2, Wh $\subset \mathbb{S}^{3}$ is an increasing union of closed solid tori $\left\{N_{k}\right\}_{k=0}^{\infty}$ so that the geometric index $I\left(N_{k}, N_{k+1}\right)=2$, for each $k$. For any $j>k$, the core $K_{k}$ of $N_{k}$ is a non-trivial knot in $N_{j}$ but unknotted in $\mathbb{S}^{3}$. In addition, the link $K_{k} \amalg \gamma_{j}$ is linked with zero linking number, for any meridian $\gamma_{j}$ of $N_{j}$.

Lemma 2.1.1. Any embedded circle $\gamma \subset \partial N_{k}$ which is the boundary of a closed embedded disc $D$ in Wh but not nullhomotopic in $\partial N_{k}$, is a meridian of $N_{k}$.

Proof. Since the disc $D$ is compact, there is some $k_{0}>k$ such that $D$ is contained in $N_{k_{0}}$.

Let $\gamma$ belong to the homology class $p\left[\gamma_{k}\right]+q\left[\theta_{k}\right]$ in $H_{1}\left(\partial N_{k}\right)$, where $\gamma_{k}$ and $\theta_{k}$ are a meridian and a longitude of $N_{k}$. Since $N_{k}$ is an unknotted solid torus in $\mathbb{S}^{3}$ (See Remark 1.3.7), $\gamma\left(\right.$ as a knot in $\mathbb{S}^{3}$ ) is isotopic to the torus $\operatorname{knot} T_{p, q}$ in $\mathbb{S}^{3}$.

Because the knot $\gamma$ bounds an embedded disc $D$ in $N_{k_{0}}$, it is a trivial knot in $N_{k_{0}} \subset \mathbb{S}^{3}$. Hence, $\gamma$ is unknotted in $\mathbb{S}^{3}$. We see from Remark 1.3 .5 that $p= \pm 1$ or $q= \pm 1$.

Since the knot $\gamma$ is trivial in $N_{k_{0}}$, we use Remark 1.3 .9 to find a meridian disc $\left(D_{1}, \partial D_{1}\right) \subset\left(N_{k_{0}}, \partial N_{k_{0}}\right)$ with $D_{1} \cap \gamma=\varnothing$. Because the geometric index $I\left(N_{k}, N_{k_{0}}\right)>0$, the disc $D_{1}$ contains at least one meridian $\gamma^{\prime}{ }_{k}$ of $\partial N_{k}$ (Lemma 1.3.11 or Lemma 1.2.5).

Therefore, $\gamma^{\prime}{ }_{k} \cap \gamma$ is empty. Their intersection number on $\partial N_{k}$ must be zero.
We knows that the intersection number of $\gamma$ and $\gamma_{k}^{\prime}$ is $q$. Therefore, we knows that $p= \pm 1, q=0$. That is to say, $\gamma$ is homotopic to the meridian $\gamma_{k}^{\prime}$ on $\partial N_{k}$. This completes the proof.

Theorem 2.1.2. Any $\gamma \subset \partial N_{k}$ bounding an embedded disc $D$ in Wh satisfies one of the following:
(1) $[\gamma]$ is trivial in $\pi_{1}\left(\partial N_{k}\right)$,
(2) $D \cap$ Int $N_{l}$ has at least $I\left(N_{l}, N_{k}\right)$ components intersecting $N_{0}$, for each $l<k$.

Note that the geometric index $I\left(N_{l}, N_{k}\right)$ is equal to $2^{k-l}$.
Proof. We argue by induction on $k$.

- When $k=0$, it is trivial.
- We suppose that it holds for $N_{k-1}$.

We suppose that the closed curve $\gamma$ is not contractible in $\partial N_{k}$. From Lemma 2.1.1, it is a meridian of $N_{k}$. In addition, the linking number of $\gamma \amalg K_{k-1}$ is zero, where $\overline{K_{k-1}}$ is the core of $N_{k-1}$ (See Remark 1.3.7).

We may assume that $D$ intersects $\partial N_{k-1}$ transversally. The set $D \cap \partial N_{k-1}$ has finitely many components $C:=\left\{\gamma_{i}\right\}_{i \in I}$. Each component $\gamma_{i}$ is an embedded circle and bounds a unique closed disc $D_{i} \subset \operatorname{Int} D$.

Let $\left\{\gamma_{j}\right\}_{j \in I_{0}}$ be the set of maximal circles in $C$ where $I_{0} \subset I$. Each $\gamma_{j}$ is the boundary of the disc $D_{j}$, for $j \in I_{0}$.
Claim: There exist at least two elements in $\left\{\gamma_{j}\right\}_{j \in I_{0}}$, which are meridians of $N_{k-1}$.
By Lemma 1.3 .10 , the maps $\pi_{1}\left(\partial N_{k}\right) \rightarrow \pi_{1}\left(\overline{N_{k} \backslash N_{0}}\right)$ and $\pi_{1}\left(\partial N_{k}\right) \rightarrow \pi_{1}\left(\overline{N_{m} \backslash N_{k}}\right)$ are both injective for any $m>k$. Van-Kampen's Theorem (See Theorem 1.1.4) gives an isomorphism between $\pi_{1}\left(\overline{N_{m} \backslash N_{0}}\right)$ and $\pi_{1}\left(\overline{N_{k} \backslash N_{0}}\right) *_{\pi_{1}\left(\partial N_{k}\right)} \pi_{1}\left(\overline{N_{m} \backslash N_{k}}\right)$. We use Lemma 1.1.5 to see that the map $\pi_{1}\left(\partial N_{k}\right) \rightarrow \pi_{1}\left(\overline{N_{m} \backslash N_{0}}\right)$ is also injective. Therefore, since $\gamma$ is not contractible in $\partial N_{k}$, we can conclude that it is not contractible in $\overline{W h \backslash N_{0}}$.

If $\gamma_{j}$ is homotopically trivial in $\partial N_{k-1}$ for each $j \in I_{0}$, then one finds a disc $D_{j}^{\prime} \subset$ $\partial N_{k-1}$. Consider a new disc $D^{\prime}:=\left(\Sigma \backslash \cup_{j \in I_{0}} D_{j}\right) \cup\left(\cup_{j \in I_{0}} D_{j}^{\prime}\right)$ in $\overline{W h \backslash N_{0}}$ with boundary $\gamma$. Therefore, $\gamma$ is contractible in $\overline{W h \backslash N_{0}}$. This contradicts the last paragraph. We see that one of $\left\{\gamma_{j}\right\}_{j \in I_{0}}$ is non-contractible in $\partial N_{k-1}$. Hence, by Lemma 2.1.1, there is at least one meridian of $N_{k-1}$ in $\left\{\gamma_{j}\right\}_{j \in I_{0}}$.

In the following, we argue by contradiction.
Suppose that there is a unique meridian of $N_{k-1}$ in the set $\left\{\gamma_{j}\right\}_{j \in I_{0}}$. That is to say, there is a unique $j_{0} \in I_{0}$ such that $\gamma_{j_{0}}$ is a meridian of $N_{k-1}$. Remark that each $\gamma_{j}$ bounds a unique disc $D_{j} \subset D$.

If $\gamma_{j}$ is not contractible $\partial N_{k}$ for some $j \in I_{0} \backslash\left\{j_{0}\right\}$, Lemma 2.1.1 shows that it is a meridian, which contradicts the uniqueness of $j_{0}$. We see that $\gamma_{j}$ is nullhomotopic in $\partial N_{k-1}$, for each $j \in I_{0} \backslash\left\{j_{0}\right\}$.

Consider a meridian disc $\hat{D}_{j_{0}}$ of $N_{k-1}$ with boundary $\gamma_{j_{0}}$, which intersects the core $K_{k-1}$ of $N_{k-1}$ transversally at one point. For $j \in I_{0} \backslash\left\{j_{0}\right\}$, there exists a disc $\hat{D}_{j} \subset \partial N_{k-1}$ with boundary $\gamma_{j}$.

Define a new disc $\hat{D}:=\left(D \backslash \cup_{j \in I_{0}} D_{j}\right) \cup_{j \in I_{0}}\left(\cup_{\gamma_{j}} \hat{D}_{j}\right)$ with boundary $\gamma$. It intersects $K_{k-1}$ transversally at one point, which implies that the intersection number of $\hat{D}$ and $K_{k-1}$ is $\pm 1$.

Therefore, the linking number of $\gamma \amalg K_{k-1}$ is $\pm 1$. This is in contradiction with the fact that its linking number is zero.

This completes the proof of the claim.
From the above claim, there are at least two distinct meridians, $\gamma_{j_{0}}$ and $\gamma_{j_{1}}$, of $N_{k-1}$ in $\left\{\gamma_{j}\right\}_{j \in I_{0}}$. Applying our inductive assumption to $D_{j_{0}}$ and $D_{j_{1}}$ respectively, we know that
$D_{j_{t}} \cap$ Int $N_{l}$ has at least $2^{k-1-l}$ components intersecting $N_{0}$ for $t=0,1$ for $l \leqslant k-1$. Therefore, $D \cap$ Int $N_{l}$ has at least $2^{k-l}$ components intersecting $N_{0}$.

Based on Theorem 2.1.2, we introduce a topological property.
Definition 2.1.3. A contractible genus one 3 -manifold $M$ is called to satisfy Property $P$ if for any properly embedded plane $\Sigma \subset M$, any $k>0$ and any closed curve $\gamma \subset \partial N_{k} \cap \Sigma$, it holds one of the following:
(1) $\gamma$ is contractible in $\partial N_{k}$;
(2) for $l<k, D \cap \operatorname{Int} N_{l}$ has at least $I\left(N_{l}, N_{k}\right)$ components intersecting $N_{0}$,
where $D \subset \Sigma$ is a unique disc with boundary $\gamma$ and $\left\{N_{k}\right\}_{k}$ is a sequence as described in Theorem 1.3.13,

We will show that all contractible genus one 3-manifolds satisfy Property $P$ (Theorem 2.1.6).
2.1.2. The Genus one case. In this part, we show that any contractible genus one 3 -manifold satisfies Property $P$.

First, recall some notations from Chapter 1.3.4. Any contractible genus one 3-manifold $M$ is the ascending union of closed solid tori $\left\{N_{k}\right\}_{k=0}^{\infty}$ so that $N_{k}$ is homotopically trivial in $N_{k+1}$ and the geometric index $I\left(N_{k}, N_{k+1}\right) \geqslant 2$ (See Theorem 1.3.13).

In the genus one case, Lemma 2.1.1 can be generalized as follows:
Lemma 2.1.4. A circle $\gamma \subset \Sigma \cap \partial N_{k}$, which is not contractible in $\partial N_{k}$, is a meridian of $N_{k}$, where $\Sigma \subset M$ is a properly embedded plane. Moreover, the unique disc $D \subset \Sigma$ with boundary $\gamma$ intersects the core $K_{0}$ of $N_{0}$.

Proof. We may assume that $\Sigma$ intersects $\partial N_{k}$ transversally. Since $\Sigma$ is properly embedded, $\Sigma \cap \partial N_{k}:=\left\{\gamma_{i}\right\}_{i=0}^{n}$ has finitely many components, where $\gamma_{0}=\gamma$. Each $\gamma_{i}$ bounds a unique closed disc $D_{i} \subset \Sigma$ (where $D_{0}=D$ ).

Define the set $C:=\left\{\gamma_{i} \mid \gamma_{i} \subset D_{0}\right.$ is not contractible in $\left.\partial N_{k}\right\}$. It is not empty $\left(\gamma_{0} \subset D_{0}\right)$.
Since $\left\{\gamma_{i}\right\}_{i=0}^{n}$ is a family of disjoint circles, we see that the intersection number of $\gamma$ and $\gamma_{i}$ in $\partial N_{k}$ is zero for each $i \neq 0$.

If $\left[\gamma_{i}\right]$ is not equal to $\pm[\gamma]$ in $\pi_{1}\left(\partial N_{k}\right)$ for some $\gamma_{i} \in C$, the intersection number of $\gamma$ and $\gamma_{i}$ is nonzero in $\partial N_{k}$. This contradicts the above fact. We can conclude that each $\gamma_{i} \in C$ is homotopic to $\gamma$ in $\partial N_{k}$, up to orientation.

In the following, we will show that each minimal circle $\gamma_{j}$ in $C$ is a meridian. This is to say, $\gamma$ is also a meridian of $N_{k}$.

The remaining proof is similar to the proof of Lemma 1.2.4. It is sufficient to show that $\gamma_{j}$ is homotopically trivial in $N_{j}$. We begin by constructing an immersed disc $\hat{D}_{j} \subset N_{k}$ with boundary $\gamma_{j}$.

Let us consider the set $C_{j}:=\left\{\gamma_{i} \mid \gamma_{i} \subset \operatorname{Int} D_{j}\right\} \subset C$ and the set $C_{j}^{\text {max }}$ of maximal circles in $C_{j}$. One has two cases: $C_{j}=\varnothing$ or $C_{j} \neq \varnothing$.

Case I: If $C_{j}$ is empty, we consider the set $Z:=\operatorname{Int} D_{j}$ and define the disc $\hat{D}_{j}$ as Int $D_{j}$;

Case II: If $C_{j}$ is not empty, then $C_{j}^{\max }$ is also non-empty. From the minimality of $\gamma_{j}$, each $\gamma_{i} \in C_{j}^{\max }$ is contractible in $\partial N_{k}$ and bounds a disc $D_{i}^{\prime} \subset \partial N_{k}$.

Define the set $Z:=\operatorname{Int} D_{j} \backslash \cup_{\gamma_{i} \in C_{j}^{\max }} D_{i}$ and the disc $\hat{D}_{j}:=Z \cup\left(\cup_{\gamma_{i} \in C_{j}^{\max }} D_{i}^{\prime}\right)$ with boundary $\gamma_{j}$.

Let us explain why $\hat{D}_{j}$ is contained in $N_{k}$. In any case, $\partial N_{k}$ cuts $M$ into two connected components, $M \backslash N_{k}$ and Int $N_{k}$. The set $Z$ is one of these components of Int $D_{j} \backslash \partial N_{k}$. Therefore, it is in $M \backslash N_{k}$ or Int $N_{k}$.

If $Z$ is in $M \backslash N_{k}$, then the disc $\hat{D}_{j}$ with boundary $\gamma_{j}$ is contained in $\overline{M \backslash N_{k}}$. Therefore, we see that $\left[\gamma_{j}\right]=0$ in $\pi_{1}\left(\overline{M \backslash N_{k}}\right)$. However, the map $\pi_{1}\left(\partial N_{k}\right) \rightarrow \pi_{1}\left(\overline{M \backslash N_{k}}\right)$ is injective (Lemma 1.3.10). That is to say, $\gamma_{j}$ is null-homotopic in $\partial N_{k}$. This contradicts the fact that $\left[\gamma_{j}\right] \neq 0$ in $\pi_{1}\left(\partial N_{k}\right)$. We can conclude that $Z$ is contained in Int $N_{k}$.

Therefore, $\hat{D}_{j}$ is contained in $N_{k}$. Its boundary $\gamma_{j}$ is nullhomotopic in $N_{k}$. Since $\gamma$ is homotopic to $\gamma_{j}$ in $\partial N_{k}$, it is also contractible in $N_{k}$. By Definition 1.2.1, $\gamma$ must be a meridian of $N_{k}$.

By Lemma 1.3.10, the two induced maps $\pi_{1}\left(\partial N_{k}\right) \rightarrow \pi_{1}\left(\overline{M \backslash N_{k}}\right)$ and $\pi_{1}\left(\partial N_{k}\right) \rightarrow$ $\pi_{1}\left(N_{k} \backslash K_{0}\right)$ are both injective. Van-Kampen's theorem (See Theorem 1.1.4) shows that $\pi_{1}\left(M \backslash K_{0}\right) \cong \pi_{1}\left(\overline{M \backslash N_{k}}\right) *_{\pi_{1}\left(\partial N_{k}\right)} \pi_{1}\left(N_{k} \backslash K_{0}\right)$. We see from Lemma 1.1.5 that the map $\pi_{1}\left(\partial N_{k}\right) \rightarrow \pi_{1}\left(M \backslash K_{0}\right)$ is also injective. Therefore, $[\gamma] \neq 0$ in $\pi_{1}\left(M \backslash K_{0}\right)$. We can conclude that the disc $D \subset \Sigma$ with boundary $\gamma$ must intersect the core $K_{0}$ of $N_{0}$.

REmark 2.1.5.

- In the proof, the set $\hat{D}_{j} \cap \operatorname{Int} N_{k}$ is equal to the set $Z$ and a subset of $D \cap \operatorname{Int} N_{k}$.
- The disc $\hat{D}_{j}$ may not be embedded, because $D_{i}^{\prime}$ may be contained in some $D_{i^{\prime}}^{\prime}$. When it is not an embedding, we can deform $\hat{D}_{j}$ in a small neighborhood of $\partial N_{k}$ in $N_{k}$ so that it becomes an embedded disc in $N_{k}$.

Theorem 2.1.6. Any contractible genus one 3 -manifold $M$ satisfies Property $P$.
Proof. Consider a properly embedded plane $\Sigma \subset M$. Suppose there is some closed curve $\gamma \subset \Sigma \cap \partial N_{k}$ which is not contractible in $\partial N_{k}$ for some $k \in \mathbb{N}_{>0}$. By Lemma 2.1.4, $\gamma$ is a meridian of $N_{k}$ and the unique closed disc $D \subset \Sigma$ with boundary $\gamma$ intersects $N_{0}$.

We may assume that $\Sigma$ intersects $\partial N_{k}$ transversally. The set $D \cap \partial N_{k}:=\left\{\gamma_{i}\right\}_{i=0}^{n}$ has finitely many components where $\gamma_{0}=\gamma$.

Define the set $C:=\left\{\gamma_{i} \mid\right.$ the circle $\gamma_{i} \subset D \cap \partial N_{k}$ is not contractible in $\left.\partial N_{k}\right\}$. (It is not empty because $\gamma \subset D$ ). We use Lemma 2.1.4 to see that each minimal circle $\gamma_{j}$ in $C$ is a meridian of $N_{k}$. It bounds a unique closed disc $D_{j} \subset D$. As in the proof of Lemma 2.1.4, we construct a disc $\hat{D}_{j} \subset N_{k}$ with boundary $\gamma_{j}$. Remark that $\hat{D}_{j} \cap \operatorname{Int} N_{k}$ is a subset of $D \cap \operatorname{Int} N_{k}$ (See the above Remark).

As described in the above remark, the disc $\hat{D}_{j}$ may be not embedded. If necessary, we can deform it in a small neighborhood of $\partial N_{k}$ in $N_{k}$ so that it becomes an embedded disc. For $l<k, \hat{D}_{j} \cap \operatorname{Int} N_{l}$ is still a subset of $D \cap \operatorname{Int} N_{l} \subset \Sigma$.

It is sufficient to show that $\hat{D}_{j} \cap \operatorname{Int} N_{l}$ has at least $I\left(N_{l}, N_{k}\right)$ components intersecting $N_{0}$.

We may assume that $\hat{D}_{j}$ intersects $\partial N_{l}$ transversally. The intersection $\hat{D}_{j} \cap \partial N_{l}:=$ $\left\{\gamma_{t}^{\prime}\right\}_{t \in T}$ has finitely many components. Let us consider the set $\hat{C}^{\text {max }}$ of maximal circles in $\left\{\gamma_{t}^{\prime}\right\}_{t \in T}$ and its subset $\hat{C}^{\text {non }}:=\left\{\gamma_{t}^{\prime} \in \hat{C}^{\text {max }} \mid \gamma_{t}^{\prime}\right.$ is not contractible in $\left.\partial N_{l}\right\}$.

Claim: $\left|\hat{C}^{\text {non }}\right| \geqslant I\left(N_{l}, N_{k}\right)$.
We argue by contradiction. Suppose that $\left|\hat{C}^{\text {non }}\right|<I\left(N_{i}, N_{k}\right)$. Each $\gamma_{t}^{\prime} \in \hat{C}^{m a x}$ bounds a unique disc $D_{t}^{\prime} \subset \hat{D}_{j}$.

If $\gamma_{t}^{\prime}$ is in $\hat{C}^{\text {non }}$, it is a meridian of $N_{l}$ (See Lemma 2.1.4). Therefore, we can find a meridian disc $D_{t}^{\prime \prime}$ of $N_{l}$ which intersects the core $K_{l}$ of $N_{l}$ transversally at one point. If $\gamma_{t}^{\prime} \in \hat{C}^{\max } \backslash \hat{C}^{\text {non }}, \gamma_{t}^{\prime}$ is contractible in $\partial N_{l}$ and bounds a disc $D_{t}^{\prime \prime}$ in $\partial N_{l}$.

Define a disc $\hat{D}_{j}^{\prime}$ with boundary $\gamma_{j}$

$$
\hat{D}_{j}^{\prime}:=\left(\hat{D}_{j} \backslash \cup_{\gamma_{t}^{\prime} \hat{C}} \hat{C}_{\text {max }} D_{t}^{\prime}\right) \cup\left(\cup_{\gamma_{t}^{\prime} \in \hat{C}^{\text {max }}} D_{t}^{\prime \prime}\right)
$$

The number $\#\left(\hat{D}_{j}^{\prime} \cap K_{l}\right)$ of points of $\hat{D}_{j}^{\prime} \cap K_{l}$ is less than $I\left(N_{l}, N_{j}\right)$.
As above, the disc $\hat{D}_{j}^{\prime}$ may be not embedded (because $D_{t}^{\prime \prime}$ may be contained in some $\left.D_{t^{\prime}}^{\prime \prime}\right)$. If necessary, we modify the disc $\hat{D}_{j}^{\prime}$ in a small neighborhood of $\partial N_{l}$ so that it becomes an embedded disc in $N_{k}$.

Therefore, we may assume that $\left(\hat{D}_{j}^{\prime}, \partial \hat{D}_{j}^{\prime}\right) \subset\left(\bar{N}_{k}, \partial N_{k}\right)$ is an embedded disc with boundary $\gamma_{j}$. Since $\gamma_{j}$ is a meridian of $N_{k}$ (See Lemma 2.1.4), $\hat{D}_{j}^{\prime}$ is a meridian disc of $N_{k}$ with $\#\left(\hat{D}_{j}^{\prime} \cap K_{l}\right)<I\left(N_{l}, N_{k}\right)$. However, the definition of the geometric index (See Definition 1.3.8 gives that $\#\left(\hat{D}_{j}^{\prime} \cap K_{l}\right) \geqslant I\left(N_{l}, N_{k}\right)$, a contradiction. This finishes the proof of the claim.

In the following, we will finish the proof of the theorem.
Let $\left\{\gamma_{s}^{\prime}\right\}_{s=1}^{m}$ be the circles in $\hat{C}^{\text {non }}$ and $D_{s}^{\prime} \subset \hat{D}_{j}$ the unique disc with boundary $\gamma_{s}^{\prime}$, where $m=\left|\hat{C}_{j}^{n o n}\right|$. From the maximality of $\gamma_{s}^{\prime}$ in $\left\{\gamma_{t}^{\prime}\right\}_{t \in T},\left\{D_{s}^{\prime}\right\}_{s=1}^{m}$ is a family of pairwise disjoint discs in $\hat{D}_{j}$.

We use Lemma 2.1.4 to see that each $\gamma_{s}^{\prime} \in \hat{C}^{n o n}$ is a meridian. Thus, $D_{s}^{\prime}$ intersects the core $K_{0}$ of $N_{0}$. The intersection $D_{s}^{\prime} \cap$ Int $N_{l}$ contains at least one component intersecting $N_{0}$.

We conclude that $\hat{D}_{j} \cap \operatorname{Int} N_{l}$ has at least $m$ components intersecting $N_{0}$. From the above claim, we know that $m \geqslant I\left(N_{l}, N_{k}\right)$. Therefore, $D \cap \operatorname{Int} N_{l}$ has at least $I\left(N_{l}, N_{k}\right)$ components intersecting $N_{0}$.

## Remark 2.1.7.

- The proof of Lemma 2.1.4 and Theorem 2.1.6 just depend on the injectivity of the two maps $\pi_{1}\left(\partial N_{k}\right) \rightarrow \pi_{1}\left(\overline{N_{k} \backslash N_{0}}\right)$ and $\pi_{1}\left(\partial N_{k}\right) \rightarrow \pi_{1}\left(\overline{M \backslash N_{k}}\right)$.
- Let $\left\{R_{k}\right\}_{k}$ be an increasing family of solid tori in a contractible 3-manifold with the property that
(1) $\pi_{1}\left(\partial R_{k}\right) \rightarrow \pi_{1}\left(\overline{R_{k} \backslash R_{0}}\right)$ is injective ;
(2) $\pi_{1}\left(\partial R_{k}\right) \rightarrow \pi_{1}\left(\overline{M \backslash R_{k}}\right)$ is injective .

However, the union $\cup_{k} R_{k}$ may not be equal to $M$.
From the above fact, Lemma 2.1.4 holds for each $R_{k}$. Further, as in the proof of Theorem 2.1.6, the family $\left\{R_{k}\right\}$ satisfies Property P. That is to say, for any
properly embedded plane $\Sigma \subset M$, any $k>0$ and any closed curve $\gamma \subset \partial R_{k} \cap \Sigma$, it holds one of the following:
(1) $\gamma$ is contractible in $\partial R_{k}$;
(2) for $l<k, D \cap$ Int $R_{l}$ has at least $I\left(R_{l}, R_{k}\right)$ components intersecting $R_{0}$, where $D \subset \Sigma$ is a unique disc with boundary $\gamma$.

### 2.2. Property $H$

2.2.1. Surgeries. Consider two closed handlebodies $N^{\prime}$ and $N$ in a 3-manifold $M$ with $N^{\prime} \subset \operatorname{Int} N$. We introduce two types of surgeries on handlebodies:
Type I: If there exists a meridian disc $D \subset N \backslash N^{\prime}$ of $N$, then we consider an open tubular neighborhood $N_{\epsilon}(D) \subset N \backslash N^{\prime}$ of $D$. We then have two cases:

Case (1): If $D$ is a splitting meridian disc, $N \backslash N_{\epsilon}(D)$ has two components. The closed handlebody $W_{1}$ is defined as the component containing $N^{\prime}$;

Case(2): If $D$ is a non-splitting meridian disc, $N \backslash N_{\epsilon}(D)$ is connected. The closed handlebody $W_{1}$ is defined by $N \backslash N_{\epsilon}(D)$.
Type II: If there exists an embedded disc $D_{1} \subset \overline{M \backslash N}$ satisfying that 1) Int $D_{1} \subset M \backslash N$ and 2) its boundary $\gamma \subset \partial N$ is not contractible in $\partial N$, we consider a closed tubular neighborhood $\overline{N_{\epsilon_{1}}\left(D_{1}\right)}$ of $D_{1}$ in $\overline{M \backslash N}$. Define a new handlebody $W_{2}$ as $N \cup \overline{N_{\epsilon_{1}}\left(D_{1}\right)}$.

Remark 2.2.1. For $i=1,2$, the genus $g\left(\partial W_{i}\right)$ of $\partial W_{i}$ is less than $g(\partial N)$. In addition, $\partial W_{i}$ is a union of $\partial W_{i} \cap \partial N$ and some disjoint discs. It tells us that the map $\pi_{1}\left(\partial W_{i} \cap\right.$ $\partial N) \rightarrow \pi_{1}\left(\partial W_{i}\right)$ is surjective.

Lemma 2.2.2. If $N^{\prime}$ is homotopically trivial in $N$, then $N^{\prime}$ is also homotopically trivial in $W_{i}$ for each $i$, where $W_{i}$ is obtained from the above surgeries.

Proof. For the type II surgery, we see that $N$ is contained in $W_{2}$. Therefore, $N^{\prime}$ is homotopically trivial in $W_{2}$.

For the type I surgery, it is sufficient to show that any circle $c \subset N^{\prime}$ bounds some disc $\hat{D}^{\prime} \subset W_{1}$.

The closed curve $c$ bounds an immersed disc $D^{\prime} \subset \operatorname{Int} N$. We will construct the required disc $\hat{D}^{\prime} \subset W_{1}$ from $D^{\prime}$.

We may assume that $D^{\prime}$ intersects $D^{-} \amalg D^{+}:=\operatorname{Int} N \cap \partial N_{\epsilon}\left(D^{\prime}\right)$ transversally. Each component $c_{i}$ of $D^{\prime} \cap\left(D^{+} \amalg D^{-}\right)$is a circle in $D^{\prime}$ and bounds a closed sub-disc $D_{i}^{\prime} \subset D^{\prime}$.

Since $D^{+}$and $D^{-}$are two disjoint discs, each $c_{i}$ is contractible in $D^{+} \amalg D^{-}$. It also bounds a disc $D_{i}^{\prime \prime} \subset D^{+} \amalg D^{-}$. Let $C^{\max }$ be the set of the maximal circles of $\left\{c_{i}\right\}_{i \in I}$ in $D^{\prime}$. We construct a disc

$$
\hat{D}^{\prime}:=D^{\prime} \backslash \cup_{c_{i} \in C^{\max }} D_{i}^{\prime} \cup\left(\cup_{c_{i} \in C^{\max }} D_{i}^{\prime \prime}\right)
$$

with boundary $c$. It stays in $\overline{N \backslash N_{\epsilon}\left(D^{\prime}\right)}$. That is to say, $c$ is contractible in $W_{1}$. Therefore, $N^{\prime}$ is homotopically trivial in $W_{1}$.
2.2.2. Effective Handlebodies. In the following, let us consider a contractible 3manifold $M$.

Theorem 2.2.3. Let $N^{\prime}$ and $N$ be two closed handlebodies in $M$ satisfying that 1) $N^{\prime} \subset$ Int $N$ and 2) $N^{\prime}$ is homotopically trivial in $N$. Then there exists a closed handlebody $R \subset M$ containing $N^{\prime}$ satisfying that
(1) the map $\pi_{1}(\partial R) \rightarrow \pi_{1}\left(\overline{R \backslash N^{\prime}}\right)$ is injective;
(2) the map $\pi_{1}(\partial R) \rightarrow \pi_{1}(\overline{M \backslash R})$ is injective;
(3) $N^{\prime}$ is homotopically trivial in $R$;
(4) $\partial R$ is a union of $\partial R \cap \partial N$ and some disjoint discs.

Remark. From (1), $R$ is an effective handlebody relative to $N^{\prime}$ (Lemma 1.2.4).
Proof. Suppose that either the map $i_{1}: \pi_{1}(\partial N) \rightarrow \pi_{1}\left(\overline{N \backslash N^{\prime}}\right)$ is not injective or the $\operatorname{map} i_{2}: \pi_{1}(\partial N) \rightarrow \pi_{1}(\overline{M \backslash N})$ is not injective. (If these two maps are both injective, $R$ is defined as $N$.)

If $i_{1}$ is not injective, Lemma 1.1 .2 shows that there exists a meridian disc $D_{1}$ of $N$ with $D_{1} \cap N^{\prime}=\varnothing$. We do the type I surgery on $N$ with the disc $D_{1}$ to obtain a new handlebody $W$.

If $i_{2}$ is not injective, we use Lemma 1.1 .2 to find an embedded circle $\gamma \subset \partial N$ and an embedded disc $D_{2} \subset \overline{M \backslash N}$ (Int $D_{2} \subset M \backslash N$ ) where $\gamma=\partial D_{2}$ is not nullhomotopic in $\partial N$. We do the type II surgery with the disc $D_{2}$ to get a new handlebody $W$.

In any case, we have that $g(\partial W)<g(\partial N)$. The boundary $\partial W$ is a union of $\partial W \cap \partial N$ and some disjoint discs $\left\{D_{i}^{\prime}\right\}_{i}$. Therefore, $\pi_{1}(\partial W \cap \partial N) \rightarrow \pi_{1}(\partial W)$ is surjective. In addition, we see from Lemma 2.2 .2 that $N^{\prime}$ is contractible in $W$.

When picking a circle $\gamma \subset \partial W$ which is not nullhomotopic in $\partial W$, we may assume that $\gamma$ is an embedded circle in $\partial W \cap \partial N$. Therefore, when repeating these two types of surgeries, we may assume that the new surgeries are operated away from these disjoint $\operatorname{discs}\left\{D_{i}^{\prime}\right\}$.

Iterate this process until we find a handlebody $R$ satisfying (1) and (2). At each step, the genus of the handlebody obtained from the surgery is less than the original one. Therefore, this process stops in no more than $g(N)$ steps.

As above, $N^{\prime}$ is contractible in $R$ and $\partial R$ is a union of $\partial R \cap \partial N$ and some disjoint discs.

Remark. If $N^{\prime}$ is not contained in a 3 -ball in $M$, then the genus of $R$ is greater than zero.

Lemma 2.2.4. Let $R \subset M$ be a closed effective handlebody relative to the closed handlebody $N^{\prime} \subset$ Int $R$ satisfying that $\pi_{1}(\partial R) \rightarrow \pi_{1}(\overline{M \backslash R})$ is injective. If a closed handlebody $N$ is an effective handlebody relative to $R \subset$ Int $N$, then $N$ is an effective handlebody relative to $N^{\prime}$.

Proof. Based on Lemma 1.2 .4 , it is sufficient to show that the map $\pi_{1}(\partial N) \rightarrow$ $\pi_{1}\left(\overline{N \backslash N^{\prime}}\right)$ is injective.

We use Lemma 1.2 .4 to show that the induced map $\pi_{1}(\partial R) \rightarrow \pi_{1}\left(\overline{R \backslash N^{\prime}}\right)$ is injective. Since $\pi_{1}(\partial R) \rightarrow \pi_{1}(\overline{M \backslash R})$ is injective, then the map $\pi_{1}(\partial R) \rightarrow \pi_{1}(\overline{N \backslash R})$ is also injective.

Van Kampen's theorem (Theorem 1.1.4) gives an isomorphism between $\pi_{1}\left(\overline{N \backslash N^{\prime}}\right)$ and $\pi_{1}(\overline{N \backslash R}) *_{\pi_{1}(\partial R)} \pi_{1}\left(\overline{R \backslash N^{\prime}}\right)$. A classical result (See [Theorem 11.67, Page 404] of $\operatorname{Rot12}$ or Lemma 1.1.5 shows that the induced map $\pi_{1}(\overline{N \backslash R}) \rightarrow \pi_{1}\left(\overline{N \backslash N^{\prime}}\right)$ is injective.

Lemma 1.2 .4 shows that the map $\pi_{1}(\partial N) \rightarrow \pi_{1}(\overline{N \backslash R})$ is injective. Therefore, the composition $\pi_{1}(\partial N) \rightarrow \pi_{1}(\overline{N \backslash R}) \rightarrow \pi_{1}\left(\overline{N \backslash N^{\prime}}\right)$ is also injective. This finishes the proof.
2.2.3. Property $H$. In the following, let us consider a contractible 3-manifold $M$ which is not homeomorphic to $\mathbb{R}^{3}$.

By Theorem 1.1.12, $M$ can be written as an ascending union of handlebodies $\left\{N_{k}\right\}_{k=0}^{\infty}$. Each $N_{k}$ is contractible in $N_{k+1}$. As in Remark 1.1.8, we can choose $N_{0}$ so that it is not contained in a 3 -ball in $M$ (because $M$ is not homeomorphic to $\mathbb{R}^{3}$ ).

In the genus one case, the family $\left\{N_{k}\right\}$ has several good properties. For example, each $N_{k}$ is an effective handlebody relative to $N_{0}$ and the map $\pi_{1}\left(\partial N_{k}\right) \rightarrow \pi_{1}\left(\overline{M \backslash N_{k}}\right)$ is injective (See Lemma 1.3.10 or Lemma 2.10 of Wan19a). These properties are necessary and crucial in our proof. In general, the family $\left\{N_{k}\right\}$ may not have these properties. To overcome this difficulty, we introduce a topological property, called Property H.

Definition 2.2.5. A family $\left\{R_{k}\right\}_{k}$ of handlebodies in a contractible 3-manifold $M:=$ $\cup_{k} N_{k}$ is called to have Property $H$ if it satisfies that
(1) the map $\pi_{1}\left(\partial R_{k}\right) \rightarrow \pi_{1}\left(\overline{R_{k} \backslash R_{0}}\right)$ is injective for $k>0$;
(2) the map $\pi_{1}\left(\partial R_{k}\right) \rightarrow \pi_{1}\left(\overline{M \backslash R_{k}}\right)$ is injective for $k \geqslant 0$;
(3) each $R_{k}$ is contractible in $R_{k+1}$ but not contained in a 3 -ball in $M$;
(4) there exists a sequence of increasing integers $\left\{j_{k}\right\}_{k}$, such that $\pi_{1}\left(\partial R_{k} \cap \partial N_{j_{k}}\right) \rightarrow$ $\pi_{1}\left(\partial R_{k}\right)$ is surjective.
where $\left\{N_{k}\right\}$ is assumed as in Remark 1.1.13.
For example, in a contractible genus one 3 -manifold $M:=\cup_{k} N_{k}$, the family $\left\{N_{k}\right\}_{k}$ satisfies Property H, where $\left\{N_{k}\right\}$ is assumed in Chapter 1.3.4 (See Lemma 1.3.10).

In the following, we will prove if a contractible 3 -manifold $M$ is not homeomorphic to $\mathbb{R}^{3}$, there is a family of handlebodies with Property H (See Theorem 2.2.6). However, such a family is not unique.

THEOREM 2.2.6. If a contractible 3-manifold $M:=\cup_{k} N_{k}$ (as above) is not homeomorphic to $\mathbb{R}^{3}$, then there is an ascending family $\left\{R_{k}\right\}_{k}$ of closed handlebodies in $M$ satisfying that
(1) the map $\pi_{1}\left(\partial R_{k}\right) \rightarrow \pi_{1}\left(\overline{R_{k} \backslash R_{0}}\right)$ is injective for $k>0$;
(2) the map $\pi_{1}\left(\partial R_{k}\right) \rightarrow \pi_{1}\left(\overline{M \backslash R_{k}}\right)$ is injective for $k \geqslant 0$;
(3) each $R_{k}$ is contractible in $R_{k+1}$ but not contained in a 3-ball in $M$;
(4) there exists a sequence of increasing integers $\left\{j_{k}\right\}_{k}$, such that $\pi_{1}\left(\partial R_{k} \cap \partial N_{j_{k}}\right) \rightarrow$ $\pi_{1}\left(\partial R_{k}\right)$ is surjective.

REmark 2.2.7.

- The union $\cup_{k} R_{k}$ may be not equal to $M$.
- For $k>0$, Van-Kampen's Theorem gives an isomorphism between $\pi_{1}\left(\overline{M \backslash R_{0}}\right)$ and $\pi_{1}\left(\overline{M \backslash R_{k}}\right) *_{\pi_{1}\left(\partial R_{k}\right)} \pi_{1}\left(\overline{R_{k} \backslash R_{0}}\right)$. Based on (1) and (2) in Theorem 2.2.6, we use Lemma 1.1.5 to show that the map $\pi_{1}\left(\partial R_{k}\right) \rightarrow \pi_{1}\left(\overline{M \backslash R_{0}}\right)$ is injective.
- As (4) in Theorem 2.2.3, $\partial R_{k}$ is the union of $\partial R_{k} \cap \partial N_{j_{k}}$ and disjoint discs.

Proof. First, we construct $R_{0}$. We repeatedly apply the Type II surgery to $N_{0}$, until we find a handlebody $R_{0}$ containing $N_{0}$ so that $\pi_{1}\left(\partial R_{0}\right) \rightarrow \pi_{1}\left(\overline{M \backslash R_{0}}\right)$ is injective.

From Remark 2.2.1, we see that, at each step, the genus of the handlebody obtained from the surgery is less than the original one. Therefore, this process stops in no more than $g\left(N_{0}\right)$ steps.

In addition, since $N_{0}$ is not contained in a 3 -ball in $M$, then $R_{0}$ has the same property.
It remains to construct the sequence $\left\{R_{k}\right\}_{k}$ inductively.
When $k$ is equal to 1 , we pick a handlebody $N_{j_{1}}$ containing $R_{0}$ satisfying that $R_{0}$ is homotopically trivial in $N_{j_{1}}$. Its existence is ensured by the following fact:

Because $R_{0}$ is compact, there is some handlebody $N_{j_{1}-1}$ containing $R_{0}$. Since $N_{j_{1}-1}$ is homotopically trivial in $N_{j_{1}}, R_{0}$ is contained in $N_{j_{1}}$ and contractible in $N_{j_{1}}$.

By Theorem 2.2.3, there exists a handlebody $R_{1}$ containing $R_{0}$ so that

- $\pi_{1}\left(\partial R_{1}\right) \rightarrow \pi_{1}\left(\overline{R_{1} \backslash R_{0}}\right)$ is injective;
- $\pi_{1}\left(\partial R_{1}\right) \rightarrow \pi_{1}\left(\overline{M \backslash R_{1}}\right)$ is injective;
- $R_{0}$ is contractible in $R_{1}$;
- $\partial R_{1}$ is a union of $\partial R_{1} \cap \partial N_{j_{1}}$ and some disjoint closed discs. Therefore, $\pi_{1}\left(\partial R_{1} \cap\right.$ $\left.\partial N_{j_{1}}\right) \rightarrow \pi_{1}\left(\partial R_{1}\right)$ is surjective.
In particular, since $R_{0}$ is not contained in a 3 -ball in $M, R_{1}$ has the same property.
Suppose that there exists a handlebody $R_{k-1}$ and a positive integer $j_{k-1}$ satisfying (1), (2), (3) and (4) in Theorem 2.2.6.

As the existence of $N_{j_{1}}$, there exists a handlebody $N_{j_{k}}$ containing $R_{k-1}$ satisfying that $R_{k-1}$ is homotopically trivial in $N_{j_{k}}$. We use Theorem 2.2 .3 to find an effective handlebody $R_{k}$ relative to $R_{k-1}$ satisfying (2), (3) and (4).

Since the map $\pi_{1}\left(\partial R_{k-1}\right) \rightarrow \pi_{1}\left(\overline{R_{k-1} \backslash R_{0}}\right)$ is injective, $R_{k-1}$ is an effective handlebody relative to $R_{0}$ (Lemma 1.2.4). Lemma 2.2.4 shows that $R_{k}$ is an effective handlebody relative to $R_{0}$. We apply Lemma 1.2 .4 again and get that $R_{k}$ also satisfies (1). This finishes the proof.

## Part 2

## Minimal Surfaces

## CHAPTER 3

## Minimal Surfaces

A minimal surface is a submanifold in a Riemannian manifold whose mean curvature is identically zero. It is also the critical point of the area functional (See the first variation formula, Equation (3.1.12)).

In Section 3.1, we first introduce the so-called first and second variation formulas for the area functional. Subsequently, we derive some notations, such as minimal surfaces, Morse index and the stability condition. Finally, we focus on the Plateau Problem and related results.

In Section 3.2, we discuss some local properties of minimal surfaces, including the strong maximal principle (See Corollary 3.2.3) and the monotonicity formula (See Proposition 3.2 .5 ) for the area. Particularly, the monotonicity formula gives a quantitative estimate for the area (See Corollary 3.2.6). This estimate can be generalized to the Riemannian case (See Theorem 3.2.7).

In Section 3.3, we study the topology of stable minimal surfaces. The stable minimal hypersurface is characterized by the first eigenvalue of the stable operator (See Lemma 3.3.1 and Theorem 3.3.4). In a manifold of positive scalar curvature, there are many topological constraints for stable minimals surfaces (See Proposition 3.3.5). For example, in the case of 3 -manifolds with positive scalar curvature, the geometry of stable minimal surfaces is influenced by the extrinsic version of Cohn-Vossen inequality (See Corollary 3.3.6 and Theorem 3.3.10). As an application, we give a new proof of Theorem 2 of [SY82]. Finally, we use Theorem 3.3.8 to study contractible 3-manifolds whose scalar curvature has a decay at infinity.

### 3.1. Background

3.1.1. Mean Curvature. Let us consider a $k$-dimensional submanifold $\Sigma^{k} \subset\left(M^{n}, g\right)$ possibly with boundary.

In the following, if $X$ is a vector field on $\Sigma \subset M$, then we let $X^{T}$ and $X^{N}$ denote the tangential and normal components, respectively. The covariant derivative $\nabla$ on $M$ induces a covariant derivative $\nabla^{\Sigma}$ on $\Sigma$ and the second fundamental form $A$ of $\Sigma$. More precisely, the induced covariant derivative $\nabla^{\Sigma}$ is given by

$$
\begin{equation*}
\nabla^{\Sigma}=(\nabla)^{T} \tag{3.1.1}
\end{equation*}
$$

and the vector-valued bilinear form $A$ on $\Sigma$ is given for $X, Y \in T \Sigma$ by

$$
\begin{equation*}
A(X, Y)=\left(\nabla_{X} Y\right)^{N} \tag{3.1.2}
\end{equation*}
$$

Since the Lie bracket of $X$ and $Y$ is a tangential vector field in $T \Sigma$, it is easy to see that $A$ is symmetric, i.e., $A(X, Y)=A(Y, X)$.

The mean curvature vector $H$ at a point $x \in \Sigma$ is defined

$$
H=\sum_{i=1}^{k} A\left(E_{i}, E_{i}\right),
$$

where $\left\{E_{i}\right\}$ is an orthonormal basis for $T_{x} \Sigma$. Furthermore, the squared norm of the second fundamental form at $x$ is given by

$$
\begin{equation*}
|A|^{2}=\sum_{i, j=1}^{k}\left|A\left(E_{i}, E_{j}\right)\right|^{2} \tag{3.1.3}
\end{equation*}
$$

Recall also that the Gauss equations assert if $X, Y \in T_{x} \Sigma$, then

$$
\begin{align*}
& K_{\Sigma}(X, Y)|X \wedge Y|^{2}=K_{M}(X, Y)|X \wedge Y|^{2} \\
& +g(A(X, X), A(Y, Y))-g(A(X, Y), A(X, Y)) \tag{3.1.4}
\end{align*}
$$

where $|X \wedge Y|^{2}$ is given by

$$
|X \wedge Y|^{2}=g(X, X) g(Y, Y)-g(X, Y)^{2}
$$

and $K_{X}(X, Y)$ and $K_{\Sigma}(X, Y)$ are the sectional curvatures of $M$ and $\Sigma$, respectively, in the 2-plane spanned by $X$ and $Y$.

For example, let $\Sigma^{n-1} \subset M^{n}$ be a hypersurface and $N$ a unit normal vector field in a neighborhood of $x \in \Sigma$, then

$$
\nabla_{(\cdot)} N: T_{x} \Sigma \rightarrow T_{x} \Sigma
$$

is a symmetric map (often referred to as the Weingarten map) and its eigenvalues $\left\{\kappa_{i}\right\}_{i}^{n-1}$ are called the principle curvatures. Moreover,

$$
g(H, N)=-\sum_{i=1}^{n-1} \kappa_{i} .
$$

Finally, if $X$ is a vector field over $\Sigma$, then the divergence of $X$ at $x \in \Sigma$ is defined as

$$
\begin{equation*}
\operatorname{div}_{\Sigma} X=\sum_{i=1}^{n-1} g\left(\nabla_{E_{i}}^{T} X, E_{i}\right) \tag{3.1.5}
\end{equation*}
$$

where $\left\{E_{i}\right\}$ is an orthonormal basis for $T_{x} \Sigma$. Notice that $d i v_{\Sigma}$ satisfies the Leibniz rule

$$
\operatorname{div}_{\Sigma}(f X)=g\left(\nabla_{\Sigma} f, X\right)+f \operatorname{div}_{\Sigma}(X)
$$

We can also use $\operatorname{div}_{\Sigma}$ to define the Laplace operator $\Delta_{\Sigma}$ on $\Sigma$ by

$$
\Delta_{\Sigma} f=\operatorname{div}_{\Sigma}\left(\nabla^{\Sigma} f\right)
$$

A function $f$ is a harmonic function on $\Sigma$ if $\Delta_{\Sigma} f=0$.
Remark 3.1.1. Note that

$$
\begin{align*}
\operatorname{div}_{\Sigma} Y^{N} & =\sum_{i} g\left(E_{i}, \nabla_{E_{i}} Y^{N}\right)=-\sum_{i} g\left(Y^{N}, \nabla_{E_{i}} E_{i}\right)  \tag{3.1.6}\\
& =-g\left(Y^{N}, H\right)
\end{align*}
$$

3.1.2. First Variation Formula. Let $F: \Sigma^{k} \times(-\epsilon, \epsilon) \rightarrow M^{n}$ be a variation of a $k$-dimensional submanifold $\Sigma^{k}$ with compact support and fixed boundary. That is, $F=\mathrm{Id}$ outside a compact set,

$$
F(x, 0)=x
$$

and for all $x \in \partial \Sigma$,

$$
F(x, t)=x .
$$

The vector $F_{t}$ restricted to $\Sigma$ is called the variational vector field. Now we want to compute the first variation of area for this one parameter family of surfaces. Let $\left(x_{i}\right)$ be a local coordinate on $\Sigma$. Set

$$
\begin{align*}
& g_{i j}(t)=g\left(F_{x_{i}}, F_{x_{j}}\right) \\
& \nu(t)=\sqrt{\operatorname{det}\left(g_{i j}(t)\right)} \sqrt{\operatorname{det}\left(g^{i j}(0)\right)} \tag{3.1.7}
\end{align*}
$$

where $\left(a^{i j}\right)$ denotes the inverse of the metric $\left(a_{i j}\right)$ and $1 \leqslant i, j \leqslant k$. Further, the area formula is

$$
\begin{equation*}
\operatorname{Vol}(F(\Sigma, t))=\int_{\Sigma} \nu(t) \sqrt{\operatorname{det}\left(g_{i j}(0)\right)} \tag{3.1.8}
\end{equation*}
$$

Differentiating it gives

$$
\begin{equation*}
\left.\frac{d}{d t} \operatorname{Vol}(F(\Sigma, t))\right|_{t=0}=\left.\int_{\Sigma} \frac{d}{d t} \nu(t)\right|_{t=0} \sqrt{\operatorname{det}\left(g_{i j}(0)\right)} \tag{3.1.9}
\end{equation*}
$$

We may choose an orthonormal coordinate system, i.e. so that at the point $x$

$$
g_{i j}(0)=\delta_{i j}= \begin{cases}0, & i \neq j \\ 1, & i=j\end{cases}
$$

Using that $\left[F_{t}, F_{x_{i}}\right]=0$, under this coordinate, we get at $x$,

$$
\begin{align*}
\left.\frac{d}{d t} \operatorname{det}\left(g_{i j}(0)\right)(t)\right|_{t=0} & =\left.\sum_{i j} \frac{d}{d t} g_{i j}(t)\right|_{t=0} g^{i j}(0)=\left.\sum_{i=1}^{k} \frac{d}{d t}\left(g\left(F_{x_{i}}, F_{x_{i}}\right)\right)\right|_{t=0} \\
& =2 \sum_{i=1}^{k} g\left(\nabla_{F_{t}} F_{x_{i}}, F_{x_{i}}\right)=2 \sum_{i=1}^{k} g\left(\nabla_{F_{x_{i}}} F_{t}, F_{x_{i}}\right)  \tag{3.1.10}\\
& =2 \operatorname{div}_{\Sigma}\left(F_{t}\right)
\end{align*}
$$

Therefore, from Equation (3.1.6) we have

$$
\begin{align*}
\left.\frac{d}{d t} \nu(t)\right|_{t=0} & =\operatorname{div}_{\Sigma}\left(F_{t}\right)=\operatorname{div_{\Sigma }}\left(F_{t}^{T}\right)+\operatorname{div}_{\Sigma}\left(F_{t}^{N}\right)  \tag{3.1.11}\\
& =-g\left(H, F_{t}^{N}\right)+\operatorname{div}_{\Sigma}\left(F_{t}^{T}\right)
\end{align*}
$$

Integrating Equation (3.1.11) gives the so-called first variation formula:

$$
\begin{equation*}
\frac{d}{d t} \operatorname{Vol}(F(\Sigma, t))=-\int_{\Sigma} g\left(F_{t}^{N}, H\right)=\int_{\Sigma} \operatorname{div}_{\Sigma}\left(F_{t}\right) \tag{3.1.12}
\end{equation*}
$$

Note that Stokes' Theorem was used to see that $\int_{\Sigma} \operatorname{div}_{\Sigma}\left(F_{t}^{T}\right)=0$. As a consequence of Equation (3.1.12), we see that $\Sigma$ is a critical point for the area functional if and only if the mean curvature $H$ vanishes identically.

Definition 3.1.2. An immersed submanifold $\Sigma^{k} \subset\left(M^{n}, g\right)$ is said to be minimal if the mean curvature $H$ vanishes identically.

For example, let $\Sigma$ be the graph of a function $u: \mathbb{R}^{n} \rightarrow \mathbb{R}$. The hypersurface $\Sigma$ is minimal in $\mathbb{R}^{n+1}$ if and only if $u$ satisfies

$$
\begin{equation*}
\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^{2}}}\right)=0 . \tag{3.1.13}
\end{equation*}
$$

It is the so-called minimal surface equation. Furthermore, in dimension three $(n=2)$, we know

$$
\begin{equation*}
\frac{|\operatorname{Hess}(u)|^{2}}{\left(1+|\nabla u|^{2}\right)^{3}} \leqslant|A|^{2} \leqslant 2 \frac{|\operatorname{Hess}(u)|^{2}}{1+|\nabla u|^{2}} \tag{3.1.14}
\end{equation*}
$$

See Pages 28 and 29 in CM11.
3.1.3. Second Variation Formula. Suppose now that $\Sigma^{k} \subset\left(M^{n}, g\right)$ is a minimal submanifold. We want to compute the second derivative of the area functional for a variation of $\Sigma$. Therefore, we consider a variation $F$ of $\Sigma$ with compact support. In fact, we assume that $F$ is a normal variation, that is, on $\Sigma$ we have

$$
F_{t}^{T}(\cdot, 0)=0
$$

As before, let $\left(x_{i}\right)_{i}$ be a local coordinate on $\Sigma$ and set

$$
\begin{aligned}
& g_{i j}(t)=g\left(F_{x_{i}}, F_{x_{j}}\right) \\
& \nu(t)=\sqrt{\operatorname{det}\left(g_{i j}(t)\right)} \sqrt{\operatorname{det}\left(g^{i j}(0)\right)}
\end{aligned}
$$

Differentiating the measure $\nu(t)$ gives

$$
\begin{equation*}
\left.\frac{d^{2}}{d t^{2}} \operatorname{Vol}(F(\Sigma, t))\right|_{t=0}=\left.\int_{\Sigma} \frac{d^{2}}{d t^{2}} \nu(t)\right|_{t=0} \sqrt{\operatorname{det}\left(g_{i j}(0)\right)} \tag{3.1.15}
\end{equation*}
$$

Recall that the first derivative of the measure $\nu(t)$ can be written as

$$
\begin{equation*}
2 \frac{d}{d t} \nu(t)=\sum_{i j} g_{i j}^{\prime}(t) g^{i j}(t) \nu(t) \tag{3.1.16}
\end{equation*}
$$

To evaluate $\left.\frac{d^{2}}{d t^{2}} \nu(t)\right|_{t=0}$ at some point $x \in \Sigma$, we may choose an orthonormal coordinate $\left(x_{i}\right)_{i}$ at $x$. Since the metric $\left(g_{i j}\right)$ is the identity at $x$, the vectors $F_{x_{i}}$ give an orthonormal basis for $T \Sigma$ at $x$. Differentiating Equation (3.1.16) then gives at $x$,

$$
\begin{equation*}
\left.2 \frac{d^{2}}{d t^{2}} \nu(t)\right|_{t=0}=\sum_{i} g_{i i}^{\prime \prime}(0)-\sum_{i j} g_{i j}^{\prime}(0)^{2}+1 / 2\left(\sum_{i} g_{i i}^{\prime}(0)\right)^{2} . \tag{3.1.17}
\end{equation*}
$$

We have that $g_{i i}^{\prime}(0)=g\left(\nabla_{F_{t}} F_{x_{i}}, F_{x_{i}}\right)=g\left(\nabla_{F_{x_{i}}} F_{t}, F_{x_{i}}\right)=-g\left(\nabla_{F_{x_{i}}} F_{x_{i}}, F_{t}\right)$. Since $\Sigma$ is minimal and $F_{t}^{T}=0$, we have $\sum_{i} g_{i i}^{\prime}(0)=-g\left(\sum_{i} \nabla_{F_{x_{i}}} F_{x_{i}}, F_{t}\right)=-g\left(F_{t}, H\right)=0$ as in Equation (3.1.6). Therefore, we get

$$
\begin{equation*}
\left.2 \frac{d^{2}}{d t^{2}} \nu(t)\right|_{t=0}=\sum_{i} g_{i i}^{\prime \prime}(0)-\sum_{i j} g_{i j}^{\prime}(0)^{2} \tag{3.1.18}
\end{equation*}
$$

Lemma 3.1.3. At the point $x$, we get

$$
\begin{aligned}
\sum_{i j} g_{i j}^{\prime}(0)^{2}= & 4 \sum_{i j} g\left(A\left(F_{x_{i}}, F_{x_{j}}\right), F_{t}\right)^{2} \\
\sum_{i} g_{i i}^{\prime \prime}(0)= & 2 \sum_{i j} g\left(A\left(F_{x_{i}}, F_{x_{j}}\right), F_{t}\right)^{2}+2\left|\nabla_{\Sigma}^{N} F_{t}\right|^{2} \\
& +2 \sum_{i} g\left(R_{M}\left(F_{x_{i}}, F_{t}\right) F_{t}, F_{x_{i}}\right)+2 \operatorname{div}_{\Sigma}\left(F_{t t}\right)
\end{aligned}
$$

Proof. An easy computation gives that

$$
g_{i j}^{\prime}(0)=g\left(\nabla_{F_{t}} F_{x_{i}}, F_{x_{j}}\right)+g\left(F_{x_{i}}, \nabla_{F_{t}} F_{x_{j}}\right)=-2 g\left(A\left(F_{x_{i}}, F_{x_{j}}\right), F_{t}\right)
$$

This implies the first equation.
We compute that

$$
g_{i i}^{\prime \prime}(0)=2 g\left(\nabla_{F_{t}} \nabla_{F_{t}} F_{x_{i}}, F_{x_{i}}\right)+2 g\left(\nabla_{F_{t}} F_{x_{i}}, \nabla_{F_{t}} F_{x_{i}}\right)
$$

Next use the definition of the Riemann curvature tensor $R_{M}$ of $M$ to get

$$
\begin{array}{rlr}
\sum_{i} g\left(\nabla_{F_{t}} \nabla_{F_{t}} F_{x_{i}}, F_{x_{i}}\right) & =\sum_{i} g\left(\nabla_{F_{t}} \nabla_{F_{x_{i}}} F_{t}, F_{x_{i}}\right) \quad \text { since }\left[F_{t}, F_{x_{i}}\right]=0 \\
& =\sum_{i} g\left(R\left(F_{x_{i}}, F_{t}\right) F_{t}, F_{x_{i}}\right)+\sum_{i} g\left(\nabla_{F_{x_{i}}} \nabla_{F_{t}} F_{t}, F_{x_{i}}\right) \\
& =\sum_{i} g\left(R\left(F_{x_{i}}, F_{F_{t}}\right) F_{t}, F_{x_{i}}\right)+\operatorname{div}_{\Sigma}\left(F_{t t}\right) .
\end{array}
$$

Therefore, we have

$$
\begin{aligned}
\sum_{i} g_{i i}^{\prime \prime}(0) & =2 \sum_{i} g\left(\nabla_{F_{t}}^{T} F_{x_{i}}, \nabla_{F_{t}}^{T} F_{x_{i}}\right)+2 \sum_{i} g\left(\nabla_{F_{t}}^{N} F_{x_{i}}, \nabla_{F_{t}}^{N} F_{x_{i}}\right) \\
& +2 \sum_{i} g\left(R\left(F_{x_{i}}, F_{F_{t}}\right) F_{t}, F_{x_{i}}\right)+2 \operatorname{div}_{\Sigma}\left(F_{t t}\right) \\
& =2 \sum_{i j} g\left(A\left(F_{x_{i}}, F_{x_{j}}\right), F_{t}\right)^{2}+2\left|\nabla_{\Sigma}^{N} F_{t}\right|^{2} \\
& +2 \sum_{i} g\left(R_{M}\left(F_{x_{i}}, F_{t}\right) F_{t}, F_{x_{i}}\right)+2 \operatorname{div}_{\Sigma}\left(F_{t t}\right) .
\end{aligned}
$$

The map $g\left(A(\cdot, \cdot), F_{t}\right): T_{x} \Sigma \times T_{x} \Sigma \rightarrow \mathbb{R}$ is a symmetric bilinear map. Since $\left\{F_{x_{i}}\right\}$ is an orthonormal basis of $T \Sigma$ at $x$, the squared norm $\left|g\left(A(\cdot, \cdot), F_{t}\right)\right|^{2}(x)$ is equal to $\sum_{i j} g\left(A\left(F_{x_{i}}, F_{x_{j}}\right), F_{t}\right)^{2}$. Similarly, the trace $\operatorname{Tr}_{\Sigma} g\left(R_{M}\left(\cdot, F_{t}\right) F_{t}, \cdot\right)(x)$ equals $\sum_{i} g\left(R_{M}\left(F_{x_{i}}, F_{t}\right) F_{t}, F_{x_{i}}\right)$.

Therefore, we get at $x$

$$
\begin{equation*}
\left.\frac{d^{2}}{d t^{2}} \nu(t)\right|_{t=0}=-\left|g\left(A(\cdot, \cdot), F_{t}\right)\right|^{2}+\left|\nabla_{\Sigma}^{N} F_{t}\right|^{2}-\operatorname{Tr}_{\Sigma} g\left(R_{M}\left(\cdot, F_{t}\right) \cdot, F_{t}\right)+\operatorname{div}_{\Sigma}\left(F_{t t}\right) \tag{3.1.19}
\end{equation*}
$$

Note that we used the skew symmetry of $R_{M}$ to reverse the sign.
The vector $F_{t t}(\cdot, 0)$ can be decomposed into two parts, the tangential part $F_{t t}^{T}$ and the normal partial part $F_{t t}^{N}$. We use Stokes' theorem to see that $\int_{\Sigma} \operatorname{div}_{\Sigma}\left(F_{t t}^{T}\right)=0$. From the minimality of $\Sigma$ and Equation (3.1.6), we have that $\int_{\Sigma} \operatorname{div}\left(F_{t t}^{N}\right)=-\int_{\Sigma} g\left(F_{t t}^{N}, H\right)=0$. Inserting Equation (3.1.19) into Equation (3.1.15), integrating and using the minimality of $\Sigma$, we get

$$
\begin{align*}
\frac{d^{2}}{d t^{2}} \operatorname{Vol}(F(\Sigma, t))= & -\int_{\Sigma}\left|g\left(A(\cdot, \cdot), F_{t}\right)\right|^{2} \\
& +\int_{\Sigma}\left|\nabla_{\Sigma}^{N} F_{t}\right|^{2}-\int_{\Sigma} \operatorname{Tr}_{\Sigma} g\left(R_{M}\left(\cdot, F_{t}\right) \cdot, F_{t}\right)  \tag{3.1.20}\\
= & -\int_{\Sigma} g\left(F_{t}, L\left(F_{t}\right)\right)
\end{align*}
$$

The self-adjoint operator $L$ is the so-called stability operator (or Jacobi operator) defined on a normal vector field $X$ to $\Sigma$ by

$$
L(X)=\Delta_{\Sigma}^{N} X+\operatorname{Tr}_{\Sigma} g\left(R_{M}(\cdot, X) \cdot, X\right)+\hat{A}(X)
$$

where $\hat{A}$ is Simons' operator defined by

$$
\hat{A}(X)=\sum_{i, j=1}^{k} g\left(A\left(E_{i}, E_{j}\right), X\right) A\left(E_{i}, E_{j}\right)
$$

and $\Delta_{\Sigma}^{N}$ is the Laplacian on the normal bundle, that is

$$
\Delta_{\Sigma}^{N} X=\sum_{i=1}^{k}\left(\nabla_{E_{i}} \nabla_{E_{i}} X\right)^{N}-\sum_{i=1}^{k}\left(\nabla_{\left(\nabla_{E_{i}} E_{i}\right)^{T}} X\right)^{N}
$$

A normal vector field $X$ with $L(X)=0$ is said to be a Jacobi field.
We will adopt the convention that $\lambda$ is a (Dirichlet) eigenvalue of $L$ on $\Omega \subset \Sigma$ if there exists a non-trivial normal vector field $X$ which vanishes on $\partial \Omega$ so that

$$
L(X)+\lambda X=0 .
$$

Definition 3.1.4. The Morse index of a compact minimal surface $\Sigma^{k} \subset\left(M^{n}, g\right)$ is the number of negative eigenvalues of the stability operator $L$ (counting with multiplicity) acting on the space of smooth sections of the normal bundle which vanishes on the boundary.

A minimal surface submanifold $\Sigma^{k} \subset\left(M^{n}, g\right)$ is stable if for all variations $F$ with fixed boundary and compact support

$$
\left.\frac{d^{2}}{d t^{2}} \operatorname{Vol}(F(\Sigma, t))\right|_{t=0}=-\int_{\Sigma} g\left(F_{t}, L\left(F_{t}\right)\right) \geqslant 0
$$

A complete (possibly non-compact) minimal submanifold without boundary is said to be stable if all compact subdomains are stable.

For a minimal hypersurface $\Sigma^{n-1} \subset\left(M^{n}, g\right)$ with trivial normal bundle, the stability operator simplifies significantly since, in this case, it becomes an operator on functions. Namely, if we identify a normal vector field $X=\eta N$, then

$$
\begin{equation*}
L(\eta)=\Delta_{\Sigma} \eta+|A|^{2} \eta+\operatorname{Ric}_{M}(N, N) \eta \tag{3.1.21}
\end{equation*}
$$

where $\operatorname{Ric}_{M}$ is the Ricci tensor of $M$.
Schoen and Yau SY79b pointed out that the stability operator (See Equation (3.1.21) is linked with the scalar curvature of $M$ (See Page 7 in SY79b).

Proposition 3.1.5. (See Page 7-8 of $\boldsymbol{S Y Y 9 b}])$ Let $\Sigma^{n} \subset\left(M^{n+1}, g\right)$ be a minimal surface with trivial normal bundle. Then the operator can be written as

$$
\begin{equation*}
L=\Delta_{\Sigma}-\kappa_{\Sigma}+\kappa_{M}+1 / 2|A|^{2} \tag{3.1.22}
\end{equation*}
$$

where $\kappa_{\Sigma}$ and $\kappa_{M}$ are the scalar curvature of $\Sigma$ and $M$, respectively.
Proof. Fixed a point $x \in \Sigma$, we may choose an orthonormal basis $\left\{E_{i}\right\}_{i=1}^{n+1}$ of $T_{x} M$. The unit vector $E_{n+1}$ is equal to the unit normal vector of $\Sigma$. The Gauss Equations (See Equation (3.1.4) assert

$$
\begin{equation*}
K_{\Sigma}\left(E_{i}, E_{j}\right)=K_{M}\left(E_{i}, E_{j}\right)+A_{i i} A_{j j}-A_{i j}^{2} \tag{3.1.23}
\end{equation*}
$$

where $K_{\Sigma}$ and $K_{M}$ are the sectional curvature of $\Sigma$ and $M$ respectively, $A_{i j}:=A\left(E_{i}, E_{j}\right)$.
Summing Equation (3.1.23), we have

$$
\sum_{1 \leqslant i<j \leqslant n} K_{\Sigma}\left(E_{i}, E_{j}\right)=\sum_{1 \leqslant i<j \leqslant n} K_{M}\left(E_{i}, E_{j}\right)+\sum_{1 \leqslant i<j \leqslant n} A_{i i} A_{j j}-A_{i j}^{2}
$$

Therefore, by the minimality of $\Sigma$, the scalar curvature of $M$ is

$$
\begin{align*}
\kappa_{M} & =\sum_{1 \leqslant i<j \leqslant n} K_{M}\left(E_{i}, E_{j}\right)+\operatorname{Ric}_{M}\left(E_{n+1}, E_{n+1}\right) \\
& =\sum_{1 \leqslant i<j \leqslant n} K_{\Sigma}\left(E_{i}, E_{j}\right)-\left(\sum_{1 \leqslant i<j \leqslant n} A_{i i} A_{j j}-A_{i j}^{2}\right)  \tag{3.1.24}\\
& +\operatorname{Ric}_{M}\left(E_{n+1}, E_{n+1}\right) \\
& =\kappa_{\Sigma}+1 / 2|A|^{2}+\operatorname{Ric}_{M}(N, N)
\end{align*}
$$

Therefore, putting Equation (3.1.24) into Equation (3.1.21), we get

$$
L=\Delta_{\Sigma}-\kappa_{\Sigma}+1 / 2|A|^{2}+\kappa_{M}
$$

3.1.4. Existence of Minimal surfaces. The following fundamental existence problem for minimal surfaces is known as the Plateau problem:

Given a closed curve $\Gamma$, find a minimal surface with boundary $\Gamma$.
This problem was first formulated by Lagrange in 1760 and was studied extensively by Plateau in the 19th century. This question had led to many significant developments in partial differential equations and geometric measure theory, such as Morrey's works Mor48, Mor09] and Fleming-Federer's works FF60.

There are various solutions to this problem, depending on the exact definition of a surface (parametrized disc, integral current, $\mathbb{Z}_{2}$-current or varifold ). In the following, we consider the version of the Plateau problem for parametrized discs.

In the case of $\mathbb{R}^{3}$, the solution was obtained in 1930 by J.Douglas Dou31 and simultaneously by T.Radó Rad30.

Theorem 3.1.6. (See [Theorem 4.1, Page 134] in [CM11]) Given a piecewise $C^{1}$ closed Jordan curve $\Gamma \subset \mathbb{R}^{3}$. there exists a map $u: D \subset \mathbb{R}^{2} \rightarrow \mathbb{R}^{3}$ so that
(1) $u: \partial D \rightarrow \Gamma$ is monotone and onto;
(2) $u \in C^{0}(\bar{D}) \cap W^{1,2}(D)$ and is $C^{\infty}$ in the interior of $D$;
(3) The image of $u$ minimizes area among all maps from the discs with boundary $\Gamma$.

The generalisation to the Riemannian manifold is due to C.B.Morrey Mor48 Mor09.
For fixed boundary $\Gamma$, an area-minimizing disc is a solution to the Plateau Problem. Therefore, Morrey Mor48, Mor09 used the variation method to find such a disc in a homegenous Riemannian 3-manifold. That is, take a sequence of mappings from the disc to the 3 -manifold whose area are going to the infimum and attempt to extract a convergent subsequence. Morrey Mor48, Mor09 pointed out that the limit of a subsequence of mappings is also the solution to the Dirichlet problem for the harmonic map.

For example, in $\mathbb{R}^{3}$, the existence of weak $\left(W^{1,2_{-}}\right)$solution to the Dirichlet problem for the harmonic map is ensured by the Kondrachov compactness theorem for $W^{1,2}$ (See [Theorem 7.22, Page 167] of [GT15]). Weyl's Lemma told us that the weak solution is smooth in the interior of domain. This result also follows from the standard regularity theory (See [Theorem 2.10, Page 23] of [GT15]). The above argument can be generalized to a homogenous Riemannian 3-manifold.

This solution is called Morrey's solution to the Plateau problem. In addition, the argument of Gulliver Gul73 and Osserman Oss70 pointed out that this solution has no interior branched point.

The remaining issue is the regularity of this solution up to the boundary. In the case of $\mathbb{R}^{3}$, J.J. Nitsche Nit69 gave an answer:

Theorem 3.1.7. ([Theorem 1, 315] in [Nit69])Let u be a function from $D \subset \mathbb{R}^{2}$ to $\mathbb{R}$ as in Theorem 3.1.6. If $\Gamma$ is a regular Jordon curve of class $C^{k, \alpha}$, where $k \geqslant 1$ and $0<\alpha<1$, then $u$ is $C^{k, \alpha}$ on $\bar{D}$.

This result can be generalized to the Riemannian 3-manifolds in HH70.
In various works, compact 3-manifolds with mean convex boundary were intensively studied. Let $(M, g)$ be a compact 3 -manifold with boundary. The boundary $\partial M$ is mean convex if

- $\partial M$ is a piecewise smooth 2-manifold consisting of smooth surfaces $\left\{H_{i}\right\}_{i}$
- for each $i$, the mean curvature of $H_{i}$ is nonnegative.

Theorem 3.1.8. (See [MY80, MY82] or[Theorem 6.28, Page 224] of [CM11]) Let $\left(M^{3}, g\right)$ be a compact Riemannian 3-manifold whose boundary is mean convex and $\gamma$ a simple closed curve in $\partial M$ which is null-homotopic in $M$. Then, $\gamma$ bounds an areaminimizing disc and any such least area disc is properly embedded.

This theorem will be repeatedly used in the following.

### 3.2. Local structures of Minimal Surfaces

It is classical that the minimal surface theory consists of two aspects: PDE and Geometry. These two aspects give various results about local structures of minimal surfaces, such as the maximum principle and the monotonicity formula.
3.2.1. Minimal surfaces are locally graphical. We begin with a minimal surface $\Sigma \subset \mathbb{R}^{3}$. The Gauss map is a continuous choice of a unit normal

$$
N: \Sigma \rightarrow \mathbb{S}^{2} \subset \mathbb{R}^{3}
$$

There are two choices of such a map $N$ and $-N$ corresponding to a choice of orientation of $\Sigma$. Suppose that $E_{1}, E_{2}$ is an orthonormal frame on $\Sigma$. We know that

$$
<\nabla_{E_{i}} N, E_{j}>=-A_{\Sigma}\left(E_{i}, E_{j}\right)
$$

where $A_{\Sigma}$ is the second fundamental form (See Equation (3.1.2) of $\Sigma$. Therefore,

$$
\begin{equation*}
|d N| \leqslant|A| . \tag{3.2.1}
\end{equation*}
$$

Lemma 3.2.1. (See [Lemma 2.4, Page 74] in CM11])Let $\Sigma \subset \mathbb{R}^{3}$ be an immersed minimal surface with

$$
\begin{equation*}
16 s^{2} \sup _{\Sigma}\left|A_{\Sigma}\right|^{2} \leqslant 1 \tag{3.2.2}
\end{equation*}
$$

If $x \in \Sigma$ and $d^{\Sigma}(x, \partial \Sigma) \geqslant 2 s$, then the geodesic ball $B^{\Sigma}(x, 2 s)$, centered at $x$ with radius $2 s$, can be written as a graph of a function $u$ over $T_{x} \Sigma$ with $|\nabla u| \leqslant 1$ and $\sqrt{2} s|H e s s(u)| \leqslant 1$.

Proof. We define

$$
\begin{equation*}
d_{x, y}:=d^{\mathbb{S}^{2}}(N(x), N(y)) \tag{3.2.3}
\end{equation*}
$$

Therefore, we see that

$$
\begin{equation*}
<N(x), N(y)>=\cos d_{x, y} \tag{3.2.4}
\end{equation*}
$$

Recall that $|d N| \leqslant|A|$ (See Equation (3.2.1)). Therefore, given $y \in B^{\Sigma}(x, 2 s)$, integrating (3.2.2) along a geodesic from $x$ to $y$ gives that

$$
\begin{equation*}
\sup _{y \in B^{\Sigma}(x, 2 s)} d_{x, y} \leqslant \frac{1}{2}<\frac{\pi}{4} \tag{3.2.5}
\end{equation*}
$$

Therefore, it follows that $B^{\Sigma}(x, 2 s)$ is contained in the graph of a function $u$ over a subset of $T_{x} \Sigma$.

We may choose a coordinate $\left(x_{k}\right)_{k=1}^{3}$ on $\mathbb{R}^{3}$ so that

$$
\begin{aligned}
& x=(0,0,0) \\
& T_{x} \Sigma=\left\{\left(x_{1}, x_{2}, 0\right)\right\} .
\end{aligned}
$$

Therefore, $N(x)=(0,0,1)$ and $N(y)=\frac{\left(-u_{x_{1}},-u_{x_{2}}, 1\right)}{\sqrt{1+|\nabla u|^{2}}}$. From Equation (3.2.3), we have

$$
\left(1+|\nabla u|^{2}\right)=<N_{x}, N_{y}>^{-2}=\cos ^{-2}\left(d_{x, y}\right)
$$

where the last equality comes from Equation (3.2.1).
If $y \in B^{\Sigma}(x, 2 s)$, Equation (3.2.5) implies that $|\nabla u(y)| \leqslant 1$. The Hessian estimate of $u$ comes from the gradient and curvature estimate together with Equation (3.1.14):

$$
|H e s s(u)|^{2} \leqslant\left(1+|\nabla u|^{2}\right)^{3}|A|^{2} \leqslant \frac{1}{2} s^{-2}
$$

3.2.2. Strong Maximal Principle. First note that the difference of two solutions to the minimal surface equation (See Equation (3.1.13) ) satisfies an elliptic divergent form equation (where the bound on the ellipticity depends on the gradient of the minimal graphs).

Lemma 3.2.2. If $u_{1}$ and $u_{2}$ are two solutions to the minimal surface equation (See Equation (3.1.13) ) on a domain $\Omega \subset \mathbb{R}^{n}$, then $v:=u_{1}-u_{2}$ satisfies an equation of the form:

$$
\begin{equation*}
\operatorname{div}(A(x) \nabla v)=0 \tag{3.2.6}
\end{equation*}
$$

where each eigenvalue $\lambda_{k}$ of the matrix $A(x):=\left(a_{i, j}(x)\right)_{n \times n}$ satisfies $\mu<\lambda_{k}<1 / \mu$, where $\mu$ depends only on the upper bounds for the gradient of $\left|\nabla u_{1}\right|$ and $\left|\nabla u_{2}\right|$.

Proof. Define the mapping $F: \mathbb{R}^{n} \rightarrow \mathbb{R}^{n}$ by

$$
F(X)=\frac{X}{\left(1+|X|^{2}\right)^{1 / 2}}
$$

Note that each $u_{k}$ satisfies $\operatorname{div}\left(F\left(\nabla u_{k}\right)\right)=0$. We know that

$$
\begin{align*}
F\left(\nabla u_{1}\right)-F\left(\nabla u_{2}\right) & =\int_{0}^{1} \frac{d}{d t} F\left(t \nabla u_{1}+(1-t) \nabla u_{2}\right) d t \\
& =\int_{0}^{1} d F\left(t \nabla u_{1}+(1-t) \nabla u_{2}\right) \cdot \nabla\left(u_{1}-u_{2}\right) d t  \tag{3.2.7}\\
& =\left(\int_{0}^{1} d F\left(t \nabla u_{1}+(1-t) \nabla u_{2}\right) d t\right) \cdot \nabla\left(u_{1}-u_{2}\right)
\end{align*}
$$

From this, we can conclude that $v=u_{1}-u_{2}$ satisfies an equation of the form $\operatorname{div}(A(x) \nabla v)=$ 0 , where the matrix $A(x)$ is given by Equation (3.2.7).

The remaining is to show that Equation (3.2.6) is a uniformly elliptic equation.
Given a unit vector $V \in \mathbb{S}^{n-1} \subset \mathbb{R}^{n}$ and $X \in \mathbb{R}^{n}$, we see that

$$
d F(X) V=\frac{V}{\left(1+|X|^{2}\right)^{1 / 2}}-\frac{<X, V\rangle}{\left(1+|X|^{2}\right)^{3 / 2}} X
$$

In particular, taking inner product with $V$ gives

$$
\begin{aligned}
\left(1+|X|^{2}\right)^{3 / 2}<d F(X) V, V> & =\left(1+|X|^{2}\right)-<X, V>^{2} \\
& \geqslant\left(1+|X|^{2}\right)-|X|^{2}=1 .
\end{aligned}
$$

It follows that $A(x)$ is a weighted average of positive definite matrix and thus also a positive definite matrix.

Corollary 3.2.3. Let $\Omega \subset \mathbb{R}^{n}$ be an open connected neighborhood of the origin. If $u_{1}, u_{2}: \Omega \rightarrow \mathbb{R}$ are two solutions to the minimal surface equation (See Equation (3.1.13)) with $u_{1} \leqslant u_{2}$ and $u_{1}(0)=u_{2}(0)$, then $u_{1} \equiv u_{2}$.

Proof. Lemma 3.2.2 tells us that the difference $v:=u_{1}-u_{2}$ satisfies $\operatorname{div}\left(a_{i, j} \nabla v\right)=0$, where the matrix $a_{i, j}$ is positive definite. We apply the strong maximum principle for the linear elliptic equation to $v$. (See [HL11] or [Theorem 3.5, Page 48] of [GT15]).

As in Lemma 3.2.1, a minimal hypersurface can be locally written as the graph of a solution to the minimal surface equation. As a consequence of Corollary 3.2.3 we know that

Corollary 3.2.4. If $\Sigma_{1}, \Sigma_{2} \subset \mathbb{R}^{n}$ are two complete connected minimal hypersurfaces (without boundary), $\Sigma_{1} \cap \Sigma_{2} \neq \varnothing$ and $\Sigma_{2}$ lies on one side of $\Sigma_{1}$, then $\Sigma_{1}=\Sigma_{2}$.

### 3.2.3. Monotonicity Formula and Area estimates.

Proposition 3.2.5. (The Monotonicity Formula) Suppose that $\Sigma^{k} \subset \mathbb{R}^{n}$ is a minimal submanifold and $x_{0} \in \Sigma$. Then for all $0<s<t$,

$$
\begin{equation*}
\frac{\operatorname{Vol}\left(B\left(x_{0}, t\right) \cap \Sigma\right)}{t^{k}}-\frac{\operatorname{Vol}\left(B\left(x_{0}, s\right) \cap \Sigma\right)}{s^{k}}=\int_{B\left(x_{0}, t\right) \backslash B\left(x_{0}, s\right) \cap \Sigma} \frac{\left|\left(x-x_{0}\right)^{N}\right|^{2}}{\left|x-x_{0}\right|^{k+2}}, \tag{3.2.8}
\end{equation*}
$$

where $\left(x-x_{0}\right)^{N}$ is the projection of the normal part of $\Sigma$ of the vector $\left(x-x_{0}\right)$.
See [Chapter 3.2, Page 24-26] of [CM11] for a proof.
As a consequence, we have
Corollary 3.2.6. Suppose that $\Sigma^{k} \subset \mathbb{R}^{n}$ is a minimal submanifold and $x_{0} \in \mathbb{R}^{n}$. Then the function

$$
\Theta_{x_{0}}(s)=\frac{\left.\operatorname{Vol} B\left(x_{0}, s\right) \cap \Sigma\right)}{\operatorname{Vol}\left(B(0, s) \subset \mathbb{R}^{k}\right)}
$$

is a non-decreasing function of $s$. Moreover, if $x_{0} \in \Sigma$, then $\Theta_{x_{0}}(s) \geqslant 1$ and

$$
\begin{equation*}
\operatorname{Vol}\left(\Sigma \cap B\left(x_{0}, s\right)\right) \geqslant \omega_{k} s^{k} \tag{3.2.9}
\end{equation*}
$$

where $\omega_{k}$ is the volume of the unit ball $B(0,1) \subset \mathbb{R}^{n}$.
The area estimate (3.2.9) can be generalized to the Riemannian manifolds. Before we state it, we will recall the coarea formula. This formula asserts (See, for instant, Chapter 3.2 of Fed14 for a proof) that if $(M, g)$ is a Riemannian manifold and the function

$$
h: M \rightarrow \mathbb{R}
$$

is a proper (i.e. $h^{-1}((\infty, t])$ is compact for all $\left.t \in \mathbb{R}\right)$ Lipschitz function, then for any locally integral function $f$ on $M$ and $t \in \mathbb{R}$,

$$
\begin{equation*}
\int_{h \leqslant t} f\left|\nabla_{M} h\right|=\int_{\infty}^{t} \int_{h=\tau} f d \tau \tag{3.2.10}
\end{equation*}
$$

Theorem 3.2.7. (See [Lemma 1, Page 445] of (MY80]) Let $\left(M^{n+1}, g\right)$ be a Riemannian manifold whose sectional curvature is bounded by a positive constant $K$ and $\Sigma^{n}$ a minimal submanifold. If for some point $x_{0} \in \Sigma, d^{M}\left(x_{0}, \partial M\right)$ and $d^{M}\left(x_{0}, \partial \Sigma\right)$ are both greater than a constant $\epsilon>0$, then for any $\delta \leqslant \min \{\epsilon, i(M)\}$

$$
\begin{equation*}
\operatorname{Vol}\left(B\left(x_{0}, \delta\right) \cap \Sigma\right) \geqslant C_{n} K \int_{0}^{\delta} t^{-1}(\sin (K t))^{n} d t \tag{3.2.11}
\end{equation*}
$$

where $i(M)$ is the radius of injectivity of $M$ and $C_{n}$ depends only on $n$.
Proof. Let $r(x)$ be the distance function of $M$ from $x_{0}$ to $x \in \Sigma$. If $r(x)$ is smaller than $i(M)$, the Hessian comparison theorem (See [Theorem 27, Page 175] of [PAR06]) gives

$$
\begin{equation*}
\operatorname{Hess}\left(e_{i}, e_{i}\right)(r)(x):=\nabla_{e_{i}} \nabla_{e_{i}} r-\nabla_{\nabla_{e_{i}} e_{i}}(r) \geqslant K \cot (K r) \tag{3.2.12}
\end{equation*}
$$

where $x \in \Sigma,\left\{e_{i}\right\}_{i=1}^{n}$ is an orthonormal basis of $T_{x} \Sigma$ and $N$ is a unit normal vector of $\Sigma$ at $x$.

Summing Equation (3.2.12) and using the minimality of $\Sigma$, we have

$$
\begin{array}{rlr}
\Delta_{\Sigma} r^{2}(x) & \geqslant 2 r(x) \Delta_{\Sigma} r(x)=2 r(x)\left(\sum_{i=1}^{n} \nabla_{e_{i}} \nabla_{e_{i}} r-\nabla_{\nabla_{e_{i}}^{\Sigma}\left(e_{i}\right)} r\right) \\
& =2 r(x)\left(\sum_{i=1}^{n} \operatorname{Hess}\left(e_{i}, e_{i}\right)(r)+\sum_{i=1}^{n} \nabla_{\nabla_{e_{i}}^{N}\left(e_{i}\right)} r\right), & \\
& =2 r(x)\left(\sum_{i=1}^{n} \operatorname{Hess}\left(e_{i}, e_{i}\right)(r)+\nabla_{H(x)} r\right), & \text { because } H(x)=\sum_{i=1}^{n} \nabla_{e_{i}}^{N} e_{i}  \tag{3.2.13}\\
& =2 r(x) \sum_{i=1}^{n} \operatorname{Hess}\left(e_{i}, e_{i}\right)(r), & \text { since } H(x)=0 \\
& \geqslant 2 n K r \cot (K r), &
\end{array}
$$

where $x \in \Sigma, \nabla^{\Sigma}, \nabla^{N}$ and $H(x)$ are defined as in Section 3.1, (See also the argument [Page 243] of $\mathbf{S Y 7 7}$ ). Integrating Equation (3.2.13) over $B(x, t)$ and noting that $|\nabla r| \leqslant 1$, we have,

$$
\begin{align*}
2 t \operatorname{Vol}\left(\partial\left(\Sigma \cap B\left(x_{0}, t\right)\right)\right. & \geqslant \int_{\partial\left(\Sigma \cap B\left(x_{0}, t\right)\right)}<\nabla^{\Sigma} r^{2}, \mathbf{n}>, & & \text { since }\left|\nabla_{\Sigma} r\right| \leqslant|\nabla r| \leqslant 1  \tag{3.2.14}\\
& =\int_{\Sigma \cap B\left(x_{0}, t\right)} \Delta_{\Sigma} r^{2}, & & \text { doing integration by parts, } \\
& \geqslant 2 n K \int_{B\left(x_{0}, t\right) \cap \Sigma} r \cot (K r) & & \text { from Equation(3.2.13). }
\end{align*}
$$

where $\mathbf{n}$ is the outward unit normal vector of $\partial\left(\Sigma \cap B\left(x_{0}, t\right)\right)$ in $\Sigma$.
Let $C(t)=\int_{B\left(x_{0}, t\right) \cap \Sigma} r \cot (K r)$. Then by the coarea formula (See Equation 3.2.10) and the fact that $|\nabla r| \leqslant 1$, we have that

$$
\begin{equation*}
\frac{\partial C(t)}{\partial t}=\int_{\{x \mid r(x)=t\} \cap \Sigma} \frac{r \cot (K r)}{|\nabla r|} \geqslant t \cot (K t) \operatorname{Vol}\left(\partial\left(\Sigma \cap B\left(x_{0}, t\right)\right)\right) \tag{3.2.15}
\end{equation*}
$$

Inserting Equation (3.2.14) into Equation (3.2.15), we find

$$
\begin{equation*}
\frac{\partial C(t)}{\partial t} \geqslant n K \cot (K t) C(t) \tag{3.2.16}
\end{equation*}
$$

It is easy to verify that

$$
\lim _{t \rightarrow 0} C(t) \sin (K t)^{-1}=K^{-n-1} C_{n}
$$

where $C_{n}$ is a positive constant depending only on $n$. It follows from Equation 3.2.16) that

$$
\begin{equation*}
C(t) \geqslant K^{-n-1} C_{n}(\sin (K t))^{n} \tag{3.2.17}
\end{equation*}
$$

for all $t \leqslant \min \{\epsilon, i(M)\}$. Therefore, Equation (3.2.14 shows that

$$
\begin{equation*}
\operatorname{Vol}\left(\partial\left(\Sigma \cap B\left(x_{0}, t\right)\right)\right) \geqslant n C_{n} K^{-n} t^{-1}(\sin (K t))^{n} \tag{3.2.18}
\end{equation*}
$$

By the coarea formula (See Equation (3.2.14)) again,

$$
\begin{align*}
\operatorname{Vol}\left(\Sigma \cap B\left(x_{0}, \delta\right)\right) & \geqslant \int_{0}^{\delta} \operatorname{Vol}\left(\partial\left(\Sigma \cap B\left(x_{0}, \tau\right)\right)\right) d \tau \\
& \geqslant n C_{n} K^{-n} \int_{0}^{\delta} \tau^{-1}(\sin (K \tau))^{n} d \tau \tag{3.2.19}
\end{align*}
$$

### 3.3. Stable minimal surfaces

In this section, let us consider a stable minimal surface $\Sigma^{n} \subset\left(M^{n+1}, g\right)$ with trivial normal bundle. First, the stable condition is linked with the first eigenvalue of the operator $L$ (See 3.1.21) or (3.1.22)). Second, if $(M, g)$ has positive curvature, the stability condition gives several topological restrictions on $\Sigma$.
3.3.1. Eigenvalues and Stability. The stability of a minimal surface is characterized by the first eigenvalue of $L$ (See (3.1.21) or (3.1.22)). A minimal hypersurface $\Sigma^{n} \subset(M, g)$ is stable if for any smooth function $\eta$ with compact support,

$$
\int_{\Sigma}-\eta L(\eta)=\int_{\Sigma}\left|\nabla_{\Sigma} \eta\right|^{2}-\operatorname{Ric}(N, N) \eta^{2}-|A|^{2} \eta^{2} \geqslant 0
$$

We see that the stable condition is equivalent to the first eigenvalue $\lambda_{1}(L, \Omega) \geqslant 0$ for each $\Omega \subset \Sigma$ where

$$
\begin{equation*}
\lambda_{1}(L, \Omega):=\inf \left\{\int_{\Sigma}-\eta L(\eta) \mid \eta \in W_{0}^{1,2}(\Omega) \text { and } \int_{\Sigma} \eta^{2}=1\right\} \tag{3.3.1}
\end{equation*}
$$

By standard elliptic theory ( see GT15 or HL11), we get the following:
Lemma 3.3.1. If $L$ and $\Omega \subset \Sigma$ are assumed as above and $\lambda_{1}:=\lambda_{1}(L, \Omega)$, then the eigenfunction $u \in W_{0}^{1,2}(\Omega)$ of the first eigenvalue $\lambda_{1}$ (i.e. $L(u)=-\lambda_{1} u$ ) is smooth.

It follows from the regularity theory for elliptic equations (See [Theorem 8.14, Page 188] of GT15 or HL11).

Together with the Harnack inequality, we see that any eigenfunction of the first eigenvalue can not change sign.

Lemma 3.3.2. Assume that $u$ is a smooth function on $\Omega$ that vanishes on $\partial \Omega$. If $L(u)=-\lambda_{1} u$ where $\lambda_{1}=\lambda_{1}(L, \Omega)$, then $u$ can not change sign.

Proof. We may assume that $u$ is not identically zero and $\int_{\Sigma} u^{2}=1$. Since $u$ vanishes on $\partial \Omega$, so does $|u|$. In fact, $|u|$ also achieve the minimum in (3.3.1). By Lemma 3.3.1, $|u|$ is smooth and $L(|u|)=-\lambda_{1}|u|$. Since $|u| \geqslant 0$ and $|u|$ is not identically zero, the Harnack inequality (See [Theorem 8.20, Page 199] of [GT15]) implies $|u|>0$ in $\Omega$. Hence, $u$ can not change sign.

Next, let us consider the positive solution to the stable operator $L$ (See Equation (3.1.21) or Equation (3.1.22)).

Proposition 3.3.3. Let $\Sigma^{n} \subset\left(M^{n+1}, g\right)$ be a minimal hypersurface with trivial normal bundle, $L$ its stability operator (See Equation (3.1.21) or (3.1.22)), and $\Omega \subset \Sigma$ a bounded domain. If there exists a positive function $u$ on $\Omega$ with $L(u)=0$, then $\Omega$ is a stable minimal surface.

Proof. Set $q(x)=|A|^{2}+\operatorname{Ric}_{M}(N, N)$ so that $L=\Delta_{\Sigma}+q$. Since $u>0$, the function $w:=\log u$ is well-defined and satisfies

$$
\begin{equation*}
\Delta_{\Sigma} w=-q-\left|\nabla_{\Sigma} w\right|^{2} \tag{3.3.2}
\end{equation*}
$$

Let us consider any compactly supported function $f$ on $\Omega$. Multiplying both sides of Equation (3.3.2) with $f^{2}$ and using integrating by parts give

$$
\begin{align*}
\int_{\Omega} f^{2} q+f^{2}\left|\nabla_{\Sigma} w\right|^{2} & =-\int_{\Omega} f^{2} \Delta_{\Sigma} w \leqslant 2 \int_{\Sigma} f\left|\nabla_{\Sigma} f\right|\left|\nabla_{\Sigma} w\right| \\
& \leqslant \int_{\Sigma} f^{2}\left|\nabla_{\Sigma} w\right|^{2}+\left|\nabla_{\Sigma} f\right|^{2} \tag{3.3.3}
\end{align*}
$$

where the second inequality follows from the Cauchy-Schwarz inequality. Canceling the $\int_{\Sigma} f^{2}\left|\nabla_{\Sigma} w\right|^{2}$ term, we have that

$$
\int_{S}-f L(f) \geqslant 0
$$

This finishes the proof.
We will give a characterization for a complete (non-compact) stable minimal hypersurfaces with trivial normal bundle. For such a hypersurface, the stability is equivalent to the existence of positive solution to the stability operator.

Theorem 3.3.4. (See [Theorem 1, Page 201] FCS80]) Let $\Sigma^{n} \subset(M, g)$ be a complete non-compact minimal hypersurface with trivial normal bundle, then the following are equivalent:
(1) $\lambda_{1}(L, \Omega) \geqslant 0$ for any bounded domain $\Omega \subset \Sigma$;
(2) $\lambda_{1}(L, \Omega)>0$ for any bounded domain $\Omega \subset \Sigma$;
(3) there is a positive function $u$ over $\Sigma$ with $L(u)=0$.

Proof. By Proposition 3.3.3, (3) implies (1).
Clearly, (2) implies (1). To see the equivalence of (1) and (2), we consider any bounded domain $\Omega_{0}$ and choose a strictly larger bounded domain $\Omega_{1}$. The variational characterization of the first eigenvalue (See (3.3.1) implies that

$$
\lambda_{1}\left(L, \Omega_{0}\right) \geqslant \lambda_{1}\left(L, \Omega_{1}\right) \geqslant 0
$$

where the second inequality follows from (1). Let $u_{0}$ be the first eigenfunction for $L$ in $\Omega_{0}$. We define $u_{1}$ on $\Omega_{1}$ by

$$
u_{1}(x)=\left\{\begin{array}{cc}
u_{0}(x), & \text { if } \quad x \in \Omega_{0} \\
0, & \text { otherwise }
\end{array}\right.
$$

where $u_{0} \geqslant 0$ and $u_{0}$ is not identically zero (See Lemma 3.3.2).
If we had that $\lambda_{1}\left(L, \Omega_{0}\right)=0$, then the nonnegative function $u_{1}$ is an eigenfunction of $\lambda_{1}\left(L, \Omega_{1}\right)$. Lemma 3.3.1 tells that $L\left(u_{1}\right)=\lambda_{1}(L, \Omega) u_{1}$ and $u_{1}$ is smooth. Since $u_{1}=0$ on $\Omega_{1} \backslash \Omega_{0}$, the Harnack inequality (See [Theorem 8.20, Page 199] of $\mathbf{G T 1 5}$ ) implies that $u_{1}$ is identically zero on $\Omega_{1}$. This is not possible. We can conclude that $\lambda_{1}\left(L, \Omega_{0}\right)>0$. The equivalence of (1) and (2) follows.

The remaining is to show that (2) implies (3). To do this, fixed a point $p \in \Sigma$ and any $r>0$, let $B^{\Sigma}(p, r)$ be the geodesic ball in $\Sigma$ with radius $r$ and centered at $p$. Then the first eigenvalue $\lambda_{1}\left(L, B^{\Sigma}(p, r)\right)$ is greater than zero. By Fredholm alternative (See [Theorem 6.15, Page 107] of $|\mathbf{G T 1 5}|$ ), there exists a function $v_{r}$ satisfying:

$$
\begin{equation*}
L\left(v_{r}\right)=-|A|^{2}-\operatorname{Ric}(N, N) \text { on } B^{\Sigma}(p, r) \text { and } v_{r}=0 \text { on } \partial B^{\Sigma}(p, r) . \tag{3.3.4}
\end{equation*}
$$

Setting $u_{r}=v_{r}+1$, Equation (3.3.4) gives

$$
\begin{equation*}
L\left(u_{r}\right)=0 \text { in } B^{\Sigma}(p, r) \text { and } u_{r}=1 \text { on } \partial B^{\Sigma}(p, r) \tag{3.3.5}
\end{equation*}
$$

We claim that

$$
\begin{equation*}
u_{r}>0 \text { on } B^{\Sigma}(p, r) . \tag{3.3.6}
\end{equation*}
$$

If $u_{r}$ is not a non-negative function, we choose a nonempty connected component $\Omega \subset \Sigma$ of the set

$$
\left\{x \in B^{\Sigma}(p, r) \mid u_{r}(x)<0\right\} .
$$

Then, $L\left(u_{r}\right)=0$ on $\Omega$ and $u_{r}=0$ on $\partial \Omega$. This is to say, the first eigenvalue $\lambda_{1}(L, \Omega) \leqslant$ 0 . This is in contradiction with (2). Therefore, we conclude that $u_{r} \geqslant 0$.

The claim follows from the Harnark inequality (See [Theorem 8.20, Page 199] of GT15]).

For each $r$, we define a positive function by

$$
w_{r}(x):=u_{r}(x)\left(u_{r}(p)\right)^{-1} \text { on } B^{\Sigma}(p, r)
$$

and see that $L\left(w_{r}\right)=0$ and $w_{r}(p)=1$.
Now, let us consider a compact set $K \subset B^{\Sigma}\left(p, R_{0}\right)$. We use the Harnack inequality (See [Theorem 8.20, Page 199] of GT15]) to have a positive constant $C\left(p, R_{0}\right)$, only depending on $p$ and $R_{0}$, satisfying for any $r>2 R_{0}$,

$$
\begin{equation*}
\left|w_{r}(x)\right| \leqslant C\left(p, R_{0}\right) \text { if } x \in B^{\Sigma}\left(p, R_{0}\right) \tag{3.3.7}
\end{equation*}
$$

The interior Schauder estimate (See [Theorem 6.2, Page 90] of GT15) gives that

$$
\left|w_{r}\right|_{C_{K}^{2, \alpha}} \leqslant C_{K}
$$

where the constant $C_{K}$ only depends on $K$ and $0<\alpha<1$.
To sum up, we have a uniform $C^{2, \alpha}$ estimate for any $w_{r}$ where $r>2 R_{0}$. We use the Arzela-Ascoli theorem to extract a subsequence of $w_{r}$ that converges uniformly to $w$. This convergence ensures that $L(w)=0$ and $w \geqslant 0$. The Harnack inequality (See [Theorem 8.20, Page 199] of [GT15]) tells us that $w$ is a positive function. This finishes the proof of the theorem.

### 3.3.2. Global Structure (I): the compact case.

Proposition 3.3.5. (See Page 166 of $\boldsymbol{S Y Y 9 b}])$ Let $\Sigma^{n} \subset\left(M^{n+1}, g\right)$ be a compact stable minimal surface with trivial normal bundle. If $(M, g)$ has positive scalar curvature, then $\left(\Sigma, g_{\Sigma}\right)$ is conformally equivalent to a metric of positive scalar curvature, where $g_{\Sigma}$ is the induced metric and $n>2$.

Proof. The stability of the minimal surface $\Sigma$ and Equation (3.1.22) give

$$
\begin{equation*}
\int_{\Sigma} \kappa_{M} \eta^{2}-\kappa_{\Sigma} \eta^{2}+1 / 2|A|^{2} \eta^{2} \leqslant \int_{\Sigma}\left|\nabla_{\Sigma} \eta\right|^{2} \tag{3.3.8}
\end{equation*}
$$

for any smooth function $\eta$ on $\Sigma$.
Since $\kappa_{M}>0$ on $\Sigma$, we can conclude that

$$
\begin{equation*}
-\int_{\Sigma} \kappa_{\Sigma} \eta^{2}<\int_{\Sigma}\left|\nabla_{\Sigma} \eta\right|^{2} \tag{3.3.9}
\end{equation*}
$$

for all smooth function $\eta$.

Let $\lambda$ be the first eigenvalue of the operator $\Delta_{\Sigma}-\frac{n-2}{2(n-1)} \kappa_{\Sigma}$ and $u$ an eigenfunction of $\lambda$. That is,

$$
\begin{equation*}
\Delta_{\Sigma} u-\frac{n-2}{4(n-1)} \kappa_{\Sigma} u=-\lambda u \tag{3.3.10}
\end{equation*}
$$

From (3.3.9), we can conclude that $\lambda>0$.
Otherwise $(\lambda \leqslant 0)$, multipling two sides of 3.3 .10 by $u$ and integrating, we see that

$$
\int_{\Sigma}\left|\nabla_{\Sigma} u\right|^{2}=-\frac{n-2}{4(n-1)} \int_{\Sigma} \kappa_{\Sigma} u^{2}+\lambda \int_{\Sigma} u^{2} \leqslant-\frac{n-2}{2(n-1)} \int_{\Sigma} \kappa_{\Sigma} u^{2} \leqslant \frac{n-2}{2(n-1)} \int_{\Sigma}\left|\nabla_{\Sigma} u\right|^{2}
$$

where the last inequality follows from (3.3.9). This is impossible.
As the argument in Lemma 3.3.2, we have that the eigenfunction $u$ is smooth and a positive function. Multiplying the metric $g_{\Sigma}$ by $u^{\frac{4}{n-2}}$, under the new metric, the scalar curvature of $\Sigma$ is

$$
u^{-\frac{n+2}{n-2}}\left(\kappa_{\Sigma}-\frac{4(n-1)}{n-2} \Delta_{\Sigma} u\right)=\frac{4(n-1)}{n-2} \lambda u^{-\frac{n+2}{n-2}}>0 .
$$

Next, we consider a stable minimal surface $\Sigma \subset\left(M^{3}, g\right)$.
Corollary 3.3.6. (See [Theorem 5.1, Page 139] of [SY79a])Let $\Sigma^{2} \subset\left(M^{3}, g\right)$ be a closed stable minimal surface with trivial normal bundle. If the complete manifold $\left(M^{3}, g\right)$ has positive scalar curvature, then $\Sigma$ is $\mathbb{S}^{2}$ or $\mathbb{R} P^{2}$ and

$$
\int_{\Sigma} \kappa_{M}+1 / 2|A|^{2} \leqslant 4 \pi
$$

Proof. Since $\Sigma$ is compact and has no boundary, we choose the constant function $\eta=1$ as a cut-off function. Equation (3.1.22) gives

$$
\begin{equation*}
\int_{\Sigma} \kappa_{M}+1 / 2|A|^{2} \leqslant \int_{\Sigma} K_{\Sigma}=2 \pi \chi(\Sigma) \tag{3.3.11}
\end{equation*}
$$

where the last equity follows from the Gauss-Bonnet formula. As a consequence, we see that $\chi(\Sigma)>0$. That is to say, $\Sigma$ is $\mathbb{S}^{2}$ or $\mathbb{R} P^{2}$. Therefore, $\chi(\Sigma) \leqslant 2$. It gives the inequality in the assertion.

In the next part, we will use Cohn-Vesson's inequality Coh35] to generalize (3.3.11) to the non-compact case.
3.3.3. Global Structure (II): the non-compact case. By Theorem 3.3 .4 , for a complete (non-compact) stable minimal surface $\Sigma \subset\left(M^{3}, g\right)$, there is a positive function $u$ over $\Sigma$ with $L(u)=0$.

Theorem 3.3.7. ([Theorem 2, Page 126] in [Fis85]) Let $\Sigma^{2} \subset(M, g)$ be a complete (non-compact) stable minimal hypersurface. If the complete manifold $\left(M^{3}, g\right)$ has nonnegative scalar curvature $(\kappa(x) \geqslant 0)$, then the new metric $u^{2} d^{2} s$ is a complete metric on $\Sigma$ with non-negative sectional curvature, where $d^{2} s$ is the induced metric and $u$ is the positive function with $L(u)=0$ (as in Theorem 3.3.4).

Proof. By Theorem 3.3.4, there is a positive function $u$ with $L(u)=0$. In addition, there exists a minimal geodesic ray $\gamma(t):[0, \infty) \rightarrow \Sigma$ in the metric $\tilde{d}^{2} s:=u^{2} d^{2} s$, where $t$ is arclength in the original metric $d^{2} s$. It is obtained as below:

Fixed $x \in M$ and for any $R>0$, let us consider a geodesic ball $B^{\Sigma}(x, R)$ in the complete manifold $\left(\Sigma, d^{2} s\right)$. Define $u_{R}:=u+\eta_{R}$ where $\eta_{R}$ is a smooth positive function satisfying

$$
\eta_{R}= \begin{cases}0, & |x|<R \\ 1, & |x|>2 R\end{cases}
$$

Then, since $u_{R}$ is bounded away from zero, the metric $u_{R}^{2} d^{2} s$ is complete. Therefore, there is a shortest geodesic $\gamma_{R}$ from $x$ to $\partial B^{\Sigma}(x, R)$ in the metric $u_{R}^{2} d^{2} s$. We can conclude that $\gamma_{R}$ must stay in $B^{\Sigma}(x, R)$. (Otherwise, under the metric $u_{R}^{2} d^{2} s$, there is another curve connecting $x$ to $\partial B^{\Sigma}(x, R)$ whose length is shorter than $\gamma_{R}$ 's.)

Since $u=u_{R}$ on $B^{\Sigma}(x, R), \gamma_{R}$ is also a minimizing geodesic in $\tilde{d}^{2} s$. Each $\gamma_{R}$ can be parametrized with respect to arclength in the metric $d^{2} s$. Let us consider the limit of these minimizing geodesics in $\left(\Sigma, \tilde{d}^{2} s\right)$. The sequence $\left\{\gamma_{R}\right\}$ sub-converges to a minimizing geodesic ray $\gamma(t)$ that is parametrized by arclength in the metric $d^{2} s$.

It remains to show the completeness of $\tilde{d}^{2} s$.
By the construction of $\gamma$, the completeness of $\tilde{d}^{2} s$ will follow if we can show that $\gamma$ has infinite length under the metric $\tilde{d}^{2} s$, i.e. it is sufficient to show that

$$
\begin{equation*}
\int_{0}^{\infty} u(\gamma(t)) d s=\infty \tag{3.3.12}
\end{equation*}
$$

Since $\gamma$ is a minimizing geodesic in $\tilde{d}^{2} s=u^{2} d^{2} s$, the second variation formula of acrlength gives

$$
\begin{equation*}
\int_{0}^{\infty}\left(\left(\frac{d \phi}{\tilde{d} s}\right)^{2}-\tilde{K} \phi^{2}\right) \tilde{d} s \geqslant 0 \tag{3.3.13}
\end{equation*}
$$

where the smooth function $\phi$ has compact support in $(0, \infty), \frac{d \phi}{d s}=u^{-1} \frac{d \phi}{d s}, \tilde{d} s=u d s$ and the sectional curvature $\tilde{K}$ of $\left(\Sigma, \tilde{d}^{2} s\right)$ is

$$
\begin{equation*}
\tilde{K}=u^{-2}\left(K-\Delta_{\Sigma} \log u\right) . \tag{3.3.14}
\end{equation*}
$$

In addition, $u$ is a positive function with $L(u)=0$. Namely,

$$
L(u)=\Delta_{\Sigma} u-K u+\left(\kappa(x)+1 / 2|A|^{2}\right) u=0
$$

where $\Delta_{\Sigma}$ is the Laplace-Beltrami operator respect to $\left(\Sigma, d^{2} s\right)$ and $K$ is the sectional curvature of $\left(\Sigma, d^{2} s\right)$. Since $\kappa \geqslant 0$, we see that $\Delta_{\Sigma} u \leqslant K u$. In addition,

$$
\begin{equation*}
\Delta_{\Sigma} \log u=u^{-1} \Delta_{\Sigma} u-\frac{\left|\nabla_{\Sigma} u\right|^{2}}{u^{2}} \tag{3.3.15}
\end{equation*}
$$

Note that together with (3.3.14) and (3.3.15), we can conclude that $\tilde{K} \geqslant 0$. Inserting (3.3.14), (3.3.15) into (3.3.13), we see that

$$
\begin{equation*}
\int_{0}^{\infty} \frac{\left(u^{\prime}\right)^{2}}{u^{3}} \phi^{2} d s \leqslant \int_{0}^{\infty} u^{-1}\left(K-\Delta_{\Sigma} \log u\right) \phi^{2} d s \leqslant \int_{0}^{\infty} u^{-1}\left(\phi^{\prime}(s)\right)^{2} d s \tag{3.3.16}
\end{equation*}
$$

where the first inequality follows from (3.3.14) and (3.3.15), the last inequality follows from 3.3.13) and $u(s)=u(\gamma(s))$.

We now show that $\int_{0}^{\infty} u d s=\infty$. Take $\phi=u \psi$, where the smooth function $\psi$ has compact support in $(0, \infty)$. Then $\phi^{\prime}=u^{\prime} \psi+u \psi^{\prime}$ and

$$
u^{-1}\left(\phi^{\prime}\right)^{2}=u^{-1}\left(u^{\prime}\right)^{2} \psi^{2}+u\left(\psi^{\prime}\right)^{2}+2 u^{\prime} \psi^{\prime} \psi
$$

The inequality (3.3.16) shows that

$$
\int_{0}^{\infty} \frac{\left(u^{\prime}\right)^{2}}{u^{3}} \phi^{2} d s=\int_{0}^{\infty} \frac{\left(u^{\prime}\right)^{2}}{u} \psi^{2} d s \leqslant \int_{0}^{\infty} u^{-1}\left(u^{\prime}\right)^{2} \psi^{2}+u\left(\psi^{\prime}\right)^{2}+2 u^{\prime} \psi^{\prime} \psi d s
$$

Therefore,

$$
0 \leqslant \int_{0}^{\infty} u\left(\psi^{\prime}\right)^{2}+2 u^{\prime} \psi \psi^{\prime} d s
$$

The integration by parts gives

$$
\begin{equation*}
0 \leqslant \int_{0}^{\infty}-u\left(\psi^{\prime}\right)^{2}-2 u \psi^{\prime \prime} \psi d s \tag{3.3.17}
\end{equation*}
$$

Set $\psi(s)=s \xi(s)$, where the smooth function $\xi$ has compact support in $[0, \infty)$. Then,

$$
\begin{aligned}
& \psi^{\prime}(s)=\xi(s)+s \xi^{\prime}(s) \\
& \psi^{\prime \prime}(s)=s \xi^{\prime \prime}(s)+2 \xi^{\prime}(s)
\end{aligned}
$$

Putting these equations into (3.3.17), we have

$$
\int_{0}^{\infty} \xi^{2} u d s \leqslant \int_{0}^{\infty}\left(-6 s \xi \xi^{\prime}-2 s^{2} \xi \xi^{\prime \prime}-s^{2}\left(\xi^{\prime}\right)^{2}\right) u d s
$$

Choose a smooth decreasing function $\xi$ so that

$$
\begin{aligned}
& \xi(s)=1, \text { for } \quad 0 \leqslant s \leqslant R, \\
& \xi(s)=0, \text { for } \quad s>2 R,
\end{aligned}
$$

and $\left|\xi^{\prime}\right|$ and $\left|\xi^{\prime \prime}\right|$ are bounded by $2 R^{-1}$ and $4 R^{-2}$ respectively, for $R \leqslant s \leqslant 2 R$. Then $\left|s \xi^{\prime}(s)\right| \leqslant 4$ and $\left|s \xi^{\prime \prime}(s)\right| \leqslant 16$. We see that

$$
\int_{0}^{R} u d s \leqslant \int_{0}^{\infty} u \xi^{2} d s \leqslant \int_{0}^{\infty}\left(-6 s \xi \xi^{\prime}-2 s^{2} \xi \xi^{\prime \prime}-s^{2}\left(\xi^{\prime}\right)^{2}\right) u d s \leqslant 72 \int_{R}^{\infty} u d s
$$

This inequality implies that $\int_{0}^{\infty} u d s=\infty$. That is to say, $\tilde{d}^{2} s=u^{2} d^{2} s$ is a complete metric with nonnegative sectional curvature $\tilde{K} \geqslant 0$.

We apply the above argument to stable minimal surfaces in a 3-manifolds of uniformly positive scalar curvature (i.e. its scalar curvature is bounded away from zero).

Theorem 3.3.8. (See [Theorem 10.2, Page 384] of [GL83] or [Theorem 1, Page 228] of RosO6 ) Let $\left(M^{3}, g\right)$ be a Riemannian 3-manifold with the scalar curvature $\kappa \geqslant c$
where $c$ is a positive constant. If $\Sigma$ is a stable minimal surface immersed in $M$, then for $x \in \Sigma$ :

$$
d^{\Sigma}(x, \partial \Sigma) \leqslant \frac{2 \pi}{\sqrt{3 c}}
$$

This result was firstly proved by Gromov-Lawson [GL83] and Schoen-Yau in [SY83]. The generalization to the stable $H$-surface was due to H.Rosenberg Ros06].

Proof. For any $x \in \Sigma$, let $R:=d^{\Sigma}(x, \partial \Sigma)$, where $d^{\Sigma}$ is the induced distance function in $\left(\Sigma, d^{2} s\right)$. For any $\epsilon>0$, the geodesic ball $B^{\Sigma}(x, R-\epsilon)$ in $\Sigma$, centered at $x$ with radius $R-\epsilon$, is contained in Int $\Sigma$.

It is sufficient to show that for any $\epsilon>0, R-\epsilon \leqslant \frac{2 \pi}{\sqrt{3 c}}$.
First, we solve the equation $L(u)=0$ on the $B^{\Sigma}(x, R-\epsilon)$, where $L$ is the stability operator (See Equation 3.1.22). As in the proof of Theorem 3.3.4, the first eigenvalue $\lambda_{1}\left(L, B^{\Sigma}(0, R-\epsilon)\right)>0$.

Let us consider the Dirichlet problem:

$$
\left\{\begin{array}{rlrl}
L(v) & =K-\left(\kappa+1 / 2|A|^{2}\right) & & \text { in } \quad B^{\Sigma}(0, R-\epsilon)  \tag{3.3.18}\\
v & =0 & \text { on } \quad \partial B^{\Sigma}(0, R-\epsilon) .
\end{array}\right.
$$

Since $\lambda_{1}\left(L, B^{\Sigma}(x, R-\epsilon)\right)>0$, we use Fredholm alternative (See [Theorem 6.15, Page 107] of GT15 ) to find a solution $v$. Setting $u=v+1$. Equations (3.3.8) give that

$$
L(u)=0 \text { in } B^{\Sigma}(x, R-\epsilon) \text { and } u=1 \text { on } \partial B^{\Sigma}(x, R-\epsilon)
$$

As in the proof of Theorem 3.3.4, the positivity of the operator $L$ implies that $u$ is a positive function.

Make a conformal change of the metric, $\tilde{d}^{2} s=u^{2} d^{2} s$ on $B^{\Sigma}(x, R-\epsilon)$. Let consider the minimizing geodesic $\gamma$ from $x$ to $\partial B^{\Sigma}(x, R-\epsilon)$.

Let $\tilde{a}$ and $a$ be the length of $\gamma$ in the metrics $\tilde{d}^{2} s$ and $d^{2} s$, respectively. Note that $R-\epsilon \leqslant a$.

The second variation formula of arclength shows that

$$
\begin{equation*}
\int_{0}^{\tilde{a}}\left(\left(\frac{d \phi}{\tilde{d} s}\right)^{2}-\tilde{K} \phi^{2}\right) \tilde{d} s \geqslant 0 \tag{3.3.19}
\end{equation*}
$$

where the smooth function $\phi$ has compact support in $(0, a), \tilde{d} s=u d s$ and $\frac{d \phi}{d s}=u^{-1} \frac{d \phi}{d s}$. We have that

$$
\begin{aligned}
& L(u)=\Delta_{\Sigma} u-K u+\left(\kappa(x)+1 / 2|A|^{2}\right) u=0 \\
& \tilde{K}=u^{-2}\left(K-\Delta_{\Sigma} \log u\right)
\end{aligned}
$$

where $\Delta_{\Sigma}$ is the Laplace-Beltrami operator in $\left(\Sigma, d^{2} s\right), \tilde{K}$ and $K$ are the sectional curvature of $\tilde{d}^{2} s$ and $d^{2} s$, respectively.

Therefore,

$$
\begin{equation*}
\phi^{2} \tilde{K} u=\frac{\phi^{2}}{u^{2}}\left(K u-\Delta_{\Sigma} u+\frac{\left|\nabla_{\Sigma} u\right|^{2}}{u}\right) \geqslant \frac{\phi^{2}}{u^{2}}\left(c u+\frac{\left|\nabla_{\Sigma} u\right|^{2}}{u}\right) \tag{3.3.20}
\end{equation*}
$$

where the second inequality follows from $L(u)=0$ and $\kappa \geqslant c$.

Putting (3.3.20) into (3.3.19), we see that

$$
\begin{equation*}
\int_{0}^{a} u^{-1} \phi^{2}\left(c+u^{-2}\left|u^{\prime}\right|^{2}\right) d s \leqslant \int_{0}^{a} u^{-1}\left(\frac{d \phi}{d s}\right)^{2} d s \tag{3.3.21}
\end{equation*}
$$

where $u(s)=u(\gamma(s))$. Setting $\phi:=u^{1 / 2} \psi$, where $\psi$ has compact support in $(0, a)$. We have

$$
\begin{aligned}
\int_{0}^{a} \psi^{2}\left(c+u^{-2}\left|u^{\prime}\right|^{2}\right) d s & \leqslant \int_{0}^{a} u^{-1}\left(u^{1 / 2} \psi^{\prime}+1 / 2 u^{-1 / 2} u^{\prime} \psi\right)^{2} d s \\
& =\int_{0}^{a}\left(\psi^{\prime}\right)^{2}+1 / 4\left(\frac{u^{\prime}}{u}\right)^{2} \psi^{2}+u^{-1} u^{\prime} \psi^{\prime} \psi d s \\
& \leqslant \int_{0}^{a}\left(\psi^{\prime}\right)^{2}+\left(\frac{u^{\prime}}{u}\right)^{2} \psi^{2}+1 / 3\left(\psi^{\prime}\right)^{2} d s
\end{aligned}
$$

where the last inequality follows from the Cauchy-Schwarz inequality $\left(\left|u^{-1} u^{\prime} \psi^{\prime} \psi\right| \leqslant\right.$ $\left.3 / 4\left(\frac{u^{\prime}}{u}\right)^{2} \psi^{2}+1 / 3\left(\psi^{\prime}\right)^{2}\right)$. Canceling the term $\left(\frac{u^{\prime}}{u}\right)^{2} \psi^{2}$ on the two sides of the inequality, we have

$$
\int_{0}^{a} \psi^{2} d s \leqslant \frac{4}{3 c} \int_{1}^{a}\left(\psi^{\prime}\right)^{2} d s .
$$

Choosing $\psi(s)=\sin \left(\pi a^{-1} s\right)$, we know that $1 \leqslant \frac{4}{3 c}\left(\frac{\pi}{a}\right)^{2}$. That is to say, $a \leqslant \frac{2 \pi}{\sqrt{3 c}}$.
Therefore, we see that for each $\epsilon>0$,

$$
R-\epsilon \leqslant a \leqslant \frac{2 \pi}{\sqrt{3 c}} .
$$

Namely, $d^{\Sigma}(0, \partial \Sigma)=R \leqslant \frac{2 \pi}{\sqrt{3 c}}$.
As a consequence, we have
Corollary 3.3.9. In a complete Riemannian 3-manifold of uniformly positive scalar curvature, any orientable complete stable minimal surface is compact and homeomorphic to $\mathbb{S}^{2}$.

Theorem 3.3.8 implies that a stable surface in a 3-manifold of uniformly positive scalar curvature is compact. By Corollary 3.3.6, it is homeomorphic to $\mathbb{S}^{2}$.

Together with Cohn-Vesson's inequality [Coh35], we generalize Corollary 3.3.6 to the non-compact case and obtain the so-called extrinsic Cohn-Vesson inequality (See [Theorem 5.8, Page 18] of Wan19a).

Theorem 3.3.10. (See Theorem 5.8 in Wan19a]) Let $\Sigma^{2} \subset\left(M^{3}, g\right)$ be a complete (non-compact) immersed stable minimal surface. If the complete manifold ( $M^{3}, g$ ) has non-negative scalar curvature $(\kappa(x) \geqslant 0)$, then

$$
\begin{equation*}
\int_{\Sigma} \kappa(x)+1 / 2|A|^{2} d v \leqslant 2 \pi \chi(\Sigma) \tag{3.3.22}
\end{equation*}
$$

Moreover, if $\kappa>0$ and $\Sigma$ is embedded, then $\Sigma$ is properly embedded.

Proof. From Equation (3.1.22), the stability operator can be written as $L:=\Delta_{\Sigma}-$ $K+\left(\kappa(x)+1 / 2|A|^{2}\right)$, where $\Delta_{\Sigma}$ is the Laplace-Beltrami operator of $\left(\Sigma, d^{2} s\right)$. Since the non-compact surface $\Sigma$ is stable minimal , we use Theorem 3.3.4 to find a positive fuction $u$ with $L(u)=0$.

Consider the metric $\tilde{d}^{2} s=u^{2} d^{2} s$. Let $\tilde{K}$ and $\tilde{d} v$ be its sectional curvature and its volume form. We see that

$$
\begin{equation*}
\tilde{K}=u^{-2}\left(K-\Delta_{\Sigma} \log u\right) \text { and } \tilde{d} v=u^{2} d v \tag{3.3.23}
\end{equation*}
$$

Theorem 3.3.7 shows that $\left(\Sigma, \tilde{d}^{2} s\right)$ is a complete surface with nonnegative sectional curvature $\tilde{K} \geqslant 0$. We use the Cohn-Vossen inequality Coh35 to have

$$
\begin{equation*}
\int_{\Sigma} \tilde{K} \tilde{d} v \leqslant 2 \pi \chi(\Sigma) \tag{3.3.24}
\end{equation*}
$$

Since $L(u)=0$, then $\int_{B^{\Sigma(x, R)}} u^{-1} L(u) d v=0$, where $B^{\Sigma}(x, R)$ is the geodesic ball in $\Sigma$ centered at $x \in \Sigma$ with radius $R$. We deduce that

$$
\begin{align*}
\int_{B^{\Sigma}(0, R)} \kappa(x)+\frac{1}{2}|A|^{2} d v & =\int_{B^{\Sigma}(0, R)}\left(K-u^{-1} \Delta_{\Sigma} u\right) d v \\
& =\int_{B^{\Sigma}(0, R)} K_{\Sigma}-\left(\Delta_{\Sigma} \log u+u^{-2}\left|\nabla_{\Sigma} u\right|\right) d v \\
& \leqslant \int_{B^{\Sigma}(0, R)} u^{-2}\left(K-\Delta_{\Sigma} \log u\right) u^{2} d v  \tag{3.3.25}\\
& =\int_{\left.B^{\Sigma}(0, R)\right)} \tilde{K} \tilde{d} v \\
& \leqslant \int_{\Sigma} \tilde{K} \tilde{d} v
\end{align*}
$$

Putting (3.3.24) into (3.3.25) and taking $R \rightarrow \infty$, we have that,

$$
\int_{\Sigma} \kappa(x)+1 / 2|A|^{2} d v \leqslant 2 \pi \chi(\Sigma)
$$

Remark that since $\Sigma$ admits a complete metric $\tilde{d}^{2} s$ of nonnegative sectional curvature, we see $\chi(\Sigma) \leqslant 1$ (See details in Corollary 3.3.11).

In the following, we consider the case that $\Sigma$ is embedded and $\kappa(x)>0$. We have that

$$
\begin{equation*}
\int_{\Sigma} \kappa(x) d v \leqslant 2 \pi \tag{3.3.26}
\end{equation*}
$$

Suppose that $\Sigma$ is not proper. There is an accumulation point $p$ of $\Sigma$ so that the set $B(p, r / 2) \cap \Sigma$ is a non-compact closed set in $\Sigma$. Namely, it is unbounded in $\left(\Sigma, d^{2} s\right)$. Hence, there is a sequence $\left\{p_{k}\right\}$ of points in $B(p, r / 2) \cap \Sigma$ going to infinity in $\left(\Sigma, d^{2} s\right)$.

Therefore, we may assume that the geodesic discs $B^{\Sigma}\left(p_{k}, r / 2\right)$ in $\Sigma$ are disjoint.
Define two constants $K_{0}:=\sup _{x \in B(p, r)}\left|K_{M}(x)\right| r_{0}:=\frac{1}{2} \min \left\{r, i_{0}, \frac{\pi}{\sqrt{K}}\right\}$ where $i_{0}:=$ $\inf _{x \in B(p, r)} \operatorname{Inj}_{M}(x)$ and $K_{M}$ is the sectional curvature of $(M, g)$. The geodesic ball $B^{\Sigma}\left(p_{k}, r_{0} / 2\right)$
in $\Sigma$ is contained in $B(p, r)$. Applying [Theorem 3, Appendix, Page 139] of [Fre96 to the geodesic disc $B^{\Sigma}\left(p_{k}, r_{0} / 2\right) \subset B(p, r)$, we have

$$
\operatorname{Area}\left(B^{\Sigma}\left(p_{k}, r_{0} / 2\right)\right) \geqslant C\left(r_{0}, K_{0}\right)
$$

This leads to a contradiction as follows:

$$
\begin{aligned}
2 \pi & \geqslant \int_{\Sigma} \kappa(x) d v \geqslant \int_{B(p, r) \cap \Sigma} \kappa(x) d v \\
& \geqslant \sum_{k} \int_{B^{\Sigma}\left(p_{k}, r_{0} / 2\right)} \kappa(x) \\
& \geqslant \inf _{x \in B(p, r)} \kappa(x) \cdot \sum_{k} \operatorname{Area}\left(B^{\Sigma}\left(p_{k}, r_{0} / 2\right)\right) \\
& \geqslant \inf _{x \in B(p, r)} \kappa(x) \cdot \sum_{k} C=\infty
\end{aligned}
$$

Combining Theorem 3.3.7 and Theorem 3.3.10, we will give a new proof of the result of Theorem 2 in $\mathbf{S Y 8 2}$.

Corollary 3.3.11. (See [Theorem 2, Page 211] of [SY82]) Let $\Sigma \subset(M, g)$ be an oriented complete non-compact stable minimal surface. If the complete manifold $\left(M^{3}, g\right)$ has nonnegative scalar curvature $(\kappa(x) \geqslant 0)$, then $\Sigma$ is diffeomorphic to $\mathbb{R}^{2}$ or $\mathbb{S}^{1} \times \mathbb{R}$. If the latter case occurs, then $\Sigma$ is totally geodesic and the scalar curvature $\kappa$ of $M$ is zero along $\Sigma$.

Moreover, if $\kappa(x)>0$, then $\Sigma$ is diffeomorphic to $\mathbb{R}^{2}$.
Proof. Since $\Sigma$ is stable minimal, we use Theorem 3.3 .4 to find a positive function $u$ with $L(u)=0$, where $L$ is the stability operator (See (3.1.21) and (3.1.22)). By Theorem 3.3.7. $\left(\Sigma, u^{2} d^{2} s\right)$ is a complete 2 -manifold with nonnegative sectional curvature, where $d^{2} s$ is the induced metric.

We apply the Soul theorem (See Theorem 1.11 and Theorem 2.1 in $\mathbf{C G 7 2}$ ) to $\left(\Sigma, u^{2} d^{2} s\right)$. This theorem asserts that if $(X, g)$ is a connected complete manifold with nonnegative sectional curvature, there is a compact totally convex, totally geodesic submanifold (called a soul of $(X, g)$ ) such that $X$ is diffeomorphic to the normal bundle of the submanifold.

Therefore, there is a submanifold $S \subset \Sigma$ (i.e. a soul) such that $\Sigma$ is the normal bundle of $S$. In addition, since $\Sigma$ is non-compact, we see that $\operatorname{dim}(S)<\operatorname{dim}(\Sigma)(\operatorname{dim}(S)=0$ or $\operatorname{dim}(S)=1)$.

Case (I) If $\operatorname{dim}(S)=0, S$ is a point. That is to say, $\Sigma$ is diffeomorphic to $\mathbb{R}^{2}$.
Case (II) If $\operatorname{dim}(S)=1, S$ is homeomorphic to $\mathbb{S}^{1}$. Since $\Sigma$ is oriented, the normal bundle is trivial. That is to say, $\Sigma$ is homeomorphic to $\mathbb{S}^{1} \times \mathbb{R}$. We use Theorem 3.3.10 to see that

$$
\int_{\Sigma} \kappa(x)+1 / 2|A|^{2} d v \leqslant 2 \pi \chi(\Sigma)
$$

In this case, we see that $\chi(\Sigma)=0$. Therefore, $\Sigma$ is totally geodesic and $\kappa=0$ on $\Sigma$.

If $\kappa>0$, the latter case never occurs. We can conclude that $\Sigma$ is diffeomorphic to $\mathbb{R}^{2}$.

Finally, we give an application of Lemma 3.3.8 in a complete 3-manifold.
Theorem 3.3.12 (See Theorem $A$ and Theorem 1.1 in Wan19c]). Assume that $\left(M^{3}, g\right)$ is a contractible complete 3-manifold. If there exists a number $\alpha \in(-\infty, 2)$ such that

$$
\liminf _{r(x) \rightarrow \infty} r^{\alpha}(x) \kappa(x)>0
$$

where $\kappa(x)$ is the scalar curvature of $(M, g)$ and $r(x)$ is the distance function from some point $0 \in M$ to $x$, then $M^{3}$ is diffeomorphic to $\mathbb{R}^{3}$.

Proof. From our assumption, there are two positive constants $C$ and $R_{0}$ such that if $r(x)>R_{0}$, then

$$
\begin{equation*}
\kappa(x) \geqslant \frac{C}{r^{\alpha}(x)} . \tag{3.3.27}
\end{equation*}
$$

Claim: If $R>2 \max \left\{R_{0},\left(\frac{4^{1+\alpha / 2} \pi}{(3 C)^{1 / 2}}\right)^{\frac{2}{2-\alpha}}\right\}$, then the induced map $\pi_{1}(M \backslash B(0,4 R)) \rightarrow$ $\pi_{1}(M \backslash B(0, R))$ is trivial.

Suppose the contrary that there exists some $R>2 \max \left\{R_{0},\left(\frac{4^{1+\alpha / 2} \pi}{(3 C)^{1 / 2}}\right)^{\frac{2}{2-\alpha}}\right\}$ so that the induced map $\pi_{1}(M \backslash B(0,4 R)) \rightarrow \pi_{1}(M \backslash B(0, R))$ is non-trivial. That is to say, there is a simple closed curve $\gamma \subset M \backslash B(0,4 R)$ which is not contractible in $M \backslash B(0, R)$.

We use the work of Morrey [Mor09, Mor48] to find an area-minimizing disc $\Omega$ with boundary $\gamma$. The surface $\Omega$ intersects the set $B(0, R)$. Therefore, $\Omega \cap \partial B(0,2 R)$ and $\Omega \cap \partial B(0,4 R)$ are both nonempty.

Let us consider the set $\Sigma:=\Omega \cap(B(0,4 R) \backslash B(0,2 R))$. It is a stable minimal surface in $(M, g)$ whose boundary is contained in the disjoint union of $\partial B(0,4 R)$ and $\partial B(0,2 R)$. Since $\kappa(x) \geqslant \frac{C}{(4 R)^{\alpha}}$ on $B(0,4 R) \backslash B(0,2 R)$, we use Lemma 3.3.8 to know that

$$
\Sigma \text { is contained in the } \frac{2(4 R)^{\alpha / 2} \pi}{(3 C)^{1 / 2}} \text {-neighborhood of } \partial \Sigma .
$$

Since $\partial \Sigma \cap \partial B(0,2 R)$ and $\partial \Sigma \cap \partial B(0,4 R)$ are both nonempty, we see that

$$
\begin{equation*}
2 R=d(\partial B(0,2 R), \partial B(0,4 R)) \leqslant 2 \frac{2(4 R)^{\alpha / 2} \pi}{(3 C)^{1 / 2}} \tag{3.3.28}
\end{equation*}
$$

That is to say, $R<\left(\frac{4^{1+\alpha / 2} \pi}{(3 C)^{1 / 2}}\right)^{\frac{2}{2-\alpha}}$. This is in contradiction of the choice of $R$. This finishes the proof of Claim.

From the above Claim, we see that $M$ is simply-connected at infinity. From Remark 1.1.8, we know that $M$ is diffeomorphic to $\mathbb{R}^{3}$.

Corollary 3.3.13 (See Theorem 3.5 in Wan19c ). Assume that $\left(M^{3}, g\right)$ is a simplyconnected open 3-manifold with $\pi_{2}(M)=\mathbb{Z}$. Let $0 \in M$ be a point and $r(x)$ a distance function from $x$ to 0 . If there exists a real number $\alpha \in[0,2)$, such that,

$$
\liminf _{r(x) \rightarrow \infty} r^{\alpha}(x) \kappa(x)>0,
$$

then $M^{3}$ is diffeomorphic to $\mathbb{R} \times \mathbb{S}^{2}$.
See the proof in $\overline{\mathbf{W a n 1 9 c}}$

## CHAPTER 4

## Convergence

In this chapter, we consider the convergence theory for minimal surfaces. First, we consider the convergence theory for minimal surface equations (See Equation (3.1.13)). Together with local properties of minimal surfaces, we discuss a classical theorem about convergence with finite multiplicity.

Then, we will introduce a new concept, the lamination. Our focus is the convergence theory for stable minimal laminations. Finally, we construct a family of stable minimal lamination in a complete contractible 3-manifold and discuss its limit.

### 4.1. The smooth convergence

Let $u$ be a function from the unit disc $\mathbb{B}^{2}$ to $\mathbb{R}$. Its image in $\mathbb{R}^{3}$ is a minimal surface if and only if

$$
\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^{2}}}\right)=0 .
$$

Lemma 4.1.1. Let $\left\{f_{i}\right\}_{i}$ be a sequence of functions from the unit disc $\mathbb{B}^{2}$ to $\mathbb{R}$. Each graph of $f_{i}$ in $\mathbb{R}^{3}$ is minimal. If $\left\|f_{i}\right\|_{C^{2}} \leqslant C<\infty$, then up to extracting a subsequence, $f_{i}$ converges smoothly on compact sets of $\mathbb{B}^{2}$ to $f$ and the graph of $f$ is also a minimal surface.

Proof. By Arzelà-Ascoli Lemma, we may extract a subsequence so that $f_{k}$ converges to $f$ in $C^{1, \alpha}$-topology on a compact set for $0<\alpha<1$. According to the minimal surface equation (See Equation (3.1.13) and the Schauder estimate for linear elliptic equation (See [Theorem 6.2, Page 90] of GT15), we see that this sequence converges in $C^{m, \alpha_{-}}$ topology on a compact set for each $m$. As a consequence, $f$ also satisfies the minimal surface equation (see Equation (3.1.13)). That is to say, its graph is minimal.

Definition 4.1.2. In a complete Riemannian 3-manifold $(M, g)$, a sequence $\left\{\Sigma_{n}\right\}$ of immersed minimal surfaces converges smoothly with finite multiplicity (at most $m$ ) to an immersed minimal surface $\Sigma$, if for each point $p$ of $\Sigma$, there is a disc neighborhood $D$ in $\Sigma$ of $p$, an integer $m$ and a neighborhood $U$ of $D$ in $M$ (consisting of geodesics of $M$ orthogonal to $D$ and centered at the points of $D$ ) so that for $n$ large enough, each $\Sigma_{n}$ intersects $U$ in at most $m$ connected components. Each component is a graph over $D$ in the geodesic coordinates. Moreover, each component converges to $D$ in $C^{2, \alpha}$-topology as $n$ goes to infinity.

Note that in the case that each $\Sigma_{n}$ is embedded, the surface $\Sigma$ is also embedded. The multiplicity at $p$ is equal to the number of connected component of $\Sigma_{n} \cap U$ for $n$ large enough. It remains constant on each component of $\Sigma$.

REMARK 4.1.3. Let us consider a family $\left\{\Sigma_{n}\right\}_{n}$ of properly embedded minimal surfaces converging to the minimal surface $\Sigma$ with finite multiplicity. Fix a compact simplyconnected subset $D \subset \Sigma$. Let $U$ be the tubular neighborhood of $D$ in $M$ with radius $\epsilon$ and $\pi: U \rightarrow D$ the projection from $U$ onto $D$. It follows that the restriction $\left.\pi\right|_{\Sigma_{n} \cap U}$ : $\Sigma_{n} \cap U \rightarrow D$ is a $m$-sheeted covering map for $\epsilon$ small enough and $n$ large enough, where $m$ is the multiplicity.

Therefore, the restriction of $\pi$ to each component of $\Sigma_{n} \cap U$ is also a covering map. Hence, since $D$ is simply-connected, it is bijective. Therefore, each component of $\Sigma_{n} \cap U$ is a normal graph over $D$.

Let us recall a classical theorem about convergence with finite multiplicity.
Theorem 4.1.4. (See [Theorem 4.37, Page 49] of [MRROZ] or [Compactness Theorem, Page 96] of $\left[\right.$ And85]) Let $\left\{\Sigma_{k}\right\}_{k \in \mathbb{N}}$ be a family of properly embedded minimal surfaces in a complete 3 -manifold $\left(M^{3}, g\right)$ satisfying (1) each $\Sigma_{k}$ intersects a given compact set $K_{0}$; (2) for any compact set $K$ in $M$, there are three constants $C_{1}=C_{1}(K)>0$, $C_{2}=C_{2}(K)>0$ and $k_{0}=k_{0}(K) \in \mathbb{N}$ such that for each $k \geqslant k_{0}$, it holds that
(1) $\left|A_{\Sigma_{k}}\right|^{2} \leqslant C_{1}$ on $K \cap \Sigma_{k}$, where $\left|A_{\Sigma_{k}}\right|^{2}$ is the square length of the second fundamental form of $\Sigma_{k}$,
(2) $\operatorname{Area}\left(\Sigma_{k} \cap K\right) \leqslant C_{2}$.

Then, after passing to a subsequence, $\Sigma_{k}$ converges to a properly embedded minimal surface with finite multiplicity in the $C^{\infty}$-topology.

Proof. Choose a point $p_{k} \in \Sigma_{k} \cap K_{0}$. Extracting a subsequence, the sequence $\left\{p_{k}\right\}$ converges to some point $p \in M$. We may suppose that the unit normal vector $\left.\overrightarrow{\mathbf{n}}_{\Sigma_{k}}\right|_{p_{k}}$ to the surface $\Sigma_{k}$ at $p_{k}$ converges to some unit vector in $T_{p} M$. Namely, the tangent space $T_{p_{k}} \Sigma_{k} \subset T_{p_{k}} M$ converges to some plane $T \subset T_{p} M$. From (1) and Lemma 3.2.1, we can express $\Sigma_{k}$ locally (near $p$ ) as some graphs of functions over $T$. That is to say, there is an open geodesic ball $U$ centered at $p$ such that for $k$ large enough,

- each component of $U \cap \Sigma_{k}$ is the graph of some function on $T$. The Hessian of this function is bounded by the bound of the second fundamental form (See Lemma 3.2.1 and (1));
- the number of components of $U \cap \Sigma_{k}$ is bounded by the area bound from Theorem 3.2.7 and (2);

We use Lemma 4.1.1 to extract a subsequence converging to a minimal surface. Then, a diagonal argument allows us find a subsequence converging smoothly to a minimal surface $\Sigma$. The area bound (See (2)) implies that this sequence converges with finite multiplicity.

From the strong maximal principle (See Corollary 3.2.1), the minimal surface $\Sigma$ is embedded or self-intersects transversally. Since $\Sigma_{k}$ is embedded, we can conclude that $\Sigma$ is also embedded.

In the following, we consider the convergence theory for stable minimal surfaces. For stable minimal surfaces in a 3-manifold, Schoen gave a uniform bound of the second fundamental form.

Lemma 4.1.5. (See [Theorem 3, Page 122] in [Sch83]) Let $\Sigma$ be an immersed stable surface in a Riemannian manifold $\left(M^{3}, g\right)$. Given $r_{0} \in(0,1]$, and a point $p_{0} \in \Sigma$, if $\Sigma \cap B\left(p_{0}, r_{0}\right)$ has compact closure in $\Sigma$, there are two constants $0<\epsilon_{0}<1$ and $C>0$ depending only on the metric $g$ near $p_{0}$ and the injective radius of $(M, g)$ at $p_{0}$ so that

$$
\left|A_{\Sigma}\right|^{2} \leqslant C r_{0}^{-2} \text { on } B\left(p_{0}, \epsilon_{0} r_{0}\right) \cap \Sigma
$$

As a consequence, we have
Corollary 4.1.6. Let $(M, g)$ be a complete Riemannian manifold and $\left\{\Sigma_{k}\right\}$ a sequence of complete embedded stable minimal surfaces. If for any compact set $K \subset M$, there is a constant $C_{1}$ depending on $K$, satisfying that for each $k$

$$
\operatorname{Area}\left(\Sigma_{k} \cap K\right) \leqslant C_{1}
$$

then after extracting a subsequence, $\Sigma_{k}$ converges smoothly to a complete stable minimal surface with finite multiplicity.

### 4.2. Minimal lamination

In the following, we assume that a complete contractible 3-manifold $(M, g)$ is not homeomorphic to $\mathbb{R}^{3}$. As in Remark 1.1.13, $M$ is an increasing union of closed handlebodies $\left\{N_{k}\right\}_{k}$ satisfying that for each $k$,

- $N_{k}$ is homtopically trivial in $N_{k+1}$;
- None of the $N_{k}$ is contained in a 3-ball.

In addition, for each $k$, the genus of $N_{k}$ is greater than zero. (If not, there is some handlebody $N_{k}$ of genus zero, namely a 3 -ball. That is to say, $N_{0}$ is contained in a 3 -ball $N_{k}$ which is in contradiction with the last paragraph. )
4.2.1. Construction of minimal laminations. From Lemma 1.2.7, each $N_{k}$ has a system of meridians $\left\{\gamma_{k}^{l}\right\}_{l=1}^{g\left(N_{k}\right)}$, where $g\left(N_{k}\right)$ is the genus of $N_{k}$. Our target now is to construct a lamination $\mathscr{L}_{k}:=\cup_{l} \Omega_{k}^{l} \subset N_{k}$ (i.e. a disjoint union of embedded surfaces) with $\partial \Omega_{k}^{l}=\gamma_{k}^{l}$ and "good" properties.

Let us recall a result of Meeks and Yau (See Theorem 3.1.8 or [Theorem 6.28 Page 224] of [CM11]). It provides us a geometric version of loop theorem to construct them.

Theorem. (See MY80, MY82, [Theorem 6.28 Page 224] of CM11] or Theorem 3.1.8 Let $\left(M^{3}, g\right)$ be a compact Riemannian 3-manifold whose boundary is mean convex and $\gamma$ a simple closed curve in $\partial M$ which is null-homotopic in $M$. Then, $\gamma$ bounds an area-minimizing disc and any such least area disc is properly embedded.

Remark 4.2 The boundary $\partial M$ is mean convex. That is, $\partial M$ is a piecewise smooth 2-manifold consisting of smooth surfaces $\{H\}_{i}$. On each $H_{i}$, the mean curvature is nonnegative.

Let $\Sigma$ be an embedded area-minimizing disc with boundary $\gamma$. It intersects $\partial M$ transversally. Therefore, Int $\Sigma$ is contained in Int $M$.

Our strategy is to apply this theorem to $\left(N_{k},\left.g\right|_{N_{k}}\right)$ for each $k$. However, the boundary of $N_{k}$ may be not mean convex. To overcome it, we find a new metric $g_{k}$ on $N_{k}$ so that

1) ( $N_{k}, g_{k}$ ) is a 3-manifold with mean convex boundary;
2) $\left.g_{k}\right|_{N_{k-1}}=\left.g\right|_{N_{k-1}}$.

The metric $g_{k}$ is constructed as below:
Let $h(t)$ be a positive smooth function on $\mathbb{R}$ so that $h(t)=1$, for any $t \in \mathbb{R} \backslash[-\epsilon, \epsilon]$. Consider the function $f(x):=h\left(d\left(x, \partial N_{k}\right)\right)$ and the metric $g_{k}:=\left.f^{2} g\right|_{N_{k}}$. Under $\left(N_{k}, g_{k}\right)$, the mean curvature $\hat{H}(x)$ of $\partial N_{k}$ is

$$
\hat{H}(x)=h^{-1}(0)\left(H(x)+2 h^{\prime}(0) h^{-1}(0)\right)
$$

Choosing $\epsilon>0$ small enough and a function $h$ with $h(0)=2$ and $h^{\prime}(0)>2 \max _{x \in \partial N_{k}}|H(x)|+$ 2 , one gets the metric $g_{k}$ which is the required candidate in the assertion.

In the following, we inductively construct the lamination $\mathscr{L}_{k} \subset\left(N_{k}, g_{k}\right)$.
When $l=1$, there is an embedded area-minimizing disc $\Omega_{k}^{1} \subset\left(N_{k}, g_{k}\right)$ with boundary $\gamma_{k}^{1}$ (See Theorem 3.1.8 or Theorem 6.28 of [CM11]). As in Remark 4.2, it intersects $\partial N_{k}$ transversally. Then, Int $\Omega_{k}^{1} \subset \operatorname{Int} N_{k}$.

Suppose that there are $l$ disjointly embedded stable minimal discs $\left\{\Omega_{k}^{i}\right\}_{i=1}^{l}$ with $\partial \Omega_{k}^{i}=$ $\gamma_{k}^{i}$.

Let us consider the Riemannian manifold $\left(T_{k, l},\left.g_{k}\right|_{T_{k, l}}\right)$, where $T_{k, l}:=N_{k} \backslash \amalg_{i=1}^{l} \Omega_{k}^{l}$. It is a handlebody of genus $g\left(N_{k}\right)-l$. For example, see the following figure.


Figure 4.1.
The boundary of $\left(T_{k, l},\left.g_{k}\right|_{T_{k, l}}\right)$ consists of two different parts. One is $\partial N_{k} \backslash \amalg_{i=1}^{l} \gamma_{l}^{i}$. The mean curvature is positive on this part. The other is $2 l$ disjoint discs $\left\{\Omega_{k}^{i}\right\}_{i=1}^{l}$ and $\left\{\Omega_{k}^{i+}\right\}_{i=1}^{l}$. The two discs $\Omega_{k}^{i-}$ and $\Omega_{k}^{i+}$ are two sides of the same minimal disc $\Omega_{k}^{i}$. The mean curvature vanishes on these discs.

Therefore, the boundary of $\left(T_{k, l},\left.g_{k}\right|_{T_{k, l}}\right)$ is mean convex (See Remark 4.2). In addition, $\left\{\gamma_{k}^{i}\right\}_{i>l}$ is a system of meridian of the handlebody $\left(T_{k, l},\left.g_{k}\right|_{T_{k}, l}\right)$.

Then, we use Theorem 3.1.8 and the above remark to find an embedded stable minimal surface $\Omega_{k}^{l+1}$ in the closure of $\left(T_{k, l},\left.g_{k}\right|_{T_{k, l}}\right)$ with boundary $\gamma_{k}^{l+1}$.

From Remark 4.2, $\Omega_{k}^{l+1}$ intersects the boundary of $\left(T_{k, l},\left.g_{k}\right|_{T_{k, l}}\right)$ transversally. Hence, Int $\Omega_{k}^{l+1}$ is contained in Int $T_{k, l}$. That is to say, $\left\{\Omega_{k}^{i}\right\}_{i=1}^{l+1}$ are disjoint stable minimal surfaces for $\left(N_{k}, g_{k}\right)$.

This finishes the inductive construction.
To sum up, there exist $g\left(N_{k}\right)$ disjointly embedded meridian discs $\left\{\Omega_{k}^{l}\right\}$. Define the lamination $\mathscr{L}_{k}$ by $山_{l} \Omega_{k}^{l}$. It is a stable minimal lamination for the new metric $g_{k}$ and for the original one away from $\partial N_{k}$ (near $N_{k-1}$, for example).

The set $\mathscr{L}_{k} \cap N_{k-1}$ is a stable minimal lamination in $(M, g)$. Each leaf has its boundary contained in $\partial N_{k-1}$. In addition, since $N_{0}$ is not contained in a 3-ball, We can conclude that each lamination $\mathscr{L}_{k}$ intersects $N_{0}$. The reason is below:

If the set $\mathscr{L}_{k} \cap N_{0}$ is empty, we choose a tubular neighborhood $N\left(\mathscr{L}_{k}\right)$ in $N_{k}$ with small radius so that the set $N\left(\mathscr{L}_{k}\right) \cap N_{0}$ is also empty. That is to say, $N_{0}$ lies in the handlebody $N_{k} \backslash N\left(\mathscr{L}_{k}\right)$ of genus zero (i.e. a 3 -ball). This is in contradiction with our assumption that $N_{0}$ is not contained in a 3 -ball.
4.2.2. Limits of minimal laminations. Let us consider the sequence $\left\{\mathscr{L}_{k}\right\}_{k}$ and its limit. From Lemma 4.1.5, we know that the sequence $\left\{\mathscr{L}_{k}\right\}_{k}$ satisfies the condition (1) in Theorem 4.1.4. However, it may not hold the condition (2) in Theorem 4.1.4.

For example, in the Whitehead manifold, each $N_{k}$ is of genus one. The lamination $\mathscr{L}_{k}$ is a meridian disc $\Omega_{k}^{1} \subset N_{k}$. From Theorem 2.1.2, $\Omega_{k}^{1} \cap \operatorname{Int} N_{1}$ has at least $2^{k-1}$ components intersecting $N_{0}$. We know that for $k>1$, each component $(\Sigma, \partial \Sigma) \subset\left(N_{1}, \partial N_{1}\right)$ of $\Omega_{k}^{1} \cap$ $\operatorname{Int} N_{1}$ is a stable minimal surface in $(M, g)$.

Choose $x_{0} \in \Sigma \cap N_{0}$ and $r_{0}=\frac{1}{2} \min \left\{r, i_{0}\right\}$, where $r:=\operatorname{dist}\left(\partial N_{0}, \partial N_{1}\right)$ and $i_{0}:=$ $\inf _{x \in N_{1}} \operatorname{Inj}_{M}(x)$. We see that the ball $B\left(x_{0}, r_{0}\right)$ is in $N_{1}$. We apply a result [Lemma 1, Page 445] in MY80 to $\left(N_{1}, \partial N_{1}\right)$. Hence, it follows that

$$
\operatorname{Area}(\Sigma) \geqslant \operatorname{Area}\left(\Sigma \cap B\left(x_{0}, r_{0}\right)\right) \geqslant C\left(i_{0}, r_{0}, K\right)
$$

where $K:=\sup _{x \in N_{1}}\left|K_{M}(x)\right|$ and $K_{M}$ is the sectional curvature.
Therefore, one has that Area $\left(N_{1} \cap \Omega_{k}^{1}\right) \geqslant 2^{k-1} C$. The area of $\mathscr{L}_{k} \cap N_{1}$ goes to infinity as $k$ goes to infinity. That is to say, the sequence $\left\{\mathscr{L}_{k}\right\}_{k}$ does not satisfy Condition (2).

Generally, the sequence $\left\{\mathscr{L}_{k}\right\}_{k}$ may not sub-converge with finite multiplicity. In the following, we consider the convergence toward a lamination.

Definition 4.2.1. A codimension one lamination in a 3 -manifold $M^{3}$ is a collection $\mathscr{L}$ of smooth disjoint surfaces (called leaves) such that $\bigcup_{L \in \mathscr{L}} L$ is closed in $M^{3}$. Moreover, for each $x \in M$ there exists an open neighborhood $U$ of $x$ and a coordinate chart $(U, \Phi)$, with $\Phi(U) \subset \mathbb{R}^{3}$ so that in these coordinates the leaves in $\mathscr{L}$ pass through $\Phi(U)$ in slices form

$$
\mathbb{R}^{2} \times\{t\} \cap \Phi(U)
$$

A minimal lamination is a lamination whose leaves are minimal. Finally, a sequence of laminations is said to converge if the corresponding coordinate maps converge in $C^{0, \alpha_{-}}$ topology.

For example, $\mathbb{R}^{2} \times \Lambda$ is a lamination in $\mathbb{R}^{3}$, where $\Lambda$ is a closed set in $\mathbb{R}$.
Note that any (compact) embedded surface (connected or not) is a lamination. In [Appendix B, Laminations] of [CM04], Coding and Minicozzi describ the limit of laminations with uniformly bounded curvatures.

Proposition 4.2.2. (See Proposition B.1, Page 610] in CMO4)Let $M^{3}$ be a fixed 3-manifold. If $\mathscr{L}_{i} \subset B(x, 2 R) \subset M$ is a sequence of minimal laminations with uniformly bounded curvatures (where each leaf has boundary contained in $\partial B(0,2 R)$ ), then a subsequence, $\mathscr{L}_{j}$, converges in the $C^{\alpha}$-topology for any $\alpha<1$ to a (Lipschitz) lamination $\mathscr{L}$ in $B(x, R)$ with minimal leaves.

We use Proposition 4.2.2 to show that
Theorem 4.2.3. The sequence $\left\{\mathscr{L}_{k}\right\}$ of laminations sub-converges to a lamination $\mathscr{L}$. Moreover, Each leaf in $\mathscr{L}$ is a complete minimal surface.

Proof. As constructed above, the intersection $\mathscr{L}_{j} \cap N_{k}$ is a stable minimal lamination or any $j>k$. It may have many leaves (connected components). Each leaf has boundary contained in $\partial N_{k}$. In addition, $\mathscr{L}_{j}$ intersects $N_{0}$.

From Lemma 4.1.5, there is a constant $C\left(N_{k-1}\right)$, depending on $N_{k}$ and $g$, so that for any $j>k$,

$$
\left|A_{\mathscr{L}_{j}}\right|^{2} \leqslant C\left(N_{k-1}\right) \text { on } N_{k-1} .
$$

Therefore, for $j>3,\left\{\mathscr{L}_{j} \cap N_{2}\right\}$ is a sequence of minimal laminations with uniformly bounded curvature where each leaf has boundary contained in $\partial N_{2}$. We use Proposition 4.2 .2 to extract a subsequence converging to a minimal lamination in $N_{1}$. Each leaf has boundary contained in $\partial N_{1}$.

We repeat the argument on each $N_{k}$. A diagonal argument allows us to find a subsequence of $\left\{\mathscr{L}_{k}\right\}$ converging to a lamination $\mathscr{L}$. Each leaf is a complete minimal surface.

For our convenience, we may assume that the sequence $\mathscr{L}_{k}$ converges to $\mathscr{L}$. In the following, we will show that if $(M, g)$ has positive scalar curvature, then each leaf in $\mathscr{L}$ is a (non-compact) stable minimal surface.

Lemma 4.2.4. Let $\Sigma$ be a compact minimal surface in a 3-manifold ( $X, g$ ) (possibly with boundary) and $(\hat{X}, \hat{g})$ the double cover of $(X, g)$. The lift $\hat{\Sigma}$ of $\Sigma$ is a connected minimal surface in $(\hat{X}, \hat{g})$. Then $\Sigma$ is stable minimal if an only if $\hat{\Sigma}$ is stable minimal.

Proof. Let $L$ and $\hat{L}$ be the stable operators of $\Sigma$ and $\hat{\Sigma}_{k}$ respectively. The operator $p^{*}(L)$ is equal to $\hat{L}$, where $p: \hat{\Sigma} \rightarrow \Sigma$ is the double cover. Let $\lambda_{1}$ and $\hat{\lambda}_{1}$ be the first eigenvalues of $L$ and $\hat{L}$ respectively.

It is sufficient to show that $\hat{\lambda}_{1}=\lambda_{1}$.
Let $f$ be an eigenfunction for $\lambda_{1}$ (i.e. $L(f)=-\lambda_{1} f$ ). The function $\hat{f}:=p^{*}(f)$ satisfies that $\hat{L}(\hat{f})=-\lambda_{1} \hat{f}$. Hence, $\lambda_{1} \geqslant \hat{\lambda}_{1}$.

Let $\tau$ be the desk transformation of the double cover $p$. The surface $\hat{\Sigma}$ is $\tau$-invariant, (namely $\tau(\hat{\Sigma})=\hat{\Sigma}$ ). The map $\tau$ is isometric on $\hat{\Sigma}$.

Let $\hat{h}$ be an eigenfunction for $\hat{\lambda}_{1}$ (i.e. $\hat{L}(\hat{h})=-\hat{\lambda}_{1} \hat{h}$ ). We may assume that $\hat{h}$ is $\tau$ invariant $(\tau(\hat{h})=\hat{h})$. (If not, we replace it by $\tau(\hat{h})+\hat{h}$.) The function $\hat{h}$ is equal to $p^{*}(h)$, where $h$ is a smooth function on $\Sigma$. We have that $L(h)=-\hat{\lambda}_{1} h$. Therefore, $\hat{\lambda}_{1} \geqslant \lambda_{1}$. We can conclude that $\hat{\lambda}_{1}=\lambda_{1}$.

Theorem 4.2.5. Each leaf in $\mathscr{L}$ is stable minimal.
Proof. Let $L_{t}$ be a leaf in the minimal lamination $\mathscr{L}$.
Case (I): If $L_{t}$ is a limit leaf (that is to say, the closure of $\mathscr{L} \backslash L_{t}$ contains $L_{t}$ ), we use the result of Meek, Pérez and Rosenberg (See [Theorem 1, Page 4] of [MPR08]) to have that it is stable minimal.

Case (II): If $L_{t}$ is not a limit leaf, the intersection $L_{t} \cap \overline{\mathscr{L} \backslash L_{t}}$ is empty. There is a tubular neighborhood $N\left(L_{t}\right)$ of $L_{t}$ such that the intersection $N\left(L_{t}\right) \cap \overline{\mathscr{L} \backslash L_{t}}$ is empty.

Let $\pi$ be the projection from $N\left(L_{t}\right)$ to $L_{t}$. For any point $p$ and $r>0$, we consider the geodesic disc $B^{L_{t}}(p, r)$ in $L_{t}$ and the set $N(p, r):=\pi^{-1}\left(B^{L_{t}}(p, r)\right)$.

Choose one component $\Sigma_{k}$ of $N(p, r) \cap \mathscr{L}_{k}$ for $k \geqslant 0$. Since $\mathscr{L}_{k}$ converges to $\mathscr{L}$, the sequence $\left\{\Sigma_{k}\right\}_{k}$ converges to some subset of $\mathscr{L}$. In addition, the intersection $\mathscr{L} \cap N(p, r)$ has the unique component, $B^{L_{t}}(p, r)$. Hence, $\Sigma_{k}$ converges to $B^{L_{t}}(p, r)$.

We first consider the case when $L_{t}$ is 2 -sided. In the following, we show that $B^{L_{t}}(p, r)$ is stable minimal for any $r>0$.

Step 1: Define the function $d_{k}: \Sigma_{k} \rightarrow \mathbb{R}$.
Let $\overrightarrow{\mathbf{n}}(x)$ be the unit normal vector to $L_{t}$ at $x$ and $\pi_{k}:=\left.\pi\right|_{\Sigma_{k}}$. The map $\pi_{k}: \Sigma_{k} \rightarrow$ $B^{L_{t}}(p, r)$ is a covering map for $k$ large enough.

We define the function $d_{k}: \Sigma_{k} \rightarrow \mathbb{R}$ as follows:

$$
d_{k}(x)=<\exp _{\pi_{k}(x)}^{-1}(x), \overrightarrow{\mathbf{n}}\left(\pi_{k}(x)\right)>
$$

Step 2: $\pi_{k}$ is injective.
Recall that the area of $\mathscr{L}_{k}$ is finite. The subset $\Sigma_{k}$ has a finite area. Therefore, $\pi_{k}$ is a finite cover for $k$ large enough.

We argue it by contradiction. Suppose that $\pi_{k}$ is a $m$-sheeted covering $(m>1)$.
Let us consider three sets in $\Sigma_{k}$ as follows:

$$
\begin{aligned}
& I^{\text {Top }}:=\left\{x \in \Sigma_{k} \mid d_{k}(x)=\max \left\{d_{k}\left(x^{\prime}\right) \mid x^{\prime} \in \pi_{k}^{-1}\left(\pi_{k}(x)\right\}\right\}\right. \\
& I^{\text {Med }}=\left\{x \in \Sigma_{k} \mid \min \left\{d_{k}\left(x^{\prime}\right) \mid x^{\prime} \in \pi_{k}^{-1}\left(\pi_{k}(x)\right)\right\}<d_{k}(x)<\max \left\{d_{k}\left(x^{\prime}\right) \mid x^{\prime} \in \pi_{k}^{-1}\left(\pi_{k}(x)\right\}\right\}\right. \\
& I^{B o t}=\left\{x \in \Sigma_{k} \mid d_{k}(x)=\min \left\{d_{k}\left(x^{\prime}\right) \mid x^{\prime} \in \pi_{k}^{-1}\left(\pi_{k}(x)\right)\right\}\right\}
\end{aligned}
$$

From the homotopy lifting property of $\pi_{k}$ and $m>1$, these disjoint three set are open. In addition, $\Sigma=I^{T o p} \amalg I^{M e d} \amalg I^{B o t}$. Since $\Sigma_{k}$ is connected, there are at least two empty sets in these three sets.

However, since $m>1, I^{\text {Top }}$ and $I^{\text {Bot }}$ are two nonempty sets. This is in contradiction with the last paragraph. We conclude that $\pi_{k}$ is injective.

Step 3: The stability of $B^{L_{t}}(x, r)$.

As in Step $2, \Sigma_{k}$ can be written as the graph of some function $f_{k}$ over $B^{L_{t}}(p, r)$ for $k$ large enough. The sequence $\left\{f_{k}\right\}$ converges in $C^{0, \alpha}$-topology. Since $\Sigma_{k}$ is stable minimal, we use Lemma 4.1.5 and Lemma 4.1.1 to have that the sequence $\left\{f_{k}\right\}$ converges in $C^{2, \alpha_{-}}$ topology. Namely, $\Sigma_{k}$ converges to $B^{L_{t}}(p, r)$ in $C^{2, \alpha}$-topology. Therefore, $B^{L_{t}}(p, r)$ is stable minimal for any $r>0$.

Therefore, we can conclude that $L_{t}$ is stable minimal.
If $L_{t}$ is 1 -sided, we consider the double cover $\widehat{N\left(L_{t}\right)}$ of $N\left(L_{t}\right)$ and the lift $\hat{L}_{t}$ of $L_{t}$. We choose $r$ large enough so that $B^{L_{t}}(p, r) \subset L_{t}$ is 1 -sided. The lift $\widehat{B^{L_{t}}(p, r)}$ of $B^{L_{t}}(p, r)$ is connected and 2-sided in the lift $\widehat{N(p, r)}$ of $N(p, r)$.

Let $\hat{\Sigma}_{k}$ be one component of the pre-image of $\Sigma_{k}$. It is a stable minimal surface. The reason is as follows:

The map $\hat{\Sigma}_{k} \rightarrow \Sigma_{k}$ is a $m^{\prime}$-sheeted cover map, where $m^{\prime} \leqslant 2$.
If $m^{\prime}=1$, the map $\hat{\Sigma}_{k} \rightarrow \Sigma_{k}$ is isometric. Therefore, $\hat{\Sigma}_{k}$ is stable minimal.
If $m^{\prime}=2$, we use Lemma 4.2 .4 to have that $\hat{\Sigma}_{k}$ is stable minimal.
As in Step 1, we define the projection $\hat{\pi}: \widehat{N\left(L_{t}\right)} \rightarrow \hat{L}_{t}$ and the function $\hat{d}_{k}: \hat{\Sigma}_{k} \rightarrow \mathbb{R}$.
The sequence $\left\{\hat{\Sigma}_{k}\right\}$ converges to $\widehat{B^{L_{t}}(p, r)}$. The map $\hat{\pi}_{k}:=\left.\hat{\pi}\right|_{\hat{\Sigma}_{k}}$ is a cover map for $k$ large enough. Since $\operatorname{Area}\left(\Sigma_{k}\right) \leqslant \operatorname{Area}\left(\mathscr{L}_{k}\right)<\infty$ and $\operatorname{Area}\left(\hat{\Sigma}_{k}\right) \leqslant 2 \operatorname{Area}\left(\Sigma_{k}\right), \hat{\Sigma}_{k}$ has a finite area. Therefore, $\hat{\pi}_{k}$ is a finite cover for $k$ large enlarge.

As in the above case, $\widehat{B^{L_{t}}(p, r)}$ is stable minimal. From Lemma 4.2.4. $B^{L_{t}}(p, r)$ is stable minimal for $r$ large enough. Namely, $L_{t}$ is stable minimal.

THEOREM 4.2.6. If $(M, g)$ has positive scalar curvature, each leaf in $\mathscr{L}$ is noncompact.

Recall that a component $\Omega_{k}^{l}$ of $\mathscr{L}_{k}$ is an area-minimizing disc with boundary $\partial \Omega_{k}^{l}$ in the closure of $\left(T_{k, l-1}, g_{k}\right)$, where $T_{k, l-1}=N_{k} \backslash \mathrm{\amalg}_{j=1}^{l-1} \Omega_{k}^{j}$ and $g_{k}$ is obtained by modifying the metric $g$. In addition, $\left.g_{k}\right|_{N_{k-1}}$ is equal to $\left.g\right|_{N_{k-1}}$.

Proof. We argue by contradiction. Suppose that there exists a compact leaf $L_{t}$ in $\mathscr{L}$.

Step 1: Topology of $L_{t}$
From the positivity of the scalar curvature, we use Corollary 3.3.6 to have that $L_{t}$ is a 2 -sphere or a projective plane.

If $L_{t}$ is a projective plane, $L_{t}$ is 1 -sided. Hence, $M \backslash L_{t}$ is connected. There is a an embedded curve $\gamma$ in $M$ which intersects $L_{t}$ transversally at one point. The intersection numberof $L_{t}$ and $\gamma$ is $\pm 1$.

However, from the contractibility of $M, \gamma$ is homotopically trivial. Hence, the intersection number of $\gamma$ and $L_{t}$ is zero, a contradiction.

We conclude that $L_{t}$ is a 2 -sphere.
Step 2: Area Estimate.

Since $M$ is irreducible (See Remark 1.1.1), there is a 3 -ball $B \subset M$ with boundary $L_{t}$. Let $N_{2 \epsilon}(B)$ be the tubular neighborhood of $B$ with radius $2 \epsilon$. The set $N_{2 \epsilon}\left(L_{t}\right)$ is a subset of $N_{2 \epsilon}(B)$.

Since $N_{2 \epsilon}(B)$ is relatively compact, there is a positive integer $k_{0}$, such that $N_{2 \epsilon}(B) \subset$ $N_{k_{0}-1}$.

From now on, we fix the integer $k>k_{0}$. Let $\left\{\Sigma_{k}^{j}\right\}_{j}$ be the set of components of $\mathscr{L}_{k} \cap N_{2 \epsilon}\left(L_{t}\right)$. The component $\Sigma_{k}^{j}$ is contained in some component $\Omega_{k}^{j}$ of $\mathscr{L}_{k}:=\amalg_{l} \Omega_{k}^{l}$.

In the following, we show that there is a constant $C$, independent of $k$ and $j$, so that the area of $\Sigma_{k}^{j}$ in $(M, g)$ is less than $C$

We may assume that $\Omega_{k}^{j}$ intersects $\partial N_{2 \epsilon}(B)$ transversally. The intersection $\Omega_{k}^{j} \cap$ $\partial N_{2 \epsilon}(B):=\left\{\gamma_{i}\right\}_{i}$ has finitely many components. Each component $\gamma_{i}$ is a circle and bounds a unique closed disc $D_{i}$ in $\Omega_{k}^{j}$.

Since $\partial N_{2 \epsilon}(B)$ is a 2 -sphere, there is an embedded disc $D_{i}^{\prime} \subset \partial N_{2 \epsilon}(B)$ with boundary $\gamma_{i}$.

We claim that for any $D_{i} \subset \Omega_{k}^{j}$,

$$
\operatorname{Area}\left(D_{i}, g_{k}\right) \leqslant \operatorname{Area}\left(D_{i}^{\prime}, g_{k}\right)
$$

where $\operatorname{Area}\left(D_{i}, g_{k}\right)$ is the area of $D_{i}$ in $\left(N_{k}, g_{k}\right)$.
We prove it by induction on $j$.
When $j=1, \Omega_{k}^{1}$ is an area-minimizing disc in $\left(N_{k}, g_{k}\right)$. If the claim does not hold for some $D_{i} \subset \Omega_{k}^{1}$, we consider the disc $\left(\Omega_{k}^{1} \backslash D_{i}\right) \cup_{\gamma_{i}} D_{i}^{\prime}$ with boundary $\partial \Omega_{k}^{1}$. Its area is less than the area of $\Omega_{k}^{1}$ in $\left(N_{k}, g_{k}\right)$, a contradiction.

Therefore, for any $D_{i} \subset \Omega_{k}^{1}$, we have $\operatorname{Area}\left(D_{i}, g_{k}\right) \leqslant \operatorname{Area}\left(D_{i}^{\prime}, g_{k}\right)$.
We suppose that it holds for any $l \leqslant j$ and any $D_{i} \subset \Omega_{k}^{l}$.
In the following, we consider that $D_{i}$ is contained in $\Omega_{k}^{j+1}$. If $D_{i}^{\prime} \cap\left(\amalg_{l \leqslant j} \Omega_{k}^{l}\right)$ is empty, this claim follows from the above argument.

If not, we may assume that $D_{i}^{\prime}$ intersects $\amalg_{l \leqslant j} \Omega_{k}^{l}$ transversally. The intersection $D_{i}^{\prime} \cap$ $\cup_{l \leqslant j} \Omega_{k}^{l}:=\left\{c_{m}\right\}_{c_{m} \in C_{i}}$ has finitely many components. Each component $c_{m}$ bounds a disc $D_{m}^{\prime} \subset D_{i}^{\prime}$. In addition, it also bounds a disc $D_{m} \subset \amalg_{l \leqslant j} \Omega_{k}^{j}$.

Let $C_{i}^{\text {max }}$ be the set of maximal circles of $C_{i}$ in $D_{i}^{\prime}$. These discs $\left\{D_{m}^{\prime}\right\}_{c_{m} \in C_{i}^{\max }}$ are disjoint. The set $D_{i}^{\prime} \backslash\left(\cup_{c_{m} \in C_{i}^{\max }} D_{m}^{\prime}\right)$ is contained in $N_{k} \backslash \amalg_{l \leqslant j} \Omega_{k}^{l}$.

We consider the disc

$$
D_{i}^{\prime \prime}:=D_{i}^{\prime} \backslash\left(\cup_{c_{m} \in C_{i}^{\max }} D_{m}^{\prime}\right) \cup_{c_{m} \in C_{i}^{\max }}\left(\cup_{c_{m}} D_{m}\right)
$$

It is contained in the closure of $\left(T_{k, j}, g_{k}\right)$.
Since $\Omega_{k}^{j+1}$ is an area-minimizing disc in the closure of $\left(T_{k, j}, g_{k}\right)$, we have that

$$
\operatorname{Area}\left(D_{i}, g_{k}\right) \leqslant \operatorname{Area}\left(D_{i}^{\prime \prime}, g_{k}\right)
$$

If not, we consider the disc $\left(\Omega_{k}^{j+1} \backslash D_{i}\right) \cup_{\gamma_{i}} D_{i}^{\prime \prime}$. Its area is less than the area of $\Omega_{k}^{j+1}$, a contradiction.

From the inductive hypothesis,

$$
\operatorname{Area}\left(D_{m}, g_{k}\right) \leqslant \operatorname{Area}\left(D_{m}^{\prime} g_{k}\right)
$$

for any $c_{m}$. Hence,

$$
\begin{aligned}
\operatorname{Area}\left(D_{i}^{\prime}, g_{k}\right) & =\operatorname{Area}\left(D_{i}^{\prime} \backslash\left(\cup_{c_{m} \in C_{i}^{\max }} D_{m}^{\prime}\right), g_{k}\right)+\sum \operatorname{Area}\left(D_{m}^{\prime} g_{k}\right) \\
& \geqslant \operatorname{Area}\left(D_{i}^{\prime} \backslash\left(\cup_{c_{m} \in C_{i}^{\max }} D_{m}^{\prime}\right), g_{k}\right)+\sum \operatorname{Area}\left(D_{m} g_{k}\right) \\
& =\operatorname{Area}\left(D_{i}^{\prime \prime}, g_{k}\right) \geqslant \operatorname{Area}\left(D_{i}, g_{k}\right) .
\end{aligned}
$$

Therefore, we finish the proof of the claim.
We will show that the above claim implies an area estimate.
Let $C^{\text {max }}$ be the set of maximal circles of $\left\{\gamma_{i}\right\}_{i}$ in $\Omega_{k}^{j}$. We have that

$$
\Omega_{k}^{j} \cap N_{2 \epsilon}\left(L_{t}\right) \subset \Omega_{k}^{j} \cap N_{2 \epsilon}(B) \subset \cup_{\gamma_{i} \in C^{\max }} D_{i} .
$$

Hence, $\Sigma_{j}^{k}$ is a subset of some $D_{i}$.
Recall that $\left.g_{k}\right|_{N_{2 \epsilon}(B)}=\left.g\right|_{N_{2 \epsilon}(B)}$ for $k>k_{0}$. For each $k$ and $j$, we have that

$$
\begin{aligned}
& \operatorname{Area}\left(\Sigma_{k}^{j}, g_{k}\right)=\operatorname{Area}\left(\Sigma_{k}^{j}, g\right) \\
& \operatorname{Area}\left(\partial N_{2 \epsilon}(B), g_{k}\right)=\operatorname{Area}\left(\partial N_{2 \epsilon}(B), g\right)
\end{aligned}
$$

We then have that

$$
\begin{aligned}
\operatorname{Area}\left(\Sigma_{k}^{j}, g\right) & =\operatorname{Area}\left(\Sigma_{k}^{j}, g_{k}\right) \leqslant \operatorname{Area}\left(D_{i}, g_{k}\right) \\
& \leqslant \operatorname{Area}\left(D_{i}^{\prime}, g_{k}\right) \leqslant \operatorname{Area}\left(\partial N_{2 \epsilon}(B), g_{k}\right) \\
& =\operatorname{Area}\left(\partial N_{2 \epsilon}(B), g\right)
\end{aligned}
$$

We conclude that for each $k>k_{0}$ and $j$,

$$
\operatorname{Area}\left(\Sigma_{k}^{j}, g\right) \leqslant \operatorname{Area}\left(\partial N_{2 \epsilon}(B), g\right)
$$

## Step 3: Contradiction.

Choose a point $p \in L_{t}$ and a point $p_{k} \in \mathscr{L}_{k} \cap N_{\epsilon}\left(L_{t}\right)$ so that $\lim _{k \rightarrow \infty} p_{k}=p$.
Let $\Sigma_{k}^{j_{k}}$ be the component of $\mathscr{L}_{k} \cap N_{2 \epsilon}\left(L_{t}\right)$ passing through $p_{k}$. As the proof in Step 2 , we have that for $k \geqslant k_{0}$

$$
\operatorname{Area}\left(\Sigma_{k}^{j_{k}}, g\right) \leqslant \operatorname{Area}\left(\partial N_{2 \epsilon}(B), g\right)
$$

From Lemma 4.1.5, the curvatures of these surfaces $\left\{\Sigma_{k}^{j_{k}}\right\}_{k}$ are uniformly bounded in $N_{2 \epsilon}\left(L_{t}\right)$. By Theorem 4.1.4, the sequence $\left\{\Sigma_{k}^{j_{k}}\right\}$ sub-converges smoothly to a properly embedded surface $\Sigma$ with finite multiplicity in $N_{\epsilon}\left(L_{t}\right)$.

For our convenience, we assume that $\left\{\Sigma_{k}^{j_{k}}\right\}$ converges smoothly to $\Sigma$ in $N_{\epsilon}\left(L_{t}\right)$. The limit $\Sigma \subset \mathscr{L}$ is a disjoint union of connected embedded surfaces. Its boundary is contained in $\partial N_{\epsilon}\left(L_{t}\right)$. In addition, $p$ lies in $\mathscr{L}$. Hence, $L_{t}$ is one component of $\Sigma$.

Since $\Sigma$ is properly embedded, the set $\Sigma^{\prime}:=\Sigma \backslash L_{t}$ is a closed set. The sets $\Sigma^{\prime}$ and $L_{t}$ are two disjoint closed sets. Choose $\delta<\epsilon / 2$ small enough such that

$$
N_{2 \delta}\left(L_{t}\right) \cap N_{2 \delta}\left(\Sigma^{\prime}\right)=\varnothing .
$$

Claim: For $k$ large enough, $\Sigma_{k}^{j_{k}}$ is contained in $N_{\delta}\left(L_{t}\right) \amalg\left(N_{2 \epsilon}\left(L_{t}\right) \backslash N_{2 \delta}\left(L_{t}\right)\right)$.
Since $\Sigma_{k}^{j_{k}}$ is a subset of $N_{2 \epsilon}\left(L_{t}\right), \Sigma_{k}^{j_{k}} \backslash N_{2 \delta}\left(L_{t}\right)$ is contained in $N_{2 \epsilon}\left(L_{t}\right) \backslash N_{2 \delta}\left(L_{t}\right)$. It is sufficient to show that $\Sigma_{k}^{j_{k}} \cap N_{2 \delta}\left(L_{t}\right)$ is contained in $N_{\delta}\left(L_{t}\right)$.

For $k$ large enough, $\Sigma_{k}^{j_{k}} \cap N_{\epsilon}\left(L_{t}\right)$ is contained in $N_{\delta}(\Sigma)$, because $\Sigma_{k}^{j_{k}} \cap N_{\epsilon}\left(L_{t}\right)$ converges to $\Sigma$. Hence, $\Sigma_{k}^{j_{k}} \cap N_{2 \delta}\left(L_{t}\right)$ is a subset of $N_{2 \delta}\left(L_{t}\right) \cap N_{\delta}(\Sigma)$. From the choice of $\delta$, we have that
a) $N_{\delta}(\Sigma)$ is equal to $N_{\delta}\left(L_{t}\right) \amalg N_{\delta}\left(\Sigma^{\prime}\right)$;
b) $N_{\delta}\left(\Sigma^{\prime}\right) \cap N_{2 \delta}\left(L_{t}\right)$ is empty.

By a), $N_{2 \delta}\left(L_{t}\right) \cap N_{\delta}(\Sigma)$ is equal to $N_{\delta}\left(L_{t}\right) \amalg\left(N_{\delta}\left(\Sigma^{\prime}\right) \cap N_{2 \delta}\left(L_{t}\right)\right)$. From b), it is equal to $N_{\delta}\left(L_{t}\right)$. Therefore, $\sum_{j_{k}}^{k} \cap N_{2 \delta}\left(L_{t}\right)$ is contained in $N_{\delta}\left(L_{t}\right)$.

This finishes the proof of the claim.
For $k$ large enough, $p_{k}$ is located in $N_{\delta}\left(L_{t}\right)$. Namely, $\Sigma_{k}^{j_{k}} \cap N_{\delta}\left(L_{t}\right)$ is non-empty. In addition, since $\partial \Sigma_{k}^{j_{k}} \subset \partial N_{2 \epsilon}\left(L_{t}\right)$ is non-empty, $\Sigma_{k}^{j_{k}} \cap\left(N_{2 \epsilon}\left(L_{t}\right) \backslash N_{2 \delta}\left(L_{t}\right)\right)$ is also nonempty.

The sets $N_{\delta}\left(L_{t}\right)$ and $\overline{N_{2 \epsilon}\left(L_{t}\right) \backslash N_{2 \delta}\left(L_{t}\right)}$ are disjoint. Since $\Sigma_{k}^{j_{k}}$ is connected, we use the claim to have that one of these two sets $\Sigma_{k}^{j_{k}} \cap N_{\delta}\left(L_{t}\right)$ and $\Sigma_{k}^{j_{k}} \cap\left(N_{2 \epsilon}\left(L_{t}\right) \backslash N_{2 \delta}\left(L_{t}\right)\right)$ is empty. This is in contradiction with the last paragraph.

We can conclude that each leaf $L_{t}$ is non-compact.
As an consequence, we give a new proof of [Corollary 10.8, Page 173] in [GL83.
Corollary 4.2.7. A complete contractible 3 -manifold with uniformly positive scalar curvature (i.e. that is, its scalar curvature is bounded away from zero) is homeomorphic to $\mathbb{R}^{3}$

Proof. Suppose that $M$ is not diffeomorphic to $\mathbb{R}^{3}$. As described above, there exists a complete (non-compact) stable minimal surface $\Sigma$. By Corollary $3.3 .11, \Sigma$ is conformally diffeomorphic to $\mathbb{R}^{2}$.

Since the scalar curvature $\kappa(x)$ of $M$ is uniformly positive, $\inf _{x \in M} \kappa(x)>0$. From Theorem 3.3.10, one has,

$$
\begin{aligned}
2 \pi & \geqslant \int_{\Sigma} \kappa(x) d v \\
& \geqslant \inf _{x \in M} \kappa(x) \cdot \int_{\Sigma} d v \\
& =\inf _{x \in M} \kappa(x) \cdot \operatorname{Area}(\Sigma) .
\end{aligned}
$$

Therefore, $\Sigma$ is a surface of finite area.
However, we apply the theorem of Gromov and Lawson [Theorem 8.8] in GL83]. This theorem asserts that if $(X, g)$ is a Riemannian manifold of positive scalar curvature, then any complete stable minimal surface of finite area in $X$ is homeomorphic to $\mathbb{S}^{2}$. Hence, $\Sigma$ is homeomorphic to $\mathbb{S}^{2}$, which leads to a contradiction with the topological structure of $\Sigma\left(\Sigma\right.$ is homeomorphic to $\left.\mathbb{R}^{2}\right)$.

## Part 3

## Proof of Main Theorems

## CHAPTER 5

## The Vanishing Property

In this chapter, we consider the geometry of a complete stable minimal surface and its relationship with the topological property of contractible 3-manifolds.

Let us consider a complete contractible Riemannian 3-manifold $(M, g)$ of positive scalar curvature and a complete (non-compact) embedded stable minimal surface $\Sigma \subset$ $(M, g)$. From Theorem 3.3 .10 and Corollary 3.3.11, the surface $\Sigma$ is a properly embedded plane (i.e. it is diffeomorphic to $\mathbb{R}^{2}$ ).

In the following, we assume that $M:=\cup_{k} N_{k}$ is not homeomorphic to $\mathbb{R}^{3}$ where $\left\{N_{k}\right\}$ is assumed as in Remark 1.1.13. By Theorem 2.2.6, there is an increasing family $\left\{R_{k}\right\}_{k}$ of closed handlebodies with Property H.

Definition. A complete embedded stable minimal surface $\Sigma \subset(M, g)$ is called to satisfy the Vanishing Property for $\left\{R_{k}\right\}_{k}$, if there exists a positive integer $k(\Sigma)$ so that for any $k \geqslant k(\Sigma)$, any circle in $\Sigma \cap \partial R_{k}$ is contractible in $\partial R_{k}$.

Let us consider a stable minimal lamination $\mathscr{L} \subset(M, g)$, where each leaf is a complete (non-compact) stable minimal surface. It is called to have the Vanishing Property for $\left\{R_{k}\right\}_{k}$, if there is a positive integer $k_{0}$ so that for any $k \geqslant k_{0}$ and each leaf $L_{t}$ in $\mathscr{L}$, then any circle in $L_{t} \cap \partial R_{k}$ is contractible in $\partial R_{k}$.

We will consider the Vanishing property and its relationship with Property P and the fundamental group at infinity.

### 5.1. The vanishing Property and Property $P$

In this section, we consider the case that a complete contractible genus one 3-manifold $(M, g)$. In this case, we see from Lemma 1.3.10 that the family $\left\{N_{k}\right\}$ (as in Theorem 1.3.13) satisfies Property H. Namely, $R_{k}$ is defined as $N_{k}$.

In addition, we see from Theorem 2.1 .6 that the manifold $M$ satisfies Property P (See Definition 2.1.3). If $(M, g)$ has positive scalar curvature, the geometry of a stable minimal lamination is constrained by the extrinsic Cohn-Vesson inequality ( See Theorem 3.3.10) as well as by Property P. Their relationship is clarified by the following theorem:

Theorem 5.1.1. Let $\mathscr{L}:=\cup_{t \in \Gamma} L_{t}$ be a stable minimal lamination in a complete contractible genus one 3 -manifold ( $M, g$ ). Each leaf $L_{t}$ is a complete (non-compact) stable minimal lamination. If the manifold $(M, g)$ has positive scalar curvature $(\kappa(x)>0)$, then $\mathscr{L}$ satisfies the Vanishing property for $\left\{N_{k}\right\}_{k}$, where $\left\{N_{k}\right\}_{k}$ is assumed as in Theorem 1.3.13.

Precisely, there exists a positive integer $k_{0}=k_{0}(M, g)$, such that for each $k \geqslant k_{0}$ and any $t \in \Lambda$, each embedded circle $\gamma$ in $L_{t} \cap \partial N_{k}$ is contractible in $\partial N_{k}$.

Proof. Since $(M, g)$ has positive scalar curvature, we know from Corollary 3.3.11 that each $L_{t}$ is diffeomorphic to $\mathbb{R}^{2}$.

We prove by contradiction. We suppose that there exists a sequence of increasing integers $\left\{k_{n}\right\}_{n}$ such that:
for each $k_{n}$, there exists a minimal surface $L_{t_{n}}$ in $\mathscr{L}$ and an embedded circle $c_{k_{n}} \subset$ $L_{t_{n}} \cap \partial N_{k_{n}}$ which is not contractible in $\partial N_{k_{n}}$.
Since $\lim _{n \rightarrow \infty} k_{n}=\infty$, we know that $\lim _{n \rightarrow \infty} I\left(N_{1}, N_{k_{n}}\right)=\infty$.
Because $L_{t_{n}}$ is homeomorphic to $\mathbb{R}^{2}$, there exists a unique disc $D_{n} \subset L_{t_{n}}$ with boundary $c_{k_{n}}$. From Property $P$ (Definition 2.1.3), we see that $D_{n} \cap \operatorname{Int} N_{1}$ has at least $I\left(N_{1}, N_{k_{n}}\right)$ components intersecting $N_{0}$, denoted by $\left\{\Sigma_{j}\right\}_{j=1}^{m}$.

Define the constants $r:=d^{M}\left(\partial N_{0}, \partial N_{1}\right), C:=\inf _{x \in N_{1}} \kappa(x), K:=\sup _{x \in N_{1}}\left|K_{M}\right|$ and $i_{0}:=\inf _{x \in N_{1}}\left(\operatorname{Inj}_{M}(x)\right)$, where $K_{M}$ is the sectional curvature of $(M, g)$ and $\operatorname{Inj}_{M}(x)$ is the injective radius at $x$ of $(M, g)$.

Choose $r_{0}=\frac{1}{2} \min \left\{i_{0}, r\right\}$ and $x_{j} \in \Sigma_{j} \cap N_{0}$, then $B\left(x_{j}, r_{0}\right)$ is in $N_{1}$. We apply Theorem 3.2.7 to the minimal surface $\left(\Sigma_{j}, \partial \Sigma_{j}\right) \subset\left(N_{1}, \partial N_{1}\right)$. Hence, one has that

$$
\operatorname{Area}\left(\Sigma_{j} \cap B\left(x_{j}, r_{0}\right)\right) \geqslant C_{1}\left(K, i_{0}, r_{0}\right) .
$$

From Theorem 3.3.10, we have:

$$
\begin{aligned}
2 \pi & \geqslant \int_{L_{t_{n}}} \kappa(x) d v \geqslant \sum_{j=1}^{m} \int_{\Sigma_{j}} \kappa(x) d v \geqslant \sum_{j=1}^{m} \int_{\Sigma_{j} \cap B\left(x_{j}, r_{0}\right)} \kappa(x) d v \\
& \geqslant \sum_{j=1}^{m} C \operatorname{Area}\left(\Sigma_{j} \cap B\left(x_{j}, r_{0}\right)\right) \\
& \geqslant C C_{1} m \geqslant C C_{1} I\left(N_{1}, N_{k_{n}}\right)
\end{aligned}
$$

This contradicts the fact that $\lim _{n \rightarrow \infty} I\left(N_{1}, N_{k_{n}}\right)=\infty$ and completes the proof.
REMARK 5.1.2. In the following, our proof requires that $\partial N_{k}$ intersects some leaf $L_{t}$ transversally. To overcome it, we will deform the solid torus $N_{k}$ in a small tubular neighborhood of $\partial N_{k}$ so that the boundary of the new solid torus intersects $L_{t}$ transversally.

This new solid torus also holds for Theorem 5.1.1. The reason is as follows:
The proof of Theorem 5.1.1 only depends on the extrinsic Cohn-Vossen inequality (See Theorem 3.3.8) and the geometric indexes. If we replace $N_{k}$ by a new solid torus obtained from deforming $N_{k}$, all geometric indexes remain unchanged. Therefore, $N_{k}^{\prime}$ also holds for Theorem 5.1.1.

Remark 5.1.3. Let $\left\{R_{k}\right\}_{k}$ be a family of solid tori in a complete contractible 3manifold $(M, g)$ with the properties that

$$
\lim _{k \rightarrow \infty} I\left(R_{1}, R_{k}\right)=\infty .
$$

Now we consider the case that the maps $\pi_{1}\left(\partial R_{k}\right) \rightarrow \pi_{1}\left(\overline{M \backslash R_{k}}\right)$ and $\pi_{1}\left(\partial R_{k}\right) \rightarrow$ $\pi_{1}\left(\overline{R_{k} \backslash R_{0}}\right)$ are both injective. From Remark 2.1.7, the family satisfies Property P (See the detail in Remark 2.1.7).

From the proof of Theorem 5.1.1, we have that if $(M, g)$ has positive scalar curvature, any complete stable minimal lamination in $(M, g)$ has the Vanishing Property for $\left\{R_{k}\right\}$.

### 5.2. The vanishing Property and $\pi_{1}^{\infty}$

Generally, the geometry of a stable minimal surface is constrained by some topological properties of the 3 -manifold. For example, the fundamental group at infinity. In the following, we will study a complete (non-compact) stable minimal surface $\Sigma \subset(M, g)$ and its relationship with the fundamental group at infinity.

Lemma 5.2.1. Let $(M, g)$ be a complete contractible Riemannian 3-manifold with positive scalar curvature $\kappa(x)>0$ and $\left\{R_{k}\right\}_{k}$ a family of handlebodies with Property $H$. If a complete embedded stable minimal surface $\Sigma$ does not satisfy the Vanishing Property for $\left\{R_{k}\right\}_{k}$, then $\pi_{1}^{\infty}(M)$ is non-trivial.

Roughly, there is a sequence of non-trivial circles in $\Sigma$ going to infinity. This sequence gives a non-trivial element in $\pi_{1}^{\infty}(M)$.

Proof. Since $\Sigma$ does not satisfy the Vanishing property for $\left\{R_{k}\right\}$, there exists a sequence $\left\{k_{n}\right\}_{n}$ of increasing integers so that for each $k_{n}$, there is a circle $\gamma_{n} \subset \partial R_{k_{n}} \cap \Sigma$ which is not nullhomotopic in $\partial R_{k_{n}}$. By Corollary 3.3.11, $\Sigma$ is conformally diffeomorphic to $\mathbb{R}^{2}$. Each $\gamma_{n}$ bounds a unique closed disc $D_{n} \subset \Sigma$.

However, $\gamma_{n}$ may not be a meridian. We will choose a meridian in $D_{n}$ of $R_{k_{n}}$ to replace it.

Since the map $\pi_{1}\left(\partial R_{k_{n}}\right) \rightarrow \pi_{1}\left(\overline{M \backslash R_{k_{n}}}\right)$ is injective (See Definition 2.2.5), we use Corollary 1.2 .6 to see that $D_{n}$ contains at least one meridian of $R_{k_{n}}$. Without loss of generality, we may assume that $\gamma_{n}$ is a meridian of $R_{k_{n}}$ and Int $D_{n}$ has no meridian of $R_{k_{n}}$. (If not, we can replace $\gamma_{n}$ by the meridian in Int $D_{n}$ ).

Since $\left\{\gamma_{n}\right\}_{n}$ is a collection of disjointly embedded circles in $\Sigma$, one of the following holds: for each $n^{\prime}$ and $n^{\prime}$,

- $D_{n^{\prime}} \subset D_{n}$;
- $D_{n} \subset D_{n^{\prime}}$;
- $D_{n} \cap D_{n^{\prime}}=\varnothing$.

We claim that:
$(*)$ :if any $n^{\prime}>n$, then $D_{n} \subset D_{n^{\prime}}$ or $D_{n} \cap D_{n^{\prime}}=\varnothing$.'
The reason is below: If not, $D_{n^{\prime}}$ is a subset of $D_{n}$. Since $\gamma_{n^{\prime}} \subset \partial R_{k_{n^{\prime}}}$ is not contractible in $M \backslash R_{0}$ (See Remark 2.2.7) and the map $\pi_{1}\left(\partial R_{k_{n}}\right) \rightarrow \pi_{1}\left(\overline{M \backslash R_{k_{n}}}\right)$ is injective, we use Corollary 1.2 .6 to see that $D_{n^{\prime}} \subset$ Int $D_{n}$ contains at least one meridian of $R_{k_{n}}$. This is in contradiction with the above assumption.

We will show there is an increasing subsequence of $\left\{D_{n}\right\}$. Furthermore, the boundaries of these discs in the subsequence gives a non-trivial element in $\pi_{1}^{\infty}(M)$.
Step 1: the existence of the ascending subsequence of $\left\{D_{n}\right\}$.
We argue by contradiction. Suppose these is no ascending subsequence in $\left\{D_{n}\right\}$. Consider the partially ordered set $\left(\left\{D_{n}\right\}_{n}, \subset\right)$ induced by the inclusion. Let $C$ be the set of minimal elements in $\left(\left\{D_{n}\right\}_{n}, \subset\right)$. These discs in $C$ are disjoint in $\Sigma$.

If the set $C$ is finite, we consider the integer $n_{0}:=\max \left\{n \mid D_{n} \in C\right\}$. From the above fact (*), the subsequence $\left\{D_{n}\right\}_{n>n_{0}}$ is an increasing subsequence, which contradicts our hypothesis. Therefore, we can conclude that the set $C$ is infinite. That is to say, there is a subsequence $\left\{D_{n_{s}}\right\}_{s}$ of disjointly embedded discs.

From Remark 2.2.7, the map $\pi_{1}\left(\partial R_{k_{n s}}\right) \rightarrow \pi_{1}\left(M \backslash R_{0}\right)$ is injective. Therefore, the disc $D_{n_{s}}$ intersects $N_{0}$. Since $N_{0}$ is a subset of $R_{0}$, it also intersects $R_{0}$.

Choose $x_{n_{s}} \in R_{0} \cap D_{n_{s}}$ and $r_{0}=\frac{1}{2} \min \left\{i_{0}, r\right\}$, where $r:=d^{M}\left(\partial R_{0}, \partial R_{1}\right)$ and $i_{0}:=$ $\inf _{x \in R_{1}}\left(\operatorname{Inj}_{M}(x)\right)$. Hence, the geodesic ball $B\left(x_{n_{s}}, r_{0}\right)$ in $M$ lies in $R_{1}$.

Define the constants $C:=\inf _{x \in R_{1}} \kappa(x), K:=\sup _{x \in R_{1}}\left|K_{M}\right|$ where $K_{M}$ is the sectional curvature of $(M, g)$.We apply Theorem 3.2.7 (See [Lemma 1, Page 445] of MY80]) to the minimal surface $D_{n_{s}} \cap R_{1}$ in $\left(R_{1}, \partial R_{1}\right)$ and obtain that

$$
\operatorname{Area}\left(D_{n_{s}} \cap B\left(x_{n_{s}}, r_{0}\right)\right) \geqslant C_{1}\left(K, i_{0}, r_{0}\right)
$$

This leads to a contradiction from Theorem 3.3.10 as follows:

$$
\begin{aligned}
2 \pi \geqslant \int_{\Sigma} \kappa d v & \geqslant \int_{R_{1} \cap \Sigma} \kappa d v \geqslant \sum_{s} \int_{D_{n_{s} \cap B\left(x_{n_{s}}, r_{0}\right)}} \kappa d v \\
& \geqslant \sum_{s} C \operatorname{Area}\left(D_{n_{s}} \cap B\left(x_{N_{s}}, r_{0}\right)\right) \\
& \geqslant \sum_{s} C C_{1}=\infty
\end{aligned}
$$

Therefore, we can conclude that there is an ascending subsequence of $\left\{D_{n}\right\}_{n}$.
From now on, we abuse the notation and write $\left\{D_{n}\right\}$ for an ascending subsequence.
Step 2: $\pi_{1}^{\infty}(M)$ is non-trivial.
Claim: There is an integer $N$ so that for $n \geqslant N,\left(D_{n} \backslash D_{n-1}\right) \cap R_{0}$ is empty.
We argue by contradiction. Suppose that there exists a family $\left\{n_{l}\right\}$ of increasing integers such that $D_{n_{l}} \backslash D_{n_{l-1}}$ intersects $R_{0}$.

Choose $x_{l} \in D_{n_{l}} \backslash D_{n_{l-1}} \cap R_{0}$. Hence the geodesic ball $B\left(x_{l}, r_{0}\right)$ in $M$ is contained in $R_{1}$, where $r_{0}$ is assumed as above. We apply Theorem 3.2.7 (See [Lemma 1, Page 445] in $\widehat{\mathbf{M Y 8 0}})$ to the minimal surface $D_{n_{l}} \backslash D_{n_{l-1}} \cap R_{1}$ in $\left(R_{1}, \partial R_{1}\right)$.

$$
\text { Area }\left(\left(D_{n_{l}} \backslash D_{n_{l-1}}\right) \cap B\left(x_{l}, r_{0}\right)\right) \geqslant C_{1}\left(K, i_{0}, r_{0}\right)
$$

From Theorem 3.3.10, one gets a contradiction as follows:

$$
\begin{aligned}
2 \pi & \geqslant \int_{\Sigma} \kappa d v \geqslant \int_{R_{1} \cap \Sigma} \kappa d v \\
& \geqslant \sum_{l} \int_{\left(D_{\left.n_{l} \backslash D_{n_{l-1}}\right) \cap B\left(x_{l}, r_{0}\right)} \kappa d v\right.} \\
& \geqslant \sum_{l} C \operatorname{Area}\left(B\left(x_{l}, r_{0}\right) \cap D_{n_{l}} \backslash D_{n_{l-1}}\right) \\
& \geqslant C \sum_{l} C_{1}=\infty
\end{aligned}
$$

This proves Claim 1.
Therefore, for $n>N, \gamma_{n}$ is homotopic to $\gamma_{N}$ in $M \backslash R_{0}$ and not nullhomotopic in $M \backslash R_{0}$.

Because $\cup_{k} R_{k}$ may not equal to $M$, the sequence $\left\{\gamma_{n}\right\}_{n>N}$ of circles may not go to infinity. For overcoming it, we choose a new family $\left\{\gamma_{n}^{\prime}\right\}_{n>N}$ of circles going to infinity to replace it.

The map $\pi_{1}\left(\partial R_{k_{n}} \cap N_{j_{k_{n}}}\right) \rightarrow \pi_{1}\left(\partial R_{k_{n}}\right)$ is surjective (See Theorem 2.2.6 and Definition 2.2.5). Hence, we can find a circle $\gamma_{n}^{\prime} \subset \partial N_{j_{k_{n}}} \cap \partial R_{k_{n}}$ which is homotopic to $\gamma_{n}$ in $\partial R_{k_{n}}$. The sequence of circles $\left\{\gamma_{n}^{\prime}\right\}_{n \geqslant N}$ goes to infinity.

The sequence $\left\{\gamma_{n}^{\prime}\right\}$ also have the property that for $n>N$,

- $\gamma_{n}^{\prime}$ is homotopic to $\gamma_{n+1}^{\prime}$ in $M \backslash R_{0}$;
- $\gamma_{n}^{\prime}$ is not nullhomotopic in $M \backslash R_{0}$.

From Remark 1.1.10, $\pi_{1}^{\infty}(M)$ is not trivial.
As a corollary, we have
Corollary 5.2.2. Let $(M, g)$ be a Riemannian 3-manifold of positive scalar curvature and $\left\{R_{k}\right\}_{k}$ a family of handlebodies with Property (H). If $\pi_{1}^{\infty}(M)$ is trivial, then any complete stable minimal surface in $(M, g)$ has the Vanishing property for $\left\{R_{k}\right\}_{k}$.

Theorem 5.2.3. Let $(M, g)$ be a Riemannian manifold of positive scalar curvature and a family of handlebodies $\left\{R_{k}\right\}_{k}$ with Property (H). If each leaf in a lamination $\mathscr{L}$ is a complete (non-compact) stable minimal surface satisfying the Vanishing Property for $\left\{R_{k}\right\}_{k}$, then the lamination $\mathscr{L}$ also has the Vanishing property for $\left\{R_{k}\right\}_{k}$.

Proof. We argue by contradiction. Suppose that there exists a sequence $\left\{L_{t_{n}}\right\}$ of leaves in $\mathscr{L}$ and a sequence of increasing integers $\left\{k_{n}\right\}_{n}$ so that some circle $\gamma_{n} \subset L_{t_{n}} \cap \partial R_{k_{n}}$ is not contractible in $\partial R_{k_{n}}$ for each $n$.

The leaf $L_{t_{n}}$ is a complete (non-compact) stable minimal surface. From Corollary 3.3.11 (See [Theorem 2, Page 211] of [SY82]), it is diffeomorphic to $\mathbb{R}^{2}$. The circle $\gamma_{n}$ bounds a unique closed disc $D_{n} \subset L_{t_{n}}$. Since $\gamma_{n}$ is not nullhomotopic in $\overline{M \backslash R_{0}}$ (See Remark 2.2.7), the disc $D_{n}$ intersects $R_{0}$.
Step 1: The sequence $\left\{L_{t_{n}}\right\}_{n}$ sub-converges with finite multiplicity.
Since each $L_{t_{n}}$ is a stable minimal surface, we use Lemma 4.1.5 (See [Theorem 3, Page 122] of $[\mathbf{S c h 8 3}])$ to show that, fixed a compact set $K \subset M$, there exists a constant $C_{1}=C_{1}(K, M, g)$ satisfying that

$$
\left|A_{L_{t_{n}}}\right|^{2} \leqslant C_{1} \text { on } K \cap L_{t_{n}}
$$

where $\left|A_{L_{n}}\right|^{2}$ is the squared norm of the second fundamental form of $L_{t_{n}}$.
From Theorem 3.3.10, $\int_{L_{t_{n}}} \kappa d v \leqslant 2 \pi$. Hence,

$$
\operatorname{Area}\left(K \cap L_{t_{n}}\right) \leqslant 2 \pi\left(\inf _{x \in K} \kappa(x)\right)^{-1}
$$

From Theorem4.1.4, the sequence $\left\{L_{t_{n}}\right\}_{n}$ sub-converges to a sublamination $\mathscr{L}^{\prime}$ of $\mathscr{L}$ with finite multiplicity. In addition, $\mathscr{L}^{\prime}$ is proper embedded.

The lamination $\mathscr{L}^{\prime}$ may has infinitely many components. Let $\mathscr{L}^{\prime \prime} \subset \mathscr{L}^{\prime}$ be a set of leaves intersecting $R_{0}$. Since $\mathscr{L}^{\prime \prime}$ is properly embedded, $\mathscr{L}^{\prime \prime}$ has finitely many leaves.

Since each leaf $L_{t}$ in $\mathscr{L}^{\prime}$ is homeomorphic to $\mathbb{R}^{2}$ (See Corollary 3.3.11), any embedded circle $\gamma \subset \partial R_{k} \cap L_{t}$ bounds a unique closed disc $D \subset L_{t}$ for $k>0$.

If $L_{t}$ is in $\mathscr{L}^{\prime} \backslash \mathscr{L}^{\prime \prime}$, the intersection $D \cap R_{0}$ is empty. Namely, $\gamma$ is contractible in $\overline{M \backslash R_{0}}$. Since $\pi_{1}\left(\partial R_{k}\right) \rightarrow \pi_{1}\left(\overline{M \backslash R_{0}}\right)$ is injective, then $\gamma$ is nullhomotopic in $\partial R_{k}$.

Therefore, we conclude that for each $k>0$ and any leaf $L_{t} \in \mathscr{L}^{\prime} \backslash \mathscr{L}^{\prime \prime}$, any circle in $L_{t} \cap \partial R_{k}$ is homotopically trivial in $\partial R_{k}$.
Step 2: The Vanishing property gives a contradiction.
From now on, we abuse the notation and write $\left\{L_{t_{n}}\right\}$ for a convergent sequence. In addition, we assume that the lamination $\mathscr{L}^{\prime \prime}:=\amalg_{s}^{m} L_{t_{s}}$ ( $\mathscr{L}^{\prime \prime}$ has finitely many leaves).

The Vanishing property gives an integer $k\left(L_{t_{s}}\right)$ for $L_{t_{s}}$. For $k \geqslant \sum_{s=1}^{m} k\left(L_{t_{s}}\right)$, any circle in $\partial R_{k} \cap \mathscr{L}^{\prime \prime}$ is contractible in $\partial R_{k}$. From the above fact, for $k>0$, any closed curve in $\partial R_{k} \cap \mathscr{L}^{\prime} \backslash \mathscr{L}^{\prime \prime}$ is also homotopically trivial in $\partial R_{k}$.

Therefore, for any $k \geqslant \sum_{s=1}^{m} k\left(L_{t_{s}}\right)$, any circle in $\partial R_{k} \cap \mathscr{L}^{\prime}$ is contractible in $\partial R_{k}$.
In the following, we fix the integer $k \geqslant \sum_{s=1}^{m} k\left(L_{t_{s}}\right)$ and have the following:
Claim: For $n$ large enough, any circle in $\partial R_{k} \cap L_{t_{n}}$ is homotopically trivial
we may assume that $\mathscr{L}^{\prime}$ intersects $\partial R_{k}$ transversally. Since $\mathscr{L}^{\prime}$ is properly embedded, $\partial R_{k} \cap \mathscr{L}^{\prime}$ has finitely many components. Each component of $\partial R_{k} \cap \mathscr{L}^{\prime}$ is an embedded circle. From the above fact, it is homotopically trivial in $\partial R_{k}$. That is to say,

$$
\pi_{1}\left(\partial R_{k} \cap \mathscr{L}^{\prime}\right) \rightarrow \pi_{1}\left(\partial R_{k}\right) \text { is a trivial map. }
$$

Choose an open tubular neighborhood $U$ of $\mathscr{L}^{\prime} \cap \partial R_{k}$ in $\partial R_{k}$. It is homotopic to $\mathscr{L}^{\prime} \cap \partial R_{k}$ in $\partial R_{k}$. Therefore, $\pi_{1}(U) \rightarrow \pi_{1}\left(\partial R_{k}\right)$ is a trivial map.

Since $\left\{L_{t_{n}}\right\}$ converges to $\mathscr{L}^{\prime}$, we see that $L_{t_{n}} \cap \partial R_{k}$ is contained in $U$ for $n$ large enough. Hence, the map $\pi_{1}\left(\partial R_{k} \cap L_{t_{n}}\right) \rightarrow \pi_{1}\left(\partial R_{k}\right)$ is trivial. Namely, any circle in $\partial R_{k} \cap L_{t_{n}}$ is contractible in $\partial R_{k}$. The claim follows.

The boundary $\gamma_{n} \subset \partial R_{k_{n}}$ of $D_{n}$ is non-contractible in $\partial R_{k_{n}}$. From Remark 2.2.7, it is non-contractible in $\overline{M \backslash R_{0}}$. If $k_{n}>k$, we use Corollary 1.2 .6 and Property H (See Definition 2.2.5 to find a meridian $\gamma^{\prime} \subset D_{n} \subset L_{t_{n}}$ of $R_{k}$. This is in contradiction with the above claim.

As a consequence, we have
Corollary 5.2.4. Let $(M, g)$ be a Riemannian manifold of positive scalar curvature and $\left\{R_{k}\right\}_{k}$ a family of handlebodies with Property $(H)$. If $\pi_{1}^{\infty}(M)$ is trivial, then any complete stable minimal lamination in $(M, g)$ has the Vanishing property for $\left\{R_{k}\right\}_{k}$.

## CHAPTER 6

## Proof of Main Theorems

In this chapter, we will explain the proof of the main theorems. For a contractible 3manifold, the existence of complete metrics of positive scalar curvature and its topological properties (for example, Property H) can be related through the limit of a sequence of lamination (constructed in Chapter 4). Combining all these, we will finish the proof of Theorem $B_{2}$ and Theorem $C$.

### 6.1. Proof of Main theorems

For the proof of the main theorems, we will argue by contradiction. In this chapter, we assume that $(M, g)$ is a complete contractible 3-manifold of positive scalar curvature which is not homeomorphic to $\mathbb{R}^{3}$.

As in Remark 1.1.13, $M$ is an increasing union of handlebodies $\left\{N_{k}\right\}_{k}$ with the property that for each $k$, (1) $N_{k}$ is homotopically trivial in $N_{k+1}$; (2) none of the $N_{k}$ is contained in a 3 -ball. In addition, the genus of $N_{k}$ is greater than zero for $k \geqslant 0$.

From Lemma 1.2.7, each $N_{k}$ has a system of meridians $\left\{\gamma_{k}^{l}\right\}_{l=1}^{g\left(N_{k}\right)}$. As in Chapter 4.2, there is a lamination $\mathscr{L}_{k}:=\mathrm{H}_{l} \Omega_{k}^{l} \subset N_{k}$. Each leaf $\Omega_{k}^{l}$ is a meridian disc with boundary $\gamma_{k}^{l}$. As described in Chapter 4.2, since $N_{0}$ is not contained in a 3 -ball, the lamination $\mathscr{L}_{k}$ intersects $N_{0}$.

The intersection $\mathscr{L}_{k} \cap N_{k-1}$ is a stable minimal lamination in $(M, g)$. From Chapter 4.2, the sequence $\left\{\mathscr{L}_{k}\right\}$ sub-converges to a stable minimal lamination $\mathscr{L}:=\cup_{t \in \Gamma} L_{t}$ in $(M, g)$. Each leaf $L_{t}$ is a complete (non-compact) stable minimal surface in $(M, g)$.

Since $(M, g)$ has positive scalar curvature, each leaf in $\mathscr{L}$ is a properly embedded plane (See Theorem 3.3.10 and Corollary 3.3.11).
6.1.1. Properties of $\mathscr{L}$. In the following, we consider the lamination $\mathscr{L}$ and its properties.

If one of the following holds:

- $\pi_{1}^{\infty}(M)$ is trivial,
- $M$ is a contractible genus one 3-manifold;
- $M$ is homeomorphic to $M_{1}$ (constructed in Chapter 1.3.5)
there is a family of ascending handlebodies $\left\{R_{k}\right\}_{k}$ satisfying Property H , so that
a) the lamination $\mathscr{L}$ has the Vanishing property for $\left\{R_{k}\right\}_{k}$;
b) for each $k$ and any $N_{j}$ containing $R_{k}$, the intersection $\mathscr{L}_{j} \cap \partial R_{k}$ has at least one meridian of $R_{k}$.

Remark 6.1.1.

- The three different conditions for $M$ correspond respectively to Theorem $C$, Theorem $\widehat{B_{2}}$ and Theorem 1.3.15.
- If $M$ is a contractible genus one 3-manifold, then $\pi_{1}^{\infty}(M)$ is trivial.

In the following, let us explain the above properties, a) and b).
If $\pi_{1}^{\infty}(M)$ is trivial, we know from Theorem 2.2.6 that there is an increasing family $\left\{R_{k}\right\}_{k}$ of closed handlebodies with Property H (See Definition 2.2.5). Corollary 5.2.4 shows that the lamination $\mathscr{L}$ has the Vanishing Property for this family.

In addition, none of the $R_{k}$ is contained in a 3-ball (See Definition 2.2.5). Together with Property H, we use Corollary 1.2 .8 to know that if $N_{j}$ contains $R_{k}$, the intersection $\mathscr{L}_{j} \cap \partial R_{k}$ has at least one meridian of $R_{k}$.

If $M$ is a contractible genus one 3-manifold, the family $\left\{N_{k}\right\}$ can be assumed as in Theorem 1.3.13. That is to say, the geometric index $I\left(N_{k}, N_{k+1}\right)$ is greater than zero.

From Lemma 1.3.10, Theorem 1.1.4 and Lemma 1.1.5, we know that the map $\pi_{1}\left(\partial N_{k}\right) \rightarrow$ $\pi_{1}\left(\overline{M \backslash N_{k}}\right)$ and $\pi_{1}\left(\partial N_{k}\right) \rightarrow \pi_{1}\left(\overline{N_{k} \backslash N_{0}}\right)$ are also injective. That is to say, the family $\left\{N_{k}\right\}$ satisfies Property H.

From Theorem 2.1.6, $M$ satisfies Property P. Theorem 5.1.1 implies that $\mathscr{L}$ satisfies the Vanishing property for $\left\{N_{k}\right\}_{k}$. In addition, since the geometric index $I\left(N_{j}, N_{k}\right)$ is greater than zero for $j<k, \mathscr{L}_{j} \cap \partial N_{j}$ has at least one meridian of $N_{j}$ (See Lemma 1.3.11).

If $M$ is homeomorphic to $M_{1}$ ( constructed in Chapter 1.3.5), then there is a properly embedded plane $P_{1}$ as constructed in Chapter 1.3.4. It cuts $M_{1}$ into two contractible 3-manifolds, $M_{1}^{\prime}$ and $M_{1}^{\prime \prime}$ (See Proposition 1.3.14). In addition, $M_{1}^{\prime}$ is homeomorphic to the Whitehead manifold. As in the construction of the Whitehead manifold (in Chapter 1.3.2) there is a family $\left\{R_{k}\right\}$ of solid tori in $M$ satisfying:

- the union $\cup_{k} R_{k}$ is equal to $M_{1}^{\prime}$;
- the geometric index $I\left(R_{k}, R_{k+1}\right)$ is equal to two for each $k$.

As in the genus one case, we know that $\pi_{1}\left(\partial R_{k}\right) \rightarrow \pi_{1}\left(\overline{M \backslash R_{k}}\right)$ and $\pi_{1}\left(\partial R_{k}\right) \rightarrow$ $\pi_{1}\left(\overline{R_{k} \backslash R_{0}}\right)$ are both injective (See Lemma 1.3.10). In addition, none of the $R_{k}$ is contained in a 3 -ball in $M$. That is to say, the family $\left\{R_{k}\right\}_{k}$ satisfies Property H.

From Remark 2.1.7, the family $\left\{R_{k}\right\}_{k}$ satisfies Property P. Since $\lim _{k \rightarrow \infty} I\left(R_{0}, R_{k}\right)=$ $\lim _{k \rightarrow \infty} 2^{k-1}=\infty$, we see from Remark 5.1 .3 that $\mathscr{L}$ satisfies the Vanishing property for $\left\{R_{k}\right\}$.

In addition, we know from Corollary 1.2.8, that for any $N_{k}$ containing $R_{j}, \mathscr{L}_{k}$ has at least one meridian of $R_{j}$ for $k \geqslant j$.

REMARK 6.1.2. In the following, our proof requires that $\partial R_{k}$ intersects some leaf $L_{t}$ transversally. To overcome it, we will deform the handlebody $R_{k}$ in a small tubular neighborhood of $\partial R_{k}$ so that the boundary of the new handlebody intersects $L_{t}$ transversally.

This new handlebody also satisfies a) and b). The reason is as follows:
For any handlebody $R_{k}^{\prime}$ obtained by deforming $R_{k}$, the maps $\pi_{1}\left(\partial R_{k}^{\prime}\right) \rightarrow \pi_{1}\left(\overline{R_{k}^{\prime} \backslash R_{0}}\right)$ and $\pi_{1}\left(\partial R_{k}^{\prime}\right) \rightarrow \pi_{1}\left(\overline{M \backslash R_{k}^{\prime}}\right)$ are both injective. The proof of a) and b) just depends on the injectivity of these two maps. Hence, the handlebody $R_{k}^{\prime}$ also holds a) and b).

Remark. The positivity of the scalar curvature gives the property, a). The topological properties of $M$ implies that $\mathscr{L}_{k}$ satisfies the property, b).

In the following, we just give the complete proof of Theorem $C$. The remaining proofs of Theorem $B_{2}$ and Theorem 1.3 .15 are the same as Theorem $C$.
6.1.2. The proof of Theorem $C$. From the above property a), there is a family of handlebodies $\left\{R_{k}\right\}_{k}$ so that $\mathscr{L}$ has the Vanishing property for the family $\left\{R_{k}\right\}_{k}$. That is to say,

There is a integer $k_{0}>0$ so that for any $k \geqslant k_{0}$, any circle in $\mathscr{L} \cap \partial R_{k}$ is contractible in $\partial R_{k}$.

If $\mathscr{L}:=\amalg_{t \in \Lambda} L_{t}$ has finitely many components, we may assume that each leaf $L_{t}$ intersects $\partial R_{k}$ transversally for $k \geqslant k_{0}$. Since $L_{t}$ is properly embedded, $L_{t} \cap \partial R_{k}:=\left\{\gamma_{i}^{t}\right\}_{i \in I_{t}}$ has finitely many components. Each component is a circle.

From the above fact, each $\gamma_{i}^{t}$ is null-homotopic in $\partial R_{k}$. Consider the unique closed disc $D_{i}^{t} \subset \partial R_{k}$ with boundary $\gamma_{i}^{t}$ and the partially ordered set $\left(\left\{D_{i}^{t}\right\}_{t \in \Lambda, i \in C_{t}}, \subset\right)$. Let $C$ be the set of maximal elements. In particular, it is a finite set. The set $\mathscr{L} \cap \partial R_{k}$ is contained in the disjoint union of closed discs in $C$.

In the general case, we also have a similar result.
Lemma 6.1.3. For any $k \geqslant k_{0}, \partial R_{k}(\epsilon) \cap \mathscr{L}$ is contained in a disjoint union of finitely many closed discs in $\partial R_{k}(\epsilon)$, where $R_{k}(\epsilon):=R_{k} \backslash N_{\epsilon}\left(\partial R_{k}\right), N_{\epsilon}\left(\partial R_{k}\right)$ is some tubular neighborhood of $\partial R_{k}$ in $R_{k}$.

In the general case when $\mathscr{L}$ has infinitely many components, we will prove it in Chapter 6.3.

We now finish the proof of Theorem C.
Proof. Suppose that some complete contractible 3 -manifold $(M, g)$ with positive scalar curvature and trivial $\pi_{1}^{\infty}(M)$ is not homeomorphic to $\mathbb{R}^{3}$. As above, there is an ascending family $\left\{R_{k}\right\}_{k}$ of handlebodies with Property H , so that
a) the lamination $\mathscr{L}$ has the Vanishing property for $\left\{R_{k}\right\}_{k}$;
b) for each $k$ and any $N_{j}$ containing $R_{k}(\epsilon)$, the intersection $\mathscr{L}_{j} \cap \partial R_{k}(\epsilon)$ has at least one meridian of $R_{k}(\epsilon)$.

The Vanishing property implies Lemma 6.1.3 (We will prove it in Chapter 6.3). That is to say, the intersection $\mathscr{L} \cap \partial R_{k}(\epsilon)$ is in the union of disjoint closed discs $\left\{D_{i}\right\}_{i=1}^{s}$ for $k \geqslant k_{0}$.

Choose an open neighborhood $U$ of the closed set $\mathscr{L} \cap R_{k+1}$ so that $U \cap \partial R_{k}(\epsilon)$ is contained in a disjoint union $\amalg_{i=1}^{s} D_{i}^{\prime}$, where $D_{i}^{\prime}$ is an open tubular neighborhood of $D_{i}$ in $\partial R_{k}(\epsilon)$ with small radius. Each $D_{i}^{\prime}$ is an open disc in $\partial R_{k}(\epsilon)$.

Since $\mathscr{L}_{k}$ subconverges to $\mathscr{L}$, there exists an integer $j$, large enough, satisfying

- $\mathscr{L}_{j} \cap R_{k+1} \subset U$;
- $R_{k}(\epsilon)$ is contained in $N_{j}$.

Therefore, $\mathscr{L}_{j} \cap \partial R_{k}(\epsilon)$ is contained in $U \cap \partial R_{k}(\epsilon) \subset \omega D_{i}^{\prime}$. The induced map $\pi_{1}\left(\mathscr{L}_{j} \cap\right.$ $\left.\partial R_{k}(\epsilon)\right) \rightarrow \pi_{1}\left(\amalg_{i} D_{i}^{\prime}\right) \rightarrow \pi_{1}\left(\partial R_{k}(\epsilon)\right)$ is a trivial map. We can conclude that any circle in $\mathscr{L}_{j} \cap \partial R_{k}(\epsilon)$ is contractible in $\partial R_{k}(\epsilon)$.

However, from b), there exists a meridian $\gamma \subset \mathscr{L}_{j} \cap \partial R_{k}(\epsilon)$ of $R_{k}(\epsilon)$. This contradicts the last paragraph and finishes the proof of Theorem $\mathbb{C}$.

### 6.2. Two topological lemmas

Before proving Lemma 6.1.3, we introduce two topological lemmas. These two lemmas play a crucial role in the proof of Lemma 6.1.3

Lemma 6.2.1. Let $(\Omega, \partial \Omega) \subset(N, \partial N)$ be a 2-sided embedded disc with some closed subdiscs removed, where $N$ is a closed handlebody of genus $g>0$. Each circle $\gamma_{i}$ is contractible in $\partial N$, where $\partial \Omega=\amalg_{i} \gamma_{i}$. Then $N \backslash \Omega$ has two connected components. Moreover, there is a unique component $B$ satisfying that the induced map $\pi_{1}(B) \rightarrow \pi_{1}(N)$ is trivial.

Proof. We argue by contradiction. Suppose that $N \backslash \Omega$ is path-connected. That is to say, there is an embedded circle $\sigma \subset N$ which intersects $\Omega$ transversally at one point.

Since each $\gamma_{i}$ is contractible in $\partial N$, it bounds a unique disc $D_{i} \subset \partial N$. The surface $\hat{\Omega}:=\Omega \bigcup_{i} \cup_{\gamma_{i}} D_{i}$ also intersects $\sigma$ transversally at one point. The intersection number between $\hat{\Omega}$ and $\sigma$ is $\pm 1$.

However, $\hat{\Omega}$ is the image of a continuous map $g: \mathbb{S}^{2} \rightarrow N$. It is contractible in $N$, since $\pi_{2}(N)=\{0\}$. The intersection number between $\hat{\Omega}$ and $\sigma$ must be zero, which leads to a contradiction.

Therefore, $N \backslash \Omega$ is not connected. Since $\Omega$ is 2-sided and connected, $N \backslash \Omega$ just has two components $B_{0}$ and $B_{1}$.

Remark that the surface $\hat{\Omega}:=\Omega \bigcup\left(\cup_{\gamma_{i}} D_{i}\right)$ is an immersed 2-sphere in $N$. This deduces that the map $\pi_{1}(\Omega) \rightarrow \pi_{1}(\hat{\Omega})$ is trivial map. Therefore, the map $\pi_{1}(\Omega) \rightarrow \pi_{1}(N)$ is trivial.

In the following, let us explain the existence of $B$.
Consider the partially ordered relationship over $\left\{D_{i}\right\}$ induced by inclusion. Therefore, $\cup_{i} D_{i}$ is equals to a disjoint union of maximal elements in $\left(\left\{D_{i}\right\}, \subset\right)$. The set $\partial N \backslash \cup_{i} D_{i}$ is a compact surface with some disjoint closed sub-discs removed.

Therefore, the induced map $\pi_{1}\left(\partial N \backslash \cup_{i} D_{i}\right) \rightarrow \pi_{1}(\partial N)$ is surjective. The induced map $\pi_{1}(\partial N) \rightarrow \pi_{1}(N)$ is also surjective. We can conclude that the composition of these two maps $\pi_{1}\left(\partial N \backslash \cup_{i} D_{i}\right) \rightarrow \pi_{1}(N)$ is also surjective.

The set $\partial N \backslash \cup_{i} D_{i}$ is contained in one of two components, $B_{1}$ and $B_{2}$, of $N \backslash \Omega$. Without loss of generality, we may assume that $B_{1}$ contains $\partial N \backslash \cup_{i} D_{i}$. Based on the last paragraph, the induced map $\pi_{1}\left(B_{1}\right) \rightarrow \pi_{1}(N)$ is surjective.

Let $G_{i}$ be the image of the map $\pi_{1}\left(B_{i}\right) \rightarrow \pi_{1}(N)$, a subgroup of $\pi_{1}(N)$. Van-Kampen's Theorem (See Theorem 1.1.4) gives an isomorphism between $\pi_{1}(N)$ and $\pi_{1}\left(B_{1}\right) *_{\pi_{1}(\Omega)}$ $\pi_{1}\left(B_{2}\right)$. Since the image of $\pi_{1}(\Omega) \rightarrow \pi_{1}(N)$ is trivial, $\pi_{1}(N)$ is isomorphic to $G_{1} * G_{2}$. Grushko's Theorem Gru40 shows that $\operatorname{rank}\left(G_{1}\right)+\operatorname{rank}\left(G_{2}\right)=\operatorname{rank}\left(\pi_{1}(N)\right)$. (The rank of a group is the smallest cardinality of a generating set for the group.) From the last paragraph, the image, $G_{1}$, of the map $\pi_{1}\left(B_{1}\right) \rightarrow \pi_{1}(N)$ is isomorphic to $\pi_{1}(N)$. That is to say, $\operatorname{rank}\left(G_{1}\right)=\operatorname{rank}\left(\pi_{1}(N)\right)$. Therefore, $\operatorname{rank}\left(G_{2}\right)$ is equal to zero. That is to say, $G_{2}$ is a trivial group. We know that $B:=B_{2}$ is the required candidate in the assertion.

Finally, we prove the uniqueness. Suppose that the two induced maps are both trivial. Therefore, the map $H_{1}\left(B_{i}\right) \rightarrow H_{1}(N)$ is trivial for each $i$. Applying the Mayer-Vietoris
sequence to $N=B_{0} \cup_{\Omega} B_{1}$, one has:

$$
H_{1}\left(B_{0}\right) \oplus H_{1}\left(B_{1}\right) \rightarrow H_{1}(N) \rightarrow \hat{H}_{0}(\Omega)
$$

Since $\Omega$ is connected, $\hat{H}_{0}(\Omega)$ is trivial. Therefore, $H_{1}(N)$ is also trivial. This contradicts the fact that $H_{1}(N)$ is isomorphic to $\mathbb{Z}^{g(N)}$. This completes the proof.

Consider two disjoint surfaces $\left(\Omega_{1}, \partial \Omega_{1}\right)$ and $\left(\Omega_{2}, \partial \Omega_{2}\right)$ as assumed in Lemma 6.2.1. For $t=1,2, N \backslash \Omega_{t}$ has two components. Let $B_{t}$ be the unique component of $N \backslash \Omega_{t}$ satisfying $\pi_{1}\left(B_{t}\right) \rightarrow \pi_{1}(N)$ is trivial. One has:

LEMMA 6.2.2. Let $\left(\Omega_{1}, \partial \Omega_{1}\right)$ and $\left(\Omega_{2}, \partial \Omega_{2}\right)$ be two disjoint surfaces as assumed in Lemma 6.2.1. For each $t=1,2, N \backslash \Omega_{t}$ has a unique component $B_{t}$ with the property that the map $\pi_{1}\left(B_{t}\right) \rightarrow \pi_{1}(N)$ is trivial. Then it holds one of the following:
(1) $B_{1} \cap B_{2}=\varnothing$;
(2) $B_{1} \subset B_{2}$;
(3) $B_{2} \subset B_{1}$.

Proof. Suppose $B_{1} \cap B_{2}$ and $B_{1} \backslash B_{2}$ are both nonempty. Say, there are two points $p_{1} \in B_{1} \backslash B_{2}$ and $p_{2} \in B_{1} \cap B_{2}$.

First, $\Omega_{2}$ is contained in $B_{1}$. The reason follows as below: $B_{1}$ includes a curve $\gamma$ joining $p_{1}$ and $p_{2}$ (since $B_{1}$ is connected). $\gamma$ must intersect $\Omega_{2}$ at some point(s). Hence, $\Omega_{2} \cap B_{1}$ is not empty. Since $\Omega_{1} \cap \Omega_{2}$ is empty, $\Omega_{2}$ lies in one of component of $N \backslash \Omega_{1}$. Therefore, $\Omega_{2}$ is contained in $B_{1}$.

Second, $\Omega_{2}$ cuts $B_{1}$ into two components. Otherwise, there is a circle in $B_{1}$ which intersects $\Omega_{2}$ at one point. As argued in Lemma 6.2.1, such a circle can not exist.

Finally, take the component $B$ of $B_{1} \backslash \Omega_{2}$ satisfying that $\partial B \cap \Omega_{1}$ is empty. Then, $B$ is also a component of $N \backslash \Omega_{2}$. In addition, the map $\pi_{1}(B) \rightarrow \pi_{1}\left(B_{1}\right) \rightarrow \pi_{1}(N)$ is trivial. From the uniqueness of $B_{2}$, one has that $B=B_{2}$. This implies $B_{2} \subset B_{1}$.

### 6.3. Proof of Lemma 6.1.3

In order to prove Lemma 6.1.3, we will introduce the set $S$ and prove the finiteness of $S$. The finiteness of $S$ will imply Lemma 6.1.3.
6.3.1. Definition of the set $S$. Let $(M, g), \mathscr{L},\left\{\mathscr{L}_{k}\right\}$ and $\left\{N_{k}\right\}$ be assumed as in the proof of Theorem $D$. As in Chapter 6.2 , there is an ascending family $\left\{R_{k}\right\}_{k}$ of handlebodies with Property $H$ with the property that $\mathscr{L}$ satisfies the Vanishing property for $\left\{R_{k}\right\}_{k}$. That is to say,
there is a positive integer $k_{0}$ so that for each $k \geqslant k_{0}$ and each $t \in \Lambda$, each circle in $L_{t} \cap \partial R_{k}$ is contractible in $\partial R_{k}$.

In the following, we will work on the open handlebody Int $R_{k}$ and construct the set $S$, for a fixed integer $k \geqslant k_{0}$.
6.3.1.1. Elements in $S$. Let $\left\{\Sigma_{i}^{t}\right\}_{i \in I_{t}}$ be the set of components of $L_{t} \cap \operatorname{Int} R_{k}$ for each $t \in \Lambda$. (It may be empty.) We will show that for each component $\Sigma_{i}^{t}, R_{k} \backslash \bar{\Sigma}_{i}^{t}$ has a unique component $B_{i}^{t}$ satisfying that $\pi_{1}\left(B_{i}^{t}\right) \rightarrow \pi_{1}\left(R_{k}\right)$ is trivial.

If $L_{t}$ intersects $\partial R_{k}$ transversally, the boundary $\partial \Sigma_{i}^{t} \subset L_{t} \cap \partial R_{k}$ is the union of some disjointly embedded circles. From the Vanishing property, any circles in the boundary $\partial \Sigma_{i}^{t} \subset L_{t} \cap \partial R_{k}$ is contractible in $\partial R_{k}$.

In addition, since $L_{t}$ is homeomorphic to $\mathbb{R}^{2}$ and $\Sigma_{i}^{t}$ is relatively compact, $\Sigma_{i}^{t}$ is homeomorphic to an open disc with some disjoint closed subdiscs removed. By Lemma 6.2.1, $R_{k} \backslash \overline{\Sigma_{i}^{t}}$ has a unique component $B_{i}^{t}$ satisfying that $\pi_{1}\left(B_{i}^{t}\right) \rightarrow \pi_{1}\left(R_{k}\right)$ is trivial.

In general, $L_{t}$ may not intersect $\partial R_{k}$ transversally. To overcome it, we will deform the surface $\partial R_{k}$. Precisely, for the leaf $L_{t}$, there is a new handlebody $\tilde{R}_{k}\left(\epsilon_{t}\right)$ containing $R_{k}$ so that $L_{t}$ intersects $\partial \tilde{R}\left(\epsilon_{t}\right)$ transversally, where $\tilde{R}_{k}\left(\epsilon_{t}\right)$ is a closed tubular neighborhood of $R_{k}$ in $M$.

We consider the component $\tilde{\Sigma}_{i}^{t}$ of $L_{t} \cap \operatorname{Int} \tilde{R}_{k}\left(\epsilon_{t}\right)$ containing $\Sigma_{i}^{t}$. As above, $\tilde{R}_{k}\left(\epsilon_{t}\right) \backslash \tilde{\Sigma}_{i}^{t}$ has a unique component $\tilde{B}_{i}^{t}$ so that the map $\pi_{1}\left(\tilde{B}_{i}^{t}\right) \rightarrow \pi_{1}\left(\tilde{R}_{k}\left(\epsilon_{t}\right)\right)$ is trivial.

Choose the component $B_{i}^{t}$ of $\tilde{B}_{i}^{t} \cap R_{k}$ whose boundary contains $\Sigma_{i}^{t}$. It is a component of $R_{k} \backslash \overline{\sum_{i}^{t}}$. In addition, the map $\pi_{1}\left(B_{i}^{t}\right) \rightarrow \pi_{1}\left(\tilde{B}_{i}^{t}\right) \rightarrow \pi_{1}\left(\tilde{R}_{k}\left(\epsilon_{t}\right)\right)$ is trivial. Since $R_{k}$ and $\tilde{R}_{k}\left(\epsilon_{t}\right)$ are homotopy equivalent, the map $\pi_{1}\left(B_{i}^{t}\right) \rightarrow \pi_{1}\left(R_{k}\right)$ is also trivial. This finishes the construction of $B_{i}^{t}$.
6.3.1.2. Properties of $S$. From Lemma 6.2.2, for any $B_{i}^{t}$ and $B_{i^{\prime}}^{t^{\prime}}$, it holds one of the following
(1) $B_{i}^{t} \cap B_{i^{\prime}}^{t^{\prime}}=\varnothing$;
(2) $B_{i}^{t} \subset B_{i^{\prime}}^{t^{\prime}}$;
(3) $B_{i^{\prime}}^{t^{\prime}} \subset B_{i}^{t}$,
where $t, t^{\prime} \in \Lambda, i \in I_{t}$ and $i^{\prime} \in I_{t^{\prime}}$.
Therefore, $\left(\left\{B_{i}^{t}\right\}_{t \in \Lambda, i \in I_{t}}, \subset\right)$ is a partially ordered set. We consider the set $\left\{B_{j}\right\}_{j \in J}$ of maximal elements. However this set may be infinite.

Definition 6.3.1. $S:=\left\{B_{j} \mid B_{j} \cap R_{k}(\epsilon / 2) \neq \varnothing\right.$, for any $\left.j \in J\right\}$, where $R_{k}(\epsilon / 2)$ is $R_{k} \backslash N_{\epsilon / 2}\left(\partial R_{k}\right)$ and $N_{\epsilon / 2}\left(\partial R_{k}\right)$ is a 2-sided tubular neighborhood of $\partial R_{k}$ with radius $\epsilon / 2$.

Proposition 6.3.2. Let $\Sigma_{i}^{t}$ be one component of $L_{t} \cap R_{k}$ and $B_{i}^{t}$ assumed as above. If $B_{i}^{t}$ is an element in $S$, then $\Sigma_{i}^{t} \cap R_{k}(\epsilon / 2)$ is nonempty.

Proof. We argue by contradiction. Suppose that $\Sigma_{i}^{t} \cap R_{k}(\epsilon / 2)$ is empty. As mentioned above, $\overline{\Sigma_{i}^{t}}$ cuts $R_{k}$ into two components. Hence, $R_{k}(\epsilon / 2)$ must be in one of these two components.

In addition, from the definition $S$, the component $B_{i}^{t}$ of $R_{k} \backslash \overline{\Sigma_{i}^{t}}$ must intersect $R_{k}(\epsilon / 2)$. One knows that $R_{k}(\epsilon / 2)$ is contained in $B_{i}^{t}$.

However, the composition of maps $\pi_{1}\left(R_{k}(\epsilon)\right) \rightarrow \pi_{1}\left(B_{i}^{t}\right) \rightarrow \pi_{1}\left(R_{k}\right)$ is an isomorphism. Therefore, the map $\pi_{1}\left(B_{i}^{t}\right) \rightarrow \pi_{1}\left(R_{k}\right)$ is non-trivial and surjective, which contradicts the fact that the map $\pi_{1}\left(B_{i}^{t}\right) \rightarrow \pi_{1}\left(R_{k}\right)$ is trivial. This finishes the proof.

Proposition 6.3.3. $R_{k}(\epsilon) \cap \mathscr{L} \subset \bigcup_{B_{j} \in S} \bar{B}_{j} \cap R_{k}(\epsilon)$. Moreover, $\partial R_{k}(\epsilon) \cap \mathscr{L} \subset$ $\bigcup_{B_{j} \in S} \bar{B}_{j} \cap \partial R_{k}(\epsilon)$.

Proof. Each component $\Sigma_{i}^{t}$ of $L_{t} \cap$ Int $R_{k}$ is contained in $\overline{B_{i}^{t}}$. Hence, $L_{t} \cap R_{k}$ is in $\cup_{i \in I_{t}} \overline{B_{i}^{t}}$. We can conclude that $\mathscr{L} \cap R_{k}$ is contained in $\cup \overline{B_{i}^{t}}$.

The set $\cup B_{i}^{t}$ is equal to $\cup_{j \in J} B_{j}$, because $\left\{B_{j}\right\}_{j \in J}$ is the set of all maximal elements in the partially ordered set $\left(\left\{B_{i}^{t}\right\}, \subset\right)$. Therefore, $\mathscr{L} \cap R_{k}$ is in $\cup_{j \in J} \overline{B_{j}}$.

From the definition of $S, \cup_{j \in J} \overline{B_{j}} \cap R_{k}(\epsilon)$ equals $\cup_{B_{j} \in S} \overline{B_{j}} \cap R_{k}(\epsilon)$. Therefore, $R_{k}(\epsilon) \cap$ $\mathscr{L} \subset \cup_{B_{j} \in S} \overline{B_{j}} \cap R_{k}(\epsilon)$.

Similarly, one has that $\cup_{j \in J} \overline{B_{j}} \cap \partial R_{k}(\epsilon)$ equals $\cup_{B_{j} \in S} \overline{B_{j}} \cap \partial R_{k}(\epsilon)$. Hence, $\partial R_{k}(\epsilon) \cap \mathscr{L} \subset$ $\cup_{B_{j} \in S} \overline{B_{j}} \cap \partial R_{k}(\epsilon)$
6.3.2. The finiteness of the set $S$. The set $\partial B_{j} \cap \operatorname{Int} R_{k}$ equals some $\Sigma_{i}^{t} \subset L_{t}$ for $t \in \Lambda$. Let us consider the set $S_{t}:=\left\{B_{j} \in S \mid \partial B_{j} \cap \operatorname{Int} R_{k} \subset L_{t}\right\}$. Then, $S=\mathrm{U}_{t \in \Lambda} S_{t}$. Note that each $B_{j} \in S_{t}$ is a $B_{i}^{t}$ for some $i \in I_{t}$.

In this subsection, we first show that each $S_{t}$ is finite. Then, we argue that $\left\{S_{t}\right\}_{t \in \Lambda}$ contains at most finitely many nonempty sets. These imply the finiteness of $S$.

Lemma 6.3.4. Each $S_{t}$ is finite.
Proof. We argue by contradiction. Suppose that $S_{t}$ is infinite for some $t$.
For each $B_{j} \in S_{t}$, there exists a $i \in I_{t}$ so that $B_{j}$ is equal to $B_{i}^{t}$, where $B_{i}^{t}$ is a component of $R_{k} \backslash \overline{\Sigma_{i}^{t}}$ and $\Sigma_{i}^{t}$ is one component of $L_{t} \cap \operatorname{Int} R_{k}$. By Proposition 6.3.2, $\Sigma_{i}^{t} \cap R_{k}(\epsilon / 2)$ is nonempty.

Choose $x_{j} \in \Sigma_{i}^{t} \cap R_{k}(\epsilon / 2)$ and $r_{0}=\frac{1}{2} \min \left\{\epsilon / 2, i_{0}\right\}$, where $i_{0}:=\inf _{x \in R_{k}} \operatorname{Inj}_{M}(x)$. Then the geodesic ball $B\left(x_{j}, r_{0}\right)$ in $M$ is contained in $R_{k}$.

We apply Theorem 3.2.7 to the minimal surface $\left(\Sigma_{i}^{t}, \partial \Sigma_{i}^{t}\right) \subset\left(R_{k}, \partial R_{k}\right)$. One knows that,

$$
\operatorname{Area}\left(\Sigma_{i}^{t} \cap B\left(x_{j}, r_{0}\right)\right) \geqslant C\left(r_{0}, i_{0}, K\right)
$$

where $K=\sup _{x \in R_{k}}\left|K_{M}\right|$. This leads to a contradiction from Theorem 3.3.10 as below:

$$
\begin{aligned}
2 \pi \geqslant \int_{L_{t}} \kappa(x) d v & \geqslant \sum_{B_{j} \in S_{t}} \int_{\Sigma_{i}^{t}} \kappa(x) d v \geqslant \sum_{B_{j} \in S_{t}} \int_{\Sigma_{i}^{t} \cap B\left(x_{j}, r_{0}\right)} \kappa(x) d v \\
& \geqslant \inf _{x \in R_{k}}(\kappa(x)) \sum_{B_{j} \in S_{t}} \operatorname{Area}\left(B\left(x_{j}, r_{0}\right) \cap \Sigma_{i}^{t}\right) \\
& \geqslant C \inf _{x \in R_{k}}(\kappa(x))\left|S_{t}\right|=\infty
\end{aligned}
$$

This finishes the proof.
Lemma 6.3.5. $\left\{S_{t}\right\}_{t \in \Lambda}$ contains at most finitely many nonempty sets.
Proof. We argue by contradiction. Suppose that there exists a sequence $\left\{S_{t_{n}}\right\}_{n \in \mathbb{N}}$ of nonempty sets. For an element $B_{j_{t_{n}}} \in S_{t_{n}}$, there is some $i_{n} \in I_{t_{n}}$ so that $B_{j_{n}}$ equals $B_{i_{n}}^{t_{n}}$ where $B_{i_{n}}^{t_{n}}$ is one component of $R_{k} \backslash \overline{\sum_{i_{n}}^{t_{n}}}$ and $\sum_{i_{n}}^{t_{n}}$ is one of components of $L_{t_{n}} \cap$ Int $R_{k}$. Note that $\pi_{1}\left(B_{i_{n}}^{t_{n}}\right) \rightarrow \pi_{1}\left(R_{k}\right)$ is trivial.

By Proposition 6.3.2, $\Sigma_{i_{n}}^{t_{n}} \cap R_{k}(\epsilon / 2)$ is not empty. Pick a point $p_{t_{n}}$ in $\Sigma_{i_{n}}^{t_{n}} \cap R_{k}(\epsilon / 2)$.

Step1: $\left\{L_{t_{n}}\right\}$ subconverges to a lamination $\mathscr{L}^{\prime} \subset \mathscr{L}$ with finite multiplicity.
Since $L_{t_{n}}$ is a stable minimal surface, We know from Lemma 4.1.5 that for any compact set $K \subset M$, there is a constant $C_{1}:=C_{1}(K, M, g)$ such that

$$
\left|A_{L_{t_{n}}}\right|^{2} \leqslant C_{1} \text { on } K \cap L_{t_{n}} .
$$

From Theorem 3.3.10, $\int_{L_{t_{n}}} \kappa(x) d v \leqslant 2 \pi$. Hence,

$$
\operatorname{Area}\left(K \cap L_{t_{n}}\right) \leqslant 2 \pi\left(\inf _{x \in K} \kappa(x)\right)^{-1}
$$

We use Theorem 4.1.4 (See [Compactness Theorem, Page 96] in And85]) to find a sub-sequence of $\left\{L_{t_{n}}\right\}$ subconverging to a properly embedded lamination $\mathscr{L}^{\prime}$ with finite multiplicity. Since $\mathscr{L}$ is a closed set in $M, \mathscr{L}^{\prime} \subset \mathscr{L}$ is a sublamination.

From now on, we abuse notation and write $\left\{L_{t_{n}}\right\}$ and $\left\{p_{t_{n}}\right\}$ for the convergent subsequence.


Figure 6.1.

Step 2: $\left\{\sum_{i_{n}}^{t_{n}}\right\}$ converges with multiplicity one.
Let $L_{t_{\infty}}$ be the unique component of $\mathscr{L}^{\prime}$ passing through $p_{\infty}$, where $p_{\infty}=\lim _{n \rightarrow \infty} p_{t_{n}}$. The limit of $\left\{\Sigma_{i_{n}}^{t_{n}}\right\}$ is the component $\Sigma_{\infty}$ of $L_{t_{\infty}} \cap R_{k}$ passing through $p_{\infty}$, where $\Sigma_{i_{n}}^{t_{n}}$ is the unique component of $R_{k} \cap L_{t_{n}}$ passing though $p_{t_{n}}$.

Let $D \subset L_{t_{\infty}}$ be a simply-connected subset satisfying $\Sigma_{\infty} \subset D$. Since $\left\{L_{t_{n}}\right\}$ converges smoothly to $L_{t_{\infty}}$, there exists $\epsilon_{1}>0$ and an integer $N$ such that

$$
\Sigma_{i_{n}}^{t_{n}} \subset D\left(\epsilon_{1}\right), \text { for } n>N,
$$

where $D\left(\epsilon_{1}\right)$ is the tubular neighborhood of $D$ with radius $\epsilon_{1}$ in $M$. (See Definition 4.1.2 and Remark 4.1.3).

Let $\pi: D\left(\epsilon_{1}\right) \rightarrow D$ be the projection. For $n$ large enough, the restriction of $\pi$ to each component of $L_{t_{n}} \cap D\left(\epsilon_{1}\right)$ is injective (See Remark 4.1.3).

Hence, $\left.\pi\right|_{\Sigma_{i_{n}}^{t_{n}}}: \Sigma_{i_{n}}^{t_{n}} \rightarrow D$ is injective. That is to say, $\Sigma_{i_{n}}^{t_{n}}$ is a normal graph over a subset of $D$. Therefore, $\left\{\Sigma_{i_{n}}^{t_{n}}\right\}$ converges to $\Sigma_{\infty}$ with multiplicity one (See Definition 4.1.2). That is to say, there is a geodesic disc $B^{\Sigma_{\infty}}\left(p_{\infty}\right) \subset \Sigma_{\infty}$ centered at $p_{\infty}$ with small raduis so that
$(* *)$ : the set $\pi^{-1}\left(B^{\Sigma_{\infty}}\left(p_{\infty}\right)\right) \cap \sum_{i_{n}}^{t_{n}}$ is connected and a normal graph over $B^{\Sigma_{\infty}}\left(p_{\infty}\right)$, for large $n$.

## Step 3: Get a contradiction.

There exists a neighborhood $U$ of $p_{\infty}$ and a coordinate map $\Phi$, such that each component of $\Phi(\mathscr{L} \cap U)$ is $\mathbb{R}^{2} \times\{x\} \cap \Phi(U)$ for some $x \in \mathbb{R}$. (See Definition 4.2.1 or Appendix B of CM11.) Choose the disc $B^{\Sigma_{\infty}}\left(p_{\infty}\right)$ and $\epsilon_{1}$ small enough such that $\pi^{-1}\left(B^{\Sigma_{\infty}}\left(p_{\infty}\right)\right) \subset U$. We may assume that $U=\pi^{-1}\left(B^{\Sigma_{\infty}}\left(p_{\infty}\right)\right)$.

From (**), $\Sigma_{i_{n}}^{t_{n}} \cap U \subset L_{t_{n}}$ is connected and a graph over $B^{\Sigma_{\infty}}\left(p_{\infty}\right)$, for $n$ large enough. Since $\partial B_{j_{t_{n}}} \cap U \subset L_{t_{n}}$ equals $\Sigma_{i_{n}}^{t_{n}} \cap U$, it is also connected. Therefore $\Phi\left(\partial B_{j_{t_{n}}} \cap U\right)$ is the set $\mathbb{R}^{2} \times\left\{x_{t_{n}}\right\} \cap \Phi(U)$ for some $x_{t_{n}} \in \mathbb{R}$. In addition, $\Phi\left(\Sigma_{\infty} \cap U\right)$ equals $\mathbb{R}^{2} \times\left\{x_{\infty}\right\} \cap \Phi(U)$ for some $x_{\infty} \in \mathbb{R}$. Since $\lim _{n \rightarrow \infty} p_{t_{n}}=p_{\infty}$, we have $\lim _{n \rightarrow \infty} x_{t_{n}}=x_{\infty}$.


Figure 6.2.

The set $U \backslash \partial B_{j_{t_{n}}}$ has two components. Therefore, $\Phi\left(B_{j_{t_{n}}} \cap U\right)$ is $\Phi(U) \cap\left\{x \mid x_{3}>\right.$ $\left.x_{t_{n}}\right\}$ or $\Phi(U) \cap\left\{x \mid x_{3}<x_{t_{n}}\right\}$. For $n$ large enough, there exists some $n^{\prime} \neq n$ such that $\mathbb{R}^{2} \times\left\{x_{t_{n^{\prime}}}\right\} \cap \Phi(U) \subset \Phi\left(B_{j_{t_{n}}} \cap U\right)$. This implies that $B_{j_{t_{n}}} \cap B_{j_{t_{n^{\prime}}}}$ is non-empty.

Since $S$ consists of maximal elements in $\left(\left\{B_{i}^{t}\right\}, \subset\right)$, the set $B_{j_{t_{n}}} \cap B_{j_{t_{n^{\prime}}}}$ is empty which leads to a contradiction. This finishes the proof.
6.3.3. The finiteness of $S$ implies Lemma 6.1.3. We will explain how to deduce Lemma 6.1.3 from the finiteness of $S$.

Proof. Since $S$ is finite, we may assume that $\partial B_{j}$ intersects $\partial R_{k}(\epsilon)$ transversally for each $B_{j} \in S$. Remark that each $B_{j}$ is equal to some $B_{i}^{t}$ and $\partial B_{j} \cap \partial R_{k}(\epsilon)$ equals $\Sigma_{i}^{t} \cap \partial R_{k}(\epsilon)$. Since each $\Sigma_{i}^{t}$ is properly embedded, $\left\{c_{i}\right\}_{i \in I}:=\partial R_{k}(\epsilon) \cap\left(\cup_{B_{j} \in S} \partial B_{j}\right)$ has finitely many components. Each component is an embedded circle.

The Vanishing property of $\mathscr{L}$ and Remark 6.1 .2 show that each $c_{i}$ is contractible in $\partial R_{k}(\epsilon)$ and bounds a unique closed disc $D_{i} \subset \partial \overline{R_{k}(\epsilon)}$ (since $k \geqslant k_{0}$ ). The set $\left(D_{i}, \subset\right)$ is a partially ordered set. Let $\left\{D_{j^{\prime}}\right\}_{j^{\prime} \in J^{\prime}}$ be the set of maximal elements. The set $J^{\prime}$ is finite .

Since the boundary of $\partial R_{k}(\epsilon) \cap \bar{B}_{j}$ is a subset of $\partial B_{j} \cap \partial R_{k}(\epsilon) \subset \amalg_{i \in I} c_{i}$, it is contained in $\amalg_{j^{\prime} \in J^{\prime}} D_{j^{\prime}}$.

Next we show that for any $B_{j} \in S, \partial R_{k}(\epsilon) \cap \overline{B_{j}}$ is contained in $\amalg_{j^{\prime} \in J^{\prime}} D_{j^{\prime}}$.
If not, $\partial R_{k}(\epsilon) \backslash \amalg_{j^{\prime} \in J^{\prime}} D_{j^{\prime}}$ is contained in $\partial R_{k}(\epsilon) \cap \bar{B}_{j}$ for some $B_{j} \in S$. This implies that the composition of two maps $\pi_{1}\left(\partial R_{k}(\epsilon) \backslash\left(\amalg_{j^{\prime} \in J^{\prime}} D_{j^{\prime}}\right)\right) \rightarrow \pi_{1}\left(\bar{B}_{j}\right) \rightarrow \pi_{1}\left(R_{k}\right)$ is not a zero map. However, the induced map $\pi_{1}\left(\bar{B}_{j}\right) \rightarrow \pi_{1}\left(R_{k}\right)$ is trivial. This is impossible. We conclude that for each $B_{j} \in S, \partial R_{k}(\epsilon) \cap \bar{B}_{j}$ is contained in $\amalg_{j^{\prime} \in J^{\prime}} D_{j^{\prime}}$.

Therefore, $\cup_{B_{j} \in S} \bar{B}_{j} \cap \partial R_{k}(\epsilon)$ is contained in $\amalg_{j^{\prime} \in J^{\prime}} D_{j^{\prime}}$. From Proposition 6.3.3, $\mathscr{L} \cap$ $\partial R_{k}(\epsilon)$ is contained in a disjoint union of finite discs $\left\{D_{j^{\prime}}\right\}_{j^{\prime} \in J^{\prime}}$. This completes the proof.

### 6.4. Deformation to Positive Scalar curvature

This section follows Kazdan's result Kaz82. In this section, we show that a complete non-Ricci-flat metric of nonnegative scalar curvature can deformed to be a complete metric of positive scalar curvature.

Let $\left(M^{n}, g\right)$ be a complete $n$-manifold. We consider the operator

$$
L(u)=-\Delta u+f u
$$

where $f$ is a smooth function on $M$.
For a bounded open set $\Omega \subset M$ with smooth boundary and outer normal derivative $\partial / \partial \nu$ on $\partial \Omega$, let $\mu_{1}(L, \Omega)$ be the lowest eigenvalue of $L$ with Neumann boundary conditions, $\partial u / \partial \nu=0$ on $\partial \Omega$. One has the well-known variational characterization of $\mu_{1}(\Omega)$

$$
\mu_{1}(L, \Omega)=\inf _{v} \frac{\int_{\Omega}\left(|\nabla v|^{2}+f v^{2}\right) d x}{\int_{\Omega} v^{2} d x}
$$

where $d x$ is the volume form and the infimum is taken over all $v$ in the Sobolev space $H^{1}(\Omega)$.

Lemma 6.4.1. (See [Theorem A, Page 228] in Kaz82]) Assume there is a bounded open set $\Omega_{0} \subset M$ such that $\mu_{1}\left(L, \Omega_{0}\right)>0$ and $f \geqslant 0$ on $M \backslash \Omega_{0}$. Then there is a solution $u>0$ on $M$ of $L(u)>0$; in fact one can find a solution of $L(u)>0$ satisfying $0<C_{1}<$ $u<C_{2}$, where $C_{1}$ and $C_{2}$ are two constants.

In the following, we consider that the conformal Laplacian $L_{g}$,

$$
L_{g}:=-\frac{4(n-1)}{n-2} \Delta_{g}+\kappa_{g} .
$$

ThEOREM 6.4.2. A complete non-Ricci-flat metric of nonnegative scalar curvature can be deformed to be a complete metric of positive scalar curvature.

Proof. Assume that $\left(M, g_{0}\right)$ is a complete non-Ricci-flat manifold with non-negative $\left(\kappa_{g_{0}} \geqslant 0\right)$. Let $p$ be a point in $\left(M, g_{0}\right)$ satisfying that

$$
\begin{equation*}
\operatorname{Ric}_{g_{0}}(p) \neq 0 \tag{6.4.1}
\end{equation*}
$$

Choose the geodesic ball $B(p, r)$ centered at $p$ with radius $r$, where $r$ is less than the injective radius of $(M, g)$ at $p$. Since $\kappa_{g_{0}} \geqslant 0$, then $\mu_{1}\left(L_{g_{0}}, B(p, r)\right)$ is non-negative.

If $\mu_{1}\left(L_{g_{0}}, B(p, r)\right)>0$, we use Lemma 6 6.4.1 to find a positive function $u$ with the following properties:

- $L_{g_{0}}(u)>0$;
- there is a positive constant $C$ such that $u \geqslant C$.

Choose the new metric $g:=u^{\frac{4}{n-2}} g_{0}$. It is a complete metric over $M$. The scalar curvature is

$$
\kappa_{g}=L_{g_{0}}(u) u^{-\frac{n+2}{n-2}}>0 .
$$

The metric $g$ is the required candidate as in the statement.
If $\mu_{1}\left(L_{g_{0}}, B(p, r)\right)=0$, we have that the scalar curvature $\kappa_{g_{0}}=0$ on $B(p, r)$ and the eigenfunction $\phi$ for $\mu_{1}\left(L_{g_{0}}, B(p, r)\right)$ is a constant function. The reason is as follows:

We have that

$$
\left\{\begin{array}{c}
-\Delta_{g_{0}} \phi+\kappa_{g_{0}} \phi=0 \\
\partial \phi / \partial \nu=0
\end{array}\right.
$$

Dong integration by parts, we have that $\int_{B(p, r)}|\nabla \phi|^{2}+\kappa_{g_{0}} \phi^{2}=0$. Thus, $\phi$ is a constant function and $\kappa_{g_{0}}=0$ on $B(p, r)$.

From [Lemma 3.3, Page 232] in Kaz82, there exists a new metric $g_{t_{0}}$ so that
(1) $\mu_{1}\left(L_{g_{t_{0}}}, B(p, r)\right)>0$;
(2) $g_{t_{0}}$ is equal to $g_{0}$ outside $B(p, r)$.

As the above case, we could find a complete metric with positive scalar curvature.
The metric $g_{t_{0}}$ is constructed as follows:
Pick a a function $\eta \in C_{0}^{\infty}\left(B(p, r), \mathbb{R}^{\geqslant 0}\right)$ with $\eta(p)>0$ and consider a family of metrics

$$
g_{t}=g_{0}-t \cdot \eta \cdot \operatorname{Ric}_{g_{0}}
$$

Since $\kappa_{g_{0}}=0$ on $B(x, r)$, the first variation formal (See [Page 233] of Kaz82] or KW75]) gives that

$$
\begin{equation*}
\left.\frac{d}{d t} \mu_{1}\left(L_{g_{t}}, B(x, r)\right)\right|_{t=0}=\frac{1}{\operatorname{Vol}(B(p, r))} \int_{B(x, r)} \eta|R i c|^{2} \tag{6.4.2}
\end{equation*}
$$

Since $\eta(p)>0$, Equations (6.4.1) and (6.4.2 gives

$$
\left.\frac{d}{d t} \mu_{1}\left(L_{g_{t}}, B(p, r)\right)\right|_{t=0}>0
$$

Since $\mu_{1}\left(L_{g_{0}}, B(p, r)\right)=0$, we find that $\mu_{1}\left(L_{g_{t_{0}}}, B(x, r)\right)>0$ for some $t_{0}>0$.
The metric $g_{t_{0}}$ is the required metric.
As a consequence, we have that
Corollary 6.4.3. No contractible genus one 3-manifold admits a complete metric of nonnegative scalar curvature.

Proof. We argue by contradiction. Suppose that $\left(M^{3}, g\right)$ is a complete contractible genus one 3-manifold of nonnegative scalar curvature.

If $g$ is Ricci-flat, it is a flat metric. Thus, $M$ is homeomorphic to $\mathbb{R}^{3}$. This is in contradiction with Theorem 1.3 .13 ( $M$ is not homeomorphic to $\mathbb{R}^{3}$ ).

If not, $g$ can be deformed to be a complete metric of positive scalar curvature. This contradicts Theorem $B_{2}$.

As the above argument, we have that
Corollary 6.4.4. A complete contractible 3-manifold with non-negative scalar curvature and trivial $\pi_{1}^{\infty}$ is homeomorphic to $\mathbb{R}^{3}$.

### 6.5. Further questions

6.5.1. The General Case. In Wan19a, Wan19b], we verified Question 1 for 3manifolds with trivial $\pi_{1}^{\infty}$. The remaining case is contractible 3-manifolds with non-trivial $\pi_{1}^{\infty}$.

In this case, we require more techniques combining minimal surfaces theory and topological surgeries on 3 -manifolds.

The key point is to understand stable minimal surfaces (as constructed in Chapter 4.2) and its relationships with the fundamental group at infinity.

Based on this relationship, we attempt to devise a new topological surgery which can reduce Question 1 to Theorem $B_{2}$.
6.5.2. RCD metrics. By [Liu13], the Whitehead manifold does not admit any complete metric with positive Ricci curvature. In metric geometry, Riemannian manifolds with lower bounds on Ricci curvature correspond to the RCD spaces. It would be interesting to know whether the Whitehead manifold has a $\operatorname{RCD}(0,3)$ metric.

### 6.5.3. Spherical Decomposition.

Definition 6.5.1. An embedded 2-sphere $S$ in a 3 -manifold $M$ is called compressible if $S$ bounds a 3 -ball in $M$.

A spherical decomposition $\mathcal{S}$ of a 3 -manifold is a locally finite collection of (possibly non-separating) pairwise disjoint embedded 2 -sphere in $M$ such that the operation of cutting $M$ along $\mathcal{S}$ and gluing a ball to each boundary component of the resulting manifold yields a collection of irreducible manifolds.

A 3 manifold $M$ is irreducible if and only if all 2 -spheres in $M$ are compressible.
If $\mathcal{S}$ is a spherical decomposition, then the collection of sphere obtained by removing compressible spheres in $\mathcal{S}$ is still a spherical decomposition.

The prime decomposition theorem for 3-manifold (See Hat00 or Kne29) is equivalent to the statement that every compact 3-manifold has a spherical decomposition. This result does not generalize to open manifolds (See $\mathbf{S c o 7 7}$ and Mai07, Mai08]). The first example was given by Scott [Sco77].

A question posed by Maillot is the following:
Question: Does a complete 3-manifold of uniformly positive scalar curvature have a spherical decomposition?

For this question, a possible approach is to use the minimal surface theory to study the geometry of incompressible spheres.

This question is related to the open problem, introduced by Bessières, Besson and Maillot BBM11]:
Question :(See BBM11) Let $M$ be a connected, orientable 3-manifold which admits a complete metric of uniformly positive scalar curvature. Is M a connected sum of spherical manifolds and copies of $\mathbb{S}^{2} \times \mathbb{S}^{1}$ ?

Generally, we attempt to use the spherical decomposition to study the following question:
Question: (See Problem 27 in Yau82]) Classify 3-manifolds admitting complete Riemannian metrics of positive (resp. nonnegative) scalar curvature up to diffeomorphism.

## Part 4

## Closed aspherical 4-Manifolds

## CHAPTER 7

## Gromov-Lawson Conjecture

### 7.1. Aspherical 4-manifolds

Definition 7.1.1. A space $M$ is called aspherical if it is path-connected and all its higher homotopy groups vanish (i.e. $\pi_{k}(M)$ is trivial for $k \geqslant 2$ ).

For example, a CW complex $M$ is aspherical if and only if its universal cover is contractible.

Lemma 7.1.2. If $M^{n}$ is an aspherical manifold, then its fundamental group is torsionfree.

However, if $M^{n}$ is an aspherical manifold, the first homology group $H_{1}(M)$ may be not torsion-free. For example, the Klein bottle $K^{2}$ is aspherical (since its universal cover is $\left.\mathbb{R}^{2}\right)$. However, $H_{1}\left(K^{2}\right) \cong \mathbb{Z} \oplus \mathbb{Z} / 2 \mathbb{Z}$.

Conjecture 7.1.3. (Gromov-Lawson Conjecture) No closed aspherical manifold has a metric of positive scalar curvature.

For example, a $n$-dimensional torus $\mathbb{T}^{n}$ has no metric of positive scalar curvature for $n \geqslant 2$. It was proved by Gromov and Lawson GL83 and by Schoen and Yau [SY82, SY17].

It is well-known from [GL83] that this conjecture holds for the 3-dimensional case. In dimension four, Gromov and Lawson GL83 gave a partial solution requiring an additional hypothesis about incompressible surfaces.

We prove that
Theorem. $F$ No closed aspherical 4-manifold with nonzero first Betti number has a metric of positive scalar curvature.

Remark that there is some closed aspherical 4-manifold whose first homology group vanishes (See RT05).

### 7.2. The non-existence result

### 7.2.1. Topological preliminary.

Lemma 7.2.1. Let $M^{4}$ be a closed 4-manifold and $\Sigma$ a closed 3-manifold. The 2-sphere $S \subset \Sigma$ cuts $\Sigma$ into $\Sigma_{1}^{\prime}$ and $\Sigma_{2}^{\prime}$ and the map $f: \Sigma \rightarrow M$ is an immersed. If $f(S)$ bounds an immersed 3 -ball $B \subset M$, then $f\left(\Sigma_{i}\right) \cup_{f(S)} B$ is a singular 3 -cycle in $M$ for $i=1,2$. Furthermore, one has

$$
[f(\Sigma)]=\left[f\left(\Sigma_{1}\right) \cup_{f(S)} B\right]+\left[f\left(\Sigma_{2}\right) \cup_{f(S)} B\right] \in H_{3}(M, \mathbb{Z})
$$

Proof. We have that $f(\Sigma)$ is an singular 3-cycle. The sets, $f\left(\Sigma_{1}\right)$ and $f\left(\Sigma_{2}\right)$, are two elements in the group, $S_{3}(M, \mathbb{Z})$, of singular 3-chain. The boundary $\partial f\left(\Sigma_{1}\right)$ is equal to $f(S)$ in $S_{2}(M, \mathbb{Z})$. We have that $\partial f\left(\Sigma_{2}\right)=-f(S) \in S_{2}(M, \mathbb{Z})$.

In the group $S_{2}(M, \mathbb{Z})$, we have that

$$
\partial\left(f\left(\Sigma_{1}\right)-B\right)=\partial\left(\Sigma_{1}\right)-\partial B=f(S)-f(S)=0
$$

In addition, we have that $f\left(\Sigma_{1}\right) \cup_{f(S)} B$ is equal to $f\left(\Sigma_{1}\right)-B$ in $S_{3}(M, \mathbb{Z})$. Therefore, $f\left(\Sigma_{1}\right) \cup_{f(S)} B$ is a 3 -cycle of $X$.

Similarly, we have that $f\left(\Sigma_{2}\right) \cup_{f(S)} B$ is equal to $f\left(\Sigma_{2}\right)+B$ in $S_{3}(M, \mathbb{Z})$. Therefore, $f\left(\Sigma_{2}\right) \cup_{f(S)} B$ is a 3 -cycle of $X$.

In $S_{3}(M, \mathbb{Z})$, we have that

$$
\begin{aligned}
f(\Sigma) & =f\left(\Sigma_{1}\right)+f\left(\Sigma_{2}\right) \\
& =f\left(\Sigma_{1}\right)-B+f\left(\Sigma_{2}\right)+B \\
& =f\left(\Sigma_{1}\right) \cup_{f(S)} B+f\left(\Sigma_{2}\right) \cup_{f(S)} B
\end{aligned}
$$

We can conclude that $[f(\Sigma)]=\left[f\left(\Sigma_{1}\right) \cup_{f(S)} B\right]+\left[f\left(\Sigma_{2}\right) \cup_{f(S)} B\right] \in H_{3}(M, \mathbb{Z})$.
Lemma 7.2.2. Let $M^{4}$ be a closed 4-manifold and $\Sigma a \mathbb{S}^{2} \times \mathbb{S}^{1}$. The set $\Sigma \backslash S$ is homeomorphic to $\mathbb{S}^{2} \times(0,1)$, where $S$ is a 2 -sphere in $\Sigma$. The map $f: \Sigma \rightarrow M$ is an immersion. If $f(S)$ bounds an immersed 3-ball, then there is a map $g: \mathbb{S}^{3} \rightarrow M$ so that

$$
[f(\Sigma)]=\left[g\left(\mathbb{S}^{3}\right)\right] \in H_{3}\left(M^{4}, \mathbb{Z}\right)
$$

Proof. The set $\Sigma \backslash S$ is an $\mathbb{S}^{2} \times(0,1)$ with boundary $S^{+} \amalg S^{-}$. In $S_{2}(M, \mathbb{Z}), f\left(S^{+}\right)=$ $f(S)$ and $f\left(S^{-}\right)=-f(S)$.

Let $h(x): \mathbb{B}^{3} \rightarrow B^{\prime}$ be the immersion with $f\left(\partial \mathbb{B}^{3}\right)=f(S)$. Consider the 2 -sphere $\Sigma \backslash S \cup_{S^{+}} \mathbb{B}_{-}^{3} \cup_{S^{-}} \mathbb{B}_{+}^{3}$, where $\mathbb{B}_{+}^{3}$ and $\mathbb{B}_{-}^{3}$ are two 3-balls. We define the map $g$ from the sphere to $M$ as follows:

$$
h(x)=\left\{\begin{array}{l}
f(x), \text { where } x \in \Sigma \backslash S \\
h(x), \text { where } x \in \mathbb{B}_{+}^{3} \\
h(x), \text { where } x \in \mathbb{B}_{-}^{3}
\end{array}\right.
$$

In the group $S_{3}(M, \mathbb{Z})$, we have that $h\left(\mathbb{S}^{3}\right)=f(\Sigma)-B+B=f(\Sigma)$. We have that $[f(\Sigma)]=\left[g\left(\mathbb{S}^{3}\right)\right] \in H_{3}\left(M^{4}, \mathbb{Z}\right)$.
7.2.2. Proof of Theorem $F$. We begin by a compact manifold $\left(M^{4}, g\right)$ with $b_{1}(M)>$ 0 . There is an embedded circle $\gamma \subset M$ with the property that $[\gamma]$ is a torsion-free element in $H_{1}(M)$. We use the Poincaré duality to find a class $u \in H_{3}\left(M^{4}\right) \cong H^{1}\left(M^{4}\right)$ satisfying that $([\gamma], u)=1$.

We apply a theorem of Fleming-Federer (See $\mid \overline{\mathbf{F F 6 0}} \|$ ) and the regularity theory for area-minimizing currents (See Chapter 7 of [Sim83]). This result asserts that, in a Riemannian manifold $\left(X^{n}, g\right)$, for a non-trivial class in $H_{n-1}(X, \mathbb{Z})$, there is a volumeminimizing hypersurface in the class satisfying that it is smooth outside a set of Hausdorff dimension $\leqslant n-8$. Therefore, there is a stable minimal hypersurface $\Sigma^{3} \subset\left(M^{4}, g\right)$ in the class $u$. In addition, the intersection number $(\gamma, \Sigma)$ of $\Sigma$ and $\gamma$ is +1 .

If $\left(M^{4}, g\right)$ has positive scalar curvature, then $\Sigma$ admits a metric of positive scalar curvature (See Proposition 3.3.5). In this case, $\Sigma$ is a connected sum of some spherical manifolds and some copies of $\mathbb{S}^{1} \times \mathbb{S}^{2}\left(\right.$ See $\left[\right.$ Per02a $\mid$ Per02b, Per03], $[\mathbf{M T 0 7}],\left[\mathbf{B B B}^{+} \mathbf{1 0} \mid\right)$.

In the following, we prove Theorem $F$.
Proof. We argue by contradiction. Suppose a closed aspherical manifold $M^{4}$ with $b_{1}\left(M^{4}\right)>0$ has a metric $g$ of positive scalar curvature.

Let $\gamma$ and $\Sigma$ be constructed as above. The intersection number $(\Sigma, \gamma)$ of $\Sigma$ and $\gamma$ is not zero. In addition, $\Sigma$ has a metric of positive scalar curvature. By Per02a, Per02b, Per03, we see that

$$
\Sigma \cong \mathbb{S}^{3} / \Gamma_{1} \# \ldots \# \mathbb{S}^{3} / \Gamma_{j} \#\left(\mathbb{S}^{1} \times \mathbb{S}^{2}\right) \# \ldots \#\left(\mathbb{S}^{1} \times \mathbb{S}^{2}\right)
$$

where each $\Gamma_{i}$ is a finite subgroup of $O(4)$ for $1 \leqslant i \leqslant j$.
There exists a family of disjointly embedded splitting 2 -spheres $\left\{S_{i}\right\}_{i=1}^{n-1}$ in $\Sigma$. They cut $M$ into $n$-components, denoted by $\left\{X_{i}\right\}_{i=1}^{n}$. That is,

$$
\Sigma \cong X_{1} \cup_{S_{1}} X_{2} \cup_{S_{2}} \cdots \cup_{S_{n-1}} X_{n}
$$

Set

$$
X_{k}^{\prime}=\left\{\begin{array}{cc}
X_{1} \cup_{S_{1}} \mathbb{B}^{3} & k=1 \\
X_{n} \cup_{S_{n-1}} \mathbb{B}^{3} & k=n \\
X_{k} \cup_{S_{k-1}} \mathbb{B}^{3} \cup_{S_{k}} \mathbb{B}^{3} & \text { otherwise },
\end{array}\right.
$$

where $\mathbb{B}^{3}$ is a unit ball in $\mathbb{R}^{3}$. Therefore, $\Sigma \cong X_{1}^{\prime} \# \ldots \# X_{n}^{\prime}$. Each $X_{k}^{\prime}$ is a spherical manifold or $\mathbb{S}^{1} \times \mathbb{S}^{2}$. (Note that $X_{k}^{\prime}$ is not a subset of $M^{4}$.)

Since $\pi_{2}(M)$ is trivial, each 2-sphere $S_{k}$ bounds an immersed 3-ball $B_{k} \subset M^{4}$. Let us consider a submanifold

$$
X_{k}^{\prime \prime}=\left\{\begin{array}{cc}
X_{1} \cup_{S_{1}} B_{1} & k=1 \\
X_{n} \cup_{S_{n-1}} B_{n-1} & k=n \\
X_{k} \cup_{S_{k-1}} B_{k-1} \cup_{S_{k}} B_{k} & \text { otherwise }
\end{array}\right.
$$

Each $X_{k}^{\prime \prime} \subset M^{4}$ can be viewed as the image of some map $f_{k}$ from $X_{k}^{\prime}$ to $M^{4}$. By Lemma 7.2.1, we know that $[\Sigma]=\sum_{k=1}^{n}\left[X_{k}^{\prime \prime}\right]$ in $H_{3}\left(M^{4}, \mathbb{Z}\right)$.

Since the intersection number $(\gamma, \Sigma)$ is not equal to zero, there is some $k$ such that the intersect number ( $\gamma, X_{k}^{\prime \prime}$ ) is not zero. As mentioned above, $X_{k}^{\prime}$ is a spherical 3-manifold or $\mathbb{S}^{1} \times \mathbb{S}^{2}$.

Case I: If $X_{k}^{\prime}$ is a spherical manifold (that is, it is $\mathbb{S}^{3} / \Gamma_{k}$ ), $X_{k}^{\prime \prime}$ can be considered as the image of the map from $\mathbb{S}^{3}$ to $M$. However, since $\pi_{3}\left(M^{4}\right)$ is trivial, we see that $X_{k}^{\prime \prime}$ is contractible in $M^{4}$. Therefore, the intersection number $\left(\gamma, X_{k}^{\prime \prime}\right)$ is zero. This is in contradiction with the last paragraph.

Case II: If $X_{k}^{\prime}$ is a $\mathbb{S}^{1} \times \mathbb{S}^{2}$, we can find a 2 -sphere $S \subset X_{k}$ so that $X_{k}^{\prime} \backslash f_{k}^{-1}(S)$ is a $\mathbb{S}^{2} \times(0,1)$. Because $\pi_{2}(M)=\{1\}, S$ bounds an immersed 3-ball. We use Lemma 7.2.2 to find the map $h: \mathbb{S}^{3} \rightarrow M$ so that $\left[h\left(\mathbb{S}^{3}\right)\right]=\left[X_{k}^{\prime \prime}\right] \in H_{3}(M, \mathbb{Z})$. Therefore, the intersection
number of $h$ and $\gamma$ is not equal to zero. However, $h$ is homotopic to a constant map, (since $\pi_{3}(M)=\{1\}$ ). The intersection number is equal to zero, a contradiction.

This finishes the proof of Theorem $F$.

### 7.3. Further questions

7.3.1. Stable minimal hypersurfaces in $\mathbb{R}^{4}$. When using the minimal hypersurfaces to study 4-manifolds, understanding the geometry of minimal hypersurfaces is necessary and crucial. However, the geometry of stable minimal hypersurfaces in a 4-manifold is not known. R. Schoen has conjectured:

Conjecture. (See [Conjecture 2.12, Page 79] of CM11]):If $\Sigma^{3} \subset \mathbb{R}^{4}$ is a complete immersed stable minimal hypersurface with trivial bundle, then $\Sigma$ is flat.
7.3.2. Closed Aspherical 4-manifolds. Generally, Gromov-Lawson conjecture is still unknown, particularly for 4-manifolds with zero first Betti number.

The proof of Theorem $F$ involves stable minimal hypersurfaces and the geometrization conjecture. In the general case, the main issue is the existence of stable minimal hypersurfaces. In the proof of Theorem $F$, its existence is ensured by our hypothesis that the first Betti number is non-zero.

In order to overcome it, we attempt to find a covering space whose first betti number is nonzero. We next use topological conditions to construct a complete stable minimal surface. Then one is led to study stable minimal surface in a 4-manifold with uniformly positive scalar curvature.

Combining with the argument in [GL83 and some metric inequalities in $\overline{\mathbf{G r o 1 8}]}$, we plan to argue by contradiction. We expect to show that such a minimal surface does not exist, which would lead to a contradiction.
7.3.3. Exotic $\mathbb{R}^{4}$. An exotic $\mathbb{R}^{4}$ is a differential manifold that is homeomorphic but not diffeomorphic to the Euclidean space $\mathbb{R}^{4}$. The first example were found by Freedman (See $[\mathbf{F Q 1 4}]$ ). Actually, there are infinitely many non-diffeomorphic differential structures of $\mathbb{R}^{4}$, as was shown first by Taubes [Tau87].

An interesting question is whether an exotic $\mathbb{R}^{4}$ admits a complete metric of positive scalar (or Ricci) curvature.

For this question, we might follow the following scheme. First, based on the construction of an exotic $\mathbb{R}^{4}$, we attempt to construct a complete stable minimal hypersurface. Then, one is led to understand the geometry of such a hypersurface and its relationship with the differential structures and the positivity of scalar (or Ricci) curvature.

We plan to argue by contradiction. Combining the geometry of the stable minimal hypersurface, we expect to show that such a hypersurface does not exist, which would lead to a contradiction.

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