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Extensions of sampling theory:
sampling on spaces of homogeneous type
and sampling along curves

Sous la direction de Philippe JAMING

Soutenue le 11 Juin 2020

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Resumé

Dans cette thèse, nous étudions différentes variations des inégalités d'échantillonnage. Tout d'abord, en reflétant un résultat dans [56], nous donnons des conditions pour l'échantillonnage des fonctions de Besov définies sur des variétés Riemanniennes compactes et des espaces de type homogène. Les techniques utilisées pour prouver ces résultats sont basées sur la décomposition des fonctions lisses en ondelettes disponibles dans ces deux contextes. De plus, comme dans le cas de l'eulidien, cette caractérisation par une expansion en ondelettes permet d'approfondir l'étude des espaces de Besov, obtenant ainsi un théorème de trace et des résultats sur leur régularité locale (inspirés des stratégies développées dans [21, 54]). Enfin, nous passons à travailler dans le cadre classique de la théorie de l'échantillonnage, mais en changeant la façon dont les échantillons sont pris: au lieu de prendre un ensemble de points discrets, nous considérons un certain type de courbes. En particulier, nous déterminons la fréquence de Nyquist pour les spirales lorsque nous échantillons des fonctions à bande limitée. Nous montrons ensuite qu'en dessous de cette fréquence, la quantité de sous-échantillonnage que les signaux compressibles admettent lorsqu'ils sont échantillonnés en spirale est limitée.

Mots clés: Théorie d'Échantillonnage, espaces de Besov, espaces de type homogène.

Abstract

In this thesis we study different variations of sampling inequalities. First, mirroring a result in [56], we give the conditions for sampling-like inequalities for Besov functions on compact Riemannian manifolds and spaces of homogeneous type. The techniques used to prove these results are based on the decomposition of smooth functions into wavelets available in both of these settings. Further, as in the euclidean case, this characterization through a wavelet expansion allows us to deepen the study of Besov spaces, obtaining a trace theorem and results about their local regularity (inspired in the strategies developed in [21, 54]). Finally we shift to work within the classic setting of sampling theory but changing the way samples are taken: instead of taking a discrete set of points we consider certain type of curves. In particular we determine the Nyquist rate for spirals when sampling bandlimited functions. We then show that, below this rate, the amount of undersampling that compressible signals admit when sampled along spirals is limited.

Key words: Sampling Theory, Besov spaces, spaces of homogeneous type.

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Chapter 1

Introduction

1.1 Version française

Le concept unificateur des différents problèmes traités dans cette thèse est l'étude des inégalités d'échantillonnage sur une variété de cadres. De manière générale, un problème d'échantillonnage consiste à reconstruire une fonction à partir de ses valeurs sur un ensemble de points suffisamment bien répartis. Ce problème a été bien étudié dans le cadre euclidien, où le résultat le plus connu est probablement le théorème de Shannon-Whittaker-Kotelnikov qui affirme qu'une fonction à bande limitée sur \mathbb{R} peut être reconstruite à partir de ses échantillons réguliers: si $f \in L^2(\mathbb{R})$ est telle que $\text{supp}(\hat{f}) \subset [-W, W]$ pour un certain $W > 0$, alors

$$f(t) = \sum_{n \in \mathbb{Z}} f(n/2W) \frac{\sin \pi(2Wt - n)}{\pi(2Wt - n)}. \quad (1.1)$$

De plus, on a l'identité

$$\|f\|_2^2 = \sum_{n \in \mathbb{Z}} |f(n/2W)|^2. \quad (1.2)$$

Ce résultat peut naturellement être étendu à des dimensions plus élevées, à différents espaces de fonctions sur \mathbb{R}^d , à des échantillons non uniformes et à d'autres variations (voir par exemple [87] et les références qui y figurent). En outre, la recherche d'une théorie équivalente sur des cadres autres que le cas euclidien est devenue un domaine de recherche très actif au cours de la dernière décennie. Parmi les cas traités ces dernières années, se trouvent le cas des variétés Riemanniennes [76], des groupes localement compacts [27], des espaces de Dirichlet [20] (pour une liste plus complète, voir [26] et les références qui y figurent). Dans cette thèse, nous travaillerons d'abord avec des variétés Riemanniennes compactes et des espaces de type homogène (espaces quasi-métriques avec une mesure borélienne doublante).

Le problème auquel on est confronté lorsqu'on traite des cas autre que le euclidien est que les techniques de l'analyse de Fourier - sur lesquelles la théorie classique de l'échantillonnage repose fortement - peuvent ne pas être disponibles. Une façon de contourner ce problème est de développer une généralisation de l'analyse de Littlewood-Paley et des formules de reproduction de Calderón, et à partir de là, de construire un système d'ondelettes. En effet, ces outils se sont avérés être un puissant substitut à la transformée de Fourier lorsqu'on essaie de décrire le comportement global et local des fonctions. Expliquons brièvement comment cela se fait dans le cas de référence de \mathbb{R}^d . Nous considérons la famille des opérateurs de convolution $S_j(f) = f * \phi_j$ où $\phi_j(x) := 2^{jd}\phi(2^jx)$ et ϕ est une fonction de la classe Schwartz avec $\int \phi(x) dx = 1$. Ensuite, pour toute $f \in L^2(\mathbb{R}^d)$, l'inégalité de Minskowski donne

$$S_j f \rightarrow f, \quad j \rightarrow +\infty.$$

Puis, en posant $\varphi_0(x) := \phi(x)$ et $\varphi_j(x) := 2^{jd}\phi(2^jx) - 2^{(j-1)d}\phi(2^{(j-1)}x)$ pour $j \geq 1$, les opérateurs $D_j(f) = f * \varphi_j$ forment une approximation de l'unité:

$$I = \sum_{j \in \mathbb{N}} D_j.$$

Si on demande en plus que $\sum_{j \in \mathbb{N}} |\widehat{\varphi_j}(\zeta)|^2 = 1$ pour $\zeta \neq 0$ alors il est possible de montrer que

$$\|f\|_2^2 = \sum_{j \in \mathbb{N}} \|D_j f\|_2^2, \quad \forall f \in L^2(\mathbb{R}^d). \quad (1.3)$$

Ensuite, la partition de \mathbb{R}^d en cubes dyadiques $(2^{-j}k, 2^{-j}(k+1)]^d$ permet de discrétiser chaque opérateur D_j et d'obtenir ce que l'on appelle un système d'ondelettes. C'est-à-dire une expansion dyadique en fonctions localisées et lisses. Dans ce cas, si nous définissons $\varphi_{j,k}(x) := 2^{-jd/2}\varphi_j(x - 2^{-j}k)$, et que la fonction ϕ a quelques propriétés supplémentaires souhaitables, alors (1.3) entraîne

$$f = \sum_{j \in \mathbb{N}} \sum_{k \in \mathbb{Z}} \langle f, \varphi_{j,k} \rangle \varphi_{j,k}, \quad \forall f \in L^2(\mathbb{R}^d). \quad (1.4)$$

Une caractéristique clé des variétés Riemanniennes et des espaces de type homogène est que, effectivement, des partitions similaires à (1.3) et la discrétisation dyadique par ondelettes qui s'ensuit (1.4) peuvent être obtenues. En ce qui concerne l'expansion des ondelettes, il est toutefois important de noter que les constructions que nous allons utiliser ne constituent pas nécessairement une base orthonormée mais plutôt une frame: au lieu de (1.4), on obtient un système $\{\varphi_{j,k}\}_{j,k}$ tel que

$$A\|f\|_2^2 \leq \sum_{j,k} |\langle f, \varphi_{j,k} \rangle|^2 \leq B\|f\|_2^2, \quad \forall f \in L^2,$$

où A, B sont des constantes fixes positives. Ensuite, la théorie des frames nous assure de l'existence d'une famille duale $\{\tilde{\varphi}_{j,k}\}_{j,k}$ telle que

$$f = \sum_{j,k} \langle f, \tilde{\varphi}_{j,k} \rangle \varphi_{j,k} = \sum_{j,k} \langle f, \varphi_{j,k} \rangle \tilde{\varphi}_{j,k}, \quad \forall f \in L^2.$$

Les reconstructions directement à partir de ce type de représentations pourraient cependant être difficiles à réaliser dans la pratique, car le calcul des coefficients $\langle f, \varphi_{j,k} \rangle$ (ou $\langle f, \tilde{\varphi}_{j,k} \rangle$) est généralement une opération coûteuse. On espère plutôt récupérer f à partir de ses valeurs dans un ensemble donné de points, ce qui est précisément le but de la théorie de l'échantillonnage. L'idée générale est que si une fonction a une oscillation modérée, alors elle peut être bien approximée dans le voisinage d'un point par sa valeur en ce point. Si ces voisinages couvrent suffisamment bien l'espace, l'échantillon devrait permettre de décrire la fonction globalement. Vu de cette façon, un théorème d'échantillonnage est, en gros, caractérisé par deux paramètres : la distribution (ou la fréquence) des échantillons et l'oscillation de la fonction.

Nous allons voir que, cette distribution d'échantillonnage est en quelque sorte conditionnée par la géométrie de l'espace ambiant, alors qu'on dispose de plusieurs moyens de mesurer l'oscillation. L'hypothèse standard serait, comme dans le théorème de Shannon-Whittaker-Kotelnikov, de travailler avec des fonctions à bande limitée. Mais, comme nous l'avons laissé entendre précédemment, il n'est pas toujours évident de savoir si et comment ce concept peut être reproduit sans la pleine puissance de l'analyse de Fourier. Nous cherchons plutôt à travailler avec les espaces qui peuvent être décrits à l'aide d'ondelettes.

Inspirés par un résultat de Strichartz [81] pour les espaces Sobolev, Jaming et Malinnikova [56] ont donné une inégalité d'échantillonnage en \mathbb{R}^d pour les fonctions dans les espaces Besov. Leur point de départ est le suivant:

Théorème ([56]). *Soit $1 \leq p \leq \infty$. Alors il existe des constantes $c_1, c_2, c_3 > 0$ telles que pour tout $r > 0$ et toute suite $\{a_n\}_{n \in \mathbb{Z}}$ avec $r/2 \leq a_{n+1} - a_n \leq r$*

$$c_1 r^{-1/p} \|f\|_p \leq \left(\sum_{n \in \mathbb{Z}} |f(a_n)|^p \right)^{1/p} \leq c_2 r^{-1/p} \|f\|_p$$

tient pour toute $f \in B_{p,1}^{1/p}(\mathbb{R})$ avec $\|f\|_{B_{p,1}^{1/p}} \leq c_3 r^{-1/p} \|f\|_{L^p}$.

Écrit de cette façon, ce résultat ressemble plus à (1.2) qu'à (1.1). Cependant, et toujours grâce à la théorie des frames, il est possible d'obtenir une reconstruction comme (1.1) à partir de cette inégalité lorsque nous nous limitons à travailler avec les espaces Paley-Wiener (qui sont toujours injectés dans n'importe quel espace

de Besov). Dans ce sens, on peut considérer ce type de résultats comme une généralisation aux théorèmes d'échantillonnage classiques.

La preuve de ce théorème utilise extensivement la décomposition en ondelettes en \mathbb{R} et est basée sur la caractérisation des espaces de Besov en termes de coefficients d'ondelettes comme l'a fait Meyer [68]. En effet, en prenant un système d'ondelettes en \mathbb{R} comme avant, les propriétés de la classe de Schwartz permettent alors de caractériser plusieurs espaces de fonctions couramment utilisés tels que les espaces de Sobolev, Hölder, Hardy, BMO et -d'un intérêt particulier pour nous- les espaces de Besov. Plus précisément, il peut être démontré que la norme d'une fonction $f \in B_{p,q}^s(\mathbb{R})$ est équivalente à

$$\left[\sum_{j \in \mathbb{N}} 2^{jq(s+1/2-1/p)} \left(\sum_{k \in \mathbb{N}} |\langle f, \varphi_{j,k} \rangle|^p \right)^{q/p} \right]^{1/q},$$

où $s \in \mathbb{R}$, $1 \leq p, q \leq \infty$ et $\varphi_{j,k}$ est un système comme dans (1.4) avec assez de régularité (au moins s moments nuls).

D'un point de vue abstrait, puisque cette caractérisation ne concerne qu'un système d'ondelettes, on peut l'utiliser pour définir des espaces de Besov sur des variétés Riemanniennes compactes et des espaces de type homogène. En outre, comme dans le cas réel, on peut montrer que, si les ondelettes ont des propriétés de régularité et de localisation suffisamment bonnes (en substitution de la classe de Schwartz), cette définition est également équivalente aux autres extensions naturelles des espaces de Besov dans ces cadres [20, 21, 36, 38, 37, 47, 69, 70, 84]. Il est toutefois important de noter que pour les types plus généraux d'espaces de type homogène, l'indice de régularité s peut être limité par le fait que dans certains cas, il n'existe pas de fonctions non triviales dont la régularité dépasse un certain niveau [39].

Ainsi, structurellement, nous disposons des mêmes outils que ceux utilisés par Jaming et Malinnikova. Cela nous permet d'adapter leur résultat à nos cadres. Pour les variétés de Riemann compactes, l'inégalité d'échantillonnage se lit comme suit:

Théorème. *Soit \mathbf{M} une variétés de Riemann compacte de dimension \mathbf{d} , et soit $1 \leq p \leq \infty$. Alors, il existe $c_1, c_2, c_3 > 0$ telles que pour tout $r > 0$ et tout ensemble $\{\zeta_1, \dots, \zeta_{N_r}\}$ de points r -distribué sur \mathbf{M}*

$$c_1 r^{-\mathbf{d}/p} \|f\|_p \leq \left(\sum_{n=1}^{N_r} |f(\zeta_n)|^p \right)^{1/p} \leq c_2 r^{-\mathbf{d}/p} \|f\|_p$$

tient pour toute $f \in B_{p,1}^{\mathbf{d}/p}(\mathbf{M})$ avec $\|f\|_{B_{p,1}^{\mathbf{d}/p}} \leq c_3 \|f\|_{L^p}$.

Ici, r -distribué fait référence à une condition très similaire à celle de [56] dans \mathbb{R} : un recouvrement de boules à rayon r qui deviennent disjointes si on réduit légèrement r (disons, à $r/2$).

La principale raison pour laquelle nous pouvons adapter le résultat de Jaming et Malinnikova aux variétés est l'existence d'une décomposition en ondelettes. L'autre facteur sous-jacent, est que la preuve dans [56] n'exige aucune propriété particulière de $\mathbb{R}^{\mathbf{d}}$ autre que le fait que la mesure de Lebesgue d'une boule de rayon r est proportionnelle à $r^{\mathbf{d}}$. Ce qui permet d'aller plus loin et d'étendre le résultat à des espaces de type homogène. En effet, nous savons grâce aux travaux de Macías et Segovia [64] que, en changeant éventuellement la pseudo métrique pour une métrique équivalente, la mesure dans un espace de type homogène peut être supposée avoir la même propriété. En fait, l'exposant correspondant \mathbf{d} s'appelle la dimension homogène. On obtient ainsi:

Théorème. *Soit X un espace de type homogène de dimension \mathbf{d} , et soit $1 \leq p \leq \infty$. Alors, il existe des constantes $c_1, c_2, c_3 > 0$ telles que pour tous les $r > 0$ et toutes les familles $\{y_n\}_n$ de points r -distribué sur X*

$$c_1 r^{-\mathbf{d}/p} \|f\|_p \leq \left(\sum_{n=1}^{N_r} |f(\zeta_n)|^p \right)^{1/p} \leq c_2 r^{-\mathbf{d}/p} \|f\|_p$$

tient pour toute $f \in B_{p,1}^{\mathbf{d}/p}(X)$ avec $\|f\|_{B_{p,1}^{\mathbf{d}/p}} \leq c_3 r^{-\mathbf{d}/p} \|f\|_{L^p}$.

Les résultats exacts sont un peu plus précis, voir Théorèmes 2.4 et 3.6 respectivement.

Le côté droit de ces inégalités d'échantillonnage peut également être interprété comme des théorèmes de trace pour les fonctions de Besov (sur un ensemble discret de points). Plus largement, la régularité des fonctions lorsqu'elles sont limitées à un sous-ensemble particulier (discret ou non) est un problème qui a ses propres intérêts, notamment en ce qui concerne les EDP et, plus précisément, les problèmes de valeurs au bord [66]. C'est pourquoi plusieurs théorèmes de trace ont été donnés pour les fonctions de Besov (et de Sobolev) définies dans des espaces métriques sous différents types d'hypothèses [38, 65, 67, 77]. Ici, en utilisant des techniques similaires à celles de la preuve du Théorème 3.6, nous proposons une nouvelle preuve d'un résultat de trace pour les fonctions de Besov sous des hypothèses relativement générales. Plus précisément, nous prouvons ce qui suit:

Théorème. *Soit X un espace de type homogène et $Y \subset X$ un sous-espace. Soient $\mathbf{d}_X, \mathbf{d}_Y$ la dimension de X et Y respectivement.*

Soit $1 \leq p < \infty, 1 \leq q \leq \infty, s \in \mathbb{R}$ et fixons $s' := s - \frac{\mathbf{d}_X - \mathbf{d}_Y}{p}$. Si $0 < s' < \theta$ alors l'opérateur de trace

$$\text{Tr} : B_{p,q}^s(X) \rightarrow B_{p,q}^{s'}(Y), \quad \text{Tr} f = f|_Y$$

est bien définie et continue.

Par *sous-espace* nous voulons dire que Y est un espace de type homogène avec la métrique résultant de la restriction de celui-ci sur X . C'est l'hypothèse également considérée dans [67]. La contribution que nous apportons à la littérature existante est que, ici, nous considérons une classe d'indices s plus large. Une autre nouveauté est la preuve elle-même, qui repose essentiellement sur le système des ondelettes, suivant de près les stratégies développées par exemple dans [21]. En bref, l'idée est d'écrire $f \in B_{p,q}^s(X)$ dans son développement en ondelettes afin de pouvoir exprimer les coefficients d'ondelettes de $f|_Y$ en termes de ceux de X ; les bonnes propriétés de localisation et de régularité des ondelettes permettent alors d'obtenir un résultat de comparaison, souvent appelé une *propriété de quasi-orthogonalité*, dont découle le théorème de trace.

De plus, ce n'est pas seulement la régularité des fonctions (Besov ou autre) qui peut être étudiée avec des ondelettes, mais aussi leurs variations locales. Dans \mathbb{R}^d , en prenant les ondelettes comme décrit précédemment, il a été montré pour la première fois par Meyer [68] qu'une fonction f appartient à l'espace de Hölder d'ordre s si et seulement si

$$|\langle f, \varphi_{j,k} \rangle| \leq C_f 2^{-j(d/2+s)} \quad \forall j, k \quad (1.5)$$

où $C_f > 0$ est une constante dépendant uniquement de f . De même pour la régularité locale, Jaffard [53] a donné une condition presque équivalente: si f est Hölder d'ordre s en x , alors

$$|\langle f, \varphi_{j,k} \rangle| \leq C_f 2^{-j d/2} (2^{-j} + |x - 2^{-j}k|)^s \quad \forall j, k, \quad (1.6)$$

et inversement si on a (1.13) pour tous les j, k et que f a au moins une certaine régularité globale de Hölder alors f est un Hölder d'ordre s' en x pour tout $0 < s' < s$.

Pour les espaces de type homogène, puisque nous avons toujours une (pseudo)métrique, nous pouvons définir la continuité de Hölder (locale et globale) de la même manière que nous le faisons dans \mathbb{R}^d . Et comme les preuves euclidiennes pour (1.5) et (1.6) reposent essentiellement sur des arguments métriques (en utilisant les propriétés de Schwartz des ondelettes), les mêmes résultats peuvent être reproduits dans des espaces de type homogène (à condition, là encore, que les ondelettes soient soigneusement construites). En effet, l'équivalent de (1.5) a déjà été prouvé par exemple dans [21], et nous montrons ici que (1.6) peut également être reproduit.

En particulier, la caractérisation des propriétés de régularité locale, au moins d'un point de vue abstrait, ouvre la porte au développement et à l'étude de l'analyse multifractale équivalente à celle de \mathbb{R}^d comme fait par Jaffard [54, 55].

C'est-à-dire, une étude sur la façon dont les ordres locaux de régularité d'une fonction varient tout au long de notre espace ambiant. Plus précisément, on définit l'exposant de Hölder d'une fonction f sur un point x comme

$$h_f(x) := \sup\{s : f \in \mathcal{C}^s(x)\},$$

et on essaie ensuite de comprendre la distribution géométrique des points ayant le même exposant à l'aide de la dimension de Hausdorff de ces ensembles: c'est ce qu'on appelle le *spectre des singularités*

$$d_f(s) := \dim_{\mathcal{H}}\{x : h_f(x) = s\}.$$

L'un des arguments pour choisir cette quantité est dû au type de fonctions habituellement étudiées dans cette théorie. Notamment, les espaces de Besov se sont avérés être un modèle pour lequel plusieurs résultats peuvent être obtenus. Et la raison principale en est, encore une fois, leur caractérisation par des ondelettes. Ainsi, en effet, on pourrait espérer reproduire de tels résultats dans le cadre d'espaces de type homogène. En particulier, nous obtenons la limite supérieure suivante pour le spectre des singularités:

Théorème. *Let $1 \leq p < \infty, 1 \leq q \leq \infty$ et $\mathbf{d}/p < s < \mathbf{d}/p + \theta$. Puis pour tous les $f \in B_{p,q}^s(X)$ et tous les $s - \mathbf{d}/p \leq \alpha < \theta$*

$$d_f(\alpha) \leq \min(\mathbf{d}, \mathbf{d} + (\alpha - s)p).$$

Et $d_f(\alpha) = -\infty$ si $\alpha < s - \mathbf{d}/p$.

Ce résultat reflète celui de [54] dans $\mathbb{R}^{\mathbf{d}}$. En outre, dans le cas de euclidien, il a été montré que cette limite est atteinte pour “presque toute” fonction dans $B_{p,q}^s(\mathbb{R}^{\mathbf{d}})$ au sens de la prévalence [29] (un concept pour les espaces vectoriels de dimension infinie qui se substitue à la notion “presque partout” au sens de Lebesgue [49]). Les arguments utilisés pour montrer ce résultat reposent sur des estimations précises des dimensions de Hausdorff qui ne peuvent être obtenues que grâce à une description fine de la géométrie euclidienne [25]. Dans les espaces de type homogène, ces résultats pourraient être plus difficiles à adapter car on n'a a priori qu'une idée approximative de la forme et de la taille de l'espace. Cette question est laissée pour des travaux futurs.

Dans la dernière partie de cette thèse, nous revenons à l'étude des inégalités d'échantillonnage mais cette fois, au lieu de changer le domaine des fonctions du cadre euclidien au non-euclidien, nous gardons le cadre euclidien mais nous modifions la façon dont les échantillons sont distribués. En particulier, nous nous intéresserons aux inégalités d'échantillonnage dans \mathbb{R}^2 où les échantillons sont placés -pas nécessairement de manière uniforme- sur certains types de courbes.

Ceci est naturellement lié au problème de *l'échantillonnage continu*: au lieu de prendre un ensemble de points discrets, nous considérons une courbe Γ (ou une famille de courbes) et essayons d'obtenir des inégalités du type suivant:

$$A\|f\|_p \leq \left(\int_{\Gamma} |f(x)|^p d\mu_{\Gamma} \right)^{1/p} \leq B\|f\|_p, \quad (1.7)$$

pour toute f dans l'espace d'échantillonnage correspondant, $\mu_{\Gamma} := \mathcal{H}^1|_{\Gamma}$ indique la mesure unidimensionnelle de Hausdorff (longueur d'arc) et A, B sont des constantes positives. L'un des principaux résultats que nous montrons est que, sous des hypothèses de régularité modérée, le problème continu et le problème discret sont les mêmes: (1.7) est équivalent à l'existence d'un ensemble discret $\Lambda \subset \Gamma$ tel que

$$A\|f\|_p \leq \left(\sum_{\lambda \in \Lambda} |f(\lambda)|^p \right)^{1/p} \leq B\|f\|_p.$$

Le résultat exact est plus précis, voir le théorème D ci-dessous.

En fait, dans le même article, Jaming et Malinnikova [56] donnent également une contrepartie continue à leur inégalité d'échantillonnage discrete –avec des preuves très similaires. Ils montrent que si une courbe régulière Γ couvre suffisamment bien l'espace, alors (1.7) est valable pour les fonctions de Besov.

Un cas illustratif est celui des lignes parallèles équidistantes, disons avec direction $\vec{v} \in \mathbb{S}^1$ et séparation r :

$$\Gamma = \{t\vec{v} + rk\vec{v}_{\perp} : t \in \mathbb{R}, k \in \mathbb{Z}\},$$

où \vec{v}_{\perp} désigne le vecteur unité perpendiculaire à \vec{v} dans le sens antihoraire. Dans ce cas précis, le résultat de Jaming et Malinnikova implique qu'il existe une constante $C > 0$ telle que (1.7) est vérifié pour toute $f \in B_{p,1}^{1/p}(\mathbb{R}^2)$ avec $\|f\|_p \leq Cr^{-1/p}\|f\|_{B_{p,1}^{1/p}}$.

Cela peut être amélioré quand on travaille dans le cadre d'un théorème d'échantillonnage classique avec des espaces de Paley-Wiener. En effet, si Ω est un corps symétrique convexe et qu'on échantillonne des fonctions sur $PW(\Omega)$ alors (1.7) tient si $\mathcal{H}^1(\Omega \cap \{t\vec{v}_{\perp} : t \in \mathbb{R}\}) < r^{-1/2}$ mais échoue pour quelques $f \in PW(\Omega)$ si $\mathcal{H}^1(\Omega \cap \{t\vec{v}_{\perp} : t \in \mathbb{R}\}) > r^{-1/2}$ (voir [85, 86]). Ainsi, $r^{-1/2}$ fonctionne effectivement comme une valeur de transition, souvent appelée la *fréquence de Nyquist*.

Cette fréquence peut également être déterminée pour des familles arbitraires de lignes parallèles [42] –pas seulement les équidistantes. Et il est certain que les résultats de [56] s'appliquent à une plus grande famille de courbes. Cependant, déterminer la fréquence exacte de Nyquist d'une courbe arbitraire est une tâche beaucoup plus subtile et il est moins évident de savoir ce que l'on peut dire en toute généralité.

La contribution que nous apportons ici est de donner des conditions nécessaires et suffisantes pour que (1.7) tient sur une certaine famille paramétrique de courbes, que nous appelons spirale. Les principaux exemples de ces courbes sont la spirale d'Archimède

$$A^\eta := \{(\eta\theta \cos 2\pi\theta, \eta\theta \sin 2\pi\theta) : \theta \geq 0\}$$

et la collection de cercles concentriques

$$O^\eta := \{(x, y) : x^2 + y^2 = \eta^2 k^2, k \in \mathbb{N}\}.$$

Du point de vue des applications, la reconstruction à partir d'échantillons dans ce type de courbes est un problème qui apparaît, par exemple, dans l'imagerie par résonance magnétique (IRM), où des capteurs mobiles capturent l'anatomie et la physiologie d'un patient. Du point de vue théorique, la raison principale pour laquelle nous pouvons obtenir des résultats pour des courbes en spirale est qu'elles se comportent qualitativement comme des unions de lignes parallèles. Ainsi, comme pour les lignes parallèles, nous sommes en mesure de déterminer leur fréquence de Nyquist exact:

Théorème. *Soit $\Omega \subset \mathbb{R}^2$ un corps symétrique centré et convexe.*

- (i) *Si $\text{diam}(\Omega)\eta < 1$, alors la spirale d'Archimède A^η et la collection de cercles concentriques O^η sont des trajectoires d'échantillonnage pour $PW^2(\Omega)$.*
- (ii) *Si $\text{diam}(\Omega)\eta > 1$, alors ni la spirale d'Archimède A^η ni la collection de cercles concentriques O^η sont des trajectoires d'échantillonnage pour $PW^2(\Omega)$.*

La preuve fait largement appel à la théorie de l'échantillonnage de Beurling [8, 9]. Le point (i) découle d'un théorème de Beurling [8], tel qu'il est appliqué dans [7]. Le point (ii) s'appuie sur la caractérisation de l'échantillonnage de Beurling en termes de limites faibles [8, 9], en montrant qu'une suite adéquate de translations d'une spirale converge vers une union de lignes parallèles, localement dans la métrique de Hausdorff.

Ayant identifié la fréquence de Nyquist des courbes en spirale, nous nous attaquons au problème de *sous-échantillonnage*. C'est-à-dire, nous considérons des spirales légèrement moins denses, mais nous limitons l'espace échantillonnage à des fonctions qui sont représentées de manière compacte dans certains dictionnaires (motivé par le fait que, dans la pratique, beaucoup de signaux que l'on essaie de récupérer sont hautement compressibles et que l'on peut donc se permettre d'échantillonner en dessous de la fréquence de Nyquist). L'une de ces restrictions est une limite de variation:

$$\mathcal{F}(W) := \{f \in PW(\Omega) : \text{var}(\hat{f}) \leq W\}.$$

On peut ensuite estimer le nombre de coefficients d'ondelettes actifs qui représentent f [18, 23]. La stabilité de l'échantillonnage de ces signaux est exprimée par le conditionnement inverse

$$\inf \left\{ \|f - g\|_{L^2(\mu_\Gamma)} : f, g \in \mathcal{F}(W), \|\hat{f} - \hat{g}\|_2 = \varepsilon \right\},$$

qui mesure jusqu'à où la distance entre deux signaux est représentée par leurs échantillons. Puisque, $\varepsilon^{-1}\mathcal{F}(W) = \mathcal{F}(\varepsilon^{-1}W)$, et $\mathcal{F}(W) \subset \mathcal{F}(W) - \mathcal{F}(W) \subset \mathcal{F}(2W)$, l'analyse du conditionnement pour les petits ε se réduit à l'asymptotique en W grand de la *marge de stabilité*:

$$A(\Gamma, \mathcal{F}(W)) := \inf \left\{ \|f\|_{L^2(\mu_\Gamma)} : \|\hat{f}\|_2 = 1, f \in \mathcal{F}(W) \right\}.$$

Le résultat d'échantillonnage que nous obtenons pour A^η et O^η nous indique que la valeur critique dans le cas de référence du carré unitaire $\Omega = [-1/2, 1/2]^2$ est $\eta = \frac{\sqrt{2}}{2}$. Nous considérons alors les spirales dont la densité est légèrement inférieure à ce valeur critique et prouvons ce qui suit.

Théorème. *Soit $\eta = (1 + \varepsilon)\sqrt{2}/2$ avec $\varepsilon \in (0, 1)$, et $\Gamma = A^\eta$ ou $\Gamma = O^\eta$. Puis pour $W > 0$,*

$$A(\Gamma, \mathcal{F}(W)) \leq C(\varepsilon W)^{-1/2}(\ln^2(\varepsilon W) + 1),$$

où $C > 0$ est une constante universelle.

En effet, cela fixe une limite à la capacité des spirales à acquérir tous les signaux compressibles en dessous de la fréquence de Nyquist. Heuristiquement, cela veut dire qu'en cas de sous-échantillonnage par un petit facteur $(1 - \varepsilon)$, on ne peut récupérer les fonctions jusqu'à la résolution $W \approx \varepsilon^{-1}$ de façon stable.

Comme indiqué précédemment, il existe une relation entre la variation d'une fonction et le nombre de coefficients d'ondelettes large [18] (ce nombre est essentiellement contrôlé par la variation W). En exploitant cette relation, nous pouvons formuler le résultat précédent en termes de coefficients d'ondelettes (de Haar). Plus précisément, si nous désignons par $\Sigma_{N,J}$ la classe de fonctions sur $[-1/2, 1/2]^2$ avec N coefficients de Haar non nuls, tous pris avec une échelle au plus égale à J , nous obtenons alors l'estimation suivante:

Théorème. *Soit $W \geq 1$, $\eta = (1 + \varepsilon)\sqrt{2}/2$ avec $\varepsilon \in (0, 1)$ et $\Gamma = A^\eta$ ou $\Gamma = O^\eta$. Puis pour $N \geq 1$,*

$$A(\Gamma, \Sigma_{N,J}) \leq CN^{-1/6}\varepsilon^{-1}\ln^4(CN),$$

où $J = C \ln(\varepsilon^{-1}N)$ et $C > 0$ est une constante universelle.

Informellement, ce théorème dit que lorsqu'on sous-échantillonne d'un petit facteur $(1 - \varepsilon)$, on peut récupérer au plus $N \approx \varepsilon^{-6}$ coefficients de Haar de façon stable.

1.2 English version

The unifying concept of the different problems treated in this thesis is the study of sampling inequalities on a variety of settings. Broadly speaking, a sampling problem consist on reconstructing functions from its values on a well enough distributed set of points. This problem has been well studied in the euclidean case, where the most common result is the Shannon-Whittaker-Kotelnikov theorem which states that a bandlimited function on \mathbb{R} can be reconstructed from its regular samples: if $f \in L^2(\mathbb{R})$ is such that $\text{supp}(\hat{f}) \subset [-W, W]$ for some $W > 0$ then

$$f(t) = \sum_{n \in \mathbb{Z}} f(n/2W) \frac{\sin \pi(2Wt - n)}{\pi(2Wt - n)}. \quad (1.8)$$

Moreover we have the identity

$$\|f\|_2^2 = \sum_{n \in \mathbb{Z}} |f(n/2W)|^2. \quad (1.9)$$

This result can be naturally extended to higher dimensions, different function spaces on \mathbb{R}^d , non-uniform samples and further variations (see e.g. [87] and references therein). In addition, the search for an equivalent theory on settings other than the usual euclidean case has become a very active field of research over the past decade. Among the cases treated in recent years there are different types of Riemannian manifolds [76], locally compact groups [27], Dirichlet spaces [20] (for a more complete list see [26] and the references there in). In this thesis we will first work with compact Riemannian manifolds and spaces of homogeneous type (quasi-metric spaces together with a Borelian doubling measure).

The problem one faces when dealing with cases other than the euclidean one is that the techniques of Fourier analysis -on which classic Sampling Theory heavily relies- might not be available. A way to circumvent this problem is to develop a generalization of the Littlewood-Paley analysis and Calderón reproducing formulas, and from there to construct a wavelet system. Indeed, these tools have been proven to be a powerful substitute to the Fourier transform when trying to describe global and local behavior of functions. Let us briefly explain how this is done in the reference case of \mathbb{R}^d . We consider the family of convolution operators $S_j(f) = f * \phi_j$ where $\phi_j(x) := 2^{jd} \phi(2^j x)$ and ϕ is a function in the Schwartz class with $\int \phi(x) dx = 1$. Then, for any $f \in L^2(\mathbb{R}^d)$, Minskowski's inequality yields

$$S_j f \rightarrow f, \quad j \rightarrow +\infty.$$

Thus, setting $\varphi_0(x) := \phi(x)$ and $\varphi_j(x) := 2^{jd} \phi(2^j x) - 2^{(j-1)d} \phi(2^{(j-1)} x)$ for $j \geq 1$, the operators $D_j(f) = f * \varphi_j$ constitute an approximation of the unity:

$$I = \sum_{j \in \mathbb{N}} D_j.$$

If we further require $\sum_{j \in \mathbb{N}} |\widehat{\varphi}_j(\zeta)|^2 = 1$ for $\zeta \neq 0$ then it can be proved that

$$\|f\|_2^2 = \sum_{j \in \mathbb{N}} \|D_j f\|_2^2, \quad \forall f \in L^2(\mathbb{R}^d). \quad (1.10)$$

Next, the partition of \mathbb{R}^d into the dyadic cubes $(2^{-j}k, 2^{-j}(k+1)]^d$ allows to discretize each operator D_j and to obtain what is called a wavelet system. That is, a dyadic expansion into smooth localized functions. In this case, if we set $\varphi_{j,k}(x) := 2^{-jd/2} \varphi_j(x - 2^{-j}k)$, and the function ϕ has some additional desirable properties, (1.10) entails

$$f = \sum_{j \in \mathbb{N}} \sum_{k \in \mathbb{Z}} \langle f, \varphi_{j,k} \rangle \varphi_{j,k}, \quad \forall f \in L^2(\mathbb{R}^d). \quad (1.11)$$

A key feature for the cases of compact Riemannian manifolds and spaces of homogeneous type is that, indeed, partitions similar to (1.10) and the subsequent wavelet dyadic discretization (1.11) can be obtained. With respect to the wavelet expansion however, it is important to remark that the constructions we are going to use do not necessarily constitute an orthonormal basis but rather a frame: instead of (1.11), one gets a system $\{\varphi_{j,k}\}_{j,k}$ such that

$$A\|f\|_2^2 \leq \sum_{j,k} |\langle f, \varphi_{j,k} \rangle|^2 \leq B\|f\|_2^2, \quad \forall f \in L^2,$$

where A, B are fixed positive constants. Then, Frame Theory ensures us of the existence of dual family $\{\tilde{\varphi}_{j,k}\}_{j,k}$ such that

$$f = \sum_{j,k} \langle f, \tilde{\varphi}_{j,k} \rangle \varphi_{j,k} = \sum_{j,k} \langle f, \varphi_{j,k} \rangle \tilde{\varphi}_{j,k}, \quad \forall f \in L^2.$$

Reconstructions directly from this type of representations, however, might be hard to achieve in practice since computing the coefficients $\langle f, \varphi_{j,k} \rangle$ (or $\langle f, \tilde{\varphi}_{j,k} \rangle$) is usually a costly operation. Instead one hopes to recover f from its values in a given set of points, which is precisely the goal of sampling theory. The general idea is that if a function has moderate oscillation, then it can be well approximated in the neighborhood of a point by its value at that point. If those neighborhoods cover sufficiently well the space, the sample should allow to describe the function globally. Viewed this way, a sampling theorem is roughly characterized by two parameters: the distribution (or rate) of the samples and the oscillation of the function.

While the sampling rate is somehow conditioned by the geometry of the ambient space, oscillation can be measured by different means. The standard assumption

would be, as in the Shannon-Whittaker-Kotelnikov theorem, to work with bandlimited functions. But, as we implied before, it is not always clear if and how this concept can be reproduced without the full power of Fourier analysis. Instead, we look to work with those spaces that can be described with the help of wavelets.

Inspired by a result of Strichartz [81] for Sobolev spaces, Jaming and Malinikova [56] gave a sampling inequality in \mathbb{R}^d for functions in Besov spaces. Their starting point is the following:

Theorem ([56]). *Let $1 \leq p \leq \infty$. Then, there exist constants $c_1, c_2, c_3 > 0$ such that for all $r > 0$ and all sequence $\{a_n\}_{n \in \mathbb{Z}}$ with $r/2 \leq a_{n+1} - a_n \leq r$*

$$c_1 r^{-1/p} \|f\|_p \leq \left(\sum_{n \in \mathbb{Z}} |f(a_n)|^p \right)^{1/p} \leq c_2 r^{-1/p} \|f\|_p$$

holds for all $f \in B_{p,1}^{1/p}(\mathbb{R})$ with $\|f\|_{B_{p,1}^{1/p}} \leq c_3 r^{-1/p} \|f\|_{L^p}$.

As stated, this result looks closer to (1.9) than (1.8). But, again thanks to frame theory, it is possible obtain a reconstruction like (1.8) from this inequality when we restrict ourselves to work with Paley-Wiener spaces (which are always embedded into any Besov space). In this sense one could understand this type of results as a generalization to classic sampling theorems.

The proof of this theorem makes extensive use of the wavelet decomposition in \mathbb{R} and is based on the characterization of Besov spaces in terms of wavelets coefficients as done by Meyer [68]. Indeed, taking a wavelet system in \mathbb{R} as before, the Schwartz class properties then allow to characterize several commonly used function spaces such as Sobolev, Hölder, Hardy, BMO and -of particular interest to us- Besov spaces. Specifically, it can be proven that the norm of a function $f \in B_{p,q}^s(\mathbb{R})$ is equivalent to

$$\left[\sum_{j \in \mathbb{N}} 2^{jq(s+1/2-1/p)} \left(\sum_{k \in \mathbb{N}} |\langle f, \varphi_{j,k} \rangle|^p \right)^{q/p} \right]^{1/q},$$

where $s \in \mathbb{R}, 1 \leq p, q \leq \infty$ and $\varphi_{j,k}$ is a system as in (1.11) with enough regularity (at least s vanishing moments).

From an abstract point of view, since this characterization only involves a wavelet system, one can use it to define Besov spaces on compact Riemannian manifolds and spaces of homogeneous type. Furthermore, as in the real case it can be shown that, if wavelets have good enough smoothness and localization properties (in substitution for the Schwartz class), this definition is also equivalent to the other natural reproductions of Besov spaces in these settings [20, 21, 36, 38,

37, 47, 69, 70, 84]. (It is important to remark, however, that for the more general types of spaces of homogeneous type the regularity index s might be restricted by the fact that in some cases there are no non-trivial functions with regularity above a certain level [39]).

Hence, structurally, we have at our disposal the same tools used by Jaming and Malinnikova. This allows us to adapt their result to our frameworks. In compact Riemannian manifolds our sampling inequality reads as follows:

Theorem. *Let \mathbf{M} be a \mathbf{d} -dimensional compact Riemannian manifold, and let $1 \leq p \leq \infty$. Then, there exists constants $c_1, c_2, c_3 > 0$ such that for all $r > 0$ and every set $\{\zeta_1, \dots, \zeta_{N_r}\}$ of r -distributed on \mathbf{M}*

$$c_1 r^{-\mathbf{d}/p} \|f\|_p \leq \left(\sum_{n=1}^{N_r} |f(\zeta_n)|^p \right)^{1/p} \leq c_2 r^{-\mathbf{d}/p} \|f\|_p$$

holds for all $f \in B_{p,1}^{\mathbf{d}/p}(\mathbf{M})$ with $\|f\|_{B_{p,1}^{\mathbf{d}/p}} \leq c_3 \|f\|_{L^p}$.

Here r -distributed refers to a very similar condition to that of [56] in \mathbb{R} : a cover of balls with radius r which become disjoint if we reduce r slightly (say, to $r/2$).

The main reason we can adapt the result of Jaming and Malinnikova to manifolds is indeed the existence of a wavelet decomposition in compact Riemannian manifolds. The other underlying factor, is that the proof in [56] does not require any particular property of $\mathbb{R}^{\mathbf{d}}$ other than the fact that the Lebesgue measure of a ball with radius r is proportional to $r^{\mathbf{d}}$. Which in turn, then allows us to go further and extend the result to spaces of homogeneous type. Indeed, we know from the work of Macías and Segovia [64] that, up to changing the pseudo metric for an equivalent one, the measure in a space of homogeneous type can be assumed to have the same property. In fact, the corresponding exponent \mathbf{d} is called the *homogeneous dimension*. We thus obtain:

Theorem. *Let X be a space of homogeneous type of dimension \mathbf{d} , and let $1 \leq p \leq \infty$. Then, there exists constants $c_1, c_2, c_3 > 0$ such that for all $r > 0$ and every family $\{y_n\}_n$ of r -distributed points on X*

$$c_1 r^{-\mathbf{d}/p} \|f\|_p \leq \left(\sum_{n=1}^{N_r} |f(\zeta_n)|^p \right)^{1/p} \leq c_2 r^{-\mathbf{d}/p} \|f\|_p$$

holds for all $f \in B_{p,1}^{\mathbf{d}/p}(X)$ with $\|f\|_{B_{p,1}^{\mathbf{d}/p}} \leq c_3 r^{-\mathbf{d}/p} \|f\|_{L^p}$.

The actual results are slightly more precise, they will take the form of Theorems 2.4 and 3.6 respectively.

The right-hand-side of these sampling inequalities can also be interpreted as trace theorems for Besov functions (onto a discrete set of points). More broadly, the regularity of functions when restricted to a particular subset (discrete or not) is a problem which has its own interests, most notably with respect to PDEs and, specifically, boundary value problems [66]. Due to this, several trace theorems have been given for Besov (and Sobolev) functions defined in metric spaces under different types of assumptions [38, 65, 67, 77]. Here, using similar techniques to those of the proof of Theorem 3.6, we offer a new proof of a trace result for Besov functions under relatively general assumptions. Specifically, we prove the following:

Theorem. *Let X be an space of homogeneous type and $Y \subset X$ a subspace. Denote $\mathbf{d}_X, \mathbf{d}_Y$ for the dimension of X and Y respectively.*

Let $1 \leq p < \infty, 1 \leq q \leq \infty, s \in \mathbb{R}$ and set $s' := s - \frac{\mathbf{d}_X - \mathbf{d}_Y}{p}$. If $0 < s' < \theta$ then the trace operator

$$\mathrm{Tr} : B_{p,q}^s(X) \rightarrow B_{p,q}^{s'}(Y), \quad \mathrm{Tr} f = f|_Y$$

is well defined and continuous.

With *subspace* we mean that Y is a space of homogeneous type with the metric resulting from the restriction of that on X . This is the assumption also considered in [67]. The contribution we make to the existing literature is that, here, we consider a larger class of indexes s . Another novelty is the proof itself, which essentially relies on the wavelet system, closely following strategies developed in, e.g. [21]. In short, the idea is to write $f \in B_{p,q}^s(X)$ in its wavelet expansion so that one can express the wavelet coefficients of $f|_Y$ in terms of those on X ; the good localization and smoothness properties of wavelets then allow to obtain a comparison result, often called a *quasi-orthogonality property*, from which the trace theorem follows.

Further, it is not only the *global* regularity of functions (Besov or otherwise) that can be studied with wavelets but also their *local* variations. In $\mathbb{R}^{\mathbf{d}}$, taking wavelets as described before, it was first shown by Meyer [68] that a function f belongs to the Hölder space of order s if and only if

$$|\langle f, \varphi_{j,k} \rangle| \leq C_f 2^{-j(\mathbf{d}/2+s)} \quad \forall j, k \quad (1.12)$$

where $C_f > 0$ is a constant only depending on f . Similarly for local regularity, Jaffard [53] gave an almost equivalent condition: if f is Hölder of order s at x , then

$$|\langle f, \varphi_{j,k} \rangle| \leq C_f 2^{-j\mathbf{d}/2} (2^{-j} + |x - 2^{-j}k|)^s \quad \forall j, k, \quad (1.13)$$

and conversely if (1.13) holds for all j, k and f has at least some global Hölder regularity then f is Hölder of order s' at x for any $0 < s' < s$.

For spaces of homogeneous type, since we still have a (pseudo)metric, we can define Hölder (local and global) continuity in the same way as we do in $\mathbb{R}^{\mathbf{d}}$. And since the euclidean proofs for (1.12) and (1.13) essentially rely on metric arguments (using the Schwartz properties of wavelets), the same results can be reproduced in spaces of homogeneous type (provided again that wavelets are carefully constructed). Indeed, the equivalent to (1.12) has already been proven in e.g. [21], and here we show that also (1.13) can be reproduced.

In particular the characterization of local regularity properties, at least from an abstract point of view, opens the door to the development and study of the equivalent multifractal analysis in $\mathbb{R}^{\mathbf{d}}$ as done by Jaffard [54, 55]. This is: a study of how the local orders of regularity of a function varies across our framework set. Specifically, one defines the *Hölder exponent* of a function f at a point x as

$$h_f(x) := \sup\{s : f \in \mathcal{C}^s(x)\},$$

and then one tries to understand the geometric distribution of points with the same exponent by means of the Hausdorff dimension of such sets: this is, the *spectrum of singularities*

$$d_f(s) := \dim_{\mathcal{H}}\{x : h_f(x) = s\}.$$

One of the arguments for choosing this quantity is because of the type of functions usually studied in this theory. Notably, Besov spaces have been proven to be a model for which several results can be obtained. And the main reason for this is, again, its wavelet characterization. Thus, indeed, one could hope to reproduce such results in the framework of spaces of homogeneous type. In particular we obtain the following upper bound for the spectrum of singularities:

Theorem. *Let $1 \leq p < \infty, 1 \leq q \leq \infty$ and $\mathbf{d}/p < s < \mathbf{d}/p + \theta$. Then for all $f \in B_{p,q}^s(X)$ and all $s - \mathbf{d}/p \leq \alpha < \theta$*

$$d_f(\alpha) \leq \min(\mathbf{d}, \mathbf{d} + (\alpha - s)p).$$

And $d_f(\alpha) = -\infty$ if $\alpha < s - \mathbf{d}/p$.

This results mirrors that of [54] in $\mathbb{R}^{\mathbf{d}}$. Further, in the euclidean case, it has been proven that this bound is attained for “almost every” function in $B_{p,q}^s(\mathbb{R}^{\mathbf{d}})$ in the sense of prevalence [29] (a concept more effective for vector spaces of infinite dimension than the Lebesgue “almost everywhere” [49]). The arguments used to prove this result rely on sharp Hausdorff dimensions estimates that can only be obtained thanks to a fine description of the euclidean geometry [25]. In spaces of homogeneous type those results might be harder to adapt since a priori one only has an approximate idea of how the shape and size of the space is. This is left for future work.

In the last part of this thesis we return to the study of sampling inequalities but this time, instead of changing the domain of functions from euclidean to non-euclidean we keep the euclidean setting but change the way the samples are distributed. In particular, we will be interested on sampling inequalities in \mathbb{R}^2 where samples are placed -not necessarily in a uniform manner- on certain types of curves.

This is naturally linked to the problem of *continuous sampling*: instead of taking a discrete set of points we consider a curve Γ (or a family of curves) and try to determine when the following type of inequalities hold:

$$A\|f\|_p \leq \left(\int_{\Gamma} |f(x)|^p d\mu_{\Gamma} \right)^{1/p} \leq B\|f\|_p, \quad (1.14)$$

for all f in the corresponding sampling space and where A, B are positive constants and $\mu_{\Gamma} := \mathcal{H}^1|_{\Gamma}$ denotes the one-dimensional Hausdorff (length) measure. One of the key results we show is that, under mild regularity assumptions, the continuous and discrete problem are the same: (1.14) is equivalent to the existence of a discrete set $\Lambda \subset \Gamma$ such that

$$A\|f\|_p \leq \left(\sum_{\lambda \in \Lambda} |f(\lambda)|^p \right)^{1/p} \leq B\|f\|_p.$$

The actual result is more precise, see Theorem D below.

Actually, in the same article, Jaming and Malinnikova [56] also give a continuous counterpart to their discrete sampling inequality –with very similar proofs. They show that if a regular curve Γ covers the space sufficiently well then (1.14) holds for Besov functions.

An illustrative case is that of equispaced parallel lines, say with direction $\vec{v} \in \mathbb{S}^1$ and separation r :

$$\Gamma = \{t\vec{v} + rk\vec{v}_{\perp} : t \in \mathbb{R}, k \in \mathbb{Z}\},$$

where \vec{v}_{\perp} denotes the counterclockwise perpendicular unit vector to \vec{v} . Specifically for this case Jaming and Malinnikova's result implies that there exists a constant $C > 0$ such that (1.14) holds for all $f \in B_{p,1}^{1/p}(\mathbb{R}^2)$ with $\|f\|_p \leq Cr^{-1/p}\|f\|_{B_{p,1}^{1/p}}$.

This can be improved when we work in the classical setting of a sampling theorem with Paley-Wiener spaces. Indeed, if Ω is a convex symmetric body and we are sampling functions on $PW(\Omega)$ then (1.14) holds if $\mathcal{H}^1(\Omega \cap \{t\vec{v}_{\perp} : t \in \mathbb{R}\}) < r^{-1/2}$ but fails for some $f \in PW(\Omega)$ if $\mathcal{H}^1(\Omega \cap \{t\vec{v}_{\perp} : t \in \mathbb{R}\}) > r^{-1/2}$ (see [85, 86]). Thus, $r^{-1/2}$ effectively works a transition value, often called the *Nyquist rate*.

This rate can also be determined for arbitrary families of parallel lines [42] –not just equispaced ones. And certainly the results in [56] apply to a larger family of

curves. However, determining the exact Nyquist rate of an arbitrary curve is a much more subtle task and it is less clear what can be said in full generality.

The contribution we make here is to give sharp necessary and sufficient conditions for (1.14) to hold on a certain parametric family of curves, that we call spiraling. The main examples of these curves are the Archimedes spiral

$$A^\eta := \{(\eta\theta \cos 2\pi\theta, \eta\theta \sin 2\pi\theta) : \theta \geq 0\}$$

and the collection of concentric circles

$$O^\eta := \{(x, y) : x^2 + y^2 = \eta^2 k^2, k \in \mathbb{N}\}.$$

From the point of view of applications, reconstruction from samples in these types of curves is a problem relevant, for example, in magnetic resonance imaging (MRI), where moving sensors capture the anatomy and physiology of a patient. On the theoretical side, the key underlying reason why we can obtain sampling results for spiraling curves is that they behave qualitatively like unions of parallel lines. Thus, as with parallel lines, we are able to determine their exact Nyquist rate:

Theorem. *Let $\Omega \subset \mathbb{R}^2$ be a convex centered symmetric body.*

- (i) *If $\text{diam}(\Omega)\eta < 1$, then the Archimedes spiral A^η and the collection of concentric circles O^η are sampling trajectories for $PW^2(\Omega)$.*
- (ii) *If $\text{diam}(\Omega)\eta > 1$, then neither the Archimedes spiral A^η nor the collection of concentric circles O^η are sampling trajectories for $PW^2(\Omega)$.*

The proof makes extensive use of Beurling's sampling theory [8, 9]. Point (i) follows from Beurling's gap covering theorem [8] as done in [7]. Point (ii) relies on Beurling's characterization of sampling in terms of *weak limits* [8, 9], by showing that an adequate sequence of translates of a spiral converges to a union of parallel lines, locally in the Hausdorff metric.

Having identified the Nyquist rate of spiraling curves, we look into *undersampling*. This is, we consider slightly less dense spirals but restrict the reconstruction problem to functions that are compactly represented in certain dictionaries (motivated from the fact that, in practice, many of the signals one tries to recover are highly compressible and we can thus afford to sample below the Nyquist rate). One of such restrictions is a variation bound:

$$\mathcal{F}(W) := \{f \in PW(\Omega) : \text{var}(\hat{f}) \leq W\}.$$

This can then be translated into the number of active wavelet coefficients that represent f [18, 23]. The stability for sampling such signals is expressed by the inverse condition number

$$\inf \left\{ \|f - g\|_{L^2(\mu_\Gamma)} : f, g \in \mathcal{F}(W), \|\hat{f} - \hat{g}\|_2 = \varepsilon \right\},$$

which measures the extent to which the distance between two signals is exhibited by their samples. Since, $\varepsilon^{-1}\mathcal{F}(W) = \mathcal{F}(\varepsilon^{-1}W)$, and $\mathcal{F}(W) \subset \mathcal{F}(W) - \mathcal{F}(W) \subset \mathcal{F}(2W)$, the analysis of the condition number for small ε reduces to the large W asymptotics of the *stability margin*:

$$A(\Gamma, \mathcal{F}(W)) := \inf \left\{ \|f\|_{L^2(\mu_\Gamma)} : \|\hat{f}\|_2 = 1, f \in \mathcal{F}(W) \right\}.$$

The sampling result we obtain for A^η and O^η tells us that the critical value in the reference case of the unit square $\Omega = [-1/2, 1/2]^2$ is $\eta = \frac{\sqrt{2}}{2}$. We consider then spirals with density slightly under the critical value and prove the following.

Theorem. *Let $\eta = (1 + \varepsilon)\sqrt{2}/2$ with $\varepsilon \in (0, 1)$, and $\Gamma = A^\eta$ or $\Gamma = O^\eta$. Then for $W > 0$,*

$$A(\Gamma, \mathcal{F}(W)) \leq C(\varepsilon W)^{-1/2}(\ln^2(\varepsilon W) + 1),$$

where $C > 0$ is a universal constant.

Effectively, this sets a limit to the capacity of spirals to acquire all compressible signals below the Nyquist rate. Informally, it says that when undersampling by a small factor $(1 - \varepsilon)$, one can only recover functions up to resolution $W \approx \varepsilon^{-1}$ with a stable condition number.

As said before, there is a relation between the variation of a function and the number of large wavelet coefficients [18] (this number is essentially controlled by the variation W). Exploiting that relation we can formulate the previous result in terms of (Haar) wavelet coefficients. Precisely, if we denote $\Sigma_{N,J}$ as the class of functions on $[-1/2, 1/2]^2$ with N non-zero Haar coefficients, all of them taken with scale at most J , we then get the following estimate:

Theorem. *Let $W \geq 1$, $\eta = (1 + \varepsilon)\sqrt{2}/2$ with $\varepsilon \in (0, 1)$ and $\Gamma = A^\eta$ or $\Gamma = O^\eta$. Then for $N \geq 1$,*

$$A(\Gamma, \Sigma_{N,J}) \leq CN^{-1/6}\varepsilon^{-1}\ln^4(CN),$$

where $J = C \ln(\varepsilon^{-1}N)$ and $C > 0$ is a universal constant.

Informally this theorem says that when undersampling by a small factor $(1 - \varepsilon)$, one can recover at most $N \approx \varepsilon^{-6}$ Haar coefficients with a stable condition number.

Chapter 2

Sampling on compact Riemannian manifolds

In this chapter we show our sampling result for Besov functions on compact Riemannian manifolds. We will use the same techniques, and indeed almost the same proofs, as in [57] where we treated the spherical case.

Specifically, the main tool we will use is a wavelet-type system of highly localized functions and whose associated coefficients allow to characterize Besov spaces. This type of systems has been constructed in various ways [20, 35, 76]. In all cases the idea is essentially to discretize a partition of the unity provided by the machinery of Spectral Theory applied to the Laplace-Beltrami operator.

2.1 Notation and preliminaries

Throughout this chapter \mathbf{M} will denote a smooth, connected, compact Riemannian manifold without boundary of dimension \mathbf{d} . We will denote ρ as the Riemannian metric and μ as the manifold (volume) measure. Next, consider Δ the *Laplace-Beltrami operator*, that can be written in local coordinates as

$$\Delta = \Delta_g = \frac{-1}{\sqrt{|\det g|}} \sum_{j,i=1}^{\mathbf{d}} \partial_j \left(g^{j,i} \sqrt{|\det g|} \partial_i \right),$$

where $g = (g_{j,i})_{j,i}$ is the metric tensor, and $g^{j,i}$ are the coefficients of the inverse matrix. It is a well known fact that this operator is self-adjoint and non-negative (see e.g. [15]). We can then use the Spectral Theorem [11] to define another self-adjoint operator, $\mathcal{D} := \sqrt{\Delta}$. Since \mathbf{M} is compact and without boundary and \mathcal{D} is elliptic, the spectrum of \mathcal{D} is a real sequence

$$0 = \lambda_0 < \lambda_1 < \lambda_2 < \dots$$

which converges to $+\infty$. Furthermore, there exists an orthonormal basis in $L^2(\mathbf{M})$ made of eigenfunctions of \mathcal{D} (all these facts follow from standard Spectral Theory, see e.g. [74]). For each eigenvalue λ we set $u_{\lambda,1}, u_{\lambda,2}, \dots$ a basis for the eigenspace associated to λ . Weyl's asymptotic formula for the dimension of eigenspaces of Δ (see e.g. [50]) implies that there is a finite number of such vectors. The representation given by the Spectral Theorem for a measurable function F can then be written as

$$F(\mathcal{D})f = \sum_{\lambda} \sum_n F(\lambda) \langle f, u_{\lambda,n} \rangle u_{\lambda,n}, \quad (2.1)$$

for all f where $F(\mathcal{D})$ is defined. Naturally, for bounded F 's this gives a bounded operator in $L^2(\mathbf{M})$. Also, real valued functions are mapped into selfadjoint operators.

Finally let us recall the notion of bandlimitedness in \mathbf{M} as done in [76]. Specifically, given $\omega > 0$ the *Paley-Wiener space* $PW_{\omega}(\mathcal{D})$ is defined as the vector space spanned by all eigenfunctions with associated eigenvalues not greater than ω . By (2.1), this space can also be identified as the image of $\mathbb{1}_{[0,\omega]}(\mathcal{D})$. Further, defining the ‘‘Fourier transform’’ in this context as the unitary transformation of the spectral theorem associated to \mathcal{D} , one could reproduce the language of bandlimitedness in the exact same way as in the euclidean case [75].

2.2 Paley-Wiener frames

2.2.1 Partition of the unity

The first step is to construct a partition of the unity from the operator \mathcal{D} . Here we follow the standard strategy mentioned at the beginning of this chapter, see e.g. [20, 26].

Let us consider an auxiliary smooth function g with $\text{supp}(g) \subset [0, 2]$ and such that $g(\lambda) = 1$ for $\lambda \in [0, 1]$, $0 \leq g(\lambda) \leq 1$ for all $\lambda \in \mathbb{R}$. Further we set $h(\lambda) := g(\lambda) - g(2\lambda)$. Finally define

$$\begin{aligned} F_0(\lambda) &:= g(\lambda), \\ F_j(\lambda) &:= h(2^{-j}\lambda), \quad j \geq 1. \end{aligned}$$

As these functions are all bounded, the operators $F_j(\mathcal{D})$ are defined on all $L^2(\mathbf{M})$. Moreover, since $\text{supp}(F_j) \subset [0, 2^{j+1}]$, (2.1) and the definition of the Paley-Wiener spaces then read

$$F_j(\mathcal{D})f \in PW_{2^{j+1}}(\mathcal{D}), \quad \forall f \in L^2(\mathbf{M}), \quad (2.2)$$

which in turn makes the operator $F_j(\mathcal{D})$ as a sort of projection onto the corresponding Paley-Wiener space. But more importantly, they also decompose the

identity. Indeed, first note that for all $N \in \mathbb{N}$ and $\lambda \in \mathbb{R}$

$$\sum_{j=0}^N F_j(\lambda) = g(2^{-N}\lambda),$$

so that

$$\sum_{j \in \mathbb{N}} F_j(\lambda) = \mathbb{1}_{[0, +\infty)}(\lambda)$$

with uniform convergence on compact sets. Then, the Spectral Theorem applied to \mathcal{D} reads

$$f = \sum_{j \in \mathbb{N}} F_j(\mathcal{D})f, \quad \forall f \in L^2(\mathbf{M}). \quad (2.3)$$

Similarly we can obtain an L^2 -norm equivalence, since

$$\frac{1}{2} \leq \sum_{j \in \mathbb{N}} F_j^2(\lambda) \leq 1, \quad \forall \lambda \geq 0, \quad (2.4)$$

and the operators $F_j(\mathcal{D})$ are always self-adjoint (since F_j is real-valued for all j). Indeed, (2.4) and the Spectral Theorem then read

$$\frac{1}{2} \|f\|_2^2 \leq \sum_{j \in \mathbb{N}} \|F_j(\mathcal{D})f\|_2^2 \leq \|f\|_2^2, \quad \forall f \in L^2(\mathbf{M}). \quad (2.5)$$

2.2.2 Lattices

The next step is to decompose each operator $F_j(\mathcal{D})$ into smaller, localized functions. To this end, we will need to partition our manifold \mathbf{M} into cells of approximate size 2^{-j^d} .

Lemma 2.1 ([26, 76]). *Let $r > 0$. There exists a disjoint partition $\{U_1, \dots, U_{N_r}\}$ of \mathbf{M} together with a set of points $x_1, \dots, x_{N_r} \in \mathbf{M}$ with the properties:*

(i) *for every $k = 1, \dots, N_r$,*

$$B(x_k, r/2) \subset U_k \subset B(x_k, r),$$

(ii) *the size of the partition is bounded by*

$$N_r \leq \max(1, Tr^{-d}),$$

where $T = T(\mathbf{M})$ is a constant depending only on \mathbf{M} .

Proof. If $r > \text{diam}(\mathbf{M})$ the lemma is obvious; we therefore assume $r \leq \text{diam}(\mathbf{M})$. In this case, and since \mathbf{M} is compact, there exists a constant $A = A(\mathbf{M})$ such that for all $x \in \mathbf{M}$

$$\mu(B(x, r)) \geq Ar^{\mathbf{d}}. \quad (2.6)$$

Now, by Zorn's lemma we can construct a maximal set of points x_1, x_2, \dots with respect to

$$\rho(x_k, x_m) \geq r, \quad k \neq m,$$

where ρ is the Riemannian distance on \mathbf{M} . The triangle inequality then implies that the balls $B(x_k, r/2)$ must be mutually disjoint. By using (2.6), the number of such points is bounded by $A^{-1}\mu(\mathbf{M})r^{-\mathbf{d}}$. We denote those points as x_1, \dots, x_{N_r} .

Finally, we define inductively

$$U_1 = B(x_1, r) \setminus \bigcup_{m \neq 1} B(x_m, r/2),$$

$$U_k = B(x_k, r) \setminus \left(\bigcup_{m < k} U_m \bigcup_{m' \neq k} B(x_{m'}, r/2) \right), \quad 1 < k \leq N_r.$$

Clearly the U_k 's are mutually disjoint and $U_k \subset B(x_k, r)$ for all k . Further, from the fact that the balls $B(x_k, r/2)$ are mutually disjoint, we obtain that $B(x_k, r/2) \subset U_k$ for all k and

$$\bigcup_k U_k = \bigcup_k B(x_k, r) = \mathbf{M},$$

which completes the proof. \square

We call this grid of points x_1, \dots, x_{N_r} an r -lattice of \mathbf{M} and U_1, \dots, U_{N_r} its associated covering.

2.2.3 Sampling on Paley-Wiener spaces

We are now ready to show how a wavelet type system can be constructed from the operators $F_j(\mathcal{D})$. The idea is to obtain a sampling theorem for Paley-Wiener spaces and then use (2.2) to apply it onto the image of each $F_j(\mathcal{D})$. Here we essentially follow [20].

First we need a technical lemma concerning heat-type kernels. Remember that from (2.1), given a smooth compactly supported function F , we can write for any $t > 0$

$$F(t\mathcal{D})f = \sum_{\lambda} \sum_n F(t\lambda) \langle f, u_{\lambda, n} \rangle u_{\lambda, n}, \quad f \in L^2(\mathbf{M}). \quad (2.7)$$

And thus, if we define the kernel

$$K_{F,t}(x, y) := \sum_{\lambda} \sum_n F(t\lambda) u_{\lambda,n}(x) u_{\lambda,n}(y),$$

we have, for all $f \in L^2(\mathbf{M})$,

$$F(t\mathcal{D})f(x) = \int_{\mathbf{M}} K_{F,t}(x, y) f(y) d\mu(y). \quad (2.8)$$

In this situation the following estimates can be proven.

Lemma 2.2 ([35]). *For all $0 < t \leq 1$ and all $N > 0$ there exists a constant C_N such that for all $x, y \in \mathbf{M}$*

$$|K_{F,t}(x, y)| \leq C_N t^{-\mathbf{d}} \left(\frac{t}{t + \rho(x, y)} \right)^N, \quad (2.9)$$

and for all $x, x', y \in \mathbf{M}$ with $\rho(x, x') \leq t$

$$|K_{F,t}(x, y) - K_{F,t}(x', y)| \leq C_N \frac{\rho(x, x')}{t} \cdot t^{-\mathbf{d}} \left(\frac{t}{t + \rho(x, y)} \right)^N. \quad (2.10)$$

The proof relies on the Theory of Pseudo-differential Operators. Anyhow, this paves the way to prove a sampling result for Paley-Wiener space.

Theorem 2.3 ([20]). *Let $\omega \geq 1$ and $r \leq \omega^{-1}$. There exist a constant $C = C(\mathbf{M})$ such that for any r -lattice x_1, \dots, x_{N_r} the inequalities*

$$\sum_{k=1}^{N_r} \int_{U_k} |f(x) - f(x_k)|^2 d\mu(x) \leq C(\omega r)^2 \|f\|_2^2 \quad (2.11)$$

holds for all $f \in PW_\omega$, and where U_1, \dots, U_{N_r} is the associated covering.

Proof. Fix an r -lattice x_1, \dots, x_{N_r} and its associated covering U_1, \dots, U_{N_r} . Take $f \in PW_\omega$. Let F be a non-negative smooth compactly supported function such that $F(\lambda) = 1$ for $\lambda \in [0, 1]$. Then (2.7) and the definition of Paley-Wiener spaces give $F(\omega^{-1}\mathcal{D})f = f$ for all $f \in PW_\omega$. Thus

$$\begin{aligned} & \sum_{k=1}^{N_r} \int_{U_k} |f(x) - f(x_k)|^2 d\mu(x) \\ & \leq \sum_{k=1}^{N_r} \int_{U_k} \left| \int_{\mathbf{M}} (K_{F,\omega^{-1}}(x, y) - K_{F,\omega^{-1}}(x_k, y)) f(y) d\mu(y) \right|^2 d\mu(x) \\ & \leq \sum_{k=1}^{N_r} \int_{U_k} \left(\int_{\mathbf{M}} |K_{F,\omega^{-1}}(x, y) - K_{F,\omega^{-1}}(x_k, y)| |f(y)|^2 d\mu(y) \right. \\ & \quad \left. \times \int_{\mathbf{M}} |K_{F,\omega^{-1}}(x, y) - K_{F,\omega^{-1}}(x_k, y)| d\mu(y) \right) d\mu(x) \end{aligned}$$

by using Hölder's inequality in \mathbf{M} for the last line. Next, since $U_k \subset B(x_k, r)$ and $r \leq \omega^{-1}$ we can apply (2.10) with $N = \mathbf{d} + 1$ to obtain

$$\begin{aligned} & \sum_{k=1}^{N_r} \int_{U_k} \left(\int_{\mathbf{M}} |K_{F, \omega^{-1}}(x, y) - K_{F, \omega^{-1}}(x_k, y)| |f(y)|^2 d\mu(y) \right. \\ & \quad \left. \times \int_{\mathbf{M}} |K_{F, \omega^{-1}}(x, y) - K_{F, \omega^{-1}}(x_k, y)| d\mu(y) \right) d\mu(x) \\ & \lesssim \sum_{k=1}^{N_r} \int_{U_k} \left(\int_{\mathbf{M}} \omega \rho(x, x_k) \cdot \omega^{\mathbf{d}} \left(\frac{\omega^{-1}}{\omega^{-1} + \rho(x, y)} \right)^{\mathbf{d}+1} |f(y)|^2 d\mu(y) \right. \\ & \quad \left. \times \int_{\mathbf{M}} \omega \rho(x, x_k) \cdot \omega^{\mathbf{d}} \left(\frac{\omega^{-1}}{\omega^{-1} + \rho(x, y)} \right)^{\mathbf{d}+1} d\mu(y) \right) d\mu(x). \end{aligned}$$

For any $x \in \mathbf{M}$, by dividing into annuli around x one can get the estimate

$$\int_{\mathbf{M}} \left(\frac{\omega^{-1}}{\omega^{-1} + \rho(x, y)} \right)^{\mathbf{d}+1} d\mu(y) \lesssim \omega^{-\mathbf{d}} \quad (2.12)$$

(see for example the proof of Lemma 2.5 below). Thus,

$$\begin{aligned} & \sum_{k=1}^{N_r} \int_{U_k} \left(\int_{\mathbf{M}} \omega^{\mathbf{d}+1} \rho(x, x_k) \left(\frac{\omega^{-1}}{\omega^{-1} + \rho(x, y)} \right)^{\mathbf{d}+1} |f(y)|^2 d\mu(y) \right. \\ & \quad \left. \times \int_{\mathbf{M}} \omega^{\mathbf{d}+1} \rho(x, x_k) \left(\frac{\omega^{-1}}{\omega^{-1} + \rho(x, y)} \right)^{\mathbf{d}+1} d\mu(y) \right) d\mu(x) \\ & \lesssim \sum_{k=1}^{N_r} \int_{U_k} \int_{\mathbf{M}} (\omega \rho(x, x_k))^2 \cdot \omega^{\mathbf{d}} \left(\frac{\omega^{-1}}{\omega^{-1} + \rho(x, y)} \right)^{\mathbf{d}+1} |f(y)|^2 d\mu(y) d\mu(x) \end{aligned}$$

Finally since the sets U_k partition \mathbf{M} and $U_k \subset B(x_k, r)$, applying Tonelli's theorem and (2.12) for any fixed y gives us

$$\begin{aligned} & \sum_{k=1}^{N_r} \int_{U_k} (\omega \rho(x, x_k))^2 \int_{\mathbf{M}} \omega^{\mathbf{d}} \left(\frac{\omega^{-1}}{\omega^{-1} + \rho(x, y)} \right)^{\mathbf{d}+1} |f(y)|^2 d\mu(y) d\mu(x) \\ & \leq (\omega r)^2 \int_{\mathbf{M}} \left(\sum_{k=1}^{N_r} \int_{U_k} \omega^{\mathbf{d}} \left(\frac{\omega^{-1}}{\omega^{-1} + \rho(x, y)} \right)^{\mathbf{d}+1} d\mu(x) \right) |f(y)|^2 d\mu(y) \\ & \leq (\omega r)^2 \int_{\mathbf{M}} \left(\int_{\mathbf{M}} \omega^{\mathbf{d}} \left(\frac{\omega^{-1}}{\omega^{-1} + \rho(x, y)} \right)^{\mathbf{d}+1} d\mu(x) \right) |f(y)|^2 d\mu(y) \\ & \leq (\omega r)^2 \|f\|_2^2. \end{aligned}$$

Altogether this yields (2.11). \square

Note that this result derives a sampling inequality. Indeed, since (i) in Lemma 2.1 implies that the U_k 's have measure comparable to $r^{\mathbf{d}}$, then (2.11) derives the following: there exist two constants c_1, c_2 depending only on \mathbf{M} such that, if $r \leq (C/2)^{-1/2}\omega^{-1}$ where C is the constant in (2.11) then

$$c_1 \|f\|_2^2 \leq r^{\mathbf{d}} \sum_{k=1}^{N_r} |f(x_k)|^2 \leq c_2 \|f\|_2^2, \quad \forall f \in PW_\omega.$$

Now let $j \in \mathbb{N}$, take $\omega_j = 2^{j+1}$ and set $r_j := (C/2)^{-1/2}2^{-j-1}$. Let $x_{j,1}, \dots, x_{j,N_j}$ be an r_j -lattice of \mathbf{M} . Then, using (2.2) we have,

$$c_1 \|F_j(\mathcal{D})f\|_2^2 \leq r_j^{\mathbf{d}} \sum_{k=1}^{N_j} |F_j(\mathcal{D})f(x_{j,k})|^2 \leq c_2 \|F_j(\mathcal{D})f\|_2^2, \quad \forall f \in L^2(\mathbf{M}).$$

Set $\varphi_{j,k}(y) := r_j^{\mathbf{d}/2} K_j(x_{j,k}, y)$ for each $k = 1, \dots, N_j$ and where K_j denotes the kernel of F_j . Thus, we can rewrite the previous inequality as

$$c_1 \|F_j(\mathcal{D})f\|_2^2 \leq \sum_{k=1}^{N_j} |\langle f, \varphi_{j,k} \rangle|^2 \leq c_2 \|F_j(\mathcal{D})f\|_2^2, \quad \forall f \in L^2(\mathbf{M}).$$

Running the sum over all j , (2.5) reads

$$\frac{c_1}{2} \|f\|_2^2 \leq \sum_{k=1}^{N_j} |\langle f, \varphi_{j,k} \rangle|^2 \leq c_2 \|f\|_2^2, \quad \forall f \in L^2(\mathbf{M}).$$

In other words, the system $\{\varphi_{j,k}\}_{j,k}$ constitutes a frame. Moreover, from (2.9) and (2.10) we can derive the following regularity properties: for all $N > 0$ there exists a constant C_N such that

— (*Size condition*) for all $j \in \mathbb{N}$, $k = 1, \dots, N_j$ and $x \in \mathbf{M}$

$$|\varphi_{j,k}(x)| \leq C_N 2^{j\mathbf{d}/2} \left(\frac{2^{-j}}{2^{-j} + \rho(x, x_{j,k})} \right)^N, \quad (2.13)$$

— (*Smoothness condition*) for all $j \in \mathbb{N}$, $k = 1, \dots, N_j$ and $x, x' \in \mathbf{M}$ with $\rho(x, x') \leq 2^{-j}$

$$|\varphi_{j,k}(x) - \varphi_{j,k}(x')| \leq C_N 2^{j\mathbf{d}/2} 2^j \rho(x, x') \left(\frac{2^{-j}}{2^{-j} + \rho(x, x_{j,k})} \right)^N. \quad (2.14)$$

Additionally, using a small perturbation of the identity in $L^2(\mathbf{M})$ Coulhon, Kerkyacharian and Petrushev [20] show that one can construct a dual basis $\tilde{\varphi}_{j,k}$ that enjoys the same regularity properties. From now on, we will consider such wavelet system. This is,

$$f = \sum_{j \in \mathbb{N}} \sum_k \langle f, \varphi_{j,k} \rangle \tilde{\varphi}_{j,k}, \quad \forall f \in L^2(\mathbf{M}), \quad (2.15)$$

and where both $\varphi_{j,k}, \tilde{\varphi}_{j,k}$ satisfy (2.13) and (2.14).

2.3 Sampling inequality for Besov spaces

As in the euclidean case, the partition of unity given by the family $\{F_j(\mathcal{D})\}_j$ allows to describe various functions spaces. Indeed, as the operators $F_j(\mathcal{D})$ behave as some sort of projection onto Paley-Wiener spaces (2.2), then (2.3) can be seen as an equivalent to a Littlewood-Paley decomposition on $\mathbb{R}^{\mathbf{d}}$, and thus similar characterizations of Sobolev, Hölder and Hardy spaces can be obtained.

Here we will work with *Besov spaces*. Given $1 \leq p, q \leq \infty$ and $s \in \mathbb{R}$, the Besov space $B_{p,q}^s(\mathbf{M})$ is defined as the set of all distributions f such that the norm

$$\|f\|_{B_{p,q}^s} := \left(\sum_{j \in \mathbb{N}} \left(2^{js} \|F_j^2(\mathcal{D})f\|_p \right)^q \right)^{1/q}$$

is finite. This description is equivalent to all other definitions that can be done for Besov spaces on manifolds [20, 36, 37, 70, 84]. In particular, one can obtain a characterization in terms of the wavelet coefficients. Indeed one can substitute $\|f\|_{B_{p,q}^s}$ with the equivalent quantity

$$\left[\sum_{j \in \mathbb{N}} 2^{jq(s+\mathbf{d}/2-\mathbf{d}/p)} \left(\sum_{k=1}^{N_j} |\langle f, \varphi_{j,k} \rangle|^p \right)^{q/p} \right]^{1/q}.$$

We will use this last characterization. We are now ready to prove our sampling inequality.

Theorem 2.4. *Let $1 \leq p \leq \infty$, $s \geq \mathbf{d}/p$ and $\alpha = \frac{s-\mathbf{d}/p}{1+s-\mathbf{d}/p}$. Fix $0 < \varepsilon \leq 1$ and let $y_1, \dots, y_{N_\varepsilon}$ be a ε -lattice. There exists constants $C = C(p, \mathbf{M})$ and $K = K(p, \mathbf{M})$ such that if $f \in B_{p,1}^s(\mathbf{M})$ with $\|f\|_{B_{p,1}^s} \leq (K\varepsilon^\alpha)^{-1} \|f\|_p$ then*

$$\frac{1}{2C\varepsilon^{\mathbf{d}/p}} \|f\|_p \leq \left(\sum_{l=1}^{N_\varepsilon} |f(y_l)|^p \right)^{1/p} \leq \frac{3}{2C\varepsilon^{\mathbf{d}/p}} \|f\|_p.$$

Let us start with the following technical estimate.

Lemma 2.5. *Let $1 \leq p \leq \infty$. There exists a constant $C = C(p, \mathbf{M})$ such that*

— for all $j \in \mathbb{N}$ and all $x \in \mathbf{M}$

$$\left(\sum_k |\varphi_{j,k}(x)|^p \right)^{1/p} \leq C 2^{j\mathbf{d}/2}, \quad (2.16)$$

— for all $j \in \mathbb{N}$ and all $x, x' \in \mathbf{M}$ with $\rho(x, x') \leq 2^{-j}$

$$\left(\sum_k |\varphi_{j,k}(x) - \varphi_{j,k}(x')|^p \right)^{1/p} \leq C 2^{j\mathbf{d}/2} 2^j \rho(x, x'). \quad (2.17)$$

And the same inequalities hold for the dual family $\{\tilde{\varphi}_{j,k}\}_{j,k}$.

Proof. Let $x \in \mathbf{M}$. Using the size condition (2.13) for $\varphi_{j,k}$ with $N = \mathbf{d} + 1$ we have that

$$\left(\sum_k |\varphi_{j,k}(x)|^p \right)^{1/p} \lesssim 2^{j\mathbf{d}/2} \left(\sum_k \left[\frac{2^{-j}}{2^{-j} + \rho(x_{j,k}, x)} \right]^{(\mathbf{d}+1)p} \right)^{1/p}. \quad (2.18)$$

Therefore, to prove (2.16) it is enough to prove that the sum on the right-hand-side is bounded by a constant depending only on \mathbf{M} and p . To that end, let us consider $U_{j,1}, \dots, U_{j,N_j}$ a partition associated to $x_{j,1}, \dots, x_{j,N_j}$. Using the triangle inequality and the fact that for all j, k , $U_{j,k} \subset B(x_{j,k}, c2^{-j})$ for some constant $c > 0$, we get that

$$2^{-j} + \rho(x_{j,k}, x) \geq \frac{1}{1+c} (2^{-j} + \rho(y, x)), \quad \forall y \in U_{j,k}.$$

Then, as $U_{j,1}, \dots, U_{j,N_j}$ is a family of disjoint sets of measure $\approx 2^{-j\mathbf{d}}$, summing over all k gives us

$$\begin{aligned} \sum_k \left(\frac{2^{-j}}{2^{-j} + \rho(x_{j,k}, x)} \right)^{(\mathbf{d}+1)p} &\leq \sum_k \frac{(1+c)^{(\mathbf{d}+1)p}}{\mu(U_{j,k})} \\ &\quad \times \int_{U_{j,k}} \left(\frac{2^{-j}}{2^{-j} + \rho(y, x)} \right)^{(\mathbf{d}+1)p} d\mu(y) \\ &\lesssim 2^{j\mathbf{d}} \int_{\mathbf{M}} \left(\frac{2^{-j}}{2^{-j} + \rho(y, x)} \right)^{(\mathbf{d}+1)p} d\mu(y) \end{aligned}$$

Diving \mathbf{M} into annuli gives

$$\begin{aligned} & \int_{\mathbf{M}} \left(\frac{2^{-j}}{2^{-j} + \rho(y, x)} \right)^{(\mathbf{d}+1)p} d\mu(y) \\ & \leq \int_{B(x, 2^{-j})} d\mu(y) + \sum_{j' < j} \int_{B(x, 2^{-j'}) \setminus B(x, 2^{-j'-1})} 2^{-j(\mathbf{d}+1)p} \rho(y, x)^{-(\mathbf{d}+1)p} d\mu(y) \\ & \lesssim 2^{-j\mathbf{d}} + 2^{-j(\mathbf{d}+1)p} \sum_{j' < j} 2^{-j'\mathbf{d}} 2^{-(-j'-1)(\mathbf{d}+1)p} \lesssim 2^{-j\mathbf{d}}, \end{aligned}$$

where the implicit constant depends only on p and \mathbf{M} . Plugged into the previous inequality, this completes the proof of (2.16). The estimate of (2.17) follows exactly the same using (2.14) instead of (2.13) in (2.18). \square

Proof of Theorem 2.4. From (2.15) we may write $f \in B_{p,1}^s(\mathbf{M})$ in its wavelet decomposition as

$$f = \sum_{j \in \mathbb{N}} \sum_k \langle f, \varphi_{j,k} \rangle \tilde{\varphi}_{j,k},$$

where the convergence is in L^2 . Moreover, using (2.16) and the fact that $f \in B_{p,1}^s$ with $s \geq \mathbf{d}/p$ we see that the sum on the right converges also with L^∞ norm: for any $x \in \mathbf{M}$

$$\begin{aligned} \sum_{j \in \mathbb{N}} \sum_k |\langle f, \varphi_{j,k} \rangle| |\tilde{\varphi}_{j,k}(x)| & \leq \sum_{j \in \mathbb{N}} \left(\sum_k |\langle f, \varphi_{j,k} \rangle|^p \right)^{1/p} \left(\sum_k |\tilde{\varphi}_{j,k}(x)|^{p'} \right)^{1/p'} \\ & \lesssim \sum_{j \in \mathbb{N}} 2^{j\mathbf{d}/2} \left(\sum_k |\langle f, \varphi_{j,k} \rangle|^p \right)^{1/p} \\ & \lesssim \sum_{j \in \mathbb{N}} 2^{j(s+\mathbf{d}/2-\mathbf{d}/p)} \left(\sum_k |\langle f, \varphi_{j,k} \rangle|^p \right)^{1/p} = \|f\|_{B_{p,1}^s}. \end{aligned}$$

Hence, as (2.14) implies the continuity of the wavelets $\varphi_{j,k}$, it follows that f is well defined and continuous all over \mathbf{M} .

Let now $U_1, \dots, U_{N_\varepsilon}$ be a partition of \mathbf{M} associated to $y_1, \dots, y_{N_\varepsilon}$, take an arbitrary $l \in \{1, \dots, N_\varepsilon\}$ and consider $x \in U_l$. By Hölder's inequality we have

$$\begin{aligned} |f(x) - f(y_l)| & \leq \sum_{j \in \mathbb{N}} \sum_k |\langle f, \varphi_{j,k} \rangle| |\tilde{\varphi}_{j,k}(x) - \tilde{\varphi}_{j,k}(y_l)| \\ & \leq \sum_{j \in \mathbb{N}} \left(\sum_k |\tilde{\varphi}_{j,k}(x) - \tilde{\varphi}_{j,k}(y_l)|^{p'} \right)^{1/p'} S_j(f) \end{aligned} \quad (2.19)$$

where for each $j \in \mathbb{N}$

$$S_j(f) := \left(\sum_k |\langle f, \varphi_{j,k} \rangle|^p \right)^{1/p}.$$

Now let $j_0 \in \mathbb{N}$ with $2^{-j_0} \geq \varepsilon$. The precise value of j_0 will be fixed later. Then, we can divide the sum on the right-hand-side of (2.19) by applying (2.17) when $0 \leq j \leq j_0$ and (2.16) when $j > j_0$:

$$\begin{aligned} |f(x) - f(y_l)| &\lesssim \sum_{j=0}^{j_0} 2^{j\mathbf{d}/2} 2^j \rho(x, y_l) S_j(f) + 2C \sum_{j>j_0} 2^{j\mathbf{d}/2} S_j(f) \\ &\lesssim \sum_{j=0}^{j_0} 2^{j\mathbf{d}/2} 2^j \rho(x, y_l) S_j(f) + 2C 2^{-j_0(s-\mathbf{d}/p)} \sum_{j>j_0} 2^{j(s+\mathbf{d}/2-\mathbf{d}/p)} S_j(f), \end{aligned}$$

where we have used again that $s \geq \mathbf{d}/p$. Next, taking the L^p -norm over U_l and using the triangle inequality in $L^p(U_l)$, we get

$$\begin{aligned} \left(\int_{U_l} |f(x) - f(y_l)|^p d\mu(x) \right)^{1/p} &\lesssim \sum_{j=0}^{j_0} 2^{j\mathbf{d}/2} 2^j \left(\int_{U_l} \rho(x, y_l)^p d\mu(x) \right)^{1/p} S_j(f) \\ &\quad + 2^{-j_0(s-\mathbf{d}/p)} \mu(U_l)^{1/p} \sum_{j>j_0} 2^{j(s+\mathbf{d}/2-\mathbf{d}/p)} S_j(f) \\ &\lesssim \sum_{j=0}^{j_0} 2^{j\mathbf{d}/2} 2^j \varepsilon^{\mathbf{d}/p+1} S_j(f) \\ &\quad + 2^{-j_0(s-\mathbf{d}/p)} \varepsilon^{\mathbf{d}/p} \sum_{j>j_0} 2^{j(s+\mathbf{d}/2-\mathbf{d}/p)} S_j(f) \end{aligned}$$

where we have used that $U_l \subset B(y_l, \varepsilon)$. We now take j_0 such that both $2^{j_0} \geq \varepsilon^{\alpha-1}$ and $2^{-j_0} \geq \varepsilon$ are satisfied, e.g. $j_0 = \lfloor \frac{\ln 1/\varepsilon}{\ln 2} (1 - \alpha/2) \rfloor + 1$. So $2^{-j_0(s-\mathbf{d}/p)} \lesssim \varepsilon^\alpha$ and thus

$$\begin{aligned} \left(\int_{U_l} |f(x) - f(y_l)|^p d\mu(x) \right)^{1/p} &\lesssim 2^{j_0} \varepsilon \varepsilon^{\mathbf{d}/p} \sum_{j=0}^{j_0} 2^{j(s+\mathbf{d}/2-\mathbf{d}/p)} S_j(f) \\ &\quad + 2^{-j_0(s-\mathbf{d}/p)} \varepsilon^{\mathbf{d}/p} \sum_{j>j_0} 2^{j(s+\mathbf{d}/2-\mathbf{d}/p)} S_j(f) \\ &\lesssim \varepsilon^\alpha \varepsilon^{\mathbf{d}/p} \|f\|_{B_{p,1}^s} \end{aligned}$$

which combined with the triangle inequality (again in $L^p(U_l)$) gives us

$$\left| \left(\int_{U_l} |f(x)|^p d\mu(x) \right)^{1/p} - |f(y_l)| \mu(U_l)^{1/p} \right| \lesssim \varepsilon^\alpha \varepsilon^{\mathbf{d}/p} \|f\|_{B_{p,1}^s},$$

So, by taking ℓ^p -norms and using the triangle inequality in that norm, we get

$$\begin{aligned}
& \left| \|f\|_p - \left(\sum_{l=1}^{N_\varepsilon} |f(y_l)|^p \mu(U_l) \right)^{1/p} \right| \\
&= \left| \left\| \left(\int_{U_l} |f(x)|^p d\mu(x) \right)^{1/p} \right\|_{\ell^p(N_\varepsilon)} - \left\| |f(y_l)| \mu(U_l)^{1/p} \right\|_{\ell^p(N_\varepsilon)} \right| \\
&\leq \left\| \left(\int_{U_l} |f(x)|^p d\mu(x) \right)^{1/p} - |f(y_l)| \mu(U_l)^{1/p} \right\|_{\ell^p(N_\varepsilon)} \\
&\lesssim \varepsilon^\alpha \varepsilon^{\mathbf{d}/p} \|f\|_{B_{p,1}^s} N_\varepsilon^{1/p} \\
&\lesssim \varepsilon^\alpha \|f\|_{B_{p,1}^s}
\end{aligned}$$

where we have used (ii) of Lemma 2.1 in the last inequality. Then, for some independent constant $C > 0$, we have

$$\left(\sum_{l=1}^{N_\varepsilon} |f(y_l)|^p \mu(U_l) \right)^{1/p} \leq C \varepsilon^\alpha \|f\|_{B_{p,1}^s} + \|f\|_p$$

and

$$\left(\sum_{l=1}^{N_\varepsilon} |f(y_l)|^p \mu(U_l) \right)^{1/p} \geq \|f\|_p - C \varepsilon^\alpha \|f\|_{B_{p,1}^s}.$$

Therefore, choosing $K = 2C$, we have that

$$\frac{1}{2} \|f\|_p \leq \left(\sum_{l=1}^{N_\varepsilon} |f(y_l)|^p \mu(U_l) \right)^{1/p} \leq \frac{3}{2} \|f\|_p$$

whenever $\|f\|_{B_{p,1}^s} \leq (M\varepsilon^\alpha)^{-1} \|f\|_p$. Since by Lemma 2.1 we know that $\mu(U_l) \approx \varepsilon^{\mathbf{d}}$, this proves the theorem. \square

Chapter 3

Sampling on spaces of homogeneous type

Compact Riemannian manifolds are part of a larger type of spaces, often used in Harmonic Analysis. These are the spaces of homogeneous type introduced by Coifman and Weiss [19]. This family also includes less smooth examples such as Lipschitz domains and certain fractals sets.

The first goal of this chapter is to show how, with tools similar to those used in the previous chapter, we can recover the same sampling inequality for Besov functions. However, the potential lack of smoothness in the ambient space, leads to a restriction on the smoothness index of the Besov spaces. This result is proved in §3.2.1 and has appeared in [58].

Secondly, in the spirit of viewing the right-hand-side sampling inequality of §3.2.1 as a trace theorem (onto a set discrete of points), we give a trace theorem for Besov functions restricted to (another) subspace of homogeneous type. This result is proved at the end of this chapter and its proof also relies heavily on the wavelet decomposition and in particular the localization properties of these systems which allow to obtain a *quasi-orthogonality* property between different wavelets.

3.1 Preliminaries

3.1.1 General definitions

A quasi-metric ρ on a set X is a function $\rho : X \times X \rightarrow [0, \infty)$ satisfying

- $\rho(x, y) = 0$ if and only if $x = y$,
- $\rho(x, y) = \rho(y, x)$ for all $x, y \in X$,

— there exists a constant $\mathbf{A} > 0$ such that for all $x, y, z \in X$

$$\rho(x, y) \leq \mathbf{A}(\rho(x, z) + \rho(z, y)). \quad (3.1)$$

Any quasi-metric defines a topology on X for which the balls $B(x, r) = \{y \in X : \rho(x, y) < r\}$ form a base. Note that (3.1) forces $\mathbf{A} \geq 1$. When $\mathbf{A} = 1$ we have a usual distance, but when $\mathbf{A} > 1$ the balls need not be open.

Following the definition of Coifman and Weiss [19], a *space of homogeneous type* (X, ρ, μ) is a set X endowed with a quasi-metric ρ and a non-negative Borelian measure μ which is *doubling*

$$0 < \mu(B(x, 2r)) \leq C\mu(B(x, r)) < \infty, \quad \forall x \in X, r > 0,$$

for some positive constant C .

Macias and Segovia [64] proved that it always exists an equivalent quasi-metric $\tilde{\rho}$ -in the sense that $A\rho(x, y) \leq \tilde{\rho}(x, y) \leq B\rho(x, y)$ for all $x, y \in X$ - such that for some constants $\mathbf{d}, \theta, \mathbf{C} > 0$ the following are verified

$$\mathbf{C}^{-1}r^{\mathbf{d}} \leq \mu(B_{\tilde{\rho}}(x, r)) \leq \mathbf{C}r^{\mathbf{d}}, \quad \forall x \in X, 0 < r \leq \text{diam}(X), \quad (3.2)$$

$$|\tilde{\rho}(x, y) - \tilde{\rho}(x', y)| \leq \mathbf{C}\tilde{\rho}(x, x')^\theta(\tilde{\rho}(x, y) + \tilde{\rho}(x', y))^{1-\theta}, \quad \forall x, x', y \in X. \quad (3.3)$$

Actually, (3.3) follows from the fact that $\tilde{\rho}^{1/\theta}$ is a distance. In particular with this topology, balls are open sets. The constant \mathbf{d} is often called the *homogeneous dimension*, and we will refer to θ as the *regularity of the space*. Also, from now on we will assume that ρ meets both (3.2) and (3.3).

Before continuing with the preliminaries, let us now provide some examples of spaces of homogeneous type.

- (i) $X = \mathbb{R}^{\mathbf{d}}$, $\rho(x, y) = \left(\sum_{j=1}^n |x_j - y_j|^2\right)^{1/2}$ and μ equals the Lebesgue measure.
- (ii) $X = \mathbb{R}^{\mathbf{d}}$, $\rho(x, y) = \sum_{j=1}^n |x_j - y_j|^{\alpha_j}$ where $\alpha_1, \dots, \alpha_n$ are positive numbers and μ equals the Lebesgue measure.
- (iii) X is the boundary of a Lipschitz domain in $\mathbb{R}^{\mathbf{d}}$, ρ is the Euclidean distance and μ is the harmonic measure.
- (iv) Any C^∞ compact Riemannian manifold with the Riemannian metric and volume.
- (v) Let $n \geq 1$ be an integer and $\mathbf{d} \in (0, n]$ a real number. Let $E \subset \mathbb{R}^n$ be a closed subset such that its \mathbf{d} -Hausdorff measure $\mathcal{H}_{\mathbf{d}}(E)$ is finite and positive. Suppose that $\mathcal{H}_{\mathbf{d}}(E \cap B(x, r)) \sim r^{\mathbf{d}}$ holds for all $x \in E$. Then equipped with the Euclidean metric and $\mu = \mathcal{H}_{\mathbf{d}}$, E is a space of homogeneous type. This includes several types of fractals such as Cantor sets and the Sierpiński triangle.

3.1.2 Wavelet expansion

As with the case of manifolds, the first step to construct a wavelet system on $L^2(X)$ is to decompose the identity into localized operators. The difference in this case is that, a priori, nothing ensures the existence of a self-adjoint elliptic operator and thus we cannot rely machinery of Spectral Theory (though one could artificially add such operators as an hypothesis [20]). Instead, the partition of unity here is directly defined in such a way that most of the usual desirable properties are preserved. For this part we essentially follow [21, §1.3].

Take h a smooth non-negative function with $\text{supp}(h) \subset [0, 2b]$ and $h(\lambda) = 1$ for $\lambda \in [c, b]$, where b and c are constants to be fixed later. Also, let $\delta \in (0, 1)$ a constant to be fixed later. Next, for each $j \in \mathbb{N}$ we define H_j as the operator with integral kernel $\delta^{-j\mathbf{d}}h(\delta^{-j}\rho(x, y))$. If $\mathbb{1}$ denotes the function that is constant 1, then (3.2) yields

$$H_j(\mathbb{1})(x) = \int_X \delta^{-j\mathbf{d}}h(\delta^{-j}\rho(x, y)) \, d\mu(y) \leq \int_{B(x, 2b\delta^j)} C\delta^{-j\mathbf{d}} \, d\mu(y) \leq CC(2b)^{\mathbf{d}},$$

and

$$\begin{aligned} H_j(\mathbb{1})(x) &= \int_X \delta^{-j\mathbf{d}}h(\delta^{-j}\rho(x, y)) \, d\mu(y) \\ &\geq \int_{B(x, b\delta^j)} \delta^{-j\mathbf{d}} \, d\mu(y) - \int_{B(x, c\delta^j)} \delta^{-j\mathbf{d}} \, d\mu(y), \\ &\geq C^{-1}b^{\mathbf{d}} - Cc^{\mathbf{d}} \geq 1 \end{aligned}$$

by choosing $b = \min(\text{diam}(X)/\delta, C^{1/\mathbf{d}})$ and $c = \frac{1}{2} \min(\text{diam}(X)/\delta, C^{-1/\mathbf{d}})$. We define then $m_j(x) := (H_j(\mathbb{1})(x))^{-1}$ for all $x \in X$. By the same reasoning we will also have $H_j(m_j)(x) \approx 1$, so that $w_j(x) := (H_j(m_j)(x))^{-1}$ is also well defined for all $x \in X$. Let M_j and W_j the operators that consist in multiplying by m_j and w_j respectively. Finally we set $S_j := M_j H_j W_j H_j M_j$. The kernel of S_j can be then written as

$$S_j f(x, y) = m_j(x)m_j(y) \int_X \delta^{-j\mathbf{d}}h(\delta^{-j}\rho(x, z))\delta^{-j\mathbf{d}}h(\delta^{-j}\rho(z, y))w_j(z) \, d\mu(z). \quad (3.4)$$

These kernels then have the following properties.

Proposition 3.1. *There exists a constant $C > 0$ such that the family $\{S_j\}_j$ verifies*

- $S_j(x, y) = 0$ if $\rho(x, y) > C\delta^j$ and $|S_j(x, y)| \leq C\delta^{-j\mathbf{d}}$ for all $x, y \in X$,
- $|S_j(x, y) - S_j(x', y)| \leq C\delta^{-j(\mathbf{d}+\theta)}\rho(x, x')^\theta$ for all $x, x', y \in X$,

— $S_j(x, y) = S_j(y, x)$ for all $x, y \in X$, and

$$\int_X S_j(x, y) d\mu(x) = 1, \quad \forall y \in X.$$

Proof. From (3.4) and the hypothesis on the function h we have

$$S_j f(x, y) = m_j(x)m_j(y) \int_{B(x, 2b\delta^j) \cap B(y, 2b\delta^j)} \delta^{-j\mathbf{d}} h(\delta^{-j}\rho(x, z)) \\ \times \delta^{-j\mathbf{d}} h(\delta^{-j}\rho(z, y)) w_j(z) d\mu(z).$$

By the quasi-triangle inequality if $z \in B(x, 2b\delta^j) \cap B(y, 2b\delta^j)$ then

$$\rho(x, y) \leq 4\mathbf{A}b\delta^j.$$

Therefore, if $\rho(x, y) > 4\mathbf{A}b\delta^j$, $S_j(x, y) = 0$.

Since the functions h, m_j, w_j are uniformly bounded then (3.4) reads

$$|S_j(x, y)| \leq C \int_{B(x, 2b\delta^j) \cap B(y, 2b\delta^j)} \delta^{-2j\mathbf{d}} d\mu(z) \leq C' \delta^{-j\mathbf{d}},$$

for all $x, y \in X$.

For the second point note that, for what we have already proven, we only need to show the inequality for $\rho(x, x') < \delta^j$. And this follows by applying the smoothness of h and (3.3):

$$|S_j(x, y) - S_j(x', y)| \leq m_j(y) \int_X \delta^{-j\mathbf{d}} |h(\delta^{-j}\rho(x, z)) - h(\delta^{-j}\rho(x', z))| \\ \times \delta^{-j\mathbf{d}} h(\delta^{-j}\rho(z, y)) w_j(z) d\mu(z) \\ \lesssim \int_X \delta^{-2j\mathbf{d}} \delta^{-j} |\rho(x, z) - \rho(x', z)| d\mu(z) \\ \lesssim \int_X \delta^{-2j\mathbf{d}} \delta^{-j} \rho(x, x')^\theta (\rho(x, z) + \rho(x', z))^{1-\theta} d\mu(z).$$

Finally, note that since $S_j(x, y)$ and $S_j(x', y)$ are zero when either $\rho(x, y) \geq C\delta^j$ or $\rho(x', y) \geq C\delta^j$ and we are assuming $\rho(x, x') < \delta^j$, then we may restrain the domain of the integral above to $z \in B(x, C\delta^j) \cap B(x', C\delta^j)$. This then gives

$$|S_j(x, y) - S_j(x', y)| \lesssim \int_{B(x, C\delta^j)} \delta^{-j(2\mathbf{d}+\theta)} \rho(x, x')^\theta d\mu(z) \lesssim \delta^{-j(\mathbf{d}+\theta)} \rho(x, x')^\theta.$$

Finally, for the third point, symmetry is obvious and the mean property follows from the definition of the functions m_j, w_j :

$$\begin{aligned} \int_X S_j(x, y) \, d\mu(x) &= m_j(y) \int_X \left(\int_X m_j(x) \delta^{-j\mathbf{d}} h(\delta^{-j}\rho(x, z)) \, d\mu(x) \right) \\ &\quad \times \delta^{-j\mathbf{d}} h(\delta^{-j}\rho(z, y)) w_j(z) \, d\mu(z) \\ &= m_j(y) \int_X H_j(m_j)(z) w_j(z) \delta^{-j\mathbf{d}} h(\delta^{-j}\rho(z, y)) \, d\mu(z) \\ &= m_j(y) \int_X \delta^{-j\mathbf{d}} h(\delta^{-j}\rho(z, y)) \, d\mu(z) = m_j(y) H_j(\mathbb{1})(y) = 1. \end{aligned}$$

□

As a corollary we obtain that $\lim_{j \rightarrow +\infty} S_j = I$ in the strong operator topology on $L^2(X)$ (convergence with L^2 -norm for each $f \in L^2(X)$). Thus, if we set $D_0 := S_0$ and $D_j := S_j - S_{j-1}$ for $j \geq 1$, we have that $I = \sum_{j \in \mathbb{N}} D_j$ in the strong operator topology. We can then write for any $N > 0$

$$I = I^2 = \sum_{j'} D_{j'} \sum_j D_j = \sum_{|j'-j| \leq N} D_{j'} D_j + \sum_{|j-j'| > N} D_{j'} D_j := T_N + R_N.$$

Further we set for each j and N , $D_j^N := \sum_{|j'-j| \leq N} D_{j'}$, so that $T_N = \sum_j D_j^N D_j$. The following Calderón's identity formula has been proven in the context of more general partitions of the unity in [43, 44].

Theorem 3.2. *The operator T_N converges to the identity in $L^2(X)$ as $N \rightarrow +\infty$. In particular for N large, T_N is invertible. Furthermore for such N 's, if we set $\tilde{D}_j := (T_N)^{-1} D_j^N$ and $\tilde{\tilde{D}}_j := D_j^N (T_N)^{-1}$, we then have*

$$I = \sum_{j \in \mathbb{N}} \tilde{D}_j D_j = \sum_{j \in \mathbb{N}} D_j \tilde{\tilde{D}}_j,$$

in the strong operator topology on $L^2(X)$.

As before, the next step to create a wavelet frame is to decompose X in the same way as we do with the dyadic cubes for $\mathbb{R}^{\mathbf{d}}$. This type of constructions on spaces of homogeneous type have been first given by Christ [17] as a decomposition up to sets of measure zero, and more recently in an exact form by Hytönen and Kairema [48]. We present the later.

Theorem 3.3. *Let $\delta > 0$ be a constant such that $\delta \leq \frac{1}{12\mathbf{A}^3}$ where \mathbf{A} is the constant of (3.1). For each $j \in \mathbb{N}$ let $\{x_{j,k}\}_{k \in I_j}$ be a countable family of points in X maximal with respect to the condition*

$$\rho(x_{j,k}, x_{j,k'}) \geq \delta^j, \quad k \neq k'. \quad (3.5)$$

Then, there exists a countable family of measurable sets $\{Q_{j,k}\}_{k \in I_j}$ with the properties

— for every $j \in \mathbb{N}$

$$X = \bigcup_{k \in I_j} Q_{j,k},$$

— there are constants $r_0, r_1 > 0$ such that for all pairs (j, k)

$$B(x_{j,k}, r_0 \delta^j) \subset Q_{j,k} \subset B(x_{j,k}, r_1 \delta^j), \quad (3.6)$$

— if $i \geq j$ then

$$\text{either } Q_{i,k'} \subset Q_{j,k} \text{ or } Q_{i,k'} \cap Q_{j,k} = \emptyset. \quad (3.7)$$

From now on we fix a value $\delta \in (0, \frac{1}{12A^3}]$ and call it the *dyadic parameter* of X . Also, we set the same parameter for the construction of the operators S_j . The sets $Q_{j,k}$ are called *dyadic cubes* and we will refer to the points $x_{j,k}$ as their *centers*. In short, we might also refer to the set of points $\{x_{j,k}\}_{j \in \mathbb{N}, k \in I_j}$ as the dyadic partition. We say that the partition is *nested* if $\{x_{j,k}\}_{k \in I_j} \subset \{x_{j+1,k}\}_{k \in I_{j+1}}$ for all $j \in \mathbb{N}$ (this can be constructed inductively by using Zorn's Lemma over (3.5)).

Now, from Theorem 3.2 we know that, taking N large enough, for any $f \in L^2(X)$

$$f(x) = \sum_{j \in \mathbb{N}} \int_X D_j^N(x, y) D_j f(y) d\mu(y),$$

which, by doing a Riemann sum over a dyadic partition, gives the following discrete version

$$f(x) \approx \sum_{j \in \mathbb{N}} \sum_{k \in I_j} \mu(Q_{j,k}) D_j^N(x, x_{j,k}) D_j f(x_{j,k}). \quad (3.8)$$

Actually, for the sum on the right to converge, a more refined partition is necessary: this idea is what leads to the construction of a wavelet system on $L^2(X)$ [21, §3.5].

Theorem 3.4. *Let $\{x_{j,k}\}_{j,k}$ be a dyadic partition. There exists two families of functions $\{\varphi_{j,k}\}_{j,k}$, $\{\tilde{\varphi}_{j,k}\}_{j,k}$ and a constant $C_\varphi > 0$ such that*

— (Size condition) for all $j \in \mathbb{N}$, $k \in I_j$

$$\varphi_{j,k}(x) = 0 \text{ if } \rho(x, x_{j,k}) \geq C_\varphi \delta^j \quad \text{and} \quad \|\varphi_{j,k}\|_\infty \leq C_\varphi, \quad (3.9)$$

and for all $j \in \mathbb{N}$, $k \in I_j$, $0 < \theta' < \theta$ there exists $C_{\theta'}$ such that

$$|\tilde{\varphi}_{j,k}(x)| \leq C_\varphi C_{\theta'} \left(\frac{\delta^j}{\delta^j + \rho(x, x_{j,k})} \right)^{\mathbf{d} + \theta'} \quad (3.10)$$

— (Smoothness condition) for all $j \in \mathbb{N}$, $k \in I_j$

$$|\varphi_{j,k}(x) - \varphi_{j,k}(x')| \leq C_\varphi \delta^{-j\theta} \rho(x, x')^\theta, \quad \forall x, x' \in X, \quad (3.11)$$

and for all $j \in \mathbb{N}$, $k \in I_j$ and $0 < \theta' < \theta$ there exists $C_{\theta'}$ such that

$$|\tilde{\varphi}_{j,k}(x) - \tilde{\varphi}_{j,k}(x')| \leq C_\varphi C_{\theta'} \left(\frac{\rho(x, x')}{\delta^j + \rho(x, x_{j,k})} \right)^{\theta'} \left(\frac{\delta^j}{\delta^j + \rho(x, x_{j,k})} \right)^{\mathbf{d} + \theta'},$$

if $\rho(x, x') \leq \frac{1}{2\mathbf{A}}(\delta^j + \rho(x, x_{j,k}))$,

— (Mean zero condition) there exists $j_\varphi \in \mathbb{N}$ such that

$$\int_X \varphi_{j,k}(x) d\mu(x) = \int_X \tilde{\varphi}_{j,k}(x) d\mu(x) = 0, \quad \forall j \geq j_\varphi, k \in I_j, \quad (3.12)$$

— the families are dual in the sense that

$$f = \sum_{j \in \mathbb{N}} \sum_{k \in I_j} \delta^{-j\mathbf{d}} \langle f, \tilde{\varphi}_{j,k} \rangle \varphi_{j,k} = \sum_{j \in \mathbb{N}} \sum_{k \in I_j} \delta^{-j\mathbf{d}} \langle f, \varphi_{j,k} \rangle \tilde{\varphi}_{j,k}$$

holds true for any $f \in L^2(X)$.

From now on, write the *wavelet decomposition* of $f \in L^2(X)$ as

$$f = \sum_{j \in \mathbb{N}} \sum_{k \in I_j} q_{j,k}(f) \varphi_{j,k} \quad (3.13)$$

where

$$q_{j,k}(f) := \delta^{-j\mathbf{d}} \langle f, \tilde{\varphi}_{j,k} \rangle. \quad (3.14)$$

Note that here, in contrast to the previous chapter, we are “normalizing” wavelets in the L^∞ -norm instead of the L^2 -norm. The reason for this rather cosmetic change, is that certain regularity properties we will study in the next chapter are better described by the coefficients set as in (3.14).

As (3.8) suggest the wavelets are defined by $\varphi_{j,k}(x) := \delta^{j\mathbf{d}} D_j(x_{j,k}, x)$, and thus its properties come from those of the kernels $D_j(x, y)$ (cf. Proposition 3.1). But since (3.8) is not exact, the dual functions $\tilde{\varphi}_{j,k}$ cannot be set as $D_j^N(x, x_{j,k})$ and instead, we have to use the inverse of the approximation operator (which is a sum of T_N and the difference with the Riemann sum). This is the reason why the compact support property cannot be retained for the functions $\tilde{\varphi}_{j,k}$ and instead we get a rapid decay around the point $x_{j,k}$.

Like the wavelet system we constructed for a compact manifold, here the family $\{\varphi_{j,k}\}_{j,k}$ is not necessarily an orthogonal system nor a tight frame. In particular

their supports are not necessarily disjoint. However, the size condition (3.9) together with (3.6) and (3.7) imply that there exist a constant $T = T(\varphi) > 0$ for which given any $j \in \mathbb{Z}$ the supports of $\{\varphi_{j,k}\}_{k \in I_j}$ have finite multiplicity T .

Lastly, let us mention that the weaker size condition for the dual family functions (3.10) here also gives a similar inequality to (2.16) for manifolds. Indeed, we have the following:

Lemma 3.5. *For each $j \in \mathbb{N}$, $k \in I_j$ let $\phi_{j,k}(x) := \frac{\delta^j}{\delta^j + \rho(x, x_{j,k})}$ and take $N > \mathbf{d}$. The following properties are verified*

— for any $1 \leq p < \infty$

$$\|(\phi_{j,k})^N\|_p \lesssim \delta^{j\mathbf{d}/p},$$

— for all $x \in X$

$$\sum_{k \in I_j} \phi_{j,k}(x)^N \lesssim 1.$$

Proof. Take any $1 \leq p < \infty$, and divide X for each j, k as

$$X = B(x_{j,k}, \delta^j) \cup \left(\bigcup_{j' < j} B(x_{j,k}, \delta^{j'}) \setminus B(x_{j,k}, \delta^{j'+1}) \right).$$

We then write

$$\begin{aligned} \|(\phi_{j,k})^N\|_p^p &= \int_X \left(\frac{\delta^j}{\delta^j + \rho(x, x_{j,k})} \right)^{Np} d\mu(x) \\ &\leq \int_{B(x_{j,k}, \delta^j)} d\mu(x) + \sum_{j' < j} \int_{B(x_{j,k}, \delta^{j'}) \setminus B(x_{j,k}, \delta^{j'+1})} \delta^{jNp} \rho(x, x_{j,k})^{-Np} d\mu(x) \\ &\lesssim \delta^{j\mathbf{d}} \end{aligned}$$

by using (3.2) (here we use the hypothesis $N > \mathbf{d}$, $p \geq 1$). This proves the first inequality. Let us now prove the second one. Fix $x \in X$. By using (3.6) and the quasi-triangle inequality we have that

$$\delta^j + \rho(x, x_{j,k}) \gtrsim \delta^j + \rho(x, y), \quad \forall y \in B(x_{j,k}, r_0 \delta^j).$$

Next, running the sum over $k \in I_j$ and using (3.7) we get

$$\begin{aligned} \sum_{k \in I_j} \phi_{j,k}(x)^N &= \sum_{k \in I_j} \left(\frac{\delta^j}{\delta^j + \rho(x, x_{j,k})} \right)^N \\ &\lesssim \delta^{-j\mathbf{d}} \sum_{k \in I_j} \int_{B(x_{j,k}, r_0 \delta^j)} \left(\frac{\delta^j}{\delta^j + \rho(x, y)} \right)^N d\mu(y) \\ &\leq \delta^{-j\mathbf{d}} \int_X \left(\frac{\delta^j}{\delta^j + \rho(x, y)} \right)^N d\mu(y) \lesssim 1, \end{aligned}$$

by using the first inequality we proved. \square

As a simple but useful corollary from we obtain, using (3.10) with any $0 < \theta' < \theta$,

$$\|\tilde{\varphi}_{j,k}\|_p \lesssim \delta^{j\mathbf{d}/p}, \quad (3.15)$$

$$\sum_{k \in I_j} \tilde{\varphi}_{j,k}(x) \lesssim 1, \quad \forall x \in X. \quad (3.16)$$

for all $1 \leq p < \infty$ and where $\tilde{\varphi}_{j,k}$ is any function of a dual family like in Theorem 3.4.

3.2 Besov spaces

Let us now define Besov spaces in this context. Following [47], given $1 \leq p, q \leq \infty$ and $s \in \mathbb{R}$ we will say that a function $f \in L^1_{loc}(X)$ belongs to the Besov space $B^s_{p,q}(X)$ if the norm

$$\|f\|_{B^s_{p,q}} := \left[\sum_{j \in \mathbb{N}} \delta^{-jq(s-\mathbf{d}/p)} \left(\sum_{k \in I_j} |q_{j,k}(f)|^p \right)^{q/p} \right]^{1/q}$$

is finite, with the usual adaptations when $p, q = \infty$ and where $q_{j,k}(f)$ are the coefficients defined in (3.14) and δ is the dyadic parameter as before.

This definition of Besov spaces arises from the one that can be given in terms of the partitions of the unity D_j , see e.g. [21, 47]. Here again, other characterizations have been given, see e.g. [69] for a definition of Besov spaces based in modulus of continuity or [38] for a version given by an interpolation space between L^p and an adequate Sobolev-type space. Furthermore, when $s \in (0, \theta)$ all this definitions are equivalent, and thus in particular the definition of $B^s_{p,q}$ is independent from

the chosen wavelet system (we can even replace the coefficients $q_{j,k}$ with those of the dual basis representation). This restriction on the regularity index s is the main difference with the previous chapter: while now we are working with a larger family of underlying spaces, the results apply to a smaller class of Besov spaces.

Also, from this definition, we get the following embedding

$$B_{p,q_1}^s(X) \subset B_{p,q_2}^s(X), \quad s \in \mathbb{R}, 1 \leq p \leq \infty, 1 \leq q_1 \leq q_2 \leq \infty, \quad (3.17)$$

which then yields

$$B_{p,q_1}^{s+\varepsilon}(X) \subset B_{p,q_2}^s(X), \quad s \in \mathbb{R}, \varepsilon > 0, 1 \leq p \leq \infty, 1 \leq q_1, q_2 \leq \infty.$$

And further from Littlewood-Paley theory one can obtain $B_{p,1}^0(X) = L^p(X)$ for any $1 \leq p \leq \infty$. Altogether this implies that

$$B_{p,1}^s(X) \subset L^p(X), \quad \forall s > 0, 1 \leq p \leq \infty. \quad (3.18)$$

The proof for any of these results can be found in e.g. [46, 47].

3.2.1 Sampling result

We now present the sampling inequality in this context.

Theorem 3.6. *Let $1 \leq p < \infty$, and set $\alpha = \max(1, \frac{\mathbf{d}}{\theta p})$, $\beta = \frac{1}{\ln(1/\delta)} \max(\frac{p}{\mathbf{d}}, \frac{1}{\theta})$. For every $l \in \mathbb{N}$, fix a collection of dyadic cubes $\{Q_{l,n}\}_{n \in I_l}$ with centers $\{a_{l,n}\}_{n \in I_l}$.*

Then, given $0 < \varepsilon < 1$ and $K > 0$, there exist a constant $\kappa = \kappa(p, \mathbf{d}, \theta)$ such that if $l \geq \beta \ln(\frac{\kappa K}{\varepsilon^\alpha})$

$$\left(\int_X \left| f(x) - \sum_{n \in I_l} f(a_{l,n}) \mathbb{1}_{Q_{l,n}}(x) \right|^p d\mu(x) \right)^{1/p} \leq \varepsilon \|f\|_{L^p}$$

holds true for all $f \in B_{p,1}^{\mathbf{d}/p}(X)$ with $\|f\|_{B_{p,1}^{\mathbf{d}/p}} \leq K \|f\|_{L^p}$. In particular, this implies that

$$(1 - \varepsilon) \|f\|_{L^p} \leq \left(\sum_{n \in I_l} |f(a_{l,n}) \mu(Q_{l,n})|^p \right)^{1/p} \leq (1 + \varepsilon) \|f\|_{L^p}$$

whenever $l \geq \beta \ln(\frac{\kappa K}{\varepsilon^\alpha})$ and $f \in B_{p,1}^{\mathbf{d}/p}(X)$ is such that $\|f\|_{B_{p,1}^{\mathbf{d}/p}} \leq K \|f\|_{L^p}$.

Proof. Let us first note that

$$\begin{aligned} & \left(\int_X \left| f(x) - \sum_{n \in I_l} f(a_{l,n}) \mathbb{1}_{Q_{l,n}}(x) \right|^p d\mu(x) \right)^{1/p} \\ &= \left\| \left(\int_{Q_{l,n}} |f(x) - f(a_{l,n})|^p d\mu(x) \right)^{1/p} \right\|_{\ell^p_{I_l}} \end{aligned} \quad (3.19)$$

so the L^p -norm of $f(x) - \sum_{n \in I_l} f(a_{l,n}) \mathbb{1}_{Q_{l,n}}(x)$ can be computed as the ℓ^p -norm of the sequence $\left\{ \|f - f(a_{l,n})\|_{L^p(Q_{l,n})} \right\}_{n \in I_l}$.

Now, take an arbitrary $n \in I_l$ and consider $x \in Q_{l,n}$. From (3.13) we may write

$$f(x) - f(a_{l,n}) = \sum_{j \in \mathbb{N}} \sum_{k \in I_j} q_{j,k}(f) (\varphi_{j,k}(x) - \varphi_{j,k}(a_{n,l})) \quad (3.20)$$

where $\{\varphi_{j,k}\}_{j,k}$ is a wavelet system like in Theorem 3.4 and $q_{j,k}(f)$ are the coefficients defined in (3.14). The convergence (in the L^∞ norm) of the sum on (3.20) is assured by the computations below.

Recall that by (3.9) if $\varphi_{j,k}(x) \neq 0$ then $\rho(x, x_{j,k}) \leq C_\varphi \delta^j$, where $\{x_{j,k}\}_{j,k}$ is the associated dyadic partition. We then introduce

$$I_{j,x} := \{k \in I_j : \rho(x, x_{j,k}) \leq C_\varphi \delta^j\}, \quad I_{j,n,x} := I_{j,a_{l,n}} \cup I_{j,x}, \quad I_{j,n} := \bigcup_{x \in Q_{l,n}} I_{j,x}.$$

First note that if $k \in I_{j,x}$ and $z \in Q_{j,k}$, then from (3.1) and (3.6) we get

$$\rho(x, z) \leq \mathbf{A}(\rho(z, x_{j,k}) + \rho(x, x_{j,k})) \leq \mathbf{A}(C_\varphi + r_1) \delta^j.$$

Therefore $Q_{j,k} \subset B(x, C\delta^j)$ with $C = \mathbf{A}(C_\varphi + r_1)$. But since the cubes $Q_{j,k}$'s are disjoint of volume $\approx \delta^{j\mathbf{d}}$ a measure counting argument shows that $\#I_{j,x} \cdot \delta^{j\mathbf{d}} \lesssim \delta^{j\mathbf{d}}$, and thus $I_{j,x}$ is a finite set with $\#I_{j,x} \lesssim 1$ where the implicit constant is independent from j and x .

Further, let us introduce $E_{j,n}(f) := \left(\sum_{k \in I_{j,n}} |q_{j,k}(f)|^p \right)^{1/p}$. By applying Hölder's inequality in (3.20) we obtain

$$\begin{aligned} |f(x) - f(a_{l,n})| &\leq \sum_{j \in \mathbb{N}} \sup_{k \in I_j} |\varphi_{j,k}(x) - \varphi_{j,k}(a_{l,n})| \sum_{k \in I_{j,n,x}} |q_{j,k}(f)| \\ &\lesssim \sum_{j \in \mathbb{N}} \sup_{k \in I_j} |\varphi_{j,k}(x) - \varphi_{j,k}(a_{l,n})| E_{j,n}(f). \end{aligned} \quad (3.21)$$

From (3.9) and (3.11) we know that

$$|\varphi_{j,k}(x) - \varphi_{j,k}(a_{l,n})| \leq \begin{cases} 2C_\varphi, \\ C_\varphi \delta^{-j\theta} \rho(x, a_{l,n})^\theta. \end{cases}$$

The second inequality improves over the first one when $\delta^{-j}\rho(x, a_{l,n}) \lesssim 1$. Then we can split the sum of (3.21) in two parts to obtain

$$|f(x) - f(a_{l,n})| \lesssim \sum_{j \leq l} \delta^{-j\theta} \rho(x, a_{l,n})^\theta E_{j,n}(f) + \sum_{j > l} E_{j,n}(f).$$

Next, taking the L^p -norm over $Q_{l,n}$ and using the triangle inequality, we get

$$\begin{aligned} \left(\int_{Q_{l,n}} |f(x) - f(a_{l,n})|^p \, d\mu(x) \right)^{1/p} &\lesssim \sum_{j \leq l} \delta^{-j\theta} \left(\int_{Q_{l,n}} \rho(x, a_{l,n})^{\theta p} \, d\mu(x) \right)^{1/p} E_{j,n}(f) \\ &\quad + \sum_{j > l} \mu(Q_{l,n})^{1/p} E_{j,n}(f) \\ &\lesssim \sum_{j \leq l} \delta^{-j\theta} \delta^{l(\theta + \mathbf{d}/p)} E_{j,n}(f) + \sum_{j > l} \delta^{l\mathbf{d}/p} E_{j,n}(f) \end{aligned}$$

where we used that $\mu(Q_{l,n}) \subset B(a_{n,l}, r_1 \delta^{l\mathbf{d}})$ in the last inequality. So when we take the ℓ^p -norm over I_l we have that

$$\begin{aligned} \left\| \left(\int_{Q_{l,n}} |f(x) - f(a_{l,n})|^p \, d\mu(x) \right)^{1/p} \right\|_{\ell_{I_l}^p} &\lesssim \sum_{j \leq l} \delta^{-j\theta} \delta^{l(\theta + \mathbf{d}/p)} \|E_{j,n}(f)\|_{\ell_{I_l}^p} \\ &\quad + \sum_{j > l} \delta^{l\mathbf{d}/p} \|E_{j,n}(f)\|_{\ell_{I_l}^p}. \end{aligned} \quad (3.22)$$

To estimate the ℓ^p norm of $E_{j,n}(f)$ we write for each $k \in I_j$, $\Lambda_{j,k,l} := \{n \in I_l : k \in I_{j,n}\}$, so that

$$\begin{aligned} \|E_{j,n}(f)\|_{\ell_{I_l}^p} &= \left(\sum_{n \in I_l} (E_{j,n}(f))^p \right)^{1/p} = \left(\sum_{n \in I_l} \sum_{k \in I_{j,n}} |q_{j,k}(f)|^p \right)^{1/p} \\ &= \left(\sum_{k \in I_j} \sum_{n \in \Lambda_{j,k,l}} |q_{j,k}(f)|^p \right)^{1/p}. \end{aligned} \quad (3.23)$$

The same arguments to estimate the cardinal of $I_{j,n,x}$ imply that

$$\#(\Lambda_{j,k,l}) \lesssim \begin{cases} \delta^{-(l-j)\mathbf{d}} & \text{if } j \leq l, \\ 1 & \text{if } j > l. \end{cases}$$

Plugging this into (3.23) gives us

$$\|E_{j,n}(f)\|_{\ell_{I_l}^p} \lesssim \begin{cases} \delta^{-(l-j)\mathbf{d}/p} \left(\sum_{k \in I_j} |q_{j,k}(f)|^p \right)^{1/p} & \text{if } j \leq l, \\ \left(\sum_{k \in I_j} |q_{j,k}(f)|^p \right)^{1/p} & \text{if } j > l, \end{cases}$$

and going back to (3.22) we obtain that

$$\begin{aligned} & \left\| \left(\int_{Q_{l,n}} |f(x) - f(a_{l,n})|^p \, d\mu(x) \right)^{1/p} \right\|_{\ell_{I_l}^p} \\ & \lesssim \sum_{j \leq l} \delta^{-j(\theta - \mathbf{d}/p) + l\theta} \left(\sum_{k \in I_j} |q_{j,k}(f)|^p \right)^{1/p} + \sum_{j > l} \delta^{l\mathbf{d}/p} \left(\sum_{k \in I_j} |q_{j,k}(f)|^p \right)^{1/p} \\ & \qquad \qquad \qquad =: I + II. \end{aligned} \quad (3.24)$$

The second term on the right-hand-side is simply bounded by

$$II \lesssim \delta^{l\mathbf{d}/p} \|f\|_{B_{p,1}^{\mathbf{d}/p}}. \quad (3.25)$$

As for the first term, we divide the sum over $j \leq l$ into two: $j < j_0$ and $j_0 \leq j \leq l$, where $j_0 < l$ is to be fixed later. We then write

$$\begin{aligned} I &= \sum_{j < j_0} \delta^{l\theta} \delta^{-j(\theta - \mathbf{d}/p)} \left(\sum_{k \in I_j} |q_{j,k}(f)|^p \right)^{1/p} + \sum_{j=j_0}^l \delta^{l\theta} \delta^{-j(\theta - \mathbf{d}/p)} \left(\sum_{k \in I_j} |q_{j,k}(f)|^p \right)^{1/p} \\ & \qquad \qquad \qquad =: I_a + I_b. \end{aligned}$$

For the sum I_a , expanding $q_{j,k}(f)$ and applying Hölder's inequality reads

$$\begin{aligned} I_a &= \sum_{j < j_0} \delta^{l\theta} \delta^{-j(\theta + \mathbf{d} - \mathbf{d}/p)} \left(\sum_{k \in I_j} |\langle f, \tilde{\varphi}_{j,k} \rangle|^p \right)^{1/p} \\ & \leq \sum_{j < j_0} \delta^{l\theta} \delta^{-j(\theta + \mathbf{d} - \mathbf{d}/p)} \left(\sum_{k \in I_j} \int_X |\tilde{\varphi}_{j,k}(x)| |f(x)|^p \, d\mu(x) \right. \\ & \qquad \qquad \qquad \left. \times \left(\int_X |\tilde{\varphi}_{j,k}(x)| \, d\mu(x) \right)^{p/p'} \right)^{1/p}. \end{aligned}$$

Using first (3.15) and then (3.16) we then obtain

$$I_a \lesssim \sum_{j < j_0} \delta^{l\theta} \delta^{-j(\theta + \mathbf{d} - \mathbf{d}/p - \mathbf{d}/p')} \left(\int_X \sum_{k \in I_j} |\tilde{\varphi}_{j,k}(x)| |f(x)|^p \, d\mu(x) \right)^{1/p} \\ \lesssim \sum_{j < j_0} \delta^{-(j-l)\theta} \|f\|_{L^p}.$$

Running the sum over $j < j_0$ finally gives us $I_a \lesssim \delta^{-(j_0-l)\theta} \|f\|_{L^p}$. And for I_b we have

$$I_b \lesssim \delta^{l\theta} (\delta^{-j_0(\theta - \mathbf{d}/p)} + \delta^{-l(\theta - \mathbf{d}/p)}) \|f\|_{B_{p,1}^{\mathbf{d}/p}} \\ \lesssim \delta^{l\mathbf{d}/p} (\delta^{-(j_0-l)(\theta - \mathbf{d}/p)} + 1) \|f\|_{B_{p,1}^{\mathbf{d}/p}}.$$

Altogether we get

$$I \lesssim \delta^{-(j_0-l)\theta} \|f\|_{L^p} + \delta^{l\mathbf{d}/p} (\delta^{-(j_0-l)(\theta - \mathbf{d}/p)} + 1) \|f\|_{B_{p,1}^{\mathbf{d}/p}}. \quad (3.26)$$

Adding (3.25) and (3.26) in (3.24) yields

$$\left\| \left(\int_{Q_{l,n}} |f(x) - f(a_{l,n})|^p \, d\mu(x) \right)^{1/p} \right\|_{\ell_{I_l}^p} \\ \lesssim \delta^{-(j_0-l)\theta} \|f\|_{L^p} + \delta^{l\mathbf{d}/p} (\delta^{-(j_0-l)(\theta - \mathbf{d}/p)} + 2) \|f\|_{B_{p,1}^{\mathbf{d}/p}}.$$

But as we saw at the beginning in (3.19) this is the same to say that

$$\left(\int_X \left| f(x) - \sum_{n \in I_l} f(a_{l,n}) \mathbb{1}_{Q_{l,n}}(x) \right|^p \, d\mu(x) \right)^{1/p} \\ \lesssim \delta^{-(j_0-l)\theta} \|f\|_{L^p} + \delta^{l\mathbf{d}/p} (\delta^{-(j_0-l)(\theta - \mathbf{d}/p)} + 2) \|f\|_{B_{p,1}^{\mathbf{d}/p}}. \quad (3.27)$$

Denote R the implicit constant in (3.27) and choose $j_0 := l - \frac{\ln(2R/\varepsilon)}{\theta \ln \delta}$ so that $R\delta^{-(j_0-l)\theta} \leq \frac{\varepsilon}{2}$.

If $p \geq \mathbf{d}/\theta$ we have that $R\delta^{l\mathbf{d}/p} (\delta^{-(j_0-l)(\theta - \mathbf{d}/p)} + 2) \leq 3R\delta^{l\mathbf{d}/p}$, and thus taking $l \geq \frac{p}{d \ln(1/\delta)} \ln \left(\frac{6R\|f\|_{B_{p,1}^{\mathbf{d}/p}}}{\varepsilon\|f\|_{L^p}} \right)$, (3.27) reduces to

$$\left(\int_X \left| f(x) - \sum_{n \in I_l} f(a_{l,n}) \mathbb{1}_{Q_{l,n}}(x) \right|^p \, d\mu(x) \right)^{1/p} \leq \varepsilon \|f\|_{L^p} \quad (3.28)$$

from where the theorem follows.

If $p < \mathbf{d}/\theta$, then

$$\begin{aligned} R\delta^{\mathbf{d}/p} (\delta^{-(j_0-l)(\theta-\mathbf{d}/p)} + 2) &\leq 2R\delta^{\mathbf{d}/p} \delta^{-(j_0-l)(\theta-\mathbf{d}/p)} \\ &\leq 2R \max \left(\delta^{l\theta}, \left(\frac{\varepsilon}{2R} \right)^{1-\frac{\mathbf{d}}{\theta p}} \delta^{\mathbf{d}/p} \right) \\ &\leq \frac{\kappa}{2} \varepsilon^{1-\frac{\mathbf{d}}{\theta p}} \delta^{l\theta} \end{aligned}$$

with $\kappa := 4R \max \left(1, (2R)^{\frac{\mathbf{d}}{\theta p}-1} \right)$. Thus, taking $l \geq \frac{1}{\theta \ln(1/\delta)} \ln \left(\frac{\kappa \|f\|_{B_{p,1}^{\mathbf{d}/p}}}{\varepsilon^{\frac{\mathbf{d}}{\theta p}} \|f\|_{L^p}} \right)$ we again obtain (3.28). \square

3.2.2 Trace theorem

Let us now consider a subset $Y \subset X$ such that there exist a measure μ_Y for which $(Y, \rho|_{Y \times Y}, \mu_Y)_{\mathbf{d}_Y, \theta}$ is a space of homogeneous type. Remark that we are keeping the quasi-metric ρ and thus we can take the same θ in (3.3) and the same δ of Theorem 3.3 for both X and Y (we could have also used in Y a quasi-metric $\tilde{\rho}$ equivalent to $\rho|_{Y \times Y}$; we choose $\rho|_{Y \times Y}$ for simplicity). In addition, and to avoid any confusion, in this subsection we will denote μ_X to the measure on X . Further we will use $\mathbf{d}_X, \mathbf{d}_Y$ for the dimensions of X and Y respectively.

Lastly, we will assume the dyadic partitions arising from Theorem 3.3 in X and Y are constructed in the following way: first for each $j \in \mathbb{N}$ we take a countable family of points $\{z_{j,k}\}_{k \in I_j(Y)}$ in Y maximal with respect to the condition (3.5) in Y ; next we extend each of this families to a family of points $\{z_{j,k}\}_{k \in I_j(X)}$ again maximal with respect to (3.5) in X .

For the wavelet systems we will use the notation $\{\psi_{X,j,k}\}_{j \in \mathbb{N}, k \in I_j(X)}$ and $\{\psi_{Y,j',k'}\}_{j' \in \mathbb{N}, k' \in I_{j'}(Y)}$.

With this setting we can obtain what has been called an *almost orthogonal property*: we compare wavelets on Y against those on X following the ideas of a similar result in [21] when comparing wavelets defined on the same space.

Lemma 3.7. *Let j_{ψ_Y} be the index of the mean zero property (3.12) in Y . Then*

(i) *For all pairs $j \in \mathbb{N}, k \in I_j(X)$ and $j' \geq j_{\psi_Y}, k' \in I_{j'}(Y)$*

$$|\langle \psi_{X,j,k}, \psi_{Y,j',k'} \rangle_Y| \lesssim \delta^{|j'-j|\theta + (j+j'-j \wedge j')\mathbf{d}_Y},$$

where $j \wedge j' := \min(j, j')$. When $j' < j_{\psi_Y}$ we have $|\langle \psi_{X,j,k}, \psi_{Y,j',k'} \rangle_Y| \lesssim \delta^{(j+j'-j \wedge j')\mathbf{d}_Y}$.

(ii) For all $j \in \mathbb{N}, k \in I_j(X)$ and $j' \in \mathbb{N}, k' \in I_{j'}(Y)$, the sets

$$\begin{aligned} I_j(X, j', k') &:= \{\tilde{k} \in I_j(X) : \langle \psi_{X,j,\tilde{k}}, \psi_{Y,j',k'} \rangle_Y \neq 0\}, \\ I_{j'}(Y, j, k) &:= \{\tilde{k} \in I_{j'}(Y) : \langle \psi_{X,j,k}, \psi_{Y,j',\tilde{k}} \rangle_Y \neq 0\}, \end{aligned}$$

are finite with $\#I_j(X, j', k') \lesssim \delta^{(j \wedge j' - j)\mathbf{d}_Y}$ and $\#I_{j'}(Y, j, k) \lesssim \delta^{(j \wedge j' - j')\mathbf{d}_Y}$.

Proof. Let us begin with (i) and assume first $j' \geq j_{\psi_Y}$ so that $\psi_{Y,j',k'}$ has mean zero on Y . When $j' \geq j$ we write

$$\begin{aligned} \langle \psi_{X,j,k}, \psi_{Y,j',k'} \rangle_Y &= \int_Y \psi_{X,j,k}(y) \psi_{Y,j',k'}(y) \, d\mu_Y(y) \\ &= \int_Y (\psi_{X,j,k}(y) - \psi_{X,j,k}(z_{j',k'})) \psi_{Y,j',k'}(y) \, d\mu_Y(y). \end{aligned}$$

Next, using (3.9) for $\psi_{Y,j',k'}$ and (3.11) for $\psi_{X,j,k}$ we obtain

$$\begin{aligned} |\langle \psi_{X,j,k}, \psi_{Y,j',k'} \rangle_Y| &\lesssim \int_{B_Y(z_{j',k'}, C_\psi \delta^{j'})} \delta^{-j\theta} \rho(y, z_{j',k'})^\theta \, d\mu_Y(y) \\ &\lesssim \delta^{-j\theta} \delta^{j'\theta} \delta^{j'\mathbf{d}_Y} = \delta^{|j'-j|\theta + (j+j'-j \wedge j')\mathbf{d}_Y} \end{aligned}$$

since $j' \geq j$. Now consider the case $j > j'$ (and $j' \geq j_{\psi_Y}$). Since $\psi_{X,j,k}$ is supported on $B_X(z_{j,k}, C_\psi \delta^j)$, and $\psi_{Y,j',k'}$ has mean zero on Y we may write

$$\begin{aligned} \langle \psi_{X,j,k}, \psi_{Y,j',k'} \rangle_Y &= \int_{B_X(z_{j,k}, C_\psi \delta^j) \cap Y} \psi_{X,j,k}(y) \psi_{Y,j',k'}(y) \, d\mu_Y(y) \\ &= \int_{B_X(z_{j,k}, C_\psi \delta^j) \cap Y} \left(\psi_{X,j,k}(y) - \int_{B_X(z_{j,k}, C_\psi \delta^j) \cap Y} \psi_{X,j,k}(z) \, d\mu_Y(z) \right) \\ &\quad \times \psi_{Y,j',k'}(y) \, d\mu_Y(y), \end{aligned}$$

where we are using the notation $\int_Q f(z) \, d\mu_Y(z) = \frac{1}{\mu_Y(Q)} \int_Q f(z) \, d\mu_Y(z)$. Then, we can repeat the argument with the function in brackets (since it also has mean zero) to obtain:

$$\begin{aligned} \langle \psi_{X,j,k}, \psi_{Y,j',k'} \rangle_Y &= \int_{B_X(z_{j,k}, C_\psi \delta^j) \cap Y} \psi_{X,j,k}(y) \psi_{Y,j',k'}(y) \, d\mu_Y(y) \\ &= \int_{B_X(z_{j,k}, C_\psi \delta^j) \cap Y} \left(\psi_{X,j,k}(y) - \int_{B_X(z_{j,k}, C_\psi \delta^j) \cap Y} \psi_{X,j,k}(z) \, d\mu_Y(z) \right) \\ &\quad \times (\psi_{Y,j',k'}(y) - \psi_{Y,j',k'}(z_{j,k})) \, d\mu_Y(y). \end{aligned}$$

Using (3.9) for $\psi_{X,j,k}$ and (3.11) for $\psi_{Y,j',k'}$ we obtain

$$|\langle \psi_{X,j,k}, \psi_{Y,j',k'} \rangle_Y| \lesssim \int_{B_X(z_{j,k}, C_\psi \delta^j) \cap Y} \delta^{-j'\theta} \rho(y, z_{j,k})^\theta d\mu_Y(y).$$

Assuming $B_X(z_{j,k}, C_\psi \delta^j) \cap Y \neq \emptyset$ we can then find $y_{j,k} \in Y$ such that $B_X(z_{j,k}, C_\psi \delta^j) \cap Y$ is contained in $B_Y(y_{j,k}, 2\mathbf{A}C_\psi \delta^j)$. Thus

$$\begin{aligned} |\langle \psi_{X,j,k}, \psi_{Y,j',k'} \rangle_Y| &\lesssim \int_{B_Y(y_{j,k}, 2\mathbf{A}C_\psi \delta^j)} \delta^{-j'\theta} \rho(y, z_{j,k})^\theta d\mu_Y(y) \\ &\lesssim \delta^{-j'\theta} \delta^{j(\mathbf{d}_Y + \theta)} = \delta^{|j' - j|\theta + (j + j' - j \wedge j')\mathbf{d}_Y} \end{aligned}$$

since $j > j'$. The case for $j' < j_{\psi_Y}$ follows by the same techniques but only using (3.9).

We move to (ii). To begin with, note that we clearly have

$$\begin{aligned} I_j(X, j', k') &\subset \{k \in I_j(X) : \text{supp}(\psi_{X,j,k}) \cap \text{supp}(\psi_{Y,j',k'}) \neq \emptyset\} := \tilde{I}_j(X, j', k'), \\ I_{j'}(Y, j, k) &\subset \{k' \in I_{j'}(Y) : \text{supp}(\psi_{X,j,k}) \cap \text{supp}(\psi_{Y,j',k'}) \neq \emptyset\} := \tilde{I}_j(X, j', k'), \end{aligned}$$

so it suffices to estimate the sizes of $\tilde{I}_j(X, j', k')$ and $\tilde{I}_j(X, j', k')$.

Let us begin by $\tilde{I}_j(X, j', k')$ for a fixed triple $j', j \in \mathbb{N}$ and $k' \in I_{j'}(Y)$. Suppose first $j \geq j'$. Due to (3.9), if $k \in \tilde{I}_j(X, j', k')$, there exists a point $y_{j,k} \in Y$ such that $\rho(y_{j,k}, z_{j',k'}) \lesssim \delta^{j'}$ and $\rho(y_{j,k}, z_{j,k}) \lesssim \delta^j$. Then, as we are supposing $j \geq j'$, there exist a universal constant $C > 0$ such that $B_Y(y_{j,k}, \delta^j) \subset B_Y(z_{j',k'}, C\delta^{j'})$ and $B_X(y_{j,k}, \delta^j) \subset B_X(z_{j,k}, C\delta^j)$. Since the family of balls $\{B_X(z_{j,k}, C\delta^j)\}_{k \in I_j(X)}$ has finite multiplicity, cf. (3.6) and (3.7), then so must the family $\{B_X(y_{j,k}, \delta^j)\}_{k \in \tilde{I}_j(X, j', k')}$ and thus also $\{B_Y(y_{j,k}, \delta^j)\}_{k \in \tilde{I}_j(X, j', k')}$. Summing over k gives us

$$\begin{aligned} \#\tilde{I}_j(X, j', k') \cdot \delta^{j\mathbf{d}_Y} &\lesssim \sum_{k \in \tilde{I}_j(X, j', k')} \mu_Y(B_Y(y_{j,k}, \delta^j)) \\ &\leq \sum_{k \in I_j(X, j', k')} \mu_Y(B_Y(y_{j,k}, \delta^j)) \\ &\lesssim \mu_Y(\cup_{k \in I_j(X, j', k')} B_Y(y_{j,k}, \delta^j)) \\ &\leq \mu_Y(B_Y(z_{j',k'}, C\delta^{j'})) \\ &\lesssim \delta^{j'\mathbf{d}_Y}, \end{aligned}$$

and thus $\#\tilde{I}_j(X, j', k') \lesssim \delta^{(j'-j)\mathbf{d}_Y}$.

If $j' \geq j$, then there exists a constant $C > 0$ such that for all $k \in \tilde{I}_j(X, j', k')$, $B_X(z_{j',k'}, \delta^{j'}) \subset B_X(z_{j,k}, C\delta^j)$. Which, since $B_X(z_{j,k}, C\delta^j)$ has finite multiplicity, reads $\#\tilde{I}_j(X, j', k') \lesssim 1$.

To estimate the size of $\tilde{I}_{j'}(Y, j, k)$ we now fix $j, j' \in \mathbb{N}$ and $k \in I_j(X)$. We assume $\tilde{I}_{j'}(Y, j, k) \neq \emptyset$. Suppose first $j' \geq j$. Since we are assuming $\tilde{I}_{j'}(Y, j, k) \neq \emptyset$, then in particular there exist a point $y_{j,k} \in Y$ such that $\rho(y_{j,k}, z_{j,k}) \lesssim \delta^j$. By (3.9) and using the quasi-triangle inequality (3.1), for any $k' \in \tilde{I}_{j'}(Y, j, k)$ we have that

$$\rho(y_{j,k}, z_{j',k'}) \lesssim \rho(z_{j,k}, z_{j',k'}) + \rho(y_{j,k}, z_{j,k}) \lesssim \delta^j + \delta^j \lesssim \delta^j.$$

This means that for a large enough constant C , $B_Y(z_{j',k'}, \delta^{j'}) \subset B_Y(y_{j,k}, C\delta^j)$ holds for all $k' \in \tilde{I}_{j'}(Y, j, k)$. The same argument as before then yields $\#\tilde{I}_{j'}(Y, j, k) \lesssim \delta^{(j-j')\mathbf{d}_Y}$.

Finally, when $j \geq j'$ we use again the finite multiplicity argument to claim $\#\tilde{I}_{j'}(Y, j, k) \lesssim 1$. \square

Using this comparison we obtain our trace theorem.

Theorem 3.8. *Let $1 \leq p < \infty, 1 \leq q \leq \infty, s \in \mathbb{R}$ and set $s' := s - \frac{\mathbf{d}_X - \mathbf{d}_Y}{p}$. If $0 < s' < \theta$ then the trace operator*

$$\mathrm{Tr} : B_{p,q}^s(X) \rightarrow B_{p,q}^{s'}(Y), \quad \mathrm{Tr}f = f|_Y$$

is well defined and continuous.

Proof. In order to apply Lemma 3.7 we will have to use the following characterizations of Besov norm in X and Y : for $f \in B_{p,q}^s(X)$ we will continue to use as a norm

$$\left[\sum_{j \in \mathbb{N}} \delta^{-jq(s - \mathbf{d}_X/p)} \left(\sum_{k \in I_j} |q_{X,j,k}(f)|^p \right)^{q/p} \right]^{1/q}$$

where $q_{X,j,k}(f) = \delta^{-j\mathbf{d}} \langle f, \tilde{\psi}_{X,j,k} \rangle_X$, and $\tilde{\psi}_{X,j,k}$ is the dual basis. But for a function $f \in B_{p,q}^{s'}(Y)$ we will use instead the dual coefficients, i.e. setting $\tilde{q}_{Y,j,k}(f) := \delta^{-j\mathbf{d}} \langle f, \psi_{Y,j,k} \rangle_Y$ we can use

$$\left[\sum_{j \in \mathbb{N}} \delta^{-jq(s' - \mathbf{d}_Y/p)} \left(\sum_{k \in I_j} |\tilde{q}_{Y,j,k}(f)|^p \right)^{q/p} \right]^{1/q}$$

as a norm in $B_{p,q}^{s'}(Y)$. Indeed, as we mentioned in the beginning of this section when $s' \in (0, \theta)$ it has been proven that this yields an equivalent norm [21].

Now, let $f \in B_{p,q}^s(X)$ and write the wavelet decomposition from (3.13)

$$\begin{aligned} \langle f, \psi_{Y,j',k'} \rangle_Y &= \sum_{j \in \mathbb{N}} \sum_{k \in I_j(X)} q_{X,j,k}(f) \cdot \langle \psi_{X,j,k}, \psi_{Y,j',k'} \rangle_Y \\ &= \sum_{j \in \mathbb{N}} \sum_{k \in I_j(X, j', k')} q_{X,j,k}(f) \cdot \langle \psi_{X,j,k}, \psi_{Y,j',k'} \rangle_Y, \end{aligned}$$

where $I_j(X, j', k')$ is the set defined in Lemma 3.7 (ii). Next, using (i) of Lemma 3.7 we get for all $j' \in \mathbb{N}$, $k' \in I_{j'}(Y)$

$$\begin{aligned} \delta^{-j'(s'-\mathbf{d}_Y/p)} |\tilde{q}_{Y,j',k'}(f)| &\lesssim \sum_{j \in \mathbb{N}} \sum_{k \in I_j(X, j', k')} \delta^{\gamma_{j,j'}} \\ &\quad \times \left(\delta^{-(j-j')\mathbf{d}_Y/p} \delta^{-j(s-\mathbf{d}_X/p)} |q_{X,j,k}(f)| \right) \end{aligned} \quad (3.29)$$

where $\gamma_{j,j'} := -(j' - j)s' + |j' - j|\theta \mathbb{1}_{\{j' \geq j_{\psi_Y}\}} + (j - j \wedge j')\mathbf{d}_Y$. We have also used the identity $s' = s - \frac{\mathbf{d}_X - \mathbf{d}_Y}{p}$. Using the estimate for the size of $I_j(X, j', k')$ we get that

$$\sum_{j \in \mathbb{N}} \sum_{k \in I_j(X, j', k')} \delta^{\gamma_{j,j'}} \lesssim \sum_{j \in \mathbb{N}} \delta^{\gamma_{j,j'}} \delta^{(j \wedge j' - j)\mathbf{d}_Y} \leq \sum_{j \in \mathbb{N}} \delta^{-(j'-j)s' + |j'-j|\theta \mathbb{1}_{\{j' \geq j_{\psi_Y}\}}} \lesssim 1,$$

by using that $s' < \theta$ when $j' \geq j_{\psi_Y}$, and $s' > 0$ when $j < j_{\psi_Y}$. Plugged into (3.29) Hölder's inequality then reads

$$\begin{aligned} \sum_{k' \in I_{j'}(Y)} \left(\delta^{-j'(s'-\mathbf{d}_Y/p)} |\tilde{q}_{Y,j',k'}(f)| \right)^p &\lesssim \sum_{k' \in I_{j'}(Y)} \left(\sum_{j \in \mathbb{N}} \sum_{k \in I_j(X, j', k')} (\delta^{\gamma_{j,j'}})^{1/p'} \right. \\ &\quad \left. \times (\delta^{\gamma_{j,j'}})^{1/p} \delta^{-(j-j')\mathbf{d}_Y/p} \delta^{-j(s-\mathbf{d}_X/p)} |q_{X,j,k}(f)| \right)^p \\ &\lesssim \sum_{k' \in I_{j'}(Y)} \sum_{j \in \mathbb{N}} \delta^{\gamma_{j,j'}} \delta^{-(j-j')\mathbf{d}_Y} \\ &\quad \times \sum_{k \in I_j(X, j', k')} (\delta^{-j(s-\mathbf{d}_X/p)} |q_{X,j,k}(f)|)^p \end{aligned}$$

Using the set $I_{j'}(Y, j, k)$ defined in Lemma 3.7 (ii) together with its size estimate, we can invert the order of summation to obtain

$$\begin{aligned} \sum_{k' \in I_{j'}(Y)} \left(\delta^{-j'(s'-\mathbf{d}_Y/p)} |\tilde{q}_{Y,j',k'}(f)| \right)^p &\lesssim \sum_{j \in \mathbb{N}} \delta^{\gamma_{j,j'}} \delta^{(j \wedge j' - j)\mathbf{d}_Y} \\ &\quad \times \sum_{k \in I_j(X)} (\delta^{-j(s-\mathbf{d}_X/p)} |q_{X,j,k}(f)|)^p. \end{aligned}$$

The final step to estimate the norm of f on $B_{p,q}^{s'}(Y)$ is to use again Hölder's inequality or the sub-additive property for q/p . But in order to do that we need the following two upper bounds which we will prove after showing they imply the result.

Claim. For all $j' \in \mathbb{N}$

$$\sum_{j \in \mathbb{N}} \delta^{\gamma_{j,j'} p} \delta^{(j \wedge j' - j) \mathbf{d}_Y} \lesssim 1,$$

and for all $j \in \mathbb{N}$

$$\sum_{j' \in \mathbb{N}} \delta^{\gamma_{j,j'} p} \delta^{(j \wedge j' - j) \mathbf{d}_Y} \lesssim 1.$$

Indeed, with this claim and applying Hölder's inequality when $q/p > 1$ or the sub-additive property $(a + b)^{q/p} \leq a^{q/p} + b^{q/p}$ when $q/p \leq 1$ we obtain

$$\begin{aligned} \|f\|_{B_{p,q}^{s'}(Y)} &= \left(\sum_{j' \in \mathbb{N}} \left[\sum_{k' \in I_{j'}(Y)} \left(\delta^{-j'(s' - \mathbf{d}_Y/p)} |\langle f, \psi_{Y,j',k'} \rangle| \right)^p \right]^{q/p} \right)^{1/q} \\ &\lesssim \left(\sum_{j' \in \mathbb{N}} \left[\sum_{j \in \mathbb{N}} \delta^{\gamma_{j,j'} p} \delta^{(j \wedge j' - j) \mathbf{d}_Y} \sum_{k \in I_j(X)} \left(\delta^{-j(s - \mathbf{d}_X/p)} |q_{X,j,k}(f)| \right)^p \right]^{q/p} \right)^{1/q} \\ &\lesssim \left(\sum_{j' \in \mathbb{N}} \sum_{j \in \mathbb{N}} \left[\delta^{\gamma_{j,j'} p} \delta^{(j \wedge j' - j) \mathbf{d}_Y} \right]^{(q/p \wedge 1)} \left[\sum_{k \in I_j(X)} \left(\delta^{-j(s - \mathbf{d}_X/p)} |q_{X,j,k}(f)| \right)^p \right]^{q/p} \right)^{1/q} \\ &\lesssim \left(\sum_{j' \in \mathbb{N}} \left[\sum_{k \in \tilde{I}_{j'}} \left(\delta^{-j(s - \mathbf{d}_X/p)} |q_{X,j,k}(f)| \right)^p \right]^{q/p} \right)^{1/q} = \|f\|_{B_{p,q}^s(Y)}, \end{aligned}$$

which proves the theorem.

It remains to prove the estimates of the Claim. Let us begin with the first sum running on j for a fixed j' . Suppose first $j' \geq j_{\psi_Y}$. Decomposing $\gamma_{j,j'}$ and separating the sum into cases yields

$$\begin{aligned} \sum_{j \in \mathbb{N}} \delta^{\gamma_{j,j'} p} \delta^{(j \wedge j' - j) \mathbf{d}_Y} &= \sum_{j \in \mathbb{N}} \delta^{[-(j' - j)s' + |j' - j|\theta + (j - j \wedge j') \mathbf{d}_Y] p} \delta^{(j \wedge j' - j) \mathbf{d}_Y} \\ &= \sum_{j \leq j'} \delta^{(j' - j)(\theta - s') p} + \sum_{j > j'} \delta^{(j - j')(s' + \theta) p} < +\infty \end{aligned}$$

by using that $s' < \theta$ when $j \leq j'$, and $s' > 0$ when $j > j'$. When $j' < j_{\psi_Y}$, diving the sum as before gives

$$\begin{aligned} \sum_{j \in \mathbb{N}} \delta^{\gamma_{j,j'} p} \delta^{(j \wedge j' - j) \mathbf{d}_Y} &= \sum_{j \in \mathbb{N}} \delta^{[-(j' - j)s' + (j - j \wedge j') \mathbf{d}_Y] p} \delta^{(j \wedge j' - j) \mathbf{d}_Y} \\ &= \sum_{j \leq j'} \delta^{(j - j')s' p} + \sum_{j > j'} \delta^{(j - j')s' p}. \end{aligned}$$

The second sum is the same as before. For the first one we use the bound $j_{\psi_Y} \delta^{-j_{\psi_Y} s'}$ which holds since $s' > 0$. The estimate for the sum running on j' for a fixed j follows similarly. \square

To end this chapter let us note how, indeed, Theorem 3.8 can be viewed as a generalization of the right-hand-side sampling inequality of Theorem 3.6. To see this, using the notation of Theorem 3.6, take Y as the subspace of discrete points $Y := \{a_{l,n}\}_{n \in I_l}$ with the same quasi-metric ρ as in X and the weighted delta measure $\mu_Y := \sum_n \mu(Q_{l,n}) \delta_{a_{l,n}}$. This does not constitute a space of homogeneous type as we have defined them since μ_Y does not verify (3.2) (it is, however, a doubling measure), but we could force the definition $B_{p,1}^0(Y) := \ell_p(Y, \mu_Y)$ so that (3.27) in Theorem 3.6 would yield

$$\|f\|_{B_{p,1}^0(Y)} \leq C \left(\|f\|_{L^p(X)} + \|f\|_{B_{p,1}^{\mathbf{d}/p}(X)} \right)$$

for some constant C and all $f \in B_{p,1}^{\mathbf{d}/p}(X)$. Then, from embedding results (3.18) we know that $B_{p,1}^{\mathbf{d}/p}(X) \subset L^p(X)$, and thus the previous inequality is equivalent to

$$\|f\|_{B_{p,1}^0(Y)} \leq C' \|f\|_{B_{p,1}^{\mathbf{d}/p}(X)},$$

which can be seen as a trace theorem to a subspace of dimension 0. In the notation of Theorem 3.8 this would cover the case $s = \frac{\mathbf{d}}{p}$, $s' = s - \frac{\mathbf{d}-0}{p} = 0$ (which cannot be proven with the tools we used). It is thus, under this perspective, that one could say Theorem 3.8 “extends” Theorem 3.6 to the cases of subspaces Y with positive dimension.

Chapter 4

Local regularity of Besov spaces

In some sense, the previous chapter showed the global regularity of traces for Besov functions. Now we focus in their local regularity in the sense of Hölder. Specifically we will look at the multifractal behavior of functions in the Besov space, i.e.: the geometric distribution of their singularities.

Once again, from the pioneering work of Meyer [68] in the euclidean case, this study can be carried out by the force of wavelets. Following the notion of global Hölder regularity as understood in [21] one can naturally define the point-wise Hölder exponent of regularity which then allows to develop the theory of multifractal analysis in spaces of homogeneous type as it is done in $\mathbb{R}^{\mathbf{d}}$ [54, 55].

4.1 Additional preliminaries

First let us stress that we will work on the same framework laid out in the first section of the previous chapter. This is, our ambient space will be a space of homogeneous type (X, ρ, μ) with dimension \mathbf{d} , regularity θ and dyadic parameter δ . Also we will use the wavelet system $\{\varphi_{j,k}\}_{j,k}$ of Theorem 3.4 and assume that the associated dyadic partition is nested.

4.1.1 Hausdorff dimension

Before we begin the study of regularities for functions defined on X , let us introduce a couple of concepts needed in multiresolution analysis. First, the Hausdorff dimension of a subset in X . Given $t \geq 0$ the t -dimensional Hausdorff measure of a set $E \subset X$ is defined as

$$\mathcal{H}^t(E) := \lim_{r \downarrow 0} \mathcal{H}_r^t(E),$$

where

$$\mathcal{H}_r^t(E) := \inf \left\{ \sum_i \text{diam}(E_i)^t : E \subset \bigcup_i E_i, \text{diam}(E_i) < r \right\}.$$

Then, its *Hausdorff dimension* is defined as

$$\dim_{\mathcal{H}} E := \inf\{t \geq 0 : \mathcal{H}^t(E) = 0\} = \sup\{t \geq 0 : \mathcal{H}^t(E) = +\infty\}.$$

It is worth point out that, when defining \mathcal{H}_r^t , one could consider instead smaller families of coverings and still obtain the same dimension. Notably, one could ask the sets E_i to be balls or dyadic cubes. Using the latter it is possible to show that the Hausdorff dimension of a space of homogeneous type coincides with its homogeneous dimension.

4.1.2 Hölder regularity

Let us now define Hölder regularity. Given $0 < s \leq \theta$ we say that a function $f : X \rightarrow \mathbb{C}$ is s -regular, if the quantity

$$\|f\|_{\mathcal{C}^s} := \sup_{x \neq y} \frac{|f(x) - f(y)|}{\rho(x, y)^s}$$

is finite. The set of such functions is denoted by $\mathcal{C}^s(X)$ and, modulo constant functions, constitutes a Banach space with norm $\|\cdot\|_{\mathcal{C}^s}$. One can verify that $\mathcal{C}^s(X)$ is non-trivial for any $0 < s \leq \theta$ using the following type of constructions: let $f \in \mathcal{C}^1(\mathbb{R})$ with compact support, fix $x_0 \in X$ and set $g(x) := f(\rho(x, x_0))$, then (3.3) implies $g \in \mathcal{C}^s(X)$ for all $0 < s \leq \theta$.

Global regularity, however, can be too general and fail to describe the behavior of a function as we move along X . Thus, we will be also interested in local regularity. Let $x \in X$ and $0 < s < \theta$. A function $f : X \rightarrow \mathbb{C}$ is said to belong to $\mathcal{C}^s(x)$ if there exists a constant $C > 0$ such that

$$|f(x) - f(y)| \leq C\rho(x, y)^s$$

holds for all y in a neighborhood of x . The *Hölder exponent* of f at x is defined as maximum order of regularity for x

$$h_f(x) := \sup\{\alpha : f \in \mathcal{C}^\alpha(x)\}. \quad (4.1)$$

This exponent may change erratically for one point to another. Thus if we want to understand how the regularity of a function varies or which are its most significant singularities it is more useful to look at what are called its *iso-Hölder sets*:

$$E_f(\alpha) := \{x : h_f(x) = \alpha\}.$$

The quantity that best captures the geometry of such sets is its Hausdorff dimension, which is called the *spectrum of singularities*

$$d_f(\alpha) := \dim_{\mathcal{H}} E_f(\alpha) = \dim_{\mathcal{H}} \{x : h_f(x) = \alpha\}, \quad (4.2)$$

adopting the convention $\dim_{\mathcal{H}} \emptyset = -\infty$. The rationale of using Hausdorff measures instead of the one provided by the ambient space (in this case μ), is that for many of the functions f usually looked at in multifractal analysis one expects that there is a value α_f for which $E_f(\alpha_f)$ is a set of full measure in our space. Hence, one tries to determine an approximate notion of a “fractal” dimension, and the best way to do that is with Hausdorff dimension.

As we said at beginning of this chapter, the main tool we will use to study regularity properties is the wavelet decomposition. Specifically, as it is the case for Besov spaces, we will characterize Hölder-regularity through the wavelet coefficients $q_{j,k}(f)$ as defined in (3.14). Global regularity in particular can be completely determined in this way:

Proposition 4.1 ([21]). *Let $f \in L^2(X)$ and $0 < s < \theta$. If $f \in \mathcal{C}^s(X)$, there exists a constant $C = C(f, s) > 0$ such that*

$$|q_{j,k}(f)| \leq C \delta^{js}, \quad \forall j, k. \quad (4.3)$$

Conversely if (4.3) holds for all j, k then $f \in \mathcal{C}^s(X)$. Further, the constant C can be written as $\|f\|_{\mathcal{C}^s}$ up to multiplication with another constant independent from f .

Note that the condition (4.3) is equivalent to $f \in B_{\infty, \infty}^s(X)$. Moreover, we could reformulate this proposition as the equality (as Banach spaces) of $\mathcal{C}^s(X)$ and $B_{\infty, \infty}^s(X)$.

Proof. Suppose first that $f \in \mathcal{C}^s(X)$ and let $j \in \mathbb{N}$, $k \in I_j$. When $0 \leq j < j_\varphi$, where j_φ is as in (3.12), the right-hand-side of (4.3) is bounded from below by $C \delta^{j_\varphi s}$. Therefore, it is enough to show that $q_{j,k}(f)$ is uniformly bounded for $0 \leq j < j_\varphi$. And this follows from (3.15) and the Cauchy-Schwartz inequality:

$$|q_{j,k}(f)| = \delta^{-j\mathbf{d}} |\langle \tilde{\varphi}_{j,k}, f \rangle| \leq \delta^{-j\mathbf{d}} \|\tilde{\varphi}_{j,k}\|_2 \|f\|_2 \lesssim \delta^{-j_\varphi \mathbf{d}/2}, \quad \forall 0 \leq j < j_\varphi.$$

We assume next that $j \geq j_\varphi$. Thus $\tilde{\varphi}_{j,k}$ has mean zero and we can write

$$\langle f, \tilde{\varphi}_{j,k} \rangle = \int_X f(x) \tilde{\varphi}_{j,k}(x) \, d\mu(x) = \int_X (f(x) - f(x_{j,k})) \tilde{\varphi}_{j,k}(x) \, d\mu(x).$$

Using s -Hölder regularity for f we then get

$$|\langle f, \tilde{\varphi}_{j,k} \rangle| \leq \|f\|_{\mathcal{C}^s} \int_X \rho(x, x_{j,k})^s |\tilde{\varphi}_{j,k}(x)| \, d\mu(x). \quad (4.4)$$

The size condition (3.10) for $\tilde{\varphi}_{j,k}(x)$ with any $\theta' \in (s, \theta)$ reads

$$\begin{aligned} \rho(x, x_{j,k})^s |\tilde{\varphi}_{j,k}(x)| &\lesssim \rho(x, x_{j,k})^s \left(\frac{\delta^j}{\delta^j + \rho(x, x_{j,k})} \right)^{\mathbf{d}+\theta'} \\ &\lesssim \rho(x, x_{j,k})^s \left(\frac{\delta^j}{\delta^j + \rho(x, x_{j,k})} \right)^s \left(\frac{\delta^j}{\delta^j + \rho(x, x_{j,k})} \right)^{\mathbf{d}+\theta'-s} \\ &\lesssim \left(\frac{\delta^j \rho(x, x_{j,k})}{\rho(x, x_{j,k})} \right)^s \left(\frac{\delta^j}{\delta^j + \rho(x, x_{j,k})} \right)^{\mathbf{d}+\theta'-s} \\ &\lesssim \delta^{js} \left(\frac{\delta^j}{\delta^j + \rho(x, x_{j,k})} \right)^{\mathbf{d}+\theta'-s}. \end{aligned}$$

This, plugged into (4.4) and applying the first estimation of Lemma 3.5 reads

$$|\langle f, \tilde{\varphi}_{j,k} \rangle| \lesssim \|f\|_{C^s} \delta^{js} \int_X \left(\frac{\delta^j}{\delta^j + \rho(x, x_{j,k})} \right)^{\mathbf{d}+\theta'-s} d\mu(x) \lesssim \|f\|_{C^s} \delta^{j(s+\mathbf{d})}$$

and hence $|q_{j,k}(f)| = |\delta^{-j\mathbf{d}} \langle f, \tilde{\varphi}_{j,k} \rangle| \lesssim \|f\|_{C^s} \delta^{js}$.

Suppose now that (4.3) holds for all j, k . For each $j \in \mathbb{N}$ define $f_j := \sum_{k \in I_j} q_{j,k}(f) \varphi_{j,k}$. Since the supports of the collection $\{\varphi_{j,k}\}_{k \in I_j}$ have finite multiplicity T , the size condition (3.9) gives for all $x \in X$

$$\begin{aligned} |f_j(x)| &\leq \sum_{k \in I_j} |q_{j,k}(f) \varphi_{j,k}(x)| \leq C_\varphi C \delta^{js} \sum_{k \in I_j} \mathbb{1}_{\text{supp}(\varphi_{j,k})}(x) \\ &\leq C_\varphi C T \delta^{js}. \end{aligned} \tag{4.5}$$

Then using (4.3) we get that

$$\sum_{j \in \mathbb{N}} |f_j(x)| \leq T C_\varphi C \sum_{j \in \mathbb{N}} \delta^{js} = T C_\varphi C (1 - \delta^s)^{-1}, \quad \text{for all } x \in X.$$

This means that $\sum_j f_j$ converges in L^∞ . Further, as we know from (3.13) that $f = \sum_j f_j$ in L^2 , then $f = \sum_j f_j$ also in L^∞ .

Now take $x, y \in X$ different and set $m \in \mathbb{Z}$ such that $\delta^{m+1} \leq \rho(x, y) < \delta^m$. We then divide the sum as $f = \sum_{j \leq m} f_j + \sum_{j > m} f_j$.

Let us start estimating the first sum. If $m < 0$, this sum is 0 so we assume that $m \geq 0$ (that is $\rho(x, y) \leq \delta$). For each $j \leq m$ we apply (4.3) the smoothness

condition (3.11) to get

$$\begin{aligned}
|f_j(x) - f_j(y)| &\leq \sum_{k \in I_j} |q_{j,k}(f)| |\varphi_{j,k}(x) - \varphi_{j,k}(y)| \\
&\leq C \delta^{js} C_\varphi \delta^{-j\theta} \rho(x, y)^\theta \left(\sum_{k \in I_j} \mathbb{1}_{\text{supp}(\varphi_{j,k})}(x) + \mathbb{1}_{\text{supp}(\varphi_{j,k})}(y) \right) \\
&\leq 2C_\varphi CT \rho(x, y)^\theta \delta^{j(s-\theta)}.
\end{aligned}$$

Running the sum over all $j \leq m$ we then have

$$\begin{aligned}
\sum_{j \leq m} |f_j(x) - f_j(y)| &\leq 2C_\varphi CT \rho(x, y)^\theta \sum_{j \leq m} \delta^{j(s-\theta)} \\
&\leq 2C_\varphi CT \rho(x, y)^\theta \left(\frac{\delta^{(m+1)(s-\theta)} - 1}{\delta^{s-\theta} - 1} \right) \\
&\leq 2C_\varphi CT (\delta^{s-\theta} - 1)^{-1} \rho(x, y)^\theta \delta^{(m+1)(s-\theta)} \\
&\leq 2C_\varphi CT \delta^{s-\theta} (\delta^{s-\theta} - 1)^{-1} \rho(x, y)^s \tag{4.6}
\end{aligned}$$

since $s < \theta$ and $\rho(x, y) < \delta^m$. Of course, the bound (4.6) is also valid when $m < 0$ as the left-hand-side is 0.

For $j > m$ we just use again (4.5):

$$\begin{aligned}
\sum_{j > m} |f_j(x) - f_j(y)| &\leq 2C_\varphi CT \sum_{j > m} \delta^{js} \\
&\leq 2C_\varphi CT (1 - \delta^s) \delta^{(m+1)s} \\
&\leq 2C_\varphi CT (1 - \delta^s) \rho(x, y)^s
\end{aligned}$$

since $\delta^{m+1} \leq \rho(x, y)$.

Altogether, this implies that there exist a constant K , independent from f , such that

$$\frac{|f(x) - f(y)|}{\rho(x, y)^s} \leq K,$$

holds for all $x \neq y$. □

For local regularity however, the situation is not so straightforward. Imitating a result from Jaffard [53] in \mathbb{R}^d , the equivalent to Proposition 4.1 for local regularity would be as follows:

Proposition 4.2. *Let $0 < s < s_0 < \theta$ and $x_0 \in X$. Let f be a square summable function in $\mathcal{C}^s(X) \cap \mathcal{C}^{s_0}(x_0)$. Then*

$$|q_{j,k}(f)| \lesssim (\delta^j + \rho(x_0, x_{j,k}))^{s_0}, \tag{4.7}$$

where the implicit constant is independent from j and k . Reciprocally, if $f \in \mathcal{C}^s(X)$ and (4.7) holds for all j, k then $f \in \mathcal{C}_{\ln}^{s_0}(x_0)$, i.e.

$$|f(x) - f(x_0)| \lesssim \frac{\ln \rho(x, x_0)}{\ln \delta} \rho(x, x_0)^{s_0} \quad (4.8)$$

for all x in a neighborhood of x_0 ; and in particular, $f \in \mathcal{C}^{s'}(x_0)$ for any $s' < s_0$.

Before proceeding to the proof of the proposition, for technical reasons, it is useful to find a reformulation of (4.7) that looks closer to (4.3). To that end, we reproduce the notion of *wavelet leaders* introduced by Jaffard in the euclidean case [55]. Let $f \in L_{loc}^1(X)$ and $j \in \mathbb{N}, k \in I_j$. The wavelet coefficient leader of f at (j, k) is defined as

$$D_{j,k}f := \sup\{|q_{j',k'}(f)| : Q_{j',k'} \subset MQ_{j,k}\}$$

with $M := 3\mathbf{A}^2(r_1 + 1)$, using the notation $MQ_{j,k} = \{Mx : x \in Q_{j,k}\}$ and where \mathbf{A} is the constant from (3.1) and r_1 the one from (3.6). Note that if $Q_{j',k'} \subset MQ_{j,k}$ then, by (3.6), there must exist a constant $c > 0$ (depending on M, r_0, r_1 and X) such that $j' \geq j - c$.

Given $x \in X$ we denote $D_j f(x)$ for the coefficient $D_{j,k}f$ where $k \in I_j$ is uniquely determined by $x \in Q_{j,k}$.

Proposition 4.3. *For any square summable function f , $x_0 \in X$ and $0 < s_0 < \theta$ the following are equivalent:*

- (i) $|q_{j,k}(f)| \lesssim (\delta^j + \rho(x_0, x_{j,k}))^{s_0}$ holds for all j, k ,
- (ii) $D_j f(x_0) \lesssim \delta^{js_0}$ holds for all j , with the implicit constant independent of j .

Proof. Let us begin by proving that (i) implies (ii). Let $j \in \mathbb{N}$ and take $k \in I_j$ such that $x_0 \in Q_{j,k}$. Further, let j_0 and k_0 with $Q_{j_0,k_0} \subset MQ_{j,k}$. Then (i) implies that

$$|q_{j_0,k_0}(f)| \lesssim (\delta^{j_0 s_0} + \rho(x_0, x_{j_0,k_0}))^{s_0}. \quad (4.9)$$

Since $Q_{j_0,k_0} \subset MQ_{j,k}$ then, by using the quasi-triangle inequality (3.1), we get

$$\rho(x_0, x_{j_0,k_0}) \lesssim \rho(x_0, x_{j,k}) + \rho(x_{j_0,k_0}, x_{j,k}) \lesssim \delta^j,$$

which plugged into (4.9) yields $|q_{j_0,k_0}(f)| \lesssim (\delta^{j_0} + \delta^j)^{s_0}$. Finally, as noted, $j_0 \geq j - c$ must hold whenever $Q_{j_0,k_0} \subset MQ_{j,k}$, and thus $|q_{j_0,k_0}(f)| \lesssim \delta^{js_0}$. Since the implicit constants are independent from j_0, k_0 (or even j) we obtain $D_j f(x_0) \lesssim \delta^{js_0}$.

We now prove the converse implication. Consider a pair j, k . If $x_0 \in MQ_{j,k}$ then by construction

$$|q_{j,k}(f)| \leq D_j f(x_0) \lesssim \delta^{js_0} \leq (\delta^j + \rho(x_0, x_{j,k}))^{s_0}.$$

If not, the inclusions of (3.6) imply that $\rho(x_0, x_{j,k}) \geq Mr_0\delta^j \geq r_0\delta^j$. In this case, let $j_0 \leq j$ be such that $r_0\delta^{j_0+1} \leq \rho(x_0, x_{j,k}) < r_0\delta^{j_0}$ and let $k_0 \in I_{j_0}$ be such that $x_0 \in Q_{j_0, k_0}$. The quasi-triangle inequality (3.1) and (3.6) again yield, for all $x \in Q_{j,k}$,

$$\begin{aligned} \rho(x, x_{j_0, k_0}) &\leq \mathbf{A}(\rho(x_0, x_{j_0, k_0}) + \rho(x_0, x)) \\ &\leq \mathbf{A}(r_0\delta^{j_0} + \mathbf{A}(\rho(x_0, x_{j,k}) + \rho(x, x_{j,k}))) \\ &\leq \mathbf{A}(r_0\delta^{j_0} + \mathbf{A}(r_0\delta^{j_0} + r_1\delta^j)) \leq 3\mathbf{A}^2 r_1 \delta^{j_0} \end{aligned}$$

and thus $Q_{j,k} \subset MQ_{j_0, k_0}$ since $M = 3\mathbf{A}^2(r_1 + 1)$. Altogether

$$|q_{j,k}(f)| \leq D_{j_0}(x_0) \lesssim \delta^{j_0 s_0} \lesssim \rho(x_0, x_{j,k})^{s_0} \leq (\delta^j + \rho(x_0, x_{j,k}))^{s_0},$$

which is what we wanted to show. \square

We are now ready to prove Proposition 4.2.

Proof of Proposition 4.2. Let us first assume that $f \in C^s(X) \cap C^{s_0}(x_0)$. For $0 \leq j < j_\varphi$ we use the same argument as in the proof of Proposition 4.1: the right-hand-side of (4.7) is bounded from below by $\delta^{j_\varphi s_0}$ and the left-hand-side is uniformly bounded by Cauchy-Schwartz.

For $j \geq j_\varphi$ we use the fact that $\tilde{\varphi}_{j,k}$ has mean zero:

$$\begin{aligned} q_{j,k}(f) &= \delta^{-j\mathbf{d}} \langle \tilde{\varphi}_{j,k}, f \rangle = \delta^{-j\mathbf{d}} \int_X \tilde{\varphi}_{j,k}(x) f(x) \, d\mu(x) \\ &= \delta^{-j\mathbf{d}} \int_X \tilde{\varphi}_{j,k}(x) (f(x) - f(x_0)) \, d\mu(x). \end{aligned}$$

Since $f \in C^s(X) \cap C^{s_0}(x_0)$ and $s < s_0$, there exists a positive constant $C = C(f)$ such that $|f(x) - f(x_0)| \leq C\rho(x, x_0)^{s_0}$ for all $x \in X$. Combined with the quasi-triangle inequality this gives us

$$\begin{aligned} |q_{j,k}(f)| &\leq \delta^{-j\mathbf{d}} \int_X |\tilde{\varphi}_{j,k}(x)| |f(x) - f(x_0)| \, d\mu(x) \\ &\lesssim \delta^{-j\mathbf{d}} \int_X |\tilde{\varphi}_{j,k}(x)| \rho(x, x_0)^{s_0} \, d\mu(x) \\ &\lesssim \delta^{-j\mathbf{d}} \int_X (\rho(x, x_{j,k}) + \rho(x_0, x_{j,k}))^{s_0} |\tilde{\varphi}_{j,k}(x)| \, d\mu(x). \end{aligned}$$

As in the proof of Proposition 4.1, we may write

$$\begin{aligned}
& (\rho(x, x_{j,k}) + \rho(x_0, x_{j,k}))^{s_0} |\tilde{\varphi}_{j,k}(x)| \\
& \lesssim \left(\frac{[\rho(x, x_{j,k}) + \rho(x_0, x_{j,k})] \delta^j}{\delta^j + \rho(x, x_{j,k})} \right)^{s_0} \left(\frac{\delta^j}{\delta^j + \rho(x, x_{j,k})} \right)^{\mathbf{d} + \theta' - s_0} \\
& \lesssim \left(\frac{\delta^j \rho(x, x_{j,k})}{\delta^j + \rho(x, x_{j,k})} + \frac{\delta^j \rho(x_0, x_{j,k})}{\delta^j + \rho(x, x_{j,k})} \right)^{s_0} \left(\frac{\delta^j}{\delta^j + \rho(x, x_{j,k})} \right)^{\mathbf{d} + \theta' - s_0} \\
& \lesssim \left(\frac{\delta^j \rho(x, x_{j,k})}{\rho(x, x_{j,k})} + \frac{\delta^j \rho(x_0, x_{j,k})}{\delta^j} \right)^{s_0} \left(\frac{\delta^j}{\delta^j + \rho(x, x_{j,k})} \right)^{\mathbf{d} + \theta' - s_0} \\
& \lesssim (\delta^j + \rho(x_0, x_{j,k}))^{s_0} \left(\frac{\delta^j}{\delta^j + \rho(x, x_{j,k})} \right)^{\mathbf{d} + \theta' - s_0}.
\end{aligned}$$

The estimation (3.15) finally implies

$$\delta^{-j\mathbf{d}} \int_X (\rho(x, x_{j,k}) + \rho(x_0, x_{j,k}))^{s_0} |\tilde{\varphi}_{j,k}(x)| \, d\mu(x) \lesssim (\delta^j + \rho(x_0, x_{j,k}))^{s_0},$$

which then yields (4.7).

We assume now that $f \in \mathcal{C}^s(X)$ and that (4.7) holds for all j, k . Let $x \in B(x_0, 1)$, take j_0 such that $\delta^{j_0+1} \leq \rho(x, x_0) < \delta^{j_0}$ and set $j_1 := \lfloor \frac{j_0 s_0}{s} \rfloor$. Next, as in Proposition 4.1, we write f as $f = \sum_j f_j$ where $f_j := \sum_k q_{j,k}(f) \varphi_{j,k}$. Further, we divide the sum over j as

$$\begin{aligned}
|f(x) - f(x_0)| & \leq \sum_{j \leq j_0} |f_j(x) - f_j(x_0)| + \sum_{j_0 < j \leq j_1} (|f_j(x)| + |f_j(x_0)|) + 2 \sum_{j > j_1} \|f_j\|_\infty \\
& := S_1 + S_2 + S_3.
\end{aligned}$$

Let us begin with the estimation of S_1 . Using the smoothness condition (3.11) for $\varphi_{j,k}$ and the property (4.7) we get

$$\begin{aligned}
|f_j(x) - f_j(x_0)| & \leq \sum_{k \in I_j} |q_{j,k}(f)| |\varphi_{j,k}(x) - \varphi_{j,k}(x_0)| \\
& \lesssim \sum_{k \in I_j} (\delta^j + \rho(x_0, x_{j,k}))^{s_0} \delta^{-j\theta} \rho(x, x_0)^{s_0} (\mathbb{1}_{\text{supp}(\varphi_{j,k})}(x) + \mathbb{1}_{\text{supp}(\varphi_{j,k})}(x_0)).
\end{aligned}$$

If $x_0 \in \text{supp}(\varphi_{j,k})$ then $\rho(x_0, x_{j,k}) \lesssim \delta^j$. If $x \in \text{supp}(\varphi_{j,k})$ then the quasi-triangle inequality reads $\rho(x_0, x_{j,k}) \lesssim \delta^j + \rho(x, x_0)$. Thus, in any case, as $\delta^j \geq \delta^{j_0} > \rho(x, x_0)$, we obtain $\rho(x_0, x_{j,k}) \lesssim \delta^j$. Plugged into the previous inequality this gives us

$$\begin{aligned}
|f_j(x) - f_j(x_0)| & \lesssim \sum_{k \in I_j} \delta^{j(s_0 - \theta)} \rho(x, x_0)^{s_0} (\mathbb{1}_{\text{supp}(\varphi_{j,k})}(x) + \mathbb{1}_{\text{supp}(\varphi_{j,k})}(x_0)) \\
& \lesssim \delta^{j(s_0 - \theta)} \rho(x, x_0)^{s_0}
\end{aligned}$$

by the finite multiplicity property of the supports of $\{\varphi_{j,k}\}_{k \in I_j}$. Finally, since $s_0 < \theta$ and $\delta^{j_0} \leq \frac{1}{8}\rho(x, x_0)$, we have

$$S_1 = \sum_{j \leq j_0} |f_j(x) - f_j(x_0)| \lesssim \delta^{j_0(s_0 - \theta)} \rho(x, x_0)^\theta \lesssim \rho(x, x_0)^{s_0}.$$

We now estimate S_2 . Using the size condition (3.9) we get, for all $j_0 < j \leq j_1$,

$$|f_j(x)| \lesssim \sum_{k \in I_j} (\delta^j + \rho(x_0, x_{j,k}))^{s_0} (\mathbb{1}_{\text{supp}(\varphi_{j,k})}(x) + \mathbb{1}_{\text{supp}(\varphi_{j,k})}(x_0)).$$

Reasoning as for S_1 , but noticing that this time we are taking $j > j_0$, we get that $\delta^j + \rho(x_0, x_{j,k}) \lesssim \delta^{j_0}$ whenever either x or x_0 are in the support of $\varphi_{j,k}$. Thus $|f_j(x)| \lesssim \delta^{j_0 s_0} \lesssim \rho(x, x_0)^{s_0}$ for any $j_0 < j \leq j_1$. Counting terms we get $j_1 - j_0 \leq (s_0/s - 1)j_0 \lesssim \frac{\ln \rho(x, x_0)}{\ln \delta}$, so that $S_2 \lesssim \frac{\ln \rho(x, x_0)}{\ln \delta} \rho(x, x_0)^{s_0}$.

Finally we estimate S_3 . Here we use the hypothesis $f \in \mathcal{C}^s(X)$ by applying Proposition 4.1 which, together with the inequality (3.15), gives $|f_j(x)| \lesssim \delta^{js}$ and then we get $S_3 \lesssim \sum_{j > j_1} \delta^{js} \lesssim \delta^{j_1 s} \lesssim \rho(x, x_0)^{s_0}$. \square

4.2 Regularity of Besov functions

We are now ready to prove the upper bound on the spectrum of singularities for Besov functions in this context. This is the counterpart to the result proven in \mathbb{R}^d [54].

Theorem 4.4. *Let $1 \leq p < \infty$ and $\mathbf{d}/p < s < \mathbf{d}/p + \theta$. Then for all $f \in B_{p,\infty}^s(X)$ and all $s - \mathbf{d}/p \leq \alpha < \theta$, the spectrum of singularities $d_f(\alpha)$ as defined in (4.2) satisfies*

$$d_f(\alpha) \leq \min(\mathbf{d}, \mathbf{d} + (\alpha - s)p).$$

And $d_f(\alpha) = -\infty$ if $\alpha < s - \mathbf{d}/p$.

Although we stated the theorem for $B_{p,q}^s$ with $q = \infty$, let us remark that from (3.17) this can be extended to any $1 \leq q < \infty$.

Since the proof relies heavily on Proposition 4.2, we will need to prove that functions in our Besov space have at least some global regularity. In other words, we are looking at a Sobolev-type embedding theorem in this context:

Lemma 4.5. *$B_{p,\infty}^s(X)$ is embedded into $\mathcal{C}^{s-\mathbf{d}/p}(X)$ whenever $\mathbf{d}/p < s < \mathbf{d}/p + \theta$ with $0 < p \leq \infty$.*

Proof. By definition if $f \in B_{p,\infty}^s(X)$ then

$$\left(\sum_{k \in I_j} |q_{j,k}(f)|^p \right)^{1/p} \leq \|f\|_{B_{p,\infty}^s} \delta^{j(s-d/p)}$$

holds for all j . In particular $|q_{j,k}(f)| \leq \|f\|_{B_{p,\infty}^s} \delta^{j(s-d/p)}$ for all j, k , which by Proposition 4.1 implies the result. \square

Proof of Theorem 4.4. First note that since $d/p < s < d/p + \theta$ and $f \in B_{p,\infty}^s(X)$, Lemma 4.5 effectively reads $f \in \mathcal{C}^{s-d/p}(X)$. Therefore $d_f(\alpha) = -\infty$ whenever $\alpha < s - d/p$. Secondly note that if $\alpha \geq s$ then $\min(d, d + (\alpha - s)p) = d$ and the theorem follows from the fact that the Hausdorff dimension of X is d .

Thus we can suppose that $s - d/p \leq \alpha < s$. Let $\varepsilon > 0$ and define

$$E^{\alpha+\varepsilon} := \{x : f \notin \mathcal{C}^{\alpha+\varepsilon}(x)\}.$$

By definition (4.1), $\{x : h_f(x) = \alpha\} \subset E^{\alpha+\varepsilon}$. Hence, from (4.2), we have

$$d_f(\alpha) = \dim_{\mathcal{H}}\{x : h_f(x) = \alpha\} \leq \dim_{\mathcal{H}} E^{\alpha+\varepsilon}. \quad (4.10)$$

We will show that $\dim_{\mathcal{H}} E^{\alpha+\varepsilon} \leq d + (\alpha - s)p + o(\varepsilon)$, and then the theorem follows by making $\varepsilon \rightarrow 0$.

Now, let $x \in E_{\alpha+\varepsilon}$. Since f has at least some global Hölder regularity, by using the contrapositive statement of Proposition 4.2, we know that (4.7) cannot hold with exponent $\alpha + 2\varepsilon$ at x . Equivalently by Proposition 4.3 this means that for any constant $C > 0$ there exists j such that $D_j f(x) \geq C \delta^{j(\alpha+2\varepsilon)}$. Thus, in particular, for any j_0 there must exist $j \geq j_0$ such that $D_j f(x) \geq \delta^{j(\alpha+2\varepsilon)}$. We define for each j

$$I_j^{\alpha+2\varepsilon} := \{k \in I_j : D_{j,k} f \geq \delta^{j(\alpha+2\varepsilon)}\},$$

and then set

$$S_j^{\alpha+2\varepsilon} := \bigcup_{k \in I_j^{\alpha+2\varepsilon}} Q_{j,k},$$

$$S^{\alpha+2\varepsilon} := \limsup_j S_j^{\alpha+2\varepsilon} = \bigcap_{j_0} \bigcup_{j \geq j_0} S_j^{\alpha+2\varepsilon}.$$

Therefore, with this notation, $E^{\alpha+\varepsilon} \subset S^{\alpha+2\varepsilon}$. In particular

$$\dim_{\mathcal{H}} E^{\alpha+\varepsilon} \leq \dim_{\mathcal{H}} S^{\alpha+2\varepsilon}. \quad (4.11)$$

To estimate the Hausdorff dimension of $S^{\alpha+2\varepsilon}$ let us estimate the size of $I_j^{\alpha+2\varepsilon}$. By definition, for each wavelet leader $D_{j,k}$ there exist j', k' such that $Q_{j',k'} \subset MQ_{j,k}$

and $|q_{j',k'}(f)| \geq \frac{1}{2}D_{j,k}$. Remember that in such case $j' \geq j - c$ for some constant $c > 0$. Altogether this implies

$$\#I_j^{\alpha+2\varepsilon} \leq \sum_{j' \geq j-c} \# \left\{ k' \in I_{j'} : |q_{j',k'}(f)| \geq \frac{1}{2}\delta^{j(\alpha+2\varepsilon)} \right\}. \quad (4.12)$$

Additionally, as $f \in B_{p,\infty}^s$, for any $j' \in \mathbb{N}$ we have that

$$\sum_{k' \in I_{j'}} |q_{j',k'}(f)|^p \lesssim \delta^{j'(s-\mathbf{d}/p)p},$$

which yields

$$\# \left\{ k' \in I_{j'} : |q_{j',k'}(f)| \geq \frac{1}{2}\delta^{j(\alpha+2\varepsilon)} \right\} \lesssim \delta^{j'(sp-\mathbf{d})} \delta^{-j(\alpha p+2\varepsilon p)}.$$

Plugging this into the right-hand-side of (4.12) we get

$$\#I_j^{\alpha+2\varepsilon} \lesssim \sum_{j' \geq j-c} \delta^{j'(sp-\mathbf{d})} \delta^{-j(\alpha p+2\varepsilon p)} \lesssim \delta^{j(sp-\mathbf{d}-\alpha p-2\varepsilon p)}.$$

Finally we cover each $S_j^{\alpha+2\varepsilon}$ with the cubes $Q_{j,k}$ -which, by (3.6), have diameter $\leq 2r_1\delta^j$. Thus, given an arbitrary $t \geq 0$, we have for all $j \in \mathbb{N}$

$$\mathcal{H}^t(S_j^{\alpha+2\varepsilon}) \leq \mathcal{H}_{2r_1\delta^j}^t(S_j^{\alpha+2\varepsilon}) \leq \sum_{k \in I_j^{\alpha+2\varepsilon}} \text{diam}(Q_{j,k})^t \lesssim \delta^{jt} \delta^{j(sp-\mathbf{d}-\alpha p-2\varepsilon p)}. \quad (4.13)$$

From $S^{\alpha+2\varepsilon} = \limsup_j S_j^{\alpha+2\varepsilon}$ we know that, for any $t \geq 0$, $\mathcal{H}^t(S^{\alpha+2\varepsilon}) \leq \sum_{j \geq j_0} \mathcal{H}^t(S_j^{\alpha+2\varepsilon})$. And thus, choosing $t = t_\varepsilon := \mathbf{d} - sp + \alpha p + 3\varepsilon p$ in (4.13) we get

$$\mathcal{H}^{t_\varepsilon}(S^{\alpha+2\varepsilon}) \lesssim \sum_{j \geq j_0} \delta^{jt_\varepsilon} \delta^{j(sp-\mathbf{d}-\alpha p-2\varepsilon p)} \lesssim \delta^{j_0\varepsilon p},$$

for any $j_0 \in \mathbb{N}$. Making $j_0 \rightarrow +\infty$ then yields $\mathcal{H}^{t_\varepsilon}(S^{\alpha+2\varepsilon}) = 0$. Which, by definition of the Hausdorff dimension, reads $\dim_{\mathcal{H}}(S^{\alpha+2\varepsilon}) \leq t_\varepsilon = \mathbf{d} - sp + \alpha p + 3\varepsilon p$. Plugged into (4.11), we obtain

$$\dim_{\mathcal{H}} E^{\alpha+\varepsilon} \leq \mathbf{d} - sp + \alpha p + 3\varepsilon p.$$

As $\varepsilon > 0$ is arbitrary, the result then follows from (4.10). \square

Chapter 5

Sampling on spiraling curves

The following chapter has appeared in [59].

5.1 Introduction

5.1.1 The mobile sampling problem

In this article, we consider the reconstruction of a compactly supported function from samples of its Fourier transform taken along certain curves, that we call spiraling. This problem is relevant, for example, in magnetic resonance imaging (MRI), where the anatomy and physiology of a person are captured by moving sensors.

The Fourier sampling problem is equivalent to the sampling problem for bandlimited functions - that is, functions whose Fourier transform are supported on a given compact set. The most classical setting concerns functions of one real variable with Fourier transform supported on the unit interval $[-1/2, 1/2]$, and sampled on a grid $\eta\mathbb{Z}$, with $\eta > 0$. The sampling rate η determines whether every bandlimited function can be reconstructed from its samples: reconstruction fails if $\eta > 1$ and succeeds if $\eta \leq 1$ [87]. The transition value $\eta = 1$ is known as the Nyquist sampling rate, and it is the benchmark for all sampling schemes: modern sampling strategies that exploit the particular structure of a certain class of signals are praised because they achieve sub-Nyquist sampling rates.

The sampling theory for bandlimited functions extends to high dimension and irregular sampling geometries [8, 24, 60], and it is instrumental in the analysis of sampling schemes arising from continuous curves [7]. The key notion is the Beurling density of a set, which measures the average number of samples per unit volume.

Beurling's density, however, does not properly reflect the acquisition cost when

samples are taken along continuous trajectories. In this case, a more relevant metric is the average *length* covered by a curve, as a proxy for scanning times [13, 16, 79, 85, 86]. For example, when sampling a function bandlimited to a compact set $\Omega \subset \mathbb{R}^2$ along equispaced parallel lines with direction $\vec{v} \in \mathbb{S}^1$,

$$L_{\vec{v},\eta} = \{t\vec{v} + \eta k\vec{v}_\perp : t \in \mathbb{R}, k \in \mathbb{Z}\},$$

the critical sampling rate is dictated by the separation between lines $\eta > 0$, and by the measure of the maximal cross section of Ω by hyperplanes perpendicular to \vec{v} [85, 86]. With the introduction of an adequate notion of *path-density*, similar results hold also for arbitrary families of parallel lines [42].

The analysis of general sampling trajectories in terms of length and density is very subtle and challenging, and little can be said in full generality [42]. Nevertheless, a solution to the mobile sampling problem is expected to be possible for concrete parametric families of curves.

5.1.2 Necessary and sufficient recovery guarantees for spiraling curves

A first contribution of this article is to give sharp necessary and sufficient conditions for Fourier sampling for a parametric family of curves, that we call spiraling. The main examples of these curves are the Archimedes spiral

$$A^\eta := \{(\eta\theta \cos 2\pi\theta, \eta\theta \sin 2\pi\theta) : \theta \geq 0\} \quad (5.1)$$

and the collection of concentric circles

$$O^\eta := \{(x, y) : x^2 + y^2 = \eta^2 k^2, k \in \mathbb{N}\}, \quad (5.2)$$

see Figure 5.1.

We identify the precise Nyquist rate of these curves in terms of the density parameter η . To be specific, we say that Γ , the image of a curve, is a *Fourier sampling trajectory* for Ω - or a *sampling trajectory* for the Paley-Wiener space $PW^2(\Omega)$ - if the following continuous sampling inequality holds:

$$A\|f\|_2^2 \leq \int_\Gamma |\widehat{f}(\xi)|^2 d\mathcal{H}^1(\xi) \leq B\|f\|_2^2, \quad f \in L^2(\Omega), \quad (5.3)$$

where $A, B > 0$ are stability constants, and \mathcal{H}^1 is the one dimensional Hausdorff (length) measure (see [56, 73]). Equivalently, Γ is a sampling trajectory if it contains a discrete sampling set (see Section 5.1.4).

Our first result reads as follows.

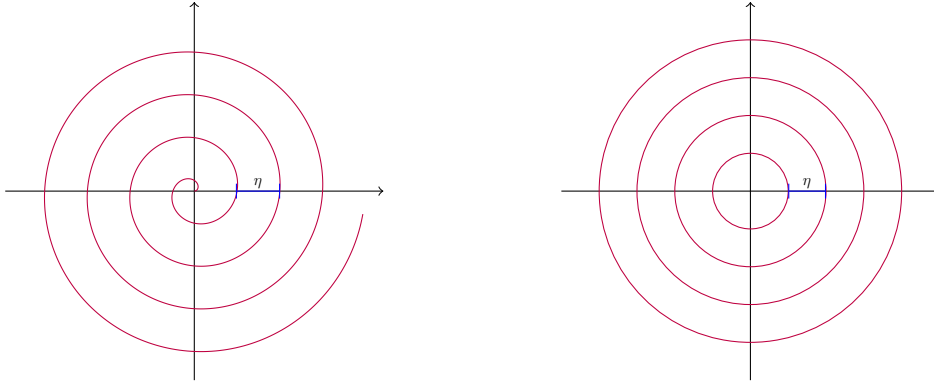


Figure 5.1: Archimedes spiral (left) and concentric circles (right) with separation η .

Theorem A. Let $\Omega \subset \mathbb{R}^2$ be a convex centered symmetric body.

- (i) If $\text{diam}(\Omega)\eta < 1$, then the Archimedes spiral A^η and the collection of concentric circles O^η are sampling trajectories for $PW^2(\Omega)$.
- (ii) If $\text{diam}(\Omega)\eta > 1$, then neither the Archimedes spiral A^η nor the collection of concentric circles O^η are sampling trajectories for $PW^2(\Omega)$.

Part (i) in Theorem A is due to Benedetto and Wu in the context of pointwise sampling [7]. Our contribution is mainly in (ii).

5.1.3 Sampling compressible signals below the Nyquist rate

Having identified the Nyquist rate of spiraling curves, we look into undersampling. Modern sampling schemes exploit the fact that many signals of interest are highly compressible, and this information is leveraged to sample below the Nyquist rate. For example, functions defined on the unit square, and obeying a variation bound

$$\mathcal{F}(W) := \{f \in L^2([-1/2, 1/2]^2) : \text{var}(f) \leq W\}$$

are compactly represented in a wavelet basis. Here, the resolution parameter W essentially controls the number of active wavelet coefficients [18, 23].

The stability of sampling schemes restricted to such signals is expressed by the inverse condition number

$$\inf \left\{ \left\| \hat{f} - \hat{g} \right\|_{L^2(\mu_\Gamma)} : f, g \in \mathcal{F}(W), \|f - g\|_2 = \varepsilon \right\},$$

See (5.24) for the definition of $\text{var}(f)$.

In standard terminology, the condition number of the sampling problem $f|_\Gamma \mapsto f$ is related to the *reciprocal* of this quantity.

where $\mu_\Gamma := \mathcal{H}^1|_\Gamma$ is the arc-measure, which measures the extent to which the distance between two signals is exhibited by their Fourier samples. Since, $\varepsilon^{-1}\mathcal{F}(W) = \mathcal{F}(\varepsilon^{-1}W)$, and $\mathcal{F}(W) \subset \mathcal{F}(W) - \mathcal{F}(W) \subset \mathcal{F}(2W)$, the analysis of the condition number for small ε reduces to the large W asymptotics of the *stability margin*:

$$A(\Gamma, \mathcal{F}(W)) := \inf \left\{ \left\| \hat{f} \right\|_{L^2(\mu_\Gamma)} : \|f\|_2 = 1, f \in \mathcal{F}(W) \right\}.$$

According to Theorem A, the critical value for the reconstruction of functions defined on the unit square with either A^η or O^η is $\eta = \sqrt{2}/2$. We consider spirals with density slightly under the critical value and prove the following.

Theorem B. Let $\eta = (1 + \varepsilon)\sqrt{2}/2$ with $\varepsilon \in (0, 1)$, and $\Gamma = A^\eta$ or $\Gamma = O^\eta$. Then for $W > 0$,

$$A(\Gamma, \mathcal{F}(W)) \leq C(\varepsilon W)^{-1/2}(\ln^2(\varepsilon W) + 1),$$

where $C > 0$ is a universal constant.

Theorem B thus sets a limit to the capacity of spirals to acquire all compressible signals below the Nyquist rate. Informally, it says that when undersampling by a small factor $(1 - \varepsilon)$, one can only recover functions up to resolution $W \approx \varepsilon^{-1}$ with a stable condition number.

A variant of Theorem B can be formulated in terms of the Haar wavelet. Let $\Sigma_{N,J}$ be the class of functions on $[-1/2, 1/2]^2$ with N non-zero Haar coefficients, all of them taken with scale at most J . We have the following estimate.

Theorem C. Let $W \geq 1$, $\eta = (1 + \varepsilon)\sqrt{2}/2$ with $\varepsilon \in (0, 1)$ and $\Gamma = A^\eta$ or $\Gamma = O^\eta$. Then for $N \geq 1$,

$$A(\Gamma, \Sigma_{N,J}) \leq CN^{-1/6}\varepsilon^{-1}\ln^4(CN),$$

where $J = C \ln(\varepsilon^{-1}N)$ and $C > 0$ is a universal constant.

Informally, Theorem C says that when undersampling by a small factor $(1 - \varepsilon)$, one can recover at most $N \approx \varepsilon^{-6}$ Haar coefficients with a stable condition number.

Theorem C complements related results that limit the wavelet-sparsity of discrete signals that can be sampled on unions of parallel lines [10]. Let us mention that the sparsity model $\Sigma_{N,J}$ is rather crude. Modern results in sparse recovery exploit the fine multiscale structure of the wavelet coefficients of natural signals [4].

The same fundamental stability restrictions expressed by Theorems C and D also apply to *any possible discretization of the continuous sampling trajectories* – see Theorem D below.

5.1.4 Overview

The problem of sampling the Fourier transform of a compactly supported function is equivalent to the sampling problem for the Paley-Wiener space of bandlimited functions. We make essential use of Beurling's sampling theory. The sufficient sampling condition in Theorem A follows from Beurling's gap covering Theorem [8], as done in [7]. The necessary condition in Theorem A is more challenging: little of the ample literature on necessary conditions for sampling [6, 28, 30, 31, 51, 52, 60, 61, 71] is applicable to sampling on curves, because the new relevant metric (length) is fundamentally different from the one corresponding to pointwise sampling (cardinality) [42].

Our main results show that spirals behave qualitatively like unions of parallel lines. While the analysis of sampling on parallel lines is based on periodization arguments [85, 86] and cross sections of the Fourier spectrum [42], in order to treat spirals, we develop approximate versions of those tools. The main observation is that an adequate sequence of translates of a spiral converges to a union of parallel lines, locally in the Hausdorff metric. This allows us to apply Beurling's characterization of sampling in terms of *weak limits* [8, 9, 78].

In order to apply weak-limit techniques to curves, we first need to connect pointwise and continuous sampling. We provide a variation of a result from Ortega-Cerdà on *sampling measures* [73], and show that, under mild regularity assumptions, the continuous sampling inequality (5.3) is equivalent to the existence of a sampling set contained in the sampling trajectory Γ - see Section 5.3.1 for precise definitions.

Theorem D. Let $\Gamma \subset \mathbb{R}^d$ be a regular trajectory, and $\Omega \subset \mathbb{R}^d$ bounded with positive measure. Then Γ is a sampling trajectory for $PW^2(\Omega)$ if and only if there exists a discrete set $\Lambda \subset \Gamma$ that is a sampling set for $PW^2(\Omega)$.

Theorem D shows that two common formulations of the mobile sampling problem are equivalent [85, 86]. As a further consequence of Theorem D, the sampling relation (5.3) also expresses the stability of a vast collection of *sampling schemes*, where functions are sampled on finite portions of the sampling trajectory Γ , and are reconstructed within a precisely described numerical accuracy [1, 2, 82, 83], and leads to well-understood truncation errors [51, 52] and implementation strategies [32, 40, 33, 88]. Moreover, any sufficiently dense set $\Lambda \subset \Gamma$ is an adequate discretization - cf. Remark 5.17.

As a second step we show that spirals suffer from *approximate aliasing*. Aliasing is the name given in signal processing to the artifacts produced by sampling on a lattice below the Nyquist rate. Aliasing is also the most obvious obstruction to subsampling compressible signals, and, heuristically, the success of sub-Nyquist sampling schemes relies on the fact that they avoid regular patterns [14, 62]. We

quantify the rate of convergence of spirals to parallel lines and derive approximate aliasing for the curves A^n (5.1) and O^n (5.2). As a consequence we obtain an upper bound on the stability margin for the reconstruction of functions of bounded variation (Theorem B) or of functions that have few active Haar coefficients (Theorem C). These results underscore the need for a certain level of randomness in structured sampling [12] and for refined multiscale models [3, 4] that apply to generic signals.

This article is organized as follows: in Section 2 we review standard notions from sampling theory and provide preliminary results; in Section 3 we derive the characterization of sampling trajectories (Theorem D) and introduce spiraling curves; in Section 4 we give necessary and sufficient conditions for sampling on spiraling curves, and prove Theorem A; in Section 5 quantify the rate of convergence of spirals to collections of parallel lines and explore consequences on approximate aliasing; and in Section 6 we prove the results on sub-Nyquist sampling, Theorems B and C.

5.2 Preliminaries on pointwise sampling

5.2.1 Notation

Throughout this paper, we will adopt the following notation: on \mathbb{R}^d , $|\cdot|$ and $\langle \cdot, \cdot \rangle$ are the usual Euclidean norm and scalar product. For sets we will use $\text{diam} E = \sup_{x,y \in E} |x - y|$, $\text{dist}(E, F) = \inf_{x \in E, y \in F} |x - y|$. The balls are denoted by $B_r(x) := \{y \in \mathbb{R}^d : |y - x| < r\}$. For points on \mathbb{R}^2 we will use the notation $x = (x_1, x_2)$, and additionally, vectors on \mathbb{S}^1 will be written as \vec{d}, \vec{l} , and if $\vec{d} = (x_1, x_2)$ we set $\vec{d}_\perp = (-x_2, x_1)$. Clockwise rotations in \mathbb{R}^2 will be denoted by $\mathcal{R}_{2\pi\theta}$ with $\theta \in [0, 1)$ and where $2\pi\theta$ is the angle of rotation. Unless otherwise stated, measures on \mathbb{R}^d are assumed to take values in $[0, +\infty]$.

Throughout the paper, for $A, B \in \mathbb{R}$, $A \lesssim B$ means that there exists a constant $C > 0$ independent from A and B such that $A \leq CB$. For functions (or measures) f, g , $f \lesssim g$ means that $f(x) \leq Cg(x)$ for all x where f and g are defined. Further, $A \asymp B$, means $A \lesssim B \lesssim A$ (and the same for functions or measures). When we want to particularly stress the dependence of the implicit constant C on other factors we may write instead $C_p, C_\gamma, C_{\Omega,p}, \dots$

5.2.2 Convex bodies

A set $\Omega \subset \mathbb{R}^d$ is called a *convex body* if it is convex, compact and has non-empty interior. A convex body is called *centered* if $0 \in \Omega^\circ$ and *symmetric* if $\Omega = -\Omega$. We

will frequently use the fact that, for a convex centered symmetric body Ω ,

$$\Omega \subset (1 + \varepsilon)\Omega^\circ \text{ and } (1 - \varepsilon)\Omega \subset \Omega^\circ, \quad \varepsilon \in (0, 1). \quad (5.4)$$

5.2.3 Paley-Wiener spaces

Let us begin by recalling the definition of the standard function spaces involved in sampling theorems. Here and thereafter we will consider the normalized version of the Fourier transform:

$$\hat{f}(\xi) = \int_{\mathbb{R}^d} e^{-2\pi i \xi \cdot x} f(x) \, dx$$

for $f : \mathbb{R}^d \rightarrow \mathbb{R}$ integrable. The Fourier transform is then extended to Schwartz distributions in the usual way.

Definition 5.1. *Let $\Omega \subset \mathbb{R}^d$ be a compact set of positive measure and $1 \leq p \leq \infty$. We define $PW^p(\Omega)$ as the subset of $L^p(\mathbb{R}^d)$ consisting of Fourier transforms of distributions supported in Ω .*

The classical *Paley-Wiener space* corresponds to $p = 2$, while $p = \infty$ yields the *Bernstein space*. This latter space models possibly non-decaying bandlimited signals, although it has some disadvantages for signal processing, such as lack of invariance under the Hilbert transform. (A remedy to some of these obstacles has been proposed in [63].)

Functions in Bernstein spaces are *entire functions of exponential type*, as follows from the direct side of Paley-Wiener's Theorem [80, Theorem 4.9]:

Proposition 5.2. *Given a convex centered symmetric body $\Omega \subset \mathbb{R}^d$ and $1 \leq p \leq \infty$, there exists a constant $c > 0$ depending only on Ω such that every $f \in PW^p(\Omega)$ can be extended to an entire function with*

$$|f(x + iy)| \lesssim e^{c|y|}, \quad x, y \in \mathbb{R}^d.$$

Functions of exponential type enjoy the following norm control of their analytic extensions on horizontal lines.

Proposition 5.3. *Let f be an entire function in \mathbb{C}^d with $|f(x + iy)| \leq M e^{c|y|}$ for all $x, y \in \mathbb{R}^d$ and where $M, c > 0$ are constants. If $f(x) \in L^p(\mathbb{R}^d)$ with $1 \leq p \leq \infty$ then for all $y \in \mathbb{R}^d$,*

$$\left(\int_{\mathbb{R}^d} |f(x + iy)|^p \, dx \right)^{1/p} \leq e^{c|y|} \left(\int_{\mathbb{R}^d} |f(x)|^p \, dx \right)^{1/p}$$

with the usual modifications when $p = \infty$.

For $1 \leq p < \infty$, Proposition 5.3 is referred as the Plancharel-Pólya inequality and for $p = \infty$ as the Phragmén-Lindelöf principle; see, e.g., [80, Lemma 4.3 and 4.11].

5.2.4 Sampling sets and Beurling's gap theorem

A set $\Lambda \subset \mathbb{R}^d$ is said to be *sampling* for $PW^p(\Omega)$ if

$$\|f\|_p \asymp \|f\|_{\ell^p(\Lambda)}, \quad f \in PW^p(\Omega),$$

where $\|f\|_{\ell^p(\Lambda)} = (\sum_{\lambda \in \Lambda} |f(\lambda)|^p)^{1/p}$, if $p < \infty$, and $\|f\|_{\ell^\infty(\Lambda)} = \sup_{\lambda \in \Lambda} |f(\lambda)|$.

A set $\Lambda \subset \mathbb{R}^d$ is *separated* if its *separation*

$$\inf_{\lambda, \lambda' \in \Lambda, \lambda \neq \lambda'} |\lambda - \lambda'|$$

is positive, and it is *relatively dense* if its *gap* (or hole)

$$\text{gap}(\Lambda) := \sup_{x \in \mathbb{R}^d} \inf_{\lambda \in \Lambda} |x - \lambda|$$

is finite.

The most effective sufficient condition for sampling bandlimited functions in high dimension is formulated in terms of gaps, and is due to Beurling [8, 9] - see also [7, 72].

Theorem 5.4. *Let $\Lambda \subset \mathbb{R}^d$ and $R > 0$. If $\text{gap}(\Lambda) < 1/(2R)$ then Λ is a sampling set for $PW^\infty(\bar{B}_{R/2}(0))$.*

The value $1/(2R)$ in Theorem 5.4 is critical in the sense that there exists a set Λ with $\text{gap}(\Lambda) = 1/(2R)$ that is not sampling for the spectrum $\bar{B}_{R/2}(0)$. On the other hand, as examples of Theorem 5.4 we note that

$$\text{gap}(A^\eta) = \text{gap}(O^\eta) = \eta/2, \tag{5.5}$$

see e.g. [7, Example 2], and hence these sets are sampling for $PW^\infty(\bar{B}_{R/2}(0))$ whenever $\eta R < 1$.

While Theorem 5.4 applies to arbitrary sets, sometimes it is convenient to work with separated sets. We state without proof the following elementary lemma.

Lemma 5.5. *Let $\Lambda \subset \mathbb{R}^d$, and $R > \text{gap}(\Lambda)$. Then there exists a separated set $\Lambda' \subset \Lambda$ such that $\text{gap}(\Lambda') \leq R$.*

5.2.5 Sampling with different norms

The following classical result shows that the sampling problems associated with PW^2 and PW^∞ are almost equivalent. See [72, Theorem 2.1] for a simple proof.

Theorem 5.6. *Let $\Lambda \subset \mathbb{R}^d$ be a separated set, $\Omega \subset \mathbb{R}^d$ a compact set of positive measure and $\varepsilon > 0$.*

(i) If Λ is sampling for $PW^2(\Omega + \bar{B}_\varepsilon(0))$ then it is sampling for $PW^\infty(\Omega)$.

(ii) If Λ is sampling for $PW^\infty(\Omega + \bar{B}_\varepsilon(0))$ then it is sampling for $PW^2(\Omega)$.

As an application, we obtain the following corollary for convex bodies.

Corollary 5.7. *Let $\Lambda \subset \mathbb{R}^d$ be a separated set, $\Omega \subset \mathbb{R}^d$ a convex centered symmetric body and $\varepsilon \in (0, 1)$.*

(i) *If Λ is sampling for $PW^2(\Omega)$, then it is sampling for $PW^\infty((1 - \varepsilon)\Omega)$.*

(ii) *If Λ is sampling for $PW^\infty(\Omega)$, then it is sampling for $PW^2((1 - \varepsilon)\Omega)$.*

Proof. By (5.4), $(1 - \varepsilon)\Omega \subset \Omega^\circ$. Since Ω is compact, this implies that

$$\text{dist}((1 - \varepsilon)\Omega, (\Omega^\circ)^c) > \varepsilon',$$

and, therefore, $(1 - \varepsilon)\Omega + \bar{B}_{\varepsilon'}(0) \subset \Omega$. The conclusions now follow from Theorem 5.6. \square

5.2.6 Characterization of sampling with weak limits

Definition 5.8. *A set $\Lambda \subset \mathbb{R}^d$ is called a set of uniqueness for $PW^p(\Omega)$ if $f \in PW^p(\Omega)$ with $f|_\Lambda = 0$ implies $f \equiv 0$.*

Sampling sets are sets of uniqueness. The converse is not true, but it is a remarkable insight due to Beurling, that it is still possible to characterize a sampling set through the uniqueness of what are called its *weak limits*.

Definition 5.9. *Let $\Lambda \subset \mathbb{R}^d$ be a closed set and let $\{\Lambda_n\}_{n \geq 1}$ be a sequence of closed sets. Then we say that $\{\Lambda_n\}_{n \geq 1}$ converges weakly to Λ if for all $R, \varepsilon > 0$ there exist $n_{R, \varepsilon}$ such that*

$$\begin{aligned} \Lambda_n \cap (-R, R)^d &\subset \Lambda + B_\varepsilon(0), \\ \Lambda \cap (-R, R)^d &\subset \Lambda_n + B_\varepsilon(0), \end{aligned}$$

hold for all $n \geq n_{R, \varepsilon}$. In this case we write $\Lambda_n \xrightarrow{w} \Lambda$.

Weak convergence can also be formulated in terms of the *Hausdorff distance* between two sets $X, Y \subset \mathbb{R}^d$:

$$d_{\mathcal{H}}(X, Y) := \inf \{ \varepsilon > 0 : X \subset Y + B_\varepsilon(0), Y \subset X + B_\varepsilon(0) \}.$$

Indeed, $\Lambda_n \xrightarrow{w} \Lambda$ if and only if for all $x \in \mathbb{R}^d$ and $R > 0$

$$d_{\mathcal{H}}((\Lambda_n \cap \bar{B}_R(x)) \cup \partial B_R(x), (\Lambda \cap B_R(x)) \cup \partial B_R(x)) \rightarrow 0.$$

(To appreciate the role of the boundary of the ball in the last equation, consider the following example in dimension $\mathbf{d} = 1$: $\Lambda_n := \{1 + 1/n\}$, $\Lambda := \{1\}$ and $B_R(x) = (0, 1)$.)

The type of weak limits that are needed to characterize sampling sets are those resulting from translates. We will denote the set of weak limits of translates of Λ by $W(\Lambda)$. Hence, $\Lambda' \in W(\Lambda)$ if and only if there exists a sequence $\{x_n\}_{n \in \mathbb{N}} \subset \mathbb{R}^{\mathbf{d}}$ such that $\Lambda + x_n \xrightarrow{w} \Lambda'$.

Theorem 5.10 (Beurling, [9, Theorem 3, pg. 345]). *Let $\Omega \subset \mathbb{R}^{\mathbf{d}}$ be a convex centered symmetric body and let $\Lambda \subset \mathbb{R}^{\mathbf{d}}$. Then Λ is a sampling set for $PW^\infty(\Omega)$ if and only if for all weak limits $\Lambda' \in W(\Lambda)$, Λ' is a set of uniqueness for $PW^\infty(\Omega)$.*

We will use the following compactness result. See, e.g., [41, Section 4] for proofs.

Lemma 5.11. *Let $\Lambda \subset \mathbb{R}^{\mathbf{d}}$ be a separated set and $\{x_n\}_{n \geq 1} \subset \mathbb{R}^{\mathbf{d}}$. Then there exist a subsequence $\{x_{n_k}\}_{k \geq 1}$ and a separated set $\Lambda' \subset \mathbb{R}^{\mathbf{d}}$ such that $\Lambda + x_{n_k} \xrightarrow{w} \Lambda'$.*

5.2.7 Sampling measures

A Borel measure μ on $\mathbb{R}^{\mathbf{d}}$ is said to be *sampling* for $PW^p(\Omega)$ if

$$\|f\|_p \asymp \|f\|_{L^p(\mu)}, \quad f \in PW^p(\Omega),$$

where $\|f\|_{L^p(\mu)} = (\int_{\mathbb{R}^{\mathbf{d}}} |f(x)|^p d\mu)^{1/p}$ if $1 \leq p < \infty$, and $\|f\|_{L^\infty(\mu)} = \text{ess sup}_\mu |f|$.

Viewed in this way, a set Λ is sampling when the associated point measure $\delta_\Lambda := \sum_{\lambda \in \Lambda} \delta_\lambda$ is a sampling measure. Let us first notice that sampling measures are uniformly bounded:

Lemma 5.12. *Let $\Omega \subset \mathbb{R}^{\mathbf{d}}$ be a convex centered symmetric body and $1 \leq p < \infty$. Let μ be a sampling measure for $PW^p(\Omega)$. Then, for all $R > 0$, $\sup_{x \in \mathbb{R}^{\mathbf{d}}} \mu(B_R(x)) < \infty$.*

Proof. First note that it is enough to construct $f_0 \in PW^p(\Omega)$ such that $|f_0| \gtrsim \chi_{(-R,R)^{\mathbf{d}}}$, where the implied constant may depend on R . Indeed, once f_0 is given, we define $f_x(t) := f_0(t - x)$ and note that $\widehat{f_x}(\xi) = e^{-2\pi i x \xi} \widehat{f_0}(\xi)$ so that $f_x \in PW^p(\Omega)$. Moreover, since μ is sampling and $1 \leq p < \infty$, we get

$$\mu(B_R(x)) \leq \mu((x - R, x + R)^{\mathbf{d}}) \lesssim \|f_x\|_{L^p(\mu)}^p \asymp \|f_x\|_p^p = \|f_0\|_p^p.$$

Next, to construct f_0 , we take $\varepsilon > 0$ such that $(-\varepsilon, \varepsilon)^{\mathbf{d}} \subset \Omega$. If we find $\varphi \in PW^p(-\varepsilon, \varepsilon)$ such that, for every R , $|\varphi| \gtrsim \chi_{(-R,R)}$ then $f_0(x_1, \dots, x_d) = \varphi(x_1) \times \dots \times \varphi(x_d)$ will do.

Now let $\psi := \chi_{(-\varepsilon/2, \varepsilon/2)} * \chi_{(-\varepsilon/2, \varepsilon/2)}$ so that

— ψ has support $(-\varepsilon, \varepsilon)$;

— $\psi = \widehat{\varphi_0}$ where $\varphi_0(t) = \left(\frac{\sin(\pi \varepsilon t)}{\pi t}\right)^2$; hence, in particular, $\varphi_0 \in PW^p(-\varepsilon, \varepsilon)$;

— φ_0 is continuous, nonnegative, and $\varphi_0(x) = 0$ if and only if $x = k/\varepsilon$, $k \in \mathbb{Z} \setminus \{0\}$.

Finally, we set $\varphi(x) := \varphi_0(x) + \varphi_0(x + 1/2\varepsilon)$. Then φ is a continuous function in $PW^p(-\varepsilon, \varepsilon)$ that never vanishes. This means that, for every $R > 0$, $\varphi \gtrsim \chi_{(-R, R)}$. The proof is thus complete. \square

We now show that for any sampling measure we can extract a weighted sampling set. The argument mirrors that of [73] for functions on the Bargmann-Fock space.

Theorem 5.13. *Let $\Omega \subset \mathbb{R}^d$ be a convex centered symmetric body, $1 \leq p < \infty$ and μ a Borelian measure in \mathbb{R}^d . Let $r > 0$ and $\{Q_n^r\}_n$ a (measurable) covering of \mathbb{R}^d with finite multiplicity and $\sup_n \text{diam} Q_n^r \leq r$, and set $\mu_r^* := \sum_n \mu(Q_n^r) \delta_{a_n^r}$ where $a_n^r \in Q_n^r$ are arbitrary points.*

Then there exists a constant $C_{\Omega, p} > 0$ such that μ is sampling for $PW^p(\Omega)$ if and only if μ_r^ is sampling for $PW^p(\Omega)$ when $r < C_{\Omega, p}$.*

Remark 5.14. *Note that, in any case, $\sup_{x \in \mathbb{R}^d} \mu(B_R(x)) < \infty$ holds for all $R > 0$. Indeed, if μ is sampling for $PW^p(\Omega)$, this is Lemma 5.12. On the other hand, if μ_r^* is sampling for $PW^p(\Omega)$, Lemma 5.12 applied to μ_r^* reads $\sup_{x \in \mathbb{R}^d} \mu_r^*(B_R(x)) < \infty$ for all $R > 0$, and thus*

$$\mu(B_\rho(x)) \leq \sum_{Q_n^r \cap B_\rho(x) \neq \emptyset} \mu(Q_n^r) \leq \mu_r^*(B_{\rho+r}(x)) \leq C_{\rho, r}, \quad \forall \rho > 0, x \in \mathbb{R}^d,$$

since $\{Q_n^r\}_n$ is a covering with $\sup_n \text{diam} Q_n^r \leq r$.

Proof of Theorem 5.13. Let $f \in PW^p(\Omega)$. First note that since $\{Q_n^r\}_n$ is a covering of \mathbb{R}^d of finite multiplicity then

$$\int_{\mathbb{R}^d} |f(x)|^p d\mu(x) \asymp \sum_n \int_{Q_n^r} |f(x)|^p d\mu(x). \quad (5.6)$$

It follows that, μ is sampling if

$$\int_{\mathbb{R}^d} |f(x)|^p dx \asymp \sum_n \int_{Q_n^r} |f(x)|^p d\mu(x), \quad (5.7)$$

and by definition μ_r^* is sampling if

$$\int_{\mathbb{R}^d} |f(x)|^p dx \asymp \sum_n |f(a_n^r)|^p \mu(Q_n^r). \quad (5.8)$$

Next, using the inequality $(a + b)^p \leq 2^{p-1}(a^p + b^p)$ in each Q_n^r , we see that

$$\begin{aligned} \sum_n \int_{Q_n^r} |f(x)|^p d\mu(x) &\leq 2^{p-1} \sum_n \int_{Q_n^r} |f(x) - f(a_n^r)|^p + |f(a_n^r)|^p d\mu(x) \\ &\lesssim I(r) + \sum_n |f(a_n^r)|^p \mu(Q_n^r) \end{aligned}$$

where $I(r) := \sum_n \int_{Q_n^r} |f(x) - f(a_n^r)|^p d\mu(x)$. Similarly,

$$\begin{aligned} \sum_n |f(a_n^r)|^p \mu(Q_n^r) &= \sum_n \int_{Q_n^r} |f(a_n^r)|^p d\mu(x) \\ &\leq 2^{p-1} \sum_n \int_{Q_n^r} |f(x) - f(a_n^r)|^p + |f(x)|^p d\mu(x) \\ &\lesssim I(r) + \sum_n \int_{Q_n^r} |f(x)|^p d\mu(x). \end{aligned}$$

Hence, to prove that (5.7) and (5.8) are equivalent for some small value of r it is enough to show that $I(r)$ is also sufficiently small. Precisely, we will now show that there exists a function $\phi(r)$ with $\phi(r) \xrightarrow{r \rightarrow 0} 0$ such that

$$I(r) \leq \phi(r) \int_{\mathbb{R}^d} |f(x)|^p dx. \quad (5.9)$$

We proceed to prove (5.9). Since $f \in PW^p(\Omega)$ then, by Proposition 5.2, it has a complex extension $f(x + iy)$ which is an entire function of exponential type (with constants depending on Ω). Hence, in particular, f is harmonic on \mathbb{C}^d and satisfies the mean value theorem that we can write in the form of $f = f * \frac{1}{|B_1(0)|} \chi_{B_1(0)}$ (convolution is in \mathbb{C}^d). Further, iterating this formula, we get $f = f * \varphi$ with $\varphi = \frac{1}{|B_1(0)|} \chi_{B_1(0)} * \cdots * \frac{1}{|B_1(0)|} \chi_{B_1(0)}$. Finally, notice that if we iterate sufficiently many times, φ is a compactly supported function of class \mathcal{C}^1 . Thus, for each n and $x \in Q_n^r$ we may write

$$\begin{aligned} |f(x) - f(a_n^r)|^p &= |(f * \varphi)(x) - (f * \varphi)(a_n^r)|^p \\ &\leq \left(\int_{\mathbb{R}^{2d}} |f(y + iz)(\varphi(x - y - iz) - \varphi(a_n^r - y - iz))| dy dz \right)^p. \end{aligned}$$

Let $r_0 > 0$ be such that $\text{supp}(\varphi) \subset B_{r_0}(0)$. Since $|x - a_n^r| \leq \text{diam} Q_n^r \leq r$ and both x and a_n^r are real, then the function $\varphi(x - y - iz) - \varphi(a_n^r - y - iz)$ is supported in $A_r(x) := \{y + iz \in \mathbb{C}^d : |y - x| < r + r_0, |z| < r_0\}$. Together with Hölder's

inequality, this gives us

$$|f(x) - f(a_n^r)|^p \leq \|\varphi(x - \cdot - i\cdot) - \varphi(a_n^r - \cdot - i\cdot)\|_{L^{p'}(A_r(x))}^p \times \int_{A_r(x)} |f(y + iz)|^p dy dz,$$

where $1/p' + 1/p = 1$. We bound $\|\varphi(x - \cdot - i\cdot) - \varphi(a_n^r - \cdot - i\cdot)\|_{L^{p'}(A_r(x))}^p$ by using the fact that $\varphi \in C_c^1$ and then applying the Mean Value Theorem

$$|\varphi(x - y - iz) - \varphi(a_n^r - y - iz)| \leq r \|\nabla \varphi\|_\infty,$$

which holds for all $y + iz \in \mathbb{C}^d$. Therefore

$$\begin{aligned} \|\varphi(x - \cdot, \cdot) - \varphi(a_n^r - \cdot, \cdot)\|_{L^{p'}(A_r(x))}^p &\lesssim r^p |A_r(x)|^{p/p'} \\ &\lesssim r^p (r + r_0)^{dp/p'} r_0^{dp/p'} =: \phi(r). \end{aligned}$$

Running the sum over all n we get

$$\begin{aligned} I(r) &= \sum_n \int_{Q_n^r} |f(x) - f(a_n^r)|^p d\mu(x) \\ &\leq \phi(r) \sum_n \int_{Q_n^r} \int_{A_r(x)} |f(y + iz)|^p dy dz d\mu(x) \\ &\lesssim \phi(r) \int_{\mathbb{R}^d} \int_{A_r(x)} |f(y + iz)|^p dy dz d\mu(x) \\ &= \phi(r) \int_{\mathbb{R}^d} \int_{|z| < r_0} \int_{|x-y| < r+r_0} |f(y + iz)|^p dy dz d\mu(x). \end{aligned}$$

since $\{Q_n^r\}_n$ has finite multiplicity. Tonelli's theorem then implies

$$\begin{aligned} I(r) &\lesssim \phi(r) \int_{\mathbb{R}^d} \int_{|z| < r_0} \int_{|x-y| < r+r_0} d\mu(x) |f(y + iz)|^p dz dy \\ &\lesssim \phi(r) \int_{\mathbb{R}^d} \int_{|z| < r_0} |f(y + iz)|^p dz dy \end{aligned}$$

since $\sup_{x \in \mathbb{R}^d} \mu(B_{r+r_0}(x)) < \infty$ (cf. Remark 5.14). Finally applying Proposition 5.3 we get

$$I(r) \lesssim \phi(r) \int_{\mathbb{R}^d} |f(x)|^p dx.$$

Eventually multiplying $\phi(r)$ by constants, this gives (5.9). \square

Remark 5.15. Note that the only impediment to extend Theorem 5.13 to the case $p = \infty$ is Remark 5.14 (which does not apply for $p = \infty$). However, if we suppose in addition that $\sup_{x \in \mathbb{R}^d} \mu(B_R(x)) < \infty$ for some $R > 0$ (which is a weaker condition to the conclusion of Remark 5.14) then we get back the result of Theorem 5.13.

5.3 Sampling trajectories and spiraling curves

5.3.1 Sampling trajectories

A *curve* is a measurable map $\gamma : \mathbb{R} \rightarrow \mathbb{R}^d$ (which we do not require to be continuous). A *trajectory* Γ is the image of a curve: $\Gamma := \gamma(\mathbb{R})$. The restriction of the one-dimensional Hausdorff measure to Γ is denoted $\mu_\Gamma := \mathcal{H}_\Gamma^1$. A trajectory Γ is called *regular* if there exists $r_0 > 0$ such that for all $r \leq r_0$ there is a constant $c_r > 0$ for which

$$\inf_{x \in \Gamma} \mu_\Gamma(B_r(x)) \geq c_r, \quad (5.10)$$

and

$$\sup_{x \in \mathbb{R}^d} \mu_\Gamma(B_{r_0}(x)) < +\infty. \quad (5.11)$$

A trajectory Γ is called a *sampling trajectory* for $PW^p(\Omega)$ if μ_Γ is a sampling measure, i.e.,

$$\|f\|_p^p \asymp \int_\Gamma |f(x)|^p d\mathcal{H}^1(x), \quad f \in PW^p(\Omega),$$

with the usual modification for $p = \infty$.

5.3.2 Characterization of sampling trajectories

Using the general result we proved for sampling measures, Theorem 5.13, we can show how to extract a sampling set from a sampling trajectory. This is Theorem D presented in the Introduction and that we recall here for the convenience of the reader:

Theorem D. Let $\Gamma \subset \mathbb{R}^d$ be a regular trajectory, $\Omega \subset \mathbb{R}^d$ bounded with positive measure. Then Γ is a sampling trajectory for $PW^2(\Omega)$ if and only if there exists $\Lambda \subset \Gamma$ that is a sampling set for $PW^2(\Omega)$. Moreover, Λ can always be chosen to be separated.

Remark 5.16. Although the statement concerns PW^2 , we remark that the following proof is still valid for any PW^p with $1 \leq p \leq \infty$.

Proof. Due to Theorem 5.13 it is enough to show that there exists a separated set $\Lambda \subset \Gamma$ such that its point measure $\delta_\Lambda = \sum_{\lambda \in \Lambda} \delta_\lambda$ is equivalent to $\mu_r^* = \sum_n \mu_\Gamma(Q_n^r) \delta_{a_n^r}$ where $a_n^r \in Q_n^r$ and $\{Q_n^r\}_n$ is a covering of finite multiplicity of \mathbb{R}^d with $\sup_n \text{diam} Q_n^r < r$, and r is small enough. Moreover, in this case we can discard those Q_n which do not meet Γ (since $\mu_\Gamma(Q_n) = 0$ when $Q_n \cap \Gamma = \emptyset$), and just consider coverings of Γ .

Given $r > 0$, let $\{a_n^r\}_n \subset \Gamma$ be maximal with respect to $|a_n^r - a_m^r| \geq r, n \neq m$. Hence, the family $\{B_r(a_n^r)\}_n$ is a covering of Γ with finite multiplicity (the covering number being bounded by 4^d). Next, set $\mu_r^* := \sum_n \mu_\Gamma(B_r(a_n^r)) \delta_{a_n^r}$ and $\Lambda_r := \{a_n^r\}_n$. Let r_0 be the constant (related to Γ) in (5.10). Taking $r < r_0$ reads $\mu_\Gamma(B_r(a_n^r)) \geq c_r$ for all n , and therefore $\mu_r^* \gtrsim \sum_n \delta_{a_n^r} = \delta_{\Lambda_r}$. On the other hand, when $r < r_0$ the condition (5.11) reads $\sup_n \mu(B_r(a_n^r)) < \infty$ so that also $\mu_r^* \lesssim \delta_{\Lambda_r}$. In sum for all $r < r_0$ we can construct a finite multiplicity r -covering $\{Q_n^r\}_n$ together with a separated set Λ_r such that $\mu_r^* = \sum_{\lambda \in \Lambda_r} \mu_\Gamma(Q_n^r) \delta_\lambda \asymp \sum_{\lambda \in \Lambda_r} \delta_\lambda = \delta_{\Lambda_r}$. \square

Remark 5.17. *The proof of Theorem D shows that if Γ is a sampling trajectory, then any separated and sufficiently dense $\Lambda \subset \Gamma$ is sampling for $PW^2(\Omega)$.*

Theorem D shows the equivalence of two possible models for the mobile sampling problem [42, 85, 86]. As a consequence, sampling trajectories lead to concrete reconstruction strategies and numerical implementations, where a finite set of samples is used, and reconstruction is achieved within a precise numerical accuracy [1, 2, 32, 82, 83, 88].

5.3.3 Spiraling curves

Let us now describe precisely what we mean by a *spiraling* trajectory. The properties we list below have been chosen so that several natural sampling strategies are covered. Concentric circles and the Archimedes spiral will be shown to satisfy these properties (see Proposition 5.19). Other examples are depicted in Figure 5.3.

A regular trajectory Γ is called spiraling if the following conditions hold:

- (i) (*Escape cone*). There exist $\alpha \in (0, 1/4)$ and $\beta \in (0, 1)$ such that the portion of Γ contained in the cone

$$S_{\alpha, \beta} := \{(\rho \cos 2\pi\theta, \rho \sin 2\pi\theta) : \rho \geq 0, \beta - \alpha \leq \theta \leq \beta + \alpha\} \quad (5.12)$$

can be parametrized in polar coordinates as

$$\gamma(\theta) = (\rho(\theta) \cos 2\pi\theta, \rho(\theta) \sin 2\pi\theta) \quad (5.13)$$

with $\theta \in \bigcup_{k \in \mathbb{N}} [k + \beta - \alpha, k + \beta + \alpha]$ and where $\rho(\theta)$ is a non negative function of class C^2 in each interval. In particular, this means that, inside the escape

cone, each piece of the trajectory $\{\gamma(\theta) : \theta \in [k + \beta - \alpha, k + \beta + \alpha]\}$, joins one of the boundary lines of the cone to the other one.

We denote by $\vec{l} = (\cos 2\pi\beta, \sin 2\pi\beta)$ the bisector vector of the escape cone. Every piece of the trajectory in the escape cone intersects the half-line $\mathbb{R}_+\vec{l}$ once.

(ii) (*Asymptotic radial monotonicity*). There exists k_ρ such that for any $\theta \in [\beta - \alpha, \beta + \alpha]$ the sequence $\rho(\theta + k)$ is strictly increasing for $k \geq k_\rho$.

In particular, inside the escape cone, two different pieces of the trajectory, $\{\gamma(\theta) : \theta \in [k + \beta - \alpha, k + \beta + \alpha]\}$ and $\{\gamma(\theta) : \theta \in [l + \beta - \alpha, l + \beta + \alpha]\}$ with $l \neq k$ and $l, k \geq k_\rho$, do not intersect.

(iii) (*Asymptotic flatness*). The curvature of $\gamma(\theta)$, denoted by $\kappa(\theta)$, tends to 0 as $\theta \rightarrow +\infty$, i.e. for all $\varepsilon > 0$ there exists $k_\varepsilon \in \mathbb{N}$ such that $\kappa(\theta) < \varepsilon$ whenever $\theta \in [k + \beta - \alpha, k + \beta + \alpha]$ with $k \geq k_\varepsilon$.

(iv) (*Asymptotic equispacing*). There are two parameters $\eta, \rho_0 > 0$ such that the sequence $\eta_k = \rho(k + \beta)$ has the property

$$\lim_k \eta_k - \eta k = \rho_0.$$

(v) (*Asymptotic velocity*). There exists a direction $\vec{d} \in \mathbb{S}^1$ non-collinear with \vec{l} such that

$$\lim_{k \rightarrow +\infty} \frac{\gamma'(k + \beta)}{|\gamma'(k + \beta)|} = \vec{d}.$$

The number $\tau := \eta\sqrt{1 - \langle \vec{l}, \vec{d} \rangle^2}$ is called the *asymptotic separation* of Γ . For short, we say that Γ is a spiraling trajectory with asymptotic velocity \vec{d} and asymptotic separation τ . Note that those parameters may not be unique.

Remark 5.18. *The class of spiraling curves is invariant under rotations. Indeed, if we rotate a spiraling curve by angle of $2\pi\theta_0$ with $\theta_0 \in [0, 1)$ then the resulting curve is spiraling with parametrization in the escape cone with $\tilde{\beta} = \beta - \theta_0$, $\tilde{\alpha} = \alpha$, and*

$$\tilde{\gamma}(\theta) = (\tilde{\rho}(\theta) \cos 2\pi\theta, \tilde{\rho}(\theta) \sin 2\pi\theta), \quad \theta \in \bigcup_k [k + \tilde{\beta} - \alpha, k + \tilde{\beta} + \alpha],$$

where $\tilde{\rho}(\theta) = \rho(\theta + \theta_0)$. The rotated curve has asymptotic velocity $R_{2\pi\theta_0}\vec{d}$, while the parameters of asymptotic equispacing η , ρ_0 and the asymptotic separation τ remain unaltered.

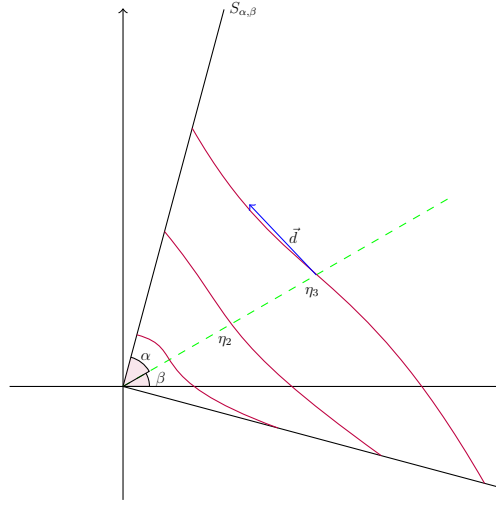


Figure 5.2: Sketch of an spiraling curve.

Further, spiraling curves are also invariant under some reasonable smooth perturbations of the escape cone. More precisely, let us first assume that $\beta = 0$ so that the escape cone is $S_{\alpha,0}$ and let $f : S_{\alpha,0} \rightarrow S_{\alpha',0}$ be a \mathcal{C}^2 one-to-one function. Assume the following:

- (i) $f((\rho \cos 2\pi\alpha, \pm\rho \sin 2\pi\alpha)) = (\rho \cos 2\pi\alpha', \pm\rho \sin 2\pi\alpha')$, i.e. f sends the boundary of $S_{\alpha,0}$ to the boundary of $S_{\alpha',0}$.
- (ii) $f(x_1, 0) = (\varphi(x_1), 0)$ for an asymptotically affine function $\varphi : \mathbb{R} \rightarrow \mathbb{R}$, that is, $\varphi(x_1) - (ax_1 + b) \rightarrow 0$ when $x_1 \rightarrow +\infty$ for some constants $a > 0$, $b \geq 0$.
- (iii) If $\rho : [-2\pi\alpha, 2\pi\alpha] \rightarrow [0, +\infty)$ then the curve $f((\rho(\theta) \cos 2\pi\theta, \rho(\theta) \sin 2\pi\theta))$ admits a parametrization in polar coordinates $(\tilde{\rho}(\theta) \cos 2\pi\theta, \tilde{\rho}(\theta) \sin 2\pi\theta)$ with $\tilde{\rho} : [-2\pi\alpha', 2\pi\alpha'] \rightarrow [0, +\infty)$.

Note that, as f is one-to-one, sends the bisector on itself and it behaves asymptotically like the increasing linear function $ax_1 + b$ in $(x_1, 0)$, f asymptotically preserves radial monotonicity.

- (iv) The Jacobian of f is uniformly bounded from above and below, i.e. there exist $A, B > 0$ such that $A|y| \leq |J_x f(y)| \leq B|y|$ for all $x \in S_{\alpha,0}$, $y \in \mathbb{R}^2$. Moreover $J_{(x_1,0)} f \rightarrow M$ when $x_1 \rightarrow +\infty$, and $M\vec{d} \neq (1, 0)$.
- (v) The Hessian of f goes to 0 when $x \rightarrow \infty$, i.e. $H_x f \rightarrow 0$ when $|x| \rightarrow +\infty$.

Then, if Γ is a spiraling trajectory with asymptotic velocity \vec{d} and asymptotic equispacing η , $f(\Gamma)$ is a spiraling trajectory with asymptotic equispacing $a\eta$ and

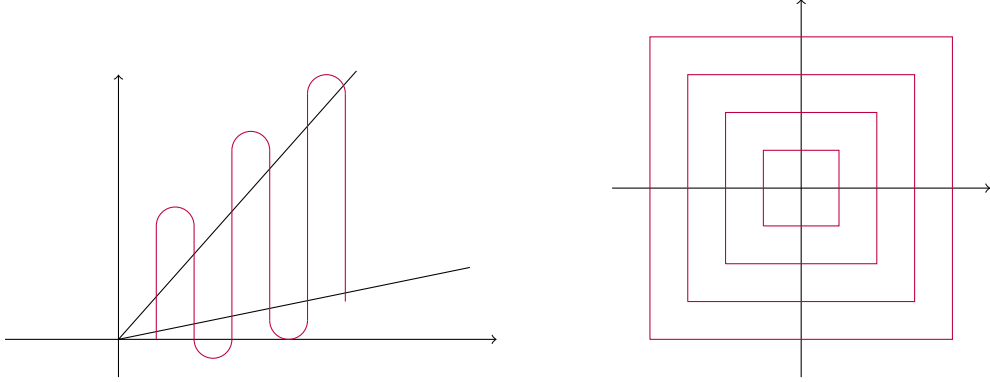


Figure 5.3: A spiraling curve based on a set of parallel lines in a sector (left) and a spiraling curve made of concentric squares (right).

asymptotic velocity $\frac{M\vec{d}}{|M\vec{d}|}$. As a consequence, we have for example that, combining these smooth perturbations with rotations, spiraling curves are invariant by any linear invertible transformation in \mathbb{R}^2 .

5.3.4 Examples of spiraling curves

Proposition 5.19. *Let $\eta > 0$. Then, the Archimedes spiral A^η and the union of circles O^η are spiraling trajectories with asymptotic separation η . Further, any $\vec{d} \in \mathbb{S}^1$ can be taken as the asymptotic velocity.*

Proof. Step 1 (Regularity). We show first that A^η and O^η are regular trajectories in the sense of § 5.3.1. We define $\rho_1(\theta) := \eta\theta$, $\rho_2(\theta) := \sum_{k \in \mathbb{N}} \eta k \chi_{[k, k+1)}(\theta)$ so that

$$\begin{aligned} A^\eta &= \{(\rho_1(\theta) \cos 2\pi\theta, \rho_1(\theta) \sin 2\pi\theta) : \theta \geq 0\}, \\ O^\eta &= \{(\rho_2(\theta) \cos 2\pi\theta, \rho_2(\theta) \sin 2\pi\theta) : \theta \geq 0\}. \end{aligned}$$

Let us begin by proving that (5.10) holds for A^η . Take $r \in (0, 1)$ and $x \in A^\eta$ and assume initially that $|x| \leq r/2$. Then

$$\begin{aligned} \mu(B_r(x)) &\geq \mu(B_{r/2}(0)) = \int_0^{r/2\eta} \sqrt{\rho_1'(\theta)^2 + (2\pi\rho_1(\theta))^2} d\theta \\ &\geq \int_0^{r/2\eta} 2\pi\rho_1(\theta) d\theta = \pi r^2/4\eta^2. \end{aligned}$$

For $|x| > r/2$, we proceed as follows. Let $y \in B_r(x) \cap A^\eta$ and write

$$\begin{aligned} x &= (\rho_1(\theta_0) \cos 2\pi\theta_0, \rho_1(\theta_0) \sin 2\pi\theta_0), \quad \theta_0 \geq 0, \\ y &= (\rho_1(\theta) \cos 2\pi\theta, \rho_1(\theta) \sin 2\pi\theta), \quad \theta \geq 0. \end{aligned}$$

Therefore,

$$\begin{aligned}
|x - y| &\leq |\rho_1(\theta_0) \cos 2\pi\theta_0 - \rho_1(\theta) \cos 2\pi\theta| + |\rho_1(\theta_0) \sin 2\pi\theta_0 - \rho_1(\theta) \sin 2\pi\theta| \\
&\leq |\rho_1(\theta_0)|(|\cos 2\pi\theta_0 - \cos 2\pi\theta| + |\sin 2\pi\theta_0 - \sin 2\pi\theta|) \\
&\quad + |\rho_1(\theta_0) - \rho_1(\theta)|(|\cos 2\pi\theta| + |\sin 2\pi\theta|) \\
&\leq (\eta + 4\pi\rho_1(\theta_0))|\theta_0 - \theta|
\end{aligned}$$

by the Mean Value Theorem. Let us set $r_0 := \frac{r}{\eta + 4\pi\rho_1(\theta_0)}$, so that $|\theta_0 - \theta| \leq r_0$ implies $|x - y| < r$. This allows us to bound the arc-length of $B_r(x) \cap A^\eta$ from below by

$$\begin{aligned}
\mu(B_r(x)) &\geq \int_{\theta_0 - r_0}^{\theta_0 + r_0} \sqrt{\rho_1'(\theta)^2 + (2\pi\rho_1(\theta))^2} d\theta \\
&\geq \int_{\theta_0 - r_0}^{\theta_0 + r_0} 2\pi\rho_1(\theta) d\theta = 4\eta\pi\theta_0 r_0 \\
&\gtrsim \frac{\rho_1(\theta_0)}{1 + \rho_1(\theta_0)} r = \frac{|x|}{1 + |x|} r \geq \frac{r^2}{2 + r} \geq \frac{r^2}{3}
\end{aligned}$$

since $|x| > r/2$, and $r \in (0, 1)$. In conclusion, for every $r < 1$ and every x , $\mu(B_r(x)) \gtrsim r^2$.

The argument for O^η is similar, this time replacing $\rho_1(\theta)$ with $\rho_2(\theta)$. We now show (5.11), beginning again with case of A^η . Note first that it is enough to bound $\mu_{A^\eta}(B_r(x))$ uniformly for $x \in A^\eta$, and some $r > 0$. Indeed, since $\text{gap}(A^\eta) = \eta/2$, for every $y \in \mathbb{R}^2$, there exists $x \in A^\eta$ such that $B_r(y) \subset B_{r+\eta/2}(x)$. In addition, any ball of radius $R > r$ can be covered with $(2R/r + 2)^2$ balls of radius r (by taking a covering $\{B_r(a_n)\}_n$ maximal with respect to $|a_n - a_m| \geq r$ as in the proof of Theorem D).

Let $x = (\rho_1(\theta_0) \cos 2\pi\theta_0, \rho_1(\theta_0) \sin 2\pi\theta_0)$ and let $r := \min\{\eta/100, 1\}$. If $|x| \leq 100$ we simply bound

$$\mu_{A^\eta}(B_r(x)) \leq \mu_{A^\eta}(B_{101}(0)).$$

Let us assume then $|x| \geq 100$. We claim that

$$B_r(x) \cap A^\eta \subset \{(\rho_1(\theta) \cos 2\pi\theta, \rho_1(\theta) \sin 2\pi\theta) : \theta \in (\theta_0 - 1/|x|, \theta_0 + 1/|x|)\}. \quad (5.14)$$

Indeed, if $y = (\rho_1(\theta) \cos 2\pi\theta, \rho_1(\theta) \sin 2\pi\theta) \in B_r(x) \cap A^\eta$,

$$\eta|\theta_0 - \theta| = |\rho_1(x) - \rho_1(y)| = ||y| - |x|| \leq |x - y| < r \leq \eta/100,$$

and therefore $|\theta_0 - \theta| \leq 1/100$. Secondly, since $|x - y| < r$, a clockwise rotation by the angle of y gives

$$|x - y| = \sqrt{|\rho_1(\theta_0) \cos 2\pi(\theta_0 - \theta) - \rho_1(\theta)|^2 + |\rho_1(\theta_0) \sin 2\pi(\theta_0 - \theta)|^2} < r,$$

and in particular $|\rho_1(\theta_0) \sin 2\pi(\theta_0 - \theta)| = |x| |\sin 2\pi(\theta_0 - \theta)| < r \leq 1$. Using the bound $|\sin 2\pi(\theta - \theta_0)| \geq |\theta - \theta_0|$ - valid for $|\theta - \theta_0| \leq 1/100$, we conclude that

$$|\theta_0 - \theta| \leq |\sin 2\pi(\theta - \theta_0)| \leq |x|^{-1},$$

and (5.14) follows. Now we can estimate,

$$\begin{aligned} \mu_{A^\eta}(B_r(x)) &\leq \int_{\theta_0-1/|x|}^{\theta_0+1/|x|} \sqrt{\rho_1'(\theta)^2 + (2\pi\rho_1(\theta))^2} d\theta \\ &\leq \int_{\theta_0-1/|x|}^{\theta_0+1/|x|} \rho_1'(\theta) + 2\pi\rho_1(\theta) d\theta = \int_{\theta_0-1/\pi|x|}^{\theta_0+1/\pi|x|} \eta + 2\pi\eta\theta d\theta \\ &\leq \frac{2\eta}{|x|} + \frac{4\pi\eta\theta_0}{|x|} \lesssim 1. \end{aligned}$$

Since $|x| \geq 100$ and $\eta\theta_0 = |x|$. The proof for O^η follows similarly.

Step 2 (The other conditions). We choose $\vec{d} \in \mathbb{S}^1$ and verify that A^η and O^η satisfy the conditions of §5.3.3 with asymptotic velocity \vec{d} . Write $\vec{d} = (-\sin 2\pi\theta_0, \cos 2\pi\theta_0)$ with $\theta_0 \in [0, 1)$, and $\beta = 2\pi\theta_0$. Thus $\vec{l} = \vec{d}_\perp = (\cos 2\pi\theta_0, \sin 2\pi\theta_0)$.

We start with A^η . For the escape cone (5.12), we choose any $\alpha \in (0, 1/4)$, and parametrize the portion of the curve inside the cone with $\gamma = \gamma_{A^\eta}$ as in (5.13), and using the function $\rho_1(\theta) = \eta\theta$ restricted to each $[k + \theta_0 - \alpha, k + \theta_0 + \alpha]$, $k \in \mathbb{N}$. Then $\rho_1(\theta)$ is strictly increasing and in particular the monotonicity condition is satisfied. Also, the curvature

$$\kappa_{A^\eta}(\theta) = \frac{2 + (2\pi\theta)^2}{\eta(1 + (2\pi\theta)^2)^{3/2}}$$

converges to 0 as $\theta \rightarrow \infty$. Hence, the asymptotic flatness condition holds. For the asymptotic equispacing condition, we let $\rho_0 := \eta\theta_0$ and simply note that

$$\rho_1(k + \beta) = \rho_1(k + \theta_0) = \eta(k + \theta_0) = \eta k + \rho_0,$$

for all $k \in \mathbb{N}$, so there is nothing to prove. Finally, we check that \vec{d} is the asymptotic velocity:

$$\begin{aligned} \lim_k \frac{\gamma'_{A^\eta}(k + \beta)}{|\gamma'_{A^\eta}(k + \beta)|} &= \lim_k \frac{\gamma'_{A^\eta}(k + \theta_0)}{|\gamma'_{A^\eta}(k + \theta_0)|} \\ &= \lim_k \frac{\rho_1'(k + \theta_0)(\cos 2\pi\theta_0, \sin 2\pi\theta_0) + 2\pi\rho_1(k + \theta_0)(-\sin 2\pi\theta_0, \cos 2\pi\theta_0)}{\sqrt{\rho_1'(k + \theta_0)^2 + (2\pi\rho_1(k + \theta_0))^2}} \\ &= \lim_k \frac{(\cos 2\pi\theta_0, \sin 2\pi\theta_0) + 2\pi(k + \theta_0)(-\sin 2\pi\theta_0, \cos 2\pi\theta_0)}{\sqrt{1 + 2\pi(k + \theta_0)^2}} \\ &= (-\sin 2\pi\theta_0, \cos 2\pi\theta_0) = \vec{d}. \end{aligned}$$

We now consider O^η . Since this curve is rotation invariant, we may assume that $\vec{d} = (0, 1)$ and $\vec{l} = (1, 0)$. The escape cone (5.12) is then parametrized with $\beta = 0$ and any $\alpha \in (0, 1/4)$ by $\rho_2(\theta) = \sum_k \eta^k \chi_{[k-\alpha, k+\alpha]}(\theta) \chi_{[k, k+1]}(\theta)$. Therefore, the function $\rho_2(\theta + k)$ is increasing for any $\theta \in [-\alpha, \alpha]$ and $\kappa_{O^\eta}(\theta) = \sum_k \frac{1}{\eta^k} \chi_{[k-\alpha, k+\alpha]}(\theta) \chi_{[k, k+1]}(\theta)$. This yields the monotonicity and asymptotic flatness conditions. The asymptotic equispacing condition it is also verified with $\rho_0 = 0$ because $\rho_2(k) = \eta^k$ for all $k \in \mathbb{Z}$. Lastly, we calculate the asymptotic velocity:

$$\lim_k \frac{\gamma'_{O^\eta}(k)}{|\gamma'_{O^\eta}(k)|} = \lim_k \frac{\rho_2'(k)(1, 0) + 2\pi\rho_2(k)(0, 1)}{\sqrt{\rho_2'(k)^2 + (2\pi\rho_2(k))^2}} = \lim_k \frac{2\pi k(0, 1)}{2\pi k} = (0, 1) = \vec{d}.$$

In summation both the Archimedes spiral and the union of circles are spiraling curves for any asymptotic velocity. \square

5.3.5 Bessel bounds

The following Bessel bounds follow from [56, Theorem 3.2].

Proposition 5.20. *Let Γ be either the spiral A^η or the concentric circles O^η . Then, there exists an independent constant $C > 0$ such that for all $1 \leq p \leq \infty$, $R > 0$ and $f \in PW^p(\bar{B}_R(0))$*

$$\|f\|_{L^p(\mu_\Gamma)} \leq C(\eta^{-1/p} + R^{1/p})\|f\|_p$$

with the usual modifications when $p = \infty$. Moreover, the same conclusion holds for the rotated Archimedes spirals $\mathcal{R}_{2\pi\theta_0}A^\eta$.

5.4 Necessary conditions for sampling on spiraling curves

5.4.1 General results

In this section we derive necessary conditions for sampling on spiraling curves, comparing the asymptotic equispacing of such a curve to the diameter of the Fourier spectrum. The following key lemma shows that spiraling curves can be locally approximated by unions of lines.

Lemma 5.21. *Let Γ be a spiraling trajectory with asymptotic velocity \vec{d} and asymptotic separation τ . Then the collection of equispaced parallel lines*

$$L_{\vec{d}, \tau} := \{t\vec{d} + \tau k\vec{d}_\perp : t \in \mathbb{R}, k \in \mathbb{Z}\} \quad (5.15)$$

is a weak limit of translates of Γ , i.e. $L_{\vec{d}, \tau} \in W(\Gamma)$.

Proof. Throughout the proof we use the notation of §5.3.3.

Step 1. Reduction of the problem.

Let us first slightly simplify the setting. Since the notion of spiraling trajectory is invariant under rotation, we may assume that the escape cone is $S_{\alpha,0}$, thus, $\beta = 0$ and $\vec{l} = \vec{i} := (1, 0)$. We then write $\vec{d} = (d_1, d_2)$. As $1 = |\vec{d}|^2 = d_1^2 + d_2^2$ and \vec{d} is not collinear with \vec{l} , then $d_2 \neq 0$. Thus $\vec{i} = d_1\vec{d} - d_2\vec{d}_\perp$ and $\vec{d}_\perp = \frac{d_1}{d_2}\vec{d} - \frac{1}{d_2}\vec{i}$. It follows that, if $\eta = \tau/|d_2|$, then

$$L_{\vec{d},\tau} = \tilde{L}_{\vec{d},\eta} := \{s\vec{d} + \eta j\vec{i} : s \in \mathbb{R}, j \in \mathbb{Z}\}.$$

Note that $\tau = \eta|d_2| = \eta\sqrt{1 - d_1^2} = \eta\sqrt{1 - \langle \vec{d}, \vec{l} \rangle^2}$.

Step 2. Reparametrization of the trajectory.

We set $\Lambda_k := \Gamma - (\eta_k, 0)$ and, from now on, we fix $R, \varepsilon > 0$. Our aim is to show that there exists k_0 (depending on R, ε) such that; for all $k \geq k_0$,

$$\Lambda_k \cap (-R, R)^2 \subset \tilde{L}_{\vec{d},\eta} + B_\varepsilon(0), \quad (5.16)$$

$$\tilde{L}_{\vec{d},\eta} \cap (-R, R)^2 \subset \Lambda_k + B_\varepsilon(0). \quad (5.17)$$

First, a simple computation shows that

$$\tilde{L}_{\vec{d},\eta} \cap (-R, R)^2 \subset L_0 := \{t\vec{d} + \eta j\vec{i} : |t| \leq T_R, j = -J_R, \dots, J_R\}$$

with $T_R = \frac{R}{|d_2|}$ and $J_R = \left\lceil \frac{2R}{\eta} \left(1 + \left|\frac{d_1}{d_2}\right|\right) \right\rceil$. Instead of (5.17), we will thus prove

$$L_0 \subset \Lambda_k + B_\varepsilon(0). \quad (5.18)$$

Next, as $\eta_k \rightarrow +\infty$ and L_0 is a bounded set, there is a k_1 such that, if $k \geq k_1$, the translate of the escape cone by η_k contains L_0 . Moreover, as $\eta_k - k\eta \rightarrow \rho_0$, we can assume that, for $k \geq k_1$, $|\eta_k - k\eta - \rho_0| < \varepsilon/2$. From now on, we will assume that $k \geq k_1$.

For each n let $\psi_n : I_n \rightarrow \mathbb{R}^2$ be a re-parametrization by arc-length of $\gamma(\theta)$ on the interval $[n - \alpha, n + \alpha]$ such that $0 \in I_n$ and $\psi_n(0) = \gamma(n) = (\eta_n, 0)$. Write $I_n^\pm = I_n \cap \mathbb{R}^\pm$. As the restriction of ψ_n to I_n^- joins the line $\mathbb{R}(\cos \alpha, -\sin \alpha)$ to the point $(\eta_n, 0)$ and $\eta_n \rightarrow +\infty$, it follows that the length of I_n^- , $|I_n^-| \rightarrow +\infty$. Similarly, $|I_n^+| \rightarrow +\infty$. (See Figure 5.4).

Therefore, there exists n_1 such that, for all $n \geq n_1$,

$$[-2T_R, 2T_R] \subset I_n.$$

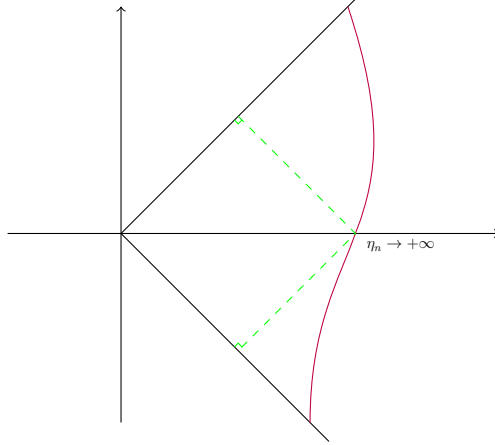


Figure 5.4: The arc-length of ψ_n is at least as large as twice the distance of η_n to $S_{\alpha,0}$.

A Taylor expansion of ψ_n at 0 reads

$$|\psi_n(t) - (\eta_n, 0) - t\psi'_n(0)| \leq \frac{|t|^2}{2} \sup_{s \in I_n} |\psi''_n(s)|.$$

This implies

$$\left| \psi_n(t) - (\eta_n, 0) - t\vec{d} \right| \leq |t| \cdot \left| \psi'_n(0) - \vec{d} \right| + \frac{|t|^2}{2} \sup_{s \in I_n} |\psi''_n(s)|. \quad (5.19)$$

On the other hand, by definition we know that

$$\psi'_n(0) = \frac{\gamma'(n)}{|\gamma'(n)|}, \quad \sup_{s \in I_n} |\psi''_n(s)| = \sup_{s \in [n-\alpha, n+\alpha]} \kappa(s).$$

As γ is a spiraling curve, $\lim_n \left| \psi'_n(0) - \vec{d} \right| = \lim_k \sup_{s \in I_n} |\psi''_n(s)| = 0$ and $\lim_n \eta_n - \eta n = \rho_0$. Therefore, there exists $n_2 \geq n_1$ such that, for $n \geq n_2$

$$|\eta_n - \eta n - \rho_0| < \varepsilon/6, \quad \left| \psi'_n(0) - \vec{d} \right| < \varepsilon/(12T_R), \quad \sup_{s \in I_n} |\psi''_n(s)| < \varepsilon/(12T_R^2).$$

It then follows from (5.19) that, for all $t \in [-2T_R, 2T_R]$, for $n \geq n_2$

$$\left| \psi_n(t) - (\eta n + \rho_0, 0) - t\vec{d} \right| < \varepsilon/2. \quad (5.20)$$

In other words, the trajectory stays at distance at most $\varepsilon/2$ from the segment joining the lines $\{x_2 = -R\}$ to the line $\{x_2 = R\}$, passing through the point $(\eta n + \rho_0, 0)$ and directed by \vec{d} . See Figure 5.5.

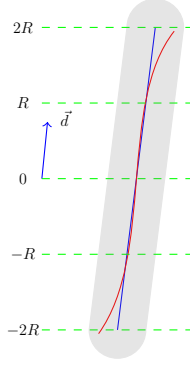


Figure 5.5: Illustration of (5.20).

Step 3. Proof of (5.18).

We are now in position to prove (5.18). Let $j \in \{-J_R, \dots, J_R\}$ and $n = k + j$. If $k \geq k_1 + J_R$ then $n \geq k_1$ so that (5.20) holds, that is

$$\left| \psi_{k+j}(t) - ((k+j)\eta + \rho_0, 0) - t\vec{d} \right| < \varepsilon/2$$

for $t \in [-T_R, T_R]$. Additionally, if $k \geq n_2 + J_R$, $|\eta_k - k\eta - \rho_0| < \varepsilon/2$. Altogether,

$$\left| t\vec{d} + j\eta\vec{i} - (\psi_{k+j}(t) - \eta_k\vec{i}) \right| \leq \left| t\vec{d} + ((k+j)\eta + \rho_0, 0) - \psi_{k+j}(t) \right| + |k\eta + \rho_0 - \eta_k| < \varepsilon$$

which is exactly (5.18).

Step 4. Proof of (5.16).

It is a bit more complicated to identify the pieces of the trajectory that go through a given square. The first part of this step consists in identifying those pieces.

We want to prove that, given $R, \varepsilon > 0$ for k sufficiently large

$$(\Gamma - (\eta_k, 0)) \cap (-R, R)^2 \subset \tilde{L}_{\vec{d}, \eta} + B_\varepsilon(0).$$

Defining $Q_{k,R} := (\eta_k - R, \eta_k + R) \times (-R, R)$, this can be rewritten as

$$\Gamma \cap Q_{k,R} \subset \{t\vec{d} + (\eta j + \eta_k)\vec{i} : t \in \mathbb{R}, j \in \mathbb{Z}\} + B_\varepsilon(0). \quad (5.21)$$

First, let n_0 be the smallest $n \in \mathbb{N}$ for which $\{\psi_n(t) : t \in I_n\} \cap Q_{k,R} \neq \emptyset$. In particular $(\eta_k, 0)$ is at distance at most R from the trajectory of $\psi_{n_0}(t)$. As $\eta_k \rightarrow +\infty$ when $k \rightarrow \infty$, this implies that $n_0 \rightarrow \infty$ when $k \rightarrow \infty$. Then, we can take k large enough so that $n_0 \geq k_\rho$, where k_ρ is the constant of radial monotonicity. This means that

$$\Gamma \cap Q_{k,R} \subset \bigcup_{n \geq k_\rho} \{\psi_n(t) : t \in I_n\} \cap Q_{k,R},$$

and thus we can use the monotonicity property to order from left to right all trajectories $\{\psi_n(t) : t \in I_n\}$ which intersect $Q_{k,R}$ (see Figure 5.6).

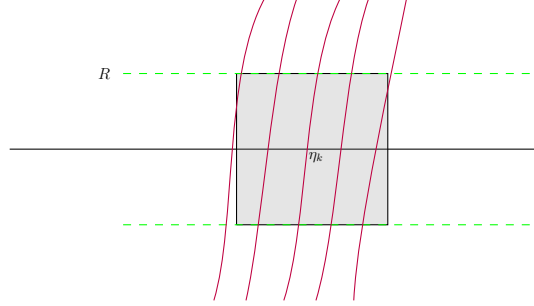


Figure 5.6: Curves in $Q_{k,R}$ ordered from left to right.

From now on, we will assume that $n \geq k_\rho$ for any piece of trajectory $\{\psi_n(t) : t \in I_n\}$ considered.

Now, let us reduce the picture. For this, we temporarily fix k and write $z = \eta_k$, $z_\pm = z \pm R$, $\tau = z_+/z_-$. As $\eta_k \rightarrow +\infty$, we can choose k so that τ is as near to 1 as we want.

Let $\mathcal{D}_\pm = \mathbb{R}_+(z_\pm, \pm R) \cup \{(0, 0)\}$ for the half-lines starting at 0 and through the left-corners of $Q_{k,R}$ and let S be the subcone delimited by these half-lines (and containing the positive x_1 -axis).

Let $a = z_- - \varepsilon - 2R|d_1/d_2|$. The choice of a has been made as follows: let ℓ_a be the line through $(a, 0)$ and directed by \vec{d} , and set $A := \ell_a \cap \{|x_2| \leq 2R\}$. Then $A \subset \{x_1 \leq z_- - \varepsilon\}$ and $A \cap \{x_1 = z_- - \varepsilon\} \neq \emptyset$.

Similarly, let $b = z_+ + \varepsilon + 2R|d_1/d_2| = z_- + \varepsilon + 2R(1 + |d_1/d_2|)$. Again, let ℓ_b be the line directed by \vec{d} through $(b, 0)$ and set $B := \ell_b \cap \{|x_2| \leq 2R\}$. Additionally, here we define $C := \ell_b \cap S$. Then

— B is at distance at least ε on the right of $Q_{k,R}$, that is, $B \subset \{x_1 \geq z_+ + \varepsilon\}$. Moreover $B \cap \{x_1 = z_+ + \varepsilon\} \neq \emptyset$. Also $B \subset \{x_1 \leq b + 2R|d_2/d_1|\}$ and $B \cap \{x_1 = b + 2R|d_2/d_1|\} \neq \emptyset$. In particular, B joins the boundaries of the strip $\{z_+ + \varepsilon \leq x_1 \leq b + 2R|d_2/d_1|\}$.

— C is included in the strip $\{\frac{bd_2}{z_-d_2+Rd_1}R \leq x_1 \leq \frac{bd_2}{z_-d_2-Rd_1}R\}$. Note that, when $k \rightarrow +\infty$, $b, z_- \rightarrow +\infty$ but $b/z_- \rightarrow 1$. It follows that, for k large enough, $C \subset B$.

Note also that, if k is large enough, then A, B, C are included in the larger escape cone, see Figure 5.7.

Next, recall from (5.20) that, if n is large enough, then for $|t| \leq 2T_R$,

$$\left| \psi_n(t) - (\eta n + \rho_0, 0) - t\vec{d} \right| < \varepsilon/2.$$

In particular, as $a, b \rightarrow +\infty$ when $k \rightarrow +\infty$, if k is large enough, then this holds for $a/\eta < n < b/\eta$. In other words, for those n 's, the part of the trajectory

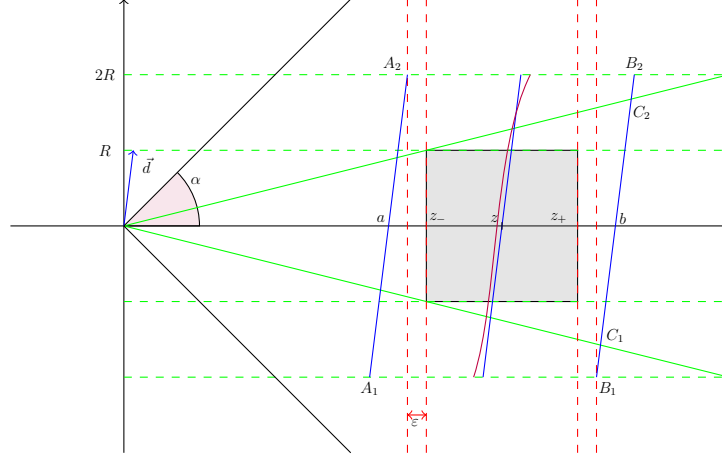


Figure 5.7: The segments are $A = [A_1, A_2]$, $B = [B_1, B_2]$, $C = [C_1, C_2]$.

$\{\psi_n(t) : |t| \leq 2T_R\}$ stays at distance less than $\varepsilon/2$ of the segments $\{(n\eta + \rho_0, 0) + t\vec{d} : |t| \leq 2T_R\}$. Those segments are all included in the rhombus delimited by A, B and the lines $\{x_2 = \pm 2R\}$. In particular, they all join the 2 boundaries of the cone S . Further, the monotonicity property shows that the curves $\{\psi_n(t) : |t| \leq 2T_R\} \cap S$ are ordered from the left to the right when n goes from a/η to b/η .

Now let $\{\psi_m(t) : t \in I_m\}$ be a piece of the trajectory that intersects Q_R . It therefore also intersects S . From the monotonicity property $\{\psi_m(t) : t \in I_m\}$ is either

- on the left of $\{\psi_{\lceil a/\eta \rceil}(t) : t \in I_{\lceil a/\eta \rceil}\} \cap S$
- or on the right of $\{\psi_{\lfloor b/\eta \rfloor}(t) : t \in I_{\lfloor b/\eta \rfloor}\} \cap S$
- or is one of $\{\psi_n(t) : t \in I_n\} \cap S$, $a/\eta < n < b/\eta$.

But, in the first two cases, $\{\psi_m(t) : t \in I_m\}$ would not intersect Q_R so that the only trajectories that may intersect Q_R are $\{\psi_n(t) : t \in I_n\} \cap S$, $a/\eta < n < b/\eta$.

Further note that, when $|t| = 2T_R$, (5.20) implies that $|\psi_n(t)| \in \{|x_2| \geq 2R - \varepsilon/2\} \subset \{|x_2| > R\}$ provided we choose $\varepsilon < 2R$. It follows that

$$\Gamma \cap Q_{k,R} \subset \bigcup_{a/\eta < n < b/\eta} \{\psi_n(t) : |t| \leq 2T_R\}.$$

In summary

$$\Gamma \cap Q_{k,R} \subset \bigcup_{a/\eta < n < b/\eta} \{(\eta n + \rho_0, 0) + t\vec{d} : |t| \leq 2T_R\} + B_{\varepsilon/2}(0).$$

To finish, note that since $\lim_k \frac{\eta n + \eta_k}{\eta(n+k) + \rho_0} = 1$ uniformly for any $n \in (a/\eta, b/\eta)$ then, for k large enough,

$$\Gamma \cap Q_{k,R} \subset \bigcup_{a/\eta < n < b/\eta} \{(\eta(n+k) + \eta_k, 0) + t\vec{d} : |t| \leq 2T_R\} + B_\varepsilon(0),$$

which is (5.21). Then (5.18) follows. \square

Proposition 5.22. *Let Γ be a spiraling trajectory with asymptotic velocity $\vec{d} \in \mathbb{S}^1$ and asymptotic separation $\tau > 0$. Let Ω be a convex centered symmetric body such that*

$$\{t\vec{d}_\perp : t \in [-1/2\tau, 1/2\tau]\} \subset \Omega^\circ.$$

Then Γ is not a sampling trajectory for $PW^2(\Omega)$.

Proof. Suppose on the contrary that Γ is a sampling trajectory for $PW^2(\Omega)$. By Theorem D, there exists a separated set $\Lambda \subset \Gamma$ that is a sampling set for $PW^2(\Omega)$. By Lemma 5.21, $L_{\vec{d}, \tau} \in W(\Gamma)$. Therefore, there exists a sequence $\{x_k : k \geq 1\} \subset \mathbb{R}^2$ such that $\Gamma + x_k \xrightarrow{w} L_{\vec{d}, \tau}$. By Lemma 5.11, we may pass to a subsequence, and assume that $\Lambda + x_k \xrightarrow{w} \Lambda'$, for some set $\Lambda' \subset \mathbb{R}^2$. Since $\Lambda \subset \Gamma$, it follows that $\Lambda' \subset L_{\vec{d}, \tau}$.

Since $\{t\vec{d}_\perp : t \in [-1/2\tau, 1/2\tau]\} \subset \Omega^\circ$, there exists $r > 1/2\tau$ such that $\{t\vec{d}_\perp : t \in [-r, r]\} \subset \Omega$, and we can take $0 < \varepsilon < 1$ such that

$$\{t\vec{d}_\perp : t \in [-1/2\tau, 1/2\tau]\} \subset \{t\vec{d}_\perp : t \in [-(1-\varepsilon)r, (1-\varepsilon)r]\} \subset (1-\varepsilon)\Omega.$$

As Λ is a sampling set for $PW^2(\Omega)$, by Corollary 5.7, it is also a sampling set for $PW^\infty((1-\varepsilon)\Omega)$. Therefore, by Theorem 5.10, Λ' is a uniqueness set for $PW^\infty((1-\varepsilon)\Omega)$. However, the non-zero function

$$f(x) := \sin\left(\frac{\pi}{\tau}\langle x, \vec{d}_\perp \rangle\right)$$

satisfies $f \equiv 0$ on $L_{\vec{d}, \tau}$ – cf. (5.15) – and therefore on Λ' , and $\text{supp}(\widehat{f}) \subset \{t\vec{d}_\perp : t \in [-1/2\tau, 1/2\tau]\} \subset (1-\varepsilon)\Omega$. This contradiction shows that Γ cannot be a sampling trajectory for $PW^2(\Omega)$. \square

5.4.2 Application to concrete curves

We now have all the elements to prove our sampling result, Theorem A, for the Archimedes spiral and the concentric circles. Let us recall the statement and then prove it.

Theorem A. Let $\Omega \subset \mathbb{R}^2$ be a convex centered symmetric body.

- (i) If $\text{diam}(\Omega)\eta < 1$, then the Archimedes spiral A^η and the collection of concentric circles O^η are sampling trajectories for $PW^2(\Omega)$.
- (ii) If $\text{diam}(\Omega)\eta > 1$, then neither the Archimedes spiral A^η nor the collection of concentric circles O^η are sampling trajectories for $PW^2(\Omega)$.

Proof. Let Γ be either A^η or O^η . Note first that

$$\Omega \subset \bar{B}_{\text{diam}(\Omega)/2}(0). \quad (5.22)$$

Indeed, if $x \in \Omega$ then by symmetry $-x \in \Omega$ and thus, $2\|x\| = \|x - (-x)\| \leq \text{diam}(\Omega)$.

For (i), assume that $\text{diam}(\Omega)\eta < 1$, and let $\varepsilon > 0$ and $\eta' > \eta$ be such that $\eta'(1 + \varepsilon)\text{diam}(\Omega) < 1$. We know from (5.5) that $\text{gap}(\Gamma) = \eta/2$. By Lemma 5.5, there exists a separated set $\Lambda \subset \Gamma$ with $\text{gap}(\Lambda) \leq \eta'/2$. Since

$$\text{gap}(\Lambda)\text{diam}(\bar{B}_{(1+\varepsilon)\text{diam}(\Omega)/2}(0)) \leq \frac{\eta'}{2}(1 + \varepsilon)\text{diam}(\Omega) < \frac{1}{2}.$$

Theorem 5.4 implies that Λ is a sampling set of $PW^\infty(\bar{B}_{(1+\varepsilon)\text{diam}(\Omega)/2}(0))$. Finally, applying Theorem 5.6, we have that Λ is a sampling set for the space $PW^2(\bar{B}_{\text{diam}(\Omega)/2}(0))$ and thus for $PW^2(\Omega)$. We invoke Theorem D to conclude that Γ is a sampling trajectory for $PW^2(\Omega)$.

For (ii), we first note that exists $\vec{d} \in \mathbb{S}^1$ such that

$$\{t\vec{d}_\perp : t \in [-\text{diam}(\Omega)/2, \text{diam}(\Omega)/2]\}$$

is contained in Ω . Indeed, by compactness, we can select $x \in \Omega$ with maximal norm. Then $\Omega \subset \bar{B}_{\|x\|}(0)$, and hence $\text{diam}(\Omega) \leq 2\|x\| = \|x - (-x)\| \leq \text{diam}(\Omega)$. Thus, $\|x\| = \text{diam}(\Omega)/2$. Letting $\vec{d}_\perp := x/\|x\|$, convexity reads

$$\{t\vec{d}_\perp : t \in [-\text{diam}(\Omega)/2, \text{diam}(\Omega)/2]\} \subset \Omega.$$

Now suppose that $\text{diam}(\Omega)\eta > 1$ and take $\varepsilon \in (0, 1)$ such that

$$(1 - \varepsilon)^{-1}1/(2\eta) = \text{diam}(\Omega)/2.$$

Therefore,

$$\{t\vec{d}_\perp : t \in [-1/(2\eta), 1/(2\eta)]\} \subset (1 - \varepsilon)\Omega \subset \Omega^\circ$$

where we used (5.4). By Proposition 5.19, Γ is spiraling with asymptotic velocity \vec{d} and asymptotic separation $\tau = \eta$. We invoke Proposition 5.22 and conclude that Γ is not a sampling trajectory for $PW^2(\Omega)$. \square

5.5 Approximate aliasing

5.5.1 Rates of convergence for weak limits

We introduce the following class of curves.

Definition 5.23. Let $\eta > 0$, $\lambda > 0$ and $\Omega \subset \mathbb{R}^2$ a convex centered symmetric body. We say that a regular trajectory Γ belongs to the class $\mathcal{C}(\eta, \lambda, \Omega)$ if there exists a constant C_Γ for which, given $\varepsilon, R > 0$ there exists $y \in \mathbb{R}^2$ such that

- (i) $(\Gamma - y) \cap (-R, R)^2 \subset (\eta\mathbb{Z} \times \mathbb{R}) + B_{\eta\varepsilon}(0)$,
- (ii) $|y| \leq C_\Gamma \frac{R^2}{\varepsilon}$,
- (iii) $\lambda^{-1/p} \|f\|_{L^p(\mu_\Gamma)} \leq \|f\|_p$ for all $f \in PW^p(\Omega)$ with $1 \leq p \leq \infty$.

We now prove that the curves A^η and O^η belong to this type of classes. For technical reasons we extend this result to any rotation $\mathcal{R}_{2\pi\theta_0}A^\eta$. As a first step, we show the following lemma, which quantifies the convergence in Lemma 5.21.

Lemma 5.24. Let $\eta > \sqrt{2}/2$ and $\theta_0 \in [0, 1)$. Then there exists a constant C independent from η and θ_0 such that given $R \geq 1$ and $0 < \varepsilon < 1/2$ the inclusion

$$(\mathcal{R}_{2\pi\theta_0}A^\eta - (\eta(n + \theta_0), 0)) \cap (-R, R)^2 \subset \eta\mathbb{Z} \times \mathbb{R} + B_\varepsilon(0)$$

holds for all $n \geq C\varepsilon^{-1}R^2$. The same inclusion holds for O^η , translating instead by $(\eta n, 0)$ and eventually taking a larger constant.

Proof. Let $x = (\eta\theta \cos 2\pi(\theta - \theta_0), \eta\theta \sin 2\pi(\theta - \theta_0)) \in \mathcal{R}_{2\pi\theta_0}A^\eta \cap (\mathbb{R} \times (-R, R))$ with $x_1 > \eta(\theta_0 + 1/4)$. Then, there exists $k \in \mathbb{N}$ such that $k - 1/4 < \theta - \theta_0 < k + 1/4$. We will prove

$$|x_1 - \eta(k + \theta_0)| \leq \frac{33R^2}{k}. \quad (5.23)$$

Assuming this for a moment, let us show how the lemma would then follow. Let $n \geq 2R$ and take $y \in (\mathcal{R}_{2\pi\theta_0}A^\eta - (\eta(n + \theta_0), 0)) \cap (-R, R)^2$. Write $y = x - (\eta(n + \theta_0), 0)$ with $x \in \mathcal{R}_{2\pi\theta_0}A^\eta$. Then $x_1 > \eta(n + \theta_0) - R$, and since $n \geq 2R$, $R \geq 1$ and $2\eta > 1$, we have $x_1 > \eta(\theta_0 + 1/4)$. Further, taking the same k as before,

$$\eta(n + \theta_0) - R < x_1 = \eta\theta \cos 2\pi(\theta - \theta_0) \leq \eta\theta |\cos 2\pi(\theta - \theta_0)| \leq \eta\theta \leq \eta(k + 1/4 + \theta_0).$$

This yields

$$k > n - 1/4 - R/\eta > n/2 - R^2/\varepsilon$$

since $n \geq 1$ and $\eta > 1/2 > \varepsilon$. Hence, taking $n \geq \frac{68R^2}{\varepsilon} \geq 2R$ we get $k > \frac{33R^2}{\varepsilon}$ and then (5.23) reads $|y_1 - \eta(n - k)| = |x_1 - \eta(k + \theta_0)| < \varepsilon$. Therefore, $y \in \eta\mathbb{Z} \times \mathbb{R} + B_\varepsilon(0)$, as claimed.

Let us now prove (5.23). Using the same notation as before for $x \in \mathcal{R}_{2\pi\theta_0}A^\eta \cap (\mathbb{R} \times (-R, R))$ with $x_1 > \eta(\theta_0 + 1/4)$, we have

$$\begin{aligned} k - 1/4 < \theta - \theta_0 < k + 1/4, \\ -R < \eta\theta \sin 2\pi(\theta - \theta_0) < R. \end{aligned}$$

In particular, since $\arcsin(\theta)$ is a strictly increasing function and $|\arcsin(\theta)| \leq 2|\theta|$ for $\theta \in [-1, 1]$,

$$|\theta - (k + \theta_0)| < \frac{1}{2\pi} \cdot \frac{2R}{\eta\theta} < \frac{R}{\eta\pi(k - 1/4 + \theta_0)} < \frac{2R}{\eta\pi k}$$

since $\theta_0 \geq 0$, $k \geq 1$. Next, using the triangle inequality and the fact that $|\cos \theta - 1| \leq |\theta|^2/2$ we get

$$\begin{aligned} |x_1 - \eta(k + \theta_0)| &= |\eta\theta \cos 2\pi(\theta - \theta_0) - \eta(k + \theta_0)| \\ &\leq |\eta\theta \cos 2\pi(\theta - \theta_0) - \eta(k + \theta_0) \cos 2\pi(\theta - \theta_0)| \\ &\quad + |\eta(k + \theta_0) \cos 2\pi(\theta - \theta_0) - \eta(k + \theta_0)| \\ &\leq \eta|\theta - (k + \theta_0)| + \eta(k + \theta_0)|\cos 2\pi(\theta - k - \theta_0) - 1| \\ &\leq \eta|\theta - (k + \theta_0)| + 2\pi^2\eta(k + \theta_0)|\theta - (k + \theta_0)|^2 \\ &\leq \frac{2R}{\pi k} + 4\pi^2\eta k \frac{4R^2}{(\eta\pi k)^2} \leq \frac{R}{k} + \frac{16R^2}{\eta k} \leq \frac{33R^2}{k}, \end{aligned}$$

since $2\eta, R, k \geq 1 > \theta_0$. The proof for O^η is similar. \square

Proposition 5.25. *Let $\eta > \sqrt{2}/2$, $\theta_0 \in [0, 1)$, and $R_0 > 0$. Then, there exists a constant $C > 0$ independent of η and θ_0 such that the curves $\mathcal{R}_{2\pi\theta_0}A^\eta$ and O^η belong to the class $\mathcal{C}(\eta, C(R_0 + 1), \bar{B}_{R_0}(0))$. Moreover, the constant $C_\gamma = C_{\mathcal{R}_{2\pi\theta_0}A^\eta}, C_{O^\eta}$ in Definition 5.23 is also independent of η, θ_0 .*

Remark 5.26. *The proof below also works for $0 < \eta \leq \sqrt{2}/2$ but then the constants $C_{\mathcal{R}_{2\pi\theta_0}A^\eta}$ and C_{O^η} depend on η . The reader may check that they satisfy a bound of the form $C\eta^{-1}$.*

Proof of Proposition 5.25. We treat only $\mathcal{R}_{2\pi\theta_0}A^\eta$, the case of O^η being similar. Condition (iii) of Definition 5.23 follows from Proposition 5.20: since $\eta > \sqrt{2}/2$, for all $f \in PW^p(\bar{B}_{R_0}(0))$

$$\|f\|_{L^p(\mu_\Gamma)} \lesssim (\eta^{-1/p} + R_0^{1/p})\|f\|_p \lesssim (R_0 + 1)^{1/p}\|f\|_p.$$

Conditions (i) and (ii) follow from the Lemma 5.24. Indeed, let $\varepsilon, R > 0$ be given and set $\varepsilon' := \eta\varepsilon$. Without loss of generality, we can assume that $\varepsilon' \in (0, 1/2)$ and $R \geq 1$. Taking $y = (\eta(n + \theta_0), 0)$ with $n = \lceil C(\eta\varepsilon')^{-1}R^2 \rceil$, Lemma 5.24 gives

$$\begin{aligned} (\mathcal{R}_{2\pi\theta_0}A^\eta - y) \cap (-R, R)^2 &\subset \eta\mathbb{Z} \times \mathbb{R} + B_{\eta\varepsilon}(0), \\ |y| &\leq 2C\varepsilon^{-1}R^2, \end{aligned}$$

as desired. \square

5.5.2 Quantitative aliasing

Having quantified the convergence in Lemma 5.21, we turn into the quantification of aliasing. While a union of lines with sub-Nyquist density leads to aliasing, we show that spirals suffer from approximate aliasing.

Let us recall the definition of *variation* of a function $f \in L^1(\Omega)$:

$$\text{var}_\Omega(f) := \sup \left\{ \left| \int_\Omega f \operatorname{div} h \right| : h \in C_c^1(\Omega, \mathbb{C}^2), \|h\|_\infty \leq 1 \right\}. \quad (5.24)$$

When f belongs to the Sobolev space $W^{1,1}(\Omega)$ integration by parts shows that $\text{var}_\Omega(f) = \|\nabla f\|_{L^1(\Omega)}$.

Lemma 5.27. *Let $\eta > \sqrt{2}/2$, $\lambda > 0$ and consider $\Gamma \in \mathcal{C}(\eta, \eta\lambda, \bar{B}_2(0))$. Let Q be the square of vertices $(0, 0)$, $(\sqrt{2}/4, \sqrt{2}/4)$, $(\sqrt{2}/2, 0)$ and $(\sqrt{2}/4, -\sqrt{2}/4)$. Then given $\zeta > 0$, there exists $g \in PW^2(Q \cup -Q)$ such that*

$$(i) \quad \|g\|_2 = 1,$$

$$(ii) \quad \eta^{-1/2} \|g\|_{L^2(\mu_\Gamma)} \leq \zeta,$$

$$(iii) \quad \text{var}(\hat{g}) \leq C \left(\max((\eta - \sqrt{2}/2)^{-1}, \eta) \frac{\lambda}{\zeta^2} \ln^4 \left(C \frac{\lambda}{\zeta^2} + 1 \right) + 1 \right),$$

$$(iv) \quad \|\hat{g}\|_\infty \leq C \max((\eta - \sqrt{2}/2)^{-1}, \eta),$$

where $C > 0$ is a universal constant.

Proof. Let Q_0 be the square of vertices $(-1, 0)$, $(0, 1)$, $(1, 0)$ and $(0, -1)$. Take $\phi \in PW^2(Q_0)$ such that $\|\phi\|_2 = \sqrt{2}/2$ and

$$|\phi(x)| \leq C_1 e^{-|x|^{1/2}} \quad (5.25)$$

for some constant $C_1 > 0$. Note that, integrating in polar coordinates

$$\|\phi\|_1 \leq 2\pi C_1 \int_0^\infty e^{-r^{1/2}} r \, dr = 4\pi \int_0^\infty e^{-s} s^3 \, ds = 24\pi C_1.$$

Also, as $\phi \in PW^2(Q_0)$ and the area of Q_0 is 2 and $\|\hat{\phi}\|_2 = \|\phi\|_2 = \sqrt{2}/2$, then Cauchy-Schwartz inequality gives $\|\hat{\phi}\|_1 \leq |Q_0|^{1/2} \|\hat{\phi}\|_2 = 1$.

Define $g_0(x) := \beta \sin\left(\frac{\pi}{\eta} x_1\right) \phi(\beta x)$ with $\beta = \min(1/(2\eta), \sqrt{2}/2 - 1/(2\eta))$. Note that, as $\eta > \sqrt{2}/2$, $\beta > 0$. For $\varepsilon, R > 0$ to be fixed later take $y = y(\varepsilon, R)$ as in

Definition 5.23 and set $g(x) := g_0(x - y)$. Then

$$\begin{aligned} \widehat{g}(\xi) &= \frac{e^{-2\pi iy\xi}}{2\beta i} \left[\widehat{\phi}(\beta^{-1}(\xi_1 - 1/(2\eta)), \beta^{-1}\xi_2) - \widehat{\phi}(\beta^{-1}(\xi_1 + 1/(2\eta)), \beta^{-1}\xi_2) \right] \\ &=: \phi_1(\xi) - \phi_2(\xi). \end{aligned} \quad (5.26)$$

As $\phi \in PW^2(Q_0)$, the support of ϕ_1 is included in $Q_\eta := \beta Q_0 + (1/(2\eta), 0)$ and the support of ϕ_2 in $-Q_\eta$. We claim that $Q_\eta \subset Q$. To show this, we argue by cases on $1/(2\eta)$. If $1/(2\eta) \leq \sqrt{2}/4$ this means that $\beta = 1/(2\eta)$ and hence the points $(1/(2\eta), \beta)$ and $(1/(2\eta), -\beta)$ belong to the square Q . Also since $1/(2\eta) - \beta = 0$ and $1/(2\eta) + \beta \leq \sqrt{2}/2$, both $(1/(2\eta) - \beta, 0)$ and $(1/(2\eta) + \beta, 0)$ belong to Q . By convexity this yields $Q_\eta \subset Q$. If $1/(2\eta) \geq \sqrt{2}/4$ this means that $\beta = \sqrt{2}/2 - 1/(2\eta)$ and hence the points $(1/(2\eta), \beta)$ and $(1/(2\eta), -\beta)$ belong to Q . Also in this case $1/(2\eta) - \beta \geq 0$ and $1/(2\eta) + \beta = \sqrt{2}/2$, so that, again, $(1/(2\eta) - \beta, 0)$ and $(1/(2\eta) + \beta, 0)$ belong to Q . Thus $Q_\eta \subset Q$ also in this case. Thus, in any case, $\text{supp}(\phi_1) \subset Q$ and $\text{supp}(\phi_2) \subset -Q$, where Q is the cube defined in the hypothesis. Altogether, $g \in PW^2(Q \cup -Q)$.

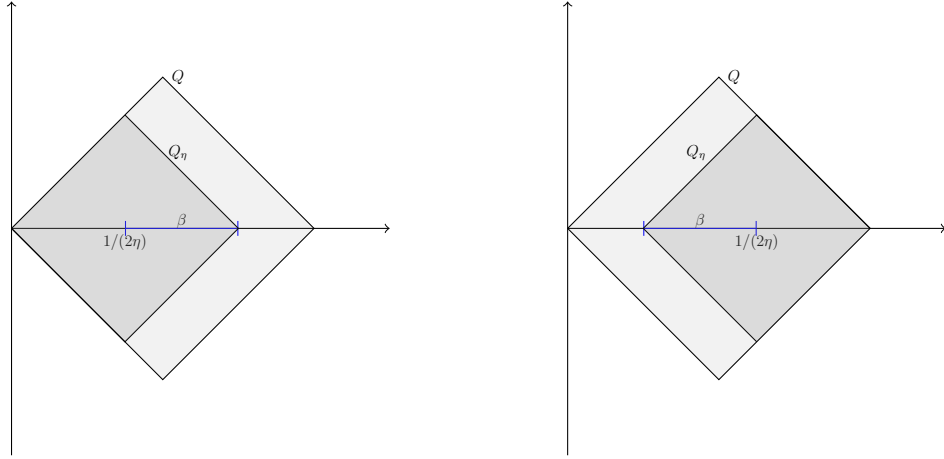


Figure 5.8: Sketch of both cases: $1/(2\eta) \leq \sqrt{2}/4$ (left), $1/(2\eta) \geq \sqrt{2}/4$ (right).

Since Q and $-Q$ only intersect at 0, then ϕ_1 and ϕ_2 have disjoint support up to a set of measure zero. Hence, using Plancharel we obtain

$$\|g\|_2^2 = \|\widehat{g}\|_2^2 = \|\phi_1\|_2^2 + \|\phi_2\|_2^2 = 2\|\phi\|_2^2 = 1,$$

which proves (i).

We now show that $\eta^{-1}\|g\|_{L^2(\mu_\Gamma)}$ is small enough for an appropriate choice of ε and R . First note that $\sup_{x \in \eta\mathbb{Z} \times \mathbb{R} + \bar{B}_{\eta\varepsilon}(0)} |g_0(x)| \leq \beta\varepsilon\|\phi\|_{L^\infty(\mathbb{R}^2)} \leq \beta\varepsilon C_1$. Then,

combining this with condition (i) of Definition 5.23 and (5.25) we have

$$\begin{aligned} \sup_{x \in \Gamma} |g(x)| &= \sup_{x \in \Gamma-y} |g_0(x)| \leq \sup_{x \in (\Gamma-y) \cap (-R, R)^2} |g_0(x)| + \sup_{x \notin B_R(0)} |g_0(x)| \\ &\leq \sup_{x \in \eta\mathbb{Z} \times \mathbb{R} + \bar{B}_{\eta\varepsilon}(0)} |g_0(x)| + \beta C_1 e^{-\beta^{1/2} R^{1/2}} \\ &\leq \beta C_1 (\varepsilon + e^{-\beta^{1/2} R^{1/2}}). \end{aligned}$$

Since $Q \cup -Q \subset \bar{B}_2(0)$, we can use the condition (iii) of Definition 5.23 together with interpolation to get

$$\begin{aligned} \eta^{-1} \|g\|_{L^2(\mu_\Gamma)}^2 &\leq \eta^{-1} \|g\|_{L^1(\mu)} \sup_{x \in \Gamma} |g(x)| \\ &\leq \lambda \|g\|_1 \sup_{x \in \Gamma} |g(x)| = \lambda \|g_0\|_1 \sup_{x \in \Gamma} |g(x)| \\ &\leq \frac{\lambda \|\phi\|_1}{\beta} \sup_{x \in \Gamma} |g(x)| \leq 24\pi\lambda C_1^2 (\varepsilon + e^{-\beta^{1/2} R^{1/2}}). \end{aligned}$$

And finally taking

$$\varepsilon = \frac{\zeta^2}{48\pi\lambda C_1^2}, \quad R = \beta^{-1} \ln^2(48\pi\lambda C_1^2 / \zeta^2),$$

we conclude that $\eta^{-1/2} \|g\|_{L^2(\mu_\Gamma)} \leq \zeta$, which is point (ii).

Note that

$$\frac{R^2}{\varepsilon} = C_2 \frac{\lambda}{\zeta^2 \beta^2} \ln^4(C_2 \lambda / \zeta^2)$$

with $C_2 = 48\pi C_1^2$.

To estimate the variation of \hat{g} we use again (5.26) and the fact that $\hat{\phi}$ is smooth and rapidly decreasing combined with Leibniz's rule:

$$\text{var}(\hat{g}) = \|\nabla \hat{g}\|_1 \leq \pi |y| \beta \left\| \hat{\phi} \right\|_1 + \left\| \nabla \hat{\phi} \right\|_1 \leq C_3 \frac{\lambda}{\zeta^2 \beta} \ln^4(C_2 \lambda / \zeta^2) + C_4$$

where $C_3 = \pi C_2 C_\Gamma$ and $C_4 = \left\| \nabla \hat{\phi} \right\|_1$. Also, (5.26) implies

$$\|\hat{g}\|_\infty \leq \beta^{-1} \left\| \hat{\phi} \right\|_\infty.$$

Finally we see that $\beta^{-1} \leq 2 \max((\eta - \sqrt{2}/2)^{-1}, \eta)$ and thus parts (iii) and (iv) follow by taking $C = \max(C_2, 2C_3, C_4, 2 \left\| \hat{\phi} \right\|_\infty)$. \square

Proposition 5.28. *Let $\eta = (1 + \varepsilon)\sqrt{2}/2$ with $\varepsilon \in (0, 1)$, and $\Gamma = A^\eta$ or $\Gamma = O^\eta$. Then given $\zeta > 0$, there exists $f \in L^2([-1/2, 1/2]^2)$ such that*

- (i) $\|f\|_2 = 1$,
- (ii) $\eta^{-1/2} \left\| \widehat{f} \right\|_{L^2(\mu_\Gamma)} \leq \zeta$,
- (iii) $\text{var}(f) \leq C(\varepsilon^{-1} \zeta^{-2} \ln^4(C\zeta^{-2}) + 1)$,
- (iv) $\|f\|_\infty \leq C\varepsilon^{-1}$,

where $C > 0$ is a universal constant.

Proof. By Proposition 5.25, $\mathcal{R}_{\pi/4} A^\eta \in \mathcal{C}(\eta, C, \bar{B}_2(0))$ for some constant $C > 0$. Set $\lambda := C\eta^{-1}$ so that $\mathcal{R}_{\pi/4} A^\eta \in \mathcal{C}(\eta, \eta\lambda, \bar{B}_2(0))$. Then using Lemma 5.27, we can construct $g \in L^2(\mathbb{R}^2)$ associated with $\mathcal{R}_{\pi/4} A^\eta$ and a given constant $\zeta > 0$. Define $f := \widehat{g} \circ \mathcal{R}_{-\pi/4}$. We will prove that f satisfies the conditions (i) – (iv) of the Proposition.

As in Lemma 5.27, let Q be the cube defined by the vertices $(0, 0)$, $(\sqrt{2}/4, \sqrt{2}/4)$, $(\sqrt{2}/2, 0)$ and $(\sqrt{2}/4, -\sqrt{2}/4)$. Since $\text{supp}(\widehat{g}) \subset Q \cup -Q$ and

$$\mathcal{R}_{-\pi/4}^{-1}(Q) = \mathcal{R}_{\pi/4}(Q) = [0, 1/2] \times [0, -1/2], \quad \mathcal{R}_{-\pi/4}^{-1}(-Q) = [-1/2, 0] \times [0, 1/2],$$

we have

$$\text{supp}(f) \subset [0, 1/2] \times [0, -1/2] \cup [-1/2, 0] \times [0, 1/2] \subset [-1/2, 1/2]^2.$$

Point (i) follows from the fact that rotations are norm-invariant transformations and $\|\widehat{g}\|_2 = 1$. To see (ii) note that $\widehat{f} = g \circ \mathcal{R}_{-\pi/4}^T$ and $\mathcal{R}_{-\pi/4}^T(A^\eta) = \mathcal{R}_{-\pi/4}^{-1}(A^\eta) = \mathcal{R}_{\pi/4} A^\eta$.

The variation of f can be estimated by the chain rule and (iii) of Lemma 5.27:

$$\begin{aligned} \text{var}(f) &= \|\nabla f\|_1 = \|\nabla(\widehat{g} \circ \mathcal{R}_{-\pi/4})\|_1 \leq \|(\nabla \widehat{g}) \circ \mathcal{R}_{-\pi/4}\|_1 \|\nabla \mathcal{R}_{-\pi/4}\|_\infty \\ &\lesssim \|\nabla \widehat{g}\|_1 = \text{var}(\widehat{g}) \lesssim \max((\eta - \sqrt{2}/2)^{-1}, \eta) \frac{\lambda}{\zeta^2} \ln^4 \left(C \frac{\lambda}{\zeta^2} \right) + 1. \end{aligned}$$

Then (iii) follows by using the identities $\eta = (1 + \varepsilon)\sqrt{2}/2$ and $\lambda = C\eta^{-1}$ so that $\max((\eta - \sqrt{2}/2)^{-1}, \eta) = (\eta - \sqrt{2}/2)^{-1} = \varepsilon^{-1}$ and $\lambda \asymp \eta^{-1} \asymp \eta \asymp 1$. Since $\|f\|_\infty = \|\widehat{g}\|_\infty$, this argument also yields (iv). \square

5.6 Compressibility and sub-Nyquist sampling

5.6.1 Analog compressibility

We will now work on the cube $[-1/2, 1/2]^2$. We recall the class $\mathcal{F}(W)$ defined in the introduction:

$$\mathcal{F}(W) := \{f \in L^2([-1/2, 1/2]^2) : \text{var}(f) \leq W\}.$$

The relevant *stability margin* is

$$A(\Gamma, \mathcal{F}(W)) := \inf\left\{\left\|\widehat{f}\right\|_{L^2(\mu_\Gamma)} : \|f\|_2 = 1, f \in \mathcal{F}(W)\right\}.$$

Let us now restate and prove Theorem B.

Theorem B. *Let $\eta = (1 + \varepsilon)\sqrt{2}/2$ with $\varepsilon \in (0, 1)$, and $\Gamma = A^\eta$ or $\Gamma = O^\eta$. Then for $W > 0$,*

$$A(\Gamma, \mathcal{F}(W)) \leq K(\varepsilon W)^{-1/2}(\ln^2(\varepsilon W) + 1), \quad (5.27)$$

where $K > 0$ is a universal constant.

Proof. Let $T > 0$, set $\zeta := \varepsilon^{-1/2}T^{-1/2}$ and take f given by Proposition 5.28 associated to ζ . Then there exists a constant $C > 0$ such that

$$\text{var}(f) \leq \Phi(T),$$

where

$$\Phi(T) := C(T \ln^4(C\varepsilon T) + 1).$$

Since $\|f\|_2 = 1$, $\eta^{-1/2}\left\|\widehat{f}\right\|_{L^2(\mu_\Gamma)} \leq \zeta$ and $\eta \asymp 1$, we have that

$$A(\Gamma, \mathcal{F}(\Phi(T))) \leq \left\|\widehat{f}\right\|_{L^2(\mu_\Gamma)} \leq \eta^{1/2}\zeta \lesssim \varepsilon^{-1/2}T^{-1/2}. \quad (5.28)$$

We claim that

$$T \gtrsim \frac{\Phi(T)}{\ln^4(\varepsilon\Phi(T)) + 1}, \quad \text{if } T \geq C^{-1}\varepsilon^{-1}e. \quad (5.29)$$

Indeed, if $C\varepsilon T \geq e$, then, since $\varepsilon \in (0, 1)$,

$$CT \leq \Phi(T) \quad \text{and} \quad T \geq e/C.$$

Consequently,

$$\Phi(T) \lesssim T \ln^4(C\varepsilon T) + 1 \lesssim T \ln^4(\varepsilon\Phi(T)) + T$$

from which (5.29) follows. Combining (5.28) and (5.29), we conclude that

$$A(\Gamma, \mathcal{F}(\Phi(T))) \lesssim \varepsilon^{-1/2}\Phi(T)^{-1/2}(\ln^2(\varepsilon\Phi(T)) + 1), \quad (5.30)$$

provided that $T \geq C^{-1}\varepsilon^{-1}e$.

Note that $\lim_{T \rightarrow +\infty} \Phi(T) = +\infty$, while $\Phi(C^{-1}\varepsilon^{-1}e) = \varepsilon^{-1}e + C$. This means that any number $W \geq \varepsilon^{-1}e + C$ can be represented as $W = \Phi(T)$, for some $T \geq C^{-1}\varepsilon^{-1}e$. Therefore, (5.30) reads (5.27) for all $W \geq \varepsilon^{-1}e + C$.

Finally, if $W \leq \varepsilon^{-1}e + C$, then $\varepsilon W \lesssim 1$, and the right-hand side of (5.27) is $\gtrsim 1$. On the other hand, the Bessel bound in Proposition 5.20 implies that

$$A(\Gamma, \mathcal{F}(W)) \leq \left\|\widehat{f}\right\|_{L^2(\mu_\Gamma)} \lesssim (\eta^{-1/2} + (1/2)^{1/2})\left\|\widehat{f}\right\|_2 \lesssim \|f\|_2 \lesssim 1,$$

since $\eta \asymp 1$. This completes the proof. \square

5.6.2 Sampling wavelet-sparse signals

We work with the Haar basis in $L^2([-1/2, 1/2]^2)$ constructed from the one in $L^2([0, 1])$ by translation and tensorization: from $h^0 = \chi_{[0,1]}$ and $h^1 = \chi_{[0,1/2]} - \chi_{[1/2,1]}$ one defines

$$h_{j,k}^e(x) = 2^{j/2} h^{e_1}(2^j(x_1 + 1/2) - k_1) 2^{j/2} h^{e_2}(2^j(x_2 + 1/2) - k_2)$$

with $j \geq 0$, $(k_1, k_2) \in \mathbb{Z}^2 \cap 2^j[0, 1]^2$, $e \in \{0, 1\}^2 \setminus \{(0, 0)\}$. We denote I the set of all such triples, and then define the sparsity classes as

$$\Sigma_{N,J} := \left\{ \sum_{(j,k,e) \in I} c_{j,k}^e h_{j,k}^e : c_{j,k}^e \in \mathbb{C}, \#I \leq N, 0 \leq j \leq J \right\}$$

and the corresponding stability margin

$$A_{N,J}(\Gamma) := \inf \left\{ \left\| \widehat{f} \right\|_{L^2(\mu_\Gamma)} : \|f\|_2 = 1, f \in \Sigma_{N,J} \right\}.$$

Let us now restate and prove Theorem C.

Theorem C. *Let $\eta = (1 + \varepsilon)\sqrt{2}/2$ with $\varepsilon \in (0, 1)$, and $\Gamma = A^\eta$ or $\Gamma = O^\eta$. Then for $N \geq 1$,*

$$A_{N,J}(\Gamma) \leq KN^{-1/6} \varepsilon^{-1} \ln^4(KN^{1/3}), \quad (5.31)$$

where $J = K \ln(\varepsilon^{-1}N)$, and $K > 0$ is a universal constant.

Proof. Fix N and let $\zeta = N^{-1/6}$ and take $f \in L^2([-1/2, 1/2]^2)$ as in Proposition 5.28. Then

$$\|f\|_2 = 1 \quad \text{and} \quad \left\| \widehat{f} \right\|_{L^2(\mu_\Gamma)} \lesssim N^{-1/6}, \quad (5.32)$$

$$\text{var}(f) \lesssim L := \varepsilon^{-1} N^{1/3} \ln^4(CN^{1/3}) + 1 \quad \text{and} \quad \|f\|_\infty \lesssim \varepsilon^{-1}. \quad (5.33)$$

Step 1. Let us consider the class

$$\Sigma_N := \left\{ \sum_{(j,k,e) \in I} c_{j,k}^e h_{j,k}^e : c_{j,k}^e \in \mathbb{C}, \#I \leq N \right\}$$

of functions with at most N active Haar coefficients *without* restrictions on the scale. Let f_N the best approximation of f in Σ_N . Since the variation of f is bounded, we can use the following inequality from [18, Theorem 8.2]:

$$\|f - f_N\|_2 \leq KN^{-1/2} \text{var}(f) \lesssim N^{-1/2} L, \quad (5.34)$$

where $K > 0$ is some universal constant. Let P_J be the orthogonal projection onto the span of wavelets with $0 \leq j \leq J$. Then $\|f - P_J f\|_1 \leq 2^{-J} \text{var}(f)$ - see for example [18, Eq. 2.10] and the references therein, or [22]. In addition, $\|f - P_J f\|_\infty \lesssim \|f\|_\infty$, and therefore interpolation yields

$$\|f - P_J f\|_2 \lesssim 2^{-J/2} \text{var}(f)^{1/2} \|f\|_\infty^{1/2}.$$

Plugging (5.33) we get

$$\|f - P_J f\|_2 \lesssim 2^{-J/2} \varepsilon^{-1} L^{1/2} \leq 2^{-J/2} \varepsilon^{-1} L \quad (5.35)$$

where we have also used the fact that $L \geq 1 > \varepsilon$. Now set $f_{N,J} := P_J f_N \in \Sigma_{N,J}$, and combine (5.35) and (5.34) to obtain

$$\begin{aligned} \|f - f_{N,J}\|_2 &\leq \|f - P_J f\|_2 + \|P_J f - P_J f_N\|_2 \\ &\leq \|f - P_J f\|_2 + \|f - f_N\|_2 \lesssim (2^{-J/2} \varepsilon^{-1/2} + N^{-1/2}) L. \end{aligned}$$

Hence, choosing $J \asymp \ln(\varepsilon^{-1} N)$,

$$\|f - f_{N,J}\|_2 \lesssim N^{-1/2} L. \quad (5.36)$$

Step 2. Write $A_{N,J} = A_{N,J}(\Gamma)$. Then, using (5.32) and the definition of $A_{N,J}$,

$$\begin{aligned} A_{N,J} &= A_{N,J} \|f\|_2 \leq A_{N,J} \|f_{N,J}\|_2 + A_{N,J} \|f_{N,J} - f\|_2 \\ &\leq \left\| \widehat{f_{N,J}} \right\|_{L^2(\mu_\Gamma)} + A_{N,J} \|f_{N,J} - f\|_2 \\ &\leq \left\| \widehat{f} \right\|_{L^2(\mu_\Gamma)} + \left\| \widehat{f_{N,J}} - \widehat{f} \right\|_{L^2(\mu_\Gamma)} + A_{N,J} \|f_{N,J} - f\|_2 \\ &\leq K_0 N^{-1/6} + K_0 \|f_{N,J} - f\|_2 + A_{N,J} \|f_{N,J} - f\|_2, \quad (5.37) \end{aligned}$$

for some constant $K_0 > 0$ and where we have also applied Proposition 5.20 to estimate $\left\| \widehat{f_{N,J}} - \widehat{f} \right\|_{L^2(\mu_\Gamma)}$. Now from (5.36), if

$$N \gtrsim L^2,$$

and the implicit constant is large enough, then $\|f - f_{N,J}\|_2 < 1/2$. Going back to (5.37) and re-applying (5.36) we get

$$A_{N,J} \lesssim N^{-1/6} + N^{-1/2} L \lesssim N^{-1/6} \varepsilon^{-1} \ln^4(CN^{1/3}).$$

Note that since $L = \varepsilon^{-1} N^{1/3} \ln^4(CN^{1/3}) + 1$, then, for $N \gtrsim L^2$ to hold it is sufficient to have $N \gtrsim \varepsilon^{-2} N^{2/3} \ln^8(CN^{1/3})$, or, equivalently,

$$N \gtrsim \varepsilon^{-6} \ln^{24}(CN^{1/3}).$$

Hence, there is a constant $C' > 0$ for which (5.31) holds whenever we have $N \geq C' \varepsilon^{-6} \ln^{24}(C' N^{1/3})$. On the other hand, if $N \leq C' \varepsilon^{-6} \ln^{24}(C' N^{1/3})$ then to prove (5.31) is enough to show $A_{N,J} \lesssim 1$, which, as in the proof of Theorem B, follows from the Bessel bounds in Proposition 5.20. \square

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