Regularized integrals and L-functions of modular forms via the Rogers-Zudilin method

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Chapter 0

Introduction

0.1 Regulator Integrals

The most central objects of this thesis are regulator integrals. In his fundamental paper [1], Beilinson defined a regulator map and formulated his famous conjectures connecting special L-values to regulators. Let X be a smooth quasi-projective variety defined over $\mathbb{Q}$. Given two integers $n \geq 0$ and p, Beilinson's regulator

$$r_{\mathcal{D}} : H^n_{\mathcal{M}}(X, \mathbb{Q}(p)) \to H^n_{\mathcal{D}}(X(\mathbb{C}), \mathbb{R}(p)),$$

is a $\mathbb{Q}$-linear map from the motivic cohomology ('an arithmetic invariant') of X to the Deligne–Beilinson cohomology ('an analytic invariant') of X. The Deligne–Beilinson cohomology depends only on the complex analytic variety $X(\mathbb{C})$. Beilinson's conjectures predict the special L-values at integers, up to a rational factors, in terms of the determinants of regulators of some rational structures in the motivic cohomology groups.

Let $N \geq 3$ be an integer and $\mathcal{H} = \{\tau \in \mathbb{C} \,|\, \mathrm{Im}(\tau) > 0\}$ be the Poincaré upper half-plane. Let $Y(N) = \Gamma(N)\backslash\mathcal{H}$ be the open modular curve associated to the congruence subgroup

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \,\middle|\, a \equiv d \equiv 1 \pmod{N},\, b \equiv c \equiv 0 \pmod{N} \right\},$$

where the action of $\Gamma(N)$ on $\mathcal{H}$ is given by Möbius transformations. In this case, for $j \geq 0$, the Deligne–Beilinson cohomology group of $Y(N)$

$$H^2_{\mathcal{D}}(Y(N), \mathbb{R}(j+2)) \simeq H^1_{dR}(Y(N), (2\pi i)^{j+1}\mathbb{R})$$

is simply the the de Rham cohomology group with twisted coefficients $(2\pi i)^{j+1}$.

In [2], Beilinson constructed a special cohomology class

$$\mathrm{Eis}^{0,0,j}_{\mathcal{D}}(u_1, u_2) \in H^2_{\mathcal{D}}(Y(N), \mathbb{R}(j+2))$$

in the Deligne–Beilinson cohomology, where $u_i \in (\mathbb{Z}/N\mathbb{Z})^2$. These classes are the image of certain special elements in the motivic cohomology group $H^2_{\mathcal{M}}(Y(N), \mathbb{Q}(j+2))$ under

the regulator map. In the case $j = 0$, they are called Beilinson-Kato elements, which are constructed using cup-products of certain modular functions on $Y(N)$ called Siegel units. In general, the class $\mathrm{Eis}_{\mathcal{D}}^{0,0,j}(u_1, u_2)$ is - loosely speaking - constructed by taking the product of a real-analytic Eisenstein series with a holomorphic Eisenstein series.

Beilinson also proved in [2] the following formula with the Rankin–Selberg method. He showed that the integrals of these classes are related to some special L-values of modular forms.

Theorem 0.1.1 (Beilinson [2]). *Let f be a cusp eigenform of weight 2 on $\Gamma_1(N)$. Let K_f be the coefficient field of f. Note $\omega_f = 2\pi i f(\tau)d\tau$ the holomorphic form associated to f. Then*

(1) For any $u_1, u_2 \in (\mathbb{Z}/N\mathbb{Z})^2$, we have

$$\int_{Y(N)} \mathrm{Eis}_{\mathcal{D}}^{0,0,j}(u_1, u_2) \wedge \omega_f \in (2\pi i)^{j+1}\Omega_f^{(-1)^j} L'(f, -j) \cdot K_f,$$

where $\Omega_f^{\pm}$ denotes Deligne's real or imaginary periods of f.

(2) There exist a level N' divisible by N and a class $\mathrm{Eis}_{\mathcal{D}}^{0,0,j}(u_1, u_2)$ in level N' with $u_1, u_2 \in (\mathbb{Z}/N'\mathbb{Z})^2$ such that the integral as in (1), computed in level N', is nonzero.

However, the constant factor in K_f and the level N' are not given explicitly in Beilinson's formula.

A new and more explicit calculation is recently done by Zudilin [49] and Brunault [11] for $j = 0$. Instead of integration over $Y(N)$, they considered the following integral of the regulator along the imaginary axis (i.e. the modular symbol $\{0, \infty\}$)

$$\int_0^{\infty} \mathrm{Eis}_{\mathcal{D}}^{0,0,0}(u_1, u_2),$$

With a powerful method of Rogers–Zudilin [48], they were able to show that

Theorem 0.1.2 (Zudilin [49], Brunault [10]). *Let $N \geq 3$ be an integer. Let $u_1 = (a_1, b_1)$, $u_2 = (a_2, b_2) \in (\mathbb{Z}/N\mathbb{Z})^2$ be nonzero vectors. Then we have*

$$\int_0^{\infty} \mathrm{Eis}_{\mathcal{D}}^{0,0,0}(u_1, u_2) = \frac{4\pi i}{N^2}\Lambda^* \left(G^{(1)}_{b_1,-a_2} G^{(1)}_{b_2,a_1} - G^{(1)}_{b_1,a_2} G^{(1)}_{b_2,-a_1},\, 0 \right),$$

where the functions $G^{(1)}$ are certain Eisenstein series of weight 1 level $\Gamma(N)$ with rational coefficients (see Section 1.3 for definition), and $\Lambda^*(f, s)$ denotes the regularized value of the completed L-function $\Lambda(f, s)$.

With the help of the method of Rogers–Zudilin, we are able to generalize Theorem 0.1.2 to arbitrary integer $j \geq 0$. Compared to Theorem 0.1.1, our formula is more precise and does not rely on a higher level N'.

Theorem 0.1.3. *Let $N \geq 3$ and $j \geq 0$ be integers. Let $u_1 = (a_1, b_1)$, $u_2 = (a_2, b_2) \in (\mathbb{Z}/N\mathbb{Z})^2$ be nonzero vectors. Then we have*

$$\int_0^\infty \mathrm{Eis}_{\mathcal{D}}^{0,0,j}(u_1, u_2) = \frac{j!(j+2)^2\pi}{N^{j+2}} i^{(j+1)^2} \Lambda^* \left(G^{(1)}_{b_1,-a_2} G^{(1)}_{b_2,a_1} - (-1)^j G^{(1)}_{b_1,a_2} G^{(1)}_{b_2,-a_1}, -j \right).$$

Generally, we have classes $\mathrm{Eis}_{\mathcal{D}}^{k_1,k_2,j}(u_1, u_2)$ living in the Deligne–Beilinson cohomology of fiber products of the universal elliptic curves. A universal elliptic curve E is a complex-analytic manifold endowed with a fibration $p : E \to Y(N)$, with the property that the fiber of p over a point $[\tau] \in Y(N)$ is exactly the elliptic curve $E_\tau = \mathbb{C}/\mathbb{Z} + \mathbb{Z}\tau$. Denote by E^w the w-fold fiber product of E over $Y(N)$. Over each point $[\tau] \in Y(N)$, the fiber of E^w over $[\tau]$ is just the w-th power E_τ^w of the elliptic curve.

Let k_1, k_2, j be non-negative integers and $u_1, u_2 \in (\mathbb{Z}/N\mathbb{Z})^2$ with $u_i \neq (0, 0)$ if $k_i = 0$. Set $w = k_1 + k_2$. Deninger–Scholl [22] and Gealy [28] generalized the construction of Beilinson by defining the element

$$\mathrm{Eis}_{\mathcal{D}}^{k_1,k_2,j}(u_1, u_2) \in H_{\mathcal{D}}^{w+2}(E^w, \mathbb{R}(w + j + 2))$$

in the Deligne–Beilinson cohomology of E^w. Deninger–Scholl and Gealy also generalized Beilinson's formula to higher weight case with the elements $\mathrm{Eis}_{\mathcal{D}}^{k_1,k_2,j}(u_1, u_2)$ and cusp eigenforms of weight $w + 2$.

In [11], Brunault considered the following regularized integral of regulator

$$\int^*_{X^w\{0,\infty\}} \mathrm{Eis}_{\mathcal{D}}^{k_1,k_2}(u_1, u_2),$$

where $X^w\{0, \infty\}$ is a certain $(w + 1)$-chain on E^w, called Shokurov cycle (see Section 3.6 for definitions). Again with the method of Rogers–Zudilin, he gave the following formula

$$\int^*_{X^w\{0,\infty\}} \mathrm{Eis}_{\mathcal{D}}^{k_1,k_2}(u_1, u_2) = C_{k_1,k_2} \Lambda^* \left(G^{(k_1+1)}_{b_1,-a_2} G^{(k_2+1)}_{b_2,a_1} - G^{(k_1+1)}_{b_1,a_2} G^{(k_2+1)}_{b_2,-a_1}, 0 \right),$$

where

$$C_{k_1,k_2} = \frac{(k_1 + 2)(k_2 + 2)}{2N^{w+2}} (2\pi)^{w+1} i^{k_1-k_2+1} \in (2\pi i)^{w+1}\mathbb{Q}.$$

In Chapter 6 we generalize his formula and compute more general regulator integrals. We obtain the following result (see Section 6.1)

Theorem 0.1.4. *Let k_1, k_2, j be nonnegative integers with $w = k_1 + k_2$. Let $N \geq 3$ and $u_1 = (a_1, b_1)$, $u_2 = (a_2, b_2) \in (\mathbb{Z}/N\mathbb{Z})^2$, suppose that $(a_i, b_i) \neq (0, 0)$ if $k_i = 0$ and $b_i \neq 0$ if $k_i = 1$, then*

$$\int^*_{X^w\{0,\infty\}} \mathrm{Eis}_{\mathcal{D}}^{k_1,k_2,j}(u_1, u_2) = C_{k_1,k_2,j} \Lambda^* \left(G^{(k_1+1)}_{b_1,-a_2} G^{(k_2+1)}_{b_2,a_1} - (-1)^j G^{(k_1+1)}_{b_1,a_2} G^{(k_2+1)}_{b_2,-a_1}, -j \right)$$

with the constant

$$C_{k_1,k_2,j} = \frac{j!(k_1 + j + 2)(k_2 + j + 2)}{2N^{w+j+2}} i^{k_1-k_2+(j+1)^2} (2\pi)^{w+1} \in (2\pi i)^{w+1}\mathbb{Q}.$$

It can be shown that the appearance of the power of $2\pi i$ and the L-value at $s=-j$ within our formula are in accordance with Beilinson's conjectures.

For integrals over more general Shokurov cycles we have

Theorem 0.1.5. *Let $w \geq m \geq k_1, k_2$. Assume that if $m = k_2$ then $k_1 = 0$. Let $N \geq 3$ and $u_1 = (a_1, b_1)$, $u_2 = (a_2, b_2) \in (\mathbb{Z}/N\mathbb{Z})^2$, suppose that $(a_i, b_i) \neq (0,0)$ if $k_i = 0$ and $b_i \neq 0$ if $k_i = 1$, then the regulator integral*

$$\int_{X^m Y^{w-m}\{0,\infty\}}^{*} \mathrm{Eis}_{\mathcal{D}}^{k_1,k_2,j}(u_1,u_2)$$

is a linear combination of L-values of quasi-modular forms (see Section 1.1) with rational coefficients .

In fact, in the higher weight case, the regulator integral usually does not converge. We need a theory of regularized integrals to solve this. It is given in the following manner.

0.2 Generalized Mellin Transform and Regularized Integral

Chapter 3 is devoted to establish a more general theory of Mellin transforms and regularized integrals.

Fixing an integer $k \geq 2$, we denote by $S_k(\Gamma_1(N))$ the space of holomorphic cusp forms of weight k level $\Gamma_1(N)$. Given a cusp form $f(\tau) = \sum_{n\geq 1} a_f(n) e^{2\pi n i \tau} \in S_k(\Gamma_1(N))$, the completed L-function associated to f is essentially the Mellin transform of f

$$\Lambda(f,s) = \int_0^\infty f(iy) y^s \frac{dy}{y} = (2\pi)^{-s}\Gamma(s) \sum_{n\geq 1} \frac{a_f(n)}{n^s}.$$

However, for many modular functions, their Mellin transforms do not exist anymore. To deal with some of these functions, the method of generalized Mellin transforms has been proposed in various literature, such as [46], [17, Chapter 1] and [19, Section 8.6].

Our theory of generalized Mellin transform handles more general functions. We investigate first functions with exp-poly-log expansions and define their generalized Mellin transforms and also their regularized integrals (see Section 3.2 and Section 3.3). The theory of generalized Mellin transform is used among other things in defining L-functions. For instance, given a weakly holomorphic cusp form $f = \sum_{n\geq n_0} a_f(n) e^{2\pi n i \tau} \in S_k^!(\mathrm{SL}_2(\mathbb{Z}))$ of weight k, we are able to recover the definition of L-function of f by Bringmann, Fricke and Kent [5]

$$\Lambda(f,s) = \sum_{n\geq n_0} \frac{a_f(n)\Gamma(s, 2\pi n t_0)}{(2\pi n)^s} + i^k \sum_{n\geq n_0} \frac{a_f(n)\Gamma(k-s, \frac{2\pi n}{t_0})}{(2\pi n)^{k-s}}.$$

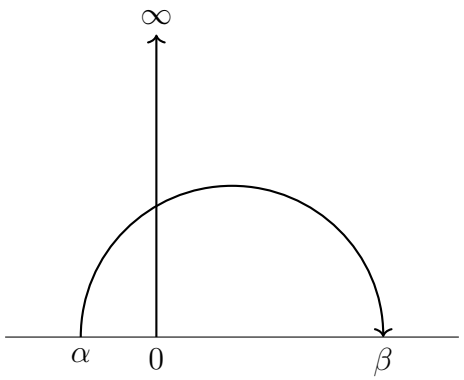

Figure 1. Modular symbols

Let $\alpha, \beta \in \mathbb{P}^1(\mathbb{Q})$ be two distinct rationals. Let $f(\tau) \in S_2(\Gamma_1(N))$ be a cusp form of weight 2 level $\Gamma_1(N)$. A modular symbol $\{\alpha, \beta\}$ is an oriented geodesic from α to β on the upper half-plane $\mathcal{H}$ (depicted in Figure 1). According to the idea of Birch (also independently by Manin), we can pair the closed form $f(\tau)d\tau$ with $\{\alpha, \beta\}$ in the following way

$$\langle f(\tau)d\tau, \{\alpha, \beta\}\rangle = \int_{\{\alpha,\beta\}} f(\tau)d\tau.$$

In particular, from Eichler–Shimura theory (see for example Kohnen–Zagier [31]), we find the period of the cusp form

$$r_0(f) = \Lambda(f, 1) = -i\langle f(\tau)d\tau, \{0, \infty\}\rangle.$$

By Stokes' theorem, we have the following 3-term relation

$$\int_{\{\alpha,\beta\}} f(\tau)d\tau + \int_{\{\beta,\gamma\}} f(\tau)d\tau + \int_{\{\gamma,\alpha\}} f(\tau)d\tau = 0.$$

For a general modular function f, the closed form $f(\tau)d\tau$ probably does not vanish at cusps and has nonzero residues. Consequently, the integration of $f(\tau)d\tau$ along a modular symbol may not converge and the 3-term relations may not hold as well.

We need the terminology of modular caps by Stevens [44] (see Section 2.1) to handle closed forms with exp-poly-log expansions. A modular cap $[\gamma, \beta]_\alpha$ is the segment of the infinitesimal horocycle at α cut by two modular symbols $\{\alpha, \beta\}$ and $\{\gamma, \alpha\}$ (depicted in Figure 2). With our theory of regularization, we succeed in defining the regularized integrals of a given closed form ω along modular symbols and modular caps in Section 3.6. The 3-term relations is replaced by the 6-term relations (see again Figure 2)

$$\int^*_{\{\alpha,\beta\}} \omega + \int^*_{[\alpha,\gamma]_\beta} \omega + \int^*_{\{\beta,\gamma\}} \omega + \int^*_{[\beta,\alpha]_\gamma} \omega + \int^*_{\{\gamma,\alpha\}} \omega + \int^*_{[\gamma,\beta]_\alpha} \omega = 0,$$

where the * indicates regularized integrals.

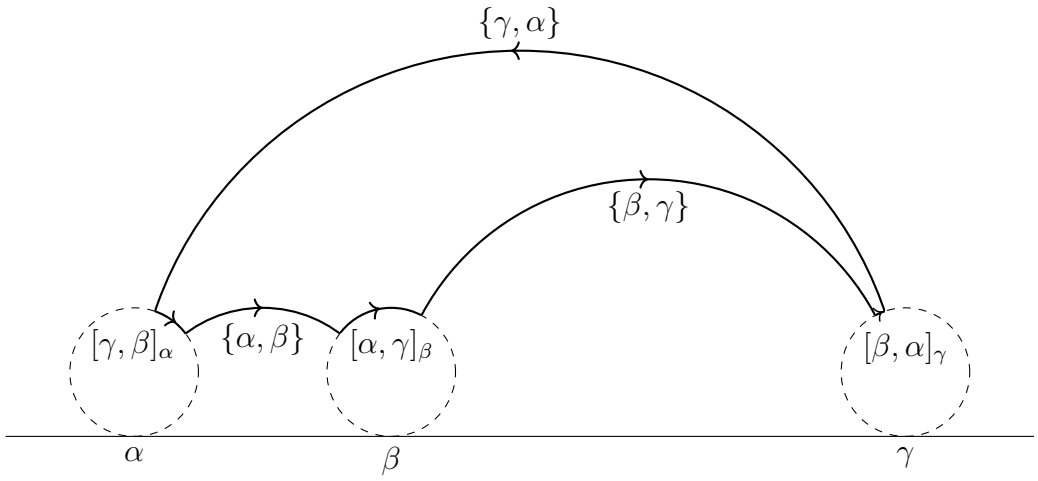

Figure 2. Modular caps

These results can be summarized in the following theorem

Theorem 0.2.1. *There is a well-defined integration pairing*

$$\Omega^1_{epl}(\mathcal{H}, \mathbb{C}) \times \mathbb{K}_2 \longrightarrow \mathbb{C},$$

where $\mathbb{K}_2$ is the space of modular symbols and modular caps, Ω^1_{epl} denotes the space of closed forms on the Poincaré upper half-plane $\mathcal{H}$ with some growth conditions at cusps.

This integration pairing can also be generalized to higher weight cases, see Section 3.6 for more details.

0.3 Double L-functions

In [33], Manin constructed multiple L-functions of holomorphic cusp forms. In this thesis we shall focus double modular L-functions. Let $f = \sum_{n\geq 0} a_n e^{2\pi n i\tau}$ and $g = \sum_{m\geq 0} b_m e^{2\pi m i\tau}$ be two modular forms with respect to a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$. Their double L-function is the following double Dirichlet series

$$L(f, g, s_1, s_2) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} \frac{a_n b_m}{n^{s_1}(n+m)^{s_2}}.$$

Manin studied also the iterated Mellin transform of cusp forms. The iterated Mellin transform of f and g is given by the following integral

$$\Lambda(f,g,s_1,s_2)=\int_0^\infty g(it_2)t_2^{s_2-1}dt_2\int_{t_2}^\infty f(it_1)t_1^{s_1-1}dt_1,$$

which is also called a double L-function of f and g.

Based on our theory of regularized integrals, we give a tentative generalization of double L-function to weakly holomorphic modular forms. Let $f=\sum_{n\geq n_0}a_nq^n$ and $g=\sum_{m\geq m_0}b_mq^m$ be two weakly holomorphic modular forms in level $\mathrm{SL}_2(\mathbb{Z})$ of weight $k_1\geq 2$ and $k_2\geq 2$ respectively. Fix an integer $0<s_1<k_1$. We define the double L-function $\Lambda(f,g,s_1,s_2)$ using generalized Mellin transform and show in Section 4.2 that

Theorem 0.3.1. *The double L-function $\Lambda(f,g,s_1,s_2)$, as a function of s_2, extends to a meromorphic function on the whole complex plane. It has possibly poles when s_2 is an integer from $-s_1$ to 0 or from k_2 to $k_2+k_1-s_1$, and is holomorphic elsewhere.*

We have also included some of its residues, for example,

$$\operatorname*{Res}_{s_2=-s_1}\Lambda(f,g,s_1,s_2)=\frac{a_0b_0}{s_1}.$$

When $a_0=0$, $k_1=k_2=k$ we shall have

$$\operatorname*{Res}_{s_2=0}\Lambda(f,g,k-1,s_2)=-\frac{(k-2)!}{(2\pi)^{k-1}}\{f,g\},$$

where $\{f,g\}$ is the Bruinier-Funke pairing (see [7, (1.15)] for definition).

0.4 Double L-values with Rogers–Zudilin Method

Rogers and Zudilin introduced a new powerful method in the proof of Boyd's conjectures on Mahler measures (see [48]). A reinterpretation of their method via correspondence of modular forms can be found in Diamantis–Neururer–Strömberg [23]. In [41], Shinder and Vlasenko use Rogers–Zudilin method to compute an explicit example of double L-value of Eisenstein-like series.

Inspired by the example of Shinder–Vlasenko, we look for more general identities of double L-values in Section 4.4. Write the double L-function

$$\mathbb{L}(f,g,s_1,s_2)=(2\pi)^{-s_2}\Gamma(s_2)L(f,g,s_1,s_2).$$

Then for fixed $s_1\in\mathbb{Z}$, the function $\mathbb{L}(f,g,s_1,s_2)$ is meromorphic in s_2. From our theory of generalized Mellin transform, the regularized value $\mathbb{L}^*(f,g,s_1,s_2)$ in s_2 always exist. Applying Rogers–Zudilin method, we show that

Theorem 0.4.1. *Let $N \geq 1$ be an integer. Let $a_1, a_2, b_1, b_2 \in \mathbb{Z}/N\mathbb{Z}$ and $k_1 \geq 2$, $k_2 \geq 2$ be positive integers. Suppose that $1 \leq s_1 \leq k_1 - 1$, $1 \leq s_2 \leq k_2 - 1$ are integers with $k_1 \leq s_1 + s_2$. Then the double L-value*

$$\mathbb{L}^*\left(G^{(k_1)}_{a_1,b_1}, H^{(k_2)}_{b_2,a_2}, k_1 - s_1, k_2 - s_2\right) + (-1)^{s_1+s_2-1}\mathbb{L}^*\left(G^{(k_1)}_{-a_1,b_1}, H^{(k_2)}_{b_2,-a_2}, k_1 - s_1, k_2 - s_2\right)$$

is a linear combination of L-values of certain quasi-modular forms with coefficients in $\mathbb{Q}(\zeta_N)$ of level $\Gamma_1(N^2)$. Here G and H are certain Eisenstein series with coefficients in $\mathbb{Q}(\zeta_N)$ (see Section 1.3 for definitions).

In particular in level $N = 1$, set $\mathbb{G}_k(\tau) = -\frac{B_k}{2k} + \sum_{n=1}^{\infty} \sigma_{k-1}(n)q^n$ to be the Eisenstein series of weight k. We deduce that

Theorem 0.4.2. *Let $k_1 \geq 4, k_2 \geq 4$ be even number. Let $1 \leq s_1 \leq k_1 - 1$ and $1 \leq s_2 \leq k_2 - s_1$ be integers with opposite parity. Set $p = \min\{k_1 - s_1, s_2\} - 1$. Then*

$$\begin{aligned} i^{s_1+s_2-1}\mathbb{L}^*\left(\mathbb{G}_{k_1}, \mathbb{G}_{k_2}, s_1, s_2\right) = \Lambda^*\left(D^p\mathbb{G}_{|k_1-s_1-s_2|+1} \cdot \mathbb{G}_{k_2-s_1-s_2+1}, 1 - s_1\right) \\ + \delta_{s_1+s_2=k_2-1}(4\pi)^{-1}\Lambda^*\left(D^p\mathbb{G}_{|k_1-s_1-s_2|+1}, -s_1\right). \end{aligned}$$

Furthermore, the double L-value $\mathbb{L}^*(\mathbb{G}_{k_1}, \mathbb{G}_{k_2}, s_1, s_2)$ is a $\mathbb{Q}[1/\pi]$-linear combination of L-values of modular forms with rational coefficients and L-values of $\mathbb{G}_2$

$$\begin{aligned} \mathbb{L}^*(\mathbb{G}_{k_1}, \mathbb{G}_{k_2}, s_1, s_2) \in \sum_{l=0}^{p} \frac{\mathbb{Q}\,\Lambda\left(\left[\mathbb{G}_{|k_1-s_1-s_2|+1}, \mathbb{G}_{k_2-s_1-s_2+1}\right]^{mod}_{p-l}, 1 - s_1 - l\right)}{\pi^l} \\ + \delta_{s_1+s_2=k_2-1}\frac{\mathbb{Q}\,\Lambda\left(\mathbb{G}_{|k_1-s_1-s_2|+1}, -s_1 - p\right)}{\pi^{p+1}} + \delta_{s_1+s_2=k_1\pm1}\frac{\mathbb{Q}\,\Lambda\left(\mathbb{G}_{k_2-s_1-s_2+1}, -s_1 - p\right)}{\pi^{p+1}}. \end{aligned}$$

Example 0.4.3. *Here we consider an example $(k_1, k_2) = (8, 10)$ and $(s_1, s_2) = (1, 4)$. We have $p = 3$. As we shall see, this double L-value $\mathbb{L}^*(\mathbb{G}_8, \mathbb{G}_{10}, 1, 4)$ can be made explicitly*

$$\begin{aligned} \mathbb{L}^*(\mathbb{G}_8, \mathbb{G}_{10}, 1, 4) &= \Lambda^*(D^3\mathbb{G}_4 \cdot \mathbb{G}_6, 0) \\ &= -\frac{1}{6720\pi^3}\Lambda(\mathbb{G}_{10}, -3) - \frac{3}{3640\pi^2}\Lambda(\Delta_{12}, -2) - \frac{1}{1872}\Lambda(\Delta_{16}, 0), \end{aligned}$$

where Δ_{12} and Δ_{16} are the unique normalized cusp forms of weight 12 and 16 respectively.

0.5 Mordell–Tornheim Double Eisenstein Series and Cohen Series

This part is a joint work with Zhang. Let k_1, k_2 and k_3 be non-negative integers. Tornheim [45] considered the following double series

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{1}{n^{k_1}m^{k_2}(n+m)^{k_3}},$$

which is now called the Mordell–Tornheim double zeta function.

Following this pattern, in this thesis we define the Mordell–Tornheim double Eisenstein series

$$G(\tau; k_1, k_2, k_3; \omega_1, \omega_2) = \sum_{\tau_1,\tau_2 \in \mathbb{Z}+\mathbb{Z}\tau}{}' \frac{1}{\tau_1^{k_1}\tau_2^{k_2}(\omega_1\tau_1+\omega_2\tau_2)^{k_3}},$$

where k_1, k_2, k_3 are non-negative integers and ω_1, ω_2 are two integers, the primed summation means the terms which τ_1, τ_2 or $\omega_1\tau_1+\omega_2\tau_2$ vanishes are omitted.

To find an explicit form of Mordell–Tornheim double Eisenstein series, we need to introduce Cohen series. Given $0 \le n \le k-2$, define $R_n \in S_k$ to be the following series

$$R_n = c_{k,n}^{-1} \sum_{\substack{a,b,c,d\in\mathbb{Z}\\ ad-bc=1}} (a\tau+b)^{-n-1}(c\tau+d)^{n+1-k} \quad \text{with } c_{k,n} = \frac{\pi\binom{k-2}{n}}{2^{k-2}i^{n+1-k}}.$$

The series R_n can be also described as the unique cusp form such that

$$\langle f, R_n\rangle = r_n(f) \quad \text{for all } f \in S_k,$$

where $\langle , \rangle$ indicates the Petersson inner product. The Cohen series is later extended by Diamantis and O'Sullivan [24] to more general settings. Using their theory of Cohen series, we are successful to give an explicit formula of Mordell–Tornheim double Eisenstein series in Section 5.4. Loosely speaking, we prove that

Theorem 0.5.1. *The Mordell–Tornheim double Eisenstein series is*

$$G(\tau; k_1, k_2, k_3; \omega_1, \omega_2) = G_{eis} + G_{cusp},$$

where G_{eis} is an explicit Eisenstein series and G_{cusp} is a linear combination of (modified) products of Eisenstein series.

In the case $\omega_1 = \omega_2 = 1$ setting $G(\tau; k_1, k_2, k_3) \coloneqq (-1)^{k_3} G(\tau; k_1, k_2, k_3; 1, 1)$ we get

$$G(\tau; k_1, k_2, k_3) = \sum_{\substack{\tau_1+\tau_2+\tau_3=0\\ \tau_1,\tau_2,\tau_3\in\mathbb{Z}+\mathbb{Z}\tau}}{}' \frac{1}{\tau_1^{k_1}\tau_2^{k_2}\tau_3^{k_3}},$$

which is symmetric in k_1, k_2 and k_3. We obtain the following formula

Theorem 0.5.2. *Let k_1, k_2 and k_3 be nonnegative integers with $k_1+k_3 > 2$, $k_2+k_3 > 2$, $k_1+k_2 > 2$ and $k = k_1+k_2+k_3 > 4$. Then the Mordell–Tornheim double Eisenstein series is the following modular form of weight k*

$$\begin{aligned} G(\tau; k_1, k_2, k_3) = (-1)^{k_3} \Bigg(& \sum_{\mu=0}^{k_1-2} \delta_{\mu\equiv k_1(2)} \binom{k_2+\mu-1}{\mu} G_{k_1-\mu} G_{k_2+k_3+\mu} \\ & + \sum_{\nu=0}^{k_2-2} \delta_{\nu\equiv k_2(2)} \binom{k_1+\nu-1}{\nu} G_{k_2-\nu} G_{k_1+k_3+\nu} \\ & - \frac{4\pi^2}{k-2} \binom{k_1+k_2-2}{k_1-1} DG_{k-2} - \binom{k_1+k_2}{k_1} G_k \Bigg). \end{aligned}$$

Hence the form $\frac{1}{\pi^k} G(\tau; k_1, k_2, k_3)$ has rational coefficients.

Example 0.5.3. *As an instance, let* $(k_1, k_2, k_3) = (2, 3, 7)$*, with a direct computation we get*

$$G(\tau; 2, 3, 7) = \frac{34\pi^{12}}{127702575}\left(E_{12}(\tau) + \frac{62270208}{11747}\Delta(\tau)\right).$$

0.6 Outline of this Thesis

Chapter 1 and Chapter 2 provide an introduction to the objects and theories needed in this thesis. In particular, we recall the theory of quasi-modular forms and give a definition of their L-functions, we also give a short introduction of Eisenstein symbols.

In Chapter 3, we review briefly the classical theory of Mellin transform. In the rest of this chapter, we develop a theory of generalized Mellin transforms and regularized integrals. These are the basic tools of the rest parts.

The Rogers–Zudilin method is introduced in Chapter 4. We study the double L-functions of weakly holomorphic modular forms. Certain double L-values of Eisenstein series are computed with Rogers–Zudilin method at the end of this chapter.

The last chapter contains our final results on regulator integrals. All the computation of regulator, involving periods and residues, is included in Chapter 6.

Chapter 5 is an independent chapter, which includes a collaborative work on Mordell–Tornheim double Eisenstein series with Zhang.

Chapter 1

Preliminaries

In this chapter, we review the basic background knowledge about modular forms, L-functions and incomplete gamma functions. We also introduce some preliminary results which we will be using in this thesis.

1.1 Modular Forms and Quasi-Modular Forms

This section provides a brief introduction for the readers who may not be familiar with the theory of quasi-modular forms or (weakly) holomorphic modular forms. Clear and systematic references of quasi-modular forms are [19] and [47].

We fix the notations first. By $\mathcal{H} = \{\tau \in \mathbb{C} \mid \mathrm{Im}(\tau) > 0\}$ we denote the Poincaré upper half-plane. Write $\overline{\mathcal{H}} = \mathcal{H} \cup \mathbb{P}^1(\mathbb{Q})$. An element $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2^+(\mathbb{Q})$ acts on $\overline{\mathcal{H}}$ by

$$\gamma\tau = \frac{a\tau + b}{c\tau + d}.$$

We will write for brevity $j(\gamma, \tau) \coloneqq c\tau + d$ and $q \coloneqq e^{2\pi i \tau}$.

In this thesis we usually focus on the following subgroups of $\Gamma_1 = \mathrm{SL}_2(\mathbb{Z})$

$$\Gamma_1(N) \coloneqq \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1 \,\middle|\, a \equiv d \equiv 1 \pmod{N},\, c \equiv 0 \pmod{N} \right\}$$
$$\Gamma(N) \coloneqq \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(N) \,\middle|\, b \equiv 0 \pmod{N} \right\}$$

for a positive integer N. The notation Γ usually indicates a *congruence subgroup*, that is, a subgroup of $\mathrm{SL}_2(\mathbb{Z})$ containing the special subgroup $\Gamma(N)$ for some N.

Let f be a function from $\mathcal{H}$ to $\mathbb{C}$ and $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, the *slash operator of weight* k is defined as

$$(f|_k\gamma)(\tau) = (c\tau + d)^{-k} f\left(\frac{a\tau + b}{c\tau + d}\right) = j(\gamma, \tau)^{-k} f(\gamma\tau).$$

1.1.1 Definitions and Examples

Let Γ be a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$ and k be a positive integer throughout this section.

The *cusps* of Γ are a collection of left coset representatives of $\Gamma\backslash\mathbb{P}^1(\mathbb{Q})$. For every cusp $\alpha \in \Gamma\backslash\mathbb{P}^1(\mathbb{Q})$ we have an element $\sigma_\alpha \in \Gamma\backslash \mathrm{SL}_2(\mathbb{Z})$ with $\sigma_\alpha\infty = \alpha$.

Definition 1.1.1. Let h be the smallest positive integer such that $\left(\begin{smallmatrix}1 & h\\ 0 & 1\end{smallmatrix}\right) \in \Gamma$. Let $f(\tau + h) = f(\tau)$ be a periodic function. Then

(1) We say that f is *holomorphic (resp. meromorphic) at* ∞, if the function $g : \mathbb{D} - \{0\} \to \mathbb{C}$ extends holomorphically (resp. meromorphically) to 0, where $g(e^{\frac{2\pi i\tau}{h}}) = f(\tau)$ and $\mathbb{D} = \{z \in \mathbb{C} \,|\, |z| = 1\}$ is the unit disk.

(2) We say that f is *holomorphic (resp. meromorphic) at a cusp* α, if $f|_k\sigma_\alpha(\tau)$ is holomorphic (resp. meromorphic) at ∞.

Definition 1.1.2. A function $f : \mathcal{H} \to \mathbb{C}$ is called a *modular form* of weight k, level Γ if

(1) f is holomorphic on $\mathcal{H}$.

(2) f is modular, i.e. for all $\gamma = \left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right) \in \Gamma$, $f|_k\gamma(\tau) = f(\tau)$.

(3) f is holomorphic at every cusp $\alpha \in \Gamma\backslash\mathbb{P}^1(\mathbb{Q})$.

If we replace (3) by

(3') f is meromorphic at every cusp of $\alpha \in \Gamma\backslash\mathbb{P}^1(\mathbb{Q})$,

then f is called a *weakly holomorphic modular form*.

A weakly holomorphic modular form has a Laurent Fourier expansion at a cusp α

$$f|_k\sigma_\alpha(\tau) = \sum_{n\geq n_0} a_n^\alpha(f)q^{n/h}.$$

Thus a weakly holomorphic modular form has possibly exponential growth at cusps. We call f *cuspidal* if f has constant term $a_0^\alpha(f) = 0$ at every cusp α of Γ. Denote by later $M_k(\Gamma)$, $S_k(\Gamma)$, $M_k^!(\Gamma)$ and $S_k^!(\Gamma)$ the space of holomorphic modular forms, holomorphic cusp forms, weakly holomorphic modular forms and weakly holomorphic cusp forms of weight k respectively. When $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ we omit it from the notations.

Example 1.1.3. *Let $k \geq 4$ be an even integer, then we have Eisenstein series*

$$G_k(\tau) = \sum_{(c,d)\neq(0,0)}{}' \frac{1}{(c\tau + d)^k},$$

they are modular forms of level $\mathrm{SL}_2(\mathbb{Z})$ *and weight* k*. We can compute their Fourier expansions, which are given by the normalized Eisenstein series*

$$E_k(\tau) = \frac{G_k(\tau)}{2\zeta(k)} = 1 - \frac{2k}{B_k}\sum_{n=1}^{\infty}\sigma_{k-1}(n)q^n,$$

where B_k *is the* k*-th Bernoulli number defined by*

$$\frac{x}{e^x - 1} = \sum_{k\geq 0}\frac{B_k}{k!}x^k$$

and $\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}$*.*

Definition 1.1.4. A *quasi-modular form* of level Γ and weight k is a holomorphic function f on $\mathcal{H}$ with a collection of functions $f_0, \ldots, f_p$ over $\mathcal{H}$ such that

(1) each f_i is holomorphic over $\mathcal{H}$.

(2) f is quasi-modular,

$$(f|_k\gamma)(\tau) = \sum_{j=0}^{p} f_j(\tau)\left(\frac{c}{c\tau + d}\right)^j \tag{1.1}$$

for any $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$.

(3) f has at most polynomial growth, i.e. there exists a constant $N > 0$ such that $f(\tau) = O\left(y^{-N}(1 + |\tau|^2)^N\right)$ as $y \to \infty$ and $y \to 0$.

If f_p is nonzero in (1.1), the degree p is called the *depth* of f.

Denote by $QM_k(\Gamma)$ the space of quasi-modular forms of weight k and by $QM_k^{\leq p}(\Gamma)$ its subspace containing quasi-modular forms of weight k with depth $\leq p$.

Example 1.1.5. *It is also possible to define Eisenstein series of weight* $k = 2$*. Set*

$$E_2(\tau) = \frac{G_2(\tau)}{2\zeta(2)} \coloneqq 1 - 24\sum_{n=1}^{\infty}\sigma_1(n)q^n.$$

In this case, the function E_2 *is in fact quasi-modular of depth* 1*. It verifies the following transformation rule (see for example [19, Corollary 5.2.17])*

$$(c\tau + d)^{-2}E_2\left(\frac{a\tau + b}{c\tau + d}\right) = E_2(\tau) - \frac{6i}{\pi}\left(\frac{c}{c\tau + d}\right)$$

with $f = f_0 = E_2$ *and* $f_1 = -\frac{6i}{\pi}$ *in* (1.1)*.*

Generally speaking, quasi-modular forms come from the derivatives of modular forms. Let $D = q\frac{d}{dq} = \frac{1}{2\pi i}\frac{d}{d\tau}$ be the differential operator, then the following result is well-known.

Proposition 1.1.6 (Zagier [47, Proposition 20])**.** *(1) The space of quasi-modular forms $QM_k(\Gamma)$ is closed under the differentiation D.*

(2) Every quasi-modular form can be written uniquely as a linear combination of derivatives of modular forms and E_2. More precisely, let $QM_k^{\leq p}(\Gamma)$ denote the subspace of quasi-modular forms of depth $\leq p$, then

$$QM_k^{\leq p}(\Gamma) = \begin{cases} \bigoplus_{j=0}^{p} D^j\left(M_{k-2j}(\Gamma)\right) & p < \frac{k}{2}, \\ \bigoplus_{j=0}^{k/2-1} D^j(\widetilde{M}_{k-2j}(\Gamma)) & p \geq \frac{k}{2}, \end{cases}$$

where $\widetilde{M}_k(\Gamma) = M_k(\Gamma)$ for $k \neq 2$ and $\widetilde{M}_2(\Gamma) = M_2(\Gamma) \oplus \mathbb{C}E_2$.

Remark 1.1.7. It follows from Proposition 1.1.6 that, given a quasi-modular form f in $QM_k^{\leq p}(\Gamma)$ of depth at most $p \geq 1$, we have a unique decomposition

$$f = \sum_{j=0}^{\min\{p, \lfloor \frac{k-1}{2} \rfloor\}} D^j(F_j) \quad \text{with} \quad F_j \in \widetilde{M}_{k-2j}.$$

As an example, for the quasi-modular form E_2^2 of weight 4 and depth 2, we have the following identity (observed originally by Ramanujan)

$$E_2^2 = E_4 + 12DE_2.$$

1.1.2 L-functions of Quasi-Modular Forms

In the rest of this section we offer a definition of L-functions for quasi-modular forms. Fix now $\Gamma = \Gamma_1(N)$ with $N \geq 1$ an integer.

Given a modular form $f = \sum_{n \geq 0} a_n(f) q^n \in M_k(\Gamma)$, recall that the *$L$-function* of f is defined as the Dirichlet series $L(f, s) = \sum_{n=1}^{\infty} a_n(f) n^{-s}$. This Dirichlet series can be analytically continued via the *completed L-function*

$$\Lambda(f, s) = N^{s/2}(2\pi)^{-s}\Gamma(s)L(f, s) = N^{s/2} \int_0^{\infty} \left(f(iy) - a_0\right) y^s \frac{dy}{y}.$$

Let $W_N : M_k(\Gamma) \to M_k(\Gamma)$ be the Atkin–Lehner involution

$$(W_N f)(\tau) = i^k N^{-k/2} \tau^{-k} f\left(-\frac{1}{N\tau}\right),$$

then we have functional equation

$$\Lambda(f, s) = \Lambda(W_N f, k - s).$$

Moreover, the function $\Lambda(f, s) + \frac{a_0(f)}{s} + \frac{a_0(W_N f)}{k-s}$ is entire in s. Thus $\Lambda(f, s)$ has only possible poles at $s = 0$ and $s = k$. We refer readers to [37, Section 4.3] for more details about L-functions of modular forms.

Definition 1.1.8. We define the L-function of E_2 via its Dirichlet series

$$L(E_2, s) = \sum_{n=1}^{\infty} \frac{\sigma_1(n)}{n^s} = \zeta(s)\zeta(s-1)$$

and we define its completed L-function as

$$\Lambda(E_2, s) = N^{s/2}(2\pi)^{-s}\Gamma(s)\zeta(s)\zeta(s-1).$$

A derivative $D^j f$ has Fourier expansion $D^j f = \sum_{n\geq 1} n^j a_n(f) q^n$. We can define its L-function via the Dirichlet series

$$\begin{aligned} L(D^j f, s) =& \sum_{n=1}^{\infty} \frac{a_n(f)}{n^{s-j}} \\ =& L(f, s-j), \end{aligned}$$

and herewith we have its completed L-function

$$\Lambda(D^j f, s) = N^{s/2}(2\pi)^{-s}\Gamma(s)L(D^j f, s) = \frac{N^{j/2}(s-1)_j}{(2\pi)^j}\Lambda(f, s-j),$$

where the Pochhammer symbol $(s-1)_j$ is the falling factorial $(s-j)(s-j+1)\dots(s-1)$. In general, we define the L-functions of a quasi-modular form as follows

Definition 1.1.9. Let $f \in QM_k(\Gamma)$ be a quasi-modular form. Given a decomposition $f = \sum_{j=0}^{p} D^j(F_j)$ for $F_j \in \widetilde{M}_{k-2j}$, we define the L-function of f to be

$$L(f, s) = \sum_{j=0}^{p} L(F_j, s-j),$$

and the completed L-function to be

$$\Lambda(f, s) = \sum_{j=0}^{p} \frac{N^{j/2}(s-1)_j}{(2\pi)^j}\Lambda(F_j, s-j).$$

Proposition 1.1.10. *Let $f \in QM_k^{\leq p}(\Gamma)$ be quasi-modular form of depth at most p, then $\Lambda(f, s)$ extends to a meromorphic function on whole complex plane, which has only possibly simple poles when $s = 0$ or when s is an integer from $k-p$ to k.*

Proof. Suppose that f is given by the decomposition $f = \sum_{j=0}^{p} D^j(F_j)$ for $F_j \in \widetilde{M}_{k-2j}$, then

$$\Lambda(f, s) = \sum_{j=0}^{p} \frac{N^{j/2}(s-1)_j}{(2\pi)^j}\Lambda(F_j, s-j).$$

If F_j is modular, then $(s-1)_j\Lambda(F_j, s-j)$ has only possibly simple poles at $s = k-j$ if $j > 0$ and has only possibly simple poles at $s = 0$ and $s = k$ if $j = 0$. We only need to focus on the remaining issue

$$\Lambda(E_2, s) = N^{s/2}(2\pi)^{-s}\Gamma(s)\zeta(s)\zeta(s-1).$$

Since zeta function vanishes at negative even integers, the function $\Lambda(E_2, s)$ has only simple poles at $s = 0, 1, 2$. If $j = 0$, then we have $k = 2$ and $(s-1)_j\Lambda(E_2, s-j) = \Lambda(E_2, s)$. If $j > 0$, then $(s-1)_j\Lambda(E_2, s-j)$ has only possibly simple poles at $s = k-j$. □

Definition 1.1.11. Let f and g be two smooth functions on $\mathcal{H}$ and k and l are fixed integers, recall that the Rankin–Cohen bracket (see [19, Definition 5.3.23]) is

$$[f, g]_n = \sum_{j=0}^{n}(-1)^j\binom{k+n-1}{j}\binom{l+n-1}{n-j}D^{n-j}fD^jg.$$

Also, when f or g is an Eisenstein series of weight 2 we modify the construction of Rankin–Cohen bracket to

$$\begin{aligned}
[E_2, g]_n^{mod} &= [E_2, g]_n - (-1)^n\frac{12}{n+l}D^{n+1}g,\\
[f, E_2]_n^{mod} &= [f, E_2]_n - \frac{12}{n+k}D^{n+1}f,\\
[E_2, E_2]_n^{mod} &= [E_2, E_2]_n - (1+(-1)^n)\frac{12}{n+2}D^{n+1}E_2.
\end{aligned}$$

Proposition 1.1.12 ([19, Theorem 5.3.24, Proposition 5.3.27])**.** *Let $f \in \widetilde{M_k}(\Gamma)$ and $g \in \widetilde{M_l}(\Gamma)$ be two forms. Then*

(1) The modified Rankin–Cohen bracket $[f, g]_n^{mod}$ is a modular form of weight $k+l+2n$. In general, the Rankin–Cohen bracket $[f, g]_n^{mod}$ is always a cusp form for $n > 0$.

(2) When f and g have rational Fourier coefficients, so does $[f, g]_n^{mod}$.

(3) We have $[g, f]_n = (-1)^n[f, g]_n$ and $[g, f]_n^{mod} = (-1)^n[f, g]_n^{mod}$.

Let $\delta_k = D - \frac{k}{4\pi\,\mathrm{Im}(\tau)}$ be the Maass–Shimura differential operator. For $j > 0$, set $\delta_k^j = \delta_{k+2j-2} \circ \delta_{k+2j-4} \circ \cdots \circ \delta_k$ and δ_k^0 to be the identity operator. Given two modular forms $f \in M_k(\Gamma)$ and $g \in M_l(\Gamma)$, Lanphier [32] gave the following formula

$$\delta_k^n f \cdot g = \sum_{j=0}^{n}\frac{\binom{n}{j}\binom{k+n-1}{j}}{\binom{k+l+2n-2j-2}{n-j}\binom{k+l+2n-j-1}{j}}\delta_k^j[f, g]_{n-j}.$$

Lemma 1.1.13. *Let $f \in \widetilde{M}_k(\Gamma)$ and $g \in \widetilde{M}_l(\Gamma)$ be two forms. Then*

$$D^n f \cdot g = \sum_{j=0}^{n} \frac{\binom{n}{j}\binom{k+n-1}{j}}{\binom{k+l+2n-2j-2}{n-j}\binom{k+l+2n-j-1}{j}} D^j [f,g]_{n-j}. \tag{1.2}$$

Proof. The proof follows exactly the same as Lanphier [32, Theorem 1]. It is worth noticing that his proof is purely combinatorial. We take holomorphic part in his proof and everything carries over to the operator D and forms f and g.

□

From this lemma we get immediately the following identities of L-functions

Lemma 1.1.14. *Let $f \in \widetilde{M}_k(\Gamma)$ and $g \in \widetilde{M}_l(\Gamma)$ be two forms. Then*

$$L\left(D^n f \cdot g, s\right) = \sum_{j=0}^{n} a_{k,l}^n(j) L\left([f,g]_{n-j}, s-j\right),$$

$$\Lambda\left(D^n f \cdot g, s\right) = \sum_{j=0}^{n} a_{k,l}^n(j) N^{j/2} (s-1)_j (2\pi)^{-j} \Lambda\left([f,g]_{n-j}, s-j\right),$$

where

$$a_{k,l}^n(j) = \frac{\binom{n}{j}\binom{k+n-1}{j}}{\binom{k+l+2n-2j-2}{n-j}\binom{k+l+2n-j-1}{j}}.$$

Remark 1.1.15. Using Lemma 1.1.14 we are able to decompose $\Lambda\left(D^n f \cdot g, s\right)$ into a linear combination of L-functions of Rankin–Cohen brackets. This process plays a significant role in Chapter 4 and Chapter 6.

In Section 3.4 we will tackle the L-functions of weakly holomorphic modular forms.

1.2 Zeta Functions and L-functions

Let $x \in \mathbb{R}/\mathbb{Z}$ and $\mathrm{Re}(s) > 1$. The *Hurwitz zeta function* is the absolutely convergent series

$$\zeta(x,s) = \sum_{\substack{y>0 \\ y \equiv x\,(1)}} \frac{1}{y^s}.$$

The Hurwitz zeta function can be extended to a meromorphic function for all $s \in \mathbb{C}$. It has only a simple pole at $s = 1$ with residue 1.

Definition 1.2.1. Let $\alpha : \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}$ be a complex function. We define the *L-function of α* to be the series

$$L(\alpha, s) \coloneqq \sum_{n=1}^{\infty} \frac{\alpha(n)}{n^s} = \sum_{m \in \mathbb{Z}/N\mathbb{Z}} \alpha(m) N^{-s} \zeta\left(\frac{m}{N}, s\right).$$

We write $\hat{\alpha}(n) = \sum_{m \in \mathbb{Z}/N\mathbb{Z}} \alpha(m)\zeta_N^{-mn}$ for the Fourier transform of α.

Remark 1.2.2. For $a \in \mathbb{Z}/N\mathbb{Z}$, let δ_a and $\hat{\delta}_a$ be the following functions from $\mathbb{Z}/N\mathbb{Z}$ to $\mathbb{C}$

$$\delta_a(n) := \begin{cases} 1 & n \equiv a\,(N), \\ 0 & \text{else}, \end{cases}$$

$$\hat{\delta}_a(n) := \zeta_N^{-an}.$$

Then

$$L(\delta_a, s) = N^{-s}\zeta\left(\frac{a}{N}, s\right),$$

and

$$L(\hat{\delta}_a, s) = \hat{\zeta}\left(-\frac{a}{N}, s\right),$$

where $\hat{\zeta}(x, s)$ is the periodic zeta function

$$\hat{\zeta}(x, s) = \sum_{n=1}^{\infty} \frac{e^{2\pi i n x}}{n^s}.$$

We write $\alpha^-(n)$ for the function $n \mapsto \alpha(-n)$. We say that the function α is *even* (resp. *odd*) if $\alpha^- = \alpha$ (resp. $\alpha^- = -\alpha$) holds. For even or odd α, the following functional equation holds

Theorem 1.2.3. *Let $\alpha : \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}$ be a complex function and $\hat{\alpha}$ be the Fourier transform of α. If α is even then*

$$L(\alpha, 1-s) = \frac{1}{\pi}\left(\frac{2\pi}{N}\right)^{1-s} \Gamma(s) \cos\left(\frac{\pi s}{2}\right) L(\hat{\alpha}, s).$$

If α is odd then

$$L(\alpha, 1-s) = \frac{i}{\pi}\left(\frac{2\pi}{N}\right)^{1-s} \Gamma(s) \sin\left(\frac{\pi s}{2}\right) L(\hat{\alpha}, s).$$

Proof. It is proved in [30, Corollary 2 (b)] that we have the following functional equation of Hurwitz zeta function

$$\zeta(x, 1-s) = \frac{\Gamma(s)}{(2\pi)^s}(e^{\frac{-i\pi s}{2}}\hat{\zeta}(x, s) + e^{\frac{i\pi s}{2}}\hat{\zeta}(-x, s)).$$

The proof is completely straightforward after summation over all $x \in \frac{1}{N}\mathbb{Z}/N\mathbb{Z}$. □

Also we have

Theorem 1.2.4. *If n is a negative integer, then*

$$L(\alpha, n) = (-1)^{n+1} L(\alpha^-, n).$$

Moreover,

$$L(\alpha, 0) + L(\alpha^-, 0) = -\alpha(0).$$

In particular, if α is even, then $L(\alpha, s)$ vanishes at $-2, -4, \cdots$, if α is odd, then $L(\alpha, s)$ vanishes at $-1, -3, \cdots$.

Proof. For $s = 0$ we have the following Hurwitz zeta values (see [11])

$$\zeta(x, 0) = \begin{cases} \frac{1}{2} - \{x\} & \text{if } x \neq 0, \\ -\frac{1}{2} & \text{if } x = 0. \end{cases}$$

Sum over all $x \in \frac{1}{N}\mathbb{Z}/N\mathbb{Z}$ then we get the two identities. □

1.3 Eisenstein Series and S-series

Let N be a positive integer. Brunault used certain Eisenstein series of level $\Gamma(N)$ and $\Gamma_1(N^2)$ in [11]. Following his notations and definitions, we will briefly review some facts about Eisenstein series and their L-functions.

Given two functions $\alpha, \beta : \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}$ and $t, u \in \mathbb{C}$, we define the *S-series*

$$S_{\alpha,\beta}^{t,u}(\tau) = \sum_{m \geq 1} \sum_{n \geq 1} \alpha(m)\beta(n) m^t n^u q_N^{mn},$$

where $q_N = e^{\frac{2\pi i \tau}{N}}$. We see

$$D^k S_{\alpha,\beta}^{t,u} = \frac{1}{N^k} S_{\alpha,\beta}^{t+k,u+k}.$$

The following three kinds of Eisenstein series will be useful in our later computations.

Lemma 1.3.1. *Let $k \geq 1$ be an integer and $(a, b) \in (\mathbb{Z}/N\mathbb{Z})^2$. Suppose $(a, b) \neq (0, 0)$ in the case $k = 2$. Define*

$$F_{a,b}^{(k)}(\tau) = a_0(F_{a,b}^{(k)}) + N^{1-k} \left(S_{\hat{\delta}_{-b}, \delta_a}^{0,k-1}(\tau) + (-1)^k S_{\hat{\delta}_b, \delta_{-a}}^{0,k-1}(\tau) \right),$$

where

$$a_0(F_{a,b}^{(1)}) = \begin{cases} 0 & \text{if } a = b = 0, \\ \frac{1}{2} \frac{1+\zeta_N^b}{1-\zeta_N^b} & \text{if } a = 0 \text{ and } b \neq 0, \\ \frac{1}{2} - \left\{ \frac{a}{N} \right\} & \text{if } a \neq 0, \end{cases}$$

and for $k \geq 2$

$$a_0(F_{a,b}^{(k)}) = \zeta(\frac{a}{N}, 1 - k).$$

Then $F_{a,b}^{(k)}(\tau)$ is an Eisenstein series of level $\Gamma(N)$ weight k.

Lemma 1.3.2. *Let $k \geq 1$ be an integer and $(a,b) \in (\mathbb{Z}/N\mathbb{Z})^2$. Suppose $a \neq 0$ in the case $k = 2$. Define*

$$G_{a,b}^{(k)}(\tau) = a_0(G_{a,b}^{(k)}) + \left(S_{\delta_b,\delta_a}^{0,k-1}(N\tau) + (-1)^k S_{\hat{\delta}_{-b},\delta_{-a}}^{0,k-1}(N\tau)\right),$$

where

$$a_0(G_{a,b}^{(1)}) = \begin{cases} 0 & \text{if } a = b = 0, \\ \frac{1}{2} - \left\{\frac{b}{N}\right\} & \text{if } a = 0 \text{ and } b \neq 0, \\ \frac{1}{2} - \left\{\frac{a}{N}\right\} & \text{if } a \neq 0 \text{ and } b = 0, \\ 0 & \text{if } a \neq 0 \text{ and } b \neq 0, \end{cases}$$

and for $k \geq 2$

$$a_0(G_{a,b}^{(k)}) = \begin{cases} N^{k-1}\zeta(\frac{a}{N}, 1-k) & \text{if } b = 0, \\ 0 & \text{if } b \neq 0. \end{cases}$$

Then $G_{a,b}^{(k)}(\tau)$ is an Eisenstein series of level $\Gamma_1(N^2)$ weight k.

Lemma 1.3.3. *Let $k \geq 1$ be an integer and $(a,b) \in (\mathbb{Z}/N\mathbb{Z})^2$. Suppose $a \neq 0$ in the case $k = 2$. Define*

$$H_{a,b}^{(k)}(\tau) = a_0(H_{a,b}^{(k)}) + \left(S_{\hat{\delta}_a,\hat{\delta}_b}^{0,k-1}(N\tau) + (-1)^k S_{\hat{\delta}_a,\hat{\delta}_b}^{0,k-1}(N\tau)\right),$$

where

$$a_0(H_{a,b}^{(1)}) = \begin{cases} 0 & \text{if } a = b = 0, \\ -\frac{1}{2}\frac{1+\zeta_N^b}{1-\zeta_N^b} & \text{if } a = 0 \text{ and } b \neq 0, \\ -\frac{1}{2}\frac{1+\zeta_N^a}{1-\zeta_N^a} & \text{if } a \neq 0 \text{ and } b = 0, \\ -\frac{1}{2}\left(\frac{1+\zeta_N^a}{1-\zeta_N^a} + \frac{1}{2}\frac{1+\zeta_N^b}{1-\zeta_N^b}\right) & \text{if } a \neq 0 \text{ and } b \neq 0, \end{cases}$$

and for $k \geq 2$

$$a_0(H_{a,b}^{(k)}) = \hat{\zeta}(-\frac{b}{N}, 1-k).$$

Then $H_{a,b}^{(k)}(\tau)$ is an Eisenstein series of level $\Gamma_1(N^2)$ weight k.

Let W_{N^2} be the Atkin–Lehner involution of level $\Gamma_1(N^2)$, then we have ([11, Lemma 3.10])

$$W_{N^2}(G_{a,b}^{(k)}) = \frac{i^k}{N} H_{a,b}^{(k)} \qquad \text{if } (a,k) \neq (0,2),$$

with $a, b, b' \in \mathbb{Z}/N\mathbb{Z}$.

We also introduce the following real analytic Eisenstein series, which will give us the Fourier expansion of Eisenstein symbols.

Definition 1.3.4 (Brunault)**.** Let $a, b \geq 0$ be integers and $u_1, u_2 \in \mathbb{Z}/N\mathbb{Z}$. We define the following real analytic Eisenstein series

$$\begin{aligned}
&F^{a,b}_{(u_1,u_2)}(\tau) = \hat{\zeta}\left(\frac{u_2}{N}, a+b+2\right) + (-1)^{a+b}\hat{\zeta}\left(-\frac{u_2}{N}, a+b+2\right) \\
&+ (-1)^b 2\pi i \binom{a+b}{a} \delta_{u_2=0} (2iy)^{-a-b-1} \left(\hat{\zeta}\left(\frac{u_1}{N}, a+b+1\right) + (-1)^{a+b}\hat{\zeta}\left(-\frac{u_1}{N}, a+b+1\right)\right) \\
&+ \frac{(-1)^{b+1}N}{b!} \sum_{j=0}^{a} \frac{(a+b-j)!}{j!(a-j)!} \left(-\frac{2\pi i}{N}\right)^{j+1} (2iy)^{-a-b-1+j} \left(S^{j-a-b-1,j}_{\hat{\delta}_{-u_1},\delta_{-u_2}}(\tau) + (-1)^{a+b} S^{j-a-b-1,j}_{\hat{\delta}_{u_1},\delta_{u_2}}(\tau)\right) \\
&+ \frac{(-1)^{a+1}N}{a!} \sum_{j=0}^{b} \frac{(a+b-j)!}{j!(b-j)!} \left(-\frac{2\pi i}{N}\right)^{j+1} (2iy)^{-a-b-1+j} \left(\overline{S}^{j-a-b-1,j}_{\hat{\delta}_{-u_1},\delta_{-u_2}}(\tau) + (-1)^{a+b} \overline{S}^{j-a-b-1,j}_{\hat{\delta}_{u_1},\delta_{u_2}}(\tau)\right).
\end{aligned}$$

These Eisenstein series have constant terms which are usually complicated to compute. To solve this, given an S-series $S^{t,u}_{\alpha,\beta}(\tau)$ with t, u two integers, we introduce the following notation in this thesis

Definition 1.3.5. Let $\alpha, \beta : \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}$ be two functions and $t, u \in \mathbb{Z}$, we define

$$\mathcal{S}^{t,u}_{\alpha,\beta}(\tau) = \begin{cases} S^{0,u}_{\alpha,\beta}(\tau) + \frac{1}{2}\alpha(0)L(\beta, -u) & t = 0, \\ S^{t,u}_{\alpha,\beta}(\tau) & t \neq 0. \end{cases}$$

Set

$$G^{(k)}_{\alpha,\beta}(\tau) \coloneqq \sum_{(a,b)\in(\mathbb{Z}/N\mathbb{Z})^2} \alpha(a)\beta(b) G^{(k)}_{a,b}(\tau),$$

if $k = 2$ we assume further that $\alpha(0)\sum_{n\in\mathbb{Z}/N\mathbb{Z}} \beta(n) = 0$. Then $G^{(k)}_{\alpha,\beta}(\tau)$ is an Eisenstein series of level $\Gamma_1(N^2)$ weight k. In particular, we have

$$\begin{aligned}
G^{(k)}_{a,b}(\tau) =& G^{(k)}_{\delta_a,\delta_b}(\tau) \\
H^{(k)}_{a,b}(\tau) =& G^{(k)}_{\hat{\delta}_b,\hat{\delta}_a}(\tau).
\end{aligned}$$

The Eisenstein series become simpler via S-series with constant terms

Lemma 1.3.6. *We have the following identity*

$$\mathcal{S}^{0,u}_{\alpha,\beta}(iNy) + (-1)^{u+1}\mathcal{S}^{0,u}_{\alpha^-,\beta^-}(iNy) = \begin{cases} G^{(u+1)}_{\beta,\alpha}(iy) & u > 0, \\ G^{(1)}_{\beta,\alpha}(iy) - \frac{1}{2}\alpha(0)\beta(0) - \beta(0)L(\alpha, 0) & u = 0. \\ = G^{(1)}_{\beta,\alpha}(iy) - \frac{1}{2}\beta(0)L(\alpha - \alpha^-, 0) & \end{cases}$$

Proof. Note that for $k \geq 1$ we have

$$L(\delta_a, 1-k) = N^{k-1}\zeta(\frac{a}{N}, 1-k).$$

Also

$$L(\delta_a, 0) = \begin{cases} \frac{1}{2} - \left\{\frac{a}{N}\right\} & \text{if } a \neq 0, \\ -\frac{1}{2} & \text{if } a = 0. \end{cases}$$

The lemma follows straightforwardly by summation over all $a, b \in \mathbb{Z}/N\mathbb{Z}$. □

Their Atkin–Lehner involutions are

Lemma 1.3.7. *Let $k \geq 1$ and $\alpha, \beta : \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}$. If $k = 2$, assume further that $\alpha(0) \sum_{n \in \mathbb{Z}/N\mathbb{Z}} \beta(n) = 0$. Then*

$$W_{N^2}\left(G^{(k)}_{\alpha,\beta}\right) = \frac{i^k}{N} G^{(k)}_{\hat{\beta},\hat{\alpha}}.$$

A quick computation shows

Lemma 1.3.8. *Let $k \geq 1$ and $\alpha, \beta : \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}$. If $k = 2$, assume further that $\alpha(0) \sum_{n \in \mathbb{Z}/N\mathbb{Z}} \beta(n) = 0$. Then*

$$\begin{aligned} &\Lambda\left(G^{(k)}_{\alpha,\beta}, s\right) = N^s (2\pi)^{-s} \Gamma(s) \Big(L(\alpha, s-k+1) L(\beta, s) + (-1)^k L(\alpha^-, s-k+1) L(\beta^-, s) \Big) \\ &= i^k N^{k-s-1} (2\pi)^{s-k} \Gamma(k-s) \Big(L(\hat{\alpha}, k-s) L(\hat{\beta}, 1-s) + (-1)^k L(\hat{\alpha}^-, k-s) L(\hat{\beta}^-, 1-s) \Big). \end{aligned}$$

1.4 Incomplete Gamma Function and Generalized Exponential Integral

The incomplete gamma functions have an important role in our later definition of generalized Mellin transform. The main goal of this section is to give an introduction to them. As a precise reference of incomplete gamma functions, see [38, Chapter 8].

We recall initially the definitions of incomplete gamma functions. The *incomplete gamma functions* are defined as the following integrals

$$\begin{aligned} \Gamma(s, z) &= \int_z^\infty t^{s-1} e^{-t} dt, \\ \gamma(s, z) &= \int_0^z t^{s-1} e^{-t} dt, \end{aligned}$$

for $\mathrm{Re}(s) > 0$ and $z \in \mathbb{C}$. Let $\gamma^*(s, z) = z^{-s} \gamma(s, z) / \Gamma(s)$. Then the function $\gamma^*(s, z)$ has the following power series expansion

$$\gamma^*(s, z) = e^{-z} \sum_{k=0}^\infty \frac{z^k}{\Gamma(s+k+1)},$$

it can be extended to an entire function in both s and z. With the relation $\Gamma(s, z) = \Gamma(s)(1 - z^s \gamma^*(s, z))$, we also have the analytic continuation of $\Gamma(s, z)$. In the case s is a nonpositive integer, we take limits of s to fill missing values.

Example 1.4.1. *Let $s = 0$, the incomplete gamma function $\Gamma(0, z)$, i.e. the exponential integral $E_1(z)$ (see below), is the following multivalued function*

$$\Gamma(0, z) = -\gamma - \operatorname{Log} z - \sum_{k=1}^{\infty} \frac{(-z)^k}{k(k!)}.$$

With the recurrence relation $\Gamma(s+1, z) = s\Gamma(s, z)+z^s e^{-z}$ we can derive the values $\Gamma(-n, z)$ for positive n.

Lemma 1.4.2. *The function $\Gamma(s, z)$ can be extended to*

(1) an entire function in z, when $s \in \mathbb{Z}_{>0}$.

(2) a multivalued function (due to the multivalueness of $\operatorname{Log} z$) in z with branching point at $z = 0$, holomorphic in each sector, when $s \notin \mathbb{Z}_{>0}$.

(3) an entire function in s, when z is nonzero.

For $z \in \mathbb{R}_{<0}$ and $s \in \mathbb{C}$, the incomplete gamma function has exponential growth asymptotic expansion

$$\Gamma(s, z) \sim z^{s-1} e^{-z} \left(1 + \frac{s-1}{z} + \frac{(s-1)(s-2)}{z^2} + \dots\right)$$

as $|z| \to \infty$ (see [38, Section 8.11 (i)]).

Let j be a nonnegative integer. The *generalized exponential integral* defined in Milgram [36] is the following integral

$$E_s^j(z) = \frac{1}{\Gamma(j+1)} \int_1^{\infty} (\log t)^j t^{-s} e^{-zt} dt.$$

For $j = 0$, it is known as the exponential integral

$$\begin{aligned} E_s(z) =& \int_1^{\infty} t^{-s} e^{-zt} dt \\ =& z^{s-1} \Gamma(1-s, z). \end{aligned}$$

Since for nonzero z the exponential integral $E_s(z)$ is entire in s, by

$$E_s^j(z) = \frac{(-1)^j}{j!} \frac{\partial^j}{\partial s^j} E_s(z),$$

the derivative $E_s^j(z)$ can be continued to an entire function in s. For $z = 0$, we have special value ([36])

$$E_s^j(0) = \left(\frac{1}{s-1}\right)^{j+1} \quad \text{for } \operatorname{Re}(s) > 1.$$

We hereby add the definition $E_s^j(0) \coloneqq 1/(s-1)^{j+1}$ for all $s \in \mathbb{C}\backslash\{1\}$. Then the function $E_s^j(z)$ is defined for all $(s,z) \in (\mathbb{C}\backslash\{1\}) \times \mathbb{C}$.

Milgram computed explicitly the function $E_s^j(z)$ with power series expansion and logarithms. In general for $j > 0$, $E_s^j(z)$ is multivalued, holomorphic on each branch of $\operatorname{Log} z$.

To sum up, we have

Lemma 1.4.3. *The function $E_s^j(z)$ is*

(1) an entire function in z, when $s \in \mathbb{Z}_{<0}$ and $j = 0$.

(2) e^{-z}/z, thus meromorphic in z with only a simple pole at $z = 0$, when $s = 0$ and $j = 0$.

(3) a multivalued function (due to the multivalueness of $\operatorname{Log} z$) in z with branching point at $z = 0$, holomorphic in each sector, when in the else cases for s and j.

(4) an entire function in s, when z is nonzero.

(5) $1/(s-1)^{j+1}$, thus meromorphic in s with a pole of order $j+1$ at $s = 1$, when $z = 0$.

Chapter 2

Eisenstein Symbols

Chapter 2 is dedicated to give a quick introduction and definition of Eisenstein symbols. The readers are invited to obtain more details in Deninger–Scholl [22], Deninger [21] and the book of Brunault–Zudilin [13]. We need to introduce several objects. In Section 2.1, we will recall the definition of universal elliptic curve. We will discuss in Section 2.2 the Deligne–Beilinson cohomology, in Section 2.3.2 and Section 2.4 the Beilinson conjecture and the construction and realization of Eisenstein symbols and the Beilinson–Deninger–Scholl elements.

2.1 Universal Elliptic Curve

In this section we give a brief introduction on universal elliptic curve. The notations are dispersed in different literature, here in this thesis we will follow the conventions in [12].

Let $N \geq 3$ be an integer. Let $Y(N)$ be the modular curve over $\mathbb{Q}$ with full level N structure. From [29, (1.8)], the complex points of $Y(N)$ are described as follows

$$Y(N)(\mathbb{C}) \simeq \mathrm{SL}_2(\mathbb{Z})\backslash \left(\mathcal{H} \times \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})\right),$$

where the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathcal{H}$ is given by Möbius transformations, on $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$ is given by left multiplications. It is endowed with the left action of $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$ by $\gamma \cdot (\tau; g) = (\tau; g\gamma^{\intercal})$. This curve is not geometrically connected, there is an isomorphism of Riemann surfaces

$$\begin{aligned}(\mathbb{Z}/N\mathbb{Z})^{\times} \times \Gamma(N)\backslash\mathcal{H} &\xrightarrow{\sim} Y(N)(\mathbb{C}) \\ (a, [\tau]) &\mapsto \left[\left(\tau; \begin{pmatrix} 0 & -1 \\ a & 0 \end{pmatrix}\right)\right].\end{aligned}$$

Let E be the universal elliptic curve over $Y(N)$. Let E^w be the w-th fiber product of E over $Y(N)$. Then the complex points of E^w can be described by the isomorphism ([21, 3.4])

$$E^w(\mathbb{C}) \simeq \left(\mathbb{Z}^{2w} \rtimes \mathrm{SL}_2(\mathbb{Z})\right) \setminus \left(\mathcal{H} \times \mathbb{C}^w \times \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})\right), \tag{2.1}$$

where the left action of $\mathrm{SL}_2(\mathbb{Z})$ is

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot (\tau; z_1, \ldots, z_w; g) = \left(\frac{a\tau + b}{c\tau + d}; \frac{z_1}{c\tau + d}, \ldots, \frac{z_w}{c\tau + d}; \begin{pmatrix} a & b \\ c & d \end{pmatrix} g \right),$$

the left action of $\mathbb{Z}^{2w}$ is

$$(m_1, n_1, \ldots, m_w, n_w) \cdot (\tau; z_1, \ldots, z_w; g) = (\tau; z_1 + m_1 - n_1\tau, \ldots, z_w + m_w - n_w\tau; g).$$

The group $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$ acts on $E^w(\mathbb{C})$ on the left side by

$$\gamma \cdot (\tau; z_1, \ldots, z_w; g) = (\tau; z_1, \ldots, z_w; g\gamma^{\intercal}).$$

Here in the semidirect product $\mathbb{Z}^{2w} \rtimes \mathrm{SL}_2(\mathbb{Z})$, the group $\mathrm{SL}_2(\mathbb{Z})$ acts on $\mathbb{Z}^{2w}$ on the left side by $\gamma \cdot (z_1, \ldots, z_w) = (z_1\gamma^{-1}, \ldots, z_w\gamma^{-1})$, regarding each element $z_i \in \mathbb{Z}^2$ as a row vector. There is an isomorphism of complex analytic manifolds

$$\begin{aligned} (\mathbb{Z}/N\mathbb{Z})^{\times} \times \left(\mathbb{Z}^{2w} \rtimes \Gamma(N)\right) \backslash (\mathcal{H} \times \mathbb{C}^w) &\xrightarrow{\sim} E^w(\mathbb{C}) \\ (a, [\tau; z_1, \ldots, z_w]) &\mapsto \left[\left(\tau; z_1, \ldots, z_w; \begin{pmatrix} 0 & -1 \\ a & 0 \end{pmatrix} \right) \right]. \end{aligned}$$

For a point $[(\tau; g)] \in Y(N)$, the fiber of the projection $E^w(\mathbb{C}) \to Y(N)(\mathbb{C})$ is exactly the w-th product of elliptic curve E_τ, where $E_\tau = \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$.

2.2 Deligne–Beilinson Cohomology

The purpose of this section is to give a short description and to review some properties of Deligne–Beilinson cohomology. Classically it is defined as the hypercohomology of the Deligne–Beilinson complex (cf. [22, Section 2]). The definition of Deligne–Beilinson cohomology which is more convenient for us to use in this thesis comes from Burgos. Further details can be found in Burgos-Kramer-Kühn [16], Burgos [15, Section 2] and Brunault-Zudilin [13, Section 8.1].

For a subring R of $\mathbb{R}$ we set $R(n) = (2\pi i)^n R$. Let X be a smooth quasi-projective complex variety. Suppose that $j : X \hookrightarrow \overline{X}$ is a smooth compactification of X with normal crossing divisor $D = \overline{X} \backslash X$. For $\Lambda \in \{\mathbb{R}, \mathbb{C}\}$, we denote by $E^n_{\log,\Lambda}(X)$ the space of Λ-valued smooth differential n-forms on $\overline{X}$ with logarithmic singularities along D.

The complex $E^*_{\log,\mathbb{C}}(X)$ is bigraded by

$$E^n_{\log,\mathbb{C}}(X) = \bigoplus_{p'+q'=n} E^{p',q'}_{\log,\mathbb{C}}(X),$$

where $E^{p',q'}$ denotes the subspace of forms of type (p, q) in $E^n_{\log,\mathbb{C}}(X)$. The differential $d : E^n \to E^{n+1}$ can be decomposed as $d = \partial + \overline{\partial}$ with $\partial : E^{p',q'} \to E^{p'+1,q'}$ and $\overline{\partial} : E^{p',q'} \to E^{p',q'+1}$.

Definition 2.2.1 ([15, Theorem 2.6], also [13, Definition 8.3])**.** For an integer $p \geq 0$, let the complex $E_p(X)^* = (E_p(X)^n, d_E^n)_{n\geq 0}$ be

$$E_p(X)^n = \begin{cases} (2\pi i)^{p-1} E^{n-1}_{\log,\mathbb{R}}(X) \cap \left(\bigoplus_{\substack{p'+q'=n-1 \\ p',q'<p}} E^{p',q'}_{\log,\mathbb{C}}(X) \right) & \text{if } n \leq 2p-1, \\ (2\pi i)^{p} E^{n}_{\log,\mathbb{R}}(X) \cap \left(\bigoplus_{\substack{p'+q'=n \\ p',q'\geq p}} E^{p',q'}_{\log,\mathbb{C}}(X) \right) & \text{if } n \geq 2p. \end{cases}$$

and

$$d_E^n \omega = \begin{cases} -\pi(d\omega) & \text{if } n \leq 2p-1, \\ -2\partial\overline{\partial}\omega & \text{if } n = 2p-1, \\ d\omega & \text{if } n \geq 2p, \end{cases}$$

where π is the projection $\bigoplus_{p',q'} \to \bigoplus_{p',q'<p}$.

Definition 2.2.2. Let X be a smooth quasi-projective complex variety. The *Deligne–Beilinson cohomology groups* of X are defined as

$$H^n_{\mathcal{D}}(X, \mathbb{R}(p)) = H^n(E_p(X)).$$

There is also a real version of Deligne–Beilinson cohomology. Let X be a smooth quasi-projective variety defined over $\mathbb{R}$. Let $F_\infty : X(\mathbb{C}) \to X(\mathbb{C})$ be the complex conjugation (real Frobenius) on the complex points of X. Given a complex differential form ω, we have the de Rham conjugation $F_{dR} : \omega \mapsto F_\infty^*(\overline{\omega})$. For an integer $p \geq 0$, we set the complexes $E_p(X/\mathbb{R})$ to be the invariants under the de Rham conjugation

$$E_p(X/\mathbb{R}) \coloneqq E_p(X(\mathbb{C}))^{F_{dR}}.$$

With the complex $E_p(X/\mathbb{R})$ we can define

Definition 2.2.3. Let X be a smooth quasi-projective variety defined over $\mathbb{R}$. The *Deligne–Beilinson cohomology groups* of X are defined as

$$H^n_{\mathcal{D}}(X/\mathbb{R}, \mathbb{R}(p)) = H^n(E_p(X/\mathbb{R})).$$

Remark 2.2.4. Our definition here relies on the smooth compactification $\overline{X}$. However, it follows from [14] that the Deligne–Beilinson cohomology does not depend on the choice of the compactification $\overline{X}$.

When $p \geq n$, the Deligne–Beilinson cohomology groups $H^n_{\mathcal{D}}(X/\mathbb{R}, \mathbb{R}(p))$ are given by

Proposition 2.2.5. *Let S_X^* be the complex of real-valued smooth differential forms over $X(\mathbb{C})$ invariant under the de Rham conjugation on $X(\mathbb{C})$. Let $\Omega^*_{\overline{X}}(\log D)$ be the complex of holomorphic forms on $X(\mathbb{C})$ with logarithmic singularities along D. Set $\pi_n(\omega) = \frac{1}{2}(\omega + (-1)^n\overline{\omega})$. For integers $n \geq 2$ and $p > n$, we have*

$$H^n_{\mathcal{D}}(X/\mathbb{R}, \mathbb{R}(n)) \simeq \frac{\{\varphi \in S_X^{n-1} \otimes \mathbb{R}(n-1) \,|\, d\varphi = \pi_{n-1}(\omega) \text{ with } \omega \in \Omega^n_{\overline{X}}(\log D)\}}{d(S_X^{n-2} \otimes \mathbb{R}(n-1))},$$

$$H^n_{\mathcal{D}}(X/\mathbb{R}, \mathbb{R}(p)) \simeq \frac{\{\varphi \in S_X^{n-1} \otimes \mathbb{R}(p-1) \,|\, d\varphi = 0\}}{d(S_X^{n-2} \otimes \mathbb{R}(p-1))}.$$

Proof. Notice that the complex $E^*_{\log,\mathbb{R}}(X)$ actually computes the de Rham cohomology $H^n_{dR}(X,\mathbb{R})$. The rest follows by direct computation with the complexes $E_p(X/\mathbb{R})$. □

Remark 2.2.6. When $p > n$, we see that the Deligne–Beilinson cohomology groups $H^n_{\mathcal{D}}(X,\mathbb{R}(p))$ are simply the de Rham cohomology groups $H^{n-1}_{dR}(X,\mathbb{R}(p-1))$ with twisted coefficients.

The cup product of Deligne–Beilinson cohomology (see [16, Definition 5.14], also in [22]) is a nature homomorphism, which is contravariant functorial, associative and graded

$$\cup:\ H^n_{\mathcal{D}}(X/\mathbb{R},\mathbb{R}(p))\otimes H^m_{\mathcal{D}}(X/\mathbb{R},\mathbb{R}(q))\to H^{n+m}_{\mathcal{D}}(X/\mathbb{R},\mathbb{R}(p+q)).$$

In particular, for two classes $[\varphi_n]\in H^n_{\mathcal{D}}(X/\mathbb{R},\mathbb{R}(n))$ and $[\varphi_m]\in H^m_{\mathcal{D}}(X/\mathbb{R},\mathbb{R}(m))$ associated to ω_n, resp. ω_m, their cup product is represented by

$$\varphi_n\cup\varphi_m=\varphi_n\wedge\pi_m(\omega_m)+(-1)^n\pi_n(\omega_n)\wedge\varphi_m.$$

We also need to introduce the pullback and pushforward morphisms of Deligne–Beilinson cohomology.

The Deligne–Beilinson cohomology groups $H^n_{\mathcal{D}}(X/\mathbb{R},\mathbb{R}(p))$ are contravariant functorial in X. Let $f:X\to Y$ be a morphism between smooth quasi-projective complex varieties. For all nonnegative integer n and integer p, there is a pullback morphism

$$f^*:H^n_{\mathcal{D}}(Y/\mathbb{R},\mathbb{R}(p))\to H^n_{\mathcal{D}}(X/\mathbb{R},\mathbb{R}(p))$$

given by

$$f^*[\varphi]=[f^*\varphi],$$

where the inner f^* is the pullback of differential forms.

Let $f:X\to Y$ be a proper morphism between smooth quasi-projective complex varieties of relative dimension e. With Poincaré duality and the covariance of Deligne–Beilinson homology ([16, Section 5.5]), for all nonnegative integer n and integer p there is a pushforward morphism

$$f_*:H^n_{\mathcal{D}}(X/\mathbb{R},\mathbb{R}(p))\to H^{n-2e}_{\mathcal{D}}(Y/\mathbb{R},\mathbb{R}(p-e)).$$

If $0\le n\le p$ and $0\le n-2e\le p-e$, such pushforward morphism is given by

$$f_*[\varphi]=[f_*\varphi],$$

where the inner f_* is the integration along the fiber (see [3, Definition before Proposition 6.14.1]), it is given by the following differential form

$$f_*\varphi=\frac{1}{(2\pi i)^e}\int_f\varphi.$$

For reader's convenience, we give at the end of this section a summary of properties of Deligne–Beilinson cohomology.

Theorem 2.2.7. *(1) The functors $X \mapsto H^n_{\mathcal{D}}(X/\mathbb{R}, \mathbb{R}(p))$ are contravariant in the category of smooth quasi-projective complex varieties. Given a morphism $f : X \to Y$ between smooth quasi-projective varieties, we have the pullback morphism with contravariant functoriality*

$$f^* : H^n_{\mathcal{D}}(Y/\mathbb{R}, \mathbb{R}(p)) \to H^n_{\mathcal{D}}(X/\mathbb{R}, \mathbb{R}(p)),$$

such morphism is given by the pullback of differential forms.

(2) Let $f : X \to Y$ be a proper morphism between smooth quasi-projective complex varieties of relative dimension e, we have the pushforward morphism with covariant functoriality

$$f_* : H^n_{\mathcal{D}}(X/\mathbb{R}, \mathbb{R}(p)) \to H^{n-2e}_{\mathcal{D}}(Y/\mathbb{R}, \mathbb{R}(p-e)).$$

If $0 \leq n \leq p$ and $0 \leq n - 2e \leq p - e$, such morphism is given by the integration along the fiber of differential forms.

(3) There is a cup product $\cup$ which is contravariant functorial, associative and graded.

2.3 Eisenstein Symbols

2.3.1 Beilinson's Conjectures

Beilinson's conjectures describe, up to rational factors, the special L-values of varieties over number fields at integers. The central concept is a regulator map from the motivic cohomology to the Deligne–Beilinson cohomology. For explicit descriptions of motivic cohomology and regulator map, see [34] and [13, Section A.1, Section A.2].

Let X be a smooth quasi-projective variety over a field k. Let n be a nonnegative integer and p be an integer, we have the *motivic cohomology group* $H^n_{\mathcal{M}}(X, \mathbb{Q}(p))$ of X. It has properties as follows

Theorem 2.3.1 ([22, (1.3)], [34, Lecture 3])**.** *(1) The functors $X \mapsto H^n_{\mathcal{M}}(X, \mathbb{Q}(p))$ are contravariant in the category of smooth quasi-projective varieties over k. Given a morphism $f : X \to Y$, we have the pullback morphism with contravariant functoriality*

$$f^* : H^n_{\mathcal{M}}(Y, \mathbb{Q}(p)) \to H^n_{\mathcal{M}}(X, \mathbb{Q}(p)).$$

(2) Let $f : X \to Y$ be a proper morphism between smooth quasi-projective varieties over k of relative dimension e, we have the pushforward morphism with covariant functoriality

$$f_* : H^n_{\mathcal{M}}(X, \mathbb{Q}(p)) \to H^{n-2e}_{\mathcal{M}}(Y, \mathbb{Q}(p-e)).$$

(3) There is a cup product $\cup$ which is contravariant functorial, associative and graded.

Let $k = \mathbb{R}$ or $\mathbb{C}$. Let X be a smooth quasi-projective variety over k. There is a *regulator map* (see [22, (2.6)], also [13, Section A.2]), defined by Beilinson

$$r_{\mathcal{D}} : H^n_{\mathcal{M}}(X, \mathbb{Q}(p)) \to H^n_{\mathcal{D}}(X/\mathbb{R}, \mathbb{R}(p)),$$

which commutes with cup products, pullbacks and pushforwards. Let k be a number field and X be a smooth quasi-projective variety over k. Write $X/\mathbb{R} = X \otimes_{\mathbb{Q}} \mathbb{R}$. The regulator map associated to X is defined as the composition

$$r_{\mathcal{D}} : H^n_{\mathcal{M}}(X, \mathbb{Q}(p)) \longrightarrow H^n_{\mathcal{M}}(X/\mathbb{R}, \mathbb{Q}(p)) \longrightarrow H^n_{\mathcal{D}}(X/\mathbb{R}, \mathbb{R}(p)),$$

where the first map is obtained by base change and the second map is the regulator map of $X/\mathbb{R}$.

Example 2.3.2. *Let X be a smooth quasi-projective complex variety. In the case $n = p = 1$, we have isomorphism*

$$H^1_{\mathcal{M}}(X, \mathbb{Q}(1)) \simeq \mathcal{O}^{\times}(X) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

The map $r_{\mathcal{D}}$ sends any invertible function f to $\log |f|$.

Assume that X is a smooth quasi-projective variety defined over $\mathbb{Q}$. Let $H^i_{\mathcal{M}}(X, \mathbb{Q}(j))_{\mathbb{Z}}$ be the integral part of the motivic cohomology (see [2, 2.4.2] or [22, (1.6), (1.7)]). Beilinson defined a natural $\mathbb{Q}$-structure $\mathcal{B}_{i,j}$ in $\det_{\mathbb{R}} H^{i+1}_{\mathcal{D}}(X/\mathbb{R}, \mathbb{R}(j))$ (see [2, 3.2] or [22, 2.3.2]). Then he formulated the following conjecture on the L-value $L(H^{i+1}(X), s)$

Conjecture 2.3.3 (Beilinson [1])**.** *Let X be a smooth projective variety defined over $\mathbb{Q}$. Let $0 \leq i \leq 2 \dim X$ and let $j > \frac{i}{2} + 1$ be integers. Then*

(1) The regulator map induces an isomorphism

$$r_{\mathcal{D}} : H^{i+1}_{\mathcal{M}}(X, \mathbb{Q}(j))_{\mathbb{Z}} \xrightarrow{\sim} H^{i+1}_{\mathcal{D}}(X/\mathbb{R}, \mathbb{R}(j)).$$

(2) We have

$$r_{\mathcal{D}}(\det H^{i+1}_{\mathcal{M}}(X, \mathbb{Q}(j))_{\mathbb{Z}}) = L^*(H^{i+1}(X), i+1-j) \cdot \mathcal{B}_{i,j},$$

where $L^*(H^{i+1}(X), i+1-j)$ denotes the leading coefficient of the Taylor expansion at $s = i+1-j$.

2.3.2 Construction of Eisenstein Symbols

Here we give a short review on the construction of Eisenstein symbols. Eisenstein symbols live in the motivic cohomology of fiber products of the universal elliptic curve. Their images under the regulator map in Deligne–Beilinson cohomology can be described with real-analytic Eisenstein series. For references and more explicit definitions see [22, Section 4] and [2].

Let $N \geq 3$ be an integer and $Y(N)$ be the modular curve with level N structure. Let $X(N)$ be its usual compactification, obtained by adjoining some cusps. The set of cusps $C_N = X(N) - Y(N)$ is a finite set of closed points with bijections

$$C_N \simeq \left\{ \begin{pmatrix} * & * \\ 0 & \pm 1 \end{pmatrix} \right\} \backslash \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z}).$$

The motivic cohomology group $H^1_{\mathcal{M}}(Y(N), \mathbb{Q}(1)) \simeq \mathcal{O}(Y(N))^\times \otimes \mathbb{Q}$ is the group of modular units defined over $\mathbb{Q}$. There is a divisor map

$$\mathrm{Res}^0_{\mathcal{M}} : H^1_{\mathcal{M}}(Y(N), \mathbb{Q}(1)) \to \mathbb{Q}[C_N]^{(0)},$$

where $\mathbb{Q}[C_N]^{(0)}$ is the $\mathbb{Q}$-linear space of divisors of degree 0 over C_N.

Let n be a positive integer. We define $\mathbb{Q}[C_N]^{(n)}$ to be the space

$$\mathbb{Q}[C_N]^{(n)} = \left\{ f : \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z}) \to \mathbb{Q} \,\middle|\, f\left(\begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} g \right) = f(g) = (-1)^n f(-g) \text{ for all } g \right\}.$$

Recall that E^n is the n-th fiber product of the universal elliptic curve E over $Y(N)$. Beilinson defined a *residue map* (see [2, (2.1.2)], also [22, (4.3.3)])

$$\mathrm{Res}^n_{\mathcal{M}} : H^{n+1}_{\mathcal{M}}(E^n, \mathbb{Q}(n+1)) \to \mathbb{Q}[C_N]^{(n)}.$$

There is an *Eisenstein symbol map* defined by Beilinson [2] (also [22, (4.6)]), which is a canonical right inverse of $\mathrm{Res}^n_{\mathcal{M}}$ for $n \geq 0$. Beilinson constructed the Eisenstein symbol map $\mathcal{E}^n_{\mathcal{M}}$ with cup-products of certain elliptic functions which have divisors on N-torsion sections of E. It is a map

$$\mathcal{E}^n_{\mathcal{M}} : \mathbb{Q}[C_N]^{(n)} \to H^{n+1}_{\mathcal{M}}(E^n, \mathbb{Q}(n+1))$$

satisfying $\mathrm{Res}^n_{\mathcal{M}} \circ \mathcal{E}^n_{\mathcal{M}} = \mathrm{id}$. In particular, the residue map $\mathrm{Res}^n_{\mathcal{M}}$ is a surjective map.

Let the *horospherical map* $\lambda^n_N : \mathbb{Q}[(\mathbb{Z}/N\mathbb{Z})^2] \to \mathbb{Q}[C_N]^{(n)}$ be the following family of map

$$\lambda^n_N(\phi)(g) = \sum_{(v_1, v_2) \in (\mathbb{Z}/N\mathbb{Z})^2} \phi\left(g^{-1}(v_1, v_2)\right) B_{n+2}\left(\left\{ \frac{v_2}{N} \right\} \right),$$

where B_{n+2} denotes the Bernoulli polynomial and $\{x\}$ is the fractional part of x.

Definition 2.3.4. For $u \in (\mathbb{Z}/N\mathbb{Z})^2$, assume further $u \neq 0$ if $n = 0$, we define the *Eisenstein symbol* to be

$$\mathrm{Eis}^n(u) = \mathcal{E}^n_{\mathcal{M}} \circ \lambda^n_N(\phi_u) \in H^{n+1}_{\mathcal{M}}(E^n, \mathbb{Q}(n+1)),$$

where ϕ_u is the characteristic function at u.

The group $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$ acts (right) on $H^{n+1}_{\mathcal{M}}(E^n, \mathbb{Q}(n+1))$. Since the map $\mathrm{Res}^n_{\mathcal{M}}$ and $\mathcal{E}^n_{\mathcal{M}}$ are $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$-equivariant, we have

Lemma 2.3.5. *For all $g \in \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$ and all $u \in (\mathbb{Z}/N\mathbb{Z})^2$ with $u \neq 0$ when $n = 0$, we have*

$$g^* \operatorname{Eis}^n(u) = \operatorname{Eis}^n(ug).$$

Let k_1, k_2, j be non-negative integers and set $w = k_1 + k_2$. Let $k = w + 2$. Consider the following diagram

$$\begin{array}{ccccc} & & E^{k_1+j+k_2} & & \\ & \swarrow^{p_1} & \downarrow{\scriptstyle p} & \searrow^{p_2} & \\ E^{k_1+j} & & E^{k_1+k_2} & & E^{j+k_2} \end{array},$$

where $p_1 : E^{k_1+j+k_2} \to E^{k_1+j}$ is the projection on the first $k_1 + j$ components, $p_2 : E^{k_1+j+k_2} \to E^{j+k_2}$ is the projection on the last $k_2 + j$ components, and $p : E^{k_1+j+k_2} \to E^{k_1+k_2}$ is the projection by omitting the middle j components. Deninger–Scholl [22] and also Gealy [28] constructed the following element

Definition 2.3.6. Let $u_1 = (a_1, b_1)$, $u_2 = (a_2, b_2) \in (\mathbb{Z}/N\mathbb{Z})^2$ with $u_i \neq 0$ when $k_i = 0$. Then the *Beilinson–Deninger–Scholl element* is

$$\begin{aligned} \operatorname{Eis}^{k_1,k_2,j}(u_1, u_2) &= p_*(p_1^* \operatorname{Eis}^{k_1+j}(u_1) \cup p_2^* \operatorname{Eis}^{k_2+j}(u_2)) \\ &\in H^{w+2}_{\mathcal{M}}(E^w, \mathbb{Q}(w + j + 2)). \end{aligned}$$

Example 2.3.7. *In the case $k_1 = k_2 = j = 0$, we have $\operatorname{Eis}^0(u) = g_u \otimes (2/N)$ where g_u is the Siegel unit on $Y(N)$. After taking cup products we get the Beilinson-Kato element (see [29]) $\operatorname{Eis}^{0,0,0}(u_1, u_2) = 4/N^2\{g_{u_1}, g_{u_2}\}$ in the K-group $K_2(Y(N)) \otimes \mathbb{Q}$.*

2.4 Realization of the Beilinson–Deninger–Scholl Elements

The aim of this section is to provide an explicit formula for the realization (i.e. the image under regulator map) of the Beilinson–Deninger–Scholl element in the Deligne–Beilinson cohomology.

Denote by $(\tau; z_1, \ldots, z_n)$ the coordinates on $E^n(\mathbb{C})$. For all integers $0 \leq a \leq n$ we define the following n-form on $\mathbb{C}^n$

$$\psi_{a,n-a} = \frac{1}{n!} \sum_{\sigma \in S_n} d\overline{z}_{\sigma(1)} \wedge \ldots d\overline{z}_{\sigma(a)} \wedge dz_{\sigma(a+1)} \cdots \wedge dz_{\sigma(n)}.$$

After [2] and [11, Section 8], we have the following proposition

Proposition 2.4.1. *Let $u \in (\mathbb{Z}/N\mathbb{Z})^2$. Assume $u \neq 0$ if $n = 0$. Then the element $r_{\mathcal{D}}(\operatorname{Eis}^n(u))$ is represented by the following real analytic n-form*

$$\operatorname{Eis}^n_{\mathcal{D}}(u) = -\frac{n!(n+2)}{2\pi N} \operatorname{Im}(\tau) \sum_{a=0}^{n} F^{a,n-a}_{gu_1}(\tau) \psi_{a,n-a} \mod d\tau, d\overline{\tau}.$$

Moreover, we have $\mathrm{d}\,\mathrm{Eis}^n_{\mathcal{D}}(u) = \pi_n(\mathrm{Eis}^n_{hol}(u))$*, where*

$$\mathrm{Eis}^n_{hol}(u) = (-1)^{n+1}\frac{n+2}{N}(2i\pi)^{n+1}F^{(n+2)}_{\sigma g u_1}(\tau)d\tau\wedge\psi_{n,0},$$

where the Fourier expansions of the functions $F^{a,n-a}_*$ *and* $F^{(n+2)}_*$ *are given in Section 1.3.*

Let $u_1=(a_1,b_1)$, $u_2=(a_2,b_2)\in(\mathbb{Z}/N\mathbb{Z})^2$. If $k_i=0$ we assume $u_i\neq 0$. Recall that we have the following Beilinson–Deninger–Scholl element

$$\begin{aligned}\mathrm{Eis}^{k_1,k_2,j}(u_1,u_2) &= p_*(p_1^*\,\mathrm{Eis}^{k_1+j}(u_1)\cup p_2^*\,\mathrm{Eis}^{k_2+j}(u_2))\\ &\in H^{w+2}_{\mathcal{M}}(E^w,\mathbb{Q}(w+j+2)).\end{aligned}$$

With the formula of cup product, pullback and pushforward morphisms of Deligne–Beilinson cohomology, we can get an explicit formula of the regulator of the Beilinson–Deninger–Scholl element

Proposition 2.4.2. *The element* $\mathrm{Eis}^{k_1,k_2,j}_{\mathcal{D}}(u_1,u_2)=r_{\mathcal{D}}(\mathrm{Eis}^{k_1,k_2,j}(u_1,u_2))$ *is represented by the following differential form*

$$\begin{aligned}\mathrm{Eis}^{k_1,k_2,j}_{\mathcal{D}}(u_1,u_2) = p_*\big(&p_1^*\,\mathrm{Eis}^{k_1+j}_{\mathcal{D}}(u_1)\wedge\pi_{k_2+j+1}(p_2^*\,\mathrm{Eis}^{k_2+j}_{hol}(u_2))\\ &+(-1)^{k_1+j+1}\pi_{k_1+j+1}(p_1^*\,\mathrm{Eis}^{k_1+j}_{hol}(u_1))\wedge p_2^*\,\mathrm{Eis}^{k_2+j}_{\mathcal{D}}(u_2)\big).\end{aligned}$$

It is also possible to define the regulator of Beilinson–Deninger–Scholl element in level $N=1,2$. In these cases, the universal elliptic curve $E(N)$ of level N does not exist anymore. Letting N' divisible by N with $N'\geq 3$, we have the universal elliptic curve $E(N')$ over $Y(N')$. The group $\mathrm{GL}_2(\mathbb{Z}/N'\mathbb{Z})$ acts on the complex points of the universal elliptic curve $E(N')(\mathbb{C})$ and the Deligne–Beilinson cohomology of $E(N')^w$. Write K for the kernel of $\mathrm{GL}_2(\mathbb{Z}/N'\mathbb{Z})\to\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$. We define

Definition 2.4.3. Let $N=1,2$. Given the integer $N'\geq 3$ with $N|N'$, the Deligne–Beilinson cohomology of $E(N)^w$ is formally defined as the following K-invariant

$$H^n_{\mathcal{D}}(E(N)^w/\mathbb{R},\mathbb{R}(p)) := H^n_{\mathcal{D}}(E(N')^w/\mathbb{R},\mathbb{R}(p))^K.$$

Observe that for $g\in\mathrm{GL}_2(\mathbb{Z}/N'\mathbb{Z})$ we have $g^*\,\mathrm{Eis}^{k_1,k_2,j}_{\mathcal{D}}(u_1,u_2)=\mathrm{Eis}^{k_1,k_2,j}_{\mathcal{D}}(u_1g,u_2g)$. A regulator of level N can be constructed from a regulator of level N' which is invariant under K. We make the following definition

Definition 2.4.4. Let $N=1,2$ and $u_1,u_2\in(\mathbb{Z}/N\mathbb{Z})^2$. Let $k_1,k_2\geq 2$ and j be integers. Suppose that the integer $N'\geq 3$ is divisible by N. The element $\mathrm{Eis}^{k_1,k_2,j}_{\mathcal{D}}(u_1,u_2)$ of level N is defined as the following element of level N'

$$\left(\frac{N'}{N}\right)^{w+2j+2}\mathrm{Eis}^{k_1,k_2,j}_{\mathcal{D}}\left(\frac{N'}{N}u_1,\frac{N'}{N}u_2\right)\in H^w_{\mathcal{D}}\left(E(N)^w/\mathbb{R},\mathbb{R}(w+j+2)\right).$$

Note that our definition relies on the choice of N'. However, we will later recognize that all our arguments and computations in Chapter 6 pass over directly in level $N = 1, 2$.

In the next chapter, we will define the Shokurov cycles. In general, the differential form $\mathrm{Eis}_{\mathcal{D}}^{k_1,k_2,j}(u_1, u_2)$ has nonzero constant terms in its Fourier expansion (cf. Section 1.3). So the integral of the regulator $\mathrm{Eis}_{\mathcal{D}}^{k_1,k_2,j}(u_1, u_2)$ over a Shokurov cycle usually does not converge. We will build in Chapter 3 a theory of regularized integrals to solve this problem.

Chapter 3

Regularization and Mellin Transform

The history of Mellin transformation can be traced back to Riemann. In his famous study of ζ-function, he gave the following formula

$$\Gamma(s)\zeta(s) = \int_0^\infty \frac{1}{e^x - 1}\, x^{s-1} dx.$$

In general setting, let $f(x)$ be a complex-valued function with positive variable x, we have the integral

$$\mathcal{M}(f, s) = \int_0^\infty f(x)\, x^{s-1} dx$$

with complex variable s. These integrals were later systematically analyzed by Mellin, after whom the name of theory was derived. Mellin transform appears everywhere in the theory of L-functions. For example, the Mellin transform of a modular form $f \in M_k$ along the imaginary axis is the completed L-function associated to f, as witnessed in Subsection 1.1.2.

We aim to build a generalization of Mellin transform to more general modular functions. This chapter is structured in two parts. The first part concerns about generalized Mellin transform and the latter concerns about periods and residues over extended modular symbols. In the first part, we start by reviewing the classical theory of Mellin transform, and then in Section 3.2 and Section 3.3 we define generalized Mellin transforms and regularized integrals, the main tool of this thesis. Several examples of L-functions are given in Section 3.4. The second part is oriented towards regulator integrals. We recall the extended modular symbols defined by Stevens [44] and formulate afterwards in Section 3.6 a theory of periods and residues of certain closed forms, such as our regulator $\mathrm{Eis}_{\mathcal{D}}^{k_1,k_2,j}(u_1, u_2)$.

3.1 Mellin Transform

In this section, we retrieve the classical theory of Mellin transform, which can be found in typical textbooks involving integral transforms. The Mellin transform is a basic tool in

analyzing zeta functions and L-functions. References of this section include [20, Chapter 8] and [26].

Definition 3.1.1. Let $f(x)$ be a continuous complex-valued function on $(0,\infty)$. The *Mellin transform* of f is given as

$$\mathcal{M}(f,s)=\int_0^\infty f(x)\,x^{s-1}dx.$$

Let us recall that we have an *existence strip* when the asymptotic conditions at 0 and ∞ are given.

Lemma 3.1.2. *If we have two real constants α and β with $\alpha<\beta$ such that*

$$f(x)=O(x^{-\alpha}) \quad \textit{when } x\to 0,$$

$$f(x)=O(x^{-\beta}) \quad \textit{when } x\to \infty,$$

then the Mellin transform $\mathcal{M}(f,s)$ exists on the open strip $\alpha<\mathrm{Re}(s)<\beta$.

Also, we have an *inversion formula* for Mellin transform.

Lemma 3.1.3. *Let $F(s)=\mathcal{M}(f,s)$ be the Mellin transform of $f(t)$ on the strip $\alpha<\mathrm{Re}(s)<\beta$. Assume that $F(c+it)$ is integrable with respect to t for all $\alpha<c<\beta$, then we have equality*

$$f(x)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}F(s)\,x^{-s}ds$$

for all x on $(0,\infty)$.

We also list some basic formulas about Mellin transform.

Lemma 3.1.4. *Let $F(s)=\mathcal{M}(f,s)$ (resp. $G(s)=\mathcal{M}(g,s)$) be the Mellin transform of $f(x)$ (resp. $g(x)$) on the strip $\alpha<\mathrm{Re}(s)<\beta$ (resp. $\alpha'<\mathrm{Re}(s)<\beta'$). Then we have the following table about Mellin transforms.*

Original functions	Mellin transforms	Existence strips
$f(ax)$, $a>0$	$a^{-s}F(s)$	$\alpha<\mathrm{Re}(s)<\beta$
$f(x^{-1})$	$-F(-s)$	$-\beta<\mathrm{Re}(s)<-\alpha$
$x^z f(x)$, $z\in\mathbb{C}$	$F(s+z)$	$\alpha<\mathrm{Re}(s+z)<\beta$
$D^k f(x)$, k positive integer	$(-\frac{1}{2\pi i})^k(s-k)_k F(s-k)$	$\alpha+k<\mathrm{Re}(s)<\beta+k$
$\int_0^\infty f(t)g(x/t)\frac{dt}{t}$	$F(s)G(s)$	$\max\{\alpha,\alpha'\}<\mathrm{Re}(s)<\min\{\beta,\beta'\}$

3.2 Generalized Mellin Transformation and Regularized Integral I

The classical Mellin transform deals with function relying on some crucial conditions at boundaries. However, these conditions do not hold for many modular forms that we encounter in this thesis. The technique of generalized Mellin transform, which appears in many literature such as [26] and [46], are widely used to treat more general functions. In a nutshell, the idea is that there is a connection between the asymptotic expansions of a function and the poles of its Mellin transform. Similar ideas can also be found in [19, Section 8.6].

Definition 3.2.1. Let $f(x)$ be a complex-valued function of positive real variable x. We say that f has *poly-log asymptotic expansion at* ∞ if for any real N

$$f(x) = \sum_{\mathrm{Re}(s_n)\leq N} c_n\, x^{-s_n}(\log x)^{l_n} + o(x^{-N}) \quad \text{when } x \to \infty,$$

or in short

$$f(x) \sim \sum_{n} c_n\, x^{-s_n}(\log x)^{l_n} \quad \text{when } x \to \infty,$$

where s_n is a series of complex numbers with non-decreasing real parts $\mathrm{Re}(s_0) \leq \mathrm{Re}(s_1) \leq \cdots \leq \mathrm{Re}(s_n) \leq \cdots$ such that either there are finitely many s_n or $\mathrm{Re}(s_n)$ tends to infinity as $n \to \infty$, and l_n are nonnegative integers. If there are only finitely many (s_n, l_n), we call f has *finite poly-log asymptotic expansion*. We write $a(f;\infty)$ as the formal sum $\sum_n c_n\, x^{-s_n}(\log x)^{l_n}$ for the poly-log expansion of f at ∞.

We can also consider functions with poly-log asymptotic expansion at 0.

Definition 3.2.2. One says that $f(x)$ has *(finite) poly-log asymptotic expansion at* 0 if $f(\frac{1}{x})$ has (resp. finite) poly-log asymptotic expansion at ∞. This is to say, for any real N we have

$$f(x) = \sum_{\mathrm{Re}(s'_n)\geq N} c'_n\, x^{-s'_n}(\log x)^{l'_n} + o(x^{-N}) \quad \text{when } x \to 0$$

with s'_n a series of complex numbers with non-increasing real parts $\mathrm{Re}(s'_0) \geq \mathrm{Re}(s'_1) \geq \cdots \geq \mathrm{Re}(s'_n) \geq \cdots$ such that either there are finitely many s'_n or $\mathrm{Re}(s'_n)$ tends to negative infinity as $n \to \infty$, and l_n nonnegative integers. Similarly, we write formally $a(f;0) = \sum_n c'_n\, x^{-s'_n}(\log x)^{l'_n}$ for its poly-log expansion at 0.

We will see soon, given a function $f(x)$ with poly-log asymptotic expansions at 0 and ∞, we can truncate its asymptotic expansions to define its generalized Mellin transform $F(s)$. Generally speaking, the asymptotic expansions of $f(x)$ at 0 and ∞ correspond to the poles of the meromorphic continuation of $F(s)$.

We are about to define the regularized value and regularized integral for a function with poly-log asymptotic expansions. The idea is to take only the constant term in a given expansion.

Definition 3.2.3. Let $f(x)$ be a continuous function on $(0,\infty)$ with poly-log asymptotic expansion at ∞. The *regularized value* of f at ∞ is defined as the constant in $a(f;\infty)$ (i.e. those terms with $(s_n, l_n) = (0,0)$), denoted by $f^*(\infty)$. In a similar way, we can also define the regularized value $f^*(0)$.

Let $F(s)$ be a meromorphic function on the whole complex plane with discrete poles. The *regularized value* of F at $a \in \mathbb{C}$ is defined as

$$F^*(a) \coloneqq \begin{cases} \lim_{s\to a}(F(s) - P_a(s)) & \text{if } a \text{ is a pole,} \\ F(a) & \text{else,} \end{cases} \tag{3.1}$$

where $P_a(s)$ is the principal part in the Laurent expansion of $F(s)$ at $s = a$.

Let $f(x)$ be a continuous complex-valued function on $(0,\infty)$. Assume that f has poly-log asymptotic expansion at ∞. Let $s \in \mathbb{C}$. Suppose that $g(x)$ is a primitive of $f(x)x^{s-1}$, that is,

$$dg(x) = f(x)x^{s-1}dx,$$

Then we claim that $g(x)$ has also poly-log asymptotic expansions at 0 and ∞. Notice that we have

$$\int x^s(\log x)^l\,dx = \begin{cases} \frac{(-1)^l\, l!}{(s+1)^{l+1}} x^{s+1} e_l\left(-(s+1)\log x\right) + C & s \neq -1, \\ \frac{(\log x)^{l+1}}{l+1} + C & s = -1, \end{cases}$$

where $e_l(x) = \sum_{i=0}^{l} \frac{x^i}{i!}$ is a polynomial. Hence there exists a poly-log asymptotic expansion $Q(x, \log x)$ with zero constant term such that

$$g(x) \sim g^*(\infty) + Q(x, \log x) \quad \text{when } x \to \infty.$$

Similarly we have the same argument at 0.

Definition 3.2.4. Let $f(x)$ be a continuous function on $(0,\infty)$ with poly-log asymptotic expansions at 0 and ∞. Let $g(x)$ be a primitive of $f(x)x^{s-1}$ for a given $s \in \mathbb{C}$, then the *regularized integrals* of $f(x)$ are defined as

$$\int_0^{t,*} f(x)x^{s-1}\,dx = g(t) - g^*(0),$$

$$\int_t^{\infty,*} f(x)x^{s-1}\,dx = g^*(\infty) - g(t),$$

and

$$\int_0^{\infty,*} f(x)x^{s-1}\,dx = g^*(\infty) - g^*(0),$$

where t is an arbitrary positive real number.

Example 3.2.5. *The regularized integral of a poly-log function is always zero. In fact,*

$$\begin{aligned}&\int_0^{\infty,*} x^s(\log x)^l\,dx = \int_0^{1,*} x^s(\log x)^l\,dx + \int_1^{\infty,*} x^s(\log x)^l\,dx\\ =&\begin{cases}\frac{(-1)^l\,l!}{(s+1)^{l+1}}x^{s+1}e_l\left(-(s+1)\log x\right)|_{x=1} - \frac{(-1)^l\,l!}{(s+1)^{l+1}}x^{s+1}e_l\left(-(s+1)\log x\right)|_{x=1} & s\neq -1,\\ \frac{(\log x)^{l+1}}{l+1}|_{x=1} - \frac{(\log x)^{l+1}}{l+1}|_{x=1} & s=-1,\end{cases}\\ =&0.\end{aligned}$$

Now let $f(x)$ be a continuous function with poly-log asymptotic expansions

$$\begin{aligned}f(x)\sim&\sum_n c_n\,x^{-s_n}(\log x)^{l_n} \quad \text{when } x\to\infty,\\ f(x)\sim&\sum_n c'_n\,x^{-s'_n}(\log x)^{l'_n} \quad \text{when } x\to 0\end{aligned} \tag{3.2}$$

as in Definition 3.2.1 and Definition 3.2.2. To give a definition for the Mellin transform of $f(x)$, we split it into two parts with a cutting point $t_0\in(0,\infty)$

$$\begin{aligned}\int_0^\infty f(x)x^{s-1}dx &= \int_{t_0}^\infty f(x)x^{s-1}dx + \int_0^{t_0} f(x)x^{s-1}dx\\ &=: L(s)+R(s).\end{aligned}$$

The first part $L(s)$ converges absolutely for $\mathrm{Re}(s)\ll 0$ and the second part $R(s)$ converges absolutely for $\mathrm{Re}(s)\gg 0$. In fact $L(s)$ can be continued meromorphically to the whole complex plane. For arbitrary real α and $\mathrm{Re}(s)\ll 0$, we have

$$\begin{aligned}L(s) &= \int_{t_0}^\infty f(x)x^{s-1}dx\\ &= \int_1^\infty (f(x)-a_{\le\alpha}(f,\infty))x^{s-1}dx - \int_1^{t_0} f(x)x^{s-1}dx\\ &\quad + \sum_{\mathrm{Re}(s_n)\le\alpha}\frac{(-1)^{l_n+1}c_n\,l_n!}{(s-s_n)^{l_n+1}},\end{aligned} \tag{3.3}$$

where

$$a_{\le\alpha}(f,\infty) = \sum_{\mathrm{Re}(s_n)\le\alpha} c_n\,x^{-s_n}(\log x)^{l_n}.$$

Therefore $L(s)$ can be extended meromorphically to the half-plane $\mathrm{Re}(s)<\alpha$. All the same also works for $R(s)$. It can be seen from (3.3) that their sum $L(s)+R(s)$ does not depend on the choice of t_0. In this manner, we can define

Definition 3.2.6. The *generalized Mellin transform* of $f(x)$ is the sum of meromorphic continuations of two parts

$$\mathcal{M}(f,s) \coloneqq L(s)+R(s).$$

Typical examples of generalized Mellin transforms are completed L-functions of modular forms, see Example 3.4.1 for details.

Conversely, the meromorphic continuation of Mellin transform also encodes the information of poly-log asymptotic expansions at 0 and ∞.

Lemma 3.2.7 ([26, Theorem 4, (i) and (ii)]). *Let $f(x)$ be a continuous complex-valued function with positive real variable x. Let $F(s) = \mathcal{M}(f, s)$ be its Mellin transform with nonempty existence strip $\alpha < \mathrm{Re}(s) < \beta$. Suppose $F(s)$ extends to a meromorphic function on the strip $\gamma < \alpha < \mathrm{Re}(s) < \beta < \delta$ with finitely many poles, and $F(s)$ is analytic on $\mathrm{Re}(s) = \gamma$ and $\mathrm{Re}(s) = \delta$. Let the sum of principle parts in the Laurent expansion $F(s)$ at all its poles in the strip $\gamma < \mathrm{Re}(s) < \delta$ be*

$$\sum_n r_n \frac{1}{(s - s_n)^{l_n+1}}.$$

If there exist $\eta_1, \eta_2 \in (\alpha, \beta)$ such that for $\mathrm{Re}(s) \in (\gamma, \eta_1) \cup (\eta_2, \delta)$,

$$|F(s)| = O(|s|^{-1-\varepsilon}) \quad \textit{with } \varepsilon > 0$$

as $|s| \to \infty$, then $f(x)$ has a poly-log boundary conditions

$$f(x) = \sum_{\mathrm{Re}(s_n) \le \alpha} r_n \left(\frac{(-1)^{l_n+1}}{l!} x^{-s_n} (\log x)^{l_n} \right) + O(x^{-\gamma}) \quad \textit{when } x \to 0,$$

and

$$f(x) = \sum_{\mathrm{Re}(s_n) \ge \beta} -r_n \left(\frac{(-1)^{l_n+1}}{l!} x^{-s_n} (\log x)^{l_n} \right) + O(x^{-\delta}) \quad \textit{when } x \to \infty.$$

Theorem 3.2.8. *Suppose that $f(x)$ is a continuous function on $(0, \infty)$ with poly-log asymptotic expansions*

$$f(x) \sim \sum_n c_n \, x^{-s_n} (\log x)^{l_n} \quad \textit{when } x \to \infty,$$

and

$$f(x) \sim \sum_n c'_n \, x^{-s'_n} (\log x)^{l'_n} \quad \textit{when } x \to 0.$$

Let $s \in \mathbb{C}$ and assume that $g(x)$ is a primitive of $f(x)x^{s-1}$. Let $F(s) = \mathcal{M}(f, s)$ be the generalized Mellin transform of $f(x)$. Then the following quantities are equal

(1) the regularized integral

$$\int_0^{\infty,*} f(x) x^{s-1} \, dx = g^*(\infty) - g^*(0),$$

(2) the regularized value $h^*(\infty)$ of the function

$$h(x) = \int_{1/x}^{x} f(t)t^{s-1}\,dt,$$

(3) the following convergent integral obtained by removing poly-log parts

$$\begin{aligned}
&\int_0^t \left(f(x) - a_{\geq s}(f;0)\right) x^{s-1}dx + \int_t^\infty \left(f(x) - a_{\leq s}(f;\infty)\right) x^{s-1}dx \\
&- \int_t^\infty a_{>s}(f;0)x^{s-1}dx - \int_0^t a_{<s}(f;\infty)x^{s-1}dx \\
&- \int_t^1 a_{=s}(f;0)x^{s-1}dx - \int_1^t a_{=s}(f;\infty)x^{s-1}dx
\end{aligned}$$

where

$$\begin{aligned}
a_{\leq s}(f,\infty) &= \sum_{s_n \leq s} c_n\, x^{-s_n}(\log x)^{l_n}, \\
a_{<s}(f,\infty) &= \sum_{s_n < s} c_n\, x^{-s_n}(\log x)^{l_n}, \quad \textit{and} \\
a_{=s}(f,\infty) &= \sum_{s_n = s} c_n\, x^{-s_n}(\log x)^{l_n}
\end{aligned}$$

(similar for $a_{\geq s}(f,0)$ and so forth) are finite series and t is an arbitrary positive real number.

(4) the regularized value $F^*(z)|_{z=s}$ at s.

Proof. By $h(x) = g(1/x) - g(x)$ we see (1) and (2) are equal. Notice that there is some $\epsilon > 0$ such that $f(x) - a_{\geq s}(f;0) = O(x^{-s+\epsilon})$ and $a_{<s}(f;\infty) = O(x^{-s+\epsilon})$ when x tends to 0, $f(x) - a_{\leq s}(f;\infty) = O(x^{-s-\epsilon})$ and $a_{>s}(f;0) = O(x^{-s-\epsilon})$ when x tends to ∞. Recall that the regularized integral of a poly-log function is always zero. So the regularized integral in (1) equals

$$\begin{aligned}
&\int_0^t \left(f(x) - a_{\geq s}(f;0)\right) x^{s-1}dx + \int_t^\infty \left(f(x) - a_{\leq s}(f;\infty)\right) x^{s-1}dx \\
&- \int_t^\infty a_{>s}(f;0)x^{s-1}dx - \int_0^t a_{<s}(f;\infty)x^{s-1}dx \\
&- \int_t^{\infty,*} a_{=s}(f;0)x^{s-1}dx - \int_0^{t,*} a_{=s}(f;\infty)x^{s-1}dx.
\end{aligned}$$

Since $x^{-1}(\log x)^l$ has primitive $\frac{(\log x)^{l+1}}{l+1}$ which vanishes at $x = 1$ and has regularized values 0 at both $x = 0$ and $x = \infty$, we conclude that (1) and (3) are the same. It remains to

show that (3) coincides with (4). Take $t = 1$ in (3), we get

$$\begin{aligned}
&\int_0^1 (f(x) - a_{\geq s}(f;0))\, x^{s-1}dx + \int_1^\infty (f(x) - a_{\leq s}(f;\infty))\, x^{s-1}dx \\
&- \int_1^\infty a_{>s}(f;0)x^{s-1}dx - \int_0^1 a_{<s}(f;\infty)x^{s-1}dx \\
&= \int_0^1 (f(x) - a_{\geq s}(f;0))\, x^{s-1}dx + \int_1^\infty (f(x) - a_{\leq s}(f;\infty))\, x^{s-1}dx \\
&- \sum_{s'_n > s} c'_n \frac{(-1)^{l'_n} l'_n!}{(s - s'_n)^{l'_n+1}} - \sum_{s_n < s} c_n \frac{(-1)^{l_n} l_n!}{(s - s_n)^{l_n+1}},
\end{aligned}$$

which is exactly the regularized value $F^*(s)$. □

3.3 Generalized Mellin Transformation and Regularized Integral II

In Section 3.2 we considered functions with poly-log expansions at 0 and ∞. This includes a fair amount of functions. But there are also many usual functions having exponential growth at cusps, such as the j-invariant

$$j(\tau) = \frac{1}{q} + 744 + 196884q + 21493760q^2 + \dots$$

To cover these functions, we define

Definition 3.3.1. Let $f(x)$ be a complex-valued function of positive real variable x. We say that $f(x)$ has *exp-poly-log expansion at* ∞ if for any real N

$$f(x) = \sum_{m=1}^{M} a_m\, x^{\sigma_m} e^{\lambda_m x} + \sum_{\mathrm{Re}(s_n) \geq N} b_n\, x^{-s_n} (\log x)^{l_n} + o(x^{-N}) \quad \text{as } x \to \infty,$$

or in short

$$f(x) \sim \sum_m a_m\, x^{\sigma_m} e^{\lambda_m x} + \sum_n b_n\, x^{-s_n} (\log x)^{l_n} \quad \text{as } x \to \infty,$$

where $\sigma_m \in \mathbb{C}$ and $\lambda_m \in \mathbb{R}_{>0}$ are finite series, s_n and l_n satisfy the same conditions as in Definition 3.2.1. If the asymptotic expansion has no log terms, then we say that $f(x)$ has *exp-poly expansion.*

Similarly as before we define

Definition 3.3.2. We say that $f(x)$ has *exp-poly-log expansion at* 0 if $f(\frac{1}{x})$ has exp-poly-log asymptotic expansion at ∞. So the function f has asymptotic expansion

$$f(x) \sim \sum_m a_m' \, x^{\sigma_m'} e^{\lambda_m'/x} + \sum_n b_n' \, x^{-s_n'} (\log x)^{l_n'} \quad \text{when } \ x \to 0,$$

where $\sigma_m' \in \mathbb{C}$ and $\lambda_m' \in \mathbb{R}_{>0}$ are finite series, s_n' and l_n' satisfy the same conditions as in Definition 3.2.2.

Let $f(x)$ be a continuous function on $(0, \infty)$ with exp-poly-log asymptotic expansions

$$f(x) \sim \sum_m a_m \, x^{\sigma_m} e^{\lambda_m x} + \text{some poly-log terms} \quad \text{as } \ x \to \infty,$$

and

$$f(x) \sim \sum_m a_m' \, x^{\sigma_m'} e^{\lambda_m'/x} + \text{some poly-log terms} \quad \text{as } \ x \to 0$$

as in Definition 3.3.1 and Definition 3.3.2. We consider again the following decomposition

$$\int_0^\infty f(x) x^{s-1} dx = \int_{t_0}^\infty f(x) x^{s-1} dx + \int_0^{t_0} f(x) x^{s-1} dx,$$

both integrals do not converge since they grow exponentially at 0 or ∞. Inspired by the method of regularization of Petersson inner products in Bringmann–Diamantis–Ehlen [4], we introduce

$$L(s;\omega) + R(s;\omega) \coloneqq \int_{t_0}^\infty f(x) x^s e^{-\omega x} \frac{dx}{x} + \int_0^{t_0} f(x) x^s e^{-\omega/x} \frac{dx}{x}$$

with a complex variable ω. It suffices to consider only the first integral $L(s, \omega)$. For $\mathrm{Re}(\omega) \gg 0$, it converges absolutely. We see for $\mathrm{Re}(\omega) \gg 0$

$$\begin{aligned} L(s;\omega) &= \int_{t_0}^\infty f(x) x^s e^{-\omega x} \frac{dx}{x} \\ &= \int_{t_0}^\infty \left(f(x) - \sum_m a_m x^{\sigma_m} e^{\lambda_m x} \right) x^s e^{-\omega x} \frac{dx}{x} \\ &\quad + \sum_m a_m (-\lambda_m + \omega)^{-\sigma_m - s} \Gamma(\sigma_m + s, (-\lambda_m + \omega) t_0) \\ &=: L_1(s;\omega) + L_2(s;\omega). \end{aligned}$$

Taking $\omega = 0$, the first line $L_1(s;0)$ is an integral of function with poly-log expansion at infinity. As we showed before, it can be continued to a meromorphic function on whole s-plane. We also want a definition of $L_2(s;0)$, the typical way is to define it with analytic continuation with respect to ω. Following [4], we take only one branch of $\mathrm{Log}(z)$ with the branch cut to be the ray $\{re^{i\theta} \,|\, r \in \mathbb{R}_{>0}\}$ to avoid problems on negative real axis, where $\theta \in (\pi, \frac{3}{2}\pi)$ is a fixed angle.

Lemma 3.3.3. *Let ω and s be complex numbers. For $\mathrm{Re}(\omega) \gg 0$, the second part $L_2(s;\omega)$ defines an entire function in s. It can be continued to a holomorphic function of (ω, s) in the open domain $\mathbb{C}\backslash\{\lambda_n + re^{i\theta} \mid r \in \mathbb{R}_{\geq 0}\} \times \mathbb{C}$. Moreover, the function $L_2(s;0)$ is entire in s and is independent of the choice of θ.*

Proof. Recall that the incomplete gamma function $\Gamma(s,z)$ can be extended to a holomorphic function in both s and z in the sector $\{z \in \mathbb{C} \mid -2\pi + \theta < \arg z < \theta\}$. As long as $\theta \in (\pi, \frac{3}{2}\pi)$, the negative real axis is always contained in this sector. Accordingly, we have an analytic continuation of $L_2(s;\omega)$ within the open set $\mathbb{C}\backslash\{\lambda_m + re^{i\theta} \mid r \in \mathbb{R}_{\geq 0}\} \times \mathbb{C}$. Since $-\lambda_m t_0$ are negative reals, the incomplete gamma functions $\Gamma(\sigma_m + s, -\lambda_m t_0)$ are independent of the choice of θ. □

Write $L(s;0) = L_1(s;0) + L_2(s;0)$. In the same spirit we can define $R(s;0)$.

Definition 3.3.4. The *generalized Mellin transform* of $f(x)$ is the sum of two parts

$$\mathcal{M}(f,s) \coloneqq L(s;0) + R(s;0).$$

This definition is independent of the selection of t_0. We observe that

$$L(s;\omega) = \int_1^\infty \left(f(x) - \sum_m a_m x^{\sigma_m} e^{\lambda_m x} \right) x^s e^{-\omega x} \frac{dx}{x} - \int_1^{t_0} f(x) x^s e^{-\omega x} \frac{dx}{x} + \sum_m a_m (-\lambda_m + \omega)^{-\sigma_m - s} \Gamma(\sigma_m + s, -\lambda_m + \omega),$$

summing up $L(s;0)$ and $R(s;0)$ together we see that the part involving t_0 vanishes.

Remark 3.3.5. In $L_1(s;\omega)$, recall that we have the generalized exponential integral function (see Section 1.4)

$$\Gamma(l_n + 1) E^{l_n}_{s_n - s + 1}(\omega) = \int_1^\infty (\log t)^{l_n} t^{s - s_n - 1} e^{-\omega t} dt$$

and its special value

$$\Gamma(l_n + 1) E^{l_n}_{s_n - s + 1}(0) = \frac{(-1)^{l_n+1} l_n!}{(s - s_n)^{l_n}},$$

which is exactly the principle part coming from the Laurent expansion of $L(s;0)$ at the pole $s = s_n$.

We offer in Section 3.4 several examples about generalized Mellin transforms of modular functions with exponential growth.

Let $f(x)$ be a continuous function with exp-poly-log expansion

$$f(x) \sim \sum_m a_m \, x^{\sigma_m} e^{\lambda_m x} + \sum_n b_n \, x^{-s_m} (\log x)^{l_n} \quad \text{as } x \to \infty.$$

For $\sigma \in \mathbb{C}$ and $\lambda > 0$ we have

$$\int x^\sigma e^{\lambda x}\, dx = -(-\lambda)^{-\sigma-1}\Gamma(\sigma+1, -\lambda x) + C.$$

Given $s \in \mathbb{C}$, let $g(x)$ be a primitive of $f(x)x^{s-1}$ then

$$g(x) \sim -\sum_m a_m(-\lambda_m)^{-\sigma_m - s}\Gamma(\sigma_m + s, -\lambda_m x) + Q(x, \log x) \quad \text{as } x \to \infty$$

for some poly-log asymptotic expansion $Q(x, \log x)$.

Remark 3.3.6. Note that for positive integer σ we have

$$\Gamma(\sigma, -\lambda x) = (\sigma-1)! e^{\lambda x} e_{\sigma-1}(-\lambda x).$$

Thus if all $\sigma_m + s$ are positive integers, $\Gamma(\sigma_m + s, -\lambda_n x)$ are nothing but polynomials of x and $e^{-\lambda_n x}$. In this case the primitive $g(x)$ has also exp-poly-log expansion.

Definition 3.3.7. Let $g(x)$ be a complex-valued function of positive real variable x. Suppose $g(x)$ has asymptotic expansion

$$g(x) \sim \sum_{m=1}^{M} \left(a_m\Gamma(\sigma_m, -\lambda_m x) + a'_m x^{\sigma_m - 1} e^{\lambda_m x}\right) + \sum_{(s_n, l_n) \neq (0,0)} b_n\, x^{-s_n}(\log x)^{l_n} + c \quad \text{as } x \to \infty,$$

where σ_m, λ_n, s_m and l_n satisfy the same conditions as in Definition 3.3.1. The *regularized value* of $g(x)$ at ∞, denoted also by $g^*(\infty)$, is defined as taking $x = 1$ in the exponential part and omitting the poly-log terms

$$g^*(\infty) = \sum_{\sigma_m, \lambda_n} \left(a_m\Gamma(\sigma_m, -\lambda_n) + a'_m e^{\lambda_n}\right) + c.$$

Similarly we can define $g^*(0)$.

Remark 3.3.8. Note that we have infinite asymptotic expansions

$$\Gamma(\sigma, -\lambda x) \sim (-\lambda x)^{\sigma-1} e^{\lambda x}\left(1 + \frac{(\sigma-1)}{x} + \frac{(\sigma-1)(\sigma-2)}{x^2} + \dots\right).$$

The finite sum $\sum_m (a_m\Gamma(\sigma_m, -\lambda_m x) + a'_m x^{\sigma_m - 1} e^{\lambda_m x})$ must have exponential growth (if nonzero). Hence the regularized value is well-defined.

This leads us to the definition of regularized integral.

Definition 3.3.9. Let $f(x)$ be a continuous function on $(0, +\infty)$ with exp-poly-log asymptotic expansions at 0 and ∞. Let $g(x)$ be a primitive of $f(x)x^{s-1}$ where $s \in \mathbb{C}$, then the *regularized integrals* of $f(x)$ are defined as

$$\int_0^{t,*} f(x)x^{s-1}\,dx = g(t) - g^*(0),$$

$$\int_t^{\infty,*} f(x)x^{s-1}\,dx = g^*(\infty) - g(t),$$

and

$$\int_0^{\infty,*} f(x)x^{s-1}\,dx = g^*(\infty) - g^*(0),$$

where t is an arbitrary positive real number.

An analogue of Theorem 3.2.8 is given here

Theorem 3.3.10. *Let $f(x)$ be a continuous function on $(0, \infty)$ with exp-poly-log asymptotic expansions. Let $s \in \mathbb{C}$ and assume that $g(x)$ is a primitive of $f(x)x^{s-1}$. Suppose that $F(s) = \mathcal{M}(f, s)$ is the generalized Mellin transform of $f(x)$. Then the following quantities are equal*

(1) the regularized integral

$$\int_0^{\infty,*} f(x)x^{s-1}\,dx = g^*(\infty) - g^*(0),$$

(2) the regularized value $h^*(\infty)$ of the function

$$h(x) = \int_{1/x}^{x} f(t)t^{s-1}\,dt,$$

(3) the regularized value $F^*(z)|_{z=s}$ at s.

Proof. The incomplete gamma function $\Gamma(\sigma_m + s, -\lambda_n)$ is always entire in s. Hence, everything follows directly from Theorem 3.2.8. □

Remark 3.3.11. The generalized Mellin transform defined in this section lacks some properties, comparing to the classical one. For example, the generalized Mellin transform of $f(ax)$ may not be $a^{-s}F(s)$. However one can still verify the followings

$$\begin{aligned}\mathcal{M}\left(f(x^{-1}), s\right) &= -\mathcal{M}\left(f(x), -s\right),\\ \mathcal{M}\left(x^z f(x), s\right) &= \mathcal{M}\left(f(x), s+z\right).\end{aligned}$$

3.4 Examples of L-functions

Let $k \geq 2$ be an integer. We work with only level $\mathrm{SL}_2(\mathbb{Z})$ throughout this section, though everything carries over to arbitrary levels.

Example 3.4.1. *Given a holomorphic modular form $f = \sum_{n\geq 0} a_f(n)q^n \in M_k$, we see the generalized Mellin transform of f is exactly the L-function of f*

$$\begin{aligned}\Lambda(f,s) = &-\frac{a_f(0)}{s} - i^k\frac{a_f(0)}{k-s}\\ &+\int_{t_0}^{\infty}\sum_{n=1}^{\infty} a_f(n)e^{-2\pi ny}y^s\frac{dy}{y} + i^k\int_{1/t_0}^{\infty}\sum_{n=1}^{\infty} a_f(n)e^{-2\pi ny}y^{k-s}\frac{dy}{y}.\end{aligned}$$

We know that the function $\Lambda(f,s)$ is a meromorphic function of s with possibly poles at $s=0$ and $s=k$. It verifies the functional equation

$$\Lambda(f,s) = i^k\Lambda(f,k-s).$$

Example 3.4.2. *Let $f = \sum_{n\geq n_0} a_f(n)q^n \in S_k^!$ be a weakly holomorphic cusp form of weight k. In [5], Bringmann, Fricke and Kent defined the L-function of f to be*

$$\Lambda(f,s) = \sum_{n\geq n_0}\frac{a_f(n)\Gamma(s,2\pi nt_0)}{(2\pi n)^s} + i^k\sum_{n\geq n_0}\frac{a_f(n)\Gamma(k-s,\frac{2\pi n}{t_0})}{(2\pi n)^{k-s}}.$$

Our regularization recovers their definition. In fact the generalized Mellin transform of f is

$$\begin{aligned}&\sum_{n_0\leq n<0}\frac{a_f(n)\Gamma(s,2\pi nt_0)}{(2\pi n)^s} + i^k\sum_{n_0\leq n<0}\frac{a_f(n)\Gamma(k-s,\frac{2\pi n}{t_0})}{(2\pi n)^{k-s}}\\ &+\int_{t_0}^{\infty}\sum_{n=1}^{\infty} a_f(n)e^{-2\pi ny}y^s\frac{dy}{y} + i^k\int_{1/t_0}^{\infty}\sum_{n=1}^{\infty} a_f(n)e^{-2\pi ny}y^{k-s}\frac{dy}{y}.\end{aligned}$$

We see that $\Lambda(f,s)$ is an entire function of s. It satisfies the functional equation

$$\Lambda(f,s) = i^k\Lambda(f,k-s).$$

Example 3.4.3. *Let $f = \sum_{n\geq n_0} a_f(n)q^n \in M_k^!$ be a weakly holomorphic form of weight k with $a_f(0)\neq 0$. Then the L-function of f is*

$$\begin{aligned}\Lambda(f,s) = &\sum_{n\geq n_0}\frac{a_f(n)\Gamma(s,2\pi nt_0)}{(2\pi n)^s} + i^k\sum_{n\geq n_0}\frac{a_f(n)\Gamma(k-s,\frac{2\pi n}{t_0})}{(2\pi n)^{k-s}}\\ &-\frac{a_f(0)}{s} - i^k\frac{a_f(0)}{k-s}.\end{aligned}$$

The last line comes form the constant term of f. We see that the function $\Lambda(f,s)$ is a meromorphic function of s with poles only at $s=0$ and $s=k$. It satisfies the functional equation

$$\Lambda(f,s) = i^k \Lambda(f, k-s).$$

Remark 3.4.4. We can also write the L-function of f as

$$\Lambda(f,s) = \sum_{n \geq n_0} a_f(n) E_{1-s}(2\pi n t_0) + i^k \sum_{n \geq n_0} a_f(n) E_{1-k+s}(2\pi n/t_0),$$

in view of the special value of exponential integral function $E_{1-s}(0) = -s^{-1}$.

Example 3.4.5. *Let f be a harmonic Maass form of weight $2-k$ (see [6] for the definition of a harmonic Maass form) with Fourier expansion*

$$f(\tau) = \sum_{n \geq n_0} a_f^+(n) q^n + \sum_{n<0} a_f^-(n) \Gamma(k-1, -4\pi n y) q^n.$$

Then f has exp-poly expansions at 0 and ∞. The generalized Mellin transform of f is

$$\begin{aligned}\Lambda(f,s) = &\sum_{n \geq n_0} \frac{a_f^+(n)\Gamma(s, 2\pi n t_0)}{(2\pi n)^s} + i^{2-k} \sum_{n \geq n_0} \frac{a_f^+(n)\Gamma(2-k-s, \frac{2\pi n}{t_0})}{(2\pi n)^{2-k-s}} \\ &+ \int_{t_0}^{\infty} \sum_{n=1}^{\infty} a_f^-(-n)\Gamma(k-1, 4\pi n y) e^{2\pi n y} y^s \frac{dy}{y} \\ &+ i^{2-k} \int_{t_0}^{\infty} \sum_{n=1}^{\infty} a_f^-(-n)\Gamma(k-1, 4\pi n y) e^{2\pi n y} y^{2-k-s} \frac{dy}{y}.\end{aligned}$$

Writing explicitly we get

$$\begin{aligned}\Lambda(f,s) + \frac{a_f^+(0)}{s} + i^{2-k}\frac{a_f^+(0)}{2-k-s} = &\\ \sum_{n \geq n_0} \frac{a_f^+(n)\Gamma(s, 2\pi n t_0)}{(2\pi n)^s} + i^{2-k} &\sum_{n \geq n_0} \frac{a_f^+(n)\Gamma(2-k-s, \frac{2\pi n}{t_0})}{(2\pi n)^{2-k-s}} \\ + (k-2)! \sum_{l=0}^{k-2} \frac{2^l}{l!} \left(\sum_{n=0}^{\infty} \frac{a_f^-(n)\Gamma(l+s, 2\pi n t_0)}{(2\pi n)^s} + i^{2-k} \sum_{n=0}^{\infty} \frac{a_f^-(n)\Gamma(l+2-k-s, \frac{2\pi n}{t_0})}{(2\pi n)^{2-k-s}} \right)&.\end{aligned}$$

This gives us one way to define the L-function of f. The function $\Lambda(f,s)$ has possibly poles at 0 and $2-k$. It verifies the functional equation

$$\Lambda(f,s) = i^{2-k}\Lambda(f, 2-k-s).$$

3.5 Modular Symbols and Modular Caps

In this section we introduce the theory of extended modular symbols, including modular symbols and modular caps, originally defined by Stevens [44].

Let $\mathcal{H}$ be the Poincaré upper half-plane. Let $\overline{\mathcal{H}}^{BS}$ be the Borel–Serre completion of $\mathcal{H}$ obtained by adding to each rational cusp a horocycle at infinity H_α. Let $\alpha \in \mathbb{P}^1(\mathbb{Q})$ be a cusp and $x \in \mathbb{P}^1(\mathbb{R}) - \{\alpha\}$, we add the 'initial point' $\pi_\alpha(x)$ of the oriented geodesic from α to x to the boundary component of α. For each $\alpha \in \mathbb{P}^1(\mathbb{Q})$ and $y > 0$, we can assign α with a horocycle H_α^y. At the infinity cusp, the horocycle H_∞^y is the straight line $\operatorname{Im}\tau = y$. Given $\alpha = p/q \in \mathbb{Q}$ with p and q coprime, H_α^y is the horocycle with Euclidean diameter $1/(q^2 y)$. We may think of the horocycle at infinity H_α as the limit of the horocycle H_α^y when $y \to \infty$.

Given a pair of distinct cusps, recall a *modular symbol* $\{\alpha, \beta\}$ is a 1-chain on $\overline{\mathcal{H}}^{BS}$ represented by the closed oriented geodesic from α to β (illustrated in Figure 3) . It has boundary $\pi_\beta(\alpha) - \pi_\alpha(\beta)$. We write $\pi_\alpha^y(\beta)$ for the intersection of $\{\alpha, \beta\}$ and H_α^y. Let $\{\alpha, \beta\}^y$ be the segment of $\{\alpha, \beta\}$ joining $\pi_\alpha^y(\beta)$ and $\pi_\beta^y(\alpha)$. Then the modular symbol can be viewed as the limit of 1-chain $\{\alpha, \beta\}^y$, as $y \to \infty$.

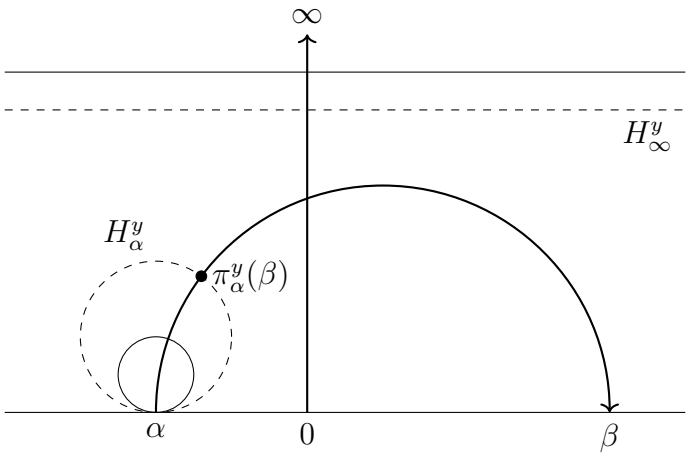

Figure 3. Modular symbols and Horocycles at infinity

The *modular cap*, defined by Stevens, is the 1-chain represented by the segment of a horocycle at infinity cut by two distinct modular symbols (illustrated in Figure 4). Write $[\beta, \gamma]_\alpha$ for the segment of H_α joining $\pi_\alpha(\beta)$ to $\pi_\alpha(\gamma)$. Let $[\beta, \gamma]_\alpha^y$ be the segment of H_α^y joining $\pi_\alpha^y(\beta)$ and $\pi_\alpha^y(\gamma)$. Then a modular cap $[\beta, \gamma]_\alpha$ can be seen as the limit of $[\beta, \gamma]_\alpha^y$, as $y \to \infty$.

We reformulate the definition of extended modular symbols by Stevens with relations on modular symbols and caps.

Definition 3.5.1. Let $\mathbb{K}_2$ be the free abelian group spanned by modular symbols and

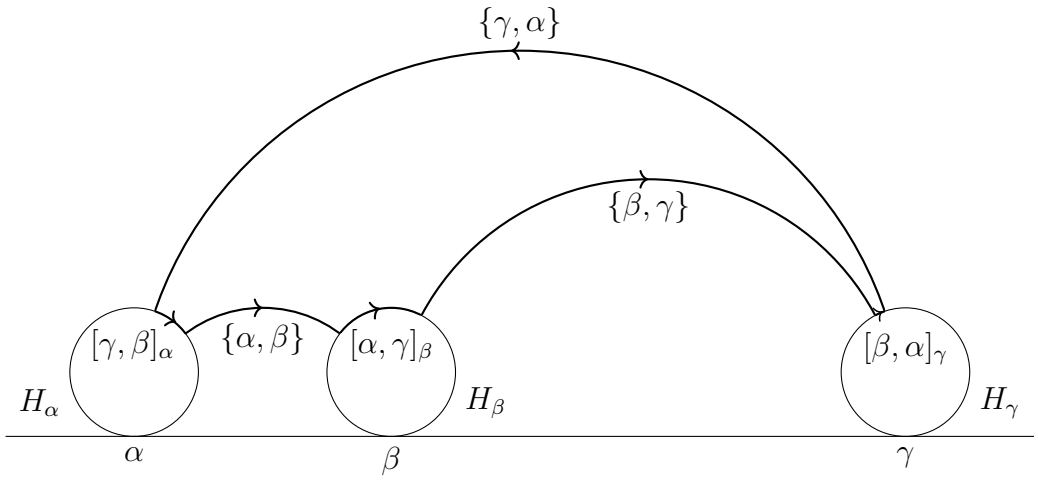

Figure 4. The 6-term relation

caps, modulo the 6-term relations

$$\{\alpha,\beta\}+\{\beta,\gamma\}+\{\gamma,\alpha\}=[\alpha,\beta]_\gamma+[\beta,\gamma]_\alpha+[\gamma,\alpha]_\beta,$$

3-term relations

$$[\alpha,\beta]_\delta+[\beta,\gamma]_\delta+[\gamma,\alpha]_\delta=0$$

and relations

$$\begin{aligned}\{\alpha,\alpha\}&=0,\\ \{\alpha,\beta\}&=-\{\beta,\alpha\},\\ [\beta,\beta]_\alpha&=0,\\ [\beta,\gamma]_\alpha&=-[\gamma,\beta]_\alpha.\end{aligned}$$

(The 6-term relation is depicted in Figure 4.)

There is a left action of $\mathrm{GL}_2^+(\mathbb{Q})$ on $\mathbb{K}_2$ given by

$$\begin{aligned}g\{\alpha,\beta\}&=\{g\alpha,g\beta\},\\ g[\beta,\gamma]_\alpha&=[g\beta,g\gamma]_{g\alpha}.\end{aligned}$$

This action is compatible with the action of $\mathrm{GL}_2^+(\mathbb{Q})$ on the Borel–Serre completion $\overline{\mathcal{H}}^{BS}$.

Let Γ be a subgroup of $\mathrm{GL}_2^+(\mathbb{Q})$. Then Γ acts on $\mathbb{K}_2$. The space $\mathbb{K}_2(\Gamma)$ is defined as the group of Γ-coinvariants, i.e. $\mathbb{K}_2/\langle x-\gamma x \,|\, x\in\mathbb{K}_2,\gamma\in\Gamma\rangle$, and modulo all torsions.

Theorem 3.5.2 (Stevens [44, (1.8)]). *Let α be a rational number with continued fraction in the form*

$$\alpha = a_0 - \cfrac{1}{a_1 - \cfrac{1}{\ddots - \cfrac{1}{a_n}}},$$

where a_0 is an integer and a_i are positive integers.

Let $\frac{p_{-1}}{q_{-1}} = \frac{1}{0}, \frac{p_0}{q_0} = \frac{a_0}{1}, \frac{p_1}{q_1}, \ldots, \frac{p_n}{q_n} = \alpha$ with $q_i > 0$ be the successive convergents and set

$$\gamma_{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \gamma_0 = \begin{pmatrix} -p_0 & p_{-1} \\ -q_0 & q_{-1} \end{pmatrix}, \cdots, \gamma_n = \begin{pmatrix} -p_n & p_{n-1} \\ -q_n & q_{n-1} \end{pmatrix}.$$

Then we have the following decomposition

$$\{\infty, \alpha\} = -\sum_{k=0}^{n} \gamma_k\{\infty, 0\} + \gamma_n[0, \frac{q_{n-1}}{q_n}]_\infty + \sum_{k=0}^{n} \gamma_{k-1}[0, a_k]_\infty - [0, \alpha]_\infty.$$

Unlike modular symbols, extended modular symbols are unlikely to be finitely generated since there are too many caps. From Theorem 3.5.2, we deduce immediately

Theorem 3.5.3. *Let $N \geq 1$ be an integer and $r_0, \ldots, r_m$ be the right coset representatives of $\Gamma(N)$ in $\mathrm{SL}_2(\mathbb{Z})$. Then $\mathbb{K}_2(N) \coloneqq \mathbb{K}_2(\Gamma(N))$ is generated by the Manin symbols $r_i\{0, \infty\}$ and caps $r_i[0, \alpha]_\infty$, where $\alpha \in \mathbb{Q}$.*

3.6 Integration over Extended Modular Symbols

Observe that the regulator $\mathrm{Eis}_{\mathcal{D}}^{k_1,k_2,j}$ is represented by a closed form which has usually polynomial growth at cusps. Its integral along the Shokurov cycle (see Definition 3.6.11 later)

$$\int_{X^m Y^{w-m}\{0,\infty\}} \mathrm{Eis}_{\mathcal{D}}^{k_1,k_2,j}(u_1, u_2)$$

will not converge in general cases.

Another example comes from the weakly holomorphic quasi-modular form $f = E_2 j$ of weight 2. It has Fourier expansions

$$f(\tau) = \frac{1}{q} + 720 + 178956q + 16714880q^2 + \ldots$$

and exp-poly growth at certain cusps. To define a period of $E_2 j$ means to inspect the integral of closed form $f(\tau)d\tau$ along the modular symbol $\{g^{-1}\infty, \infty\}$ for $g \in \mathrm{SL}_2(\mathbb{Z})$

$$r_f(g) = \int_{g^{-1}\infty}^{\infty} E_2 j(\tau) d\tau.$$

Unfortunately this integral is not convergent.

Let ω be a real-analytic closed 1-form on $\mathcal{H}$. Usually ω is set to be invariant under some congruence subgroup, then ω is real-analytic modular of weight 2. One may pose the following questions

(1) Is it possible to define the period of ω, namely, for every modular symbol $\{\alpha, \beta\}$, a (regularized) integral $\int_\alpha^{\beta,*} \omega$?

(2) If so, will we have 3-term relations like Stokes' formula?

(3) Do we have periods in higher weights?

3.6.1 Weight 2 Case

Let us use our theory of regularization to introduce a definition of regularized integrals along extended modular symbols. The first thing that we need is a moderate growth of the closed form ω at every cusp.

Let ω be a real-analytic closed 1-form on $\mathcal{H}$. Any element $g \in \mathrm{GL}_2^+(\mathbb{Q})$ acts on $\mathcal{H}$ and we have $g^*\omega$ as the pullback of the differential form ω. Given any $\alpha = p/q \in \mathbb{Q}$ with p and q coprime, we set

$$\sigma_\alpha = \begin{pmatrix} p & u \\ q & v \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}),$$

then $\sigma_\alpha \infty = \alpha$. The element σ_α is unique modulo the parabolic subgroup at infinity. For two such elements σ_α and σ'_α we have

$$\sigma_\alpha'^{-1}\sigma_\alpha \in \left\{ \pm \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \;\middle|\; m \in \mathbb{Z} \right\}.$$

Definition 3.6.1. Let $\Omega^1_{epl}(\mathcal{H}, \mathbb{C})$ be the space of real-analytic complex-valued closed 1-forms ω which satisfy the following conditions

(1) for every $g \in \mathrm{SL}_2(\mathbb{Z})$, the 1-form $g^*\omega$ has exp-poly-log expansions at infinity with respect to y. This is to say, let $\tau = x + iy \in \mathcal{H}$ and $g^*\omega = f_1(x, y)d\tau + f_2(x, y)d\overline{\tau}$, then for every x, the functions $f_1(x, \cdot)$ and $f_2(x, \cdot)$ have exp-poly-log asymptotic expansions at ∞.

(2) for every $g \in \mathrm{SL}_2(\mathbb{Z})$ and $x \in \mathbb{R}$, the integral $\int_{[0,x]_\infty^y} g^*\omega$ has exp-poly-log expansions at infinity with respect to y, where $[0, x]_\infty^y$ is the line segment from iy to $x + iy$.

Indeed, the first condition guarantees the existence of integrals of ω along modular symbols (*periods*) and the second condition guarantees the existence of integrals of ω along modular caps (*residues*).

Remark 3.6.2. If $f(y)$ has exp-poly-log expansion at infinity, it is evident that $f(ay)$ has also exp-poly-log expansion at infinity for $a \in \mathbb{R}_{>0}$. This indicates that we can replace in the definition $\mathrm{SL}_2(\mathbb{Z})$ by $\mathrm{GL}_2^+(\mathbb{Q})$ since

$$\begin{pmatrix} r & s \\ 0 & t \end{pmatrix} \cdot \tau = \frac{r\tau + s}{t}$$

only changes the imaginary part $y = \mathrm{Im}(\tau)$ to its multiple.

Proposition 3.6.3. *Let $\omega \in \Omega^1_{epl}(\mathcal{H}, \mathbb{C})$ be a closed 1-form and $\{\alpha, \beta\}$ be a modular symbol. Then the function*

$$I(y) = \int_{\{\alpha,\beta\}^y} \omega$$

has exp-poly-log expansion at infinity. In particular, the regularized value $I^*(\infty)$ exists.

Proof. The integral is taken from $\pi^y_\alpha(\beta)$ to $\pi^y_\beta(\alpha)$ (see Figure 3). Take any cutting point t between them on $\{\alpha, \beta\}$, we denote by $\{\pi^y_\alpha(\beta), t\}$(resp. $\{t, \pi^y_\beta(\alpha)\}$) the segment joining $\pi^y_\alpha(\beta)$ and t (resp. t and $\pi^y_\beta(\alpha)$). Then

$$\begin{aligned} \int_{\{\alpha,\beta\}^y} \omega &= \int_{\{\pi^y_\alpha(\beta),t\}} \omega + \int_{\{t,\pi^y_\beta(\alpha)\}} \omega \\ &= \int_{\mathrm{Re}(\sigma_\alpha^{-1}t)+iy}^{\sigma_\alpha^{-1}t} \sigma_\alpha^*\omega + \int_{\sigma_\beta^{-1}t}^{\mathrm{Re}(\sigma_\beta^{-1}t)+iy} \sigma_\beta^*\omega. \end{aligned}$$

Since $\sigma_\alpha^*\omega$ and $\sigma_\beta^*\omega$ have exp-poly-log expansions at ∞, the regularized values of both integrals at ∞ are well-defined. □

For integrals over modular caps, let us use the same idea.

Proposition 3.6.4. *Let $\omega \in \Omega^1_{epl}(\mathcal{H}, \mathbb{C})$ be a closed 1-form. Then the function*

$$J(y) = \int_{[\beta,\gamma]^y_\alpha} \omega.$$

has exp-poly-log expansion at infinity. In particular, the regularized value $J^*(\infty)$ exists.

Proof. By $\sigma_\alpha^{-1}[\beta, \gamma]^y_\alpha = [\sigma_\alpha^{-1}\beta, \sigma_\alpha^{-1}\gamma]^y_\infty$, we have

$$\begin{aligned} \int_{[\beta,\gamma]^y_\alpha} \omega &= \int_{[\sigma_\alpha^{-1}\beta,\sigma_\alpha^{-1}\gamma]^y_\infty} \sigma_\alpha^*\omega \\ &= \int_{[0,\sigma_\alpha^{-1}\gamma]^y_\infty} \sigma_\alpha^*\omega - \int_{[0,\sigma_\alpha^{-1}\beta]^y_\infty} \sigma_\alpha^*\omega. \end{aligned}$$

□

Definition 3.6.5. Let $\omega \in \Omega^1_{epl}(\mathcal{H}, \mathbb{C})$ and let $\{\alpha, \beta\}, [\beta, \gamma]_\alpha$ be a modular symbol and a modular cap. We define the regularized integrals of ω along the modular symbol $\{\alpha, \beta\}$ and cap $[\beta, \gamma]_\alpha$

$$\int^*_{\{\alpha,\beta\}} \omega\,, \quad \int^*_{[\beta,\gamma]_\alpha} \omega$$

to be the regularized values of $\int_{\{\alpha,\beta\}^y} \omega$ and $\int_{[\beta,\gamma]^y_\alpha} \omega$ as $y \to \infty$.

It is imperative that we can perform change of variable on our regularized integrals. This is to say, given an extended modular symbol $\kappa \in \mathbb{K}_2$, for all $g \in \mathrm{SL}_2(\mathbb{Z})$ we should have the identity

$$\int^*_\kappa g^*\omega = \int^*_{g\kappa} \omega.$$

In particular, if ω is invariant under some congruence subgroup Γ, then we can pair it with elements in the extended modular symbol space $\mathbb{K}_2(\Gamma)$. Luckily, this is not a problem with our definition of horocycles H^y_*.

Lemma 3.6.6. *An element $g \in \mathrm{SL}_2(\mathbb{Z})$ preserves the horocycles H^y_*, that is,*

$$gH^y_\alpha = H^y_{g\alpha}.$$

In particular, we have $\sigma_\alpha H^y_\infty = H^y_\alpha$.

Proof. Let $\alpha = p/q \in \mathbb{Q}$ with p and q coprime. Then

$$\begin{aligned} w \in \sigma_\alpha H^y_\infty &\iff \mathrm{Im}\, \sigma_\alpha^{-1} w = y \iff \frac{\mathrm{Im}(w)}{|p - qw|^2} = y \\ &\iff \left(\mathrm{Re}(w) - p/q\right)^2 + \left(\mathrm{Im}(w) - 1/(2q^2y)\right)^2 = \left(1/(2q^2y)\right)^2 \\ &\iff w \in H^y_\alpha. \end{aligned}$$

For general $g \in \mathrm{SL}_2(\mathbb{Z})$ we have

$$gH^y_\alpha = (g\sigma_\alpha)H^y_\infty = H^y_{g\alpha}.$$

□

Lemma 3.6.7. *Let $\omega \in \Omega^1_{epl}(\mathcal{H}, \mathbb{C})$ and $g \in \mathrm{SL}_2(\mathbb{Z})$. Let $\{\alpha, \beta\}, [\beta, \gamma]_\alpha$ be a modular symbol and a modular cap. Then*

$$\begin{aligned} \int^*_{\{\alpha,\beta\}} g^*\omega &= \int^*_{\{g\alpha, g\beta\}} \omega \\ \int^*_{[\beta,\gamma]_\alpha} g^*\omega &= \int^*_{[g\beta, g\gamma]_{g\alpha}} \omega \end{aligned}$$

Proof. From Lemma 3.6.6 we find that $g \in \mathrm{SL}_2(\mathbb{Z})$ preserves the horocycles, that is,

$$gH_\alpha^y = H_{g\alpha}^y, \quad gH_\beta^y = H_{g\beta}^y.$$

Consequently,

$$\int_{\{\alpha,\beta\}^y} g^*\omega = \int_{\{g\alpha,g\beta\}^y} \omega$$
$$\int_{[\beta,\gamma]_\alpha^y} g^*\omega = \int_{[g\beta,g\gamma]_{g\alpha}^y} \omega.$$

Take both side regularized value we win. □

Theorem 3.6.8. *There is a well-defined pairing*

$$\begin{aligned} \Omega^1_{epl}(\mathcal{H},\, \mathbb{C}) \times \mathbb{K}_2 &\longrightarrow \mathbb{C} \\ (\omega, \{\alpha,\beta\}) &\longmapsto \int_{\{\alpha,\beta\}}^* \omega \\ (\omega, [\beta,\gamma]_\alpha) &\longmapsto \int_{[\beta,\gamma]_\alpha}^* \omega \end{aligned}$$

given by the regularized integrals over modular symbols and caps.

Proof. We only need to check the 6-term relations. The rests are obvious. Let $\alpha, \beta, \gamma \in \mathbb{P}^1(\mathbb{Q})$ be distinct cusps. Integrating ω over the closed 1-chain $\{\alpha,\beta\}^y + [\alpha,\gamma]_\beta^y + \{\beta,\gamma\}^y + [\beta,\alpha]_\gamma^y + \{\gamma,\alpha\}^y + [\gamma,\beta]_\alpha^y$ (see Figure 3), by Stokes' theorem we get

$$\int_{\{\alpha,\beta\}^y} \omega + \int_{[\alpha,\gamma]_\beta^y} \omega + \int_{\{\beta,\gamma\}^y} \omega + \int_{[\beta,\alpha]_\gamma^y} \omega + \int_{\{\gamma,\alpha\}^y} \omega + \int_{[\gamma,\beta]_\alpha^y} \omega = 0.$$

Take regularized value we see

$$\int_{\{\alpha,\beta\}}^* \omega + \int_{[\alpha,\gamma]_\beta}^* \omega + \int_{\{\beta,\gamma\}}^* \omega + \int_{[\beta,\alpha]_\gamma}^* \omega + \int_{\{\gamma,\alpha\}}^* \omega + \int_{[\gamma,\beta]_\alpha}^* \omega = 0.$$

□

Let Γ be a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$. We write $\Omega^1_{epl}(\mathcal{H},\mathbb{C})^\Gamma$ for the subspace of closed 1-forms invariant under the action of Γ. Immediately we have

Theorem 3.6.9. *There is a well-defined pairing*

$$\begin{aligned} \Omega^1_{epl}(\mathcal{H},\mathbb{C})^\Gamma \times \mathbb{K}_2(\Gamma) &\longrightarrow \mathbb{C} \\ (\omega,\, \{\alpha,\beta\}) &\longmapsto \int_{\{\alpha,\beta\}}^* \omega \\ (\omega,\, [\beta,\gamma]_\alpha) &\longmapsto \int_{[\beta,\gamma]_\alpha}^* \omega \end{aligned}$$

given by the regularized integrals over modular symbols and caps.

Remark 3.6.10. With continued fraction decomposition (see Theorem 3.5.2), to determine all the periods and residues of $\omega \in \Omega^1_{epl}(\mathcal{H}, \mathbb{C})$, it suffices to compute for every $g \in \mathrm{SL}_2(\mathbb{Z})$ the integrals

$$\int_0^{\infty,*} g^*\omega, \qquad \int_{[0,\alpha]_\infty}^{*} g^*\omega.$$

3.6.2 Higher Weight Case

The way to deal with higher weight forms is using Shokurov cycles [42] (see also [35]). Let $k \geq 2$ be an integer and set $w = k - 2$. Let $\mathbb{Z}\langle X, Y\rangle_w$ be the linear subspace of the free $\mathbb{Z}$-algebra generated by X and Y which consists of non-commutative homogeneous polynomials of degree w. Define the left action of $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{Z}\langle X, Y\rangle_w$ by

$$(gP)(X, Y) = P(dX - bY, -cX + aY).$$

The linear space $\mathbb{Z}\langle X, Y\rangle_w$ has dimension 2^w and is generated by the polynomials $\prod_{i=1}^w (a_i X + b_i Y)$ with $a_i, b_i \in \mathbb{Z}$.

Definition 3.6.11. Let $0 \leq m \leq w$ be an integer and $\alpha, \beta \in \mathbb{P}^1(\mathbb{Q})$ be two distinct cusps. Let $\sigma = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. Given a non-commutative polynomial $P = \prod_{i=1}^w (a_i X + b_i Y)$ with $a_i, b_i \in \mathbb{Z}$, the *Shokurov cycle* is the following $(w+1)$-cycle on $E^w(\mathbb{C})$

$$P\{\alpha, \beta\} = \{(\tau; t_1(a_1\tau + b_1), \ldots, t_w(a_w\tau + b_w); \sigma) \,|\, \tau \in \{\alpha, \beta\},\, t_1, \ldots, t_w \in [0, 1]\},$$

where $\{\alpha, \beta\}$ is the modular symbol.

Denote by $\gamma_{\tau,P}$ the fiber of projection $P\{\alpha, \beta\} \to \{\alpha, \beta\}$ at τ. If $P = \prod_{i=1}^w (a_i X + b_i Y)$ with $a_i, b_i \in \mathbb{Z}$ then

$$\gamma_{\tau,P} = \{(\tau; t_1(a_1\tau + b_1), \ldots, t_w(a_w\tau + b_w); \sigma) \,|\, t_1, \ldots, t_w \in [0, 1]\}.$$

In particular, the fiber of the projection $X^m Y^{w-m}\{\alpha, \beta\} \to \{\alpha, \beta\}$ at τ, is

$$\gamma_{\tau,m} = \{(\tau; t_1\tau, \ldots, t_m\tau, t_{m+1}, \ldots, t_w; \sigma) \,|\, t_1, \ldots, t_w \in [0, 1]\}.$$

and the fiber of $X^m Y^{w-m}\{0, \infty\} \to \{0, \infty\}$ at iy is

$$\gamma_{y,m} = \{(iy; it_1 y, \ldots, it_m y, t_{m+1}, \ldots, t_w; \sigma) \,|\, t_1, \ldots, t_w \in [0, 1]\}.$$

If $\omega = 0$ then the Shokurov cycles are nothing but modular symbols.

Definition 3.6.12. The torsion-free abelian group $\mathbb{K}_k = \mathbb{Z}\langle X, Y\rangle_w \otimes_{\mathbb{Z}} \mathbb{K}_2$ is call the space of *weight k extended modular symbols*. We endow the space of extended modular symbols $\mathbb{K}_k$ with the tensor product action of $\mathrm{SL}_2(\mathbb{Z})$.

The abelian group $\mathbb{K}_k$ is generated by the modular symbols $P\{\alpha,\beta\}$ and caps $P[\beta,\gamma]_\alpha$, where $P=\prod_{i=1}^w(a_iX+b_iY)$ is a polynomial in $\mathbb{Z}\langle X,Y\rangle_w$. Let $\mathbb{K}_k(N)$ be the quotient of $\mathbb{K}_k$ by all $\gamma x-x$ for $x\in\mathbb{K}_k$, $\gamma\in\Gamma(N)$, that is, the $\Gamma(N)$-coinvariants, and then modular any torsion.

Let ν be the following holomorphic map

$$\begin{aligned}\nu:\mathcal{H}\times\mathbb{C}^w&\to E^w(\mathbb{C})\\(\tau;z_1,\dots,z_k)&\mapsto[(\tau;z_1,\dots,z_k;\sigma)]\end{aligned}$$

which maps to a connected component of $E^w(\mathbb{C})$. Let ω be a complex-valued real-analytic closed $(w+1)$-form on $E^w(\mathbb{C})$. To integrate ω along a Shokurov cycle $P\{\alpha,\beta\}$, it is equivalent to integrate ω on the fiber $\gamma_{\tau,P}$ first and then on the modular symbol $\tau\in\{\alpha,\beta\}$

$$\int_{P\{\alpha,\beta\}}\omega=\int_{\{\alpha,\beta\}}\int_{\gamma_{\tau,P}}\omega.$$

Since integration along the fiber commutes with the exterior differential (see [3, Proposition 6.14.1]), the form $\int_{\gamma_{\tau,P}}\omega$ is a closed form on $\mathcal{H}$. Seeing this, we define

Definition 3.6.13. Let $\Omega^{w+1}_{epl}(E^w,\mathbb{C})$ be the collection of complex-valued real-analytic closed $(w+1)$-form ω on $E^w(\mathbb{C})$ which verifies the following conditions

(1) $\nu^*\omega$ is of the form

$$\nu^*\omega=\sum_{\epsilon_1,\dots,\epsilon_w\in\{0,1\}}\omega_{\epsilon_1,\dots,\epsilon_w}\wedge dz_1^{(\epsilon_1)}\wedge\cdots\wedge dz_w^{(\epsilon_n)},$$

where $\omega_{\epsilon_1,\dots,\epsilon_n}$ is a 1-form on $\mathcal{H}$ and $dz_i^{(0)}=dz_i$, $dz_i^{(1)}=d\overline{z}_i$.

(2) We have $\int_{\gamma_{\tau,P}}\omega\in\Omega^1_{epl}(\mathcal{H},\mathbb{C})$ for every $P=\prod_{i=1}^w(a_iX+b_iY)\in\mathbb{Z}\langle X,Y\rangle_w$. That is, integration along the fiber of ω always gives us a closed 1-form with exp-poly-log expansions.

Here the first condition ensures the linearity of integration along extended modular symbols. The second condition allows us to define the regularized integrals. With these conditions, we can define

Definition 3.6.14. For a differential form $\omega\in\Omega^{w+1}_{epl}(E^w,\mathbb{C})$ and $P\{\alpha,\beta\}$, $P[\beta,\gamma]_\alpha$ a modular symbol and a modular cap in $\mathbb{K}_k$, we define symbolically the regularized integrals

$$\begin{aligned}\int^*_{P\{\alpha,\beta\}}\omega&=\sum_j a_j\int^*_{\{\alpha,\beta\}}\int_{\gamma_{\tau,P_j}}\omega,\\\int^*_{P[\beta,\gamma]_\alpha}\omega&=\sum_j a_j\int^*_{[\beta,\gamma]_\alpha}\int_{\gamma_{\tau,P_j}}\omega,\end{aligned}$$

where $P=\sum_j a_jP_j$ is a decomposition of P with $P_j=\prod_{i=1}^w(a_{j,i}X+b_{j,i}Y)$ for some $a_{j,i},b_{j,i}\in\mathbb{Z}$.

Now we arrive at

Lemma 3.6.15. *Let $\omega \in \Omega^{w+1}_{epl}(E^w, \mathbb{C})$ and $g \in \Gamma(N)$. Let $P\{\alpha, \beta\}$ and $P[\beta, \gamma]_\alpha$ be a modular symbol and modular cap in $\mathbb{K}_k$. Then*

$$\int^*_{P\{\alpha,\beta\}} \omega = \int^*_{(gP)\{g\alpha,g\beta\}} \omega,$$
$$\int^*_{P[\beta,\gamma]_\alpha} \omega = \int^*_{(gP)[g\beta,g\gamma]_{g\alpha}} \omega.$$

Proof. Given $P = \prod_{i=1}^w (a_i X + b_i Y)$ with $a_i, b_i \in \mathbb{Z}$ we denote the $(w+1)$-cycle

$$P\{\alpha, \beta\}^y = \{(\tau; t_1(a_1\tau + b_1), \ldots, t_w(a_w\tau + b_w); \sigma) \,|\, \tau \in \{\alpha, \beta\}^y,\, t_1, \ldots, t_w \in [0, 1]\}.$$

Now ω is invariant under the action of an element $g \in \Gamma(N)$. Thus

$$\int^*_{P\{\alpha,\beta\}} g^*\omega = \int^*_{P\{\alpha,\beta\}} \omega.$$

Then our regularized integral $\int^*_{P\{\alpha,\beta\}} \omega$ is the regularized value of the function

$$I(y) = \int_{P\{\alpha,\beta\}^y} \omega = \int_{g(P\{\alpha,\beta\}^y)} g^*\omega$$

as $y \to \infty$. Let $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma(N)$. Setting $\tau' = g\tau$, we find that $g(P\{\alpha, \beta\}^y)$ is the following $(\omega + 1)$-cycle

$$\left\{(\tau'; t_1 \frac{a_1 g^{-1}\tau' + b_1}{cg^{-1}\tau' + d}, \ldots, t_w \frac{a_w g^{-1}\tau' + b_w}{cg^{-1}\tau' + d}; g\sigma) \,\middle|\, \tau' \in \{g\alpha, g\beta\}^y,\, t_1, \ldots, t_w \in [0, 1]\right\}.$$

Since $t_i \frac{a_i g^{-1}\tau' + b_i}{cg^{-1}\tau' + d} = t_i(a_i(d\tau' - b) + b_i(-c\tau' + a))$, we have $g(P\{\alpha, \beta\}^y) = (gP)\{g\alpha, g\beta\}^y$. Thus,

$$I(y) = \int_{P\{\alpha,\beta\}^y} \omega = \int_{(gP)\{g\alpha,g\beta\}^y} \omega.$$

Take regularized value as $y \to \infty$ we see the identity. The same works for modular caps. □

Following the same pattern,

Theorem 3.6.16. *There is a well-defined pairing*

$$\begin{aligned} \Omega^{w+1}_{epl}(E^w, \mathbb{C}) \times \mathbb{K}_k(N) &\longrightarrow \mathbb{C} \\ (\omega, P\{\alpha, \beta\}) &\longmapsto \int^*_{P\{\alpha,\beta\}} \omega \\ (\omega, P[\beta, \gamma]_\alpha) &\longmapsto \int^*_{P[\beta,\gamma]_\alpha} \omega. \end{aligned}$$

Proof. All that remains for us to check is the 6-term relations. It follows directly from the 6-term relations of the form $\int_{\gamma_{\tau,P}} \omega \in \Omega^1_{epl}(\mathcal{H}, \mathbb{C})$. □

3.7 Application to Eisenstein Symbols

Proposition 3.7.1. *Let $t, u \in \mathbb{C}$. The S-series $S^{t,u}_{\alpha,\beta}(iy)$ has poly-log asymptotic expansions at $y = 0$ and ∞. Moreover, if t and u are both integers, the S-series $S^{t,u}_{\alpha,\beta}(iy) + (-1)^{t+u} S^{t,u}_{\alpha^-,\beta^-}(iy)$ has finite poly-log asymptotic expansions.*

Proof. When $\mathrm{Re}(s) \gg 0$, the S-series $S^{t,u}_{\alpha,\beta}(iy)$ has Mellin transform (see [11, Lemma 7.1])

$$F(s) = \left(\frac{2\pi}{N}\right)^{-s} \Gamma(s) L(\alpha, s-t) L(\beta, s-u).$$

It continues to a meromorphic function on the whole complex plane with only finite many poles from two L-functions, and it may have poles (possibly infinite many) of large negative integers from the gamma factor. But in either case, the Mellin transform has finitely many poles on every bounded strip and verifies the condition of Lemma 3.2.7, thus f has poly-log asymptotic expansions at $y = 0$ and ∞.

The Mellin transform of $S^{t,u}_{\alpha,\beta}(iy) + (-1)^{t+u} S^{t,u}_{\alpha^-,\beta^-}(iy)$ is

$$\left(\frac{2\pi}{N}\right)^{-s} \Gamma(s) \Big(L(\alpha, s-t) L(\beta, s-u) + (-1)^{t+u} L(\alpha^-, s-t) L(\beta^-, s-u) \Big).$$

If t and u are both integers, the L-value $L(\alpha, s-t)L(\beta, s-u) + (-1)^{t+u} L(\alpha^-, s-t)L(\beta^-, s-u)$ vanishes for large negative integer s, thus $S^{t,u}_{\alpha,\beta}(iy) + (-1)^{t+u} S^{t,u}_{\alpha^-,\beta^-}(iy)$ has finite poly-log asymptotic expansions. □

Since $S^{t,u}_{\alpha,\beta}(iy)$ has poly-log asymptotic expansions we see

Proposition 3.7.2. *For functions $\alpha, \beta : \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}$ and $t, u \in \mathbb{C}$. Let $s \in \mathbb{C}$, we have*

$$\int_0^{\infty,*} S^{t,u}_{\alpha,\beta}(iy) y^s \frac{dy}{y} = \left(\left(\frac{2\pi}{N}\right)^{-z} \Gamma(z) L(\alpha, z-t) L(\beta, z-u) \right) \Big|^*_{z=s}.$$

Recall the real analytic Eisenstein series

$$\mathrm{Eis}^n_{\mathcal{D}}(u_1) = -\frac{n!(n+2)}{2\pi N} \mathrm{Im}(\tau) \sum_{a=0}^{n} F^{a,n-a}_{gu_1}(\tau) \psi_{a,n-a} \mod d\tau, d\overline{\tau}$$

and the holomorphic Eisenstein series

$$\mathrm{Eis}^n_{hol}(u_1) = (-1)^{n+1} \frac{n+2}{N} (2i\pi)^{n+1} F^{(n+2)}_{\sigma g u_1}(\tau) d\tau \wedge \psi_{n,0}$$

can be expressed with S-series.

Proposition 3.7.3. *The element* $\mathrm{Eis}_{\mathcal{D}}^{k_1,k_2,j}(u_1,u_2)$ *belongs to the space* $\Omega^{w+1}_{epl}(E^w,\mathbb{C})$. *So there is a well-defined linear map*

$$\int_{-}^{*} \mathrm{Eis}_{\mathcal{D}}^{k_1,k_2,j}(u_1,u_2) : \mathbb{K}_k(N) \to \mathbb{C}.$$

Proof. Since for $g \in \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$ we have $g^* \mathrm{Eis}_{\mathcal{D}}^{k_1,k_2,j}(u_1,u_2) = \mathrm{Eis}_{\mathcal{D}}^{k_1,k_2,j}(u_1g,u_2g)$, it is enough to show that for all $u_1,u_2 \in (\mathbb{Z}/N\mathbb{Z})^2$ (with $u_i \neq 0$ if $k_i + j = 0$), the differential form $\int_{\gamma_{\tau,P}} \mathrm{Eis}_{\mathcal{D}}^{k_1,k_2,j}(u_1,u_2)$ has exp-poly-log expansion and residues at infinity. Let $P \in \mathbb{Z}\langle X,Y\rangle_w$ be a polynomial. Integrating the forms $dz_1^{(\epsilon_1)} \wedge \cdots \wedge dz_w^{(\epsilon_n)}$ on the fiber $\gamma_{\tau,P}$ gives us polynomials of τ and $\overline{\tau}$.

The form $\int_{\gamma_{\tau,P}} \mathrm{Eis}_{\mathcal{D}}^{k_1,k_2,j}(u_1,u_2)$ is a linear combination of products of τ, $\overline{\tau}$ and $S_{*,*}^{t,u}(\tau)$, $S_{*,*}^{t,u}(\overline{\tau})$. Since S-series decreasing exponentially at infinity ([23, Section 3]), the differential form $\int_{\gamma_{\tau,P}} \mathrm{Eis}_{\mathcal{D}}^{k_1,k_2,j}(u_1,u_2)$ has also poly-log expansion at ∞.

Every term in the Fourier expansion involving $q^{n/N} = e^{2\pi n\tau/N}$ with $n \geq 1$ tends to 0 as $y \to \infty$. Thus, the residue comes only from the constant terms of $\mathrm{Eis}_{\mathcal{D}}$ and Eis_{hol}. To compute a residue of a regulator at infinity is hence equivalent to compute the residue of a polynomial of τ and $\overline{\tau}$ at infinity. Thus, the residues always exist. □

Chapter 4

Double L-values

In this chapter, we discuss the double L-functions and the method of Rogers–Zudilin. We define and study the double L-functions of weakly holomorphic modular forms in Section 4.2, with our theory of generalized Mellin transform. In Section 4.3 we explain the method of Rogers–Zudilin. In the last Section 4.4, we use Rogers–Zudilin method to show that many double L-values of Eisenstein series are linear combinations of modular L-values.

4.1 Generalized Iterated Mellin Transform

In [33] Manin defined the multiple L-functions for several modular forms. Some special double L-values of Eisenstein series are discussed in Brown [9] and Shinder–Vlasenko [41]. We start first with introducing their definitions.

Let $f = \sum_{n\geq 0} a_n q^n$ and $g = \sum_{m\geq 0} b_m q^m$ be two modular forms of some congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$. Their *double L-function* is the following double Dirichlet series

$$L(f, g, s_1, s_2) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \frac{a_n b_m}{n^{s_1}(n+m)^{s_2}}.$$

We write $\mathbb{L}(f, g, s_1, s_2) = (2\pi)^{-s_2}\Gamma(s_2)L(f, g, s_1, s_2)$.

If $a_0 = 0$, there is another *double L-function* of f and g given by the following iterated integral (see [33])

$$\Lambda(f, g, s_1, s_2) = \int_0^{\infty} g(it_2)t_2^{s_2-1}dt_2 \int_{t_2}^{\infty} f(it_1)t_1^{s_1-1}dt_1.$$

The connection of two L-functions is given by Sreekantan's formula [43]. Let $a_0 = 0$ and s_1 be a positive integer. Then

$$\mathbb{L}(f, g, s_1, s_2) = \frac{(2\pi)^{s_1}}{\Gamma(s_1)} \sum_{r=0}^{s_1-1} \Lambda(f, g, s_1 - r, s_2 + r). \tag{4.1}$$

The double L-function is an *iterated Mellin transform* of the two functions f and g. In general the iterated Mellin transform is the following iterated integral

$$\mathcal{M}(f,g,s_1,s_2)=\int_0^\infty g(it_2)t_2^{s_2-1}dt_2\int_{t_2}^\infty f(it_1)t_1^{s_1-1}dt_1.$$

We manage to generalize this definition for functions with more general growth conditions.

Let $f(x)$ be a function with poly-log expansions. Recall that the primitive of $f(x)x^{s-1}$ also has poly-log expansions. We define

Definition 4.1.1. Let f and g be two functions with poly-log expansions at 0 and ∞. Then the *generalized iterated Mellin transform* of f and g is the following iterated regularized integral

$$\mathcal{M}(f,g,s_1,s_2)=\int_0^\infty g(it_2)t_2^{s_2-1}dt_2\int_{t_2}^{\infty,*} f(it_1)t_1^{s_1-1}dt_1.$$

Let $f(x)$ be a function with exp-poly expansions and $s\in\mathbb{C}$ such that the following condition is met

$$\begin{aligned} f(x)&\sim\sum_{\sigma_m,\lambda_n} a_{m,n}x^{\sigma_m}e^{\lambda_n x}+\text{some powers of } x \quad \text{as } x\to\infty,\\ f(x)&\sim\sum_{\sigma'_m,\lambda'_n} a'_{m,n}x^{\sigma'_m}e^{\lambda'_n/x}+\text{some powers of } x \quad \text{as } x\to 0 \qquad (\ddagger)\\ &\text{such that } \sigma_m+s \text{ and } -\sigma'_m-s \text{ are nonnegative integers.}\end{aligned}$$

We know from Remark 3.3.6 that the primitive of $f(x)x^{s-1}$ also has exp-poly expansions. So we define

Definition 4.1.2. Let f and g be two functions with exp-poly expansions. Let $s_1\in\mathbb{C}$ and f satisfying the condition (‡). Then the *generalized iterated Mellin transform* of f and g is the following iterated regularized integral

$$\mathcal{M}(f,g,s_1,s_2)=\int_0^\infty g(it_2)t_2^{s_2-1}dt_2\int_{t_2}^{\infty,*} f(it_1)t_1^{s_1-1}dt_1.$$

4.2 Double L-functions of Weakly Holomorphic Modular Forms

In this section we give an analogue of double L-functions to weakly holomorphic modular forms. The essential tool that we use is the generalized iterated Mellin transform introduced in the former section.

Let $f=\sum_{n\geq n_0}a_nq^n$ and $g=\sum_{m\geq m_0}b_mq^m$ be two weakly holomorphic modular forms of level $\mathrm{SL}_2(\mathbb{Z})$, with weight $k_1\geq 2$ and $k_2\geq 2$ respectively. Notice that the forms f and g have exp-poly expansions at infinity. Moreover, if $0<s_1<k_1$ is an integer, then f and s_1 satisfy the condition (‡). Hence, we define

Definition 4.2.1. Let $f, g \in M_k^!$ be two weakly holomorphic modular forms of level $\mathrm{SL}_2(\mathbb{Z})$ with weight $k_1 \geq 2$ and $k_2 \geq 2$ respectively. Let $0 < s_1 < k_1$ be an integer. The *double L-function* of f and g is the following generalized iterated Mellin transform

$$\Lambda(f, g, s_1, s_2) = \int_0^\infty g(it_2) t_2^{s_2-1} dt_2 \int_{t_2}^{\infty,*} f(it_1) t_1^{s_1-1} dt_1.$$

To compute double L-function, we split it again as before

$$\begin{aligned}\Lambda(f, g, s_1, s_2) &= \int_0^\infty g(it_2) t_2^{s_2-1} dt_2 \int_{t_2}^{\infty,*} f(it_1) t_1^{s_1-1} dt_1 \\ &= \int_1^\infty g(it_2) t_2^{s_2-1} dt_2 \int_{t_2}^{\infty,*} f(it_1) t_1^{s_1-1} dt_1 + \int_0^1 g(it_2) t_2^{s_2-1} dt_2 \int_{t_2}^{\infty,*} f(it_1) t_1^{s_1-1} dt_1.\end{aligned}$$

Here both integrals are evaluated in the sense of generalized Mellin transform. The left one is

$$\begin{aligned}L &= \int_1^\infty g(it_2) t_2^{s_2-1} dt_2 \int_{t_2}^{\infty,*} f(it_1) t_1^{s_1-1} dt_1 \\ &= \int_1^\infty g(it_2) t_2^{s_2-1} \left(\sum_{n \neq 0} \frac{a_n \Gamma(s_1, 2\pi n t_2)}{(2\pi n)^{s_1}} - a_0 \frac{t_2^{s_1}}{s_1} \right) dt_2.\end{aligned}$$

Since $\Gamma(s_1, x) = (s_1 - 1)! e^{-x} e_{s_1-1}(x)$, we see that L is

$$-\frac{a_0}{s_1} \int_1^\infty g(it_2) t_2^{s_1+s_2-1} dt_2 + (s_1 - 1)! \sum_{n,m} a_n b_m \sum_{r=0}^{s_1-1} \frac{1}{(2\pi n)^{s_1-r} r!} \int_1^\infty e^{-2\pi(n+m)t} t^{r+s_2-1} dt.$$

Explicitly,

$$\begin{aligned}L = &\sum_{r=0}^{s_1-1} \frac{1}{r!} \frac{(s_1-1)!}{(2\pi)^{s_1+s_2}} \sum_{\substack{n+m \neq 0 \\ n \neq 0}} \frac{a_n b_m \Gamma(s_2 + r, 2\pi(m+n))}{n^{s_1-r}(m+n)^{s_2+r}} - \sum_{r=0}^{s_1-1} \frac{1}{r!} \frac{(s_1-1)!}{(2\pi)^{s_1-r}} \sum_{\substack{n+m=0 \\ n \neq 0}} \frac{a_n b_m}{n^{s_1-r}} \frac{1}{s_2+r} \\ &- \frac{a_0}{(2\pi)^{s_1+s_2}} \sum_{m \neq 0} \frac{b_m \Gamma(s_1+s_2, 2\pi m)}{m^{s_1+s_2}} \frac{1}{s_1} + a_0 b_0 \frac{1}{s_1(s_1+s_2)}.\end{aligned}$$

Expressing with exponential integral it becomes

$$L = \sum_{r=0}^{s_1-1} \frac{(s_1-1)!}{r!} \sum_{\substack{n,m \\ n \neq 0}} \frac{a_n b_m E_{1-s_2-r}(2\pi(m+n))}{(2\pi n)^{s_1-r}} - \frac{1}{s_1} \sum_m a_0 b_m E_{1-s_1-s_2}(2\pi m),$$

For the right one we have

$$\begin{aligned}R =& \int_0^1 g(it_2)t_2^{s_2-1}dt_2 \int_{t_2}^{\infty,*} f(it_1)t_1^{s_1-1}dt_1\\ =& \int_0^1 g(it_2)t_2^{s_2-1}dt_2 \left(\int_0^{\infty,*} f(it_1)t_1^{s_1-1}dt_1 - \int_0^{t_2,*} f(it_1)t_1^{s_1-1}dt_1\right)\\ =& i^{k_2}\Lambda(f,s_1)\int_1^\infty g(it_2)t_2^{k_2-s_2-1}dt_2 - i^{k_1}\int_0^1 g(it_2)t_2^{s_2-1}dt_2\int_{1/t_2}^\infty f(it_1)t_1^{k_1-s_1-1}dt_1.\end{aligned}$$

So

$$\begin{aligned}R =& i^{k_2}\Lambda(f,s_1)\sum_{m\neq 0}\frac{b_m\Gamma(k_2-s_2,2\pi m)}{(2\pi m)^{k_2-s_2}} - i^{k_2}\Lambda(f,s_1)b_0\frac{1}{k_2-s_2}\\ &- i^{k_1}\int_0^1 g(it_2)t_2^{s_2-1}dt_2\int_{1/t_2}^\infty f(it_1)t_1^{k_1-s_1-1}dt_1\end{aligned}$$

The last line is

$$\begin{aligned}&- i^{k_1}\int_0^1 g(it_2)t_2^{s_2-1}\left(\sum_{n\neq 0}\frac{a_n\Gamma\left(k_1-s_1,\frac{2\pi n}{t_2}\right)}{(2\pi n)^{k_1-s_1}} - a_0\frac{t_2^{s_1-k_1}}{k_1-s_1}\right)dt_2\\ =& -i^{k_1+k_2}\int_1^\infty g(it_2)t_2^{k_2-s_2-1}\left(\sum_{n\neq 0}\frac{a_n\Gamma\left(k_1-s_1,2\pi n t_2\right)}{(2\pi n)^{k_1-s_1}} - a_0\frac{t_2^{k_1-s_1}}{k_1-s_1}\right)dt_2.\end{aligned}$$

Similar to L, we have

$$\begin{aligned}R =& i^{k_2}\Lambda(f,s_1)\sum_{m\neq 0}\frac{b_m\Gamma(k_2-s_2,2\pi m)}{(2\pi m)^{k_2-s_2}} - i^{k_2}\Lambda(f,s_1)b_0\frac{1}{k_2-s_2}\\ &- i^{k_1+k_2}\sum_{r=0}^{k_1-s_1-1}\frac{1}{r!}\frac{(k_1-s_1-1)!}{(2\pi)^{k_1+k_2-s_1-s_2}}\sum_{\substack{n+m\neq 0\\ n\neq 0}}\frac{a_nb_m\Gamma(k_2-s_2+r,2\pi(m+n))}{n^{k_1-s_1-r}(m+n)^{k_2-s_2+r}}\\ &+ i^{k_1+k_2}\sum_{r=0}^{k_1-s_1-1}\frac{1}{r!}\frac{(k_1-s_1-1)!}{(2\pi)^{k_1-s_1-r}}\sum_{\substack{n+m=0\\ n\neq 0}}\frac{a_nb_m}{n^{k_1-s_1-r}}\frac{1}{k_2-s_2+r}\\ &+ i^{k_1+k_2}\frac{a_0}{(2\pi)^{k_1+k_2-s_1-s_2}}\sum_{m\neq 0}\frac{b_m\Gamma(k_1+k_2-s_1-s_2,2\pi m)}{m^{k_1+k_2-s_1-s_2}}\frac{1}{k_1-s_1}\\ &- i^{k_1+k_2}a_0b_0\frac{1}{(k_1-s_1)(k_1+k_2-s_1-s_2)}.\end{aligned}$$

The result can be shorter with exponential integral functions

$$\begin{aligned} R =& i^{k_2}\Lambda(f,s_1)\sum_m b_m E_{1-k_2+s_2}(2\pi m) \\ &- i^{k_1+k_2}\sum_{r=0}^{k_1-s_1}\frac{(k_1-s_1-1)!}{r!}\sum_{\substack{n,m\\ n\neq 0}}\frac{a_n b_m E_{1-k_2+s_2-r}\left(2\pi(m+n)\right)}{(2\pi n)^{k_1-s_1-r}} \\ &+ \frac{i^{k_1+k_2}}{k_1-s_1}\sum_m a_0 b_m E_{1+s_1+s_2-k_1-k_2}(2\pi m). \end{aligned}$$

In conclusion,

Theorem 4.2.2. *Let $f=\sum_{n\geq n_0}a_nq^n$ and $g=\sum_{m\geq m_0}b_mq^m$ be two weakly holomorphic modular forms of weight $k_1\geq 2$ and $k_2\geq 2$ respectively. Let $0<s_1<k_1$ be an integer. Then*

1. *The double L-function $\Lambda(f,g,s_1,s_2)$, as a function of s_2, extends to a meromorphic function on the whole s_2-plane.*
2. *As a function of s_2, $\Lambda(f,g,s_1,s_2)$ has possibly poles when s_2 is an integer from $-s_1$ to 0 or from k_2 to $k_2+k_1-s_1$, and $\Lambda(f,g,s_1,s_2)$ is holomorphic elsewhere.*
3. *We have residues*

$$\operatorname*{Res}_{s_2=-s_1}\Lambda(f,g,s_1,s_2)=\frac{a_0b_0}{s_1},\qquad \operatorname*{Res}_{s_2=k_2+k_2-s_1}\Lambda(f,g,s_1,s_2)=i^{k_1+k_2}\frac{a_0b_0}{k_1-s_1}.$$

4. *Suppose $a_0=0$, $k_1=k_2=k$ then*

$$\operatorname*{Res}_{s_2=0}\Lambda(f,g,k-1,s_2)=-\frac{(k-2)!}{(2\pi)^{k-1}}\{f,g\},$$

$$\operatorname*{Res}_{s_2=k_2}\Lambda(f,g,1,s_2)=-\frac{(k-2)!}{(2\pi)^{k-1}}\{f,g\}+i^k\Lambda(f,1)b_0,$$

where $\{f,g\}=\sum_{n\in\mathbb{Z}}\frac{a_nb_{-n}}{n^{k-1}}$ is the Bruinier-Funke pairing of f and g defined in [7, (1.15)].

Remark 4.2.3. It is interesting to observe that Bruinier-Funke pairing appears in the residue. Here we give a clue. The pairing $\{f,g\}=0$ for all $g\in S_k^!$ is equivalent to say that $D^{1-k}f$ is a weakly holomorphic form of weight $2-k$. Then $g\cdot D^{1-k}f$ is exactly a weakly holomorphic form of weight 2, which must be cuspidal. Therefore there is no pole in the L-function $\Lambda(f,g,k-1,s_2)$.

Following the identity (4.1), when $a_0 = 0$ we define another double L-function of f and g to be

$$\mathbb{L}(f, g, s_1, s_2) = \frac{(2\pi)^{s_1}}{\Gamma(s_1)} \sum_{t=0}^{s_1-1} \Lambda(f, g, s_1 - t, s_2 + t).$$

We have immediately

Theorem 4.2.4. *Let $f = \sum_{n\geq n_0} a_n q^n$ and $g = \sum_{m\geq m_0} b_m q^m$ be two weakly holomorphic modular forms of weight $k_1 \geq 2$ and $k_2 \geq 2$ respectively. Suppose $a_0 = 0$. Let $0 < s_1 < k_1$ be an integer. Then as a function of s_2, the double L-function $\mathbb{L}(f, g, s_1, s_2)$ can be extended meromorphically to the entire s_2-plane. It has possibly poles when s_2 is an integer from $-s_1$ to 0 or from k_2 to $k_2 + k_1 - s_1$, and holomorphic elsewhere.*

4.3 Rogers–Zudilin Method

Rogers and Zudilin [40] introduce a powerful tool to calculate Mahler measures and L-series. Their method has various applications, especially in evaluating double L-values of Eisenstein series. We will introduce in this section their method.

Lemma 4.3.1. *Let $t_1, t_2, u_1, u_2, s \in \mathbb{C}$ be complex numbers and $\alpha_1, \alpha_2, \beta_1, \beta_2 : \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}$ be functions. We have*

$$\int_0^\infty S^{t_1,u_1}_{\alpha_1,\beta_1}\left(\frac{i}{y}\right) S^{t_2,u_2}_{\alpha_2,\beta_2}(iy) y^s \frac{dy}{y} = \int_0^\infty S^{t_1+s,t_2}_{\alpha_1,\alpha_2}(iy) S^{u_1,u_2-s}_{\alpha_2,\beta_2}\left(\frac{i}{y}\right) y^s \frac{dy}{y}.$$

Proof. (See [24, Theorem 3.2]) This is equivalent to

$$\int_0^\infty S^{t_1,u_1}_{\alpha_1,\beta_1}\left(\frac{i}{Ny}\right) S^{t_2,u_2}_{\alpha_2,\beta_2}(iNy) y^s \frac{dy}{y} = \int_0^\infty S^{t_1+s,t_2}_{\alpha_1,\alpha_2}(iNy) S^{u_1,u_2-s}_{\alpha_2,\beta_2}\left(\frac{i}{Ny}\right) y^s \frac{dy}{y}.$$

Upon the change of variable $y \to \frac{m_1}{n_2} y$, we find

$$\begin{aligned}\int_0^\infty \alpha_1(m_1)\beta_1(n_1)\alpha_2(m_2)\beta_2(n_2) e^{-2\pi \frac{m_1 n_1}{N^2 y}} e^{-2\pi m_2 n_2 y} y^s \frac{dy}{y} \\ = \left(\frac{m_1}{n_2}\right)^s \int_0^\infty \alpha_1(m_1)\alpha_2(m_2)\beta_1(n_1)\beta_2(n_2) e^{-2\pi m_1 m_2 y} e^{-2\pi \frac{n_1 n_2}{N^2 y}} y^s \frac{dy}{y}.\end{aligned}$$

We finish the proof by summing up every term. □

Rogers–Zudilin Method can be seen as a way to exchange L-functions of Eisenstein series. We provide a simple heuristic interpretation of this. Consider the following Mellin transforms of S-series

$$F_1(v) = \int_0^\infty S^{t_1,u_1}_{\alpha_1,\beta_1}(iy) y^v \frac{dy}{y} = \left(\frac{2\pi}{N}\right)^{-v} \Gamma(v) L(\alpha_1, v - t_1) L(\beta_1, v - u_1)$$

and

$$F_2(v)=\int_0^\infty S^{t_2,u_2}_{\alpha_2,\beta_2}(iy)y^v\frac{dy}{y}=\left(\frac{2\pi}{N}\right)^{-v}\Gamma(v)L(\alpha_2,v-t_2)L(\beta_2,v-u_2).$$

With inverse Mellin transform we see their convolution is

$$\int_0^\infty S^{t_1,u_1}_{\alpha_1,\beta_1}\left(\frac{i}{y}\right)S^{t_2,u_2}_{\alpha_2,\beta_2}(iy)y^s\frac{dy}{y}=\int_{c-i\infty}^{c+i\infty}F_1(-v)F_2(v+s)dv.$$

Pay attention that the Mellin transforms $F_1(-v)$ and $F_2(v+s)$ should both exist when $\mathrm{Re}(v)=c$ (this may not happen in general!). Exchanging the L-functions we get

$$\begin{aligned}&\left(\frac{2\pi}{N}\right)^{-s}\int_{c-i\infty}^{c+i\infty}\Gamma(-v)L(\alpha_1,-v-t_1)L(\beta_1,-v-u_1)\\&\qquad\cdot\Gamma(v+s)L(\alpha_2,v+s-t_2)L(\beta_2,v+s-u_2)dv\\=&\left(\frac{2\pi}{N}\right)^{-s}\int_{c-i\infty}^{c+i\infty}\Gamma(-v)L(\beta_1,-v-u_1)L(\beta_2,v+s-u_2)\\&\qquad\cdot\Gamma(v+s)L(\alpha_1,-v-t_1)L(\alpha_2,v+s-t_2)dv.\end{aligned}$$

The latter is exactly

$$\int_0^\infty S^{t_1+s,t_2}_{\alpha_1,\alpha_2}(iy)S^{u_1,u_2-s}_{\beta_1,\beta_2}\left(\frac{i}{y}\right)y^s\frac{dy}{y}.$$

We give now the following variation of Rogers–Zudilin method which can simplify a lot of things in our later calculations.

Lemma 4.3.2. *Let $\alpha_1,\alpha_2,\beta_1,\beta_2:\mathbb{Z}/N\mathbb{Z}\to\mathbb{C}$ be functions. Let t_1,u_1,u_2 and s be integers with $1\le -u_1\le s\le u_2$ and $t_1\ge 0$. Suppose further that $L(\beta_1,z)$ has no poles when $u_1=-1$ and $L(\alpha_2,z)$ has no poles when $s=1$. Then*

$$\begin{aligned}&\int_0^{\infty,*}\left(\mathcal{S}^{t_1,u_1}_{\alpha_1,\beta_1}+(-1)^{t_1+u_1+1}\mathcal{S}^{t_1,u_1}_{\alpha_1^-,\beta_1^-}\right)\left(\frac{i}{y}\right)\left(\mathcal{S}^{0,u_2}_{\alpha_2,\beta_2}+(-1)^{u_2+1}\mathcal{S}^{0,u_2}_{\alpha_2^-,\beta_2^-}\right)(iy)\,y^s\\&+(-1)^{t_1+s}\left(\mathcal{S}^{t_1,u_1}_{\alpha_1^-,\beta_1}+(-1)^{t_1+u_1+1}\mathcal{S}^{t_1,u_1}_{\alpha_1,\beta_1^-}\right)\left(\frac{i}{y}\right)\left(\mathcal{S}^{0,u_2}_{\alpha_2^-,\beta_2}+(-1)^{u_2+1}\mathcal{S}^{0,u_2}_{\alpha_2,\beta_2^-}\right)(iy)\,y^s\frac{dy}{y}\\=&\int_0^{\infty,*}\left(\mathcal{S}^{0,u_1+s}_{\alpha_2,\beta_1}+(-1)^{u_1+s+1}\mathcal{S}^{0,u_1+s}_{\alpha_2^-,\beta_1^-}\right)(iy)\left(\mathcal{S}^{t_1,u_2-s}_{\alpha_1,\beta_2}+(-1)^{t_1+u_2+s+1}\mathcal{S}^{t_1,u_2-s}_{\alpha_1^-,\beta_2^-}\right)\left(\frac{i}{y}\right)y^s\\&+(-1)^{t_1+s}\left(\mathcal{S}^{0,u_1+s}_{\alpha_2^-,\beta_1}+(-1)^{u_1+s+1}\mathcal{S}^{0,u_1+s}_{\alpha_2,\beta_1^-}\right)(iy)\left(\mathcal{S}^{t_1,u_2-s}_{\alpha_1^-,\beta_2}+(-1)^{t_1+u_2+s+1}\mathcal{S}^{t_1,u_2-s}_{\alpha_1,\beta_2^-}\right)\left(\frac{i}{y}\right)y^s\frac{dy}{y}.\end{aligned}$$

Proof. Note that

$$\int_0^{\infty,*}\left(S^{t_1,u_1}_{\alpha_1,\beta_1}\left(\frac{i}{y}\right)+\frac{1}{2}\delta_{t_1=0}\alpha_1(0)L(\beta_1,-u_1)\right)\left(\mathcal{S}^{0,u_2}_{\alpha_2,\beta_2}(iy)+\frac{1}{2}\alpha_2(0)L(\beta_2,-u_2)\right)y^s\frac{dy}{y}$$

is the sum of the integral

$$\int_0^\infty S^{t_1,u_1}_{\alpha_1,\beta_1}\left(\frac{i}{y}\right) S^{0,u_2}_{\alpha_2,\beta_2}(iy)\, y^s \frac{dy}{y}$$

and two regularized values

$$\left(\frac{1}{2}\left(\frac{2\pi}{N}\right)^z \Gamma(-z)L(\alpha_1,-z-t_1)L(\beta_1,-z-u_1)L(\beta_2,-u_2)\alpha_2(0)\right)\Big|^*_{z=s}$$
$$+\left(\frac{1}{2}\left(\frac{2\pi}{N}\right)^{-z} \Gamma(z)\delta_{t_1=0}L(\beta_1,-u_1)L(\alpha_2,z)L(\beta_2,z-u_2)\alpha_1(0)\right)\Big|^*_{z=s}.$$

There are exactly eight products like this, they sum altogether to become

$$\int_0^{\infty,*}\left(S^{t_1,u_1}_{\alpha_1,\beta_1}+(-1)^{t_1+u_1+1}S^{t_1,u_1}_{\alpha_1^-,\beta_1^-}\right)\left(\frac{i}{y}\right)\left(S^{0,u_2}_{\alpha_2,\beta_2}+(-1)^{u_2+1}S^{0,u_2}_{\alpha_2^-,\beta_2^-}\right)(iy)\,y^s\frac{dy}{y}$$
$$+(-1)^{t_1+s}\int_0^{\infty,*}\left(S^{t_1,u_1}_{\alpha_1^-,\beta_1}+(-1)^{t_1+u_1+1}S^{t_1,u_1}_{\alpha_1,\beta_1^-}\right)\left(\frac{i}{y}\right)\left(S^{0,u_2}_{\alpha_2^-,\beta_2}+(-1)^{t_1+u_2+s+1}S^{0,u_2}_{\alpha_2,\beta_2^-}\right)(iy)\,y^s\frac{dy}{y}$$

and two regularized values

$$\begin{aligned}&\frac{1}{2}\left(\frac{2\pi}{N}\right)^s \alpha_2(0)L(\beta_2+(-1)^{u_2+1}\beta_2^-,-u_2)\\ &\qquad\cdot\left(\Gamma(-z)L(\alpha_1+(-1)^{t_1+s}\alpha_1^-,-z-t_1)L(\beta_1+(-1)^{u_1+s+1}\beta_1^-,-z-u_1)\right)\Big|^*_{z=s}\\ &+\frac{1}{2}\left(\frac{2\pi}{N}\right)^{-s}\delta_{t_1=0}\alpha_1(0)L(\beta_1+(-1)^{u_1+1}\beta_1^-,-u_1)\\ &\qquad\cdot\left(\Gamma(z)L(\alpha_2+(-1)^{s}\alpha_2^-,z)L(\beta_2+(-1)^{u_1+s+1}\beta_2^-,z-u_2)\right)\Big|^*_{z=s}.\end{aligned}\tag{4.2}$$

Applying Rogers–Zudilin method to the products $S^{t_1,u_1}_{*,*}(\frac{i}{y})S^{u_2,0}_{*,*}(iy)$, we see that the integral becomes

$$\int_0^{\infty,*}\left(S^{0,u_1+s}_{\alpha_2,\beta_1}+(-1)^{u_1+s+1}S^{0,u_1+s}_{\alpha_2^-,\beta_1^-}\right)(iy)\left(S^{t_1,u_2-s}_{\alpha_1,\beta_2}+(-1)^{t_1+u_2+s+1}S^{t_1,u_2-s}_{\alpha_1^-,\beta_2^-}\right)\left(\frac{i}{y}\right)y^s\frac{dy}{y}$$
$$+(-1)^{t_1+s}\int_0^{\infty,*}\left(S^{0,u_1+s}_{\alpha_2^-,\beta_1}+(-1)^{u_1+s+1}S^{0,u_1+s}_{\alpha_2,\beta_1^-}\right)(iy)\left(S^{t_1,u_2-s}_{\alpha_1^-,\beta_2}+(-1)^{t_1+u_2+s+1}S^{t_1,u_2-s}_{\alpha_1,\beta_2^-}\right)\left(\frac{i}{y}\right)y^s\frac{dy}{y}.$$

Since $L(\alpha_1+(-1)^{t_1+s}\alpha_1^-,-s-t_1)=0$ and the only poles in (4.2) comes from $\Gamma(-z)$, we can exchange the L-functions

$$\begin{aligned}&\alpha_2(0)L(\beta_2+(-1)^{u_2+1}\beta_2^-,-u_2)\\ &\qquad\cdot\left(\Gamma(-z)L(\alpha_1+(-1)^{t_1+s}\alpha_1^-,-z-t_1)L(\beta_1+(-1)^{u_1+s+1}\beta_1^-,-z-u_1)\right)\Big|^*_{z=s}\\ =&\alpha_2(0)L(\beta_1+(-1)^{u_1+s+1}\beta_1^-,-s-u_1)\\ &\qquad\cdot\left(\Gamma(-z)L(\alpha_1+(-1)^{t_1+s}\alpha_1^-,-z-t_1)L(\beta_2+(-1)^{u_2+1}\beta_2^-,-z-u_2+s)\right)\Big|^*_{z=s}.\end{aligned}$$

Similarly,

$$\begin{aligned}
&\alpha_1(0)L(\beta_1+(-1)^{u_1+1}\beta_1^-,-u_1)\\
&\qquad\cdot\left(\Gamma(z)L(\alpha_2+(-1)^s\alpha_2^-,z)L(\beta_2+(-1)^{u_1+s+1}\beta_2^-,z-u_2)\right)\Big|_{z=s}^*\\
=&\alpha_1(0)L(\beta_2+(-1)^{t_1+u_1+s+1}\beta_2^-,s-u_2)\\
&\qquad\cdot\left(\Gamma(z)L(\alpha_2+(-1)^s\alpha_2^-,z)L(\beta_1+(-1)^{u_1+1}\beta_1^-,z-u_1-s)\right)\Big|_{z=s}^*.
\end{aligned}$$

After collecting everything we get indeed

$$\begin{aligned}
&\int_0^{\infty,*}\left(\mathcal{S}^{0,u_1+s}_{\alpha_2,\beta_1}+(-1)^{u_1+s+1}\mathcal{S}^{0,u_1+s}_{\alpha_2^-,\beta_1^-}\right)(iy)\left(\mathcal{S}^{t_1,u_2-s}_{\alpha_1,\beta_2}+(-1)^{t_1+u_2+s+1}\mathcal{S}^{t_1,u_2-s}_{\alpha_1^-,\beta_2^-}\right)\left(\frac{i}{y}\right)y^s\frac{dy}{y}\\
+&(-1)^{t_1+s}\int_0^{\infty,*}\left(\mathcal{S}^{0,u_1+s}_{\alpha_2^-,\beta_1}+(-1)^{u_1+s+1}\mathcal{S}^{0,u_1+s}_{\alpha_2,\beta_1^-}\right)(iy)\left(\mathcal{S}^{t_1,u_2-s}_{\alpha_1^-,\beta_2}+(-1)^{t_1+u_2+s+1}\mathcal{S}^{t_1,u_2-s}_{\alpha_1,\beta_2^-}\right)\left(\frac{i}{y}\right)y^s\frac{dy}{y}.
\end{aligned}$$

□

4.4 Double L-values of Eisenstein Series

With Rogers–Zudilin method we are able to give some interesting identities about double L-values.

Let $N\geq 1$ be an integer. Let $f=\sum_{n\geq 0}a_nq^n$ and $g=\sum_{m\geq 0}b_mq^m$ be two modular forms of level $\Gamma_1(N)$, weight k_1 and k_2 respectively. Their double L-function

$$\begin{aligned}
\mathbb{L}(f,g,s_1,s_2)=&\frac{\Gamma(s_2)}{(2\pi)^{s_2}}L(f,g,s_1,s_2)\\
=&\frac{\Gamma(s_2)}{(2\pi)^{s_2}}\sum_{n=1}^{\infty}\frac{a_nb_m}{n^{s_1}(n+m)^{s_2}}
\end{aligned}$$

converges for s_1 and s_2 sufficiently large.

Following Shinder and Vlasenko [41], given $0<s_1<k_1$ and $s_2\in\mathbb{C}$ with $\mathrm{Re}(s)\gg 0$, we are capable to write the double L-function $\mathbb{L}(f,g,s_1,s_2)$ as an integral

$$\begin{aligned}
\mathbb{L}(f,g,s_1,s_2)=&\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\frac{a_nb_m}{n^{s_1}(n+m)^{s_2}}\int_0^{\infty}y^{s_2-1}e^{-2\pi(m+n)y}dy\\
=&\int_0^{\infty}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\frac{a_nb_m}{n^{s_1}}e^{-2\pi(m+n)y}y^{s_2}\frac{dy}{y}\\
=&\int_0^{\infty}\left(g\cdot D^{-s_1}f^*\right)(iy)y^{s_2}\frac{dy}{y},
\end{aligned}$$

where $D^{-s_1}f^*=\sum_{n>0}n^{-s_1}a_nq^n$ is the repeated primitive of $f^*=f-a_0$.

Noticing a quasi-modular form or a repeated primitive of modular form must have poly-log expansions at 0 and ∞, we make the following definition

Definition 4.4.1. Fixing $s_1 \in \mathbb{Z}$, the double L-function $\mathbb{L}(f, g, s_1, s_2)$ is defined as the generalized Mellin transform

$$\mathbb{L}(f, g, s_1, s_2) = \int_0^\infty \left(g \cdot D^{-s_1} f^*\right)(iy) y^{s_2} \frac{dy}{y}.$$

Generally $\mathbb{L}(f, g, s_1, s_2)$ is meromorphic as a function of s_2. Denote by $\mathbb{L}^*(f, g, s_1, s_2)$ as its regularized value at a given s_2.

A direct corollary of the Rogers–Zudilin method in Lemma 4.3.2 is

Theorem 4.4.2. *Let $a_1, a_2, b_1, b_2 \in \mathbb{Z}/N\mathbb{Z}$ and $k_1 \geq 2$, $k_2 \geq 2$ be positive integers. Let $1 \leq s_1 \leq k_1 - 1$, $1 \leq s_2 \leq k_2 - 1$ be integers with $k_1 \leq s_1 + s_2$. If $s_1 + s_2 \leq k_2$ then*

$$\begin{aligned}
&i^{k_2} N^{k_2-1} \mathbb{L}^*\left(G^{(k_1)}_{a_1,b_1}, H^{(k_2)}_{b_2,a_2}, k_1 - s_1, k_2 - s_2\right) \\
&\quad + (-1)^{s_1+s_2-1} i^{k_2} N^{k_2-1} \mathbb{L}^*\left(G^{(k_1)}_{-a_1,b_1}, H^{(k_2)}_{b_2,-a_2}, k_1 - s_1, k_2 - s_2\right) \\
=& i^{s_1+s_2-k_1+1} N^{s_1+s_2-k_1} \mathbb{L}^*\left(G^{(k_2-s_2-s_1+1)}_{b_2,a_1}, H^{(s_1+s_2-k_1+1)}_{b_1,a_2}, 1 - s_1, s_1 - k_1 + 1\right) \\
&\quad + (-1)^{s_1+s_2-1} i^{s_1+s_2-k_1+1} N^{s_1+s_2-k_1} \mathbb{L}^*\left(G^{(k_2-s_2-s_1+1)}_{b_2,-a_1}, H^{(s_1+s_2-k_1+1)}_{b_1,-a_2}, 1 - s_1, s_1 - k_1 + 1\right) \\
&\quad + \frac{1}{2}\delta_{s_1+s_2}(k_1)\delta_{b_1}(0) L(\delta_{a_2} - \delta_{-a_2}, 0) \Lambda^*\left(D^{s_1-1}\left(G^{(k_2-k_1+1)}_{b_2,a_1} + (-1)^{k_1} G^{(k_2-k_1+1)}_{b_2,-a_1}\right), s_1 - k_1\right).
\end{aligned}$$

If $s_1 + s_2 > k_2$ then

$$\begin{aligned}
&i^{k_2} N^{k_2-1} \mathbb{L}^*\left(G^{(k_1)}_{a_1,b_1}, H^{(k_2)}_{b_2,a_2}, k_1 - s_1, k_2 - s_2\right) \\
&\quad + (-1)^{s_1+s_2-1} i^{k_2} N^{k_2-1} \mathbb{L}^*\left(G^{(k_1)}_{-a_1,b_1}, H^{(k_2)}_{b_2,-a_2}, k_1 - s_1, k_2 - s_2\right) \\
=& i^{s_1+s_2-k_1+1} N^{s_1+s_2-k_1} \mathbb{L}^*\left(G^{(s_1+s_2-k_2+1)}_{a_1,b_2}, H^{(s_1+s_2-k_1+1)}_{b_1,a_2}, s_2 - k_2 + 1, s_1 - k_1 + 1\right) \\
&\quad + (-1)^{s_1+s_2-1} i^{s_1+s_2-k_1+1} N^{s_1+s_2-k_1} \mathbb{L}^*\left(G^{(s_1+s_2-k_2+1)}_{-a_1,b_2}, H^{(s_1+s_2-k_1+1)}_{b_1,-a_2}, s_2 - k_2 + 1, s_1 - k_1 + 1\right) \\
&\quad + \frac{1}{2}\delta_{s_1+s_2}(k_1)\delta_{b_1}(0) L(\delta_{a_2} - \delta_{-a_2}, 0) \Lambda^*\left(D^{k_2-s_1-1}\left(G^{(k_1-k_2+1)}_{a_1,b_2} + (-1)^{k_1} G^{(k_1-k_2+1)}_{b_2,-a_1}\right), s_1 - k_1\right).
\end{aligned}$$

All the G's and H's here are supposed to be Eisenstein series.

Remark 4.4.3. It is worth noticing that the latter double L-values in the above proposition are nothing but L-values of quasi-modular forms. Moreover, if $k_i = 1$ or some G, H are quasi-modular of weight 2, Theorem 4.4.2 remains true, in the sense of modulo L-values of Eisenstein series. In these cases there will be some lost constant terms in Eisenstein series of weight 1 or 2.

In particular if $N = 1$ let $\mathbb{G}_k(\tau) = -\frac{B_k}{2k} + \sum_{n=1}^\infty \sigma_{k-1}(n) q^n$ be the Eisenstein series of weight k. With Theorem 4.4.2 and Lanphier's formula (1.2), we obtain

Theorem 4.4.4. *Let $k_1 \geq 4, k_2 \geq 4$ be even number. Let $1 \leq s_1 \leq k_1 - 1$ and $1 \leq s_2 \leq k_2 - s_1$ be integers with opposite parity. Set $p = \min\{k_1 - s_1, s_2\} - 1$. Then*

$$i^{s_1+s_2-1}\mathbb{L}^*\left(\mathbb{G}_{k_1}, \mathbb{G}_{k_2}, s_1, s_2\right) = \Lambda^*\left(D^p\mathbb{G}_{|k_1-s_1-s_2|+1} \cdot \mathbb{G}_{k_2-s_1-s_2+1}, 1 - s_1\right) + \delta_{s_1+s_2=k_2-1}(4\pi)^{-1}\Lambda^*\left(D^p\mathbb{G}_{|k_1-s_1-s_2|+1}, -s_1\right).$$

Moreover, the double L-value $\mathbb{L}^*(\mathbb{G}_{k_1}, \mathbb{G}_{k_2}, s_1, s_2)$ is a $\mathbb{Q}[1/\pi]$-linear combination of L-values of modular forms with rational coefficients and L-values of $\mathbb{G}_2$

$$\mathbb{L}^*(\mathbb{G}_{k_1}, \mathbb{G}_{k_2}, s_1, s_2) \in \sum_{l=0}^{p} \frac{\mathbb{Q}\,\Lambda\left(\left[\mathbb{G}_{|k_1-s_1-s_2|+1}, \mathbb{G}_{k_2-s_1-s_2+1}\right]_{p-l}^{mod}, 1 - s_1 - l\right)}{\pi^l} + \delta_{s_1+s_2=k_2-1}\frac{\mathbb{Q}\,\Lambda\left(\mathbb{G}_{|k_1-s_1-s_2|+1}, -s_1 - p\right)}{\pi^{p+1}} + \delta_{s_1+s_2=k_1\pm1}\frac{\mathbb{Q}\,\Lambda\left(\mathbb{G}_{k_2-s_1-s_2+1}, -s_1 - p\right)}{\pi^{p+1}}.$$

Here we give some concrete examples of double L-values in level $\mathrm{SL}_2(\mathbb{Z})$.

Example 4.4.5. *Let $k_1 = 6$, $k_2 = 6$. Noticing there is no cusp form of weight 10, we have*

$$\begin{aligned}\mathbb{L}^*(\mathbb{G}_6, \mathbb{G}_6, 2, 1) =& -\mathbb{L}^*(\mathbb{G}_4, \mathbb{G}_4, 0, -1)\\ =& -\Lambda^*(\mathbb{G}_4 \cdot \mathbb{G}_4, -1)\\ =& -\frac{1}{120}\Lambda(\mathbb{G}_8, -1)\end{aligned}$$

and

$$\begin{aligned}\mathbb{L}^*(\mathbb{G}_6, \mathbb{G}_6, 2, 3) =& \mathbb{L}^*(\mathbb{G}_2, \mathbb{G}_2, -2, -1) + (4\pi)^{-1}\Lambda^*(D^2\mathbb{G}_2, -2)\\ =& \Lambda^*(D^2\mathbb{G}_2 \cdot \mathbb{G}_2, -1) + (4\pi)^{-1}\Lambda^*(D^2\mathbb{G}_2, -2)\\ =& \frac{5}{4\pi^3}\Lambda(\mathbb{G}_2, -4) + \frac{3}{16\pi^2}\Lambda(\mathbb{G}_4, -3).\end{aligned}$$

Example 4.4.6. *Let $k_1 = 6$, $k_2 = 8$. Then*

$$\begin{aligned}\mathbb{L}^*(\mathbb{G}_6, \mathbb{G}_8, 1, 2) =& -\mathbb{L}^*(\mathbb{G}_4, \mathbb{G}_6, -1, 0)\\ =& -\mathbb{L}^*(D\mathbb{G}_4 \cdot \mathbb{G}_6, 0)\\ =& \frac{11}{25200\pi}\Lambda(\mathbb{G}_{10}, -1) + \frac{1}{350}\Lambda(\Delta, 0)\end{aligned}$$

and

$$\begin{aligned}\mathbb{L}^*(\mathbb{G}_6, \mathbb{G}_8, 1, 4) =& \mathbb{L}^*(\mathbb{G}_2, \mathbb{G}_4, -3, 0)\\ =& \mathbb{L}^*(D^3\mathbb{G}_2 \cdot \mathbb{G}_4, 0)\\ =& -\frac{3}{560\pi^4}\Lambda(\mathbb{G}_4, -4) - \frac{3}{640\pi^3}\Lambda(\mathbb{G}_6, -3) + \frac{1}{672}\Lambda(\Delta, 0),\end{aligned}$$

where Δ is the unique normalized Hecke eigenform of weight 12.

Example 4.4.7. *Let s_1 and s_2 verify the conditions in Theorem 4.4.4. Assume that $s_1 = k_1 - 1$ or $s_2 = 1$. Then*

$$\mathbb{L}^*(\mathbb{G}_{k_1}, \mathbb{G}_{k_2}, s_1, 1) = i^{s_1}\Lambda(\mathbb{G}_{k_1-s_1} \cdot \mathbb{G}_{k_2-s_1}, 1 - s_1) - \delta_{s_1=k_2-2} i^{k_2}(4\pi)^{-1}\Lambda(\mathbb{G}_{k_1-s_1}, -s_1),$$

and

$$\begin{aligned}\mathbb{L}^*(\mathbb{G}_{k_1}, \mathbb{G}_{k_2}, k_1 - 1, s_2) = -i^{k_1+s_2}\Lambda(\mathbb{G}_{s_2} \cdot \mathbb{G}_{k_2-k_1-s_2+2}, 2 - k_1) \\ - \delta_{s_2=k_2-k_1} i^{k_2}(4\pi)^{-1}\Lambda(\mathbb{G}_{k_2-k_1-s_2+2}, 1 - k_1).\end{aligned}$$

A list of extra examples about double L-values is given in Appendix A.

Remark 4.4.8. If s_1 and s_2 have same parity, then the double L-value $\mathbb{L}^*(\mathbb{G}_{k_1}, \mathbb{G}_{k_2}, s_1, s_2)$ maybe not modular. Such examples can be find in [9].

Chapter 5

Mordell–Tornheim Double Eisenstein Series

The Mordell–Tornheim double zeta function, originally defined by Tornheim [45], is the double series

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{n^{k_1}m^{k_2}(n+m)^{k_3}},$$

where k_1, k_2 and k_3 are both non-negative integers. He studied in detail the convergence condition of such series and their values. It can be seen as a double version of the Riemann ζ-function.

We manage to construct a double version of the original Eisenstein series

$$G_k(\tau)=\sum_{\omega\in\mathbb{Z}+\mathbb{Z}\tau}{}'\frac{1}{\omega^k}.$$

The double Eisenstein series of Mordell–Tornheim type is the following series

$$G\left(\tau;k_1,k_2,k_3;\omega_1,\omega_2\right)=\sum_{\tau_1,\tau_2\in\mathbb{Z}+\mathbb{Z}\tau}{}'\frac{1}{\tau_1^{k_1}\tau_2^{k_2}(\omega_1\tau_1+\omega_2\tau_2)^{k_3}},$$

where k_1,k_2,k_3 are non-negative integers and ω_1, ω_2 are two integers, the primed summation means the terms which τ_1,τ_2 or $\omega_1\tau_1+\omega_2\tau_2$ vanishes are omitted. In the later part of this chapter, it is always convenient to assume that ω_1 and ω_2 are coprime positive integers.

To investigate the Mordell–Tornheim double Eisenstein series, we introduce the theory of Cohen series developed by Diamantis and O'Sullivan [24] in Section 5.3. With the technique of partial fraction decomposition and Cohen series, we are able to give in Section 5.4 an explicit formula of Mordell–Tornheim double Eisenstein series into an Eisenstein part and a cusp part. The formula we obtained in the specific case $\omega_1=\omega_2=1$ (Theorem 5.4.2) has some interesting applications, some identities on divisor functions will be given in Section 5.5.

5.1 Convergence of Mordell–Tornheim Double Series

In this section, we will give the sufficient and necessary conditions of the convergence of both double zeta series and Eisenstein series of Mordell–Tornheim type. We recall firstly a necessary and sufficient condition of convergence for Mordell–Tornheim zeta function, given in Tornheim [45].

Lemma 5.1.1 (Tornheim)**.** *Let k_1, k_2, k_3 be reals. Then the Mordell–Tornheim double zeta series*

$$\zeta_{MT}(k_1,k_2,k_3)=\sum_{l_1=1}^{\infty}\sum_{l_2=1}^{\infty}\frac{1}{l_1^{k_1}l_2^{k_2}\left(l_1+l_2\right)^{k_3}}$$

converges if and only if $k_1+k_3>1$, $k_2+k_3>1$ and $k_1+k_2+k_3>2$.

We now state the following theorem which concerns the convergence of Mordell–Tornheim double zeta series.

Theorem 5.1.2. *Let k_1,k_2,k_3 be non-negative integers. Then the symmetrized Mordell–Tornheim double zeta series*

$$\zeta_{MT,\mathbb{Z}}(k_1,k_2,k_3;\omega_1,\omega_2)=\sum_{l_1,l_2\in\mathbb{Z}}{}'\frac{1}{l_1^{k_1}l_2^{k_2}\left(\omega_1l_1+\omega_2l_2\right)^{k_3}}$$

converge absolutely if and only if $k_1+k_3>1$, $k_2+k_3>1$, $k_1+k_2>1$ and $k_1+k_2+k_3>2$.

Proof. Up to some sign change, it is equivalent that we consider the following two summation

$$\sum_{l_1=1}^{\infty}\sum_{l_2=1}^{\infty}\frac{1}{l_1^{k_1}l_2^{k_2}\left(\omega_1l_1+\omega_2l_2\right)^{k_3}}$$

and

$$\sum_{l_1=1}^{\infty}\sum_{l_2=1}^{\infty}\frac{1}{l_1^{k_1}l_2^{k_2}|\omega_1l_1-\omega_2l_2|^{k_3}}.$$

The convergence of first series is equivalent to

$$\sum_{l_1=1}^{\infty}\sum_{l_2=1}^{\infty}\frac{1}{l_1^{k_1}l_2^{k_2}\left(l_1+l_2\right)^{k_3}},$$

which is exactly the Lemma 5.1.1. We divide the second summation into two parts by considering $\omega_1l_1-\omega_2l_2>0$ and $\omega_1l_1-\omega_2l_2<0$. For the first part, it is equivalent to

$$\sum_{l_2=1}^{\infty}\sum_{l_3=1}^{\infty}\frac{1}{\left(l_3+\omega_2l_2\right)^{k_1}l_2^{k_2}l_3^{k_3}},$$

which can be also deduced from Lemma 5.1.1. The second part is similar. □

Theorem 5.1.3. *Let k_1, k_2, k_3 be non-negative integers and $\tau \in \mathcal{H}$. Then the Mordell–Tornheim double Eisenstein series*

$$G(\tau; k_1, k_2, k_3; \omega_1, \omega_2) = \sum_{\tau_1, \tau_2 \in \mathbb{Z}+\mathbb{Z}\tau}{}' \frac{1}{\tau_1^{k_1}\tau_2^{k_2}\left(\omega_1\tau_1 + \omega_2\tau_2\right)^{k_3}}$$

converge absolutely if and only if $k_1+k_3 > 2$, $k_2+k_3 > 2$, $k_1+k_2 > 2$ and $k_1+k_2+k_3 > 4$.

Proof. As well-known, given a fixed $\tau \in \mathcal{H}$, for any $m, n \in \mathbb{Z}$,

$$|m + n\tau| \ll_\tau |m| + |n| \quad \text{and} \quad |m| + |n| \ll_\tau |m + n\tau|.$$

Therefore we know that the absolute convergence of the series is equivalent to the convergence of the following positive series

$$\sum_{m_1, m_2, n_1, n_2}{}' \frac{1}{\left(|m_1| + |n_1|\right)^{k_1}\left(|m_2| + |n_2|\right)^{k_2}\left(|\omega_1 m_1 + \omega_2 m_2| + |\omega_1 n_1 + \omega_2 n_2|\right)^{k_3}}. \tag{5.1}$$

Clearly the series (5.1)

$$\ll \sum_{m_1, m_2 \in \mathbb{Z}}{}' \frac{1}{|m_1|^{k_1/2}|m_2|^{k_2/2}|\omega_1 m_1 + \omega_2 m_2|^{k_3/2}} \sum_{n_1, n_2 \in \mathbb{Z}}{}' \frac{1}{|n_1|^{k_1/2}|n_2|^{k_2/2}|\omega_1 n_1 + \omega_2 n_2|^{k_3/2}}.$$

Note that some terms are missing in the inequality, in fact they converge obviously under our condition, so we omit them for simplification.

According to Theorem 5.1.2, both two sums are convergent if $k_1 + k_2 > 2$, $k_1 + k_3 > 2$, $k_2 + k_3 > 2$ and $k_1 + k_2 + k_3 > 4$. This gives us the if part.

Conversely, we can assume that ω_1 and ω_2 are positive. Then the sum (5.1) has a subseries

$$\sum_{m_1, m_2, n_1, n_2 \geq 0}{}' \frac{1}{\left(m_1 + n_1\right)^{k_1}\left(m_2 + n_2\right)^{k_2}\left(\omega_1 m_1 + \omega_2 m_2 + \omega_1 n_1 + \omega_2 n_2\right)^{k_3}}.$$

By setting $l_1 = m_1 + n_1$, and $l_2 = m_2 + n_2$, it becomes

$$\sum_{l_1, l_2 > 0} \frac{(l_1 + 1)(l_2 + 1)}{l_1^{k_1} l_2^{k_2}\left(\omega_1 l_1 + \omega_2 l_2\right)^{k_3}}.$$

Note that both ω_1 and ω_2 are positive, by Lemma 5.1.1 we find that the sum (5.1) converges only if $(k_1 - 1) + k_3 > 1$, $(k_2 - 1) + k_3 > 1$ and $(k_1 - 1) + (k_2 - 1) + k_3 > 2$. Recall that the Mordell–Tornheim double Eisenstein series have the same convergence as the sum (5.1). Thus the series $G(\tau; k_1, k_2, k_3; \omega_1, \omega_2)$ converges only if $k_1 + k_3 > 2$, $k_2 + k_3 > 2$ and $k_1 + k_2 + k_3 > 4$.

Since ω_1 and ω_2 are coprime, setting $\tau_3 = \omega_1\tau_1 + \omega_2\tau_2$ we find

$$\begin{aligned} G(\tau; k_1, k_2, k_3; \omega_1, \omega_2) &= \sum_{\tau_1,\tau_2\in\mathbb{Z}+\mathbb{Z}\tau}{}' \frac{1}{\tau_1^{k_1}\tau_2^{k_2}(\omega_1\tau_1+\omega_2\tau_2)^{k_3}} \\ &= (-1)^{k_2}\omega_1^{k_1} \sum_{\tau_2,\tau_3\in\mathbb{Z}+\mathbb{Z}\tau}{}' \frac{1}{(\tau_3^{k_1}+\omega_2\tau_2)^{k_1}\tau_2^{k_2}\tau_3^{k_3}} \\ &= (-1)^{k_2}\omega_1^{k_1} G(\tau; k_2, k_3, k_1; 1, \omega_2). \end{aligned}$$

This implies that $G(\tau; k_1, k_2, k_3; \omega_1, \omega_2)$ converges only when all the four symmetric conditions $k_1 + k_2 > 2$, $k_1 + k_3 > 2$, $k_2 + k_3 > 2$ and $k_1 + k_2 + k_3 > 4$ hold. This gives us the only if part. □

5.2 Double Zeta Functions and Partial Fraction Decomposition

The aim of this section is to determine the Mordell–Tornheim double zeta values. In fact, they can be made explicitly with products of Hurwitz zeta values. We will see the key ingredient in the proof is the following partial fraction decomposition.

Lemma 5.2.1. *Let k_1, k_2 and k_3 be nonnegative integers. Then we have the following partial fraction decomposition*

$$\begin{aligned} \frac{1}{l_1^{k_1}l_2^{k_2}(l_1+l_2)^{k_3}} &= \sum_{\mu=0}^{k_1-1} \frac{\binom{k_2+\mu-1}{\mu}}{l_1^{k_1-\mu}(l_1+l_2)^{k_2+k_3+\mu}} + \sum_{\nu=0}^{k_2-1} \frac{\binom{k_1+\nu-1}{\nu}}{l_2^{k_2-\nu}(l_1+l_2)^{k_1+k_3+\nu}} \\ &= \sum_{\mu=0}^{k_1-1} \frac{(-1)^{\mu}\binom{k_3+\mu-1}{\mu}}{l_1^{k_1-\mu}l_2^{k_2+k_3+\mu}} + \sum_{\nu=0}^{k_3-1} \frac{(-1)^{k_1}\binom{k_1+\nu-1}{\nu}}{(l_1+l_2)^{k_3-\nu}l_2^{k_1+k_2+\nu}}. \end{aligned}$$

Let $x \in \mathbb{R}$, we will need the following Hurwitz zeta function

$$\zeta_{\mathbb{Z}}(s, x) = \sum_{n\in\mathbb{Z}}{}' \frac{1}{(n+x)^s}.$$

Here $\sum'$ means the summation omit the term $n = -x$ when $x \in \mathbb{Z}$. The function $\zeta_{\mathbb{Z}}(s, x)$ converges absolutely for $\mathrm{Re}(s) > 1$. It can be also defined for $s = 1$ as a Cauchy principal value. In this case, it reduces to the cotangent function

$$\zeta_{\mathbb{Z}}(1, x) = \lim_{N\to\infty} \sum_{|n|\le N}{}' \frac{1}{n+x} = \begin{cases} 0 & \text{if } x \in \mathbb{Z} \\ \pi\cot(\pi x) & \text{else} \end{cases}$$

Proposition 5.2.2. *Let k_1, k_2, k_3 be nonnegative integers which verifies the condition in Theorem 5.1.2. Then Mordell–Tornheim double zeta value $\zeta_{MT,\mathbb{Z}}(k_1, k_2, k_3; \omega_1, \omega_2)$ is a $\mathbb{Q}$-linear combinations of $\zeta(k)$ and Hurwitz zeta values $\sum_{r\,(\omega_2)} \zeta_{\mathbb{Z}}(m, r/\omega_2)\zeta_{\mathbb{Z}}(k-m, \omega_1 r/\omega_2)$ with $1 \le m \le k-1$, here the sum is taken for all r modulo ω_2.*

Proof. With partial fraction decomposition, a general Mordell–Tornheim double zeta value can be reduced to the case where k_1 or k_2 equals 0.

$$\begin{aligned}
&\sum_{l_1,l_2\in\mathbb{Z}}{}' \frac{1}{l_1^{k_1} l_2^{k_2} (\omega_1 l_1+\omega_2 l_2)^{k_3}} \\
&= \sum_{l_1,l_2\in\mathbb{Z}}{}' \left(\sum_{u=0}^{k_1-1} \frac{\omega_1^u \omega_2^{k_2} \binom{k_2+u-1}{u}}{l_1^{k_1-u}(\omega_1 l_1+\omega_2 l_2)^{k_2+k_3+u}} + \sum_{v=0}^{k_2-1} \frac{\omega_1^{k_1} \omega_2^{v} \binom{k_1+v-1}{v}}{l_2^{k_2-v}(\omega_1 l_1+\omega_2 l_2)^{k_1+k_3+v}} \right) \\
&= \sum_{u=0}^{k_1-1} \omega_1^u \omega_2^{k_2} \binom{k_2+u-1}{u} \zeta_{MT,\mathbb{Z}}(k_1-u,0,k_2+k_3+u;\omega_1,\omega_2) \\
&\quad + \sum_{v=0}^{k_2-1} \omega_1^{k_1} \omega_2^{v} \binom{k_1+v-1}{v} \zeta_{MT,\mathbb{Z}}(0,k_2-v,k_1+k_3+v;\omega_1,\omega_2).
\end{aligned}$$

It remains to focus on the case $k_2 = 0$. Taking modulo ω_2, we are able to decompose each term into a product of two Hurwitz zeta values.

$$\begin{aligned}
&\sum_{l_1,l_2\in\mathbb{Z}}{}' \frac{1}{l_1^{k_1} (\omega_1 l_1+\omega_2 l_2)^{k_3}} \\
&= \sum_{r=0}^{\omega_2-1} \sum_{\substack{l_1\equiv r\,(\omega_2) \\ l_3\equiv \omega_1 r\,(\omega_2)}}{}' \frac{1}{l_1^{k_1} l_3^{k_3}} - \omega_1^{-k_3} \sum_{l_1\in\mathbb{Z}}{}' \frac{1}{l_1^{k_1+k_3}} \\
&= \sum_{r=0}^{\omega_2-1} \omega_2^{-k_1-k_3} \zeta_{\mathbb{Z}}(k_1, r/\omega_2) \zeta_{\mathbb{Z}}(k_3, \omega_1 r/\omega_2) - \omega_1^{-k_3} \zeta_{\mathbb{Z}}(k_1+k_3).
\end{aligned}$$

Note that this series is not convergent if k_1 or k_3 equals 1, but we can interpret this sum as a Cauchy principal value and everything still works. □

5.3 Cohen Series and Periods of Cusp Forms

In this section we introduce the Cohen series originally defined by Cohen [18], roughly speaking, these series represent the linear functionals arising from the periods. We will also generalize these series to a more general twisted version.

Let $\Gamma_1 = \mathrm{SL}_2(\mathbb{Z})$ be the full modular group. Let $\Gamma(N) \subset \Gamma_1$ be the congruence group of level N and let α be a cusp of $\Gamma(N)$. The parabolic subgroup $\Gamma(N)_\alpha$ contains all the elements in $\Gamma(N)$ that fix α. Let $\sigma_\alpha \in \mathrm{SL}_2(\mathbb{R})$ be the element such that $\sigma_\alpha \infty = \alpha$ and

$$\sigma_\alpha^{-1} \Gamma(N)_\alpha \sigma_\alpha \cdot \{\pm 1\} = \left\{ \pm \begin{pmatrix} 1 & N \\ 0 & 1 \end{pmatrix}^m \;\middle|\; m \in \mathbb{Z} \right\}.$$

A cusp form $f \in S_k(\Gamma(N))$ have a Fourier expansion at α

$$f|_k \sigma_\alpha(\tau) = \sum_{m=1}^{\infty} a_{\alpha,f}(m) q^{m/N}.$$

Let p/q be rational, recall that the twisted L-function associate to f is

$$L_\alpha(f,s;p/q)=\sum_{m=1}^{\infty}\frac{a_{\alpha,f}(m)\zeta_{qN}^{pm}}{m^s},$$

this Dirichlet series can be continued to the whole complex plane via the completed L-function

$$\Lambda_\alpha(f,s;p/q)=(2\pi)^{-s}N^s\Gamma(s)L_\alpha(f,s;p/q)=\int_0^\infty (f|_k\sigma_\alpha)\left(iy+\frac{p}{q}\right)y^s\frac{dy}{y}.$$

In particular, for $0\leq n\leq k-2$, we have the twisted n-th periods of f

$$r_n(f;p/q):=\Lambda_\infty(f,n+1;p/q).$$

Consider the cusp form R_n in $S_k(\Gamma_1)$ which represents the periods of cusp forms, that is, for any $f\in S_k(\Gamma_1)$

$$\langle f,R_n\rangle=r_n(f),$$

Cohen in [18] (see also Kohnen–Zagier [31]) gave an explicit characterization of R_n which is given via the so-called Cohen series

$$R_n=c_{k,n}^{-1}\sum_{\substack{a,b,c,d\in\mathbb{Z}\\ ad-bc=1}}\frac{1}{(a\tau+b)^{n+1}(c\tau+d)^{k-n-1}},$$

where $c_{k,n}=i^{k-1-n}2^{2-k}\pi\binom{k-2}{n}$.

They also showed that there exists a unique cusp form $X_{m,n}\in S_k(\Gamma_1)$ such that for any Hecke eigenform f, we have the following Rankin-Selberg type identity

$$\langle f,X_{m,n}\rangle=r_m(f)r_n(f),$$

where m,n be two integers with opposite parity. An explicit formula for $X_{m,n}$ in term of Eisenstein series is also given. Let $f,g\in M_k(\Gamma(N))$ be two modular forms of weight k_1,k_2, recall the Rankin–Cohen bracket of index l is given by

$$[f,g]_l=\sum_{\mu+\nu=l}(-1)^\nu\binom{k_1+l-1}{\nu}\binom{k_2+l-1}{\mu}D^\mu fD^\nu g.$$

This Rankin–Cohen bracket is in general a modular form of weight k_1+k_2+2l, and is a cusp form if l is nonzero. In the case when f or g is Eisenstein series of weight 2, recall that the modified Rankin–Cohen bracket is defined by

$$[f,G_2]_l^{mod}=[f,G_2]_l-\frac{4\pi^2}{k_1+l}D^{l+1}f,$$

$$[G_2,G_2]_l^{mod}=[G_2,G_2]_l-(1+(-1)^l)\frac{4\pi^2}{2+l}D^{l+1}f.$$

Let $0 \leq m < n < k/2$ with opposite parity, then $X_{m,n}$ is given by

$$\begin{aligned} X_{m,n} =& (-1)^{(k+n-m+1)/2} \frac{1}{8(\pi i)^k \binom{k-2}{m}} \\ & \cdot \Big((2\pi i)^{2m} \Gamma(k-n-m-1)\Gamma(n-m+1)[G_{k-n-m-1}, G_{n-m+1}]_m^{mod} \\ & - \delta_{m,0} \frac{\Gamma(k+1)}{B_k} \frac{B_{n+1}B_{k-n-1}}{(n+1)(k-n-1)} G_k \Big). \end{aligned}$$

With the obvious relation $X_{m,n} = (-1)^{k/2} X_{m,k-2-n}$, all other $X_{m,n}$ can be determined in the same manner.

We follow Diamantis–O'Sullivan [24] to define the generalized Cohen series

$$\mathcal{C}_{k,N,\alpha}(\tau, s; p/q) := \sum_{\gamma \in \Gamma(N)} \frac{1}{(\sigma_\alpha^{-1}\gamma\tau + p/q)^s j(\sigma_\alpha^{-1}\gamma, \tau)^k},$$

where $s \in \mathbb{C}$. Then we state the following theorem, which is originally from Diamantis–O'Sullivan, with a slight modification to arbitrary width N.

Theorem 5.3.1. *Let $k > 2$ be an integer and $p/q \in \mathbb{Q}$. Then the series $\mathcal{C}_{k,N,\alpha}(\tau, s; p/q)$ is absolutely convergent when s lies in the strip $1 < \mathrm{Re}(s) < k-1$. It has a meromorphic continuation to all $s \in \mathbb{C}$ as a cusp form in $S_k(\Gamma(N))$, which satisfies the following Petersson inner product identity for all cusp form $f \in S_k(\Gamma(N))$*

$$\langle f, \mathcal{C}_{k,N,\alpha}(\tau, \overline{s}; p/q) \rangle = \epsilon_N 2^{2-k} e^{is\pi/2} \pi \frac{\Gamma(k-1)}{\Gamma(s)\Gamma(k-s)} \Lambda_\alpha(f, k-s; -p/q), \tag{5.2}$$

where $\epsilon_N = 2$ when $N \in \{1, 2\}$ and $\epsilon_N = 1$ when $N > 2$.

For simplicity, we write $\mathcal{C}_k(\tau, s; p/q) = \mathcal{C}_{k,1,\infty}(\tau, s; p/q)$. The Cohen series $\mathcal{C}_k(\tau, s; p/q)$ satisfies

$$\mathcal{C}_k(\tau, s; p/q) = \mathcal{C}_k(\tau, s; p/q + n), \quad n \in \mathbb{Z}.$$

With generalized Cohen series, we now state the twisted version for kernel of the period map.

Theorem 5.3.2. *Let $k > 2$ be an even integer and $p/q \in \mathbb{Q}$, then there is a unique cusp form $R_n(p/q) \in S_k(\Gamma_1)$ such that*

$$\langle f, R_n(p/q) \rangle = r_n(f; p/q)$$

given by

$$R_n(p/q) = c_{k,n}^{-1} \mathcal{C}_k(\tau, k-n-1; -p/q).$$

Let $\rho_m : S_k \to S_k$ be the linear functional defined by

$$f \mapsto r_m(f)f, \quad f \text{ any normalized Hecke eigenform.}$$

We define the cusp form

$$X_{m,n}(p/q) := \rho_m(R_n(p/q) + (-1)^{n-m+1}R_n(-p/q)).$$

Given a basis of Hecke eigenforms, the cusp form $R_n(p/q) + (-1)^{n-m+1}R_n(-p/q)$ decomposes as

$$\begin{aligned} &R_n(p/q) + (-1)^{n-m+1}R_n(-p/q) \\ &= \sum_f \langle R_n(p/q) + (-1)^{n-m+1}R_n(-p/q), f\rangle \langle f, f\rangle^{-1} f \\ &= \sum_f \left(r_n(f; p/q) + (-1)^{n-m+1} r_n(f; -p/q)\right) \langle f, f\rangle^{-1} f. \end{aligned}$$

Apply the functional ρ_m we see

$$X_{m,n}(p/q) = \sum_f r_m(f) \left(r_n(f; p/q) + (-1)^{n-m+1} r_n(f; -p/q)\right) \langle f, f\rangle^{-1} f.$$

So we conclude that

Proposition 5.3.3. *Let $k > 2$ be an even integer and $p/q \in \mathbb{Q}$, then there exists a unique cusp form $X_{m,n}(p/q) \in S_k(\Gamma_1)$ such that for any normalized Hecke eigenform f*

$$\left\langle f, X_{m,n}(p/q)\right\rangle = r_m(f) \left(r_n(f; p/q) + (-1)^{n-m+1} r_n(f; -p/q)\right).$$

The cusp form $X_{m,n}(p/q)$ can also be made explicitly in terms of Rankin–Cohen brackets of Eisenstein series, however, of higher level rather than level 1.

Let $N > 1$ be an integer and $\overline{u} \in (\mathbb{Z}/N\mathbb{Z})^2$. Recall an Eisenstein series of weight k and level $\Gamma(N)$ is the following series

$$G_k^{\overline{u}}(\tau) := \sum_{(c,d) \equiv \overline{u}\,(N)} \frac{1}{(c\tau + d)^k}.$$

Lemma 5.3.4. *Let $n \geq 1, m \geq 1$ and l be positive integers. Let $N > 1$ be an integer and $\omega \in (\mathbb{Z}/N\mathbb{Z})^\times$. Suppose that $n + m > 4$ and $k = n + m + 2l$ is an even number. Let $f \in S_k(\Gamma_1)$ be a cusp form, then*

$$\begin{aligned} \left\langle f, \sum_{\substack{\overline{u},\overline{v} \in (\mathbb{Z}/N\mathbb{Z})^2 \\ \overline{v} = \omega\overline{u}}} [G_m^{\overline{u}}, G_n^{\overline{v}}]_l^{mod} \right\rangle =& \frac{2^{2-2l} i^{k-n} \pi^{m+n}}{\Gamma(m)\Gamma(n)} \binom{k-2}{l} N^{-n} \\ & \cdot r_l(f) \left(r_{m+l-1}(f; -\omega/N) + (-1)^m r_{m+l-1}(f; \omega/N)\right). \end{aligned}$$

Here $\sum_{\substack{\overline{u},\overline{v} \in (\mathbb{Z}/N\mathbb{Z})^2 \\ \overline{v} = \omega\overline{u}}} [G_m^{\overline{u}}, G_n^{\overline{v}}]_l^{mod}$ is a modular form of level 1.

Proof. Let $\overline{u} \in (\mathbb{Z}/N\mathbb{Z})^2$ and $\overline{v} = \omega\overline{u}$. A direct computation shows that

$$\begin{aligned}[G_m^{\overline{u}}, G_n^{\overline{v}}]_l =&(2\pi i)^{-l} \sum_{\mu+\nu=l} (-1)^\mu \binom{l}{\mu} \frac{\Gamma(m+l)\Gamma(n+l)}{\Gamma(n)\Gamma(m)\Gamma(l+1)} \\ &\cdot \sum_{\substack{c,d\in\mathbb{Z} \\ (c,d)\equiv u}}{}' \left(\frac{c^\mu}{(c\tau+d)^{m+\mu}} \sum_{a,b\in\mathbb{Z}}{}' \frac{(aN+\omega c)^\nu}{(aN\tau+bN+\omega(c\tau+d))^{n+\nu}} \right) \\ =&(2\pi i)^{-l} \frac{\Gamma(m+l)\Gamma(n+l)}{\Gamma(n)\Gamma(m)\Gamma(l+1)} N^{-n} \\ &\cdot \sum_{\substack{a,b,c,d\in\mathbb{Z} \\ (c,d)\equiv\overline{u}}}{}' \frac{(ad-bc)^l}{(c\tau+d)^{m+l}(a\tau+b+\frac{\omega}{N}(c\tau+d))^{n+l}}. \end{aligned} \tag{5.3}$$

Summing around all $\overline{u} \in (\mathbb{Z}/N\mathbb{Z})^2$, they in (5.3) splits into three parts

$$\begin{aligned}&\sum_{\substack{a,b,c,d\in\mathbb{Z} \\ (c,d)\equiv u}}{}' \frac{(ad-bc)^l}{(c\tau+d)^{m+l}(a\tau+b+\frac{\omega}{N}(c\tau+d))^{n+l}} \\ &= \sum_{\substack{a,b,c,d\in\mathbb{Z} \\ ad-bc=0}}{}' + \sum_{\substack{a,b,c,d\in\mathbb{Z} \\ ad-bc>0}}{}' + \sum_{\substack{a,b,c,d\in\mathbb{Z} \\ ad-bc<0}}{}' .\end{aligned}$$

the first sum is either 0 (when $l > 0$) or an Eisenstein series of level 1 (when $l = 0$) which makes no contribution in the Petersson product. The second and the third sum altogether turn out to be a cusp form of level 1

$$\sum_{h=1}^{\infty} \frac{1}{h^{m+n+l-1}} \left(T_h\mathcal{C}_k(\tau, n+l; \omega/N) + (-1)^m T_h\mathcal{C}_k(\tau, n+l; -\omega/N)\right).$$

Hence by Theorem 5.3.1,

$$\begin{aligned}&\sum_{h=1}^{\infty} \frac{1}{h^{m+n+l-1}} \left\langle f, T_h\mathcal{C}_k(\tau, n+l; \omega/N) + (-1)^m T_h\mathcal{C}_k(\tau, n+l; -\omega/N) \right\rangle \\ &= \sum_{h=1}^{\infty} \frac{a_f(h)}{h^{k-l-1}} \overline{c}_{k,m+l-1} \left(r_{m+l-1}(f; -\omega/N) + (-1)^m r_{m+l-1}(f; \omega/N)\right) \\ &= \frac{(2\pi)^{k-l-1}\overline{c}_{k,m+l-1}}{\Gamma(k-l-1)} r_{k-l-2}(f) \left(r_{m+l-1}(f; -\omega/N) + (-1)^m r_{m+l-1}(f; \omega/N)\right).\end{aligned}$$

In the case where m or n equals 1 or 2, we use Hecke's trick to add $|c\tau+d|^s$ in the summation. The corresponding Rankin–Cohen bracket is changed to the modified version. □

Corollary 5.3.5. *Let $N > 2$, we have*

$$X_{0,n}(-\omega/N) = i^{n+1}\frac{\Gamma(k-n-1)\Gamma(n+1)}{4\pi^k}N^{k-n-1}\Big(\sum_{\substack{\overline{u},\overline{v}\in(\mathbb{Z}/N\mathbb{Z})^2\\ \overline{v}=\omega\overline{u}}}[G^{\overline{v}}_{n+1}, G^{\overline{u}}_{k-n-1}]^{mod}_0 - \frac{\sum_{r\,(N)}\zeta_{\mathbb{Z}}(n+1,\omega r/N)\zeta_{\mathbb{Z}}(k-n+1,r/N)}{2N^k\zeta(k)}G_k\Big).$$

In particular, we recover Zagier's identity

$$X_{0,n} = i^{n+1}\frac{\Gamma(k-n-1)\Gamma(n+1)}{8\pi^k}\Big([G_{n+1}, G_{k-n-1}]^{mod}_0 + \frac{\Gamma(k+1)}{B_k}\frac{B_{n+1}B_{k-n-1}}{\Gamma(n+2)\Gamma(k-n)}G_k\Big).$$

We now define further the double Cohen series

$$\mathcal{C}_k(\tau, s_1, s_2; \omega_1, \omega_2) := \sum_{\gamma\in\Gamma_1}\frac{1}{(\gamma\tau+\omega_1)^{s_1}(\gamma\tau+\omega_2)^{s_2}j(\gamma,\tau)^k}, \tag{5.4}$$

where $\omega_1, \omega_2 \in \mathbb{Q}$ and s_1, s_2 are integers such that $1 < s_1 + s_2 < k-1$.

Lemma 5.3.6. *Let $\omega_1, \omega_2 \in \mathbb{Q}$ and s_1, s_2 be integers with $1 < s_1 + s_2 < k-1$. Then the double Cohen series $\mathcal{C}_k(\tau, s_1, s_2; \omega_1, \omega_2)$ converges absolutely.*

Proof. We omit the detail and refer to [24]. The treatment is similar as there. □

Proposition 5.3.7. *Let $\omega_1, \omega_2 \in \mathbb{Q}$ and s_1, s_2 be integers with $1 < s_1 + s_2 < k-1$. Then the double Cohen series is a cusp form in S_k, more precisely*

$$\mathcal{C}_k(\tau, s_1, s_2; \omega_1, \omega_2) = \sum_{\mu=0}^{s_1-1}\frac{(-1)^\mu\binom{s_2+\mu-1}{\mu}}{(\omega_2-\omega_1)^{s_2+\mu}}\mathcal{C}_k(\tau, s_1-\mu; \omega_1) + \sum_{\nu=0}^{s_2-1}\frac{(-1)^\nu\binom{s_1+\nu-1}{\nu}}{(\omega_1-\omega_2)^{s_1+\nu}}\mathcal{C}_k(\tau, s_2-\nu; \omega_2).$$

Proof. We proceed with Lemma 5.2.1 to get

$$\begin{aligned}&\sum_{\gamma\in\Gamma}\frac{1}{(\gamma\tau+\omega_1)^{s_1}(\gamma\tau+\omega_2)^{s_2}j(\gamma,\tau)^k}\\ =&\sum_{\gamma\in\Gamma}\sum_{\mu=0}^{s_1-1}\frac{(-1)^\mu\binom{s_2+\mu-1}{\mu}}{(\gamma\tau+\omega_1)^{s_1-\mu}(\omega_2-\omega_1)^{s_2+\mu}}\frac{1}{j(\gamma,\tau)^k}\\ &+\sum_{\gamma\in\Gamma}\sum_{\nu=0}^{s_2-1}\frac{(-1)^\nu\binom{s_1+\nu-1}{\nu}}{(\gamma\tau+\omega_2)^{s_2-\nu}(\omega_1-\omega_2)^{s_1+\nu}}\frac{1}{j(\gamma,\tau)^k}.\end{aligned}$$

In both two sums each term is a Cohen series except for the cases $\mu = s_1 - 1$, $\nu = s_2 - 1$. The exceptional cases contribute as

$$(-1)^{s_1-1}\frac{\binom{s_1+s_2-2}{s_1-1}}{(\omega_2-\omega_1)^{s_1+s_2-1}}\sum_{\gamma\in\Gamma}\left(\frac{1}{\gamma\tau+\omega_1}-\frac{1}{\gamma\tau+\omega_2}\right)\frac{1}{j(\gamma,\tau)^k}.$$

This series should converge absolutely since the else converge absolutely by Theorem 5.3.1. To compute this term, we need the following Cauchy principle value when $u \in \mathbb{C}\backslash\mathbb{Z}$

$$\lim_{N\to\infty}\sum_{n=-N}^{N}\frac{1}{n+u} = \pi\cot\pi u.$$

By rearranging the summation, we have

$$\begin{aligned}
&\sum_{\gamma\in\Gamma}\left(\frac{1}{\gamma\tau+\omega_1}-\frac{1}{\gamma\tau+\omega_2}\right)\frac{1}{j(\gamma,\tau)^k}\\
&=\sum_{\gamma\in\Gamma_\infty\backslash\Gamma}\sum_{n\in\mathbb{Z}}\left(\frac{1}{\gamma\tau+n+\omega_1}-\frac{1}{\gamma\tau+n+\omega_2}\right)\frac{2}{j(\gamma,\tau)^k} \qquad (5.5)\\
&=\sum_{\gamma\in\Gamma_\infty\backslash\Gamma}\left(\cot\pi(\gamma\tau+\omega_1)-\cot\pi(\gamma\tau+\omega_2)\right)\frac{2\pi}{j(\gamma,\tau)^k}.
\end{aligned}$$

On the other hand, let $q = e^{2\pi i\gamma\tau}$ and $q_j = e^{2\pi i\omega_j}$, $j = 1, 2$, we have

$$\cot(\pi(\gamma\tau+\omega_j)) = i\frac{q_jq+1}{q_jq-1} = -i(1+2q_jq+2q_j^2q^2+\cdots).$$

Hence the equation (5.5) gives us the cusp form

$$\sum_{\gamma\in\Gamma_\infty\backslash\Gamma}\sum_{m=1}^{\infty}\frac{-4\pi i(q_1^m-q_2^m)q^m}{j(\gamma,\tau)^k} = -4\pi i\sum_{m=1}^{\infty}(q_1^m-q_2^m)\sum_{\gamma\in\Gamma_\infty\backslash\Gamma}\frac{q^m}{j(\gamma,\tau)^k}.$$

Let f be a cusp form, as in the proof of Theorem 5.3.1, we can compute the following Petersson inner product

$$\Big\langle f, -4\pi i\sum_{m=1}^{\infty}(q_1^m-q_2^m)\sum_{\gamma\in\Gamma_\infty\backslash\Gamma}\frac{q^m}{j(\gamma,\tau)^k}\Big\rangle = 4\pi i\Gamma(k-1)\sum_{m=1}^{\infty}\frac{a_f(m)}{(4\pi m)^{k-1}}(q_1^{-m}-q_2^{-m}).$$

On the other hand,

$$\begin{aligned}
\langle f, \mathcal{C}_k(\tau,1;\omega_j)\rangle &= 2^{3-k}\pi i\Lambda_\infty(f,k-1;-\omega_j)\\
&= 4\pi i\Gamma(k-1)\sum_{m\geq 1}\frac{a_f(m)}{(4\pi m)^{k-1}}q_j^{-m}.
\end{aligned}$$

Hence by comparing these two Petersson inner products, we find

$$\sum_{\gamma\in\Gamma}\left(\frac{1}{\gamma\tau+\omega_1}-\frac{1}{\gamma\tau+\omega_2}\right)\frac{1}{j(\gamma,\tau)^k} = \mathcal{C}_k(\tau,1;\omega_1)-\mathcal{C}_k(\tau,1;\omega_2). \qquad (5.6)$$

□

5.4 Mordell–Tornheim double Eisenstein Series

Let k_1, k_2 and k_3 be positive integers with $k_1+k_3>2$, $k_2+k_3>2$, $k_1+k_2>2$ and $k_1+k_2+k_3>4$. Then by Theorem 5.1.3, the Mordell–Tornheim double Eisenstein series

$$G(\tau;k_1,k_2,k_3;\omega_1,\omega_2)=\sum_{\tau_1,\tau_2\in\mathbb{Z}+\mathbb{Z}\tau}{}' \frac{1}{\tau_1^{k_1}\tau_2^{k_2}(\omega_1\tau_1+\omega_2\tau_2)^{k_3}}$$

converges absolutely. In this section, we will give an explicit formula for the Mordell–Tornheim double Eisenstein series in terms of classical Eisenstein series. More precisely, we have the following theorem

Theorem 5.4.1. *Let k_1,k_2,k_3 be positive integers, $\omega_1,\omega_2\in\mathbb{Z}$ and $\tau\in\mathcal{H}$, then we have*

$$G(\tau;k_1,k_2,k_3;\omega_1,\omega_2)=G_{eis}+G_{cusp}$$

Here G_{eis} is an Eisenstein series which is given by

$$G_{eis}=\frac{\zeta_{MT,\,\mathbb{Z}}(k_1,k_2,k_3;\omega_1,\omega_2)}{2\zeta(k)}\cdot G_k.$$

The G_{cusp} is a cusp form given by

$$\begin{aligned}G_{cusp}=&\sum_{\mu=0}^{k_1-2}\frac{(-1)^{k_1}\binom{k_3+\mu-1}{\mu}}{\omega_2^{k_3+\mu}\omega_1^{-\mu}}\delta_{\mu\equiv k_1(2)}\\&\cdot\Big([G_{k_1-\mu},G_{k_2+k_3+\mu}]_0^{mod}-2\frac{\zeta(k_1-\mu)\zeta(k_2+k_3+\mu)}{\zeta(k)}G_k\Big)\\&+\sum_{\nu=0}^{k_3-1}\frac{(-1)^{k_1}\binom{k_1+\nu-1}{\nu}}{\omega_2^{k_1+\nu}\omega_1^{-k_1}}\\&\cdot\Big(\sum_{\substack{\overline{u},\overline{v}\in(\mathbb{Z}/\omega_1\mathbb{Z})^2\\ \overline{v}=\omega_2\overline{u}}}[G^{\overline{v}}_{k_3-\nu},G^{\overline{u}}_{k_1+k_2+\nu}]_0^{mod}-\frac{\sum_{r(\omega_1)}\zeta_{\mathbb{Z}}(k_3-\nu,\omega_2 r/\omega_1)\zeta_{\mathbb{Z}}(k_1+k_2+\nu,r/\omega_1)}{2\omega_1^k\zeta(k)}G_k\Big),\end{aligned}$$

here the modified Rankin–Cohen brackets in last line involves only Eisenstein series of even weights when ω_1 is 1 or 2.

Proof. Let $\tau_1=a\tau+b$ and $\tau_2=c\tau+d$, we write

$$\det(\tau_1,\tau_2)=\det\begin{pmatrix}a&b\\c&d\end{pmatrix}.$$

First, we consider the part

$$G_{eis}=\sum_{\det(\tau_1,\tau_2)=0}{}'\frac{1}{\tau_1^{k_1}\tau_2^{k_2}(\omega_1\tau_1+\omega_2\tau_2)^{k_3}}.$$

Since τ_1, τ_2 and $\omega_1\tau_1+\omega_2\tau_2$ are proportional,

$$\begin{aligned} G_{eis} &= \sum_{\det(\tau_1,\tau_2)=0}{}' \frac{1}{\tau_1^{k_1}\tau_2^{k_2}(\omega_1\tau_1+\omega_2\tau_2)^{k_3}} \\ &= \sum_{n,m\in\mathbb{Z}}{}' \frac{1}{n^{k_1}m^{k_2}(\omega_1 n+\omega_2 m)^{k_3}} \sum_{\gcd(c,d)=1}{}' \frac{1}{(c\tau+d)^k} \\ &= \zeta_{MT,\mathbb{Z}}(k_1,k_2,k_3;\omega_1,\omega_2)/2\zeta(k)\cdot G_k. \end{aligned}$$

Now we consider the remaining part

$$G_{cusp} = \sum_{\det(\tau_1,\tau_2)\neq 0} \frac{1}{\tau_1^{k_1}\tau_2^{k_2}(\omega_1\tau_1+\omega_2\tau_2)^{k_3}}.$$

We split again the sums into two parts

$$\begin{aligned} &\sum_{\det(\tau_1,\tau_2)\neq 0} \frac{1}{\tau_1^{k_1}\tau_2^{k_2}(\omega_1\tau_1+\omega_2\tau_2)^{k_3}} \\ &= \sum_{\det(\tau_1,\tau_2)>0} \frac{1}{\tau_1^{k_1}\tau_2^{k_2}(\omega_1\tau_1+\omega_2\tau_2)^{k_3}} + \sum_{\det(\tau_1,\tau_2)<0} \frac{1}{\tau_1^{k_1}\tau_2^{k_2}(\omega_1\tau_1+\omega_2\tau_2)^{k_3}}. \end{aligned}$$

Changing the variable by $\tau_2\to-\tau_2$ in the second sum, we see it is enough to calculate only the first one. This sum can be computed with double Cohen series and Hecke operator

$$\begin{aligned} &\sum_{\det(\tau_1,\tau_2)>0} \frac{1}{\tau_1^{k_1}\tau_2^{k_2}(\omega_1\tau_1+\omega_2\tau_2)^{k_3}} \\ &= \sum_{\substack{a,b,c,d\in\mathbb{Z}\\ ad-bc>0}} \frac{1}{(a\tau+b)^{k_1}(c\tau+d)^{k_2}(\omega_1(a\tau+b)+\omega_2(c\tau+d))^{k_3}} \\ &= \omega_1^{-k_3}\sum_{n=1}^{\infty}\sum_{\substack{a,b,c,d\in\mathbb{Z}\\ ad-bc=n}} \frac{1}{\left(\frac{a\tau+b}{c\tau+d}\right)^{k_1}\left(\frac{a\tau+b}{c\tau+d}+\frac{\omega_2}{\omega_1}\right)^{k_3}(c\tau+d)^k} \\ &= \omega_1^{-k_3}\sum_{n=1}^{\infty}\sum_{\det\gamma=n} \frac{1}{(\gamma\tau)^{k_1}(\gamma\tau+\omega_2/\omega_1)^{k_3}j(\gamma,\tau)^k} \\ &= \omega_1^{-k_3}\sum_{n=1}^{\infty}\frac{1}{n^{k-1}}T_n\left(\mathcal{C}_k(\tau,k_1,k_3;0,\omega_2/\omega_1)\right). \end{aligned}$$

Hence, G_{cusp} equals

$$\omega_1^{-k_3}\sum_{n=1}^{\infty}\frac{1}{n^{k-1}}\left(T_n\mathcal{C}_k(\tau,k_1,k_3;0,\omega_2/\omega_1)+(-1)^{k_2}T_n\mathcal{C}_k(\tau,k_1,k_3;0,-\omega_2/\omega_1)\right).$$

Recalling Proposition 5.3.7 we see that

$$\begin{aligned}
&\sum_{n=1}^{\infty}\frac{1}{n^{k-1}}\left(T_n\mathcal{C}_k(\tau,k_1,k_3;0,\omega_2/\omega_1)+(-1)^{k_2}T_n\mathcal{C}_k(\tau,k_1,k_3;0,-\omega_2/\omega_1)\right)\\
=&\sum_{\mu=0}^{k_1-2}(-1)^{\mu}\binom{k_3+\mu-1}{\mu}(\omega_1/\omega_2)^{k_3+\mu}\\
&\cdot\sum_{n=1}^{\infty}\frac{1}{n^{k-1}}\left(T_n\mathcal{C}_k(\tau,k_1-\mu;0)+(-1)^{k_1-\mu}T_n\mathcal{C}_k(\tau,k_1-\mu;0)\right)\\
&+\sum_{\nu=0}^{k_3-1}(-1)^{k_1}\binom{k_1+\nu-1}{\nu}(\omega_2/\omega_1)^{k_1+\nu}\\
&\cdot\sum_{n=1}^{\infty}\frac{1}{n^{k-1}}\left(T_n\mathcal{C}_k(\tau,k_3-\nu;\omega_2/\omega_1)+(-1)^{k_3-\nu}T_n\mathcal{C}_k(\tau,k_3-\nu;-\omega_2/\omega_1)\right).
\end{aligned}$$

If ω_1 is 1 or 2, the last line vanishes when $k_3-\nu$ is odd. As has been evaluated in the proof of Lemma 5.3.3, the sum G_{cusp} can be computed clearly with Rankin–Cohen brackets. □

In the case $\omega_1=\omega_2=1$, the Mordell–Tornheim double Eisenstein series

$$G(\tau;k_1,k_2,k_3):=\sum_{\substack{\tau_1+\tau_2+\tau_3=0\\ \tau_1,\tau_2,\tau_3\in\mathbb{Z}+\mathbb{Z}\tau}}{}'\ \frac{1}{\tau_1^{k_1}\tau_2^{k_2}\tau_3^{k_3}},$$

symmetric in k_1, k_2 and k_3, may be expressed in a more brief form.

Theorem 5.4.2. *Let k_1, k_2 and k_3 be positive integers which satisfy the condition in Theorem 5.1.3. Then we have the following*

$$\begin{aligned}
G(\tau;k_1,k_2,k_3)=&(-1)^{k_3}\Bigg(\sum_{\mu=0}^{k_1-2}\delta_{\mu\equiv k_1(2)}\binom{k_2+\mu-1}{\mu}G_{k_1-\mu}G_{k_2+k_3+\mu}\\
&+\sum_{\nu=0}^{k_2-2}\delta_{\nu\equiv k_2(2)}\binom{k_1+\nu-1}{\nu}G_{k_2-\nu}G_{k_1+k_3+\nu}\\
&-\frac{4\pi^2}{k-2}\binom{k_1+k_2-2}{k_1-1}DG_{k-2}-\binom{k_1+k_2}{k_1}G_k\Bigg).
\end{aligned}$$

Hence $\frac{1}{\pi^k}G(\tau;k_1,k_2,k_3)$ has rational coefficients.

5.5 Examples and Fourier Coefficients of Mordell–Tornheim Double Eisenstein Series

Recall that we have the normalized Eisenstein series E_k

$$E_k(\tau) = \frac{G_k(\tau)}{2\zeta(k)} = 1 - \frac{2k}{B_k}\sum_{n=1}^{\infty}\sigma_{k-1}(n)q^n.$$

Example 5.5.1. *Let $k = 6$. All the Mordell–Tornheim double Eisenstein series of weight 6 is listed as follows*

$$\begin{aligned} G(\tau;1,2,3) &= -\frac{\pi^6}{945}E_6(\tau),\\ G(\tau;2,2,2) &= \frac{2\pi^6}{945}E_6(\tau). \end{aligned}$$

Example 5.5.2. *Let $k = 8$. All the Mordell–Tornheim double Eisenstein series of weight 6 is listed as follows*

$$\begin{aligned} G(\tau;1,2,5) &= -\frac{\pi^8}{14175}E_8(\tau),\\ G(\tau;1,3,4) &= -\frac{2\pi^8}{14175}E_8(\tau),\\ G(\tau;2,2,4) &= \frac{2\pi^8}{14175}E_8(\tau),\\ G(\tau;2,3,3) &= 0. \end{aligned}$$

Example 5.5.3. *Let $k = 12$ and $k_1 = k_2 = k_3 = 4$. Then we have a Mordell–Tornheim double Eisenstein series of weight 12*

$$G(\tau;4,4,4) = \frac{38\pi^{12}}{91216125}\left(E_{12}(\tau) - \frac{111196800}{13129}\Delta(\tau)\right),$$

where Δ is the unique Hecke cusp form of weight 12.

Example 5.5.4. *For any integer $k \geq 1$, we have $G(\tau; 2k, 2k+1, 2k+1) = 0$. In fact, we have*

$$\begin{aligned} &3G(\tau;2k,2k+1,2k+1)\\ &= \sum_{\tau_1+\tau_2+\tau_3=0}{}' \frac{\tau_1+\tau_2+\tau_3}{\tau_1^{2k+1}\tau_2^{2k+1}\tau_3^{2k+1}} = 0. \end{aligned}$$

Using Theorem 4 and evaluating the coefficients of q^n, we have the following formula

$$2\sum_{\substack{\mu=1\\2\nmid\mu}}^{2k-1}\binom{2k+\mu}{\mu}\Bigg(\binom{6k+1}{4k+\mu}B_{2k+1-\mu}\sigma_{4k+\mu}(n)+\binom{6k+1}{4k+1+\mu}B_{4k+1+\mu}\sigma_{2k-\mu}(n)$$
$$-2(6k+1)\binom{6k}{2k-\mu}\sum_{m=1}^{n-1}\sigma_{2k-\mu}(m)\sigma_{4k+\mu}(n-m)\Bigg)$$
$$=(6k+1)\binom{4k}{2k}n\sigma_{6k-1}(n)-\binom{4k+2}{2k+1}\sigma_{6k+1}(n).$$

When $k=1$, we recover the classical formula from Ramanujan

$$504\sum_{m=1}^{n-1}\sigma_1(m)\sigma_5(n-m)=20\sigma_7(n)+21\sigma_5(n)-42n\sigma_5(n)+\sigma_1(n).$$

Example 5.5.5. *For any integer $k\geq 1$, $G(\tau;1,2k,2k+1)$ is an Eisenstein series. In fact, we have*

$$G(\tau;1,2k,2k+1)+G(\tau;1,2k+1,2k)$$
$$=\sum_{\tau_1+\tau_2+\tau_3=0}{}'\frac{\tau_2+\tau_3}{\tau_1\,\tau_2^{2k+1}\tau_3^{2k+1}}=-G_{4k+2}(\tau).$$

Evaluating the coefficients of q^n, we get

$$\sum_{\substack{\mu=0\\2\mid\mu}}^{2k-2}\Bigg(B_{2k-\mu}\binom{4k+1}{2k-\mu}\sigma_{2k+1+\mu}(n)+B_{2k+2+\mu}\binom{4k+1}{2k-\mu-1}\sigma_{2k-\mu-1}(n)$$
$$-2(4k+1)\binom{4k}{2k+\mu+1}\sum_{m=1}^{n-1}\sigma_{2k-1-\mu}(m)\sigma_{2k+1+\mu}(n-m)\Bigg)$$
$$=(4k+1)n\sigma_{4k-1}(n)-(4k+3)\sigma_{4k+1}(n)/2.$$

When $k=1$, we recover the classical formula from Ramanujan

$$240\sum_{m=1}^{n-1}\sigma_1(m)\sigma_3(n-m)=21\sigma_5(n)-30n\sigma_3(n)+10\sigma_3(n)-\sigma_1(n).$$

More examples of Mordell–Tornheim double Eisenstein series $G(\tau;k_1,k_2,k_3)$ can be found in Appendix B.

Chapter 6

Final Computations

In Chapter 2 we introduced the element $\mathrm{Eis}_{\mathcal{D}}^{k_1,k_2,j}(u_1,u_2)$ and then in Chapter 3 we established a theory of regularized integrals which is applicable for this regulator. So the following regulator integrals do make sense

$$\int_{P\{0,\infty\}}^{*} \mathrm{Eis}_{\mathcal{D}}^{k_1,k_2,j}(u_1,u_2) \quad \text{and} \quad \int_{P[0,\alpha]_\infty}^{*} \mathrm{Eis}_{\mathcal{D}}^{k_1,k_2,j}(u_1,u_2).$$

In this chapter we aim to compute these integrals. We begin first by stating our final results on computation of regulator integrals, including periods and residues of the regulator $\mathrm{Eis}_{\mathcal{D}}^{k_1,k_2,j}(u_1,u_2)$ for $P = X^m Y^{w-m}$ for certain $m \geq k_1, k_2$. In Section 6.3 and Section 6.2, the regulator integral is clearly delineated. We will need certain preparatory calculations about regulators, given in Section 6.4 and Section 6.6. The whole computation and our final proof are included in Section 6.7 and Section 6.8. The vital tool is the Rogers–Zudilin method given by Lemma 4.3.2.

6.1 Results

We compute explicitly the periods of the regulator. The period

$$\int_{X^w\{0,\infty\}}^{*} \mathrm{Eis}_{\mathcal{D}}^{k_1,k_2,j}(u_1,u_2)$$

is exactly a $\mathbb{Q}(w+1)$-multiple of L-value of a modular form with rational coefficients. More precisely we obtain

Theorem 6.1.1. *Let k_1, k_2, j be nonnegative integers with $w = k_1 + k_2$. Let $N \geq 3$ and $u_1 = (a_1, b_1)$, $u_2 = (a_2, b_2) \in (\mathbb{Z}/N\mathbb{Z})^2$, suppose that $(a_i, b_i) \neq (0,0)$ if $k_i = 0$ and $b_i \neq 0$ if $k_i = 1$, then*

$$\int_{X^w\{0,\infty\}}^{*} \mathrm{Eis}_{\mathcal{D}}^{k_1,k_2,j}(u_1,u_2) = C_{k_1,k_2,j}\Lambda^*\left(G_{b_1,-a_2}^{(k_1+1)}G_{b_2,a_1}^{(k_2+1)} - (-1)^j G_{b_1,a_2}^{(k_1+1)}G_{b_2,-a_1}^{(k_2+1)}, -j\right)$$

with the constant

$$C_{k_1,k_2,j} = \frac{j!(k_1+j+2)(k_2+j+2)}{2N^{w+j+2}} i^{k_1-k_2+(j+1)^2}(2\pi)^{w+1} \in \mathbb{Q}(w+1).$$

Remark 6.1.2. The appearance of the L-value at $s=-j$ is in accordance with the Beilinson's conjectures that we discussed in Subsection 2.3.1.

Also we compute the residues

$$\int^*_{X^w[0,x]_\infty} \mathrm{Eis}_{\mathcal{D}}^{k_1,k_2,j}(u_1,u_2),$$

the result is given below

Theorem 6.1.3. *Let k_1, k_2, j be nonnegative integers with $w=k_1+k_2$. Let $N\geq 3$ and $u_1=(a_1,b_1)$, $u_2=(a_2,b_2)\in(\mathbb{Z}/N\mathbb{Z})^2$, suppose that $(a_i,b_i)\neq(0,0)$ if $k_i=0$ and $b_i\neq 0$ if $k_i=1$, then*

$$\int^*_{X^w[0,x]_\infty} \mathrm{Eis}_{\mathcal{D}}^{k_1,k_2,j}(u_1,u_2) = R^{k_1,k_2,j}(u_1,u_2) + (-1)^{k_1+k_2+j+1}R^{k_2,k_1,j}(u_2,u_1),$$

where

$$R^{k_1,k_2,j}(u_1,u_2) = (-1)^{\frac{1}{2}j(j-1)+k_2}\delta_0(a_1)\frac{(k_1+j)!(k_1+j+2)(k_2+j+2)}{2(k_2+1)N^{k_2+j+3}}(2\pi i)^{k_2+1}$$
$$\cdot L(\delta_{-a_2},-k_2-j-1)L(\delta_{b_1}+(-1)^{k_1+j}\delta_{-b_1},k_1+j+1)x^{k_2+1}.$$

Here we exhibit several examples of regulator integrals.

Example 6.1.4. *Let $j=0$, then we recover the following Brunault's result [11, Theorem 1.1]*

$$\int^*_{X^w\{0,\infty\}} \mathrm{Eis}_{\mathcal{D}}^{k_1,k_2}(u_1,u_2) =$$
$$\frac{(k_1+2)(k_2+2)}{2N^{w+2}}(2\pi)^{w+1}i^{k_1-k_2+1}\Lambda^*\left(G^{(k_1+1)}_{b_1,-a_2}G^{(k_2+1)}_{b_2,a_1} - G^{(k_1+1)}_{b_1,a_2}G^{(k_2+1)}_{b_2,-a_1},0\right).$$

Example 6.1.5. *Let $k_1=k_2=0$ and $j=1$ then we have period*

$$\int^*_{\{0,\infty\}} \mathrm{Eis}_{\mathcal{D}}^{0,0,1}(u_1,u_2) = \frac{9\pi}{N^3}\Lambda\left(G^{(1)}_{b_1,-a_2}G^{(1)}_{b_2,a_1} + G^{(1)}_{b_1,a_2}G^{(1)}_{b_2,-a_1},-1\right)$$

and residue

$$\int^*_{[0,x]_\infty} \mathrm{Eis}_{\mathcal{D}}^{0,0,1}(u_1,u_2) = R^{0,0,1}(u_1,u_2) + R^{0,0,1}(u_2,u_1)$$

with

$$R^{0,0,1}(u_1,u_2) = \delta_0(a_1)\frac{9\pi i}{N^4}L(\delta_{-a_2},-2)L(\delta_{b_1}-\delta_{-b_1},2)x.$$

Example 6.1.6. *Let $k_1 = 0$, $k_2 = 1$ and $j = 1$ then we have periods*

$$\int_{X\{0,\infty\}}^{*} \mathrm{Eis}_{\mathcal{D}}^{0,1,1}(u_1,u_2) = -\frac{24\pi^2 i}{N^4}\Lambda\left(G^{(1)}_{b_1,-a_2}G^{(2)}_{b_2,a_1} + G^{(1)}_{b_1,a_2}G^{(2)}_{b_2,-a_1}, -1\right)$$

and

$$\begin{aligned}\int_{Y\{0,\infty\}}^{*} \mathrm{Eis}_{\mathcal{D}}^{0,1,1}(u_1,u_2) &= \int_{\sigma_* X\{\infty,0\}}^{*} \mathrm{Eis}_{\mathcal{D}}^{0,1,1}(u_1,u_2)\\ &= -\int_{X\{0,\infty\}}^{*} \mathrm{Eis}_{\mathcal{D}}^{0,1,1}(u_1\sigma,u_2\sigma)\\ &= \frac{24\pi^2 i}{N^4}\Lambda\left(G^{(1)}_{-a_1,-b_2}G^{(2)}_{-a_2,b_1} + G^{(1)}_{-a_1,b_2}G^{(2)}_{-a_2,-b_1}, -1\right).\end{aligned}$$

In more general, for certain $k_1, k_2 \le m \le w$, the period

$$\int_{X^m Y^{w-m}\{0,\infty\}}^{*} \mathrm{Eis}_{\mathcal{D}}^{k_1,k_2,j}(u_1,u_2)$$

is a linear combination of L-values of quasi-modular forms with rational coefficients.

Theorem 6.1.7. *Let k_1, k_2, j be nonnegative integers with $w = k_1 + k_2$. Let m be an integer with $k_1, k_2 \le m \le w$. Assume that if $m = k_2$ then $k_1 = 0$. Let $N \ge 3$ and $u_1 = (a_1, b_1)$, $u_2 = (a_2, b_2) \in (\mathbb{Z}/N\mathbb{Z})^2$, suppose that $(a_i, b_i) \ne (0,0)$ if $k_i = 0$ and $b_i \ne 0$ if $k_i = 1$, then*

$$\begin{aligned}&\int_{X^m Y^{w-m}\{0,\infty\}}^{*} \mathrm{Eis}_{\mathcal{D}}^{k_1,k_2,j}(u_1,u_2) = \frac{j!(k_1+j+2)(k_2+j+2)}{4N^{w+j+2}} i^{k_1-k_2+(j+1)^2}(2\pi)^{w+1}\\
&\cdot\left(\Lambda^*\left(D^{w-m}\left(G^{(m-k_2+1)}_{b_1,-a_2} - (-1)^{w-m}G^{(m-k_2+1)}_{b_1,a_2}\right)\left(G^{(m-k_1+1)}_{b_2,a_1} + (-1)^j G^{(m-k_1+1)}_{b_2,-a_1}\right), -j\right)\right.\\
&+\sum_{l=1}^{j}\frac{1}{l!}\left(\frac{4\pi}{N}\right)^l\\
&\left.\cdot\Lambda^*\left((-1)^{w+m+1}D^{w-m+l}G^{(m-k_2+1)}_{b_1,a_2}G^{(m-k_1+1)}_{b_2,a_1} + (-1)^j D^{w-m+l}G^{(m-k_2+1)}_{b_1,-a_2}G^{(m-k_1+1)}_{b_2,-a_1}, -j+l\right)\right.\\
&-(-1)^j\sum_{l=0}^{w-m}\frac{(j+l)!}{j!}\left(\frac{4\pi}{N}\right)^{-l}C^{(1)}_{k_2,m-k_1,k_2-l}\\
&\left.\Lambda^*\left(D^{w-m-l}G^{(m-k_1+1)}_{b_2,-a_1}G^{(m-k_2+1)}_{b_1,a_2} + (-1)^{w-m+j+1}D^{w-m-l}G^{(m-k_1+1)}_{b_2,a_1}G^{(m-k_2+1)}_{b_1,-a_2}, -j-l\right)\right.\\
&+(-1)^j\sum_{l=-j}^{w-m}\frac{(j+l)!}{j!}\left(\frac{4\pi}{N}\right)^{-l}C^{(2)}_{k_2,m-k_1,k_2-l,j}\\
&\left.\Lambda^*\left(D^{w-m-l}G^{(m-k_1+1)}_{b_2,a_1}G^{(m-k_2+1)}_{b_1,a_2} + (-1)^{w-m+j+1}D^{w-m-l}G^{(m-k_1+1)}_{b_2,-a_1}G^{(m-k_2+1)}_{b_1,-a_2}, -j-l\right)\right),\end{aligned}$$

where the $C^{(1)}$ and $C^{(2)}$ are integers defined in Definition 6.4.4.

Remark 6.1.8. Thanks to Lanphier's identity, we are able to rewrite the result in Theorem 6.1.7 as linear combination of L-values of modular forms (also G_2) with rational coefficients.

After an inspection we can find out that Theorem 6.1.7 extends to level $N = 1, 2$. In particular, we get

Theorem 6.1.9. *Let $k_1 \geq 2$, $k_2 \geq 2$ be even integers with $w = k_1 + k_2$. Let the integers $k_1, k_2 \leq m \leq w$ be odd and $j \geq 0$ be even, we have in level 1*

$$\int_{X^m Y^{w-m}\{0,\infty\}}^{*} \mathrm{Eis}_{\mathcal{D}}^{k_1,k_2,j}(0,0) = i^{k_1-k_2+(j+1)^2} 2(2\pi)^{w+1} j!(k_1+j+2)(k_2+j+2)$$

$$\cdot \Big(2\Lambda^* \left(D^{w-m} \mathbb{G}_{m-k_2+1} \cdot \mathbb{G}_{m-k_1+1}, -j \right) + \sum_{l=1}^{j} \frac{(4\pi)^l}{l!} \Lambda^* \left(D^{w-m+l} \mathbb{G}_{m-k_2+1} \cdot \mathbb{G}_{m-k_1+1}, l-j \right)$$

$$- \sum_{l=0}^{w-m} \frac{(j+l)!}{j!} (4\pi)^{-l} C^{(1)}_{k_2,m-k_1,k_2-l} \Lambda^* \left(D^{w-m-l} \mathbb{G}_{m-k_1+1} \cdot \mathbb{G}_{m-k_2+1}, -j-l \right)$$

$$+ \sum_{l=-j}^{w-m} \frac{(j+l)!}{j!} (4\pi)^{-l} C^{(2)}_{k_2,m-k_1,k_2-l,j} \Lambda^* \left(D^{w-m-l} \mathbb{G}_{m-k_1+1} \cdot \mathbb{G}_{m-k_2+1}, -j-l \right) \Big).$$

6.2 The Regulator Integrals

Recall in Section 2.3.2 we introduced the regulator

$$\begin{aligned} \mathrm{Eis}_{\mathcal{D}}^{k_1,k_2,j}(u_1,u_2) &= p_*(p_1^* \mathrm{Eis}_{\mathcal{D}}^{k_1+j}(u_1) \cup p_2^* \mathrm{Eis}_{\mathcal{D}}^{k_2+j}(u_2)) \\ &\in H_{\mathcal{D}}^{w+2}(E^w/\mathbb{R}, \mathbb{R}(w+j+2)), \end{aligned}$$

where p, p_1 and p_2 are the projections

$$\begin{array}{ccccc} & & E^{k_1+j+k_2} & & \\ & \swarrow^{p_1} & \downarrow{\scriptstyle p} & \searrow^{p_2} & \\ E^{k_1+j} & & E^{k_1+k_2} & & E^{j+k_2} \end{array} .$$

Write θ for the isomorphism on E^{k+j} defined by

$$\theta : E^{k_2} \times_{Y(N)} E^j \times_{Y(N)} E^{k_1} \xrightarrow{\sim} E^{k_1} \times_{Y(N)} E^j \times_{Y(N)} E^{k_2}.$$

Note that θ also restricts to an isomorphism $E^{k_2+k_1} \to E^{k_1+k_2}$ we write it also as θ by an abuse of notation.

Set $q = \theta^{-1} \circ p \circ \theta$, $q_1 = p_1 \circ \theta$ and $q_2 = p_2 \circ \theta$. We have commutative diagram

$$\begin{array}{ccc} E^{k_2+j+k_1} & \xrightarrow{\theta} & E^{k_1+j+k_2} \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle p} \\ E^{k_2+k_1} & \xrightarrow{\theta} & E^{k_1+k_2} \end{array} .$$

Since $\theta^* p_* = (\theta^{-1})_* p_* = q_* (\theta^{-1})_* = q_* \theta^*$ and $\theta^* p_i^* = q_i^*$, we have

$$\theta^* \operatorname{Eis}_{\mathcal{D}}^{k_1,k_2,j}(u_1,u_2) = q_*(q_2^* \operatorname{Eis}_{\mathcal{D}}^{k_2+j}(u_2) \cup q_1^* \operatorname{Eis}_{\mathcal{D}}^{k_1+j}(u_1)).$$

Definition 6.2.1. Let $\eta_{y,m}$ be the following fiber of $(\theta^{-1})_* X^m Y^{w-m}\{0,\infty\}$ on $E^{k_2} \times_{Y(N)} E^{k_1}$

$$\eta_{y,m} = \{(iy; it_1 y, \dots, it_{m-k_1} y, t_{m-k_1+1}, \dots, t_{k_2}, it_{k_2+1} y, \dots, it_w y, \sigma) \,|\, t_1, \dots, t_w \in [0,1]\}.$$

We endow $\eta_{y,m}$ with the orientation induced by the product orientation $[0,1]^{k_2} \times [0,1]^{k_1}$.

With the isomorphism (2.1), we write $(\tau; z_1, \dots, z_{k_1}, t_1, \dots, t_j, z'_1, \dots, z'_{k_2})$ for the coordinates on $E^{w+j}(\mathbb{C})$. We take the canonical orientation $dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_{k_1} \wedge d\bar{z}_{k_1} \wedge dt_1 \wedge d\bar{t}_1 \wedge \cdots \wedge dt_j \wedge d\bar{t}_j \wedge dz'_1 \wedge \cdots \wedge d\bar{z}'_{k_2}$ on $E^{k_1+j+k_2}$.

We divide the whole regulator integral into two parts

$$\begin{aligned}\int_{X^m Y^{w-m}\{0,\infty\}}^* \operatorname{Eis}_{\mathcal{D}}^{k_1,k_2,j}(u_1,u_2) = &\int_{X^m Y^{w-m}\{0,\infty\}}^* p_*\big(p_1^* \operatorname{Eis}_{\mathcal{D}}^{k_1+j}(u_1) \wedge \pi_{k_2+j+1}(p_2^* \operatorname{Eis}_{hol}^{k_2+j}(u_2))\big) \\ &+ \int_{X^m Y^{w-m}\{0,\infty\}}^* (-1)^{k_1+j+1} p_*\big(\pi_{k_1+j+1}(p_1^* \operatorname{Eis}_{hol}^{k_1+j}(u_1)) \wedge p_2^* \operatorname{Eis}_{\mathcal{D}}^{k_2+j}(u_2)\big).\end{aligned}$$

For the second integral we have

$$\begin{aligned}&(-1)^{k_1+j+1} \int_{X^m Y^{w-m}\{0,\infty\}}^* p_*\big(\pi_{k_1+j+1}(p_1^* \operatorname{Eis}_{hol}^{k_1+j}(u_1)) \wedge p_2^* \operatorname{Eis}_{\mathcal{D}}^{j+k_2}(u_2)\big) \\ =&(-1)^{k_1+j+1} \int_{(\theta^{-1})_* X^m Y^{w-m}\{0,\infty\}}^* \theta^*\big(p_*\big(\pi_{k_1+j+1}(p_1^* \operatorname{Eis}_{hol}^{k_1+j}(u_1)) \wedge p_2^* \operatorname{Eis}_{\mathcal{D}}^{j+k_2}(u_2)\big)\big) \\ =&(-1)^{(k_1+j+1)(k_2+j+1)} \int_{(\theta^{-1})_* X^m Y^{w-m}\{0,\infty\}}^* \theta^*\big(p_*\big(p_2^* \operatorname{Eis}_{\mathcal{D}}^{j+k_2}(u_2) \wedge \pi_{k_1+j+1}(p_1^* \operatorname{Eis}_{hol}^{k_1+j}(u_1))\big)\big) \\ =&(-1)^{(k_1+j+1)(k_2+j+1)+k_1 j+k_2 j} \int_{(\theta^{-1})_* X^m Y^{w-m}\{0,\infty\}}^* q_*\big(q_2^* \operatorname{Eis}_{\mathcal{D}}^{k_2+j}(u_2) \wedge \pi_{k_1+j+1}(q_1^* \operatorname{Eis}_{hol}^{j+k_1}(u_1))\big).\end{aligned}$$

The isomorphism θ change the orientation of $X^m Y^{w-m}\{0,\infty\}$ with a sign $(-1)^{k_1 k_2}$, so it is

$$(-1)^{k_1+k_2+j+1} \int_0^{\infty,*} \int_{\eta_{y,m}} q_*\big(q_2^* \operatorname{Eis}_{\mathcal{D}}^{k_2+j}(u_2) \wedge \pi_{k_1+j+1}(q_1^* \operatorname{Eis}_{hol}^{j+k_1}(u_1))\big).$$

For brevity, we set

$$I^{k_1,k_2,j}(u_1,u_2) = \int_0^{\infty,*} \int_{\gamma_{y,m}} p_*\big(p_1^* \operatorname{Eis}_{\mathcal{D}}^{k_1+j}(u_1) \wedge \pi_{k_2+j+1}(p_2^* \operatorname{Eis}_{hol}^{k_2+j}(u_2))\big),$$

and

$$J^{k_1,k_2,j}(u_1,u_2) = \int_0^{\infty,*} \int_{\eta_{y,m}} q_*\big(q_2^* \operatorname{Eis}_{\mathcal{D}}^{k_2+j}(u_2) \wedge \pi_{k_1+j+1}(q_1^* \operatorname{Eis}_{hol}^{j+k_1}(u_1))\big).$$

Finally, the full regulator is given as

$$\begin{aligned}\int^*_{X^mY^{w-m}\{0,\infty\}} \mathrm{Eis}_{\mathcal{D}}^{k_1,k_2,j}(u_1,u_2) &= \int^*_{X^mY^{w-m}\{0,\infty\}} p_*\big(p_1^*\,\mathrm{Eis}_{\mathcal{D}}^{k_1+j}(u_1)\wedge \pi_{k_2+j+1}(p_2^*\,\mathrm{Eis}_{hol}^{k_2+j}(u_2))\big)\\ &+\int^*_{X^mY^{w-m}\{0,\infty\}} (-1)^{k_1+j+1}p_*\big(\pi_{k_1+j+1}(p_1^*\,\mathrm{Eis}_{hol}^{k_1+j}(u_1))\wedge p_2^*\,\mathrm{Eis}_{\mathcal{D}}^{k_2+j}(u_2)\big)\\ &= I^{k_1,k_2,j}(u_1,u_2)+(-1)^{k_1+k_2+j+1}J^{k_1,k_2,j}(u_1,u_2).\end{aligned}$$

6.3 Explicit Regulators

Let us start by recalling our object of interest.
Given $u_1=(a_1,b_1)$, $u_2=(a_2,b_2)\in(\mathbb{Z}/N\mathbb{Z})^2$, we have the differential forms

$$\begin{aligned}\mathrm{Eis}_{\mathcal{D}}^{k_1+j}(u_1) &= -\frac{(k_1+j)!(k_1+j+2)}{2\pi N}\,\mathrm{Im}(\tau)\sum_{a=0}^{k_1+j}F_{gu_1}^{a,k_1+j-a}(\tau)\psi_{a,k_1+j-a} \mod d\tau,d\overline{\tau}\\ &= -\frac{(k_1+j)!(k_1+j+2)}{2\pi N}\,\mathrm{Im}(-\frac{1}{\tau})\sum_{a=0}^{k_1+j}F_{\sigma gu_1}^{a,k_1+j-a}(-\frac{1}{\tau})\tau^{-a}\overline{\tau}^{-k_1+a}\psi_{a,k_1+j-a} \mod d\tau,d\overline{\tau}\end{aligned}$$

and

$$\mathrm{Eis}_{hol}^{k_2+j}(u_2)=(-1)^{k_2+j+1}\frac{k_2+j+2}{N}(2i\pi)^{k_2+j+1}F_{\sigma gu_2}^{(k_2+j+2)}(\tau)d\tau\wedge\psi_{k_2+j,0}.$$

Let $\pi:E^{k_1+j+k_2}(\mathbb{C})\to\mathcal{H}$ and $\pi':E^{k_2+j+k_1}(\mathbb{C})\to\mathcal{H}$ be the canonical projections onto the upper half-plane. Let $\nu:\mathcal{H}\to Y(N)(\mathbb{C})$ be the map onto the connected component of $Y(N)$ given by $\tau\mapsto[(\tau,\sigma)]$. Given $t\in\mathbb{R}$, set $W=\pi^{-1}(\{0,\infty\})$, $W'=\pi'^{-1}(\{0,\infty\})$, $V=\pi^{-1}([0,x]^y_\infty)$ and $V'=\pi'^{-1}([0,x]^y_\infty)$. They can be seen as subvarieties of $E^k(\mathbb{C})$. Then we see directly that

Lemma 6.3.1. *We have differential forms*

$$\begin{aligned}p_1^*\,\mathrm{Eis}_{\mathcal{D}}^{k_1+j}(u_1)|_W &= C_1y^{-k_1-j-1}\sum_{a=0}^{k_1+j}(-1)^{k_1+j-a}F_{-u_1}^{a,k_1+j-a}(\frac{i}{y})p_1^*\psi_{a,k_1+j-a} \mod dy\\ p_1^*\,\mathrm{Eis}_{\mathcal{D}}^{k_1+j}(u_1)|_V &= D_1y\sum_{a=0}^{k_1+j}F_{\sigma u_1}^{a,k_1+j-a}(x+iy)p_1^*\psi_{a,k_1+j-a} \mod d\tau,d\overline{\tau}\end{aligned}$$

and

$$\begin{aligned}p_2^*\,\mathrm{Eis}_{hol}^{k_2+j}(u_2)|_W &= 2C_2dy\wedge F_{-u_2}^{(k_2+j+2)}(iy)p_2^*\psi_{k_2+j,0}\\ p_2^*\,\mathrm{Eis}_{hol}^{k_2+j}(u_2)|_V &= 2D_2dx\wedge F_{-u_2}^{(k_2+j+2)}(x+iy)p_2^*\psi_{k_2+j,0},\end{aligned}$$

where

$$C_1 = -\, i^{-k_1-j}\frac{(k_1+j)!(k_1+j+2)}{2\pi N},\quad C_2 = i\frac{k_2+j+2}{2N}(-2i\pi)^{k_2+j+1},$$
$$D_1 = -\,\frac{(k_1+j)!(k_1+j+2)}{2\pi N},\quad D_2 = \frac{k_2+j+2}{2N}(-2i\pi)^{k_2+j+1}.$$

Symmetrically we get

Lemma 6.3.2. *We have differential forms*

$$q_2^*\,\mathrm{Eis}_{\mathcal{D}}^{k_2+j}(u_2)|_{W'} = C_1' y^{-k_2-j-1}\sum_{a=0}^{k_2+j}(-1)^{k_2+j-a}F_{-u_2}^{a,k_2+j-a}(\frac{i}{y})q_2^*\psi_{a,k_2+j-a} \mod dy$$
$$q_2^*\,\mathrm{Eis}_{\mathcal{D}}^{k_2+j}(u_2)|_{V'} = D_1' y\sum_{a=0}^{k_2+j}F_{\sigma u_2}^{a,k_2+j-a}(x+iy)q_2^*\psi_{a,k_2+j-a} \mod d\tau, d\overline{\tau}$$

and

$$q_1^*\,\mathrm{Eis}_{hol}^{k_1+j}(u_1)|_{W'} = 2C_2' dy\wedge F_{-u_1}^{(k_1+j+2)}(iy)q_1^*\psi_{k_1+j,0}$$
$$q_1^*\,\mathrm{Eis}_{hol}^{k_1+j}(u_1)|_{V'} = 2D_2' dx\wedge F_{-u_1}^{(k_1+j+2)}(x+iy)q_1^*\psi_{k_1+j,0},$$

where

$$C_1' = -\, i^{-k_2-j}\frac{(k_2+j)!(k_2+j+2)}{2\pi N},\quad C_2' = i\frac{k_1+j+2}{2N}(-2i\pi)^{k_1+j+1},$$
$$D_1' = -\,\frac{(k_2+j)!(k_2+j+2)}{2\pi N},\quad D_2' = \frac{k_1+j+2}{2N}(-2i\pi)^{k_1+j+1}.$$

With the Fourier development in Lemma 1.3.4, we are able to use S-series to demonstrate the differential form $p_1^*\,\mathrm{Eis}_{\mathcal{D}}^{k_1+j}(u_1)$.

Lemma 6.3.3. *We have*

$$p_1^*\,\mathrm{Eis}_{\mathcal{D}}^{k_1+j}(u_1)|_W = \eta_0+\eta_1$$

with

$$\eta_0 = C_1\sum_{a=0}^{k_1+j}(-1)^{k_1+j-a}\alpha y^{-k_1-j-1}p_1^*\psi_{a,k_1+j-a},$$

$$\eta_1 = (-i)^{k_1+j}\frac{N}{2^{k_1+j+2}}C_1\sum_{l=0}^{k_1+j}\left(\mathfrak{S}_{\delta_{b_1},\hat{\delta}_{a_1}}^{l,l-k_1-1-j}\left(\frac{i}{y}\right)+(-1)^{k_1+j}\mathfrak{S}_{\delta_{-b_1},\hat{\delta}_{-a_1}}^{l,l-k_1-1-j}\left(\frac{i}{y}\right)\right)\Omega_l$$
$$+(-i)^{k_1+j}\frac{N}{2^{k_1+j+2}}C_1\sum_{l=0}^{k_1+j}\left(\overline{\mathfrak{S}}_{\delta_{-b_1},\hat{\delta}_{-a_1}}^{l,l-k_1-1-j}\left(\frac{i}{y}\right)+(-1)^{k_1+j}\overline{\mathfrak{S}}_{\delta_{b_1},\hat{\delta}_{a_1}}^{l,l-k_1-1-j}\left(\frac{i}{y}\right)\right)\overline{\Omega}_l \mod dy,$$

where

$$\alpha = L(\delta_{b_1}, k_1 + j + 2) + (-1)^{k_1+j} L(\delta_{-b_1}, k_1 + j + 2)$$

and

$$\Omega_l = \frac{(k_1 + j - l)!}{l!} \left(\frac{4\pi}{N}\right)^{l+1} y^{-l} \sum_{a=l}^{k_1+j} \frac{p_1^* \psi_{a,k_1+j-a}}{(k_1 + j - a)!(a - l)!}.$$

Proof. (See [11, (25)]) Note that we write the Fourier expansion with the $\mathcal{S}$-series here for the sake of simplicity. This will prevent us from some intricate discussions of constant terms later. □

After exchanging k_1 and k_2,

Lemma 6.3.4. *We have*

$$q_2^* \operatorname{Eis}_{\mathcal{D}}^{k_2+j}(u_1)|_{W'} = \eta_0' + \eta_1'$$

with

$$\eta_0' = C_1' \sum_{a=0}^{k_2+j} (-1)^{k_2+j-a} \alpha' y^{-k_2-j-1} q_2^* \psi_{a,k_2+j-a},$$

$$\begin{aligned}\eta_1' = &(-i)^{k_2+j} \frac{N}{2^{k_2+j+2}} C_1 \sum_{l=0}^{k_2+j} \left(\mathcal{S}_{\delta_{b_2}, \hat{\delta}_{a_2}}^{l, l-k_2-1-j} \left(\frac{i}{y}\right) + (-1)^{k_2+j} \mathcal{S}_{\delta_{-b_2}, \hat{\delta}_{-a_2}}^{l, l-k_2-1-j} \left(\frac{i}{y}\right) \right) \Omega_l' \\ &+ (-i)^{k_2+j} \frac{N}{2^{k_2+j+2}} C_1 \sum_{l=0}^{k_2+j} \left(\overline{\mathcal{S}}_{\delta_{-b_2}, \hat{\delta}_{-a_2}}^{l, l-k_2-1-j} \left(\frac{i}{y}\right) + (-1)^{k_2+j} \overline{\mathcal{S}}_{\delta_{b_2}, \hat{\delta}_{a_2}}^{l, l-k_2-1-j} \left(\frac{i}{y}\right) \right) \overline{\Omega'}_l \mod dy,\end{aligned}$$

where

$$\alpha' = L(\delta_{b_2}, k_2 + j + 2) + (-1)^{k_2+j} L(\delta_{-b_2}, k_2 + j + 2)$$

and

$$\Omega_l' = \frac{(k_2 + j - l)!}{l!} \left(\frac{4\pi}{N}\right)^{l+1} y^{-l} \sum_{a=l}^{k_2+j} \frac{q_2^* \psi_{a,k_2+j-a}}{(k_2 + j - a)!(a - l)!}.$$

6.4 Binomial Identities

We first prove some identities involving binomial coefficients which we will need in the future computations of the regulator integral.

Definition 6.4.1. Let a, b, c be nonnegative integers such that $b, c \leq a$. The numbers $H_{a,b,c}$ are the hypergeometric sums

$$\begin{aligned}H_{a,b,c} = &\sum_{\substack{\epsilon_1+\cdots+\epsilon_a=c \\ \epsilon_i \in \{0,1\}}} (-1)^{\epsilon_1+\cdots+\epsilon_b} = \sum_{s=0}^{\infty} (-1)^s \binom{b}{s} \binom{a-b}{c-s} \\ = &\binom{a-b}{c} {}_2F_1\left(\begin{matrix} -c, & -b \\ a-b-c+1 \end{matrix}; -1\right).\end{aligned}$$

One see directly that

Lemma 6.4.2. *Let k, m, a be non-negative integers and $a, m \leq k$, we have*

$$H_{k,m,k-a} = (-1)^m H_{k,m,a}.$$

The following variation of Chu-Vandermonde identity is needed in the computation.

Lemma 6.4.3. *Let n, a, b be positive integers,*

$$\sum_{j=0}^{a} (-1)^{a-j} \binom{a}{j} \binom{b+j}{c} = \begin{cases} 0 & c < a, \\ \binom{b}{c-a} & c \geq a. \end{cases}$$

We will need also the following constants.

Definition 6.4.4. Let k, m, l, j be non-negative integers and $m \leq k$, we define the constants

$$C^{(1)}_{k,m,l} := \sum_{a=l}^{k} \binom{a}{l} H_{k,m,k-a},$$

$$C^{(2)}_{k,m,l,j} := \sum_{a=\max\{j,l\}}^{k+j} \binom{a}{l} H_{k,m,k+j-a}.$$

Lemma 6.4.5. *Let k, m, l, j be non-negative integers and $m \leq k$. Then the constants $C^{(1)}_{k,m,l}$ and $C^{(2)}_{k,m,l,j}$ will vanish if $l < m$.*

Proof. Let us show that $C^{(2)}_{k,m,l,j} = 0$ when $l \leq j$, the proof of the other case $l > j$ follows in exactly similar way, and $C^{(1)}_{k,m,l}$ follows as well. Note that

$$C^{(2)}_{k,m,l,j} x^k = \sum_{a=0}^{k} \sum_{s=0}^{m} (-1)^s \binom{a+j}{l} \binom{m}{s} x^{a+s} \binom{k-m}{k-a-s} x^{k-a-s},$$

thus $C^{(2)}_{k,m,l,j}$ is the k-th coefficient in the power series of product of

$$(1-x)^m {}_2F_1\left(\begin{matrix} 1, j+1 \\ j-l+1 \end{matrix}; x\right) + o(x^k) = \sum_{a=0}^{k} \sum_{s=0}^{m} (-1)^s \binom{a+j}{l} \binom{m}{s} x^{a+s}$$

and

$$(1+x)^{k-m} = \sum_{t=0}^{k-m} \binom{k-m}{t} x^t.$$

Meanwhile,

$$(1+x)^{k-m}(1-x)^m {}_2F_1\left(\begin{matrix} 1, j+1 \\ j-l+1 \end{matrix}; x\right) = (1+x)^{k-m}(1-x)^{m-l-1} {}_2F_1\left(\begin{matrix} j-l, -l \\ j-l+1 \end{matrix}; x\right)$$

is a polynomial of degree $k-1$. So we conclude $C^{(2)}_{k,m,l,j} = 0$ for $l < m$. □

We shall need the following lemma to simplify some L-values of quasi-modular forms with Lanphier's formula.

Lemma 6.4.6. *Let $m, n \geq 1$ and $r, p \geq 0$ be integers. Set*

$$a_{m,n}^{l}(t) = \frac{\binom{l}{t}\binom{m+l-1}{t}}{\binom{m+n+2l-2t-2}{l-t}\binom{m+n+2l-t-1}{t}}$$

to be the coefficient which appears in Lanphier's formula (1.2)*. Then*

$$\sum_{t=0}^{r}(-2)^t\frac{(r)_t}{(t+p)!}\left(a_{m,n}^{t+p}(t)+(-1)^{r+1}a_{n,m}^{t+p}(t)\right)=0.$$

Proof. We see

$$\sum_{t=0}^{r}(-2)^t\frac{(r)_t}{(t+p)!}a_{m,n}^{t+p}(t)=\frac{(m+n+p-2)!(m+n+2p-1)}{(m+p-1)!} \cdot\sum_{t=0}^{r}(-2)^t\frac{(m+t+p-1)!r!}{(m+n+t+2p-1)!t!(r-t)!}.$$

This is exactly the hypergeometric value

$$\frac{(m+m+p-2)!}{(m+n+2p-2)!}{}_2F_1\left(\begin{matrix}-r, m+p\\ m+n+2p\end{matrix};2\right).$$

With the relation

$${}_2F_1\left(\begin{matrix}-r, m+p\\ m+n+2p\end{matrix};z\right)=(1-z)^r{}_2F_1\left(\begin{matrix}-r, n+p\\ m+n+2p\end{matrix};z\right)$$

we conclude. □

Lemma 6.4.7. *Let k, m be non-negative integers and $m \leq k$, then*

$$\sum_{a=0}^{k}H_{k,m,a}=\begin{cases}0 & m\neq 0,\\ 2^k & m=0.\end{cases}$$

Proof. Exchange the order of sum, we see

$$\begin{aligned}\sum_{a=0}^{k}H_{k,m,a}&=\sum_{a=0}^{k}\sum_{s=0}^{\infty}(-1)^s\binom{m}{s}\binom{k-m}{a-s}\\ &=\sum_{s=0}^{\infty}(-1)^s\binom{m}{s}\sum_{a=s}^{k-m+s}\binom{k-m}{a-s}\\ &=\sum_{s=0}^{\infty}(-1)^s\binom{m}{s}2^{k-m}\\ &=\delta_{m=0}2^{k-m}.\end{aligned}$$

□

6.5 Some Preparations I

In preparation to computing the regulator integral, we first evaluate their integration along the fibers $\gamma_{y,m}$ and $\eta_{y,m}$. Let us start by the following integrals.

Lemma 6.5.1. *When $m \geq k_1$,*

$$\int_{\gamma_{\tau,m}} p_*(p_1^*\psi_{a,k_1+j-a} \wedge p_2^*\psi_{k_2+j,0}) = \begin{cases} C_3 \cdot (-1)^{k_1-a}\binom{k_1+j-a}{j} y^{m+j} & 0 \leq a \leq k_1, \\ 0 & k_1 < a \leq k_1+j, \end{cases}$$

$$\int_{\gamma_{\tau,m}} p_*(p_1^*\psi_{a,k_1+j-a} \wedge p_2^*\psi_{0,k_2+j}) = \begin{cases} C_3 \cdot (-1)^{m-a}\binom{a}{j} y^{m+j} & j \leq a \leq k_1+j, \\ 0 & 0 \leq a < j, \end{cases}$$

$$\int_{\gamma_{\tau,m}} p_*(p_1^*\psi_{k_1+j-a,a} \wedge p_2^*\psi_{k_2+j,0}) = \begin{cases} C_3 \cdot (-1)^{j-a}\binom{a}{j} y^{m+j} & j \leq a \leq k_1+j, \\ 0 & 0 \leq a < j, \end{cases}$$

and

$$\int_{\gamma_{\tau,m}} p_*(p_1^*\psi_{k_1+j-a,a} \wedge p_2^*\psi_{0,k_2+j}) = \begin{cases} C_3 \cdot (-1)^{m-k_1+j-a}\binom{k_1+j-a}{j} y^{m+j} & 0 \leq a \leq k_1, \\ 0 & k_1 < a \leq k_1+j, \end{cases}$$

where

$$C_3 = (-1)^{\frac{1}{2}j(j-1)} \frac{i^m k_1! j!}{\pi^j (k_1+j)!}.$$

Proof. Recall that we take the canonical orientation $dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_{k_1} \wedge d\bar{z}_{k_1} \wedge dt_1 \wedge d\bar{t}_1 \wedge \cdots \wedge dt_j \wedge d\bar{t}_j \wedge dz'_1 \wedge \cdots \wedge d\bar{z}'_{k_2}$ on $E^{k_1+j+k_2}$. For the first identity, when $0 \leq a \leq k_1$, we have

$$\begin{aligned} & p_1^*\psi_{a,k_1+j-a} \wedge p_2^*\psi_{k_2+j,0} \\ =& \binom{k_1+j}{a}^{-1} \sum_{\Sigma\varepsilon_i = k_1+j-a} dz_1^{(\varepsilon_1)} \wedge \cdots \wedge dz_{k_1}^{(\varepsilon_{k_1})} \wedge dt_1^{(\varepsilon_{k_1+1})} \wedge \cdots \wedge dt_j^{(\varepsilon_{k_1+j})} \\ & \wedge dt_1 \wedge \cdots \wedge dt_j \wedge dz'_1 \wedge \cdots \wedge dz'_{k_2} \\ =& (-1)^{\frac{1}{2}j(j+1)} \binom{k_1+j}{a}^{-1} \sum_{\Sigma\varepsilon_i = k_1-a} dz_1^{(\varepsilon_1)} \wedge \cdots \wedge dz_{k_1}^{(\varepsilon_{k_1})} \wedge dz'_1 \wedge \cdots \wedge dz'_{k_2} \\ & \wedge (dt_1 \wedge d\bar{t}_1) \cdots \wedge (dt_j \wedge d\bar{t}_j), \end{aligned}$$

and when $a \geq k_1$, this form vanishes because we have at least one $dt_i \wedge dt_i = 0$. Integrating along each fiber of the projection $p : E^{k_1+j+k_2} \to E^{k_1+k_2}$ while noticing the fact $\int_{E_\tau} dt \wedge$

$d\bar{t} = -2i \operatorname{Im} \tau$, we get

$$\begin{aligned}
&\int_{\gamma_{\tau,m}} p_*(p_1^*\psi_{a,k_1+j-a} \wedge p_2^*\psi_{k_2+j,0}) \\
&= \int_{\gamma_{\tau,m}} \frac{1}{(2\pi i)^j} \int_{E_\tau^j} p_1^*\psi_{a,k_1+j-a} \wedge p_2^*\psi_{k_2+j,0} \\
&= (-1)^{\frac{1}{2}j(j+1)}(2\pi i)^{-j}\binom{k_1+j}{a}^{-1}\binom{k_1}{k_1-a}\overline{\tau}^{k_1-a}\tau^{m-k_1+a}(-2iy)^j \\
&= C_3 \cdot (-1)^{k_1-a}\binom{k_1+j-a}{j} y^j.
\end{aligned}$$

The second one is similar, when $j \le a$,

$$\begin{aligned}
&\int_{\gamma_{y,m}} p_*(p_1^*\psi_{a,k_1+j-a} \wedge p_2^*\psi_{0,k_2+j}) \\
&= \int_{\gamma_{y,m}} \frac{1}{(2\pi i)^j} \int_{E_\tau^j} p_1^*\psi_{a,k_1+j-a} \wedge p_2^*\psi_{0,k_2+j} \\
&= \int_{\gamma_{y,m}} \frac{1}{(2\pi i)^j} \int_{E_\tau^j} \binom{k_1+j}{a}^{-1} \sum_{\Sigma\varepsilon_i = k_1+j-a} dz_1^{(\varepsilon_1)} \wedge \cdots \wedge dz_{k_1}^{(\varepsilon_{k_1})} \wedge dt_1^{(\varepsilon_{k_1+1})} \wedge \cdots \wedge dt_j^{(\varepsilon_{k_1+j})} \\
&\wedge d\bar{t}_1 \wedge \cdots \wedge d\bar{t}_j \wedge d\overline{z'}_1 \wedge \cdots \wedge d\overline{z'}_{k_2} \\
&= (-1)^{\frac{1}{2}j(j-1)}(2\pi i)^{-j}\binom{k_1+j}{a}^{-1}\binom{k_1}{k_1+j-a}(-1)^{m+j-a}(iy)^k(-2iy)^j \\
&= C_3 \cdot (-1)^{m-a}\binom{a}{j} y^{m+j}.
\end{aligned}$$

Take $k_1 + j - a$ instead of a and we get the rests. □

For integral over $\eta_{y,m}$ we have similar results.

Lemma 6.5.2. *When $m \ge k_1$,*

$$\int_{\eta_{y,m}} q_*(q_2^*\psi_{a,k_2+j-a} \wedge q_1^*\psi_{k_1+j,0}) = \begin{cases} C_3' \cdot \frac{a!(k_2+j-a)!}{j!k_2!} H_{k_2,m-k_1,k_2-a} y^{m+j} & 0 \le a \le k_2, \\ 0 & k_2 < a \le k_2 + j, \end{cases}$$

$$\int_{\eta_{y,m}} q_*(q_2^*\psi_{a,k_2+j-a} \wedge q_1^*\psi_{0,k_1+j}) = \begin{cases} C_3' \cdot (-1)^{k_1+j} \frac{a!(k_2+j-a)!}{j!k_2!} H_{k_2,m-k_1,k_2+j-a} y^{m+j} & j \le a \le k_2 + j, \\ 0 & 0 \le a < j, \end{cases}$$

$$\int_{\eta_{y,m}} q_*(q_2^*\psi_{k_2+j-a,a} \wedge q_1^*\psi_{k_1+j,0}) = \begin{cases} C_3' \cdot \frac{a!(k_2+j-a)!}{j!k_2!} H_{k_2,m-k_1,a-j} y^{m+j} & j \le a \le k_2 + j, \\ 0 & 0 \le a < j, \end{cases}$$

and

$$\int_{\eta_{y,m}} q_*(q_2^*\psi_{k_2+j-a,a} \wedge q_1^*\psi_{0,k_1+j}) = \begin{cases} C_3' \cdot (-1)^{k_1+j} \frac{a!(k_2+j-a)!}{j!k_2!} H_{k_2,m-k_1,a} y^{m+j} & 0 \le a \le k_2, \\ 0 & k_2 < a \le k_2 + j, \end{cases}$$

where

$$C_3' = (-1)^{\frac{1}{2}j(j-1)} \frac{i^m k_2! j!}{\pi^j (k_2+j)!}.$$

Proof. The proof follows exactly the same as Lemma 6.5.1. We take the first one as an example.

The form

$$\begin{aligned}
&q_1^* \psi_{a,k_2+j-a} \wedge q_2^* \psi_{k_1+j,0} \\
=&\binom{k_2+j}{a}^{-1} \sum_{\Sigma \varepsilon_i = k_2+j-a} dz_1'^{(\varepsilon_1)} \wedge \cdots \wedge dz_{k_2}'^{(\varepsilon_{k_2})} \wedge dt_1^{(\varepsilon_{k_2+1})} \wedge \cdots \wedge dt_j^{(\varepsilon_{k_2+j})} \\
&\wedge dt_1 \wedge \cdots \wedge dt_j \wedge dz_1 \wedge \cdots \wedge dz_{k_1} \\
=&(-1)^{\frac{1}{2}j(j+1)} \binom{k_2+j}{a}^{-1} \sum_{\Sigma \varepsilon_i = k_2-a} dz_1'^{(\varepsilon_1)} \wedge \cdots \wedge dz_{k_2}'^{(\varepsilon_{k_2})} \wedge dz_1 \wedge \cdots \wedge dz_{k_1} \\
&\wedge (dt_1 \wedge d\bar{t}_1) \cdots \wedge (dt_j \wedge d\bar{t}_j)
\end{aligned}$$

vanishes precisely if $0 \leq a \leq k_1$. Recalling the constant

$$H_{k_2,m-k_1,k_2-a} = \sum_{\substack{\epsilon_1+\cdots+\epsilon_{k_2}=k_2-a \\ \epsilon_i \in \{0,1\}}} (-1)^{\epsilon_1+\cdots+\epsilon_{m-k_1}},$$

we integrate this form on $\eta_{y,m}$ and get the first formula. □

With Lemma 6.5.1 we can get

Lemma 6.5.3. *When $m \geq k_1$, for $0 \leq l \leq k_1 + j$, we have*

$$\int_{\gamma_{y,m}} p_*(\Omega_l \wedge p_2^* \psi_{k_2+j,0}) = \begin{cases} C_3 \cdot \frac{1}{k_1!} \left(\frac{4\pi}{N}\right)^{k_1+1} y^{m-k_1+j} & l = k_1, \\ 0 & l \neq k_1, \end{cases}$$

$$\int_{\gamma_{y,m}} p_*(\Omega_l \wedge p_2^* \psi_{0,k_2+j}) = \begin{cases} C_3 \cdot (-1)^{m-k_1+j} \frac{1}{k_1!(l-k_1)!} \left(\frac{4\pi}{N}\right)^{l+1} y^{m+j-l} & k_1 \leq l \leq k_1+j, \\ 0 & 0 \leq l < k_1, \end{cases}$$

$$\int_{\gamma_{y,m}} p_*(\overline{\Omega}_l \wedge p_2^* \psi_{k_2+j,0}) = \begin{cases} C_3 \cdot (-1)^{k_1} \frac{1}{k_1!(l-k_1)!} \left(\frac{4\pi}{N}\right)^{l+1} y^{m+j-l} & k_1 \leq l \leq k_1+j, \\ 0 & 0 \leq l < k_1, \end{cases}$$

$$\int_{\gamma_{y,m}} p_*(\overline{\Omega}_l \wedge p_2^* \psi_{0,k_2+j}) = \begin{cases} C_3 \cdot (-1)^{m+j} \frac{1}{k_1!} \left(\frac{4\pi}{N}\right)^{k_1+1} y^{m-k_1+j} & l = k_1, \\ 0 & l \neq k_1. \end{cases}$$

Proof. For the first one,

$$\begin{aligned}\int_{\gamma_{y,m}} p_*(\Omega_l \wedge p_2^*\psi_{k_2+j,0}) &= C_3 \cdot \frac{(k_1+j-l)!}{j!l!}\left(\frac{4\pi}{N}\right)^{l+1} y^{m+j-l}\sum_{a=l}^{k_1}\frac{(-1)^{k_1-a}}{(k_1-a)!(a-l)!}\\ &= C_3 \cdot \frac{(k_1+j-l)!}{j!l!}\left(\frac{4\pi}{N}\right)^{l+1} y^{m+j-l}\sum_{a=0}^{k_1-l}\frac{(-1)^{k_1-l-a}}{(k_1-l-a)!a!}\\ &= C_3 \cdot \frac{(k_1+j-l)!}{j!l!(k_1-l)!}\left(\frac{4\pi}{N}\right)^{l+1} y^{m+j-l}\sum_{a=0}^{k_1-l}(-1)^{k_1-l-a}\binom{k_1-l}{a}\\ &= \begin{cases}0, & l\neq k_1,\\ C_3\cdot\frac{1}{k_1!}\left(\frac{4\pi}{N}\right)^{k_1+1} y^{m-k_1+j} & l=k_1.\end{cases}\end{aligned}$$

For the second one

$$\begin{aligned}&\int_{\gamma_{y,m}} p_*(\Omega_l \wedge p_2^*\psi_{0,k_2+j})\\ &\quad= C_3 \cdot \frac{(k_1+j-l)!}{j!l!}\left(\frac{4\pi}{N}\right)^{l+1} y^{m+j-l}\sum_{a=\max\{j,l\}}^{k_1+j}\frac{(-1)^{m-a}a!}{(k_1+j-a)!(a-l)!(a-j)!}.\end{aligned}$$

We consider the case $l \geq j$ first, then

$$\begin{aligned}&\int_{\gamma_{y,m}} p_*(\Omega_l \wedge p_2^*\psi_{0,k_2+j})\\ &\quad= C_3 \cdot \frac{(k_1+j-l)!}{j!l!}\left(\frac{4\pi}{N}\right)^{l+1} y^{m+j-l}\sum_{a=0}^{k_1+j-l}\frac{(-1)^{m-l-a}(a+l)!}{(k_1+j-l-a)!a!(a+l-j)!}\\ &\quad= C_3 \cdot \frac{1}{l!}\left(\frac{4\pi}{N}\right)^{l+1} y^{k+j-l}\sum_{a=0}^{k_1+j-l}(-1)^{m-a-l}\binom{k_1+j-l}{a}\binom{a+l}{j}.\end{aligned}$$

With Lemma 6.4.3 we know this equals

$$\begin{cases}0, & 0\leq l<k_1,\\ C_3\cdot(-1)^{m-k_1+j}\frac{1}{k_1!(l-k_1)!}\left(\frac{4\pi}{N}\right)^{l+1} y^{k+j-l} & k_1\leq l\leq k_1+j.\end{cases}$$

For the case $l < j$, we have

$$\begin{aligned}
&\int_{\gamma_{y,m}} p_*(\Omega_l \wedge p_2^*\psi_{0,k_2+j}) \\
&\quad = C_3 \cdot \frac{(k_1+j-l)!}{j!l!}\left(\frac{4\pi}{N}\right)^{l+1} y^{k+j-l}\sum_{a=0}^{k_1}\frac{(-1)^{k-j-a}(a+j)!}{(k_1-a)!a!(a-l+j)!} \\
&\quad = C_3 \cdot \frac{(k_1+j-l)!}{j!k_1!}\left(\frac{4\pi}{N}\right)^{l+1} y^{k+j-l}\sum_{a=0}^{k_1}(-1)^{k-j-a}\binom{k_1}{a}\binom{a+j}{l},
\end{aligned}$$

which leads to the same result as $l \geq j$. The rest calculations are similar. □

Along the same lines of Lemma 6.5.3, we have the followings results.

Lemma 6.5.4. *When $m \geq k_1$, for $0 \leq l \leq k_2 + j$, we have*

$$\int_{\eta_{y,m}} q_*(\Omega_l' \wedge q_1^*\psi_{k_1+j,0}) = \begin{cases} C_3' \cdot \frac{(k_2+j-l)!}{j!k_2!}\left(\frac{4\pi}{N}\right)^{l+1} C^{(1)}_{k_2,m-k_1,l} y^{m+j-l} & m-k_1 \leq l \leq k_2, \\ 0 & \textit{else}, \end{cases}$$

$$\int_{\eta_{y,m}} q_*(\Omega_l' \wedge q_1^*\psi_{0,k_1+j}) = \begin{cases} C_3' \cdot (-1)^{k_1+j}\frac{(k_2+j-l)!}{j!k_2!}\left(\frac{4\pi}{N}\right)^{l+1} C^{(2)}_{k_2,m-k_1,l,j} y^{m+j-l} & m-k_1 \leq l \leq k_2+j, \\ 0 & \textit{else}, \end{cases}$$

$$\int_{\eta_{y,m}} q_*(\overline{\Omega}_l' \wedge q_1^*\psi_{k_1+j,0}) = \begin{cases} C_3' \cdot (-1)^{m-k_1}\frac{(k_2+j-l)!}{j!k_2!}\left(\frac{4\pi}{N}\right)^{l+1} C^{(2)}_{k_2,m-k_1,l,j} y^{m+j-l} & m-k_1 \leq l \leq k_2+j, \\ 0 & \textit{else}, \end{cases}$$

$$\int_{\eta_{y,m}} q_*(\overline{\Omega}_l' \wedge q_1^*\psi_{0,k_1+j}) = \begin{cases} C_3' \cdot (-1)^{m+j}\frac{(k_2+j-l)!}{j!k_2!}\left(\frac{4\pi}{N}\right)^{l+1} C^{(1)}_{k_2,m-k_1,l} y^{m+j-l} & m-k_1 \leq l \leq k_2, \\ 0 & \textit{else}. \end{cases}$$

6.6 Some Preparations II

We compute in this section some explicit regularized L-values of Eisenstein series and their derivatives.

Lemma 6.6.1. *Let $l > 0$ and $k \geq 0$ be integers, then*

$$\begin{aligned}
\int_0^{\infty,*} y^l\left(F_{a,b}^{(k+2)}(iy) + (-1)^{l-1}\overline{F}_{a,b}^{(k+2)}(iy)\right)\frac{dy}{y} = \\
(2\pi)^{-l}\Gamma(l)N^{l-k-1}L(\hat{\delta}_{-b} + (-1)^{l-1}\hat{\delta}_b, l)L(\delta_a + (-1)^{l+k}\delta_{-a}, l-k-1).
\end{aligned}$$

In particular, if $l \leq k$, this regularized integral will vanish.

Proof. Let $a_0(F_{a,b}^{(k+2)})$ be the constant term in the Fourier expansion of $F_{a,b}^{(k+2)}$. We have $\overline{F}_{a,b}^{(k+2)}(iy) = F_{a,-b}^{(k+2)}(iy)$. These lead to

$$\begin{aligned}
H(s) :=& \int_0^\infty y^s \left(F_{a,b}^{(k+2)}(iy) + (-1)^{l-1}\overline{F}_{a,b}^{(k+2)}(iy)\right)\frac{dy}{y} \\
=& \int_0^\infty y^s \left(F_{a,b}^{(k+2)}(iy) + (-1)^{l-1}F_{a,-b}^{(k+2)}(iy)\right)\frac{dy}{y} \\
=& \int_0^\infty y^s \left(F_{a,b}^{(k+2)}(iy) + (-1)^{l-1}F_{a,-b}^{(k+2)}(iy) - a_0(F_{a,b}^{(k+2)}) + (-1)^l a_0(F_{a,-b}^{(k+2)})\right)\frac{dy}{y} \\
=& N^{-k-1}\int_0^\infty y^s \left(S^{0,k+1}_{\hat{\delta}_{-b},\delta_a}(iy) + (-1)^{k+1}S^{0,k+1}_{\hat{\delta}_b,\delta_{-a}}(iy)\right. \\
&\left. + (-1)^{l-1}S^{0,k+1}_{\hat{\delta}_b,\delta_a}(iy) + (-1)^{l+k}S^{0,k+1}_{\hat{\delta}_{-b},\delta_{-a}}(iy)\right)\frac{dy}{y} \\
=& N^{-k-1}\int_0^\infty y^s \left(S^{0,k+1}_{\hat{\delta}_{-b}+(-1)^{l-1}\hat{\delta}_b,\delta_a+(-1)^{l+k}\delta_{-a}}(iy)\right)\frac{dy}{y}.
\end{aligned}$$

Therefore by Proposition 3.7.2,

$$H(s) = \frac{(2\pi)^{-s}\Gamma(s)}{N^{k+1-s}}L(\hat{\delta}_{-b} + (-1)^{l-1}\hat{\delta}_b, s)L(\delta_a + (-1)^{l+k}\delta_{-a}, s-k-1).$$

This function $H(s)$ extends to a meromorphic function on the whole s-plane. So the regularized integral has the value $H*(l)$. If $l \le k$ then $L(\delta_a + (-1)^{k-l}\delta_{-a}, l-k-1) = 0$ holds. Hence $H(l) = 0$ in this case. □

Lemma 6.6.2. *Let k be a nonnegative integer, then*

$$\int_0^{\infty,*}\left(F_{a,b}^{(k+2)}(iy) - \overline{F}_{a,b}^{(k+2)}(iy)\right)\frac{dy}{y} = (2\pi)^{-k-2}\Gamma(k+2)i^{k+2}NL(\hat{\delta}_a + (-1)^{k+1}\hat{\delta}_{-a}, k+2)L(\delta_b - \delta_{-b}, 1).$$

Proof. Notice that $F_{a,b}^{(k+2)}(i/y) = (iy)^{k+2}F_{b,-a}^{(k+2)}(iy)$. Therefore,

$$\begin{aligned}
&\int_0^{\infty,*} y^s\left(F_{a,b}^{(k+2)}(iy) - \overline{F}_{a,b}^{(k+2)}(iy)\right)\frac{dy}{y} = \int_0^{\infty,*} y^{-s}\left(F_{a,b}^{(k+2)}(i/y) - F_{a,-b}^{(k+2)}(i/y)\right)\frac{dy}{y} \\
=& \int_0^{\infty,*} y^{-s}(iy)^{k+2}\left(F_{b,-a}^{(k+2)}(iy) - F_{-b,-a}^{(k+2)}(iy))\right)\frac{dy}{y} \\
=& N^{-k-1}i^{k+2}\int_0^{\infty,*} y^{k+2-s}S^{0,k+1}_{\hat{\delta}_a+(-1)^{k+1}\hat{\delta}_{-a},\delta_b-\delta_{-b}}(iy)\frac{dy}{y} \\
=& \frac{(2\pi)^{s-k-2}\Gamma(k+2-s)i^{k+2}}{N^{s-1}}L(\hat{\delta}_a + (-1)^{k+1}\hat{\delta}_{-a}, k+2-s)L(\delta_b - \delta_{-b}, 1-s).
\end{aligned}$$

Take $s = 0$ we get the result. □

Lemma 6.6.3. *Let $k_1 \geq k_2$ and $s \geq 1$ be nonnegative integers. Let $\alpha, \beta : \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}$ be functions. Then*

$$\begin{aligned}
&\int_0^{\infty,*} \left(\mathcal{S}_{\alpha,\beta}^{k_1,k_2} + (-1)^{k_1+k_2+1}\mathcal{S}_{\alpha^-,\beta^-}^{k_1,k_2}\right)(iNy)y^{-s}\frac{dy}{y} \\
&+ (-1)^{k_1+s+1}\int_0^{\infty,*} \left(\mathcal{S}_{\alpha^-,\beta}^{k_1,k_2} + (-1)^{k_1+k_2+1}\mathcal{S}_{\alpha,\beta^-}^{k_1,k_2}\right)(iNy)y^{-s}\frac{dy}{y} \\
=&i^{k_1+k_2+1}N^{k_1+k_2+2s}(2\pi)^{-k_1-k_2-s-1}\frac{(k_1+s)!(k_2+s)!}{s!} \\
&\cdot L(\hat{\alpha}+(-1)^{k_1+s+1}\hat{\alpha}^-, k_1+s+1)L(\hat{\beta}+(-1)^{k_2+s}\hat{\beta}^-, k_2+s+1).
\end{aligned}$$

Proof. Notice that when $z \to 0$, we have $\Gamma(z+l-s) \sim (-1)^{s-l}\frac{1}{(j+1-l)!}z^{-1}$ and $\Gamma(z-k_2-s) \sim (-1)^{k_2+s}\frac{1}{(k_2+s)!}z^{-1}$. Thus,

$$\begin{aligned}
&\int_0^{\infty,*} \left(\mathcal{S}_{\alpha,\beta}^{k_1,k_2} + (-1)^{k_1+k_2+1}\mathcal{S}_{\alpha^-,\beta^-}^{k_1,k_2}\right)(iNy)y^{-s}\frac{dy}{y} \\
&\quad + (-1)^{k_1+s+1}\int_0^{\infty,*} \left(\mathcal{S}_{\alpha^-,\beta}^{k_1,k_2} + (-1)^{k_1+k_2+1}\mathcal{S}_{\alpha,\beta^-}^{k_1,k_2}\right)(iNy)y^{-s}\frac{dy}{y} \\
=&(2\pi)^s \lim_{z\to 0}\left(\Gamma(z-s)L(\beta+(-1)^{k_2+s}\beta^-, z-k_2-s)\right)\cdot L(\alpha+(-1)^{k_1+s+1}\alpha^-, z-k_1-s) \\
=&(-1)^{k_2}(2\pi)^s\frac{(k_2+s)!}{s!}\lim_{z\to 0}\left(\Gamma(z-k_2-s)L(\beta+(-1)^{k_2+s}\beta^-, z-k_2-s)\right) \\
&\quad \cdot L(\alpha+(-1)^{k_1+s+1}\alpha^-, z-k_1-s) \\
=&(-1)^{k_2}(2\pi)^{-k_2}\frac{(k_2+s)!}{s!}N^{k_2+s}\Lambda(G_{\alpha,\beta}^{(k_1-k_2+1)} + (-1)^{k_2+s}G_{\alpha,\beta^-}^{(k_1-k_2+1)}, -k_2-s).
\end{aligned}$$

With Atkin–Lehner involution in Lemma 1.3.7 it becomes

$$(-1)^{k_2}(2\pi)^{-k_2}\frac{(k_2+s)!}{s!}N^{k_2+s}\Lambda(G_{\hat{\beta},\hat{\alpha}}^{(k_1-k_2+1)} + (-1)^{k_2+s}G_{\hat{\beta}^-,\hat{\alpha}}^{(k_1-k_2+1)}, k_1+s+1)$$

Finally applying Lemma 1.3.8 we find that the regularized integral is

$$\begin{aligned}
&(-1)^{k_2}(2\pi)^{-k_1-k_2-s-1}\frac{(k_1+s)!(k_2+s)!}{s!}i^{k_1-k_2+1}N^{k_1+k_2+2s} \\
&\qquad \cdot L(\hat{\alpha}+(-1)^{k_1+s+1}\hat{\alpha}^-, k_1+s+1)L(\hat{\beta}+(-1)^{k_2+s}\hat{\beta}^-, k_2+s+1)
\end{aligned}$$

□

6.7 Final Computations

Now we manage to calculate the full regulator integral. To do this, we will separate the full integral into four parts I_1, J_1, I_2 and J_2 in the Subsection 6.7.1. In the case

$m > k_1$ and $m > k_2$, we will find that I_1 and J_1 actually vanish. Via Rogers–Zudilin method, we will be able to write I_2 and J_2 precisely with some modular L-values. For the rest case, we will inspect carefully the remaining terms, then we will be aware that they will not appear in the final result. At last we will show how to evaluate the residues of the regulator in the Subsection 6.7.4.

6.7.1 Full Regulator Integrals

Consider first the regulator integral

$$I = \int_{X^m Y^{w-m}\{0,\infty\}}^{*} p_* \left(p_1^* \operatorname{Eis}_{\mathcal{D}}^{k_1+j}(u_1) \wedge \pi_{k_2+j+1} \left(p_2^* \operatorname{Eis}_{\mathcal{D}}^{k_2+j}(u_2) \right) \right),$$

we see

$$\begin{aligned}
&\pi_{k_2+j+1} \left(p_2^* \operatorname{Eis}_{\mathcal{D}}^{k_2+j}(u_2) \right) |_W = C_2 dy \wedge \left(F_{-u_2}^{(k_2+j+2)}(iy) p_2^* \psi_{k_2+j,0} - \overline{F}_{-u_2}^{(k_2+j+2)}(iy) p_2^* \psi_{0,k_2+j} \right) \\
&= dy \wedge C_2 \left(F_{-a_2,-b_2}^{(k_2+j+2)}(iy) p_2^* \psi_{k_2+j,0} - F_{-a_2,b_2}^{(k_2+j+2)}(iy) p_2^* \psi_{0,k_2+j} \right).
\end{aligned}$$

The integral I divides into two parts I_1 and I_2.

$$\begin{aligned}
I_1 =& \int_0^{\infty,*} \int_{\gamma_{y,m}} p_* \left(\eta_0 \wedge \pi_{k_2+j+1}(p_2^* \operatorname{Eis}_{hol}^{k_2+j}(u_2)) \right) \\
=& (-1)^{k_1+j} C_1 C_2 \int_0^{\infty,*} dy \int_{\gamma_{y,m}} \sum_{a=0}^{k_1+j} (-1)^{k_1+j-a} \alpha_1 y^{-k_1-j-1} \\
& \cdot p_* \left(p_1^* \psi_{a,k_1+j-a} \wedge (F_{-u_2}^{(k_2+j+2)}(iy) p_2^* \psi_{k_2+j,0} - \overline{F}_{-u_2}^{(k_2+j+2)}(iy) p_2^* \psi_{0,k_2+j}) \right).
\end{aligned}$$

Precisely, with Lemma 6.5.1 we have

$$\begin{aligned}
I_1 =& (-1)^{k_1+j} \alpha_1 C_1 C_2 C_3 \int_0^{\infty,*} dy \Bigg((-1)^j \sum_{a=0}^{k_1} \binom{k_1+j-a}{j} y^{m-k_1-1} F_{-u_2}^{(k_2+j+2)}(iy) \\
& - (-1)^{m-k_1+j} \sum_{a=j}^{k_1+j} \binom{a}{j} y^{m-k_1-1} \overline{F}_{-u_2}^{(k_2+j+2)}(iy) \Bigg).
\end{aligned}$$

Hence I_1 is the following regularized integral

$$(-1)^{k_1} \alpha_1 C_1 C_2 C_3 \binom{k_1+j+1}{j+1} \int_0^{\infty,*} y^{m-k_1} \left(F_{-u_2}^{(k_2+j+2)}(iy) + (-1)^{m-k_1-1} \overline{F}_{-u_2}^{(k_2+j+2)}(iy) \right) \frac{dy}{y}.$$

For the other part I_2 we have

$$\begin{aligned}
I_2 &= \int_0^{\infty,*} \int_{\gamma_{y,m}} p_* \left(\eta_1 \wedge \pi_{k_2+j+1}(p_2^* \operatorname{Eis}_{hol}^{k_2+j}(u_2)) \right) \\
&= (-1)^{k_1+j} C_2 N^{-k_2-j-1} \int_0^{\infty,*} dy \int_{\gamma_{y,m}} \left(\mathcal{S}^{0,k_2+j+1}_{\hat{\delta}_{b_2},\delta_{-a_2}}(iy) + (-1)^{k_2+j} \mathcal{S}^{0,k_2+j+1}_{\hat{\delta}_{-b_2},\delta_{a_2}}(iy) \right) p_* \left(\eta_1 \wedge p_2^* \psi_{k_2+j,0} \right) \\
&- (-1)^{k_1+j} C_2 N^{-k_2-j-1} \int_0^{\infty,*} dy \int_{\gamma_{y,m}} \left(\mathcal{S}^{0,k_2+j+1}_{\hat{\delta}_{-b_2},\delta_{-a_2}}(iy) + (-1)^{k_2+j} \mathcal{S}^{0,k_2+j+1}_{\hat{\delta}_{b_2},\delta_{a_2}}(iy) \right) p_* \left(\eta_1 \wedge p_2^* \psi_{0,k_2+j} \right).
\end{aligned}$$

Writing explicitly with Lemma 6.3.3 we see I_2 is a sum of four integrals

$$\begin{aligned}
I_2 &= i^{k_1+j} \frac{1}{2^{k_1+j+2}} N^{-k_2-j} C_1 C_2 \left(\int_0^{\infty,*} dy \sum_{l=0}^{k_1+j} \left(\mathcal{S}^{l,\, l-k_1-1-j}_{\delta_{b_1},\hat{\delta}_{a_1}} \left(\frac{i}{y} \right) + (-1)^{k_1+j} \mathcal{S}^{l,\, l-k_1-1-j}_{\delta_{-b_1},\hat{\delta}_{-a_1}} \left(\frac{i}{y} \right) \right) \right. \\
&\cdot \left(\mathcal{S}^{0,k_2+j+1}_{\hat{\delta}_{b_2},\delta_{-a_2}}(iy) + (-1)^{k_2+j} \mathcal{S}^{0,k_2+j+1}_{\hat{\delta}_{-b_2},\delta_{a_2}}(iy) \right) \int_{\gamma_{y,m}} p_* \left(\Omega_l \wedge p_2^* \psi_{k_2+j,0} \right) \\
&+ \int_0^{\infty,*} dy \sum_{l=0}^{k_1+j} \left(\mathcal{S}^{l,\, l-k_1-1-j}_{\delta_{-b_1},\hat{\delta}_{a_1}} \left(\frac{i}{y} \right) + (-1)^{k_1+j} \mathcal{S}^{l,\, l-k_1-1-j}_{\delta_{b_1},\hat{\delta}_{-a_1}} \left(\frac{i}{y} \right) \right) \\
&\cdot \left(\mathcal{S}^{0,k_2+j+1}_{\hat{\delta}_{b_2},\delta_{-a_2}}(iy) + (-1)^{k_2+j} \mathcal{S}^{0,k_2+j+1}_{\hat{\delta}_{-b_2},\delta_{a_2}}(iy) \right) \int_{\gamma_{y,m}} p_* \left(\overline{\Omega}_l \wedge p_2^* \psi_{k_2+j,0} \right) \\
&- \sum_{l=0}^{k_1+j} \left(\mathcal{S}^{l,\, l-k_1-1-j}_{\delta_{b_1},\hat{\delta}_{a_1}} \left(\frac{i}{y} \right) + (-1)^{k_1+j} \mathcal{S}^{l,\, l-k_1-1-j}_{\delta_{-b_1},\hat{\delta}_{-a_1}} \left(\frac{i}{y} \right) \right) \\
&\cdot \left(\mathcal{S}^{0,k_2+j+1}_{\hat{\delta}_{-b_2},\delta_{-a_2}}(iy) + (-1)^{k_2+j} \mathcal{S}^{0,k_2+j+1}_{\hat{\delta}_{b_2},\delta_{a_2}}(iy) \right) \int_{\gamma_{y,m}} p_* \left(\Omega_l \wedge p_2^* \psi_{0,k_2+j} \right) \\
&- \sum_{l=0}^{k_1+j} \left(\mathcal{S}^{l,\, l-k_1-1-j}_{\delta_{-b_1},\hat{\delta}_{a_1}} \left(\frac{i}{y} \right) + (-1)^{k_1+j} \mathcal{S}^{l,\, l-k_1-1-j}_{\delta_{b_1},\hat{\delta}_{-a_1}} \left(\frac{i}{y} \right) \right) \\
&\left. \cdot \left(\mathcal{S}^{0,k_2+j+1}_{\hat{\delta}_{-b_2},\delta_{-a_2}}(iy) + (-1)^{k_2+j} \mathcal{S}^{0,k_2+j+1}_{\hat{\delta}_{b_2},\delta_{a_2}}(iy) \right) \int_{\gamma_{y,m}} p_* \left(\overline{\Omega}_l \wedge p_2^* \psi_{0,k_2+j} \right) \right).
\end{aligned}$$

Also we have the other part of the regulator

$$J = \int_{(\theta^{-1})_* X^m Y^{w-m}\{0,\infty\}}^* q_* \left(q_2^* \operatorname{Eis}_{\mathcal{D}}^{k_2+j}(u_2) \wedge \pi_{k_1+j+1} \left(q_1^* \operatorname{Eis}_{\mathcal{D}}^{k_1+j}(u_1) \right) \right),$$

it divides into the following two integrals J_1 and J_2 as well. Applying Lemma 6.4.7 we

get the first part

$$\begin{aligned}J_1 &= \int_0^{\infty,*}\int_{\eta_{y,m}} q_*\left(\eta_0' \wedge \pi_{k_1+j+1}(q_1^*\,\mathrm{Eis}_{hol}^{k_1+j}(u_1))\right)\\ &= (-1)^{k_2+j}C_1'C_2'\int_0^{\infty,*} dy\sum_{a=0}^{k_2+j}(-1)^{k_2+j-a}\alpha' y^{-k_2-j-1}\\ &\quad\cdot\int_{\eta_{y,m}} q_*\left(q_2^*\psi_{a,k_2+j-a}\wedge(F_{-u_1}^{(k_1+j+2)}(iy)q_1^*\psi_{k_1+j,0}-\overline{F}_{-u_1}^{(k_1+j+2)}(iy)q_1^*\psi_{0,k_1+j})\right)\end{aligned}$$

It follows from Lemma 6.5.1 that J_1 is

$$\begin{aligned}&(-1)^{k_2+j}\alpha' C_1'C_2'C_3'\Bigg(\sum_{a=0}^{k_2}(-1)^{k_2+j-a}\frac{a!(k_2+j-a)!}{j!k_2!}H_{k_2,m-k_1,k_2-a}\int_0^{\infty,*} y^{m-k_2}F_{-u_1}^{(k_1+j+2)}(iy)\frac{dy}{y}\\ &+\sum_{a=0}^{k_2}(-1)^{m+j-a+1}\frac{a!(k_2+j-a)!}{j!k_2!}H_{k_2,m-k_1,k_2-a}\int_0^{\infty,*} y^{m-k_2}\overline{F}_{-u_1}^{(k_1+j+2)}(iy)\frac{dy}{y}\Bigg).\end{aligned}$$

Therefore,

$$\begin{aligned}J_1 = \alpha' C_1'C_2'C_3'&\sum_{a=0}^{k_2}(-1)^a\frac{a!(k_2+j-a)!}{j!k_2!}H_{k_2,m-k_1,k_2-a}\\ &\cdot\int_0^{\infty,*} y^{m-k_2-1}\left(F_{-u_1}^{(k_1+j+2)}(iy)dy+(-1)^{m-k_2-1}\overline{F}_{-u_1}^{(k_1+j+2)}(iy)\right)dy.\end{aligned}$$

The second part is

$$\begin{aligned}J_2 &= \int_0^{\infty,*}\int_{\eta_{y,m}} q_*\left(\eta_1' \wedge \pi_{k_1+j+1}(q_1^*\,\mathrm{Eis}_{hol}^{k_1+j}(u_1))\right)\\ &= (-1)^{k_2+j}C_2'N^{-k_1-j-1}\left(\int_0^{\infty,*} dy\int_{\eta_{y,m}}\left(\mathcal{S}^{0,k_1+j+1}_{\hat{\delta}_{b_1},\delta_{-a_1}}(iy)+(-1)^{k_1+j}\mathcal{S}^{0,k_1+j+1}_{\hat{\delta}_{-b_1},\delta_{a_1}}(iy)\right)q_*\left(\eta_1\wedge q_1^*\psi_{k_1+j,0}\right)\right.\\ &\left.-\int_0^{\infty,*} dy\int_{\eta_{y,m}}\left(\mathcal{S}^{0,k_1+j+1}_{\hat{\delta}_{-b_1},\delta_{-a_1}}(iy)+(-1)^{k_1+j}\mathcal{S}^{0,k_1+j+1}_{\hat{\delta}_{b_1},\delta_{a_1}}(iy)\right)q_*\left(\eta_1\wedge q_1^*\psi_{0,k_1+j}\right)\right).\end{aligned}$$

Likewise we have

$$
\begin{aligned}
J_2 = i^{k_2+j}\frac{1}{2^{k_2+j+2}}N^{-k_1-j}C_1'C_2'\Bigg(&\int_0^{\infty,*}dy\sum_{l=0}^{k_2+j}\left(\mathcal{S}^{l,\,l-k_2-1-j}_{\delta_{b_2},\hat{\delta}_{a_2}}\left(\frac{i}{y}\right)+(-1)^{k_2+j}\mathcal{S}^{l,\,l-k_2-1-j}_{\delta_{-b_2},\hat{\delta}_{-a_2}}\left(\frac{i}{y}\right)\right)\\
&\cdot\left(\mathcal{S}^{0,k_1+j+1}_{\hat{\delta}_{b_1},\delta_{-a_1}}(iy)+(-1)^{k_1+j}\mathcal{S}^{0,k_1+j+1}_{\hat{\delta}_{-b_1},\delta_{a_1}}(iy)\right)\int_{\eta_{y,m}}q_*\left(\Omega_l'\wedge q_1^*\psi_{k_1+j,0}\right)\\
&+\int_0^{\infty,*}dy\sum_{l=0}^{k_2+j}\left(\mathcal{S}^{l,\,l-k_2-1-j}_{\delta_{-b_2},\hat{\delta}_{a_2}}\left(\frac{i}{y}\right)+(-1)^{k_2+j}\mathcal{S}^{l,\,l-k_2-1-j}_{\delta_{b_2},\hat{\delta}_{-a_2}}\left(\frac{i}{y}\right)\right)\\
&\cdot\left(\mathcal{S}^{0,k_1+j+1}_{\hat{\delta}_{b_1},\delta_{-a_1}}(iy)+(-1)^{k_1+j}\mathcal{S}^{0,k_1+j+1}_{\hat{\delta}_{-b_1},\delta_{a_1}}(iy)\right)\int_{\eta_{y,m}}q_*(\overline{\Omega}_l'\wedge q_1^*\psi_{k_1+j,0})\\
&-\sum_{l=0}^{k_2+j}\left(\mathcal{S}^{l,\,l-k_2-1-j}_{\delta_{b_2},\hat{\delta}_{a_2}}\left(\frac{i}{y}\right)+(-1)^{k_2+j}\mathcal{S}^{l,\,l-k_2-1-j}_{\delta_{-b_2},\hat{\delta}_{-a_2}}\left(\frac{i}{y}\right)\right)\\
&\cdot\left(\mathcal{S}^{0,k_1+j+1}_{\hat{\delta}_{-b_1},\delta_{-a_1}}(iy)+(-1)^{k_1+j}\mathcal{S}^{0,k_1+j+1}_{\hat{\delta}_{b_1},\delta_{a_1}}(iy)\right)\int_{\eta_{y,m}}q_*(\Omega_l'\wedge q_1^*\psi_{0,k_1+j})\\
&-\sum_{l=0}^{k_2+j}\left(\mathcal{S}^{l,\,l-k_2-1-j}_{\delta_{-b_2},\hat{\delta}_{a_2}}\left(\frac{i}{y}\right)+(-1)^{k_2+j}\mathcal{S}^{l,\,l-k_2-1-j}_{\delta_{b_2},\hat{\delta}_{-a_2}}\left(\frac{i}{y}\right)\right)\\
&\cdot\left(\mathcal{S}^{0,k_1+j+1}_{\hat{\delta}_{-b_1},\delta_{-a_1}}(iy)+(-1)^{k_1+j}\mathcal{S}^{0,k_1+j+1}_{\hat{\delta}_{b_1},\delta_{a_1}}(iy)\right)\int_{\eta_{y,m}}q_*(\overline{\Omega}_l'\wedge q_1^*\psi_{0,k_1+j})\Bigg).
\end{aligned}
$$

In later subsections we will compute thoroughly I_1, I_2, J_1 and J_2. The part I_1 and J_1 can be evaluated with the regularized L-values introduced in Section 6.6. As we said before, we do not expect that they appear in our final results after some cancellations. The more important parts I_2 and J_2, we will show they give us modular L-values by dint of Rogers–Zudilin method.

6.7.2 Case $m > k_1$, $m > k_2$

With the regularized L-values of F in Lemma 6.6.1 we find in this case $I_1 = J_1 = 0$, so $I = I_2$ and $J = J_2$. According to Lemma 6.5.3, the integral I_2 is a sum of four integrals

$$
\begin{aligned}
I_2 = {} & i^{k_1+j}\frac{(4\pi)^{k_1+1}}{2^{k_1+j+2}N^{k_1+k_2+j+1}k_1!}C_1C_2C_3 \\
& \cdot\left(\int_0^{\infty,*} y^{m-k_1+j}dy\left(\mathcal{S}^{k_1,-1-j}_{\delta_{b_1},\hat{\delta}_{a_1}}\left(\frac{i}{y}\right)+(-1)^{k_1+j}\mathcal{S}^{k_1,-1-j}_{\delta_{-b_1},\hat{\delta}_{-a_1}}\left(\frac{i}{y}\right)\right)\right. \\
& \left.\cdot\left(\mathcal{S}^{0,k_2+j+1}_{\hat{\delta}_{b_2},\delta_{-a_2}}(iy)+(-1)^{k_2+j}\mathcal{S}^{0,k_2+j+1}_{\hat{\delta}_{-b_2},\delta_{a_2}}(iy)\right)\right. \\
& +(-1)^{k_1}\sum_{l=0}^{j}\frac{1}{l!}\left(\frac{4\pi}{N}\right)^l\int_0^{\infty,*} y^{m-k_1+j-l}dy\left(\mathcal{S}^{l+k_1,l-1-j}_{\delta_{-b_1},\hat{\delta}_{a_1}}\left(\frac{i}{y}\right)+(-1)^{k_1+j}\mathcal{S}^{l+k_1,l-1-j}_{\delta_{b_1},\hat{\delta}_{-a_1}}\left(\frac{i}{y}\right)\right) \\
& \cdot\left(\mathcal{S}^{0,k_2+j+1}_{\hat{\delta}_{b_2},\delta_{-a_2}}(iy)+(-1)^{k_2+j}\mathcal{S}^{0,k_2+j+1}_{\hat{\delta}_{-b_2},\delta_{a_2}}(iy)\right) \\
& +(-1)^{m+k_1+j+1}\sum_{l=0}^{j}\frac{1}{l!}\left(\frac{4\pi}{N}\right)^l\int_0^{\infty,*} y^{m-k_1+j-l}dy\left(\mathcal{S}^{l+k_1,l-1-j}_{\delta_{b_1},\hat{\delta}_{a_1}}\left(\frac{i}{y}\right)+(-1)^{k_1+j}\mathcal{S}^{l+k_1,l-1-j}_{\delta_{-b_1},\hat{\delta}_{-a_1}}\left(\frac{i}{y}\right)\right) \\
& \cdot\left(\mathcal{S}^{0,k_2+j+1}_{\hat{\delta}_{-b_2},\delta_{-a_2}}(iy)+(-1)^{k_2+j}\mathcal{S}^{0,k_2+j+1}_{\hat{\delta}_{b_2},\delta_{a_2}}(iy)\right) \\
& +(-1)^{m+j+1}\int_0^{\infty,*} y^{m-k_1+j}dy\left(\mathcal{S}^{k_1,-1-j}_{\delta_{-b_1},\hat{\delta}_{a_1}}\left(\frac{i}{y}\right)+(-1)^{k_1+j}\mathcal{S}^{k_1,-1-j}_{\delta_{b_1},\hat{\delta}_{-a_1}}\left(\frac{i}{y}\right)\right) \\
& \left.\cdot\left(\mathcal{S}^{0,k_2+j+1}_{\hat{\delta}_{-b_2},\delta_{-a_2}}(iy)+(-1)^{k_2+j}\mathcal{S}^{0,k_2+j+1}_{\hat{\delta}_{b_2},\delta_{a_2}}(iy)\right)\right).
\end{aligned}
$$

We take out the eight products of the first, last integrals and the eight products of the terms $l = 0$ within the second and third integrals. They are

$$
\begin{aligned}
& \left(\mathcal{S}^{k_1,-1-j}_{\delta_{b_1},\hat{\delta}_{a_1}}\left(\frac{i}{y}\right)+(-1)^{k_1+j}\mathcal{S}^{k_1,-1-j}_{\delta_{-b_1},\hat{\delta}_{-a_1}}\left(\frac{i}{y}\right)\right)\cdot\left(\mathcal{S}^{0,k_2+j+1}_{\hat{\delta}_{b_2},\delta_{-a_2}}(iy)+(-1)^{k_2+j}\mathcal{S}^{0,k_2+j+1}_{\hat{\delta}_{-b_2},\delta_{a_2}}(iy)\right) \\
& +(-1)^{k_1}\left(\mathcal{S}^{k_1,-1-j}_{\delta_{-b_1},\hat{\delta}_{a_1}}\left(\frac{i}{y}\right)+(-1)^{k_1+j}\mathcal{S}^{k_1,-1-j}_{\delta_{b_1},\hat{\delta}_{-a_1}}\left(\frac{i}{y}\right)\right)\cdot\left(\mathcal{S}^{0,k_2+j+1}_{\hat{\delta}_{b_2},\delta_{-a_2}}(iy)+(-1)^{k_2+j}\mathcal{S}^{0,k_2+j+1}_{\hat{\delta}_{-b_2},\delta_{a_2}}(iy)\right) \\
& +(-1)^{m+k_1+j+1}\left(\mathcal{S}^{k_1,-1-j}_{\delta_{b_1},\hat{\delta}_{a_1}}\left(\frac{i}{y}\right)+(-1)^{k_1+j}\mathcal{S}^{k_1,-1-j}_{\delta_{-b_1},\hat{\delta}_{-a_1}}\left(\frac{i}{y}\right)\right)\left(\mathcal{S}^{0,k_2+j+1}_{\hat{\delta}_{-b_2},\delta_{-a_2}}(iy)+(-1)^{k_2+j}\mathcal{S}^{0,k_2+j+1}_{\hat{\delta}_{b_2},\delta_{a_2}}(iy)\right) \\
& +(-1)^{m+j+1}\left(\mathcal{S}^{k_1,-1-j}_{\delta_{-b_1},\hat{\delta}_{a_1}}\left(\frac{i}{y}\right)+(-1)^{k_1+j}\mathcal{S}^{k_1,-1-j}_{\delta_{b_1},\hat{\delta}_{-a_1}}\left(\frac{i}{y}\right)\right)\left(\mathcal{S}^{0,k_2+j+1}_{\hat{\delta}_{-b_2},\delta_{-a_2}}(iy)+(-1)^{k_2+j}\mathcal{S}^{0,k_2+j+1}_{\hat{\delta}_{b_2},\delta_{a_2}}(iy)\right).
\end{aligned}
$$

Via Rogers–Zudilin method as in Lemma 4.3.2, the sixteen products of S-series can be collected together to provide us the following integral

$$
\int_0^{\infty,*}\mathcal{S}^{k_1,w-m}_{\delta_{b_1}+(-1)^{k_1}\delta_{-b_1},\delta_{-a_2}+(-1)^{w-m+1}\delta_{a_2}}\left(\frac{i}{y}\right)\mathcal{S}^{0,m-k_1}_{\hat{\delta}_{b_2}+(-1)^{m-k_1+j+1}\hat{\delta}_{-b_2},\hat{\delta}_{a_1}+(-1)^j\hat{\delta}_{-a_1}}(iy).
$$

For each $1 < l \le j$, we have eight products of S-series

$$
\begin{aligned}
&(-1)^{k_1}\int_0^{\infty,*} y^{m-k_1+j-l}dy\left(\mathcal{S}^{l+k_1,l-1-j}_{\delta_{-b_1},\hat{\delta}_{a_1}}\left(\frac{i}{y}\right)+(-1)^{k_1+j}\mathcal{S}^{l+k_1,l-1-j}_{\delta_{b_1},\hat{\delta}_{-a_1}}\left(\frac{i}{y}\right)\right)\\
&\cdot\left(\mathcal{S}^{0,k_2+j+1}_{\hat{\delta}_{b_2},\delta_{-a_2}}(iy)+(-1)^{k_2+j}\mathcal{S}^{0,k_2+j+1}_{\hat{\delta}_{-b_2},\delta_{a_2}}(iy)\right)\\
&+(-1)^{m+k_1+j+1}\int_0^{\infty,*} y^{m-k_1+j-l}dy\left(\mathcal{S}^{l+k_1,l-1-j}_{\delta_{b_1},\hat{\delta}_{a_1}}\left(\frac{i}{y}\right)+(-1)^{k_1+j}\mathcal{S}^{l+k_1,l-1-j}_{\delta_{-b_1},\hat{\delta}_{-a_1}}\left(\frac{i}{y}\right)\right)\\
&\cdot\left(\mathcal{S}^{0,k_2+j+1}_{\hat{\delta}_{-b_2},\delta_{-a_2}}(iy)+(-1)^{k_2+j}\mathcal{S}^{0,k_2+j+1}_{\hat{\delta}_{b_2},\delta_{a_2}}(iy)\right)
\end{aligned}
$$

Applying Rogers–Zudilin method in a similar manner, they become the following integral

$$
\begin{aligned}
&\int_0^{\infty,*}(-1)^{k_1+k_2+m+1}\left(\mathcal{S}^{0,m-k_1}_{\hat{\delta}_{b_2},\hat{\delta}_{a_1}}+(-1)^{m-k_1+1}\mathcal{S}^{0,m-k_1}_{\hat{\delta}_{-b_2},\hat{\delta}_{-a_1}}\right)(iy)\\
&\qquad\cdot\left(\mathcal{S}^{l+k_1,l+w-m}_{\delta_{b_1},\delta_{a_2}}+(-1)^{m+k_2+1}\mathcal{S}^{l+k_1,l+w-m}_{\delta_{-b_1},\delta_{-a_2}}\right)\left(\frac{i}{y}\right)y^{m-k_1+j-l+1}\frac{dy}{y}\\
&+\int_0^{\infty,*}(-1)^{j}\left(\mathcal{S}^{0,m-k_1}_{\hat{\delta}_{b_2},\hat{\delta}_{-a_1}}+(-1)^{m-k_1+1}\mathcal{S}^{0,m-k_1}_{\hat{\delta}_{-b_2},\hat{\delta}_{a_1}}\right)(iy)\\
&\qquad\cdot\left(\mathcal{S}^{l+k_1,l+w-m}_{\delta_{b_1},\delta_{-a_2}}+(-1)^{m+k_2+1}\mathcal{S}^{l+k_1,l+w-m}_{\delta_{-b_1},\delta_{a_2}}\right)\left(\frac{i}{y}\right)y^{m-k_1+j-l+1}\frac{dy}{y}.
\end{aligned}
$$

Thus we get

$$
\begin{aligned}
I=&(-1)^{k_2+\frac{1}{2}j(j+1)+1}\frac{i^{m+k_2+j}(2\pi)^{k_1+k_2+1}j!(k_1+j+2)(k_2+j+2)}{4N^{k_1+k_2+j+3}}\\
&\cdot\left(\int_0^{\infty,*}\mathcal{S}^{k_1,w-m}_{\delta_{b_1}+(-1)^{k_1}\delta_{-b_1},\delta_{-a_2}+(-1)^{w-m+1}\delta_{a_2}}\left(\frac{i}{y}\right)\mathcal{S}^{0,m-k_1}_{\hat{\delta}_{b_2}+(-1)^{m-k_1+j+1}\hat{\delta}_{-b_2},\hat{\delta}_{a_1}+(-1)^{j}\hat{\delta}_{-a_1}}(iy)\right.\\
&\left.\cdot\,y^{m-k_1+j+1}\frac{dy}{y}\right.\\
&+\sum_{l=1}^{j}\frac{1}{l!}\left(\frac{4\pi}{N}\right)^l\int_0^{\infty,*}(-1)^{k_1+k_2+m+1}\left(\mathcal{S}^{0,m-k_1}_{\hat{\delta}_{b_2},\hat{\delta}_{a_1}}+(-1)^{m-k_1+1}\mathcal{S}^{0,m-k_1}_{\hat{\delta}_{-b_2},\hat{\delta}_{-a_1}}\right)(iy)\\
&\cdot\left(\mathcal{S}^{l+k_1,l+w-m}_{\delta_{b_1},\delta_{a_2}}+(-1)^{m+k_2+1}\mathcal{S}^{l+k_1,l+w-m}_{\delta_{-b_1},\delta_{-a_2}}\right)\left(\frac{i}{y}\right)y^{m-k_1+j-l+1}\frac{dy}{y}\\
&+\sum_{l=1}^{j}\frac{1}{l!}\left(\frac{4\pi}{N}\right)^l\int_0^{\infty,*}(-1)^{j}\left(\mathcal{S}^{0,m-k_1}_{\hat{\delta}_{b_2},\hat{\delta}_{-a_1}}+(-1)^{m-k_1+1}\mathcal{S}^{0,m-k_1}_{\hat{\delta}_{-b_2},\hat{\delta}_{a_1}}\right)(iy)\\
&\cdot\left(\mathcal{S}^{l+k_1,l+w-m}_{\delta_{b_1},\delta_{-a_2}}+(-1)^{m+k_2+1}\mathcal{S}^{l+k_1,l+w-m}_{\delta_{-b_1},\delta_{a_2}}\right)\left(\frac{i}{y}\right)y^{m-k_1+j-l+1}\frac{dy}{y}.
\end{aligned}
$$

Changing variable $y \mapsto \frac{1}{Ny}$ we see

$$\begin{aligned}
I = &(-1)^{k_2+\frac{1}{2}j(j+1)+1}\frac{i^{m+k_2+j}(2\pi)^{k_1+k_2+1}j!(k_1+j+2)(k_2+j+2)}{4N^{k_1+k_2+j+3}}\cdot N^{-m+k_1-j-1}\\
&\cdot\left(\int_0^{\infty,*}\mathcal{S}^{k_1,w-m}_{\delta_{b_1}+(-1)^{k_1}\delta_{-b_1},\delta_{-a_2}+(-1)^{w-m+1}\delta_{a_2}}(iNy)\mathcal{S}^{0,m-k_1}_{\hat{\delta}_{b_2}+(-1)^{m-k_1+j+1}\hat{\delta}_{-b_2},\hat{\delta}_{a_1}+(-1)^j\hat{\delta}_{-a_1}}\left(\frac{i}{Ny}\right)\right.\\
&\left.\cdot y^{-m+k_1-j-1}\frac{dy}{y}\right.\\
&+\sum_{l=1}^{j}\frac{(4\pi)^l}{l!}\int_0^{\infty,*}(-1)^{k_1+k_2+m+1}\left(\mathcal{S}^{l+k_1,l+w-m}_{\delta_{b_1},\delta_{a_2}}+(-1)^{m+k_2+1}\mathcal{S}^{l+k_1,l+w-m}_{\delta_{-b_1},\delta_{-a_2}}\right)(iNy)\\
&\cdot\left(\mathcal{S}^{m-k_1,0}_{\hat{\delta}_{a_1},\hat{\delta}_{b_2}}+(-1)^{m-k_1+1}\mathcal{S}^{m-k_1,0}_{\hat{\delta}_{-a_1},\hat{\delta}_{-b_2}}\right)\left(\frac{i}{Ny}\right)y^{-m+k_1-j+l-1}\frac{dy}{y}\\
&+\sum_{l=1}^{j}\frac{(4\pi)^l}{l!}\int_0^{\infty,*}(-1)^{j}\left(\mathcal{S}^{l+k_1,l+w-m}_{\delta_{b_1},\delta_{-a_2}}+(-1)^{m+k_2+1}\mathcal{S}^{l+k_1,l+w-m}_{\delta_{-b_1},\delta_{a_2}}\right)(iNy)\\
&\cdot\left(\mathcal{S}^{m-k_1,0}_{\hat{\delta}_{-a_1},\hat{\delta}_{b_2}}+(-1)^{m-k_1+1}\mathcal{S}^{m-k_1,0}_{\hat{\delta}_{a_1},\hat{\delta}_{-b_2}}\right)\left(\frac{i}{Ny}\right)y^{-m+k_1-j+l-1}\frac{dy}{y}\Bigg).
\end{aligned}$$

We have always $w-m\leq k_1$. If $m=w$, we claim that $\mathcal{S}^{k_1,w-m}_{\delta_{b_1}+(-1)^{k_1}\delta_{-b_1},\delta_{-a_2}+(-1)^{w-m+1}\delta_{a_2}}(iNy)$ is already an Eisenstein series. In fact, if $m=w$ then

$$\begin{aligned}
&\mathcal{S}^{k_1,w-m}_{\delta_{b_1}+(-1)^{k_1}\delta_{-b_1},\delta_{-a_2}+(-1)^{w-m+1}\delta_{a_2}}(iNy)\\
=&\mathcal{S}^{k_1,0}_{\delta_{b_1}+(-1)^{k_1}\delta_{-b_1},\delta_{-a_2}-\delta_{a_2}}(iNy)\\
=&G^{(k_1+1)}_{b_1,-a_2}(iy)-G^{(k_1+1)}_{b_1,a_2}(iy).
\end{aligned}$$

By Lemma 1.3.2 and Lemma 1.3.3 we have the following identities about Eisenstein series and their derivatives

$$\mathcal{S}^{k_1,w-m}_{\delta_{b_1}+(-1)^{k_1}\delta_{-b_1},\delta_{-a_2}+(-1)^{w-m+1}\delta_{a_2}}(iNy)=D^{w-m}\left(G^{(m-k_2+1)}_{b_1,-a_2}(iy)-(-1)^{w-m}G^{(m-k_2+1)}_{b_1,a_2}(iy)\right),$$

$$\mathcal{S}^{0,m-k_1}_{\hat{\delta}_{b_2}+(-1)^{m-k_1+j+1}\hat{\delta}_{-b_2},\hat{\delta}_{a_1}+(-1)^j\hat{\delta}_{-a_1}}\left(\frac{i}{Ny}\right)=H^{(m-k_1+1)}_{b_2,a_1}\left(\frac{i}{N^2y}\right)+(-1)^jH^{(m-k_1+1)}_{b_2,-a_1}\left(\frac{i}{N^2y}\right).$$

Also, we have

$$\begin{aligned}
\mathcal{S}^{l+k_1,l+w-m}_{\delta_{b_1},\delta_{a_2}}(iNy)+(-1)^{m+k_2+1}\mathcal{S}^{l+k_1,l+w-m}_{\delta_{-b_1},\delta_{-a_2}}(iNy)&=D^{w-m+l}G^{(m-k_2+1)}_{b_1,a_2}(iy),\\
\mathcal{S}^{l+k_1,l+w-m}_{\delta_{b_1},\delta_{-a_2}}(iNy)+(-1)^{m+k_2+1}\mathcal{S}^{l+k_1,l+w-m}_{\delta_{-b_1},\delta_{a_2}}(iNy)&=D^{w-m+l}G^{(m-k_2+1)}_{b_1,-a_2}(iy)
\end{aligned}$$

and

$$\mathcal{S}^{0,m-k_1}_{\hat{\delta}_{b_2},\hat{\delta}_{a_1}}\left(\frac{i}{Ny}\right)+(-1)^{m-k_1+1}\mathcal{S}^{0,m-k_1}_{\hat{\delta}_{-b_2},\hat{\delta}_{-a_1}}\left(\frac{i}{Ny}\right)=H^{(m-k_1+1)}_{b_2,a_1}\left(\frac{i}{N^2y}\right),$$
$$\mathcal{S}^{0,m-k_1}_{\hat{\delta}_{b_2},\hat{\delta}_{-a_1}}\left(\frac{i}{Ny}\right)+(-1)^{m-k_1+1}\mathcal{S}^{0,m-k_1}_{\hat{\delta}_{-b_2},\hat{\delta}_{a_1}}\left(\frac{i}{Ny}\right)=H^{(m-k_1+1)}_{b_2,-a_1}\left(\frac{i}{N^2y}\right).$$

At last, we perform the Atkin–Lehner involution $W_{N^2}(H^{(k)}_{a,b})=i^{-k}NG^{(k)}_{a,b}$ to all the Eisenstein series H, then the integral I becomes a sum of L-values of quasi-modular forms. Eventually we get

$$\begin{aligned}I=&(-1)^{k_2+\frac{1}{2}j(j+1)}\frac{i^{k_1+k_2+j+1}(2\pi)^{k_1+k_2+1}j!(k_1+j+2)(k_2+j+2)}{4N^{k_1+k_2+j+2}}\\&\cdot\left(\Lambda^*\left(D^{w-m}\left(G^{(m-k_2+1)}_{b_1,-a_2}-(-1)^{w-m}G^{(m-k_2+1)}_{b_1,a_2}\right)\left(G^{(m-k_1+1)}_{b_2,a_1}+(-1)^jG^{(m-k_1+1)}_{b_2,-a_1}\right),-j\right)\right.\\&+\sum_{l=1}^{j}\frac{1}{l!}\left(\frac{4\pi}{N}\right)^l\Lambda^*\left((-1)^{w+m+1}D^{w-m+l}G^{(m-k_2+1)}_{b_1,a_2}G^{(m-k_1+1)}_{b_2,a_1}\right.\\&\left.\left.+(-1)^jD^{w-m+l}G^{(m-k_2+1)}_{b_1,-a_2}G^{(m-k_1+1)}_{b_2,-a_1},l-j\right)\right).\end{aligned}\tag{6.1}$$

The computation of J_2 is the same as I_2. We can separate J_2 into two integrals, each contains a sum of eight products of Eisenstein series

$$\begin{aligned}&i^{k_2+j}\frac{(4\pi)^{k_2+1}}{2^{k_2+j+2}N^{k_1+k_2+j+1}k_2!}C'_1C'_2C'_3\sum_{l=0}^{w-m}\frac{(j+l)!}{j!}\left(\frac{4\pi}{N}\right)^{-l}C^{(1)}_{k_2,m-k_1,k_2-l}\\&\cdot\left(\int_0^{\infty,*}\left(\mathcal{S}^{-l-1-j,-l+k_2}_{\hat{\delta}_{a_2},\delta_{b_2}}\left(\frac{i}{y}\right)+(-1)^{k_2+j}\mathcal{S}^{-l-1-j,-l+k_2}_{\hat{\delta}_{-a_2},\delta_{-b_2}}\left(\frac{i}{y}\right)\right)\right.\\&\cdot\left(\mathcal{S}^{0,k_1+j+1}_{\hat{\delta}_{b_1},\delta_{-a_1}}(iy)+(-1)^{k_1+j}\mathcal{S}^{0,k_1+j+1}_{\hat{\delta}_{-b_1},\delta_{a_1}}(iy)\right)y^{m-k_2+j+l}dy\\&+(-1)^{m+j+1}\int_0^{\infty,*}\left(\mathcal{S}^{-l-1-j,-l+k_2}_{\hat{\delta}_{a_2},\delta_{-b_2}}\left(\frac{i}{y}\right)+(-1)^{k_2+j}\mathcal{S}^{-l-1-j,-l+k_2}_{\hat{\delta}_{-a_2},\delta_{b_2}}\left(\frac{i}{y}\right)\right)\\&\left.\cdot\left(\mathcal{S}^{0,k_1+j+1}_{\hat{\delta}_{-b_1},\delta_{-a_1}}(iy)+(-1)^{k_1+j}\mathcal{S}^{0,k_1+j+1}_{\hat{\delta}_{b_1},\delta_{a_1}}(iy)\right)y^{m-k_2+j+l}dy\right)\end{aligned}$$

and

$$
\begin{aligned}
& i^{k_2+j}\frac{(4\pi)^{k_2+1}}{2^{k_2+j+2}N^{k_1+k_2+j+1}k_2!}C_1'C_2'C_3'\sum_{l=-j}^{w-m}\frac{(j+l)!}{j!}\left(\frac{4\pi}{N}\right)^{-l}C^{(2)}_{k_2,m-k_1,k_2-l}\\
& \cdot\left((-1)^{m-k_1}\int_0^{\infty,*}\left(\mathcal{S}^{-l-1-j,-l+k_2}_{\hat{\delta}_{a_2},\delta_{-b_2}}\left(\frac{i}{y}\right)+(-1)^{k_2+j}\mathcal{S}^{-l-1-j,-l+k_2}_{\hat{\delta}_{-a_2},\delta_{b_2}}\left(\frac{i}{y}\right)\right)\right.\\
& \cdot\left(\mathcal{S}^{0,k_1+j+1}_{\hat{\delta}_{b_1},\delta_{-a_1}}(iy)+(-1)^{k_1+j}\mathcal{S}^{0,k_1+j+1}_{\hat{\delta}_{-b_1},\delta_{a_1}}(iy)\right)y^{m-k_2+j+l}dy\\
& +(-1)^{k_1+j+1}\int_0^{\infty,*}\left(\mathcal{S}^{-l-1-j,-l+k_2}_{\hat{\delta}_{a_2},\delta_{b_2}}\left(\frac{i}{y}\right)+(-1)^{k_2+j}\mathcal{S}^{-l-1-j,-l+k_2}_{\hat{\delta}_{-a_2},\delta_{-b_2}}\left(\frac{i}{y}\right)\right)\\
& \left.\cdot\left(\mathcal{S}^{0,k_1+j+1}_{\hat{\delta}_{-b_1},\delta_{-a_1}}(iy)+(-1)^{k_1+j}\mathcal{S}^{0,k_1+j+1}_{\hat{\delta}_{b_1},\delta_{a_1}}(iy)\right)y^{m-k_2+j+l}dy\right).
\end{aligned}
$$

Applying Rogers–Zudilin method in the same way as I_2, J_2 becomes

$$
\begin{aligned}
J_2 = {} & i^{k_2+j}\frac{(4\pi)^{k_2+1}}{2^{k_2+j+2}N^{k_1+k_2+j+1}k_2!}C_1'C_2'C_3'\sum_{l=0}^{w-m}\frac{(j+l)!}{j!}\left(\frac{4\pi}{N}\right)^{-l}C^{(1)}_{k_2,m-k_1,k_2-l}\\
& \int_0^{\infty,*}y^{m-k_2+j+l}dy\left(\mathcal{S}^{-l+k_2,-l-m+w}_{\delta_{b_2},\delta_{-a_1}}\left(\frac{i}{y}\right)+(-1)^{m-k_1+1}\mathcal{S}^{-l+k_2,-l-m+w}_{\delta_{-b_2},\delta_{a_1}}\left(\frac{i}{y}\right)\right)\\
& \cdot\left(\mathcal{S}^{0,m-k_2}_{\hat{\delta}_{b_1},\hat{\delta}_{a_2}}(iy)+(-1)^{m-k_2+1}\mathcal{S}^{0,m-k_2}_{\hat{\delta}_{-b_1},\hat{\delta}_{-a_2}}(iy)\right)\\
& +(-1)^{w+m+j+1}\left(\mathcal{S}^{-l+k_2,-l-m+w}_{\delta_{b_2},\delta_{a_1}}\left(\frac{i}{y}\right)+(-1)^{m-k_1+1}\mathcal{S}^{-l+k_2,-l-m+w}_{\delta_{-b_2},\delta_{-a_1}}\left(\frac{i}{y}\right)\right)\\
& \cdot\left(\mathcal{S}^{0,m-k_2}_{\hat{\delta}_{b_1},\hat{\delta}_{-a_2}}(iy)+(-1)^{m-k_2+1}\mathcal{S}^{0,m-k_2}_{\hat{\delta}_{-b_1},\hat{\delta}_{a_2}}(iy)\right)\\
& +i^{k_2+j}\frac{(4\pi)^{k_2+1}}{2^{k_2+j+2}N^{k_1+k_2+j+1}k_2!}C_1'C_2'C_3'\sum_{l=-j}^{w-m}\frac{(j+l)!}{j!}\left(\frac{4\pi}{N}\right)^{-l}C^{(2)}_{k_2,m-k_1,k_2-l,j}\\
& \int_0^{\infty,*}y^{m-k_2+j+l}dy\cdot(-1)\left(\mathcal{S}^{-l+k_2,-l-m+w}_{\delta_{b_2},\delta_{a_1}}\left(\frac{i}{y}\right)+(-1)^{m-k_1+1}\mathcal{S}^{-l+k_2,-l-m+w}_{\delta_{-b_2},\delta_{-a_1}}\left(\frac{i}{y}\right)\right)\\
& \cdot\left(\mathcal{S}^{0,m-k_2}_{\hat{\delta}_{b_1},\hat{\delta}_{a_2}}(iy)+(-1)^{m-k_2+1}\mathcal{S}^{0,m-k_2}_{\hat{\delta}_{-b_1},\hat{\delta}_{-a_2}}(iy)\right)\\
& +(-1)^{w+m+j}\left(\mathcal{S}^{-l+k_2,-l-m+w}_{\delta_{b_2},\delta_{-a_1}}\left(\frac{i}{y}\right)+(-1)^{m-k_1+1}\mathcal{S}^{-l+k_2,-l-m+w}_{\delta_{-b_2},\delta_{a_1}}\left(\frac{i}{y}\right)\right)\\
& \cdot\left(\mathcal{S}^{0,m-k_2}_{\hat{\delta}_{b_1},\hat{\delta}_{-a_2}}(iy)+(-1)^{m-k_2+1}\mathcal{S}^{0,m-k_2}_{\hat{\delta}_{-b_1},\hat{\delta}_{a_2}}(iy)\right).
\end{aligned}
$$

In the case $l=w-m$, we claim that all the S-series $S^{-l+k_2,-l-m+w}_{*,*}+(-1)^{m-k_1+1}S^{-l+k_2,-l-m+w}_{*,*}$ in the integral are already Eisenstein series. If $a_1=0$, then the constant term of $G_{b_2,-a_1}-G_{b_2,a_1}$ vanishes. If $a_1\neq 0$, then the constant terms of both $G_{b_2,-a_1}$ and G_{b_2,a_1} are zero.

Thus

$$\begin{aligned}J &= (-1)^{k_1+\frac{1}{2}j(j+1)}\frac{i^{k_1+k_2+j+1}(2\pi)^{k_1+k_2+1}j!(k_1+j+2)(k_2+j+2)}{4N^{k_1+k_2+j+2}}\\ &\cdot\sum_{l=0}^{w-m}\frac{(j+l)!}{j!}\left(\frac{4\pi}{N}\right)^{-l}C^{(1)}_{k_2,m-k_1,k_2-l}\\ &\Lambda^*\left(D^{w-m-l}G^{(m-k_1+1)}_{b_2,-a_1}G^{(m-k_2+1)}_{b_1,a_2}+(-1)^{w+m+j+1}D^{w-m-l}G^{(m-k_1+1)}_{b_2,a_1}G^{(m-k_2+1)}_{b_1,-a_2},-j-l\right)\\ &-\sum_{l=-j}^{w-m}\frac{(j+l)!}{j!}\left(\frac{4\pi}{N}\right)^{-l}C^{(2)}_{k_2,m-k_1,k_2-l,j}\\ &\Lambda^*\left(D^{w-m-l}G^{(m-k_1+1)}_{b_2,a_1}G^{(m-k_2+1)}_{b_1,a_2}+(-1)^{w+m+j+1}D^{w-m-l}G^{(m-k_1+1)}_{b_2,-a_1}G^{(m-k_2+1)}_{b_1,-a_2},-j-l\right).\end{aligned}$$

6.7.3 Case $m = k_1 > k_2$ or $m = k_2$, $k_1 = 0$

In the case $m = k_1 > k_2$ or $m = k_2$ but $k_1 = 0$ we are facing two problems. The first one is that there are lost integrals I_1 and J_1 which may not vanish. The second one is that there are Eisenstein series of weight 1 having some lost constant terms. We will compute exactly those lost integrals and constant terms in this section and show they actually do not appear in the final result.

Evaluation of the first part I. Recall that the integral

$$\begin{aligned}I_1 = (-1)^{k_1}\alpha C_1C_2C_3\int_0^{\infty,*}&\binom{k_1+j+1}{j+1}y^{m-k_1-1}\\ &\cdot\left(F^{(k_2+j+2)}_{-u_2}(iy)+(-1)^{m-k_1-1}\overline{F}^{(k_2+j+2)}_{-u_2}(iy)\right)dy.\end{aligned}$$

By Lemma 6.6.1, I_1 vanishes unless $m = k_1$. If $m = k_1$ happens, with the regularized L-value of F in Lemma 6.6.2 we have

$$I_1 = \delta_{m=k_1}\alpha C_1C_2C_3\binom{k_1+j+1}{j+1}\cdot\int_0^{\infty,*}\left(F^{(k_2+j+2)}_{-u_2}(iy)-\overline{F}^{(k_2+j+2)}_{-u_2}(iy)\right)\frac{dy}{y}$$

Namely

$$\begin{aligned}I_1 =&\,\delta_{m=k_1}(-1)^{\frac{1}{2}j(j-1)+k_1}\frac{i^j(k_1+j+2)!(k_2+j+2)!}{8(j+1)\pi^{j+2}N}L(\hat{\delta}_{b_1}+(-1)^{k_1+j}\hat{\delta}_{-b_1},k_1+j+2)\\ &\cdot L(\hat{\delta}_{-a_2}+(-1)^{k_2+j+1}\hat{\delta}_{a_2},k_2+j+2)L(\delta_{-b_2}-\delta_{b_2},1)\\ =&\,\delta_{m=k_1}(-1)^{\frac{1}{2}j(j+1)}\frac{i^j(k_1+j+2)!(k_2+j+2)!}{8(j+1)N\pi^{j+2}}L(\hat{\delta}_{-b_1}+(-1)^{k_1+j}\hat{\delta}_{b_1},k_1+j+2)\\ &\cdot L(\hat{\delta}_{-a_2}+(-1)^{k_2+j+1}\hat{\delta}_{a_2},k_2+j+2)L(\delta_{-b_2}-\delta_{b_2},1).\end{aligned}$$

Next we are going to evaluate I_2, recall

$$\begin{aligned}
I_2 =&(-1)^{k_2+\frac{1}{2}j(j+1)+1}\frac{i^{m+k_2+j}(2\pi)^{k_1+k_2+1}j!(k_1+j+2)(k_2+j+2)}{4N^{k_1+k_2+j+3}}\cdot N^{-m+k_1-j-1}\\
&\left(\int_0^{\infty,*}\left(\mathcal{S}^{k_1,w-m}_{\delta_{b_1},\delta_{-a_2}}+(-1)^{m+k_2+1}\mathcal{S}^{k_1,w-m}_{\delta_{-b_1},\delta_{a_2}}\right)(iNy)\right.\\
&\cdot\left(\mathcal{S}^{0,m-k_1}_{\hat{\delta}_{b_2},\hat{\delta}_{a_1}}+(-1)^{m-k_1+1}\mathcal{S}^{0,m-k_1}_{\hat{\delta}_{-b_2},\hat{\delta}_{-a_1}}\right)\left(\frac{i}{Ny}\right)y^{-m+k_1-j-1}\frac{dy}{y}\\
&+(-1)^{w+m+j+1}\int_0^{\infty,*}\left(\mathcal{S}^{k_1,w-m}_{\delta_{b_1},\delta_{a_2}}+(-1)^{m+k_2+1}\mathcal{S}^{k_1,w-m}_{\delta_{-b_1},\delta_{-a_2}}\right)(iNy)\\
&\cdot\left(\mathcal{S}^{0,m-k_1}_{\hat{\delta}_{b_2},\hat{\delta}_{-a_1}}+(-1)^{m-k_1+1}\mathcal{S}^{m-k_1,0}_{\hat{\delta}_{-b_2},\hat{\delta}_{a_1}}\right)\left(\frac{i}{Ny}\right)y^{-m+k_1-j-1}\frac{dy}{y}\\
&+\sum_{l=0}^{j}\frac{(4\pi)^l}{l!}\int_0^{\infty,*}(-1)^{w+m+1}\left(\mathcal{S}^{l+k_1,l+w-m}_{\delta_{b_1},\delta_{a_2}}+(-1)^{m+k_2+1}\mathcal{S}^{l+k_1,l+w-m}_{\delta_{-b_1},\delta_{-a_2}}\right)(iNy)\\
&\cdot\left(\mathcal{S}^{0,m-k_1}_{\hat{\delta}_{b_2},\hat{\delta}_{a_1}}+(-1)^{m-k_1+1}\mathcal{S}^{0,m-k_1}_{\hat{\delta}_{-b_2},\hat{\delta}_{-a_1}}\right)\left(\frac{i}{Ny}\right)y^{-m+k_1+l-j-1}\frac{dy}{y}\\
&+\sum_{l=0}^{j}\frac{(4\pi)^l}{l!}\int_0^{\infty,*}(-1)^{j}\left(\mathcal{S}^{l+k_1,l+w-m}_{\delta_{b_1},\delta_{-a_2}}+(-1)^{m+k_2+1}\mathcal{S}^{l+k_1,l+w-m}_{\delta_{-b_1},\delta_{a_2}}\right)(iNy)\\
&\left.\cdot\left(\mathcal{S}^{0,m-k_1}_{\hat{\delta}_{b_2},\hat{\delta}_{-a_1}}+(-1)^{m-k_1+1}\mathcal{S}^{0,m-k_1}_{\hat{\delta}_{-b_2},\hat{\delta}_{a_1}}\right)\left(\frac{i}{Ny}\right)y^{-m+k_1+l-j-1}\frac{dy}{y}\right).
\end{aligned}\tag{6.2}$$

The evaluation of I_2 is exactly the same compared to the case $m>k_1$ and $m>k_2$, unless we have some lost constant terms in S-series. In fact if $k_1=0$ then

$$\begin{aligned}
&\mathcal{S}^{0,w-m}_{\delta_{b_1}+\delta_{-b_1},\delta_{-a_2}+(-1)^{w-m+1}\delta_{a_2}}(iNy)\\
=&\mathcal{S}^{0,0}_{\delta_{b_1}+\delta_{-b_1},\delta_{-a_2}-\delta_{a_2}}(iNy)\\
=&G^{(1)}_{b_1,-a_2}(iy)-G^{(1)}_{b_1,a_2}(iy).
\end{aligned}$$

Thus the lost constant terms only come from the S-series $\mathcal{S}^{0,m-k_1}_{*,*}$. This happens only when $m=k_1$. If so we see

$$\begin{aligned}
\mathcal{S}^{0,0}_{\hat{\delta}_{b_2},\hat{\delta}_{a_1}}\left(\frac{i}{Ny}\right)-\mathcal{S}^{0,0}_{\hat{\delta}_{-b_2},\hat{\delta}_{-a_1}}\left(\frac{i}{Ny}\right)=H^{(1)}_{b_2,a_1}\left(\frac{i}{N^2y}\right)-\frac{1}{2}-L(\hat{\delta}_{b_2},0),\\
\mathcal{S}^{0,0}_{\hat{\delta}_{b_2},\hat{\delta}_{-a_1}}\left(\frac{i}{Ny}\right)-\mathcal{S}^{0,0}_{\hat{\delta}_{-b_2},\hat{\delta}_{a_1}}\left(\frac{i}{Ny}\right)=H^{(1)}_{b_2,-a_1}\left(\frac{i}{N^2y}\right)-\frac{1}{2}-L(\hat{\delta}_{b_2},0),
\end{aligned}$$

they both share the same constant term

$$\frac{1}{2}+L(\hat{\delta}_{b_2},0)=\frac{1}{2}L(\hat{\delta}_{b_2}-\hat{\delta}_{-b_2},0)=\frac{N}{2\pi i}L(\delta_{b_2}-\delta_{-b_2},1).$$

Then we have

$$I_2 = A + B + C,$$

where the part A is the one which is identical to (6.1), the part B is coming from the lost term in the first two integrals of (6.2) when $m = k_1$, and the part C is from the lost term in the last two integrals of (6.2) when $m = k_1$.

The part A is

$$\begin{aligned} A = &(-1)^{k_2+\frac{1}{2}j(j+1)} \frac{i^{k_1+k_2+j+1}(2\pi)^{k_1+k_2+1} j!(k_1+j+2)(k_2+j+2)}{4N^{k_1+k_2+j+2}} \\ &\cdot \Bigg(\Lambda^* \left(D^{w-m} \left(G_{b_1,-a_2}^{(m-k_2+1)} - (-1)^{w-m} G_{b_1,a_2}^{(m-k_2+1)} \right) \left(G_{b_2,a_1}^{(m-k_1+1)} + (-1)^j G_{b_2,-a_1}^{(m-k_1+1)} \right), -j \right) \\ &+ \sum_{l=1}^{j} \frac{1}{l!} \left(\frac{4\pi}{N} \right)^l \Lambda^* \Big((-1)^{w+m+1} D^{w-m+l} G_{b_1,a_2}^{(m-k_2+1)} G_{b_2,a_1}^{(m-k_1+1)} \\ &+ (-1)^j D^{w-m+l} G_{b_1,-a_2}^{(m-k_2+1)} G_{b_2,-a_1}^{(m-k_1+1)}, l-j \Big) \Bigg). \end{aligned}$$

With the regularized L-value in Lemma 6.6.3, we can compute B and C. We have

$$\begin{aligned} B = &\delta_{m=k_1} (-1)^{k_2+\frac{1}{2}j(j+1)+1} \frac{i^{k_1+k_2+j}(2\pi)^{k_1+k_2+1} j!(k_1+j+2)(k_2+j+2)}{4N^{k_1+k_2+2j+4}} \\ &\cdot (-1)^{k_1+j} \frac{N}{2\pi i} L(\hat{\delta}_{-b_1} + (-1)^{k_1+j}\hat{\delta}_{b_1}, k_1+j+2) \\ &\cdot \Bigg(\int_0^{\infty,*} \left(\mathcal{S}_{\delta_{b_1},\delta_{-a_2}}^{k_1,k_2} + (-1)^{k_1+k_2+1} \mathcal{S}_{\delta_{-b_1},\delta_{a_2}}^{k_1,k_2} \right) (iNy) y^{-m+k_1-j-1} \frac{dy}{y} \\ &+ (-1)^{w+m+j+1} \int_0^{\infty,*} \left(\mathcal{S}_{\delta_{b_1},\delta_{a_2}}^{k_1,k_2} + (-1)^{k_1+k_2+1} \mathcal{S}_{\delta_{-b_1},\delta_{-a_2}}^{k_1,k_2} \right) (iNy) y^{-m+k_1-j-1} \frac{dy}{y} \Bigg), \end{aligned}$$

that is,

$$\begin{aligned} B = &\delta_{m=k_1} (-1)^{\frac{1}{2}j(j-1)+1} \frac{i^j (k_1+j+2)!(k_2+j+2)!}{2^{j+4}\pi^{j+2}(j+1)N} L(\hat{\delta}_{-b_1} + (-1)^{k_1+j}\hat{\delta}_{b_1}, k_1+j+2) \\ &\cdot L(\hat{\delta}_{-a_2} + (-1)^{k_2+j+1}\hat{\delta}_{a_2}, k_2+j+2) L(\delta_{-b_2} - \delta_{b_2}, 1). \end{aligned}$$

In a similar manner we have

$$\begin{aligned} C = &\delta_{m=k_1} (-1)^{\frac{1}{2}j(j+1)+1} \frac{i^j (k_1+j+2)!(k_2+j+2)!}{2^{j+4}\pi^{j+2}(j+1)N} \sum_{l=0}^{j} \frac{(-1)^l}{(2\pi)^l} (4\pi)^l \binom{j+1}{l} \\ &\cdot L(\hat{\delta}_{-b_1} + (-1)^{k_1+j}\hat{\delta}_{b_1}, k_1+j+2) L(\hat{\delta}_{-a_2} + (-1)^{k_2+j+1}\hat{\delta}_{a_2}, k_2+j+2) \\ &\cdot L(\delta_{-b_2} - \delta_{b_2}, 1), \end{aligned}$$

that is,

$$\begin{aligned}C =& \delta_{m=k_1}(-1)^{\frac{1}{2}j(j-1)}\frac{i^j(k_1+j+2)!(k_2+j+2)!}{2^{j+4}\pi^{j+2}(j+1)N}\left(1-2^{j+1}\right)\\ &\cdot L(\hat{\delta}_{-b_1}+(-1)^{k_1+j}\hat{\delta}_{b_1},k_1+j+2)L(\hat{\delta}_{-a_2}+(-1)^{k_2+j+1}\hat{\delta}_{a_2},k_2+j+2)\\ &\cdot L(\delta_{-b_2}-\delta_{b_2},1).\end{aligned}$$

Sum up altogether we see immediately $I_1+B+C=0$. Eventually only the main part A will survive. We conclude that $I=A$.

Evaluation of the second part J. We are now left to compute the other part J. Like our computation of I, only if $m=k_2$ do we have nontrivial J_1 and lost constant terms in J_2. Then $m=k_2$ and $k_1=0$ in this case. If so we have

$$\begin{aligned}J^{k_1,k_2,j}(u_1,u_2)=&\int_0^{\infty,*}\int_{\eta_{y,m}}q_*\big(q_2^*\operatorname{Eis}_{\mathcal{D}}^{k_2+j}(u_2)\wedge\pi_{k_1+j+1}(q_1^*\operatorname{Eis}_{hol}^{j+k_1}(u_1))\big)\\ =&I^{k_2,k_1,j}(u_2,u_1).\end{aligned}$$

6.7.4 Residues of the Regulators

In the present subsection we compute the residues of $\operatorname{Eis}_{\mathcal{D}}^{k_1,k_2,j}(u_1,u_2)$ along the modular caps. We will compute only in the case $m=w$, its residue along $X^w[0,x]_\infty$. For general cases $m<w$, we can compute its residues along $X^mY^{w-m}[0,x]_\infty$ exactly the same way.

In the beginning we have the following integrals. Their calculations are exactly the same compared to Lemma 6.5.1.

Lemma 6.7.1. *We have*

$$\int_{\gamma_{\tau,w}}p_*(p_1^*\psi_{a,k_1+j-a}\wedge p_2^*\psi_{k_2+j,0})=\begin{cases}D_3\cdot\binom{k_1+j-a}{j}y^j\overline{\tau}^{k_1-a}\tau^{k_2+a} & 0\le a\le k_1,\\ 0 & k_1<a\le k_1+j,\end{cases}$$

$$\int_{\gamma_{\tau,w}}p_*(p_1^*\psi_{a,k_1+j-a}\wedge p_2^*\psi_{0,k_2+j})=\begin{cases}D_3\cdot(-1)^j\binom{a}{j}y^j\overline{\tau}^{w+j-a}\tau^{a-j} & j\le a\le k_1+j,\\ 0 & 0\le a<j,\end{cases}$$

$$\int_{\gamma_{\tau,w}}p_*(p_1^*\psi_{k_1+j-a,a}\wedge p_2^*\psi_{k_2+j,0})=\begin{cases}D_3\cdot\binom{a}{j}y^j\overline{\tau}^{a-j}\tau^{w+j-a} & j\le a\le k_1+j,\\ 0 & 0\le a<j,\end{cases}$$

and

$$\int_{\gamma_{\tau,w}}p_*(p_1^*\psi_{k_1+j-a,a}\wedge p_2^*\psi_{0,k_2+j})=\begin{cases}D_3\cdot(-1)^j\binom{k_1+j-a}{j}y^j\overline{\tau}^{k_2+a}\tau^{k_1-a} & 0\le a\le k_1,\\ 0 & k_1<a\le k_1+j,\end{cases}$$

where

$$D_3=(-1)^{\frac{1}{2}j(j-1)}\frac{k_1!j!}{\pi^j(k_1+j)!}.$$

Let us look back on the regulator

$$p_1^* \operatorname{Eis}_{\mathcal{D}}^{k_1+j}(u_1)|_V = D_1 y \sum_{a=0}^{k_1+j} F_{\sigma u_1}^{a,k_1+j-a}(x+iy) p_1^* \psi_{a,k_1+j-a} \mod d\tau, d\overline{\tau}$$

and

$$p_2^* \operatorname{Eis}_{hol}^{k_2+j}(u_2)|_V = 2D_2 dx \wedge F_{-u_2}^{(k_2+j+2)}(x+iy) p_2^* \psi_{k_2+j,0},$$

where $V = \pi^{-1}([0,x]_\infty^y)$. Since

$$\begin{aligned}
&\pi_{k_2+j+1}\left(p_2^* \operatorname{Eis}_{\mathcal{D}}^{k_2+j}(u_2)\right)|_V = D_2 dx \wedge \left(F_{-u_2}^{(k_2+j+2)}(iy) p_2^* \psi_{k_2+j,0} + \overline{F}_{-u_2}^{(k_2+j+2)}(iy) p_2^* \psi_{0,k_2+j}\right) \\
&= dx \wedge D_2 \left(F_{-a_2,-b_2}^{(k_2+j+2)}(x+iy) p_2^* \psi_{k_2+j,0} + F_{-a_2,b_2}^{(k_2+j+2)}(x+iy) p_2^* \psi_{0,k_2+j}\right),
\end{aligned}$$

we have

$$\begin{aligned}
R(y) :=& \int_{X^w[0,x]_\infty^y}^* p_* \left(p_1^* \operatorname{Eis}_{\mathcal{D}}^{k_1+j}(u_1) \wedge \pi_{k_2+j+1}\left(p_2^* \operatorname{Eis}_{\mathcal{D}}^{k_2+j}(u_2)\right)\right) \\
=& (-1)^{k_1+j} D_1 D_2 \int_{[0,x]_\infty^y}^* dx \int_{\gamma_{\tau,m}} \sum_{a=0}^{k_1+j} \left(a_0 y + (-1)^{k_1+j-a} \binom{k_1+j}{a} b_0 y^{-k_1-j}\right) \\
&\cdot p_*\left(p_1^* \psi_{a,k_1+j-a} \wedge c_0 (p_2^* \psi_{k_2+j,0} + p_2^* \psi_{0,k_2+j})\right) + O(e^{-2\pi y/N}),
\end{aligned}$$

where

$$\begin{aligned}
a_0 =& L(\delta_{-a_1}, k_1+j+2) + (-1)^{k_1+j} L(\delta_{a_1}, k_1+j+2), \\
b_0 =& \delta_{a_1}(0)(-2i)^{-k_1-j}\pi \left(L(\delta_{-b_1}, k_1+j+1) + (-1)^{k_1+j} L(\delta_{b_1}, k_1+j+1)\right), \\
c_0 =& N^{-k_2-j-1} L(\delta_{-a_2}, -k_2-j-1)
\end{aligned}$$

are constant terms which appear in the Fourier expansions of F_{-b_1,a_1}^{a,k_1+j-a} and $F_{-a_2,\pm b_2}^{(k_2+j+2)}$.

After Lemma 6.7.1 we have

$$\begin{aligned}
R(y) =& (-1)^{k_1+j} c_0 D_1 D_2 D_3 \int_{[0,x]_\infty^y} dx \\
&\cdot \left(\sum_{a=0}^{k_1} \left(a_0 y + (-1)^{k_1+j-a} \binom{k_1+j}{a} b_0 y^{-k_1-j}\right) \binom{k_1+j-a}{j} y^j (x-iy)^{k_1-a} (x+iy)^{k_2+a}\right. \\
&\left. + (-1)^j \sum_{a=0}^{k_1} \left(a_0 y + (-1)^a \binom{k_1+j}{a} b_0 y^{-k_1-j}\right) \binom{k_1+j-a}{j} y^j (x-iy)^{k_2+a} (x+iy)^{k_1-a}\right) \\
&+ O(e^{-2\pi y/N}).
\end{aligned}$$

Viewing $R(y)$ as a polynomial of y, we only need to consider its constant terms. Therefore $R^*(\infty)$ is the coefficient of y^{k_1} in

$$\begin{aligned}(-1)^{k_1+j} b_0 c_0 D_1 D_2 D_3 \int_{[0,x]_\infty^y} dx \\ \cdot \Bigg(\sum_{a=0}^{k_1} (-1)^{k_1+j-a} \binom{k_1+j}{a} \binom{k_1+j-a}{j} (x-iy)^{k_1-a} (x+iy)^{k_2+a} \\ + \sum_{a=0}^{k_1} (-1)^{a+j} \binom{k_1+j}{a} \binom{k_1+j-a}{j} (x-iy)^{k_2+a} (x+iy)^{k_1-a} \Bigg).\end{aligned}$$

Observe that this polynomial is

$$\begin{aligned}&(-1)^{k_1} b_0 c_0 D_1 D_2 D_3 \binom{k_1+j}{j} \int_{[0,x]_\infty^y} dx \\ &\cdot \sum_{a=0}^{k_1} \binom{k_1}{a} \left((-x+iy)^{k_1-a} (x+iy)^a (x+iy)^{k_2} - (x-iy)^{k_2} (-x+iy)^a (x+iy)^{k_1-a} \right) \\ =&(-1)^{k_1} b_0 c_0 D_1 D_2 D_3 \binom{k_1+j}{j} (2iy)^{k_1} \int_{[0,x]_\infty^y} \left((x+iy)^{k_2} + (x-iy)^{k_2} \right) dx.\end{aligned}$$

This yields

$$\begin{aligned}R^*(\infty) &= (-1)^{k_1} 2 b_0 c_0 D_1 D_2 D_3 \binom{k_1+j}{j} (2i)^{k_1+1} \int_{[0,x]_\infty^y}^* x^{k_2} dx \\ &= (-1)^{k_1} 2 b_0 c_0 D_1 D_2 D_3 \binom{k_1+j}{j} (2i)^{k_1} (k_2+1)^{-1} x^{k_2+1}.\end{aligned}$$

Set $R^{k_1,k_2,j}(u_1,u_2) = (-1)^{k_1} 2 b_0 c_0 D_1 D_2 D_3 \binom{k_1+j}{j} (2i)^{k_1} (k_2+1)^{-1} x^{k_2+1}$ then we can obtain another integral

$$\begin{aligned}S^{k_1,k_2,j}(u_1,u_2) &= \int_{[0,x]_\infty}^* \int_{\eta_{\tau,m}} q_* \left(q_2^* \operatorname{Eis}_{\mathcal{D}}^{k_2+j}(u_2) \wedge \pi_{k_1+j+1} \left(q_1^* \operatorname{Eis}_{\mathcal{D}}^{k_1+j}(u_1) \right) \right) \\ &= R^{k_2,k_1,j}(u_2,u_1).\end{aligned}$$

6.8 Proof of Results

Proof of Theorem 6.1.7 and Theorem 6.1.3. The full regulator is given as

$$\begin{aligned}\int_{X^m Y^{w-m}\{0,\infty\}}^* \operatorname{Eis}_{\mathcal{D}}^{k_1,k_2,j}(u_1,u_2) &= \int_{X^m Y^{w-m}\{0,\infty\}}^* p_* \big(p_1^* \operatorname{Eis}_{\mathcal{D}}^{k_1+j}(u_1) \wedge \pi_{k_2+j+1}(p_2^* \operatorname{Eis}_{hol}^{k_2+j}(u_2)) \big) \\ &+ \int_{X^m Y^{w-m}\{0,\infty\}}^* (-1)^{k_1+j+1} p_* \big(\pi_{k_1+j+1}(p_1^* \operatorname{Eis}_{hol}^{k_1+j}(u_1)) \wedge p_2^* \operatorname{Eis}_{\mathcal{D}}^{k_2+j}(u_2) \big) \\ &= I^{k_1,k_2,j}(u_1,u_2) + (-1)^{k_1+k_2+j+1} J^{k_1,k_2,j}(u_1,u_2).\end{aligned}$$

And when $m = w$ the residue is

$$\begin{aligned}\int_{X^w[0,x]_\infty}^{*} \mathrm{Eis}_{\mathcal{D}}^{k_1,k_2,j}(u_1,u_2) &= \int_{X^w[0,x]_\infty}^{*} p_*\big(p_1^*\,\mathrm{Eis}_{\mathcal{D}}^{k_1+j}(u_1) \wedge \pi_{k_2+j+1}(p_2^*\,\mathrm{Eis}_{hol}^{k_2+j}(u_2))\big)\\ &+ \int_{X^w[0,x]_\infty}^{*} (-1)^{k_1+j+1} p_*\big(\pi_{k_1+j+1}(p_1^*\,\mathrm{Eis}_{hol}^{k_1+j}(u_1)) \wedge p_2^*\,\mathrm{Eis}_{\mathcal{D}}^{k_2+j}(u_2)\big)\\ &= R^{k_1,k_2,j}(u_1,u_2) + (-1)^{k_1+k_2+j+1} S^{k_1,k_2,j}(u_1,u_2).\end{aligned}$$

We get the result directly from the computations of periods I and J and residues R and S we did in Section 6.7. □

If $m = w$ we have much simpler result with Lanphier's formula.

Proof of Theorem 6.1.1. If $m = w$ then we have the constants $C^{(1)}_{k_2,k_2,k_2} = 1$ and $C^{(2)}_{k_2,k_2,k_2+l,j} = \binom{j}{l}$. We deduce instantly from Theorem 6.1.7 that

$$\begin{aligned}&\int_{X^w\{0,\infty\}}^{*} \mathrm{Eis}_{\mathcal{D}}^{k_1,k_2,j}(u_1,u_2) = (-1)^{\frac{1}{2}j(j+1)} \frac{j!(k_1+j+2)(k_2+j+2)}{4N^{w+j+2}} i^{k_1-k_2+j+1}(2\pi)^{w+1}\\ &\cdot \Bigg(2\Lambda^*\left(G^{(k_1+1)}_{b_1,-a_2}G^{(k_2+1)}_{b_2,a_1} - (-1)^j G^{(k_1+1)}_{b_1,a_2}G^{(k_2+1)}_{b_2,-a_1}, -j\right)\\ &+ \sum_{l=0}^{j} \frac{1}{l!}\left(\frac{4\pi}{N}\right)^l \Lambda^*\left(-D^l G^{(k_1+1)}_{b_1,a_2}G^{(k_2+1)}_{b_2,a_1} + (-1)^j D^l G^{(k_1+1)}_{b_1,-a_2}G^{(k_2+1)}_{b_2,-a_1}, l-j\right)\\ &+ \sum_{l=0}^{j} \frac{1}{l!}\left(\frac{4\pi}{N}\right)^l \Lambda^*\left((-1)^j D^l G^{(k_2+1)}_{b_2,a_1}G^{(k_1+1)}_{b_1,a_2} - D^l G^{(k_2+1)}_{b_2,-a_1}G^{(k_1+1)}_{b_1,-a_2}, l-j\right)\Bigg).\end{aligned}$$

We claim that the last two lines actually vanish. With Lanphier's formula we can rewrite the L-value of quasi-modular forms

$$\sum_{l=0}^{j} \frac{1}{l!}\left(\frac{4\pi}{N}\right)^l \Lambda^*\left(-D^l G^{(k_1+1)}_{b_1,a_2}G^{(k_2+1)}_{b_2,a_1} + (-1)^j D^l G^{(k_2+1)}_{b_2,a_1}G^{(k_1+1)}_{b_1,a_2}, l-j\right)$$

to the following explicit sum of L-values of Rankin–Cohen brackets

$$\sum_{p=0}^{j} B(p)\left(\frac{4\pi}{N}\right)^p \Lambda^*\left(\left[G^{(k_1+1)}_{b_1,a_2}, G^{(k_2+1)}_{b_2,a_1}\right]_p, p-j\right)$$

with

$$B(p) = \sum_{t=0}^{j-p} (-2)^t \frac{(j-p)_t}{(t+p)!}\left(a^{t+p}_{k_1+1,k_2+1}(t) + (-1)^{j-p+1} a^{t+p}_{k_2+1,k_1+1}(t)\right).$$

It is indicated in Lemma 6.4.6 that we have actually $B(p) \equiv 0$, this gives us the desired result. □

Appendices

Appendix A

Table of $\mathbb{L}^*(\mathbb{G}_{k_1}, \mathbb{G}_{k_2}, s_1, s_2)$ with $k_1 + k_2 \leq 14$

Let the L-functions be

$$\Lambda(\mathbb{G}_k, s) = (2\pi)^{-s}\Gamma(s)\zeta(s)\zeta(s-k+1)$$

and

$$\Lambda(\Delta, s) = (2\pi)^{-s}\Gamma(s)\prod_p \left(1 - \tau(p)p^{-s} + p^{11-2s}\right)^{-1},$$

they satisfy the functional equations

$$\Lambda(\mathbb{G}_k, s) = i^k\Lambda(\mathbb{G}_k, k-s) \quad \text{and} \quad \Lambda(\Delta, s) = \Lambda(\Delta, 12-s).$$

Double L-values $\mathbb{L}^*(\mathbb{G}_{k_1}, \mathbb{G}_{k_2}, s_1, s_2)$ with $1 \leq s_1 \leq k_1$ and $1 \leq s_2 \leq k_2 - s_1$.

$k_1 = 4$, $k_2 = 4$

$$\mathbb{L}^*(\mathbb{G}_4, \mathbb{G}_4, 1, 2) = \tfrac{3}{8\pi^2}\Lambda(\mathbb{G}_2, -2) + \tfrac{5}{48\pi}\Lambda(\mathbb{G}_4, -1)$$

$$\mathbb{L}^*(\mathbb{G}_4, \mathbb{G}_4, 2, 1) = -\tfrac{3}{4\pi}\Lambda(\mathbb{G}_2, -2) - \tfrac{5}{12}\Lambda(\mathbb{G}_4, -1)$$

$k_1 = 4$, $k_2 = 6$

$$\mathbb{L}^*(\mathbb{G}_4, \mathbb{G}_6, 1, 2) = \tfrac{1}{80\pi^2}\Lambda(\mathbb{G}_4, -2) + \tfrac{7}{480\pi}\Lambda(\mathbb{G}_6, -1)$$

$$\mathbb{L}^*(\mathbb{G}_4, \mathbb{G}_6, 1, 4) = \tfrac{1}{2\pi^3}\Lambda(\mathbb{G}_2, -3) + \tfrac{1}{16\pi^2}\Lambda(\mathbb{G}_4, -2)$$

$$\mathbb{L}^*(\mathbb{G}_4, \mathbb{G}_6, 2, 1) = -\tfrac{1}{8\pi}\Lambda(\mathbb{G}_4, -2) - \tfrac{7}{80}\Lambda(\mathbb{G}_6, -1)$$

$$\mathbb{L}^*(\mathbb{G}_4, \mathbb{G}_6, 2, 3) = -\tfrac{3}{4\pi^2}\Lambda(\mathbb{G}_2, -3) - \tfrac{5}{24\pi}\Lambda(\mathbb{G}_4, -2)$$

$$\mathbb{L}^*(\mathbb{G}_4, \mathbb{G}_6, 3, 2) = \tfrac{1}{\pi}\Lambda(\mathbb{G}_2, -3) + \tfrac{5}{12}\Lambda(\mathbb{G}_4, -2)$$

$$\boldsymbol{k_1 = 6,\ k_2 = 4}$$

$$\mathbb{L}^*(\mathbb{G}_6, \mathbb{G}_4, 1, 2) = \tfrac{3}{10\pi^2}\Lambda(\mathbb{G}_4, -2) + \tfrac{7}{240\pi}\Lambda(\mathbb{G}_6, -1)$$

$$\mathbb{L}^*(\mathbb{G}_6, \mathbb{G}_4, 2, 1) = -\tfrac{3}{8\pi}\Lambda(\mathbb{G}_4, -2) - \tfrac{7}{80}\Lambda(\mathbb{G}_6, -1)$$

$$\boldsymbol{k_1 = 4,\ k_2 = 8}$$

$$\mathbb{L}^*(\mathbb{G}_4, \mathbb{G}_8, 1, 2) = \tfrac{1}{168\pi^2}\Lambda(\mathbb{G}_6, -2) + \tfrac{5}{1008\pi}\Lambda(\mathbb{G}_8, -1)$$

$$\mathbb{L}^*(\mathbb{G}_4, \mathbb{G}_8, 1, 4) = \tfrac{1}{168\pi^3}\Lambda(\mathbb{G}_4, -3) + \tfrac{1}{160\pi^2}\Lambda(\mathbb{G}_6, -2)$$

$$\mathbb{L}^*(\mathbb{G}_4, \mathbb{G}_8, 1, 6) = -\tfrac{7}{16\pi^3}\Lambda(\mathbb{G}_4, -3) - \tfrac{1}{48\pi^2}\Lambda(\mathbb{G}_6, -2)$$

$$\mathbb{L}^*(\mathbb{G}_4, \mathbb{G}_8, 2, 1) = -\tfrac{1}{12\pi}\Lambda(\mathbb{G}_6, -2) - \tfrac{5}{126}\Lambda(\mathbb{G}_8, -1)$$

$$\mathbb{L}^*(\mathbb{G}_4, \mathbb{G}_8, 2, 3) = -\tfrac{3}{80\pi^2}\Lambda(\mathbb{G}_4, -3) - \tfrac{7}{240\pi}\Lambda(\mathbb{G}_6, -2)$$

$$\mathbb{L}^*(\mathbb{G}_4, \mathbb{G}_8, 2, 5) = \tfrac{21}{40\pi^2}\Lambda(\mathbb{G}_4, -3) + \tfrac{7}{120\pi}\Lambda(\mathbb{G}_6, -2)$$

$$\mathbb{L}^*(\mathbb{G}_4, \mathbb{G}_8, 3, 2) = \tfrac{3}{16\pi}\Lambda(\mathbb{G}_4, -3) + \tfrac{7}{80}\Lambda(\mathbb{G}_6, -2)$$

$$\mathbb{L}^*(\mathbb{G}_4, \mathbb{G}_8, 3, 4) = -\tfrac{7}{16\pi}\Lambda(\mathbb{G}_4, -3) - \tfrac{7}{80}\Lambda(\mathbb{G}_6, -2)$$

$$\boldsymbol{k_1 = 6,\ k_2 = 6}$$

$$\mathbb{L}^*(\mathbb{G}_6, \mathbb{G}_6, 1, 2) = \tfrac{1}{480\pi}\Lambda(\mathbb{G}_8, -1)$$

$$\mathbb{L}^*(\mathbb{G}_6, \mathbb{G}_6, 1, 4) = -\tfrac{15}{16\pi^4}\Lambda(\mathbb{G}_2, -4) - \tfrac{1}{16\pi^3}\Lambda(\mathbb{G}_4, -3)$$

$$\mathbb{L}^*(\mathbb{G}_6, \mathbb{G}_6, 2, 1) = -\tfrac{1}{120}\Lambda(\mathbb{G}_8, -1)$$

$$\mathbb{L}^*(\mathbb{G}_6, \mathbb{G}_6, 2, 3) = \tfrac{5}{4\pi^3}\Lambda(\mathbb{G}_2, -4) + \tfrac{3}{16\pi^2}\Lambda(\mathbb{G}_4, -3)$$

$$\mathbb{L}^*(\mathbb{G}_6, \mathbb{G}_6, 3, 2) = -\tfrac{5}{4\pi^2}\Lambda(\mathbb{G}_2, -4) - \tfrac{5}{16\pi}\Lambda(\mathbb{G}_4, -3)$$

$$\mathbb{L}^*(\mathbb{G}_6, \mathbb{G}_6, 4, 1) = \tfrac{5}{4\pi}\Lambda(\mathbb{G}_2, -4) + \tfrac{5}{12}\Lambda(\mathbb{G}_4, -3)$$

$$\boldsymbol{k_1 = 8,\ k_2 = 4}$$

$$\mathbb{L}^*(\mathbb{G}_8, \mathbb{G}_4, 1, 2) = \tfrac{2}{7\pi^2}\Lambda(\mathbb{G}_6, -2) + \tfrac{5}{336\pi}\Lambda(\mathbb{G}_8, -1)$$

$$\mathbb{L}^*(\mathbb{G}_8, \mathbb{G}_4, 2, 1) = -\tfrac{1}{3\pi}\Lambda(\mathbb{G}_6, -2) - \tfrac{5}{126}\Lambda(\mathbb{G}_8, -1)$$

$$\boldsymbol{k_1 = 4,\ k_2 = 10}$$

$$\mathbb{L}^*(\mathbb{G}_4, \mathbb{G}_{10}, 1, 2) = -\tfrac{1}{225}\Lambda(\Delta, 0) + \tfrac{1}{288\pi^2}\Lambda(\mathbb{G}_8, -2) + \tfrac{11}{4800\pi}\Lambda(\mathbb{G}_{10}, -1)$$

$$\mathbb{L}^*(\mathbb{G}_4, \mathbb{G}_{10}, 1, 4) = -\tfrac{1}{432}\Lambda(\Delta, 0) + \tfrac{1}{448\pi^3}\Lambda(\mathbb{G}_6, -3) + \tfrac{5}{3024\pi^2}\Lambda(\mathbb{G}_8, -2)$$

$$\mathbb{L}^*(\mathbb{G}_4, \mathbb{G}_{10}, 1, 6) = -\tfrac{1}{540}\Lambda(\Delta, 0) - \tfrac{1}{864\pi^2}\Lambda(\mathbb{G}_8, -2)$$

$$\mathbb{L}^*(\mathbb{G}_4, \mathbb{G}_{10}, 1, 8) = -\tfrac{1}{432}\Lambda(\Delta, 0) + \tfrac{27}{64\pi^3}\Lambda(\mathbb{G}_6, -3) + \tfrac{5}{432\pi^2}\Lambda(\mathbb{G}_8, -2)$$

$$\mathbb{L}^*(\mathbb{G}_4, \mathbb{G}_{10}, 2, 1) = -\tfrac{1}{16\pi}\Lambda(\mathbb{G}_8, -2) - \tfrac{11}{480}\Lambda(\mathbb{G}_{10}, -1)$$

$$\mathbb{L}^*(\mathbb{G}_4, \mathbb{G}_{10}, 2, 3) = -\tfrac{1}{56\pi^2}\Lambda(\mathbb{G}_6, -3) - \tfrac{5}{504\pi}\Lambda(\mathbb{G}_8, -2)$$

$$\mathbb{L}^*(\mathbb{G}_4, \mathbb{G}_{10}, 2, 5) = \tfrac{1}{240\pi}\Lambda(\mathbb{G}_8, -2)$$

$$\mathbb{L}^*(\mathbb{G}_4, \mathbb{G}_{10}, 2, 7) = -\tfrac{27}{56\pi^2}\Lambda(\mathbb{G}_6, -3) - \tfrac{5}{168\pi}\Lambda(\mathbb{G}_8, -2)$$

$$\mathbb{L}^*(\mathbb{G}_4, \mathbb{G}_{10}, 3, 2) = \tfrac{1}{8\pi}\Lambda(\mathbb{G}_6, -3) + \tfrac{5}{126}\Lambda(\mathbb{G}_8, -2)$$

$$\mathbb{L}^*(\mathbb{G}_4, \mathbb{G}_{10}, 3, 4) = -\tfrac{1}{120}\Lambda(\mathbb{G}_8, -2)$$

$$\mathbb{L}^*(\mathbb{G}_4, \mathbb{G}_{10}, 3, 6) = \tfrac{3}{8\pi}\Lambda(\mathbb{G}_6, -3) + \tfrac{5}{126}\Lambda(\mathbb{G}_8, -2)$$

$$\boldsymbol{k_1 = 6,\ k_2 = 8}$$

$$\mathbb{L}^*(\mathbb{G}_6, \mathbb{G}_8, 1, 2) = \tfrac{1}{350}\Lambda(\Delta, 0) + \tfrac{11}{25200\pi^2}\Lambda(\mathbb{G}_{10}, -1)$$

$$\mathbb{L}^*(\mathbb{G}_6, \mathbb{G}_8, 1, 4) = \tfrac{1}{672}\Lambda(\Delta, 0) - \tfrac{3}{560\pi^4}\Lambda(\mathbb{G}_4, -4) - \tfrac{3}{640\pi^3}\Lambda(\mathbb{G}_6, -3)$$

$$\mathbb{L}^*(\mathbb{G}_6, \mathbb{G}_8, 1, 6) = \tfrac{1}{840}\Lambda(\Delta, 0) - \tfrac{9}{4\pi^5}\Lambda(\mathbb{G}_2, -5) - \tfrac{5}{56\pi^4}\Lambda(\mathbb{G}_4, -4)$$

$$\mathbb{L}^*(\mathbb{G}_6, \mathbb{G}_8, 2, 1) = -\tfrac{11}{5040}\Lambda(\mathbb{G}_{10}, -1)$$

$$\mathbb{L}^*(\mathbb{G}_6, \mathbb{G}_8, 2, 3) = \tfrac{1}{40\pi^3}\Lambda(\mathbb{G}_4, -4) + \tfrac{3}{160\pi^2}\Lambda(\mathbb{G}_6, -3)$$

$$\mathbb{L}^*(\mathbb{G}_6, \mathbb{G}_8, 2, 5) = \tfrac{45}{16\pi^4}\Lambda(\mathbb{G}_2, -5) + \tfrac{1}{4\pi^3}\Lambda(\mathbb{G}_4, -4)$$

$$\mathbb{L}^*(\mathbb{G}_6, \mathbb{G}_8, 3, 2) = -\tfrac{3}{40\pi^2}\Lambda(\mathbb{G}_4, -4) - \tfrac{7}{160\pi}\Lambda(\mathbb{G}_6, -3)$$

$$\mathbb{L}^*(\mathbb{G}_6, \mathbb{G}_8, 3, 4) = -\tfrac{5}{2\pi^3}\Lambda(\mathbb{G}_2, -5) - \tfrac{3}{8\pi^2}\Lambda(\mathbb{G}_4, -4)$$

$$\mathbb{L}^*(\mathbb{G}_6, \mathbb{G}_8, 4, 1) = \tfrac{1}{4\pi}\Lambda(\mathbb{G}_4, -4) + \tfrac{7}{80}\Lambda(\mathbb{G}_6, -3)$$

$$\mathbb{L}^*(\mathbb{G}_6, \mathbb{G}_8, 4, 3) = \tfrac{15}{8\pi^2}\Lambda(\mathbb{G}_2, -5) + \tfrac{5}{12\pi}\Lambda(\mathbb{G}_4, -4)$$

$$\mathbb{L}^*(\mathbb{G}_6, \mathbb{G}_8, 5, 2) = -\tfrac{3}{2\pi}\Lambda(\mathbb{G}_2, -5) - \tfrac{5}{12}\Lambda(\mathbb{G}_4, -4)$$

$k_1 = 8$, $k_2 = 6$

$$\mathbb{L}^*(\mathbb{G}_8, \mathbb{G}_6, 1, 2) = -\tfrac{1}{350}\Lambda(\Delta, 0) + \tfrac{11}{16800\pi}\Lambda(\mathbb{G}_{10}, -1)$$

$$\mathbb{L}^*(\mathbb{G}_8, \mathbb{G}_6, 1, 4) = -\tfrac{1}{672}\Lambda(\Delta, 0) - \tfrac{6}{7\pi^4}\Lambda(\mathbb{G}_4, -4) + \tfrac{3}{128\pi^3}\Lambda(\mathbb{G}_6, -3)$$

$$\mathbb{L}^*(\mathbb{G}_8, \mathbb{G}_6, 2, 1) = -\tfrac{11}{5040}\Lambda(\mathbb{G}_{10}, -1)$$

$$\mathbb{L}^*(\mathbb{G}_8, \mathbb{G}_6, 2, 3) = \tfrac{1}{\pi^3}\Lambda(\mathbb{G}_4, -4) + \tfrac{1}{16\pi^2}\Lambda(\mathbb{G}_6, -3)$$

$$\mathbb{L}^*(\mathbb{G}_8, \mathbb{G}_6, 3, 2) = -\tfrac{4}{5\pi^2}\Lambda(\mathbb{G}_4, -4) - \tfrac{7}{80}\Lambda(\mathbb{G}_6, -3)$$

$$\mathbb{L}^*(\mathbb{G}_8, \mathbb{G}_6, 4, 1) = \tfrac{1}{2\pi}\Lambda(\mathbb{G}_4, -4) + \tfrac{7}{80}\Lambda(\mathbb{G}_6, -3)$$

$k_1 = 10$, $k_2 = 4$

$$\mathbb{L}^*(\mathbb{G}_{10}, \mathbb{G}_4, 1, 2) = \tfrac{1}{225}\Lambda(\Delta, 0) + \tfrac{5}{18\pi^2}\Lambda(\mathbb{G}_8, -2) + \tfrac{11}{1200\pi}\Lambda(\mathbb{G}_{10}, -1)$$

$$\mathbb{L}^*(\mathbb{G}_{10}, \mathbb{G}_4, 2, 1) = -\tfrac{5}{16\pi}\Lambda(\mathbb{G}_8, -2) - \tfrac{11}{480}\Lambda(\mathbb{G}_{10}, -1)$$

Appendix B

Table of $G(\tau; k_1, k_2, k_3)$ with weight lower than 12

Mordell–Tornheim double Eisenstein series $G(\tau; k_1, k_2, k_3)$ with $k = k_1 + k_2 + k_3$.

$\boldsymbol{k = 6}$
$G(\tau; 1, 2, 3) = -\frac{\pi^6}{945} E_6(\tau)$
$G(\tau; 2, 2, 2) = \frac{2\pi^6}{945} E_6(\tau)$
$\boldsymbol{k = 8}$
$G(\tau; 1, 2, 5) = -\frac{\pi^8}{14175} E_8(\tau)$
$G(\tau; 1, 3, 4) = -\frac{2\pi^8}{14175} E_8(\tau)$
$G(\tau; 2, 2, 4) = \frac{2\pi^8}{14175} E_8(\tau)$
$G(\tau; 2, 3, 3) = 0$
$\boldsymbol{k = 10}$
$G(\tau; 1, 2, 7) = -\frac{\pi^{10}}{155925} E_{10}(\tau)$
$G(\tau; 1, 3, 6) = -\frac{\pi^{10}}{66825} E_{10}(\tau)$
$G(\tau; 1, 4, 5) = -\frac{\pi^{10}}{93555} E_{10}(\tau)$

$$G(\tau;2,2,6) = \tfrac{2\pi^{10}}{155925}E_{10}(\tau)$$

$$G(\tau;2,3,5) = \tfrac{\pi^{10}}{467775}E_{10}(\tau)$$

$$G(\tau;2,4,4) = \tfrac{4\pi^{10}}{467775}E_{10}(\tau)$$

$$G(\tau;3,3,4) = -\tfrac{2\pi^{10}}{467775}E_{10}(\tau)$$

$k = 12$

$$G(\tau;1,2,9) = -\tfrac{404\pi^{12}}{638512875}\left(E_{12}(\tau) - \tfrac{51891840}{69791}\Delta(\tau)\right)$$

$$G(\tau;1,3,8) = -\tfrac{326\pi^{12}}{212837625}\left(E_{12}(\tau) + \tfrac{34594560}{112633}\Delta(\tau)\right)$$

$$G(\tau;1,4,7) = -\tfrac{643\pi^{12}}{638512875}\left(E_{12}(\tau) + \tfrac{1089728640}{444313}\Delta(\tau)\right)$$

$$G(\tau;1,5,6) = -\tfrac{739\pi^{12}}{638512875}\left(E_{12}(\tau) - \tfrac{1089728640}{510649}\Delta(\tau)\right)$$

$$G(\tau;2,2,8) = \tfrac{808\pi^{12}}{638512875}\left(E_{12}(\tau) - \tfrac{51891840}{69791}\Delta(\tau)\right)$$

$$G(\tau;2,3,7) = \tfrac{34\pi^{12}}{127702575}\left(E_{12}(\tau) + \tfrac{62270208}{11747}\Delta(\tau)\right)$$

$$G(\tau;2,4,6) = \tfrac{43\pi^{12}}{58046625}\left(E_{12}(\tau) + \tfrac{42456960}{29713}\Delta(\tau)\right)$$

$$G(\tau;2,5,5) = \tfrac{38\pi^{12}}{91216125}\left(E_{12}(\tau) - \tfrac{111196800}{13129}\Delta(\tau)\right)$$

$$G(\tau;3,3,6) = -\tfrac{68\pi^{12}}{127702575}\left(E_{12}(\tau) + \tfrac{62270208}{11747}\Delta(\tau)\right)$$

$$G(\tau;3,4,5) = -\tfrac{19\pi^{12}}{91216125}\left(E_{12}(\tau) - \tfrac{111196800}{13129}\Delta(\tau)\right)$$

$$G(\tau;4,4,4) = \tfrac{38\pi^{12}}{91216125}\left(E_{12}(\tau) - \tfrac{111196800}{13129}\Delta(\tau)\right)$$

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