AdS /CFT and quantum gravity

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Introduction and Summary

The development of holographic duality [6] [93] has been highly effective and substantial in addressing major problems in theoretical particle physics. The significance of this development relies on the establishment of the connection between weakly and strongly coupled theories. This fact allows the study of long standing questions on quantum gravitational phenomena , gives access to the strong coupling regime of quantum field theories, while it allows for the study of non-perturbative string theory through weakly coupled gauge theories.

Regarding quantum field theory, enormous progress has been achieved with the use of powerful computational techniques based on supersymmetry which allow for obtaining exact results for supersymmetric observables. One of the leading exact result methods in gauge theories is the one of supersymmetric localization [114] [99], which relies on reducing an infinite dimensional path integral to a finite-dimensional integral. This method has found application in supersymmetric gauge theories in various dimensions with some notable mentions being the two-dimensional $\mathcal{N} = (2, 2)$ theories (partition function on round and squashed S^2 [27][60][70]), three-dimensional $\mathcal{N} = 2$ theories [79] (supersymmetric Wilson loops in Chern-Simons matter theories) and the four-dimensional $\mathcal{N} = 2$ theories (Localization on round S^4[99], Nekrasov partition function [96]).

The theoretical background of this work is the holographic duality between three-dimensional $\mathcal{N} = 4$ superconformal theories and type IIB supergravity on the warped background $AdS_4 \times_w M_6$, where the six-dimensional manifold is comprised by two two-spheres wraped over a two-dimensional Riemann surface: $S^2 \times \hat{S}^2 \times \Sigma_{(2)}$. Both the supergravity solutions and this class of superconformal theories have been subjects of intensive study and offer a rich ground for exploring various directions such as interesting dualities or new approaches to important questions in quantum gravity. This correspondence has been developed in [12][13] and has been tested throughout [14]. In the present work, we study specific questions in the theories of both sides of the aforementioned duality.

The first main axis of this work regards the solutions of the gravitational side of the correspondence and in particular the study of massive $\mathcal{N} = 4$ (super)gravity in four dimensional anti-de Sitter space. Relativistic theories of massive gravity [47] [105] [76] [48], have received a great amount of attention over the last years among other theories of modified gravity. The question that lies in the core of these theories, is a long-standing one of theoretical physics and regards whether the graviton can obtain a mass.

General relativity (GR), is an accurate and physically elegant theory of gravity at low energies. It is given in terms of the Einstein-Hilbert (EH) action and describes the non-linear self-interactions of a massless spin-2 field. Modifications of gravity are mainly motivated by important questions in cosmology [50] [37] , with the most significant being undoubtedly the cosmological constant problem [112]. Well promising directions are the ones where gravity is modified at large distances, or equivalently at low energies. An example of such an IR modification of gravity, is a theory of massive gravity. Consistently modifying GR into a theory of massive gravity would simply be based on adding a mass term for the graviton to the EH action so that in the limit of vanishing graviton mass ($m \to 0$), GR would be recovered. Nevertheless, this logic hides significant inconsistencies and pathologies. The first attempt of writing an action of a theory with a massive graviton, was made in [63]. Indeed, in this case, in the limit of vanishing graviton mass the predictions of the obtained theory are far from those of GR. A particular example , is the case of the linear theory coupled to matter, in the work [111], the massless limit of which gives a different prediction for light bending from the one GR gives. This is one of the typical disagreements in physical predictions one confronts in theories of massive gravity in the massless limit, a fact that is characterized as a van Dam-Veltman-Zakharov discontinuity [87][100] . The source of this problem has to do with the degrees of freedom the mass term introduces. This can be seen from the decomposition of the Pauli-Fierz mass term: the four-dimensional massive graviton possesses five degrees of freedom out of which two correspond to the helicity-2 mode (massless graviton), another two to the helicity-1 mode (massless vector) and finally one to the helicity-0 mode (massless scalar). The scalar provides the coupling to matter, as it couples to the trace of stress-tensor of the theory. In the massless limit, this coupling remains, and therefore, the theory obtained in this limit is not GR but rather GR plus a massless scalar, which turns out to be the source for the aforementioned physical parameter discontinuities.

The problem of such disconituities is resolved by the so called Vainshtein screening [109]. This concept suggests that a massive gravity theory becomes strongly coupled in the IR, in the massless limit. Therefore the linear theory does not provide the complete description but rather a first order description of a non-linear theory. These non-linearities which become dominant in the massless limit, are the ones that finally compensate the discontinuities. However it has been shown that the non-linear theories suffer from the presence of ghost-degrees of freedom. This regards the work [33], where the studied non-linear theory appeared to possess instead of five degrees of freedom, an extra one, which corresponds to a scalar with negative kinetic term, called Boulware-Desser ghost. In flat background, this mode decouples but on the contrary around non-trivial backgrounds its mass remains finite and hence it is a part of the theory.

Nevertheless, the attention this interesting subject has attracted has led to significant developments which provided solutions to certain of the aforementioned pathologies, at some extent. These include developements in effective field theory [10][35][51][26][49] where regimes of validity and properties of such theories have been studied and the constructions of non-linear extentions of the Fierz-Pauli actions which are ghost free [48][74][36].

Questions regarding consistency and validity of these theories could be answered by embedding massive gravity in string theory, which is ultraviolet (UV) complete. String theory is a consistent theory and therefore it is expected that the effective four-dimensional theories obtained by the embeddings studied in this work will be free of ghosts and other pathologies. An important detail is that most of the problems that regard massive gravity theories, like the vDVZ discontinuity, are absent in anti-de Siter space.[75][100][80] Therefore studies of string theory embeddings of massive gravity in AdS space could be very promising, apart from the fact that are far from realistic, apparently due to the negative cosmological constant and of supersymmetry.

A model that concerned four dimensional massive gravity in AdS space, is the one intrduced by Karch and Randall [82][81]. This work regards the study of metric fluctuations in AdS_5 in the presence of a thin AdS_4 brane on which the four dimensional graviton would be localized (two slices of AdS_5 glued along the thin AdS_4 brane). It was shown that if the ratio of the AdS radii of curvature is small ($L_5/L_4 \ll 1$), this result to a slightly massive graviton localized on the AdS_4 brane along with two infinite towers of AdS_5 modes. The string-theory embedding of the Karch-Randall model is given by the exact solutions describing intersecting D3, D5 and NS5 branes [52][54]. Nevertheless, this thin-brane approximation fails, as it is going to be explained in the main text [20] and therefore the Kaluza-Klein scale, beyond which any effective four-dimensional description breaks down, is the AdS_4 radius (L_4) and not the AdS_5 (L_5).

Other proposals regarding string theory embeddings of massive AdS gravity are based on transparent boundary conditions in AdS [102] [101] [61] or on multitrace deformations of the dual CFT [85][4] [86] and they share the characteristic that the graviton mass arises as a quantum effect.

The second main axis of this work regards the gauge theory side of the correspondence. Specifically, the objective is the study of exactly marginal deformations of the superconformal theories at hand.

One way of deforming a given conformal field theory is by adding an appropriate term in its action. The term is of the form $\delta S_0 = \lambda \int d^d x \mathcal{O}$, where λ a coupling constant and $\mathcal{O}$ is a local operator of the original theory. The study of deformations actually relies on the inspection of the conformal multiplets of the theory: In particular the deforming local operator is a conformal primary of the corresponding multiplet (and moreover a scalar so that Lorentz symmetry is preserved). The type of deformation depends on the scaling dimension ($\Delta_{\mathcal{O}}$) of the local operator. Irrelevant deformations are initiated in the case where $\Delta_{\mathcal{O}} > d$ and then the CFT is an IR fixed point of the renormalization group flow along which the coupling λ flows to zero. Relevant deformations are initiated if $\Delta_{\mathcal{O}} < d$, where the original CFT is the UV fixed point of the renormalization group flow and the coupling grows towards the IR . Apparently in both above cases conformal symmetry is broken at leading order in the coupling. Finally, marginal deformations are initiated in the case $\Delta_{\mathcal{O}} = d$ and preserve conformal invariance at leading order in the coupling. Hence marginal deformations lead to a neighboring CFT. Next to leading order, when the coupling receives corrections, these are divided into marginally irrelevant, marginally relevant and exactly marginal, depending on whether conformal symmetry is broken at higher order in λ or not.

In the case of superconformal field theories, the logic is similar: Superconformal deformations are the ones that preserve supersymmetry. They are activated by conformal primaries that are annihilated by (all) Poincaré supercharges, up to a total derivative. These operators are characterized as ”top components” of the multiplet, in order to stress the difference between them and the ”bottom component”-namely the superconformal primary of the multiplet -which does not meet the above requirement of initiating a superconformal deformation. An important detail is that top components may appear both at the top level of a multiplet (namely by the action of all Q-supercharges on the conformal primary) but also at intermediate levels of a multiplet [38]. Therefore the enumeration of all possible superconformal deformations of a given theory boils down to the enumeration of all possible top components, which demands a detailed inspection of the superconformal multiplets of the theory. Regarding exactly marginal superconformal deformations, the deforming operators are top components of short (absolutely protected A) multiplets of the theory.

The observable which encodes all the information about the short multiplets of the theory, is the superconformal index. First studied in [84] and [30], this object has the definition of a Witten index variant in radial quantization (Hamiltonian definition), as a trace over the space of local operators of the given theory, constructed to receive contributions only from short multiplets. This fact indicates that the index is invariant under continuous deformations of the theory. Moreover, the superconformal index has a definition as a partition function on $S^{d-1} \times S^1$ (Lagrangian definition) and can be computed using supersymmetric localization. This approach has been quite successful in the computation of protected spectra of superconformal theories as well as in the study of dualities in various dimensions [77] [83] [103].

The outline of this work is as follows. In the first chapter, we introduce in detail the three-dimensional $\mathcal{N} = 4$ superconformal theories. The starting point is the relevant supersymmetry algebra and its superconformal extention. We then describe how the content of these theories is organized in $\mathcal{N} = 2$ multiplets and we write down the supersymmetric actions that describe them. The theories considered in this work are linear quiver theories [65], whose structure is explained in detail throughout this chapter. These theories can be understood as low-energy limits of theories living in the worldvolume type-IIB string theory branes [73]: $D3$-branes ending on $NS5$-branes and being intersected by $D5$-branes. The brane picture is very important for establishing the connection with the dual supergravity. The final section is devoted to the moduli space of vacua and the symmetries of these theories [11][72][43]. We present the Higgs, Coulomb and mixed branches, and comment on their structure. For selected examples of quiver theories, the brane configurations that encode the information on their moduli space are presented. The final addition to the section regards the global symmetry of the linear quiver theories and three-dimesional Mirror symmetry [78]. It is explained how the mirror symmetry acts on the branches of the moduli space of the theories and how the mirror dual of a theory can be obtained, using the IIB string theory analogue of mirror symmetry, which is the S-duality. Finally, the full global symmetry of these theories is presented, as the product of the unitary groups rotating the fundamental hypermultiplets of the the theory.

The second chapter is devoted to the IIB dual supergravity solutions, which are the aforementioned warped geometries. [52][54]. In the case where the Riemann surface $\Sigma_{(2)}$ is compact, string theory on this background geometry provides a realization of four-dimensional quantum gravity. In the first section of the chapter we provide a detailed presentation of these solutions. They are parametrized by two harmonic functions on the Riemann surface, on the boundary of which they have (admissible) singularities, interpreted as sources of D5 and NS5 branes. Additionally, the singularities carry D3 brane charge and therefore the supergravity solutions have a brane description similar to the dual superconformal theories. Throughout the second section of the chapter, we present the precise holographic dictionary developed in [12]. With this chapter, we close Part I and we proceed to the main part of the work.

In Part II, we present some new proposals for top-down string theory embeddings of massive four-dimensional anti-de Sitter gravity and of corresponding bigravity models[22][21]. The problem we encounter is to look for supergravity solutions that allow for the lowest-lying AdS_4 graviton to acquire a small mass. The solutions are the ones presented in the first part, with the difference that the six dimensional internal manifold is non-compact: semi-infinite throats of Janus geometry are attached to the compact manifold and are the characteristics of the geometry that lead to a slightly massive graviton. Although the main analysis is carried out in supergravity, the underlying holographic duality presented in Part I, provides instructive insights on the Higgsing mechanism and motivates the construction of the supergravity solutions in question. This is explained in the first section of Chapter 3, where the problem of the small graviton mass is reformulated in field theory as the problem of the small anomalous dimension for the dual stress tensor. The sections that follow include the main analysis, which relies on the study and search of metric fluctuations, around the considered supergravity solutions, that can acquire a small mass. The main result of this study is a quantized formula for the graviton mass, which depends on the effective gravitational couplng of the compact manifold along with the quantized charge of the semi-infinite throat and the variation of the dilaton throughout the throat.

The final chapter of Part II regards a more general setting, the one of an AdS_4 bimetric theory. Holographically this is motivated by the connection of two initially disjoint three-dimensional linear quiver theories, by gauging of a common global symmetry. This corresponds to an extra gauge node of low rank, connecting the two quivers. The two initially conserved stress-tensors of the disjoint quivers eventually mix into a combination which remains conserved and one which is not conserved and hence acquires a small anomalous dimension. The dual supergravity picture is the one of two initially decoupled $AdS_4 \times M_6$ spacetimes, which are connected by a thin throat with $AdS_5 \times S^5$ or Janus geometry, depending on whether the dilaton varies throughout the throat or not. This depends on the characteristics of each of the two supergravity solutions. The two initially massless gravitons mix and the final result is bigravity theory, which includes one massless and one massive graviton. As in the previous chapter, the mass of the graviton is computed and it depends as before on the effective gravitational couplings of the two spacetimes as well as on the quantized charge of the throat and on the dilaton variation. The massive AdS_4 gravity of the previous chapter, turns out to be just a special case of this more general framework: In the limit where one of the two effective gravitational couplings vanishes or when the AdS_4 radius of one of the two spacetimes diverges, the massless graviton decouples and only the massive one remains, verifying that for our solutions, the massive gravity is a special decoupling limit of the bigravity theory.

The content of Part III is almost entirely focused on the three-dimensional linear quiver theories and their operator content [23] . The objective is to map out the superconformal manifold of such theories, namely the family of such theories generated by superconformal (partially or fully supersymmetry preserving) deformations that are exactly marginal. Therefore, we are focusing on scalar top components of short $\mathcal{N} = 4$ multiplets with scaling dimension $\Delta = 3$. However, top components preserving $\mathcal{N} = 4$ supersymmetry do not exist. The maximal supersymmetry that allows the existence of such operators is $\mathcal{N} = 2$. The first part of the analysis relies on "detecting" such exactly marginal operators by the analysis of representations of the superconformal algebra in three dimensions with $\mathcal{N} = 2, 4$ supersymmetry. In particular, after determining in which representations these operators sit, they are extracted and counted with the help of the superconformal index. The index is computed via Coulomb branch localization, namely as a complex multiple integral over the Coulomb branch, summed over monopole charges. The calculation is given in full detail throughout section 4 of Chapter 5. Consequently the Coulomb branch integral is re expressed in terms of superconformal characters, and in this way the relevant superconformal moduli are clearly identified. The main result is that the marginal (chiral) operators of the $T_{\rho}^{\hat{\rho}}[SU(N)]$ theories, transform in the $S^2(\mathrm{Adj}\mathbb{G}, \mathrm{Adj}\hat{\mathbb{G}})$ representation of the electric and magnetic flavor symmetry, plus length-4 strings modulo redundancies for linear quivers with abelian nodes. In particular the the mixed marginal operators transform in the $(\mathrm{Adj}, \mathrm{Adj}, 0)$ representation of the global symmetry (5.1.4), up to some overcounting the quivers of have abelian gauge nodes. The last addition to the chapter regards the holographic interpretation of the results.

Part I

$3d\,\mathcal{N}=4$ Superconformal Theories and type IIB Supergravity Duals

Chapter 1

$3d\,\mathcal{N}=4$ Superconformal Theories

1.1 $\mathcal{N}=4$ supersymmetric gauge theories in three dimensions

In (2+1)-dimensions, the $\mathcal{N}=4$ supersymmetry algebra has 8 real Poincaré supercharges. This fact renders it half-maximal, given that the maximal rank of supersymmetry in three-dimensions is $\mathcal{N}=8$ which corresponds to 16 real supercharges. The algebra is written in terms of four real spinors $\{\mathcal{Q}_\alpha^A\}$:

$$\{\mathcal{Q}_\alpha^A, \mathcal{Q}_\beta^B\} = 2\sigma^\mu_{\alpha\beta}\delta^{AB}P_\mu + 2\epsilon_{\alpha\beta}\mathcal{Z}^{AB} \tag{1.1.1}$$

, with $A, B = (1,2,3,4)$ and $\mu = (0,1,2)$. This algebra is obtained through dimensional reduction from the $\mathcal{N}=2$ supersymmetry algebra in (3+1)-dimensions. The matrices σ^μ generate the three-dimensional Clifford algebra, whereas the central term $\mathcal{Z}$, is a real and antisymmetric matrix, which commutes with the algebra generators. The superconformal extention of the algebra is given by $\mathfrak{osp}(4|4)$ and has in total 16 real supercharges, as it contains 8 additional real conformal supercharges. Finally, the supercharges are rotated by the $SO(4) \simeq SO(3)_R \times SO(3)_{R'}$ R-symmetry group.

The field content of a 3d $\mathcal{N}=4$ super Yang-Mills with gauge group $\mathbb{G}$ is organized in hypermultiplets ($\mathbb{H}$) and a vector multiplet ($\mathbb{V}$), defined in terms of $\mathcal{N}=2$ multiplets [7] [46]. A $3d$ $\mathcal{N}=4$ hypermultiplet is expressed in terms of two $\mathcal{N}=2$ chiral multiplets in conjugate representations $(\mathcal{R}, \bar{\mathcal{R}})$:

$$\mathbb{H} : \{\underbrace{\phi, \psi_\alpha, F_\Phi}_{\Phi^{\mathcal{N}=2}_{\mathcal{R}}} \quad ; \quad \underbrace{\tilde{\phi}, \tilde{\psi}_\alpha, \tilde{F}_{\tilde{\Phi}}}_{\tilde{\Phi}^{\mathcal{N}=2}_{\bar{\mathcal{R}}}}\} \tag{1.1.2}$$

, with $(\phi, \tilde{\phi})$ and $(F_\Phi, \tilde{F}_{\tilde{\Phi}})$ being complex scalars and complex auxiliary scalars accordingly, whereas $(\psi_\alpha, \tilde{\psi}_\alpha)$ are two-component complex spinors.

A $\mathcal{N}=4$ vectormultiplet is written in terms of an $\mathcal{N}=2$ chiral and $\mathcal{N}=2$ vector multiplet, in the adjoint representation of the gauge group ($Adj\mathbb{G}$):

$$\mathbb{V}:\{\underbrace{A_\mu,\lambda_\alpha,\sigma,F}_{V^{\mathcal{N}=2}_{Adj(\mathbb{G})}}\quad;\quad\underbrace{\tilde{\phi},\xi_\alpha,\tilde{F}}_{\Phi^{\mathcal{N}=2}_{Adj(\mathbb{G})}}\} \tag{1.1.3}$$

One notable mention is that in three dimensions there is a duality between the abelian vector-field and a real scalar field:

$$F_{\mu\nu}=\epsilon_{\mu\nu\rho}\partial^\rho\varphi \tag{1.1.4}$$

This scalar is refered to as dual photon and is real and periodic ($\varphi\sim\varphi+g$).

The $\mathcal{N}=2$ vectormultiplets and chiral multiplets in terms of which the $\mathcal{N}=4$ vectormultiplets nd hypermultiplets are expressed, can be introduced in the form of vector ($\mathcal{V}$) and chiral (Φ) superfields, accordingly:

$$\begin{aligned}&\mathcal{V}=-\theta^\alpha\sigma^\mu_{\alpha\beta}\bar{\theta}^\beta A^\mu-\theta\bar{\theta}\sigma+i\theta^2\bar{\theta}\bar{\lambda}-i\bar{\theta}^2\theta\lambda+\frac{1}{2}\theta^2\bar{\theta}^2F\\&\Phi=\phi+\sqrt{2}\theta\psi+\theta^2F_\Phi,\quad \bar{D}_\alpha\Phi=0\end{aligned}$$

The supersymmetric action- in flat space- for a three-dimensional $\mathcal{N}=4$ gauge theory is comprised of the pieces for the $\mathcal{N}=4$ vector and hypermultiplets. These can be written in terms of the above superfields. The first piece is the action for the vector multiplet:

$$S^{\mathcal{N}=4}_{\mathbb{V}}=S^{\mathcal{N}=2}_V+\frac{1}{g^2}S^{\mathcal{N}2}_\Phi=\frac{1}{g^2}\int d^3xd^2\theta d^2\bar{\theta}Tr\Big(W^2_\alpha-\Phi^\dagger e^{2\mathcal{V}}\Phi\Big)+\text{h.c} \tag{1.1.5}$$

, with $W_\alpha=-1/4\bar{D}\bar{D}e^{-\mathcal{V}}D_\alpha e^{\mathcal{V}}$, being the chiral field strength and g the three-dimensional coupling. The two pieces of the expression written in component form read:

$$\begin{aligned}&S^{\mathcal{N}=2}_V=\frac{1}{2g^2_{YM}}\int d^3xTr\Big(\frac{1}{2}F_{\mu\nu}F^{\mu\nu}+D_\mu\sigma D^\mu\sigma+D^2+i\bar{\lambda}\gamma^\mu D_\mu\lambda+i\bar{\lambda}[\sigma,\lambda]\Big)\\&S^{\mathcal{N}2}_\Phi=-\int d^3x\Big(D_\mu\bar{\phi}D^\mu\phi+\bar{\phi}\sigma^2\phi+\bar{F}F-i\bar{\psi}\gamma^\mu D_\mu\psi+i\bar{\psi}\sigma\psi+i\bar{\psi}+\bar{\psi}\lambda\phi-i\bar{\phi}\bar{\lambda}\psi\Big)\end{aligned}$$

The second block of the total supersymmetric action is the one regarding the matter content, namely hypermultiplets:

$$S^{\mathcal{N}=4}_{\mathbb{H}}=-\int d^3xd^2\theta d^2\bar{\theta}\sum_{\{\phi\}}\Big(\Phi^\dagger e^{2\mathcal{V}}\Phi+\tilde{\Phi}^\dagger e^{-2\mathcal{V}}\tilde{\Phi}\Big) \tag{1.1.6}$$

,with the two chiral superfields in conjugate representations and the sum over the matter content ($\{\phi\}$) of the theory. The last piece is the action for the $\mathcal{N}=4$ superpotential, $\mathcal{W}_{\mathcal{N}=4}$ which reads:

$$S_{\mathcal{W}_{\mathcal{N}=4}} = -i\sqrt{2}\int d^3x d^2\theta \sum_{\{\phi\}} \left(\tilde{\phi}\Phi\phi\right) + \text{h.c} \tag{1.1.7}$$

The final addition to this section regards the deformations of the above supersymmetric actions. For each abelian factor of the gauge group there can be a Fayet-Iliopoulos term. The three real parameters transform as a $SU(2)_R$-triplet and they can be considered as the bottom components of a background $\mathcal{N}=4$ abelian vector multiplet $\tilde{\mathbb{V}}_0$ coupled to the topological currents $\mathcal{J}_t$ associated with the abelian factors of the gauge group.

$$S_{\text{FI},\mathcal{N}=4} = \int d^3x d^2\theta d^2\bar{\theta}\text{Tr}(\sum V_{FI}) + \int d^3d^2\theta \text{Tr}(\Phi\Phi_{\text{FI}} + \text{h.c}) \tag{1.1.8}$$

, here the deformed Lagrangian is obtained by $V_{\text{FI}} \sim \eta\bar{\theta}\theta$ and $\Phi_{\text{FI}} = 0$:

$$S_{FI} \sim \eta_{FI}\int d^3xD \tag{1.1.9}$$

For the hypermultiplets there can be real and complex mass terms. In this case the three real parameters transform as a triplet of $SU(2)_{R'}$ and can be considered as the bottom components of a background $\mathcal{N}=4$ abelian vector multiplet $\tilde{\mathbb{V}}_0$ coupled to the flavor symmetry currents $\mathcal{J}_{\text{flav}}$:

$$S_{m,\mathcal{N}=4} = -\int d^3x d^2\theta d^2\bar{\theta}\sum_{\{\phi\}}(\phi^\dagger e^{2V_m}\phi + \tilde{\phi}^\dagger e^{2V_m}\tilde{\phi}) - i\sqrt{2}\int d^3x d^2\theta \sum_{\{\phi\}}(\tilde{\phi}\Phi_m\phi) + \text{h.c} \tag{1.1.10}$$

Here, m stands for the real mass parameter, the complex mass is rotated to zero and the deformed Lagrangian is obtained by setting $|\phi_m = 0$ and $V_m \sim m\bar{\theta}\theta$. It then reads in component form:

$$\begin{aligned} S_{m,\mathcal{N}=4} &= S_{\mathcal{N}=2}(\phi, m) + S_{\mathcal{N}=2}(\tilde{\phi}, -m) = \\ &= \int d^3x(D_\mu\bar{\phi}D^\mu\phi + m^2\bar{\phi}\phi + \bar{F}_\Phi F_\Phi - i\bar{\psi}\gamma^\mu D_\mu\psi + im\bar{\psi}\psi) \end{aligned} \tag{1.1.11}$$

1.2 Linear quivers and their Brane Realizations

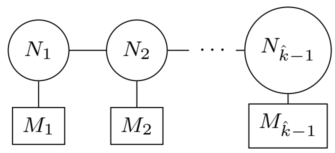
1.2.1 $T^{\rho}_{\hat{\rho}}[SU(N)]$ theories

In this work we are interested in a specific class of three-dimensional $\mathcal{N}=4$ theories [73] . These theories arise as non-trivial infrared (IR) fixed-points of the renormalization group (RG) flow of *linear quiver theories* [65]. Designated as $T^{\rho}_{\hat{\rho}}[SU(N)]$, with $\rho, \hat{\rho}$ being integer partitions of N, these theories under certain conditions which will be presented shortly, flow in the IR to 3d $\mathcal{N}=4$ super Yang-Mills theory with gauge group $SU(N)$.

The full information on the gauge and matter content of these theories is conveniently packaged in a quiver diagram, as it is going to be explained in detail below. The gauge group $\mathbb{G}$ of the $T^{\rho}_{\hat{\rho}}[SU(N)]$ theories is a product of unitary groups:

$$\mathbb{G} = U(N_1) \times U(N_2) \times ... \times U(N_{\hat{k}-1}) \tag{1.2.1}$$

To each unitary factor $U(N_i)$ corresponds a $\mathcal{N}=4$ vector multiplet and $N_{f,i}$-$\mathcal{N}=4$ hypermultiplets in the fundamental representation of the gauge group. On top of that, there is a $\mathcal{N}=4$ hypermultiplet transforming in the bi-fundamental representation of each $U(N_i) \times U(N_{i+1})$ product. In the quiver notation, a node is assigned to each unitary group, a box contains the number M_i of the hypermultiplets transforming in the fundamental of $U(N_i)$ and finally a horizontal segment connecting two nodes represents a bi-fundamental hypermultiplet



The above data that characterize the gauge theory, are completely determined by the three labels $(\rho, \hat{\rho}; N)$ for which:

$$\rho : (l_1, l_2, ..., l_k), \quad N = \sum_{i=1}^{k} l_i$$
$$\hat{\rho} : (\hat{l}_1, \hat{l}_2, ..., \hat{l}_{\hat{k}}), \quad N = \sum_{\hat{i}=1}^{\hat{k}} \hat{l}_{\hat{i}}$$

, where the integers $\{l_i\}$ and $\{\hat{l}_{\hat{i}}\}$ are ordered so that $l_i \geq l_{i+1}$ and $\hat{l}_{\hat{i}} \leq \hat{l}_{\hat{i}+1}$, a fact that has a particular meaning as we will see in what follows. The two partitions apparently correspond to two Young diagrams, with N-boxes each. The i^{th} row of the ρ-diagram is of length l_i whereas the length of the $\hat{i}^{\text{th}}$ is $\hat{l}_{\hat{i}}$. Apart from these two Young diagrams, it is instrumental at this point to introduce also the transpose diagrams $\rho^T, \hat{\rho}^T$, as it will be useful for later analysis. Regarding the ρ^T Young diagram, the length of its $\hat{j}^{\text{th}}$ row is denoted as $l^T_{\hat{j}}$ whereas the length of the j^{th} row of the $\hat{\rho}^T$ Young diagram is denoted as $\hat{l}^T_j$. The notation we use for the transposed diagrams is carefully chosen; the meaning of this will become clear soon.

The most substantial part of this analysis regards to the IR-fixed point of the RG flow itself. It is to be made precise that not all of $T^{\rho}_{\hat{\rho}}[SU(N)]$ theories flow to a non-trivial fixed point in the IR. Some of them flow to theories with decoupled sectors with free hypermultiplets or free vector multiplets. On the contrary, we focus on the class of theories which have no such decoupled sectors in the IR. In [65] it has been conjectured that such theories have a non-trivial IR fixed point if and only if each unitary gauge group factor $U(N_i)$ of $\mathbb{G}$ has at least $2N_i$ fundamental hypermultiplets:

$$N_{f,i} \geq 2N_i \tag{1.2.2}$$

where $N_{f,i}$ stands for the total number of flavors/hypermultiplets of the $U(N_i)$ node, which for a quiver diagram is translated to:

$$N_{i-1} + N_{f,i} + N_{i+1} \geq 2N_i \tag{1.2.3}$$

When this condition is met for each node of a linear quiver, the gauge symmetry group of the theory can be fully Higgsed a fact that corresponds, as introduced above, to the existence of a non-trivial IR fixed point. Theories which exhibit this trait are characterized as 'good'. Theories for which the inequality is satisfied, are called 'balanced' and more will be presented about them in the discussion of the moduli space of the theories in question, in following chapters. On the contrary, when this condition is not met, the corresponding theory is called 'bad' and does not have a non-trivial fixed point in the IR. A last type of theories are the ones for which $N_f = 2N_i - 1$, that are called 'ugly': these flow on the IR to a free twisted hypermultiplet. The present work though, mostly regards 'good' theories ; the rest is mentioned for consistency.

This IR-flow condition has an analogue expression in terms of the two integer partitions $\rho, \hat{\rho}$ of N which completely determine the quiver data, as it will be immediately presented in what follows. In [65] the condition on the partitions is translated to be:

$$\rho^T > \hat{\rho} \tag{1.2.4}$$

and the equivalent of this condition is that the rank of each of the unitary gauge groups in the quiver, is positive: $\{N_i\} \geq 0$.In what follows, we will see how the above partitions are defined and how the this inequality is obtained by the brane configuration from which we can read-off a linear quiver theory.

1.2.2 Brane Configurations

The theories we just introduced are realized as low-energy limits of brane configurations involving $D3 - D5$ and $NS5$ branes. It will become clear how the 'good' theory conditions can be obtained from such brane pictures. In order to retain $\mathcal{N} = 4$ supersymmetry, the branes extend in the various spacetime directions as dictated in the table below:

	0	1	2	3	4	5	6	7	8	9
D3	$\odot$	$\odot$	$\odot$	$\odot$						
D5	$\odot$	$\odot$	$\odot$		$\odot$	$\odot$	$\odot$			
NS5	$\odot$	$\odot$	$\odot$					$\odot$	$\odot$	$\odot$

Let us explain gradually and in detail, how a simple quiver is obtained from such a configuration. Consider a $U(N)$ theory with M-fundamental hypermultiplets, as in Figure 1.1. The $U(N)$ theory is the one living in the worldvolume of N-coincident $D3$ branes, which have an infinite extent along the (012)-directions while being suspended between two $NS5$'s along the (3)-direction.

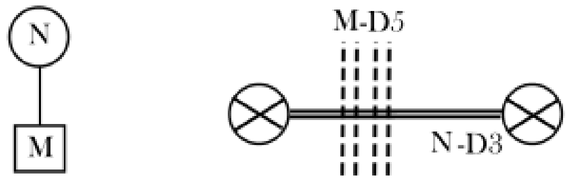


Figure 1.1: A U(N) theory with M-fundamental hypermultiplets. The U(N) theory lives in worldvolume of the D3 branes and the hypermultiplets are provided by the strings stretched between the D3 and the M-D5 branes.

In the low energy limit, this finite separation becomes irrelevant in comparison to the large wavelengths we consider and hence, effectively the theory becomes three-dimensional. The M-hypermultiplets in the fundamental of $U(N)$ are obtained by the strings stretched between the $D3$ and the $M - D5$ branes intersecting them (3-5 strings). This is generalized to longer quivers, by considering consequent NS5-D3,5-NS5 ”blocks” as shown below, in Figure 1.2.

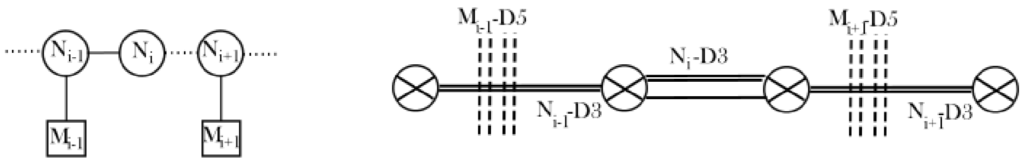


Figure 1.2: A general linear quiver and the corresponding brane configuration

In this case, the above fields are supplemented by hypermultiplets transforming in the bifundamental hypermultiplet of $U(N_i) \times U(N_{i+1})$, which correspond to strings stretched from the $N_i - D3$ brane stack to the neighboring $N_{i+1} - D3$ brane stack. Since in the low energy limit the (3)-direction becomes irrelevant, we can consider horizontal moves of the $D5$ branes along this direction, the result any of which, describes the same quiver theory. However, these brane moves involve some non-trivial string dynamics: each time a $D5$ brane crosses an $NS5$, a new $D3$ brane is created between them. This is known as a Hanany-Witten transition [73]. Such brane configurations can be encoded in a way invariant under these moves, in terms of $D5$ and $NS5$-brane linking numbers. These quantities are defined in the following way:

$$l_i = -n_{i,\#} + R_i^{NS5} \tag{1.2.5}$$
$$\hat{l}_i = \hat{n}_{j,\#} + L_j^{D5} \tag{1.2.6}$$

, where with $n_{i,\#}$ and $\hat{n}_{j,\#}$ we denote the net number of $D3$'s ending on the i^{th} $D5$ and on the j^{th} $NS5$ brane accordingly, where R_i^{NS5} and L_j^{D5} denote the number of $NS5$ branes on the right of the i^{th} $D5$ and the number of $D5$ branes on the left of j^{th} $NS5$. There are two details which should be taken into account regarding this labeling. First, by convention the $D5$ branes are labeled from left to right whereas the $NS5$ ones are labeled conversely. Moreover, the linking numbers are non-increasing, with all the $D5$ ones having the extra trait that are always non-negative, while automatically $\hat{l}_{\hat{k}} = N_{\hat{k}-1}$:

$$l_i \geq l_{i+1} \quad ; \quad \hat{l}_j \geq \hat{l}_{j+1} \tag{1.2.7}$$

Now, let's see what information does the inequality satisfied by the $NS5$ linking numbers gives us. If we consider the linear quiver of the picture above the inequality for the linking numbers of the j^{th} and $(j+1)^{\text{th}}$ $NS5$ branes results to the conjectured relation presented above for the existence of a non-trivial IR fixed point. Note that in terms of the brane picture, the bifundamental hypermultiplets of $U(N_i) \times U(N_{i+1})$ and $U(N_{i+1}) \times U(N_{i+2})$, are $N_{i+2} + N_i$ hypermultiplets transforming in the fundamental of $U(N_{i+1})$.

Moving gradually all the $D5$ branes of the above configuration on the right of all $NS5$ ones, by taking simultaneously into account the $D3$ brane creation at each crossing with a D5, results to a final configuration with a total number of $N - D3$ branes, being suspended between $NS5$ on the left and $D5$s on the right. Then:

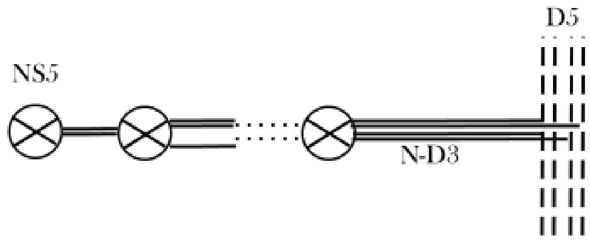


Figure 1.3: The resulting brane picture after moving all the D5 branes on the right of the NS5 branes. The linking numbers of the fivebranes comprise the partitions ρ and $\hat{\rho}$ of N, where N is the total number of D3 branes suspended between the fivebranes.

$$\sum_{i=1}^{k} l_i = \sum_{j=1}^{\hat{k}} \hat{l}_j = N \tag{1.2.8}$$

and hence the five-brane linking numbers define the two ordered integer partitions of N, $(\rho, \hat{\rho})$, introduced in the previous section, which correspond to two Young diagrams. The number of fundamental hypermultiplets is then given, according to their definition, by the multiplicity of each integer in ρ and $\hat{\rho}$:

$$\begin{aligned}\rho &= l_1 + l_2 + ... + l_{\hat{k}+1} \\ &= \underbrace{1 + .. + 1}_{M_1} + \underbrace{2 + .. + 2}_{M_2} + ...\end{aligned}$$

$$\begin{aligned}\hat{\rho} &= \hat{l}_1 + \hat{l}_2 + ... + \hat{l}_{k+1} \\ &= \underbrace{1 + .. + 1}_{\hat{M}_1} + \underbrace{2 + .. + 2}_{\hat{M}_2} + ...\end{aligned}$$

,where $\hat{M}j$ are the number of fundamental hypermultiplets of the magnetic (mirror dual) quiver, which will be introduced in one of the following sections.
These are exactly the brane configurations that realize the $T^{\rho}_{\hat{\rho}}[SU(N)]$ theories.

Before we turn on two interesting examples, now that we have described the notion of the linking numbers, we return to the general linear quiver of the above figure and its brane picture, in the form where the fivebranes are factorized, as shown in Figure 1.3: The brane configuration corresponding to any (good) linear quiver can be brought to the form where D3 branes end up on a number of NS5 branes on the left and on a number of D5 branes on the right. Excatly from this picturewe read directly the labels $(\rho, \hat{\rho}, N)$ of the theory.

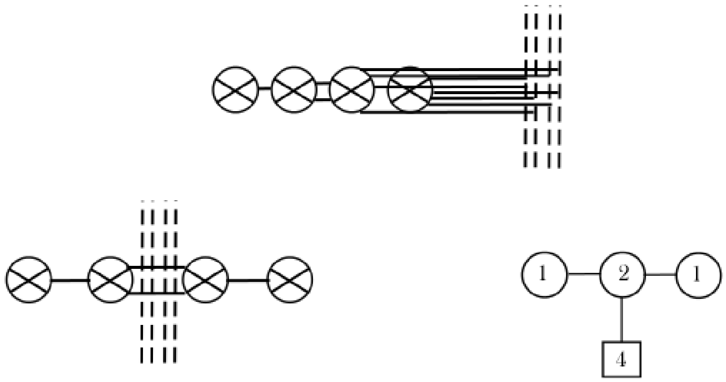


Figure 1.4: Quiver diagram and brane configurations of $T^{(3,3,1,1)}_{(2,2,2,2)}[SU(8)]$

We study the move of each D5 brane, towards the left, starting with the innermost one and by taking into account the vanishing of a D3 brane each time a D5 crosses an NS5. Supersymmetry requires that no more than one D3 branes can suspend between a D5 and an NS5: this is the so called s-rule. As shown in detail in [12], respecting the s-rule results to a number of inequalities for the linking numbers which are actually the 'good' IR flow conditions in terms of the two partitions of N/Young tableaux, in a weaker form than the one given in (1.2.4).

$$\hat{\rho}^T \geq \rho \tag{1.2.9}$$

We close this section with a simple example of a linear quiver. Consider $T^{(3,3,1,1)}_{(2,2,2,2)}[SU(8)]$, which is a good theory. From the partitions, we have that the brane picture includes four $D5$s with linking numbers (2,2,2,2) and four $NS5$s, with linking numbers (3,3,1,1). The brane configuration is given in Figure 1.4. Moving the $D5$s until the net number of $D3$s ending on them is zero, we obtain the quiver diagram, displayed at the lower part of Figure 1.4. Note that from the brane picture found at the upper part of Figure 1.4 one reads directly $(\rho, \hat{\rho}, N)$, whereas from the one at the lower-left part of the same figure, the quiver diagram is directly extracted.

1.3 Moduli Space and Symmetries

1.3.1 Moduli space of vacua: Higgs, Coulomb and Mixed branches

Three-dimensional $\mathcal{N}=4$ gauge theories have a moduli space which is parametrized by the vacuum expectation values (vevs) of the vector multiplet scalars $\langle\sigma,\varphi\rangle$ and hypermultiplet scalars $\langle\phi,\tilde{\phi}\rangle$. There are three distinct cases of parametrization. The first is the one where the vevs of the vector multiplet scalars vanish while the ones of the hypermultiplet scalars are non-zero, the latter parametrize the so called *Higgs* ($\mathcal{C}_{\mathbf{H}}$) branch of the moduli space. In the opposite case, when the vevs of the hypermultiplet scalars vanish and the ones of the vectormultiplet scalars are non zero, then the latter parametrize the *Coulomb* ($\mathcal{C}_{\mathbf{C}}$) branch. Finally, apart from purely Higgs or Coulomb branches, there also exist the ones that are parametrized by vevs from both sets of scalar fields and those are called *Mixed* branches. A general description of the the full moduli space of a three-dimensional $\mathcal{N}=4$ theory with gauge group $U(N)$ is as a union of these mixed branches [106]:

$$\mathcal{M}=\bigcup_r \mathcal{C}_{\mathbf{H},r}\times\mathcal{C}_{\mathbf{C},r} \tag{1.3.1}$$

We start by the description of the Higgs branch of a theory with N_f flavors. The bosonic fields of the hypermultiplets are pairs of complex scalars $\mathcal{Q}_\alpha=(\phi_\alpha,\tilde{\phi}^{\alpha\dagger})^T$ with $\alpha=1,2,...,N_f$, which transform as $SU(2)_R$ doublets. The vevs of these scalars parametrize the Higgs branch and they satisfy a triplet of D-term equations $\mathrm{Tr}[\mathcal{Q}\mathcal{Q}^\dagger\sigma_i]=0$ modulo gauge transformations, which defines the Higgs branch as a hyper-Kähler quotient. The gauge group is completely broken at the Higgs branch and the theory in the IR is that of free hypermultiplets. On the contrary, when complete Higgsing is not possible, the low-energy theory contains also free vector multiplets. Finally the Higgs branch is classically exact, it does not receive any quantum corrections and hence can be studies from the classical Lagrangian of the theory. This is implied from the fact that once the gauge coupling is promoted to a superfield the scalars of the superfield are $SU(2)_R$ singlets, transform only under the $SU(2)_H$ and so they appear only in the Coulomb branch [78].

The Coulomb branch is a hyper-Kähler manifold, parametrized by the vevs of the vector multiplet scalars. These are diagonal matrices breaking the gauge group to its maximal torus, $U(1)^r$, with $r=\mathrm{rank}\mathbb{G}$. The matter fields are massive and the IR theory is the one of free (abelian) vector multiplets. On the contrary with the Higgs branch, the Coulomb branch receives quantum corrections that modify the geometry of its classical descritpion. The quantum corrections are induced by *monopole* operators. These operators that belong to the general category of defect operators, are extra chiral (with respect to an $\mathcal{N}=2$ subalgebra of the $\mathcal{N}=4$) operators subject to quantum relations not derived from a superpotential. They are defined by the prescription of singular boundary conditions in the path integral. Specifically the insertion of a monopole operator $\mathcal{V}_{\mathbf{m}}(x)$ at point x, corresponds to performing the path integral over the gauge field configurations with a Dirac monopole singularity at x: $A_\pm\sim\frac{m}{2}(\pm1-\cos\theta)d\phi$, where we use the spherical coordinates (r,θ,ϕ) and $A_\pm$ denotes the gauge connection on the two patches of a two-sphere enclosing the monopole insertion point x. The monopole singularity is given by an embedding $U(1)\hookrightarrow\mathbb{G}$, determined by the monopole charge, taking values in the weight lattice GNO dual group modulo Weyl reflections, $m\in\Gamma_{\mathbb{G}^V}/\mathcal{W}$, alongwith the Dirac quantization condition $e^{2\pi i m}=1_{\mathbb{G}}$.

In order for the monopole operator to preserve $\mathcal{N}=2$ supersymmetry, the boundary condtition is imposed for the $\mathcal{N}=2$ vectormultiplet scalar: $\sigma\sim m/2r,(r\to0)$. In this way the inserted operator is a $\frac{1}{2}$-BPS monopole operator sitting in a chiral multiplet of the theory, and therefore they are counted along with the other chiral multiplets of the theory.

Having already introduced the Higgs and Coulomb branches we now focus on the mixed branches. These subregions of the moduli space are generated when scalars from both the vector multiplets and hypermultiplets acquire non-vanishing vacuum expectation values. The description of the mixed branches can be given in terms of type-IIB brane configurations [34] , as the ones introduced in previous section.

One important detail regarding the brane configurations in this discussion, is to specify the boundary conditions of the bosonic fields living on a D3 brane which ends on either kind of fivebrane. A three-dimensional scalar can obey either Dirichlet boundary conditions and vanish at the boundary, or Dirichlet, where its value at the boundary is free. For the three dimensional vector, we have to take into account the dimensional reduction from four dimensions: he vector in four dimensions reduced to a three-dimensional vector , which is set to zero by Neuman boundary conditions and a scalar, vanishing for Dirichlet boundary conditions. On the D3 branes the effective three-dimensional theory includes a scalar plus the fluctuations along three transverse directions as the bosonic massless modes of the $\mathcal{N} = 4$ hypermultiplet and finally the three-dimensional vector and the fluctuations along the rest of the transverse directions as the bosonic part of the $\mathcal{N} = 4$ vectormultiplet. Therefore, the boundary conditions imposed when a D3 brane ends on a D5 brane, set to zero the bosonic part of the vector multiplet and the only surviving modes are the hypermultiplet scalars. On the contrary, the vectormultiplet scalars are the only ones surviving when a D3 brane is suspended between two NS5 branes.

This information can be combined with the discussion of the previous section regarding Higgs and Coulomb branches. D3 branes suspended between two NS5 branes and moving freely along the NS5 directions, correspond different values of the vectormultiplet scalars and namely to Coulomb branch moduli. On the other hand, motions of D3 branes along the D5 directions (while being suspended between them) correspond to different values for the hypermultiplet scalars and hence to Higgs branch moduli. Finally, and D3 brane suspended between a D5 and an NS5 , has no moduli as half of them are set to zero by Dirichlet and the other half of them by Neuman boundary conditions. In a given brane construction, by counting the number of mobile D3 branes per case one can determine the dimension of a Coulomb or Higgs branch of the theory.

The mixed branches are determined by a partition

$$\lambda = (\lambda_1, ..., \lambda_k) \quad ; \quad \sum_{i=1}^{k} \lambda_i = N$$

the entries of which determine the number of D3 branes ending on each D5 of the brane configuration describing our theory. Moreover,the "good" theory condition holds: $\hat{\rho}^T \geq \lambda \geq \rho$. The case $\hat{\rho}^T = \lambda$ is the full Higgs branch of our theory, whereas $\lambda = \rho$ corresponds to the full Coulomb branch. The intermediate λ-partitions correspond to the various mixed branches of the theory. In this sense, the full moduli space of vacua of a 3d $\mathcal{N} = 4$ theory is:

$$\mathcal{M} = \bigcup_{\hat{\rho}^T \geq \lambda \geq \rho} \mathcal{C}_\lambda \times \mathcal{H}_\lambda$$
$$\mathcal{M} = \bigcup_{\hat{\rho}^T \geq \lambda \geq \rho} \mathcal{C}_\lambda \times \mathcal{C}_{\lambda^T} \quad ; \quad \mathcal{M} = \bigcup_{\hat{\rho}^T \geq \lambda \geq \rho} \mathcal{H}_{\lambda^T} \times \mathcal{H}_\lambda$$

where the second line just uses the fact that the two branches are exchanged by mirror symmetry, which characterizes superconformal theories in three dimensions and will be explained in detail in the following section. Therefore, the Higgs branch of a theory the mixed branch of

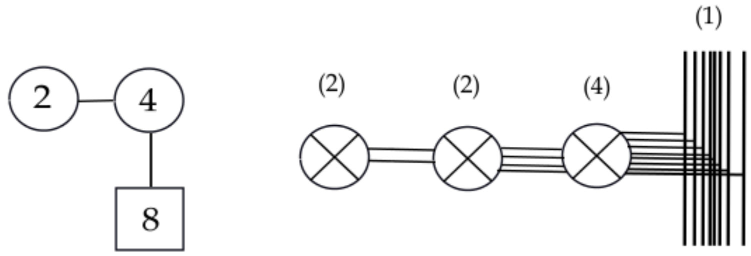


Figure 1.5: The quiver diagram and the brane configuration of $T^{(1,\ldots,1)}_{(4,2,2)}[SU(8)]$. The linking numbers of the fivebranes are specified in the parentheses above each brane.

which is determined by the partition λ is mapped to the Coulomb branch of the theory the mixed branch of which is determined by the partition dual to λ (which is its transpose).

We can proceed to an example and answer two questions:

- For a particular λ, which is the theory whose full Coulomb branch coincides with the Coulomb branch part of the above mixed branch, $\mathcal{C}_\lambda$? Accordingly for the Higgs branch.
- Can we determine the dimensions of the Coulomb and Higgs branches?

Example: Mixed branches of $T^{(1,\ldots,1)}_{(4,2,2)}[SU(8)]$

The quiver diagram and the brane configuration of this theory are found in Figure 1.5. As indicated by the linking numbers above the fivebranes, this theory is labelled by two partitions $\rho = (1,..,1)$ and $\hat{\rho} = (4,4,2)$, whereas $\rho^T = (8)$ and $\hat{\rho}^T = (3,3,1,1)$. The various branches can be given by integer partitions of 8 obeying $\hat{\rho}^T \geq \lambda \geq \rho$. As noted above, the marginal cases correspond to full Coulomb and full Higgs branches, so here we can pick an intermediate partition to study the case of a mixed branch.

$$\lambda = (2,2,1,1,1,1) \quad \lambda^T = (6,2)$$

In this case, two D3 branes are frozen on the innermost D5, another two on the next one and so on. The mixed branch brane configuration is the following:

The Coulomb branch part of the mixed branch is generated by the mobile D3 branes (red), stretched between the NS5 branes and the Higgs branch part of the mixed branch is generated by the mobile D3 branes (blue), between the D5s. The D3 branes suspended between the two different kinds of fivebranes have no moduli, as explained in the previous paragraph.

By giving infinitely large vevs to the Higgs branch moduli, namely by moving the blue D3 branes to infinity, we can read off the quiver theory the full Coulomb brnch of which coincides exactly with the Coulomb branch part of the above mixed branch. We get the quiver theory below which is labeled by the partitions λ and $\hat{\rho}$.

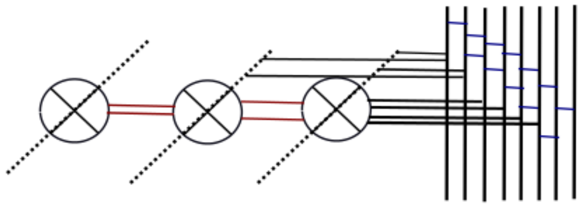
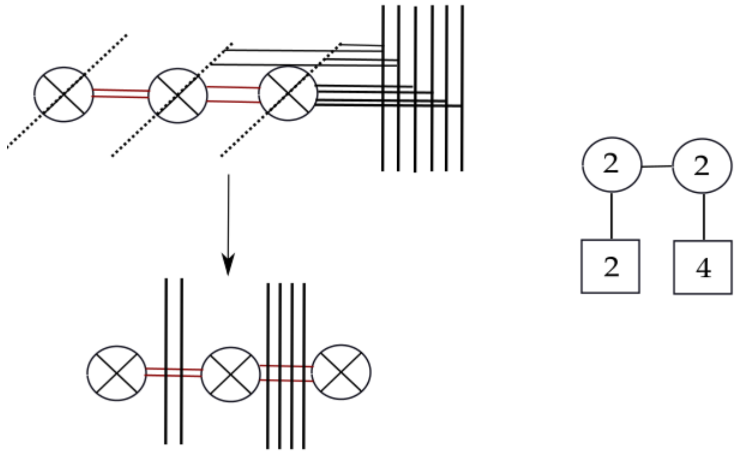


Figure 1.6: The brane configuration corresponding to the mixed branch $\lambda = (2,2,1,1,1,1) \quad \lambda^T = (6,2)$ of $T^{(1,\ldots,1)}_{(4,2,2)}[SU(8)]$



We see that the full Coulomb branch of this theory is exactly the Coulomb branch part of the Mixed branch labeled by λ and therefore $\mathcal{C}_\lambda \equiv \mathcal{C}[T^{\lambda}_{\hat{\rho}}]$.

Acting accordingly for the Higgs branch, we see that it coincides with the full Higgs branch of the theory with gauge group $U(2)$ and eight fundamental hypermultiplets, labeled by the partitions ρ and λ^T. Therefore $\mathcal{H}_\lambda \equiv \mathcal{H}[T^{\rho}_{\lambda^T}]$ and finally:

$$\mathcal{M} = \bigcup_{\hat{\rho}^T \geq \lambda \geq \rho} \mathcal{C}_\lambda \times \mathcal{H}_\lambda = \bigcup_{\hat{\rho}^T \geq \lambda \geq \rho} \mathcal{C}[T^{\lambda}_{\hat{\rho}}] \times \mathcal{H}[T^{\rho}_{\lambda^T}] = \bigcup_{\hat{\rho}^T \geq \lambda \geq \rho} \mathcal{C}[T^{\lambda}_{\hat{\rho}}] \times \mathcal{C}[T^{\lambda^T}_{\rho}] =$$

Given that we know the partitions which label the theories the full Coulomb and Higgs branches of which coincide with the corresponding factors of the Mixed branch of the initial theory, we can directly compute their dimensions.

- $\mathbf{d}_{\mathcal{C}[T^{\lambda}_{\hat{\rho}}]} = \frac{1}{2}\sum_i (\hat{\rho}_i^{T2} - \lambda_i^2)$
- $\mathbf{d}_{\mathcal{H}[T^{\rho}_{\lambda^T}]} = \mathbf{d}_{\mathcal{C}[T^{\lambda^T}_{\rho}]} = \frac{1}{2}\sum_i (\rho_i^{T2} - \lambda_i^{T2})$

For the above cases, we obtain the correct results, namely $(\mathbf{d}_{\mathcal{C}[T^{\lambda}_{\hat{\rho}}]}, \mathbf{d}_{\mathcal{C}[T^{\lambda^T}_{\rho}]}) = (4, 12)$ and these are exactly the number of mobile D3 branes per case.

1.3.2 Mirror and Global Symmetry

Three-dimensional $\mathcal{N} = 4$ theories are characterized by *Mirror* symmetry [78] [45]. This is a particular symmetry which exchanges the Higgs with the Coulomb branch of two theories, exchanges the Fayet-Iliopoulos and mass parameters and finally the $SU(2)_R$ and $SU(2)_{R'}$ of the $SO(4)$ R-symmetry. Regarding the above brief discussion regarding the moduli space of vacua, we see that the Coulomb branch of a theory, which is in pronciple difficult to study due to the fact that receives quantum corrections, can be studied from the classical Higgs branch of the mirror theory.

Regarding the linear quivers studied in this work, mirror symmetry exchanges the integer partitions ρ and $\hat{\rho}$. Of course this implies that we have another IR flow condition which should be satisfied in order for the mirror symmetric quiver to have a non trivial fixed point in the IR:

$$T^{\rho}_{\hat{\rho}}[SU(N)] \leftrightarrow T^{\hat{\rho}}_{\rho}[SU(N)], \quad \rho^T > \hat{\rho} \leftrightarrow \hat{\rho}^T > \rho \tag{1.3.2}$$

Specifically, both theories of the mirror pair flow in the IR to the same fixed point. Apart from mirror symmetry, these theories are characterized by additional global symmetries, which are determined by ρ and $\hat{\rho}$. From the previous discussion on the general linear quivers, these global symmetries correspond to the ones rotating the fundamental hypermultiplets of the theory: $H_\rho = \prod_i U(M_i)$ and $H_{\hat{\rho}} = \prod_i U(\hat{M}_i)$ for the mirror theory. Therefore at the fixed point, the total global symmetry is expected to be:

$$H_\rho \times H_{\hat{\rho}} \tag{1.3.3}$$

Mirror symmetry is realized as S-duality in type IIB string theory. We can go back to the brane description of the quiver theories and apply S-duality. Such action exchanges D5 with NS5 branes while leaving the D3 branes intact. Since the the 5 brane data (linking numbers) are encolded in the Young diagrams ρ and $\hat{\rho}$, it is obvious that S-duality acts actually by exchanging them. In this way, by starting from a given quiver (e.g *electric/magnetic*), we can obtain its mirror dual (e.g *magnetic/electric*).

At this point all of the above can be summarized in an instructive example. The first step to start from a given (electric) linear quiver. By writing down its brane picture and factorizing the 5 branes as explained througout the text, we can retrieve its full data, in terms of the two integer partitions.

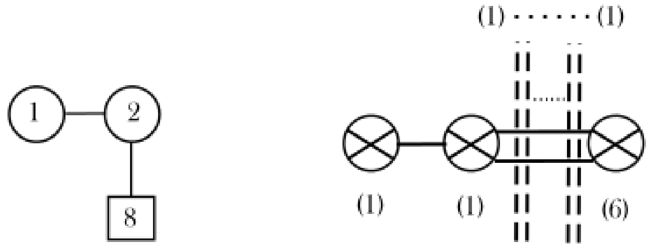
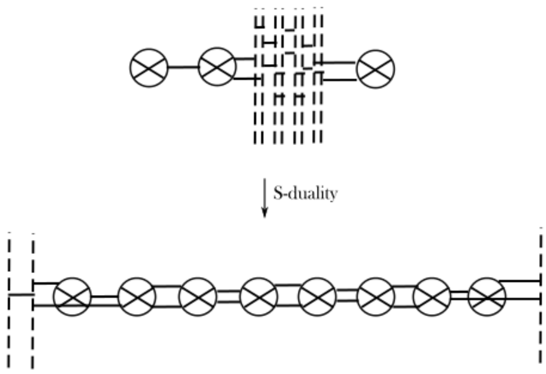


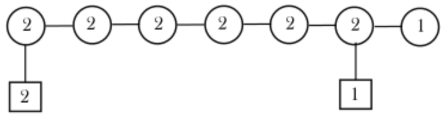
Figure 1.7: Quiver diagram of $T^{(6,1,1)}_{(1,..1)}[SU(8)]$ along with the brane construciton with the fivebrane-linking numbers indicated

This is a theory with eight fundamental hypermultiplets provided by the eight D5 branes intersecting the second set of D3s. The linking numbers are computed using the definition given in the previous section and are given in parentheses for each type of five-brane (under each NS5 and above each one of the D5s). Below the data that identify this quiver theory are given; it can be verified that the good- theory conditions are automatically satisfied:

$$(\rho, \hat{\rho}, N): \quad \left\{ \ydiagram{1,1,1,1,1,1,1,1}, \ydiagram{6,1,1}, 8 \right\} \tag{1.3.4}$$

After having identified the electric theory, the next step is to proceed to obtaining the magnetic theory. The first step is to break all the D3 branes between the D5s and then apply S-duality, which, as stated above, exchanges D5 with NS5 branes while keeping D3 branes invariant. Finally, by moving the (converted) D5s towards the interior of the brane configuration until there are no D3 branes attached on them, gives the picture from which the magnetic quiver can be obtained.





This is the magnetic theory, which is of course good and flows to the same fixed point in the IR as the electric theory does. From the brane configuration one can verify that the integer partitions that label the magnetic theory are the exchanged partitions of the electric theory. In Part III of this work, the discussion on the moduli space is continued, with focus on mixed branches of dual pairs of such linear quiver theories.

Chapter 2

Holographic Duals: IIB Supergravity on $AdS_4 \times S^2 \times \hat{S}^2 \ltimes \Sigma_{(2)}$

2.1 The supergravity solutions

The second part of the correspondence regards a particular class of solutions of type IIB supergravity whose global structure was developed in [12]. The local solutions preserving $\mathfrak{osp}(4|4)$ superconformal symmetry were constructed as holographic duals to BPS domain walls in four-dimensional $\mathcal{N}=4$ super Yang-Mills theory [52][54] [1]. The solutions were constructed to have $OSp(4|4)$ symmetry, by imposing that the type IIB Killing spinor equations are solved by Killing spinors generating this symmetry. The global solutions which be discussed in the main part of the chapter, where constructed in [12], and an exact holographic dictionary was developed. We start with presenting the local solutions. The first step of the construction is that the bosonic part of the symmetry group is encoded in the following metric ansatz:

$$ds^2_{10d} = L_4^2 ds^2_{\mathrm{AdS}_4} + f^2 ds^2_{S^2} + \hat{f}^2 ds^2_{\hat{S}^2} + 4\rho^2 dz d\bar{z} \tag{2.1.1}$$

which describes an AdS_4 fibration over the six-dimensional manifold $S^2 \times \hat{S}^2 \ltimes \Sigma_{(2)}$, where Σ_2 stands for a two-dimensional Riemann surface with the topology of a disk. In this analysis we focus on solutions with $\Sigma_{(2)}$ being the infinite strip:

$$\Sigma_{(2)} = \{z = x + iy \in \mathbb{C} | 0 \leq \mathrm{Imz} \leq \frac{\pi}{2}\} \tag{2.1.2}$$

In this way, indeed, the $SO(2,3) \times SO(3) \times SO(3)$ bosonic part of $OSp(4|4)$ is given as the AdS_4 isometries times the isometry groups of the two-spheres. The scale factors $L_4, f, \hat{f}, \rho$, are real functions of the local complex coordinates $(z, \bar{z})$, which parametrize the strip and are expressed in terms of two real harmonic functions: $h(z, \bar{z})$ and $\hat{h}(z, \bar{z})$. Along with these, their dual functions are defined, which will be necessary in determining the gauge potentials of the setup, as we will see in what follows.

$$h(z,\bar z) = -i(\mathcal{A} - \bar{\mathcal{A}}), \qquad \hat h(z,\bar z) = \hat{\mathcal{A}} + \bar{\hat{\mathcal{A}}}$$
$$h^D(z,\bar z) = \mathcal{A} + \bar{\mathcal{A}}, \qquad \hat h^D(z,\bar z) = i(\hat{\mathcal{A}} - \bar{\hat{\mathcal{A}}})$$

,where $\mathcal{A}, \hat{\mathcal{A}}$ are holomorphic and analytic in the interior of the Riemann surface. Below it will be seen that they may have singularities on the two boundaries of the strip, the nature of which will be explained in detail. The explicit expressions for the metric factors are:

$$L_4^8 = 16\frac{\mathcal{U}\hat{\mathcal{U}}}{W^2}, \qquad \rho^8 = \frac{\mathcal{U}\hat{\mathcal{U}}W^2}{h^4\hat h^4} \tag{2.1.3}$$
$$f^8 = 16h^8\frac{\hat{\mathcal{U}}W^2}{\mathcal{U}^3}, \qquad \hat f^8 = 16\hat h^8\frac{\mathcal{U}W^2}{\hat{\mathcal{U}}^3} \tag{2.1.4}$$

where $\mathcal{U}$,$\hat{\mathcal{U}}$ and W, are auxiliary functions defined as:

$$\mathcal{U} = 2h\hat h|\partial_z h|^2 - h^2 W, \quad \hat{\mathcal{U}} = 2h\hat h|\partial_z \hat h|^2 - \hat h^2 W, \quad \text{where} \quad W = \partial_z\partial_{\bar z}(h\hat h) \tag{2.1.5}$$

The solutions include non-vanishing R-R and NS-NS three-form field strengths:

$$F_{(3)} = \hat\omega_{(2)} \wedge db_2, \quad H_{(3)} = \omega_{(2)} \wedge db_1 \tag{2.1.6}$$

where $\hat\omega_{(2)}$, $\omega_{(2)}$ stand for the volume forms of $\hat S^2, S^2$ and the $b_{1,2}$ are one forms which depend on the harmonic functions. The remaining components of the supergravity solution are the R-R 5-form and the dilaton:

$$F_5 = -4L_4^4\omega_{(4)} \wedge \mathcal{F} + 4f^2\hat f^2\omega_{(2)} \wedge \hat\omega_{(2)} \wedge (\star_{(2)}\mathcal{F}), \qquad e^{4\phi} = \frac{\hat{\mathcal{U}}}{\mathcal{U}} \tag{2.1.7}$$

,with $\omega_{(4)}$ being the AdS_4-volume form and $\mathcal{F}$ being a 1-form on the infinite strip with the property that the quantity $L_4^4\mathcal{F}$ is closed. It is straightforward to prove the following identities that hold between the metric factors and the dilaton: $L_4^2 f^8 = 4e^{2\phi}h^2$, $L_4^2\hat f^8 = 4e^{-2\phi}\hat h^2$ and $\rho^6 f^2\hat f^2 = 4W^2$. These will be important for the analysis in Part II, where metric fluctuations around these solutions will be studied. The details of most of the above functions are given throughout [12][52][54] but are not of practical use and in terms of simplicity, are not mentioned in this text.

The regularity of the solutions is ensured by the fact that the harmonic functions $h, \hat h$ obey particular boundary conditions on the two strip-boundaries. The first harmonic function vanishes on the lower boundary along with the normal derivative of the second harmonic function while the converse holds for the upper boundary. In this sense the $\Sigma_{(2)}$-boundary is realized in terms of divisions:

$$h|_{y=0} = \partial_\perp \hat{h}|_{y=0} = 0 \quad \text{and} \quad \hat{h}|_{y=\frac{\pi}{2}} = \partial_\perp h|_{y=\frac{\pi}{2}} = 0 \tag{2.1.8}$$

The above conditions imply that the f-factor vanishes in the lower strip-boundary whereas the $\hat{f}$-factor vanishes on the upper strip-boundary. This, in turn, corresponds to the fact that the radius of the S^2 vanishes on the lower strip boundary and accordingly the radius of the $\hat{S}^2$ vanishes on the upper boundary. Therefore, the points on the $\Sigma_{(2)}$-boundary are interior points of the full ten-dimensional geometry; the only boundary in this framework is the conformal boundary of AdS_4.

These conditions, along with the requirement that both the dilaton and the metric factors be regular in the interior of $\Sigma_{(2)}$ but divergent at isolated points of the boundaries, form a set of global consistency conditions that severely constrain the supergravity solutions and directly the form of the harmonic functions $(h, \hat{h})$. Regarding the aforementioned isolated points of the $\Sigma_{(2)}$ boundary, there are three kinds of admissible singularities for the harmonic functions, interpreted to be sourced by type IIB branes: D5, NS5 and D3 branes. The ones interpreted as fivebranes, are logarithmic-cut singularities and in what follows we will see that they also carry D3 brane charge. As regards the ones interpreted purely as D3 brane sources, they are square-root singularities[52].

We can proceed to the detailed description of the simplest solution which contains brane singularities. The form of the two harmonic functions is given below:

$$h = -i\alpha \sinh(z-\beta) - \underline{\gamma \log\Big[\tanh\big(\frac{i\pi}{4} - \frac{z-\delta}{2}\big)\Big]} + \text{c.c} \tag{2.1.9}$$

$$\hat{h} = \hat{\alpha}\cosh(z-\hat{\beta}) - \underline{\hat{\gamma}\log\Big[\tanh\big(\frac{z-\hat{\delta}}{2}\big)\Big]} + \text{c.c} \tag{2.1.10}$$

The two harmonic functions are labelled by two sets of four real parameters, $(\alpha, \beta, \gamma, \delta)$ and the corresponding hatted ones. The underlined part of the expressions corresponds to the fivebrane singularities: δ and $\hat{\delta}$ are the positions of the D5 and NS5 sources on the upper and lower strip-boundaries. Obviously this solution describes just one stack of D5 branes on the upper boundary at $z = \delta + \frac{i\pi}{2}$ and one of NS5s on the lower boundary, at $z = \hat{\delta}$. Except of these singular points, all the rest of the boundary belongs to the interior of the full ten-dimensional geometry. Regarding the rest of the parameters, we will see that $(\beta, \hat{b})$ determine the dilaton and in particular the dilaton variation between the two asymptotic region of the strip. This is explained in detail in the corresponding Appendix A, but also through the Part II of this work. The parameters $(\alpha, \hat{\alpha})$ and $(\gamma, \hat{\gamma})$ play an important role in the physics and the geometry of our setup and are going to be explained below. It is notable that they are assumed to be all non-negative [52].

Let us now examine the physics in the vicinity of the D5 branes. Recall that the two sphere $\hat{S}^2$ shrinks to zero radius on the upper boundary. Consider a segment I surrounding the singularity at $z = \hat{\delta} + \frac{i\pi}{2}$ and the fibration of the $\hat{S}^2$ over this segment: $I \times \hat{S}^2$ is a non-contractible three-cycle. It has topology of a three-sphere given the fact that on the upper boundary the second harmonic function and the second two-sphere form factor vanish ($\hat{h} = \hat{f} = 0$). It can be verified that this three cycle supports a non-vanishing RR three-form flux, which corresponds to the total number of the D5 branes at the particular point (or equivalently to the total D5 brane charge).

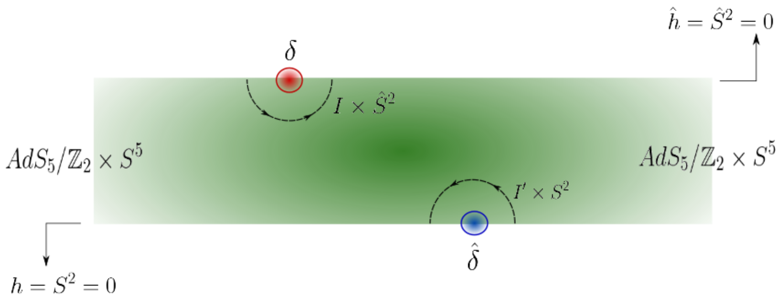


Figure 2.1: Solution with one D5 and one NS5 brane: The indicated three-cycles at the region of each point singularity, are non-contractible and support R-R and NS-NS three-form fluxes, as explained in the text. The boundary conditions for the harmonic functions are specified, along with the asymptotic $AdS_5/\mathbb{Z}_2 \times S^5$ regions.

Accordingly, focusing on the lower-boundary singularity, we have the non-contractible three-cycle $I' \times S^2$obtained as a fibration of S^2 two-sphere over a segment I', which surrounds the singularity at $z = \hat{\delta}$. This three-cycle (which also has the topology of a three-sphere as on the lower boundary $h = f = 0$) supports non-vanishing NS-NS three-form flux, which counts the total number of NS5 branes at that point (or equivalently the total NS5 brane charge).

$$N_{D5} = \frac{1}{4\pi^2\alpha'}\int_{I\times\hat{S}^2} F_{(3)} = \frac{4}{\alpha'}\gamma, \quad N_{NS5} = \frac{1}{4\pi^2\alpha'}\int_{I'\times S^2} H_{(3)} = \frac{4}{\alpha'}\hat{\gamma} \tag{2.1.11}$$

, from where the meaning of the real parameters $\gamma, \hat{\gamma}$ becomes evident: they count (in appropriate normalization) the number of fivebranes per stack. Moreover, it should be noted that the fivebrane charge is quantized in units of $2\kappa_0 T_5$, with $2\kappa_0^2 = (2\pi)^7\alpha'^4$ being the gravitational coupling, and with the fivebrane tension being $T_5 = \frac{1}{(2\pi)^5\alpha'^3}$, while since the dilaton is being kept arbitraty, one has the freedom to choose for the string coupling $g_s = 1$ which renders the tensions of the two fivebrane types equal.

What remains to be presented regarding the singularities is the D3 brane charge of these solutions. First of all, the aforementioned singularities carry D3 brane charge, apart from the fivebrane charge already described. Focusing on the upper-boundary supporting singularities, except from the F_3 flux supported by the three-cycle $I \times \hat{S}^2$, there is also a five-form flux threading the non-contractible five-cycle which is obtained as a fibration of the S^2 over the above three-cycle: $(I \times \hat{S}^2) \times S^2$. A significant detail is that this five-form is not just simply the R-R F_5, but rather the five-form $\tilde{F}_5 = F_5 - B_2 \wedge F_3$, with B_2 being the gauge potential of the NS-NS three-form. Accordingly, regarding the lower-boundary singularities, except from the H_3 flux supported by the three-cycle $I' \times S^2$, there is also a five-form flux threading the non-contractible five-cycle which is obtained as a fibration of the $\hat{S}^2$ over the above three-cycle: $(I' \times S^2) \times \hat{S}^2$. The five-form here is the gauge-variant combination $\tilde{F}'_5 = F_5 + C_2 \wedge H_3$, with C_2 being the gauge potential of the R-R three-form:

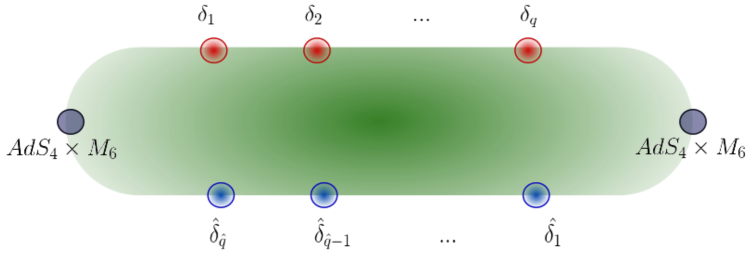


Figure 2.2: Compactification limit: As $\alpha, \hat{\alpha} \to \infty$, the asymptotic regions close and are replaced by regions homeomorphic to $AdS_4 \times M_6$. Here, solutions with multiple fivebrane singularities are depicted.

$$
\begin{aligned}
N_{D3}^{D5} &= \int_{(I\times \hat{S}^2)\times S^2} F_5 - B_2 \wedge F_3 \\
N_{D3}^{NS5} &= \int_{(I'\times S^2)\times \hat{S}^2} F_5 + C_2 \wedge H_3
\end{aligned}
$$

The above D3 charges are called *Page* charges. This subtlety in defining the D3 brane charge is explained in detail in [12]. Briefly, it turns out that in the presence of the two kinds of fivebranes, the RR five-form, which is gauge-invariant, corresponds to a non-conserved current generating a non-conserved D3-brane charge. However, a particular definition, of a gauge-variant this time, five form as a combination of the RR fiveform and the gauge potentials for the RR and NS NS three-forms, results to a local, conserved and quantized but gauge-variant D3 brane charge, which is called Page charge. in this setup, there are two ways to define such a five-form, respecting the criterion of which out of the two gauge potentials can be globally defined on each of the non-contractible five-cycles in introduced above. Apart from being supported on fivebrane singularities, D3 brane charge can also be found at the asymptotic regions of the strip.

In the figure above, there is a new piece of information regarding the geometry of the asymptotic regions. Substituting the form of the harmonic functions introduced above in the metric, results to asymptotic regions with geometry $AdS_5 \times S^5$. The details of the solution, are explained in the corresponding Appendix B . $AdS_5 \times S^5$ is the near horizon geometry of a number of D3 branes and therefore there is also charge associated with these regions. Nevertheless, this charge will not play any role, given that the focus is given to the case where these asymptotic $AdS_5 \times S^5$ regions are capped-off. The limit of interest in the presented solutions is the one where $\alpha, \hat{\alpha} \to \infty$, which is smooth. In this limit the asymptotic $AdS_5 \times S^5$ regions cap-off and the points at infinity become regular interior points of the full ten-dimensional geometry. This has been given in detail in [12] The harmonic functions that describe this setting are introduced below:

$$
h = -\sum_{i=1}^{q} \gamma_i \log\Big[\tanh\big(\frac{i\pi}{4} - \frac{z-\delta_i}{2}\big)\Big] + \text{c.c}, \quad \hat{h} = -\sum_{j=1}^{\hat{q}} \hat{\gamma}_j \log\Big[\tanh\big(\frac{z-\hat{\delta}_j}{2}\big)\Big] + \text{c.c} \tag{2.1.12}
$$

, where for the positions of the singularities have opposite ordering. This choice of harmonic functions corresponds to the compact geometry $AdS_4 \times M_6$ as it can be verified by substituting them into the metric ansatz. Regarding the fivebrane charges, we still have that the number of D5 (NS5) branes at each stack is proportional (or equal, depending on normalization) to the corresponding γ_i ($\hat{\gamma}_j$)-parameter of the solution: $N_{D5}^{(i)} \sim \gamma_i$ and $N_{NS5}^{(j)} \sim \hat{\gamma}_j$. Finally, the D3 brane charge is obtained by the expressions for the page charges defined above, for which the substitution of the two harmonic functions gives:

$$N_{D3}^{(i)} = N_{D5}^{(i)} \sum_{j=1}^{\hat{q}} N_{NS5}^{(j)} \frac{2}{\pi} \mathbf{arctan}(e^{\hat{\delta}_j - \delta_i}), \quad N_{D3}^{(j)} = N_{NS5}^{(j)} \sum_{i=1}^{q} N_{D5}^{(i)} \frac{2}{\pi} \mathbf{arctan}(e^{\hat{\delta}_j - \delta_i}) \tag{2.1.13}$$

These expressions show that the particular solutions have the description of near horizon geometries of a brane configuration involving D3 branes being suspended between stacks of NS5 branes and D5 branes. This brane picture is of paramount importance for establishing the duality between these solutions and the three dimensional theories presented in the previous chapter: Recall that all the data that label a linear quiver theory can be obtained by the exactly same type of brane configuration.

2.2 Holographic Dictionary

The conjectured duality presented in [12] is between the IR limit of three-dimensional $\mathcal{N} = 4$ theories and type IIB supergravity on $AdS_4 \times M_6$. A common description of these holographically dual theories, is in terms of the specific type-IIB brane configurations described above. Regarding the three dimensional theories, the brane picture is comprised by N-D3 branes, suspended between k-D5 branes and $\hat{k}$-NS5 branes. In the supergravity side, these quantities are related to the numbers of fivebranes per point singularity on the strip boundaries and to the corresponding charge of D3 branes emanating from or ending on each stack:

$$\begin{cases} k = \sum_{i=1}^{q} N_{D5}^{(i)} \\ \hat{k} = \sum_{j=1}^{\hat{q}} N_{NS5}^{(j)} \end{cases} \qquad N = \sum_{i=1}^{q} N_{D3}^{(i)} = - \sum_{j=1}^{(\hat{q})} N_{D3}^{(j)} \tag{2.2.1}$$

, with the relation on the right hand side implying charge conservation. The labels that remain are the integer partitions of N,namely ρ and $\hat{\rho}$. The linking numbers for the fivebranes, defined in the first chapter, give the net number of D3 branes attached on each one of them. In the supergravity side, the partitions are given in terms of the usual linking numbers, which are now defined as the ratios:

$$l^{(i)} = \frac{N_{D3}^{i}}{N_{D5}^{i}} = \sum_{j=1}^{\hat{q}} N_{NS5}^{(j)} \frac{2}{\pi} \mathbf{arctan}(e^{\hat{\delta}_j - \delta_i}), \qquad \rho = (\overbrace{l^{(1)}, ..., l^{(1)}}^{N_{D5}^{(1)}}, ..., \overbrace{l^{(p)}, ..., l^{(p)}}^{N_{D5}^{(p)}})$$

$$\hat{l}^{(j)} = -\frac{N_{D3}^{(j)}}{N_{NS5}^{(j)}} = \sum_{i=1}^{q} N_{D5}^{(i)} \frac{2}{\pi} \mathbf{arctan}(e^{\hat{\delta}_j - \delta_i}) \qquad \hat{\rho} = (\underbrace{\hat{l}^{(1)}, ...\hat{l}^{(1)}}_{N_{NS5}^{(1)}}, ..., \underbrace{\hat{l}^{(\hat{p})}, ..., \hat{l}^{(\hat{p})}}_{N_{NS5}^{(\hat{p})}})$$

, where here the linking number $l^{(i)}$ stands for the number of D3 branes ending on each D5 brane of the i^{th} brane stack and accordingly $\hat{l}^{(j)}$ gives the number of D3 branes emanating from each NS5 of the j^{th} brane stack. These quantities should be integer and this imposes that the parameters $\delta_i, \hat{\delta}_j$ are quantized. With this addition, the two integer partitions of N in the supergravity side, are written above. We see that the theories at both sides of the correspondence are practically labeled by the following : number of D5 branes per stack ($N_{D5}^{(i)}$), number of NS5 branes per stack ($N_{NS5}^{(j)}$) and the linking numbers $l^{(i)}, \hat{l}^{(j)}$. From the form of the partitions we see that the number of D5 branes per stack matches with the number of fundamental hypermultiplets per gauge node of the $T^{\rho}_{\hat{\rho}}[SU(N)]$ while the number of NS5 branes per stack matches with the number of fundamental hypermultiplets per gauge node of the mirror theory, $T^{\hat{\rho}}_{\rho}[SU(N)]$.

$$N_{D5}^{(i)} = M_i, \quad N_{NS5}^{(j)} = \hat{M}_j \tag{2.2.2}$$

This, along with the charge conservation condition given in the beginning of this section, gives an overall matching of parameters between the theories in the two sides of the correspondence.

Furthermore, the inequalities $\rho^T > \hat{\rho}$ that are necessary to be met in order for the three-dimensional theory to have a non-trivial IR fixed point, are also satisfied in the supergravity side by the defined linking numbers, so there is also s matching of constraints.

Finally, the duality map is completed by the matching of symmetries. The starting point is the three-dimensional theories: first of all, the superconformal symmetry $OSp(4|4)$ the bosonic symmetries of which, $SO(2,3) \times SO(3) \times SO(3)$ are realized as isometries of $AdS_4 \times S^2 \times \hat{S}^2$, as stated in the beginning of this chapter. Moreover, in the final section of the previous chapter, it was explained that in the IR these theories have a global symmetry given by the product of the groups that rotate the fundamental hypermultiplets of the dual pair of theories: $H_\rho \times H_{\hat{\rho}} = \prod_i U(M_i) \times \prod_j U(\hat{M}_j)$. In holographic duality, the global symmetries of the boundary theory, correspond to gauge symmetries of the bulk theory. The supergravity solutions presented in this chapter, are characterized by singularities interpreted as fivebranes in string theory. The gauge theory that lives in the worldvolume of a stack of $N_{D5}^{(i)}$ D5 branes is $U(N_{D5}^{(i)})$ and accordingly for a stack of $N_{NS5}^{(j)}$ branes, $U(N_{NS5}^{(j)})$. Therefore, the full gauge symmetry in the bulk reads: $\prod_i U(N_{D5}^{(i)}) \times \prod_j U(N_{NS5}^{(j)})$. Combining this with the parameter matching of the previous paragraph, we have that indeed the full global symmetry of the three-dimensional theory matches the gauge symmetry of the bulk supergravity:

$$H_\rho \times H_{\hat{\rho}} = \prod_i U(N_{D5}^{(i)}) \times \prod_j U(N_{NS5}^{(j)}) \tag{2.2.3}$$

With this final section we close the Part I, which was a presentation of the theoretical background on which the work introduced throughout the following chapters is based. Some important individual parts are left as Appendices (Elements of Representation theory for three-dimensional $\mathcal{N} = 2$ and $\mathcal{N} = 4$ theories, $AdS_5 \times S^5$ and its Janus deformation) as they are used in all the chapters that follow.

Part II

String Theory embeddings of Massive AdS_4 Gravity and Bimetric Models

Chapter 3

Massive AdS_4 gravity from String Theory

3.1 The holographic viewpoint

The holographic duality described in the first part of this work, is instrumental for the treatment of the problem of the AdS_4 graviton Higgsing, the details of which have been introduced in the corresponding part of the general introduction. Although the main analysis is carried out in the gravitational side of the holographic correspondence, starting the study of the problem from the point of view of the dual sCFT$_3$ is instructive regarding the construction of the massive AdS_4 supergravity solutions.

The question of the presence of a bulk massive graviton is related to the energy-momentum conservation in the dual boudary CFT. Holographically the AdS_4 graviton is dual to the stress tensor of the dual sCFT$_3$, the scaling dimension of which is related to the graviton mass [6]:

$$m^2 L_4^2 = \Delta(\Delta - 3) \tag{3.1.1}$$

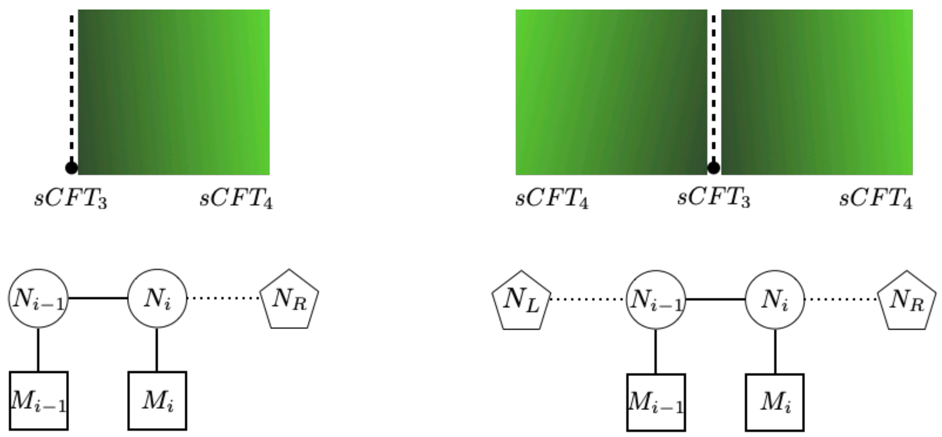
, with L_4 being the AdS_4 radius and Δ the scaling dimension of the stress-tensor. The representation of the three-dimensional conformal algebra $\mathfrak{so}(2,3)$ which contains $T_{\alpha\beta}$ and its conformal descendants, is actually short, due to the conservation of the stress-tensor which results to three null states (null descendants). The scaling dimension of the conserved stress-tensor does not receive quantum corrections and hence it is canonical: $\Delta = 3$. From the above holographic expression, it becomes obvious that a conserved stress tensor of the boundary three-dimensional CFT corresponds to a massless AdS_4 graviton. Therefore, a slightly massive graviton would correnspond to a non-conserved stress-tensor.

In this case, its scaling dimension would receive quantum corrections in the form of a small anomalous dimension [5], $\epsilon \ll 1$, which would then correspond to a small graviton mass:

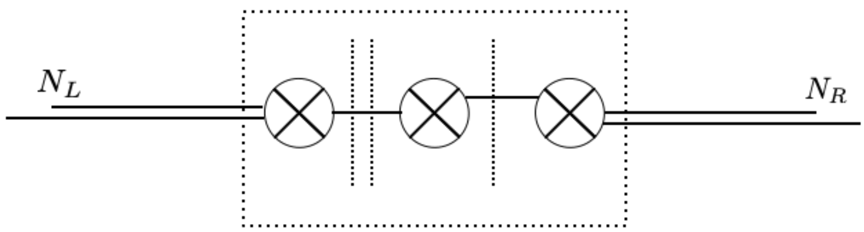
$$m^2 L_4^2 \sim \mathcal{O}(\epsilon) \tag{3.1.2}$$

The above scenario can be realised in a setup where the three-dimensional theory is a boundary of a four-dimensional one. The bulk theory is a $\mathcal{N} = 4$ super Yang-Mills $SU(N)$ theory.

The conservation of the stress-tensor fails, due to its non-zero component along the extra dimension. Indeed, in this case there is no shortening condition and the stress-tensor acquires an anomalous dimension, which corresponds to a mass for the dual graviton. A similar physical setting is the one where the initial three-dimensional theory is a defect of a four-dimensional one and accordingly we have two contributions of anomalous dimensions, from the left and the right of the defect. Both cases are depicted below, along with their generic quiver form:



Nevertheless, we are interested in a slightly massive graviton, and hence in a small anomalous dimension. This requirement suggests that the stress tensor should dissipate *weakly* in the extra dimension. Weak dissipation is ensured by the scarcity of the degrees of freedom of the four-dimensional theory in comparison to the three dimensional boundary one. A direct way to see this is from the corresponding brane configurations. The above linear quivers can be read off from the brane diagrams below. The conditions $\{(N_i, M_i)\} \gg N_R$ and $\{(N_i, M_i)\} \gg N_R, N_L$, where $N_{L,R}$ is the number of semi-infinite D3 branes attached on the leftmost and rightmost NS5 branes (in the world-volume of which the 4d theories live), are the ones which finally guarantee the weak dissipation of the stress tensor, a fact which as explained above corresponds to a small anomalous dimension and hence to a small graviton mass. In the final part of the analysis we will see a more concrete way to realise the weak dissipation of the stress tensor, which involves the expression of the anomalous dimension in terms as a ratio of free energies of the four-dimensional and three-dimensional theories.



From the brane picture and from what we already know regarding the dual supergravity solutions from Part I, we can comment on the geometry of the solutions with a massive graviton: The semi-infinite D3 branes corresonding to the four-dimensional theories, have an $AdS_5 \times S^5$ near horizon geometry, whereas the three-dimensional theory is dual to an AdS_4 space fibered over a six-dimensional compact manifold, M_6. Therefore we already got a hint for the gravity side from the dual CFT picture.

Apart from the above discussed mechanism, there is another physical setup which can eventually lead to a slightly massive AdS_4-graviton. This would be the case where two initially decoupled three-dimensional theories are coupled *weakly* while comformal symmetry is preserved[22]. The coupling is mediated by messenger degrees of freedom. The gravity dual of this picture is a connection of two initially decoupled AdS_4 universes, with two initially masseless gravitons, via a thin throat of particular geometry. The result is a bimetric AdS_4 theory, where a massless and a massive graviton are both included. This possibility will be examined in detail in the final chapter of this section.

3.2 Higgsing in Representation theory

Before moving to geometry, let us discuss the Higgsing from the point of view of representation theory. The notation and the conventions, along with selected aspects of representation theory for three-dimensional $\mathcal{N}=2$ and $\mathcal{N}=4$ theories are given in the Appendix A

Let $D(\Delta, s)$ denote a unitary highest-weight representation of $\mathfrak{so}(2,3)$ with conformal primary of spin s and scaling dimension Δ. Massive gravitons belong to long representations of the algebra. The decomposition of a long spin-s representation at the unitarity threshold reads [101]

$$D(s+1+\epsilon; s) \quad \xrightarrow{\epsilon\to 0} \quad D(s+1;s) \oplus D(s+2; s-1)\ . \tag{3.2.1}$$

Thus the AdS$_4$ graviton ($s=2$) acquires a mass by eating a massive Goldstone vector. In the $10d$ supergravity this vector must be the combination of off-diagonal components of the metric and tensor fields that is dual to the CFT operator T_{a4}.

Since we will here deal with $\mathcal{N}=4$ backgrounds, fields and dual operators fit in representations of the larger superconformal algebra $\mathfrak{osp}(4|4)$. These have been all classified under mild assumptions [59][39]. In the notation of [39] (slightly retouched in [19]) the supersymmetric extension of the above decomposition reads

$$L[0]^{(0;0)}_{1+\epsilon} \quad \xrightarrow{\epsilon\to 0} \quad A_2[0]^{(0;0)}_1 \oplus B_1[0]^{(1;1)}_2\ , \tag{3.2.2}$$

where L denotes a long representation, A_i (B_i) a short representation that is marginally (absolutely) protected, and $[s]^{(j;j')}_{\Delta}$ denotes a superconformal primary with spin s, scaling dimension Δ and $\mathfrak{so}(4)$ R-symmetry quantum numbers $(j;j')$. The above decomposition (or recombination) describes the Higgsing of the $\mathcal{N}=4$ graviton multiplet in AdS$_4$. That this is at all possible is not automatic. For instance $\mathcal{N}=4$ supersymmetry forbids the Higgsing of ordinary gauge symmetries because conserved vector currents transform in absolutely protected representations of $\mathfrak{osp}(4|4)$ [90][38].

The bosonic field content of the above $\mathcal{N}=4$ multiplets is as follows:

$$A_2[0]_1^{(0;0)} = [0]_1^{(0;0)} \oplus [0]_2^{(0;0)} \oplus [1]_2^{(1;0)\oplus(0;1)} \oplus [2]_3^{(0;0)} \oplus \text{fermions} \ , \tag{3.2.3}$$

$$\begin{aligned} B_1[0]_2^{(1;1)} \ = \ & [0]_2^{(1;1)} \oplus [1]_3^{(1;1)\oplus(1;0)\oplus(0;1)} \oplus [0]_3^{(2;0)\oplus(0;2)\oplus(1;0)\oplus(0;1)\oplus(1;1)\oplus(0;0)} \\ & \oplus[1]_4^{(0;0)\oplus(1;0)\oplus(0;1)} \oplus [0]_4^{(1;1)\oplus(0;0)} \oplus [0]_5^{(0;0)} \oplus \text{fermions} \ . \end{aligned} \tag{3.2.4}$$

The supergraviton multiplet A_2 has in addition to the spin-2 boson, six vectors and two scalar fields, making a total of 16 physical states. The eaten Goldstone multiplet B_1 contains 112 physical bosonic states and as many fermions. These latter include massive spin-3/2 states which are not part of the spectrum of gauged $4d$ supergravity [19]. Higgsing with that much supersymmetry is thus necessarily a higher dimensional process.

3.3 Massive spin-2 on $AdS_4 \times M_6$

The main analysis regards in confronting the problem of the AdS_4-graviton Higgsing in the gravitational side of our holographic framework. This problem is hence going to be considered in the supergravity background which was introduced in detail in the previous section. Working in a collective approach as regards the base manifold $M_6 = (S^2 \times \hat{S}^2) \ltimes \Sigma_{(2)}$, we start with the metric of our warped geometry:

$$ds^2_{10} = L_4^2(Y)\bar{g}_{\mu\nu}(X)dX^\mu dX^\nu + g_{ij}(Y)dY^i dY^j \tag{3.3.1}$$

where $L_4(Y)$ is the radius of the AdS_4-fiber at a point-Y of M_6, the coordinates X, Y parametrize AdS_4 and M_6 respectively, while $\mu, \nu : \{0, 1, 2, 3\}$ and $(i, j) : \{4, 5, 6, 7, 8, 9\}$. Note that the base manifold, can be either compact ($\bar{M}_6$) or non-compact (M_6). In the first case, $AdS_4 \times \bar{M}_6$ is a standard compactification, where the string spectrum contains a massless spin-2 mode. On the contrary, in the following we will see how a non-compact internal manifold, corresponds to the case where the lowest-lying graviton acquires a small mass. In this case, M_6 is comprised by semi-finite throats of radius L_5, attached to a central compact 'bag' of size $L_{`bag'} \gg L_5$, which are the ones contributing to the small graviton-mass, as we will see from what follows. The typical size of the bag, is parametrically bound to the $AdS_4$ radius, $L_{\text{`bag'}}$ L_4. This is a general characteristic property of $AdS_d \times M_D$ warped backgrounds, widely known as the scale non-separation problem [108][67], where the size of the internal manifold if of the same order as the radius of the AdS_d space.

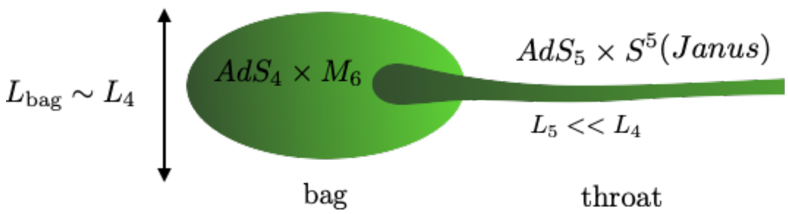


Figure 3.1: The manifold described in the text, is comprised by one (or two at most) semi-infinite $AdS_5 \times S^5$ or Janus throat of radius L_5 attached on a compact 'bag' of typical size $L_{\text{`bag'}} \sim L_4 \gg L_5$.

The full ten-dimensional geometry of the semi-infinite throats is $AdS_5 \times S^5$, or its supersymmetric Janus generalization. Both geometries are presented in detail in the Appendix B. While in the $AdS_5 \times S^5$ case the dilaton remains constant throughout the throat,the Janus solution includes a dilaton whose value interpolates between two constant ones: the one at the refion where the throats attache on the bag, ϕ_{bag} and the value at infinity, ϕ_∞. Therefore, the variation $\Delta\phi = \phi_\infty - \phi_{bag}$ of the dilaton is a parameter which characterizes the throat, along with its radius, L_5.

We are interested in metric perturbations around our AdS_4 vacua, which have a small mass and are of factorized form:

$$ds_{10}^2 = L_4^2(Y)(\bar{g}_{\mu\nu}(X) + \underbrace{h_{\mu\nu}(X,Y)}_{=h_\lambda(X)\psi_\lambda(Y)})dX^\mu dX^\nu + g_{ij}(Y)dY^i dY^j \tag{3.3.2}$$

where $\psi_\lambda(Y)$ is the internal-space wavefunction and $h_\lambda(X)$ is a solution of the Pauli-Fierz equations for a massive spin-2 mode in AdS_4:

$$(\Box_X^{(2)} - \lambda)h_{\mu\nu;\lambda} = 0 \tag{3.3.3}$$

$$\nabla^\mu h_{\mu\nu;\lambda} = 0 = \bar{g}^{\mu\nu}h_{\mu\nu;\lambda} \tag{3.3.4}$$

,where $\Box_X^{(2)}$ is the Lichnerowicz-Laplace operator, acting on two-index tensors in AdS_4 whereas from the last two equations it is evident that these spin-2 excitations are transverse and traceless. Finally, the eigenvalue λ is related to the invariant square mass.

$$\lambda + 2 = m^2(Y)L_4^2(Y) \tag{3.3.5}$$

It is significant to note that while both the mass and the AdS_4 radius may vary along the internal space, their product is constant, and this is a characteristic of warped solutions. A substitution of the mass eigenstates $h_{\mu\nu}(X,Y)$ along with the Pauli-Fierz relations, in the linearised Einstein equations according to [20], results to a second order differential equation for the wavefunction:

$$\underbrace{-\frac{L_4^{-2}(Y)}{\sqrt{g}}(\partial_i\sqrt{g}g^{ij}L_4^4(Y)\partial_j)\psi_\lambda(Y)}_{\mathcal{M}^2} = (\lambda+2)\psi_\lambda(Y) \to$$

$$\to \mathcal{M}^2\psi_\lambda(Y) = m^2(Y)L_4^2(Y)\psi_\lambda(Y) \tag{3.3.6}$$

, with $\mathcal{M}^2$ being the Laplace-Beltrami operator on M_6. It is notable that the above spin-2 eigenmode equation depends only on geometric data and not on details of matter field backgrounds, in contrast with the case of lower-spin modes, the linearized equations of which do not depend on metric details of the background fields.

A complete definition of the spectral problem, recquires the introduction of a norm for the wavefunctions, which is obtained from its Kaluza-Klein reduction from ten dimensions:

$$\langle\psi_1|\psi_2\rangle = \int_{M_6} d^6Y\sqrt{g}L_4^2\psi_1^*\psi_2 \tag{3.3.7}$$

The last two relations, summarize the full spin-2 mass-eigenvalue problem in the considered geometry. A significant detail is that the mass-squared operator is hermitean and that its eigenvalue is non-negative, a fact that can be easily checked with an integration by parts:

$$\langle \psi, \mathcal{M}^2 \psi \rangle = \int\limits_{M_6} d^6Y \sqrt{g} L_4^4 \partial_i \psi^* \partial^i \psi \geq 0 \tag{3.3.8}$$

Nevertheless, the ultimate goal is not the solution of the full eigenvalue problem -which is a highly complicated project- but rather the computation of the mass of the lowest lying AdS_4 graviton, namely the computation of the minimal eigenvalue of the mass-squared operator.

As we will see in detail in the following analysis, the contributions to the graviton mass are only made by the regions of the semi-infinite throats. Recall that the warped background of interest is dual to a three-dimensional linear quiver theory. This suggests, as explained in the previous chapter, that this theory can be coupled as a boundary to a four-dimensional theory, or can be a defect, separating two four-dimensional theories. Holographically, this corresponds to at most two semi-infinite throats attached to the compact 'bag'. This can of course change for other cases of theories, such as for backgrounds dual to star-shaped quivers, where more than two throats may be attached and hence contribute to the graviton mass.

The strategy therefore is to concentrate to the computation of the smallest eigenvalue of the mass squared operator. This is computed by the minimization of the expression (3.3.8) complemented by the normalization condition for the norm:

$$m_0^2 L_4^2 = \min_\psi [\int_{M_6} d^6Y \sqrt{g} L_4^4 (g^{ij} \partial_i \psi^* \partial_j \psi)] \tag{3.3.9}$$

$$\int_{M_6} d^6Y \sqrt{g} L_4^2 |\psi|^2 = 1 \tag{3.3.10}$$

Since in the case where the internal manifold is compact ($\bar{M}_6$) the lowest-lying graviton is massless and hence it is described by a constant wavefunction. Once the semi-infinite throats are introduced, a distinction has to be made and the wavefunction in the 'bag' will be designated as ψ_{bag}. Minimizing the mass in the 'bag' region sets the wavefunction to a constant :

$$\psi_0(Y) \equiv \psi_{bag} = [\int_{\bar{M}_6} d^6Y \sqrt{g} L_4^2]^{-\frac{1}{2}} = \text{const.} \tag{3.3.11}$$

A crucial point is that once the semi-infinite throats are introduced, the integral which describes the norm of the wavefunction diverges and hence the wavefunction is non-normalizable. This fact implies that the wavefunction has to vanish inside the throats. This should happen though in accordance with the minimization of the mass eigenvalue.

This can be indeed quantified. We go back to the variational problem and start from the expression for the mass eigenvalue. Specifically, we focus on the region of the throats and hence our initial problem which regards the full internal manifold, is reduced to the one regarding only the throat region, the geometry of which is the Janus one B.

Given that for the scalar wave operator we have $g^{ij}\partial_i\partial_j = \rho^{-2}\bar{\partial}\partial$:

$$\int_{M_6} d^6Y\sqrt{g}L_4^4 g^{ij}\partial_i\psi^*\partial_j\psi = \int_{M_6} d^6Y\sqrt{g}L_4^4\rho^{-2}|\partial\psi|^2 \tag{3.3.12}$$

The metric on the strip is $4\rho^2 dzd\bar{z}$ and hence the volume element reads $4\rho^2 dxdy = \sqrt{g}dzd\bar{z}$, while for the two-spheres we have $d\Omega d\hat{\Omega} f^2\hat{f}^2$:

$$\int_{M_6} d^6Y\sqrt{g}L_4^4\rho^{-2}|\partial\psi|^2 = 4\cdot(4\pi)^2\int dxdyL_4^4 f^2\hat{f}^2|\partial\psi|^2 \tag{3.3.13}$$

Moreover, by using the definitions (2.1.3) and the corresponding identities we have $L_4^4 f^2\hat{f}^2 = 16h^2\hat{h}^2$. Finally by combining the two parts:

$$\sqrt{g}L_4^4\rho^{-2} = L_4^4 f^2\hat{f}^2 = 16h^2\hat{h}^2 \tag{3.3.14}$$

We now turn to the quantity which characterizes the norm of the wavefunction,$\sqrt{g}L_4^2$. The above identities are used, along with the form of the Janus harmonic functions and the fact that the radius of the throat is given by $L_5 = 2\cosh^{\frac{1}{4}}(\delta\phi)$ (throughout the analysis we have set for convenience $\alpha = \hat{\alpha} = 1$). It is straightforward in this context to obtain the expression for $16h^2\hat{h}^2$:

$$16h^2\hat{h}^2 = 16\sin^2(2y)(\cosh(2x) + \cosh(\delta\phi))^2 = \frac{L_5^8}{16}\sin^2(2y)\mathcal{G}(x), \tag{3.3.15}$$

$$\mathcal{G}(x) := \Big|\frac{h\hat{h}}{W}\Big|^2 = (\frac{\cosh(2x) + \cosh(cosh)(\delta\phi)}{\cosh(\delta\phi)})^2 \tag{3.3.16}$$

Therefore for the quantity of interest we have:

$$\sqrt{g}L_4^2 = 16h^2\hat{h}^2\frac{\rho}{L_4^2} = \frac{L_5^8}{16}\sin^2(2y)\mathcal{G}(x)\Big|\frac{h\hat{h}}{W}\Big| \tag{3.3.17}$$

and hence we see that in the region of the throats it reaches the minimum value L_5, while it blows up at infinity. Therefore as claimed above, the graviton wavefunction has to vanish inside the throat in order to be normalizable. In particular, this should happen in the region where the quantity which characterizes the mass eigenvalue, $\sqrt{g}L_4^4$, becomes minimal. The above are summarized in the the following plot, where the AdS_4 radius L_4 and the graviton wavefunction which minimizes the mass are plotted as functions of the Janus coordinate x.

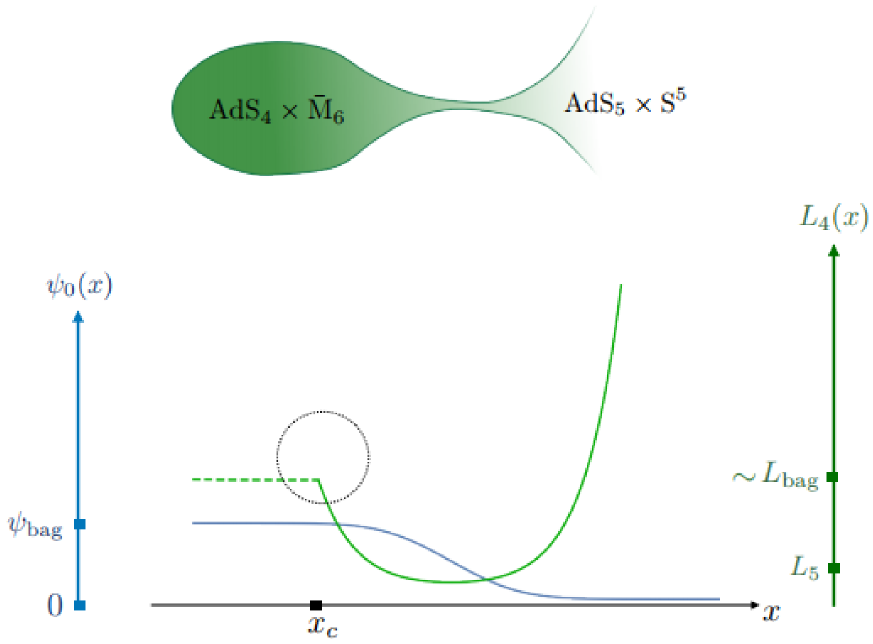


Figure 3.2: The AdS_4 radius, L_4 becomes minimal inside the throat and diverges at infinity, where the full ten dimensional geometry asymptotes to $AdS_5/\mathbb{Z}_2 \times S^5$. The graviton wavefunction is constant in the compact region and vanishes exponentially at infinity

Starting from the 'bag' part of M_6, we have a nearly constant value for the AdS_4 radius (dashed line) which, due to the non-separation of scales, is parametrically bound to the size of the 'bag', $L_{\text{'bag'}} \sim L_4$. The region where the throat is attached to the 'bag' is the one circled in the plot and there $\sqrt{g}L_4^2$ vanishes exponentially fast. Finally, in the throat region , the wavefunction vanishes exponentially fast, in the region of minimal $\sqrt{g}L_4^4$. The AdS_4 radius reaches a minimum of L_5 in the throat, while it diverges at infinity. Therefore, only the throat contributes to the graviton mass whereas in leading order to L_5/L_4 only the 'bag'-region contributes to the norm of the wavefunction.

In what follows we will derive the form of the minimizing wavefunction and an expression for the mass of the lowest-lying. The variational problem in the Janus geometry reads:

$$m_0^2 L_4^2 = \simeq \min_\psi \Big[\int_{\text{throats}} d^6Y \sqrt{g} L_4^4 g^{ij} \partial_i \psi^* \partial_j \psi \Big], \qquad \psi \to \begin{cases} \psi_{\text{'bag'}} & \text{in matching region} \\ 0 & \text{at infinity} \end{cases} \tag{3.3.18}$$

Integrating over the two-spheres gives:

$$m_0^2 L_4^2 \simeq \min_\psi \Big[\frac{\pi^3}{4} L_5^8 \int_{x_c}^{\infty} dx \mathcal{G}(x) \Big(\frac{d\psi}{dx}\Big)^2 \Big], \qquad \psi \to \begin{cases} \psi_{\text{'bag'}} & x = x_c \\ 0 & x \to \infty \end{cases} \tag{3.3.19}$$

The cutoff value x_c is just a large negative value, at the boundary of the matching region, in other case it could be replaced by $-\infty$. The result is independent of this value and its role is just to remind us that the wavefunction would have been non-normalizable in the complete Janus geometry.

The expression for the wavefunction which minimizes the mass inside the throats is obtained by solving the above variational problem.

Specifically, the analysis followed in [20] regarding the spin-2 spectrum provides a significant simplification in Janus geometry: the graviton wavefunction can be expanded in a basis of spherical harmonics on the two-spheres. Therefore, the first step is to substitute the wavefunction in its expanded form in the eigenmode equation. Choosing $(l_1, l_2) = (0,0)$ since we are focused on the lowest lying graviton, the eigenvalue equation is reduced to the form (6.1) of [20] :

$$-\frac{1}{2\cosh(\delta\phi)}\Big[\frac{d^2}{dx^2} - 4(n+1)^2\Big](\cosh(\delta\phi) + \cosh(2x))\psi(x) = (2+m^2)\psi(x) \tag{3.3.20}$$

, where in the above expression n is the relic of the Y-dependent piece of the wavefunction, decomposed in a basis of trigonometric functions [20] . Demanding the lowest-lying graviton to be nearly massless, we impose the condition $(m,n) = (0,0)$ and the above expression takes the form:

$$-\frac{1}{4\cosh(\delta\phi)}\Big(\frac{d^2}{dx^2} - 4\Big)(\cosh(\delta\phi) + \cosh(2x))\psi(x) = \psi(x) \tag{3.3.21}$$

For $\delta\phi = 0$, the obtained solution is the minimized wavefunction inside a semi-infinite $AdS_5 \times S^5$ throat. This will be presented in the next chapter and here the attention is given only in the Janus case. The differential equation for $\delta\phi \neq 0$ finally boils down to:

$$\frac{d}{dx}\Big(\mathcal{G}(x)\Big(\frac{d\psi_0}{dx}\Big)\Big) \to \psi_0(x) = c_1 + c_2\int_0^x \frac{dx'}{\mathcal{G}(x')} \tag{3.3.22}$$

with c_1, c_2 being integration constants whose form is fixed by the boundary conditions of the problem. By performing the integral analytically while setting $\cosh(\delta\phi) = \alpha$ and by fixing the lower integration limit so that the integral is an odd function of x, we finally obtain:

$$\begin{aligned} I(x,\alpha) &:= \int_0^x \frac{\alpha^2 dx'}{(\cosh(2x') + a)^2} = \\ &= \frac{\alpha^3}{2(\alpha^2-1)^{3/2}}\log\Big[\frac{\sqrt{\alpha+1} + \sqrt{\alpha-1}\tanh(x)}{\sqrt{\alpha+1} - \sqrt{\alpha-1}\tanh(x)}\Big] - \frac{\alpha^2}{(\alpha^2-1)}\frac{\tanh(x)}{[(\alpha+1) - (\alpha-1)\tanh^2(x)]} \end{aligned} \tag{3.3.23}$$

And hence the expression for the minimizing wavefunction in the throat region reads:

$$\psi_0(x,\alpha) \simeq \frac{1}{2}\psi_{\text{'bag'}}\Big[1 - \frac{I(x,\alpha)}{I(\infty,\alpha)}\Big] \tag{3.3.24}$$

The integral approaches its limiting values exponentially. Therefore up to exponentially small corrections $\psi_0(x_c) \simeq \psi_{\text{'bag'}}$. Regarding the the norm and the mass, this result indicates that the contribution of this wavefunction to the norm can be neglected in comparison to the one of the $\psi_{\text{'bag'}}$, while it is a dominating contribution to the mass.

Having obtained the expression for the wavefunction, the next step is to plug it back in the mass eigenmode equation. Taking into account that the integrand is actually a total derivative and moreover that $\mathcal{G}(x) d\psi_0 dx = -\psi_{\text{'bag'}}/2I(\infty, \alpha)$ and $[\psi_0]_{x_c}^{\infty} = -\psi_{\text{'bag'}}$, we obtain:

$$m_0^2 L_4^2 \simeq \frac{\pi^3}{8} L_5^8 \int_{x_c}^{\infty} dx \mathcal{G}(x) \Big(\frac{d\psi_0}{dx}\Big)^2 = \frac{\pi^3}{4} L_5^8 \Big[\mathcal{G}(x) \frac{d\psi_0}{dx} \psi_0\Big]_{x_c}^{\infty} = \tag{3.3.25}$$

$$= \frac{\pi^3}{4} L_5^8 \frac{\psi_{\text{'bag'}}}{I(\infty, \alpha)} \to \boxed{m_0^2 L_4^2 \simeq \frac{3\pi^3}{4} L_5^8 \psi_{\text{'bag'}}^2 J(\alpha)} \tag{3.3.26}$$

We have finally obtained the main result of this analysis: a new formula for the mass of the lowest-lying AdS_4 graviton, with dependence on the parameters of the throat (radius L_5 and dilaton variation $\delta\phi$) and on the size of the internal manifold, through $\psi_{\text{'bag'}}$.

The function $J(\alpha)$ in the above result, is actually a correction factor that reflects the effect of the Janus geometry in the value of the mass:

$$J(\alpha)^{-1} := 6I(\infty, \alpha) = \frac{3\alpha^3}{(\alpha^2 - 1)^{3/2}} \log\Big[\alpha + \sqrt{\alpha^2 - 1}\Big] - \frac{3\alpha^2}{(\alpha^2 - 1)} \tag{3.3.27}$$

The behaviour of this function is given in the plot below and indicates that as the dilaton variation increases, the graviton mass is suppressed. Of course its effect becomes trivial in the case where the geometry of the throats is $AdS_5 \times S^5$ and the value of the dilaton does not vary throughout the throat ($\delta\phi = 0$)

Regarding the the value of the dilaton at the matching region (entrance of the throat), it is fixed by the parameters of the bag. The value at infinity, determines the coupling constant (g_{YM}) of the dual 4d $\mathcal{N} = 4$ super Yang-Mills theory. The limit $|\delta\phi| \to \infty$ corresponds to the limit where the four-dimensional theory decouples from the three-dimensional one $g_{YM} \to 0$, which results to the restoration of the conservation of the stress-tensor $T_{\alpha\beta}$ and to a vanishing graviton mass. Nevertheless, as shown both numerically and analytically in [20] in this limit the low-lying mass spectrum becomes discrete and hence no continuous Higgsing can be achieved.

A final insertion to this section is the rewriting of the obtained expression of the graviton mass in terms of other physical parameters of the solutions.

We start from the minimizing wavefuntion in the 'bag' region, which can be re-expressed in terms of the AdS_4 radius and the volume of the 'bag':

$$\psi_{\text{'bag'}}^{-2} = \int_{\bar{M}_6} d^6 Y \sqrt{g} L_4^2 = \frac{\int_{\bar{M}_6} d^6 Y \sqrt{g} L_4^2}{\int_{\bar{M}_6} d^6 Y \sqrt{g}} \int_{\bar{M}_6} d^6 Y \sqrt{g} \to \psi_{\text{'bag'}}^{-2} = \langle L_4^2 \rangle_{\text{'bag'}} V_6 \tag{3.3.28}$$

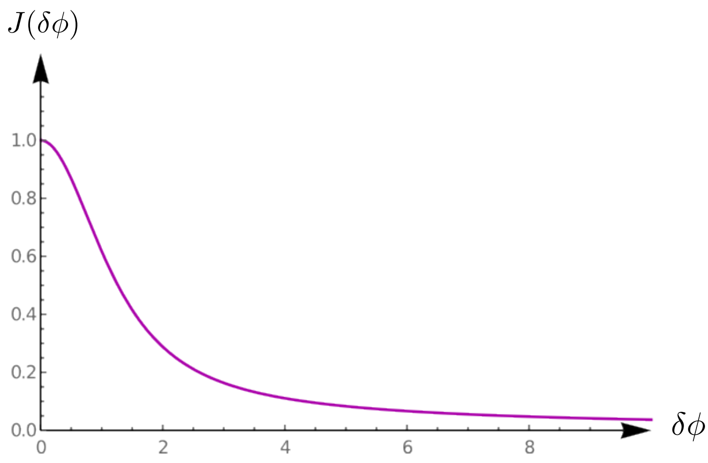


Figure 3.3: The Janus correction factor, $J(\delta\phi)$

Substituting this to the obtained result for the graviton mass, we get an expression in terms of geometric data:

$$\boxed{m_0^2 L_4^2 \simeq \frac{3\pi^3 L_5^8}{4V_6 \langle L_4^2 \rangle_{\textbf{`bag'}}} \times J(\cosh(\delta\phi))} \tag{3.3.29}$$

This rewriting indeed gives an interesting result. First it should be noted that due to the non-separation of scales - explained in the beginning of the chapter - the six-dimensional volume can be rewritten as $V_6^{1/3} \sim L_{bag}^2 \sim \langle L_4^2 \rangle_{\textbf{`bag'}}$, which indicates that the graviton mass is suppressed by $(L_5/L_4)^8$. This is the main difference in comparison to the result obtained from the Karch-Randall model, which is the ratio of the radii in the second instead of the eighth power, $m^2 \sim (L_5/L_4)^2$ [82] and is a hint of the failure of the thin-brane approximation followed in that context.

Another way to rewrite the result (3.3.25), involves - apart from the above rewriting of $\psi_{\text{`bag'}}$- the compactification volume and the four-dimensional effective gravitational coupling. Using the expressions for the ten-dimensional gravitational coupling in terms of the above parameters as well as of the string coupling (λ_s) and Regge slope (α')

$$\begin{cases} \kappa_{10}^2 = V_6 \kappa_4^2 \\ 2\kappa_{10} = (2\pi)^7 \alpha'^4 \lambda_s^2 \end{cases} \qquad m_0^2 L_4^2 \simeq \frac{3L_5^8}{2^8 \pi^4 \alpha'^4 \lambda_s^2} \frac{\kappa_4^2}{\langle L_4^2 \rangle_{\text{`bag'}}} \tag{3.3.30}$$

Now, inserting the expression for the radius of the Janus throat B $L_5^4 = 4\pi\alpha'^2 \lambda_s n$, with n being the D3 brane charge of the throat, the above expression takes the form:

$$\boxed{m_0^2 L_4^2 \simeq \frac{3n^2}{16\pi^2} \frac{\kappa_4^2}{\langle L_4^2 \rangle_{\text{`bag'}}} \times J(\cosh(\delta\phi))} \tag{3.3.31}$$

The important information this expression provides, is that the mass is quantized, since it is proportional to the quantized D3 brane charge. Therefore, Higgsing is not a continuous process. Moreover, this expression is similar to the one obtained in [101]. It is interesting as the approach followed in those works regards the graviton mass as a quantum effect, whereas in our approach tha mass is rather obtained from a small-fluctuation analysis around our classical supergravity solution.

This section closes with a rewritting of the main result for the graviton mass, this time within the framework of the dual CFT. In particular, the concept of interest is the degrees of freedom of both the three-dimensional theory and the bulk four-dimensional one. In the first section of this chapter, it was explained that the weak dissipation of the three-dimensional stress tensor into the fourth dimension -and hence the appearance of small anomalous dimension which holographically corresponds to a small graviton mass- is ensured by the scarcity of the degrees of freedom of the four-dimensional CFT with respect to the ones of the three-dimensional one. The quantity that measures the degrees of freedom in three dimensions is the free energy on the S^3. The calculation for arbitrary dimension [69] is given below, along with the specialization to four dimensions and the free energy in three dimensions (holographic computation [14] and CFT computation [28] [97]):

$$\begin{cases} \tilde{F}_d = \sin(\frac{\pi d}{2}) \log \mathcal{Z}(S^d) \to \tilde{F}_4 = \frac{\alpha\pi}{2} \\ \tilde{F}_3 = \frac{4\pi^2 \langle L_4^2 \rangle}{\kappa_4^2} \end{cases} \tag{3.3.32}$$

Finally, working out the main result gives:

$$\boxed{3\epsilon \simeq m_0^2 L_4^2 \simeq \frac{6\pi^3 \tilde{F}_4}{\tilde{F}_3} \times J(\cosh(\delta\phi))} \tag{3.3.33}$$

The anomalous dimension is thus proportional to the ration of the degrees of freedom of the bulk theory over the ones of the three-dimensionsal defect. The weak dissipation corresponds to $\epsilon \ll 1$ from which we obtain:

$$\tilde{F}_4 \ll \tilde{F}_3 \tag{3.3.34}$$

,which is exactly the argued condition. In practice this can be realized as a small number of semi infinite D3 branes attached on the outtermost NS5 brane of a configuration involving a much larger number of D3 branes suspended between NS5s and intersected by a large number of D5 branes.

3.4 Conclusions and perspectives

In this section we proposed a top-down string theory embedding of massive AdS_4 gravity, with the main results of the analysis being:

- that Massive AdS_4 gravity lies in the string landscape, namely this theory can be successfully embedded in string theory
- a new quasi-universal quantized formula for the mass of the lowest-lying AdS_4-graviton ,obtained by studying metric fluctuations around classical supergravity solutions (3.3.25)

the fact that such theory can be embedded in a UV-complete theory like string theory, indicates that the corresponding four-dimensional gravity theory will be free of pathologies, such ghosts or physical parameter discontinuities. The relevant energies in which one could compare the effective theory with the already introduced ghost-free actions, lie between the lowest-lying graviton mass and the inverse AdS_4 radius, at which the effective theory is considered to break down. This is connected with the non-separation of scales in AdS compactifications, due to which the AdS radius-as explained in the first section of the chapter-is bound parametrically to the size of the internal space and to the Kaluza-Klein scale.

Nevertheless, it is highly non-trivial to write down an effective action from string theory. One possible strategy could be to exploit the remarkable fact that the massive $\mathcal{N}=4$ supergraviton, has the same degrees of freedom as the massless $\mathcal{N}=8$ supergraviton. This suggests that the above Higgsing process could be described in four dimensions as an $\mathcal{N}=4$ deformation of the maximal $\mathcal{N}=8$ of Cremmer, Julia and Scherk [40]. Note that since the Goldstone multiplet $B_1[0]_2^{(1;1)}$ contains extra gravitini, it is not part of the spectrum of the usual $\mathcal{N}=4$ (gauged) supergravity in four dimensions. Extra degrees of freedom could be alternatively introduced by adding $\mathcal{N}=4$ higher-derivative terms as in [62]. The authors of this reference analyzed the massive spectrum in Minkowski spacetime and found it to contain ghosts. It could be interesting to repeat their analysis in Anti-de Sitter spacetime, especially if the hierarchy $m_0 \ll L_4^{-1}$ can be achieved.

Apart from the problem of pinning down the effective action for the already studied embedding, there are more open questions in this direction.

First, given that apparently the main study of the problem took place in the gravitational side of the holographic duality. Therefore an interesting task would be to perform the calculation purely on the CFT side, namely compute the small anomalous dimension of the almost-conserved three-dimensional stress tensor and match with our result for the graviton mass.

A third open direction, is the study of embeddings of AdS massive gravity in other dimensions and with other amounts of supersymmetry. This is a priori constrained though [18] : Many exact AdS_D solutions with $D > 4$ and half-maximal supersymmetry are known by now, for instance for AdS_7 [9], for AdS6 [56] [57] [8], and for AdS5 [64]. But among them there are cases where the stress tensor belongs to protected supermultiplets and hence cannot receive an anomalous dimension, ruling out in this way the corresponding massive gravity. An example is the AdS_7 supergravity, whose dual 6d $\mathcal{N}=1$ theory possesses a stress tensor that belongs to a protected B-series multiplet Thus massive AdS_7 supergravity is a priori excluded. The same holds for AdS_6 with $\mathcal{N}=1$, and for supergravities that are dual to theories with more-than-half-maximal supersymmetry such as $\mathcal{N}\rangle 2$ in 4d and $\mathcal{N}\rangle 4$ in 3d. However, a situation with no protection is $\mathcal{N}=2, AdS_5$ and hence it would be interesting to search for embeddings of massive AdS supergravity in this case.

Chapter 4

Stringy AdS_4 Bigravity

This chapter includes the article "*Quantum Gates to other Universes*" by C. Bachas and the author [22]. The additional features are minimal modifications regarding the notation of the two last sections as well as an addition of an extra final section.

Abstract
We present a microscopic model of a bridge connecting two large Anti-de-Sitter Universes. The Universes admit a holographic description as three-dimensional $\mathcal{N}=4$ supersymmetric gauge theories based on large linear quivers, and the bridge is a small rank-n gauge group that acts as a messenger. On the gravity side, the bridge is a piece of a highly-curved $\mathrm{AdS}_5\times\mathrm{S}^5$ throat carrying n units of five-form flux. We derive a universal expression for the mixing of the two massless gravitons: $M^2 \simeq 3n^2(\kappa_4^2+\kappa_4'^2)/16\pi^2$, where M is the mass splitting of the gravitons, $\kappa_4^2, \kappa_4'^2$ are the effective gravitational couplings of the AdS_4 Universes, and n is the quantized charge of the gate. This agrees with earlier results based on double-trace deformations, with the important difference that the effective coupling is quantized. We argue that the apparent non-localities of holographic double-trace theories are resolved by integrating-in the (scarce) degrees of freedom of the gate.

4.1 Introduction

One of the tantalizing aspects of General Relativity is the possibility of connecting disjoint Universes. Most of the attention has been captured by wormholes which are pointlike contacts between Universes. But one can in principle consider a wormbrane or Wp-brane, that is a bridge whose entry and exit are of spacetime dimension $p+1$. In this language the usual wormholes are $W(-1)$-branes.

When the Universes are AdS_{d+1}, holographic duality offers a different perspective of such objects as bridges between two decoupled d-dimensional field theories. Consistency requires that non-traverseable wormholes correspond to pure entanglement of the theories [92], while traverseable bridges must also involve a Hamiltonian coupling [66]-[110]. The generic deformation is given by a double-trace coupling

$$\delta\mathcal{L} \sim \int d^p\zeta \; \mathcal{O}(x(\zeta))\, \mathcal{O}'(x'(\zeta))\, K(x,x') \;, \tag{4.1.1}$$

where $\mathcal{O}, \mathcal{O}'$ are single-trace operators in the two theories, $x(\zeta)$ and $x'(\zeta)$ parametrize the boundary submanifolds sewed together by the coupling, and $K(x,x')$ is an interaction kernel. If one insists on conformal invariance the coupling will extend at all scales, and the bridge will have codimension $(d-p)$ both in the boundary and in the bulk. [1] This excludes the case $p=-1$. In this paper we will focus on the other extreme, $p=d$, where the entry and the exit of the bridge are the entire AdS spacetime. From a higher-dimensional perspective on the other hand, they look like entries to localized defects.

Double-trace deformations were introduced in [3]-[29] and used to model two or more interacting gravitons in [102]-[86]. Because of the absence of the van Dam-Veltman-Zakharov (DVZ) discontinuity in Anti-de-Sitter spacetime [87][100] a linear combination of the massless gravitons can obtain an arbitrarily-small mass M. An interesting feature of these double-trace models is that M comes from a one-loop quantum-gravity effect. However, although double-trace deformations have been understood as boundary conditions in the supergravity limit [115], their status in string theory is less clear. Their presence seems to introduce non-localities both in the target spacetime and on the worldsheet [3][29].

The gates presented in this paper share one key feature with these earlier models: M is suppressed by two powers of the effective gravitational coupling κ_{d+1}. Contrary, however, to double-trace models, our gates have a good semiclassical limit and are perfectly local when viewed both from the boundary and from the bulk. The price to pay (as usual) for locality is that the continuous double-trace coupling must be traded for an integer charge.

The basic idea is illustrated in figure 4.5 . One starts with two large-quiver gauge theories that are dual to two large AdS_4 spacetimes. The number of degrees of freedom in these quivers is measured by the inverse-squared effective couplings, κ_4^{-2} and $\kappa_4'^{-2}$. The bridge is then an additional 'messenger' node representing a small gauge group with rank $n \ll \kappa_4^{-2}, \kappa_4'^{-2}$. We here consider quivers corresponding to 'good' 3d $\mathcal{N}=4$ supersymmetric gauge theories at the origin of their Higgs or Coulomb branches [73, 65] for which a detailed holographic dictionary is available [12]-[19] . The idea is however more general. When $n \gg 1$ (but still much smaller than $\kappa_4^{-2}, \kappa_4'^{-2}$) the bridge admits a smooth gravitational description as a $\mathrm{AdS}_5\times\mathrm{S}^5$ throat of radius $L \sim n^{1/4}$. This had been noticed already in [12][13]. Excising the throat is equivalent to integrating out the messenger degrees of freedom leading to two effective descriptions of the bridge, either as the gluing of two AdS_4 spacetimes or as a multitrace deformation of the boundary theories. In our example both these effective descriptions are highly non-local because one integrates out massless fields. But it should be obvious that this apparent non-locality is a red herring.

To make the field-theory deformation quasi-local one may give mass to the hypermultiplets represented by the two links that join the $U(n)$ node to the quivers. The dual geometry should now exhibit a characteristic scale below which the bridge between the Universes disappears. Taking the formal $m \to \infty$ limit makes the double-trace deformation local, but the geometry is singular. This explains the tension between locality in field theory and in string theory. To resolve it one must simply integrate back-in the gate fields.

Quivers like those of figure 4.5 actually make sense for any $p \leq d$ (including $p=-1$) and can serve as definitions of Wp-branes. In most cases the dual geometries are singular, and the problem is further complicated by infrared divergences. The question of what constitutes a 'weak link' (as opposed to a full-fledged interface) must be in particular carefully reexamined. These issues will be discussed elsewhere.

The plan of the present paper is as follows: In section 4.2 we review some relevant features of the 3d $\mathcal{N}=4$ quiver gauge theories that we need. We recall in particular how the data for good quivers can be repackaged efficiently in an ordered pair of Young diagrams $(\rho, \hat{\rho})$. In section 4.3 we describe the microscopic gate of figure 4.5 as the rearrangement of n boxes in

[1] In general, after taking account of the backreaction the bridge will be fat rather than delta-function localized in the transverse dimensions. Its worldvolume can be either Euclidean or Lorentzian.

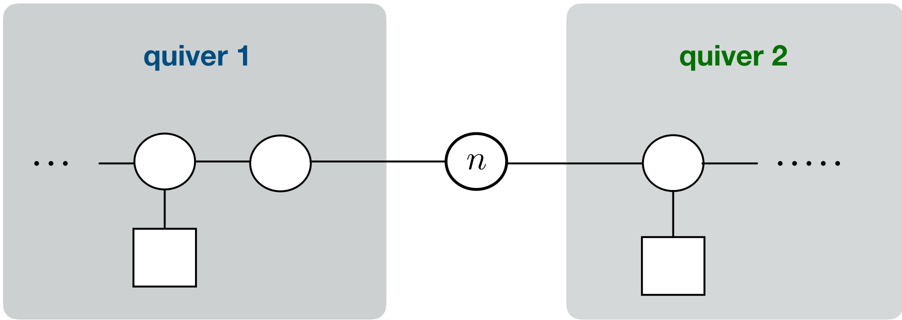


Figure 4.1: Two large quivers corresponding to two large AdS Universes joined by a gate which is a low-rank gauge theory coupling via bifundamental matter to the quivers.

the Young diagrams. In section 4.4 we present the dual type-IIB supergravity solutions before and after the construction of the bridge. The mixing of the gravitons due to the introduction of the gate is calculated in the semiclassical limit $1 \ll n \ll \kappa_4^{-2}, \kappa_4'^{-2}$ in section 4.5, and shown to agree parametrically with the double-trace models of [102]-[86]. One can interpret our result as a rule of quantization of the double-trace coupling. Finally, in section 4.7 we comment on future directions.

4.2 Partitions for good quivers

The field theories of our holographic setup are three-dimensional $\mathcal{N} = 4$ gauge theories that can be engineered with D3-branes suspended between D5-branes and NS5-branes [73]. Let $A, N, \hat{N}$ be respectively the number of these three types of brane. To define the gauge theory one must give two ordered partitions of A in N or $\hat{N}$ positive integers

$$A = \; l_1 + l_2 + \cdots + l_N \;\; = \; \hat{l}_1 + \hat{l}_2 + \cdots \hat{l}_{\hat{N}} \; , \tag{4.2.1}$$

where $l_i \geq l_{i+1}$ and $\hat{l}_{\hat{\imath}} \geq \hat{l}_{\hat{\imath}+1}$. These describe the distribution of the D3-branes among NS5-branes on the left and D5-branes on the right. Equivalently, the partitions define two Young diagrams, ρ and $\hat{\rho}$, both with the same number A of boxes. The diagram ρ has l_i boxes in the i^{th} row, and $\hat{\rho}$ has $\hat{l}_{\hat{\jmath}}$ boxes in the $\hat{\jmath}^{\,th}$ row. We label the rows of the transposed Young diagram ρ^T (i.e. the columns of ρ) by hatted Latin letters, and the rows of the transposed Young diagrams $\hat{\rho}^T$ by unhatted letters. The reason for this notation will soon be clear. The length of the $\hat{\jmath}^{th}$ row in ρ^T is $l^T_{\hat{\jmath}}$, and the length of the j^{th} row in $\hat{\rho}^T$ is $\hat{l}^T_j$.

Quivers whose gauge symmetry can be entirely Higgsed correspond to pairs obeying the ordering condition $\rho^T > \hat{\rho}$. It was conjectured by Gaiotto and Witten [65] that at the origin of their Higgs branch such 'good theories' flow to strongly-coupled supersymmetric CFTs that are irreducible with no free-field factors. We can put the ordering condition in compact form by introducing the integrated row lengths

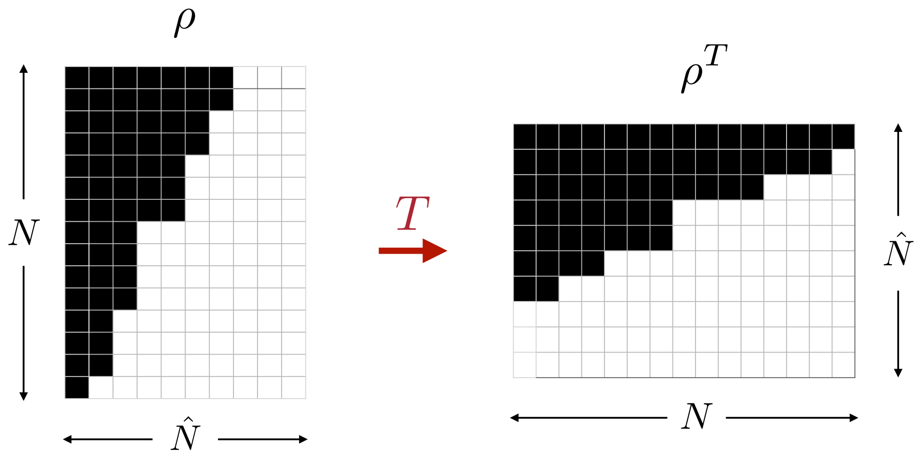


Figure 4.2: The Young diagram ρ and its transpose ρ^T inscribed in their respective grids.

$$L_j = \sum_{i=1}^{j} l_i \quad , \qquad L^T_{\hat{j}} = \sum_{\hat{i}=1}^{\hat{j}} l^T_{\hat{i}} \quad , \qquad \hat{L}_{\hat{j}} = \sum_{\hat{i}=1}^{\hat{j}} \hat{l}_{\hat{i}} \quad , \qquad \hat{L}^T_j = \sum_{i=1}^{j} \hat{l}^T_i \quad , \tag{4.2.2}$$

which count the total number of boxes in the first j or $\hat{j}$ rows of the corresponding diagrams. The condition $\rho^T > \hat{\rho}$ is then equivalent to the following set of strict inequalities

$$L^T_{\hat{j}} > \hat{L}_{\hat{j}} \qquad \text{for all} \quad \hat{j} = 1, 2 \cdots, \hat{N} - 1 \ . \tag{4.2.3}$$

In words, the first $\hat{j}$ rows of ρ^T contain more boxes than the first $\hat{j}$ non-empty rows of $\hat{\rho}$, for all $\hat{j}$. The mirror statement $\hat{\rho}^T > \rho$ can be shown to be mathematically equivalent.

The first of the above inequalities implies that $N > \hat{l}_1$, while its mirror statement is $\hat{N} > l_1$. It follows that the Young diagrams ρ^T and $\hat{\rho}$ are contained in a $\hat{N} \times N$ grid, and the diagrams ρ and $\hat{\rho}^T$ are both contained in a $N \times \hat{N}$ grid, see figure 4.2. This justifies our use of the same labelling for the rows of ρ^T and $\hat{\rho}$, and also for the rows of ρ and $\hat{\rho}^T$. When viewed as directed walks ρ and $\hat{\rho}$ end at the lower left corner of their respective grids, while the transposed walks begin at the upper right corner of their grids.

The linear-quiver theories defined by such partitions are called $T^{\hat{\rho}}_{\rho}[SU(A)] \equiv T^{\rho}_{\hat{\rho}}[SU(A)]$ where '$\equiv$' denotes mirror symmetry. Their quivers are shown in figure 4.3. We call electric the quiver with $\hat{N} - 1$ nodes (for which the gauge group is realized on D3-branes suspended on

NS5-branes) and magnetic the quiver with $N-1$ nodes (with the D3-branes suspended on D5-branes). To minimize the occurence of hatted symbols we choose to show here the magnetic quiver. The ranks n_j and m_j of the gauge and the flavor groups can be read from the row-lengths of ρ and $\hat{\rho}^T$ via the relations

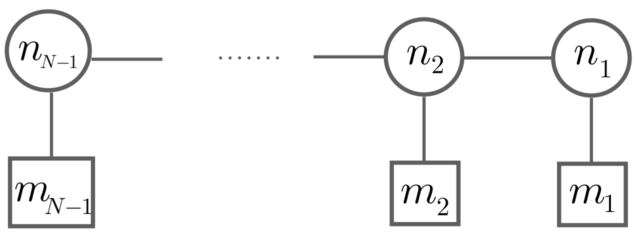
$$n_i = \hat{L}_j^T - L_j \ , \qquad m_j = \hat{l}_j^T - \hat{l}_{j+1}^T \qquad\qquad (j = 1, \cdots, N-1) \ . \tag{4.2.4}$$

The ordering condition $\hat{\rho}^T > \rho$ ensures that all gauge-group factors have positive rank, while for the flavor groups this is automatic. The dual electric quiver can be expressed likewise in terms of the row lengths of ρ^T and $\hat{\rho}$.

Besides mirror symmetry which exchanges ρ and $\hat{\rho}$, good pairs of Young diagrams admit one other involution (C) which replaces ρ by its complement ρ^c inside the $N \times \hat{N}$ grid, and $\hat{\rho}$ by its compliment $\hat{\rho}^c$ inside the $\hat{N} \times N$ grid, as in figure 4.4 . The lengths of the rows in the transformed diagrams are

$$l_j^c = \hat{N} - l_{N-j} \qquad \text{and} \qquad \hat{l}_{\hat{j}}^c = N - \hat{l}_{\hat{N}-\hat{j}} \ . \tag{4.2.5}$$

The reader is invited to check that this operation amounts to a reflection of the electric and magnetic quivers. In the underlying string theory this flips the orientation of the suspended D3-branes. Since the $\mathcal{N}=4$ gauge theories are not chiral, C is a symmetry of the problem. [2]



$$\boxed{\begin{aligned} n_j &= \hat{L}_j^T - L_j \\ m_j &= \hat{l}_j^T - \hat{l}_{j+1}^T \end{aligned}}$$

Figure 4.3: The magnetic quiver for the ordered pair of partitions $(\rho, \hat{\rho})$. The gauge-group ranks n_j and the flavor-group ranks m_j can be expressed in terms of the row lengths of ρ and $\hat{\rho}^T$. The inequality $\hat{\rho}^T > \rho$ guarantees the positivity of all n_j.

[2] C changes A to $(N\hat{N} - A)$, in apparent violation of the D3-brane charge. It is however known that this latter

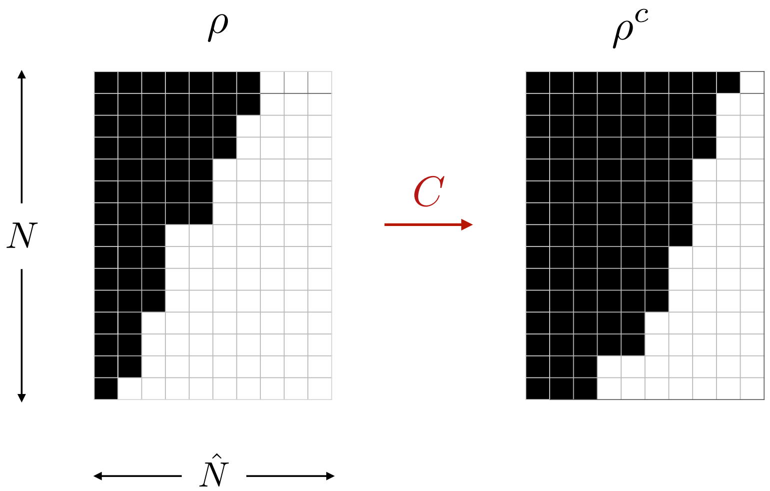


Figure 4.4: The operation C that replaces ρ by its complement inside the $N \times \hat{N}$ grid, and $\hat{\rho}$ by its complement inside the $N \times \hat{N}$ grid. C changes black to white and rotates the diagram by 180^o.

4.3 Quantum gates as box moves

Consider now two decoupled theories, described by the good pairs $(\rho_1, \hat{\rho}_1)$ and $(\rho_2, \hat{\rho}_2)$. We assume that all the brane charges are large, so that the dual AdS_4 spacetimes can be described accurately by type-IIB supergravity. We would like to couple these theories weakly, as shown in the figure below. 'Weak' means that the node joining the quivers has a gauge group of low rank n. The weakest bridge has $n = 1$. When the quivers in the picture are magnetic, we call this an 'elementary magnetic bridge' between theory 1 and theory 2. Since we may join either of the two ends of each quiver, there exist four different magnetic bridges between two theories, and also four different electric bridges.

Before describing the bridge, let us first construct the partition pair $(\rho, \hat{\rho})$ in the decoupled case, $n = 0$. Together the two quivers have $(N_1 + N_2)$ D5-branes and $(\hat{N}_1 + \hat{N}_2)$ NS5-branes, so the grid containing ρ must have dimensions $(N_1+N_2)\times(\hat{N}_1+\hat{N}_2)$, and the grid containing $\hat{\rho}$ must have dimensions $(\hat{N}_1 + \hat{N}_2) \times (N_1 + N_2)$. The partitions corresponding to the product theory are shown in figure 4.5 . Their Young diagrams contain all the boxes in the black upper-left blocks of the grids, and none of the boxes in the white lower-right blocks. The off-diagonal blocks contain the diagrams of theory 1 and 2, as shown in the figure.

To construct the magnetic quiver of the composite theory one looks at eqs. (4.2.4). The first $(N_1 - 1)$ nodes reproduce the quiver of theory 1, but at the next node one finds a gauge group of zero rank, $n_{N_1} = \hat{L}^T_{N_1} - L_{N_1} = 0$. The remaining nodes, $j > N_1$, reproduce the quiver of theory 2. The fact that the bridge has $n = 0$ rank means that the partitions ρ and $\hat{\rho}$ fail to obey strict ordering at the N_1^{th} node where the theories decouple. [3]

is only defined modulo large gauge transformations, and can be shifted by $N\hat{N}$. This shift changes the charge to $-A$ consistently with the fact that C reverses the orientation of the D3-branes.

[3]If n were negative, the corresponding node would have anti-D3 branes breaking supersymmetry.

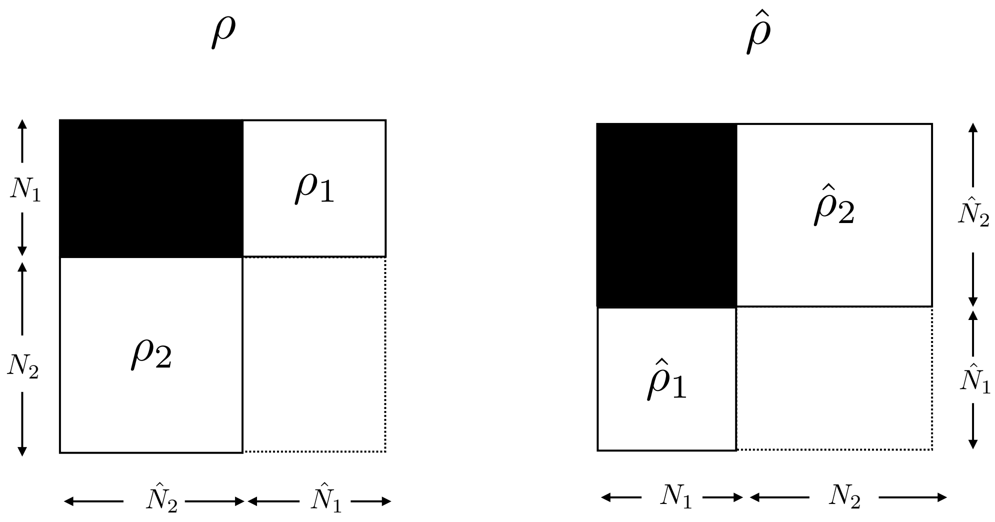


Figure 4.5: The Young diagrams $(\rho, \hat{\rho})$ corresponding to two decoupled theories $(\rho_1, \hat{\rho}_1)$ and $(\rho_2, \hat{\rho}_2)$.

It should now be clear how to create a bridge. We must crank up the rank of the N_1^{th} gauge-group factor by rearranging a few boxes of these diagrams. The rearrangement should restore the strict inequalities $\rho^T > \hat{\rho}$ that characterize irreducible quivers. To visualize the construction of the bridge let us assume that the Young diagrams of the original theories are rectangular blocks. [4] It will soon become clear that the construction is general and does not depend on this simplifying assumption. For now take ρ_p $(p = 1, 2)$ to be rectangular $N_p \times l_{1,p}$ blocks, and $\hat{\rho}_p$ rectangular $\hat{N}_p \times \hat{l}_{1,p}$ blocks, where $l_{1,p}$ and $\hat{l}_{1,p}$ are the sizes of the longest rows, i.e. with our simplifying assuption of all rows. Recall that these lengths are bounded respectively by $\hat{N}_p$ and N_p.

Figure 4.6 shows the Young diagrams ρ and $\hat{\rho}^T$ before and after the construction of a bridge. The initial diagrams have the general form of figure 4.5. A magnetic bridge can be constructed by moving n boxes of the diagram ρ from the N_1^{th} to the $(N_1 + 1)^{th}$ row. This increases the rank of the N_1^{th} node from 0 to n, leaving all other quantum numbers in the magnetic quiver unchanged, see equations (4.2.4). The rank of the gauge group at the connecting node is bounded by

$$2n \le \hat{N}_2 - l_{1,2} + l_{1,1} \ . \tag{4.3.1}$$

Since the right-hand-side is at least equal to 2, elementary bridges are always allowed. For $n > 1$ there exist several rearrangements of the boxes that respect the strict ordering. They all look indistinguishable at leading order in $n/N\hat{N}$ as will become clear in the following sections. The elementary $n = 1$ bridge requires the rearrangement of a single box and can be considered as the quantum of a gate.

The reader can easily convince herself that the simplifying assumption of rectangular Young diagrams plays no role, and that elementary magnetic bridges between good theories always exist. It is also straightforward to exhibit the electric quiver of the composite theory with a

[4] In the quiver theories, this assumption leads to single-factor flavor groups. In the dual type-IIB solutions, it corresponds to single stacks of 5-brane sources of each type.

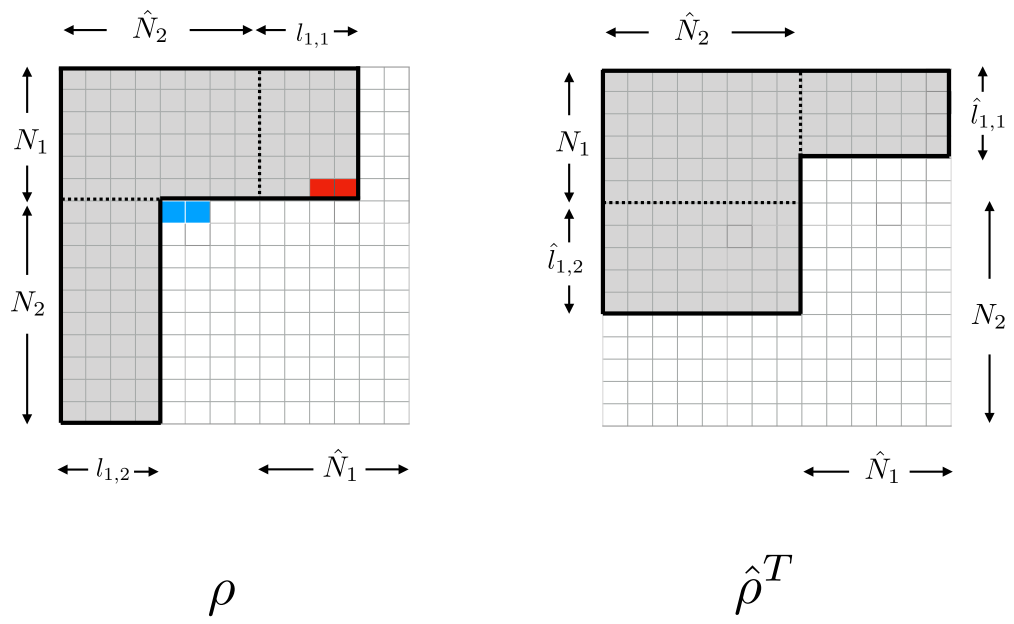


Figure 4.6: The Young diagrams ρ and $\hat{\rho}^T$ obtained by merging two single-stack quivers, as discussed in the text. A magnetic bridge is created by moving n boxes from the red to the blue positions in the diagram ρ, while leaving $\hat{\rho}$ the same. [In this example $N_1 = \hat{N}_1 = 6, N_2 = 10, \hat{N}_2 = 8$ and $l_{1,1} = l_{1,2} = \hat{l}_{1,1} = 4$, $\hat{l}_{1,2} = 5$. The original diagrams contain $A_1 = 24$ and $A_2 = 40$ boxes, so for the merged diagrams $A = 64$] .

magnetic bridge, but its detailed form is not particularly illuminating. The new bridge has small readjustments of both gauge-group and flavor-group ranks at several nodes of the originally-decoupled electric quivers.

4.4 Geometry of the gates

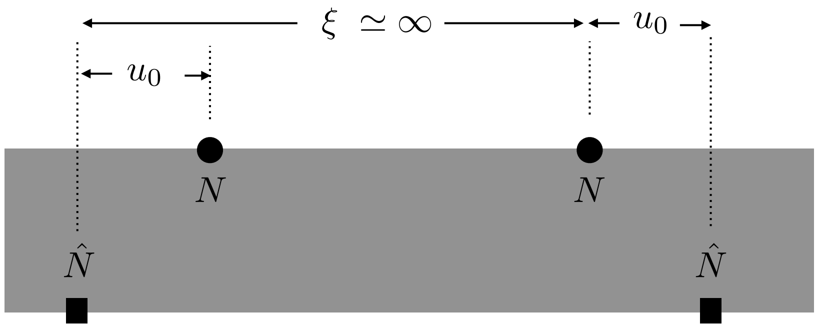
The type IIB solutions dual to the $\mathcal{N} = 4$ quiver theories were found in [12, 13]. The geometry has the warped form $(\mathrm{AdS}_4 \times \mathrm{S}^2 \times \hat{\mathrm{S}}^2) \times_w \Sigma$, with Σ the infinite strip $0 \leq \mathrm{Im} z \leq \pi/2$ [53, 55]. The S^2 fiber degenerates at the lower boundary of the strip and the $\hat{\mathrm{S}}^2$ fiber degenerates at the upper boundary, but these are mere coordinate singularities. Points where the AdS_4 fiber degenerates, on the other hand, are positions of 5-brane sources. The D5-branes which wrap the 2-sphere S^2 are localized at $z = \delta_j + \frac{i\pi}{2}$ on the upper boundary of Σ, while the NS5-branes which wrap the second sphere $\hat{\mathrm{S}}^2$ are localized at $z = \hat{\delta}_{\hat{j}}$ on the lower boundary. The relation of the five-brane positions to their linking numbers is [12]

$$l_j = \sum_{\hat{j}=1}^{\hat{N}} \vartheta(\hat{\delta}_{\hat{j}} - \delta_j) \ , \qquad \hat{l}_{\hat{j}} = \sum_{j=1}^{N} \vartheta(\hat{\delta}_{\hat{j}} - \delta_j) \ , \tag{4.4.1}$$

where ϑ is the function

$$\vartheta(u) = \frac{2}{\pi} \arctan(e^{-u}) \tag{4.4.2}$$

which extrapolates between 1 and 0 as u goes from $-\infty$ and ∞, and the five-brane singularities have been labeled in clockwise order in order to respect our convention that $\{l_j\}$ and $\{\hat{l}_{\hat{j}}\}$ are non-increasing sequences.



Rectangular Young diagrams correspond to solutions with a single stack of N D5-branes all at the same position $z = \delta + i\pi/2$, and a single stack of NS5-branes all at the same position $z = \hat{\delta}$. In this case (4.4.1) reduce to two equations

$$l = \hat{N}\,\vartheta(\hat{\delta} - \delta)\ , \qquad \hat{l} = N\,\vartheta(\hat{\delta} - \delta)\ , \tag{4.4.3}$$

which are related by the conservation law $Nl = \hat{N}\hat{l}$. Requesting that both linking numbers be integers can make this system of equations overconstrained. The general solution is of the form

$$l = \frac{\hat{N}m}{\text{gcd}}\ , \qquad \hat{l} = \frac{Nm}{\text{gcd}}\ , \qquad \text{where} \quad 0 < m < \text{gcd} \tag{4.4.4}$$

and gcd is the greatest common divisor of N and $\hat{N}$. If N and $\hat{N}$ are relatively prime there is no solution whatsoever, if $\text{gcd}(N, \hat{N}) = 2$ there is a unique isolated solution $m = 1 \Longleftrightarrow \hat{\delta} = \delta$ etc etc. The fact that the solutions to (4.4.3) depend on detailed arithmetic properties of N and $\hat{N}$ is physically unreasonnable, and is actually an artifact of the assumption of single-stack five-branes. By allowing the stacks to split one finds a large number of nearby solutions when the five-brane charges N and $\hat{N}$ are large.

Let us assume now that we have found a solution of (4.4.3) with $\delta - \hat{\delta} = u_0$. To describe two decoupled quiver theories we take two copies of the above five-brane stacks with infinite separation along the Rez axis as in figure 4.7 . To simplify the calculation, we take the symmetric arrangement shown in the figure: two stacks of N D5-branes are separated by $\xi - u_0$, and two stacks of $\hat{N}$ NS5-branes are separated by $\xi + u_0$, so that the entire configuration is invariant under reflection of the Rez- axis.

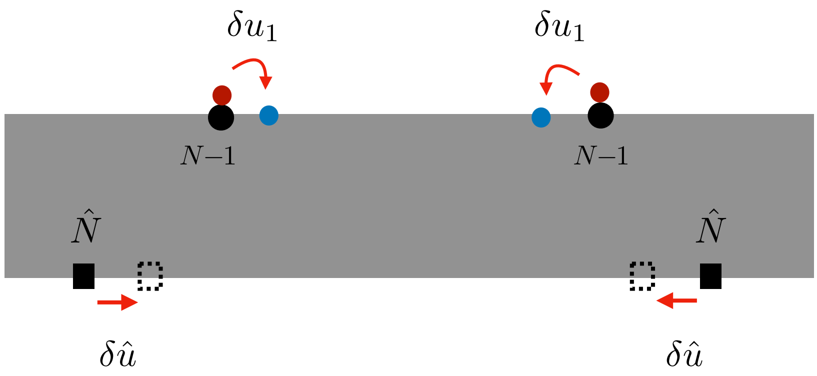


Figure 4.7: The initial geometry (upper part of the figure) which is dual to the two decoupled quiver theories has singularities separated by $\xi = \infty$. The quantum bridge obtained by the rearrangement of boxes in ρ shown in figure 4.6 corresponds to taking large but finite ξ and making the moves shown in the lower part of the above figure. The entire NS5-brane stacks, and one brane detached from each D5-brane stack, are respectively displaced by $\delta\hat{u}$ and δu_1 towards the center of the strip.

Using equations (4.4.1) one finds in the $\xi = \infty$ limit

$$l_1 = \hat{N}(1+\vartheta_0)\,, \quad l_2 = \hat{N}(1-\vartheta_0)\,, \quad \hat{l}_1 = N(2-\vartheta_0)\,, \quad \hat{l}_2 = N\vartheta_0\ . \tag{4.4.5}$$

where $\vartheta_0 = \frac{2}{\pi}\arctan(\exp(-u_0))$. These linking numbers match those of the Young diagrams for two decoupled quivers, see figure 4.6, if one identifies $N_1 = N_2 = N$, $\hat{N}_1 = \hat{N}_2 = \hat{N}$, and

$$l_{1,1} = \hat{N}\vartheta_0\,, \quad \hat{l}_{1,1} = N\vartheta_0\,, \quad l_{1,2} = \hat{N}(1-\vartheta_0)\,, \quad \hat{l}_{1,2} = N(1-\vartheta_0)\ .$$

Notice that theory 2 is the C-transform of theory 1 defined in figure 4.4. This is expected since the two theories are obtained by Rez reflection from each other. Of course the $\mathcal{N}=4$ theory is self-conjugate, so our choice of relative orientation just indicates by which ends we chose to join the two decoupled quivers.

We want now to find a new solution obtained from this initial configuration by (i) taking ξ large but finite, and (ii) making some small five-brane moves. The two moves that create the elementary bridge of the previous section are shown in the lower part of figure 4.7. The entire NS5-brane stacks are displaced by $\delta\hat{u}$ towards the center of the figure, while only a single D5-brane is detached from each D5-brane stack and displaced by δu_1 in the same direction. To match the Young diagrams of figure 4.6, all linking numbers except those of the detached D5 branes should stay the same after these moves, while the detached D5-branes should transfer n units of linking number to each other. This gives three equation for the three unknown parameters ($\xi, \delta\hat{u}$ and δu_1) of the new solution

$$-\hat{N}(\delta u_1 - \delta\hat{u})\sin\pi\vartheta_0 \simeq -\pi n\ , \qquad \hat{N}\delta\hat{u}\sin\pi\vartheta_0 - 2\hat{N}e^{-\xi} \simeq 0\ ,$$

$$N\delta\hat{u}\sin\pi\vartheta_0 - \delta u_1\sin\pi\vartheta_0 + 2Ne^{-\xi} \simeq 0\ , \tag{4.4.6}$$

where we have neglected terms that are subleading in the limit $N, \hat{N} \gg n$. The solution of these leading-order equations is

$$e^{-\xi} \simeq \frac{\pi n}{4N\hat{N}}\,, \qquad \delta\hat{u} \simeq \frac{\pi n}{2N\hat{N}\sin\pi\vartheta_0}\,, \qquad \delta u_1 \simeq \frac{\pi n}{\hat{N}\sin\pi\vartheta_0}\,. \tag{4.4.7}$$

Note that all dispacements are proportional to the rank n of the additional gauge group in the magnetic quiver, and that the displacement of the detached D5-branes is parametrically larger than that of the NS5-branes in the large $N, \hat{N}$ limit.

The metric of the ten-dimensional type-IIB solution is [53, 55]

$$\frac{4}{\alpha'} ds^2 = L_4^2 \, ds^2_{\rm AdS} + f^2 \, ds^2_{(1)} + \hat{f}^2 \, ds^2_{(2)} + 4\rho^2 \, dz d\bar{z} \ , \tag{4.4.8}$$

where $ds^2_{(i)} = d\vartheta_i^2 + \sin\vartheta_i^2 d\varphi_i^2$ are the metrics of the unit-radius 2-spheres, $ds^2_{\rm AdS}$ is the metric of the unit-radius AdS$_4$ spacetime, α' is the Regge slope parameter, and the four scale factors are given by

$$L_4^8 = 16 \, \frac{\mathcal{U}\hat{\mathcal{U}}}{W^2} \ , \quad f^8 = 16 h^8 \, \frac{\hat{\mathcal{U}} W^2}{\mathcal{U}^3} \ , \quad \hat{f}^8 = 16 \hat{h}^8 \, \frac{\mathcal{U} W^2}{\hat{\mathcal{U}}^3} \ , \quad \rho^8 = \frac{\mathcal{U}\hat{\mathcal{U}} W^2}{h^4 \hat{h}^4} \ . \tag{4.4.9}$$

In the above expressions

$$W = \partial_z \partial_{\bar{z}} (h\hat{h}) \ , \qquad \mathcal{U} = 2h\hat{h} |\partial_z h|^2 - h^2 W \ , \qquad \hat{\mathcal{U}} = 2h\hat{h} |\partial_z \hat{h}|^2 - \hat{h}^2 W \tag{4.4.10}$$

and h_1, $\hat{h}$ are harmonic functions on the z-strip obtained by summing, respectively, over the D5-brane and the NS5-brane singularities. For the configuration of figure 4.7 these harmonic functions read: [12]

$$\begin{aligned} h = \ & -(N-1) \log\tanh\left(\frac{i\pi}{4} - \frac{z}{2} + \frac{\xi - u_0}{4}\right) - (N-1) \log\tanh\left(\frac{i\pi}{4} - \frac{z}{2} - \frac{\xi - u_0}{4}\right) \\ & - \log\tanh\left(\frac{i\pi}{4} - \frac{z}{2} + \frac{\xi - u_0}{4} - \frac{\delta u_1}{2}\right) - \log\tanh\left(\frac{i\pi}{4} - \frac{z}{2} - \frac{\xi - u_0}{4} + \frac{\delta u_1}{2}\right) \ + \ c.c. \ , \end{aligned}$$

$$\hat{h} = -\hat{N} \log\tanh\left(\frac{z}{2} - \frac{\xi + u_0}{4} + \frac{\delta\hat{u}}{2}\right) - \hat{N} \log\tanh\left(\frac{z}{2} + \frac{\xi + u_0}{4} - \frac{\delta\hat{u}}{2}\right) \ + \ c.c. \ . \tag{4.4.11}$$

The solutions also have a non-trivial dilaton

$$e^{\Phi} = \left(\frac{\hat{\mathcal{U}}}{\mathcal{U}}\right)^{1/4} \ , \tag{4.4.12}$$

Setting $z = x + iy$ and expanding these harmonic functions near the center of the strip ($|x| \ll \xi$) gives after a little calculation

$$h \simeq 8Ne^{-\xi/2}\cosh x \sin y \ , \qquad \hat{h} \simeq 8\hat{N}e^{-\xi/2}\cosh x \cos y \ , \tag{4.4.13}$$

where we dropped terms of order $O(e^{-3|\xi-x|/2})$ which are subleading in the $\xi \to \infty$ limit. Plugging these expansions in (4.4.8)-(4.4.10) gives the $\mathrm{AdS}_5 \times \mathrm{S}^5$ metric expressed as an AdS_4 foliation over x. The radius L and the constant dilaton Φ_0 read

$$L^4 = 4\pi\alpha'^2\, n \ , \qquad e^{\Phi_0} = \left(\frac{\hat{N}}{N}\right)^{1/4} \ . \tag{4.4.14}$$

As expected, the radius only depends on the number n of D3-branes that created the AdS_5 throat/bridge. We are here working in units $g_s = 1$ where the NS5-branes and the D5-branes have equal tension. The AdS_5 throat does not of course extend out to infinity, it is cut off at $x \sim \pm\xi/2$ where the AdS_5 boundary is capped.

4.5 Mixing of the gravitons

We will compute the mixing of the gravitons in the regime $1 \ll n \ll N\hat{N}$, in which the bridge is thin compared to the AdS spacetimes on either side, but supergravity can be trusted. The general expression for the spectrum of spin-2 excitations in any warped supergravity background was given in [20]. The relevant eigenvalue problem depends only on the metric $g_{(6)}$ of the compact space $\mathcal{M}_6$, and on the warp factor $e^A \equiv \rho_4$. The mass-squared operator and the norm of wavefunctions read

$$M^2 = -\frac{e^{-2A}}{\sqrt{g_{(6)}}}\,\partial_a\sqrt{g_{(6)}}\,e^{4A}g^{ab}\partial_b \ , \qquad \|\psi\|^2 \ = \int_{\mathcal{M}_6}\sqrt{g_{(6)}}\,e^{2A}\,\psi^*\psi \ , \tag{4.5.1}$$

where ψ is a scalar wavefunction on $\mathcal{M}_6$. Here M^2 is the dimensionless mass, which is the eigenvalue of the Lichnerowicz Laplacian (the spin-2 wave operator) on the unit-radius AdS_4 spacetime. It is related to the scaling dimension of the dual operator by the well-known formula $\Delta(\Delta-3) = M^2$. For the case at hand $\mathcal{M}_6 = (\mathrm{S}^2 \times \mathrm{S}^{2\prime}) \times_w \Sigma$, and using our expressions for the scale factors we find:

$$\|\psi\|^2 \ = \ (4\pi)^2\int_\Sigma dxdy\,(4\rho^2 f^2\hat{f}^2L_4^2)\,|\psi|^2 \ = \ 2^9\pi^2\int_\Sigma dxdy\,h\hat{h}\,|\bar{\partial}\partial(h\hat{h})|\,|\psi|^2 \ , \tag{4.5.2}$$

$$\langle\psi|M^2|\psi\rangle = (4\pi)^2\int_\Sigma dxdy\,(4f^2\hat{h}^2L_4^4)\,(\bar{\partial}\psi^*)\partial_z\psi \;=\; 2^{10}\pi^2\int_\Sigma dxdy\,(h_1h_2)^2\,|\partial_z\psi|^2\;. \tag{4.5.3}$$

These expressions are valid for any of the AdS$_4$ solutions in [12, 13], we will now specialize to the nearly-factorized configurations (4.4.11).

Consider first the decoupling limit $\xi\to\infty$. Each AdS$_4$ spacetime has a massless graviton with constant wavefunction ψ_0, and a tower of massive excitations with $M\sim O(1)$. The normalized wavefunction of the massless graviton is

$$\psi_0 = V_6^{-1/2} \quad \text{with} \quad V_6 = 2^9\pi^2\int_\Sigma dxdy\,h\hat{h}\left|\partial\bar{\partial}(h\hat{h})\right| \;:=\; (N\hat{N})^2 v_6\;. \tag{4.5.4}$$

Here v_6 is a number $\sim O(1)$ that depends on the details of each decoupled theory, and whose precise value is not important. It can be computed by keeping in $h,\hat{h}$ only the five-branes near $x\sim\xi/2$ for the theory on the right of the bridge, or only those near $x\sim-\xi/2$ for the theory on the left. In the example the two theories are identical.

It is useful to express this compactification volume in terms of an effective four-dimensional gravitational coupling. Following ref. [14] one defines a consistent truncation to four-dimensional gravity with effective action $S_{\rm eff} = -(1/2\kappa_4^2)\int d^4x\sqrt{g_{(4)}}\,(R_{(4)}+6)$ which admits the unit-radius AdS$_4$ as solution. The relation of κ_4 to V_6 is

$$\kappa_4^2 \;=\; \kappa_{10}^2\,V_6^{-1}(\frac{\alpha'}{4})^{-4}\;, \qquad \text{where} \quad 2\kappa_{10}^2 = (2\pi)^7(\alpha')^4 \tag{4.5.5}$$

is the type-IIB gravitational coupling. This parametrization is particularly convenient when comparing the on-shell supergravity action with the free energy of the quiver gauge theory on the 3-sphere [14]. [5]

Let us consider next the configuration with a bridge. The two previously massless gravitons will now mix, so that the graviton with constant wavefunction ψ_0 remains massless, while the orthogonal combination ψ_1 obtains a small mass. To find these new wavefunctions, note that the AdS$_5\times$S^5 bridge makes a parametrically-small contribution to the compactification volume. Indeed, cutting off the throat at $x=\pm x_0$ we find

$$\text{Volume}_{(\text{throat})} \;\sim\; L^8\int_{-x_0}^{x_0}\cosh^4 x\,dx \;\sim\; n^2e^{4x_0}\;, \tag{4.5.6}$$

[5] When comparing to [14] and to earlier references, note that we have here rescaled the harmonic functions by $\alpha'/4$, so that the coefficients of the $\log\tanh$ contributions are integer.

which should be compared to the volume of the five-brane regions $\sim (N\hat{N})^2$. From (4.4.7) one sees that the two volumes are of the same order if $x_0 \simeq \xi/2$, i.e. when the AdS$_5$ cutoff reaches the five-brane regions, as should be expected. Here we take $\xi/2 \gg x_0 \gg 1$ so that the throat volume stays parametrically small and can be ignored. The two wavefunctions at this leading order are then given by

$$\psi_0 \simeq (2V_6)^{-1/2} \; , \qquad \psi_1 \;\simeq\; \begin{cases} (2V_6)^{-1/2} & \text{for} \quad x > x_0 \, , \\ \;\psi_1(x) & \text{for} \quad -x_0 < x < x_0 \, , \\ -(2V_6)^{-1/2} & \text{for} \quad x < -x_0 \; . \end{cases} \tag{4.5.7}$$

Here $\psi_1(x)$ is an interpolating function in the throat region which must be chosen so as to minimize the mass. Note that under reflection $x \to -x$, ψ_0 is even and ψ_1 is odd as in the double-well potential of quantum mechanics.

From (4.5.3) it follows that the only contribution to the mass of the ψ_1 state comes from the throat region where the geometry is approximately AdS$_5\times$S^5,

$$L^{-2}ds^2 \;\simeq\; dx^2 + \cosh^2 x \, ds^2(\mathrm{AdS}_4) + ds^2(\mathrm{S}^5) \; .$$

The function ψ_1 that minimizes the mass in this cut-off AdS$_5$ throat is a solution to the differential equation

$$\frac{d}{dx}(\cosh^4 x \, \frac{d\psi_1}{dx}) = 0 \quad \Longrightarrow \quad \psi_1(x) \simeq \frac{3}{2}(\tanh x - \frac{1}{3}\tanh^3 x)(2V_6)^{-1/2} \; . \tag{4.5.8}$$

In infinite AdS$_5$ spacetime this would have been a non-normalizable solution, but in our capped off geometry it is normalized by imposing a smooth interpolation between the two asymptotic values $\pm(2V_6)^{-1/2}$. Inserting this wavefunction in (4.5.3) and using the harmonic functions (4.4.13) leads to the following expression for the mass

$$\langle\psi_1|M^2|\psi_1\rangle \simeq 2^{16}\pi^3(N\hat{N}e^{-\xi})^2\int_{-x_0}^{x_0} dx \; \cosh^4 x(\frac{d\psi_1}{dx})^2 \simeq 2^{16}\pi^3(N\hat{N}e^{-\xi})^2 \times \frac{3}{2V_6} \; . \tag{4.5.9}$$

Using finally (4.4.7) and the relation (4.5.5) of V_6 to the effective gravitational coupling we arrive at the main result of this paper:

$$M^2 \;=\; \frac{3}{8\pi^2}\,\kappa_4^2\, n^2 \qquad (n = 1, 2, \cdots) \; . \tag{4.5.10}$$

If one restores the AdS$_4$ radius R in this formula, one finds $M^2 = (3G_N/\pi R^4)\, n^2$, where G_N is the four-dimensional Newton's constant.

It is straightforward to extend this calculation to the case of a bridge connecting AdS_4 Universes of unequal size. The properly normalized wavefunction orthogonal to the massless graviton in this case reads

$$\mathcal{N}^{-1}\psi_1(x) \;\simeq\; \frac{3}{4}(V_6' + V_6)(\tanh x - \frac{1}{3}\tanh^3 x) + \frac{1}{2}(V_6' - V_6)\ ,$$

$$\text{where} \qquad \mathcal{N}^{-1} = \sqrt{V_6 V_6'(V_6 + V_6')} \tag{4.5.11}$$

and V_6' (V_6) is the compactification volume of the Universe on the left (right) side of the bridge. Note that this wavefunction extrapolates between $\mathcal{N}\,V_6'$ at $x \to \infty$, and $-\mathcal{N}\,V_6$ at $x \to -\infty$. Inserting it in the expression for the mass gives

$$M^2 = \frac{3}{16\pi^2}(\kappa_4^2 + \kappa_4'^2)\, n^2\ , \tag{4.5.12}$$

where κ_4 an d κ_4' are the effective gravitational couplings for the two theories. For identical Universes this reduces to (4.5.10). Note that for unequal Universes the mixing is dominated by the smaller Universe whose effective Newton's constant is the strongest.

4.6 Bimetric and Massive AdS_4 gravity

In this section, we will initially present a generalization of the main result of this chapter, based on the work carried out in the previous chapter. In particular, we will consider the case where the two $AdS_4 \times M_6$ spacetimes are connected through a Janus semi-infinite throat insted of an $AdS_5 \times S^5$ one. Finally, it is going to be verified for our solutions that the massive AdS_4 gravity can be obtained from bigravity in a spacial decoupling limit.

We consider The manifold M_6 now consists of a Janus throat capped-off on both sides by two bags, $\overline{\mathrm{M}}_6$ and $\overline{\mathrm{M}}_6{}'$. For economy of notation we introduce the parameters

$$v := \int_{\overline{\mathrm{M}}_6} \sqrt{g}\, L_4^2 \qquad \text{and} \qquad v' := \int_{\overline{\mathrm{M}}_6'} \sqrt{g}\, L_4^2 \ .$$

Note that v is just a short-hand for the parameter $V_6 \langle L_4^2 \rangle_{\mathrm{bag}} = \psi_{\mathrm{bag}}^{-1/2}$ of the previous chapter. Using the inner product $\langle \psi_1 | \psi_2 \rangle = \int_{\mathrm{M}_6} \sqrt{g}\, L_4^2\, \psi_1^* \psi_2$ one finds easily two orthogonal, low-lying spin-2 states. A massless state with constant wavefunction throughout M_6 (which is normalizable because M_6 is now compact), and a massive state whose wavefunction is approximately constant in the bags,

$$\psi_0(x) \simeq (v+v')^{-1/2} \times \begin{cases} \sqrt{v'/v} & \text{in } \overline{\mathrm{M}}_6, \\ -\sqrt{v/v'} & \text{in } \overline{\mathrm{M}}_6' \ . \end{cases} \tag{4.6.1}$$

Since the throat makes a subleading contribution to the inner product, the above wavefunction is clearly orthogonal to the constant one, i.e. to the wavefunction of the massless graviton. This second mode is necessarily massive because ψ_0 is forced to vary inside the Janus throat in order to extrapolate between the above values at the exits.

By repeating the calculation carried out in the previous chapter regarding the minimizing wavefunction in the Janus throat, with the only difference in the boundary conditions, we obtain the result:

$$\psi_0 \ \simeq \ \frac{1}{2\sqrt{vv'(v-v')}} \left[(v'-v) - (v'+v) \frac{I(x,a)}{I(\infty,a)} \right] \ , \tag{4.6.2}$$

where $I(x,a)$ has been defined in eq. (3.3.23). Inserting the above wavefunction in (3.3.25), and reexpressing v and v' in terms of radii and effective couplings gives

$$m_{\mathrm{g}}^2\, L_4^2 \ = \ \frac{3\, n^2}{16\pi^2} \left[\frac{\kappa_4^2}{\langle L_4^2 \rangle_{\mathrm{bag}}} + \frac{\kappa_4^2{}'}{\langle L_4^2 \rangle_{\mathrm{bag}'}} \right] \times J(\cosh \delta\phi) \ . \tag{4.6.3}$$

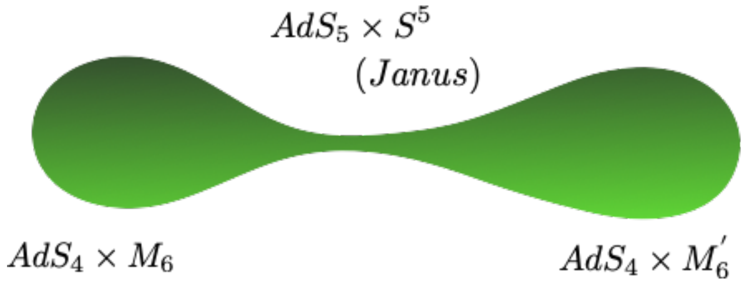


Figure 4.8: Two universes, $AdS_4 \times M_6$ and $AdS_4 \times M_6'$ connected via a thin throat with Janus or $AdS_5 \times S^5$ geometry. When initially decoupled, each spacetime includes a massless low-lying graviton. After the coupling, the two gravitons mix, resulting to one massive and one massless combination.

This result agrees with the one of the previous section, if we go back to the $AdS_5 \times S^5$ case by setting the dilaton variation to zero and hence for $J(\cosh(\delta\phi) = 0)$. Moreover, in this calculation we have reintroduced the warp factor L_4 in the formulae, as throughout the previous sections of this chapter we chose to work in units of $L_4 = 1$ and absorbed $\langle L_4^2 \rangle$ in the definition of κ_4^2.

Apparently how this result reduces to the formula for the graviton mass (3.3.25) in the decoupling limit $\kappa_4' \to 0$ or equivalently $\langle L_4^2 \rangle_{\mathrm{bag}'} \to \infty$. In this limit the massless graviton has vanishing wavefunction and decouples, whereas ψ_0 is concentrated entirely in the (unprimed) bag $\overline{\mathrm{M}}_6$ and in the throat.

From the perspective of the dual field theory, these bigravity solutions are not $4d$ defect CFTs, but rather $3d$ CFTs of a special kind. They are superconformal gauge theories based on linear quivers with a low-rank 'weak' node [22]. Removing this node breaks the quiver into two disjoint quivers. One could in principle integrate out the scarce messenger fields, thereby generating multitrace couplings between disjoint theories in the spirit of [85][4]. In contrast with these references, the couplings are however non-local (they are generated by massless messengers) and exactly scale invariant (the AdS_4 symmetry is manifest). Conversely, integrating back in the messenger fields restores the interpretation of the multitrace couplings in terms of a classical supergravity background, and resolves the conflicts with string-theory locality discussed in refs. [3][2].

Similar comments apply to the relation of our models with the transparent boundary conditions of [101][61]. These could conceivably mimic the effects of the semi-infinite throats, but they are obscuring the issues of locality and scale invariance. It is nevertheless interesting that they lead to the same parametric dependence of m_{g} on the effective gravitational coupling κ_4.

4.7 Concluding Remarks

We may compare our result for graviton mixing with the one obtained by Aharony et al [4] in the double-trace deformation model. Their field theory calculation gives a mass that depends on a continuous double-trace coupling h (in which we reabsorbed numerical factors) and on the central charges of the two theories via the combination

$$M^2 = h^2(\frac{1}{c_1} + \frac{1}{c_2}) \ . \tag{4.7.1}$$

This is of the same form as (4.5.11) if one notes that the central charges c_1, c_2, defined as the coefficients in the two-point function of the energy-momentum tensors, can be identified with κ_4^{-2} and $\kappa_4'^{-2}$. The important difference is that in our model h is quantized. It would be interesting to see if this quantization rule can be also found by studying RG flows in the space of double-trace coupling. Note that if one views the quantum bridge as the minimal allowed coupling between two mutually-hidden sectors of a theory, the quantization of charge ensures that the mixing cannot be weaker than $\sim \kappa_4 \kappa_4'$, in harmony with the general spirit of the weak gravity conjecture.

To an observer in Universe 1 the gate looks like a D3-brane with AdS worldvolume. By conservation of five-form flux, the exit looks like an anti-D3 brane in Universe 2. Since the two Universes are invariant under charge-conjugation, only an observer travelling through the throat can compare the charges of entry and exit.

The D3-branes are special because they have a regular extremal horizons, but other defects can serve as entries and exits of a bridge. The simplest case is that of a D-instanton, which was identified as a wormhole solution of type-IIB supergravity in [68] and should be revisited in the light of our present discussion. Another interesting question was raised by the recent paper which counted the number of conserved energy-momentum tensors in class-S theories by means of an index. It would be interesting to find a way of counting the number of nearly conserved energy-momentum tensors, i.e. of the dual spin-2 gravitons with mass much below the mass gap of $O(1)$.

Finally an obvious question is whether, like D-branes, quantum gates can also be described on the string worldsheet by a modification of the rules of string perturbation theory. Ideas include sigma models that flow to topological theories in the infrared [17], zero size worm-holes in the 2d gravity of the worldsheet [3], or worldsheets with conformal interfaces [24]. Viewing the gates as weak quiver links may give a new breadth to these earlier efforts.

Part III

$T^{\hat{\rho}}_{\rho}[SU(N)]$ Superconformal Manifolds

Chapter 5

Exactly marginal Deformations

This chapter includes the article "*Marginal Deformations of 3d $\mathcal{N}=4$ Linear Quiver Theories*" by C. Bachas, Bruno Le Floch and the author [23], along with an extra appendix E.1.

Abstract We study superconformal deformations of the $T_\rho^{\hat\rho}[SU(N)]$ theories of Gaiotto-Hanany-Witten, paying special attention to mixed-branch operators with both electrically -and magnetically-charged fields. We explain why all marginal $\mathcal{N}=2$ operators of an $\mathcal{N}=4$ CFT$_3$ can be extracted unambiguously from the superconformal index. Computing the index at the appropriate order we show that the mixed moduli in $T_\rho^{\hat\rho}[SU(N)]$ theories are double-string operators transforming in the (Adj,Adj) representation of the electric and magnetic flavour groups, up to some overcounting for quivers with abelian gauge nodes. We comment on the holographic interpretation of the results, arguing in particular that gauged supergravities can capture the entire moduli space if, in addition to the (classical) parameters of the background solution, one takes into account the (quantization) moduli of boundary conditions.

5.1 Introduction

Superconformal field theories (SCFT) often have continuous deformations preserving some superconformal symmetry. The space of such deformations is a Riemannian manifold (the 'superconformal manifold') which coincides with the moduli space of supersymmetric Anti-de Sitter (AdS) vacua when the SCFT has a holographic dual. Mapping out such moduli spaces is of interest both for field theory and for the study of the string-theory landscape.

In this paper we will be interested in superconformal manifolds in the vicinity of the 'good' theories $T_\rho^{\hat\rho}[SU(N)]$ whose existence was conjectured by Gaiotto and Witten [65]. These are three-dimensional $\mathcal{N}=4$ SCFTs arising as infrared fixed points of a certain class of quiver gauge theories introduced by Hanany and Witten [73]. Their holographic duals are four-dimensional Anti-de Sitter (AdS$_4$) solutions of type-IIB string theory [12]-[91]. Our main motivation in this work was to extract features of these moduli spaces not readily accessible from the gravity side. We build on the analysis of ref. [19] which we complete and amend in significant ways.

Superconformal deformations of a d-dimensional theory $T_\star$ are generated by the set of marginal operators $\{\mathcal{O}_i\}$ that preserve some or all of its supersymmetries. [1] The existence of such operators is constrained by the analysis of representations of the superconformal algebra [38]. In particular, unitary SCFTs have no moduli in $d=5$ or 6 dimensions, whereas in the case $d=3$ of interest here moduli preserve at most $\mathcal{N}=2$ supersymmetries. Those preserving only $\mathcal{N}=1$ belong to long ('D-term') multiplets whose dimension is not protected against quantum corrections.

[1]One exception to this general rule is the gauging of a global symmetry with vanishing β function in four dimensions.

The existence of such $\mathcal{N}=1$ moduli (and of non-supersymmetric ones) is fine tuned and thus accidental. For this reason we focus here on the $\mathcal{N}=2$ moduli.

The general local structure of $\mathcal{N}=2$ superconformal manifolds in three dimensions (and of the closely-related case $\mathcal{N}=1$ in $d=4$) has been described in [89]-[71]. These manifolds are Kähler quotients of the space $\{\lambda^i\}$ of marginal supersymmetry-preserving couplings modded out by the complexified global (flavor) symmetry group G_{global},

$$\mathcal{M}_{\text{SC}} \simeq \{\lambda^i\}/G^{\mathbb{C}}_{\text{global}} \simeq \{\lambda^i | D^a = 0\}/G_{\text{global}} \ . \tag{5.1.1}$$

The meaning of this is as follows: Marginal scalar operators $\mathcal{O}_i$ fail to be *exactly* marginal if and only if they combine with conserved-current multiplets of G_{global} to form long (unprotected) current multiplets. Requesting this not to happen imposes the moment-map conditions

$$D^a = \lambda^i T^a_{i\bar{j}} \bar{\lambda}^{\bar{j}} + O(\lambda^3) \ = \ 0 \ , \tag{5.1.2}$$

where T^a are the generators of G_{global} in the representation of the couplings. The second quotient by G_{global} in (5.1.1) identifies deformations that belong to the same orbit. The complex dimension of the moduli space is therefore equal to the difference

$$\dim \mathcal{M}_{\text{SC}} \ = \ \#\{\mathcal{O}_i\} - \dim G_{\text{global}} \ . \tag{5.1.3}$$

In the dual gauged supergravity (when one exists) the fields dual to $\mathcal{O}_i$ are $\mathcal{N}=2$ hypermultiplets, and $D^a = 0$ are D-term conditions [44].

The global flavour symmetry of the $T^{\hat{\rho}}_{\rho}[SU(N)]$ theories, viewed as $\mathcal{N}=2$ SCFTs, is a product

$$G_{\text{global}} = G \times \hat{G} \times U(1) \ , \tag{5.1.4}$$

where G and $\hat{G}$ are the flavour groups of the electric and magnetic theories that are related by mirror symmetry, and $U(1)$ is the subgroup of the $SO(4)_R$ symmetry which commutes with the unbroken $\mathcal{N}=2$. To calculate the local moduli space we must then list all marginal supersymmetric operators and the G_{global}-representation(s) in which they transform. Many of these deformations are standard superpotential deformations involving hypermultiplets of either the electric theory or its magnetic mirror. Some marginal operators involve, however, both kinds of hypermultiplets and do not admit a local Lagrangian description. We refer to such deformations as 'mixed'. They are specific to three dimensions, and will be the focus of our paper.

Marginal deformations belong to three kinds of $\mathcal{N}=4$ superconformal multiplet [19]. The electric and magnetic superpotentials belong, respectively, to spin-2 representations of $SO(3)_H$

and $SO(3)_C$, where $SO(3)_H \times SO(3)_C \simeq SO(4)_R$ is the $\mathcal{N} = 4$ R symmetry. [2] The mixed marginal operators on the other hand transform in the $(J^H, J^C) = (1, 1)$ representation. In the holographic dual supergravity the $(2, 0)$ and $(0, 2)$ multiplets describe massive $\mathcal{N} = 4$ vector bosons, while the $(1, 1)$ multiplets contain also spin-$\frac{3}{2}$ fields. These latter are also special for another reason: they are Stueckelberg fields capable of rendering the $\mathcal{N} = 4$ graviton multiplet massive [22][21]. In representation theory they are the unique short multiplets that can combine with the conserved energy-momentum tensor into a long multiplet. This monogamous relation will allow us to identify them unambiguously in the superconformal index.

More generally, one cannot distinguish in the superconformal index the contribution of the $\mathcal{N} = 2$ chiral ring, which contains scalar operators with arbitrary (J^H, J^C), from contributions of other short multiplets. Two exceptions to this rule are the pure Higgs- and pure Coulomb-branch chiral rings whose R-symmetry quantum numbers are $(J^H, 0)$ and $(0, J^C)$. The corresponding multiplets are absolutely protected, i.e. they can never recombine to form long representations of the $\mathcal{N} = 4$ superconformal algebra [39]. These two subrings of the chiral ring can thus be unambiguously identified. Their generating functions (known as the Higgs-branch and Coulomb-branch Hilbert series [72]-[41]) are indeed simple limits of the superconformal index [104]. Arbitrary elements of the chiral ring, on the other hand, are out of reach of presently-available techniques. [3] Fortunately this will not be an obstacle for the marginal $(1, 1)$ operators of interest here.

The result of our calculation has no big surprises. As we will show, the mixed marginal operators transform in the $(\mathrm{Adj}, \mathrm{Adj}, 0)$ representation of the global symmetry (5.1.4), up to some overcounting when (and only when) the quivers of $T^{\hat\rho}_{\rho}[SU(N)]$ have abelian gauge nodes. More generally, the set of all marginal $\mathcal{N} = 2$ operators is of the form

$$S^2(\mathrm{Adj}\, G + \mathrm{Adj}\, \hat{G}) \; + \; [\mathrm{length} - 4 \text{ strings}] \; - \; \text{redundant} \; , \tag{5.1.5}$$

where S^2 is the symmetrized square of representations, the 'length-4 strings' correspond to quartic superpotentials made out of the hypermultiplets of the electric or the magnetic theory only, while redundancies arise exclusively from the F-term conditions at abelian gauge nodes. Calculating these redundancies is the main technical result of our paper. On the way, we will provide also some new checks of $3d$ mirror symmetry.

Our calculation settles one issue about the dual AdS moduli that was left open in ref. [19]. As is standard in holography, the global symmetries G and $\hat{G}$ of the SCFT are realized as gauge symmetries on the gravity side. The corresponding $\mathcal{N} = 4$ vector bosons live on stacks of magnetized D5-branes and NS5-branes which wrap two different 2-spheres (S^2_H and S^2_C) in the ten-dimensional spacetime [12]. The R-symmetry spins J^H and J^C are the angular momenta on these spheres. As was explained in [19], the Higgs-branch superconformal moduli correspond to open-string states on the D5-branes: either non-excited single strings with $J^H = 2$, or bound states of two $J^H = 1$ strings. The Coulomb branch superconformal moduli correspond likewise to open D-string states on NS5-branes. For mixed moduli ref. [19] suggested two possibilities: either bound states of a $J^H = 1$ open string on the D5-branes with a $J^C = 1$ D-string from the NS5 branes, or single closed-string states that are scalar partners of massive gravitini. Our results here seem to rule out the second possibility, at least for the solutions dual to linear quivers. [4]

[2] $SO(3)_H$ and $SO(3)_C$ act on the chiral rings of the pure Higgs and pure Coulomb branches of the theory, whence their names. They are exchanged by mirror symmetry.

[3] Though there do exist some interesting suggestions [43][34] on which we will comment at the end of this paper.

[4] Of course the uncharged open-string states mix in the interacting theory with closed strings. A more precise statement is that, for linear quivers, the latter do not contribute new states to the $\Delta = 2$ chiral ring.

It was also noted in ref. [19] that although gauged $\mathcal{N}=4$ supergravity can in principle account for the $(2,0)$ and $(0,2)$ moduli that are scalar partners of spontaneously-broken gauge bosons, it has no massive spin-$\frac{3}{2}$ multiplets to account for single-particle $(1,1)$ moduli. But if all (1,1) moduli are 2-particle states, they can be in principle accounted for by modifying the AdS_4 boundary conditions along the lines of [115][29]. The dismissal in ref. [19] of gauged supergravity, as not capturing the entire moduli space, is therefore premature. Note however that changing the boundary conditions does not affect the classical AdS solution but only the fluctuations around it. Put differently these moduli show up only upon quantization. The analysis of AdS moduli spaces in gauged supergravity [44] must be extended to incorporate such 'quantization moduli.'

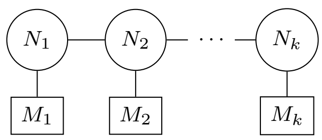
This paper is organized as follows: Section 5.2 reviews some generalities about good $T_\rho^{\hat\rho}[SU(N)]$ theories, and exhibits their superconformal index written as a multiple integral and sum over Coulomb-branch moduli and monopole fluxes. Our aim is to recast this expression into a sum of superconformal characters with fugacities restricted as pertaining to the index. These restricted characters and the linear relations that they obey are derived in section 5.3 . We also explain in this section why the ambiguities inherent in the decomposition of the index as a sum over representations do not affect us for the problem at hand.

Section 5.4 contains our main calculation. We first expand the determinants so as to only keep contributions from operators with scaling dimension $\Delta \leq 2$, and then perform explicitly the integrals and sums. The result is reexpressed as a sum of characters of $OSp(4|4)\times G\times \hat{G}$ in section 5.5 . We identify the superconformal moduli, comment on their holographic interpretation (stressing the role of a stringy exclusion principle) and conclude. Some technical material is relegated to appendices. Appendix **??** sketches the derivation of the superconformal index as a localized path integral over the Coulomb branch. This is standard material included for the reader's convenience. In appendix C we prove a combinatorial lemma needed in the main calculation. Lastly a closed-form expression for the index of $T[SU(2)]$, which is sQED_3 with two 'selectrons', is derived in appendix E.2 . This renders manifest a general property (which we do not use in this paper), namely the factorization of the index in holomorphic blocks [98]-[25].

5.2 Superconformal index of $T_\rho^{\hat\rho}[SU(N)]$

5.2.1 Generalities on $T_\rho^{\hat\rho}[SU(N)]$

We consider the 3d $\mathcal{N}=4$ gauge theories [73] based on the linear quivers of the figure. Circle nodes in these quivers stand for unitary gauge groups $U(N_i)$, squares designate fundamental hypermultiplets and horizontal links stand for bifundamental hypermultiplets. One can generalize to theories with



orthogonal and symplectic gauge groups and to quivers with non-trivial topology, but we will not consider such complications here. We are interested in the infrared limit of 'good theories' [65] for which $N_{j-1}+N_{j+1}+M_j \geq 2N_j \ \forall j$. These conditions ensure that at a generic point of the Higgs branch the gauge symmetry is completely broken.

The theories are defined in the ultraviolet (UV) by the standard $\mathcal{N}=4$ Yang-Mills plus matter $3d$ action. All masses and Fayet-Iliopoulos terms are set to zero and there are no Chern-Simons terms. We choose the vacuum at the origin of both the Coulomb and Higgs branches, where all scalar expectation values vanish. Thus the only continuous parameters of the theory are the dimensionful gauge couplings g_i, which flow to infinity in the infrared.

Every good linear quiver has a mirror which is also a good linear quiver and whose discrete data we denote by hats, $\{\hat{N}_{\hat{j}}, \hat{M}_{\hat{j}}, \hat{k}\}$. A useful parametrization of both quivers is in terms of an ordered pair of partitions, $(\rho, \hat{\rho})$ with $\rho^T > \hat{\rho}$, see appendix C . The SCFT has global (electric and magnetic) flavour symmetries,

$$G \times \hat{G} = (\prod_j U(M_j))/U(1) \ \times \ (\prod_{\hat{j}} U(\hat{M}_{\hat{j}}))/U(1) \ , \tag{5.2.1}$$

with $\operatorname{rank} G = \hat{k}$ and $\operatorname{rank} \hat{G} = k$. In the string-theory embedding the flavour symmetries are realized on $(\hat{k}+1)$ D5-branes and $(k+1)$ NS5-branes [73]. The symmetry G is manifest in the microscopic Lagragian of the electric theory, as is the Cartan subalgebra of $\hat{G}$ which is the topological symmetry whose conserved currents are the dual field strengths $\operatorname{tr}{}^*F_{(j)}$. The non-abelian extension of $\hat{G}$ is realized in the infrared by monopole operators [32][31].

In addition to $G \times \hat{G}$ the infrared fixed-point theory has global superconformal symmetry. The $\mathcal{N}=4$ superconformal group in three dimensions is $OSp(4|4)$. It has eight real Poincaré supercharges transforming in the $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ representation of $SO(1,2) \times SO(3)_H \times SO(3)_C$. The two-component 3d Lorentzian spinors can be chosen real. The marginal deformations studied in this paper leave unbroken a $\mathcal{N}=2$ superconformal symmetry $OSp(2|4) \subset OSp(4|4)$. This is generated by two out of the four real $SO(1,2)$ spinors, so modulo $SO(4)_R$ rotations the embedding is unique. Let $Q^{(\pm\pm)}$ be a complex basis for the four Poincaré supercharges, where the superscripts are the eigenvalues of the diagonal R-symmetry generators J_3^H and J_3^C. Without loss of generality we can choose the two unbroken supercharges to be the complex pair $Q^{(++)}$ and $Q^{(--)}$, so that the $\mathcal{N}=2$ R-symmetry is generated by $J_3^H + J_3^C$ and the extra commuting $U(1)$ by $J_3^H - J_3^C$. We use this same basis in the definition of the superconformal index.

5.2.2 Integral expression for the index

There is a large literature on the $\mathcal{N}=2$ superconformal index in three dimensions, for a partial list of references see [30]-[113]. The index is defined as a weighted sum over local operators of the SCFT, or equivalently over all quantum states on the two-sphere,

$$\mathcal{Z}_{S^2\times S^1} = \operatorname{Tr}_{\mathcal{H}_{S^2}} (-1)^F q^{\frac{1}{2}(\Delta+J_3)} t^{J_3^H - J_3^C} e^{-\beta(\Delta - J_3 - J_3^H - J_3^C)} \ . \tag{5.2.2}$$

In this formula F is the fermion number of the state, J_3 the third component of the spin, Δ the energy, and $q, t, e^{-\beta}$ are fugacities. Only states for which $\Delta = J_3 + J_3^H + J_3^C$ contribute to the index which is therefore independent of the fugacity β.

The non-abelian R symmetry guarantees (for good theories) that the $U(1)_R$ of the $\mathcal{N}=2$ subalgebra is the same in the ultraviolet and the infrared. We can therefore compute $\mathcal{Z}_{S^2\times S^1}$ in the UV where the $3d$ gauge theory is free. The index can be further refined by turning on fugacities for the flavour symmetries, and background fluxes on S^2 for the flavour groups In our calculation we will include flavour fugacities but set the flavour fluxes to zero.

The superconformal index eq. (5.2.2) is the appropriately twisted partition function of the theory on $S^2 \times S^1$. It can be computed by supersymmetric localization of the functional integral, for a review see ref. [113]. For each gauge-group factor $U(N_j)$ there is a sum over monopole charges $\{m_{j,\alpha}\} \in \mathbb{Z}^{N_j}$ and an integral over gauge fugacities (exponentials of gauge holonomies) $\{z_{j,\alpha}\} \in U(1)^{N_j}$. The calculation is standard and is summarized in appendix **??**. The result is most conveniently expressed with the help of the plethystic exponential (PE) symbol,

$$
\begin{aligned}
\mathcal{Z}_{S^2\times S^1} = \prod_{j=1}^{k}\left[\frac{1}{N_j!}\sum_{m_j\in\mathbb{Z}^{N_j}}\int\prod_{\alpha=1}^{N_j}\frac{dz_{j,\alpha}}{2\pi i z_{j,\alpha}}\right]&\Bigg\{(q^{\frac{1}{2}}t^{-1})^{\Delta(\mathbf{m})}\prod_{j=1}^{k}\Bigg[\prod_{\alpha=1}^{N_j} w_j^{m_{j,\alpha}}\prod_{\alpha\neq\beta}^{N_j}(1-q^{\frac{1}{2}|m_{j,\alpha}-m_{j,\beta}|}z_{j,\beta}z_{j,\alpha}^{-1})\Bigg]\\
\times\,\mathrm{PE}\Bigg(\sum_{j=1}^{k}\sum_{\alpha,\beta=1}^{N_j}&\frac{q^{\frac{1}{2}}(t^{-1}-t)}{1-q}\,q^{|m_{j,\alpha}-m_{j,\beta}|}z_{j,\beta}z_{j,\alpha}^{-1}\\
&+\frac{(q^{\frac{1}{2}}t)^{\frac{1}{2}}(1-q^{\frac{1}{2}}t^{-1})}{1-q}\sum_{j=1}^{k}\sum_{p=1}^{M_j}\sum_{\alpha=1}^{N_j}q^{\frac{1}{2}|m_{j,\alpha}|}\sum_{\pm}z_{j,\alpha}^{\mp1}\mu_{j,p}^{\pm1}\\
&+\frac{(q^{\frac{1}{2}}t)^{\frac{1}{2}}(1-q^{\frac{1}{2}}t^{-1})}{1-q}\sum_{j=1}^{k-1}\sum_{\alpha=1}^{N_j}\sum_{\beta=1}^{N_{j+1}}q^{\frac{1}{2}|m_{j,\alpha}-m_{j+1,\beta}|}\sum_{\pm}z_{j,\alpha}^{\mp1}z_{j+1,\beta}^{\pm1}\Bigg)\Bigg\}\,.
\end{aligned}
\tag{5.2.3}
$$

Here $z_{j,\alpha}$ is the S^1 holonomy of the $U(N_j)$ gauge field and $m_{j,\alpha}$ its 2-sphere fluxes (*viz.* the monopole charges of the corresponding local operator in $\mathbb{R}^3$) with α labeling the Cartan generators; $\mu_{j,p}$ are flavour fugacities with $p = 1, \cdots, M_j$, and w_j is a fugacity for the topological $U(1)$ whose conserved current is $\mathrm{tr}\,{}^*F_{(j)}$. The plethystic exponential of a function $f(v_1, v_2, \cdots)$ is given by

$$
\mathrm{PE}(f) = \exp\left(\sum_{n=1}^{\infty}\frac{1}{n}f(v_1^n, v_2^n, \cdots)\right)\,. \tag{5.2.4}
$$

Finally $\mathbf{m}$ denotes collectively all magnetic charges, and the crucial exponent $\Delta(\mathbf{m})$ reads

$$
\Delta(\mathbf{m}) = -\frac{1}{2}\sum_{j=1}^{k}\sum_{\alpha,\beta=1}^{N_j}|m_{j,\alpha}-m_{j,\beta}| + \frac{1}{2}\sum_{j=1}^{k}M_j\sum_{\alpha=1}^{N_j}|m_{j,\alpha}| + \frac{1}{2}\sum_{j=1}^{k-1}\sum_{\alpha=1}^{N_j}\sum_{\beta=1}^{N_{j+1}}|m_{j,\alpha}-m_{j+1,\beta}|\,. \tag{5.2.5}
$$

Note that the smallest power of q in any given monopole sector is $\frac{1}{2}\Delta(\mathbf{m})$. Since the contribution of any state to the index is proportional to $q^{\frac{1}{2}(\Delta+J_3)}$, we see that $\Delta(\mathbf{m})$ is the Casimir energy of the ground state in the sector $\mathbf{m}$, or equivalently the scaling dimension [and the $SO(3)_C$ spin] of the corresponding monopole operator [32][31]. As shown by Gaiotto and Witten [65] this dimension is strictly positive for all the good theories that interest us here. cf equations (2.16-17) in Gaiotto-Witten

We would now like to extract from the index (5.2.3) the number, flavour representations and $U(1)$ charges of all marginal $\mathcal{N} = 2$ operators. To this end we need to rewrite the index as a sum over characters of the global $OSp(4|4) \times G \times \hat{G}$ symmetry,

$$
\mathcal{Z}_{S^2\times S^1} = \sum_{(\mathfrak{R},\mathbf{r},\hat{\mathbf{r}})}\mathcal{I}_{\mathfrak{R}}(q,t)\,\chi_{\mathbf{r}}(\mu)\chi_{\hat{\mathbf{r}}}(\hat{\mu}) \tag{5.2.6}
$$

where the sum runs over all triplets of representations $(\mathfrak{R}, \mathbf{r}, \hat{\mathbf{r}})$, $\chi_{\mathbf{r}}$ and $\chi_{\hat{\mathbf{r}}}$ are characters of G and $\hat{G}$, and $\mathcal{I}_{\mathfrak{R}}$ are characters of $OSp(4|4)$ with fugacities restricted as pertaining for the index. To proceed we must now make a detour to review the unitary representations of the $\mathcal{N} = 4$ superconformal algebra in three dimensions.

5.3 Characters of $OSp(4|4)$ and Hilbert series

5.3.1 Representations and recombination rules

All unitary highest-weight representations of $OSp(4|4)$ have been classified in refs. [59][39]. As shown in these references, in addition to the generic long representations there exist three series of short or BPS representations:

$$A_1[j]^{(j^H,j^C)}_{1+j+j^H+j^C} \quad (j>0)\ , \qquad A_2[0]^{(j^H,j^C)}_{1+j^H+j^C}\ , \quad \text{and} \quad B_1[0]^{(j^H,j^C)}_{j^H+j^C}\ . \tag{5.3.1}$$

We follow the notation of [39] where $[j]^{(j^H,j^C)}_{\delta}$ denotes a superconformal primary with energy δ, and $SO(1,2)\times SO(3)_H\times SO(3)_C$ spin quantum numbers j, j^H, j^C.[5] We use lower-case symbols for the quantum numbers of the superconformal primaries in order to distinguish them from those of arbitrary states in the representation. The subscripts labelling A and B indicate the level of the first null states in the representation.

The A-type representations lie at the unitarity threshold ($\delta_A = 1+j+j^H+j^C$) while those of B-type are separated from this threshold by a gap, $\delta_B = \delta_A - 1$. Since for short representations the primary dimension δ is fixed by the spins and the representation type, we will from now on drop it in order to make the notation lighter.

The general character of $OSp(4|4)$ is a function of four fugacities, corresponding to the eigenvalues of the four commuting bosonic generators J_3, J_3^H, J_3^C and Δ. For the index one fixes the fugacity of J_3 and then a second fugacity automatically drops out. More explicitly

$$\begin{aligned}\mathcal{I}_{\mathfrak{R}}(q,t) = \ & \chi_{\mathfrak{R}}(e^{i\pi}, q, t, e^{\beta}) \\ & \text{where} \quad \chi_{\mathfrak{R}}(w,q,t,e^{\beta}) = \mathrm{Tr}_{\mathfrak{R}}\, w^{2J_3} q^{\frac{1}{2}(\Delta+J_3)} t^{J_3^H - J_3^C} e^{-\beta(\Delta - J_3 - J_3^H - J_3^C)}\ .\end{aligned} \tag{5.3.2}$$

Although general characters are linearly-independent functions, this is not the case for indices. The index of long representations vanishes, and the indices of short representations that can recombine into a long one sum up to zero. This is why, as is well known, $\mathcal{Z}_{S^2\times S^1}$ does not determine (even) the BPS spectrum of the theory unambiguously. Fortunately, we can avoid this difficulty for our purposes here, as we will now explain.

In any $3d$, $\mathcal{N} = 4$ SCFT the ambiguity in extracting the BPS spectrum from the index can be summarized by the following recombination rules [39]

[5] A small difference from [39] is that we use spins rather than Dynkin labels.

$$L[0]^{(j^H,j^C)} \to A_2[0]^{(j^H,j^C)} \oplus B_1[0]^{(j^H+1,j^C+1)} , \tag{5.3.3a}$$

$$L[\tfrac{1}{2}]^{(j^H,j^C)} \to A_1[\tfrac{1}{2}]^{(j^H,j^C)} \oplus A_2[0]^{(j^H+\frac{1}{2},j^C+\frac{1}{2})} , \tag{5.3.3b}$$

$$L[j\geq 1]^{(j^H,j^C)} \to A_1[j]^{(j^H,j^C)} \oplus A_1[j-\tfrac{1}{2}]^{(j^H+\frac{1}{2},j^C+\frac{1}{2})} . \tag{5.3.3c}$$

The long representations on the left-hand side are taken at the unitarity threshold $\delta \to \delta_A$. From these recombination rules one sees that the characters of the B-type multiplets form a basis for contributions to the index. Simple induction indeed gives

$$(-)^{2j}\,\mathcal{I}_{A_1[j]^{(j^H,j^C)}} = \mathcal{I}_{A_2[0]^{(j^H+j,\,j^C+j)}} = -\mathcal{I}_{B_1[0]^{(j^H+j+1,\,j^C+j+1)}} . \tag{5.3.4}$$

We need therefore to compute the index only for B-type multiplets. The decomposition of these latter into highest-weight representations of the bosonic subgroup $SO(2,3)\times SO(4)$ can be found in ref. [39]. Using the known characters of $SO(2,3)$ and $SO(4)$ and taking carefully the limit $w \to e^{i\pi}$ leads to the following indices

$$\mathcal{I}_{B_1[0]^{(0,0)}} = 1 , \tag{5.3.5a}$$

$$\mathcal{I}_{B_1[0]^{(j^H>0,0)}} = (q^{\frac{1}{2}}t)^{j^H}\,\frac{(1-q^{\frac{1}{2}}t^{-1})}{(1-q)} , \tag{5.3.5b}$$

$$\mathcal{I}_{B_1[0]^{(0,j^C>0)}} = (q^{\frac{1}{2}}t^{-1})^{j^C}\,\frac{(1-q^{\frac{1}{2}}t)}{(1-q)} , \tag{5.3.5c}$$

$$\mathcal{I}_{B_1[0]^{(j^H>0,j^C>0)}} = q^{\frac{1}{2}(j^H+j^C)}t^{j^H-j^C}\,\frac{(1-q^{\frac{1}{2}}(t+t^{-1})+q)}{(1-q)} . \tag{5.3.5d}$$

Note that all superconformal primaries of type B are scalar fields with $\delta = j^H + j^C$, so one of them saturates the BPS bound $\delta = j_3 + j_3^H + j_3^C$ and contributes the leading power $q^{\frac{1}{2}(j^H+j^C)}$ to the index. Things work differently for type-A multiplets whose primary states have $\delta = 1+j+j^H+j^C > j_3+j_3^H+j_3^C$, so they cannot contribute to the index. Their descendants can however saturate the BPS bound and contribute, because even though a Poincaré supercharge raises the dimension by $\frac{1}{2}$, it can at the same time increase $J_3 + J_3^H + J_3^C$ by as much as $\frac{3}{2}$.

5.3.2 Protected multiplets and Hilbert series

General contributions to the index can be attributed either to a B-type or to an A-type multiplet. There exists, however, a special class of *absolutely protected* B-type representations which do not appear in the decomposition of any long multiplet. Their contribution to the index can therefore be extracted unambiguously. Inspection of (5.3.3) gives the following list of multiplets that are

$$\underline{\text{absolutely protected}}: \qquad B_1[0]^{(j^H,j^C)} \qquad \text{with} \quad j^H \le \frac{1}{2} \quad \text{or} \quad j^C \le \frac{1}{2}\,. \tag{5.3.6}$$

Consider in particular the $B_1[0]^{(j^H,0)}$ series.[6] The highest-weights of these multiplets are chiral $\mathcal{N}=2$ scalar fields that do not transform under $SO(3)_C$ rotations. This is precisely the Higgs-branch chiral ring consisting of operators made out of $\mathcal{N}=4$ hypermultiplets of the electric quiver. It is defined entirely by the classical F-term conditions. Likewise the highest-weights of the $B_1[0]^{(0,j^C)}$ series, which are singlets of $SO(3)_H$, form the chiral ring of the Coulomb branch whose building blocks are magnetic hypermultiplets. Redefine the fugacities as follows

$$x_\pm = q^{\frac{1}{4}} t^{\pm\frac{1}{2}}\ . \tag{5.3.7}$$

It follows then immediately from (5.3.5) that in the limit $x_-=0$ the index only receives contributions from the Higgs-branch chiral ring, while in the limit $x_+=0$ it only receives contributions from the chiral ring of the Coulomb branch.

The generating functions of these chiral rings, graded according to their dimension and quantum numbers under global symmetries, are known as Hilbert series (HS). In the context of $3d$ $\mathcal{N}=4$ theories elegant general formulae for the Higgs-branch and Coulomb-branch Hilbert series were derived in refs. [72]-[42], see also [41] for a review. It follows from our discussion that

$$\mathcal{Z}_{S^2\times S^1}\Big|_{x_-=0} = \mathrm{HS}^{\mathrm{Higgs}}(x_+) \qquad \text{and} \qquad \mathcal{Z}_{S^2\times S^1}\Big|_{x_+=0} = \mathrm{HS}^{\mathrm{Coulomb}}(x_-)\ . \tag{5.3.8}$$

These relations between the superconformal index and the Hilbert series were established in ref. [104] by matching the corresponding integral expressions. Here we derive them directly from the $\mathcal{N}=4$ superconformal characters.

What about other operators of the chiral ring ? The complete $\mathcal{N}=2$ chiral ring consists of the highest weights in <u>all</u> $B_1[0]^{(j^H,j^C)}$ multiplets of the theory.[7] As seen, however, from eq. (5.3.4) the mixed-branch operators (those with both j^H and $j^C \ge 1$) cannot be extracted unambiguously from the index. This shows that there is no simple relation between the Hilbert series of the full chiral ring and the superconformal index. The Hilbert series is better adapted for studying supersymmetric deformations of a SCFT, but we lack a general method to compute it (see however [43][34] for interesting ideas in this direction). Fortunately these complications will not be important for the problem at hand.

The reason is that marginal deformations exist only in the restricted set of multiplets:

$$\underline{\text{marginal}}: \qquad B_1[0]^{(j^H,j^C)} \qquad \text{with} \quad j^H + j^C = 2\ . \tag{5.3.9}$$

[6] The representations $B_1[0]^{(j^H,\frac{1}{2})}$ and $B_1[0]^{(\frac{1}{2},j^C)}$ only appear in theories with free hypermultiplets and play no role for good theories.

[7] The A-type multiplets do not contribute to the chiral ring, since none has scalar states that saturate the BPS bound (i.e. $\Delta = J_3^H + J_3^C$ *and* $J=0$).

These are in the absolutely protected list (5.3.6) with the exception of $B_1[0]^{(1,1)}$, a very interesting multiplet that contains also four spin-3/2 fields in its spectrum. This multiplet is not absolutely protected, but it is part of a 'monogamous relation': its unique recombination partner is $A_2[0]^{(0,0)}$ and vice versa. Furthermore $A_2[0]^{(0,0)}$ is the $\mathcal{N}=4$ multiplet of the conserved energy-momentum tensor [39], [8] which is unique in any irreducible SCFT. As a result the contribution of $B_1[0]^{(1,1)}$ multiplets can be also unambiguously extracted from the index.

A similar though weaker form of the argument actually applies to all $\mathcal{N}=2$ SCFT. Marginal chiral operators belong to short $OSp(2|4)$ multiplets whose only recombination partners are the conserved $\mathcal{N}=2$ vector-currents. We already alluded to this fact when explaining why the $3d$ $\mathcal{N}=2$ superconformal manifold has the structure of a moment-map quotient [71]. If the global symmetries of the SCFT are known (which may not be always easy), one can extract unambiguously its marginal deformations from the index.

5.4 Calculation of the index

We turn now to the main calculation of this paper, namely the expansion of the expression (5.2.2) in terms of characters of the global symmetry $OSp(4|4)\times G\times\hat{G}$. Since we are only interested in the marginal multiplets (5.3.9) whose contribution starts at order $O(q)$, it will be sufficient to expand the index to this order. In terms of the fugacities $x_\pm$ we must keep terms up to order $O(x^4)$. As we have just seen, each of the terms in the expansion to this order can be unambiguously attributed to a $OSp(4|4)$ representation.

We will organize the calculation in terms of the magnetic Casimir energy eq. (5.2.5). We start with the zero-monopole sector, and then proceed to positive values of $\Delta(\mathbf{m})$.

5.4.1 The zero-monopole sector

In the $\mathbf{m}=0$ sector all magnetic fluxes vanish and the gauge symmetry is unbroken. The expression in front of the plethystic exponential in (5.2.2) reduces to

$$\prod_{j=1}^{k}\left[\frac{1}{N_j!}\int\prod_{\alpha=1}^{N_j}\frac{dz_{j,\alpha}}{2\pi i z_{j,\alpha}}\prod_{\alpha\neq\beta}^{N_j}(1-z_{j,\beta}z_{j,\alpha}^{-1})\right]. \tag{5.4.1}$$

This can be recognized as the invariant Haar measure for the gauge group $\prod_{j=1}^{k}U(N_j)$. The measure is normalized so that for any irreducible representation R of $U(N)$

$$\frac{1}{N!}\int\prod_{\alpha=1}^{N}\frac{dz_\alpha}{2\pi i z_\alpha}\prod_{\alpha\neq\beta}^{N}(1-z_\beta z_\alpha^{-1})\,\chi_R(z) \;=\; \delta_{R,0}\,. \tag{5.4.2}$$

Thus the integral projects to gauge-invariant states, as expected. We denote this operation on any combination, X, of characters as $X\big|_{\text{singlet}}$.

[8] In the dual gravity theory, this recombination makes the $\mathcal{N}=4$ supergraviton massive. Thus $B_1[0]^{(1,1)}$ is a Stueckelberg multiplet for the 'Higgsing' of $\mathcal{N}=4$ AdS supergravity [22][21].

Since we work to order $O(q)$ we may drop the denominators $(1-q)$ in the plethystic exponential. The contribution of the $\mathbf{m}=0$ sector to the index can then be written as

$$\mathcal{Z}^{\mathbf{m}=0}_{S^2\times S^1} = \mathrm{PE}\Big(x_+(1-x_-^2)X + (x_-^2 - x_+^2)Y\Big)\Big|_{\text{singlet}} + O(x^5) \tag{5.4.3}$$

with

$$X = \sum_{j=1}^{k}(\Box_j\overline{\Box}_j^{\mu} + \overline{\Box}_j\Box_j^{\mu}) + \sum_{j=1}^{k-1}(\Box_j\overline{\Box}_{j+1} + \overline{\Box}_j\Box_{j+1}) \qquad \text{and} \quad Y = \sum_{j=1}^{k}\overline{\Box}_j\Box_j\ . \tag{5.4.4}$$

The notation here is as follows: $\Box_j$ denotes the character of the fundamental representation of the jth unitary group, and $\overline{\Box}_j$ that of the anti-fundamental. To distinguish gauge from global (electric) flavour groups we specify the latter with the symbol of the corresponding fugacities μ, while for the gauge group the dependence on the fugacities z is implicit. The entire plethystic exponent can be considered as a character of $\mathcal{G}\times G\times U(1)\times\mathbb{R}^+$, where $\mathcal{G}$ is the gauge group and $U(1)\times\mathbb{R}^+\subset \mathrm{OSp}(4|4)$ are the superconformal symmetries generated by $J_3^H - J_3^C$ and by $\Delta + J_3$. The "singlet" operation projects on singlets of the gauge group only.

The plethystic exponential is a sum of powers $\mathbb{S}^k\chi$ of characters, where $\mathbb{S}^k$ is a multiparticle symmetrizer that takes into account fermion statistics. For instance

$$\mathbb{S}^2(a+b-c-d) = \mathrm{S}^2a + ab + \mathrm{S}^2b - (a+b)(c+d) + \Lambda^2c + cd + \Lambda^2d \tag{5.4.5}$$

where S^k and Λ^k denote standard symmetrization or antisymmetrization. Call Ω the exponent in eq. (5.4.3) . To the quartic order that we care about we compute

$$\begin{aligned}
&\mathbb{S}^2\Omega = x_+^2\mathrm{S}^2X + x_+(x_-^2 - x_+^2)XY + x_-^4\mathrm{S}^2X + x_+^4\Lambda^2Y - x_+^2x_-^2(X^2+Y^2)\ ,\\
&\mathbb{S}^3\Omega = x_+^3\mathrm{S}^3X + x_+^2(x_-^2 - x_+^2)Y\mathrm{S}^2X\ \ ,\\
&\mathbb{S}^4\Omega = x_+^4\mathrm{S}^4X\ .
\end{aligned} \tag{5.4.6}$$

Upon projection on the gauge-invariant sector one finds

$$X\big|_{\text{singlet}} = XY\big|_{\text{singlet}} = 0 \qquad \text{and} \qquad Y\big|_{\text{singlet}} = k\ . \tag{5.4.7}$$

Second powers of Y also give (μ-independent) pure numbers,

$$Y^2\Big|_{\text{singlet}} = \mathrm{S}^2Y\Big|_{\text{singlet}} + \Lambda^2Y\Big|_{\text{singlet}}$$

with

$$\mathrm{S}^2 Y\Big|_{\text{singlet}} = \frac{1}{2}k(k+1) + \sum_{j=1}^{k} \delta_{N_j \neq 1} \;, \qquad \Lambda^2 Y\Big|_{\text{singlet}} = \frac{1}{2}k(k-1) \;. \tag{5.4.8}$$

The remaining terms in the expansion require a little more work with the result

$$\begin{aligned} X^2\Big|_{\text{singlet}} &= 2\,\mathrm{S}^2 X\Big|_{\text{singlet}} = 2\left(k - 1 + \sum_{j=1}^{k} \overline{\square}_j^{\mu} \square_j^{\mu}\right) , \\ \mathrm{S}^3 X\Big|_{\text{singlet}} &= \sum_{j=1}^{k-1} (\square_j^{\mu} \overline{\square}_{j+1}^{\mu} + \overline{\square}_j^{\mu} \square_{j+1}^{\mu}) \;, \\ Y\mathrm{S}^2 X\Big|_{\text{singlet}} &= k^2 + k - 2 + \delta_{N_1=1} + \delta_{N_k=1} + \sum_{j=1}^{k} \Big[(k + \delta_{N_j \neq 1}) \overline{\square}_j^{\mu} \square_j^{\mu} - 2\delta_{N_j=1} \Big] \;, \end{aligned} \tag{5.4.9}$$

and finally (and most tediously)

$$\begin{aligned} \mathrm{S}^4 X\Big|_{\text{singlet}} = &\sum_{j=2}^{k-1} \delta_{N_j \neq 1} + \sum_{j=1}^{k-1} \delta_{N_j \neq 1} \delta_{N_{j+1} \neq 1} + \frac{(k-1)k}{2} + \sum_{j=1}^{k-2} (\square_j^{\mu} \overline{\square}_{j+2}^{\mu} + \overline{\square}_j^{\mu} \square_{j+2}^{\mu}) \\ &+ \sum_{j=1}^{k} \delta_{N_j \neq 1} (2 - \delta_{j=1} - \delta_{j=k}) |\square_j^{\mu}|^2 + (k-1) \sum_{j=1}^{k} |\square_j^{\mu}|^2 \\ &+ \sum_{j<j'}^{k} |\square_j^{\mu}|^2 |\square_{j'}^{\mu}|^2 + \sum_{j=1}^{k} |\square\!\square_j^{\mu}|^2 + \sum_{j=1}^{k} \delta_{N_j \neq 1} \left| \begin{array}{c}\square\\[-6pt]\square\end{array}_j^{\mu} \right|^2 \end{aligned} \tag{5.4.10}$$

where in the last equation we used the shorthand $|R|^2$ for the character of $R \otimes \overline{R}$, and denoted the (anti)symmetric representations of $U(M_j)$ by Young diagrams.

Let us explain how to compute the singlets in $Y\mathrm{S}^2 X$. One obtains gauge-invariant contributions to that term in three different ways: the product of a gauge-invariant from Y and one from $\mathrm{S}^2 X$, or the product of an $SU(N_j)$ adjoint in Y with either a fundamental and an antifundamental, or a pair of bifundamentals, coming from $\mathrm{S}^2 X$. This gives three terms:

$$Y\mathrm{S}^2 X\Big|_{\text{singlet}} = k\left(k-1+\sum_{j=1}^{k} |\square_j^{\mu}|^2\right) + \left(\sum_{j=1}^{k} \delta_{N_j \neq 1} |\square_j^{\mu}|^2\right) + \left(-\delta_{N_1 \neq 1} - \delta_{N_k \neq 1} + \sum_{j=1}^{k} 2\delta_{N_j \neq 1}\right) \tag{5.4.11}$$

where we used that the $SU(N_j)$ adjoint is absent when $N_j = 1$, and that the outermost nodes have a single bifundamental hypermultiplet rather than two. After a small rearrangement, this is the same as the last line of (5.4.9).

For $\mathrm{S}^4 X|_{\text{singlet}}$ we organized terms according to how many bifundamentals they involve. First, four bifundamentals can be connected in self-explanatory notation as or or $\{\,{}^2\}$. Next, two bifundamentals and two fundamentals of different gauge groups can be connected as , while for the same group they can be either connected as or , or disconnected as a pair of bifundamentals and a flavour current (see below). When the node is abelian the first two terms are already included in the third and should not be counted separately. Finally, four fundamental hypermultiplets can form two pairs at different nodes, or if they come from the same node they should be split in two conjugate pairs, $Q^p_{j,\alpha} Q^r_{j,\beta}$ and $\tilde{Q}^{\bar{p}}_{j,\alpha} \tilde{Q}^{\bar{r}}_{j,\beta}$, with each pair separately symmetrized or antisymmetrized. When the gauge group is abelian the antisymmetric piece is absent.

5.4.2 Higgs-branch chiral ring

As a check, let us use the above results to calculate the Hilbert series of the Higgs branch. We have explained in section 5.3.2 that this is equal to the index evaluated at $x_- = 0$. Non-trivial monopole sectors make a contribution proportional to $x_-^{2\Delta(\mathbf{m})}$ and since $\Delta(\mathbf{m}) > 0$ they can be neglected. The Higgs-branch Hilbert series therefore reads

$$\mathrm{HS}^{\mathrm{Higgs}}(x_+) = \mathcal{Z}^{\mathbf{m}=0}_{S^2\times S^1}\Big|_{x_-=0} \,. \tag{5.4.12}$$

Setting $x_- = 0$ in eqs. (5.4.3) and (5.4.6) we find

$$\begin{aligned} \mathrm{HS}^{\mathrm{Higgs}}(x_+) \;=\; & 1 + x_+^2\,(\mathrm{S}^2 X - Y)\Big|_{\mathrm{singlet}} + x_+^3\,\mathrm{S}^3 X\Big|_{\mathrm{singlet}} \\ & + x_+^4\,(\mathrm{S}^4 X + \Lambda^2 Y - Y\mathrm{S}^2 X)\Big|_{\mathrm{singlet}} + O(x_+^5)\,. \end{aligned} \tag{5.4.13}$$

Inserting now (5.4.7)-(5.4.10) gives after some straightforward algebra

$$\begin{aligned} \mathrm{HS}^{\mathrm{Higgs}}(x_+) \;=\; & 1 \,+\, x_+^2\,\underbrace{\Big(\sum_{j=1}^{k} |\Box^\mu_j|^2 - 1\Big)}_{\mathrm{Adj}G} \,+\, x_+^3\,\underbrace{\sum_{j=1}^{k-1} (\Box^\mu_j \overline{\Box}^\mu_{j+1} + \overline{\Box}^\mu_j \Box^\mu_{j+1})}_{\mathrm{length=3\ strings}} \\ & + x_+^4 \Bigg[\, \underbrace{\sum_{j<j'}^{k} |\Box^\mu_j|^2 |\Box^\mu_{j'}|^2 + \sum_{j=1}^{k}\Big(|\Box\Box^\mu_j|^2 + \delta_{N_j\neq 1}\Big|\begin{smallmatrix}\Box\\[-2pt]\Box\end{smallmatrix}{}^\mu_j\Big|^2 - |\Box^\mu_j|^2\Big)}_{\mathrm{double-string\ operators}} \\ & \quad + \underbrace{\sum_{j=2}^{k-1}\Big(\Box^\mu_{j-1}\overline{\Box}^\mu_{j+1} + \overline{\Box}^\mu_{j-1}\Box^\mu_{j+1} + (1-\delta_{N_j=1})|\Box^\mu_j|^2\Big) - \Delta n_{\mathrm{H}}}_{\mathrm{length=4\ strings}} \Bigg] \,+\, O(x_+^5)\,. \end{aligned} \tag{5.4.14}$$

where

$$\Delta n_{\mathrm{H}} = 1 + \sum_{j=2}^{k-1} \delta_{N_j=1} - \sum_{j=1}^{k-1} \delta_{N_j=1}\delta_{N_{j+1}=1} \,. \tag{5.4.15}$$

This agrees with expectations. Recall that the Higgs-branch Hilbert series counts chiral operators made out of the scalar fields, Q_j^p and $\tilde{Q}_j^{\bar{p}}$, of the (anti)fundamental hypermultiplets, and the scalars of the bifundamental hypermultiplets $Q_{j,j+1}$ and $\tilde{Q}_{j+1,j}$ (the gauge indices are here suppressed). Gauge-invariant products of these scalar fields can be drawn as strings on the quiver diagram [19], and they obey the following matrix relations derived from the $\mathcal{N}=4$ superpotential,

$$Q_{j,j+1}\tilde{Q}_{j+1,j} + \tilde{Q}_{j,j-1}Q_{j-1,j} + \sum_{p,\bar{p}=1}^{M_j} Q_j^p \tilde{Q}_j^{\bar{p}} \delta_{p\bar{p}} = 0 \qquad \forall j = 1, \cdots, k \; . \tag{5.4.16}$$

The length of each string gives the $SO(3)_H$ spin and scaling dimension of the operator, and hence the power of x_+ in the index. Since good theories have no free hypermultiplets there are no contributions at order x_+. At order x_+^2 one finds the scalar partners of the conserved flavour currents that transform in the adjoint representation of G. Higher powers come either from single longer strings or, starting at order x_+^4, from multistring 'bound states'. One indeed recognizes the second line in (5.4.14) as the symmetrized product of strings of length two,

$$S^2\chi_{\mathrm{Adj}G} \; = \; S^2(\sum_{j=1}^{k} |\Box_j^{\mu}|^2 - 1) \; , \tag{5.4.17}$$

modulo the fact that for abelian gauge nodes some of the states are absent. These and the additional single-string operators of length 3 and 4 can be enumerated by diagrammatic rules, we refer the reader to [19] for details. Note also that the correction term Δn_{H} is the number of disjoint parts of the quiver when all abelian nodes are deleted. For each such part one neutral length-4 operator turns out to be redundant by the F-term conditions.

The quartic term of the Hilbert series counts marginal Higgs-branch operators. When the electric flavour-symmetry group G is large, the vast majority of these are double-string operators. Their number far exceeds the number ($\dim G$) of moment-map constraints, eq. (5.1.2), so generic $T_\rho^{\hat{\rho}}$ theories have a large number of double-string $\mathcal{N}=2$ moduli.

5.4.3 Contribution of monopoles

Going back to the full superconformal index, we separate it in three parts as follows

$$\mathcal{Z}_{S^2\times S^1} = -1 + \mathrm{HS}^{\mathrm{Higgs}}(x_+, \mu) \; + \; \mathrm{HS}^{\mathrm{Coulomb}}(x_-, \hat{\mu}) \; + \; \mathcal{Z}^{\mathrm{mixed}}(x_+, x_-, \mu, \hat{\mu}) \tag{5.4.18}$$

where the remainder $\mathcal{Z}^{\mathrm{mixed}}$ vanishes if either $x_- = 0$ or $x_+ = 0$. The Higgs-branch Hilbert series only depends on the electric-flavour fugacities $\mu_{j,p}$, and the Hilbert series of the Coulomb branch only depends on the magnetic-flavour fugacities w_j. To render the notation mirror-symmetric these latter should be redefined as follows

$$w_j = \hat{\mu}_j \hat{\mu}_{j+1}^{-1} \,. \tag{5.4.19}$$

Note that since the index (5.2.3) only depends on ratios of the $\hat{\mu}_j$, the last fugacity $\hat{\mu}_{k+1}$ is arbitrary and can be fixed at will. This reflects the fact that a phase rotation of all fundamental magnetic quarks is a gauge rather than global symmetry.

Mirror symmetry predicts that $\mathrm{HS}^{\mathrm{Coulomb}}$ is given by the same expression (5.4.14) with x_+ replaced by x_- and all other quantities replaced by their hatted mirrors. We will assume that this is indeed the case[9] and focus on the mixed piece $\mathcal{Z}^{\mathrm{mixed}}$.

As opposed to the two Hilbert series, which only receive contributions from B-type primaries, $\mathcal{Z}^{\mathrm{mixed}}$ has contributions from both A-type and B-type multiplets, and from both superconformal primaries and descendants. Let us first collect for later reference the terms of the $\mathbf{m} = 0$ sector that were not included in the Higgs-branch Hilbert series. From the results in section 5.4.1 one finds

$$\begin{aligned}
\mathcal{Z}^{\mathbf{m}=0}_{S^2\times S^1} - HS^{\mathrm{Higgs}} \;&=\; \Big[x_-^2 Y \,+\, x_-^4 \mathrm{S}^2 Y \,+\, x_+^2 x_-^2 \,(Y \mathrm{S}^2 X - X^2 - Y^2)\Big]_{\mathrm{singlet}} \\
&=\; x_-^2 k \,+\, x_-^4 \Big(\frac{1}{2}k(k+1) + \sum_{j=1}^{k} \delta_{N_j \neq 1}\Big) \\
+\; x_+^2 x_-^2 \Big(\sum_{j=1}^{k} &(k-1-\delta_{N_j=1})|\Box_j^{\mu}|^2 - 2k - \sum_{j=1}^{k} \delta_{N_j=1} + \delta_{N_1=1} + \delta_{N_k=1}\Big) \;+ O(x^5)\,.
\end{aligned} \tag{5.4.20}$$

The two terms in the second line contribute to the Coulomb-branch Hilbert series, while the third line is a contribution to the mixed piece.

We turn next to non-trivial monopole sectors whose contributions are proportional to $x_-^{2\Delta(\mathbf{m})}$. At the order of interest we can restrict ourselves to sectors with $0 < \Delta(\mathbf{m}) \leq 2$. Finding which monopole charges contribute to a generic value of $\Delta(\mathbf{m})$ is a hard combinatorial problem. For the lowest values $\Delta(\mathbf{m}) = \frac{1}{2}, 1$ and for good theories it was solved in ref. [65].

Fortunately this will be sufficient for our purposes here since, to the order of interest, the sectors $\Delta(\mathbf{m}) = 2$ and $\Delta(\mathbf{m}) = \frac{3}{2}$ only contribute to the Coulomb-branch Hilbert series, not to the mixed piece. This is obvious for $\Delta(\mathbf{m}) = 2$, while for $\Delta(\mathbf{m}) = \frac{3}{2}$ subleading terms in (5.2.3) with a single additional power of $q^{1/4}$ have unmatched gauge fugacities $z_{j,\alpha}$, and vanish after projection to the invariant sector (see below). In addition, good theories have no monopole operators with $\Delta(\mathbf{m}) = \frac{1}{2}$. Such operators would have been free twisted hypermultiplets, and there are none in the spectrum of good theories. This leaves us with $\Delta(\mathbf{m}) = 1$.

The key concept for describing monopole charges is that of *balanced* quiver nodes, defined as the nodes that saturate the 'good' inequality $N_{j-1} + N_{j+1} + M_j \geq 2N_j$. Let $\mathcal{B}_\xi$ denote the sets of consecutive balanced nodes, i.e. the disconnected parts of the quiver diagram after non-balanced nodes have been deleted. As shown in [65] each such set corresponds to a non-abelian flavor

[9] It is straightforward to verify the assertion at the quartic order computed here. Mirror symmetry of the complete index can be proved by induction (I. Lavdas and B. Le Floch, work in progress).

group $U(|\mathcal{B}_\xi|+1)$ in the mirror magnetic quiver.[10] Monopole charges in the sector $\Delta(\mathbf{m})=1$ are necessarily of the following form:

$$m_{j_1,\alpha_1} = m_{j_1+1,\alpha_2} = \cdots = m_{j_1+\ell,\alpha_\ell} = \pm 1 \qquad \text{with} \quad [j_1, j_1+\ell] \subseteq \ \mathcal{B}_\xi \tag{5.4.21}$$

for one choice of color indices at each gauge node, and for one given set of balanced nodes, $\mathcal{B}_\xi$. Up to permutations of the color indices we can choose $\alpha_1=\alpha_2=\cdots=\alpha_\ell=1$.

Define $j_1+\ell\equiv j_2$, and let Γ be the sequence of gauge nodes $\Gamma=\{j_1, j_1+1, \cdots j_1+\ell\equiv j_2\}$. To calculate the contribution of (5.4.21) to the index, we first note that the above assignement of magnetic fluxes breaks the gauge symmetry down to

$$\mathcal{G}_\Gamma = \prod_{j\notin\Gamma} U(N_j) \times \prod_{j\in\Gamma} \left[U(N_j-1)\times U(1)\right] \ . \tag{5.4.22}$$

Let us pull out of the integral expression (5.2.3) the fugacities $\prod_{j\in\Gamma} w_j^{\pm}$ and the overall factor x_-^2. Setting $q=0$ everywhere else and summing over equivalent permutations of color indices gives precisely the invariant measure of $\mathcal{G}_\Gamma$, normalized so that it integrates to 1. To calculate all terms systematically we must therefore expand the integrand in powers of $q^{1/4}$, and then project on the $\mathcal{G}_\Gamma$ invariant sector. To the order of interest we find

$$\mathcal{Z}^{\Delta(\mathbf{m})=1}_{S^2\times S^1} \ = \ x_-^2 \sum_{\mathcal{B}_\xi} \prod_{\Gamma\subseteq\mathcal{B}_\xi} \big(\prod_{j\in\Gamma} w_j + \prod_{j\in\Gamma} w_j^{-1}\big)\, \mathrm{PE}\Big(x_+ X' + (x_-^2 - x_+^2) Y'\Big)\bigg|_{\mathcal{G}_\Gamma\,\text{singlet}} + O(x^5) \tag{5.4.23}$$

where

$$\begin{aligned} X' &= \sum_{j=1}^{k} (\overline{\square}^{\mu}_j \square'_j + \square^{\mu}_j \overline{\square}'_j) + \sum_{j=1}^{k-1} (\overline{\square}'_j \square'_{j+1} + \square'_j \overline{\square}'_{j+1}) + \sum_{j,j+1\in\Gamma} (z_{j,1} z^{-1}_{j+1,1} + z^{-1}_{j,1} z_{j+1,1}) \ , \\ Y' &= \sum_{j=1}^{k} \overline{\square}_j \square_j = \sum_{j=1}^{k} \overline{\square}'_j \square'_j + (j_2 - j_1 + 1) \ , \end{aligned} \tag{5.4.24}$$

and in these expressions $\square'_j$ denotes the fundamental of $U(N_j-1)$ if $j\in\Gamma$, and the fundamental of $U(N_j)$ if $j\notin\Gamma$. By convention $\square'_j=0$ when $N'_j=N_j-1=0$.

Performing the projection onto $\mathcal{G}_\Gamma$ singlets gives

$$X'|_{\mathcal{G}_\Gamma\,\text{singlet}} = 0 \,, \qquad Y'|_{\mathcal{G}_\Gamma\,\text{singlet}} = (j_2-j_1+1) + \sum_{j=1}^{k} \delta_{N'_j\neq 0} \ ,$$

[10] As a result ξ ranges over the different components of the magnetic flavour group, i.e. the subset of gauge nodes $(\hat{\jmath}=1,\cdots\hat{k})$ in the mirror quiver of the magnetic theory for which $\hat{M}_{\hat{\jmath}}>1$.

and

$$\mathrm{S}^2 X'|_{\mathcal{G}_\Gamma\,\mathrm{singlet}} \;=\; \sum_{j=1}^{k} \delta_{N'_j\neq 0}\,\overline{\square}^{\mu}_j \square^{\mu}_j + (j_2 - j_1) \;+ \sum_{j=1}^{k-1} \delta_{N'_j\neq 0}\delta_{N'_{j+1}\neq 0}\;. \tag{5.4.25}$$

Collecting and rearranging terms gives

$$\begin{aligned}\mathcal{Z}^{\Delta(\mathbf{m})=1}_{S^2\times S^1} \;=\; &\sum_{\mathcal{B}_\xi} \prod_{[j_1,j_2]\subseteq\mathcal{B}_\xi} (\hat\mu_{j_1}\hat\mu^{-1}_{j_2+1} + \hat\mu^{-1}_{j_1}\hat\mu_{j_2+1})\Bigg[x_-^2 \;+\; x_-^4\,(k + \sum_{j\in\Gamma}\delta_{N_j\neq 1}) \\ &+ x_-^2 x_+^2 \left(\sum_{j=1}^{k}\delta_{N'_j\neq 0}\,\overline{\square}^{\mu}_j\square^{\mu}_j + \sum_{j=1}^{k-1}\delta_{N'_j\neq 0}\delta_{N'_{j+1}\neq 0} - \sum_{j=1}^{k}\delta_{N'_j\neq 0} - 1\right)\Bigg] \;+\; O(x^5)\;.\end{aligned} \tag{5.4.26}$$

The terms that do not vanish for $x_+ = 0$ are contributions to the Hilbert series of the Coulomb branch. For a check let us consider the leading term. Combining it with the one from eq. (5.4.20) gives the adjoint representation of $\hat{G}$, as predicted by mirror symmetry

$$\mathrm{HS}^{\mathrm{Coulomb}} \;=\; 1 + x_-^2\,\underbrace{\Big[k + \sum_{\mathcal{B}_\xi}\prod_{j_1,j_2\in\mathcal{B}_\xi}(\hat\mu_{j_1}\hat\mu^{-1}_{j_2+1} + \hat\mu^{-1}_{j_1}\hat\mu_{j_2+1})\Big]}_{\mathrm{Adj}\hat{G}} \;+\; O(x_-^3)\;. \tag{5.4.27}$$

Note that the k Cartan generators of $\hat{G}$ (those corresponding to the topological symmetry) contribute to the index in the $\mathbf{m} = 0$ sector. The monopole operators that enhance this symmetry in the infrared to the full non-abelian magnetic group enter in the sector $\Delta(\mathbf{m}) = 1$.

5.4.4 The mixed term

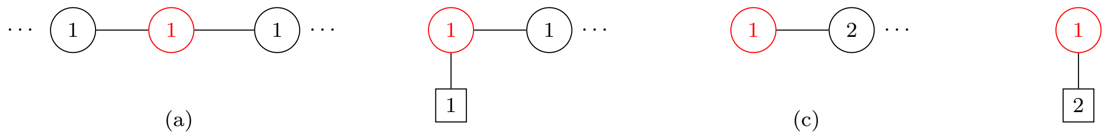
Let us now put together the mixed terms from eqs. (5.4.20) and (5.4.26). If the quiver has no abelian nodes all $N_j > 1$ and all $N'_j > 0$, and our expressions simplify enormously. The last line in eq. (5.4.20) collapses to $(k-1)\sum_j|\square^{\mu}_j|^2 - 2k$, and the last line of (5.4.26) collapses to $\sum_j|\square^{\mu}_j|^2 - 2$. Combining the two gives the following result for quivers with

<u>No abelian nodes</u> :

$$\begin{aligned}\mathcal{Z}^{\mathrm{mixed}} \;&=\; x_+^2 x_-^2\Big[(\sum_j|\square^{\mu}_j|^2 - 2)\Big(k - 1 + \sum_{\mathcal{B}_\xi}\sum_{\epsilon=\pm}\prod_{[j_1,j_2]\subseteq\mathcal{B}_\xi}(\hat\mu_{j_1}\hat\mu^{-1}_{j_2+1})^{\epsilon}\Big) - 2\Big] \;+\; O(x^5) \\ &= x_+^2 x_-^2\,\Big[(\chi_{\mathrm{Adj}\,G} - 1)(\chi_{\mathrm{Adj}\hat{G}} - 1) - 2\Big] \;+\; O(x^5)\;.\end{aligned} \tag{5.4.28}$$

We will interpret this result in the following section. But first let us consider the corrections coming from abelian nodes.

The μ-dependent correction in the $\mathbf{m} = 0$ sector, eq. (5.4.20), is a sum over all abelian gauge nodes of $|\Box_j^{\mu}|^2$ which should be subtracted from the above result. We expect, by mirror symmetry, a similar subtraction for abelian gauge nodes of the magnetic quiver. To see how this comes about note first that $N'_j = 0$ in (5.4.26) implies that j is an abelian balanced node in $\Gamma = [j_1, j_2] \subseteq \mathcal{B}_\xi$. Now an abelian balanced node has exactly two fundamental hypermultiplets, so it is necessarily one of the following four types:



(a) (b) (c) (d)

The balanced node is drawn in red, and the dots indicate that the [good] quiver extends beyond the piece shown in the figure, with extra flavour and/or gauge nodes. The set $\mathcal{B}_\xi$ may contain several balanced nodes, as many as the rank of the corresponding non-abelian factor of the magnetic-flavour symmetry. Notice however that abelian nodes of type (c) cannot coexist in the same $\mathcal{B}_\xi$ with abelian nodes of the other types. So we split the calculation of the $\Delta(\mathbf{m}) = 1$ sector according to whether $\mathcal{B}_\xi$ contains abelian nodes of type (a) and/or (b), or nodes of type (c). The case (d) corresponds to a single theory called $T[SU(2)]$ and will be treated separately.

Replacing $\delta_{N'_j \neq 0}$ by $1 - \delta_{N'_j = 0}$ in the last line of (5.4.26) and doing the straightforward algebra leads to the following result for the $x_+^2 x_-^2$ piece:

$$\sum_{j=1}^{k} \delta_{N'_j \neq 0} \overline{\Box}_j^{\mu} \Box_j^{\mu} + \sum_{j=1}^{k-1} \delta_{N'_j \neq 0} \delta_{N'_{j+1} \neq 0} - \sum_{j=1}^{k} \delta_{N'_j \neq 0} - 1 \;=\; \sum_{j=1}^{k} \overline{\Box}_j^{\mu} \Box_j^{\mu} - 2 - \begin{cases} 1 & \text{(a)} + \text{(b)} \\ 0 & \text{(c)} \end{cases} \tag{5.4.29}$$

The term in front of the ‘cases’ on the right-hand side was already accounted for in (5.4.28). The extra subtraction vanishes when $\mathcal{B}_\xi$ is of type (c), and equals -1 when $\mathcal{B}_\xi$ is of type (a) and/or (b). This is precisely what one expects from mirror symmetry. Indeed, as shown in appendix C , the two cases in eq. (5.4.29) correspond to the $\hat{M}_\xi = |\mathcal{B}_\xi| + 1$ magnetic flavours being charged under a non-abelian, respectively abelian gauge group in the magnetic quiver ($\hat{N}_\xi > 1$, respectively $\hat{N}_\xi = 1$). In the first case there is no correction to (5.4.28), while in the second summing over all monopole-charge assignements in $\mathcal{B}_\xi$ reconstructs, up to a fugacity-independent term equal to the rank, the adjoint character of the non-abelian magnetic-flavour symmetry.

Putting everything together we finally get for

Arbitrary quivers :

$$\begin{aligned} \mathcal{Z}^{\text{mixed}} &= x_+^2 x_-^2 \Big[(\chi_{\mathrm{Adj}\,G}(\mu) - 1)(\chi_{\mathrm{Adj}\,\hat{G}}(\hat{\mu}) - 1) - 2 \\ &\quad - \sum_{j | N_j = 1} |\Box_j^{\mu}|^2 - \sum_{\hat{j} | \hat{N}_{\hat{j}} = 1} |\Box_{\hat{j}}^{\hat{\mu}}|^2 + \Delta n_{\text{mixed}} \Big] + O(x^5) \ , \end{aligned} \tag{5.4.30}$$

where the fugacity-independent correction reads

$$\Delta n_{\rm mixed} = \sum_{\hat{j}|\hat{N}_{\hat{j}}=1} \hat{M}_{\hat{j}} \ + \delta_{N_1=1} + \delta_{N_k=1} - \sum_{j=1}^{k} \delta_{N_j=1} \ . \tag{5.4.31}$$

We leave it as an exercise for the reader to show that $\Delta n_{\rm mixed}$ is (like the rest of the expression) mirror symmetric, albeit not manifestly so.

For completeness we give finally the result for $T[SU(2)]$, the theory described by the quiver (d). This is a self-dual abelian theory with global symmetry $SU(2) \times \widehat{SU}(2)$. In self-explanatory notation the result for this case reads

$$\underline{\mathbf{T[SU(2)]}}: \qquad \mathcal{Z}^{\rm mixed} \ = x_+^2 x_-^2 \left(-3 - \mu - \mu^{-1} - \hat{\mu} - \hat{\mu}^{-1}\right) + O(x^5) \ . \tag{5.4.32}$$

It turns out that for this theory the full superconformal index can be expressed in closed form, in terms of the q-hypergeometric function. This renders manifest a general property of the index, its factorization in holomorphic blocks [98]-[25]. Since we are not using this feature in our paper, the calculation is relegated to appendix E.2 .

This completes our calculation of the mixed quartic terms of the superconformal index. We will next rewrite the index as a sum of characters of $OSp(4|4)$ and interpret the result.

5.5 Counting the $\mathcal{N} = 2$ moduli

The full superconformal index up to order $O(q) \sim O(x^4)$ is given by (5.4.18) together with expressions (5.4.14)-(5.4.15) for the Higgs branch Hilbert series, their mirrors for the Coulomb branch Hilbert series, and expressions (5.4.30)-(5.4.31) for the mixed term. Collecting everything and using also (5.3.5) for the indices of individual representations of the superconformal algebra $OSp(4|4)$ leads to the main result of this paper

$$\begin{aligned}
\mathcal{Z}_{S^2\times S^1} =& 1 + \underbrace{x_+^2(1-x_-^2)}_{\mathcal{I}_{B_1[0]^{(1,0)}}} \chi_{{\rm Adj}\, G} + \underbrace{x_-^2(1-x_+^2)}_{\mathcal{I}_{B_1[0]^{(0,1)}}} \chi_{{\rm Adj}\hat{G}} + \underbrace{x_+^3}_{\mathcal{I}_{B_1[0]^{(3/2,0)}}} \chi_{\ell=3} + \underbrace{x_-^3}_{\mathcal{I}_{B_1[0]^{(0,3/2)}}} \hat{\chi}_{\ell=3} \\
&+ \underbrace{x_+^4}_{\mathcal{I}_{B_1[0]^{(2,0)}}} \boxed{\left(S^2\chi_{{\rm Adj}\, G} + \chi_{\ell=4} - \Delta\chi^{(2,0)}\right)} + \underbrace{x_-^4}_{\mathcal{I}_{B_1[0]^{(0,2)}}} \boxed{\left(S^2\chi_{{\rm Adj}\hat{G}} + \hat{\chi}_{\ell=4} - \Delta\chi^{(0,2)}\right)} \\
&+ \underbrace{x_+^2 x_-^2}_{\mathcal{I}_{B_1[0]^{(1,1)}}} \boxed{\left(\chi_{{\rm Adj}\, G}\, \chi_{{\rm Adj}\hat{G}} - \Delta\chi^{(1,1)}\right)} + \underbrace{(-x_+^2 x_-^2)}_{\mathcal{I}_{A_2[0]^{(0,0)}}} + O(x^5) \ .
\end{aligned} \tag{5.5.1}$$

where $\chi_{\ell=\mathrm{n}}$ counts independent single strings of length $n = 3, 4$ on the electric quiver, $\hat{\chi}_{\ell=\mathrm{n}}$ counts likewise single strings on the magnetic quiver, and the correction terms coming from abelian (electric and magnetic) gauge nodes are given by

$$\begin{aligned}
\Delta\chi^{(2,0)} = \sum_{j=1}^{k} \delta_{N_j=1} \left|\boxminus_j^{\mu}\right|^2 \ , \qquad \Delta\chi^{(0,2)} = \sum_{\hat{j}=1}^{\hat{k}} \delta_{\hat{N}_{\hat{j}}=1} \left|\boxminus_{\hat{j}}^{\hat{\mu}}\right|^2 \ , \\
\text{and} \qquad \Delta\chi^{(1,1)} = \sum_{j|N_j=1} |\Box_j^{\mu}|^2 + \sum_{\hat{j}|\hat{N}_{\hat{j}}=1} |\Box_{\hat{j}}^{\hat{\mu}}|^2 - \Delta n_{\mathrm{mixed}} \ .
\end{aligned} \tag{5.5.2}$$

Notice that we have used here the fact that the SCFT has a unique energy-momentum tensor which is part of the $A_2[0]^{(0,0)}$ multiplet, and that all the other $OSp(4|4)$ multiplets can be unambiguously identified at this order.

5.5.1 Examples and interpretation

The marginal $\mathcal{N} = 2$ deformations are the terms enclosed in boxes in (5.5.1). Those in the second line are standard quartic superpotentials involving only the $\mathcal{N} = 4$ hypermultiplets of the electric quiver, or only their twisted cousins of the magnetic quiver. The electric superpotentials (counted in the Higgs-branch Hilberts series) are of two kinds: (i) single strings of length 4 that transform in the adjoint of each gauge-group factor $U(M_j)$, or in the bifundamental of next-to-nearest neighbour flavour groups $U(M_j) \times U(M_{j+2})$; and (ii) double-string operators in the $S^2(\mathrm{Adj}G)$ representation. If there are abelian gauge nodes some of these operators are absent. The same statements of course hold for magnetic superpotentials and the mirror quiver.

The more interesting deformations, the ones made out of both types of hypermultiplets, are in the third line of (5.5.1). For quivers with no abelian nodes, these mixed operators are all possible $|\mathrm{Adj}G| \times |\mathrm{Adj}\hat{G}|$ gauge-invariant products of two fundamental hypermultiplets and two fundamental twisted hypermultiplets [11]

$$\mathcal{O}_{j;\hat{j}}^{(\bar{p},r;\bar{\hat{p}},\hat{r})} = (\tilde{Q}_j^{\bar{p}} Q_j^r)(\tilde{\hat{Q}}_{\hat{j}}^{\bar{\hat{p}}} \hat{Q}_{\hat{j}}^{\hat{r}}) \ , \tag{5.5.3}$$

where hats denote the scalars of the (twisted) hypermultiplets.

Some of the above operators can be identified with superpotential deformations involving both hypermultiplets and vector multiplets. Consider, in particular, the following gauge-invariant chiral operators of the electric theory

$$\mathcal{O}_{j,j'}^{(\bar{p},r)} = (\tilde{Q}_j^{\bar{p}} Q_j^r)\, \mathrm{tr}(\Phi_{j'}) \ , \tag{5.5.4}$$

where Φ_j is the $\mathcal{N} = 2$ chiral field in the $\mathcal{N} = 4$ vector multiplet at the jth gauge-group node. It can be easily shown that $\mathrm{tr}(\Phi_j)$ is the scalar superpartner of the jth topological $U(1)$ current,

[11] More precisely, all but the overall combination $\sum_j \sum_{p,\bar{p}} Q_j^p \tilde{Q}_j^{\bar{p}} \delta_{p\bar{p}}$ and its mirror. These are the scalar partners of the two missing $U(1)$ flavour symmetries that are gauged.

so that the operators (5.5.4) are the same as the operators (5.5.3) when these latter are restricted to the Cartan subalgebra of $\hat{G}$. Similarly, projecting (5.5.3) onto the Cartan subalgebra of G gives mixed superpotential deformations of the magnetic Lagrangian. The remaining $(|\mathrm{Adj}G| - \mathrm{rank}G) \times (|\mathrm{Adj}\hat{G}| - \mathrm{rank}\hat{G})$ deformations involve both charged hypermultiplets and monopole operators and have a priori no Lagrangian description.

We can also understand why some mixed operators are absent when the quiver has abelian nodes. Recall that the $\mathcal{N}=4$ superpotential reads

$$W = \sum_{j=1}^{k}\Big(Q_{j,j-1}\Phi_j\tilde{Q}_{j,j-1} + \tilde{Q}_{j,j+1}\,\Phi_j Q_{j,j+1} + \sum_{p,\bar{p}=1}^{M_j} \tilde{Q}_j^{\bar{p}}\,\Phi_j Q_j^p \delta_{p\bar{p}}\Big)\ , \tag{5.5.5}$$

from which one derives the following F-term conditions : $\tilde{Q}_j^{\bar{p}}\,\Phi_j = \Phi_j Q_j^p = 0$ for all j, p and $\bar{p}$. Note that Φ_j is an $N_j \times N_j$ matrix, while $Q_j^{\bar{p}}$ and $Q_j^p = 0$ are bra and ket vectors. If (and only if) j is an abelian node, these conditions imply $\mathcal{O}_{j,j}^{(\bar{p},r)} = 0$ so that these operators should be subtracted. This explains the first of the three terms in the subtraction $\delta\chi^{(1,1)}$, eq. (5.5.2). The second is likewise explained by the F-term conditions at abelian nodes of the magnetic quiver. Finally $\Delta n_{\mathrm{mixed}}$ corrects some overcounting in these abelian-node subtractions.

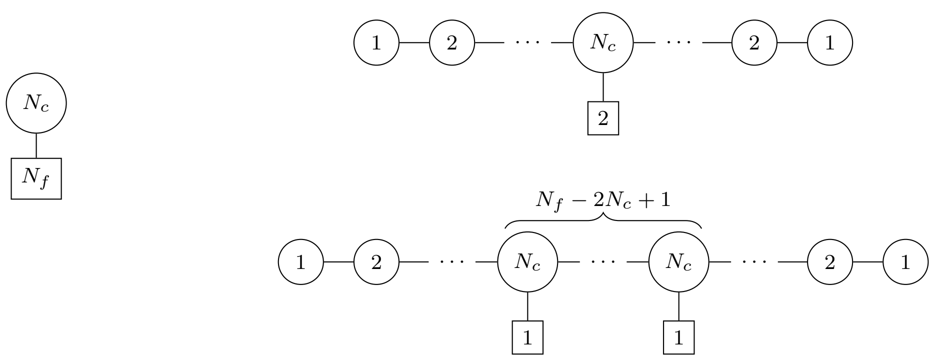
We may summarize the discussion as follows:

Marginal chiral operators of $T_\rho^{\hat{\rho}}[SU(N)]$ transform in the $S^2(\mathrm{Adj}G + \mathrm{Adj}\hat{G})$ representation of the electric and magnetic flavour symmetry, plus strings of length 4 (in either adjoints or bifundamentals of individual factors), modulo redundancies for quivers with abelian nodes.

Note that the above logic could be extended to chiral operators of arbitrary dimension $\Delta = n$. Operator overcounting arises, however, in this case at electric or magnetic gauge nodes of rank $\leq n-1$, making the combinatorial problem considerably harder.

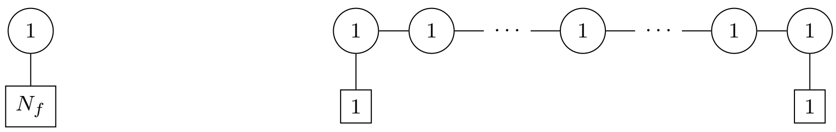
We now illustrate these results with selected examples:

sQCD$_3$: The electric theory has gauge group $U(N_c)$ with $N_c \geq 2$, and $N_f \geq 2N_c$ fundamental flavours. Its electric and magnetic quivers are drawn below. The magnetic quiver with $N_f = 2N_c$ (upper right figure) differs from the one for $N_f > 2N_c$ (lower right figure). Both have $N_f - 1$ balanced nodes, corresponding to the electric $SU(N_f)$ flavour symmetry, but their magnetic symmetry is, respectively, $SU(2)$ and $U(1)$:



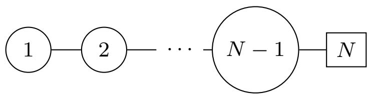
The $N_f > 2N_c$ theories have $\frac{1}{2}N_f^2(N_f^2-1)$ electric, one magnetic, and (N_f^2-1) mixed marginal operators from 2-string states. There are no extra marginal operators from length-4 strings, except in the special case $N_f = 2N_c+1$, and no abelian-node redundancies. The number of D-term conditions is N_f^2+1, so that the complex dimension of the superconformal manifold is $\dim\mathcal{M}_{SC} = \frac{1}{2}N_f^2(N_f^2-1)-1$. When $N_f = 2N_c$ the number of electric operators is the same, but there are now six magnetic operators, $3(N_f^2-1)$ mixed ones, three length-4 strings, and N_f^2+3 D-term conditions.

sQED$_3$: This is a $U(1)$ theory with $N_f > 2$ charged hypermultiplets. The magnetic quiver has N_f-1 abelian balanced nodes and one charged hypermultiplet at each end of the chain:



This theory has $\frac{1}{4}N_f^2(N_f+1)^2$ marginal electric operators (because the antisymmetric combination $Q^{[p}Q^{r]}$ vanishes), one magnetic operator, and no mixed ones. To prove this latter assertion one computes $\Delta n_{\text{mixed}} = 3$ from eq. (5.4.31) [checking in passing that the expression is mirror symmetric]. In the special case $N_f = 4$ there is in addition a length-4 magnetic string. Note that for $N_f \gg 1$ the dimension of the superconformal manifold of sQED$_3$ is reduced by a factor two compared to the superconformal manifold of sQCD$_3$.

$\mathbf{T[SU(N)]}$: This theory is defined by the self-dual fully-balanced quiver shown below.



For $N \geq 3$ there are $\frac{1}{2}N^2(N^2-1)$ electric operators, as many magnetic operators, and $(N^2-1)^2$ mixed ones. The dimension of the superconformal manifold is $\dim\mathcal{M}_{SC} = (N^2-1)(2N^2-3)-1$. The case $T[SU(2)]$ was discussed already separately.

5.5.2 The holographic perspective

In this last part we discuss the relation to string theory and sketch some directions for future work.

As discussed in the introduction, the $T_\rho^{\hat\rho}[SU(N)]$ theories are holographically dual to type IIB string theory in the supersymmetric backgrounds of refs. [12][13]. The geometry has a $\mathrm{AdS}_4\times\mathrm{S}^2_H\times\mathrm{S}^2_C$ fiber over a basis which is the infinite strip Σ. The $SO(2,3)\times SO(3)_H\times SO(3)_C$ symmetry of the SCFT is realized as isometries of the fiber. The solution features singularities on the upper (lower) boundary of the strip which correspond to D5-brane sources wrapping S^2_H (NS5-brane sources wrapping S^2_C). These two-spheres are trivial in homology, yet the branes are stabilized by non-zero worldvolume fluxes that counterbalance the negative tensile stress [16].

There is a total of $k+1$ NS5-branes and $\hat{k}+1$ D5-branes. Their position along the boundary of the strip is a function of their linking number, which increases from left to right for D5-branes and decrease for NS5-branes [12]. Branes with the same linking number overlap giving non-abelian flavour symmetries. The linking number of a fivebrane can be equivalently defined as

- the D3-brane charge dissolved in the fivebrane ;
- the worldvolume flux on the wrapped two-sphere;
- the node of the corresponding quiver, for instance the $\hat{\imath}$th D5-brane provides a fundamental hypermultiplet at the $\hat{l}_{\hat{\imath}} = i$ node of the electric quiver (see appendix C).

The R-symmetry spins J^H, J^C are the angular momenta of a state on the two spheres. Given the above dictionary, can we understand the results of this paper from the string-theory side?

Consider first the Higgs-branch chiral ring which consists of the highest weights of all $B_1[0]^{(j^H,0)}$ multiplets. When decomposed in terms of conformal primaries these multiplets read [39]

$$B_1[0]^{(j^H,0)}_{j^H} = [0]^{(j^H,0)}_{j^H} \oplus [0]^{(j^H-1,1)}_{j^H+1} \oplus [1]^{(j^H-1,0)}_{j^H+1} \oplus \text{fermions}_{j^H+\frac{1}{2}} \ . \tag{5.5.6}$$

Note that the top component includes a vector boson with scaling dimension $\Delta = j^H+1$. This is a massless gauge boson in AdS_4 for $j^H = 1$ ('conserved current' multiplet) and a massive gauge boson for $j^H > 1$. As explained in ref. [19], both massless and massive vector bosons are states of fundamental open strings on the D5-branes. Their vertex operators include a scalar wavefunction on S^2_H with angular momentum $J^H = j^H - 1$. Consider such an open string stretching between two D5-branes with linking numbers ℓ and ℓ'. Since these latter are magnetic-monopole fields on S^2_H, the open string couples to a net field $(\ell - \ell')$. Its wavefunction is therefore given by the well-known monopole spherical harmonics with [12]

$$j^H - 1 = \frac{1}{2}|\ell - \ell'| + \mathbb{N} \tag{5.5.7}$$

where $\mathbb{N}$ are the natural numbers. Recalling that the linking numbers also designate the nodes of the electric quiver, we understand why the Higgs-branch chiral ring includes strings of minimal length $|\ell - \ell'| + 2$ transforming in the bi-fundamental of $U(M_\ell) \times U(M_{\ell'})$ for all $k \geq \ell' > \ell > 0$ [19]. The bifundamental strings of length 3 and 4 in eq. (5.4.14) are of this kind.

The $\Delta = 2$ chiral ring also includes strings of length 4 in the adjoint of $U(M_j)$ for all $k > j > 1$, see (5.4.14). The corresponding open-string vector bosons on the ith stack of D5-branes do not feel a monopole field ($\ell = \ell' = i$) but have angular momentum $j^H - 1 = 1$. Notice however that

[12] This celebrated result goes back to the early days of quantum mechanics [107]. We have used it implicitly when expressing determinants as q-Pochammer symbols. For an amusing real-time manifestation of the effect see [15] .

these length-4 operators are missing at the two ends of the quiver, i.e. for $i = 1$ and for $i = k$. How can one understand this from the string theory side ?

A plausible explanation comes from a well-known effect dubbed 'stringy exclusion principle' in ref. [94]. The relevant setup features K NS5-branes and a set of probe D-branes ending on them. The worldsheet theory in this background has an affine algebra $\mathfrak{su}(2)_K$ [13] and D-branes (Cardy states) labelled by the set of dominant affine weights $\lambda = 0, 1, \cdots, K-1$. The ground states of open strings stretched between two such D-branes have weights ν in the interval

$$[\,|\lambda-\lambda'|, \min(\lambda+\lambda', 2K-\lambda-\lambda')\,]$$

and in steps of two [58]. Translating $\lambda = \ell - 1$ (see [15]), $\mu = 2(j^H - 1)$ and $K = k+1$ (the total number of NS5-branes) gives in replacement of (5.5.7)

$$j^H - 1 \;=\; \frac{1}{2}|\ell-\ell'|\,,\ \frac{1}{2}|\ell-\ell'|+1\,,\ \cdots\ ,\min\Big(\frac{1}{2}(\ell+\ell'-2)\,, k-\frac{1}{2}(\ell+\ell')\Big)\ . \qquad (5.5.8)$$

The intuitive understanding of the upper cutoff is that a string cannot remain in its ground state if its angular momentum exceeds the size of the sphere. It follows that for $\ell = \ell' = 1$ or k, only the $j^H = 1$ states survive, in agreement with our findings for the Higgs-branch chiral ring.

To be sure this is just an argument, not a proof, because in the solutions dual to $T^{\hat\rho}_{\rho}[SU(N)]$ the 3-sphere threaded by the NS5-brane flux is highly deformed by the strong back reaction of the D-branes. The perfect match with the field theory side suggests, however, that the detailed geometry does not matter when it comes to the above stringy effect.

The superconformal index brings to light other exclusion effects associated to abelian gauge nodes of the electric and magnetic quivers, as summarized in eqs. (5.5.1) and (5.5.2). For higher elements of the chiral ring, these effects are more generally related to the finite ranks of the gauge groups. This is a ubiquitous phenomenon in holography – McGreevy et al coined the name 'giant graviton' for it in the prototypical $\mathrm{AdS}_5\times\mathrm{S}^5$ example [95]. We did not manage to find a simple explanation for giant-graviton exclusions in the problem at hand. Part of the difficulty is that, as opposed to the 5-brane linking numbers, the gauge group ranks have a less direct meaning on the gravitational side of the AdS/CFT correspondence. [14]

We conclude our discussion of the AdS side with a remark about gauged $\mathcal{N} = 4$ supergravity. In addition to the graviton, this has n vector multiplets and global $SL(2)\times SO(6,n)$ symmetry, part of which may be gauged. Insisting that the gauged theory have a supersymmetric AdS_4 vacuum restricts the form of the gauge group to be $G_H \times G_C \times G_0 \subset SO(6,n)$, where the (generally) non-compact G_H and G_C contain the R-symmetries $SO(3)_H$ and $SO(3)_C$ [90].

The vector bosons of spontaneously-broken gauge symmetries belong to B-type multiplets with $(j^H, j^C) = (2,0)$ or $(0,2)$. These can describe the length-4 marginal operators in the Higgs-branch or Coulomb-branch chiral rings. As noted on the other hand in ref. [19], there is no room for elementary $(1,1)$ multiplets in $\mathcal{N} = 4$ supergravity, because such multiplets have extra spin-$\frac{3}{2}$ fields. But we have just seen that linear-quiver theories have no single-string $(1,1)$ operators, so the above limitation does not apply. All mixed marginal deformations correspond to double-string operators that can be described effectively by modifying the boundary conditions of their single-string constituents [115][29]. Note that boundary conditions change the quantization, not the solution. So

[13] The bosonic subalgebra has level $K-2$ and an extra factor 2 is added by fermions.

[14] Note that two theories with the same flavour symmetry, i.e. the same disposition of five-branes, can have very different gauge-group ranks. This feature (called 'fine print' in ref. [19]) is best illustrated by sQCD_3 with a fixed number of flavours, N_f, but an arbitrary number of colors $N_c \in (2, [N_f/2]-1)$, see section 5.5.1.

Gauged $\mathcal{N} = 4$ supergravity has the necessary ingredients to describe the complete moduli space of the $T_\rho^{\hat\rho}[SU(N)]$ theories, provided one considers both classical and quantization moduli.

This quells, at least for linear quivers, the concern raised in [19] that reduction of string theory to gauged $4d$ supergravity may truncate away part of the moduli space.

5.5.3 One last comment

We end with a remark about the Hilbert series of $T_\rho^{\hat\rho}[SU(N)]$ theories. As we explained in section 5.3, the full chiral ring consists of the highest-weights of all B-type multiplets in the theory with arbitrary (j^H, j^C). The relevant and marginal operators can be identified unambiguously in the index, as can the entire Higgs-branch and Coulomb-branch subrings. But general mixed elements (with $j^H, j^C > 1$) cannot be extracted unambiguously. A calculation that does not rely on the superconformal index would therefore be of great interest.

A natural conjecture for the full Hilbert series [34] is that it is the coordinate ring of the union of all branches B_σ (for the $T_\rho^{\hat\rho}$ theory, σ ranges over partitions between ρ and $\hat\rho^T$),

$$\mathrm{HS}\Big(\bigcup_\sigma B_\sigma \;\Big|\; x_+, x_-\Big) = \sum_\Lambda (-1)^{|\Lambda|-1}\,\mathrm{HS}\Big(\bigcap_{\sigma\in\Lambda} B_\sigma \;\Big|\; x_+, x_-\Big) \tag{5.5.9}$$

where Λ runs over all non-empty subsets of the branches of the theory. In words, the full Hilbert series would be the sum of Hilbert series of every branch, minus corrections due to pairwise intersections and so on. It can be checked that this conjecture is consistent with the Higgs branch and Coulomb branch limits ($q^{1/4}t^{\mp 1/2} \to 0$ with $q^{1/4}t^{\pm 1/2}$ fixed). One can also compare the number of $B_1[0]^{(1,1)}$ multiplets suggested by this conjecture to the number extracted from the index. In the limited set of examples that we checked [with zero or one mixed branch] we found an exact match. Finding a better way to confirm or falsify this conjecture is an interesting problem.

Appendix

Appendix A

Elements of representation theory for 3d $\mathcal{N} = 2$ and $\mathcal{N} = 4$ theories

In this appendix, we present some basic aspects of representation theory for three-dimensional $\mathcal{N} = 2$ and $\mathcal{N} = 4$ theories. A classification of unitary representations of the various superconformal algebras has been done in [39], the notation of which is followed.

The three-dimensional superconformal algebra with $\mathcal{N}$ amount of supersymmetry is $\mathfrak{osp}(\mathcal{N}|4)$, with its maximal bosonic subalgebra being the direct product of the conformal algebra in three-dimensions and of the the R-symmetry algebra:

$$\mathfrak{osp}(\mathcal{N}|4) \supset \mathfrak{so}(3,2) \times \mathfrak{so}(\mathcal{N})_R \tag{A.0.1}$$

and hence for the $\mathcal{N} = 2, 4$ theories:

$$\mathfrak{osp}(2|4) \supset \mathfrak{so}(3,2) \times \mathfrak{so}(2)_r, \quad \mathfrak{so}(2)_r \sim \mathfrak{u}(1)_r \tag{A.0.2}$$

$$\mathfrak{osp}(4|4) \supset \mathfrak{so}(3,2) \times \mathfrak{so}(4)_R, \quad \mathfrak{so}(4)_R \sim \mathfrak{su}(2)_R \times \mathfrak{su}(2)_{R'} \tag{A.0.3}$$

A superconformal multiplet is completely determined by the quantum numbers of its superconformal primary operator, namely the one with the lowest scaling dimension, which is annihilated by the generators of superconformal boosts ($\mathcal{S}, \Delta_S = -\frac{1}{2}$) and of special conformal transformations ($K_\mu, \Delta_{K_\mu} = -1$). The rest of the operators comprising the superconformal multiplet, are obtained by the action on the superconformal primary by the supercharges $\mathcal{Q}, \Delta_{\mathcal{Q}} = \frac{1}{2}$ and are called superconformal descendant operators. A descendant obtained by the action of l-supercharges on the superconformal primary, is at the *level-l* of the multiplet.

Heighest-weight states of a given representation of the $\mathfrak{osp}(2|4)$ or $\mathfrak{osp}(4|4)$ are denoted as $[j]_\Delta^r$ and $[j]_\Delta^{R,R'}$, with j being the spacetime spin, r ,(R, R') being the R-symmetry spins and of course Δ being the scaling dimension. The requirement for unitarity is that all operators comprising a superconformal multiplet have non-negative norm this condition is expressed in the form of *unitarity bounds* for the scaling dimension, $\Delta \geq f(j, R)$. When the bound is saturated, the multiplet includes descendants with vanishing norm, called *null* states, which can be consistently removed.

Multiplets for which the inequality is strict, are called *Long* multiplets, all states have positive norm and they are denoted by $L[j]_{\Delta}^{R,R'}$, where the labels are the quantum numbers of the superconformal primary.

Multiplets containing null states are called *short.* There is a distinction between short mutliplets at the *threshold* where the above unitarity bound is saturated and *isolated* short multiplets, which are separated from the other types of multiplets, that have the same quantum numbers, by a gap. These two categories of short multiplets are denoted as follows:

$$A_l[j]_{\Delta_A}^{\{R\}}, \quad \text{short at threshold} \tag{A.0.4}$$

$$X_l[j]_{\Delta_X}^{\{R\}}, \quad \text{isolated short} \tag{A.0.5}$$

where $X \in (B, C, D)$ (different kinds of isolated short multiplets), with $\Delta_A > \Delta_B > \Delta_C > \Delta_D$ and l, as stated above, denotes the level where the first null state appears. Finally, it is important to present the concept of recombination of representations. In the limit where the scaling dimension of a long multiplets approaches the unitarity bound from above, it breaks into an $A_l[j]_{\Delta_A}^{\{R\}}$ multiplet plus another short multiplet that can be either short of the same kind or an isolated one, which contains the null states of the prior short multiplet. In this sense the two short multiplets recombine into the long one at the unitarity threshold:

$$L[j]_{\Delta_L \to \Delta_A}^{\{R\}} \to A_l[j]_{\Delta_A}^{\{R\}} \oplus X_l[j]_{\Delta_X}^{\{R\}} \tag{A.0.6}$$

, where here $X \in (A, B, C, D)$ and the multiplets that contribute to the recombination have the same Lorentz and R-symmetry quantum numbers. Note that there are certain kinds of short multiplets that are never involved in such recombinations: these are called absolutely protected multiplets.

Below, we present the multiplet content and the shortening conditions for the $\mathcal{N} = 2$ and $\mathcal{N} = 4$ theories.

Starting from the $\mathcal{N} = 2$ theories, the four independent Poincaré supercharges are labeled by the projections of the spacetime spin and the R-symmetry spin:

$$\{Q = [\pm\frac{1}{2}]^{(-1)}, \bar{Q} = \pm\frac{1}{2}]^{(1)}\} \tag{A.0.7}$$

The unitary superconformal multiplets, are obtained by imposing independent shortening conditions for the above supercharges. The shortening conditions and the a full list of the superconformal multiplets are given in [39]. Here we refer to some of the characteristic $\mathcal{N} = 2$ multiplets:

$$L\bar{B}_1[0]_r^{(r>0)} = [0]_r^{(r)} \oplus [\frac{1}{2}]_{r+\frac{1}{2}}^{(r-1)} \oplus [0]_{r+1}^{(r-2)} \tag{A.0.8}$$

$$A_1\bar{A}_1[1]_2^{(0)} = [1]_2^{(0)} \oplus [\frac{3}{2}]_{\frac{5}{2}}^{(\pm 1)} \oplus [2]_3^{(0)} \tag{A.0.9}$$

$$A_2\bar{A}_2[0]_1^{(0)} = [0]_1^{(0)} \oplus [\frac{1}{2}]_{\frac{3}{2}}^{(\pm 1)} \oplus [0]_2^{(0)} \oplus [1]_2^{(0)} \tag{A.0.10}$$

the first is the *superpotential* multiplet, the second the *stress-tensor* multiplet and the last one the *vector current* multiplet.

Regarding finally the $\mathcal{N} = 4$ multiplets, again the shortening conditions and the full list of multiplets, is found in [39]. There are eight independent Poincaré supercharges:

$$\{[\pm\frac{1}{2}]_{1/2}^{(1,1)}, [\pm\frac{1}{2}]_{1/2}^{(1,-1)}, [\pm\frac{1}{2}]_{1/2}^{(-1,1)}, [\pm\frac{1}{2}]_{1/2}^{(-1,-1)}\} \tag{A.0.11}$$

and the short superconformal multiplets are of the short-at-threshold, which can be involved in recombinations and of B-short-isolated type, which are absolutely protected:

$$A_1[j]_{1+j+R+R'}^{(R,R')}(j > 0), \quad A_2[0]_{1+R+R'}^{R,R'}, \quad B_1[0]_{R+R'}^{R,R'} \tag{A.0.12}$$

These are the objects of interest throughout Part III.

Appendix B

The supersymmetric Janus solution

In the second chapter of Part I, we have introduced the pair of harmonic functions $h, \hat{h}$, which gave a solution of the supergravity equations on $AdS_4 \times M_6$ with $AdS_5 \times S^5$ aymptotic regions and point singularities supporting fivebrane and threebrane charge. Here we will present the pure $AdS_5 \times S^5$ solution and its Janus generalization.

Setting $\gamma = \hat{\gamma} = 0$ along with $\beta = \hat{\beta} = 0$, results to the simple form for the harmonic functions:

$$h = -i\alpha \sinh(z) + \text{c.c} \tag{B.0.1}$$

$$\hat{h} = \hat{\alpha} \cosh(z) + \text{c.c} \tag{B.0.2}$$

This solution is characterized by the absence of fivebrane charges as well as by a constant dilaton ($\delta\phi = 0$) and a non-vanishing five-form background. Substituting $h, \hat{h}$ in the metric ansatz by computing the various metric factors of the solution, results to the $AdS_5 \times S^5$ metric:

$$ds^2_{(10)} = L_5^2 \Big(\cosh^2(x) ds^2_{AdS_4} + (dx^2 + dy^2) + \sin^2(y) ds^2_{S^2} + \cos^2(y) ds^2_{\hat{S}^2} \Big) \tag{B.0.3}$$

Its radius is given in terms of the only parameters of the solution, $L_5 = 2(\alpha\hat{\alpha})^{1/4}$.

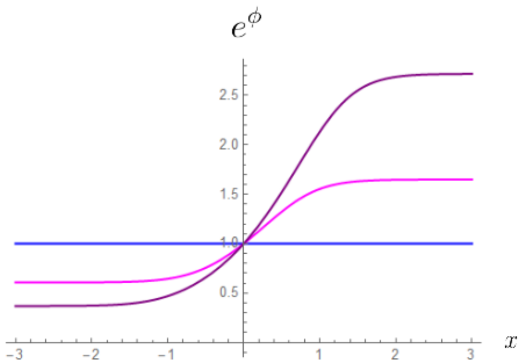
Reintroducing the parameters $\beta, \hat{\beta}$ in the above solution, accounts for the deformation which results to its so-called *supersymmetric Janus* generalization:

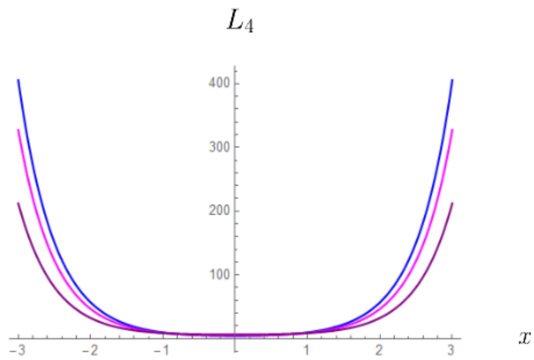
$$h = -i\alpha \sinh(z - \beta) + \text{c.c} \tag{B.0.4}$$

$$\hat{h} = \hat{\alpha} \cosh(z - \hat{\beta}) + \text{c.c} \tag{B.0.5}$$

,where the deformation corresponds to a translation along the strip $\Sigma_{(2)}$ which preserves the boundary conditions of the two harmonic functions. This is a supersymmetric domain wall between two asymptotic regions with $AdS_5 \times S^5$ geometry. The dilaton interpolates between two constant values at the asymptotic regions (ϕ_+, ϕ_- at $\pm\infty$) and the above parameters denote the difference between these values :$\beta = -\hat{\beta} = \delta\phi/2$. The radius of the Janus solution depends now on the dilaton variation, just by a rescaling by a factor $\cosh^{1/4}(\delta\phi)$: $L_5 = 2(\alpha\hat{\alpha}\cosh(\delta\phi))^{1/4}$, whereas the dilaton in the two asymptotic regions is given by $e^{2\phi} = \hat{\alpha} e^{\pm\delta\phi}/\alpha$.

The plot at the left hand side presents the dilaton, plotted for different asymptotic values, as a function of the horizontal strip coordinate. The right hand side plot, is the one of the AdS_4 warp factor for the set of dilaton variations of the left plot ($\delta\phi = 0, 1, 2$ for the blue, magenta and purple curves accordingly). Both plots are along $y = \pi/4$. As the dilaton variation increases, the warp factor broadens and flattens. Finally, in the limit of large dilaton variation, $\delta\phi \to$ the geometry approaches $AdS_4 \times R$ [20]





Appendix C

Combinatorics of linear quivers

We collect here formulae for the different parametrizations of the discrete data of the good linear quivers, and we establish a lemma used in section 5.4.4 of the main text.

The mirror-symmetric parametrization of the quiver is in terms of two partitions $(\rho, \hat{\rho})$ with an equal total number N of boxes, if these partitions are viewed as Young diagrams. We label entries of these partitions and of their transposes as

$$\begin{aligned}
\rho &= (l_1, l_2, \ldots, l_{k+1}) \quad \text{with} \quad l_1 \geq l_2 \geq \cdots \geq l_{k+1} \geq 1, \\
\rho^T &= (l_1^T, l_2^T, \ldots, l_{l_1}^T) \quad \text{with} \quad l_1^T \geq l_2^T \geq \cdots \geq l_{l_1}^T \geq 1, \\
\hat{\rho} &= (\hat{l}_1, \hat{l}_2, \ldots, \hat{l}_{\hat{k}+1}) \quad \text{with} \quad \hat{l}_1 \geq \hat{l}_2 \geq \cdots \geq \hat{l}_{\hat{k}+1} \geq 1, \\
\hat{\rho}^T &= (\hat{l}_1^T, \hat{l}_2^T, \ldots, \hat{l}_{\hat{l}_1}^T) \quad \text{with} \quad \hat{l}_1^T \geq \hat{l}_2^T \geq \cdots \geq \hat{l}_{\hat{l}_1}^T \geq 1,
\end{aligned} \tag{C.0.1}$$

where we used the fact that the number of rows of ρ^T is given by the longest row l_1 of ρ, we denoted the number of rows of ρ as $l_1^T = k+1 \geq 2$, and likewise for hatted quantities. To simplify formulae, the sequences (l_j), $(l_{\hat{\jmath}}^T)$, $(\hat{l}_{\hat{\jmath}})$, $(\hat{l}_j^T)$ are extended with zeros when j or $\hat{\jmath}$ goes beyond the last entry. The total number of boxes is $\sum_j l_j = \sum_{\hat{\jmath}} l_{\hat{\jmath}}^T = \sum_{\hat{\jmath}} \hat{l}_{\hat{\jmath}} = \sum_j \hat{l}_j^T = N$.

In the string-theory embedding ρ and $\hat{\rho}$ describe how N D3-branes end on two sets of five-branes: on $k+1$ NS5-branes to the left and on $\hat{k}+1$ D5-branes to the right.[1] The number of D3-branes ending on the jth NS5-brane (or its linking number which is invariant under brane moves) is l_j, and likewise for the hatted quantities A useful alternative parametrization of these partitions is in terms of the numbers of their same-length rows

$$\begin{aligned}
\rho &= (\underbrace{1 + \cdots + 1}_{\hat{M}_1} + \cdots + \underbrace{\ell + \cdots + \ell}_{\hat{M}_\ell} + \cdots + \underbrace{\hat{k} + \cdots + \hat{k}}_{\hat{M}_{\hat{k}}}) \, , \\
\hat{\rho} &= (\underbrace{1 + \cdots + 1}_{M_1} + \cdots + \underbrace{\ell + \cdots + \ell}_{M_\ell} + \cdots + \underbrace{k + \cdots + k}_{M_k}) \, ,
\end{aligned} \tag{C.0.2}$$

where we used the good property $\hat{\rho}^T > \rho$ which implies that $l_1 \leq \hat{k}$ and $\hat{l}_1 \leq k$. Note that here some of the M_ℓ and $\hat{M}_\ell$ may vanish, when there are no fundamental hypermultiplets at the corresponding gauge-group nodes. Note also that the label ξ for groups of balanced nodes in

[1] In some of the earlier literature, especially ref. [12], ρ designated the partition of D3-branes among D5-branes and $\hat{\rho}$ the partition among NS5-branes. Our flipped convention here is chosen so as to remove all hats from the data of the electric quiver, defined as the theory whose manifest flavour symmetry is realized on D5-branes. Note in particular that in the parametrization (C.0.2) the number of same-length rows of $\hat{\rho}$ runs over $j = 1, \cdots, k$.

section 5.4.4 runs over stacks of NS5-branes with $\hat{M}_\ell > 1$, i.e. over nodes in the magnetic quiver with non-abelian flavour groups.

The electric and magnetic gauge groups are $\prod_{j=1}^{k} U(N_j)$ and $\prod_{\hat{\jmath}=1}^{\hat{k}} U(\hat{N}_{\hat{\jmath}})$:

$$\begin{array}{ll}
\begin{tikzcd}[column sep=small] N_1 \arrow[r, dash] \arrow[d, dash] & N_2 \arrow[r, dash] \arrow[d, dash] & \cdots \arrow[r, dash] & N_k \arrow[d, dash] \\ \boxed{M_1} & \boxed{M_2} & & \boxed{M_k} \end{tikzcd} & \begin{aligned} N_j &= N_{j-1} + \hat{l}_j^T - l_j \text{ with } N_0 = 0, \\ M_j &= \hat{l}_j^T - \hat{l}_{j+1}^T, \end{aligned} \\
\begin{tikzcd}[column sep=small] \hat{N}_1 \arrow[r, dash] \arrow[d, dash] & \hat{N}_2 \arrow[r, dash] \arrow[d, dash] & \cdots \arrow[r, dash] & \hat{N}_{\hat{k}} \arrow[d, dash] \\ \boxed{\hat{M}_1} & \boxed{\hat{M}_2} & & \boxed{\hat{M}_{\hat{k}}} \end{tikzcd} & \begin{aligned} \hat{N}_{\hat{\jmath}} &= \hat{N}_{\hat{\jmath}-1} + l_{\hat{\jmath}}^T - \hat{l}_{\hat{\jmath}} \text{ with } \hat{N}_0 = 0, \\ \hat{M}_{\hat{\jmath}} &= l_{\hat{\jmath}}^T - l_{\hat{\jmath}+1}^T. \end{aligned}
\end{array} \tag{C.0.3}$$

The full 3d $\mathcal{N} = 4$ flavour group is $G \times \hat{G}$ with $G = (\prod_{j=1}^{k} U(M_j))/U(1)$ and $\hat{G} = (\prod_{\hat{\jmath}=1}^{k} U(\hat{M}_{\hat{\jmath}}))/U(1)$. By definition of transposition, $\hat{l}_j^T$ counts rows of $\hat{\rho}$ with at least j boxes, so the following difference counts rows of $\hat{\rho}$ with exactly j boxes:

$$\begin{aligned} M_j &= \hat{l}_j^T - \hat{l}_{j+1}^T = \#\{\hat{\imath} \mid \hat{l}_{\hat{\imath}} = j\}, \\ \text{and likewise} \quad \hat{M}_{\hat{\jmath}} &= l_{\hat{\jmath}}^T - l_{\hat{\jmath}+1}^T = \#\{i \mid l_i = \hat{\jmath}\}. \end{aligned} \tag{C.0.4}$$

We restrict our attention to *good theories*: those with all $N_j \geq 1$ and $\hat{N}_{\hat{\jmath}} \geq 1$. In particular, $1 \leq \hat{N}_1 = l_1^T - \hat{l}_1 = k + 1 - \hat{l}_1$, namely $\hat{l}_1 \leq k$. Likewise, $l_1 \leq \hat{k}$.

An important quantity is the *balance* of a node. It takes a very simple form in terms of the partitions:

$$\begin{aligned} N_{j+1} + N_{j-1} + M_j - 2N_j &= (N_{j+1} - N_j) - (N_j - N_{j-1}) + M_j \\ &= \hat{l}_{j+1}^T - l_{j+1} - \hat{l}_j^T + l_j + \hat{l}_j^T - \hat{l}_{j+1}^T = l_j - l_{j+1}. \end{aligned} \tag{C.0.5}$$

The node j is *balanced* if this vanishes. An interval $\mathcal{B} \subseteq [1, k]$ of balanced nodes of the electric quiver thus corresponds to $|\mathcal{B}| + 1$ consecutive l_j equal to the same value $\hat{\jmath}$. In terms of the transposed partition, this means $\hat{M}_{\hat{\jmath}} = l_{\hat{\jmath}}^T - l_{\hat{\jmath}+1}^T = |\mathcal{B}| + 1$. This is the well-known $SU(|\mathcal{B}| + 1)$ flavour symmetry enhancement.

Lemma C.0.1

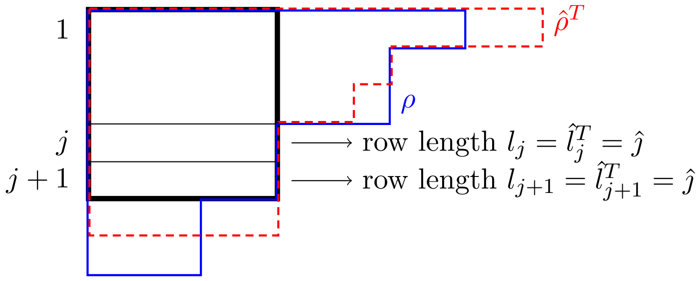
If the electric quiver has a balanced abelian node $N_j = 1$ then one of the following possibilities holds:

1. *$1 < j < k$ and $M_j = 0$ and $N_{j-1} = N_{j+1} = 1$;*
2. *$j = k = 1$ and $M_1 = 2$ (this is the $T[SU(2)]$ theory);*
3. *$j = 1$ and $M_1 = 1$ and $N_2 = 1$;*
4. *$j = k$ and $M_k = 1$ and $N_{k-1} = 1$;*
5. *$j = 1$ and $M_1 = 0$ and $N_2 = 1$;*
6. *$j = k$ and $M_k = 0$ and $N_{k-1} = 2$.*

The corresponding magnetic gauge group (at position $\hat{\jmath} := l_j$) is abelian in the first four cases and nonabelian in the last two.

Proof: The balance condition reads $N_{j-1} + N_{j+1} + M_j = 2N_j = 2$. This implies that (N_{j-1}, M_j, N_{j-1}) are $(1,0,1)$, $(0,2,0)$, $(0,1,1)$, $(1,1,0)$, $(0,0,2)$ or $(2,0,0)$. For each case where $N_{j-1} = 0$ we deduce $j = 1$ because all nodes in $[1,k]$ have non-zero rank. Similarly, $N_{j+1} = 0$ implies $j = k$. We then work out the rank of the magnetic gauge group in each case.

Case 1. From $N_j - N_{j-1} = 0$ and $M_j = 0$ and $N_{j+1} - N_j = 0$ we see that $l_j = \hat{l}^T_j = \hat{l}^T_{j+1} = l_{j+1}$ (we denote this $\hat{\jmath}$). Thus the intersection of ρ (drawn in blue below) and $\hat{\rho}^T$ (drawn in red and dashed) includes a $(j+1) \times \hat{\jmath}$ rectangle (drawn as thick black lines), and the two partitions share a boundary



By definition, $\hat{N}_{\hat{\jmath}}$ counts boxes in rows 1 through $\hat{\jmath}$ of ρ^T, minus those in the same rows of $\hat{\rho}$. Removing the common rectangle, this compares the numbers of boxes of the two partitions below the rectangle. Since the total numbers of boxes in both partitions are the same, it is equivalent to comparing boxes above the lower edge of the rectangle, hence $\hat{N}_{\hat{\jmath}} = N_{j+1} = 1$.

Case 2. $T[SU(2)]$ is self-mirror and abelian.

Cases 3. and 5. $N_1 = 1$ gives $\hat{l}^T_1 = l_1 + 1$. Thus, $\hat{N}_{l_1}$ counts boxes of ρ^T (this partition has l_1 rows) minus all boxes of $\hat{\rho}$ except its last ($\hat{l}^T_1$-th) row. Since $|\rho^T| = |\hat{\rho}|$, we conclude that the rank we care about is $\hat{N}_{l_1} = \hat{l}_{\hat{l}^T_1}$. This in turn is equal to the number of entries of $\hat{\rho}^T$ equal to $\hat{l}^T_1$. Note now that $\hat{l}^T_1 = \hat{l}^T_2 + M_1$. If $M_1 > 0$ (case 3) then $\hat{l}^T_2 < \hat{l}^T_1$ so $\hat{N}_{l_1} = 1$. If $M_1 = 0$ (case 5) then $\hat{l}^T_2 = \hat{l}^T_1$ so $\hat{N}_{l_1} \geq 2$.

Cases 4. and 6. $N_k = 1$ (and $N_{k+1} = 0$) gives $l_{k+1} = \hat{l}^T_{k+1} + 1$, while balance gives $l_k = l_{k+1}$. On general grounds, $1 \leq \hat{N}_1 = l^T_1 - \hat{l}_1 = k + 1 - \hat{l}_1$ so the number of rows $\hat{l}_1$ of $\hat{\rho}^T$ is $\leq k$, hence in particular $\hat{l}^T_{k+1} = 0$. From all this we deduce that $l_k = l_{k+1} = 1$ and that we want to know $\hat{N}_1$. Now use $\hat{l}^T_k = \hat{l}^T_{k+1} + M_k$. If $M_k = 0$ then this vanishes so $\hat{\rho}^T$ has at most $k-1$ rows, so $\hat{N}_1 = k + 1 - \hat{l}_1 \geq 2$. If $M_k > 0$ then $\hat{\rho}^T$ has k rows, namely $\hat{N}_1 = k + 1 - \hat{l}_1 = 1$.

Appendix D

Index and plethystic exponentials

The twisted partition function on $S^2 \times S^1$ of the $T_\rho^{\hat\rho}$ theory is given by a multiple sum over monopole charges and a multiple integral over gauge fugacities, see e.g. [113]

$$
\begin{aligned}
\mathcal{Z}_{S^2\times S^1} = \prod_{j=1}^{k}\left[\frac{1}{N_j!}\sum_{m_j\in\mathbb{Z}^{N_j}}\int\prod_{\alpha=1}^{N_j}\frac{dz_{j,\alpha}}{2\pi i z_{j,\alpha}}\right]\Bigg\{\prod_{j=1}^{k}\prod_{\alpha=1}^{N_j} w_j^{m_{j,\alpha}} Z_{j,\alpha}^{\text{vec,diag}} \\
\prod_{j=1}^{k}\prod_{\alpha\neq\beta}^{N_j} Z_{j,\alpha,\beta}^{\text{vec,off-diag}}\prod_{j=1}^{k}\prod_{p=1}^{M_j}\prod_{\alpha=1}^{N_j} Z_{j,p,\alpha}^{\text{fund,hyp}}\prod_{j=1}^{k-1}\prod_{\alpha=1}^{N_j}\prod_{\beta=1}^{N_{j+1}} Z_{j,\alpha,j+1,\beta}^{\text{bifund,hyp}}\Bigg\}
\end{aligned}
\tag{D.0.1}
$$

where

$$
Z_{j,\alpha}^{\text{vec,diag}} = \frac{(q^{\frac12}t;q)_\infty}{(q^{\frac12}t^{-1};q)_\infty} \tag{D.0.2a}
$$

$$
\begin{aligned}
Z_{j,\alpha,\beta}^{\text{vec,off-diag}} &= (q^{\frac12}t^{-1})^{-\frac12|m_{j,\alpha}-m_{j,\beta}|}(1-q^{\frac12|m_{j,\alpha}-m_{j,\beta}|}z_{j,\beta}z_{j,\alpha}^{-1}) \\
&\quad\times\frac{(tq^{\frac12+|m_{j,\alpha}-m_{j,\beta}|}z_{j,\beta}z_{j,\alpha}^{-1}\,;\,q)_\infty}{(t^{-1}q^{\frac12+|m_{j,\alpha}-m_{j,\beta}|}z_{j,\beta}z_{j,\alpha}^{-1}\,;\,q)_\infty}
\end{aligned}
\tag{D.0.2b}
$$

$$
Z_{j,p,\alpha}^{\text{fund,hyp}} = (q^{\frac12}t^{-1})^{\frac12|m_{j,\alpha}|}\frac{(t^{-\frac12}q^{\frac34+\frac12|m_{j,\alpha}|}z_{j,\alpha}^{\pm1}\mu_{j,p}^{\mp1}\,;\,q)_\infty}{(t^{\frac12}q^{\frac14+\frac12|m_{j,\alpha}|}z_{j,\alpha}^{\mp1}\mu_{j,p}^{\pm1}\,;\,q)_\infty} \tag{D.0.2c}
$$

$$
Z_{j,\alpha,j+1,\beta}^{\text{bifund,hyp}} = (q^{\frac12}t^{-1})^{\frac12|m_{j,\alpha}-m_{j+1,\beta}|}\frac{(t^{-\frac12}q^{\frac34+\frac12|m_{j,\alpha}-m_{j+1,\beta}|}z_{j,\alpha}^{\pm1}z_{j+1,\beta}^{\mp1}\,;\,q)_\infty}{(t^{\frac12}q^{\frac14+\frac12|m_{j,\alpha}-m_{j+1,\beta}|}z_{j,\alpha}^{\mp1}z_{j+1,\beta}^{\pm1}\,;\,q)_\infty}\,. \tag{D.0.2d}
$$

The expressions (D.0.2) are the one-loop determinants of the $\mathcal{N}=4$ multiplets of $T_\rho^{\hat\rho}$, namely the Cartan and charged vector multiplets, and the fundamental and bifundamental hypermultiplets. The variables q, t are the fugacities defined in eq. (5.2.2), $z_{j,\alpha}$ (where α labels the Cartan generators) are the S^1 holonomies of the $U(N_j)$ gauge field and $m_{j,\alpha}$ its 2-sphere fluxes, *viz.* the monopole charges of the corresponding local operator in $\mathbb{R}^3$. Furthermore $\mu_{j,p}$ are flavor fugacities, w_j is a fugacity for the topological $U(1)$ symmetry whose conserved current is $\operatorname{tr}{}^*F_{(j)}$, while the q-Pochhammer symbols $(a;q)_\infty$ are defined by

$$(a;q)_\infty = \prod_{n=0}^{\infty}(1-aq^n) \qquad \text{and} \quad (\ldots a^{\pm 1}b^{\mp 1};q)_\infty = (\ldots ab^{-1};q)_\infty(\ldots a^{-1}b;q)_\infty \,. \tag{D.0.3}$$

Compared to the expressions in ref. [113] we have here replaced the background flux coupling to any given multiplet by its absolute value. This is allowed because the $\mathcal{N}=4$ multiplets are self-conjugate, so their one-loop determinants are insensitive to the sign of the flux. The theory is also free from parity anomalies, so that the overall signs are unambiguous. [1]

At leading order in the q expansion, the contribution of each monopole sector $\mathbf{m}=\{m_{j,\alpha}\}$ to the superconformal index is $(q^{\frac{1}{2}}t^{-1})^{\Delta(\mathbf{m})}$, where

$$2\Delta(\mathbf{m}) = \sum_{j=1}^{k}\sum_{\alpha,\beta=1}^{N_j} -|m_{j,\alpha}-m_{j,\beta}| + \sum_{j=1}^{k} M_j \sum_{\alpha=1}^{N_j}|m_{j,\alpha}| + \sum_{j=1}^{k-1}\sum_{\alpha=1}^{N_j}\sum_{\beta=1}^{N_{j+1}}|m_{j,\alpha}-m_{j+1,\beta}|. \tag{D.0.4}$$

The sphere Casimir energy $\Delta(\mathbf{m})$ is the scaling dimension [and the $SO(3)_C$ spin] of the corresponding monopole operator [32][31]. It is known that in $\mathcal{N}=4$ theories monopole-operator dimensions are one-loop exact, and that they are strictly positive for good linear quivers [65]. The index (D.0.1) admits therefore an expansion in positive powers of q.

It is useful to rewrite the superconformal index in terms of the plethystic exponential (PE) which is defined, for any function $f(v_1, v_2, \cdots)$ of arbitrarily many variables, by the following expression

$$\mathrm{PE}(f) = \exp\left(\sum_{n=1}^{\infty}\frac{1}{n}f(v_1^n, v_2^n, \cdots)\right) \,. \tag{D.0.5}$$

The reader can verify the following simple identities:

$$\mathrm{PE}(f+g) = \mathrm{PE}(f)\,\mathrm{PE}(g)\,, \quad \mathrm{PE}(-v) = (1-v)\,, \quad \mathrm{PE}((a,q)_\infty) = \mathrm{PE}(-\frac{a}{1-q}) \,. \tag{D.0.6}$$

Using these identities one can bring the index to the following form

$$\begin{aligned}
\mathcal{Z}_{S^2\times S^1} = \prod_{j=1}^{k}\left[\frac{1}{N_j!}\sum_{m_j\in\mathbb{Z}^{N_j}}\int\prod_{\alpha=1}^{N_j}\frac{dz_{j,\alpha}}{2\pi i z_{j,\alpha}}\right]\Bigg\{&(q^{\frac{1}{2}}t^{-1})^{\Delta(\mathbf{m})}\prod_{j=1}^{k}\left[\prod_{\alpha}^{N_j}w_j^{m_{j,\alpha}}\prod_{\alpha\neq\beta}^{N_j}(1-q^{\frac{1}{2}|m_{j,\alpha}-m_{j,\beta}|}z_{j,\beta}z_{j,\alpha}^{-1})\right]\\
\times\,\mathrm{PE}\Bigg(&\sum_{j=1}^{k}\sum_{\alpha,\beta=1}^{N_j}\frac{q^{\frac{1}{2}}(t^{-1}-t)}{1-q}q^{|m_{j,\alpha}-m_{j,\beta}|}z_{j,\beta}z_{j,\alpha}^{-1}\\
&+\frac{(q^{\frac{1}{2}}t)^{\frac{1}{2}}(1-q^{\frac{1}{2}}t^{-1})}{1-q}\sum_{j=1}^{k}\sum_{p=1}^{M_j}\sum_{\alpha=1}^{N_j}q^{\frac{1}{2}|m_{j,\alpha}|}\sum_{\pm}z_{j,\alpha}^{\mp 1}\mu_{j,p}^{\pm 1}\\
&+\frac{(q^{\frac{1}{2}}t)^{\frac{1}{2}}(1-q^{\frac{1}{2}}t^{-1})}{1-q}\sum_{j=1}^{k-1}\sum_{\alpha=1}^{N_j}\sum_{\beta=1}^{N_{j+1}}q^{\frac{1}{2}|m_{j,\alpha}-m_{j+1,\beta}|}\sum_{\pm}z_{j,\alpha}^{\mp 1}z_{j+1,\beta}^{\pm 1}\Bigg)\Bigg\}\,.
\end{aligned} \tag{D.0.7}$$

[1] There exists a subtle sign $(-)^{e\cdot m}$ related to the change of spin of dyonic states with charges (e,m). The $T_\rho^{\hat\rho}$ theory has no Chern-Simons terms, so the flux ground states have no electric charge, e, and contribute with plus signs to the index. For excited states in the flux background this sign can be absorbed in the fugacities $z_{j,\alpha}$; it is in the end irrelevant since the $z_{j,\alpha}$ integrations project to gauge-invariant states.

This is equation (5.2.3) in the main text. Notice that after extracting some factors, the contributions of vector, fundamental and bifundamental multiplets add up in the argument of the plethystic exponential, as they would in the standard exponential function.

The usefulness of the above rewriting can be illustrated with a simple example, that of a free hypermultiplet whose superconformal index is

$$\mathcal{Z}^{\text{free hyp}}_{S^2\times S^1} = \frac{(t^{-\frac{1}{2}}q^{\frac{3}{4}}\mu^{\mp 1}\,;\,q)_\infty}{(t^{\frac{1}{2}}q^{\frac{1}{4}}\mu^{\pm 1}\,;\,q)_\infty} \;=\; \mathrm{PE}\left(\frac{(q^{\frac{1}{4}}t^{\frac{1}{2}} - q^{\frac{3}{4}}t^{-\frac{1}{2}})}{1-q}(\mu+\mu^{-1})\right)\,. \tag{D.0.8}$$

One recognizes in the PE exponent the contributions of the charge-conjugate $\mathcal{N}=2$ chiral multiplets, each contributing to the index with one scalar ($\Delta = J_3^H = \frac{1}{2}$ and $J_3 = J_3^C = 0$) and one fermionic state (with $\Delta = 1$, $J_3^H = 0$ and $J_3 = J_3^C = \frac{1}{2}$). As for the factor of $(1-q)$, this sums up descendant states obtained by the action of the derivative that raises both Δ and J_3 by one unit. Multiparticle states (created by products of fields) are taken care of by the plethystic exponential, the informaiton in them is in this simple case redundant.

Of course in interacting theories supersymmetric multiparticle states may be null, due for example to F-term conditions. The plethystic exponent must in this case be interpreted appropriately, as we discuss in the main text.

Appendix E

Superconformal index of $T[SU(2)]$

E.1 Analytical computation of the index

In the main text, it has been described in detail how one can obtain the superconformal index for 'good' 3d $\mathcal{N}=4$ theories, in terms of the relevant fugacities of the theory as well as in terms of a supercharacter expansion. A last addition to this already rich picture, is an exact, analytical computation of the $S^2 \times S^1$ partition function, which gives a closed form result. This is realized by a brute-force evaluation of the contour integrals by employing q-special function techniques [88] and is of course doable for all the theories presented above.

A simple and interesting theory to work out at this point is $T[SU(2)]$. $T[SU(2)]$ is special in the sense that both the electric and the magnetic flavor currents are non-abelian symmetry currents, a trait that is not observed in $N_f \geq 3$ SQED theories, as one can easily check ,there are relations we cannot deduce from the F-term conditions.

Here we present all three approaches to the superconformal index with focus on the analytical calculation and in the next section of this appendix we display the factorization property of the $T[SU(2)]$ index which also characterizes 3d $\mathcal{N}=2$ theories.

The expression for the superconformal index of this theory reads:

$$\mathcal{Z}_{S^2\times S^1}^{T[SU(2)]} = \sum_{m\in\mathbb{Z}} \oint_{S^1(1)} \frac{dz}{2\pi i z} \frac{(q^{\frac{1}{2}}t;q)_\infty}{(q^{\frac{1}{2}}t^{-1};q)_\infty} (q^{1/2}t^{-1})^{|m|} \prod_{j=1}^{2} \frac{(t^{-\frac{1}{2}}q^{\frac{3}{4}+\frac{|m|}{2}}z^{\pm}\mu_j^{\mp};q)_\infty}{(t^{\frac{1}{2}}q^{\frac{1}{4}+\frac{|m|}{2}}z^{\mp}\mu_j^{\pm};q)_\infty} \tag{E.1.1}$$

,with the compact notation for the q-Pochhammer symbols: $(a^{\pm};q)_\infty = (a;q)_\infty(a^{-1};q)_\infty$ and μ_j as the electric flavor fugacity.

The first approach is based on summing over monopole charges and expanding the integrand into q and z-series. In particular we expand the integrand around $q=0$ to first order, then expand around $z=0$ and finally pick the corresponding z-series coefficient. The result reads:

$$\mathcal{Z}_{S^2\times S^1}^{T[SU(2)]} = 1 + q^{\frac{1}{2}}(3t+3t^{-1}) + q(5t^2 - 7 + 5t^{-2}) + \mathcal{O}(q^{3/2}) \tag{E.1.2}$$

The expression is manifestly mirror symmetric, namely symmetric under the inversion $t \to t^{-1}$, as expected (and this has been checked up to $\mathcal{O}(q^{10})$). A more informative form of this result is obtained by rewriting this expression in terms of supercharacters, using directly the relations given in the main text. In this way one can clearly extract the representation content for $T[SU(2)]$:

$$\mathcal{Z}^{T[SU(2)]}_{S^2\times S^1} = 1 + 3(\chi_{B_1[0]_1^{(1,0)}} + \chi_{B_1[0]_1^{(0,1)}}) + 5(\chi_{B_1[0]_2^{(2,0)}} + \chi_{B_1[0]_2^{(0,2)}}) - \chi_{B_1[0]_2^{(1,1)}} + \ldots \tag{E.1.3}$$

We observe that the index indeed enumerates the absolutely protected B-type multiplets which contain the sought for marginal operators. The rest of the terms in the expansion correspond to the mixed branch and are just products of characters of the above B-type multiplets.

Of course, the term of interest is the q-coefficient, which contains for a given quiver theory all products of electric and magnetic flavor currents along with corrections coming from completing the expressions for the $B_1[0]_1^{(1,0),(0,1)}$ characters, the contribution from the stress-tensor and corrections from F-term relations. This coefficient comes partly from the zero monopole sector ($P(m) = 0$) and partly from the $P(m) = 2$ sector and is obtained by following the procedure of analytically studying contributions to the index from each monopole sector. The result for $T[SU(2)]$ reads:

$$\mathcal{Z}^{T[SU(2)]}_{S^2\times S^1}|_q = -3 - \mu - \mu^{-1} - \tilde{w} - \tilde{w}^{-1} \tag{E.1.4}$$

, where $\mu, \tilde{w}$ stand for the electric and magnetic flavor fugacities. At this point we see that this matches exactly with the q-term in the above result.

We can now proceed and display the procedure which leads to a closed form result for the index of $T[SU(2)]$. The integration contour is the maximal torus of the gauge group and hence in our case this corresponds to the unit circle. Since $|q| < 1$ and by assuming that the same holds for the R-symmetry fugacity t, it follows that the contribution to the poles comes only from the $(t^{\frac{1}{2}}q^{\frac{1}{4}+\frac{|m|}{2}}z^{-1}\mu; q)_\infty$ and $(t^{\frac{1}{2}}q^{\frac{1}{4}+\frac{|m|}{2}}z^{-1}\mu^{-1}; q)_\infty$ denominators of (E.2.1) and therefore:

$$\mathcal{Z}^{T[SU(N)]}_{S^2\times S^1} = \sum_{j=0}^{\infty} \mathrm{Res}[z_j = t^{\frac{1}{2}}\mu q^{\frac{1}{4}+\frac{|m|}{2}+j}] + \sum_{\hat{j}=0}^{\infty} \mathrm{Res}[z_{\hat{j}} = t^{\frac{1}{2}}\mu^{-1} q^{\frac{1}{4}+\frac{|m|}{2}+\hat{j}}] \tag{E.1.5}$$

These infinite sets of residues correspond to the infinite first order poles from the products in the denominators.

By working out the first few pole contributions, we produce the expression for the fist infinite sum, which can be easily brought to the form

$$\begin{aligned}&\frac{(q^{\frac{1}{2}}t; q)_\infty}{(q^{\frac{1}{2}}t^{-1}; q)_\infty}\frac{(t^{-1}q^{\frac{1}{2}}\mu^{-2}; q)_\infty}{(tq^{\frac{1}{2}+|m|}\mu^2; q)_\infty} \cdot \frac{(q^{1+|m|}; q)_\infty}{(q; q)_\infty} \cdot \frac{(t^{-1}q^{\frac{1}{2}}; q)_\infty}{(tq^{\frac{1}{2}+|m|}; q)_\infty} \\ &\qquad \cdot \frac{(q^{1+|m|}\mu^2; q)_\infty}{(\mu^{-2}; q)_\infty}\, {}_3\phi_4\left[\begin{matrix} t\mu^2 q^{\frac{1}{2}}, t\mu^2 q^{\frac{1}{2}+|m|}, tq^{\frac{1}{2}}, tq^{\frac{1}{2}+|m|} \\ q^{1+|m|}, \mu^2 q^{1+|m|}, \mu^2 q \end{matrix}; q, t^{-2}q\right]\end{aligned} \tag{E.1.6}$$

where along the way we have used the expression for the finite q-Pochhammer symbol:

$$
(z;q)_n = \begin{cases} 1,\, n=0 \\ \prod_{i=0}^{n-1}(1-zq^i), n>0 \end{cases} \qquad \text{(E.1.7)}
$$

as well as the definition of the q-hypergeometric series:

$$
{}_r\phi_{r+1}\left[\begin{matrix} a_1,...,a_{r+1} \\ b_1,...,b_r \end{matrix} ;q,z\right] = \sum_{n=0}^{\infty} \frac{(a_1,...,a_{r+1};q)_n}{(q,b_1,...,b_r;q)_n} z^n \qquad \text{(E.1.8)}
$$

and the expression for the first set of residues reads:

$$
\sum_{m\in\mathbb{Z}} \frac{(q^{\frac{1}{2}}t;q)_\infty}{(q^{\frac{1}{2}}t^{-1};q)_\infty}(t^{-1}q^{\frac{1}{2}})^{|m|} \frac{(t^{-1}q^{\frac{1}{2}}\mu^{-2};q)_\infty}{(tq^{\frac{1}{2}+|m|}\mu^2;q)_\infty} \cdot \frac{(q^{1+|m|};q)_\infty}{(q;q)_\infty} \cdot \frac{(t^{-1}q^{\frac{1}{2}};q)_\infty}{(tq^{\frac{1}{2}+|m|};q)_\infty} \cdot \frac{(q^{1+|m|}\mu^2;q)_\infty}{(\mu^{-2};q)_\infty}
$$

$$
\sum_{n=0}^{\infty} \frac{(t\mu^2q^{\frac{1}{2}}, t\mu^2q^{\frac{1}{2}+|m|}, tq^{\frac{1}{2}}, tq^{\frac{1}{2}+|m|};q)_n}{(q, q^{1+|m|}, \mu^2q^{1+|m|}, \mu^2q;q)_n}(t^{-2}q)^n \qquad \text{(E.1.9)}
$$

At this point we proceed by rearranging the sums and working out the expression by using the property:

$$
(Az^m;q)_n = \frac{(Az^m;q)_\infty}{(Az^{m+n})_\infty} \qquad \text{(E.1.10)}
$$

What one achieves in this way is to incorporate the sum over monopole charges in defining q-hypergeometric series. The process after some work finally gives:

$$
\sum_{n=0}^{\infty} \frac{(q^{\frac{1}{2}}t;q)_\infty}{(q^{\frac{1}{2}}t^{-1};q)_\infty}(t^{-1}q^{\frac{1}{2}})^n \frac{(t\mu^2q^{\frac{1}{2}};q)_n}{(q;q)_n} \frac{(tq^{\frac{1}{2}};q)_n}{(\mu^2q;q)_n} \frac{(t^{-1}\mu^{-2}q^{\frac{1}{2}};q)_\infty}{(\mu^{-2};q)_\infty} \frac{(t^{-1}q^{\frac{1}{2}};q)_\infty}{(t\mu^2q^{\frac{1}{2}};q)_\infty} \frac{(\mu^2q;q)_\infty}{(tq^{\frac{1}{2}};q)_\infty}
$$

$$
{}_2\phi_1\left[\begin{matrix} t\mu^2q^{\frac{1}{2}}, tq^{\frac{1}{2}} \\ \mu^2q \end{matrix} ;q,t^{-1}q^{\frac{1}{2}}\right] \qquad \text{(E.1.11)}
$$

Apparently the first few terms of the above expression, correspond to the same q-hypergeometric series. Therefore the final result for the first set of residues reads:

$$
\frac{(q^{\frac{1}{2}}t;q)_\infty}{(q^{\frac{1}{2}}t^{-1};q)_\infty} \frac{(t^{-1}\mu^{-2}q^{\frac{1}{2}};q)_\infty}{(\mu^{-2};q)_\infty} \frac{(t^{-1}q^{\frac{1}{2}};q)_\infty}{(t\mu^2q^{\frac{1}{2}};q)_\infty} \frac{(\mu^2q;q)_\infty}{(tq^{\frac{1}{2}};q)_\infty} {}_2\phi_1\left[\begin{matrix} t\mu^2q^{\frac{1}{2}}, tq^{\frac{1}{2}} \\ \mu^2q \end{matrix} ;q,t^{-1}q^{\frac{1}{2}}\right]^2 \qquad \text{(E.1.12)}
$$

We have therefore the final expression for the first set of residues. The second and last set, differs only by the fact that instead of the fugacity μ, its expression include the inverse fugacity μ^{-1}. Finally the full result for the superconformal index of $T[SU(2)]$ reads:

$$\boxed{\mathcal{Z}^{T[SU(2)]}_{S^2\times S^1} = \frac{(t^{-1}\mu^{-2}q^{\frac{1}{2}};q)_\infty}{(t\mu^2q^{\frac{1}{2}};q)_\infty}\frac{(\mu^2q;q)_\infty}{(\mu^{-2};q)_\infty}\,{}_2\phi_1\left[\begin{matrix} t\mu^2q^{\frac{1}{2}}, tq^{\frac{1}{2}} \\ \mu^2 q \end{matrix}\;;q,t^{-1}q^{\frac{1}{2}}\right]^2 + \left(\mu\to\mu^{-1}\right)} \tag{E.1.13}$$

Of course, expanding around $q=0$ leads us directly back to (E.1.2) and hence everything is consistent. Although this expression is practically mirror symmetric, this fact is not manifest in the expression and perhaps is a relic of (E.2.1).

Finally, an interesting trait of this result is its factorized form, similar to the one encountered in $3d\,\mathcal{N}=2$ theories. In that case, superconformal indices (as well as sphere partition functions) are written as sums of products of basic building blocks, refered to as holomorphic blocks, which are partition functions on $\mathcal{D}^2\times S^1$. This is introduced in the following section.

E.2 $T[SU(2)]$ index as holomorphic blocks

As well-known from the study of 3d $\mathcal{N}=2$ theories [98]-[25] (see also [104] for the $\mathcal{N}=4$ case), superconformal indices (and various other partition functions) are bilinear combinations of basic building blocks, refered to as (anti)holomorphic blocks, which are partition functions on $\mathcal{D}^2 \times S^1$. We here work out this factorization for $T[SU(2)]$, and then verify that the resulting closed-form expression (E.2.8) reproduces our expansion of the superconformal index at order $O(q)$. The structure generalizes but we did not find it useful in concrete calculations, because for generic theories this factorized form contains a large number of terms.

The expression for the full superconformal index of $T[SU(2)]$ reads:

$$\mathcal{Z}^{T[SU(2)]}_{S^2\times S^1} = \frac{(q^{\frac{1}{2}}t;q)_\infty}{(q^{\frac{1}{2}}t^{-1};q)_\infty} \sum_{m\in\mathbb{Z}} (q^{\frac{1}{2}}t^{-1})^{|m|} w^m \oint_{S^1} \frac{dz}{2\pi i z} \prod_{p=1}^{2} \prod_{\pm} \frac{(t^{-\frac{1}{2}} q^{\frac{3}{4}+\frac{|m|}{2}} z^{\pm} \mu_p^{\mp}; q)_\infty}{(t^{\frac{1}{2}} q^{\frac{1}{4}+\frac{|m|}{2}} z^{\mp} \mu_p^{\pm}; q)_\infty} , \tag{E.2.1}$$

where m is the unique monopole charge, and z runs over the unit circle in the classical Coulomb branch $\mathbb{C}$. The integrand has poles at[1]

$$z = z_{s,j} := \mu_s t^{\frac{1}{2}} q^{\frac{1}{4}+\frac{|m|}{2}+j} \quad \text{and} \quad z = \mu_s \big(t^{\frac{1}{2}} q^{\frac{1}{4}+\frac{|m|}{2}+j}\big)^{-1} \quad \text{for } s=1,2 \text{ and integer } j\geq 0. \tag{E.2.2}$$

We calculate the index as an expansion in powers of q, hence $|q|<1$, with $|t|=|\mu_s|=1$. The poles that we named $z_{s,j}$ thus lie inside the $|z|=1$ contour and other poles outside.

To warm up, compute the contribution to $\mathcal{Z}^{T[SU(2)]}_{S^2\times S^1}$ from the pole at $z_{s,0}$ for $m=0$:

$$C_s := \prod_{p\neq s} \frac{(q\mu_s\mu_p^{-1};q)_\infty}{(\mu_s^{-1}\mu_p;q)_\infty} \frac{(q^{\frac{1}{2}}t^{-1}\mu_s^{-1}\mu_p;q)_\infty}{(q^{\frac{1}{2}}t\mu_s\mu_p^{-1};q)_\infty}. \tag{E.2.3}$$

Before moving on to other residues, we note that the identity

$$(iq^{\frac{1}{8}}a^{\frac{1}{2}})^{|m|} \frac{(q^{\frac{3}{4}+\frac{|m|}{2}}a;q)_\infty}{(q^{\frac{1}{4}+\frac{|m|}{2}}a^{-1};q)_\infty} = (iq^{\frac{1}{8}}a^{\frac{1}{2}})^{m} \frac{(q^{\frac{3}{4}+\frac{m}{2}}a;q)_\infty}{(q^{\frac{1}{4}+\frac{m}{2}}a^{-1};q)_\infty} \tag{E.2.4}$$

allows us to replace $|m|\to m$ throughout (E.2.1). The resulting expression involves both positive and negative powers of q, which would make our lives harder if we wanted to expand in powers of q, but leads to nicer residues. We compute the contribution from the $z_{s,j}$ pole for any m:

[1]At first sight there is also a pole at $z=0$, but in fact the q-Pochhammer factors tend to zero there.

$$
\begin{aligned}
&\frac{(q^{\frac{1}{2}}t;q)_\infty}{(q^{\frac{1}{2}}t^{-1};q)_\infty}(q^{\frac{1}{2}}t^{-1}w)^m\prod_{p=1}^{2}\frac{(q^{1+j+\frac{|m|+m}{2}}\mu_s\mu_p^{-1};q)_\infty(q^{\frac{1}{2}-j-\frac{|m|-m}{2}}t^{-1}\mu_s^{-1}\mu_p;q)_\infty}{((q^{-j-\frac{|m|-m}{2}}\mu_s^{-1}\mu_p;q)_\infty)'(q^{\frac{1}{2}+j+\frac{|m|+m}{2}}t\mu_s\mu_p^{-1};q)_\infty}\\
&=C_s\,(q^{\frac{1}{2}}t^{-1}w)^{k_+-k_-}\prod_{p=1}^{2}\frac{(q^{\frac{1}{2}}t\mu_s\mu_p^{-1};q)_{k_+}(q^{\frac{1}{2}-k_-}t^{-1}\mu_s^{-1}\mu_p;q)_{k_-}}{(q\mu_s\mu_p^{-1};q)_{k_+}(q^{-k_-}\mu_s^{-1}\mu_p;q)_{k_-}}
\end{aligned}
\tag{E.2.5}
$$

where the prime in the first line denotes the removal of the vanishing factor in the q-Pochhammer symbol for $p=s$, and we then used finite q-Pochhammer $(a;q)_k=(a;q)_\infty/(aq^k;q)_\infty$ and changed variables to $k_\pm\coloneqq j+\frac{|m|\pm m}{2}\geq 0$. Altogether

$$
\mathcal{Z}^{T[SU(2)]}_{S^2\times S^1}=\sum_{s=1}^{2}C_s\prod_{\pm}\Bigg(\sum_{k_\pm\geq 0}(q^{\frac{1}{2}}t^{-1}w^{\pm1})^{k_\pm}\prod_{p=1}^{2}\frac{(q^{\frac{1}{2}}t\mu_s\mu_p^{-1};q)_{k_\pm}}{(q\mu_s\mu_p^{-1};q)_{k_\pm}}\Bigg). \tag{E.2.6}
$$

We recognize here the q-hypergeometric series

$$
{}_2\phi_1\left[\begin{matrix}a,b\\c\end{matrix}\;\middle|\;q,z\right]\coloneqq\sum_{k\geq 0}\frac{(a;q)_k(b;q)_k}{(q;q)_k(c;q)_k}z^k. \tag{E.2.7}
$$

In terms of $\mu\coloneqq\mu_1\mu_2^{-1}$ and $\hat{\mu}\coloneqq w$

$$
\mathcal{Z}^{T[SU(2)]}_{S^2\times S^1}=\frac{(q\mu;q)_\infty}{(\mu^{-1};q)_\infty}\frac{(q^{\frac{1}{2}}t^{-1}\mu^{-1};q)_\infty}{(q^{\frac{1}{2}}t\mu;q)_\infty}\prod_{\pm}\left({}_2\phi_1\left[\begin{matrix}q^{\frac{1}{2}}t,q^{\frac{1}{2}}t\mu\\q\mu\end{matrix}\;\middle|\;q,q^{\frac{1}{2}}t^{-1}\hat{\mu}^{\pm1}\right]\right)+(\mu\leftrightarrow\mu^{-1}). \tag{E.2.8}
$$

This is the factorized form of the index. It is possible to show, using complicated identities obeyed by q-hypergeometric series, that this result is mirror-symmetric

To compare with the main text we expand in powers of q and organize the series in terms of supercharacters so as to extract the representation content:

$$
\begin{aligned}
\mathcal{Z}^{T[SU(2)]}_{S^2\times S^1}&=1+q^{\frac{1}{2}}t\chi_3(\mu)+q^{\frac{1}{2}}t^{-1}\chi_3(\hat{\mu})+qt^2\chi_5(\mu)+qt^{-2}\chi_5(\hat{\mu})-q(1+\chi_3(\mu)+\chi_3(\hat{\mu}))+O(q^{\frac{3}{2}})\\
&=1+\chi_3(\mu)\,\mathcal{I}_{(1,0)}+\chi_3(\hat{\mu})\,\mathcal{I}_{(0,1)}+\boxed{\chi_5(\mu)\,\mathcal{I}_{(2,0)}}-\mathcal{I}_{(1,1)}+\boxed{\chi_5(\hat{\mu})\,\mathcal{I}_{(0,2)}}+O(q^{\frac{3}{2}})
\end{aligned}
\tag{E.2.9}
$$

where $\chi_3(\mu)\coloneqq\mu+1+\mu^{-1}$ and $\chi_5(\mu)\coloneqq\mu^2+\mu+1+\mu^{-1}+\mu^{-2}$ are characters of $SU(2)$, and we used the short-hand notation for the superconformal indices $\mathcal{I}_{(J^H,J^C)}\coloneqq\mathcal{I}_{B_1[0]^{(J^H,J^C)}}(q,t)$. This agrees with eq. (5.4.32) of section 5.4.

As explained in the paper the following BPS multiplets can be unambiguously identified:

- 1: the identity;
- $\chi_3(\mu)\mathcal{I}_{(1,0)}$: the $SU(2)$ electric-flavour currents;
- $\chi_3(\hat{\mu})\mathcal{I}_{(0,1)}$: the $S\widehat{U}(2)$ magnetic-flavour currents;
- $\chi_5(\mu)\mathcal{I}_{(2,0)}$: products of two electric currents;
- $\chi_5(\hat{\mu})\mathcal{I}_{(0,2)}$: products of two magnetic currents;
- $-\mathcal{I}_{(1,1)}$: the energy-momentum tensor multiplet $A_2[0]^{(0,0)}$.

The bottom component $\tilde{Q}^{\bar{p}}Q^r$ of an electric-current multiplet is the product of an antifundamental and a fundamental chiral scalar (the F-term condition imposes $\tilde{Q}^1Q^1 + \tilde{Q}^2Q^2 = 0$). Since the gauge group is abelian, $\tilde{Q}^{\bar{p}}Q^r$ has rank 1 hence zero determinant. This removes one of the six products of two electric currents, thus explaining why there are only five such products in (E.2.9).

Altogether we see that the $T[SU(2)]$ theory has no mixed marginal (or relevant) chiral operators. All exactly-marginal deformations are purely electric or purely magnetic superpotentials. After imposing the D-term conditions the supeconformal manifold has dimension $10 - 7 = 3$.

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RÉSUMÉ

Dans le cadre de la dualité holographique entre une vaste famille de vides 1/2-maximalement supersymétriques Anti-de Sitter à quatre dimensions (AdS_4) et des théories des champs superconformes $\mathcal{N}$=4 supersymétriques à trois dimensions (sCFT$_3$), nous étudions des questions théoriques majeures de gravité quantique et de théories de jauge. Ce travail a deux directions principales: La premiere partie est consacrée aux mécanismes par lesquels le graviton AdS_4 peut acquérir une petite masse, tandis que la seconde partie concerne la cartographie de la variété superconforme des sCFT$_3$ considérées. En ce qui concerne la question du mecanisme de Higgs pour le graviton d'AdS_4, nous proposons un nouveau mécanisme qui repose sur le couplage "faible" de deux sCFT$_3$s, initialement découplées, en jaugent une symmétrie globale commune. Les deux tenseurs de stress initialement conservés se mélangent et le résultat est une combinaison conservée et une combinaison orthogonale, dont la dimension acquiert une petite dimension anormale. Holographiquement, cette configuration correspond à la connexion de deux univers AdS_4 initialement découplés via un $AdS_5 \times S^5$ fin, autrement appelé une "gorge" de Janus. Le résultat est une théorie AdS4-bimétrique, avec un graviton sans masse et un graviton massif, dont la petite masse correspond à la dimension anormale de la combinaison duale de tenseurs de stress. Nous calculons la masse du graviton, qui est exprimée en termes de données géométriques de la "gorge" de Janus et de l'univers AdS_4 considéré. Une limite particulière de découplage de cette théorie, où le couplage gravitationnel effectif à quatre dimensions de l'un des deux univers tend vers zéro, résulte en une théorie de gravité massive dans AdS_4. En ce qui concerne la deuxième direction de ce travail, les déformations superconformes des sCFT$_3$s considérées qui génèrent la variété superconformale sont des déformations préservant $\mathcal{N} = 2$ supersymétrie, générées par des opérateurs exactement marginaux. Nous présentons comment tous ces opérateurs peuvent être systématiquement extraits de l'index superconforme. Les opérateurs de branche de Coulomb et de Higgs sont pris en compte, tandis qu'une attention particulière est accordée aux opérateurs mixtes. On montre que les modules de branches mixtes de ces théories sont des opérateurs à double-corde qui se transforment dans la représentation (Adj, Adj) des groupes de saveurs électriques et magnétiques, modulo un surcomptage pour les quivers avec des nœuds de jauge abéliens. Enfin, nous commentons sur l'interprétation holographique des résultats, en affirmant que les supergravités mesurées peuvent capturer l'espace des modules tout entier si, outre les paramètres de la solution d'arrière-plan, les modules de quantification des conditions aux limites sont également pris en compte.

MOTS CLÉS

Theorie des cordes, Theories superconformes des champs, Dualité Holographique, Gravité Massive, Localization Supersymetrique, Indice superconforme, theories quiver, varieté superconforme

ABSTRACT

Based on the holographic duality between a large class of half-maximally supersymmetric four-dimensional Anti-de Sitter (AdS_4) vacua and three-dimensional $\mathcal{N} = 4$ superconformal field theories (sCFT$_3$), we study quantum gravitational and gauge theoretic questions. This work has two main directions: The first part is devoted to the mechanisms through which the low-lying AdS_4-graviton can acquire a small mass whereas the second part regards the mapping of the superconformal manifold of the considered $sCFT_3$s. Regarding the question of the graviton Higgsing in AdS_4, we propose a new mechanism which relies on "weakly" coupling two initially decoupled $sCFT_3$s, by gauging a common global symmetry. The two initially conserved stress tensors mix and the result of this mixing is a conserved combination and an orthogonal combination, the scaling dimension of which acquires a small anomalous dimension. Holographically, this setup is dual to connecting two initially decoupled AdS_4 universes via a thin $AdS_5 \times S^5$ or Janus "throat". The result is an AdS_4-bimetric theory, with one massless and one massive graviton, the small mass of which corresponds to the anomalous dimension of the dual stress tensor combination. We compute the mass of the graviton, which is expressed in terms of the geometric data of the Janus "throat" and of the considered AdS_4 universe. A special decoupling limit of this theory, where the effective four-dimensional gravitational coupling of one of the two universes vanishes, results to an AdS_4-Massive gravity theory. Regarding the second direction of this work, superconformal deformations of the considered sCFT$_3$s which generate the superconformal manifold, are $\mathcal{N} = 2$ supersymmetry preserving deformations, generated by exactly marginal operators. We present how all these operators can be consistently extracted from the superconformal index. Coulomb and Higgs branch operators are considered, while particular attention is payed to mixed-branch operators. It is shown that the mixed-branch moduli of these theories are double-string operators transforming in the (Adj,Adj) representation of the electric and magnetic flavour groups, up to overcounting for quivers with abelian gauge nodes. Finally, we comment on the holographic interpretation of the results, arguing that gauged supergravities can capture the entire moduli space if, in addition to the parameters of the background solution, quantization moduli of boundary conditions are also taken into account.

KEYWORDS

String Theory, Superconformal field theories, Holographic Duality, Massive Gravity, Supersymmetric Localization, Superconformal Index, Quiver theories, Superconformal Manifold