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Reliability and Maintenance Analysis of Multi-component Systems with Stochastic Dependence

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Nan ZHANG

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présentée et soutenue par

Nan ZHANG

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Reliability and Maintenance Analysis of Multi-component Systems with Scholastic Dependence

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**Reliability and maintenance analysis of
multi-component systems with stochastic
dependence**

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Nan Zhang

Abstract

In modern society, systems composed of many equipments which may interact with each other play a vital role and become more and more important. For instance, one can enumerate the transport systems, communication systems, utilities, manufacturing plants, universities, hospitals and so forth. It is not a simple task to ensure the normal operation of systems since they are often subject to deterioration and wear and operate in a changing environment. Maintenance actions can be carried out in order to avoid the failure of the system which may result in economic losses, environmental contaminations, loss of personnel, etc.

This thesis considers maintenance problems and reliability evaluations of multi-component systems. At first, we study a two-component system with various types of stochastic dependencies. Three models have been considered. In the first model, two components share a given load during a mission and the load is carried by only one component during the failure of its counterpart, which results in the increasing of its failure rate. In the second model, the failure of component 1 may impact the degradation of the other component. The system is down if component 2 fails. In the third model, the failure of component 1 may lead to failure of the other component and therefore the system.

For each of the three models above, several maintenance policies are proposed. Under each policy, the explicit expression of the long-term average cost as well as the cost optimality are studied. In addition, for the last two models, several warranty policies are proposed and the allocation of the warranty cost between the manufacturer and the consumer are studied.

Finally, a M -out-of- N system which operates in a dynamic environment is studied. Performance measures such as the reliability of the system, the remaining useful life (RUL), the asymptotic availability are calculated.

In conclusion, it is shown that in multi-component systems, stochastic dependence between components may affect both system performance measures and the optimization of maintenance policies.

Key words: Stochastic processes, Reliability (Engineering), Dependence (statistic), Maintenance, Warranty

Résumé

Dans la société moderne, les systèmes composés de plusieurs composants qui peuvent interagir entre eux, par exemple, les systèmes de transport, les systèmes de communication, les services publics, les usines de fabrication, les universités, les hôpitaux, jouent un rôle essentiel et ont une grande importance. Les systèmes sont souvent sujet à détérioration et usure et opèrent dans un environnement changeant, c'est pourquoi assurer leur fonctionnement n'est pas une tâche simple. Des actions de maintenance sont mises en place afin d'éviter les défaillances du système qui peuvent être catastrophiques et entraîner des pertes économiques, des contaminations environnementales, des pertes de personnels.

Cette thèse considère le problème de la maintenance des systèmes multi-composants. Dans un premier temps, on considère un système à deux composants avec dépendance stochastique. Trois modèles de dépendance ont été considérés. Dans le premier modèle les composants partagent une charge donnée durant une mission et à la défaillance d'un des composants la charge est portée par le composant opérationnel, ce qui le fragilise et augmente son taux de défaillance. Dans le deuxième modèle, la défaillance du composant 1 peut impacter la dégradation de l'autre composant. Le système est en panne si le composant deux est en panne. Dans le troisième modèle, la défaillance du composant 1 peut induire la défaillance de l'autre composant et donc du système.

Pour chacun des trois modèles, plusieurs politiques de maintenance sont proposées et leur optimalité est étudié. Pour chaque politique, l'expression explicite du coût moyen à long terme est présentée. De plus, pour les deux derniers modèles, plusieurs politiques de garantie sont proposées et l'expression explicite de leur coût est dérivée.

Pour finir, un système M parmi N est considéré et les mesures de performance telles que la fiabilité du système, la durée de vie utile restante (RUL), la disponibilité asymptotique sont calculées. Il est montré que dans les systèmes multi-composants, la dépendance stochastique entre les composants affecte à la fois les mesures de performance et l'optimisation des politiques de maintenance.

Mots-clés :

Processus stochastiques, Fiabilité, Dépendance (statistique), Entretien, Garantie

General introduction

In modern society, systems composed of many equipments which may interact with each other play a vital role and become more and more important. For example, the transport systems (trains, buses, underground, ships and aeroplanes), communication systems (television, telephone and internet), utilities, manufacturing plants, universities, hospitals and so forth. Every system is not reliable in the sense that it deteriorates with time/usage, experiences shocks, operates in the dynamic environment not the ideal laboratory conditions. When system failures occur, some of them are catastrophic and may inflict economic losses, environmental contaminations, personnel casualties, etc. Thankfully, the system failure can be avoided and be treated by maintenance actions. Therefore, the evaluation of the system reliability and the scheme of system maintenance strategies are significant which motivates us to maintain the smooth operation of the system and reduce the financial loss.

In Reliability Engineering and Operation Research, the system reliability assessment and maintenance cost optimization are extensively studied in the last 50 years. Generally speaking, the 'black box' method considering the complex system as a single-unit system is a widely used method as it is tractable technically. However, it may ignore the individual dependencies as well as the structures of complex systems. To address this limitation, this thesis takes a small step in modeling multi-component systems by taking into consideration the stochastic interaction between components. It is organized as follows.

In **Chapter 1**, a brief review of the literature on the multi-component maintenance optimization is presented. The existing stochastic models, maintenance policies, mathematical methodologies, the dependency classification of multi-component systems and our motivation are introduced.

In **Chapter 2**, a two-component load-sharing system inspired by the steel wire rope in mining system is proposed. It is assumed that the failure rates of the two components are time dependent and load dependent. Whenever one fails, it is imperfectly repaired with a time delay during which the failure rate of the survival component increases because of the resulting overload. Three maintenance policies are proposed considering imperfect preventive maintenance and system replacement. The optimal average costs in the long run under different maintenance policies are derived from the theoretical propositions.

In **Chapter 3**, we study a two-component system with failure dependence where the deterioration sub-system is involved. Component 1 failure causes random damages to component 2. While the failure of component 2 is catastrophic and it induces the failure of component 1. Component 2 fails when its damages reach or firstly exceed a predetermined level. The virtual age method is applied in the corrective maintenance of component 1 and the system is renewed whenever component 2 fails. The explicit expressions of the long run average cost under different preventive maintenance policies are developed. Sufficient conditions of the existence of the long-run expected costs are derived theoretically. Furthermore, from the points of view of the manufacturer and the owner of the system, the

warranty profits to the manufacturer and the costs to the owner under different warranty policies (the renewing free repair policy and the non-renewing free repair policy) are derived respectively. It is shown that the failure dependence between components has effect to both the manufacture profit and the consumer cost.

In **Chapter 4**, a similar model as presented in Chapter 3 is developed. Here the failure interaction between components are: the component 1 failure induces the component 2 failure with probability $\bar{r}(= 1 - r)$, $0 < r < 1$ while the component 2 failure causes the failure of component 1. Also, three different preventive maintenance policies are proposed. The system long-run average maintenance costs, the optimization of the maintenance cost are derived as well as the manufacture profits and the owner costs in the context of warranty.

It is seen that in the first three types of interactions, we mainly focus on the optimizations of the long-run average maintenance costs and the expected short-run maintenance costs of two-component systems under different maintenance policies. In **Chapter 5**, the reliability of a M -out-of- N system under dynamic environment conditions is considered. It is assumed that the failure rates of the components are time-dependent and environment-state-dependent. The dependence between components is generated by the common operating environment conditions that they share which is described by a continuous-time Markov chain. The system reliability-based measures like the system reliability, the remaining useful lifetime (RUL), the limiting average availability under different system inspection policies are calculated.

The above models on multi-component systems permit us to glimpse the effect of stochastic dependence between components on the system reliability and the maintenance costs. However, they are still a long way from the management the treatment of complex systems. In **Chapter 6**, the conclusion is made and future perspectives are discussed.

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Chapter 1

Introduction

1.1 Problem statement

1.1.1 Introduction

In modern society, systems composed of many equipments which may interact with each other play a critical role and become more and more important [1]. For example, the transport systems (trains, buses, underground, ships and aeroplanes), communication systems (television, telephone and internet), utilities, manufacturing plants, universities, hospitals and so forth. Every system is not reliable in the sense that it deteriorates with time/usage, experiences shocks, operates in a dynamic environment. When system failures occur, some of them are catastrophic and may inflict economic losses, environmental contaminations, personnel casualties, etc. Thankfully, the system failure can be avoided and be treated by maintenance actions.

The area of maintenance has been extensively studied in the last 50 years. Before the second world war, the maintenance action was mainly regarded as the corrective maintenance behavior which brought the system back to its operational state by trained technician. It was in the military field during the second world war when for the first time that maintenance received increasing attention. On the one hand, military equipments were designed with features like operational mobility, accuracy, aggressiveness, destructiveness, manoeuvrability which result in more complex, and in consequence, less reliable products. On the other hand, the development of military equipments are time-limited-scheduled in the battlefield which is not amenable to repeating experiments. Therefore it is necessary to incorporate the system reliability and maintenance behaviors into the design, the development, the manufacture, the production etc. of the equipments. Especially after the Korean War, the Army's prevailing attitude had been shifted from the "use it up and replace it" to the "highlight of the preventive maintenance". Since then the system reliability analysis and the scheme of maintenance policies have been extended from the military field to the computer science, the economic domain, the industrial and agricultural production, etc.

In Reliability Engineering and Operation Research, the 'black box' method treating the multi-component system as a 'single-component' system is generally selected in the mathematical models [2]. The reason is partly due to that the method is technically tractable. However, the assumption that the composition and internal structures of the system can be ignored is seldom valid. On the one hand, potential interactions between components (homogeneous or heterogeneous) complicate the evaluation of the system reliability/availability as well as the modeling and the optimization of maintenance actions. On the other hand, the corrective re-condition/replacement of separate components provides the opportunity of group maintenance which may be more economic. Therefore, the evaluation of the system reliability and the scheme of system maintenance strategies of multi-component systems by considering the dependence between components are significant which permit us to maintain the smooth operation of the system and reduce the financial loss.

The primary objectives of this thesis are to provide:

- Modeling of the stochastic dependence in the multi-component system;
- Explicit expressions of the long-run average maintenance costs of systems under various preventive maintenance policies;
- Theoretically, sufficient conditions of the existence of the long-run expected maintenance costs;
- Explicit expressions of the cost allocations between the manufacture and the equipment owner under various types of warranty policies;
- Explicit expressions of the reliability-based measures of the M -out-of- N system with heterogeneous components when it operates in a dynamic environment.

1.1.2 modeling of the multi-component system

Generally speaking, in the context of the optimization of maintenance policies and the evaluation of the reliability measures of a multi-component system, the following system characteristics should be taken into consideration [3]:

- Evaluation of the behavior of each component: we model the failure process of each component either by its lifetime distribution function or its degradation distribution function.
- Identification of the Structure and the way that components are assembled to fulfill a function. It is necessary to present the structure of the system, for example, components are in series, in parallel, etc. and the relation between the state of the system (functioning or failed) and the state of each component (functioning or failed).

- Recognition of the stochastic dependence as well as the structure dependence existing between components. It is known that the assumption of stochastic independence in the multi-component system is seldom valid for example, in a load-sharing system. The incorporation of the stochastic dependence in the modeling contributes to the accurate estimation of the system failure time and hence results in the effectiveness of the optimization of maintenance policies.

1.1.3 Our objective

To highlight the impact of the dependence between components on the system reliability as well as on the maintenance cost, our first objective is to optimize maintenance costs of systems with various failure interactions between components in this thesis. Both corrective maintenance policy and preventive maintenance policy are carried out in the component-level as well as in the system-level respectively. According to the component/system recovery state after the maintenance, the minimal repair, imperfect repair and the replacement of the component/system are executed. Explicit expressions of the long-run average maintenance costs of systems under various preventive maintenance policies; Theoretically, sufficient conditions of the existence of the long-run expected maintenance costs. Explicit expressions of the cost allocations between the manufacture and the equipment owner under various types of warranty policies.

Subsequently, reliability, remaining time to failure (RUL) and availability of multi-component systems are important system measures which provide the evaluation of the system ability to function at an interval of time or a specified moment. In consequence, our second objective is to carry out this measures of a M -out-of- N system with non-identical components operating in dynamic environment conditions. The explicit expressions of the system reliability, the remaining useful lifetime and the asymptotic average availability of the system under different inspection policies are derived.

1.2 State of the art

1.2.1 Introduction

The objective of this section is to provide a state of the art with respect to the system reliability and maintenance problems which are relevant to this thesis. Section 1.2.2 contributes to the introduction of reliability measures of the single-component system like the system reliability, availability, mean time to failure, remaining useful lifetime. In Section 1.2.3, the characteristics in the modeling of multi-component systems are presented in detail. Section 1.2.4 and Section 1.2.5 are devoted to the study of the maintenance and the warranty respectively.

1.2.2 Reliability measures

In the areas of the system reliability and maintenance policies, basically, the obvious questions are: how long a system can operate without failure? what is the possibility that a system remains fully functioning in a given time interval or a particular moment? what are the responses to these questions if the system is not new at its initial operation time epoch? In the next, we attempt to answer these questions.

1.2.2.1 Reliability and Mean time to failure

Denote by $X(X \geq 0)$ a random variable which represents the failure time of a component with cumulative distribution function $F(t) = \mathbb{P}(X \leq t)$. The survival distribution of X is

$$R(t) = \mathbb{P}(X > t) = 1 - F(t)$$

which is also referred to as the **reliability function** of the component.

Further more, if the density function $f(t)$ of X exists, the mean of X is

$$\mathbb{E}(X) = \int_0^{\infty} tf(t)dt = \int_0^{\infty} R(t)dt$$

which is known as **mean time to failure** in reliability theory.

It is seen that reliability function describes the component's ability to operation for a given time period from its initial new moment (time 0). Mean time to failure describes the expected operating time of a component before its first failure.

1.2.2.2 Availability

Availability is generally regarded as the effectiveness/readiness indicator for repairable systems because it takes both reliability and maintainability into account. There exist various definitions of availability in the literature. For example, it was defined as the probability that the system is operating satisfactorily at any point in time under stated conditions in [4], it is referred to as the ratio of the component's expected uptime to the aggregate of the expected values of up and down time. We present here the definitions of system availability introduced by Nakagawa [5]. Let be

$$Z(t) = \begin{cases} 1 & \text{if the system is operating at time } t, \\ 0 & \text{if the system fails at time } t. \end{cases}$$

Then

- **Pointwise availability** is the probability that the system operates at a given instant of time which is defined as

$$A(t) = \mathbb{P}(Z(t) = 1) = \mathbb{E}(Z(t)).$$

- **Interval availability** is the expected fraction of a given interval that the system can operate

$$\frac{1}{t} \int_0^t A(\theta) d\theta.$$

- **Limiting interval availability** which is also referred to as the limiting average availability describes the up-rate of the system in the long-run which is noted as

$$A \equiv \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t A(\theta) d\theta.$$

1.2.2.3 Remaining time to failure and Mean residual life

Remaining time to failure is an important characteristic both practically and theoretically which is defined as the time interval from the current time to the end of the useful life. Let X be the life time of the system and suppose that the system has survived by time t . Let $X_t = X - t$ be the **remaining time to failure** of the system at time t . The **Mean residual life** (also referred to as the mean remaining life) is defined as

$$\mu(t) = \mathbb{E}(X - t \mid X > t) = \left(\int_t^\infty \bar{F}(x) dx \right) / \bar{F}(t)$$

where $F(\cdot)$ is the lifetime distribution with respect to X , $\bar{F}(\cdot) = 1 - F(\cdot)$.

1.2.3 The modeling of the multi-component system

As shown in Section 1.1.2, in the context of the modeling of the multi-component system as we are concerned, generally the reliability of each component in the system, the structure of the system and the way that components are assembled to fulfill a function, the stochastic dependence between components should be identified. These qualities are presented in detail in the next part. In Section 1.2.3.1, two common types of modeling of the component reliability: failure-rate modeling and the degradation modeling are presented. We will introduce the coherent systems [6, 7] in Section 1.2.3.2. The stochastic dependence between components of the multi-component system in the bibliography are summarized in Section 1.2.3.3.

1.2.3.1 The reliability modeling of components

1.2.3.1.a Failure-rate modeling

Besides the reliability measures which are mentioned in section 1.2.2, failure rate is also an important quantity for the system behavior in reliability theory. In this framework, it is considered that the system has only two states: functioning or failed and no information concerning the 'physical condition' of the system is available during its operating period. In consequence, it is necessary to estimate the possibility that the system fails in the near future provided that it is functioning right now. Denote by X the lifetime of a component with lifetime distribution $F(\cdot)$ and density function $f(\cdot)$ (if it exists). Let $r(t)$ be the failure rate of a system at time t , then the probability that the system fails in $(t, t + \delta t]$ given that it has been survived by time t is $r(t)\delta t$, which means that

$$\mathbb{P}(X \leq t + \delta t \mid X > t) = \frac{F(t + \delta t) - F(t)}{1 - F(t)} \approx \frac{f(t)\delta t}{1 - F(t)} = r(t)\delta t.$$

Therefore the **failure rate function** is as follows

$$r(t) = \frac{f(t)}{1 - F(t)}.$$

It is clear that the failure rate function describes the frequency with which a system fails but it is not a probability. Some common continuous lifetime distributions $F(\cdot)$, their density functions $f(\cdot)$ and failure rates $r(\cdot)$ are listed in the following.

- **The Exponential distribution** is extensively studied in the reliability theory because of its technical simplicity. The density function of the exponential distribution can be used to describe the time interval between two consecutive events in a Poisson process. It has the property of being memoryless which means the failure rate is time-independent. Its lifetime distribution, density function and the failure rate are respectively:

$$\begin{aligned} F(t; \lambda) &= 1 - e^{-\lambda t} \quad \text{for } t \geq 0 \text{ and } 0 \text{ otherwise,} \\ f(t; \lambda) &= \lambda e^{-\lambda t} \quad \text{for } t \geq 0 \text{ and } 0 \text{ otherwise,} \\ r(t; \lambda) &= \lambda \quad \text{for } t \geq 0 \text{ and } 0 \text{ otherwise.} \end{aligned}$$

- **The Weibull distribution** As pointed out in the above that the Exponential distribution has constant failure rate which indicates that a component with exponentially distributed lifetime does not age with time. For systems without this characteristic, extensions of exponential distribution are required, for instance, the Weibull distribution. It is named after Swedish mathematician Waloddi Weibull who proposed it to describe the breaking strength of materials in 1939 [8]. Since then it is frequently applied in the reliability theory and various of its properties and applications have been

developed.

$$\begin{aligned} F(t; \alpha, \beta) &= 1 - e^{-(\alpha t)^\beta} \quad \text{for } t \geq 0 \text{ and } 0 \text{ otherwise,} \\ f(t; \alpha, \beta) &= \alpha\beta(\alpha t)^{\beta-1} e^{-(\alpha t)^\beta} \quad \text{for } t \geq 0 \text{ and } 0 \text{ otherwise,} \\ r(t; \alpha, \beta) &= f(t)/(1 - F(t)) = \alpha\beta(\alpha t)^{\beta-1} \quad \text{for } t \geq 0 \text{ and } 0 \text{ otherwise.} \end{aligned}$$

It is easily seen that Weibull distribution reduces to exponential distribution when $\beta = 1$; the failure rate is increasing with respect to t when $\beta > 1$ and decreasing when $\beta < 1$. Besides, if a random variable X has a geometric distribution, i.e., $\mathbb{P}(X \geq k) = q^k$, the survival function distribution of a random variable $X^{1/\alpha}$ for $\alpha > 0$ can be seen as the discrete Weibull distribution.

- **The Gamma distribution** is also an extension of Exponential distribution which has been extensively applied in reliability theory, insurance theory, wireless communication, neuroscience, bacterial gene expression etc. Assume that a system experiences a series of shocks which arrive according to the homogeneous Poisson process with parameter λ . Let T be the first failure time of the system as well as the exactly arrival time of the k th shock. Then T is gamma distributed with parameters (k, λ) denoted as $T \sim \text{Gamma}(k, \lambda)$. Its density function is:

$$f(t; k, \lambda) = \frac{\lambda}{\Gamma(k)} (\lambda t)^{k-1} e^{-\lambda t}$$

where $\Gamma(\cdot)$ is the gamma function, $k > 0$, $t > 0$, $\lambda > 0$.

1.2.3.1.b Degradation modeling

In Section 1.2.3.1.a, we introduced failure rate modeling, which is usually applied to describe systems with only two states: functioning or failed. For systems that the information of intermediate states between the perfectly functioning and the failed state are available, degradation modeling is commonly considered. Let $X(t)$ be the degradation level of a system by time t . In this situation, the system is considered as entering into the failure state when its degradation level reaches or exceeds a specified threshold. In the reliability assessment as well as the life-time prognostic [9, 10, 11], it seems that the degradation data generally provide more information than just the failure time data. Because in modern society, the rapid development of the monitoring techniques permits to provide the information continuously concerning the system health and state, the operating environment conditions, etc. It is possible to estimate the future behavior and take measures to prevent the system failure by taking advantage of the monitoring information. In consequence, the random degradation modeling has been paid increasing attention in the reliability theory (especially in the context of conditional based maintenance policy). Here we briefly introduce some common degradation model in reliability analysis.

Markov process is a stochastic process extensively applied in the degradation modeling because of its non-hereditary property [12] which means that given the value

of $X(t)$, the present value of $X(\tau)$, where $\tau > t$, are independent of the past $X(u)$, $u < t$. In reliability analysis, the discrete-time Markov processes with finite/countable state spaces (referred to as Markov chains) and continuous-time Markov processes with independent increments such as the Brownian motion with drift), the compound Poisson process, and the gamma process are frequently used. [13].

- **Brownian motion with drift** with parameter (μ, σ) is a stochastic process $\{X(t), t \geq 0\}$ with the following properties [14]:
 - $X(t)$ is normally distributed with mean μt and variance $\sigma^2 t$ for all $t \geq 0$;
 - $X(t)$ has independent and stationary increments;
 - $X(t)$ is continuous at $t = 0$.

It is widely used in the modeling of systems with non-monotone degradations for instance the exchange value of shares and the movement of small particles in fluids and air. In the stress-strength model, the explicit expression of the density function is available when the stress and the strength are supposed to be independent Brownian motions with drift [15]. The parameter estimations using the Maximum likelihood and Bayesian estimation method are obtainable.

- **Gamma process** As mentioned before, the Brownian motion with drift process is not proper in modeling monotonically increasing degradation. Gamma process can be applied to address this limitation in the modeling of continuous monotone degradation. Gamma process is a stochastic process with independent, non-negative increments following a gamma distribution with an identical scale parameter. It was firstly proposed by Abdel-Hameed [16] to describe the system deterioration in random times. It has been widely used and successfully data-fitted in describing system degradation on the account of wear, fatigue, corrosion, crack growth, erosion, consumption, creep, swell, degrading health index, etc. [13, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27]. Other applications are in the theory of water storage by dams and the theory of risk of ruin due to aggregate insurance claims, etc. $X(t)$ is a Gamma process if it has the following properties.

- $X(0) = 0$,
- $X(t)$ has independent increments,
- For $t > 0$ and $h > 0$, $X(t+h) - X(h)$ follows a Gamma distribution with shape parameter $\alpha(t+h) - \alpha(h)$ and scale parameter β . The random variable following Gamma distribution with shape parameter $\alpha(t)$ and scale parameter β has density function

$$f_{\alpha(t),\beta}(x) = \frac{\beta}{\Gamma(\alpha(t))} (\beta x)^{\alpha(t)-1} \exp(-\beta x) \quad \text{for } x \geq 0.$$

The expectation and the variance of $X(t)$ are $\alpha(t)/\beta$ and $\alpha(t)/\beta^2$ respectively. The process develops into a stationary Gamma process when $\alpha(t) = \alpha t$.

- **Compound Poisson process** is also widely applied in the modeling of the system failure for instance the fatigue failures in aircraft fuselage. It is assumed that the system degradation is a discrete monotone process which is caused by discrete randomly-distributed shocks. Compared to the Gamma process which describes infinite number of jumps in finite time intervals, the Compound Poisson process mainly depicts finite numbers of jumps in finite time intervals. It is usually supposed that the shocks occur according to a Poisson process with intensity λ . The amount of damage induced by the i th shock are identically and independently distributed with distribution function $G(\cdot)$. The system fails when the amount of damages reach or firstly exceed a pre-determined threshold L . Therefore the reliability function of the system at time t can be represented by [28]:

$$R(t) = \sum_{k=1}^{\infty} e^{-\lambda t} \frac{(\lambda t)^k}{k!} G^{(k)}(L).$$

1.2.3.2 System structure

In terms of practical application, various system configurations are considered for example: the series system which is functioning if and only if all components are operating; the parallel system which is functioning if and only if at least one component is operating; the k -out-of- n system which is functioning if and only if the number of operating components is no less than k ; the series-parallel system which is composed by series subsystems in parallel; etc. It is noticed that all the systems mentioned above share a point in common: the states of the system/components can be classified into functioning or failed; the state of the system can be totally determined by the state of each component as well as the system configuration. Besides, it seems that each component may contribute to the system reliability and no one is redundant in this context. The properties draw forth the **coherent system**.

Firstly we introduce the concept of structure function [29]. For a n -component system, let be $\mathbf{x} = \{x_1, x_2, \dots, x_n\} \in \{0, 1\}^n$ where for each i , $x_i = 1$ if component i is functioning and 0 if it is failed. Consider the space $\{0, 1\}^n$ of all possible state vectors for the system, the **structure function** $\phi : \{0, 1\}^n \rightarrow \{0, 1\}$ is a mapping that associates those state vectors \mathbf{x} for which the system works with the value 1 and those state vectors \mathbf{x} for which the system fails with the value 0. For example, the structure function of the series system is $\phi(\mathbf{x}) = \prod_{i=1}^n x_i = \min(x_1, x_2, \dots, x_n)$, while for the parallel system, $\phi(\mathbf{x}) = 1 - \prod_{i=1}^n (1 - x_i) = \max(x_1, x_2, \dots, x_n)$.

Besides, component i is said to be irrelevant if the structure function of the system satisfies

$$\phi\{x_1, \dots, x_{i-1}, 0, x_i + 1, \dots, x_n\} = \phi\{x_1, \dots, x_{i-1}, 1, x_i + 1, \dots, x_n\}$$

for all possible values of $\{x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n\} \in \{0, 1\}^{n-1}$. Further more, let be

$\mathbf{x} \geq \mathbf{y}$ when $x_i \geq y_i$ for $i = 1, 2, \dots, n$;

$\mathbf{x} > \mathbf{y}$ when $\mathbf{x} \geq \mathbf{y}$ and $x_j > y_j$ for some $j = 1, 2, \dots, n$.

We call a system **monotone** if its structure function satisfies

$$\phi(\mathbf{x}) \leq \phi(\mathbf{y})$$

whenever $\mathbf{x} \leq \mathbf{y}$.

A system is said to be **coherent** if each of its components is relevant and if its structure function is monotone.

1.2.3.3 Stochastic dependence

To model a multi-component system in the context of the optimization of maintenance policies and the calculation of system reliability measures, besides the modeling of the reliability of each component presented in section 1.2.3.1, the system configuration introduced in section 1.2.3.2, the modeling of the dependence between components is also required which is demonstrated in this section.

Firstly, the stochastic dependence induced by the changing environment under which a system operates is presented.

1.2.3.3.a Stochastic dependence due to the operating environment

Over the last few years, the modeling of the reliability of systems with consideration of the dynamic operating environment [30] plays a role in the reliability analysis and maintenance study. This is due to that the system lifetime distribution is generally obtained by the statistical life testing procedures executed under given fixed, ideal laboratory conditions. However, the internal stresses in a system induced by its operating environment may have influence on the system lifetime. For instance, the environment can impact the change of the failure/degradation rate or the failure mode. Consider a jet engine [31], the changes of atmosphere conditions like pressure, temperature, humidity during its take-off, landing may have effect on its deterioration. Or the blade of offshore wind turbine, its deterioration depends on the salt concentration in the air [32, 33, 34].

In literature, most researchers take the model of Esary et al. [28] as the first environment-related work where the successive damages caused by random shocks were neither independent nor identically distributed but were time-dependent relating to environment conditions. Several properties about the system survival time were obtained. Various failure models regarding systems in dynamic environment were established since then [30].

Çinlar and Özekici [35] presented the **intrinsic age process** where the concept of intrinsic age was proposed to represent the cumulative hazard accumulated in time with varying environment during its operation period. A number of intrinsic ageing models can be found in [36, 37, 38, 39] for instance. ‘While these models encompass quite general laws for deterioration and are theoretically appealing, they do not readily lend themselves to computational analysis’, as Kiessler et al. [40]

pointed out. In reality, the most common method to handle the component/system dependence due to the dynamic environment is the proportional hazard model.

The proportional hazard model was introduced by Cox [41] in order to estimate the effects of different covariates influencing the times to the failures in biomedicine. In reliability analysis, it is applied to incorporate the effect of covariates such as the operational environment of the system (e.g. compressure, pressure, humidity, etc.), the operating history of the system (e.g. overhauls, lubrications, etc.) and other factors in the failure rate. It is assumed that the system failure rate can be represented as follows.

$$\lambda(t; z) = g(z)\lambda_0(t)$$

where z is a vector consisting of covariates associated with the system, $\lambda_0(t)$ is the baseline hazard function which represents the system failure rate when the covariates have no influence on the system failure pattern. Banjevic and Jardine [42] calculated the reliability function and the remaining useful lifetime of the system when its failure rate was governed by the external environment. Zhao et al. [43] discussed the optimal maintenance strategies of the degradation system where the impact of the environment to the degradation process was modeled by covariates via the Cox proportional hazards model. Lawless et al. [44] considered a gamma-process model to describe the crack growth by incorporating random effects. Other related works can be found in [45, 46, 47, 48]. Kiessler et al. [40] investigated the single-component system whose deterioration was driven by its operating environment which was described by a Markov chain. The system limiting average availability was derived. Kharoufeh et al. [49] extended the work of [40] by considering external random shocks to the system, the Laplace-Stieltjes transforms of the unconditional and conditional system lifetime distributions as well as the limiting average availability were carried out.

In addition to the modeling of the environment as well as its influence on the reliability of the system, the failure/degradation interaction is also extensively applied to describe the stochastic dependence between components in the multi-component system.

1.2.3.3.b Stochastic dependence due to failure interactions between components

In reliability engineering, failure interaction means that the state (failure rate/degradation level) of a component has influence on the lifetime of others. For example, a propeller may come off of an airplane and pierce the fuselage, inducing additional damage and safety risks; the deterioration of hydrodynamic bearings may loosen the primary transmission shafts, which may raise the vibration levels in the gearbox. The failure interaction was first introduced by Murthy et al. ([50],[51]) where they introduced two types of dependencies. When the evolution of each component is modelled by its lifetime distribution, for the two-component system, the two types of failure interactions are presented in the following.

- Type I failure interaction means that the failure of component i may induce the instantaneous failure of component $3 - i$ with probability p_i , $i = 1, 2$.
- Type II failure interaction manifested the failure of a component can affect the failure rate of the other one.

An extensive literature has been developed until now to formalize these dependencies, to propose relevant models to assess their impact on the system reliability and maintenance costs. Jhang and Sheu [52] considered the problem of analyzing age replacement policies in a multi-component system with type I failure interaction. Each component had two types of failures: the minor failure which was corrected with the minimal repair and the major failure which may induce the system failure and was removed with system replacement. The optimal maintenance policies were discussed. Scarf and Deara [53] studied a two-component systems with both failure interaction and economic dependence. Long-run average costs were calculated and the management implications for the implementation of the various policies were discussed. Zequeira and Bérenguer [54] studied the reliability of the two-component in parallel system with type II failure interaction under staggered periodic inspection. Barros et al. [55] introduced imperfect monitoring in a two-component parallel system. It was assumed that the failure of component i was detected with probability $1 - p_i$ and was not detected with probability p_i . The optimization of the total expected discounted maintenance cost was taken into consideration.

It is worth mentioning that the type II failure interaction between components is common in the **load-sharing systems** [56, 57, 58]. In material testing, software reliability, population sampling, mechanical engineering the load can strongly impact the component state (failure rate, reliability, availability, damage level etc.). Because in a load-sharing system, when a component fails, the static or time-varying workload is undertaken by the non-failed components. Therefore the state of the survival components are affected by the increased load they bear. For example, in a distributed computer system, servers work together to complete the workload enforced on the system; in a gear system, according to certain load sharing rule, the workload is shared by each mesh gear pair; in a high voltage system, a transmitter has to undertake more range if some of its counterparts fail, etc. Load-sharing systems have been extensively studied in the framework of statistic inference and reliability characteristics. Kim et al. [59] proposed the classical maximum-likelihood estimation of the system parameters where all components had identical exponential distribution. Park et al. [60] extended the model of Kim et al. [59] by considering the parallel system with Weibull distributed lifetime distribution. The closed-form Maximum Likelihood Estimator and conditional Best Unbiased Estimator of the Weibull rate parameters were derived. Singh et al. [61] developed the reliability of a parallel system with Lindy lifetime distribution components and the system parameters estimation were presented. Jain and Gupta [62] obtained the reliability and mean time to failure of a load sharing M -out-of- N system with non-identical and non-repairable components. Amari et al. [63] gave an overview of the load-sharing systems. More recent papers have considered load sharing systems for instance [64, 65, 66, 67, 68].

When the evolution of each component is modelled by its degradation, the stochastic dependence between components has also been studied in the literature. The common-mode deterioration was [69] utilized to describe the situation that several components could deteriorate at the same time epoch due to, for example, similar working conditions. Wang et al. [70] studied a system with multiple degradation measures in which the joint probability density function in terms of the degradation measures was estimated for the prediction of the system reliability. Bian et al. [71] presented a stochastic methodology for modeling degradation interactions of a n -component system as well as the prediction of their respective residual lifetimes. Rasmekomen and Parlikad [72] optimized the condition-based maintenance policy for an industrial cold box in a petrochemical plant where the state of certain components could affect the rate of degradation of other components.

In reality, it is common that the deterioration of one component can increase the load on the other. Rasmekomen et al. [73] considered a N -component system where the state of certain components could affect the rate of degradation of other components, i.e., state-rate degradation interactions; Do [74] studied a two-component system where the deterioration speed of each component depended not only on its own state (deterioration level) but also on the state of the other. Both of the two articles applied the threshold maintenance actions and it was shown that the stochastic dependence between components had significant influence to the maintenance cost which should be taken into consideration in the evaluation of the system reliability.

Besides, Nakagawa et al. [75] proposed another type of failure interaction which was also called the shock damage interaction in the literature. It implied that in a two-component system, a random accumulated damage may be induced to component 2 due to the failure of the other one. Satow and Osaki [76] considered the maintenance optimization problem of the system consisting of two components with shock-damage interaction. It was assumed that component 1 was repairable and it was minimally repaired at failure. The failure of component 1 occurred according to a Non-Homogeneous Poisson Process. Whenever the component 1 failed, it induced a random amount of damage to component 2. The damage was additive and component 2 failed whenever the total damage exceeded a certain failure level. More detailed studies can be found in [77, 78, 79, 80, 81].

It is pointed out that in the maintenance models regarding the system with failure interaction, economic dependence and structural dependence are always incorporated in the scheme of maintenance policies [82]. Economic dependence indicates that the cost of joint maintenance of a number of components does not equal to the sum of their individual maintenance cost. For instance, the group maintenance of traffic control lights may result in a reduction in the maintenance costs due to the economizing on set-up costs (transportation of manpower and equipment.) Structural dependence implies that some components are structurally or functionally bonded such that one has to dismantle some operating components for the purpose of replace or repair failed ones. More details will be presented in section 1.2.4.

We have introduced various types of stochastic dependence in the multi-component system in the above. However, in some situations, it is difficult to construct these dependencies because of the lack of knowledge with respect to the internal

impacts between components and the external impacts to the system. The common approach to address these limitations is to consider the stochastic dependence in the framework of multivariate distributions/copulas functions.

1.2.3.3.c Stochastic dependence and Multivariate distributions/ Copulas functions

Multivariate distributions seem to be a natural method to describe the stochastic dependence between components in the multi-component system. Besides, it can be applied to depict the dependence between various indicators of single-component systems. This is partly due to that one single indicator may be inadequate to measure the system deterioration. As presented in [83], the railway track geometry was measured by the longitudinal (NL) and transversal (NT) levelling indicators. These two dependent measurements were modelled through a bivariate Gamma process. Mercier et al. [84] proposed the bivariate non-decreasing Lévy process to describe the system deterioration. The system failure occurred when its bivariate deterioration level reached a so called failure zone. The maintenance cost with respect to various parameter settings were discussed. Singpurwalla et al. [85] proposed the two-dimensional renewal processes to describe the bivariate failure indexed by age and usage in the context of warranty analysis. Since then a bulk of papers have been carried out with consideration of the cost analysis of two-dimensional warranty [86, 87, 88, 89, 90, 91]. In [92], Singpurwalla and Youn-gren developed families of multivariate life distributions for systems operating in dynamic environment conditions. Both gamma process and the shot-noise process were respectively applied to depict the impact of the environment to the system lifetime.

Copulas have been extensively studied in finance to address multivariate models [93, 94, 95, 96] as it lifts the restriction of bivariate wear subordinator and allows various types of marginal process (for instance in a two-component system, the deterioration of the two components may be described by Gamma process and inverse Gaussian process respectively). Besides, it allows to separate the effect of dependence from the effects of the marginal distributions.

In the framework of maintenance analysis, Hong et al. [97] considered three Copulas functions to model the degradation dependency between components. The optimization of condition-based maintenance decisions were discussed as well as the necessity of considering dependency in stochastic degradations for accurate estimating of the system cost rate and the failure probability. Li et al. [98] considered the condition based maintenance policies for multi-component systems where the stochastic dependence between components was described by Lévy copulas. Wang et al. [99] considered a s-dependent competing risk model for systems subject to multiple degradation processes and random shocks where the dependence between the degradation processes was described by a time-varying copulas. In [100], the copula function was applied to depict the dependence between different degradation measurements of the system reliability of LED lamp. More references can be seen in [101, 99, 102, 103]

1.2.4 Maintenance

Maintenance is usually categorized into the following two types according to the state of the system (functioning or failed) at the maintenance [104]:

- **Corrective maintenance** means all actions performed as a result of failure, to restore an item to a specified condition (MIL-STD-721B).
- **Preventive maintenance** implies that all actions performed in an attempt to retain an item in specified condition by providing systematic inspection, detection, and prevention of incipient failures (MIL-STD-721B). It benefits the system by avoiding or mitigating the sudden failure through premeditated operations. For the failure-rate modeling that no system information before the system failure is available, the preventive maintenance is usually taken at predetermined age. For instance, the preventive maintenance is carried out when the system age arrives at T , $T > 0$ is a predetermined constant. For the degradation modeling, the system degradation is usually taken into consideration in the preventive maintenance. For instance, the system is preventively repaired when its degradation level reaches or firstly exceed M , $0 < M < L$ where L is the system failure threshold. More detailed information regarding maintenance policies of deteriorating systems can be found in [105].

Maintenance can also be classified into the following three types according to the degree to which the functioning conditions of an item is restored by maintenance:

- **Perfect repair** A maintenance action restores the system to an as-good-as-new state in the sense that the system has the same lifetime distribution comparing to its initial operating state [21, 106].
- **Minimal repair** A maintenance action brings the failed system back to its operating state without altering its failure rate. It is also referred to as the as-bad-as-old maintenance behavior. Changing a safety belt/lubricating a car is regarded as the minimal repair behavior [107, 108, 109].
- **Imperfect repair** A maintenance action restores the system to an operating state between as-good-as-new and as-old-as-bad. Generally speaking, imperfect repair improves the operating state of the system but not brings it to the initially new state [110, 111, 112, 113, 114, 115, 116, 117].

Many methods have been proposed to describe the imperfect maintenance degree.

- **(p, q) rule** Nakagawa [118] introduced the (p, q) rule under which the system was perfectly repaired with probability p and minimally repaired with $q = 1 - p$. Wang et al. [119] derived the optimal imperfect maintenance policies of the single-component system. Liao et al. [120] considered an economic production quantity (EPQ) model that concerned a deteriorating

system by integrating maintenance and production programs. The probability that the system was perfectly/minimally repaired was not constant but depended on the number of imperfect repair. The optimal time interval between preventive maintenance was examined.

- **$(p(t), q(t))$ rule** Block et al. [121] stretched the (p, q) rule to $(p(t), q(t))$ rule: a component was renewed with probability $p(t)$ and minimal repaired with not a $q(t) = 1 - p(t)$ where t was the age of the component since the last replacement. Iyer [122] calculated the steady-state availability when the system repair time was non-negligible.
- **Geometric process** Lin [123] presented the geometric process to describe the imperfect maintenance under which the system life time X was renewed and reduced to αX where $0 < \alpha < 1$. It is also referred to as the quasi renewal process by Wang and Pham [124] in which they introduced the properties of the process and studied the optimization maintenance policies of three imperfect maintenance models with respect to various system parameter settings. Biswas and Sarkar [125] examined the system limiting availability when it is imperfectly repaired with the geometric process and non-negligible repair time. Bai and Pham [126] proposed the repair-limit risk-free warranty policies under which the system failures less or equal to m within the warranty period were imperfectly repaired which was modeled by the geometric process. The expected warranty-cost and the variance of the warranty-cost per unit sold were obtained. More related works can be seen in [127, 128, 129, 130, 131].
- **The virtual age method** proposed by Kijima et al. [132, 133, 134] which have been widely studied in the literature. Instead of considering the system age as the time elapsed since it was new, they assumed the virtual age (or effective age) as the real condition of the system which was reduced after the repair. They developed two imperfect maintenance models. Let B_n, A_n, X_n be the component virtual age after the n th repair, the repair degree of the n th repair and the time between the $(n - 1)$ th and n th repair. In Kijima model 1, it was assumed that the repair could reduce the damage or the age emerged only during the last survival period which yielded $B_n = B_{n-1} + A_n X_n, B_0 = 0$. While in Kijima model 2, the maintenance effect decreased all damages before the n th repair which yielded: $B_n = A_n(B_{n-1} + X_n), B_0 = 0$. Jack [135] analysed the effect of imperfect maintenance using the concept of virtual age. Bartholomew-Biggs et al. [136] derived a performance criteria to explore the effect of imperfect preventive maintenance with the virtual age method. Instead of the general cost-centered models over the finite time horizon, Marais [137] built a discrete semi Markov structure dealing with the maximum revenue of the system in finite horizon with virtual age method. Scarsini and Shaked [138] developed a benefit rate model in which each component generated benefits depending on its virtual age. They gave stochastic comparisons and monotonicity properties proofs in their model.
- **ARI and ARA models** Doyen and Gaudoin [111] proposed two classes of imperfect repair models: the so-called ARI and ARA models. The impact of the maintenance was characterized by the reduction of the system failure

intensity. In ARI models, the maintenance reduced the system failure rate. In ARA models, the maintenance reduced the system virtual age.

For the component which deteriorates with time, Van et al. [139, 140] considered the imperfect repair which reduced the system degradation to a non-zero level. The reduction degree was described by a truncated normal distribution. In the work of Wu et al. [141], they assumed that the imperfect repair restored the system degradation level to a constant level between the initial condition and the potential failure threshold.

1.2.5 Warranty

In Section 1.2.4, we mainly focus on the introduction of maintenance policies under which generally the objective is either the calculation of maintenance cost in the long/short time horizons or the evaluation of system reliability measures. In the framework of the former, basically it is supposed that the product owner takes full charge of the maintenance cost within its usage. However, in the second half of twentieth century, due to the awakening of consumer awareness of self-protection [142]; the perfection of the laws and regulations of protection of the rights and interests of consumers, more and more manufactures take these situations into consideration by utilizing warranty as a powerful marketing means. Warranty is a contract between the manufacturer and the consumer which requests the manufacturer to repair, replace the product or to compensate the consumer if the product fails before a pre-determined time period, which is referred to as warranty period. Since then, warranty plays an increasingly important role to both the consumer and the manufacturer. From the buyer's point of view, warranty is protectional which supports them to mitigate or even avoid losses if the item is out of function when properly used. Also, warranty is informational as the buyers generally deduce an item with longer warranty is more reliable and has better quality. From the manufacturer's point of view, warranty is also protectional and promotional because the conditional warranty prevents the misuse of products (for example, a broken-down HTC mobile phone within warranty period is not supposed to be repaired if the failure is caused by rough handling, exposure to moisture, dampness or extreme thermal etc.) and can be seen as an advertisement since the buyers believe a longer warranty is more reliable when other circumstances are settled. Besides, warranty is an instrument which can be used in competitions with other manufacturers.

In the next, various types of warranty policies and correspondingly the warranty cost analysis are presented.

1.2.5.1 Warranty policies

Basically, there are three common types of warranties in the literature: the free replacement warranty (FRW), the pro rata warranty (PRW) and the rebate policy.

Under the FRW, the manufacturer covers the total cost of repair/replace of the product before the expiration of the warranty. It is usually applied to the consumer products from inexpensive items to expensive repairable (automobiles, refrigerators, large screen color TVs) to non-repairable items (microchips and other electronic components).

Under the PRW, a replacement product at prorated cost $(1 - \frac{X}{W})C$ within a pro rata warranty is offered where X is the component age at replacement, W and C are the warranty period and selling cost respectively.

Under the rebate policy, the manufacture provides a refund with amount αC if the item fails prior to time W (applied field is similar as FRW).

There are many extensions and variations of warranty policy according to the combination of the above, the renewing/non-renewing and the dimension that are considered. We give a brief introduction in the following.

1.2.5.1.a One dimensional warranty policies

For the one dimensional warranty policies, it is assumed that the failure models are indexed by a single variable, such as the product age or its usage. The age is measured by calendar time such as one month, one year, and the usage is quantified by real operating time in terms of mileage, number of copies, etc. In this framework, the following warranty policies are generally considered in literature.

- Non-renewing FRW means that the warranty period is fixed.
- Renewing FRW implies that the failed product is replaced by an identical one with a new warranty W . (It is extensively applied to inexpensive electrical, electronic and mechanical products such as coffee grinder, alarm clocks, photocopier, automobile battery etc.)
- Renewing PRW manifests that the manufacturer offers replacement at a prorated cost and renew the warranty at the replace time (It is usually put into use in non-repairable item field like auto tires, batteries).
- Non-renew/partial renewing/renewing combination FRW/PRW means that the sellers provide a maintenance/replace free of charge until W_1 and a prorated refund in (W_1, W) . (It is widely used in consumer products)
- Pro-rata rebate policy where the rebate depends on the age. (It is implemented to the inexpensive non-repairable components such as batteries, tires, ceramics, and so on.)

1.2.5.1.b Two dimensional warranty policies

In reality, for some products, it is not sufficient to take only one characteristic as criteria for judging the warranty eligibility of failed products. For example, for

automobiles, sometimes warranty coverage has both restrictions on the product age and mileage. In this circumstance, the warranty is not characterised by an interval but by a spatial space where one axis represents the age of the product and the other represents its usage in general. The policies can be extended naturally.

- Two-dimensional non-renewing FRW means that the product repair/replacement is free of charge up to time W or usage U whichever occurs first. (Nearly all auto manufacturers offer this type of policy)
- Two-dimensional combination FRW/PRW means two-dimensional FRW in (W_1, U_1) and PRW otherwise in the warranty region.

1.2.5.1.c Cumulative Warranties

It is usually applied when items are sold as a lot with n items ensemble.

- Cumulative FRW: A lot of n items is warranted for a total period nW . The n items in the lot are used one at a time. If $S_n < nW$, where S_n is the sum of the service times of the n items, free replacement items are supplied, also one at a time, until the first instant when the total lifetimes of all failed items plus the service time of the item then in use is at least nW . (applied for industrial or commercial equipment bought a lot as spares and just use one each time such as bearings and drill bits, mechanical or electronic modules in airborne units.)

1.2.5.1.d Reliability Improvement Warranties (RIW)

It has been widely used by US military [143] which is beneficial for both the seller and the consumer. Through the warranty, the sellers are offered with a monetary incentive for improving the production design therefore enhancing the system reliability during the warranty. At the same time, they should promise to repair/replace the system if failure occurs during the warranty.

Noting that no matter what types a warranty is, there is no doubt that the warranty arouses extra expense except those related to the design, the manufacture and the sale, etc. As mentioned in [144], generally the warranty cost varies from 1% to 10% of total sales depending on the product reliability and the warranty types. Therefore, it is important to have an efficient evaluation of the warranty cost which permits the decision-maker to allocate the warranty budget as well as to maximize the profit.

1.2.5.2 Warranty cost analysis

As mentioned in [145], the following costs are of importance for both sellers and consumers.

- warranty cost per item sale
- Warranty cost over the lifetime of an item
- Warranty costs over the product life cycle
- Cost per unit time

Basically, the one-dimensional warranty case analysis is similar to that of the general maintenance oriented analysis. The related issues and models are:

- Extended warranties (because of the diversity attitude to risk of heterogeneous customers): Padmanabhan and Rao [146] examined the extended warranty with different risk attitude described by a utility function. Ashgarizadeh and Murthy [147] used a game theory (the Stackelberg model) to deal with the optimal choice of both the agent and the consumer because of the asymmetric competition between them.
- Cost per unit time: the product usage rate is various because of the diversity of consumer. Kim et al. [148] explored the expected warranty cost per unit of both repairable and non-repairable products where the product lifetime distribution was affected by the usage intensity. In the model, the usage was a random variable which was described by continuous and discrete distribution respectively under FRW.
- Phase type distributions: it is generally used because of its advantage of easy calculation comparing to the renewal process when dealing with non-repairable product under FRW. See [149, 150].
- One item with multi working state or some dependency: Zuo et al. [151] examined the cost per item with multi-state deteriorating product where the sojourn time in each state was exponential distributed.
- Multi-component systems: Ritchken et al. [152] analysed the average cost rate of a parallel non-repairable hot system with exponential lifetime distribution under rebate warranty and age replacement policy. Bai et al. [153] explored a new warranty policy RFSW for multi-component system. Under this policy, the failed components were replaced while the rest were maintained to reduce the failure probability. The system warranty costs and variances in one cycle were given for series/parallel/series-parallel/parallel-series systems. Liu et al. [154] developed the warranty cost per cycle for both series system and parallel system under renew FRW where there was type 1 failure interaction between components. The model was similar to [153].

For the case of two-dimensional warranty cost analysis, there exist three main approaches which are listed in the following.

- Approach 1: the time to first failure is described by a bivariate distribution $F(x, u)$ (see [155, 88])

- Approach 2: under this circumstance, the age and usage are combined by a variable (like $z = ax + bu$) which is analysed in the modeling. (see [91, 156])
- Approach 3: the usage is a random variable Y because of consumer diversity. However, it is fixed for a given consumer and there is a relation function between the age and the usage of the system. So it is can be transferred to one-dimensional problem. (see [157])

1.3 Conclusion

In this chapter, first, the objectives of this thesis are introduced in section 1.1.1. After which, important reliability measures which will be used in the following in section 1.2.2 are presented. To model a multi-component system with stochastic dependence, the failure rate/degradation modeling, the system structure and various types of stochastic dependencies in the literature are summarized. In the domains of the maintenance optimization and warranty cost analysis, several maintenance policies and warranty policies are illustrated respectively. In the next section, inspired by the hoisting ropes in the mining system, a two-component load sharing system will be studied. Three types of maintenance policies are proposed. The system maintenance cost per unit of time in the long time horizon as well as its optimization problem will be discussed.

Chapter 2

Maintenance of a load sharing system

2.1 Problem statement and notations

There are many load sharing parallel systems in industry and engineering field. For instance, one can enumerate the sensors which take the workload together in a distributed computer system; the pumps sharing the workload in a hydraulic system; the welded joints in a bridge support [158]; the cables in a suspension bridge system [159] and the hoisting ropes in the mining system. Hoisting rope plays a significant role in mining system as its tensile strength and lifetime affect directly the system reliability and the system operation state. According to the literature [160, 161], the break of ropes is relevant to the fretting wear, mechanical damage, operating environment like temperature, corrosive gas, distortion, etc. Therefore, regular inspections, lubrication and overhaul are necessary for the enterprises to increase the effective operation of systems and decrease the probability of failure. Our model can be applied to a mining hoist system with two hoisting rope. The two ropes share the system load uniformly. Whenever one fails,

- the survival component bears the whole system load.
- the sudden component failure can be regarded as a shock which increases the failure rate of the survival one.

Different maintenance policies are provided to slow down the rope deterioration and to maintain the rope in a good condition. Furthermore, for the safety, the two components can be replaced together when the age of the system arrives at a predetermined time limit.

The main goal of this chapter is to focus on a specific case of stochastic dependency (load sharing) and to propose optimal maintenance decision rules in this context. The system under study is supposed to be a parallel system described by two main equivalent and interacting sub-systems. The failure rates of the two components

are time dependent and load dependent. Whenever one fails, it is imperfectly repaired with a time delay during which the failure rate of the survival component increases because of the resulting overload. Three maintenance policies are proposed considering imperfect preventive maintenance and system replacement. We aim at investigating how maintenance actions can compensate for the negative effects of the load sharing. A first strategy (Policy 1) is proposed which is very conservative and very simple to implement: inspections and preventive actions are performed for each component, and the whole system is either preventively renewed if one component fails or preventively renewed at a pre-determined age T . Hence, the effect of the load sharing is assumed to be so risky that the whole system is replaced as soon as possible after the first failure. The second proposed strategy (Policy 2) tends to be less conservative: the whole system is renewed only after a given age T or after a short delay of the system failure. At last, a third strategy (Policy 3) will add the possibility to increase the inspections and preventive actions after the first failure of a component. In each situation, the preventive maintenance age T is optimized to balance between the risk of total failure and the cost of over-renewal. Hence, the effects of the load sharing are carefully taken into account from two main perspectives:

- how long the load sharing system is supposed to be in usage?
- what is the optimal inspection periodicity before and after the first failure of one of the redundant sub-systems?

To make it clear, the notations involved in this section are listed.

X_i	lifetime of component i , $i = 1, 2$.
$h_i(t)$	failure rate of component i at time t , $i = 1, 2$.
$l_i(t)$	load undertaken by component i at time t , $i = 1, 2$.
$\beta_i(t)$	nominal failure rate of component i at time t in absence of the load
τ_0	duration of one mission
τ	duration between two consecutive imperfect repair in policy 1, $\tau = k_2\tau_0, k_2 \in \mathbb{N}^*$
τ'	duration between two consecutive imperfect repair of the long-lived component after the failure of its counterpart in policy 3, $\tau' = k'_2\tau_0 < \tau$
λ	failure rate reduction factor, $0 < \lambda < 1$
$\lfloor x \rfloor$	maximum integer not greater than x
$\lceil x \rceil$	minimum integer not smaller than x
$r_1(t)$	failure rate of the short-lived component under policy 1
$r_i(t)$	failure rate of the long-lived component under policy i , $i = 2, 3$
T	system preventive replacement time under each policy, $T = k_3\tau$, $k_3 \in \mathbb{N}^*$
p_k	probability that the system is replaced at $k\tau_0$ in policy 1
q_k	probability that both components fail in the period $((k-1)\tau_0, k\tau_0]$ in policy 1

$F(x)$	lifetime distribution of component i when both of them are functional, $i = 1, 2$.
n_x	$\lfloor \frac{x}{\tau} \rfloor$
m_x	$\lfloor \frac{x}{\tau_0} \rfloor$
M_x	$\lceil \frac{x}{\tau_0} \rceil$
\hat{n}_j	$\lfloor \frac{j}{k_2} \rfloor, j \in \mathbb{N}^*$
\tilde{n}_j	$\lfloor \frac{j}{k_2} \rfloor, j \in \mathbb{N}^*$
$p_{i,k}$	probability that the two components fail in $((i-1)\tau_0, i\tau_0]$, $((k-1)\tau_0, k\tau_0]$ respectively before the system preventive replacement under policy 2
P_{ik_3}	probability that one component survives at $k_3\tau_0$ while one fails in $((i-1)\tau_0, i\tau_0]$, $1 \leq i \leq k_3$ under policy 2.
$p_{i,k}^{(3)}$	probability that the two components fail in $((i-1)\tau_0, i\tau_0]$, $((k-1)\tau_0, k\tau_0]$ respectively before the system preventive replacement under policy 3
$P_{ik_3}^{(3)}$	probability that one component survives at $k_3\tau_0$ while one fails in $((i-1)\tau_0, i\tau_0]$, $1 \leq i \leq k_3$ under policy 3.

2.2 Model descriptions

In the work of Birnbaum et al. [162], they assumed that the system failure is caused by two different causes: the system load and deterioration factors independent of the load. Under these hypotheses, the example of lifetime estimation of the 6061-T6 aluminum sheeting is addressed. Moreover, in [163], the author considered a system where the failure rate depends on the load and a constant deterioration.

Similarly to Birnbaum et al. [162], in our study, we consider that the failure rate depends on the load and a deterioration factor. And contrary to [163], we consider that the deterioration is time dependent. More precisely, the failure rate of component i at time t is defined as follows:

$$h_i(t) = \beta_i(t)l_i(t)$$

where $l_i(t)$ is the load it undertakes at time t , $\beta_i(t)$ is the nominal failure rate in absence of load representing the deterioration or corrosion related factor of component i at time t , $i = 1, 2$.

Furthermore, it is assumed that in absence of load, according to the number of survival components, the lifetime of component i follows a Weibull distribution with a scale parameter equal to one. In other words,

$$\beta_i(t) = \begin{cases} at^{a-1} & \text{if both components are operational at time } t \\ a_1t^{a_1-1} & \text{if component } 3-i \text{ fails before time } t \end{cases}$$

where, $i = 1, 2$ and $a_1 \geq a \geq 1$. So the component deterioration has a positive dependence on the load it bears. The system load $2l$ is shared uniformly by both

components when they operate. If one component fails at time t , the system is still functional with the survival one who takes the whole load. The duration of one mission of the system is noted as τ_0 . During a mission, that is to say within a time horizon $((k\tau_0, (k+1)\tau_0]$ for any $k \in \mathbb{N}^*$, the system cannot be maintained if failure occurs. Therefore, the maintenance operations can be carried out only at the end of missions at time $\tau_0, 2\tau_0, \dots$. To avoid failure and therefore a period of unavailability and loss of production, different maintenance operations are carried out. We propose and analyze three policies in our study. The maintenance operations, their impacts and scheduling are described as follows. Before the presentation of each policy, we define the age of the parallel system by the time passes by from its brand new state without taking account of the number of operational components.

Policy 1: component based policy

Under policy 1, when both component 1 and 2 are functioning, before age T , they undergo

- preventive imperfect repairs after every k_2 missions where k_2 is a constant. In other words, preventive imperfect repairs are carried out at age $\tau, 2\tau, \dots$ where $\tau = k_2\tau_0, k_2 \in \mathbb{N}^*$. The approach of Arithmetic Reduction of Intensity with memory one (ARI₁) [111] is carried out to describe the imperfect repair action which yields

$$\beta_i((j\tau)^+) = \beta_i(j\tau) - \lambda[\beta_i(j\tau) - \beta_i((j-1)\tau)^+]$$

where $(j\tau)^+$ is the right limit of $j\tau$, $j \in \mathbb{N}^*$, $0 < \lambda < 1$, $i = 1, 2$. It is also assumed that the imperfect repair has no effect on the wear-out speed of the system.

In case that the age of each component is not less than T or that one may fail before age T , then

- preventive replacement actions are implemented at system age T or at the end of mission after the first component failure which occurs first. In other words, The system is preventively replaced at $T = k_3\tau_0$ or at $i\tau_0$, where the **first component** failure occurs in $((i-1)\tau_0, i\tau_0]$, $i = 1, 2, \dots, (k_3 - 1)$, which comes first. The constant k_3 is a decision parameter, $k_3 \geq k_2$.

Besides,

- The replacement after failure is not instantaneous. More precisely, there is a delay $i\tau_0 - t_f$ when a component fails at $t_f \in ((i-1)\tau_0, i\tau_0]$, $i = 1, 2, \dots, (k_3 - 1)$.
- The cost of system imperfect repair is c_2 ($\frac{c_2}{2}$ for each component), and the cost for renewing system is c_r each time. Besides, there is a penalty c_p when both components fail in the same time period $((i-1)\tau_0, i\tau_0]$, $i = 1, 2, \dots, k_3$.

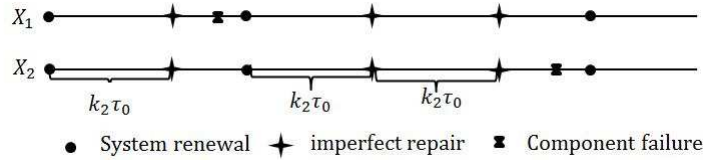


Figure 2.1: A possible operational process of the system under policy 1

An example of such policy is depicted in Figure 2.1.

Policy 2: system based policy

- The system is imperfectly repaired as in policy 1.
- The system is replaced at age T ($T = k_3\tau_0$) or at $k\tau_0$ when **the whole system** fails in $((k-1)\tau_0, k\tau_0]$ which occurs first with cost c_r , $k = 1, 2, \dots, (k_3 - 1)$.
- There is a penalty c_p if both the two components fail by time T .

Comparing to policy 1, under which the system is replaced with a time delay when the short-lived component fails, in policy 2, the system keeps operating with the long-lived component until it fails. Since the replacement is not instantaneous, there is a period of unavailability. An example of policy 2 is depicted in Figure 2.2.

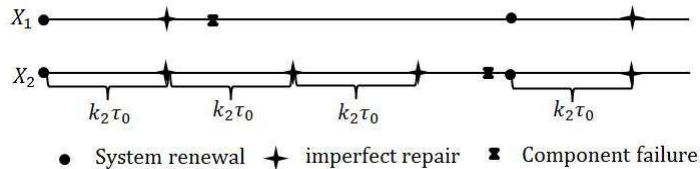


Figure 2.2: A possible operational process of the system under policy 2

Policy 3: component based variant policy

- When both components are operational as in policy 1, preventive imperfect repairs are carried out at age $\tau, 2\tau, \dots$. After the first component failure, the surviving component is imperfectly repaired as in policy 1 but more frequently at intervals of $\tau' = k'_2\tau_0$, $k'_2 < k_2$. An imperfect repair for each component incurs a cost $\frac{c_2}{2}$. The aim is to prevent the effect of the load-sharing on the surviving component.
- The replacement policies (both the corrective and the preventive replacement) are similar to policy 1.

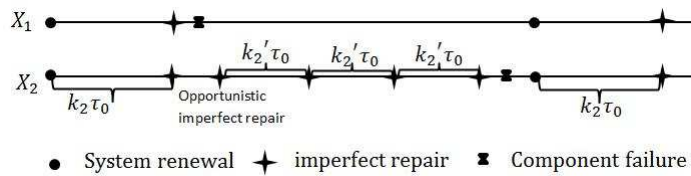


Figure 2.3: A possible operational process of the system under policy 3

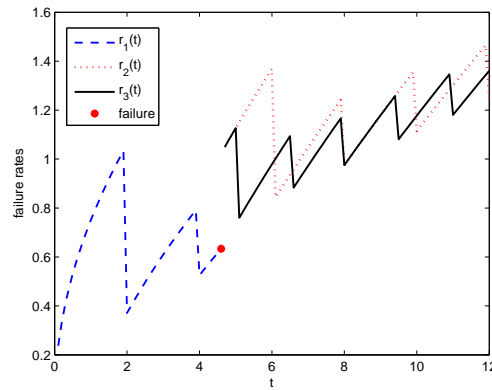


Figure 2.4: failure rates

Possible operational processes of the system policy 3 is depicted in Figure 2.3. Let be $r_1(t)$ the failure rate of the short-lived component, $r_2(t)$ (resp. $r_3(t)$) be the failure rate of the long-lived component in policy 2 (resp. policy 3). Figure 2.4 gives an example of their variation tendencies where the red point represents the failure of the short-lived component.

2.3 Maintenance policy evaluation

In this section, the long run average maintenance costs under different maintenance policies are calculated.

2.3.1 Average cost evaluation under policy 1

Failure rate and lifetime distribution

It is easily seen that the failure rate of the short-lived component under policy 1 can be given as

$$r_1(t) = at^{a-1}l - \lambda a(i\tau)^{a-1}l, i\tau < t \leq (i+1)\tau.$$

Let $\bar{F}(x)$ be the survival function of the short-lived component. Denote $n_x := \lfloor \frac{x}{\tau} \rfloor$

for any $0 \leq x < T$, then we have:

$$\bar{F}(x) = \exp(-x^a l + \lambda l a \tau^{a-1} \sum_{i=1}^{n_x} z_i i^{a-1}) \quad (2.1)$$

where $z_i = x - n_x \tau$ if $i = n_x$ and τ otherwise.

System replacement and failure probability

Denote p_k be the probability that the system is replaced at time $k\tau_0$, q_k be the probability that both components fail in the same period $((k-1)\tau_0, k\tau_0]$, $k = 1, 2, \dots, k_3$. As

$$p_k = \mathbb{P}(X_1 > (k-1)\tau_0, X_2 > (k-1)\tau_0) - \mathbb{P}(X_1 > k\tau_0, X_2 > k\tau_0), \quad k = 1, 2, \dots, (k_3 - 1).$$

One can deduce

$$p_k = \bar{F}^2((k-1)\tau_0) - \bar{F}^2(k\tau_0), \quad k = 1, 2, \dots, (k_3 - 1). \quad (2.2)$$

$$p_{k_3} = 1 - \sum_{k=1}^{k_3-1} p_k. \quad (2.3)$$

Similarly, we have q_k as follows:

$$\begin{aligned} q_k &= 2\mathbb{P}\left((k-1)\tau_0 < X_1 \leq X_2 < k\tau_0\right) \\ &= p_k - 2\mathbb{P}\left((k-1)\tau_0 < X_1 < k\tau_0, X_2 > k\tau_0\right) \\ &= p_k - 2 \int_{(k-1)\tau_0}^{k\tau_0} f(x) \bar{F}(x) \exp\left(-2l[(k\tau_0)^{a1} - x^{a1}] + \lambda l a (n_x \tau)^{a-1} (k\tau_0 - x)\right) dx \end{aligned} \quad (2.4)$$

where $n(x) = \lfloor \frac{x}{\tau} \rfloor$, $F(x) = 1 - \bar{F}(x)$, $\bar{F}(x)$ is given in equation (2.1), and $f(x) = \frac{dF(x)}{dx} = (lax^{a-1} - \lambda l a (n_x \tau)^{a-1}) \bar{F}(x)$ for $k = 1, 2, \dots, k_3$.

Average long-run cost per unit time

The mean cost of one renewal cycle is:

$$c_r + c_2 \sum_{k=1}^{k_3} p_k \hat{n}_{k-1} + c_p \sum_{k=1}^{k_3} q_k$$

where $\hat{n}_{k-1} = \lfloor \frac{k-1}{k_2} \rfloor$. The average length of a lifetime cycle is defined:

$$\sum_{k=1}^{k_3} p_k k \tau_0$$

According to the renewal reward process, the long-run cost per unit time $C(k_3)$ under policy 1 can be given by

$$C(k_3) = \frac{c_r + c_2 \sum_{k=1}^{k=k_3} p_k \hat{n}_{k-1} + c_p \sum_{k=1}^{k=k_3} q_k}{\sum_{k=1}^{k_3} p_k k \tau_0} \quad (2.5)$$

The average cost can be obtained by substituting equations (2.2), (2.3) and (2.4) into equation (2.5). By utilizing the similar method, the cost rate in the long run under policy 2 and 3 are derived in the following respectively.

2.3.2 Average cost evaluation under policy 2

Failure rate

Given that $\min(X_1, X_2) = x$, the failure rate of the long-lived component $r_2(t)$ under policy 2 can be represented as

$$r_2(t) = \begin{cases} r_1(t) & t \leq x \\ 2\lambda a_1 t^{a_1-1} - \lambda a(n_x \tau)^{a-1} & x < t \leq (n_x + 1)\tau \\ 2\lambda a_1 t^{a_1-1} - 2\lambda a_1 ((n_x + i)\tau)^{a_1-1} & (n_x + i)\tau < t \leq (n_x + i + 1)\tau, \end{cases}$$

for $i = 1, 2, \dots$

System replacement and failure probability

Denote by $p_{i,k}$ the probability that the two components fail in $((i-1)\tau_0, i\tau_0]$, $((k-1)\tau_0, k\tau_0]$ respectively, where $1 \leq i < k \leq k_3$, one can deduce:

$$\begin{aligned} p_{i,k} &= 2\mathbb{P}\left((i-1)\tau_0 < X_1 \leq i\tau_0, (k-1)\tau_0 < X_2 \leq k\tau_0\right) \\ &= 2 \int_{(i-1)\tau_0}^{i\tau_0} f(x) \bar{F}(x) \left(\exp\left(-\int_x^{(k-1)\tau_0} r_2(t) dt\right) - \exp\left(-\int_x^{k\tau_0} r_2(t) dt\right) \right) dx \\ &= 2 \int_{(i-1)\tau_0}^{i\tau_0} f(x) \bar{F}(x) (K_{(k-1)\tau_0}(x) - K_{k\tau_0}(x)) dx \end{aligned} \quad (2.6)$$

where

$$K_t(x) = \begin{cases} \exp\left[-2l(t^{a_1} - x^{a_1}) + \lambda a(n_x \tau)^{a-1}(t-x)\right], & n_x = n_t \\ \exp\left[-2l(t^{a_1} - x^{a_1}) + \lambda a(n_x \tau)^{a-1}((n_x + 1)\tau - x) + 2l\lambda \sum_{j=1}^{n_t - n_x} z_j a_1 ((n_x + j)\tau)^{a_1-1}\right], & \text{otherwise} \end{cases} \quad (2.7)$$

where $n_x = \lfloor \frac{x}{\tau} \rfloor$, $F(\cdot)$ is the system lifetime distribution defined in equation (2.1) and $f(\cdot)$ is its intensity function. $z_j = t - n_t \tau$ when $j = n_t - n_x$ and τ otherwise.

Similarly, denote P_{ik_3} be the probability that one component survives at $k_3\tau_0$ while one fails in $((i-1)\tau_0, i\tau_0]$, $1 \leq i \leq k_3$. Then

$$P_{ik_3} = 2 \int_{(i-1)\tau_0}^{i\tau_0} f(x) \bar{F}(x) K_{k_3\tau_0}(x) dx \quad (2.8)$$

where $K_{k_3\tau_0}$ is given as in equation (2.7).

Let $p_{k,k} = q_k$, denote $p_k^{(2)}$ be the probability that the system is renewed at $k\tau_0$ which yields

$$p_k^{(2)} = \sum_{i=1}^k p_{i,k}, \quad k = 1, 2, \dots, k_3 - 1;$$

$$p_{k_3}^{(2)} = 1 - \sum_{i=1}^{k_3-1} p_k^{(2)}.$$

Average long-run cost per unit time

The cost rate in the long run $C^{(2)}(k_3)$ under maintenance policy 2 is therefore:

$$\begin{aligned} C^{(2)}(k_3) &= \frac{1}{\sum_{k=1}^{k_3} p_k^{(2)} k \tau_0} \left(c_r + \frac{c_2}{2} \sum_{k=1}^{k_3} \sum_{i=1}^k p_{i,k} (\hat{n}_{k-1} + \hat{n}_{i-1}) \right) \\ &+ \frac{1}{\sum_{k=1}^{k_3} p_k^{(2)} k \tau_0} \frac{c_2}{2} \sum_{i=1}^{k_3} P_{i,k_3} (\hat{n}_{i-1} + \hat{n}_{k_3-1}) \\ &+ \frac{1}{\sum_{k=1}^{k_3} p_k^{(2)} k \tau_0} \left(c_2 \bar{F}^2(k_3 \tau_0) \hat{n}_{k_3-1} + c_p \sum_{k=1}^{k_3} \sum_{i=1}^k p_{i,k} \right) \end{aligned} \quad (2.9)$$

2.3.3 Average cost evaluation under policy 3

Failure rate

Under this policy, the imperfect repair is carried out at time $n\tau$, $n = 1, 2, \dots$ if both components are functional. Moreover, we assume that if one component fails in $[(i-1)\tau_0, i\tau_0]$, then the survival component will be repaired imperfectly at $i\tau_0 + j\tau'$, $j = 1, 2, \dots$, $\tau' = k_2'\tau_0 < \tau$. Other conditions are similar as in policy 2. The failure rate of the long-lived component under policy 3 given that $\min(X_1, X_2) = x$ can be represented as

$$r_3(t) = \begin{cases} r_1(t) & t \leq x \\ 2\lambda a_1 t^{a_1-1} - \lambda \lambda a (n_x \tau)^{a-1} & x < t \leq M_x \tau_0 \\ 2\lambda a_1 (t^{a_1-1} - \lambda (M_x \tau_0 + i\tau')^{a_1-1}) & M_x \tau_0 + i\tau' < t \leq M_x \tau_0 + (i+1)\tau' \end{cases}$$

for $i = 0, 1, \dots$, where $M_x = \lceil \frac{x}{\tau_0} \rceil$

System replacement and failure probability

Define $p_{i,k}^{(3)}$ be the probability that the component failures occur in $((i-1)\tau_0, i\tau_0]$, $((k-1)\tau_0, k\tau_0]$ respectively under policy 3, $P_{i,k_3}^{(3)}$ be the probability that one component survives at $k_3\tau_0$ while one fails in $((i-1)\tau_0, i\tau_0]$, $1 \leq i \leq k_3$ under policy

3. One can deduce

$$\begin{aligned} p_{i,k}^{(3)} &= 2\mathbb{P}((i-1)\tau_0 < X_1 \leq i\tau_0, (k-1)\tau_0 < X_2 \leq k\tau_0) \\ &= 2 \int_{(i-1)\tau_0}^{i\tau_0} f(x)\bar{F}(x)(K_{(k-1)\tau_0}^{(3)}(x) - K_{k\tau_0}^{(3)}(x))dx \end{aligned} \quad (2.10)$$

where

$$K_t^{(3)}(x) \begin{cases} e^{-2l(t^{a_1-x^{a_1}}) + \lambda la(n_x\tau)^{a_1-1}(t-x)}, & n(t) < 0 \\ e^{-2l(t^{a_1-x^{a_1}}) + \lambda la(n_x\tau)^{a_1-1}(i\tau_0-x) + 2l\lambda \sum_{j=0}^{n(t)} \delta_j a_1 (i\tau_0 + j\tau')^{a_1-1}}, & \text{otherwise} \end{cases} \quad (2.11)$$

where $n_x = \lfloor \frac{x}{\tau} \rfloor$, $n(t) = \lfloor \frac{t-i\tau_0}{\tau} \rfloor$, $\delta_j = (t - i\tau_0 - n(t)\tau')$ when $j = n(t)$ and τ' otherwise. Utilizing the same method we have

$$P_{i,k_3}^{(3)} = 2 \int_{(i-1)\tau_0}^{i\tau_0} f(x)\bar{F}(x)K_{k_3\tau_0}^{(3)}(x)dx \quad (2.12)$$

where $K_{k_3\tau_0}^{(3)}$ is defined in equation (2.11), $F(\cdot)$ is the system lifetime distribution defined in equation (2.1) and $f(\cdot)$ is its intensity function.

Similarly, let $p_{k,k}^{(3)} = q_k$, denote by $p_k^{(3)}$ the probability that the system is renewed at $k\tau_0$, then

$$\begin{aligned} p_k^{(3)} &= \sum_{i=1}^k p_{i,k}^{(3)}, \quad k = 1, 2, \dots, k_3 - 1; \\ p_{k_3}^{(3)} &= 1 - \sum_{i=1}^{k_3-1} p_k^{(3)}. \end{aligned}$$

Average long-run cost per unit time

The cost rate in the long run $C^{(3)}(k_3)$ under maintenance policy 3 is:

$$\begin{aligned} C^{(3)}(k_3) &= \frac{1}{\sum_{k=1}^{k_3} p_k^{(3)} k\tau_0} (c_r + \frac{c_2}{2} \sum_{k=1}^{k_3} \left(\sum_{i=1}^{k-1} p_{i,k}^{(3)} (\tilde{n}_{k-1-i} + 1 + 2\hat{n}_{i-1}) + 2q_k \hat{n}_{k-1} \right) \\ &\quad + \frac{c_2}{2} \left(\sum_{i=1}^{k_3-1} P_{i,k_3} (2\hat{n}_{i-1} + \tilde{n}_{k_3-1-i} + 1) + 2P_{k_3,k_3} \hat{n}_{k_3-1} \right) \\ &\quad + c_2 \bar{F}^2(k_3\tau_0) \hat{n}_{k_3-1} + c_p \sum_{k=1}^{k_3} \sum_{i=1}^k p_{i,k}^{(3)}) \end{aligned} \quad (2.13)$$

2.4 The properties of minimal average costs

In order to minimize the average cost rate under each maintenance policy, the following theorems are presented.

Theorem 2.1 *The optimal cost rate $C(k_3)$ satisfies the following equation.*

$$\lim_{k_3 \rightarrow \infty} (C(k_3 + 1) - C(k_3)) = 0$$

See Appendix 7.1.1 for the proof.

Theorem 2.2 *The optimal cost rate $C^{(2)}(k_3)$ satisfies the following equation.*

$$\lim_{k_3 \rightarrow \infty} (C^{(2)}(k_3 + 1) - C^{(2)}(k_3)) = 0.$$

See Appendix 7.1.2 for the proof.

Theorem 2.3 *The optimal cost rate $C^{(3)}(k_3)$ satisfies the following equation.*

$$\lim_{k_3 \rightarrow \infty} (C^{(3)}(k_3 + 1) - C^{(3)}(k_3)) = 0.$$

See Appendix 7.1.3 for the proof.

2.5 Parameter estimation

To develop maintenance policies to equilibrate the owner costs and the system safety, the primary issue is to obtain efficient evaluation of the system failure-related properties which implies the estimation of a , l and a_1 respectively in this study.

We propose a two-step method to estimate the system parameters a and a_1 . Suppose that the test number is n and the failure times of the short-lived component (resp. the long-lived component) are t_i (resp. \hat{t}_i), $i = 1, 2, \dots, n$ where $t_i < \hat{t}_i$ and $\{t_i, i = 1, 2, \dots, n\}$ (resp. $\{\hat{t}_i, i = 1, 2, \dots, n\}$) are independent. The system load is known and equal to $2l$. Therefore, according to the failure rates, the likelihood function of $\{t_i, i = 1, 2, \dots, n\}$ is given as follows:

$$f(t_1, t_2, \dots, t_n; a, l) = (al)^n \prod_{i=1}^n t_i^{a-1} \exp(-l \sum_{i=1}^n t_i^a)$$

By calculating respectively the first derivative of the log-likelihood function with respect to a and l the maximum-likelihood estimate \hat{a} and \hat{l} can be obtained from

$$\frac{n}{\hat{a}} + \sum_{i=1}^n \log t_i - \hat{l} \hat{a} \sum_{i=1}^n t_i^{\hat{a}-1} = 0 \quad (2.14)$$

$$\hat{l} = \frac{n}{\sum_{i=1}^n t_i^{\hat{a}}} \quad (2.15)$$

Besides, it is reasonable to assume that \hat{t}_i is independent of \hat{t}_j ($i \neq j$) given the value of t_i for any $i, j \in \{1, 2, \dots, n\}$. Therefore, the conditional likelihood function of $\{\hat{t}_i, i = 1, 2, \dots, n\}$ is as follows:

$$f(\hat{t}_1, \hat{t}_2, \dots, \hat{t}_n \mid t_1, t_2, \dots, t_n; a_1, \hat{l}) = (2\hat{l}a_1)^n \prod_{i=1}^n t_i^{a_1-1} \exp(-2\hat{l} \sum_{i=1}^n t_i^{a_1} + 2\hat{l} \sum_{i=1}^n t_i^{a_1})$$

Similarly, the estimate \hat{a}_1 can be obtained from

$$\frac{n}{\hat{a}_1} + \log \sum_{i=1}^n \hat{t}_i - 2\hat{l}\hat{a}_1 \sum_{i=1}^n (t_i^{\hat{a}_1-1} - t_i^{\hat{a}_1-1}) = 0 \quad (2.16)$$

2.6 Numerical analysis

2.6.1 Failure probability calculation by two methods

For illustration, considering $k_1 = 1$, $k_2 = 4$, $a = 1.2$, $a_1 = 1.6$, $l = 0.006$, $\lambda = 0.9$, $k_3 = 200$, the failure probabilities p_k , q_k are calculated by both exact calculation and Monte Carlo methods. For each probability, 100000 Monte Carlo simulations are carried out. For $1 < k < k_3 = 200$, the probabilities p_k , q_k are depicted respectively in Figure 2.5 and Figure 2.6. The results show the consistency of calculations with the two methods. Based on the above and the fact that the time used for numerical results (about 0.3s) is tiny compared to which of the simulation (about 2 days), in the following the analyses of different maintenance policies are carried out only by numerical calculations.

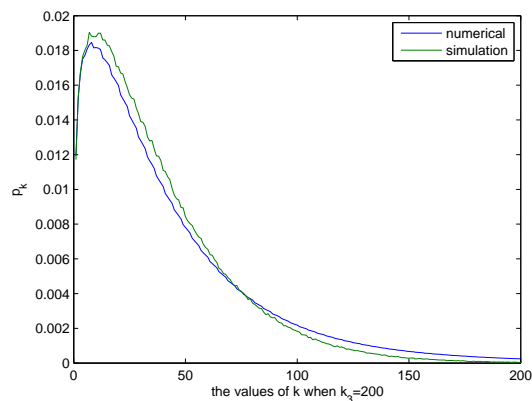


Figure 2.5: p_k in function of k when $k_3 = 200$, Monte Carlo (green), exact method (blue)

In the following, we present a numerical example to illustrate the system behavior. Afterward, for different maintenance policies, we analyse the impact of different parameters on the long run average total cost.

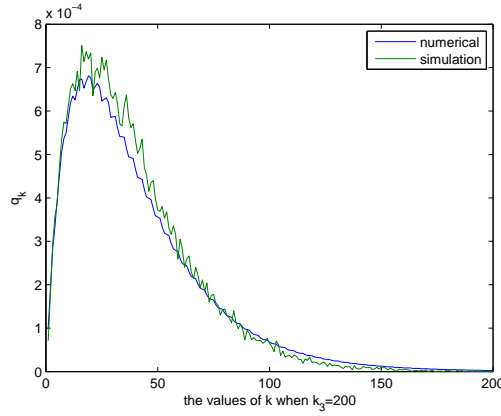


Figure 2.6: q_k in function of k when $k_3 = 200$, Monte Carlo (green), exact method (blue)

2.6.2 Optimization Algorithm

Since the existence of the minimal cost is proved the maintenance optimization can be carried out with different methods.

For the numerical minimization the following algorithm is implemented.

1. Set parameters $\tau, \tau_0, \tau', a, a_1, l, \lambda, c_2, c_r, c_p$, and the accuracy ε
2. Set $k_3 = 1$.
3. for an arbitrary k_3 presented in Section 2.3 calculate $C(k_3)$ and $C(k_3 + 1)$
 - if $|C(k_3 + 1) - C(k_3)| < \varepsilon$ and $|C(k_3 + 2) - C(k_3 + 1)| < \varepsilon$ go to 4
 - else $k_3 = k_3 + 1$ repeat 3
4. Compare the value from $C(1)$ to $C(k_3 + 1)$ and let the optimal value be $C^* = \min\{C(i)\}, i = 1, 2, \dots, k_3 + 1$.

In the following, let $C^{(i)*}$ and $N^{(i)*}\tau_0$ be the optimal maintenance cost and the corresponding maintenance time under policy $i, i = 1, 2, 3$.

2.6.3 Policy 1: component based policy

Let $\tau = 4, \tau_0 = 1, c_2 = 25, c_r = 100, c_p = 220, l = 0.04, \lambda = 0.5, a = 1.3, a_1 = 2, \varepsilon = 0.0001$. Tables 2.1-2.4 show the variation of the optimal long run cost rate under policy 1 with one parameter while other system parameters are unchanged.

The numerical results in Tables 2.1, 2.2, 2.3 and 2.4 can be interpreted as follows.

optimal	$l = 0.03$	0.04	0.06	$\lambda = 0.4$	0.5	0.6
$N^{(1)*}$	20	16	12	16	16	24
$C^{(1)*}$	19.8975	24.0127	32.1823	25.1940	24.0127	22.6421

Table 2.1: The optimal cost rate with different load l and maintenance effect λ under policy 1.

optimal	$a = 1.3$	1.4	1.5	$a_1 = 1.8$	2	2.1
$N^{(1)*}$	16	12	12	20	16	16
$C^{(1)*}$	24.0127	26.6954	29.3361	21.6656	24.0127	25.1899

Table 2.2: The optimal cost rate with different deterioration parameters a and a_1 under policy 1.

optimal	$c_2 = 20$	25	30	$c_r = 90$	100	120
$N^{(1)*}$	16	16	12	16	16	20
$C^{(1)*}$	23.1970	24.0127	24.8285	22.7558	24.0127	26.4438

Table 2.3: The optimal cost rate with different maintenance cost units c_2 and c_r under policy 1.

optimal	$c_p = 150$	220	250
$N^{(1)*}$	24	16	16
$C^{(1)*}$	21.5265	24.0127	25.0171

Table 2.4: The optimal cost rate with different penalty cost unit c_p under policy 1.

- Table 2.1 shows that the total maintenance cost and maintenance frequency increase with the load. Moreover, the closer is the imperfect maintenance to the perfect maintenance the cheaper is the total maintenance cost. As λ increases, the system replacement become less frequent.
- It can be noticed in Table 2.2 that, long run average total maintenance cost increases as the components deteriorate faster. More replacements are required for fast deteriorations.
- When a , a_1 , λ , l are high, the system deteriorates faster, it is then sensible to carry out the preventive maintenance more often. The increasing parameter a of Weibull distribution, which is positively correlated with the failure rate of the system, impacts lightly the optimal average costs, while it is more sensitive to the variation of l .
- The optimal average costs are quite robust in the sense that they don't vary significantly with the degradation parameters.
- Unsurprisingly, the optimal cost rate C^* is increasing with imperfect maintenance cost c_2 , system renewal cost c_r and failure penalty cost c_p .

2.6.4 Policy 2: system based policy

Under policy 2, we take the same parameters as in policy 1. The optimal average long-run cost $C^{(2)*}$ under different parameter settings are illustrated in Tables 2.5, 2.6, 2.7, 2.8. The following features can be pointed out:

- Similarly to policy 1, it can be noticed that the $C^{(2)*}$ decreases with the system load l and shows decreasing tendency with λ . Because the larger λ is, the better is the repair. Therefore, the system is more robust and is liable to survive. The higher is the system load, the larger is the system failure rate. Thus the system fails more frequently which causes an increase of the maintenance cost.
- Table 2.6 considers the variation of $C^{(2)*}$ under different deterioration parameters a and a_1 . In our example, the long-run average total cost is not very sensitive to the variations of a_1 . It is more sensitive to small values of a .
- In Tables 2.7-2.8 the impact of maintenance costs units variations on $C^{(2)*}$ are presented. Obviously $C^{(2)*}$ increases with respect to maintenance costs units. It is shown in Table 2.8 that the system is replaced earlier when the penalty is high. Therefore, the system owner are recommended to consider the potential risk he or she should undertake when the consequence of system failure is serious.

optimal	$l = 0.03$	0.04	0.06	$\lambda = 0.4$	0.5	0.6
$N^{(2)*}$	24	20	16	12	20	28
$C^{(2)*}$	28.6937	34.6901	45.9031	36.6135	34.6901	32.2203

Table 2.5: The optimal cost rate with different load l and maintenance effect λ under policy 2.

optimal	$a = 1.3$	1.4	1.5	$a_1 = 1.8$	2	2.1
$N^{(2)*}$	20	8	4	8	20	20
$C^{(2)*}$	34.6901	38.6017	39.9732	32.3945	34.6901	35.3785

Table 2.6: The optimal cost rate with different deterioration parameters a and a_1 under policy 2.

optimal	$c_2 = 20$	25	30	$c_r = 90$	100	120
$N^{(2)*}$	20	20	16	16	20	24
$C^{(2)*}$	33.8679	34.6901	35.5110	33.6349	34.6901	36.7052

Table 2.7: The optimal cost rate with different maintenance costs units c_2 and c_r under policy 2.

optimal	$c_p = 150$	220	250
$N^{(2)*}$	28	20	16
$C^{(2)*}$	28.1650	34.6901	37.4398

Table 2.8: The optimal cost rate with different c_p under policy 2.

2.6.5 Policy 3: component based policy, variant

Under policy 3, by adopting the parameters setting as in the cost analysis under policy 1 and assuming that $k'_2 = 3$, Tables 2.9, 2.10, 2.11 and 2.12 elucidate the similar average cost variation as in policy 2.

optimal	$l = 0.03$	0.04	0.06	$\lambda = 0.4$	0.5	0.6
$N^{(3)*}$	20	16	16	8	16	28
$C^{(3)*}$	28.6769	34.6156	45.2158	36.3411	34.6156	32.0843

Table 2.9: The optimal cost rate with different load l and maintenance efficiency λ under policy 3.

optimal	$a = 1.3$	1.4	1.5	$a_1 = 1.8$	2	2.1
$N^{(3)*}$	16	4	4	8	16	20
$C^{(3)*}$	34.6156	37.2893	38.4369	31.5371	34.6156	35.4033

Table 2.10: The optimal cost rate with different deterioration parameters a and a_1 under policy 3.

optimal	$c_2 = 20$	25	30	$c_r = 90$	100	120
$N^{(3)*}$	20	16	16	8	16	24
$C^{(3)*}$	33.7100	34.6156	35.4971	33.4508	34.6156	36.6353

Table 2.11: The optimal cost rate with different maintenance unit costs c_2 and c_r under policy 3.

optimal	$c_p = 150$	220	250
$N^{(3)*}$	28	16	8
$C^{(3)*}$	28.2491	34.6156	37.2236

Table 2.12: The optimal cost rate with different penalty c_p under policy 3.

It is pointed out that in our example policy 1 is the most economical one compared to policy 2 and 3. In most cases, policy 3 is a second-best choice which indicates that it is necessary to consider the period to carry out the imperfect maintenance when only one component is functional in the system. In all policies, the maintenance costs are very sensitive to the system load. In sum, under high loads, low quality system and non efficient maintenance operations, the maintenance policies are very costly. The penalty cost may have influence on the policy preference. For example when $c_p = 150$ or 220, the cost difference between policy 2 and policy 3 is tiny while policy 3 is more favored when the failure consequence is serious.

2.7 Conclusion

In this chapter, different maintenance policies for a two-component load-sharing system are proposed. To avoid system failure, imperfect preventive maintenance and preventive system replacement are applied. The long-run average cost of the system under different maintenance policies are obtained. Numerical examples are illustrated and it is shown that policy 1 is the most cost saving policy. It is recommended to the decision maker to consider an equilibrium between the average maintenance cost and the system reliability.

Chapter 3

Maintenance and warranty of a system with type 3 stochastic dependence

In this chapter, we consider the maintenance optimization problem and the warranty cost of a two-component system: the evolutions of the two components are modelled by its lifetime distribution (component 1) and its degradation respectively. Component 2 degenerates with time which fails when its degradation level reaches or first exceeds a predetermined level. The dependencies between the two components are: whenever component 1 fails, it causes a random amount of damage to component 2 which may accelerate its failure; whenever component 2 fails, it is catastrophic and induces the failure of component 1. This system can be used to model the interaction between the brake pad and the disc rotor in a brake system where the automobile and motorcycle decelerate through friction [76]. If brake pad fails, it causes damages to the disc rotor because of the excessive wear comparing to the normal wear. The brake system fails when disc rotor wears out. In the following, we will study such a system in the contexts of the maintenance costs in section 3.1 as well as the warranty costs respectively in section 3.2.

3.1 Maintenance cost analysis

To guarantee the normal operation of the system, both corrective maintenance and preventive maintenance policies are carried out. It is assumed that component 1 undergoes imperfect repair with virtual age method when it fails and the system is renewed when component 2 failure occurs. Three preventive maintenance policies are considered respectively: (1) the system is replaced at a planned time T ; (2) the system is replaced at the N th failure of component 1; (3) the system is replaced at the planned time T or at the N th failure of component 1 which occurs first.

Our primary objective is to provide explicit expressions of the system long-run expected costs per unit of time under different maintenance policies. Sufficient conditions of the existence of the long-run expected costs are derived theoretically.

Numerical results and Monte Carlo simulations are presented to illustrate the policies. Sensitivity analysis are discussed to elucidate the trend variation of the average cost under different system parameter settings.

3.1.1 Model description and system analysis

In the following, first, the model description is given.

General assumptions

- For a brand-new component 1, in absence of the maintenance action, it has a lifetime distribution $F(\cdot)$, $F(0) = 0$ and density function $f(\cdot)$. Assume that component 1 is repairable.
- Whenever component 1 fails, a random amount of damage is induced to component 2. We assume the damages Z_j ($j = 1, 2, \dots$) are additive and identical, independent random variables with distribution $H(\cdot)$, $H(\cdot)$ is differentiable. Let $H^{*(k)}(\cdot)$ be the k fold convolution of $H(\cdot)$ with itself.
- Component 2 is non-repairable and it fails when its degradation level exceeds a predetermined threshold L . Assume that the degradation level is nondecreasing. The failure of component 2 induces component 1 into simultaneous failure.
- In absence of the damage caused by component 1, let be $\{Y(t), t \geq 0\}$ the natural degradation level of component 2 at time t and σ_L be the time at which the degradation level reaches or first exceeds L , $L > 0$. Then its distribution function is

$$G_{\sigma_L}(t) = \mathbb{P}(\sigma_L \leq t) = \mathbb{P}(Y(t) \geq L), t \geq 0.$$

$G_{\sigma_L}(t)$ is integrable.

- The repair time and system renewal time are negligible.

Now, some assumptions on component 1 and its maintenance are listed

- Let us denote X_i and a , respectively the inter-maintenance time between the $(i - 1)$ th and the i th repair and the age-reducing factor of component 1.
- The effective age of component 1 after the i th repair is $B_i = B_{i-1} + aX_i$, $i = 1, 2, \dots$, where the initial age $B_0 = 0$ and the reduction degree $0 \leq a \leq 1$.
- Let $V_n(\cdot)$ be the distribution function of virtual age B_n . $V_n(\cdot)$ is differentiable.

- Let $N(t)$ be the number of component 1 failures by time t and $p_n(t) = \mathbb{P}\{N(t) = n\}$. It is easy to verify that according to [132], the latter probability mass function is given as follows:

$$p_n(t) = \int_0^{at} \frac{\bar{F}(y + \frac{at-y}{a})}{\bar{F}(y)} v_n(y) dy \quad (3.1)$$

where $\bar{F}(\cdot) = 1 - F(\cdot)$, $v_n(x) = \frac{d}{dx} V_n(x)$, $v_1(x) = \frac{1}{a} f(\frac{x}{a})$ and $v_{n+1}(x) = \frac{1}{a} \int_0^x \frac{f(y + \frac{x-y}{a})}{\bar{F}(y)} v_n(y) dy$ for $n \geq 1$ [132]. $\mathbb{E}N(t) = \sum_{n=0}^{\infty} n p_n(t)$ is differentiable.

Similarly, let be $N_s(t)$ the number of system failure by time t and $p_n^s\{t\} = \mathbb{P}\{N_s(t) = n\}$. We have the following proposition.

Proposition 1 *Let $F_s(t)$ be the lifetime distribution function of the system, then*

$$F_s(t) = \sum_{k=0}^{\infty} p_k(t) \int_0^{\infty} G_{\sigma_{L-z}}(t) dH^{*(k)}(z). \quad (3.2)$$

See Appendix 7.2.1 for detailed proof.

In the next, by assuming the repair cost of component 1 is c_1 , the replacement cost of the system is c_2 , we provide results on the system long run average maintenance cost under different preventive policies.

3.1.2 Maintenance cost derivation

In this part, the average long run maintenance costs are derived considering the renewal reward theorem. Indeed, since the system is replaced as good as new after the failure of component 2, the system replacement intervals are independent and identically distributed. Therefore, the interval Δ between two replacements can be considered as a renewal cycle and the long run average maintenance cost $\mathbb{E}C_{\infty}$ can be obtained considering the average maintenance cost on a renewal cycle $\mathbb{E}(C(\Delta))$ as follows: $\mathbb{E}C_{\infty} = \frac{\mathbb{E}(C(\Delta))}{\mathbb{E}(\Delta)}$.

3.1.2.1 Age- T -based policy

Let's consider an age based policy, where the system is replaced either at its failure or when it reaches the age T , whichever comes first. Let $C_{\infty}(T)$ be the long run average maintenance cost associated to the age-based policy.

Theorem 3.1 *The long run average maintenance cost of the age- T -based policy is*

$$C_{\infty}(T) = \frac{c_2 + c_1 \mathbb{E}N(T) \bar{F}_s(T) + c_1 \int_0^T \mathbb{E}N(t) dF_s(t)}{T \bar{F}_s(T) + \int_0^T t dF_s(t)} \quad (3.3)$$

where $\mathbb{E}N(t) = \sum_{n=0}^{\infty} np_n(t)$ is the expected failure number of component 1 by time t , $p_n(t)$ is given in equation (3.1), $F_s(t)$ is the system lifetime distribution function given in equation (3.2).

See the Appendix 7.2.2 for detailed proof.

3.1.2.2 Failure-number- N -based maintenance policy

Under this policy, we replace the whole system at the N th failure of component 1 or at the failure of component 2 whichever occurs first. Let $C_{\infty}(N)$ be the average long run maintenance cost.

Theorem 3.2 *The average long run maintenance cost under the failure-number- N -based policy $C_{\infty}(N)$ is calculated as follows*

$$C_{\infty}(N) = \frac{c_2 + c_1 \sum_{k=1}^{N-1} \int_0^{\infty} R_{k-1}(t) dV_k(at)}{\int_0^{\infty} \sum_{k=0}^{N-1} p_k(t) R_k(t) dt} \quad (3.4)$$

where $R_i(t) = \int_0^L (1 - G_{\sigma_L-z}(t)) dH^{*(i)}(z)$, $i = 1, 2, \dots, N$, $R_0(t) = 1 - G_{\sigma_L}(t)$, $G_{\sigma_L}(t)$ is the probability distribution function of the first hitting time of level L by the component 2 degradation, $H(\cdot)$ is the cumulative distribution function of the total damage caused by component 1 to component 2, $H^{*(n)}(\cdot)$ is the n -fold convolution of $H(t)$ with itself, $p_k(t)$ is the probability that k component 1 failures occur in $[0, t]$, $V_k(t)$ is the distribution function of the virtual age after the k th repair.

See the Appendix 7.2.3 for the proof.

3.1.2.3 (N, T) -based maintenance policy

In the (N, T) policy, the system is replaced at age T , or at the N th failure of component 1, or at the time of component 2 failure whichever occurs first. Assume that the long run average maintenance cost under this circumstance is $C_{\infty}(N, T)$.

Theorem 3.3 *The average long run maintenance cost under (N, T) policy $C_{\infty}(N, T)$ is as follows:*

$$C_{\infty}(N, T) = \frac{c_2 + c_1 \sum_{k=1}^{N-1} \int_0^T R_{k-1}(t) dV_k(at)}{\int_0^T \sum_{k=0}^{N-1} p_k(t) R_k(t) dt}$$

where $R_i(t) = \int_0^L (1 - G_{\sigma_L-z}(t)) dH^{*(i)}(z)$, $i = 1, 2, \dots, N$, $R_0(t) = 1 - G_{\sigma_L}(t)$, $G_{\sigma_L}(t)$ is the probability distribution function of the first hitting time of level L by the component 2 total degradation at time t , $H(t)$ is the lifetime distribution of damage caused by component 1 to component 2 until time t , $H^{*(n)}(t)$ is the n -fold convolution of $H(t)$ with itself, $p_k(t)$ is the probability of k failures of component

1 occur in $[0, t]$, $V_k(t)$ is the distribution function of the component 1 virtual age after the k th repair.

The proof is given in the Appendix 7.2.4.

In the following section, the existences of optimal T^* and N^* minimizing $C_\infty(T)$ and $C_\infty(N)$ respectively are discussed.

3.1.3 Maintenance policy optimization

In this paragraph, some conditions on the existence of the optimal maintenance age- T -based and failure-number- N -based policies are derived.

Theorem 3.4 *Under the age- T -based policy, the optimal and unique age T^* which minimizes the long run average maintenance cost exists if the following assumptions are satisfied:*

$$\lim_{T \rightarrow \infty} \frac{d\mathbb{E}N(T)}{dT} \int_0^T \bar{F}_s(t) dt - \mathbb{E}N(T)\bar{F}_s(T) - \int_0^T \mathbb{E}N(t) dF_s(t) > \frac{c_2}{c_1}, \quad (3.5)$$

and

$$\frac{d^2\mathbb{E}N(t)}{d^2t} > 0, t > 0. \quad (3.6)$$

When component 1 has a Weibull lifetime distribution $F(t) = 1 - \exp(-\lambda t^b)$ with minimal repair ($a = 1$), the optimal $T^* = \infty$ when $b \leq 1$ which means there is no preventive maintenance in the optimal policy.

See the Appendix 7.2.5 for the proof .

Theorem 3.5 *Under the failure-number- N -based maintenance policy, the optimal N^* exists if the following assumptions are satisfied:*

$$\lim_{N \rightarrow \infty} \frac{d_1(N) \sum_{k=1}^{N-1} d_2(k)}{d_2(N)} - \sum_{k=1}^{N-1} d_1(k) > \frac{c_2}{c_1}, \quad (3.7)$$

and

$$\frac{d_1(N)}{d_2(N)} \text{ is a convex function with respect to } N. \quad (3.8)$$

Where $d_1(k) = \int_0^\infty \bar{R}_{k-1}(y) a v_k(ay) dy$ and $d_2(k) = \int_0^\infty p_k(t) R_k(t) dt$.

See the Appendix 7.2.6. for the proof .

3.1.4 Numerical examples

Here we assume component 1 has a Weibull cumulative distribution function $F(t) = 1 - e^{-\lambda t^b}$, $t > 0$. The damage is exponentially distributed with expectation μ . The deterioration of component 2 follows a homogeneous Gamma process which has been widely used and successfully data-fitted in describing system degradation on the account of erosion, corrosion, crack growth, etc. [13]. Its density function is as follows:

$$g_{\alpha t, \beta}(u) = \frac{\beta^{\alpha t} u^{\alpha t - 1} e^{-\beta u}}{\Gamma(\alpha t)}, \quad \alpha > 0, \beta > 0 \quad (3.9)$$

where

$$\Gamma(\alpha) = \int_0^{\infty} u^{\alpha-1} e^{-u} du. \quad (3.10)$$

Therefore the first hitting time has the following distribution function:

$$G_{\sigma_L}(t) = \frac{\Gamma(\alpha t, L\beta)}{\Gamma(\alpha t)}, t \geq 0 \quad (3.11)$$

where the lower incomplete Gamma function is defined as follows:

$$\Gamma(\alpha, x) = \int_x^{\infty} z^{\alpha-1} e^{-z} dz. \quad (3.12)$$

To begin with, let be $\lambda = 0.01$, $b = 2$, $a = 0.6$, $T = 10$, $N = 2$, $L = 20$, $\lambda = 0.01$, $b = 2$, $\alpha = 4$, $\beta = 2$, $c_1 = 50$, $c_2 = 250$, $c_3 = 300$. Table 3.1 shows some quantities obtained by their formulas and Monte Carlo simulations with 10^5 histories and 95% confidence intervals (CI). All results are coherent. For the sensitivity analysis, the

	Formula	Simulation	95% confidence interval
$p_0(T)$	0.3679	0.3670	[0.3576, 0.3764]
$p_1(T)$	0.4211	0.4234	[0.4137, 0.4331]
$p_2(T)$	0.1648	0.1626	[0.1554, 0.1689]
$F_s(T)$	0.5820	0.5800	[0.5770, 0.5831]
$C_{\infty}(T, N)$	34.2762	34.2715	[34.1976, 34.3454]

Table 3.1: Calculations of various quantities by their formulas and the Monte Carlo simulations ($N = 10^5$) respectively

baseline parameters are chosen according to the results obtained in the previous paragraph and figures. Considering the Gamma process parameters setting $\alpha = 4$, $\beta = 2$, the average time to cross the safety level $L = 20$ is 10 time units. The system reliability function at time $t = 10$ is 0.4 which means for these parameters setting the threshold $L = 20$ can be considered as a security level not to cross otherwise the system is not functioning correctly. For the optimization of the long run average maintenance cost, as it is mentioned in the obtained theorems, the existence of the optimal value depends on the system parameters and the cost ratio: $\frac{c_2}{c_1}$. The smaller is $\frac{c_2}{c_1}$ the higher is the possibility of the existence of a

optimum. Therefore, under the constraint $c_2 > c_1$, we chose close cost values for c_1 and c_2 in order to assure the existence of the optimum. It should be mentioned that this is an example presenting the system properties and optimal maintenance cost rates under different maintenance policies. In reality, parameters selections are based on the real data and the parameter estimation etc. We set $\alpha = 4, \beta = 2, \lambda = 0.1, b = 2, a = 0.6, L = 20, c_1 = 500, c_2 = 600, \mu = 1$. In the following, one parameter is changed to evaluate the variation of the average cost while other parameters remain unchanged.

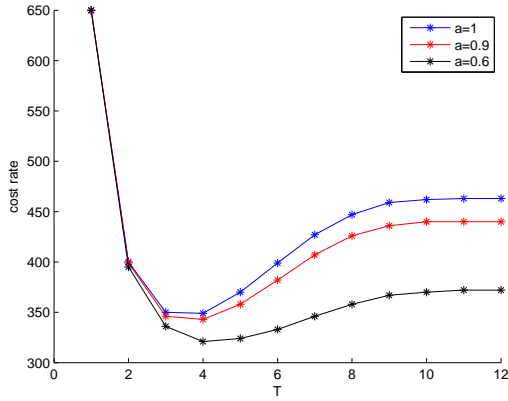


Figure 3.1: $C_\infty(T)$ with different a

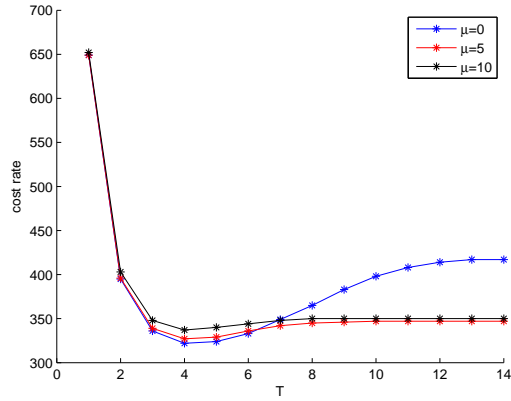


Figure 3.2: $C_\infty(T)$ with different μ

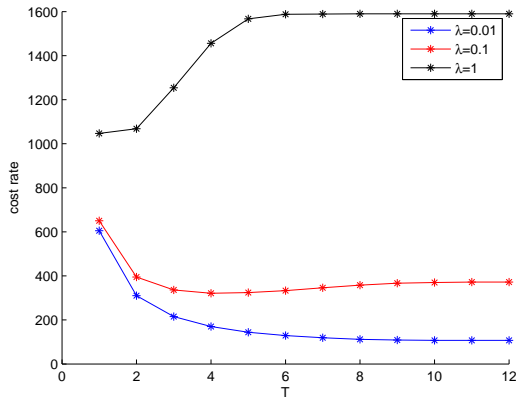


Figure 3.3: $C_\infty(T)$ with different λ

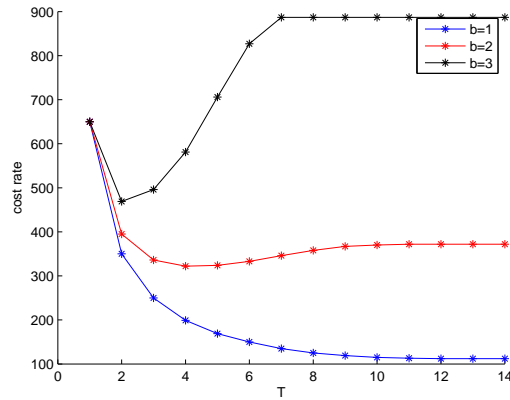


Figure 3.4: $C_\infty(T)$ with different b

α	β	$C_\infty(T^*)$	T^*
4	2	322	4
4	1	323	4
6	1	335	∞

c_1	c_2	$C_\infty(T^*)$	T^*
500	600	322	4
300	600	240	6
500	900	383	5

Table 3.2: The optimal average cost rate under age- T -based policy with different gamma process parameters (left) and different repair cost (right)

Figures 3.1-3.4 and Table 3.2 show the long run average maintenance cost $C_\infty(T)$ with different parameters setting. The following behaviors are pointed out.

- The optimal expected cost rate increases with a and μ . The larger a is, the

worse is the repair. Therefore, there are more damages are caused to component 2 which induces the system failure. As μ represents the expectation of damage, for large values of a and μ the system fails more often. Under these settings, the optimal cost is less sensitive to the variations of μ . This is due to the fact that the failure is mostly due to the deterioration of the system which is faster and more significant in comparison to damages.

- Preventive maintenance is not necessary for small values of λ and b . Indeed, in this case, component 1 is rarely failed and its failure barely damages component 2. The failure is more due to the natural deterioration of component 2. On the contrary, when λ and b are very large, the system is preventively repaired after a short operational period which increases the average long run cost. Of course it is then more worthwhile to replace the whole system when component 1 often fails and its maintenance cost is expensive.
- It can be noticed in Table 3.2, the optimal average cost increases as α , c_1 , c_2 grows. In this case, the deterioration of the component 2 is faster and the maintenance are more costly.

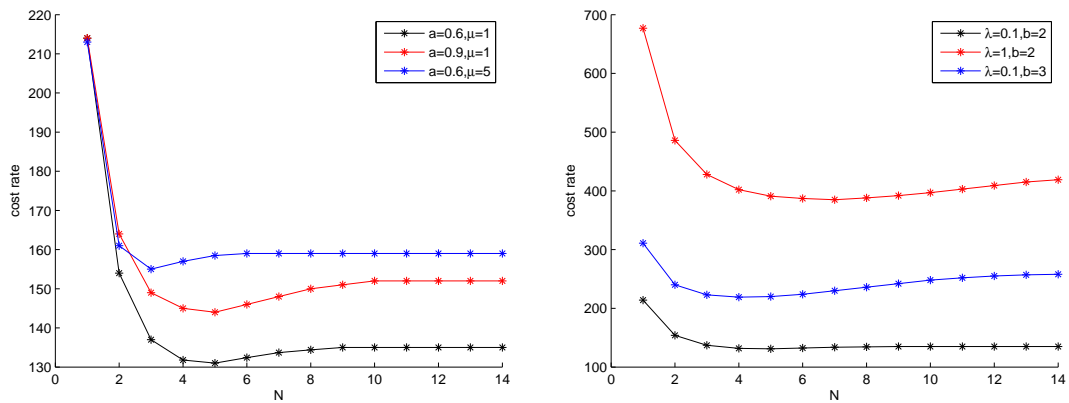


Figure 3.5: $C_\infty(N)$ with different parameters

Figures 3.5 and Table 3.3 show the long run average maintenance cost under failure-number- N based policy with different parameters setting. The repair cost of unit is set to $c_1 = 100$ and other conditions are as in age- T -based policy. As expected it can be noticed that under failure-number- N -based policy the average cost rate increases with b , μ and λ . A similar behavior as in the age- T -based policy can be pointed out. Figure 3.6 shows the cost rate under (N, T) policy with the

α	β	$C_\infty(N^*)$	N^*
4	2	131	5
4	1	167	4
6	1	213	3

c_1	c_2	$C_\infty(N^*)$	N^*
100	600	131	5
100	500	117	4
500	600	213	1

Table 3.3: The optimal average cost rate under failure-number- N -based policy with different gamma process parameters (left) and component repair cost (right)

parameters given as in the T policy.

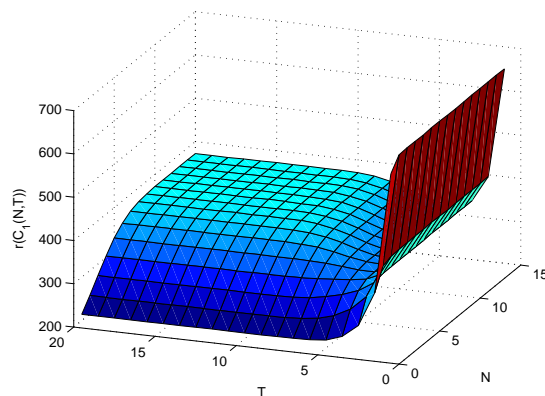


Figure 3.6: $C_\infty(T, N)$ with different parameters policy

3.2 Warranty cost analysis

In this section, we examine the expected warranty costs from the perspectives of the manufacturer and the consumer respectively. By considering the product service time, the warranty costs allocations between the manufacturer and the consumer are presented. Numerical examples are given to demonstrate the applicability of the methodology. It is shown that, independent of the type of the warranty policy, the failure interaction between components has a non-negligible impact on the manufacturer profits and the consumer costs. The initial warranty length has impact on the product quality preferences to both the consumer and the manufacturer.

Two warranty policies: the renewing free-repair warranty (RFRW) and the non-renewing free-repair warranty (non-renewing FRW) are studied respectively. More precisely, it means, during the initial warranty period W ,

- under the non-renewing FRW
 - component 1 is imperfectly repaired at its failure at time T_{f1} and the warranty of the system remains valid during the remaining $W - T_{f1}$ period.
 - the system is replaced at the failure of component 2 at time T_{f2} and the warranty of the system remains valid only during the remaining $W - T_{f2}$ period.
- under the RFRW,
 - component 1 is imperfectly repaired at its failure at time T_{f1} and the warranty of the system remains valid during the remaining $W - T_{f1}$ period.
 - the system is replaced at the failure of component 2 at time T_{f2} and a full system warranty period of length W is provided.

Suppose that to the manufacturer, the component 1 repair cost and the system replace cost are c_1 and c_2 respectively. First, under the two warranty policies, from the perspective of the manufacturer, the expected warranty costs are formulated.

3.2.1 Manufacturer's expected warranty costs

For simplicity, we call the system failure as minor failure if it contains only component 1 failure and major failure if both the two components in the system fail.

3.2.1.1 Expected manufacturer's cost under the non-renewing FRW

Under the non-renewing FRW, the manufacturer covers the repair or replacement cost up to a duration W from the initial product purchase time. Denote $\mathbb{E}(C(W))$ be the manufacturer's expected warranty cost under non-renewing FRW within warranty period W . The following theorem gives the expected manufacturer cost under non-renewing FRW.

Theorem 3.6 *Under the non-renewing FRW, the expected manufacturer cost $\mathbb{E}(C(W))$ within the warranty period W is given by:*

$$\mathbb{E}(C(W)) = \int_0^W [1 + M(W - t)]k(t)dt$$

where $M(t)$ is the renewal function related with $F_s(t)$ which can be given by $M(t) = \sum_{n=1}^{\infty} F_s^{(n)}(t)$, $k(t) = c_1 \bar{F}_s(t) \sum_{n=0}^{\infty} n[av_n(at) - \int_0^{at} \frac{f(y + \frac{at-y}{a})}{F(y)} v_n(y) dy] + c_2 f_s(t)$, $F_s(t)$ ($f_s(t)$) (given in equation (3.2)), $F(t)$ ($f(t)$) are the lifetime distribution (density) functions of component 2 and component 1 respectively, $\bar{F}_s(t) = 1 - F_s(t)$, a is the imperfect maintenance degree of component 1, $v_n(t)$ is given in equation (3.1).

See Appendix 7.2.7 for the detailed proof.

3.2.1.2 Expected manufacturer's cost under RFRW policy

Under the RFRW, the major failure within the warranty period is replaced by a new one (repaired as good as new) with a full system warranty period W . Let us denote T_r the warranty cycle which is a time interval from the system purchase time until the expiration of the warranty. It is obvious that in this situation the warranty cycle depends on the initial warranty period W , the system renewal times and the time interval between two consecutive system renewal time. Denote $N_r(W)$ be the system renewal times when the initial warranty period is W , and J_i , $i = 1, 2, \dots, N_r(W)$ the corresponding time interval between the $(i - 1)$ th and the i th system renewal time which are identically and independently distributed

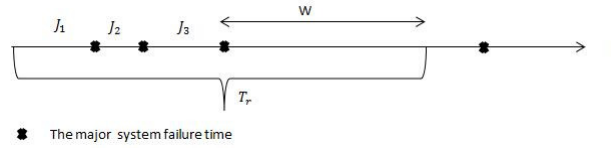


Figure 3.7: A possible system warranty period under RFRW policy

random variables. Figure 3.7 gives an example of total system warranty length under RFRW. It can be noticed that:

$$T_r = J_1 + J_2 + \cdots + J_{N_r(W)} + W$$

Let $\mathbb{E}(C^R(W))$ be the expected manufacturer warranty cost under RFRW. We have the following theorem.

Theorem 3.7 *Under the RFRW, the expected system cost is*

$$\mathbb{E}(C^R(W)) = \sum_{n=0}^{\infty} \bar{F}_s(W) F_s^n(W) \left(nc_2 + c_1 \mathbb{E}(N(W)) + nc_1 \frac{\int_0^W \mathbb{E}(N(t)) dF_s(t)}{F_s(W)} \right)$$

where $F_s(t)$ is the component 2 lifetime distribution function, $\bar{F}_s(t) = 1 - F_s(t)$, $\mathbb{E}(N(t)) = \sum_{n=1}^{\infty} np_n(t)$ which is the expected failure times of component 1 by time t before system failure.

See Appendix 7.2.8 for detailed proof.

By now, we have formulated the expected warranty costs of the manufacturer under the non-renewing FRW and the RFRW. One step further, if we take the consumer's behaviour into consideration, assume that he/she intends to put the product in service until T , $W < T < \infty$. After which, the system is not repaired when failure occurs. Besides, after the product warranty, the consumer chose coming back to the original manufacturer (or the supplier, retailer, seller etc. here we do not distinguish them) for the maintenance. Denoted c_{11} , c_{22} be the component 1 repair cost and the system renewal cost respectively to the consumer. It is rational to set $c_{ii} > c_i$, $i = 1, 2$ as the manufacturer earns from the difference between the prices c_{11} , c_{22} and its costs c_1 , c_2 . In the next, the warranty cost and profit to the consumer and the manufacturer respectively are examined.

3.2.2 Expected warranty profit and cost in $(0, T]$

In the following, by assuming that the product service time is T , $W < T < \infty$. The warranty cost analysis of different warranty policies in $(0, T]$ are explored. Hereafter we assume that component 1 is minimally repaired when failure occurs.

3.2.2.1 Expected warranty profit and cost in $(0, T]$ under the RFRW

Let $C^s(T)$ and $TP^s(T)$ be the total warranty cost of the consumer (owner) and the profit of the manufacturer respectively, let $\mathbb{E}(C^s(T))$ and $\mathbb{E}(TP^s(T))$ be their expectations. They are formulated in the following theorem.

Theorem 3.8 *Under the RFRW, the expected maintenance cost of the consumer and the expected total profit of the manufacturer in $[0, T]$ are as follows:*

$$\begin{aligned}\mathbb{E}(C^s(T)) &= \int_W^T (1 + M(T - u)) dh(u) \\ \mathbb{E}(TP^s(T)) &= \mathbb{E}(C^s(T)) - \mathbb{E}(C^1(T)) - c_2 M(T)\end{aligned}$$

where

$$\begin{aligned}h(T) &= c_{22}(F_s(T) - F_s(W)) + c_{11} \int_W^T \bar{F}_s(t) d\mathbb{E}N(t) \text{ for } T \geq W \\ \mathbb{E}(C^1(T)) &= h_1(T) + \int_0^T h_1(T - t) dM(t) \\ h_1(T) &= c_1 \mathbb{E}(N(T)) - c_1 \int_0^T F_s(t) d\mathbb{E}(N(t))\end{aligned}$$

$M(t) = \sum_{n=1}^{\infty} F_s^{(n)}(t)$ is the system renewal function related with $F_n(t)$ and $\mathbb{E}N(t) = \sum_{n=1}^{\infty} np_n(t)$ is the expected failure number of component 1 in $[0, t]$ before system replacement. $F_s(t)(f_s(t))$, $F(t)(f(t))$ are the lifetime distribution (density) functions of component 2 and component 1 respectively, $\bar{F}_s(t) = 1 - F_s(t)$.

See Appendix 7.2.9 for the proof.

3.2.2.2 Expected warranty profit and cost under the non-renewing FRW

Under the non-renewing FRW, the manufacturer pays the full warranty cost if the system fails within the warranty period and the warranty is not renewed. It is reasonable to assume that the consumer is provided with a new system with a full warranty if the major failure occurs in $(W, T]$ and he/she covers himself/herself the full purchase cost.

Let $C^{sn}(T)$ and $TP^{sn}(T)$ be the total warranty cost of the consumer and the profit of the manufacturer respectively. Let $\mathbb{E}(C^{sn}(T))$, $\mathbb{E}(TP^{sn}(T))$ be their expectations.

Theorem 3.9 *Under the non-renewing FRW, when component 1 failures occur according to a Poisson process, the expected warranty cost of the consumer and the*

expected total profit of the manufacturer in $[0, T]$ are derived as follows:

$$\begin{aligned}\mathbb{E}(C^{sn}(T)) &= h_n(T) + \int_0^T h_n(T-u) dM_U(u) \\ \mathbb{E}(TP^{sn}(T)) &= \mathbb{E}(C^{sn}(T)) - c_1 \mathbb{E}N(T) - c_2 M(T)\end{aligned}$$

where

$$\begin{aligned}F_U(t) &= F_s(t) - \int_0^W \bar{F}(t-x) dM(x) \quad \text{for } t \geq W \text{ and } 0 \text{ otherwise} \\ h_n(t) &= c_{22} F_U(t) + c_{11} \int_W^t \bar{F}_U(\theta) d\mathbb{E}N(\theta) \quad \text{for } t \geq W \text{ and } 0 \text{ otherwise}\end{aligned}$$

$M_U(t)(M(t))$ is the system renewal function related with $F_U(t)(F_n(t))$ and $\mathbb{E}N(t) = \sum_{n=1}^{\infty} np_n(t)$ is the expected failure number of component 1 in $[0, t]$ before system replacement. $F_s(t)(f_s(t)), F(t)(f(t))$ are the lifetime distribution (density) functions of component 2 and component 1 respectively, $\bar{F}_s(t) = 1 - F_s(t)$.

See Appendix 7.2.10 for the detailed proof.

3.2.3 Numerical examples

In this section, the warranty cost under respectively RFRW and non renewing FRW are presented. The impact of system parameter setting associated to the warranty costs are explored. The consumer cost and the manufacturer profit under the two warranty policies are compared.

As shown in section 3.1.4, We keep suppose that component 1 is Weibull distributed with lifetime distribution function $F(t) = 1 - e^{(-\lambda t^b)}, t > 0$. The damage induced to component 2 by component 1 failure has exponential distribution with expectation μ . The natural deterioration of component 2 follows a homogeneous Gamma process. We keeps the notations in section 3.1.4. Firstly, let $S = \{\lambda = 1/64, b = 2, a = 1, \mu = 4, \alpha = 4, \beta = 2, L = 20, c_1 = 5, c_2 = 25, c_{11} = 1, c_{22} = 4, T = 20\}$. In the following, the impacts of some parameters involved in the model are investigated by changing one parameter each time and comparing with the results derived by using the original data set S .

The impacts of the initial warranty length and the failure interaction

Figure 3.8 shows the consumer's warranty costs and the manufacturer's profits under the RFRW with various values of W and μ . As expected, the difference between the total expected manufacturer profit $\mathbb{E}(TP^s)$ and the expected cost of the consumer $\mathbb{E}(C^s)$ is constant.

In both cases of short-run and long-run horizon, the expected warranty cost of the consumer (or the expected profit of the manufacturer) is a decreasing function of the warranty period. It is easy to understand that as the the warranty increases,

the manufacturer has to cover more warranty costs. Consequently the expected manufacturer profit $\mathbb{E}(TP^s)$ or the expected cost of the consumer $\mathbb{E}(C^s)$ decrease as the warranty period grows.

In presence of interaction between components, for short warranty period, the manufacturer profit increases. As the warranty period gets longer, the average total manufacturer benefit (or the consumer total average cost) decreases. Which means in presence of interaction between components, a long warranty period is more cost efficient for the consumer. Indeed, the component dependency has a stronger impact on the long term where due to the aging of component 1 more and more maintenance are required. The component or system failure usually occurs after the warranty period resulting in a maintenance cost to the consumer and so a profit to the manufacturer. When W gets larger, the failure interaction between components accelerates the failure within warranty leading to a gratis system maintenance for the consumer which decreases the income of the manufacturer. Therefore, in presence of dependence, the manufacturer profit (consumer cost) decreases for long warranty periods.

The impact of the repair degree of component 1

Figure 3.9 shows the expected manufacturer cost within the warranty period under RFRW with different repair degree of component 1 ($a = 0.2$, $a = 0.6$ and $a = 1$). It shows that the manufacturer warranty cost increases with a . The larger a is, the worse is the repair of component 1. More damages are induced to component 2 which accelerates its degradation and so its failure. Therefore more system maintenance and replacement costs are induced to the manufacturer expenses. It is noted that the manufacturer cost is less sensitive when W is small. This is due to the fact that in our example, the occurrence of component 1 failure and the system failure are rare within small W even component 1 is minimally repaired at failure.

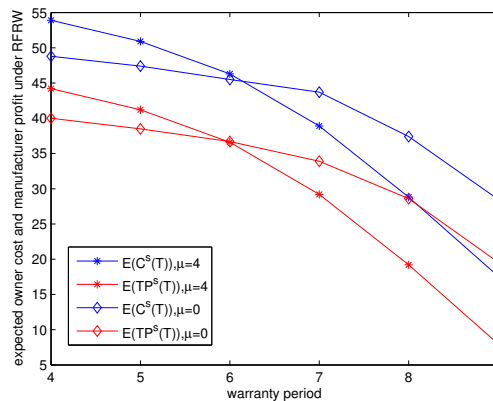


Figure 3.8: The expected consumer cost and manufacturer profit under RFRW with various values of W and μ

In the following, We reset $S = \{W = 5, \lambda = 1/8, b = 1, a = 1, \mu = 5, \alpha = 4, \beta = 2, L = 20, c_1 = 5, c_2 = 25, c_{11} = 1, c_{22} = 4, T = 20\}$.

The impact of component 1 quality

Figure 3.10 shows that when the warranty period is small, both consumer cost and

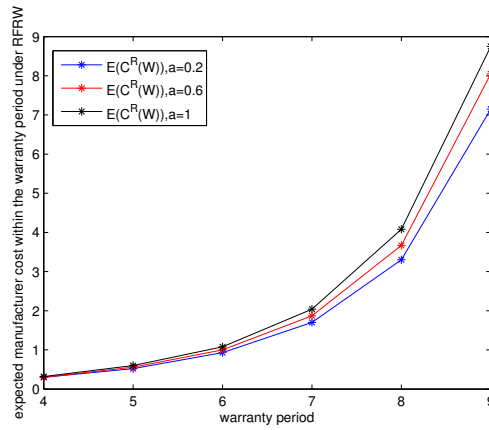


Figure 3.9: The expected manufacturer cost $\mathbb{E}(C^R(W))$ under RFRW with different repair degree of component 1 ($a = 0.2$, $a = 0.6$ and $a = 1$).

manufacturer profit increase with λ . The larger is λ , the shorter is the lifetime of component 1. Therefore more component/system failure are occurred within the service time T . Hence for short warranty period, the consumer pays more and the manufacturer loses less. On the contrary, when the warranty period W is large, during the period T , more failure costs are covered by the manufacturer which decreases its profit and the consumer cost.

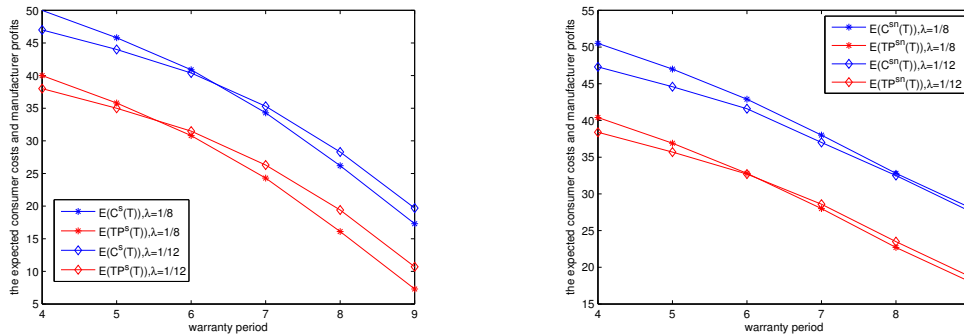


Figure 3.10: The expected consumer costs and manufacturer profits with different λ ($\lambda = 1/8$, $\lambda = 1/12$).

The impact of component 2 natural deterioration

Similarly, it can be noticed in Figure 3.11 that the manufacturer profit and the consumer cost increase with α when W is small and decrease with α when W is large. With large α , component 2 deteriorates faster and its lifetime is shorter. It is then more beneficial to the consumer to have long warranty period. For a short warranty period with these parameters setting, $\alpha \in \{4, 6\}$, the failure is not mainly due to the natural deterioration, henceforth the total maintenance cost is not very sensitive to the changes of α .

From the sensitivity analysis of the impacts of component 1 and component 2, it is noticed that the initial warranty period W has impact on the manufacturer's and the consumer's product quality preferences.

The impact of cost units

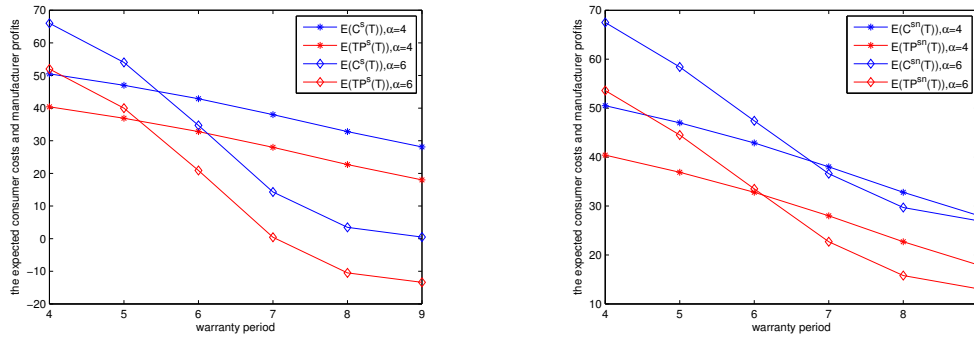


Figure 3.11: The expected consumer cost and manufacturer profits with different α ($\alpha = 4, \alpha = 6$).

parameters cost/profit	$c_{11} = 5, c_{22} = 25$			$c_{11} = 5, c_{22} = 15$			$c_{11} = 10, c_{22} = 25$		
	$W = 4$	$W = 6$	$W = 9$	$W = 4$	$W = 6$	$W = 9$	$W = 4$	$W = 6$	$W = 9$
$\mathbb{E}(C^s(T))$	50.0	40.9	17.3	31.5	25.0	10.5	53.6	42.5	17.5
$\mathbb{E}(TP^s(T))$	40.0	30.8	7.3	21.5	15.0	0.45	43.6	32.4	7.5
$\mathbb{E}(C^{sn}(T))$	50.5	42.9	28.1	29.6	25.6	17.0	51.3	43.3	29.0
$\mathbb{E}(TP^{sn}(T))$	40.4	32.8	18.0	20.6	16.7	8.2	41.3	34.3	20.1

Table 3.4: The warranty costs and profits under the RFRW and the non-renewing FRW with different values of c_{11} and c_{22}

The manufacture profits and consumer costs under both RFRW and non renewing FRW with different maintenance costs are presented in Tables 3.4 and 3.5. As expected, both the manufacturer profit and the consumer cost decrease with the value of W . As W gets larger, more maintenance costs are covered by the manufacturer which induces the manufacture profit and the consumer cost. Both the consumer cost and the manufacturer profit increase with the consumers maintenance cost units c_{11} and c_{22} . Since c_1 and c_2 are maintenance costs units of the manufacturer, the consumer maintenance cost is unchanged with c_1 and c_2 . However, the manufacturer profits under both the RFRW and non renewing FRW decrease as c_1 and c_2 increase. It is obvious that the RFRW is more economic for the consumer and the non-renewing FRW is more favorable to the manufacturer. In addition, the most profitable policy for the manufacturer is no warranty policy under which the consumer covers the whole cost during its usage.

It is worth mentioning that the estimation of the warranty costs/profits to the consumer/manufacturer under different warranty policies could be references paving the way for lucrative market strategies. What important is the overall interests to

parameters cost/profit	$c_1 = 1, c_2 = 4$			$c_1 = 1, c_2 = 8$			$c_1 = 3, c_2 = 8$		
	$W = 4$	$W = 6$	$W = 9$	$W = 4$	$W = 6$	$W = 9$	$W = 4$	$W = 6$	$W = 9$
$\mathbb{E}(C^s(T))$	50.0	40.9	17.3	50.0	40.9	17.3	50.0	40.9	17.3
$\mathbb{E}(TP^s(T))$	40.0	30.8	7.3	31.7	22.7	-1.0	27.9	18.6	-4.7
$\mathbb{E}(C^{sn}(T))$	50.5	42.9	28.1	50.5	42.9	28.1	50.5	42.9	28.1
$\mathbb{E}(TP^{sn}(T))$	40.4	32.8	18.0	32.2	24.8	10.0	28.3	20.6	7.6

Table 3.5: The warranty costs and profits under the RFRW and the non-renewing FRW with different values of c_1 and c_2

the manufacturer rather than the profits/costs induced by the warranty individual.

3.3 Conclusion

In this chapter, we have considered a two-component system with type 3 stochastic dependence. In the first scenario, different preventive maintenance policies have been proposed and correspondingly, expected maintenance costs in the long time horizon have been calculated. The impact of the stochastic dependence on the system maintenance cost have been analyzed numerically. In the second scenario, the warranty costs of the manufacturer and the consumer have been studied. Two warranty policies: the non-renewing free replacement policy and the renewing free replacement policy have been considered. Under each policy, in the short-time horizon as well as in the long-time horizon, the expected warranty costs of the manufacturer and the consumer have been calculated respectively. Numerical examples show that it would be beneficial to the manufacturer to mitigate the stochastic dependence between components in product design phase. Therefore it is recommended to the decision-maker to consider the warranty cost budget in the development of the warranty strategy, to take the warranty cost as a reference in the evaluation of the product profit, etc.

Chapter 4

Maintenance and warranty of a system with type 1 stochastic dependence

In this chapter, we consider a two-component system which is similar to the one in Chapter 3 in terms of the evolutions of components. The difference is the stochastic dependence between components. Here we suppose that whenever component 1 failure occurs, it causes the failure of component 2 with probability $\bar{r}(= 1 - r)$, $0 < r < 1$ and has no impact on component 2 with probability r ; whenever component 2 fails, it causes the failure of component 1. The same maintenance policies and warranty policies as in chapter 3 are carried out in the following. Besides, we reuse the notations in Chapter 3.

4.1 Evaluation of long run average maintenance costs

Assume T_{sI} and $F_{sI}(t)$ be the system lifetime and its cumulative probability distribution respectively. Denote $\bar{F}_{sI}(t) = 1 - F_{sI}(t)$, $\bar{G}_{\sigma_L}(t) = 1 - G_{\sigma_L}(t)$. Note that

$$\begin{aligned}\bar{F}_{sI}(t) &= p_0(t)\mathbb{P}(Y(t) < L) + \sum_{k=1}^{\infty} \mathbb{P}\left(Y(t) + \sum_{i=1}^{N(t)} Z_i < L \mid N(t) = k\right) \mathbb{P}(N(t) = k) \\ &= p_0(t)\bar{G}_{\sigma_L}(t) + \sum_{k=1}^{\infty} \mathbb{P}\left(Y(t) + \sum_{i=1}^k Z_i < L \mid \sum_{i=1}^k Z_i = 0\right) \mathbb{P}\left(\sum_{i=1}^k Z_i = 0\right) p_k(t) \\ &= p_0(t)\bar{G}_{\sigma_L}(t) + \sum_{k=1}^{\infty} \mathbb{P}(Y(t) < L) r^k p_k(t) \\ &= \sum_{k=0}^{\infty} r^k p_k(t) \bar{G}_{\sigma_L}(t)\end{aligned}$$

where $p_n(t)$ is the probability that the failure number of component 1 is n by time t which is given in equation (3.1). Therefore

$$F_{sI}(t) = 1 - \sum_{k=0}^{\infty} r^k p_k(t) \bar{G}_{\sigma_L}(t) \quad (4.1)$$

4.1.1 Maintenance cost derivation

4.1.1.1 Age- T -based policy

Denote by $C_{\infty I}(T)$ the average long run cost when it undergoes age- T -based policy. The following theorem gives its expression.

Theorem 4.1 *The average long run cost $C_{\infty I}(T)$ under age- T -based policy is*

$$C_{\infty I}(T) = \frac{c_2 + \sum_{n=1}^{\infty} (n-1) c_1 r^{n-1} \bar{r} \int_0^T \bar{G}_{\sigma_L}(t) dV_n(at)}{\int_0^T \bar{F}_{sI}(t) dt} + \frac{\sum_{n=0}^{\infty} n r^n c_1 (\int_0^T p_n(t) dG_{\sigma_L}(t) + p_n(T) \bar{G}_{\sigma_L}(T))}{\int_0^T \bar{F}_{sI}(t) dt}$$

where $F_{sI}(t)$ is the distribution of the system failure time with type 1 failure interaction between components given in equation (4.1), $p_n(t)$ is given in equation (3.1), $\mathbb{E}N(t) = \sum_{n=0}^{\infty} n p_n(t)$, $V_k(t)$ is the distribution function of the component 1 virtual age after the k th repair given in equation (3.1), $G_{\sigma_L}(t)$ is the probability distribution function of the first hitting time of level L by the component 2 total degradation at time t ,

See the Appendix 7.3.1 for the proof.

4.1.1.2 Failure-number- N based policy

Theorem 4.2 *Suppose that $C_{\infty I}(N)$ is the system average long run cost under failure-number- N policy, then*

$$C_{\infty I}(N) = \frac{c_2 + \sum_{n=1}^{N-1} (n-1) c_1 r^{n-1} \bar{r} \int_0^{\infty} \bar{G}_{\sigma_L}(t) dV_n(at) + \sum_{n=0}^{N-1} n r^n c_1 \int_0^{\infty} p_n(t) dG_{\sigma_L}(t)}{\int_0^{\infty} \sum_{k=0}^{N-1} r^k p_k(t) \bar{G}_{\sigma_L}(t) dt} + \frac{\int_0^{\infty} r^{N-1} (N-1) c_1 \bar{G}_{\sigma_L}(t) dV_N(at)}{\int_0^{\infty} \sum_{k=0}^{N-1} r^k p_k(t) \bar{G}_{\sigma_L}(t) dt}$$

where $p_n(t)$ is given in equation (3.1), $V_k(t)$ is the distribution function of the component 1 virtual age after the k th repair given in equation (3.1), $G_{\sigma_L}(t)$ is the probability distribution function of the first hitting time of level L by the component 2 total degradation at time t .

See the Appendix 7.3.2 for the proof.

4.1.1.3 (N, T) -based maintenance policy

Theorem 4.3 *Suppose that $C_{\infty I}(N, T)$ is the average long run system cost under (N, T) policy, then*

$$C_{\infty I}(N, T) = \frac{c_2 + \sum_{n=1}^{N-1} (n-1)c_1 r^{n-1} \bar{r} \int_0^T \bar{G}_{\sigma_L}(t) dV_n(at) + \sum_{n=0}^{N-1} nr^n c_1 \int_0^T p_n(t) dG_{\sigma_L}(t)}{\int_0^T \sum_{k=0}^{N-1} r^k p_k(t) \bar{G}_{\sigma_L}(t) dt} + \frac{\int_0^T r^{N-1} (N-1) c_1 \bar{G}_{\sigma_L}(t) dV_N(at) + \sum_{n=1}^{N-1} nc_1 \bar{G}_{\sigma_L}(T) p_n(T)}{\int_0^T \sum_{k=0}^{N-1} r^k p_k(t) \bar{G}_{\sigma_L}(t) dt}$$

where $p_n(t)$ is given in equation (3.1), $V_k(t)$ is the distribution function of the component 1 virtual age after the k th repair given in equation (3.1), r is the probability that component 1 failure has no impact on component 2, $G_{\sigma_L}(t)$ is the probability distribution function of the first hitting time of level L by the component 2 total degradation at time t .

Since the result is a combination of results obtained in Theorem 4.1 and 4.2, the proof is omitted

4.1.2 Maintenance policy optimization

When component 1 failure occurs according to a nonhomogeneous Poisson process, we have the following Theorem.

Theorem 4.4 *Suppose that the failure rate of component 1 at t is $h_1(t)$. When component 1 undergoes minimal repair ($a = 1$), the optimal and unique age T^* which minimizes $C_{\infty I}(T)$ exists if*

$$\lim_{T \rightarrow \infty} h_1(T) \int_0^T \bar{F}_{sI}(t) dt - \int_0^T \bar{F}_{sI}(t) h_1(t) dt > \frac{c_2}{rc_1} \quad (4.2)$$

And

$$\frac{dh_1(t)}{dt} > 0, t > 0 \quad (4.3)$$

When component 1 has Weibull lifetime $F(t) = 1 - \exp(-\lambda t^b)$, the optimal $T^* = \infty$ when $b \leq 1$ which means no preventive maintenance is the optimal policy. Otherwise inequation (4.3) always holds and hence the optimal and unique age T^* exists if only condition (4.2) holds.

See the Appendix 7.3.3 for the proof.

4.1.3 Numerical examples

Similar to the numerical analysis in section 3.1.4, we also assume component 1 has a Weibull cumulative distribution function $F(t) = 1 - e^{-\lambda t^b}, t > 0$. The

deterioration of component 2 follows a homogeneous Gamma process. Here we set $\alpha = 4, \beta = 2, \lambda = 0.1, b = 2, a = 0.6, L = 20, c_1 = 500, c_2 = 600$ and $r = 0.8$. Figures 4.1 and 4.2 show the cost rate $C_{\infty I}(T)$ when we change only one parameter.

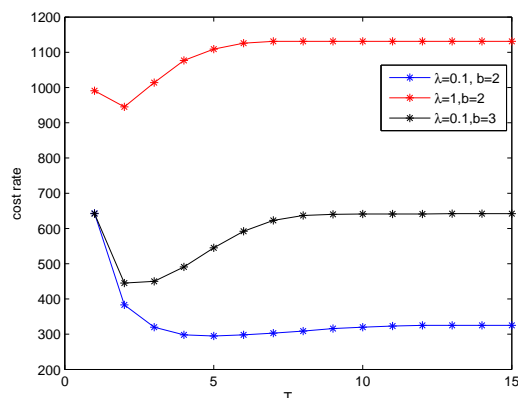
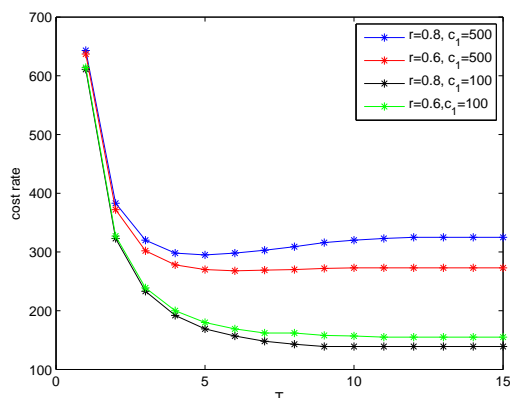


Figure 4.1: $C_{\infty I}(T)$ with different r and c_1 Figure 4.2: $C_{\infty I}(T)$ with different λ and b

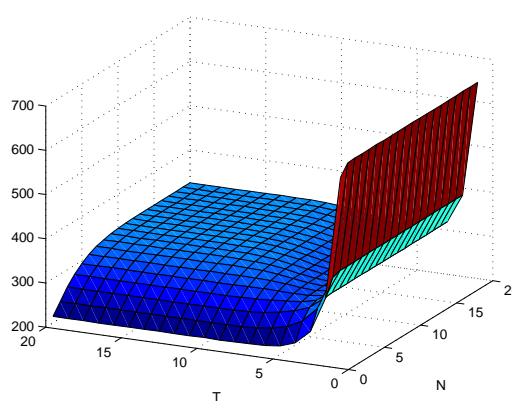
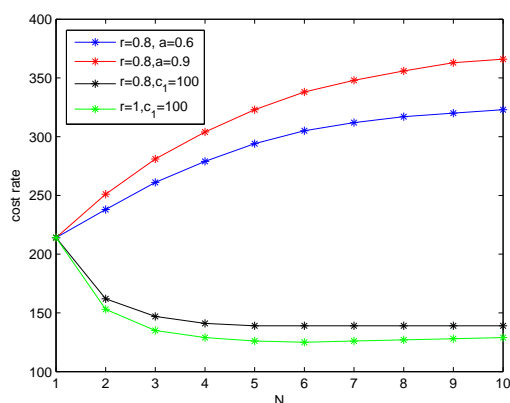


Figure 4.3: $C_{\infty I}(N)$ with different parameters Figure 4.4: $C_{\infty I}(N, T)$ with different N and T

Similarly, the variation of the expected cost rate under the failure-number- N -based policy and (N, T) policy are illustrated in Figures 4.3 and 4.4. The following results can be pointed out.

- When the imperfect repair cost is high, the optimal expected cost increases with r . Because the greater r is, the more imperfect repairs are carried out. However, the imperfect repair is not effective as it improves slightly the component 1 ($a = 0.6$) with a high cost ($c_1 = 500$) comparing to the system replacement with a similar cost ($c_2 = 600$).
- The optimal average cost increases when λ and b increase. The larger these parameters get, the more often the component 1 is failed and repaired.
- The optimal $C_{\infty I}(N)$ increases with N when the imperfect repair cost c_1 is small.

4.2 Evaluation of the warranty costs

Two warranty policies, the renewing free-repair warranty (RFRW) and the non-renewing free-repair warranty (non-renewing FRW) are examined in this section. In the following, to simplify the calculation, we assume that component 1 is minimally repaired when it fails. Let $N(t)$ be the number of component 1 failures by time t and $p_n(t) = \mathbb{P}\{N(t) = n\}$. Then it follows a non-homogeneous Poisson process with

$$p_n(t) = \frac{H(t)^n}{n!} e^{-H(t)} \quad (4.4)$$

where $H(t) = \int_0^t h(\theta) d\theta$.

First, the expected warranty costs of the manufacturer within the warranty period are derived.

4.2.1 Warranty costs of the manufacturer

Two theorems regarding the warranty cost of the manufacturer under the non-renewing FRW and the RFRW respectively are presented.

Theorem 4.5 *Under the non-renewing FRW, the expected manufacturer cost $\mathbb{E}(C(W))$ within the warranty period W is given by*

$$\mathbb{E}(C(W)) = \int_0^W [1 + M(W - t)] k(t) dt$$

where $M(t)$ is the renewal function related with $F_{sI}(t)$ denoted by $M(t) = \sum_{n=1}^{\infty} F_{sI}^{(n)}(t)$, $F_{sI}(t)$ is the distribution of the system failure time with type 1 failure interaction between components given in equation (4.1), $k(t) = c_1 r h(t) \bar{F}_{sI}(t) + c_2 f_{sI}(t)$, $f_{sI}(t)$ is the intensity function of $F_{sI}(t)$, $h(t)$ is the failure rate of component 1 by time t .

See Appendix 7.3.4 for the proof.

Theorem 4.6 *Under the RFRW, from the viewpoint of the manufacturer, the expected warranty cost within the warranty cycle is*

$$\mathbb{E}(C^R(W)) = \sum_{n=0}^{\infty} \bar{F}_{sI}(W) F_{sI}^n(W) \left(n c_2 + c_1 r H(W) + n r c_1 \frac{\int_0^W H(t) dF_{sI}(t)}{F_{sI}(W)} \right)$$

$F_{sI}(t)$ is the distribution of the system failure time with type 1 failure interaction between components given in equation (4.1), $h(t)$ and $H(t)$ are the component 1 failure rate and cumulative hazard function respectively.

See Appendix 7.3.5 for the proof.

In the next paragraph, the warranty cost allocation to the consumer and the manufacturer are examined.

Theorem 4.7 *Under the RFRW, the expected maintenance cost of the ownership and the expected total profit of the manufacturer in $[0, T]$ are as follows:*

$$\begin{aligned}\mathbb{E}(C(T)) &= \int_W^T (1 + M(T - u))k(u)du \\ \mathbb{E}(P(T)) &= \mathbb{E}(C(T)) - \int_0^T (1 + M(T - t))k_1(t)dt - c_2M(T)\end{aligned}$$

where $F_{sI}(t)$ is the system renewal time distribution given in equation (4.1), $k(t) = c_{22}f_{sI}(t) + c_{11}\bar{F}_{sI}(t)rh(t)$ for $t \geq W$ and 0 otherwise; $k_1(t) = c_1r\bar{F}_{sI}(t)h(t)$, $t > 0$ and $M(t)$ is the renewal function related with $F_{sI}(t)$ given as $M(t) = \sum_{n=1}^{\infty} F_{sI}^{(n)}(t)$, $f_{sI}(t)$ is the derivative function of $F_{sI}(t)$.

See Appendix 7.3.6 for the proof.

4.2.1.0.a Expected maintenance cost analysis under the non-renewing FRW

Theorem 4.8 *Under the non-renewing FRW, the expected warranty cost of the consumer in $[0, T]$ is derived as*

$$\mathbb{E}(C_{no}(T)) = K_0(T) + \int_0^T K_0(T - u)dM_U(u)$$

where $K_0(t) = c_{22}F_U(t) + c_{11}r \int_W^t \bar{F}_U(\theta)h(\theta)d\theta$ for $t \geq W$ and 0 otherwise; $F_U(t) = F_{sI}(t) - \int_0^W \bar{F}(t - x)dM(x)$ for $t \geq W$ and 0 otherwise; and $M_U(t)$ is the renewal function related with $F_U(t)$.

If $F \in IFR$, the manufacturer profit $E_{no}(P(T))$ satisfies

$$\mathbb{E}(P_{no}(T)) \geq \mathbb{E}_{low}(P(T))$$

where $\mathbb{E}_{low}(P(T)) = \mathbb{E}(C_{no}(T)) - c_1rH(T) - c_2M(T)$ and the ' \geq ' becomes '=' when $h(t)$ is a constant.

See Appendix 7.3.7 for the proof.

4.2.2 Numerical examples

Here we also consider that component 1 has Weibull-distributed lifetime $F(t) = 1 - \exp(-\lambda t^b)$, $t > 0$. The deterioration of component 2 is described by a homogeneous Gamma process. We keep the notations in section 3.1.4 in Chapter 3. Let $\alpha =$

$2, \beta = 2, \lambda = \frac{1}{64}, b = 2, L = 8, c_1 = 1, c_2 = 5$. Table 4.1 and 4.2 show the manufacturer's expected warranty costs within the warranty cycle under different warranty policies with respect to the failure interaction factor \bar{r} and the initial warranty length W respectively.

\bar{r}	0	0.1	0.2	0.3	0.4	0.5
$\mathbb{E}(C(W))$	0.60	0.75	0.88	1.02	1.15	1.28
$\mathbb{E}(C^R(W))$	0.62	0.78	0.95	1.13	1.33	1.53

Table 4.1: The warranty costs of the manufacture under the non-renewing FRW and RFRW with respect to different values of \bar{r} when $W = 5$

We can observe that the expected warranty costs increase continuously with the failure interaction factor and the initial warranty length respectively. It is more sensitive in regards of the RFRW policy, for example, when $\bar{r} = 0.5$, the warranty cost under the RFRW is $(1.53 - 0.62)/0.62 = 1.48$ -fold larger than the cost of the system without failure interaction ($\bar{r} = 0$). For the non-renewing FRW case, this figure stands at $(1.28 - 0.60)/0.60 = 1.13$. It is easy to understand that as \bar{r} (or W) increases, more system major failure occurs within the warranty cycle which results in a rise of the warranty cost. Besides, the non-renewing FRW is always more economic than the RFRW. The reason is due to that under the non-renewing FRW, the manufacturer pays in full of the warranty costs during a fixed time period W , whereas under the RFRW, a longer warranty cost coverage period is required.

It should deserve mentioning that the warranty cost during the warranty cycle could be served as a reference for the warranty budget. It is also important to consider the consumer's behaviour (for example his or her expected usage time period of the product) and the manufacturer's market share (for example the number of the consumers who prefers to come back for the product maintenance after the warranty expiration) for a general estimation of the warranty cost.

Suppose that the product is put into use for a time period T . The consumer comes back to the manufacturer (or the retailer, the seller, here we held the same regard between them) for the product maintenance after the expiration of the warranty cycle. He or she is charged c_{11}, c_{22} respectively for the component 1 maintenance and the system replacement. Let $W = 5, T = 10, \bar{r} = 0.3, \alpha = 2, \beta = 2, \lambda = \frac{1}{64}, b = 1, L = 8, c_1 = 1, c_2 = 5, c_{11} = 4, c_{22} = 12$ be the basic parameter settings. The following figures and tables demonstrate the sensibilities of the manufacturer profits and the consumer costs by varying different system indicators and remaining others unchanged.

It can be seen that

W	4	5	6	7	8
$\mathbb{E}(C(W))$	0.58	1.02	1.70	2.65	3.72
$\mathbb{E}(C^R(W))$	0.61	1.13	2.14	4.13	8.09

Table 4.2: The warranty costs of the manufacture under the non-renewing FRW and RFRW with respect to different values of W when $\bar{r} = 0.3$

- Figures 4.5 and 4.6 illustrate the warranty profits and costs for the manufacturer and the consumer with different values of W and \bar{r} respectively. Both the manufacturer profits and the consumer costs decrease with the initial warranty length W and the failure interaction between components \bar{r} respectively. It is easy to understand that as the warranty period increases, the manufacturer has to cover more warranty costs. Consequently the expected manufacturer profit or the expected cost of the consumer decrease as the warranty period grows. The larger is \bar{r} , the shorter is the expected system lifetime. Therefore more system maintenances are required during the warranty period W and the product service time T reducing the manufacturer warranty profit and the consumer's cost. Under each warranty policy, the difference between the consumer cost and the manufacturer profit is independent of W . The reason is that the difference between the consumer's cost and the manufacturer's profit is actually the manufacturer's maintenance cost during the product service time T which is independent of W . The failure interaction between components has negative effect to the manufacturer profit. It is recommended to the engineers to eliminate the failure interaction between components.
- Table 4.3 describes the variations of the manufacturer profits and the consumer costs with respect to λ, b the parameters in the Weibull distribution and the product service time T . For fixed T , under both the non-renewing FRW and RFRW, the warranty costs of the consumer are decreasing functions of λ and b . It shows that given the product service time T , the consumer prefers better quality and reliability of component 1. Besides, from the perspective of the manufacturer, under the non-renewing FRW, for short product service time ($T = 10$), the manufacturer benefits less from component 1 with good quality (the manufacturer's profit is 4.41 when the expected life time of component 1 is 12 ($\lambda = \frac{1}{12}, b = 1$)) comparing to a bad quality one (the manufacturer's profit is more than 4.69 when the expected lifetime of component 1 is 6.93 ($\lambda = \frac{1}{12}, b = 2$)). On the contrary, the corresponding manufacturer profits are -69.12 and 21.56 when the product service time is $T = 50$. Therefore the service time of the product has impact on the warranty profits which need to be well estimated in the devising of warranty strategies. It might be improper or even counterproductive to assume that the consumer uses the product for a long-run time horizon ($T = \infty$).
- Table 4.4 represents the sensibility of the manufacturer profits and the consumer costs with different component 2 deterioration rates. The average deterioration rate is α/β in this case. It indicates that under the non-renewing FRW, both the manufacturer and the consumer prefer component 2 with slow deterioration speed. However, it is more economic for the consumer owning a fast deteriorate speed when the RFRW is offered. Because in the situation that $\alpha = 2, \beta = 1, F_s(W) = 0.72$ which means with an opportunity of more than 70% the system is replaced by the manufacture with no charge and then the process repeats. However, this figure descends strikingly to 0.12 when $\alpha = 1, \beta = 1$. Besides, $F_s(T) = 0.73$ which implies that the system major failure occurs frequently during the period $[W, T]$ during which the consumer has in charge the system replacement costs.

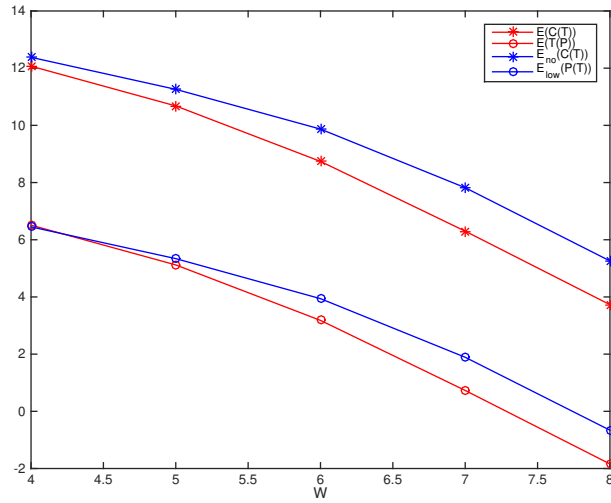


Figure 4.5: The manufacturer's warranty profits $\mathbb{E}(P(T))$ and $\mathbb{E}_{low}(P(T))$ and the consumer's warranty costs $\mathbb{E}(C(T))$ and $\mathbb{E}_{no}(C(T))$ with different values of W

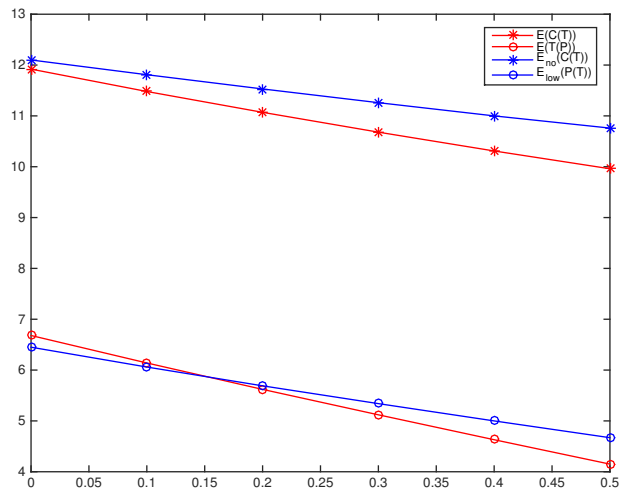


Figure 4.6: the manufacturer's warranty profits $\mathbb{E}(P(T))$ and $\mathbb{E}_{low}(P(T))$ and the consumer's warranty costs $\mathbb{E}(C(T))$ and $\mathbb{E}_{no}(C(T))$ with different values of \bar{r}

Parameters	$\mathbb{E}(C(T))$	$\mathbb{E}(P(T))$	$\mathbb{E}_{no}(C(T))$	$\mathbb{E}_{no}(P(T))$
$\lambda = \frac{1}{64}, b = 2, T = 10$	10.68	5.12	11.26	5.34
$\lambda = \frac{1}{12}, b = 2, T = 10$	11.29	0.45	18.36	4.69
$\lambda = \frac{1}{12}, b = 1, T = 10$	9.55	4.16	9.84	4.41
$\lambda = \frac{1}{64}, b = 2, T = 50$	74.02	38.26	62.39	3.24
$\lambda = \frac{1}{12}, b = 2, T = 50$	82.67	20.56	123.79	-69.12
$\lambda = \frac{1}{12}, b = 1, T = 50$	66.81	33.18	56.01	21.56

Table 4.3: The warranty cost and profit under RFRW and the non-renewing FRW policies respectively with different values of λ , b and T

Parameters	$\mathbb{E}(C(T))$	$\mathbb{E}(P(T))$	$\mathbb{E}_{no}(C(T))$	$\mathbb{E}_{no}(P(T))$
$\alpha = 2, \beta = 1$	4.98	-4.81	11.20	1.40
$\alpha = 1, \beta = 1$	7.65	3.65	7.82	3.81

Table 4.4: The warranty cost and profit under RFRW and the non-renewing FRW policies respectively with different values of α and β .

4.3 Conclusion

We have studied a two-component system with type 1 stochastic dependence in this chapter. Similar to problems considered in chapter 4, the long run average maintenance costs under three preventive policies have been calculated. After which, the warranty costs of the manufacturer and the consumer in the short time horizon have been derived. The effect of the dependence to the system has been discussed numerically.

Chapter 5

Reliability-based measures in dynamic environment

In this chapter, we focus on the reliability-based measures of multi-component systems operating in a dynamic environment. The system reliability and the remaining useful lifetime for the M -out-of- N system configuration are analysed. By assuming that the system failures are non-self-announcing and the repair times are non-negligible, under the periodic inspection policy, the system limiting average availability is calculated. We aim at providing explicit expressions for this system performance measures. Numerical examples indicate that the environment condition has significant effects on the system reliability and the remaining useful lifetime.

Notation

Φ	$\{1, 2, \dots, N\}$
\mathbf{S}	$\{1, 2, \dots, m\}$
\emptyset	empty set
\mathbf{I}	the $m \times m$ identity matrix
$E_1 \setminus E_2$	the set-theoretic difference of set E_1 and set E_2
$\{c_1, c_2, \dots, c_k\}$	subset of Φ with elements $c_1, c_2, \dots, c_k, k = 1, 2, \dots, N$
$\underline{\{c_1, c_2, \dots, c_k\}}$	$\Phi \setminus \{c_1, c_2, \dots, c_k\}, k = 1, 2, \dots, N$
$\overline{T_i}$	the lifetime of component $i, i \in \Phi$
T	the lifetime of the system
$W(t)$	the environment state at time t
$h_i(t, j)$	the failure rate of component i at time t when the environment state is $j, i \in \Phi, j \in \mathbf{S}$
$\mathbf{H}_i(t)$	the matrix with elements $H_i(k, k) = h_i(t, k), i \in \Phi, k \in \mathbf{S}$ on the primary diagonal and 0 otherwise
\mathbf{H}_i	the matrix $\mathbf{H}_i(t)$ when it is time-independent, $i \in \Phi$
$\mathbf{H}_0(t)$	the matrix $\mathbf{H}_i(t)$ when it is independent of $i, i \in \Phi$
\mathbf{H}_0	the matrix $\mathbf{H}_0(t)$ when it is time-dependent
$\mathbf{H}(t)$	the sum of $\mathbf{H}_i(t), i \in \mathbf{S}$, i.e. $\mathbf{H}(t) = \sum_{i \in \mathbf{S}} \mathbf{H}_i(t)$

\mathbf{Q}	the transition rate matrix of the continuous-time Markov chain
q_{ij}	the element of \mathbf{Q} , $i, j \in \mathbf{S}$
$\text{diag}([a_1, a_2, \dots, a_n])$	The diagonal matrix with elements a_1, a_2, \dots, a_n in the main diagonal.
$B_{ij}^\varnothing(t)$	the probability that each component survives by time t when the environment condition is j given its initial value i at time 0, $i, j \in \mathbf{S}$
$\mathbf{B}^\varnothing(t)$	the matrix with elements $B_{ij}^\varnothing(t)$, i.e. $\mathbf{B}^\varnothing(t) = [B_{ij}^\varnothing(t)]$, $i, j \in \mathbf{S}$
$B_{ij}^{\{c_1, c_2, \dots, c_k\}}(t)$	the probability that components c_1, c_2, \dots, c_k fail at time t while the rest are survival at time t when the environment state is j given its initial value i at time 0, $c_i \in \Phi, i = 1, 2, \dots, k$.
$\mathbf{B}^{\{c_1, c_2, \dots, c_k\}}(t)$	the matrix with elements $B_{ij}^{\{c_1, c_2, \dots, c_k\}}(t)$, $i, j \in \mathbf{S}$
$B_{ij}^{(l)}(t)$	the probability that there are l components fail by time t with the environment state j where the initial environment state is i , $l = 0, 1, \dots, N - 1$ when all components have identical failure rates under each environment state
$\mathbf{B}^{(l)}(t)$	the matrix with elements $B_{ij}^{(l)}(t)$, $i, j \in \mathbf{S}$
$B_{ij}(t)$	the reliability of the M -out-of- N system at time t when the environment state is j given the initial environment state i , $i, j \in \mathbf{S}$
$\mathbf{B}(t)$	the matrix with elements $B_{ij}(t)$, $i, j \in \mathbf{S}$
\mathbf{e}	the $m \times 1$ matrix of 1s
\mathbf{e}_i	the $1 \times m$ matrix whose i th element is 1 and others are 0 respectively
$R(t)$	the reliability function of the system
$R_i(t)$	the conditional reliability function of the system given that the initial environment state is i , $i \in \mathbf{S}$
$F(t)$	the lifetime distribution of the system
$F_i(t)$	the conditional lifetime distribution of the system given that the initial environment state is i , $i \in \mathbf{S}$
$T_{t,i}^{\{c_1, c_2, \dots, c_k\}}$	the remaining useful lifetime of the system given that components i , $i \in \{c_1, c_2, \dots, c_k\}$ fail by time t the environment state i
$CR_i(u; t, \{c_1, c_2, \dots, c_k\})$	the conditional reliability of the system given that component l fails by time t , $l \in \{c_1, c_2, \dots, c_k\} \subset \Phi$ and the environment state at time t is i
$r_i(t; \{c_1, c_2, \dots, c_k\})$	the expected remaining useful lifetime (RUL) of the system at time t given the environment state $W(t)$ is i and components l fail by time t , $l \in \{c_1, c_2, \dots, c_k\}$
$L_{ij}^{\{d_1, d_2, \dots, d_r\}}(x, t; \{c_1, c_2, \dots, c_k\})$	the probability that component l , $l \in \{d_1, d_2, \dots, d_r\}$ fail by time t where the environment state is j given that component m , $m \in \{c_1, c_2, \dots, c_k\}$ fail at time x with environment state i
$\mathbf{L}^{\{d_1, d_2, \dots, d_r\}}(x, t; \{c_1, c_2, \dots, c_k\})$	the matrix with elements $L_{ij}^{\{d_1, d_2, \dots, d_r\}}(x, t; \{c_1, c_2, \dots, c_k\})$
$L_{ij}^{l_0, l}(\theta, t)$	the probability that the number of failed components is l at time t with environment state j given that at time θ the number of failed components is l_0 with environment state i for the system with identical components possessing constant hazard rates
$\mathbf{L}^{l_0, l}(\theta, t)$	the matrix with elements $L_{ij}^{l_0, l}(\theta, t)$, $i, j \in \mathbf{S}$

$X(t)$	the state of the system, $X(t) = 1$ means it is functional and 0 means it fails at time t
Y	the system repair time with distribution function $G(\cdot)$ and density function $g(y)$
τ	the system inspection period
A_s	the asymptotic average availability of the system when it is inspected under the equidistant inspection policy

5.1 Model description

The mathematical model is described in the following.

- Consider a M -out-of- N system operating if at least M out of the N components are functional. It is put into service at time 0 with perfect working state. Suppose that the components are labelled as component 1, component 2, \dots , component N . The system lifetime is T . The lifetime of component i is denoted by T_i , $i \in \Phi$, $\Phi = \{1, 2, \dots, N\}$. Components are independent under the fixed environment.
- The operations of components are driven by an external environment which is described by a continuous-time Markov chain $\mathcal{W} = \{W(t), t \geq 0\}$ with a finite state space $\mathcal{S} = \{1, 2, \dots, m\}$, infinitesimal generator \mathbf{Q} and transition probability $\pi_{ij}(t)$, $i, j \in \mathcal{S}$. In effect, the environment can be regarded as the working condition which affects the system state (its failure rate). For example, the environment is mild, normal, dangerous and so forth to the system.
- Component i , $i \in \Phi$ has failure rate $\lambda_i(t) = h_i(t, W(t))$ where $h_i(t, j)$ is the hazard rate of component i at age t when the environment state is j and $\int_0^\infty h_i(t, j) dt = \infty$, $i \in \Phi$, $j \in \mathcal{S}$. Without loss of generality, we further assume that $h_i(t, 1) < h_i(t, 2) < \dots < h_i(t, m)$ for any $t \geq 0$ and $i \in \Phi$, $j \in \mathcal{S}$.

5.2 Reliability function and lifetime distribution

We consider here that there is no maintenance actions. In order to obtain the system reliability as well as the system lifetime distribution, firstly denote by

$$B_{ij}^{\mathcal{Q}}(t) = \mathbb{P}(T_k > t, W(t) = j, \forall k \in \Phi, | W(0) = i) \quad (5.1)$$

the probability that no component failures occur by time t when the environment state is j given the initial environment state i , $i, j \in \mathcal{S}$. Then we have the following proposition.

Proposition 2 Denoted be the $m \times m$ matrices $\mathbf{B}^\varnothing(t) = [B_{ij}^\varnothing(t)]_{m \times m}$ and $\mathbf{H}(t) = \text{diag}[\sum_{l \in \Phi} h_l(t, j)]_{m \times m}$ which represents the diagonal matrix with elements $\mathbf{H}_{jj}(t) = \sum_{l \in \Phi} h_l(t, j)$ in the main diagonal, the following equation is valid.

$$\frac{d\mathbf{B}^\varnothing(t)}{dt} = \mathbf{B}^\varnothing(t)(\mathbf{Q} - \mathbf{H}(t)) \quad (5.2)$$

In particular, when the hazard rate is time-independent, i.e. $\mathbf{H}(t) = \mathbf{H}$, then it is easily seen that

$$\mathbf{B}^\varnothing(t) = \exp((\mathbf{Q} - \mathbf{H})t)$$

See Appendix 7.4.1 for the proof.

It is clear that until now, we can obtain the reliability of the N -component series system by $\mathbf{B}^\varnothing(t)$ and the probability vector of the initial environment state. However, for the general M -out-of- N system, it is also necessary to evaluate the state of each component (failed or working) at given time epoches. Let be $\{c_1, c_2, \dots, c_k\}$ a subset of Φ which is implemented to record the label of the k failed components. Correspondingly denote $\{c_1, c_2, \dots, c_k\}$ the absolute complement of $\{c_1, c_2, \dots, c_k\}$ in Φ which records the labels of the survival components. For instance, suppose that $\Phi = \{1, 2, 3, 4, 5\}$, then $\{c_1, c_2\} = \{2, 5\}$ ($\{c_1, c_2\} = \{1, 3, 4\}$) if the failed components are component 2 and 5; $\{c_1, c_2, c_3, c_4\} = \{1, 2, 4, 5\}$ ($\{c_1, c_2, c_3, c_4\} = \{3\}$) if the failed components are component 1, 2, 4 and 5, etc.

Further more, let be

$$B_{ij}^{\{c_1, c_2, \dots, c_k\}}(t) = \mathbb{P}\left(T_l > t, T_p < t, W(t) = j, \forall p \in \{c_1, c_2, \dots, c_k\}, \quad (5.3)\right. \\ \left. \forall l \in \{c_1, c_2, \dots, c_k\} \mid W(0) = i\right)$$

the probability that component labelled p , $p \in \{c_1, c_2, \dots, c_k\}$ fail while the rest work by time t with environment state j when the initial environment state is i , $\{c_1, c_2, \dots, c_k\} = \Phi \setminus \{c_1, c_2, \dots, c_k\}$. Denoted by $\mathbf{B}^{\{c_1, c_2, \dots, c_k\}}(t) = [B_{ij}^{\{c_1, c_2, \dots, c_k\}}(t)]_{m \times m}$. We have the following theorem.

Theorem 5.1 For any $k < N$, the probability matrix $\mathbf{B}^{\{c_1, c_2, \dots, c_k\}}(t)$ satisfies

$$\frac{d\mathbf{B}^{\{c_1, c_2, \dots, c_k\}}(t)}{dt} = \mathbf{B}^{\{c_1, c_2, \dots, c_k\}}(t)\left(\mathbf{Q} - \sum_{j \in \{c_1, c_2, \dots, c_k\}} \mathbf{H}_j(t)\right) \quad (5.4) \\ + \sum_{l=1}^k \mathbf{B}^{\{c_1, c_2, \dots, c_k\} \setminus \{c_l\}}(t) \mathbf{H}_{c_l}(t)$$

where $\mathbf{B}^{\{i_m\} \setminus \{i_m\}}(t) = \mathbf{B}^\varnothing(t)$, and $\mathbf{H}_i(t) = \text{diag}[h_i(t, j)]_{m \times m}$ is $m \times m$ matrix, $i \in \Phi$, $j \in \mathcal{S}$.

See Appendix 7.4.2 for the proof.

It is seen that the probability matrix $\mathbf{B}^{\{c_1, c_2, \dots, c_k\}}(t)$ can be derived by Theorem 5.1 for the general case that components are heterogeneous in the regard of their failure rates under different environment states. In particular, for the M -out-of- N system consisting only time-independent components or identical distributed components under each environment state, the following two corollaries are valid.

Corollary 1 *When the hazard rates of components are time-independent, i.e. $\mathbf{H}_i(t) = \mathbf{H}_i$, $i \in \Phi$, the matrix $\mathbf{B}^{\{c_1, c_2, \dots, c_k\}}(t)$ in Theorem 5.1 can be represented as*

$$\mathbf{B}^{\{c_1, c_2, \dots, c_k\}}(t) = \sum_{m=0}^k (-1)^m \sum_{\substack{l_u \in \{c_1, c_2, \dots, c_k\}, \\ u=1, 2, \dots, m \\ l_i \neq l_j \text{ if } i \neq j}} \exp\left(\left(\mathbf{Q} - \sum_{j \in \{c_1, c_2, \dots, c_k\}} \mathbf{H}_j - \sum_{i=1}^m \mathbf{H}_{l_i}\right)t\right) \quad (5.5)$$

where $\sum_{i=1}^0 = 0$.

See Appendix 7.4.3 for the proof.

When the N components have identical lifetime distributions under given environments, i.e. $\mathbf{H}_i(t) = \mathbf{H}_0(t)$ for any $i \in \Phi$. Let $B_{ij}^{(l)}(t)$ be the probability that there are l components fail by time t with the environment state j where the initial environment state is i , $l = 1, 2, \dots, N-1$. The matrix expression is $\mathbf{B}^{(l)}(t) = [B_{ij}^{(l)}(t)]_{m \times m}$. The following corollary can be derived.

Corollary 2 *For any $l = 1, 2, \dots, N-1$, $\mathbf{B}^{(l)}(t)$ satisfies*

$$\frac{d\mathbf{B}^{(l)}(t)}{dt} = \mathbf{B}^{(l)}(t)(\mathbf{Q} - (N-l)\mathbf{H}_0(t)) + \mathbf{B}^{(l-1)}(t)(N-l+1)\mathbf{H}_0(t) \quad (5.6)$$

where $\mathbf{B}^{(l)}(t) = \mathbf{B}^{\emptyset}(t)$ when $l = 0$.

In particular, when the hazard rates are time-independent, i.e. $\mathbf{H}_0(t) = \mathbf{H}_0$ we have

$$\mathbf{B}^{(l)}(t) = \binom{N}{l} \sum_{i=0}^l (-1)^{l-i} \binom{l}{i} \exp((\mathbf{Q} - (N-i)\mathbf{H}_0)t) \quad (5.7)$$

where $\mathbf{H}_0 = \text{diag}[h(j)]_{m \times m}$.

Corollary 2 can be easily verified by Theorem 5.1 and Corollary 1.

In the following, the reliability function $R(t)$ and the lifetime distribution function $F(t)$ of the M -out-of- N system are derived respectively. Let be

$$B_{ij}(t) = B_{ij}^{\emptyset}(t) + \sum_{k=1}^{N-M} \sum_{1 \leq c_1 < c_2 < \dots < c_k \leq N} B_{ij}^{\{c_1, c_2, \dots, c_k\}}(t) \quad (5.8)$$

and the matrix form

$$\mathbf{B}(t) = \mathbf{B}^\emptyset(t) + \sum_{k=1}^{N-M} \sum_{1 \leq c_1 < c_2 < \dots < c_k \leq N} \mathbf{B}^{\{c_1, c_2, \dots, c_k\}}(t) \quad (5.9)$$

where $\sum_{l=1}^0 = 0$. Equation (5.9) can also be rewritten as follows.

$$\mathbf{B}(t) = \mathbf{B}^\emptyset(t) + \sum_{l=1}^{N-M} \mathbf{B}^{(l)}(t)$$

when the N components have identical lifetime distributions under given environment conditions.

Assume that the initial probability row vector of the environment process is given by $\boldsymbol{\alpha} = [\alpha_i]$, where $\alpha_i = \mathbb{P}(W(0) = i)$, $i \in \mathcal{S}$. Let \mathbf{e} be a column vector of 1s, \mathbf{e}_i be a $1 \times m$ matrix whose i th element is 1 and others are 0 respectively. The reliability function $R(t)$, the lifetime distribution function $F(t)$ of the M -out-of- N system are given as

$$R(t) = \boldsymbol{\alpha} \mathbf{B}(t) \mathbf{e} \quad (5.10)$$

$$F(t) = 1 - \boldsymbol{\alpha} \mathbf{B}(t) \mathbf{e} \quad (5.11)$$

The conditional reliability $R_i(t)$ and the conditional distribution of the system $F_i(t)$ given the initial environment state i are

$$R_i(t) = \mathbf{e}_i \mathbf{B}(t) \mathbf{e} \quad (5.12)$$

$$F_i(t) = 1 - \mathbf{e}_i \mathbf{B}(t) \mathbf{e} \quad (5.13)$$

5.3 Remaining useful lifetime

Besides the reliability, the remaining useful lifetime (RUL) is also an important criterion considered extensively in the reliability analysis. Accurate estimation of RUL permits us to predict the system failure process by taking advantage of the system monitoring information. It is beneficial to the formulation of maintenance policies. There are many definitions of the RUL, and here we apply the definition of RUL as in [42] which means that the surviving function of the RUL is

$$\begin{aligned} & \mathbb{P}(T_{t,i}^{\{c_1, c_2, \dots, c_p\}} > h) \\ &= \mathbb{P}\left(T - t > h \mid T_u < t, T_v > t, \forall u \in \{c_1, c_2, \dots, c_p\}, \forall v \in \underline{\{c_1, c_2, \dots, c_p\}}, W(t) = i\right) \end{aligned} \quad (5.14)$$

where T is the system lifetime, $\underline{\{c_1, c_2, \dots, c_p\}} = \Phi \setminus \{c_1, c_2, \dots, c_p\}$.

To calculate the expected remaining useful lifetime of the system, it is necessary to record the current condition of each component (failed or work). To do this,

denote

$$L_{ij}^{\{d_1, d_2, \dots, d_r\}}(x, t; \{c_1, c_2, \dots, c_k\}) = \mathbb{P}\left(T_s > t, T_p < t, \forall s \in \{c_1, c_2, \dots, c_k\} \setminus \{d_1, d_2, \dots, d_r\}, \right. \\ \left. \forall p \in \{d_1, d_2, \dots, d_r\}, W(t) = j \mid T_u > x, T_v < x, \forall v \in \{c_1, c_2, \dots, c_k\}, \forall u \in \{c_1, c_2, \dots, c_k\}, \right. \\ \left. W(x) = i\right)$$

be the probability that component $s, s \in \{d_1, d_2, \dots, d_r\}$ fails in the time interval $(x, t]$ and the environment state is j at time t given that component $v, v \in \{c_1, c_2, \dots, c_k\}$ fails by time x when the environment state is i where $\{d_1, d_2, \dots, d_r\} \subseteq \{c_1, c_2, \dots, c_k\}$. The following lemma is given before the calculation of the system remaining useful lifetime.

Proposition 3 Let be $\mathbf{L}^{\{d_1, d_2, \dots, d_r\}}(x, t; \{c_1, c_2, \dots, c_k\})$ the $m \times m$ matrix form:

$$\mathbf{L}^{\{d_1, d_2, \dots, d_r\}}(x, t; \{c_1, c_2, \dots, c_k\}) = [L_{ij}^{\{d_1, d_2, \dots, d_r\}}(x, t; \{c_1, c_2, \dots, c_k\})]_{m \times m}$$

It can be derived that

$$\frac{\partial \mathbf{L}^{\{d_1, d_2, \dots, d_r\}}(x, t; \{c_1, c_2, \dots, c_p\})}{\partial t} = \mathbf{L}^{\{d_1, d_2, \dots, d_r\}}(x, t; \{c_1, c_2, \dots, c_p\})(\mathbf{Q} - \sum_{u \in \sigma(p, r)} \mathbf{H}_u(t)) \\ + \sum_{l=1}^r \mathbf{L}^{\{d_1, d_2, \dots, d_r\} \setminus \{d_l\}}(x, t; \{c_1, c_2, \dots, c_p\}) \mathbf{H}_{d_l}(t) \quad (5.15)$$

where $\sigma(p, r) = \Phi \setminus \{c_1, \dots, c_p, d_1, \dots, d_r\}$.

The proof is omitted as the same method is similar as in the proof of Theorem 5.1.

It is pointed out that both $\{c_1, c_2, \dots, c_p\}$ and $\{d_1, d_2, \dots, d_r\}$ can be \emptyset . In particular,

- $\{c_1, c_2, \dots, c_p\} = \emptyset$ means that there is no components failures by time x and $\mathbf{L}^\emptyset(x, x; \emptyset) = \mathbf{I}$;
- $\{d_1, d_2, \dots, d_r\} = \emptyset$ means there is no components failures in the interval $(x, t]$, in this case equation (5.15) can be represented as

$$\frac{\partial \mathbf{L}^\emptyset(x, t; \{c_1, c_2, \dots, c_p\})}{\partial t} = \mathbf{L}^\emptyset(x, t; \{c_1, c_2, \dots, c_p\})(\mathbf{Q} - \sum_{u \in \sigma(p)} \mathbf{H}_u(t))$$

where $\sigma(p) = \Phi \setminus \{c_1, c_2, \dots, c_p\}$

- $\{c_1, c_2, \dots, c_p\} = \{d_1, d_2, \dots, d_r\} = \emptyset$ means no components failures occur by time t and $\mathbf{L}^\emptyset(x, x; \emptyset) = \mathbf{I}$.

Given the environment state at time x and the initial values that where \mathbf{I} is the $m \times m$ identity matrix, $\mathbf{L}^{\{d_1, d_2, \dots, d_r\}}(t, x; \{c_1, c_2, \dots, c_p\})$ can be calculated by the

product integration method [164] which is showed in section 5.5. Let be

$$\mathbf{L}(x, t; \{c_1, c_2, \dots, c_p\}) = \sum_{r=0}^{N-M-k} \sum_{\substack{d_1 < d_2 < \dots < d_r \\ d_i \in \{c_1, c_2, \dots, c_p\} \\ i=1, 2, \dots, r}} \mathbf{L}^{\{d_1, d_2, \dots, d_r\}}(x, t; \{c_1, c_2, \dots, c_p\})$$

the matrix to record the probability that the system is survival at time t given that the components c_1, c_2, \dots, c_k fail at time x . where $\{c_1, c_2, \dots, c_p\} = \Phi$ when $\{c_1, c_2, \dots, c_p\} = \emptyset$ and $\{d_1, d_2, \dots, d_r\} = \emptyset$ when $r = 0$.

Denoted by $CR_i(u; t, W(t), \{c_1, c_2, \dots, c_p\})$ the conditional reliability of the system given that component l fails by time t , $l \in \{c_1, c_2, \dots, c_k\} \subset \Phi$ and the environment state at time t is i ; $r_i(t; \{c_1, c_2, \dots, c_k\})$ the expected remaining useful lifetime (RUL) of the system at time t given the environment state $W(t)$ is i and components l fail by time t , $l \in \{c_1, c_2, \dots, c_k\}$. The conditional reliability and the expected RUL of the system can be expressed as

$$CR_i(u; t, W(t), \{c_1, c_2, \dots, c_p\}) = \mathbf{e}_i \mathbf{L}(t, u; \{c_1, c_2, \dots, c_p\}) \mathbf{e} \quad (5.16)$$

$$r_i(t, W(t); \{c_1, c_2, \dots, c_p\}) = \int_t^\infty \mathbf{e}_i \mathbf{L}(t, u; \{c_1, c_2, \dots, c_p\}) \mathbf{e} du \quad (5.17)$$

where \mathbf{e} is a column vector of 1s, \mathbf{e}_i is a $1 \times m$ matrix whose i th element is 1 and others are 0 respectively.

For the special case when all the components of the M -out-of- N system have identical, constant failure rates under each environment state, let $r_i^{l_0}(\theta)$ be the expected remaining useful lifetime of the M -out-of- N system given that the system environment is i at time θ and the number of component failures is l_0 . The following corollary is validated.

Corollary 3 For the M -out-of- N system with identical components possessing constant hazard rates under given environment conditions, $r_i^{l_0}(\theta)$ satisfies

$$r_i^{l_0}(\theta) = \sum_{l=l_0}^{N-M} \int_\theta^\infty \mathbf{e}_i (\mathbf{L}^{l_0, l}(\theta, t)) \mathbf{e} dt$$

where

$$\mathbf{L}^{l_0, l}(\theta, t) = \binom{N-l_0}{l-l_0} \sum_{i=0}^{l-l_0} (-1)^{l-l_0-i} \binom{l-l_0}{i} \exp\left(\left(\mathbf{Q} - (N-l_0-i)\mathbf{H}_0\right)(t-\theta)\right)$$

$$0 < \theta < t, l_0 \leq l < N, \mathbf{L}^{l_0, l_0}(\theta, \theta) = \mathbf{I}, \mathbf{L}^{l_0, l}(\theta, \theta) = \mathbf{0}, l_0 < l.$$

See 7.4.4 for the proof.

5.4 Asymptotic average availability

In the following, we intend to derive the limiting average availability of the system undergoing periodic inspections. Let $\mathcal{X} := X(t)$ be the state of the system where $X(t) = 1$ if the system is in the up-state and $X(t) = 0$ if it is in the down-state. Other assumptions and notations are presented as follows.

- The system is new at time $t = 0$ and the duration between two consecutive inspections length is τ ;
- The system failure is not self-announcing and can be revealed only by system inspections;
- Upon inspection, it is perfectly repaired with a random time of length Y with distribution function $G(y)$ (density function $g(y)$) if the system failure is diagnosed; However, nothing is done if the system is in the up-state.

Thus the system limiting average availability can be defined as [165] as follows.

$$A_s = \lim_{t \rightarrow \infty} \frac{\int_0^t \mathbb{E}[X(s)] ds}{t}$$

Denoted by U_i and R_i the i th system failure epoch and the i th system renewal epoch respectively. According to [40], $\{(W_{R_n}, R_n), n = 1, 2, \dots\}$ is a Markov renewal process. We further define $W_n = W_{R_n}$ the environment state at the n th system replacement epoch. It is seen that $\{W_n, n = 1, 2, \dots\}$ is an irreducible, discrete-time Markov chain with one-step transition probability matrix \mathbf{P} and stationary distribution $\mathbf{p} = [p_i]$, $i \in \mathbf{S}$ which satisfy

$$\begin{aligned} p_j &= \sum_{i \in \mathbf{S}} p_i P_{ij}, j \in \mathbf{S} \\ \sum_{i \in \mathbf{S}} p_i &= 1 \end{aligned} \quad (5.18)$$

Therefore according to [166], the system limiting average availability A_s can be obtained:

$$A_s = \frac{\sum_{k \in \mathbf{S}} p_k \mathbb{E}_k(U_1)}{\sum_{k \in \mathbf{S}} p_k \mathbb{E}_k(R_1)} \quad (5.19)$$

The following theorems show the limiting average availability of the M -out-of- N system.

Theorem 5.2 *Under the equidistant inspection policy, the limiting average availability of the M -out-of- N system can be given as*

$$A_s = \frac{\sum_{k \in \mathbf{S}} p_k \int_0^\infty (1 - F_k(t)) dt}{\sum_{k \in \mathbf{S}} p_k (\sum_{i=0}^\infty \tau (1 - F_k(i\tau)) + \int_0^\infty (1 - G(y)) dy)}$$

where $F_k(t)$ is the system lifetime distribution when the initial environment state is k given in equation (5.13), $G(y)$ is the distribution function of the length of the repair time, p_k is the stationary probability derived from the transition probability with element

$$P_{ij} = \sum_{m \in \mathcal{S}} \sum_{l \in \mathcal{S}} \sum_{k=1}^{\infty} \int_0^{\infty} \pi_{lj}(y) dG(y) \sum_{p=0}^{N-M} \left(\sum_{1 \leq c_1 < c_2 < \dots < c_p \leq N} B_{im}^{\{c_1, c_2, \dots, c_p\}}((k-1)\tau) \right. \\ \left. \times \left(\pi_{ml}(\tau) - \sum_{r=0}^{N-M-p} \sum_{\substack{d_1 < d_2 < \dots < d_r \\ d_i \in \{c_1, c_2, \dots, c_p\} \\ i=1, 2, \dots, r}} L_{ml}^{\{d_1, d_2, \dots, d_r\}}((k-1)\tau, k\tau; \{c_1, c_2, \dots, c_p\}) \right) \right)$$

where $B_{im}^{\{c_1, c_2, \dots, c_p\}}(t)$, $L_{ml}^{\{d_1, d_2, \dots, d_r\}}((k-1)\tau, k\tau; \{c_1, c_2, \dots, c_p\})$, $i, m, l \in \mathcal{S}$ are defined in equations (5.3) and (5.15) respectively, $B_{im}^{\{c_1, c_2, \dots, c_p\}}(t) = B_{im}^{\emptyset}(t)$ when $p = 0$, $\pi_{ij}(t)$ is the transition probability of the continuous time Markov environment, $G(y)$ is the distribution function of the length of the repair time.

See Appendix 7.4.5 for the proof.

5.5 Numerical examples

5.5.1 Reliability illustration by two methods

To show the advantage of our method, the system reliability is illustrated in the following example. Consider a system consisting of 5 identical components where the operational environment is described by the continuous time Markov process with infinitesimal generator

$$Q = \begin{pmatrix} -4 & 2 & 1 & 1 \\ 1 & -3 & 1 & 1 \\ 1 & 1 & -2.5 & 0.5 \\ 2 & 1 & 1 & -4 \end{pmatrix}$$

The failure rate Matrix of component i is given as

$$H_i(t) = \text{diag}([0.002, 0.01, 0.005, 0.007])$$

for $i \in \{1, 2, 3, 4, 5\}$. Let be the initial environment probability vector $\alpha = [0, 1, 0, 0]$. Table 5.1 presents the run-times of MATLAB2010b in calculating the system reliability of M -out-of-5 system by time $t = 40$ through our proposed method and the Monte-Carlo simulation (the number of simulation times is 10000) respectively under different values of M . It is seen that our calculation is more efficient and practical especially in dealing with large complex systems consisting of numerous components.

	M=4		M=3	
	$R(t)$	run-time	$R(t)$	run-time
our method	0.6963	0.000479s	0.9255	0.00089s
simulation method	0.6909	54.923931s	0.9230	57.749855s

Table 5.1: The system reliability and the calculation run-time of M -out-of-5 system by time $t = 40$ through our method and the Monte Carlo simulations

5.5.2 System reliability

It is seen that we always confront the following equation in the calculation of the system reliability and the remaining useful lifetime.

$$\begin{aligned}\frac{d\rho(t)}{dt} &= \rho(t)\mathbf{M}(t) + \mathbf{N}(t) \\ \rho(a) &= \rho_0, a < t\end{aligned}\quad (5.20)$$

where $\rho(t)$, $\mathbf{M}(t)$ and $\mathbf{N}(t)$ are $n \times n$ matrices. The equation solution is easily obtained when $\mathbf{N}(t) = \mathbf{0}$ and $\mathbf{M}(t)$ is independent of t ($\mathbf{M}(t) = \mathbf{M}$). In this circumstance, we have

$$\rho(t) = \rho_0 \exp(\mathbf{M} \times (t - a))$$

Otherwise, the closed-form solution is nearly impossible to get. Here the product-integration method proposed by Vito Volterra is implemented for the approximately numerical solution. Details about the method is presented by Gill et al [167]. According to which, for the homogeneous case when $\mathbf{N}(t) = 0$, the solution of equation (5.20) is

$$\rho(t) = \rho_0 \prod_{i=\frac{a}{\delta}}^{\frac{t}{\delta}} \left(I + \mathbf{M}(i\delta)\delta \right)$$

where δ is the step size in the calculation. More details can be found in [164].

For the non-homogeneous case, the solution is

$$\rho(t) = \rho_0 \mathbf{Z}(t) + \int_a^t \mathbf{N}(x) \prod_{i=\frac{x}{\delta}}^{\frac{t}{\delta}} (I + \mathbf{M}(i\delta)\delta) dx$$

where $\mathbf{Z}(t) = \prod_{i=\frac{a}{\delta}}^{\frac{t}{\delta}} (I + \mathbf{M}(i\delta)\delta)$.

Consider a three-component system operates under an external environment described by a continuous time Markov process with infinitesimal generator

$$\mathbf{Q} = 10^{-3} \times \begin{pmatrix} -3 & 1 & 1 & 1 \\ 2 & -4 & 1 & 1 \\ 1 & \frac{1}{2} & -2 & \frac{1}{2} \\ 2 & 1 & 2 & -5 \end{pmatrix}$$

The component failure rates under different environment states are given as

$$\begin{aligned}\mathbf{H}_1 &= \text{diag}([0.002, 0.005, 0.009, 0.012]); \\ \mathbf{H}_2 &= \text{diag}([0.003, 0.007, 0.012, 0.017]); \\ \mathbf{H}_3 &= \text{diag}([0.006, 0.009, 0.014, 0.020]);\end{aligned}$$

The initial probability vector of the environment process is $\alpha = [0.45, 0.25, 0.15, 0.15]$. Table 5.2 shows the reliability of the k -out-of-3 system with different values of k . Table 5.3 presents the conditional reliability of the 2-out-of-3 system under different initial environment state respectively.

t	$t = 2$	$t = 4$	$t = 6$	$t = 8$	$t = 10$	$t = 12$	$t = 14$	$t = 16$	$t = 20$
$k = 3$	0.9558	0.9141	0.8748	0.8377	0.8027	0.7696	0.7382	0.7086	0.6537
$k = 2$	0.9991	0.9966	0.9927	0.9875	0.9813	0.9741	0.9660	0.9573	0.9380
$k = 1$	1.0000	0.9999	0.9998	0.9996	0.9992	0.9987	0.9981	0.9973	0.9951

Table 5.2: The reliability of the k -out-of-3 system with different values of k

t	$t = 2$	$t = 4$	$t = 6$	$t = 8$	$t = 10$	$t = 12$	$t = 14$	$t = 16$	$t = 20$
$R_1(t)$	0.9999	0.9994	0.9987	0.9976	0.9962	0.9945	0.9925	0.9902	0.9846
$R_2(t)$	0.9994	0.9978	0.9951	0.9915	0.9871	0.9817	0.9756	0.9688	0.9533
$R_3(t)$	0.9985	0.9941	0.9871	0.9780	0.9670	0.9542	0.9400	0.9246	0.8907
$R_4(t)$	0.9970	0.9888	0.9763	0.9601	0.9411	0.9197	0.8966	0.8720	0.8203

Table 5.3: The reliability of the 2-out-of-3 system with different initial environment state $k \in \{1, 2, 3, 4\}$ respectively

5.5.3 System conditional reliability and expected remaining useful lifetime

Let

$$\mathbf{Q} = 10^{-4} \times \begin{pmatrix} -2 & 2 & 0 \\ 3 & -7 & 4 \\ 0 & 4 & -4 \end{pmatrix}$$

Assume that the system consists three components with failure rates

$$h_i(t, j) = \frac{\beta_i}{\lambda_i} \left(\frac{t}{\lambda_i}\right)^{\beta_i-1} env(i, j), i = 1, 2, 3; j = 1, 2, 3.$$

where $env(i, j)$ is the environment related factor with respect to component i under environment j and

$$\begin{aligned}[\beta_1, \beta_2, \beta_3] &= [1.5, 2, 2.5] \\ [\lambda_1, \lambda_2, \lambda_3] &= [10^4, 0.5 \times 10^4, 0.3 \times 10^4] \\ [env(i, 1), env(i, 2), env(i, 3)] &= [1, 5, 10], i = 1, 2, 3.\end{aligned}$$

Table 5.4 gives $CR_i(t; x, W(x), \emptyset)$ the conditional reliability of the three-component in series system given the initial survival time $x = 1000$ with different environment state at t respectively.

Tables 5.5-5.8 show the conditional reliabilities $CR_i(t; x, W(x), \{c_2\})$, $CR_i(t; x, W(x), \{c_1\})$ and $CR_i(t; x, W(x), \emptyset)$ ($x = 1000$) of the parallel system with different initial environment conditions respectively. It can be seen that under each situation, the initial environment state has significant impact on the system reliability. It seems that environment state 1 is the most friendly environment to the system. Environment 3 is very dangerous under which the reliability of the system is the smallest compared to which under environment 1 and 2.

$W(x)/t$	$t = 1100$	$t = 1200$	$t = 1300$	$t = 1400$	$t = 1500$	$t = 1600$	$t = 1700$
$W(x) = 1$	0.9778	0.9527	0.9248	0.8945	0.8621	0.8279	0.7924
$W(x) = 2$	0.8968	0.7961	0.7002	0.6108	0.5292	0.4560	0.3913
$W(x) = 3$	0.8069	0.6425	0.5056	0.3938	0.3039	0.2330	0.1777

Table 5.4: The conditional reliability of the series system given the initial survival time $x = 1000$ and different initial environment state $W(x)$

$W(x)/t$	$t = 1100$	$t = 1200$	$t = 1300$	$t = 1400$	$t = 1500$	$t = 1600$	$t = 1700$
$W(x) = 1$	0.9778	0.9527	0.9248	0.8945	0.8621	0.8279	0.7924
$W(x) = 2$	0.8968	0.7961	0.7002	0.6108	0.5292	0.4560	0.3913
$W(x) = 3$	0.8069	0.6425	0.5056	0.3938	0.3039	0.2330	0.1777

Table 5.5: The conditional reliability of the 2-out-of-3 system given that component 1 fails by time $x = 1000$ and different initial environment state

$W(x)/t$	$t = 1100$	$t = 1200$	$t = 1300$	$t = 1400$	$t = 1500$	$t = 1600$	$t = 1700$
$W(x) = 1$	0.9783	0.9533	0.9254	0.8951	0.8627	0.8285	0.7931
$W(x) = 2$	0.8990	0.7982	0.7021	0.6126	0.5307	0.4573	0.3924
$W(x) = 3$	0.8108	0.6458	0.5082	0.3958	0.3055	0.2342	0.1786

Table 5.6: The conditional reliability of the 2-out-of-3 system system given that component 2 fails by time $x = 1000$ and different initial environment state

$W(x)/t$	$t = 1100$	$t = 1200$	$t = 1300$	$t = 1400$	$t = 1500$	$t = 1600$	$t = 1700$
$W(x) = 1$	0.9792	0.9543	0.9266	0.8964	0.8640	0.8300	0.7945
$W(x) = 2$	0.9030	0.8020	0.7058	0.6160	0.5339	0.4601	0.3949
$W(x) = 3$	0.8178	0.6517	0.5132	0.3998	0.3087	0.2367	0.1806

Table 5.7: The conditional reliability of the 2-out-of-3 system system given that component 3 fails by time $x = 1000$ and different initial environment state

$W(x)/t$	$t = 1100$	$t = 1200$	$t = 1300$	$t = 1400$	$t = 1500$	$t = 1600$	$t = 1700$
$W(x) = 1$	0.9996	0.9990	0.9982	0.9967	0.9941	0.9903	0.9848
$W(x) = 2$	0.9936	0.9788	0.9545	0.9209	0.8787	0.8293	0.7746
$W(x) = 3$	0.9791	0.9306	0.8614	0.7784	0.6887	0.5978	0.5104

Table 5.8: The conditional reliability of the 2-out-of-3 system given that all components survive by time $x = 1000$ and different initial environment state

Similarly, Figures 5.1-5.5 illustrate the expected RUL of the series system and the 2-out-of-3 system with different components states (fail or work) and environment conditions ($W(x) = 1, 2, 3$) respectively. It is seen that under each situation, the expected RUL of the system decreases with time. When the initial environment is mild $W(x) = 1$, the expected RUL is large. When the environment is dangerous $W(x) = 3$, the expected RUL is less.

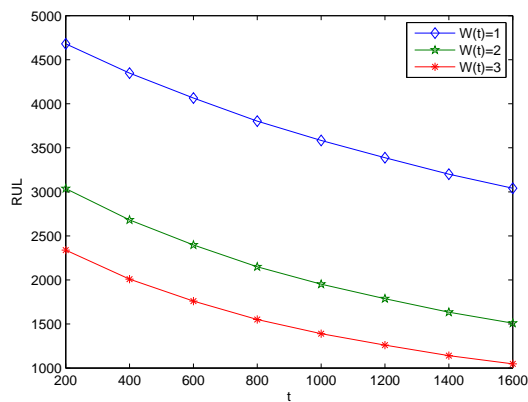


Figure 5.1: The expected remaining useful lifetimes $r_i(t, W(t); \emptyset)$ of the series system with different initial survival time t and environment state $W(t) = 1, 2, 3$.

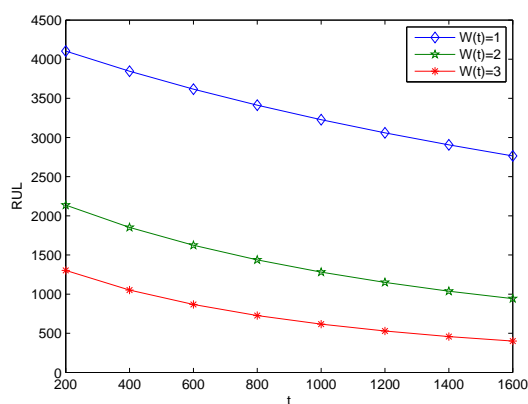


Figure 5.2: The expected remaining useful lifetimes $r_i(t, W(t); \{c_2\})$ of the 2-out-of-3 system with different initial survival time t and environment state $W(t) = 1, 2, 3$.

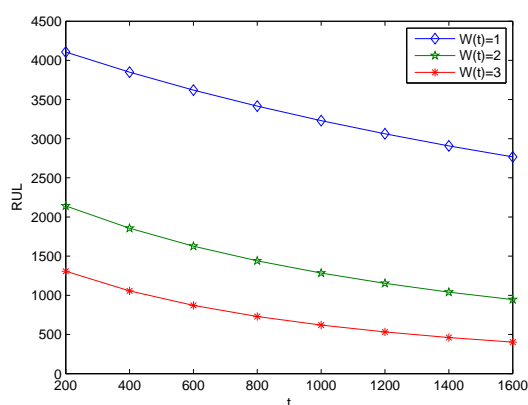


Figure 5.3: The expected remaining useful lifetimes $r_i(t, W(t); \{c_1\})$ of the 2-out-of-3 system with different initial survival time t and environment state $W(t) = 1, 2, 3$.

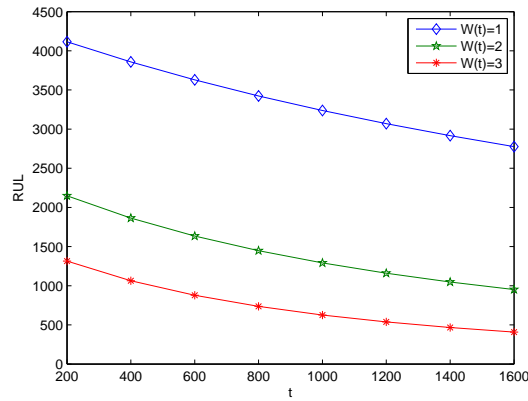


Figure 5.4: The expected remaining useful lifetimes $r_i(t, W(t); \{c_1\})$ of the 2-out-of-3 system with different initial survival time t and environment state $W(t) = 1, 2, 3$.

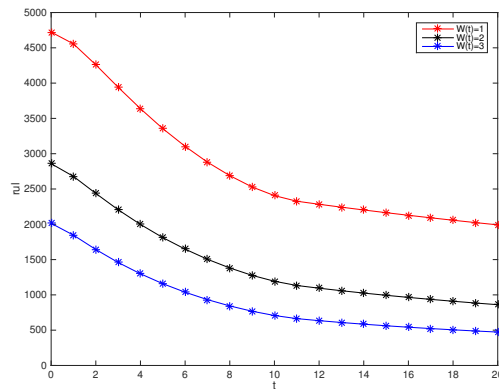


Figure 5.5: The expected remaining useful lifetimes $r_i(t, W(t); \emptyset)$ of the parallel system with different initial survival time t and environment state $W(t) = 1, 2, 3$.

5.5.4 Asymptotic average availability

Firstly, denote here

$$Q = \begin{pmatrix} -0.1 & 0.1 \\ 0.2 & -0.2 \end{pmatrix}$$

Let

$$h_i(j, t) = 0.05 \times j \times i^2, i = 1, 2, 3; j = 1, 2.$$

Assume that the repair time is exponentially distributed with mean $\lambda = \frac{1}{2}$. Table 5.9 illustrates the asymptotic average availabilities of the k -out-of-3 system under the periodic inspection policy with respect to different values of τ . It is seen that the smaller the inspection period is, the larger is the asymptotic average availability. Also, value of the initial environment has significant impact on the system's expected RUL.

τ	0.2	0.4	0.8	1	1.2	1.5	1.8	2	2.5
$A_s(k=1)$	0.9636	0.9579	0.9465	0.9410	0.9355	0.9274	0.9194	0.9142	0.9015
$A_s(k=2)$	0.8560	0.8363	0.7994	0.7822	0.7656	0.7418	0.7192	0.7047	0.6701
$A_s(k=3)$	0.6448	0.6062	0.5366	0.5054	0.4765	0.4371	0.4021	0.3810	0.3348

Table 5.9: The asymptotic average availabilities of the k -out-of-3 system under the equidistant inspection policy with different values of the inspection period τ

5.6 Conclusion

In this chapter, we have presented a framework of the M -out-of- N system operating in dynamic environment, where the objective is to evaluate the system reliability-based measures such as the system reliability, the remaining useful lifetime, the limiting average availability and so on. It is shown that the environment state has significant influence to the system reliability (Tables 5.4-5.8) and the expected remaining useful lifetime (Figures 5.1-5.5). The results obtained here are very useful which permit the heterogeneity of components possessing different failure rates as well as the flexibility of the system configuration such as the series system or the parallel system. Even still, it may be interesting to take into consideration of the observation of the environment states at inspection epochs and investigate preventive maintenance policies in the model in our future work.

Chapter 6

Conclusion

In this thesis, we have studied the maintenance policies, warranty policies and reliability measures of multi-component systems with various types of stochastic dependencies.

In Chapter 2, a two-component load-sharing system inspired by the steel wire rope in mining system is proposed. It is assumed that the failure rates of the two components are time dependent and load dependent. Whenever one fails, it is imperfectly repaired with a time delay during which the failure rate of the survival component increases because of the resulting overload. Three maintenance policies are proposed considering imperfect preventive maintenance and system replacement. The optimal average costs in the long run under different maintenance policies are derived from the theoretical propositions.

In Chapter 3, we study a two-component system with failure dependence where the deterioration sub-system is involved in. Component 1 failure causes random damages to component 2. While the failure of component 2 is catastrophic and it induces the failure of component 1. Component 2 fails when its damages reach or firstly exceed a predetermined level. The virtual age method is applied in the correctively repair of component 1 and the system is renewed when component 2 fails. The explicit expressions of the long run average cost under different preventive maintenance policies are developed. Sufficient conditions of the existence of the long-run expected costs are derived theoretically. Further more, from both the points of view of the manufacturer and the owner of the system, the warranty profits to the manufacturer and the costs to the owner under different warranty policies (the renewing free repair policy and the non-renewing free repair policy) are derived respectively. It is shown that the failure dependence between components has effect to both the manufacture profit and the consumer cost.

In Chapter 4, a similar model as presented in Chapter 3 is developed. Here the failure interaction between components are: the component 1 failure induces the component 2 failure with probability $\bar{r} = 1 - r$, $0 < r < 1$ while the component 2 failure causes the failure of component 1. Also, three different preventive maintenance policies are proposed. The system long-run average maintenance costs, the optimization of the maintenance cost are derived as well as the manufacture profits and the owner costs in the context of warranty.

In Chapter 5, the reliability measures of a $M - out - of - N$ system under dynamic environment conditions is considered. It is assumed that the failure rates of the components are time-dependent and environment-state-dependent. The dependence between components is generated by the common operating environment conditions that they share which is described by a continuous-time Markov chain. The system reliability-based measures like the system reliability, the mean time to failure (MTTF), the remaining useful lifetime (RUL), the limiting average availability under different system inspection policies are calculated.

The above models on multi-component systems permit us to glimpse the effect of stochastic dependence between components on the system reliability and the maintenance costs. However, there are still a lot of significant issues and problems to be developed in the future works.

First, we have developed models with two-component systems in Chapter 2-4. It is more challenging and interesting to consider multicomponent systems which may have various types of dependencies among them. For instance, the economic dependence, the structure dependence.

Secondly, in our models, the stochastic dependencies among components are constructed by assuming, either that whenever one component fails, its impact on the lifetime of the long-lived component is known (in Chapter 2-4), or that the impact of the environment on each component is known (in Chapter 5). However, it is sensible to study a multi-component system by utilizing multivariate distributions or Copulas functions in the reliability analysis as well as the optimization of maintenance policies.

Thirdly, we can further develop condition based maintenance policies in the models in Chapter 3-4 in which the evolution of one component is modeled by its degradation. The effect of the condition based maintenance to the system reliability and the maintenance cost are worth studying. Besides, in Chapter 5, it may be interesting to propose some preventive maintenance policies to the system when its expected RUL falls below a threshold .

Finally, we would like to apply real deterioration/failure time data and implement statistical inference to estimate different model parameters.

Chapter 7

Appendix

7.1 Proofs of theorems and propositions in Chapter 2

7.1.1 The proof of Theorem 2.1

proof 1 *As mentioned in [111], the failure rate of the short-lived component satisfies*

$$r_1(t) \leq (1 - \lambda)at^{a-1}l$$

which yields

$$\lim_{k \rightarrow \infty} p_k \leq \lim_{k \rightarrow \infty} \exp(-(1 - \lambda)((k - 1)\tau_0)^a l) = 0$$

Since $0 \leq q_k \leq p_k$, so p_k and q_k go to 0 when k goes to infinity.

Let be $C(N)$ and $C(N + 1)$ the long-run cost rate when the preventive replacement is carried out at $N\tau_0$ and $(N + 1)\tau_0$ respectively. Denoted by p_i, p'_j, q_i, q'_j the probabilities that the system is renewed at $i\tau_0$ and $j\tau_0$, that both components fail in $[(i - 1)\tau_0, i\tau_0], [(j - 1)\tau_0, j\tau_0]$ respectively, $i = 1, 2, \dots, N$ and $j = 1, 2, \dots, N + 1$. Let $A_k = \lfloor \frac{k}{k_2} \rfloor c_2 + c_r$, $x = c_r \sum_{i=1}^N q_i + \sum_{i=1}^N p_i A_i$, $y = \sum_{i=1}^N p_i i \tau_0$. Then

$$C(N + 1) - C(N) = \frac{x + c_p q'_{N+1} + p'_{N+1} (A_{N+1} - A_N)}{y + p'_{N+1} \tau_0} - \frac{x}{y}$$

As p'_{N+1} and q'_{N+1} go to 0 when N goes to infinity and $(A_{N+1} - A_N) \leq c_2$ is bounded, therefore

$$\lim_{k_3 \rightarrow \infty} (C(N + 1) - C(N)) = 0$$

◇

7.1.2 The proof of Theorem 2.2

proof 2 Similarly, denoted $C^{(2)}(N)$ and $C^{(2)}(N+1)$ the long-run cost rate when the preventive replacement is carried out at $N\tau_0$ and $(N+1)\tau_0$ respectively under policy 2. Let $x^{(2)}$ and $y^{(2)}$ be the numerator and denominator of $C^{(2)}(k_3)$ respectively in equation (2.9). When the system preventive replacements are carried out at $(N+1)\tau_0$, let p'_{N+1} be the probability that the system is replaced at $(N+1)\tau_0$ under policy 1, $p^{(2)'}_{N+1}$ be the probability that the system is replaced at $(N+1)\tau_0$ under policy 2, $p'_{i,N+1}$ be the probability that the two components fail in $((i-1)\tau_0, i\tau_0]$ and $((N\tau_0, (N+1)\tau_0)$ under policy 2 respectively, $B_N = c_2(\hat{n}_N - \hat{n}_{N-1})$. Then we have

$$C^{(2)}(N+1) - C^{(2)}(N) = \frac{x^{(2)} + B_N \sum_{i=1}^N P_{iN} + 2c_2 p'_{N+1} \hat{n}_{N-1} + c_p \sum_{i=1}^{N+1} p'_{i,N+1}}{y^{(2)} + p^{(2)'}_{N+1} \tau_0} - \frac{x^{(2)}}{y^{(2)}}$$

where $\hat{n}_N = \lfloor \frac{N}{k_2} \rfloor$. It can be seen that

$$\begin{aligned} \lim_{N \rightarrow \infty} p^{(2)'}_{N+1} &\leq \lim_{N \rightarrow \infty} \bar{F}(N\tau_0) \leq \lim_{N \rightarrow \infty} \exp(-(1-\lambda)l(N\tau_0)^a) = 0 \\ \lim_{N \rightarrow \infty} p'_{N+1} \hat{n}_{N-1} &\leq \lim_{N \rightarrow \infty} \frac{N}{\bar{F}^2(N\tau_0)} \leq \lim_{N \rightarrow \infty} \frac{N}{\exp(2(1-\lambda)l(N\tau_0)^a)} = 0 \\ \lim_{N \rightarrow \infty} \sum_{i=1}^N P_{iN} &\leq \lim_{N \rightarrow \infty} \bar{F}((N-1)\tau_0) = 0 \end{aligned}$$

Therefore $C^{(2)}(N)$ satisfies the following equation.

$$\lim_{k_3 \rightarrow \infty} (C^{(2)}(N+1) - C^{(2)}(N)) = 0.$$

◇

7.1.3 The proof of Theorem 2.3

proof 3 Under policy 3, denoted $C^{(3)}(N)$ and $C^{(3)}(N+1)$ the long-run cost rate when the preventive replacements are carried out at $N\tau_0$ and $(N+1)\tau_0$ respectively. Let $x^{(3)}$ and $y^{(3)}$ be the numerator and denominator of $C^{(3)}(k_3)$ respectively in equation (2.13). Then

$$C^{(3)}(N+1) - C^{(3)}(N) = \frac{x^{(3)} + \sum_{i=1}^{N-1} P_{iN} \tilde{B}(N, i) + 2c_2 \bar{F}^2(N\tau_0) B_N + c_p \sum_{i=1}^{N+1} p'_{i,N+1}{}^{(3)}}{y^{(3)} + p'_{N+1}{}^{(3)} \tau_0} - \frac{x^{(3)}}{y^{(3)}}$$

where $\tilde{B}(N, i) = c_2(\tilde{n}_{N-i} - \tilde{n}_{N-1-i})$, $B_N = c_2(\hat{n}_N - \hat{n}_{N-1})$ $p'_{N+1}{}^{(3)}$ is the probability that the system is replaced at $(N+1)\tau_0$ under policy 3, $p'_{i,N+1}{}^{(3)}$ is the probability that the two components fail in $((i-1)\tau_0, i\tau_0]$ and $((N\tau_0, (N+1)\tau_0)$ under policy 3

respectively. By the similar method as in policy 2, it can be proved that

$$\begin{aligned}\lim_{N \rightarrow \infty} \sum_{i=1}^N p_{iN}'^{(3)} &= 0 \\ \lim_{N \rightarrow \infty} p_N'^{(3)} &= 0 \\ \lim_{N \rightarrow \infty} \bar{F}^2(N\tau_0)B_N &= 0\end{aligned}$$

Therefore $C^{(3)}(N)$ satisfies the following equation.

$$\lim_{k_3 \rightarrow \infty} (C^3(N+1) - C^3(N)) = 0$$

7.2 Proofs of theorems and propositions in Chapter 3

7.2.1 The proof of Proposition 1

proof 4 To prove Proposition 1, let T_i , $i = 1, 2, \dots$, be the i th system renewal time. It is obvious that they are identically and independently distributed. Then

$$\begin{aligned}\mathbb{P}\{T_1 \leq t\} &= p_0(t)\mathbb{P}(Y(t) > L) + \sum_{k=1}^{\infty} \mathbb{P}(Y(t) + \sum_{i=1}^{N(t)} Z_i > L \mid N(t) = k)\mathbb{P}(N(t) = k) \\ &= p_0(t)\mathbb{P}(Y(t) > L) + \sum_{k=1}^{\infty} p_k(t) \int_0^{\infty} \mathbb{P}(Y(t) > (L - z))dH^{*(k)}(z) \\ &= p_0(t)G_{\sigma_L}(t) + \sum_{k=1}^{\infty} p_k(t) \int_0^{\infty} G_{\sigma_{L-z}}(t)dH^{*(k)}(z)\end{aligned}$$

where $G_{\sigma_x}(t) = 1$ when $x < 0$. $H^{*(n)}(t)$ is the n -fold convolution of $H(t)$ with itself. Therefore

$$F_s(t) = p_0(t)G_{\sigma_L}(t) + \sum_{k=1}^{\infty} p_k(t) \int_0^{\infty} G_{\sigma_{L-z}}(t)dH^{*(k)}(z)$$

◇

7.2.2 The proof of Theorem 3.1

proof 5 Let Δ_i^T, U_i^T be respectively the length of the i th replacement cycle and the cost incurred during this period, $i = 1, 2, \dots$. Then $\{\Delta_i^T, U_i^T\}$ constitutes a renewal reward process which yields

$$C_{\infty}(T) = \frac{\mathbb{E}(U_1^T)}{\mathbb{E}(\Delta_1^T)}.$$

As

$$\begin{aligned} E(U_1^T) &= \mathbb{P}(T_1 > T)(c_2 + c_1 \mathbb{E}N(T)) + \int_0^T (c_2 + c_1 \mathbb{E}N(t)) d\mathbb{P}(T_1 \leq t) \\ E(\Delta_1^T) &= T\mathbb{P}(T_1 > T) + \int_0^T t d\mathbb{P}(T_1 \leq t) \end{aligned}$$

where $\mathbb{P}(T_1 \leq T) = F_s(T)$. Therefore equation (3.3) is straightforward. \diamond

7.2.3 The proof of Theorem 3.2

proof 6 Let S_n^1 be the n th failure time of component 1 which is $S_n^1 = \sum_{i=0}^n X_i$. One shall denote $\varphi(t)$ the degradation level of unit 2 at time t , Δ_1^N the time elapsed in one replacement cycle, U_1^N the total cost in one replacement cycle and $\mathbb{E}(\Delta_1^N)$, $\mathbb{E}(U_1^N)$ be the expectations of Δ_1^N and U_1^N respectively. The following equality is verified

$$\begin{aligned} \mathbb{P}(U_1^N > t) &= \mathbb{P}(S_N^1 > t, \varphi(t) < L) \\ &= \mathbb{P}(N(t) < N) \mathbb{P}(\varphi(t) < L | N(t) < N) \\ &= \sum_{k=0}^{N-1} p_k(t) \mathbb{P}(Y(t) + \sum_{i=0}^k Z_i < L) \\ &= \sum_{k=0}^{N-1} p_k(t) \int_0^L (1 - G_{\sigma_{L-z}}(t)) dH^{*(k)}(z) \end{aligned}$$

Thus:

$$\mathbb{E}(U_1^N) = \int_0^\infty \sum_{k=0}^{N-1} p_k(t) \int_0^L (1 - G_{\sigma_{L-z}}(t)) dH^{*(k)}(z) dt \quad (7.1)$$

Note that the expected number of component 1 imperfect repairs is

$$\sum_{k=1}^{N-1} \int_0^\infty \bar{F}_s(t) dV_k(at)$$

Therefore,

$$\mathbb{E}(\Delta_1^N) = c_2 + c_1 \sum_{k=1}^{N-1} \int_0^\infty \bar{F}_s(t) dV_k(at) \quad (7.2)$$

By the renewal theory $C_\infty(N) = \frac{\mathbb{E}(U_1^N)}{\mathbb{E}(\Delta_1^N)}$, and considering equations (7.1) and (7.2) the Theorem is proved. \diamond

7.2.4 The proof of Theorem 3.3

proof 7 Denote Δ_1^{NT} , U_1^{NT} be the length and the cost of one replacement cycle respectively, then

$$\Delta_1^{NT} = \begin{cases} T, & \Delta_1^N > T \\ \Delta_1^N, & \Delta_1^N \leq T \end{cases}$$

where Δ_1^N is the renewal cycle under failure-number- N -based policy defined by equation (7.1) in the proof of Theorem 3.2. Therefore,

$$\begin{aligned} \mathbb{E}(\Delta_1^{NT}) &= \int_0^T t d\mathbb{P}(\Delta_1^N \leq t) + T\mathbb{P}(\Delta_1^N > T) = \int_0^T \mathbb{P}(\Delta_1^N > t) dt \\ &= \int_0^T \sum_{k=0}^{N-1} p_k(t) \int_0^L (1 - G_{\sigma_{L-z}}(t)) dH^{*(k)}(z) dt \end{aligned}$$

As

$$\mathbb{E}(U_1^{NT}) = c_2 + c_1 \sum_{n=1}^{N-1} \int_0^T \bar{F}_s(t) dV_n(at)$$

The expected long run maintenance cost is derived from the expressions of $\mathbb{E}(\Delta_1^{NT})$ and $\mathbb{E}(U_1^{NT})$ and the renewal reward theorem. \diamond

7.2.5 The proof of Theorem 3.4

proof 8 By differentiating $C_\infty(T)$ with respect to T , it can be seen that

$$\frac{dC_\infty(T)}{dT} \geq (\leq) 0 \iff \frac{dEN(T)}{dT} \int_0^T \bar{F}_s(t) dt - \int_0^T \bar{F}(t) dEN(t) \geq (\leq) \frac{c_2}{c_1}.$$

Let be

$$G_{au1}(T) = \frac{dEN(T)}{dT} \int_0^T \bar{F}_s(t) dt - \int_0^T \bar{F}(t) dEN(t).$$

Since

$$\frac{dG_{au1}(T)}{dT} = \frac{d^2EN(T)}{dT^2} \int_0^T \bar{F}_s(t) dt,$$

$G_{au1}(T)$ is an increasing function of T if $\frac{d^2EN(T)}{dT^2} > 0$ for all T . Besides, if $\lim_{T \rightarrow \infty} G_{au1}(T) > \frac{c_2}{c_1}$, note that $G_{au1}(0) = 0$, then there exists T_0 such that

$$G_{au1}(t) < 0 \quad \text{if } t \leq T_0 \quad \text{and} \quad G_{au1}(t) > 0 \quad \text{if } t > T_0.$$

So the derivative of $C_\infty(T)$ must less than 0 until T_0 than greater than 0. Therefore there is a finite and unique T^* which minimizes the average long run cost $C_\infty(T)$.

Additionally, when unit 1 has a Weibull lifetime distribution $F(t) = 1 - \exp(-\lambda t^b)$ and its failure occurs according to a non-homogeneous Poisson process, then $\mathbb{E}N(t) =$

λt^b which yields $\frac{d^2 \mathbb{E}N(t)}{d^2 t} = \lambda b(b-1)t^{b-2}$. Therefore $\frac{d^2 \mathbb{E}N(t)}{d^2 t} \leq 0$ for $b \leq 1$ which implies G_{au1} is a decreasing function of T . Hence $T^* = \infty$. On the other hand, for $b > 1$, $\frac{d^2 \mathbb{E}N(t)}{d^2 t} > 0$ and the condition in equation (3.6) always holds. \diamond

7.2.6 The proof of Theorem 3.5

proof 9 By calculating the difference between $C_\infty(N+1)$ and $C_\infty(N)$, we have

$$C_\infty(N+1) - C_\infty(N) > (\leq) 0 \iff \frac{d_1(N) \sum_{k=1}^{N-1} d_2(k)}{d_2(N)} - \sum_{k=1}^{N-1} d_1(k) > (\leq) \frac{c_2}{c_1}. \quad (7.3)$$

Let be

$$G_{au2}(N) = \frac{d_1(N) \sum_{k=1}^{N-1} d_2(k)}{d_2(N)} - \sum_{k=1}^{N-1} d_1(k).$$

Then

$$G_{au2}(N+1) - G_{au2}(N) = \sum_{k=1}^N d_2(k) \left(\frac{d_1(N+1)}{d_2(N+1)} - \frac{d_1(N)}{d_2(N)} \right).$$

Therefore $G_{au2}(N)$ is a convex function of N if $\frac{d_1(N)}{d_2(N)}$ is a convex function of N . Besides, if the following condition

$$\lim_{N \rightarrow \infty} \frac{d_1(N) \sum_{k=1}^{N-1} d_2(k)}{d_2(N)} - \sum_{k=1}^{N-1} d_1(k) > \frac{c_2}{c_1} \quad (7.4)$$

holds, from condition (7.3), there exists N_0 such that $\forall n > N_0, C_\infty(n+1) > C_\infty(n)$. Therefore if $C(1) < \frac{c_2}{c_1}$, $C_\infty(N)$ is convex with respect to N which implies the existence of the minimum of the series $C_\infty(N)$. Otherwise, $\exists N_0 < N_1$ such that $C_\infty(1) < C_\infty(2) \cdots < C_\infty(N_0)$, $C_\infty(N_0) > C_\infty(N_0+1) > \cdots > C_\infty(N_1)$ and $\cdots < C_\infty(N_1) < \cdots < C_\infty(N_1+1) < \cdots < C_\infty(N_1+2) < \cdots$. So the minimum cost can be decided by $\min(C_\infty(1), C_\infty(N_1))$. Therefore the minimum exists whichever the case is. \diamond

7.2.7 The proof of Theorem 3.6

proof 10 By conditioning on the first renewal time of the system Y_{T_1} , we have:

$$\mathbb{E}(C(W)|Y_{T_1} = t) = \begin{cases} c_1 \mathbb{E}(N(W)) & t > W \\ c_2 + c_1 \mathbb{E}(N(t)) + \mathbb{E}(C(W-t)) & t \leq W \end{cases}$$

where $\mathbb{E}(N(t)) = \sum_{n=0}^{\infty} np_n(t)$ is the expected maintenance times of component 1 in $[0, t]$ before system replacement. Based on the law of total probability:

$$\begin{aligned}\mathbb{E}(C(W)) &= c_1 \bar{F}_s(W) \mathbb{E}(N(W)) + \int_0^W \{c_2 + c_1 \mathbb{E}(N(t)) + \mathbb{E}(C(W-t))\} dF_s(t) \\ &= K(W) + \int_0^W \mathbb{E}(C(W-t)) dF_s(t)\end{aligned}\quad (7.5)$$

where $K(W) = c_1 \bar{F}_s(W) \mathbb{E}(N(W)) + \int_0^W \{c_2 + c_1 \mathbb{E}(N(t))\} dF_s(t)$. From the renewal property, equation (7.5) is equal to

$$\mathbb{E}(C(W)) = K(W) + \int_0^W K(W-x) dM(x)$$

where $M(t)$ is the renewal function related with $F_s(t)$. The above equation is equal to

$$\mathbb{E}(C(W)) = \int_0^W [1 + M(W-t)] k(t) dt$$

where $k(t) = \frac{dK(t)}{dt} = c_1 \bar{F}_s(t) \sum_{n=0}^{\infty} n [av_n(at) - \int_0^{at} \frac{f(y+\frac{at-y}{a})}{\bar{F}(y)} v_n(y) dy] + c_2 f_s(t)$. \diamond

7.2.8 The proof of Theorem 3.7

proof 11 It can be noticed that $\mathbb{P}(N_r(W) = n) = F_s^n(W) \bar{F}_s(W)$. Denote $F_{J_i}(t)$ the distribution function of J_i where $N_r(W)$ is the system renewal times when the initial warranty period is W which has been defined in section 3.2.1.2.

$$\begin{aligned}F_{J_i}(t) &= \mathbb{P}(J_i \leq t) = \mathbb{P}(T_i \leq t \mid T_i \leq W) \\ &= \begin{cases} \frac{F_s(t)}{F_s(W)} & \text{if } t < W \\ 1 & \text{if } t \geq W \end{cases}\end{aligned}$$

Let $C_{nW}(T)$ be the total system warranty cost by time T under RFRW with warranty period W given that the number of major system failure n . Then

$$\begin{aligned}\mathbb{E}(C^R(W)) &= \mathbb{E}[\mathbb{E}(C^R(W) \mid N_r(W) = n)] \\ &= \sum_{n=0}^{\infty} F_s^n(W) \bar{F}_s(W) \mathbb{E}[C_{nW}(J_1 + J_2 + \dots + J_n + W)] \\ &= \sum_{n=0}^{\infty} F_s^n(W) \bar{F}_s(W) \\ &\quad \times \int_0^W \dots \int_0^W (nc_2 + c_1 \sum_{i=1}^n \mathbb{E}(N(j_i)) + c_1 \mathbb{E}(N(W))) dF_{J_1}(j_1) \dots dF_{J_n}(j_n) \\ &= \sum_{n=0}^{\infty} \bar{F}_s(W) \left(nc_2 F_s^n(W) + c_1 \mathbb{E}(N(W)) F_s^n(W) + nc_1 F_s^{n-1}(W) \int_0^W \mathbb{E}(N(t)) dF_s(t) \right) \\ &= \sum_{n=0}^{\infty} \bar{F}_s(W) F_s^n(W) \left(nc_2 + c_1 \mathbb{E}(N(W)) + nc_1 \frac{\int_0^W \mathbb{E}(N(t)) dF_s(t)}{F_s(W)} \right)\end{aligned}$$

Therefore we obtain the warranty cost under RFRW for the manufacturer. \diamond

7.2.9 The proof of Theorem 3.8

proof 12 *Similar to the proof of Theorem 3.6, by conditioning on the first system failure time Y_{T_1} , we have*

$$\mathbb{E}(C^s(T) \mid Y_{T_1} = t) = \begin{cases} c_{11}(\mathbb{E}N(T) - \mathbb{E}N(W)) & t > T \\ c_{22} + c_{11}(\mathbb{E}N(t) - \mathbb{E}N(W)) + \mathbb{E}(C^s(T - t)) & W < t \leq T \\ \mathbb{E}(C^s(T - t)) & 0 < t \leq W \end{cases}$$

By the law of total probability:

$$\mathbb{E}(C^s(T)) = h(T) + \int_0^T h(T - u) dM(u) = \int_W^T (1 + M(T - u)) dh(u)$$

where $h(T) = c_{22}(F_s(T) - F_s(W)) + c_{11} \int_W^T \bar{F}_s(t) d\mathbb{E}N(t)$ for $T \geq W$ and 0 otherwise. Also, let be $\mathbb{E}(C^1(T))$ the expected total maintenance cost of component 1 by time t from the perspective of the manufacturer. Then

$$\mathbb{E}(C^1(T) \mid Y_{T_1} = t) = \begin{cases} c_1 \mathbb{E}N(T) & t > T \\ c_1 \mathbb{E}N(t) + \mathbb{E}(C^1(T - t)) & 0 < t \leq T \end{cases}$$

Therefore

$$\mathbb{E}(C^1(T)) = h_1(T) + \int_0^T h_1(T - t) dM(t)$$

where $h_1(T) = c_1 \mathbb{E}N(T) - c_1 \int_0^T F_s(t) d\mathbb{E}N(t)$. Noting that the warranty profit of the manufacturer is derived from the consumer's cost minus the manufacturer's cost in $[0, T]$, therefore the manufacturer profit given is deduced. \diamond

7.2.10 The proof of Theorem 3.9

proof 13 *Let $U (U > W)$ be the time interval between two consecutive system purchase time by the consumer. That is to say that the system renewal cost are covered by the consumer rather than the manufacturer. Denote $\gamma(W)$ be the residual life to the system at time W , then $U = W + \gamma(W)$. Let $F_\gamma(t)$, $F_U(t)$ be the distribution function of $\gamma(W)$ and U then from the renewal theory we know:*

$$F_\gamma(t) = F_s(W + t) - \int_0^W \bar{F}_s(W + t - x) dM(x) \quad (7.6)$$

$$E(U) = (M(W) + 1) \int_0^\infty \bar{F}_s(t) dt \quad (7.7)$$

where $M(W) = \sum_{n=1}^\infty F_s^{(n)}(W)$ and

$$F_U(t) = \begin{cases} 0 & 0 \leq t \leq W \\ F_\gamma(t - W) & W < t \end{cases}$$

Thus by conditioning on the first purchase time of the consumer, his/her expected total cost can be derived. we have:

$$\mathbb{E}(C^{sn}(T) | U = u) = \begin{cases} c_{11}(\mathbb{E}N(T) - \mathbb{E}N(W)) & u > T \\ c_{22} + c_{11}(\mathbb{E}N(u) - \mathbb{E}N(W)) + \mathbb{E}(C^{sn}(T - u)) & W < u \leq T \end{cases}$$

Therefore the expected total cost of the consumer is derived by the law of total probability. The expected profit of the manufacturer is then obtained as the revenue minus the maintenance cost by time T . \diamond

7.3 The proofs of Theorems and propositions in Chapter 4

7.3.1 The proof of Theorem 4.1

proof 14 Let $P_1^{cm}(T)$ and $P_2^{cm}(T)$ be the probabilities that the system is replaced before T due to the shock caused by component 1 and due to the natural degradation of component 2 respectively. Let $P^{pm}(T)$ be the probability that the system is replaced at time T . These probabilities are given as follows:

$$\begin{aligned} P_1^{cm}(T) &= \int_0^T \sum_{n=1}^{\infty} r^{n-1} \bar{r} \bar{G}_{\sigma_L}(t) dV_n(at) \\ P_2^{cm}(T) &= \int_0^T \sum_{n=0}^{\infty} r^n p_n(t) dG_{\sigma_L}(t) \\ P^{pm}(T) &= \sum_{n=0}^{\infty} r^n p_n(T) \bar{G}_{\sigma_L}(T) \end{aligned}$$

As a result, the expected cycle cost is calculated as follows:

$$c_2 + \sum_{n=1}^{\infty} (n-1)c_1 r^{n-1} \bar{r} \int_0^T \bar{G}_{\sigma_L}(t) dV_n(at) + \sum_{n=0}^{\infty} nr^n c_1 \left(\int_0^T p_n(t) dG_{\sigma_L}(t) + p_n(T) \bar{G}_{\sigma_L}(T) \right)$$

As the expected length of one replacement cycle is

$$T \bar{F}_{sI}(T) + \int_0^T t dF_{sI}(t) = \int_0^T \bar{F}_{sI}(t) dt$$

So the average long run cost under age- T -based policy is obtained by the renewal reward theorem. \diamond

7.3.2 The proof of Theorem 4.2

proof 15 Let $P_1^{cm}(N)$ and $P_2^{cm}(N)$ be the probabilities that the system is respectively replaced due to the system failure induced by component 1 or due to the

natural system failure respectively. Let $P^{pm}(N)$ be the probability that the system is replaced at the N th unit 1 failure. They are given by

$$\begin{aligned} P_1^{cm}(N) &= \int_0^\infty \sum_{n=1}^{N-1} r^{n-1} \bar{r} \bar{G}_{\sigma_L}(t) dV_n(at) \\ P_2^{cm}(N) &= \int_0^\infty \sum_{n=0}^{N-1} r^n p_n(t) dG_{\sigma_L}(t) \\ P^{pm}(N) &= \int_0^\infty r^{N-1} \bar{G}_{\sigma_L}(t) dV_N(at) \end{aligned}$$

So the expected cost over a replacement cycle is given by

$$\begin{aligned} \mathbb{E}(C(\Delta_I)) &= c_2 + \sum_{n=1}^{N-1} (n-1)c_1 r^{n-1} \bar{r} \int_0^\infty \bar{G}_{\sigma_L}(t) dV_n(at) + \sum_{n=0}^{N-1} nr^n c_1 \int_0^T p_n(t) dG_{\sigma_L}(t) \\ &\quad + \int_0^\infty r^{N-1} (N-1)c_1 \bar{G}_{\sigma_L}(t) dV_N(at) \end{aligned}$$

As the total time elapsed in one cycle Δ_I satisfies

$$\mathbb{P}(\Delta_I > t) = \sum_{k=0}^{N-1} r^k p_k(t) \bar{G}_{\sigma_L}(t)$$

which implies

$$E(\Delta_I) = \int_0^\infty \sum_{k=0}^{N-1} r^k p_k(t) \bar{G}_{\sigma_L}(t) dt$$

From the renewal reward theory, the calculation of $C_{\infty I}(N)$ is straightforward. \diamond

7.3.3 The proof of Theorem 4.4

proof 16 In the following, we call component 1 failure as minor failure when the damage caused to component 2 is 0 and major failure when the damage is L respectively. Component 1 failure occurs according to a non-homogeneous Poisson process when it undergoes minimal repair while the system is supposed to be replaced when component 2 fails. Assume $h_1(t)$ to be the component 1 failure rate at time t , from the decomposition property of the Poisson process, the minor failure of component 1 occurs according to a non-homogeneous Poisson process with intensity rate $rh_1(t)$. Denote S_n^M be the n th minor failure time of component 1, therefore

$$\mathbb{P}(S_n^M \leq t) = \sum_{i=n}^{\infty} \frac{(rH_1(t))^i \exp(-rH_1(t))}{i!}$$

where $H_1(t) = \int_0^t h_1(\theta)d\theta$. By the similar method as we mentioned in Theorem 3.1, $C_{\infty I}(T)$ can be rewritten as

$$C_{\infty I}(T) = \frac{c_2 + c_1 \sum_{n=1}^{\infty} \int_0^T \bar{F}_{sI}(t) rh_1(t) dt}{\int_0^T \bar{F}_{sI}(t) dt}$$

Therefore the optimal condition of T^* is derived by adopting the method in Theorem 3.1. \diamond

7.3.4 The proof of Theorem 4.5

proof 17 As component 1 failure occurs according to the non-homogeneous Poisson process with intensity rate $h(t)$, from the decomposition property of the non-homogeneous Poisson process, the number of component 1 failure inducing the minor system failure follows a non-homogeneous Poisson process with intensity rate $rh(t)$. Therefore before the major system failure, the expected failure number of component 1 by time t can be given as $\mathbb{E}_1(t) = rH(t)$. By conditioning on the first renewal time of the system T_1 , we have

$$\mathbb{E}(C(W)|T_1 = t) = \begin{cases} c_1 r H(W) & t > W \\ c_2 + c_1 r H(t) + \mathbb{E}(C(W - t)) & t \leq W \end{cases}$$

where $\mathbb{E}(N(t)) = \sum_{n=0}^{\infty} n p_n(t)$ is the expected maintenance cost of component 1 in $[0, t]$. Based on the law of total probability

$$\begin{aligned} \mathbb{E}(C(W)) &= c_1 \bar{F}_s(W) r H(W) + \int_0^W \{c_2 + c_1 r H(t) + \mathbb{E}(C(W - t))\} dF_s(t) \\ &= K(W) + \int_0^W \mathbb{E}(C(W - t)) dF_s(t) \end{aligned} \quad (7.8)$$

where $K(W) = c_1 \bar{F}_s(W) r H(W) + \int_0^W (c_2 + c_1 r H(t)) dF_s(t)$. From the renewal property, equation (7.8) is equal to

$$\mathbb{E}(C(W)) = K(W) + \int_0^W K(W - x) dM(x)$$

where $M(t)$ is the renewal function related to $F_s(t)$. The above equation is equal to

$$\mathbb{E}(C(W)) = \int_0^W [1 + M(W - t)] k(t) dt$$

where $k(t) = c_1 r h(t) \bar{F}_s(t) + c_2 f_s(t)$. Therefore we obtained the expected warranty cost of the manufacturer under the non-renewing FRW. \diamond

7.3.5 The proof of Theorem 4.6

proof 18 It can be noticed that $\mathbb{P}(N_w = n) = F_s^n(W)\bar{F}_s(W)$. Let $\mathbb{E}(C(t))$ be the expected warranty cost of the manufacturer during time period t from the initial product sold moment. Then we have

$$\begin{aligned}
\mathbb{E}(C^R(W)) &= \mathbb{E}[\mathbb{E}(C^R(W|N_w = n))] \\
&= \sum_{n=0}^{\infty} F_s^n(W)\bar{F}_s(W)\mathbb{E}[C(\theta_1 + \theta_2 + \dots + \theta_n + W)] \\
&= \sum_{n=0}^{\infty} F_s^n(W)\bar{F}_s(W) \int_0^W \dots \int_0^W (nc_2 + c_1r \sum_{i=1}^n H(t_i) + c_1rH(W)dF_{\theta}(t_1) \dots dF_{\theta}(t_n)) \\
&= \sum_{n=0}^{\infty} \bar{F}_s(W) \left(nc_2F_s^n(W) + c_1rH(W)F_s^n(W) + nrc_1F_s^{n-1}(W) \int_0^W H(t)dF_s(t) \right) \\
&= \sum_{n=0}^{\infty} \bar{F}_s(W)F_s^n(W) \left(nc_2 + c_1rH(W) + nrc_1 \frac{\int_0^W H(t)dF_s(t)}{F_s(W)} \right)
\end{aligned}$$

Therefore we obtain the expected warranty cost of the manufacturer under the RFRW. \diamond

7.3.6 The proof of Theorem 4.7

proof 19 By conditioning on the first system renewal time T_1 , we have

$$\mathbb{E}(C(T) | T_1 = t) = \begin{cases} c_{11}r(H(T) - H(W)) & t > T \\ c_{22} + c_{11}r(H(t) - H(W)) + \mathbb{E}(C(T - t)) & W < t \leq T \\ \mathbb{E}(C(T - t)) & 0 < t \leq W \end{cases}$$

By the law of total probability:

$$\mathbb{E}(C(T)) = K(T) + \int_0^T K(T - u)dM(u) = \int_W^T (1 + M(T - u))k(u)du$$

where $K(t) = c_{22}(F_s(t) - F_s(W)) + c_{11} \int_W^t \bar{F}_s(\theta)rh(\theta)d\theta$ for $t \geq W$ and 0 otherwise. $k(u)$ is the derivative of $K(u)$ given as $k(u) = \frac{dK(u)}{du} = c_{22}f_s(u) + c_{11}\bar{F}_s(u)rh(u)$.

In addition, let be $\mathbb{E}(C_0(T))$ the expected warranty cost caused by minor system failure by time t for the manufacturer. By the similar method as in the calculation of $\mathbb{E}(C(T))$, we have

$$\mathbb{E}(C_0(T)) = \int_0^T (1 + M(T - t))k_1(t)dt$$

where $k_1(t) = \bar{F}_s(t)h(t)$, $t > 0$ and $M(t)$ is the renewal function related with $F_s(t)$. As the warranty profit of the manufacture is equivalent to the income gaining from the consumer's warranty payment minus the total warranty cost in $[0, T]$. So the manufacturer profit is deduced by substituting $\mathbb{E}(C(T))$, $\mathbb{E}(C_0(T))$ and $c_2M(T)$ in the equation

$$\mathbb{E}(P(T)) = \mathbb{E}(C(T)) - \mathbb{E}(C_0(T)) - c_2 M(T) \diamond$$

7.3.7 The proof of Theorem 4.8

proof 20 Let U , ($U > W$) be the time epoch when the system is firstly renewed after W and $F_U(t)$ be its lifetime distribution function. Denote $\gamma(W)$ be the residual life to the system at time W , then $U = W + \gamma(W)$ and $F_\gamma(t)$ be its distribution function. Then

$$F_\gamma(t) = F_s(W + t) - \int_0^W \bar{F}_s(W + t - x) dM(x)$$

and so

$$F_U(t) = \begin{cases} 0 & 0 \leq t \leq W \\ F_s(t) - \int_0^W \bar{F}_s(t - x) dM(x) & W < t \end{cases}$$

where $M(W) = \sum_{n=1}^{\infty} F_s^{(n)}(W)$. By conditioning on the first consumer purchase time U we have:

$$\mathbb{E}_{no}(C(T)) = K_0(T) + \int_W^T K_0(T - x) dM_U(x)$$

where $K_0(t) = c_{22} F_U(t) + c_{11} \int_W^t \bar{F}_U(\theta) r h(\theta) d\theta$ for $t > W$ and 0 otherwise, $M_U(t)$ is the renewal function related with $F_U(t)$.

Therefore the expected total cost of the ownership is derived by the law of total probability. The expected profit of the manufacturer is then obtained by as the revenue minus the maintenance cost by time T . It is easily seen that the system renewal cost during T is $M(T)$. The minimal repair cost of component 1 is less or equal to $rH(T)$ when the component 1 lifetime distribution $F \in IFR$. For example, assume that the system is renewed once at T_1 during T . Let $T_2 = T - T_1$, then the minimal repair cost is

$$r(H(T_1) + rH(T_2)) = r \left(\int_0^{T_1} h(t) dt + \int_{T_1}^T h(t - T_1) dt \right) \leq r \int_0^T h(t) t = H(T)$$

and ' \geq ' becomes '=' only when $h(t)$ is a constant. \diamond

7.4 Proofs of theorems and propositions in Chapter 5

7.4.1 The proof of Proposition 2

From the definition of $B_{ij}^\varnothing(t)$ it is easily seen that

$$\begin{aligned}
 B_{ij}^\varnothing(t + \Delta t) &= \mathbb{P}(T_l > t + \Delta t, W(t + \Delta t) = j, \forall l \in \Phi \mid W(0) = i) \\
 &= \sum_{k \in \mathcal{S}} \mathbb{P}(T_l > t + \Delta t, \forall l \in \Phi, W(t + \Delta t) = j, \mid T_l > t, W(t) = k, \forall l \in \Phi) \\
 &\quad \times \mathbb{P}(T_l > t, W(t) = k, \forall l \in \Phi \mid W(0) = i) \\
 &= \sum_{k \in \mathcal{S}} \exp\left(-\int_t^{t+\Delta t} \sum_{l \in \Phi} h_l(\theta, k) d\theta\right) \pi_{kj}(\Delta t) B_{ik}^\varnothing(t)
 \end{aligned}$$

As we know that when Δt is very small,

$$\pi_{kj}(\Delta t) = \begin{cases} 1 + q_{jj}\Delta t + o(\Delta t) & k = j \\ q_{kj}\Delta t + o(\Delta t) & k \neq j \end{cases}$$

where q_{kj} are the entries of \mathbf{Q} . Therefore

$$\frac{dB_{ij}^\varnothing(t)}{dt} = -B_{ij}^\varnothing(t) \left(\sum_{l \in \Phi} h_l(t, j) \right) + \sum_{k \in \mathcal{S}} B_{ik}^\varnothing(t) q_{kj} \quad (7.9)$$

which can also be represented in the matrix form as

$$\frac{d\mathbf{B}^\varnothing(t)}{dt} = \mathbf{B}^\varnothing(t)(\mathbf{Q} - \mathbf{H}(t)) \quad (7.10)$$

where $\mathbf{B}^\varnothing(t) = [B_{ij}^\varnothing(t)]_{m \times m}$ and $\mathbf{H}(t) = \text{diag}[\sum_{l \in \Phi} h_l(t, j)]$ which represents the diagonal matrix with elements $\mathbf{H}_{jj}(t) = \sum_{l \in \Phi} h_l(t, j)$ in the main diagonal. In particular, when the hazard rate is time-independent, i.e. $\mathbf{H}(t) = \mathbf{H}$, then it is easily seen that

$$\mathbf{B}^\varnothing(t) = \exp((\mathbf{Q} - \mathbf{H})t)$$

◇

7.4.2 The proof of Theorem 5.1

proof 21 *The similar method as utilized in the calculation of $\mathbf{B}^\varnothing(t)$ in Proposition 2 is implemented here. From the definition of $B_{ij}^{\{c_1, c_2, \dots, c_k\}}(t)$, we have*

$$\begin{aligned}
& B_{ij}^{\{c_1, c_2, \dots, c_k\}}(t + \Delta t) \\
= & \mathbb{P}\left(T_l > t + \Delta t, T_p < t + \Delta t, W(t + \Delta t) = j, \forall p \in \{c_1, c_2, \dots, c_k\}, \forall l \in \underline{\{c_1, c_2, \dots, c_k\}} \right. \\
& \left. | W(0) = i\right) \\
= & \sum_{m \in \mathcal{S}} \mathbb{P}\left(T_l > t + \Delta t, T_p < t + \Delta t, \forall p \in \{c_1, c_2, \dots, c_k\}, \forall l \in \underline{\{c_1, c_2, \dots, c_k\}}, \right. \\
& W(t + \Delta t) = j, | T_l > t, T_p < t, \forall p \in \{c_1, c_2, \dots, c_k\}, \forall l \in \underline{\{c_1, c_2, \dots, c_k\}}, \\
& \left. W(t) = m\right) B_{im}^{\{c_1, c_2, \dots, c_k\}}(t) \\
+ & \sum_{m \in \mathcal{S}} \sum_{v=1}^k \mathbb{P}\left(T_l > t + \Delta t, \forall l \in \underline{\{c_1, c_2, \dots, c_k\}}, T_p < t + \Delta t, \forall p \in \{c_1, c_2, \dots, c_k\}, \right. \\
& W(t + \Delta t) = j, | T_l > t, \forall l \in \underline{\{c_1, c_2, \dots, c_k\}} \cup \{c_v\}, T_p < t, \forall p \in \{c_1, c_2, \dots, c_k\} \setminus c_v, \\
& \left. W(t) = m\right) B_{im}^{\{c_1, c_2, \dots, c_k\} \setminus \{c_v\}}(t) + o((\Delta t)) \\
= & \sum_{m \in \mathcal{S}} \pi_{mj}(\Delta t) \exp\left(-\sum_{u \in \underline{\{c_1, c_2, \dots, c_k\}}} \int_t^{t+\Delta t} h_u(\theta, m) d\theta\right) B_{im}^{\{c_1, c_2, \dots, c_k\}}(t) \\
+ & \sum_{m \in \mathcal{S}} \sum_{v=1}^k \pi_{mj}(\Delta t) \exp\left(-\sum_{u \in \underline{\{c_1, c_2, \dots, c_k\}}} \int_t^{t+\Delta t} h_u(\theta, m) d\theta\right) \\
& \times \left(1 - \exp\left(-\int_t^{t+\Delta t} h_{c_v}(\theta, m) d\theta\right)\right) B_{im}^{\{c_1, c_2, \dots, c_k\} \setminus \{c_v\}}(t) + o((\Delta t))
\end{aligned}$$

where $\lim_{t \rightarrow \infty} \frac{o(\Delta t)}{\Delta t} = 0$. It is known that when Δt is very small,

$$\pi_{kj}(\Delta t) = \begin{cases} 1 + q_{jj}\Delta t + o((\Delta t)) & k = j \\ q_{kj}\Delta t + o((\Delta t)) & k \neq j \end{cases}$$

where q_{kj} are the entries of \mathbf{Q} . By implementing the similar method as in the calculation of $B_{ij}^\varnothing(t)$ in equation (7.9), we have

$$\begin{aligned}
\frac{dB_{ij}^{\{c_1, c_2, \dots, c_k\}}(t)}{dt} &= \sum_{m \in \mathcal{S}} B_{im}^{\{c_1, c_2, \dots, c_k\}}(t) q_{mj} + \sum_{l=1}^k B_{ij}^{\{c_1, c_2, \dots, c_k\} \setminus \{c_l\}}(t) h_{c_l}(t, j) \\
&\quad - B_{ij}^{\{c_1, c_2, \dots, c_k\}}(t) \sum_{l \in \underline{\{c_1, c_2, \dots, c_k\}}} h_l(t, j)
\end{aligned}$$

which can also be represented by the matrix form

$$\frac{d\mathbf{B}^{\{c_1, c_2, \dots, c_k\}}(t)}{dt} = \mathbf{B}^{\{c_1, c_2, \dots, c_k\}}(t) \left(\mathbf{Q} - \sum_{j \in \{c_1, c_2, \dots, c_k\}} \mathbf{H}_j(t) \right) + \sum_{l=1}^k \mathbf{B}^{\{c_1, c_2, \dots, c_k\} \setminus \{c_l\}}(t) \mathbf{H}_{c_l}(t)$$

where $\mathbf{B}^{\{c_1, c_2, \dots, c_k\}}(t) = [B_{ij}^{\{c_1, c_2, \dots, c_k\}}(t)]_{m \times m}$ and $\mathbf{H}_i(t) = \text{diag}[h_i(t, j)]_{m \times m}$, $i \in \Phi$, $j \in \mathcal{S} \diamond$

7.4.3 The proof of Corollary 1

proof 22 The mathematical induction method is carried out in the following proof.

When $k = 1$, from Theorem 5.1, we have

$$\frac{d\mathbf{B}^{\{c_1\}}(t)}{dt} = \mathbf{B}^{\{c_1\}}(t) \left(\mathbf{Q} - \sum_{j \in \Phi \setminus \{c_1\}} \mathbf{H}_j \right) + \mathbf{B}^\emptyset(t) \mathbf{H}_{c_1}(t)$$

where $\mathbf{B}^\emptyset(t) = \exp((\mathbf{Q} - \mathbf{H})t)$. As $\mathbf{B}^{\{c_1\}}(0) = \mathbf{0}$, On can deduce that

$$\mathbf{B}^{\{c_1\}}(t) = \exp \left(\left(\mathbf{Q} - \sum_{j \in \Phi \setminus \{c_1\}} \mathbf{H}_j \right) t \right) - \exp((\mathbf{Q} - \mathbf{H})t)$$

Assume that when $k = p$, equation (5.5) works which means

$$\mathbf{B}^{\{c_1, c_2, \dots, c_p\}}(t) = \sum_{m=0}^p (-1)^m \sum_{\substack{l_u \in \{c_1, c_2, \dots, c_p\} \\ u=1, 2, \dots, m \\ l_i \neq l_j \text{ if } i \neq j}} \exp \left(\left(\mathbf{Q} - \sum_{j \in \{c_1, c_2, \dots, c_p\}} \mathbf{H}_j - \sum_{i=1}^m \mathbf{H}_{l_i} \right) t \right) \quad (7.11)$$

When $k = p + 1$, as

$$\begin{aligned} \frac{d\mathbf{B}^{\{c_1, c_2, \dots, c_{p+1}\}}(t)}{dt} &= \mathbf{B}^{\{c_1, c_2, \dots, c_{p+1}\}}(t) \left(\mathbf{Q} - \sum_{j \in \{c_1, c_2, \dots, c_{p+1}\}} \mathbf{H}_j \right) \quad (7.12) \\ &\quad + \sum_{l=1}^{p+1} \mathbf{B}^{\{c_1, c_2, \dots, c_{p+1}\} \setminus \{c_l\}}(t) \mathbf{H}_{c_l} \end{aligned}$$

where $\{c_1, c_2, \dots, c_{p+1}\} = \Phi \setminus \{c_1, c_2, \dots, c_{p+1}\}$. It can be seen that the expression of $\mathbf{B}^{\{c_1, c_2, \dots, c_{p+1}\} \setminus \{c_l\}}(t)$, $l \in \{c_1, c_2, \dots, c_{p+1}\}$ can be derived in equation (7.11) as one can notice that it is the case when the number of failed components is p . Therefore

by substituting the expression of $B^{\{c_1, c_2, \dots, c_{p+1}\}} \setminus c_1(t)$ into equation (7.12) we have

$$\begin{aligned}
& \frac{dB^{\{c_1, c_2, \dots, c_{p+1}\}}(t)}{dt} \\
&= B^{\{c_1, c_2, \dots, c_{p+1}\}}(t) \left(Q - \sum_{j \in \{c_1, c_2, \dots, c_{p+1}\}} H_j \right) \\
&+ \sum_{l=1}^{p+1} \sum_{m=0}^p (-1)^m \sum_{\substack{l_u \in \{c_1, c_2, \dots, c_{p+1}\} \setminus \{c_l\} \\ u=1, 2, \dots, m \\ l_i \neq l_j \text{ if } i \neq j}} \exp \left(\left(Q - \sum_{j \in \{c_1, c_2, \dots, c_{p+1}\}} H_j - H_{c_l} - \sum_{i=1}^m H_{l_i} \right) t \right) H_{c_l} \\
&= B^{\{c_1, c_2, \dots, c_{p+1}\}}(t) \left(Q - \sum_{j \in \{c_1, c_2, \dots, c_{p+1}\}} H_j \right) \\
&+ \sum_{m=0}^p (-1)^m \sum_{\substack{l_u \in \{c_1, c_2, \dots, c_{p+1}\} \\ u=1, 2, \dots, m+1 \\ l_i \neq l_j \text{ if } i \neq j}} \exp \left(\left(Q - \sum_{j \in \{c_1, c_2, \dots, c_{p+1}\}} H_j - \sum_{i=1}^{m+1} H_{l_i} \right) t \right) \sum_{i=1}^{m+1} H_{l_i}
\end{aligned} \tag{7.13}$$

Let be $\mathbf{a} = Q - \sum_{j \in \{c_1, c_2, \dots, c_{p+1}\}} H_j$, $\mathbf{a}_m(l_1, l_2, \dots, l_{m+1}) = Q - \sum_{j \in \{c_1, c_2, \dots, c_{p+1}\}} H_j - \sum_{i=1}^{m+1} H_{l_i}$, $\mathbf{k}_m = (-1)^m \sum_{i=1}^{m+1} H_{l_i}$, by solving equation (7.13) we have

$$\begin{aligned}
& B^{\{c_1, c_2, \dots, c_{p+1}\}}(t) \\
&= \sum_{m=0}^p \sum_{\substack{l_i \in \{c_1, c_2, \dots, c_{p+1}\} \\ i=1, 2, \dots, m+1 \\ l_i \neq l_j \text{ if } i \neq j}} \left(\exp(\mathbf{a}t) - \exp(\mathbf{a}_m(l_1, l_2, \dots, l_{m+1})t) \right) \mathbf{k}_m (\mathbf{a} - \mathbf{a}_m(l_1, l_2, \dots, l_{m+1}))^{-1} \\
&= \sum_{m=0}^p \sum_{\substack{l_i \in \{c_1, c_2, \dots, c_{p+1}\} \\ i=1, 2, \dots, m+1 \\ l_i \neq l_j \text{ if } i \neq j}} (-1)^m \left(\exp(\mathbf{a}t) - \exp(\mathbf{a}_m(l_1, l_2, \dots, l_{m+1})t) \right) \\
&= \sum_{m=0}^p (-1)^m \binom{p+1}{m+1} \exp(\mathbf{a}t) + \sum_{m=0}^p \sum_{\substack{l_i \in \{c_1, c_2, \dots, c_{p+1}\} \\ i=1, 2, \dots, m+1 \\ l_i \neq l_j \text{ if } i \neq j}} (-1)^{m+1} \exp(\mathbf{a}_m(l_1, l_2, \dots, l_{m+1})t) \\
&= \exp(\mathbf{a}t) + \sum_{m=1}^{p+1} \sum_{\substack{l_i \in \{c_1, c_2, \dots, c_{p+1}\} \\ i=1, 2, \dots, m+1 \\ l_i \neq l_j \text{ if } i \neq j}} (-1)^m \exp(\mathbf{a}_{m-1}(l_1, l_2, \dots, l_m)t) \\
&= \sum_{m=0}^{p+1} (-1)^m \sum_{\substack{l_i \in \{c_1, c_2, \dots, c_{p+1}\} \\ i=1, 2, \dots, m+1 \\ l_i \neq l_j \text{ if } i \neq j}} \exp \left(\left(Q - \sum_{j \in \{c_1, c_2, \dots, c_{p+1}\}} H_j - \sum_{i=1}^m H_{l_i} \right) t \right)
\end{aligned}$$

where $\binom{p+1}{m+1}$ is the $(m+1)$ -combination of the set with $p+1$ distinguished elements. It is seen that equation (5.5) holds for the $B^{\{c_1, c_2, \dots, c_{p+1}\}}(t)$ and therefore the proof is complete. \diamond

7.4.4 The proof of Corollary 3

proof 23 *Denoted*

$$L_{ij}^{l_0,l}(\theta, t) = \mathbb{P}(l \text{ components fail at } t, W(t) = j \mid l_0 \text{ components fail at } \theta, W(\theta) = i)$$

with its matrix form $\mathbf{L}^{l_0,l}(\theta, t) = [L_{ij}^{l_0,l}(\theta, t)]_{m \times m}$ where $\theta > 0$, $l_0 \leq l < N$. It is obviously that $\mathbf{L}^{l_0,l_0}(\theta, \theta) = \mathbf{I}$, $\mathbf{L}^{l_0,l}(\theta, \theta) = \mathbf{0}$ for $l_0 < l$. By applying the similar method, we have

$$\frac{d\mathbf{L}^{l_0,l}(\theta, t)}{dt} = \mathbf{L}^{l_0,l-1}(\theta, t)(N-l+1)\mathbf{H}_0 + \mathbf{L}^{l_0,l}(\theta, t)(\mathbf{Q} - (N-l)\mathbf{H}_0)$$

where $\mathbf{L}^{l_0,l_0-1}(\theta, t) = \mathbf{0}$ which yields

$$\mathbf{L}^{l_0,l}(\theta, t) = \binom{N-l_0}{l-l_0} \sum_{i=0}^{l-l_0} (-1)^{l-l_0-i} \binom{l-l_0}{i} \exp\left((\mathbf{Q} - (N-l_0-i)\mathbf{H}_0)(t-\theta)\right)$$

Therefore $r_i^{l_0}(\theta)$ can be obtained.

$$r_i^{l_0}(\theta) = \sum_{l=l_0}^{N-M} \int_{\theta}^{\infty} \mathbf{e}_i(\mathbf{L}^{l_0,l}(\theta, t)) \mathbf{e} dt$$

where \mathbf{e} is a column vector of 1s, \mathbf{e}_i is a $1 \times m$ matrix whose i th element is 1 and others are 0 respectively.

◇

7.4.5 The proof of Theorem 5.2

proof 24 *It is easily seen that the expectation of the system lifetime distribution when the initial environment is k , $k \in \mathbf{S}$ can be derived as*

$$\mathbb{E}_k(U_1) = \int_0^{\infty} (1 - F_k(t)) dt$$

where $F_k(t)$ is the conditional system lifetime distribution given is equation (5.13).

Furthermore, the length of a renewal cycle when the initial environment is k can be obtained:

$$\begin{aligned} \mathbb{E}_k(R_1) &= \sum_{i=1}^{\infty} i\tau \mathbb{P}((i-1)\tau < U_1 \leq i\tau \mid W_0 = k) + \mathbb{E}Y \\ &= \tau \sum_{i=0}^{\infty} (1 - F_k(i\tau)) + \int_0^{\infty} (1 - G(y)) dy \end{aligned}$$

Denoted by $h(t)$ the system failure rate at time t . Define

$$H_{\min}(t) = \int_0^t \sum_{i \in \Phi} h_i(t, 1) dt$$

where $h_i(t, 1)$ is the failure rate of component i under environment 1. Then

$$1 - F_k(t) \leq \exp(-H_{\min}(t))$$

As $\int_0^\infty h_i(t, j) dt = \infty$ for $i \in \Phi, j \in \mathbf{S}$ and $\mathbb{E}_k(U_1) = \int_0^\infty (1 - F_k(t)) dt$ therefore $\mathbb{E}_k(U_1)$ is convergent. Also from the expression of $E_k(R_1)$, it is easily seen the convergence of $\mathbb{E}_k(R_1)$. So $\frac{\mathbb{E}_k(U_1)}{\mathbb{E}_k(R_1)}$ exists for any $k \in \mathbf{S}$.

In the next, we focus on the calculation of the transition kernel $Q_{ij}(t)$. Let us define

$$\mathcal{L}\{Q_{ij}\}(s) = \mathbb{E}[e^{-sR_1} \mathbf{1}_{W_{R_1}=j} \mid W_0 = i]$$

where $\mathbf{1}_A(x)$ is the indicator function which equals 1 when $x \in A$ and 0 otherwise. By conditioning on the repair time Y and the first failure diagnosed time U_1 we may write

$$\begin{aligned} & \mathcal{L}\{Q_{ij}\}(s) \\ = & \int_0^\infty \mathbb{E}[e^{-s(U_1+y)} \mathbf{1}_{W_{U_1+y}=j} \mid W_0 = i] dGy \\ = & \sum_{l \in \mathbf{S}} \sum_{k=1}^\infty \mathbb{P}(U_1 = k\tau, W_{k\tau} = l \mid W_0 = i) \int_0^\infty \pi_{lj}(y) e^{-s(k\tau+y)} dGy \\ = & \sum_{l \in \mathbf{S}} \sum_{m \in \mathbf{S}} \sum_{k=1}^\infty \sum_{p=0}^{N-M} \mathbb{P}(U_1 = k\tau, W_{k\tau} = l \mid T_u > (k-1)\tau, T_v < (k-1)\tau, \\ & \forall u \in \{c_1, c_2, \dots, c_p\}, \forall v \in \{c_1, c_2, \dots, c_p\}, W_{(k-1)\tau} = m) \\ \times & \mathbb{P}(T_u > (k-1)\tau, T_v < (k-1)\tau, \forall u \in \{c_1, c_2, \dots, c_p\}, \forall v \in \{c_1, c_2, \dots, c_p\}, \\ & W_{(k-1)\tau} = m, \mid W_0 = i) \int_0^\infty \pi_{lj}(y) e^{-s(k\tau+y)} dGy \\ = & \sum_{l \in \mathbf{S}} \sum_{m \in \mathbf{S}} \sum_{k=1}^\infty \sum_{p=0}^{N-M} \int_0^\infty \pi_{lj}(y) e^{-s(k\tau+y)} dGy B_{im}^{\{c_1, c_2, \dots, c_p\}}((k-1)\tau) \\ & \times \mathbb{P}(U_1 = k\tau, W_{k\tau} = l \mid T_u > (k-1)\tau, T_v < (k-1)\tau, \forall u \in \{c_1, c_2, \dots, c_p\}, \\ & \forall v \in \{c_1, c_2, \dots, c_p\}, W_{(k-1)\tau} = m) \\ = & \sum_{m \in \mathbf{S}} \sum_{l \in \mathbf{S}} \sum_{k=1}^\infty \int_0^\infty \pi_{lj}(u) e^{-s(k\tau+y)} dG(y) \sum_{p=0}^{N-M} \left(\sum_{1 \leq c_1 < c_2 < \dots < c_p \leq N} B_{im}^{\{c_1, c_2, \dots, c_p\}}((k-1)\tau) \right. \\ & \left. \times \left(\pi_{ml}(\tau) - \sum_{r=0}^{N-M-p} \sum_{\substack{d_1 < d_2 < \dots < d_r \\ d_i \in \{c_1, c_2, \dots, c_p\} \\ i=1, 2, \dots, r}} L_{ml}^{\{d_1, d_2, \dots, d_r\}}((k-1)\tau, k\tau; \{c_1, c_2, \dots, c_p\}) \right) \right) \end{aligned}$$

Therefore, the (i, j) th element of the transition probability matrix \mathbf{P} can be given

as:

$$\begin{aligned}
P_{ij} &= \lim_{s \rightarrow 0} \mathcal{L}\{Q_{ij}\}(s) \\
&= \sum_{m \in \mathcal{S}} \sum_{l \in \mathcal{S}} \sum_{k=1}^{\infty} \int_0^{\infty} \pi_{lj}(u) dG(u) \sum_{p=0}^{N-M} \left(\sum_{1 \leq c_1 < c_2 < \dots < c_p \leq N} B_{im}^{\{c_1, c_2, \dots, c_p\}}((k-1)\tau) \right. \\
&\quad \left. \times \left(\pi_{ml}(\tau) - \sum_{r=0}^{N-M-p} \sum_{\substack{d_1 < d_2 < \dots < d_r \\ d_i \in \{c_1, c_2, \dots, c_p\} \\ i=1, 2, \dots, r}} L_{ml}^{\{d_1, d_2, \dots, d_r\}}((k-1)\tau, k\tau; \{c_1, c_2, \dots, c_p\}) \right) \right)
\end{aligned}$$

where $B_{im}^{\{c_1, c_2, \dots, c_p\}}(t) = B_{im}^{\emptyset}(t)$ when $p = 0$ which is given in Equation (7.9). \diamond

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Doctorat : Optimisation et Sûreté des Systèmes

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Analyse de fiabilité et de maintenance de systèmes multi-composants avec dépendances stochastiques

Dans la société moderne, les systèmes composés de plusieurs composants qui peuvent interagir entre eux jouent un rôle essentiel et ont une grande importance. Les systèmes sont souvent sujet à détérioration et usure et opèrent dans un environnement changeant, c'est pourquoi assurer leur fonctionnement n'est pas une tâche simple. Des actions de maintenance sont mises en place afin d'éviter les défaillances du système.

Cette thèse considère le problème de la maintenance des systèmes multi-composants. Dans un premier temps, on considère un système à deux composants avec dépendance stochastique. Trois modèles de dépendance ont été considérés.

Pour chacun des trois modèles, plusieurs politiques de maintenance sont proposées et leur optimalité est étudié. Pour chaque politique, l'expression explicite du coût moyen à long terme est présentée. De plus, pour les deux derniers modèles, plusieurs politiques de garantie sont proposées et l'expression explicite de leur coût est dérivée.

Pour finir, un système M parmi N est considéré et les mesures de performance telles que la fiabilité du système, la durée de vie utile restante (RUL), la disponibilité asymptotique sont calculées. Il est montré que dans les systèmes multi-composants, la dépendance stochastique entre les composants affecte à la fois les mesures de performance et l'optimisation des politiques de maintenance.

Mots clés : processus stochastiques - dépendance (statistique) - fiabilité - entretien - garantie.

Reliability and Maintenance Analysis of Multi-component Systems with Stochastic Dependence

In modern society, systems composed of many components which may interact with each other play a vital role and become more and more important. It is not a simple task to ensure the normal operation of systems since that they are often subject to deterioration and wear and operate in a changing environment. Maintenance actions can be carried out in order to avoid the failure of the system.

This thesis considers maintenance problems and reliability evaluations of multi-component systems. At first, we study a two-component system with various types of stochastic dependencies. Three models have been considered. For each of the three models above, several maintenance policies are proposed. Under each policy, the explicit expressions of the long-term average cost as well as the cost optimality are studied. In addition, for the last two models, several guarantee policies are proposed and the allocation of the warranty cost between the manufacturer and the consumer are studied.

Finally, an M -out-of- N system which operates in a dynamic environment is studied. Performance measures such as the reliability of the system, the remaining useful life (RUL), the asymptotic availability are calculated.

In conclusion, it is shown that in multi-component systems, stochastic dependence between components may affect both system performance measures and the optimization of maintenance policies.

Keywords: stochastic processes - dependence (statistics) - reliability (engineering) - maintenance - warranty.

Thèse réalisée en partenariat entre :

