Existence results for nonlinear wave equations

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après avis des rapporteurs

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Contents

Introduction

1 Classical topics on the wave-Klein-Gordon equations

1.1 Introduction of the wave equation

We are mainly interested in the second order partial differential equations of the form

$$-\Box u + m^2 u = f, \tag{1.1}$$

with its prescribed initial data on some hypersurface $t = t_0$

$$u(t_0, \cdot) = u_0, \qquad \partial_t u(t_0, \cdot) = u_1. \tag{1.2}$$

We have used the notation

$$\Box := \eta^{\alpha\beta}\partial_\alpha\partial_\beta, \tag{1.3}$$

which is called the wave operator, and

$$\eta := \mathrm{diag}(-1, 1, \cdots, 1, 1) \tag{1.4}$$

is the metric in the Minkowski spacetime $\mathbb{R}^{1+d}$, with d the space dimension. We denote the Cartesian coordinates of a point $(t, x) = (x^\alpha)$, and its radius $r := \left(x_a x^a\right)^{1/2}$. Throughout, we use Roman letters to represent spacetime indices and Latin letters for space indices, and Einstein summation convention is adopted unless specified. In the above f is called the source term: if f is a function independent of u and its partial derivatives, we call (1.1) a linear wave equaton; if f is a function of u or its partial derivatives, we call (1.1) a nonlinear wave equation. In the above, $m \geqslant 0$ is some constant, and (1.1) is called a wave equation if $m = 0$, while it is called a Klein-Gordon equation with mass m when $m \neq 0$. But we might refer to the equation (1.1) with $m \geqslant 0$ as a wave equation if no confusion arises.

Wave equation is first discovered by d'Alembert in one space dimension in 1746, and later by Euler in three space dimension. We emphasize that we consider wave

equations in the whole space instead of restricting them to some bounded domain with prescribed boundary condition. We proceed by introducing several well-known notions and results.

Definition 1.1. *Consider two regular functions u and v, and we call $T^{\alpha\beta}\partial_\alpha u\partial_\beta v$ as a quadratic null form if*

$$T^{\alpha\beta}\xi_\alpha\xi_\beta = 0 \tag{1.5}$$

for all null vectors, i.e. vectors satisfying $\xi^\alpha\xi_\alpha = 0$.

We list some specific examples of null forms.

- $$Q_0(u,v) := \partial^\alpha u\partial_\alpha v.$$
- $$Q_{ab}(u,v) := \partial_a u\partial_b v - \partial_b u\partial_a v.$$
- $$Q_{0a}(u,v) := \partial_t u\partial_a v - \partial_a u\partial_t v.$$

The last two types of null forms are called strong null forms by Georgiev [20], because they have a certain compatibility with the Klein-Gordon components.

There are several groups of vector fields which play very important role in the following analysis.

- Translations:
$$\partial_\alpha := \partial_{x^\alpha}.$$
- Rotations:
$$\Omega_{ab} := x_a\partial_b - x_b\partial_a.$$
- Lorentz boosts:
$$L_a := t\partial_a + x_a\partial_t.$$
- Scaling:
$$L_0 := t\partial_t + r\partial_r.$$

We have the following commutation property between the vector fields above and the wave (or Klein-Gordon) operator.

Proposition 1.2. *It holds*

$$\begin{aligned} &[\partial_\alpha, -\Box + m^2] = 0, \qquad [L_a, -\Box + m^2] = 0,\\ &[\Omega_{ab}, -\Box + m^2] = 0, \qquad [L_0, -\Box] = 2\Box. \end{aligned} \tag{1.6}$$

1.2 What are we interested in wave equations

There are several natural questions coming to mind when $f = f(t, x)$ is a prescribed function independent of u and its derivatives.

- Whether the wave (or Klein-Gordon) equation (1.1) admits a solution?
- What is a solution?
- If the solution exists, does the solution exist for all time or only for finite time?
- If the solution exists, how fast does it decay, concerning wave equation is one kind of dispersive equation?

When f is a function of u or its derivatives, the situation is more complicated.

- In different dimensions, what kind of forms of $f = f(u, \partial u, \partial\partial u)$ lead to (small data) a global-in-time solution?
- If the solution exists to the nonlinear wave equation, how fast does the solution decay?
- What can we say about the systems of wave (or Klein-Gordon or coupled wave-Klein-Gordon) equations?

For some classical results about linear wave equations, one refers to the beginning part of the monograph [69] or the lecture note [56] for more details.

We give some well-known related problems on the nonlinear wave equations.

Example 1.3 (Strauss conjecture)**.** *Consider the semilinear wave equation*

$$-\Box u = |u|^{p_c+\alpha}, \qquad \big(u, \partial_t u\big)(t_0, \cdot) = (u_0, u_1), \tag{1.7}$$

in which p_c is the positive root of the equation

$$(d-1)p^2 - (d+1)p - 2 = 0.$$

Global existence and well-posedness of solutions with small initial data is expected by Strauss [71], which was proved by Georgiev, Lindblad, and Sogge [22] and other works (we are not going to be exhausitive), for $\alpha > 0$, while counterexamples of finite time blow-up solutions exist when $\alpha < 0$. Some recent well-known works on this topic include [70], [51], [62], and many others.

Example 1.4 (Weak null conjecture)**.** *In the papers [52] and [53], Lindblad and Rodnianski first came up with the notion of weak null condition, which generalised the notion of the classical null condition. Utilising this idea, they first proved the global stability of Minkowski spacetime in the wave gauge.*

Whether weak null condition is a sufficient condition or a necessary condition to the existence of global solutions to wave equations is still open. For the advances on this subject, one refers to [3], [11], [37], and many others.

1.3 What tools do people have to study wave equations

The study of nonlinear wave equations is an active research field since 40 years ago. Numerous of great results come out.

A broad class of wave equations with quadratic nonlinearities satisfying the so-called null condition, as shown independently by Klainerman [39, 40] and Christodoulou [8], do admit global-in-time solutions. The vector field method, due to Klainerman, and the conformal method, due to Christodoulou, have been two major approaches to studying wave equations. Later on, Rodnianski and Lindblad [53, 54] come up with the notion of weak null condition, and are able to treat a larger class of nonlinearities. But whether the weak null condition is sufficient or necessary conditions to the existence of global solutions to wave equations is still an open problem. Other generalisations of the null form have also been used to great effect, see for example Pusateri and Shatah [65]. Recently, the r^p weighted method by Dafermos and Rodnianski [9] is able to treat wave equations in the flat metric or metrics with black holes; the hyperboloidal foliation method by LeFloch and Ma [44, 46], and extended in [45], to treat coupled wave-Klein-Gordon systems turns out to be a big success.

By contrast, the Klein-Gordon equation requires a different analysis from the wave equation. One key obstruction is that the scaling vector field $S = t\partial_t + x^a\partial_a$ does not commute with the Klein-Gordon operator $-\Box + 1$, which thus does not allow us to apply the Klainerman-Sobolev inequality directly. Pioneering works by Klainerman using the vector field method in [41], and by Shatah employing a normal form method in [66], led the way in treating a wide class of Klein-Gordon-type equations.

In [20], Georgiev studies a coupled wave-Klein-Gordon system where he came up with the notion of strong null forms, which are consistent with the Klein-Gordon equations, and he obtained global stability results for this system.

Besides, the method of spacetime resonance, introduced by Shatah, Germain, and Masmoudi in the series of papers [25], [23], [27], [26], [24], etc., appears to be a powerful tool in studying nonlinear dispersive equations, including wave equations, Klein-Gordon equations, Schordinger equations (see for instance [28]), and many others. As commented, spacetime resonance method bridges the vector field method and normal

form method in a certain way. It was used in [65] to prove the global solutions of wave equations, which also generalised the classical null forms.

The method we employ in this thesis is the so-called hyperboloidal foliation method which is first introduced by LeFloch and Ma [44] in 2014. Later on they apply this method to a more generalised coupled wave-Klein-Gordon system in [45] and succeed in tackling the Einstein-Klein-Gordon system in [48]. Besides, Ma proves a class of quasilinear wave-Klein-Gordon equations in dimension two [59] and [58], and in dimension one [60], and Ma and Huang provided a conformal type energy adapted to the hyperboloidal foliation which is useful to obtain L^2-type estimates for wave components. Worth to mention, Fajman, Joudioux, and Smulevici apply this method to the Einstein-Vlasov system in [17].

1.4 What do we already know about wave equations

After decades of investigations, lots of important results come out. In the linear case, the solution can be written out explicitly in an integral form (for example in dimension 3). Since wave equation is one kind of dispersive equation, people are very interested in exploring its decay property. Various of methods (like Fourier analysis, utilisation of the integral formulation of the solution, and energy method) can be used to prove the following result for free wave (or Klein-Gordon) equations.

Theorem 1.5. *Let u be the solution to the wave equation*

$$\Box u + m^2 u = 0, \quad u(t_0, \cdot) = u_0, \quad \partial_t u(t_0, \cdot) = u_1,$$

in which u_0, u_1 are compactly supported smooth functions, then it holds [1]

$$\begin{aligned} |u| &\lesssim t^{-(d-1)/2} \qquad if\, m = 0, \\ |u| &\lesssim m^{-1} t^{-d/2} \qquad if\, m \neq 0, \end{aligned} \tag{1.8}$$

in which d is the space dimension.

For general nonlinear wave equations, the local existence results are classical. We cite this result from [69].

Theorem 1.6. *Consider the quasilinear wave equation of the form*

$$\begin{aligned} &g^{\alpha\beta}(u, \partial u)\partial_\alpha \partial_\beta u = F(u, \partial u), \\ &u(t_0, \cdot) = u_0, \qquad \partial_t u(t_0, \cdot) = u_1, \end{aligned} \tag{1.9}$$

[1] Throughout, $A \lesssim B$ means $A \leqslant CB$ with some generic constant C, and similar convention is true for the notation $\simeq$.

and we assume $g^{\alpha\beta}$ and F are smooth functions with all derivatives $O(1)$. We also assume that

$$F(0,0) = 0, \qquad \sum_{\alpha\beta} \left|g^{\alpha\beta} - \eta^{\alpha\beta}\right| < 1/2. \tag{1.10}$$

Let $N \geqslant d+2$ and $(u_0, u_1) \in H^{N+1} \times H^N$. Then there is a $T > t_0$, depending on the norm of the data, so that the initial value problem above admits a unique solution, satisfying

$$\sum_{|\alpha| \leqslant N+1} \|\partial^\alpha u(t,\cdot)\|_{L^2} < +\infty, \qquad t_0 \leqslant t \leqslant T. \tag{1.11}$$

Also, if

$$T_* := \sup\{t : \sum_{|\alpha| \leqslant N+1} \|\partial^\alpha u(t,\cdot)\|_{L^2} < +\infty\},$$

then

- *either $T_* = +\infty$,*
- *or*

$$\sum_{|\alpha| \leqslant (N+3)/2} |\partial^\alpha u(t,x)| \notin L^\infty\big([t_0, T_*] \times \mathbb{R}^n\big).$$

For nonlinear wave equations with null forms, Klainerman and Christodoulou proved independently the following result.

Theorem 1.7 (Klainerman and Christodoulou, 1986)**.** *Consider the wave equation* (1.1) *in dimension $\mathbb{R}^{1+3}$ with*

$$f = T^{\alpha\beta}\partial_\alpha u \partial_\beta u$$

a null form, and let N be a sufficiently large integer. There exists $\epsilon_0 > 0$ such that for all $\epsilon \in (0, \epsilon_0)$, and all initial data (u_0, u_1) satisfying the smallness condition

$$\|u_0\|_{H^{N+1}} + \|u_1\|_{H^N} \leqslant \epsilon, \tag{1.12}$$

the initial value problem (1.1)–(1.2) *admits a global-in-time solution u.*

2 Physical models related to wave-Klein-Gordon equations

2.1 Examples of wave equations

We first provide some examples of wave equations in Luk [56].

Example 2.1 (Maxwell's equation, Maxwell, 1860s)**.** *Let $E : I \times R^3 \to R^3$ and $B : I \times R^3 \to R^3$ be (time-dependent) vector fields representing the electric and magnetic field. The Maxwell's equations are given by*

$$\begin{aligned} \partial_t E &= \nabla \times B, \\ \partial_t B &= -\nabla \times E, \\ \nabla \cdot E &= 0, \\ \nabla \cdot B &= 0. \end{aligned} \tag{2.1}$$

A priori, they may not look like wave equations (they are not even second order!). However, if we differentiate the first equation by ∂_t, use the second and third equations, we get

$$-\Box E_i = 0, \qquad i = 1, 2, 3. \tag{2.2}$$

Similarly, it is also true that

$$-\Box B_i = 0, \qquad i = 1, 2, 3. \tag{2.3}$$

Thus given initial data

$$\big(E, B\big)(t_0, \cdot) = (E^0, B^0), \tag{2.4}$$

which are divergence free, then the Maxwell's equations in (2.1) *are equivalent to the following wave equations*

$$\begin{aligned} &-\Box E_i = 0, \qquad -\Box B_i = 0, \\ &\big(E_i, \partial_t E_i\big)(t_0, \cdot)\big(E_i^0, (\nabla \times B^0)_i\big), \qquad \big(B_i, \partial_t B_i\big)(t_0, \cdot)\big(B_i^0, -(\nabla \times E^0)_i\big). \end{aligned} \tag{2.5}$$

Example 2.2 (Irrotational compressible fluids, Euler, 1752)**.** *A fluid in $I \times R^3$ is described by a vector field $v : I \times R^3 \to R^3$ describing the velocity of the fluid and a non-negative function $h : I \times R^3 \to R$ describing the enthalpy.*

Define the pressure p to be a function of the enthalpy $p = p(h)$ such that

- $p > 0$,
- $\rho := \frac{dp}{dh} > 0$,
- $\eta^2 := \rho(\frac{d^2p}{dh^2})^{-1} > 0$.

We call ρ the density of the fluid and η the speed of sound. The Euler equations are given by

$$\begin{aligned} \partial_t v_i + (v \cdot \nabla) v_i &= -\partial_i h, \\ \partial_t \rho + \nabla \cdot (\rho v) &= 0, \end{aligned} \tag{2.6}$$

for $i = 1, 2, 3$. We say that a flow is irrotational if $\nabla \times v = 0$. In that case, we can write $v = -\nabla\phi$, where ϕ is defined up to adding a function of time.

The first equation above gives

$$\nabla\big(\partial_t\phi - (1/2)|\nabla\phi|^2 - h\big) = 0.$$

Since we have the freedom to add a function of time to ϕ (which does not change v), we can choose

$$\partial_t\phi - (1/2)|\nabla\phi|^2 - h = 0.$$

We can get from the second equation in (2.6) *that*

$$\partial_t\partial_t\phi - 2\partial^i\phi\partial_t\partial_i\phi + \partial^i\phi\partial^j\phi\partial_i\partial_j\phi - \eta^2\partial^i\partial_i\phi = 0. \tag{2.7}$$

Example 2.3 (Einstein vacuum equations, Einstein, 1915)**.** *The Einstein vacuum equations describe the propagate of gravitational waves in the absence of matter and take the form*

$$Ric(g) = 0, \tag{2.8}$$

where the Lorentzian metric g is the unknown and Ric is the Ricci curvature. In a coordinate system, these equations take the form

$$Ric(g)_{\mu\nu} = -\frac{1}{2}g^{\alpha\beta}\partial_\alpha\partial_\beta g_{\mu\nu} - \frac{1}{2}g^{\alpha\beta}\partial_\mu\partial_\nu g_{\alpha\beta} + \frac{1}{2}g^{\alpha\beta}\partial_\alpha\partial_\nu g_{\beta\mu} + \frac{1}{2}g^{\alpha\beta}\partial_\beta\partial_\mu g_{\alpha\nu} + F_{\mu\nu}(g,\partial g), \tag{2.9}$$

where $F_{\mu\nu}(g,\partial g)$ is a function of g and its derivatives.

This does not look like a wave equation (because of the second to fourth terms)! However, a more careful choice of coordinates allows one to rewrite this system as a system of nonlinear wave equations.

Example 2.4 (Wave map equations)**.** *Let $\phi : I \times R^n \to S^m = \{x \in R^{m+1} : |x| = 1\}$. The wave map equation is given by the following system of $(m+1)$ equations:*

$$\begin{aligned} -\Box\phi &= \phi\big(\partial_t\phi^T\partial_t\phi - \partial^i\phi^T\partial_i\phi\big), \\ \big(\phi, \partial_t\phi\big)(t_0,\cdot) &= (\phi_0,\phi_1), \end{aligned} \tag{2.10}$$

where ϕ^T denotes the transpose of the vector ϕ.

One can show that suppose $|\phi_0|^2 = 1$ and $\phi_0^T\phi_1 = 0$, then, if a solution ϕ exists in $I \times R^n$, then $|\phi|^2 = 1$, i.e., ϕ is indeed a map to the sphere.

Note again that the above examples are extracted from the lecture note by Luk [56].

2.2 Examples of Klein-Gordon equations

We demonstrate a few examples of Klein-Gordon equations.

Example 2.5 (Higgs equation)**.** *Let $\Phi : \mathbb{R}^{1+3} \to \mathbb{C}^2$, and $V(p) := (p - v^2)^2$ with $v > 0$ some fixed constant, we call the following equation the (massive) homogeneous Higgs equation*

$$-\Box\Phi + V'(\Phi^*\Phi)\Phi = 0. \tag{2.11}$$

This equation is derived from the standard model of electroweak interactions, see for instance [15] or Chapter 2.

Example 2.6 (Massive Dirac-Proca equations)**.** *The field equations for the massive Dirac-Proca model with unknowns A^μ and ψ read*

$$\begin{aligned} \Box A^\nu - m^2 A^\nu &= -\psi^*\gamma^0\gamma^\nu(P_L\psi), \\ -i\gamma^\mu\partial_\mu\psi + M\psi &= -\gamma^\mu A_\mu(P_L\psi). \end{aligned} \tag{2.12}$$

in which $P_L = \frac{1}{2}(I_4 - \gamma^5)$, γ^α the Gamma matrices, and $m > 0$, $M > 0$. Here $\psi : \mathbb{R}^{3+1} \to \mathbb{C}^4$ represents a massive Dirac fermion with spin $1/2$ and $A^\mu : \mathbb{R}^{3+1} \to \mathbb{R}$ represents a massive boson (the Proca field) of mass m^2 with spin 1.

The equation for the spinor ψ does not seem to be a wave equation unless we act the Dirac operator $-i\gamma^\mu\partial_\mu$ one more time on it and apply the fact

$$-\Box = \big(-i\gamma^\mu\partial_\mu\big)^2. \tag{2.13}$$

2.3 Examples of coupled wave-Klein-Gordon equations

Several coupled wave-Klein-Gordon equations are listed here.

Example 2.7 (Dirac-Proca equations)**.** *The field equations for the Dirac-Proca model with unknowns A^μ and ψ read*

$$\begin{aligned} \Box A^\nu - m^2 A^\nu + \partial^\nu(\partial_\mu A^\mu) &= -\psi^*\gamma^0\gamma^\nu(P_L\psi), \\ -i\gamma^\mu\partial_\mu\psi &= -\gamma^\mu A_\mu(P_L\psi). \end{aligned} \tag{2.14}$$

with again $P_L = \frac{1}{2}(I_4 - \gamma^5)$ and $m > 0$. Here $\psi : \mathbb{R}^{3+1} \to \mathbb{C}^4$ represents a massless Dirac fermion with spin $1/2$ and $A^\mu : \mathbb{R}^{3+1} \to \mathbb{R}$ represents a massive boson (the Proca field) of mass m^2 with spin 1.

It does not look like a Klein-Gordon system for the gauge fields A^ν, but one observes that the initial constraints

$$\big(\partial_\mu A^\mu\big)(t_0, \cdot) = 0, \qquad \partial_t\big(\partial_\mu A^\mu\big)(t_0, \cdot) = 0, \tag{2.15}$$

are propagated by the equations in (2.14), and thus the Dirac-Proca equations now

become

$$\begin{aligned}\Box A^\nu - m^2 A^\nu &= -\psi^* \gamma^0 \gamma^\nu (P_L \psi),\\ -i\gamma^\mu \partial_\mu \psi &= -\gamma^\mu A_\mu (P_L \psi).\end{aligned}$$

Example 2.8 (Klein-Gordon-Zakharov equations)**.** *Recall the Klein-Gordon-Zakharov equations*

$$\begin{aligned}-\Box u &= \sum_a \Delta |v_a|^2,\\ -\Box v_a + v_a &= u v_a,\end{aligned} \tag{2.16}$$

where the unknowns u is real valued and v_a are complex valued for $a = 1, 2, 3$. The initial data are denoted by

$$\big(u, \partial_t u\big)(t = 2, \cdot) = \big(u^{(0)}, u^{(1)}\big), \qquad \big(v_a, \partial_t v_a\big)(t = 2, \cdot) = \big(v_a^{(0)}, v_a^{(1)}\big), \tag{2.17}$$

We rewrite the equations (3.1.7) *in the following form where we can only consider real valued functions*

$$\begin{aligned}-\Box u &= \sum_a \partial_i \Big(\partial^i \big(x_a^2 + y_a^2\big)\Big),\\ -\Box x_a + x_a &= u x_a,\\ -\Box y_a + y_a &= u y_a,\end{aligned} \tag{2.18}$$

in which we use the notations $x_a := Re(v_a)$ and $y_a := Im(v_a)$ to denote the real part and the imaginary part of a complex number z, respectively.

We note that the regularity of u is one order less than that of v_a. This can be seen from the initial data, where we consider the norms

$$\|u^{(0)}\|_{H^{N_0}}, \quad \|u^{(1)}\|_{H^{N_0-1}}, \quad \|v_a^{(0)}\|_{H^{N_0+1}}, \quad \|v_a^{(1)}\|_{H^{N_0}},$$

with N_0 some large integer. Thus equations (3.1.7) *are semilinear equations.*

Example 2.9 (Einstein-Klein-Gordon equations)**.** *This example is extracted from [45].*

We consider the Einstein equations for an unknown spacetime (M, g):

$$Ric_{\alpha\beta} - \frac{R}{2} g_{\alpha\beta} = 8\phi T_{\alpha\beta}, \tag{2.19}$$

where $Ric^{\alpha\beta}$ denotes the Ricci curvature tensor and $R = g^{\alpha\beta} Ric_{\alpha\beta}$ denotes the scalar curvature. The matter is taken to be a massive scalar field with potential $V = V(\phi)$ and stress-energy tensor

$$T_{\alpha\beta} := \nabla_\alpha \phi \nabla_\beta \phi - \Big(\frac{1}{2} \nabla_\gamma \phi \nabla^\gamma \phi + V(\phi)\Big) g_{\alpha\beta}, \tag{2.20}$$

and specifically,

$$V(\phi) := \frac{c^2}{2}\phi^2, \tag{2.21}$$

where $c^2 > 0$ represents the mass of the scalar field. By applying ∇^α to (2.19) and using the Bianchi identity

$$\nabla^\alpha\Big(Ric_{\alpha\beta} - \frac{R}{2}g_{\alpha\beta}\Big) = 0,$$

one checks that the Einstein-scalar field system implies

$$\begin{aligned} Ric_{\alpha\beta} &= 8\pi\big(\nabla_\alpha\phi\nabla_\beta\phi + V(\phi)g_{\alpha\beta}, \\ \Box_g\phi &= V'(\phi), \end{aligned} \tag{2.22}$$

in which $\Box_g := g^{\alpha\beta}\partial_\alpha\partial_\beta$. A suitable choice of coordinates make the first equation also a system of quasilinear wave equations.

Example 2.10 (Standard model of electroweak interactions)**.** *This example is extracted from Chapter 2, see also [15].*

We need some notations for the fields of the theory. Let $\sigma_a \in \mathbb{C}^{2\times 2}$ be the standard Pauli matrices where $a = 1, 2, 3$, which form a basis for the Lie algebra $\mathfrak{su}(2)$ which we normalize by defining $\tau_a := \frac{1}{2}\sigma_a \in \mathbb{C}^{2\times 2}$. The five fields in the theory are the following:

$\Phi \in \mathbb{C}^2$	*$SU(2)$-doublet Higgs field,*
$Y = (Y^\mu)$	*$U(1)$-gauge field,*
$W = (W_a^\mu)$	*$SU(2)$-gauge field,*
$(e_L, e_R) \in \mathbb{C}^2 \times \mathbb{C}^2$	*(left- and right-handed) electron spinor,*
$\nu_L \in \mathbb{C}^2$	*left-handed neutrino spinor.*

(2.23)

Derived from the Lagrangian density of electroweak standard model, the evolution equations are

$$\begin{aligned} D_\mu D^\mu\Phi &= V'(\Phi^*\Phi)\Phi + g_e e_R^* L, \\ D_\nu F^{\mu\nu} &= -\frac{g_Y}{2}j^\mu, \\ D_\nu H_a^{\mu\nu} &= -g_W\, J_a^\mu, \\ i\sigma_R^\mu D_\mu e_R &= g_e\Phi^* L, \\ i\sigma_L^\mu D_\mu L &= g_e\Phi e_R, \end{aligned} \tag{2.24}$$

in which

$$\begin{aligned} j^\nu &:= L^*\sigma_L^\nu L + 2e_R^*\sigma^\nu e_R + i(D^\nu\Phi)^*\Phi - i\Phi^*(D^\nu\Phi), \\ J_a^\nu &:= Re[L^*\sigma_L^\nu\tau_a L] - i(D^\nu\Phi)^*\tau_a\Phi + i\Phi^*\tau_a(D^\nu\Phi). \end{aligned} \tag{2.25}$$

We introduce various notations we use in the system of equations

$$\begin{aligned} H_{\mu\nu a} &:= \partial_\mu W_{\nu a} - \partial_\nu W_{\mu a} + g_W c_{abc} W_{\mu b} W_{\nu c}, \\ F_{\mu\nu} &:= \partial_\mu Y_\nu - \partial_\nu Y_\mu. \end{aligned} \tag{2.26}$$

Here the structure constants c_{abc} for $SU(2)$ are the usual 3-dimensional totally anti-symmetric Levi-Civita symbols normalised so that $c_{123} \equiv 1$. We have used the following covariant derivatives

$$\begin{aligned} D_\mu \Phi &:= \big(\partial_\mu - i g_W W_{\mu a} \tau_a + i \frac{g_Y}{2} Y_\mu\big)\Phi, \\ D_\mu e_R &:= \big(\partial_\mu - i g_Y Y_\mu\big) e_R, \\ D_\mu L &:= \big(\partial_\mu - i g_W W_{\mu a} \tau_a - i \frac{g_Y}{2} Y_\mu\big) L. \end{aligned} \tag{2.27}$$

For completeness the covariant derivatives on the field tensors $F^{\mu\nu}$ and $H_a^{\mu\nu}$ are the following

$$\begin{aligned} D_\mu F^{\nu\rho} &:= \partial_\mu F^{\nu\rho}, \\ D_\mu H_a^{\nu\rho} &:= \partial_\mu H_a^{\nu\rho} + g_W c_{abc} W_{\mu b} H_c^{\nu\rho}. \end{aligned} \tag{2.28}$$

3 Revisit of the hyperboloidal foliation method

3.1 Hyperboloidal foliation of Minkowski spacetime

Instead of considering the flat foliation, we use the hyperboloidal foliation of the Minkowski spacetime, which for instance was introduced and used by Klainerman [41] and LeFloch-Ma [44]. We need to introduce and study the energy functional for wave or Klein-Gordon components on hyperboloids, and it is necessary to first recall some notations from [44] concerning the hyperboloidal foliation method. We consider here the $(3+1)$-dimensional Minkowski spacetime with signature $(-,+,+,+)$, and in Cartesian coordinates we adopt the notation of one point $(t,x) = (x^0, x^1, x^2, x^3)$, with its spatial radius $r := |x| = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}$. Partial derivatives are denoted by ∂_α (for $\alpha = 0, 1, 2, 3$), and

$$L_a := x^a \partial_t + t \partial_a, \qquad a = 1, 2, 3 \tag{3.1}$$

represent the Lorentz boosts. Throughout, the functions considered are defined in the interior of the future light cone

$$\mathcal{K} := \{(t,x) : r < t - 1\},$$

whose vertex is $(1,0,0,0)$. We denote the hyperboloidal hypersurfaces by

$$\mathcal{H}_s := \{(t,x) : t^2 - r^2 = s^2\}, \qquad s > 1.$$

Besides the subsets of $\mathcal{K}$ limited by two hyperboloids $\mathcal{H}_{s_0}$ and $\mathcal{H}_{s_1}$ with $s_0 \leqslant s_1$ are denoted by

$$\mathcal{K}_{[s_0,s_1]} := \{(t,x) : s_0^2 \leqslant t^2 - r^2 \leqslant s_1^2; r < t-1\}.$$

We now introduce the semi-hyperboloidal frame

$$\underline{\partial}_0 := \partial_t, \qquad \underline{\partial}_a := \frac{L_a}{t} = \frac{x^a}{t}\partial_t + \partial_a. \tag{3.2}$$

We notice that the vectors $\underline{\partial}_a$ generate the tangent space of the hyperboloids. Besides, the vector field

$$\underline{\partial}_\perp := \partial_t + (x^a/t)\partial_a$$

is orthogonal to the hyperboloids and is proportional to the scaling vector field $S := t\partial_t + x^a\partial_a$.

The dual of the semi-hyperboloidal frame is given by

$$\underline{\theta}^0 := dt - (x^a/t)dx^a, \qquad \underline{\theta}^a := dx^a.$$

The (dual) semi-hyperboloidal frame and the (dual) natural Cartesian frame are connected by the following relations

$$\begin{aligned} \underline{\partial}_\alpha &= \Phi_\alpha^{\alpha'}\partial_{\alpha'}, \qquad & \partial_\alpha &= \Psi_\alpha^{\alpha'}\underline{\partial}_{\alpha'}, \\ \underline{\theta}^\alpha &= \Psi_{\alpha'}^{\alpha} dx^{\alpha'}, \qquad & dx^\alpha &= \Phi_{\alpha'}^{\alpha}\underline{\theta}^{\alpha'}, \end{aligned} \tag{3.3}$$

where the transition matrix (Φ_α^β) and its inverse (Ψ_α^β) are given by

$$(\Phi_\alpha^\beta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ x^1/t & 1 & 0 & 0 \\ x^2/t & 0 & 1 & 0 \\ x^3/t & 0 & 0 & 1 \end{pmatrix} \tag{3.4}$$

and

$$(\Psi_\alpha^\beta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -x^1/t & 1 & 0 & 0 \\ -x^2/t & 0 & 1 & 0 \\ -x^3/t & 0 & 0 & 1 \end{pmatrix}. \tag{3.5}$$

3.2 Energy estimates on hyperboloids

Following [45] and considering in the Minkowski background, we introduce the energy functional E_m for a nice function $\phi = \phi(t,x)$ defined on the hyperboloid $\mathcal{H}_s$

$$E_m(s,\phi) := \int_{\mathcal{H}_s} \Big(\big(\partial_t \phi\big)^2 + \sum_a \big(\partial_a \phi\big)^2 + 2(x^a/t)\partial_t\phi\partial_a\phi + m^2\phi^2 \Big)\, dx, \tag{3.6}$$

which has two other equivalent (and more useful) expressions

$$\begin{aligned} E_m(s,\phi) &= \int_{\mathcal{H}_s} \Big(\big((s/t)\partial_t \phi\big)^2 + \sum_a \big(\underline{\partial}_a \phi\big)^2 + m^2\phi^2 \Big)\, dx \\ &= \int_{\mathcal{H}_s} \Big(\big(\underline{\partial}_\perp \phi\big)^2 + \sum_a \big((s/t)\partial_a \phi\big)^2 + \sum_{a<b} \big(t^{-1}\Omega_{ab}\phi\big)^2 + m^2\phi^2 \Big)\, dx, \end{aligned} \tag{3.7}$$

in which

$$\Omega_{ab} := x^a\partial_b - x^b\partial_a$$

are the rotational vector fields, and $\underline{\partial}_\perp = \partial_t + (x^a/t)\partial_a$ is the orthogonal vector field. It is helpful to point it out that each term in the expressions (3.7) are non-negative, which is vital in estimating the energies of wave or Klein-Gordon equations. We use the notation

$$E(s,\phi) := E_0(s,\phi)$$

for simplicity. In the above, the integral in $L^1(\mathcal{H}_s)$ is defined from the standard (flat) metric in $\mathbb{R}^3$, i.e.

$$\|\phi\|_{L^1_f(\mathcal{H}_s)} := \int_{\mathcal{H}_s} |\phi|\, dx = \int_{\mathbb{R}^3} \big|\phi(\sqrt{s^2+r^2},x)\big|\, dx. \tag{3.8}$$

Next, we recall the energy estimates for the wave-Klein-Gordon equations.

Proposition 3.1 (Energy estimates for wave-Klein-Gordon equations)**.** *For all $m \geqslant 0$ and $s \geqslant 2$, it holds that*

$$E_m(s,u)^{1/2} \leqslant E_m(2,u)^{1/2} + \int_2^s \| -\Box u + m^2 u\|_{L^2_f(\mathcal{H}_{s'})}\, ds' \tag{3.9}$$

for all sufficiently regular function u, which is defined and supported in the region $\mathcal{K}_{[2,s]}$.

For the proof, one refers to [45].

3.3 Estimates for commutators and null forms

We briefly review how the commutators and null forms are estimated, but the precise statements are not given here.

Heuristically speaking, the following relations are valid:

$$\begin{aligned} \big|\partial\partial^I L^J\phi\big| &\simeq \big|\partial^I L^J\partial\phi\big|, \\ \big|\partial^I L^J(T^{\alpha\beta}\partial_\alpha\phi\partial_\beta\psi)\big| &\lesssim (s/t)\big|\partial^I L^J(\partial\phi\partial\psi)\big|, \end{aligned} \tag{3.10}$$

in which ϕ, ψ are two good functions, and $T^{\alpha\beta}$ is a null form.

4 Motivation and organisation of the thesis

4.1 Overview and the co-authorships

As we described above, the greatest interest of studying nonlinear wave equations lies in proving the existence of global-in-time solutions and in applying this result to some models in physics if possible. Hence we will mainly deal with the coupled wave-Klein-Gordon-type equations from the $U(1)$ electroweak standard model, the Dirac-Proca system, the Klein-Gordon-Zakharov model and beyond. On the other hand, we also focus on equations of possibly only mathematical interest or on some other aspects in the field of analysis.

In Chapters 1–2, the work is joint with Philippe G. LeFloch and Zoe Wyatt; in Chapter 3, the work is joint with Zoe Wyatt; in Chapters 4, 5 and 6, the work is joint with Philippe G. LeFloch.

4.2 Main work in Part I

In this first part, we study the evolution equations for electromagnetic and weak interactions between elementary particles coming from the Standard Model in physics. In particular we first treat in Chapter 1 the Higgs mechanism applied to an abelian $U(1)$ model, where we study the Euler-Lagrangian equations and prove the global stability results for this model. Next in Chapter 2 we will look at the Higgs mechanism applied to the more complicated non-abelian $SU(2) \times U(1)$ gauge group of the full Glashow-Weinberg-Salam (GSW) model, where we will only treat some preliminary aspects of the system of equations of this full model, but the proof of the global stability results is still ongoing. Nevertheless, we prove the global stability results for some subcases, which are obtained by setting certain parameters to be zero.

In Chapter 1 we study the equations coming from the $U(1)$ electroweak standard

model, which comprise Klein-Gordon equations as well as massive and massless Dirac equations. A simplified illustration of the model is

$$\begin{aligned} -\Box v + v &= \psi^* \psi, \\ -i\gamma^\mu \partial_\mu \psi + m_g \psi &= v\psi, \end{aligned} \tag{4.1}$$

in which $m_g \geqslant 0$, and ψ is the Dirac field. In particular, we investigate there the Dirac equation and its hyperboloidal energy functional. We also provide a decay result for the Dirac equation which is uniform in the mass coefficient $m_g \in [0, 1]$

$$|\psi| \lesssim \min\{t^{-1}, m_g^{-1} t^{-3/2}\}, \tag{4.2}$$

and thus allow for the mass coefficient to be arbitrarily small. Our energy bounds are uniform (modulo a logarithm growth) with respect to the hyperboloidal time variable.

In Chapter 2 we first derive the system of equations from the Lagrangian density of the GSW theory. Next we do several linear combinations on the fields to disclose their nature, for example we transform the four mixed vague gauge vector fields to three massive vector bosons and one massless vector photon. Later on we make some gauge choice and prove it is propagated by the system of equations. Besides, we extract some simplified models by letting certain parameters vanish, and prove these models are globally stable.

4.3 Main work in Part II

In Part II (Chapters 3–4) , we prove the stability results for a series of wave-Klein-Gordon equations in dimension 3.

In Chapter 3, motivated by the Klein-Gordon-Zakharov model, we study a more general coupled wave-Klein-Gordon system of the form

$$\begin{aligned} -\Box v + v &= uv, \\ -\Box u &= uv + u\partial v. \end{aligned} \tag{4.3}$$

Compared with the Klein-Gordon-Zakharov model as demonstrated in Example 2.8, we do not need to require the nonlinearities in the wave equation are of divergence form. Note that the Klein-Gordon-Zakharov equations are essentially of the form

$$\begin{aligned} -\Box v' + v' &= \partial u' v', \\ -\Box u' &= v'^2, \end{aligned} \tag{4.4}$$

which is easier to handle since $\partial u'$ behaves better than u' when u' is a wave component. However the system (4.3) we consider is more subtle to deal with, which is due to the absence of derivatives on the wave components. Relying on a careful analysis and an

induction argument, we will prove the existence of global-in-time solution to (4.3) which enjoys sharp pointwise decay property

$$|u| \lesssim t^{-1}, \qquad |v| \lesssim t^{-3/2}. \tag{4.5}$$

In Chapter 4, we prove a unified pointwise decay result, for a Klein-Gordon system, with respect to its mass. This is motivated by the study of the electroweak $U(1)$ model in Chapter 1, where we obtain a unified pointwise decay result on the Dirac component, also regarding its possibly vanishing mass. We consider the system

$$\begin{aligned} -\Box u + m^2 u &= P^\alpha \partial_\alpha \big(uv + v^2 - u^3\big), \\ -\Box v + v &= u^2 + uv, \end{aligned} \tag{4.6}$$

in which the mass $m \in [0, 1]$. We note the nonlinearities in the u equation are of divergence form, hence the study of (4.6) can be regarded as a generalisation of the Dirac equations in this direction. The difficulties here include that: 1) we cannot use the L^2-type estimates and sup-norm estimates obtained from the energy estimates on the mass term $m^2 u$, which is due to the bad factor m^{-1}; 2) unless $m = 0$, the scaling and the conformal vector fields do not commute with the operator $-\Box + m^2$; 3) the sup-norm estimate for the wave equation introduced in [2] and [45] cannot be applied any longer here. By a simple employment of the Fourier analysis method, we are able to get the unified estimates on some part of u component. Since the rest part is easier to handle, we finally obtain

$$|u| \lesssim \min\{t^{-1}, m^{-1}t^{-3/2}\}. \tag{4.7}$$

4.4 Main work in Part III

This part is devoted to obtain a new L^2-type of estimates for the wave components on the hyperboloids, where we rely on the Fourier transform method in the hyperbolic space.

As we mentioned that in recent years, LeFloch and Ma [44, 45] demonstrated the relevance of hyperbolic foliations for the study of nonlinear wave problems and, for several classes of other systems. In order to have a complementary understanding of that method, a natural idea is to investigate the wave equation by using the Fourier transform method on the hyperboloids, which we call the hyperbolic Fourier transform, and to see whether we can obtain new properties of wave equations on the hyperboloids. Following [6, 29], the hyperbolic Fourier transform of a nice function $\phi = \phi(t, x)$ on the

hyperboloid $\mathcal{H}_s$ is defined by

$$\begin{aligned}&\widehat{\phi}(s,\lambda,\omega) := s^{-i\lambda s+m}\int_{\mathcal{H}_s}\langle -(t,x),(1,\omega)\rangle^{i\lambda s-m}\phi(t,x)\,d\mathcal{H}_s,\\&(\lambda,\omega)\in\mathbb{R}_+\times S^{n-1}\subset\mathbb{R}^{n+1},\end{aligned}\tag{4.8}$$

in which

$$\langle X,Y\rangle := -X^0Y^0+\sum_a X^aY^a,\qquad X,Y\in\mathbb{R}^{n+1},$$

and

$$\int_{\mathcal{H}_s}\phi(t,x)\,d\mathcal{H}_s=\int_{\mathbb{R}^3}\phi(\sqrt{s^2+|x|^2},x)\frac{s}{\sqrt{s^2+|x|^2}}\,dx.\tag{4.9}$$

Interesting enough, we indeed discover a new L^2-type estimate for wave components. To the best of our acknowledge, the existing methods on obtaining L^2-type estimates for waves on the hyperboloids include: 1) to apply Hardy inequality on the wave components, see for instance [45]; 2) to rely on the conformal vector field and the conformal energy estimates, which was first introduced by Morawetz [63] in the flat setting, see [61] for the efforts in the hyperboloidal setting.

As a simple application, we prove the existence of global-in-time solutions to the semilinear wave equation

$$-\Box u=u^3$$

in $\mathbb{R}^{1+3}$.

In Chapter 6, mimicing the proof in [57] for the 1-dimensional wave equation with null nonlinearities

$$-\partial_t\partial_t u+\partial_x\partial_x u=(\partial_t u)^2-(\partial_x u)^2,\tag{4.10}$$

we simplify a little bit of their proof, i.e. we do not need to estimate the energy on the null segments. We conduct the analysis in the hyperbolic space, and prove the existence of the global-in-time solution u. In the end, based on the new L^2-type estimates in Chapter 5 we also provide a sup-norm estimate on the solution u

$$|u|\lesssim C_1\epsilon t^{\delta}.\tag{4.11}$$

Part I

Evolution of the Higgs Boson: Global nonlinear stability and energy bounds

Chapter 1

Study of the U(1) model

[1]

Contents

[1]This chapter is a joint work with Philippe G. LeFloch and Zoe Wyatt.

1.1 Introduction

Main objective. Our primary objective is to study the equations of motion arising from the Higgs mechanism applied to an abelian $U(1)$ gauge theory on a Minkowski background after spontaneous symmetry breaking. We view this model as a stepping-stone towards the full non-abelian Glashow-Weinberg-Salam theory (GSW), also known as the electroweak Standard Model. For background physics information on the models treated in this chapter, see for example [1].

In short, we study here a class of nonlinear wave equations which involve the first-order Dirac equation coupled to second-order wave or Klein-Gordon equations. We are interested in the initial value problem for such systems, when the initial data have sufficiently small Sobolev-type norm. We provide here a new application of the hyperboloidal foliation method introduced for such coupled systems by LeFloch and Ma [44], which has been successfully used to establish global-in-time existence results for nonlinear systems of coupled wave and Klein-Gordon equations. This method takes its root in pioneering work by Friedrich [19, 18] on the vacuum Einstein equations and by Klainerman [38, 41] and Hörmander [30] on the (uncoupled) Klein-Gordon equation, as well as Katayama [35, 34]. Taking into account nonlinear interaction terms that couple wave and Klein-Gordon equations together, was a challenge tackled in [44] and in the subsequent developments on the Einstein equations [45, 49, 48] and [17, 68].

The model of interest. In the **abelian $U(1)$ gauge model**, the set of unknowns consists of a spinor field $\psi : \mathbb{R}^{3+1} \to \mathbb{C}^4$ representing a fermion of mass m_g with spin 1/2, a vector field $A = (A^\mu)$ representing a massive boson of mass m_q with spin 1, and a complex scalar field χ representing the perturbation from the constant minimum ϕ_0 of the complex Higgs field $\phi = \phi_0 + \chi$. Given the ground state ϕ_0 with norm denoted

by $v^2 := \phi_0^*\phi_0$ and the three physical parameters m_q, m_λ, m_g, the equations of motion in a modified Lorenz gauge consist of three evolution equations

$$\begin{aligned} \big(\Box - m_q^2\big)A^\nu &= Q_{A^\nu}, \\ \Box\chi - m_q^2\,\frac{\phi_0}{2v^2}\big(\phi_0^*\chi - \chi^*\phi_0\big) - m_\lambda^2\,\frac{\phi_0}{2v^2}\big(\phi_0^*\chi - \chi^*\phi_0\big) &= Q_\chi, \\ i\gamma^\mu\partial_\mu\psi - m_g\psi &= Q_\psi, \end{aligned} \tag{1.1.1a}$$

and a constraint equation

$$\mathrm{div}A + i\frac{m_q}{v\sqrt{2}}\big(\phi_0^*\chi - \chi^*\phi_0\big) = 0. \tag{1.1.1b}$$

Here, the quadratic nonlinearities Q_{A^ν}, Q_χ, and Q_ψ are to be defined later, and ϕ_0 is a constant complex number with magnitude $|\phi_0| = v$. Throughout, we use the signature $(-,+,+,+)$ for the wave operator $\Box := \eta^{\mu\nu}\partial_\mu\partial_\nu = -\partial_t^2 + \Delta$, while the matrices γ^μ and γ_5 are the standard Dirac matrices (cf. Section 1.2.1).

Some important physical parameters are provided together with the mass coefficients above, namely

$$m_q^2 := 2q^2v^2 > 0, \qquad m_\lambda^2 := 4\lambda v^2 > 0, \qquad m_g := gv^2 \geqslant 0, \tag{1.1.2}$$

themselves depend on given **coupling constants**, denoted by λ, g, q, as well as the so-called "vacuum expectation value" of the Higgs field denoted by v.

The nonlinear stability of the Higgs field. Our main result concerning the system (1.1.1) is a proof of the global-in-time existence of solutions for sufficiently small perturbations away from the **constant vacuum state**, defined by the conditions

$$A^\mu \equiv 0, \quad \phi \equiv \phi_0, \quad \psi \equiv 0. \tag{1.1.3}$$

It is convenient to work with the perturbed Higgs field $\chi = \phi - \phi_0$ as our main unknown. The initial data set are denoted by

$$\big(A^\nu, \chi, \psi\big)(t_0,\cdot) = \big(A_0^\nu, \chi_0, \psi_0\big), \qquad \big(\partial_t A^\nu, \partial_t\chi\big)(t_0,\cdot) = \big(A_1^\nu, \chi_1\big), \tag{1.1.4}$$

and these data are said to be **Lorenz compatible** if

$$\begin{aligned} \partial_a A_0^a &= -A_1^0 - iq\big(\phi_0^*\chi_0 - \chi_0^*\phi_0\big), \\ \Delta A_0^0 - m_q^2 A_0^0 &= -\partial_i A_1^i - iq\big(\phi_0^*\chi_1 - \chi_1^*\phi_0\big) + iq\big(\chi_0^*\chi_1 - \chi_1^*\chi_0\big) \\ &\quad + 2q^2A_0^0\big(\phi_0^*\chi_0 + \chi_0^*\phi_0 + \chi_0^*\chi_0\big) + q\psi_0^*\psi_0. \end{aligned} \tag{1.1.5}$$

This elliptic-type system consists of two equations for 11 functions. In particular, it is easily checked that it admits non-trivial solutions, for instance with compact support. Observe also that throughout we use the convention that Greek indices take values in $\{0,1,2,3\}$ and Latin indices in $\{1,2,3\}$. In the following statement, we have $m_g \geqslant 0$, and the coefficient m_g^{-1} is interpreted as $+\infty$ when $m_g = 0$.

Theorem 1.1.1 (Nonlinear stability of the ground state for the Higgs boson)**.** *Consider the system* (1.1.1) *with parameters* $m_q, m_\lambda > 0$ *and* $m_g \in [0, \min(m_q, m_\lambda)]$*, and let* N *be a sufficiently large integer. There exists* $\epsilon_0 > 0$*, which is independent of* m_g*, such that for all* $\epsilon \in (0, \epsilon_0)$ *and all compactly supported, Lorenz compatible initial data (in the sense of* (1.1.5)*) satisfying the smallness condition*

$$\|A_0, \chi_0, \psi_0\|_{H^{N+1}(\mathbb{R}^3)} + \|A_1, \chi_1\|_{H^N(\mathbb{R}^3)} \leqslant \epsilon, \tag{1.1.6}$$

the initial value problem of (1.1.1) *admits a global-in-time solution* (A, χ, ψ) *with, moreover,*

$$|A| \lesssim \epsilon t^{-3/2}, \qquad |\chi| \lesssim \epsilon t^{-3/2}, \qquad |\psi| \lesssim \epsilon \min\left(t^{-1}, m_g^{-1} t^{-3/2}\right). \tag{1.1.7}$$

A simpler model: the Dirac-Proca equations. Keeping (1.1.1) in mind, it is interesting to first understand a simplified subset of (1.1.1), called the Dirac-Proca equations. In the Lorenz gauge the equations of motion for this model read

$$\begin{aligned} \Box A^\nu - m^2 A^\nu &= -\psi^* \gamma^0 \gamma^\nu (P_L \psi), \\ -i\gamma^\mu \partial_\mu \psi + M\psi &= -\gamma^\mu A_\mu (P_L \psi), \end{aligned} \tag{1.1.8}$$

where $P_L = \frac{1}{2}(I_4 - \gamma^5)$. This system describes a spinor field $\psi : \mathbb{R}^{3+1} \to \mathbb{C}^4$ representing a fermion of mass M with spin $1/2$ and a vector field $A = (A^\mu)$ representing a massive boson of mass m with spin 1. We allow the mass parameters M, m to be positive or zero and, in particular, we will be interested in the case $M = 0$ (i.e. the standard Dirac-Proca equations) as well as in the case $M > 0$ (massive field). Observe that the Dirac-Proca system (1.1.8) contains no Higgs field and so the masses M, m are introduced 'artificially" in the model, instead of arising from the Higgs mechanism.

A set of initial data set for (1.1.8), say,

$$\left(A^\nu, \partial_t A^\nu, \psi\right)(t_0, \cdot) = \left(a^\nu, b^\nu, \psi_0\right), \tag{1.1.9}$$

is called Lorenz compatible if one has

$$\begin{aligned} b^0 + \partial_j a^j &= 0, \\ \Delta a^0 - m^2 a^0 &= -\psi_0^* P_L \psi_0 - \partial_j b^j. \end{aligned} \tag{1.1.10}$$

We postpone the statement of our second main result, for this system, to Theorem 1.4.1

below.

Strategy of proof. The small data global existence problem for (1.1.8) with $m_g \equiv 0$ and for (1.1.1) was solved in Tsutsumi in [76, 77]. Yet, it is very interesting to revisit this system via the hyperboloidal foliation method developed in [44] as it provides a different perspective on the problem and somewhat sharper estimates. We thus introduce a hyperboloidal foliation which covers the interior of a light cone in Minkowski spacetime, and we then construct the solutions of interest in the future of an initial hyperboloid. For compactly-supported initial data this is equivalent to solving the initial value problem for a standard t=constant initial hypersurface.

In the case of the Dirac-Proca model, our theorem refines the results in [76] in several directions. First, the global-in-time existence is established under a lower regularity assumption on the initial data. Namely, the boundedness of the initial data in the norm $\|\cdot\|_{H^N}$ ($N \geqslant 20$) is needed in [76], while we only require $N \geqslant 6$. Most importantly, in the main statement in [76], a slow growth of the energy of high-order derivatives may take place. With our method of proof, with the exception of the highest order term, *we do not have any growth factor* in the L^2 or L^∞ norms. We refer to Section 1.7 for a further discussion of these improved estimates. In contrast to [76], for simplicity we assume that the initial data are compactly supported. Uniform energy bounds were also established in [50].

Moreover, in the case $0 \leqslant M \ll m$ we analyze the Dirac equation (expressed in a first-order form) and derive the following energy functional on hyperboloids

$$E^{\mathcal{H}}(s,\psi) := \int_{\mathcal{H}_s} \Big(\psi^*\psi - \frac{x_i}{t}\psi^*\gamma^0\gamma^i\psi\Big)\, dx. \tag{1.1.11}$$

We can show that this energy is positive definite and controls the norm $\|(s/t)\psi\|_{L^2}$, which provides us with a notion of energy on hyperboloids and should be compared with the more standard approach based on the standard t = constant hypersurfaces; cf. for example [5].

Challenges for the global analysis. There are several difficulties in dealing with the system of coupled wave–Klein–Gordon equations (1.1.1). The first and most well-known one is that the standard Klein-Gordon equation does not commute with the scaling Killing field of Minkowski spacetime, which prevents us from applying the standard vector field method in a direct manner.

Second, the nonlinearities in the Klein–Gordon equation (or the Proca equation) include a bad term describing ψ–ψ interactions, which can be regarded as a wave–wave interaction term and does not have good decay. Tsutsumi [76] was able to overcome this difficulty by defining a new variable whose quadratic nonlinearity only involves the

so-called 'strong null forms' $Q_{\mu j}$ (first introduced in [20]) compatible with the scaling vector field. A similar, but complicated new variable, was also defined in [77]. Due to our use of the hyperboloidal foliation method we do not need to find new variables leading solely to strong null forms.

Furthermore, it is challenging to establish a global stability result uniformly for all $m_g \geqslant 0$, while it is relative easy to complete the proof for either $m_g = 0$ or m_g a large constant. In the case where the mass is small, say $m_g = \epsilon^2$, if we treat ψ as a Klein-Gordon field then the field decays like

$$|\psi| \lesssim C_1 \epsilon^{-1} t^{-3/2}.$$

However with this decay we cannot arrive at the improved estimate $(1/2)C_1\epsilon$ if we start from the *a priori* estimate $C_1\epsilon$, where C_1 is some large constant introduced in the bootstrap method. This is because the "improved" estimates we find are $\epsilon + C_1^2$ instead of $\epsilon + (C_1\epsilon)^2$. Hence ψ behaves more like a wave component when $m_g \ll 1$, but since the mass m_g may be very small but non-zero, we cannot apply techniques for wave equations. We find it possible to overcome these difficulties by analysing the first–order Dirac equation, which admits a positive energy functional, and this energy plays a key role in the whole analysis.

The coupling constants. For the mass parameters (1.1.2) appearing in the $U(1)$–model we assume that

$$m_q \simeq m_\lambda > 0, \qquad m_g \geqslant 0, \tag{1.1.12}$$

which depend on the coupling constants q, g and the Higgs constants λ, v. On the other hand, for the mass parameters appearing in the Dirac–Proca model we assume[2]

$$m > 0, \qquad M \geqslant 0.$$

Outline of this chapter. In Section 1.2, we will introduce the hyperboloidal foliation method, the Dirac equation, and the energy for the Dirac component. In Sections 1.2.1 and 1.2.2 we give basic definitions for the Dirac equation and hyperboloidal foliation, the latter taken from [44]. In Section 1.2.3 we define the energy for the Dirac field on hyperboloids and give an energy estimate in Proposition 1.2.4. Note this will be complemented by other formulations of the energy functional in Section 1.3. Finally in Section 1.2.4 we convert the Dirac equation into a second order wave/Klein-Gordon equation and define the appropriate energy functional. Further properties about the hyperboloidal energy functionals are established in Section 1.3. Therein we provide complementing views on the Dirac energy $E^{\mathcal{H}}$. The first comes from using a Cholesky

[2]There is no Higgs field to generate the masses, and we do not need to use subscripts for the mass coefficients.

decomposition for the energy integrand in Section 1.3.1 and next using a Weyl decomposition for the Dirac spinor in Section 1.3.2. Then we study the Dirac–Proca model and the abelian model in Section 1.4 and Section 1.5, respectively. Next, in Section 1.6 we discuss the system of equations (1.1.1) and its nonlinearities are studied. Finally, in Section 1.7 we rely on a bootstrap argument and prove the desired the stability result.

1.2 Hyperboloidal energy functionals for the Dirac operator

1.2.1 Dirac spinors and matrices

This section is devoted to analyzing energy functionals for the Dirac equation with respect to a hyperboloid foliation of Minkowski spacetime. Recall the Dirac equation for the unknown $\psi : \mathbb{R}^{3+1} \to \mathbb{C}^4$

$$-i\gamma^\mu \partial_\mu \psi + M\psi = F, \tag{1.2.1}$$

with prescribed $\mathbb{C}^4$-valued right-hand side F and mass $M \in \mathbb{R}$. To make sense of this equation we need to define various complex vectors and matrices.

For a complex vector $z = (z_0, z_1, z_2, z_3)^T \in \mathbb{C}^4$ let $\bar{z}$ denote the conjugate, and $z^* := (\bar{z}_0, \bar{z}_1, \bar{z}_2, \bar{z}_3)$ denotes the conjugate transpose. If also $w \in \mathbb{C}^4$ then the conjugate inner product is defined by

$$\langle z, w \rangle := z^* w = \sum_{\alpha=1}^{4} \bar{z}_\alpha w_\alpha.$$

The Hermitian conjugate[3] of a matrix A is denoted by A^*, meaning

$$(A^*)_{\alpha\beta} := (\bar{A})_{\beta\alpha}.$$

The Dirac matrices γ^μ for $\mu = 0, 1, 2, 3$ are 4×4 matrices satisfying the identities

$$\begin{aligned} &\{\gamma^\mu, \gamma^\nu\} := \gamma^\mu\gamma^\nu + \gamma^\mu\gamma^\nu = -2\eta^{\mu\nu} I, \\ &(\gamma^\mu)^* = -\eta_{\mu\nu}\gamma^\nu, \end{aligned} \tag{1.2.2}$$

where $\eta = \mathrm{diag}(-1, 1, 1, 1)$. The Dirac matrices give a matrix representation of the Clifford algebra. We will use the Dirac representation, so that the Dirac matrices take

[3] In the physics literature, A^* is often denoted by $A^\dagger$.

the following form

$$\gamma^0 = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \tag{1.2.3}$$

where σ^i's are the standard Pauli matrices:

$$\sigma^1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1.2.4}$$

One often uses the following product of the Gamma matrices

$$\gamma_5 := i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}. \tag{1.2.5}$$

The γ_5 matrix squares to I_4 and so we also define the following projection operators

$$P_L := \frac{1}{2}\left(I_4 - \gamma_5\right), \qquad P_R := \frac{1}{2}\left(I_4 + \gamma_5\right) \tag{1.2.6}$$

to extract the 'left-handed' and 'right-handed' parts of a spinor, that is, to extract its chiral parts. We also note the following useful identities[4]

$$(\gamma^\mu)^* = \gamma^0\gamma^\mu\gamma^0, \quad (\gamma^0\gamma^\mu)^* = \gamma^0\gamma^\mu, \quad \{\gamma_5, \gamma^\mu\} = 0. \tag{1.2.7}$$

For further details on Dirac matrices and the representation considered, see [1].

1.2.2 Hyperboloidal foliation of Minkowski spacetime

In order to introduce the energy formula of the Dirac component ψ on hyperboloids, we first need to recall some notation from [44] concerning the hyperboloidal foliation method. We denote the point $(t, x) = (x^0, x^1, x^2, x^3)$ in Cartesion coordinates, with its spatial radius $r := |x| = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}$. We write $\partial_\alpha := \partial_{x^\alpha}$ (for $\alpha = 0, 1, 2, 3$) for partial derivatives and

$$L_a := x^a\partial_t + t\partial_a, \qquad a = 1, 2, 3 \tag{1.2.8}$$

for the Lorentz boosts. Throughout this chapter, we consider functions defined in the interior of the future light cone $\mathcal{K} := \{(t, x) : r < t - 1\}$, with vertex $(1, 0, 0, 0)$. We consider hyperboloidal hypersurfaces $\mathcal{H}_s := \{(t, x) : t^2 - r^2 = s^2\}$ with $s > 1$. Also $\mathcal{K}_{[s_0,s_1]} := \{(t, x) : s_0^2 \leqslant t^2 - r^2 \leqslant s_1^2; r < t - 1\}$ is used to denote subsets of $\mathcal{K}$ limited by two hyperboloids.

[4] In physics, the notation $\bar{\psi} := \psi^\dagger\gamma^0$ is often used, but we will avoid this here.

The semi-hyperboloidal frame is defined by

$$\underline{\partial}_0 := \partial_t, \qquad \underline{\partial}_a := \frac{L_a}{t} = \frac{x^a}{t}\partial_t + \partial_a. \tag{1.2.9}$$

Observe that the vectors $\underline{\partial}_a$ generate the tangent space to the hyperboloids. We also introduce the vector field $\underline{\partial}_\perp := \partial_t + \frac{x^a}{t}\partial_a$ which is orthogonal to the hyperboloids.

For this semi-hyperboloidal frame above, the dual frame is given by $\underline{\theta}^0 := dt - \frac{x^a}{t}dx^a$, $\underline{\theta}^a := dx^a$. The (dual) semi-hyperboloidal frame and the (dual) natural Cartesian frame are connected by the relation

$$\underline{\partial}_\alpha = \Phi_\alpha^{\alpha'}\partial_{\alpha'}, \quad \partial_\alpha = \Psi_\alpha^{\alpha'}\underline{\partial}_{\alpha'}, \quad \underline{\theta}^\alpha = \Psi_{\alpha'}^{\alpha} dx^{\alpha'}, \quad dx^\alpha = \Phi_{\alpha'}^{\alpha}\underline{\theta}^{\alpha'}, \tag{1.2.10}$$

where the transition matrix (Φ_α^β) and its inverse (Ψ_α^β) are given by

$$(\Phi_\alpha^\beta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ x^1/t & 1 & 0 & 0 \\ x^2/t & 0 & 1 & 0 \\ x^3/t & 0 & 0 & 1 \end{pmatrix}, \qquad (\Psi_\alpha^\beta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -x^1/t & 1 & 0 & 0 \\ -x^2/t & 0 & 1 & 0 \\ -x^3/t & 0 & 0 & 1 \end{pmatrix}. \tag{1.2.11}$$

Throughout, we use roman font E to denote energies coming from a first-order PDE (see below) and, calligraphic font $\mathcal{E}$ to denote energies coming from a second-order PDE. In the Minkowski background, we first introduce the energy $\mathcal{E}$ for scalar-valued or vector-valued maps ϕ defined on a hyperboloid $\mathcal{H}_s$:

$$\begin{aligned} \mathcal{E}_m(s,\phi) &:= \int_{\mathcal{H}_s} \Big(|\partial_t\phi|^2 + \sum_a |\partial_a\phi|^2 + (x^a/t)(\partial_t\phi^*\partial_a\phi + \partial_t\phi\partial_a\phi^*) + m^2|\phi|^2 \Big)\, dx \\ &= \int_{\mathcal{H}_s} \Big(|(s/t)\partial_t\phi|^2 + \sum_a |\underline{\partial}_a\phi|^2 + m^2|\phi|^2 \Big)\, dx \\ &= \int_{\mathcal{H}_s} \Big(|\underline{\partial}_\perp\phi|^2 + \sum_a |(s/t)\partial_a\phi|^2 + \sum_{a<b} |t^{-1}\Omega_{ab}\phi|^2 + m^2|\phi|^2 \Big)\, dx, \end{aligned} \tag{1.2.12}$$

where

$$\Omega_{ab} := x^a\partial_b - x^b\partial_a \tag{1.2.13}$$

denotes the rotational vector field. We also write $\mathcal{E}(s,\phi) := \mathcal{E}_0(s,\phi)$ for simplicity. All of our integrals in L^1, L^2, etc. are defined from the standard (flat) metric in $\mathbb{R}^3$, so

$$\|\phi\|_{L^1_f(\mathcal{H}_s)} = \int_{\mathcal{H}_s} |\phi|\, dS := \int_{\mathbb{R}^3} \big|\phi(\sqrt{s^2+r^2}, x)\big|\, dx. \tag{1.2.14}$$

1.2.3 Hyperboloidal energy of the Dirac equation

We now derive a hyperboloidal energy for the Dirac equation (1.2.1). Premultiplying the PDE (1.2.1) by $\psi^*\gamma^0$ gives

$$\psi^*\partial_0\psi + \psi^*\gamma^0\gamma^j\partial_j\psi + iM\psi^*\gamma^0\psi = i\psi^*\gamma^0 F\,. \tag{1.2.15}$$

The conjugate of (1.2.1) is

$$(\partial_\mu\psi^*)(\gamma^\mu)^* - i\psi^* M = -iF^*.$$

Multiplying this equation by ψ gives

$$(\partial_0\psi^*)\psi + (\partial_j\psi^*)\gamma^0\gamma^j\psi - iM\psi^*\gamma^0\psi = -iF^*\gamma^0\psi. \tag{1.2.16}$$

Adding (1.2.15) and (1.2.16) together yields

$$\partial_0(\psi^*\psi) + \partial_j(\psi^*\gamma^0\gamma^j\psi) = i\psi^*\gamma^0 F - iF^*\gamma^0\psi. \tag{1.2.17}$$

Note the mass term does not appear in (1.2.17). Moreover recalling $2\mathrm{Re}[z] = z + \bar{z}$ for some $z \in \mathbb{C}$ then we see that (1.2.17) is the real part of (1.2.15). It would appear however if we subtracted (1.2.15) from (1.2.16), that is the imaginary part of (1.2.15), then we find

$$\psi^*\partial_0\psi - \partial_0\psi^*\cdot\psi + \psi^*\gamma^0\gamma^j\partial_j\psi - \partial_j\psi^*\cdot\gamma^0\gamma^j\psi + 2iM\psi^*\gamma^0\psi = i\psi^*\gamma^0 F + iF^*\gamma^0\psi.$$

However such an expression does not appear to be useful. We return to (1.2.17) and integrate over regions in spacetime to obtain energy inequalities. These give the following two definitions of first-order energy functionals.

Integrating (1.2.17) over $[t_0, t] \times \mathbb{R}^3$, and assuming spatially compactly supported initial data, gives the following result, see for instance [78].

Lemma 1.2.1. *On $t = const$ slices, define the energy*

$$E^{flat}(t,\psi) := \int_{\mathbb{R}^3} (\psi^*\psi)(t,x)dx. \tag{1.2.18}$$

Then it holds

$$E^{flat}(t,\psi) = E^{flat}(t_0,\psi) + \int_{t_0}^{t}\int_{\mathbb{R}^3} \left(i\psi^*\gamma^0 F - iF^*\gamma^0\psi\right) dxdt. \tag{1.2.19}$$

Such functionals on a constant-time foliation have been considered frequently in the literature, see for instant [10]. The following Lemma gives a new perspective using a hyperboloidal foliation.

Lemma 1.2.2. *On hyperboloidal slices $\mathcal{H}_s$ define the energy*

$$E^{\mathcal{H}}(s,\psi) := \int_{\mathcal{H}_s} \Big(\psi^*\psi - \frac{x_i}{t}\psi^*\gamma^0\gamma^i\psi\Big)\, dx. \tag{1.2.20}$$

For solutions to (1.2.17) *this satisfies*

$$E^{\mathcal{H}}(s,\psi) = E^{\mathcal{H}}(s_0,\psi) + \int_{s_0}^{s}\int_{\mathcal{H}_{\bar{s}}} (\bar{s}/t)(i\psi^*\gamma^0 F - iF^*\gamma^0\psi)\, dx d\bar{s}. \tag{1.2.21}$$

Proof. Integrating (1.2.17) over $\mathcal{K}_{[s_0,s]}$ gives

$$\begin{aligned}&\int_{\mathcal{H}_s} \big(\psi^*\psi, \psi^*\gamma^0\gamma^j\psi\big)\cdot n\, d\sigma - \int_{\mathcal{H}_{s_0}} \big(\psi^*\psi, \psi^*\gamma^0\gamma^j\psi\big)\cdot n\, d\sigma \\ &= \int_{\mathcal{K}_{[s_0,s]}} (i\psi^*\gamma^0 F - iF^*\gamma^0\psi) dt dx.\end{aligned} \tag{1.2.22}$$

Here n and $d\sigma$ are the unit normal and induced Lebesgue measure on the hyperboloids respectively

$$n = (t^2+r^2)^{-1/2}(t,-x^i), \quad d\sigma = t^{-1}(t^2+r^2)^{1/2}dx. \tag{1.2.23}$$

Using this explicit form of n and $d\sigma$ gives the result. □

To our knowledge the hyperboloidal Dirac energy (1.2.20) is new. We now show that this energy $E^{\mathcal{H}}(s,\psi)$ is indeed positive definite.

Proposition 1.2.3 (Hyperboloidal energy of the Dirac equation. I)**.** *By defining*

$$E^{+}(s,\psi) := \int_{\mathcal{H}_s} \Big(\psi - \frac{x_i}{t}\gamma^0\gamma^i\psi\Big)^* \Big(\psi - \frac{x_j}{t}\gamma^0\gamma^j\psi\Big)\, dx, \tag{1.2.24a}$$

one has

$$E^{\mathcal{H}}(s,\psi) = \frac{1}{2}E^{+}(s,\psi) + \frac{1}{2}\int_{\mathcal{H}_s}\frac{s^2}{t^2}\psi^*\psi\, dx, \tag{1.2.24b}$$

which in particular implies the positivity of the energy

$$E^{\mathcal{H}}(s,\psi) \geqslant \frac{1}{2}\int_{\mathcal{H}_s}\frac{s^2}{t^2}\psi^*\psi\, dx \geqslant 0. \tag{1.2.24c}$$

Proof. Expanding out the bracket in $E^{+}(s,\psi)$ gives

$$\begin{aligned}E^{+}(s,\psi) &:= \int_{\mathcal{H}_s}\Big(\psi^*\psi - 2\frac{x_i}{t}\psi^*\gamma^0\gamma^i\psi + \psi^*\frac{x^i x^k}{t^2}\gamma^0\gamma^j\gamma^0\gamma^l\psi\delta_{ij}\delta_{kl}\Big)\, dx \\ &= \int_{\mathcal{H}_s}\Big(\psi^*\psi - 2\frac{x_i}{t}\psi^*\gamma^0\gamma^i\psi - \psi^*\frac{x_i x_j}{t^2}\gamma^{(i}\gamma^{j)}\psi\Big)\, dx\end{aligned}$$

$$= \int_{\mathcal{H}_s} \Big(\psi^*\psi - 2\frac{x_i}{t}\psi^*\gamma^0\gamma^i\psi + \psi^*\psi\frac{x^i x^j}{t^2}\delta_{ij} \Big)\, dx$$
$$= \int_{\mathcal{H}_s} \Big(\psi^*\psi - 2\frac{x_i}{t}\psi^*\gamma^0\gamma^i\psi + \psi^*\psi\frac{t^2-s^2}{t^2} \Big)\, dx = 2E^{\mathcal{H}}(s,\psi) - \int_{\mathcal{H}_s} \frac{s^2}{t^2}\psi^*\psi\, dx,$$

and re-arranging gives (1.2.24b). □

Using the above positivity property we can now establish the following energy inequality.

Proposition 1.2.4 (Hyperboloidal energy estimate for Dirac spinor)**.** *For the massive Dirac spinor described by* (1.2.1) *the following estimate holds*

$$E^{\mathcal{H}}(s,\psi)^{1/2} \leqslant E^{\mathcal{H}}(s_0,\psi)^{1/2} + \int_{s_0}^{s} \|F\|_{L^2_f(\mathcal{H}_{\bar{s}})}\, d\bar{s}. \tag{1.2.25}$$

Proof. Differentiating (1.2.21) with respect to s yields

$$E^{\mathcal{H}}(\bar{s},\psi)^{1/2}\frac{d}{d\bar{s}}E^{\mathcal{H}}(\bar{s},\psi)^{1/2} \leqslant \frac{1}{2}\int_{\mathcal{H}_{\bar{s}}} (\bar{s}/t)\big(|F^*\gamma^0\psi| + |\psi^*\gamma^0 F|\big)\, dx$$
$$\leqslant \|(\bar{s}/t)\psi\|_{L^2_f(\mathcal{H}_{\bar{s}})}\|F\|_{L^2_f(\mathcal{H}_{\bar{s}})}.$$

Recalling in Proposition 1.2.3, we have $\|(\bar{s}/t)\psi\|_{L^2_f(\mathcal{H}_{\bar{s}})} \leqslant E^{\mathcal{H}}(\bar{s},\psi)^{1/2}$. Thus we have

$$\frac{d}{d\bar{s}}E^{\mathcal{H}}(\bar{s},\psi)^{1/2} \leqslant \|F\|_{L^2_f(\mathcal{H}_{\bar{s}})},$$

and the conclusion follows by integrating over $[s_0, s]$. □

1.2.4 Hyperboloidal energy based on the second-order formulation

Finally in this section we will convert the Dirac equation into a second-order PDE and define associated energy functionals. Apply the first-order Dirac operator $-i\gamma^\nu\partial_\nu$ to the Dirac equation (1.2.1) and use the identity (1.2.2) to obtain

$$\eta^{\mu\nu}\partial_\mu\partial_\nu\psi + M(-i\gamma^\nu\partial_\nu\psi) = -i\gamma^\nu\partial_\nu F. \tag{1.2.26}$$

Substituting the PDE into the bracketed term gives the following second-order PDE

$$\Box\psi - M^2\psi = -MF - i\gamma^\nu\partial_\nu F. \tag{1.2.27}$$

This provides us with another approach for deriving an energy estimate for the Dirac equation. We now check the hyperboloidal energy coming from (1.2.27).

Lemma 1.2.5 (Second-order hyperboloidal energy estimate for the Dirac equation)**.** *For all solution $\psi : \mathbb{R}^{3+1} \to \mathbb{C}^4$ of*

$$\Box\psi - M^2\psi = G, \tag{1.2.28a}$$

one has

$$\mathcal{E}_M(s,\psi)^{1/2} \leqslant \mathcal{E}_M(s_0,\psi)^{1/2} + \int_{s_0}^{s} \|G\|_{L^2(\mathcal{H}_\tau)} d\tau, \tag{1.2.28b}$$

where

$$\mathcal{E}_M(s,\psi) := \int_{\mathcal{H}_s} \Big(|\partial_t\psi|^2 + \sum_i |\partial_i\psi|^2 + M^2|\psi|^2 + 2\sum_i \frac{x^i}{t} \mathrm{Re}\big[\partial_t\psi\partial_i\psi^*\big] \Big) dx. \tag{1.2.28c}$$

Proof. The conjugate of (1.2.28a) reads $\Box\psi^* - M^2\psi^* = G^*$. Using $-\partial_t\psi^*$ and $-\partial_t\psi$ as multipliers on (1.2.28a) and its conjugate respectively we obtain

$$\begin{aligned}
\partial_t\psi^*\partial_t^2\psi - \partial_t\psi^* \sum_i \partial_i^2\psi + M^2\partial_t\psi^*\cdot\psi &= -\partial_t\psi^* G, \\
\partial_t^2\psi^*\partial_t\psi - \sum_i \partial_i^2\psi^*\partial_t\psi + M^2\partial_t\psi\cdot\psi^* &= -G^*\partial_t\psi.
\end{aligned}$$

Adding these two equations together gives

$$\begin{aligned}
&\partial_t\big(\partial_t\psi^*\partial_t\psi + \partial^i\psi^*\partial_i\psi + M^2\psi^*\psi\big) - \sum_i \partial_i\big(\partial_t\psi^*\partial_i\psi + \partial_t\psi\partial_i\psi^*\big) \\
&= -\big(\partial_t\psi^* G + G^*\partial_t\psi\big) = -2\mathrm{Re}\big[G^*\partial_t\psi\big].
\end{aligned}$$

Integrating this equation in the region $\mathcal{K}_{[s_0,s]}$, we have

$$\begin{aligned}
&\int_{\mathcal{H}_s} \Big(|\partial_t\psi|^2 + \sum_i |\partial_i\psi|^2 + M^2|\psi|^2, -(\partial_t\psi^*\partial_i\psi + \partial_i\psi^*\partial_t\psi) \Big) \cdot n \, d\sigma \\
&- \int_{\mathcal{H}_{s_0}} \Big(|\partial_t\psi|^2 + \sum_i |\partial_i\psi|^2 + M^2|\psi|^2, -(\partial_t\psi^*\partial_i\psi + \partial_i\psi^*\partial_t\psi) \Big) \cdot n \, d\sigma \\
&= -2\int_{\mathcal{K}_{[s_0,s]}} \mathrm{Re}\big[G^*\partial_t\psi\big] dt dx.
\end{aligned}$$

Using the explicit form of n and $d\sigma$ given in (1.2.23) and noting that $2\mathrm{Re}[\partial_i\psi^*\partial_t\psi] = \partial_t\psi^*\partial_i\psi + \partial_i\psi^*\partial_t\psi$ we find

$$\mathcal{E}_M(s,\psi) - \mathcal{E}_M(s_0,\psi) = -2\int_{s_0}^{s}\int_{\mathcal{H}_t} \mathrm{Re}\big[G^*\partial_t\psi\big] dt dx. \tag{1.2.29}$$

We next use the change of variable formula $\partial_i = \underline{\partial}_i - (x^i/t)\partial_t$ to rewrite the energy term:

$$\int_{\mathcal{H}_s} \Big(|\partial_t \psi|^2 + \sum_i |\partial_i \psi|^2 + M^2|\psi|^2 + \frac{x^i}{t}\big(\partial_t\psi^* \partial_i \psi + \partial_i \psi^* \partial_t \psi\big)\Big) dx$$
$$= \int_{\mathcal{H}_s} \Big(|(s/t)\partial_t \psi|^2 + \sum_i |\underline{\partial}_i \psi|^2 + M^2|\psi|^2 \Big) dx.$$

We can now estimate the nonlinearity on the RHS using the change of variables $\tau = (t^2 - r^2)^{1/2}$ and $dtdx = (\tau/t)d\tau dx$. In particular we have

$$-2\int_{\mathcal{H}_\tau} \mathrm{Re}\big[G^* \partial_t \psi\big](\tau/t)\, dx \leqslant 2\|(\tau/t)\partial_t\psi\|_{L^2(\mathcal{H}_\tau)} \|G\|_{L^2(\mathcal{H}_\tau)}$$
$$\leqslant 2\mathcal{E}_M(\tau,\psi)^{1/2} \|G\|_{L^2(\mathcal{H}_\tau)}.$$

Thus by differentiating (1.2.29) and using the above we have

$$\frac{d}{d\tau}\mathcal{E}_M(\tau,\psi)^{1/2} \leqslant \|G\|_{L^2(\mathcal{H}_\tau)}.$$

Integrating this expression over $[s_0, s]$ gives the desired result. □

We end with a short remark here on positive and negative mass spinors. In the original first-order Dirac equation (1.2.1) the mass M was defined as a real parameter with no sign restriction. Furthermore the mass did not appear in the hyperboloidal energy $E^{\mathcal{H}}$ defined in (1.2.20). This implies that spinors with equal masses but opposite signs ($\pm M$) would still obey the same energy estimates of Proposition 1.2.4.

This is consistent with the second-order equation (1.2.27) for the Dirac field. In this equation the mass appears squared, so spinors with equal masses, but of opposite signs, obey the *same* second order equation. Moreover the mass M^2 appears in the second-order hyperboloidal energy expression $\mathcal{E}_M$ in (1.2.28c).

Thus either the mass M should not appear in $E^{\mathcal{H}}$, as we have found, or if the mass M were to appear in $E^{\mathcal{H}}$, it would necessarily need to be invariant under a sign change, so as to agree with the second-order energy estimates involving $\mathcal{E}_M$.

1.3 Additional properties of Dirac spinors on hyperboloids

1.3.1 Hyperboloidal energy based on a Cholesky decomposition

Our first task is to obtain a hyperboloidal energy for the Dirac field ψ expressed in terms of a product of complex vectors $z(\psi)^* z(\psi)$. Such an expression is then easily seen to be positive semi-definite which, clearly, is in contrast to the form given in (1.2.20) and Proposition 1.2.3.

Recall the standard Cholesky decomposition: any Hermitian, positive-definite matrix A can be decomposed in a unique way as

$$A = P^* P, \tag{1.3.1}$$

where P is a lower triangular matrix with real and positive diagonal entries. In particular if A is positive semi-definite then the decomposition exists however one loses uniqueness and the diagonal entries of P may be zero.

We now prove the following result.

Proposition 1.3.1 (Hyperboloidal energy for the Dirac equation. II)**.** *There exists a lower triangular matrix P with real and positive diagonal entries such that*

$$E^{\mathcal{H}}(s,\psi) = \int_{\mathcal{H}_s} (P\psi)^*(P\psi) dx, \tag{1.3.2}$$

and specifically

$$P = \begin{pmatrix} s/t & 0 & 0 & 0 \\ 0 & s/t & 0 & 0 \\ x^3/t & x^1/t - ix^2/t & 1 & 0 \\ x^1/t + ix^2/t & -x^3/t & 0 & 1 \end{pmatrix}, \tag{1.3.3}$$

which can also be expressed as

$$P = \frac{(s/t)+1}{2} I_4 + \frac{(s/t)-1}{2}\gamma^0 + \delta_{ij}\frac{x^i}{t}\gamma^0\gamma^j. \tag{1.3.4}$$

The above expression is quite natural and resembles what is known for the wave equation: the factor x^i/t comes from Stokes' theorem applied to hyperboloids and we cannot expect to fully control the standard L^2 norm, namely $\int_{\mathcal{H}_s} \psi^*\psi \, dx$.

Proof. **Step 1. Existence of the decomposition.** Before we proceed with the derivation of the identity, we present an argument showing that such a decomposition

exists by proving positive semi-definiteness. For simplicity of notation, let $N_i := x^i/t$. The integrand of $E^{\mathcal{H}}(s,\psi)$ can be written as $\psi^* A\psi$ where $A := I_4 + N_j\gamma^0\gamma^j$. Here the spatial indices are contracted with δ_{ij}, so that $N_j\gamma^j = \delta_{ij}N^i\gamma^j$. Note A is hermitian since $A^* = I_4 + N_j(\gamma^0\gamma^j)^* = A$. Also

$$(N_j\gamma^0\gamma^j)(N_k\gamma^0\gamma^k) = -N_jN_k\gamma^0\gamma^0\gamma^j\gamma^k = -N_jN_k\gamma^{(j}\gamma^{k)} = N_jN^jI_4.$$

Then for all $z \in \mathbb{C}^4$ we have

$$\begin{aligned} 0 \leqslant \big(Az\big)^*\big(Az\big) &= (1 + N_jN^j)z^*I_4z + 2z^*N_j\gamma^0\gamma^jz \\ &\leqslant 2\left(z^*I_4z + z^*N_j\gamma^0\gamma^jz\right) \\ &= 2z^*Az. \end{aligned}$$

We used that $N_jN^j = (r/t)^2 \leqslant 1$ which holds in the light-cone $\mathcal{K}$. Thus A is positive semi-definite.

Step 2. Computing the matrix P. With respect to the Dirac representation (1.2.3) we have

$$\begin{aligned} A &= I_4 + N_j\begin{pmatrix} I_2 & 0_2 \\ 0_2 & -I_2 \end{pmatrix}\begin{pmatrix} 0_2 & \sigma^j \\ -\sigma^j & 0_2 \end{pmatrix} \\ &= \begin{pmatrix} I_2 & 0_2 \\ 0_2 & I_2 \end{pmatrix} + N_j\begin{pmatrix} 0_2 & \sigma^j \\ \sigma^j & 0_2 \end{pmatrix}. \end{aligned}$$

Here I_2 and 0_2 represent the 2×2 identity and zero matrices respectively. Calculate the second term above using the Pauli matrices:

$$N_j\begin{pmatrix} 0 & \sigma^j \\ \sigma^j & 0 \end{pmatrix} = N_1\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + N_2\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + N_3\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} N_3 & N_1 - iN_2 \\ N_1 + iN_2 & -N_3 \end{pmatrix}.$$

Define $\omega := N_1 + iN_2$ and recall $N_i \in \mathbb{R}$. Thus we have

$$A = \begin{pmatrix} & I_2 & N_3 & \bar{\omega} \\ & & \omega & -N_3 \\ N_3 & \bar{\omega} & & \\ \omega & -N_3 & & I_2 \end{pmatrix}.$$

Consider now 2×2 complex matrices B, C, D such that

$$\begin{pmatrix} B & 0 \\ C & D \end{pmatrix}^*\begin{pmatrix} B & 0 \\ C & D \end{pmatrix} = \begin{pmatrix} & I_2 & N_3 & \bar{\omega} \\ & & \omega & -N_3 \\ N_3 & \bar{\omega} & & \\ \omega & -N_3 & & I_2 \end{pmatrix}.$$

This implies the following identities

$$\begin{aligned} D^*D &= I_2, \\ C^*D = D^*C &= \begin{pmatrix} N_3 & \bar{\omega} \\ \omega & -N_3 \end{pmatrix}, \\ B^*B + C^*C &= I_2. \end{aligned}$$

If we let $D = I_2$ and $C = \begin{pmatrix} N_3 & \bar{\omega} \\ \omega & -N_3 \end{pmatrix}$ then we must solve

$$B^*B = I_2 - \begin{pmatrix} N_3 & \bar{\omega} \\ \omega & -N_3 \end{pmatrix}^* \begin{pmatrix} N_3 & \bar{\omega} \\ \omega & -N_3 \end{pmatrix} = \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

where $\lambda := 1 - (N_3^2 + \bar{\omega}\omega) = 1 - (N_1^2 + N_2^2 + N_3^2)$. Indeed $\lambda = 1 - (r/t)^2 = (s/t)^2 \geqslant 0$ so we can take $B = \sqrt{\lambda} I_2 = (s/t) I_2$. □

1.3.2 Hyperboloidal energy based on the Weyl spinor representation

Yet one more approach in deriving energy estimates is obtained by expressing the Dirac spinors in terms of Weyl spinors and then studying the energy of Weyl spinors (1.3.7) instead. This provides another convenient way to study Dirac equations. Decompose the spinor ψ and source term F as

$$\psi = \begin{pmatrix} u + v \\ u - v \end{pmatrix}, \quad F = \begin{pmatrix} F_+ + F_- \\ F_+ - F_- \end{pmatrix}, \tag{1.3.5}$$

where $u, v : \mathbb{R}^{1+3} \to \mathbb{C}^2$ are Weyl spinors and $F_\pm$ are $\mathbb{C}^2$-valued right hand sides. Defining $\partial_\pm := \partial_0 \pm \sigma^i \partial_i$ the PDE (1.2.1) can be shown to be equivalent to

$$\begin{aligned} \partial_- v + iMu &= iF_+, \\ \partial_+ u + iMv &= iF_-. \end{aligned} \tag{1.3.6}$$

A Dirac-Klein-Gordon system with respect to such a Weyl spinor decomposition has been studied, albeit in the low-regularity setting, by Bournaveas [5]. Following a similar approach to Section 1.2.3 we find an analogous hyperboloidal Weyl spinor energy:

$$E^\sigma_\pm(s, u) := \int_{\mathcal{H}_s} \left(u^* u \pm \frac{x_j}{t} u^* \sigma^j u \right) dx. \tag{1.3.7}$$

Similar to Propositions 1.2.3 and 1.2.4 we can prove positivity and an energy estimate for $E^\sigma_\pm$.

Proposition 1.3.2. *For a $\mathbb{C}^2$-valued function w the following holds:*

$$E^{\sigma}_{\pm}(s,w) \geqslant \frac{1}{2}\int_{\mathcal{H}_s} \frac{s^2}{t^2} w^* w \, dx \geqslant 0. \tag{1.3.8}$$

Furthermore for solutions u, v to (1.3.6) *we have*

$$\big(E^{\sigma}_{+}(s,u) + E^{\sigma}_{-}(s,v)\big)^{1/2} \leqslant \big(E^{\sigma}_{+}(s_0,u) + E^{\sigma}_{-}(s_0,v)\big)^{1/2} + \int_{s_0}^{s} \|F_+\|_{L^2(\mathcal{H}_{\bar{s}})} + \|F_-\|_{L^2(\mathcal{H}_{\bar{s}})} d\bar{s}. \tag{1.3.9}$$

Proof. **Step 1.** Using the Dirac representation (1.2.3) and the decomposition (1.3.5), the PDE (1.2.1) becomes

$$\begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix} \partial_0 \begin{pmatrix} u+v \\ u-v \end{pmatrix} + \begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix} \partial_j \begin{pmatrix} u+v \\ u-v \end{pmatrix} + iM \begin{pmatrix} u+v \\ u-v \end{pmatrix} = i \begin{pmatrix} F_+ + F_- \\ F_+ - F_- \end{pmatrix}. \tag{1.3.10}$$

Defining $\partial_{\pm} := \partial_0 \pm \sigma^i \partial_i$ this becomes

$$\begin{pmatrix} \partial_+ u + \partial_- v \\ \partial_- v - \partial_+ u \end{pmatrix} + iM \begin{pmatrix} u+v \\ u-v \end{pmatrix} = i \begin{pmatrix} F_+ + F_- \\ F_+ - F_- \end{pmatrix}. \tag{1.3.11}$$

Adding and subtracting the two rows above gives the following

$$\begin{aligned} \partial_- v + iMu &= iF_+, \\ \partial_+ u + iMv &= iF_-. \end{aligned}$$

Following a similar approach to deriving (1.2.17), we multiply the first and second equation by v^* and u^* respectively.

$$\begin{aligned} u^*\partial_0 u + u^*\sigma^j\partial_j u + iMu^*v &= iu^*F_-, \\ v^*\partial_0 v - v^*\sigma^j\partial_j v + iMu^*v &= -iv^*F_+. \end{aligned}$$

One then adds these equations to their conjugate to obtain the following:

$$\begin{aligned} \partial_0(u^*u) + \partial_j(u^*\sigma^j u) + iM(u^*v - v^*u) &= iu^*F_- - iF_-^*u, \\ \partial_0(v^*v) - \partial_j(v^*\sigma^j v) + iM(v^*u - u^*v) &= iv^*F_+ - iF_+^*v. \end{aligned}$$

Note the mass terms appear above. However if add these equations together we obtain

$$\partial_0(u^*u + v^*v) + \partial_j\big(u^*\sigma^j u - v^*\sigma^j v\big) = iu^*F_- - iF_-^*u + iv^*F_+ - iF_+^*v, \tag{1.3.12}$$

which does not contain a term involving M. This equation is the analogous Weyl spinor version of (1.2.17). Clearly integrating (1.3.12) over $\mathcal{K}_{[s_0,s]}$ gives the energy functional $E^{\sigma}_{\pm}(s,u)$ defined in (1.3.7).

Step 2. Next we establish that for a $\mathbb{C}^2$-valued function w the following holds:

$$E^\sigma_\pm(s,w) \geqslant \frac{1}{2}\int_{\mathcal{H}_s} \frac{s^2}{t^2} w^* w \, dx \geqslant 0.$$

The idea is in the spirit of Proposition 1.2.3. Observe that the sigma matrices are Hermitian and satisfy the following anti-commutator relation: $\{\sigma^i, \sigma^j\} = 2\delta^{ij} I_2$. Then we have

$$\begin{aligned}\int_{\mathcal{H}_s} \Big(w \pm \frac{x_j}{t}\sigma^j w\Big)^* \Big(w \pm \frac{x_j}{t}\sigma^j w\Big)\, dx &= \int_{\mathcal{H}_s} \Big(w^* w \pm 2\frac{x_j}{t} w^* \sigma^j w + \frac{x_j x_k}{t^2} w^* \sigma^{(j}\sigma^{k)} w\Big)\, dx \\ &= \int_{\mathcal{H}_s} \Big(w^* w (1 + (r/t)^2) \pm 2\frac{x_j}{t} w^* \sigma^j w\Big)\, dx \\ &= 2E^\sigma_\pm(s,w) - \int_{\mathcal{H}_s} \frac{s^2}{t^2} w^* w \, dx.\end{aligned}$$

Thus we have

$$E^\sigma_\pm(s,w) = \frac{1}{2}\int_{\mathcal{H}_s} \Big(w \pm \frac{x_j}{t}\sigma^j w\Big)^* \Big(w \pm \frac{x_j}{t}\sigma^j w\Big)\, dx + \frac{1}{2}\int_{\mathcal{H}_s} \frac{s^2}{t^2} w^* w \, dx \geqslant 0.$$

Step 3. Next, let us show that the following hyperboloidal energy inequality holds for the Weyl spinor equation (1.3.6)

$$\big(E^\sigma_+(s,u) + E^\sigma_-(s,v)\big)^{1/2} \leqslant \big(E^\sigma_+(s_0,u) + E^\sigma_-(s_0,v)\big)^{1/2} + \int_{s_0}^{s} \|F_+\|_{L^2(\mathcal{H}_{\bar{s}})} + \|F_-\|_{L^2(\mathcal{H}_{\bar{s}})} d\bar{s}.$$

Namely, integrating (1.3.12) over $\mathcal{K}_{[s_0,s]}$ we obtain

$$\begin{aligned}&E^\sigma_+(s,u) + E^\sigma_-(s,v) \\ &= E^\sigma_+(s_0,u) + E^\sigma_-(s_0,v) + \int_{s_0}^{s} d\bar{s} \int_{\mathcal{H}_{\bar{s}}} (\bar{s}/t)(iu^* F_- - iF_-^* u + iv^* F_+ - iF_+^* v)\, dx.\end{aligned}$$

Differentiating in s and noting that $\|(s/t)u, (s/t)v\|_{L^2(\mathcal{H}_s)} \leqslant \|u,v\|_{L^2(\mathcal{H}_s)} \leqslant (E^\sigma_+(s,u) + E^\sigma_-(s,v))^{1/2}$ gives

$$\frac{d}{ds}\Big(E^\sigma_+(s,u) + E^\sigma_-(s,v)\Big)^{1/2} \leqslant \|F_+\|_{L^2(\mathcal{H}_\tau)} + \|F_-\|_{L^2(\mathcal{H}_\tau)}.$$

□

1.4 Nonlinear stability of the ground state for the Dirac-Proca model

1.4.1 The Dirac-Proca model as a PDE system

Using the tools of Section 1.2, we will now discuss the Dirac-Proca model, the gauge condition and choice of initial data. This leads us to the second main stability Theorem 1.4.1 below, which can be proved using the methods of Sections 1.6 and 1.7. Without fixing a gauge, the field equations for the Dirac-Proca model with unknowns A^μ and ψ read

$$\begin{aligned} \Box A^\nu - m^2 A^\nu + \partial^\nu(\partial_\mu A^\mu) &= -\psi^* \gamma^0 \gamma^\nu (P_L \psi), \\ -i\gamma^\mu \partial_\mu \psi &= -\gamma^\mu A_\mu (P_L \psi). \end{aligned} \tag{1.4.1}$$

Recall $P_L = \frac{1}{2}(I_4 - \gamma^5)$ was defined in (1.2.6). Here $\psi : \mathbb{R}^{3+1} \to \mathbb{C}^4$ represents a *massless* Dirac fermion with spin 1/2 and $A^\mu : \mathbb{R}^{3+1} \to \mathbb{R}$ represents a massive boson (the Proca field) of mass m^2 with spin 1. As discussed in [76], (1.4.1) is equivalent to the following system

$$\begin{aligned} \Box A^\nu - m^2 A^\nu &= -\psi^* \gamma^0 \gamma^\nu (P_L \psi), \\ -i\gamma^\mu \partial_\mu \psi &= -\gamma^\mu A_\mu (P_L \psi), \\ \partial_\mu A^\mu &= 0. \end{aligned} \tag{1.4.2}$$

It can easily be shown that $\partial_\mu A^\mu$ satisfies a homogeneous Klein-Gordon equation. Thus we may treat $\partial_\mu A^\mu = 0$ as a constraint provided we specify the initial data set $\big(A_0^\nu, A_1^\nu, \psi_0\big)$ at some time $t_0 > 0$:

$$A^\nu(t_0, \cdot) = a^\nu, \qquad \partial_t A^\nu(t_0, \cdot) = b^\nu, \qquad \psi(t_0, \cdot) = \psi_0. \tag{1.4.3}$$

to satisfy the following two 'Lorenz compatibility' conditions

$$\begin{aligned} b^0 + \partial_j a^j &= 0, \\ \Delta a^0 - m^2 a^0 &= -\psi_0^* P_L \psi_0 - \partial_j b^j. \end{aligned} \tag{1.4.4}$$

For more generality, see also [4], we consider (1.4.1) with an artificial mass M added to the spinor:

$$\begin{aligned} \Box A^\nu - m^2 A^\nu &= -\psi^* \gamma^0 \gamma^\nu (P_L \psi), \\ -i\gamma^\mu \partial_\mu \psi + M\psi &= -\gamma^\mu A_\mu (P_L \psi). \end{aligned} \tag{1.4.5}$$

The mass parameters $M \geqslant 0$ and $m > 0$ are constants, and we will study both cases $M = 0$ and $M > 0$. Again the initial data will be taken to satisfy (1.4.4). This elliptic-type system of two equations for nine scalar functions admits non-trivial compactly supported solutions. For example one may choose: $A_0^j \in C_c(\mathbb{R}^{3+1})$ for $j = 1, 2, 3$, $\psi_0 \in C_c(\mathbb{R}^{3+1})$ such that each component of ψ_0 is the same, and all remaining initial

data is trivial.

1.4.2 Main result for the Dirac-Proca model

We now state our result for the Dirac-Proca model. The proof of the theorem below will clearly follow from the the proof we will develop for our main result (Theorem 1.7.1), and so we omit it.

For the model under consideration in this section, the *ground state* of the theory is simply given by

$$A \equiv 0, \qquad \psi \equiv 0. \tag{1.4.6}$$

Theorem 1.4.1. *Consider the Dirac-Proca system* (1.1.8) *with* $M \geqslant 0$, $m > 0$, *and let* N *be a sufficiently large integer. There exists* $\epsilon_0 > 0$, *which is independent of* M, *such that for all* $\epsilon \in (0, \epsilon_0)$ *and all compactly supported, Lorenz compatible initial data in the sense of* (1.1.10), *satisfying the smallness condition*

$$\|a^\nu, \psi_0\|_{H^{N+1}} + \|b^\nu\|_{H^N} \lesssim \epsilon, \tag{1.4.7}$$

the initial value problem of (1.1.8) *admits a unique global-in-time solution* (ψ, A^ν). *Furthermore, the following decay result holds*

$$|A| \lesssim \epsilon t^{-3/2}, \qquad |\psi| \lesssim \epsilon \min\left(t^{-1}, M^{-1}t^{-3/2}\right). \tag{1.4.8}$$

1.5 Nonlinear stability of the ground state for the $U(1)$ model

1.5.1 The U(1) model as a PDE system

We now treat the Higgs mechanism applied to a $U(1)$ abelian gauge field. This gives some exposure to the problems coming from the Higgs field that we will meet when dealing with the full GSW model in future work. From the Lagrangian of this theory; see (1.5.6) below, and in a suitably modified Lorenz gauge; see (1.5.5) below, the field equations for this model read

$$\begin{aligned} (\Box - m_q^2)A^\nu &= Q_{A^\nu}, \\ \Box\chi - m_q^2 \frac{\phi_0}{2v^2}\left(\phi_0^*\chi - \chi^*\phi_0\right) - m_\lambda^2 \frac{\phi_0}{2v^2}\left(\phi_0^*\chi + \chi^*\phi_0\right) &= Q_\chi, \\ i\gamma^\mu\partial_\mu\psi - m_g\psi &= Q_\psi, \end{aligned} \tag{1.5.1}$$

with quadratic or higher order terms given by

$$\begin{aligned} Q_{A^\nu} &:= iq\big(\chi^*(\partial^\nu\chi) - (\partial^\nu\chi^*)\chi\big) + 2q^2A^\nu\big(\chi^*\phi_0 + \phi_0^*\chi + \chi^*\chi\big) + q\psi^*\gamma^0\gamma^\nu\psi, \\ Q_\chi &:= 2iqA_\mu\partial^\mu\chi + q^2\chi\big(\phi_0^*\chi - \chi^*\phi_0\big) + q^2A^\mu A_\mu\big(\phi_0+\chi\big) \\ &\quad + 2\lambda\chi^*\chi\phi_0 + 2\lambda\chi\big(\phi_0^*\chi + \chi^*\phi_0 + \chi^*\chi\big) - g\big(\phi_0+\chi\big)\psi^*\gamma^0\psi, \\ Q_\psi &:= g\big(\phi_0^*\chi + \chi^*\phi_0 + \chi^*\chi\big)\psi - q\gamma^\mu A_\mu\psi. \end{aligned} \tag{1.5.2}$$

Here $\chi := \phi - \phi_0 : \mathbb{R}^{1+3} \to \mathbb{C}$ is the perturbation from a constant vacuum state ϕ_0 satisfying $\phi_0^*\phi_0 = v^2$ and $\partial_\mu\phi_0 = 0$. The field $\psi : \mathbb{R}^{3+1} \to \mathbb{C}^4$ represents a fermion of mass m_g with spin $1/2$ and $A_\mu : \mathbb{R}^{1+3} \to \mathbb{R}$ represents a massive boson of mass m_q^2 with spin 1. Furthermore the mass coefficients

$$m_q^2 = 2q^2v^2, \quad m_\lambda^2 = 4\lambda v^2, \quad m_g = gv^2 \tag{1.5.3}$$

depend themselves on given coupling constants denoted by q, g, λ, as well as the so-called "vacuum expectation value" of the Higgs field, denoted by v.

The initial data set are denoted by

$$\big(A^\nu, \chi, \psi\big)(t_0,\cdot) = \big(A_0^\nu, \chi_0, \psi_0\big), \qquad \big(\partial_t A^\nu, \partial_t\chi\big)(t_0,\cdot) = \big(A_1^\nu, \chi_1\big), \tag{1.5.4}$$

which are said to be 'Lorenz compatible' if they satisfy

$$\begin{aligned} \partial_a A_0^a &= -A_1^0 - iq\big(\phi_0^*\chi_0 - \chi_0^*\phi_0\big), \\ \Delta A_0^0 - m_q^2A_0^0 &= -\partial_iA_1^i - iq\big(\phi_0^*\chi_1 - \chi_1^*\phi_0\big) + iq\big(\chi_0^*\chi_1 - \chi_1^*\chi_0\big) \\ &\quad + 2q^2A_0^0\big(\phi_0^*\chi_0 + \chi_0^*\phi_0 + \chi_0^*\chi_0\big) + q\psi_0^*\psi_0. \end{aligned} \tag{1.5.5}$$

The derivation of (1.5.5) will follow from results of the following Section 1.5.2, in particular Lemma 1.5.3. Similar to the constraint equations (1.4.4) for the Dirac-Proca model, we also obtain in (1.5.5) an elliptic-type system of only two equations for eleven functions. Clearly non-trivial solutions with compact support can be constructed.

Our main result was stated in the introduction (Theorem 1.1.1) and the proof will be provided in Section 1.7.

1.5.2 The abelian action and $U(1)$ invariance

The Lagrangian we consider is

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - (D_\mu\phi)^*D^\mu\phi - V(\phi^*\phi) - i\psi^*\gamma^0\gamma^\mu D_\mu\psi + g\phi^*\phi\psi^*\gamma^0\psi, \tag{1.5.6}$$

where we use the following definitions for the Higgs potential, gauge curvature and gauge covariant derivatives:

$$\begin{aligned} V(\phi^*\phi) &:= \lambda(\phi^*\phi - v^2)^2, \\ F_{\mu\nu} &:= \partial_\mu A_\nu - \partial_\nu A_\mu, \\ D_\mu\phi &:= (\partial_\mu - iqA_\mu)\phi, \\ D_\mu\psi &:= (\partial_\mu - iqA_\mu)\psi. \end{aligned} \tag{1.5.7}$$

Furthermore λ, v, g, q are constants. A calculation shows that the Euler-Lagrange equations for (1.5.6) are the following

$$\begin{aligned} \partial_\mu F^{\mu\nu} &= iq\phi^*(D^\nu\phi) - iq(D^\nu\phi)^*\phi + q\psi^*\gamma^0\gamma^\nu\psi, \\ D^\mu D_\mu\phi &= V'(\phi^*\phi)\phi - g\phi\psi^*\gamma^0\psi, \\ i\gamma^\mu D_\mu\psi &= g\phi^*\phi\psi, \end{aligned} \tag{1.5.8}$$

together with (1.5.7). These PDEs can also be expressed as

$$\begin{aligned} \Box A^\nu - \partial^\nu(\mathrm{div}A) - 2q^2A^\nu\phi^*\phi &= iq\big(\phi^*(\partial^\nu\phi) - (\partial^\nu\phi^*)\phi\big) + q\psi^*\gamma^0\gamma^\nu\psi, \\ \Box\phi - V'(\phi^*\phi)\phi - iq\phi\partial_\mu A^\mu &= 2iqA_\mu\partial^\mu\phi + q^2A^\mu A_\mu\phi - g\phi\psi^*\gamma^0\psi, \\ i\gamma^\mu\partial_\mu\psi - g\phi^*\phi\psi &= -q\gamma^\mu A_\mu\psi. \end{aligned} \tag{1.5.9}$$

Lemma 1.5.1. *$\mathcal{L}$ has a $U(1)$-gauge symmetry.*

Proof. The $U(1)$ gauge symmetry induces the following transformations

$$A_\mu \mapsto A'_\mu := A_\mu + \partial_\mu\alpha, \quad \phi \mapsto \phi' := e^{iq\alpha}\phi, \quad \psi \mapsto \psi' := e^{iq\alpha}\psi, \tag{1.5.10}$$

where $\alpha = \alpha(t, \mathbf{x})$ is some arbitrary function of space and time. This implies

$$\begin{aligned} D_\mu\phi \mapsto D'_\mu\phi' &= e^{iq\alpha}D_\mu\phi, \\ D_\mu\psi \mapsto D'_\mu\psi' &= \partial_\mu(e^{iq\alpha}\psi) - iq(A_\mu + \partial_\mu\alpha)e^{iq\alpha}\psi = e^{iq\alpha}D_\mu\psi. \end{aligned} \tag{1.5.11}$$

By the commutation of partial derivatives we immediately see that $F_{\mu\nu}$ is invariant under $U(1)$ gauge transformations. Inserting these transformations into the Lagrangian

one obtains

$$\begin{aligned}\mathcal{L}' &= -i\psi'^*\gamma^0\gamma^\mu D'_\mu\psi' - \frac{1}{4}F'_{\mu\nu}F'^{\mu\nu} - (D'_\mu\phi')^*D'^\mu\phi' - V((\phi')^*\phi') + \phi'^*\phi' g\psi'^*\gamma^0\psi' \\ &= -ie^{-iq\alpha}\psi^*\gamma^0\gamma^\mu\partial_\mu(e^{iq\alpha}\psi) - i\big(e^{iq\alpha}\psi, -iq\gamma^\mu(A_\mu+\partial_\mu\alpha)e^{iq\alpha}\psi\big) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \\ &\quad - (D_\mu\phi)^*e^{-iq\alpha}e^{iq\alpha}D_\mu\phi - V(\phi^*e^{-iq\alpha}e^{iq\alpha}\phi) + g\phi^*\phi e^{-iq\alpha}\psi^*\gamma^0e^{iq\alpha}\psi \\ &= -i\psi^*\gamma^0\gamma^\mu D_\mu\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - (D_\mu\phi)^*D^\mu\phi - V(\phi^*\phi) + g\phi^*\phi\psi^*\gamma^0\psi = \mathcal{L}.\end{aligned} \tag{1.5.12}$$

Thus $\mathcal{L}$ is invariant under the transformation (1.5.10). □

1.5.3 Propagation of an inhomogeneous Lorenz gauge

Next, with a similar aim to that of Section 1.4.1, we will turn the PDE (1.5.9) into one of definite type by specifying a particular gauge for the vector field A^μ.

Lemma 1.5.2 (The inhomogeneous Lorenz gauge)**.** *Let X be a suitably regular scalar field. Consider the modified system*

$$\begin{aligned}\Box A^\nu - 2q^2A^\nu\phi^*\phi &= \partial^\nu X + iq\big(\phi^*(\partial^\nu\phi) - (\partial^\nu\phi^*)\phi\big) + q\psi^*\gamma^0\gamma^\nu\psi, \\ \Box\phi - V'(\phi^*\phi)\phi - iq\phi X &= 2iqA_\mu\partial^\mu\phi + q^2A^\mu A_\mu\phi - g\phi\psi^*\gamma^0\psi, \\ i\gamma^\mu\partial_\mu\psi - g\phi^*\phi\psi &= -q\gamma^\mu A_\mu\psi.\end{aligned} \tag{1.5.13}$$

Suppose the initial data for (1.5.13) *satisfy*

$$\begin{aligned}\big(div A - X\big)(t_0,\cdot) &= 0, \\ \partial_t\big(div A - X\big)(t_0,\cdot) &= 0.\end{aligned} \tag{1.5.14}$$

Then as long as the solution to (1.5.13) *exists, it will satisfy $\div A = X$.*

Proof. To propagate the gauge choice $\div A = X$ imposed on the initial data we will take the divergence of the first equation in (1.5.13) and use the second evolution equation (1.5.13):

$$\begin{aligned}\Box(\mathrm{div}A - X) &= iq(\phi^*\Box\phi - \Box\phi^*\phi) + 2q^2\partial_\nu(A^\nu\phi^*\phi) + q\partial_\nu(\psi^*\gamma^0\gamma^\nu\psi) \\ &= iq\phi^*\Big(2iqA_\mu\partial^\mu\phi + iq\phi\partial_\mu A^\mu + q^2A_\mu A^\mu\phi + V'\phi - g\phi\psi^*\gamma^0\psi\Big) \\ &\quad - iq\Big(-2iqA_\mu\partial^\mu\phi^* - iq\phi^*\partial_\mu A^\mu + q^2A_\mu A^\mu\phi^* + (V'\phi)^* - g\phi^*\psi^*\gamma^0\psi\Big)\phi \\ &\quad + 2q^2\partial_\nu(A^\nu\phi^*\phi) + q\psi^*\gamma^0\gamma^\mu\partial_\mu\psi + q\partial_\mu\psi^*\gamma^0\gamma^\mu\psi \\ &= iq\big(\phi^*V'\phi - (V'\phi)^*\phi\big) = 0,\end{aligned}$$

where in the last line we used the specific Higgs potential (1.5.7). □

1.5.4 Gauge choice for the abelian model

The Higgs potential has a non-zero minimum at $\phi_0 := ve^{i\theta/v}$ where

$$V(\phi_0^*\phi_0) = 0, \tag{1.5.15}$$

and $\theta : \mathbb{R}^{1+3} \to \mathbb{R}$ is arbitrary. There is ambiguity in this minimum state due to the $U(1)$ symmetry represented by θ. A particular choice of θ will break this $U(1)$ symmetry, and such a scenario is termed 'spontaneous symmetry breaking'. We consider perturbations of the form

$$\chi := \phi - \phi_0,$$

where ϕ_0 is *constant* in space and time and has magnitude $|\phi_0| = v$. The following result is a consequence of Lemma 1.5.2 by choosing $X = -iq(\phi_0^*\chi - \chi^*\phi_0)$.

Lemma 1.5.3. *Suppose the initial data satisfy the following gauge condition*

$$\begin{aligned} \big(divA + iq(\phi_0^*\chi - \chi^*\phi_0)\big)(t_0, \cdot) &= 0, \\ \partial_t\big(divA + iq(\phi_0^*\chi - \chi^*\phi_0)\big)(t_0, \cdot) &= 0. \end{aligned} \tag{1.5.16}$$

Then the Euler-Lagrange equations for (1.5.6) *are equivalent to those written in* (1.5.1).

1.6 Structure and nonlinearity of the models

1.6.1 Aim of this section

In this section we will use various techniques to treat the variables in (1.5.1) and their nonlinearities. The nonlinearities, defined in (1.5.2), which require additional work to control are of the following two types

$$\psi^*\gamma^0\gamma^\nu\psi, \qquad \psi^*\gamma^0\psi, \tag{1.6.1}$$

appearing in Q_{A^ν} and Q_χ respectively. If the mass of the Dirac spinor is zero, $m_g = 0$, then the nonlinearities in (1.6.1) will have insufficiently fast decay. In this case it is sufficient to employ a transformation introduced by [76], see (1.6.3) and (1.6.9) below, to overcome such problems. Finally although the equation satisfied by χ is of ambiguous type, we find that χ can be decomposed into two components, with each component satisfying a Klein-Gordon equation.

1.6.2 Hidden null structure from Tsutsumi

Following Tsutsumi [76] we can now uncover a null structure using a particular transformation. However we will not need to reduce to the 'strong null forms' via any complicated transformations [77, eq (2.6)].

For complex-valued functions $\Phi(t,x), \Psi(t,x) : \mathbb{R}^{3+1} \to \mathbb{C}^n$, recall the null form

$$Q_0(\Phi,\Psi) := (\partial_0\Phi)^*\partial_0\Psi - (\nabla\Phi)^*\nabla\Psi. \tag{1.6.2}$$

Define a new variable

$$\widetilde{A}^\nu := A^\nu + \frac{q}{m_q^2}\psi^*\gamma^0\gamma^\nu\psi. \tag{1.6.3}$$

This satisfies the non-linear Klein-Gordon equation

$$\big(\Box - m_q^2\big)\widetilde{A}^\nu = Q_{\widetilde{A}^\nu}, \tag{1.6.4}$$

in which the nonlinearities are

$$\begin{aligned}
Q_{\widetilde{A}^\nu} &:= \frac{2q}{m_q^2}Q_0(\psi,\gamma^0\gamma^\nu\psi) + \frac{q}{m_q^2}G_\psi^*\gamma^0\gamma^\nu\psi + \frac{q}{m_q^2}\psi^*\gamma^0\gamma^\nu G_\psi + \frac{2q}{m_q^2}m_g^2\psi^*\gamma^0\gamma^\nu\psi,\\
G_\psi &:= -m_g Q_\psi - i\gamma^\nu\partial_\nu Q_\psi\\
&= -m_g\big(g(\phi_0^*\chi + \chi^*\phi_0 + \chi^*\chi)\psi - q\gamma^\mu A_\mu\psi\big)\\
&\quad - ig\gamma^\nu\partial_\nu(\phi_0^*\chi + \chi^*\phi_0 + \chi^*\chi)\psi + iq\gamma^\nu\gamma^\mu\partial_\nu A_\mu\psi\\
&\quad - g(\phi_0^*\chi + \chi^*\phi_0 + \chi^*\chi)(m_g\psi + Q_\psi) - 2igA^\mu\partial_\mu\psi - gA_\mu\gamma^\mu(m_g\psi + Q_\psi),
\end{aligned}$$

where (1.2.27) and (1.5.2) were used to compute G_ψ. Note the nonlinearity $\psi^*\gamma^0\gamma^\nu\psi$ in (1.6.4) now appears with a good factor m_g^2 to compensate for the lack of derivatives hitting ψ.

In the case $0 \leqslant m_g \ll \min(m_q, m_\lambda)$, the second order formulation (1.2.27) of the Dirac equation is more like a nonlinear wave equation. In this case we do not have good bounds for either the L_f^2 or L^∞ norm of ψ and this is why the null structure of (1.6.4) and the factor m_g^2 in front of the ψ–ψ interaction are needed. However, in the case $m_g \gtrsim \min(m_q, m_\lambda)$, better bounds are available and we do not require such a null structure any more.

1.6.3 Decomposition of χ

In order to study the behaviour of χ, we find it more convenient to consider equations for the following two variables

$$\chi_\pm := \phi_0^*\chi \pm \chi^*\phi_0. \tag{1.6.5}$$

Since the following identity holds

$$|\chi_+|^2 + |\chi_-|^2 = 2v^2|\chi|^2, \tag{1.6.6}$$

it is equivalent to estimate either χ or $\chi_\pm$. These new variables satisfy the Klein-Gordon equations

$$\Box\chi_+ - m_\lambda^2\chi_+ = Q_{\chi_+}, \tag{1.6.7}$$
$$\Box\chi_- - m_q^2\chi_- = Q_{\chi_-}, \tag{1.6.8}$$

with

$$\begin{aligned} Q_{\chi_+} &:= 2iqA_\mu\partial^\mu\chi_- + q^2\chi_-^2 + q^2A^\mu A_\mu\big(2v^2+\chi_+\big) \\ &\quad + 4\lambda v^2\chi^*\chi + 2\lambda\chi_+^2 + 2\lambda\chi_+\chi^*\chi - g\big(2v^2+\chi_+\big)\psi^*\gamma^0\psi, \\ Q_{\chi_-} &:= 2iqA_\mu\partial^\mu\chi_+ + q^2\chi_-\chi_+ + q^2A_\mu A^\mu\chi_- + 2\lambda\chi_-\chi_+ + 2\lambda\chi_-\chi^*\chi - g\chi_-\psi^*\gamma^0\psi. \end{aligned}$$

Similar to the previous section, we introduce the new variable

$$\widetilde{\chi}_+ := \chi_+ - \frac{2m_g}{m_\lambda^2}\psi^*\gamma^0\psi, \tag{1.6.9}$$

which satisfies the following Klein-Gordon equation

$$\Box\widetilde{\chi}_+ - m_\lambda^2\widetilde{\chi}_+ = Q_{\widetilde{\chi}_+} \tag{1.6.10}$$

with the nonlinearity

$$Q_{\widetilde{\chi}_+} := -4\frac{m_g}{m_\lambda^2}Q_0(\psi,\gamma^0\psi) + 2\frac{m_g}{m_\lambda^2}G_\psi^*\gamma^0\psi + 2\frac{m_g}{m_\lambda^2}\psi^*\gamma^0G_\psi - 4\frac{m_g^3}{m_\lambda^2}\psi^*\gamma^0\psi + Q_{\chi_+} + 2m_g\psi^*\gamma^0\psi.$$

The final term $2m_g\psi^*\gamma^0\psi$ here cancels the problematic term in Q_{χ_+} and any other nonlinearities of the form $\psi^*\gamma^0\psi$ now appear with a good factor of m_g^3 in front.

1.7 Bootstrap argument

1.7.1 Overview

This section is devoted to using a bootstrap argument to prove Theorem 1.1.1. After the treatment of the equations in Section 1.6, we remind one that in the following analysis we will deal with the unknowns

$$\widetilde{A}^\nu = A^\nu + \frac{q}{m_q^2}\psi^*\gamma^0\gamma^\nu\psi, \quad \widetilde{\chi}_+ = \chi_+ - \frac{2m_g}{m_\lambda^2}\psi^*\gamma^0\psi, \quad \chi_-, \quad \psi, \tag{1.7.1}$$

which satisfy equations

$$\begin{aligned}\Box\widetilde{A}^{\nu}-m_q^2\widetilde{A}^{\nu}&=Q_{\widetilde{A}^{\nu}},\\ \Box\widetilde{\chi}_{+}-m_{\lambda}^2\widetilde{\chi}_{+}&=Q_{\widetilde{\chi}_{+}}\\ \Box\chi_{-}-m_q^2\chi_{-}&=Q_{\chi_{-}},\\ i\gamma^{\mu}\partial_{\mu}\psi-m_g\psi&=Q_{\psi}.\end{aligned}$$

We also provide here the following theorem, which details the L^2 and L^{∞} estimates of the unknowns.

Theorem 1.7.1. *Under the smallness assumptions in Theorem 1.1.1 the solution satisfies the following energy estimates*

$$\begin{aligned}\big\|(s/t)\partial^I L^J\partial_{\mu}(A_{\nu},\chi)\big\|_{L^2_f(\mathcal{H}_s)}+\big\|(s/t)\partial_{\mu}\partial^I L^J(A_{\nu},\chi)\big\|_{L^2_f(\mathcal{H}_s)}&\lesssim C_1\epsilon,\quad |I|+|J|\leqslant N,\\ \big\|\partial^I L^J(A_{\nu},\chi)\big\|_{L^2_f(\mathcal{H}_s)}&\lesssim C_1\epsilon,\quad |I|+|J|\leqslant N,\\ \big\|(s/t)\partial^I L^J\psi\big\|_{L^2_f(\mathcal{H}_s)}&\lesssim C_1\epsilon,\quad |I|+|J|\leqslant N,\\ \big\|(s/t)\partial_{\alpha}\partial^I L^J\psi\big\|_{L^2_f(\mathcal{H}_s)}&\lesssim C_1\epsilon\log s,\quad |I|+|J|=N,\end{aligned}\tag{1.7.2}$$

and the following L^{∞} estimates

$$\begin{aligned}\sup_{(t,x)\in\mathcal{H}_s}\Big(t^{1/2}s\big|\partial_{\alpha}\partial^I L^J(A_{\nu},\chi)\big|+t^{1/2}s\big|\partial^I L^J\partial_{\alpha}(A_{\nu},\chi)\big|\Big)&\lesssim C_1\epsilon,\quad |I|+|J|\leqslant N-2,\\ \sup_{(t,x)\in\mathcal{H}_s}\Big(t^{3/2}\big|\partial^I L^J A_{\nu},\partial^I L^J\chi\big|\Big)&\lesssim C_1\epsilon,\quad |I|+|J|\leqslant N-2,\\ \sup_{(t,x)\in\mathcal{H}_s}\Big(t^{1/2}s\big|\partial^I L^J\psi\big|\Big)&\lesssim C_1\epsilon,\quad |I|+|J|\leqslant N-2,\\ \sup_{(t,x)\in\mathcal{H}_s}\Big(t^{1/2}s\big|\partial_{\alpha}\partial^I L^J\psi\big|\Big)&\lesssim C_1\epsilon\log s,\quad |I|+|J|=N-2,\end{aligned}\tag{1.7.3}$$

where C_1 is some constant to be determined later.

Strategy of the proofs of Theorems 1.1.1 and 1.7.1. The proofs are based on a bootstrap argument. In Section 1.7.2 we recall the estimates for null terms and various commutators. The bootstrap assumptions will be made in (1.7.10). These bootstraps, combined with some standard commutator estimates and Sobolev-type inequalities, will lead to certain weak estimates in (1.7.12) and (1.7.14). In Section 1.7.4 we will use our first-order hyperboloidal energy to upgrade our estimates for the Dirac component, namely in Theorem 1.7.7 and Corollary 1.7.8. In Section 1.7.5 we obtain estimates for the transformed variables $\tilde{A}^{\nu}$ and $\tilde{\chi}_{+}$ defined above. Putting all of this together we finally are able to close our bootstrap assumptions.

1.7.2 Estimates for null forms and commutators

We first illustrate estimates for the quadratic null terms, which, roughly speaking, reveal the following bounds

$$\big|\partial^\alpha u\partial_\alpha v\big| \lesssim \big(s^2/t^2\big)\big|\partial u\big|\big|\partial v\big|.$$

Lemma 1.7.2. *We have the following estimate for the quadratic null term $T^{\alpha\beta}\partial_\alpha u\partial_\beta v$ with constants $T^{\alpha\beta}$ and u, v sufficiently regular.*

$$\begin{aligned}
&\big|\partial^I L^J(T^{\mu\nu}\partial_\mu u\partial_\nu v)\big|\\
\lesssim &\sum_{\substack{|I_1|+|I_2|\leqslant|I|,\\ |J_1|+|J_2|\leqslant|J|,\\ a,\beta}}\Big(\big|\partial^{I_1}L^{J_1}\underline{\partial}_a u\big|\big|\partial^{I_2}L^{J_2}\underline{\partial}_\beta v\big| + \big|\partial^{I_1}L^{J_1}\underline{\partial}_\beta u\big|\big|\partial^{I_2}L^{J_2}\underline{\partial}_a v\big|\Big)\\
&+ (s/t)^2\sum_{\substack{|I_1|+|I_2|\leqslant|I|,\\ |J_1|+|J_2|\leqslant|J|}}\big|\partial^{I_1}L^{J_1}\partial_t u\big|\big|\partial^{I_2}L^{J_2}\partial_t v\big|.
\end{aligned}\tag{1.7.4}$$

One refers to [44] for the proof. We next recall the estimates for commutators, also proved in [44]. Heuristically speaking, the following lemma provides the relation

$$\big|\Gamma\partial^I L^J u\big| \simeq \big|\partial^I L^J\Gamma u\big|,$$

in which Γ takes values from the set

$$\{\partial_\alpha, \underline{\partial}_\alpha, \partial_\alpha\partial_\beta, (s/t)\partial_\alpha\}.$$

Lemma 1.7.3. *Assume a function u defined in the region $\mathcal{K}$ is regular enough, then we have*

$$\big|[\partial^I L^J, \partial_\alpha]u\big| \leqslant C(|I|,|J|)\sum_{|J'|<|J|,\beta}\big|\partial_\beta\partial^I L^{J'}u\big|,\tag{1.7.5}$$

$$\big|[\partial^I L^J, \underline{\partial}_a]u\big| \leqslant C(|I|,|J|)\Big(\sum_{|I'|<|I|,|J'|<|J|,b}|\underline{\partial}_b\partial^{I'}L^{J'}u| + t^{-1}\sum_{|I'|\leqslant|I|,|J'|\leqslant|J|}|\partial^{I'}L^{J'}u|\Big),\tag{1.7.6}$$

$$\big|[\partial^I L^J, \underline{\partial}_\alpha]u\big| \leqslant C(|I|,|J|)\Big(\sum_{|I'|<|I|,|J'|<|J|,\beta}\big|\partial_\beta\partial^{I'}L^{J'}u\big| + t^{-1}\sum_{|I'|\leqslant|I|,|J'|\leqslant|J|,\beta}\big|\partial_\beta\partial^{I'}L^{J'}u\big|\Big),\tag{1.7.7}$$

$$\big|[\partial^I L^J, \partial_\alpha\partial_\beta]u\big| \leqslant C(|I|,|J|)\sum_{|I'|\leqslant|I|,|J'|<|J|,\gamma,\gamma'}\big|\partial_\gamma\partial_{\gamma'}\partial^{I'}L^{J'}u\big|,\tag{1.7.8}$$

$$\big|\partial^I L^J((s/t)\partial_\alpha u)\big| \leqslant \big|(s/t)\partial_\alpha \partial^I L^J u\big| + C(|I|,|J|) \sum_{|I'|\leqslant |I|,|J'|\leqslant |J|,\beta} \big|(s/t)\partial_\beta \partial^{I'} L^{J'} u\big|. \tag{1.7.9}$$

1.7.3 Bootstrap assumptions and basic estimates

By the local well-posedness of semilinear PDEs, there exists an $s_1 > s_0$ in which the following bootstrap assumptions hold for all $s \in [s_0, s_1]$

$$\begin{aligned} \mathcal{E}_{m_q}\big(s, \partial^I L^J A_\nu\big)^{1/2} + \mathcal{E}_{m_\lambda}\big(s, \partial^I L^J \chi\big)^{1/2} &\leqslant C_1\epsilon, && |I|+|J| \leqslant N, \\ \mathcal{E}_{m_g}(s, \partial^I L^J \psi)^{1/2} &\leqslant C_1\epsilon, && |I|+|J| \leqslant N-1, \\ \mathcal{E}_{m_g}(s, \partial^I L^J \psi)^{1/2} &\leqslant C_1\epsilon \log s, && |I|+|J| = N. \end{aligned} \tag{1.7.10}$$

If we can prove the refined estimates

$$\begin{aligned} \mathcal{E}_{m_q}\big(s, \partial^I L^J A_\nu\big)^{1/2} + \mathcal{E}_{m_\lambda}\big(s, \partial^I L^J \chi\big)^{1/2} &\leqslant \frac{1}{2}C_1\epsilon, && |I|+|J| \leqslant N, \\ \mathcal{E}_{m_g}(s, \partial^I L^J \psi)^{1/2} &\leqslant \frac{1}{2}C_1\epsilon, && |I|+|J| \leqslant N-1, \\ \mathcal{E}_{m_g}(s, \partial^I L^J \psi)^{1/2} &\leqslant \frac{1}{2}C_1\epsilon \log s, && |I|+|J| = N, \end{aligned} \tag{1.7.11}$$

then we are able to assert that s_1 cannot be finite, which in turn implies a global existence result for (1.1.1).

Combining bootstrap assumptions (1.7.10) and the estimates for commutators in Lemma 3.3.1, the following sets of estimates are immediately obtained:

$$\begin{aligned} \big\|(s/t)\partial^I L^J \partial_\mu (A_\nu, \chi)\big\|_{L^2_f(\mathcal{H}_s)} + \big\|(s/t)\partial_\mu \partial^I L^J (A_\nu, \chi)\big\|_{L^2_f(\mathcal{H}_s)} &\leqslant C_1\epsilon, && |I|+|J| \leqslant N. \\ \big\|\partial^I L^J (A_\nu, \chi)\big\|_{L^2_f(\mathcal{H}_s)} &\leqslant C_1\epsilon, && |I|+|J| \leqslant N, \\ m_g\big\|\partial^I L^J \psi\big\|_{L^2_f(\mathcal{H}_s)} + \big\|(s/t)\partial_\mu \partial^I L^J \psi\big\|_{L^2_f(\mathcal{H}_s)} &\leqslant C_1\epsilon, && |I|+|J| \leqslant N-1, \\ m_g\big\|\partial^I L^J \psi\big\|_{L^2_f(\mathcal{H}_s)} + \big\|(s/t)\partial_\mu \partial^I L^J \psi\big\|_{L^2_f(\mathcal{H}_s)} &\leqslant C_1\epsilon \log s, && |I|+|J| = N. \end{aligned} \tag{1.7.12}$$

Furthermore we can obtain L^∞ estimates by recalling the following Sobolev-type inequality on hyperboloids [44].

Proposition 1.7.4 (Sobolev-type inequality on hyperboloids)**.** *For all sufficiently smooth functions $u = u(t,x)$ supported in the region $\{(t,x) : |x| < t-1\}$, then for $s \geqslant 2$ one has*

$$\sup_{\mathcal{H}_s} \big|t^{3/2} u(t,x)\big| \lesssim \sum_{|J|\leqslant 2} \big\|L^J u\big\|_{L^2_f(\mathcal{H}_s)}, \tag{1.7.13}$$

where the summation is over Lorentz boosts L.

Combined with the estimates (1.7.12) the following also hold:

$$\begin{aligned}
\sup_{(t,x)\in\mathcal{H}_s}\Big(t^{1/2}s\big|\partial_\alpha\partial^I L^J(A_\nu,\chi)\big|+t^{1/2}s\big|\partial^I L^J\partial_\alpha(A_\nu,\chi)\big|\Big)&\lesssim C_1\epsilon,\quad |I|+|J|\leqslant N-2,\\
\sup_{(t,x)\in\mathcal{H}_s}\Big(t^{3/2}\big|\partial^I L^J A_\nu,\partial^I L^J\chi\big|\Big)&\lesssim C_1\epsilon,\quad |I|+|J|\leqslant N-2,\\
\sup_{(t,x)\in\mathcal{H}_s}\Big(m_g t^{3/2}\big|\partial^I L^J\psi\big|+t^{1/2}s\big|\partial_\alpha\partial^I L^J\psi,\partial^I L^J\partial_\alpha\psi\big|\Big)&\lesssim C_1\epsilon,\quad |I|+|J|=N-3,\\
\sup_{(t,x)\in\mathcal{H}_s}\Big(m_g t^{3/2}\big|\partial^I L^J\psi\big|+t^{1/2}s\big|\partial^I L^J\psi,\partial^I L^J\partial_\alpha\psi\big|\Big)&\lesssim C_1\epsilon\log s,\quad |I|+|J|\leqslant N-2.
\end{aligned}\tag{1.7.14}$$

1.7.4 First-order energy estimate for the Dirac field

To obtain decay estimates for the Dirac component ψ, a standard method is to analyse the second-order form of the Dirac equation (1.2.27). This is then a semilinear Klein-Gordon equation with mass m_g^2 and so there are now standard techniques to estimate the nonlinearity; see for example [2] and [45]. However, the right-hand side term appearing in our wave equation (1.2.27) does not decay sufficiently fast for this argument to close, which is due to the possibly vanishing mass $m_g^2\geqslant 0$. Thus at this point we recall Proposition 1.2.3 and the lower bound (1.2.24c) for the energy $E^{\mathcal{H}}$. This motivates us to analyse the first-order form of the Dirac equation.

We first adapt our Sobolev-type inequality on hyperboloids to include boosts which commute with the Dirac operator $i\gamma^\nu\partial_\nu$. Following Bachelot [4] we introduce modified boosts $\widehat{L}_a$ that differ from L_a by a constant matrix

$$\widehat{L}_a := L_a+\frac{1}{2}\gamma_0\gamma_a.\tag{1.7.15}$$

It then holds that $[\widehat{L}_a, i\gamma^\nu\partial_\nu]=0$. The following result is a simple extension of Proposition 1.7.4.

Corollary 1.7.5. *Suppose $\psi=\psi(t,x)$ is a sufficiently smooth spinor field supported in the region $\{(t,x):|x|<t-1\}$, then it holds for $s\geqslant 2$*

$$\sup_{\mathcal{H}_s}\big|t^{3/2}\psi(t,x)\big|\lesssim\sum_{|J|\leqslant 2}\big\|\widehat{L}^J\psi\big\|_{L^2_f(\mathcal{H}_s)},\tag{1.7.16}$$

where $\widehat{L}$ denotes a modified Lorentz boost.

We will also make use of the following Gronwall inequality, and one can find it in [14].

Lemma 1.7.6 (Gronwall-type inequality)**.** *Let $u(t)$ be a non-negative function that satisfies the integral inequality*

$$u(t) \leqslant C + \int_{t_0}^{t} b(s)u(s)^{1/2}\,ds, \quad C \geqslant 0, \tag{1.7.17}$$

where $b(t)$ is non-negative function for $t \geqslant t_0$. Then it holds

$$u(t) \leqslant C + \left(\int_{t_0}^{t} b(s)\,ds\right)^2. \tag{1.7.18}$$

We are now in a position to obtain certain improved energy and sup norm estimates for ψ.

Theorem 1.7.7. *Under the same assumptions as Theorem 1.7.1, the Dirac field ψ satisfies*

$$\left\|(s/t)\widehat{L}^J\psi\right\|_{L^2_f(\mathcal{H}_s)} \lesssim \epsilon + (C_1\epsilon)^2, \quad |J| \leqslant N, \tag{1.7.19}$$

$$\sup_{\mathcal{H}_s}\left|t^{1/2}s\widehat{L}^J\psi\right| \lesssim \epsilon + (C_1\epsilon)^2, \quad |J| \leqslant N-2. \tag{1.7.20}$$

As a consequence, one has the following sup-norm estimates for ψ:

$$\sup_{\mathcal{H}_s}\left|t\partial^I\widehat{L}^J\psi\right| \lesssim \epsilon + (C_1\epsilon)^2, \qquad |I| + |J| \leqslant N-2. \tag{1.7.21}$$

Proof. **Step 1.** Recall the equation in (1.5.1) for the Dirac field

$$\begin{aligned} &i\gamma^\mu\partial_\mu\psi - m_g\psi = H\psi,\\ &H := g(\phi_0^*\chi + \chi^*\phi_0 + \chi^*\chi) - g\gamma^\mu A_\mu. \end{aligned}$$

Since $i\psi^*\gamma^0 H - iH^*\gamma^0\psi = 0$ we have the following conserved energy

$$E^{\mathcal{H}}(s,\psi) = E^{\mathcal{H}}(s_0,\psi), \tag{1.7.22}$$

then according to the inequality (1.2.24c) in Proposition 1.2.3, we are able to initialise the induction argument by

$$\|(s/t)\psi\|_{L^2_f(\mathcal{H}_s)} \lesssim \epsilon.$$

Step 2. For induction purposes, assume

$$\left\|(s/t)\widehat{L}^J\psi\right\|_{L^2_f(\mathcal{H}_s)} \lesssim \epsilon + (C\epsilon)^2$$

holds for $0 \leqslant |J| \leqslant k-1 \leqslant N-3$, and now consider the case $1 \leqslant |J| = k \leqslant N-2$. Act

$\widehat{L}^J$ on the Dirac equation above to obtain

$$\gamma^\mu\partial_\mu(\widehat{L}^J\psi)+im_g\widehat{L}^J\psi=-iH(\widehat{L}^J\psi)-iR,$$

with

$$R:=\widehat{L}^J(H\psi)-H(\widehat{L}^J\psi),\qquad H=g(\phi_0^*\chi+\chi^*\phi_0+\chi^*\chi)-g\gamma^\mu A_\mu.$$

Observe that R contains only terms, up to some constant matrices, of type

$$\widehat{L}^{J_1}H\cdot\widehat{L}^{J_2}\psi,\quad |J_1|+|J_2|\leqslant|J|,\quad |J_2|\leqslant k-1. \tag{1.7.23}$$

Using (1.2.25) and consequently by Lemma 1.7.4 and the induction assumption

$$\begin{aligned}E^{\mathcal{H}}(s,\widehat{L}^J\psi)^{1/2}&\leqslant E^{\mathcal{H}}(s_0,\widehat{L}^J\psi)^{1/2}+\int_{s_0}^s\|R\|_{L^2_{\underline{f}}(\mathcal{H}_{\bar{s}})}d\bar{s}\\&\lesssim\epsilon+\int_{s_0}^s\sum_{\substack{|J_1|+|J_2|\leqslant|J|\\|J_2|\leqslant|J|-1}}\big\|(t/s)\widehat{L}^{J_1}H\big\|_{L^\infty(\mathcal{H}_{\bar{s}})}\big\|(s/t)\widehat{L}^{J_2}\psi\big\|_{L^2_{\underline{f}}(\mathcal{H}_{\bar{s}})}\Big)\,d\bar{s}\\&\lesssim\epsilon+(C_1\epsilon)^2\int_{s_0}^s\bar{s}^{-3/2}\,d\bar{s},\end{aligned} \tag{1.7.24}$$

which gives

$$\big\|(s/t)\widehat{L}^J\psi\big\|_{L^2_{\underline{f}}(\mathcal{H}_s)}\lesssim\epsilon,\quad |J|\leqslant k.$$

Step 3. The above analysis shows for $|J|\leqslant N-4$

$$\sum_{|J'|\leqslant2}\big\|(s/t)\widehat{L}^{J'}\widehat{L}^J\psi\big\|_{L^2_{\underline{f}}(\mathcal{H}_s)}\lesssim\epsilon+(C_1\epsilon)^2.$$

Thus by the Sobolev inequality (3.2.19), we deduce

$$\sup_{\mathcal{H}_s}\big|t^{1/2}s\widehat{L}^J\psi\big|\lesssim\epsilon+(C_1\epsilon)^2,\quad |J|\leqslant N-4.$$

Step 4. We now consider the case $|J|=N-1$. An energy estimate yields

$$\begin{aligned}E^{\mathcal{H}}(s,\widehat{L}^J\psi)^{1/2}&\leqslant\epsilon+\int_{s_0}^s\Big(\sum_{\substack{|J_1|+|J_2|\leqslant N-1,\\|J_1|>|J_2|,|J_2|\leqslant N-4}}\big\|\widehat{L}^{J_1}H\big\|_{L^2_{\underline{f}}(\mathcal{H}_{\bar{s}})}\big\|\widehat{L}^{J_2}\psi\big\|_{L^\infty(\mathcal{H}_{\bar{s}})}\\&\quad+\sum_{\substack{|J_1|+|J_2|\leqslant N-1,\\|J_1|\leqslant|J_2|,|J_2|\leqslant N-2}}\big\|(t/s)\widehat{L}^{J_1}H\big\|_{L^\infty(\mathcal{H}_{\bar{s}})}\big\|(s/t)\widehat{L}^{J_2}\psi\big\|_{L^2_{\underline{f}}(\mathcal{H}_{\bar{s}})}\Big)\,d\bar{s}\\&\leqslant\epsilon+(C_1\epsilon)^2\int_{s_0}^s\bar{s}^{-3/2}\,d\bar{s},\end{aligned} \tag{1.7.25}$$

which implies

$$\big\|(\bar{s}/t)\widehat{L}^J\psi\big\|_{L^2_f(\mathcal{H}_{\bar{s}})} \lesssim \epsilon + (C_1\epsilon)^2. \tag{1.7.26}$$

The same analysis also applies to the case $|J| = N$. And repeating Step 3 gives (1.7.20) for $|J| \leqslant N-2$. □

As a consequence, we have the following sup-norm estimates for ψ.

Corollary 1.7.8. *It holds that*

$$\sup_{\mathcal{H}_s} \big|t^{1/2}s\partial^I L^J\psi\big| \lesssim \epsilon + (C_1\epsilon)^2, \qquad |I| + |J| \leqslant N-2. \tag{1.7.27}$$

1.7.5 Refined estimates

In this final subsection we close our bootstrap argument. For this to work we move to the transformed vector field $\widetilde{A}^\nu$ defined in (1.6.3) and the transformed scalar field $\widetilde{\chi}_+$ defined in (1.6.9), which are heuristically of the form

$$A^\nu = \widetilde{A}^\nu + \mathcal{O}(|\psi|^2), \qquad \chi_+ = \widetilde{\chi}_+ + \mathcal{O}(|\psi|^2).$$

Thus using the estimates for A^ν and χ_+ coming from (1.7.12) and (1.7.14), together with the previous energy and sup-norm estimates for ψ, the following estimates for $\widetilde{A}^\nu$ and $\widetilde{\chi}_+$ hold

$$\begin{aligned}
\big\|(s/t)\partial^I L^J\partial_\mu(\widetilde{A}_\nu,\widetilde{\chi}_+)\big\|_{L^2_f(\mathcal{H}_s)} + \big\|(s/t)\partial_\mu\partial^I L^J(\widetilde{A}_\nu,\widetilde{\chi}_+)\big\|_{L^2_f(\mathcal{H}_s)} &\lesssim C_1\epsilon, \qquad |I|+|J| \leqslant N,\\
\big\|\partial^I L^J(\widetilde{A}_\nu,\widetilde{\chi}_+)\big\|_{L^2_f(\mathcal{H}_s)} &\lesssim C_1\epsilon, \qquad |I|+|J| \leqslant N,\\
\sup_{\mathcal{H}_s}\Big(t^{1/2}s\big|\partial_\alpha\partial^I L^J(\widetilde{A}_\nu,\widetilde{\chi}_+)\big| + t^{1/2}s\big|\partial^I L^J\partial_\alpha(\widetilde{A}_\nu,\widetilde{\chi}_+)\big|\Big) &\lesssim C_1\epsilon, \quad |I|+|J| \leqslant N-2,\\
\sup_{\mathcal{H}_s}\Big(t^{3/2}\big|\partial^I L^J(\widetilde{A}_\nu,\widetilde{\chi}_+)\big|\Big) &\lesssim C_1\epsilon, \quad |I|+|J| \leqslant N-2.
\end{aligned} \tag{1.7.28}$$

We first look at the energy for ψ in the case $|I|+|J| = N$

$$\mathcal{E}_{m_g}(s,\partial^I L^J\psi)^{1/2} \leqslant \epsilon + C\sum_{\nu,\mu}\int_{s_0}^s \big(\big\|\partial^I L^J((\partial_\mu H)\psi)\big\|_{L^2_f(\mathcal{H}_{\bar{s}})} + \big\|\partial^I L^J(H\partial_\mu\psi)\big\|_{L^2_f(\mathcal{H}_{\bar{s}})}\big)\,d\bar{s}.$$

By noting

$$\begin{aligned}
&\sum_{\mu} \|\partial^I L^J(\partial_\mu H\psi)\|_{L^2_{\bar f}(\mathcal{H}_{\bar s})} \\
\leqslant &\sum_{\substack{I_1+I_2=I, J_1+J_2=J \\ |I_1|+|J_1|\leqslant N-2,\,\mu}} \big\|(t/s)\partial^{I_1}L^{J_1}\partial_\mu H\big\|_{L^\infty(\mathcal{H}_{\bar s})} \big\|(s/t)\partial^{I_2}L^{J_2}\psi\big\|_{L^2_{\bar f}(\mathcal{H}_{\bar s})} \\
&+ \sum_{\substack{I_1+I_2=I, J_1+J_2=J \\ |I_1|+|J_1|\geqslant N-1,\,\mu}} \big\|(s/t)\partial^{I_1}L^{J_1}\partial_\mu H\big\|_{L^2_{\bar f}(\mathcal{H}_{\bar s})} \big\|(t/s)\partial^{I_2}L^{J_2}\psi\big\|_{L^\infty(\mathcal{H}_{\bar s})} \\
\lesssim &\,(C_1\epsilon)^2\bar s^{-3/2}\log\bar s + (C_1\epsilon)^2\bar s^{-1} \lesssim (C_1\epsilon)^2\bar s^{-1},
\end{aligned} \tag{1.7.29}$$

we obtain

$$\mathcal{E}_{m_g}(s,\partial^I L^J\psi)^{1/2} \leqslant \epsilon + C(C_1\epsilon)^2\log s. \tag{1.7.30}$$

Similarly for $|I|+|J|\leqslant N-1$ we obtain

$$\mathcal{E}_{m_g}(s,\partial^I L^J\psi)^{1/2} \leqslant \epsilon + C(C_1\epsilon)^2. \tag{1.7.31}$$

In order to obtain estimates for A^ν, we first bound the energy for $\widetilde{A}^\nu$

$$\begin{aligned}
\mathcal{E}_m(s,\partial^I L^J\tilde A^\nu)^{1/2} &\leqslant \epsilon + \int_{s_0}^{s} \big\|\partial^I L^J(\langle\gamma^0\gamma^\nu\partial_\mu\psi,\partial^\mu\psi\rangle)\big\|_{L^2_{\bar f}(\mathcal{H}_{\bar s})} + m_g^2\big\|\partial^I L^J(\psi^*\gamma^0\gamma^\nu\psi)\big\|_{L^2_{\bar f}(\mathcal{H}_{\bar s})} \\
&\quad + \big\|\partial^I L^J(\psi^2\partial A + \psi A\partial\psi + \psi^3\partial\psi)\big\|_{L^2_{\bar f}(\mathcal{H}_{\bar s})}\, d\bar s \\
&\leqslant \epsilon + C(C_1\epsilon)^2.
\end{aligned} \tag{1.7.32}$$

Next, recalling definition (1.6.3) we use Young's inequality to obtain for all $|I|+|J|\leqslant N$

$$\begin{aligned}
\mathcal{E}_{m_q}(s,\partial^I L^J A^\nu)^{1/2} &\leqslant (3/2)\mathcal{E}_{m_q}(s,\partial^I L^J\widetilde{A}^\nu)^{1/2} + (3/2)\mathcal{E}_{m_q}(s,\partial^I L^J(\psi)^2)^{1/2} \\
&\leqslant (3/2)\epsilon + C\sum_{\substack{I_1+I_2=I, J_1+J_2=J \\ I_1+J_1\leqslant N-2}} \big\|\partial^{I_1}L^{J_1}\psi\big\|_{L^\infty(\mathcal{H}_s)}\mathcal{E}(s,\partial^{I_2}L^{J_2}\psi)^{1/2} \\
&\lesssim \epsilon + (C_1\epsilon)^2.
\end{aligned} \tag{1.7.33}$$

A similar procedure gives the refined estimates for χ_+

$$\mathcal{E}_{m_\lambda}(s,\partial^I L^J\chi_+)^{1/2} \leqslant (3/2)\epsilon + C(C_1\epsilon)^2, \tag{1.7.34}$$

while the refined estimates for χ_-

$$\mathcal{E}_{m_q}(s,\partial^I L^J\chi_-)^{1/2} \leqslant \epsilon + C(C_1\epsilon)^2 \tag{1.7.35}$$

can be obtained directly. A combination of (1.7.34) and (1.7.35) gives the refined

estimates for χ

$$\mathcal{E}_{m_\lambda}(s, \partial^I L^J \chi)^{1/2} \lesssim \epsilon + (C_1\epsilon)^2. \tag{1.7.36}$$

By choosing C_1 sufficiently large and ϵ sufficiently small, we arrive at the refined bounds (1.7.11). This shows global existence and thus completes the proof of Theorem 1.7.1.

Chapter 2

Study of the GSW model

[1]

Contents

[1] This chapter is an ongoing work with Philippe G. LeFloch and Zoe Wyatt.

2.1 Introduction

2.1.1 The fields and Lagrangian of the Standard Model

Matter and gauge fields of the theory. We study the evolution equations for electromagnetic and weak interactions between elementary particles coming from the Standard Model. In particular we will look at the Higgs mechanism applied to the non-abelian $SU(2) \times U(1)$ gauge group of the Glashow-Weinberg-Salam (GSW) model.

Let us begin by presenting the Lagrangian for the GSW model. We start with four massless gauge bosons[2] coupled to a complex Higgs field and twelve fundamental fermions. The quartic potential for the Higgs field has a 'Mexican hat' shape, meaning that the non-zero minimum of this potential gives a non-zero vacuum expectation value for the Higgs. After spontaneous symmetry breaking, and on a fixed Minkowski background, the vacuum solution will now involve one massless and three massive gauge bosons. The key issue is whether the Higgs field, and indeed all the variables, are nonlinearly stable under perturbations of the vacuum solution.

We need some notation for the (five) fields of the theory which are

$$\begin{aligned}
&\Phi \in \mathbb{C}^2 && SU(2)\text{-doublet Higgs field},\\
&Y = (Y^\mu) && U(1)\text{-gauge field},\\
&W = (W_a^\mu) && SU(2)\text{-gauge field},\\
&(e_L, e_R) \in \mathbb{C}^2 \times \mathbb{C}^2 && \text{(left- and right-handed) electron spinor},\\
&\nu_L \in \mathbb{C}^2 && \text{left-handed neutrino spinor}.
\end{aligned} \tag{2.1.1}$$

Here, $\mu = 0, 1, 2, 3$ is a spacetime index and $a = 1, 2, 3$ and the fields Y, W_a are vector fields in Minkowski space while e_L, e_R, ν_L are complex-valued vector fields. For simplicity in our presentation, we consider only two fermions, but it would be straightforward to include the full set of fundamental fermions. It is convenient also to collect the left-handed spinors into an $SU(2)$-doublet

$$L = \begin{pmatrix} \nu_L \\ e_L \end{pmatrix} \in \mathbb{C}^4 \qquad \text{left-handed } SU(2)\text{-doublet}. \tag{2.1.2}$$

In addition to the fields, the GSW model also involves five coupling constants denoted by g_Y, g_W, g_e, and λ, v. The first three parameters describe the relative strength of the interactions while the last two are parameters coming from the Higgs mechanism. Certain combinations of these constants can be measured in experiments, as we discuss below.

[2]From a physical standpoint, the classical physics description of the bosonic fields is quite meaningful, however for fermionic fields it would be desirable to take quantum effects into account.

Matter and gauge interaction Lagrangian. Recall the Standard Model describes three kinds of fundamental forces: the electromagnetic, weak, and strong interactions. We are only interested in the electroweak sector of the Standard Model, which for conciseness we also refer to as the Standard Model. Following [15], we are going to derive the evolution equations for GSW theory on a flat Minkowski background with, however, the signature $(-,+,+,+)$.

The GSW theory is based on two important notions:

- Covariant derivative operators describing nonlinear interactions between the fields defined by
$$\begin{aligned} D_\mu \Phi &:= \big(\partial_\mu - i g_W W_{\mu a}\tau_a + i\frac{g_Y}{2}Y_\mu\big)\Phi, \\ D_\mu e_R &:= \big(\partial_\mu - i g_Y Y_\mu\big)e_R, \\ D_\mu L &:= \big(\partial_\mu - i g_W W_{\mu a}\tau_a - i\frac{g_Y}{2}Y_\mu\big)L, \end{aligned} \tag{2.1.3}$$
in which g_Y, g_W are physical constants. Let $\sigma_a \in \mathbb{C}^{2\times 2}$ be the standard Pauli matrices where $a = 1,2,3$, which form a basis for the Lie algebra $\mathfrak{su}(2)$ which we normalize by defining $\tau_a := \frac{1}{2}\sigma_a \in \mathbb{C}^{2\times 2}$.

- A nonlinear potential is assumed for the complex scalar field Φ which is denoted by V and satisfies
$$V(\Phi^*\Phi) \geqslant 0, \qquad V(v^2) = V'(v^2) = 0, \qquad V''(v^2) > 0. \tag{2.1.4}$$
in which $v > 0$ is physical constant.

First, the Lagrangian density for the GSW theory contains four interaction terms

$$\mathcal{L}_{GSW} = \mathcal{L}_\Phi + \mathcal{L}_Y + \mathcal{L}_W + \mathcal{L}_\psi, \tag{2.1.5}$$

in which

$$\begin{aligned} \mathcal{L}_\Phi &:= -(D^\mu\Phi)^*(D_\mu\Phi) - V(\Phi^*\Phi), \\ \mathcal{L}_Y &:= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \\ \mathcal{L}_W &:= -\frac{1}{4}H_{\mu\nu a}H_a^{\mu\nu}, \\ \mathcal{L}_\psi &:= -Re\Big(iL^*\sigma_L^\mu D_\mu L\Big) - Re\Big(ie_R^*\sigma_R^\mu D_\mu e_R\Big) - g_e(L^*\Phi e_R + e_R^*\Phi^* L), \end{aligned} \tag{2.1.6}$$

where g_e is a physical constant. The terms $\mathcal{L}_W$ and $\mathcal{L}_Y$ involve the field strength $H_{\mu\nu a}$

for the $SU(2)$-gauge fields W_a^μ and the field strength $F_{\mu\nu}$ for the $U(1)$-gauge field Y_μ:

$$\begin{aligned} H_{\mu\nu a} &:= \partial_\mu W_{\nu a} - \partial_\nu W_{\mu a} + g_W c_{abc} W_{\mu b} W_{\nu c}, \\ F_{\mu\nu} &:= \partial_\mu Y_\nu - \partial_\nu Y_\mu. \end{aligned} \tag{2.1.7}$$

The last term in the expression of $H_{\mu\nu a}$ involves the structure constants c_{abc} for the $SU(2)$-symmetry, which are nothing but the 3-dimensional totally antisymmetric Levi-Civita symbols, normalized so that $c_{123} = 1$. The notation Re stands for the real part of a complex number.

For completeness the covariant derivatives on the field tensors $F^{\mu\nu}$ and $H_a^{\mu\nu}$ are the following

$$\begin{aligned} D_\mu F^{\nu\rho} &:= \partial_\mu F^{\nu\rho}, \\ D_\mu H_a^{\nu\rho} &:= \partial_\mu H_a^{\nu\rho} + g_W c_{abc} W_{\mu b} H_c^{\nu\rho}. \end{aligned} \tag{2.1.8}$$

2.1.2 Main nonlinear stability statement

A preliminary version of our main result is now stated. A more precise statement will be provided with the regularity and decay of solutions specified. A trivial solution to the system is given by taking

$$\begin{aligned} &\Phi \text{ is a constant with } \Phi^*\Phi = v^2, \\ &Y = 0, \qquad W = 0, \\ &L = 0, \quad e_R = 0. \end{aligned} \tag{2.1.9}$$

Theorem 2.1.1 (Nonlinear stability of the vacuum state in the Glashow-Salam-Weinberg theory)**.** *The Higgs Boson is nonlinearly stable under small perturbations, in the sense that any initial perturbations to the fields* (2.1.9) *leads to a solution to the standard model which is globally defined in time with* Φ *approaching a constant value* Φ_∞ *with*

$$\lim_{t\to+\infty} \big(\Phi - \Phi_\infty, Y, W, L, e_R\big) = 0 \tag{2.1.10}$$

and

$$\lim_{t\to+\infty} \Phi^*\Phi = \Phi_\infty^*\Phi_\infty = v^2. \tag{2.1.11}$$

Our analysis will be presented on simplified models derived by taking some formal choices of the physical parameters. To be more precise, we only provide the proof of Theorem 2.1.1 under the restriction

$$m_Y = g_Y = 0. \tag{2.1.12}$$

A generalization of the Standard Model. In the GSW model, it is well known that the right-handed neutrino is taken to be $\nu_R = 0$ and the left-handed neutrino ν_L is massless and satisfies the equation

$$i\sigma_L^\mu \partial_\mu \nu_L = F_L.$$

However, in the physical experiments the left-handed neutrino has a tiny but strictly positive mass. Hence it is artificial but of mathematically and physically interest to study a more general model, where there involves right-handed neutrino and a mass parameter $m_n \geqslant 0$

$$i\sigma_L^\mu \partial_\mu \nu_L + m_n \nu_R = F_L, \qquad i\sigma_R^\mu \partial_\mu \nu_R + m_n \nu_L = F_R,$$

which can be written equivalently as

$$i\gamma^\mu \partial_\mu \psi_n + m_n \psi_n = F_{\psi_n}, \tag{2.1.13}$$

with

$$\psi_n = \begin{pmatrix} \nu_R \\ \nu_L \end{pmatrix}, \quad F_{\psi_n} := \begin{pmatrix} F_L \\ F_R \end{pmatrix}.$$

Observe that we can recover the GSW model by letting $m_n = 0$ and $\nu_R = 0$. Moreover, we are able to prove an existence result which is uniform in the mass parameter $0 \leqslant m_n \ll \min\{m_H, m_W, m_Z, m_e\}$.

2.1.3 Role of the physical constants

Further notations. Throughout this chapter we will be interested in various bilinear forms involving complex vectors and matrices. Let $\bar{z}_j$ denote the conjugate of a complex vector $z = (z_1, z_2, z_3, z_4)^T \in \mathbb{C}^4$, and $z^* := (\bar{z}_1, \bar{z}_2, \bar{z}_3, \bar{z}_4)$ denote its conjugate transpose. If also $w \in \mathbb{C}^4$ then the inner product is defined by

$$\langle z, w \rangle := z^* w = \sum_{j=1}^{4} \bar{z}_j w_j. \tag{2.1.14}$$

The Hermitian conjugate[3] of a matrix A is denoted by A^*, that is

$$(A^*)_{ij} := (\bar{A})_{ji}. \tag{2.1.15}$$

The chiral nature of GSW implies particular relationships between certain components of the spinor for the electron ψ_e and neutrino ψ_n. This is made clearer by using the Chiral representation of the Dirac matrices. First recall that the Dirac matrices γ^μ for

[3]In the physics literature, A^* is often denoted by $A^\dagger$.

$\mu = 0, 1, 2, 3$ are 4×4 matrices satisfying the identities

$$\begin{aligned} &\{\gamma^\mu, \gamma^\nu\} := \gamma^\mu\gamma^\nu + \gamma^\mu\gamma^\nu = -2\eta^{\mu\nu}, \\ &(\gamma^\mu)^* = -\eta_{\mu\nu}\gamma^\nu, \end{aligned} \tag{2.1.16}$$

where $\eta = \text{diag}(-1, 1, 1, 1)$. The Dirac matrices give a matrix representation of the Clifford algebra. We obtain the chiral representation of the Dirac matrices by first recalling the standard Pauli matrices

$$\sigma^1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.1.17}$$

These matrices can be placed into a 4-vector as

$$\sigma_R^\mu := (I_2, \sigma^i), \qquad \sigma_L^\mu := (I_2, -\sigma^i), \tag{2.1.18}$$

where $i = 1, 2, 3$.

From this the chiral representation of the gamma matrices is

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma_R^\mu \\ \sigma_L^\mu & 0 \end{pmatrix}. \tag{2.1.19}$$

Using the first gamma matrix we define the following bilinear form

$$(z, w) := z^*\gamma^0 w, \tag{2.1.20}$$

where $z, w \in \mathbb{C}^4$. In particular, this is defined in the physics literature as $\bar{\psi}\psi = \psi^*\gamma^0\psi = (\psi, \psi)$. Furthermore in the chiral basis the 5th gamma matrix is diagonal

$$\gamma_5 := i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -I_2 & 0 \\ 0 & I_2 \end{pmatrix}. \tag{2.1.21}$$

The γ_5 matrix squares to I_4 and so one often uses the following projection operators, which in this basis take the simple form,

$$P_R := \frac{1}{2}(I_4 + \gamma_5) = \begin{pmatrix} I_2 & 0 \\ 0 & 0 \end{pmatrix}, \qquad P_L := \frac{1}{2}(I_4 - \gamma_5) = \begin{pmatrix} 0 & 0 \\ 0 & I_2 \end{pmatrix}. \tag{2.1.22}$$

For a spinor ψ the projections

$$\psi_L = P_L\psi, \quad \psi_R = P_R\psi \tag{2.1.23}$$

are respectively called the left-handed and right-handed spinors. In the GSW model

we will be interested in the following 2-component spinors

$$\begin{aligned} e_R &:= P_R\psi_e \in \mathbb{C}^2 : \text{ right-handed electron,} \\ e_L &:= P_L\psi_e \in \mathbb{C}^2 : \text{ left-handed electron,} \\ \nu_L &:= P_L\psi_n \in \mathbb{C}^2 : \text{ left-handed neutrino.} \end{aligned} \tag{2.1.24}$$

Note the non-existent right-handed neutrino is not included, i.e.

$$\nu_R := P_R\psi_n = 0. \tag{2.1.25}$$

In the following we use ψ and ν_L to denote the collection of fields e_R, e_L, ν_L or equivalently e_R, L.

The gauge coupling constants. Recall there are five independent coupling constants g_Y, g_W, g_e, and λ, v in the Lagrangian density of the Standard Model. From the parameters g_Y and g_W we can compute the Weinberg angle θ_W defined by

$$\frac{g_Y^2}{g_W^2} =: \tan^2\theta_W \simeq 0.2858. \tag{2.1.26}$$

This implies that $g_Y \simeq (0.5346)g_W$ have the same order of magnitude.

The evolution of the Higgs field is described by a potential function which is often taken to be

$$V(\Phi^*\Phi) := \lambda\big(\Phi^*\Phi - v^2\big)^2, \tag{2.1.27}$$

where $\lambda, v > 0$ are constants and the following conditions clearly hold:

$$V \geqslant 0, \quad V(v^2) = 0, \qquad V'(v^2) = 0, \qquad V'' = 2\lambda > 0. \tag{2.1.28}$$

The two constants λ, v cannot be directly measured experimentally, but the expression

$$m_H := 2v\sqrt{\lambda} \tag{2.1.29}$$

can be measured and represents the mass of the Higgs field. In the physics literature, the so-called vacuum expectation value for the Higgs field is the positive constant v, and the Higgs field is expected to generate the masses for the W and Z bosons, according to the definition:

$$m_W := \frac{1}{2}vg_W, \qquad m_Z := \frac{1}{2}v\sqrt{g_W^2 + g_Y^2}. \tag{2.1.30}$$

We also define for convenience the non-physical mass parameter

$$m_Y := \frac{1}{2}vg_Y. \tag{2.1.31}$$

On the other hand, the mass of the electron is given in the form

$$m_e := \frac{1}{\sqrt{2}} v g_e, \tag{2.1.32}$$

in which the electron coupling constant g_e is found to be very small in comparison to g_Y, g_W. Since the W boson has a large mass $m_W \simeq 80$ GeV/c^2 compared to the small mass of the electron $m_e \simeq 0.51$ MeV/$c^2 = 51 \times 10^{-5}$ GeV/c^2, we thus have

$$g_e \ll g_W, g_Y. \tag{2.1.33}$$

In conclusion, the relation between the sizes of these mass parameters is

$$m_n \ll m_e \ll m_Y \simeq m_W \simeq m_Z \simeq m_H, \tag{2.1.34}$$

in which m_n denotes the mass of neutrinos, which does not appear in the Lagrangian density of the Standard Model though.

The latest data [16] for the masses of the different particles measured at CERN are provided in the following two tables. Note also that the vacuum expectation value of the Higgs is $v \simeq 246 GeV$.

	Higgs boson	Z boson	W boson
PDG measurements GeV/c^2	125.18 ± 0.16	91.1876 ± 0.0021	80.379 ± 0.012
GSW value	$m_H = 2v\sqrt{\lambda}$	$m_Z = \frac{1}{2} v \sqrt{g_W^2 + g_Y^2}$	$m_W = \frac{1}{2} v g_W$

Table 2.1: Masses of the gauge and Higgs bosons

	Electron	Left-handed neutrino
PDG measurements MeV/c^2	$0.5109989461 \pm 0.0000000031$	$0 < m_\nu < 2 \times 10^{-6}$
GSW value	$m_e = \frac{1}{\sqrt{2}} v g_e$	0

Table 2.2: Masses of the electron and neutrino fermions

2.1.4 Strategy and outline of this chapter

Coupling constants and simplified models. We first recall that the relation between the sizes of the mass parameters appearing in the Lagrangian density (2.1.5) is

$$m_e \ll m_Y \simeq m_W \simeq m_Z \simeq m_H,$$

while a similar relation holds if we consider the coupling constants

$$g_e \ll g_Y \simeq g_W \simeq \sqrt{\lambda}. \tag{2.1.35}$$

Next we deduce simplified models by setting certain parameters to be zero, and then analyse them. It might not be physical to do so, but this manipulation helps us to understand the structure of the full system.

CASE I. We set

$$m_Y = 0, \tag{2.1.36}$$

which is equivalent to set $g_Y = 0$. We recall again the complicated expression of the Lagrangian density

$$\begin{aligned}
\mathcal{L}_{GSW} &= \mathcal{L}_\Phi + \mathcal{L}_Y + \mathcal{L}_W + \mathcal{L}_\psi,\\
\mathcal{L}_\Phi &= -(D^\mu\Phi)^*(D_\mu\Phi) - V(\Phi^*\Phi),\\
\mathcal{L}_Y &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu},\\
\mathcal{L}_W &= -\frac{1}{4}H_{\mu\nu a}H_a^{\mu\nu},\\
\mathcal{L}_\psi &= -Re\Big(iL^*\sigma_L^\mu D_\mu L\Big) - Re\Big(ie_R^*\sigma_R^\mu D_\mu e_R\Big) - g_e(L^*\Phi e_R + e_R^*\Phi^* L),
\end{aligned}$$

with

$$\begin{aligned}
H_{\mu\nu a} &= \partial_\mu W_{\nu a} - \partial_\nu W_{\mu a} + g_W c_{abc} W_{\mu b} W_{\nu c},\\
F_{\mu\nu} &= \partial_\mu Y_\nu - \partial_\nu Y_\mu,
\end{aligned}$$

as well as the definition of the covariant derivatives

$$\begin{aligned}
D_\mu\Phi &= \big(\partial_\mu - ig_W W_{\mu a}\tau_a + i\frac{g_Y}{2}Y_\mu\big)\Phi,\\
D_\mu e_R &= \big(\partial_\mu - ig_Y Y_\mu\big)e_R,\\
D_\mu L &= \big(\partial_\mu - ig_W W_{\mu a}\tau_a - i\frac{g_Y}{2}Y_\mu\big)L,\\
D_\mu F^{\nu\rho} &= \partial_\mu F^{\nu\rho},\\
D_\mu H_a^{\nu\rho} &= \partial_\mu H_a^{\nu\rho} + g_W c_{abc} W_{\mu b} H_c^{\nu\rho}.
\end{aligned}$$

Due to the choice of $g_Y = 0$, the gauge field Y^μ interacts with no other fields but itself. According to (2.2.2) below the Euler-Lagrange equations are

$$\begin{aligned}
D_\mu D^\mu\Phi &= V'(\Phi^*\Phi)\Phi + g_e e_R^* L,\\
D_\nu F^{\mu\nu} &= -\frac{g_Y}{2}j^\mu = 0,\\
D_\nu H_a^{\mu\nu} &= -g_W\, J_a^\mu,\\
i\sigma_R^\mu D_\mu e_R &= g_e\Phi^* L,
\end{aligned} \tag{2.1.37}$$

$$i\sigma_L^\mu D_\mu L = g_e \Phi e_R,$$

with

$$J_a^\nu := Re[L^* \sigma_L^\nu \tau_a L] - i(D^\nu \Phi)^* \tau_a \Phi + i\Phi^* \tau_a (D^\nu \Phi),$$

and in the above the gauge field Y^μ only appears in the second equation (the $F^{\mu\nu}$ equation).

CASE II. We consider another simple case and set

$$m_Y = m_W = 0, \tag{2.1.38}$$

which is equivalent to set $g_Y = g_W = 0$. All of the covariant derivatives in the Lagrangian density now become partial derivatives, the gauge field Y^μ only interact with itself, and so do the gauge fields W_a.

Similarly according to (2.2.2) the Euler-Lagrange equations are

$$\begin{aligned}
\partial_\mu \partial^\mu \Phi &= V'(\Phi^* \Phi)\Phi + g_e e_R^* L, \\
\partial_\nu F^{\mu\nu} &= -\frac{g_Y}{2} j^\mu = 0, \\
\partial_\nu H_a^{\mu\nu} &= -g_W \, J_a^\mu = 0, \\
i\sigma_R^\mu \partial_\mu e_R &= g_e \Phi^* L, \\
i\sigma_L^\mu \partial_\mu L &= g_e \Phi e_R.
\end{aligned} \tag{2.1.39}$$

CASE III. We consider an even simpler case which is obtained by setting

$$m_e = m_Y = m_W = 0, \tag{2.1.40}$$

which is equivalent to $g_e = g_Y = g_W = 0$.

Similarly according to (2.2.2) the Euler-Lagrange equations are

$$\begin{aligned}
\partial_\mu \partial^\mu \Phi &= V'(\Phi^* \Phi)\Phi + g_e e_R^* L = V'(\Phi^* \Phi)\Phi, \\
\partial_\nu F^{\mu\nu} &= -\frac{g_Y}{2} j^\mu = 0, \\
\partial_\nu H_a^{\mu\nu} &= -g_W \, J_a^\mu = 0, \\
i\sigma_R^\mu \partial_\mu e_R &= g_e \Phi^* L = 0, \\
i\sigma_L^\mu \partial_\mu L &= g_e \Phi e_R = 0.
\end{aligned} \tag{2.1.41}$$

We note that all of the equations are linear.

The Dirac field on hyperboloidal slices. In Section 1.2 of Chapter 1 we obtain a new energy functional for the Dirac field defined on hyperboloids

$$E^{\mathcal{H}}(s,\psi) = \int_{\mathcal{H}_s} \Big(\psi^*\psi - \frac{x_i}{t}\psi^*\gamma^0\gamma^i\psi\Big)\, dx. \tag{2.1.42}$$

Significantly, we are able to show that this energy is positive definite with a useful coercive property on the standard L^2 norm. It is useful for a massless spinor or a spinor with tiny mass, but for the case of a spinor with large mass we anticipate that the more useful estimates come from looking at the Dirac equation as a second-order Klein-Gordon type PDE.

Structure of this Chapter. In Section 2.2 we study the system of equations and choose the gauge. Next in Section 2.3 a detailed statement of the main result is given. Then we analyse the system of the simplified model in Section 2.4. In the end we prove the global stability result of the simplified model.

2.2 Formulation and gauge choice for the GSW model

2.2.1 Derivation of the field equations

It is a standard matter to derive evolution equations by expressing the conditions for a critical point of the Lagrangian. It will be convenient to introduce the currents

$$\begin{aligned} j^\nu &:= L^*\sigma_L^\nu L + 2e_R^*\sigma^\nu e_R + i(D^\nu\Phi)^*\Phi - i\Phi^*(D^\nu\Phi),\\ J_a^\nu &:= Re\big[L^*\sigma_L^\nu\tau_a L\big] - i(D^\nu\Phi)^*\tau_a\Phi + i\Phi^*\tau_a(D^\nu\Phi). \end{aligned} \tag{2.2.1}$$

Lemma 2.2.1. *The Euler-Lagrange equations associated with the Lagrangian $\mathcal{L}_{GSW}$ defined in* (2.1.5) *read as follows:*

$$\begin{aligned} D_\mu D^\mu\Phi &= V'(\Phi^*\Phi)\Phi + g_e e_R^* L,\\ D_\nu F^{\mu\nu} &= -\frac{g_Y}{2} j^\mu,\\ D_\nu H_a^{\mu\nu} &= -g_W\, J_a^\mu,\\ i\sigma_R^\mu D_\mu e_R &= g_e\Phi^* L,\\ i\sigma_L^\mu D_\mu L &= g_e\Phi e_R, \end{aligned} \tag{2.2.2}$$

in which the field strengths and covariant derivatives are defined in (2.1.7) *and* (2.1.3)*, respectively.*

We are interested in obtaining tractable PDEs for the unknowns $(e_R, L, \Phi, Y^\nu, W_a^\nu)$.

Thus using the expressions for $F_{\mu\nu}$ and $H_{\mu\nu a}$ in terms of Y_μ and $W_{\mu a}$ from (2.1.7), we obtain the following explicit form of the system:

$$\begin{aligned}
\Box\Phi &= -ig_W\Big(2W_{\mu a}\tau_a\partial^\mu\Phi + \mathrm{div}W_a\tau_a\Phi\Big) + ig_Y\Big(Y_\mu\partial^\mu\Phi + \frac{1}{2}\mathrm{div}Y\Phi\Big) \\
&\quad - \frac{g_W^2}{4}W_{\mu a}W_a^\mu\Phi + g_Y g_W Y_\mu W_a^\mu\tau_a\Phi - \frac{g_Y^2}{4}Y_\mu Y^\mu\Phi \\
&\quad + V'(\Phi^*\Phi)\Phi + g_e e_R^* L, \qquad (2.2.3a)\\
\Box Y^\nu - \partial^\nu(\mathrm{div}Y) &= -\frac{g_Y}{2}\Big(\big(L^*\sigma_L^\nu L + 2e_R^*\sigma_L^\nu e_R\big) + i\big((\partial^\nu\Phi^*)\Phi - \Phi^*\partial^\nu\Phi\big)\Big) \\
&\quad + g_Y g_W W_a^\nu\Phi^*\tau_a\Phi - \frac{g_Y^2}{2}Y^\nu\Phi^*\Phi, \qquad (2.2.3b)\\
\Box W_a^\nu - \partial^\nu(\mathrm{div}W_a) &= -g_W\Big(-c_{abc}\partial^\mu(W_{\mu b}W_c^\nu) + Re\big[L^*\sigma_L^\nu\tau_a L\big] \\
&\quad - c_{abc}W_b^\mu\big(\partial_\mu W_c^\nu - \partial^\nu W_{\mu c}\big) - i\big(\partial^\nu\Phi^*\tau_a\Phi - \Phi^*\tau_a\partial^\nu\Phi\big)\Big) \\
&\quad + g_W^2 c_{abc}c_{cde}W_b^\mu W_{\mu e}W_e^\nu - \frac{g_W^2}{2}W_a^\nu\Phi^*\Phi + g_W g_Y Y^\nu\Phi^*\tau_a\Phi, \\
&\qquad (2.2.3c)\\
i\sigma_R^\mu\partial_\mu e_R &= g_e\Phi^* L - g_Y e_R\sigma_R^\mu Y_\mu, \qquad (2.2.3d)\\
i\sigma_L^\mu\partial_\mu L &= g_e\Phi e_R - \frac{g_Y}{2}\sigma_L^\mu Y_\mu L + g_W W_{\mu a}\sigma_L^\mu\tau_a L. \qquad (2.2.3e)
\end{aligned}$$

Observe that these equations are underdetermined without a specific gauge choice. In the following we will discuss which gauge choice we would like to impose.

One should keep in mind that, by definition,

$$\Phi^* L = \Phi_1^*\nu_L + \Phi_2^* e_L, \qquad \Phi e_R = \begin{pmatrix}\Phi_1 e_R\\ \Phi_2 e_R\end{pmatrix}, \qquad (2.2.4)$$

as well as

$$\sigma_L^\mu L = \begin{pmatrix}\sigma_L^\mu\nu_L\\ \sigma_L^\mu e_L\end{pmatrix}, \qquad \tau_a L = \begin{pmatrix}\tau_{a11}\nu_L + \tau_{a12}e_L\\ \tau_{a21}\nu_L + \tau_{a22}e_L\end{pmatrix}, \qquad (2.2.5)$$

where τ_{aAB} denotes the A-th row and B-th column component of the matrix τ_a.

2.2.2 Energy identities leading to gauge choice

By looking at the energy functionals associated to the following equations we will see that a fairly specific gauge choice is suggested to be chosen.

$$\begin{aligned}
\Box\Phi &= -2i(g_W W_{\mu a}\tau_a - \frac{g_Y}{2}Y_\mu)\partial^\mu\Phi - i(g_W \mathrm{div}W_a\tau_a - \frac{g_Y}{2}\mathrm{div}Y)\Phi \\
&\quad - \frac{g_W^2}{4}W_{\mu a}W_a^\mu\Phi + g_Y g_W Y_\mu W_a^\mu\tau_a\Phi - \frac{g_Y^2}{4}Y_\mu Y^\mu\Phi \\
&\quad + V'(\Phi^*\Phi)\Phi + g_e e_R^* L, \\
\Box Y^\nu &= \partial^\nu(\mathrm{div}Y) - \frac{g_Y}{2}\big(L^*\sigma_L^\nu L + 2e_R^*\sigma_L^\nu e_R\big) - \frac{ig_Y}{2}\big((\partial^\nu\Phi^*)\Phi - \Phi^*\partial^\nu\Phi\big) \\
&\quad + g_Y g_W W_a^\nu\Phi^*\tau_a\Phi - \frac{g_Y^2}{2}Y^\nu\Phi^*\Phi, \\
\Box W_a^\nu &= \partial^\nu(\mathrm{div}W_a) + g_W c_{abc}\partial^\mu(W_{\mu b}W_c^\nu) - g_W Re\big[L^*\sigma_L^\nu\tau_a L\big] \\
&\quad + g_W c_{abc}W_b^\mu\big(\partial_\mu W_c^\nu - \partial^\nu W_{\mu c} + g_W c_{cde}W_{\mu e}W_e^\nu\big) + ig_W\big(\partial^\nu\Phi^*\tau_a\Phi - \Phi^*\tau_a\partial^\nu\Phi\big) \\
&\quad - \frac{g_W^2}{2}W_a^\nu\Phi^*\Phi + g_W g_Y Y^\nu\Phi^*\tau_a\Phi, \\
i\sigma_R^\mu\partial_\mu e_R &= g_e\Phi^* L - g_Y e_R\sigma_R^\mu Y_\mu, \\
i\sigma_L^\mu\partial_\mu L &= g_e\Phi e_R - \frac{g_Y}{2}\sigma_L^\mu Y_\mu L + g_W W_{\mu a}\sigma_L^\mu\tau_a L, \\
\mathrm{div}Y &= f, \\
\mathrm{div}W_a &= g_a.
\end{aligned} \tag{2.2.6}$$

We first define the following energy for the W and Y fields by

$$E_{YW}(t) := \int_{\mathbb{R}^3}\Big(\sum_{\alpha,\nu}|\partial_\alpha Y^\nu|^2 + \sum_{\alpha,\nu,a}|\partial_\alpha W_a^\nu|^2 + H(Y,W,\Phi)\Big)\,dx,$$

where

$$H(Y,W,\Phi) := \frac{1}{4}\Phi^*\Big(g_Y^2\sum_\nu Y^\nu Y^\nu - 2g_W g_Y\sum_{a,\nu}\sigma^a Y^\nu W_a^\nu + g_W^2\sum_{a,\nu}W_a^\nu W_a^\nu\Big)\Phi. \tag{2.2.7}$$

Lemma 2.2.2. *With the notation $C_a = \Phi^*\sigma_a\Phi$ the following identity holds*

$$H(Y,W,\Phi) = \frac{1}{4}\frac{1}{\Phi^*\Phi}\big|g_Y\Phi^*\Phi Y - 2g_W\sum_a C_a W_a\big|^2 + \frac{g_W^2}{\Phi^*\Phi}\sum_{a<b}\big|C_a W_b - C_b W_a\big|^2. \tag{2.2.8}$$

Proof.

$$\begin{aligned}
&H(Y,W,\Phi) - \frac{1}{4\Phi^*\Phi}\big|g_Y\Phi^*\Phi Y - 2g_W\sum_a C_aW_a\big|^2\\
&= \frac{1}{4}g_W^2\Phi^*\Phi\Big[\sum_a|W_a|^2 - \frac{4}{(\Phi^*\Phi)^2}\big|\sum_a C_aW_a\big|^2\Big]\\
&= \frac{1}{4}g_W^2\Phi^*\Phi\Big[\sum_a|W_a|^2 - \frac{4}{(\Phi^*\Phi)^2}\sum_{a,b}C_b^2|W_a|^2 + \frac{4}{(\Phi^*\Phi)^2}\sum_{a<b}\big|C_aW_b - C_bW_a\big|^2\Big]\\
&= \frac{g_W^2}{\Phi^*\Phi}\sum_{a<b}\big|C_aW_b - C_bW_a\big|^2 + g_W^2\Phi^*\Phi\sum_a|W_a|^2\Big(1 - \frac{4}{(\Phi^*\Phi)^2}\sum_b C_b^2\Big).
\end{aligned}$$

Hence in order to verify (2.2.8) it suffices to prove

$$\sum_b C_b^2 = \frac{1}{4}(\Phi^*\Phi)^2. \tag{2.2.9}$$

First recall the completeness relation for the Pauli matrices

$$\sum_{a=1}^{3}(\sigma^a)^\alpha{}_\beta(\sigma^a)^\gamma{}_\tau = 2\delta_\tau^\alpha\delta_\beta^\gamma - \delta_\beta^\alpha\delta_\tau^\gamma.$$

This identity implies the following

$$\begin{aligned}
4\sum_a C_a^2 &= \sum_a(\Phi^*\sigma^a\Phi)^2 = \sum_a\Phi^\beta(\sigma^a)^\alpha{}_\beta(\Phi)_\alpha(\Phi)^\tau(\sigma^a)^\gamma{}_\tau(\Phi)_\gamma\\
&= \Phi^\beta(\Phi)_\alpha(\Phi)^\tau(\Phi)_\gamma\big(2\delta_\tau^\alpha\delta_\beta^\gamma - \delta_\beta^\alpha\delta_\tau^\gamma\big) = (\Phi^*\Phi)^2.
\end{aligned}$$

□

Lemma 2.2.3. *Assume Y and W_a are solutions to* (2.2.6)*. Then*

$$\begin{aligned}
\frac{d}{dt}E_{YW}(t) =& \sum_\nu\int_{\mathbb{R}^3}\Big(-\partial^\nu f + \frac{ig_Y}{2}\big((\partial^\nu\Phi^*)\Phi - \Phi^*\partial^\nu\Phi\big)\Big)\partial_tY^\nu\,dx\\
&+\sum_{\nu,a}\int_{\mathbb{R}^3}\Big(-\partial^\nu g_a - ig_W\big(\partial^\nu\Phi^*\tau_a\Phi - \Phi^*\tau_a\partial^\nu\Phi\big)\Big)\partial_tW_a^\nu\,dx + Q_{YW},
\end{aligned} \tag{2.2.10}$$

in which the high order terms are given by

$$\begin{aligned} Q_{YW} =& \sum_{\nu} \int_{\mathbb{R}^3} \Big(\frac{g_Y}{2} \big(L^* \sigma_L^\nu L + 2 e_R^* \sigma_L^\nu e_R \big) \Big) \partial_t Y^\nu \, dx \\ &+ \sum_{\nu,a} \int_{\mathbb{R}^3} \Big(- g_W c_{abc} \partial^\mu (W_{\mu b} W_c^\nu) + g_W Re\big[L^* \sigma_L^\nu \tau_a L \big] \Big) \partial_t W_a^\nu \, dx \\ &+ \sum_{\nu,a} \int_{\mathbb{R}^3} \Big(\frac{g_Y^2}{2} Y^\nu Y^\nu \partial_t (\Phi^* \Phi) - g_W g_Y Y^\nu W_a^\nu \partial_t (\Phi^* \tau_a \Phi) + \frac{g_W^2}{2} W_a^\nu W_a^\nu \partial_t (\Phi^* \Phi) \Big) \, dx. \end{aligned}$$

Since we do not want low order terms to appear in the right hand side of the energy identity (2.2.10), the gauge choice should let

$$-\partial^\nu f + \frac{i g_Y}{2} \big((\partial^\nu \Phi^*) \Phi - \Phi^* \partial^\nu \Phi \big) \quad \text{and} \quad -\partial^\nu g_a - i g_W \big(\partial^\nu \Phi^* \tau_a \Phi - \Phi^* \tau_a \partial^\nu \Phi \big) \tag{2.2.11}$$

be at least quadratic.

Lemma 2.2.4. *Assume* Φ *is a solution to* (2.2.6) *then it holds*

$$\begin{aligned} &\frac{d}{dt} \int_{\mathbb{R}^3} \Big(|\partial \Phi|^2 + V(\Phi^* \Phi) \Big) \, dx \\ =& \int_{\mathbb{R}^3} \Big(i \frac{g_Y}{2} f \big(\partial_t \Phi^* \Phi - \Phi^* \partial_t \Phi \big) - i g_W g_a \big(\partial_t \Phi^* \tau_a \Phi - \Phi^* \tau_a \partial_t \Phi \big) \Big) \, dx \\ &+ \int_{\mathbb{R}^3} \Big(- \frac{g_W^2}{4} W_{\mu a} W_a^\mu \partial_t \big(\Phi^* \Phi \big) + g_Y g_W Y_\mu W_a^\mu \partial_t \big(\Phi^* \tau_a \Phi \big) - \frac{g_Y^2}{4} Y_\mu Y^\mu \partial_t \big(\Phi^* \Phi \big) \Big) \, dx. \end{aligned} \tag{2.2.12}$$

We note the first term in the right hand side (2.2.12) seems problematic. But this motivates us to choose the gauge so that

$$\begin{aligned} & i \frac{g_Y}{2} f \big(\partial_t \Phi^* \Phi - \Phi^* \partial_t \Phi \big) - i g_W g_a \big(\partial_t \Phi^* \tau_a \Phi - \Phi^* \tau_a \partial_t \Phi \big) \\ =& - \frac{d}{dt} \big(\text{positive term} \big) + \text{cubic term}. \end{aligned} \tag{2.2.13}$$

2.2.3 The gauge choice for the GSW model

We will primarily be interested in the following vacuum solutions to (2.2.2)

$$(Y^\mu = 0, W_a^\mu = 0, \Phi = \Phi_0, e_R = 0, L = 0), \tag{2.2.14}$$

such that $\Phi_0^*\Phi_0 = v^2$ and $\partial_\mu\Phi_0 = 0$. If we denote the perturbation of the Higgs field by

$$\chi := \Phi - \Phi_0,$$

and adopt the notations $\mathring{C}_a := \phi_0^*\tau_a\phi_0$, then the evolution equations take the following form.

- Equations of the Higgs field:

$$\begin{aligned}
&-\Box\chi - V'(\Phi^*\Phi)\Phi_0 + g_Y^2(\chi^*\Phi_0 - \Phi_0^*\chi)\Phi_0 + g_W^2(\chi^*\tau_a\Phi_0 - \Phi_0^*\tau_a\chi)\tau_a\Phi_0 \\
&= ig_W\Big(2W_{\mu a}\tau_a\partial^\mu\chi + \mathrm{div}W_a\tau_a\chi\Big) + ig_Y\Big(Y_\mu\partial^\mu\chi + \frac{1}{2}\mathrm{div}Y\chi\Big) \\
&\quad - \frac{g_W^2}{4}W_{\mu a}W_a^\mu\Phi + g_Y g_W Y_\mu W_a^\mu\tau_a\Phi - \frac{g_Y^2}{4}Y_\mu Y^\mu\Phi + g_e e_R^* L + V'(\Phi^*\Phi)\chi.
\end{aligned} \tag{2.2.15a}$$

- Equations of the gauge fields:

$$\begin{aligned}
&-\Box Y^\nu + \frac{g_Y^2v^2}{2}Y^\nu - g_W g_Y\mathring{C}_a W_a^\nu \\
&= \frac{g_Y}{2}\Big(\big(L^*\sigma_L^\nu L + 2e_R^*\sigma_L^\nu e_R\big) + i\big((\partial^\nu\chi^*)\chi - \chi^*\partial^\nu\chi\big)\Big) \\
&\quad - g_Y g_W W_a^\nu\big(\Phi_0^*\tau_a\chi + \chi^*\tau_a\Phi_0 + \chi^*\tau_a\chi\big) + \frac{g_Y^2}{2}Y^\nu\big(\Phi^*\Phi - v^2\big), \\
&\quad -\Box W_a^\nu + \frac{g_W^2v^2}{2}W_a^\nu - g_W g_Y\mathring{C}_a Y^\nu \\
&= g_W\Big(-c_{abc}\partial^\mu(W_{\mu b}W_c^\nu) + Re\big[L^*\sigma_L^\nu\tau_a L\big] - c_{abc}W_b^\mu\big(\partial_\mu W_c^\nu - \partial^\nu W_{\mu c}\big) \\
&\quad - i\big(\partial^\nu\chi^*\tau_a\chi - \chi^*\tau_a\partial^\nu\chi\big)\Big) - g_W^2c_{abc}c_{cde}W_b^\mu W_{\mu e}W_e^\nu + \frac{g_W^2}{2}W_a^\nu\big(\Phi^*\Phi - v^2\big) \\
&\quad - g_W g_Y Y^\nu\big(\chi^*\tau_a\Phi_0 + \Phi_0^*\tau_a\chi + \chi^*\tau_a\chi\big).
\end{aligned} \tag{2.2.15b}$$

- Equations of the spinors:

$$\begin{aligned}
i\sigma_R^\mu\partial_\mu e_R &= g_e\Phi_0^*L - g_Y\sigma_R^\mu Y_\mu e_R + g_e\chi^*L, \\
i\sigma_L^\mu\partial_\mu L &= g_e\Phi_0 e_R - \frac{g_Y}{2}\sigma_L^\mu Y_\mu L + \frac{g_W}{2}W_{\mu a}\big(\sigma_L^\mu\tau_a - \tau_a\sigma_L^\mu\big)L + g_e\chi e_R.
\end{aligned} \tag{2.2.15c}$$

- The gauge conditions:

$$\mathrm{div}Y = \frac{ig_Y}{2}\big(\partial^\nu\chi^*\Phi_0 - \Phi_0^*\partial^\nu\chi\big), \qquad \mathrm{div}W_a = -ig_W\big(\partial^\nu\chi^*\tau_a\Phi_0 - \Phi_0^*\tau_a\partial^\nu\chi\big). \tag{2.2.15d}$$

In the above the terms of quadratic order or higher in the evolution equations are grouped together and put in the right hand sides of the equations. We also note the gauge choice in (2.2.15d) meet the requirements in (2.2.11) and (2.2.13).

2.2.4 Propagation of the gauge

Analogous to the Lemma 1.5.2 of $U(1)$ model in Chapter 1 we impose the following gauge conditions on the initial data

$$\mathrm{div} Y = f, \qquad \mathrm{div} W_a = g_a, \tag{2.2.16}$$

and show that such a gauge choice propagates under the evolution equations (2.2.18) below. When $f, g \equiv 0$ the gauge choice (2.2.16) is often called the Lorenz gauge. Such a gauge was chosen for the Yang-Mills field considered in [7]. The interest of the following result is that it produces immediately a coupled system of wave and Dirac equaitons for which the initial value problem is well-known to be well-posed for sufficiently small times.

Proposition 2.2.5 (Hyperbolic formulation of the GSW model in the inhomogeneous Lorenz gauge)**.** *Consider the following initial data*

$$\begin{aligned} (div Y - f)|_{t=0} = 0, \quad \partial_t (div Y - f)|_{t=0} = 0, \\ (div W_a - g_a)|_{t=0} = 0, \quad \partial_t (div W_a - g_a)|_{t=0} = 0, \end{aligned} \tag{2.2.17}$$

together with the system

$$\begin{aligned} -\Box \Phi &= -2i(g_W W_{\mu a}\tau_a - \frac{g_Y}{2} Y_\mu)\partial^\mu \Phi - i(g_W g_a \tau_a - \frac{g_Y}{2} f)\Phi \\ &\quad - (g_W W_{\mu a}\tau_a - \frac{g_Y}{2} Y_\mu)(g_W W_b^\mu \tau_b - \frac{g_Y}{2} Y^\mu)\Phi + V'(\Phi^* \Phi) + g_e e_R^* L, \\ -\Box Y^\nu &= -\partial^\nu f - \frac{g_Y}{2} j^\nu, \\ -\Box W_a^\nu &= -\partial^\nu g_a + Q_a^\nu, \\ i\sigma_R^\mu \partial_\mu e_R &= g_e \Phi^* L - g_Y e_R \sigma^\mu Y_\mu, \\ i\sigma_L^\mu \partial_\mu L &= g_e \Phi e_R - \frac{g_Y}{2}\sigma_L^\mu Y_\mu L + \frac{g_W}{2} W_{\mu a}\big(\sigma_L^\mu \tau_a - \tau_a \sigma_L^\mu\big) L, \end{aligned} \tag{2.2.18}$$

in which, with the definition (2.2.1)*,*

$$Q_a^\nu := g_W c_{abc}\big(W_c^\mu \partial_\mu W_b^\nu + W_b^\nu g_c\big) - g_W J_a^\nu. \tag{2.2.19}$$

Then as long as the solution to (2.2.18) *exists, it satisfies*

$$div Y = f, \qquad div W_a = g_a. \tag{2.2.20}$$

Proof. Step 1. To show the propagation of the gauge choice for Y^ν we first take the divergence of the evolution equation for Y^ν. This yields

$$-\Box(\mathrm{div}Y - f) = \partial_\nu j^\nu. \tag{2.2.21}$$

Thus it suffices to show

$$\partial_\nu j^\nu = \partial_\nu\big(L^*\sigma_L^\nu L + 2e_R^*\sigma^\nu e_R + i(D^\nu\Phi)^*\Phi - i\Phi^*(D^\nu\Phi)\big) = 0, \tag{2.2.22}$$

using the evolution equations (2.2.18). We now introduce the short-hand notations

$$D_\mu\Phi = \partial_\mu\Phi + i\Lambda_\mu\Phi, \quad \Lambda_\mu := -g_W W_{\mu a}\tau_a + \frac{g_Y}{2}Y_\mu \tag{2.2.23}$$

with the property $\Lambda_\mu = \Lambda_\mu^*$. We first compute

$$\begin{aligned}
&\partial_\nu\Big(i(D^\nu\Phi)^*\Phi - i\Phi^*(D^\nu\Phi)\Big)\\
&= i\partial_\nu(D^\nu\Phi)^*\Phi + i(D^\nu\Phi)^*\partial_\nu\Phi - i\Phi^*\partial_\nu(D^\nu\Phi) - i\partial_\nu\Phi^*(D^\nu\Phi)\\
&= i((D^\nu - i\Lambda_\nu)D^\nu\Phi)^*\Phi + i(D^\nu\Phi)^*(D^\nu - i\Lambda_\nu)\Phi\\
&\quad - i\Phi^*(((D^\nu - i\Lambda_\nu))D^\nu\Phi) - i((D^\nu - i\Lambda_\nu)\Phi)^*(D^\nu\Phi)\\
&= i(D_\nu D^\nu\Phi)^*\Phi - (D^\nu\Phi)^*\Lambda_\nu\Phi + i(D^\nu\Phi)^*D^\nu\Phi + (D^\nu\Phi)^*\Lambda_\nu\Phi\\
&\quad - i\Phi^*(D_\nu D^\nu\Phi) - \Phi^*\Lambda_\nu D^\nu\Phi - i(D^\nu\Phi)^*D^\nu\Phi + \Phi^*\Lambda_\nu D^\nu\Phi,\\
&= -i\big(V'(\Phi^*\Phi)\Phi + g_e e_R^* L\big)^*\Phi + i\Phi^*\big(V'(\Phi^*\Phi)\Phi + g_e e_R^* L\big)\\
&= -ig_e L^* e_R\Phi + ig_e\Phi^* e_R^* L,
\end{aligned}$$

in which we used the evolution equations (2.2.2). The remaining terms of (2.2.22) give

$$\begin{aligned}
&\partial_\nu\Big(L^*\sigma_L^\nu L + 2e_R^*\sigma^\nu e_R\Big)\\
&= (\partial_\nu L)^*\sigma_L^\nu L + L^*\sigma_L^\nu\partial_n uL + 2(\partial_\nu e_R)^*\sigma^\nu e_R + 2e_R^*\sigma^\nu(\partial_\nu e_R)\\
&= -ig_W W_{\nu a}L^*\tau_a\sigma_L^\nu L - i\frac{g_Y}{2}Y_\nu L^*\sigma_L^\nu L + ig_e e_R^*\Phi^* L - i\frac{g_W}{2}W_{\nu a}L^*\big(\sigma_L^\nu\tau_a - \tau_a\sigma_L^\nu\big)L\\
&\quad + ig_W W_{\nu a}L^*\sigma_L^\nu\tau_a L + i\frac{g_Y}{2}Y_\nu L^*\sigma_L^\nu L - ig_e L^*\Phi e_R - i\frac{g_W}{2}W_{\nu a}L^*\big(\sigma_L^\nu\tau_a - \tau_a\sigma_L^\nu\big)L\\
&\quad - 2ig_Y Y_\nu e_R^*\sigma_R e_R + 2ig_e L^*\Phi e_R + 2ig_Y Y_\nu e_R^*\sigma_R^\nu e_R - 2ig_e e_R^*\Phi^* L\\
&= ig_e L^*\Phi e_R - ig_e e_R^*\Phi^* L.
\end{aligned}$$

Putting these results together gives $\partial_\nu j^\nu = 0$ and this proves (2.2.22). Before moving to the second part of the proof, note that the result we have shown is equivalent to proving

$$D_\nu D_\mu F^{\mu\nu} = D_\nu j^\nu = \partial_\nu j^\nu = 0. \tag{2.2.24}$$

Step 2. We now turn to prove the propagation of the gauge choice for W_a^ν. Using

the last expression (2.2.24) as inspiration, we first prove

$$D_\mu D_\nu H_a^{\mu\nu} = 0. \tag{2.2.25}$$

For simplicity, we will sum over the a index, so $\tau_a c_{abc} X_b Y_c = [X,Y]_a \tau_a = [X,Y]$. Using the Jacobi identity for commutators one has

$$\begin{aligned} D_\nu D_\mu H^{\mu\nu} &= \partial_\nu\big(\partial_\mu H^{\mu\nu} + [W_\mu, H^{\mu\nu}]\big) + [W_\nu, \partial_\mu H^{\mu\nu}] + [W_\nu, [W_\mu, H^{\mu\nu}]] \\ &= \partial_\nu\partial_\mu H^{\mu\nu} + [W_\mu, \partial_\nu H^{\mu\nu}] + [W_\nu, \partial_\mu H^{\mu\nu}] + [\partial_\nu W_\mu, H^{\mu\nu}] + [W_\nu, [W_\mu, H^{\mu\nu}]] \\ &= [\partial_\nu W_\mu, H^{\mu\nu}] + [W_\nu, [W_\mu, H^{\mu\nu}]] \\ &= [\partial_\nu W_\mu, H^{\mu\nu}] + \frac{1}{2}[[W_\nu, W_\mu], H^{\mu\nu}] \\ &= [H_{\mu\nu}, H^{\mu\nu}] = 0 \end{aligned} \tag{2.2.26}$$

where the Jacobi identity reads

$$[W_\nu, [W_\mu, H^{\mu\nu}]] = -[W_\mu, [H^{\mu\nu}, W_\nu]] - [H^{\mu\nu}, [W_\nu, W_\mu]].$$

The identity (2.2.25) implies $D_\mu J_a^\mu = 0$, which will be useful when propagating the gauge.

Step 3. First we take the divergence of the evolution equation for W_a^ν. This yields

$$-\Box(\mathrm{div} W_a - g_a) = \partial_\nu Q_a^\nu. \tag{2.2.27}$$

Thus it suffices to prove $\partial_\nu Q_a^\nu = 0$.

$$\begin{aligned} &\partial_\nu\Big(g_W c_{abc}\big(W_c^\mu \partial_\mu W_b^\nu + W_b^\nu g_c\big) - J_a^\nu\Big) \\ &\quad= g_W c_{abc}\big(W_c^\mu \partial_\mu(\mathrm{div} W_b) + \partial_\mu W_b^\nu \partial_\nu W_c^\mu + \mathrm{div} W_b\, \mathrm{div} W_c + W_b^\nu \partial_\nu(\mathrm{div} W_c)\big) \\ &\qquad - D_\nu J_a^\nu + g_W c_{abc} W_{\nu b} j_c^\nu \\ &\quad= g_W c_{abc}\Big(W_c^\mu \partial_\mu(\mathrm{div} W_b) + \partial_\mu W_b^\nu \partial_\nu W_c^\mu + \mathrm{div} W_b\, \mathrm{div} W_c + W_b^\nu \partial_\nu(\mathrm{div} W_c) \\ &\qquad + W_{\nu b}\big(\partial_\mu H_c^{\nu\mu} + g_W c_{cij} W_{\mu i} H_j^{\nu\mu}\big)\Big) \\ &\quad= g_W c_{abc}\big(W_c^\mu \partial_\mu \mathrm{div} W_b + W_b^\nu \partial_\nu(\mathrm{div} W_c) + \mathrm{div} W_b\, \mathrm{div} W_c\big) \\ &\qquad + g_W^2 c_{abc} c_{cij}\big(\partial_\nu W_{\mu b} W_i^\nu W_j^\mu + \partial_\nu W_{j\mu} W_b^\nu W_i^\mu - \partial_\mu W_{\nu j} W_b^\nu W_i^\mu\big) \\ &\qquad + g_W^3 c_{abc} c_{cij} c_{jkl} W_{\nu b} W_{\mu i} W_k^\nu W_l^\mu \\ &\quad= 0 \end{aligned} \tag{2.2.28}$$

where we have used the antisymmetry in certain Roman indices and also the following identities: $c_{abc}c_{cij} = (\delta_{ai}\delta_{bj} - \delta_{aj}\delta_{bi})$ and $c_{abc}c_{cij}c_{jkl} = \delta_{ai}c_{bkl} - \delta_{bi}c_{akl}$. For example

explicitly in the first term we relabel indices to obtain

$$\begin{aligned} & c_{abc}\big(W_c^\mu \partial_\mu(\mathrm{div}W_b) + W_b^\nu \partial_\nu(\mathrm{div}W_c)\big) \\ & = c_{abc}W_c^\mu \partial_\mu(\mathrm{div}W_b) - c_{acb}W_b^\mu \partial_\mu \mathrm{div}W_c \\ & = c_{abc}W_c^\mu \partial_\mu(\mathrm{div}W_b) - c_{abc}W_c^\mu \partial_\mu \mathrm{div}W_b = 0 \end{aligned} \tag{2.2.29}$$

and also

$$c_{abc}\mathrm{div}W_b\,\mathrm{div}W_c = -c_{acb}\mathrm{div}W_b\,\mathrm{div}W_c = -c_{abc}\mathrm{div}W_b\,\mathrm{div}W_c. \tag{2.2.30}$$

□

Since this proposition is true for any choice of f, g_a, specifically its result applies to our gauge choice (2.2.15d). Hence we only need to impose the gauge conditions (2.2.15d) on the initial data, and then they propagate in the evolution equations.

2.3 Nonlinear stability statement for the GSW model

2.3.1 Parametrization of the ground state

We find it convenient, and with no loss of any generality, to choose

$$\Big(Y^\mu = 0, W_a^\mu = 0, \Phi_0 = \begin{pmatrix} 0 & v \end{pmatrix}^T, e_R = 0, L = 0\Big) \tag{2.3.1}$$

to be the ground state solution. Then correspondingly the gauge choice is given by

$$\mathrm{div}Y = \frac{ig_Y}{2}\big(\partial^\nu \chi^* \Phi_0 - \Phi_0^* \partial^\nu \chi\big), \qquad \mathrm{div}W_a = -ig_W\big(\partial^\nu \chi^* \tau_a \Phi_0 - \Phi_0^* \tau_a \partial^\nu \chi\big). \tag{2.3.2}$$

In order to get this gauge choice propagated, according to Proposition 2.2.5, it suffices to propose

$$\begin{aligned} \mathrm{div}Y &= \frac{ig_Y}{2}\big(\partial^\nu \chi^* \Phi_0 - \Phi_0^* \partial^\nu \chi\big), & \mathrm{div}W_a &= -ig_W\big(\partial^\nu \chi^* \tau_a \Phi_0 - \Phi_0^* \tau_a \partial^\nu \chi\big), \\ \partial_t \mathrm{div}Y &= \frac{ig_Y}{2}\partial_t\big(\partial^\nu \chi^* \Phi_0 - \Phi_0^* \partial^\nu \chi\big), & \partial_t \mathrm{div}W_a &= -ig_W \partial_t\big(\partial^\nu \chi^* \tau_a \Phi_0 - \Phi_0^* \tau_a \partial^\nu \chi\big), \end{aligned}$$

initially, and we denote the initial data set by

$$\begin{aligned} \Big(\chi, Y^\mu, W_a^\mu, e_R, L\Big)(t_0, \cdot) &= \Big(\chi_0, Y_0^\mu, W_{a0}^\mu, e_{R0}, L_0\Big), \\ \partial_t\Big(\chi, Y^\mu, W_a^\mu, e_R, L\Big)(t_0, \cdot) &= \Big(\chi_1, Y_1^\mu, W_{a1}^\mu, e_{R1}, L_1\Big), \end{aligned} \tag{2.3.3}$$

which are called compatible if they satisfy (2.2.15d).

And finally we get the system of equations written as

$$\begin{aligned}
-\Box\chi &= V'(\Phi^*\Phi)\Phi_0 - g_Y^2(\chi^*\Phi_0 - \Phi_0^*\chi)\Phi_0 - g_W^2(\chi^*\tau_a\Phi_0 - \Phi_0^*\tau_a\chi)\tau_a\Phi_0 \\
&\quad ig_W\Big(2W_{\mu a}\tau_a\partial^\mu\chi + \mathrm{div}W_a\tau_a\chi\Big) + ig_Y\Big(Y_\mu\partial^\mu\chi + \frac{1}{2}\mathrm{div}Y\chi\Big) \\
&\quad - \frac{g_W^2}{4}W_{\mu a}W_a^\mu\Phi + g_Yg_WY_\mu W_a^\mu\tau_a\Phi - \frac{g_Y^2}{4}Y_\mu Y^\mu\Phi + g_ee_R^*L + V'(\Phi^*\Phi)\chi, \\
-\Box Y^\nu &= -\frac{g_Y^2v^2}{2}Y^\nu + g_Wg_Y\mathring{C}_aW_a^\nu \\
&\quad \frac{g_Y}{2}\Big(\big(L^*\sigma_L^\nu L + 2e_R^*\sigma_L^\nu e_R\big) + i\big((\partial^\nu\chi^*)\chi - \chi^*\partial^\nu\chi\big)\Big) \\
&\quad - g_Yg_WW_a^\nu\big(\Phi_0^*\tau_a\chi + \chi^*\tau_a\Phi_0 + \chi^*\tau_a\chi\big) + \frac{g_Y^2}{2}Y^\nu\big(\Phi^*\Phi - v^2\big), \\
-\Box W_a^\nu &= -\frac{g_W^2v^2}{2}W_a^\nu + g_Wg_Y\mathring{C}_aY^\nu \\
&\quad g_W\Big(-c_{abc}\partial^\mu(W_{\mu b}W_c^\nu) + Re\big[L^*\sigma_L^\nu\tau_aL\big] - c_{abc}W_b^\mu\big(\partial_\mu W_c^\nu - \partial^\nu W_{\mu c}\big) \\
&\quad - i\big(\partial^\nu\chi^*\tau_a\chi - \chi^*\tau_a\partial^\nu\chi\big)\Big) - g_W^2c_{abc}c_{cde}W_b^\mu W_{\mu e}W_e^\nu + \frac{g_W^2}{2}W_a^\nu\big(\Phi^*\Phi - v^2\big) \\
&\quad - g_Wg_YY^\nu\big(\chi^*\tau_a\Phi_0 + \Phi_0^*\tau_a\chi + \chi^*\tau_a\chi\big), \\
i\sigma_R^\mu\partial_\mu e_R &= g_e\Phi_0^*L - g_Y\sigma_R^\mu Y_\mu e_R + g_e\chi^*L, \\
i\sigma_L^\mu\partial_\mu L &= g_e\Phi_0e_R - \frac{g_Y}{2}\sigma_L^\mu Y_\mu L + \frac{g_W}{2}W_{\mu a}\big(\sigma_L^\mu\tau_a - \tau_a\sigma_L^\mu\big)L + g_e\chi e_R.
\end{aligned} \tag{2.3.4}$$

2.3.2 Statement of the nonlinear stability result

We are ready here to provide the statement of the nonlinear stability result of the full GSW model (2.3.4)–(2.3.3).

Theorem 2.3.1. *Consider the GSW system* (2.3.4) *with gauge condition* (2.3.2) *and vacuum equilibrium* (2.3.1)*, which has compatible initial data set* (2.3.3)*,and let* N *be a sufficiently large integer. There exists* $\epsilon_0 > 0$ *such that for all* $\epsilon \in (0, \epsilon_0)$ *and all compactly supported compatible initial data satisfying the smallness condition*

$$\|\chi_0, Y_0^\mu, W_{a0}^\mu, e_{R0}, L_0\|_{H^{N+1}} + \|\chi_1, Y_1^\mu, W_{a1}^\mu, e_{R1}, L_1\|_{H^N} \leqslant \epsilon, \tag{2.3.5}$$

the initial value problem (2.3.4)–(2.3.3) *admits a unique global-in-time solution* $\big(\chi, Y^\mu, W_a^\mu, e_R, L\big)$.

Note again we will only provide the proof of this theorem under the assumption

$$m_Y = g_Y = 0.$$

2.3.3 Mixed and pure systems

At the beginning of this section we made the special choice $\Phi_0 = (0, v)^*$, but we provide a study on the system (2.3.4), which is true for all constants Φ_0 satisfying $\Phi_0^*\Phi_0 = v^2$.

Mixed and pure Klein-Gordon systems Suggested by Lemma 2.2.2 above, we introduce the following five new unknowns:

$$\begin{aligned} N_{ab}^\nu &:= \mathring{C}_a W_b^\nu - \mathring{C}_b W_a^\nu, \qquad a < b, \\ N_+^\nu &:= g_Y \Phi_0^* \Phi_0 Y^\nu + 2 g_W \sum_a \mathring{C}_a W_a^\nu, \\ N_-^\nu &:= g_W v^2 Y^\nu - 2 g_Y \sum_a \mathring{C}_a W_a^\nu, \end{aligned} \tag{2.3.6}$$

in which $\mathring{C}_a = \frac{1}{2}\Phi_0^* \sigma_a \Phi_0$.

These new variables satisfy the following equations

$$\begin{aligned} -\Box N_{ab}^\nu + \frac{g_W^2 v^2}{2} N_{ab}^\nu &= \mathring{C}_a Q_{Wb}^\nu - \mathring{C}_a Q_{Wb}^\nu, \quad a < b, \\ -\Box N_+^\nu &= g_W v^2 Q_Y^\nu + 2 g_Y \sum_a \mathring{C}_a Q_{Wa}^\nu, \\ -\Box N_-^\nu + \frac{1}{2} v^2 (g_W^2 + g_Y^2) N_-^\nu &= g_W v^2 Q_Y^\nu - 2 g_Y \sum_a \mathring{C}_a Q_{Wa}^\nu. \end{aligned} \tag{2.3.7}$$

More details are given in the following lemma.

Lemma 2.3.2. *There exists some $a_0 \in \{1, 2, 3\}$ such that $|C_{a_0}| > v/6$ and thus the variables $\{N_{a_0 b}^\nu, N_{a_0 c}^\nu, N_+, N_-\}$ fully determine the original unknowns $\{Y^\nu, W_a^\nu\}$. That is,*

$$span\{Y^\nu, W_a^\nu\} = span\{N_{a_0 b}^\nu, N_{a_0 c}^\nu, N_+, N_-\}.$$

Proof. The identity (2.2.8) suggests we consider the following linear combinations:

$$\begin{pmatrix} N_{12}^\nu \\ N_{13}^\nu \\ N_{23}^\nu \\ N_-^\nu \end{pmatrix} := \begin{pmatrix} -\mathring{C}_2 & \mathring{C}_1 & 0 & 0 \\ -\mathring{C}_3 & 0 & \mathring{C}_1 & 0 \\ 0 & -\mathring{C}_3 & \mathring{C}_2 & 0 \\ -2g_W \mathring{C}_1 & -2g_W \mathring{C}_2 & -2g_W \mathring{C}_3 & g_Y v^2 \end{pmatrix} \begin{pmatrix} W_1^\nu \\ W_2^\nu \\ W_3^\nu \\ Y^\nu \end{pmatrix} \tag{2.3.8}$$

The determinant of this linear transformation is zero and so it is non-invertible. More-

over the kernel is

$$g_W v^2\Big(\frac{2g_Y}{g_W v^2}\mathring{C}_1, \frac{2g_Y}{g_W v^2}\mathring{C}_2, \frac{2g_Y}{g_W v^2}\mathring{C}_3, 1\Big), \tag{2.3.9}$$

This suggests we consider the following combination of vector fields

$$N_+^\nu := g_W v^2 Y^\nu + 2g_Y \mathring{C}_a W_a^\nu.$$

By the construction N_+^ν is independent of (N_{ab}^ν, N_-^ν) and from (2.3.4) satisfies the following inhomogeneous wave equation

$$-\Box N_+^\nu = g_W^2 v^2 Q_Y^\nu + 2g_Y \mathring{C}_a Q_{Wa}^\nu. \tag{2.3.10}$$

Note we used the fact that $\sum_a \mathring{C}_a\mathring{C}_a = \frac{v^4}{4}$. □

Finally we can also derive the new variables (2.3.6) using the structure of the original system of equations (2.2.3). This leads us to the following definition of mixed and pure Klein-Gordon systems.

Definition 2.3.3. *Given a vector-valued unknown $U = (U_1, U_2, \cdots, U_n)^T \in \mathbb{C}^n$ and a semi-positive definite Hermitian matrix $A \in \mathbb{C}^{n\times n}$, then the system*

$$-\Box U - AU = 0, \tag{2.3.11}$$

is called a mixed Klein-Gordon and wave system if $\det A = 0$. *The system is called a pure Klein-Gordon system if* $\det A > 0$.

Lemma 2.3.4. *Given a system of the form* (2.3.11)*, there exists a linear combination of the variables U such that* $\dim(\ker A)$ *of these variables satisfy wave equations, and the remaining* $n - \dim(\ker A)$ *variables satisfy Klein-Gordon equations.*

Proof. Observe that since A is Hermitian we can decompose it as $A = P^*\Lambda P$ where Λ is a diagonal matrix, P is a unitary matrix and P^* its conjugate transpose. This implies

$$-\Box U = P^*\Lambda P U \Rightarrow -\Box(PU) = \Lambda(PU) \tag{2.3.12}$$

Finally recall that the number of diagonal entries in Λ is precisely given by $\dim(\ker A)$. □

Lemma 2.3.5. *By neglecting the higher order source terms in* (2.3.4)*, the following linear equations*

$$-\Box Y^\nu = \frac{g_Y^2 v^2}{2} Y^\nu - g_W g_Y \mathring{C}_a W_a^\nu, \quad -\Box W_a^\nu = \frac{g_W^2 v^2}{2} W_a^\nu - g_W g_Y \mathring{C}_a Y^\nu, \tag{2.3.13}$$

can be shown to form a mixed Klein-Gordon and wave system. Furthermore the linear combination of variables from (2.3.4) *which satisfy strict wave and Klein-Gordon equations are* $\{N_{13}^\nu, N_{12}^\nu, N_\pm^\nu\}$ *as defined in* (2.3.6).

Proof. The equations (2.3.13) can be expressed in the form (2.3.11) with $U^T = (W_1^\nu, W_2^\nu, W_3^\nu, Y^\nu)$ and Hermitian matrix

$$A := \begin{pmatrix} \frac{g_W^2 v^2}{2} & 0 & 0 & -g_W g_Y \mathring{C}_1 \\ 0 & \frac{g_W^2 v^2}{2} & 0 & -g_W g_Y \mathring{C}_2 \\ 0 & 0 & \frac{g_W^2 v^2}{2} & -g_W g_Y \mathring{C}_3 \\ -g_W g_Y \mathring{C}_1 & -g_W g_Y \mathring{C}_2 & -g_W g_Y \mathring{C}_3 & \frac{g_Y^2 v^2}{2} \end{pmatrix}, \tag{2.3.14}$$

with

$$\det A = \Big(\frac{g_W^2 v^2}{2}\Big)^4 \Big(\big(\frac{g_W^2 v^2}{2}\big) \tan^2 \theta_W - g_W^2 g_Y^2 \sum_a \mathring{C}_a^2 \Big) = 0.$$

In order to determine which linear combinations of U give Klein-Gordon or wave equations, diagonalise the matrix $A = P^* \Lambda P$, where P is the matrix of eigenvalues of A. This leads to the following new variables

$$\begin{pmatrix} N_{13} \\ N_{12} \\ N_+ \\ N_- \end{pmatrix} =: PU = \begin{pmatrix} -\mathring{C}_3 & 0 & \mathring{C}_1 & 0 \\ -\mathring{C}_2 & \mathring{C}_1 & 0 & 0 \\ 2g_Y \mathring{C}_1 & 2g_Y \mathring{C}_2 & 2g_Y \mathring{C}_3 & g_W v^2 \\ -2g_Y \mathring{C}_1 & -2g_Y \mathring{C}_2 & -2g_Y \mathring{C}_3 & g_W v^2 \end{pmatrix} \begin{pmatrix} W_1^\nu \\ W_2^\nu \\ W_3^\nu \\ Y^\nu \end{pmatrix}, \tag{2.3.15}$$

and associated eigenvalues containing the masses

$$\Lambda(\Phi) := \begin{pmatrix} \frac{g_W^2 v^2}{2} & 0 & 0 & 0 \\ 0 & \frac{g_W^2 v^2}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} v^2 (g_W^2 + g_Y^2) \end{pmatrix}. \tag{2.3.16}$$

□

Mixed and pure Dirac systems There is an analogous study of the mixed and pure Dirac systems, but we omit it since the procedure is very similar.

2.4 Revisit of the model with $g_Y = 0$

Full system of equations From this section on, we always adopt the assumption (2.1.12), i.e.

$$m_Y = g_Y = 0.$$

Before we go to the system of equations, let us adopt some notations which will

make the expressions shorter and clearer. First denote the sets of Klein-Gordon and wave fields by

$$\mathbb{K} := \{\chi, U^{\pm}, V_a^-, W_1^\nu, W_2^\nu, W_3^\nu, e_R, e_L\}, \qquad \mathbb{W} := \{\nu_L\}. \tag{2.4.1}$$

Then we adopt that the following notation

$$Q(\mathbb{K}, \mathbb{W}, \partial\mathbb{K}, \partial\mathbb{W}; \mathbb{K}, \partial\mathbb{K}) := \sum_{s\in\mathbb{K}\bigcup\mathbb{W}\bigcup\partial\mathbb{K}\bigcup\partial\mathbb{W},\ k\in\mathbb{K}\bigcup\partial\mathbb{K}} c_{sk} sk$$

represents a general quadratic nonlinearity of the type sk with some constant coefficient c_{sk}, and we have denoted $\partial\mathbb{K} := \{\partial k : k \in \mathbb{K}\}$. And similarly the the following notation

$$C(\mathbb{K}; \mathbb{K}; \mathbb{W}) := \sum_{k,k'\in\mathbb{K},\ s\in\mathbb{W}} c_{kk's} kk's$$

represents a general cubic nonlinearity.

In this case, the system of equations becomes (with $A = 1, 2$)

$$\begin{aligned}
-\Box\chi &= -V'(\Phi^*\Phi)\Phi_0 - g_W^2(\chi^*\tau_a\Phi_0 - \Phi_0^*\tau_a\chi)\tau_a\Phi_0 + Q(\mathbb{K}, \partial\mathbb{K}; \mathbb{K}) + Q(\mathbb{K}; \mathbb{W}),\\
-\Box Y^\nu &= 0,\\
-\Box W_A^\nu &= -\frac{g_W^2 v^2}{2} W_A^\nu + C(\mathbb{K}; \mathbb{K}; \mathbb{K}) + Q(\mathbb{K}, \partial\mathbb{K}; \mathbb{K}) + Q(\mathbb{K}; \mathbb{W}),\\
-\Box W_3^\nu &= -\frac{g_W^2 v^2}{2} W_3^\nu + C(\mathbb{K}; \mathbb{K}; \mathbb{K}) + Q(\mathbb{K}, \partial\mathbb{K}; \mathbb{K}) + \nu_L^*\sigma_L^\nu\nu_L,\\
i\sigma_R^\mu\partial_\mu e_R &= g_e\Phi_0^* L + g_e\chi^* L,\\
i\sigma_L^\mu\partial_\mu L &= g_e\Phi_0 e_R + \frac{g_W}{2} W_{\mu a}\big(\sigma_L^\mu\tau_a - \tau_a\sigma_L^\mu\big)L + g_e\chi e_R,
\end{aligned} \tag{2.4.2}$$

in which we recall $\Phi_0 = (0, v)^T$.

We note that the variables Y^ν satisfy homogeneous wave equations, and they do not interact with other fields. Due to this reason, we will drop Y^ν equations in the following analysis.

Decomposition of the Higgs equation It is preferred to use variables

$$U^{\pm} := \chi^*\Phi_0 \pm \Phi_0^*\chi, \qquad V_a^- := \chi^*\tau_a\Phi_0 - \Phi_0^*\tau_a\chi$$

in the analysis since they satisfy Klein-Gordon equations. Thus the equation for the Higgs field is decomposed into four equations

$$\begin{aligned} -\Box U^+ + 4\lambda v^2 U^+ &= Q(\mathbb{K}, \partial\mathbb{K}; \mathbb{K}) + Q(\mathbb{K}; \mathbb{W}), \\ -\Box U^- + \frac{g_W^2 v^2}{2} U^- &= 2g_W^2 \mathring{C}_a V_a^- + Q(\mathbb{K}, \partial\mathbb{K}; \mathbb{K}) + Q(\mathbb{K}; \mathbb{W}), \\ -\Box V_a^- + \frac{g_W^2 v^2}{2} V_a^- &= \frac{g_y^2}{2} \mathring{C}_a U^- + Q(\mathbb{K}, \partial\mathbb{K}; \mathbb{K}) + Q(\mathbb{K}; \mathbb{W}). \end{aligned} \tag{2.4.3}$$

We also note that the first order terms in the right hand sides of the last two equations do not cause trouble, which is thanks to the relations

$$U^- = -2V_3^-, \qquad \mathring{C}_1 = \mathring{C}_2 = 0, \qquad \mathring{C}_3 = -v^2/4. \tag{2.4.4}$$

A new variable for the field W_3^ν We note that in the nonlinearities of the W_3^ν equations there is a term of type ν-ν interaction, which does not decay fast enough. Following the work of Tsutsumi [76] we introduce the new variables

$$\widetilde{W}_3^\nu := W_3^\nu + \frac{2}{g_w^2 v^2} \nu_L^* \sigma_L^\nu \nu_L, \tag{2.4.5}$$

which are close to W_3^ν in terms of the energy norms, and which satisfy Klein-Gordon equations with good nonlinearities

$$-\Box \widetilde{W}_3^\nu + \frac{g_W^2 v^2}{2} \widetilde{W}_3^\nu = C(\mathbb{K}, \mathbb{W}; \mathbb{K}; \mathbb{K}) + Q(\mathbb{K}, \partial\mathbb{K}; \mathbb{K}) + \partial_\mu \nu_L^* \sigma_L^\nu \partial^\mu \nu_L. \tag{2.4.6}$$

Klein-Gordon equations for the electron field ψ_e. On the other hand, we find it easier for the analysis if we transform the Dirac equation of $\psi_e = (e_L, e_R)$ to a Klein-Gordon equation

$$-\Box \psi_e + m_e^2 \psi_e = Q(\mathbb{K}, \partial\mathbb{K}; \mathbb{K}) + Q(\partial\mathbb{K}; \mathbb{W}) + Q(\mathbb{K}; \partial\mathbb{W}), \tag{2.4.7}$$

for the reason that Klein-Gordon components give faster decay by utilising the role of the mass.

Dirac equation for the neutrino field ν_L. We now demonstrate two forms of the Dirac equation for the neutrino ν_L.

- First order Dirac equation

$$i\sigma_L^\mu \partial_\mu \nu_L = Q(\mathbb{K}; \mathbb{K}) + \frac{g_W}{2} W_{\mu 3} \sigma_L^\mu \nu_L. \tag{2.4.8}$$

- Second order wave equation

$$-\Box\nu_L = Q(\mathbb{K}, \partial\mathbb{K}; \mathbb{K}) + Q(\partial\mathbb{K}; \mathbb{W}) + Q(\mathbb{K}; \partial\mathbb{W}). \tag{2.4.9}$$

2.5 Proof of the main theorem for the case $g_Y = 0$

2.5.1 Bootstrap assumptions and basic estimates

This last section is devoted to relying on the bootstrap method to prove Theorem 2.3.1 with the choice of parameter $g_Y = 0$.

With the small data assumption (2.3.5) we may assume $E^{\mathcal{H}}(s_0, \nu_L) \lesssim \epsilon$. By the local well-posedness of semilinear wave equations, it holds for some small $\delta > 0$ and for all $s \in [s_0, s_1]$ such that [4]

$$\begin{aligned}
\mathcal{E}(s, \partial^I L^J \nu_L)^{1/2} &\leqslant C_1 \epsilon s^\delta, && |I| + |J| \leqslant N, \\
\mathcal{E}(s, \partial^I L^J \nu_L)^{1/2} &\leqslant C_1 \epsilon, && |I| + |J| \leqslant N-1, \\
\sum_{k\in\mathbb{K}} \mathcal{E}_m(s, \partial^I L^J k)^{1/2} &\leqslant C_1 \epsilon s^\delta, && |I| + |J| \leqslant N, \\
\sum_{k\in\mathbb{K}} \mathcal{E}_m(s, \partial^I L^J k)^{1/2} &\leqslant C_1 \epsilon, && |I| + |J| \leqslant N-1,
\end{aligned} \tag{2.5.1}$$

in which C_1 is some large positive constant to be determined,

$$s_1 := \sup\{s : (2.5.1)\ \textit{holds}\}, \tag{2.5.2}$$

and

$$m := \min\{m_e, m_W, m_H\}. \tag{2.5.3}$$

If we can prove the refined estimates

$$\begin{aligned}
\mathcal{E}(s, \partial^I L^J \nu_L)^{1/2} &\leqslant \frac{1}{2} C_1 \epsilon s^\delta, && |I| + |J| \leqslant N, \\
\mathcal{E}(s, \partial^I L^J \nu_L)^{1/2} &\leqslant \frac{1}{2} C_1 \epsilon, && |I| + |J| \leqslant N-1, \\
\sum_{k\in\mathbb{K}} \mathcal{E}_m(s, \partial^I L^J k)^{1/2} &\leqslant \frac{1}{2} C_1 \epsilon s^\delta, && |I| + |J| \leqslant N, \\
\sum_{k\in\mathbb{K}} \mathcal{E}_m(s, \partial^I L^J k)^{1/2} &\leqslant \frac{1}{2} C_1 \epsilon, && |I| + |J| \leqslant N-1,
\end{aligned} \tag{2.5.4}$$

[4] We adopt notations in Chapter 1

then we are able to assert that s_1 cannot be finite, which implies a global existence result for the system (2.4.2).

Combining bootstrap assumptions (2.5.1) and the estimates for commutators in Lemma 3.3.1 in Chapter 1, the following set of estimates are immediately obtained:

$$\begin{aligned}
\|(s/t)\partial_\mu\partial^I L^J \nu_L, (s/t)\partial^I L^J \partial_\mu \nu_L\|_{L^2_f(\mathcal{H}_s)} &\lesssim C_1\epsilon s^\delta, \qquad |I|+|J| = N,\\
\|(s/t)\partial_\mu\partial^I L^J \nu_L, (s/t)\partial^I L^J \partial_\mu \nu_L\|_{L^2_f(\mathcal{H}_s)} &\lesssim C_1\epsilon, \qquad |I|+|J| \leqslant N-1,\\
\sum_{k\in\mathbb{K}} \|(s/t)\partial^I L^J \partial_\mu k, (s/t)\partial_\mu\partial^I L^J k, \partial^I L^J k\|_{L^2_f(\mathcal{H}_s)} &\lesssim C_1\epsilon s^\delta, \qquad |I|+|J| \leqslant N,\\
\sum_{k\in\mathbb{K}} \|\partial_\mu\partial^I L^J k, \partial^I L^J \partial_\mu k\|_{L^2_f(\mathcal{H}_s)} &\lesssim C_1\epsilon s^\delta, \quad |I|+|J| \leqslant N-1.
\end{aligned} \tag{2.5.5}$$

Furthermore we can obtain L^∞ estimates by recalling the Sobolev-type inequality on hyperboloids [44] stated in Proposition 1.7.4.

Combined with the estimates (2.5.5) the following also holds:

$$\begin{aligned}
\sup_{\mathcal{H}_s} t^{1/2} s|\partial_\alpha\partial^I L^J \nu_L, \partial^I L^J \partial_\alpha \nu_L| &\lesssim C_1\epsilon s^\delta, \quad |I|+|J| = N-2,\\
\sup_{\mathcal{H}_s} t^{1/2} s|\partial_\alpha\partial^I L^J \nu_L, \partial^I L^J \partial_\alpha \nu_L| &\lesssim C_1\epsilon, \qquad |I|+|J| \leqslant N-3,\\
\sup_{k\in\mathbb{K},\ \mathcal{H}_s} \Big(t^{1/2} s|\partial_\alpha\partial^I L^J k, \partial^I L^J \partial_\alpha k| + t^{3/2}|\partial^I L^J k|\Big) &\lesssim C_1\epsilon s^\delta, \quad |I|+|J| \leqslant N-2,\\
\sup_{k\in\mathbb{K},\ \mathcal{H}_s} t^{3/2}\big|\partial_\mu\partial^I L^J k, \partial^I L^J \partial_\mu k, \partial^I L^J k\big| &\lesssim C_1\epsilon s^\delta, \quad |I|+|J| \leqslant N-3.
\end{aligned} \tag{2.5.6}$$

2.5.2 First-order energy estimate for the neutrino field ν_L

To obtain decay estimates for the neutrino component ν_L, a standard method is to analyse the second-order wave equation (2.4.9). This is then a semilinear wave equation and so there are known techniques to estimate the nonlinearity, see for example [2] and [45]. However, the right-hand side term appearing in our wave equation (2.4.9) does not decay sufficiently fast for this argument to close. At this point however, we recall Proposition 1.2.3 and the lower bound $\|(s/t)\psi\|_{L^2_f}$ for the energy $E^{\mathcal{H}}$. This motivates us to analyse the first-order form of the Dirac equation (2.4.8).

On the other hand, instead of considering the Weyl equation for the left-handed neutrino

$$i\sigma_L^\mu \partial_\mu \nu_L = Q(\mathbb{K};\mathbb{K}) + \frac{g_W}{2} W_{\mu 3}\sigma_L^\mu \nu_L =: F_{\nu_L},$$

we find it more convenient to deal with the Dirac equation for the neutrino ψ_n

$$i\gamma^\mu \partial_\mu \psi_n = F_{\psi_n}, \tag{2.5.7}$$

with

$$\psi_n = \begin{pmatrix} \nu_R \\ \nu_L \end{pmatrix}, \quad F_{\psi_n} := \begin{pmatrix} F_{\nu_L} \\ 0 \end{pmatrix}.$$

It is obvious that all estimates true for ψ_n are automatically true for ν_L due to

$$|\nu_L| \leqslant |\psi_n|.$$

Next we need to adapt the Sobolev-type inequality on hyperboloids (see Proposition 1.7.4 in Chapter 1) to include boosts which commute with the Dirac operator $i\gamma^\nu\partial_\nu$. In [4] Bachelot introduced the modified Lorentz boosts $\widehat{L}_a$ that differ from L_a by a constant matrix

$$\widehat{L}_a := L_a + \frac{1}{2}\gamma_0\gamma_a, \tag{2.5.8}$$

which enjoy the property $[\widehat{L}_a, i\gamma^\nu\partial_\nu] = 0$. And the following modified Sobolev-type inequality holds.

Proposition 2.5.1. *Suppose $\psi = \psi(t,x)$ is a sufficiently smooth spinor field supported in the region $\{(t,x) : |x| < t-1\}$, then it holds for $s \geqslant 2$*

$$\sup_{\mathcal{H}_s} \left|t^{3/2}\psi(t,x)\right| \lesssim \sum_{|J|\leqslant 2} \left\|\widehat{L}^J\psi\right\|_{L^2_f(\mathcal{H}_s)}, \tag{2.5.9}$$

where $\widehat{L}$ denotes a modified Lorentz boost.

Theorem 2.5.2. *Under the bootstrap assumptions* (2.5.1)*, the neutrino field ψ_n in* (2.5.7)*, which satisfies*

$$i\gamma^\mu\partial_\mu\psi_n = \begin{pmatrix} Q(\mathbb{K};\mathbb{K}) + \frac{g_W}{2}W_{\mu 3}\sigma_L^\mu\nu_L \\ 0 \end{pmatrix},$$
$$\psi_n(t_0,\cdot) = \begin{pmatrix} 0 \\ \nu_{L0} \end{pmatrix},$$

admits the following estimates

$$\|(s/t)\widehat{L}^J\psi_n\|_{L^2_f(\mathcal{H}_s)} \lesssim \epsilon + (C\epsilon)^2, \quad |J| \leqslant N, \tag{2.5.10}$$
$$\sup_{\mathcal{H}_s} t^{1/2}s|\widehat{L}^J\psi_n| \lesssim \epsilon + (C\epsilon)^2, \quad |J| \leqslant N-2. \tag{2.5.11}$$

We revisit the proof of Theorem 1.7.7 in Chapter 1.

Proof. Step 1 : By multiplying $\psi_n^*\gamma^0$ and $\gamma^0\psi_n$ to the Dirac equation of ψ_n and its complex conjugate respectively, we obtain that

$$\partial_t(\psi_n^*\psi) + \partial_a(\psi_n^*\gamma^0\gamma^a\partial_\mu\psi_n)\psi_n^*\gamma^0\gamma^\mu\partial_\mu\psi_n = 0,$$

which indicates

$$\|(s/t)\psi_n\|_{L^2_f(\mathcal{H}_s)} \lesssim \epsilon. \tag{2.5.12}$$

Step 3 : By an induction argument, we consider the case $1 \leqslant |J| \leqslant N-2$. Act $\widehat{L}^J$ on (2.5.7) to get

$$\gamma^\mu \partial_\mu (\widehat{L}^J \psi_n) = -Z_\mu \gamma^\mu \widehat{L}^J \psi_n + \widetilde{R}, \tag{2.5.13}$$

with

$$\widetilde{R} := \sum_{|J_1|+|J_2|\leqslant |J|, |J_1|\leqslant |J|-1} Q(\widehat{L}^{J_1}\mathbb{K}, \widehat{L}^{J_1}\{\psi_n\}; L^{J_2}\mathbb{K}).$$

Using the energy estimates and the induction assumption

$$\begin{aligned}
&E^{\mathcal{H}}(s, \widehat{L}^J\psi_n) \\
&\leqslant E^{\mathcal{H}}(s_0, \widehat{L}^J\psi_n) + \int_{s_0}^{s}\int_{\mathcal{H}_{\bar{s}}} (s/t)\big(|(\widehat{L}^J\psi_n)^*\gamma^0\tilde{R}| + |\tilde{R}^*\gamma^0(\widehat{L}^J\psi_n)|\big)\, dx d\bar{s} \\
&\lesssim \epsilon^2 + \int_{s_0}^{s} \|(\bar{s}/t)\widehat{L}^J\psi_n\|_{L^2_f(\mathcal{H}_{\bar{s}})} \sum_{\substack{k,k'\in\mathbb{K}, |J_1|+|J_2|\leqslant |J| \\ |J_2|\leqslant |J|-1}} \Big(\|(t/s)\widehat{L}^{J_1}k\|_{L^\infty(\mathcal{H}_{\bar{s}})} \\
&\qquad\qquad \big(\|(s/t)\widehat{L}^{J_2}\psi_n\|_{L^2_f(\mathcal{H}_{\bar{s}})} + \|(s/t)\widehat{L}^{J_2}k'\|_{L^2_f(\mathcal{H}_{\bar{s}})}\big)\Big)\, d\bar{s} \\
&\lesssim \epsilon^2 + \epsilon^2 \int_{s_0}^{s} \bar{s}^{-3/2}\bar{s}^{\delta}\|(\bar{s}/t)\widehat{L}^J\psi_n\|_{L^2_f(\mathcal{H}_{\bar{s}})}\, d\bar{s}.
\end{aligned}$$

Recall that lower bound of $E^{\mathcal{H}}$ implies

$$\|(s/t)\widehat{L}^J\psi_n\|^2_{L^2_f(\mathcal{H}_s)} \lesssim \epsilon^2 + (C_1\epsilon)^2 \int_{s_0}^{s} \bar{s}^{-3/2}\bar{s}^{\delta}\|(\bar{s}/t)\widehat{L}^J\psi_n\|_{L^2_f(\mathcal{H}_{\bar{s}})}\, d\bar{s}. \tag{2.5.14}$$

Thus it follows that (with $C_1\epsilon \ll 1/2$)

$$\|(s/t)\widehat{L}^J\psi_n\|_{L^2_f(\mathcal{H}_s)} \lesssim \epsilon, \quad |J| \leqslant N-2. \tag{2.5.15}$$

Step 4 : The above analysis shows for $|J| \leqslant N-4$

$$\sum_{|I|\leqslant 2} \|(s/t)\widehat{L}^I\widehat{L}^J\psi_n\|_{L^2_f(\mathcal{H}_s)} \lesssim \epsilon. \tag{2.5.16}$$

Thus by the Sobolev inequality in Proposition (2.5.1), we deduce

$$\sup_{\mathcal{H}_s} t^{1/2} s|\widehat{L}^J\psi_n| \lesssim C\epsilon, \quad |J| \leqslant N-4. \tag{2.5.17}$$

Step 5 : We now consider the case $|J| = N-1$. An energy estimate yields

$$\begin{aligned}
&E^{\mathcal{H}}(s, \widehat{L}^J\psi_n)\\
&\leqslant \epsilon^2 + \int_{s_0}^{s} \|(\bar{s}/t)\widehat{L}^J\psi_n\|_{L^2_f(\mathcal{H}_{\bar{s}})}\Big(\sum_{\substack{k\in\mathbb{K},|J_1|+|J_2|\leqslant N-1,\\ |J_1|>|J_2|,|J_2|\leqslant N-4}} \|\widehat{L}^{J_1}k\|_{L^2_f(\mathcal{H}_{\bar{s}})}\|\widehat{L}^{J_2}\psi_n\|_{L^\infty(\mathcal{H}_{\bar{s}})}\\
&\qquad\qquad + \sum_{\substack{k\in\mathbb{K},|J_1|+|J_2|\leqslant N-1,\\ |J_1|\leqslant|J_2|,|J_2|\leqslant N-2}} \|(t/s)\widehat{L}^{J_1}k\|_{L^\infty(\mathcal{H}_{\bar{s}})}\|(s/t)\widehat{L}^{J_2}\psi_n\|_{L^2_f(\mathcal{H}_{\bar{s}})}\Big)\, d\bar{s}\\
&\leqslant \epsilon^2 + (C_1\epsilon)^2\int_{s_0}^{s} \bar{s}^{-3/2}\bar{s}^{\delta}\|(\bar{s}/t)\widehat{L}^J\psi_n\|_{L^2_f(\mathcal{H}_{\bar{s}})}\, d\bar{s}.
\end{aligned}$$

By Gronwall inequality this implies

$$\|(\bar{s}/t)\widehat{L}^J\psi_n\|_{L^2_f(\mathcal{H}_{\bar{s}})} \lesssim \epsilon + (C_1\epsilon)^2. \tag{2.5.18}$$

The same analysis also applies to the case $|J| = N$. And repeating *Step* 4 gives (2.5.11) in the case $|J| \leqslant N-2$. □

As a consequence, we have the following estimates for ν_L.

Corollary 2.5.3. *The neutrino ν_L satisfies*

$$\begin{aligned}
\|(s/t)L^J\nu_L\|_{L^2_f(\mathcal{H}_s)} &\lesssim \epsilon + (C_1\epsilon)^2, \quad |J| \leqslant N,\\
\sup_{\mathcal{H}_s} t^{1/2}s|L^J\nu_L| &\lesssim \epsilon + (C_1\epsilon)^2, \quad |J| \leqslant N-2.
\end{aligned} \tag{2.5.19}$$

2.5.3 Refined estimates

In this subsection we derive the refined estimates (2.5.4) and close our bootstrap argument. For this to work we move to the transformed field $\widetilde{W}_3^\nu$ defined in (2.4.5), which heuristically is of the form

$$\widetilde{W}_3^\nu = W_3^\nu + \mathcal{O}(|\nu_L|^2), \tag{2.5.20}$$

and thus using the estimates for W_3^ν coming from (2.5.5) and (2.5.6), together with the previous energy and sup-norm estimates for ν_L, the following estimates for $\widetilde{W}_3^\nu$ hold

$$\begin{aligned}
\|(s/t)\partial^I L^J \partial_\mu \widetilde{W}_3^\nu, (s/t)\partial_\mu \partial^I L^J \widetilde{W}_3^\nu, \partial^I L^J \widetilde{W}_3^\nu\|_{L^2_f(\mathcal{H}_s)} &\lesssim C_1 \epsilon s^\delta, \qquad |I| + |J| \leqslant N, \\
\|\partial^I L^J \partial_\mu \widetilde{W}_3^\nu, \partial_\mu \partial^I L^J \widetilde{W}_3^\nu\|_{L^2_f(\mathcal{H}_s)} &\lesssim C_1 \epsilon s^\delta, \quad |I| + |J| \leqslant N - 1, \\
\sup_{\mathcal{H}_s} \Big(t^{1/2} s \big|\partial_\alpha \partial^I L^J \widetilde{W}_3^\nu, \partial^I L^J \partial_\alpha \widetilde{W}_3^\nu\big| + t^{3/2} \big|\partial^I L^J \widetilde{W}_3^\nu\big| \Big) &\lesssim C_1 \epsilon s^\delta, \quad |I| + |J| \leqslant N - 2, \\
\sup_{\mathcal{H}_s} \Big(t^{3/2} \big|\partial_\alpha \partial^I L^J \widetilde{W}_3^\nu, \partial^I L^J \partial_\alpha \widetilde{W}_3^\nu\big| \Big) &\lesssim C_1 \epsilon s^\delta, \quad |I| + |J| \leqslant N - 3.
\end{aligned} \tag{2.5.21}$$

Moreover, (2.4.5) indicates that any energy bounds (either of L^2-type or of L^∞-type) for W_3^ν are automatically true for $\widetilde{W}_3^\nu$ and vice verse.

Proof of refined estimates (2.5.4). We first look at the energy for ν_L in the case $|I| + |J| = N$

$$\begin{aligned}
&\mathcal{E}(s, \partial^I L^J \nu_L)^{1/2} \\
&\leqslant \mathcal{E}(s_0, \partial^I L^J \nu_L)^{1/2} + C \sum_{k \in \mathbb{K}} \int_{s_0}^s \big(\|\partial^I L^J (\partial k \nu_L)\|_{L^2_f(\mathcal{H}_{\bar{s}})} + \|\partial^I L^J (k \partial \nu_L)\|_{L^2_f(\mathcal{H}_{\bar{s}})} \big) \, d\bar{s}.
\end{aligned}$$

By noting

$$\begin{aligned}
&\sum_{k \in \mathbb{K}} \|\partial^I L^J (\partial k \nu_L)\|_{L^2_f(\mathcal{H}_{\bar{s}})} \\
&\leqslant \sum_{\substack{I_1 + I_2 = I, J_1 + J_2 = J \\ |I_1| + |J_1| \leqslant N-2, k \in \mathbb{K}}} \|(t/s)\partial^{I_1} L^{J_1} \partial k\|_{L^\infty(\mathcal{H}_{\bar{s}})} \|(s/t)\partial^{I_2} L^{J_2} \nu_L\|_{L^2_f(\mathcal{H}_{\bar{s}})} \\
&\quad + \sum_{\substack{I_1 + I_2 = I, J_1 + J_2 = J \\ |I_1| + |J_1| \geqslant N-1, k \in \mathbb{K}}} \|(s/t)\partial^{I_1} L^{J_1} \partial k\|_{L^2_f(\mathcal{H}_{\bar{s}})} \|(t/s)\partial^{I_2} L^{J_2} \nu_L\|_{L^\infty(\mathcal{H}_{\bar{s}})} \\
&\lesssim (C_1\epsilon)^2 \bar{s}^{-3/2} \bar{s}^\delta + (C_1\epsilon)^2 \bar{s}^{-1+\delta} \\
&\lesssim (C_1\epsilon)^2 \bar{s}^{-1+\delta},
\end{aligned}$$

and the estimates

$$\begin{aligned}
&\sum_{k\in\mathbb{K}} \|\partial^I L^J(k\partial\nu_L)\|_{L^2_f(\mathcal{H}_{\bar{s}})}\\
&\leqslant \sum_{\substack{I_1+I_2=I, J_1+J_2=J\\ |I_1|+|J_1|\leqslant N-2,\, k\in\mathbb{K}}} \|(t/s)\partial^{I_1}L^{J_1}k\|_{L^\infty(\mathcal{H}_{\bar{s}})}\|(s/t)\partial^{I_2}L^{J_2}\partial\nu_L\|_{L^2_f(\mathcal{H}_{\bar{s}})}\\
&\quad+\sum_{\substack{I_1+I_2=I, J_1+J_2=J\\ |I_1|+|J_1|\geqslant N-1,\, k\in\mathbb{K}}} \|\partial^{I_1}L^{J_1}k\|_{L^2_f(\mathcal{H}_{\bar{s}})}\|(t/s)\partial^{I_2}L^{J_2}\partial\nu_L\|_{L^\infty(\mathcal{H}_{\bar{s}})}\\
&\lesssim (C_1\epsilon)^2\bar{s}^{-5/4},
\end{aligned}$$

we obtain

$$\mathcal{E}(s,\partial^I L^J\nu_L)^{1/2}\lesssim \epsilon+(C_1\epsilon)^2 s^\delta,\quad |I|+|J|=N. \tag{2.5.22}$$

Similarly for $|I|+|J|\leqslant N-1$ we obtain

$$\mathcal{E}(s,\partial^I L^J\nu_L)^{1/2}\lesssim \epsilon+(C_1\epsilon)^2. \tag{2.5.23}$$

Next, in order to obtain estimates for W_3^ν, we first bound the energy for $\widetilde{W}_3^\nu$

$$\begin{aligned}
\mathcal{E}_m(s,\partial^I L^J\widetilde{W}_3^\nu)^{1/2}&\leqslant \mathcal{E}_m(s_0,\partial^I L^J\widetilde{W}_3^\nu)^{1/2}+\int_{s_0}^s \|\partial^I L^J(\partial_\mu\nu_L\partial^\mu\nu_L)\|_{L^2_f(\mathcal{H}_{\bar{s}})}\\
&\quad+\|\partial^I L^J Q(\mathbb{K},\partial\mathbb{K};\mathbb{K})\|_{L^2_f(\mathcal{H}_{\bar{s}})}\\
&\lesssim \epsilon+(C_1\epsilon)^2.
\end{aligned}$$

Next, recalling the definition (2.4.5) or the relation (2.5.20) we use Young's inequality to obtain for all $|I|+|J|\leqslant N$

$$\begin{aligned}
&\mathcal{E}_m(s,\partial^I L^J W_3^\nu)^{1/2}\\
&\leqslant 2\mathcal{E}_m(s,\partial^I L^J\widetilde{W}_3^\nu)^{1/2}+2\mathcal{E}_m(s,\partial^I L^J(\nu_L^*\nu_L))^{1/2}\\
&\lesssim \epsilon+(C_1\epsilon)^2.
\end{aligned} \tag{2.5.24}$$

Till now the remaining terms need to handle are

$$k\in\mathbb{K}_1:=\mathbb{K}/\{W_3^\nu\}.$$

But we find that the same way we estimate ν_L in the beginning of the proof applies to all $k\in\mathbb{K}_1$. Hence we get

$$\begin{aligned}
&\mathcal{E}(s,\partial^I L^J\nu_L)^{1/2}\lesssim \epsilon+(C_1\epsilon)^2 s^\delta\quad |I|+|J|=N,\\
&\mathcal{E}(s,\partial^I L^J\nu_L)^{1/2}\lesssim \epsilon+(C_1\epsilon)^2,\quad |I|+|J|\leqslant N-1.
\end{aligned} \tag{2.5.25}$$

By choosing C_1 sufficiently large and ϵ sufficiently small such that $C_1\epsilon \ll 1/2$, we arrive at the refined bounds (2.5.4). This shows global existence and completes the proof of Theorem 2.1.1 in the case $g_Y = 0$. $\square$

Part II

Stability of a class of wave-Klein-Gordon equations

Chapter 3

Global stability in 3 dimensional space

[1]

Contents

[1] This chapter is a joint work with Zoe Wyatt [13].

3.1 Introduction

Model problem. Systems of wave equations and Klein-Gordon equations are of great importance in mathematics and physics. Examples in the field include the semilinear Dirac-Proca equations and Klein-Gordon-Zakharov equations, and the quasilinear Einstein-Klein-Gordon equations. In this chapter we will study the following semilinear coupled wave-Klein-Gordon system using the hyperboloidal foliation method of LeFloch-Ma [44]. Consider[2]

$$\begin{aligned} -\Box u &= F_u := uv + u\partial_t v, \\ -\Box v + v &= F_v := uv, \end{aligned} \tag{3.1.1}$$

with initial data prescribed on the time slice $t = 2$

$$\begin{aligned} \big(u, \partial_t u\big)(t=2,\cdot) &= \big(u_0, u_1\big), \\ \big(v, \partial_t v\big)(t=2,\cdot) &= \big(v_0, v_1\big). \end{aligned} \tag{3.1.2}$$

Our aim is to prove that initial data, sufficiently small in some norm, yield global-in-time solutions that decay back to the trivial solution. The main difficulty is that there are no derivatives on the wave component u on the right-hand-side terms F_u and F_v of equation (3.1.1), and thus the nonlinearities appear to decay insufficiently fast. To be more precise, the best we can expect is that

$$\|F_u\|_{L^2} = \|uv + u\partial_t v\|_{L^2} \sim t^{-1}, \qquad \|F_v\|_{L^2} = \|uv\|_{L^2} \sim t^{-1}, \tag{3.1.3}$$

both of which are not integrable.

Previous work and motivation. Before we demonstrate our techniques for treating (3.1.1), let us briefly discuss some previous work in the literature. Recall, in the

[2] $\Box := \eta^{\alpha\beta}\partial_\alpha\partial_\beta$, with $\eta = \mathrm{diag}(-1,1,1,1)$. Unless specified, Roman letters and Latin letters take values in $\{0,1,2,3\}$ and $\{1,2,3\}$ respectively, and Einstein summation convention is adopted.

celebrated counterexample by John [33], that there exist wave equations with certain nonlinearities that are quadratic in derivatives but which do not admit global-in-time solutions. Nonetheless, a broad class of wave equations with nonlinearities, quadratic in derivatives, satisfying the so-called null condition, as shown independently by Klainerman [40] and Christodoulou [8], do admit global-in-time solutions. The vector field method, due to Klainerman, and the conformal method, due to Christodoulou, have been two major approaches to studying wave equations. Other related versions of the null condition have also been used to great effect, see for example [53, 54] and [**?**].

By contrast, the Klein-Gordon equation requires a different analysis from the wave equation. One key obstruction is that the scaling vector field $S = t\partial_t + x^a\partial_a$ does not commute with the Klein-Gordon operator $-\Box+1$, which thus prevents us from applying the Klainerman-Sobolev inequality directly. Pioneering works by Klainerman using the vector field method in [41], and by Shatah employing a normal form method in [66], led the way in treating a wide class of Klein-Gordon-type equations.

Furthermore our study of the PDE (3.1.1) was motivated by other coupled wave-Klein-Gordon systems in the literature and future work on Dirac-Klein-Gordon systems [12]. For example, Tsutsumi and his collaborators studied the Dirac-Proca system in [76] and the Klein-Gordon-Zakharov system in [64]. Katayama also investigated a coupled wave-Klein-Gordon system with a large class of quadratic nonlinearities in [35]. With these in mind, our aim is to utilise the hyperboloidal foliation method developed by LeFloch and Ma in [45], where the authors studied a quasilinear coupled wave-Klein-Gordon system. See also the work of Ionescu and Pausader [31] for other efforts in this direction.

Main result. Returning to our system (3.1.1), we find that we can treat the uv nonlinearity appearing in F_u by transforming the variable u in a similar way to the work of Tsutsumi in [76]. Note this is only at the expense of bringing a null form into the new wave equation. As for the nonlinear term $u\partial_t v$ in F_u, we rewrite it as two terms $u\partial_t v = \partial_t(uv) - v\partial_t u$, in which the former is a total derivative and the latter is easier to deal with due to the derivative on the wave component. Then, following [35], we split the wave equation into two new wave equations, and the strategy for handling the uv-type nonlinearity applies once more. On the other hand, to treat the uv term appearing in F_v of the Klein-Gordon equation, the novel idea is that we move the term to the left hand side and treat v as a Klein-Gordon field with varying mass $m = \sqrt{1-u}$. This enables us to apply the techniques in [45].

We are now ready to state the main theorem.

Theorem 3.1.1 (Nonlinear stability of a wave-Klein-Gordon model)**.** *Consider the system* (3.1.1) *and let* $N \geqslant 8$ *be an integer. Then there exists* $\epsilon_0 > 0$ *such that for all* $\epsilon \in (0, \epsilon_0)$ *and all compactly supported initial data* (u_0, u_1, v_0, v_1) *satisfying the smallness*

condition

$$\|u_0, v_0\|_{H^{N+1}(\mathbb{R}^3)} + \|u_1, v_1\|_{H^N(\mathbb{R}^3)} \leqslant \epsilon, \tag{3.1.4}$$

the initial value problem (3.1.1)–(3.1.2) *admits a global-in-time solution* (u, v) *with*

$$|u(t,x)| \lesssim t^{-1}, \quad |v(t,x)| \lesssim t^{-3/2}. \tag{3.1.5}$$

For the proof of the main theorem, we employ the strategy introduced by LeFloch and Ma in [45], which allows us to obtain robust pointwise decay for both wave and Klein-Gordon components. We also apply a hyperboloidal conformal-type energy estimate for the wave component, which was first introduced by Ma and Huang in [61]. This enables us to obtain good L^2-type bounds for the wave component u. All together, our proof is shorter and yields better energy bounds for both wave and Klein-Gordon components compared to those in [45]. The restriction to compact initial data can also be removed, see for example [48, 47] or [42].

Generalisations of the main result. One can also easily show, though we will not explicitly do so here, that Theorem 3.1.1 is also true for the following more general system

$$\begin{aligned} -\Box u &= Q(u, v, \partial v; v, \partial v), \\ -\Box v + v &= Q(u; u, v) + Q(\partial u, v, \partial v; v, \partial v), \end{aligned} \tag{3.1.6}$$

where we use the short-hand notation $Q(\cdots;\cdots)$ to denote quadratic nonlinearities involving interactions between one term from each side of the semicolon. Note further that compared to the work of [64] and [35] a wider class of nonlinearities can be treated. In [35, (2.14)] any nonlinearity for the wave equation involving at most one derivative, needed to be of divergence form. This is not needed in our setting. We also remind one that the u–u interaction term above was treated by Tsutsumi in [76]. Finally, it was speculated in [44] that nonlinear interaction terms of the form

$$Q(u; v, \partial v)$$

may lead to finite time blow-up. Thus this chapter partially answers their question by showing that certain terms of this form do not lead to finite time blow-up.

The Klein-Gordon-Zakharov equations. Our result allows us to also deal with the Klein-Gordon-Zakharov equations, which have been studied before using constant time slices or phase-space methods in [64, 35, 75]. Moreover we could also treat some Dirac-Klein-Gordon and Dirac-Proca type equations but we will discuss this in future

work [12]. Recall the Klein-Gordon-Zakharov equations

$$\begin{aligned} -\Box u &= \sum_a \Delta |v_a|^2, \\ -\Box v_a + v_a &= u v_a, \end{aligned} \tag{3.1.7}$$

where the unknown u is real valued and v_a are complex valued for $a = 1, 2, 3$. The initial data are denoted by

$$\big(u, \partial_t u\big)(t = 2, \cdot) = \big(u^{(0)}, u^{(1)}\big), \qquad \big(v_a, \partial_t v_a\big)(t = 2, \cdot) = \big(v_a^{(0)}, v_a^{(1)}\big), \tag{3.1.8}$$

In order to apply the strategy of Theorem 3.1.1, we rewrite the equations (3.1.7) in the following form

$$\begin{aligned} -\Box u &= \sum_a \partial_i \Big(\partial^i \big(x_a^2 + y_a^2 \big) \Big), \\ -\Box x_a + x_a &= u x_a, \\ -\Box y_a + y_a &= u y_a, \end{aligned} \tag{3.1.9}$$

in which we use the notations $x_a := \mathrm{Re}(v_a)$ and $y_a := \mathrm{Im}(v_a)$ to denote the real part and imaginary part of a complex number z, respectively.

We note that the regularity of u is one order less than that of v_a. This can be seen from the initial data, which we consider the norm

$$\|u^{(0)}\|_{H^{N_0}}, \quad \|u^{(1)}\|_{H^{N_0-1}}, \quad \|v_a^{(0)}\|_{H^{N_0+1}}, \quad \|v_a^{(1)}\|_{H^{N_0}},$$

with N_0 some large integer. Thus equations (3.1.7) are semilinear equations. Note also that the wave nonlinearity in (3.1.9) is of divergence form, and thus easier to handle than that those in Theorem 3.1.1. Thus our method of proof applies to this system in a very similar way and for which reason we omit the details.

Outline The rest of this chapter is organised as follows. In Section 3.2, we revisit the basics of the hyperboloidal foliation method; next, the estimates for commutators and null forms are given in Section 3.3; later on, we illustrate the techniques obtaining pointwise decay estimates for wave and Klein-Gordon components in Section 3.4; in Section 3.5, by initialising the bootstrap method, we provide some basic estimates needed afterwards; we then derive refined estimates for Klein-Gordon and wave components in Section 3.6 and Section 3.7 respectively; in the last section, we demonstrate the proof of the main theorem, and give some remarks.

3.2 Basics of the hyperboloidal foliation method

3.2.1 Hyperboloidal foliation of Minkowski spacetime

In order to introduce an energy functional for wave or Klein-Gordon components on hyperboloids, we first need to recall some notation from [44] concerning the hyperboloidal foliation method. We adopt the signature $(-,+,+,+)$ in the $(3+1)$–dimensional Minkowski spacetime, and we denote the point $(t,x)=(x^0,x^1,x^2,x^3)$ in Cartesion coordinates, with its spatial radius $r:=|x|=\sqrt{(x^1)^2+(x^2)^2+(x^3)^2}$. We write ∂_α (for $\alpha=0,1,2,3$) for partial derivatives and

$$L_a := x^a\partial_t + t\partial_a, \qquad a=1,2,3 \tag{3.2.1}$$

represent the Lorentz boosts. Throughout this chapter, we consider functions defined in the interior of the future light cone $\mathcal{K}:=\{(t,x): r<t-1\}$, with vertex $(1,0,0,0)$. We consider hyperboloidal hypersurfaces $\mathcal{H}_s:=\{(t,x): t^2-r^2=s^2\}$ with $s>1$. Also $\mathcal{K}_{[s_0,s_1]}:=\{(t,x): s_0^2\leqslant t^2-r^2\leqslant s_1^2; r<t-1\}$ is used to denote subsets of $\mathcal{K}$ limited by two hyperboloids $\mathcal{H}_{s_0}$ and $\mathcal{H}_{s_!}$ with $s_0\leqslant s_1$.

The semi-hyperboloidal frame is defined by

$$\underline{\partial}_0 := \partial_t, \qquad \underline{\partial}_a := \frac{L_a}{t} = \frac{x^a}{t}\partial_t + \partial_a. \tag{3.2.2}$$

Note that the vectors $\underline{\partial}_a$ generate the tangent space to the hyperboloids. We also introduce the vector field $\underline{\partial}_\perp := \partial_t + (x^a/t)\partial_a$, which is orthogonal to the hyperboloids.

For the semi-hyperboloidal frame above, the dual frame is given by $\underline{\theta}^0 := dt - (x^a/t)dx^a$ and $\underline{\theta}^a := dx^a$. The (dual) semi-hyperboloidal frame and the (dual) natural Cartesian frame are connected by the relation

$$\underline{\partial}_\alpha = \Phi_\alpha^{\alpha'}\partial_{\alpha'}, \quad \partial_\alpha = \Psi_\alpha^{\alpha'}\underline{\partial}_{\alpha'}, \quad \underline{\theta}^\alpha = \Psi^\alpha_{\alpha'}dx^{\alpha'}, \quad dx^\alpha = \Phi^\alpha_{\alpha'}\underline{\theta}^{\alpha'}, \tag{3.2.3}$$

where the transition matrix (Φ_α^β) and its inverse (Ψ_α^β) are given by

$$(\Phi_\alpha^\beta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ x^1/t & 1 & 0 & 0 \\ x^2/t & 0 & 1 & 0 \\ x^3/t & 0 & 0 & 1 \end{pmatrix} \tag{3.2.4}$$

and

$$(\Psi_\alpha^\beta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -x^1/t & 1 & 0 & 0 \\ -x^2/t & 0 & 1 & 0 \\ -x^3/t & 0 & 0 & 1 \end{pmatrix}. \tag{3.2.5}$$

3.2.2 Energy estimates on hyperboloids

Following [45], we first introduce the energy E_m, in the Minkowski background, for a function ϕ defined on a hyperboloid $\mathcal{H}_s$:

$$\begin{aligned} E_m(s,\phi) &:= \int_{\mathcal{H}_s} \Big(\big(\partial_t \phi\big)^2 + \sum_a \big(\partial_a \phi\big)^2 + 2(x^a/t)\partial_t\phi\partial_a\phi + m^2\phi^2 \Big)\, dx \\ &= \int_{\mathcal{H}_s} \Big(\big((s/t)\partial_t \phi\big)^2 + \sum_a \big(\underline{\partial}_a \phi\big)^2 + m^2\phi^2 \Big)\, dx \\ &= \int_{\mathcal{H}_s} \Big(\big(\underline{\partial}_\perp \phi\big)^2 + \sum_a \big((s/t)\partial_a \phi\big)^2 + \sum_{a<b} \big(t^{-1}\Omega_{ab}\phi\big)^2 + m^2\phi^2 \Big)\, dx, \end{aligned} \tag{3.2.6}$$

in which $\Omega_{ab} := x^a\partial_b - x^b\partial_a$ the rotational vector field, $\underline{\partial}_\perp := \partial_t + (x^a/t)\partial_a$ the orthogonal vector field, and we denote $E(s,\phi) := E_0(s,\phi)$ for simplicity. In the above, the integral in $L^1(\mathcal{H}_s)$ is defined from the standard (flat) metric in $\mathbb{R}^3$, i.e.

$$\|\phi\|_{L^1_{\mathring{f}}(\mathcal{H}_s)} := \int_{\mathcal{H}_s} |\phi|\, dx = \int_{\mathbb{R}^3} \big|\phi(\sqrt{s^2+r^2}, x)\big|\, dx. \tag{3.2.7}$$

Next, we adapt the energy estimates to our situation.

Proposition 3.2.1 (Energy estimate for wave equation)**.** *For all $s \geqslant 2$, it holds that*

$$E(s,u)^{1/2} \leqslant E(2,u)^{1/2} + \int_2^s \|\Box u\|_{L^2_{\mathring{f}}(\mathcal{H}_{s'})}\, ds' \tag{3.2.8}$$

for every sufficiently regular function u, which is defined and supported in the region $\mathcal{K}_{[2,s]}$.

For the proof, one refers to [45].

Proposition 3.2.2 (Energy estimate for Klein-Gordon equation with varying mass)**.** *Let v be a solution to the Klein-Gordon equation with mass 1*

$$-\Box v + v = uv + f, \tag{3.2.9}$$

which can also be regarded as a Klein-Gordon equation with varying mass $1-u$

$$-\Box v + (1-u)v = f, \tag{3.2.10}$$

defined and supported in the region $\mathcal{K}_{[2,s]}$, and u is a sufficiently regular function defined and supported in the same region $\mathcal{K}_{[2,s]}$, which is assumed to be small

$$|u| \leqslant \frac{1}{10}. \tag{3.2.11}$$

Then the energy on the hyperboloid $\mathcal{H}_s$ can be controlled by either

$$E_1(s,v)^{1/2} \leqslant E_1(2,v)^{1/2} + \int_2^s \Big(\|uv\|_{L^2_{\mathfrak{f}}(\mathcal{H}_{s'})} + \|f\|_{L^2_{\mathfrak{f}}(\mathcal{H}_{s'})} \Big) ds', \tag{3.2.12}$$

or

$$E_1(s,v)^{1/2} \leqslant 2E_1(2,v)^{1/2} + 2\int_2^s \Big(\|(s'/t)\partial_t uv\|_{L^2_{\mathfrak{f}}(\mathcal{H}_{s'})} + \|f\|_{L^2_{\mathfrak{f}}(\mathcal{H}_{s'})} \Big) ds'. \tag{3.2.13}$$

The energy estimate (3.2.13) is better than (3.2.12) in the cases where $\partial_t u$ decays faster than u, which is the case when u is a solution to some wave equation.

Proof. The proof of the energy estimate (3.2.12) is standard and we omit it. In order to prove the energy estimate (3.2.13), we first test the equation (3.2.10) by the multiplier $\partial_t v$ and write the resulting equation in the following favorable form

$$\frac{1}{2}\partial_t\big((\partial_t v)^2 + \sum_a (\partial_a v)^2 + (1-u)v^2\big) + \sum_a \partial_a\big(- \partial_t v \partial_a v\big) = -\frac{1}{2}v^2\partial_t u + \partial_t v f. \tag{3.2.14}$$

We then integrate the identity (3.2.14) over the region $\mathcal{K}_{[2,s]}$ and do integration by parts to arrive at

$$\begin{aligned} & E_{\sqrt{1-u}}(s,v)^{1/2}\frac{d}{ds}E_{\sqrt{1-u}}(s,v)^{1/2} \\ = & \int_{\mathcal{H}_s} (s/t)\big(-\frac{1}{2}v^2\partial_t u + \partial_t v f\big)\, dx \\ \leqslant & \|(s/t)\partial_t uv\|_{L^2_{\mathfrak{f}}(\mathcal{H}_s)}\|v\|_{L^2_{\mathfrak{f}}(\mathcal{H}_s)} + \|f\|_{L^2_{\mathfrak{f}}(\mathcal{H}_s)}\|(s/t)\partial_t v\|_{L^2_{\mathfrak{f}}(\mathcal{H}_s)}. \end{aligned} \tag{3.2.15}$$

Next by recalling the assumption that $|u| \leqslant 1/10$, we have

$$\frac{9}{10}E_1(s,v)^{1/2} \leqslant E_{\sqrt{1-u}}(s,v)^{1/2} \leqslant \frac{11}{10}E_1(s,v)^{1/2},$$

which leads to

$$E_{\sqrt{1-u}}(s,v)^{1/2} \leqslant E_{\sqrt{1-u}}(2,v)^{1/2} + \frac{11}{10}\int_2^s \Big(\|v\partial_t u\|_{L^2(\mathcal{H}_{s'})} + \|f\|_{L^2(\mathcal{H}_{s'})} \Big) ds',$$

and finally (3.2.13). □

3.2.3 Conformal-type energy estimates on hyperboloids

We now introduce a conformal-type energy which is adapted to the hyperboloidal foliation setting, which is due to Ma and Huang in [61]. This lemma will be key to a

robust estimate of the L^2-type norm for the wave component u.

Lemma 3.2.3. *Define the conformal-type energy of a sufficiently regular function u, which is supported in the region $\mathcal{K} = \{(t,x) : |x| < t-1\}$, by*

$$E_{con}(u,s) := \int_{\mathcal{H}_s} \Big(\sum_a \big(s\underline{\partial}_a u\big)^2 + \big(Ku + 2u\big)^2 \Big)\, dx, \tag{3.2.16}$$

in which we used the notation of the weighted inverted time translation

$$Ku := \big(s\partial_s + 2x^a \underline{\partial}_a\big)u.$$

Then it holds

$$E_{con}(u,s)^{1/2} \leqslant E_{con}(u,s_0)^{1/2} + 2\int_{s_0}^{s} s' \|\Box u\|_{L^2_f(\mathcal{H}_{s'})}\, ds', \tag{3.2.17}$$

with moreover

$$\big\|(s/r)u\big\|_{L^2_f(\mathcal{H}_s)} \leqslant E_{con}(u,s)^{1/2}. \tag{3.2.18}$$

3.2.4 Sobolev-type and Hardy-type inequality

We first state a Sobolev-type inequality adapted to the hyperboloids, which is of vital importance for proving sup-norm estimates for both wave and Klein-Gordon components. For the proof, one refers to either [44] or [45] for details.

Proposition 3.2.4. *For all sufficient smooth functions $u = u(t,x)$ supported in $\{(t,x) : |x| < t-1\}$ and for all $s \geqslant 2$, one has*

$$\sup_{\mathcal{H}_s} \big|t^{3/2} u(t,x)\big| \lesssim \sum_{|J|\leqslant 2} \|L^J u\|_{L^2_f(\mathcal{H}_s)}, \tag{3.2.19}$$

in which the symbol L denotes the Lorentz boosts and J is a multi-index. We will also frequently make use of the following identity which follows from (3.2.19) *and standard commutator estimates:*

$$\sup_{\mathcal{H}_s} \big|st^{1/2} u(t,x)\big| \lesssim \sum_{|J|\leqslant 2} \|(s/t)L^J u\|_{L^2_f(\mathcal{H}_s)}, \tag{3.2.20}$$

In order to control the L^2–type of norm for the wave component u, we need the following Hardy-type inequality on the hyperboloidal foliation, see [44] for instance.

Lemma 3.2.5. *Assume the function u is defined and supported in the region $\{(t,x) :$*

$|x| < t-1\}$ *and is sufficiently regular, then for all* $s \geqslant 2$, *one has*

$$\|r^{-1}u\|_{L^2_f(\mathcal{H}_s)} \lesssim \sum_a \|\underline{\partial}_a u\|_{L^2_f(\mathcal{H}_s)}. \tag{3.2.21}$$

3.3 Estimates for commutators and null forms

3.3.1 Commutator estimates

We restate the estimates for the commutators, which are proven in [44] and [45].

Lemma 3.3.1. *Assume a function* u *defined in the region* $\mathcal{K}$ *is regular enough, then with the generic constant* $C(|I|,|J|)$, *we have*

$$\big|[\partial^I L^J, \partial_\alpha]u\big| \leqslant C(|I|,|J|) \sum_{|J'|<|J|,\beta} \big|\partial_\beta \partial^I L^{J'} u\big|, \tag{3.3.1}$$

$$\big|[\partial^I L^J, \underline{\partial}_a]u\big| \leqslant C(|I|,|J|)\Big(\sum_{|I'|<|I|,|J'|<|J|,b} \big|\underline{\partial}_b \partial^{I'} L^{J'} u\big| + t^{-1} \sum_{|I'|\leqslant|I|,|J'|\leqslant|J|} \big|\partial^{I'} L^{J'} u\big|\Big), \tag{3.3.2}$$

$$\big|[\partial^I L^J, \underline{\partial}_\alpha]u\big| \leqslant C(|I|,|J|)\Big(\sum_{|I'|<|I|,|J'|<|J|,\beta} \big|\partial_\beta \partial^{I'} L^{J'} u\big| + t^{-1} \sum_{|I'|\leqslant|I|,|J'|\leqslant|J|,\beta} \big|\partial_\beta \partial^{I'} L^{J'} u\big|\Big), \tag{3.3.3}$$

$$\big|[\partial^I L^J, \partial_\alpha\partial_\beta]u\big| \leqslant C(|I|,|J|) \sum_{|I'|\leqslant|I|,|J'|<|J|,\gamma,\gamma'} \big|\partial_\gamma\partial_{\gamma'} \partial^{I'} L^{J'} u\big|, \tag{3.3.4}$$

$$\big|\partial^I L^J((s/t)\partial_\alpha u)\big| \leqslant |(s/t)\partial_\alpha \partial^I L^J u| + C(|I|,|J|) \sum_{|I'|\leqslant|I|,|J'|\leqslant|J|,\beta} \big|(s/t)\partial_\beta \partial^{I'} L^{J'} u\big|. \tag{3.3.5}$$

Recall here that Greek indices $\alpha,\beta \in \{0,1,2,3\}$ *and Latin indices* $a,b \in \{1,2,3\}$.

3.3.2 Null form estimates

The estimates for null forms are also restated for convenience.

Lemma 3.3.2. *For the quadratic null term* $\partial^\alpha u \partial_\alpha v$ *with sufficiently regular functions*

u and v, one has

$$\begin{aligned}\left|\partial^I L^J(\partial^\alpha u \partial_\alpha v)\right| \lesssim & \sum_{\substack{|I_1|+|I_2|\leqslant|I|,\\ |J_1|+|J_2|\leqslant|J|,\\ a,\beta}} \left(\left|\partial^{I_1} L^{J_1} \underline{\partial}_a u \partial^{I_2} L^{J_2} \underline{\partial}_\beta v\right| + \left|\partial^{I_1} L^{J_1} \underline{\partial}_\beta u \partial^{I_2} L^{J_2} \underline{\partial}_a v\right|\right) \\ & + (s/t)^2 \sum_{\substack{|I_1|+|I_2|\leqslant|I|,\\ |J_1|+|J_2|\leqslant|J|}} \left|\partial^{I_1} L^{J_1} \partial_t u \partial^{I_2} L^{J_2} \partial_t v\right|.\end{aligned} \tag{3.3.6}$$

One refers to [44] for the proof.

3.4 Tools for pointwise estimates for wave and Klein-Gordon components

3.4.1 Sup-norm estimates for wave components

We recall the following lemma from [45], which is essential in proving the sup-norm bound for wave components. An alternative proof of Lemma 3.4.1 is also found in [2].

Lemma 3.4.1 (Pointwise estimates for wave components)**.** *Suppose u is a spatially compactly supported solution to the wave equation*

$$\begin{aligned} -\Box u &= f, \\ u(t_0, x) &= \partial_t u(t_0, x) = 0, \end{aligned} \tag{3.4.1}$$

with f spatially compactly supported and satisfying

$$|f| \leqslant C_f t^{-2-\nu}(t-r)^{-1+\mu}, \tag{3.4.2}$$

for $0 < \mu \leqslant 1/2$ and $0 < \nu \leqslant 1/2$. Then we have

$$|u(t,x)| \lesssim \frac{C_f}{\nu\mu}(t-r)^{\mu-\nu}t^{-1}, \tag{3.4.3}$$

where C_f is some constant.

3.4.2 Sup-norm estimates for Klein-Gordon components

Following the pointwise estimates for Klein-Gordon components in the hyperboloidal foliation setting, which were first introduced in [45], we adapt it to our case where the mass of the Klein-Gordon field varies.

Proposition 3.4.2 (Pointwise estimates for Klein-Gordon components with varying mass)**.** *Assume v is a sufficiently regular and spatially compactly supported solution to the Klein-Gordon equation*

$$\begin{aligned} -\Box v + (1-u)v &= f, \\ v|_{\mathcal{H}_2} = v_0, \quad \partial_t v|_{\mathcal{H}_2} &= v_1, \end{aligned} \tag{3.4.4}$$

with the assumption $|u| \leqslant 1/10$, then one has

$$s^{3/2}\big|v(t,x)\big| + (s/t)^{-1}s^{3/2}\big|\underline{\partial}_{\perp}v(t,x)\big| \lesssim V(t,x), \tag{3.4.5}$$

with

$$V(t,x) := \begin{cases} e^{\int_{s_0}^{s}|\frac{d}{d\lambda}u(\lambda t/s,\lambda x/s)|\,d\lambda}\Big(\|v_0\|_{L^\infty(\mathcal{H}_2)} + \|v_1\|_{L^\infty(\mathcal{H}_2)} + F(s)\Big), & r/t \leqslant 3/5, \\ e^{\int_{s_0}^{s}|\frac{d}{d\lambda}u(\lambda t/s,\lambda x/s)|\,d\lambda}F(s), & 3/5 \leqslant r/t \leqslant 1, \end{cases} \tag{3.4.6}$$

and

$$s_0 := \begin{cases} 2, & r/t \leqslant 3/5, \\ \sqrt{\frac{t+r}{t-r}}, & 3/5 \leqslant r/t \leqslant 1, \end{cases} \tag{3.4.7}$$

and

$$F(s) := \int_{s_0}^{s}\Big|R[v](\lambda t/s,\lambda x/s) + \lambda^{3/2}f(\lambda t/s,\lambda x/s)\Big|\,d\lambda, \tag{3.4.8}$$

where

$$R[v] := s^{3/2}\sum \underline{\partial}_a\underline{\partial}_a v + \frac{x^a x^b}{s^{1/2}}\underline{\partial}_a\underline{\partial}_b v + \frac{3}{4s^{1/2}}v + \frac{3x^a}{s^{1/2}}\underline{\partial}_a v. \tag{3.4.9}$$

The proof of Proposition 3.4.2 is based on the decomposition result in Lemma 3.4.3 and an estimate of ODE in Lemma 3.4.4, both stated below. We refer to [45] for the detailed proofs, but give a simpler proof of Lemma 3.4.4 below, which provides a neater expression of the estimate for the ODE.

Lemma 3.4.3. *Assume v is a sufficiently regular solution to the Klein-Gordon equation* (3.4.4)*, and let*

$$w_{t,x}(\lambda) := \lambda^{3/2}v(\lambda t/s,\lambda x/s), \quad (t,x) \in \mathcal{K},$$

then the following second-order ODE with respect to λ holds

$$\frac{d^2}{d\lambda^2}w_{t,x}(\lambda) + \big(1-u(\lambda t/s,\lambda x/s)\big)w_{t,x}(\lambda) = \big(R[v] + s^{3/2}f\big)(\lambda t/s,\lambda x/s). \tag{3.4.10}$$

Lemma 3.4.4. *Consider the second-order ODE*

$$\begin{aligned} &z''(\lambda) + \big(1-G(\lambda)\big)z(\lambda) = k(\lambda), \\ &z(s_0) = z_0, \quad z'(s_0) = z_1, \quad |G(\lambda)| \leqslant 1/10, \end{aligned} \tag{3.4.11}$$

in which k is assumed to be integrable, then we have the following pointwise estimate

$$\big((z')^2(s)+(1-G(s))z^2(s)\big)^{1/2} \lesssim e^{\int_{s_0}^s |G'(\lambda)|\,d\lambda}\Big(\big((z')^2(0)+z^2(0)\big)^{1/2}+\int_{s_0}^s k(\lambda)\,d\lambda\Big). \quad (3.4.12)$$

Proof. We set $Y(\lambda) = \big((z')^2(\lambda) + (1 - G(\lambda))z^2(\lambda)\big)^{1/2}$, and then by multiplying $z'(\lambda)$ in (3.4.11), we get

$$\begin{aligned} \frac{d}{d\lambda}Y^2(\lambda) &= z'(\lambda)k(\lambda) - G'(\lambda)z^2(\lambda) \\ &\leqslant Y(\lambda)\big(k(\lambda) + |G'|Y(\lambda)\big). \end{aligned} \quad (3.4.13)$$

In order to proceed, we divide $Y(\lambda)$ in the above inequality and, integrate to get

$$Y(s) \leqslant Y(s_0) + \int_{s_0}^s \big(k(\lambda) + |G'|Y(\lambda)\big)\,d\lambda. \quad (3.4.14)$$

Finally, we apply Gronwall-type inequality from Lemma 3.4.5 to end the proof. □

We have used the following standard Gronwall inequality.

Lemma 3.4.5. *Let $u(t)$ be continuous and nonnegative in $[0, T]$, and satisfy*

$$u(t) \leqslant A + \int_0^t \Big(a(s)u(s) + b(s)\Big)\,ds, \quad (3.4.15)$$

where $a(t)$ and $b(t)$ are nonnegative integrable functions in $[0, T]$ and A is nonnegative constant. Then it holds

$$u(t) \leqslant \Big(A + \int_0^t b(s)\,ds\Big)e^{\int_0^t a(s)\,ds}, \quad t \in [0, T]. \quad (3.4.16)$$

3.5 Bootstrap method

Before beginning the bootstrap argument, we recall the theorem we will be proving.

Theorem 3.5.1. *Consider the system*

$$\begin{aligned} -\Box u &= uv + u\partial_t v, \\ -\Box v + v &= uv, \end{aligned} \quad (3.5.1)$$

whose initial data are prescribed on the time slice $t = 2$

$$\begin{aligned} \big(u, \partial_t u\big)(2, \cdot) &= \big(u_0, u_1\big), \\ \big(v, \partial_t v\big)(2, \cdot) &= \big(v_0, v_1\big). \end{aligned} \quad (3.5.2)$$

Let N be a sufficiently large integer, for example $N \geqslant 8$ suffices. Then there exists $\epsilon_0 > 0$ such that for all $\epsilon \in (0, \epsilon_0)$ and all compactly supported initial data (u_0, u_1, v_0, v_1) satisfying the smallness condition

$$\|u_0, v_0\|_{H^{N+1}(\mathbb{R}^3)} + \|u_1, v_1\|_{H^N(\mathbb{R}^3)} \leqslant \epsilon, \tag{3.5.3}$$

the initial value problem (3.1.1)–(3.1.2) *admits a global-in-time solution (u, v) with*

$$|u(t,x)| \lesssim t^{-1}, \quad |v(t,x)| \lesssim t^{-3/2}. \tag{3.5.4}$$

3.5.1 Bootstrap assumption

We assume that the following bootstrap assumptions hold in the interval $[2, s_1]$

$$E(s, \partial^I L^J u)^{1/2} \leqslant C_1 \epsilon, \qquad |I| + |J| \leqslant N - 1, \tag{3.5.5a}$$
$$E(s, \partial^I u)^{1/2} \leqslant C_1 \epsilon s^{\delta}, \qquad |I| = N, \tag{3.5.5b}$$
$$E(s, \partial^I L^J u)^{1/2} \leqslant C_1 \epsilon s^{|J|\delta}, \qquad |I| + |J| \leqslant N, |J| \geqslant 1 \tag{3.5.5c}$$
$$E_1(s, \partial^I L^J v)^{1/2} \leqslant C_1 \epsilon s^{|J|\delta}, \qquad |I| + |J| \leqslant N, \tag{3.5.5d}$$
$$\|(s/t)\partial^I L^J u\|_{L^2_f(\mathcal{H}_s)} \leqslant C_1 \epsilon s^{1/2+|J|\delta}, \qquad |I| + |J| \leqslant N, \tag{3.5.5e}$$
$$|\partial^I L^J u| \leqslant C_1 \epsilon t^{-1} s^{|J|\delta}, \qquad |I| + |J| \leqslant N - 4, \tag{3.5.5f}$$
$$|\partial^I L^J v| \leqslant (C_1 \epsilon)^{1/2} t^{-3/2} s^{|J|\delta}, \qquad |I| + |J| \leqslant N - 4, \tag{3.5.5g}$$

in which C_1 is some big constant which is fixed once and for all, δ is some fixed small constant, i.e. $0 < \delta \ll 1$, and s_1 is defined by

$$s_1 := \sup\{s : (3.5.5)\ \textit{hold}\}.$$

We recall that the fact $s_1 > 2$ follows from the local existence result, which is classical, see for example [44, Section 11]. And importantly, we note that C_1 and δ are independent of s_1.

In order to prove the stability result stated in Theorem 3.1.1, it suffices to demon-

strate the refined energy bounds below

$$\begin{aligned}
E(s,\partial^I L^J u)^{1/2} &\leqslant \frac{1}{2}C_1\epsilon, && |I|+|J|\leqslant N-1,\\
E(s,\partial^I u)^{1/2} &\leqslant \frac{1}{2}C_1\epsilon s^{\delta}, && |I|=N,\\
E(s,\partial^I L^J u)^{1/2} &\leqslant \frac{1}{2}C_1\epsilon s^{|J|\delta}, && |I|+|J|\leqslant N, |J|\geqslant 1,\\
E_1(s,\partial^I L^J v)^{1/2} &\leqslant \frac{1}{2}C_1\epsilon s^{|J|\delta}, && |I|+|J|\leqslant N,\\
\|(s/t)\partial^I L^J u\|_{L^2_f(\mathcal{H}_s)} &\leqslant \frac{1}{2}C_1\epsilon s^{1/2+|J|\delta}, && |I|+|J|\leqslant N,\\
|\partial^I L^J u| &\leqslant \frac{1}{2}C_1\epsilon t^{-1}s^{|J|\delta}, && |I|+|J|\leqslant N-4,\\
|\partial^I L^J v| &\leqslant \frac{1}{2}(C_1\epsilon)^{1/2}t^{-3/2}s^{|J|\delta}, && |I|+|J|\leqslant N-4.
\end{aligned} \tag{3.5.6}$$

Note that the bounds in (3.5.6) indicate that s_1 cannot be of finite value, which thus completes the proof of a global-in-time solution stated in the main Theorem 3.1.1.

3.5.2 Direct estimates

Direct consequences of (3.5.5a) and (3.5.5d) are the following:

$$\begin{aligned}
|\partial^I L^J \partial u| + |\partial\partial^I L^J u| &\lesssim C_1\epsilon t^{-1/2}s^{-1}, && |I|+|J|\leqslant N-3,\\
|\partial^I L^J v| &\lesssim C_1\epsilon t^{-3/2}s^{(|J|+2)\delta}, && |I|+|J|\leqslant N-2.
\end{aligned} \tag{3.5.7}$$

These follow from the Sobolev–type inequality of Proposition 3.2.4 and estimates for commutators in Lemma 3.3.1.

Assumptions (3.5.5a)–(3.5.5c) also imply the following L^2–type estimates

$$\begin{aligned}
\|(s/t)\partial^I L^J \partial u\|_{L^2_f(\mathcal{H}_s)} + \|(s/t)\partial\partial^I L^J u\|_{L^2_f(\mathcal{H}_s)} &\lesssim C_1\epsilon, && |I|+|J|\leqslant N-1,\\
\|(s/t)\partial^I L^J \partial u\|_{L^2_f(\mathcal{H}_s)} + \|(s/t)\partial\partial^I L^J u\|_{L^2_f(\mathcal{H}_s)} &\lesssim C_1\epsilon s^{\delta}, && |I|=N,\\
\|(s/t)\partial^I L^J \partial u\|_{L^2_f(\mathcal{H}_s)} + \|(s/t)\partial\partial^I L^J u\|_{L^2_f(\mathcal{H}_s)} &\lesssim C_1\epsilon s^{|J|\delta}, && |I|+|J|=N, |J|\geqslant 1.
\end{aligned} \tag{3.5.8}$$

3.6 Refined estimates for the Klein-Gordon component

3.6.1 Refined energy estimates for v

We show here the refined estimates for the Klein-Gordon component, and we will see that the most difficult part is to get the refined ones for $\partial^I v$. The difficulty comes from the integral of

$$\int_2^s s'^{-1}\, ds'$$

diverges, but we can circumvent it by moving the nonlinear term uv in the Klein-Gordon equation in (3.1.1) to the left hand side and then regarding the mass of v as the varying one $1-u$.

Lemma 3.6.1. *By utilising the notation of commutators $[A,B]u := A(Bu) - B(Au)$, we have*

$$\big\|[1-u,\partial^I L^J]v\big\|_{L^2_f(\mathcal{H}_s)} \lesssim (C_1\epsilon)^{3/2} s^{-1+|J|\delta}, \quad |I|+|J| \leqslant N, \tag{3.6.1}$$

and furthermore, we have

$$\big\|[1-u,\partial^I]v\big\|_{L^2_f(\mathcal{H}_s)} \lesssim (C_1\epsilon)^{3/2} s^{-3/2}, \quad |I| \leqslant N. \tag{3.6.2}$$

Proof. First note the expansion of the commutator

$$[1-u,\partial^I L^J]v = \sum_{\substack{I_1+I_2=I, J_1+J_2=J \\ |I_1|+|J_1|\geqslant 1}} \partial^{I_1} L^{J_1} u \partial^{I_2} L^{J_2} v.$$

For the case of $|J| \geqslant 1$, we conduct the following

$$\begin{aligned}\big\|[1-u,\partial^I L^J]v\big\|_{L^2_f(\mathcal{H}_s)} \lesssim & \sum_{\substack{I_1+I_2=I, J_1+J_2=J \\ |I_1|+|J_1|\geqslant |I_2|+|J_2|}} \|(s/t)\partial^{I_1} L^{J_1} u\|_{L^2_f(\mathcal{H}_s)} \|(t/s)\partial^{I_2} L^{J_2} v\|_{L^\infty(\mathcal{H}_s)} \\ & + \sum_{\substack{I_1+I_2=I, J_1+J_2=J \\ 1\leqslant |I_1|+|J_1|\leqslant |I_2|+|J_2|}} \|\partial^{I_1} L^{J_1} u\|_{L^\infty(\mathcal{H}_s)} \|\partial^{I_2} L^{J_2} v\|_{L^2_f(\mathcal{H}_s)},\end{aligned}$$

and the L^2–type estimates for u in (3.5.5) verifies

$$\big\|[1-u,\partial^I L^J]v\big\|_{L^2_f(\mathcal{H}_s)} \lesssim \sum_{J_1+J_2=J} C_1\epsilon s^{1/2+|J_1|\delta}(C_1\epsilon)^{1/2} t^{-1/2} s^{-1+|J_2|\delta} + C_1\epsilon t^{-1} s^{|J_1|\delta} C_1\epsilon s^{|J_2|\delta},$$

which leads to (3.6.1).

For the proof of (3.6.2), we proceed in the same way but pay attention to the fact that

$$\begin{aligned}\left\|(s/t)\partial^{I_1}u\right\|_{L^2_f(\mathcal{H}_s)} &\lesssim C_1\epsilon, \qquad 1\leqslant |I_1|\leqslant N,\\ \left\|(s/t)\partial^{I_1}u\right\|_{L^\infty(\mathcal{H}_s)} &\lesssim C_1\epsilon t^{-3/2}, \qquad 1\leqslant |I_1|\leqslant N-4.\end{aligned}$$

□

Proposition 3.6.2 (Refined energy estimates for v)**.** *Consider the Klein-Gordon equation in* (3.1.1) *and assume the bounds in* (3.5.5) *hold, then we have the following refined ones*

$$E_1(s,\partial^I L^J v)^{1/2}\lesssim \epsilon+(C_1\epsilon)^{3/2}s^{|J|\delta},\quad |I|+|J|\leqslant N. \tag{3.6.3}$$

Proof. We first act $\partial^I L^J$ on the Klein-Gordon equation in (3.1.1) to get

$$-\Box\partial^I L^J v+(1-u)\partial^I L^J v=\sum_{\substack{I_1+I_2=I,J_1+J_2=J\\ |I_1|+|J_1|\geqslant 1}}\partial^{I_1}L^{J_1}u\partial^{I_2}L^{J_2}v.$$

We then apply the energy estimate (3.2.13) for Klein-Gordon equations with varying masses and use Lemma 3.6.1 to show

$$\begin{aligned}&E_1(s,\partial^I L^J v)^{1/2}\\ &\leqslant 2E(2,\partial^I L^J v)^{1/2}+2\int_2^s\Big(\|(s'/t)\partial_t u\partial^I L^J v\|_{L^2_f(\mathcal{H}_{s'})}+\left\|[1-u,\partial^I L^J]v\right\|_{L^2_f(\mathcal{H}_s)}ds'\\ &\lesssim \epsilon+\int_2^s\Big(\|\partial_t u\|_{L^\infty(\mathcal{H}_{s'})}\|\partial^I L^J v\|_{L^2_f(\mathcal{H}_{s'})}+\sum_{\substack{I_1+I_2=I,J_1+J_2=J\\ |I_1|+|J_1|\geqslant 1}}\|\partial^{I_1}L^{J_1}u\partial^{I_2}L^{J_2}v\|_{L^2_f(\mathcal{H}_{s'})}\Big)\,ds'.\end{aligned} \tag{3.6.4}$$

Successively, in the case of $|J|\geqslant 1$, it is true that

$$E_1(s,\partial^I L^J v)^{1/2}\lesssim \epsilon+(C_1\epsilon)^{3/2}\int_2^s s'^{-1+|J|\delta}\,ds'\lesssim \epsilon+(C_1\epsilon)^{3/2}s^{|J|\delta}, \tag{3.6.5}$$

while in the case of $|J|=0$, better estimates on $\partial^{I_1}u$ with $|I_1|\geqslant 1$ enable us to obtain

$$E_1(s,\partial^I L^J v)^{1/2}\lesssim \epsilon+(C_1\epsilon)^{3/2}\int_2^s s'^{-3/2+\delta}\,ds'\lesssim \epsilon+(C_1\epsilon)^{3/2}, \tag{3.6.6}$$

which finishes the proof. □

3.6.2 Refined pointwise estimates for v

We now prove the refined sup-norm bounds for the Klein-Gordon component v,and we first prepare some lemmas which will be of help.

Lemma 3.6.3. *The solution u to our wave equation satisfies*

$$e^{\int_{s_0}^{s} |\frac{d}{d\lambda} u(\lambda t/s, \lambda x/s)|\, d\lambda} \lesssim 1. \tag{3.6.7}$$

Proof. We observe that

$$\frac{d}{d\lambda} u(\lambda t/s, \lambda x/s) = (t/s) \underline{\partial}_{\perp} u(\lambda t/s, \lambda x/s),$$

and, on the other hand, we have

$$\underline{\partial}_{\perp} u(t,x) = \frac{s^2}{t^2} \partial_t u(t,x) + \frac{x^a}{t^2} L_a u(t,x).$$

Hence by recalling the pointwise bootstrap (3.5.5f) of u that

$$|L_a u(t,x)| \leqslant C_1 \epsilon t^{-1} s^{\delta},$$

we find

$$\big|(t/s) \underline{\partial}_{\perp} u(t,x)\big| \lesssim C_1 \epsilon s^{-3/2}.$$

This implies that

$$\Big|\frac{d}{d\lambda} u(\lambda t/s, \lambda x/s)\Big| \lesssim C_1 \epsilon \lambda^{-3/2},$$

and hence the completeness of the proof. □

Lemma 3.6.4. *We have the estimate for $R[\partial^I L^J v]$ in the region $\mathcal{K}_{[2,s_1]}$ that*

$$\big|R[\partial^I L^J v](\lambda t/s, \lambda x/s)\big| \lesssim C_1 \epsilon (s/t)^{3/2} \lambda^{-3/2+N\delta}, \quad |I| + |J| \leqslant N - 4. \tag{3.6.8}$$

The proof can be found in [45].

One last ingredient is the commutator estimate stated below.

Lemma 3.6.5. *The following estimates for the the commutator are valid*

$$\big|\big([1-u, \partial^I L^J] v\big)(\lambda t/s, \lambda x/s)\big| \lesssim (C_1\epsilon)^{3/2} (s/t)^{5/2} \lambda^{-5/2+|J|\delta}, \quad |I| + |J| \leqslant N - 4, \tag{3.6.9}$$

moreover, in the case of $|J| = 0$, one has

$$\big|\big([1-u, \partial^I] v\big)(\lambda t/s, \lambda x/s)\big| \lesssim (C_1\epsilon)^{3/2} (s/t)^{2} \lambda^{-3}, \quad |I| \leqslant N - 4. \tag{3.6.10}$$

Proof. First recall the expansion of the commutator

$$[1-u, \partial^I L^J] v = - \sum_{\substack{I_1+I_2=I, J_1+J_2=J \\ |I_1|+|J_1| \geqslant 1}} \partial^{I_1} L^{J_1} u \partial^{I_2} L^{J_2} v.$$

Next recall the pointwise estimates in (3.5.5) and they give

$$\begin{aligned}\big([1-u,\partial^I L^J]v\big)(t,x) &\lesssim \sum_{J_1+J_2=J} C_1\epsilon t^{-1}s^{|J_1|\delta}(C_1\epsilon)^{1/2}t^{-3/2}s^{|J_2|\delta}\\ &\lesssim (C_1\epsilon)^{3/2}t^{-5/2}s^{|J|\delta} = (C_1\epsilon)^{3/2}(s/t)^{5/2}s^{-5/2+|J|\delta},\end{aligned}$$

which finishes the proof of (3.6.9).

For the proof of (3.6.10), we proceed in the same way but recall the estimate below from (3.5.7)

$$|\partial^{I_1}u| \lesssim C_1\epsilon t^{-1/2}s^{-1}, \quad 1 \leqslant |I_1| \leqslant N-4.$$

□

We are in a position to give the proof of the refined sup-norm bounds for the Klein-Gordon component.

Proposition 3.6.6 (Refined pointwise estimates for v)**.** *The following estimates are valid*

$$\big|\partial^I L^J v\big| + \big|(t/s)\underline{\partial}_\perp \partial^I L^J v\big| \lesssim C_1\epsilon t^{-3/2}s^{|J|\delta}, \quad |I|+|J| \leqslant N-4. \tag{3.6.11}$$

Proof. We act $\partial^I L^J$ on the Klein-Gordon equation in (3.1.1) to get

$$-\Box\partial^I L^J v + (1-u)\partial^I L^J v = [1-u,\partial^I L^J]v.$$

We have

$$F(s) \leqslant \int_{s_0}^{s} \Big(\big|R[\partial^I L^J v](\lambda t/s,\lambda x/s)\big| + \lambda^{3/2}\big|[1-u,\partial^I L^J]v\big|(\lambda t/s,\lambda x/s)\Big)\,d\lambda,$$

in which $F(s)$ was defined in (3.4.8) in Proposition 3.4.2. Then by recalling the estimate (3.6.8) and the commutator estimates (3.6.10) from the previous two Lemmas, we have

$$F(s) \lesssim C_1\epsilon(s/t)^{3/2}s^{|J|\delta},$$

which leads to the bound

$$|\partial^I L^J v(t,x)| \lesssim s^{-3/2}|F| \lesssim C_1\epsilon t^{-3/2}s^{|J|\delta}.$$

□

As a consequence, we have

$$|\partial\partial^I L^J v| \lesssim C_1\epsilon t^{-1/2}s^{-1+|J|\delta}, \quad |I|+|J| \leqslant N-4, \tag{3.6.12}$$

which is due to the following two identities (see also [45]):

$$\partial_t = \frac{t^2}{s^2}\big(\underline{\partial}_\perp - (x^a/t)\underline{\partial}_a\big), \qquad \partial_a = -\frac{tx^a}{s^2}\underline{\partial}_\perp + \frac{x^a x^b}{t^2}\underline{\partial}_b + \underline{\partial}_a.$$

We note that (3.6.12) is used when we estimate the pointwise decay of the null form $\partial_\alpha u \partial^\alpha v$ in (3.7.10) below.

3.7 Refined estimates for the wave component

3.7.1 Overview of the strategy on treating u

If we deal directly with the nonlinearity uv for the wave equation in (3.1.1), it is very difficult to get either desired energy estimates or pointwise estimates. Due to this difficulty, we are motivated to do a transformation and seek for a new unknown which satisfies a wave equation with good nonlinearity, and which meanwhile is close to the original wave component u up a higher order correction term. The idea to treat the Klein-Gordon field is similar as the use of a normal form transformation by Shatah [66] combined with the technique used to deal with *wave–wave* interaction used by Tsutsumi [76]. But before we do the transformation, we find it necessary to first split the wave equation into two, which agrees with the special structure of the equation for u.

Proposition 3.7.1. *Let (u, v) be a solution to the model problem* (3.1.1)

$$\begin{aligned} -\Box u &= uv + u\partial_t v, \\ -\Box v + v &= uv, \\ \big(u, \partial_t u\big)(2,\cdot) = \big(u_0, u_1\big), &\quad \big(v, \partial_t v\big)(2,\cdot) = \big(v_0, v_1\big), \end{aligned}$$

then we can split u into the following form

$$u = U_1 + \partial_t U_2, \tag{3.7.1}$$

in which U_1 and U_2 are solutions to the two wave equations below:

$$\begin{aligned} -\Box U_1 &= uv - v\partial_t u, \\ \big(U_1, \partial_t U_1\big)(2,\cdot) &= \big(u_0, u_1 + u_0 v_0\big), \end{aligned} \tag{3.7.2}$$

and

$$\begin{aligned} -\Box U_2 &= uv, \\ \big(U_2, \partial_t U_2\big)(2,\cdot) &= \big(0, 0\big). \end{aligned} \tag{3.7.3}$$

We recall that this key observation of splitting as in (3.7.1) is due to Katayama [35].

Next, we do a transformation to make the nonlinearities in the U_1 and U_2 equations easier to deal with.

Proposition 3.7.2. *Consider the wave equations of U_1 and U_2 in Proposition 3.7.1, and set*

$$\widetilde{U}_1 := U_1 + uv, \qquad \widetilde{U}_2 := U_2 + uv,$$

then the new unknowns $\widetilde{U}_1$ and $\widetilde{U}_2$ satisfy wave equations with new nonlinearities, which are easy to handle, i.e.

$$-\Box\widetilde{U}_1 = -\partial^\alpha u\partial_\alpha v - v\partial_t u + u^2 v + uv^2, \tag{3.7.4}$$

and

$$-\Box\widetilde{U}_2 = -\partial^\alpha u\partial_\alpha v + u^2 v + uv^2. \tag{3.7.5}$$

Proof. The proof follows by simple calculations. We only do it for $\widetilde{U}_2$

$$-\Box\widetilde{U}_2 = -\Box(U_2 + uv) = -\Box U_2 - \partial^\alpha u\partial_\alpha v + (-\Box u)v + u(-\Box v + v) - uv,$$

then by utilising the equations in (3.7.3), we finally arrive at (3.7.5). □

The following consequences follow immediately, which say about that U's are very close to $\widetilde{U}$'s.

Lemma 3.7.3. *Assume U_1 and U_2 are solutions to* (3.7.2) *and* (3.7.3) *respectively, and let the bootstrap assumptions in* (3.5.5) *hold, then it verifies for all $s \in [2, s_1]$ that*

$$\begin{aligned}
\frac{1}{2}E(s,\partial^I L^J U_p)^{1/2} \leqslant & E(s,\partial^I L^J \widetilde{U}_p)^{1/2} \leqslant 2E(s,\partial^I L^J U_p)^{1/2}, && |I|+|J| \leqslant N,\\
\frac{1}{2}E_{con}(s,\partial^I L^J U_p)^{1/2} \leqslant & E_{con}(s,\partial^I L^J \widetilde{U}_p)^{1/2} \leqslant 2E_{con}(s,\partial^I L^J U_p)^{1/2}, && |I|+|J| \leqslant N,\\
& |\partial^I L^J (U_p - \widetilde{U}_p)| \leqslant C_1\epsilon t^{-3/2}, && |I|+|J| \leqslant N-4,
\end{aligned} \tag{3.7.6}$$

for $p = 1, 2$.

Proof. The proof follows by the fact that the difference between U_p and $\widetilde{U}_p$ is a quadratic term uv, which has very good decay property. □

3.7.2 Estimates of the U_1 part

We are now about to derive various estimates for U_1, which will be based on the analysis of the new unknown $\widetilde{U}_1$. We start by a simple lemma, estimating $v\partial_t u$.

Lemma 3.7.4. *Let the bootstrap assumptions in* (3.5.5) *be true, then it holds*

$$\big\|\partial^I L^J(v\partial_t u)\big\|_{L^2_f(\mathcal{H}_s)} \lesssim (C_1\epsilon)^{3/2} s^{-3/2+|J|\delta}, \quad |I|+|J| \leqslant N, \tag{3.7.7}$$

and

$$\big|\partial^I L^J(v\partial_t u)\big| \lesssim (C_1\epsilon)^{3/2} t^{-2} s^{-1+|J|\delta}, \quad |I|+|J| \leqslant N-4. \tag{3.7.8}$$

Proof. We directly do the estimates

$$\begin{aligned}
\big\|\partial^I L^J(v\partial_t u)\big\|_{L^2_f(\mathcal{H}_s)} &\leqslant \sum_{\substack{I_1+I_2=I\\ J_1+J_2=J}} \big\|\partial^{I_1} L^{J_1}\partial_t u \partial^{I_2} L^{J_2} v\big\|_{L^2_f(\mathcal{H}_s)} \\
&\leqslant \sum_{\substack{I_1+I_2=I, J_1+J_2=J\\ |I_1|+|J_1|\leqslant |I_2|+|J_2|}} \big\|\partial^{I_1} L^{J_1}\partial_t u\|_{L^\infty(\mathcal{H}_s)} \big\|\partial^{I_2} L^{J_2} v\big\|_{L^2_f(\mathcal{H}_s)} \\
&+ \sum_{\substack{I_1+I_2=I, J_1+J_2=J\\ |I_1|+|J_1|\geqslant |I_2|+|J_2|}} \big\|(s/t)\partial^{I_1} L^{J_1}\partial_t u\|_{L^2_f(\mathcal{H}_s)} \big\|(t/s)\partial^{I_2} L^{J_2} v\big\|_{L^\infty(\mathcal{H}_s)},
\end{aligned}$$

and finally the basic estimates in Subsection 3.5.2 completes the proof of (3.7.7).

For the sup-norm bound, note that

$$\big|\partial^I L^J(v\partial_t u)\big| \leqslant \sum_{\substack{I_1+I_2=I\\ J_1+J_2=J}} \big|\partial^{I_1} L^{J_1}\partial_t u \partial^{I_2} L^{J_2} v\big|,$$

and then it follows from the bootstrap assumptions (3.5.5) as well as the pointwise estimates (3.5.7) for $\partial^{I_1} L^{J_1}\partial_t u$. □

Lemma 3.7.5. *We have*

$$\Big\|\partial^I L^J\big(-\partial^\alpha u\partial_\alpha v - v\partial_t u + u^2 v + uv^2\big)\Big\|_{L^2_f(\mathcal{H}_s)} \lesssim (C_1\epsilon)^{3/2} s^{-3/2+|J|\delta}, \quad |I|+|J| \leqslant N, \tag{3.7.9}$$

as well as

$$\Big|\partial^I L^J\big(-\partial^\alpha u\partial_\alpha v - v\partial_t u + u^2 v + uv^2\big)\Big| \lesssim (C_1\epsilon)^{3/2} t^{-2} s^{-1+|J|\delta}, \quad |I|+|J| \leqslant N-4. \tag{3.7.10}$$

Proof. The terms are either null, $\partial_t uv$ or cubic. Since $\partial_t uv$ is already treated in Lemma 3.7.4, one refers to Lemma 3.3.2 for more details on treating null forms. □

Proposition 3.7.6 (Energy estimates for U_1)**.** *Consider the wave equation in* (3.7.2) *and assume the bounds in* (3.5.5) *hold, then we have the following energy estimates for* U_1

$$E(s, \partial^I L^J U_1)^{1/2} \lesssim \epsilon + (C_1\epsilon)^{3/2}, \quad |I|+|J| \leqslant N. \tag{3.7.11}$$

Proof. Firstly, by (3.7.6), we know

$$E(2,\partial^I L^J\widetilde{U}_1)^{1/2}\leqslant 2\epsilon.$$

Then recall the energy estimates (3.2.8) for wave equations and we easily obtain

$$\begin{aligned}&E(s,\partial^I L^J\widetilde{U}_1)^{1/2}\\&\leqslant E(2,\partial^I L^J\widetilde{U}_1)^{1/2}+\int_2^s\Big\|\partial^I L^J\big(-\partial^\alpha u\partial_\alpha v-\partial_t uv+u^2v+uv^2\big)\Big\|_{L^2_f(\mathcal{H}_{s'})}ds'\\&\lesssim\epsilon+(C_1\epsilon)^{3/2},\end{aligned}$$

in which the last inequality is due the estimate (3.7.9). By recalling the equivalence relation (3.7.6) between U_1 and $\widetilde{U}_1$ we complete the proof. □

The ideas of the proofs for the two propositions below are very similar to the one above, i.e. we can get good estimates for the auxiliary unknown $\widetilde{U}_1$ easily, and then an application of the equivalence relation (3.7.6) in turn gives us good estimates of the unknown U_1. And we omit the proofs for the following two propositions.

Proposition 3.7.7 (Conformal-type energy estimates for U_1)**.** *The conformal-type energy introduced in Subsection 3.2.3 satisfies*

$$E_{con}(s,\partial^I L^J U_1)^{1/2}\lesssim\epsilon+(C_1\epsilon)^{3/2}s^{1/2+|J|\delta},\quad |I|+|J|\leqslant N.\tag{3.7.12}$$

Consequently, we have

$$\big\|(s/r)\partial^I L^J U_1\big\|_{L^2_f(\mathcal{H}_s)}\lesssim\epsilon+(C_1\epsilon)^{3/2}s^{1/2+|J|\delta},\quad |I|+|J|\leqslant N,\tag{3.7.13}$$

which is due to the conformal–type bounds for U_1 above and the Hardy–type inequality (3.2.21).

Proposition 3.7.8 (Pointwise estimates for U_1)**.** *We have*

$$|\partial^I L^J U_1|\lesssim\big(\epsilon+(C_1\epsilon)^{3/2}\big)t^{-1}s^{|J|\delta},\quad |I|+|J|\leqslant N-4.\tag{3.7.14}$$

The proof of this Proposition clearly follows from Lemma 3.4.1 and the sup-estimate obtained in (3.7.10).

3.7.3 Estimates of the U_2 part

We state the following propositions about estimates of U_2, but we do not provide proofs as they are either the same as or easier than those of U_1.

Proposition 3.7.9 (Energy estimates for U_2)**.** *Consider the wave equation in* (3.7.3) *and assume the bounds in* (3.5.5) *hold, then we have the following energy estimates for* U_2

$$E(s,\partial^I L^J U_2)^{1/2} \lesssim \epsilon + (C_1\epsilon)^{3/2}, \quad |I|+|J| \leqslant N. \tag{3.7.15}$$

As a consequence, it gives us

$$\big\|(s/t)\partial_t\partial^I L^J U_2\big\|_{L^2_f(\mathcal{H}_s)} + \big\|(s/t)\partial^I L^J \partial_t U_2\big\|_{L^2_f(\mathcal{H}_s)} \lesssim \epsilon + (C_1\epsilon)^{3/2}, \quad |I|+|J| \leqslant N. \tag{3.7.16}$$

Proposition 3.7.10 (Pointwise estimates for U_2)**.** *We have*

$$|\partial_t\partial^I L^J U_2| + |\partial^I L^J \partial_t U_2| \lesssim \big(\epsilon + (C_1\epsilon)^{3/2}\big) t^{-1/2} s^{-1}, \quad |I|+|J| \leqslant N-4. \tag{3.7.17}$$

The proof of this Proposition clearly follows from Lemma 3.4.1 and the Sobolev embedding of Proposition 3.2.4.

3.7.4 Refined estimates for u

We are now about to derive the refined estimates for u, which will be based on the analysis of the new unknown U.

Proposition 3.7.11 (Refined energy estimates for u)**.** *Consider the wave equation in* (3.1.1) *and assume the bounds in* (3.5.5) *hold, then we have the following refined ones*

$$\begin{aligned} E(s,\partial^I L^J u)^{1/2} &\lesssim \epsilon + (C_1\epsilon)^{3/2}, && |I|+|J| \leqslant N-1, \\ E(s,\partial^I u)^{1/2} &\lesssim \epsilon + (C_1\epsilon)^{3/2} s^{\delta}, && |I| = N, \\ E(s,\partial^I L^J u)^{1/2} &\lesssim \epsilon + (C_1\epsilon)^{3/2} s^{|J|\delta}, && |I|+|J| \leqslant N, |J| \geqslant 1. \end{aligned} \tag{3.7.18}$$

Proof. For $|I|+|J| \leqslant N-1$, we have

$$E(s,\partial^I L^J u)^{1/2} \lesssim E(s,\partial^I L^J U_1)^{1/2} + E(s,\partial^I L^J \partial_t U_2)^{1/2},$$

then the energy estimates of U_1 and U_2 and the commutators give the desired result.

Next, for the case of $|I|+|J| = N$ with $|J| \geqslant 1$, we recall the original equation in (3.1.1) and have

$$-\Box\partial^I L^J u = \sum_{\substack{I_1+I_2=I \\ J_1+J_2=J}} \Big(\partial^{I_1} L^{J_1} u \partial^{I_2} L^{J_2} v + \partial^{I_1} L^{J_1} u \partial^{I_2} L^{J_2} \partial_t v\Big). \tag{3.7.19}$$

Then by the energy estimates for wave components (3.2.8), it is true that

$$\begin{aligned}&E(s,\partial^I L^J u)^{1/2}\\ &\leqslant E(2,\partial^I L^J u)^{1/2}+\int\sum_{\substack{I_1+I_2=I\\ J_1+J_2=J}}\Big\|\partial^{I_1}L^{J_1}u\partial^{I_2}L^{J_2}v+\partial^{I_1}L^{J_1}u\partial^{I_2}L^{J_2}\partial_t v\Big\|_{L^2_f(\mathcal{H}_{s'})}ds'.\end{aligned}$$

Successively, we arrive at

$$E(s,\partial^I L^J u)^{1/2}\lesssim\epsilon+(C_1\epsilon)^{3/2}s^{|J|\delta},$$

which is based on the estimates we already have obtained. The case of $|I|=N$ can be treated in a similar way, and hence the proof is done. □

Proposition 3.7.12 (Refined L^2-type energy estimates for u)**.** *It validates that*

$$\big\|(s/t)\partial^I L^J u\big\|_{L^2_f(\mathcal{H}_s)}\lesssim\epsilon+(C_1\epsilon)^{3/2}s^{1/2+|J|\delta},\quad |I|+|J|\leqslant N. \tag{3.7.20}$$

Proof. We simply have

$$\big\|(s/t)\partial^I L^J u\big\|_{L^2_f(\mathcal{H}_s)}\lesssim\big\|(s/r)\partial^I L^J U_1\big\|_{L^2_f(\mathcal{H}_s)}+\big\|(s/t)\partial^I L^J\partial_t U_2\big\|_{L^2_f(\mathcal{H}_s)},$$

and finish the proof by recalling the estimates (3.7.13) and (3.7.16). □

Proposition 3.7.13 (Refined pointwise estimates for u)**.** *We have*

$$|\partial^I L^J u|\lesssim\big(\epsilon+(C_1\epsilon)^{3/2}\big)t^{-1}s^{|J|\delta},\quad |I|+|J|\leqslant N-4. \tag{3.7.21}$$

Proof. It is true that

$$|\partial^I L^J u|\leqslant|\partial^I L^J U_1|+|\partial^I L^J\partial_t U_2|,$$

and the proof is done by the use of (3.7.14) and (3.7.17). □

3.8 Proof of the stability result and further remarks

Proof of the stability result We first close the bootstrap method, which immediately gives the proof of the main theorem.

Proof of Theorem 3.1.1. By collecting all of the refined estimates for wave and Klein-Gordon components, which are stated in the propositions in Section 3.6 and Subsection 3.7.4, we choose large $C_1\gg 1$ and small $\epsilon\ll 1$ such that $C_1\epsilon\ll 1$, then we arrive at the desired estimates in (3.5.6). Furthermore, as explained at the end of Subsection 3.5.1, we also have provided the proof of Theorem 3.1.1. □

Concluding remarks. Motivated by the Klein-Gordon-Zakharov system studied in [64, 35, 75], other classes of coupled wave-Klein-Gordon systems from [35] and also [44], we have studied the system

$$-\Box u = uv + u\partial_t v,$$
$$-\Box v + v = uv.$$

By relying on the strategy introduced in [45], we obtained global stability results and sharp decay estimates

$$|u| \lesssim t^{-1}, \qquad |v| \lesssim t^{-3/2}.$$

This system is part of a broarder class of systems where one studies nonlinearities with critical exponents and whether this leads to global stability or finite time blow-up. See for example [32], [22] for the Strauss conjecture of wave equations, [55, 36] for Klein-Gordon equations, [76] for Dirac-Proca systems. We end this chapter by asking the following questions for possible future work:

- What are the critical cases of nonlinearities for a wave-Klein-Gordon system in general dimensions?
- Depending on the critical cases, does the solution to the system exist globally or blow up in finite time?

Chapter 4

Zero mass problem in 3 dimensional space

[1]

Contents

4.1 Introduction and main Result

The coupled wave-Klein-Gordon systems have attracted a lot of attention since decades ago, which are motivated by some important models from physics. They include the Dirac-Klein-Gordon equations, the Dirac-Proca equations, the Einstein-Klein-Gordon equations, the Klein-Gordon-Zakharov equations, the massive Maxwell-Klein-Gordon equations and many others.

[1]This chapter is a joint work with Philippe G. LeFloch.

Recall that Klainerman [41] and Shatah [66] were first able to treat Klein-Gordon equations with quadratic nonlinearities in $\mathbb{R}^{1+3}$. In [41] Klainerman applied the powerful vector field method in the light cone to show the existence of global-in-time solutions to Klein-Gordon equations with compactly supported initial data. On the other hand, Shatah proved in [66] the same result by using a normal form method, where the compactness assumption on the initial data is not needed. By relying on these two main methods, various results came out.

Later on, LeFloch-Ma in [44] in 2014 introduced the hyperboloidal foliation method on treating wave and Klein-Gordon equation in one framework, where the compactness of the initial data is needed. Recently, in [48] LeFloch-Ma were able to get rid the compactness assumption by introducing the Euclidian-hyperboidal foliation method. Utilising this method, Ma in [60] proved the small data global existence result on the 1 dimensional wave-Klein-Gordon equations.

Motivated by the hyperboloidal foliation method, we are interested in studying here the behavior of the Klein-Gordon component when its mass m tends to 0. It is conjectured that the solution to the Klein-Gordon equation, within a short range of time, behaves more like waves when the mass is very small. We provide a rigorous proof for the Klein-Gordon equations with a certain class of nonlinearities which are of divergence form. Consider

$$\begin{aligned} -\Box u + m^2 u &= P^\alpha \partial_\alpha \big(uv + v^2 - u^3\big), \\ -\Box v + v &= u^2 + uv, \end{aligned} \tag{4.1.1}$$

in which P^α are fixed constants and $m \in [0,1]$ is the mass parameter, and with prescribed initial data at $t = 2$

$$\big(u, \partial_t u\big)(2,\cdot) = \big(u_0, u_1\big), \qquad \big(v, \partial_t v\big)(2,\cdot) = \big(v_0, v_1\big). \tag{4.1.2}$$

Our goal is to prove small data global existence result with pointwise decay results, which are uniform in term of the mass parameter m. The main theorem is stated now.

Theorem 4.1.1. *Consider the system* (4.1.1) *with P^α fixed constants and the mass parameter $m \in [0,1]$, and let $N \geqslant 8$ be an integer. Then there exists $\epsilon_0 > 0$, which is notably independent of m, such that for all $\epsilon \in (0, \epsilon_0)$ and all compactly supported initial data (u_0, u_1, v_0, v_1) satisfying the smallness condition*

$$\|u_0, v_0\|_{H^{N+5}(\mathbb{R}^3)} + \|u_1, v_1\|_{H^{N+4}(\mathbb{R}^3)} \leqslant \epsilon, \tag{4.1.3}$$

the initial value problem (4.1.1)–(4.1.2) *admits a global-in-time solution (u, v) with*

$$|u(t,x)| \lesssim \min\{t^{-1}, m^{-1}t^{-3/2}\}, \quad |v(t,x)| \lesssim t^{-3/2}. \tag{4.1.4}$$

There is a loss of regularity here, which will be seen in the proof of this theorem. We also note that Theorem 4.1.1 is much easier to prove at the end points $m = 0$ and $m = 1$, in which cases the equation of u is a wave equation and a Klein-Gordon equation respectively. But there are several difficulties appearing if we want to treat the system (4.1.1) uniformly in terms of $m \in [0, 1]$: 1) the L^2-type estimates and sup-norm estimates obtained by the estimates on the mass term m^2u cannot be used due to the bad factor m^{-1}; 2) the scaling vector field and the conformal vector field do not commute with the Klein-Gordon operator $-\Box + m^2$; 3) the tricks in Chapter 3 Section 3.4 in obtaining pointwise estimates for wave or Klein-Gordon components cannot be applied here due to the possibly vanishing mass m; 4) it is hard to get either a uniform L^2-type estimate or a uniform sup-norm estimate on u.

There are few works on this subject which study the Klein-Gordon equations with possibly vanishing mass. To the best of our acknowledge the only existing such results are due to [12], where the authors studied the problem of the Dirac equation with vanishing mass. We recall that in [12] the proof highly relied on the special structure of the Dirac equations. The result in Theorem 4.1.1 can be regarded as a generalisation of the results in [12].

Remark 4.1.2. *Note that the system of equations*

$$\begin{aligned} -\Box u + m^2 u &= P^\alpha \partial_\alpha \big(uv\big), \\ -\Box v + v &= u^2, \end{aligned} \tag{4.1.5}$$

is corresponding the Dirac-Proca model (when $m = 0$), while the system of equations

$$\begin{aligned} -\Box u + m^2 u &= P^\alpha \partial_\alpha \big(v^2\big), \\ -\Box v + v &= uv, \end{aligned} \tag{4.1.6}$$

is corresponding to the Klein-Gordon-Zakharov model (when $m = 0$). Our main result in Theorem 4.1.1 (possibly a weaker result) is true for both of the models and the proof of Theorem 4.1.1 also applies to them.

The reason why we add the term $P^\alpha \partial_\alpha(-u^3)^3$ in the nonlinearities of u equation is that it gives us some cancellations to the bad terms, the details are given in the proof.

The rest of this chapter is planned as follows. In Section 4.2 we revisit the hyperboloidal foliation method and some useful estimates. Next in Section 4.3 we prove the unified pointwise decay estimates for homogeneous Klein-Gordon equations. Then we analyse the system of equations and treat the nonlinearities in Section 4.4. Finally we prove the main result relying on the bootstrap method in the last section.

4.2 Revisit of the hyperboloidal foliation method

The basics of the hyperboloidal foliation method, which were introduced by LeFloch-Ma in [44], are introduced several times in this thesis, so one refers to Section 1.2 in Chapter 1 or Section 3.2 in Chapter 3 for more details. For convenience, we give only the following heuristic ideas on how to estimate the null forms and commutators. Roughly speaking, we have

$$\begin{aligned} \big|\partial^\alpha\phi\partial_\alpha\psi\big| &\lesssim \frac{s}{t}|\partial\phi||\partial\psi|, \\ \big|\partial\partial^I L^J\phi\big| &\simeq \big|\partial^I L^J\partial\phi\big|, \end{aligned} \tag{4.2.1}$$

for two nice functions ϕ, ψ. For the rigorous statements, one refers to Section 1.7 in Chapter 1 or Section 3.3 in Chapter 3.

4.3 Unified decay results for homogeneous Klein-Gordon equations

We first consider a simple homogeneous Klein-Gordon equation

$$\begin{aligned} &-\Box w + m^2 w = 0, \\ &\big(w, \partial_t w\big)(0,\cdot) = \big(w_0, w_1\big), \end{aligned} \tag{4.3.1}$$

and prove the following theorem.

Theorem 4.3.1 (Unified decay results for homogeneous Klein-Gordon equations)**.** *Consider the initial value problem* (4.3.1)*, and assume the initial data are compactly supported and satisfy*

$$\|w_0\|_{H^5(\mathbb{R}^3)} + \|w_1\|_{H^4(\mathbb{R}^3)} \leqslant \epsilon, \tag{4.3.2}$$

then the following unified decay result is valid

$$|w| \lesssim \epsilon \min\{(t+2)^{-1}, m^{-1}(t+2)^{-3/2}\}. \tag{4.3.3}$$

The proof relies on a simple utilization of the Fourier method, which is from the lecture note by Luk [56] in treating homogeneous wave equations. We first revisit some basics in Fourier analysis before the proof.

Recall the Fourier transform of a nice function $\phi = \phi(x)$ is defined by

$$\widehat{\phi}(\xi) := \int_{\mathbb{R}^3} \phi(x) e^{-2\pi i x\cdot\xi}\, dx,$$

and the inverse Fourier transform of a nice function $\psi = \psi(\xi)$ is defined by

$$\check{\psi}(x) := \int_{\mathbb{R}^3} \psi(\xi) e^{2\pi i x\cdot\xi}\, d\xi.$$

Next we recall some basic but important facts in Fourier analysis.

Proposition 4.3.2. *The following properties hold for a nice function $\phi = \phi(x)$:*

- *Inverse formula.*
$$\phi = \check{\widehat{\phi}}. \tag{4.3.4}$$
- *Relation between partial derivatives and multipliers.*
$$\widehat{\partial_a \phi}(\xi) = 2\pi i \xi_a \widehat{\phi}(\xi). \tag{4.3.5}$$
- *Plancheral identity.*
$$\|\phi\|_{L^2(\mathbb{R}^3)} = \big\|\widehat{\phi}\big\|_{L^2(\mathbb{R}^3)}. \tag{4.3.6}$$

Proof of Theorem 4.3.1. In the Fourier space, the equation (4.3.1) can be written as

$$\partial_t \partial_t \widehat{w}(t,\xi) + \xi_m^2 \widehat{w}(t,\xi) = 0,$$

with initial data

$$\big(\widehat{w}, \partial_t \widehat{w}\big)(0,\cdot) = \big(\widehat{w}_0, \widehat{w}_1\big),$$

in which we used the notation

$$\xi_m := \big(4\pi^2|\xi|^2 + m^2\big)^{1/2}.$$

Next by solving the ordinary differential equation above, we get the explicit solution in the Fourier space

$$\widehat{w}(t,\xi) = \cos(2\pi t \xi_m)\widehat{w}_0(\xi) + \frac{\sin(2\pi t\xi_m)}{2\pi\xi_m}\widehat{w}_1(\xi), \tag{4.3.7}$$

which can also be expressed by the following four terms.

$$\widehat{w}(t,\xi) = e^{2\pi i t\xi_m}\Big(\frac{\widehat{w}_0(\xi)}{2} + \frac{\widehat{w}_1(\xi)}{2\pi i\xi_m}\Big) + e^{2\pi i t\xi_m}\Big(\frac{\widehat{w}_0(\xi)}{2} + \frac{\widehat{w}_1(\xi)}{2\pi i\xi_m}\Big). \tag{4.3.8}$$

Then we estimate the inverse Fourier transform of those four terms above, but we notice that it suffices to estimate the first two terms. We denote by the inverse Fourier

transform of the second term

$$I_1 := \int_{\mathbb{R}^3} e^{2\pi i(t\xi_m + x\cdot\xi)} \frac{\widehat{w}_1(\xi)}{2\pi i \xi_m}\, d\xi.$$

Without loss of any generality, we assume

$$x = (0, 0, |x|),$$

and we use the polar coordinates for the first two components of ξ, i.e.

$$\xi = \big(\rho\cos\xi_\theta, \rho\sin\xi_\theta, \xi_3\big), \qquad (\rho, \xi_\theta) \in [0, +\infty) \times [0, 2\pi),$$

and thus

$$d\xi = \rho d\rho d\xi_\theta d\xi_3.$$

It also helps to note that

$$\frac{\partial|\xi|}{\partial\rho} = \frac{\rho}{|\xi|}, \qquad \frac{\partial\xi_m}{\partial\rho} = \frac{\rho}{\xi_m},$$

as well as

$$\partial_\rho e^{2\pi i t\xi_m} = 2\pi i t \frac{\rho}{\xi_m} e^{2\pi i t\xi_m}.$$

Now relying on these results we further arrive at

$$\begin{aligned} I_1 &= -\frac{1}{4\pi^2 t}\int_{\mathbb{R}}\int_0^{2\pi}\int_0^{+\infty} \partial_\rho e^{2\pi i t\xi_m} e^{2\pi i x\cdot\xi}\widehat{w}_1\, d\rho d\xi_\theta d\xi_3 \\ &= \frac{1}{4\pi^2 t}\int_{\mathbb{R}}\int_0^{2\pi}\int_0^{+\infty} e^{2\pi i(t\xi_m + x\cdot\xi)}\partial_\rho\widehat{w}_1\, d\rho d\xi_\theta d\xi_3 - \frac{1}{2\pi t}\int_{\mathbb{R}} e^{2\pi i(t\xi_m + |x|\xi_3)}\widehat{w}_1(\rho = 0, \xi_3)\, d\xi_3 \\ &=: I_{11} + I_{12}, \end{aligned}$$

where we did integration by parts in the second step. Observe that

$$\int_{\mathbb{R}}\int_0^{2\pi}\int_0^{+\infty} \frac{1}{(1+|\xi|^2)^2}\, d\rho d\xi_\theta d\xi_3 \lesssim 1,$$

as well as

$$\begin{aligned} \sum_a (1+|\xi|^2)^2 |\partial_a \widehat{w}_1| &\lesssim \sum_a \|(1-\Delta)^2(x_a w_1)\|_{L^1(\mathbb{R}^3)} \\ &\leqslant \|w_1\|_{H^4(\mathbb{R}^3)}, \end{aligned}$$

where we used the fact in the last step that

$$L^p(\Omega) \subset L^1(\Omega), \qquad p \geqslant 1,$$

when $\Omega \subset \mathbb{R}^d$ is a compact set. Thus we arrive at

$$|I_{11}| \lesssim t^{-1}\|w_1\|_{H^4(\mathbb{R}^3)},$$

and similarly we can show

$$|I_{12}| \lesssim t^{-1}\|w_1\|_{H^4(\mathbb{R}^3)}.$$

To conclude, we have

$$|I_1| \lesssim t^{-1}\|w_1\|_{H^4(\mathbb{R}^3)}. \tag{4.3.9}$$

Next we do the same analysis on the inverse Fourier transform of the first term, which we denote by

$$I_0 := \int_{\mathbb{R}^3} e^{2\pi i(t\xi_m + x\cdot\xi)} \frac{\widehat{w}_0(\xi)}{2}\, d\xi.$$

By adopting the same setting, we proceed and get

$$\begin{aligned} I_0 &= \frac{1}{4\pi i t}\int_{\mathbb{R}}\int_0^{2\pi}\int_0^{+\infty} \partial_\rho e^{2\pi i t\xi_m}\xi_m e^{2\pi i x\cdot\xi}\widehat{w}_0\, d\rho d\xi_\theta d\xi_3 \\ &= -\frac{1}{4\pi i t}\int_{\mathbb{R}}\int_0^{2\pi}\int_0^{+\infty} e^{2\pi i t\xi_m}\frac{\rho}{\xi_m} e^{2\pi i x\cdot\xi}\widehat{w}_0\, d\rho d\xi_\theta d\xi_3 \\ &\quad - \frac{1}{4\pi i t}\int_{\mathbb{R}}\int_0^{2\pi}\int_0^{+\infty} e^{2\pi i t\xi_m}\xi_m e^{2\pi i x\cdot\xi}\partial_\rho\widehat{w}_0\, d\rho d\xi_\theta d\xi_3 \\ &\quad - \frac{1}{2it}\int_{\mathbb{R}} \left(\xi_3^2 + m^2\right)^{1/2} e^{2\pi i(t\xi_m + |x|\xi_3)}\widehat{w}_0(\rho = 0, \xi_3)\, d\xi_3. \end{aligned}$$

Similarly, we conclude that

$$|I_0| \lesssim t^{-1}\|w_0\|_{H^4(\mathbb{R}^3)}. \tag{4.3.10}$$

A combination of (4.3.9) and (4.3.10) gives

$$|w(t,x)| \lesssim (t+2)^{-1}\big(\|w_0\|_{H^4(\mathbb{R}^3)} + \|w_1\|_{H^4(\mathbb{R}^3)}\big), \qquad t \geqslant 2. \tag{4.3.11}$$

On the other hand, we observe that it is easy to show

$$|w(t,x)| \lesssim \|w_0\|_{H^4(\mathbb{R}^3)} + \|w_1\|_{H^4(\mathbb{R}^3)}, \qquad 0 \leqslant t \leqslant 2. \tag{4.3.12}$$

Hence we arrive at (4.3.3) since the bound $m^{-1}(t+2)^{-3/2}$ is trivial to prove. □

We note again that the proof above is from Luk [56].

4.4 Analysis on the equations and nonlinearities

Recall the system of equations in (4.1.1)

$$\begin{aligned} -\Box u + m^2 u &= P^\alpha \partial_\alpha \big(uv + v^2 - u^3\big) =: Q_u, \\ -\Box v + v &= u^2 + uv =: Q_v, \\ \big(u, \partial_t u\big)(2,\cdot) = \big(u_0, u_1\big), &\qquad \big(v, \partial_t v\big)(2,\cdot) = \big(v_0, v_1\big). \end{aligned}$$

Our goal in this section is to introduce some new variables, which will transform some terms we cannot handle to terms we can.

We introduce the following six new variables

$$U^\alpha, \qquad U^5, \qquad V^u := v + u^2, \tag{4.4.1}$$

which are solutions to the wave-Klein-Gordon equations below:

$$\begin{aligned} -\Box U^\alpha + m^2 U^\alpha &= P^\alpha \big(uv + v^2 - u^3\big), \\ \big(U^\alpha, \partial_t U^\alpha\big)(2,\cdot) &= (0,0), \end{aligned} \tag{4.4.2}$$

$$\begin{aligned} -\Box U^5 + m^2 U^5 &= 0, \\ \big(U^5, \partial_t U^5\big)(2,\cdot) &= \big(u_0, P^0(u_0 v_0 + v_0^2 - u_0^3)\big), \end{aligned} \tag{4.4.3}$$

and

$$\begin{aligned} -\Box V^u + V^u &= uv + 2m^2 u^2 + 2\partial_\alpha u \partial^\alpha u - 2uQ_u, \\ \big(V^u, \partial_t V^u\big)(2,\cdot) &= \big(v_0 - u_0^2, v_1 - 2u_0 u_1\big). \end{aligned} \tag{4.4.4}$$

We then further introduce four other variables

$$\widetilde{U}^\alpha := U^\alpha + P^\alpha uv,$$

which are solutions to the following equations

$$\begin{aligned} -\Box \widetilde{U}^\alpha + m^2 \widetilde{U}^\alpha &= P^\alpha \big(- m^2 uv - \partial^\mu u \partial_\mu v + v^2 - u^3 + vQ_u + uQ_v\big), \\ \big(\widetilde{U}^\alpha, \partial_t \widetilde{U}^\alpha\big)(2,\cdot) &= \big(P^\alpha u_0 v_0, P^\alpha u_0 v_1 + P^\alpha u_1 v_0\big). \end{aligned} \tag{4.4.5}$$

Recall in Remark 4.1.2 we have mentioned that the term $P^\alpha \partial_\alpha (-u^3)^3$ would cancel some bad terms, and now it is clearer to see that the term $P^\alpha \partial_\alpha (-u^3)^3$ cancels the bad term hidden in uQ_v. On the other hand, we also emphasize that the term $m^2 uv$ can be regarded as Klein-Gordon and Klein-Gordon interaction term thanks to the factor m^2, while the term uv in the U^α equations is regarded as wave and Klein-Gordon interaction

term. Hence the term m^2uv is a good one now.

In conclusion, by introducing several intermediate variables, we can first estimate $\widetilde{U}^\alpha, V^u$ and then get estimates on u, v by the relations

$$u = U^5 + \partial_\alpha U^\alpha, \qquad \widetilde{U}^\alpha = U^\alpha + P^\alpha uv, \qquad V^u = v + u^2.$$

The important thing is that the nonlinearities appearing in the equations of $\widetilde{U}^\alpha$ and V^u are easy to handle.

4.5 Bootstrap method

4.5.1 Basic setting

Recall the local well-posed results on nonlinear wave-Klein-Gordon equations, and we assume that the following bounds validate on the interval $[2, s_1]$

$$\begin{aligned}
\sup_{\mathcal{H}_s} \left|t\partial^I L^J u\right| &\leqslant C_1\epsilon, \qquad |I| + |J| \leqslant N-2,\\
E_m(s, \partial^I L^J u)^{1/2} &\leqslant C_1\epsilon, \qquad |I| + |J| \leqslant N-1,\\
E_m(s, \partial^I L^J u)^{1/2} &\leqslant C_1\epsilon s^\delta, \qquad |I| + |J| = N,\\
E_1(s, \partial^I L^J v)^{1/2} &\leqslant C_1\epsilon s^\delta, \qquad |I| + |J| \leqslant N,
\end{aligned} \tag{4.5.1}$$

in which $0 < \delta \ll 1/10$ is an arbitrary fixed constant, C_1 is some (large) constant to be determined. We take

$$s_1 := \sup\{s : (4.5.1)\ \textit{holds}\}.$$

In order to prove the small data global existence result, it suffices to show the following refined estimates

$$\begin{aligned}
\sup_{\mathcal{H}_s} \left|t\partial^I L^J u\right| &\leqslant \frac{1}{2}C_1\epsilon, \qquad |I| + |J| \leqslant N-2,\\
E_m(s, \partial^I L^J u)^{1/2} &\leqslant \frac{1}{2}C_1\epsilon, \qquad |I| + |J| \leqslant N-1,\\
E_m(s, \partial^I L^J u)^{1/2} &\leqslant \frac{1}{2}C_1\epsilon s^\delta, \qquad |I| + |J| = N,\\
E_1(s, \partial^I L^J v)^{1/2} &\leqslant \frac{1}{2}C_1\epsilon s^\delta, \qquad |I| + |J| \leqslant N.
\end{aligned} \tag{4.5.2}$$

Since the refined estimates allow one to extend the hyperbolic time interval $[2, s_1]$ to a strictly larger one, which contradicts the definition of s_1 unless $s_1 = +\infty$.

As a direct consequence, we have the following estimates for all $s \in [2, s_1]$

$$\begin{aligned}
\big\|r^{-1}\partial^I L^J u\big\|_{L^2_f(\mathcal{H}_s)} &\lesssim C_1\epsilon s^\delta, \qquad |I|+|J| \leqslant N,\\
\big\|(s/t)\partial\partial^I L^J u, (s/t)\partial^I L^J \partial u\big\|_{L^2_f(\mathcal{H}_s)} + m\big\|\partial^I L^J u\big\|_{L^2_f(\mathcal{H}_s)} &\leqslant C_1\epsilon s^\delta, \qquad |I|+|J| \leqslant N,\\
\big\|(s/t)\partial\partial^I L^J v, (s/t)\partial^I L^J \partial v\big\|_{L^2_f(\mathcal{H}_s)} + \big\|\partial^I L^J v\big\|_{L^2_f(\mathcal{H}_s)} &\leqslant C_1\epsilon s^\delta, \qquad |I|+|J| \leqslant N,\\
\sup_{\mathcal{H}_s}\big|st^{1/2}\partial\partial^I L^J u, st^{1/2}\partial^I L^J\partial u\big| + m\sup_{\mathcal{H}_s}\big|t^{3/2}\partial^I L^J u\big| &\leqslant C_1\epsilon s^\delta, \qquad |I|+|J| \leqslant N-2,\\
\sup_{\mathcal{H}_s}\big|st^{1/2}\partial\partial^I L^J v, st^{1/2}\partial^I L^J\partial v\big| + \sup_{\mathcal{H}_s}\big|t^{3/2}\partial^I L^J v\big| &\leqslant C_1\epsilon s^\delta, \qquad |I|+|J| \leqslant N-2.
\end{aligned} \tag{4.5.3}$$

4.5.2 Refined estimates on the wave-Klein-Gordon component u

In order to obtain the refined estimates on u, we first derive the estimates on the intermediate variables $\widetilde{U}^\alpha$, then by the relation $\widetilde{U}^\alpha = U^\alpha + P^\alpha uv$ we get estimates on the intermediate variables U^α, and finally we improve the estimates on u by the relation $u = U^5 + \partial_\alpha U^\alpha$.

We begin with the estimates on U^5, which satisfies the homogeneous wave-Klein-Gordon equation.

Lemma 4.5.1. *Under the assumptions in* (4.5.1)*, it holds for all* $s \in [2, s_1]$ *that*

$$\begin{aligned}
\big\|r^{-1}\partial^I L^J U^5\big\|_{L^2_f(\mathcal{H}_s)} &\lesssim \epsilon, \qquad |I|+|J| \leqslant N,\\
\big|\partial^I L^J U^5\big| &\lesssim \epsilon t^{-1}, \qquad |I|+|J| \leqslant N.
\end{aligned} \tag{4.5.4}$$

Proof. The pointwise estimates on U^5 are thanks to Theorem 4.3.1.

Next we obtain easily from the energy estimates that

$$E_m(s, \partial^I L^J U^5)^{1/2} \leqslant E_m(2, \partial^I L^J U^5)^{1/2} \lesssim \epsilon.$$

Hence the first inequality in (4.5.4) follows from the Hardy inequality. □

We proceed by estimating the intermediate variables $\widetilde{U}^\alpha$.

Lemma 4.5.2. *Assume the estimates in* (4.5.1) *hold, then we have*

$$\begin{aligned}
\big\|(s/t)\partial\partial^I L^J \widetilde{U}^\alpha, (s/t)\partial^I L^J\partial\widetilde{U}^\alpha\big\|_{L^2_f(\mathcal{H}_s)} &\lesssim (C_1\epsilon)^2, \qquad |I|+|J| \leqslant N,\\
\big|\partial_\alpha\partial^I L^J\widetilde{U}^\alpha, \partial^I L^J\partial_\alpha\widetilde{U}^\alpha\big| &\lesssim C_1\epsilon t^{-1/2}s^{-1}, \qquad |I|+|J| \leqslant N-2.
\end{aligned} \tag{4.5.5}$$

Proof. By recalling the equations (4.4.5) for $\widetilde{U}^\alpha$ and the energy estimates

$$\begin{aligned}&E_m(s,\partial^I L^J\widetilde{U}^\alpha)^{1/2}\\ &\leqslant E_m(2,\partial^I L^J\widetilde{U}^\alpha)^{1/2}+\int_2^s \big\|\partial^I L^J P^\alpha\big(-m^2uv-\partial^\mu u\partial_\mu v+v^2-u^3+vQ_u+uQ_v\big)\big\|_{L^2_f(\mathcal{H}_{s'})}\,ds',\end{aligned}$$

it suffices to prove

$$\big\|\partial^I L^J P^\alpha\big(-m^2uv-\partial^\mu u\partial_\mu v+v^2-u^3+vQ_u+uQ_v\big)\big\|_{L^2_f(\mathcal{H}_s)}\lesssim (C_1\epsilon)^2 s^{-5/4},$$

for all $|I|+|J|\leqslant N$.

Since the components v and ∂u has very good L^2-type and L^∞-type bounds according to (4.5.3), and cubic terms involved are harmless, so it is easy to get the above estimate and we omit the details. □

Next we move to the estimates on the intermediate variables U^α.

Lemma 4.5.3. *It holds the following estimates*

$$\begin{aligned}\big\|(s/t)\partial\partial^I L^J U^\alpha,(s/t)\partial^I L^J\partial U^\alpha\big\|_{L^2_f(\mathcal{H}_s)}&\lesssim (C_1\epsilon)^2,\qquad |I|+|J|\leqslant N,\\ \big|\partial_\alpha\partial^I L^J U^\alpha,\partial^I L^J\partial_\alpha U^\alpha\big|&\lesssim C_1\epsilon t^{-1/2}s^{-1},\qquad |I|+|J|\leqslant N-2.\end{aligned}\tag{4.5.6}$$

Proof. Recall the relation $\widetilde{U}^\alpha=U^\alpha+P^\alpha uv$, and the proof follows from the observations below

$$\begin{aligned}\big\|\partial^I L^J P^\alpha(uv)\big\|_{L^2_f(\mathcal{H}_s)}&\lesssim (C_1\epsilon)^2,\qquad |I|+|J|\leqslant N,\\ \sup_{\mathcal{H}_s}\big|t^{3/2}\partial^I L^J P^\alpha(uv)\big|&\lesssim (C_1\epsilon)^2,\qquad |I|+|J|\leqslant N-2.\end{aligned}$$

□

In conclusion, we obtain the refined estimates on the component u.

Proposition 4.5.4 (Refined estimates on u)**.** *It is valid that*

$$\begin{aligned}\sup_{\mathcal{H}_s}\big|t\partial^I L^J u\big|&\lesssim \epsilon+(C_1\epsilon)^2,\qquad |I|+|J|\leqslant N-2,\\ E_m(s,\partial^I L^J u)^{1/2}&\leqslant \epsilon+(C_1\epsilon)^2,\qquad |I|+|J|\leqslant N-1,\\ E_m(s,\partial^I L^J u)^{1/2}&\leqslant \epsilon+(C_1\epsilon)^2 s^\delta,\qquad |I|+|J|=N.\end{aligned}\tag{4.5.7}$$

Proof. For $|I|+|J|\leqslant N-1$ it is true that

$$E_m(s,\partial^I L^J u)^{1/2}\lesssim E_m(s,\partial^I L^J U^5)+E_m(s,\partial^I L^J\partial_\alpha U^\alpha),$$

and by gathering the estimates established for U^5 and U^α we have

$$E_m(s,\partial^I L^J u)^{1/2} \lesssim \epsilon + (C_1\epsilon)^2.$$

One remaining thing is to estimate for the case $|I|+|J|=N$. Hence by energy estimates

$$E_m(s,\partial^I L^J u)^{1/2} \leqslant E_m(2,\partial^I L^J u)^{1/2} + \int_2^s \big\|\partial^I L^J P^\alpha \partial_\alpha\big(uv+v^2-u^3\big)\big\|_{L^2_f(\mathcal{H}_{s'})}\,ds',$$

we only need to show

$$\big\|\partial^I L^J P^\alpha \partial_\alpha\big(uv+v^2-u^3\big)\big\|_{L^2_f(\mathcal{H}_s)} \lesssim (C_1\epsilon)^2 s^{-1+\delta}.$$

We notice that the only troublesome term is

$$\partial^I L^J P^\alpha\big(u\partial_\alpha v\big),$$

which is the bad part of the term $\partial^I L^J P^\alpha \partial_\alpha\big(uv\big)$. Recall the chain rule, and we get

$$\begin{aligned}
\big\|\partial^I L^J P^\alpha\big(u\partial_\alpha v\big)\big\|_{L^2_f(\mathcal{H}_s)} \leqslant & \sum_{I_1+I_2=I, J_1+J_2=J} \big\|P^\alpha \partial^{I_1} L^{J_1} u \partial^{I_2} L^{J_2}\partial_\alpha v\big\|_{L^2_f(\mathcal{H}_s)} \\
\leqslant & \sum_{I_1+I_2=I, J_1+J_2=J} \big\|(t/s)\partial^{I_1} L^{J_1} U^5\big\|_{L^\infty(\mathcal{H}_s)} \big\|(s/t)P^\alpha \partial^{I_2} L^{J_2}\partial_\alpha v\big\|_{L^2_f(\mathcal{H}_s)} \\
& + \sum_{\substack{I_1+I_2=I, J_1+J_2=J \\ |I_1|+|J_1|\leqslant |I_2|+|J_2|}} \big\|(t/s)\partial^{I_1} L^{J_1}\partial_\beta U^\beta\big\|_{L^\infty(\mathcal{H}_s)} \big\|(s/t)P^\alpha \partial^{I_2} L^{J_2}\partial_\alpha v\big\|_{L^2_f(\mathcal{H}_s)} \\
& + \sum_{\substack{I_1+I_2=I, J_1+J_2=J \\ |I_1|+|J_1|\geqslant |I_2|+|J_2|}} \big\|(s/t)\partial^{I_1} L^{J_1}\partial_\beta U^\beta\big\|_{L^2_f(\mathcal{H}_s)} \big\|(t/s)P^\alpha \partial^{I_2} L^{J_2}\partial_\alpha v\big\|_{L^\infty(\mathcal{H}_s)},
\end{aligned}$$

and by inserting the bounds for each term we arrive at

$$\big\|\partial^I L^J P^\alpha\big(u\partial_\alpha v\big)\big\|_{L^2_f(\mathcal{H}_s)} \lesssim \epsilon + (C_1\epsilon)^2 s^{-1+\delta}.$$

The refined pointwise estimates are thanks to the decomposition $u = U^5 + \partial_\alpha U^\alpha$ and the application of Sobolev-type inequalities on the hyperboloids. □

4.5.3 Refined estimates on the Klein-Gordon component v

In order to improve the energy estimates on the Klein-Gordon component v, we first deduce the refined estimates on the intermediate variable V^u.

Lemma 4.5.5. *It holds that*

$$E_1(s,\partial^I L^J V^u)^{1/2} \lesssim \epsilon + (C_1\epsilon)^2 s^\delta, \qquad |I|+|J| \leqslant N. \tag{4.5.8}$$

Proof. We begin by applying the energy estimates

$$E_1(s,\partial^I L^J V^u)^{1/2} \leqslant E_1(2,\partial^I L^J V^u)^{1/2} + \int_2^s \big\|\partial^I L^J\big(uv+2m^2u^2+2\partial_\alpha u\partial^\alpha u-2uQ_u\big)\big\|_{L^2_f(\mathcal{H}_{s'})}\,ds',$$

and we find it is easy to get

$$\big\|\partial^I L^J\big(2m^2u^2+2\partial_\alpha u\partial^\alpha u-2uQ_u\big)\big\|_{L^2_f(\mathcal{H}_{s'})} \lesssim (C_1\epsilon)^2 s'^{-5/4}, \qquad |I|+|J|\leqslant N,$$

and the only problematic term is $\partial^I L^J\big(uv\big)$. We now estimate this part

$$\begin{aligned}
\big\|\partial^I L^J\big(uv\big)\big\|_{L^2_f(\mathcal{H}_{s'})} \leqslant & \sum_{I_1+I_2=I, J_1+J_2=J} \big\|\partial^{I_1}L^{J_1}\big(U^5+\partial_\alpha U^\alpha\big)\partial^{I_2}L^{J_2}v\big\|_{L^2_f(\mathcal{H}_{s'})} \\
\leqslant & \sum_{I_1+I_2=I, J_1+J_2=J} \big\|\partial^{I_1}L^{J_1}U^5\big\|_{L^\infty(\mathcal{H}_{s'})}\big\|\partial^{I_2}L^{J_2}v\big\|_{L^2_f(\mathcal{H}_{s'})} \\
& + \sum_{\substack{I_1+I_2=I, J_1+J_2=J \\ |I_1|+|J_1|\leqslant |I_2|+|J_2|}} \big\|\partial^{I_1}L^{J_1}\big(\partial_\alpha U^\alpha\big)\big\|_{L^\infty(\mathcal{H}_{s'})}\big\|\partial^{I_2}L^{J_2}v\big\|_{L^2_f(\mathcal{H}_{s'})} \\
& + \sum_{\substack{I_1+I_2=I, J_1+J_2=J \\ |I_1|+|J_1|\geqslant |I_2|+|J_2|}} \big\|(s/t)\partial^{I_1}L^{J_1}\big(\partial_\alpha U^\alpha\big)\big\|_{L^2(\mathcal{H}_{s'})}\big\|(t/s)\partial^{I_2}L^{J_2}v\big\|_{L^\infty(\mathcal{H}_{s'})},
\end{aligned}$$

which gives us

$$\big\|\partial^I L^J\big(uv\big)\big\|_{L^2_f(\mathcal{H}_{s'})} \lesssim (C_1\epsilon)^2 s'^{-1+\delta} \qquad |I|+|J|\leqslant N.$$

Hence we arrive at (4.5.8). □

As a consequence, we obtain the refined estimates for v.

Proposition 4.5.6 (Refined estimates on v)**.** *It holds that*

$$E_1(s,\partial^I L^J v)^{1/2} \lesssim \epsilon + (C_1\epsilon)^2 s^\delta, \qquad |I|+|J|\leqslant N. \tag{4.5.9}$$

Proof. Recall the relation $V^u = v + u^2$, hence it suffices to show

$$\big\|\partial^I L^J(u^2)\big\|_{L^2_f(\mathcal{H}_s)} \lesssim (C_1\epsilon)^2 s^\delta.$$

Next recall $u = U^5 + \partial_\alpha U^\alpha$, and expand u^2 in terms of U^5, U^α, which gives us

$$u^2 = (U^5)^2 + 2U^5\partial_\alpha U^\alpha + \partial_\alpha U^\alpha \partial_\beta U^\beta.$$

We estimate them term by term. First it holds for $|I| + |J| \leqslant N$

$$\begin{aligned}\left\|\partial^I L^J (U^5)^2\right\|_{L^2_f(\mathcal{H}_s)} &\leqslant \sum_{I_1+I_2=I, J_1+J_2=J} \left\|r\partial^{I_1} L^{J_1} U^5\right\|_{L^\infty(\mathcal{H}_s)} \left\|r^{-1}\partial^{I_1} L^{J_1} U^5\right\|_{L^2_f(\mathcal{H}_s)} \\ &\lesssim \epsilon^2.\end{aligned}$$

We find that

$$\begin{aligned}\left\|\partial^I L^J (U^5\partial_\alpha U^\alpha)\right\|_{L^2_f(\mathcal{H}_s)} &\leqslant \sum_{I_1+I_2=I, J_1+J_2=J} \left\|(t/s)\partial^{I_1} L^{J_1} U^5\right\|_{L^\infty(\mathcal{H}_s)} \left\|(s/t)\partial^{I_1} L^{J_1}\partial_\alpha U^\alpha\right\|_{L^2_f(\mathcal{H}_s)} \\ &\lesssim (C_1\epsilon)^2 s^{-1+\delta}, \qquad |I| + |J| \leqslant N.\end{aligned}$$

Similarly, we can easily get

$$\left\|\partial^I L^J (\partial_\alpha U^\alpha \partial_\beta U^\beta)\right\|_{L^2_f(\mathcal{H}_s)} \lesssim (C_1\epsilon)^2 s^{-1+\delta}, \qquad |I| + |J| \leqslant N.$$

Finally, we combine the above estimates and arrive at (4.5.9). □

4.5.4 Closure of the bootstrap method

Proof of Theorem 4.1.1. Recall the refined estimates for u and v components which are stated in Propositions 4.5.4 and 4.5.6 respectively. We arrive at (4.5.2) by choosing C_1 sufficiently large, and ϵ sufficiently small (such that $C_1\epsilon \ll 1/2$). As a consequence, we also provide the proof for Theorem 4.1.1. □

Part III

Hyperboloidal Fourier transform and a new L^2–type estimate for waves

Chapter 5

A new L^2–type estimate for waves and its application

[1]

Contents

5.1 Introduction

Background and motivation A few years ago, LeFloch and Ma [44, 45] illustrated the method of hyperboloidal foliations (review of that method can be found in [44] or in the introduction and etc.) for the study of nonlinear wave problems and, for several other systems of equations. In order to have a complementary understanding of that

[1] This chapter is a joint work with Philippe G. LeFloch.

method, a natural idea is to investigate the wave equation by using the Fourier transform method on the hyperboloids, and to see whether we can obtain more properties of wave equations on the hyperboloids. More details about this method, which we call here the hyperbolic Fourier transform method, can be found in [21, 6, 73, 74].

Interesting enough, we indeed discover a new L^2-type estimate for wave components. To the best of our acknowledge, the existing methods on obtaining L^2-type estimates for waves on the hyperboloids include: 1) applying Hardy inequality on the wave components, see for instance [45]; 2) relying on the conformal vector field and the conformal energy estimates, which was first introduced by Morawetz [63] in the flat setting, see [61] for the study in the hyperboloidal setting.

According to the global stability problems of some physical models, it happens a lot that terms in the nonlinearities of the equations are made of wave components while there are no derivatives hitting on the wave components. Such systems include the Einstein-Klein-Gordon [45] model, the Klein-Gordon-Zakharov model [64], and so fourth. Hence it is important and necessary in such cases to estimate the L^2-type norms of wave components.

Main result We are interested in the wave equation

$$-\Box u = f, \tag{5.1.1}$$

with initial data u_0, u_1 prescribed at the time slice $t = 1$:

$$\big(u, \partial_t u\big)(1, \cdot) = \big(u_0, u_1\big). \tag{5.1.2}$$

We establish here a new L^2-type estimates for the solution u in terms of its initial data and the source function.

Theorem 5.1.1 (L^2-type energy bound)**.** *Consider the spacetime $\mathbb{R}^{d+1}$ with $d \geqslant 1$, and let u be the solution to the initial value problem* (5.1.1)–(5.1.2)*. Assume the source $f = f(t, x)$ satisfies*

$$\big\|f\big\|_{L^2(\mathcal{H}_s)} \leqslant C_f s^{-p}, \tag{5.1.3}$$

then it holds

$$\Big\|\frac{u}{t^{1/2}}\Big\|_{L^2_f(\mathcal{H}_s)} \lesssim s^{\delta} + C_f s^{3/2-p+\delta}, \tag{5.1.4}$$

in which C_f is a constant depending on f, and $\delta > 0$ can be any fixed small constant.

In the above, we have used the notation (for any function $\phi : [1, +\infty) \times \mathbb{R}^d \to \mathbb{R}$)

$$\|\phi\|_{L^2_f(\mathcal{H}_s)} := \Big(\int_{\mathcal{H}_s} \big|u(\sqrt{s^2 + |x|^2}, x)\big|^2 \, dx \Big)^{1/2}. \tag{5.1.5}$$

It will be convenient to also use the geometric definition:

$$\|\phi\|_{L^2(\mathcal{H}_s)} := \Big(\int_{\mathcal{H}_s} \big|u(\sqrt{s^2+|x|^2}, x)\big|^2 \frac{s}{\sqrt{s^2+|x|^2}}\, dx \Big)^{1/2}. \tag{5.1.6}$$

To the best of our knowledge, the L^2-type estimate in Theorem 5.1.1 is new. One can also obtain L^2-type estimates by employing the conformal-type energy estimates, but that result can not deduce ours. By applying the Sobolev inequality on hyperboloids (see for instance Proposition 1.7.4 in Chapter 1 or Proposition 3.2.4 in Chapter 3), we arrive at the following pointwise estimate for waves.

Corollary 5.1.2. *Adopting the same assumptions in Theorem 5.1.1, with moreover*

$$\big\|L^J f\big\|_{L^2(\mathcal{H}_s)} \leqslant C_f s^{-p}, \qquad |J| \leqslant 2 \tag{5.1.7}$$

we have

$$\sup_{\mathcal{H}_s} t|u| \lesssim s^{\delta} + C_f s^{3/2-p+\delta}. \tag{5.1.8}$$

Organization of this chapter In Section 5.2, we revisit the definition and some properties of the hyperbolic Fourier transform. Then we provide the proof the main theorem in Section 5.3. Finally in Section 5.4 we prove the small data global existence result of a wave equation with cubic nonlinearities as an application of Theorem 5.1.1.

5.2 The hyperbolic Fourier transform: definition and basic properties

5.2.1 The hyperboloidal foliation of Minkowski spacetime

The spacetime foliation of interest

We consider Minkowski spacetime $M = \mathbb{R}^{d+1}$ in dimension $d+1 \geqslant 2$, whose metric in Cartesian coordinates reads

$$g := -dt^2 + \sum_{a=1}^{d} (dx^a)^2, \qquad (t, x) = (t, x^a). \tag{5.2.1}$$

By convention, Latin indices $a, b, \ldots$ describe $1, \ldots, d$ and are raised or lowered with the Minkowski metric, so that for instance $x^a = x_a$. It is convenient to introduce the radial variable r by $r^2 := \sum_{a=1}^{d} (x^a)^2$. These coordinates determine a foliation of M by spacelike hypersurfaces of constant time t, which of course have vanishing curvature.

On the other hand, the interior of the light cone defined (from the origin) by

$$\mathcal{K} = \big\{ r = |x| < t \big\} \tag{5.2.2}$$

can also be foliated by the following family of spacelike hyperboloids:

$$\mathcal{H}_s := \big\{ t > 0,\, t^2 - r^2 = s^2 \big\}, \qquad s > 0, \tag{5.2.3}$$

which are curved hypersurfaces with constant (negative) curvature.

The initial value problem (for a suitable class of second- or first-order equations) can be posed by prescribing an initial data on the hyperboloid $\mathcal{H}_{s_0}$ (for some $s_0 > 0$) and solving within the future of this hyperboloid, that is, within the domain covered by the foliation

$$\mathcal{J}_{s_0} := \big(\mathcal{H}_s\big)_{s \geqslant s_0}. \tag{5.2.4}$$

Within $\mathcal{J}_{s_0}$ (and more generally within the light cone $\mathcal{K}$), it is convenient to introduce the *hyperboloidal variables*

$$s := \sqrt{t^2 - r^2}, \qquad \overline{x}^a := x^a, \tag{5.2.5}$$

so that $(t, x) \equiv (s, \overline{x})$ determines a point on the hyperboloid $\mathcal{H}_s$. It is also convenient to introduce similarly the *normalized variables*

$$\big(\widetilde{t}, \widetilde{x}\big) := \Big(\frac{t}{s}, \frac{x}{s}\Big), \qquad (\widetilde{s}, \widetilde{x}) := (s, x/s), \tag{5.2.6}$$

so that the point $(\widetilde{t}, \widetilde{x}) \equiv (s, \overline{x})$ is still regarded as a geometric point of the hyperboloid $\mathcal{H}_s$ but *parametrized* by variables belonging to the unit hyperboloid $\mathcal{H}_1$. We will explain our notation explicitly below whenever this terminology could lead to confusion. Abusing notation, we express a function $\phi = \phi(t, x)$ in either forms

$$\phi(t, x) = \overline{\phi}(s, \overline{x}) = \widetilde{\phi}(\widetilde{s}, \widetilde{x}) \tag{5.2.7}$$

and, for the sake of simplicity,

$$\phi(t, x) = \phi(s, \overline{x}) = \phi(\widetilde{s}, \widetilde{x}). \tag{5.2.8}$$

We also write the partial derivatives as

$$\partial_\alpha \phi = \partial_\alpha \phi(t, x), \qquad \overline{\partial}_\alpha \phi = \overline{\partial}_\alpha \phi(s, x). \tag{5.2.9}$$

The *hyperboloidal frame* (in dimension 3), by definition, is

$$\begin{aligned}\overline{\partial}_0 &:= \partial_s = \frac{s}{t}\partial_t = \frac{1}{t}\sqrt{t^2-r^2}\,\partial_t,\\ \overline{\partial}_a &:= \underline{\partial}_a = \frac{x^a}{t}\partial_t + \partial_a\end{aligned} \tag{5.2.10}$$

and, the dual version of this frame is $ds := (t/s)dt - (x^a/s)dx^a$ and $d\overline{x}^a := dx^a$. We will need the expression of the transition matrices between the hyperboloidal and Cartesian frames, that is,

$$\overline{\partial}_\alpha = \overline{\Phi}^\beta_\alpha \partial_\beta, \qquad \partial_\alpha = \overline{\Psi}^\beta_\alpha \overline{\partial}_\beta \tag{5.2.11}$$

and

$$(\overline{\Phi}^\beta_\alpha) = \begin{pmatrix} s/t & 0 & 0 & 0\\ x^1/t & 1 & 0 & 0\\ x^2/t & 0 & 1 & 0\\ x^3/t & 0 & 0 & 1\end{pmatrix}, \qquad (\overline{\Psi}^\beta_\alpha) = \begin{pmatrix} t/s & 0 & 0 & 0\\ -x^1/s & 1 & 0 & 0\\ -x^2/s & 0 & 1 & 0\\ -x^3/s & 0 & 0 & 1\end{pmatrix}. \tag{5.2.12}$$

On the other hand, our *normalized frame* reads

$$\begin{aligned}\widetilde{\partial}_s &:= \partial_{\widetilde{s}} = \frac{t}{\sqrt{t^2-r^2}}\partial_t + \frac{x^a}{\sqrt{t^2-r^2}}\partial_a,\\ \widetilde{\partial}_a &:= \partial_{\widetilde{x}_a} = \frac{\sqrt{t^2-r^2}x_a}{t}\partial_t + \sqrt{t^2-r^2}\partial_a\end{aligned} \tag{5.2.13}$$

and, we have

$$\widetilde{\partial}_s = \partial_s + \frac{x^a}{s}\overline{\partial}_a, \qquad \widetilde{\partial}_a = s\overline{\partial}_a. \tag{5.2.14}$$

The transition matrices connecting normalized and Cartesian frame are found to be

$$\widetilde{\partial}_\alpha = \widetilde{\Phi}^\beta_\alpha \partial_\beta, \qquad \partial_\alpha = \widetilde{\Psi}^\beta_\alpha \widetilde{\partial}_\beta \tag{5.2.15}$$

and

$$\begin{aligned}(\widetilde{\Phi}^\beta_\alpha) &= \begin{pmatrix} t/\widetilde{s} & x^1/\widetilde{s} & x^2/\widetilde{s} & x^3/\widetilde{s}\\ \widetilde{s}x^1/t & \widetilde{s} & 0 & 0\\ \widetilde{s}x^2/t & 0 & \widetilde{s} & 0\\ \widetilde{s}x^3/t & 0 & 0 & \widetilde{s}\end{pmatrix},\\ (\widetilde{\Psi}^\beta_\alpha) &= \begin{pmatrix} \widetilde{t} & -\widetilde{t}\widetilde{x}^1/\widetilde{s} & -\widetilde{t}\widetilde{x}^2/\widetilde{s} & -\widetilde{t}\widetilde{x}^3/\widetilde{s}\\ -\widetilde{x}^1 & (1+(\widetilde{x}^1)^2)/\widetilde{s} & \widetilde{x}^1\widetilde{x}^2/\widetilde{s} & \widetilde{x}^1\widetilde{x}^3/\widetilde{s}\\ -\widetilde{x}^2 & \widetilde{x}^2\widetilde{x}^1/\widetilde{s} & (1+(\widetilde{x}^2)^2)/\widetilde{s} & \widetilde{x}^2\widetilde{x}^3/\widetilde{s}\\ -\widetilde{x}^3 & \widetilde{x}^3\widetilde{x}^1/\widetilde{s} & \widetilde{x}^3\widetilde{x}^2/\widetilde{s} & (1+(\widetilde{x}^3)^2)/\widetilde{s}\end{pmatrix}.\end{aligned} \tag{5.2.16}$$

Integration on hyperboloids

The (flat) Minkowski metric induces, on each hypersurface $\mathcal{H}_s$, a (curved) d-dimensional Riemannian metric, denoted by g_{ab} and given by

$$g_{ab} = \delta_{ab} - \frac{x_a x_b}{s^2 + r^2}, \tag{5.2.17}$$

whose inverse is

$$g^{ab} = \delta^{ab} + \frac{x^a x^b}{s^2}. \tag{5.2.18}$$

Recall our convention $x^a = x_a$. The determinant of this metric is

$$\det(g_{ab}) = \frac{s^2}{s^2 + r^2} = \frac{s^2}{t^2}. \tag{5.2.19}$$

These expressions allow us to express the integral of a function $f = f(t, x)$ restricted on any hyperboloid $\mathcal{H}_s$, as follows:

$$\begin{aligned} \int_{\mathcal{H}_s} f \, d\mathcal{H}_s &= \int_{\mathbb{R}^d} f(s, x) \, \frac{s \, dx}{\sqrt{s^2 + |x|^2}} \\ &= \widetilde{s}^d \int_{\mathbb{R}^d} \widetilde{f}(\widetilde{s}, \widetilde{x}) \, \frac{d\widetilde{x}}{\sqrt{1 + |\widetilde{x}|^2}}, \qquad s \geqslant s_0. \end{aligned} \tag{5.2.20}$$

5.2.2 The hyperbolic Fourier transform

We introduce a generalization of the Fourier transform [6, 29], as follows. From now on, we set $m := (d-1)/2$ and $S^{d-1} \subset \mathbb{R}^d$ denotes the unit $(d-1)$-dimensional sphere embedded in $\mathbb{R}^d$. We will also use the bracket notation

$$\langle X, Y \rangle = -X^0 Y^0 + \sum_a X^a Y^a, \qquad X, Y \in \mathbb{R}^{d+1}, \tag{5.2.21}$$

which is nothing but the scalar product of two vectors X, Y in Minkowski spacetime. Given any function $\phi = \phi(t, x) = \phi(s, x)$ and restricting it to a hyperboloid $\mathcal{H}_s$ for some $s > 0$, we define the *hyperbolic Fourier transform* $\widehat{\phi} = \widehat{\phi}(s, \lambda, \omega)$ by

$$\begin{aligned} &\widehat{\phi}(s, \lambda, \omega) := s^{-i\lambda s + m} \int_{\mathcal{H}_s} \langle -(t, x), (1, \omega) \rangle^{i\lambda s - m} \phi(t, x) \, d\mathcal{H}_s, \\ &(\lambda, \omega) \in \mathbb{R}_+ \times S^{d-1} \subset \mathbb{R}^{d+1}. \end{aligned} \tag{5.2.22}$$

Since $t = t(s, x) = \sqrt{s^2 + |x|^2}$ is a function of x on the hyperboloid $\mathcal{H}_s$, we can

express the hyperbolic Fourier transform in either forms:

$$\begin{aligned}\widehat{\phi}(s,\lambda,\omega) &= \int_{\mathbb{R}^d} s^{-i\lambda s+m}\Big(t-x^a\omega_a\Big)^{i\lambda s-m}\phi(t,x)\frac{s}{t}\,dx\\ &= \int_{\mathbb{R}^d} s^{-i\lambda s+m}\Big(t(s,x)-x^a\omega_a\Big)^{i\lambda s-m}\phi(s,x)\frac{s}{t(s,x)}\,dx\\ &= \int_{\mathbb{R}^d} s^{-i\lambda s+m}\Big(\sqrt{s^2+|x|^2}-x^a\omega_a\Big)^{i\lambda s-m}\phi(s,x)\frac{s}{\sqrt{s^2+|x|^2}}\,dx.\end{aligned} \tag{5.2.23}$$

Note in passing the following identity which relates the hyperbolic Fourier transform expressed in our two possible choices of coordinates:

$$\mathcal{F}_{\mathcal{H}_s}\big(\phi\big)(s,\lambda,\omega) = s^d\mathcal{F}_{\mathcal{H}_1}\big(\widetilde{\phi}\big)(1,\lambda s,\omega), \tag{5.2.24}$$

with $\phi=\phi(s,x)=\widetilde{\phi}(\widetilde{s},\widetilde{x})$.

Next, we define the *inverse hyperbolic Fourier transform* of a function $\psi=\psi(s,\lambda,\omega)$ as

$$\check{\psi}(s,x) = \int_0^{+\infty}\int_{S^{d-1}} s^{i\lambda s-m}\Big(t(s,x)-x^a\omega_a\Big)^{-i\lambda s-m}\psi(s,\lambda,\omega)\,\frac{d\lambda d\omega}{|A(\lambda s)|^2}, \tag{5.2.25}$$

in which the so-called Harish-Chandra c-function is (up to an irrelevant multiplicative constant)

$$A(\lambda) := \sqrt{2}(2\pi)^{d/2}\,\frac{\Gamma(i\lambda)}{\Gamma(i\lambda+m)}, \tag{5.2.26}$$

Γ being the usual Gamma function. As might be expected, we have the following elementary properties.

Proposition 5.2.1. *For every function $\phi=\phi(s,x)$ and $\psi=\psi(s,\lambda,\omega)$, one has (for all relevant s, x, λ, ω)*

$$\check{\widehat{\phi}}(s,x)=\phi(s,x), \qquad \widehat{\check{\psi}}(s,\lambda,\omega)=\psi(s,\lambda,\omega). \tag{5.2.27}$$

2. The hyperbolic Fourier transform is related to the standard Fourier transform $\mathcal{F}_{\mathbb{R}^d}$ as follows:

$$\widehat{\phi}(s,\lambda,\omega)\simeq\big(\mathcal{F}_{\mathbb{R}^d}\phi\big)(+\infty,\lambda\omega) \quad \text{ when } s\to+\infty, \tag{5.2.28}$$

where $\phi(+\infty,\cdot)$ denoted the limit of $\phi(s,\cdot)$ when $s\to+\infty$ (whenever this limit exists).

Proof. The property (5.2.27) is a standard matter; see for instance [21]. To establish

(5.2.28) we formally expand the expression under consideration as $s \to +\infty$:

$$\begin{aligned}\widehat{\phi}(s,\lambda,\omega) &= \int_{\mathbb{R}^d} \Big(\frac{\sqrt{s^2+|x|^2}}{s} - \frac{x^a}{s}\omega_a\Big)^{i\lambda s - m} \phi(s,x) \frac{s}{\sqrt{s^2+|x|^2}}\, dx \\ &\simeq \int_{\mathbb{R}^d} \Big(\frac{\sqrt{s^2+|x|^2}}{s} - \frac{x^a}{s}\omega_a\Big)^{i\lambda s} \phi(s,x)\, dx,\end{aligned}$$

and therefore

$$\begin{aligned}\widehat{\phi}(s,\lambda,\omega) &\simeq \int_{\mathbb{R}^d} e^{i\lambda s \log\left(\frac{\sqrt{s^2+|x|^2}}{s} - \frac{x^a}{s}\omega_a\right)} \phi(+\infty,x)\, dx \\ &\simeq \int_{\mathbb{R}^d} e^{-i\lambda x^a \omega_a} \phi(+\infty,x)\, dx = \big(\mathcal{F}_{\mathbb{R}^d}\phi\big)(+\infty,\lambda\omega). \qquad \square\end{aligned}$$

5.2.3 Further properties of the hyperbolic Fourier transform

Basic algebraic properties

The following statements are easily checked by elementary calculations and we thus omit the proof.

Proposition 5.2.2. *The following algebraic identities are satisfied by the hyperbolic Fourier transform:*

- Reflection property: *with* $\tau\phi(t,x) := \phi(t,-x)$ *one has*

$$\mathcal{F}_{\mathcal{H}_1}\big(\phi\big)(\lambda,-\omega) = \mathcal{F}_{\mathcal{H}_1}\big(\tau\phi\big)(\lambda,\omega). \tag{5.2.29}$$

Approximation for Harish-Chandre c-function

In the expresion of the inverse transform above, we might need the explicit expression

$$2(2\pi)^d |A(s)|^{-2} = \begin{cases} \prod_{k=0}^{(d-3)/2} \big(k^2+s^2\big), & d \text{ odd}, \\ s\tanh(\pi s) \prod_{k=0}^{d/2} \Big((k-3/2)^2+s^2\Big), & d \text{ even}. \end{cases} \tag{5.2.30}$$

For the proof, one refers to [21].

In dimension three, we have the much simpler expression

$$2(2\pi)^3 |A(s)|^{-2} = s^2, \qquad d=3, \tag{5.2.31}$$

while for general dimensions we have the asymptotic formula

$$2(2\pi)^d|A(s)|^{-2} \simeq s^{d-1}, \qquad s \to +\infty. \tag{5.2.32}$$

Plancherel identity

Proposition 5.2.3. *For some universal constant $C > 0$, the following Plancherel identity*

$$\int_{\mathcal{H}_1} |f(t,x)|^2 \, d\mathcal{H}_1 = C \int_0^{+\infty} \int_{S^2} |\widehat{Q}(\lambda,\omega)|^2 \lambda^2 \, d\omega d\lambda, \tag{5.2.33}$$

in which $\widehat{Q} = \widehat{Q}(\lambda,\omega)$ is the hyperbolic Fourier transform of a function $f = f(t,x)$ defined on $\mathcal{H}_1$.

Proof. From the definition, we find

$$\begin{aligned}\int_{\mathcal{H}_1} |f(t,x)|^2 \, d\mathcal{H}_1 &= \int_{\mathbb{R}^3} \overline{f}(1,x) f(1,x) \, \frac{dx}{t} \\ &= \int_{\mathbb{R}^3} \overline{f}(1,x) \int_0^{+\infty} \int_{S^2} \widehat{Q}(\lambda,\omega)(t - x^a\omega_a)^{-i\lambda-1} |A(\lambda)|^{-2} \, d\omega d\lambda \frac{dx}{t}.\end{aligned}$$

Exchanging the order of the integrals, we obtain

$$\begin{aligned}\int_{\mathcal{H}_1} |f(t,x)|^2 \, d\mathcal{H}_1 &= \int_0^{+\infty} \int_{S^2} \widehat{Q}(\lambda,\omega) |A(\lambda)|^{-2} \int_{\mathbb{R}^3} \overline{f}(1,x)(t - x^a\omega_a)^{-i\lambda-1} \, \frac{dx}{t} d\omega d\lambda \\ &= \int_0^{+\infty} \int_{S^2} \widehat{Q}(\lambda,\omega) \overline{\widehat{Q}}(\lambda,\omega) |A(\lambda)|^{-2} \, d\omega d\lambda,\end{aligned}$$

which, thanks to (5.2.31), completes the proof. □

Remark 5.2.4. *While λ is positive in our definition so far, the definition of the Fourier transform can be extended so that λ takes complex values. For instance, this is relevant in view of the Paley–Wiener theorem which provides decay properties for the hyperbolic Fourier transform of a function.*

5.2.4 The hyperbolic Laplace-Beltrami operator

The Laplace-Beltrami operator is defined, for any function $f : \mathcal{H}_s \to \mathbb{R}$, by

$$\begin{aligned}\Delta_{\mathcal{H}_s} f &:= g^{-1/2}\overline{\partial}_a\big(g^{1/2}g^{ab}\overline{\partial}_b f\big)\\ &= \frac{t}{s}\overline{\partial}_a\Big(\frac{s}{t}\Big(\overline{\partial}^a f + \frac{x^a x^b}{s^2}\overline{\partial}_b f\Big)\Big) = t\overline{\partial}_a\Big(\frac{1}{t}\Big(\overline{\partial}^a f + \frac{x^a x^b}{s^2}\overline{\partial}_b f\Big)\Big)\\ &= \overline{\partial}^a\overline{\partial}_a f + \frac{x^a x^b}{s^2}\overline{\partial}_a\overline{\partial}_b f + d\frac{x^a}{s^2}\overline{\partial}_a f.\end{aligned} \tag{5.2.34}$$

Given any $(\lambda,\omega) \in \mathbb{R}_+ \times S^{d-1} \subset \mathbb{R}^{d+1}$, the following complex-valued function (arising in the definition (5.2.22))

$$(t,x) \in \mathcal{H}_s \mapsto E(s,x;\lambda,\omega) := \Big(\frac{t}{s} - \frac{x^a}{s}\omega_a\Big)^{i\lambda s - m} \tag{5.2.35}$$

enjoys the following properties.

Proposition 5.2.5. *1. The functions $E : (s,x) \mapsto E(s,x;\lambda,\omega)$ are the* eigenfunctions of the Laplace-Beltrami operator $\Delta_{\mathcal{H}_s}$. *In other words, one has*

$$-\Delta_{\mathcal{H}_s} E(s,x;\lambda,\omega) = \Big(\lambda^2 + \frac{m^2}{s^2}\Big)E(s,x;\lambda,\omega) \tag{5.2.36}$$

and, therefore, the eigenvalues *of the operator $\Delta_{\mathcal{H}_s}$ are*

$$\lambda^2 + \frac{m^2}{s^2} = \lambda^2 + \frac{(d-1)^2}{4s^2}. \tag{5.2.37}$$

2. The eigenfunctions satisfy the orthogonality-like property

$$\int_{\mathbb{R}^d} E(s,x;-\lambda',\omega')E(s,x;\lambda,\omega)\frac{s\,dx}{t} = \delta(\lambda'-\lambda,\omega'-\omega), \tag{5.2.38}$$

where δ denotes the Dirac measure.

The above results will play an essential in our analysis since it provides a way to diagonalizing $\Delta_{\mathcal{H}_s}$ and, in turn, to diagonalize the wave operator on $\mathcal{H}_s$. Comparing the expressions in the unit hyperboloid $\mathcal{H}_1$ and in an arbitrary hyperboloid $\mathcal{H}_s$, we easily find the relation

$$\Delta_{\mathcal{H}_s} = s^{-2}\widetilde{\Delta}_{\mathcal{H}_1}, \tag{5.2.39}$$

in the sense that $\Delta_{\mathcal{H}_s}\psi(t,x) = \frac{1}{s^2}\widetilde{\Delta}_{\mathcal{H}_1}\widetilde{\psi}(\widetilde{t},\widetilde{x})$ for any function $\psi = \psi(t,x) = \widetilde{\psi}(\widetilde{t},\widetilde{x})$.

Proof. 1. For the proof we use the short-hand notation $E(s,x;\lambda,\omega) = B^{i\lambda s-m}$, with

$$B := -\frac{1}{s}\langle (t,x),(1,\omega)\rangle = \frac{t}{s} - \frac{x^a}{s}\omega_a.$$

In view of (5.2.34), we have

$$\Delta_{\mathcal{H}_s} E(s,x;\lambda,\omega) = t\overline{\partial}_a\Big(\frac{1}{t}\Big(\overline{\partial}^a E(s,x;\lambda,\omega) + \frac{x^a x^b}{s^2}\overline{\partial}_b E(s,x;\lambda,\omega)\Big)\Big).$$

We first calculate

$$\overline{\partial}^a E(s,x;\lambda,\omega) = (i\lambda s - m)B^{i\lambda s-m-1}\Big(\frac{x^a}{st} - \frac{\omega^a}{s}\Big)$$

and

$$\begin{aligned}\frac{x^a x^b}{s^2}\overline{\partial}_b E(s,x;\lambda,\omega) =& (i\lambda s - m)\frac{x^a}{s^2}B^{i\lambda s-m-1}\Big(\frac{x^b x_b}{st} - \frac{x^b}{s}\omega_b\Big)\\ &= (i\lambda s - m)\frac{x^a}{s^2}B^{i\lambda s-m-1}\Big(\frac{t^2-s^2}{st} - \frac{x^b}{s}\omega_b\Big)\\ &= (i\lambda s - m)\frac{x^a}{s^2}E(s,x;\lambda,\omega) - (i\lambda s - m)\frac{x^a}{st}B^{i\lambda s-m-1}.\end{aligned}$$

Consequently, we obtain

$$\begin{aligned}&\overline{\partial}^a E(s,x;\lambda,\omega) + \frac{x^a x^b}{s^2}\overline{\partial}_b E(s,x;\lambda,\omega)\\ &= (i\lambda s - m)\frac{x^a}{s^2}E(s,x;\lambda,\omega) - (i\lambda s - m)B^{i\lambda s-m-1}\frac{\omega^a}{s}\end{aligned}$$

and, successively, we compute

$$\begin{aligned}&\Delta_{\mathcal{H}_s} E(s,x;\lambda,\omega)\\ &= \overline{\partial}_a\Big(\overline{\partial}^a E(s,x;\lambda,\omega) + \frac{x^a x^b}{s^2}\overline{\partial}_b E(s,x;\lambda,\omega)\Big) - \frac{\overline{\partial}_a t}{t}\Big(\overline{\partial}^a E(s,x;\lambda,\omega) + \frac{x^a x^b}{s^2}\overline{\partial}_b E(s,x;\lambda,\omega)\Big)\\ &= d(i\lambda s - m)\frac{1}{s^2}E(s,x;\lambda,\omega) + (i\lambda s - m)^2\frac{x^a}{s^2}B^{i\lambda s-m-1}\Big(\frac{x_a}{st} - \frac{\omega_a}{s}\Big)\\ &\quad - (i\lambda s - m)(i\lambda s - m - 1)B^{i\lambda s-m-2}\Big(\frac{x_a}{st} - \frac{\omega_a}{s}\Big)\frac{\omega^a}{s}\\ &\quad - (i\lambda s - m)\frac{x^a x_a}{s^2 t^2}E(s,x;\lambda,\omega) + (i\lambda s - m)B^{i\lambda s-m-1}\frac{x_a\omega^a}{st^2},\end{aligned}$$

so

$$\begin{aligned}
&\Delta_{\mathcal{H}_s} E(s,x;\lambda,\omega)\\
&= d(i\lambda s - m)\frac{1}{s^2}E(s,x;\lambda,\omega) + (i\lambda s - m)^2\frac{1}{s^2}B^{i\lambda - m} - (i\lambda s - m)^2\frac{1}{st}B^{i\lambda s - m - 1}\\
&\quad + (i\lambda s - m)(i\lambda s - m - 1)\frac{1}{st}B^{i\lambda s - m - 1}\\
&\quad - (i\lambda s - m)\frac{1}{s^2}E(s,x;\lambda,\omega) + (i\lambda s - m)\frac{1}{t^2}E(s,x;\lambda,\omega)\\
&\quad - (i\lambda s - m)\frac{1}{t^2}B^{i\lambda s - m} + (i\lambda s - m)\frac{1}{st}B^{i\lambda s - m - 1}\\
&= (i\lambda s - m)(i\lambda s + d - m - 1)\frac{1}{s^2}E(s,x;\lambda,\omega).
\end{aligned}$$

This motivates our choice of $m = (d-1)/2$ so that $d - m - 1 = m$ and we arrive at the first conclusion of the proposition.

2. We need to prove, say for $s = 1$,

$$\delta(\lambda' - \lambda, \omega' - \omega) = \int_{\mathbb{R}^d} \big(t - x^a\omega'_a\big)^{-i\lambda' - 1}\big(t - x^a\omega_a\big)^{i\lambda - 1}\frac{dx}{t}. \tag{5.2.40}$$

From the inverse Fourier transform goven in Proposition 5.2.1, we can write (with $s = 1$)

$$\begin{aligned}
\widehat{\phi}(\lambda,\omega) &= \int_{\mathbb{R}^d}\phi(x)\big(t - x^a\omega_a\big)^{i\lambda - 1}\frac{dx}{t}\\
&= \int_{\mathbb{R}^d}\int_0^{+\infty}\int_{S^{d-1}}\widehat{\phi}(\lambda',\omega')\big(t - x^a\omega'_a\big)^{-i\lambda' - 1}|A(\lambda')|^{-2}\,d\omega' d\lambda'\big(t - x^a\omega_a\big)^{i\lambda - 1}\frac{dx}{t}.
\end{aligned}$$

By re-arranging the terms, we obtain

$$\widehat{\phi}(\lambda,\omega) = \int_0^{+\infty}\int_{S^{d-1}}\widehat{\phi}(\lambda',\omega')|A(\lambda')|^{-2}C(\lambda,\omega,\lambda',\omega')\,d\omega' d\lambda',$$

with

$$C(\lambda,\omega,\lambda',\omega') := \int_{\mathbb{R}^d}\big(t - x^a\omega'_a\big)^{-i\lambda' - 1}\big(t - x^a\omega_a\big)^{i\lambda - 1}\frac{dx}{t}.$$

Obviously, only the Dirac mass $C(\lambda,\omega,\lambda',\omega') = \delta(\lambda' - \lambda, \omega' - \omega)$ satisfies this equation on $\widehat{\phi}$ and we obtain (5.2.40) and thus (5.2.38). □

Space	coord.	operator	eigenfunctions	eigenvalues	profile	symmetries
$\mathbb{R}^d$	x	$\Delta_{\mathbb{R}^d}$	$e^{ix\xi}$	$-\lvert\xi\rvert^2$	$e^{\pm it\lvert\xi\rvert}$	translations ∂_a rotations $x_a\partial_b - x_b\partial_a$
$\mathcal{H}_s$	(t,x) $= \big(\sqrt{s^2+\lvert x\rvert^2}, x\big)$	$\Delta_{\mathcal{H}_s}$	$\big(\frac{t}{s} - \frac{x^a}{s}\omega_a\big)^{i\lambda s - m}$	$-\lambda^2 - \frac{m^2}{s^2}$	$s^{\pm i\lambda - m}$	boosts $t\partial_a + x_a\partial_t$ rotations $x_a\partial_b - x_b\partial_a$
$\mathcal{H}_1$	$(\widetilde{t},\widetilde{x})$ $= \big(\sqrt{1+\lvert\widetilde{x}\rvert^2}, \widetilde{x}\big)$	$\widetilde{\Delta}_{\mathcal{H}_1}$	$\big(\widetilde{t} - \widetilde{x}^a\omega_a\big)^{i\lambda - m}$	$-\lambda^2 - m^2$	$s^{\pm i\lambda - m}$	boosts $\widetilde{t}\widetilde{\partial}_a + \widetilde{x}_a\widetilde{\partial}_t$ rotations $\widetilde{x}_a\widetilde{\partial}_b - \widetilde{x}_b\widetilde{\partial}_a$

Table 5.1: Euclidian versus hyperbolic transforms

5.2.5 Equivalent parametrizations for hyperboloids

We now look at different parametrizations for hyperboloids $\mathcal{H}_s$, which follow from [21].

Parametrization I. We already introduced one parametrization of the hyperboloids $\mathcal{H}_s$, that is,

$$(t,x) \in \mathcal{H}_s, \quad t = \sqrt{s^2+|x|^2}, \tag{5.2.41}$$

whose metric $g_{\mathcal{H}_s} = g_{ab}dx^a dx^b$ is given by with

$$g_{\mathcal{H}_s,ab} = \delta_{ab} - \frac{x_a x_b}{s^2+r^2}. \tag{5.2.42}$$

Parametrization II. Another natural choice of coordinates on the hyperboloid $\mathcal{H}_s$ is provided by the polar coordinates in the hyperbolic space, that is,

$$\Omega = (s\cosh r, s\,\omega\sinh r) \in \mathcal{H}_s, \tag{5.2.43}$$

in which

$$(r,\omega) \in \mathbb{R}_+ \times S^{d-1}.$$

A further calculation gives us the expression of the metric

$$g_{\mathcal{H}_s} = s^2\big(dr^2 + \sinh^2 r d\omega^2\big). \tag{5.2.44}$$

Parametrization III One more model for the hyperboloids will be useful. Let us fix some n-dimensional disk $B_R := \big\{y \in \mathbb{R}^d : |y| < R\big\}$ with given radius $R > 0$, and for each $b > 0$ we endow it with the metric

$$g_b := \frac{4b^2}{(R^2-|y|^2)^2}dy^2. \tag{5.2.45}$$

By chossing $b = sR$, we can check that the mapping

$$\mathbb{R}_+ \times S^{d-1} \ni (r, \omega) \mapsto y = R\omega \frac{\sinh(r/2)}{\cosh(r/2)} \tag{5.2.46}$$

provides us with an isometry between the disk B_R endowed with metric (5.2.45) and the hyperboloid $\mathcal{H}_s$ (with its metric (5.2.44)).

5.3 Proof of the main theorem

Revisit of the main result We now recall what we are going to prove. Consider the wave equation

$$-\Box u = f,$$
$$\big(u, \partial_t u\big)(1, \cdot) = \big(u_0, u_1\big),$$

and we want to estimate the L^2-type norm of u by the information from the source function f (and its initial data). Here we are interested in the weighted L^2 norm of u with no partial derivatives on u. To be more precise, we want to show

$$\Big\| \frac{u}{t^{1/2}} \Big\|_{L^2_f(\mathcal{H}_s)} \lesssim s^\delta + C_f s^{3/2-p+\delta},$$

for arbitrary small $\delta > 0$, under the L^2-type assumption on the source f

$$\big\| f \big\|_{L^2(\mathcal{H}_s)} \leqslant C_f s^{-p}.$$

Proof of Theorem 5.1.1 In order to derive Theorem 5.1.1, we first recall the following ingredients about the hyperbolic Fourier transform, see for instance [72, 67].

Lemma 5.3.1. *Relying on the coordinates $\big(\tau, \widetilde{x}\big) = \big(\log s, x/s\big)$ in the hyperbolic space $\mathcal{H}_s$, with $s = \sqrt{t^2 - |x|^2}$, the wave operator on the unit hyperboloid $\mathcal{H}_1$*

$$\Box_{\mathcal{H}_1} := -\partial_\tau \partial_\tau + \Delta_{\mathcal{H}_1} + \frac{d-1}{2} \tag{5.3.1}$$

takes the form, for any sufficiently regular function ϕ,

$$\Box_{\mathcal{H}_1} \big(e^\tau \phi\big)(\tau, \widetilde{x}) = \big(t^2 - |x|^2\big)^{1+(d-1)/4} \Box \phi(t, x). \tag{5.3.2}$$

The following scaling property for the norm $\| \cdot \|_{L^2(\mathcal{H}_s)}$ will also be used.

Lemma 5.3.2. *Given a sufficiently regular function $\phi = \phi(t, x) = \phi(s, x) = \phi(\tau, \widetilde{x})$, it holds*

$$\|\phi(s, \cdot)\|_{L^2(\mathcal{H}_s)} = s^{d/2} \|\phi(\tau, \cdot)\|_{L^2(\mathcal{H}_1)}. \tag{5.3.3}$$

Second, we recall some properties of the hyperbolic Fourier transform. Given a sufficiently regular function $\phi = \phi(t,x) = \phi(s,x) = \phi(\tau,\widetilde{x})$, then the following holds.

- Define the hyperbolic Fourier transform by

$$\widehat{\phi}(\tau,\lambda,\omega) = \int_{\mathbb{R}^d} \left(\sqrt{1+|\widetilde{x}|^2} - \widetilde{x}^a\omega_a\right)^{i\lambda-(d-1)/2} \phi(\tau,\widetilde{x}) \frac{1}{\sqrt{1+|\widetilde{x}|^2}}\, d\widetilde{x}, \tag{5.3.4}$$

  then its inverse formula is given by

$$\phi(\tau,\widetilde{x}) = \int_0^{+\infty}\int_{S^{d-1}} \left(\sqrt{1+|\widetilde{x}|^2} - \widetilde{x}^a\omega_a\right)^{-i\lambda-(d-1)/2} \widehat{\phi}(\tau,\lambda,\omega)\, \frac{d\lambda d\omega}{|A(\lambda)|^2}, \tag{5.3.5}$$

  in which the so-called Harish-Chandra c-function is (up to an irrelevant multiplicative constant)

$$A(\lambda) := \sqrt{2}(2\pi)^{d/2}\, \frac{\Gamma(i\lambda)}{\Gamma(i\lambda+m)}, \tag{5.3.6}$$

  Γ being the usual Gamma function.

- The Plancherel identity holds

$$\|\phi(\tau,\widetilde{x})\|_{L^2(\mathcal{H}_1)} = C\int_0^{+\infty}\int_{S^{d-1}} |\widehat{\phi}(\tau,\lambda,\omega)|^2 |A(\lambda)|^{-2}\, d\omega d\lambda, \tag{5.3.7}$$

  with some constant C.

- The wave operator in the hyperbolic space $\Box_{\mathcal{H}_1}$ takes the following formula after performing the hyperbolic Fourier transform

$$-\widehat{\Box_{\mathcal{H}_1}\phi}(\tau,\lambda,\omega) = \left(\partial_\tau^2\widehat{\phi} + \lambda^2\widehat{\phi}\right)(\tau,\lambda,\omega). \tag{5.3.8}$$

Proof of Theorem 5.1.1. We only give the proof for the case of $(3+1)$ dimension, but the same proof applies to general dimensions.

We first rewrite equation for u in (5.1.1) in the hyperbolic space with coordinates $\left(\tau,\widetilde{x}\right)$

$$-\Box_{\mathcal{H}_1} U(\tau,\widetilde{x}) = e^{3\tau} f(\tau,\widetilde{x}), \tag{5.3.9}$$

in which $U := e^{\tau}u(\tau)$. Taking the hyperbolic Fourier transform, we get

$$\partial_\tau\partial_\tau\widehat{U}(\tau,\lambda,\omega) + \lambda^2\widehat{U}(\tau,\lambda,\omega) = e^{3\tau}\widehat{f}(\tau,\lambda,\omega). \tag{5.3.10}$$

Solving the ordinary differential equation (5.3.10), we obtain explicitly

$$\begin{aligned}\widehat{U}(\tau,\lambda,\omega) = \widehat{U}(\tau_0,\lambda,\omega)\cos(\lambda\tau) + \widehat{\partial_\tau U}(\tau_0,\lambda,\omega)\frac{\sin(\lambda\tau)}{\lambda} \\ + \frac{1}{\lambda}\int_{\tau_0}^{\tau} e^{3\tau'}\widehat{f}(\tau',\lambda,\omega)\sin\big(\lambda(\tau-\tau')\big)\,d\tau',\end{aligned} \tag{5.3.11}$$

for all $\lambda \geqslant 0$. We take L^2-norm in the space $\mathbb{R}_+ \times S^2$ for $\widehat{U}(\tau,\lambda,\omega)$ to get

$$\begin{aligned}&\int_0^{+\infty}\int_{S^2} |\widehat{U}(\tau,\lambda,\omega)|^2\lambda^2\,d\omega d\lambda \\ &\leqslant \int_0^{+\infty}\int_{S^2} |\widehat{U}(\tau_0,\lambda,\omega)|^2\lambda^2\,d\omega d\lambda + \tau\int_0^{+\infty}\int_{S^2} |\widehat{\partial_\tau U}(\tau_0,\lambda,\omega)|^2\lambda^2\,d\omega d\lambda \\ &\quad + \int_{\tau_0}^{\tau}\tau e^{3\tau'}\int_0^{+\infty}\int_{S^2} |\widehat{f}(\tau',\lambda,\omega)|^2\lambda^2\,d\omega d\lambda d\tau',\end{aligned}$$

where we have used the fact $\sin|p| \leqslant |p|$. Then Plancherel identity gives us

$$\|U(\tau)\|_{L^2(\mathcal{H}_1)} \leqslant \|U(\tau_0)\|_{L^2(\mathcal{H}_1)} + \tau\|\partial_\tau U(\tau_0)\|_{L^2(\mathcal{H}_1)} + \tau\int_{\tau_0}^{\tau} e^{3\tau'}\|f(\tau')\|_{L^2(\mathcal{H}_1)}\,d\tau'. \tag{5.3.12}$$

In view of the scaling property (5.3.3), we easily get

$$\|U(\tau)\|_{L^2(\mathcal{H}_1)} = \Big\|\frac{u(s)}{t^{1/2}}\Big\|_{L^2_f(\mathcal{H}_s)}$$

and

$$\|f(\tau)\|_{L^2(\mathcal{H}_1)} = e^{-3/2\tau}\|f(s)\|_{L^2(\mathcal{H}_s)} \lesssim C_f e^{-3/2\tau - p\tau}.$$

We complete the proof of Theorem 5.1.1 by observing that $\tau \lesssim e^{\delta/2\tau}$, as well as

$$\int_{\tau_0}^{\tau} e^{3/2\tau' - p\tau'}\,d\tau' \lesssim e^{3/2\tau - p\tau + \delta/2\tau}.$$

□

5.4 Application to a semilinear wave equation

As a simple application, we prove small initial data global existence result for the following semilinear wave equation in $\mathbb{R}^{1+3}$

$$-\Box u = u^3, \qquad \big(u, \partial_t u\big)(t_0,\cdot) = (u_0, u_1). \tag{5.4.1}$$

Theorem 5.4.1. *Consider the wave equation in* (5.4.1)*, and let N be a sufficiently large integer. Then there exists ϵ_0, such that for all $\epsilon \in (0, \epsilon_0)$ and all compactly supported initial data (u_0, u_1), satisfying the smallness condition*

$$\|u_0\|_{H^{N+1}} + \|u_1\|_{H^N} \leqslant \epsilon,$$

the initial value problem (5.4.1) *admits a global-in-time solution u.*

For the proof we rely on the bootstrap method. We assume for $s \in [s_0, s_1]$ it holds

$$\begin{aligned} \left\|t^{-1/2}\partial^I L^J u\right\|_{L^2_f(\mathcal{H}_s)} &\leqslant C_1 \epsilon s^{\delta}, \quad |I| + |J| \leqslant N, \\ \left|\partial^I L^J u\right| &\leqslant C_1 \epsilon t^{-2/3+\delta} s^{-1/3-2\delta}, \quad |I| + |J| \leqslant N-2, \end{aligned} \tag{5.4.2}$$

for some small $\delta > 0$ and some C_1 to be fixed, and

$$s_1 := \sup\{s : (5.4.2)\ \textit{holds}\}.$$

If we can prove the refined estimate

$$\begin{aligned} \left\|t^{-1/2}\partial^I L^J u\right\|_{L^2_f(\mathcal{H}_s)} &\leqslant \frac{1}{2} C_1 \epsilon s^{\delta}, \quad |I| + |J| \leqslant N, \\ \left|\partial^I L^J u\right| &\leqslant \frac{1}{2} C_1 \epsilon t^{-2/3+\delta} s^{-1/3-2\delta}, \quad |I| + |J| \leqslant N-2, \end{aligned} \tag{5.4.3}$$

then we can infer that s_1 cannot be of finite value, which implies existence of global-in-time solutions.

We need the following lemma from [45] to improve the pointwise estimates of u. An alternative proof of Lemma 5.4.2 is given in [2].

Lemma 5.4.2 (Pointwise estimates for wave components)**.** *Suppose u is a spatially compactly supported solution to the wave equation*

$$\begin{aligned} -\Box u &= f, \\ u(t_0, x) &= \partial_t u(t_0, x) = 0, \end{aligned} \tag{5.4.4}$$

with f spatially compactly supported and satisfying

$$|f| \leqslant C_f t^{-2-\nu}(t-r)^{-1+\mu}, \tag{5.4.5}$$

for $0 < \mu \leqslant 1/2$ and $0 < \nu \leqslant 1/2$. Then we have

$$|u(t,x)| \lesssim \frac{C_f}{\nu\mu}(t-r)^{\mu-\nu} t^{-1}, \tag{5.4.6}$$

where C_f is some constant.

Direct consequences from the bootstrap assumption (5.4.2) and the supnorm estimates for waves give the refined pointwise estimates of u.

$$\left|\partial^I L^J u\right| \lesssim \epsilon t^{-1/2} s^{-1} + (C_1\epsilon)^3 t^{-1+\delta} s^{-2\delta}, \quad |I| + |J| \leqslant N - 2. \tag{5.4.7}$$

The following proposition gives the refined L^2-type estimates.

Proposition 5.4.3 (Refined L^2-type estimates)**.** *Assume the bounds in* (5.4.2)*, then it holds*

$$\left\|t^{-1/2}\partial^I L^J u\right\|_{L^2_f(\mathcal{H}_s)} \lesssim \epsilon + (C_1\epsilon)^3 s^{\delta}. \tag{5.4.8}$$

Proof. According to Theorem 5.1.1, it suffices to show

$$\left\|(s/t)^{1/2}\partial^I L^J (u^3)\right\|_{L^2_f(\mathcal{H}_s)} \lesssim (C_1\epsilon)^3 s^{-3/2}, \qquad |I| + |J| \leqslant N.$$

By the product rule, we have for $|I| + |J| \geqslant 1$ that

$$\begin{aligned}\left\|(s/t)^{1/2}\partial^I L^J (u^3)\right\|_{L^2_f(\mathcal{H}_s)} &\leqslant \sum_{I_1+I_2+I_3=I, J_1+J_2+J_3=J.} \left\|\partial^{I_1} L^{J_1} u \partial^{I_2} L^{J_2} u \partial^{I_3} L^{J_3} u\right\|_{L^2_f(\mathcal{H}_s)} \\ &\lesssim (C_1\epsilon)^3 s^{-3/2}.\end{aligned}$$

For the case $|I| + |J| = 0$, we can obtain the desired estimates in the same way. □

Remark 5.4.4. *We compare here those two kinds of derivatives*

$$\partial_a, \qquad \underline{\partial}_a.$$

What we already know for the wave component u is that

$$\left\|(s/t)\partial_a u\right\|_{L^2_f(\mathcal{H}_s)} \lesssim C_1\epsilon, \qquad |\partial_a u| \lesssim C_1\epsilon t^{-1/2} s^{-1}.$$

From [61], one possibly obtains that

$$\left\|\underline{\partial}_a u\right\|_{L^2_f(\mathcal{H}_s)} \lesssim C_1\epsilon s^{-1/2}, \qquad |\underline{\partial}_a u| \lesssim C_1\epsilon t^{-3/2} s^{-1/2}.$$

On the other hand, by using the analysis here, on can arrive at the bound

$$\left\|t^{1/2}\underline{\partial}_a u\right\|_{L^2_f(\mathcal{H}_s)} \lesssim C_1\epsilon s^{\delta}, \qquad |\underline{\partial}_a u| \lesssim C_1\epsilon t^{-2+\delta},$$

with $\delta > 0$.

In the hyperboloidal setting, the conclusion for the wave component u is that $\underline{\partial}_a u$ behaves better than $(s/t)\partial_a u$. And our L^2-type estimate can control $\underline{\partial}_a u$ better near the light cone, where $t \sim s^2$.

Chapter 6

Global stability in 1 dimensional space

[1]

Contents

6.1 Overview

The global stability results for wave equations with null form nonlinearities in 1 dimension were first proved by Lucilt, Yang, and Yu [57]. Later on a coupled wave-Klein-Gordon system in 1 dimension was studied by Ma [60], where Ma also removed the compactness assumption on the initial data of the hyperboloidal foliation method. The wave equation of interest here with initial data is the following

$$\begin{aligned} -\partial_t\partial_t u + \partial_x\partial_x u &= (\partial_t u)^2 - (\partial_x u)^2, \\ \big(u, \partial_t u\big)(t_0, \cdot) &= \big(u_0, u_1\big). \end{aligned} \tag{6.1.1}$$

[1] This chapter is a joint work with Philippe G. LeFloch.

We are interested here in simplifying the proof in [57]. To be more precise, we provide a proof mimicing the one in [57], where we do not need to estimate the energies on the null curve segments. Besides, we also provide here a pointwise estimates on the wave component u without any derivatives on it, which is based on the new L^2-type estimates established in Chapter 5 and the Sobolev-type estimates in [44]. We prefer to conduct the proof in the hyperbolic space, while the proof also works in the whole flat space. We might not recall the notations and inequalities introduced before.

The main theorem is stated now.

Theorem 6.1.1. *Consider the wave equation with null nonlinearity* (6.1.1) *in 1 dimension, and let $N \geqslant 1$ be an integer. Then there exists $\epsilon_0 > 0$, such that for all compactly supported initial data satisfying*

$$\|u_1\|_{H^N} + \|u_0\|_{H^{N+1}} < \epsilon \leqslant \epsilon_0, \tag{6.1.2}$$

the Cauchy problem (6.1.1) *admits a global-in-time solution u. Furthermore, if $N \geqslant 2$ we have*

$$|u| \lesssim C_1 \epsilon s^{\delta}, \tag{6.1.3}$$

with $\delta > 0$ some arbitrarily small number and s the hyperbolic time.

The rest of this chapter is organised as follows. In Section 6.2, we introduce some Sobolev-type inequalities. Next, we rewrite wave equations in the hyperbolic space in Section 6.3. Later on in Section 6.4 we initialise the bootstrap method. Then in Section 6.5, we prove the refined estimates which prove the existence of global-in-time solution. Finally we prove the pointwise estimates in the last section.

6.2 Sobolev-type inequality

We first provide a complete proof of the global stability result for $\mathbb{R}^{1+1}$ wave equation with null non-linearity. We assume the initial data have compact support and thus we can rewrite it in the hyperbolic space with the new variables

$$(\tau, \widetilde{x}) := \Big(\frac{1}{2}\log(t^2 - x^2), \frac{x}{(t^2 - x^2)^{1/2}}\Big)$$

and, then conduct a bootstrap argument. Again, an alternative approach to this problem can be found in [57].

We work with the one-dimensional unit hyperboloid

$$\mathcal{H}_1 := \big\{(t, x) : t^2 = x^2 + 1,\, (t, x) \in \mathbb{R}^2\big\}. \tag{6.2.1}$$

Definition 6.2.1. *The hyperbolic Fourier transform of a function $f = f(t,x)$ defined on $\mathcal{H}_1$ is defined as*

$$\widehat{f}(\lambda) := \int_{-\infty}^{+\infty} f\big(\sqrt{1+x^2},x\big)\big(\sqrt{1+x^2}-x\big)^{i\lambda}\,\frac{dx}{\sqrt{1+x^2}}, \qquad \lambda\in\mathbb{R}. \tag{6.2.2}$$

The above formula and the ones we now derive hold for all (suitably) integrable functions. The L^2 and H^1 norms of a function $f:\mathcal{H}_1\to\mathbb{R}$ are defined as

$$\|f\|_{L^2(\mathcal{H}_1)} := \Big(\int_{-\infty}^{+\infty}\frac{|f(x)|^2dx}{\sqrt{1+x^2}}\Big)^{1/2} \tag{6.2.3}$$

and

$$\|f\|_{H^1(\mathcal{H}_1)} := \Big(\int_{-\infty}^{+\infty}\frac{\big(|f(x)|^2+|\sqrt{1+x^2}\partial_x f(x)|^2\big)dx}{\sqrt{1+x^2}}\Big)^{1/2} \tag{6.2.4}$$

respectively.

Proposition 6.2.2. • **Inverse formula***: From the hyperbolic Fourier transform of a function $f = f(t,x) = f(x)$ defined on $\mathcal{H}_1$, we can recover the function by the formula) (up to a constant)*

$$f(x) = \int_{-\infty}^{+\infty}\widehat{f}(\lambda)\big(\sqrt{1+x^2}-x\big)^{-i\lambda}\,d\lambda. \tag{6.2.5}$$

- **Plancherel identity:** *For a sufficiently regular function $f = f(t,x)$ on $\mathcal{H}_1$ with its hyperbolic Fourier transform $\widehat{f} = \widehat{f}(\lambda)$, it holds*

$$\|f\|_{L^2(\mathcal{H}_1)} = \|\widehat{f}\|_{L^2(\mathbb{R})}.$$

Like the standard Fourier transforms, derivatives can be transformed to multipliers.

Lemma 6.2.3. *For any function f defined on $\mathcal{H}_1$ one has*

$$\mathcal{F}_{\mathcal{H}_1}\Big(\big(\sqrt{1+x^2}\partial_x\big)^n f(x)\Big)(\lambda) = (i\lambda)^n\widehat{f}(\lambda), \qquad n=0,1,2,\dots, \tag{6.2.6}$$

with, moreover,

$$\big\|\big(\sqrt{1+x^2}\partial_x\big)^n f(x)\big\|_{L^2(\mathcal{H}_1)} = \|\lambda^n\widehat{f}(\lambda)\|_{L^2(\mathbb{R})}.$$

Now we are ready to give the statement and proof of the following Sobolev-type inequality needed afterwards.

Lemma 6.2.4. *We define the function space*

$$H^N(\mathcal{H}_1) := \Big\{f : \int_{-\infty}^{+\infty}|\hat{f}(\lambda)|^2(1+\lambda^{2N})\,d\lambda < +\infty\Big\}.$$

Given a function $f(x) \in H^1(\mathcal{H}_1)$, then it is bounded, i.e.

$$\sup_{x\in\mathcal{H}_1} |f(x)| \leqslant \|f\|_{H^1(\mathcal{H}_1)}. \tag{6.2.7}$$

Proof. Recall the inverse hyperbolic Fourier transform formula

$$f(x) = \int_{-\infty}^{+\infty} \widehat{f}(\lambda)\big(\sqrt{1+x^2}-x\big)^{-i\lambda}\, d\lambda.$$

It is obvious that

$$\begin{aligned} |f(x)| &\leqslant \int_{-\infty}^{+\infty} |\widehat{f}(\lambda)|\, d\lambda \\ &\lesssim \|\widehat{f}\|_{L^2(\mathbb{R})}^{1/2} \|\lambda\widehat{f}\|_{L^2(\mathbb{R})}^{1/2} = \|f\|_{L^2(\mathcal{H}_1)}^{1/2} \|\sqrt{1+x^2}\partial_x f\|_{L^2(\mathcal{H}_1)}^{1/2}, \end{aligned}$$

where we used the inequality $\|f(x)\|_{L^1(\mathbb{R})} \lesssim \|f(x)\|_{L^2(\mathbb{R})}^{1/2} \|xf(x)\|_{L^2(\mathbb{R})}^{1/2}$ which can be found in [43]. □

6.3 Wave equations in hyperbolic spaces

Lemma 6.3.1. *Consider the wave equation with null quadratic non-linearity*

$$\begin{aligned} &\partial_t^2 u - \Delta u = \partial_t u \partial_t u - \partial_x u \partial_x u, \\ &u(1,x) = u_0(x), \qquad \partial_t u(1,x) = u_1(x). \end{aligned} \tag{6.3.1}$$

Then in the new coordinates $(\tau, \widetilde{x}) := \big(\log\sqrt{t^2-x^2}, x/\sqrt{t^2-x^2}\big)$ it will be expressed as

$$\partial_\tau^2 u - \widetilde{\Delta}_{\mathcal{H}_1} u = (\partial_\tau u)^2 - \big(\sqrt{1+\widetilde{x}^2}\widetilde{\partial}_x u\big)^2, \tag{6.3.2}$$

in which

$$\widetilde{\Delta}_{\mathcal{H}_1} = \sqrt{1+\widetilde{x}^2}\widetilde{\partial}_x$$

is the Laplace-Beltrami operator on the unit hyperboloid $\mathcal{H}_1$ and

$$\widetilde{\partial}_x := \partial_{\widetilde{x}}.$$

Proof. It is helpful to first look at the vector fields (∂_t, ∂_x) expressed in the new frame:

$$\begin{aligned} \partial_t &= e^{-\tau}\sqrt{1+\widetilde{x}^2}\partial_\tau - e^{-\tau}\widetilde{x}\sqrt{1+\widetilde{x}^2}\widetilde{\partial}_x, \\ \partial_x &= -e^{-\tau}\widetilde{x}\partial_\tau + e^{-\tau}(1+\widetilde{x}^2)\widetilde{\partial}_x. \end{aligned}$$

Successively, we can express the null term $(\partial_t u)^2 - (\partial_x u)^2$ in the new coordinates:

$$\begin{aligned}(\partial_t u)^2 - (\partial_x u)^2 &= \big(e^{-\tau}\sqrt{1+\widetilde{x}^2}\partial_\tau u - e^{-\tau}\widetilde{x}\sqrt{1+\widetilde{x}^2}\widetilde{\partial}_x u\big)^2 - \big(-e^{-\tau}\widetilde{x}\partial_\tau u + e^{-\tau}(1+\widetilde{x}^2)\widetilde{\partial}_x u\big)^2 \\ &= e^{-2\tau}(\partial_\tau u)^2 - e^{-2\tau}\big(\sqrt{1+\widetilde{x}^2}\widetilde{\partial}_x u\big)^2.\end{aligned}$$

Next recall the relation

$$\Box\Big(e^{-m\tau(t,x)}\phi(t,x)\Big) = e^{-(2+m)\tau}\Box_{\mathcal{H}_1}\phi(\tau,\widetilde{x}),$$

where the wave operator in hyperbolic space is given by

$$\Box_{\mathcal{H}_1} := -\partial_\tau\partial_\tau + \widetilde{\Delta}_{\mathcal{H}_1} + m^2, \tag{6.3.3}$$

hence (6.3.2) can be obtained by noticing that $m = (n-1)/2 = 0$ in $\mathbb{R}^{1+1}$. □

Theorem 6.3.2. *Consider the wave equation* (6.3.1)*, and assume the initial data are compactly supported in* $B_{1/2}(0)$*, then there exists* $\epsilon_0 > 0$*, such that for any initial data satisfying*

$$\|u_1\|_{H^1} + \|u_0\|_{H^2} < \epsilon \leqslant \epsilon_0,$$

equation (6.3.1) *admits a global-in-time solution* $u = u(t,x)$*.*

Remark 6.3.3. *In order to prove* (6.3.1) *has a global-in-time solution, it suffices to show* (6.3.2) *admits a global-in-time solution in terms of the time variable* τ*.*

6.4 Bootstrap assumption

We first introduce two differential operators which will be used frequently in the following contents:

$$D_- := \partial_\tau - \sqrt{1+\widetilde{x}^2}\widetilde{\partial}_x, \qquad D_+ := \partial_\tau + \sqrt{1+\widetilde{x}^2}\widetilde{\partial}_x.$$

We notice that the wave operator can be rewritten as

$$-\Box_{\mathcal{H}_1} = D_-D_+ = D_+D_-,$$

while the null non-linearity is expressed to be

$$(\partial_\tau u)^2 - \big(\sqrt{1+\widetilde{x}^2}\widetilde{\partial}_x u\big)^2 = D_-uD_+u.$$

Now we pose the bootstrap assumptions in the interval $[0,\tau_1]$:

$$\begin{aligned}\sum_{k=0,1}\|\omega_-^{\delta_2}D_-(\sqrt{1+\widetilde{x}^2}\widetilde{\partial}_x)^k u\|_{L^2(\mathcal{H}_1)}+\sum_{k=0,1}\|\omega_+^{\delta_2}D_+(\sqrt{1+\widetilde{x}^2}\widetilde{\partial}_x)^k u\|_{L^2(\mathcal{H}_1)}\leqslant C_1\epsilon e^{\delta_1\tau},\\ \sum_{k=0,1}\|D_-(\sqrt{1+\widetilde{x}^2}\widetilde{\partial}_x)^k u\|_{L^2(\mathcal{H}_1)}+\sum_{k=0,1}\|D_+(\sqrt{1+\widetilde{x}^2}\widetilde{\partial}_x)^k u\|_{L^2(\mathcal{H}_1)}\leqslant C_1\epsilon,\end{aligned}\tag{6.4.1}$$

in which the weights are

$$\omega_-:=e^{\tau}(\sqrt{1+\widetilde{x}^2}-\widetilde{x})+1,\qquad \omega_+:=e^{\tau}(\sqrt{1+\widetilde{x}^2}+\widetilde{x})+1,$$

and the parameters C_1 is some big constant to be determined, $\delta_1>0$ and $\delta_2>0$ are fixed constants with the restriction

$$\delta_2>2\delta_1.$$

The upper bound τ_1 is taken to be the largest number such that the bootstrap assumptions in (6.4.1) are satisfied, and we know τ_1 is strictly positive due to the smallness of initial data.

We are going to prove that the upper bounds in (6.4.1) can be refined to be

$$\begin{aligned}\sum_{k=0,1}\|\omega_-^{\delta_2}D_-(\sqrt{1+\widetilde{x}^2}\widetilde{\partial}_x)^k u\|_{L^2(\mathcal{H}_1)}+\sum_{k=0,1}\|\omega_+^{\delta_2}D_+(\sqrt{1+\widetilde{x}^2}\widetilde{\partial}_x)^k u\|_{L^2(\mathcal{H}_1)}\leqslant \frac{1}{2}C_1\epsilon e^{\delta_1\tau},\\ \sum_{k=0,1}\|D_-(\sqrt{1+\widetilde{x}^2}\widetilde{\partial}_x)^k u\|_{L^2(\mathcal{H}_1)}+\sum_{k=0,1}\|D_+(\sqrt{1+\widetilde{x}^2}\widetilde{\partial}_x)^k u\|_{L^2(\mathcal{H}_1)}\leqslant \frac{1}{2}C_1\epsilon,\end{aligned}\tag{6.4.2}$$

which ensures the existence of the global-in-time solution to (6.3.2).

A direct consequence from the bootstrap assumptions (6.4.1) are the following sup-norm bounds.

Lemma 6.4.1. *Assume* (6.4.1) *holds, then*

$$\begin{aligned}\sup\big|\omega_-^{\delta_2}D_-u\big|+\sup\big|\omega_+^{\delta_2}D_+u\big|\lesssim C_1\epsilon e^{\delta_1\tau},\\ \sup\big|D_-u\big|+\sup\big|D_+u\big|\lesssim C_1\epsilon.\end{aligned}\tag{6.4.3}$$

Proof. Observe that

$$\big|\sqrt{1+\widetilde{x}^2}\widetilde{\partial}_x\omega_\pm^{\delta_2}\big|\leqslant\delta_2\omega_\pm^{\delta_2},$$

then an application of the Sobolev-type inequality (6.2.7) implies (6.4.3). □

6.5 Refined energy estimates

Lemma 6.5.1. *Consider the wave equation* (6.3.2)

$$\partial_\tau^2 u - \widetilde{\Delta}_{\mathcal{H}_1} u = (\partial_\tau u)^2 - \big(\sqrt{1+\widetilde{x}^2}\widetilde{\partial}_x u\big)^2,$$

then it holds for any smooth enough multiplier Z that

$$\begin{aligned}
&\int_{\mathbb{R}} ZD_-u \frac{d\widetilde{x}}{\sqrt{1+\widetilde{x}^2}}(\tau)\\
\leqslant &\int_{\mathbb{R}} ZD_-u \frac{d\widetilde{x}}{\sqrt{1+\widetilde{x}^2}}(0) + \int_0^\tau \Big(\|D_+ZD_-u\|_{L^1(\mathcal{H}_1)} + \|ZD_-uD_+u\|_{L^1(\mathcal{H}_1)} \Big) d\tau,\\
&\int_{\mathbb{R}} ZD_+u \frac{d\widetilde{x}}{\sqrt{1+\widetilde{x}^2}}(\tau)\\
\leqslant &\int_{\mathbb{R}} ZD_+u \frac{d\widetilde{x}}{\sqrt{1+\widetilde{x}^2}}(0) + \int_0^\tau \Big(\|D_-ZD_+u\|_{L^1(\mathcal{H}_1)} + \|ZD_-uD_+u\|_{L^1(\mathcal{H}_1)} \Big) d\tau.
\end{aligned} \tag{6.5.1}$$

Proof. Consider (6.3.2) in the form

$$D_+D_-u = D_-uD_+u,$$

then we obtain after testing the multiplier Z

$$\int_0^\tau \int_{\mathbb{R}} ZD_+D_-u \frac{d\widetilde{x}}{\sqrt{1+\widetilde{x}^2}} d\tau = \int_0^\tau \int_{\mathbb{R}} ZD_-uD_+u \frac{d\widetilde{x}}{\sqrt{1+\widetilde{x}^2}} d\tau.$$

Successively, we rewrite

$$ZD_+D_-u = \partial_\tau\big(ZD_-u\big) + \sqrt{1+\widetilde{x}^2}\widetilde{\partial}_x\big(ZD_-u\big) - D_-ZD_+u,$$

thus we have the first energy estimate in (6.5.1), due to the term $\sqrt{1+\widetilde{x}^2}\widetilde{\partial}_x\big(ZD_-u\big)$ does not contribute with respect to the integral $\int_{\mathbb{R}} \cdot\, d\widetilde{x}/\sqrt{1+\widetilde{x}^2}$. □

We also state the following consequence, with specific choices of the multiplier Z, which are more useful.

Lemma 6.5.2. *Consider the wave equation* (6.3.2)*, then it holds*

$$\begin{aligned}
\|D_-u\|^2_{L^2(\mathcal{H}_1)}(\tau) &\lesssim \|D_-u\|^2_{L^2(\mathcal{H}_1)}(0) + \int_0^\tau \int_{\mathbb{R}} (D_-u)^2 D_+u \frac{d\widetilde{x}}{\sqrt{1+\widetilde{x}^2}} d\tau,\\
\|\omega_-^{\delta_2} D_-u\|^2_{L^2(\mathcal{H}_1)}(\tau) &\lesssim \|\omega_-^{\delta_2} D_-u\|^2_{L^2(\mathcal{H}_1)}(0) + \int_0^\tau \int_{\mathbb{R}} \omega_-^{2\delta_2} (D_-u)^2 D_+u \frac{d\widetilde{x}}{\sqrt{1+\widetilde{x}^2}} d\tau,\\
\|D_-(\sqrt{1+\widetilde{x}^2}\widetilde{\partial}_x u)\|^2_{L^2(\mathcal{H}_1)}(\tau) &\lesssim \|D_-(\sqrt{1+\widetilde{x}^2}\widetilde{\partial}_x u)\|^2_{L^2(\mathcal{H}_1)}(0) + \int_0^\tau \int_{\mathbb{R}} \Big(\big(D_-(\sqrt{1+\widetilde{x}^2}\widetilde{\partial}_x u)\big)^2 D_+u
\end{aligned}$$

$$+ D_-(\sqrt{1+\widetilde{x}^2}\widetilde{\partial}_x u)D_-uD_+(\sqrt{1+\widetilde{x}^2}\widetilde{\partial}_x u)\Big)\frac{d\widetilde{x}}{\sqrt{1+\widetilde{x}^2}}d\tau,$$

$$\begin{aligned}\|\omega_-^{\delta_2}D_-(\sqrt{1+\widetilde{x}^2}\widetilde{\partial}_x u)\|^2_{L^2(\mathcal{H}_1)}(\tau) \lesssim\ & \|\omega_-^{\delta_2}D_-(\sqrt{1+\widetilde{x}^2}\widetilde{\partial}_x u)\|^2_{L^2(\mathcal{H}_1)}(0)\\ &+\int_0^\tau\int_{\mathbb{R}}\omega_-^{2\delta_2}\big(D_-(\sqrt{1+\widetilde{x}^2}\widetilde{\partial}_x u)\big)^2D_+u\frac{d\widetilde{x}}{\sqrt{1+\widetilde{x}^2}}d\tau\\ &+\int_0^\tau\int_{\mathbb{R}}\omega_-^{2\delta_2}D_-(\sqrt{1+\widetilde{x}^2}\widetilde{\partial}_x u)D_-uD_+(\sqrt{1+\widetilde{x}^2}\widetilde{\partial}_x u)\frac{d\widetilde{x}}{\sqrt{1+\widetilde{x}^2}}d\tau.\end{aligned} \tag{6.5.2}$$

Proof. For the first energy estimate, it is easy to have by taking

$$Z = D_-u.$$

We then choose

$$Z = \omega_-^{2\delta_2}D_-u$$

and, the second one can also be achieved with no difficulty by noting

$$D_+\big(\omega_-^{2\delta_2}D_-u\big) = \omega_-^{2\delta_2}D_+D_-u.$$

In order to prove the last two estimates, we first act $\sqrt{1+\widetilde{x}^2}\widetilde{\partial}_x$ on both sides of (6.3.2)

$$D_+D_-\big(\sqrt{1+\widetilde{x}^2}\widetilde{\partial}_x u\big) = D_+\big(\sqrt{1+\widetilde{x}^2}\widetilde{\partial}_x u\big)D_-u + D_+uD_-\big(\sqrt{1+\widetilde{x}^2}\widetilde{\partial}_x u\big).$$

Finally, a similar argument by taking

$$Z = D_-\big(\sqrt{1+\widetilde{x}^2}\widetilde{\partial}_x u\big)$$

and

$$Z = \omega_-^{2\delta_2}D_-\big(\sqrt{1+\widetilde{x}^2}\widetilde{\partial}_x u\big)$$

respectively completes the proof. □

Now we are in a position to prove the refined estimates (6.4.2) and thus close the bootstrap argument.

Proof of the refined estimates (6.4.2). We will only provide the proof for energies with D_-u term, since the rest ones with D_+u term can be bounded in a similar way.

Firstly, by the energy estimates in Lemma (6.5.2), we have

$$\|D_-u\|^2_{L^2(\mathcal{H}_1)}(\tau) \lesssim \|D_-u\|^2_{L^2(\mathcal{H}_1)}(0) + \int_0^\tau\int_{\mathbb{R}}(D_-u)^2D_+u\frac{d\widetilde{x}}{\sqrt{1+\widetilde{x}^2}}d\tau$$

$$
\begin{aligned}
&\leqslant \epsilon^2 + \int_0^\tau \int_{\mathbb{R}} \omega_-^{2\delta_2} (D_- u)^2 \omega_-^{-2\delta_2} C_1\epsilon\, \omega_+^{-\delta_2} e^{\delta_1\tau'} \frac{d\widetilde{x}}{\sqrt{1+\widetilde{x}^2}} d\tau' \\
&\leqslant \epsilon^2 + C_1\epsilon \int_0^\tau \|\omega_-^{\delta_2} D_- u\|^2_{L^2(\mathcal{H}_1)} \|\omega_-^{-\delta_2}\omega_+^{-\delta_2}\|_{L^\infty}\, d\tau' \\
&\lesssim \epsilon^2 + (C_1\epsilon)^3 \int_0^\tau e^{(2\delta_1-\delta_2)\tau'}\, d\tau',
\end{aligned}
$$

in which we used the sup-norm estimate for $\omega_+^{\delta_2} D_+ u$ in the second line and the estimate for weighted Sobolev norm of $D_- u$ in the last line. Due to that fact $\delta_2 > 2\delta_1$, we obtain

$$
\|D_- u\|^2_{L^2(\mathcal{H}_1)}\big|_\tau \lesssim \epsilon^2 + (C_1\epsilon)^3.
$$

Next we bound the weighted $L^2(\mathcal{H}_1)$ norm for $D_- u$. Note Lemma (6.5.2) implies

$$
\begin{aligned}
\|\omega_-^{\delta_2} D_- u\|^2_{L^2(\mathcal{H}_1)}(\tau) &\lesssim \|\omega_-^{\delta_2} D_- u\|^2_{L^2(\mathcal{H}_1)}(0) + \int_0^\tau \int_{\mathbb{R}} \omega_-^{2\delta_2} (D_- u)^2 D_+ u \frac{d\widetilde{x}}{\sqrt{1+\widetilde{x}^2}} d\tau' \\
&\lesssim \epsilon^2 + C_1\epsilon \int_0^\tau \|\omega_-^{\delta_2} D_- u\|^2_{L^2(\mathcal{H}_1)}\, d\tau' \\
&\lesssim \epsilon^2 + (C_1\epsilon)^3 \int_0^\tau e^{2\delta_1\tau'}\, d\tau' \\
&\lesssim \epsilon^2 + (C_1\epsilon)^3 e^{2\delta_1\tau}
\end{aligned}
$$

where we applied sup-norm estimate for $D_+ u$ in the second line and weighted $L^2(\mathcal{H}_1)$ norm estimate for $D_- u$ in the third line.

Successively, we estimate the term $\|D_-(\sqrt{1+\widetilde{x}^2}\widetilde{\partial}_x u)\|_{L^2(\mathcal{H}_1)}(\tau)$. From Lemma (6.5.2), it holds

$$
\begin{aligned}
&\|D_-(\sqrt{1+\widetilde{x}^2}\widetilde{\partial}_x u)\|^2_{L^2(\mathcal{H}_1)}(\tau) \\
&\lesssim \|D_-(\sqrt{1+\widetilde{x}^2}\widetilde{\partial}_x u)\|^2_{L^2(\mathcal{H}_1)}(0) + \int_0^\tau \int_{\mathbb{R}} \Big(\big(D_-(\sqrt{1+\widetilde{x}^2}\widetilde{\partial}_x u)\big)^2 D_+ u \\
&\qquad\qquad + D_-(\sqrt{1+\widetilde{x}^2}\widetilde{\partial}_x u) D_- u D_+(\sqrt{1+\widetilde{x}^2}\widetilde{\partial}_x u)\Big) \frac{d\widetilde{x}}{\sqrt{1+\widetilde{x}^2}} d\tau \\
&\lesssim \epsilon^2 + \int_0^\tau \Big(\|\omega_-^{\delta_1\tau'} D_-(\sqrt{1+\widetilde{x}^2}\widetilde{\partial}_x u)\|^2_{L^2(\mathcal{H}_1)} \|\omega_-^{\delta_1\tau'} D_+ u\|_{L^\infty} \\
&+ \|\omega_-^{\delta_1\tau'} D_-(\sqrt{1+\widetilde{x}^2}\widetilde{\partial}_x u)\|_{L^2(\mathcal{H}_1)} \|\omega_+^{\delta_1\tau'} D_+(\sqrt{1+\widetilde{x}^2}\widetilde{\partial}_x u)\|_{L^2(\mathcal{H}_1)} \|\omega_-^{\delta_1\tau'}\omega_+^{\delta_1\tau'} D_+ u\|_{L^\infty} \Big)\, d\tau' \\
&\lesssim \epsilon^2 + (C_1\epsilon)^3 \int_0^\tau e^{(2\delta_1-\delta_2)\tau'}\, d\tau',
\end{aligned}
$$

where we used estimates in the bootstrap assumptions and sup-norm bounds. Then we

find

$$\|D_-(\sqrt{1+\widetilde{x}^2}\widetilde{\partial}_x u)\|^2_{L^2(\mathcal{H}_1)}(\tau) \lesssim \epsilon^2 + (C_1\epsilon)^3.$$

Finally, we come to bound the term $\|\omega_-^{\delta_2} D_-(\sqrt{1+\widetilde{x}^2}\widetilde{\partial}_x u)\|_{L^2(\mathcal{H}_1)}(\tau)$. By applying the estimates in Lemma (6.5.2) we have

$$\begin{aligned}
\|\omega_-^{\delta_2} D_-(\sqrt{1+\widetilde{x}^2}\widetilde{\partial}_x u)\|^2_{L^2(\mathcal{H}_1)}(\tau) \lesssim\ & \|\omega_-^{\delta_2} D_-(\sqrt{1+\widetilde{x}^2}\widetilde{\partial}_x u)\|^2_{L^2(\mathcal{H}_1)}(0) \\
&+ \int_0^\tau \int_{\mathbb{R}} \omega_-^{2\delta_2} \big(D_-(\sqrt{1+\widetilde{x}^2}\widetilde{\partial}_x u)\big)^2 D_+ u \frac{d\widetilde{x}}{\sqrt{1+\widetilde{x}^2}} d\tau \\
&+ \int_0^\tau \int_{\mathbb{R}} \omega_-^{2\delta_2} D_-(\sqrt{1+\widetilde{x}^2}\widetilde{\partial}_x u) D_- u D_+(\sqrt{1+\widetilde{x}^2}\widetilde{\partial}_x u) \frac{d\widetilde{x}}{\sqrt{1+\widetilde{x}^2}} d\tau \\
&\lesssim \epsilon^2 + \int_0^\tau \big\|\omega_-^{\delta_2} D_-(\sqrt{1+\widetilde{x}^2}\widetilde{\partial}_x u)\big\|^2_{L^2(\mathcal{H}_1)} \|D_+ u\|_{L^\infty}\, d\tau' \\
&+ \int_0^\tau \big\|\omega_-^{\delta_2} D_-(\sqrt{1+\widetilde{x}^2}\widetilde{\partial}_x u)\big\|_{L^2(\mathcal{H}_1)} \|\omega_-^{\delta_2} D_- u\|_{L^\infty} \|D_+ u\|_{L^2(\mathcal{H}_1)}\, d\tau' \\
&\lesssim \epsilon^2 + (C_1\epsilon)^3 \int_0^\tau e^{2\delta_1 \tau'}\, d\tau',
\end{aligned}$$

in which we applied energy estimates from the bootstrap assumptions and their consequences in the last line. Thus we obtain

$$\|\omega_-^{\delta_2} D_-(\sqrt{1+\widetilde{x}^2}\widetilde{\partial}_x u)\|^2_{L^2(\mathcal{H}_1)}(\tau) \lesssim \epsilon^2 + (C_1\epsilon)^3 e^{2\delta_1 \tau}.$$

We are now able to get the refined estimates by choosing C_1 very big and ϵ sufficiently small, and thus we claim the wave equations (6.3.2) and hence (6.3.1) obtain global-in-time solutions. □

6.6 Pointwise estimates of the wave component

In this section we study the pointwise estimates of the wave component u, with no derivatives on it, which is the second part of the Theorem 6.1.1. In order to prove it we rely on the new L^2-type estimates established in Chapter 5 and the Sobolev-type estimates in [44].

We start by recalling the following L^2-type estimate of u.

Lemma 6.6.1. *It holds*

$$\|t^{-1/2} u\|_{L^2_f(\mathcal{H}_s)} \lesssim \epsilon s^\delta + (C_1\epsilon)^2 s^\delta. \tag{6.6.1}$$

Proof. Recall the sup-norm estimates in (6.4.3) and the L^2-type estimates in (6.4.1),

we have

$$\|D_+uD_-u\|_{L^2(\mathcal{H}_1)}(\tau) \lesssim (C_1\epsilon)^2 e^{-\delta_1\tau},$$

which is integrable.

Hence from the L^2-type estimates established in Theorem 5.1.1 (and its proof), we arrive at the desired result. □

We also recall some facts about null forms.

Lemma 6.6.2. *It holds that*

$$L\big(\partial_t u\partial_t v - \partial_x u\partial_x v\big) = \partial_t u\partial_t(Lv) - \partial_x u\partial_x(Lv) + \partial_t(Lu)\partial_t v - \partial_x(Lu)\partial_x v, \tag{6.6.2}$$

in which $L = x\partial_t + t\partial_x$ is the Lorentz boost in 1 dimension.

Relying on (6.6.2) we further have the following estimate.

Lemma 6.6.3. *Assume $N \geqslant 3$ in Theorem 6.1.1, then it holds*

$$\|t^{-1/2}L^J u\|_{L^2_f(\mathcal{H}_s)} \lesssim \epsilon s^\delta + (C_1\epsilon)^2 s^\delta, \qquad |J| \leqslant 1. \tag{6.6.3}$$

Next we recall the following Sobolev-type estimates which was first introduced in [44].

Lemma 6.6.4. *For all sufficiently smooth functions $u = u(t,x)$ supported in the region $\{(t,x) : |x| < t-1\}$, then for $s \geqslant 2$ one has*

$$\sup_{\mathcal{H}_s} \big|t^{1/2}u(t,x)\big| \lesssim \sum_{|J|\leqslant 1} \big\|L^J u\big\|_{L^2_f(\mathcal{H}_s)}, \tag{6.6.4}$$

where the summation is over Lorentz boosts $L = x\partial_t + t\partial_x$.

Proof of the pointwise estimate (6.1.3). A combination of the L^2-type estimates in (6.6.3) and the Sobolev-type inequality (6.6.4), we obtain the pointwise estimate (6.1.3)

$$|u| \lesssim C_1\epsilon s^\delta.$$

□

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Existence results for nonlinear wave equations

Abstract :
This thesis is devoted to showing the existence of global-in-time solutions to some nonlinear equations, including the wave-Klein-Gordon equations and the Dirac equations. For the wave-Klein-Gordon-Dirac equations, based on the hyperboloidal foliation method we establish several global stability results, explore the asymptotic behaviors of these solutions, and study how the solutions are affected when some mass parameters go to certain limits. As an application, we can prove that several physical models are globally stable: the Dirac-Klein-Gordon model, the Dirac-Proca model, the Klein-Gordon-Zakharov model, the $U(1)$ model of electroweak interactions and so on. In Part I, we study the electroweak standard model. We first prove the global nonlinear stability results for the $U(1)$ model, where we obtain uniform energy bounds (modulo a slow logarithm growth). Next we move to the full standard model, and get global stability results in some special cases. In Part II, we analyse a class of coupled wave-Klein-Gordon equations with critical nonlinearities, and prove the existence of global-in-time solutions which enjoy sharp pointwise decay property. Besides we study a class of Klein-Gordon equations with possibly vanishing mass, and prove sharp pointwise decay results which are uniform in its mass. In Part III, we mainly investigate the hyperbolic Fourier transform, and derive a new L^2-type estimate for waves.
Keywords : Hyperboloidal foliation method, wave-Klein-Gordon equations, standard model, mass parameter limit, critical nonlinearities, hyperbolic Fourier transform

Résultats d'existence pour des équations d'ondes non-linéaires

Résumé :
Cette thèse est consacrée à démontrer l'existence des solutions globales en temps pour certaines équations non-linéaires, y compris les équations d'ondes et de Klein-Gordon et les équations de Dirac. Pour les équations d'ondes qui sont basées sur la méthode de la feuilletage hyperboloïde, nous établissons plusieurs résultats de stabilité globale, explorons les comportements asymptotiques de ces solutions, et étudions comment les solutions sont affectées quand des paramètres de masse atteignent certaines limites. Comme application, nous prouvons que plusieurs modèles physiques sont globalement stables : le modèle de Dirac-Klein-Gordon, le modèle de Dirac-Proca, le modèle de Klein-Gordon-Zakharov, le modèle $U(1)$ des interactions électro-faibles etc.. Dans la partie I, nous étudions le modèle standard électro-faible. Nous prouvons la stabilité globale non-linéaire du modèle $U(1)$, dans laquelle nous obtenons des bornes uniformes (modulo une croissance lente de logarithme). Puis, pour le modèle standard complet, nous obtenons la stabilité globale dans certains cas spéciaux. Dans la partie II, nous analysons une sorte d'équations d'ondes et de Klein-Gordon avec non-linéalités critiques, et prouvons l'existence des solutions globales qui possède une propriété de décroissance pointue. D'ailleurs, nous étudions une sorte d'equation de Klein-Gordon avec une masse qui pourrait s'annuler, et prouvons la décroissance qui est uniforme en terme de sa masse. Dans la partie III, nous examinons principalement la transformation (hyperboloïde) de Fourier, et dérivons une nouvelle estimation de type-L^2 pour les ondes.
Mots clés : La méthode du feuilletage hyperboloïdal, des équations d'ondes et des équations de Klein–Gordon, modèle standard de la physique des particules, limite de paramètre de masse, non-linéarités critiques, la transformation (hyperboloïde) de Fourier