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Design of interval observers for uncertain distributed systems

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Par

Tatiana KHARKOVSKAIA

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**Design of interval observers for uncertain
distributed systems**

**Conception d'observateurs par intervalle pour les systèmes à paramètres
distribués avec incertitudes**

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Notations

- \mathbb{R} is the Euclidean space ($\mathbb{R}_+ = \{\tau \in \mathbb{R} : \tau \geq 0\}$), the Euclidean norm for a vector $x \in \mathbb{R}^n$ will be denoted as $|x|$.
- If X is a normed space with norm $\|\cdot\|_X$, $\Omega \subset \mathbb{R}^n$ is an open set for some $n \geq 1$ and $\phi : \Omega \rightarrow X$, define

$$\|\phi\|_{L^2(\Omega, X)}^2 = \int_{\Omega} \|\phi(s)\|_X^2 ds, \quad \|\phi\|_{L^\infty(\Omega, X)} = \operatorname{ess\,sup}_{s \in \Omega} \|\phi(s)\|_X.$$

- By $L^\infty(\Omega, X)$ and $L^2(\Omega, X)$ denote the set of functions $\Omega \rightarrow X$ with the properties $\|\cdot\|_{L^\infty(\Omega, X)} < +\infty$ and $\|\cdot\|_{L^2(\Omega, X)} < +\infty$, respectively.
- In special cases for $u : \mathbb{R}_+ \rightarrow \mathbb{R}^p$ the symbol $\|u\|_{[t_0, t_1]}$ denotes its L_∞ norm for $t \in [t_0, t_1]$, the set of all $u \in \mathbb{R}^p$ with the property $\|u\|_{[0, +\infty)} < \infty$ will be denoted as \mathcal{L}_∞^p .
- Denote $I = [0, \ell]$ for some $\ell > 0$, let $C^k(I, \mathbb{R})$ be the set of functions having continuous derivatives at least up to the order $k \geq 0$ on I .
- $\mathcal{C}_\tau^n = C([-\tau, 0], \mathbb{R}^n)$ is the set of continuous maps from $[-\tau, 0]$ into \mathbb{R}^n for $n \geq 1$; $\mathcal{C}_{\tau+}^n = \{y \in \mathcal{C}_\tau^n : y(s) \in \mathbb{R}_+^n, s \in [-\tau, 0]\}$; x_t is an element of \mathcal{C}_τ^n defined as $x_t(s) = x(t+s)$ for all $s \in [-\tau, 0]$.
- For any $q > 0$ and an open interval $I' \subset I$ define $W^{q, \infty}(I', \mathbb{R})$ as a subset of functions $y \in C^{q-1}(I', \mathbb{R})$ with an absolutely continuous $y^{(q-1)}$ and with $y^{(q)}$ essentially bounded on I' , $\|y\|_{W^{q, \infty}} = \sum_{i=0}^q \|y^{(i)}\|_{L^\infty(I', \mathbb{R})}$.
- By $H^q(I, \mathbb{R})$ with $q \geq 0$ denote the Sobolev space of functions with derivatives through order q in $L^2(I, \mathbb{R})$, and for $q < 0$ the corresponding dual spaces, while by $H_0^q(I, \mathbb{R})$ a closure of C^∞ functions having compact support in I with respect to the norm in $H^q(I, \mathbb{R})$.
- The symbols I_n , $E_{n \times m}$ and E_p denote the identity matrix with dimension $n \times n$, the matrix with all elements equal 1 with dimensions $n \times m$ and $p \times 1$, respectively.

- $a \mathcal{R} b$ corresponds to an elementwise relation $\mathcal{R} \in \{<, >, \leq, \geq\}$ (a and b are vectors or matrices): for example $a < b$ (vectors) means $\forall i : a_i < b_i$; for $\phi, \varphi \in \mathcal{C}_\tau$ the relation $\phi \mathcal{R} \varphi$ has to be understood elementwise for all domain of definition of the functions, *i.e.* $\phi(s) \mathcal{R} \varphi(s)$ for all $s \in [-\tau, 0]$.
- The relation $P < 0$ ($P > 0$) means that the matrix $P = P^T \in \mathbb{R}^{n \times n}$ is negative (positive) definite.
- For two functions $\phi_1, \phi_2 : I \rightarrow \mathbb{R}$ their relation $\phi_1 \leq \phi_2$ has to be understood as $\phi_1(x) \leq \phi_2(x)$ for almost all $x \in I$, the inner product is defined in a standard way:

$$\langle \phi_1, \phi_2 \rangle = \int_0^\ell \phi_1(x) \phi_2(x) dx.$$

General Introduction

For engineers, a wide variety of information is not directly obtained through measurement. Some parameters (constants of an electrical actuator, time delay in communication, etc.) or internal variables (robot's posture, torques applied to a robot, localization of a mobile, temperature at some point in a tank, etc.) are unknown or are not measured. Similarly, signals from sensors are often distorted and tainted by measurement noises. To control such processes, and to extract information conveyed by the signals, one often has to identify a model and estimate parameters or variables. The present work will focus on the state estimation.

Most engineering systems present dynamic behaviours, i.e. their input-output relations depend on time. When the time variation is considered in a continuous way, differential equations constitute the background for dynamic systems.

Input and output variables are not sufficient for the description of dynamic systems, and one has to consider additional descriptive variables, taking into consideration the memory of the dynamical system. These variables constitute the state, which is a set of variables expressed at the present time and which collects sufficient system's history to characterize his future. In general, this set is preferred to be chosen in a minimal way.

In differential equations, the notion of state is linked to the initial conditions one has to fix in order to determine the uniqueness of a solution. It can also be related to the variables representing some energy storage, such as the load in a capacitor or an inductor in electric devices. Some systems can be represented with a state vector belonging to some Euclidean space. This is the case for devices, the elements of which can be considered as concentrated at some points of the device (pointwise capacitor/inductor in an electric circuit, angular position in a rigid robot, etc.). Such dynamics can be studied via ordinary differential equations (ODEs). However, in many cases, the distribution of elements along the space cannot be neglected (long transmission lines, robots made with soft materials, heat propagation, etc.). In such cases, the energy is continuously varying along the space, and the state becomes a function, operating in some infinite-dimensional, non-Euclidean space. The underlying mathematics involve partial derivatives with regards to more than

one variable (time and space, generally), leading to partial differential equations (PDEs). Studying PDEs is more difficult than for ODEs for several reasons. Topology of functional spaces is rather tricky (for instance, contrarily to the finite-dimensional case, there is no general equivalence between norms). Boundary conditions can be defined in various ways (Dirichlet, Neumann, mixed). Also, the analysis strongly depend on the particular class of considered PDE (parabolic, hyperbolic and others).

Estimation issues in such infinite dimensional dynamical systems constitute the framework of this thesis.

More precisely, this work is devoted to the study of interval observers, which form a subclass of set-membership estimators and whose design is based on the monotone systems theory, to their applications in estimation and control of uncertain distributed systems, and to the analysis of their properties and restrictions.

In opposite to a conventional observer, which in the absence of measurement noise and uncertainties has to converge to the exact value of the state of the estimated system (it gives a pointwise estimation of the state), the interval observers evaluate at each time instant a set of admissible values for the state, consistently with the measured output (i.e. they provide an interval estimation). This idea of interval observer design has been proposed rather recently in Gouzé et al., 2000, but it has already received numerous extensions for various classes of dynamical models.

The advantages of the interval observers are that they are well adapted for observer design in highly uncertain systems (if the intervals of admissible values for unknown terms are given) and that they are capable to provide asymptotically rather tight bounds on the estimation accuracy, since the interval of admissible values for the state at each instant of time is evaluated. Besides, this technique takes advantage of an approximate knowledge of the initial condition and gives information on the unknown state of the system studied at any instant of time, whereas classical observers only provide a useful information asymptotically, that is, from an engineering point of view, for sufficiently large time-values.

Interval observers, presented in literature, mainly was constructed for dynamical systems, described by ordinary differential equations, for example, for linear time-invariant systems Gouzé et al., 2000; Mazenc and Bernard, 2010, 2011, linear parameter-varying systems (LPV) Chebotarev et al., 2015; Efimov et al., 2012, 2013e,f, nonlinear systems Raïssi et al., 2012, 2010, discrete-time systems Efimov et al., 2013a,b; Mazenc et al., 2013, 2014, 2012a, descriptor systems or systems with algebraic constraints Efimov et al., 2014.

Besides these numerous works on interval observers for ODEs, there are some papers, which extended this method for time-delay systems Efimov et al., 2016, 2013c, 2015a; Mazenc et al., 2012b, described by functional differential equations. For such models,

which are infinite-dimensional in contrast with ordinary differential equations (ODEs), the analysis and design are much more complicated and require specially developed concepts and algorithms Richard, 2003. For instance, the observability and methods for estimation for delayed systems with unknown inputs and nonlinearities are considered in Califano et al., 2011; Zheng et al., 2011, application of an algebraic approach for observer design in LPV time-delay systems is presented in Briat et al., 2011, an estimation problem for positive systems with time-varying unknown delays is studied in Rami et al., 2013. The existing solutions for the interval observers in the field Efimov et al., 2013c, 2015a; Mazenc et al., 2012b; Polyakov et al., 2013 are based on the delay-independent conditions of positivity from Ait Rami, 2009; Haddad and Chellaboina, 2004.

Some results on interval observer design for uncertain time-varying delay can be found in Efimov et al., 2013c; Rami et al., 2013. In Efimov et al., 2016 the delay-dependent conditions on positivity are introduced for the case with equal delay in the state and in the output, that may correspond to a delay-free system with delayed measurements.

To fill this gap, in Chapter 4 a new approach is introduced to cope with the above-mentioned obstacle, when the measurement delays are different from state delays.

Moreover, this thesis goal is to extend the interval observer framework to the infinite-dimensional systems, by considering not only systems with delays, but also distributed in space ones, which described by partial differential equations (PDE).

Sound, heat, electrostatics, electrodynamics, fluid flow, elasticity, or quantum mechanics, as well as the models of other physical phenomena, can be formalized similarly in terms of PDEs, whose distributed nature introduces an additional level of intricacy. That is why control and estimation of PDEs is a very popular direction of research nowadays Barje et al., 2013; Bredies et al., 2013; Demetriou, 2004; Hasan et al., 2016; Kamran and Drakunov, 2015; Krstic, 2009; Meurer, 2013; Nguyen, 2008; Russell, 2003; Smyshlyaev and Krstic, 2005; Smyshlyaev and Krstic, 2010. In this class of models, where the system state is a function of the space at each instant of time, the problem of its explicit measurement is natural, since only pointwise and discrete space measurements are realizable by a sensor Jorgensen et al., 1984; Vande Wouwer et al., 2000. Frequently, in order to design a state estimator, the finite-dimensional approximation approach is used Alvarez and Stephanopoulos, 1982; Dochain, 2000; Hagen and I., 2003; Vande Wouwer and Zeitz, 2002, then the observation problem is addressed with the well-known tools available for finite-dimensional systems, while the convergence assessment has to be performed with respect to the solutions of the original distributed system. Analysis and design in the original distributed coordinates are more complicated, but also attract attention of many researchers Ahmed-Ali et al., 2015; Fridman and Bar Am, 2013; Fridman, 2013; Hidayat et al., 2011; Liu and Fridman, 2012; Schaum et al., 2014; Selivanov and Fridman, 2018;

Smyshlyaev and Krstic, 2010. In Pisano and Orlov, 2017 a stabilizing control design with a proportional-discontinuous feedback is proposed for a parabolic PDE with pointwise collocated sensing and actuation, and with in-domain distributed disturbances. The work Wang et al., 2018 presents a Luenberger-type observer-based distributed control with non-collocated sensors and actuators.

Nevertheless, as far as the author knows, there was no work on interval observers for PDEs. In chapter 2 of the present thesis, a method to design an interval observer was developed for a parabolic PDE with Dirichlet boundary conditions, which is based on a finite-element Galerkin approximation. This method is also called early-lumping approach for a distributed system, when the first step is the discretization of the given system to a lumped one, described by the set of ODEs, and the second step is to apply already existing solutions of observer design.

Using the discretization error estimates from Wheeler, 1973, the enveloping interval for solutions of PDE was evaluated. An interesting feature of this approach is that being applied to a nonlinear PDE, assuming that all nonlinearities are bounded and treated as perturbations, then the proposed interval observer is linear and can be easily implemented providing bounds on solutions of the originally nonlinear PDE (under the hypothesis that these solutions exist).

Beside that, based on constructed observer for the lumped system, an output stabilizing control was developed in Section 2.3. The proposed control strategy disposes a similar feature, since it is designed for a finite-dimensional model, but guaranteeing boundedness of trajectories for an uncertain distributed dynamics. These results were published in Kharkovskaia et al., 2016, 2018b.

At Chapter 3 an interval observer design is proposed for the same class of system as in Chapter 2, described by a parabolic PDE with non-zero Dirichlet boundary conditions, but it is not based on any approximation, and the observer has the same distributed nature. Using the conditions of positivity of solutions of parabolic PDEs presented in Nguyen and Coron, 2016, an interval observer is constructed governed by PDE, whose estimation error dynamics (also distributed) is positive. The stability analysis from Fridman and Blighovsky, 2012 is extended to the considered scenario with non-zero measurement noise and boundary conditions. The conditions under which the distributed parameter system possesses the input-to-state stability property is found, where boundary conditions influence the external disturbance and the initial conditions as well. Also the main restriction for stability property is imposed by parameters of the system and by the intervals between the measurement points. To overcome this restriction the feedback controller design is developed in Section 3.3 using the received interval estimates. It is assumed that the control is spatially distributed influencing the system dynamics

through piecewise constant shape functions, such a hypothesis is introduced to respect the implementation feasibility of the designed control law, since infinitesimal in space variations of the actuator signal cannot be realized in practice. These methods were presented in Kharkovskaia et al., 2017, 2020.

The outline of this thesis is as follows. Chapter 1 presents some preliminaries on interval analysis and displays the main approaches of interval observer design, which were used when writing this thesis, for continuous-time linear systems in Section 1.2 and for time-delay systems in Section 1.4. Preliminaries on partial differential equations are shown in Section 1.5 with some existing results on estimation and control for PDEs. Chapter 2 is devoted to interval observers for PDEs using finite-element approximations with additional control in Section 2.3. Chapter 3 presents a PDE interval observer for uncertain non-homogeneous heat equations, which has the same distributed nature as the considered system. And again, Section 3.3 shows an output stabilizing control. Chapter 4 presents the interval estimation for second-order delay differential equations with delayed measurements and uncertainties. Finally, concluding remarks and a discussion on future works are given in Chapter 5.

Theoretical framework

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This chapter presents basic concepts of the interval observer construction, main restrictions and conditions for the design, notation and preliminaries for partial differential equations (PDEs) and its analysis, as well as for systems with delays.

The idea of interval observer design has been proposed in Gouzé et al., 2000. “Given bounds on the uncertainties in the model, we are looking for dynamic bounds on the estimation of the variables. Because bounds are usually given by intervals, we shall build what we call ‘interval observers’.”

Consider a dynamical system:

$$\begin{aligned}\dot{x}(t) &= f(t, x(t)), & x(0) &= x_0 \\ y(t) &= h(t, x(t)),\end{aligned}\tag{1.1}$$

where the state $x \in X \subset \mathbb{R}^n$ and $y \in \mathbb{R}^p$ is the output.

Definition 1.1. A pair of systems:

$$\begin{aligned}\dot{z}^-(t) &= g^-(t, z^-(t), y(t)) \\ x^-(t) &= l^-(t, z^-(t), z^+(t), y(t))\end{aligned}\tag{1.2}$$

and

$$\begin{aligned} \dot{z}^+(t) &= g^+(t, z^+(t), y(t)) \\ x^+(t) &= l^+(t, z^-(t), z^+(t), y(t)) \end{aligned} \quad (1.3)$$

is an interval estimator for system S_0 if for any (relatively) compact set $X_0 \subset X$, there exist $(z_-(0), z_+(0))$ such that the coupled system (1.1), (1.2), (1.3) verifies:

$$x^-(t) \leq x(t) \leq x^+(t), \quad \forall t \geq 0$$

for any initial condition $x(0) \in X_0$.

In the case of uncertainties this supposed interval estimation becomes more attainable: an observer can be constructed such that using the input-output information it evaluates the set of admissible values (interval) for the state at each instant of time. The interval width is proportional to the size of the model uncertainty (it has to be minimized by tuning the observer parameters).

Starting in 2000 with the work of Gouzé et al., 2000, the interval analysis for constructing an estimator for uncertain and nonlinear systems becomes more popular.

For example, the paper Jaulin, 2002 studies the application of interval analysis of Moore, 1966; Moore et al., 2009 to state estimation of nonlinear continuous-time systems in a bounded-error context, and checked on a famous Lotka-Volterra predator-prey model.

In Bernard and Gouzé, 2004 the authors extended the proposed method for the biological reactor, using a set of interval observers and closed loop mass balance observers to construct interval ones. They run simultaneously a broad set of interval observers and select the best ones, therefore it allows to cope with large uncertainties, but realistic for biotechnological processes.

1.1 Interval analysis

Given a matrix $A \in \mathbb{R}^{m \times n}$, define $A^+ = \max\{0, A\}$, $A^- = A^+ - A$ (similarly for vectors) and $|A| = A^+ + A^-$.

Lemma 1.1. Efimov et al., 2012 *Let $x \in \mathbb{R}^n$ be a vector variable, $\underline{x} \leq x \leq \bar{x}$ for some $\underline{x}, \bar{x} \in \mathbb{R}^n$.*

(1) *If $A \in \mathbb{R}^{m \times n}$ is a constant matrix, then*

$$A^+ \underline{x} - A^- \bar{x} \leq Ax \leq A^+ \bar{x} - A^- \underline{x}. \quad (1.4)$$

(2) If $A \in \mathbb{R}^{m \times n}$ is a matrix variable and $\underline{A} \leq A \leq \bar{A}$ for some $\underline{A}, \bar{A} \in \mathbb{R}^{m \times n}$, then

$$\begin{aligned} \underline{A}^+ \underline{x}^+ - \bar{A}^+ \underline{x}^- - \underline{A}^- \bar{x}^+ + \bar{A}^- \bar{x}^- &\leq Ax \\ &\leq \bar{A}^+ \bar{x}^+ - \underline{A}^+ \bar{x}^- - \bar{A}^- \underline{x}^+ + \underline{A}^- \underline{x}^-. \end{aligned} \quad (1.5)$$

Furthermore, if $-\bar{A} = \underline{A} \leq 0 \leq \bar{A}$, then the inequality (1.5) can be simplified: $-\bar{A}(\bar{x}^+ + \underline{x}^-) \leq Ax \leq \bar{A}(\bar{x}^+ + \underline{x}^-)$.

1.2 Nonnegative continuous-time linear systems

A matrix $A \in \mathbb{R}^{n \times n}$ is called Hurwitz if all its eigenvalues have negative real parts, and it is called Metzler if all its elements outside the main diagonal are nonnegative. Any solution of the linear system

$$\begin{aligned} \dot{x} &= Ax + B\omega(t), \quad \omega: \mathbb{R}_+ \rightarrow \mathbb{R}_+^q, \\ y &= Cx + D\omega(t), \end{aligned} \quad (1.6)$$

with $x \in \mathbb{R}^n$, $y \in \mathbb{R}^p$ and a Metzler matrix $A \in \mathbb{R}^{n \times n}$, is elementwise nonnegative for all $t \geq 0$ provided that $x(0) \geq 0$ and $B \in \mathbb{R}_+^{n \times q}$ Farina and Rinaldi, 2000; Kaczorek, 2002; Smith, 1995. The output solution $y(t)$ is nonnegative if $C \in \mathbb{R}_+^{p \times n}$ and $D \in \mathbb{R}_+^{p \times q}$. Such a dynamical system is called cooperative (monotone) or nonnegative if only initial conditions in \mathbb{R}_+^n are considered Farina and Rinaldi, 2000; Kaczorek, 2002; Smith, 1995.

For a Metzler matrix $A \in \mathbb{R}^{n \times n}$ its stability can be checked verifying a Linear Programming (LP) problem $A^T \lambda < 0$ for some $\lambda \in \mathbb{R}_+^n \setminus \{0\}$, or the Lyapunov matrix equation $A^T P + PA < 0$ for a diagonal matrix $P \in \mathbb{R}^{n \times n}$, $P > 0$ (in the general case the matrix P should not be diagonal). The L_1 and L_∞ gains for nonnegative systems (1.6) have been studied in Briat et al., 2011; Ebihara et al., 2011, for this kind of systems these gains are interrelated.

Lemma 1.2. Briat et al., 2011; Ebihara et al., 2011 *Let the system (1.6) be nonnegative (i.e. A is Metzler, $B \geq 0$, $C \geq 0$ and $D \geq 0$), then it is asymptotically stable if and only if there exist $\lambda \in \mathbb{R}_+^n \setminus \{0\}$ and a scalar $\gamma > 0$ such that the following LP problem is feasible:*

$$\begin{bmatrix} A^T \lambda + C^T E_p \\ B^T \lambda - \gamma E_q + D^T E_p \end{bmatrix} < 0.$$

Moreover, in this case L_1 gain of the operator $\omega \rightarrow y$ is lower than γ .

Lemma 1.3. Briat et al., 2011; Ebihara et al., 2011 *Let the system (1.6) be cooperative (i.e. A is Metzler, $B \geq 0$, $C \geq 0$ and $D \geq 0$), then it is asymptotically stable if and only if there exist*

$\lambda \in \mathbb{R}_+^n \setminus \{0\}$ and a scalar $\gamma > 0$ such that the following LP problem is feasible:

$$\begin{bmatrix} A\lambda + BE_q \\ C\lambda - \gamma E_p + DE_q \end{bmatrix} < 0.$$

Moreover, in this case L_∞ gain of the transfer $\omega \rightarrow y$ is lower than γ .

The conventional results and definitions on the L_2/L_∞ stability for linear systems can be found in Khalil, 2002.

1.3 Interval observer for continuous-time linear systems

Consider the following system

$$\dot{x}(t) = Ax(t) + d(t), \quad y(t) = Cx(t) + v(t), \quad t \in \mathbb{R}_+ \quad (1.7)$$

where $x(t) \in \mathbb{R}^n$ is the state, $y(t) \in \mathbb{R}^p$ is the output, $d(t) \in \mathbb{R}^n$ is the disturbance, $d \in \mathcal{L}_\infty^n$; $v(t) \in \mathbb{R}^p$ is the measurement noise, $v \in \mathcal{L}_\infty^p$; the matrices A, C have appropriate dimensions. This model has three sources of uncertainty: initial conditions for $x(0)$, instant values of d and v . It is assumed that all these uncertain factors belong to known intervals.

Assumption 1.1. Let $x(0) \in [\underline{x}_0, \bar{x}_0]$ for some known $\underline{x}_0, \bar{x}_0 \in \mathbb{R}^n$, let also two functions $\underline{d}, \bar{d} \in \mathcal{L}_\infty^n$ and a constant $V > 0$ be given such that

$$\underline{d}(t) \leq d(t) \leq \bar{d}(t), \quad |v(t)| \leq V \quad \forall t \geq 0.$$

Thus, by this assumption three intervals $[\underline{x}_0, \bar{x}_0]$, $[\underline{d}(t), \bar{d}(t)]$ and $[-V, V]$ determine uncertainty of (0), $d(t)$ and $v(t)$ respectively. It is required to calculate two estimates $\underline{x}, \bar{x} \in \mathcal{L}_\infty^n$, using the available information on these intervals and $y(t)$, such that

$$\underline{x}(t) \leq x(t) \leq \bar{x}(t) \quad \forall t \geq 0. \quad (1.8)$$

Interval observer is a solution to this problem:

$$\begin{aligned} \dot{\underline{x}}(t) &= A\underline{x}(t) + L[y(t) - C\underline{x}(t)] - |L|E_p V + \underline{d}(t), \\ \dot{\bar{x}}(t) &= A\bar{x}(t) + L[y(t) - C\bar{x}(t)] + |L|E_p V + \bar{d}(t), \\ \underline{x}(0) &= \underline{x}_0, \bar{x}(0) = \bar{x}_0, \end{aligned} \quad (1.9)$$

where $L \in \mathbb{R}^{n \times p}$ is the observer gain to be designed. The conditions to satisfy for L are given below.

Theorem 1.1. Gouzé et al., 2000 *Let Assumption 1.1 hold and $x \in \mathcal{L}_\infty^n$, then in the system (1.7) with the interval observer (1.9) the relations (1.8) are satisfied provided that the matrix $A - LC$ is Metzler. In addition, $\underline{x}, \bar{x} \in \mathcal{L}_\infty^n$ if $A - LC$ is Hurwitz.*

To understand the main ideas behind this result (that is very representative for all kinds of interval observers) consider its proof.

Proof 1.1. Define two estimation errors

$$\underline{e}(t) = x(t) - \underline{x}(t), \quad \bar{e}(t) = \bar{x}(t) - x(t),$$

which yield differential equations:

$$\begin{aligned} \dot{\underline{e}}(t) &= [A - LC]\underline{e}(t) + Lv(t) + |L|E_p V + d(t) - \underline{d}(t), \\ \dot{\bar{e}}(t) &= [A - LC]\bar{e}(t) - Lv(t) + |L|E_p V + \bar{d}(t) - d(t). \end{aligned}$$

By Assumption 1.1, for all $t \geq 0$

$$|L|E_p V \pm Lv(t) \geq 0, \quad d(t) - \underline{d}(t), \quad \bar{d}(t) - d(t) \geq 0.$$

If $A - LC$ is a Metzler matrix, since all inputs of $\underline{e}(t), \bar{e}(t)$ are positive and $\underline{e}(0) \geq 0, \bar{e}(0) \geq 0$, then $\underline{e}(t) \geq 0, \bar{e}(t) \geq 0$ for all $t \geq 0$ Farina and Rinaldi, 2000; Smith, 1995. The property (1.8) follows from these relations. If $A - LC$ is Hurwitz, since all inputs of $\underline{e}(t), \bar{e}(t)$ are bounded, then $\underline{e}, \bar{e} \in \mathcal{L}_\infty^n$ and boundedness of \underline{x}, \bar{x} is implied by boundedness of x .

Thus, from this proof we can conclude that the idea for design of an interval observer is to guarantee nonnegativity of the estimation error dynamics. The observer gain L has to be designed in a way that the matrix $A - LC$ would be Metzler and Hurwitz. In addition, in order to optimize the width of estimated interval $[\underline{x}(t), \bar{x}(t)]$ the problem of L_1 or L_∞ optimization of that gain value can be posed. Using lemmas 1.2 and 1.3 this problem can be formulated as a LP computational procedure, for example, for L_1 optimization it is necessary to find $\lambda \in \mathbb{R}^n, w \in \mathbb{R}^p$ and a diagonal matrix $M \in \mathbb{R}^{n \times n}$ such that

$$\begin{aligned} \begin{bmatrix} A^T \lambda + C^T w + E_n \\ \lambda - \gamma E_n \end{bmatrix} &< 0, \\ \lambda > 0, M &\geq 0, A^T \lambda + C^T w + M \lambda \geq 0, \end{aligned} \tag{1.10}$$

then $w = L^T \lambda$ Chebotarev et al., 2015.

Formulation (1.3) provides an effective computational tool to design interval observers, but it is only sufficient condition and in some cases this LP problem may have no solution,

but it does not imply that it is not possible to design an interval observer. Roughly speaking in this case it is not possible to find L such that $A - LC$ is simultaneously Metzler and Hurwitz. It is well known fact that Hurwitz property of a matrix is preserved under a similarity transformation of coordinates, then to overcome the issue it is possible to design the gain L such that the matrix $A - LC$ is Hurwitz and next to find a nonsingular matrix $S \in \mathbb{R}^{n \times n}$ such that in the new coordinates $z = Sx$ the state matrix $D = S(A - LC)S^{-1}$ is Metzler (it is Hurwitz by construction). The conditions of existence of such a real transformation matrix S are given in the following lemma.

Lemma 1.4. Raïssi et al., 2012 *Given the matrices $A \in \mathbb{R}^{n \times n}$, $D \in \mathbb{R}^{n \times n}$ and $C \in \mathbb{R}^{p \times n}$. If there is a matrix $L \in \mathbb{R}^{n \times p}$ such that the matrices $A - LC$ and D have the same eigenvalues, then there is a matrix $S \in \mathbb{R}^{n \times n}$ such that $D = S(A - LC)S^{-1}$ provide that the pairs $(A - LC, \chi_1)$ and (D, χ_2) are observable for some $\chi_1, \chi_2 \in \mathbb{R}^{1 \times n}$.*

Note that (1.7) can be rewritten as follows:

$$\dot{x}(t) = (A - LC)x(t) + Ly(t) - Lv(t) + d(t).$$

Under conditions of this lemma and in the new coordinates $z = Sx$ the system (1.7) takes the form:

$$\dot{z}(t) = Dz(t) + SLy(t) + \delta(t), \quad \delta(t) = S[d(t) - Lv(t)]. \quad (1.11)$$

And using Lemma 1.1 we obtain that

$$\underline{\delta}(t) \leq \delta(t) \leq \bar{\delta}(t),$$

where $\underline{\delta}(t) = S^+ \underline{d}(t) - S^- \bar{d}(t) - |SL|E_p V$ and $\bar{\delta}(t) = S^+ \bar{d}(t) - S^- \underline{d}(t) + |SL|E_p V$. Next, for the system (1.11) all conditions of Theorem 1.1 are satisfied and an interval observer similar to (1.9) can be designed:

$$\begin{aligned} \dot{\underline{z}}(t) &= D\underline{z}(t) + LSy(t) + \underline{\delta}(t), \\ \dot{\bar{z}}(t) &= D\bar{z}(t) + LSy(t) + \bar{\delta}(t), \\ \underline{z}(0) &= S^+ \underline{x}_0 - S^- \bar{x}_0, \quad \bar{z}(0) = S^+ \bar{x}_0 - S^- \underline{x}_0, \\ \underline{x}(t) &= (S^{-1})^+ \underline{z}(t) - (S^{-1})^- \bar{z}(t), \\ \bar{x}(t) &= (S^{-1})^+ \bar{z}(t) - (S^{-1})^- \underline{z}(t), \end{aligned} \quad (1.12)$$

where the relations (1.4) are used to calculate the initial conditions for \underline{z}, \bar{z} and the estimates \underline{x}, \bar{x} . It is easy to show that in (1.12) the inclusion (1.8) is satisfied and $\underline{x}, \bar{x} \in \mathcal{L}_\infty^n$. If conditions of Lemma 1.4 are not satisfied, then it is possible also to ask that transformation

to new coordinates be time-varying as in Mazenc and Bernard, 2011 or Combastel, 2013. The interval observer then has a form similar to (1.12) with $S = S(t)$, but its realisation needs more computations than for (1.12) due to time-varying transformation. Another way to have a Metzler matrix for the interval observer was presented in Cacace et al., 2015. It is shown there, that in general any system may be embedded in its internal positive representation, which has dimension $2n$ (then interval observer has dimension $4n$). For example, a matrix can be decomposed as a difference of Metzler and nonnegative matrices:

$$A - LC = (A - LC)_{\setminus} + (A - LC)_{\times} = (A - LC)_{\setminus} + (A - LC)_{\times}^+ - (A - LC)_{\times}^-,$$

where $(A - LC)_{\setminus}$ is the diagonal matrix composed by elements of $(A - LC)$ on the main diagonal and $(A - LC)_{\times} = A - LC - (A - LC)_{\setminus}$. Then the following interval observer can be proposed for (1.7):

$$\begin{aligned} \dot{\underline{x}}(t) &= [(A - LC)_{\setminus} + (A - LC)_{\times}^+] \underline{x}(t) - (A - LC)_{\times}^- \bar{x}(t) + Ly(t) - |L|E_p V + \underline{d}(t), \\ \dot{\bar{x}}(t) &= [(A - LC)_{\setminus} + (A - LC)_{\times}^+] \bar{x}(t) - (A - LC)_{\times}^- \underline{x}(t) + Ly(t) + |L|E_p V + \bar{d}(t), \\ \underline{x}(0) &= \underline{x}_0, \bar{x}(0) = \bar{x}_0, \end{aligned} \quad (1.13)$$

Theorem 1.2. Mazenc and Bernard, 2010 *Let Assumption 1.1 hold and $x \in \mathcal{L}_{\infty}^n$, then in the system (1.7) with the interval observer (1.13) the relations (1.8) are satisfied. In addition, $\underline{x}, \bar{x} \in \mathcal{L}_{\infty}^n$ if*

$$R = \begin{bmatrix} (A - LC)_{\setminus} + (A - LC)_{\times}^+ & (A - LC)_{\times}^- \\ (A - LC)_{\times}^- & (A - LC)_{\setminus} + (A - LC)_{\times}^+ \end{bmatrix}$$

is Hurwitz.

As we can conclude the requirement on Metzler property of the matrix $A - LC$ is completely avoided, and the main difficulty in application of the last theorem consists in finding conditions under which the Metzler matrix R is Hurwitz. It is also difficult to formulate some LMIs to find L in this setting.

The main approaches to design interval observers for LTI systems are presented. If it is possible to find L such that $A - LC$ is Hurwitz and Metzler (looking for a solution of LP problem (1.3)), then the interval observer (1.9) is a solution. If such a gain L does not exist, then different transformations of coordinates from Lemma 1.4 and Combastel, 2013; Mazenc and Bernard, 2011 can be tested. In this case L also can be found as a solution of LP problem or LMIs, and the interval observer (1.12) can be applied. An alternative solution (1.13) also can be tested. Note that developed observers (1.9), (1.12) can be easily applied for some nonlinear systems, where all nonlinear terms are contained in $d(t) = d(t, u(t), x(t))$

in a manner that Assumption 1.1 is satisfied for some $\underline{d}, \bar{d} \in \mathcal{L}_\infty^n$, $\underline{d}(t) = \underline{d}(t, u(t), y(t), V)$ and $\bar{d}(t) = \bar{d}(t, u(t), y(t), V)$.

By this section, the basic approaches to design interval observer for linear continuous time system was presented, and they will be used to construct an observer for PDEs after the approximation procedure.

1.4 Interval observers for time-delay systems

Delays appear in many control systems at different levels of conception: at the state dynamics, in the control channel, or in the measurement signals. Sensors, actuators, and communication networks that are involved in the feedback loops usually introduce delays. Arising in differential equations, they may cause instability and oscillation of the solutions of the considered system.

This section presents main results on existing solutions on constructing interval observers for time-delay systems, then in Chapter 4 a new approach is introduced for the case, when the measurement delays are different from state delays.

Consider a linear system with constant delays

$$\dot{x}(t) = A_0 x(t) + \sum_{i=1}^N A_i x(t - \tau_i) + \omega(t), \quad (1.14)$$

where $x(t) \in \mathbb{R}^n$ is the state, $x_t \in \mathcal{C}_\tau^n$ for $\tau = \max_{1 \leq i \leq N} \tau_i$ where $\tau_i \in \mathbb{R}_+$ are the delays ($\mathcal{C}_\tau = C([- \tau, 0], \mathbb{R})$ is the set of continuous maps from $[- \tau, 0]$ into \mathbb{R} , $\mathcal{C}_{\tau+} = \{y \in \mathcal{C}_\tau : y(s) \in \mathbb{R}_+, s \in [- \tau, 0]\}$); a piecewise continuous function $\omega \in \mathcal{L}_\infty^n$ is the input; the constant matrices A_i , $i = \overline{0, N}$ have appropriate dimensions. The system (4.1) is called cooperative or nonnegative Haddad and Chellaboina, 2004 if it admits $x(t) \in \mathbb{R}_+^n$ for all $t \geq t_0$ provided that $x_{t_0} \in \mathcal{C}_{\tau+}^n$ and $\omega : \mathbb{R} \rightarrow \mathbb{R}_+^n$.

Lemma 1.5. Dambrine and Richard, 1993; Haddad and Chellaboina, 2004 *The system (1.14) is nonnegative for all $\tau \in \mathbb{R}_+$ iff A_0 is Metzler and A_i , $i = \overline{1, N}$ are nonnegative matrices.*

Consider the system (1.14) with an output $y \in \mathbb{R}^p$ subject to a bounded noise $v \in \mathcal{L}_\infty^p$:

$$y = Cx, \quad \psi = y + v(t), \quad (1.15)$$

where $C \in \mathbb{R}^{p \times n}$. Below the relation $a \leq b$ for $a, b \in \mathcal{C}_\tau^n$ is understood in the sense that $a(\theta) \leq b(\theta)$ for all $\theta \in [- \tau, 0]$.

Assumption 1.2. Let $x \in \mathcal{L}_\infty^n$ with $\underline{x}_0 \leq x_{t_0} \leq \bar{x}_0$ for some $\underline{x}_0, \bar{x}_0 \in \mathcal{C}_\tau^n$; $\|v\| \leq V$ for a given $V > 0$; and $\underline{\omega}(t) \leq \omega(t) \leq \bar{\omega}(t)$ for all $t \geq t_0$ for some known $\underline{\omega}, \bar{\omega} \in \mathcal{L}_\infty^n$.

In this assumption it is supposed that the state of the system (1.14) is bounded with an unknown upper bound, but with a specified admissible set for initial conditions $[\underline{x}_0, \bar{x}_0]$. Uncertainty of the system is collected in the external input ω with known bounds on the incertitude $\underline{\omega}, \bar{\omega}$, the measurement noise v and the interval of initial conditions $[\underline{x}_0, \bar{x}_0]$.

As we see above, interval observers have an enlarged dimension (the examples given above have $2n$ variables to estimate n states). Thus, design of reduced order interval observers is of a big importance for applications. A reduced order interval observer for time-delay system (1.14), (1.15) has been proposed in Efimov et al., 2013c,d, those ideas are explained below.

For the system (1.14), (1.15) there exists a nonsingular matrix $S \in \mathbb{R}^{n \times n}$ such that $x = S[y^T z^T]^T$ for an auxiliary variable $z \in \mathbb{R}^{n-p}$ (define $S^{-1} = [C^T Z^T]^T$ for a matrix $Z \in \mathbb{R}^{(n-p) \times n}$), then

$$\begin{aligned} \dot{y}(t) &= R_1 y(t) + R_2 z(t) + \sum_{i=1}^N [D_{1i} y(t - \tau_i) + D_{2i} z(t - \tau_i)] + C \omega(t), \\ \dot{z}(t) &= R_3 y(t) + R_4 z(t) + \sum_{i=1}^N [D_{3i} y(t - \tau_i) + D_{4i} z(t - \tau_i)] + Z \omega(t), \end{aligned} \quad (1.16)$$

for some matrices R_k, D_{ki} , $k = \overline{1, 4}$, $i = \overline{1, N}$ of appropriate dimensions. Introducing a new variable $w = z - Ky = Ux$ for a matrix $K \in \mathbb{R}^{(n-p) \times p}$ with $U = Z - KC$, from (1.16) the following equation is obtained

$$\begin{aligned} \dot{w}(t) &= G_0 \psi(t) + M_0 w(t) + \sum_{i=1}^N [G_i \psi(t - \tau_i) + M_i w(t - \tau_i)] + \beta(t), \\ \beta(t) &= U \omega(t) - G_0 v(t) - \sum_{i=1}^N G_i v(t - \tau_i), \end{aligned} \quad (1.17)$$

where $\psi(t)$ is defined in (1.15), $G_0 = R_3 - KR_1 + (R_4 - KR_2)K$, $M_0 = R_4 - KR_2$, and $G_i = D_{3i} - KD_{1i} + \{D_{4i} - KD_{2i}\}K$, $M_i = D_{4i} - KD_{2i}$ for $i = \overline{1, N}$. Under Assumption 1.2 and using

the relations (1.4) the following inequalities follow:

$$\begin{aligned}\underline{\beta}(t) &\leq \beta(t) \leq \overline{\beta}(t), \\ \underline{\beta}(t) &= U^+ \underline{\omega}(t) - U^- \overline{\omega}(t) - \sum_{i=0}^N |G_i| E_p V, \\ \overline{\beta}(t) &= U^+ \overline{\omega}(t) - U^- \underline{\omega}(t) + \sum_{i=0}^N |G_i| E_p V.\end{aligned}$$

Then the next interval reduced-order observer can be proposed for (1.14):

$$\begin{aligned}\dot{\underline{w}}(t) &= G_0 \psi(t) + M_0 \underline{w}(t) + \sum_{i=1}^N [G_i \psi(t - \tau_i) + M_i \underline{w}(t - \tau_i)] + \underline{\beta}(t), \\ \dot{\overline{w}}(t) &= G_0 \psi(t) + M_0 \overline{w}(t) + \sum_{i=1}^N [G_i \psi(t - \tau_i) + M_i \overline{w}(t - \tau_i)] + \overline{\beta}(t), \\ \underline{w}_0 &= U^+ \underline{x}_0 - U^- \overline{x}_0, \quad \overline{w}_0 = U^+ \overline{x}_0 - U^- \underline{x}_0.\end{aligned}\tag{1.18}$$

The applicability conditions for (1.18) are given below.

Theorem 1.3. Efimov et al., 2013c *Let Assumption 1.2 be satisfied and the matrices $M_0, M_i, i = \overline{1, N}$ form an asymptotically stable cooperative system. Then $\underline{x}, \overline{x} \in \mathcal{L}_\infty^n$ and*

$$\underline{x}(t) \leq x(t) \leq \overline{x}(t) \quad \forall t \geq 0,$$

where

$$\begin{aligned}\underline{x}(t) &= S^+ [\underline{y}(t)^T \underline{z}(t)^T]^T - S^- [\overline{y}(t)^T \overline{z}(t)^T]^T, \\ \overline{x}(t) &= S^+ [\overline{y}(t)^T \overline{z}(t)^T]^T - S^- [\underline{y}(t)^T \underline{z}(t)^T]^T,\end{aligned}\tag{1.19}$$

$$\begin{aligned}\underline{y}(t) &= \psi(t) - V, \quad \overline{y}(t) = \psi(t) + V, \\ \underline{z}(t) &= \underline{w}(t) + K^+ \underline{y} - K^- \overline{y}, \quad \overline{z}(t) = \overline{w}(t) + K^+ \overline{y} - K^- \underline{y}.\end{aligned}\tag{1.20}$$

The main condition of Theorem 1.3 is rather straightforward: the matrices $M_0, M_i, i = \overline{1, N}$ have to form a stable cooperative system. It is a standard LMI problem to find a matrix K such that the system composed by $M_0, M_i, i = \overline{1, N}$ is stable, but to find a matrix K making the system stable and cooperative simultaneously could be more complicated. However, the advantage of Theorem 1.3 is that its main condition can be reformulated

using LMIs following the idea of Rami et al., 2007.

Proposition 1.1. Efimov et al., 2013c *Let there exist $\varsigma \in \mathbb{R}_+$, $p \in \mathbb{R}_+^{n-p}$, $q \in \mathbb{R}_+^{n-p}$ and $B \in \mathbb{R}^{(n-p) \times p}$ such that the following LMIs are satisfied:*

$$\begin{aligned} p^\top \Pi_0 - E_{n-p}^\top B \Pi_1 + q^\top &\leq 0, \quad p > 0, \quad q > 0; \\ \text{diag}[p]R_4 - BR_2 + \varsigma I_{n-p} &\geq 0, \quad \varsigma > 0; \\ \text{diag}[p]D_{4i} - BD_{2i} &\geq 0, \quad i = \overline{1, N}; \\ \Pi_0 = R_4 + \sum_{i=1}^N D_{4i}, \quad \Pi_1 = R_2 + \sum_{i=1}^N D_{2i}, \end{aligned}$$

then $K = \text{diag}[p]^{-1}B$ and the matrices $M_0 = R_4 - KR_2$, $M_i = D_{4i} - KD_{2i}$, $i = \overline{1, N}$ represent a stable cooperative system in (1.18).

If these LMIs are not satisfied, the assumption that the matrix M_0 is Metzler and the matrices M_i , $i = \overline{1, N}$ are nonnegative can be relaxed using Lemma 1.4 Efimov et al., 2013c.

1.5 Partial differential equations

In this section, partial differential equations of parabolic type and its well-posedness are considered. Then, we consider a stabilizing controller design for a one-dimensional semilinear diffusion equation, where the measurements of the state are taken in a finite number of fixed sampling spatial points. The diffusion coefficient and the nonlinearity may be unknown, but they satisfy some bounds. The considered controller can be implemented by a finite number of stationary sensors and actuators. In chapters 2 and 3 some of these results are used for analysis and comparison with proposed methods.

Parabolic PDEs

Consider the following semilinear scalar diffusion equation

$$\begin{aligned} z_t(x, t) &= \frac{\partial}{\partial x} [a(x)z_x(x, t)] + \phi(z(x, t), x, t)z(x, t) + u(x, t), \\ t &\geq t_0, \quad x \in [0, \ell], \ell > 0 \end{aligned} \tag{1.21}$$

with Dirichlet boundary conditions

$$z(0, t) = z(\ell, t) = 0, \tag{1.22}$$

or with mixed boundary conditions

$$z_x(0, t) = \gamma z(0, t), z(\ell, t) = 0, \gamma \geq 0, \quad (1.23)$$

where subindexes denote the corresponding partial derivatives and γ may be unknown. $u(x, t)$ is the control input. The functions a and ϕ are of class C^1 and may be unknown. These functions satisfy the inequalities

$$a \geq a_0 > 0, \phi_m \geq \phi \geq \phi_m,$$

where a_0, ϕ_m and ϕ_M are known bounds.

It is well-known that the open-loop system (1.21) under the above boundary conditions may become unstable if ϕ_M is big enough (see Curtain and Zwart, 1995 for $\phi \equiv \phi_M$). Moreover, a linear infinite-dimensional state feedback $u(x, t) = Kz(x, t)$ with big enough $K > 0$ exponentially stabilizes the system (see Proposition). Let the points $0 = x_0 < x_1 < \dots < x_N = I$ divide $[0, \ell]$ into N sampling intervals. We assume that N sensors are placed in the middle $\tilde{x}_j = \frac{x_{j+1} + x_j}{2}$ ($j = 0, \dots, N - 1$) of these intervals. The sampling intervals in space may be variable but bounded

$$x_{j+1} - x_j \leq \Delta. \quad (1.24)$$

The goal here is to design for (1.21) an exponentially stabilizing sampled data in space controller Fridman and Blighovsky, 2012

$$u(x, t) = -Kz(\tilde{x}_j, t), \tilde{x}_j = \frac{x_{j+1} + x_j}{2}, x_j \leq x < x_{j+1}, j = 0, \dots, N - 1 \quad (1.25)$$

with the gain $K > 0$. The closed-loop system (1.21), (1.25) has the form:

$$z_t(x, t) = \frac{\partial}{\partial x} [a(x)z_x(x, t)] + \phi(z(x, t), x, t)z(x, t) - Kz(\tilde{x}_j, t), \quad (1.26)$$

$$x_j \leq x < x_{j+1}, j = 0, \dots, N - 1.$$

By using the relation $z(\tilde{x}_j, t) = z(x, t) - \int_{\tilde{x}_j}^x z_\zeta(\zeta, t) d\zeta$, (1.26) can be represented as

$$z_t(x, t) = \frac{\partial}{\partial x} [a(x)z_x(x, t)] + \phi(z(x, t), x, t)z(x, t) \quad (1.27)$$

$$-K[z(x, t) - \int_{\tilde{x}_j}^x z_\zeta(\zeta, t) d\zeta], x_j \leq x < x_{j+1}, j = 0, \dots, N - 1.$$

Note that, since the feedback control here is defined on each interval $I_j = x_{j+1} - x_j \leq$

Δ , $j = 0, \dots, N - 1$, the system (1.21) is better to rewrite like Selivanov and Fridman, 2018

$$z_t(x, t) = \frac{\partial}{\partial x} [a(x)z_x(x, t)] + \phi(z(x, t), x, t)z(x, t) + \sum_{j=0}^p b_j(x)u_j(t), \quad (1.28)$$

$$t \geq t_0, x \in I = [0, \ell],$$

where the control signals $u_j(t)$ are applied through the shape functions $b_j \in L^2(I, [0, 1])$ such that

$$\begin{cases} b_j(x) = 0 & x \notin I_j, \\ b_j(x) = 1 & x \in I_j. \end{cases} \quad (1.29)$$

The static output feedback control (1.25) method presented in this section will be used in Chapter 3 for analysis and comparison with interval observer feedback control.

For later use, we need the following inequalities:

Lemma 1.6. Hardy et al., 1988 Wirtinger's Inequality. *Let $z \in H^1(I, \mathbb{R})$, then*

$$\int_0^\ell z^2(\xi) d\xi \leq \frac{b\ell^2}{\pi^2} \int_0^\ell \left[\frac{dz(\xi)}{d\xi} \right]^2 d\xi, \quad (1.30)$$

and if $z(0) = z(\ell) = 0$, then $b = 1$; if only $z(0) = 0$ or $z(\ell) = 0$, then $b = 4$.

Lemma 1.7. Bar Am and Fridman, 2014 Poincare's Inequality. *Let $z \in H^1(I, \mathbb{R})$ with $\int_0^\ell z(\xi) d\xi = 0$, then*

$$\int_0^\ell z^2(\xi) d\xi \leq \frac{\ell^2}{\pi^2} \int_0^\ell \left[\frac{dz(\xi)}{d\xi} \right]^2 d\xi. \quad (1.31)$$

Well-posedness of the closed-loop system

Let us establish the well-posedness of the closed-loop system (1.21) under the continuous in time controller (1.25)

$$z_t(x, t) = \frac{\partial}{\partial x} [a(x)z_x(x, t)] + \phi(z(x, t), x, t)z(x, t) - Kz(x, t) + K \int_{\tilde{x}_j}^x z_\zeta(\zeta, t) d\zeta,$$

$$x_j \leq x < x_{j+1}, \tilde{x}_j = \frac{x_{j+1} + x_j}{2}, j = 0, \dots, N - 1, t \geq t_0, z(x, t_0) = z^{(0)}(x) \quad (1.32)$$

and under the Dirichlet boundary conditions (1.22). The well-posedness under the mixed conditions (1.23) can be proved similarly. Introduce the Hilbert space $H = L^2(I, \mathbb{R})$, $I = [0, \ell]$ with the norm $\|\cdot\|_{L^2}$ and with the scalar product $\langle \cdot, \cdot \rangle$. The boundary-value problem (1.32)

can be rewritten as a differential equation

$$\dot{\omega}(t) = A\omega(t) + F(t, \omega(t)), \quad t \geq t_0 \quad (1.33)$$

in H where the operator $A = \frac{\partial[a(x)\frac{\partial}{\partial x}]}{\partial x}$ has the dense domain

$$\mathcal{D}(A) = \{\omega \in H^2(I, \mathbb{R}) : \omega(0) = \omega(\ell) = 0\},$$

and the nonlinear term $F : \mathbb{R} \times H^1(I, \mathbb{R}) \rightarrow L^2(I, \mathbb{R})$ is defined on functions $\omega(\cdot, t)$ according to

$$F(t, \omega(\cdot, t)) = \phi(\omega(x, t), x, t)\omega(x, t) - K\omega(x, t) + K \int_{\tilde{x}_j}^x \omega_\zeta(\zeta, t) d\zeta.$$

It is well-known that A generates a strongly continuous exponentially stable semigroup T , which satisfies the inequality $\|T(t)\| \leq \kappa e^{-\delta t}$, ($t \geq 0$) with some constant $\kappa \geq 1$ and decay rate $\delta > 0$ (see, e.g. Curtain and Zwart, 1995 for details). The domain $H^1 = \mathcal{D}(A) = A^{-1}H$ forms another Hilbert space with the graph inner product $\langle x, y \rangle_1 = \langle Ax, Ay \rangle$, $x, y \in H^1$. The domain $\mathcal{D}(A)$ is dense in H and the inequality $\|A\omega\|_{L^2} \geq \mu\|\omega\|_{L^2}$ holds for all $\omega \in \mathcal{D}(A)$ and some constant $\mu > 0$. Operator $-A$ is positive, so that its square root $(-A)^{\frac{1}{2}}$ with

$$H^{\frac{1}{2}} = \mathcal{D}((-A)^{\frac{1}{2}}) = \{\omega \in H^1(I, \mathbb{R}) : \omega(0) = \omega(\ell) = 0\}$$

is well defined. Moreover, $H^{\frac{1}{2}}$ is a Hilbert space with the scalar product

$$\langle u, v \rangle_{\frac{1}{2}} = \langle (-A)^{\frac{1}{2}}u, (-A)^{\frac{1}{2}}v \rangle.$$

Denote by $H^{-\frac{1}{2}}$ the dual of $H^{\frac{1}{2}}$ with respect to the pivot space H . Then A has an extension to a bounded operator $A : H^{\frac{1}{2}} \rightarrow H^{-\frac{1}{2}}$. We have $H^1 \subset H^{\frac{1}{2}} \subset H$ with continuous embedding and the following inequality

$$\|(-A)^{\frac{1}{2}}\omega\|_{L^2} \geq \mu\|\omega\|_{L^2} \quad \text{for all } \omega \in H^{\frac{1}{2}} \quad (1.34)$$

holds. All relevant material on fractional operator degrees can be found, e.g., in Tucsnak and Weiss, 2009. A function $\omega : [t_0, T) \rightarrow H^{\frac{1}{2}}$ is called a strong solution of (1.33) if

$$\omega(t) - \omega(t_0) = \int_{t_0}^t [A\omega(s) + F(s, \omega(s))] ds \quad (1.35)$$

holds for all $t \in [t_0, T)$. Here, the integral is computed in $H^{-\frac{1}{2}}$. Differentiating (1.35) we obtain (1.33).

Since the function ϕ of class C^1 , the following Lipschitz condition

$$\|F(t_1, \omega_1) - F(t_2, \omega_2)\|_{L^2} \leq C[|t_1 - t_2| + \|(-A)^{\frac{1}{2}}(\omega_1 - \omega_2)\|_{L^2}] \quad (1.36)$$

with some constant $C > 0$ holds locally in $(t_i, \omega_i) \in \mathbb{R}_+ \times H^{\frac{1}{2}}$, $i = 1, 2$. Thus, Theorem 3.3.3 of Henry, 1993 is applicable to (1.33), and by applying this theorem, a unique strong solution $\omega(t) \in H^{\frac{1}{2}}$ of (1.33), initialized with $z^{(0)} \in H^{\frac{1}{2}}$, exists locally. Since ϕ is bounded, there exists $C^1 > 0$ such that

$$\|F(t, \omega)\|_{L^2} \leq C_1 \|(-A)^{\frac{1}{2}} \omega\|_{L^2}, \quad \forall \omega \in H^{\frac{1}{2}}.$$

Hence, the strong solution initialized with $z^{(0)} \in H^{\frac{1}{2}}$ exists for all $T \geq t_0$ Henry, 1993.

LMIs for the exponential stabilization

Consider the stabilization problem via sampled-data in spatial variable controller which is continuous in time. In this case let us assume that ϕ is upper bounded with $\phi \leq \phi_M < \infty$ (thus $\phi_m = -\infty$). Consider the closed-loop system (1.32) under the mixed boundary conditions (1.23). By using the Lyapunov function

$$V(t) = \int_0^\ell z^2(x, t) dx, \quad (1.37)$$

derive conditions that guarantee $\dot{V}(t) + 2\delta V(t) \leq 0$ along (1.32), (1.23). The latter inequality yields $V(t) \leq e^{-2\delta(t-t_0)} V(t_0)$ or

$$\int_0^\ell z^2(x, t) dx \leq e^{-2\delta(t-t_0)} \int_0^\ell z^2(x, t_0) dx \quad (1.38)$$

for the strong solutions of (1.32), (1.23) initialized with

$$z(\cdot, t_0) \in H^1(0, \ell) : z_x(0, t_0) = \gamma z(0, t_0), \quad z(\ell, t_0) = 0. \quad (1.39)$$

If (1.38) holds, then the system (1.32) under boundary conditions (1.23) is exponentially stable with the decay rate δ .

Differentiating V along (1.32) we find

$$\begin{aligned}\dot{V}(t) &= 2 \int_0^\ell z(x, t) z_t(x, t) dx = \\ & 2 \int_0^\ell z(x, t) \times \left[\frac{\partial}{\partial x} [a(x) z_x(x, t)] + \phi(z(x, t), x, t) z(x, t) - K z(x, t) \right] dx \\ & + 2 \sum_{j=0}^{N-1} \int_{x_j}^{x_{j+1}} K z(x, t) [z(x, t) - z(\tilde{x}_j, t)] dx.\end{aligned}$$

Integration by parts and substitution of the boundary conditions (1.23) lead to

$$\begin{aligned}2 \int_0^\ell z(x, t) \frac{\partial}{\partial x} [a(x) z_x(x, t)] dx &= 2a(x) z(x, t) z_x(x, t) \Big|_0^\ell \\ &- 2 \int_0^\ell a(x) z_x^2(x, t) dx \leq -2a_0 \int_0^\ell z_x^2(x, t) dx\end{aligned}\tag{1.40}$$

Therefore,

$$\begin{aligned}\dot{V}(t) &\leq -2a_0 \int_0^\ell z_x^2(x, t) dx + 2 \int_0^\ell (\phi_M - K) z^2(x, t) dx \\ &+ 2 \sum_{j=0}^{N-1} \int_{x_j}^{x_{j+1}} K z(x, t) [z(x, t) - z(\tilde{x}_j, t)] dx\end{aligned}\tag{1.41}$$

By Young's inequality, for any scalar $\bar{R} > 0$ the following holds:

$$\begin{aligned}&-2K \sum_{j=0}^{N-1} \int_{x_j}^{x_{j+1}} z(x, t) [z(x, t) - z(\tilde{x}_j, t)] dx \\ &\leq K \left[\bar{R} \int_0^\ell z^2(x, t) dx + \bar{R}^{-1} \sum_{j=0}^{N-1} \int_{x_j}^{x_{j+1}} [z(x, t) - z(\tilde{x}_j, t)]^2 dx \right].\end{aligned}\tag{1.42}$$

Then, application of Wirtinger's inequality (1.30) yields

$$\begin{aligned}\int_{x_j}^{x_{j+1}} [z(x, t) - z(\tilde{x}_j, t)]^2 dx &= \int_{x_j}^{\tilde{x}_j} [z(x, t) - z(\tilde{x}_j, t)]^2 dx + \\ &\int_{\tilde{x}_j}^{x_{j+1}} [z(x, t) - z(\tilde{x}_j, t)]^2 dx \leq \frac{\Delta^2}{\pi^2} \int_{x_j}^{x_{j+1}} z_x^2(x, t) dx.\end{aligned}\tag{1.43}$$

Choosing next $\bar{R} = \frac{\Delta}{\pi}R$, we find from (1.41)-(1.43) that

$$\begin{aligned} \dot{V}(t) + 2\delta V(t) &\leq \left(R^{-1}K\frac{\Delta}{\pi} - 2a_0 \right) \int_0^\ell z_x^2(x,t)dx \\ &\quad + \left(RK\frac{\Delta}{\pi} + 2\delta + 2(\phi_M - K) \right) \int_0^\ell z^2(x,t)dx. \end{aligned} \quad (1.44)$$

By Wirtinger's inequality (1.30), $\dot{V}(t) + 2\delta V(t) \leq 0$ if

$$\begin{aligned} R^{-1}K\frac{\Delta}{\pi} - 2a_0 &\leq 0, \\ RK\frac{\Delta}{\pi} + 2\delta + 2(\phi_M - K) + \frac{\pi^2}{b\ell^2} \left(R^{-1}K\frac{\Delta}{\pi} - 2a_0 \right) &\leq 0, \end{aligned} \quad (1.45)$$

where $b = 4$. Under the Dirichlet boundary conditions (1.22), application of (1.30) leads to the same conclusion with $b = 1$ in (1.45). Note that inequalities (1.45) are feasible for small enough $\delta > 0$, $\Delta > 0$ iff $K > \phi_M - \frac{a_0\pi^2}{b\ell^2}$. By this, the following proposition has been proved

Proposition 1.2. (i) Given $b = 4$, $K > \phi_M - \frac{a_0\pi^2}{b\ell^2}$, $R > 0$, let there exist $\Delta > 0$ and $\delta > 0$ such that the linear scalar inequalities (1.45) are feasible. Then the closed-loop system (1.21), (1.25) under the mixed boundary conditions (1.23) is exponentially stable with the decay rate δ (in the sense of (1.38)).

(ii) If the conditions of (i) hold with $b = 1$, then the closed-loop system (1.21), (1.25) under the Dirichlet boundary conditions (1.22) is exponentially stable with the decay rate δ .

(iii) The state-feedback controller $u = -Kz(x,t)$ exponentially stabilizes (1.21) with the decay rate $\delta > 0$ if $K > \phi_M - \frac{a_0\pi^2}{b\ell^2} + \delta$, where $b = 1$ corresponds to (1.22) and $b = 4$ to (1.23).

Remark 1.1. The condition (1.45) of Proposition 1.2 cannot be improved for the diffusion equation

$$z_t(x,t) = z_{xx}(x,t), \quad (1.46)$$

where $x \in [0, \pi]$ under the mixed boundary conditions $z_x(0,t) = z(\pi,t) = 0$. The feasibility of (1.45) with $K = 0$, $a = 1$ guarantees the exponential decay rate $\delta = 0.25$ of the system. This is the exact decay rate since -0.25 is the rightmost eigenvalue of the operator $A = \frac{\partial^2}{\partial x^2}$ with the domain (Tucsnak and Weiss, 2009)

$$\mathcal{D}(A) = \{\omega \in H^2(I, \mathbb{R}) : \omega_x(0) = \omega(\pi) = 0\}.$$

The same conclusion is true for the Dirichlet boundary conditions with $\delta = 1$.

Interval observers and controls for PDEs using finite-element approximations

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The first result on interval observers for systems described by PDEs has been proposed in Kharkovskaia et al., 2016, where the finite-dimensional approximation approach was used. Then in Kharkovskaia et al., 2018b it was extended and the additional design of an output stabilizing control based on constructed interval observer was proposed. This chapter describes the framework of interval estimation for PDEs using early lumping approach.

2.1 Approximation of distributed systems

In this section basic facts on finite-dimensional approximations of a PDE and some auxiliary results are given.

Following Wheeler, 1973, consider the following PDE with homogeneous Dirichlet

boundary conditions:

$$\begin{aligned} \rho(x) \frac{\partial z(x,t)}{\partial t} &= L[x, z(x,t)] + r(x,t) \quad \forall (x,t) \in I \times (0, T), \\ z(x,0) &= z_0(x) \quad \forall x \in I, \\ 0 &= z(0,t) = z(1,t) \quad \forall t \in (0, T), \end{aligned} \tag{2.1}$$

where $I = [0, 1]$ and $T > 0$,

$$L(x, z) = \frac{\partial}{\partial x} \left(a(x) \frac{\partial z}{\partial x} \right) - b(x) \frac{\partial z}{\partial x} - q(x)z,$$

$r \in L^\infty(I \times [0, T], \mathbb{R})$, $a, b, q, \rho \in L^\infty(I, \mathbb{R})$ and there exist $a_0, a_1, \rho_0, \rho_1 \in \mathbb{R}_+$ such that

$$0 < a_0 \leq a(x) \leq a_1, \quad 0 < \rho_0 \leq \rho(x) \leq \rho_1 \quad \forall x \in I,$$

and $a', b' \in L^2(I, \mathbb{R})$, where $a' = \partial a(x)/\partial x$.

Let $\Delta = \{x_j\}_{j=0}^{N'}$ for some $N' > 0$, where $0 = x_0 < x_1 < \dots < x_{N'} = 1$, and $I_j = (x_{j-1}, x_j)$, $h_j = x_j - x_{j-1}$, $h = \max_{1 \leq j \leq N'} h_j$. Let $P_s(I')$ be the set of polynomials of the degree less than $s + 1$, $s > 0$ on an interval $I' \subseteq I$, then adopt the notation:

$$\begin{aligned} M^{s,\Delta} &= \{v \in C^0(I, \mathbb{R}) : v(x) = v_j(x) \quad \forall x \in I_j, \\ &\quad v_j \in P_s(I_j) \quad \forall 1 \leq j \leq N'\} \end{aligned}$$

and $M = M_0^{s,\Delta} = \{v \in M^{s,\Delta} : v(0) = v(1) = 0\}$.

Introduce a bilinear form:

$$\mathcal{L}(y, v) = -\langle ay', v' \rangle - \langle by', v \rangle - \langle qy, v \rangle \quad y, v \in H^1(I, \mathbb{R}),$$

and define

$$\lambda \geq \frac{1}{2a_0} (\text{ess sup}_{x \in I} b^2(x) - \text{ess inf}_{x \in I} q(x)).$$

The continuous-time Galerkin approximation $Z(\cdot, t) \in M$ to the solution $z(x, t)$ of the parabolic system (2.1) is defined by

$$\begin{aligned} \left\langle \rho \frac{\partial Z}{\partial t}, \Phi \right\rangle &= \mathcal{L}(Z, \Phi) + \langle r, \Phi \rangle \quad \forall \Phi \in M, \quad \forall t \in (0, T); \\ \mathcal{L}(Z - z_0, \Phi) - \lambda \langle Z - z_0, \Phi \rangle &= 0 \quad \forall \Phi \in M, \quad t = 0. \end{aligned} \tag{2.2}$$

Assumption 2.1. *There exist $s > 0$, $l_1 > 0$ and $l_2 > 0$ such that the solution z of (2.1) belongs*

to $L^\infty([0, T], W^{s+1, \infty}(I, \mathbb{R}))$ and $\partial z / \partial t \in L^2([0, T], H^{s+1}(I, \mathbb{R}))$,

$$\|z\|_{L^\infty([0, T], W^{s+1, \infty}(I, \mathbb{R}))} \leq l_1, \quad \|\partial z / \partial t\|_{L^2([0, T], H^{s+1}(I, \mathbb{R}))} \leq l_2.$$

Proposition 2.1. Wheeler, 1973 *Let Assumption 2.1 be satisfied, then there is $\rho > 0$ such that*

$$\|Z - z\|_{L^\infty(I \times (0, T), \mathbb{R})} \leq \rho h^{s+1} (l_1 + l_2),$$

where z and Z are solutions of (2.1) and (2.2), respectively.

Remark 2.1. Since the operator $A = \frac{\partial}{\partial x} \left(a(x) \frac{\partial}{\partial x} \right) : L^2(I, \mathbb{R}) \rightarrow D(A) \subset L^2(I, \mathbb{R})$ with $D(A) = H_0^1(I, \mathbb{R}) \cap H^2(I, \mathbb{R})$ is closed, then for $z_0 \in D(A)$ and $\frac{\partial r}{\partial t} \in C([0, T], L^2(I, \mathbb{R}))$ using, for example, Pazy, 1983, Ch. 1, Corollary 2.5 we derive $z \in C^0([0, T], D(A))$. Taking into account that $Z \in C^0([0, T], D(A))$ we conclude that in the latter case the obtained estimate on $Z - z$ holds for all $t \in [0, T]$ and $x \in I$.

Remark 2.2. The constants l_1 and l_2 depend on the original solution z and may be evaluated *a priori* from the domain of application, while ρ needs a numeric experimentation to be estimated. Thus, in order to be applied, the result of this proposition can also be interpreted as the existence for any $\bar{\rho} > 0$ a sufficiently small discretization step $h > 0$ such that $\rho h^{s+1} (l_1 + l_2) \leq \bar{\rho}$.

In order to calculate Z , let $\Phi_j \in M$, $1 \leq j \leq N$ with $N \geq N'$ be a basis in M , then following the Galerkin method Thomée, 2006 the solution $Z(x, t)$ of (2.2) can be presented as

$$Z(x, t) = \sum_{j=1}^N \xi_j(t) \Phi_j(x),$$

where $\xi = [\xi_1 \dots \xi_N]^T \in \mathbb{R}^N$ is the vector of coefficients satisfying the ODEs for all $1 \leq j \leq N$:

$$\begin{aligned} \left\langle \rho \sum_{i=1}^N \dot{\xi}_i \Phi_i, \Phi_j \right\rangle &= \mathcal{L} \left(\sum_{i=1}^N \xi_i \Phi_i, \Phi_j \right) + \langle r, \Phi_j \rangle \quad \forall t \in (0, T); \\ \mathcal{L} \left(\sum_{i=1}^N \xi_i(0) \Phi_i - z_0, \Phi_j \right) - \lambda \left\langle \sum_{i=1}^N \xi_i(0) \Phi_i - z_0, \Phi_j \right\rangle &= 0, \end{aligned}$$

which finally can be presented in the form (*a.a.* means “for almost all”):

$$\Upsilon \dot{\xi}(t) = \Lambda \xi(t) + \bar{r}(t) \quad a.a. \ t \in (0, T); \quad \Psi \xi(0) = \omega,$$

where for all $1 \leq i, j \leq N$

$$\begin{aligned}\Upsilon_{j,i} &= \langle \rho \Phi_i, \Phi_j \rangle, \Lambda_{j,i} = \mathcal{L}(\Phi_i, \Phi_j), \bar{r}_j = \langle r, \Phi_j \rangle, \\ \Psi_{j,i} &= \mathcal{L}(\Phi_i, \Phi_j) - \lambda \langle \Phi_i, \Phi_j \rangle, \bar{\omega}_j = \mathcal{L}(z_0, \Phi_j) - \lambda \langle z_0, \Phi_j \rangle.\end{aligned}$$

Under the introduced restrictions on (2.1) and by construction of the basis functions Φ_j , we assume that the matrices Υ and Ψ are nonsingular, therefore

$$\dot{\xi}(t) = A\xi(t) + G\bar{r}(t) \quad a.a. \ t \in (0, T), \quad \xi(0) = \xi_0, \quad (2.3)$$

where $A = \Upsilon^{-1}\Lambda \in \mathbb{R}^{N \times N}$, $G = \Upsilon^{-1}$, $\xi_0 = \Psi^{-1}\bar{\omega} \in \mathbb{R}^N$ and $\bar{r} \in L^\infty([0, T], \mathbb{R}^N)$. Then for any $\xi_0 \in \mathbb{R}^N$ the corresponding solution $\xi \in C^0([0, T], \mathbb{R}^N)$ to Cauchy problem (2.3) can be easily calculated.

Interval estimates

For $\phi \in \mathbb{R}$ define two operators \cdot^+ and \cdot^- as follows:

$$\phi^+ = \max\{0, \phi\}, \quad \phi^- = \phi^+ - \phi.$$

Lemma 2.1. *Let $s, \underline{s}, \bar{s} : I \rightarrow \mathbb{R}$ admit the relations $\underline{s} \leq s \leq \bar{s}$, then for any $\phi : I \rightarrow \mathbb{R}$ we have*

$$\langle \underline{s}, \phi^+ \rangle - \langle \bar{s}, \phi^- \rangle \leq \langle s, \phi \rangle \leq \langle \bar{s}, \phi^+ \rangle - \langle \underline{s}, \phi^- \rangle.$$

Proof 2.1. By definition,

$$\langle s, \phi \rangle = \langle s, \phi^+ - \phi^- \rangle = \langle s, \phi^+ \rangle - \langle s, \phi^- \rangle$$

and the functions ϕ^+, ϕ^- take only positive values, then $\langle \underline{s}, \phi^+ \rangle \leq \langle s, \phi^+ \rangle \leq \langle \bar{s}, \phi^+ \rangle$, $\langle \underline{s}, \phi^- \rangle \leq \langle s, \phi^- \rangle \leq \langle \bar{s}, \phi^- \rangle$ and the result follows by substitution.

Lemma 2.2. *Let there exist $\underline{\xi}, \bar{\xi} \in C^0([0, T], \mathbb{R}^N)$ such that for the solution ξ of (2.3) we have*

$$\underline{\xi}(t) \leq \xi(t) \leq \bar{\xi}(t) \quad \forall t \in [0, T],$$

then for the solution Z of (2.2),

$$\underline{Z}(x, t) \leq Z(x, t) \leq \bar{Z}(x, t) \quad \forall (x, t) \in I \times [0, T] \quad (2.4)$$

and $\underline{Z}, \bar{Z} \in C^0(I \times [0, T], \mathbb{R})$, where

$$\begin{aligned} \underline{Z}(x, t) &= \sum_{j=1}^N (\underline{\xi}_j(t) \Phi_j^+(x) - \bar{\xi}_j(t) \Phi_j^-(x)), \\ \bar{Z}(x, t) &= \sum_{j=1}^N (\bar{\xi}_j(t) \Phi_j^+(x) - \underline{\xi}_j(t) \Phi_j^-(x)). \end{aligned} \quad (2.5)$$

Proof 2.2. The result follows from the definitions of $\Phi_j^+(x), \Phi_j^-(x)$ and Lemma 1.1:

$$\begin{aligned} Z(x, t) &= \sum_{j=1}^N \xi_j(t) \Phi_j(x) = \sum_{j=1}^N \xi_j(t) [\Phi_j^+(x) - \Phi_j^-(x)] \\ &\leq \sum_{j=1}^N \bar{\xi}_j(t) \Phi_j^+(x) - \underline{\xi}_j(t) \Phi_j^-(x) = \bar{Z}(x, t), \end{aligned}$$

similarly for $\underline{Z}(x, t)$. The needed continuity of \underline{Z}, \bar{Z} is deduced from similar properties of Φ_j and $\underline{\xi}, \bar{\xi}$ since by construction $\xi \in C^0([0, T], \mathbb{R}^N)$.

The result Lemma 2.2 connects the interval estimates obtained for a real vector ξ and the approximated solution Z , and can be extended to z as follows:

Lemma 2.3. *Let Assumption 2.1 be satisfied and there exist $\underline{Z}, \bar{Z} \in L^\infty(I \times [0, T], \mathbb{R})$ such that (2.4) be true for the solution Z of (2.2), then there is $\rho > 0$ such that for the solution z of (2.1),*

$$\underline{z}(x, t) \leq z(x, t) \leq \bar{z}(x, t) \quad (2.6)$$

for all $x \in I$ and almost all $t \in [0, T]$, where $\underline{z}, \bar{z} \in L^\infty(I \times [0, T], \mathbb{R})$ given by

$$\begin{aligned} \underline{z}(x, t) &= \underline{Z}(x, t) - \rho h^{s+1} (l_1 + l_2), \\ \bar{z}(x, t) &= \bar{Z}(x, t) + \rho h^{s+1} (l_1 + l_2). \end{aligned} \quad (2.7)$$

Proof 2.3. The result can be justified by applying the estimates on the error $Z - z$ given in Proposition 2.1:

$$\begin{aligned} z(t, x) &\leq Z(t, x) + \rho h^{s+1} (l_1 + l_2) \\ &\leq \bar{Z}(x, t) + \rho h^{s+1} (l_1 + l_2) = \bar{z}(x, t) \end{aligned}$$

for almost all $(x, t) \in I \times [0, T]$, and similarly for $\underline{z}(x, t)$.

Therefore, according to lemmas 2.2 and 2.3, in order to calculate interval estimates for (2.1) it is enough to design an interval observer for (2.3).

2.2 Interval observer design

Assume that the state $z(x, t)$ is available for measurements in certain points $x_i^m \in I$ for $1 \leq i \leq p$:

$$y_i(t) = z(x_i^m, t) + v_i(t), \quad (2.8)$$

where $y(t), v(t) \in \mathbb{R}^p$, $v \in L^\infty(\mathbb{R}_+, \mathbb{R}^p)$ is the measurement noise. Under Assumption 2.1 from Proposition 2.1, for a finite-element approximation we can assign

$$y_i(t) = Z(x_i^m, t) + v_i(t) + e_i(t),$$

where $\|e\|_{L^\infty([0, T], \mathbb{R}^p)} \leq \rho h^{s+1} (l_1 + l_2)$ for some $\rho > 0$, $e = [e_1 \dots e_p]^T$. Next,

$$y_i(t) = \sum_{j=1}^N \xi_j(t) \Phi_j(x_i^m) + v_i(t) + e_i(t)$$

and

$$y(t) = C\xi(t) + v(t), \quad (2.9)$$

with $v(t) = v(t) + e(t) \in \mathbb{R}^p$ being the new measurement noise, and $C \in \mathbb{R}^{p \times N}$ is the appropriate matrix:

$$C = \begin{bmatrix} \Phi_1(x_1^m) & \dots & \Phi_N(x_1^m) \\ \vdots & \ddots & \vdots \\ \Phi_1(x_p^m) & \dots & \Phi_N(x_p^m) \end{bmatrix}.$$

We will also assume that in (2.1),

$$r(x, t) = \sum_{k=1}^m r_{1k}(x) u_k(t) + r_0(x, t),$$

where $u(t) \in \mathbb{R}^m$ is a control (known input), $r_{1k} \in L^\infty(I, \mathbb{R})$ and $r_0 \in L^\infty(I \times [0, T], \mathbb{R})$. Then

in (2.3):

$$\begin{aligned} G\bar{r}(t) &= G \begin{bmatrix} \langle \sum_{k=1}^m r_{1k}(x)u_k(t) + r_0(x,t), \Phi_1 \rangle \\ \vdots \\ \langle \sum_{k=1}^m r_{1k}(x)u_k(t) + r_0(x,t), \Phi_N \rangle \end{bmatrix} \\ &= Bu(t) + Gd(t), \end{aligned}$$

where

$$B = G \begin{bmatrix} \langle r_{11}, \Phi_1 \rangle & \dots & \langle r_{1m}, \Phi_1 \rangle \\ \vdots & \ddots & \vdots \\ \langle r_{11}, \Phi_N \rangle & \dots & \langle r_{1m}, \Phi_N \rangle \end{bmatrix} \in \mathbb{R}^{N \times m}, \quad d(t) = \begin{bmatrix} \langle r_0, \Phi_1 \rangle \\ \vdots \\ \langle r_0, \Phi_N \rangle \end{bmatrix} \in \mathbb{R}^N$$

is an external unknown disturbance.

The idea of the work consists in design of an interval observer for the approximation (2.3), (2.9) with the aim to calculate an interval estimate for the state of (2.1), (2.8) taking into account the approximation error evaluated in Proposition 2.1 and the results of lemmas 2.2 and 2.3. For this purpose we need the following hypothesis.

Assumption 2.2. Let $\underline{z}_0 \leq z_0 \leq \bar{z}_0$ for some known $\underline{z}_0, \bar{z}_0 \in L^\infty(I, \mathbb{R})$, two functions $\underline{r}_0, \bar{r}_0 \in L^\infty(I \times [0, T], \mathbb{R})$ and a constant $\nu_0 > 0$ be given such that

$$\underline{r}_0(x, t) \leq r_0(x, t) \leq \bar{r}_0(x, t), \quad |\nu(t)| \leq \nu_0 \text{ a.a. } (x, t) \in I \times (0, T).$$

Assumption 2.3. There are a matrix $L \in \mathbb{R}^{N \times p}$ and a Metzler matrix $D \in \mathbb{R}^{N \times N}$ s.t. the matrices $A - LC$ and D have the same eigenvalues and the pairs $(A - LC, \chi_1)$, (D, χ_2) are observable for some $\chi_1 \in \mathbb{R}^{1 \times N}$, $\chi_2 \in \mathbb{R}^{1 \times N}$.

Thus, by Assumption 2.2 three intervals $[\underline{z}_0, \bar{z}_0]$, $[\underline{r}_0(x, t), \bar{r}_0(x, t)]$ and $[-\nu_0, \nu_0]$ determine for all $(x, t) \in I \times [0, T]$ in (2.1), (2.8) uncertainty of values of z_0 , $r_0(x, t)$ and $\nu(t)$, respectively. Using Lemma 2.1 we obtain:

$$\begin{aligned} \underline{d}(t) &\leq d(t) \leq \bar{d}(t) \quad \forall t \in [0, T], \\ \underline{d}(t) &= \begin{bmatrix} \langle \underline{r}_0, \Phi_1^+ \rangle - \langle \bar{r}_0, \Phi_1^- \rangle \\ \vdots \\ \langle \underline{r}_0, \Phi_N^+ \rangle - \langle \bar{r}_0, \Phi_N^- \rangle \end{bmatrix}, \quad \bar{d}(t) = \begin{bmatrix} \langle \bar{r}_0, \Phi_1^+ \rangle - \langle \underline{r}_0, \Phi_1^- \rangle \\ \vdots \\ \langle \bar{r}_0, \Phi_N^+ \rangle - \langle \underline{r}_0, \Phi_N^- \rangle \end{bmatrix} \end{aligned}$$

and under Assumption 2.1

$$-V \leq \nu(t) \leq V = \nu_0 + \rho h^{s+1} (l_1 + l_2).$$

Finally,

$$\underline{\xi}_0 \leq \xi(0) \leq \bar{\xi}_0,$$

where

$$\underline{\xi}_0 = (\Psi^{-1})^+ \underline{\omega} - (\Psi^{-1})^- \bar{\omega}, \quad \bar{\xi}_0 = (\Psi^{-1})^+ \bar{\omega} - (\Psi^{-1})^- \underline{\omega}$$

and $\underline{\omega}_j \leq \omega_j \leq \bar{\omega}_j$ for all $1 \leq j \leq N$ with

$$\begin{aligned} \bar{\omega}_j &= \lambda[\langle \bar{z}_0, \Phi_j^- \rangle - \langle z_0, \Phi_j^+ \rangle] - \langle az'_0, \Phi_j'^+ \rangle + \langle a\bar{z}'_0, \Phi_j'^- \rangle \\ &\quad - \langle b^+ z'_0 - b^- \bar{z}'_0, \Phi_j^+ \rangle + \langle b^+ \bar{z}'_0 - b^- z'_0, \Phi_j^- \rangle \\ &\quad - \langle q^+ z_0 - q^- \bar{z}_0, \Phi_j^+ \rangle + \langle q^+ \bar{z}_0 - q^- z_0, \Phi_j^- \rangle, \\ \underline{\omega}_j &= \lambda[\langle z_0, \Phi_j^- \rangle - \langle \bar{z}_0, \Phi_j^+ \rangle] - \langle a\bar{z}'_0, \Phi_j'^+ \rangle + \langle az'_0, \Phi_j'^- \rangle \\ &\quad - \langle b^+ \bar{z}'_0 - b^- z'_0, \Phi_j^+ \rangle + \langle b^+ z'_0 - b^- \bar{z}'_0, \Phi_j^- \rangle \\ &\quad - \langle q^+ \bar{z}_0 - q^- z_0, \Phi_j^+ \rangle + \langle q^+ z_0 - q^- \bar{z}_0, \Phi_j^- \rangle, \end{aligned}$$

According to Assumption 2.3 (which is always satisfied if the pair (A, C) is observable, for example) and Raïssi et al., 2012 there is a nonsingular matrix $S \in \mathbb{R}^{N \times N}$ such that $D = S(A - LC)S^{-1}$. Now, applying the results of Chebotarev et al., 2015; Gouzé et al., 2000 two bounded estimates $\underline{\xi}, \bar{\xi} \in C^0([0, T], \mathbb{R}^N)$ can be calculated, based on the available information on these intervals and $y(t)$, such that

$$\underline{\xi}(t) \leq \xi(t) \leq \bar{\xi}(t) \quad \forall t \in [0, T]. \quad (2.10)$$

For this purpose, following Chebotarev et al., 2015; Gouzé et al., 2000, rewrite (2.3):

$$\dot{\xi}(t) = (A - LC)\xi(t) + Bu(t) + Ly(t) - Lv(t) + Gd(t).$$

In the new coordinates $\zeta = S\xi$, (2.3) takes the form:

$$\dot{\zeta}(t) = D\zeta(t) + SBu(t) + SLy(t) + \delta(t), \quad (2.11)$$

$\delta(t) = S[Gd(t) - Lv(t)]$. And using Lemma 1.1 we obtain

$$\underline{\delta}(t) \leq \delta(t) \leq \bar{\delta}(t),$$

where $\underline{\delta}(t) = (SG)^+ \underline{d}(t) - (SG)^- \bar{d}(t) - |SL|E_p V$ and $\bar{\delta}(t) = (SG)^+ \bar{d}(t) - (SG)^- \underline{d}(t) + |SL|E_p V$.

Next, for the system (2.11) an interval observer can be proposed:

$$\begin{aligned}
\dot{\underline{\zeta}}(t) &= D\underline{\zeta}(t) + SBu(t) + SLy(t) + \underline{\delta}(t), \\
\dot{\overline{\zeta}}(t) &= D\overline{\zeta}(t) + SBu(t) + SLy(t) + \overline{\delta}(t), \\
\underline{\zeta}(0) &= S^+ \underline{\xi}_0 - S^- \overline{\xi}_0, \quad \overline{\zeta}(0) = S^+ \overline{\xi}_0 - S^- \underline{\xi}_0, \\
\underline{\xi}(t) &= (S^{-1})^+ \underline{\zeta}(t) - (S^{-1})^- \overline{\zeta}(t), \\
\overline{\xi}(t) &= (S^{-1})^+ \overline{\zeta}(t) - (S^{-1})^- \underline{\zeta}(t),
\end{aligned} \tag{2.12}$$

where the relations (1.4) are used to calculate the initial conditions for $\underline{\zeta}, \overline{\zeta}$ and the estimates $\underline{\xi}, \overline{\xi}$.

Proposition 2.2. *Let assumptions 2.1, 2.2 and 2.3 be satisfied. Then for (2.3), (2.9) with the interval observer (2.12) the relations (2.10) are fulfilled and $\underline{\xi}, \overline{\xi} \in C^0([0, T], \mathbb{R}^N)$. In addition, $\underline{\xi}, \overline{\xi} \in L^\infty([0, T], \mathbb{R}^N)$ if $A - LC$ is Hurwitz.*

Proof 2.4. By Assumption 2.1 $z \in L^\infty([0, T], W^{s+1, \infty}(I, \mathbb{R}))$, then $\xi, \zeta, \underline{\xi}, \overline{\xi} \in C^0([0, T], \mathbb{R}^N)$ and $\xi, \zeta \in L^\infty([0, T], \mathbb{R}^N)$ by construction. Define two estimation errors

$$\underline{e}(t) = \zeta(t) - \underline{\zeta}(t), \quad \overline{e}(t) = \overline{\zeta}(t) - \zeta(t),$$

which yield the differential equations:

$$\dot{\underline{e}}(t) = D\underline{e}(t) + \delta(t) - \underline{\delta}(t), \quad \dot{\overline{e}}(t) = D\overline{e}(t) + \overline{\delta}(t) - \delta(t).$$

By Assumption 2.2 and the previous calculations,

$$\delta(t) - \underline{\delta}(t) \geq 0, \quad \overline{\delta}(t) - \delta(t) \geq 0 \quad \forall t \in [0, T].$$

If D is a Metzler matrix, since all inputs of $\underline{e}(t), \overline{e}(t)$ are positive and $\underline{e}(0) \geq 0, \overline{e}(0) \geq 0$, then $\underline{e}(t) \geq 0, \overline{e}(t) \geq 0$ for all $t \geq 0$ Farina and Rinaldi, 2000; Smith, 1995. The property (2.10) follows from these relations. If $A - LC$ is Hurwitz, D possesses the same property, since all inputs $\delta(t) - \underline{\delta}(t), \overline{\delta}(t) - \delta(t)$ are bounded, then $\underline{e}, \overline{e} \in L^\infty([0, T], \mathbb{R}^N)$ and the boundedness of $\underline{\xi}, \overline{\xi}$ is followed by the boundedness of ξ .

Remark 2.3. In order to regulate the estimation accuracy it is worth to strengthen the conditions of stability for $\underline{\xi}, \overline{\xi}$ (Hurwitz property of the matrix $A - LC$) to a requirement that the L_∞ gain of the transfer $\begin{bmatrix} \delta - \underline{\delta} \\ \overline{\delta} - \delta \end{bmatrix} \rightarrow \begin{bmatrix} \overline{e} \\ \underline{e} \end{bmatrix}$ is less than γ for some $\gamma > 0$. To this end, coupling this restriction with the conditions of Assumption 2.3 the following nonlinear

matrix inequalities can be obtained:

$$\begin{bmatrix} W^T + W + I_N & P \\ P & \gamma^2 I_N \end{bmatrix} \leq 0, \quad (2.13)$$

$$W + Z \geq 0, P > 0, Z > 0, \quad (2.14)$$

$$SA - FC = P^{-1}WS, \quad (2.15)$$

which have to be solved with respect to diagonal matrices $P \in \mathbb{R}^{N \times N}$ and $Z \in \mathbb{R}^{N \times N}$, nonsingular matrices $S \in \mathbb{R}^{N \times N}$ and $W \in \mathbb{R}^{N \times N}$, some $F \in \mathbb{R}^{N \times p}$ and $\gamma > 0$. Then $D = P^{-1}W$ and $L = S^{-1}F$. It is easy to see that this system can be easily solved iteratively: first, a solution $P^{-1}W$ of the LMIs (2.13), (2.14) can be found for given $N > 0$ with optimally tuned $\gamma > 0$, second, the existence of a solution S and F of the LMI (2.15) can be checked. If such a solution does not exist, then another iteration can be performed for some other values of N .

Theorem 2.1. *Let assumptions 2.1, 2.2 and 2.3 be satisfied and the matrix $A - LC$ be Hurwitz. Then for (2.1), (2.8) with the interval observer (2.5), (2.7), (2.12) the relations (2.6) are fulfilled and $\underline{z}, \bar{z} \in L^\infty(I \times [0, T], \mathbb{R})$.*

Proof 2.5. Since all conditions of Proposition 2.2 are satisfied, then the property (2.10) for $\xi(t)$ is true. Next, all restrictions of Lemma 2.2 are verified and the interval estimate (2.4) for $Z(t)$ is justified. Finally, the needed interval estimates for $z(t)$ can be obtained by applying Lemma 2.3.

Remark 2.4. The designed interval observer can also be applied to a nonlinear PDE. If in Assumption 2.2,

$$\underline{r}_0(x, t) \leq r_0(x, t, z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial t}) \leq \bar{r}_0(x, t)$$

for some known $\underline{r}_0, \bar{r}_0 \in L^\infty(I \times [0, T], \mathbb{R})$ for all $x \in I$, $t \in [0, T)$ and the corresponding solutions $z(x, t)$ (provided that they exist for such a nonlinear PDE and the Galerkin method can be applied), then the interval observer (2.5), (2.7), (2.12) preserves its form and the result of Theorem 2.1 stays correct. In such a case the proposed interval observer can be used for a fast and reliable calculation of envelopes for solutions of nonlinear PDEs.

2.3 Control design

In this section the interval observer (2.12) is used to design a control law ensuring stabilization of the finite-dimensional approximation (2.3), (2.9) in the spirit of Efimov et al., 2013f, which implies also (under additional mild restrictions) the stabilization of (2.1).

In Theorem 2.1 the gain L together with the transformation matrix S have been used to guarantee the properties of positivity and stability for the dynamics of estimation errors $\underline{e}(t)$, $\bar{e}(t)$. The positivity property has been obtained uniformly in $u(t)$. Thus, the control design can be applied in order to ensure boundedness of the observer estimates $\underline{z}(x, t)$, $\bar{z}(x, t)$, that in its turn (since $\underline{z}(x, t) \leq z(x, t) \leq \bar{z}(x, t)$ for almost all $(x, t) \in I \times [0, T)$, see (2.6)) will provide boundedness of $z(x, t)$. An advantage of this approach is that the system (2.1) is uncertain, distributed and the state of that system cannot be measured (it is infinite-dimensional), while the observer (2.12) together with (2.5), (2.7) is a completely known linear system with the accessible state $\underline{\zeta}(t)$, $\bar{\zeta}(t)$ Efimov et al., 2013f. An obstacle is that the dimension of the state of (2.12) is $2N$, while the dimension of the control is m , similarly to (2.3).

In our work, the control is chosen as a conventional state linear feedback:

$$u(t) = \underline{K}\underline{\zeta}(t) + \bar{K}\bar{\zeta}(t) \quad (2.16)$$

where $\underline{K}, \bar{K} \in \mathbb{R}^{m \times N}$ are two feedback matrix gains to be designed. To this end, let us consider the combined system, which consists of (2.11), (2.12) and (2.16):

$$\dot{\eta}(t) = (\tilde{A} + \tilde{B}K\Gamma)\eta(t) + \tilde{\Delta}(t), \quad (2.17)$$

where $\eta = [\zeta^T \underline{\zeta}^T \bar{\zeta}^T]^T \in \mathbb{R}^{3N}$ is the combined state and

$$\tilde{A} = \begin{bmatrix} SAS^{-1} & 0 & 0 \\ SLCS^{-1} & D & 0 \\ SLCS^{-1} & 0 & D \end{bmatrix}, \tilde{B} = \begin{bmatrix} SB \\ SB \\ SB \end{bmatrix}, K = [0 \ \underline{K} \ \bar{K}],$$

$$\Gamma = \begin{bmatrix} 0 & 0 & 0 \\ 0 & I_N & 0 \\ 0 & 0 & I_N \end{bmatrix}, \tilde{\Delta}(t) = \begin{bmatrix} SGd(t) \\ \underline{\delta}(t) + SLv(t) \\ \bar{\delta}(t) + SLv(t) \end{bmatrix}.$$

Proposition 2.3. *Let assumptions 2.2 and 2.3 be satisfied. Then for (2.3), (2.9) with the interval observer (2.12) and the control law (2.16) the relations (2.10) are satisfied. In addition, $\underline{\xi}, \bar{\xi}, \bar{\xi} \in L^\infty([0, T), \mathbb{R}^N)$ if there exists a matrix $\tilde{X} \in \mathbb{R}^{3N \times 3N}$ in the form*

$$\tilde{X} = \begin{bmatrix} X_P & 0 & 0 \\ 0 & & \\ 0 & X_Q & \end{bmatrix}, X_P = X_P^T \in \mathbb{R}^{N \times N}, X_Q = X_Q^T \in \mathbb{R}^{2N \times 2N} \quad (2.18)$$

and $\tilde{Y} \in \mathbb{R}^{m \times 2N}$ that satisfy the matrix inequalities

$$\tilde{A}\tilde{X} + \tilde{X}\tilde{A}^T + \tilde{B} \begin{bmatrix} 0 & \tilde{Y} \end{bmatrix} + \begin{bmatrix} 0 & \tilde{Y} \end{bmatrix}^T \tilde{B}^T < 0, \quad X_P > 0, \quad X_Q > 0, \quad (2.19)$$

then $[\underline{K} \ \bar{K}] = \tilde{Y}X_Q^{-1}$ and (2.3), (2.9), (2.12), (2.16) is stable.

Proof 2.6. The relations (2.10) can be substantiated repeating the same arguments as previously in Proposition 2.2 (they are independent in control).

Substitution of the control (2.16) into the equations of the interval observer (2.12) together with the actual system (2.11) will give us the equations of the combined system (2.17), in which $\tilde{\Delta}(t)$ is bounded since the signals $d(t)$, $\underline{\delta}(t)$, $\bar{\delta}(t)$ and $v(t)$ are bounded by Assumption 2.2 and previous calculations made in Section 2.2. Calculating derivative of the Lyapunov function $V(\eta) = \eta^T \tilde{P} \eta$ we obtain

$$\begin{aligned} \dot{V} &= \eta^T [(\tilde{A} + \tilde{B}K\Gamma)^T \tilde{P} + \tilde{P}(\tilde{A} + \tilde{B}K\Gamma)] \eta + 2\eta^T \tilde{P} \tilde{\Delta} \\ &\leq \eta^T [(\tilde{A} + \tilde{B}K\Gamma)^T \tilde{P} + \tilde{P}(\tilde{A} + \tilde{B}K\Gamma) + \chi I_{3N}] \eta + \chi^{-1} \tilde{\Delta}^T \tilde{P}^2 \tilde{\Delta} \end{aligned}$$

for some $\chi > 0$. Therefore, to prove the proposition we need to ensure stability of the matrix $\tilde{A} + \tilde{B}K\Gamma$ by verifying the Lyapunov equation for the matrix $\tilde{P} = \tilde{X}^{-1}$:

$$(\tilde{A} + \tilde{B}K\Gamma)^T \tilde{P} + \tilde{P}(\tilde{A} + \tilde{B}K\Gamma) < 0,$$

which can be rewritten as

$$\tilde{A}\tilde{X} + \tilde{X}\tilde{A}^T + \tilde{B}K\Gamma\tilde{X} + \tilde{X}\Gamma^T K^T \tilde{B}^T < 0,$$

and after the transformation

$$\tilde{B}K\Gamma\tilde{X} = \tilde{B}K \begin{bmatrix} 0 & 0 & 0 \\ 0 & & \\ 0 & X_Q & \end{bmatrix} = \tilde{B} \begin{bmatrix} 0 & \tilde{Y} \end{bmatrix}$$

it is equivalent to the stated LMI (2.19) for the variable $\tilde{Y} = [\underline{K} \ \bar{K}]X_Q$.

Remark 2.5. In order to regulate the estimation accuracy it is worth to strengthen the conditions of stability for η to a requirement that the H_∞ gain of the transfer $\tilde{\Delta} \rightarrow z$ is less than γ for some $\gamma > 0$, where $z = H\eta$ is an auxiliary performance output (for example, $z = \bar{\zeta} - \underline{\zeta}$ characterizes the interval estimation accuracy). To this end, consider again the

Lyapunov function $V(\eta) = \eta^T \tilde{P} \eta$ whose derivative can be rewritten as follows:

$$\dot{V} = \begin{pmatrix} \eta \\ \tilde{\Delta} \end{pmatrix} \begin{pmatrix} (\tilde{A} + \tilde{B}K\Gamma)^T \tilde{P} + \tilde{P}(\tilde{A} + \tilde{B}K\Gamma) + H^T H & \tilde{P} \\ \tilde{P} & -\gamma^2 I_{3N} \end{pmatrix} \begin{pmatrix} \eta \\ \tilde{\Delta} \end{pmatrix} - z^T z + \gamma^2 \tilde{\Delta}^T \tilde{\Delta}.$$

As in the proof of Proposition 2.3 denote $\tilde{P}^{-1} = \tilde{X}$, then the following linear matrix equality can be obtained:

$$\begin{pmatrix} \tilde{X} & 0 \\ 0 & I_{3N} \end{pmatrix} \begin{pmatrix} (\tilde{A} + \tilde{B}K\Gamma)^T \tilde{P} + \tilde{P}(\tilde{A} + \tilde{B}K\Gamma) + H^T H & \tilde{P} \\ \tilde{P} & -\gamma^2 I_{3N} \end{pmatrix} \begin{pmatrix} \tilde{X} & 0 \\ 0 & I_{3N} \end{pmatrix} \\ = \begin{pmatrix} \tilde{A}\tilde{X} + \tilde{X}\tilde{A}^T + \tilde{B} \begin{bmatrix} 0 & \tilde{Y} \end{bmatrix} + \begin{bmatrix} 0 & \tilde{Y} \end{bmatrix}^T \tilde{B}^T + \tilde{X}H^T H \tilde{X} & I_{3N} \\ I_{3N} & -\gamma^2 I_{3N} \end{pmatrix}$$

where

$$\begin{bmatrix} 0 & \tilde{Y} \end{bmatrix} = K\Gamma\tilde{X}$$

and $\tilde{Y} \in \mathbb{R}^{m \times 2N}$ is a new matrix variable. Finally, using Schur complement we derive an LMI:

$$\begin{bmatrix} \tilde{A}\tilde{X} + \tilde{X}\tilde{A}^T + \tilde{B} \begin{bmatrix} 0 & \tilde{Y} \end{bmatrix} + \begin{bmatrix} 0 & \tilde{Y} \end{bmatrix}^T \tilde{B}^T & I_{3N} & \tilde{X}H \\ I_{3N} & -\gamma^2 I_{3N} & 0_{3N} \\ H\tilde{X} & 0_{3N} & -I_{3N} \end{bmatrix} < 0, \quad (2.20)$$

$$X_P > 0, X_Q > 0$$

which has to be solved with respect to the matrices X_P , X_Q and \tilde{Y} , then $[\underline{K} \ \bar{K}] = \tilde{Y}X_Q^{-1}$ as in Proposition 2.3.

Remark 2.6. The required gains \underline{K} and \bar{K} exist if the matrix pair (A, B) is controllable (stabilizable). Indeed, impose a restriction that $\underline{K} = \bar{K} = 0.5KS^{-1}$, where $K \in \mathbb{R}^{m \times N}$ now is a new controller gain to find, and consider two auxiliary variables

$$e(t) = \zeta(t) - \frac{\underline{\zeta}(t) + \bar{\zeta}(t)}{2}, \quad w(t) = \bar{\zeta}(t) - \underline{\zeta}(t),$$

which correspond to a regulation error with respect to the middle value of the estimated interval and the interval width, and whose dynamics is as follows:

$$\dot{e}(t) = De(t) + \delta(t) - \frac{\underline{\delta}(t) + \bar{\delta}(t)}{2}, \quad \dot{w}(t) = Dw(t) + \bar{\delta}(t) - \underline{\delta}(t).$$

Obviously, to study the stability property of (2.3), (2.9), (2.12), (2.16), instead of analysis of the vector η its linear transformation $\eta^* = [\zeta^T \ e^T \ w^T]^T \in \mathbb{R}^{3N}$ can be considered, then $u(t) = KS^{-1}(\zeta(t) - e(t))$ and

$$\dot{\eta}^*(t) = A^* \eta^*(t) + \Delta^*(t),$$

where

$$A^* = \begin{bmatrix} S(A+BK)S^{-1} & -SBKS^{-1} & 0 \\ 0 & D & 0 \\ 0 & 0 & D \end{bmatrix}, \Delta^*(t) = \begin{bmatrix} SGd(t) \\ \delta(t) - \frac{\delta(t) + \bar{\delta}(t)}{2} \\ \bar{\delta}(t) - \underline{\delta}(t) \end{bmatrix}.$$

As before, the vector $\Delta^*(t)$ is bounded, and the system stability follows the same property of the matrix A^* , which has an upper-triangular structure and if the matrix D is stable (*i.e.* the pair (A, C) is observable), then as in the conventional case the separation principle holds and the required conclusion can be justified for a stable matrix $A + BK$. Note that the use in the control of both bounds, $\underline{\zeta}$ and $\bar{\zeta}$, allows to compensate the dependence of the dynamics of ζ on the interval width w , contrarily the case when only one bound is used, as in Mazenc et al., 2013.

Now, in order to prove boundedness of the state of (2.1) with application of the control (2.16), let us replace Assumption 2.1 with the following one:

Assumption 2.4. *Let the coefficients of (2.1), $a, b, q, p \in L^\infty(I, \mathbb{R})$, be smooth on I , $z_0(x) \in H_0^1(I, \mathbb{R})$, $r_0 \in L^2([0, T], L^2(I, \mathbb{R}))$, $r_1 \in L^2(I, \mathbb{R})$ and for any $u \in C^1([0, T], \mathbb{R})$ there exists a weak solution of (2.1) $z \in L^2([0, T], H_0^1(I, \mathbb{R}))$ with $\dot{z}_t \in L^2([0, T], H^{-1}(I, \mathbb{R}))$.*

Using this regularity hypothesis it is possible to substantiate stabilization by the control (2.16) of the distributed-parameter system (2.1):

Theorem 2.2. *Let assumptions 2.2, 2.3 and 2.4 be satisfied. Then for the system (2.1), (2.8) with the interval observer (2.5), (2.12) and the control (2.16), the relations (2.6) are satisfied, and $z, \underline{z}, \bar{z} \in L^\infty(I \times [0, T], \mathbb{R})$.*

Proof 2.7. According to Evans, 1998 with the restrictions on the coefficients, input and initial conditions of the system (2.1) introduced in Assumption 2.4, it holds that

$$\begin{aligned} z &\in L^2([0, T], H^2(I, \mathbb{R})) \cap L^\infty([0, T], H_0^1(I, \mathbb{R})), \\ \dot{z}_t &\in L^2([0, T], L^2(I, \mathbb{R})), \end{aligned}$$

and we have the estimate

$$\begin{aligned} \operatorname{esssup}_{0 \leq t \leq T} \|z(\cdot, t)\|_{H_0^1(I, \mathbb{R})} + \|z(\cdot, t)\|_{L^2([0, T], H^2(I, \mathbb{R}))} + \|\dot{z}_t(\cdot, t)\|_{L^2([0, T], L^2(I, \mathbb{R}))} \\ \leq c \left(\|r(\cdot, t)\|_{L^2([0, T], L^2(I, \mathbb{R}))} + \|z_0\|_{H_0^1(I, \mathbb{R})} \right), \end{aligned}$$

where the constant c is depending only on I , T and the coefficient functions a, b, q, p . Note that the above inequality does not imply boundedness of z , and it only states its existence on the interval of time $[0, T)$ (a kind of forward completeness in time). Therefore, Assumption 2.1 is valid for some $l_1 > 0$ and $l_2 > 0$, then we can apply the result of Proposition 2.1 to estimate the error of the approximation. Finally, since all conditions are satisfied, the results of Proposition 2.3 and Lemma 2.3 are true, then the conclusion on boundedness of $z, \underline{z}, \bar{z}$ follows.

2.4 Example

Academic example with control

Consider an unstable academic example of (2.1) with

$$\begin{aligned} \rho(x) &= 0.7 \sin(0.67x), \quad a(x) = 1.5 + 1.5 \cos(0.2x^{0.25}), \quad b(x) = -2 + \sin(2\sqrt{x}), \\ q(x) &= -0.8 - x^2 \cos(3x), \quad r_1(x) = x^3 + 2.5, \\ r_0(x, t) &= r_{01}(x)r_{02}(t), \quad r_{01}(x) = 0.1 \cos(3\pi x), \quad |r_{02}(t)| \leq 1, \end{aligned}$$

and $T = 10$, then $\lambda = 1$ is an admissible choice and r_{02} is an uncertain part of the input r_0 (for simulation $r_{02}(t) = \cos(5t)$), then

$$\underline{r}_0(x, t) = -|r_{01}(x)|, \quad \bar{r}_0(x, t) = |r_{01}(x)|.$$

The uncertainty of initial conditions is given by

$$\underline{z}_0(x) = z_0(x) - 1, \quad \bar{z}_0(x) = z_0(x) + 1,$$

where $z_0(x) = \sin(\pi x)$ is the function used as the initial condition for simulation. Take $\Delta = \{0, h, 2h, \dots, 1 - h, 1\}$ with $h = 1/N'$, and a pyramidal basis

$$\Phi_i(x) = \begin{cases} 0 & x \leq x_{i-1}, \\ \frac{x-x_{i-1}}{x_i-x_{i-1}} & x_{i-1} < x \leq x_i, \\ \frac{x_{i+1}-x}{x_{i+1}-x_i} & x_i < x \leq x_{i+1}, \\ 0 & x \geq x_{i+1} \end{cases} \quad (2.21)$$

for $i = 0, \dots, N = N'$ (it is assumed $x_{-1} = -h$ and $x_{N+1} = 1 + h$). For simulation we took $N = 10$, then the approximated dynamics (2.3), (2.9) is an observable system, and assume that $\rho h^{s+1}(l_1 + l_2) \leq \bar{\rho} = 0.1$. Let $p = 3$ with $x_1^m = 0.2$, $x_2^m = 0.5$, $x_3^m = 0.8$, and

$$v(t) = 0.1[\sin(20t) \sin(15t) \cos(25t)]^T,$$

then $v_0 = 0.14$. For calculation of the scalar product in space or for simulation of the approximated PDE in time, the explicit Euler method has been used with the step 0.01. The matrix L is selected to ensure distinct eigenvalues of the matrix $A - LC$ in the interval $[-10.22, -1.4]$, then S^{-1} is composed by eigenvectors of the matrix $A - LC$ and the matrix D is chosen diagonal.

To calculate the control matrix $[\underline{K} \ \bar{K}]$ the LMIs (2.20) has been used with YALMIP optimization toolbox in Matlab, and it is found with $\gamma = 1.1505$ that

$$\underline{K} = \bar{K} = [-0.034 \ 0.126 \ 0.122 \ 0.076 \ 0.185 \ -0.018 \ -0.508 \ -0.022].$$

The results of the interval estimation and control are shown in the Fig. 2.1 for different instances of time, where red lines corresponds to $Z(x, t)$, while green and blue ones represent $\underline{z}(x, t)$ and $\bar{z}(x, t)$, respectively (20 and 40 points are used for plotting in space and in time).

Black–Scholes model

The Black–Scholes PDE governs the price evolution of an option under the so-called Black–Scholes model (a mathematical model of a financial market containing derivative

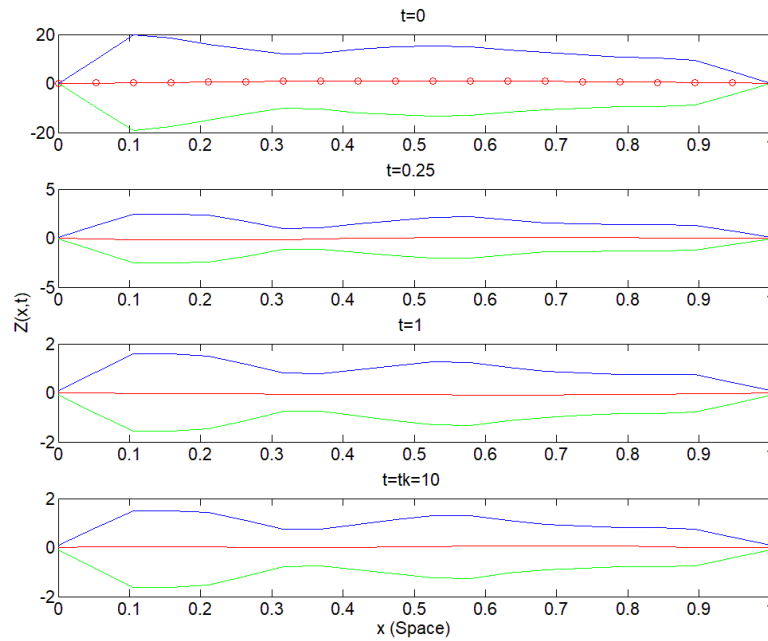


Figure 2.1 – The results of the interval estimation of the academic example for different instants of time: $t = 0, 0.25, 1, 10$ for $N = 10$

investment instruments¹):

$$\begin{aligned} \frac{\partial \mathcal{V}(S, t)}{\partial t} &= -\frac{1}{2} \sigma^2 S^2 \frac{\partial^2 \mathcal{V}(S, t)}{\partial S^2} - (r(t) - q(t)) S \frac{\partial \mathcal{V}(S, t)}{\partial S} \\ &\quad + r(t) \mathcal{V}(S, t) + g(S) \quad \forall (S, t) \in I \times (0, T), \\ \mathcal{V}(S, 0) &= \mathcal{V}_0 \quad \forall S \in I, \\ 0 &= \mathcal{V}(0, t) = \mathcal{V}(1, t) \quad \forall t \in (0, T), \end{aligned}$$

where $\mathcal{V}(S, t)$ is the price of the option, S is the stock price belonging a given interval of admissible prices I ; r is the risk-free interest rate, q is the dividend rate of the underlying asset, σ is the volatility of the stock; and $g(S)$ is an inhomogeneous term Butler and Schachter, 1986; Hyong-Chol et al., 2016. Obviously, this equation can be presented in the form of (2.1) with the following parameters ($x = S$ for a normalized price):

$$\begin{aligned} a(x) &= -0.16x^2(2 + \sin(x))\cos(x) - 0.16x(2 + \sin(x))^2, \\ \rho(x) &= 1, \quad b(x) = -0.06x, \quad q(x) = 0.06, \quad r_1(x) = 0.8x^2 - 1, \\ r_0(x, t) &= r_{01}(x)r_{02}(t), \quad r_{01}(x) = 0.2\sin^2(3\pi x), \quad |r_{02}(t)| \leq 1, \end{aligned}$$

¹https://en.wikipedia.org/wiki/Black-Scholes_equation

<https://www.theguardian.com/science/2012/feb/12/black-scholes-equation-credit-crunch>

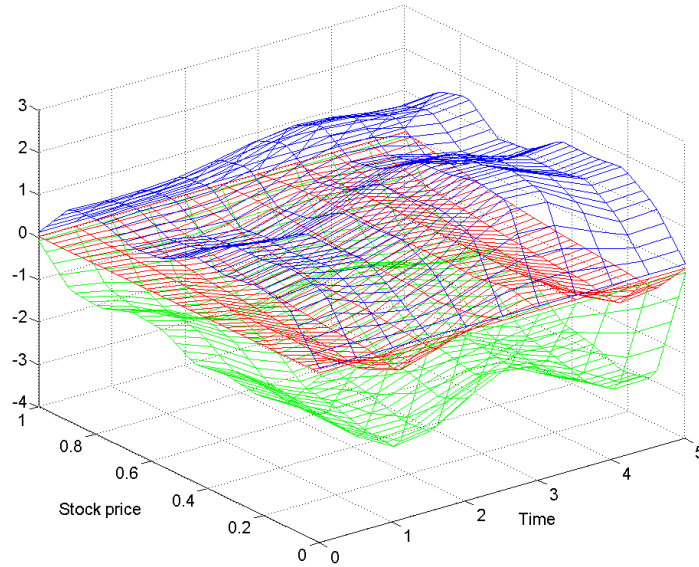


Figure 2.2 – The results of the interval estimation for the Black-Scholes model

with $T = 5$, then $\lambda = 1$ is an admissible choice and r_{02} is an uncertain part of the input r_0 , the uncertainty of initial conditions (for simulation $z_0(x) = \max(x - 25e^{-0.06x}, 0)$) is given by the interval

$$\underline{z}_0(x) = \max(z_0(x) - 0.1, 0), \quad \bar{z}_0(x) = z_0(x) + 0.1.$$

The decomposition basis (2.21) is taken as in the previous example, points for measurements with $p = 3$ are $x_1^m = 0.2$, $x_2^m = 0.5$, $x_3^m = 0.8$, and

$$v(t) = 0.2[\cos(2t) \quad \sin(1.8t) \quad \cos(3t)]^T,$$

then $v_0 = 0.217$. For calculation of the scalar product in space or for simulation of the approximated PDE in time, the implicit Euler method is used with the step 0.01. The matrix L is selected to ensure distinct eigenvalues of the matrix $A - LC$ in the interval $[-8.63, -0.72]$, then S^{-1} is composed by eigenvectors of $A - LC$ and the matrix D is chosen diagonal. The results of the interval estimation are shown in Fig. 2.2, where the red surface corresponds to $Z(x, t)$, while green and blue ones represent $\underline{z}(x, t)$ and $\bar{z}(x, t)$, respectively (20 and 40 points are used for plotting in space and in time). In the Fig. 2.3 the interval estimates are shown for different instants of time.

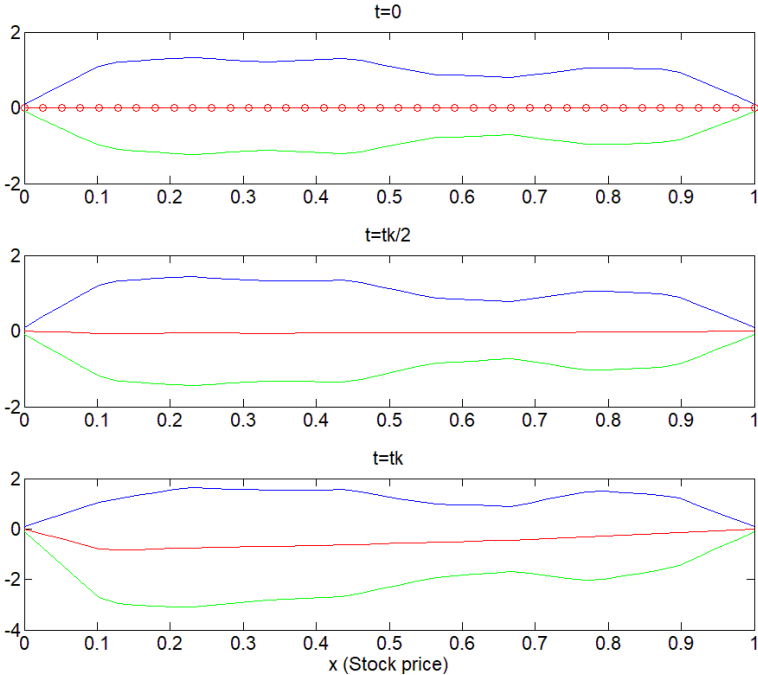


Figure 2.3 – The results of the interval estimation of the Black Scholes model at instant of time $t = 0, \frac{T}{2}, T$

Interval observer design and control of uncertain non-homogeneous heat equations

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The main structure of the present chapter is as follows. First, an interval observer described by PDEs without applying finite-element approximations is proposed for uncertain distributed parameter systems. Second, an additional design of an output stabilizing control is performed based on interval observations. The estimation error dynamics (also distributed) of the proposed interval observer is guaranteed to be positive following the conditions of positivity of solutions of parabolic PDEs presented in Nguyen and Coron, 2016. The stability analysis from Liu and Fridman, 2012 is also extended to the considered scenario with non-zero measurement noise and boundary conditions, and further applied for a stabilizing control synthesis for an unstable PDE. An advantage of using interval observers, over Luenberger type observers of Liu and Fridman, 2012 and approximation-based interval observer of Kharkovskaia et al., 2018b, consists in calculation on-line of accurate bounds explicitly on the given distributed trajectories. It is assumed that the control is spatially distributed influencing the system dynamics through shape functions.

Such a hypothesis is introduced to respect the implementation feasibility of the designed control law, since infinitesimal in space variations of the actuator signal cannot be realized in practice. It is worth to highlight that here such a restriction on shape functions is not related with any early lumping procedure.

3.1 Input-to-state stability and positivity of non-homogeneous heat equation

In this section the basic facts on heat equation and positivity of its solutions are given.

Heat equation

Consider the following PDE with associated boundary conditions:

$$\begin{aligned} \frac{\partial z(x,t)}{\partial t} &= L[x, z(x,t)] + r(x,t) + \sum_{j=0}^p b_j(x)u_j(t) \quad \forall (x,t) \in I \times \mathcal{T}, \\ z(x, t_0) &= z_0(x) \quad \forall x \in I, \\ z(0, t) &= \alpha(t), \quad z(\ell, t) = \beta(t) \quad \forall t \in \mathcal{T}, \end{aligned} \quad (3.1)$$

where $I = [0, \ell]$ with $0 < \ell < +\infty$, $\mathcal{T} = [t_0, t_0 + T)$ for $t_0 \in \mathbb{R}$ and $T > 0$,

$$L(x, z) = \frac{\partial}{\partial x} \left(a(x) \frac{\partial z}{\partial x} \right) + q(x)z,$$

$a \in C^1(I, \mathbb{R})$, $q \in C(I, \mathbb{R})$ and there exist $a_{\min}, a_{\max} \in \mathbb{R}_+$ such that

$$0 < a_{\min} \leq a(x) \leq a_{\max} \quad \forall x \in I;$$

the boundary conditions $\alpha, \beta \in C^2(\mathcal{T}, \mathbb{R})$ and the external input $r \in C^1(I \times \mathcal{T}, \mathbb{R})$; the initial conditions $z_0 \in \mathcal{Z}_0 = \{z_0 \in H^2(I, \mathbb{R}) : z_0(0) = \alpha(0), z_0(\ell) = \beta(0)\}$; the controls $u_j : \mathcal{T} \rightarrow \mathbb{R}$ are Lipschitz continuous functions. The space domain I is divided into $p + 1$ subdomains I_j for $j = 0, 1, \dots, p$, where the control signals $u_j(t)$ are applied through the shape functions $b_j \in L^2(I, [0, 1])$ such that

$$\begin{cases} b_j(x) = 0 & x \notin I_j, \\ b_j(x) = 1 & x \in I_j. \end{cases} \quad (3.2)$$

The controls u_j are designed in Section 3.3, in Sections 3.1 and 3.2 they are assumed to be given and $u_j \in L^\infty(\mathcal{T}, \mathbb{R})$ for all $j = 0, 1, \dots, p$.

Proposition 3.1. *Assume*

$$a_{\min} \frac{\pi^2}{\ell^2} > q_{\max}, \quad (3.3)$$

where $q_{\max} = \sup_{x \in I} q(x)$, then for the solutions of (3.1) the following estimate is satisfied for all $t \in \mathcal{T}$:

$$\frac{1}{2} \int_0^\ell z^2(x, t) dx \leq e^{-\chi(t-t_0)} \int_0^\ell w_0^2(x) dx + \chi^{-2} \int_0^\ell \tilde{r}^2(x, t) dx + \frac{\ell}{2} [\alpha^2(t) + \beta^2(t)], \quad (3.4)$$

where $\chi = a_{\min} \frac{\pi^2}{\ell^2} - q_{\max}$, $w_0(x) = z_0(x) - \delta(x, t_0)$, $\delta(x, t) = \alpha(t) + \frac{x}{\ell}(\beta(t) - \alpha(t))$, and

$$\tilde{r}(x, t) = r(x, t) + \frac{1}{\ell} \frac{\partial a(x)}{\partial x} (\beta(t) - \alpha(t)) + q(x) \delta(x, t) - \delta_t(x, t) + \sum_{j=0}^p b_j(x) u_j(t). \quad (3.5)$$

Proof 3.1. Denote $w(x, t) = z(x, t) - \delta(x, t)$, then

$$\frac{\partial w(x, t)}{\partial t} = L[x, w(x, t)] + \tilde{r}(x, t) \quad \forall (x, t) \in I \times \mathcal{T},$$

$$w(x, t_0) = w_0(x) \quad \forall x \in I, \quad (3.6)$$

$$w(0, t) = w(\ell, t) = 0 \quad \forall t \in \mathcal{T}. \quad (3.7)$$

We start with the well-posedness analysis of the system (3.6) under Dirichlet boundary conditions (3.7). The boundary-value problem (3.6) can be represented as an abstract differential equation

$$\dot{\zeta}(t) = A\zeta(t) + F(t, \zeta(t)), \quad t \geq t_0, \quad \zeta(t_0) = \zeta_0 \quad (3.8)$$

in the Hilbert space $L^2(I, \mathbb{R})$, where the operator $A = \frac{\partial}{\partial x} \left(a(x) \frac{\partial}{\partial x} \right)$ has the dense domain $\mathcal{D}(A) = \{\zeta \in H^2(I, \mathbb{R}) : \zeta(0) = \zeta(\ell) = 0\}$. The nonlinear term $F : \mathcal{T} \times L^2(I, \mathbb{R}) \rightarrow L^2(I, \mathbb{R})$ is defined on functions $\zeta(\cdot, t)$ according to

$$F(t, \zeta(x, t)) = q(x) \zeta(x, t) + \tilde{r}(x, t),$$

where $\tilde{r}(x, t)$ is given in the equation (3.5). It is a well-known fact that A generates a strongly continuous exponentially stable semigroup Φ , which satisfies the inequality $\|\Phi(t)\| \leq \kappa e^{-\rho t}$ for all $t \geq 0$ with some constant $\kappa \geq 1$ and decay rate $\rho > 0$.

By introduced restrictions on the initial and boundary conditions $\alpha(t)$, $\beta(t)$ and $\delta(x, t)$

in the PDE (3.6) and if $u_j(t)$ is Lipschitz continuous in t , then $F(t, \zeta)$ is Lipschitz continuous in both variables:

$$\|F(t_1, \zeta_1) - F(t_2, \zeta_2)\|_{L^2(I, \mathbb{R})} \leq L_1 |t_1 - t_2| + L_2 \|\zeta_1 - \zeta_2\|_{L^2(I, \mathbb{R})}$$

for all $t_1, t_2 \in \mathcal{T}$ and $\zeta_1, \zeta_2 \in L^2(I, \mathbb{R})$, with some $L_1 > 0$ and $L_2 > 0$. Therefore, for all $\zeta_0 \in \mathcal{D}(A)$ there exists a strong solution of the initial value problem (3.8) in $C(\mathcal{T}, L^2(I, \mathbb{R}))$ by Pazy, 1983, Theorem 6.1.6.

Now consider for (3.6) the following Lyapunov function

$$V(t) = \int_0^\ell w^2(x, t) dx.$$

We have

$$\dot{V}(t) = 2 \int_0^\ell w(x, t) \left[\frac{\partial}{\partial x} (a(x)w_x(x, t)) + q(x)w(x, t) + \tilde{r}(x, t) \right] dx.$$

Integrating by parts and substituting the boundary conditions of $w(x, t)$ lead to

$$\begin{aligned} \dot{V}(t) &= 2a(x)w(x, t)w_x(x, t)|_0^\ell - 2 \int_0^\ell a(x)w_x^2(x, t) dx \\ &\quad + 2 \int_0^\ell q(x)w^2(x, t) + w(x, t)\tilde{r}(x, t) dx \\ &= 2 \int_0^\ell q(x)w^2(x, t) - a(x)w_x^2(x, t) + w(x, t)\tilde{r}(x, t) dx. \end{aligned}$$

Using Wirtinger's inequality (1.30) and Young's inequality Hardy et al., 1988,

$$2w(x, t)\tilde{r}(x, t) \leq \chi w^2(x, t) + \chi^{-1} \tilde{r}^2(x, t),$$

we obtain (recall that $\chi = a_{\min} \frac{\pi^2}{\ell^2} - q_{\max}$, see the formulation of the proposition):

$$\begin{aligned} \dot{V}(t) &\leq -2(a_{\min} \frac{\pi^2}{\ell^2} - q_{\max}) \int_0^\ell w^2(x, t) dx + 2 \int_0^\ell w(x, t)\tilde{r}(x, t) dx \\ &\leq -\chi V(t) + \chi^{-1} \int_0^\ell \tilde{r}^2(x, t) dx. \end{aligned}$$

Therefore, if $\chi > 0$ then the system (3.6) has bounded solutions:

$$\begin{aligned} \int_0^\ell z^2(x,t)dx &\leq 2V(t) + 2 \int_0^\ell \delta^2(x,t)dx \\ &\leq 2(e^{-\chi(t-t_0)}V(t_0) + \chi^{-2} \int_0^\ell \tilde{r}^2(x,t)dx + \frac{\ell}{2}[\alpha^2(t) + \beta^2(t)]) \end{aligned}$$

for all $t \in \mathcal{T}$, that completes the proof.

Consequently, Proposition 3.1 fixes the conditions under which the distributed parameter system (3.1) possesses the input-to-state stability (ISS) property Dashkovskiy et al., 2011; Dashkovskiy and Mironchenko, 2013 with respect to the boundary conditions α , β , the external disturbance r and the control signals u_j . The main restriction of that proposition is (3.3) and can be easily validated for a sufficiently small ℓ .

Note that after a straightforward calculus the estimate from Proposition 3.1 can be rewritten as follows for all $t \in \mathcal{T}$:

$$\|z(\cdot, t)\|_{L^2(I, \mathbb{R})}^2 \leq 4e^{-\chi(t-t_0)}[\|z_0\|_{L^2(I, \mathbb{R})}^2 + \rho(t_0)] + 8\chi^{-2}\|r(\cdot, t)\|_{L^2(I, \mathbb{R})}^2 + \gamma(t),$$

where $\rho(t) = \frac{\ell}{2}[\alpha^2(t) + \beta^2(t)]$ (weighted norm of the boundary conditions), $\gamma(t) = 8\chi^{-2}\rho'(t) + 2(1 + 4\frac{q_{\max}^2}{\chi^2} + 16\frac{\partial a_{\max}^2}{\chi^2\ell^2})\rho(t)$ and $\rho'(t) = \frac{\ell}{2}[\dot{\alpha}^2(t) + \dot{\beta}^2(t)]$ (weighted norm of derivative of the boundary conditions) are all bounded functions of time $t \in \mathcal{T}$, $\partial a_{\max} = \sup_{x \in I} \frac{\partial a(x)}{\partial x}$.

Positivity of solutions

In general, the solution $z(\cdot, t)$ of (3.1) takes its values in \mathbb{R} and it can change sign with $(x, t) \in I \times \mathcal{T}$. For brevity of presentation of the results of this subsection we will always assume that $u_j(t) = 0$ for all $t \in \mathcal{T}$ and $j = 0, 1, \dots, p$.

Definition 3.1. The system (3.1) with $u_j(t) = 0$ for all $j = 0, 1, \dots, p$ is called nonnegative (positive) on the interval \mathcal{T} if for

$$\alpha(t) \geq 0, \beta(t) \geq 0, r(x, t) \geq 0 \quad \forall (x, t) \in I \times \mathcal{T}$$

the implication $z_0(x) \geq 0 \Rightarrow z(x, t) \geq 0$ ($z_0(x) > 0 \Rightarrow z(x, t) > 0$) holds for all $(x, t) \in I \times \mathcal{T}$ and for all $z_0 \in \mathcal{Z}_0$.

A well-known example of a nonnegative system is non-homogeneous heat equation

defined over $x \in (-\infty, +\infty)$:

$$\begin{aligned} \frac{\partial \zeta(x, t)}{\partial t} &= a \frac{\partial^2 \zeta(x, t)}{\partial^2 x} + r(x, t) \quad \forall (x, t) \in \mathbb{R} \times \mathcal{T}, \\ \zeta(x, 0) &= \zeta_0(x) \quad \forall x \in \mathbb{R}, \end{aligned} \quad (3.9)$$

where $a > 0$ is a constant, $q = 0$ and $\zeta_0 : \mathbb{R} \rightarrow \mathbb{R}_+$, whose solution can be calculated analytically using Green's function (fundamental solution or the heat kernel) Thomée, 2006:

$$\begin{aligned} \zeta(x, t) &= \frac{1}{2\sqrt{\pi a t}} \int_{-\infty}^{+\infty} e^{-\frac{(x-y)^2}{4at}} \zeta_0(y) dy \\ &+ \int_0^t \int_{-\infty}^{+\infty} \frac{e^{-\frac{(x-y)^2}{4a(t-s)}}}{2\sqrt{\pi a(t-s)}} r(y, s) dy ds. \end{aligned}$$

It is straightforward to verify that for nonnegative ζ_0 and r the expression in the right-hand side stays nonnegative for all $(x, t) \in \mathbb{R} \times (0, +\infty)$. This conclusion is valid for the case $x \in \mathbb{R}$. However, if $x \in I$, even the homogenous heat equation (3.9) with $r(x, t) = 0$ for all $(x, t) \in I \times \mathcal{T}$, and with the boundary condition

$$0 = \zeta(0, t) = \zeta(\ell, t) \quad \forall t \in \mathcal{T} \quad (3.10)$$

admits the solution in the form Thomée, 2006:

$$\begin{aligned} \zeta(x, t) &= \sum_{n=1}^{+\infty} D_n \sin\left(\frac{n\pi x}{\ell}\right) e^{-a \frac{n^2 \pi^2}{\ell^2} t}, \\ D_n &= \frac{2}{\ell} \int_0^\ell \zeta_0(x) \sin\left(\frac{n\pi x}{\ell}\right) dx, \end{aligned}$$

whose positivity is less trivial to establish.

For this reason, using Maximum principle Friedman, 1964 the following general result has been proven in Nguyen and Coron, 2016:

Proposition 3.2. *Let $\alpha, \beta \in L^2(\mathcal{T}, \mathbb{R}_+)$, $r \in L^2(I \times \mathcal{T}, \mathbb{R}_+)$ and $z_0 \in H^1(I, \mathbb{R}_+)$, then*

$$z(x, t) \geq 0 \quad \forall (x, t) \in I \times \mathcal{T},$$

i.e. (3.1) with $u_j(t) = 0$ for all $j = 0, 1, \dots, p$ is nonnegative on the interval \mathcal{T} .

Therefore, if boundary and initial conditions, and external inputs, take only nonnegative values, then the solutions of (3.1) possess the same property.

3.2 Interval observer design for the heat equation

Consider (3.1) with some uncertain boundary conditions $\alpha, \beta \in C^2(\mathcal{T}, \mathbb{R})$, an uncertain external input $r \in C(I \times \mathcal{T}, \mathbb{R})$ and initial conditions $z_0 \in \mathcal{Z}_0$, and assume that the state $z(x, t)$ is available for measurements in certain points $0 < x_1^m < x_2^m < \dots < x_p^m < \ell$:

$$y_j(t) = z(x_j^m, t) + v_j(t), \quad j = 1, \dots, p, \quad (3.11)$$

where $y(t) = [y_1(t), \dots, y_p(t)]^T \in \mathbb{R}^p$ is the measured output signal, $v(t) = [v_1(t), \dots, v_p(t)] \in \mathbb{R}^p$ is the output disturbance (measurement noise). Design of a conventional observer under similar conditions has been studied in Liu and Fridman, 2012; Schaum et al., 2014. Further, to simplify the technical presentation (to simplify the proof of well-posedness of the estimation error dynamics) we assume differentiability of the output disturbance:

Assumption 3.1. Let $v \in C^2(\mathcal{T}, \mathbb{R}^p)$.

A goal of the work consists in design of interval observers for the distributed parameter system (3.1), (3.11). For this purpose we need the following hypothesis.

Assumption 3.2. Let $\underline{z}_0 \leq z_0 \leq \bar{z}_0$ for some known $\underline{z}_0, \bar{z}_0 \in \mathcal{Z}_0$, let also functions $\underline{\alpha}, \bar{\alpha}, \underline{\beta}, \bar{\beta} \in C^2(\mathcal{T}, \mathbb{R})$, $\underline{r}, \bar{r} \in C^1(I \times \mathcal{T}, \mathbb{R})$ and a constant $\nu_0 > 0$ be given such that for all $(x, t) \in I \times \mathcal{T}$:

$$\begin{aligned} \underline{\alpha}(t) \leq \alpha(t) \leq \bar{\alpha}(t), \quad \underline{\beta}(t) \leq \beta(t) \leq \bar{\beta}(t), \\ \underline{r}(x, t) \leq r(x, t) \leq \bar{r}(x, t), \quad |v(t)| \leq \nu_0. \end{aligned}$$

Thus, by Assumption 3.2 five intervals, $[\underline{\alpha}(t), \bar{\alpha}(t)]$, $[\underline{\beta}(t), \bar{\beta}(t)]$, $[\underline{z}_0, \bar{z}_0]$, $[\underline{r}(x, t), \bar{r}(x, t)]$ and $[-\nu_0, \nu_0]$, determine for all $(x, t) \in I \times \mathcal{T}$ in (3.1), (3.11) the uncertainty of the values for $\alpha(t)$, $\beta(t)$, z_0 , $r(x, t)$ and $v(t)$, respectively.

Remark 3.1. These imperfections can be related with various reasons, *e.g.* unknown parameters, external signals, nonlinearities, *etc.*, but they have to be included in the corresponding intervals. For example, consider even more complicated case, let

$$r(z, x, t) = \theta_1 \tilde{r}(x, t) + \theta_2(z, x, t),$$

where

$$\theta_1 \in [\underline{\theta}_1, \bar{\theta}_1]$$

is an unknown parameter taking values in the given interval $[\underline{\theta}_1, \bar{\theta}_1]$, $\tilde{r} : I \times \mathcal{T} \rightarrow \mathbb{R}_+$ is a known function and $\theta_2 : L^2(I, \mathbb{R}) \times I \times \mathcal{T} \rightarrow [\underline{\theta}_2, \bar{\theta}_2]$ is an unknown function taking values

in the given set $[\underline{\theta}_2, \bar{\theta}_2]$. Then

$$r(z, x, t) \in [\underline{\theta}_1 \bar{r}(x, t) + \underline{\theta}_2, \bar{\theta}_1 \bar{r}(x, t) + \bar{\theta}_2] = [\underline{r}(x, t), \bar{r}(x, t)],$$

and this case also can be studied in the same way as (3.1).

The simplest interval observer for (3.1) under the introduced assumptions is as follows for $j = 0, 1, \dots, p$:

$$\begin{aligned} \frac{\partial \bar{z}(x, t)}{\partial t} &= L[x, \bar{z}(x, t)] + \bar{r}(x, t) + b_j(x)u_j(t) \quad \forall (x, t) \in I_j \times \mathcal{T}, \\ \bar{z}(x, t_0) &= \bar{z}_0(x) \quad \forall x \in I_j, \\ \bar{z}(x_j^m, t) &= \bar{Z}_j(t), \quad \bar{z}(x_{j+1}^m, t) = \bar{Z}_{j+1}(t) \quad \forall t \in \mathcal{T}; \end{aligned} \quad (3.12)$$

$$\begin{aligned} \frac{\partial \underline{z}(x, t)}{\partial t} &= L[x, \underline{z}(x, t)] + \underline{r}(x, t) + b_j(x)u_j(t) \quad \forall (x, t) \in I_j \times \mathcal{T}, \\ \underline{z}(x, t_0) &= \underline{z}_0(x) \quad \forall x \in I_j, \\ \underline{z}(x_j^m, t) &= \underline{Z}_j(t), \quad \underline{z}(x_{j+1}^m, t) = \underline{Z}_{j+1}(t) \quad \forall t \in \mathcal{T}, \end{aligned}$$

where $\bar{z} \in C(\mathcal{T}, L^2(I, \mathbb{R}))$ and $\underline{z} \in C(\mathcal{T}, L^2(I, \mathbb{R}))$ are upper and lower estimates of the solution $z(x, t)$; $I_j = [x_j^m, x_{j+1}^m]$ with $x_0^m = 0$ and $x_{p+1}^m = \ell$; the upper and lower estimates for the boundary conditions are

$$\begin{aligned} \bar{Z}(t) &= [\bar{Z}_0(t), \dots, \bar{Z}_p(t)]^T = [\bar{\alpha}(t), y_1(t) + \nu_0, \dots, y_p(t) + \nu_0, \bar{\beta}(t)]^T, \\ \underline{Z}(t) &= [\underline{Z}_0(t), \dots, \underline{Z}_p(t)]^T = [\underline{\alpha}(t), y_1(t) - \nu_0, \dots, y_p(t) - \nu_0, \underline{\beta}(t)]^T. \end{aligned}$$

Therefore, the domain I of the solution of (3.1) is divided on $p + 1$ subdomains with appropriate boundary conditions. It is related with the manner the output injection is applied. In (3.12) the use of the output injection directly in the observer right-hand side is avoided since the analysis of positivity of the estimation error dynamics, which is obligatory for an interval observer and given below, is straightforward if the output injection is present at the boundaries, but it is more evolved in other cases.

The upper and the lower interval estimation errors for (3.1) and (3.12) can be introduced as follows:

$$\bar{e}(x, t) = \bar{z}(x, t) - z(x, t), \quad \underline{e}(x, t) = z(x, t) - \underline{z}(x, t), \quad (3.13)$$

whose dynamics take the form for $j = 0, 1, \dots, p$:

$$\begin{aligned}
\frac{\partial \bar{e}(x, t)}{\partial t} &= L[x, \bar{e}(x, t)] + \bar{r}(x, t) - r(x, t) \quad \forall (x, t) \in I_j \times \mathcal{T}, \\
\bar{e}(x, t_0) &= \bar{z}_0(x) - z_0(x) \quad \forall x \in I_j, \\
\bar{e}(x_j^m, t) &= \bar{Z}_j(t) - z(x_j^m, t) \quad \forall t \in \mathcal{T}, \\
\bar{e}(x_{j+1}^m, t) &= \bar{Z}_{j+1}(t) - z(x_{j+1}^m, t) \quad \forall t \in \mathcal{T}; \\
\frac{\partial \underline{e}(x, t)}{\partial t} &= L[x, \underline{e}(x, t)] + r(x, t) - \underline{r}(x, t) \quad \forall (x, t) \in I_j \times \mathcal{T}, \\
\underline{e}(x, t_0) &= z_0(x) - \underline{z}_0(x) \quad \forall x \in I_j, \\
\underline{e}(x_j^m, t) &= z(x_j^m, t) - \underline{Z}_j(t) \quad \forall t \in \mathcal{T}, \\
\underline{e}(x_{j+1}^m, t) &= z(x_{j+1}^m, t) - \underline{Z}_{j+1}(t) \quad \forall t \in \mathcal{T}.
\end{aligned} \tag{3.14}$$

Theorem 3.1. *Let assumptions 3.1 and 3.2 be satisfied, then in (3.1), (3.12):*

$$\underline{z}(x, t) \leq z(x, t) \leq \bar{z}(x, t) \quad \forall (x, t) \in I \times \mathcal{T}. \tag{3.15}$$

In addition, if

$$\Delta x^m < \pi \sqrt{\frac{a_{\min}}{q_{\max}}}, \tag{3.16}$$

where $\Delta x^m = \max_{j \in \{0, 1, \dots, p\}} (x_{j+1}^m - x_j^m)$, then for all $t \in \mathcal{T}$:

$$\begin{aligned}
\|\bar{z}(\cdot, t) - z(\cdot, t)\|_{L^2(I, \mathbb{R})}^2 &\leq 4e^{-\chi(t-t_0)} [\|\bar{z}_0 - z_0\|_{L^2(I, \mathbb{R})}^2 + \bar{\rho}(t_0)] \\
&\quad + 8\chi^{-2} \|\bar{r}(\cdot, t) - r(\cdot, t)\|_{L^2(I, \mathbb{R})}^2 + \bar{\gamma}(t), \\
\|z(\cdot, t) - \underline{z}(\cdot, t)\|_{L^2(I, \mathbb{R})}^2 &\leq 4e^{-\chi(t-t_0)} [\|z_0 - \underline{z}_0\|_{L^2(I, \mathbb{R})}^2 + \underline{\rho}(t_0)] \\
&\quad + 8\chi^{-2} \|r(\cdot, t) - \underline{r}(\cdot, t)\|_{L^2(I, \mathbb{R})}^2 + \underline{\gamma}(t),
\end{aligned}$$

where $\bar{\rho}(t) = \ell \|\bar{Z}(t) - Z(t)\|^2$, $\bar{\rho}'(t) = \ell \|\dot{\bar{Z}}(t) - \dot{Z}(t)\|^2$, $\bar{\gamma}(t) = 8\chi^{-2} \bar{\rho}'(t) + 2(1 + 4\frac{q_{\max}^2}{\chi^2} + 16\frac{\partial a_{\max}^2}{\chi^2 \ell^2}) \bar{\rho}(t)$, $\underline{\rho}(t) = \ell \|Z(t) - \underline{Z}(t)\|^2$, $\underline{\rho}'(t) = \ell \|\dot{Z}(t) - \dot{\underline{Z}}(t)\|^2$, $\underline{\gamma}(t) = 8\chi^{-2} \underline{\rho}'(t) + 2(1 + 4\frac{q_{\max}^2}{\chi^2} + 16\frac{\partial a_{\max}^2}{\chi^2 \ell^2}) \underline{\rho}(t)$ and

$$Z(t) = [\alpha(t), y^T(t) - v^T(t), \beta(t)]^T.$$

Proof 3.2. Under Assumption 3.2, for all $(x, t) \in I \times \mathcal{T}$, in (3.14) the external inputs

$$\bar{r}(x, t) - r(x, t) \geq 0, \quad r(x, t) - \underline{r}(x, t) \geq 0,$$

the initial conditions

$$\bar{z}_0(x) - z_0(x) \geq 0, z_0(x) - \underline{z}_0(x) \geq 0,$$

the boundary conditions

$$\begin{aligned} \bar{e}(x_0^m, t) &= \bar{\alpha}(t) - \alpha(t) \geq 0, \\ \bar{e}(x_i^m, t) &= y_i(t) + \nu_0 - z(x_i^m, t) = \nu(t) + \nu_0 \geq 0, \quad i = 1, \dots, p, \\ \bar{e}(x_{p+1}^m, t) &= \bar{\beta}(t) - \beta(t) \geq 0; \\ \underline{e}(x_0^m, t) &= \alpha(t) - \underline{\alpha}(t) \geq 0, \\ \underline{e}(x_i^m, t) &= z(x_i^m, t) - y_i(t) + \nu_0 = \nu_0 - \nu(t) \geq 0, \quad i = 1, \dots, p, \\ \underline{e}(x_{p+1}^m, t) &= \beta(t) - \underline{\beta}(t) \geq 0, \end{aligned} \tag{3.17}$$

are all nonnegative. Therefore, according to Proposition 3.2 the PDE (3.14) is nonnegative on the interval \mathcal{T} , which implies the required interval estimates by the definition of \bar{e} and \underline{e} .

Boundedness of \bar{z}, \underline{z} for all $t \geq t_0$ follows from Proposition 3.1 and the condition (3.16) under Assumption 3.1.

Remark 3.2. Following the idea from Fridman, 2013, the well-posedness of (3.12) can be established by showing the well-posedness of the estimation errors (3.13), which satisfy the equations (3.14). By the introduced constraints on the system parameters, $r(x, t)$, $\underline{r}(x, t)$ and $\bar{r}(x, t)$; initial conditions $z_0(x)$, $\underline{z}_0(x)$ and $\bar{z}_0(x)$, and boundary conditions for the error dynamics (3.14) (recall (3.17) for $\alpha, \beta, \underline{\alpha}, \bar{\alpha}, \underline{\beta}, \bar{\beta} \in C^2(\mathcal{T}, \mathbb{R})$ and $\nu \in C^2(\mathbb{R}_+, \mathbb{R}^p)$ by assumptions 3.1 and 3.2), and for $\bar{z}_0 - z_0, z_0 - \underline{z}_0 \in \mathcal{D}(A)$ there exists a *strong* solution $\underline{e}, \bar{e} \in C(\mathcal{T}, L^2(I, \mathbb{R}))$ of initial value problem (3.14) with $\underline{e}(t, \cdot), \bar{e}(t, \cdot) \in \mathcal{D}(A)$ by Pazy, 1983, Corollary 4.2.5. Therefore, if $\underline{e}(t, \cdot), \bar{e}(t, \cdot) \in \mathcal{D}(A)$ and $z(t, \cdot) \in \mathcal{D}(A)$, then there exists a unique solution $\underline{z}, \bar{z} \in C(\mathcal{T}, L^2(I, \mathbb{R}))$ to the interval observer system (3.12) with $\underline{z}(t, \cdot), \bar{z}(t, \cdot) \in \mathcal{D}(A)$ for all $t \in \mathcal{T}$.

It is a well-known fact that the system (3.14) can be unstable if the function q takes sufficiently big values Curtain and Zwart, 1995. In Liu and Fridman, 2012 it has been proven, for $\alpha(t) = \beta(t) = 0$ and $\nu(t) = 0$, that the observer (3.12) is asymptotically stable if the difference Δx^m is sufficiently small (*i.e.* there are sufficient quantity of sensors uniformly distributed in I). The presented Theorem 3.1 ensures positiveness of the interval estimation errors and boundedness of the interval estimates \bar{z} and \underline{z} in the presence of non-zero boundary conditions $\alpha(t)$, $\beta(t)$ and measurement noise $\nu(t)$.

3.3 Stabilizing control

In this section the interval observer (3.12) is used for design of a control law ensuring stabilization of (3.1).

The main restriction on stability for the system (3.1) is $q_{\max} < a_{\min} \frac{\pi^2}{\ell^2}$. The inequality (3.16) imposes the same property for the interval observer (3.12): if the difference Δx^m is sufficiently small, which means that the quantity of measurement points is sufficiently high, then the observer estimation error is bounded, but it does not imply stability of the original system. To overcome this restriction, let us consider together the system (3.1) and the interval observer (3.12), designed in the Section 3.2, both endowed with control input $u_j(t) \in H^1(\mathcal{T}, \mathbb{R})$ through the shape functions $b_j(x) \in L^2(I, \mathbb{R})$ on each space subdomain I_j , where the control is chosen as an interval observer state feedback:

$$u_j(t) = -\frac{K_j}{\Delta x_j^m} \int_{x_j^m}^{x_{j+1}^m} (\underline{z}(\xi, t) + \bar{z}(\xi, t)) d\xi, \quad j = 0, \dots, p, \quad (3.18)$$

where K_j are the sequential feedback gains to be designed on each I_j , $K_j > 0$ and

$$\Delta x_j^m = (x_{j+1}^m - x_j^m) \quad \forall j \in \{0, 1, \dots, p\}.$$

Remark 3.3. For brevity we consider the same number of sensors and actuators with collocated subintervals I_j . It is not difficult to extend our results to the non-collocated case by modifying arguments of Selivanov and Fridman, 2018. This is because our design is based on separation of the controller and the observer designs. While the observer part of this paper is completely new, the controller part is based on a modification of the existing controller method from Fridman and Bar Am, 2013. Our modification of the existing controller design is as follows: we use transformation to move boundary disturbances into the right-hand side of PDE and employ a special structure of the controller based on the interval observer. Then the ISS analysis of the closed-loop system follows the existing method for controller design. Thus, by modifying arguments of Section 2 of Selivanov and Fridman, 2018, it is possible to achieve ISS by using a boundary controller at $x = \ell$ via the backstepping.

Thus, the control is applied in order to ensure boundedness of the observer estimates $\underline{z}(x, t)$, $\bar{z}(x, t)$, that in its turn (since $\underline{z}(x, t) \leq z(x, t) \leq \bar{z}(x, t)$ for all $(x, t) \in I \times \mathcal{T}$, see Theorem 3.1) will provide boundedness of $z(x, t)$ as in Efimov et al., 2013f. Recall the shape functions (3.2) $b_j(x) = 1$ on I_j and $b_j(x) = 0$ if $x \notin I_j$ and substitute the control (3.18) in (3.1)

on interval I_j for all $j = 0, \dots, p$:

$$\begin{aligned} \frac{\partial z(x, t)}{\partial t} &= \frac{\partial}{\partial x} (a(x)z_x(x, t)) + q(x)z(x, t) + r(x, t) \\ &- \frac{K_j}{\Delta} x_j^m \int_{x_j^m}^{x_{j+1}^m} (\underline{z}(\xi, t) + \bar{z}(\xi, t)) d\xi, \quad \forall (x, t) \in I_j \times \mathcal{T}. \end{aligned} \quad (3.19)$$

We consider the same shift for the system as before $\delta(x, t) = \alpha(t) + \frac{x}{\ell}(\beta(t) - \alpha(t))$, then the new state variable (as in the proof of Proposition 3.1) is $w(x, t) = z(x, t) - \delta(x, t)$, and it satisfies the following PDE with zero boundary conditions:

$$\begin{aligned} \frac{\partial w(x, t)}{\partial t} &= \frac{\partial}{\partial x} (a(x)w_x(x, t)) + q(x)w(x, t) + \tilde{r}(x, t) \\ &- \sum_{j=0}^p b_j(x) \frac{K_j}{\Delta} x_j^m \int_{x_j^m}^{x_{j+1}^m} (\underline{z}(\xi, t) + \bar{z}(\xi, t)) d\xi \quad \forall (x, t) \in I \times \mathcal{T}, \\ w(x, t_0) &= w_0(x) \quad \forall x \in I, \\ w(0, t) &= w(\ell, t) = 0 \quad \forall t \in \mathcal{T}. \end{aligned}$$

where $\tilde{r}(x, t) = r(x, t) + \frac{1}{\ell} \frac{\partial a(x)}{\partial x} (\beta(t) - \alpha(t)) + q(x)\delta(x, t) - \delta_t(x, t)$ (before this auxiliary perturbation also included the control part $\sum_{j=0}^p b_j(x)u_j(t)$).

Consider the interval observer error dynamics (3.14), which is nonnegative by Theorem 3.1 and bounded if the condition (3.16) is satisfied. Recall the relations $\underline{z}(x, t) = z(x, t) - \underline{e}(x, t)$ and $\bar{z}(x, t) = z(x, t) + \bar{e}(x, t)$ and substitute them into the dynamics of $w(x, t)$:

$$\begin{aligned} \frac{\partial w(x, t)}{\partial t} &= \frac{\partial}{\partial x} (a(x)w_x(x, t)) + q(x)w(x, t) + \tilde{r}^*(x, t) \\ &+ \sum_{j=0}^p b_j(x) \frac{K_j}{\Delta} x_j^m \int_{x_j^m}^{x_{j+1}^m} (\underline{e}(\xi, t) - \bar{e}(\xi, t)) d\xi \\ &- 2 \sum_{j=0}^p b_j(x) \frac{K_j}{\Delta} x_j^m \int_{x_j^m}^{x_{j+1}^m} w(\xi, t) d\xi \quad \forall (x, t) \in I \times \mathcal{T}, \end{aligned}$$

where $\tilde{r}^*(x, t) = \tilde{r}(x, t) - 2 \sum_{j=0}^p b_j(x) \frac{K_j}{\Delta} x_j^m \int_{x_j^m}^{x_{j+1}^m} \delta(\xi, t) d\xi$. Since $\underline{e}(x, t) \geq 0, \bar{e}(x, t) \geq 0$ and bounded under the condition (3.16), the terms $\int_{x_j^m}^{x_{j+1}^m} (\underline{e}(\xi, t) - \bar{e}(\xi, t)) d\xi$ can be made a part of a new disturbance

$$R(x, t) = \tilde{r}^*(x, t) + \sum_{j=0}^p b_j(x) \frac{K_j}{\Delta} x_j^m \int_{x_j^m}^{x_{j+1}^m} (\underline{e}(\xi, t) - \bar{e}(\xi, t)) d\xi,$$

then

$$\begin{aligned} \frac{\partial w(x,t)}{\partial t} &= \frac{\partial}{\partial x} (a(x)w_x(x,t)) + q(x)w(x,t) + R(x,t) \\ &- 2 \sum_{j=0}^p b_j(x) \frac{K_j}{\Delta} x_j^m \int_{x_j^m}^{x_{j+1}^m} w(\xi,t) d\xi \quad \forall (x,t) \in I \times T. \end{aligned}$$

In order to analyze the influence of the integral feedback, let us use the relation

$$\frac{1}{\Delta} x_j^m \int_{x_j^m}^{x_{j+1}^m} w(\xi,t) d\xi = w(x,t) - f(x,t), \quad x \in I_j,$$

proposed in Fridman and Bar Am, 2013, where

$$f(x,t) \triangleq \frac{1}{\Delta} x_j^m \int_{x_j^m}^{x_{j+1}^m} [w(x,t) - w(\xi,t)] d\xi$$

is a piecewise continuous function and $\frac{\partial f}{\partial x} = \frac{\partial w}{\partial x}$. Finally, the following closed-loop system has been obtained:

$$\begin{aligned} \frac{\partial w(x,t)}{\partial t} &= \frac{\partial}{\partial x} (a(x)w_x(x,t)) + q(x)w(x,t) + R(x,t) \\ &- 2 \sum_{j=0}^p K_j b_j(x) w(x,t) + 2 \sum_{j=0}^p K_j b_j(x) f(x,t). \end{aligned} \quad (3.20)$$

Validity of the interval inclusion (3.15) can be proven repeating the same arguments as in Theorem 3.1 since the observer design is independent on the form of control. To analyze stability of the closed-loop system (3.20) let us consider a Lyapunov function:

$$V(t) = \int_0^\ell w^2(x,t) dx,$$

whose derivative takes the form for any $\gamma > 0$ and $\kappa > 0$:

$$\begin{aligned} \dot{V}(t) + 2\kappa V(t) - \gamma^2 \int_0^\ell R(x,t)^2 dx &= 2 \int_0^\ell w(x,t) \left[\frac{\partial}{\partial x} (a(x)w_x(x,t)) \right. \\ &\quad \left. + q(x)w(x,t) + R(x,t) \right] dx + 2\kappa \int_0^\ell w^2(x,t) dx - \gamma^2 \int_0^\ell R^2(x,t) dx \\ &- 4 \int_0^\ell \left[\sum_{j=0}^p K_j b_j(x) w(x,t) \right] w(x,t) dx + 4 \int_0^\ell \left[\sum_{j=0}^p K_j b_j(x) f(x,t) \right] w(x,t) dx. \end{aligned} \quad (3.21)$$

Integration by parts and substitution of the boundary conditions for $w(x, t)$ lead to

$$2 \int_0^\ell w(x, t) \frac{\partial}{\partial x} (a(x) w_x(x, t)) dx = 2a(x)w(x, t)w_x(x, t)|_0^\ell - 2 \int_0^\ell a(x)w_x^2(x, t) dx \leq -2a_{\min} \int_0^\ell w_x^2(x, t) dx.$$

The function $f(x, t)$ has the zero average $\int_{x_j^m}^{x_{j+1}^m} f(x, t) dx = 0$ and $f_x = w_x$, and by applying the Poincaré's inequality (1.31) on subdomains I_j the following upper estimate is obtained:

$$-2a_{\min} \int_{x_j^m}^{x_{j+1}^m} w_x^2(x, t) dx \leq -2a_{\min} \frac{\pi^2}{(\Delta x_j^m)^2} \int_{x_j^m}^{x_{j+1}^m} f^2(x, t) dx,$$

then

$$\begin{aligned} -2a_{\min} \int_0^\ell w_x^2(x, t) dx &= -2a_{\min} \sum_{j=0}^p \int_{x_j^m}^{x_{j+1}^m} w_x^2(x, t) dx \\ &\leq -2a_{\min} \frac{\pi^2}{(\Delta x_j^m)^2} \sum_{j=0}^p \int_{x_j^m}^{x_{j+1}^m} f^2(x, t) dx. \end{aligned}$$

The next term of (3.21) can be rewritten using the fact that $b_j(x) = 1$ on I_j in (3.2) and under a mild simplifying restriction that $K_j = K$ for all $j = 0, \dots, p$:

$$-4 \int_0^\ell \left[\sum_{j=0}^p K_j b_j(x) w^2(x, t) \right] dx = -4K \sum_{j=0}^p \int_{x_j^m}^{x_{j+1}^m} w^2(x, t) dx.$$

And the cross term of (3.21) can be treated in the same way:

$$4 \int_0^\ell \left[\sum_{j=0}^p K_j b_j(x) f(x, t) w(x, t) \right] dx = 4K \sum_{j=0}^p \int_{x_j^m}^{x_{j+1}^m} w(x, t) f(x, t) dx.$$

Therefore, using an upper bound $\int_0^\ell q(x) w^2(x, t) \leq q_{\max} \int_0^\ell w^2(x, t)$ and denoting $\eta^\top = [w(x, t) f(x, t) R(x, t)]$, we get

$$\dot{V}(t) + 2\kappa V(t) - \gamma^2 \int_0^\ell R(x, t)^2 dx \leq \sum_{j=0}^p \int_{x_j^m}^{x_{j+1}^m} \eta^\top \Phi \eta dx \leq 0$$

provided that

$$\Phi \triangleq \begin{bmatrix} 2(\kappa + q_{\max} - 2K) & 2K & 1 \\ 2K & \frac{-2a_{\min}\pi^2}{(\Delta x^m)^2} & 0 \\ 1 & 0 & -\gamma^2 \end{bmatrix} \leq 0$$

for $\Delta x^m = \max_{j \in \{0,1,\dots,p\}} \Delta x_j^m$. Using the Schur complement the above inequality is satisfied if

$$\begin{bmatrix} \frac{2a_{\min}\pi^2}{\Delta x^m} & 0 \\ 0 & \gamma^2 \end{bmatrix} > 0, \quad 2K - \frac{(\Delta x^m)^2}{a_{\min}\pi^2} K^2 - \kappa - q_{\max} - \frac{1}{2}\gamma^{-2} \geq 0,$$

where the first property is valid by proposed construction and the last one is a quadratic inequality with respect to K . Using the imposed restriction (3.16) there exists $\rho > 0$ such that

$$\frac{(\Delta x^m)^2}{a_{\min}\pi^2} = \frac{1}{q_{\max} + \rho},$$

then the needed inequality holds if

$$2K - \frac{1}{q_{\max} + \rho} K^2 - \kappa - q_{\max} - \frac{1}{2}\gamma^{-2} \geq 0,$$

that always has a solution for

$$\kappa + \frac{1}{2}\gamma^{-2} \leq \rho.$$

In particular, for $\kappa + \frac{1}{2}\gamma^{-2} = \rho$ we obtain:

$$K = q_{\max} + \rho = \frac{a_{\min}\pi^2}{(\Delta x^m)^2}.$$

The inequality

$$\dot{V}(t) + 2\kappa V(t) - \gamma^2 \int_0^\ell R(x,t)^2 dx \leq 0$$

implies boundedness of the solutions $w(x,t)$ as in the proof of Proposition 3.1. We have proved the following theorem.

Theorem 3.2. *Let assumptions 3.1 and 3.2 be satisfied. Let there exist $\kappa > 0$, $K > 0$, $\gamma > 0$ and $\Delta x^m < \pi \sqrt{\frac{a_{\min}}{q_{\max}}}$ that satisfy the LMI*

$$\Phi \leq 0.$$

Then for the solutions of the closed-loop system (3.19), the interval inclusion (3.15) and the

estimates on $\|z(\cdot, t) - \underline{z}(\cdot, t)\|_{L^2(I, \mathbb{R})}$, $\|\bar{z}(\cdot, t) - z(\cdot, t)\|_{L^2(I, \mathbb{R})}$ from Theorem 3.1 are valid and

$$\begin{aligned} \frac{1}{2} \int_0^\ell z^2(x, t) dx &\leq e^{-2\kappa(t-t_0)} \int_0^\ell w_0^2(x) dx + \frac{\gamma^2}{2\kappa} \int_0^\ell R(x, t)^2 dx \\ &+ \frac{\ell}{2} (\alpha^2(t) + \beta^2(t)) \quad \forall (x, t) \in I \times \mathcal{T}. \end{aligned}$$

Remark 3.4. Note that qualitatively the above L^2 boundedness estimate for z can also be obtained using static output feedback, however it can be rather conservative, and using the on-line calculated upper and lower observer bounds \underline{z} and \bar{z} we can deduce a tighter interval estimate on the state. This can be an important advantage for applications dedicated to state constrained problems (e.g. in reactors).

3.4 Examples

In this section we will consider two applications of the proposed interval observer in order to compare the obtained results with the interval observer from Kharkovskaia et al., 2018b and the control from Liu and Fridman, 2012.

Controller based on the interval observer

Consider an academic example of (3.1) for

$$\begin{aligned} a(x) &= \frac{1}{4} \left(1 + \frac{3}{4} \sin(2\pi x)\right), \quad q(x) = 5 + \frac{1}{2} \cos(\pi x), \\ r(x, t) &= \sin(\pi x) [\cos(2t) + \epsilon(t)], \quad |\epsilon(t)| \leq 1, \end{aligned}$$

with $T = 2$ and $\ell = 1$, then ϵ is an uncertain part of the input r (for simulation $\epsilon(t) = \cos(10t)$), and

$$\underline{r}(x, t) = \sin(\pi x) [\cos(2t) - 1], \quad \bar{r}(x, t) = \sin(\pi x) [\cos(2t) + 1].$$

The uncertainty of initial conditions is given by the interval

$$\underline{z}_0(x) = z_0(x) - 1, \quad \bar{z}_0(x) = z_0(x) + 1,$$

where $z_0(x) = 5 \sin(\pi x)$, and for boundary initial conditions

$$\begin{aligned}\underline{\alpha}(t) &= \sin(2t) - 1, \quad \overline{\alpha}(t) = \sin(2t) + 1, \\ \underline{\beta}(t) &= \sin(5t) - 1, \quad \overline{\beta}(t) = \sin(5t) + 1,\end{aligned}$$

where $\alpha(t) = \sin(2t)$ and $\beta(t) = \sin(5t)$. Let $p = 3$ with $x_1^m = 0.3$, $x_2^m = 0.6$, $x_3^m = 0.8$, and

$$v(t) = 0.1[\sin(20t) \sin(15t) \cos(25t)]^\top,$$

then $\nu_0 = 0.173$. In this case $a_{\min} = \frac{1}{16}$, $q_{\max} = 5\frac{1}{2}$. With these parameters, q_{\max} is larger than $a_{\min} \frac{\pi^2}{\ell^2}$, which means that the system is unstable (the conditions of Proposition 3.1 fail to satisfy). The maximum distance between sensors is $\Delta x^m = 0.3$, and the restriction (3.16) for the interval observer is still verified. Therefore, Theorem 3.1 can be used to construct an observer for the unstable system (3.1). Then, to stabilize it, following the conditions of Theorem 3.2, the control gain $K = 3.2865$ was calculated, and the controls $u_j(t)$ on each interval $I_i = [x_i^m, x_{i+1}^m]$, $i = \overline{0, p}$ with $x_0^m = 0$ and $x_{p+1}^m = \ell$ were computed by (3.18).

For calculation of scalar product in space and for simulation of the discretized PDE in time, the implicit Euler method has been used with the step size $dt = 0.01$. The results of a simultaneous interval estimation and control are shown in Fig. 3.1, where the red surface corresponds to $z(x, t)$, while green and blue ones represent $\underline{z}(x, t)$ and $\overline{z}(x, t)$, respectively (20 and 100 points are used for plotting in space and in time).

In order to compare the proposed interval observer based control (3.18) with a static output feedback control

$$u_j(t) = -K^* y_j(t), \tag{3.22}$$

the feedback gain $K^* = 4.8832$ is calculated following the result of the work Liu and Fridman, 2012. Since the system (3.1) contains uncertainties in disturbances $r(x, t)$, $v(t)$ and boundary conditions $\alpha(t)$, $\beta(t)$, the static output feedback can guarantee only input-to-state stability in the sense of Proposition 3.1 with respect to the input $\tilde{r}(x, t)$, which contains all this incertitude. To compare the precision ensured by both controllers in our example, first, the L^2 upper estimate of $z(x, t)$ for this feedback control is calculated as follows. Note that

$$V(t) = \|z\|_{L^2(I, \mathbb{R})} \leq \ell \overline{z}^2(t),$$

where

$$\overline{z}(t) = \max_{x \in I} |z(x, t)|$$

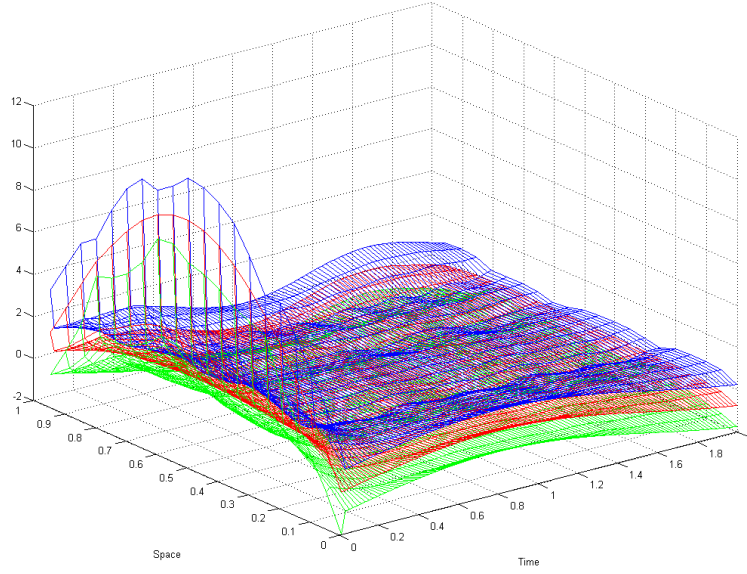


Figure 3.1 – The results of the interval observer based control of the heat equation for $N = 20$: the lower bound $\underline{z}(x, t)$, the state $z(x, t)$ and the upper bound $\bar{z}(x, t)$.

Clearly,

$$z(t, x) \in [-\bar{z}(t), \bar{z}(t)] \quad \forall (x, t) \in I \times \mathcal{T}.$$

From another side, the obtained L^2 estimates can be presented as

$$V(t) \leq e^{-2\delta(t-t_0)}V(t_0) + \gamma \int_0^\ell |\tilde{r}(x, t)|^2 dx = \bar{V}(t),$$

where $\bar{V}(t)$ can be calculated on-line for the given gain K^* (it determines the values of parameters $\delta > 0$ and $\gamma > 0$) and the imposed upper bounds on $\tilde{r}(x, t)$. Second, for illustration we assume that $\bar{V}(t) = \ell \bar{z}^2(t)$, then the obtained bounds $[-\bar{z}(t), \bar{z}(t)] = [-\sqrt{\ell^{-1}\bar{V}(t)}, \sqrt{\ell^{-1}\bar{V}(t)}]$ are shown in the Fig. 3.2 (black solid lines) together with the interval estimates of the proposed observer (3.12) (green and blue ones) for different instances of time. Red curves in the Fig. 2 represent the simulation of the stabilized heat equation (3.1) state using the interval observer, while the black dashed curves represent the state of (3.1) stabilized by output feedback (3.22). As we can conclude from this evaluation, the guaranteed bounds given by the interval observer based control are almost always more accurate than provided by the static feedback from L^2 estimates.

Remark 3.5. Note that since for calculation of solutions the finite-element discretization/approximation methods are used, then their error of approximation has to be taken into account in the final estimates in order to ensure the desired interval inclusion property

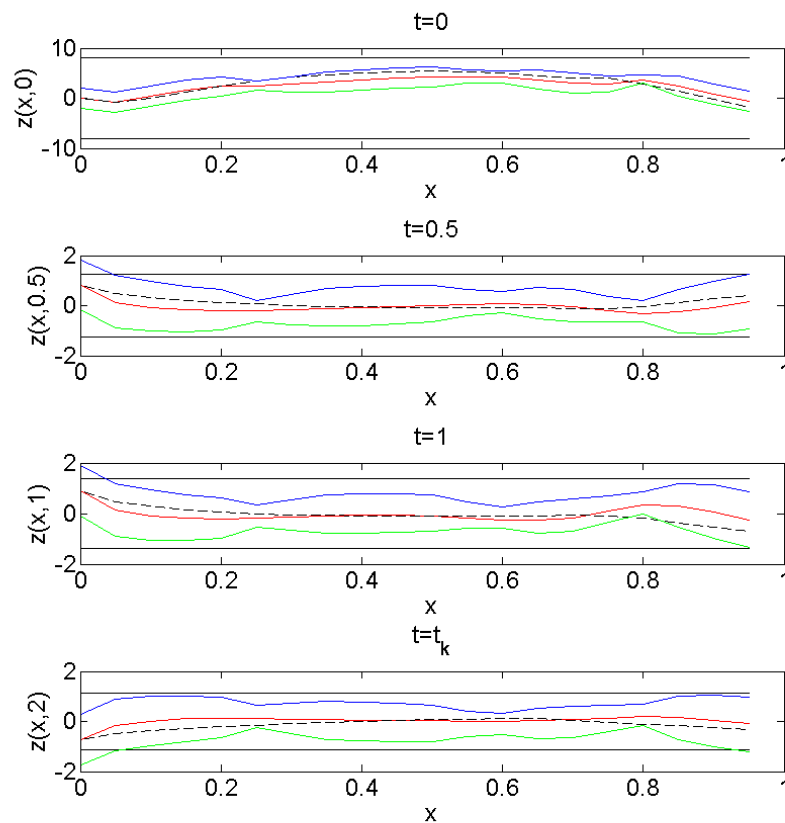


Figure 3.2 – The results of the interval observer based control and the L^2 estimate of the static output feedback control for the heat equation for different instants of time: $t = 0, 0.5, 1, 2$ for $N = 20$. Here $\underline{z}(x, \cdot)$ and $\bar{z}(x, \cdot)$ represent the interval observer bounds, $z(x, \cdot)$ is the stabilized state using observer, black dashed lines represent a state of (3.1) stabilized by output feedback (3.22), black solid lines are $[-\bar{z}(t), \bar{z}(t)]$.

for all $x \in I$ and $t \in \mathcal{T}$, see Kharkovskaia et al., 2018b where the result from Wheeler, 1973 was applied for an evaluation of this error.

Remark 3.6. As mentioned in Liu and Fridman, 2012, there are no advantages of the Luenberger observer-based controller in the case of collocated sensors and actuators over the corresponding static output-feedback. However, as it is shown in this example, interval observer allows to achieve essentially lower state bounds than the corresponding static output-feedback.

The interval observer comparison

Consider a heat equation (3.1) with:

$$\begin{aligned} a(x) &= 2 + 0.7 \sin(\pi x), \quad q(x) = 0.5 \sin(0.5x), \\ r(x, t) &= r_1(x)r_2(t), \quad r_1(x) = 2 \cos(3\pi x), \quad |r_2(t)| \leq 1, \end{aligned}$$

$T = 10$ and $\ell = 1$. Here r_2 is an uncertain part of the input r (for simulation $r_2(t) = \cos(15t)$), and

$$\underline{r}(x, t) = -|r_1(x)|, \quad \bar{r}(x, t) = |r_1(x)|.$$

The uncertainty of initial conditions is given by the interval

$$\underline{z}_0(x) = z_0(x) - 1, \quad \bar{z}_0(x) = z_0(x) + 1,$$

where $z_0(x) = \cos(5\pi x)$, and the boundary conditions $\alpha(t)$ and $\beta(t)$ are assumed to be 0, since the approach from Kharkovskaia et al., 2018b does not employ nonzero conditions. Let $p = 3$ with $x_1^m = 0.3$, $x_2^m = 0.5$, $x_3^m = 0.8$, and

$$v(t) = 0.2[\sin(20t) \sin(15t) \cos(25t)]^T,$$

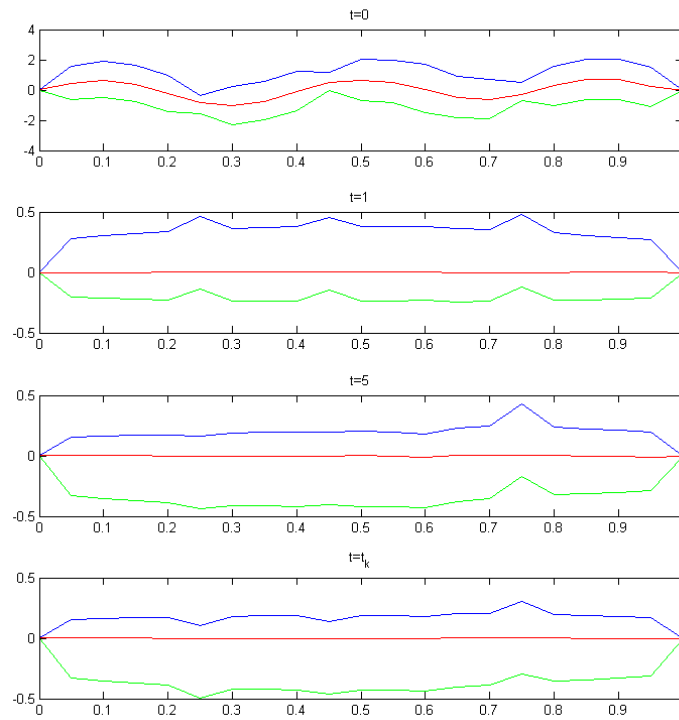
then $v_0 = 0.2$. In this case $\Delta x^m = 0.3$, $a_{\min} = 1.3$, $q_{\max} = 0.5$ and the restriction (3.16) is satisfied. Take $\Delta = \{0, h, 2h, \dots, 1 - h, 1\}$ with $h = 1/N'$, and a pyramidal basis

$$\Phi_i(x) = \begin{cases} 0 & x \leq x_{i-1}, \\ \frac{x-x_{i-1}}{x_i-x_{i-1}} & x_{i-1} < x \leq x_i, \\ \frac{x_{i+1}-x}{x_{i+1}-x_i} & x_i < x \leq x_{i+1}, \\ 0 & x \geq x_{i+1} \end{cases}$$

for $i = 0, \dots, N = N'$ (it is assumed $x_{-1} = -h$ and $x_{N+1} = 1 + h$). For simulation we took $N = 20$, then the approximated dynamics from Kharkovskaia et al., 2018b is an observable system, and assume that the error of approximation for both approaches $\rho h^{s+1}(l_1 + l_2) = 0.1$. For the Galerkin approximation approach Kharkovskaia et al., 2018b the matrix \mathcal{L} has been chosen to ensure distinct eigenvalues of the matrix $A - \mathcal{L}C$ in the interval $[-30.9, -0.67]$, then S^{-1} has been composed by eigenvectors of the matrix $A - \mathcal{L}C$ and the matrix D has been selected diagonal (all these matrices are defined in Kharkovskaia et al., 2018b).

As before, for the calculation of scalar product in space and for simulation of the discretized PDE in time, the implicit Euler method has been used with the step size $dt = 0.01$ for the PDE interval observer, and the explicit one with the same step for the approximation approach. The results of comparison of the two approaches, the present and the approximation one from Kharkovskaia et al., 2018b, are shown in Fig. 3.3, where the red lines corresponds to $z(x, \cdot)$, while green and blue ones represent $\underline{z}(x, \cdot)$ and $\bar{z}(x, \cdot)$, respectively, at the instances $t = 0, 1, 5, 10$. From this figure one can clearly notice that the obtained interval for the state is more precise with the PDE interval observer approach (3.12).

1. PDE interval observer



2. Approximation approach for IO design

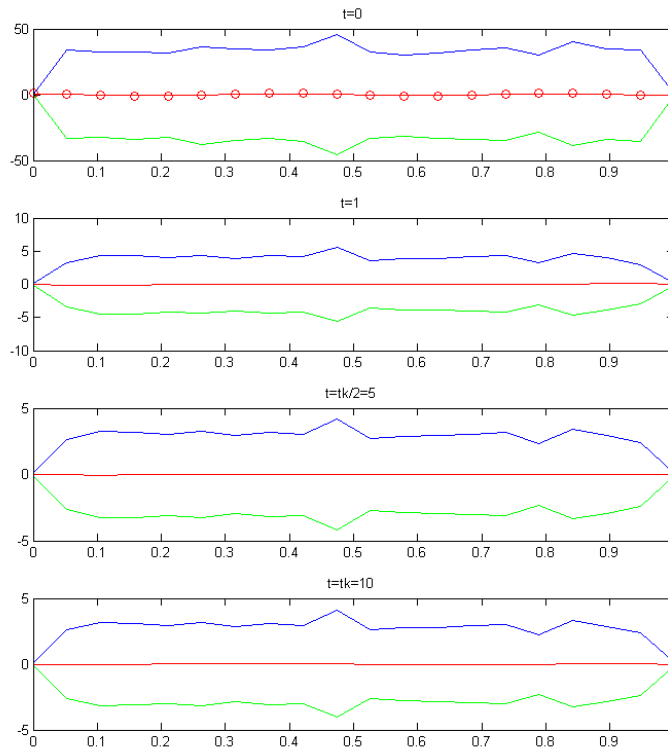


Figure 3.3 – The results of 1) the PDE interval observer (3.12) and 2) the approximation approach interval observer from Kharkovskaia et al., 2016, $N = 20$. Here the lower bound is $\underline{z}(x, \cdot)$, the state is $z(x, \cdot)$ and the upper bound is $\bar{z}(x, \cdot)$.

Interval estimation for second-order delay differential equations with delayed measurements and uncertainties

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In this chapter a simple benchmark problem is investigated of an unstable second order delayed system with delayed measurements (*e.g.* a delayed model for motion of a single mass point)

$$\begin{aligned}\ddot{x}(t) &= -ax(t - \tau) + f(t), \quad t \geq 0, \\ y(t) &= x(t - \theta) + v(t), \quad \tau \neq \theta,\end{aligned}$$

where $x(t) \in \mathbb{R}$ is the position, $y(t) \in \mathbb{R}$ is the measured output signal, $f : \mathbb{R}_+ \rightarrow \mathbb{R}$ and $v : \mathbb{R}_+ \rightarrow \mathbb{R}$ are state perturbation and measurement noise (unknown bounded signals), $\tau > 0$ and $\theta > 0$ are the delays. Our goal is to design an interval observer for this system, but the main issue is that beside stability conditions, to construct an interval observer it is necessary to check that the estimation error dynamics possess the positivity property.

4.1 Delay-independent positivity

Consider a time-invariant linear system with time-varying delay:

$$\dot{x}(t) = A_0 x(t) + A_1 x(t - \tau(t)) + Bf(t), \quad t \in [0, +\infty), \quad (4.1)$$

$$x(h) = \phi(h) \text{ for } -\bar{\tau} \leq h \leq 0, \quad \phi \in \mathcal{C}_{\bar{\tau}}^n, \quad (4.2)$$

where $x(t) \in \mathbb{R}^n$, $x_t \in \mathcal{C}_{\bar{\tau}}^n$ is the state function; $\tau : \mathbb{R}_+ \rightarrow [-\bar{\tau}, 0]$ is the time-varying delay, a Lebesgue measurable function of time, $\bar{\tau} \in \mathbb{R}_+$ is the maximum delay; $f \in \mathcal{L}_{\infty}^m$ is the input; the constant matrices A_0 , A_1 and B have appropriate dimensions; $\phi : (-\bar{\tau}, 0] \rightarrow \mathbb{R}^n$ is a Borel measurable bounded function of initial conditions.

Definition 4.1. Agarwal et al., 2012 Function $x : \mathbb{R} \rightarrow \mathbb{R}^n$, which is locally absolutely continuous on $[0, \infty)$, is called a solution of problem (4.1), (4.2) if it satisfies (4.1) for almost all $t \in [0, \infty)$ and equality (4.2) for $t \leq 0$.

The matrix A_0 is called Metzler if all its off-diagonal elements are nonnegative. The system (4.1) is called *positive* if for $\phi \in \mathcal{C}_{\bar{\tau}_+}^n$ it has the corresponding solution $x(t) \geq 0$ for all $t \geq 0$.

Lemma 4.1. Ait Rami, 2009; Haddad and Chellaboina, 2004 *The system (4.1) is positive iff A_0 is Metzler, $A_1 \geq 0$ and $Bf(t) \geq 0$ for all $t \geq 0$. A positive system (4.1) is asymptotically stable with $f(t) \equiv 0$ for all $t \in \mathbb{R}_+$ iff there are $p, q \in \mathbb{R}_+^n$ ($p > 0$ and $q > 0$) such that*

$$p^T [A_0 + A_1] + q^T = 0.$$

Under conditions of the above lemma the system has bounded solutions for $f \in \mathcal{L}_{\infty}^m$ Fridman, 2014. Note that for linear time-invariant systems the conditions of positive invariance of polyhedral sets have been similarly given in Dambrine et al., 1995, as well as conditions of asymptotic stability in the nonlinear case have been considered in Borne et al., 2003; Dambrine and Richard, 1993; Dambrine and Richard, 1994.

4.2 Representation of the solution for delay differential equations

Since $\bar{\tau}$ is the maximum delay for $\tau(t)$ define a bounded set $\mathcal{T} = \{s \in (0, \bar{\tau}] : s \leq \tau(s)\}$, then the non-zero initial value problem (4.1), (4.2) can be rewritten to have zero initial

conditions for $t < 0$ and with the same solution $x(t)$ for all $t \geq 0$:

$$\begin{aligned} \dot{x}(t) &= A_0x(t) + A_1x(t - \tau(t)) + Bf(t) + f^*(t), \quad t \in [0, +\infty), \\ x(h) &= 0 \text{ for } h < 0, \end{aligned} \quad (4.3)$$

where

$$f^*(t) = \begin{cases} A_1\phi(t - \tau(t)) & t \in \mathcal{T} \\ 0 & \text{otherwise} \end{cases}.$$

In addition to the problem (4.1), (4.2), where x , f and ϕ are vector signals or functions, we will consider a problem where the solution is a $n \times n$ matrix function. For example, the $n \times n$ matrix function $C(t, s) = X(t)X^{-1}(s)$, where $X(t)$ satisfies a homogeneous initial value problem

$$\begin{aligned} \dot{X}(t) &= A_0X(t) + A_1X(t - \tau(t)), \quad t \geq s, \\ X(\theta) &= 0 \text{ for } \theta < s, \quad X(s) = I_n \end{aligned} \quad (4.4)$$

for each $s \geq 0$, is called the Cauchy matrix of (4.4). By construction $C(t, s) = 0$ for $0 \leq t < s$. Using this Cauchy matrix $C(t, 0)$, a unique solution of non-homogeneous system (4.3) will take the form

$$x(t) = C(t, 0)x(0) + \int_0^t C(t, s)(Bf(s) + f^*(s))ds, \quad (4.5)$$

which is also the solution for the representation (4.1) with the initial conditions (4.2).

Based on this idea, for non-zero initial function in Efimov et al., 2016 the conditions on delay-dependent positivity are introduced by verifying an additional constrain on the first interval $t \in [0, \bar{\tau}]$:

Lemma 4.2. Efimov et al., 2016 *The system (4.1) with $Bf(t) \geq 0$ for all $t \geq 0$, $x(0) \in \mathbb{R}_+^n$, with a Metzler matrix A_0 , $A_1 \geq 0$ and $0 \leq (A_0)_{i,i} \leq e(A_1)_{i,i} < (A_0)_{i,i} + \bar{\tau}^{-1}$ for all $i = 1, \dots, n$, has the corresponding solution $x(t) \geq 0$ for all $t \geq 0$ provided that*

$$Bf(t) \geq -f^*(t) \quad \forall t \in [0, \bar{\tau}].$$

4.3 Conditions on positivity of a second order system

Following the result of Domoshnitsky, 2014, consider the second-order delay differential equation with an input signal:

$$\begin{aligned} \ddot{x}(t) + a(t)x(t - \tau) - b(t)x(t - \theta) &= f(t) \quad t \in [0, +\infty), \\ x(h) &= \phi(h) \quad \text{for } h \leq 0, \dot{x}(0) \in \mathbb{R} \end{aligned} \quad (4.6)$$

with constant delays $\tau, \theta \geq 0$ and nonnegative functions $a, b, f \in \mathcal{L}_\infty$ and $\phi \in \mathcal{C}_{\max\{\tau, \theta\}}$. The corresponding homogeneous equation is considered as

$$\ddot{x}(t) + a(t)x(t - \tau) - b(t)x(t - \theta) = 0 \quad t \in [0, +\infty). \quad (4.7)$$

For a signal $q \in \mathcal{L}_\infty$ denote further the following short hands

$$q_* = \operatorname{ess\,inf}_{t \geq 0} q(t), \quad q^* = \operatorname{ess\,sup}_{t \geq 0} q(t).$$

Theorem 4.1. Domoshnitsky, 2014 *Assume that $0 \leq \tau < \theta$ and there exists $\varepsilon > 0$ such that the inequalities*

$$\varepsilon \leq \{a(t) - b(t)\} \leq \frac{1}{4} b_*^2 (\theta - \tau)^2, \quad \forall t \in [0, +\infty) \quad (4.8)$$

and

$$\frac{1}{\sqrt{a^*} \exp\left\{\frac{b_*(\theta - \tau)^2}{4}\right\}} \arctan \frac{b_*(\theta - \tau)}{2\sqrt{a^*} \exp\left\{\frac{b_*(\theta - \tau)^2}{4}\right\}} > \theta - \tau \quad (4.9)$$

are fulfilled. Then

(1) the elements $C_{11}(t, s)$ and $C_{12}(t, s)$ of the Cauchy matrix

$$C(t, s) = \begin{bmatrix} C_{11}(t, s) & C_{12}(t, s) \\ C_{21}(t, s) & C_{22}(t, s) \end{bmatrix} \in \mathbb{R}^{2 \times 2}$$

of (4.7) are nonnegative for $0 \leq s < t < +\infty$;

(2) the Cauchy function $C_{12}(t, s)$ of (4.7) satisfies the exponential estimate

$$|C_{12}(t, s)| \leq N e^{-\alpha(t-s)} \quad \forall 0 \leq s \leq t < +\infty$$

for some $N > 0, \alpha > 0$ and the integral estimate

$$\sup_{t \geq 0} \int_0^t C_{12}(t, s) ds \leq \frac{1}{\varepsilon};$$

(3) if there exists $\lim_{t \rightarrow \infty} \{a(t) - b(t)\} = k$, then equality

$$\lim_{t \rightarrow \infty} \int_0^t C_{12}(t,s) ds = \frac{1}{k}$$

is fulfilled.

According to this theorem, the solution $x(t)$ of the equation (4.6) being written in the form (4.5) has nonnegative elements $C_{11}(t,s)$ and $C_{12}(t,s)$ of the Cauchy matrix function, and with nonnegative $f(t)$ and the proper choice of parameters $a(t)$, $b(t)$ and delays τ , θ the system (4.6) will have a bounded solution. The Theorem 4.1 also establishes the exponential estimate for the Cauchy function $C_{12}(t,s)$ and its convergence rate. This result concerns the exponential stability of (4.7), which are based on the maximum principles for the second order delay differential equation (4.7) Domoshnitsky, 2014. Using the representation of solutions (4.5) (see also (4.10) below), we can see that this principle is reduced to positivity of the Cauchy matrix element $C_{11}(t,s)$ and $C_{12}(t,s)$ for nonnegative initial conditions.

Proposition 4.1. *Let $0 < \tau < \theta$ in the system (4.6) and the conditions (4.8), (4.9) of the Theorem 4.1 hold. If $f(t) + f^*(t) \geq 0$ for all $t \geq 0$, $\dot{x}(0) \in \mathbb{R}_+$ and $\phi(h) \geq 0$ for all $h \in [-\theta, 0]$, where*

$$f^*(t) = \begin{cases} -a(t)\phi(t-\tau) + b(t)\phi(t-\theta) & t \in [0, \tau] \\ b(t)\phi(t-\theta) & t \in (\tau, \theta], \\ 0 & t > \theta \end{cases},$$

then the corresponding solution satisfies $x(t) \geq 0$ for all $t \geq 0$.

Proof. Following the expression (4.5), the position solution for the system (4.6) can be written as follows:

$$x(t) = C_{11}(t,0)x(0) + C_{12}(t,0)\dot{x}(0) + \int_0^t C_{12}(t,s)(f(s) + f^*(s)) ds, \quad (4.10)$$

where due to the result of Theorem 4.1 all terms in the right hand-side are nonnegative. ■

4.4 Problem statement

The main object of study in this note is the delay differential equation of the second order, which represents an unstable delayed model of motion of a single mass point:

$$\ddot{x}(t) = -a(t)x(t - \tau) + f(t), \quad t \in [0, +\infty) \quad (4.11)$$

where $x(t) \in \mathbb{R}$ is the position of the point, $\dot{x}(t) \in \mathbb{R}$ and $\ddot{x}(t) \in \mathbb{R}$ are the velocity and the acceleration of the point motion, respectively; $\tau \geq 0$ is a constant state delay, $a(t) \in \mathbb{R}_+$ with $a \in \mathcal{L}_\infty$ is the parameter function, $f \in \mathcal{L}_\infty$ is the external input. The equation (4.11) can be presented in a time-varying version of (4.1) with the corresponding matrices:

$$A_0 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad A_1(t) = \begin{bmatrix} 0 & 0 \\ -a(t) & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \quad (4.12)$$

The initial conditions for (4.11) are considered as a scalar function only for position:

$$x(h) = \phi(h) \text{ for } -\tau \leq h \leq 0, \quad \phi \in \mathcal{C}_\tau; \quad \dot{x}(0) \in \mathbb{R}, \quad (4.13)$$

then the solution (4.5) can be rewritten to describe only the position of (4.11) as follows:

$$\begin{aligned} x(t) &= C_{11}(t, 0)x(0) + C_{12}(t, 0)\dot{x}(0) \\ &\quad + \int_0^t C_{12}(t, s)(f(s) + f^*(s)) ds, \end{aligned} \quad (4.14)$$

where

$$C(t, s) = \begin{bmatrix} C_{11}(t, s) & C_{12}(t, s) \\ C_{21}(t, s) & C_{22}(t, s) \end{bmatrix} \in \mathbb{R}^{2 \times 2}$$

with the initial condition $C(0, 0) = I_2$, and

$$f^*(t) = \begin{cases} -a(t)\phi(t - \tau) & t \in [0, \tau] \\ 0 & t > \tau \end{cases}.$$

Furthermore, we consider that the position of (4.11) is available for measurements with some constant delay θ , $\theta \geq \tau$:

$$y(t) = x(t - \theta) + v(t), \quad (4.15)$$

where $y(t) \in \mathbb{R}$ is an output with measurement noise $v \in \mathcal{L}_\infty$. It is worth stressing that since $\theta \geq \tau$, the initial conditions for (4.11), $x(h) = \phi(h)$, should be defined for $h \in [-\theta, 0]$. To

continue the analysis with this data, we also need to introduce the following hypothesis:

Assumption 4.1. *There exist known functions $\underline{\phi}, \bar{\phi} \in \mathcal{C}_\theta$ such that $\underline{\phi}(h) \leq \phi(h) \leq \bar{\phi}(h)$ for all $h \in [-\theta, 0]$, and $\underline{\dot{x}}_0 \leq \dot{x}(0) \leq \bar{\dot{x}}_0$ for some known $\underline{\dot{x}}_0, \bar{\dot{x}}_0 \in \mathbb{R}$.*

The assumption about a known set $[\underline{\phi}, \bar{\phi}]$ for the initial conditions ϕ is standard for the interval or set-membership estimation theory Efimov et al., 2013c; Gouzé et al., 2000; Jaulin, 2002; Kieffer and Walter, 2004; Moisan et al., 2009. We will assume that the parameter a is known and the instant values of the signals $f(t)$ and $v(t)$ are unavailable:

Assumption 4.2. *There exist known signals $\underline{f}, \bar{f} \in \mathcal{L}_\infty$ and a constant $\nu_0 > 0$ such that $\underline{f}(t) \leq f(t) \leq \bar{f}(t)$ and $|\nu(t)| \leq \nu_0$ for all $t \geq 0$.*

Therefore, the uncertain inputs $f(t)$ and $\nu(t)$ in (4.11) and (4.15) belong to the known intervals $[\underline{f}(t), \bar{f}(t)]$ and $[-\nu_0, \nu_0]$ respectively for all $t \geq 0$.

The goal is to design an interval observer for (4.11), (4.15)

$$\begin{aligned} \dot{\xi}(t) &= F[\xi_t, \underline{f}(t), \bar{f}(t), \nu_0, y(t)], \quad \xi_t \in \mathcal{C}_\theta^s, \quad s > 0, \\ \begin{bmatrix} \underline{x}(t) \\ \underline{\dot{x}}(t) \end{bmatrix} &= \underline{G}[\xi_t, \underline{f}(t), \bar{f}(t), \nu_0, y(t)], \\ \begin{bmatrix} \bar{x}(t) \\ \bar{\dot{x}}(t) \end{bmatrix} &= \bar{G}[\xi_t, \underline{f}(t), \bar{f}(t), \nu_0, y(t)] \end{aligned}$$

such that for all $t \geq 0$

$$\begin{aligned} \underline{x}(t) &\leq x(t) \leq \bar{x}(t), \\ \underline{\dot{x}}(t) &\leq \dot{x}(t) \leq \bar{\dot{x}}(t) \end{aligned}$$

provided that assumptions 4.1 and 4.2 are satisfied and $x - \underline{x}, \bar{x} - x, \dot{x} - \underline{\dot{x}}, \bar{\dot{x}} - \dot{x} \in \mathcal{L}_\infty$. A similar problem has been studied in Efimov et al., 2013c; Efimov et al., 2015b; Mazenc et al., 2012b.

4.5 Interval observer design

In this section we will present two steps to design interval observers for the system (4.11) and (4.15). First, using the result of Theorem 4.1 an observer will be given for interval estimation of the position $x(t)$. Second, another interval observer will be designed to obtain the interval inclusion for $\dot{x}(t)$.

The first observer for the position

The corresponding homogeneous equation

$$\ddot{x}(t) = -a(t)x(t - \tau), \quad t \in [0, +\infty)$$

for (4.11) has unbounded solutions in case of a constant parameter a Domoshnitsky, 2014, which means that (4.11) is unstable. For the output $y(t)$ given in (4.15) the observer for (4.11) can be constructed as follows:

$$\ddot{x} = -a(t)x(t - \tau) + \ell(t)x(t - \theta) + f(t) - \ell(t)y(t) + \ell(t)v(t), \quad (4.16)$$

or in the state space form:

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -a(t)x_1(t - \tau) + \ell(t)x_1(t - \theta) + f(t) - \ell(t)y(t) + \ell(t)v(t), \end{aligned}$$

where $\ell \in \mathcal{L}_\infty$ is an observer gain to be designed.

Remark 4.1. Note that, as it has been explained in Section 4.4, the equation (4.11) can be presented in the form (4.1) with time-varying matrices A_0, A_1 and B as in (4.12), and with the output in form (4.15) with different measurement delay $\theta \geq \tau$, then the matrix A_1 can be nonnegative only when $\theta = \tau$, in other cases it is always $A_1 < 0$, then the system (4.11) and (4.16) do not possess the delay-independent or delay-dependent positivity properties according to lemmas 4.1 and 4.2, respectively.

Let us consider the first observer for the position $x(t)$ of the system (4.16) in the form:

$$\begin{aligned} \dot{\bar{x}}^-(t) &= -a(t)\bar{x}^-(t - \tau) + \ell(t)\bar{x}^-(t - \theta) \\ &\quad + \underline{f}(t) - \ell(t)y(t) - \ell(t)v_0 - \rho(t), \\ \dot{\bar{x}}^+(t) &= -a(t)\bar{x}^+(t - \tau) + \ell(t)\bar{x}^+(t - \theta) \\ &\quad + \overline{f}(t) - \ell(t)y(t) + \ell(t)v_0 + \rho(t), \end{aligned} \quad (4.17)$$

where $\bar{x}^-(t), \bar{x}^+(t) \in \mathbb{R}$ are the estimates for the position of motion (4.11) for $t \in [0, +\infty)$ with initial conditions

$$\begin{aligned} \bar{x}^-(h) &= \underline{\phi}(h), \quad \bar{x}^+(h) = \overline{\phi}(h) \quad \forall h \in [-\theta, 0], \\ \dot{\bar{x}}^-(0) &= \underline{\dot{x}}_0, \quad \dot{\bar{x}}^+(0) = \overline{\dot{x}}_0 \end{aligned}$$

from Assumption 4.1 and

$$\rho(t) = \begin{cases} a(t)[\bar{\phi}(t-\tau) - \underline{\phi}(t-\tau)] & t \leq \tau \\ 0 & t > \tau \end{cases}.$$

Proposition 4.2. *Let the measurement delays satisfy the relation $\theta \geq \tau > 0$, and assumptions 4.1, 4.2 be satisfied. For the system (4.11) with initial conditions (4.13) and the observer (4.17) select the observer gain $\ell(t) \geq 0$ to satisfy the conditions (4.8), (4.9) of Theorem 4.1 with $b(t) = \ell(t)$ for all $t \geq 0$. Then its position satisfies the interval inclusion*

$$x^-(t) \leq x(t) \leq x^+(t) \quad \forall t \in [0, +\infty) \quad (4.18)$$

and $x^+ - x, x - x^- \in \mathcal{L}_\infty$.

Proof. Define two estimation errors $e^+(t) = x^+(t) - x(t)$ and $e^-(t) = x(t) - x^-(t)$, $e^+(t), e^-(t) \in \mathbb{R}$, then their dynamics take the form:

$$\begin{aligned} \ddot{e}^-(t) &= -a(t)e^-(t-\tau) + \ell(t)e^-(t-\theta) \\ &\quad + f(t) - \underline{f}(t) + \ell(t)(v_0 + v(t)) + \rho(t), \\ \ddot{e}^+(t) &= -a(t)e^+(t-\tau) + \ell(t)e^+(t-\theta) \\ &\quad + \bar{f}(t) - f(t) + \ell(t)(v_0 - v(t)) + \rho(t). \end{aligned} \quad (4.19)$$

With the proper choice of the gain $\ell(t)$ such that the restrictions (4.8) and (4.9) of Theorem 4.1 are satisfied under the substitution $\ell(t) = b(t)$, the solutions of (4.19) can be written as in (4.14):

$$\begin{aligned} e^-(t) &= C_{11}(t, 0)e^-(0) + C_{12}(t, 0)\dot{e}^-(0) + \int_0^t C_{12}(t, s)\tilde{f}^-(s)ds, \\ e^+(t) &= C_{11}(t, 0)e^+(0) + C_{12}(t, 0)\dot{e}^+(0) + \int_0^t C_{12}(t, s)\tilde{f}^+(s)ds, \end{aligned}$$

where

$$\begin{aligned} \tilde{f}^-(s) &= f(t) - \underline{f}(t) + \ell(t)(v_0 + v(t)) + \rho(t) + \rho^-(t), \\ \tilde{f}^+(s) &= \bar{f}(t) - f(t) + \ell(t)(v_0 - v(t)) + \rho(t) + \rho^+(t), \end{aligned}$$

where $\rho^-(t)$ and $\rho^+(t)$ defined according to Proposition 4.1:

$$\rho^-(t) = \begin{cases} -a(t)[\phi(t-\tau) - \underline{\phi}(t-\tau)] + \ell(t)[\phi(t-\theta) - \underline{\phi}(t-\theta)] & t \in [0, \tau] \\ \ell(t)[\phi(t-\theta) - \underline{\phi}(t-\theta)] & t \in (\tau, \theta], \\ 0 & t > \theta \end{cases}$$

$$\rho^+(t) = \begin{cases} -a(t)[\bar{\phi}(t-\tau) - \phi(t-\tau)] + \ell(t)[\bar{\phi}(t-\theta) - \phi(t-\theta)] & t \in [0, \tau] \\ \ell(t)[\bar{\phi}(t-\theta) - \phi(t-\theta)] & t \in (\tau, \theta]. \\ 0 & t > \theta \end{cases}$$

According to Theorem 4.1, the functions $C_{11}(t, 0)$ and $C_{12}(t, 0)$ are nonnegative, and following Proposition 4.1 the variables $e^-(t) \geq 0$ and $e^+(t) \geq 0$ for all $t \in [0, +\infty)$ if the inputs $\tilde{f}^-(t)$ and $\tilde{f}^+(t)$ are nonnegative, which yields to interval inclusion (4.18) for the first state of the system (4.11). Note that $f(t) - \underline{f}(t)$, $\bar{f}(t) - f(t)$ and $v_0 + v(t)$, $v_0 - v(t)$ is positive by Assumption 4.2, and by the definition of the function ρ

$$\tilde{f}^-(s) \geq \rho(t) + \rho^-(t) = \begin{cases} a(t)[\bar{\phi}(t-\tau) - \phi(t-\tau)] + \ell(t)[\phi(t-\theta) - \underline{\phi}(t-\theta)] & t \in [0, \tau], \\ \ell(t)[\phi(t-\theta) - \underline{\phi}(t-\theta)] & t \in (\tau, \theta], \\ 0 & t > \theta, \end{cases}$$

$$\tilde{f}^+(s) \geq \rho(t) + \rho^+(t) = \begin{cases} a(t)[\phi(t-\tau) - \underline{\phi}(t-\tau)] + \ell(t)[\bar{\phi}(t-\theta) - \phi(t-\theta)] & t \in [0, \tau], \\ \ell(t)[\bar{\phi}(t-\theta) - \phi(t-\theta)] & t \in (\tau, \theta], \\ 0 & t > \theta, \end{cases}$$

where $\bar{\phi}(t-\tau) - \phi(t-\tau)$ and $\phi(t-\theta) - \underline{\phi}(t-\theta)$ are nonnegative by Assumption 4.1, and the result follows (boundedness of $x^+ - x$, $x - x^-$ is a consequence of Assumption 4.2 and the part 2) of Theorem 4.1). ■

Using this observer it is possible to derive the interval estimates for the position $x(t)$ without delay, but the velocity is not yet estimated since the matrix $A_1 < 0$.

The second observer for the velocity

As mentioned above, the second order delay differential equation (4.11) can be presented in form of (4.1) with matrices (4.12). Let us consider the delayed term $-a(t)x(t-\tau)$ as a disturbance and rewrite (4.1) for this case:

$$\dot{x}(t) = Ax(t) + \rho(t, x(t), x(t-\tau), f(t)), \quad (4.20)$$

where $x(t) = [x_1(t) \ x_2(t)]^T$ is the state vector, $x_1(t)$ is the estimated position by (4.18), $x_2(t)$ is a velocity of motion (4.11);

$$A = A_0 - KC = \begin{bmatrix} -k_1 & 1 \\ -k_2 & 0 \end{bmatrix},$$

$$\rho(t, x(t), x(t-\tau), f(t)) = \begin{bmatrix} k_1 x_1(t) \\ k_2 x_1(t) + f(t) - a(t)x_1(t-\tau) \end{bmatrix},$$

$K = [k_1 \ k_2]^T$ is a new observer gain, $C = \begin{bmatrix} 1 & 0 \end{bmatrix}$.

Assumption 4.3. *There are $K \in \mathbb{R}^2$ and a Metzler matrix $D \in \mathbb{R}^{2 \times 2}$ such that the matrices $A_0 - KC$ and D have the same eigenvalues and the pairs $(A_0 - KC, \chi_1)$ and (D, χ_2) are observable for some $\chi_1 \in \mathbb{R}^{1 \times 2}$, $\chi_2 \in \mathbb{R}^{1 \times 2}$.*

According to this assumption there is a nonsingular matrix $S \in \mathbb{R}^{2 \times 2}$ such that $D = S(A_0 - KC)S^{-1}$ Raïssi et al., 2012, and the new coordinates $z = Sx$ can be introduced transforming the system (4.20) to the form:

$$\dot{z}(t) = Dz(t) + S\rho(t, x(t), x(t-\tau), f(t)). \quad (4.21)$$

Using Lemma 1.1 we obtain that

$$\underline{\rho}(t) \leq \rho(t, x(t), x(t-\tau), f(t)) \leq \bar{\rho}(t),$$

where the functions $\underline{\rho}(t)$ and $\bar{\rho}(t)$ depend only on available information (the variables $x_1^-(t)$, $x_1^+(t)$, $x_1^-(t-\tau)$ and $x_1^+(t-\tau)$ are given by the first observer (4.17) for all $t \in [0, +\infty)$):

$$\underline{\rho}(t) = \begin{bmatrix} k_1 x_1^-(t) \\ k_2 x_1^-(t) + \underline{f}(t) - a(t)x_1^+(t-\tau) \end{bmatrix},$$

$$\bar{\rho}(t) = \begin{bmatrix} k_1 x_1^+(t) \\ k_2 x_1^+(t) + \bar{f}(t) - a(t)x_1^-(t-\tau) \end{bmatrix}.$$

Now, applying the results of Chebotarev et al., 2015; Gouzé et al., 2000 two estimates $\underline{z}, \bar{z} \in \mathbb{R}^2$ can be calculated based on the available information on these intervals (the interval inclusion (4.18) for $x_1(t)$ without delay), such that

$$\underline{z}(t) \leq z(t) \leq \bar{z}(t) \quad \forall t \in [0, +\infty). \quad (4.22)$$

In other words, an interval observer can be designed for the transformed dynamics (4.21):

$$\begin{aligned}
\dot{\underline{z}}(t) &= D\underline{z}(t) + S^+\underline{\rho}(t) - S^-\overline{\rho}(t), \\
\dot{\overline{z}}(t) &= D\overline{z}(t) + S^+\overline{\rho}(t) - S^-\underline{\rho}(t); \\
\underline{z}(h) &= S^+\underline{\Phi}(h) - S^-\overline{\Phi}(h), \\
\overline{z}(h) &= S^+\overline{\Phi}(h) - S^-\underline{\Phi}(h), \quad \forall h \in [-\tau, 0]; \\
\underline{x}(t) &= (S^{-1})^+\underline{z}(t) - (S^{-1})^-\overline{z}(t), \\
\overline{x}(t) &= (S^{-1})^+\overline{z}(t) - (S^{-1})^-\underline{z}(t),
\end{aligned} \tag{4.23}$$

where

$$\underline{\Phi}(h) = \begin{bmatrix} \underline{\phi}(h) \\ \begin{cases} 0 & h < 0 \\ \underline{\dot{x}}_0 & h = 0 \end{cases} \end{bmatrix}, \quad \overline{\Phi}(h) = \begin{bmatrix} \overline{\phi}(h) \\ \begin{cases} 0 & h < 0 \\ \overline{\dot{x}}_0 & h = 0 \end{cases} \end{bmatrix}$$

and the relations (1.4) are used to calculate the initial conditions for $\underline{z}(h)$, $\overline{z}(h)$ at $h \in [-\theta, 0]$ and the estimates \underline{x} , \overline{x} .

Proposition 4.3. *Let assumptions 4.1, 4.2 and 4.3 be satisfied. Then for the second order delay equation (4.11), presented in form (4.1), with initial conditions 4.2, and with the interval observer (4.23) the relations for the velocity*

$$\underline{x}_2(t) \leq x_2(t) \leq \overline{x}_2(t), \quad x(t) \in \mathbb{R}^2, \quad \forall t \in [0, +\infty) \tag{4.24}$$

are fulfilled provided that the conditions of Proposition 4.2 are verified.

Proof. Introduce two estimation errors $\underline{e}(t) = z(t) - \underline{z}(t)$ and $\overline{e}(t) = \overline{z}(t) - z(t)$ for the observer (4.23), which yield the differential equations:

$$\begin{aligned}
\dot{\underline{e}}(t) &= D\underline{e}(t) + \delta(t) - \underline{\delta}(t), \\
\dot{\overline{e}}(t) &= D\overline{e}(t) + \overline{\delta}(t) - \delta(t),
\end{aligned}$$

where $\delta(t) = S\rho(t, x(t), x(t-\tau), f(t))$ and, by using Lemma 1.1, $\underline{\delta}(t) = S^+\underline{\rho}(t) - S^-\overline{\rho}(t)$, $\overline{\delta}(t) = S^+\overline{\rho}(t) - S^-\underline{\rho}(t)$, which provide the inclusion $\underline{\delta}(t) \leq \delta(t) \leq \overline{\delta}(t)$ for all $t \geq 0$ by the form of $\underline{\rho}$, $\overline{\rho}$, the assumption 4.2 and by the interval estimates obtained for $x_1(t)$ from the observer (4.17) (all conditions of Proposition 4.2 are true). Then it gives for all $t \in [0, +\infty)$:

$$\delta(t) - \underline{\delta}(t) \geq 0, \quad \overline{\delta}(t) - \delta(t) \geq 0,$$

and if D is a Metzler matrix, since all inputs of $\underline{e}(t)$, $\overline{e}(t)$ are positive and the initial condi-

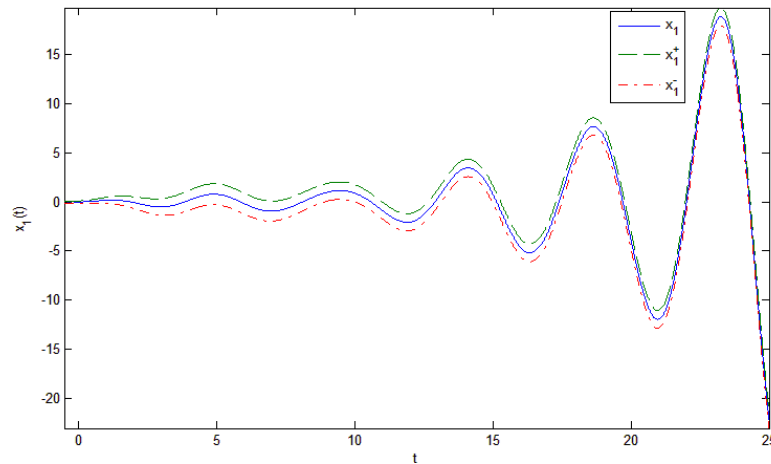


Figure 4.1 – The result of simulation of observer (4.17) for the position $x(t)$ of (4.11).

tions $\underline{e}(h) \geq 0$, $\bar{e}(h) \geq 0$ for $\forall h \in [-\tau, 0]$ by the Assumption 4.1 and the above calculations, then $\underline{e}(t) \geq 0$, $\bar{e}(t) \geq 0$ for all $t \geq 0$. The property (4.24) follows from these relations. ■

4.6 Example

To show the efficiency of the proposed observers we consider the motivation example (4.11):

$$\dot{x}(t) = -ax(t - \tau) + f(t), \quad t \in [0, +\infty)$$

with the values of parameter $a = 2$, the internal delay $\tau = 0.2$, the perturbation $f(t) = 0.5(\cos(2t) + 0.3\cos(10t))$; the measurement delay $\theta = 0.5$ and the noise $v(t) = v_0 \sin(60t)$ for $v_0 = 0.07$. The initial conditions (4.2) $x(h) = \phi(h)$ for $h \in [-\theta, 0]$, where $\phi(h) = 0.1 \sin(\frac{1}{4\pi\theta}t)$, $\dot{x}(0) = 0$. The Assumption 4.2 is satisfied for $\underline{f}(t) = 0.5(\cos(2t) - 0.3)$, $\bar{f}(t) = 0.5(\cos(2t) + 0.3)$; and the bounds on initial conditions are given as $\underline{\phi}(h) = -0.1$, $\bar{\phi}(h) = 0.1$, $\bar{\dot{x}}_0 = -\underline{\dot{x}}_0 = 0.1$. For $\ell = 1.8$ the conditions (4.8), (4.9) are satisfied. For simulation the explicit Euler method with the step $T = 10^{-3}$ was used. The results of simulation of the first observer (4.17) for position $x(t)$ of (4.11) are shown in Fig. 4.1 for $t \in [0, 25]$. In Fig. 4.2 the errors of the estimation are presented: for position $e_1(t) \geq 0$ for $t \geq 0$; and for the velocity $e_2(t)$ is not positive (as it is supposed to be). For the second observer (4.23) the observer gain $K = [3 \ 1]$ with the matrix $D = \text{diag}[-2.618; -0.382]$ satisfies the Assumption 4.3 with the transformation matrix $S = \begin{bmatrix} -1.25 & 0.48 \\ 0.48 & -1.25 \end{bmatrix}$. The estimates derived for the velocity of (4.11) by the second observer (4.23) are shown at the Fig. 4.3.

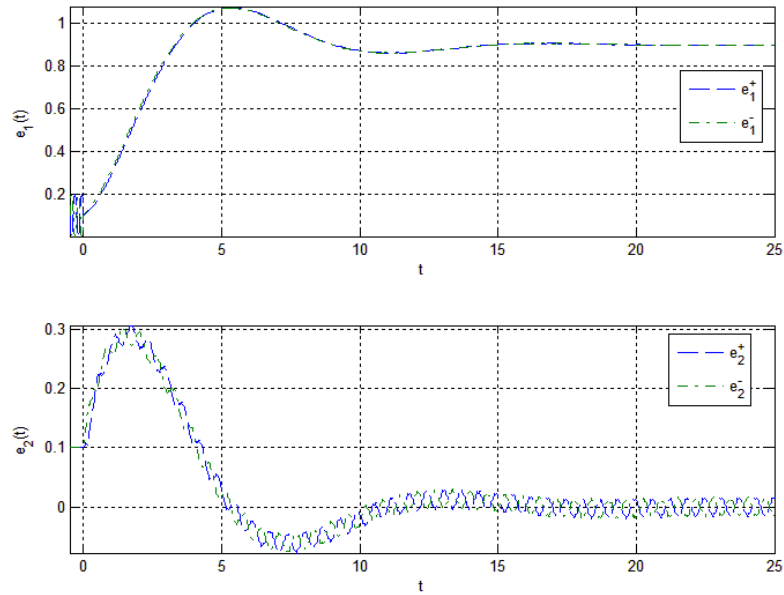


Figure 4.2 – Errors of estimation $e_1(t)$, $e_2(t)$ of the first observer (4.17).

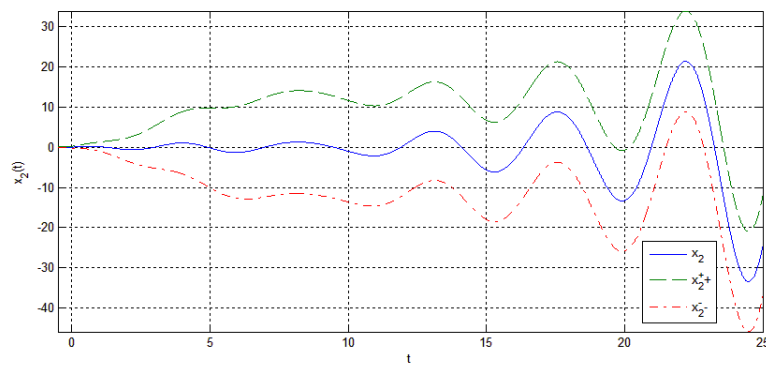


Figure 4.3 – The result of simulation of observer (4.23) for the velocity $\dot{x}(t)$ of (4.11).

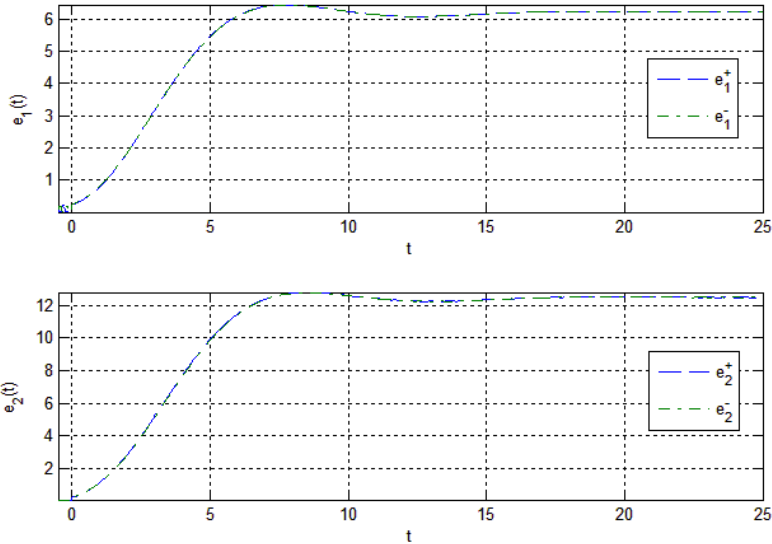


Figure 4.4 – Errors of estimation $e_1(t), e_2(t)$ of the second observer (4.23).

Conclusions

The work here presented can be mainly seen as contributions to the study of interval observers of infinite dimensional systems. The results obtained can be divided into two groups: observer design for PDEs and time-delay systems.

The second chapter dealt, using early lumping approach, with a parabolic PDE under Dirichlet boundary conditions and uncertainties, presented by bounded distributed disturbance. A method of interval observer design was proposed, which is based on a finite-element Galerkin approximation. The errors of discretization given in Wheeler, 1973 were taken into account by the interval estimates. The proposed interval observer was used for control of an uncertain PDE system. The efficiency of the proposed interval observer and control was demonstrated through numerical experiments with the Black–Scholes model.

The third chapter addressed to a method of PDE-interval observer, which is not based on a finite-element approximation. The proposed observer has the same distributed nature. The design employs the positivity of solutions of the heat equation proposed in Nguyen and Coron, 2016. The proposed interval observer is used for stabilization of an uncertain heat equation. The result was compared with static output feedback controller.

Finally, the last chapter presents the interval estimation approach for a second-order delay differential equation with position delayed measurements, uncertain input and initial conditions. The proposed approach consists in two consecutively connected interval observers. The first one estimates the set of admissible values for the position without delay using new delay-dependent conditions on positivity of a second order system. Then derived interval estimates of the position are used to design the second observer evaluating an interval of admissible values for the velocity of the considered dynamical system.

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Appendix

Publications

A.1 Conference Articles

NOLCOS16, Monterey, USA

T. Kharkovskaia, D. Efimov, A. Polyakov, and J.-P. Richard [2016]. “Interval observers for PDEs: approximation approach”. In: *Proc. 10th IFAC Symposium on Nonlinear Control Systems (NOLCOS)*. Monterey

IFAC17, Toulouse, France

T. Kharkovskaia, D. Efimov, E. Fridman, A. Polyakov, and J.-P. Richard [July 2017]. “On design of interval observers for parabolic PDEs”. In: *Proc. 20th IFAC WC 2017*. Toulouse, France

CDC2018, Miami Beach, Florida, USA

T. Kharkovskaia, D. Efimov, E. Fridman, A. Polyakov, and J.-P. Richard [2018a]. “Interval estimation for second-order delay differential equations with delayed measurements and uncertainties”. In: *CDC 2018 - 57th IEEE Conference on Decision and Control*. Fontainebleau (FL), United States

A.2 Journal Articles

Automatica, 2018

T. Kharkovskaia, D. Efimov, A. Polyakov, and J.-P. Richard [2018b]. “Design of interval observers and controls for PDEs using finite-element approximations”. In: *Automatica* 93, pp. 302–310

Automatica, January 2020

T. Kharkovskaia, D. Efimov, E. Fridman, A. Polyakov, and J.-P. Richard [Jan. 2020]. “Interval observer design and control of uncertain non-homogeneous heat equations”. In: *Automatica* 111, p. 0

Substantial summaries

B.1 Résumé substantiel en français

La thèse est consacrée à l'étude des observateurs par intervalle, qui forment une sous-classe d'estimateurs d'appartenance à un ensemble et dont la conception est basée sur la théorie des systèmes monotones, à leurs applications dans l'estimation et le contrôle de systèmes distribués incertains, ainsi qu'à l'analyse de leurs propriétés et de leurs restrictions.

Contrairement à un observateur conventionnel qui, en l'absence de bruit de mesure et d'incertitudes, doit converger vers la valeur exacte de l'état du système estimé (il donne une estimation ponctuelle de l'état), les observateurs par intervalle évaluent à chaque instant un ensemble de valeurs admissibles pour l'état, de manière cohérente avec la sortie mesurée (c'est-à-dire qu'ils fournissent une estimation d'intervalle). Cette idée de conception d'observateur d'intervalle a été proposée assez récemment dans Gouzé et al., 2000, mais elle a déjà reçu de nombreuses extensions pour diverses classes de modèles dynamiques.

Les avantages des observateurs par intervalle sont qu'ils sont bien adaptés à la conception d'observateur dans des systèmes très incertains (si les intervalles de valeurs admissibles pour les termes inconnus sont donnés) et qu'ils sont capables de fournir des limites asymptotiquement assez serrées sur la précision de l'estimation, car l'intervalle des valeurs admissibles pour l'état à chaque instant est évalué. De plus, cette technique tire parti d'une connaissance approximative de la condition initiale et donne des informations sur l'état inconnu du système étudié à tout instant, tandis que les observateurs classiques ne fournissent qu'une information utile asymptotiquement, c'est-à-dire d'un point de vue technique, pour des valeurs temporelles suffisamment grandes.

Les observateurs par intervalle, présentés dans la littérature, ont été principalement construits pour des systèmes dynamiques, décrits par des équations différentielles ordinaires, par exemple pour des systèmes linéaires invariants dans le temps Gouzé et al., 2000; Mazenc and Bernard, 2010, 2011, systèmes linéaires à variation de paramètre (LVP) Chebotarev et al., 2015; Efimov et al., 2012, 2013e,f, systèmes non linéaires Raïssi et al., 2012, 2010, systèmes à temps discret Efimov et al., 2013a,b; Mazenc et al., 2013, 2014, 2012a, systèmes de descripteurs ou systèmes avec contraintes algébriques Efimov et al., 2014.

En outre, ces nombreux travaux sur les observateurs par intervalle pour les EDO, il y a quelques articles qui ont étendu cette méthode pour les systèmes à retardement Efimov et al., 2016, 2013c, 2015a; Mazenc et al., 2012b, décrit par des équations différentielles fonctionnelles. Pour ces modèles, qui sont infiniment dimensionnels contrairement aux équations différentielles ordinaires (EDO), l'analyse et la conception sont beaucoup plus compliquées et nécessitent des concepts et des algorithmes spécialement développés Richard, 2003. Par exemple, l'observabilité et les méthodes d'estimation de systèmes retardés à entrées et non linéarités inconnues sont prises en compte dans Califano et al., 2011; Zheng et al., 2011, l'application d'une approche algébrique pour la conception par observateur dans les systèmes à retardement LPV est présentée dans Briat et al., 2011, un problème d'estimation pour les systèmes positifs avec des retards inconnus variant dans le temps est étudié dans Rami et al., 2013. Les solutions existantes pour les observateurs d'intervalle dans le domaine Efimov et al., 2013c, 2015a; Mazenc et al., 2012b; Polyakov et al., 2013 sont basées sur les conditions de positivité indépendantes du délai de Ait Rami, 2009; Haddad and Chellaboina, 2004.

Quelques résultats sur la conception d'observateurs par intervalle pour un système de retard variable dans le temps peuvent être trouvés dans Efimov et al., 2013c; Rami et al., 2013. Dans Efimov et al., 2016, les conditions de positivité dépendant du retard sont introduites pour le cas avec un retard égal dans l'état et dans la sortie, ce qui peut correspondre à un système sans retard avec des mesures retardées.

Pour combler cette lacune, dans le chapitre 4, une nouvelle approche est introduite pour faire face à l'obstacle susmentionné, lorsque les retards de mesure sont différents des retards d'état.

De plus, l'objectif de la thèse est d'étendre d'observateur par intervalle aux systèmes de dimension infinie, en considérant non seulement les systèmes avec des retards, mais également distribués dans les systèmes spatiaux, qui sont décrits par des équations aux dérivées partielles (EDP).

Le son, la chaleur, l'électrostatique, l'électrodynamique, l'écoulement des fluides, l'élasticité ou la mécanique quantique, ainsi que les modèles d'autres phénomènes physiques, peuvent être formalisés de la même manière en termes d'EDP, dont la nature distribuée introduit un niveau supplémentaire de complexité. C'est pourquoi le contrôle et l'estimation des EDP sont aujourd'hui une direction de recherche très populaire Barje et al., 2013; Bredies et al., 2013; Demetriou, 2004; Hasan et al., 2016; Kamran and Drakunov, 2015; Krstic, 2009; Meurer, 2013; Nguyen, 2008; Russell, 2003; Smyshlyaev and Krstic, 2005; Smyshlyaev and Krstic, 2010. Dans cette classe de modèles, où l'état du système est fonction de l'espace à chaque instant du temps, le problème de sa mesure explicite est naturel, car seules des mesures spatiales ponctuelles et discrètes sont réalisables par un capteur Jorgensen et al., 1984; Vande Wouwer et al., 2000. Fréquemment, afin de concevoir un estimateur d'état, l'approche d'approximation de dimension finie est utilisée Alvarez and Stephanopoulos, 1982; Dochain, 2000; Hagen and I., 2003; Vande Wouwer and Zeitz, 2002, ensuite, le problème d'observation est abordé avec les outils bien connus disponibles pour les systèmes de dimension finie, tandis que l'évaluation de la convergence doit être effectuée par rapport aux solutions du système distribué d'origine. L'analyse et la conception dans les coordonnées distribuées d'origine sont plus compliquées, mais attirent également l'attention de nombreux chercheurs Ahmed-Ali et al., 2015; Fridman

and Bar Am, 2013; Fridman, 2013; Hidayat et al., 2011; Liu and Fridman, 2012; Schaum et al., 2014; Selivanov and Fridman, 2018; Smyshlyaev and Krstic, 2010. Dans Pisano and Orlov, 2017 une conception de contrôle de stabilisation avec une rétroaction proportionnelle discontinue est proposée pour une EDP parabolique avec détection et actionnement colocalisés ponctuellement, et avec des perturbations distribuées dans le domaine. Le travail Wang et al., 2018 présente un contrôle distribué basé sur des observateurs de type Luenberger avec des capteurs et des actionneurs non colocalisés.

Néanmoins, à la connaissance de l'auteur, aucun travail n'a été effectué sur les observateurs par intervalle pour les EDP. Dans le chapitre 2 de la présente thèse, une méthode pour concevoir un observateur par intervalle a été développée pour une EDP parabolique avec des conditions aux limites de Dirichlet, qui est basée sur une approximation de Galerkin par éléments finis. Cette méthode est également appelée approche précoce grumelage pour un système distribué, lorsque la première étape consiste à discrétiser le système donné à un système lumpé, décrit par l'ensemble des EDO, et la deuxième étape consiste à appliquer des solutions déjà existantes de conception d'observateurs.

En utilisant les estimations d'erreur de discrétisation de Wheeler, 1973, l'intervalle enveloppant pour les solutions de EDP a été évalué. Une caractéristique intéressante de cette approche est que, appliquée à une EDP non linéaire, en supposant que toutes les non-linéarités sont limitées et traitées comme des perturbations, l'observateur par intervalle proposé est linéaire et peut être facilement mis en œuvre en fournissant des limites sur les solutions de la EDP non linéaire d'origine (sous la hypothèse que ces solutions existent).

À côté de cela, sur la base d'un observateur construit pour le système lumpé, un contrôle de stabilisation de la sortie a été développé dans la section 2.3. La stratégie de contrôle proposée dispose d'une caractéristique similaire, car elle est conçue pour un modèle de dimension finie, mais en garantissant la limitation des trajectoires pour une dynamique distribuée incertaine. Ces résultats ont été publiés dans Kharkovskaia et al., 2016, 2018b.

Dans le chapitre 3 un plan d'observation par intervalle est proposé pour la même classe de système qu'au chapitre 2, décrite par une EDP parabolique avec des conditions aux limites de Dirichlet non nulles, mais elle n'est basée sur aucune approximation, et l'observateur a la même nature distribuée. En utilisant les conditions de positivité des solutions de EDP paraboliques présentées dans Nguyen and Coron, 2016, un observateur par intervalle est construit gouverné par EDP, dont la dynamique d'erreur d'estimation (également distribuée) est positive. L'analyse de stabilité de Fridman and Blighovsky, 2012 est étendue au scénario considéré avec un bruit de mesure et des conditions aux limites non nulles. On trouve les conditions dans lesquelles le système de paramètres distribués possède la propriété de stabilité entrée-à-état, où les conditions aux limites influencent également la perturbation externe et les conditions initiales. De plus, la principale restriction de la propriété de stabilité est imposée par les paramètres du système et par les intervalles entre les points de mesure. Pour surmonter cette restriction, la conception du contrôleur de rétroaction est développée dans la Section 3.3 en utilisant les estimations d'intervalle reçues. On suppose que la commande est distribuée spatialement, influençant la dynamique du système par des fonctions de forme constantes par morceaux, une telle hypothèse est introduite pour respecter la faisabilité de la mise en œuvre de la loi de commande conçue, car les variations infinitésimales dans l'espace du signal de

l'actionneur ne peuvent pas être réalisées en pratique. Ces méthodes ont été présentées dans Kharkovskaia et al., 2017, 2020.

Dans le chapitre 2 la EDP suivante avec des conditions aux limites de Dirichlet homogènes est considérée:

$$\begin{aligned} \rho(x) \frac{\partial z(x, t)}{\partial t} &= L[x, z(x, t)] + r(x, t) \quad \forall (x, t) \in I \times (0, T), \\ z(x, 0) &= z_0(x) \quad \forall x \in I, \\ 0 &= z(0, t) = z(1, t) \quad \forall t \in (0, T), \end{aligned} \tag{B.1}$$

où $I = [0, 1]$ et $T > 0$,

$$L(x, z) = \frac{\partial}{\partial x} \left(a(x) \frac{\partial z}{\partial x} \right) - b(x) \frac{\partial z}{\partial x} - q(x)z,$$

$r \in L^\infty(I \times [0, T], \mathbb{R})$, $a, b, q, \rho \in L^\infty(I, \mathbb{R})$ et il existe $a_0, a_1, \rho_0, \rho_1 \in \mathbb{R}_+$ tel que

$$0 < a_0 \leq a(x) \leq a_1, \quad 0 < \rho_0 \leq \rho(x) \leq \rho_1 \quad \forall x \in I,$$

et $a', b' \in L^2(I, \mathbb{R})$, où $a' = \partial a(x)/\partial x$.

Ensuite, l'approximation de Galerkin en temps continu $Z(\cdot, t)$ (2.2) est présenté dans la section 2.1, avec base d'approximation Φ_j , $1 \leq j \leq N$ avec $N \geq N'$, puis sa solution $Z(x, t)$ de (2.2) peut être présenté comme (Thomée, 2006)

$$Z(x, t) = \sum_{j=1}^N \xi_j(t) \Phi_j(x),$$

où $\xi = [\xi_1 \dots \xi_N]^T \in \mathbb{R}^N$ est le vecteur de coefficients satisfaisant l'ensemble final d'EDO suivant (*a.a.* signifie "pour presque tous"):

$$\dot{\xi}(t) = A\xi(t) + G\bar{r}(t) \quad a.a. \ t \in (0, T), \quad \xi(0) = \xi_0, \tag{B.2}$$

où les matrices $A \in \mathbb{R}^{N \times N}$, $G \in \mathbb{R}^N$, conditions initiales $\xi_0 \in \mathbb{R}^N$ et $\bar{r} \in L^\infty([0, T], \mathbb{R}^N)$ sont calculés dans la section 2.1. Alors pour tout $\xi_0 \in \mathbb{R}^N$ la solution correspondante $\xi \in C^0([0, T], \mathbb{R}^N)$ au problème de Cauchy (B.2) peut être facilement calculé.

Pour estimer l'erreur de l'approximation, nous supposons qu'il existe $s > 0$, $l_1 > 0$ et $l_2 > 0$ de telle sorte que la solution z de (B.1) et son dérivé de première dans le temps sont bornés:

$$\|z\|_{L^\infty([0, T], W^{s+1, \infty}(I, \mathbb{R}))} \leq l_1, \quad \|\partial z / \partial t\|_{L^2([0, T], H^{s+1}(I, \mathbb{R}))} \leq l_2.$$

Ensuite, sous cette Assomption 2.1 et (Wheeler, 1973) l'erreur d'approximation Z est bornée

$$\|Z - z\|_{L^\infty(I \times (0, T), \mathbb{R})} \leq \rho h^{s+1} (l_1 + l_2), \tag{B.3}$$

où $\rho > 0$.

Pour établir des liens entre les estimations d'intervalle pour ξ et Z , et entre Z et

l'état de (B.1) z , les lemmes 2.2 et 2.3 a été proposé. S'il existe $\underline{\xi}, \bar{\xi} \in C^0([0, T], \mathbb{R}^N)$ et $\underline{Z}, \bar{Z} \in C^0(I \times [0, T], \mathbb{R})$, puis

$$\begin{aligned}\underline{Z}(x, t) &= \sum_{j=1}^N (\underline{\xi}_j(t) \Phi_j^+(x) - \bar{\xi}_j(t) \Phi_j^-(x)), \\ \bar{Z}(x, t) &= \sum_{j=1}^N (\bar{\xi}_j(t) \Phi_j^+(x) - \underline{\xi}_j(t) \Phi_j^-(x)).\end{aligned}\tag{B.4}$$

La preuve est basée sur le lemme 2.1. Si pour $s, \underline{s}, \bar{s} : I \rightarrow \mathbb{R}$, $\underline{s} \leq s \leq \bar{s}$, alors pour tout $\phi : I \rightarrow \mathbb{R}$ avec les opérateurs $\phi^+ = \max\{0, \phi\}$, $\phi^- = \phi^+ - \phi$:

$$\langle \underline{s}, \phi^+ \rangle - \langle \bar{s}, \phi^- \rangle \leq \langle s, \phi \rangle \leq \langle \bar{s}, \phi^+ \rangle - \langle \underline{s}, \phi^- \rangle.$$

Le lien entre les estimations d'intervalle pour z et Z sont établis sous l'Assomption 2.1 et (B.3):

$$\begin{aligned}\underline{z}(x, t) &= \underline{Z}(x, t) - \rho h^{s+1} (l_1 + l_2), \\ \bar{z}(x, t) &= \bar{Z}(x, t) + \rho h^{s+1} (l_1 + l_2)\end{aligned}\tag{B.5}$$

pour tous $x \in I$ et presque tous $t \in [0, T]$, où $\underline{z}, \bar{z} \in L^\infty(I \times [0, T], \mathbb{R})$.

Supposons que l'État $z(x, t)$ est disponible pour des mesures en certains points $x_i^m \in I$:

$$y_i(t) = z(x_i^m, t) + v_i(t), \quad j = 1, \dots, p,\tag{B.6}$$

où $y(t), v(t) \in \mathbb{R}^p$, $v \in L^\infty(\mathbb{R}_+, \mathbb{R}^p)$ est le bruit de mesure. Sous l'Hypothèse 2.1 de Proposition 2.1, et en appliquant à nouveau l'approximation des éléments finis, nous obtenons

$$y(t) = C \xi(t) + v(t),\tag{B.7}$$

avec $v(t) = v(t) + e(t) \in \mathbb{R}^p$ étant le nouveau bruit de mesure. Nous supposons également que (B.1),

$$r(x, t) = \sum_{k=1}^m r_{1k}(x) u_k(t) + r_0(x, t),$$

où $u(t) \in \mathbb{R}^m$ est un contrôle (entrée connue), $r_{1k} \in L^\infty(I, \mathbb{R})$ et $r_0 \in L^\infty(I \times [0, T], \mathbb{R})$. Puis dans (B.2) $G\bar{r}(t) = Bu(t) + Gd(t)$, $d(t) \in \mathbb{R}^N$ est une perturbation externe inconnue. Les matrices $B \in \mathbb{R}^{N \times m}$ and $C \in \mathbb{R}^{p \times N}$ sont calculés dans la section 2.2.

Dans l'Hypothèse 2.2 trois intervalles $[\underline{z}_0, \bar{z}_0]$, $[\underline{r}_0(x, t), \bar{r}_0(x, t)]$ et $[-v_0, v_0]$ déterminent pour tous $(x, t) \in I \times [0, T]$ dans (2.1), (2.8) incertitude des valeurs de z_0 , $r_0(x, t)$ et $v(t)$, respectivement.

Selon l'Hypothèse 2.3, il y a une matrice $L \in \mathbb{R}^{N \times p}$ et une matrice de Metzler $D \in \mathbb{R}^{N \times N}$ de telle sorte que les matrices $A - LC$ et D avoir les mêmes valeurs propres, où L est le gain de l'observateur pour assurer la propriété Hurwitz pour la matrice $A - LC$ (il peut être trouvé à l'aide d'IML (2.13), (2.14), (2.15)), alors il est possible de faire une transformation de coordonnées $\zeta = S\xi$ avec une matrice de transformation non singulière $S \in \mathbb{R}^{N \times N}$ tel

que $D = S(A - LC)S^{-1}$.

Ensuite, pour le système (B.2) un observateur par intervalle peut être proposé:

$$\begin{aligned}\dot{\underline{\zeta}}(t) &= D\underline{\zeta}(t) + SBu(t) + SLy(t) + \underline{\delta}(t), \\ \dot{\overline{\zeta}}(t) &= D\overline{\zeta}(t) + SBu(t) + SLy(t) + \overline{\delta}(t), \\ \underline{\zeta}(0) &= S^+\underline{\xi}_0 - S^-\overline{\xi}_0, \quad \overline{\zeta}(0) = S^+\overline{\xi}_0 - S^-\underline{\xi}_0, \\ \underline{\xi}(t) &= (S^{-1})^+\underline{\zeta}(t) - (S^{-1})^-\overline{\zeta}(t), \quad \overline{\xi}(t) = (S^{-1})^+\overline{\zeta}(t) - (S^{-1})^-\underline{\zeta}(t),\end{aligned}\tag{B.8}$$

où les relations (1.4) de Lemme 1.1 sont utilisés pour calculer les conditions initiales pour $\underline{\zeta}, \overline{\zeta}$ et les estimations $\underline{\xi}, \overline{\xi}$, aussi bien que $\underline{\delta}(t), \overline{\delta}(t)$ de l'entrée $\delta(t) = S[Gd(t) - Lv(t)]$. L'application de cet observateur par intervalle conduit à une inclusion d'intervalle pour le (B.2) $\underline{\xi}(t) \leq \xi(t) \leq \overline{\xi}(t) \forall t \in [0, T]$. Puis par Proposition 2.2 et le Théorème 2.1 avec l'utilisation de Lemmas 2.2 et 2.3 nous trouvons les estimations d'intervalle pour (B.1)

$$\underline{z}(x, t) \leq z(x, t) \leq \overline{z}(x, t).$$

Dans la section 2.3 l'observateur par intervalle (B.8) est utilisé pour concevoir une loi de contrôle assurant la stabilisation de l'approximation de dimension finie (B.2), (B.7) dans l'esprit de Efimov et al., 2013f, ce qui implique également (sous de légères restrictions supplémentaires) la stabilisation (B.1). La commande est choisie comme une rétroaction linéaire d'état conventionnelle:

$$u(t) = \underline{K}\underline{\zeta}(t) + \overline{K}\overline{\zeta}(t)$$

où $\underline{K}, \overline{K} \in \mathbb{R}^{m \times N}$ sont deux gains de matrice de rétroaction, qui peuvent être calculés par les inégalités de matrice

$$\widetilde{A}\widetilde{X} + \widetilde{X}\widetilde{A}^T + \widetilde{B} \begin{bmatrix} 0 & \widetilde{Y} \end{bmatrix} + \begin{bmatrix} 0 & \widetilde{Y} \end{bmatrix}^T \widetilde{B}^T < 0, \quad X_P > 0, \quad X_Q > 0,$$

où

$$\widetilde{X} = \begin{bmatrix} X_P & 0 & 0 \\ 0 & & \\ 0 & X_Q & \end{bmatrix} \in \mathbb{R}^{3N \times 3N}, \quad X_P = X_P^T \in \mathbb{R}^{N \times N}, \quad X_Q = X_Q^T \in \mathbb{R}^{2N \times 2N}$$

et $\widetilde{Y} \in \mathbb{R}^{m \times 2N}$, puis $[\underline{K} \quad \overline{K}] = \widetilde{Y}X_Q^{-1}$.

La Section 2.4 présente des tests d'algorithmes proposés d'estimation par intervalle et de stabilisation sur un exemple académique et un modèle de Black-Scholes.

Contribution principale du Chapitre 3 est comme suit. Tout d'abord, un observateur par intervalles décrit par les EDP sans appliquer d'approximations par éléments finis est proposé pour les systèmes de paramètres distribués incertains. Deuxièmement, une conception supplémentaire d'un contrôle de stabilisation de sortie est effectuée sur la base d'observations d'intervalle.

La PDE considérée avec les conditions aux limites associées a une forme (B.1) similaire,

dotée d'un contrôle:

$$\begin{aligned} \frac{\partial z(x,t)}{\partial t} &= L[x, z(x,t)] + r(x,t) + \sum_{j=0}^p b_j(x)u_j(t) \quad \forall (x,t) \in I \times \mathcal{T}, \\ z(x, t_0) &= z_0(x) \quad \forall x \in I, \\ z(0, t) &= \alpha(t), \quad z(\ell, t) = \beta(t) \quad \forall t \in \mathcal{T}, \end{aligned} \quad (\text{B.9})$$

où $I = [0, \ell]$ avec $0 < \ell < +\infty$, $\mathcal{T} = [t_0, t_0 + T)$ pour $t_0 \in \mathbb{R}$ et $T > 0$,

$$L(x, z) = \frac{\partial}{\partial x} \left(a(x) \frac{\partial z}{\partial x} \right) + q(x)z,$$

$a \in C^1(I, \mathbb{R})$, $q \in C(I, \mathbb{R})$ et il existe $a_{\min}, a_{\max} \in \mathbb{R}_+$ tel que $0 < a_{\min} \leq a(x) \leq a_{\max} \forall x \in I$; les conditions aux limites $\alpha, \beta \in C^2(\mathcal{T}, \mathbb{R})$ et l'entrée externe $r \in C^1(I \times \mathcal{T}, \mathbb{R})$; les conditions initiales $z_0 \in \mathcal{Z}_0 = \{z_0 \in H^2(I, \mathbb{R}) : z_0(0) = \alpha(0), z_0(\ell) = \beta(0)\}$; les commandes $u_j : \mathcal{T} \rightarrow \mathbb{R}$ sont les fonctions continues de Lipschitz. Le domaine spatial I est divisé en $p + 1$ sous-domaines I_j pour $j = 0, 1, \dots, p$, où les signaux de commande $u_j(t)$ sont appliqués via les fonctions de forme $b_j \in L^2(I, [0, 1])$ tel que

$$\begin{cases} b_j(x) = 0 & x \notin I_j, \\ b_j(x) = 1 & x \in I_j. \end{cases} \quad (\text{B.10})$$

Les commandes u_j sont conçus dans la Section 3.3, dans les Sections 3.1 et 3.2 ils sont supposés être donnés et $u_j \in L^\infty(\mathcal{T}, \mathbb{R})$ pour tous $j = 0, 1, \dots, p$.

La Section 3.1 est consacré à une analyse de stabilité entrée-à-état, présentée dans la Proposition 3.1, où la principale restriction à la stabilité de l'équation de chaleur non homogène (B.9) est:

$$a_{\min} \frac{\pi^2}{\ell^2} > q_{\max}, \quad (\text{B.11})$$

où $q_{\max} = \sup_{x \in I} q(x)$, puis pour les solutions de (B.9) l'estimation suivante est satisfaite pour tous $t \in \mathcal{T}$:

$$\frac{1}{2} \int_0^\ell z^2(x,t) dx \leq e^{-\chi(t-t_0)} \int_0^\ell w_0^2(x) dx + \chi^{-2} \int_0^\ell \tilde{r}^2(x,t) dx + \frac{\ell}{2} [\alpha^2(t) + \beta^2(t)],$$

où $\chi = a_{\min} \frac{\pi^2}{\ell^2} - q_{\max}$, $w_0(x) = z_0(x) - \delta(x, t_0)$, $\delta(x, t) = \alpha(t) + \frac{x}{\ell}(\beta(t) - \alpha(t))$, et

$$\tilde{r}(x, t) = r(x, t) + \frac{1}{\ell} \frac{\partial a(x)}{\partial x} (\beta(t) - \alpha(t)) + q(x)\delta(x, t) - \delta_t(x, t) + \sum_{j=0}^p b_j(x)u_j(t).$$

En général, la solution $z(\cdot, t)$ de (B.9) prend ses valeurs \mathbb{R} et il peut changer de signe avec $(x, t) \in I \times \mathcal{T}$. Mais pour construire un observateur par intervalle, nous avons besoin $z(\cdot, t) \geq 0$.

Le système (B.9) est appelé non négatif (positif) sur l'intervalle \mathcal{T} si les conditions aux limites et initiales et les entrées externes ne prennent que des valeurs non négatives (positives) pour tous $(x, t) \in I \times \mathcal{T}$. La Proposition 3.2 avec Nguyen and Coron, 2016 établit un résultat plus général pour (B.9). Il est non négatif sur l'intervalle \mathcal{T} avec $u_j(t) = 0$ pour tous $j = 0, 1, \dots, p$, si $\alpha, \beta \in L^2(\mathcal{T}, \mathbb{R}_+)$, $r \in L^2(I \times \mathcal{T}, \mathbb{R}_+)$ et $z_0 \in H^1(I, \mathbb{R}_+)$.

Comme dans le Chapitre 2 nous supposons que l'état $z(x, t)$ est disponible pour les mesures (B.6) sur certains points $x_i^m \in I$, $0 < x_1^m < x_2^m < \dots < x_p^m < \ell$. Supposons également la différentiabilité de la perturbation de sortie $v \in C^2(\mathcal{T}, \mathbb{R}^p)$ et délimitation des incertitudes. Par Hypothèse 3.2 cinq intervalles, $[\underline{\alpha}(t), \bar{\alpha}(t)]$, $[\underline{\beta}(t), \bar{\beta}(t)]$, $[\underline{z}_0, \bar{z}_0]$, $[\underline{r}(x, t), \bar{r}(x, t)]$ et $[-v_0, v_0]$, déterminé pour tous $(x, t) \in I \times \mathcal{T}$ dans (B.9), (B.6) l'incertitude des valeurs de $\alpha(t)$, $\beta(t)$, z_0 , $r(x, t)$ et $v(t)$, respectivement.

L'observateur par intervalle le plus simple pour (B.9) selon les assumptions introduites est la suivante pour $j = 0, 1, \dots, p$:

$$\begin{aligned} \frac{\partial \bar{z}(x, t)}{\partial t} &= L[x, \bar{z}(x, t)] + \bar{r}(x, t) + b_j(x)u_j(t) \quad \forall (x, t) \in I_j \times \mathcal{T}, \\ \bar{z}(x, t_0) &= \bar{z}_0(x) \quad \forall x \in I_j, \\ \bar{z}(x_j^m, t) &= \bar{Z}_j(t), \quad \bar{z}(x_{j+1}^m, t) = \bar{Z}_{j+1}(t) \quad \forall t \in \mathcal{T}; \end{aligned} \quad (\text{B.12})$$

$$\begin{aligned} \frac{\partial \underline{z}(x, t)}{\partial t} &= L[x, \underline{z}(x, t)] + \underline{r}(x, t) + b_j(x)u_j(t) \quad \forall (x, t) \in I_j \times \mathcal{T}, \\ \underline{z}(x, t_0) &= \underline{z}_0(x) \quad \forall x \in I_j, \\ \underline{z}(x_j^m, t) &= \underline{Z}_j(t), \quad \underline{z}(x_{j+1}^m, t) = \underline{Z}_{j+1}(t) \quad \forall t \in \mathcal{T}, \end{aligned}$$

où $\bar{z} \in C(\mathcal{T}, L^2(I, \mathbb{R}))$ et $\underline{z} \in C(\mathcal{T}, L^2(I, \mathbb{R}))$ sont des estimations supérieures et inférieures de la solution $z(x, t)$; $I_j = [x_j^m, x_{j+1}^m]$ avec $x_0^m = 0$ et $x_{p+1}^m = \ell$; les estimations supérieures et inférieures des conditions aux limites sont

$$\begin{aligned} \bar{Z}(t) &= [\bar{Z}_0(t), \dots, \bar{Z}_p(t)]^T = [\bar{\alpha}(t), y_1(t) + v_0, \dots, y_p(t) + v_0, \bar{\beta}(t)]^T, \\ \underline{Z}(t) &= [\underline{Z}_0(t), \dots, \underline{Z}_p(t)]^T = [\underline{\alpha}(t), y_1(t) - v_0, \dots, y_p(t) - v_0, \underline{\beta}(t)]^T. \end{aligned}$$

Le domaine I de la solution de (3.1) est divisé sur $p + 1$ sous-domaines avec des conditions aux limites appropriées. Elle est liée à la manière dont l'injection de sortie est appliquée. Dans (B.12) l'utilisation de l'injection de sortie directement dans le côté droit de l'observateur est évitée car l'analyse de la positivité de la dynamique d'erreur d'estimation, qui est obligatoire pour un observateur par intervalle, est simple si l'injection de sortie est présente aux limites, mais elle est plus évoluée dans d'autres cas.

La condition principale pour l'observateur par intervalle (B.12) est donné dans le Théorème 3.1:

$$\Delta x^m < \pi \sqrt{\frac{a_{\min}}{q_{\max}}}, \quad (\text{B.13})$$

où $\Delta x^m = \max_{j \in \{0, 1, \dots, p\}} (x_{j+1}^m - x_j^m)$, alors pour tous $t \in \mathcal{T}$ les erreurs d'estimation $\bar{z}(\cdot, t) - z(\cdot, t)$,

$z(\cdot, t) - \underline{z}(\cdot, t)$ sont bornés non négatifs et dans (B.9), (B.12):

$$\underline{z}(x, t) \leq z(x, t) \leq \bar{z}(x, t) \quad \forall (x, t) \in I \times \mathcal{T}.$$

La principale restriction à la stabilité du système(B.9) est (B.11). L'inégalité (B.13) impose la même propriété à l'observateur par intervalle (B.12): si la différence Δx^m est suffisamment petit, ce qui signifie que la quantité de points de mesure est suffisamment élevée, alors l'erreur d'estimation de l'observateur est limitée, mais elle n'implique pas la stabilité du système d'origine. Pour surmonter cette restriction, considérons ensemble le système (B.9)et l'observateur par intervalle (B.12), conçu dans la Section 3.2, tous deux dotés d'une entrée de commande $u_j(t) \in H^1(\mathcal{T}, \mathbb{R})$ à travers les fonctions de forme $b_j(x) \in L^2(I, \mathbb{R})$ sur chaque sous-domaine de l'espace I_j , où le contrôle est choisi comme un retour d'état d'observateur par intervalle:

$$u_j(t) = -\frac{K_j}{\Delta x_j^m} \int_{x_j^m}^{x_{j+1}^m} (\underline{z}(\xi, t) + \bar{z}(\xi, t)) d\xi, \quad j = 0, \dots, p, \quad (\text{B.14})$$

où K_j sont les gains de rétroaction séquentielle à concevoir sur chaque I_j , $K_j > 0$ et $\Delta x_j^m = (x_{j+1}^m - x_j^m) \forall j \in \{0, 1, \dots, p\}$. Par souci de concision, nous considérons le même nombre de capteurs et d'actionneurs avec des sous-intervalles colocalisés I_j . Ainsi, le contrôle est appliqué afin d'assurer le caractère borné des estimations des observateurs $\underline{z}(x, t)$, $\bar{z}(x, t)$, qu'à son tour (depuis $\underline{z}(x, t) \leq z(x, t) \leq \bar{z}(x, t)$ pour tous $(x, t) \in I \times \mathcal{T}$, voir le Théorème 3.1) fournira la délimitation de $z(x, t)$ comme dans Efimov et al., 2013f. Le gain de contrôle K est trouvé en résolvant l'IML du théorème 3.2:

$$\Phi \triangleq \begin{bmatrix} 2(\kappa + q_{\max} - 2K) & 2K & 1 \\ 2K & \frac{-2a_{\min}\pi^2}{(\Delta x^m)^2} & 0 \\ 1 & 0 & -\gamma^2 \end{bmatrix} \leq 0$$

pour $\kappa > 0$, $K > 0$, $\gamma > 0$ et (B.13).

Il convient de noter que qualitativement l'estimation de la limite ci-dessus L^2 pour z peut également être obtenu en utilisant un retour de sortie statique, mais il peut être plutôt conservateur, et en utilisant les limites d'observation supérieures et inférieures calculées en ligne \underline{z} et \bar{z} on peut en déduire une estimation d'intervalle plus serrée sur l'état. Cela peut être un avantage important pour les applications dédiées aux problèmes de contraintes d'état (*par exemple* dans les réacteurs). Tests du contrôleur de stabilisation (B.14) et comparaison avec un retour de sortie $u_j(t) = -K^* y_j(t)$ sont donnés dans la Section 3.4, ainsi que la comparaison entre l'approche basée sur l'approximation (B.8) et observateur par intervalle distribués (B.12).

Le Chapitre 4 étudie un problème de référence simple d'un système retardé instable du second ordre avec des mesures retardées (un modèle retardé pour le mouvement d'un seul point de masse)

$$\ddot{x}(t) = -ax(t - \tau) + f(t), \quad t \geq 0, \quad (\text{B.15})$$

$$y(t) = x(t - \theta) + v(t), \quad \tau \leq \theta, \quad (\text{B.16})$$

où $x(t) \in \mathbb{R}$ est la position, $y(t) \in \mathbb{R}$ est le signal de sortie mesuré, $f : \mathbb{R}_+ \rightarrow \mathbb{R}$ and $v : \mathbb{R}_+ \rightarrow \mathbb{R}$ sont des perturbations d'état et du bruit de mesure (signaux bornés inconnus), $\tau > 0$ et $\theta > 0$ sont les retards.

Les conditions initiales de (B.15) sont considérés comme une fonction scalaire uniquement pour la position:

$$x(h) = \phi(h) \text{ for } -\tau \leq h \leq 0, \phi \in \mathcal{C}_\tau; \dot{x}(0) \in \mathbb{R}, \quad (\text{B.17})$$

Il convient de souligner que depuis $\theta \geq \tau$, les conditions initiales pour (B.15), $x(h) = \phi(h)$, doit être défini pour $h \in [-\theta, 0]$.

Par Hypothèse 4.1 introduire quelques fonctions connues $\underline{\phi}, \bar{\phi} \in \mathcal{C}_\theta$ tel que $\underline{\phi}(h) \leq \phi(h) \leq \bar{\phi}(h)$ pour tous $h \in [-\theta, 0]$, et $\underline{\dot{x}}_0 \leq \dot{x}(0) \leq \bar{\dot{x}}_0$ pour certains connus $\underline{\dot{x}}_0, \bar{\dot{x}}_0 \in \mathbb{R}$. L'Assomption d'un ensemble connu $[\underline{\phi}, \bar{\phi}]$ pour les conditions initiales ϕ est standard pour la théorie d'estimation de par intervalle ou de l'appartenance à un ensemble Efimov et al., 2013c; Gouzé et al., 2000; Jaulin, 2002; Kieffer and Walter, 2004; Moisan et al., 2009. Nous supposons également que le paramètre a est connue et les valeurs instantanées des signaux $f(t)$ et $v(t)$ ne sont pas disponibles. Par conséquent, les entrées incertaines $f(t)$ et $v(t)$ dans (B.15) et (B.16) appartiennent aux intervalles connus $[\underline{f}(t), \bar{f}(t)]$ et $[-v_0, v_0]$ respectivement pour tous $t \geq 0$.

Nous proposons le premier observateur pour de la position $x(t)$ du système (B.15) sous la forme:

$$\begin{aligned} \ddot{x}^-(t) &= -a(t)x^-(t-\tau) + \ell(t)x^-(t-\theta) + \underline{f}(t) - \ell(t)y(t) - \ell(t)v_0 - \rho(t), \\ \ddot{x}^+(t) &= -a(t)x^+(t-\tau) + \ell(t)x^+(t-\theta) + \bar{f}(t) - \ell(t)y(t) + \ell(t)v_0 + \rho(t), \end{aligned} \quad (\text{B.18})$$

où $x^-(t), x^+(t) \in \mathbb{R}$ sont les estimations de la position de mouvement (B.15) pour $t \in [0, +\infty)$ avec conditions initiales

$$\begin{aligned} x^-(h) &= \underline{\phi}(h), \quad x^+(h) = \bar{\phi}(h) \quad \forall h \in [-\theta, 0], \\ \dot{x}^-(0) &= \underline{\dot{x}}_0, \quad \dot{x}^+(0) = \bar{\dot{x}}_0 \end{aligned}$$

de l'Assomption 4.1 et

$$\rho(t) = \begin{cases} a(t)[\bar{\phi}(t-\tau) - \underline{\phi}(t-\tau)] & t \leq \tau \\ 0 & t > \tau \end{cases}.$$

Ses performances dépendent des conditions du Théorème 4.1 Domoshnitsky, 2014 et Proposition 4.1 sur la positivité de la solution $x(t)$ pour l'observateur par intervalle (B.18), où $\ell(t)$ peut être trouvé comme

$$\varepsilon \leq \{a(t) - \ell(t)\} \leq \frac{1}{4} \ell_*^2 (\theta - \tau)^2, \quad \forall t \in [0, +\infty)$$

$$\frac{1}{\sqrt{a^*} \exp\left\{\frac{\ell_*(\theta-\tau)^2}{4}\right\}} \arctan \frac{\ell_*(\theta-\tau)}{2\sqrt{a^*} \exp\left\{\frac{\ell_*(\theta-\tau)^2}{4}\right\}} > \theta - \tau,$$

où les notations $\ell_* = \operatorname{ess\,inf}_{t \geq 0} \ell(t)$, $a^* = \operatorname{ess\,sup}_{t \geq 0} a(t)$. En utilisant cet observateur, il est possible de dériver les estimations d'intervalle pour la position $x(t)$ sans délai:

$$x^-(t) \leq x(t) \leq x^+(t) \quad \forall t \in [0, +\infty), \quad (\text{B.19})$$

mais pour la vitesse, nous avons besoin d'un deuxième observateur par intervalle.

Considérons le terme retardé $-a(t)x(t-\tau)$ comme une perturbation et réécrire (B.15) pour ce cas sous forme d'espace d'état:

$$\dot{x}(t) = Ax(t) + \rho(t, x(t), x(t-\tau), f(t)), \quad (\text{B.20})$$

où $x(t) = [x_1(t) \ x_2(t)]^T$ est le vecteur d'état, $x_1(t)$ est la position estimée par (B.19), $x_2(t)$ est une vitesse de mouvement (B.15);

$$A = A_0 - KC = \begin{bmatrix} -k_1 & 1 \\ -k_2 & 0 \end{bmatrix},$$

$$\rho(t, x(t), x(t-\tau), f(t)) = \begin{bmatrix} k_1 x_1(t) \\ k_2 x_1(t) + f(t) - a(t)x_1(t-\tau) \end{bmatrix},$$

$K = [k_1 \ k_2]^T$ est un nouveau gain d'observateur, $C = \begin{bmatrix} 1 & 0 \end{bmatrix}$.

Selon l'Assomption 4.3 il y a une matrice non singulière $S \in \mathbb{R}^{2 \times 2}$ tel que $D = S(A_0 - KC)S^{-1}$ Raïssi et al., 2012, et les nouvelles coordonnées $z = Sx$ peuvent être introduites transformant le système (B.20) à la forme:

$$\dot{z}(t) = Dz(t) + S\rho(t, x(t), x(t-\tau), f(t)). \quad (\text{B.21})$$

Utiliser le Lemme 1.1 nous obtenons cela

$$\underline{\rho}(t) \leq \rho(t, x(t), x(t-\tau), f(t)) \leq \bar{\rho}(t),$$

où les fonctions $\underline{\rho}(t)$ et $\bar{\rho}(t)$ ne dépendent que des informations disponibles (les variables $x_1^-(t)$, $x_1^+(t)$, $x_1^-(t-\tau)$ et $x_1^+(t-\tau)$ sont donnés par le premier observateur (B.18) pour tous $t \in [0, +\infty)$). Deux estimations $\underline{z}, \bar{z} \in \mathbb{R}^2$ peut être calculé sur la base des informations disponibles sur ces intervalles (l'inclusion de l'intervalle (B.19) pour $x_1(t)$ sans délai), et

un observateur par intervalle peut être conçu pour la dynamique transformée (B.21):

$$\begin{aligned}
\dot{\underline{z}}(t) &= D\underline{z}(t) + S^+ \underline{\rho}(t) - S^- \overline{\rho}(t), \\
\dot{\overline{z}}(t) &= D\overline{z}(t) + S^+ \overline{\rho}(t) - S^- \underline{\rho}(t); \\
\underline{z}(h) &= S^+ \underline{\Phi}(h) - S^- \overline{\Phi}(h), \\
\overline{z}(h) &= S^+ \overline{\Phi}(h) - S^- \underline{\Phi}(h), \quad \forall h \in [-\tau, 0]; \\
\underline{x}(t) &= (S^{-1})^+ \underline{z}(t) - (S^{-1})^- \overline{z}(t), \\
\overline{x}(t) &= (S^{-1})^+ \overline{z}(t) - (S^{-1})^- \underline{z}(t),
\end{aligned} \tag{B.22}$$

où les conditions initiales $\underline{\Phi}(h)$, $\overline{\Phi}(h)$ sont des conditions jointes de l'Assomption 4.1. Les relations (1.4) de Lemma 1.1 sont utilisés pour calculer les conditions initiales pour $\underline{z}(h)$, $\overline{z}(h)$ à $h \in [-\theta, 0]$ et les estimations \underline{x} , \overline{x} . Ensuite, il est prouvé dans la Proposition 4.3:

$$\underline{x}_2(t) \leq x_2(t) \leq \overline{x}_2(t), \quad x(t) \in \mathbb{R}^2, \quad \forall t \in [0, +\infty),$$

que la vitesse est limitée par des estimations d'intervalle de l'observateur (B.22). La simulation de l'algorithme d'estimation proposé est présentée sur l'exemple de motivation (B.15) dans la Section 4.6.

En conclusion, le présent travail a été consacré à l'estimation des systèmes de dimension infinie. Les chapitres 2 et 3 présentent l'estimation et le contrôle de l'intervalle pour les EDP incertaines, en utilisant différentes approches. La deuxième approche montre de meilleures performances et fournit des limites plus strictes pour l'état d'un EDP, en évitant les erreurs d'approximation de Galerkin. Le chapitre 4 présente de nouvelles conditions de positivité pour les équations différentielles de retard avec des mesures retardées. Ces résultats peuvent être étendus pour de futures recherches dans plusieurs directions. Par exemple, étendre la méthode d'estimation par intervalle proposée pour les PDE avec conditions aux limites Neumann, mixtes; étudier différents types de EDP: hyperbolique, non linéaire; explorer d'autres moyens d'approximation des EDP. La mise en œuvre dans un exemple du monde réel (comme une pile à combustible à oxyde solide) peut être plus informative. Concernant les systèmes retardés, on peut étendre l'algorithme proposé à un système LTV plus général; envisager une chaîne d'observateurs par intervalle pour faire face aux retards de valeur élevés. L'estimation par intervalle peut être appliquée pour une installation avec un réseau de capteurs distribués interconnectés.

Abstract

This work presents new results on interval state estimation for uncertain distributed systems, the state of which has an infinite dimension and is described by partial (PDEs) or (FDEs) functional differential equations. An interval observer evaluates at each time instant a set of admissible values for the state (an interval), consistently with the measured output. Chapters 2 and 3 focus on an interval observer design for a parabolic PDE with Dirichlet boundary conditions. The method in Chapter 2 is based on a finite-element Galerkin approximation, the interval inclusion of the state is calculated using the error estimates of the approximation. Chapter 3 presents an interval observer in the form of PDEs without Galerkin projection. In both chapters, the obtained interval estimates are applied to the design of a dynamic output feedback stabilizing controller. Chapter 4 deals with a second-order delay differential equation with uncertainties, which has form of an FDE. The proposed method contains two consecutive interval observers. The first one estimates, at each instant of time, the interval for the delay-free position using new delay-dependent conditions on positivity. Then, derived estimates of the position are used to design the second observer providing an interval for the velocity. All the obtained results are supported by numerical simulations. In particular, Chapter 2 includes experiments on the Black–Scholes model.

Keywords: interval observer, uncertainty, infinite dimension, time-delay system, dynamic system, PDE

CONCEPTION D’OBSERVATEURS PAR INTERVALLE POUR LES SYSTÈMES À PARAMÈTRES DISTRIBUÉS AVEC INCERTITUDES

Résumé

Ce travail présente de nouveaux résultats sur l’estimation d’état par intervalle pour des systèmes distribués incertains, qui sont des systèmes de dimension infinie : leur état, fonctionnel, est régi par des équations aux dérivées partielles (EDP) ou fonctionnelles (EDF). Le principe de l’observation par intervalle est d’estimer à chaque instant un ensemble de valeurs admissibles pour l’état (un intervalle), de manière cohérente avec la sortie mesurée. Les chapitres 2 et 3 se concentrent sur la conception d’observateurs par intervalle pour une EDP parabolique avec des conditions aux limites de type Dirichlet. Dans le chapitre 2, on utilise une approximation en dimension finie (éléments finis de type Galerkin), l’intervalle d’inclusion tenant compte des erreurs de l’approximation. Le chapitre 3 présente un observateur par intervalle sous la forme d’EDP sans projection de Galerkin. Dans ces deux chapitres, les estimations par intervalle obtenues sont utilisées pour concevoir un contrôleur stabilisant par retour de sortie dynamique. Le chapitre 4 envisage le cas des systèmes différentiels fonctionnels (EDF) à retards, à travers une équation différentielle de deuxième ordre avec incertitudes. La méthode proposée contient deux observateurs par intervalle consécutifs : le premier calcule à chaque instant l’intervalle pour la position non retardée grâce à de nouvelles conditions de positivité dépendantes du retard. Le deuxième observateur calcule un intervalle pour la vitesse, grâce à une estimation de dérivée. Tous les résultats obtenus sont vérifiés par des simulations numériques. En particulier, le chapitre 2 inclut des expériences sur le modèle Black – Scholes.

Mots clés : observateur par intervalle, incertitude, dimension infinie, système à retard, système dynamique, EDP

