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Alexis Léculier

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THÈSE

En vue de l'obtention du

DOCTORAT DE L'UNIVERSITÉ DE TOULOUSE

Délivré par : *l'Université Toulouse 3 Paul Sabatier (UT3 Paul Sabatier)*

Présentée et soutenue le *Le 7 Juillet 2020* par :

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**Modèles de diffusion non-conventionnelle en écologie et
biologie évolutive impliquant des environnements
fragmentés**

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Chapter 1

Introduction

A. Objet de la thèse

Ce manuscrit de thèse s'inscrit dans le domaine de l'étude mathématique de la dynamique des populations en écologie et biologie évolutive. Plus précisément, nous nous intéresserons à la dynamique d'une seule population modélisée par sa densité n . Nous porterons une attention particulière au rôle que peut jouer une diffusion non-locale dans un habitat fragmenté.

Dans les deux prochains paragraphes, nous expliquerons brièvement quelles sont les principales équations et questions abordées dans cette thèse. Dans un second temps, pour chacune de ces deux équations, nous donnerons une description précise des modèles en jeu. Ensuite, nous rappellerons le contexte scientifique dans lequel ce manuscrit s'inscrit avant de présenter les principaux résultats obtenus.

Les premiers travaux de cette thèse décrivent la dynamique spatiale d'une population soumise à l'équation de réaction diffusion fractionnaire suivante

$$\begin{cases} \partial_t n + (-\Delta)^\alpha n = f(x, n) & \text{pour } x \in \Omega, t > 0, \\ n = 0 & \text{pour } x \in \Omega^c, t > 0. \end{cases} \quad (1.1)$$

Pour fixer les idées, nous prendrons dans toute cette introduction $f(x, n) = n - n^2$ ou $f(x, n) = \mu(x)n - n^2$: une non-linéarité de type Fisher-KPP. L'opérateur de diffusion $(-\Delta)^\alpha$ est défini comme suit :

$$(-\Delta)^\alpha n(x) = C_{d,\alpha} \int_{\mathbb{R}^d} \frac{n(x) - n(y)}{|x - y|^{d+2\alpha}} dy \quad \text{avec} \quad C_{d,\alpha} = \frac{4^\alpha \Gamma(\frac{d}{2} + \alpha)}{\pi^{\frac{d}{2}} |\Gamma(-\alpha)|}. \quad (1.2)$$

Ce type d'opérateur décrit un processus de diffusion longue portée avec une grande fréquence. Nous nous intéresserons particulièrement au critère de survie de l'espèce ainsi qu'aux phénomènes de colonisation voir d'invasion du territoire Ω .

Dans cette thèse, nous étudions également un modèle venant de la biologie évolutive, où nous considérons une population structurée phénotypiquement à l'équilibre et sujette à deux types de déplacement spatiaux : des déplacements locaux et non locaux. De plus, nous prenons en compte deux autres phénomènes. Le premier est la prise en compte de mutations qui modifient légèrement le trait phénotypique θ de la progéniture, ce phénomène crée de la variabilité dans la population. Le second phénomène est la compétition entre les individus, qui mène à

une sélection naturelle.

Nous cherchons à observer l'émergence de traits phénotypiques dominants lorsque les mutations deviennent petites. L'équation choisie pour modéliser les phénomènes biologiques impliqués est :

$$\begin{cases} -\partial_{xx}n_\varepsilon - \varepsilon^2\partial_{\theta\theta}n_\varepsilon + Ln_\varepsilon = n_\varepsilon[R(x, \theta) - \rho_\varepsilon] & \text{pour } x \in \Omega, -A < \theta < A, \\ \rho_\varepsilon(x) = \int_{-A}^A n_\varepsilon(x, \theta)d\theta & \text{pour } x \in \Omega, \\ \partial_{\nu_x}n_\varepsilon = \partial_{\nu_\theta}n_\varepsilon = 0 & \text{pour } (x, \theta) \in \partial(\Omega^c \times]-A, A[). \end{cases} \quad (1.3)$$

La diffusion spatiale est ici modélisée par le Laplacien par rapport à la variable x pour la diffusion locale et un opérateur intégral L qui sera décrit par la suite. Les mutations sont modélisées par le Laplacien par rapport à la variable θ . Enfin, les termes R et ρ désignent respectivement le taux de croissance intrinsèque des individus, et la taille totale de la population. Nous prenons en compte en effet via ce dernier terme un taux de mortalité dû à la compétition entre les individus.

B. L'équation de Fisher-KPP fractionnaire dans un environnement fragmenté

Rappelons l'équation considérée dans toute cette partie :

$$\begin{cases} \partial_t n + (-\Delta)^\alpha n = f(x, n) & \text{pour } x \in \Omega, t > 0, \\ n = 0 & \text{pour } x \in \Omega^c, t > 0. \end{cases} \quad (1.1)$$

B.1 Description du modèle et motivations biologiques

B.1.a Les motivations biologiques

Nous considérons ici une espèce biologique se développant dans un environnement fragmenté. Nous supposons que cette espèce biologique est sujette à une dispersion longue portée.

La dispersion longue portée désigne un mode de dispersion où les individus ne se déplacent pas seulement de proche en proche mais aussi sur de grandes distances dans un laps de temps relativement court. De plus, la fréquence de ces sauts est supposée être grande. Ce type de phénomène a été observé par des biologistes comme par exemple dans l'article [70] qui introduit de multiples exemples d'espèces biologiques ayant une telle dispersion tel que le scarabée des céréales. En effet, la dispersion de ce scarabée est accélérée par l'action de l'homme. Un autre exemple est celui d'arbres présenté dans l'article [74] dont le pollen peut être dispersé sur de grandes distances sous l'effet de grands vents.

Les environnements fragmentés sont considérés pour illustrer une réalité tangible : les espèces biologiques ne peuvent pas s'implémenter partout. De ce fait, leurs habitats peuvent être séparés par un (ou des) obstacle(s) et s'en retrouver fragmentés. Par exemple, une espèce

d'arbres ne pourra s'implémenter dans un environnement urbain. Un autre exemple est celui d'un archipel pour une espèce terrestre. Remarquons que sous l'effet de la dispersion longue portée, l'ensemble de l'environnement peut être colonisé par l'espèce. Ceci est faux si l'on suppose que les individus se déplacent seulement de proche en proche.

Pour de tels phénomènes de diffusion longue portée, l'article [70] met en évidence des vitesses d'invasion exponentielles en temps. Au contraire, si la diffusion est une dispersion standard (i.e. déplacement des individus uniquement de proche en proche) alors les vitesses d'invasion sont constantes (c'est à dire que le front d'invasion avance de façon linéaire). Enfin, soulignons que pour parler d'*invasion* il faut deux phénomènes : une colonisation de l'ensemble du territoire et une autosuffisance (ou autonomie) des individus sur les territoires colonisés.

Ainsi, les questions biologiques précises motivant cette étude et prenant en compte l'ensemble de ces phénomènes sont :

1. Peut on mettre en évidence un critère assurant la survie/l'extinction de l'espèce biologique ?
2. Lorsqu'il y a survie de l'espèce, peut on prédire la distribution asymptotique de la population ?
3. Si une petite quantité d'individus arrive dans un environnement vierge, est ce que l'espèce va coloniser cet environnement ?
4. Si il y a colonisation, dans quel cas peut on parler d'invasion ? Quelles sont les caractéristiques de cette invasion ?

Enfin, soulignons que nous tenterons d'apporter des réponses à ces questions à travers un modèle simplifié. Ce modèle ne prend pas en compte quantités de facteurs biologiques. Cependant, cette simplification permet de mettre en évidence des réponses qualitatives. Ces réponses qualitatives se retrouvent ensuite quantitativement dans certains exemples issus de la biologie comme des phénomènes d'accélération de la propagation. Evidemment, pour coller à un jeu spécifique de données, il faudrait pousser plus loin le travail de modélisation en prenant en compte dans (1.1) des phénomènes biologiques plus fins. Ce type de travail de modélisation n'est pas fait ici.

Je tiens à souligner que j'ai eu au cours de ma thèse un bref aperçu de ce type de travail de modélisation. En effet, lors de mon stage au CEMRACS, nous avons modélisé la dynamique de moustiques tigres (infectés ou non par la bactérie *Wolbachia*) en prenant en compte certains phénomènes propre à l'insecte comme l'incompressibilité cytoplasmique. Nous renvoyons à la partie D pour une première présentation du modèle.

B.1.b Description du modèle

Nous étudions ici l'équation (1.1). Elle modélise la dynamique au cours du temps d'une densité de population biologique n sujette à une dispersion longue portée dans un domaine fragmenté Ω .

Nous parlerons de domaine fragmenté lorsque le domaine Ω modélisant l'environnement de vie de l'espèce biologique est composé de plusieurs composantes connexes (*patches*). Comme la densité de population est supposée nulle en dehors du domain Ω (i.e. dans Ω^c), nous imposons une condition de Dirichlet extérieur. Ceci est à mettre en opposition avec une dispersion locale. Dans le cas d'une diffusion locale, l'espèce peut coloniser seulement les composantes connexes où elle est présente initialement.

L'hypothèse commune pour modéliser une dispersion longue portée avec une grande fréquence

est de considérer une équation intégral-différentielles dont le noyau de l'opérateur de diffusion K est à *queue lourde*. Le Laplacien fractionnaire, défini comme :

$$(-\Delta)^\alpha n(x) = C_{d,\alpha} \int_{\mathbb{R}^d} \frac{n(x) - n(y)}{|x - y|^{d+2\alpha}} dy \quad \text{avec} \quad C_{d,\alpha} = \frac{4^\alpha \Gamma(\frac{d}{2} + \alpha)}{\pi^{\frac{d}{2}} |\Gamma(-\alpha)|},$$

rentre dans cette famille d'opérateur à queue lourde (l'intégrale est à prendre ici au sens de la valeur principale). Notons que le noyau est singulier à l'origine : la fréquence de dispersion proche de x est très grande. Du point de vue de l'espèce, les individus vont se déplacer de proche en proche et avoir des sauts en espace avec une grande fréquence. D'un point de vu stochastique, la dispersion suit un processus de type Lévy.

Le terme de réaction dans l'équation (1.1) est de type Fisher-KPP. Ce type de terme de réaction sert à modéliser une croissance logistique. Par exemple, si $f(x, n) = \mu(x)n - n^2$ alors $\mu(x)n$ est un terme de naissance et $-n^2$ est un terme de mort modélisant la compétition pour les ressources. Les premières études d'une équation de type Fisher-KPP ($\partial_t n - \Delta n = n - n^2$) sont dues à Fisher [58] et Kolmogorov, Petrovsky et Piskunov [73] en 1937.

B.2 Etat de l'art

Nous décomposons l'état de l'art en deux sous-parties : les articles traitant de l'existence et de l'unicité d'une solution stationnaire bornée à une équation de Fisher-KPP et ceux discutant des phénomènes d'invasion. Il est évident qu'un même article peut rentrer dans les deux catégories, néanmoins, il est pratique de distinguer les deux littératures afin de présenter les outils en jeu.

B.2.a Existence et unicité de la solution stationnaire de Fisher-KPP

Lors de la preuve de la convergence vers un front d'invasion, un des ingrédients essentiels de la preuve est l'existence et l'unicité de la solution stationnaire positive bornée.

L'existence s'appuie souvent sur un argument spectral. En effet, il suffit de regarder le signe de la valeur propre principale $\lambda_0(\Omega, \mu)$ de l'opérateur $-\Delta - \mu$ dans Ω ou dans un sous domaine avec des conditions de Dirichlet aux bords (respectivement extérieur).

Si $\lambda_0(\Omega, \mu) < 0$ alors il existe une unique solution stationnaire positive et bornée (construite par une méthode de sous et sur-solutions [102]).

Sinon pour $\lambda_0(\Omega, \mu) \geq 0$, l'unique solution stationnaire positive bornée est la solution triviale. Cette valeur propre principale existe si l'on a des hypothèses de compacité sur Ω ou de périodicité sur μ . A l'initiative de Berestycki, des travaux [21], [18], [12], [13] définissent une généralisation de cette valeur propre principale lorsque celle-ci n'est pas bien définie comme dans le cas d'un domaine non borné. Cette nouvelle notion de valeur propre principale généralisée n'affecte pas le critère d'existence d'une solution stationnaire bornée non-triviale. Dans le cas d'une diffusion fractionnaire et d'un terme de naissance μ périodique, Berestycki, Roquejoffre et Rossi établissent l'existence d'une valeur propre principale dans [20].

Pour un opérateur de diffusion (locale ou fractionnaire), l'unicité de la solution stationnaire de (1.1) pour $\Omega = \mathbb{R}^d$ découle de l'application du principe du maximum / principe de comparaison (voir [10]). Remarquons que pour un opérateur intégral du type le Laplacien fractionnaire, le principe du maximum est trivial tandis que le lemme de Hopf découle de la

comparaison avec des fonctions barrières bien choisies (voir les travaux de Ros-Oton et Serra [97] ou de Grecco et Servadei [65]).

Enfin, pour des opérateurs intégraux dont le noyau est moins singulier, Berestycki, Coville et Vo [12] ou Coville, Davilla et Martinez [46] ont montré l'existence et l'unicité d'une solution stationnaire d'une équation de type Fisher-KPP non-locale. Le critère d'existence est basé sur le signe d'une valeur propre principale (généralisée ou non) tandis que l'unicité est montrée par la continuité et l'intégrabilité du noyau de l'opérateur de diffusion.

Dans la suite nous parlerons de domaine Ω et de taux de naissance μ *variables* lorsque $\lambda_0(\Omega, \mu) < 0$. Si $\Omega = \mathbb{R}^d$ (respectivement $\mu = 1$), on écrira alors $\lambda_0(\Omega, \mu) = \lambda_0(\mu)$ (respectivement $\lambda_0(\Omega, \mu) = \lambda_0(\Omega)$).

B.2.b Phénomène d'invasion

Comme mentionné précédemment, l'équation (1.1) a été introduite par Fisher [58] et Kolmogorov, Petrovsky et Piskunov [73] en 1937 dans le cas d'une diffusion standard ($\partial_t n - \Delta n = n - n^2$) avec pour domaine $\Omega = \mathbb{R}^d$. Kolmogorov, Petrovsky et Piskunov prouvent l'existence d'ondes planes progressives. Elles sont un type de solution particulière connectant la solution stationnaire stable 1 à la solution stationnaire instable 0. Ces dernières sont de la forme $n(x, t) = N(x - ct)$ avec $c \geq c_* = 2\sqrt{f'(0)}$. En 1978, Aronson et Weinberger [5] prouvent que n'importe quelle condition initiale positive à support compact engendre un phénomène d'invasion à vitesse constante c_* . Cela peut se traduire ainsi : si un observateur se déplace le long d'une direction donnée à une vitesse constante $c < c_*$ alors, au bout d'un certain temps, il verra l'environnement saturé partout autour de lui. En revanche, si l'observateur se déplace à une vitesse $c > c_*$ alors, au bout d'un certain temps, il ne verra personne autour de lui. Aronson et Weinberger prouvent leurs résultats en construisant des sous et sur-solutions adaptées à leurs problèmes. Dans [54], Evans et Souganidis retrouvent ce résultat en utilisant une approche provenant de l'optique géométrique. Ils exhibent naturellement les deux états, à savoir, saturation de l'environnement ($n = 1$) ou absence de population ($n = 0$) en introduisant un changement de variable qui suit le front

$$n_\varepsilon(x, t) = n\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}\right).$$

En effet, n_ε est solution de $\partial_t n_\varepsilon - \varepsilon \Delta n_\varepsilon = n_\varepsilon - n_\varepsilon^2$. Afin d'obtenir des estimations sur n_ε et passer à la limite en ε , Evans et Souganidis utilisent une transformation de Hopf-Cole en introduisant

$$u_\varepsilon = \varepsilon \log(n_\varepsilon).$$

Ce type d'approche sera utilisé dans les chapitres 2 et 3 de la thèse. Il est à noter que le résultat d'invasion précédent est isotropique : aucune direction n'est privilégiée. Ceci est dû à l'isotropie des termes de naissance et de diffusion. Si on considère un terme de naissance périodique viable de la forme $\mu(x)n$ (qui privilégie certaines directions plutôt que d'autres), alors Freidlin et Gartner ont montré dans [63] qu'une condition initiale positive à support compact entraîne un phénomène d'invasion avec une vitesse d'invasion $c_*(e)$ qui dépend de la direction choisie $e \in S^{d-1}$. La preuve de l'invasion de Freidlin et Gartner est basée sur une approche probabiliste. Rossi propose dans [98] une preuve déterministe de ce dernier résultat (voir aussi [14]). La Figure 1.1 est une illustration du phénomène d'invasion dans le cas d'un environnement homogène (à gauche) ou périodique (à droite).

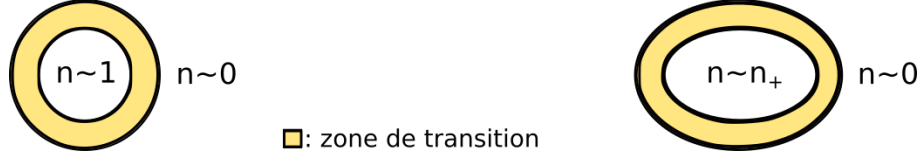


FIGURE 1.1 – Illustration d’un phénomène d’invasion pour $f(x, n) = n - n^2$ (à gauche) et $f(x, n) = \mu(x)n - n^2$ (à droite).

Avant d’évoquer les résultats connus sur les équations de type Fisher-KPP fractionnaires, nous mentionnons deux derniers articles impliquant une diffusion locale. Le premier de Hamel et Guo [66] traite d’une équation de type Fisher-KPP dans un environnement périodiquement hostile. Les auteurs démontrent qu’une invasion est toujours possible dans les composantes connexes du domaine non-hostiles et viables qui interceptent le support de la donnée initiale. Le deuxième article de Berestycki et Nadin [18] donnent des résultats d’invasion très généraux. Ils établissent l’existence de deux ensembles étoilés $\underline{\mathcal{S}} \subset \overline{\mathcal{S}} \subset \mathbb{R}^d$ tel que pour tout compact $K \subset \underline{\mathcal{S}}$ et fermé $F \subset \mathbb{R}^d \setminus \overline{\mathcal{S}}$ on a

$$\lim_{t \rightarrow +\infty} \sup_{x \in tK} |n(x, t) - n_+(x)| = 0 \quad \text{et} \quad \lim_{t \rightarrow +\infty} \sup_{x \in tF} |n(x, t)| = 0.$$

Ces deux ensembles existent lorsque qu’une version généralisée de la valeur propre principale prend un signe négatif. Dans le cas simple $\partial_t n - \Delta n = n - n^2$, $\Omega = \mathbb{R}^d$, il est clair que nous retrouvons $\underline{\mathcal{S}} = \overline{\mathcal{S}} = \{|x| = c_*\}$.

Les phénomènes d’accélération du front de propagation dans des équations Fisher-KPP fractionnaires ont d’abord été observés numériquement (voir [47], [82] et [70]). En effet, ces investigations numériques laissent penser que la vitesse d’invasion est exponentielle en temps. Ce résultat a été démontré par Cabré et Roquejoffre [36] pour l’équation $\partial_t n + (-\Delta)^\alpha n = n - n^2$ dans $\mathbb{R}^d \times \mathbb{R}_+$. Pour une telle équation, les auteurs démontrent en construisant des sous et sur-solutions adaptées que

$$\forall c > 1, \quad \lim_{t \rightarrow +\infty} \sup_{|x| < e^{\frac{c-1}{d+2\alpha}t}} |n(x, t) - 1| = 0 \quad \text{et} \quad \lim_{t \rightarrow +\infty} \sup_{|x| > e^{\frac{ct}{d+2\alpha}}} |n(x, t)| = 0.$$

Ce résultat est retrouvé par une approche d’optique géométrique par Mirrahimi et Méléard [85]. L’idée est d’introduire un changement de variable qui suit le front $n_\varepsilon(x, t) = n(|x|^{\frac{1}{\varepsilon}-1}x, \frac{t}{\varepsilon})$ et ensuite de montrer la convergence de la suite à l’aide d’une transformation de Hopf-Cole. Ce type de phénomène d’accélération a aussi été retrouvé dans le cadre d’opérateurs à queues lourdes réguliers par Garnier [62]. Ce premier résultat d’invasion est établi dans un milieu homogène. Il a été étendu aux milieux hétérogènes périodiques (i.e. aux termes de réaction de la forme $f(x, n) = \mu(x)n - n^2$) par Cabré, Coulon et Roquejoffre [35]. Sous l’hypothèse que μ est un taux de naissance viable (i.e. $\lambda_0(\mu) < 0$), il existe une unique solution stationnaire bornée n_+ et il s’ensuit

$$\forall c > 1, \quad \lim_{t \rightarrow +\infty} \sup_{|x| < e^{\frac{c-1|\lambda_0(\mu)|t}{d+2\alpha}}} |n(x, t) - n_+(x)| = 0 \quad \text{et} \quad \lim_{t \rightarrow +\infty} \sup_{|x| > e^{\frac{c|\lambda_0(\mu)|t}{d+2\alpha}}} |n(x, t)| = 0.$$

Remarquons que contrairement au cas d’une diffusion locale, la diffusion non-locale induit toujours une propagation isotropique même dans un environnement hétérogène. Soulignons enfin que [35] donne une description précise des lignes de niveaux en temps longs.

Nous clôturons l'état de l'art sur les phénomènes d'invasion avec plusieurs remarques. Premièrement, pour une équation avec une diffusion locale ($\partial_t n - \Delta n = n - n^2$), en partant d'une donnée initiale ayant des queues lourdes (i.e. $n_0(x) \sim \frac{1}{|x|^{d+2\alpha}}$), il a été démontré par Hamel et Roques [69] que la vitesse d'invasion est exponentielle en temps. De façon heuristique, lorsque la donnée initiale est à support compact, nous verrons que le principal rôle de la diffusion fractionnaire est « au début ». En effet, contrairement à la diffusion locale qui donne une solution en temps court avec des queues exponentielles (i.e. $n(x, t = 1) \sim e^{-c|x|^2}$), le Laplacien fractionnaire entraîne une solution en temps court à queue lourde (i.e. $n(x, t = 1) \sim \frac{1}{|x|^{d+2\alpha}}$). Ces queues lourdes amènent ensuite à une propagation exponentielle en temps. Deuxièmement, d'autres questions plus complexes peuvent être considérées à partir de l'équation de Fisher-KPP comme

- l'équation de Fisher-KPP avec une ligne de diffusion rapide ([19], [11], [94]),
- la caractérisation du second membre d'un développement asymptotique du front d'invasion (voir Barmson [31] ou Hamel, Nolen, Roquejoffre, Ryznik [67], [68] et [93]),
- la stabilité du front d'invasion (voir Sattinger [99], Gallay [61] ou Faye et Holzer [55]).

Enfin nous avons dans cet état de l'art seulement mentionné les équations de réaction-diffusion (fractionnaire ou non) de type Fisher-KPP. Il existe bien entendu des équations de réaction diffusion moins simplistes comme les équations de type combustions, bistables... Nous renvoyons les lecteurs intéressés aux livres de Smoller [102] et de Berestycki et Hamel (à paraître). Les phénomènes d'invasion pour ce type d'équation existent mais sont plus complexes à étudier.

B.3 Présentations des résultats pour l'équation de Fisher-KPP fractionnaire

Les principaux résultats concernant l'équation de Fisher-KPP fractionnaire sont introduits ici. Ces résultats sont regroupés par chapitre de thèse.

B.3.a Chapitre 2- Equation de Fisher-KPP fractionnaire dans un environnement périodique

Le premier résultat de ce manuscrit introduit dans le chapitre 2 est un résultat d'invasion

Théorème 1.1 (L. (Chapitre 2)). *Soit n la solution de (1.1) avec $\Omega = \mathbb{R}^d$, $f(x, n)$ de la forme $\mu(x)n - n^2$ où μ est périodique. Si λ_0 la valeur propre principale de $(-\Delta)^\alpha - \mu$ est négative, alors, on a*

1. $\forall c < \frac{|\lambda_0|}{d+2\alpha}, \lim_{t \rightarrow +\infty} \sup_{|x| < e^{ct}} |n(x, t) - n_+(x)| = 0,$
2. $\forall C > \frac{|\lambda_0|}{d+2\alpha}, \lim_{t \rightarrow +\infty} \sup_{|x| > e^{Ct}} |n(x, t)| = 0.$

Ce résultat (tel qu'énoncé dans le manuscrit) est étendu à des opérateurs intégraux singuliers plus généraux que le Laplacien fractionnaire. De même, le terme de réaction est de type KPP périodique plus général que $\mu(x)n - n^2$.

Nous pouvons décomposer la preuve de ce résultat en 4 grandes étapes :

- A- S'assurer de l'existence et de l'unicité d'une solution stationnaire de (1.1).

- B- Donner une estimation de la solution au temps $t = 1$.
- C- Trouver une sous-solution qui va « pousser » la solution jusqu'à n_+ dans $\{|x| < e^{ct}\}$ ainsi qu'une sur-solution qui va « écraser » la solution proche de 0 dans $\{|x| > e^{ct}\}$.
- D- Conclure.

L'intérêt du Théorème 1.1 réside principalement dans la preuve des étapes C et D. En effet, le Théorème 1.1 est une extension du résultat de Cabré, Coulon et Roquejoffre [35] à des opérateurs plus généraux via l'approche développée par Mirrahimi et Méléard [85] pour le Laplacien fractionnaire. La preuve du Théorème 1.1 montre la robustesse des techniques de solutions de viscosités employées pour l'étude d'équations générales de type Fisher-KPP fractionnaires.

L'étape A (existence et unicité d'une solution stationnaire) s'appuie sur le signe de la valeur propre principale et le principe du maximum. Tandis que l'étape B (estimation de la solution en temps courts) repose sur une estimation du noyau de la chaleur fractionnaire (comme énoncé dans Bogdan [25] ou Chen et Kumagai [43]).

L'étape C peut être abordée via une approche issue de l'optique géométrique (comme Evans et Souganidis [51]). L'idée générale est d'effectuer un changement de variable qui suit le front d'invasion. Ainsi, nous considérerons l'équation satisfaite par $n_\varepsilon(x, t) = n(|x|^{\frac{1}{\varepsilon}-1}x, \frac{t}{\varepsilon})$ (à l'image du changement de variable effectué par Méléard et Mirrahimi dans [85]). Après ce changement de variable, une sous (respectivement sur)-solution de ce problème peut être trouvée aisément en considérant $\phi_0 \times G$ où G est une solution d'une équation différentielle ordinaire perturbée et ϕ_0 une fonction propre principale associée à l'opérateur $(-\Delta)^\alpha - \mu$. En effet, lorsque $\varepsilon \rightarrow 0$, les termes de diffusions fractionnaires tendent vers 0 laissant formellement une simple équation différentielle ordinaire. Une autre méthode développée par Cabré, Coulon et Roquejoffre (voir [36] et [35]) aurait été de considérer l'équation satisfaite par $\frac{n}{\phi_0(r(t)\cdot)}$ où $r(t)$ décroît exponentiellement vite.

L'étape D peut aussi être abordée de deux façons différentes. Cabré Coulon Roquejoffre ([35]) propose une preuve directe. La preuve proposée dans ce manuscrit s'appuie sur des techniques provenant de la théorie des solutions de viscosité et des fonctions tests perturbées.

Il est à noter que cette approche a été développée en même temps par Souganidis et Tarfulea [103] et Bouin, Garnier, Henderson et Patout [27]. Souganidis et Tarfulea prouvent un résultat de propagation pour une famille d'opérateurs intégraux singuliers semblables au Laplacien fractionnaire. Cette étude est sensiblement proche du travail présenté. Le travail de Bouin, Garnier, Henderson et Patout [27] montrent la robustesse des techniques sur une famille d'opérateurs intégraux à queues lourdes non-singuliers dans un environnement homogène.

Soulignons que le résultat précédent est établi pour $\Omega = \mathbb{R}^d$: l'environnement considéré n'est pas fragmenté. Toutefois, la périodicité de μ permet formellement de diviser l'environnement en deux : un domaine où l'environnement est favorable à l'espèce $\Omega_- = \{x \in \mathbb{R}^d \mid \mu(x) \geq 0\}$ et un domaine défavorable $\Omega_+ = \{x \in \mathbb{R}^d \mid \mu(x) < 0\}$ (les signes en indice de Ω_+ et Ω_- renvoient aux signes de la valeur propre principale associée à chacun des domaines : $\lambda(\Omega_+) > 0$ et $\lambda(\Omega_-) < 0$). Si on considère une suite de taux de naissance μ_k telle que $\mu_k(x) \xrightarrow[k \rightarrow +\infty]{} -\infty$ lorsque $x \in \Omega_+$, alors l'environnement défavorable à l'espèce devient létal lorsque $k \rightarrow +\infty$. Ainsi, si l'ensemble Ω_- est fragmenté (i.e. composé de composantes connexes bornées), il s'ensuit formellement qu'à la limite $k \rightarrow +\infty$, l'espèce vit dans l'environnement fragmenté Ω_- avec des conditions de Dirichlet dans Ω_+ . L'étude de phénomènes d'invasions dans un environnement périodique fragmenté Ω est réalisée dans le Chapitre 3.

B.3.b Chapitre 3- Equation de Fisher-KPP fractionnaire dans un environnement périodique fragmenté

Nous dirons qu'un environnement est périodique et fragmenté s'il existe $\Omega_0 \subset \mathbb{R}^d$ borné vérifiant l'hypothèse de la boule intérieure et extérieure uniforme tel que

$$\Omega = \bigcup_{k \in \mathbb{Z}^d} (\Omega_0 + a_k),$$

avec a_k une suite périodique de \mathbb{R}^d telle que

$$(\Omega_0 + a_i) \cap (\Omega_0 + a_j) \neq \emptyset \iff i = j.$$

Nous rappelons que la condition de la boule intérieure (respectivement extérieure) uniforme réfère à l'existence d'un rayon $r_0 > 0$ tel que

$$\begin{aligned} & \forall x \in \Omega, \quad \exists z_x \in \Omega \text{ tel que } B(z_x, r_0) \subset \Omega \\ & \text{(respectivement } \forall x \in \Omega^c, \quad \exists z_x \in \Omega^c \text{ tel que } B(z_x, r_0) \subset \Omega^c). \end{aligned}$$

Le résultat d'invasion est alors le suivant :

Théorème 1.2 (L.-Mirrahimi-Roquejoffre (Chapitre 3)). *Soit n la solution de (1.1) avec Ω un domaine périodique fragmenté et $f(x, n) = n - n^2$. Si la valeur propre principale λ_0 de $(-\Delta)^\alpha - Id$ dans Ω_0 avec des conditions de Dirichlet extérieures est négative, alors il existe une unique solution stationnaire bornée positive non-triviale n_+ .*

De plus, pour toute solution n avec $n(t=0)$ positive, non-triviale à support compacte dans $\bar{\Omega}$, nous avons que

1. $\forall c < \frac{|\lambda_0|}{d+2\alpha}, \quad \lim_{t \rightarrow +\infty} \sup_{|x| < e^{ct}} |n(x, t) - n_+(x)| = 0,$
2. $\forall C > \frac{|\lambda_0|}{d+2\alpha}, \quad \lim_{t \rightarrow +\infty} \sup_{|x| > e^{Ct}} |n(x, t)| = 0.$

Le schéma de preuve du Théorème 1.2 suit la progression A-B-C-D énoncée plus haut. Toutefois de nouvelles difficultés apparaissent du fait que la solution n et la solution stationnaire n_+ s'annulent dans Ω^c . De ce fait, la preuve d'unicité de la solution stationnaire ne repose pas seulement sur le principe du maximum. Elle s'appuie également sur un Lemme de Hopf fractionnaire (voir [65]) et une estimation précise par au-dessus et en-dessous de n_+ . Cette estimation est la suivante

$$c\delta(x)^\alpha \leq n_+(x) \leq C\delta(x)^\alpha \quad \text{avec } c, C > 0 \quad \text{et} \quad \delta(x) = \text{dist}(x, \partial\Omega)1_\Omega(x). \quad (1.4)$$

L'inégalité (1.4) est obtenue via le principe de comparaison et fonctions barrières développées par [97].

Ceci permet de conclure à la preuve de l'étape A du plan de démonstration annoncée à la suite du Théorème 1.1. L'étape B pourrait être résolue en appliquant des résultats généraux sur les estimations du noyau de la chaleur du Laplacien fractionnaire développés par Bogdan, Grywny et Ryznar [24] ou Chen, Kim et Song [42]. Malgré cela, nous développons notre propre estimation du noyau de la chaleur. Cette dernière est moins générale que celles énoncées dans [24] et [42]. En effet, elle ne s'applique uniquement en temps courts pour des données initiales à support compact. Toutefois, la preuve est relativement courte et s'appuie uniquement sur un raisonnement déterministe (contre un raisonnement probabiliste dans [24] et [42]).

La preuve du point C peut encore une fois être développée de deux manières différentes : via une approche inspirée par [35] ou par [85]. Nous avons choisi d'utiliser la seconde approche pour les mêmes raisons évoquées précédemment. Cependant, nous avons amélioré la précision des sous et sur-solutions afin d'obtenir une estimation précise des lignes de niveaux. Nous proposons une preuve pour l'étape D qui est une adaptation de [35] aux nouvelles difficultés qu'engendrent la présence d'un domaine hostile. Notons que dans cette preuve, le changement de variable

$$(x, t) \mapsto \left(|x|^{\frac{1}{\varepsilon}-1}, \frac{t}{\varepsilon} \right)$$

n'intervient pas. La deuxième preuve est une preuve où le changement de variable est préservé. Pour chacune d'entre elles, le fait que n et n_+ s'annulent sur Ω^c entraîne de nouvelles difficultés par rapport à ce qui a été fait dans les travaux antérieurs ([35], [85] ou [77]).

B.3.c Chapitre 4- Equation de Fisher-KPP fractionnaire dans un environnement fragmenté général

Le troisième résultat présenté dans cette introduction est un résultat d'existence et d'unicité de solution stationnaire bornée de l'équation (1.1) pour des domaines Ω généraux inclus dans \mathbb{R}^d et $f(x, n) = n - n^2$. Ces problèmes sont abordés dans le chapitre 4 de ce manuscrit.

Le résultat d'existence repose sur l'existence d'un sous-domaine $\underline{\Omega} \subset \Omega$ borné *viable* (i.e. de valeur propre principale $\lambda_0(\underline{\Omega})$ négative). Si un tel domaine existe, il est facile de donner une sous-solution (une fonction propre positive associée à la valeur propre $\lambda_0(\underline{\Omega})$) et une sur-solution (une constante suffisamment grande). A partir de ces sous-solutions et sur-solutions, la construction d'une solution stationnaire non-triviale s'ensuit (voir [102]). Il est donc important de savoir donner des critères donnant le signe de $\lambda_0(\Omega)$. Il est bien connu que pour un domaine connexe, la valeur propre du domaine est décroissante par rapport à la taille du domaine (i.e. $\lambda_0(t\Omega)$ décroît lorsque t grandit). Cependant, comme le Laplacien fractionnaire est un opérateur non-local, la valeur propre principale est aussi définie pour des domaines non-connexes. Le premier résultat du Chapitre 4 répond aux questions suivantes pour des intervalles bornés dans \mathbb{R} :

Si l'on considère $a > 0$ tel que le segment $]0, a[$ est viable et le segment $]0, \frac{a}{2}[$ est non-viable, est ce qu'il existe une distance $d_0 > 0$ tel que pour toute distance $d < d_0$, l'ensemble $]0, \frac{a}{2}[\cup] \frac{a}{2} + \mu, a + \mu[$ soit viable? En d'autres termes, est ce que l'application $(\mu \in \mathbb{R}^+ \mapsto \lambda_\alpha(]0, \frac{a}{2}[\cup] \frac{a}{2} + \mu, a + \mu[)$ est continue en 0?

Le Théorème suivant répond à ces deux questions :

Théorème 1.3 (L.-Roquejoffre (Chapitre 4)). *L'application $(\mu \in \mathbb{R}^{+*} \mapsto \lambda_\alpha(]0, \frac{a}{2}[\cup] \frac{a}{2} + \mu, a + \mu[)$ est continue croissante sur tout son ensemble de définition. De plus, elle peut être prolongée par continuité en 0.*

La preuve en dehors de 0 repose essentiellement sur le quotient de Rayleigh. La preuve en 0 pour $\alpha < \frac{1}{2}$ repose également sur un quotient de Rayleigh et sur la densité des fonctions de $H^\alpha(]0, a[)$ qui vérifient $\phi(\frac{a}{2}) = 0$. En revanche pour $\alpha \geq \frac{1}{2}$, la preuve revient à montrer que $\frac{a}{2}$ est une singularité effaçable de $(-\Delta)^\alpha - Id$ dans $]0, a[\setminus \{\frac{a}{2}\}$. Pour cela, nous utilisons le relèvement du Laplacien fractionnaire au demi plan supérieur introduit par Caffarelli et Silvestre [37].

Remarquons que l'importance de la dimension 1, ne se retrouve que pour montrer la continuité en $\mu = 0$. Ainsi, nous généralisons certains cas du théorème précédent :

Théorème 1.4 (L.-Roquejoffre (Chapitre 4)). *Soient Ω_1 et Ω_2 deux sous-ensembles réguliers et bornés de \mathbb{R}^d . Soit $\Omega_{\mu,e}$ l'ensemble défini par*

$$\Omega_{\mu,e} = \Omega_1 \cup (\Omega_2 + t_\mu e) \text{ avec } (t_\mu, e) \in \mathbb{R}^+ \times \mathbb{S}^{d-1} \text{ tel que } \text{dist}(\Omega_1, (\Omega_2 + t_\mu e)) = \mu.$$

Alors l'application $(\mu \in \mathbb{R}^{+} \mapsto \lambda_\alpha(\Omega_{\mu,e}))$ est continue et croissante sur tout son ensemble de définition.*

Une conséquence du Théorème 1.4 est que la distance entre les différentes composantes connexes de Ω joue un rôle primordial dans l'établissement d'un sous-domaine $\underline{\Omega}$ viable.

Le résultat d'unicité (établi dans \mathbb{R}^d) est plus complexe à démontrer que celui d'existence. Il s'appuie sur l'encadrement de n'importe quelle solution stationnaire bornée non-triviale par en-dessous et au-dessus par la même fonction \mathcal{G} (à une multiplication par une constante près)

$$\text{i.e. } \exists \mathcal{G} : \Omega \rightarrow \mathbb{R}^+ \text{ et } c, C > 0, \quad \text{tel que} \quad c \mathcal{G}(x) \leq n_+(x) \leq C \mathcal{G}(x). \quad (1.5)$$

Par exemple, si $\Omega = \mathbb{R}^d$ alors $\mathcal{G} = 1$. Si Ω est périodique et fragmenté, l'inégalité (1.4) donne $\mathcal{G}(x) = \delta(x)^\alpha$.

Le but de ce chapitre est de démontrer l'assertion (1.5) sous des hypothèses sur Ω les plus larges possibles. Nous ne donnons pas ici les hypothèses les plus générales que l'on ait trouvées (nous renvoyons au chapitre 4 pour plus de détails) mais nous donnons un aperçu de certaines hypothèses nécessaires. D'autres hypothèses plus techniques seront considérées dans le chapitre 4. Voici le type d'hypothèses à considérer :

1. Ω satisfait la condition de la boule extérieure et intérieure.
2. Ω peut être divisé en deux parties : un sous-ensemble non-vidé viable Ω_- et un sous-ensemble non-viable Ω_+ .
3. D'autres hypothèses plus techniques et restrictives sont nécessaires notamment sur la distance entre Ω_- et Ω_+ et entre les composantes de Ω_+ .

Par domaine viable, on entend qu'en tout point $x \in \Omega_-$, il existe un sous-domaine $\underline{\Omega}_-$ tel que

$$\lambda_\alpha(\underline{\Omega}_-) < 0 \quad \text{et} \quad x \in \underline{\Omega}_- \subset \Omega.$$

De plus, ces sous-ensembles viables $\underline{\Omega}_-$ sont de nombre fini (à translation près).

Par domaine non-viable, on entend qu'il existe $\lambda_0 > 0$, tel que pour tout point $z \in \mathbb{R}^d$ et pour tout rayon $R > 0$, on a

$$\lambda_\alpha(\Omega_+ \cap B(z, R)) > \lambda_0.$$

Insistons sur le fait que le signe en indice de Ω_- renvoie au signe de la valeur propre principale d'un sous-ensemble viable inclus dans Ω_- . De même, le signe de Ω_+ renvoie au signe de la valeur propre principale de $\lambda_\alpha(\Omega_+ \cap B(z, R))$ pour tout centre $z \in \mathbb{R}^d$ et R positif.

La figure 1.2 donne une illustration de ces hypothèses. Ces trois hypothèses permettent déjà de comprendre le type d'ensemble Ω que nous pouvons considérer. Si les conditions des boules intérieures et extérieures sont satisfaites et si Ω contient une zone viable, les hypothèses optimales considérées dans le Chapitre 4 permettent d'englober : les ensembles bornés, les domaines périodiques (fragmentés ou non), les domaines non bornés avec une zone viable Ω_- borné ou non...

Moralement, par le caractère non-viable de Ω_+ , n'importe quelle solution à l'équilibre n_+ est « alimentée » par Ω_- : cela s'apparente à un effet « source-puit ». Ainsi l'estimation générale de n_+ est la suivante

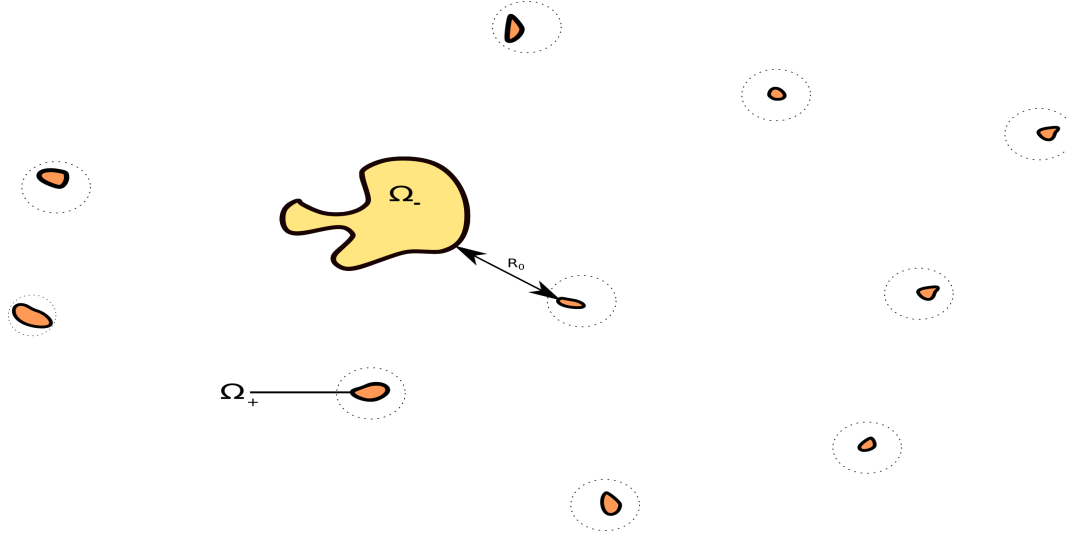


FIGURE 1.2 – Illustration des hypothèses précédentes.

Théorème 1.5 (L.-Roquejoffre (Chapitre 4)). *Sous les hypothèses précédentes, soit $\mathcal{G} : \Omega \rightarrow \mathbb{R}^+$ définie par*

$$\mathcal{G}(x) = \min(\delta(x)^\alpha, 1) \times \left(1_{\Omega_-}(x) + \int_{\Omega_-} \frac{1_{\Omega_+}(y)}{|x-y|^{d+2\alpha}} dy \right)$$

alors pour toute solution stationnaire bornée non-triviale n_+ il existe $c, C > 0$ tel que

$$c \mathcal{G}(x) \leq n_+(x) \leq C \mathcal{G}(x).$$

Conséquence : Il existe une unique solution stationnaire bornée non-triviale de l'équation (1.1).

Soulignons que le Théorème 1.5 implique que selon le choix de Ω , le comportement asymptotique de n_+ peut varier complètement. Nous présentons dans cette introduction trois exemples où les comportements asymptotiques diffèrent. Fixons la dimension $d = 1$. Si Ω est périodique fragmenté alors $\mathcal{G}(x) = \delta(x)^\alpha$. Si Ω_- est borné et Ω_+ non borné alors n_+ décroît comme $\frac{\delta(x)^\alpha}{1+|x|^{1+2\alpha}}$. Si $\Omega_- = \mathbb{R}_-$ et Ω_+ est non borné alors n_+ décroît comme $\frac{\delta(x)^\alpha}{1+\max(0,x)^{2\alpha}}$.

Remarquons également que Ω_- et Ω_+ ne sont pas fixés. Sous les hypothèses plus générales énoncées dans le chapitre 4, nous pouvons toujours définir de nouveaux ensembles Ω'_+ et Ω'_- différents de Ω_+ et Ω_- mais vérifiant toujours les hypothèses requises par le Theorème 1.5. Par exemple, si Ω_+ est borné alors on peut définir $\Omega'_- = \Omega$ et $\Omega'_+ = \emptyset$.

Enfin, pour des domaines comportant des composantes non-viables (i.e. Ω_+ ne peut pas être réduit à l'ensemble vide), nous faisons la distinction entre invasion et colonisation. Si Ω_+ ne peut pas être réduit à l'ensemble vide, alors l'espèce colonise le territoire qui lui est disponible mais il n'y a pas de phénomènes d'invasion. En effet, on ne peut pas éviter un phénomène de type *source-puit*.

B.4 Retour sur les motivations biologiques

Au vue des résultats mathématiques présentés, nous répondons ici aux questions biologiques énoncées dans la section B.1.a. Si l'espèce biologique considérée rentre dans les hypothèses

de modélisation, alors un critère de survie de l'espèce est donné par le signe de la valeur propre principale $\lambda_\alpha(\mathcal{O})$ où \mathcal{O} est un sous-domaine bien choisie de Ω . De plus, sous certaines hypothèses sur l'environnement, nous pouvons prédire la distribution en temps grand de l'espèce en fonction de l'environnement considéré. Enfin, lorsque la structure de l'environnement est particulière (par exemple, un environnement périodique fragmenté), alors, nous pouvons parler d'invasion avec une vitesse d'invasion exponentielle en temps.

C. Un problème d'évolution Darwinienne dans un environnement spatial fragmenté prenant en compte une dispersion non-locale.

Rappelons que l'équation considérée est :

$$\begin{cases} -\partial_{xx}n_\varepsilon - \varepsilon^2\partial_{\theta\theta}n_\varepsilon + Ln_\varepsilon = n_\varepsilon[R(x, \theta) - \rho_\varepsilon] & \text{pour } x \in \Omega, -A < \theta < A, \\ \rho_\varepsilon(x) = \int_{-A}^A n_\varepsilon(x, \theta)d\theta & \text{pour } x \in \Omega, \\ \partial_{\nu_x}n_\varepsilon = \partial_{\nu_\theta}n_\varepsilon = 0 & \text{pour } (x, \theta) \in \partial(\Omega^c \times]-A, A[) \end{cases} \quad (1.3)$$

avec $\Omega \subset \mathbb{R}$.

C.1. Description du modèle et motivations biologiques

L'équation (1.3) modélise l'état d'équilibre d'une densité de population biologique asexuée n_ε qui dépend d'une position en espace x et d'un trait phénotypique continue θ .

Nous supposons que les individus se dispersent spatialement selon deux phénomènes : ils suivent un mouvement brownien ou ils « sautent » (bougent rapidement) d'un point à l'autre. Du point de vue macroscopique, le premier type de dispersion se modélise par un Laplacien $-\partial_{xx}$ tandis que la dispersion non-locale est modélisée par un opérateur intégral L . Nous supposerons que L est un opérateur intégral non-singulier de la forme

$$L(n)(x) = \int_{\Omega} [n(x) - n(y)]K(x - y)dy \quad \text{avec } K(x) = K(-x), 0 < c < K < C \text{ et régulier.}$$

Un tel opérateur non-local, nous permet de considérer des environnements fragmentés. Toutefois, nous restreindrons l'étude à un domaine de \mathbb{R} , borné et comprenant un nombre fini de composantes connexes :

$$\text{i.e. } \Omega = \bigcup_{i=1}^m]a_i, b_i[\quad \text{avec } a_1 < b_1 < a_2 < \dots < a_m < b_m.$$

Le choix de décomposer la modélisation de la diffusion spatiale en deux opérateurs, à savoir un local et un non-local, plutôt qu'un opérateur intégral singulier prenant en compte les deux dynamiques (comme le Laplacien fractionnaire) est essentiellement technique. Nous reviendrons sur ce point dans les perspectives de la thèse.

Nous supposons que le trait phénotypique du parent est transmis à la descendance. Toutefois, il est possible que ce trait subisse de petites mutations que nous modélisons par un Laplacien $-\varepsilon^2 \partial_{\theta\theta}$.

Le terme $R(x, \theta)$ est un taux de croissance prenant en compte la natalité et la mortalité intrinsèque. A titre d'exemple, nous pouvons prendre comme taux de croissance

$$R(x, \theta) = r - g(bx - \theta)^2.$$

Le terme r est un taux de croissance maximale. Le trait optimal à la position x est donné par $\theta_o(x) = bx$. Lorsqu'un individu a un trait θ différent de ce trait optimal, il y a un coût pour son taux de croissance qui est donné par $(\theta - \theta_o(x))^2$ fois la constante g qu'on appellera la pression de la sélection. Les individus fortement inadaptés à la position x auront un taux de naissance faible (voir négatif). La constante b indique à quelle vitesse le trait optimal varie en tant que fonction de l'espace.

Enfin le terme ρ_ε désigne la taille totale de la population. Via le terme $-\rho_\varepsilon$ nous prenons en compte un taux de mortalité dû à la compétition entre les individus pour les ressources disponibles.

Les conditions aux bords de Neumann par rapport à x modélisent le fait qu'aucun individu ne peut ni entrer ni sortir du territoire. De même, les conditions aux bords de Neumann par rapport à θ se traduisent par le fait qu'aucun mutant ne puisse apparaître en dehors de l'intervalle $] - A, A[$.

Le but du chapitre 5 est de présenter nos premiers résultats obtenus sur l'apparition de trait(s) dominant(s) (voir Figure 1.3 en exemple) lorsque le terme de mutation devient petit (i.e. $\varepsilon \rightarrow 0$).

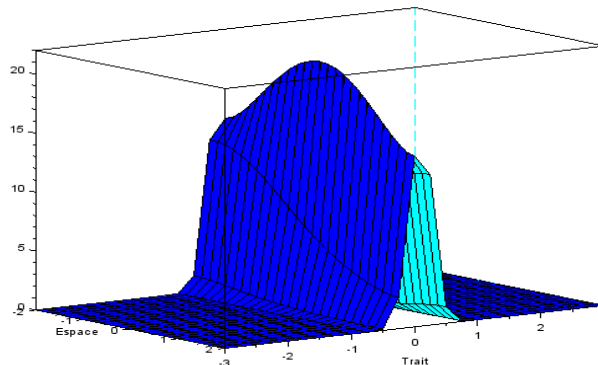


FIGURE 1.3 – Simulation numérique de n_ε pour $\Omega =] - 2, 2[$, $A = 3$ et $r = 5, b = 1, g = 5, \varepsilon = 0.005$.

C.2. Etat de l'art

De nombreux travaux étudient des modèles de sélection-mutation en utilisant des équations intégro-différentiels (voir par exemple [81], [39] ou [48]). Ici nous nous intéressons à une approche basée sur des équations de Hamilton-Jacobi initiée en 2005 par Dieckmann, Jabin,

Mishler et Perthame [50]. Les auteurs utilisent une approche via les équations de Hamilton-Jacobi pour étudier la dynamique des solutions d'équations de réaction-diffusion modélisant une population biologique dans un régime de petites mutations. Grâce à cette méthode, Barles et Perthame obtiennent des résultats de concentration pour une équation similaire à (1.3) ne faisant pas intervenir de structure spatiale. En effet, ils démontrent que sous certaines hypothèses sur les coefficients, à mesure que les mutations deviennent petites (i.e. $\varepsilon \rightarrow 0$), la solution $n_\varepsilon(\theta)$ converge vers une masse de Dirac $\delta_{\bar{\theta}}$. En d'autres termes, lorsque les mutations deviennent petites, la population tend vers une population monomorphe. Ce résultat a été étendu par Barles, Mirrahimi et Perthame [8] à des hypothèses plus générales. Cette approche via des équations de Hamilton-Jacobi pour démontrer l'émergence de traits dominants à mesure que les mutations deviennent petites est robuste. En effet, elle a été adaptée à de multiples situations comme par exemple : l'étude d'une population évoluant dans un environnement changeant de façon périodique au cours du temps (tel qu'un cycle jours/nuit) ou linéaire (changement climatique) par Figueroa Iglesias et Mirrahimi [56] et [57], l'étude d'une population évoluant dans un environnement organisé spatialement de façon discrète (présence de deux habitats avec des termes de migrations entre les habitats) par Mirrahimi [88] et Mirrahimi et Gandon [91] ou l'étude d'une population évoluant dans un environnement spatiale continu et convexe par Perthame et Souganidis [95]. Les travaux présentés dans cette partie sont dans la continuité de [88] et [95]. Dans la suite, nous présentons les premiers travaux sur les modèles de sélection-mutation prenant en compte un environnement spatial hétérogène avant de revenir plus précisément sur les résultats de [88] et [95].

Un premier modèle prenant en compte une diffusion locale en espace et en trait a d'abord été développé par Champagnat et Méléard dans [41]. Dans cet article, les auteurs dérivent d'un modèle stochastique une première équation de réaction-diffusion semblable à (1.3). Arnold, Desvillettes et Prévost [4] démontrent l'existence d'une solution stationnaire non-triviale à un système de réaction-diffusion similaire à celui développé par Champagnat et Méléard [41] via un théorème de point fixe de Schaudder. Dans [88], l'auteur considère une population pouvant vivre dans deux habitats hétérogènes avec un terme d'échange entre deux habitats. En fonction du terme de reproduction dans chacun des habitats, l'auteur met en évidence la possible apparition du polymorphisme dans la population (émergence de deux traits dominants) à mesure que les mutations deviennent petites. Souganidis et Perthame dans [95] considèrent un environnement spatial convexe continu. Ils mettent en évidence l'émergence d'un trait dominant lorsque les mutations deviennent petites. Le même type de résultat (i.e. apparition d'un trait dominant pour une espèce organisée spatialement et phénotypiquement lorsque les mutations deviennent petites) a été établi en même temps que [95] par Lam et Lou dans [75]. Cependant, ce dernier article n'adopte pas une approche via des équations de Hamilton-Jacobi.

De multiples autres questions peuvent être abordées par l'étude d'équations de réaction-diffusion modélisant une espèce biologique dépendant d'une variable spatiale x et d'une autre variable phénotypique θ . Nous présentons ici brièvement le type de questions qui ont été traitées. L'existence de *front d'invasion* a été montrée par Alfaro, Coville et Raoul [2] et par Berestycki, Jin et Silvestre [17]. L'étude de persistance dans un environnement changeant a aussi été étudié par Alfaro, Berestycki et Raoul [1] pour un environnement sujet à un changement climatique linéaire au cours du temps. Citons également les travaux traitant de l'invasion du crapaud buffle en Australie ([109], [26], [16] ou encore [28]). Des études biolo-

giques ont relevé que la vitesse d'invasion accélérerait au cours du temps. Cette accélération serait due à un changement phénotypique : l'allongement de la taille des jambes accélérant la diffusivité des individus.

Il est également possible de modéliser des mutations par un noyau intégral, pour prendre en compte des sauts non-locaux (voir par exemple [8] ou [48] qui étudient de tels modèles de sélection-mutation dans des environnements homogènes). Champagnat et Méléard toujours dans [41] dérivent également un modèle impliquant un terme intégral pour modéliser les mutations. Ce second modèle a aussi fait l'objet de plusieurs travaux : Arnold, Desvillettes et Prévost dans [4] ont démontré l'existence d'une solution stationnaire pour un tel modèle. Lorsque le terme de mutation est modélisé par un Laplacien fractionnaire, l'étude de l'émergence de traits dominants par une approche de Hamilton-Jacobi avec contraintes est le sujet de [85] par Méléard et Mirrahimi et de [90] par Mirrahimi. Notons que Berestycki, Jin et Silvestre [17] considèrent aussi un modèle impliquant le Laplacien fractionnaire pour modéliser les mutations.

C.3 Présentation des résultats

L'étude porte sur le comportement asymptotique de (1.3) lorsque $\varepsilon \rightarrow 0$. Nous cherchons à démontrer l'existence de trait(s) dominant(s) : nous nous attendons au fait que n_ε converge au sens de la mesure vers une somme masses de Dirac en θ

$$\text{i.e.} \quad n_\varepsilon(x, \theta) \xrightarrow{\varepsilon \rightarrow 0} \sum_{i=1}^K N_i(x) \delta_{\bar{\theta}_i}(\theta)$$

pour certains $\bar{\theta}_i$ dans $] -A, A[$. Par exemple si $K = 1$, nous dirons que la population limite est monomorphe de trait optimal $\bar{\theta}$: tous les individus de la population limite auront le trait $\bar{\theta}$. De plus, dans cet exemple, il est facile de vérifier que si n_ε tend vers $N(x) \delta_{\bar{\theta}}(\theta)$ alors N est un vecteur propre principal de l'opérateur $-\partial_{xx} + L - [R(\cdot, \bar{\theta}) - \rho]$ avec des conditions aux bords de type Neumann. Notez que la valeur propre principale associée est nulle. De plus, il est facile de remarquer à l'aide d'un quotient de Rayleigh que dans cet exemple précis, la valeur propre principale $\lambda(\bar{\theta}, 0)$ associée à $-\partial_{xx} + L - R(\cdot, \bar{\theta})$ avec des conditions de Neumann aux bords

$$\text{i.e.} \quad \begin{cases} -\partial_{xx}\psi + L\psi - R(\cdot, \bar{\theta})\psi = \lambda(\bar{\theta}, 0)\psi & \text{dans } \Omega, \\ \partial_{\nu_x}\psi = 0 & \text{sur } \partial\Omega, \end{cases} \quad (1.6)$$

est strictement négative. Ainsi, une condition nécessaire et suffisante pour assurer l'existence d'une solution à (1.3) pour ε assez petit est la condition suivante sur la valeur propre principale $\lambda(\theta, 0)$:

$$\min_{\theta \in]-A, A[} \lambda(\theta, 0) < 0.$$

Soulignons que nous retrouvons un critère d'existence semblable à celui déjà exhibé pour l'équation de Fisher-KPP fractionnaire : dans un régime de petites mutations au moins quelque(s) trait(s) doive(nt) être viable(s).

Sachant que nous voulons démontrer la convergence au sens de la mesure de n_ε vers une somme de masses de Dirac, une transformation de Hopf-Cole $u_\varepsilon = \varepsilon \log(n_\varepsilon)$ est une technique adaptée. En effet, tandis que la limite de n_ε , lorsque $\varepsilon \rightarrow 0$, est une mesure, la limite de u_ε serait une fonction continue, solution d'une équation de Hamilton-Jacobi. L'idée principale dans cette méthode est d'étudier d'abord la limite de u_ε et ensuite en dériver des informations

sur la limite de n_ε .

Ainsi nous démontrons le théorème suivant

Théorème 1.6 (L.-Mirrahimi (Chapitre 5)). *Sous les hypothèses précédentes et le long d'une sous-suite, les résultats suivants sont vrais*

1. La famille ρ_ε converge uniformément vers une fonction ρ qui vérifie

$$\exists c, C > 0, \text{ telles que } \forall x \in \Omega, c < \rho(x) < C.$$

2. La famille u_ε converge uniformément vers une fonction u . De plus u ne dépend que de θ et est une solution de viscosité de l'équation de Hamilton-Jacobi suivante

$$\begin{cases} -|\partial_\theta u(\theta)|^2 = -\lambda(\theta, \rho), \\ \max_{\theta \in [-A, A]} u(\theta) = 0, \end{cases} \quad (1.7)$$

avec $\lambda(\theta, \rho)$ la valeur propre principale associée à l'opérateur $-\partial_{xx} + L - [R(\cdot, \theta) - \rho]$ avec des conditions aux bords de Neumann.

3. La famille n_ε converge au sens de la mesure vers une mesure non-triviale n . De plus, le support de n est inclu dans $\Omega \times \{\theta \in]-A, A[\mid u(\theta) = 0\}$.

Pour montrer la convergence de u_ε , nous utilisons le fait que u_ε est solution de

$$\begin{cases} -\frac{1}{\varepsilon} \partial_{xx} u_\varepsilon - \frac{|\partial_x u_\varepsilon|^2}{\varepsilon^2} - \varepsilon \partial_{\theta\theta} u_\varepsilon - |\partial_\theta u_\varepsilon|^2 \\ \quad + \int_{\Omega} \left[1 - e^{\frac{u_\varepsilon(y) - u_\varepsilon(x)}{\varepsilon}} \right] K(x-y) dy = (R(x, \theta) - \rho_\varepsilon(x)) & \text{dans } \Omega \times]-A, A[, \\ \rho_\varepsilon(x) = \int_{]-A, A[} n_\varepsilon(x, \theta) d\theta & \text{dans }]-A, A[, \\ \partial_{\nu_x} u_\varepsilon(x, \theta) = 0 \text{ sur } \partial\Omega \times]-A, A[, \quad \partial_{\nu_\theta} u_\varepsilon(x, \pm A) = 0. \end{cases} \quad (1.8)$$

Premièrement, donnons un argument heuristique qui permet de retrouver (1.7). Faisons un développement asymptotique de u_ε et ρ_ε :

$$u_\varepsilon(x, \theta) = u_0(x, \theta) + \varepsilon u_1(x, \theta) + o(\varepsilon) \quad \text{et} \quad \rho_\varepsilon(x) = \rho_0(x) + o_\varepsilon(1).$$

Si nous remplaçons u_ε et ρ_ε par de tels développement asymptotique dans (1.8), il s'ensuit

$$\begin{aligned} & \frac{1}{\varepsilon} \left(-\partial_{xx} u_0 - \frac{|\partial_x u_0|^2}{\varepsilon} - 2|\partial_x u_0 \partial_x u_1| \right) + \int_{\Omega} \left[1 - e^{\frac{u_0(y, \theta) - u_0(x, \theta)}{\varepsilon} + u_1(y, \theta) - u_1(x, \theta) + o_\varepsilon(1)} \right] K(x-y) dy \\ & - |\partial_\theta u_0|^2 - \partial_{xx} u_1 - |\partial_x u_1|^2 - [R - \rho_0] + o_\varepsilon(1) = 0. \end{aligned}$$

Cette équation suggère que $u_0(x, \theta) = u_0(\theta)$. De plus, en passant formellement à la limite $\varepsilon \rightarrow 0$, nous trouvons

$$-|\partial_\theta u_0(\theta)|^2 = [R(x, \theta) - \rho_0(x)] + \partial_{xx} u_1(x, \theta) + |\partial_x u_1(x, \theta)|^2 - \int_{\omega} [1 - e^{u_1(y, \theta) - u_1(x, \theta)}] K(x-y) dy. \quad (1.9)$$

Or comme la partie gauche de l'équation (1.9) ne dépend pas de x , cela laisse à penser que $u_1 = \log(\psi^\theta)$ avec ψ^θ la fonction propre principale (prise positive) de l'opérateur $-\partial_{xx} - L - [R(\cdot, \theta) - \rho]$ et des conditions aux bords de Neumann. Si cette affirmation est vraie, il s'ensuit

$$-|\partial_\theta u_0(\theta)|^2 = \left([R(x, \theta) - \rho_0(x)] \psi^\theta(x) + \partial_{xx} \psi^\theta(x) - L(\psi^\theta)(x) \right) (\psi^\theta(x))^{-1}.$$

Nous pouvons alors formellement conclure que la limite u_0 est solution de l'équation de Hamilton-Jacobi (1.7).

La preuve détaillée du Théorème 1.6 s'appuie principalement sur une inégalité de type Harnack et des estimations Lipschitz de u_ε . Cette inégalité et ces estimations sont introduites de façon précise dans le Chapitre 5 de la thèse.

D. Présentation du projet de CEMRACS

Durant l'été de ma première année de thèse, j'ai participé au CEMRACS dont le thème était « *Modèles numériques et mathématiques pour des applications médicales : descriptions déterministes, probabilistes et statistiques* ». Le but était de réaliser une première étude numérique sur le contrôle d'une population de *Aedes aegypti* (moustique tigre) en prenant en compte une dispersion spatiale. L'enjeu est d'ordre sanitaire : cette espèce de moustique est le principal vecteur de la Dengue. La motivation biologique est de relâcher une population de moustiques « transformée » par une bactérie tueuse de virus : la bactérie *Wolbachia*. En effet, les moustiques infectés par *Wolbachia* ne transmettent plus la Dengue. De plus, la reproduction entre des moustiques mâles (infectés ou non) avec des femelles infectées entraînent une descendance infectée, tandis que la proportion d'oeufs issus de la reproduction entre mâles infectés et femelles saines a une faible probabilité d'éclore ($1 - s_H$ avec $s_H \in]0, 1[$). En partant de ce constat, le but est de remplacer la population de moustiques résidente par une population de moustique infectée par *Wolbachia*.

L'équation considérée est la suivante

$$\begin{cases} \partial_t p - \Delta p = f(p) - ug(p) & \text{avec } x \in \Omega, t \in]0, T[, \\ \partial_\nu p = 0 & \text{avec } x \in \partial\Omega, \\ p(x, 0) = 0, \end{cases} \quad (1.10)$$

où Ω est un convexe borné de \mathbb{R}^2 et

$$f(p) = p(1-p) \frac{d_1 b_2 - b_2 d_1 (1 - s_{HP})}{b_1 (1-p)(1 - s_{HP}) + b_2 p} \quad \text{et} \quad g(p) = \frac{1}{\kappa} \frac{b_1 (1-p)(1 - s_{HP})}{b_1 (1-p)(1 - s_{HP}) + b_2 p}.$$

La variable p représente la proportion de moustiques infectés, u est le contrôle (i.e. la proportion de moustiques infectés relâchés dans Ω), b_1 et d_1 (respectivement b_2 et d_2) sont des taux de morts et de naissances d'insectes sains (respectivement infectés) et $\frac{1}{\kappa}$ est la capacité de charge du territoire. L'objectif est de minimiser la fonctionnelle

$$J(u) = \int_{\Omega} (1 - p(x, T))^2 dx,$$

(i.e. $p(x, T) = 1$ signifie que la population de moustique au temps T est composée seulement de moustiques infectés) avec u tel que

$$u \in L^\infty(\Omega \times [0, T]), \quad 0 \leq u \leq M, \quad a.e. \quad \text{et} \quad \int_0^T \int_{\Omega} u(x, t) dx dt < C,$$

pour des données positives M et C . La dérivation du problème est détaillée dans le chapitre 6 (et dans les références données à l'intérieur). Citons tout de même l'article [3] par Almeida,

Privat, Strugarek et Vauchelet. Dans cet article, les auteurs considèrent une équation différentielle ordinaire semblable à (1.10) (absence de structure spatiale et avec un terme de réaction identique à (1.10)). Les auteurs démontrent que la méthode optimale de relâché est « *bang-bang* » (saturation des contraintes). Nous citons également [105] par Strugarek, Vauchelet et Zubelli qui traitent d'un problème organisé en temps et en espace. Cependant, ce travail s'intéresse à une stratégie de remplacement en temps long contrairement à ce qui a été considéré au CEMRACS.

Pendant le CEMRACS, nous avons supposé que les temps de libération de u étaient discrets et donnés. Nous avons prouvé l'existence théorique d'un minimiseur à notre problème. Nous n'avons cependant pas réussi à le retrouver numériquement. De ce fait, nous avons réduit la complexité du problème en supposant que les lâchés de moustiques sont modélisés par un nombre fini de distributions Gaussiennes. Les jeux de données considérées semblent tendre vers un remplacement de la population non-infectée par une population infectée par *Wolbachia*.

E. Plan du manuscrit

Le chapitre 2 est consacré à la preuve du Théorème 1.1 : l'étude de l'invasion d'un territoire périodique pour une dynamique dictée par une équation de type Fisher-KPP fractionnaire. Le chapitre 3 est dédié à la démonstration du Théorème 1.2 : l'étude de l'invasion d'un territoire périodique et fragmenté lorsque la dynamique est dictée par une équation de Fisher-KPP fractionnaire.

Le chapitre 4 se concentre sur les résultats d'existence et d'unicité de solutions stationnaires aux équations de Fisher-KPP fractionnaires dans des domaines fragmentés (Théorèmes 1.3 et 1.5).

Le chapitre 5 se concentre sur l'étude de problèmes d'évolutions dans des domaines fragmentés (Théorème 1.6).

Enfin le chapitre 6 présente les résultats sur le remplacement de moustiques *Aedes aegypti* sains par des moustiques infectés par la bactérie *Wolbachia*.

La présentation des chapitres de cette thèse adopte la présentation d'articles scientifiques. En effet, chaque chapitre fait l'objet d'une publication (chapitres 2, 3 et 6) ou d'un dépôt de soumission (chapitre 4). L'étude faite dans le chapitre 5 n'est pas encore aboutie (nous détaillons dans les perspectives de la thèse les points que nous aimerions aborder avant de soumettre nos résultats à une revue scientifique). Enfin, nous cloturons ce manuscrit par la présentation des conclusions et des perspectives. Les perspectives se composent de nouvelles questions ouvertes apparues dans le cadre de ce travail ou de prolongements naturels que j'aimerais aborder.

Chapter 2

A singular limit in a fractional reaction-diffusion equation with periodic coefficients

Ce chapitre a fait l'objet d'une publication scientifique dans Communication in Mathematical Sciences (Volume 17).

2.1 Introduction

2.1.1 The equation.

We are interested in the following equation :

$$\begin{cases} \partial_t n(x, t) + L^\alpha(n)(x, t) = \mu(x)n(x, t) - n(x, t)^2, & (x, t) \in \mathbb{R}^d \times [0, +\infty) \\ n(x, 0) = n_0(x) \in C_c^\infty(\mathbb{R}^d, \mathbb{R}^+). \end{cases} \quad (2.1)$$

In the above setting, μ is a 1-periodic function, $\alpha \in (0, 1)$ is given and the term L^α denotes a fractional elliptic operator which is defined as follows :

$$L^\alpha(n)(x, t) := -PV \int_{\mathbb{R}^d} (n(x+h, t) - n(x, t)) \beta(x, \frac{h}{|h|}) \frac{dh}{|h|^{d+2\alpha}}, \quad (2.2)$$

where $\beta : \mathbb{R}^d \times \mathbb{S}^{d-1} \rightarrow \mathbb{R}$ is a 1-periodic smooth function such that for all $(x, \theta) \in \mathbb{R}^d \times \mathbb{S}^{d-1}$

$$\beta(x, \theta) = \beta(x, -\theta) \quad \text{and} \quad 0 < b \leq \beta(x, \theta) \leq B,$$

with b and B positive constants. When β is constant, we recover the classical fractional Laplacian $(-\Delta)^\alpha$.

The main aim of this paper is to describe the propagation front associated with (2.1). We show that the stable state invades the unstable state with an exponential speed.

2.1.2 The motivation

Equation (2.1) models the growth and the invasion of a species subject to non-local dispersion in a heterogeneous environment. Such models describe the situations where individuals can jump (move rapidly) from one point to the other, for instance because of the wind for

seeds or human transportation for animals. The function n stands for the density of the population in position x at time t . The diffusion term represented by the operator L^α describes the motions of individuals. The "logistic term" $\mu(x)n(x, t) - n(x, t)^2$ represents the growth rate of the population. The heterogeneity of the environment is modeled by the periodic function μ . The regions where μ is positive represent areas where the species are favored whereas μ negative prevents the growth of the species. Conversely, the term $-n^2$ characterizes the death term because of some "logistic" considerations, as for example the quantity of food.

The operator L^α generalizes the fractional Laplacian $(-\Delta)^\alpha$ which models "homogeneous" jumps : the individuals jump in every direction with the same frequency. Whereas the operator L^α models "heterogeneous" jumps : the individuals prefer to jump in the direction where β is high. Also, the frequency of jumps will depend on the position x of the individuals. Note that for the one dimensional case, for a regular bounded function n , $(1 - \alpha)L^\alpha(n)(x)$ tends to $-\frac{\beta(x)}{4}n''(x)$ as α tends to 1^- which corresponds to a heterogeneous local diffusion. Moreover, the function β will affect the principal eigenvalue λ_1 of $L^\alpha - \mu(x)Id$ (and consequently the negativity of λ_1 which is a criterion for the existence of a positive bounded stationary state). However, the hypothesis $0 < b \leq \beta \leq B$ implies that the techniques used for the fractional Laplacian are robust and can be extended to the case of the operator L^α .

Equation (2.1) was first introduced by Fisher in [58] (1937) and Kolmogorov, Petrovskii and Piscunov in [73] (1938) in the particular case of a homogeneous environment ($\mu = 1$) and a standard diffusion ($L^\alpha = -\Delta$) which corresponds to the case $\alpha = 1$ and $\beta = 1$. In [5], Aronson and Weinberger proved a first similar result to our result for the case introduced by Fisher and Kolmogorov, Petrovskii and Piscunov. In this case, the propagation is with a constant speed independent of the direction. In [63], Freidlin and Gärtner studied the question with a standard Laplacian in a heterogeneous environment (μ periodic). Using a probabilistic approach, they showed that the speed of the propagation is dependent on the chosen direction $e \in \mathbb{S}^{d-1}$. But, the speed $c(e)$ in the direction e is constant. Other proofs of this result, using PDE tools, can be found in [15] and [98]. In the case of the fractional Laplacian and a constant environment, Cabré and Roquejoffre in [36] proved the front position is exponential in time (see also for instance [47] for some heuristic and numerical works predicting such behavior and [85] for an alternative proof). Then in [35], Cabré, Coulon and Roquejoffre investigate the speed of propagation in a periodic environment modeled by equation (2.1) but considering the fractional Laplacian instead of the operator L^α . One should underline the fact that in the fractional case, the speed of propagation does not depend anymore on the direction. They proved that the speed of propagation is exponential in time with a precise exponent depending on a periodic principal eigenvalue.

The objective of this work is to provide an alternative proof of this property using an asymptotic approach known as "approximation of geometric optics". We will be interested in the long-time behavior of the solution n . We demonstrate that in the set

$$\left\{ (x, t) \mid |x| < e^{\frac{|\lambda_1|t}{d+2\alpha}} \right\},$$

as t tends to infinity, n converges to a stationary state n_+ and outside of this domain n tends to zero. The main idea in this approach is to perform a long time-long range rescaling to catch the effective behavior of the solution (see for instance [60], [54] and [6] for the classical Laplacian case). This paper is closely related to [85] where the authors Méléard and Mirrahimi have introduced such an "approximation" for a model with the fractional Laplacian and a simpler reaction term $(n - n^2)$. A very recent work, [27], uses also the techniques introduced in [85] (known as the introduction of an adapting rescaling and the investigation of adapted

sub and super solution) to investigate an integro-differential homogeneous Fisher-KPP-type equation : the operator L^α is replaced by $J * n - n$ where the kernel J is fat tailed but does not have singularity at the origin.

This paper was initially written with a fractional Laplacian. At its completion, we became aware of a preprint by Souganidis and Tarfulea [103] which proves a result quite close to ours in the case of spatially periodic stable operators. Our proof is quite different since their approach is based more on the theory of viscosity solutions. We have verified that our approach works for the model treated in [103] with no additional idea. We present our result with the operator L^α given by (2.2), where the proof for the fractional Laplacian applies almost word by word. In the course of the paper, we explain the points of our proof that allow to reach the generality of [103].

2.1.3 The assumptions.

For the initial data we will assume

$$n_0 \in C_c^\infty(\mathbb{R}^d, \mathbb{R}^+), \quad n_0 \not\equiv 0. \quad (\text{H1})$$

The function μ is a 1-periodic function, i.e.

$$\forall k \in \{1, \dots, d\}, \quad \mu(x_1, \dots, x_k + 1, \dots, x_d) = \mu(x_1, \dots, x_d). \quad (\text{H2})$$

Under the assumptions on β , the operator $L^\alpha - \mu(x)Id$ admits a principal eigenpair (ϕ_1, λ_1) by the Krein-Rutman Theorem (see [20]) that is

$$\begin{cases} L^\alpha \phi_1(x) - \mu(x)\phi_1(x) = \lambda_1 \phi_1(x), & x \in \mathbb{R}^d, \\ \phi_1 \text{ periodic, } \phi_1 > 0, \quad \|\phi_1\| = 1. \end{cases} \quad (2.3)$$

To assure the existence of a bounded, positive and periodic steady solution n_+ for (2.1), we will assume that the principal eigenvalue λ_1 is negative :

$$\lambda_1 < 0. \quad (\text{H3})$$

Note that such stationary solution is unique in the class of positive and periodic stationary solutions (see [10]).

In section 4, we will study a more general equation :

$$\begin{cases} \partial_t n(x, t) + L^\alpha(n)(x, t) = F(x, n(x, t)), & (x, t) \in \mathbb{R}^d \times [0, +\infty) \\ n(x, 0) = n_0(x) \in C_c^\infty(\mathbb{R}^d, \mathbb{R}^+). \end{cases} \quad (2.4)$$

We make the following assumptions for F :

$$\begin{cases} (i) \quad \forall s \in \mathbb{R}, \quad x \mapsto F(x, s) \text{ is periodic,} \\ (ii) \quad F(x, 0) = 0, \\ (iii) \quad \exists \underline{c}, \bar{C} > 0 \text{ such that } \forall (x, s) \in \mathbb{R}^d \times \mathbb{R}, \quad -\underline{c} \leq \partial_s \left(\frac{F(x, s)}{s} \right) \leq -\bar{C}, \\ (iv) \quad \exists M > 0, \forall (x, s) \in \mathbb{R}^d \times [M, +\infty[, \quad F(x, s) < 0. \end{cases} \quad (\text{H4})$$

We will denote $\partial_s(F)(x, 0)$ by $\mu(x)$ and we still denote by (λ_1, ϕ_1) the principal eigenvalue and eigenfunction of $L^\alpha - \mu(x)Id$. We also still suppose (H3) (i.e. λ_1 is strictly negative).

2.1.4 The main result and the method.

We introduce the following rescaling

$$(x, t) \mapsto \left(|x|^{\frac{1}{\varepsilon}} \frac{x}{|x|}, \frac{t}{\varepsilon} \right). \quad (2.5)$$

We perform this rescaling because one expects that for t large enough, n is close to the stationary state n_+ in the following set $\left\{ (x, t) \in \mathbb{R}^d \times \mathbb{R}^+ \mid |x| < e^{\frac{|\lambda_1|t}{d+2\alpha}} \right\}$ and n is close to 0 in the set $\left\{ (x, t) \in \mathbb{R}^d \times \mathbb{R}^+ \mid |x| > e^{\frac{|\lambda_1|t}{d+2\alpha}} \right\}$. The change of variable (2.5) will therefore respect the geometries of these sets. We then rescale the solution of (2.1) as follows

$$n_\varepsilon(x, t) = n\left(|x|^{\frac{1}{\varepsilon}} \frac{x}{|x|}, \frac{t}{\varepsilon}\right)$$

and a new steady state

$$n_{+, \varepsilon}(x) = n_+\left(|x|^{\frac{1}{\varepsilon}} \frac{x}{|x|}\right).$$

We prove :

Theorem 2.1. *Assuming (H1), (H2) and (H3), let n be the solution of (2.1). Then*

- (i) $n_\varepsilon \rightarrow 0$, locally uniformly in $\mathcal{A} = \left\{ (x, t) \in \mathbb{R}^d \times (0, \infty) \mid |\lambda_1| t < (d + 2\alpha) \log |x| \right\}$,
- (ii) $\frac{n_\varepsilon}{n_{+, \varepsilon}} \rightarrow 1$, locally uniformly in $\mathcal{B} = \left\{ (x, t) \in \mathbb{R}^d \times (0, \infty) \mid |\lambda_1| t > (d + 2\alpha) \log |x| \right\}$.

To provide the main idea to prove Theorem 2.1, we first explain the main element of the proof in the case of constant environment which was introduced in [85].

A central argument to prove such a result in the case of a constant environment, is that, using the rescaling (2.5), as $\varepsilon \rightarrow 0$, the term $((-\Delta)^\alpha(n)n^{-1})\left(|x|^{\frac{1}{\varepsilon}-1}x, \frac{t}{\varepsilon}\right)$ vanishes. More precisely, one can provide a sub and a super-solution to the rescaled equation which are indeed a sub and a super-solution to a perturbation of an ordinary differential equation derived from (2.1) by omitting the term $(-\Delta)^\alpha$. They also have the property that when one applies the operator $f \mapsto (-\Delta)^\alpha(f)f^{-1}$ to such functions, the outcome is very small and of order $\mathcal{O}(\varepsilon^2)$ as ε tends to 0.

In the case of periodic μ , we use the same idea. However, in this case, the sub and super-solutions are multiplied by the principal eigenfunction and, the term $(L^\alpha(n)n^{-1})\left(|x|^{\frac{1}{\varepsilon}-1}x, \frac{t}{\varepsilon}\right)$ will not just tend to 0 as in [85] but also compensate the periodic media. To prove the convergence of n_ε , dealing with this periodic term, we use the method of perturbed test functions from the theory of viscosity solutions and homogenization (introduced by Evans in [51] and [52]). Note that we also generalize the arguments of [85] to deal with a more general integral term L^α while in [85], only the case of the fractional Laplacian was considered. In the last part, we will also generalize Theorem 2.1 to the case of Fisher-KPP reaction term :

Theorem 2.2. *Assuming (H1), (H2), (H3) and (H4), let n be the solution of (2.4). Then*

- (i) $n_\varepsilon \rightarrow 0$, locally uniformly in $\mathcal{A} = \left\{ (x, t) \in \mathbb{R}^d \times (0, \infty) \mid |\lambda_1| t < (d + 2\alpha) \log |x| \right\}$,
- (ii) $\frac{n_\varepsilon}{n_{+, \varepsilon}} \rightarrow 1$, locally uniformly in $\mathcal{B} = \left\{ (x, t) \in \mathbb{R}^d \times (0, \infty) \mid |\lambda_1| t > (d + 2\alpha) \log |x| \right\}$.

The proof of this Theorem follows from an adaptation of the proof of Theorem 2.1.

In section 2, we introduce preliminary results and technical tools. In section 3, after the rescaling, we provide a sub and a super-solution and demonstrate Theorem 2.1. In section 4, we provide the points of the proof of Theorem 2.2 that differ from the proof of Theorem 2.1.

2.2 Preliminary results

We first state a classical result on the fractional heat kernel.

Proposition 2.1. [43] *There exists a positive constant C larger than 1 such that the heat kernel $p_\alpha(x, y, t)$ associated with the operator $\partial_t + L^\alpha$ verifies the following inequalities for $t > 0$:*

$$C^{-1} \times \min(t^{-\frac{d}{2\alpha}}, \frac{t}{|x-y|^{d+2\alpha}}) \leq p_\alpha(x, y, t) \leq C \times \min(t^{-\frac{d}{2\alpha}}, \frac{t}{|x-y|^{d+2\alpha}}). \quad (2.6)$$

The proof of this proposition is given in [43].

Now we use this proposition to demonstrate that beginning with a positive compactly supported initial data leads to a solution with algebraic tails.

Proposition 2.2. *Assuming (H1), then there exist two constants c_m and c_M depending on n_0 , d , α and μ such that :*

$$\frac{c_m}{1 + |x|^{d+2\alpha}} \leq n(x, 1) \leq \frac{c_M}{1 + |x|^{d+2\alpha}}.$$

Proof. First, we define $M := \max(\max n_0, \max |\mu|)$, we easily note that the constant functions 0 and M are respectively sub and super-solution to our problem. Then, thanks to the comparison principle (which is given in [36]), we have the following inequalities, for all $(x, t) \in \mathbb{R}^d \times [0, +\infty[$:

$$0 \leq n(x, t) \leq M.$$

Let \underline{n} and \bar{n} be the solutions of the two following systems :

$$\begin{cases} \partial_t \underline{n} + L^\alpha \underline{n} = -2M \underline{n}, \\ \underline{n}(x, 0) = n_0(x), \end{cases} \quad (2.7)$$

and

$$\begin{cases} \partial_t \bar{n} + L^\alpha \bar{n} = \max |\mu| \bar{n}, \\ \bar{n}(x, 0) = n_0(x). \end{cases} \quad (2.8)$$

Thanks to Proposition 2.1, we can solve (2.7) and find

$$\underline{n}(x, t) = e^{-2Mt} \int_{\mathbb{R}^d} p_\alpha(x, y, t) n_0(y) dy,$$

Thus for any $t > 0$, we obtain

$$\begin{aligned} e^{-2Mt} \int_{\text{supp}(n_0)} C^{-1} \times n_0(y) \min(t^{-\frac{d}{2\alpha}}, \frac{t}{|x-y|^{d+2\alpha}}) dy &\leq \underline{n}(x, t) \\ \Rightarrow e^{-2M} \int_{\text{supp}(n_0)} C^{-1} \times n_0(y) \min(1, \frac{1}{|x-y|^{d+2\alpha}}) dy &\leq \underline{n}(x, 1). \end{aligned}$$

Thanks to the dominated convergence theorem, we have :

$$(1 + |x|^{d+2\alpha}) e^{-2M} \int_{\text{supp}(n_0)} C^{-1} n_0(y) \min(1, \frac{1}{|x-y|^{d+2\alpha}}) dy \xrightarrow{|x| \rightarrow \infty} e^{-2M} \int_{\text{supp}(n_0)} C^{-1} n_0(y) dy.$$

Therefore, we conclude by a compactness argument that for any $x \in \mathbb{R}^d$:

$$\frac{e^{-2M} C^{-1}}{(1 + |x|^{d+2\alpha})} \leq \underline{n}(x, 1), \quad (2.9)$$

where the last C is a new constant depending only on n_0 . Moreover thanks to the comparison principle, we have that for any $t \geq 0$

$$\underline{n}(x, t) \leq n(x, t) \Rightarrow \frac{e^{-2Mt}C^{-1}}{(1 + |x|^{d+2\alpha})} \leq n(x, 1). \quad (2.10)$$

In the same way, we can solve (2.8) and the solution is

$$\bar{n}(x, t) = e^{\max|\mu|t} \int_{\mathbb{R}^d} n_0(y) \times p_\alpha(t, x, y) dy.$$

Using similar arguments, we get that for all $x \in \mathbb{R}^d$,

$$n(x, 1) \leq \bar{n}(x, 1) \leq \frac{Ce^{\max|\mu|t}}{(1 + |x|^{d+2\alpha})}. \quad (2.11)$$

By combining (2.10) and (2.11) together, we finally obtain

$$\frac{c_m}{1 + |x|^{d+2\alpha}} \leq n(x, 1) \leq \frac{c_M}{1 + |x|^{d+2\alpha}}. \quad (2.12)$$

□

We next provide a technical lemma which will be useful all along the article. The main ideas of the proof of the lemma come from [85] by S. Méléard and S. Mirrahimi for Point (i) and [45] by A.C. Coulon Chalmin for Point (ii). To this end, we first introduce the computation of L^α of a product of functions :

$$L^\alpha(fg)(x, t) = f(x, t)L^\alpha g(x, t) + g(x, t)L^\alpha f(x, t) - \widetilde{K}(f, g)(x, t),$$

with,

$$\widetilde{K}(f, g)(x, t) := C' PV \int_{\mathbb{R}^d} \frac{(f(x, t) - f(x + h, t))(g(x, t) - g(x + h, t))}{|h|^{d+2\alpha}} \beta(x, \frac{h}{|h|}) dy.$$

Lemma 2.1. *Let γ be a positive constant such that*

$$\gamma \in \begin{cases} [0, 2\alpha[& \text{if } \alpha < \frac{1}{2} \\]2\alpha - 1, 1[& \text{if } \frac{1}{2} \leq \alpha < 1, \end{cases}$$

$\chi : \mathbb{R} \rightarrow \mathbb{R}^d$ be a C^1 periodic, strictly positive function and $g(x) := \frac{1}{1+|x|^{d+2\alpha}}$. Then there exists a positive constant C , which does not depend on x , such that, for all $x \in \mathbb{R}^d$:

(i) for all $a > 0$,

$$|L^\alpha g(ax)| \leq a^{2\alpha} C g(ax),$$

(ii) for all $a \in]0, 1]$,

$$|\widetilde{K}(g(a \cdot), \chi)(x)| \leq \frac{Ca^{2\alpha-\gamma}}{1 + (a|x|)^{d+2\alpha}} = Ca^{2\alpha-\gamma} g(ax).$$

The proof is given in the Appendix. Note that we will not use the assumption $b \leq \beta$ in the proof of Lemma 2.1 (but only $\beta \leq B$). The assumption $b \leq \beta$ is necessary for Proposition 2.1 and also to ensure the existence and the positiveness of ϕ_1 .

Remark. *If we want to reach the same level of generality as in [103], we just have to adapt the previous Lemma to an operator L^α with a kernel β of the form $\beta(x, y)$ where β is a 1-periodic with respect to x , smooth function from $\mathbb{R}^d \times \mathbb{R}^d$ such that for all $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d$*

$$\beta(x, y) = \beta(x, -y) \quad \text{and} \quad 0 < b \leq \beta(x, y) \leq B,$$

with b and B positive constants. The interested reader can verify that the proof of Lemma 2.1 is robust enough and can easily be adapted to such kernels.

2.3 The proof of Theorem 2.1

In this section we will provide the proof of Theorem 2.1. Let us rewrite (2.1) with respect to the rescaling given by (2.5)

$$\varepsilon \partial_t \left(n \left(|x|^{\frac{1}{\varepsilon}} \frac{x}{|x|}, \frac{t}{\varepsilon} \right) \right) = -L^\alpha(n) \left(|x|^{\frac{1}{\varepsilon}} \frac{x}{|x|}, \frac{t}{\varepsilon} \right) + n \left(|x|^{\frac{1}{\varepsilon}} \frac{x}{|x|}, \frac{t}{\varepsilon} \right) [\mu \left(|x|^{\frac{1}{\varepsilon}} \frac{x}{|x|} \right) - n \left(|x|^{\frac{1}{\varepsilon}} \frac{x}{|x|}, \frac{t}{\varepsilon} \right)]. \quad (2.13)$$

Notation. For any function $v : \mathbb{R}^d \times \mathbb{R}^+ \rightarrow \mathbb{R}$ and $w : \mathbb{R}^d \rightarrow \mathbb{R}$ we denote by v_ε and w_ε the rescaled functions given by :

$$v_\varepsilon(x, t) := v \left(|x|^{\frac{1}{\varepsilon}} \frac{x}{|x|}, \frac{t}{\varepsilon} \right) \quad \text{and} \quad w_\varepsilon(x) = w \left(|x|^{\frac{1}{\varepsilon}-1} x \right).$$

One can write the first term in the right-hand side of (2.13) in terms of n_ε in the following way.

$$\begin{aligned} L^\alpha(n) \left(|x|^{\frac{1}{\varepsilon}-1} x, \frac{t}{\varepsilon} \right) &= -PV \int_{\mathbb{R}^d} \frac{n \left(|x|^{\frac{1}{\varepsilon}-1} x + h, \frac{t}{\varepsilon} \right) - n \left(|x|^{\frac{1}{\varepsilon}-1} x, \frac{t}{\varepsilon} \right)}{|h|^{2\alpha+d}} \times \beta \left(|x|^{\frac{1}{\varepsilon}-1} x, \frac{h}{|h|} \right) dh \\ &= -PV \int_{\mathbb{R}^d} \left(n_\varepsilon \left(\left| |x|^{\frac{1}{\varepsilon}-1} x + h \right|^\varepsilon \frac{\left(|x|^{\frac{1}{\varepsilon}-1} x + h \right)}{\left| |x|^{\frac{1}{\varepsilon}-1} x + h \right|}, t \right) - n_\varepsilon(x, t) \right) \frac{\beta_\varepsilon(x, \frac{h}{|h|}) dh}{|h|^{2\alpha+d}}. \end{aligned}$$

We can hence define :

$$L_\varepsilon^\alpha(n_\varepsilon)(x, t) := L^\alpha(n) \left(|x|^{\frac{1}{\varepsilon}} \frac{x}{|x|}, \frac{t}{\varepsilon} \right),$$

which allows us to write (2.13) as below :

$$\varepsilon \partial_t n_\varepsilon(x, t) = -L_\varepsilon^\alpha n_\varepsilon(x, t) + n_\varepsilon(x, t) [\mu_\varepsilon(x) - n_\varepsilon(x, t)]. \quad (2.14)$$

In the same way we define

$$\widetilde{K}_\varepsilon(n_\varepsilon, \chi_\varepsilon)(x, t) := \widetilde{K}(n, \chi) \left(|x|^{\frac{1}{\varepsilon}-1} x, \frac{t}{\varepsilon} \right).$$

Moreover, according to the inequalities (2.12), we can consider $n(x, 1)$ as our initial data instead of $n(x, 0)$. So we can replace the assumption (H1) by :

$$\frac{C_m}{1 + |x|^{d+2\alpha}} \leq n_0(x) \leq \frac{C_M}{1 + |x|^{d+2\alpha}} \Rightarrow \frac{C_m}{1 + |x|^{\frac{d+2\alpha}{\varepsilon}}} \leq n_{0,\varepsilon}(x) \leq \frac{C_M}{1 + |x|^{\frac{d+2\alpha}{\varepsilon}}}. \quad (\text{H1}')$$

In the next subsection we are going to provide sub and super-solutions to (2.14) which will allow us to demonstrate Theorem 2.1 in a second subsection.

2.3.1 Sub and super-solution to (2.14).

Theorem 2.3. We assume (H2) and (H3) and we choose positive constants $C_m < \frac{|\lambda_1|}{\max \phi_1}$ and $C_M > \frac{|\lambda_1|}{\min \phi_1}$ and δ such that

$$0 < \delta \leq \min \left(\sqrt{C_M \min \phi_1 - |\lambda_1|}, \sqrt{|\lambda_1| - C_m \max \phi_1} \right).$$

Then there exists a positive constant $\varepsilon_0 < \delta$ such that for all $\varepsilon \in]0, \varepsilon_0[$ we have :

$$(i) \quad f_\varepsilon^M(t, x) = \phi_{1,\varepsilon}(x) \times \frac{C_M}{1 + e^{-\frac{t}{\varepsilon} (|\lambda_1| + \varepsilon^2) - \frac{\delta}{\varepsilon} |x|^{\frac{d+2\alpha}{\varepsilon}}}} \quad \text{is a super-solution of (2.14),}$$

(ii) $f_\varepsilon^m(x, t) = \phi_{1,\varepsilon}(x) \times \frac{C_m e^{-\frac{\delta}{\varepsilon}}}{1 + e^{-\frac{t}{\varepsilon}(|\lambda_1| - \varepsilon^2) - \frac{\delta}{\varepsilon}|x|^{\frac{d+2\alpha}{\varepsilon}}}}$ is a sub-solution of (2.14).

(iii) Moreover, if we assume (H1') and $C_m < \frac{c_m}{\max |\phi_1|}$ and $C_M > \frac{c_M}{\min |\phi_1|}$ where c_m and c_M are given by (H1') then for all $(x, t) \in \mathbb{R}^d \times [0, +\infty[$,

$$\phi_{1,\varepsilon}(x) \times \frac{C_m e^{-\frac{\delta}{\varepsilon} - \varepsilon t}}{1 + e^{-\frac{|\lambda_1|t + \delta}{\varepsilon} |x|^{\frac{d+2\alpha}{\varepsilon}}}} \leq n_\varepsilon(x, t) \leq \phi_{1,\varepsilon}(x) \times \frac{C_M e^{\varepsilon t}}{1 + e^{-\frac{|\lambda_1|t + \delta}{\varepsilon} |x|^{\frac{d+2\alpha}{\varepsilon}}}}. \quad (2.15)$$

Proof. Since the proofs of (i) and (ii) follow from similar arguments, we will only provide the proof of (i) and (iii).

Proof of (i). We define :

$$\psi(x, t) := \frac{C_M}{1 + e^{-t(|\lambda_1| + \varepsilon^2) - \frac{\delta}{\varepsilon}|x|^{\frac{d+2\alpha}{\varepsilon}}}}. \quad (2.16)$$

Then, noticing that ϕ_1 is independent of t , we first bound $\partial_t \psi_\varepsilon$ from below,

$$\begin{aligned} \partial_t \psi_\varepsilon(x, t) &= \frac{C_M \frac{(|\lambda_1| + \varepsilon^2)}{\varepsilon} e^{-t \frac{(|\lambda_1| + \varepsilon^2)}{\varepsilon} - \frac{\delta}{\varepsilon}|x|^{\frac{d+2\alpha}{\varepsilon}}}}{(1 + e^{-t \frac{(|\lambda_1| + \varepsilon^2)}{\varepsilon} - \frac{\delta}{\varepsilon}|x|^{\frac{d+2\alpha}{\varepsilon}}})^2} \\ &= \frac{\psi_\varepsilon(x, t)}{\varepsilon} \left[(|\lambda_1| + \varepsilon^2) \frac{e^{-t \frac{(|\lambda_1| + \varepsilon^2)}{\varepsilon} - \frac{\delta}{\varepsilon}|x|^{\frac{d+2\alpha}{\varepsilon}}}}{1 + e^{-t \frac{(|\lambda_1| + \varepsilon^2)}{\varepsilon} - \frac{\delta}{\varepsilon}|x|^{\frac{d+2\alpha}{\varepsilon}}}} \right] \\ &\geq \frac{\psi_\varepsilon(x, t)}{\varepsilon} [|\lambda_1| + \varepsilon^2 - \psi_\varepsilon(x, t) \phi_{1,\varepsilon}(x)]. \end{aligned} \quad (2.17)$$

The last inequality is obtained from the definition of C_M and ε . Actually, for such C_M and ε , we have, for any positive non-null constant A , the following relation :

$$\frac{A(|\lambda_1| + \varepsilon^2)}{1 + A} \geq |\lambda_1| + \varepsilon^2 - \frac{C_M \min \phi_1}{1 + A},$$

because,

$$\begin{aligned} |\lambda_1| + \varepsilon^2 - \frac{C_M \min \phi_1}{1 + A} &= \frac{(1 + A)(|\lambda_1| + \varepsilon^2) - C_M \min \phi_1}{1 + A} \\ &= \frac{A(|\lambda_1| + \varepsilon^2) - (C_M \min \phi_1 - |\lambda_1| - \varepsilon^2)}{1 + A} \\ &\leq \frac{A(|\lambda_1| + \varepsilon^2)}{1 + A}. \end{aligned}$$

We also compute $L_\varepsilon^\alpha(f_\varepsilon^M)(x, t)$ as a fractional Laplacian of a product of functions,

$$L_\varepsilon^\alpha(f_\varepsilon^M)(x, t) = \phi_{1,\varepsilon}(x) L_\varepsilon^\alpha \psi_\varepsilon(x, t) + \psi_\varepsilon(x, t) L_\varepsilon^\alpha \phi_{1,\varepsilon}(x) - \widetilde{K}_\varepsilon(\psi_\varepsilon, \phi_{1,\varepsilon})(x, t) \quad (2.18)$$

with \widetilde{K} given in section 2. Replacing this in equation (2.14) and using the two previous results (2.17) and (2.18), we find :

$$\begin{aligned} &\varepsilon \partial_t f_\varepsilon^M(x, t) + L_\varepsilon^\alpha f_\varepsilon^M(x, t) - f_\varepsilon^M(x, t) [\mu_\varepsilon(x) - f_\varepsilon^M(x, t)] \\ &\geq f_\varepsilon^M(x, t) (|\lambda_1| + \varepsilon^2 - f_\varepsilon^M(x, t)) + \phi_{1,\varepsilon}(x) L_\varepsilon^\alpha \psi_\varepsilon(x, t) + \psi_\varepsilon(x, t) L_\varepsilon^\alpha \phi_{1,\varepsilon}(x) \\ &\quad - \widetilde{K}_\varepsilon(\psi_\varepsilon, \phi_{1,\varepsilon})(x, t) - \mu_\varepsilon(x) f_\varepsilon^M(x, t) + f_\varepsilon^M(x, t)^2 \\ &= \varepsilon^2 f_\varepsilon^M(x, t) + \phi_{1,\varepsilon}(x) L_\varepsilon^\alpha \psi_\varepsilon(x, t) - \widetilde{K}_\varepsilon(\psi_\varepsilon, \phi_{1,\varepsilon})(x, t), \end{aligned}$$

where we have used (2.3) and (H3) for the last equality.

In order to control $L_\varepsilon^\alpha \psi_\varepsilon(x, t)$ and $\widetilde{K}_\varepsilon(\psi_\varepsilon, \phi_{1,\varepsilon})(x, t)$, we are going to use Lemma 2.1. For, $L_\varepsilon^\alpha \psi_\varepsilon(x, t)$, noticing that $\psi_\varepsilon(x, t) = C_M g(e^{\frac{-t(|\lambda_1|+\varepsilon^2)-\delta}{\varepsilon(1+2\alpha)}} |x|^{\frac{1}{\varepsilon}-1} x)$, and thanks to the point (i) of Lemma 2.1, we obtain :

$$-C e^{-2\alpha \frac{t(|\lambda_1|+\varepsilon^2)+\delta}{\varepsilon}} \psi_\varepsilon(t, x) \leq L_\varepsilon^\alpha \psi_\varepsilon(t, x).$$

But, comparing the growths, there exists $\varepsilon_1 > 0$ such that for $\varepsilon < \varepsilon_1$ and for all $t \geq 0$:

$$C_M \times C e^{-2\alpha \frac{t(|\lambda_1|+\varepsilon^2)+\delta}{\varepsilon(1+2\alpha)}} - \frac{\varepsilon^2}{3} \leq 0,$$

hence :

$$-\frac{\varepsilon^2}{3} \psi_\varepsilon(x, t) \leq L_\varepsilon^\alpha \psi_\varepsilon(x, t).$$

Now we deal with $\widetilde{K}_\varepsilon(\psi_\varepsilon, \phi_{1,\varepsilon})(x, t)$ in a similar fashion. Thanks to Lemma 2.1 (ii), we find :

$$\begin{aligned} \widetilde{K}_\varepsilon(\psi_\varepsilon, \phi_{1,\varepsilon})(x, t) &= \widetilde{K}(\psi, \phi_1)\left(|x|^{\frac{1}{\varepsilon}} \frac{x}{|x|}, \frac{t}{\varepsilon}\right) \\ &\leq C e^{-\frac{(2\alpha-\gamma)[t(|\lambda_1|+\varepsilon^2)+\delta]}{\varepsilon}} \psi\left(|x|^{\frac{1}{\varepsilon}} \frac{x}{|x|}, \frac{t}{\varepsilon}\right) \\ &= C e^{-\frac{(2\alpha-\gamma)[t(|\lambda_1|+\varepsilon^2)+\delta]}{\varepsilon}} \psi_\varepsilon(x, t). \end{aligned}$$

Then, noticing that for any choice of α , $2\alpha - \gamma$ is strictly positive, we deduce that there exists $\varepsilon_2 > 0$ such that for all $\varepsilon < \varepsilon_2$:

$$C e^{-(2\alpha-\gamma) \frac{t(|\lambda_1|+\varepsilon^2)+\delta}{\varepsilon}} - \frac{\varepsilon^2 \min \phi_1}{3} \leq 0.$$

We deduce that

$$\widetilde{K}_\varepsilon(\psi_\varepsilon, \phi_{1,\varepsilon})(x, t) \leq \frac{\varepsilon^2}{3} \psi_\varepsilon(x, t) \min \phi_1 \leq \frac{\varepsilon^2}{3} \psi_\varepsilon(x, t) \phi_{1,\varepsilon}(x).$$

We set :

$$\varepsilon_0 = \min(\varepsilon_1, \varepsilon_2).$$

Then, we conclude that for $\varepsilon \leq \varepsilon_0$, we have :

$$\begin{aligned} &\varepsilon \partial_t f_\varepsilon^M(x, t) + L_\varepsilon^\alpha f_\varepsilon^M(x, t) - f_\varepsilon^M(x, t) [\mu_\varepsilon(x) - f_\varepsilon^M(x, t)] \\ &\geq \varepsilon^2 f_\varepsilon^M(x, t) + \phi_{1,\varepsilon}(x) L_\varepsilon^\alpha \psi_\varepsilon(x, t) - \widetilde{K}_\varepsilon(\psi, \phi_1)(x, t) \\ &\geq \varepsilon^2 f_\varepsilon^M(x, t) - \frac{\varepsilon^2}{3} \phi_{1,\varepsilon}(x) \psi_\varepsilon(x, t) - \frac{\varepsilon^2}{3} \phi_{1,\varepsilon}(x) \psi_\varepsilon(x, t) \\ &\geq \frac{\varepsilon^2}{3} f_\varepsilon^M(x, t) \\ &\geq 0. \end{aligned}$$

Therefore f_ε^M is a super-solution of (2.14) and this concludes the proof of the point (i).

Proof of (iii). From (H1'), since $\max |\phi_1| C_m < c_m$ and $c_M < C_M \min |\phi_1|$, we have :

$$f_\varepsilon^m(x, 0) = \frac{\phi_{1,\varepsilon}(x) \times C_m e^{-\frac{\delta}{\varepsilon}}}{1 + e^{-\frac{\delta}{\varepsilon}} |x|^{\frac{d+2\alpha}{\varepsilon}}} = \frac{\phi_{1,\varepsilon}(x) \times C_m}{e^{\frac{\delta}{\varepsilon}} + |x|^{\frac{d+2\alpha}{\varepsilon}}} \leq \frac{c_m}{1 + |x|^{\frac{d+2\alpha}{\varepsilon}}} \leq n_\varepsilon(x, 0) \leq f_\varepsilon^M(x, t).$$

Then, according to the comparison principle, we obtain :

$$\phi_{1,\varepsilon}(x) \times \frac{C_m e^{-\frac{\delta}{\varepsilon}}}{1 + e^{-\frac{t}{\varepsilon}(|\lambda_1| - \varepsilon^2) - \frac{\delta}{\varepsilon}} |x|^{\frac{d+2\alpha}{\varepsilon}}} \leq n_\varepsilon(x, t) \leq \phi_{1,\varepsilon}(x) \times \frac{C_M}{1 + e^{-\frac{t}{\varepsilon}(|\lambda_1| + \varepsilon^2) - \frac{\delta}{\varepsilon}} |x|^{\frac{d+2\alpha}{\varepsilon}}},$$

and hence

$$\phi_{1,\varepsilon}(x) \times \frac{C_m e^{-\frac{\delta}{\varepsilon} - \varepsilon t}}{1 + e^{-\frac{|\lambda_1|t + \delta}{\varepsilon}} |x|^{\frac{d+2\alpha}{\varepsilon}}} \leq n_\varepsilon(x, t) \leq \phi_{1,\varepsilon}(x) \times \frac{C_M e^{\varepsilon t}}{1 + e^{-\frac{|\lambda_1|t + \delta}{\varepsilon}} |x|^{\frac{d+2\alpha}{\varepsilon}}}. \quad (2.15)$$

□

2.3.2 Convergence to the stationary state.

Thanks to the inequalities (2.15), we can now prove Theorem 2.1. To prove this theorem, we are going to follow the ideas of Méléard and Mirrahimi in [85].

Proof. (Proof of Theorem 2.1.) First, we perform a Hopf-Cole transformation

$$u_\varepsilon(x, t) := \varepsilon \log n_\varepsilon(x, t) \quad \text{and} \quad u_{+,\varepsilon}(x) := \varepsilon \log n_{+,\varepsilon}(x). \quad (2.19)$$

Taking the logarithm in (2.15) and multiplying by ε , we find :

$$\begin{aligned} -\varepsilon^2 t + \varepsilon \log C_m \phi_{1,\varepsilon} - \varepsilon \log(1 + e^{-\frac{|\lambda_1|t + \delta}{\varepsilon}} |x|^{\frac{d+2\alpha}{\varepsilon}}) - \delta &\leq u_\varepsilon(x, t) \\ \text{and } u_\varepsilon(x, t) &\leq \varepsilon^2 t + \varepsilon \log C_M \phi_{1,\varepsilon} - \varepsilon \log(1 + e^{-\frac{|\lambda_1|t + \delta}{\varepsilon}} |x|^{\frac{d+2\alpha}{\varepsilon}}). \end{aligned}$$

Define

$$\underline{u}(x, t) = \liminf_{\varepsilon \rightarrow 0} u_\varepsilon(x, t), \quad \bar{u}(x, t) = \limsup_{\varepsilon \rightarrow 0} u_\varepsilon(x, t), \quad \text{for all } (x, t) \in \mathbb{R}^d \times (0, +\infty).$$

Letting $\varepsilon \rightarrow 0$, we obtain

$$\begin{aligned} \min(0, |\lambda_1| t + \delta - (d + 2\alpha) \log |x|) - \delta &\leq \underline{u}(x, t) \\ &\leq \bar{u}(x, t) \leq \min(0, |\lambda_1| t + \delta - (d + 2\alpha) \log |x|). \end{aligned}$$

We then let $\delta \rightarrow 0$ and we obtain

$$u(x, t) := \underline{u}(x, t) = \bar{u}(x, t) = \min(0, |\lambda_1| t - (d + 2\alpha) \log |x|).$$

We deduce that u_ε converges locally uniformly in $\mathbb{R}^d \times [0, +\infty[$ to u since the above limits are locally uniform in ε .

Proof of (i). For any compact set K in \mathcal{A} , there exists a positive constant a such that for all $(x_0, t_0) \in K$, we have $u(x_0, t_0) < -a$. It is thus immediate from (2.19) that n_ε converges uniformly to 0 in $K \subset \mathcal{A}$. This concludes the proof of (i).

Proof of (ii). We divide (2.14) by n_ε and we obtain

$$\partial_t u_\varepsilon + L_\varepsilon^\alpha n_\varepsilon n_\varepsilon^{-1} = \mu_\varepsilon - n_\varepsilon,$$

that we rewrite as below,

$$n_\varepsilon = n_{+, \varepsilon} + (-\partial_t u_\varepsilon - L_\varepsilon^\alpha n_\varepsilon n_\varepsilon^{-1} + \mu_\varepsilon - n_{+, \varepsilon}). \quad (2.20)$$

Step 1 : $\frac{n_\varepsilon(x_0, t_0)}{n_{+, \varepsilon}(x_0)} \geq 1 + o(1)$ in every compact set of \mathcal{B} .

Let K be a compact set of \mathcal{B} and $(x_0, t_0) \in K$. We choose ν a positive constant small enough such that for all $(y, s) \in K$,

$$(d + 2\alpha) \log |y| < |\lambda_1|s - 2\nu \quad \text{and} \quad 2\nu < |\lambda_1|s. \quad (2.21)$$

First, we define

$$\underline{\varphi}(x, t) := \min(0, -(d + 2\alpha) \log |x| + |\lambda_1|t_0 - \nu) - (t - t_0)^2.$$

It is easy to verify that $u - \underline{\varphi}$ achieves a local strict in t and a global in x minimum at (x_0, t_0) . Then, we define

$$\underline{\varphi}_\varepsilon(x, t) := -\varepsilon \log(1 + e^{-\frac{|\lambda_1|t_0 - \nu}{\varepsilon}} |x|^{\frac{d+2\alpha}{\varepsilon}}) - (t - t_0)^2.$$

Thus, $(\underline{\varphi}_\varepsilon)_\varepsilon$ converges locally uniformly to $\underline{\varphi}$. Moreover, since n_+ is periodic and strictly positive, we have that $u_{+, \varepsilon}$ converges to 0, hence $u_\varepsilon - (\underline{\varphi}_\varepsilon + u_{+, \varepsilon}) \xrightarrow{\varepsilon \rightarrow 0} u - \underline{\varphi}$ locally uniformly. Thus, there exists $(x_\varepsilon, t_\varepsilon) \in \mathbb{R}^d \times [0, +\infty[$ such that $(x_\varepsilon, t_\varepsilon)$ is a minimum point (local in t and global in x) of $(u_\varepsilon - \underline{\varphi}_\varepsilon - u_{+, \varepsilon})$ and $(u_\varepsilon - \underline{\varphi}_\varepsilon - u_{+, \varepsilon})(x_\varepsilon, t_\varepsilon) \rightarrow 0$. Since (x_0, t_0) is a strict in t local minimum of $u - \underline{\varphi}$, one can choose t_ε such that $t_\varepsilon \rightarrow t_0$. We deduce that

$$\partial_t u_\varepsilon(x_\varepsilon, t_\varepsilon) = \partial_t \underline{\varphi}_\varepsilon(x_\varepsilon, t_\varepsilon) = -2(t_\varepsilon - t_0) = o(1). \quad (2.22)$$

One should ensure that $(x_\varepsilon)_{\varepsilon \rightarrow 0}$ have all their accumulation points in $\overline{B}(0, e^{\frac{|\lambda_1|t_0 - \nu}{d+2\alpha}})$ as ε tends to 0. This is the case because, at time $t = t_0$, in $\overline{B}(0, e^{\frac{|\lambda_1|t_0 - \nu}{d+2\alpha}})$, $u_\varepsilon - \underline{\varphi}_\varepsilon - u_{+, \varepsilon}$ tends to 0, whereas in $\overline{B}(0, e^{\frac{|\lambda_1|t_0 - \nu}{d+2\alpha}})^c$, $u_\varepsilon - \underline{\varphi}_\varepsilon - u_{+, \varepsilon}$ tends to a strictly positive function.

We deduce that there exists $\varepsilon_1 > 0$ such that for all $\varepsilon < \varepsilon_1$ we have $x_\varepsilon \in \overline{B}(0, e^{\frac{|\lambda_1|t_0 - \frac{\nu}{2}}{d+2\alpha}})$. Then we continue by proving $(-L_\varepsilon^\alpha(n_\varepsilon)n_\varepsilon^{-1} + \mu_\varepsilon - n_{+, \varepsilon})(x_\varepsilon, t_\varepsilon) \geq o(1)$,

$$-L_\varepsilon^\alpha(n_\varepsilon)n_\varepsilon^{-1}(x_\varepsilon, t_\varepsilon) = \int_{\mathbb{R}^d} (e^{\frac{u_\varepsilon \left(\left| |x_\varepsilon|^{\frac{1}{\varepsilon} - 1} x_\varepsilon + h \right|^{\varepsilon - 1} \right)_{(|x_\varepsilon|^{\frac{1}{\varepsilon} - 1} x_\varepsilon + h), t_\varepsilon}}{\varepsilon} - u_\varepsilon(x_\varepsilon, t_\varepsilon)} - 1) \frac{\beta_\varepsilon(x, \frac{h}{|h|}) dh}{|h|^{d+2\alpha}}.$$

From the definition of $(x_\varepsilon, t_\varepsilon)$, we have for all $y \in \mathbb{R}^d$:

$$(u_\varepsilon - \underline{\varphi}_\varepsilon - u_{+, \varepsilon})(x_\varepsilon, t_\varepsilon) \leq (u_\varepsilon - \underline{\varphi}_\varepsilon - u_{+, \varepsilon})(y, t_\varepsilon),$$

and thus

$$(\underline{\varphi}_\varepsilon + u_{+, \varepsilon})(y, t_\varepsilon) - (\underline{\varphi}_\varepsilon + u_{+, \varepsilon})(x_\varepsilon, t_\varepsilon) \leq u_\varepsilon(y, t_\varepsilon) - u_\varepsilon(x_\varepsilon, t_\varepsilon).$$

Therefore, from (2.19) we have

$$-L_\varepsilon^\alpha(e^{\frac{\underline{\varphi}_\varepsilon}{\varepsilon}} n_{+, \varepsilon})(e^{\frac{\underline{\varphi}_\varepsilon}{\varepsilon}} n_{+, \varepsilon})^{-1}(x_\varepsilon, t_\varepsilon) \leq -L_\varepsilon^\alpha(n_\varepsilon)n_\varepsilon^{-1}(x_\varepsilon, t_\varepsilon).$$

Finally, using that $n_{+, \varepsilon}$ is a solution of the stationary equation, we obtain

$$\begin{aligned}
 (-L_\varepsilon^\alpha(n_\varepsilon)n_\varepsilon^{-1} + \mu_\varepsilon - n_{+, \varepsilon})(x_\varepsilon, t_\varepsilon) &\geq (-L_\varepsilon^\alpha(e^{\frac{\varphi_\varepsilon}{\varepsilon}}n_{+, \varepsilon}))((e^{\frac{\varphi_\varepsilon}{\varepsilon}}n_{+, \varepsilon})^{-1} + \mu_\varepsilon - n_{+, \varepsilon})(x_\varepsilon, t_\varepsilon) \\
 &= (-L_\varepsilon^\alpha(e^{\frac{\varphi_\varepsilon}{\varepsilon}})(e^{-\frac{\varphi_\varepsilon}{\varepsilon}}) - L_\varepsilon^\alpha(n_{+, \varepsilon})(n_{+, \varepsilon})^{-1} \\
 &\quad + \widetilde{K}_\varepsilon(e^{\frac{\varphi_\varepsilon}{\varepsilon}}, n_{+, \varepsilon})((e^{\frac{\varphi_\varepsilon}{\varepsilon}}n_{+, \varepsilon})^{-1} + \mu_\varepsilon - n_{+, \varepsilon})(x_\varepsilon, t_\varepsilon) \\
 &= -L_\varepsilon^\alpha(e^{\frac{\varphi_\varepsilon}{\varepsilon}})(e^{-\frac{\varphi_\varepsilon}{\varepsilon}})(x_\varepsilon, t_\varepsilon) \\
 &\quad + \widetilde{K}_\varepsilon(e^{\frac{\varphi_\varepsilon}{\varepsilon}}, n_{+, \varepsilon})(e^{\frac{\varphi_\varepsilon}{\varepsilon}}n_{+, \varepsilon})^{-1}(x_\varepsilon, t_\varepsilon).
 \end{aligned}$$

In order to control the last two terms of the above inequality, we are going to use Lemma 2.1. Note that, we have the following link between $e^{\frac{\varphi_\varepsilon}{\varepsilon}}$ and $g(x) = \frac{1}{1+|x|^{d+2\alpha}}$:

$$e^{\frac{\varphi_\varepsilon}{\varepsilon}}(x, t) = \frac{e^{-\frac{(t-t_0)^2}{\varepsilon}}}{1 + e^{-\frac{|\lambda_1|t_0 - \nu}{\varepsilon}|x|^{\frac{d+2\alpha}{\varepsilon}}}} = e^{-\frac{(t-t_0)^2}{\varepsilon}} g(e^{-\frac{|\lambda_1|t_0 - \nu}{(d+2\alpha)\varepsilon}} |x|^{\frac{1}{\varepsilon}-1} x).$$

And so, we can deduce from Lemma 2.1 that :

$$o(1) = -C e^{-\frac{2\alpha(|\lambda_1|t_0 - \nu)}{(d+2\alpha)\varepsilon}} \leq -L_\varepsilon^\alpha(e^{\frac{\varphi_\varepsilon}{\varepsilon}})(e^{-\frac{\varphi_\varepsilon}{\varepsilon}})(x_\varepsilon, t_\varepsilon),$$

and,

$$o(1) = -C e^{-\frac{(2\alpha-\gamma)(|\lambda_1|t_0 - \nu)}{\varepsilon}} n_{+, \varepsilon}(x_\varepsilon, t_\varepsilon)^{-1} \leq \widetilde{K}_\varepsilon(e^{\frac{\varphi_\varepsilon}{\varepsilon}}, n_{+, \varepsilon})(e^{\frac{\varphi_\varepsilon}{\varepsilon}}n_{+, \varepsilon})^{-1}(x_\varepsilon, t_\varepsilon).$$

We deduce that :

$$o(1) \leq (-L_\varepsilon^\alpha(n_\varepsilon)n_\varepsilon^{-1} + \mu_\varepsilon - n_{+, \varepsilon})(x_\varepsilon, t_\varepsilon). \quad (2.23)$$

Finally, combining the above inequality with (2.20) and (2.22), we obtain that

$$1 + o(1) \leq \frac{n_\varepsilon(x_\varepsilon, t_\varepsilon)}{n_{+, \varepsilon}(x_\varepsilon)}.$$

Now, we want to bring back this inequality at the point (x_0, t_0) . There are two cases :

Case 1 : $|x_\varepsilon| \geq |x_0|$

Because of the definition of $\underline{\varphi}_\varepsilon$, we have :

$$\underline{\varphi}_\varepsilon(x_\varepsilon, t_\varepsilon) \leq \underline{\varphi}_\varepsilon(x_0, t_0).$$

Since $(x_\varepsilon, t_\varepsilon)$ is a minimum point of $u_\varepsilon - (\underline{\varphi}_\varepsilon + u_{\varepsilon,+})$, we deduce that

$$u_\varepsilon(x_\varepsilon, t_\varepsilon) - u_{+, \varepsilon}(x_\varepsilon) \leq u_\varepsilon(x_0, t_0) - u_{+, \varepsilon}(x_0).$$

Thanks to (2.19), it follows that

$$1 + o(1) \leq \frac{n_\varepsilon(x_\varepsilon, t_\varepsilon)}{n_{+, \varepsilon}(x_\varepsilon)} \leq \frac{n_\varepsilon(x_0, t_0)}{n_{+, \varepsilon}(x_0)}.$$

Case 2 : $|x_\varepsilon| < |x_0|$

In this case, since $(x_0, t_0) \in K$ and thanks to (2.21), we have that

$$-|\lambda_1|t_0 + \nu + (d + 2\alpha) \log(|x_\varepsilon|) \leq -|\lambda_1|t_0 + \nu + (d + 2\alpha) \log(|x_0|) \leq -\nu < 0,$$

and thus $e^{-\frac{|\lambda_1|t_0 + \nu + (d+2\alpha)\log(|x_\varepsilon|)}{\varepsilon}} = o(1)$. We deduce that

$$e^{-\frac{\varphi_\varepsilon(x_\varepsilon, t_\varepsilon)}{\varepsilon}} = e^{\frac{(t_\varepsilon - t_0)^2}{\varepsilon}} \left(1 + e^{-\frac{|\lambda_1|t_0 + \nu + (d+2\alpha)\log(|x_\varepsilon|)}{\varepsilon}}\right) \geq 1.$$

Moreover, following similar computations, we obtain that $e^{\frac{\varphi_\varepsilon(x_0, t_0)}{\varepsilon}} = 1 + o(1)$. Hence, from the definition of $(x_\varepsilon, t_\varepsilon)$, we get

$$u_\varepsilon(x_\varepsilon, t_\varepsilon) - u_{+, \varepsilon}(x_\varepsilon) + \underline{\varphi}_\varepsilon(x_0, t_0) - \underline{\varphi}_\varepsilon(x_\varepsilon, t_\varepsilon) \leq u_\varepsilon(x_0, t_0) - u_{+, \varepsilon}(x_0).$$

Thanks to (2.19), we obtain that

$$1 + o(1) \leq \frac{n_\varepsilon(x_\varepsilon, t_\varepsilon)}{n_{+, \varepsilon}(x_\varepsilon)} \times \frac{e^{\frac{\varphi_\varepsilon(x_0, t_0)}{\varepsilon}}}{e^{\frac{\varphi_\varepsilon(x_\varepsilon, t_\varepsilon)}{\varepsilon}}} \leq \frac{n_\varepsilon(x_0, t_0)}{n_{+, \varepsilon}(x_0)}.$$

So we have proved, in all cases

$$1 + o(1) \leq \frac{n_\varepsilon(x_0, t_0)}{n_{+, \varepsilon}(x_0)}.$$

Step 2 : $\frac{n_\varepsilon(x_0, t_0)}{n_{+, \varepsilon}(x_0)} \leq 1 + o(1)$ in every compact set of \mathcal{B} .

This step is very similar to the first one.

We pick $(x_0, t_0) \in \mathcal{B}$ and let ν be a positive constant. As before, we define

$$\bar{\varphi}(x, t) := \min(0, |\lambda_1|t_0 + \nu - (d + 2\alpha)\log|x|) + (t - t_0)^2.$$

It is easy to verify that $u - \bar{\varphi}$ achieves a local and strict in t and a global in x maximum at (x_0, t_0) . Then, defining

$$\bar{\varphi}_\varepsilon(x, t) := -\varepsilon \log\left(1 + e^{-\frac{|\lambda_1|t_0 + \nu}{\varepsilon}} |x|^{\frac{d+2\alpha}{\varepsilon}}\right) + (t - t_0)^2,$$

we have that $(\bar{\varphi}_\varepsilon)_\varepsilon$ converges locally uniformly to $\bar{\varphi}$. Moreover, we know that $u_{+, \varepsilon}$ tends to 0 and so $u_\varepsilon - (\bar{\varphi}_\varepsilon + u_{+, \varepsilon}) \xrightarrow{\varepsilon \rightarrow 0} 0$ uniformly in \mathcal{B} . Thus, there exists $(x_\varepsilon, t_\varepsilon) \in \mathbb{R}^d \times [0, +\infty[$ such that $(x_\varepsilon, t_\varepsilon)$ is a maximum point, global in x and local in t , of $(u_\varepsilon - \bar{\varphi}_\varepsilon - \varepsilon u_{+, \varepsilon})$ and

$$(u_\varepsilon - \bar{\varphi}_\varepsilon - \varepsilon u_{+, \varepsilon})(x_\varepsilon, t_\varepsilon) \rightarrow 0. \quad (2.24)$$

Since (x_0, t_0) is a strict in t local maximum of $u - \bar{\varphi}$, one can choose t_ε such that $t_\varepsilon \rightarrow t_0$. We deduce that

$$\partial_t u_\varepsilon(x_\varepsilon, t_\varepsilon) = \partial_t \bar{\varphi}_\varepsilon(x_\varepsilon, t_\varepsilon) = 2(t_\varepsilon - t_0) = o(1). \quad (2.25)$$

One should ensure that $(x_\varepsilon)_{\varepsilon \rightarrow 0}$ have all their accumulation points in $\overline{B}(0, e^{\frac{|\lambda_1|t_0 + \frac{\nu}{4}}{d+2\alpha}})$. This is the case because for ε small enough, in $\overline{B}(0, e^{\frac{|\lambda_1|t_0}{d+2\alpha}})$, $u_\varepsilon - \bar{\varphi}_\varepsilon - \varepsilon u_{+, \varepsilon}$ tends to 0 whereas in $\overline{B}(0, e^{\frac{|\lambda_1|t_0 + \frac{\nu}{4}}{d+2\alpha}})^c$, $u_\varepsilon - \bar{\varphi}_\varepsilon - \varepsilon u_{+, \varepsilon}$ is lower than a strictly negative function.

We deduce that there exists $\varepsilon_2 > 0$ such that for all $\varepsilon < \varepsilon_2$ we have

$$x_\varepsilon \in \overline{B}(0, e^{\frac{|\lambda_1|t_0 + \frac{\nu}{4}}{d+2\alpha}}). \quad (2.26)$$

Then we continue by showing $(-L_\varepsilon^\alpha(n_\varepsilon)n_\varepsilon^{-1} + \mu_\varepsilon - n_{+, \varepsilon}) \leq o(1)$. With similar computations as in step 1, we obtain :

$$(-L_\varepsilon^\alpha(n_\varepsilon)n_\varepsilon^{-1} + \mu_\varepsilon - n_{+, \varepsilon})(x_\varepsilon, t_\varepsilon) \leq -L_\varepsilon^\alpha\left(e^{\frac{\bar{\varphi}_\varepsilon}{\varepsilon}}\right)\left(e^{-\frac{\bar{\varphi}_\varepsilon}{\varepsilon}}\right)(x_\varepsilon, t_\varepsilon)$$

$$+ \widetilde{K}_\varepsilon(e^{\frac{\overline{\varphi}_\varepsilon}{\varepsilon}}, n_{+, \varepsilon})(e^{\frac{\overline{\varphi}_\varepsilon}{\varepsilon}} n_{+, \varepsilon})^{-1}(x_\varepsilon, t_\varepsilon).$$

Since

$$e^{\frac{\overline{\varphi}_\varepsilon}{\varepsilon}}(x, t) = \frac{e^{\frac{(t-t_0)^2}{\varepsilon}}}{1 + e^{-\frac{|\lambda_1|t_0 + \nu}{\varepsilon}|x|^{\frac{d+2\alpha}{\varepsilon}}}} = e^{\frac{(t-t_0)^2}{\varepsilon}} g(e^{-\frac{|\lambda_1|t_0 + \nu}{\varepsilon(d+2\alpha)}} |x|^{\frac{1}{\varepsilon}-1} x),$$

we can deduce thanks to Lemma 2.1 that :

$$(-L_\varepsilon^\alpha(e^{\frac{\overline{\varphi}_\varepsilon}{\varepsilon}})(e^{-\frac{\overline{\varphi}_\varepsilon}{\varepsilon}})(x_\varepsilon, t_\varepsilon) \leq C e^{-\frac{2\alpha(|\lambda_1|t_0 + \nu)}{(d+2\alpha)\varepsilon}} = o(1),$$

and,

$$\widetilde{K}_\varepsilon(e^{\frac{\overline{\varphi}_\varepsilon}{\varepsilon}}, n_{+, \varepsilon})(e^{\frac{\overline{\varphi}_\varepsilon}{\varepsilon}} n_{+, \varepsilon})^{-1}(x_\varepsilon, t_\varepsilon) \leq C e^{-\frac{(2\alpha-\gamma)(|\lambda_1|t_0 + \nu)}{\varepsilon(d+2\alpha)}} n_{+, \varepsilon}(x_\varepsilon, t_\varepsilon)^{-1} = o(1).$$

Finally, combining the two previous inequalities and (2.25) in (2.20) we have obtained

$$\frac{n_\varepsilon(x_\varepsilon, t_\varepsilon)}{n_{+, \varepsilon}(x_\varepsilon)} \leq 1 + o(1).$$

Then, there are two cases to bring it back to the point (x_0, t_0) :

Case 1 : $|x_\varepsilon| \leq |x_0|$ By definition of $\overline{\varphi}_\varepsilon$, we have :

$$\overline{\varphi}_\varepsilon(x_0, t_0) \leq \overline{\varphi}_\varepsilon(x_\varepsilon, t_\varepsilon).$$

Since $(x_\varepsilon, t_\varepsilon)$ is a maximum point of $u_\varepsilon - (\overline{\varphi}_\varepsilon + u_{\varepsilon,+})$, we deduce that

$$u_\varepsilon(x_0, t_0) - u_{+, \varepsilon}(x_0) \leq u_\varepsilon(x_\varepsilon, t_\varepsilon) - u_{+, \varepsilon}(x_\varepsilon).$$

Thanks to (2.19), it follows that,

$$\frac{n_\varepsilon(x_0, t_0)}{n_{+, \varepsilon}(x_0)} \leq \frac{n_\varepsilon(x_\varepsilon, t_\varepsilon)}{n_{+, \varepsilon}(x_\varepsilon)} \leq 1 + o(1).$$

Case 2 : $|x_\varepsilon| > |x_0|$ Thanks to (2.26), there exists ε_2 such that for all positive $\varepsilon < \varepsilon_2$ there holds

$$|x_\varepsilon| \leq e^{\frac{|\lambda_1|t_0 + \frac{\nu}{2}}{\varepsilon}} \Rightarrow -|\lambda_1|t_0 - \nu + (d+2\alpha) \log |x_\varepsilon| < -\frac{\nu}{2}.$$

And thus,

$$e^{-\frac{\overline{\varphi}_\varepsilon(x_\varepsilon, t_\varepsilon)}{\varepsilon}} = e^{-\frac{(t_\varepsilon - t_0)^2}{\varepsilon}} (1 + e^{\frac{-|\lambda_1|t_0 - \nu + (d+2\alpha) \log(|x_\varepsilon|)}{\varepsilon}}) \leq 1 + e^{-\frac{\nu}{2\varepsilon}} \leq 1 + o(1).$$

Moreover, we know by definition that $e^{\frac{\overline{\varphi}_\varepsilon(x_0, t_0)}{\varepsilon}} = 1 + o(1)$.

Furthermore, by definition of $(x_\varepsilon, t_\varepsilon)$, we have

$$u_\varepsilon(x_0, t_0) - u_{+, \varepsilon}(x_0) \leq u_\varepsilon(x_\varepsilon, t_\varepsilon) - u_{+, \varepsilon}(x_\varepsilon) + \overline{\varphi}_\varepsilon(x_0, t_0) - \overline{\varphi}_\varepsilon(x_\varepsilon, t_\varepsilon),$$

Combining the above inequalities and thanks to (2.19) we obtain that

$$\frac{n_\varepsilon(x_0, t_0)}{n_{+, \varepsilon}(x_0)} \leq \frac{n_\varepsilon(x_\varepsilon, t_\varepsilon)}{n_{+, \varepsilon}(x_\varepsilon)} \times \frac{e^{\frac{\overline{\varphi}_\varepsilon(x_0, t_0)}{\varepsilon}}}{e^{\frac{\overline{\varphi}_\varepsilon(x_\varepsilon, t_\varepsilon)}{\varepsilon}}} \leq 1 + o(1).$$

So we have proved, in all cases

$$\frac{n_\varepsilon(x_0, t_0)}{n_{+, \varepsilon}(x_0)} \leq 1 + o(1).$$

Passing up to the limit, we finally obtain the result of (ii). □

2.4 Generalization to KPP-type reaction terms

We can generalize our result to a model with a reaction term $F(x, s)$ which verifies the Fisher KPP assumptions given by (H4).

Example 2.1. Obviously we can take as before

$$F(x, s) = \mu(x)s - s^2.$$

Example 2.2. We can generalize it to the classical example :

$$F(x, s) = \mu(x)s - \omega(x)s^2.$$

Where μ is a continuous periodic function, and ω is a continuous periodic strictly positive function.

Of course, we keep the main idea of the previous proof : the rescaling (2.5). So the equation (2.4) becomes :

$$\varepsilon \partial_t n_\varepsilon(x, t) = -L_\varepsilon^\alpha n_\varepsilon(x, t) + F_\varepsilon(x, n_\varepsilon(x, t)). \quad (2.27)$$

As before, according to the comparison principle, the point (ii) and (iii) of (H4) imply

$$\mu(x)n - \underline{c}n^2 \leq F(x, n) \leq \mu(x)n - \overline{C}n^2. \quad (2.28)$$

If we associate this result with (2.6), one can still obtain that the solution will have algebraic tails at time $t = 1$ and hence one can replace the assumption (H1) by (H1') :

$$\frac{c_m}{1 + |x|^{\frac{d+2\alpha}{\varepsilon}}} \leq n_{0,\varepsilon}(x) \leq \frac{C_M}{1 + |x|^{\frac{d+2\alpha}{\varepsilon}}}. \quad (H1')$$

Therefore, we still have the same sub and super-solutions :

Theorem 2.4. We assume (H2), (H3) and (H4) and if we choose $C_m < \frac{|\lambda_1|}{\underline{c} \max \phi_1}$ and $C_M > \frac{|\lambda_1|}{\overline{C} \min \phi_1}$ where \underline{c} and \overline{C} are given by the assumptions (iii) of (H4) and a positive constant δ such that

$$0 < \delta \leq \min(\sqrt{\overline{C}C_M \min \phi_1 - |\lambda_1|}, \sqrt{|\lambda_1| - \underline{c}C_m \max \phi_1});$$

then there exists a positive constant $\varepsilon_0 < \delta$ such that for all $\varepsilon \in]0, \varepsilon_0[$ we have :

(i) $f_\varepsilon^M(x, t) = \phi_{1,\varepsilon}(x) \times \frac{C_M}{1 + e^{-\frac{t}{\varepsilon}(|\lambda_1| + \varepsilon^2) - \frac{\delta}{\varepsilon}|x|^{\frac{d+2\alpha}{\varepsilon}}}}$ is a super-solution of (2.27),

(ii) $f_\varepsilon^m(x, t) = \phi_{1,\varepsilon}(x) \times \frac{C_m e^{-\frac{\delta}{\varepsilon}}}{1 + e^{-\frac{t}{\varepsilon}(|\lambda_1| - \varepsilon^2) - \frac{\delta}{\varepsilon}|x|^{\frac{d+2\alpha}{\varepsilon}}}}$ is a sub-solution of (2.27).

(iii) Moreover, if we assume (H1') and $C_m < \frac{c_m}{\max |\phi_1|}$ and $C_M > \frac{C_M}{\min |\phi_1|}$ with c_m and C_M given by (H1') then for all $(x, t) \in \mathbb{R}^d \times [0, +\infty[$,

$$\phi_{1,\varepsilon}(x) \times \frac{C_m e^{-\frac{\delta}{\varepsilon} - \varepsilon t}}{1 + e^{-\frac{|\lambda_1|t + \delta}{\varepsilon}|x|^{\frac{d+2\alpha}{\varepsilon}}}} \leq n_\varepsilon(x, t) \leq \phi_{1,\varepsilon}(x) \times \frac{C_M e^{\varepsilon t}}{1 + e^{-\frac{|\lambda_1|t + \delta}{\varepsilon}|x|^{\frac{d+2\alpha}{\varepsilon}}}}. \quad (2.29)$$

Proof. Here is the main step of the proof of the point (i). As in the proof of Theorem 2.3, we put :

$$f_\varepsilon^M(t, x) = \phi_{1,\varepsilon}(x) \times \psi_\varepsilon(x, t),$$

where ψ_ε is given by (2.16), but with a constant C_M given in Theorem 2.4. Then, with similar computations as before, we find :

$$\partial_t f_\varepsilon^M \geq \frac{f_\varepsilon^M}{\varepsilon} (|\lambda_1| + \varepsilon^2 - \overline{C} f_\varepsilon^M).$$

Therefore, using (2.28) and Lemma 2.1, we get :

$$\begin{aligned} & \varepsilon \partial_t f_\varepsilon^M(x, t) + L_\varepsilon^\alpha f_\varepsilon^M(x, t) - F_\varepsilon(x, f_\varepsilon(x, t)) \\ & \geq f_\varepsilon^M(x, t) (|\lambda_1| + \varepsilon^2 - \overline{C} f_\varepsilon^M(x, t)) + \phi_{1,\varepsilon}(x) L_\varepsilon^\alpha \psi_\varepsilon(x, t) + \psi_\varepsilon(x, t) L_\varepsilon^\alpha \phi_{1,\varepsilon}(x) \\ & \quad - \widetilde{K}_\varepsilon(\psi, \phi_1)(x, t) - \mu_\varepsilon(x) f_\varepsilon^M(x, t) + \overline{C} f_\varepsilon^M(x, t)^2 \\ & \geq \varepsilon^2 f_\varepsilon^M(x, t) - \frac{\varepsilon^2}{3} f_\varepsilon^M(x, t) - \frac{\varepsilon^2}{3} f_\varepsilon^M(x, t) + \psi_\varepsilon(x, t) [L_\varepsilon^\alpha \phi_{1,\varepsilon}(x) - (\lambda_1 + \mu_\varepsilon(x)) \phi_{1,\varepsilon}(x)] \\ & \geq 0. \end{aligned}$$

Thus, we have demonstrated the point (i). The proof of the point (ii) follows similar arguments. We do not give the proof of the point (iii) because this is similar to the proof of (iii) of Theorem 2.3 : the main argument is the comparison principle. \square

Thus, we can perform the Hopf-Cole transformation (2.19) and we obtain that u_ε converges locally uniformly to :

$$u(x, t) = \min(0, |\lambda_1| t - (d + 2\alpha) \log |x|).$$

Therefore the part (i) of Theorem 2.2 can be proved following similar arguments as in the proof of (i) of Theorem 2.1. The proof of (ii) changes a little bit so we are going to provide the demonstration.

Proof. (Proof of (ii) of Theorem 2.2.) Dividing by n_ε in (2.14), we obtain

$$\partial_t u_\varepsilon + L_\varepsilon^\alpha n_\varepsilon n_\varepsilon^{-1} = \frac{F_\varepsilon(x, n_\varepsilon)}{n_\varepsilon}. \quad (2.30)$$

Step 1 : $\frac{n_\varepsilon(x_0, t_0)}{n_{+,\varepsilon}(x_0)} \geq 1 + o(1)$ in every compact set of \mathcal{B} .

The main difference with the proof of Theorem 2.1 is that from (2.30), we do not obtain directly $\frac{n_\varepsilon(x_\varepsilon, t_\varepsilon)}{n_{+,\varepsilon}(x_\varepsilon)} \geq 1 + o(1)$ but we deduce $\frac{F_\varepsilon(x_\varepsilon, n_{+,\varepsilon})}{n_{+,\varepsilon}}(x_\varepsilon, t_\varepsilon) - \frac{F_\varepsilon(x_\varepsilon, n_\varepsilon)}{n_\varepsilon}(x_\varepsilon, t_\varepsilon) \geq o(1)$.

Let K be a compact set of \mathcal{B} and $(x_0, t_0) \in K$. We choose ν a positive constant small enough such that for all $(y, s) \in K$,

$$(d + 2\alpha) \log |y| < |\lambda_1| s - 2\nu \quad \text{and} \quad 2\nu < |\lambda_1| s. \quad (2.31)$$

First, we define

$$\underline{\varphi}(t, x) := \min(0, -(d + 2\alpha) \log |x| + |\lambda_1| t_0 - \nu) - (t - t_0)^2.$$

It is easy to verify that $u - \underline{\varphi}$ achieves a local strict in t and a global in x minimum at (x_0, t_0) . Then, we define

$$\underline{\varphi}_\varepsilon(x, t) := -\varepsilon \log(1 + e^{-\frac{|\lambda_1| t_0 - \nu}{\varepsilon}} |x|^{\frac{d+2\alpha}{\varepsilon}}) - (t - t_0)^2.$$

Thus, $(\underline{\varphi}_\varepsilon)_\varepsilon$ converges locally uniformly to $\underline{\varphi}$. We know that $u_{+,\varepsilon}$ tends to 0 and so $u_\varepsilon - (\underline{\varphi}_\varepsilon + u_{+,\varepsilon}) \xrightarrow{\varepsilon \rightarrow 0} u - \underline{\varphi}$ locally uniformly. Thus, there exists $(x_\varepsilon, t_\varepsilon) \in \mathcal{B}$ such that $(x_\varepsilon, t_\varepsilon)$ is a

minimum point of $(u_\varepsilon - \underline{\varphi}_\varepsilon - u_{+, \varepsilon})$ and $(u_\varepsilon - \underline{\varphi}_\varepsilon - u_{+, \varepsilon})(x_\varepsilon, t_\varepsilon) \rightarrow 0$. Since (x_0, t_0) is a strict local minimum of $u - \underline{\varphi}$ in t , we can choose t_ε such that $t_\varepsilon \rightarrow t_0$. Then

$$\partial_t u_\varepsilon(x_\varepsilon, t_\varepsilon) = \partial_t \underline{\varphi}_\varepsilon(x_\varepsilon, t_\varepsilon) = -2(t_\varepsilon - t_0) = o(1). \quad (2.32)$$

With the same reasoning as in the proof of Theorem 2.1, we get that there exists $\varepsilon_1 > 0$ such that for $\varepsilon < \varepsilon_1$, $x_\varepsilon \in \overline{B}(0, e^{\frac{|\lambda_1|t_0 - \frac{\nu}{2}}{d+2\alpha}})$.

Then we continue by proving

$$(-L_\varepsilon^\alpha(n_\varepsilon)n_\varepsilon^{-1} + \frac{F_\varepsilon(x, n_\varepsilon)}{n_\varepsilon})(x_\varepsilon, t_\varepsilon) \geq \left(\frac{F_\varepsilon(x, n_\varepsilon)}{n_\varepsilon} - \frac{F_\varepsilon(x_\varepsilon, n_{+, \varepsilon})}{n_{+, \varepsilon}} \right)(x_\varepsilon, t_\varepsilon) + o(1).$$

We know that

$$-L_\varepsilon^\alpha(n_\varepsilon)n_\varepsilon^{-1}(x_\varepsilon, t_\varepsilon) = \int_{\mathbb{R}^d} \left(e^{\frac{u_\varepsilon \left(\left| |x_\varepsilon|^{\frac{1}{\varepsilon}-1} x_\varepsilon + h \right|^{\varepsilon-1} \right)_{(|x_\varepsilon|^{\frac{1}{\varepsilon}-1} x_\varepsilon + h), t_\varepsilon} - u_\varepsilon(x_\varepsilon, t_\varepsilon)}{\varepsilon}} - 1 \right) \frac{\beta_\varepsilon(x_\varepsilon, \frac{h}{|h|}) dh}{|h|^{d+2\alpha}}.$$

Note that, from the definition of $(x_\varepsilon, t_\varepsilon)$, we have for all $y \in \mathbb{R}^d$:

$$(u_\varepsilon - \underline{\varphi}_\varepsilon - u_{+, \varepsilon})(x_\varepsilon, t_\varepsilon) \leq (u_\varepsilon - \underline{\varphi}_\varepsilon - u_{+, \varepsilon})(y, t_\varepsilon),$$

and thus by (2.19)

$$-L_\varepsilon^\alpha(e^{\frac{\varphi_\varepsilon}{\varepsilon}} n_{+, \varepsilon})(e^{\frac{\varphi_\varepsilon}{\varepsilon}} n_{+, \varepsilon})^{-1}(x_\varepsilon, t_\varepsilon) \leq -L_\varepsilon^\alpha(n_\varepsilon)n_\varepsilon^{-1}(x_\varepsilon, t_\varepsilon).$$

Finally, we obtain

$$\begin{aligned} (-L_\varepsilon^\alpha(n_\varepsilon)n_\varepsilon^{-1} + \frac{F_\varepsilon(x, n_\varepsilon)}{n_\varepsilon})(x_\varepsilon, t_\varepsilon) &\geq (-L_\varepsilon^\alpha(e^{\frac{\varphi_\varepsilon}{\varepsilon}} n_{+, \varepsilon}))((e^{\frac{\varphi_\varepsilon}{\varepsilon}} n_{+, \varepsilon})^{-1} + \frac{F_\varepsilon(x, n_\varepsilon)}{n_\varepsilon})(x_\varepsilon, t_\varepsilon) \\ &\geq (-L_\varepsilon^\alpha(e^{\frac{\varphi_\varepsilon}{\varepsilon}}))(e^{-\frac{\varphi_\varepsilon}{\varepsilon}}) - \frac{F_\varepsilon(x, n_{+, \varepsilon})}{n_{+, \varepsilon}} \\ &\quad + \widetilde{K}_\varepsilon(e^{\frac{\varphi_\varepsilon}{\varepsilon}}, n_{+, \varepsilon})((e^{\frac{\varphi_\varepsilon}{\varepsilon}} n_{+, \varepsilon})^{-1} + \frac{F_\varepsilon(x, n_\varepsilon)}{n_\varepsilon})(x_\varepsilon, t_\varepsilon) \\ &\geq (o(1) + \frac{F_\varepsilon(x, n_\varepsilon)}{n_\varepsilon} - \frac{F_\varepsilon(x, n_{+, \varepsilon})}{n_{+, \varepsilon}})(x_\varepsilon, t_\varepsilon). \end{aligned} \quad (2.33)$$

We have to note that thanks to Lemma 2.1, in the last inequality, we have controlled the terms:

$$o(1) \leq \widetilde{K}_\varepsilon(e^{\frac{\varphi_\varepsilon}{\varepsilon}}, n_{+, \varepsilon})(e^{\frac{\varphi_\varepsilon}{\varepsilon}} n_{+, \varepsilon})^{-1}(x_\varepsilon, t_\varepsilon) \quad \text{and} \quad o(1) \leq -L_\varepsilon^\alpha(e^{\frac{\varphi_\varepsilon}{\varepsilon}})(e^{-\frac{\varphi_\varepsilon}{\varepsilon}})(x_\varepsilon, t_\varepsilon).$$

Finally combining (2.32) and (2.33), we obtain

$$o(1) \leq \frac{F_\varepsilon(x_\varepsilon, n_{+, \varepsilon})}{n_{+, \varepsilon}}(x_\varepsilon, t_\varepsilon) - \frac{F_\varepsilon(x_\varepsilon, n_\varepsilon)}{n_\varepsilon}(x_\varepsilon, t_\varepsilon). \quad (2.34)$$

We are going to prove by contradiction that (2.34) implies $o(1) + n_{+, \varepsilon}(x_\varepsilon) \leq n_\varepsilon(x_\varepsilon, t_\varepsilon)$. Let's suppose that there exists a subsequence $(\varepsilon_k)_{k \in \mathbb{N}}$ and a positive constant C such that

$$n_{\varepsilon_k}(x_{\varepsilon_k}, t_{\varepsilon_k}) + C < n_{+, \varepsilon_k}(x_{\varepsilon_k}).$$

Then, thanks to the strict monotony of the function $s \mapsto \frac{F(x,s)}{s}$ (assumption (iv) in (H4)) and the mean value Theorem, there exists a sequence y_k such that

$$\begin{aligned} o_{\varepsilon_k}(1) &\leq \frac{F(x_{\varepsilon_k}, n_{+, \varepsilon_k})}{n_{+, \varepsilon_k}}(x_{\varepsilon_k}, t_{\varepsilon_k}) - \frac{F(x_{\varepsilon_k}, n_{\varepsilon_k})}{n_{\varepsilon_k}}(x_{\varepsilon_k}, t_{\varepsilon_k}) \\ &= \partial_s \left(\frac{F(x_{\varepsilon_k}, s)}{s} \right) (y_{\varepsilon_k}) (n_{+, \varepsilon_k} - n_{\varepsilon_k})(x_{\varepsilon_k}, t_{\varepsilon_k}) \\ &\leq -\bar{C} \times C. \end{aligned}$$

This is a contradiction. Therefore, for ε small enough,

$$n_{+, \varepsilon}(x_\varepsilon) + o(1) \leq n_\varepsilon(x_\varepsilon, t_\varepsilon) \Rightarrow 1 + o(1) \leq \frac{n_\varepsilon(x_\varepsilon, t_\varepsilon)}{n_{+, \varepsilon}(x_\varepsilon)}.$$

To bring back this inequality at the point (x_0, t_0) , we use exactly the same arguments as for the proof of Theorem 2.1 by considering a disjunction of cases $|x_\varepsilon| < |x_0|$ and $|x_0| < |x_\varepsilon|$. We do not provide the details of this disjunctions of cases since they are the same. So we have proved, in all cases

$$1 + o(1) \leq \frac{n_\varepsilon(x_0, t_0)}{n_{+, \varepsilon}(x_0)}.$$

The second step can also be proved following similar arguments as in the previous step, thus we do not provide the demonstration. \square

Appendix - Proof of Lemma 2.1

All along the appendix, we will denote by C positive constants that can change from line to line.

Proof of Lemma 2.1. Proof of (i). We are going to follow the appendix A of [85].

Let $\delta < \frac{1}{2}$ be a positive constant. By a compactness argument, we only have to prove it for $|x| > 1$. We compute

$$\begin{aligned} \left| \frac{L^\alpha(g)(x)}{g(x)} \right| &= \left| \int_{\mathbb{R}^d} \left(\frac{1 + |x|^{d+2\alpha}}{1 + |x+h|^{d+2\alpha}} - 1 \right) \frac{\beta(x, \frac{h}{|h|}) dh}{|h|^{d+2\alpha}} \right| \\ &\leq \int_{\mathbb{R}^d \setminus [B(-x, \delta|x|) \cup B(0, \delta)]} \left| \frac{1 + |x|^{d+2\alpha}}{1 + |x+h|^{d+2\alpha}} - 1 \right| \frac{\beta(x, \frac{h}{|h|}) dh}{|h|^{d+2\alpha}} \\ &\quad + \int_{B(-x, \delta|x|) \setminus B(0, \delta)} \left| \frac{1 + |x|^{d+2\alpha}}{1 + |x+h|^{d+2\alpha}} - 1 \right| \frac{\beta(x, \frac{h}{|h|}) dh}{|h|^{d+2\alpha}} \\ &\quad + \int_{B(0, \delta)} \left| \frac{1 + |x|^{d+2\alpha}}{1 + |x+h|^{d+2\alpha}} - 1 \right| \frac{\beta(x, \frac{h}{|h|}) dh}{|h|^{d+2\alpha}} \\ &= I_1 + I_2 + I_3. \end{aligned}$$

Let us begin by approximating I_1 .

$$I_1 = \int_{\mathbb{R}^d \setminus [B(-x, \delta|x|) \cup B(0, \delta)]} \left| \frac{1 + |x|^{d+2\alpha}}{1 + |x+h|^{d+2\alpha}} - 1 \right| \frac{\beta(x, \frac{h}{|h|}) dh}{|h|^{d+2\alpha}}$$

$$\begin{aligned}
 &\leq \int_{\mathbb{R}^d \setminus [B(-x, \delta|x|) \cup B(0, \delta)]} \left| \frac{C}{\delta^{d+2\alpha}} - 1 \right| \frac{\beta(x, \frac{h}{|h|}) dh}{|h|^{d+2\alpha}} \\
 &\leq \frac{(C+1)}{\delta^{d+2\alpha}} B \operatorname{mes}(\mathbb{S}^{d-1}) \int_{\delta}^{+\infty} \frac{dh}{|h|^{1+2\alpha}} = \frac{C}{\delta^{d+4\alpha}}.
 \end{aligned}$$

For I_2 , we write :

$$\begin{aligned}
 I_2 &= \int_{B(-x, \delta|x|) \setminus B(0, \delta)} \left| \frac{1 + |x|^{d+2\alpha}}{1 + |x+h|^{d+2\alpha}} - 1 \right| \frac{\beta(x, \frac{h}{|h|}) dh}{|h|^{d+2\alpha}} \\
 &\leq B \int_{B(-x, \delta|x|) \setminus B(0, \delta)} \left| \frac{1 + |x|^{d+2\alpha}}{1 + |x+h|^{d+2\alpha}} - 1 \right| \frac{dh}{|h|^{d+2\alpha}} \\
 &\leq B \int_{B(-x, \delta|x|) \setminus B(0, \delta)} \frac{||x|^{d+2\alpha} - |x+h|^{d+2\alpha}|}{1 + |x+h|^{d+2\alpha}} \frac{dh}{|h|^{d+2\alpha}} \\
 &\leq B \int_{B(-x, \delta|x|) \setminus B(0, \delta)} \frac{|x|^{d+2\alpha} + |x+h|^{d+2\alpha}}{1 + |x+h|^{d+2\alpha}} \frac{dh}{|h|^{d+2\alpha}} \\
 &\leq B \int_{B(-x, \delta|x|) \setminus B(0, \delta)} \frac{|x|^{d+2\alpha} + |\delta x|^{d+2\alpha}}{1 + |x+h|^{d+2\alpha}} \frac{dh}{|h|^{d+2\alpha}} \\
 &\leq C \int_{B(-x, \delta|x|) \setminus B(0, \delta)} \frac{1}{1 + |x+h|^{d+2\alpha}} \frac{|x|^{d+2\alpha}}{|h|^{d+2\alpha}} dh.
 \end{aligned}$$

But we know that $h \in B(-x, \delta|x|) \setminus B(0, \delta)$, using that $\delta < \frac{1}{2} < |x|$, we deduce that

$$|x|(1 - \delta) \leq |h| \leq (1 + \delta)|x| \Rightarrow \left| \frac{x}{h} \right| \leq \left| \frac{1}{1 - \delta} \right|.$$

Thus, we deduce

$$\begin{aligned}
 I_2 &\leq \frac{C}{(1 - \delta)^{d+2\alpha}} \int_{B(-x, \delta|x|) \setminus B(0, \delta)} \frac{1}{1 + |x+h|^{d+2\alpha}} dh \\
 &\leq \frac{C}{(1 - \delta)^{d+2\alpha}} \int_0^{\delta|x|} \frac{r^{d-1}}{1 + r^{d+2\alpha}} dr \\
 &\leq \frac{C}{(1 - \delta)^{d+2\alpha}} \int_0^{\infty} \frac{r^{d-1}}{1 + r^{d+2\alpha}} dr.
 \end{aligned}$$

To control I_3 , we write I_3 in the following form :

$$I_3 = C \left| \int_{B(0, \delta)} \left(\frac{1 + |x|^{d+2\alpha}}{1 + |x+h|^{d+2\alpha}} + \frac{1 + |x|^{d+2\alpha}}{1 + |x-h|^{d+2\alpha}} - 2 \right) \frac{\beta(x, \frac{h}{|h|}) dh}{|h|^{d+2\alpha}} \right|.$$

Next, we define

$$f(x, h) := \frac{1 + |x|^{d+2\alpha}}{1 + |x+h|^{d+2\alpha}}.$$

Since for all $x \in \mathbb{R}^d$, the map that $(h \mapsto f(x, h))$ is $C^{1+2\alpha}$, we know that I_3 is well defined. Moreover for every $h \in B(0, \delta) \setminus \{0\}$, when the parameter $|x|$ tends to ∞ , we have that $\frac{(f(x, h) + f(x, -h) - 2)\beta(x, \frac{h}{|h|})}{|h|^{d+2\alpha}}$ tends to 0. So we deduce thanks to the dominated convergence Theorem, that $(x \mapsto I_3(x))$ tends to 0 when $|x|$ tends to ∞ . According to the continuity of the maps $(x \mapsto f(x, h))$ and $(x \mapsto \beta(x, \theta))$, we deduce that the map $(x \mapsto I_3(x))$ is continuous

and so we conclude that I_3 is bounded independently of x . We refer to [85] for more details (see the Annex A1).

Combining the above inequalities, we obtain that there exists a constant C such that for all $x \in \mathbb{R}^d$,

$$|L^\alpha g(x)| \leq Cg(x). \quad (2.35)$$

Using the above inequality, we can conclude with a change of variable $z = ay$:

$$\begin{aligned} |L^\alpha g(ax)| &= \left| \int_{\mathbb{R}^d} \frac{g(ax) - g(ax + ay)}{|y|^{d+2\alpha}} \beta(ax, \frac{y}{|y|}) dy \right| \\ &= \left| \int_{\mathbb{R}^d} \frac{g(ax) - g(ax + z)}{|a^{-1}z|^{d+2\alpha}} a^{-d} \beta(ax, \frac{a^{-1}z}{|a^{-1}z|}) dz \right| \\ &= a^{2\alpha} |L^\alpha(g)(ax)| \\ &\leq Ca^{2\alpha} g(ax). \end{aligned}$$

Finally, we obtain

$$|L^\alpha g(ax)| \leq Ca^{2\alpha} g(ax).$$

Proof of (ii). Since all the functions involved in \widetilde{K} are differentiable, and thanks to the dominated convergence theorem, we deduce that \widetilde{K} is continuous. We can note the following fact :

$$|\nabla g(x)| = O(|x|^{-(d+2\alpha+1)}) \text{ as } |x| \rightarrow +\infty. \quad (2.36)$$

With the change of variable $\tilde{x} = ay$, we find :

$$\widetilde{K}(g(a \cdot), \chi)(x) = a^{2\alpha} C' PV \int_{\mathbb{R}^d} \frac{(g(ax) - g(\tilde{x}))(\chi(x) - \chi(a^{-1}\tilde{x}))}{|ax - \tilde{x}|^{d+2\alpha}} \beta(x, \frac{ax - \tilde{x}}{|ax - \tilde{x}|}) d\tilde{x}. \quad (2.37)$$

Since $\chi \in C^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, $\beta \in L^\infty(\mathbb{R}^d \times \mathbb{S}^{d-1})$ and $g \in C^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ this integral converges in \mathbb{R}^d . For $x \in \mathbb{R}^d$, we have to estimate

$$J(x) = a^{2\alpha} C' PV \int_{\mathbb{R}^d} \frac{(g(x) - g(\tilde{x}))(\chi(a^{-1}x) - \chi(a^{-1}\tilde{x}))}{|x - \tilde{x}|^{d+2\alpha}} \beta(a^{-1}x, \frac{x - \tilde{x}}{|x - \tilde{x}|}) d\tilde{x}$$

at point ax . We define for $x \in \mathbb{R}^d$

$$J_1(x) = a^{2\alpha} C' PV \int_{B(x,1)} \frac{(g(x) - g(\tilde{x}))(\chi(a^{-1}x) - \chi(a^{-1}\tilde{x}))}{|x - \tilde{x}|^{d+2\alpha}} \beta(a^{-1}x, \frac{x - \tilde{x}}{|x - \tilde{x}|}) d\tilde{x},$$

$$\text{and } J_2(x) = a^{2\alpha} C' PV \int_{\mathbb{R}^d \setminus B(x,1)} \frac{(g(x) - g(\tilde{x}))(\chi(a^{-1}x) - \chi(a^{-1}\tilde{x}))}{|x - \tilde{x}|^{d+2\alpha}} \beta(a^{-1}x, \frac{x - \tilde{x}}{|x - \tilde{x}|}) d\tilde{x},$$

so that $J = J_1 + J_2$. We split the proof in two parts : when $|x| \leq M$ and when $|x| > M$ with M a positive constant arbitrarily large.

For $|x| \leq M$, according to (2.37)

$$\widetilde{K}(g(a \cdot), \chi)(x) = a^{2\alpha} J(ax).$$

First we prove the existence of a constant C large enough such that

$$\forall x \in \overline{B}(0, M), |J(x)| \leq Cg(x). \quad (2.38)$$

Since $|J|$ is continuous, we deduce that in $\overline{B}(0, M)$, $|J|$ is bounded by a constant D . Thus, since g is decreasing, if we take C larger than $\frac{D}{a} \times (1 + M^{d+2\alpha})$, the assertions (2.38) holds true. Since $a < 1$, we conclude that for all $x \in \overline{B}(0, M)$:

$$|\widetilde{K}(g(a \cdot), \chi)(x)| = a^{2\alpha} |J(ax)| \leq a^{2\alpha} \frac{C}{1 + |ax|^{d+2\alpha}} \leq a^{2\alpha-\gamma} C \frac{1 + |x|^{d+2\alpha}}{1 + |ax|^{d+2\alpha}} g(x) \leq a^{2\alpha-\gamma} C g(x).$$

For $|x| > M$, we first study J_1 and then J_2 .

Estimate of J_1 : From the formula (2.36), for $|x| > M$, since χ is $C^1(\mathbb{R}^d)$ and periodic, and since $\gamma < 1$ and $2\alpha - \gamma$ is strictly positive, we have :

$$\begin{aligned} |J_1(x)| &\leq CB \int_{B(x,1)} \frac{a^{2\alpha-\gamma} |x - \tilde{x}|^\gamma}{|x - \tilde{x}|^{d+2\alpha}} \sup_{z \in [x; \tilde{x}]} |\nabla g(z)| |x - \tilde{x}| d\tilde{x} \\ &\leq C \frac{a^{2\alpha-\gamma}}{|x|^{d+2\alpha}} \int_{B(x,1)} \frac{1}{|x - \tilde{x}|^{d+2\alpha-\gamma-1}} d\tilde{x} \\ &\leq \frac{a^{2\alpha-\gamma} D_1}{1 + |x|^{d+2\alpha}}. \end{aligned}$$

Estimate of J_2 : Since χ is bounded and $a < 1$, we obtain :

$$\begin{aligned} |J_2(x)| &\leq a^{2\alpha} CB \int_{|y| \geq 1} \frac{g(x)}{|y|^{d+2\alpha}} dy + a^{2\alpha} CB \int_{|y| \geq 1} \frac{g(x+y)}{|y|^{d+2\alpha}} dy \\ &\leq a^{2\alpha} CB g(x) + a^{2\alpha} CB \int_{|y| \geq \frac{|x|}{2}} \frac{g(x+y)}{|y|^{d+2\alpha}} dy + a^{2\alpha} CB \int_{1 \leq |y| \leq \frac{|x|}{2}} \frac{g(x+y)}{|y|^{d+2\alpha}} dy \\ &\leq a^{2\alpha} \frac{CB}{|x|^{d+2\alpha}} + a^{2\alpha} \frac{2^{d+2\alpha} CB}{|x|^{d+2\alpha}} \int_{\mathbb{R}^d} g(y) dy + a^{2\alpha} CB \int_{1 \leq |y| \leq \frac{|x|}{2}} \frac{g(\frac{x}{2})}{|y|^{d+2\alpha}} dy \\ &\leq a^{2\alpha} \frac{CB}{|x|^{d+2\alpha}} + a^{2\alpha} \frac{2^{d+2\alpha} CB}{|x|^{d+2\alpha}} \int_{\mathbb{R}^d} g(y) dy + a^{2\alpha} CB 2^{d+2\alpha} g(x) \int_{|y| \geq 1} \frac{1}{|y|^{d+2\alpha}} dy \\ &\leq a^{2\alpha-\gamma} \frac{D_2}{1 + |x|^{d+2\alpha}}. \end{aligned}$$

The third line is obtained noting that for $|y| \leq \frac{|x|}{2}$, we have $\frac{|x|}{2} \leq |x| - |y| \leq |x + y|$.

Putting all together we find the existence of C such that (ii) holds. \square

Chapter 3

Propagation in a fractional reaction-diffusion equation in a periodically hostile environment

Ce chapitre va être publié dans Journal of Dynamics and Differential Equations dans un article co-signé avec Sepideh Mirrahimi et Jean-Michel Roquejoffre.

3.1 Introduction

3.1.1 Model and question

We focus on the following equation :

$$\begin{cases} \partial_t n(x, t) + (-\Delta)^\alpha n(x, t) = n(x, t)(1 - n(x, t)) & \text{for } (x, t) \in \Omega \times]0, \infty[, \\ n(x, t) = 0 & \text{for } (x, t) \in \Omega^c \times [0, \infty[, \\ n(x, 0) = n_0(x), \end{cases} \quad (3.1)$$

where Ω is a periodic domain of \mathbb{R}^d that will be specified later on, n_0 a compactly supported initial data and $(-\Delta)^\alpha$ the fractional Laplacian with $\alpha \in]0, 1[$ which is defined as follows :

$$\forall (x, t) \in \mathbb{R}^d \times]0, +\infty[, \quad (-\Delta)^\alpha n(x, t) = C_\alpha \text{PV} \int_{\mathbb{R}^d} \frac{n(x, t) - n(y, t)}{|x - y|^{d+2\alpha}} dy$$

where $C_\alpha = \frac{4^\alpha \Gamma(\frac{d}{2} + \alpha)}{\pi^{\frac{d}{2}} |\Gamma(-\alpha)|}$. The main aim of this paper is to describe the propagation front associated to (3.1). We show that the stable state invades the unstable state with an exponential speed.

Equation (3.1) models the dynamic of a species subject to a non-local dispersion in a periodically hostile environment. The quantity $n(x, t)$ stands for the density of the population at position x and time t . The fractional Laplacian describes the motion of individuals, it takes into account the possibility of "large jump" (move rapidly) of individuals from one point to another with a high rate, for instance because of human activities for animals or because of the wind for seeds. The term $(1 - n(x, t))$ represents the growth rate of the population at position x and time t . The originality of this model is the following, the reachable areas for the species are disconnected and periodic. Here, we assume that the regions where the species can develop itself are homogeneous.

Many works deal with the case of a standard diffusion ($\alpha = 1$, see [49] for a proof of the passage from the non-local to the local character of $(-\Delta)^\alpha$) with homogenous or heterogeneous environment (see [58], [73], [5] and [63]). Closer to this article, Guo and Hamel in [66] focus on a Fisher-KPP equation with periodically hostile regions and a standard diffusion. The authors prove that the stable state invades the unstable state in the connected component of the support of the initial data. In our work, thanks to the non-local character of the fractional Laplacian, contrary to what happens in [66], we show that there exists a unique non-trivial positive bounded stationary state, supported everywhere in the domain. Moreover, this steady state invades the unstable state 0 with an exponential speed.

3.1.2 Assumptions, notations and results

The domain Ω is a smooth non-connected periodic domain of \mathbb{R}^d

$$\text{i.e. } \Omega = \bigcup_{k \in \mathbb{Z}^d} \Omega_0 + a_k, \text{ with } \Omega_0 \text{ a smooth bounded domain of } \mathbb{R}^d \text{ and } a_k \in \mathbb{R}^d. \quad (3.2)$$

We assume that

$$(\Omega_0 + a_i) \cap (\Omega_0 + a_j) \neq \emptyset \quad \text{if and only if} \quad i = j.$$

Moreover, if we denote e_i the i^{th} vector of the canonical basis of \mathbb{R}^d then we assume that for all $k \in \mathbb{Z}^d$ there holds $a_{k+e_i} - a_k = a_{e_i}$. Moreover, we assume that the principal eigenvalue λ_1 of the Dirichlet operator $(-\Delta)^\alpha - Id$ in Ω_0 is negative

$$\text{i.e. } \lambda_1 < 0. \quad (\text{H1})$$

We also introduce the eigenvalue problem associated to the whole domain Ω . It is well known (thanks to the Krein Rutman theorem) that the principal eigenvalue λ_0 of the Dirichlet operator $(-\Delta)^\alpha - Id$ in Ω is simple in the algebraic and geometric sense and moreover, the associated principal eigenfunction ϕ_0 , solves

$$\text{i.e. } \begin{cases} ((-\Delta)^\alpha - Id) \phi_0 = \lambda_0 \phi_0 & \text{in } \Omega, \\ \phi_0 = 0 & \text{in } \Omega^c, \\ \phi_0 \text{ has a constant sign that can be chosen positive.} \end{cases} \quad (3.3)$$

The first result of this paper ensures the existence and the uniqueness of a positive bounded stationary state n_+ of (3.1) :

$$\text{i.e. } \begin{cases} (-\Delta)^\alpha n_+ = n_+ - n_+^2 & \text{in } \Omega, \\ n_+ = 0 & \text{in } \Omega^c. \end{cases} \quad (3.4)$$

Theorem 3.1. *Under the assumption (H1), there exists a unique positive and bounded stationary state n_+ to (3.1). Moreover, we have $0 \leq n_+ \leq 1$ and n_+ is periodic.*

The existence is due to the negativity of the principal eigenvalue of the Dirichlet operator $(-\Delta)^\alpha - Id$ in Ω_0 which allows to construct by an iterative method a stationary state (see [102] for more details). As for the uniqueness, the main step is to prove that thanks to the non-local character of the fractional Laplacian, any positive bounded stationary state behaves like

$$\delta(x)^\alpha = \text{dist}(x, \partial\Omega)^\alpha 1_\Omega(x). \quad (3.5)$$

Then, a classical argument (see [10] and [16]) relying on the maximum principle and the Hopf lemma provides the result. We should underline that the uniqueness is clearly due to the non-local character of the operator $(-\Delta)^\alpha$, and it does not hold in the case of a standard diffusion term ($\alpha = 1$). A direct consequence of the existence of a stationary solution is

Corollary 3.1. *The principal eigenvalue λ_0 of the Dirichlet operator $(-\Delta)^\alpha - Id$ in Ω is negative.*

Once we have established a unique candidate to be the limit of $n(x, t)$ as t tends to $+\infty$, we prove the invasion phenomena. First, we prove that starting from

$$n_0 \in C_0^\infty(\Omega) \cap C_c(\mathbb{R}^d) \quad \text{and} \quad n_0 \not\equiv 0 \quad (\text{H2})$$

the solution has algebraic tails at time $t = 1$. To prove it, we provide an estimate of the heat kernel at time $t = 1$ for a general multi-dimensional domain which satisfies the uniform interior and exterior ball condition :

Definition 3.1 (The uniform interior and exterior ball condition). *A set $\mathcal{O} \subset \mathbb{R}^d$ with $d \geq 1$ satisfies the uniform interior and exterior ball condition if there exists $r_1 > 0$ such that*

$$\begin{aligned} \forall x \in \partial\mathcal{O}, \exists y_x \in \mathcal{O} \text{ such that } x \in \partial B(y_x, r_1) \text{ and } B(y_x, r_1) \subset \mathcal{O}, \\ \text{and } \forall z \in \mathcal{O}^c, \exists y_z \in \mathcal{O}^c \text{ such that } z \in B(y_z, r_1) \text{ and } B(y_z, r_1) \subset \mathcal{O}^c. \end{aligned}$$

Theorem 3.2. *Let \mathcal{O} be a smooth domain of \mathbb{R}^d with $d \geq 1$ satisfying the uniform interior and exterior ball condition. If we define p as the solution of the following equation*

$$\begin{cases} \partial_t p(x, t) + (-\Delta)^\alpha p(x, t) = 0 & \text{for all } (x, t) \in \mathcal{O} \times]0, +\infty[, \\ p(x, t) = 0 & \text{for all } (x, t) \in \mathcal{O}^c \times [0, +\infty[, \\ p(x, t = 0) = n_0(x) \in C_0^\infty(\mathcal{O}, \mathbb{R}^+) \cap C_c^0(\mathbb{R}^d), \end{cases} \quad (3.6)$$

then there exists $c > 0$ and $C > 0$ such that for all $x \in \mathcal{O}$,

$$\frac{c \times \min(\delta(x)^\alpha, 1)}{1 + |x|^{d+2\alpha}} \leq p(x, t = 1) \leq \frac{C \times \min(\delta(x)^\alpha, 1)}{1 + |x|^{d+2\alpha}}. \quad (3.7)$$

Once Theorem 3.2 is established, we are able to state the main result of the paper.

Theorem 3.3. *Assume (H1) and (H2). Then for all $\mu > 0$ there exists a time $t_\mu > 0$ such that :*

(i) *for all $c < \frac{|\lambda_0|}{d+2\alpha}$ and all $(x, t) \in \{|x| < e^{ct}\} \times]t_\mu, +\infty[$*

$$|n(x, t) - n_+(x)| \leq \mu.$$

(ii) *for all $C > \frac{|\lambda_0|}{d+2\alpha}$ and all $(x, t) \in \{|x| > e^{Ct}\} \times]t_\mu, +\infty[$*

$$|n(x, t)| \leq \mu.$$

We detail the general strategy to prove Theorem 3.3 in the next section. Our last result concerns the level sets of the solution n .

Theorem 3.4. *Let $\nu > 0$ and $\Omega_\nu := \{x \in \Omega \mid \delta(x) > \nu\}$. There exists a constant c_ν which depends only on ν , n_0 , Ω such that for all $\mu \in]0, c_\nu[$, there exist $t_\mu, c_\mu, C_\mu > 0$ such that*

$$\{(x, t) \in \Omega_\nu \times]t_\mu, +\infty[\mid n(x, t) = \mu\} \subset \{(x, t) \in \Omega_\nu \times]t_\mu, +\infty[\mid c_\mu e^{|\lambda_0|t} \leq |x|^{d+2\alpha} \leq C_\mu e^{|\lambda_0|t}\}.$$

3.1.3 Discussion on the main results

Theorem 3.2 is an application of general results about the fractional Dirichlet heat kernel estimates given for instance in [42] or in [25]. Both of the two cited articles use a probabilistic approach. We propose in this work a deterministic proof of the lower bound of the fractional Dirichlet kernel estimates. Our proof is quite simple but the result is not as general as those presented in [42] and [25]. In particular, it is only valid for finite time. It relies on a well adapted decomposition of the fractional Laplacian. We do not provide the proof of the upper bound of the fractional Dirichlet kernel estimates since there is no difficulties to obtain such bound.

Theorem 3.3 can be seen as a generalisation of the results of [45] or [77]. Indeed, if we study a non-local Fisher KPP equation in the whole domain \mathbb{R}^d with a reaction term depending on a parameter such that the reaction term becomes more and more unfavorable in Ω^c then we recover Theorem 3.3. This is fully in the spirit of [66]. In fact, if we study the equation :

$$\begin{cases} \partial_t n + (-\Delta)^\alpha n = \mu_\delta(x)n - n^2 & \text{in } \mathbb{R}^d \times]0, +\infty[, \\ n(x, t = 0) = n_0(x), \end{cases}$$

with

$$\mu_\delta(x) = \begin{cases} 1 & \text{if } x \in \Omega, \\ 1 - (\delta + 1)\text{dist}(x, \Omega) & \text{if } 0 < \text{dist}(x, \Omega) \leq \frac{1}{\delta}, \\ -\frac{1}{\delta} & \text{if } \frac{1}{\delta} < \text{dist}(x, \Omega). \end{cases}$$

Then, denoting by λ_δ the principal eigenvalue of the operator $((-\Delta)^\alpha - \mu_\delta)$ we claim that

$$\lambda_\delta \xrightarrow{\delta \rightarrow 0} \lambda_0. \quad (3.8)$$

It is then possible to obtain the result of Theorem 3.3 from such approximate problems in the spirit of [66]. Although we do not use such method, similar difficulties would arise to treat the problems with this approximation procedure. Our method can indeed be adapted to study those problems in a uniform way.

3.2 Strategy, comparison tools and outline of the paper

3.2.1 The general strategy

The general strategy to establish the results of Theorem 3.3 is the following :

- A- Identify the unique candidate to be the limit. This is the content of Theorem 3.1.
- B- Starting from a compactly supported initial data, the solution n has algebraic tails immediatly after $t = 0$. This is the content of Theorem 3.2.
- C- Establish a sub and a super-solution which bound the solution n from below and above.
- D- Use the sub-solution to "push" the solution n to the unique non-trivial stationary state n_+ in $\left\{ |x| < e^{\frac{|\lambda_0|t}{d+2\alpha}} \right\}$ and use the super-solution to "crush" the solution n to 0 in $\left\{ |x| > e^{\frac{|\lambda_0|t}{d+2\alpha}} \right\}$.

The proof of C can be done with two different approaches. The first one is introduced in [35] by Cabré, Coulon and Roquejoffre. The idea is to consider the quantity

$$v(x, t) = \phi_0(r(t)x)^{-1}n(r(t)x, t)$$

where the eigenfunction ϕ_0 is introduced in (3.3) and $r(t)$ decreases exponentially fast. Next, the problem can be formally reduced to a transport equation leading to the fact that v is of the form $\frac{\phi_0(x)}{1+b(t)|x|^{d+2\alpha}}$. The idea is therefore to look for a sub-solution \underline{v} and a supersolution \bar{v} of the form

$$\underline{v}(x, t) = \frac{\underline{a}\phi_0(x)}{1 + \underline{b}(t)|x|^{d+2\alpha}} \quad \text{and} \quad \bar{v}(x, t) = \frac{\bar{a}\phi_0(x)}{1 + \bar{b}(t)|x|^{d+2\alpha}}$$

(where the positive constants \underline{a}, \bar{a} and the function \underline{b}, \bar{b} have to be adjusted).

The second approach is introduced in [85] by Méléard and Mirrahimi (in order to extend the Geometric optics approach of [59] and [60], put to work in the PDE framework in [54]). The main idea is to perform the following scaling on equation (3.1)

$$(x, t) \mapsto \left(|x|^{\frac{1}{\varepsilon}} \frac{x}{|x|}, \frac{t}{\varepsilon} \right), \quad \varepsilon > 0 \text{ small.} \quad (3.9)$$

One of the interest of this scaling is to catch the effective behavior of the solution. Indeed, this scaling keeps invariant the set

$$\mathcal{B} = \left\{ (x, t) \in \mathbb{R} \times \mathbb{R}^+ \mid (d + 2\alpha) \log |x| < |\lambda_0|t \right\}$$

where λ_0 is defined by (3.3). In the region \mathcal{B} , we expect that the solution n is close to the stationary state n_+ and in the region \mathcal{B}^c , we expect that the solution n is closed to 0. Then, we look for sub/super-solutions on the form

$$\phi_0(x) \times G(x, t)$$

where G needs to be determined. Taking G with an algebraic tail gives that, once the scaling is performed, the fractional Laplacian of G vanishes as the parameter ε tends to 0. Indeed, the sub and super solutions are just perturbations of a simple ODE and are valid only for small ε .

We choose the second method for the two reasons. First, the scaling reveals two invariant regions (\mathcal{B} and \mathcal{B}^c) which catch the effective behavior of the solutions. The alternative approach does not account for these features as clearly. Secondly, this method gives very clear indications to construct sub- and super-solutions. It highlights the important terms in the equation to establish sub- and super-solutions. Once the scaling is performed, the fractional Laplacian terms in the computations vanish as the parameter ε tends to 0. This means that the only role of the fractional Laplacian in determining the invasion speed is at initial time where it determines the algebraic tails of the solution. This is indeed very different from the classical Fisher-KPP equation where the diffusion not only determines the exponential tails of the solution but it also modifies the invasion speed in positive times (see [85]). This is why in the asymptotic study of the classical Fisher KPP equation, one obtains a Hamilton-Jacobi equation [54] while in the fractional KPP equation the limit is a simple ordinary differential equation. Thus scaling the equation as in (3.9) is not a mere artefact, it really helps in the understanding of the phenomenon. Moreover, we improve the precision of the sub and super-solutions compared with the ones obtained in [85]. Therefore, the exhibited sub and super-solutions and the ones that could be obtained with the alternative approach have the same level of precision.

The proof of D can be achieved with the rescaled solution $n\left(|x|^{\frac{1}{\varepsilon}} \frac{x}{|x|}, \frac{t}{\varepsilon}\right)$ using the method of perturbed test functions from the theory of viscosity solutions and homogenization (introduced by Evans in [51] and [52] and by Mirrahimi and Méléard in [85] for the fractional Laplacian). Since the proof is technical, long and not easy to grasp (the domain moves also with the parameter ε), we prefer to drop the scaling and to perform the inverse scaling on our sub and super solutions. Therefore, we provide a direct proof of D by adapting the proof of Theorem 1.6 in [45]. In this proof, the author proves thanks to a subsolution that there exists $\sigma > 0$ and $t_\sigma > 0$ such that

$$\sigma < \inf_{(x,t) \in \left\{ |x| < e^{\frac{(|\lambda_0| - \delta)t}{d+2\alpha}} \right\} \times]t_\sigma, +\infty[} n(x, t).$$

This last claim is obviously false in our case since the solution vanishes on the boundary. This is the main new difficulty that we will encounter. We overcome it by establishing the same kind of estimates away from the boundary.

Theorem 3.4 is the consequence of the precision of the sub and super-solutions established in step C. Note that such level sets can not be established with the sub and super-solutions involved in [85] or [77]. However, the new ingredients presented here to establish the sub and the super-solutions can be adapted to the framework of [85] or [77]. In the framework of [77], we recover Theorem 1.1.5 of [45].

3.2.2 The comparison tools and some notations

All along the article, we will use many times the comparison principle. We recall here what we mean by comparison principle.

Theorem (The comparison principle). *Let f be a smooth function, $a \in [0, +\infty[$ and $b \in]0, +\infty]$. If \underline{n} and \bar{n} are such that*

$$\begin{aligned} \forall (x, t) \in \Omega \times]a, b[, \quad \partial_t \underline{n} + (-\Delta)^\alpha \underline{n} &\leq f(\underline{n}), & \text{and} & \quad \partial_t \bar{n} + (-\Delta)^\alpha \bar{n} \geq f(\bar{n}), \\ \forall (x, t) \in \Omega^c \times]a, b[, \quad \underline{n}(x, t) &\leq \bar{n}(x, t), & \text{and} & \quad \forall x \in \mathbb{R}^d, \quad \underline{n}(x, t = a) \leq \bar{n}(x, t = a) \end{aligned}$$

then

$$\forall (x, t) \in \Omega \times]a, b[, \quad \underline{n}(x, t) \leq \bar{n}(x, t).$$

In the same spirit, we recall the fractional Hopf Lemma stated in [65].

Lemma (The fractional Hopf Lemma [65]). *Let $\mathcal{O} \subset \mathbb{R}^d$ be an open set satisfying the uniform interior and exterior ball condition at $z \in \partial\mathcal{O}$ and let $c \in L^\infty(\mathcal{O})$. Consider a positive lower semi-continuous function $u : \mathbb{R}^d \mapsto \mathbb{R}$ satisfying $(-\Delta)^\alpha u \geq c(x)u$ point-wise in \mathcal{O} . Then, either u vanishes identically in \mathcal{O} , or there holds*

$$\liminf_{\substack{x \rightarrow z \\ x \in \mathcal{O}}} \frac{u(x)}{\delta(x)^\alpha} > 0.$$

All along the article, for any set \mathcal{U} and any positive constant ν , we introduce the following new sets :

$$\mathcal{U}_\nu = \{x \in \mathcal{U} \mid \text{dist}(x, \partial\mathcal{U}) > \nu\}, \quad \mathcal{U}_{-\nu} = \{x \in \mathbb{R}^d \mid \text{dist}(x, \mathcal{U}) < \nu\}. \quad (3.10)$$

The constants denoted by c or C may change from one line to another when there is no confusion possible. Also, we drop the constant C_α and the Cauchy principal value $P.V.$ in front of the fractional Laplacian for better readability.

3.2.3 Outline of the paper

In section 3, we demonstrate Theorem 3.1. Next, section 4 is dedicated to the proof of Theorem 3.2. The first part of section 5 introduces the scaling and provides the sub and super-solutions. Finally, the second part of section 5 is devoted to the proof of Theorems 3.3 and 3.4.

3.3 Uniqueness of the stationary state n_+

First, we state a proposition which gives the shape of any non-trivial bounded sub and super-solution to (3.4) near the boundary. Then, we use this result to prove the uniqueness result. Since the proof of the existence is classical we do not provide it.

Proposition 3.1. (i) *If u is a smooth positive bounded function such that $u(x) = 0$ for all $x \in \Omega^c$ and $(-\Delta)^\alpha u(x) \leq u(x) - u(x)^2$ for all $x \in \Omega$, then there exists $C > 0$ such that for all $x \in \mathbb{R}^d$*

$$u(x) \leq C\delta(x)^\alpha.$$

(ii) *If v is a smooth positive bounded function such that $v(x) = 0$ for all $x \in \Omega^c$, $(-\Delta)^\alpha v(x) \geq v(x) - v(x)^2$ for all $x \in \Omega$ and $v \not\equiv 0$ then there exists $c > 0$ such that for all $x \in \mathbb{R}^d$*

$$c\delta(x)^\alpha \leq v(x).$$

Proof of Proposition 3.1. Proof of (i). Let u be a continuous positive bounded function such that $u = 0$ in Ω^c and $(-\Delta)^\alpha u \leq u - u^2$ in Ω . Let x be a point of the boundary. Let $z_x \in \mathbb{R}^d$ and $r_1 > 0$ be the elements provided by the uniform exterior ball condition such that

$$B(z_x, r_1) \subset \Omega^c \quad \text{and} \quad x \in \overline{B(z_x, r_1)} \cap \partial\Omega.$$

We rescale and translate a barrier function (provided for instance in Annex B of [97]). This barrier function $\bar{\phi}$ satisfies the following properties :

$$\begin{cases} (-\Delta)^\alpha \bar{\phi} \geq 1 & \text{in } B(z_x, 4r_1) \setminus B(z_x, r_1), \\ \bar{\phi} \equiv 0 & \text{in } B(z_x, r_1), \\ 0 \leq \bar{\phi} \leq C(|z_x - x| - r)^\alpha & \text{in } B(z_x, 4r_1) \setminus B(z_x, r_1), \\ \max u \leq \bar{\phi} \leq C & \text{in } \mathbb{R}^d \setminus B(z_x, 4r_1). \end{cases} \quad (3.11)$$

We prove that $u \leq \bar{\phi}$ in \mathbb{R}^d . By construction we have $u \leq \bar{\phi}$ in $(B(z_x, 4r_1) \setminus B(z_x, r_1))^c$. Assume by contradiction that there exists $x_0 \in (B(z_x, 4r_1) \setminus B(z_x, r_1)) \cap \Omega$ such that $(\bar{\phi} - u)(x_0) < 0$. Then, there exists $x_1 \in (B(z_x, 4r_1) \setminus B(z_x, r_1)) \cap \Omega$ such that $(\bar{\phi} - u)(x_1) = \min_{x \in \mathbb{R}^d} (\bar{\phi} - u)(x) < 0$.

Thus, we obtain

$$(-\Delta)^\alpha (\bar{\phi} - u)(x_1) < 0 \quad \text{and} \quad (-\Delta)^\alpha (\bar{\phi} - u)(x_1) \geq 1 - u(x_1) + u(x_1)^2 \geq 0,$$

a contradiction.

Proof of (ii). Let v be a continuous positive bounded function such that $v = 0$ in Ω^c and $(-\Delta)^\alpha v \geq v - v^2$ in Ω . An easy but important remark is the following : thanks to the

non-local character of the fractional Laplacian, since $v \not\equiv 0$, we deduce that $v > 0$ in the whole domain Ω . Otherwise, the following contradiction holds true :

$$\exists \underline{x} \in \Omega \text{ such that } v(\underline{x}) = 0 \quad \text{and} \quad (-\Delta)^\alpha v(\underline{x}) - v(\underline{x}) + v(\underline{x})^2 = - \int_{\mathbb{R}^d} \frac{v(y)}{|x-y|^{d+2\alpha}} dy < 0.$$

Next, let k be any element of \mathbb{Z}^d . We introduce $\underline{w}_k : (x, t) \in \mathbb{R}^d \times [0, +\infty[\mapsto \underline{w}_k(x, t) \in \mathbb{R}$ as the solution of

$$\begin{cases} \partial_t \underline{w}_k + (-\Delta)^\alpha \underline{w}_k = \underline{w}_k - \underline{w}_k^2 & \text{in } (\Omega_0 + a_k) \times]0, +\infty[, \\ \underline{w}_k(x, t) = 0 & \text{in } \mathbb{R}^d \setminus (\Omega_0 + a_k) \times [0, +\infty[\\ \underline{w}_k(x, 0) = v(x) & \text{in } (\Omega_0 + a_k), \end{cases} \quad (3.12)$$

where Ω_0 and a_k are introduced in (3.2). Thanks to the remark above, and recalling (H1), we deduce thanks to Theorem 5.1 in [20] that $\underline{w}_k(\cdot, t) \xrightarrow{t \rightarrow +\infty} \underline{w}_{stat}(\cdot)$ with \underline{w}_{stat} the solution of

$$\begin{cases} (-\Delta)^\alpha \underline{w}_{stat} = \underline{w}_{stat} - \underline{w}_{stat}^2, & \text{in } (\Omega_0 + a_k), \\ \underline{w}_{stat} = 0 & \text{in } \mathbb{R}^d \setminus (\Omega_0 + a_k). \end{cases} \quad (3.13)$$

Note that the above \underline{w}_{stat} does not depend on the choice of k , i.e. $\underline{w}_k(\cdot, t)$ converges as t tends to $+\infty$ to the same \underline{w}_{stat} (up to a translation). Then, we conclude thanks to the comparison principle that

$$\underline{w}_{stat}(x) \leq v(x), \quad \forall x \in \mathbb{R}^d.$$

Since, $(\Omega_0 + a_k)$ is bounded, we apply the results of [97] to find that there exists a constant $c > 0$ such that

$$c\delta(x)^\alpha 1_{(\Omega_0 + a_k)}(x) \leq \underline{w}_{stat}(x) \leq v(x).$$

The previous analysis holds for every $k \in \mathbb{Z}^d$. We conclude that

$$c\delta(x)^\alpha \leq v(x). \quad (3.14)$$

□

Proof of Theorem 3.1. The argument relies on the fact that two steady solutions are comparable everywhere thanks to Proposition 3.1. This is in the spirit of [10] and [16] in the context of standard diffusion. Let u and v be two bounded steady solutions of (3.4). By the maximum principle, we easily have that for all $x \in \mathbb{R}^d$,

$$u(x) \leq 1 \quad \text{and} \quad v(x) \leq 1.$$

We will assume that

$$v(x_0) \leq u(x_0) \quad \text{where } x_0 \in \Omega_0. \quad (3.15)$$

Thanks to Proposition 3.1, we deduce the existence of two constants $0 < c \leq C$ such that :

$$c\delta(x)^\alpha \leq u(x) \leq C\delta(x)^\alpha \quad \text{and} \quad c\delta(x)^\alpha \leq v(x) \leq C\delta(x)^\alpha.$$

Thus there exists a constant $\lambda > 1$ such that for all $x \in \mathbb{R}^d$,

$$u(x) \leq \lambda v(x). \quad (3.16)$$

We set $l_0 := \inf \{ \lambda \geq 1 \mid \forall x \in \mathbb{R}^d, u(x) \leq \lambda v(x) \}$. The point is to prove by contradiction that $l_0 = 1$. It implies that x_0 is a contact point, and will allow us to conclude thanks to the fractional maximum principle that $u = v$.

We assume by contradiction that $l_0 > 1$. Next, we define :

$$\tilde{w} = \inf_{x \in \Omega} \frac{(l_0 v - u)(x)}{\delta(x)^\alpha} \geq 0. \quad (3.17)$$

There are two cases to be considered.

Case 1 : $\tilde{w} > 0$.

We show in this case that we can construct $l_1 < 1$ such that $u(x) \leq l_1 l_0 v(x)$ for all $x \in \mathbb{R}^d$: a contradiction. If $\tilde{w} > 0$, we claim that there exists $\mu \in]0, 1[$ and $\nu > 0$ such that for all $x \in \Omega \setminus \Omega_\nu$ (we recall that Ω_ν is defined by (3.10)),

$$\frac{\tilde{w}}{2} \leq \frac{(\mu l_0 v - u)(x)}{\delta(x)^\alpha}. \quad (3.18)$$

Indeed, if there does not exist such couple (μ, ν) , we deduce that for all $n \in \mathbb{N}$, there exists $(x_n)_{n \in \mathbb{N}} \in \Omega$, such that $\delta(x_n) \leq \frac{1}{n}$ and

$$\frac{((1 - \frac{1}{n})l_0 v - u)(x_n)}{\delta(x_n)^\alpha} < \frac{\tilde{w}}{2}.$$

Passing to the liminf we get the following contradiction :

$$0 < \tilde{w} \leq \frac{\tilde{w}}{2}.$$

And so, the existence of the couple (μ, ν) implies that

$$(\mu l_0 v - u)(x) \geq 0, \quad \forall x \in \Omega \setminus \Omega_\nu. \quad (3.19)$$

Next, we claim that

$$\exists \rho > 0 \text{ such that } \forall x \in \Omega_\nu, \text{ we have } \rho \leq (l_0 v - u)(x). \quad (3.20)$$

Indeed, if such ρ does not exist then there exists a sequence $(x_n)_{n \in \mathbb{N}} \in \Omega$ such that $\delta(x_n) \geq \nu$ and $(l_0 v - u)(x_n) \xrightarrow[n \rightarrow +\infty]{} 0$. Then we obtain

$$\frac{(l_0 v - u)(x_n)}{\delta(x_n)^\alpha} \leq \frac{(l_0 v - u)(x_n)}{\nu^\alpha} \xrightarrow[n \rightarrow +\infty]{} 0$$

which is in contradiction with the hypothesis $\tilde{w} > 0$. The existence of such ρ implies that for all $x \in \Omega_\nu$

$$\left(\left(1 - \frac{\rho}{\max l_0 v} \right) l_0 v - u \right) (x) \geq 0. \quad (3.21)$$

Finally, if we define $l_1 = \max(\mu, 1 - \frac{\rho}{\max l_0 v + 1})$ then we obtain the desired contradiction. Therefore this case cannot occur.

Case 2 : $\tilde{w} = 0$.

We consider $(x_n)_{n \in \mathbb{N}}$ a minimizing sequence of \tilde{w} . There are 3 subcases : a subsequence of $(x_n)_{n \in \mathbb{N}}$ converges to $x_0 \in \Omega$, a subsequence of $(x_n)_{n \in \mathbb{N}}$ converges to $x_b \in \partial\Omega$ and any subsequence of $(x_n)_{n \in \mathbb{N}}$ diverges.

Subcase a : *There exists $x_0 \in \Omega$, such that $\frac{(l_0 v - u)(x_0)}{\delta(x_0)^\alpha} = 0$.*

Since $x_0 \in \Omega$ we deduce that $(l_0 v - u)(x_0) = 0$. Hence, by the maximum principle, $u = l_0 v$. We deduce that $l_0 v$ is a solution of (3.4) and we conclude that :

$$l_0(v - v^2) = l_0(-\Delta)^\alpha(v) = (-\Delta)^\alpha(l_0 v) = l_0 v - (l_0 v)^2. \quad (3.22)$$

This equation leads to $l_0 = 1$, a contradiction.

Subcase b : *There exists $x_b \in \partial\Omega$, such that $\liminf_{\substack{x \rightarrow x_b, \\ x \in \Omega}} \frac{(l_0 v - u)(x)}{\delta(x)^\alpha} = 0$.*

Here is a summary of what we know :

- (i) $l_0 v - u \geq 0$,
- (ii) $(-\Delta)^\alpha(l_0 v - u) \geq -l_0(l_0 v - u)$,
- (iii) $(l_0 v - u)(x_b) = 0$.

According to the fractional Hopf Lemma, the previous assumptions leads to

$$\liminf_{\substack{x \rightarrow x_b, \\ x \in \Omega}} \frac{(l_0 v - u)(x)}{\delta(x)^\alpha} > 0.$$

However, we have assumed that $\liminf_{\substack{x \rightarrow x_b, \\ x \in \Omega}} \frac{(l_0 v - u)(x)}{\delta(x)^\alpha} = 0$, a contradiction.

Subcase c : *There exists a minimizing sequence $(x_n)_{n \in \mathbb{N}}$ such that $|x_n|$ tends to the infinity.*

First, we set

$$\bar{x}_k = x_k - a_{\lfloor x_k \rfloor},$$

where $\lfloor x \rfloor \in \mathbb{Z}^d$ is such that $x \in \Omega_0 + a_{\lfloor x \rfloor}$. Since $\bar{x}_k \in \Omega_0$, we deduce that up to a subsequence \bar{x}_k converges to $\bar{x}_\infty \in \bar{\Omega}_0$. Then we define :

$$u_k(x) = u(x + \bar{x}_k) \text{ and } v_k(x) = v(x + \bar{x}_k).$$

We also define the following set :

$$\Omega_\infty = \{x \in \mathbb{R} \mid x + \bar{x}_\infty \in \Omega\}.$$

By fractional elliptic regularity (see [96]), we deduce that up to a subsequence $(u_n)_{n \in \mathbb{N}}$ and $(v_n)_{n \in \mathbb{N}}$ converges to u_∞ and v_∞ solutions that verifies

$$\begin{aligned} \forall x \in \Omega_\infty, \quad & (-\Delta)^\alpha u_\infty(x) = u_\infty(x) - u_\infty(x)^2, \quad (-\Delta)^\alpha v_\infty(x) = v_\infty(x) - v_\infty(x)^2 \\ \text{and } \forall x \in \Omega_\infty^c, \quad & u_\infty(x) = v_\infty(x) = 0. \end{aligned}$$

Remark that

$$l_0 v_\infty - u_\infty \geq 0 \text{ and } \liminf_{\substack{x \rightarrow 0 \\ x \in \Omega_\infty}} \frac{(l_0 v_\infty - u_\infty)(x)}{\text{dist}(x, \partial\Omega_\infty)^\alpha} = 0.$$

Hence, if $\bar{x}_\infty \in \Omega_0$ then $0 \in \Omega_\infty$ and we fall in the subcase a). If $\bar{x}_\infty \notin \Omega_0$ then $0 \in \partial\Omega_\infty$ and we fall in the subcase b). Both cases lead to a contradiction.

Thus, we conclude that $l_0 = 1$. □

Remark. Noticing that for all $(x, k) \in \Omega \times \mathbb{Z}^d$, we have

$$(-\Delta)^\alpha(n_+(\cdot + a_k))(x) = \int_{\mathbb{R}} \frac{n_+(x + a_k) - n_+(y + a_k)}{|x + a_k - (y + a_k)|^{d+2\alpha}} dy = n_+(x + a_k) - n_+(x + a_k)^2,$$

we deduce by uniqueness of the solution of (3.4) that n_+ is periodic.

3.4 The fractional heat kernel and the preparation of the initial data

We first introduce some requirements in order to achieve the proof of the lower bound of Theorem 3.2. Once we have established Theorem 3.2, we apply it to the initial data. Let $u \in C^\infty(\mathbb{R}^d \times]0, +\infty[)$, then we set for all $(x, t) \in \mathbb{R}^d \times]0, +\infty[$

$$L^\alpha(u)(x, t) = \int_{B(0, \nu)} \frac{u(x, t) - u(y, t)}{|y|^{d+2\alpha}} dy. \quad (3.23)$$

We also introduce $\tilde{\phi}_\nu$ as the principal positive eigenfunction of the operator L^α associated to the principal eigenvalue μ_ν

$$\text{i.e. } \begin{cases} L^\alpha \tilde{\phi}_\nu = \mu_\nu \tilde{\phi}_\nu & \text{in } B(0, \nu) \\ \tilde{\phi}_\nu = 0 & \text{in } B(0, \nu)^c, \\ \tilde{\phi}_\nu \geq 0, \|\tilde{\phi}_\nu\|_\infty = 1. \end{cases}$$

Next, we state two intermediate technical results.

Lemma 3.1. *Let w be the solution of the equation*

$$\begin{cases} \partial_t w + L^\alpha w = 1 & \text{in } B(0, \nu) \times]0, +\infty[\\ w(x, t) = 0 & \text{in } B(0, \nu)^c \times [0, +\infty[, \\ w(x, t = 0) = 0 & \text{in } B(0, \nu). \end{cases} \quad (3.24)$$

Then there exists a constant $c_\nu > 0$ such that

$$c_\nu \times \tilde{\phi}_\nu(x) \leq w(x, t = 1).$$

Proof. We define $\tau(t) = \frac{1}{\mu_\nu}(1 - e^{-\mu_\nu t})$ such that

$$\begin{cases} \tau'(t) + \mu_\nu \tau(t) = 1, \\ \tau(0) = 0. \end{cases}$$

Thanks to this choice of $\tau(t)$, the application $\underline{w}(x, t) := \tau(t) \times \tilde{\phi}_\nu(x)$ is a sub-solution to (3.24). Actually, we have

$$(\partial_t + L^\alpha)(\underline{w}) - 1 = \tau' \tilde{\phi}_\nu + \mu_\nu \tau \tilde{\phi}_\nu - 1 \leq \tau' \tilde{\phi}_\nu + \mu_\nu \tau \tilde{\phi}_\nu - \tilde{\phi}_\nu = \tilde{\phi}_\nu(\tau' + \mu_\nu \tau - 1) = 0.$$

Since $\underline{w}(t = 0) = 0 \leq w(t = 0)$, we can conclude thanks to the comparison principle that for all $(x, t) \in \mathbb{R}^d \times [0, +\infty[$, we have $\underline{w}(x, t) \leq w(x, t)$. Setting the time $t = 1$ in the last inequality leads to

$$\underline{w}(x, 1) = \frac{1}{\mu_\nu}(1 - e^{-\mu_\nu}) \phi_\nu(x) = c_\nu \phi_\nu(x) \leq w(x, 1).$$

□

Next, we establish a barrier function for L^α in the spirit of the one introduced in [97].

Lemma 3.2. *There exists a function $\underline{\psi}$ such that*

$$\begin{cases} L^\alpha \underline{\psi} \leq 0 & \text{in } B(0, \nu) \setminus B(0, \frac{\nu}{2}), \\ \underline{\psi} = 0 & \text{in } B(0, \nu)^c, \\ \underline{\psi} \leq 1 & \text{in } B(0, \frac{\nu}{2}), \\ c(\nu - |x|)^\alpha \leq \underline{\psi} & \text{in } B(0, \nu), \\ \underline{\psi} \text{ is continuous in } B(0, \nu) \setminus B(0, \frac{\nu}{2}). \end{cases} \quad (3.25)$$

Proof. Choose C large enough such that the first point and the third point of (3.25) holds true with the following $\underline{\psi}$:

$$\underline{\psi}(x) := \left(\frac{1}{C}(\nu^2 - |x|^2)^\alpha + \frac{1}{2}1_{B(0, \frac{\nu}{4})}(x) \right) 1_{B(0, \nu)}(x).$$

Indeed, defining $f(x) := (\nu^2 - |x|^2)^\alpha$, we have for C large enough and $x \in B(0, \nu) \setminus B(0, \frac{\nu}{2})$

$$L^\alpha \underline{\psi}(x) \leq \frac{L^\alpha f(x)}{C} - \frac{1}{2} \int_{B(0, \frac{\nu}{4})} \frac{1}{|x-y|^{d+2\alpha}} dy \leq \frac{\sup_{B(0, \nu) \setminus B(0, \frac{\nu}{2})} |L^\alpha f|}{C} - \frac{m(B(0, \frac{\nu}{4}))}{2} \times \left(\frac{4}{\nu} \right)^{d+2\alpha} < 0.$$

The other conditions follow. \square

Proof of Theorem 3.2. The aim is to prove that there exists a constant $c > 0$ such that

$$\forall x \in \mathcal{O}, \text{ we have } \frac{c \min(\delta(x)^\alpha, 1)}{1 + |x|^{d+2\alpha}} \leq p(x, 1). \quad (3.26)$$

To achieve the proof, there will be 4 steps.

First, up to a translation and possibly a scaling of n , we prove (3.26) in $\{|x| < 1 + 2\nu\}$ where $\nu = \min(\frac{1}{4}, r_1)$ (with r_1 the radius provided by the uniform interior ball). Next, we introduce a suitable decomposition of the fractional Laplacian (involving L^α) to prove the existence of $c_1 > 0$ such that

$$\begin{cases} \frac{c_1}{1 + |x|^{d+2\alpha}} \leq \partial_t p(x, t) + L^\alpha p(x, t) + \lambda p(x, t) & \text{for all } (x, t) \in (\mathcal{O} \setminus \{|x| > 1 + \nu\}) \times]0, 1], \\ p(x, t) \geq 0 & \text{for all } (x, t) \in (\mathcal{O} \setminus \{|x| > 1 + \nu\})^c \times [0, 1], \\ p(x, t = 0) = n_0(x) \in C_0^\infty(\mathcal{O}, \mathbb{R}^+) \cap C_c(\mathbb{R}^d) \end{cases} \quad (3.27)$$

where L^α is defined by (3.23) and $\lambda = \int_{\mathbb{R}^d \setminus B(0, \nu)} \frac{1}{|y|^{d+2\alpha}} dy$. In a third step, we will show that

$$\exists c_2 > 0 \text{ such that } \frac{c_2}{1 + |x|^{d+2\alpha}} \leq p(x, t = 1) \text{ for all } x \in (\mathcal{O}_\nu \cap \{|x| > 1 + 2\nu\}). \quad (3.28)$$

Finally, we prove the same kind of result near the boundary :

$$\exists c_3 > 0 \text{ such that } \frac{c_3 \delta(x)^\alpha}{1 + |x|^{d+2\alpha}} \leq p(x, t = 1) \text{ for all } x \in (\mathcal{O} \setminus \mathcal{O}_\nu \cap \{|x| > 1 + 2\nu\}). \quad (3.29)$$

Step 1. First, note that thanks to a translation and possibly a scaling, we can suppose the following hypothesis :

$$\exists \sigma > 0 \text{ such that } \sigma < n_0(x) \text{ for all } x \in B(0, 2). \quad (3.30)$$

Next, we claim that

$$\inf_{\substack{t \in (0,1) \\ z \in B(0,1+2\nu)}} p(z, t) > 0. \quad (3.31)$$

Indeed, let ϕ_2 be the first positive eigenfunction of the Dirichlet fractional Laplacian in $B(0, 2)$ and λ_2 the associated eigenvalue

$$\text{i.e. } \begin{cases} (-\Delta)^\alpha \phi_2 = \lambda_2 \phi_2 & \text{for } x \in B(0, 2), \\ \phi_2 = 0 & \text{for } x \in B(0, 2)^c, \\ \|\phi_2\|_\infty = 1. \end{cases}$$

Then the function

$$\underline{p}(x, t) := \sigma \times \phi_2(x) \times e^{-\lambda_2 t}$$

is a sub-solution to (3.6) (where σ is defined by (3.30)). According to the comparison principle, we have for all $(x, t) \in B(0, 1 + 2\nu) \times [0, 1]$

$$0 < \min_{\substack{s \in [0,1] \\ y \in B(0,1+2\nu)}} \underline{p}(y, s) = \sigma \times \min_{B(0,1+2\nu)} \phi_2 \times e^{-|\lambda_2|} \leq \underline{p}(x, t) \leq p(x, t).$$

We deduce that if c is small enough, then (3.26) holds true for all $x \in B(0, 1 + 2\nu)$.

Step 2. In this step we prove (3.27) which is a key element to prove (3.26) for $x \in \{|x| > 1 + 2\nu\} \cap \mathcal{O}$

Then, we focus on $\{|x| > 1 + \nu\}$. We split the fractional Laplacian into 2 parts :

$$(-\Delta)^\alpha p(x, t) = \int_{\mathbb{R}^d \setminus B(0, \nu)} \frac{p(x, t) - p(x + y, t)}{|y|^{d+2\alpha}} dy + L^\alpha p(x, t) = I_1(x, t) + L^\alpha p(x, t). \quad (3.32)$$

For I_1 , we obtain :

$$I_1(x, t) = \int_{\mathbb{R}^d \setminus B(0, \nu)} \frac{p(x, t) - p(x + y, t)}{|y|^{d+2\alpha}} dy = \lambda p(x, t) - \int_{\mathbb{R}^d \setminus B(0, \nu)} \frac{p(x + y, t)}{|y|^{d+2\alpha}} dy.$$

Since $|x| > 1 + \nu$, we have

$$\inf_{\substack{t \in (0,1) \\ z \in B(0,1+\nu)}} p(z, t) \int_{B(0,1)} \frac{1}{|z - x|^{d+2\alpha}} dz \leq \int_{B(-x,1)} \frac{p(x + y, t)}{|y|^{d+2\alpha}} dy \leq \int_{\mathbb{R}^d \setminus B(0, \nu)} \frac{p(x + y, t)}{|y|^{d+2\alpha}} dy. \quad (3.33)$$

Equation (3.33) ensures the existence of a positive constant $c_1 > 0$ such that we have for all $(x, t) \in (\mathcal{O} \cap \{|x| > 1 + \nu\}) \times [0, 1]$

$$\frac{c_1}{1 + |x|^{d+2\alpha}} \leq \int_{\mathbb{R}^d \setminus B(0, \nu)} \frac{p(x + y, t)}{|y|^{d+2\alpha}} dy.$$

It follows that

$$I_1(x, t) \leq \lambda p(x, t) - \frac{c_1}{1 + |x|^{d+2\alpha}}. \quad (3.34)$$

Equations (3.32) and (3.34) lead to (3.27). Moreover, if we define $v(x, t) = e^{\lambda t} \times p(x, t)$, we find the following system :

$$\begin{cases} \frac{c_1}{1 + |x|^{d+2\alpha}} \leq \partial_t v(x, t) + L^\alpha v(x, t) & \text{for } (x, t) \in (\mathcal{O} \cap \{|x| > 1 + \nu\}) \times]0, 1], \\ v(x, t) \geq 0 & \text{for } (x, t) \in (\mathcal{O} \cap \{|x| > 1 + \nu\})^c \times [0, 1], \\ v(x, t = 0) = n_0(x) \in C_0^\infty(\mathcal{O}, \mathbb{R}^+). \end{cases} \quad (3.35)$$

Step 3. By uniform continuity of $(x \mapsto \frac{1}{1+|x|^{d+2\alpha}})$ in \mathbb{R}^d , we deduce the existence of $c'_1 > 0$ such that for all $x_0 \in (\mathcal{O}_\nu \cap \{|x| > 1 + \nu\})$ and all $(x, t) \in (\mathcal{O}_\nu \cap \{|x| > 1 + 2\nu\}) \times]0, 1]$ we have

$$\frac{c'_1}{1 + |x_0|^{d+2\alpha}} 1_{B(0, \nu)}(x - x_0) \leq \frac{c_1}{1 + |x|^{d+2\alpha}} \leq \partial_t v(x, t) + L^\alpha v(x, t). \quad (3.36)$$

Inequality (3.36) gives that for all $(x, t) \in (\mathcal{O}_\nu \cap \{|x| > 1 + \nu\}) \times]0, 1]$

$$1_{B(0, \nu)}(x - x_0) \leq \partial_t \left(\frac{1 + |x_0|^{d+2\alpha}}{c'_1} v(x, t) \right) + L^\alpha \left(\frac{1 + |x_0|^{d+2\alpha}}{c'_1} v(x, t) \right).$$

Then, according to the comparison principle and Lemma 3.1, we deduce that

$$\forall x \in (\mathcal{O}_\nu \cap \{|x| > 1 + \nu\}), \quad c_\nu \tilde{\phi}_\nu(x - x_0) \leq \frac{1 + |x_0|^{d+2\alpha}}{c'_1} v(x, t = 1). \quad (3.37)$$

If we evaluate (3.37) at $x = x_0$, we obtain

$$\frac{c_\nu c'_1 e^{-\lambda} \tilde{\phi}_\nu(0)}{1 + |x_0|^{d+2\alpha}} \leq p(x_0, t = 1).$$

Defining $c_2 = c_\nu c'_1 e^{-\lambda} \tilde{\phi}_\nu(0)$ leads to (3.28).

Step 4. As in the proof of Proposition 3.1, we can show by contradiction that there exists a positive constant c_0 such that for all $x \in \mathbb{R}^d$,

$$c_0 \underline{\psi}(x) \leq \tilde{\phi}_\nu(x)$$

where $\underline{\psi}$ is defined in Lemma 3.2. Then we take $x_1 \in (\mathcal{O} \setminus \mathcal{O}_\nu) \cap \{|x| > 1 + 2\nu\}$. Since \mathcal{O} satisfies the uniform interior ball condition, there exists $x_0 \in \partial \mathcal{O}_\nu$ such that $x_1 \in B(x_0, \nu)$, $B(x_0, \nu) \subset \mathcal{O} \cap \{|x| > 1 + \nu\}$ and $\nu - |x_1 - x_0| = \delta(x_1)$. Thanks to (3.37) and the fourth point of Lemma 3.2, we deduce

$$c_\nu c_0 c_\nu \delta(x_1)^\alpha \leq c_\nu c_0 \underline{\psi}(x_1 - x_0) \leq c_\nu \phi_\nu(x_1 - x_0) \leq \frac{(|x_0| + 1)^{d+2\alpha}}{c'_1} v(x_1, t = 1).$$

We deduce that there exists $c_3 > 0$ such that (3.29) holds true.

Combining (3.28), (3.29) and (3.31) yields the conclusion of the Theorem. \square

We apply Theorem 3.2 to show that starting from $n(x, 0) \in C_0^\infty(\Omega) \cap C_c^\infty(\mathbb{R})$, the solution of (3.1) $n(\cdot, t = 1)$ has algebraic tails.

Proposition 3.2. *There exists two constants c_m and c_M depending on n_0 such that for all $x \in \Omega$, we have*

$$\frac{c_m \delta(x)^\alpha}{1 + |x|^{d+2\alpha}} \leq n(x, 1) \leq \frac{c_M \delta(x)^\alpha}{1 + |x|^{d+2\alpha}}. \quad (3.38)$$

Proof. Defining $M := \max(\max n_0, 1)$, the solution n belongs to the set $[0, M]$ (0 is a sub-solution and M is a super-solution).

We begin with the proof that $\frac{c_m \delta(x)^\alpha}{1 + |x|^{d+2\alpha}} \leq n(x, 1)$.

Let \underline{n} be the solution of :

$$\begin{cases} \partial_t \underline{n}(x, t) + (-\Delta)^\alpha \underline{n}(x, t) = -M \underline{n}(x, t) & \text{for all } (x, t) \in \Omega \times]0, +\infty[, \\ \underline{n}(x, t) = 0 & \text{for all } (x, t) \in \Omega^c \times [0, +\infty[, \\ \underline{n}(x, 0) = n_0(x) & \text{for all } x \in \mathbb{R}^d, \end{cases} \quad (3.39)$$

Thanks to the comparison principle, we deduce that for all $(x, t) \in \mathbb{R} \times [0, +\infty[$, we have

$$\underline{n}(x, t) \leq n(x, t).$$

Moreover, if we define $\underline{p}(x, t) = e^{Mt} \underline{n}(x, t)$, we find that \underline{p} is solution of (3.6). Since Ω fulfils the uniform interior and exterior ball condition, we deduce thanks to Theorem 3.2 that there exists $c_m > 0$ such that

$$\frac{c_m \delta(x)^\alpha}{1 + |x|^{d+2\alpha}} \leq \underline{n}(x, t = 1) \leq n(x, t = 1). \quad (3.40)$$

The proof works the same for the other bound. \square

3.5 The proof of Theorem 3.3

3.5.1 Rescaling and preparation

The aim of this subsection is to establish the following Theorem.

Theorem 3.5. *We assume (H1) and (H2) then for all $\nu > 0$, the following holds true*

1. *For all $c < \frac{|\lambda_0|}{d+2\alpha}$, there exists a constant $\sigma > 0$ and a time $t_\sigma > 0$ such that*

$$\forall (x, t) \in \left(\Omega_\nu \cap \{|x| < e^{ct}\} \right) \times]t_\sigma, +\infty[\text{ we have } \sigma < n(x, t). \quad (3.41)$$

2. *For all $C > \frac{|\lambda_0|}{d+2\alpha}$, there exists three constants $\bar{C}_1, \bar{C}_2, \kappa > 0$ such that we have for all $(x, t) \in \{|x| > e^{Ct}\} \times]1, +\infty[$*

$$n(x, t) \leq \frac{\bar{C}_1}{1 + \bar{C}_2 e^{\kappa t}}. \quad (3.42)$$

First we establish sub and super-solutions by performing the rescaling (3.9). Finally, we prove Theorem 3.5 by performing the inverse of this rescaling on the sub and super-solutions.

We rescale the solution of (3.1) as follows :

$$n_\varepsilon(x, t) = n\left(|x|^{\frac{1}{\varepsilon}} \frac{x}{|x|}, \frac{t}{\varepsilon}\right). \quad (3.43)$$

Next, the equation becomes

$$\begin{cases} \varepsilon \partial_t n_\varepsilon + (-\Delta)_\varepsilon^\alpha n_\varepsilon = n_\varepsilon(1 - n_\varepsilon) & \text{for } (x, t) \in \Omega^\varepsilon \times]0, +\infty[, \\ n_\varepsilon(x, t) = 0 & \text{for } (x, t) \in \Omega^{\varepsilon^c} \times]0, +\infty[, \\ n_\varepsilon(x, 0) = n_{0,\varepsilon}(x) & \in C_0^\infty(\Omega^\varepsilon, \mathbb{R}^+). \end{cases} \quad (1_\varepsilon)$$

where $(-\Delta)_\varepsilon^\alpha n_\varepsilon(x, t) = (-\Delta)^\alpha n\left(|x|^\frac{1}{\varepsilon} \frac{x}{|x|}, \frac{t}{\varepsilon}\right)$ and $\Omega^\varepsilon = \{x \in \mathbb{R}^d \mid |x|^\frac{1}{\varepsilon} x \in \Omega\}$.
Next, we set

$$g(x) := \frac{1}{1 + |x|^{d+2\alpha}}.$$

We state the behavior of g under the fractional Laplacian in the spirit of [35].

Lemma 3.3. *Let γ be a positive constant in $]0, \alpha[$ such that $2\alpha - \gamma < 1$. Let $\chi \in C^\alpha(\mathbb{R}^d)$ be a periodic positive function. Then there exists a positive constant C , such that we have for all $x \in \mathbb{R}^d$:*

(i) for all $a > 0$,

$$|(-\Delta)^\alpha g(ax)| \leq a^{2\alpha} C g(ax),$$

(ii) for all $a \in]0, 1]$,

$$|\widetilde{K}(g(a \cdot), \chi)(x)| \leq \frac{Ca^{2\alpha-\gamma}}{1 + (a|x|)^{d+2\alpha}} = Ca^{2\alpha-\gamma} g(a|x|),$$

where $\widetilde{K}(u, v)(x) = \int_{\mathbb{R}^d} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{d+2\alpha}} dy$ is such that

$$(-\Delta)^\alpha(u \times v) = (-\Delta)^\alpha(u) \times v + u \times (-\Delta)^\alpha(v) - \widetilde{K}(u, v).$$

Since, the same kind of result can be found in the appendix A of [77], we do not provide the proof of this lemma. Note that here, the lemma is stated with less regularity on χ such than in [77]. Nevertheless, there is no difficulty to adapt the proof.

Notation. As we have introduced $(-\Delta)_\varepsilon^\alpha n_\varepsilon(x, t) = (-\Delta)^\alpha n\left(|x|^\frac{1}{\varepsilon} \frac{x}{|x|}, \frac{t}{\varepsilon}\right)$, we introduce

$$\widetilde{K}_\varepsilon(u, v)(x, t) = \widetilde{K}(u, v)\left(|x|^\frac{1}{\varepsilon} x, \frac{t}{\varepsilon}\right).$$

For any application $h : \mathbb{R}^d \rightarrow \mathbb{R}$, we define

$$\begin{aligned} h_\varepsilon : \mathbb{R}^d &\rightarrow \mathbb{R} \\ x &\mapsto h\left(|x|^\frac{1}{\varepsilon} x\right). \end{aligned}$$

For any set \mathcal{U} , we will denote

$$\mathcal{U}^\varepsilon = \left\{x \in \mathbb{R}^d \mid |x|^\frac{1}{\varepsilon} x \in \mathcal{U}\right\}. \quad (3.44)$$

For reasons of brevity, we will always denote $(\mathcal{U}_\nu)^\varepsilon$ by $\mathcal{U}_\nu^\varepsilon$.

In the following, we denote by c_0 and C_0 the positive constants provided by [97] such that

$$c_0 \delta(x)^\alpha \leq \phi_0(x) \leq C_0 \delta(x)^\alpha. \quad (3.45)$$

Proposition 3.3. *We assume (H1) and (H2). If we set $C_m = \min\left(\frac{|\lambda_0|}{2(\max\{\phi_0+1\}, C_0)}, \frac{c_m}{C_0}, 1\right)$ and $C_M = \frac{2|\lambda_0|+c_M}{\min(1, c_0)}$ where c_0, c_m, C_0 and C_M are introduced in (3.45) and (3.38) then there exists $\varepsilon_0 > 0$ such that for all $\varepsilon < \varepsilon_0$, the following holds true.*

1. For $t \in]0, \frac{4}{\varepsilon^2}[$, if f_ε^m is defined as

$$f_\varepsilon^m(x, t) = \frac{C_m \min\left(e^{-\frac{1}{\varepsilon} + \frac{\varepsilon t}{4}}, 1\right)}{1 + e^{\frac{-(|\lambda_0| - \varepsilon^2)t}{\varepsilon} - \frac{1}{\varepsilon} |x|^{d+2\alpha}}} \times (\phi_{0,\varepsilon}(x) + \varepsilon),$$

then it is a sub-solution of (1_ε) in $\Omega^\varepsilon \times]0, \frac{4}{\varepsilon^2}[$.

2. For $t \in]\frac{4}{\varepsilon^2}, +\infty[$, if f_ε^m is defined as

$$f_\varepsilon^m(x, t) = \frac{C_m}{1 + e^{\frac{-|\lambda_0|t}{\varepsilon} + \varepsilon \arctan(t - \frac{4}{\varepsilon^2}) + \frac{3}{\varepsilon} |x|^{\frac{d+2\alpha}{\varepsilon}}}} \times (\phi_{0,\varepsilon}(x) + \varepsilon),$$

then it is a sub-solution of (1_ε) in $\Omega^\varepsilon \times]\frac{4}{\varepsilon^2}, +\infty[$.

3. If f_ε^M is defined as

$$f_\varepsilon^M(x, t) = \frac{C_M \times \phi_{0,\varepsilon}(x)}{1 + e^{\frac{-|\lambda_0|t}{\varepsilon} - \varepsilon \arctan(t) - \frac{1}{\varepsilon} |x|^{\frac{d+2\alpha}{\varepsilon}}}},$$

then it is a super-solution of (1_ε) in $\Omega^\varepsilon \times]0, +\infty[$.

4. For all $(x, t) \in \mathbb{R}^d \times [0, +\infty[$, $f_\varepsilon^m(x, t) \leq n_\varepsilon(x, t + \varepsilon) + \varepsilon$ and $n_\varepsilon(x, t + \varepsilon) \leq f_\varepsilon^M(x, t)$.

Remark. 1. In the establishment of the sub and the super solutions, the choice of \arctan is not primordial. We only need a positive increasing and smooth function h which satisfies

$$e^{\frac{-t}{\varepsilon}} \leq \varepsilon h'(t)$$

for all $t > t_0$ and ε small enough. In [85], $h(t) = t$ but it does not allow to recover the level set of the solution as precisely as in [45]. The choice of a bounded function h (such as \arctan) allows to recover the same level of precision in the establishment of the level sets.

2. The study (and the definition) of f_ε^m is split into two parts. For small time, the term $e^{-\frac{1}{\varepsilon}}$ in the denominator is necessary in order to control the term $(-\Delta)^\alpha f_\varepsilon^m$ for small time. But, to use the comparison principle (and establish 4.), we have to check that the initial data are ordered in the right way. This is why, the term $e^{-\frac{1}{\varepsilon}}$ is needed in the numerator. However, this last term is an obstacle to establish the level sets result. Therefore, the trick is to "kill" this term for large time by replacing it by $\min(e^{\frac{\varepsilon t}{4} - \frac{1}{\varepsilon}}, 1)$. This is why, we split the study of f_ε^m into two parts : when t is small (i.e. $t < \frac{4}{\varepsilon^2}$) and when t is large (i.e. $t > \frac{4}{\varepsilon^2}$).

Proof. We begin by proving (1). Let (x, t) be in $\Omega^\varepsilon \times]0, \frac{4}{\varepsilon^2}[$. We define :

$$\psi_\varepsilon(x, t) = \frac{C_m}{1 + e^{-\frac{t}{\varepsilon}(|\lambda_0| - \varepsilon^2) - \frac{1}{\varepsilon} |x|^{\frac{d+2\alpha}{\varepsilon}}}} = C_m g_\varepsilon(e^{-\frac{t(|\lambda_0| - \varepsilon^2) - 1}{d+2\alpha}} x) \quad \text{and} \quad a(t) = e^{-\frac{1}{\varepsilon} + \frac{\varepsilon t}{4}}$$

thus $f_\varepsilon^m(x, t) = a(t) \times \psi_\varepsilon(x, t) \times (\phi_{0,\varepsilon}(x) + \varepsilon)$.

First, we bound $\varepsilon \partial_t \psi_\varepsilon$ from above :

$$\begin{aligned} \varepsilon \partial_t \psi_\varepsilon(x, t) &= \varepsilon \frac{C_m \frac{(|\lambda_0| - \varepsilon^2)}{\varepsilon} e^{-\frac{t}{\varepsilon}(|\lambda_0| - \varepsilon^2) - \frac{1}{\varepsilon} |x|^{\frac{d+2\alpha}{\varepsilon}}}}{(1 + e^{-\frac{t}{\varepsilon}(|\lambda_0| - \varepsilon^2) - \frac{1}{\varepsilon} |x|^{\frac{d+2\alpha}{\varepsilon}}})^2} \\ &= \psi_\varepsilon(x, t) \left[\frac{(|\lambda_0| - \varepsilon^2) e^{-\frac{t}{\varepsilon}(|\lambda_0| - \varepsilon^2) - \frac{1}{\varepsilon} |x|^{\frac{d+2\alpha}{\varepsilon}}}}{1 + e^{-\frac{t}{\varepsilon}(|\lambda_0| - \varepsilon^2) - \frac{1}{\varepsilon} |x|^{\frac{d+2\alpha}{\varepsilon}}}} \right] \\ &\leq \psi_\varepsilon(x, t) [|\lambda_0| - \varepsilon^2 - \psi_\varepsilon(x, t) (\phi_{0,\varepsilon}(x) + \varepsilon)] \\ &\leq \psi_\varepsilon(x, t) [|\lambda_0| - \varepsilon^2 - f_\varepsilon(x, t)]. \end{aligned} \tag{3.46}$$

The last inequalities hold because $a(t) \leq 1$ and denoting by $D = e^{-\frac{t}{\varepsilon}(|\lambda_0| - \varepsilon^2) - \frac{1}{\varepsilon}|x|^{\frac{d+2\alpha}{\varepsilon}}}$ and using the definition of C_m , we obtain for all $\varepsilon < \min(\sqrt{\frac{|\lambda_0|}{2}}, 1)$

$$\begin{aligned} |\lambda_0| - \varepsilon^2 - \psi_\varepsilon(\phi_{0,\varepsilon} + \varepsilon) - (|\lambda_0| - \varepsilon^2) \frac{D}{1+D} &= \frac{|\lambda_0| - \varepsilon^2 - C_m(\phi_{0,\varepsilon} + \varepsilon)}{1+D} \\ &\geq \frac{|\lambda_0| - \varepsilon^2 - \frac{|\lambda_0|}{2}}{1+D} \\ &\geq 0. \end{aligned}$$

Next, we compute $(-\Delta)_\varepsilon^\alpha f_\varepsilon^m(x, t)$

$$\begin{aligned} (-\Delta)_\varepsilon^\alpha f_\varepsilon^m(x, t) &= a(t)(\phi_{0,\varepsilon}(x) + \varepsilon)(-\Delta)_\varepsilon^\alpha \psi_\varepsilon(x, t) + a(t)\psi_\varepsilon(x, t)(-\Delta)_\varepsilon^\alpha \phi_{0,\varepsilon}(x) \\ &\quad - a(t)\widetilde{K}_\varepsilon(\psi, (\phi_0 + \varepsilon))(x, t). \end{aligned}$$

Combining (3.46) and the above equality we find :

$$\begin{aligned} &\varepsilon \partial_t f_\varepsilon^m(x, t) + (-\Delta)_\varepsilon^\alpha f_\varepsilon^m(x, t) - f_\varepsilon^m(x, t)(1 - f_\varepsilon^m(x, t)) \\ &\leq f_\varepsilon^m(x, t)(|\lambda_0| - \frac{3\varepsilon^2}{4} - f_\varepsilon^m(x, t)) + a(t)(\phi_{0,\varepsilon}(x) + \varepsilon)(-\Delta)_\varepsilon^\alpha \psi_\varepsilon(x, t) \\ &\quad + a(t)\psi_\varepsilon(x, t)(-\Delta)_\varepsilon^\alpha \phi_{0,\varepsilon}(x) - a(t)\widetilde{K}_\varepsilon(\psi, \phi_0 + \varepsilon)(x, t) - f_\varepsilon^m(x, t)(1 - f_\varepsilon^m(x, t)) \\ &= f_\varepsilon^m(x, t)(|\lambda_0| - \frac{3\varepsilon^2}{4}) + a(t)(\phi_{0,\varepsilon}(x) + \varepsilon)(-\Delta)_\varepsilon^\alpha \psi_\varepsilon(x, t) + a(t)(\lambda_0 + 1)\psi_\varepsilon(x, t)\phi_{0,\varepsilon}(x) \\ &\quad - f_\varepsilon^m(x, t) - a(t)\widetilde{K}_\varepsilon(\psi, \phi_0 + \varepsilon)(x, t) \\ &= f_\varepsilon^m(x, t)(|\lambda_0| - \frac{3\varepsilon^2}{4}) + a(t)(\lambda_0 + 1)\psi_\varepsilon(x, t)(\phi_{0,\varepsilon}(x) + \varepsilon)(x, t) - f_\varepsilon^m(x, t) \\ &\quad - a(t)\varepsilon(\lambda_0 + 1)\psi_\varepsilon(x, t) + a(t)(\phi_{0,\varepsilon}(x) + \varepsilon)(-\Delta)_\varepsilon^\alpha \psi_\varepsilon(x, t) - a(t)\widetilde{K}_\varepsilon(\psi, \phi_0 + \varepsilon)(x, t) \\ &= -\frac{3\varepsilon^2}{4}f_\varepsilon^m(x, t) - a(t)\varepsilon(\lambda_0 + 1)\psi_\varepsilon(x, t) + a(t)(\phi_{0,\varepsilon}(x) + \varepsilon)(-\Delta)_\varepsilon^\alpha \psi_\varepsilon(x, t) \\ &\quad - a(t)\widetilde{K}_\varepsilon(\psi, \phi_0 + \varepsilon)(x, t). \end{aligned} \tag{3.47}$$

Thanks to Lemma 3.3, we obtain

$$\begin{aligned} |(-\Delta)_\varepsilon^\alpha \psi_\varepsilon(x, t)| &= |C_m(-\Delta)_\varepsilon^\alpha (g_\varepsilon(e^{-\frac{t(|\lambda_0| - \varepsilon^2) + 1}{d+2\alpha}})) (x)| \\ &\leq |C_m e^{-\frac{2\alpha[t(|\lambda_0| - \varepsilon^2) + 1]}{\varepsilon(d+2\alpha)}} (g_\varepsilon)(e^{-\frac{t(|\lambda_0| - \varepsilon^2) + 1}{d+2\alpha}} x)|. \end{aligned}$$

We deduce that there exists $\varepsilon_1 > 0$ such that for all $\varepsilon < \varepsilon_1$:

$$|(-\Delta)_\varepsilon^\alpha \psi_\varepsilon(x, t)| \leq \frac{\varepsilon^2}{4} \psi_\varepsilon(x, t). \tag{3.48}$$

Since $(\phi_0 + \varepsilon)$ is periodic, positive and C^α according to [97] (Proposition 1.1), we conclude from Lemma 3.3 that there exists $\gamma \in]0, \alpha[$ and a constant C such that

$$|\widetilde{K}_\varepsilon(\psi, \phi_0 + \varepsilon)(x, t)| \leq C e^{-\frac{[t(|\lambda_0| - \varepsilon^2) + 1](2\alpha - \gamma)}{\varepsilon(d+2\alpha)}} \psi_\varepsilon(x, t).$$

We deduce the existence of $\varepsilon_2 > 0$ such that for all $\varepsilon < \varepsilon_2$, we have

$$|\widetilde{K}_\varepsilon(\psi, \phi_0 + \varepsilon)(x, t)| \leq \frac{\varepsilon^3}{4} \psi_\varepsilon(x, t) = \frac{\varepsilon^2 \min(\phi_{0,\varepsilon} + \varepsilon)}{4} \psi_\varepsilon(x, t). \tag{3.49}$$

Noticing that $(\lambda_0 + 1) > 0$, inserting (3.48) and (3.49) into (3.47), we conclude that for all $\varepsilon < \varepsilon_0 := \min(\varepsilon_1, \varepsilon_2, \sqrt{\frac{|\lambda_0|}{2}}, 1)$ and $(x, t) \in \Omega^\varepsilon \times]0, \frac{4}{\varepsilon^2}[$ we have :

$$\begin{aligned}
 & \varepsilon \partial_t f_\varepsilon^m(x, t) + (-\Delta)_\varepsilon^\alpha f_\varepsilon^m(x, t) - f_\varepsilon^m(x, t) + f_\varepsilon^m(x, t)^2 \\
 & \leq -\frac{3\varepsilon^2}{4} f_\varepsilon^m(x, t) - a(t)\varepsilon(\lambda_0 + 1)\psi_\varepsilon(x, t) \\
 & \quad + a(t)(\phi_{0,\varepsilon} + \mu)(x)(-\Delta)_\varepsilon^\alpha \psi_\varepsilon(x, t) - a(t)\widetilde{K}_\varepsilon(\psi, \phi_0 + \varepsilon)(x, t) \\
 & \leq -\frac{3\varepsilon^2}{4} f_\varepsilon^m(x, t) + \frac{\varepsilon^2}{4} f_\varepsilon(x, t) + \frac{\varepsilon^2}{4} f_\varepsilon(x, t) \\
 & \leq -\frac{\varepsilon^2}{4} f_\varepsilon^m(x, t) \\
 & \leq 0.
 \end{aligned}$$

Therefore, f_ε^m is a sub-solution of (1_ε) for $(x, t) \in \Omega^\varepsilon \times]0, \frac{4}{\varepsilon^2}[$. It concludes the proof of (1).

We continue by proving (2). Let (x, t) be in $\Omega^\varepsilon \times]\frac{4}{\varepsilon^2}, +\infty[$. We define :

$$\begin{aligned}
 \psi_\varepsilon(x, t) &= \frac{C_m}{1 + e^{-\frac{|\lambda_0|t}{\varepsilon} + \varepsilon \arctan(t - \frac{4}{\varepsilon^2}) + \frac{3}{\varepsilon}|x|^{\frac{d+2\alpha}{\varepsilon}}}} = C_m g_\varepsilon(e^{-\frac{|\lambda_0|t + \varepsilon^2 \arctan(t - \frac{4}{\varepsilon^2}) + 3}{d+2\alpha}} x) \\
 \text{thus } f_\varepsilon^m(x, t) &= \psi_\varepsilon(x, t) \times (\phi_{0,\varepsilon}(x) + \varepsilon).
 \end{aligned}$$

First, we bound $\varepsilon \partial_t \psi_\varepsilon$ from above :

$$\begin{aligned}
 \varepsilon \partial_t \psi_\varepsilon(x, t) &= \varepsilon \frac{C_m \frac{(|\lambda_0| - \frac{\varepsilon^2}{1 + (t - \frac{4}{\varepsilon^2})^2})}{\varepsilon} e^{-\frac{|\lambda_0|t}{\varepsilon} + \varepsilon \arctan(t - \frac{4}{\varepsilon^2}) + \frac{3}{\varepsilon}|x|^{\frac{d+2\alpha}{\varepsilon}}}}{(1 + e^{-\frac{|\lambda_0|t}{\varepsilon} + \varepsilon \arctan(t - \frac{4}{\varepsilon^2}) + \frac{3}{\varepsilon}|x|^{\frac{d+2\alpha}{\varepsilon}}})^2} \\
 &= \psi_\varepsilon(x, t) \left[\left(|\lambda_0| - \frac{\varepsilon^2}{1 + (t - \frac{4}{\varepsilon^2})^2} \right) \frac{e^{-\frac{|\lambda_0|t}{\varepsilon} + \varepsilon \arctan(t - \frac{4}{\varepsilon^2}) + \frac{3}{\varepsilon}|x|^{\frac{d+2\alpha}{\varepsilon}}}}{1 + e^{-\frac{|\lambda_0|t}{\varepsilon} + \varepsilon \arctan(t - \frac{4}{\varepsilon^2}) + \frac{3}{\varepsilon}|x|^{\frac{d+2\alpha}{\varepsilon}}}} \right] \quad (3.50) \\
 &\leq \psi_\varepsilon(x, t) \left[|\lambda_0| - \frac{\varepsilon^2}{1 + (t - \frac{4}{\varepsilon^2})^2} - \psi_\varepsilon(x, t)(\phi_{0,\varepsilon}(x) + \varepsilon) \right] \\
 &\leq \psi_\varepsilon(x, t) \left[|\lambda_0| - \frac{\varepsilon^2}{1 + (t - \frac{4}{\varepsilon^2})^2} - f_\varepsilon(x, t) \right].
 \end{aligned}$$

Denoting by $D = e^{-\frac{|\lambda_0|t}{\varepsilon} + \varepsilon \arctan(t - \frac{4}{\varepsilon^2}) + \frac{3}{\varepsilon}|x|^{\frac{d+2\alpha}{\varepsilon}}}$ and using the definition of C_m , we obtain indeed for all $\varepsilon < \min(\sqrt{\frac{|\lambda_0|}{2}}, 1)$

$$\begin{aligned}
 & |\lambda_0| - \frac{\varepsilon^2}{1 + (t - \frac{4}{\varepsilon^2})^2} - \psi_\varepsilon(\phi_{0,\varepsilon} + \varepsilon) - \left(|\lambda_0| - \frac{\varepsilon^2}{1 + (t - \frac{4}{\varepsilon^2})^2} \right) \frac{D}{1 + D} \\
 &= \frac{|\lambda_0| - \frac{\varepsilon^2}{1 + (t - \frac{4}{\varepsilon^2})^2} - C_m(\phi_{0,\varepsilon} + \varepsilon)}{1 + D} \\
 &\geq \frac{|\lambda_0| - \varepsilon^2 - \frac{|\lambda_0|}{2}}{1 + D} \\
 &\geq 0.
 \end{aligned}$$

Next, we compute $(-\Delta)_\varepsilon^\alpha f_\varepsilon^m(x, t)$

$$(-\Delta)_\varepsilon^\alpha f_\varepsilon^m(x, t) = (\phi_{0,\varepsilon} + \varepsilon)(x)(-\Delta)_\varepsilon^\alpha \psi_\varepsilon(x, t) + \psi_\varepsilon(x, t)(-\Delta)_\varepsilon^\alpha \phi_{0,\varepsilon}(x) - \widetilde{K}_\varepsilon(\psi, (\phi_0 + \varepsilon))(x, t).$$

Combining (3.50) and the above equality we find, following similar computations as in (1),

$$\begin{aligned} & \varepsilon \partial_t f_\varepsilon^m(x, t) + (-\Delta)_\varepsilon^\alpha f_\varepsilon^m(x, t) - f_\varepsilon^m(x, t)(1 - f_\varepsilon^m(x, t)) \\ & \leq -\frac{\varepsilon^2}{1 + (t - \frac{4}{\varepsilon^2})^2} f_\varepsilon^m(x, t) + (\phi_{0,\varepsilon}(x) + \varepsilon)(-\Delta)_\varepsilon^\alpha \psi_\varepsilon(x, t) - \widetilde{K}_\varepsilon(\psi, \phi_0 + \varepsilon)(x, t). \end{aligned} \quad (3.51)$$

Thanks to Lemma 3.3, we obtain

$$\begin{aligned} |(-\Delta)_\varepsilon^\alpha \psi_\varepsilon(x, t)| &= |C_m(-\Delta)_\varepsilon^\alpha (g_\varepsilon(e^{\frac{[-|\lambda_0|t + \varepsilon^2 \arctan(t - \frac{4}{\varepsilon^2}) + 3]}{d+2\alpha}}))|(x)| \\ &\leq |C_m e^{-\frac{2\alpha[-|\lambda_0|t + \varepsilon^2 \arctan(t - \frac{4}{\varepsilon^2}) + 3]}{\varepsilon(d+2\alpha)}} (g_\varepsilon)(e^{\frac{[-|\lambda_0|t + \varepsilon^2 \arctan(t - \frac{4}{\varepsilon^2}) + 3]}{d+2\alpha}} x)|. \end{aligned}$$

Noticing that since $t \geq \frac{4}{\varepsilon^2}$, there exists $\varepsilon_3 > 0$ such that for all $\varepsilon < \varepsilon_3$

$$-|\lambda_0|t + \varepsilon^2 \arctan(t - \frac{4}{\varepsilon^2}) + 3 \leq \frac{-|\lambda_0|t}{2} - \frac{2|\lambda_0|}{\varepsilon^2} + \varepsilon^2 \frac{\pi}{2} + 3 \leq \frac{-|\lambda_0|t}{2}.$$

We deduce the existence of $\varepsilon_4 < \varepsilon_3$ such that for all $\varepsilon < \varepsilon_4$:

$$|(-\Delta)_\varepsilon^\alpha \psi_\varepsilon(x, t)| \leq e^{\frac{-\alpha|\lambda_0|t}{\varepsilon(d+2\alpha)}} \psi_\varepsilon(x, t) \leq \frac{\varepsilon^2}{3(1 + (t - \frac{4}{\varepsilon^2})^2)} \psi_\varepsilon(x, t). \quad (3.52)$$

Following similar computations, we deduce the existence of $\varepsilon_5 > 0$ such that for all $\varepsilon < \varepsilon_5$, we have

$$|\widetilde{K}_\varepsilon(\psi, \phi_0 + \varepsilon)(x, t)| \leq \frac{\varepsilon^3}{3(1 + (t - \frac{4}{\varepsilon^2})^2)} \psi_\varepsilon(x, t) = \frac{\varepsilon^2 \min(\phi_{0,\varepsilon} + \varepsilon)}{3(1 + (t - \frac{4}{\varepsilon^2})^2)} \psi_\varepsilon(x, t). \quad (3.53)$$

Inserting (3.52) and (3.53) into (3.51) and defining $\varepsilon_0 := \min(\varepsilon_3, \varepsilon_4, \varepsilon_5, \sqrt{\frac{|\lambda_0|}{2}}, 1)$, we conclude that for all $\varepsilon < \varepsilon_0$ and all $(x, t) \in \Omega^\varepsilon \times]\frac{4}{\varepsilon^2}, +\infty[$ we have :

$$\begin{aligned} & \varepsilon \partial_t f_\varepsilon^m(x, t) + (-\Delta)_\varepsilon^\alpha f_\varepsilon^m(x, t) - f_\varepsilon^m(x, t) + f_\varepsilon^m(x, t)^2 \\ & \leq -\frac{\varepsilon^2}{(1 + (t - \frac{4}{\varepsilon^2})^2)} f_\varepsilon^m(x, t) + (\phi_{0,\varepsilon} + \mu)(x)(-\Delta)_\varepsilon^\alpha \psi_\varepsilon(x, t) - \widetilde{K}_\varepsilon(\psi, \phi_0 + \varepsilon)(x, t) \\ & \leq -\frac{\varepsilon^2}{3(1 + (t - \frac{4}{\varepsilon^2})^2)} f_\varepsilon^m(x, t) \\ & \leq 0. \end{aligned}$$

Therefore, f_ε^m is a sub-solution of (1_ε) for $(x, t) \in \Omega^\varepsilon \times]\frac{4}{\varepsilon^2}, +\infty[$.

The proof of (3) follows the same arguments as the proof of (2).

For the proof of (4), we have to check that the initial data are ordered in the right way. According to (3.38), (3.45) and the definition of C_m , we have that for all $x \in \Omega^\varepsilon$,

$$f_\varepsilon^m(x, 0) = \frac{C_m(\phi_{0,\varepsilon}(x) + \varepsilon)}{e^{\frac{1}{\varepsilon}} + |x|^{\frac{d+2\alpha}{\varepsilon}}} \leq \frac{c_m \delta_\varepsilon(x)^\alpha}{1 + |x|^{\frac{d+2\alpha}{\varepsilon}}} + \varepsilon \leq n_\varepsilon(x, \varepsilon) + \varepsilon.$$

Furthermore,

$$\forall (x, t) \in (\Omega^\varepsilon)^c \times [0, +\infty[, \text{ we know that } f_\varepsilon^m(x, t) \leq \varepsilon \leq n_\varepsilon(x, t + \varepsilon) + \varepsilon. \quad (3.54)$$

Thus we conclude from the comparison principle that for all $(x, t) \in \mathbb{R}^d \times [0, \frac{4}{\varepsilon^2}[$, we have

$$f_\varepsilon^m(x, t) \leq n_\varepsilon(x, t + \varepsilon) + \varepsilon. \quad (3.55)$$

Since, we have that for all $x \in \mathbb{R}^d$

$$\lim_{t \rightarrow \frac{4}{\varepsilon^2}, t < \frac{4}{\varepsilon^2}} f_\varepsilon^m(x, t) = \lim_{t \rightarrow \frac{4}{\varepsilon^2}, t > \frac{4}{\varepsilon^2}} f_\varepsilon^m(x, t)$$

and recalling that f_ε^m is also a subsolution in $\Omega^\varepsilon \times]\frac{4}{\varepsilon^2}, +\infty[$ and the inequality (3.54), we deduce thanks to the comparison principle that for all $(x, t) \in \mathbb{R}^d \times [0, +\infty[$

$$f_\varepsilon^m(x, t) \leq n_\varepsilon(x, t + \varepsilon) + \varepsilon. \quad (3.56)$$

The other inequality can be obtained following similar arguments. \square

A direct consequence of (3.56) is that if ε fulfills the assumption of Proposition 3.3 then

$$\forall (x, t) \in \mathbb{R}^d \times]\frac{4}{\varepsilon^2}, +\infty[\quad \frac{C_m \times \phi_{0,\varepsilon}(x)}{1 + e^{\frac{-|\lambda_0|t}{\varepsilon} + \varepsilon \arctan(t - \frac{4}{\varepsilon^2}) + \frac{3}{\varepsilon} |x|^{\frac{d+2\alpha}{\varepsilon}}}} \leq n_\varepsilon(x, t + \varepsilon) + \varepsilon. \quad (3.57)$$

Next, we establish some consequences of Theorem 3.3 on the solution n without the scaling (3.9).

Proof of Theorem 3.5. First, we prove the first point by using the sub-solution f_ε^m . It is sufficient to prove it for $\nu < r_0$ (where r_0 is the radius of the uniform interior ball condition satisfied by Ω_0).

Proof of 1. Set $\nu > 0$, $c < \frac{|\lambda_0|}{d+2\alpha}$ and $\varepsilon_0 > 0$ provided by Proposition 3.3. we deduce that for $\varepsilon = \frac{\min\left(\varepsilon_0, \sqrt{\frac{4(|\lambda_0| - (d+2\alpha)c)}{\frac{\pi}{2} + 3}}, \frac{C_m \min \phi_0}{4}\right)}{2}$ and for all $(x, t) \in (\Omega_\nu^\varepsilon \cap \{|x| < e^{ct}\}) \times]\frac{4}{\varepsilon^2}, +\infty[$ we have

$$\frac{C_m \phi_0 (|x|^{\frac{1}{\varepsilon} - 1} x)}{1 + e^{\frac{-|\lambda_0|t}{\varepsilon} + \varepsilon \arctan(t - \frac{4}{\varepsilon^2}) + \frac{3}{\varepsilon} |x|^{\frac{d+2\alpha}{\varepsilon}}}} \leq n_\varepsilon(x, t + \varepsilon) + \varepsilon$$

which implies

$$\frac{C_m \min_{\Omega_\nu} \phi_0}{1 + e^{\frac{t}{\varepsilon}((d+2\alpha)c - |\lambda_0|) + \varepsilon \arctan(t - \frac{4}{\varepsilon^2}) + \frac{3}{\varepsilon}}}} \leq n_\varepsilon(x, t + \varepsilon) + \varepsilon.$$

If we perform the inverse scaling to (3.9), since $\varepsilon < \sqrt{\frac{4(|\lambda_0| - (d+2\alpha)c)}{\frac{\pi}{2} + 3}}$, it follows that for all $t > \frac{4}{\varepsilon^3}$

$$t(c(d+2\alpha) - |\lambda_0|) + \varepsilon \arctan(\varepsilon t - \frac{4}{\varepsilon^2}) + \frac{3}{\varepsilon} < 0.$$

We deduce that for all $(x, t) \in (\Omega_\nu \cap \{|x| < e^{ct}\}) \times]\frac{4}{\varepsilon^3} + 1, +\infty[$

$$\frac{C_m \min_{\Omega_\nu} \phi_0}{2} \leq \frac{C_m \min_{\Omega_\nu} \phi_0}{1 + e^{t(c(d+2\alpha) - |\lambda_0|) + \varepsilon \arctan(\varepsilon t - \frac{4}{\varepsilon^2}) + \frac{3}{\varepsilon}}} \leq n(x, t + 1) + \varepsilon.$$

If we define $\sigma = \frac{C_m \min_{\Omega_\nu} \phi_0}{4}$ and $t_\sigma = \frac{4}{\varepsilon^3} + 1$, we conclude that (3.41) holds true.

We prove the second point by using the super-solution f_ε^M .

Proof of 2. Let $C > \frac{|\lambda_0|}{d+2\alpha}$ and we fix $\varepsilon = \varepsilon_0$ such that Proposition 3.3 holds true. It implies that

$$\forall (x, t) \in \mathbb{R}^d \times]0, +\infty[, \quad n_\varepsilon(x, t + \varepsilon) \leq f_\varepsilon^M(x, t).$$

If we perform the inverse scaling to (3.9), it follows that

$$n(x, t + 1) \leq \frac{C_M \times \max \phi_0}{1 + e^{-|\lambda_0|t - \varepsilon \arctan(\varepsilon t) - \frac{1}{\varepsilon}|x|} |x|^{d+2\alpha}}.$$

Then for all $(x, t) \in \{|x| > Ct\} \times]0, +\infty[$ we have

$$n(x, t + 1) \leq \frac{C_M \times \max \phi_0}{1 + e^{-\frac{\varepsilon\pi}{2} - \frac{1}{\varepsilon} e^{t[(d+2\alpha)C - |\lambda_0|]}}.$$

Defining $\bar{C}_1 := 2C_M \max \phi_0$, $\bar{C}_2 := e^{-\frac{\varepsilon\pi}{2} - \frac{1}{\varepsilon}}$ and $\kappa := (d + 2\alpha)C - |\lambda_0|$ then the conclusions follows. \square

3.5.2 The final argument

Proof of Theorem 3.3. We will prove (i) by splitting the proof into two parts : the upper bound and the lower bound. We will not provide the proof of (ii) since it is a direct application of 2. of Theorem 3.5.

Proof of (i). Let μ be a positive constant. We want to prove that there exists a time $t_\mu > 0$ such that for any $c < \frac{|\lambda_0|}{d+2\alpha}$ we have for all $(x, t) \in \{|x| < e^{ct}\} \times]t_\mu, +\infty[$

$$|n(x, t) - n_+(x)| \leq \mu.$$

First we establish that there exists a time $t_1 > 0$ such that

$$\forall (x, t) \in \Omega \times]t_1, +\infty[, \quad n(x, t) - n_+(x) \leq \mu \tag{3.58}$$

Next, we prove the existence of a time $t_2 > 0$ such that

$$\forall (x, t) \in \Omega \times]t_2, +\infty[, \quad -\mu \leq n(x, t) - n_+(x) \tag{3.59}$$

The difficult part will be to establish (3.59). This is why, we do not provide all the details of the proof of (3.58).

Proof that (3.58) holds true. Thanks to (3.38) and Proposition 3.1, we deduce the existence of a constant $C \geq 1$ such that

$$n(x, t = 1) \leq Cn_+(x).$$

Moreover, the solution \bar{n} of

$$\begin{cases} \partial_t \bar{n} + (-\Delta)^\alpha \bar{n} = \bar{n} - \bar{n} & \text{in } \Omega \times]1, +\infty[, \\ \bar{n}(x, t) = 0 & \text{in } \Omega^c \times [1, +\infty[, \\ \bar{n}(x, t = 1) = Cn_+(x) & \text{in } \Omega \end{cases}$$

is a super solution of (3.1). According to the comparison principle we deduce that

$$\forall (x, t) \in \mathbb{R} \times [1, +\infty[, \quad n(x, t) \leq \bar{n}(x, t). \tag{3.60}$$

One can easily observe that \bar{n} is periodic, decreasing in time and converges uniformly to n_+ in the whole domain Ω as $t \rightarrow +\infty$. Thus there exists a times $t_1 > 1$ such that

$$\forall (x, t) \in \Omega \times]t_1, +\infty[, \quad \bar{n}(x, t) - n_+(x) \leq \mu.$$

The conclusion follows.

Proof that (3.59) holds true. We split this part of the proof into two subparts, what happens on the boundary and what happens in the interior.

The boundary estimates. Since \bar{n} is decreasing in time and thanks to (3.60), we deduce that for all $(x, t) \in \Omega \times [1, +\infty[$

$$|n(x, t) - n_+(x)| \leq n(x, t) + n_+(x) \leq \bar{n}(x, t) + n_+(x) \leq (C + 1)n_+(x).$$

According to Proposition 3.1, we deduce that for all $(x, t) \in \Omega \times [1, +\infty[$

$$|n(x, t) - n_+(x)| \leq C(C + 1)\delta(x)^\alpha.$$

We conclude that for all $(x, t) \in \Omega \times [1, +\infty[$ such that $\delta(x) < \left(\frac{\mu}{C(C+1)}\right)^{\frac{1}{\alpha}} := \nu_1$ we have

$$|n(x, t) - n_+(x)| \leq \mu.$$

The interior estimates. Thanks to Theorem 3.1, we know that $n_+ \leq 1$ thus it is sufficient to prove the existence of $t_2 > 0$ such that

$$\forall (x, t) \in \left(\{|x| < e^{ct}\} \cap \Omega_{\nu_2}\right) \times]t_2, +\infty[\quad 1 - \mu \leq \frac{n(x, t)}{n_+(x)} \quad \text{where } \nu_2 = \min(\nu_1, r_1)$$

where ν_1 is provided by the previous step and r_1 by the uniform interior ball condition.

The idea is to approximate n_+ by the solution of (3.4) on a ball of radius M . Noticing that thanks to (H1), there exists $M_0 > 0$ such that for $M > M_0$, there exists a unique bounded positive solution $n_{M,+}$ of

$$\begin{cases} (-\Delta)^\alpha n_{M,+} = n_{M,+} - n_{M,+}^2 & \text{in } \Omega \cap B(0, M), \\ n_{M,+} = 0 & \text{in } (\Omega \cap B(0, M))^c \end{cases} \quad (3.61)$$

We claim that

$$\exists M_1 > M_0, \text{ such that } \forall M > M_1, \forall x \in \Omega_{0, \nu_2}, \quad (1 - \mu)^{\frac{1}{2}} \leq \frac{n_{M,+}(x)}{n_+(x)}. \quad (3.62)$$

The proof of this claim is postponed to the end of this paragraph. Next, we approach $n_{M,+}$ by the long time solution of the following equation :

$$\begin{cases} \partial_t n_{M,z} + (-\Delta)^\alpha n_{M,z} = n_{M,z} - n_{M,z}^2 & \text{in } (\Omega \cap B(0, M)) \times]0, +\infty[, \\ n_{M,z}(x, t) = 0 & \text{in } (\Omega \cap B(0, M))^c \times]0, +\infty[, \\ n_{M,z}(x, t = 0) = \sigma 1_{B(z, \frac{\nu_2}{4})}(x). \end{cases} \quad (3.63)$$

where σ is provided by Theorem 3.5 and $z \in \Omega_{0, \frac{\nu_2}{2}}$ will be fixed later on. We claim that

$$\exists t_\mu > 0, \text{ such that } \forall z \in \Omega_{0, \frac{\nu_2}{2}}, \forall (x, t) \in \Omega_{0, \nu_2} \times]t_\mu, +\infty[, \quad (1 - \mu)^{\frac{1}{2}} \leq \frac{n_{M,z}(x, t)}{n_{M,+}(x)}. \quad (3.64)$$

Again, the proof of this claim is postponed to the end of this section. Next, we define

$$t_\mu = t_\mu + t_\sigma \quad (3.65)$$

where t_μ is defined by (3.64) and t_σ by Theorem 3.5. Let $(x, t) \in (\Omega_{\nu_2} \cap \{|x| < e^{ct}\}) \times]t_\mu, +\infty[$ and $j \in \mathbb{Z}^d$ be such that $x \in \Omega_0 + a_j$. Since $\nu_2 < r_1$ (the radius of the uniform interior ball condition), we deduce the existence of $z_x \in \Omega_0, \frac{\nu_2}{2}$ such that

$$x \in B(z_x + a_j, \frac{\nu_2}{4}) \text{ and } \forall y \in B(z_x + a_j, \frac{\nu_2}{4}) \text{ there holds } y \in (\Omega_0 + a_j)_{\frac{\nu_2}{4}} \cap \{|y| < e^{ct}\}. \quad (3.66)$$

Remarking that $\frac{n(x,t)}{n_+(x)} = \frac{n(x,t)}{n_+(x-a_j)}$, we are going to control each terms of the following decomposition :

$$\frac{n(x,t)}{n_+(x)} = \frac{n(x,t)}{n_{M,z_x}(x-a_j, t-t_\sigma)} \times \frac{n_{M,z_x}(x-a_j, t-t_\sigma)}{n_{M,+}(x-a_j)} \times \frac{n_{M,+}(x-a_j)}{n_+(x-a_j)} = \text{I} \times \text{II} \times \text{III}$$

where n_{M,z_x} is defined in (3.63).

Control of I.

Thanks to (3.41) and (3.66), it follows that

$$\forall y \in B(z_x + a_j, \frac{\nu_2}{4}), \quad \sigma \leq n(y, t_\sigma).$$

Recalling that $n_{M,z_x}(x, 0) = \sigma 1_{B(z_x, \frac{\nu_2}{4})}(x)$, we conclude thanks to the comparison principle that

$$\forall (y, s) \in \mathbb{R}^d \times [t_\sigma, +\infty[\text{ we have } n_{M,z_x}(y-a_j, s-t_\sigma) \leq n(y, s).$$

Since $t > t_\sigma$, we conclude that

$$1 \leq \frac{n(x,t)}{n_{M,z_x}(x-a_j, t-t_\sigma)}. \quad (3.67)$$

Control of II.

Since $t - t_\sigma > t_\mu$, we deduce thanks to (3.64) that

$$(1 - \mu)^{\frac{1}{2}} \leq \frac{n_{M,z_x}(x-a_j, t-t_\sigma)}{n_{M,+}(x-a_j)}. \quad (3.68)$$

Control of III.

Since $x - a_j \in \Omega_0$, we deduce thanks to (3.62) that

$$(1 - \mu)^{\frac{1}{2}} \leq \frac{n_{M,+}(x-a_j)}{n_+(x-a_j)}. \quad (3.69)$$

Combining (3.68), (3.69) and (3.67), we conclude that for all $(x, t) \in (\Omega_{\nu_2} \times]t_\mu, +\infty[) \cap \{(x, t) \in \mathbb{R}^d \times \mathbb{R}^+ \mid |x| < e^{ct}\}$, we obtain

$$1 - \mu \leq \frac{n(x,t)}{n_{M,z_x}(x-a_j, t-t_\sigma)} \times \frac{n_{M,z_x}(x-a_j, t-t_\sigma)}{n_{M,+}(x-a_j)} \times \frac{n_{M,+}(x-a_j)}{n_+(x-a_j)} = \frac{n(x,t)}{n_+(x-a_j)} = \frac{n(x,t)}{n_+(x)}.$$

This concludes the proof of Theorem 3.3. \square

It remains to prove the claims (3.62) and (3.64). The proof of (3.62) relies on the uniqueness result stated in Theorem (3.1).

Proof of (3.62). The map $(M \in]M_0, +\infty[\mapsto n_{M,+})$ is increasing as n_M is a sub-solution to the equation for $n_{M'}$ for $M' > M$. It converges to a weak solution of (3.4). By fractional elliptic regularity, the limit is a strong solution of (3.4). We conclude thanks to the uniqueness of the solution of (3.4) stated in Theorem 3.1. \square

The proof of (3.64) relies on a compactness argument.

Proof of (3.64). For a fixed $z \in \Omega_\nu$, the proof of convergence of $n_{M,z}$ to $n_{M,+}$ is classical thanks to (H1). For each $z \in \Omega_{0,\nu}$, there exists $t_z > 0$ such that

$$\forall (x, t) \in \mathbb{R}^d \times]t_z, +\infty[, \quad (1 - \mu)^{\frac{1}{2}} \leq \frac{n_{M,z}(x, t)}{n_{M,+}(x)}.$$

We claim that $\sup_{z \in \Omega_{0,\nu}} t_z < +\infty$. This assertion is true by compactness of $\overline{\Omega}_{0,\nu}$ (otherwise there exists $\bar{z} \in \overline{\Omega}_{0,\nu}$ such that $t_{\bar{z}} = +\infty$ which is a contradiction). \square

3.5.3 The result on the level sets

In this section, we use the sub and super-solutions established in Section 3.5.1 to prove Theorem 3.4.

Proof. Let $\nu > 0$ be such that $\Omega_\nu \neq \emptyset$. We define

$$c_\nu = C_m \min_{y \in \Omega_\nu} \phi_0(y)$$

where the function C_m is defined in Proposition 3.3. Set $\mu \in]0, c_\nu[$ and we define $\varepsilon = \min\left(\frac{C_m \min_{y \in \Omega_\nu} \phi_0(y) - \mu}{2}, \varepsilon_0, \mu\right)$ where ε_0 is provided by Proposition 3.3. Next we define $t_\mu = \frac{4}{\varepsilon^3} + 1$. Let $(x, t) \in \Omega_\nu \times]t_\mu, +\infty[$ be such that $n(x, t) = \mu$. First, we prove that there exists $c > 0$ (independent of the choice of (x, t)) such that $ce^{|\lambda_0|t} \leq |x|^{d+2\alpha}$. Next, we prove the existence of $C > 0$ (independent of the choice of (x, t)) such that $|x|^{d+2\alpha} \leq Ce^{|\lambda_0|t}$. Defining $C_\mu = \max(C, c^{-1})$, the conclusion follows.

Existence of c . Thanks to Proposition 3.3, after the inverse scaling of (3.9), we obtain

$$\begin{aligned} & \frac{C_m(\phi_0(x) + \varepsilon)}{1 + e^{-|\lambda_0|(t-1) + \varepsilon \arctan(\varepsilon(t-1) - \frac{4}{\varepsilon^2}) + \frac{3}{\varepsilon}} |x|^{d+2\alpha}} \leq n(x, t) + \varepsilon = \mu + \varepsilon \\ \Rightarrow & C_m \min_{y \in \Omega_\nu} \phi_0(y) - \mu + \varepsilon(C_m - 1) \leq 2\mu e^{-|\lambda_0|(t-1) + \varepsilon \arctan(\varepsilon(t-1) - \frac{4}{\varepsilon^2}) + \frac{3}{\varepsilon}} |x|^{d+2\alpha} \\ \Rightarrow & \left(\frac{C_m \min_{y \in \Omega_\nu} \phi_0(y) - \mu}{2} e^{-\frac{\varepsilon\pi}{2} - \frac{3}{\varepsilon} - |\lambda_0|} \right) e^{|\lambda_0|t} \leq |x|^{d+2\alpha} \end{aligned}$$

If we define $c = \frac{(C_m \min_{y \in \Omega_\nu} \phi_0(y) - \mu) e^{-\frac{\varepsilon\pi}{2} - \frac{3}{\varepsilon} - |\lambda_0|}}{4\mu}$ then the conclusion follows.

Existence of C . Thanks to Proposition 3.3, after the inverse scaling of (3.9), we obtain

$$\mu = n(x, t) \leq \frac{C_M \phi_0(x)}{1 + e^{-|\lambda_0|(t-1) - \varepsilon \arctan(\varepsilon(t-1)) - \frac{1}{\varepsilon}} |x|^{d+2\alpha}}.$$

It follows

$$|x|^{d+2\alpha} \leq \left(\frac{(C_M \max \phi_0 - \mu) e^{\varepsilon \frac{\pi}{2} + \frac{1}{\varepsilon} - |\lambda_0|}}{\mu} \right) e^{|\lambda_0|t}.$$

If we define $C_\mu = \frac{(C_M \max \phi_0 - \mu) e^{\varepsilon \frac{\pi}{2} + \frac{1}{\varepsilon} - |\lambda_0|}}{\mu}$ and $c_\mu = C_\mu^{-1}$ then the conclusion follows. \square

Chapter 4

Properties of steady states for a class of non-local Fisher-KPP equations in general domains

Ce chapitre a fait l'objet d'un dépôt sur Arxiv et va faire l'objet d'une soumission dans un journal scientifique. Il a été co-écrit avec Jean-Michel Roquejoffre.

4.1 Introduction

4.1.1 Model, question, motivation

We investigate here existence and uniqueness of bounded positive solutions for the fractional Fisher-KPP equations of the form

$$\text{i.e. } \begin{cases} (-\Delta)^\alpha n_+(x) = n_+(x) - n_+^2(x) & \text{for } x \in \Omega, \\ n_+(x) = 0 & \text{for } x \in \Omega^c. \end{cases} \quad (4.1)$$

The domain Ω is an infinite union of patches, all of them but perhaps one being bounded. The operator $(-\Delta)^\alpha$ is the fractional Laplacian :

$$(-\Delta)^\alpha \phi(x) = C_\alpha PV \int_{\mathbb{R}^d} \frac{\phi(x) - \phi(y)}{|x - y|^{d+2\alpha}} dy \quad \text{with } C_\alpha = \frac{4^\alpha \Gamma(\frac{d}{2} + \alpha)}{\pi^{\frac{d}{2}} |\Gamma(-\alpha)|}, \quad (4.2)$$

we assume $\alpha < 1$ throughout the work. Clearly, existence and uniqueness would be false if $\alpha = 1$ (just think of a periodic union of large line segments), but nonlocality implies a solidarity between patches that may make existence and uniqueness become true. We will first see some effects of the nonlocality when dealing with principal eigenvalue problems, we will then try to understand how the solidarity forced by nonlocal diffusion eventually leads to existence and uniqueness. More important in our opinion, we will take advantage of the disconnectedness of the domain to derive precise estimates of possible solutions of (4.1) at infinity, that will eventually imply uniqueness.

The evolution equation

$$\begin{cases} \partial_t n(x, t) + (-\Delta)^\alpha n(x, t) = n(x, t) - n(x, t)^2 & \text{for } (x, t) \in \Omega \times]0, +\infty[, \\ n(x, t) = 0 & \text{for } (x, t) \in \Omega^c \times [0, +\infty[, \\ n(x, 0) = n_0(x), \end{cases} \quad (4.3)$$

models biological invasions. The variable n stands for a density of population. The fractional Laplacian models the fact that a species can jump from one point to another with a high rate. If a bounded solution of (4.3) converges as t tends to $+\infty$, it is either to 0, either to a non-trivial stationary state of (4.3) : a solution of (4.1). Thus, if (4.1) does not admit a bounded positive solution, we deduce that the species modeled by n will extinct. On the other hand, if there exists a unique bounded positive solution n_+ to (4.1) to which the solution n converges to n_+ as t tends to $+\infty$, the species will persist. An equation of type (4.3) was first introduced in 1937 by Fisher in [58] and by Kolmogorov, Petrovskii and Piscunov in [73] in the whole domain \mathbb{R}^d and with a standard diffusion.

Our model accounts for a situation where reproduction is allowed on some patches (perhaps an infinity of them) while the outside environment is lethal to the species. Moreover, the patches may be, individually, unfavourable to reproduction, and this may be true for all but one of them. The question is whether the species may survive in these conditions, and if such is the case, in which quantity. Of course this can only be possible because of nonlocal diffusion, survival being clearly impossible in the conditions just described, if the diffusion is the usual one. One of the main result of this work is that there is survival but if all the patches but one are individually unfavourable, the density of individuals will decay like a power of the distance to the favourable patch that we evaluate precisely.

Before that, we account for specific effects of the nonlocal diffusion for simple one-dimensional domains.

4.1.2 Some effects of nonlocality on the principal eigenvalue

We consider a domain made up of two patches of variable distance, and we wonder how the principal eigenvalue is affected by the distance between the patches. In particular, we ask whether it is continuous with respect to this distance. This question is of course irrelevant if the fractional Laplacian is replaced by the usual Laplacian, as both domains have a principal eigenvalue of their own. However, in the nonlocal case, there is a solidarity between the patches and the question becomes relevant. In the one dimensional case, we give a positive answer especially when the distance tends to 0. Our domain is of the form

$$\Omega_{1,2,\mu} = \Omega_1 \cup \Omega_2 \quad \text{and} \quad \text{dist}(\Omega_1, \Omega_2) = \inf \{|x_1 - x_2| \text{ with } x_1 \in \Omega_1, x_2 \in \Omega_2\} = 2\mu. \quad (4.4)$$

Notation. For any smooth bounded set \mathcal{O} , let $\xi_\alpha(\mathcal{O})$ be the principal eigenvalue of the fractional Laplacian with an exponent α with Dirichlet conditions outside the domain \mathcal{O}

$$i.e. \quad \begin{cases} (-\Delta)^\alpha \phi_\alpha = \xi_\alpha(\mathcal{O})\phi_\alpha & \text{in } \mathcal{O}, \\ \phi_\alpha = 0 & \text{in } \mathcal{O}^c. \end{cases} \quad (4.5)$$

The existence of such eigenvalue is ensured by the Krein-Rutman Theorem. In this part, we will adopt the following notation :

$$\underline{\xi}_{i,\alpha} = \xi_\alpha(\Omega_i).$$

We denote by $\phi_{1,2,\mu,\alpha}$ and $\underline{\phi}_{i,\alpha}$ the eigenfunctions associated respectively to $\xi_\alpha(\Omega_{1,2,\mu})$ and $\underline{\xi}_{i,\alpha}$.

The main continuity result

As previously said, a two-piece domain has a first principal eigenvalue, and this eigenvalue is continuous under the mutual distance of the two pieces is intuitively obvious, as soon as

they remain far apart. When they are put together, continuity still holds : this result has of course no equivalent in the case of the standard Laplacian. Here is the precise statement.

Theorem 4.1. *Under the previous assumptions, the function $(\mu \in]0, +\infty[\mapsto \xi_\alpha(\Omega_{1,2,\mu}))$ is increasing and continuous. Moreover, it is continuous up to $\mu = 0$ and*

$$\xi_\alpha(\Omega_{1,2,0}) = \xi_\alpha(]0, |\Omega_1| + |\Omega_2|[).$$

A first ingredient for the proof of Theorems 4.1 and the monotonicity of $(\mu \mapsto \lambda_\alpha(\Omega_{1,2,\mu}))$ is the Rayleigh quotient

$$R_\alpha(\phi) = \frac{\int_{\mathbb{R}} (-\Delta)^\alpha \phi(x) \phi(x) dx}{\int_{\mathbb{R}} \phi^2(x) dx} = \frac{\frac{1}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{(\phi(x) - \phi(y))^2}{|x-y|^{1+2\alpha}} dy dx}{\int_{\mathbb{R}} \phi^2(x) dx} := \frac{[\phi]_\alpha}{\|\phi\|_2^2}. \quad (4.6)$$

We will evaluate R_α in the following spaces, for a general set \mathcal{O} :

$$H_0^\alpha(\mathcal{O}) = \left\{ \phi \in L^2(\mathcal{O}) \text{ such that } [\phi]_\alpha < +\infty \text{ and } \phi|_{\mathbb{R} \setminus \mathcal{O}} = 0 \right\}. \quad (4.7)$$

The link between the Rayleigh quotient and the principal eigenvalues is the following :

$$\xi_\alpha(\Omega_{1,2,\mu}) = \min_{\phi \in H_0^\alpha(\Omega_{1,2,\mu}) \setminus \{0\}} R_\alpha(\phi) = R_\alpha(\phi_{1,2,\mu,\alpha}) \quad \text{and} \quad \xi_{i,\alpha} = \min_{\phi \in H_0^\alpha(\Omega_i) \setminus \{0\}} R_\alpha(\phi) = R_\alpha(\phi_{i,\alpha}). \quad (4.8)$$

The proof of the continuity of $\xi_\alpha(\Omega_{1,2,\mu})$ with respect to μ when $\mu > 0$ is a consequence of standard uniqueness/compactness arguments. The continuity at $\mu = 0$ is more involved, especially when $\alpha \geq \frac{1}{2}$. In this case, we have to prove that the contact point between the two domains Ω_1 and Ω_2 becomes a removable singularity. To achieve this result, we will use the extension on the upper half plane of the fractional Laplacian introduced by Caffarelli and Silvestre in [37]. For the case $\alpha < \frac{1}{2}$, the case $\mu = 0$ can be treated as the case $\mu > 0$ thanks to the density of the set of functions satisfying $\phi = 0$ (see [106]). We emphasise that the demonstration holds true up to $\mu = 0$ because we work in a one dimensional space. Indeed, in one dimension, there is only one way to connect two intervals. For $d > 2$, the result holds true for distances $\mu > 0$, with no real modification.

The limit $\alpha \rightarrow 1$: Consequences of Theorem 4.1

It is well known (see [49]) that for a smooth function n and for all $x \in \mathbb{R}$, the function $\alpha \mapsto (-\Delta)^\alpha n(x)$ is continuous, and that

$$(-\Delta)^\alpha n(x) \xrightarrow{\alpha \rightarrow 1^-} (-\Delta)n(x).$$

The standard compactness/uniqueness argument yields the continuity of the function $(\alpha \in]0, 1[\mapsto \xi_\alpha(\Omega_{1,2,\mu}))$. The next proposition describes, in our specific case, the dynamics of $\xi_\alpha(\Omega_{1,2,\mu})$ when α tends to 1.

Proposition 4.1. *The function $\alpha \in]0, 1[\mapsto \xi_\alpha(\Omega_{1,2,\mu})$ is continuous up to $\alpha = 1$ with*

$$\xi_1(\Omega_{1,2,\mu}) = \min(\xi_1, \xi_2). \quad (4.9)$$

We now focus on the dynamics of $\xi_\alpha(\Omega_{1,2,\mu})$ when (μ, α) converges both to $(0, 1)$. There is a competition between the non-local character which becomes "less important" as α tends to 1 and the requirement of this non-local character which becomes also "less important" as μ tends to 0. Theorem 4.1 and Proposition 4.1 imply the following statement, which has once again no equivalent in the case of local diffusion :

Theorem 4.2. *Under the previous hypothesis, for all $\xi_* \in [\xi_1(\Omega_{1,2,0}), \min(\xi_{1,1}, \xi_{2,1})]$, there exists a sequence $(\mu_k, \alpha_k)_{k \in \mathbb{N}}$ such that*

$$(\mu_k, \alpha_k) \xrightarrow[k \rightarrow +\infty]{} (0, 1) \quad \text{and} \quad \xi_{\alpha_k}(\Omega_{1,2,\mu_k}) \xrightarrow[k \rightarrow +\infty]{} \xi_*. \quad (4.10)$$

4.1.3 Existence and uniqueness of a steady solution to (4.1)

Notations and assumptions on Ω

Before giving the hypothesis and the results, we introduce some notations which will be used all along the article. Let \mathcal{O} be a general smooth domain of \mathbb{R}^d , x be a point of \mathbb{R}^d and ν be a positive constant. Then we define the sets :

$$\mathcal{O} + x = \{y \in \mathbb{R}^d \text{ such that } y - x \in \mathcal{O}\} \quad \text{and} \quad \mathcal{O}_\nu = \{y \in \mathcal{O} \text{ such that } \text{dist}(y, \partial\mathcal{O}) > \nu\} \quad (4.11)$$

where $\partial\mathcal{O}$ is the boundary of \mathcal{O} . Since the distance to the boundary of the domain Ω under study will play an important role, we will denote it by δ ,

$$\text{i.e. } \delta(x) = \text{dist}(x, \partial\Omega)1_\Omega(x). \quad (4.12)$$

When it is defined, the principal eigenvalue of the fractional Dirichlet operator $(-\Delta)^\alpha - Id$ in \mathcal{O} will play also an important role in the following. We will denote it by $\lambda_\alpha(\mathcal{O})$:

$$\text{i.e. } \begin{cases} (-\Delta)^\alpha \phi(x) - \phi(x) = \lambda_\alpha(\mathcal{O})\phi(x) & \text{for } x \in \mathcal{O}, \\ \phi(x) = 0 & \text{for } x \in \mathbb{R}^d \setminus \mathcal{O}. \end{cases} \quad (4.13)$$

We underline that $\lambda_\alpha(\mathcal{O}) = \xi_\alpha(\mathcal{O}) - 1$. The principal eigenvalue $\lambda_\alpha(\mathcal{O})$ will be a key ingredient of most of the upcoming results. Note that a such eigenvalue is well defined for instance when the domain \mathcal{O} is smooth and bounded with a finite number of components. It is also well defined if \mathcal{O} is smooth, periodic and such that the number of components in all compact sets is finite. For such domains, there is a dichotomy : either $\lambda_\alpha(\mathcal{O}) < 0$ and (4.1) admits a positive bounded non-trivial solution, either $\lambda_\alpha(\mathcal{O}) \geq 0$ and the unique positive bounded solution of (4.1) is 0.

We assume that the domain Ω may be written as

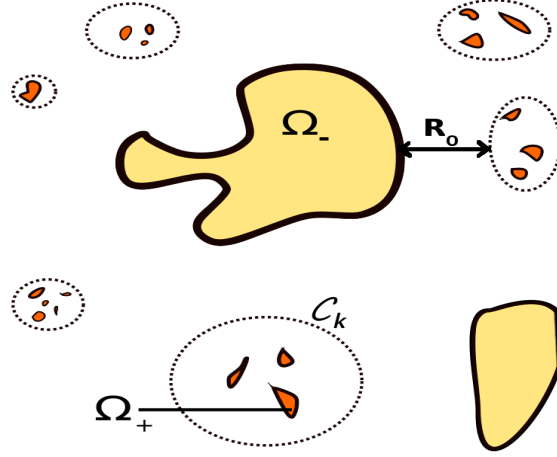
$$\Omega = \bigcup_{k \in \mathbb{N}} \Omega_k \quad (H1)$$

where the sets $(\Omega_k)_{k \in \mathbb{N}}$ are smooth, connected, bounded. Moreover, we assume that

$$\Omega \text{ satisfies the uniform interior and exterior ball condition.} \quad (H2)$$

Definition 4.1 (The interior and exterior ball condition). *A set \mathcal{O} satisfies the uniform interior and exterior ball condition if there exists $\varepsilon_0 \in]0, 1[$ such that*

$$\begin{aligned} & \forall x \in \mathcal{O}, \exists z_x \in \mathcal{O} \quad \text{such that } x \in B(z_x, \varepsilon_0) \subset \mathcal{O}, \\ \text{and } & \forall y \in \mathcal{O}^c, \exists z_y \in \mathcal{O}^c \quad \text{such that } y \in B(z_y, \varepsilon_0) \subset \mathcal{O}^c. \end{aligned} \quad (4.14)$$


 FIGURE 4.1 – Illustration of a possible domain Ω

We assume that Ω can be decomposed in the following form :

$$\Omega = \Omega_- \cup \Omega_+ = \bigcup_{k \in \mathbb{N}} (\Omega_{-,k} \cup \Omega_{+,k}). \quad (\text{H3})$$

In the following, when we pick $x \in \Omega_{\pm}$, the integer k_x will denote the only integer such that $x \in \Omega_{\pm, k_x}$.

We assume that the domain Ω_+ is composed by some uniformly bounded clusters $\mathcal{C}_{+,k}$ which are "far away" each other

$$\begin{aligned} \text{i.e.} \quad \forall k \in \mathbb{N}, \exists z_k \in \mathbb{R}^d \text{ such that } & \mathcal{C}_{+,k} \subset B(z_k, r_0), \quad |z_i - z_j| > R_0 + r_0 \text{ (for } i \neq j \text{)} \\ & \text{and } \text{dist}(\Omega_+, \Omega_-) > R_0. \end{aligned} \quad (\text{H4})$$

In what follows, the constant R_0 will be assume suitably large. Moreover, we assume that the eigenvalue of the Dirichlet operator $(-\Delta)^\alpha - Id$ in $\{x \in \mathbb{R}^d | \text{dist}(x, \mathcal{C}_{+,k}) < \varepsilon_1\}$ is uniformly bounded from below for some positive $\varepsilon_1 < \frac{\varepsilon_0}{4}$:

$$\text{i.e. } 0 < \lambda_0 < \lambda_\alpha(\{x \in \mathbb{R}^d | \text{dist}(x, \mathcal{C}_{+,k}) < \varepsilon_1\}). \quad (\text{H5})$$

For Ω_- , we assume that it is not empty and there exists a finite number of bounded sets $(\underline{\Omega}_{-,n})_{n \in \{1, \dots, N\}}$ made up with a finite number of connected components such that

$$\begin{aligned} \forall n \in \{1, \dots, N\}, \lambda_\alpha(\underline{\Omega}_{-,n}) < 0, \\ \text{and } \forall x \in \Omega_-, \exists (y, n) \in \mathbb{R}^d \times \{1, \dots, N\} \text{ such that } x \in (y + \underline{\Omega}_{-,n}) \text{ and } (y + \underline{\Omega}_{-,n}) \subset \Omega. \end{aligned} \quad (\text{H6})$$

Remark. The assumptions (H1)– (H6) on Ω cover a large case of sets from the sets with only one bounded connected component to a general unbounded with an infinite number of components.

At the end of section 4.3, we present some examples of domains which satisfy such assumptions.

Remark. The decomposition $\Omega = \Omega_- \cup \Omega_+$ is not unique. For instance if Ω_+ is bounded, one can take $\Omega'_- = \Omega$ and $\Omega'_+ = \emptyset$ as a new decomposition.

The steady solution n_+ : existence, accurate estimates, uniqueness

We begin by stating that there exists a bounded non-trivial solution n_+

Theorem 4.3. *Under the assumption (H6), there exists a bounded non-trivial solution of (4.1).*

Thanks to (H6), we can provide a subsolution : the solution of (4.1) where the domain Ω is switched with $\Omega_- \cap B(0, R)$ (with R large enough such that $\lambda_\alpha(\Omega_- \cap B(0, R)) < 0$). Next, a supersolution is the constant function 1. Finally, we construct a solution by iteration from this sub and super solution (see [102]).

A much less classical result is an estimate from above and below of any positive non-trivial bounded solutions of (4.1).

Theorem 4.4. *Assume that Ω fulfils (H1)– (H6) and that $R_0 > C_{d,\alpha,\varepsilon_0,\varepsilon_1,r_0}$ where $C_{d,\alpha,\varepsilon_0,\varepsilon_1,r_0}$ is a positive constant depending on the parameters $d, \alpha, \varepsilon_0, \varepsilon_1, r_0$ then for any bounded positive solution n_+ of (4.1), there exists two positive constants $c_1 \in]0, 1[$, $C_1 \in \mathbb{R}_+$ such that for all $x \in \mathbb{R}^d$ we have*

$$c_1 \mathcal{G}(x) \leq n_+(x) \leq \mathcal{G}(x) \tag{4.15}$$

with

$$\mathcal{G}(x) = \min(\min(C_1 \delta(x)^\alpha, \varepsilon^\alpha) \times G(x), 1). \tag{4.16}$$

and

$$G(x) = 1_{\Omega_-}(x) + \left(\int_{\Omega_-} \frac{1}{|x-y|^{d+2\alpha}} dy \right) 1_{\Omega_+}(x). \tag{4.17}$$

We underline that the function \mathcal{G} depends only on the decomposition (H3). We will devote a special section to its proof, its strategy being presented at the beginning. Thanks to Theorem 4.4, we prove uniqueness of the solution of (4.1).

Theorem 4.5. *Assuming (H1)– (H6) and if R_0 is large enough then there exists a unique bounded positive non-trivial solution of (4.1).*

The proof of uniqueness follows a general argument introduced by Berestycki in [10]. Thanks to Theorem 4.4, we compare two solutions and we conclude thanks to the maximum principle or the fractional Hopf Lemma.

A consequence is that the steady solution is a global attractor for the Cauchy Problem.

Corollary 4.1. *Under the previous hypothesis, if n is solution of (4.3) and n_0 is positive, non-trivial and compactly supported initial data with the closure of the support included in the closure of Ω then there holds*

$$\forall x \in \mathbb{R}^d, \quad \lim_{t \rightarrow +\infty} n(x, t) = n_+(x).$$

We underline that the proof will use the uniqueness result. From a biological point of view, we talk about a colonisation rather than an invasion phenomena. Indeed, invasion imply a colonisation phenomenon and an autonomy from the population in the neighborhood of the colonised area. This last fact does not hold in Ω_+ according to Theorem 4.4.

4.1.4 Discussion, comparison with existing results

Theorem 4.5 has a link with some recent results dealing with steady solutions of nonlocal Fisher-KPP equations in general environments. Closest to the result is the analysis of Berestycki, Coville and Vo [12], where the dispersal is given by a smooth integral kernel, the domain is the whole space, but the reproduction term is inhomogeneous. Under essentially two assumptions, namely that a generalised principal eigenvalue (see [21] for the general definitions) is negative, and that the medium outside a ball is unfavourable, existence and uniqueness of a steady solution under a certain subsolution is established. We note a related work by Brasseur [32] that achieves a passage to the limit of a more and more concentrated dispersal. Our general setting is less general than the situation considered in [12], in particular we have to make assumptions related to the fact that the environment is fragmented. In particular, whether Theorem 4.5 holds under the sole assumption on the sign of a generalised principal eigenvalue is an important question whose answer is unknown to us at the moment. Additional technical difficulties are present due to the fragmented environment (barriers at the boundary of the domain are sometimes tricky to devise) as well as the presence of the fractional Laplacian. On the other hand, we note that our uniqueness result holds in the whole class of bounded solutions due to the general estimate provided in Theorem 4.4, which we regard as one of the main results of this work.

An aspect of the problem, that would call for further developments, is the detailed description of the invasion, in other words quantitative estimates on the convergence to the steady solution. This would be especially interesting if infinitely many patches are favourable - a description of the steady states in this situation is, by the way, still to be developed, although not totally out of reach from the arguments of the present work. We mention the periodic setting treated in [78], where exponential invasion is proved. To understand how things have to be modified outside this setting is still, to our knowledge, open.

4.1.5 Outline of the paper

We first provide in Section 4.2 a study on the one-dimensional case : the dependence of the sign of $\xi_\alpha(\Omega_{1,2,\mu})$ on the parameters $|\Omega_{i,\alpha}|$, μ and α . Next, in Section 4.3 the proof of Theorem 4.4. Section 4.4 is devoted to the proof of Theorem 4.5. In section 4.4.2, we establish Theorem 4.1. We also provide some numerics in order to illustrate Corollary 4.1 and Theorem 4.1.

In order to be more readable, we do not write the principal eigenvalue $P.V.$ and the constant C_α in front of the fractional Laplacian. When there is no possible confusion, the constants denoted by c and C may change from one line to another. In all this work, when it is not precised, ϕ is usually used to denote a principal eigenfunction. Moreover, it is taken positive and with unit L^2 norm.

4.2 Dependence of the sign of $\xi_\alpha(\Omega_{1,2,\mu})$ on the parameters

In subsection 4.2.1, we focus on the dependence of $\xi_\alpha(\Omega_{1,2,\mu})$ on the size of Ω_i and μ . We provide the proof of Theorem 4.1 and Proposition 4.2. Then, in subsection 4.2.2, we provide the proof of Theorem 4.1. Finally, subsection 4.2.3 is devoted to the investigation of the description of the dynamics of $\xi_\alpha(\Omega_{1,2,\mu})$ when $(\mu, \alpha) \rightarrow (0, 1)$ at the same time.

4.2.1 Dependence on $|\Omega_i|$ and μ

We start the proof of Theorem 4.1 with the following classical (and useful) bounds on $\xi_\alpha(\Omega_{1,2,\mu})$.

Proposition 4.2. *Under the previous hypothesis, there holds for all $\mu \geq 0$*

$$\xi_\alpha(\Omega_{1,2,\mu}) \leq \min(\underline{\xi}_{1,\alpha}, \underline{\xi}_{2,\alpha}).$$

Proof. For all $\mu > 0$, it is straightforward that $H_0^\alpha(\Omega_i) \subset H_0^\alpha(\Omega)$. Thus, we deduce that the function $\underline{\phi}_{i,\alpha} \in H_0^\alpha(\Omega)$. We conclude thanks to the Rayleigh quotient that we have :

$$\xi_\alpha(\Omega_{1,2,\mu}) = \min_{\phi \in H_0^\alpha(\Omega) \setminus \{0\}} R_\alpha(\phi) \leq R_\alpha(\underline{\phi}_{i,\alpha}) = \underline{\xi}_{i,\alpha}.$$

□

Proof of Theorem 4.1. We first show the monotonicity. Next, we prove the continuity for $\mu > 0$ finally we demonstrate the continuity up to $\mu = 0$.

Proof of the monotonicity of $(\mu \mapsto \xi_\alpha(\Omega_{1,2,\mu}))$. Let μ_1 and μ_2 be two positive constants such that $\mu_1 < \mu_2$. We will consider $\Omega_{1,2,\mu_1}$ and $\Omega_{1,2,\mu_2}$, that we write explicitly (possibly up to a translation) in order to fix ideas :

$$\Omega_{1,2,\mu_1} =] - A_1 - \mu_1, -\mu_1[\cup] \mu_1, A_2 + \mu_1[\quad \text{and} \quad \Omega_{1,2,\mu_2} =] - A_1 - \mu_2, -\mu_2[\cup] \mu_2, A_2 + \mu_2[.$$

We define $\mu = \mu_2 - \mu_1$. We recall that for all $j \in \{1, 2\}$

$$\xi_\alpha(\Omega_{1,2,\mu_j}) = \min_{\phi \in H_0^\alpha(\Omega_{1,2,\mu_j}) \setminus \{0\}} R_\alpha(\phi) = R_\alpha(\phi_{1,2,\mu_j,\alpha}).$$

The idea is to translate each component of the support of $\phi_{1,2,\mu_2,\alpha}$ in $\Omega_{1,2,\mu_1}$. Thus we define

$$\phi^\mu(x) := \begin{cases} \phi(x - \mu_2 + \mu_1) = \phi_{1,2,\mu_2,\alpha}(x - \mu) & \text{if } x \in] - \infty, 0[, \\ \phi(x + \mu_2 - \mu_1) = \phi(x + \mu) & \text{if } x \in [0, +\infty[\end{cases} \quad (4.18)$$

for any $\phi \in H_0^\alpha(\Omega_{1,2,\mu_2})$. We easily observe that ϕ^μ belongs to $H_0^\alpha(\Omega_{1,2,\mu_1})$. Next, we remark that $\|\phi_{1,2,\mu_2,\alpha}^\mu\|_2 = \|\phi_{1,2,\mu_2,\alpha}\|_2 = 1$. The aim is to show that

$$[\phi_{1,2,\mu_2,\alpha}^\mu]_\alpha \leq [\phi_{1,2,\mu_2,\alpha}]_\alpha. \quad (4.19)$$

If (4.19) holds true, since the L^2 norm is conserved, we obtain from (4.6) that

$$R_\alpha(\phi_{1,2,\mu_2,\alpha}^\mu) = \frac{[\phi_{1,2,\mu_2,\alpha}^\mu]_\alpha}{\|\phi_{1,2,\mu_2,\alpha}^\mu\|_2^2} \leq \frac{[\phi_{1,2,\mu_2,\alpha}]_\alpha}{\|\phi_{1,2,\mu_2,\alpha}\|_2^2} = R_\alpha(\phi_{1,2,\mu_2,\alpha}).$$

It allows us to conclude that

$$\xi_\alpha(\Omega_{1,2,\mu_1}) = \min_{\phi \in H_0^\alpha(\Omega_{1,2,\mu_1}) \setminus \{0\}} R_\alpha(\phi) \leq R_\alpha(\phi_{1,2,\mu_2,\alpha}^\mu) \leq R_\alpha(\phi_{1,2,\mu_2,\alpha}) = \xi_\alpha(\Omega_{1,2,\mu_2}).$$

Thus, we prove (4.19). We will denote

$$k_\varepsilon(x) = \max(\varepsilon, |x|^{1+2\alpha}) \quad \text{such that} \quad [\phi]_\alpha = \lim_{\varepsilon \rightarrow 0} \frac{1}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{(\phi(x) - \phi(y))^2}{k_\varepsilon(x - y)} dy dx.$$

For all $\varepsilon > 0$ and for all positive $\phi \in H_0^\alpha(\Omega_{1,2,\mu_2})$ thanks to the Fubini-Tonelli theorem, we have

$$\begin{aligned} \frac{1}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{(\phi^\mu(x) - \phi^\mu(y))^2}{k_\varepsilon(x-y)} dy dx &= \frac{1}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{\phi^\mu(x)^2 + \phi^\mu(y)^2 - 2\phi^\mu(x)\phi^\mu(y)}{k_\varepsilon(x-y)} dy dx \\ &= \int_{\mathbb{R}} \phi^\mu(x)^2 \int_{\mathbb{R}} \frac{1}{k_\varepsilon(x-y)} dy dx - \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{\phi^\mu(x)\phi^\mu(y)}{k_\varepsilon(x-y)} dy dx \\ &= C_\varepsilon \int_{\mathbb{R}} \phi^\mu(x)^2 dx - \int_{\Omega_{1,2,\mu_1}} \int_{\Omega_{1,2,\mu_1}} \frac{\phi^\mu(x)\phi^\mu(y)}{k_\varepsilon(x-y)} dy dx. \end{aligned}$$

Here, $C_\varepsilon = \int_{\mathbb{R}} \frac{1}{k_\varepsilon(z)} dz$ is a constant independent of the choice of ϕ . Next we have :

$$\begin{aligned} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{(\phi^\mu(x) - \phi^\mu(y))^2}{k_\varepsilon(x-y)} dy dx &= C_\varepsilon \int_{\mathbb{R}} \phi^\mu(x)^2 dx - \int_{\Omega_1^{\mu_1}} \int_{\Omega_1^{\mu_1}} \frac{\phi^\mu(x)\phi^\mu(y)}{k_\varepsilon(x-y)} dy dx \\ &\quad - \int_{\Omega_2^{\mu_1}} \int_{\Omega_2^{\mu_1}} \frac{\phi^\mu(x)\phi^\mu(y)}{k_\varepsilon(x-y)} dy dx - 2 \int_{\Omega_2^{\mu_1}} \int_{\Omega_1^{\mu_1}} \frac{\phi^\mu(x)\phi^\mu(y)}{k_\varepsilon(x-y)} dy dx \end{aligned} \quad (4.20)$$

Thanks to a change of variable, we obtain that for all $i \in \{1, 2\}$:

$$\int_{\Omega_i^{\mu_1}} \int_{\Omega_i^{\mu_1}} \frac{\phi^\mu(x)\phi^\mu(y)}{k_\varepsilon(x-y)} dy dx = \int_{\Omega_i^{\mu_2}} \int_{\Omega_i^{\mu_2}} \frac{\phi(x)\phi(y)}{k_\varepsilon(x-y)} dy dx. \quad (4.21)$$

Since for $(x, y) \in \Omega_1^{\mu_2} \times \Omega_2^{\mu_2}$ we have $k_\varepsilon(x-y+2\mu) \leq k_\varepsilon(x-y)$, we obtain

$$\int_{\Omega_2^{\mu_2}} \int_{\Omega_1^{\mu_2}} \frac{\phi(x)\phi(y)}{k_\varepsilon(x-y)} dy dx \leq \int_{\Omega_2^{\mu_2}} \int_{\Omega_1^{\mu_2}} \frac{\phi(x)\phi(y)}{k_\varepsilon(x-y+2\mu)} dy dx = \int_{\Omega_2^{\mu_1}} \int_{\Omega_1^{\mu_1}} \frac{\phi^\mu(x)\phi^\mu(y)}{k_\varepsilon(x-y)} dy dx. \quad (4.22)$$

Inserting (4.21) and (4.22) in (4.20), we conclude that for all $\varepsilon > 0$ and all positive $\phi \in H_0^\alpha(\Omega_{1,2,\mu_2})$, we have

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \frac{(\phi^\mu(x) - \phi^\mu(y))^2}{k_\varepsilon(x-y)} dy dx \leq \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{(\phi(x) - \phi(y))^2}{k_\varepsilon(x-y)} dy dx.$$

Sending ε to 0, we conclude that (4.19) holds true and the conclusion follows.

Proof of the continuity of $(\mu \mapsto \xi_\alpha(\Omega_{1,2,\mu}))$ for $\mu > 0$. Let $\mu > 0$ and $(\mu_k)_{k \in \mathbb{N}}$ with $\mu_k \xrightarrow[k \rightarrow +\infty]{} \mu$ and $\mu_k > 0$. According to Proposition 4.2 and the Krein-Rutman Theorem, we have $0 \leq \xi_\alpha(\Omega_{1,2,\mu}) < \min(\underline{\xi}_1, \underline{\xi}_2)$. Up to a subsequence, $\xi_\alpha(\Omega_{1,2,\mu_k})$ converges to ξ_∞ . We normalise $\phi_{1,2,\mu_k,\alpha}$ such that $\|\phi_{1,2,\mu_k,\alpha}\|_{L^2} = 1$. Next, thanks to the Rayleigh quotient, we obtain that

$$[\phi_{1,2,\mu_k,\alpha}]_\alpha \leq \xi_\alpha(\Omega_{1,2,\mu_k}) \leq \min(\underline{\xi}_1, \underline{\xi}_2).$$

Thus, we deduce that $\|\phi_{1,2,\mu_k,\alpha}\|_{H^\alpha}$ is bounded. Up to a new subsequence, $\phi_{1,2,\mu_k,\alpha}$ converges strongly in $L^2(\mathbb{R})$ and weakly in $H^\alpha(\mathbb{R})$ to ϕ_∞ . It is straightforward to obtain that $\phi_\infty \geq 0$ and $\phi_\infty = 0$ in $(\Omega_{1,2,\mu})^c$. Moreover, since $\Omega_{1,2,\mu_k}$ converges to $\Omega_{1,2,\mu}$ we deduce that for all compact set K of $\Omega_{1,2,\mu}$, there exists $k_0 \in \mathbb{N}$ such that for $k > k_0$, we have

$$K \subset \Omega_{1,2,\mu_k} \text{ and } 1 - \varepsilon \leq \|\phi_{1,2,\mu_k,\alpha}\|_{L^2(K)}$$

with ε as small as we want. We deduce that $\|\phi_\infty\|_{L^2(\Omega_{1,2,\mu})} = 1$. With the same idea, we get that in all compact set K of $\Omega_{1,2,\mu}$, ϕ_∞ is a weak solution to

$$\begin{cases} (-\Delta)^\alpha \phi_\infty = \xi_\infty \phi_\infty & \text{for } x \in K, \\ \phi_\infty = 0 & \text{for } x \in \Omega_{1,2,\mu}^c, \\ \phi_\infty \geq 0, \|\phi_\infty\|_{L^2} = 1. \end{cases}$$

Since it is true in all compact set of $\Omega_{1,2,\mu}$ we conclude that

$$\begin{cases} (-\Delta)^\alpha \phi_\infty = \xi_\infty \phi_\infty & \text{for } x \in \Omega_{1,2,\mu}, \\ \phi_\infty = 0 & \text{for } x \in \Omega_{1,2,\mu}^c, \\ \phi_\infty \geq 0, \|\phi_\infty\|_{L^2} = 1. \end{cases} \quad (4.23)$$

Thanks to the fractional elliptic regularity (see [38]), we obtain that (4.23) is true in the strong sense. By uniqueness of the eigenvalue associated to a strictly positive eigenfunction, we finally conclude that $\xi_\infty = \xi_\alpha(\Omega_{1,2,\mu})$ and $\phi_\infty = \phi_{1,2,\mu,\alpha}$.

Proof of the continuity up to $\mu = 0$. Let $(\mu_k)_{k \in \mathbb{N}}$ be a sequence such that $\mu_k \xrightarrow[k \rightarrow +\infty]{} 0$ and $\mu_k > 0$. Following the same idea than in the previous part, we find that $\xi_\alpha(\Omega_{1,2,\mu_k})$ converges to some ξ_∞ and $\phi_{1,2,\mu_k,\alpha}$ converges to ϕ_∞ with ϕ_∞ a bounded solution of

$$\begin{cases} (-\Delta)^\alpha \phi_\infty = \xi_\infty \phi_\infty & \text{for } x \in]-A, A[\setminus \{0\}, \\ \phi_\infty = 0 & \text{for } x \in]-A, A[^c, \\ \phi_\infty \geq 0, \|\phi_\infty\|_{L^2} = 1. \end{cases} \quad (4.24)$$

We consider two cases : $\alpha < \frac{1}{2}$ and $\alpha \geq \frac{1}{2}$.

Case 1 $\alpha < \frac{1}{2}$. It is sufficient to remark that $\phi_{1,2,\mu_k,\alpha}$ converges to ϕ_∞ in $H^\alpha(]-A, A[)$ (see the comments after the proof of Lemma 16.1 p. 82 of [106]). Indeed, since the set of functions $\{\psi \in H_0^\alpha(]-A, A[) \mid \psi(0) = 0\}$ is dense in $H_0^\alpha(]-A, A[)$, we conclude by compactness as in the case $\mu > 0$.

Case 2 $\alpha \geq \frac{1}{2}$. The idea is to show that 0 is a removable singularity in the extended problem, as introduced in [37]. This will allow us to conclude that $\phi_\infty = \phi_{1,2,0,\alpha}$ and $\xi_\infty = \xi_\alpha(\Omega_{1,2,0})$. The inspiration for the whole proof comes from Serrin [100].

So, let v be the solution of

$$\begin{cases} -\operatorname{div}(y^a \nabla v) = 0 & \text{for } (x, y) \in \mathbb{R} \times \mathbb{R}^+ \\ v(x, 0) = \phi_\infty(x) & \text{for } x \in \mathbb{R}, \end{cases} \quad \text{with } a = 1 - 2\alpha.$$

From [37] we have that, for all $x \in]-A, A[\setminus \{0\}$:

$$\lim_{y \rightarrow 0} y^a \partial_y v(x, y) = -(-\Delta)^\alpha \phi_\infty(x) = -\xi_\infty \phi_\infty(x) = -\xi_\infty v(x, 0).$$

We define w as the solution of the following equation

$$\begin{cases} -\operatorname{div}(y^a \nabla w) = 0 & \text{for } (x, y) \in]-L, L[\times]0, L[\\ w(x, y) = v(x, y) & \text{for } (x, y) \in \partial(]-L, L[\times]0, L[) \cap \{y > 0\}, \\ \lim_{y \rightarrow 0} y^a \partial_y w(x, y) = -\lim_{y \rightarrow 0} \xi_\infty w(x, y) & \text{for } x \in]-L, L[, \end{cases} \quad (4.25)$$

where $L \in]0, A[$. That w exists and is unique is a consequence of the implicit function Theorem for L small enough. Define $W = v - w$ such that

$$\begin{cases} -\operatorname{div}(y^\alpha \nabla W) = 0 & \text{for } (x, y) \in]-L, L[\times]0, L[\\ W(x, y) = 0 & \text{for } (x, y) \in \partial(]-L, L[\times]0, L[) \cap \{y > 0\}, \\ \lim_{y \rightarrow 0} y^\alpha \partial_y W(x, y) = -\lim_{y \rightarrow 0} \xi_\infty W(x, y) & \text{for } x \in]-L, L[\setminus \{0\}. \end{cases} \quad (4.26)$$

We are going to prove that $W = 0$. For this purpose, we split W into two parts : $W = W_1 + W_2$. The function W_1 is solution of the following equation :

$$\begin{cases} -\operatorname{div}(y^\alpha \nabla W_1) = 0 & \text{for } (x, y) \in B(0, \varepsilon)^+, \\ W_1(x, y) = W(x, y) & \text{for } (x, y) \in \partial B(0, \varepsilon)^+, \\ \lim_{y \rightarrow 0} y^\alpha \partial_y W_1(x, y) = 0 & \text{for } x \in]-\varepsilon, \varepsilon[\end{cases} \quad (4.27)$$

where $\varepsilon \in]0, \frac{L}{2}[$ will be chosen later on.

Next, we focus the study of W_2 on the domain $(]-L, L[\times]0, L[) \setminus B(0, \varepsilon)^+$. The equation for W_2 is

$$\begin{cases} -\operatorname{div}(y^\alpha \nabla W_2) = 0 & \text{for } (x, y) \in (]-L, L[\times]0, L[) \setminus B(0, \varepsilon)^+ \\ W_2(x, y) = 0 & \text{for } (x, y) \in \partial(]-L, L[\times]0, L[) \cap \{y > 0\}, \\ W_2(x, y) = W_1(x, y) & \text{for } (x, y) \in \partial B(0, \varepsilon)^+ \\ \lim_{y \rightarrow 0} y^\alpha \partial_y W_2(x, y) = -\lim_{y \rightarrow 0} \xi_\infty W(x, y) & \text{for } x \in]-L, L[\setminus]-\varepsilon, \varepsilon[. \end{cases} \quad (4.28)$$

We underline that $W_2 \xrightarrow{\varepsilon \rightarrow 0} W$ weakly. Therefore, since $W_1 = W - W_2$, we deduce that $W_1 \xrightarrow{\varepsilon \rightarrow 0} 0$ weakly also. In the following, we denote by

$$\begin{aligned} \mathcal{D} &=]-L, L[\times]0, L[\setminus B(0, \varepsilon), & \Gamma_1 &= \partial(]-L, L[\times]0, L[) \setminus (]-L, L[\times \{0\}), \\ \Gamma_2 &=]-L, -\varepsilon[\cup]\varepsilon, L[, & \text{and} & \Gamma_3 &= \partial B(0, \varepsilon)^+. \end{aligned}$$



FIGURE 4.2 – The set under study

The aim of the rest of the proof is to prove that $W_2 \xrightarrow{\varepsilon \rightarrow 0} 0$. If it holds true, the conclusion follows. For this purpose, we use the superposition principle and we split the study of W_2 into two parts : W_2' and W_2'' which are the respective solutions of

$$\left\{ \begin{array}{ll} -\operatorname{div}(y^a \nabla W_2') = 0 & \text{for } (x, y) \in \mathcal{D} \\ W_2'(x, y) = 0 & \text{for } (x, y) \in \Gamma_1, \\ W_2'(x, y) = W_1(x, y) & \text{for } (x, y) \in \Gamma_3 \\ \lim_{y \rightarrow 0} y^a \partial_y W_2'(x, y) = 0 & \text{for } x \in \Gamma_2 \end{array} \right. \quad (4.29)$$

and

$$\left\{ \begin{array}{ll} -\operatorname{div}(y^a \nabla W_2'') = 0 & \text{for } (x, y) \in \mathcal{D} \\ W_2''(x, y) = 0 & \text{for } (x, y) \in \Gamma_1, \\ W_2''(x, y) = 0 & \text{for } (x, y) \in \Gamma_3, \\ \lim_{y \rightarrow 0} y^a \partial_y W_2''(x, y) = -\lim_{y \rightarrow 0} \xi_\infty W(x, y) & \text{for } x \in \Gamma_2. \end{array} \right. \quad (4.30)$$

Since, W_1 tends weakly to 0 as $\varepsilon \rightarrow 0$, it follows that W_2' tends also weakly to 0. It remains to prove that W_2'' vanishes.

Let ϕ be a test function in $H_0^1(\mathcal{D}, |y|^a)$ (see Chapter 1 of [72] for the general framework of weighted Sobolev spaces). Noticing that for all $\tau > 0$ and $(x, y) \in \mathcal{D} \cap \{y > \tau\}$, we have

$$\begin{aligned} \operatorname{div}(\phi |y|^a \nabla W_2'')(x, y) &= \nabla \phi(x, y) |y|^a \nabla W_2''(x, y) + \phi(x, y) \operatorname{div}(|y|^a \nabla W_2'')(x, y) \\ &= \nabla \phi(x, y) |y|^a \nabla W_2''(x, y). \end{aligned}$$

We deduce that

$$\begin{aligned} \int_{\mathcal{D}} \nabla \phi |y|^a \nabla W_2'' dx dy &= \int_{\mathcal{D} \cap \{y > \tau\}} \nabla \phi |y|^a \nabla W_2'' dx dy + \int_{\mathcal{D} \cap \{y \leq \tau\}} \nabla \phi |y|^a \nabla W_2'' dx dy \\ &= \int_{\mathcal{D} \cap \{y > \tau\}} \operatorname{div}(\phi |y|^a \nabla W_2'')(x, y) dx dy + \int_{\mathcal{D} \cap \{y \leq \tau\}} \nabla \phi |y|^a \nabla W_2'' dx dy \\ &= \int_{\partial(\mathcal{D} \cap \{y > \tau\})} \phi |y|^a \nabla W_2''(x, y) \cdot \nu dx + \int_{\mathcal{D} \cap \{y \leq \tau\}} \nabla \phi |y|^a \nabla W_2'' dx dy \\ &= \int_{\mathcal{D} \cap \{y = \tau\}} -\phi(x, \tau) |\tau|^a \partial_y W_2''(x, \tau) dx + \int_{\mathcal{D} \cap \{y \leq \tau\}} \nabla \phi |y|^a \nabla W_2'' dx dy. \end{aligned}$$

Next, if we let τ tends to 0, the second term of the right hand side tends to 0 thanks to the Cauchy-Schwarz inequality. Recalling that $W = W_1 + W_2' + W_2''$, we finally have obtained the following variational equation :

$$\int_{\mathcal{D}} \nabla \phi \nabla W_2'' |y|^a dx dy - \int_{\Gamma_2} \phi(x, 0) \xi_\infty W_2''(x, 0) dx = \int_{\Gamma_2} \xi_\infty (W_1(x, 0) + W_2'(x, 0)) \phi(x, 0) dx. \quad (4.31)$$

In order to prove the existence and uniqueness of a solution of (4.31), we are going to apply the Lax-Milgram theorem. The linear map is the following :

$$\Lambda : \phi \in H_0^1(\mathcal{D}, |y|^a) \rightarrow \xi_\infty \int_{\Gamma_2} (W_1(x, 0) + W_2'(x, 0)) \phi(x, 0) dx.$$

Since $(W_1 + W_2')$ tends weakly to 0 as $\varepsilon \rightarrow 0$, we deduce that

$$\|\Lambda\|_{H_0^1(\mathcal{D})'} \xrightarrow{\varepsilon \rightarrow 0} 0. \quad (4.32)$$

Next, we show that for L small enough, the bilinear form

$$b : (u, v) \in H_0^1(\mathcal{D}) \times H_0^1(\mathcal{D}) \rightarrow \int_{\mathcal{D}} \nabla u \nabla v |y|^a dx dy - \int_{\Gamma_2} u(x, 0) \xi_\infty v(x, 0) dx$$

is continuous and coercive.

First, the Poincaré inequality implies that $\phi \mapsto (\int_{\mathcal{D}} \|\nabla\phi\|^2 |y|^a dx dy)^{\frac{1}{2}}$ is a norm that is equivalent to the usual norm on $H_0^1(\mathcal{D}, |y|^a)$ (see equation (1.5) p. 9 of [72]). thus the continuity follows.

Secondly, we prove that for L small enough, b is coercive. Indeed, since $a = 1 - 2\alpha < 0$, we have

$$\begin{aligned} \int_{\Gamma_2} u(x, 0)^2 dx &= 2 \int_{\Gamma_2} \int_0^L u(x, y) \partial_y u(x, y) \frac{|y|^a}{|y|^a} dy dx \\ &\leq \frac{2}{L^a} \int_{\Gamma_2} \left(\int_0^L u(x, y)^2 |y|^a dy \right)^{\frac{1}{2}} \left(\int_0^L \partial_y u(x, y)^2 |y|^a dy \right)^{\frac{1}{2}} dx \\ &\leq \frac{2}{L^a} \int_{\Gamma_2} \left(\int_0^L u(x, y)^2 |y|^a dy \right)^{\frac{1}{2}} \left(\int_0^L \partial_y u(x, y)^2 |y|^a dy \right) dx \\ &\leq \frac{2}{L^a} \left(\int_{\Gamma_2} \int_0^L u(x, y)^2 |y|^a dy dx \right)^{\frac{1}{2}} \left(\int_{\Gamma_2} \int_0^L \partial_y u(x, y)^2 |y|^a dy dx \right)^{\frac{1}{2}} \\ &\leq 2C \text{diam}(\mathcal{D}) L^{2\alpha-1} \int_{\mathcal{D}} |\nabla u|^2 |y|^a dx dy. \end{aligned}$$

Therefore, for L small enough, we deduce that there exists $c_b > 0$ such that

$$b(u, u) > c_b \int_{\mathcal{D}} |\nabla u|^2 |y|^a dx dy.$$

We conclude thanks to the Lax-Milgram theorem that there exists a unique solution to (4.31). Moreover thanks to the estimates of the norm of the solution in the Lax-Milgram theorem and (4.32), we deduce that

$$\|W_2''\|_{H_0^1(\mathcal{D})} \leq \frac{\|\Lambda\|_{H_0^1(\mathcal{D})'}}{c_b} \xrightarrow{\varepsilon \rightarrow 0} 0.$$

The conclusion follows. □

4.2.2 Dependence on α

In this subsection, we provide the proof of Proposition 4.1.

Proof of Proposition 4.1. The continuity for $\alpha < 1$ is based on similar arguments as those presented in the proof of the continuity of $\xi_\alpha(\Omega_{1,2,\mu})$ for $\mu > 0$. Therefor, we focus on the continuity up to $\alpha = 1$.

First, we establish a monotonicity result. We claim that the function

$$(\alpha \in]0, 1] \mapsto \text{diam}(\Omega_{1,2,\mu})^{2\alpha} \xi_\alpha(\Omega_{1,2,\mu})) \text{ is increasing,}$$

(where $\text{diam}(\Omega_{1,2,\mu})$ designates the diameter of $\Omega_{1,2,\mu}$). Indeed, since if $\alpha < \beta$ there holds $H_0^\beta(\Omega_{1,2,\mu}) \subset H_0^\alpha(\Omega_{1,2,\mu})$, we deduce that

$$\begin{aligned} \xi_\alpha(\Omega_{1,2,\mu}) &\leq \min_{\phi \in H_0^\alpha(\Omega_{1,2,\mu})} \mathcal{R}_\alpha(\phi) \\ &\leq \mathcal{R}_\alpha(\phi_{1,2,\mu,\beta}) \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{(\phi_{1,2,\mu,\beta}(x) - \phi_{1,2,\mu,\beta}(y))^2}{|x-y|^{d+2\alpha}} dy dx \\
&= \frac{1}{2} \int_{\mathbb{R}} \int_{\mathbb{R}, |x-y| < \text{diam}(\Omega_{1,2,\mu})} \frac{(\phi_{1,2,\mu,\beta}(x) - \phi_{1,2,\mu,\beta}(y))^2}{|x-y|^{d+2\alpha}} dy dx \\
&\quad + \frac{1}{2} \int_{\mathbb{R}} \int_{\mathbb{R}, |x-y| \geq \text{diam}(\Omega_{1,2,\mu})} \frac{(\phi_{1,2,\mu,\beta}(x) - \phi_{1,2,\mu,\beta}(y))^2}{|x-y|^{d+2\alpha}} dy dx \\
&= \frac{\text{diam}(\Omega_{1,2,\mu})^{2(\beta-\alpha)}}{2} \int_{\mathbb{R}} \int_{\mathbb{R}, |x-y| < \text{diam}(\Omega_{1,2,\mu})} \frac{(\phi_{1,2,\mu,\beta}(x) - \phi_{1,2,\mu,\beta}(y))^2}{|x-y|^{d+2\beta}} dy dx \\
&\quad + \frac{1}{2} \int_{\mathbb{R}} \phi_{1,2,\mu,\beta}(x) \int_{\mathbb{R}, |x-y| \geq \text{diam}(\Omega_{1,2,\mu})} \frac{1}{|x-y|^{d+2\alpha}} dy dx \\
&= \frac{\text{diam}(\Omega_{1,2,\mu})^{2(\beta-\alpha)}}{2} \int_{\mathbb{R}} \int_{\mathbb{R}, |x-y| < \text{diam}(\Omega_{1,2,\mu})} \frac{(\phi_{1,2,\mu,\beta}(x) - \phi_{1,2,\mu,\beta}(y))^2}{|x-y|^{d+2\beta}} dy dx \\
&\quad + \frac{\text{diam}(\Omega_{1,2,\mu})^{-2\alpha}}{2} \int_{\mathbb{R}} \phi_{1,2,\mu,\beta}(x)^2 dx \\
&= \frac{\text{diam}(\Omega_{1,2,\mu})^{2(\beta-\alpha)}}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{(\phi_{1,2,\mu,\beta}(x) - \phi_{1,2,\mu,\beta}(y))^2}{|x-y|^{d+2\beta}} dy dx \\
&= \text{diam}(\Omega_{1,2,\mu})^{2(\beta-\alpha)} \xi_{\beta}(\Omega_{1,2,\mu}).
\end{aligned}$$

(Note that this last result holds true for $\beta = 1$ thanks to the limit (2.8) in [49]).

Next, we prove the continuity result for $\alpha = 1$. Let $(\alpha_k)_{k \in \mathbb{N}} \in]\frac{1}{2}, 1[$ be a sequence such that $\alpha_k \xrightarrow[k \rightarrow +\infty]{} 1$. By replacing $\phi_{1,2,\mu,\beta}$ by the ϕ_1 (with ϕ_1 the principal eigenfunction which corresponds to the principal eigenvalue $\min(\xi_1(\Omega_1), \xi_2(\Omega_2))$), we deduce that

$$0 < \xi_{\alpha_k}(\Omega_{1,2,\mu}) \leq \text{diam}(\Omega)^{2-2\alpha_k} \min(\xi_1(\Omega_1), \xi_2(\Omega_2)). \quad (4.33)$$

We deduce that up to an extraction, $\xi_{\alpha_k}(\Omega_{1,2,\mu})$ converges to ξ_{∞} . We recall that $\|\phi_{1,2,\mu,\alpha_k}\|_2 = 1$. Moreover, for all $k \in \mathbb{N}$, we have

$$\begin{aligned}
\langle (-\Delta)^{\alpha_k} \phi_{\alpha_k,1,2,\mu}, \phi_{\alpha_k,1,2,\mu} \rangle &= \xi_{\alpha_k}(\Omega_{1,2,\mu}) \|\phi_{\alpha_k,1,2,\mu}\|_2^2 \\
&\leq \text{diam}(\Omega_{1,2,\mu})^{2(1-\alpha_k)} \min(\xi_1(\Omega_1), \xi_1(\Omega_2)).
\end{aligned}$$

Up to an extraction and thanks to the Sobolev embedding (see [49]), $\phi_{\alpha_k,1,2,\mu}$ converges to ϕ_{∞} in $L^2(\Omega_{1,2,\mu})$. Moreover, the limit satisfies in a weak sense the following equation

$$\begin{cases} (-\Delta)\phi_{\infty}(x) = \xi_{\infty}\phi_{\infty}(x) & \text{for } x \in \Omega_{1,2,\mu}, \\ \phi_{\infty}(x) = 0 & \text{for } x \in \partial\Omega_{1,2,\mu}. \end{cases} \quad (4.34)$$

By the standard elliptic regularity, we find that (4.34) is true in a strong sense. Furthermore, thanks to (4.33), we deduce that $\xi_{\infty} \leq \min(\xi_1(\Omega_1), \xi_2(\Omega_2))$. Since the L^2 norm of ϕ_{∞} is not trivial and by positiveness of ϕ_{∞} and uniqueness of ϕ_1 (up to a constant), we conclude that $\phi_{\infty} = \phi_1$. □

4.2.3 Dependence on μ and α

For this subsection, we denote by $i \in \{1, 2\}$ the index of the larger domain (i.e. $|\Omega_i| \geq |\Omega_{3-i}|$). This subsection is devoted to the

Proof of Proposition 4.2. We split the proof of (4.10) in three parts : first, we assume $\xi_* = \xi_1(\Omega_{1,2,\mu})$, then we assume $\xi_* = \underline{\xi}_{i,1}$, and finally we do the general case.

Proof when $\xi_* = \xi_1(\Omega_{1,2,\mu})$. Let $(\alpha_k)_{k \in \mathbb{N}}$ be any increasing sequence such that $\alpha_k \rightarrow 1$. According to Theorem 4.1, there exists $\underline{\mu}_k(\alpha_k)$ such that

$$\forall \mu \leq \underline{\mu}_k(\alpha_k), \quad |\xi_{\alpha_k}(\Omega_{1,2,\mu}) - \xi_{\alpha_k}(\Omega_{1,2,0})| \leq \frac{1}{k}.$$

We conclude that $(\alpha_k, \underline{\mu}_k(\alpha_k))$ gives the result.

Proof when $\xi_* = \underline{\xi}_{i,1}$. Let $(\nu_k)_{k \in \mathbb{N}}$ be any decreasing sequence such that $\nu_k \rightarrow 0$. According to Theorem 4.1, there exists $\overline{\beta}_k(\nu_k) < 1$ such that

$$\forall \beta \geq \overline{\beta}_k(\nu_k), \quad |\xi_\beta(\Omega_{1,2,\nu_k}) - \underline{\xi}_{i,1}| \leq \frac{1}{k}.$$

We conclude that $(\nu_k, \overline{\beta}_k(\nu_k))$ gives the result.

Proof when $\xi_* \in]\xi_1(\Omega_{1,2,0}), \underline{\xi}_{i,1}[$. Let (μ_k, α_k) be defined as follows :

$$\begin{cases} (\mu_0, \alpha_0) = (\mu_1, \alpha_1) = (\mu_2, \alpha_2) = \left(\frac{1}{2}, \frac{1}{2}\right), \\ (\mu_{3k}, \alpha_{3k}) = \left(\mu_{3k-1}, \max(\overline{\beta}_k(\mu_{3k-1}), \alpha_{3k-1}, 1 - \frac{1}{k})\right), \\ (\mu_{3k+1}, \alpha_{3k+1}) = (\mu_{3k+1}, \alpha_{3k}), \\ (\mu_{3k+2}, \alpha_{3k+2}) = \left(\min(\underline{\mu}_k(\alpha_{3k}), \mu_{3k}, \frac{1}{k}), \alpha_{3k}\right), \end{cases}$$

where we will fix $\mu_{3k+1} \in [\mu_{3k}, \mu_{3k+2}]$ later on. The constants $\underline{\mu}_k(\alpha_{3k})$ and $\overline{\beta}_k(\mu_{3k-1})$ are defined respectively in the two first parts of the proof. With a such choice of (μ_k, α_k) , we have

$$\xi_{\alpha_{3k}}(\Omega_{1,2,\mu_{3k}}) \xrightarrow[k \rightarrow +\infty]{} \underline{\xi}_{i,1} \quad \text{and} \quad \xi_{\alpha_{3k+2}}(\Omega_{1,2,\mu_{3k+2}}) \xrightarrow[k \rightarrow +\infty]{} \xi_1(\Omega_{1,2,0}).$$

Let $k_0 \in \mathbb{N}$ be such that for all $k > k_0$,

$$\xi_1(\Omega_{1,2,0}) + \frac{2}{k} < \xi_* < \underline{\xi}_{i,1} - \frac{2}{k}. \quad (4.35)$$

Inequalities (4.35) implies

$$\xi_{\alpha_{3k}}(\Omega_{1,2,\mu_{3k}}) < \xi_* < \xi_{\alpha_{3k+2}}(\Omega_{1,2,\mu_{3k+2}}).$$

Next, we fix $\mu_{3k+1} = \mu_{3k}$ for $k \in \{0, \dots, k_0\}$. Since $\mu \mapsto \xi_{\alpha_k}(\Omega_{1,2,\mu})$ is continuous and increasing, and because $\alpha_{3k} = \alpha_{3k+1}$, we deduce thanks to the intermediate value theorem that there exists $\mu_{3k+1} \in]\mu_{3k}, \mu_{3k+2}[$ such that $\xi_{\alpha_{3k+1}}(\Omega_{1,2,\mu_{3k+1}}) = \xi_*$. We conclude that the sequence $(\xi_k, \gamma_k) = (\mu_{3k+1}, \alpha_{3k+1})$ gives the result. \square

4.3 Estimates for steady solutions

This section is devoted to the proof of Theorem 4.4. First, we provide the general strategy of the proof, next we prove intermediate results and finally, we prove Theorem 4.4. We underline here that

$$0 \leq n_+ \leq 1$$

is easy to obtain thanks to the comparison principle. Therefore, we prove that there exists two constants $c, C > 0$ such that, if the function $G(x)$ is given by (4.16), then we have

$$c \min(\delta(x)^\alpha, \varepsilon_0^\alpha) G(x) \leq n_+(x) \leq C \min(\delta(x)^\alpha, \varepsilon_0^\alpha) G(x).$$

We highlight that the subdomains $\underline{\Omega}_{-,k}$ which satisfy (H6) may not be connected. It is one of the interest to consider non-local diffusion instead of local one. Moreover, it may happen that the principal eigenvalue of $(-\Delta)^\alpha - Id$ defined in the connected components of $\underline{\Omega}_{-,k}$ are all positive however the principal eigenvalue of $(-\Delta)^\alpha - Id$ defined in the whole domain $\underline{\Omega}_{-,k}$ is negative.

A consequence of Theorem 4.1 is the following : if $d = 1$ and $\underline{\Omega}_- = \Omega_{1,2,\mu}$ (where $\underline{\Omega}_-$ refers to (H6) and $\Omega_{1,2,\mu}$ to (4.4)), the following assertions hold :

1. we must have $\lambda_\alpha(\Omega_{1,2,\mu}) < 0$,
2. if $\lambda_\alpha(\Omega_1) < 0$ or $\lambda_\alpha(\Omega_2) < 0$, then $\underline{\Omega}_- = \Omega_{1,2,\mu}$ holds true for all $\mu > 0$,
3. if $\lambda_\alpha(\Omega_1) > 0$ and $\lambda_\alpha(\Omega_2) > 0$, then there exists $\mu_0 > 0$ such that $\mu < \mu_0$.

The first point is a consequence of the continuity up to $\mu = 0$ and the monotonicity of $(\mu \mapsto \lambda_\alpha(\Omega_{1,2,\mu}))$ established in Theorem 4.1. The second point is a direct consequence of Proposition 4.2. Finally the last point follows thanks to the continuity of the application $\mu \mapsto \lambda_\alpha(\Omega_{1,2,\mu})$ and the following Lemma

Lemma 4.1. *Let $\Omega_{1,2,\mu}$ be a set as those introduced in (4.4). If $\lambda_\alpha(\Omega_1), \lambda_\alpha(\Omega_2) > 0$, then there exists a distance $\mu_0 \geq 0$ such that $\lambda_\alpha(\Omega_{1,2,\mu_0}) = 0$.*

The proof of this lemma involves lengthy but standard computations. Therefore, we postpone it at the end of the paper in the Appendix.

4.3.1 Strategy of the proof of Theorem 4.4

The lower part of (4.15) will be obtained from the fractional Poisson kernel in a ball (see for instance in [24]) :

$$P_r(x, y) = c_\alpha^d \left(\frac{r^2 - |x|^2}{|y|^2 - r^2} \right)^\alpha \frac{1}{|x - y|^d} 1_{\{|y| > r\}}(x). \quad (4.36)$$

With this kernel, for any smooth function v and $z \in \mathbb{R}^d$ we have that the solution of the equation

$$\begin{cases} (-\Delta)^\alpha \underline{n}(x) = 0 & x \in B(z, r), \\ \underline{n}(x) = v(x) & x \in B(z, r)^c, \end{cases} \quad \text{is} \quad \underline{n}(x) = \int_{\mathbb{R}^d} P_r(x - z, y - z) v(y) dy.$$

The difficult part of the proof will be to obtain the upper bound in Ω_+ of (4.15). First, we establish that :

Lemma 4.2. *Let n_+ be a positive bounded solution of (4.1). If $R_0 > C_{d,\alpha,\varepsilon_0,\varepsilon_1,r_0}$ then there exists a positive constant C_2 such that for all $x \in \Omega_+$*

$$n_+(x) \leq C_2 \int_{\Omega_-} \frac{1}{|x - y|^{d+2\alpha}} dy. \quad (4.37)$$

To prove Lemma 4.2, we localise the function n_+ in a cluster \mathcal{C}_k and we use hypothesis (H4) which essentially says that the cluster is at large distance from the others.

Next, the idea is to compare the solution n_+ with a translated and rescaled barrier function ψ . This particular barrier function satisfies

$$\begin{cases} (-\Delta)^\alpha \psi(x) > 1 & \text{for } x \in T(0, 1, 4), \\ \psi(x) = 0 & \text{for } x \in B(0, 1), \\ \psi(x) \leq c(|x| - 1)^\alpha & \text{for } x \in T(0, 1, 4), \\ 1 \leq \psi \leq C & \text{for } x \in B(0, 4)^c, \end{cases} \quad (4.38)$$

where $T(z, r, R)$ designates the torus of center $z \in \mathbb{R}^d$ and of inner and outer radius $0 < r < R$

$$\text{i.e. } T(z, r, R) = \{y \in \mathbb{R}^d, r < |y - z| < R\}.$$

The construction of a such barrier function can be found in Appendix B of [97]. Then, we look for "a suitable" constant C_k such that for all $x \in \mathcal{C}_{+,k}$ we have

$$n_+(x) \leq C_k \psi\left(\frac{x - z_x}{\varepsilon_0}\right)$$

where z_x and ε_0 are introduced in (H2). We claim that there exists a positive constant $C'_{d,\alpha,\varepsilon_0,\varepsilon_1,r_0}$ which depends only on the listed parameters such that

$$C_k = \sup_{x \in \mathcal{C}_{+,k}} \left(C'_{d,\alpha,\varepsilon_0,\varepsilon_1,r_0} \int_{\Omega_-} \frac{1}{|x - y|^{d+2\alpha}} dy \right). \quad (4.39)$$

Thanks to (4.39), the third property of (4.38) and some Harnack type inequalities the conclusion follows.

Remark. *In particular cases, some explicit and more tractable versions of \mathcal{G} can be found. For instance, if Ω_- is bounded, one can prove that $\mathcal{G}(x) = \frac{\min(\delta(x)^\alpha, 1)}{1 + |x|^{d+2\alpha}}$. We provide some examples at the end of this section.*

4.3.2 Harnack type properties

First, we introduce the following function, defined for $x \in \Omega_+$

$$g(x) = \int_{\Omega_-} \frac{1}{|x - y|^{d+2\alpha}} dy, \quad (4.40)$$

such that

$$G(x) = 1_{\Omega_-}(x) + g(x)1_{\Omega_+}(x).$$

We are going to prove some properties for g and G which are elementary in nature, but will be important for the sequel.

Proposition 4.3. *There exists a positive constant $C > 0$ such that for all $k \in \mathbb{N}$,*

$$\sup_{x \in \mathcal{C}_{+,k}} g(x) \leq C \inf_{x \in \mathcal{C}_{+,k}} g(x). \quad (4.41)$$

Proof. Let $k \in \mathbb{N}$ and $x_k \in \overline{\mathcal{C}_{+,k}}$ be such that

$$\sup_{x \in \mathcal{C}_{+,k}} \int_{\Omega_-} \frac{1}{|x-y|^{d+2\alpha}} dy = \int_{\Omega_-} \frac{1}{|x_k-y|^{d+2\alpha}} dy.$$

According to (H4), we have for all $(x, y) \in \mathcal{C}_{+,k} \times \Omega_-$,

$$|x-y| \leq |x-x_k| + |x_k-y| \leq 2r_0 + |x_k-y|.$$

It follows that

$$\frac{1}{(2r_0 + |x_k-y|)^{d+2\alpha}} \leq \frac{1}{|x-y|^{d+2\alpha}}.$$

Since $y \in \Omega_-$, we deduce that $R_0 \leq |x_k-y|$ and we conclude to the existence of a constant C (independent of x_k and y) such that

$$\frac{1}{|x_k-y|} \leq \frac{C}{2r_0 + |x_k-y|}.$$

The conclusion follows. \square

Corollary 4.2. *There exists $\varepsilon_2 > 0$ such that for all $\varepsilon < \varepsilon_2$ we have*

$$\inf_{\substack{x,y \in \Omega \\ |x-y| < \varepsilon_2}} \frac{G(x)}{G(y)} > 0. \quad (4.42)$$

With similar arguments, we prove

Lemma 4.3. *Let $\tilde{g} : x \in \Omega_+ \mapsto \int_{\Omega \cap B(z_x, R_0)^c} \frac{1}{|x-y|^{d+2\alpha}} dy$. There exists $C_{r_0, d, \alpha}$ depending on r_0, d, α such that*

$$\forall x \in \mathcal{C}_{+,k}, \quad \max_{y \in \mathcal{C}_{+,k}} \tilde{g}(y) \leq C_{r_0, d, \alpha} \tilde{g}(x).$$

Proposition 4.4. *The map G is uniformly continuous in Ω_+ , that is*

$$\forall \varepsilon > 0, \exists \mu_\varepsilon \text{ such that } \forall (x, y) \in \Omega_+, |x-y| < \mu_\varepsilon, \text{ then } |G(x) - G(y)| < \varepsilon.$$

Proof. Let $\varepsilon > 0$ and $R > 0$ be such that

$$\int_{|y| > R} \frac{1}{|y|^{d+2\alpha}} dy = \frac{1}{R^{2\alpha}} \leq \frac{\varepsilon}{4}. \quad (4.43)$$

Next, remarking that for all $(x, y) \in \Omega_+ \times \Omega_-$ we have $R_0 \leq |x-y|$, by the uniform continuity of the application $(t \in]R_0, +\infty[\mapsto t^{-(d+2\alpha)})$ we deduce the existence of a distance $\mu_\varepsilon > 0$ such that for all $x_1, x_2 \in \Omega_+$ with $|x_1 - x_2| \leq \mu_\varepsilon$ we have

$$\forall y \in \Omega_-, \left| \frac{1}{|x_1-y|^{d+2\alpha}} - \frac{1}{|x_2-y|^{d+2\alpha}} \right| \leq \frac{\varepsilon}{2m(B(0, R+2\mu_\varepsilon))}. \quad (4.44)$$

Next, we conclude thanks to (4.43) and (4.44) that

$$\begin{aligned} & \left| \int_{\Omega_-} \frac{1}{|x_1-y|^{d+2\alpha}} dy - \int_{\Omega_-} \frac{1}{|x_2-y|^{d+2\alpha}} dy \right| \\ & \leq \left| \int_{\Omega_- \cap B(x_1, R+2\mu_\varepsilon)} \frac{1}{|x_1-y|^{d+2\alpha}} - \frac{1}{|x_2-y|^{d+2\alpha}} dy \right| \\ & \quad + \left| \int_{\Omega_- \cap B(x_1, R+2\mu_\varepsilon)^c} \frac{1}{|x_1-y|^{d+2\alpha}} dy \right| + \left| \int_{\Omega_- \cap B(x_1, R+\mu_\varepsilon)^c} \frac{1}{|x_2-y|^{d+2\alpha}} dy \right| \\ & \leq \varepsilon. \end{aligned}$$

\square

Finally, we recall the following strong maximum principle for the fractional Laplacian :

Lemma 4.4. *For any smooth bounded domain $\mathcal{U} \subset \mathbb{R}^d$, and any smooth non-trivial function $v \in H^\alpha(\mathcal{U})$ such that*

1. $v \geq 0$,
2. $((-\Delta)^\alpha - Id)v > 0$,

it follows $v > 0$ in the interior of the domain \mathcal{U} .

4.3.3 The proof of Lemma 4.2

Step 1 : Localisation argument. Let χ_k be a function such that $\chi_k \in C_c^\infty(\mathbb{R}^d, [0, 1])$ and

$$\chi_k(x, z) = \begin{cases} 1 & \text{if } x \in B(z_k, r_0), \\ 0 & \text{if } x \in B(z_k, 2r_0)^c \end{cases} \quad (4.45)$$

where z_k and r_0 are provided by (H4). We set $n_k = n_+ \times \chi_k$. This function belongs to $H_0^\alpha(\mathcal{C}_{+,k})$

$$\begin{aligned} ((-\Delta)^\alpha - Id)n_k(x) &= \int_{\mathbb{R}^d} \frac{n_k(x) - n_k(y)}{|x - y|^{d+2\alpha}} dy - n_k(x) \\ &= \int_{\mathbb{R}^d} \frac{n_+(x) - n_+(y)}{|x - y|^{d+2\alpha}} dy + \int_{\mathbb{R}^d} \frac{n_+(y)(1 - \chi_k(y))}{|x - y|^{d+2\alpha}} dy - n_+(x) \\ &= \int_{\mathbb{R}^d} \frac{n_+(y)(1 - \chi_k(y))}{|x - y|^{d+2\alpha}} dy - n_+(x)^2 \\ &\leq \int_{B(z_k, R_0)^c \cap \Omega} \frac{n_+(y)}{|x - y|^{d+2\alpha}} dy. \end{aligned}$$

In the following, we denote by

$$\tilde{g}_k = \max_{x \in \mathcal{C}_{+,k}} \int_{B(z_k, R_0)^c \cap \Omega} \frac{n_+(y)}{|x - y|^{d+2\alpha}} dy \quad (4.46)$$

Let ϕ_k be the principal eigenfunction of the Dirichlet operator $(-\Delta)^\alpha - Id$ in the set

$$\{x \in \mathbb{R}^d \mid \text{dist}(x, \mathcal{C}_{+,k}) < \varepsilon_1\}.$$

Next, we define

$$\psi_k(x) = \frac{\tilde{g}_k}{c\lambda_0 \min(\varepsilon_1^{2\alpha}, 1)} \phi_k(x) \quad (4.47)$$

where c is a positive constant such that $\min_{y \in \mathcal{C}_{+,k}} \phi_k(y) > c \min(\varepsilon_1^{2\alpha}, 1)$. Indeed, since $\mathcal{C}_{+,k}$ is uniformly bounded with respect to k , we deduce thanks to [97] (Proposition 3.5) and thanks to the interior ball condition that c exists and is positive. Moreover, we have obviously

$$\tilde{g}_k \leq ((-\Delta)^\alpha - Id)\psi_k(x).$$

Since $n_+ \leq 1$, we deduce the existence of $C > 1$ such that

$$\forall x \in \mathcal{C}_{+,k}, \quad n_+(x) \leq C\psi_k(x).$$

Let $C^* = \inf \{C > 1 \mid n_+ \leq C\psi_k\}$. We claim that $C^* = 1$. Indeed, if we assume by contradiction that $C^* \neq 1$, it follows that there exists $x_0 \in \overline{\mathcal{C}_{+,k}}$ such that $C^*\psi_k(x_0) - n_k(x_0) = 0$. Recalling that $n_k|_{\partial\mathcal{C}_{+,k}} = 0$ and $\psi_k|_{\partial\mathcal{C}_{+,k}} \neq 0$, it follows that $x_0 \in \mathcal{C}_{+,k}$. Remarking that

$$((-\Delta)^\alpha - Id)(C^*\psi_k - n_k)(x) \geq C^*\tilde{g}_k - \tilde{g}(x) \geq 0,$$

the existence of $x_0 \in \mathcal{C}_{+,k}$ is in contradiction with Lemma 4.4. Therefore, we deduce that $C^* = 1$. We conclude thanks to Lemma 4.3 that for all $x \in \mathcal{C}_{+,k}$

$$n_+(x) \leq n_k(x) \leq \psi_k(x) = \underbrace{\frac{C_{r_0,d,\alpha}}{c\lambda_0 \min(1, \varepsilon_1^{2\alpha})}}_{:=C_0} \tilde{g}(x). \quad (4.48)$$

Step 2 : Concentration in Ω_- . Since $n_+ \leq 1$, $R_0 > C_0^{\frac{1}{2\alpha}}$ it follows by an immediate iteration of (4.48) that for all $x \in \Omega_+$:

$$n_+(x) \leq C_0 \sum_{k=0}^{+\infty} I_k(x)$$

with

$$I_k(x) = C_0 \int_{\Omega_+ \cap B(x, R_0)^c} \frac{I_{k-1}(y)}{|x-y|^{d+2\alpha}} dy, \quad I_0 = \int_{\Omega_-} \frac{1}{|x-y|^{d+2\alpha}} dy.$$

Let us prove by iteration that

$$I_k(x) \leq C_1^k \int_{\Omega_-} \frac{1}{|x-y|^{d+2\alpha}} dy \quad \text{with } C_1 < 1. \quad (4.49)$$

The conclusion that we may only use the components Ω_- will follows. It is clear that (4.49) holds true for $k = 0$. Next, we prove it for $k = 1$:

$$\begin{aligned} I_1(x) &= C_0 \int_{\Omega_+ \cap B(z_x, R_0)^c} \frac{1}{|x-y_0|^{d+2\alpha}} \int_{\Omega_-} \frac{1}{|y_0-y_1|^{d+2\alpha}} dy_1 dy_0 \\ &= C_0 \int_{(x-y'_0) \in \Omega_+, y'_0 \in B(0, R_0)^c} \frac{1}{|y'_0|^{d+2\alpha}} \int_{\Omega_-} \frac{1}{|x-y'_0-y_1|^{d+2\alpha}} dy_1 dy'_0. \end{aligned}$$

Next, by the convexity of the function ($t \mapsto |t|^{d+2\alpha}$) it follows

$$|x-y_1| \leq |x-y'_0-y_1| + |y'_0| \Rightarrow \frac{1}{|x-y'_0-y_1|^{d+2\alpha}} \leq \frac{2^{d-1+2\alpha}}{|x-y_1|^{d+2\alpha}} \left(1 + \frac{|y'_0|^{d+2\alpha}}{|x-y'_0-y_1|^{d+2\alpha}}\right).$$

Fubini's Theorem leads to

$$\begin{aligned} I_1(x) &\leq C_0 \int_{\Omega_-} \frac{1}{|x-y_1|^{d+2\alpha}} \int_{(x-y'_0) \in \Omega_+, y'_0 \in B(0, R_0)^c} \frac{2^{d-1+2\alpha}}{|y'_0|^{d+2\alpha}} \left(1 + \frac{|y'_0|^{d+2\alpha}}{|x-y'_0-y_1|^{d+2\alpha}}\right) dy'_0 dy_1 \\ &\leq C_0 \int_{\Omega_-} \frac{2^{d-1+2\alpha}}{|x-y_1|^{d+2\alpha}} \left(\int_{y'_0 \in B(0, R_0)^c} \frac{1}{|y'_0|^{d+2\alpha}} dy'_0 + \int_{(x-y'_0) \in \Omega_+} \frac{1}{|x-y'_0-y_1|^{d+2\alpha}} dy'_0 \right) dy_1 \\ &\leq \frac{2^{d+2\alpha} C_0}{R_0^{2\alpha}} \times \int_{\Omega_-} \frac{1}{|x-y_1|^{d+2\alpha}} dy_1. \end{aligned}$$

Assuming $R_0 > (2^{d+2\alpha} C_0)^{\frac{1}{2\alpha}}$ leads to the conclusion that (4.49) holds true for $k = 1$. If we assume that it is true for $k \in \mathbb{N}$, we have by the recursive hypothesis on I_k that

$$I_{k+1}(x) \leq C_0 \int_{\Omega_+ \cap B(x, R_0)^c} \frac{C_1^k}{|x-y_0|^{d+2\alpha}} \int_{\Omega_-} \frac{1}{|y_0-y_1|^{d+2\alpha}} dy_1 dy_0.$$

The conclusions follows from computations that are similar to those of the case $k = 1$.

4.3.4 Proof of Theorem 4.4

Before providing the proof of Theorem 4.4 we introduce an intermediate technical result :

Lemma 4.5. *There exist two constant $C, \sigma > 0$ such that for all $x \in \Omega_+$,*

$$\int_{\Omega_-} \frac{1}{|x-y|^{d+2\alpha}} dy \leq C \int_{(\Omega_-)_\sigma} \frac{1}{|x-y|^{d+2\alpha}} dy$$

and

$$\int_{\mathcal{C}_{+,k_x}} \frac{1}{|x-y|^{d+2\alpha}} dy \leq C \int_{(\mathcal{C}_{+,k_x})_\sigma} \frac{1}{|x-y|^{d+2\alpha}} dy,$$

where \mathcal{O}_σ is defined in (4.11) for any set \mathcal{O} .

Proof. We prove only the first inequality. The second one can be proved following similar computations. Thanks to the interior ball condition (H2), it is sufficient to prove that there exists $\sigma, C > 0$ such that there holds

$$\int_{B(z,\varepsilon_0)} \frac{1}{|x-y|^{d+2\alpha}} dy \leq C \int_{B(z,\varepsilon_0)_\sigma} \frac{1}{|x-y|^{d+2\alpha}} dy \quad (4.50)$$

for any $x \in \Omega_+$ and $B(z,\varepsilon) \subset \Omega_-$. First, we remark that there exists two constants $c_1, c_2 > 0$ such that for all $t \in]R_0, +\infty[$,

$$\frac{c_1}{t^{d+2\alpha}} \leq \frac{1}{(t+\varepsilon_0)^{d+2\alpha}} \quad \text{and} \quad \frac{1}{(t-\varepsilon_0)^{d+2\alpha}} \leq \frac{c_2}{t^{d+2\alpha}}.$$

Next, denoting by $m(E)$ the Lebesgue measure of the set E , we define

$$\sigma = \frac{\varepsilon_0}{4} \quad \text{and} \quad C = \frac{2c_2 m(B(0, \varepsilon_0))}{c_1 m(B(0, \frac{\varepsilon_0}{2}))}.$$

With a such choice of constants, it follows that

$$\begin{aligned} C \int_{B(z,\varepsilon_0)_\sigma} \frac{1}{|x-y|^{d+2\alpha}} dy - \int_{B(z,\varepsilon_0)} \frac{1}{|x-y|^{d+2\alpha}} dy &\geq \frac{C m(B(0, \frac{\varepsilon_0}{2}))}{(|x-z|+\varepsilon_0)^{d+2\alpha}} - \frac{m(B(0, \varepsilon_0))}{(|x-z|-\varepsilon_0)^{d+2\alpha}} \\ &\geq \frac{C c_1 m(B(0, \frac{\varepsilon_0}{2}))}{|x-z|^{d+2\alpha}} - \frac{c_2 m(B(0, \varepsilon_0))}{|x-z|^{d+2\alpha}} \\ &\geq \frac{c_2 m(B(0, \varepsilon_0))}{|x-z|^{d+2\alpha}} \geq 0. \end{aligned}$$

□

Proof of Theorem 4.4. We split the study into two parts : the study in Ω_+ and the study in Ω_- . Each part is split into two sub-parts :the lower and the upper bounds.

Part 1 : The study in Ω_- . Subpart A : The lower bound. Let $x \in \Omega_-$ and $z_x \in \Omega_-$, $k \in \{1, \dots, N\}$ such that (H6) holds true. Since $\lambda_\alpha(\underline{\Omega}_k) < 0$, we deduce the existence of $\underline{n}_{k,z}$ the solution of

$$\begin{cases} (-\Delta)^\alpha \underline{n}_{k,z} = \underline{n}_{k,z_x} - \underline{n}_{k,z_x}^2 & \text{in } \underline{\Omega}_k + z_x, \\ \underline{n}_{k,z_x} = 0 & \text{in } (\underline{\Omega}_k + z_x)^c. \end{cases}$$

The maximum principle implies that

$$\underline{n}_{k,z_x}(x) \leq n_+(x).$$

Moreover, since $\underline{\Omega}_k$ is bounded and regular, we deduce the existence of c_k such that

$$c_k \min(\delta(x)^\alpha, 1) \leq \underline{n}_{k,z_x}(x).$$

If we fix $c = \min_{k \in \{1, \dots, N\}} c_k$ then the conclusion follows.

Subpart B : The upper bound. From the maximum principle, it is clear that $n_+(x) \leq 1$. Therefore, we focus on what happens at the boundary : let $x \in \Omega_-$ such that $\delta(x) < \varepsilon_0$ and $\tilde{z}_x \in \Omega^c$ provided by the exterior ball condition such that

$$B(\tilde{z}_x, \varepsilon_0) \cap \Omega = \emptyset, \quad \overline{B}(\tilde{z}_x, \varepsilon_0) \cap \overline{\Omega} \neq \emptyset \quad \text{and} \quad x \in B(\tilde{z}_x, 2\varepsilon_0).$$

Then, from the maximum principle applied to n_+ and $\psi(\frac{\cdot - \tilde{z}_x}{\varepsilon_0})$ where ψ is defined by (4.38), it follows

$$n_+(x) \leq \psi\left(\frac{x - \tilde{z}_x}{\varepsilon_0}\right) \leq C\delta(x)^\alpha.$$

The conclusion follows.

Part 2 : The study in Ω_+ . Subpart A : The lower bound. We prove that for all $x \in \Omega_+$

$$c \min(\delta(x)^\alpha, \varepsilon_0^\alpha) \int_{\Omega_-} \frac{1}{|x - y|^{d+2\alpha}} dy \leq n_+(x). \quad (4.51)$$

Let $x \in \Omega_+$ and $z_x \in \Omega_+$ provided by (H2) (remark that $z_x = x$ if $\delta(x) \geq \varepsilon_0$). We define \underline{n} as the solution of

$$\begin{cases} (-\Delta)^\alpha \underline{n}(y) = 0 & \text{for } y \in B(z_x, \varepsilon_0), \\ \underline{n}(y) = n_+ & \text{for } y \in \Omega_-, \\ \underline{n}(y) = 0 & \text{for } y \in \mathbb{R}^d \setminus (B(z_x, \varepsilon_0) \cup \Omega_-). \end{cases}$$

The comparison principle gives that

$$\underline{n}(x) \leq n_+(x).$$

We recall that thanks to (H6) for all positive σ small enough, there exists $c_\sigma > 0$ such that $c_\sigma < n_+(y)$ for all $y \in (\Omega_-)_\sigma$. Formula (4.36) gives

$$\begin{aligned} \underline{n}(x) &= \int_{\Omega_-} c_\alpha^d \left(\frac{\varepsilon_0^2 - |x - z_x|^2}{|y - z_x|^2 - \varepsilon_0^2} \right)^\alpha \frac{1}{|x - y|^d} n_+(y) dy \\ &\geq c c_\alpha^d (\varepsilon_0^2 - |x - z_x|^2)^\alpha \int_{\Omega_{-, \sigma}} \frac{c_\sigma}{|y - z_x|^{d+2\alpha}} dy. \end{aligned}$$

If $\varepsilon_0 \leq \delta(x)$, we deduce that $z_x = x$ and (4.51) holds true thanks to Lemma 4.5. Otherwise, we have

$$(\varepsilon_0 \delta(x))^\alpha \leq (\varepsilon_0 + |x - z_x|)^\alpha \delta(x)^\alpha = (\varepsilon_0^2 - |x - z_x|^2)^\alpha.$$

By uniform continuity and compactness of $B(z_x, \varepsilon_0)$, we deduce that for all $y \in \Omega_-$ there holds $\frac{c}{|x-y|^{d+2\alpha}} \leq \frac{1}{|z_x-y|^{d+2\alpha}}$, thus (4.51) follows from Lemma 4.5.

Subpart B : The upper bound. Thanks to Lemma 4.2, we only have to consider $x \in \{y \in \Omega_+ \text{ such that } \delta(y) < \varepsilon_0\}$. Let $k \in \mathbb{N}$ and $x \in \mathcal{C}_{+,k}$ be such that $\delta(x) < \varepsilon_0$. Assumption (H2) ensures the existence of $\tilde{z}_x \in \mathbb{R}^d$ such that

$$B(\tilde{z}_x, \varepsilon_0) \cap \Omega = \emptyset, \quad \partial B(\tilde{z}_x, \varepsilon_0) \cap \partial\Omega \neq \emptyset \quad \text{and} \quad x \in B(\tilde{z}_x, 2\varepsilon_0) \cap \Omega. \quad (4.52)$$

As mentioned in section 2.1, the aim of the proof is to prove that the constant C_k defined by (4.39) verifies

$$n_+(x) \leq C_k \psi\left(\frac{x - \tilde{z}_x}{\varepsilon_0}\right) \quad (4.53)$$

where ψ_k is defined by (4.38). Let $\tilde{\chi}_k$ be a function of $C_c^\infty(\mathbb{R}^d, [0, 1])$ such that

$$\tilde{\chi}_k(x) = \begin{cases} 1 & \text{for } x \text{ such that } \text{dist}(x, \mathcal{C}_{+,k}) < \frac{\varepsilon_0}{4} \\ 0 & \text{for } x \text{ such that } \text{dist}(x, \mathcal{C}_{+,k}) > \frac{\varepsilon_0}{2}. \end{cases} \quad (4.54)$$

In order to prove (4.53), we prove first that for all $y \in T(\tilde{z}_x, \varepsilon_0, 4\varepsilon_0) \cap \mathcal{C}_{+,k}$

$$(-\Delta)^\alpha \left(n_+ \tilde{\chi}_k(y) - C_k \psi\left(\frac{y - \tilde{z}_x}{\varepsilon_0}\right) \right) < 0. \quad (4.55)$$

Next, we prove that for all $y \in (T(\tilde{z}_x, \varepsilon_0, 4\varepsilon_0) \cap \mathcal{C}_{+,k})^c$, we have

$$n_+ \tilde{\chi}_k(y) - C_k \psi\left(\frac{y - \tilde{z}_x}{\varepsilon_0}\right) \leq 0. \quad (4.56)$$

The conclusion follows thanks to the maximum principle.

Proof that (4.55) holds true. Let $y \in T(\tilde{z}_x, \varepsilon_0, 4\varepsilon_0) \cap \mathcal{C}_{+,k}$, then thanks to the properties of ψ (see (4.38)), and Lemma 4.2, we obtain

$$\begin{aligned} & (-\Delta)^\alpha \left(n_+ \tilde{\chi}_k(y) - C_k \psi\left(\frac{y - \tilde{z}_x}{\varepsilon_0}\right) \right) \\ & \leq (-\Delta)^\alpha n_+(y) + \int_{\mathcal{C}_{+,k}^c} \frac{n_+(y_0)(1 - \tilde{\chi}_k(y_0))}{|y - y_0|^{d+2\alpha}} dy_0 - \frac{C_k}{\varepsilon_0^{2\alpha}} \\ & \leq n_+(y) + \int_{\Omega_-} \frac{1}{|y_0 - y|^{d+2\alpha}} dy_0 + \int_{\mathcal{C}_{+,k}^c \cap \Omega_+} \frac{n_+(y_0)}{|y - y_0|^{d+2\alpha}} dy_0 - C_k \\ & \leq (C_2 + 1) \int_{\Omega_-} \frac{1}{|y - y_0|^{d+2\alpha}} dy_0 + C_2 \int_{\mathcal{C}_{+,k}^c \cap \Omega_+} \frac{1}{|y - y_0|^{d+2\alpha}} \int_{\Omega_-} \frac{1}{|y_0 - y_1|^{d+2\alpha}} dy_1 dy_0 - C_k \\ & \leq \left(C_2 \left(1 + \frac{C_{d,\alpha,r_0}}{R_0^{2\alpha}} \right) + 1 \right) \int_{\Omega_-} \frac{1}{|y - y_0|^{d+2\alpha}} dy_0 - C_k. \end{aligned}$$

We conclude thanks to (4.39) that (4.55) holds true.

Proof that (4.56) holds true. According to (4.52), it is straightforward that $(n_+ \tilde{\chi}_k(y) - C_k \psi(\frac{y - \tilde{z}_x}{\varepsilon_0})) \leq 0$ for all $y \in B(\tilde{z}_x, \varepsilon_0) \cup (B(\tilde{z}_x, 4\varepsilon_0)^c \cap \mathcal{C}_{+,k}^c)$. Therefore, we focus on $y \in B(\tilde{z}_x, 4\varepsilon_0)^c \cap \mathcal{C}_{+,k}$. Thanks to Lemma 4.2 and the properties of ψ (introduced in (4.38)), we obtain

$$n_+(y) \tilde{\chi}_k(y) - C_k \psi\left(\frac{y - \tilde{z}_x}{\varepsilon_0}\right) \leq n_+(y) - C_k \leq C_2 \int_{\Omega_-} \frac{1}{|y - y_0|^{d+2\alpha}} dy_0 - C_k < 0.$$

We deduce that (4.56) holds true.

The conclusion follows thanks to Proposition 4.3. \square

4.3.5 Some examples where \mathcal{G} is explicit

In this subsection, we detail some examples where the function \mathcal{G} is more explicit. It highlights how the shape of the steady solution is strongly connected to the domain Ω .

1. $\Omega = \mathbb{R}^d$. It is well known that in this case

$$\mathcal{G}(x) = 1.$$

2. Ω is **bounded**. In this case, we have

$$\mathcal{G}(x) = \delta(x)^\alpha.$$

We recover here a consequence of [97].

3. Ω is **unbounded and** $\Omega_+ = \emptyset$. Following Theorem 4.4, there holds

$$\mathcal{G}(x) = \min(1, \delta(x)^\alpha).$$

Remark that this case contains the periodic patchy domain (see [78] for more details) :

$$\Omega = \bigcup_{k \in \mathbb{Z}^d} (\Omega_0 + a_k)$$

where $\Omega_0 + a_k = \{x \in \mathbb{R}^d \mid x - a_k \in \Omega_0\}$, Ω_0 bounded and such that $\lambda_\alpha(\Omega_0) < 0$ and $a_k \in \mathbb{R}^d$ such that

$$a_{k+e_i} - a_k = a_{e_i} \quad \text{where } e_i \text{ is the } i^{\text{th}} \text{ element of the canonical basis of } \mathbb{Z}^d.$$

4. Ω is **unbounded with** Ω_- **bounded**. In this case, up to a translation, we can assume that $B(0, \varepsilon_0) \subset \Omega_- \subset B(0, R_1)$ (with $R_1 > 0$ large enough). Then it is easy to observe that there exists two constants $c, C > 0$ such that

$$\frac{c \min(\delta(x)^\alpha, 1)}{1 + |x|^{d+2\alpha}} \leq \mathcal{G}(x) \leq \frac{C \min(\delta(x)^\alpha, 1)}{1 + |x|^{d+2\alpha}}.$$

5. **The dimension** $d = 1$ **and** $\Omega_- = \mathbb{R}^-$. In this case, a straightforward computation leads to

$$c \min(\delta(x)^\alpha, 1) \left(1_{\Omega_-}(x) + \frac{1_{\Omega_+}(x)}{1 + |x|^{2\alpha}} \right) \leq \mathcal{G}(x) \leq C \min(\delta(x)^\alpha, 1) \left(1_{\Omega_-}(x) + \frac{1_{\Omega_+}(x)}{1 + |x|^{2\alpha}} \right).$$

Of course, the hypothesis on Ω allows more general domains. However, with these five examples, we can already observe how the behavior of the solution n_+ may change from one domain to another.

4.4 Uniqueness and attractivity of the steady state

4.4.1 Uniqueness

As mentioned before, the idea to prove uniqueness is to compare two solutions u_+ and v_+ thanks to Theorem 4.4. Before providing the details of the proof, we make an easy but important remark :

$$\forall x \in \mathbb{R}^d, \quad \mathcal{G}(x) \leq 1. \tag{4.57}$$

In the case where $\Omega_+ = \emptyset$, the proof would be very easy. Because of the possibly degeneracy of the solutions in Ω_+ , the proof is not just an adaptation of the case $\Omega_+ = \emptyset$. It is inspired by the strategy of the proof of the fractional Hopf Lemma provided by Grecco and Servadei in [65].

Proof of Theorem 4.5. Let u_+ and v_+ be two bounded non trivial solutions of (4.1). Thanks to Theorem 4.4, there exists a constant $L > 0$ such that

$$u_+ \leq Lv_+.$$

We define $l = \inf \{L > 1 \text{ such that } u_+ \leq Lv_+\}$. We are going to prove by contradiction that $l = 1$ because if $l = 1$ then $u_+ \leq v_+$ and with the same argument, $v_+ \leq u_+$ and then the conclusion follows. Thus, we assume $l > 1$ by contradiction. Next we define

$$w = \inf_{x \in \Omega} \frac{(lv_+ - u_+)(x)}{\mathcal{G}(x)}.$$

Thanks to the definition of l , we deduce that $w = 0$ (otherwise we can construct $l^* < 1$ such that $u_+ < l^*v_+$ holds true, see the proof of Theorem 1 in [78] for more details). Let $(x_n)_{n \in \mathbb{N}} \in \Omega$ be a minimizing sequence

$$\text{i.e.} \quad \frac{(lv_+ - u_+)(x_n)}{\mathcal{G}(x_n)} \xrightarrow{n \rightarrow +\infty} 0.$$

In order to localise where we use the maximum principle or the fractional Hopf Lemma, we will use the help from the bubble function $\beta_{z,r}$:

$$\beta_{z,r}(x) = C_\alpha (r^2 - |x - z|^2)^\alpha 1_{B(z,r)}(x) \quad \text{with } C_\alpha = \frac{\Gamma(\frac{d}{2})}{4^\alpha \Gamma(1 + \alpha) \Gamma(\frac{d}{2} + \alpha)}. \quad (4.58)$$

Indeed, $\beta_{z,r}$ is solution of the equation

$$\begin{cases} (-\Delta)^\alpha \beta_{z,r}(x) = 1 & \text{for } x \in B(z,r), \\ \beta_{z,r}(x) = 0 & \text{for } x \in B(z,r)^c. \end{cases} \quad (4.59)$$

We distinguish 2 cases : (up to a subsequence) $\inf \delta(x_n) > \delta_0 > 0$ and $\delta(x_n) \xrightarrow{n \rightarrow +\infty} 0$.

In the first case, we define $z_n = x_n$ and $r = \delta_0$, and in the second one, we define $r = \varepsilon_0$ and we use the interior ball condition (hypothesis (H2)) to deduce the existence of $z_n \in \Omega$ such that

$$\delta(x_n) = \varepsilon_0 - |z_n - x_n|.$$

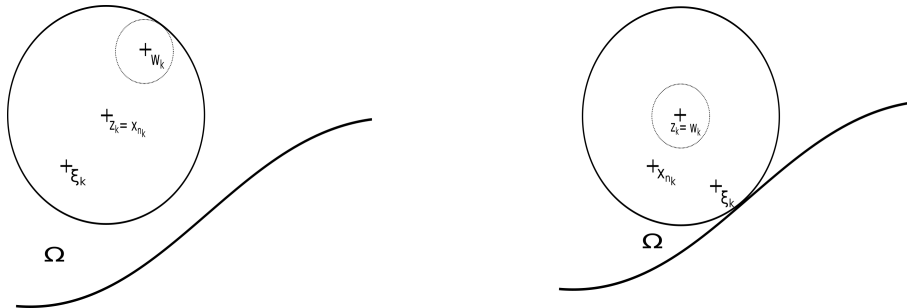


FIGURE 4.3 – The left picture illustrates the case where $\inf \delta(\xi_k) > \delta_0$ (in this case, $z_k = x_k$). The right picture illustrates the case $\inf \delta(\xi_k) = 0$. In this case, up to a subsequence, we can assume that $\delta(\xi_k) < \frac{\varepsilon_0}{4}$ and $w_k = z_k$.

We prove by contradiction that there exists $\rho > 0$ such that

$$\forall x \in \mathbb{R}^d, \quad \rho \beta_{z_n, r}(x) \leq \frac{(lv_+ - u_+)(x_n)}{G(x_n)}. \quad (4.60)$$

Indeed, if (4.60) holds true for all $n \in \mathbb{N}$, then we deduce the following contradiction

$$\begin{aligned} \rho C_\alpha r^\alpha &= \inf_{x \in B(0, \varepsilon_0)} \frac{\rho \beta_{0, r}(x)}{(r - |x|)^\alpha} = \inf_{x \in B(z_n, r)} \frac{\rho \beta_{z_n, r}(x)}{(r - |x - z_n|)^\alpha} \\ &\leq \frac{(lv_+ - u_+)(x_n)}{\min(r_0, \delta(x_n))^\alpha G(x_n)} = \frac{(lv_+ - u_+)(x_n)}{\mathcal{G}(x_n)} \xrightarrow{n \rightarrow +\infty} 0. \end{aligned}$$

Assume by contradiction that (4.60) is false. Then since $\text{supp}(\beta_{z_n, r}) = B(z_n, r)$, we deduce that

$$\forall k \in \mathbb{N}, \exists n_k > n_{k-1} \text{ and } y_k \in B(z_{n_k}, r) \text{ such that } \frac{(lv_+ - u_+)(y_k)}{G(x_{n_k})} < \frac{\beta_{z_{n_k}, r}(y_k)}{k}.$$

Since for all $x \in \mathbb{R}^d \setminus B(z_{n_k}, r)$ we have

$$\frac{\beta_{z_{n_k}, r}(x)}{k} - \frac{(lv_+ - u_+)(x)}{G(x_{n_k})} = -\frac{(lv_+ - u_+)(x)}{G(x_{n_k})} < 0$$

we deduce that $\frac{\beta_{z_{n_k}, r}}{k} - \frac{(lv_+ - u_+)}{G(x_{n_k})}$ takes its maximum at some $\xi_k \in B(z_{n_k}, r)$. Remark that since $(\beta_{z_{n_k}, r})_{k \in \mathbb{N}}$ is uniformly bounded, we have that

$$0 < \frac{(lv_+ - u_+)(\xi_k)}{G(x_{n_k})} < \frac{\sup \beta_{z_{n_k}, r}}{k} \xrightarrow{k \rightarrow +\infty} 0. \quad (4.61)$$

Next, if we compute $(-\Delta)^\alpha (lv_+ - u_+)$, we find that

$$\begin{aligned} (-\Delta)^\alpha (lv_+ - u_+) &= lv_+ - lv_+^2 - u_+ + u_+^2 \\ &= (lv_+ - u_+) - (lv_+ - u_+)(v_+ + u_+) + u_+ v_+ (l - 1) \\ &\geq (lv_+ - u_+)(1 - (u_+ + v_+)). \end{aligned}$$

Recalling that $u_+ \leq 1$ and $v_+ \leq 1$, we obtain

$$-\frac{(lv_+ - u_+)}{G(x_{n_k})} \leq (-\Delta)^\alpha \frac{(lv_+ - u_+)}{G(x_{n_k})}. \quad (4.62)$$

On one hand, if we evaluate (4.62) at ξ_k we find thanks to (4.61) and Proposition (4.3) that for k large enough

$$-\varepsilon \leq -\frac{C_1 G(\xi_k) \delta(\xi_k)^\alpha}{G(x_{n_k})} \leq -\frac{(lv_+ - u_+)(\xi_k)}{G(x_{n_k})} \leq (-\Delta)^\alpha \frac{(lv_+ - u_+)(\xi_k)}{G(x_{n_k})} \quad (4.63)$$

with ε as small as we want.

On the other hand, we claim that there exists $c_* > 0$ such that $(-\Delta)^\alpha \frac{(lv_+ - u_+)(\xi_k)}{G(x_{n_k})} \leq -c_* + \varepsilon$ with ε as small as we want. Indeed, we first introduce $w_k \in B(z_{n_k}, r)$ such that

$$B(w_k, \frac{r}{4}) \subset B(z_{n_k}, r), \quad \delta(w_k) > \frac{r}{2} \quad \text{and} \quad \forall y \in B(w_k, \frac{r}{4}), \quad |y - \xi_k| > \frac{r}{4} \quad (4.64)$$

(remark that one can take $w_k = z_{n_k}$ if $\delta(x_{n_k}) \xrightarrow[k \rightarrow +\infty]{} 0$). Next, we split $(-\Delta)^\alpha \frac{(lv_+ - u_+)(\xi_k)}{G(x_{n_k})}$ in the following way

$$\begin{aligned} (-\Delta)^\alpha \frac{(lv_+ - u_+)(\xi_k)}{G(x_{n_k})} &= \frac{1}{G(x_{n_k})} \int_{B(w_k, \frac{r}{4})} \frac{(lv_+ - u_+)(\xi_k) - (lv_+ - u_+)(y)}{|\xi_k - y|^{d+2\alpha}} dy \\ &\quad + \frac{1}{G(x_{n_k})} \int_{B(w_k, \frac{r}{4})^c} \frac{(lv_+ - u_+)(\xi_k) - (lv_+ - u_+)(y)}{|\xi_k - y|^{d+2\alpha}} dy \\ &= I_1 + I_2. \end{aligned}$$

For I_1 , according to (4.61), (4.64), Theorem 4.4 and Corollary 4.2 we deduce that for k large enough we have

$$\begin{aligned} I_1 &= \frac{m(B(0, \frac{r}{4}))(lv_+ - u_+)(\xi_k)}{G(x_{n_k})} - \int_{B(w_k, \frac{r}{4})} \frac{(lv_+ - u_+)(y)}{G(x_{n_k})|\xi_k - y|^{d+2\alpha}} dy \\ &\leq \frac{m(B(0, \frac{r}{4}))(lv_+ - u_+)(\xi_k)}{G(x_{n_k})} - \int_{B(w_k, \frac{r}{4})} \frac{(c_1 + C_1)G(y) \min(\delta(y)^\alpha, 1)}{G(x_{n_k})|\xi_k - y|^{d+2\alpha}} dy \\ &\leq \frac{\varepsilon}{2} - r^\alpha (c_1 + C_1) \inf_{y \in B(w_k, \frac{r}{4})} \frac{G(y)}{G(x_{n_k})} \inf_{\xi \in T(0, \frac{r}{4}, \frac{3r}{4})} \int_{B(0, \frac{r}{4})} \frac{1}{|\xi - y|^{d+2\alpha}} dy \\ &= \frac{\varepsilon}{2} - c_*. \end{aligned} \tag{4.65}$$

Note that c_* depends only on r and not on k . For I_2 , we recall that

$$\max_{x \in \mathbb{R}^d} \left(\frac{\beta_{z_{n_k}, r}(x)}{k} - \frac{(lv_+ - u_+)(x)}{G(x_{n_k})} \right) = \frac{\beta_{z_{n_k}, r}(\xi_k)}{k} - \frac{(lv_+ - u_+)(\xi_k)}{G(x_{n_k})}.$$

Then we deduce that for k large enough we have

$$\begin{aligned} I_2 &\leq \frac{1}{k} \int_{B(w_k, \frac{r}{4})^c} \frac{\beta_{z_{n_k}, \varepsilon_0}(\xi_k) - \beta_{z_{n_k}, \varepsilon_0}(y)}{|\xi_k - y|^{d+2\alpha}} dy \\ &= \frac{1}{k} \left((-\Delta)^\alpha \beta_{z_{n_k}, r}(\xi_k) - \int_{B(w_k, \frac{r}{4})} \frac{\beta_{z_{n_k}, r}(\xi_k) - \beta_{z_{n_k}, r}(y)}{|\xi_k - y|^{d+2\alpha}} dy \right) \\ &\leq \frac{1}{k} \left(1 + \max_{\xi \in T(0, \frac{r}{4}, \frac{3r}{4})} \int_{B(0, \frac{\varepsilon_0}{2})} \frac{\beta_{0, r}(y)}{|\xi - y|^{d+2\alpha}} dy \right) \\ &\leq \frac{\varepsilon}{2}. \end{aligned} \tag{4.66}$$

Finally, combining (4.63), (4.65) and (4.66), we deduce that for k large enough we have

$$-\varepsilon \leq (-\Delta)^\alpha \frac{(lv_+ - u_+)(\xi_k)}{G(x_{n_k})} \leq -c_* + \varepsilon$$

which is a contradiction for $\varepsilon = \frac{c_*}{4}$. □

4.4.2 Convergence to the steady state

The idea is quite classical : it consists in enclosing the solution of (4.3) between an increasing (with respect to the time) sub-solution and a decreasing (with respect to the time) super-solution. The specific ingredient is, once again, that the behaviour of the solution of the

Cauchy Problem at infinity has to match that of the steady solution. This is achieved through estimates of the heat kernel developed in [78] (Theorem 2 p.3) for general domains which satisfy the uniform interior and exterior ball condition (here assumed in the assumption (H2)). The details being otherwise standard, we just present an overview of the proof.

The super-solution. Up to a translation, there is no loss of generality to assume that

$$B(0, \varepsilon_0) \subset \Omega_-.$$

On one hand thanks to Lemma 4.1, there exists some constants $c > 0$ (that may change from line to line) such that

$$\begin{aligned} n_+(x) &\geq c \min(\delta(x)^\alpha, 1) \times \left(1_{\Omega_-}(x) + \int_{\Omega_-} \frac{1_{\Omega_+}(x)}{|x-y|^{d+2\alpha}} dy \right) \\ &\geq c \min(\delta(x)^\alpha, 1) \times \left(1_{\Omega_-}(x) + \int_{B(0, \varepsilon_0)} \frac{1_{\Omega_+}(x)}{|x-y|^{d+2\alpha}} dy \right) \\ &\geq \frac{c \min(\delta(x)^\alpha, 1)}{1 + |x|^{d+2\alpha}}. \end{aligned} \quad (4.67)$$

On the other hand, thanks to Theorem 2 in [78], there exists a constant $C > 0$ such that

$$n(x, t = 1) \leq \frac{C \min(\delta(x)^\alpha, 1)}{1 + |x|^{d+2\alpha}}. \quad (4.68)$$

From (4.67) and (4.68), we deduce that there exists a constant $C > 1$ such that

$$n(x, t = 1) \leq C n_+(x).$$

Next, we define

$$\begin{cases} \partial_t \bar{n} + (-\Delta)^\alpha \bar{n} = \bar{n} - \bar{n}^2 & \text{for } (x, t) \in \Omega \times]0, +\infty[, \\ \bar{n}(x, t) = 0 & \text{for } (x, t) \in \Omega^c \times [0, +\infty[, \\ \bar{n}(x, t = 0) = C n_+(x) & \text{for } x \in \Omega. \end{cases}$$

Since, in the distributional sense, we have

$$\lim_{t \rightarrow 0^+} \partial_t \bar{n}(x, t) = \lim_{t \rightarrow 0^+} (C \bar{n}(x, t) - C \bar{n}(x, t)^2 - (-\Delta)^\alpha C \bar{n}(x, t)) = C n_+(x) (1 - C) \leq 0.$$

We deduce that \bar{n} is decreasing. Since it is bounded from below by n_+ , it converges to a non-trivial stationary state of (4.1) and by uniqueness of n_+ we deduce that $\bar{n}(x, t) \xrightarrow{t \rightarrow +\infty} n_+(x)$.

The sub-solution. As previously, we assume that $B(0, \varepsilon_0) \subset \underline{\Omega}_1 \subset \Omega_-$ where $\underline{\Omega}_1$ is defined by (H6). Thanks to Theorem 2 in [78], there exists a constant $c > 0$ such that

$$\frac{c \min(\delta(x)^\alpha, 1)}{1 + |x|^{d+2\alpha}} \leq n(x, t = 1).$$

It follows that there exists $\sigma > 0$ such that

$$\sigma < \inf_{x \in \underline{\Omega}_1} n(x, t = 1).$$

Next, if we denote by ϕ_1 the principal eigenfunction of $(-\Delta)^\alpha - Id$ in $\underline{\Omega}_1$, it follows that the solution \underline{n} of

$$\begin{cases} \partial_t \underline{n} + (-\Delta)^\alpha \underline{n} = \underline{n} - \underline{n}^2 & \text{for } (x, t) \in \Omega \times]0, +\infty[, \\ \underline{n}(x, t) = 0 & \text{for } (x, t) \in \Omega^c \times [0, +\infty[, \\ \underline{n}(x, t = 0) = \frac{\min(\sigma, |\lambda_1|)\phi_1(x)}{\|\phi_1\|_\infty} & \text{for } x \in \Omega. \end{cases}$$

is increasing with respect to the time. Indeed, it is sufficient to verify it at time $t = 0$. For $x \in \underline{\Omega}_1$, there holds (in a distributional sense)

$$\begin{aligned} \lim_{t \rightarrow 0^+} \partial_t \underline{n}(x, t) &= \lim_{t \rightarrow 0^+} (\underline{n}(x, t) - \underline{n}(x, t)^2 - (-\Delta)^\alpha \underline{n}(x, t)) \\ &= \frac{\min(\sigma, |\lambda_1|)\phi_1(x)}{\|\phi_1\|_\infty} (|\lambda_1| - \frac{\min(\sigma, |\lambda_1|)\phi_1(x)}{\|\phi_1\|_\infty}) \\ &\geq 0. \end{aligned}$$

For $x \in \Omega \setminus \underline{\Omega}_1$, there holds (still in a distributional sense)

$$\lim_{t \rightarrow 0^+} \partial_t \underline{n}(x, t) = \lim_{t \rightarrow 0^+} (\underline{n}(x, t) - \underline{n}(x, t)^2 - (-\Delta)^\alpha \underline{n}(x, t)) = \int_{\mathbb{R}^d} \frac{\frac{\min(\sigma, |\lambda_1|)\phi_1(y)}{\|\phi_1\|_\infty}}{|x - y|^{d+2\alpha}} dy > 0.$$

Finally, \underline{n} is increasing and bounded therefore point-wise converging. By fractional elliptic regularity, the limit is a solution of (4.1) and by uniqueness of the non trivial stationary state n_+ we conclude that $\underline{n}(x, t) \xrightarrow[t \rightarrow +\infty]{} n_+(x)$.

Conclusion. Since the initial datum are right ordered

$$\text{i.e. } \underline{n}(x, t = 0) \leq n(x, t = 1) \leq \bar{n}(x, t = 0),$$

we conclude thanks to the comparison principle and the conclusions of the two lasts parts that

$$\lim_{t \rightarrow +\infty} \underline{n}(x, t) = n_+(x) \leq \lim_{t \rightarrow +\infty} n(x, t) \leq \lim_{t \rightarrow +\infty} \bar{n}(x, t) = n_+(x).$$

This ends the proof of Corollary 4.1.

4.5 Numerical illustrations, perspectives

We provide some numerical illustrations of the results developed in this section. More precisely, we investigate the large time of simulations of equation (4.3) with two 1-dimensional disconnected components modeled by a finite difference method. We vary the distance μ between the two components and the exponent of the diffusivity α . We recover that if the distance μ is to high or the constant α is to closed to 1 then the solution n of (4.3) tends to 0 which means that $\lambda_\alpha(\Omega_{1,2,\mu}) \geq 0$. Whereas if the distance μ is not to high and the constant α is not closed to 1 then the solution n tends to a non-trivial positive stationary state which means that $\lambda_\alpha(\Omega_{1,2,\mu}) < 0$.

The first simulation (Figure 4.4) shows the numerical solution n at time $T = 2000$ and $T = 4000$. We can not distinguish the difference between the two drawings, we deduce that we have reached the stationary state.

Next, in Figure 4.5 we increase α . We put $\alpha = 0.8$, and we find that the solution is almost 0.

FIGURE 4.4 – $\alpha = 0.5, \mu = 0.5, \Omega_1 = \Omega_2 =]0, 2[$

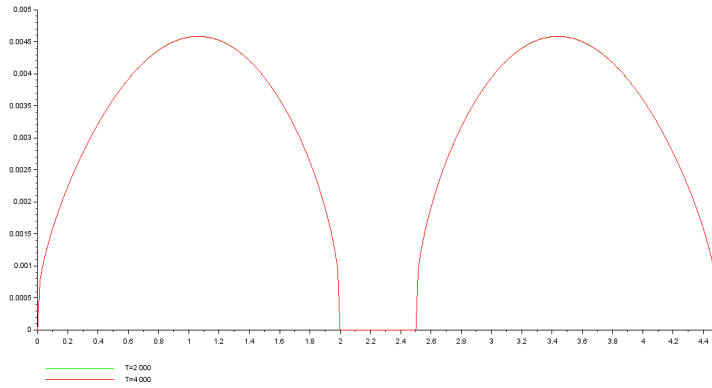
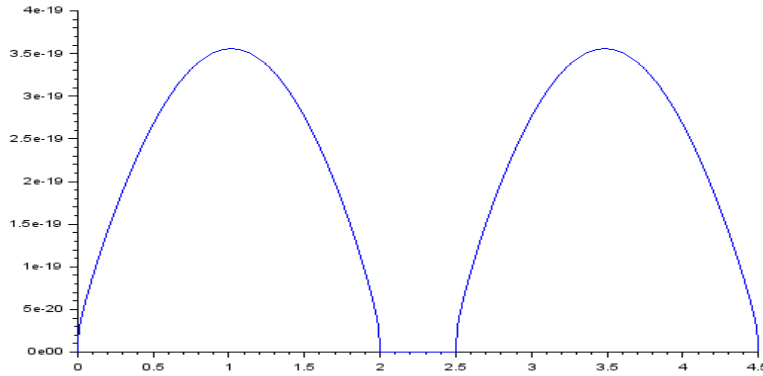


FIGURE 4.5 – $\alpha = 0.8, \mu = 0.5, \Omega_1 = \Omega_2 =]0, 2[$



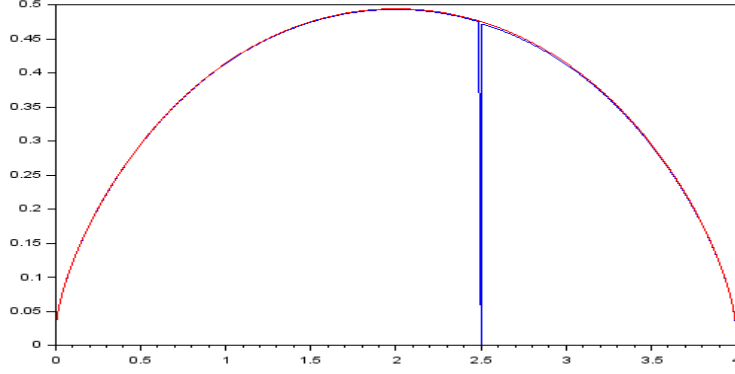
Finally, Figure 4.6 illustrate Theorem 4.1 by the following simulation :

As a conclusion to this work, let us recall that, after establishing conditions on a domain Ω which ensure the existence and the uniqueness of the stationary state of the fractional Fisher-KPP equation, we focus on the principal eigenvalue of $(-\Delta)^\alpha - Id$ in one dimension of domain composed by two bounded connected components. This study is strongly related to the issue of existence and uniqueness of the stationary state of the fractional Fisher-KPP equation.

The perspective are the followings. We would like to relax the hypothesis on the domain Ω . Indeed, rather than a minimal distance between two patches, we would like to assume that

$$\exists \lambda_0 > 0 \text{ such that } \forall R > 0, \quad \lambda_\alpha(\Omega_+ \cap B(0, R)) > \lambda_0.$$

We also expect to prove the continuity result on the principal eigenvalue in the multi-dimensional case. Finally, we would like to have a better understanding of the dynamic of $\lambda_\alpha(] - a - \mu, -\mu[\cup] \mu, a + \mu[)$ when $(\alpha, \mu) \rightarrow (1, 0)$.

FIGURE 4.6 – $\alpha = 0.5$, $\mu = 0.0001$, $\Omega_1 = \Omega_2 =]0, 2[$


Appendix - Proof of Lemma 4.1

We provide here the proof of Lemma 4.1. Before giving the proof, we introduce a new notation :

Notation. For all $\phi \in H_0^\alpha(\Omega_{1,2,\mu})$ and $i \in \{1, 2\}$, we will denote by ϕ^i the function ϕ restricted to the set Ω_i and extended by 0 outside Ω_i

$$i.e. \quad \phi^i(x) = \phi(x)1_{\Omega_i}(x).$$

We also denote by $\underline{\lambda}_i$ the principal eigenvalue of $(-\Delta)^\alpha - Id$ in Ω_i with 0 exterior Dirichlet conditions.

Remark. For all $\phi \in H_0^\alpha(\Omega_{1,2,\mu})$, we have

$$\phi(x) = \phi^1(x) + \phi^2(x) \quad \text{and} \quad \phi^i \in H_0^\alpha(\Omega_i) \quad \text{for all} \quad i \in \{1, 2\}.$$

Proof. The aim of the proof is to prove that for μ large enough, there exists $C > 0$ such that

$$\lambda_\alpha(\Omega_{1,2,\mu}) \geq \min(\lambda_{1,\alpha}, \lambda_{2,\alpha}) - \frac{C}{\mu^{1+2\alpha}}.$$

The conclusion follows by the intermediate value Theorem. We start from the Rayleigh quotient defining $\underline{\lambda}_{i,\alpha}$:

$$\begin{aligned} \underline{\lambda}_{i,\alpha} &= \frac{\int_{\mathbb{R}} \int_{\mathbb{R}} \frac{(\underline{\phi}_{i,\alpha}(x) - \underline{\phi}_{i,\alpha}(y))^2}{|x-y|^{1+2\alpha}} dy - (\underline{\phi}_{i,\alpha}(x))^2 dx}{\int_{\Omega_i} (\underline{\phi}_{i,\alpha}(x))^2 dx} \\ &= \frac{\int_{\Omega_i} \int_{\mathbb{R}} \frac{(\underline{\phi}_{i,\alpha}(x) - \underline{\phi}_{i,\alpha}(y))^2}{|x-y|^{1+2\alpha}} dy - (\underline{\phi}_{i,\alpha}(x))^2 dx}{\int_{\Omega_i} (\underline{\phi}_{i,\alpha}(x))^2 dx} + \frac{\int_{\mathbb{R} \setminus \Omega_i} \int_{\Omega_i} \frac{(\underline{\phi}_{i,\alpha}(y))^2}{|x-y|^{1+2\alpha}} dy dx}{\int_{\Omega_i} (\underline{\phi}_{i,\alpha}(x))^2 dx} \\ &= \frac{\int_{\Omega_i} \int_{\Omega_i} \frac{(\underline{\phi}_{i,\alpha}(x) - \underline{\phi}_{i,\alpha}(y))^2}{|x-y|^{1+2\alpha}} dy - (\underline{\phi}_{i,\alpha}(x))^2 dx}{\int_{\Omega_i} (\underline{\phi}_{i,\alpha}(x))^2 dx} + 2 \frac{\int_{\mathbb{R} \setminus \Omega_i} \int_{\Omega_i} \frac{(\underline{\phi}_{i,\alpha}(y))^2}{|x-y|^{1+2\alpha}} dy dx}{\int_{\Omega_i} (\underline{\phi}_{i,\alpha}(x))^2 dx}. \end{aligned}$$

We continue in the same way by rewriting $\lambda_{1,2,\mu,\alpha}$:

$$\begin{aligned}\lambda_\alpha(\Omega_{1,2,\mu}) &= \frac{\int_{\mathbb{R}} \int_{\mathbb{R}} \frac{(\phi_{1,2,\mu,\alpha}(x) - \phi_{1,2,\mu,\alpha}(y))^2}{|x-y|^{1+2\alpha}} dy - (\phi_{1,2,\mu,\alpha}(x))^2 dx}{\int_{\Omega_{1,2,\mu}} (\phi_{1,2,\mu,\alpha}(x))^2 dx} \\ &= \frac{\int_{\Omega_{1,2,\mu}} \int_{\mathbb{R}} \frac{(\phi_{1,2,\mu,\alpha}(x) - \phi_{1,2,\mu,\alpha}(y))^2}{|x-y|^{1+2\alpha}} dy - (\phi_{1,2,\mu,\alpha}(x))^2 dx}{\int_{\Omega_{1,2,\mu}} (\phi_{1,2,\mu,\alpha}(x))^2 dx} \\ &\quad + \frac{\int_{\mathbb{R} \setminus \Omega_{1,2,\mu}} \int_{\mathbb{R}} \frac{(\phi_{1,2,\mu,\alpha}(x) - \phi_{1,2,\mu,\alpha}(y))^2}{|x-y|^{1+2\alpha}} dy dx}{\int_{\Omega_{1,2,\mu}} (\phi_{1,2,\mu,\alpha}(x))^2 dx}.\end{aligned}$$

Thus, we have found :

$$\lambda_\alpha(\Omega_{1,2,\mu}) = \frac{1}{\int_{\Omega_{1,2,\mu}} (\phi_{1,2,\mu,\alpha}(x))^2 dx} (I_1 + I_2). \quad (4.69)$$

We rewrite I_1 and I_2 in order to involving the expression of $\underline{\lambda}_{1,\alpha}$ and $\underline{\lambda}_{2,\alpha}$. We begin by rewriting I_1 :

$$\begin{aligned}I_1 &= \int_{\Omega_{1,2,\mu}} \int_{\mathbb{R}} \frac{(\phi_{1,2,\mu,\alpha}(x) - \phi_{1,2,\mu,\alpha}(y))^2}{|x-y|^{1+2\alpha}} dy - (\phi_{1,2,\mu,\alpha}(x))^2 dx \\ &= \int_{\Omega_1} \int_{\mathbb{R}} \frac{(\phi_{1,2,\mu,\alpha}(x) - \phi_{1,2,\mu,\alpha}(y))^2}{|x-y|^{1+2\alpha}} dy - (\phi_{1,2,\mu,\alpha}(x))^2 dx \\ &\quad + \int_{\Omega_2} \int_{\mathbb{R}} \frac{(\phi_{1,2,\mu,\alpha}(x) - \phi_{1,2,\mu,\alpha}(y))^2}{|x-y|^{1+2\alpha}} dy - (\phi_{1,2,\mu,\alpha}(x))^2 dx \\ &= \int_{\Omega_1} \int_{\Omega_1} \frac{(\phi_{1,2,\mu,\alpha}(x) - \phi_{1,2,\mu,\alpha}(y))^2}{|x-y|^{1+2\alpha}} dy - (\phi_{1,2,\mu,\alpha}(x))^2 dx + \int_{\Omega_1} \int_{\mathbb{R} \setminus \Omega_{1,2,\mu}} \frac{(\phi_{1,2,\mu,\alpha}(x))^2}{|x-y|^{1+2\alpha}} dy dx \\ &\quad + \int_{\Omega_2} \int_{\Omega_2} \frac{(\phi_{1,2,\mu,\alpha}(x) - \phi_{1,2,\mu,\alpha}(y))^2}{|x-y|^{1+2\alpha}} dy - (\phi_{1,2,\mu,\alpha}(x))^2 dx + \int_{\Omega_2} \int_{\mathbb{R} \setminus \Omega_{1,2,\mu}} \frac{(\phi_{1,2,\mu,\alpha}(x))^2}{|x-y|^{1+2\alpha}} dy dx \\ &\quad + 2 \int_{\Omega_1} \int_{\Omega_2} \frac{(\phi_{1,2,\mu,\alpha}(x) - \phi_{1,2,\mu,\alpha}(y))^2}{|x-y|^{1+2\alpha}} dy dx \\ &= \int_{\Omega_1} \int_{\Omega_1} \frac{(\phi_{1,2,\mu,\alpha}^1(x) - \phi_{1,2,\mu,\alpha}^1(y))^2}{|x-y|^{1+2\alpha}} dy - (\phi_{1,2,\mu,\alpha}^1(x))^2 dx + \int_{\Omega_1} \int_{\mathbb{R} \setminus \Omega_1} \frac{(\phi_{1,2,\mu,\alpha}^1(x))^2}{|x-y|^{1+2\alpha}} dy dx \\ &\quad + \int_{\Omega_2} \int_{\Omega_2} \frac{(\phi_{1,2,\mu,\alpha}^2(x) - \phi_{1,2,\mu,\alpha}^2(y))^2}{|x-y|^{1+2\alpha}} dy - (\phi_{1,2,\mu,\alpha}^2(x))^2 dx + \int_{\Omega_2} \int_{\mathbb{R} \setminus \Omega_2} \frac{(\phi_{1,2,\mu,\alpha}^2(x))^2}{|x-y|^{1+2\alpha}} dy dx \\ &\quad + 2 \int_{\Omega_1} \int_{\Omega_2} \frac{(\phi_{1,2,\mu,\alpha}(x) - \phi_{1,2,\mu,\alpha}(y))^2}{|x-y|^{1+2\alpha}} dy dx \\ &\quad - \int_{\Omega_1} \int_{\Omega_2} \frac{(\phi_{1,2,\mu,\alpha}^1(x))^2}{|x-y|^{1+2\alpha}} dy dx - \int_{\Omega_2} \int_{\Omega_1} \frac{(\phi_{1,2,\mu,\alpha}^2(x))^2}{|x-y|^{1+2\alpha}} dy dx.\end{aligned}$$

Finally, we find that

$$\begin{aligned}
 I_1 &= \int_{\Omega_1} \int_{\Omega_1} \frac{(\phi_{1,2,\mu,\alpha}^1(x) - \phi_{1,2,\mu,\alpha}^1(y))^2}{|x-y|^{1+2\alpha}} dy - (\phi_{1,2,\mu,\alpha}^1(x))^2 dx + \int_{\Omega_1} \int_{\mathbb{R} \setminus \Omega_1} \frac{(\phi_{1,2,\mu,\alpha}^1(x))^2}{|x-y|^{1+2\alpha}} dy dx \\
 &+ \int_{\Omega_2} \int_{\Omega_2} \frac{(\phi_{1,2,\mu,\alpha}^2(x) - \phi_{1,2,\mu,\alpha}^2(y))^2}{|x-y|^{1+2\alpha}} dy - (\phi_{1,2,\mu,\alpha}^2(x))^2 dx + \int_{\Omega_2} \int_{\mathbb{R} \setminus \Omega_2} \frac{(\phi_{1,2,\mu,\alpha}^2(x))^2}{|x-y|^{1+2\alpha}} dy dx \\
 &+ 2 \int_{\Omega_1} \int_{\Omega_2} \frac{(\phi_{1,2,\mu,\alpha}(x) - \phi_{1,2,\mu,\alpha}(y))^2}{|x-y|^{1+2\alpha}} dy dx \\
 &- \int_{\Omega_2} \int_{\Omega_1} \frac{(\phi_{1,2,\mu,\alpha}^2(x))^2}{|x-y|^{1+2\alpha}} dy dx - \int_{\Omega_1} \int_{\Omega_2} \frac{(\phi_{1,2,\mu,\alpha}^1(x))^2}{|x-y|^{1+2\alpha}} dy dx.
 \end{aligned} \tag{4.70}$$

With similar computations, we find that for I_2 :

$$\begin{aligned}
 I_2 &= \int_{\Omega_1} \int_{\mathbb{R} \setminus \Omega_1} \frac{(\phi_{1,2,\mu,\alpha}^1(y))^2}{|x-y|^{1+2\alpha}} dx dy + \int_{\Omega_2} \int_{\mathbb{R} \setminus \Omega_2} \frac{(\phi_{1,2,\mu,\alpha}^2(y))^2}{|x-y|^{1+2\alpha}} dx dy \\
 &- \int_{\Omega_1} \int_{\Omega_2} \frac{(\phi_{1,2,\mu,\alpha}^1(y))^2}{|x-y|^{1+2\alpha}} dx dy - \int_{\Omega_2} \int_{\Omega_1} \frac{(\phi_{1,2,\mu,\alpha}^2(y))^2}{|x-y|^{1+2\alpha}} dx dy.
 \end{aligned} \tag{4.71}$$

Combining (4.70) and (4.71) in (4.69), we deduce

$$\begin{aligned}
 \lambda_\alpha(\Omega_{1,2,\mu}) &= \frac{2 \int_{\Omega_1} \int_{\Omega_2} \frac{(\phi_{1,2,\mu,\alpha}(x) - \phi_{1,2,\mu,\alpha}(y))^2}{|x-y|^{1+2\alpha}} dy dx}{\int_{\Omega_{1,2,\mu}} (\phi_{1,2,\mu,\alpha}(x))^2 dx} \\
 &+ \frac{\int_{\Omega_1} (\phi_{1,2,\mu,\alpha}^1(x))^2 dx \left(\frac{\int_{\Omega_1} \int_{\Omega_1} \frac{(\phi_{1,2,\mu,\alpha}^1(x) - \phi_{1,2,\mu,\alpha}^1(y))^2}{|x-y|^{1+2\alpha}} dy - (\phi_{1,2,\mu,\alpha}^1(x))^2 dx + 2 \int_{\Omega_1} \int_{\mathbb{R} \setminus \Omega_1} \frac{(\phi_{1,2,\mu,\alpha}^1(x))^2}{|x-y|^{1+2\alpha}} dy dx \right)}{\int_{\Omega_1} (\phi_{1,2,\mu,\alpha}^1(x))^2 dx} \\
 &+ \frac{\int_{\Omega_2} (\phi_{1,2,\mu,\alpha}^2(x))^2 dx \left(\frac{\int_{\Omega_2} \int_{\Omega_2} \frac{(\phi_{1,2,\mu,\alpha}^2(x) - \phi_{1,2,\mu,\alpha}^2(y))^2}{|x-y|^{1+2\alpha}} dy - (\phi_{1,2,\mu,\alpha}^2(x))^2 dx + 2 \int_{\Omega_2} \int_{\mathbb{R} \setminus \Omega_2} \frac{(\phi_{1,2,\mu,\alpha}^2(x))^2}{|x-y|^{1+2\alpha}} dy dx \right)}{\int_{\Omega_2} (\phi_{1,2,\mu,\alpha}^2(x))^2 dx} \\
 &+ \frac{\int_{\Omega} (\phi_{1,2,\mu,\alpha}(x))^2 dx}{2 \int_{\Omega_1} \int_{\Omega_2} \frac{(\phi_{1,2,\mu,\alpha}^1(x))^2}{|x-y|^{1+2\alpha}} dy dx + 2 \int_{\Omega_2} \int_{\Omega_1} \frac{(\phi_{1,2,\mu,\alpha}^2(x))^2}{|x-y|^{1+2\alpha}} dy dx} \\
 &\geq \frac{\int_{\Omega_1} (\phi_{1,2,\mu,\alpha}^1(x))^2 dx}{\int_{\Omega_{1,2,\mu}} (\phi_{1,2,\mu,\alpha}(x))^2 dx} \lambda_{1,\alpha} + \frac{\int_{\Omega_2} (\phi_{1,2,\mu,\alpha}^2(x))^2 dx}{\int_{\Omega_{1,2,\mu}} (\phi_{1,2,\mu,\alpha}(x))^2 dx} \lambda_{2,\alpha} \\
 &- \frac{2 \int_{\Omega_1} \int_{\Omega_2} \frac{(\phi_{1,2,\mu,\alpha}^1(x))^2}{|x-y|^{1+2\alpha}} dy dx + 2 \int_{\Omega_2} \int_{\Omega_1} \frac{(\phi_{1,2,\mu,\alpha}^2(x))^2}{|x-y|^{1+2\alpha}} dy dx}{\int_{\Omega_{1,2,\mu}} (\phi_{1,2,\mu,\alpha}(x))^2 dx} \\
 &\geq \min(\lambda_{1,\alpha}, \lambda_{2,\alpha}) - \frac{2 \int_{\Omega_1} \int_{\Omega_2} \frac{(\phi_{1,2,\mu,\alpha}^1(x))^2}{|x-y|^{1+2\alpha}} dy dx + 2 \int_{\Omega_2} \int_{\Omega_1} \frac{(\phi_{1,2,\mu,\alpha}^2(x))^2}{|x-y|^{1+2\alpha}} dy dx}{\int_{\Omega_{1,2,\mu}} (\phi_{1,2,\mu,\alpha}(x))^2 dx}.
 \end{aligned}$$

But, for $i \in \{1, 2\}$, we have for all $x \in \Omega_i$ and $y \in \Omega_{3-i}$

$$2\mu < |x - y| \Rightarrow -\frac{1}{(2\mu)^{1+2\alpha}} \leq -\frac{1}{|x - y|^{1+2\alpha}}.$$

We deduce that

$$\lambda_\alpha(\Omega_{1,2,\mu}) \geq \min(\lambda_{1,\alpha}, \lambda_{2,\alpha}) - \frac{2 \int_{\Omega_1} \int_{\Omega_2} \frac{(\phi_{1,2,\mu,\alpha}^1(x))^2}{|x-y|^{1+2\alpha}} dy dx + 2 \int_{\Omega_2} \int_{\Omega_1} \frac{(\phi_{1,2,\mu,\alpha}^2(x))^2}{|x-y|^{1+2\alpha}} dy dx}{\int_{\Omega_{1,2,\mu}} (\phi_{1,2,\mu,\alpha}(x))^2 dx}$$

$$\begin{aligned} &\geq \min(\underline{\lambda}_{1,\alpha}, \underline{\lambda}_{2,\alpha}) - \frac{2 \int_{\Omega_1} \int_{\Omega_2} \frac{(\phi_{1,2,\mu,\alpha}^1(x))^2}{\mu^{1+2\alpha}} dy dx + 2 \int_{\Omega_2} \int_{\Omega_1} \frac{(\phi_{1,2,\mu,\alpha}^2(x))^2}{\mu^{1+2\alpha}} dy dx}{\int_{\Omega_{1,2,\mu}} (\phi_{1,2,\mu,\alpha}(x))^2 dx} \\ &\geq \min(\underline{\lambda}_{1,\alpha}, \underline{\lambda}_{2,\alpha}) - \frac{4A}{(2\mu)^{1+2\alpha}} \xrightarrow{\mu \rightarrow +\infty} \min(\underline{\lambda}_{1,\alpha}, \underline{\lambda}_{2,\alpha}). \end{aligned}$$

Since $\min(\underline{\lambda}_{1,\alpha}, \underline{\lambda}_{2,\alpha}) > 0$, we deduce the existence of $\mu_1 > 0$ such that for all $\mu > \mu_1$,

$$\lambda_\alpha(\Omega_{1,2,\mu}) > 0.$$

□

Chapter 5

Adaptation to a heterogeneous environment with nonlocal dispersion

Ce chapitre est le résumé d'un travail en cours avec Sepideh Mirrahipi.

5.1 Introduction

In this work, we provide an asymptotic analysis of the equilibria of a non-local parabolic Lotka-Volterra type equation. Such equation arises in the study of adaptive evolution of phenotypically structured populations. The equation under study is the following

$$\begin{cases} -\partial_{xx}n_\varepsilon - \varepsilon^2\partial_{\theta\theta}n_\varepsilon + Ln_\varepsilon = n_\varepsilon(R(x, \theta) - \rho_\varepsilon(x)) & \text{in } \Omega \times]-A, A[, \\ \rho_\varepsilon(x) = \int_{]-A, A[} n_\varepsilon(x, \theta)d\theta & \text{in }]-A, A[, \\ Ln_\varepsilon(x, \theta) = \int_{\Omega} [n_\varepsilon(x, \theta) - n_\varepsilon(y, \theta)]K(x - y)dy & \text{in } \Omega \times]-A, A[, \\ \partial_{\nu_x}n_\varepsilon(x, \theta) = 0 \text{ on } \partial\Omega \times]-A, A[, \quad \partial_{\nu_\theta}n_\varepsilon(x, \pm A) = 0 \end{cases} \quad (\text{E})$$

with Ω a bounded subset of \mathbb{R} . We have denoted by ∂_{ν_x} , ∂_{ν_θ} the exterior derivatives with respect to the variables x and θ . Here, $n_\varepsilon(x, \theta)$ stands for the density of a population at equilibrium at position x with a phenotypical trait θ . The term $R(x, \theta)$ corresponds to the intrinsic growth rate of individuals of phenotype θ at position x . The term $\rho(x)$ corresponds to the total size of the population at position x . Via this term in the right hand side of (E), we take into account a mortality rate due to the competition between the individuals at the same position, independently of their traits. The trait of the parent is transmitted to the offspring. However the trait can be modified due to the mutations that we model by a Laplace term. We also consider that the species is subject to a local and a non-local dispersion in the space variable x . Indeed, in addition to a classical local dispersion term modeled by a Laplace term, we assume that the individuals can jump from position x to position y with a rate $K(x - y)$. From a biological point of view, non-local dispersion may have an important role for instance for the tree species and because of the effect of the wind on the seeds or the pollens [74]. Finally, the Neumann boundary condition with respect to x models the fact that the species cannot leave the domain. The Neumann boundary condition with respect to θ means that the mutants cannot be born with a trait in $] - A, A[^c$.

Several questions motivate our analysis. Is it possible to determine extinction and survival criteria for such model? Is it possible to characterize the population distribution at equilibrium? What will be the impact of the non-local dispersion and a fragmented habitat

on the phenotypical distribution of the population ?

Several works have studied the properties of phenotypically structured populations subject to selection, migration and mutation. We will describe these related works in the following paragraphs. The originality of our work lies in the non-local spatial dispersion operator L . It allows to consider non-connected domains. The non-local dispersion may have antagonistic effects on the population dynamics. On the one hand, it may allow the population to reach new favorable geographic regions which are not accessible by a local diffusion. On the other hand, it may also impede local adaptation by bringing individuals with locally maladapted traits from other regions. While the role of the nonlocal dispersion is significant in many situations, as in the adaptation of forest trees to the climate change, very few theoretical works take it into account [74].

Integro-differential equations modeling the Darwinian evolution of phenotypically structured equations have been widely studied (see for instance [39, 48, 81]). In the context of space heterogeneous environments, a model closely related to (E) but with only a classical diffusion term for dispersion was derived from a stochastic individual based model by Champagnat and Méléard in [41]. A first analysis of such deterministic model was performed by Arnold, Desvillettes and Prévost in [4], where they investigated the existence of non-trivial steady states of the model. Propagation phenomena and existence of traveling front solutions for parabolic equations close to (E) were studied by Alfaro, Coville and Raoul in [2], by Bouin and Mirrahimi in [30] and by Berestycki, Jin and Silvestre in [17]. Alfaro, Berestycki and Raoul also studied a related model taking into account the effect of the climate change. A wide number of articles focus on a closely related equation known as the "cane-toads" model where the growth rate is independent of the trait, but the trait influences the ability of dispersal leading to a θ coefficient in front of the diffusion term in space (see for instance [109, 26, 29, 28]). This equation is motivated by the propagation of the cane toads in Australia by taking into account the role of a phenotypical trait : the size of the legs of the toads. More closely to our work, the steady states of a "cane-toads" type model, in the regime of small mutations, were studied by Perthame and Souganidis in [95] and by Lam and Lou in [75]. In another related project, a model where similarly to (E) the growth rate, and not the dispersion rate, depends on the phenotype, but considering a discrete spatial structure, was studied by Mirrahimi [88, 89]. In these works an asymptotic analysis of the steady states in the regime of small mutations was provided. In particular, it was shown that the presence of spatial heterogeneity can lead to polymorphic situations, that is the emergence of several dominant traits in the population.

In this work, we will use an approach based on Hamilton-Jacobi equations, which is adapted to study the small mutation regime (ε small). A closely related approach was first introduced in [60, 54], by Friedlin using probabilistic techniques and by Evans and Souganidis using deterministic tools, to study the propagation phenomena in reaction-diffusion equations. In the context of models from evolutionary biology and in the regime of small mutations, this method was suggested by Dieckmann, Jabin, Mishler and Perthame [50]. In [9], Barles and Perthame provide the first rigorous results within this approach and obtain a concentration phenomena : as the mutations become small, the solution converges to a Dirac mass. In this case, the population at equilibrium is monomorphic (there is a single dominant trait in the population). We quote [8] which extends the main results of [50]. This approach was then widely extended to study more general models with heterogeneity. In particular, in the context of the space heterogeneous environments, the works [30, 109, 95, 88, 89] are within this framework.

In an ecological context, fragmented environments and non-local spatial dispersion phe-

nomena were studied by Léculier, Mirrahimi and Roquejoffre [78] and by Léculier and Roquejoffre [79]. Both mentioned works do not take into account any phenotypical structure. In [78], the authors study invasion phenomena in a Fisher-KPP equation involving a fractional Laplacian arising in a fragmented periodic environment with Dirichlet exterior conditions. In [79], the authors study the existence and uniqueness of bounded positive steady-states in a Fisher-KPP equation involving a fractional Laplacian in general fragmented environment with Dirichlet exterior conditions. One of the perspectives of the present work is to study models with other operators of dispersion, as the fractional Laplacian $(-\partial_{xx})^\alpha$, instead of $-\partial_{xx} + L$, and considering Dirichlet exterior conditions. The results of [78, 79] would help us to attain this goal. Considering a non-local operator with a regular kernel as L makes a significant difference with the fractional Laplacian. However, we see this work as a first step before considering the problem with a fractional Laplacian and Dirichlet exterior conditions, in term of the technical difficulties.

5.1.1 The assumptions and the notations

The domain $\Omega \subset \mathbb{R}$ is assumed to be bounded and composed of one or several connected components :

$$\Omega = \bigcup_{i=1}^m]a_i, b_i[\quad \text{with} \quad a_1 < b_1 < a_2 < \dots < a_m < b_m. \quad (\text{H1})$$

We assume that the growth rate verifies

$$R \in C^1(\overline{\Omega} \times [-A, A]) \quad \text{and} \quad \|R\|_{W^{1,\infty}(\Omega \times]-A, A])} < C_R. \quad (\text{H2})$$

Example 5.1. *A typical example of growth rate is written*

$$R(x, \theta) = r - g(bx - \theta)^2.$$

In this example, r is the maximal growth rate. The above quadratic term indicates that the optimal trait at position x is given by $\theta_o = bx$. The term b is the gradient of the environment : it indicates how fast the optimal trait varies as a function of a position in space. Moreover, g corresponds to the selection pressure. If g increases, the habitats becomes more hostile for unsuitable individuals. Remark, that depending on the choice of parameters r, b, g , the growth rate R can be negative.

We make the following assumptions on K

$$K \in C^1(\mathbb{R}), \quad K > 0, \quad K(x) = K(-x), \quad 0 < c_K < K < C_K \quad \text{and} \quad |\partial_x K| < C_K. \quad (\text{H3})$$

We introduce here two eigenvalues problems associated to the equation (E) : let $\lambda(\theta, \rho)$ be the principal eigenvalue of the operator $-\partial_{xx} - L - [R(\cdot, \theta) - \rho]Id$ and μ_ε be the principal eigenvalue of the operator $-\partial_{xx} - \varepsilon^2 \partial_{\theta\theta} - L - R$ with Neumann boundary conditions :

$$\text{i.e.} \quad \begin{cases} -\partial_{xx}\psi^\theta + L(\psi^\theta) - [R(\cdot, \theta) - \rho]\psi^\theta = \lambda(\theta, \rho)\psi^\theta & \text{in } \Omega, \\ \partial_{\nu_x}\psi^\theta = 0 & \text{in } \partial\Omega \end{cases} \quad (5.1)$$

and

$$\begin{cases} -\partial_{xx}\xi_\varepsilon - \varepsilon^2 \partial_{\theta\theta}\xi_\varepsilon + L\xi_\varepsilon - R\xi_\varepsilon = \mu_\varepsilon\xi_\varepsilon & \text{in } \Omega \times]-A, A[, \\ \partial_{\nu_x}\xi_\varepsilon = \partial_{\nu_\theta}\xi_\varepsilon = 0 & \text{on } \partial(\Omega \times]-A, A[). \end{cases} \quad (5.2)$$

All along the article, we consider that the principal eigenfunctions (such as ψ or ξ_ε) are taken positive with a L^2 norm equals to 1.

We make the following assumption :

$$\exists \theta_0 \in]-A, A[, \quad \text{such that} \quad \min_{\theta \in]-A, A[} \lambda(\theta, 0) = \lambda(\theta_0, 0) < 0. \quad (\text{H4})$$

Lemma 5.1. *Under the assumptions (H1)–(H4)*

$$\mu_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} \lambda(\theta_0, 0).$$

It follows obviously that

$$\exists \varepsilon_0 > 0, \quad \forall \varepsilon \in]0, \varepsilon_0[, \quad \mu_\varepsilon < \frac{\lambda(\theta_0, 0)}{2} < 0. \quad (5.3)$$

We postpone the proof of Lemma 5.1 to the Appendix and we make the hypothesis that (5.3) holds true.

5.1.2 The results and the strategy

First, we prove the existence of a solution of (E) for all small value ε .

Theorem 5.1. *Under the assumptions (H1)–(H4), for all $\varepsilon \in]0, \varepsilon_0[$, there exists a non-trivial positive bounded solution n_ε of (E).*

The proof of Theorem 5.1 follows the one of Theorem 2.1 by Lam and Lou in [75] which treats the case of local diffusion. This proof relies on a topological degree argument. In section 5.2, we provide the additional arguments. which allows to adapt the proof of [75] to the non-local operator L .

Next, we perform the Hopf-Cole transformation

$$n_\varepsilon(x, \theta) = e^{\frac{u_\varepsilon(x, \theta)}{\varepsilon}}. \quad (5.4)$$

This is the usual first step in the Hamilton-Jacobi approach (see [50, 9, 8]). The main idea in this approach is to first study the limit of u_ε as $\varepsilon \rightarrow 0$, and next obtain from this limit, information on the limit of the phenotypic density n_ε . The advantage of this transformation is that the limit of u_ε is usually a continuous function which solves a Hamilton-Jacobi equation, while the limit of n_ε is a measure. Performing such change of variable, we find that u_ε is solution to

$$\left\{ \begin{array}{l} -\frac{1}{\varepsilon} \partial_{xx} u_\varepsilon - \frac{|\partial_x u_\varepsilon|^2}{\varepsilon^2} - \varepsilon \partial_{\theta\theta} u_\varepsilon - |\partial_\theta u_\varepsilon|^2 \\ \quad + \int_{\Omega} \left[1 - e^{\frac{u_\varepsilon(y) - u_\varepsilon(x)}{\varepsilon}} \right] K(x - y) dy = (R(x, \theta) - \rho_\varepsilon(x)) \quad \text{in } \Omega \times]-A, A[, \\ \rho_\varepsilon(x) = \int_{]-A, A[} n_\varepsilon(x, \theta) d\theta \quad \text{in } \Omega, \\ \partial_{\nu_x} u_\varepsilon(x, \theta) = 0 \text{ on } \partial\Omega \times]-A, A[, \quad \partial_{\nu_\theta} u_\varepsilon(x, \pm A) = 0. \end{array} \right. \quad (E_{HC})$$

We prove the following

Theorem 5.2. *Under the assumptions (H1)–(H4), as $\varepsilon \rightarrow 0$ along subsequences, there holds*

1. ρ_ε converges uniformly to ρ with

$$0 < c \leq \rho \leq C,$$

2. u_ε converges uniformly to u with u a viscosity solution of

$$\begin{cases} -|\partial_\theta u(\theta)|^2 = -\lambda(\theta, \rho), \\ \max u(\theta) = 0, \end{cases} \quad (5.5)$$

where $\lambda(\theta, \rho)$ is the principal eigenvalue introduced in (5.1). Moreover, the limit u depends only on θ .

3. n_ε converges to n in the sense of measures. Moreover, $\text{supp } n \subset \{u(\theta) = 0\}$.

We present briefly heuristic arguments to understand how the second item of Theorem 5.2 can be recovered. First, we perform asymptotic developments of u_ε and ρ_ε with respect to the powers of ε

$$\text{i.e. } u_\varepsilon(x, \theta) = u_0(x, \theta) + \varepsilon u_1(x, \theta) + o(\varepsilon) \quad \text{and} \quad \rho_\varepsilon(x) = \rho_0(x) + o_\varepsilon(1).$$

Next, we implement such asymptotic developments into (E_{HC}) , it follows that

$$\begin{aligned} \frac{1}{\varepsilon} \left(-\partial_{xx} u_0 - 2|\partial_x u_0 \partial_x u_1| - \frac{|\partial_x u_0|^2}{\varepsilon} \right) + \int_\Omega [1 - e^{\frac{u_0(y, \theta) - u_0(x, \theta)}{\varepsilon} + u_1(y, \theta) - u_1(x, \theta) + o_\varepsilon(1)}] K(x - y) dy \\ - \partial_{xx} u_1 - |\partial_x u_1|^2 - |\partial_\theta u_0|^2 - [R - \rho_0] + o_\varepsilon(1) = 0. \end{aligned}$$

This equation suggests that

$$u_0(x, \theta) = u_0(\theta)$$

and passing to the limit $\varepsilon \rightarrow 0$, we deduce that

$$-|\partial_\theta u_0(\theta)|^2 = [R(x, \theta) - \rho_0(x)] + \partial_{xx} u_1(x, \theta) - |\partial_x u_1(x, \theta)|^2 - \int_\Omega [1 - e^{u_1(y, \theta) - u_1(x, \theta)}] K(x - y) dy. \quad (5.6)$$

Since the left hand side of (5.6) does not depend on x , it suggests that u_1 is such that $u_1 = \log(\psi^\theta)$ with ψ^θ the principal eigenfunction introduced in (5.1), that is

$$-|\partial_\theta u_0(\theta)|^2 = \left([R(x, \theta) - \rho_0(x)] \psi^\theta(x) + \partial_{xx} \psi^\theta(x) - L(\psi^\theta) \right) (\psi^\theta)^{-1}. \quad (5.7)$$

We deduce that u_0 is a formal solution of (5.5).

From a technical point of view, the convergence of $(u_\varepsilon)_{\varepsilon > 0}$ is proved using the Arzela-Ascoli Theorem and a perturbed test function argument (see [51]). To prove the uniform continuity, using the Bernstein's method, we prove that the first derivatives are bounded. These bounds rely on the establishment of Harnack type inequalities. Indeed, we prove the following result on the regularity of u_ε

Theorem 5.3. *Under assumptions (H1)–(H4), the following results hold true.*

1. [Harnack inequality] *There exists a constant $C > 0$ (independent of the choice of ε) such that for all intervals $I \subset]-A, A[$ with $|I| = \varepsilon$, there holds*

$$\sup_{(x, \theta) \in \Omega \times I} n_\varepsilon(x, \theta) \leq C \inf_{(x, \theta) \in \Omega \times I} n_\varepsilon(x, \theta). \quad (5.8)$$

2. [Lipschitz bounds] *There exists $C > 0$ such that for all ε small enough,*

$$|\partial_x u_\varepsilon| \leq C\varepsilon \quad \text{and} \quad |\partial_\theta u_\varepsilon| < C. \quad (5.9)$$

3. [Bounds on ρ_ε] *For all ε small enough, ρ_ε is uniformly bounded in $W^{2,p}(\Omega)$ for all $p \in [1, +\infty]$. Moreover, there exists $c, C > 0$ (independent of the choice of ε) such that*

$$c \leq \rho_\varepsilon \leq C. \quad (5.10)$$

4. [Bounds on u_ε] *The following holds true*

$$\lim_{\varepsilon \rightarrow 0} \sup_{(x,\theta) \in \Omega \times]-A, A[} u_\varepsilon \leq 0 \quad \text{and} \quad -a < \lim_{\varepsilon \rightarrow 0} \inf_{(x,\theta) \in \Omega \times]-A, A[} u_\varepsilon, \quad (5.11)$$

with $a > 0$.

Note that the combination of the local and the non-local diffusion terms makes the establishment of such regularity estimates non-standard (see for instance [8] and [88] where such types of estimates were obtained for related models with a local diffusion term).

Remark. *In 1. of Theorem 5.3, the interval I can be at the boundary of $] - A, A[$*

$$\text{i.e. } I =] - A, -A + \varepsilon[\quad \text{or} \quad I =]A - \varepsilon, A[.$$

5.1.3 Outline of the paper

In section 5.2, we provide some preliminary results and the existence of n_ε by proving Theorem 5.1. Next, in section 5.3, we prove the regularity results given by Theorem 5.3. Section 5.4 is devoted to the proof of Theorem 5.2. We provide some numerical simulations and discuss the types of outcomes that may be obtained for the asymptotic populations density.

The constants c and C are positive constants independent of the choice of ε and may change from line to line when there is no confusion possible.

5.2 Preliminary results

First, we establish a Hopf Lemma. It is obtained by a classical argument but for the sake of completeness and because of the presence of the less classical non-local operator L , we provide the proof. Next, we verify the existence of $\lambda(\theta, \rho)$ and then the existence of a non-trivial bounded solution n_ε . We close this section with the proof of the convergence of ρ_ε .

5.2.1 A Hopf Lemma

In this section we prove the following Hopf Lemma

Lemma 5.2 (Hopf Lemma). *Let u be a smooth function defined on Ω such that*

$$-\partial_{xx} n + L(n) + c(x)n \geq 0, \quad (5.12)$$

with c a positive bounded smooth function. If there exists $x_0 \in \partial\Omega$ such that $\min_{x \in \Omega} n(x) = n(x_0) < 0$ then either n is constant or

$$\partial_{\nu_x} n(x) < 0. \quad (5.13)$$

The proof is in the spirit of the classical proof of the Hopf Lemma (see [53] p.250).

Proof. Up to a scaling, there is no loss of generality if we assume that $B(0, 1) \subset \Omega$ and $x_0 \in \partial B(0, 1)$. Next, we define

$$v(x) = \left[e^{-\frac{3}{4}\lambda} - e^{-\lambda \max(0, |x|^2 - \frac{1}{4})} \right] 1_{B(0,1)}(x),$$

for λ a positive constant. We underline that $v(x) = e^{-\frac{3}{4}\lambda} - 1$ in $B(0, \frac{1}{2})$. Next, we claim that by taking λ large enough, for all $x \in B(0, 1) \setminus B(0, \frac{3}{4})$ there holds

$$\begin{aligned} -\partial_{xx}v(x) &= 2\lambda e^{-\lambda \max(0, |x|^2 - \frac{1}{4})} (2\lambda|x|^2 - 1) > 0 \\ \text{and } Lv(x) &= \int_{B(0,1)} e^{-\lambda \max(0, |y|^2 - \frac{1}{4})} K(x-y) dy - \int_{\Omega} e^{-\lambda \max(0, |x|^2 - \frac{1}{4})} K(x-y) dy > 0. \end{aligned} \quad (5.14)$$

The first inequality of (5.14) follows a straightforward computation. For the second inequality, according to the assumption (H3), we have

$$\liminf_{\lambda \rightarrow +\infty} \int_{B(0,1)} e^{-\lambda \max(0, |y|^2 - \frac{1}{4})} K(x-y) dy - \int_{\Omega} e^{-\lambda \max(0, |x|^2 - \frac{1}{4})} K(x-y) dy \geq c_K |B(0, \frac{1}{2})| > 0.$$

Therefore, if λ is large enough, (5.14) holds true.

Next, we claim that if n is not constant, the minimum can not be reached in the interior of Ω . Otherwise, we deduce the existence of $x_1 \in \Omega$ such that $n(x_1) = \min_{x \in \Omega} n < 0$. Since c is positive, we have

$$-\partial_{xx}n(x_1) \leq 0, \quad Ln(x_1) < 0 \quad \text{and} \quad c(x_1)n(x_1) \leq 0.$$

Therefore, we deduce that

$$-\partial_{xx}n(x_1) + Ln(x_1) + c(x_1)n(x_1) < 0.$$

This is in contradiction with the assumption (5.12).

We deduce that $\min_{x \in \partial B(0, \frac{3}{4})} n(x_0) - n(x) < 0$. Next, taking ε small enough, there holds that

$$\forall x \in \partial B(0, \frac{3}{4}), \quad n(x_0) - n(x) - \varepsilon v(x) < 0.$$

Since $v = 0$ on $\partial B(0, 1)$ and by definition of x_0 , it follows

$$\forall x \in \partial B(0, 1), \quad n(x_0) - n(x) - \varepsilon v(x) \leq 0.$$

Moreover, according to (5.12) and (5.14) we have for all $x \in B(0, 1) \setminus B(0, \frac{3}{4})$ that

$$-\partial_{xx}(n(x_0) - n(x) - \varepsilon v(x)) + L(n(x_0) - n(x) - \varepsilon v(x)) + c(x)(n(x_0) - n(x) - \varepsilon v(x)) \leq 0.$$

We deduce thanks to the maximum principle that $n(x_0) - n(x) - \varepsilon v(x) \leq 0$. We conclude that

$$\partial_{\nu_x} n(x_0) \leq -\partial_{\nu_x} \varepsilon v(x_0) = -\varepsilon 2\lambda e^{-\frac{3\lambda}{4}} < 0.$$

□

5.2.2 Existence of a principal eigenpair

Proposition 5.1. *For a fixed bounded smooth function ρ and a fixed value $\theta \in]-A, A[$ there exists a principal eigenvalue $\lambda(\theta, \rho)$ of the operator $-\partial_{xx}\psi + L(\psi) - (R(\cdot, \theta) - \rho)\psi$ with Neumann boundary conditions*

$$i.e. \begin{cases} -\partial_{xx}\psi + L(\psi) - (R(\cdot, \theta) - \rho)\psi = \lambda(\theta, \rho)\psi & \text{in } \Omega, \\ \partial_{\nu_x}\psi = 0 & \text{on } \partial\Omega. \end{cases} \quad (5.15)$$

The associated eigenfunction ψ has a constant sign and is unique up to multiplication by a constant. Moreover, the function $\lambda(\theta, \rho)$ is continuous with respect to θ and the function ρ .

In the following, we will consider that ψ is positive and of L^2 norm equal to 1.

Proof. First, we prove the existence of the principal eigenpair by verifying that we can apply the Krein Rutman Theorem (see [102] p 122). Since it is classical, we do not provide all the details. The cone of functions where we apply the Krein-Rutmann Theorem is

$$K = \overline{\{u \in C^{1+\alpha}(\Omega) \mid u > 0 \text{ and } \partial_{\nu_x}u = 0\}}.$$

We define $\mathcal{L}(v)$ as the unique solution of

$$\begin{cases} -\partial_{xx}\mathcal{L}(v) + \int_{\Omega} [\mathcal{L}(v)(x) - \mathcal{L}(v)(y)]K(x-y)dy - (R(\cdot, \theta) - \rho - C)\mathcal{L}(v) = v & \text{in } \Omega, \\ \partial_{\nu_x}\mathcal{L}(v) = 0 & \text{on } \partial\Omega \end{cases}$$

where $C > \sup_{x \in \Omega} (R(x, \theta) - \rho(x))$ and $v \in K$. The operator \mathcal{L} is linear, compact thanks to the elliptic estimates. We have to prove that

$$\forall v \in K \setminus \{0\}, \quad \mathcal{L}(v) \in \text{int}(K).$$

Let v be in K with v not trivial. By elliptic regularity, it follows $\mathcal{L}(v) \in C^{1+\alpha}$ and $\partial_{\nu_x}\mathcal{L}(v) = 0$. It remains to prove that $\mathcal{L}(v) > 0$.

First we prove that if $\mathcal{L}(v)$ is constant then it is necessarily a positive constant. Next we prove that if $\mathcal{L}(v)$ varies then $\mathcal{L}(v) > 0$.

Assume that $\mathcal{L}(v) = c$. Let $\bar{x} \in \Omega$ be such that $v(\bar{x}) > 0$. Moreover, the choice of C gives $-(R(\bar{x}, \theta) - \rho(\bar{x}) - C) > 0$ and since $-\partial_{xx}\mathcal{L}(v) = L(\mathcal{L}(v)) = 0$, we deduce that

$$\mathcal{L}(v)(\bar{x}) = c = \frac{v(\bar{x})}{-(R(\bar{x}, \theta) - \rho(\bar{x}) - C)} > 0.$$

Next, we suppose that $\mathcal{L}(v)$ is not constant. Assume by contradiction that there exists x such that $\mathcal{L}(v)(x) \leq 0$. Let $\bar{x}' \in \bar{\Omega}$ be such that

$$\inf_{x \in \bar{\Omega}} \mathcal{L}(v)(x) = \mathcal{L}(v)(\bar{x}').$$

Then either $\bar{x}' \in \Omega$ or $\bar{x}' \in \partial\Omega$. In the first case, we deduce that

$$-\partial_{xx}\mathcal{L}(v)(\bar{x}') \leq 0 \quad \text{and} \quad L(\mathcal{L}(v))(\bar{x}') < 0,$$

which leads to the following contradiction

$$0 \leq v(\bar{x}') = -\partial_{xx}\mathcal{L}(v)(\bar{x}') + L(\mathcal{L}(v))(\bar{x}') - (R(\bar{x}', \theta) - \rho(\bar{x}') - C)\mathcal{L}(v)(\bar{x}') < 0.$$

If $\bar{x}' \in \partial\Omega$, since $\mathcal{L}(v)$ is not constant, we deduce from Lemma 5.2 that $\partial_{\nu_x}\mathcal{L}(v)(\bar{x}') < 0$. It is in contradiction with the Neumann boundary condition. We conclude that we can apply the Krein Rutman theorem and the conclusion follows.

Next, we prove the continuity of $\lambda(\theta, \rho)$ with respect to θ . Let ρ be a smooth bounded function. By the uniform continuity of R , for any $\varepsilon > 0$, there exists $\delta > 0$ such that for all $\theta_1, \theta_2 \in]-A, A[$ with $|\theta_1 - \theta_2| < \delta$ there holds

$$\sup_{x \in \Omega} |R(x, \theta_1) - R(x, \theta_2)| < \varepsilon.$$

We prove that for a such couple of θ_1, θ_2 , we have $|\lambda(\theta_1, \rho) - \lambda(\theta_2, \rho)| < \varepsilon$. We denote ψ^{θ_i} the principal eigenfunction associated to the eigenvalue θ_i . Thanks to the Rayleigh quotient, we know that

$$\lambda(\theta_i, \rho) = \int_{\Omega} |\partial_x \psi^{\theta_i}(x)|^2 - [R(x, \theta_i) - \rho(x)] \psi^{\theta_i}(x)^2 dx + \int_{\Omega \times \Omega} \frac{[\psi^{\theta_i}(x) - \psi^{\theta_i}(y)]^2}{2} K(x - y) dy dx.$$

Moreover, if we define

$$\mathcal{R}(\theta, g) = \int_{\Omega} |\partial_x g(x)|^2 dx + \int_{\Omega \times \Omega} \frac{[g(x) - g(y)]^2}{2} K(x - y) dy dx - \int_{\Omega} (R(x, \theta) - \rho(x)) g(x)^2 dx,$$

we have that

$$\lambda(\theta_i, \rho) = \min_{g \in H^1(\Omega)} \mathcal{R}(\theta_i, g) = \mathcal{R}(\theta_i, \psi^{\theta_i}).$$

We deduce that

$$\lambda(\theta_2, \rho) \leq \mathcal{R}(\theta_2, \psi^{\theta_1}) = \lambda(\theta_1, \rho) + \int_{\Omega} [R(x, \theta_1) - R(x, \theta_2)] \psi^{\theta_1}(x)^2 dx \leq \lambda(\theta_1, \rho) + \varepsilon.$$

With similar computations, we obtain that $\lambda(\theta_1, \rho) \leq \lambda(\theta_2, \rho) + \varepsilon$. We conclude that

$$|\lambda(\theta_1, \rho) - \lambda(\theta_2, \rho)| \leq \varepsilon.$$

With similar computations, we conclude to the continuity with respect to ρ . \square

The existence of the solution of (5.2) is also due to the Krein-Rutman Theorem, therefore we do not provide the proof of existence.

5.2.3 Existence of a non-trivial solution of (E)

As mentioned in the introduction, we recall that the proof is an adaptation of the proof of Theorem 2.1 by Lam and Lou in [75]. The major difference is the presence of the integral operator L . Therefore, we only provide the main elements dealing with the integral operator L .

Proof of Theorem 5.1. We fix $\varepsilon \in]0, \varepsilon_0[$ (where ε_0 is given by (5.3)). Let $\tau \in [0, 1]$ and n_{τ} be a solution of

$$\begin{cases} -\partial_{xx} n_{\tau} - \varepsilon^2 \partial_{\theta\theta} n_{\tau} + L n_{\tau} = n_{\tau} (R - \tau \rho_{\tau} - (1 - \tau) n_{\tau}) & \text{in } \Omega \times]-A, A[, \\ \rho_{\tau}(x) = \int_{]-A, A[} n_{\tau}(x, \theta) d\theta & \text{in } \Omega, \\ \partial_{\nu_x} n_{\tau}(x, \theta) = 0 \text{ on } \partial\Omega \times]-A, A[, \quad \partial_{\nu_{\theta}} n_{\tau}(x, \pm A) = 0 & \text{in } \Omega. \end{cases} \quad (E_{\tau})$$

It is well known that for $\tau = 0$, according to (5.3), there exists a non-trivial steady solution n_0 . As in [75], we prove that there exists a constant $C_{\varepsilon} > 1$ (which may depend on ε) such that we have for any $\tau \in [0, 1]$

$$C_{\varepsilon}^{-1} \leq \int_{-A}^A \int_{\Omega} n_{\tau} dx d\theta \leq C_{\varepsilon}.$$

Then one can conclude using a topological degree arguments.

The lower bound. Let v_τ be such that $n_\tau = \xi_\varepsilon v_\tau$ (where ξ_ε is provided by (5.2)). First, we remark that

$$L(v_\tau \xi_\varepsilon) = v_\tau L(\xi_\varepsilon) + \xi_\varepsilon L(v_\tau) + \Lambda(v_\tau, \xi_\varepsilon)$$

and
$$\Lambda(v_\tau, \xi_\varepsilon)(x) = \int_{\Omega} [(v_\tau(x) - v_\tau(y))(\xi_\varepsilon(y) - \xi_\varepsilon(x))] K(x - y) dy.$$

Then v_τ is solution of

$$-\xi_\varepsilon \partial_{xx} v_\tau - 2\partial_x \xi_\varepsilon \partial_x v_\tau - \varepsilon^2 \xi_\varepsilon \partial_{\theta\theta} v_\tau - 2\varepsilon^2 \partial_\theta \xi_\varepsilon \partial_\theta v_\tau + \xi_\varepsilon L(v_\tau) + \Lambda(v_\tau, \xi_\varepsilon) + \mu_\varepsilon \xi_\varepsilon v_\tau = -v_\tau \xi_\varepsilon [\tau \rho_\tau + (1 - \tau) n_\tau].$$

If we multiply it by $\frac{\xi_\varepsilon}{v_\tau}$, we obtain

$$\frac{-\partial_x(\xi_\varepsilon^2 \partial_x v_\tau) - \varepsilon^2 \partial_\theta(\xi_\varepsilon^2 \partial_\theta v_\tau) + \xi_\varepsilon^2 L(v_\tau) + \xi_\varepsilon \Lambda(v_\tau, \xi_\varepsilon)}{v_\tau} = \xi_\varepsilon^2 (-\mu_\varepsilon - \tau \rho_\tau - (1 - \tau) n_\tau).$$

Next, we integrate over all the domain

$$\begin{aligned} & \int_{-A}^A \int_{\Omega} \frac{-\partial_x(\xi_\varepsilon^2 \partial_x v_\tau) - \varepsilon^2 \partial_\theta(\xi_\varepsilon^2 \partial_\theta v_\tau) + \xi_\varepsilon^2 L(v_\tau) + \xi_\varepsilon \Lambda(v_\tau, \xi_\varepsilon)}{v_\tau} dx d\theta \\ &= \int_{-A}^A \int_{\Omega} \frac{-\partial_x(\xi_\varepsilon^2 \partial_x v_\tau) - \varepsilon^2 \partial_\theta(\xi_\varepsilon^2 \partial_\theta v_\tau)}{v_\tau} dx d\theta + \int_{-A}^A \int_{\Omega} \frac{\xi_\varepsilon^2 L(v_\tau) + \xi_\varepsilon \Lambda(v_\tau, \xi_\varepsilon)}{v_\tau} dx d\theta \\ &= I_1 + I_2. \end{aligned}$$

We next prove that I_1 and I_2 are negative. For I_1 , by an integration by part, we have

$$I_1 = \int_{-A}^A \int_{\Omega} \frac{-\partial_x(\xi_\varepsilon^2 \partial_x v_\tau) - \varepsilon^2 \partial_\theta(\xi_\varepsilon^2 \partial_\theta v_\tau)}{v_\tau} dx d\theta = - \int_{-A}^A \int_{\Omega} \frac{\xi_\varepsilon^2}{v_\tau^2} (|\partial_x v_\tau|^2 + \varepsilon^2 |\partial_\theta v_\tau|^2) dx d\theta \leq 0.$$

For I_2 , using that K is even and the Fubini Theorem, we obtain

$$\begin{aligned} I_2 &= \int_{-A}^A \int_{\Omega} \frac{\xi_\varepsilon^2(x) L(v_\tau)(x) + \xi_\varepsilon(x) \Lambda(v_\tau, \xi_\varepsilon)(x)}{v_\tau(x)} dx d\theta \\ &= \int_{-A}^A \int_{\Omega} \frac{\xi_\varepsilon(x)}{v_\tau(x)} \int_{\Omega} [(v_\tau(x) - v_\tau(y)) \xi_\varepsilon(y)] K(x - y) dy dx d\theta \\ &= - \int_{-A}^A \int_{\Omega} \int_{\Omega} \left[\frac{\xi_\varepsilon(x) \xi_\varepsilon(y)}{v_\tau(x) v_\tau(y)} (v_\tau(y) - v_\tau(x))^2 \right] K(x - y) dy dx d\theta - I_2 \end{aligned}$$

We deduce that

$$I_2 = -\frac{1}{2} \int_{-A}^A \int_{\Omega} \int_{\Omega} \left[\frac{\xi_\varepsilon(x) \xi_\varepsilon(y)}{v_\tau(x) v_\tau(y)} (v_\tau(y) - v_\tau(x))^2 \right] K(x - y) dy dx d\theta \leq 0.$$

Therefore, we have that

$$\int_{-A}^A \int_{\Omega} \xi_\varepsilon^2 [-\mu_\varepsilon - \tau \rho_\tau - (1 - \tau) n_\tau] dx d\theta \leq 0.$$

Thanks to (5.3), we conclude that for ε small enough

$$\frac{|\lambda(\theta_0, 0)|}{2} \leq -\mu_\varepsilon = -\mu_\varepsilon \int_{-A}^A \int_{\Omega} \xi_\varepsilon^2 dx d\theta \leq \sup(\xi_\varepsilon^2) [\tau + (1 - \tau)] \int_{-A}^A \int_{\Omega} n_\tau dx d\theta.$$

The upper bound. First, we remark that thanks to the Neumann boundary conditions and the parity of K , we have that

$$\int_{-A}^A \int_{\Omega} -\partial_{xx}n_{\tau} - \partial_{\theta\theta}n_{\tau} + L(n_{\tau})dx d\theta = 0.$$

Therefore, if we integrate (E_{τ}) with respect to x and θ , we obtain

$$\begin{aligned} \left(\frac{(1-\tau)}{2A|\Omega|} + \frac{\tau}{|\Omega|} \right) \|n_{\tau}\|_{L^1}^2 &= \frac{\tau}{|\Omega|} \left(\int_{\Omega} \rho_{\tau} dx \right)^2 + \frac{(1-\tau)}{2A|\Omega|} \left(\int_{-A}^A \int_{\Omega} n_{\tau} dx d\theta \right)^2 \\ &\leq \tau \int_{\Omega} \rho_{\tau}^2 dx + (1-\tau) \int_{-A}^A \int_{\Omega} n_{\tau}^2 dx d\theta \\ &= \int_{-A}^A \int_{\Omega} R n_{\tau} dx d\theta \leq C_R \int_{-A}^A \int_{\Omega} n_{\tau} dx d\theta = C_R \|n_{\tau}\|_{L^1}. \end{aligned}$$

Conclusion. It follows that there exists a bounded non trivial solution n_{ε} of (E). However, we have indeed proved that there exist constants $c, C > 0$ such that

$$\frac{c}{\sup \xi_{\varepsilon}^2} \leq \int_{-A}^A \int_{\Omega} n_{\varepsilon} dx d\theta \leq C.$$

□

5.3 Regularity results

In this section we prove Theorem 5.3. The sub-sections correspond respectively to the proof of the item 1. 2. 3. and 4. of Theorem 5.3. But, we need an intermediate result : ρ_{ε} is uniformly bounded.

Lemma 5.3. *Under the assumptions (H1) – (H4), we have that for all $\varepsilon < \varepsilon_0$*

$$0 \leq \rho_{\varepsilon} \leq C_R,$$

(where C_R is introduced in (H2)). Moreover, there exists $C > 0$ such that for all ε small enough

$$\|\rho_{\varepsilon}\|_{W^{2,p}(\Omega)} \leq C.$$

Proof. The L^{∞} -bounds. It is obvious that $\rho_{\varepsilon} \geq 0$. If we integrate (E) with respect to θ , we obtain

$$\begin{cases} -\partial_{xx}\rho_{\varepsilon} + L\rho_{\varepsilon} = \int_{-A}^A R(\cdot, \theta)n_{\varepsilon}(\cdot, \theta)d\theta - \rho_{\varepsilon}^2 & \text{in } \Omega, \\ \partial_{\nu_x}\rho_{\varepsilon} = 0 & \text{in } \partial\Omega. \end{cases} \quad (E_{\rho})$$

Recalling the L^{∞} bounds on R (H2), it follows :

$$-\partial_{xx}\rho_{\varepsilon} + L\rho_{\varepsilon} \leq C_R\rho_{\varepsilon} - \rho_{\varepsilon}^2.$$

We conclude thanks to the maximum principle that $\rho_{\varepsilon} \leq C_R$.

The $W^{2,p}(\Omega)$ bounds. Thanks to the L^{∞} bounds on R, K, ρ (assumptions (H2), (H3) and the previous inequality), we may write (E_{ρ}) on the following form

$$-\partial_{xx}\rho_{\varepsilon} = f_{\varepsilon}$$

with $f_{\varepsilon} \in L^{\infty}(\Omega)$ uniformly bounded. The result follows from the standard elliptic estimates. □

Corollary 5.1. *There exists a constant $C > 0$ such that for all ε small enough*

$$|\partial_x \rho_{\varepsilon}| \leq C. \quad (5.16)$$

5.3.1 A Harnack inequality

The first step to prove the first item of Theorem 5.3 is to prove the result in the interior of $\Omega \times]-A, A[$.

Theorem 5.4. *For all $(x_0, \theta_0) \in \Omega \times]-A, A[$, and $R_0 > 0$ such that*

$$B_{3R_0}(x_0) \times B_{3\varepsilon R_0}(\theta_0) \subset \Omega \times]-A, A[$$

there exists $C(R_0) > 0$ such that

$$\sup_{(x,\theta) \in B_{R_0}(x_0) \times B_{\varepsilon R_0}(\theta_0)} n_\varepsilon(x, \theta) \leq C(R_0) \inf_{(x,\theta) \in B_{R_0}(x_0) \times B_{\varepsilon R_0}(\theta_0)} n_\varepsilon(x, \theta). \quad (5.17)$$

Next, we extend the above result thanks to a reflective argument (see Remark 9 p.275 in [33]).

We perform the following change of variable : $\tilde{n}(x, \theta) = n_\varepsilon(x, \varepsilon\theta)$. Therefore, we consider the following scaled equation

$$\begin{cases} -\partial_{xx}\tilde{n} - \partial_{\theta\theta}\tilde{n} + L\tilde{n} = \tilde{n}[\tilde{R} - \rho] & \text{in } \Omega \times]-\varepsilon^{-1}A, \varepsilon A[, \\ \partial_{\nu_x}\tilde{n} = \partial_{\nu_\theta}\tilde{n} = 0 & \text{in } \partial(\Omega \times]-\varepsilon^{-1}A, \varepsilon A[. \end{cases} \quad (\text{E}')$$

We have denoted by \tilde{R} the function $\tilde{R}(x, \theta) = R(x, \varepsilon\theta)$. Remark that \tilde{R} still verifies (H2).

Proof of Theorem 5.4. Let $(x_0, \theta_0) \in \Omega \times]-\varepsilon^{-1}A, \varepsilon^{-1}A[$ and a radius $R_0 > 0$ be such that $B_{3R_0}(x_0, \theta_0) \subset \Omega \times]-\varepsilon^{-1}A, \varepsilon^{-1}A[$. If we denote by $f(x, \theta) = \int_\Omega \tilde{n}(y, \theta)K(y-x)dy$, according to (H3) it follows that $f \in L^\infty(B_{2R_0}(x_0, \theta_0))$. From the classical Harnack inequality, the Theorem 9.20 and 9.22 pp. 244-246 in [64], and using (H2) we deduce the existence of $C_1 > 0$ (depending on R_0) such that

$$\begin{aligned} \sup_{(x,\theta) \in B_{R_0}(x_0, \theta_0)} \tilde{n}(x, \theta) &\leq C_1 \inf_{(x,\theta) \in B_{R_0}(x_0, \theta_0)} \tilde{n}(x, \theta) + C_1 \sup_{(x,\theta) \in B_{2R_0}(x_0, \theta_0)} |f(x, \theta)| \\ &\leq C_1 \inf_{(x,\theta) \in B_{R_0}(x_0, \theta_0)} \tilde{n}(x, \theta) + C_1 C_K \sup_{\theta \in B_{2R_0}(\theta_0)} \int_\Omega \tilde{n}(x, \theta) dx. \end{aligned} \quad (5.18)$$

The main element of the proof is to prove the following claim :

$$\exists C > 0 \quad \text{such that} \quad \sup_{\theta \in B_{2R_0}(\theta_0)} \int_\Omega \tilde{n}(x, \theta) dx \leq C \inf_{(x,\theta) \in B_{R_0}(x_0, \theta_0)} \tilde{n}(x, \theta). \quad (5.19)$$

It is clear that if (5.19) holds true, the conclusion follows.

First, we integrate (E') with respect to x . It follows thanks to the Neumann boundary conditions that for all $\theta \in B_{3R_0}(\theta_0)$ we have

$$\begin{aligned} -\partial_{\theta\theta} \int_\Omega \tilde{n}(x, \theta) dx &= \frac{\int_\Omega \tilde{n}(x, \theta) (\tilde{R}(x, \theta) - \rho(x) - \int_\Omega K(x-y)dy) dx}{\int_\Omega \tilde{n}(x, \theta) dx} \int_\Omega \tilde{n}(x, \theta) dx \\ &\quad + \frac{\int_\Omega \int_\Omega \tilde{n}(y, \theta) K(x-y) dy dx}{\int_\Omega \tilde{n}(x, \theta) dx} \int_\Omega \tilde{n}(x, \theta) dx. \end{aligned}$$

Thanks to the L^∞ -bounds on K, \tilde{R}, ρ (assumptions (H2), (H3) and Lemma 5.3) and the Fubini theorem, we have

$$-C \leq \frac{\int_\Omega \tilde{n}(x, \theta) (\tilde{R}(x, \theta) - \rho(x) - \int_\Omega K(x-y)dy) dx}{\int_\Omega \tilde{n}(x, \theta) dx} < C$$

$$\text{and } \frac{\int_{\Omega} \int_{\Omega} \tilde{n}(y, \theta) K(x-y) dy dx}{\int_{\Omega} \tilde{n}(x, \theta) dx} \leq C_K |\Omega|.$$

It follows

$$-C \int_{\Omega} \tilde{n}(x, \theta) dx \leq -\partial_{\theta\theta} \int_{\Omega} \tilde{n}(x, \theta) dx \leq C \int_{\Omega} \tilde{n}(x, \theta) dx.$$

Hence, we apply the Harnack inequality to $\theta \in B_{3R_0}(\theta_0) \mapsto \int_{\Omega} \tilde{n}(x, \theta) dx$ into the ball $B_{2R_0}(\theta_0)$ and we deduce the existence of a constant $C_2 > 0$ such that

$$\sup_{\theta \in B_{2R_0}(\theta_0)} \int_{\Omega} \tilde{n}(x, \theta) dx \leq C_2 \inf_{\theta \in B_{2R_0}(\theta_0)} \int_{\Omega} \tilde{n}(x, \theta) dx. \quad (5.20)$$

Next, thanks to the L^∞ -bounds on K, \tilde{R}, ρ (assumptions (H2), (H3) and Lemma 5.3), it follows that in $\Omega \times B_{2R_0}(\theta_0)$

$$c_K \inf_{\theta \in B_{2R_0}(\theta_0)} \int_{\Omega} \tilde{n}(y, \theta) dy \leq c_K \int_{\Omega} \tilde{n}(y, \theta) dy \leq \int_{\Omega} \tilde{n}(y, \theta) K(x-y) dy \leq (-\partial_{xx} - \partial_{\theta\theta}) \tilde{n} + C\tilde{n}.$$

From an inequality developed by Krylov (we refer to Theorem 7.1 p33 in [34] and the reference therein), we deduce the existence of a constant $C_3 > 0$ such that

$$\inf_{B_{2R_0}(\theta_0)} \int_{\Omega} \tilde{n}(x, \theta) dx \leq C_3 \inf_{B_{R_0}(x_0, \theta_0)} \tilde{n}(x, \theta). \quad (5.21)$$

Combining the previous inequality with (5.20) and (5.21) yields to

$$\sup_{\theta \in B_{2R_0}(\theta_0)} \int_{\Omega} \tilde{n}(x, \theta) dx \leq C_2 \inf_{\theta \in B_{2R_0}(\theta_0)} \int_{\Omega} \tilde{n}(x, \theta) dx \leq C_2 C_3 \inf_{(x, \theta) \in B_{R_0}(x_0, \theta_0)} \tilde{n}(x, \theta).$$

This concludes the proof. \square

5.3.2 Lipschitz estimates

We prove 2. of Theorem 5.3 by the Bernstein method.

Proof of 2. of Theorem 5.3. We recall the main equation satisfied by u_ε :

$$\frac{-\partial_{xx} u_\varepsilon}{\varepsilon} - \frac{|\partial_x u_\varepsilon|^2}{\varepsilon^2} - \varepsilon \partial_{\theta\theta} u_\varepsilon - |\partial_\theta u_\varepsilon|^2 + \int_{\Omega} [1 - e^{\frac{u_\varepsilon(y, \theta) - u_\varepsilon(x, \theta)}{\varepsilon}}] K(x-y) dy = R(x, \theta) - \rho_\varepsilon \quad (5.22)$$

with Neumann boundary conditions. The first step is to differentiate (5.22) with respect to x and multiply it by $\frac{\partial_x u_\varepsilon}{\varepsilon^2}$:

$$\begin{aligned} & - \frac{\partial_{xxx} u_\varepsilon \partial_x u_\varepsilon}{\varepsilon^3} - \frac{\partial_x \left(\frac{|\partial_x u_\varepsilon|^2}{\varepsilon^2} \right) \partial_x u_\varepsilon}{\varepsilon^2} + \int_{\Omega} e^{\frac{u_\varepsilon(y, \theta) - u_\varepsilon(x, \theta)}{\varepsilon}} K(x-y) dy \frac{\partial_x u_\varepsilon^2}{\varepsilon^3} \\ & - \frac{\partial_x |\partial_\theta u_\varepsilon|^2 \partial_x u_\varepsilon}{\varepsilon^2} - \frac{\partial_{x\theta\theta} u_\varepsilon \partial_x u_\varepsilon}{\varepsilon} = \frac{\left(\int_{\Omega} [e^{\frac{u_\varepsilon(y, \theta) - u_\varepsilon(x, \theta)}{\varepsilon}} - 1] \partial_x K(x-y) dy + \partial_x R - \partial_x \rho_\varepsilon \right) \partial_x u_\varepsilon}{\varepsilon^2}. \end{aligned}$$

Remarking that

$$\partial_{xxx} u_\varepsilon \partial_x u_\varepsilon = \frac{\partial_{xx} (|\partial_x u_\varepsilon|^2)}{2} - (\partial_{xx} u_\varepsilon)^2 \quad \text{and} \quad \partial_{x\theta\theta} u_\varepsilon \partial_x u_\varepsilon = \frac{\partial_{\theta\theta} (|\partial_x u_\varepsilon|^2)}{2} - (\partial_{\theta x} u_\varepsilon)^2$$

yields to

$$\begin{aligned}
 & -\frac{\partial_{xx}\left(\frac{|\partial_x u_\varepsilon|^2}{\varepsilon^2}\right)}{2\varepsilon} + \frac{(\partial_{xx}u_\varepsilon)^2}{\varepsilon^3} - \frac{\partial_x\left(\frac{|\partial_x u_\varepsilon|^2}{\varepsilon^2}\right)\partial_x u_\varepsilon}{\varepsilon^2} \\
 & + \int_{\Omega} e^{\frac{u_\varepsilon(y,\theta)-u_\varepsilon(x,\theta)}{\varepsilon}} K(x-y)dy \frac{(\partial_x u_\varepsilon)^2}{\varepsilon^3} - \frac{\partial_x|\partial_\theta u_\varepsilon|^2\partial_x u_\varepsilon}{\varepsilon^2} \\
 & + \frac{(\partial_{\theta x}u_\varepsilon)^2}{\varepsilon} - \frac{\varepsilon\partial_{\theta\theta}\left(\frac{|\partial_x u_\varepsilon|^2}{\varepsilon^2}\right)}{2} = \frac{\left(\int_{\Omega}[e^{\frac{u_\varepsilon(y,\theta)-u_\varepsilon(x,\theta)}{\varepsilon}} - 1]\partial_x K(x-y)dy + \partial_x R - \partial_x \rho_\varepsilon\right)\partial_x u_\varepsilon}{\varepsilon^2}.
 \end{aligned} \tag{5.23}$$

In the second step, we differentiate (5.22) with respect to θ and multiply by $\partial_\theta u_\varepsilon$. With computations similar to the ones presented above, we find

$$\begin{aligned}
 & -\frac{\partial_{xx}(|\partial_\theta u_\varepsilon|^2)}{2\varepsilon} + \frac{(\partial_{\theta x}u_\varepsilon)^2}{\varepsilon} - \partial_\theta \frac{|\partial_x u_\varepsilon|^2}{\varepsilon^2} \partial_\theta u_\varepsilon - \frac{\varepsilon}{2} \partial_{\theta\theta}(|\partial_\theta u_\varepsilon|^2) + \varepsilon(\partial_{\theta\theta}u_\varepsilon)^2 - \partial_\theta|\partial_\theta u_\varepsilon|^2\partial_\theta u_\varepsilon \\
 & + \int_{\Omega} \frac{(\partial_\theta u_\varepsilon(x,\theta))^2 - \partial_\theta u_\varepsilon(x,\theta)\partial_\theta u_\varepsilon(y,\theta)}{\varepsilon} e^{\frac{u_\varepsilon(y,\theta)-u_\varepsilon(x,\theta)}{\varepsilon}} K(x-y)dy = \partial_\theta R \partial_\theta u_\varepsilon.
 \end{aligned} \tag{5.24}$$

Next, we introduce

$$p_\varepsilon(x, \theta) = \frac{|\partial_x u_\varepsilon(x, \theta)|^2}{\varepsilon^2} + |\partial_\theta u_\varepsilon(x, \theta)|^2. \tag{5.25}$$

If we combine (5.23) and (5.24) together and we rewrite it involving p_ε , it follows

$$\begin{aligned}
 & -\frac{\partial_{xx}p_\varepsilon}{2\varepsilon} - \frac{\varepsilon\partial_{\theta\theta}p_\varepsilon}{2} + \frac{1}{\varepsilon} \int_{\Omega} [p_\varepsilon(x, \theta) - \partial_\theta u_\varepsilon(x, \theta)\partial_\theta u_\varepsilon(y, \theta)] e^{\frac{u_\varepsilon(y,\theta)-u_\varepsilon(x,\theta)}{\varepsilon}} K(x-y)dy \\
 & - \frac{\partial_x p_\varepsilon \partial_x u_\varepsilon}{\varepsilon^2} - \partial_\theta p_\varepsilon \partial_\theta u_\varepsilon + \frac{2(\partial_{x\theta}u_\varepsilon)^2}{\varepsilon} + \frac{(\partial_{xx}u_\varepsilon)^2}{\varepsilon^3} + \varepsilon(\partial_{\theta\theta}u_\varepsilon)^2 \\
 & = \left(\int_{\Omega}[e^{\frac{u_\varepsilon(y,\theta)-u_\varepsilon(x,\theta)}{\varepsilon}} - 1]\partial_x K(x-y)dy + \partial_x R - \partial_x \rho_\varepsilon\right) \frac{\partial_x u_\varepsilon}{\varepsilon^2} + \partial_\theta R \partial_\theta u_\varepsilon.
 \end{aligned} \tag{5.26}$$

Let $(x_\varepsilon, \theta_\varepsilon)$ be such that

$$\sup_{(x,\theta) \in \Omega \times]-A, A[} p_\varepsilon(x, \theta) = p_\varepsilon(x_\varepsilon, \theta_\varepsilon).$$

Thanks to the Neumann boundaries conditions, we deduce that $(x_\varepsilon, \theta_\varepsilon) \notin \partial\Omega \times \partial(]-A, A[)$. Therefore, we distinguish three cases : either $(x_\varepsilon, \theta_\varepsilon) \in \Omega \times]-A, A[$ or $(x_\varepsilon, \theta_\varepsilon) \in \partial\Omega \times]-A, A[$ or $(x_\varepsilon, \theta_\varepsilon) \in \Omega \times \{\pm A\}$.

Case 1 : $(x_\varepsilon, \theta_\varepsilon) \in \Omega \times]-A, A[$. First, we bound the right-hand-side of (5.26). Indeed, thanks to the Harnack inequality (first item of Theorem 5.3) and the L^∞ -bounds on the derivative of K , R and ρ_ε (assumptions (H2), (H3) and Corollary 5.1), it follows that

$$\left(\int_{\Omega}[e^{\frac{u_\varepsilon(y,\theta)-u_\varepsilon(x,\theta)}{\varepsilon}} - 1]\partial_x K(x-y)dy + \partial_x R - \partial_x \rho_\varepsilon\right) \frac{\partial_x u_\varepsilon}{\varepsilon^2} + \partial_\theta R \partial_\theta u_\varepsilon \leq \frac{C\sqrt{p_\varepsilon}}{\varepsilon}. \tag{5.27}$$

Next, we evaluate (5.26) at $(x_\varepsilon, \theta_\varepsilon)$. We claim that

$$\begin{aligned}
 & -\partial_{xx}p_\varepsilon(x_\varepsilon, \theta_\varepsilon) \geq 0, \quad -\partial_{\theta\theta}p_\varepsilon(x_\varepsilon, \theta_\varepsilon) \geq 0, \quad \partial_x p_\varepsilon(x_\varepsilon, \theta_\varepsilon) = \partial_\theta p_\varepsilon(x_\varepsilon, \theta_\varepsilon) = 0 \\
 \text{and} \quad & \frac{1}{\varepsilon} \int_{\Omega} [p_\varepsilon(x_\varepsilon, \theta_\varepsilon) - \partial_\theta u_\varepsilon(x_\varepsilon, \theta_\varepsilon)\partial_\theta u_\varepsilon(y, \theta_\varepsilon)] e^{\frac{u_\varepsilon(y,\theta_\varepsilon)-u_\varepsilon(x_\varepsilon,\theta_\varepsilon)}{\varepsilon}} K(x_\varepsilon - y)dy \geq 0.
 \end{aligned} \tag{5.28}$$

Indeed, the first inequalities follow easily since $p(x_\varepsilon, \theta_\varepsilon) = \max p_\varepsilon$ and the last inequality holds true thanks to the following computations

$$\frac{1}{\varepsilon} \int_{\Omega} [p_\varepsilon(x_\varepsilon, \theta_\varepsilon) - \partial_\theta u_\varepsilon(x_\varepsilon, \theta_\varepsilon)\partial_\theta u_\varepsilon(y, \theta_\varepsilon)] e^{\frac{u_\varepsilon(y,\theta_\varepsilon)-u_\varepsilon(x_\varepsilon,\theta_\varepsilon)}{\varepsilon}} K(x_\varepsilon - y)dy$$

$$\begin{aligned}
 &\geq \frac{1}{\varepsilon} \left[\int_{\Omega} p_{\varepsilon}(x_{\varepsilon}, \theta_{\varepsilon}) e^{\frac{u_{\varepsilon}(y, \theta_{\varepsilon}) - u_{\varepsilon}(x_{\varepsilon}, \theta_{\varepsilon})}{\varepsilon}} K(x_{\varepsilon} - y) dy \right. \\
 &\quad - \frac{1}{2} \int_{\Omega} \partial_{\theta} u_{\varepsilon}^2(x_{\varepsilon}, \theta_{\varepsilon}) e^{\frac{u_{\varepsilon}(y, \theta_{\varepsilon}) - u_{\varepsilon}(x_{\varepsilon}, \theta_{\varepsilon})}{\varepsilon}} K(x_{\varepsilon} - y) dy \\
 &\quad \left. - \frac{1}{2} \int_{\Omega} \partial_{\theta} u_{\varepsilon}^2(y, \theta_{\varepsilon}) e^{\frac{u_{\varepsilon}(y, \theta_{\varepsilon}) - u_{\varepsilon}(x_{\varepsilon}, \theta_{\varepsilon})}{\varepsilon}} K(x_{\varepsilon} - y) dy \right] \\
 &\geq \frac{1}{2\varepsilon} \left[\int_{\Omega} p_{\varepsilon}(x_{\varepsilon}, \theta_{\varepsilon}) e^{\frac{u_{\varepsilon}(y, \theta_{\varepsilon}) - u_{\varepsilon}(x_{\varepsilon}, \theta_{\varepsilon})}{\varepsilon}} K(x_{\varepsilon} - y) dy - \int_{\Omega} p_{\varepsilon}(y, \theta_{\varepsilon}) e^{\frac{u_{\varepsilon}(y, \theta_{\varepsilon}) - u_{\varepsilon}(x_{\varepsilon}, \theta_{\varepsilon})}{\varepsilon}} K(x_{\varepsilon} - y) dy \right] \\
 &\geq 0.
 \end{aligned}$$

We deduce thanks to (5.27) and (5.28) that

$$\begin{aligned}
 \frac{1}{2\varepsilon} \left[\frac{\partial_{xx} u_{\varepsilon}(x_{\varepsilon}, \theta_{\varepsilon})}{\varepsilon} + \varepsilon \partial_{\theta\theta} u_{\varepsilon}(x_{\varepsilon}, \theta_{\varepsilon}) \right]^2 &\leq \frac{1}{\varepsilon} \left[\left(\frac{\partial_{xx} u_{\varepsilon}(x_{\varepsilon}, \theta_{\varepsilon})}{\varepsilon} \right)^2 + (\varepsilon \partial_{\theta\theta} u_{\varepsilon}(x_{\varepsilon}, \theta_{\varepsilon}))^2 \right] \\
 &\leq \frac{C \sqrt{p_{\varepsilon}(x_{\varepsilon}, \theta_{\varepsilon})}}{\varepsilon}.
 \end{aligned}$$

Hence, using the original equation (5.22), we deduce that

$$\left[-p_{\varepsilon}(x_{\varepsilon}, \theta_{\varepsilon}) + \int_{\Omega} (1 - e^{\frac{u_{\varepsilon}(y, \theta_{\varepsilon}) - u_{\varepsilon}(x_{\varepsilon}, \theta_{\varepsilon})}{\varepsilon}}) K(x_{\varepsilon} - y) dy - R(x_{\varepsilon}, \theta_{\varepsilon}) + \rho_{\varepsilon}(x_{\varepsilon}) \right]^2 \leq C \sqrt{p_{\varepsilon}(x_{\varepsilon}, \theta_{\varepsilon})}. \quad (5.29)$$

Thanks to the L^{∞} -bounds on K , R and ρ_{ε} (assumption (H2) (H3) and Lemma 5.3), it follows that $p_{\varepsilon}(x_{\varepsilon}, \theta_{\varepsilon})$ is uniformly bounded with respect to ε . The conclusion follows.

Case 2 : $(x_{\varepsilon}, \theta_{\varepsilon}) \in \partial\Omega \times]-A, A[$. First remark that in this case, $p_{\varepsilon}(x_{\varepsilon}, \theta_{\varepsilon}) = |\partial_{\theta} u_{\varepsilon}(x_{\varepsilon}, \theta_{\varepsilon})|^2$. We claim that p_{ε} verifies also the Neumann boundary conditions at $(x_{\varepsilon}, \theta_{\varepsilon})$. Indeed, according to the Neumann boundary conditions satisfied by u_{ε} , we can use a reflective argument and differentiate p_{ε} on the boundary. We obtain

$$\partial_x p_{\varepsilon}(x_{\varepsilon}, \theta_{\varepsilon}) = \frac{2\partial_x u_{\varepsilon}(x_{\varepsilon}, \theta_{\varepsilon}) \partial_{xx} u_{\varepsilon}(x_{\varepsilon}, \theta_{\varepsilon})}{\varepsilon^2} + 2\partial_{\theta} u_{\varepsilon}(x_{\varepsilon}, \theta_{\varepsilon}) \partial_{x\theta} u_{\varepsilon}(x_{\varepsilon}, \theta_{\varepsilon}) = 0$$

because

$$\partial_x u_{\varepsilon}(x_{\varepsilon}, \theta_{\varepsilon}) = 0 \quad \text{and} \quad \partial_{x\theta} u_{\varepsilon}(x_{\varepsilon}, \theta_{\varepsilon}) = 0.$$

Since $p(x_{\varepsilon}, \theta_{\varepsilon}) = \max p_{\varepsilon}$, we deduce that

$$-\partial_{xx} p_{\varepsilon}(x_{\varepsilon}, \theta_{\varepsilon}) \geq 0.$$

We conclude that (5.28) and (5.27) hold also true in this case and the conclusion follows from the same computations as in the previous case.

Case 3 : $(x_{\varepsilon}, \theta_{\varepsilon}) \in \Omega \times \{\pm A\}$. This case is treated in the same manner as the previous case. \square

5.3.3 The bounds on ρ_{ε}

We recall the equation (E_{ρ}) satisfied by ρ_{ε} :

$$\begin{cases} -\partial_{xx} \rho_{\varepsilon} + L\rho_{\varepsilon} = \int_{-A}^A R(x, \theta) n_{\varepsilon}(x, \theta) d\theta - \rho_{\varepsilon}^2 & \text{in } \Omega, \\ \partial_{\nu_x} \rho_{\varepsilon} = 0 & \text{on } \partial\Omega. \end{cases} \quad (E_{\rho})$$

Proof of 4. of Theorem 5.3. We start by proving that $0 < c \leq \sup \rho_\varepsilon \leq C$. Next, we prove that $c < \rho_\varepsilon < C$ holds true in the whole domain Ω .

The bounds on $\sup \rho_\varepsilon$. The upper bound is already obtained in Lemma 5.3. It remains to prove the lower bound. Assume by contradiction that there exists a sequence ε_k such that

$$\varepsilon_k \xrightarrow[k \rightarrow +\infty]{} 0 \quad \text{and} \quad \sup \rho_{\varepsilon_k} \xrightarrow[k \rightarrow +\infty]{} 0.$$

Next, if we multiply (E) by ξ_{ε_k} (introduced in (5.2)) and we integrate by part, we obtain

$$\mu_\varepsilon \int_{-A}^A \int_{\Omega} n_{\varepsilon_k} \xi_{\varepsilon_k} dx d\theta = - \int_{-A}^A \int_{\Omega} \rho_{\varepsilon_k} n_{\varepsilon_k} \xi_{\varepsilon_k} dx d\theta.$$

We deduce thanks to (5.3) that for k large enough, it holds

$$\frac{|\lambda(\theta_0, 0)|}{2} \leq -\mu_{\varepsilon_k} \leq \sup \rho_{\varepsilon_k} \frac{\int_{-A}^A \int_{\Omega} n_{\varepsilon_k} \xi_{\varepsilon_k} dx d\theta}{\int_{-A}^A \int_{\Omega} n_{\varepsilon_k} \xi_{\varepsilon_k} dx d\theta}.$$

It is in contradiction with the hypothesis $\sup \rho_{\varepsilon_k} \xrightarrow[k \rightarrow +\infty]{} 0$. Therefore, there exists a constant $c > 0$ such that

$$\forall \varepsilon \in]0, \varepsilon_0[, \quad c \leq \sup \rho_\varepsilon. \quad (5.30)$$

The bounds on ρ_ε in the whole domain Ω . The upper bound is already proved in Lemma 5.3. Therefore, we concentrate on the lower bound. Let $\varepsilon < \varepsilon_0$ and $x_0 \in \Omega$ be such that

$$\rho_\varepsilon(x_0) = \sup \rho_\varepsilon.$$

We conclude thanks to (5.30) and the Lipschitz estimates obtained in the second item of Theorem 5.3 that for all $x \in \Omega$

$$\rho_\varepsilon(x) = \int_{-A}^A e^{\frac{u_\varepsilon(x, \theta)}{\varepsilon}} d\theta = \int_{-A}^A e^{\frac{u_\varepsilon(x, \theta) - u_\varepsilon(x_0, \theta) + u_\varepsilon(x_0, \theta)}{\varepsilon}} d\theta \geq \rho_\varepsilon(x_0) e^{-C} \geq c e^{-C}.$$

□

5.3.4 The bounds on u_ε

Proof of 3. of Theorem 5.3. First, we prove that there exists $a > 0$ such that $-a < u_\varepsilon$. Thanks to the third item of Theorem 5.3, we know that there exists $c > 0$ such that for all ε small enough we have

$$c < \int_{-A}^A n_\varepsilon(x, \theta) d\theta.$$

We deduce the existence of $(x_0, \theta_0) \in \Omega \times]-A, A[$ such that

$$\frac{c}{2A} \leq n_\varepsilon(x_0, \theta_0).$$

Hence, it follows

$$\varepsilon \log \left(\frac{c}{2A} \right) \leq u_\varepsilon(x_0, \theta_0).$$

We conclude thanks to the Lipschitz estimates established in the second item of Theorem 5.3 that

$$\forall (x, \theta) \in \Omega \times]-A, A[, \quad -a \leq -2CA + \varepsilon \left[\log \left(\frac{c}{2A} \right) - C|\Omega| \right] \leq u_\varepsilon(x, \theta). \quad (5.31)$$

Next, we prove that $\lim_{\varepsilon \rightarrow 0} \sup_{(x, \theta) \in \Omega \times]-A, A[} u_\varepsilon(x, \theta) \leq 0$.

We prove it by contradiction. Assume that there exists $a > 0$ and sequences $\varepsilon_k, (x_k, \theta_k)$ such that

$$\varepsilon_k \xrightarrow[k \rightarrow +\infty]{} 0 \quad \text{and} \quad a < u_{\varepsilon_k}(x_k, \theta_k).$$

Using the Lipschitz estimates provided by the second item of Theorem 5.3, it follows for all $\theta \in (B_{\frac{a}{4C}}(\theta_k) \cap]-A, A[)$

$$u_{\varepsilon_k}(x_k, \theta) = u_{\varepsilon_k}(x_k, \theta) - u_{\varepsilon_k}(x_k, \theta_k) + u_{\varepsilon_k}(x_k, \theta_k) \geq -C|\theta - \theta_k| + a \geq \frac{a}{2}$$

where C corresponds to the Lipschitz estimate given by (5.9). We deduce that

$$\rho_{\varepsilon_k}(x_k) \geq \min(2A, \frac{a}{4C})e^{\frac{a}{2\varepsilon_k}}.$$

We conclude that $\liminf_{k \rightarrow +\infty} \rho_{\varepsilon_k}(x_k) = +\infty$. This is in contradiction with the L^∞ bounds on ρ_ε established in the third item of Theorem 5.3. □

5.4 Convergence to the Hamilton Jacobi equation

Proof of Theorem 5.2. We prove here the 3 items of Theorem 5.2.

Proof of 1. Thanks to the third item of Theorem 5.3, it follows that for ε small enough $0 < c \leq \rho_\varepsilon \leq C$ and $\|\rho_\varepsilon\|_{W^{2,p}(\Omega)} \leq C$. We deduce from the classical Sobolev injection (see [33]) that ρ_ε converges, along subsequences, strongly in $W^{1,p}(\Omega)$ and in particular uniformly to ρ and ρ verifies

$$0 < c \leq \rho \leq C.$$

Proof of 2. From the Lipschitz estimates and the bounds established in the second and the fourth items of Theorem 5.3, we deduce thanks to the Arzela-Ascoli Theorem that up to a subsequence, $(u_\varepsilon)_{\varepsilon > 0}$ converges locally uniformly to some continuous function u . Moreover, the limit function u does not depend on x .

We prove that u is a viscosity solution of

$$-|\partial_\theta u|^2 = -\lambda(\theta, \rho)$$

with $\lambda(\theta, \rho)$ the principal eigenvalue of (5.1). We recall that for a fixed value θ , Proposition 5.1 provides the existence of a sequence of principal eigenvalues $\lambda(\theta, \rho_\varepsilon)$ associated with a sequence of positive eigenfunctions $(\psi_\varepsilon^\theta)_{\varepsilon > 0}$ of the operator $-\partial_{xx} + L - (R(x, \theta) - \rho_\varepsilon)$ with Neumann boundary conditions :

$$\text{i.e. } \begin{cases} -\partial_{xx}\psi_\varepsilon^\theta + L(\psi_\varepsilon^\theta) - (R(x, \theta) - \rho_\varepsilon)\psi_\varepsilon^\theta = \lambda(\theta, \rho_\varepsilon)\psi_\varepsilon^\theta & \text{in } \Omega, \\ \partial_{\nu_x}\psi_\varepsilon^\theta = 0 & \text{on } \partial\Omega. \end{cases} \quad (5.32)$$

Since $\psi_\varepsilon^\theta > 0$, we introduce

$$\Psi_\varepsilon^\theta = \ln(\psi_\varepsilon^\theta).$$

Therefore, for any test function ϕ , let $\theta \in]-A, A[$ be such that $u - \phi$ has a strict maximum at θ . Then, there exists $(x_\varepsilon, \theta_\varepsilon) \in \overline{\Omega} \times]-A, A[$ such that

$$\theta_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} \theta \quad \text{and} \quad \max_{(x, \theta) \in \overline{\Omega} \times]-A, A[} u_\varepsilon(x, \theta) - \phi(\theta) - \varepsilon \Psi_\varepsilon^{\theta_\varepsilon}(x) = u_\varepsilon(x_\varepsilon, \theta_\varepsilon) - \phi(\theta_\varepsilon) - \varepsilon \Psi_\varepsilon^{\theta_\varepsilon}(x_\varepsilon).$$

We distinguish two cases : either $x_\varepsilon \in \Omega$ or $x_\varepsilon \in \partial\Omega$.

Case 1 : $x_\varepsilon \in \Omega$. Since u_ε is a classical solution of (E_{HC}) , we deduce that it is also a viscosity solution, therefore

$$\begin{aligned} & -\frac{\partial_{xx}(\phi(\theta_\varepsilon) + \varepsilon \Psi_\varepsilon^{\theta_\varepsilon}(x_\varepsilon))}{\varepsilon} - \frac{[\partial_x(\phi(\theta_\varepsilon) + \varepsilon \Psi_\varepsilon^{\theta_\varepsilon}(x_\varepsilon))]^2}{\varepsilon^2} + \int_{\Omega} [1 - e^{\Psi_\varepsilon^{\theta_\varepsilon}(y) - \Psi_\varepsilon^{\theta_\varepsilon}(x_\varepsilon)}] K(x_\varepsilon - y) dy \\ & - \varepsilon \partial_{\theta\theta}(\phi(\theta_\varepsilon) + \varepsilon \Psi_\varepsilon^{\theta_\varepsilon}(x_\varepsilon)) - [\partial_\theta(\phi(\theta_\varepsilon) + \varepsilon \Psi_\varepsilon^{\theta_\varepsilon}(x_\varepsilon))]^2 - R(x_\varepsilon, \theta_\varepsilon) + \rho_\varepsilon(x_\varepsilon) \leq 0. \end{aligned}$$

Remarking that ϕ does not depend on x and the θ value is fixed in $\Psi_\varepsilon^{\theta_\varepsilon}$, we deduce that

$$\begin{aligned} & -\partial_{xx} \Psi_\varepsilon^{\theta_\varepsilon}(x_\varepsilon) - [\partial_x \Psi_\varepsilon^{\theta_\varepsilon}(x_\varepsilon)]^2 + \int_{\Omega} [1 - e^{\Psi_\varepsilon^{\theta_\varepsilon}(y) - \Psi_\varepsilon^{\theta_\varepsilon}(x_\varepsilon)}] K(x_\varepsilon - y) dy - R(x_\varepsilon, \theta_\varepsilon) + \rho_\varepsilon(x_\varepsilon) \\ & - \varepsilon \partial_{\theta\theta} \phi(\theta_\varepsilon) - [\partial_\theta \phi(\theta_\varepsilon)]^2 \leq 0. \end{aligned} \tag{5.33}$$

Next, we observe that (5.32) implies

$$-\partial_{xx} \Psi_\varepsilon^{\theta_\varepsilon}(x_\varepsilon) - [\partial_x \Psi_\varepsilon^{\theta_\varepsilon}(x_\varepsilon)]^2 + \int_{\Omega} [1 - e^{\Psi_\varepsilon^{\theta_\varepsilon}(y) - \Psi_\varepsilon^{\theta_\varepsilon}(x_\varepsilon)}] K(x_\varepsilon - y) dy - R(x_\varepsilon, \theta_\varepsilon) + \rho_\varepsilon(x_\varepsilon) = \lambda(\theta_\varepsilon, \rho_\varepsilon).$$

Therefore, passing to the limit $\varepsilon \rightarrow 0$, thanks to the continuity of $\lambda(\theta, \rho)$ with respect to θ and ρ (Proposition 5.1), it follows that

$$-[\partial_\theta \phi(\theta)]^2 \leq -\lambda(\theta, \rho).$$

Case 2 : $x_\varepsilon \in \partial\Omega$. First, we remark that in this case,

$$-\partial_x u_\varepsilon(x_\varepsilon, \theta_\varepsilon) = -\partial_x \Psi_\varepsilon^{\theta_\varepsilon}(x_\varepsilon) = 0.$$

Therefore, we deduce that

$$-\partial_x [u_\varepsilon(x_\varepsilon, \theta_\varepsilon) - \phi(\theta_\varepsilon) - \varepsilon \Psi_\varepsilon^{\theta_\varepsilon}(x_\varepsilon)] = 0.$$

Moreover, since $(u_\varepsilon - \phi - \varepsilon \Psi_\varepsilon^{\theta_\varepsilon})(x_\varepsilon, \theta_\varepsilon) = \max(u_\varepsilon - \phi - \varepsilon \Psi_\varepsilon^{\theta_\varepsilon})$, we have firstly by a reflective argument that

$$-\partial_{xx}(u_\varepsilon - \phi - \varepsilon \Psi_\varepsilon^{\theta_\varepsilon})(x_\varepsilon, \theta_\varepsilon) \geq 0, \tag{5.34}$$

and secondly, we have

$$u_\varepsilon(y, \theta_\varepsilon) - u_\varepsilon(x_\varepsilon, \theta_\varepsilon) \leq \varepsilon [\Psi_\varepsilon^{\theta_\varepsilon}(y) - \Psi_\varepsilon^{\theta_\varepsilon}(x_\varepsilon)]. \tag{5.35}$$

The inequalities (5.34) and (5.35) lead to

$$\begin{aligned} & -\partial_{xx} \varepsilon \Psi_\varepsilon^{\theta_\varepsilon}(x_\varepsilon) \leq -\partial_{xx} u_\varepsilon(x_\varepsilon, \theta_\varepsilon) \\ \text{and} \quad & \int_{\Omega} [1 - e^{\Psi_\varepsilon^{\theta_\varepsilon}(y) - \Psi_\varepsilon^{\theta_\varepsilon}(x_\varepsilon)}] K(x_\varepsilon - y) dy \leq \int_{\Omega} [1 - e^{\frac{u_\varepsilon(y, \theta_\varepsilon) - u_\varepsilon(x_\varepsilon, \theta_\varepsilon)}{\varepsilon}}] K(x_\varepsilon - y) dy. \end{aligned}$$

Therefore, the conclusion follows from similar computation as above.

Finally, u is a sub-solution of (5.5) in a viscosity sense. With similar arguments, u is also a super-solution. We conclude that u is a viscosity solution of (5.5).

Next, we prove that $\sup_{\theta \in [-A, A]} u(\theta) = 0$.

If it does not holds true, it follows that $\sup_{\theta \in [-A, A]} u(\theta) < -a < 0$. Hence for ε small enough, we deduce that $\max_{(x, \theta) \in \Omega \times [-A, A]} u_\varepsilon(x, \theta) < -\frac{a}{2}$, which implies that $\rho_\varepsilon < c$ for ε sufficiently small. This is in contradiction with the third item of Theorem 5.3. We conclude that $\max_{\theta \in [-A, A]} u(\theta) = 0$.

Proof of 3. Thanks to the L^∞ bounds on ρ_ε (third point of Theorem 5.3), we deduce that

$$c \leq \|n_\varepsilon\|_{L^1(\Omega \times]-A, A])} \leq C.$$

It follows that n_ε converges in the sense of the measure to n and the measure n is non-negative and not trivial. Finally, we prove that

$$\text{supp } n_\varepsilon \subset \Omega \times \{\theta \in]-A, A[\mid u(\theta) = 0\}.$$

Indeed, let $\phi \in C_c^\infty(\Omega \times]-A, A[)$ be any positive test function such that

$$\text{supp } \phi \subset \Omega \times \{\theta \in]-A, A[\mid u(\theta) = 0\}^c. \quad (5.36)$$

We prove that $\int_\Omega \int_{-A}^A \phi(x, \theta) n(x, \theta) dx d\theta = 0$.

To this end, we introduce $-a = \sup_{\text{supp } \phi} u$. According to (5.36), it follows that $a > 0$. We deduce that for all ε small enough and all $(x, \theta) \in \Omega \times \text{supp } \phi$, we have

$$u_\varepsilon(x, \theta) \leq -\frac{a}{2}. \quad (5.37)$$

We conclude that

$$\begin{aligned} \int_\Omega \int_{-A}^A \phi(x, \theta) n(x, \theta) d\theta dx &= \int_\Omega \int_{\text{supp } \phi(x, \cdot)} \phi(x, \theta) n(x, \theta) d\theta dx \\ &= \lim_{\varepsilon \rightarrow 0} \int_\Omega \int_{\text{supp } \phi(x, \cdot)} \phi(x, \theta) n_\varepsilon(x, \theta) d\theta dx \\ &= \lim_{\varepsilon \rightarrow 0} \int_\Omega \int_{\text{supp } \phi(x, \cdot)} \phi(x, \theta) e^{\frac{u_\varepsilon(x, \theta)}{\varepsilon}} d\theta dx \\ &\leq \lim_{\varepsilon \rightarrow 0} \int_\Omega \int_{\text{supp } \phi(x, \cdot)} \phi(x, \theta) e^{\frac{-a}{2\varepsilon}} d\theta dx \\ &= 0. \end{aligned}$$

This concludes the proof of 3. □

5.5 Discussions and numerics

5.5.1 Numerical resolutions

To find a numerical solution of (E), we solve numerically the following parabolic equation :

$$\begin{cases} \partial_t n_\varepsilon - \partial_{xx} n_\varepsilon - \varepsilon^2 \partial_{\theta\theta} n_\varepsilon + L(n_\varepsilon) = [R - \rho_\varepsilon] n_\varepsilon & \text{in } \mathbb{R}^+ \times \Omega \times]-A, A[, \\ \rho_\varepsilon(t, x) = \int_{-A}^A n_\varepsilon(t, x, \theta) d\theta & \text{in } \mathbb{R}^+ \times \Omega, \\ \partial_{\nu_x} n_\varepsilon = \partial_{\nu_\theta} n_\varepsilon = 0, \\ n(t = 0, x, \theta) = n_0(x, \theta). \end{cases} \quad (E_t)$$

For this numerical study, we take

$$R(x, \theta) = r - g(bx - \theta)^2.$$

We recall from Example 5.1 that in this case, r is a growth rate and $-g(bx - \theta)^2$ models the selection. The parameter g is the selection pressure whereas b is the gradient of the environment. We provide numerical examples where we vary this set of parameters.

We implement equation (E_t) by a semi-implicit finite difference method. We stop the algorithm when we find a numerical steady state of (E_t) : a numerical solution of (E) .

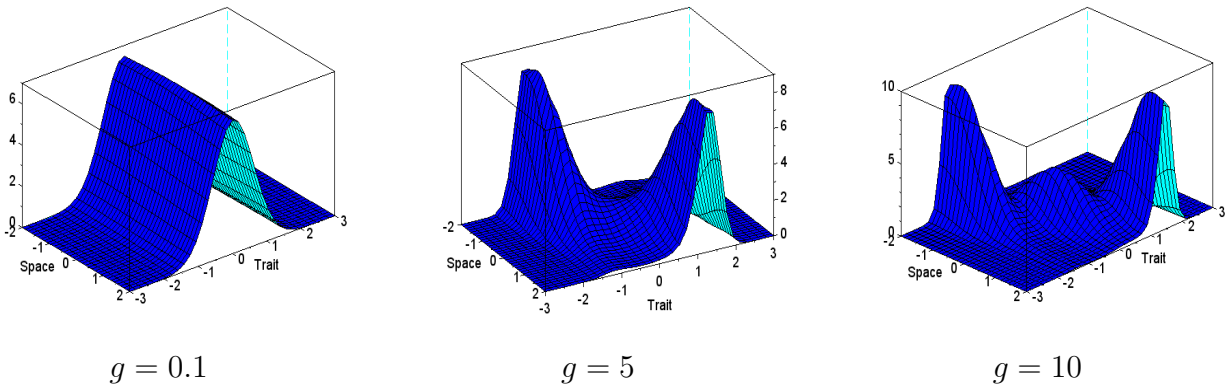


FIGURE 5.1 – Numerical solutions of (E) with 3 different parameters g ($g = 0.1, 5, 10$) and the other parameters are fixed as follows $\Omega =] - 2, 2[$, $A = 3$, $r = 10$, $b = 1$ and $\varepsilon = 0.005$.

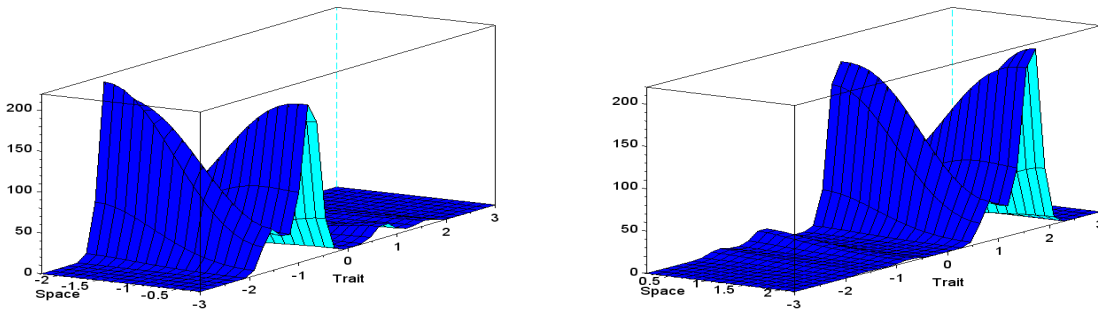


FIGURE 5.2 – Numerical resolution of (E) with the following set of parameters $r = 10$, $b = 1$, $g = 10$, $A = 3$, $\varepsilon = 0.005$ and $\Omega =] - 2.2, -0.2[\cup] 0.2, 2.2[$.

First, we underline that in all the numerical resolutions, the density of the population concentrates around one or several distinct trait(s). Moreover, these optimal traits are present everywhere in space thanks to the local and the non-local migration. However, the density of the population at the position x with an optimal trait θ_m depends on whether this trait θ_m is adapted or not to the position x .

Figure 5.1 illustrates the variation of the density of the population when Ω is connected and g varies. When the selection pressure g increases, we observe that several dominant traits appear.

Figures 5.2 and 5.3 illustrate the influence of the distance when we consider a patchy space

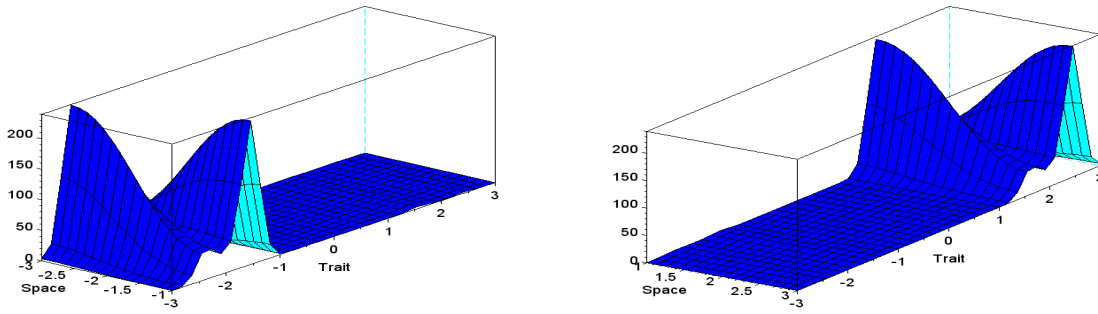


FIGURE 5.3 – Numerical resolution of (E) with the following set of parameters $r = 10$, $b = 1$, $g = 10$, $A = 3$, $\varepsilon = 0.005$ and $\Omega =] - 3, -1[\cup] 1, 3[$.

domain. Indeed, we keep, the same parameters as the third example of Figure 5.1 ($A = 3$, $r = 10$, $b = 1$, $\varepsilon = 0.005$ and $g = 10$) but the space domains are respectively $\Omega =] - 2.2, -0.2[\cup] 0.2, 2.2[$ (Figure 5.2) and $\Omega =] - 1, -3[\cup] 1, 3[$ (Figure 5.3). We observe that with a same set of parameters, considering a patchy domain leads to the apparition of more dominants traits compared with the population living in a single connected environment. Moreover, the part of the population living in a patch Ω_1 , with a dominant traits θ_2 adapted to the other patch Ω_2 , becomes weaker as the distance between both patches increases. Finally, we want to emphasize that the emergence of new dominant traits as a function of the distance between habitats depends also on the value of g . Indeed, for the same parameters value as in Figure 5.3 except for g that we take $g = 0.1$, we observe in Figure 5.4 that the stationary state is still monomorphic even though $\Omega =] - 3, -1[\cup] 1, 3[$.

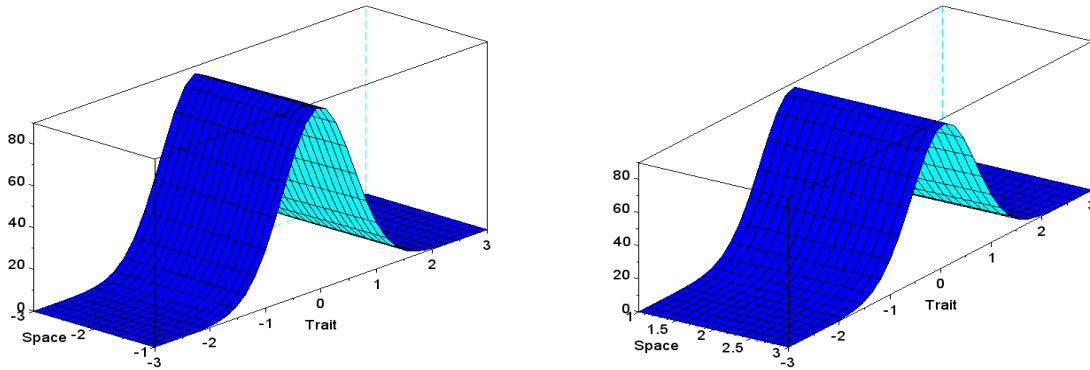


FIGURE 5.4 – Numerical resolution of (E) with the following set of parameters $r = 10$, $b = 1$, $g = 0.1$, $A = 3$, $\varepsilon = 0.005$ and $\Omega =] - 3, -1[\cup] 1, 3[$.

Appendix - Proof of Lemma 5.1

The proof of Lemma 5.1 follows essentially the steps of the proof of the convergence of u_ε (i.e. second item of Theorem 5.2). Therefore, we will only emphasize the differences between the two proofs. We made the choice to provide the proof of the convergence of u_ε rather than

the convergence of μ_ε because it is the result that motivated the current study.

Proof of Lemma 5.1. We recall the equation satisfied by μ_ε and ξ_ε

$$\begin{cases} -\partial_{xx}\xi_\varepsilon - \varepsilon^2\partial_{\theta\theta}\xi_\varepsilon + L\xi_\varepsilon - R\xi_\varepsilon = \mu_\varepsilon\xi_\varepsilon & \text{in } \Omega \times]-A, A[, \\ \partial_{\nu_x}\xi_\varepsilon = \partial_{\nu_\theta}\xi_\varepsilon = 0 & \text{on } \partial(\Omega \times]-A, A[). \end{cases} \quad (5.2)$$

The existence of ξ_ε is ensured by the Krein-Rutman Theorem. Moreover, according to the Krein-Rutman Theorem, the sign of ξ_ε is constant. Therefore, we consider that $\xi_\varepsilon > 0$, $\|\xi_\varepsilon\|_{L^2} = 1$ and we define

$$v_\varepsilon = \varepsilon \ln(\xi_\varepsilon).$$

Next, we prove that μ_ε is bounded from below and above respectively by $-\sup R$ and $-\inf R$. First, we focus on the upper bound. Let $(\bar{x}, \bar{\theta}) \in \bar{\Omega} \times [-A, A]$ be such that $\sup_{(x,\theta) \in \bar{\Omega} \times [-A, A]} \xi_\varepsilon(x, \theta) = \xi_\varepsilon(\bar{x}, \bar{\theta})$. If $(\bar{x}, \bar{\theta}) \in \Omega \times]-A, A[$, it follows

$$(-\partial_{xx}\xi_\varepsilon - \varepsilon^2\partial_{\theta\theta}\xi_\varepsilon + L(\xi_\varepsilon))(\bar{x}, \bar{\theta}) \leq 0.$$

From (5.2), we deduce that

$$\mu_\varepsilon \leq -R(\bar{x}, \bar{\theta}) \leq -\inf R.$$

If $(\bar{x}, \bar{\theta})$ belongs to $\partial(\Omega \times]-A, A[)$, we conclude with a reflective argument and the same computations as in the previous case. In any case, for all $\varepsilon > 0$ we have

$$\mu_\varepsilon < -\inf R.$$

Next, we focus on the lower bound. Let $(\underline{x}, \underline{\theta}) \in \bar{\Omega} \times [-A, A]$ be such that $\inf_{(x,\theta) \in \bar{\Omega} \times [-A, A]} \xi_\varepsilon = \xi_\varepsilon(\underline{x}, \underline{\theta})$. With similar arguments as for the upper bound, we deduce that

$$-\sup R \leq -R(\underline{x}, \underline{\theta}) \leq \mu_\varepsilon.$$

Therefore, μ_ε is uniformly bounded from below and above thus μ_ε converges along subsequences to μ .

Next, as we have established Lipschitz and uniform bounds on u_ε , we can prove that there exists a constant $C > 0$ such that

$$|\partial_x v_\varepsilon| < C\varepsilon, \quad |\partial_\theta v_\varepsilon| < C, \quad -C < \lim_{\varepsilon \rightarrow 0} \inf_{\Omega \times]-A, A[} v_\varepsilon, \quad \text{and} \quad \lim_{\varepsilon \rightarrow 0} \sup_{\Omega \times]-A, A[} v_\varepsilon \leq 0.$$

Therefore, we deduce that v_ε converges along subsequences to v . Moreover, with similar computations as in the proof of the second item of Theorem 5.2, we deduce that v is a viscosity solution of

$$\begin{cases} -[\partial_\theta v(\theta)]^2 = -\lambda(\theta, -\mu), \\ \max_{\theta \in [-A, A]} v(\theta) = 0. \end{cases} \quad (5.38)$$

Next, we claim that

$$\lambda(\theta, -\mu) = \lambda(\theta, 0) - \mu. \quad (5.39)$$

We postpone the proof of this claim to the end of this paragraph. Thanks to (5.38) and (5.39) we deduce that

$$\begin{cases} -[\partial_\theta v(\theta)]^2 = -\lambda(\theta, 0) + \mu, \\ \max_{\theta \in [-A, A]} v(\theta) = 0. \end{cases}$$

Remark that $-\lambda(\theta, 0) + \mu \leq 0$ for all $\theta \in [-A, A]$. Next, we introduce $\theta_m \in [-A, A]$ such that

$$v(\theta_m) = \max_{\theta \in [-A, A]} v(\theta).$$

It follows that $\partial_\theta v(\theta_m) = 0$ and $-\lambda(\theta_m, 0) + \mu = 0 = \max(-\lambda(\theta, 0) + \mu)$. We deduce thanks to (H4) that

$$0 = \max(-\lambda(\theta, 0) + \mu) = -\min(\lambda(\theta, 0)) + \mu = -\lambda(\theta_0, 0) + \mu.$$

We conclude that

$$\lambda(\theta_0, 0) = \mu.$$

We finish the proof by remarking that the previous convergence result holds for any subsequence of μ_ε . Therefore, we conclude that

$$\lim_{\varepsilon \rightarrow 0} \mu_\varepsilon = \lambda(\theta_0, 0).$$

It remains to prove (5.39). Let ψ_μ^θ be the principal eigenfunction associated to the principal eigenvalue of $\lambda(\theta, -\mu)$ with μ a constant

$$\begin{cases} -\partial_{xx}\psi_\mu^\theta + L\psi_\mu^\theta - [R(\cdot, \theta) + \mu]\psi_\mu^\theta = \lambda(\theta, -\mu)\psi_\mu^\theta & \text{in } \Omega, \\ \partial_{\nu_x}\psi_\mu^\theta = 0 & \text{on } \partial\Omega. \end{cases}$$

It follows that

$$-\partial_{xx}\psi_\mu^\theta + L\psi_\mu^\theta - R(\cdot, \theta)\psi_\mu^\theta = (\lambda(\theta, -\mu) + \mu)\psi_\mu^\theta.$$

Since μ is constant, $\psi_\mu^\theta > 0$ and by the uniqueness of the positive eigenfunction of $-\partial_{xx} + L - R(\cdot, \theta)$ (up to a multiplication by a scalar), we deduce that $\lambda(\theta, -\mu) + \mu = \lambda(\theta, 0)$. \square

Chapter 6

Optimal release of mosquitoes to control dengue transmission

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Introduction

Aedes aegypti is the main vector transmitting dengue viruses. This mosquito can also transmit chikungunya, yellow fever and Zika infection. According to the World Health Organization, 390 million people are infected by dengue every year and 3.9 billion people, in 128 countries, are at risk of infection by dengue viruses. As there is no treatment for dengue fever, the current method of preventing dengue virus transmission and epidemics is to target the vector, i.e. the mosquito. Beyond preventing mosquitoes from accessing egg-laying habitats by environmental management and modification, one of the most promising control techniques is to transform mosquito population with a virus-suppressing *Wolbachia* bacteria. The idea of using *Wolbachia* for disease control was first proposed in the 1960s [76] but applying it to *Aedes aegypti* population is very recent. *Wolbachia* bacterium strains were isolated from *Drosophila melanogaster* in laboratory just before 2000 [87, 83] but were introduced into *Aedes aegypti* embryos only on 2009 [84]. The capability of this bacteria to suppress dengue virus and other pathogens transmission by *Aedes aegypti* was shown in laboratory around 2010 [92, 22, 113]. It was also shown that this bacteria shortens life span [114] and most of the infected adults do not reach the infectious stage. But the most important modification induced by the bacteria is cytoplasmic incompatibility (CI) [84]. Cytoplasmic incompatibility is used by the bacteria to spread rapidly into natural population [110] by producing non-viable eggs when uninfected females mate with infected males. Reproduction between infected males and females lead to infected eggs. As this bacteria is vertically transmitted (from mother to off-springs), uninfected males mating with infected females give rise only to infected eggs.

We are interested on optimizing the release of *Wolbachia*-infected mosquitoes into a wild host population of mosquitoes. Thus, the aim of the study is to model the propagation across time and space of the density of infected mosquitos, denoted n_2 , starting from a controlled release u into an existing population of uninfected. In what follows, we will denote by n_1 the density of uninfected mosquitos.

Formally, a proportion $1 - s_h$ of uninfected female's eggs fertilized by infected males

actually hatch. Cytoplasmic incompatibility is perfect when $s_h = 1$. We denote by b_1 , respectively b_2 , the net fecundity rate of uninfected females, respectively infected females. Death rate for uninfected mosquitoes is denoted d_1 . As *Wolbachia* decreases lifespan, death rate of infected mosquitoes d_2 verifies $d_2 > d_1$. It is also observed that *Wolbachia* infected mosquitoes tend to have reduced fertility, then $b_2 \leq b_1$. Finally, we denote κ the carrying capacity. Cytoplasmic incompatibility and vertical transmission drive the spatial spread of the infected population producing a bistable dynamic of *Wolbachia* [111]. If the infected population is installed above a sufficient threshold frequency Θ compared to the uninfected population, it will spread and tend to increase to 1, otherwise it will tend to decline to zero.

For fixed maximal time $T > 0$ and domain Ω , the system of equation that we consider is the following :

$$\left\{ \begin{array}{ll} \partial_t n_1 - D\Delta n_1 = b_1 n_1 \left(1 - s_h \frac{n_2}{n_1 + n_2}\right) \left(1 - \frac{n_1 + n_2}{\kappa}\right) - d_1 n_1 & \text{in } \Omega, \\ \partial_t n_2 - D\Delta n_2 = b_2 n_2 \left(1 - \frac{n_1 + n_2}{\kappa}\right) - d_2 n_2 + u & \text{in } \Omega, \\ \partial_\nu n_1 = \partial_\nu n_2 = 0 & \text{on } \partial\Omega, \\ n_1(0, x) = n_1^0(x), \quad n_2(0, x) = n_2^0(x) & \text{in } \Omega. \end{array} \right. \quad (6.1)$$

The equations driving the dynamics of n_1 and n_2 are bistable and monostable reaction-diffusion equations, respectively. Note that in the reaction term of the first equation the term $-\frac{n_2}{n_1+n_2}$ stands for the vertical transition of the disease whereas the coefficient s_h models that this vertical transmission may or not be perfect because of the cytoplasmic incompatibility. More precisely, assuming homogeneous repartition of individuals, the probability to mate with an infected mosquito is $\frac{n_2}{n_1+n_2}$. Then, uninfected mosquitoes are generated from mating of uninfected mosquitoes with uninfected mosquito (probability $n_1 \frac{n_1}{n_1+n_2}$) or uninfected mosquitoes with infected mosquitoes but with a probability $(1 - s_h)n_1 \frac{n_2}{n_1+n_2}$. The first term in the right hand side is the sum of this latter quantities. The diffusion coefficient is denoted D ; it is assumed to be the same for both populations since both populations belongs to the same genus of mosquitoes. The last term of the second equation $+u$ stands here to model the releases of infected mosquitoes developed in laboratory : it is on this control that we will act upon. More precisely, a question we want to address in this work is to know what should be the shape of the release function u to be as close as possible to the total invasion of the infected population into the domain.

The outline of this paper is the following. In the next section, we introduce the optimal control problem and prove the existence of an optimum for this problem. In Section 6.2, we consider a toy problem, which is a very simplified version of the full problem, for which we can solve explicitly the optimal problem and find the optimum. In Section 6.3, we investigate numerically the optimization of the spatial releases of mosquitoes. Finally, we end this paper with a conclusion and perspective for future works. An appendix is devoted to recalling the reduction of system (6.1).

6.1 Optimal Control Problem

We are going to simplify the problem. Instead of studying the coupled equations (6.1), we are going to follow the proportion of mosquitoes $p(t, x) = \frac{n_2(t, x)}{n_1(t, x) + n_2(t, x)}$ as in [104]. This reduction is clearly justified in the limit of large population in [104] (see also [3, Section 2.3]). Details on the formal computation are provided in Appendix. In order to simplify the reading,

we perform the scaling $x = \frac{\tilde{x}}{\sqrt{D}}$ not to keep the diffusion coefficient along the computations. Obviously, for the numerical simulations performed in Section 6.3, we have to keep in mind this scaling.

Denoting by p the proportion of infected mosquitoes, and u the release function, the dynamics is governed by the reaction-diffusion equation

$$\begin{cases} \frac{\partial p}{\partial t} - \Delta p = f(p) + ug(p), & t \in (0, T), \quad x \in \Omega, \\ \partial_\nu p(t, x) = 0, & x \in \partial\Omega, \\ p(0, x) = 0, & x \in \Omega, \end{cases} \quad (6.2)$$

where

$$f(p) = p(1-p) \frac{d_1 b_2 - d_2 b_1 (1 - s_h p)}{b_1 (1-p)(1 - s_h p) + b_2 p} \quad \text{and} \quad g(p) = \frac{1}{\kappa} \cdot \frac{b_1 (1-p)(1 - s_h p)}{b_1 (1-p)(1 - s_h p) + b_2 p}. \quad (6.3)$$

The general optimal control problem we want to investigate involves the least-squares functional J defined by

$$\hat{J}(u) = \frac{1}{2} \int_{\Omega} (1 - p(T, x))^2 dx, \quad (6.4)$$

which models that one aims at steering the system as close as possible to the target state. In some sense, it stands for the research of a control strategy ensuring the persistence of infected mosquitoes at the time horizon T .

Of course, it is relevant from the biological point of view to impose several constraints on the control function u . Indeed, the production of *Wolbachia*-infected mosquitoes is limited, which imposes that the total number of mosquitoes released is bounded. Hence, the control function u is assumed to belong to the set

$$\mathcal{U}_{T,C,M} = \left\{ u \in L^\infty([0, T] \times \Omega), \quad 0 \leq u \leq M \text{ a.e.}, \quad \int_0^T \int_{\Omega} u(t, x) dx dt \leq C \right\}. \quad (6.5)$$

modeling an upper limit on the instantaneous number of *Wolbachia*-infected individuals released at time t , as well as on the total number of released mosquitoes.

We then deal with the following optimal control problem :

$$\boxed{\inf_{u \in \mathcal{U}_{T,C,M}} \hat{J}(u)}. \quad (\mathcal{P}_{\text{full}})$$

Since this problem involves the minimization over function depending on time and space variables, it is difficult to study. Then, we will reduce it to a simpler one by assuming that the time distribution of the control function is given.

6.1.1 Modeling of the optimal control problem

In order to weaken the difficulty of Problem $(\mathcal{P}_{\text{full}})$, we introduce a simpler, although still relevant, problem by assuming that :

- releases are done periodically in time (for instance every week) and are impulses in time¹;

1. We consider Dirac measures since at the time-level of the study (namely, some generations), the release can be considered as instantaneous.

- at each release, the largest allowed amount of mosquitoes is released, corresponding to the maximal production capacity per week (which is relevant, according to the comparison principle).

As a consequence, we will be interested in determining the optimal way of releasing spatially the infected mosquitoes. Let us denote by $t_0 = 0 < t_1 < \dots < t_N = T$, $t_i = i\Delta T$, the release times. Rewriting the L^1 constraint on the control as $\langle u, 1 \rangle_{\mathcal{D}', \mathcal{D}((0,T) \times \Omega)} \leq C$, the control function reads

$$u(t, x) = \sum_{i=0}^{N-1} u_i(x) \delta_{\{t=t_i\}}, \quad \text{with} \quad \sum_{i=0}^{N-1} \int_{\Omega} u_i(x) dx \leq C,$$

where the pointwise constraint is modified into $0 \leq u_i(\cdot) \leq M$.

The new optimal design problem reads

$$\inf_{\mathbf{u} \in \mathcal{V}_{T,C,M}} \tilde{J}(\mathbf{u}), \quad \text{where } \mathbf{u} = (u_i)_{0 \leq i \leq N-1}, \quad \tilde{J}(\mathbf{u}) = J \left(\sum_{i=0}^{N-1} u_i(\cdot) \delta_{\{t=t_i\}} \right) \quad (\mathcal{P}'_{\text{full}})$$

and

$$\mathcal{V}_{T,C,M} = \left\{ \mathbf{u} = (u_i)_{0 \leq i \leq N-1}, \text{ with } 0 \leq u_i \leq M \text{ a.e. in } \Omega \text{ and } \sum_{i=0}^{N-1} \int_{\Omega} u_i(x) dx \leq C \right\}.$$

It is possible to recast System (6.2) without source measure terms, coming from the specific form of the control functions. For the sake of simplicity, we provide here a naive formal analysis, but claim that this can be proven rigorously by using a standard variational analysis.

Let us approximate the Dirac measure at $t = t_i$ by the function $\frac{1}{\varepsilon} \mathbf{1}_{[t_i, t_i + \varepsilon]}$. Making the change of variable $t = t_i + \tau\varepsilon$, and introducing \tilde{p} given by $\tilde{p}(\tau, x) = p(t, x)$, one gets from system (6.2) that \tilde{p} solves

$$\frac{\partial \tilde{p}}{\partial \tau} - \varepsilon \Delta \tilde{p} = \varepsilon f(\tilde{p}) + u_i g(\tilde{p}). \quad \tau \in [0, 1], \quad x \in \Omega.$$

Letting formally ε go to 0 and denoting, with a slight abuse of notation, still by \tilde{p} the formal limit of the system above yields

$$\frac{\partial \tilde{p}}{\partial \tau}(\tau, x) = u_i(x) g(\tilde{p}(\tau, x)), \quad \tau \in [0, 1], \quad x \in \Omega. \quad (6.6)$$

Let us denote G the anti-derivative of $\frac{1}{g}$ vanishing at 0, namely

$$G(p) = \int_0^p \frac{dq}{g(q)}.$$

Then, by a direct integration of (6.6) on $[0, 1]$, we obtain

$$G(\tilde{p}(1, x)) = G(\tilde{p}(0, x)) + u_i(x), \quad x \in \Omega.$$

Coming back on the function p yields

$$p(t_i^+, x) = G^{-1}(G(p(t_i^-, x)) + u_i(x)), \quad x \in \Omega.$$

Hence we arrive at the system

$$\begin{cases} \frac{\partial p}{\partial t} - \Delta p = f(p), & t \in (0, T) \setminus \{t_i\}_{i \in \{1, \dots, N-1\}}, \quad x \in \Omega, \\ \partial_\nu p(t, x) = 0, & x \in \partial\Omega, \\ p(0^+, \cdot) = G^{-1}(u_0(\cdot)), \\ p(t_i^+, \cdot) = G^{-1}(G(p(t_i^-, \cdot)) + u_i(\cdot)), & i \in \{1, \dots, N-1\} \end{cases} \quad (6.7)$$

and the optimization problem reads

$$\boxed{\inf_{\mathbf{u} \in \mathcal{V}_{T,C,M}} J(\mathbf{u}) \quad \text{with } J(\mathbf{u}) = \frac{1}{2} \int_{\Omega} (1 - p(T, x))^2 dx}, \quad (\mathcal{P}_{\text{reduced}})$$

where p is the solution of (6.7). In the next Section, we investigate the existence of solutions for this problem.

6.1.2 Existence of minimizers

Theorem 6.1. *Problem $(\mathcal{P}_{\text{reduced}})$ has a solution.*

Proof. For the sake of readability, we only provide the proof in the case $N = 2$. Indeed, there is no additional difficulty to deal with the general case whose proof follows exactly the same lines. The proof is divided into several steps.

Let $u^n = \{u_i^n\}_{i \in \{1, \dots, N\}} \in (\mathcal{V}_{T,C,M})^{\mathbb{N}}$ be a minimizing sequence for Problem $(\mathcal{P}_{\text{reduced}})$.

Notice that, since u belongs to $\mathcal{V}_{T,C,M}$ and G^{-1} takes its value in $[0, 1[$, we infer from the maximum principle that $0 \leq p(t, \cdot) < 1$ for a.e. $t \in [0, T]$ so that one has for all $u \in \mathcal{V}_{T,C,M}$

$$0 \leq J(u) \leq \frac{|\Omega|}{2}.$$

It follows that $\inf_{u \in \mathcal{V}_{T,C,M}} J(u)$ belongs to $(0, \frac{|\Omega|}{2})$ and, in particular, is finite.

Step 1 : *Convergence of the minimizing sequence.*

Let p^n be the solution to (6.7) associated to the control function u^n and let us introduce

$$\begin{aligned} v_0^n(\cdot) &= u_0^n(\cdot) \\ v_1^n(\cdot) &= G^{-1}(G(p^n(t_1^-, \cdot)) + u_1^n(\cdot)). \end{aligned}$$

By induction, one easily shows that v^n is uniformly bounded in L^∞ . Since the class $\mathcal{V}_{T,C,M}$ is closed for the L^∞ weak-star topology, there exists $v^\infty \in \mathcal{V}_{T,C,M}$ such that, up to a subsequence, v^n converges weakly-star to v^∞ in L^∞ . Here and in the sequel, we will denote similarly with a slight abuse of notation a given sequence and any subsequence.

Multiplying the main equation of (6.7) by p^n and integrating by parts, we infer from the above estimates the existence of a positive constant C such that

$$\frac{1}{2} \int_0^T \int_{\Omega} \partial_t (p^n(t, x)^2) dx dt + \int_0^T \int_{\Omega} |\nabla p^n(t, x)|^2 dx dt \leq C$$

for every $n \in \mathbb{N}$, which also reads

$$\frac{1}{2} \int_{\Omega} \left([(p^n(t, x))^2]_{t=0}^{t=t_1} + [(p^n(t, x))^2]_{t=t_1}^{t=T} \right) dx + \int_0^T \int_{\Omega} |\nabla p^n(t, x)|^2 dx dt \leq C$$

for every $n \in \mathbb{N}$.

The sequence of functions p^n is uniformly bounded in $L^2([0, T], H^1(\Omega))$ thanks to the pointwise bounds on p^n . Furthermore, by using (6.7), one gets that the sequence $\partial_t p^n$ is uniformly bounded in $L^2([0, T], W^{-1,1}(\Omega))$. According to the Aubin-Lions theorem (see [101]) we infer that p^n converges (up to a subsequence) to $p^\infty \in L^2([0, T], H^1(\Omega))$, strongly in

$L^2([0, T], L^2(\Omega))$ and weakly in $L^2([0, T], H^1(\Omega))$. Passing to the limit in (6.7) yields that p^∞ is a weak solution to

$$\begin{cases} \partial_t p^\infty - \Delta p^\infty = f(p^\infty), & t \in (0, T), \quad x \in \Omega, \\ \partial_\nu p^\infty(t, x) = 0, & t \in (0, T), \quad x \in \partial\Omega, \end{cases} \quad (6.8)$$

It is standard that any solution to this bistable reaction-diffusion equation is continuous in time.

Introducing $u_0^\infty := G(p^\infty(0^+, \cdot))$ and $u_1^\infty := G(v_1^\infty) - G(p^\infty(t_1^-, \cdot))$, we claim that $p^\infty(t_1^-, \cdot) = v_1^\infty(\cdot)$. A such equality holds true by passing to the limit as $n \rightarrow +\infty$ in the variational formulation on p^n and by using adapted test-functions belonging to

$$V_1 = \left\{ q \in C^\infty([0, T], C^\infty(\Omega) \cap C^0(\bar{\Omega})) \text{ whose support is contained in } [t_i, t_{i+1}[\right\}.$$

This is a consequence of the weak convergence of p^n in $H^1(\Omega)$ to p^∞ . Notice, in particular, that $G(v_1^n)$ converges weakly star in L^∞ to $G(v_1^\infty)$. Indeed, this is a consequence of the continuity and convexity since one has $G''(p) = \kappa b^2 \frac{(1 - s_h p^2)}{((p - 1)(s_h p - 1))^2}$ which is positive whenever p belongs to $[0, 1]$.

Step 2 : Conclusion.

Let us first show that u^∞ belongs to $\mathcal{V}_{T,C,M}$. Since the derivative of G is $1/g$ which is positive, G is increasing and therefore, one has $0 \leq u^\infty \leq m$ a.e. in Ω .

For the integral condition (namely, $\int_\Omega u \leq C$), let us distinguish between two cases :

Case 1 : if $m|\Omega| \leq C$, the conclusion follows immediately.

Case 2 : if $m|\Omega| > C$, let us use that G is, as aforementioned, lower semi-continuous for the weak-star topology of L^∞ . Thus, we deduce that

$$\int_\Omega u^\infty = \int_\Omega G(v^\infty) \leq \liminf_{n \rightarrow +\infty} \int_\Omega G(v^n) = \liminf_{n \rightarrow +\infty} \int_\Omega u^n \leq C.$$

It follows that u^∞ belongs to $\mathcal{V}_{T,C,M}$ and one concludes by using the Fatou Lemma :

$$\begin{aligned} J(u^\infty) &= \frac{1}{2} \int_\Omega (1 - p^\infty(T, x))^2 dx = \frac{1}{2} \int_\Omega \liminf_{n \rightarrow +\infty} (1 - p^n(T, x))^2 dx \\ &\leq \liminf_{n \rightarrow +\infty} \frac{1}{2} \int_\Omega (1 - p^n(T, x))^2 dx = \liminf_{n \rightarrow +\infty} J(u_n) = \inf_{u \in \mathcal{V}_{T,C,M}} J(u). \end{aligned}$$

We finally infer that u^∞ solves Problem $(\mathcal{P}_{\text{reduced}})$. □

Remark. *The uniqueness issue remains open, even for simple domain. It is likely that symmetries of the release domain play an important role.*

It is interesting to notice that, in a very particular case, we have an explicit expression of the minimizer for this problem.

Proposition 6.1. *Let $N \in \mathbb{N}^*$ and $M \leq \frac{C}{|\Omega|}$. Then $u = M$ is the unique solution of Problem $(\mathcal{P}_{\text{reduced}})$.*

Proof. It is a direct application of the comparison principle. Let u^* be a solution of Problem $(\mathcal{P}_{\text{reduced}})$. By contradiction, let us assume that u^* is not identically equal to M a.e. in Ω . Then, let t_i be a release time for which the associated control function u_i^* is not identically equal to M in Ω . Recall that $u_i^* \leq M$. Let us denote by p^* the solution of the problem (6.2) associated to the control function u^* . Let u^M be the control function defined by

$$u_i^M = M \quad \text{and} \quad u_j^M = u_j^* \quad \text{for all } j \in \{0, \dots, N-1\} \setminus \{i\}.$$

Let p^M be the solution of (6.2) associated to u^M identically. Since G^{-1} is an increasing function by the comparison principle we have for all time $t \in [0, T]$ and a.e. $x \in \Omega$,

$$0 < p^*(t, x) \leq p^M(t, x) < 1.$$

Evaluating this expression at time $t = T$, the expected conclusion follows by noting that the constant function equal to M on $(0, T) \times \Omega$ belongs to $\mathcal{U}_{T,C,M}$. \square

6.1.3 Computation of derivatives

As a preliminary remark, we claim that for any element \mathbf{u} of the set $\mathcal{V}_{T,C,M}$ and any admissible perturbation \mathbf{h} , the mapping $\mathcal{V}_{T,C,M} \ni \mathbf{u} \mapsto p \in L^2(0, T, H^1(\Omega))$, where p denotes the unique weak solution of (6.7), is differentiable in the sense of Gâteaux at \mathbf{u} in the direction \mathbf{h} . Indeed, proving such a property is standard in calculus of variations and rests upon an application of the implicit function theorem. In the sequel, and with no confusion possible, we will denote by \dot{p} the Gâteaux-differential of p at \mathbf{u} in direction \mathbf{h} and by $\langle dJ(\mathbf{u}), \mathbf{h} \rangle$ the Gâteaux-differential of J at \mathbf{u} in direction \mathbf{h} , namely

$$\langle dJ(\mathbf{u}), \mathbf{h} \rangle = \lim_{\varepsilon \searrow 0} \frac{J(\mathbf{u} + \varepsilon \mathbf{h}) - J(\mathbf{u})}{\varepsilon}.$$

Let us make the cone of admissible perturbations precise. We call “admissible perturbation” any element of the tangent cone $\mathcal{T}_{\mathbf{u}, \mathcal{V}_{T,C,M}}$ to the set $\mathcal{V}_{T,C,M}$ at \mathbf{u} .

Definition 6.1. *The cone $\mathcal{T}_{\mathbf{u}, \mathcal{V}_{T,C,M}}$ is the set of N -tuples $\mathbf{h} = (h_0, \dots, h_{N-1}) \in (L^\infty(\Omega))^N$ such that, for any $i \in \{0, \dots, N-1\}$ and for any sequence of positive real numbers ε_n decreasing to 0, there exists a sequence of functions $h_i^n \in L^\infty(0, T)$ converging to h_i as $n \rightarrow +\infty$, and $u_i + \varepsilon_n h_i^n \in \mathcal{V}_{T,C,M}$ for every $n \in \mathbb{N}$ (see e.g. [44]).*

Proposition 6.2. *Assume that $N = 1$. Let $\mathbf{u} = (u_0) \in \mathcal{V}_{T,C,M}$ and $\mathbf{h} = (h_0) \in \mathcal{T}_{\mathbf{u}, \mathcal{V}_{T,C,M}}$. One has*

$$\langle dJ(\mathbf{u}), \mathbf{h} \rangle = \int_{\Omega} h(x) (G^{-1})'(u_0(x)) q(0, x) dx,$$

where q is the unique solution of the backward problem

$$\begin{cases} -\partial_t q(t, x) - \Delta q(t, x) - f'(p(t, x)) q(t, x) = 0, & (t, x) \in (0, T) \times \Omega, \\ \partial_n q(t, x) = 0, & (t, x) \in (0, T) \times \partial\Omega, \\ q(T, x) = p(T, x) - 1, & x \in \Omega. \end{cases}$$

Proof. By using the preliminary discussion, one has

$$\langle dJ(\mathbf{u}), \mathbf{h} \rangle = \int_{\Omega} \dot{p}(T, x) (p(T, x) - 1) dx, \quad (6.9)$$

where \dot{p} denotes the unique solution of the system

$$\begin{cases} \frac{\partial \dot{p}}{\partial t} - \Delta \dot{p} = f'(p)\dot{p}, & t \in (0, T), \quad x \in \Omega, \\ \partial_\nu \dot{p}(t, x) = 0, & x \in \partial\Omega, \\ \dot{p}(0^+, \cdot) = (G^{-1})'(u_0(\cdot))h. \end{cases} \quad (6.10)$$

Let us multiply the main equation of this system by q and then integrate by parts with respect to the variables t and x . By using in particular the Green formula, we get successively that

$$\begin{aligned} \int_0^T \int_\Omega q \frac{\partial \dot{p}}{\partial t} dx dt &= - \int_0^T \int_\Omega \dot{p} \frac{\partial q}{\partial t} dx dt + \int_\Omega q(T, x) \dot{p}(T, x) dx - \int_\Omega q(0, x) \dot{p}(0^+, x) dx, \\ - \int_0^T \int_\Omega q \Delta \dot{p} dx dt &= - \int_0^T \int_\Omega \dot{p} \Delta q dx dt, \end{aligned}$$

and therefore,

$$\langle dJ(\mathbf{u}), \mathbf{h} \rangle = \int_\Omega q(T, x) \dot{p}(T, x) dx = \int_\Omega q(0, x) \dot{p}(0^+, x) dx,$$

yielding the desired conclusion. \square

Remark. For practical purposes, it may be useful to notice that

$$q(t, x) = \tilde{q}(T - t, x), \quad t \in [0, T], \quad x \in \Omega,$$

where \tilde{q} denotes the solution of the initial boundary value problem

$$\begin{cases} \partial_t \tilde{q}(t, x) - \Delta \tilde{q}(t, x) - f'(p(T - t, x))\tilde{q}(t, x) = 0, & (t, x) \in (0, T) \times \Omega, \\ \partial_n \tilde{q}(t, x) = 0, & (t, x) \in (0, T) \times \partial\Omega, \\ \tilde{q}(0, x) = p(T, x) - 1, & x \in \Omega. \end{cases}$$

6.2 A toy Problem

This section is devoted to investigating a simpler version of $(\mathcal{P}_{\text{full}})$ corresponding to the case $N = 1$ with $f = 0$. More precisely, let p be the solution of

$$\begin{cases} \frac{\partial p}{\partial t} - \Delta p = 0, & t \in (0, T), \quad x \in \Omega, \\ \partial_\nu p(t, x) = 0, & x \in \partial\Omega, \\ p(0^+, \cdot) = u_0(\cdot). \end{cases} \quad (6.11)$$

Then, the optimization toy problem reads

$$\boxed{\inf_{u_0 \in \mathcal{V}_{T,C,M}} \hat{J}(u_0) \quad \text{with} \quad \hat{J}(u_0) = \frac{1}{2} \int_\Omega (1 - p(T, x))^2 dx}, \quad (\mathcal{P}_{\text{toy}})$$

where $p \in C^0([0, T], L^2(\Omega))$ is the unique solution of Equation (6.11). Note that the equation (6.11) has to be understood in a weak sense, since $u_0 \in L^\infty(\Omega) \subset L^2(\Omega)$ (see for example [108, Section 10.7]).

For this simple problem, we are able to solve explicitly the optimization problem :

Theorem 6.2. *Problem (6.11) has a unique solution u_0 , which is constant and equal to $\min(1, M, \frac{C}{|\Omega|})$.*

Proof. First, note that Problem (\mathcal{P}_{toy}) has a solution. Indeed, it is standard that the mapping $L^2(\Omega) \ni u_0 \mapsto p \in C^0([0, T], L^2(\Omega))$ is continuous. Therefore, so is \hat{J} by composition of continuous mappings. The conclusion follows by observing that $\mathcal{V}_{T, C, M}$ is a compact subset of $L^2(\Omega)$.

The proof relies on a well-adapted rewriting of the criterion \hat{J} . For that purpose, let us introduce the Neumann operator $-\Delta_N$ on Ω defined on

$$\mathcal{D}(-\Delta_N) = \{y \in H^2(\Omega) \mid \frac{\partial y}{\partial n}|_{\partial\Omega} = 0 \text{ and } \int_{\Omega} y(x) dx = 0\}.$$

According to the spectral theorem, there exists an orthonormal family $(\phi_j)_{j \geq 1}$ consisting of (real-valued) eigenfunctions of $-\Delta_N$, associated with the non-decreasing sequence positive eigenvalues $(\lambda_j)_{j \geq 1}$. Moreover, by setting $\lambda_0 = 0$ and $\phi_0 = \frac{1}{\sqrt{|\Omega|}}$, the sequence $(\phi_j)_{j \geq 0}$ is a Hilbert basis of $L^2(\Omega)$ and any solution p of (6.11) can be expanded in a unique way in $L^2(\Omega)$ as

$$p(t, x) = \sum_{j=0}^{+\infty} \langle p(0, \cdot), \phi_j \rangle_{L^2(\Omega)} e^{-\lambda_j t} \phi_j(x) = \sum_{j=0}^{+\infty} u_{0j} e^{-\lambda_j t} \phi_j(x), \quad (6.12)$$

with $u_{0j} = \langle u_0, \phi_j \rangle_{L^2(\Omega)}$. By expanding the square in the definition of \hat{J} , we then infer that

$$\begin{aligned} \hat{J}(u_0) &= \frac{|\Omega|}{2} - \int_{\Omega} p(T, x) dx + \frac{1}{2} \int_{\Omega} p(T, x)^2 dx \\ &= \frac{|\Omega|}{2} - \sqrt{|\Omega|} u_{00} + \frac{1}{2} \sum_{j=0}^{+\infty} e^{-2\lambda_j T} u_{0j}^2 \\ &= \frac{|\Omega|}{2} - \int_{\Omega} u_0(x) dx + \frac{1}{2} \sum_{j=0}^{+\infty} e^{-2\lambda_j T} \left(\int_{\Omega} u_0(x) \phi_j(x) dx \right)^2. \end{aligned}$$

Let u be a solution of Problem (6.11) and $h \in \mathcal{T}_{u_0, \mathcal{V}_{T, C, M}}$. Then, one has

$$\begin{aligned} \langle d\hat{J}(u_0), h \rangle &= - \int_{\Omega} h(x) dx + \sum_{j=0}^{+\infty} e^{-2\lambda_j T} \left(\int_{\Omega} u_0(x) \phi_j(x) dx \right) \left(\int_{\Omega} h(x) \phi_j(x) dx \right) \\ &= \int_{\Omega} h(x) \psi(x) dx, \end{aligned}$$

where $\psi(x) = -1 + \sum_{j=0}^{+\infty} e^{-2\lambda_j T} u_{0j} \phi_j(x)$.

The first order optimality conditions reads

$$\langle d\hat{J}(u_0), h \rangle \geq 0, \quad \forall h \in \mathcal{T}_{u_0, \mathcal{V}_{T, C, M}}. \quad (6.13)$$

The analysis of such optimality condition is standard in optimal control theory (see for example [80]) and yields the existence of a Lagrange multiplier $\xi \leq 0$ such that

- on $\{u_0 = M\}$, $\psi(x) \leq \xi$,
- on $\{u_0 = 0\}$, $\psi(x) \geq \xi$,
- on $\{0 < u_0 < M\}$, $\psi(x) = \xi$,
- $\xi (\int_{\Omega} u_0(x) dx - C) = 0$ (complementarity condition).

Let us investigate the optimality of constant functions. To this aim, notice that the functional \hat{J} is strictly convex². It follows that the optimality conditions (6.13) are at the same time necessary and sufficient and that Problem (6.11) has a unique solution.

Let u_0 be an admissible constant function for Problem (\mathcal{P}_{toy}). Then, $u_0 \in [0, M]$ whenever $M \leq C/|\Omega|$ and $u_0 \in [0, C/|\Omega|]$ elsewhere.

Furthermore, if u_0 is constant, then,

$$\psi(x) = -1 + u_0.$$

Let us now investigate each case separately. If $u_0 = 0$, then, from the complementarity condition, $\xi = 0$ and $\psi(x) = -1$ which is in contradiction with the optimality conditions above. Let us assume that $u_0 \neq 0$.

- If $u_0 = \frac{C}{|\Omega|}$, then this is admissible only if $C \leq M|\Omega|$. In this case we find $\psi(x) = \frac{-|\Omega|+C}{|\Omega|}$, and thus the optimality conditions are satisfied if and only if $|\Omega| \geq C$. All in all, $u_0 = \frac{C}{|\Omega|}$ is indeed a solution if, and only if, $\min(1, M)|\Omega| \geq C$.
- If $u_0 \neq \frac{C}{|\Omega|}$: then $\xi = 0$. Either $u_0 = M$, in which case the optimality conditions are satisfied only if $M \leq 1$, and this solution is admissible only if $M|\Omega| \leq C$; or $0 < u_0 < M$, in which case the optimality conditions hold only if $u_0 = 1$ (since $\psi \equiv \xi = 0$ in this case), which is admissible only if $M \geq 1$ and $|\Omega| \leq C$. All in all, $u_0 = \min(1, M)$ is a solution if and only if $\min(1, M)|\Omega| \leq C$.

The conclusion follows. □

6.3 Gaussian Releases

From a practical point of view, not all controls $u \in \mathcal{V}_{T,C,M}$ correspond to a release that could actually be conducted, as for example the constant solution of the toy problem of the previous section. To guarantee a solution that could be implemented, we restrict here the admissible controls to more accurately model the way mosquitoes are released in practice.

We thus consider that there are $K \in \mathbb{N}$ simultaneous releases and that each one results in a Gaussian distribution of mosquitoes centered around the position of the release $x_k \in \Omega$ for $k = 1, \dots, K$. Then, the feasible controls are of the form

$$u_K(x, x_1, \dots, x_K) = \sum_{k=1}^K m \exp\left(-\frac{\|x - x_k\|^2}{\sigma^2}\right), \quad (6.14)$$

where the constants m and σ are chosen such that $u_K(\cdot, x_1, \dots, x_K) \in \mathcal{V}_{T,C,M}$. In particular, we choose to saturate the constraint on the total number of mosquitoes released, i.e. we take $\int_{\Omega} u_K(x) dx = C$.

². The convexity results from the convexity of the square function combined with the linearity of $u_0 \mapsto p(T, \cdot)$. Furthermore,

$$\langle d^2 \hat{J}(u_0), h, h \rangle = \sum_{j=0}^{+\infty} e^{-2\lambda_j T} \left(\int_{\Omega} h(x) \phi_j(x) dx \right)^2 \geq 0$$

and vanishes if, and only if, $\int_{\Omega} h(x) \phi_j(x) dx = 0$ for all j , meaning that $h = 0$ since $(\phi_j)_{j \geq 1}$ is a Hilbert basis of $L^2(\Omega)$. The strict convexity of \hat{J} follows.

The goal is then to find the best position of the releases and the optimization problem becomes

$$\inf_{(x_1, \dots, x_K) \in \Omega^K} J_K(x_1, \dots, x_K) \quad \text{with } J_K(x_1, \dots, x_K) = \frac{1}{2} \int_{\Omega} (1 - p(T, x))^2 dx, \quad (\mathcal{P}_K)$$

where $p \in C^0([0, T], L^2(\Omega))$ is the unique solution of (6.7) with control $u_K(\cdot, x_1, \dots, x_K)$.

Remark. *Since Ω is a bounded domain in \mathbb{R}^2 , the question of the existence of a minimizer is trivial. But, the uniqueness is still a challenging problem.*

Proposition 6.3. *Let $(x_1, \dots, x_K) \in \Omega^K$. For $k \in \{1, \dots, K\}$, one has*

$$\frac{\partial J_K}{\partial x_k}(x_1, \dots, x_K) = \int_{\Omega} (G^{-1})'(u_K(x)) q(0, x) \frac{\partial u_K}{\partial x_k}(x, x_1, \dots, x_K) dx,$$

where q is the unique solution of the backward problem

$$\begin{cases} -\partial_t q(t, x) - \Delta q(t, x) - f'(p(t, x)) q(t, x) = 0, & (t, x) \in (0, T) \times \Omega, \\ \partial_n q(t, x) = 0, & (t, x) \in (0, T) \times \partial\Omega, \\ q(T, x) = p(T, x) - 1, & x \in \Omega. \end{cases}$$

Proof. It is an easy application of the chain rule. First, we notice that

$$J_K(x_1, \dots, x_K) = J(u_K(x, x_1, \dots, x_K)).$$

Next, using Proposition 6.2, we find thanks to the chain rule that for all $k \in \{1, \dots, K\}$

$$\begin{aligned} \nabla J_K(x_1, \dots, x_K) &= \langle dJ(u(x, x_1, \dots, x_K)), \nabla u(x, x_1, \dots, x_K) \rangle \\ &= \int_{\Omega} (G^{-1})'(u_K(x, x_1, \dots, x_K)) q(0, x) \nabla u_K(x, x_1, \dots, x_K) dx. \end{aligned}$$

We deduce the result from the last equality. \square

6.3.1 Numerical Resolution

We now present the computation of the numerical solution of (\mathcal{P}_K) . For this we use a direct method which consists in carrying out a discretization of Equation (6.7) and of the control in order to obtain a finite dimensional optimization problem with constraints. We can then compute an approximation of a local minimizer of (\mathcal{P}_K) with a numerical optimization solver. Our results were obtained with the finite element toolbox *FreeFem++* [71] which contains an implementation of the optimization routine *Ipopt* [112].

We therefore consider a finite element basis of functions $(\varphi_i)_i$ that allows us to discretize the control as $u_h(x, x_1, \dots, x_K) = \sum_i u_i \varphi_i(x)$ and the proportion of infected mosquitoes as $p_h(t, x) = \sum_i p_i(t) \varphi_i(x)$, the finite element approximation of the solution of the PDE (6.7) with initial condition $G^{-1}(u_h(x, x_1, \dots, x_K))$. The cost function can be computed with numerical integration as $J_h(x_1, \dots, x_K) = \int_{\Omega} (1 - p_h(T, x))^2 dx$. In addition, *Ipopt* requires the gradient of the cost function and thanks to Proposition 6.3 we have

$$\frac{\partial J_h}{\partial x_k} = \int_{\Omega} (G^{-1})'(u_h(x)) q_h(0, x) \frac{\partial u_h}{\partial x_k}(x, x_1, \dots, x_K) dx$$

where $q_h(0, x)$ is the finite element approximation of the solution of the backwards PDE.

Remark. Because of Proposition 6.1, we were interested in the case $M > \frac{C}{|\Omega|}$. In addition, we have fixed C such that the constant solution $u = \frac{C}{|\Omega|}$ leads to extinction (as T tends to $+\infty$) but there exists $R \in]0, \sqrt{\frac{C}{\pi M}}[$ such that the function $u(x) = M \times 1_{B(0,R)}(x)$ belongs to $\mathcal{V}_{T,C,M}$ and leads to invasion (as T tends to $+\infty$).

We now present numerical simulations for the parameters given in Table 6.1. The birth and death rates are given per day, whereas the unit of the carrying capacity is per m^2 and the diffusion coefficient is given per m^2 per day. The numerical values are taken from [3] and references therein. We consider a square domain of 1 hectare, a final time of 200 days and we set the total amount of mosquitoes released such that $C < G(\theta)|\Omega|$. In Figure 6.1 we show the control $u_K(\cdot, x_1, \dots, x_K)$ for $K = 3, 4, 5, 6$ releases and for each case the same total amount of mosquitoes is released. For the case of 6 releases we display in Figure 6.2 the time dynamics of the proportion of infected mosquitoes $p(t, \cdot)$. As expected, it leads to the total invasion of the domain.

Parameter	b_1	b_2	d_1	d_2	κ	D
Value	1.12	1.12	0.27	0.36	$6 \cdot 10^{-2}$	2.5

TABLE 6.1 – Model parameters

Our simulations seem to be very sensitive on the initial data given to *Ipopt*. Indeed, for most choices of initial datum in the optimization algorithm, the best solution provided by *Ipopt* has the "same shape" as the initial datum (more precisely, by assimilating the Gaussian releases to domains, the optimal solution seems to have the same number of connected components as the initialization). Heuristically, this suggest that the function J_K we aim at minimizing mainly penalizes a lot the final time (here $T = 200$) and does not take into account what occurs at intermediate times. Since most of the initial data lead to invasion with the set of parameter we considered (in other words, the global minimum of J_K is almost reached), the considered interior points algorithm (via the software *Ipopt*) tries some new configurations relatively close to the initial data to find out that it was already an "almost" global minimum of J_K . We have tried unsuccessfully to make tests with a lower final time, and the results are similar. In a future work, to avoid such bad boundary effects, we foresee to consider another functional J_K taking into account not only the final time but also several intermediate times.

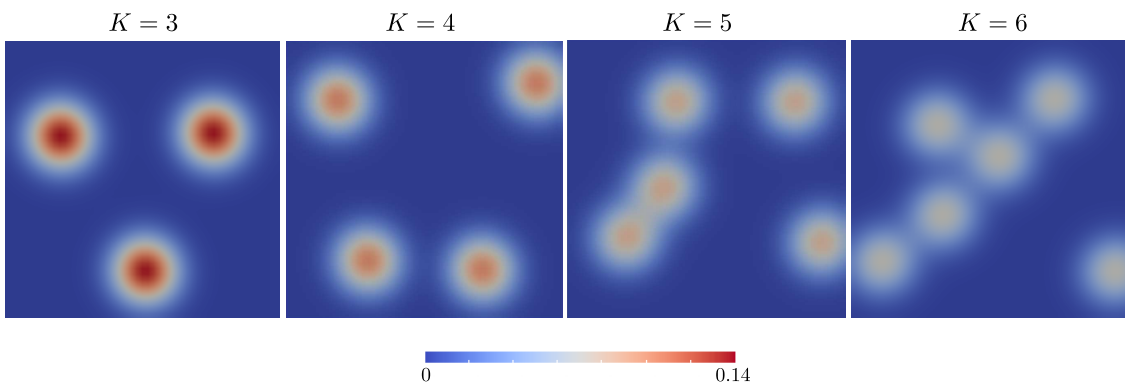


FIGURE 6.1 – $u_K(\cdot, x_1, \dots, x_K)$ for $K = 3, 4, 5, 6$ releases and $C = 0.017$.

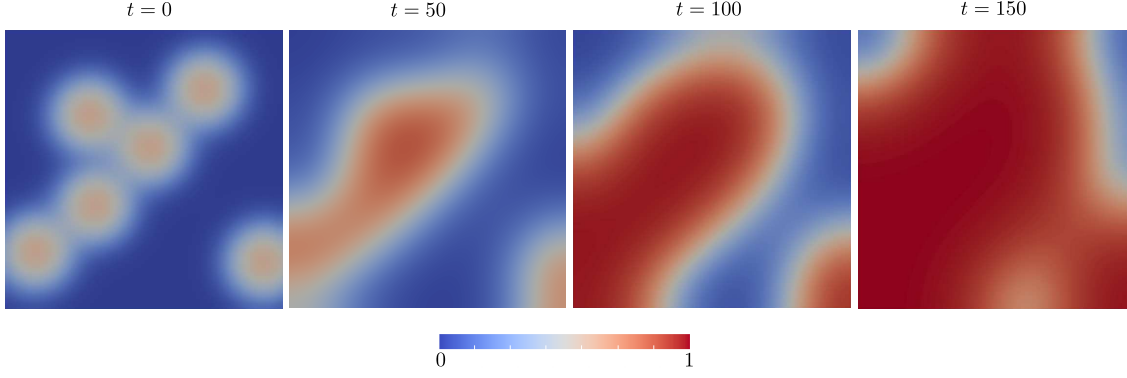


FIGURE 6.2 – $p(t, \cdot)$ for $K = 6$ releases, $t = 0, 50, 100, 150$ days and $C = 0.017$.

6.4 Conclusion

We investigate in this work the optimization of the release of *Wolbachia*-infected mosquitoes into a host population in the aim to replace the wild population by a *Wolbachia*-infected population unable to transmit several diseases to human. To conduct this study, we first reduce the optimal problem under investigation by assuming that the time distribution is given. Then we obtain existence of a minimum for this latter problem. Finally, reducing again the control problem by considering that the releases are modeled by Gaussian distributions, some numerical computations are performed.

Optimization strategies for release protocols of mosquitoes have been investigated by several authors [107, 40, 23]. However, in these papers, only the time optimization of the releases is investigated. Up to our knowledge, this work is the first attempt in optimizing spatially the releases, which is of great interest for experiments in the field. The preliminary results obtained in this paper should be continued. In particular, the optimality conditions for the system (\mathcal{P}_K) should be studied in a future work in the aim to find properties of the optimal solution. The numerical simulations should also be continued to have a better representation of what is observed in the field.

Appendix - Reduction of system (6.1)

For the sake of completeness and for reader facility, we explain briefly in this appendix how to reduce system (6.1) to system (6.2). We will not provide all the details of this reduction but only the main steps. We refer to [104] and [3, Section 2.3] for the interested reader. The starting point is to introduce a small parameter $0 < \varepsilon \ll 1$ modeling the ratio of the fertility on the death rate. Indeed for mosquitoes population, the fertility is large compared to death rates. System (6.1) reads then

$$\begin{cases} \partial_t n_1^\varepsilon - D\Delta n_1^\varepsilon = \frac{b_1}{\varepsilon} n_1^\varepsilon \left(1 - s_h \frac{n_2^\varepsilon}{n_1^\varepsilon + n_2^\varepsilon}\right) \left(1 - \frac{n_1^\varepsilon + n_2^\varepsilon}{\kappa}\right) - d_1 n_1^\varepsilon, \\ \partial_t n_2^\varepsilon - D\Delta n_2^\varepsilon = \frac{b_2}{\varepsilon} n_2^\varepsilon \left(1 - \frac{n_1^\varepsilon + n_2^\varepsilon}{\kappa}\right) - d_2 n_2^\varepsilon + u. \end{cases}$$

As $\varepsilon \rightarrow 0$, we expect from this system that $n_1^\varepsilon + n_2^\varepsilon \rightarrow \kappa$. Hence we introduce the quantity $n^\varepsilon = \frac{1}{\varepsilon} \left(1 - \frac{n_1^\varepsilon + n_2^\varepsilon}{\kappa}\right)$ and denote $p^\varepsilon = \frac{n_2^\varepsilon}{n_1^\varepsilon + n_2^\varepsilon}$ the proportion of infected mosquitoes. From

straightforward computations, we deduce the system satisfied by $(n^\varepsilon, p^\varepsilon)$:

$$\begin{cases} \partial_t n^\varepsilon - D\Delta n^\varepsilon = \frac{1}{\varepsilon} \left((1 - \varepsilon n^\varepsilon) a(p^\varepsilon) (Z(p^\varepsilon) - n^\varepsilon) - \frac{u}{\kappa} \right), \\ \partial_t p^\varepsilon - D\Delta p^\varepsilon + \frac{2\varepsilon}{1 - \varepsilon n^\varepsilon} \nabla p^\varepsilon \cdot \nabla n^\varepsilon = p^\varepsilon (1 - p^\varepsilon) [n^\varepsilon (b_2 - b_1 - b_1 s_h p^\varepsilon) + d_1 - d_2] + \frac{u(1 - p^\varepsilon)}{\kappa(1 - \varepsilon n^\varepsilon)}, \end{cases}$$

where we use the notations $a(p) = b_1(1 - p)(1 - s_h p) + b_2 p > 0$ and $Z(p) = \frac{d_1(1-p)+d_2p}{a(p)} > 0$. Assuming that the sequences $(n^\varepsilon)_\varepsilon$ and $(p^\varepsilon)_\varepsilon$ admit limits denoted n and p respectively, we deduce from the first equation that, formally,

$$n = Z(p) - \frac{u}{\kappa} a(p). \quad (6.15)$$

Passing into the limit into the equation satisfied by p , we get

$$\partial_t p - D\Delta p = p(1 - p)(n(b_2 - b_1(1 - s_h p)) + d_1 - d_2) + \frac{u}{\kappa}(1 - p),$$

Injecting the expression of n (6.15) into this latter equation, we recover the equation

$$\partial_t p - D\Delta p = f(p) + ug(p),$$

with f and g defined in (6.3)

Chapter 7

Conclusions and perspectives

A. Conclusion

Dans cette thèse nous nous sommes intéressés à une étude mathématique de problèmes issus d'écologie et biologie évolutive. En effet, nous avons étudié l'influence d'une dispersion non-locale pour une espèce biologique vivant dans un domaine fragmenté. Plus précisément, nous avons établi un critère de survie pour une espèce biologique dont la dynamique est régie par une équation de Fisher-KPP fractionnaire dans un domaine fragmenté avec des conditions extérieures de Dirichlet. Ce critère repose sur le signe de la valeur propre principale de sous-ensembles inclus dans le domaine. De plus, nous avons démontré l'existence et l'unicité de la solution positive et bornée de l'équation de Fisher-KPP fractionnaire à l'équilibre. Dans le cas d'un domaine périodique et fragmenté, nous avons démontré qu'un phénomène d'invasion à vitesse exponentielle a lieu.

Enfin, dans une seconde partie, nous avons considéré une espèce biologique organisée phénotypiquement et sujette à de petites mutations du phénotype et à une dispersion spatiale à la fois locale et non-locale. Nous avons montré l'apparition de traits dominants lorsque les mutations deviennent petites.

B. Perspectives

Je présente ici les prolongements naturels et les questions qui sont apparus à l'issue de ce travail. Dans le titre de chacune des perspectives, je mentionnerai le chapitre impliqué.

Etendre les résultats d'existence et d'unicité de la solution stationnaire de l'équation de Fisher-KPP fractionnaire au cadre de la valeur propre principale généralisée (Chapitre 4).

Une quantité qui apparaît naturelle à considérer est la valeur propre principale généralisée introduite par Berestycki et Rossi (voir [21] et [12]). Sous le spectre de cette valeur propre généralisée, il est naturel de se demander quelles sont les hypothèses minimales pour établir un résultat d'existence et d'unicité via des estimations précises comme établi dans le Chapitre 4.

Une condition nécessaire est naturellement de considérer une valeur propre principale généralisée $\lambda_\alpha(\Omega)$ strictement négative. La question est : est-ce suffisant ? Dans le chapitre 4,

l'hypothèse de boule intérieure et extérieure a été utilisée à maintes reprises afin d'obtenir des estimations uniformes au bord du domaine. Peut-on se passer d'une telle hypothèse ? De même, peut-on se passer de l'hypothèse qui paraît assez technique traitant des distances entre les *clusters* de Ω_+ et entre Ω_+ et Ω_- ? Dans le cas où les réponses sont négatives, est-ce possible d'exhiber un contre-exemple de domaine qui admet deux solutions stationnaires distinctes ?

Etendre le résultat de continuité de la valeur propre du Laplacien fractionnaire d'ordre α par rapport à la distance entre deux sous-domaines en 0 pour $\alpha \geq \frac{1}{2}$ et la dimension $d \geq 2$ (Chapitre 4).

Toujours dans le chapitre 3, nous avons établi la continuité de l'application

$$(\mu \in \mathbb{R}^+ \mapsto \lambda_\alpha([a, -\mu[\cup]\mu, b]))$$

jusqu'à $\mu = 0$ (avec $a < 0 < b$). Pour $\alpha < \frac{1}{2}$ la preuve s'appuie sur un argument de compacité et de densité des fonctions à trace nulle en $x = 0$. Un tel argument se généralise aisément à la dimension supérieure. En revanche pour $\alpha \geq \frac{1}{2}$, il a fallu montrer que le point à l'interface $x = 0$, est une singularité effaçable. L'avantage de la dimension 1 est qu'il n'y a qu'une façon d'obtenir une telle interface. À partir de la dimension 2, l'interface obtenue est moins claire. Lorsque celle-ci est un hyper-plan (ou l'image par un difféomorphisme d'un hyperplan), je m'attends à ce que l'argument de prolongement par continuité soit robuste. Il y a cependant des détails techniques qui apparaissent. En effet, considérer une interface ayant une forme d'hyper-plan en dimension $d \geq 2$ fait intervenir des « coins » dans les sous-domaines (i.e. les sous-domaines sont Lipschitziens et non C^1). A priori, ces points nécessitent un traitement particulier.

En revanche, je ne suis pas sûr que le résultat reste vrai lorsque le recollement est singulier (par exemple en dimension deux, deux disques qui se rapprochent pour obtenir $\Omega = B(-1, 1) \cup B(1, 1)$). Il ne m'apparaît pas alors évident que la méthode soit robuste.

Pousser plus loin l'étude de la dynamique de $\lambda_\alpha([a, -\mu[\cup]\mu, b])$ lorsque $\mu \rightarrow 0$ et $\alpha \rightarrow 1$ avec $a < 0 < b$ (Chapitre 4).

L'idée serait de trouver deux fonctions ($\mu \in \mathbb{R}^+ \rightarrow \underline{\alpha}(\mu)$) et ($\mu \in \mathbb{R}^+ \rightarrow \bar{\alpha}(\mu)$) telles que pour toutes suites $(\mu_n, \alpha_n) \rightarrow (0, 1)$ avec $\alpha_n < \underline{\alpha}(\mu_n)$, on ait

$$\lambda_{\alpha_n}([a, -\mu_n[\cup]\mu_n, b]) \xrightarrow{n \rightarrow +\infty} \lambda_1([a, b]),$$

tandis que pour toutes suites $(\mu'_n, \alpha'_n) \rightarrow (0, 1)$ avec $\alpha'_n > \bar{\alpha}(\mu'_n)$, on ait

$$\lambda_{\alpha'_n}([a, -\mu'_n[\cup]\mu'_n, b]) \xrightarrow{n \rightarrow +\infty} \min(\lambda_1([a, 0]), \lambda_1([0, b])).$$

Le but de l'étude serait de donner des propriétés qualitatives des fonctions $\underline{\alpha}$ et $\bar{\alpha}$.

Etudier les équilibres et les phénomènes d'invasions dans un modèle fractionnaire impliquant une non-linéarité bistable à la place d'une non linéarité du type Fisher-KPP (Chapitres 3 et 4).

Il a été démontré (voir [86]) qu'une non-linéarité bistable dans une équation impliquant une diffusion fractionnaire dans un environnement homogène n'implique pas un phénomène d'invasion avec une accélération exponentielle. Un phénomène d'invasion a toujours lieu mais la vitesse de ce dernier est constant. Il est ainsi naturel de se demander si ce résultat reste vrai dans le cadre d'un environnement fragmenté périodique.

Je me demande également si les résultats d'unicité de la solution stationnaire démontrés dans le chapitre 4 pour des équations de Fisher-KPP dans des environnements fragmentés peuvent se retrouver dans un cadre d'équations fractionnaires bistables. Etant donné que dans le cas d'équations de Fisher-KPP fractionnaire, nous avons démontré que la densité de population à l'équilibre peut être aussi petite que nous le souhaitons (pour un environnement bien choisi), ainsi dans un cadre bistable, les phénomènes de type *source-puit* pourraient être renforcés.

Avoir une meilleure compréhension de la distribution des traits optimaux (Chapitre 5).

Nous avons établi dans le chapitre 5 la convergence de n_ε vers une mesure non-triviale qui « charge » les traits optimaux. Les simulations numériques laissent à penser que la distribution de ces traits optimaux est discrète. Il serait intéressant de retrouver ce résultat théoriquement. Ainsi, la limite n serait une somme de masses de Dirac. Ce résultat dépendrait évidemment du taux de croissance R . Je me demande donc quelles sont les hypothèses à imposer pour avoir émergence de traits optimaux discrets.

Une première étape peut être de montrer que les zéros de la fonction limite u sont discrets. Si cela est vérifié alors le résultat s'ensuit car n est non-triviale et se concentre sur les zéros de la fonction limite u . Les articles de Mirrahimi [88] et de Perthame et Souganidis [95] adoptent une telle approche pour montrer l'émergence de traits optimaux.

Démontrer l'émergence de traits optimaux à mesure que les mutations deviennent petites en considérant le Laplacien fractionnaire à la place de l'opérateur de diffusion $-\partial_{xx} + L$ (Chapitre 5).

Le but de cette dernière perspective serait de remplacer l'opérateur $-\partial_{xx} + L$ par le Laplacien fractionnaire qui d'un point de vue modélisation prend en compte les déplacements locaux et non-locaux. Un tel changement ne peut se faire sans changer les conditions aux bords. Ainsi, il faudrait remplacer les conditions aux bords en espace de Neumann par des conditions aux bords de Dirichlet. Je m'attends à ce que les résultats sur la valeur propre principale du Laplacien fractionnaire dans un domaine fragmenté borné avec des conditions extérieures de Dirichlet établis dans le chapitre 4 soient utiles ici. De plus, un tel changement d'opérateur induit de nouvelles difficultés techniques : la méthode de Bernstein pour établir les bornes Lipschitz n'est pas utilisable en l'état. Une telle méthode a bien été développée pour des solutions de viscosité à une équation elliptique impliquant des opérateurs fractionnaires (voir [7] par Barles, Ley et Topp) mais l'étude proposée ici est en dehors du cadre de [7]. Une première étape serait de conserver l'opérateur de diffusion spatiale $-\partial_{xx} + L$ et de remplacer les conditions aux bords de Neumann par des conditions aux bords de Dirichlet.

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Modèles de diffusion non-conventionnelle en écologie et biologie évolutive impliquant des environnements fragmentés

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Résumé

Dans cette thèse nous nous intéressons à une étude mathématique qualitative de problèmes issus d'écologie et de biologie évolutive. Nous étudions l'influence d'une dispersion non-locale pour une espèce biologique vivant dans un environnement fragmenté. Plus précisément, dans une première partie, nous établissons un critère de survie pour une espèce biologique dont la dynamique est régie par une équation de Fisher-KPP fractionnaire dans un domaine fragmenté avec des conditions extérieures de Dirichlet. Ce critère repose sur le signe de la valeur propre principale de sous ensembles inclus dans le domaine. De plus, nous démontrons un résultat d'existence et d'unicité de la solution stationnaire d'une équation de Fisher-KPP dans des domaines fragmentés généraux appartenant à la classe des solutions positives, bornées et non-triviales. Dans le cas particulier d'un domaine périodique et fragmenté, nous établissons l'existence d'un phénomène d'invasion à vitesse exponentielle. Enfin, dans une seconde partie, nous considérons un modèle traitant d'une espèce biologique organisée phénotypiquement vivant dans un environnement fragmenté. Cette espèce est sujette à des mutations à petits effets phénotypiques ainsi qu'à une dispersion spatiale à la fois locale et non-locale. Nous démontrons l'émergence de traits phénotypiques dominants lorsque les mutations ont de petits effets.

Mots clefs : Biologie évolutive, Noyau de diffusion non-conventionnelle, Phénomène d'accélération, Analyse asymptotique, Equations integro-différentielles paraboliques

Abstract

In this thesis, we are interested in a qualitative mathematical study of problems from ecology and evolutionary biology. We study the influence of a non-local dispersion for a biological species living in a patchy environment. More precisely, we first establish a criterion whose ensures the survival of a biological species which dynamics are driven by a fractional Fisher-KPP equation in a fragmented domain with Dirichlet exterior conditions. This criterion relies on the sign of the principal eigenvalue of subsets included in the fragmented domain. Moreover, we demonstrate an existence and uniqueness result of the stationary state of a Fisher-KPP equation in general patchy domains belonging to the class of non-negative, bounded and non-trivial solutions. In the particular case of a periodic and patchy domain, we establish the existence of invasion phenomena with exponential speed. Finally, we consider a model dealing with a phenotypically structured biological species living in a patchy environment. This species is subject to small mutations of the phenotype and to local and non-local spatial dispersion. We demonstrate the emergence of phenotypical dominant traits as the mutations become small.

Key words : Evolutionary biology, Unconventional diffusion kernel, Acceleration phenomena, Asymptotic analysis, Parabolic integro-differential equations