Positivity of direct images and projective varieties with nonnegative curvature

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under the supervision of
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Ma troisième maxime était de tâcher toujours plutôt à me vaincre que la fortune, et à changer mes désirs que l'ordre du monde : et généralement de m'accoutumer à croire qu'*il n'y a rien qui soit entièrement en notre pouvoir que nos pensées*.

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一切有为法，如梦幻泡影，如露亦如电，应作如是观。

Sūtra du Diamant

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Contents

Convention and Notations

Throughout this thesis, a complex variety means a reduced irreducible complex analytic space. A(n) (analytic) fibre space is a proper morphism between complex varieties whose fibres are connected. An analytic fibre space is called an algebraic fibre space if it is also a projective morphism. An analytic fibre space $f : X \to Y$ is called a Kähler fibre space if locally over Y, X is a Kähler variety in the sense of [HP16, Definition 2.2]. A $\mathbb{Q}$-line bundle on a complex variety X means an element of $\mathrm{Pic}(X) \otimes \mathbb{Q}$ (c.f. also [Var10, Lecture 1, §8.3, Definition 8.6]) and we use "+" to denote the tensor product of two $\mathbb{Q}$-line bundles (and mix this notation with the addition of $\mathbb{Q}$-divisors). Over a complex variety, a "(analytic) Zariski open subset" signifies an open subset of the variety whose complement is a closed analytic subspace.

Dans les parties en français, un espace analytique complexe, sauf mentionné explicitement, est toujours supposé d'être irréductible et réduit, donc correspond à « complex variety » en anglais. Une variété complexe ou kählérienne est toujours supposée d'être lisse, c-à-d., correspondent aux « complex manifold » et « Kähler manifold » en anglais respectivement. En revanche, une variété projective n'est pas nécessairement lisse, c-à-d., correspond à "projective variety" en anglais. Une fibration (analytique) est un morphisme propre entre espaces analytiques dont les fibres sont connexes, c-à-d., correspond à « analytic fibre space » en anglais. Une fibration analytique est dite *algébrique* si elle est aussi un morphisme projectif. Une fibration analytique $f : X \to Y$ est dite *kählérienne* si localement au-dessus de Y, X est est un espace kählérien au sens de [HP16, Definition 2.2]. Un $\mathbb{Q}$-fibré en droites sur un espace analytique complexe signifie un élément de $\mathrm{Pic}(X) \otimes \mathbb{Q}$ (c.f. aussi [Var10, Lecture 1, §8.3, Definition 8.6]) et l'on utilise "+" pour désigner le produit tensoriel de deux $\mathbb{Q}$-fibré en droites (et l'on mélange cette notation avec l'addition des $\mathbb{Q}$-diviseurs). Sur un espace analytique complexe, un « ouvert de Zariski (analytique) » signifie un ouvert de l'espace dont le complémentaire est un sous-espace complexe fermé (non-nécessairement irréductible).

Introduction

Let k be a algebraically closed filed, one of the central problems in algebraic geometry to classify all the algebraic varieties over k up to isomorphism; when $k = \mathbb{C}$, one can also consider more generally the classification problem of complex analytic spaces (especially ones in the Fujiki class $\mathscr{C}$). This study is initiated, on one hand, by Bernhard Riemann, Henri Poincaré, etc. in their works on the uniformization of Riemann surfaces (algebraic curves) from the analytic point of view; and on the other hand, by the Italian school (Guido Castelnuovo, Federigo Enriques, Francesco Severi, etc.) in the works on minimal models of algebraic surfaces from the algebraic point of view. In the framework of modern mathematics, their ideas are further developed, and a lot of achievements have been made in the last century by the remarkable works of Kunihiko Kodaira, David Mumford, Shigeru Iitaka, Kenji Ueno, Shigefumi Mori, Eckart Viehweg, Yujiro Kawamata, János Kollár, Vyacheslav Shokurov, etc. As a fruit of these works, the principle of the classification problem is established and the problem can be divided into two aspects, namely, the aspect of birational / bimeromorphic classification and the aspect of the construction of (good compactification of) moduli spaces.

This thesis concentrates mainly on the first aspect of the classification problem, in which great progress has recently been made by the works [BCHM10; BDPP13]. In [BCHM10] the Minimal Model Program (abbr. MMP) is almost established by following the ideas of Vyacheslav Shokurov, while [BDPP13] makes a significant progress towards the Abundance by describing the positive cone of pseudoeffective divisors. Roughly speaking, by combining these results, we have that smooth (or mildly singular) projective varieties are divided into two (birationally stable) classes:

- varieties with pseudoeffective canonical divisors, which are shown to reach a minimal model (that is, a mildly singular variety with nef canonical divisor) under the MMP;
- uniruled varieties, which are shown to reach a Mori fibre space (a fibre space whose general fibre is a Fano variety of Picard number 1) under the MMP.

The general philosophy in the study of minimal varieties / uniruled varieties is to study the canonical fibrations associated to them, which reduces the study to the study of the base and of the general fibre. The main results of this thesis are developed respectively along these two major lines, as is precised below.

For minimal varieties, the most important associated canonical fibration is the Iitaka-Kodaira fibration defined by a sufficiently high multiple of the canonical divisor, whose general fibre is of Kodaira dimension 0 and which is expected, by the Abundance conjecture, to be a everywhere defined fibre space (instead of a meromorphic/rational mapping) onto a canonically polarized variety (a canonical model). Although the Abundance conjecture is largely open, much progress has been made in the proof of an important corollary of it, known as the Iitaka conjecture $C_{n,m}$, which predicts the superadditivity of the Kodaira dimension with respect to algebraic fibre spaces:

Conjecture 1 (Iitaka Conjecture $C_{n,m}$, [Uen75, §11.5, Conjecture C_n, pp. 132-133]). *Let $f : X \to Y$ be an algebraic fibre space between projective varieties with $\dim X = n$ and $\dim Y = m$, and let F be the general fibre of f. Then we have*

$$\kappa(X) \geqslant \kappa(Y) + \kappa(F).$$

Recall that the Kodaira dimension $\kappa(X)$ of a complex variety X is defined to be the dimension of the image of the aforementioned Iitaka-Kodaira fibration, or equivalently, the unique integer $\kappa \in \{-\infty, 0, 1, \cdots, \dim X\}$ such that there are constants $C_1, C_2 > 0$ independent of m satisfying

$$C_1 \cdot m^\kappa \leqslant \mathrm{h}^0(X, K_X^{\otimes m}) \leqslant C_2 \cdot m^\kappa,$$

for m sufficiently large and divisible. Recently an important special case of the $C_{n,m}$ is proved by Junyan Cao and Mihai Păun in [CP17]. Although a large part of MMP is not known for Kähler varieties, by using in depth the recent developments of complex analytic methods, especially, the Ohsawa-Takegoshi type extension theorem with optimal estimate obtained by Qi'an Guan and Xiangyu Zhou in [GZ15a, Theorem 3.21] and generalized by Cao in [Cao17, Theorem 1.1] (c.f. [ZZ18] for an alternative proof), I am able to extend the main result of [CP17] (and also a main result in [Vie83]) to the Kähler case, i.e. to prove the following:

Theorem A. *Let $f : X \to Y$ be a fibre space between compact Kähler manifolds with general fibre denoted by F. And let Δ be an effective $\mathbb{Q}$-divisor on X such that (X, Δ) is Kawamata log terminal (abbr. klt). Suppose that one of the following conditions is satisfied:*

(I) *there is an integer $m > 0$ such that $m\Delta$ is an integral divisor and that the determinant line bundle $\det f_*(K_{X/Y}^{\otimes m} \otimes \mathscr{O}_X(m\Delta))$ is big on Y;*

(II) *Y is a complex torus.*

Then

$$\kappa(X, K_X + \Delta) \geqslant \kappa(F, K_F + \Delta_F) + \kappa(Y),$$

where $\Delta_F := \Delta|_F$.

The proof of Theorem A relies on a positivity result for direct images of twisted relative pluricanonical bundles ([DWZZ18, Theorem 9.3], c.f. §2.2.2 for an alternative proof) and a Green-Lazarsfeld-Simpson type result on the cohomology jumping loci ([Wan19, Theorem D]). In [DWZZ18] a more general result on positivity for L^p-Finsler metrics on direct images of twisted relative pluricanonical bundles is established by using a new characterization of psh functions; in [Wan19] I give an alternative proof for the L^2 Hermitian metric, based on the Ohsawa-Takegoshi extension theorem with optimal estimate obtained by Qi'an Guan and Xiangyu Zhou in [GZ15a] and generalized by Junyan Cao in [Cao17] (an alternative proof is given in [ZZ18]). Let me recall that: for a vector bundle E over a complex manifold, a singular Hermitian metric on E is given by a measurable family of Hermitian functions on each fibre of E which is non-singular almost everywhere; on the direct image of the (twisted) relative canonical bundle, there is a natural L^2-Hermitian metric, which is defined by the fibrewise integrals of (twisted) differential n-forms (n is the relative dimension of the fibre space).

In the other direction, i.e. the study of uniruled varieties, instead of studying the Iitaka-Kodaira fibrations (which do not provide any information in the uniruled case), one studies the Albanese maps and the maximal rationally connected (MRC) fibrations. A general philosophy, inspired by the fundamental work of Shigefumi Mori [Mor79], is that when the anticanonical bundle or the tangent bundle of a variety admits certain positivity, these canonical fibrations should have a rigid structure (typically, being a locally

constant fibration). For a projective variety with log canonical (lc) singularities, if the anticanonical divisor is ample ($\mathbb{Q}$-Fano case) the two aforementioned fibrations are both trivial by the classical works of Kollár-Mori-Miyaoka [KMM92] and of Frédéric Campana [Cam92] (and by Qi Zhang in [Zha05] for the singular case); it is then natural to ask the same question for varieties with nef anticanonical divisors. Recall that a Cartier divisor or line bundle on a projective variety is called nef if its intersection number with any curve is nonnegative, or equivalently, if it admits smooth Hermitian metrics whose curvature forms have arbitrarily small negative parts (thus we can extend this notion to any compact complex manifold, c.f. [DPS94]). In the smooth case, the study of the Albanese maps and of the MRC fibrations is accomplished by [Cao19; CH19], in these works it is proved that for a smooth projective variety with nef anticanonical bundle the aforementioned two maps are (everywhere defined) locally constant fibrations, which implies that smooth projective varieties with nef anticanonical bundles admit Beauville-Bogomolov type decomposition: when passing to the universal cover they can be decomposed into a product of $\mathbb{C}^q$, a Calabi-Yau variety, a hyperkähler variety and a rationally connected variety (the first three components are given by the classical Beauville-Bogomolov decomposition). By the philosophy of the MMP, it is intended to generalize this structure theorem to the singular case, i.e. the following conjecture:

Conjecture 2. *Let X be a projective varieties with klt singularities and suppose that the anticanonical divisor $-K_X$ of X is nef. Then up to replacing X by a (finite) quasi-étale cover, the Albanese map and the MRC fibration of X induce a decomposition of the universal cover $\tilde{X}$ of X into a product*

$$\tilde{X} \simeq \mathbb{C}^q \times Z \times F,$$

where q is the augmented irregularity of X, Z is a klt projective variety with trivial canonical divisor and F is a rationally connected variety.

Similar to the smooth case, by applying the klt Beauville-Bogomolov decomposition theorem established by the successive works [GKP16; Dru18a; GGK19; HP19], the variety Z in the decomposition above can be further decomposed as a product of Calabi-Yau varieties and of irreducible symplectic varieties. However, different from the case of varieties with numerically trivial canonical divisor, even in the smooth case one cannot in general get a product structure up to finite (quasi-)étale cover for varieties with nef anticanonical divisor due to the appearance of the rationally connected factor, e.g. there are ruled surfaces over an elliptic curve which cannot split into a product of the elliptic curve and $\mathbb{P}^1$ up to finite étale cover(c.f. [Dru18b, Example 9.1, Example 11.2], [EIM20, Example 6.2]).

In this thesis the Conjecture 2 is partially established by generalizing the main results of [Cao19] and [CH19] to the klt singular case. Recall that a normal projective variety X is called of Fano type (resp. semi-Fano type), if there is an effective $\mathbb{Q}$-divisor Δ on X such that (X, Δ) is a klt pair and that the twisted anticanonical divisor $-(K_X + \Delta)$ is ample (resp. nef), c.f. [PS09, Definition 2.5, Lemma-Definition 2.6]. The principal results are the following:

Theorem B. *Let X be a normal projective variety of semi-Fano type. Then the Albanese map $\mathrm{alb}_X : X \dashrightarrow \mathrm{Alb}_X$ is an everywhere defined locally constant fibration, i.e. alb_X is an analytic fibre bundle with connected fibres such that X is equal to the product of the universal cover of Alb_X by the fibre of alb_X quotient by a diagonal action of $\pi_1(\mathrm{Alb}_X)$.*

Theorem C. *Let X be a normal projective variety of semi-Fano type with simply connected smooth locus X_{reg}. Then the MRC fibration of X induces a decomposition of X into a product $F \times Z$ with F rationally connected and $K_Z \sim 0$.*

Let us remark that the local triviality (also known as the "isotriviality", especially in algebraic geometry) of the Albanese map of X is obtained in the work of Zsolt Patakafalvi and Maciej Zdanowicz [PZ19, Corollary 1.17 (Corollary A.14)] under the additional assumption that X is $\mathbb{Q}$-factorial. The strategy in their paper is to show that every (closed) fibre is isomorphic by proving the numerical flatness of the direct images on every complete intersection curve. In this thesis, we can use analytic methods to prove more generally the global numerical flatness of the direct images, and thus can obtain the stronger result that the Albanese map is not only locally trivial but also a locally constant fibration.

The basic idea of the proof of this theorem is the same as [Cao19; CH19]: study the positivity of the direct images of powers of a relative very ample line bundle, and prove that up to a twisting they are numerically flat. For the Albanses map, we can directly conclude since it is everywhere defined; as for the MRC fibration, this can only provide us with a decomposition of the tangent sheaf into algebraically integrable foliations. The problem is that these foliations are singular foliations on a singular variety, then we cannot apply the general theory of (regular) foliations; to overcome this difficulty, a key observation is that the decomposition implies that the foliations are weakly regular in the sense of [Dru18b].

Methodology

In this section, let us briefly summarize the methods and main tools applied in the study of the classification problem, especially in this thesis. In order to study the classification of complex varieties, one needs both algebraic and complex analytic methods. The technical core of algebraic methods is the Hodge Theory, whose modern version has been totally rewritten by Pierre Deligne in terms of homological algebra and largely developed by Morihiko Saito from the viewpoin t of mixed Hodge modules. All the vanishing theorems and positivity results of direct images can be deduced from the Hodge Theory. The application of analytic methods to classification problem is initiated by the works of Yum-Tong Siu, Shing-Tung Yau and Jean-Pierre Demailly. The central idea is to study the singular metrics on vector bundles as well as the multiplier ideals associated to them, e.g. the Hodge metric on the direct image of the relative canonical bundle and the (singular) Kähler-Einstein metrics on (the tangent bundle of the regular locus of) complex varieties. By introducing the (semi)positivity notion for singular Hermitian metrics on vector bundles, we can formulate and prove more general (Nadel) vanishing theorems and more general (metric version of) positivity results for direct images (c.f. [PT18; HPS18; DWZZ18]), and thus in many cases the analytic methods can totally replace the algebraic ones. The proof of these results relies on the (variants of) Ohsawa-Takegoshi type extension theorems with optimal estimates, c.f. [GZ15b; Cao17]. In order to obtain more refined structure theorems for complex varieties, a very important ingredient is the foliation theory, which provides a path towards uniformization type results. Nevertheless the classical results on foliations is not sufficient for the classification problem since by the philosophy of MMP one needs to treat mildly singular varieties, in consequence much effort has been made for the development of the theory of singular foliation over (mildly) singular varieties. A paradigm of the application of this theory is the proof of the klt version of the Beauville-Bogomolov decomposition theorem as mentioned above, especially the work of Stéphane Druel in [Dru18a; Dru18b].

On the Iitaka conjecture $C_{n,m}$ for Kähler fibre spaces

Let X be a compact complex variety and let L be a ($\mathbb{Q}$-)line bundle on X, recall that the Iitaka(-Kodaira) dimension of L, denoted by $\kappa(X,L)$, is the maximum of the dimension of the image of $\bar{X}$ via the meromorphic mapping $\bar{X} \dashrightarrow \mathbb{P}\mathrm{H}^0(\bar{X}, \nu^*L^{\otimes m})$ defined by the linear series $\left|\nu^*L^{\otimes m}\right|$ for $m \in \mathbb{Z}_{>0}$ sufficiently large and divisible (if $\left|\nu^*L^{\otimes m}\right| = \varnothing$ for all $m \in \mathbb{Z}_{>0}$ then we say that $\kappa(X,L) = -\infty$), where $\nu : \bar{X} \to X$ is the normalization of X. In particular, the Kodaira dimension of a compact complex variety X, denoted by $\kappa(X)$, is the Iitaka-Kodaira dimension of the canonical bundle of any smooth model of X, and $\kappa(X)$ is known to be the most important bimeromorphic invariant of X.

The Iitaka conjecture $C_{n,m}$, in its original form, predicts the superadditivity of the Kodaira dimension with respect to algebraic fibre spaces (c.f. [Uen75, §11.5, Conjecture C_n, pp. 132-133]); more precisely, for $f : X \to Y$ a fibre space between normal projective varieties whose general fibre is denoted by F, the conjecture $C_{n,m}$ predicts that

$$\kappa(X) \geqslant \kappa(F) + \kappa(Y).$$

This conjecture is intimately related to the study of birational classification of complex algebraic varieties (the Minimal Model Program). According to the philosophy of MMP, the conjecture $C_{n,m}$ is naturally generalized to the log version, usually called $C^{\log}_{n,m}$; Moreover, Frédéric Campana further generalize $C_{n,m}$ to the setting of geometric orbifolds, called $C^{\mathrm{orb}}_{n,m}$, which is formulated in [Cam04, Conjecture 4.1] and in [Cam09, Conjecture 6.1]. In addition, by taking into consideration the variation of the fibre space, Eckart Viehweg also propose a stronger version of the $C_{n,m}$, called $C^{+}_{n,m}$, which plays a role in the study of moduli spaces.

As shown in [KMM87] (resp. [Kaw85]), the conjecture $C_{n,m}$ (resp. $C^{+}_{n,m}$) can be regarded as the consequence of the famous Minimal Model Conjecture and the Abundance Conjecture; moreover, in virtue of the superadditivity of Nakayama's numerical dimensions (c.f. [Nak04, §V.4.a, 4.1.Theorem(1), pp. 220-221]), $C^{\log}_{n,m}$ follows from the so-called generalized Abundance Conjecture (for $\mathbb{Q}$-divisors), c.f. [Fuj17, Remark 1.8].

Although initially stated for projective varieties, the conjecture $C_{n,m}$, as well as the MMP and the Abundance, are considered as still hold for complex varieties in the Fujiki class $\mathscr{C}$ (c.f. [Fuj78; Cam04; HP16; CHP16; Fuj17]); nevertheless they do not hold true in general for non-Kähler compact complex varieties, c.f. [Uen75, Remark 15.3, p. 187] for a counterexample. As mentioned above, one of the main results of this thesis is to prove the klt Kähler version of $C^{\log}_{n,m}$ in two important special cases and further generalize the second one to the geometric orbifold setting.

The conjecture $C_{n,m}$ is already known in lower dimensions (for example: $\dim X \leqslant 6$, [Bir09]; $\dim Y = 1$, [Fuj78; Kaw82]; $\dim Y = 2$, [Kaw81; Vie83; Cao18]). As for higher dimensions, it has been proved, by using the method of positivity of direct images developed by Phillip Griffiths, Takao Fujita, Yujiro Kawamata, Eckart Viehweg, Bo Berndtsson, Mihai Păun, Shigeharu Takayama, etc., in the following three important cases:

1. Y is of general type (Kawamata [Kaw81]; Viehweg [Vie83]; Campana [Cam04], in the geometric orbifold setting);
2. there exists an integer $m > 0$ such that $\det f_*(K^{\otimes m}_{X/Y})$ is big on Y, i.e. $\kappa(Y, \det f_*(K^{\otimes m}_{X/Y})) = \dim Y$ (Viehweg [Vie83]);
3. Y is an Abelian variety (Cao & Păun [CP17], the klt version).

In this thesis I treat the Kähler (log or orbifold) version of the above three cases.

Theorem A(I) generalizes [Vie83, Theorem II], which is intimately related to $C^{+}_{n,m}$ (c.f. [Vie83] for more details; this thesis, however, will not pursue in this direction); while

Part (II) generalizes [CP17, Theorem 1.1] and it will be further generalized to the setting of geometric orbifolds, in other word, we will prove $C^{\mathrm{orb}}_{n,m}$ for f when Y is a complex torus.

Moreover, by following the same strategy of the proof of Part (I), we recover the result that klt Kähler version of $C^{\log}_{n,m}$ holds for $f : (X, \Delta) \to Y$ when Y is of general type, which generalizes [Kaw81, Theorem 3]; we also further generalize this result to the geometric orbifold setting. Let us remark that the general (log canonical) version of $C^{\mathrm{orb}}_{n,m}$ for Y of general type (in the orbifold sense) has already been proved in [Cam04]; the proof is based on a weak positivity result for direct images of twisted pluricanonical bundles, for which [Cam04] only proves the projective case, and gives some hints for the Kähler case; it is established in this generality in [Fuj17].

Now let us explain the strategy of the proof of Theorem A. Generally speaking, as in the mainstream of works on $C_{n,m}$ (among others, [Fuj78; Kaw81; Kaw82; Vie83; CP17; Fuj17]), our proof is based on the positivity of relative pluricanonical bundles and of their direct images.

The key ingredient of the proof of Part (I) of Theorem A is the positivity of the relative m-Bergman kernel metric for Kähler fibre spaces, which is proved by Junyan Cao in [Cao17] by applying the Ohsawa-Takegoshi extension theorem with optimal estimate for Kähler fibre spaces (c.f. Theorem 2.1.1) also obtained in [Cao17] (c.f. also [GZ15a]), and states as follows (c.f. Theorem 2.2.1):

> Let $f : X \to Y$ be a Kähler fibre space between complex manifolds and let (L, h_L) be holomorphic line bundle on X endowed with a singular Hermitian metric whose curvature current is positive. Suppose that on the general fibre of f there exists a section of $K^{\otimes m}_{X/Y} \otimes L$ satisfying the $L^{2/m}$-integrability condition for some m, then the relative m-Bergman kernel metric $h^{(m)}_{X/Y,L}$ on $K^{\otimes m}_{X/Y} \otimes L$ has positive curvature current.

With the help of this positivity result, Part (I) of Theorem A, as well as the klt Kähler version of $C^{\log}_{n,m}$ for general type bases can both be deduced from (a global version of) the Ohsawa-Takegoshi type extension (Theorem 2.1.2) as follows:

- First by the useful Lemma 3.1.1, we can reduce the proof of the addition formula to that of the non-vanishing of the (twisted) relative pluricanonical bundle, up to adding an ample line bundle from the base.

- If Y is of general type in the orbifold sense, the non-vanishing result mentioned above follows easily from the Ohsawa-Takegoshi type extension (Theorem 2.1.2) in contrast to the proof in [Vie83; Cam04; Fuj17], where such non-vanishing results are deduced from the weak positivity of the direct images. Let us remark that: by generalizing the weak positivity theorem for f Kähler fibre space and for Δ log canonical, the general (log canonical) version is proved in [Cam04; Fuj17].

- In the situation of Part (I) of Theorem A, the proof of this non-vanishing result follows the same strategy, but requires an extra effort to establish a comparison theorem between the determinant of the direct image and the canonical bundle of X, see Theorem 3.1.4, which is a Kähler version of [CP17, Theorem 3.13].

The analytic proof given above does not explicitly involve any positivity result of direct images while it has the drawback of not being able to tackle the log canonical case.

Now we turn to the proof of Part (II) of Theorem A, for which we follow step by step the same argument in [CP17]. It is based on the positivity of the canonical L^2 metric on direct images sheaves (c.f. Theorem 2.2.4) which is stated as following:

Let $f : X \to Y$ be a Kähler fibre space between complex manifolds and let (L, h_L) be a holomorphic line bundle on X endowed with a semipositively curved singular Hermitian metric. Then the canonical L^2-Hermitian metric $g_{X/Y,L}$ on the direct image sheaf $f_*(K_{X/Y} \otimes L \otimes \mathscr{J}(h_L))$ is a semipositively curved singular Hermitian metric which satisfies the L^2 extension property.

The main strategy for the proof of the above positivity result is already implicitly comprised in [HPS18], and the result is explicitly shown in [DWZZ18] by proving a more general positivity theorem for singular L^p-Finsler metrics on direct images. In fact, this result is a consequence of the Ohsawa-Takegoshi extension theorem with optimal estimate obtained in [GZ15a] and generalized to Kähler case by [Cao17] (c.f. [ZZ18] for an alternative proof); the new feature is the L^2 extension property, which generalizes the well-known property of $\mathscr{O}$ that a L^2 holomorphic function extends across any analytic subset (compare this with the "minimal extension property" in [HPS18, Definition 20.1]). By combining the above positivity result of the canonical L^2 metric on direct images with the positivity of the relative m-Bergman kernel metric and by using the explicit construction of the relative m-Bergman kernel metric to get rid of the multiplier ideal (as in [CP17, §4, p.367]), we obtain the following positivity theorem for direct images of twisted pluricanonical bundles, which serves as a key ingredient of the proof of Theorem A(II):

Theorem D. *Let $f : X \to Y$ a Kähler fibre space with X and Y complex manifolds. Let Δ be an effective $\mathbb{Q}$-divisor on X such that the pair (X, Δ) is klt. Then for any integer $m > 0$ such that $m\Delta$ is an integral divisor, the torsion free sheaf*

$$\mathscr{F}_{m,\Delta} := f_*\left(K_{X/Y}^{\otimes m} \otimes \mathscr{O}_X(m\Delta)\right)$$

admits a canonical semi-positively curved singular Hermitian metric $g^{(m)}_{X/Y,\Delta}$ which satisfies the L^2 extension property.

Historically, the study of the positivity of direct images of (twisted) (pluri)canonical bundle(s) is initiated by the works of Phillip Griffiths on the variation of Hodge structures in the 60s, and is pursued by Fujita in [Fuj78] and by Kawamata in [Kaw81]; afterwards the study splits into two (related and complementary) main streams: the Hodge-theoretical aspect is further developed by Viehweg in the framework of weak positivity by algebro-geometric methods, while the curvature aspect is exploited by Bo Berndtsson, Mihai Păun and Shigeharu Takayama (among others) by complex-analytic methods and by introducing the notion of (semipositively curved) singular Hermitian metrics. The results mentioned above follow the philosophy of the latter stream. Let us remark that for a torsion free sheaf on a (quasi-)projective variety, the existence of a semi-positively curved singular Hermitian metric implies the weak positivity, while the reciprocal implication is not yet known (it is in fact a singular version of Griffiths's conjecture). The advantage to have such a metric is that: in case that the determinant line bundle is trivial, one can further deduce, by using the L^2 extension property, that this torsion free sheaf is a Hermitian flat vector bundle (c.f. Theorem 1.4.8). In this way we obtain a stronger regularity and our proof of Theorem A(II), like [CP17], leans on this regularity.

As a corollary of Theorem D, one finds that the induced metric $\det g^{(m)}_{X/Y,\Delta}$ on the determinant bundle $\det \mathscr{F}_{m,\Delta}$ has positive curvature current. Now let $Y = T$ be a complex torus; by an induction argument we can further assume that T is a simple torus, that is, containing no non-trivial subtori. Then by a structure theorem for pseudoeffective line bundles on complex tori [CP17, Theorem 3.3] we have the following dichotomy according to the sign of $\det \mathscr{F}_{m,\Delta}$:

- there is a integer $m > 0$ sufficiently large and divisible such that $\det \mathscr{F}_{m,\Delta}$ is ample;
- for every m sufficiently large and divisible, $\det \mathscr{F}_{m,\Delta}$ is numerically trivial.

Apparently the first case falls into the situation of the Theorem A(I). Hence we only need to tackle the second case, where one can use the L^2 extension property to further conclude that $(\mathscr{F}_{m,\Delta}, g^{(m)}_{X/Y,\Delta})$ is a Hermitian flat vector bundle. Furthermore, by a standard argument which dates back to Yujiro Kawamata, we are reduced to the case $\kappa(X, K_X+\Delta) \leqslant 0$, i.e. it is enough to prove that $\kappa(F, K_F + \Delta_F) \geqslant 1$ implies $\kappa(X, K_X + \Delta) \geqslant 1$. This reduction relies on the following a log Kähler version of [Kaw81, Theorem 1], which follows from [Cam04, Theorem 4.2] or [Fuj17, Theorem1.7] (or Theorem 3.1.2 for the klt case):

Theorem E. *Let X be a compact Kähler manifold. Suppose that there is an effective $\mathbb{Q}$-divisor Δ on X such that (X, Δ) is log canonical and that $\kappa(X, K_X + \Delta) = 0$ (i.e. X is bimeromorphically log Calabi-Yau). Then the Albanese map $\mathrm{alb}_X : X \to \mathrm{Alb}_X$ of X is a fibre space.*

The proof of this theorem will be given in §3.2, it is similar to that of [Kaw81]. In fact, when $\Delta = 0$ and X projective, the theorem is proved in [Kaw81]; for $\Delta = 0$ and X Kähler a proof is also sketched in [Kaw81, Theorem 24], but does not contain enough details. In virtue of [Fuj17, Theorem 1.7] (or Theorem 3.1.2 for the klt case) one can easily obtain Theorem E by following the strategies of [Kaw81], and it is exactly in this way our proof in §3.2 proceeds. Let us remark that a similar result with $\Delta = 0$ for special varieties in the sense of Campana is also stated in [Cam04] where the proof is sketched based on [Kaw81].

Now we are reduced to show that $\kappa(F, K_F + \Delta_F) \geqslant 1$ implies that $\kappa(X, K_X + \Delta) \geqslant 1$. $\mathscr{F}_{m,\Delta}$ being Hermitian flat, it is given by a unitary representation ρ_m of the fundamental group of T. The group $\pi_1(T)$ being Abelian, this representation is decomposed into 1-dimensional sub-representations. If the image of ρ_m is finite, then one can use the parallel transport to extend pluricanonical sections on F to X; if the image of ρ_m is infinite, then a fortiori $\kappa(X, K_X + \Delta) \geqslant 1$ by the following pluricanonical klt Kähler version of the structure theorem on cohomology jumping loci à la Green-Lazarsfeld-Simpson (c.f. [GL91; Sim93]), which is another key ingredient of the proof of Theorem A(II).

Theorem F. *Let $g : X \to Y$ be a morphism between compact Kähler manifolds. Let Δ be an effective $\mathbb{Q}$-divisor on X such that (X, Δ) is a klt pair. Then for every $m > 0$ such that $m\Delta$ is an integral divisor and for every $k > 0$, the cohomology jumping locus*

$$V^0_k\left(g_*\left(K_X^{\otimes m} \otimes \mathscr{O}_X(m\Delta)\right)\right) := \left\{ \rho \in \mathrm{Pic}^0(Y) \;\middle|\; \mathrm{h}^0(Y, g_*(K_X^{\otimes m} \otimes \mathscr{O}_X(m\Delta)) \otimes \rho) \geqslant k \right\}$$

is a finite union of torsion translates of subtori in $\mathrm{Pic}^0(Y)$.

The study of cohomology jumping loci was initiated by the works of Green-Lazarsfeld [GL87; GL91], which assure that every component of cohomology jumping loci is a translate of a subtorus, and is further developed by Carlos Simpson in [Sim93], where he proves that these translates are torsion translates. Recently, the main result of [Sim93] is generalized by Botong Wang to the Kähler case in [Wan16a], where he treats the case $g = \mathrm{id}_X$, $m = 1$ and $\Delta = 0$ in the statement of Theorem F and this is the starting point of our proof of Theorem F. In fact, when $g = \mathrm{id}_X$ and X projective, the proof of the theorem is already implicitly comprised in [CKP12] although they only explicitly state and prove in [CKP12] a result corresponding to our Corollary 3.3.9 with X smooth projective and (X, Δ) log canonical by using [Sim93]; we thus follow the strategy in [CKP12] to deduce Theorem F from the basic case treated in [Wan16a, Corollary 1.4]. Notice that [Wan16a] and hence our Theorem F require that X is "globally" Kähler; by contrast, Theorem D holds for any Kähler fibre space (X is only assumed to be locally Kähler over Y). Let us

remark that in the hypothesis of $C^{\log}_{n,m}$ it is essential to suppose that X is globally Kähler, in fact [Uen75, Remark 15.3, p. 187] provides an example of a Kähler fibre space for which $C_{n,m}$ does not hold.

Let us explain how to finish the proof of Theorem A(II) by using Theorem F. By following the argument in [CP11] one easily deduces from Theorem F (c.f. Corollary 3.3.9):

- $K_X + \Delta$ is the most effective $\mathbb{Q}$-line bundle in its numerical class.
- If $\kappa(X, K_X + \Delta) = \kappa(X, K_X + \Delta + L) = 0$ for some numerically trivial ($\mathbb{Q}$-)line bundle L, then L is a torsion point in $\mathrm{Pic}^0(X)$.

Now the proof of Theorem A(II) can be finished as follows: if $\mathrm{Im}(\rho_m)$ is infinite, by the decomposition of $\mathscr{F}_{m,\Delta}$ one sees that $K_X + \Delta$ has non-negative Kodaira dimension up to twisting a non-torsion numerically trivial ($\mathbb{Q}$-)line bundle, hence the first point above shows that $\kappa(X, K_X + \Delta) \geqslant 0$; moreover, if $\kappa(X, K_X + \Delta) = 0$ then the second point will lead to a contradiction, hence a fortiori $\kappa(X, K_X + \Delta) \geqslant 1$, thus we finish the proof of Theorem A. As a by-product of the first point above, we can prove the Kähler version of the (generalized) log Abundance Conjecture in the case of numerical dimension zero (c.f. Theorem 3.3.11) by using the divisorial Zariski decomposition obtained in [Bou04] (c.f.[Bou04, Definition 3.7]) .

Let us remark that one can follow the same strategies in [CP17, §5] to prove more generally that the $C^{\log}_{n,m}$ is true if $\det \mathscr{F}_{m,\Delta}$ is numerically trivial for some $m \in \mathbb{Z}_{>0}$ (i.e. the Kähler version of [CP17, Theorem 5.6]) by using the remarkable result of Zuo in [Zuo96, Corollary 1]. In this thesis, however, we will not further pursue in this direction.

Finally by using an induction argument and by applying the results already obtained we generalize Part (II) of Theorem A to the geometric orbifold setting:

Theorem G. *Let $f : X \to T$ be a fibre space with X compact Kähler manifold and T complex torus and let F be the general fibre of f. Let Δ be an effective $\mathbb{Q}$-divisor on X such that (X, Δ) is klt. Then*

$$\kappa(X, K_X + \Delta) \geqslant \kappa(F, \Delta_F) + \kappa(T, B_{f,\Delta}).$$

where $\Delta_F := \Delta|_F$ and $B_{f,\Delta}$ denotes the branching divisor on T w.r.t f and Δ.

In the theorem above, the branching divisor is defined as following: for any analytic fibre space $f : (X, \Delta) \to Y$ between compact complex manifolds with Δ an effective $\mathbb{Q}$-divisor on X, the branching divisor $B_{f,\Delta}$ (with respect to f and Δ) is defined as the most effective $\mathbb{Q}$-divisor on Y such that $f^* B_{f,\Delta} \leqslant R_{f,\Delta}$ modulo exceptional divisors, where the ramification divisor (w.r.t. f and Δ) is defined as $R_{f,\Delta} := R_f + \Delta$ and

$$R_f := \sum_{f(W) \text{ is a divisor on } Y} (\mathrm{Ram}_W(f) - 1)W$$

with $\mathrm{Ram}_W(f)$ denoting the ramification (in codimension 1) index of f along W. Precisely, assume the singular locus of f is contained in a (reduced) divisor $\Sigma_Y \subseteq Y$ and write

$$f^* \Sigma_Y = \sum_{i \in I} b_i W_i,$$

where W_i are prime divisors on X, then for $i \in I^{\mathrm{div}}$ where

$$I^{\mathrm{div}} := \text{ set of indices } i \in I \text{ such that } f(W_i) \text{ is a divisor on } Y,$$

we have $b_i = \mathrm{Ram}_{W_i}(f)$ and thus

$$R_f = \sum_{i \in I^{\mathrm{div}}} (b_i - 1) W_i .$$

Let us remark that the above definition of $B_{f,\Delta}$ coincides with [Cam04, Definition 1.29] (orbifold base) when Δ is lc on X, c.f. §3.5.

On the structure of klt projective varieties with nef anticanonical divisors

A general philosophy in the study of uniruled varieties is that a variety whose anticanonical bundle or the tangent bundle admits certain positivity, should exhibit certain birational rigidity, e.g. the canonical fibrations associated to them (the Albanese maps and the MRC fibrations) should have some rigid structure (typically, being locally constant fibration). This is inspired by the fundamental works of Shigefumi Mori [Mor79] and of Siu-Yau [SY80], proving the conjecture of Hartshorne-Frankel; their works characterize the projective spaces in terms of the amplitude of the tangent bundle (also true in positive characteristics), or equivalently, the positivity of the holomorphic bisectional curvature (also true for compact Kähler manifolds). An analytic generalization of Mori-Siu-Yau's result is obtained by Ngaiming Mok in [Mok88] for compact Kähler manifolds with nonnegative holomorphic bisectional curvature: he proved that the universal covers of these manifolds are decomposed into products of $\mathbb{C}^q$, of projective spaces and of (irreducible) compact Hermitian symmetric spaces of rank $\geqslant 2$. In order to establish the algebro-geometric counterpart of the main result of [Mok88], considerations are given to compact Kähler manifolds with nef tangent bundles, whose structures are settled by [DPS94], modulo the Campana-Peternell conjecture (it conjectures that smooth Fano varieties with nef tangent bundle are rationally homogeneous), by showing that the Albanese map is a locally constant fibration with Fano fibres. Then attention are further paid to smooth projective varieties (or more generally, compact Kähler manifolds) with nef anticanonical bundles. By MMP methods, the 3-dimensional case is extensively studied by Thomas Peternell and his collaborators in [PS98; BP04]. Recently the structure theorem for these varieties is established in [Cao19; CH19] by applying the method of positivity of direct images and by using the results in the previous works [Zha96; Pău97; Pău01; Zha05; LTZZ10]; moreover, the result is extended to klt pairs by [CCM19] when the variety is smooth projective. According to the general philosophy of MMP, it is then natural to extend this structure theorem to the mildly singular case, as stated in Conjecture 2.

In order to prove Conjecture 2 we follow the idea of [Cao19; CH19] and intend to show:

1. The Albanese map $\mathrm{alb}_X : X \dashrightarrow \mathrm{Alb}_X$ of X is a (everywhere defined) locally constant fibration;

2. The fundamental group of X_{reg} is of polynomial growth, equivalently (by [Gro81, Main Theorem]), $\pi_1(X_{\mathrm{reg}})$ is virtually nilpotent (i.e. admits a nilpotent subgroup of finite index);

3. If $\pi_1(X_{\mathrm{reg}}) = \{1\}$ then the maximal rationally connected (MRC) fibration of X is everywhere defined and induces a decomposition of X into a product of a rationally connected variety and of a projective variety with trivial canonical divisor.

The Points 1 and 3 above will be shown in this thesis (c.f. Theorem B and Theorem C) while the Point 2 seems quite difficult, at least the method in [Pău97] do not apply to this case. Apart from trying to prove the Point 2, there is also hope that one can directly prove the Conjecture 2 without studying the fundamental group (or at least by proving something much weaker on the fundamental group), c.f. [CCM19] and §4.4. As a consequence of Theorem B and Theorem C we can reduce Conjecture 2 to the following Conjecture 3. The detailed proof of this reduction will be given in §4.5.2.

Conjecture 3. *Let X be a normal projective variety of semi-Fano type. Then the fundamental group of X_{reg} is of polynomial growth.*

As to be shown in §4.5, this conjecture extends the Gurjar-Zhang conjecture on the finiteness of the fundamental group of the smooth locus of varieties of Fano type (c.f. [GZ94; GZ95; Zha95; Sch07; Xu14; GKP16; TX17]), which is recently settled by L.Braun in [Bra20]. It can also be regarded as a natural generalization of the following folklore conjecture (c.f. [GGK19]):

Conjecture 4. *Let X be a klt projective variety with trivial canonical divisor and vanishing augmented irregularity. Then the fundamental group of X_{reg} is finite.*

We will see in §4.5 that Conjecture 3 implies Conjecture 4. In the sequel let us briefly explain the ideas of the proof of Theorem B and Theorem C:

- First, an easy observation shows that [Cao19, 2.8.Proposition] is still valid even the total space is singular (c.f. Proposition 2.3.5), hence the problem of proving that a fibre space is a locally constant fibration can be reduced to proving that the direct images of the powers of a relative ample line bundle are numerically flat.

- By [CH19, Proposition 2.9] (c.f. Proposition 2.3.6) the proof of the numerical flatness of a reflexive sheaf can be divided into two parts: first, prove that the direct image admits weakly semipositive singular Hermitian metrics; second, prove that the determinant bundle of the direct image sheaf is numerically trivial. The first part can be deduced from the general positivity result of direct image sheaves (c.f. [CCM19, Theorem 2.2] or Corollary 2.2.13) by using the fact that $-K_X$ is nef, c.f. [CCM19, Lemma 3.4] or Proposition 4.1.3; while the second part can be established, at least birationally, with the help of the main result of [Zha05] (Proposition 4.1.2), c.f. Proposition 4.1.8.

- By using the method of [LTZZ10] we can prove that the Albanese map of X is flat, then we can further improve the aforementioned birational version of the numerical flatness result and show that the direct image of powers of some relatively very ample line bundle is numerically flat; by Proposition 2.3.5 this proves Theorem B .

- As for Theorem C, a similar yet much more subtle argument as that in [CH19, §3.C] applied to the MRC fibration of X shows that birationally X can be decomposed into a product, which gives rise to a splitting of T_X into direct sum of two algebraically integrable foliations, one having rationally connected Zariski closures of leaves, the other having trivial canonical class. However, X being singular and these foliations being singular, one cannot directly apply [Hör07, 2.11.Corollary]. To overcome this difficulty, we observe that the decomposition implies that the two foliations are weakly regular, then we can use the related results in [Dru17; Dru18b] to show that, up to a $\mathbb{Q}$-factorial terminal model, the MRC fibration is everywhere defined. In this situation, we can use a similar argument as the one in the proof of Theorem B to show the numerical flatness of the direct images up to a base change, and finally [Dru18a, Lemma 4.6] permits us to conclude.

Organization of the thesis

The thesis is organized as following: in Chapter 1 we recall some preliminary results which will be used in the proof of the main theorems; and Chapter 2 is devoted to the development of the main tools needed in this thesis, as mentioned above in the section of Methodology, especially, Theorem D is proved in §2.2. After this, the last two chapters are devoted to the proof of the main results of the thesis:

- In Chapter 3 we consider the Iitaka conjecture $C_{n,m}$ for Kähler fibre spaces, and Theorem A is proved; in particular, Theorem A(I) is proved in §3.1, Theorem E is proved in §3.2, in §3.3 we show Theorem F and in §3.4 we conclude the proof of Theorem A by combining the previous results, finally the proof of Theorem G is given in §3.5.
- In Chapter 4 we study klt projective varieties with nef anticanonical divisors; in particular, Theorem B and Theorem C are proved respectively in §4.2 and §4.3, and in §4.5 we study the fundamental groups of the smooth locus of these varieties, especially we prove that the Conjecture 2 can be reduced to the Conjecture 3. The §4.4 is added after all the other parts of Chapter 4 has been finished, where we discuss the foliations (in particular the algebraically integrable ones) with numerically trivial canonical class by following the suggestions of Stéphane Druel and give an alternative proof of Theorem C.

Introduction (Français)

Soit k un corps algébriquement clos, un des problèmes centraux en géométrie algébrique est de classifier les variétés algébriques sur k à isomorphisme près ; si $k = \mathbb{C}$, on peut aussi considérer plus généralement le problème de classification pour les espaces analytiques complexes (irréductible et réduit, en particulier ceux dans la classe $\mathscr{C}$ de Fujiki). Cette étude est initiée, d'une part par Bernhard Riemann, Henri Poincaré, etc. dans leurs travaux sur l'uniformisation des surfaces de Riemann (courbes algébriques) du point de vue analytique ; d'une autre part par l'École italienne (Guido Castelnuovo, Federigo Enriques, Francesco Severi, etc.) sur les modèles minimaux des surfaces algébriques du point de vue algébrique. Dans le cadre des mathématiques modernes, leurs idées ont été davantage développées et de nombreuses avancées ont été réalisées au cours du siècle dernier, surtout les travaux remarquables de Kunihiko Kodaira, David Mumford, Shigeru Iitaka, Kenji Ueno, Shigefumi Mori, Eckart Viehweg, Yujiro Kawamata, János Kollár, Vyacheslav Shokurov, etc.. Comme fruit de ces travaux, le principe du problème de classification est établi et, selon ce principe, le problème peut se diviser en deux aspects, à savoir, l'aspect de la classification birationelle / biméromorphe et l'aspect de la construction de (une bonne compactification de) l'espace des modules.

Cette thèse se concentre principalement sur le premier aspect du problème de classification, sur lequel de grand progrès ont été faits récemment avec les travaux de [BCHM10; BDPP13]. Dans [BCHM10] le programme des modèles minimaux (abbr. MMP) est presque établi en suivant les idées de Vyacheslav Shokurov, tandis que [BDPP13] fait un progrès significatif vers l'abondance en décrivant le cône positif des diviseurs pseudoeffectifs. En combinant ces résultats, on voit que les variétés projectives lisses (ou légèrement singulières) peuvent se diviser en deux classes (birationellement stables) :

- les variétés à diviseur canonique pseudoeffectif, pour lesquelles le MMP aboutit à un modèle minimal (c'est-à-dire, une variété légrèrement singulière à diviseur canonique nef) ;
- les variétés uniréglées, celles pour lesquelles le MMP aboutit à une fibration de Mori (une fibration dont la fibre générale est de Fano à nombre de Picard 1) sous le MMP.

La philosophie générale dans l'étude des variétés minimales / variétés uniréglées est d'étudier les fibrations canoniques qui leur sont associées, ce qui réduit cette étude à étudier la base et la fibre générale. Les résultats principaux de cette thèse sont développés le long ces deux grandes lignes, comme précisés ci-dessous.

Pour les variétés minimales, la plus importante fibration associées est la fibration d'Iitaka-Kodaira définie par un multiple suffisamment grand du diviseur canonique, dont la fibre est de dimension de Kodaira dimension 0 et qui, selon la conjecture d'abondance, devrait être un morphisme définie partout (au lieu d'une application rationnelle / méromorphe) vers une variété canoniquement polarisée (un modèle canonique). Bien que la conjecture d'abondance reste encore largement ouverte, beaucoup de progrès ont été faits dans la direction d'un corollaire important, connu sous le nom « conjecture $C_{n,m}$

d'Iitaka », qui prédit la sur-additivité de la dimension de Kodaira par rapport aux fibrations algébriques :

Conjecture 1 (Conjecture $C_{n,m}$ d'Iitaka, [Uen75, §11.5, Conjecture C_n, pp. 132-133]). *Soit $f : X \to Y$ une fibration algébrique entre variétés projective lisses avec $\dim X = n$ et $\dim Y = m$, et soit F la fibre générale de f, alors nous avons*

$$\kappa(X) \geqslant \kappa(Y) + \kappa(F).$$

Rappelons que la dimension de Kodaira $\kappa(X)$ d'un espace analytique complexe X est définie comme étant la dimension de l'image de la fibration d'Iitaka-Kodaira mentionnée ci-dessus, ou de façon équivalente, l'unique entier $\kappa \in \{-\infty, 0, 1, \cdots, \dim X\}$ tel qu'il existe des constantes $C_1, C_2 > 0$ indépendantes de m satisfaisant l'inégalité

$$C_1 \cdot m^\kappa \leqslant \mathrm{h}^0(X, K_X^{\otimes m}) \leqslant C_2 \cdot m^\kappa,$$

pour tout m suffisamment grand et divisible. Récemment un cas spécial important de la conjecture $C_{n,m}$ a été démontré par Junyan Cao and Mihai Păun [CP17]. Tandis qu'une grande partie du MMP reste inconnue pour les variétés kählérienne, en utilisant en profondeur le développement récent des méthodes analytiques, en particulier les théorème d'extension du type Ohsawa-Takegoshi obtenus par Qi'an Guan et Xiangyu Zhou dans [GZ15a, Theorem 3.21] et généralisés Cao dans [Cao17, Theorem 1.1] (c.f. [ZZ18] pour une preuve alternative), j'arrive à étendre le résultat principal de [CP17] (ainsi que l'un des résultats principaux de [Vie83]) au cas kählérien :

Théorème A. *Soit $f : X \to Y$ une fibration entre variétés kählérienne dont la fibre générale est dénotée par F, et soit Δ un $\mathbb{Q}$-diviseur effectif sur X tel que (X, Δ) soit Kawamata log terminal (abbr. klt). Supposons qu'une des conditions suivantes est satisfaite :*

(I) *Il existe un entier $m > 0$ tel que $m\Delta$ est un diviseur entier et que le fibré déterminant $\det f_*(K_{X/Y}^{\otimes m} \otimes \mathscr{O}_X(m\Delta))$ est gros sur Y ;*

(II) *Y est un tore complexe.*

Alors

$$\kappa(X, K_X + \Delta) \geqslant \kappa(F, K_F + \Delta_F) + \kappa(Y),$$

où $\Delta_F := \Delta|_F$.

La preuve du théorème A repose sur la positivité des images directes des fibrés pluricanoniques relatifs tordus ([DWZZ18, Theorem 9.3], c.f. §2.2.2 pour une preuve alternative) et un résultat du type Green-Lazarsfeld-Simpson sur les lieux de sauts de cohomologie ([Wan19, Theorem D]). Dans [DWZZ18] un résultat plus général sur la positivité des métriques L^p-finslériennes sur les images directes des fibrés pluricanoniques relatifs tordus est établie en utilisant une caractérisation nouvelle des fonctions psh ; dans [Wan19] je donne une démonstration pour les métrique L^2-hermitienne, basée sur le théorème d'extension d'Ohsawa-Takegoshi avec estimation optimale obtenu par Qi'an Guan et Xiangyu Zhou dans [GZ15a] et généralisé par Cao dans [Cao17] (une preuve alternative est donnée dans [ZZ18]). Rappelons que pour un fibré vectoriel E sur une variété complexe, une métrique hermitienne singulière sur E est donnée par un famille mesurable de fonctions hermitiennes sur chaque fibre de E qui est non-singulière presque partout ; sur l'image directe des fibrés pluricanoniques relatifs (tordus), il y a une métrique L^2-hermitienne naturelle, qui est définie par l'intégrale fibre à fibre des n-formes différentielles tordues (n désigne la dimension relative de la fibration).

Dans l'autre direction, c-à-d., dans l'étude des variétés uniréglées, au lieu d'étudier la fibration d'Iitaka-Kodaira (ce qui ne fournit aucune information pour variétés uniréglées), on étudie l'application d'Albanese et la fibration rationnellement connexe maximale (MRC). La philosophie générale, inspirée par les travaux fondamentaux de Shigefumi Mori [Mor79], est que, quand le fibré anticanonique ou le fibré tangent d'une variété admet certaine positivité, ces fibrations canoniques devraient avoir une structure rigide (typiquement, être une fibration localement constante). Pour une variété projective à singularités log canonique (lc), si son diviseur anticanonique est ample (le cas des variétés de $\mathbb{Q}$-Fano) les fibrations sont toutes triviales par les travaux classiques de Kollár-Mori-Miyaoka [KMM92] et de Frédéric Campana [Cam92] (et par Qi Zhang dans [Zha05] pour le cas singulier) ; il est donc naturel de poser la même question pour les variétés projective à diviseur anticanonique nef. Rappelons qu'un diviseur de Cartier ou un fibré en droites sur une variété projective est dit nef si son nombre d'intersection avec toute courbe est $\geqslant 0$, ou de façon équivalente, s'il admet des métriques hermitiennes lisses dont la courbure a une partie négative arbitrairement petite (donc on peut étendre cette notion à tout espace analytique complexe compact, c.f. [DPS94]). Dans le cas lisee, les études de l'application d'Albanese et de la fibration MRC sont menées à leurs termes dans [Cao19] et [CH19] respectivement. Dans ces travaux il est établi que pour une variété projective à fibré anticanonique nef, les deux applications rationnelles mentionnées ci-dessus sont des fibrations localement constante (définies partout), ce qui implique qu'une variété projective lisse à diviseur anticanonique nef admet une décomposition du type Beauville-Bogomolov : le revêtement universel d'une telle variété peut être décomposé en un produit de $\mathbb{C}^q$, des variétés de Calabi-Yau, des variétés hyperkählériennes et d'une variété rationnellement connexe (les trois premiers facteurs sont donnés par la décomposition de Beauville-Bogomolov classique). Selon la philosophie du MMP, on se propose de généraliser ce théorème de structure au cas singulier, c'est à dire de démontrer la conjecture suivante :

Conjecture 2. *Soit X une variété projective à singularités klt et supposons que le diviseur anticanonique $-K_X$ de X est nef. Alors quitte à remplacer X par un revêtement quasi-étale, l'application d'Albanese et la fibration MRC de X induisent une décomposition du revêtement universel $\tilde{X}$ de X en un produit*

$$\tilde{X} \simeq \mathbb{C}^q \times Z \times F,$$

où q désigne l'irrégularité augmentée de X, Z est une variété projective klt à diviseur canonique trivial et F est une variété rationnellement connexe.

Comme dans le cas lisse, en appliquant la version singulière (klt) de la décomposition de Beauville-Bogomolov établie par les travaux successifs [GKP16 ; Dru18a ; GGK19 ; HP19], la variété Z ci-dessus peut être décomposée davantage en un produit des variétés projectives de Calabi-Yau par des variétés irréductibles symplectiques projectives. Cependant, assez différent du cas des variétés à diviseur canonique numériquement trivial, même dans le cas lisse on ne peut en général pas obtenir une structure de produit à revêtement (quasi-)étale fini près pour les variétés à diviseur anticanonique nef à cause de l'apparition du facteur rationnellement connexe, par exemple il y a des surfaces réglées au-dessus d'une courbe elliptique qui ne peuvent pas se décomposer en un produit de la courbe elliptique par $\mathbb{P}^1$ à revêtement étale fini près (c.f. [Dru18b, Example 9.1, Example 11.2], [EIM20, Example 6.2]).

Dans cette thèse la conjecture 2 est partiellement établie en généralisant les résultats principaux de [Cao19] et de [CH19] au cas singulier klt. Rappelons qu'une variété projective normale X est dite *du type Fano* (resp. *du type semi-Fano*), s'il existe un $\mathbb{Q}$-diviseur Δ sur X tel que la paire (X, Δ) soit klt et que le diviseur anticanonique tordu soit ample (resp. nef), c.f. [PS09, Definition 2.5, Lemma-Definition 2.6]. On montre les théorèmes suivants concernant la structure des variétés du type semi-Fano :

Théorème B. *Soit X une variété projective normale du type semi-Fano. Alors son application d'Albanese $\mathrm{alb}_X : X \dashrightarrow \mathrm{Alb}_X$ est une fibration localement constante (définie partout), i.e. alb_X est une fibration localement triviale telle que X est isomorphe au produit du revêtement universel de Alb_X par la fibre de alb_X quotienté par une action diagonale de $\pi_1(\mathrm{Alb}_X)$.*

Théorème C. *Soit X une variété projective normale du type semi-Fano dont le lieu lisse X_{reg} est simplement connexe. Alors la fibration MRC de X induit une décomposition de X en un produit $F \times Z$ avec F rationnellement connexe et $K_Z \sim 0$.*

Remarquons que la trivialité locale (aussi connue sous le nom « iso-trivialité », surtout en géométrie algébrique) de l'application d'Albanese de X a été déjà obtenue dans le travail de Zsolt Patakafalvi et Maciej Zdanowicz [PZ19, Corollary 1.17 (Corollary A.14)] sous l'hypothèse supplémentaire que X est $\mathbb{Q}$-factorielle. La stratégie dans leur article est de montrer que les fibres (fermées) sont toutes isomorphes en prouvant la platitude numérique des images directes sur les courbes intersections complètes. Dans cette thèse, on peut utiliser les méthodes analytiques pour démontrer plus généralement la platitude numérique « globale » des images directes, et donc peut obtenir le résultat plus fort que l'application d'Albanese est non seulement localement triviale mais aussi une fibration localement constante.

L'idée de base de la preuve de ces deux théorèmes est la même que [Cao19 ; CH19] : étudier la positivité des images directes des puissances d'un fibré en droites relativement très ample, et prouver qu'elles sont numériquement plates à un twist près. Pour l'application d'Albanese, on peut conclure directement comme l'application est définie partout ; quant à la fibration MRC, ceci nous fournit seulement une décomposition du faisceau tangent en feuilletages algébriques. Le problème est que ces feuilletages sont singuliers sur une variété projective singulière, en conséquence, on ne peut pas appliquer directement la théorie générale des feuilletages (réguliers) ; afin de surmonter cette difficulté, une observation clé est que les feuilletages sont faiblement réguliers au sens de [Dru18b].

Méthodologie

Dans cette section, récapitulons les méthodes et les outils principaux appliqués à l'étude du problème de classification, surtout ceux dans cette thèse. Afin d'étudier le problème de classification pour les espaces analytiques complexes, on a besoin à la fois des méthodes algébriques et analytiques complexes. Le cœur technique des méthodes algébriques est la théorie de Hodge, dont la version moderne a été totalement réécrite par Pierre Deligne en termes d'algèbre homologique et largement développée par Morihiko Saito du point de vue des modules de Hodge mixtes. Tous les théorèmes d'annulation et résultats de positivité des images directes peuvent se déduire de la théorie de Hodge. L'application des méthodes analytiques au problème de classification est initiée par les travaux de Yum-Tong Siu, Shing-Tung Yau et Jean-Pierre Demailly. L'idée centrale est d'étudier les métriques singulières sur les fibrés vectoriels ainsi que les idéaux multiplicateurs qui leur sont associés, e.g. la métrique de Hodge sur l'image directe du fibré canonique relatif et les métriques de Kähler-Einstein (singulières) sur (le fibré tangent du lieu lisse de) les espaces analytiques complexes. En introduisant la notion de (semi)positivité pour les métriques hermitiennes singulières sur les fibrés vectoriels, on peut formuler et prouver des théorèmes d'annulation (de Nadel) plus généraux et (la version métrique de) des résultats plus généraux de positivité des images directes (c.f. [PT18 ; HPS18 ; DWZZ18]), et donc dans de nombreux cas les méthodes analytiques peuvent totalement remplacer les méthodes algébriques. La preuve de ces résultats repose sur (des variantes du) le théorème d'extension du type Ohsawa-Takegoshi à l'estimation optimale, c.f. [GZ15b ; Cao17]. Afin d'obtenir des théorèmes de structure plus raffinés pour les

espaces analytiques complexes, un ingrédient très important est la théorie des feuilletages, qui pointe dans la direction de résultats du type uniformisation. Pourtant les résultats classiques sur les feuilletages ne suffisent pas pour le problème de classification car la philsophie du MMP exige que l'on traite les variétés légèrement singulières, en conséquence beaucoup d'efforts ont été faits pour le développement de la théorie des feuilletages singuliers sur les variétés (légèrement) singulières. Un paradigme de l'application de cette théorie est la preuve de la version klt du théorème de décomposition de Beauville-Bogomolov mentionnée ci-dessus, en particulier le travail de Stéphane Druel dans [Dru18a ; Dru18b].

Sur la conjecture $C_{n,m}$ d'Iitaka pour les fibrations kählériennes

Soit X un espace analytique complexe compact et soit L un ($\mathbb{Q}$-)fibré en droites sur X, rappelons que la dimension d'Iitaka(-Kodaira) de L, notée par $\kappa(X, L)$, est le maximum de la dimension de l'image de $\bar{X}$ via l'application méromorphe $\bar{X} \dashrightarrow \mathbb{P}\mathrm{H}^0(\bar{X}, \nu^* L^{\otimes m})$ définie la série linéaire $\left|\nu^* L^{\otimes m}\right|$ pour $m \in \mathbb{Z}_{>0}$ suffisamment grand et divisible (si $\left|\nu^* L^{\otimes m}\right| = \varnothing$ pour tout $m \in \mathbb{Z}_{>0}$ alors on pose $\kappa(X, L) = -\infty$), où $\nu : \bar{X} \to X$ est la normalisation de X. En particulier, la dimension de Kodaira d'un espace analytique complexe compact X, notée par $\kappa(X)$, est la dimension d'Iitaka-Kodaira du fibré canonique d'un modèle lisse de X, et $\kappa(X)$ est l'invariant biméromorphe le plus important de X.

La conjecture $C_{n,m}$ d'Iitaka, dans sa forme originale, prédit la sur-additivité de la dimension de Kodaira par rapport aux fibrations algébriques (c.f. [Uen75, §11.5, Conjecture C_n, pp. 132-133]) ; plus précisément, pour $f : X \to Y$ une fibration entre variétés projectives normales dont la fibre générale est notée par F, la conjecture $C_{n,m}$ prédit que

$$\kappa(X) \geqslant \kappa(F) + \kappa(Y).$$

Cette conjecture est intimement liée à l'étude de la classification birationnelle des variétés algébriques (le programme des modèles minimaux). Selon la philosophie du MMP, la conjecture $C_{n,m}$ se généralise naturellement à sa version logarithmique, généralement appelée $C^{\log}_{n,m}$. Frédéric Campana, quant à lui, généralise davantage $C_{n,m}$ au cadre des orbifoldes géométriques, appelée $C^{\mathrm{orb}}_{n,m}$, ce qui est formulée dans [Cam04, Conjecture 4.1] et [Cam09, Conjecture 6.1]. Par ailleurs, en prenant en compte la variation de fibration, Eckart Viehweg propose aussi une version plus forte de $C_{n,m}$, nommée $C^{+}_{n,m}$.

La conjecture $C_{n,m}$ (resp. $C^{+}_{n,m}$) peut se voir, par [KMM87] (resp. [Kaw85]), comme une conséquence des fameuses conjecture des modèles minimaux et conjecture d'abondance ; de plus, en vertu de la sur-additivité de la dimension numérique de Nakayama (c.f. [Nak04, §V.4.a, 4.1.Theorem(1), pp. 220-221]), $C^{\log}_{n,m}$ se déduit de ce que l'on appelle la conjecture d'abondance généralisée (pour les $\mathbb{Q}$-diviseurs), c.f. [Fuj17, Remark 1.8].

Bien qu'elle ait été initialement énoncée pour les variétés projectives, la conjecture $C_{n,m}$, ainsi que les conjectures des modèles minimaux et d'abondance, est naturellement étendue aux espaces analytiques complexes compacts dans la classe $\mathscr{C}$ de Fujiki (c.f. [Fuj78 ; Cam04 ; HP16 ; CHP16 ; Fuj17]) ; pourtant ces conjectures ne sont pas vraies pour les variétés complexes non-kählériennes, c.f. [Uen75, Remark 15.3, p. 187] pour un contre-exemple. Comme mentionné ci-dessus, un des résultats principaux dans cette thèse est de démontrer la version kählérienne klt de $C^{\log}_{n,m}$ dans deux cas spéciaux importants et de généraliser davantage le deuxième au cadre des orbifoldes géométriques.

La conjecture $C_{n,m}$ est déjà connue dans la dimension basse (par exemple : $\dim X \leqslant 6$, [Bir09] ; $\dim Y = 1$, [Fuj78 ; Kaw82] ; $\dim Y = 2$, [Kaw81 ; Vie83 ; Cao18]). Quant aux résultats en dimension supérieure, on montre la conjecture, en appliquant la méthode de positivité des images directes développée par Phillip Griffiths, Takao Fujita, Yujiro

Kawamata, Eckart Viehweg, Bo Berndtsson, Mihai Păun, Shigeharu Takayama, etc., dans les trois cas importants suivants :

1. Y est de type général (Kawamata [Kaw81] ; Viehweg [Vie83] ; Campana [Cam04], cadre des orbifoldes géométriques) ;
2. Il existe un entier $m > 0$ tel que $\det f_*(K_{X/Y}^{\otimes m})$ soit gros sur Y, i.e. $\kappa(Y, \det f_*(K_{X/Y}^{\otimes m})) = \dim Y$ (Viehweg [Vie83]) ;
3. Y est une variété abélienne (Cao & Păun [CP17], la version klt).

Dans cette thèse, on traite la version (log ou orbifolde) kählérienne de ces trois cas.

Le théorème A(I) généralise [Vie83, Theorem II], qui est intimement lié à $C^+_{n,m}$ (c.f. [Vie83] pour les détails ; cette thèse, cependant, ne poursuivra pas dans cette direction) ; la partie (II) généralise [CP17, Theorem 1.1] et on le généralisera davantage au cadre des orbifoldes géométriques, autrement dit, on prouve la conjecture $C^{\mathrm{orb}}_{n,m}$ pour f quand Y est un tore complexe.

De plus, en suivant la même stratégie de la preuve du théorème A(I), on retrouve le résultat que la version klt kählérienne de $C^{\log}_{n,m}$ est vraie pour $f : (X, \Delta) \to Y$ quand Y est de type général, qui généralise [Kaw81, Theorem 3] ; et l'on généralisera davantage ce résultat au cadre des orbifoldes géométriques. Remarquons que la version plus générale (log canonique) de $C^{\mathrm{orb}}_{n,m}$ pour Y de type général (au sens des orbifoldes géométriques) a été démontrée [Cam04] ; la preuve repose sur un résultat de positivité faible pour les images directes des fibrés pluricanoniques tordus, pour lequel [Cam04] seulement montre le cas projectif, et donne quelques indications pour le cas kählérien ; ce résultat de positivité faible est établi dans cette généralité dans [Fuj17].

Expliquons maintenant la stratégie de la démonstration du théorème A. Gross modo, comme dans le courant principal des travaux sur la conjecture $C_{n,m}$ (parmi eux, [Fuj78 ; Kaw81 ; Kaw82 ; Vie83 ; CP17 ; Fuj17]), ma démonstration repose sur la positivité des fibrés pluricanoniques relatifs et de ses images directes.

L'ingrédient clé dans la démonstration de la partie (I) du théorème A est la positivité de la métrique du noyau de m-Bergman relatif pour les fibrations kählériennes, ce qui est prouvée par Junyan Cao dans [Cao17] en appliquant le théorème d'extension d'Ohsawa-Takegoshi avec l'estimation optimale pour les fibrations kählériennes (c.f. le théorème 2.1.1) aussi obtenue dans [Cao17] (c.f. aussi [GZ15a]), et s'énonce comme suivant (c.f. le théorème 2.2.1) :

> Soit $f : X \to Y$ une fibration kählérienne entre variétés complexes et soit (L, h_L) un fibré en droites holomorphe sur X équipé d'une métrique hermitienne singulière à courant de courbure positif. Supposons que sur la fibre générale de f il existe une section de $K_{X/Y}^{\otimes m} \otimes L$ satisfaisant la condition $L^{2/m}$-intégrabilité pour certain m, alors la métrique du noyau de m-Bergman relatif $h^{(m)}_{X/Y,L}$ sur $K_{X/Y}^{\otimes m} \otimes L$ est à courant de courbure positif.

À l'aide de ce résultat de positivité, la partie (I) du théorème A, ainsi que la version klt kählérienne de $C^{\log}_{n,m}$ pour les bases de type général, peut se déduire (d'une version globale) du théorème d'extension du type Ohsawa-Takegoshi (théorème 2.1.2) comme suit :

- D'abord par l'utile lemme 3.1.1, on peut réduire la preuve de la formule d'addition à la non-annulation du fibré pluricanonique relatif tordu, quitte à ajouter un fibré en droites ample venant de la base.

- Si Y est de type général au sens des orbifoldes géométriques, la non-annulation mentionnée ci-dessus se déduit facilement du théorème d'extension du type Ohsawa-Takegoshi (théorème 2.1.2) tandis que dans [Vie83 ; Cam04 ; Fuj17] de tels résultats de non-annulation se déduisent de la positivité faible des images directes. Remarquons que : en généralisant le théorème de positivité faible pour f fibration kählérienne et pour Δ log canonique, la version générale (log canonique) de ce résultat est démontré dans [Cam04 ; Fuj17].

- Dans la situation de la partie (I) du théorème A, la démonstration de la non-annulation suit la même stratégie, mais exige des efforts supplémentaires pour établir un théorème de comparaison entre le déterminant de l'image directe et le fibré canonique de X, voir le théorème 3.1.4, qui est la version kählérienne de [CP17, Theorem 3.13].

La preuve analytique présentée ci-dessus n'implique pas explicitement de résultat de positivité pour les images directes tandis qu'elle a l'inconvénient de ne pas pouvoir traiter le cas log canonique.

Tournons-nous maintenant vers la démonstration de la partie (II) du théorème A, pour laquelle l'on suit pas à pas le même argument que [CP17]. Elle repose sur la positivité de la métrique canonique L^2 sur les images directes (c.f. le théorème 2.2.4) s'énonçant comme ci-dessous :

> Soit $f : X \to Y$ une fibration kählérienne et soit (L, h_L) un fibré en droites holomorphe sur X équipé d'une métrique hermitienne singulière à courant de courbure positif. Alors la métrique L^2-hermitienne canonique sur l'image directe $f_*(K_{X/Y} \otimes L \otimes \mathscr{J}(h_L))$, dénotée par $g_{X/Y,L}$, est une métrique hermitienne singulière semipositive qui satisfait la propriété d'extension L^2.

La stratégie principale pour la preuve du résultat de positivité ci-dessus a été implicitement comprise dans [HPS18], et le résultat a été explicitement démontré dans [DWZZ18] en prouvant un théorème de positivité plus général pour les métrique L^p-finslériennes singulières sur les images directes. En effet, ce résultat est une conséquence du théorème d'extension d'Ohsawa-Takegoshi avec estimation optimale obtenu dans [GZ15a] et généralisé au cas kählérien par [Cao17] (c.f. [ZZ18] pour une preuve alternative) ; l'élément nouveau dans l'énoncé est la propriété d'extension L^2, ce qui généralise la propriété bien connue de $\mathscr{O}$ qu'une fonction holomorphe L^2-intégrable s'étend à travers des sous-espaces analytiques non-nécessairement irréductibles (comparer cette notion avec celle de la « propriété d'extension minimale » dans [HPS18, Definition 20.1]). En combinant le résultat ci-dessus sur la positivité pour la métrique L^2 avec la positivité de la métrique du noyau de m-Bergman relatif et en utilisant la construction explicite de la métrique du noyau de m-Bergman relatif pour se débarrasser de l'idéal multiplicateur (comme dans [CP17, §4, p.367]), on obtient le théorème suivant de positivité pour les images directes des fibrés pluricanonique relatifs tordus, qui sert d'ingrédient clé dans la démonstration du théorème A(II) :

Théorème D. *Soit $f : X \to Y$ une fibration kählérienne avec X et Y des variétés complexes. Soit Δ un $\mathbb{Q}$-diviseur effectif sur X tel que la paire (X, Δ) soit klt. Alors pour tout entier $m > 0$ tel que $m\Delta$ soit un diviseur entier, le faisceau sans torsion*

$$\mathscr{F}_{m,\Delta} := f_*\left(K_{X/Y}^{\otimes m} \otimes \mathscr{O}_X(m\Delta)\right)$$

admet une métrique hermitienne singulière canonique semipositive $g^{(m)}_{X/Y,\Delta}$ satisfaisant la propriété d'extension L^2.

Historiquement, l'étude de la positivité des images directes du fibré (pluri)canonique (tordu) est initiée par les travaux de Phillip Griffiths sur la variation des structures de Hodge dans les années 60, et est poursuivie par Takao Fujita dans [Fuj78] et par Yujiro Kawamata dans [Kaw81]; ensuite l'étude s'est divisée en deux courants principaux (liés et complémentaires l'un de l'autre) : l'aspect de la théorie de Hodge est développé davantage par Eckart Viehweg dans le cadre de la positivité faible par des méthode algro-géométriques, tandis que l'aspect de la courbure est exploité par Bo Berndtsson, Mihai Păun et Shigeharu Takayama (parmi d'autres) par des méthodes analytiques complexes en introduisant la notion de métrique hermitienne singulière (semipositive). Le résultat mentionné ci-dessus suit la philosophie de ce dernier courant. Remarquons que pour un faisceau sans torsion sur une variété (quasi-)projective, l'existence d'une métrique hermitienne singulière semipositive implique la positivité faible, mais l'implication réciproque n'est pas connue (c'est en effet une version singulière de la conjecture de Griffiths). L'avantage d'avoir une telle métrique est que : au cas où le fibré déterminant est (numériquement) trivial, on peut en déduire, en utilisant la propriété d'extension L^2, que ce faisceau sans torsion est un fibré vectoriel hermitien plat (c.f. Theorem 1.4.8). De cette façon on obtient une régularité assez forte pour les images directe, et notre preuve du théorème A(II), comme [CP17], repose sur cette régularité.

Comme un corollaire du théorème D, la métrique induite $\det g^{(m)}_{X/Y,\Delta}$ sur le fibré déterminant $\det \mathscr{F}_{m,\Delta}$ a un courant de courbure positif. Soit maintenant $Y = T$ est un tore complexe; par récurrence l'on peut supposer davantage que T est un tore simple, c'est-à-dire, n'admettant aucun sous-tore non-trivial. Alors par le théorème de structure pour les fibré en droites pseudoeffectif sur les tores complexes [CP17, Theorem 3.3] on a la dichotomie suivante selon le signe de $\det \mathscr{F}_{m,\Delta}$:

- il existe un entier $m > 0$ suffisamment grand et divisible tel que $\det \mathscr{F}_{m,\Delta}$ soit ample;
- pour tout m suffisamment grand and divisible, $\det \mathscr{F}_{m,\Delta}$ est numériquement trivial.

Évidemment le premier cas tombe dans la situation du théorème A(I). Donc il suffit de traiter le second cas, où l'on peut utiliser la propriété d'extension L^2 pour conclure que $(\mathscr{F}_{m,\Delta}, g^{(m)}_{X/Y,\Delta})$ est un fibré vectoriel hermitien plat. De plus, par un argument standard qui remonte à Yujiro Kawamata, on peut se ramener au cas où $\kappa(X, K_X + \Delta) \leqslant 0$, c-à-d., il suffit de prouver que $\kappa(F, K_F + \Delta_F) \geqslant 1$ implique $\kappa(X, K_X + \Delta) \geqslant 1$. Cette réduction repose sur la suivante version log kählérienne du [Kaw81, Theorem 1], qui se déduit de [Cam04, Theorem 4.2] ou [Fuj17, Theorem1.7] (ou du théorème 3.1.2 pour le cas klt) :

Théorème E. *Soit X une variété kählérienne compacte. Supposons qu'il existe un $\mathbb{Q}$-diviseur effectif Δ sur X tel que la paire (X, Δ) soit log canonique et que $\kappa(X, K_X + \Delta) = 0$ (i.e. X est biméromorphiquement log Calabi-Yau). Alors l'application d'Albanese $\mathrm{alb}_X : X \to \mathrm{Alb}_X$ de X est une fibration.*

La preuve de ce théorème sera donnée dans §3.2, qui est similaire à celle dans [Kaw81]. En effet, quand $\Delta = 0$ et X projective, le théorème a été démontré dans [Kaw81]; pour $\Delta = 0$ et X kählérienne une preuve a été aussi esquissée dans [Kaw81, Theorem 24], mais ne contient pas assez de détails. En vertu de [Fuj17, Theorem 1.7] (ou du théorème 3.1.2 pour le cas klt) on peut en déduire le théorème E en suivant la stratégie de [Kaw81], et c'est exactement de cette façon que nous procédons dans §3.2. Remarquons qu'un résultat similaire avec $\Delta = 0$ pour les variétés spéciales au sens de Campana est énoncé dans [Cam04] dont la preuve est esquissée en s'appuyant sur [Kaw81].

Maintenant on se ramène à prouver que $\kappa(F, K_F + \Delta_F) \geqslant 1$ implique que $\kappa(X, K_X + \Delta) \geqslant 1$. Étant hermitien plat, $\mathscr{F}_{m,\Delta}$ est donné par une représentation unitaire ρ_m du groupe fondamental de T. Le groupe $\pi_1(T)$ étant abélien, cette représentation est décomposée

en sous-représentations de dimension 1. Si l'image de ρ_m est finie, alors on peut étendre les sections pluricanoniques sur F à X par transport parallèle ; si l'image de ρ_m est infinie, alors a fortiori $\kappa(X, K_X + \Delta) \geqslant 1$ par une version pluricanonique klt du théorème de structure sur les lieux de sauts de cohomologie à la Green-Lazarsfeld-Simpson (c.f. [GL91 ; Sim93]) suivante, ce qui est un autre ingrédient clé dans la démonstration du théorème A(II).

Théorème F. *Soit $g : X \to Y$ un morphisme entre variétés kählériennes compactes. Soit Δ un $\mathbb{Q}$-diviseur effectif sur X tel que la paire (X, Δ) soit klt. Alors pour tout $m > 0$ tel que $m\Delta$ est un diviseur entier et pour tout $k > 0$, le lieu de sauts de cohomologie*

$$V_k^0\left(g_*\left(K_X^{\otimes m} \otimes \mathscr{O}_X(m\Delta)\right)\right) := \left\{ \rho \in \mathrm{Pic}^0(Y) \;\middle|\; \mathrm{h}^0(Y, g_*(K_X^{\otimes m} \otimes \mathscr{O}_X(m\Delta)) \otimes \rho) \geqslant k \right\}$$

est une réunion finie des translations de torsion des sous-tores dans $\mathrm{Pic}^0(Y)$.

L'étude des lieux de sauts de cohomologie est initiée par les travaux de Green-Lazarsfeld [GL87 ; GL91] qui affirme que chaque composante des lieux de sauts de cohomologie est une translation d'un sous-tore, et est développée davantage par Carlos Simpson dans [Sim93], où il prouve que cette translation est de torsion. Récemment, le résultat principal de [Sim93] est généralisé par Botong Wang au cas kählérien dans [Wan16a], où il traite le cas où $g = \mathrm{id}_X$, $m = 1$ et $\Delta = 0$ dans l'énoncé du théorème F, and c'est le point de départ de notre preuve du théorème F. En effet, quand $g = \mathrm{id}_X$ et X projective, la preuve du théorème a été implicitement comprise dans [CKP12] tandis qu'ils seulement énoncent explicitement et prouvent dans [CKP12] un résultat correspondant à notre corollaire 3.3.9 avec X projective lisse et (X, Δ) log canonique en appliquant [Sim93] ; on suit alors la stratégie dans [CKP12] pour déduire le théorème F du cas fondamental traité dans [Wan16a, Corollary 1.4]. Notons que [Wan16a], et donc le théorème F exige que X soit « globalement » kählérienne ; en revanche, le théorème D est vrai pour toute fibration kählérienne (X est seulement supposée d'être kählérienne localement au-dessus de Y). Remarquons que dans l'hypothèse de la conjecture $C^{\log}_{n,m}$ il est essentiel de supposer que X est « globalement » kählérienne, en effet [Uen75, Remark 15.3, p. 187] fournit un exemple de fibration kählérienne pour laquelle $C_{n,m}$ n'est pas vraie.

Expliquons comment finir la preuve du théorème A(II) en appliquant le théorème F. En suivant l'argument dans [CP11] on déduit facilement du théorème F (c.f. le corollaire 3.3.9) que :

- $K_X + \Delta$ est le $\mathbb{Q}$-fibré en droites le plus effectif dans sa classe numérique.
- Si $\kappa(X, K_X + \Delta) = \kappa(X, K_X + \Delta + L) = 0$ pour un certain ($\mathbb{Q}$-)fibré en droites numériquement trivial L, alors L est un point de torsion dans $\mathrm{Pic}^0(X)$.

Maintenant la preuve du théorème A(II) peut s'accomplir comme suivant : si $\mathrm{Im}(\rho_m)$ est infinie, alors par la décomposition de $\mathscr{F}_{m,\Delta}$ on voit que $K_X + \Delta$ a dimension de Kodaira non-négative quitte à tensoriser avec un ($\mathbb{Q}$-)fibré en droites numériquement trivial non-torsion, donc le premier point ci-dessus prouve que $\kappa(X, K_X + \Delta) \geqslant 0$; de plus, si $\kappa(X, K_X + \Delta) = 0$ alors le second point mène à une contradiction, donc a fortiori $\kappa(X, K_X + \Delta) \geqslant 1$, ce qui achève la preuve du théorème A. Comme un sous-produit du premier point ci-dessus, on obtient une version log kählérienne de la conjecture (généralisée) d'abondance au cas où la dimension numérique est zéro (c.f. le théorème 3.3.11) en utilisant la décomposition divisorielle de Zariski obtenue dans [Bou04] (c.f.[Bou04, Definition 3.7]) .

Remarquons que l'on peut suivre la même stratégie de [CP17, §5] pour démontrer plus généralement que $C^{\log}_{n,m}$ est vrai si $\det \mathscr{F}_{m,\Delta}$ est numériquement trivial pour un $m \in$

$\mathbb{Z}_{>0}$ (i.e. la version kählérienne de [CP17, Theorem 5.6]) en utilisant le résultat remarquable de Kang Zuo dans [Zuo96, Corollary 1]. Dans cette thèse, cependant, on ne poursuivra pas dans cette direction.

Pour conclure, on généralise la partie (II) du théorème A au cadre des orbifoldes géométriques en utilisant un argument de récurrence et en appliquant les résultats déjà obtenus :

Théorème G. *Soit $f : X \to T$ une fibration analytique avec X une variété kählérienne et T un tore complexe, et notons F la fibre générale de f. Soit Δ un $\mathbb{Q}$-diviseur effectif sur X tel que (X, Δ) soit klt. Alors*

$$\kappa(X, K_X + \Delta) \geqslant \kappa(F, \Delta_F) + \kappa(T, B_{f,\Delta}).$$

où $\Delta_F := \Delta|_F$ et $B_{f,\Delta}$ désigne le diviseur de branchement sur T par rapport à f et Δ.

Dans le théorème ci-dessus, le diviseur de branchement $B_{f,\Delta}$ est défini comme suivant : pour une fibration $f : (X, \Delta) \to Y$ entre variétés complexes avec Δ un $\mathbb{Q}$-diviseur effectif sur X, le diviseur de branchement $B_{f,\Delta}$ (par rapport à f et Δ) est défini comme le $\mathbb{Q}$-diviseur le plus effectif sur Y tel que $f^*B_{f,\Delta} \leqslant R_{f,\Delta}$ modulo diviseurs exceptionnels, où le diviseur de ramification (par rapport à f et Δ) est défini comme $R_{f,\Delta} := R_f + \Delta$ et

$$R_f := \sum_{f(W) \text{ est un diviseur de } Y} (\mathrm{Ram}_W(f) - 1)W$$

avec $\mathrm{Ram}_W(f)$ désigne l'indice de ramification (en codimension 1) de f le long W. Plus précisément, supposons que le lieu singulier de f est contenu dans un diviseur réduit $\Sigma_Y \subseteq Y$ et écrivons

$$f^*\Sigma_Y = \sum_{i \in I} b_i W_i,$$

où les W_i sont des diviseurs premiers sur X, alors pour $i \in I^{\mathrm{div}}$ où

$$I^{\mathrm{div}} := \text{ l'ensemble des indices } i \in I \text{ tels que } f(W_i) \text{ est un diviseur sur } Y,$$

on a $b_i = \mathrm{Ram}_{W_i}(f)$ et donc

$$R_f = \sum_{i \in I^{\mathrm{div}}} (b_i - 1)W_i.$$

Remarquons que la définition de $B_{f,\Delta}$ ci-dessus coïncide avec [Cam04, Definition 1.29] (« orbifold base ») quand Δ est lc on X, c.f. §3.5.

Sur la structure des variétés projectives klt à diviseur anticanonique nef

Une philosophie générale dans l'étude des variétés uniréglées est que les variétés dont le fibré anticanonique ou le fibré tangent admet une certaine positivité, devraient présenter une certaine rigidité birationnelle, par exeple les fibrations leur associées (l'application d'Albanese et la fibration MRC) devraient avoir une structure rigide (typiquement être une fibration localement constante). Ceci est inspirée par le travail fondamental de Shigefumi Mori [Mor79] et de Siu-Yau [SY80], ce qui montrent la conjecture de Hartshorne-Frankel ; Leurs travaux caractérisent les espaces projectifs en terme de l'amplitude du fibré tangent (qui est aussi valable en caractéristique positive), ou de façon équivalente, la positivité (stricte) de la courbure bisectionnelle holomorphe (pour les variétés kählériennes compactes). Une généralisation analytique du théorème de Mori-Siu-Yau est obtenue par Ngaiming Mok dans [Mok88] pour les variétés kählériennes

compactes à courbure bisectionnelle holomorphe : il montre que le revêtement universel d'une telle variété se décompose en un produit de $\mathbb{C}^q$, d'espaces projectifs et d'espaces hermitiens symétriques compacts (irréductibles) à rang $\geqslant 2$. Afin d'établir l'équivalent algébro-géométrique du résultat principal de [Mok88], on considère les variétés kählériennes compactes à fibré tangent nef, dont le théorème de structure est établi par [DPS94], modulo la conjecture de Campana-Peternell (elle prédit que les variétés de Fano lisse à fibré tangent nef sont rationnellement homogènes), en démontrant que l'application d'Albanese est une fibration localement constante à fibre de Fano. Puis on s'intéresse aux variétés projectives lisses (ou plus généralement, les variétés kählérienne compactes) à fibré anticanonique nef. Par les méthodes du MMP, le cas 3-dimensionnel a été étudié de façon approfondie par Thomas Peternell et ses collaborateurs dans [PS98 ; BP04]. Récemment le théorème de structure de ces variétés est établi par [Cao19 ; CH19] en appliquant la méthode de la positivité des images directes et en utilisant les résultats dans les précédents travaux [Zha96 ; Pău97 ; Pău01 ; Zha05 ; LTZZ10] ; de plus, leurs résultats sont étendus aux cas des paires klt par [CCM19] quand la variété est projective lisse. Selon la philosophie générale du MMP, il est alors naturel d'étendre ce théorème aux variétés légèrement singulières, comme énoncé dans la conjecture 2.

Afin de démontrer la conjecture 2 on suit l'idée de [Cao19 ; CH19] qui vise à prouver :

1. L'application d'Albanese $\mathrm{alb}_X : X \dashrightarrow \mathrm{Alb}_X$ de X est une fibration localement constante (définie partout) ;

2. Le groupe fondamental de X_{reg} est à croissance polynomiale, de façon équivalente (par [Gro81, Main Theorem]), $\pi_1(X_{\mathrm{reg}})$ est virtuellement nilpotent (i.e. admet un sous-groupe nilpotent d'indice finie) ;

3. Si $\pi_1(X_{\mathrm{reg}}) = \{1\}$ alors la fibration maximale rationnellement connexe (MRC) de X est définie partout et induit une décomposition de X en un produit d'une variété rationnellement connexe par une variété projective à diviseur canonique trivial.

Les points 1 and 3 ci-dessus sont démontrés dans cette thèse (c.f. le théorème B et le théorème C) tandis que le point 2 semble assez difficile, et au moins la méthode dans [Pău97] n'a pas l'air de s'appliquer à ce cas. Outre l'essai de démontrer le point 2, il est aussi espéré que l'on puisse démontrer directement la conjecture 2 sans étudier le groupe fondamental (ou au moins par montrer un résultat beaucoup moins fort sur le groupe fondamental), c.f. [CCM19] et §4.4. Comme une conséquence des théorème B et théorème C, on peut réduire la preuve de la conjecture 2 à celle de la conjecture 3 suivante. La démonstration détaillée de cette réduction est donnée dans §4.5.2.

Conjecture 3. *Soit X une variété projective normale du type semi-Fano. Alors le groupe fondamental de X_{reg} est à croissance polynomiale.*

On montrera dans §4.5 que cette conjecture généralise la conjecture de Gurjar-Zhang sur la finitude du groupe fondamental du lieu lisse des variétés du type Fano (c.f. [GZ94 ; GZ95 ; Zha95 ; Sch07 ; Xu14 ; GKP16 ; TX17 ; GGK19]), conjecture qui a été récemment établie par Lukas Braun dans [Bra20]. Elle peut se voir aussi comme une généralisation naturelle de la conjecture folklore suivante (c.f. [GGK19]) :

Conjecture 4. *Soit X une variété projective à singularités klt et à diviseur canonique trivial dont l'irrégularité augmentée est nulle. Alors le groupe fondamental de X_{reg} est fini.*

On verra dans §4.5 que la conjecture 3 implique la conjecture 4. Dans la suite expliquons brièvement les idées de la démonstration des théorème B et théorème C :

- D'abord, par une observation facile on montre que [Cao19, 2.8.Proposition] est encore valable dans le cas où l'espace est singulier (c.f. la proposition 2.3.5), donc afin de montrer qu'une fibration est une fibration localement constante on peut se ramener à montrer que les images directes des puissances d'un fibré en droites relativement amples sont numériquement plates.
- Par [CH19, Proposition 2.9] ou [Wu20, Corollary of the Main Theorem] (c.f. Proposition 2.3.6) la preuve de la platitude numériquement d'un faisceau réflexif peut se diviser en deux étapes : premièrement, prouver que l'image directe admet une suite de métriques hermitiennes singulière qui lui rend faiblement semipositive ; puis prouver que le fibré déterminant de l'image directe est numériquement trivial. La première partie se déduit du résultat général de positivité des images directes (c.f. [CCM19, Theorem 2.2] or Corollary 2.2.13) en utilisant le fait que $-K_X$ is nef, c.f. [CCM19, Lemma 3.4] ou Proposition 4.1.3 ; tandis que la seconde partie peut s'établir, au moins birationnellement, à l'aide du résultat principal de [Zha05] (c.f. Proposition 4.1.2).
- En utilisant la méthode de [LTZZ10] on peut démontrer que l'application d'Albanese de X est plate, donc on peut encore améliorer la version birationnelle du résultat de platitude numérique mentionné ci-dessus et démontrer que les image directes des puissances d'un fibré en droites relativement très ample sont numériquement plates ; par la proposition 2.3.5 ceci prouve le théorème B .
- Quant au théorème C, un argument similaire mais beaucoup plus subtile comme [CH19, §3.C] s'appliquant à la fibration MRC de X montre que birationnellement X peut se décomposer en un produit, ce qui donne lieu à un scindage de T_X en somme directe de deux feuilletages algébriques, un dont l'adhérence de la feuille générale est rationnellement connexe, l'autre à classe canonique triviale. Cependant, X étant singulière et ces feuilletages étant singuliers, on ne peut directement appliquer [Hör07, 2.11.Corollary]. Afin de surmonter ces difficulté on observe que la décomposition implique les deux feuilletages sont faiblement réguliers, et donc que l'on peut utiliser des résultats dans [Dru18b] pour montrer que la fibration MRC est définie partout. Dans cette situation, l'on peut raisonner de façon similaire à la preuve du Théorème B pour montrer la platitude numérique des images directe à un changement de base près, et finalement [Dru18a, Lemma 4.6] nous permet à conclure.

Organisation de la thèse

Cette thèse est organisée comme suit : dans le chapitre 1 l'on rappelle des résultats préliminaires qui seront utiles dans la démonstration des théorèmes principaux ; puis le chapitre 2 se consacre au développement des outils principaux nécessaire dans la thèse, comme mentionné ci-dessus dans la section de Méthodologie, en particulier le théorème D est montré dans §2.2. Ensuite, les deux chapitres qui suivent se consacrent à la preuve des théorème principaux dans la thèse :

- Dans le chapitre 3 l'on considère la conjecture $C_{n,m}$ d'Iitaka pour les fibration kählérienne, et l'on prouve le théorème A. En particulier, le théorème A(I) est démontré dans §3.1, le théorème E est démontré dans §3.2, dans §3.3 l'on démontre le théorème F et dans §3.4 on conclut la démonstration du théorème A en combinant les précédents résultats, enfin la preuve du théorème G est donnée dans §3.5.
- Dans le chapitre 4 on étudie les variétés projectives klt à diviseur anticanonique nef. En particulier, le théorème B et le théorème C sont démontrés respectivement dans

§4.2 et §4.3, et dans §4.5 on étudie le groupe fondamental du lieu lisse de telles variétés, en particulier l'on prouve que la conjecture 2 se déduit de la conjecture 3. La section §4.4 vient s'ajouter après que toutes les autre parties du chapitre 4 ont été accomplies, où l'on discute les feuilletages (surtout ceux qui sont algébrique) à classe canonique numériquement triviale en suivant les suggestions de Stéphane Druel.

Chapter 1

Preliminary results

In this chapter we collect some preliminary results which will be used for the proof of our main theorems.

1.1 An analytic geometry toolkit

In this section we recall some auxiliary results which are well-known in algebraic geometry, but whose analytic versions, as far as we know, have not yet been well formulated in literatures; we will not give the detailed proofs but instead indicate how to get rid of the algebraicity hypothesis.

(A) A Covering Lemma

First we state a covering lemma which allow us to reduce problems on pluricanonical bundles to the case of the canonical bundle.

Lemma 1.1.1. *Let X be compact complex manifold. and let L be a line bundle on X such that $\kappa(X,L) \geqslant 0$. Suppose that there exists an integer $m > 0$ such that there exists an effective divisor $D \in \left|L^{\otimes m}\right|$ whose support is SNC. Then there is a compact complex manifold V admitting a surjective generically finite projective morphism $f : V \to X$ such that the direct image of K_V admits a direct decomposition:*

$$f_* K_V \simeq \bigoplus_{i=0}^{m-1} K_X \otimes L^{\otimes i} \otimes \mathscr{O}_X(-\left\lfloor \frac{i}{m} D \right\rfloor).$$

The construction of f is done by taking a cyclic cover along D followed by a desingularization. This construction is standard. However, there are three main ingredients in this construction that need to be clarified:

(a) The construction of cyclic covers: c.f. [Laz04, §4.1.B, pp. 242-243, vol.I] and [Kol97, §2.9, p. 9], which can be easily generalized to the analytic case.

(b) Viehweg's results on rational singularities in [Vie77]:

  (b1) A finite ramified cover over a smooth projective variety with the cover space being normal and the branching locus being a SNC divisor, has quotient singularities ([Vie77, Lemma 2]); in this case, the singularity is toroidal, and the result is standard from [KKMS73].

  (b2) A quotient singularity is a rational singularity ([Vie77, Proposition 1]). This follows from Kempf's criterion on rationality of singularities (c.f. [KKMS73, §I.3, condition (d)(e) pp. 50-51]), which is essentially an analytic result.

(c) A duality theorem for canonical sheaves (the canonical sheaf of a complex variety is defined as the $(-d)$-th cohomology of the dualizing complex, where d denotes the dimension of the complex variety) on singular complex varieties, which can be proved by applying [RR70] or [BS76] combined with a spectral sequence argument.

Remark 1.1.2. For later use, we remark that the point (b2) above can be further generalized to higher relative dimension by a local computation as in [Vie83, Lemma 3.6] and by [KKMS73]: for $f : X \to Y$ be a proper flat morphism between complex manifolds such that the singular locus $\Sigma_Y \subsetneq Y$ is a smooth divisor and the preimage $f^*\Sigma_Y$ is a *reduced* SNC divisor, then for any surjective morphism $\phi : Y' \to Y$ with Y' smooth, the fibre product $X \underset{Y}{\times} Y'$ has (at most) rational singularities. C.f. also [Hör10, 3.13.Lemma].

(B) A Flattening Lemma

In order to prove Theorem A we need the following auxiliary result, which is an analytic version of [Vie83, Lemma 7.3]:

Lemma 1.1.3. *Let $p : V \to W$ a morphism of complex manifolds, then there exists a commutative diagram*

$$\begin{array}{ccc} V' & \xrightarrow{\pi_V} & V \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} \\ W' & \xrightarrow[\pi_W]{} & W \end{array}$$

with V' and W' complex manifolds, the morphisms π_W and π_V projective and bimeromorphic such that the morphism p' verifies the following propriety: every p'-exceptional (i.e. $\operatorname{codim}_{W'} p'(D') \geqslant 2$) divisor D' of V' is π_V-exceptional (i.e. $\operatorname{codim}_V(\pi_V(D')) \geqslant 2$). In addition, we can further assume that

(a) *π_W is an isomorphism over W_0, the (analytic) Zariski open subset of$W$ over which p is smooth;*

(b) *π_V is an isomorphism over $p^{-1}W_0$;*

(c) *$\Sigma_{W'} := \pi_W^{-1}(W \setminus W_0)$ and $p'^*\Sigma_{W'}$ are divisors of SNC support.*

Proof. This is simply a consequence of [Hir75, Flattening Theorem]. □

In the sense of [Cam04], the lemma above shows that any fibre space admits a (higher) bimeromorphic model which is *neat* and *prepared* (c.f. [Cam04, §1.1.3]). Moreover, Lemma 1.1.3 is well behaved with respect to klt/lc pairs, as implies the following fact:

Lemma 1.1.4. *Let X be a complex variety and Δ an effective $\mathbb{Q}$-divisor on X such that the pair (X, Δ) is klt (resp. lc). For any log resolution $\mu : X' \to X$ of (X, Δ), there is an effective $\mathbb{Q}$-divisor Δ' over X' with SNC support such that the pair (X', Δ') is also klt (resp. lc) and that $\mu_*\Delta' = \Delta$. Moreover we have $\kappa(X', K_{X'} + \Delta') = \kappa(X, K_X + \Delta)$.*

Proof. This is well known to experts of MMP, we nevertheless give a proof for the convenience of the readers. The pair (X, Δ) being klt, we can write (an isomorphism of $\mathbb{Q}$-line bundles):

$$K_{X'} + \mu_*^{-1}\Delta - \sum_{a_i<0} a_i E_i \simeq \mu^*(K_X + \Delta) + \sum_{a_i>0} a_i E_i, \tag{1.1}$$

where the E_i's are μ-exceptional prime divisors and

$$a_i := a(E_i, X, \Delta)$$

denotes the discrepancy of E_i with respect to the pair (X, Δ). Put

$$\Delta' := \mu_*^{-1}\Delta - \sum_{a_i<0} a_i E_i\,,$$

then Δ' is an effective $\mathbb{Q}$-divisor with SNC support and $\mu_*\Delta' = \Delta$. The hypothesis that (X, Δ) is klt (resp. lc) implies that $a_i > -1$ (resp. $a_i \geqslant -1$) for every i and that the coefficients of prime components in Δ are < 1 (resp. $\leqslant 1$), hence the coefficients of the prime components in Δ' are all < 1 (resp. $\leqslant 1$). By [KM98, Corollary 2.31(3), p. 53] the pair (X', Δ') is klt (resp. lc). The equality $\kappa(X', K_{X'} + \Delta') = \kappa(X, K_X + \Delta)$ results from [Debo1, Lemma 7.11, p. 175] and (1.1). □

1.2 Negativity Lemma in analytic geometry

The negativity lemma is an important auxiliary result in the study of the bimeromorphic/birational classification of complex analytic/algebraic varieties. In the algebraic setting it is well known, c.f. [KM98, Lemma 3.39, p. 102-103]. However, to our knowledge the analytic version of the lemma has never been explicitly written and proven in the literatures. In this section we will give a detailed proof. The idea of the proof comes from [BdF12, Proposition 2.12].

Lemma 1.2.1 (the Negativity Lemma)**.** *Let $h : Z \to Y$ be a proper bimeromorphic morphism between normal complex varieties. Let B be a Cartier divisor on Z such that $-B$ is h-nef. Then B is effective if and only if h_*B is effective.*

Proof. First notice that if B is effective, then h_*B is effective; hence it remains to show that h_*B is effective $\Rightarrow$ B is effective. To this end we proceed in three steps:

(A) Reduction to the case where h is a sequence of blow-ups with smooth centres
For any proper bimeromorphic morphism $f : Z' \to Z$, B is effective $\Leftrightarrow$ f^*B is effective; moreover, if we note $h' = h \circ f$, then $h'_* f^*B = h_*B$ and $-f^*B$ is h'-nef. This observation gives us the flexibility to replace Z with a higher bimeromorphic model. In particular, by Chow's Lemma ([Hir75, Corollary 2]) we can suppose that h is projective. In addition, by Hironaka's construction in [Hir75] we see that h is in fact the blow-up of an analytic subspace (a coherent ideal) of X (c.f. [Hir75, Definition 4.1]); hence by Hironaka's resolution of singularities, we can take a principalization h' of this ideal, which is constructed by a sequence of blow-ups with smooth centres, by the universal property of blow-ups, h' dominates h. C.f. also [BJ17, Lemma 4.1]. Now up to replacing h' with h, we can assume that h is a locally finite (over Y) sequence of blow-ups with smooth centres; moreover the problem being local over Y, one can further assume that h is a finite sequence. In particular, (e.g. by an induction on the number of blow-ups contained in h) there exists an effective Cartier divisor h-exceptional divisor A such that $-A$ is h-ample.

(B) Reduction to the case where $-B$ is h-ample by an approximation argument
In this step we use an approximation argument to reduce to the case where $-B$ is h-ample. To this end, assume that the lemma is true for h-anti-ample divisors. By Step (A), one gets an h-exceptional divisor A such that $-A$ is h-ample. Since $h_*A = 0$, our assumption implies that A is effective. For every $m > 0$, the Cartier divisor$-mB - A$ is h-ample; in addition, $h_*(mB + A) = mh_*B \geqslant 0$, hence by our assumption, $mB + A$ is effective.

By arguing coefficients by coefficients and by letting m tend to $+\infty$ we obtain that B is effective[1].

(C) The case where $-B$ is h-ample

By the reduction procedures (A) and (B), we can suppose that h is projective and that B is a Cartier divisor on Z such that $-B$ is h-ample. Since $-B$ is h-ample, then for any $m >> 0$, the Cartier divisor $-mB$ is relatively globally generated, i.e. we have an surjection

$$h^*h_*\mathscr{O}_Z(-mB) \twoheadrightarrow \mathscr{O}_Z(-mB).$$

In particular, $\mathscr{O}_Z(-mB) = h^{-1}\mathfrak{a}_m \cdot \mathscr{O}_Y$ where $\mathfrak{a}_m = h_*\mathscr{O}_Z(-mB)$ fractional ideal on Y (i.e. a torsion free subsheaf of rank 1 of $\mathscr{M}_Y$ the sheaf of germs of meromorphic functions on Y) since h is bimeromorphic. It remains to see that $\mathfrak{a}_m$ is an authentic ideal. To this end it suffices to consider the inclusion (by hypothesis h_*B is effective)

$$\mathfrak{a}_m = h_*\mathscr{O}_Z(-mB) \subseteq \mathscr{O}_Y(-mh_*B) \subseteq \mathscr{O}_Y,$$

where the inclusion $h_*\mathscr{O}_Z(-mB) \subseteq \mathscr{O}_Y(-mh_*B)$ above results from Lemma 1.3.2.

□

1.3 Reflexive hull of the direct image of line bundles

In this subsection we will prove the following theorem, which is nothing but an analytic version of [Nak04, III.5.10.Lemma, pp. 107-108]. The proof of the theorem is not essentially different from that in [Nak04]; except that, for the analytic case, one has to modify the arguments, especially in the Step 2 below, so that on can avoid the usage of the relative Zariski decomposition (which is not known in analytic case; even in the algebraic case, it is only established in some special cases in [Nak04] and it does not hold in general due to a counterexample in [Les16]).

Theorem 1.3.1. *Let $\pi : X \to S$ be a proper surjective morphism between normal complex varieties, and let L be a π-effective (i.e. $\pi_*L \neq 0$) line bundle on X. Then there is an effective π-exceptional (i.e. $\operatorname{codim}_S \pi(E) \geqslant 2$) Weil divisor E such that for any $k \in \mathbb{Z}_{>0}$ one has*

$$\left[\pi_*\left(L^{\otimes k}\right)\right]^{\wedge} \simeq \pi_*\left[L^{\otimes k} \otimes \mathscr{O}_X(kE)\right]. \tag{1.2}$$

Intuitively the theorem means that the vertical poles of the sections of $L^{\otimes k}$ are linearly bounded. The proof of Theorem 1.3.1 proceeds in five steps:

Step 0 First let us remark that we can always assume that X is smooth by taking a desingularization by the following observation

Lemma 1.3.2. *Let $h : Z \to Y$ a bimeromorphic morphisme between normal complex varieties. Then for every Weil divisor D on Z, we have an inclusion*

$$h_*\mathscr{O}_Z(D) \subseteq \mathscr{O}_Y(h_*D).$$

Proof. Since h is an isomorphism over a(n) (analytic) Zariski open subset of codimension $\geqslant 2$ in Y, $h_*\mathscr{O}_Z(D)$ and $\mathscr{O}_Y(h_*D)$ are isomorphic in codimension 1; $h_*\mathscr{O}_Z(D)$ being torsion

[1] Let us remark that in many cases, when there is no ambiguity, by saying that "a divisor is effective" we mean that it is *linearly equivalent to* an effective divisor; but in the statement of the Negativity Lemma 1.2.1 we take "effectivity" in its strict sense.

free and $\mathscr{O}_Y(h_*D)$ reflexive, we have (noting that on a normal complex variety reflexive sheaves are determined in codimension 1):

$$h_*\mathscr{O}_Z(D) \hookrightarrow (h_*\mathscr{O}_Z(D))^\wedge \simeq \mathscr{O}_Y(h_*D).$$

□

In fact, assume that Theorem 1.3.1 holds for X smooth, let us prove that it holds in general. To this end, let $\mu : X' \to X$ be a desingularization of X, then by our assumption, there is an effective divisor E' on X' such that

$$\left[\pi'_*\left(\mu^* L^{\otimes k}\right)\right]^\wedge = (\pi')_*\left[\mu^* L^{\otimes k} \otimes \mathscr{O}_{X'}(kE')\right],$$

hence by Lemma 1.3.2 and the projection formula we have an inclusion

$$\left[\pi_*\left(L^{\otimes k}\right)\right]^\wedge = \pi_*\left(L^{\otimes k} \otimes \mu_*\mathscr{O}_{X'}(kE')\right) \hookrightarrow \pi_*\left(L^{\otimes k} \otimes \mathscr{O}_X(kE)\right) \hookrightarrow \left[\pi_*\left(L^{\otimes k} \otimes \mathscr{O}_X(kE)\right)\right]^\wedge$$

where $E := \mu_* E'$. Since the inclusion is an isomorphism in codimension 1, it is in fact an equality. Consequently, we always assume that X is smooth in the sequel.

Step 1 By the coherence of the reflexive hull $(\pi_* L)^\wedge$ there is an π-exceptional divisor E making the equation (1.2) holds for $k = 1$ (and thus one can choose E such that (1.2) holds for a finite number of k).

Step 2 In virtue of **Step 1** we are able to prove the reflexivity criterion below:

Proposition 1.3.3 (Reflexivity Criterion). *Let $\pi : X \to S$ and L as in Theorem 1.3.1. Suppose that for every effective π-exceptional divisor G, there is a component Γ of G such that $[L \otimes \mathscr{O}_X(G)]|_\Gamma$ is not $\pi|_\Gamma$-pseudoeffective, then $\pi_* L$ is reflexive on S.*

Let us recall the notion of relative pseudoeffectivity for (ℚ-)line bundles / Cartier divisors in the analytic setting: Let $p : V \to W$ a proper surjective morphism of analytic varieties and let L be a ℚ-line bundle on V, then L is said to be p-pseudoeffective if its pull-back $L|_{\tilde{F}}$ is pseudoeffective (c.f. [Dem10, §6.A, (6.2) Definition, p. 47]) where $\tilde{F}$ denotes a desingularization of the general fibre F of p. A ℚ-Cartier divisor D on V is said to be p-pseudoeffective if its associated ℚ-line bundle $\mathscr{O}_X(D)$ is so. Before going to the proof let us first prove the following auxiliary lemma:

Lemma 1.3.4. *Let $\pi : X \to S$ and L as in Proposition 1.3.3, then for any effective π-exceptional divisor B on X, one has:*

$$\pi_* L \simeq \pi_*[L \otimes \mathscr{O}_X(B)] \tag{1.3}$$

Proof. B is effective, one can write

$$B = \sum_{i=1}^{r} b_i B_i,$$

with $b_i \in \mathbb{Z}_{>0}$ and $r \in \mathbb{N}$ ($r = 0$ simply means that $B = 0$). Note

$$b := \sum_{i=1}^{r} b_i.$$

Now let us prove (1.3) by induction on b : By our hypothesis on L (the condition in Proposition 1.3.3), $\exists\, i \in \{1, \cdots, r\}$ such that $[L \otimes \mathscr{O}_X(B)]|_{B_i}$ is non-$\pi|_{B_i}$-pseudoeffective, thus

$$(\pi|_{B_i})_*[L \otimes \mathscr{O}_X(B)]|_{B_i} = 0.$$

Consider the short exact sequence

$$0 \longrightarrow \mathscr{O}_X(-B_i) \longrightarrow \mathscr{O}_X \longrightarrow \mathscr{O}_{B_i} \longrightarrow 0.$$

By tensoring with $L \otimes \mathscr{O}_X(B)$ and applying the functor π_* one gets

$$0 \to \pi_*[L \otimes \mathscr{O}_X(B - B_i)] \longrightarrow \pi_*[L \otimes \mathscr{O}_X(B)] \longrightarrow (\pi|_{B_i})_*[L \otimes \mathscr{O}_X(B)]|_{B_i} = 0$$

hence $\pi_*[L \otimes \mathscr{O}_X(B - B_i)] \simeq \pi_*[L \otimes \mathscr{O}_X(B)]$. Apply the induction hypothesis we obtain that $\pi_*[L \otimes \mathscr{O}_X(B - B_i)] \simeq \pi_* L$, which proves the isomorphism (1.3). □

Now return to the proof of the Reflexivity Criterion 1.3.3 :

Proof of Proposition 1.3.3. By **Step 1** there is an effective π-exceptional E, such that

$$(\pi_* L)^\wedge \simeq \pi_*[L \otimes \mathscr{O}_X(E)]\,;$$

Apply Lemma 1.3.4 to E and we obtain:

$$\pi_* L \simeq \pi_*[L \otimes \mathscr{O}_X(E)] \simeq (\pi_* L)^\wedge\,,$$

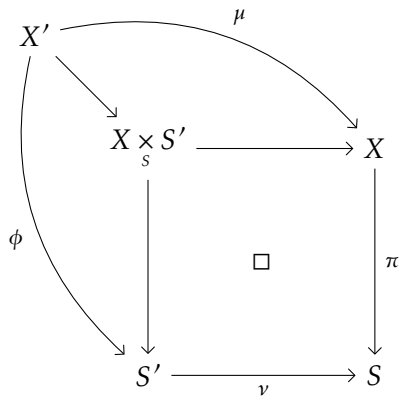
hence $\pi_* L$ is reflexive. □

Step 3 In this step, we prove that in the situation of Theorem 1.3.1 there exists a π-exceptional divisor which is not relatively pseudoeffective on each component of $\mathrm{Exc}(\pi)$. More precisely we will show:

Proposition 1.3.5. *For any $\pi : X \to S$ as in the Theorem 1.3.1, there is an effective π-exceptional divisor E such that for any π-exceptional prime divisor Γ, $E|_\Gamma$ is not $\pi|_\Gamma$-pseudoeffective.*

The proof is the same as that in [Nako4, III.5.10.Lemma, pp. 107-108]. For the convenience of the readers, we provide the details below:

Proof. The starting point of the proof is the following observation: if π is flat, then $\pi_* L$ is always reflexive. Consider thus a flattening of π (c.f. [Hir75], or for the algebraic case, [Ray72, §4.1, Theorem 1, p. 26]): let $\nu : S' \to S$ be a projective bimeromorphic morphism (a sequence of blow-ups with smooth centres) which flattens π and let X' be the normalization of the main component of $X \underset{S}{\times} S'$ equipped with morphisms $X' \xrightarrow{\mu} X$ and $X' \xrightarrow{\phi} S'$ (μ is projective and ϕ is equidimensional).



By the construction of ν, there is a π-exceptional effective (Cartier) divisor Δ such that $-\Delta$ is ν-ample. Consider the divisor $E := \mu_*(\phi^*\Delta)$. Then E is effective since Δ is effective; E is Cartier since X is smooth. Moreover, $-\Delta$ is ν-ample, hence $-\phi^*\Delta$ is μ-nef: in fact, let C be a curve contracted by μ, then ϕ_*C (which is, by definition, a curve on S' if C is not contracted by ϕ or is equal to 0 otherwise) is contracted by ν since $\pi \circ \mu = \nu \circ \phi$, hence by the projection formula we get

$$(-\phi^*\Delta \cdot C) = (-\Delta \cdot \phi_*C) \geqslant 0,$$

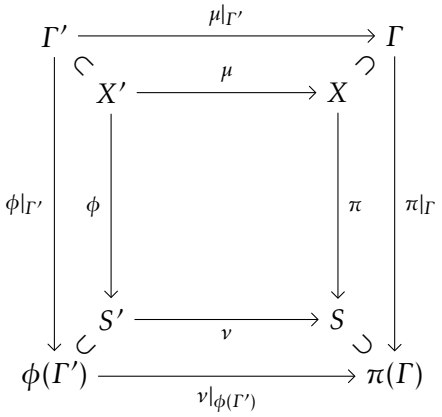
μ being projective, this implies that $-\phi^*\Delta$ is μ-nef; then so is $\mu^*E - \phi^*\Delta$. Now since

$$\mu_*(\mu^*E - \phi^*\Delta) = E - E = 0,$$

then we have $\mu^*E - \phi^*\Delta \leqslant 0$ by the Negativity Lemma 1.2.1.

Assume by contradiction that there exists a π-exceptional prime divisor Γ such that $E|_\Gamma$ is $\pi|_\Gamma$-pseudoeffective and set

$$\Gamma' := \text{ the strict transformation of } \Gamma \text{ by } \mu^{-1}.$$



Then $\mu^*E|_{\Gamma'}$ is $(\pi\circ\mu)|_{\Gamma'}$-pseudoeffective, hence $\phi^*\Delta|_{\Gamma'}$ is $(\nu\circ\phi)|_{\Gamma'}$-pseudoeffective since $\mu^*E \leqslant \phi^*\Delta$. On the other hand, by our construction $-\Delta$ is ν-ample, then $-\Delta|_{\phi(\Gamma')}$ is $\nu|_{\phi(\Gamma')}$-ample, and thus

$$-\phi^*\Delta|_{(\Gamma')} = (\phi|_{(\Gamma')})^*(-\Delta|_{\phi(\Gamma')})$$

is $(\nu \circ \phi)|_{\Gamma'}$-nef. Therefore $-\phi^*\Delta|_{(\Gamma')}$ is $(\nu \circ \phi)|_{\Gamma'}$-numerically trivial, which implies that $-\Delta|_{\phi(\Gamma')}$ is $\nu|_{\phi(\Gamma')}$-numerically trivial. But $-\Delta|_{\phi(\Gamma')}$ is $\nu|_{\phi(\Gamma')}$-ample, this cannot happen unless $\nu|_{\phi(\Gamma')} : \phi(\Gamma') \to \pi(\Gamma)$ is finite. We finally get the sought contradiction by showing that $\nu|_{\phi(\Gamma')}$ cannote be finite:

Since ϕ is the composition of a finite morphism (normalization) followed by a flat morphism, ϕ is equidimensional; in particular, $\phi(\Gamma')$ is Weil divisor on S. Moreover, $\nu(\phi(\Gamma')) = \pi \circ \mu(\Gamma') = \pi(\Gamma)$ is of codimension $\geqslant 2$, hence $\phi(\Gamma')$ is ν-exceptional; in particular, the general fibre of the morphism $\nu|_{\phi(\Gamma')} : \phi(\Gamma') \to \pi(\Gamma)$ is of dimension $\geqslant 1$. Thus we prove the proposition. □

Step 4 Let $\pi : X \to S$ and L a π-effective line bundle on X as in Theorem 1.3.1. The problem begin local, one can replace X (resp. S) by a neighbourhood of a compact in X (resp. in S); in particular the set of π-exceptional prime divisors, denoted by $\mathscr{Exc}(\pi)$, is a finite set, and thus we can write:

$$\mathscr{Exc}(\pi) = \{\Gamma_1, \cdots, \Gamma_t\}$$

By **Step 3** an effective π-exceptional divisor E such that $E|_{\Gamma_i}$ is non-$\pi|_{\Gamma_i}$-pseudoeffective. In the sequel we will deduce Theorem 1.3.1 from the Reflexivity Criterion 1.3.3:

1. E being π-exceptional effective, we can write

$$E = \sum_{i=1}^{t} a_i \Gamma_i\,,\ a_i \in \mathbb{Z}_{\geqslant 0}\,.$$

We claim that the a_i's are all strictly positive. Otherwise, there exists a j such that $a_j = 0$, implying that $\Gamma_j \not\subseteq \mathrm{Supp}(E)$, then $E|_{\Gamma_j}$ is an effective divisor, in particular it is $\pi|_{\Gamma_j}$-pseudoeffective, contradicting the hypothesis on E.

2. Moreover we claim that there is a $b \in \mathbb{Z}_{>0}$ such that $\forall \beta \geqslant b, \beta \in \mathbb{Q}_{>0}$, $\left(L+\beta E\right)\Big|_{\Gamma_i}$ is a $\mathbb{Q}$-line bundle which is non-$\pi|_{\Gamma_i}$-pseudoeffective for all $i = 1, 2, \cdots, t$. Otherwise there is a sequence of positive rational numbers $\beta_n \to +\infty$ such that for every n, $\left(L+\beta_n E\right)\Big|_{\Gamma_{i_n}}$ is a $\pi|_{\Gamma_{i_n}}$-pseudoeffective $\mathbb{Q}$-line bundle for some i_n. $\mathscr{E}textxc(\pi)$ being finite, we can assume that there exists an index i such that $\left(L+\beta_n E\right)\Big|_{\Gamma_i}$ is π_{Γ_i}-pseudoeffective for every n (up to considering a subsequence). Hence

$$\left(E + \frac{1}{\beta_n} L\right)\Big|_{\Gamma_i}$$

is an $\pi|_{\Gamma_i}$-pseudo-effective $\mathbb{Q}$-line bundle for every n. This implies (by letting $\beta_n \to +\infty$) that $E|_{\Gamma_i}$ is $\pi|_{\Gamma_i}$-pseudoeffective, contradicting to the point 1 above.

3. Let us set

$$L_k = L^{\otimes k} \otimes \mathscr{O}_X(kbE),$$

then in order to prove Theorem 1.3.1 we only need to show that $\pi_* L_k$ is reflexive. In fact, since S is normal, and since $\pi_*\left(L^{\otimes k}\right)$ and $\pi_* L_k$ are isomorphic outside an analytic subset of codimension $\geqslant 2$, therefore as soon as $\pi_* L_k$ is reflexive, we get immediately

$$\pi_* L_k \simeq \left[\pi_*\left(L^{\otimes k}\right)\right]^{\wedge}.$$

We finally prove that $\pi_* L_k$ is reflexive in virtue of Proposition 1.3.3. It suffices to check that L_k satisfies the conditions in Proposition 1.3.3: let G be an π-exceptional effective divisor, then there is a minimal $c \in \mathbb{Q}_{>0}$ such that $cE \geqslant G$. In fact, if we write

$$G = \sum_{i=1}^{t} g_i \Gamma_i\,,$$

then we can take

$$c = \max_{i=1,\cdots,t} \left\{\frac{g_i}{a_i}\right\}.$$

In particular, by the minimality of c there exists an i such that $\Gamma_i \not\subseteq \mathrm{Supp}(cE - G)$, implying that the $\mathbb{Q}$-divisor $\left(cE-G\right)\Big|_{\Gamma_i}$ is $\pi|_{\Gamma_i}$-pseudoeffective. However by the point 2 above, the $\mathbb{Q}$-line bundle

$$\left(L_k + G\right)\Big|_{\Gamma_i} + \left(cE - G\right)\Big|_{\Gamma_i} = k\left[L + \left(b + \frac{c}{k}\right)E\right]\Big|_{\Gamma_i}$$

is non-$\pi|_{\Gamma_i}$-pseudoeffective, hence a fortiori the line bundle $\left(L_k + G\right)\Big|_{\Gamma_i}$ is not $\pi|_{\Gamma_i}$-pseudoeffective. Therefore L_k satisfies the conditions in Proposition 1.3.3, thus L_k is reflexive. This ends the proof of Theorem 1.3.1. □

1.4 Singular Hermitian metrics over vector bundles

In this section we recall the notion of (Griffiths) semipositively curved singular Hermitian metrics on vector bundles / torsion free sheaves. Let us fix X a complex manifold.

Definition 1.4.1. Let E be holomorphic vector bundle on X. A (Griffiths) semipositively curved singular Hermitian metric h on E is given by a measurable family of Hermitian functions on each fibre of E, such that for every (holomorphic) local section $s \in \mathrm{H}^0(U, E^*)$ of the dual bundle E^*, the function $\log|\sigma|^2_{h^*}$ is psh on U. The vector bundle E is said semipositively curved if it admits a semipositively curved singular metric.

Remark 1.4.2. This definition implies that h is bounded almost everywhere, moreover, fix any smooth Hermitian metric h_0 on E, then as a consequence of [Pău16, 2.10.Remark, 2.18.Remark] the singular metric h is locally uniformly bounded from below by $C \cdot h_0$ for some constant $C > 0$.

The semi-positivity of singular Hermitian metrics is preserved by tensor products, pull-back by proper surjective morphisms, and by generically surjective morphisms of vector bundles (thus by symmetric and exterior products), c.f. [GG18, II.B.4] and [Pău16, 2.14.Lemma, 2.15.Lemma]. Moreover one has the following extension theorem for semipositively curved singular Hermitian metrics:

Proposition 1.4.3 (c.f. [CH19, 2.4.Proposition])**.** *Let E be a holomorphic vector bundle on X. Suppose that there is a (analytic) Zariski open subset $X_0 \neq \emptyset$ of X and a semipositively curved singular Hermitian metric h on $E|_{X_0}$. Then h extends to a semipositive singular Hermitian metric on E if one of the following two conditions is verified:*

(1) $\mathrm{codim}(X \backslash X_0) \geqslant 2$;

(2) *h is locally uniformly bounded below by a constant $C > 0$ on X_0 with respect to some smooth Hermitian metric on E.*

In virtue of Proposition 1.4.3 and [Kob87, Corollary 5.5.15, p. 147] one can extend Definition 1.4.1 to torsion free sheaves:

Definition 1.4.4. Let X be a complex manifold and let $\mathscr{F}$ be a torsion free sheaf on X. By [Kob87, Corollary 5.5.15, p. 147], $\mathscr{F}$ is locally free in codimension 1. A semipositively curved singular Hermitian metric h on $\mathscr{F}$ is a semipositively curved singular Hermitian metric on $\mathscr{F}|_U$ for some (analytic) Zariski open subset U such that $\mathrm{codim}_X U \geqslant 2$ and $\mathscr{F}|_U$ locally free. The torsion free sheaf $\mathscr{F}$ is said to be semipositively curved if it admits a semipositively curved singular Hermitian metric.

Remark 1.4.5. The notion of semipositively curved metric on torsion free sheaves can lead to some unexpected pathology, e.g. e.g. the ideal sheaf $\mathscr{I}_Z$ of a analytic subset Z of codimension $\geqslant 2$ admits a natural semipositively curved singular Hermitian metric. In order to exclude such pathology we introduce in the Definition 1.4.7 below the notion of "L^2-extension property".

Let $\mathscr{F}$ and h as in the Definition 1.4.4 above, then h induces a semipositively curved singular Hermitian metric $\det h$ on the line bundle $\det\mathscr{F}$ where the determinant bundle $\det\mathscr{F}$ is defined as

$$\det\mathscr{F} := \left(\bigwedge^r \mathscr{F}\right)^\wedge$$

with $r = \mathrm{rk}\,\mathscr{F}$ and $()^\wedge = ()^{**}$ denotes the reflexive hull (c.f. [Kob87, §5.6, pp. 149-154]).

We end this subsection by two regularity theorems:

Theorem 1.4.6. *Let (E,h) be a holomorphic vector bundle on X equipped with a semipositively curved singular Hermitian metric h. Suppose that the metric $\det h$ is locally bounded from above, then the coefficients of the Chern connection form θ_E (defined by the equation $h\theta_E = \partial h$) are L^2_{loc} on X, and in consequence the total curvature current $\Theta_h(E)$ of E is well defined and semipositive in the sense of Griffiths, which can be locally written as $\Theta_h(E) = \bar{\partial}\theta_E$. In particular, if the curvature current $\Theta_{\det h}$ vanishes, then (E,h) is Hermitian flat.*

Proof. The theorem is proved in [Rau15, Theorem 1.6] by an approximation argument (c.f. also [Pău16, 2.25.Theorem, 2.26.Corollary]). Heuristically, this is a higher rank version of the well known fact (the line bundle case) that if a psh function ϕ is L^∞_{loc}, then $\nabla\phi$ is L^2_{loc}. As for the last statement (c.f. [Pău16, 2.26.Corollary] and [CP17, 2.7.Theorem]): by our first statement the total curvature current $\Theta_h(E)$ is well defined and Griffith semipositive, then the vanishing of $\Theta_{\det h}$ implies the vanishing of $\Theta_h(E)$; the regularity of h results from the ellipticity of the Laplacian $\partial\bar{\partial}$. □

In the sequel we introduce the notion of "L^2-extension property", which is simply an analogue of the property of $\mathscr{O}$ that every L^2 holomorphic function extends. It helps to exclude certain unexpected pathology as mentioned in the Remark 1.4.5, e.g. the natural semipositively curved (generically flat) singular Hermitian metric on the ideal sheaf $\mathscr{I}_Z$ of a analytic subset Z of codimension $\geqslant 2$ does not satisfy the L^2 extension property.

Definition 1.4.7. Let $\mathscr{F}$ be a torsion free sheaf on X equipped with a singular Hermitian metric h. The metric h is said to satisfy the "L^2-extension property" if for any open subset $U \subseteq X$, for any $Z \subsetneq U$ analytic subset of U such that $\mathscr{F}$ is locally free over $U\backslash Z$ and for any section $\sigma \in \mathrm{H}^0(U\backslash Z, \mathscr{F})$ such that

$$\int_{U\backslash Z} |\sigma|^2_h d\mu < +\infty,$$

the section σ extends (uniquely) to a section $\bar{\sigma} \in \mathrm{H}^0(U,\mathscr{F})$.

This propriety is particularly useful when we consider a torsion free sheaf whose determinant bundle is numerically trivial. Nevertheless let us remark that the condition on the L^2 extension property is indispensable in the theorem above. For example, as mentioned above, the ideal sheaf $\mathscr{I}_Z$ of an analytic subset Z of codimension $\geqslant 2$ admits a natural semipositively curved singular Hermitian metric $h_{\mathscr{I}_Z}$, which equals to the flat metric of $\mathscr{O}$ on $X\backslash Z$. The determinant of $\mathscr{I}_Z$ is trivial, but definitely $\mathscr{I}_Z$ is not a (Hermitian flat) vector bundle. Notice that $(\mathscr{I}_Z, h_{\mathscr{I}_Z})$ does not satisfy the L^2 extension property: let B be a small ball in X meeting Z, then non-zero constant functions on $B\backslash Z$ (which are L^2) cannot extend across Z. Now we can state:

Theorem 1.4.8. *Let X be a connected complex manifold and let $\mathscr{F}$ be a torsion free sheaf of rank r on X equipped with a semipositively curved singular Hermitian metric h. Suppose that*

(1) *$\det\mathscr{F}$ is numerically trivial, i.e. $c_1(\det\mathscr{F}) = c_1(\mathscr{F}) = 0$;*

(2) *h satisfies the L^2-extension property as in Definition 1.4.7.*

Then $(\mathscr{F},h)$ is a Hermitian flat vector bundle.

Proof. The proof is essentially analogous to that of [CP17, Theorem 5.2]. Since h is semipositively curved, the metric $\det h$ on $\det\mathscr{F}$ is semipositively curved, thus the curvature current $\Theta_{\det h}(\det\mathscr{F})$ is positive; but $\det\mathscr{F}$ is numerically trivial, hence a fortiori $\Theta_{\det h}(\det\mathscr{F}) = 0$. Then by Theorem 1.4.6, $(\mathscr{F}|_{X_\mathscr{F}}, h|_{X_\mathscr{F}})$ is a Hermitian flat vector bundle

(i.e. $h|_{X_{\mathscr{F}}}$ is a smooth Hermitian metric $\mathscr{F}|_{X_{\mathscr{F}}}$ whose curvature vanishes) where $X_{\mathscr{F}}$ denotes the locally free locus of $\mathscr{F}$. By [Kob87, Proposition 1.4.21, p. 13] the Hermitian flat vector bundle $(\mathscr{F}|_{X_{\mathscr{F}}}, h|_{X_{\mathscr{F}}})$ is defined by a representation

$$\pi_1(X_{\mathscr{F}}) \to \mathrm{U}(r).$$

The codimension of $X \backslash X_{\mathscr{F}}$ being at least 2, the group $\pi_1(X_{\mathscr{F}})$ is isomorphic to $\pi_1(X)$ and we actually have a representation

$$\pi_1(X) \to \mathrm{U}(r),$$

which gives rise to a Hermitian vector bundle (E, h_E) of rank r on X. Then by construction we have an isometry

$$\phi : \mathscr{F}|_{X_{\mathscr{F}}} \to E|_{X_{\mathscr{F}}}.$$

By reflexivity of $\mathscr{H}om_{\mathscr{O}_X}(\mathscr{F}, E)$ this extends to an injection of sheaves $\mathscr{F} \hookrightarrow E$ which we still denote by ϕ. It remains to show that ϕ is surjective. The problem being local, we can assume that X is a small open ball, so that E is trivial. Now take $u \in \mathrm{H}^0(X, E)$ a holomorphic section of E, since h_E is a flat metric (hence smooth), $|u|_{h_E,z}$ is finite for every $z \in X$. The map $\phi|_{X_{\mathscr{F}}}$ being an isometry, there exists a section $v_0 \in \mathrm{H}^0(X_{\mathscr{F}}, \mathscr{F})$ such that $i(v_0) = u|_{X_{\mathscr{F}}}$ and $|v_0|_{h,z} = |u|_{h_E,z} < +\infty$ for all $z \in X_{\mathscr{F}}$. But $(\mathscr{F}, h)$ satisfies the L^2 extension property, v_0 extends to a section $v \in \mathrm{H}^0(X, \mathscr{F})$, thus $\phi(v) = u$, implying the surjectivity of ϕ. □

1.5 Albanese map of quasi-projective varieties

In this section, we recall some general results about the Albanese maps of smooth quasi-projective varieties. Our main reference is [Fuj15, §3], c.f. also [Kaw81, §5]. First recall the definition of semi-Abelian varieties. Let us remark that they are called "quasi-Abelian varieties" by Iitaka and [Kaw81; Fuj15] (which is different from the notion of "quasi-Abelian varieties" in [AK01]); we choose to use the name "semi-Abelian variety", which seems to be more commonly used in algebraic geometry.

Definition 1.5.1 ([Kaw81, §5, Definition, p. 271]; [Fuj15, Defniition 2.8]). Let G be a connected algebraic group and let

$$1 \to H \to G \to A \to 1$$

be the Chevalley decomposition (c.f. [Con02, Theorem 1.1]) of G, where H is a linear algebraic group and A is an Abelian variety. G is called a *semi-Abelian variety* if $H \simeq \mathbb{G}_m^{\dim H}$ where $\mathbb{G}_m$ denotes the multiplicative group $\mathbb{C}^*$.

We collect some elementary properties of semi-Abelian varieties as following:

Proposition 1.5.2 ([Fuj15, Lemma 2.11, Lemma 2.13]). *Let G be a semi-Abelian variety and let*

$$1 \to \mathbb{G}_m^d \to G \to A \to 1$$

be its Chevalley decomposition with A an Abelian variety. Then

(a) *G is a principal $\mathbb{G}_m^d$-bundle over A;*

(b) *G is a commutative group;*

(c) *the universal cover of G is $\mathbb{C}^{\dim G}$ and $G \simeq \mathbb{C}^{\dim G}/\pi_1(G)$ with $\pi_1(G)$ viewed as a lattice in $\mathbb{C}^{\dim G}$.*

Analogous to the case of Abelian varieties (or even complex tori, c.f. [Uen75, Lemma 10.1, Theorem 10.3, pp. 116-119]), the closed subvarieties of semi-Abelian varieties have the following rigidity property:

Proposition 1.5.3 ([Fuj15, Theorem 4.4])**.** *Let G be a semi-Abelian variety and let W be a closed subvariety of G. Then the logarithmic Kodaira dimension $\bar{\kappa}(W) \geqslant 0$ and $\bar{\kappa}(W) = 0$ if and only if it is a translate of a semi-Abelian subvariety of G.*

Let us recall the notion of logarithmic Kodaira dimension as mentioned in the proposition above (c.f. [Kaw81, §5, Definition, p. 271] and [Fuj15, Definition 2.2]):

Definition 1.5.4. Let V° be a smooth quasi-projective variety (or more generally an algebraic variety), and take V to be a smooth compactification of V° such that $D_V := V \setminus V^\circ$ is a (reduced) SNC divisor (the existence of such V is ensured by Nagata's compactification theorem, Chow's lemma and Hironaka's resolution of singularities). Then the *logarithmic Kodaira dimension* of V°, denoted by $\bar{\kappa}(V^\circ)$ is defined to be the Iitaka-Kodaira dimension of $K_V + D_V$, that is

$$\bar{\kappa}(V^\circ) := \kappa(V, K_V + D_V).$$

Now let us turn to the Albanese maps of smooth quasi-projective varieties:

Proposition-Definition 1.5.5 ([Fuj15, Theorem 3.16])**.** Let U be a smooth quasi-projective variety and let u be a fixed point of U. Then there is a semi-Abelian variety $\widetilde{\mathrm{Alb}}_U$ and an algebraic morphism $\widetilde{\mathrm{alb}}_U : U \to \widetilde{\mathrm{Alb}}_U$ such that $\widetilde{\mathrm{alb}}_U(u) = 0$ and that for any algebraic morphism $\alpha : U \to G$ to a semi-Abelian variety G satisfying $\alpha(u) = 0_G$, there is a unique morphism of algebraic groups $f : \widetilde{\mathrm{Alb}}_U \to G$ such that $\alpha = f \circ \widetilde{\mathrm{alb}}_U$; and $\widetilde{\mathrm{alb}}_U$ is uniquely determined by this universal property. $\widetilde{\mathrm{alb}}_U$ is called the *Albanese map* of U and $\widetilde{\mathrm{Alb}}_U$ is called the *Albanese variety* of U. Moreover, if U is compact, then $\widetilde{\mathrm{alb}}_U$ coincide with the Albanese map of U.

See [Fuj15, §3] for the construction of $\widetilde{\mathrm{Alb}}_U$ and $\widetilde{\mathrm{alb}}_U$ and be careful that in [Kaw81; Fuj15] this is called the "quasi-Albanese map". Nevertheless, we call it simply the Albanese map, because this is the only reasonable one (there is no other way to define it and hence "quasi-" is a little bit redundant). Now let us recall some basic properties of the Albanese map.

Proposition 1.5.6 ([Fuj15, Lemma 3.11])**.** *Let U be a smooth quasi-projective variety and let $\widetilde{\mathrm{alb}}_U : U \to \widetilde{\mathrm{Alb}}_U$ be its Albanese map. Then the induced morphism*

$$(\widetilde{\mathrm{alb}}_U)_* : \mathrm{H}_1(U, \mathbb{Z}) \to \mathrm{H}_1(\widetilde{\mathrm{Alb}}_U, \mathbb{Z})$$

is surjective and the kernel of $(\widetilde{\mathrm{alb}}_U)_*$ is equal to the torsion part of $\mathrm{H}_1(U, \mathbb{Z})$.

Proposition 1.5.7. *Let U be a smooth quasi-projective variety and let $\widetilde{\mathrm{alb}}_U : U \to \widetilde{\mathrm{Alb}}_U$ be its Albanese map. Take V be a compactification of U such that $V \setminus U$ is SNC divisor. Let*

$$1 \to \mathbb{G}_m^d \to \widetilde{\mathrm{Alb}}_U \xrightarrow{p} A_U \to 1$$

be the Chevalley decomposition of $\widetilde{\mathrm{Alb}}_U$. Set Z be the closure of $\mathrm{Im}(\widetilde{\mathrm{alb}}_U)$. Then

(a) *A_U is isomorphic to the Albanese variety Alb_V of V such that $\mathrm{alb}_V|_U = p \circ \widetilde{\mathrm{alb}}_U$.*

(b) *Z generates $\widetilde{\mathrm{Alb}}_U$.*

Proof. (a) simply results from the construction in [Fuj15, Lemma 3.7-3.8]. As for (b), let G be the algebraic subgroup of $\widetilde{\mathrm{Alb}}_U$ generated by Z, and set W be the image of Z in A_U; then W is the image of alb_V, and by [Uen75, Lemma 9.14, pp. 108-110] (c.f. also Proposition 4.2.1(b)) W generates $A_U = \mathrm{Alb}_V$, hence the morphism $G \hookrightarrow \widetilde{\mathrm{Alb}}_U \to A_U$ is surjective, therefore we can write the Chevalley decomposition of G as

$$1 \to H \to G \to A_U \to 1.$$

with $H \subseteq \mathbb{G}_m^d$. Since H is diagonalizable, by [Spr98, 3.2.7.Corollary, p. 45] H is a direct product of a finite Abelian group with an algebraic torus; but G is connected then so is H, hence H is an algebraic torus and thus by definition G is a semi-Abelian variety. Then the morphism $U \to G$ satisfies the universal property of the Albanese map, hence a fortiori $G = \widetilde{\mathrm{Alb}}_U$. □

1.6 Horizontal divisors and base changes

Let V be a complex variety which is fibred over another complex variety W (c.f. Definition 2.3.3). By looking at the dimension of the image of its components in W, a (Weil) divisor on V can be divided into a sum of the horizontal part plus the vertical part (c.f. [Laz04, §2.1.C, Proof of Corollary 2.1.38, p. 138]). The aim of this appendix is to show that the notion of "horizontality" for divisors on an equidimensional fibre space is stable under base change. This result is of course well known to experts, we nevertheless provide a detailed account here for the convenience of the readers. The main result is the following:

Proposition 1.6.1. *Let $f : V \to W$ be an equidimensional fibre space between complex varieties with W quasi-projective and let D be a Cartier divisor on V which is horizontal with respect to V, then for any morphism $g : W' \to W$, the pullback divisor $g_V^* D$ is horizontal with respect to the base change morphism $f' : V' \to W'$ where $V' := V \underset{W}{\times} W'$ and $g_V : V' \to V$ is the natural morphism.*

The key point in the proof of the proposition above consists in proving the following auxiliary:

Lemma 1.6.2. *Let $f : V \to W$ be an equidimensional fibre space between complex varieties of relative dimension d and let D be an effective Weil divisor on V which is horizontal with respect to f. Suppose that W is quasi-projective. Then D is equidimensional over W of relative dimension $d-1$.*

Proof. By induction on the dimension of W. If $\dim W = 0$ there is nothing to prove. For arbitrary W, the result follows if $\mathscr{O}_D$ is flat over W. In general, we apply the generic flatness [Mat70, (22.A) Lemma 1, pp. 156-158] (c.f. also [HL10, Lemma 2.1.6, p. 36] and [ACG11, Proposition (3.9), p. 18]) to f and $\mathscr{O}_D$ to find an effective divisor H on W (by using quasi-projectivity of W) such that $\mathscr{O}_D$ is flat over $W \backslash H$. Since D is horizontal and f is surjective, D is mapped surjectively onto W, hence by [Mat70, (13.B) Theorem 19(2), p. 79] D is equidimensional over $W \backslash H$ of relative dimension $d-1$. Again by horizontality of D, $f^{-1}(H)$ cannot be contained in $\mathrm{Supp}(D)$, hence $D|_{f^{-1}(H)_{\mathrm{red}}}$ is still an effective Weil divisor and is horizontal over H, then by applying the induction hypothesis to $f|_{f^{-1}(H)_{\mathrm{red}}} : f^{-1}(H)_{\mathrm{red}} \to H_{\mathrm{red}}$ which is still a equidimensional fibre space of relative dimension d we see that D is equidimensional over W of relative dimension $d-1$ (H_{red} may be reducible, yet by considering component by component we can conclude). □

Now let us turn to the proof of the Proposition 1.6.1.

Proof of the Proposition 1.6.1. By definition, it suffices to treat the case that D is a prime divisor. Suppose by contradiction that $g_V^* D$ contains a component E which is vertical with respect to f'. Since f is equidimensional, then so is f', hence $f'(E)$ is of codimension 1 in W. By [Mat70, (13.B) Theorem 19(1), p. 79] the restriction of E to any fibre of f' is either empty or of dimension d, where d denotes the relative dimension of f (hence also that of f'). In consequence D must contain a d-dimensional component of a fibre of f'. But this is impossible by the Lemma 1.6.2. □

Chapter 2

Main tools

2.1 Ohsawa-Takegoshi type extension theorems

As is explained above, the key point of the proof of Theorem 2.2.4, like many other results in complex geometry, is the Ohsawa-Takegoshi extension theorem. In this subsection we will state theorems of Ohsawa-Takegoshi type for Kähler fibre spaces in the following two forms:

Local Version For a Kähler fibre space whose base is an open ball in some $\mathbb{C}^d$, we have the following extension theorem of Ohsawa-Takegoshi type with optimal estimation:

Theorem 2.1.1 (higher dimensional version of [Cao17, Theorem 1.1 (Corollary 3.1)]). *Let $p : X \to B$ be a analytic (Kähler) fibre space with X a Kähler manifold and $B \subseteq \mathbb{C}^d$ the open ball of centre 0 and of radius R. Let (L, h_L) be a holomorphic line bundle on X equipped with h_L a singular Hermitian metric such that the curvature current of h_L is positive. Suppose that $X_0 := p^{-1}(0)$ is a smooth fibre of p, and that $h_L|_{X_0}$ is not identically $+\infty$. Then for any holomorphic section $f \in \mathrm{H}^0(X_0, K_{X_0} \otimes L|_{X_0} \otimes \mathscr{J}(h_L|_{X_0}))$, there exists a section $F \in \mathrm{H}^0(X, K_X \otimes L)$ such that $F|_{X_0} = f$ and*

$$\frac{1}{\mu(B)} \int_X |F|^2 e^{-\phi_L} \leqslant \int_{X_0} |f|^2 e^{-\phi_L},$$

where $\mu(B)$ denotes the Lebesgue measure of B.

Proof. We obtain the theorem by applying [Cao17, Theorem 1.3] to the fibre space $X \xrightarrow{p} B$ with $E = p^*\mathscr{O}_B^{\oplus d}$, $v = p^*t$ where $t = (t_1, \cdots, t_d)$ and t_i's are standard coordinates of $\mathbb{C}^d$, $A = 2d \log R$, $c_A(t) \equiv 1$, and by letting $\delta \to +\infty$ (c.f. also [GZ15a, §4.2, Lemma 4.14]). In particular, when $d = 1$ one recovers [Cao17, Theorem 1.1 (Corollary 3.1)]. □

Global Version In many cases, one needs a global version of Ohsawa-Takegoshi extension theorem for Kähler fibre spaces over projective bases; in this case, one cannot obtain an optimal estimation, but one still has an surjection of section spaces up to a twisting by a ample line bundle coming from the base, along with a weaker estimation on the L^2 norm. In fact we have the following:

Theorem 2.1.2 (Kähler version of [Den17a, Corollary 2.10]). *Let Y be a smooth projective variety of dimension d and let $f : X \to Y$ be a surjective morphism between compact Kähler manifolds with connected fibres. Let (A_Y, y) be any pair with A_Y ample line bundle on Y and $y \in Y_0$ (where Y_0 denotes the smooth locus of f), such that the Seshadri constant*

$$\epsilon(A_Y, y) > \dim Y = d.$$

Let (L,h_L) be any holomorphic line bundle on X equipped with a singular Hermitian metric h_L whose curvature current is positive, such that $h_L|_{X_y} \not\equiv +\infty$. Then for any section $u \in \mathrm{H}^0(X_y, K_{X_y}\otimes L|_{X_y}\otimes \mathscr{J}(h_L|_{X_y}))$, there is a section $\sigma \in \mathrm{H}^0(X, K_X\otimes L\otimes f^*A_Y)$ such that $\sigma|_{X_y}=u$ with an L^2 estimate independent of L.

For the proof, we refer to [Den17a, Corollary 2.10]. Just remark that: in [Den17a] this theorem is only stated for f a projective morphism. The above Kähler version holds because the proof of [Den17a, Corollary 2.10] depends only on [Dem15, (2.8)Theorem] (c.f. also [Den17a, Theorem 2.9]), which is valid for any pseudo-convex Kähler manifold.

2.2 Positivity of the twisted relative pluricanonical bundles and their direct images

Let $f : X \to Y$ be a Kähler fibre space between complex manifolds (c.f. the definitions in Convetion and Notations). Let (L,h_L) be a line bundle on X equipped with a singular Hermitian metric h_L whose curvature current $\Theta_{h_L}(L)$ is positive. The main purpose of this section is to establish the positivity result for the L-twisted relative pluricanonical bundles and their direct images mentioned in Introduction (c.f. Theorem 2.2.1 and Theorem 2.2.4). To this end, we will explain the construction of the relative m-Bergman kernel metric $h^{(m)}_{X/Y,L}$ on $K^{\otimes m}_{X/Y}\otimes L$ and of the canonical L^2 metric $g_{X/Y,L}$ on the direct image sheaf $f_*(K_{X/Y}\otimes L\otimes\mathscr{J}(h_L))$.

Let us recall briefly the history of the study of these canonical metrics. Initially, the case with h_L a smooth metric and f smooth is considered in [Ber09], where the positivity of $f_*(K_{X/Y}\otimes L)$ is proved by an explicit calculation of the curvature; as a simple consequence, one deduces the positivity of the relative Bergman kernel metric (with $m=1$), c.f. [BP08, §1, p. 348]. In the more general case where f is projective but not necessarily smooth and $f_*(K_{X/Y}\otimes L)$ is locally free, the positivity of $f_*(K_{X/Y}\otimes L)$ is proved in [BP08, Theorem 3.5] based on the work of Berndtsson; this result is in turn used in [BP08, Corollary 4.2] to prove the positivity of the relative m-Bergman kernel metric under the assumption that the direct image sheaf $f_*\left(K^{\otimes m}_{X/Y}\otimes L\right)$ is locally free. In [PT18], these positivity results are established for f projective with the locally freeness conditions for direct images removed: it is made clear that the positivity of the relative m-Bergman kernel metric can be regarded as a consequence of the Ohsawa-Takegoshi extension Theorem with the optimal estimate, and thus can be obtained independently of the positivity of direct images; while the proof of the positivity of $f_*(K_{X/Y}\otimes L)$ is based on [BP08] and is done by a semistable reduction plus an explicit calculation. A little later, it is realized that the positivity of the canonical metric is also a consequence of the Ohsawa-Takegoshi extension theorem with the optimal estimate, as is explained in [HPS18]. Therefore in order to obtain a Kähler version of this theorem, all one needs is to generalize the Ohsawa-Takegoshi extension theorem to the Kähler case. Thanks to [Cao17], this result is established and the positivity of the relative m-Bergman kernel metric is also proved in [Cao17] as a corollary; in consequence, by virtue of the main result in [Cao17] one can follow the same arguments in [HPS18] to demonstrate the positivity of the canonical L^2 metric $g_{X/Y,L}$ for f Kähler fibre space. Recently we are informed that this result is established in [DWZZ18] by following the strategy of [HPS18] and by a more general positivity theorem for singular Finsler metrics on direct images. For the convenience of the readers, we will nevertheless provide some details of the proof in §2.2.2.

2.2.1 Positivity of the relative m-Bergman kernel metrics

Let $f : X \to Y$ be an analytic fibre space between complex manifolds and let (L, h_L) be a holomorphic line bundle on X equipped with a singular Hermitian metric h_L with curvature current $\Theta_{h_L}(L) \geqslant 0$. Set $n = \dim X$, $d = \dim Y$ and $e = \dim X - \dim Y = n - d$. Let us recall the construction of the relative m-Bergman kernel metric on $K_{X/Y}^{\otimes m} \otimes L$. We will follow [CP17, §2.1] and [Cao17, §3.2]; for more details, c.f. [BP08, §1].

Let Y_0 be the (analytic) Zariski open subset of Y over which f is smooth. Let $x \in f^{-1}(Y_0)$ and let $z_1, \cdots, z_{d+e}$ be local coordinates near x; write $y = f(x) \in Y_0$ and let $t_1, \cdots, t_d$ be local coordinates near y such that $z_{j+e} = f^* t_j$. Suppose in addition that over the coordinate neighbourhood of x (resp. of y) chosen as above the line bundle L as well as the canonical bundles of X are trivial (resp. the canonical bundle of Y is trivial).

Suppose that $f_*\left(K_{X/Y}^{\otimes m} \otimes L\right) \neq 0$. We define a $L^{2/m}$-Finsler norm on $\mathrm{H}^0(X_y, K_{X_y}^{\otimes m} \otimes L|_{X_y})$ by taking the integral over the fibre

$$\|u\|_{m,y,L}^{\frac{2}{m}} := \int_{X_y} |u|^{\frac{2}{m}} e^{-\frac{1}{m}\phi_L}, \tag{2.1}$$

where ϕ_L denotes the local weight of the metric h_L (we authorize this to be $+\infty$, which is the case when $h_L|_{X_y} \equiv +\infty$). In addition, we denote by F_u the coefficient of $(dz_1 \wedge \cdots \wedge dz_{d+e})^{\otimes m}$ in the local expression of $u \wedge f^*(dt_1 \wedge \cdots \wedge t_d)^{\otimes m}$. Then local weight $\phi_{X/Y,L}^{(m)}$ of the relative m-Bergman kernel metric $h_{X/Y,L}^{(m)}$ is given by

$$e^{\phi_{X/Y,L}^{(m)}(x)} = \sup_{\|u\|_{m,y,L} \leqslant 1} |F_u(x)|^2. \tag{2.2}$$

Let us remark that if $h_L|_{X_y} \equiv +\infty$ (2.2) is equal to 0 by convention and thus $\phi_{X/Y,L}(x) = -\infty$. The metric $h_{X/Y,L}^{(m)} = e^{-\phi_{X/Y,L}^{(m)}}$ can also be described in an intrinsic way as follows: for $\xi \in (K_{X/Y}^{\otimes(-m)} \otimes L^{-1})_x$, we have

$$|\xi|_{h_{X/Y,L}^{(m)*},x} = \sup_{\|u\|_{m,y,L} \leqslant 1} |\xi(u(x))|.$$

Suppose in the sequel of this subsection that f is a Kähler fibre space with X and Y complex manifolds. By using the Ohsawa-Takegoshi extension theorem with optimal estimate (c.f. Theorem 2.1.1) Junyan Cao proved in [Cao17] that the relative m-Bergman kernel metric constructed above is semipositively curved (since the construction is local over Y, the Kähler hypothesis on X and Y in the original statement of [Cao17, Theorem 3.5] is in fact not necessary, and can be replaced by the hypothesis that f is a Kähler fibre space, in which case X is only assumed to be Kähler locally over Y):

Theorem 2.2.1 ([Cao17, Theorem 1.2 (Theorem 3.5)]). *Let $f : X \to Y$ be a Kähler fibre space with X and Y complex manifolds and (L, h_L) be a holomorphic line bundle on X equipped with a singular Hermitian metric h_L whose curvature current is positive. Let m be a positive integer. Suppose that for a general point $y_0 \in Y$ there exists a non-zero section $u \in \mathrm{H}^0(X_{y_0}, K_{X_{y_0}}^{\otimes m} \otimes L|_{X_{y_0}})$ satisfying*

$$\int_{X_{y_0}} |u|^{\frac{2}{m}} e^{-\frac{1}{m}\phi_L} < +\infty,$$

then the curvature current of the relative m-Bergman kernel metric $h_{X/Y,L}^{(m)}$ is positive. More precisely, there is an (analytic) Zariski open subset of $f^{-1}(Y_0)$ (c.f. Remark 2.2.2 below) such that the local weight $\phi_{X/Y,L}^{(m)}$ of the metric $h_{X/Y,L}^{(m)}$ defined above is a psh function uniformly bounded from above, thus it admits a unique (psh) extension on X.

Remark 2.2.2. Though we do note use this, let us make it precise the (analytic) Zariski open in the Theorem 2.2.1 above. Define for every (quasi-)psh function ϕ and for every integer $m > 0$ the ideal sheaf $\mathscr{J}_m(\phi)$ by taking

$$\mathscr{J}_m(\phi)_x := \left\{ f \in \mathscr{O}_{X,x} \,\middle|\, |f|^{\frac{2}{m}} e^{-\frac{1}{m}\phi} \in L^1_{\mathrm{loc}} \right\},$$

which is proved to be coherent in [Cao17]. Then the integrability condition in Theorem 2.2.1 is equivalent to the non-vanishing condition that $f_*(K_{X/Y}^{\otimes m} \otimes L \otimes \mathscr{J}_m(h_L)) \neq 0$. And the open subset mentioned in Theorem 2.2.1 can be taken to be $f^{-1}(U)$ where $U \subseteq Y_0$ is the (analytic) Zariski open subset consist of all point $t \in U$ such that

$$\mathrm{h}^0(X_t, (K_{X/Y}^{\otimes m} \otimes L \otimes \mathscr{J}_m(h_L)|_{X_t}) = \mathrm{rk}\, f_*(K_{X/Y}^{\otimes m} \otimes L \otimes \mathscr{J}_m(h_L)).$$

In particular by the Grauert's semi-continuity theorem [Uen75, Theorem 1.4(3), p. 6], $f_*(K_{X/Y}^{\otimes m} \otimes L \otimes \mathscr{J}_m(h_L))$ satisfies the base change property over U. For more details, c.f. [Cao17, Proof of Theorem 3.5] and Lemma 2.2.5 below.

By an explicit local calculation as in [CP17, Theorem 2.3] or [Pău16, 3.33.Theorem] we obtain (in virtue of Theorem 2.2.1 the proof in [CP17] apparently does not require the projectivity of f):

Proposition 2.2.3 (Kähler version of [CP17, Remark 2.5] or [Pău16, 3.35.Remark]). *Let $f : X \to Y$ be a Kähler fibre space with X and Y complex manifolds and (L, h_L) be a holomorphic line bundle on X equipped with a singular Hermitian metric h_L whose curvature current is positive. Let m be a positive integer. Suppose that for a general point $y_0 \in Y$ there exists a non-zero section $u \in \mathrm{H}^0(X_{y_0}, K_{X_{y_0}}^{\otimes m} \otimes L|_{X_{y_0}})$ satisfying*

$$\int_{X_{y_0}} |u|^{\frac{2}{m}} e^{-\frac{1}{m}\phi_L} < +\infty,$$

(as in the hypothesis of Theorem 2.2.1). Then we have

$$\Theta_{h^{(m)}_{X/Y,L}}(K_{X/Y}^{\otimes m} \otimes L) \geqslant m[\Sigma_f] \tag{2.3}$$

in the sense of currents, where the divisor Σ_f is defined in the Introduction. In particular, the current $\Theta_{h^{(m)}_{X/Y,L}}(K_{X/Y}^{\otimes m} \otimes L)$ is singular along the multiple fibres of f in codimension 1.

Proof. Let us remark that in [CP17] the proof of inequality (2.3) is only sketched for $m = 1$. For the convenience of the readers let us give a detailed proof for the general case here. Since a positive $(1,1)$-current extends across analytic subsets of codimension 2, it suffices to check the inequality around a general point of W_i for every $i \in I_{\mathrm{div}}$ (so that one can assume that every W_i is smooth). Say $i = 1 \in I_{\mathrm{div}}$, and let x be a general point of W_1. Take a small ball B_y (of radius < 1) around $y = f(x)$ with holomorphic local coordinates $(t_j)_{j=1,\cdots,d}$ and a small ball $\Omega_x \subset f^{-1}(B_y)$ around x with holomorphic local coordinates $(z_i)_{i=1,\cdots,n}$, such that W_1 is locally defined by the equation $z_{e+1} = 0$ and that $f(W_1)$ is defined by $t_1 = 0$. Then f is locally given by the formula (up to reordering the indices):

$$(z_1, \cdots, z_e, z_{e+1}, \cdots, z_n) \longmapsto (z_{e+1}^{b_1}, z_{e+2}, \cdots, z_n).$$

Now let $y_0 \in B_y \backslash (t_1 = 0)$, and let $u \in \mathrm{H}^0(X_{y_0}, K_{X_{y_0}}^{\otimes m} \otimes L|_{X_{y_0}})$ satisfying the $L^{2/m}$ condition as in the hypothesis; up to a normalization one can suppose that $\|u\|_{m,y_0,L} = 1$. Then by the construction of F_u we have

$$1 = \|u\|^{\frac{2}{m}}_{m,y_0,L} = \int_{X_{y_0}} |u|^{\frac{2}{m}} e^{-\frac{1}{m}\phi_L} \geqslant \int_{\Omega_x \cap X_{y_0}} \left| \frac{F_u}{z_{e+1}^{m(b_1-1)}} \right|^{\frac{2}{m}} d\mu_{X_{y_0}}$$

where $d\mu_{X_{y_0}}$ is the Lebesgue measure on X_{y_0} with respect to the z_i's. Notice that

$$\Omega_x \cap X_{y_0} = \left\{ z_{e+1}^{b_1} = t_1(y_0), z_{e+i} = t_i(y_0), 2 \leqslant i \leqslant d \right\},$$

hence by applying the Ohsawa-Takegoshi type extension theorem [DWZZ18, Theorem 4.2] (or [BP10, o.2.Proposition]) to $\Omega = \Omega_x$, $p = (z_{e+1}, \cdots, z_n)$ and $\phi = (b_1 - 1)\log|z_{e+1}|^2$, the holomorphic function F_u extends to a function G_u defined on Ω_x satisfying the following $L^{2/m}$-integrability condition:

$$\int_{\Omega_x} \left| \frac{G_u}{z_{e+1}^{m(b_1-1)}} \right|^{\frac{2}{m}} d\mu_X \leqslant \mu(B_y).$$

By valuative integrability criterion [Bou16, Theorem 10.13] the generic Lelong number of $\log|G_u|$ over W_1 is greater than $m(b_1 - 1)$, implying that

$$\log|G_u|^2 \leqslant m(b_1 - 1)\log|z_{e+1}|^2 + C_{y_0}$$

for some uniform (the section space $\mathrm{H}^0(X_{y_0}, K_{X_{y_0}}^{\otimes m} \otimes L|_{X_{y_0}})$ being finite-dimensional) constant C_{y_0} depending on y_0. Hence by the construction (2.2) we have

$$\phi_{X/Y,L}^{(m)}(z) \leqslant m(b_1 - 1)\log|z_{e+1}|^2 + C_{f(z)};$$

by the mean-value inequality the constant $C_{f(z)}$ can be chosen locally uniform. The function $\phi_{X/Y,L}^{(m)} - m(b-1-1)\log|z_{e+1}|^2$ is psh outside $z_{e+1} = 0$ (since $\log|z_{e+1}|^2$ is pluri-harmonic outside $z_{e+1} = 0$), but it is bounded, hence it is a psh function and this proves (2.3). □

2.2.2 Positivity of the canonical L^2 metric on the direct images

In this subsection, let $f : X \to Y$ be an analytic fibre space between complex manifolds and let (L, h_L) be a holomorphic line bundle on X equipped with a singular Hermitian metric h_L with curvature current $\Theta_{h_L}(L) \geqslant 0$. We will show in the sequel that the canonical L^2 metric on the direct image sheaf $f_*(K_{X/Y} \otimes L \otimes \mathscr{J}(h_L))$ is semipositively curved, that is, to prove the following theorem:

Theorem 2.2.4 (Kähler version of [PT18, Theorem 1(b)]). *let $f : X \to Y$ be a Kähler fibre space between complex manifolds and let (L, h_L) be a holomorphic line bundle on X equipped with a singular Hermitian metric h_L with curvature current $\Theta_{h_L}(L) \geqslant 0$. Then the torsion free sheaf $f_*(K_{X/Y} \otimes L \otimes \mathscr{J}(h_L))$ admits a canonical semipositively curved singular Hermitian metric $g_{X/Y,L}$ which satisfies the L^2 extension property.*

The argument is very close to that in [HPS18, §22-24]. For the convenience of the readers, we will nevertheless explain it in details.

First recall the construction of the canonical L^2 metric on the direct image of the adjoint line bundle (twisted by the multiplier ideal). Briefly speaking, it is done as following: when $Y = \mathrm{pt}$, then X is compact, and this is nothing other than the natural L^2 norm on $\mathrm{H}^0(X, K_X \otimes L \otimes \mathscr{J}(h_L))$; for the general case, we just do this construction in family.

Precisely, $g_{X/Y,L}$ is constructed as following: let Y_0 be the Zariski open of Y over which f is smooth and let $y \in Y_0$. Take a coordinate neighbourhood B of y, so that K_Y is trivial over B, then there is a nowhere vanishing holomorphic d-form η such that $K_B \simeq \mathscr{O}_B \cdot \eta$. For any section $u \in \mathrm{H}^0(B, f_*(K_{X/Y} \otimes L \otimes \mathscr{J}(h_L)))$, one can regard it as a morphism of $\mathscr{O}_B$-modules (in virtue of the projection formula)

$$u : K_B \longrightarrow f_*(K_X \otimes L \otimes \mathscr{J}(h_L))|_B$$

Thus we obtain a section $u(\eta) \in \mathrm{H}^0(B, f_*(K_X \otimes L \otimes \mathscr{J}(h_L))) = \mathrm{H}^0(f^{-1}(B), K_X \otimes L \otimes \mathscr{J}(h_L))$. Locally over $f^{-1}(B \cap Y_0)$ we can write $u(\eta) = \sigma_u \wedge f^*\eta$; whilst the choice of σ_u depends on η, its restriction to the fibre $\sigma_u|_{X_y}$ does not. The local sections $\sigma_u|_{X_y}$'s glue together to give rise to a section $\sigma_{u,y} \in \mathrm{H}^0(X_y, K_{X_y} \otimes (L \otimes \mathscr{J}(h_L))|_{X_y})$. Then we define the canonical L^2 metric as following: for two local sections u, v of $f_*(K_{X/Y} \otimes L)$ (resp. of $f_*(K_{X/Y} \otimes L \otimes \mathscr{J}(h_L))$), define

$$g_{X/Y,L}(u,v)(y) = \left(\sqrt{-1}\right)^{n^2} \int_{X_y} \sigma_{u,y} \wedge \bar{\sigma}_{v,y} e^{-\phi_L}. \tag{2.4}$$

Before proving the result, let us recall the following comparison result of the restriction of the multiplier ideal of a metric to a fibre and the multiplier ideal of the restriction of the metric to a fibre:

Lemma 2.2.5. *Let $f : X \to Y$ and (L, h_L) as in the Theorem 2.2.4 above. Suppose that f is smooth. Then for any $y \in Y$ we have*

$$\mathscr{J}(h_L|_{X_y}) \subseteq \mathscr{J}(h_L)|_{X_y}.$$

Moreover, for almost every $y \in Y$ we have

$$\mathscr{J}(h_L|_{X_y}) = \mathscr{J}(h_L)|_{X_y}.$$

Proof. The inclusion $\mathscr{J}(h_L|_{X_y}) \subseteq \mathscr{J}(h_L)|_{X_y}$ results from the local Ohsawa-Takegoshi extension theorem (see e.g. [Bło13, Theorem 1]) while the equality for a.e. $y \in Y$ is simply a consequence of the Fubini's theorem. C.f. [Pău16, 3.29.Remark] for more details. Let us remark that the same result holds for $\mathscr{J}_m$ as defined in Remark 2.2.2. □

Next let us fix some notations for later use:

Notations: Set Y_1 the (analytic) Zariski open subset of Y_0 such that

(i) $f_*(K_{X/Y} \otimes L \otimes \mathscr{J}(h_L))$ and the quotient sheaf of $f_*(K_{X/Y} \otimes L)$ by $f_*(K_{X/Y} \otimes L \otimes \mathscr{J}(h_L))$ are both locally free over Y_1 ;

(ii) $f_*(K_{X/Y} \otimes L)$ satisfies the base change property over Y_1 , i.e. $f_*(K_{X/Y} \otimes L) \otimes \kappa(y) \simeq \mathrm{H}^0(X_y, K_{X_y} \otimes L|_{X_y})$ for every $y \in Y_1$ (e.g. if the function $y \mapsto \mathrm{h}^0(X_y, K_{X_y} \otimes L|_{X_y})$ is locally constant on Y_1, c.f. [Uen75, Theorem 1.4(3), p. 6]).

Set in addition $\mathscr{G}_L := f_*(K_{X/Y} \otimes L \otimes \mathscr{J}(h_L))$. With these notations we get immediately from Lemma 2.2.5 the following:

Lemma 2.2.6. *We have inclusions*

$$\mathrm{H}^0(X_y, K_{X_y} \otimes L|_{X_y} \otimes \mathscr{J}(h_L|_{X_y})) \subseteq \mathscr{G}_L \otimes \kappa(y) \subseteq f_*(K_{X/Y} \otimes L) \otimes \kappa(y) = \mathrm{H}^0(X_y, K_{X_y} \otimes L|_{X_y})$$

for every $y \in Y_1$.

For any $y \in Y_1$, since $f_*(K_{X/Y} \otimes L)$ satisfies the base change property, the expression of the metric $g_{X/Y,L}$ is simpler: for $u \in \mathscr{G}_L \otimes \kappa(y)$, u can be regarded as a section in $\mathrm{H}^0(X_y, K_{X_y} \otimes (L \otimes \mathscr{J}(h_L))|_{X_y}) \subseteq \mathrm{H}^0(X_y, K_{X_y} \otimes L|_{X_y})$, and we have

$$|u|^2_{g_{X/Y,L},y} = \int_{X_y} |u|^2 e^{-\phi_L}. \tag{2.5}$$

In particular, $|u|^2_{g_{X/Y,L},y}$ $(y \in Y_1)$ is finite if and only if $u \in \mathrm{H}^0(X_y, K_{X_y} \otimes L|_{X_y} \otimes \mathscr{J}(h_L|_{X_y}))$. Now let us prove the following result which ensures that $g_{X/Y,L}$ is well-behaved:

Proposition 2.2.7. *The metric $g_{X/Y,L}$ defined above on $f_*(K_{X/Y}\otimes L\otimes\mathscr{J}(h_L))$ is measurable, and is non-degenerate and bounded almost everywhere.*

Proof. We check successively:

(a) $g_{X/Y,L}$ is measurable: this is surely well known to experts, but since it does not appear explicitly in the literatures we give the details for the convenience of the readers and take this chance to fix some notations for later use. Let $s\in \mathrm{H}^0(B,\mathscr{G}_L)$ be a local section on B with B a small ball in Y, we will show that $\Lambda_s := |s|^2_{g_{X/Y,L}}$ is a measurable function. To this end, we can assume that B is contained in Y_0; in addition, s can be regarded as a section in $\mathrm{H}^0(f^{-1}B, K_{X/Y}\otimes L)$, and thus $s(y)\in \mathrm{H}^0(X_y, K_{X_y}\otimes L|_{X_y})$; $s\wedge f^*\eta\in \mathrm{H}^0(f^{-1}B, K_X\otimes L)$ where η is a nowhere vanishing holomorphic d-form, giving rise to a trivialization $K_B\simeq \mathscr{O}_B\cdot\eta$. By definition, for any $y\in B\cap Y_1$ we have

$$\Lambda_s(y)=\int_{X_y}|s(y)|^2e^{-\phi_L},$$

By Ehresmann's theorem (c.f. for example [Voi02, §9.1.1, Proposition 9.3, pp. 209-210]) we have a diffeomorphism $X_0\times B\xrightarrow{\tau} f^{-1}B$ such that $\tau|_{X_0\times\{0\}}\circ i_0=\mathrm{id}_{X_0}$ where $i_y: X_0\to X_0\times B$ is the natural inclusion which identifies X_0 à $X_0\times\{y\}$ in $X_0\times B$. Then we can write

$$\Lambda_s(y)=\int_{X_0}G_s(y,\cdot)\mathrm{Vol}_{X_0} \tag{2.6}$$

where Vol_{X_0} is a fixed volume form on X_0 and G_s is a function such that

$$G_s(y,\cdot)\mathrm{Vol}_{X_0}=\left|\tau^*(s\wedge f^*\eta)\big|_{X_0\times\{y\}}\right|^2e^{-\phi_L}. \tag{2.7}$$

ϕ_L being a psh function, the function G_s is lower semi-continuous and is well defined on $X_0\times(B\cap Y_1)$, in particular it is measurable. Hence by Fubini's theorem, Λ_s is measurable.

(b) $g_{X/Y,L}$ is non-degenerate and bounded almost everywhere (c.f. also [Pău16, 3.29.Remark]): first one notices that by the formula (2.5) the metric $g_{X/Y,L}$ is non-degenerate over Y_1 since ϕ_L is a psh function. In order to show that $g_{X/Y,L}$ is bounded almost everywhere, it suffices to prove that the natural inclusion

$$\mathrm{H}^0(X_y, K_{X_y}\otimes L|_{X_y}\otimes\mathscr{J}(h_L|_{X_y}))\hookrightarrow\mathscr{G}_L\otimes\kappa(y)$$

is an isomorphism for $y\in Y_1$ almost everywhere. This simply results from Lemma 2.2.5.

□

By virtue of Proposition 2.2.7, in order to prove that $g_{X/Y,L}$ defined above extends to a semipositively curved singular Hermitian metric on $\mathscr{G}_L$, it remains to show: for $U\subseteq Y$ an open subset, and for $\alpha\in\mathrm{H}^0(U,\mathscr{G}_L^*)$ a non-zero section, $\psi_\alpha := \log|\alpha|^2_{g^*_{X/Y,L}}$ (a function well-defined on $U\cap Y_0$) extends to a psh function on U. To this end, we will successively establish (by Proposition 2.2.7, $\psi_\alpha\not\equiv-\infty$ on $U\cap Y_0$):

(A) ψ_α is locally uniformly bounded from above on $U_1 := U\cap Y_1$;

(B) ψ_α is upper semi-continuous on U_1;

(C) ψ_α satisfies the mean value inequality on any disc in U_1.

In fact, the points (B) and (C) imply that ψ_α is a psh function over U_1; and the point (A) implies moreover that $\psi_\alpha|_{U_1}$ admits a unique psh extension to U. In addition, let us remark that up to replacing Y par U, one can suppose that α is a global section; in this case ψ_α is a function well defined over Y_0. The proof of theses three points relies on the Ohsawa-Takegoshi type extension Theorem 2.1.1, which enables us to extend a section on the fibre to a neighbourhood along with an L^2 estimate (in some cases we should require this estimate to be optimal).

Proof of (A): let $y_0 \in Y$, we will prove that y_0 admits a neighbourhood such that on its intersection with Y_1 the function ψ_α is uniformly upper bounded. To this end, take a small open ball B_0 of centre y_0 in Y and denote $B_1 := \frac{1}{2}B_0$, $B = B_2 := \frac{1}{4}B_0$ and $R_0 =$ radius of B_0. We will prove in the sequel that ψ_α is uniformly upper bounded on $B \cap Y_1$. This proceeds in two steps:

(A1) Firstly we prove that

$$\psi_\alpha|_{B\cap Y_1} \leqslant \text{ punctual supremum of the family of functions } \left\{\log|\alpha(s)|^2\right\}_{s\in S_{M_0}} \tag{2.8}$$

where S_{M_0} denotes the set of sections $s \in \mathrm{H}^0(B_1, \mathscr{G}_L) = \mathrm{H}^0(f^{-1}(B_1), K_{X/Y}\otimes L\otimes \mathscr{J}(h_L))$ satisfying the following L^2 condition:

$$\int_{f^{-1}B_1} \left|s \wedge f^*\eta|_{B_1}\right|^2 e^{-\phi_L} \leqslant \left(\frac{3}{4}\right)^d \mu(B_0) := M_0, \tag{2.9}$$

where $\mu(B_0)$ denotes the Lebesgue measure of B_0 and η a nowhere vanishing holomorphic n-form on B_0 (which gives rise to a trivialization $K_{B_0} \simeq \mathscr{O}_{B_0}\cdot\eta$).

For every $y \in B \cap Y_1$ such that $h_L|_{X_y} \not\equiv +\infty$ (if $h_L|_{X_y} \equiv +\infty$, then $\psi_\alpha(y) = -\infty$ and (2.8) is automatically established at y), we have

$$\psi_\alpha(y) = \log\left|\alpha(y)\right|^2_{g^*_{X/Y,L},y} = \sup_{\|u\|_{y,L}\leqslant 1} \log\left|\alpha(y)(u)\right|^2.$$

The set $\left\{u \in \mathrm{H}^0(X_y, K_{X_y}\otimes L|_{X_y}) \,\middle|\, \|u\|_{y,L} \leqslant 1\right\}$ being compact, the supremum is attained by a vector $v_y \in \mathscr{G}_L\otimes\kappa(y)$ satisfying $\|v_y\|_{y,L} = |v_y|_{g_{X/Y,L},y} = 1$ (we denote $\|\cdot\|_{1,y,L} = \|\cdot\|_{y,L}$, compare (2.1) and (2.5)); in particular $v_y \in \mathrm{H}^0(X_y, K_{X_y}\otimes L|_{X_y}\otimes\mathscr{J}(h_L|_{X_y}))$. Consider the open ball $B_y := B(y, \frac{3}{4}R_0)$ of centre y and of radius $= \frac{3}{4}R_0$. Then $B \subseteq B_1 \subseteq B_y \subseteq B_0$. By Theorem 2.1.1 we get a section $s_y \in \mathrm{H}^0(B_y, \mathscr{G}_L)$ such that $s_y|_{X_y} = v_y$ and satisfies the following L^2 condition:

$$\int_{f^{-1}B_y} \left|s_y \wedge f^*\eta|_{B_y}\right|^2 e^{-\phi_L} \leqslant \mu(B_y)\cdot\|v_y\|_{y,L} = \mu(B_y) = \left(\frac{3}{4}\right)^d \mu(B_0) = M_0\,.$$

In particular, $s_y|_{B_1}$ satisfies Condition (2.9), then $s_y|_{B_1} \in S_{M_0}$. In addition, we have

$$\psi_\alpha(y) = \left(\log\left|\alpha(s_y)\right|^2\right)(y),$$

which proves (2.8).

(A2) By the previous step, it remains to prove that the functions $\log|\alpha(s)|^2$ ($s \in S_{M_0}$) are all uniformly upper bounded over $\bar{B}$ by a uniform constant. In fact we can prove the following more general:

Lemma 2.2.8. *For a fixed $M \geqslant 0$, define*

$$S_M := \left\{ s \in \mathrm{H}^0(B_1, \mathscr{G}_L) = \mathrm{H}^0(f^{-1}(B_1), K_{X/Y} \otimes L \otimes \mathscr{J}(h_L)) \,\middle|\, \int_{f^{-1}B_1} \left| s \wedge f^*\eta|_{B_1} \right|^2 e^{-\phi_L} \leqslant M \right\},$$

then for every compact $K \subseteq B_1$, there exists a constant $C_K \geqslant 0$ (independent of s) such that

$$\sup_K |\alpha(s)| \leqslant C_K$$

for every $s \in S_M$.

Proof. The lemma is deduced from some well known facts about the Fréchet space structure on the cohomology spaces of coherent sheaves over complex spaces, as presented in [GR56, §VIII.A, pp. 234-246]. By [GR56, §VIII.A, 8.Theorem, pp. 239-240], for any coherent sheaf on an analytic space, we can equip its section spaces with a unique Fréchet space structure, s.t. the restriction morphisms are continuous.

(a) By [GR56, §VIII.A, 7.Theorem, pp. 237-238], the section α, regarded as a morphism $\mathscr{G}_L \to \mathscr{O}_Y$, induces continuous map between Fréchet spaces

$$\alpha|_{B_1} : \mathrm{H}^0(B_1, \mathscr{G}_L) \longrightarrow \mathrm{H}^0(B_1, \mathscr{O}_X).$$

(b) By uniqueness, the a priori different topologies on the two isomorphic spaces $\mathrm{H}^0(B_1, \mathscr{G}_L)$ and $\mathrm{H}^0(f^{-1}(B_1), K_{X/Y} \otimes L \otimes \mathscr{J}(h_L))$ are homeomorpic.

(c) $S_M \subseteq \mathrm{H}^0(B_1, \mathscr{G}_L)$ is compact with respect to the Fréchet space topology. This is a result of (b) and Montel's Theorem.

(d) By [Car61, §V.4.2, Proposition 2.1, pp. 165-166] the compacts in $\mathrm{H}^0(B_1, \mathscr{O}_X)$ are closed and bounded.

By combining (a) (c) and (d) we establish the lemma. □

Proof of (B): let $y_0 \in Y_1$, and let $\{y_k\}_{k>0}$ be any sequence in Y_1 convergent to y_0, we will prove that

$$\limsup_{k \to +\infty} \psi_\alpha(y_k) \leqslant \psi_\alpha(y_0).$$

The problem being local, we can replace Y by B_0 a small open ball of centre y_0 ($y_0 = 0$ in B_0) in Y. Note $R_0 :=$ the radius of B_0 and $B_i := \left(\frac{1}{2}\right)^i B_0$. Since there is a subsequence of $\{\psi_\alpha(y_k)\}_{k>0}$ which converges to the limit superior of $\{\psi_\alpha(y_k)\}_{k>0}$, we can assume that the sequence $\{\psi_\alpha(y_k)\}_{k>0}$ is convergent. In addition, up to shifting the numbering of the sequence we can assume that $\{y_k\}_{k>0} \subseteq B_3$; we can also assume that $\psi_\alpha(y_k) \neq -\infty$, $\forall k$ (in particular, $h_L|_{X_{y_k}} \not\equiv +\infty$). As in the step (A1) above, there exists for every $k \in \mathbb{Z}_{>0}$ a vector $v_k \in \mathrm{H}^0(X_{y_k}, K_{X_{y_k}} \otimes L|_{X_{y_k}} \otimes \mathscr{J}(h_L|_{X_{y_k}}))$ such that $\|v_k\|_{y_k, L} = 1$ and

$$\psi_\alpha(y_k) = \log \left| \alpha(y_k)(v_k) \right|^2.$$

Consider $B_{y_k} := B(y_k, \frac{7}{8}R_0)$ the open ball of centre y_k and of radius $\frac{7}{8}R_0$, then $B_3 \subseteq B_2 \subseteq B_1 \subseteq B_{y_k} \subseteq B_0$. Still by Theorem 2.1.1, we obtain a section $s_k \in \mathrm{H}^0(B_{y_k}, \mathscr{G}_L) = \mathrm{H}^0(f^{-1}(B_{y_k}), K_{X/Y} \otimes L \otimes \mathscr{J}(h_L))$ such that $s_k|_{X_{y_k}} = v_k$ and

$$\int_{f^{-1}(B_{y_k})} |s_k|^2 e^{-\phi_L} \leqslant \left(\frac{7}{8}\right)^d \mu(B_0) := M_0'.$$

Denote $F_k = \alpha(s_k)|_{B_1}$ and $\theta_k := \log|F_k|^2$, then F_k is a holomorphic function on B_1 and θ_k is a psh function (with analytic singularities); in addition, we have that $\psi_\alpha(y_k) = \theta_k(y_k)$. By Lemma 2.2.8 (taking $M = M_0'$ and $K = \bar{B}_2$), there is a constant $C_{\bar{B}_2}$ independent of k such that $|F_k| \leqslant C_{\bar{B}_2}$ on $\bar{B}_2$ for every k; in consequence, the derivatives of F_k satisfy

$$|\nabla F_k|^2 \leqslant \tilde{C}_{\bar{B}_2} := \frac{16\sqrt{n}}{R_0} C_{\bar{B}_2}$$

on $\bar{B}_3$ (c.f. [Car61, §V.1.2, Lemme, p. 146]). In particular, since $\{y_k\}_{k>0} \subseteq B_3$, we have

$$\big||F_k(0)| - |F_k(y_k)|\big| \leqslant \big|F_k(0) - F_k(y_k)\big| \leqslant \tilde{C}_{\bar{B}_2}|y_k - 0| \to 0 \text{ when } k \to +\infty,$$

hence we get

$$\lim_{k\to+\infty} \theta_k(y_k) = \lim_{k\to+\infty} (\log|F_k(y_k)|) = \lim_{k\to+\infty} (\log|F_k(0)|) = \lim_{k\to+\infty} \theta_k(0) \tag{2.10}$$

By definition, we have

$$|\alpha(s_k)| \leqslant |\alpha|_{g^*_{X/Y,L}} |s_k|_{g_{X/Y,L}} \quad \Rightarrow \quad \psi_\alpha + \log\lambda_k \geqslant \theta_k,$$

where $\lambda_k := \Lambda_{s_k} = |s_k|^2_{g_{X/Y,L}}$. By passing to the limit superior we obtain (in virtue of (2.10))

$$\psi_\alpha(0) + \limsup_{k\to+\infty} (\log\lambda_k(0)) \geqslant \limsup_{k\to+\infty} \theta_k(0) = \lim_{k\to+\infty} \theta_k(0) = \lim_{k\to+\infty} \theta_k(y_k) = \lim_{k\to+\infty} \psi_\alpha(y_k).$$

It remains thus to show

$$\limsup_{k\to+\infty} (\log\lambda_k(0)) \leqslant 0,$$

and this amounts to show (the function log being increasing and continuous)

$$\limsup_{k\to+\infty} \lambda_k(0) \leqslant 1.$$

Now up to taking an extraction, we can assume that the sequence $\{\lambda_k(0)\}_{k>0}$ is convergent. By the compactness of $S_{M_0'}$ (Point (b) in the proof of Lemma 2.2.8), up to taking a subsequence, we can further assume that $\{s_k\}_{k>0}$ converges uniformly on all compacts in B_1 to a section $s \in S_{M_0'}$. By (2.6) (c.f. Point (a) in the proof of Proposition 2.2.7) we have for $y \in B_1 \cap Y_1$ that

$$\lambda_k(y) = \int_{X_0} G_{s_k}(y,\cdot)\mathrm{Vol}_{X_0},$$
$$\Lambda_s(y) = \int_{X_0} G_s(y,\cdot)\mathrm{Vol}_{X_0}.$$

By (2.7) the compact convergence $\{s_k\}_{k>0}$ implies that $\left\{G_{s_k}\right\}_{k>0}$ converges uniformly over all compacts to G_s (especially over $\bar{B}_3$). By Point (a) in the proof of Proposition 2.2.7, the G_{s_k}'s as well as G_s are all lower semi-continuous functions, thus

$$G_{s_k}(0,\cdot) \leqslant \liminf_{l\to+\infty} G_{s_k}(y_l,\cdot),$$
$$G_s(0,\cdot) \leqslant \liminf_{l\to+\infty} G_s(y_l,\cdot),$$

and in consequence (by a diagonal process)

$$G_s(0,\cdot) \leqslant \liminf_{k\to+\infty} G_{s_k}(y_k,\cdot).$$

Then Fatou's lemma implies that,

$$\lim_{k\to+\infty}\lambda_k(0)=\Lambda_s(0)=\int_{X_0}G_s(0,\cdot)\mathrm{Vol}_{X_0}\leqslant\int_{X_0}\liminf_{k\to+\infty}G_{s_k}(y_k,\cdot)\mathrm{Vol}_{X_0}$$
$$\leqslant\liminf_{k\to+\infty}\int_{X_0}G_{s_k}(y_k,\cdot)=\liminf_{k\to+\infty}\lambda_k(y_k)=1,$$

which proves the result.

Proof of (C): Let Δ be any disc contained in Y_1, we will prove that

$$\psi_\alpha(0)\leqslant\frac{1}{\mu(\Delta)}\int_\Delta\psi_\alpha d\mu. \tag{2.11}$$

We can assume that $Y=\Delta(=Y_1=Y_0)$, in particular, f is a smooth fibration. If $\psi_\alpha(0)=-\infty$, then the inequality (2.11) is automatically established; hence we can assume that $\psi_\alpha(0)\neq-\infty$, in particular $h_L|_{X_0}\not\equiv+\infty$. As in the step (A1), there is a section $v\in\mathrm{H}^0(X_0,K_{X_0}\otimes L|_{X_0}\otimes\mathscr{J}(h_L|_{X_0}))$ such that $\|v\|_{0,L}=1$ and

$$\psi_\alpha(0)=\log|\alpha(0)(v)|^2.$$

Again by Theorem 2.1.1 we get a section $s\in\mathrm{H}^0(X,K_{X/\Delta}\otimes L\otimes\mathscr{J}(h_L))$ such that $s|_{X_0}=v$ and

$$\int_\Delta\Lambda_s(t)dt=\int_X|s|^2e^{-\phi_L}\leqslant\mu(\Delta).$$

In particular $\left(\log|\alpha(s)|^2\right)(0)=\psi_\alpha(0)$. By definition we have

$$|\alpha(s)|\leqslant|\alpha|_{g^*_{X/Y,L}}|s|_{g_{X/Y,L}}\quad\Rightarrow\quad\psi_\alpha+\log\Lambda_s\geqslant\log|\alpha(s)|^2.$$

The function $\log|\alpha(s)|$ being psh on Δ, it satisfies the mean value inequality, hence we have

$$\frac{1}{\mu(\Delta)}\int_\Delta\psi_\alpha d\mu+\frac{1}{\mu(\Delta)}\int_\Delta\log\Lambda_s d\mu\geqslant\frac{1}{\mu(\Delta)}\int_\Delta\log|\alpha(s)|d\mu\geqslant\left(\log|\alpha(s)|^2\right)(0)=\psi_\alpha(0).$$

It remains to show that

$$\int_\Delta\log\Lambda_s d\mu\leqslant0,$$

but the function log being concave, this is a result of Jensen's inequality: Λ_s being integrable, we have

$$\int_\Delta\log\Lambda_s\frac{d\mu}{\mu(\Delta)}\leqslant\log\left(\int_\Delta\Lambda_s\frac{d\mu}{\mu(\Delta)}\right)=\leqslant\log1=0.$$

This proves (2.11), and thus finishes the proof of the step (C). Hence $g_{X/Y,L}$ is a semipositively curved singular Hermitian metric on $\mathscr{G}_L$.

In order to finish the proof of Theorem 2.2.4, it remains to show that $(\mathscr{G}_L,g_{X/Y,L})$ satisfies the L^2 extension property. To this end, take an open subset U of Y and Z an analytic subset of U, and take a local section $s\in\mathrm{H}^0(U\backslash Z,\mathscr{G}_L)$ satisfying the L^2 integrability condition, we will show that s extends to a section over U. The problem being local, we can replace U by a small ball B in Y (with $t_1,\cdots,t_d$ the standard coordinates). Then $s\in\mathrm{H}^0(B\backslash Z,\mathscr{G}_L)=\mathrm{H}^0(f^{-1}(B\backslash Z),K_{X/Y}\otimes L\otimes\mathscr{J}(h_L))$ satisfies the following L^2 condition:

$$M_s:=\int_{B\backslash Z}\left(|s|^2_{g_{X/Y,L}}\right)\eta=\int_{f^{-1}(B\backslash Z)}|s\wedge f^*\eta|^2e^{-\phi_L}<+\infty,$$

where $\eta = dt_1 \wedge \cdots \wedge dt_d$ is a nowhere vanishing holomorphic d-form (giving rise to a trivialization $K_B \simeq \mathscr{O}_B \cdot \eta$). Then it is an elementary consequence of Riemann extension that s extends to a section in $\mathrm{H}^0(f^{-1}(B), K_{X/Y} \otimes L \otimes \mathscr{J}(h_L)) = \mathrm{H}^0(B, \mathscr{G}_L)$. For the convenience of the readers, let us gives the details of the argument. Let $x \in f^{-1}(Z)$ and let Ω be the ball centred at x of radius ϵ such that $\Omega' = 2\Omega$ is contained in $f^{-1}B$ (with ϵ sufficiently small). Set $\mu_\epsilon = \mu(\Omega)$ and set $z_1, \cdots, z_n$ the local coordinates of Ω'. In addition, let σ_L be a local basis of L (i.e. $L|_{\Omega'} \simeq \mathscr{O}_{\Omega'} \cdot \sigma_L$) , then $e^{-\phi_L}|_{\Omega'} = |\sigma_L|^2_{h_L}$ and we can write:

$$s \wedge f^*\eta = \Phi_s \cdot (dz_1 \wedge \cdots \wedge dz_n) \otimes \sigma_L$$

for some holomorphic function Φ_s on $\Omega' \cap f^{-1}(B \backslash Z)$. The function ϕ_L is psh, in particular it is uniformly bounded from above over Ω' (since it is sufficiently small), hence there is a constant C such that $\phi_L \leqslant C$ over Ω'. $\forall y \in \Omega \cap f^{-1}(B \backslash Z)$, we have $B(y, \epsilon) \subseteq \Omega'$ and therefore

$$|\Phi_s(y)|^2 \leqslant \frac{1}{\mu(B(y,\epsilon))} \int_{B(y,\epsilon)\cap f^{-1}(B\backslash Z)} |\Phi_s|^2 dz \leqslant \frac{e^C}{\mu_\epsilon} \int_{B(y,\epsilon)\cap f^{-1}(B\backslash Z)} |s \wedge f^*\eta|^2 e^{-\phi_L} \leqslant \frac{Me^C}{\mu_\epsilon} < +\infty.$$

Hence Φ_s is uniformly bounded on Ω, then by Riemann extension Φ_s extends to a holomorphic function over Ω. This implies that s extends to a section in $\mathrm{H}^0(f^{-1}B, K_{X/Y} \otimes L \otimes \mathscr{J}(h_L)) = \mathrm{H}^0(B, \mathscr{G}_L)$, meaning that $(\mathscr{G}_L, g_{X/Y,L})$ satisfies the L^2 extension property. This finishes the proof of Theorem 2.2.4. □

2.2.3 Positivity of direct images of twisted relative pluricanonical bundles

In this subsection, we will apply Theorem 2.2.1 and Theorem 2.2.4 to prove Theorem D, which will serve as a key ingredient in the proof of Theorem A.

Proof of Theorem D. Recall that

$$\mathscr{F}_{m,\Delta} := f_*\Big(K^{\otimes m}_{X/Y} \otimes \mathscr{O}_X(m\Delta)\Big).$$

If $\mathscr{F}_{m,\Delta} = 0$, then there is nothing to prove; hence we assume that $\mathscr{F}_{m,\Delta} \neq 0$. Since (X, Δ) is klt (implying that (X_y, Δ_y) is klt for y general by [Laz04, §9.5.D, Theorem 9.5.35, pp. 210-211, vol.II]) and $\mathscr{F}_{m,\Delta} \neq 0$, the condition in the hypothesis of Theorem 2.2.1 is satisfied for $L = \mathscr{O}_X(m\Delta)$ and $h_L = h^{\otimes m}_\Delta$ where h_Δ is the canonical (singular) Hermitian metric defined by the local equations of Δ, then we obtain a singular Hermitian metric $h^{(m)}_{X/Y,m\Delta}$ over $K^{\otimes m}_{X/Y} \otimes \mathscr{O}_X(m\Delta)$ whose curvature current is positive. However one cannot directly apply Theorem 2.2.4 to obtain a semipositively curved singular Hermitian metric on $\mathscr{F}_{m,\Delta}$. In order to overcome this difficulty, we introduce the line bundle

$$L_{m-1} = K^{\otimes(m-1)}_{X/Y} \otimes \mathscr{O}_X(m\Delta),$$

equipped with the metric

$$h_{L_{m-1}} := (h^{(m)}_{X/Y,m\Delta})^{\otimes\frac{m-1}{m}} \otimes h_\Delta.$$

Then the curvature current of $h_{L_{m-1}}$ is positive. We are now ready to apply Theorem 2.2.4 to $L = L_{m-1}$, except that we need to establish in addition that the natural inclusion

$$f_*\Big(K_{X/Y} \otimes L_{m-1} \otimes \mathscr{J}(h_{L_{m-1}})\Big) \hookrightarrow \mathscr{F}_{m,\Delta}$$

is generically an isomorphism.

To this end, let Y_2 be the (analytic) Zariski open subset of Y_0 satisfying Conditions (i)(ii) in the definition of Y_1 for $L = L_{m-1}$ (see Notations) and such that the pair (X_y, Δ_y) is

klt for $\forall y \in Y_2$ (c.f. [Laz04, §9.5.D, Theorem 9.5.35, pp. 210-211, vol.II]). By virtue of the base change property of $\mathscr{F}_{m,\Delta}$ over Y_2 and of Lemma 2.2.6, it suffices to prove that the natural inclusion

$$\mathrm{H}^0(X_y, K_{X_y} \otimes L_{m-1}|_{X_y} \otimes \mathscr{J}(h_{L_{m-1}}|_{X_y})) \hookrightarrow \mathrm{H}^0(X_y, K_{X_y} \otimes L_{m-1}|_{X_y}) \tag{2.12}$$

is an isomorphism for $y \in Y_2$. But this results from the following Lemma 2.2.9. □

Lemma 2.2.9. *Let $f : X \to Y$ be a Kähler fibre space between complex manifolds and let N be a $\mathbb{Q}$-line bundle endowed with a semipositively curved singular Hermitian metric h_N such that $\mathscr{J}(h_N|_{X_y}) = \mathscr{O}_{X_y}$ for almost every $y \in Y_0$ (which is the case, e.g. if $\mathscr{J}(h_N) = \mathscr{O}_X$, by Lemma 2.2.5). If the direct image sheaf $f_*(K_{X/Y}^{\otimes m} \otimes N^{\otimes m}) \neq 0$, then by Theorem 2.2.1 one can construct the relative m-Bergman kernel metric $h^{(m)}_{X/Y,mN}$ on $K_{X/Y}^{\otimes m} \otimes N^{\otimes m}$ whose curvature current is positive. Set*

$$N_{m-1} := K_{X/Y}^{\otimes(m-1)} \otimes N^{\otimes m}$$

and

$$h_{N_{m-1}} := (h^{(m)}_{X/Y,mN})^{\otimes \frac{m-1}{m}} \otimes h_N.$$

Then the natural inclusion

$$\mathrm{H}^0(X_y, K_{X_y} \otimes N_{m-1}|_{X_y} \otimes \mathscr{J}(h_{N_{m-1}}|_{X_y})) \hookrightarrow \mathrm{H}^0(X_y, K_{X_y} \otimes N_{m-1}|_{X_y})$$

is an isomorphism (or equivalently, surjective) for a.e. $y \in Y_0$.

Proof. Let $y \in Y_0$ be a point such that $\mathscr{J}(h_N|_{X_y}) = \mathscr{O}_{X_y}$ and let $v \in \mathrm{H}^0(X_y, K_{X_y} \otimes N_{m-1}|_{X_y}) = \mathrm{H}^0(X_y, K_{X_y}^{\otimes m} \otimes N^{\otimes m}|_{X_y})$, then with the same notations as in §2.2.1 we can write

$$v \wedge (dt_1 \wedge \cdots \wedge dt_d)^{\otimes m} = F_v \cdot (dz_1 \wedge \cdots \wedge dz_n)^{\otimes m}.$$

Since $\mathscr{J}(h_N|_{X_y}) = \mathscr{O}_{X_y}$, we have

$$\|v\|^{\frac{2}{m}}_{m,y,mN} = \int_{X_y} |v|^{\frac{2}{m}} e^{-\phi_N} = \int_{X_y} \left(|F_v|^{\frac{2}{m}} e^{-\phi_N}\right) \mathrm{Vol}_{X_y} < +\infty, \tag{2.13}$$

where ϕ_N denotes the local weight of the metric h_N. By (2.2) (c.f. also [BP10, §A.2, p. 8]) the local weight $\phi^{(m)}_{X/Y,mN}$ satisfies

$$\phi^{(m)}_{X/Y,mN} = \log\left(\sup_{\|u\|_{m,y,mN} \leqslant 1} |F_u|^2\right) \geqslant \log\left(\frac{|F_v|^2}{\|v\|^2_{m,y,mN}}\right),$$

and thus

$$\log|F_v|^2 \leqslant \phi^{(m)}_{X/Y,mN} + O(1) \quad \Rightarrow \quad |F_v|^{\frac{2(m-1)}{m}} e^{-\frac{m-1}{m}\phi^{(m)}_{X/Y,mN}} \leqslant O(1). \tag{2.14}$$

Regarded as a holomorphic n-form with values in the line bundle $N_{m-1}|_{X_y}$, the section v satisfies

$$|v|^2 e^{-\phi_{N_{m-1}}} = |F_v|^2 e^{-\frac{m-1}{m}\phi^{(m)}_{X/Y,mN} - \phi_N} \cdot \mathrm{Vol}_{X_y},$$

where $\phi_{N_{m-1}}$ denotes the local weight of the metric $h_{L_{m-1}}$, hence by (2.13) and (2.14) we have

$$\begin{aligned}
\|v\|^2_{y,N_{m-1}} &= \int_{X_y} |v|^2 e^{-\phi_{N_{m-1}}} = \int_{X_y} \left(|F_v|^2 e^{-\frac{m-1}{m}\phi^{(m)}_{X/Y,N} - \phi_N}\right) \mathrm{Vol}_{X_y} \\
&= \int_{X_y} \left(|F_v|^{\frac{2}{m}} e^{-\phi_N}\right) \cdot \left(|F_v|^{\frac{2(m-1)}{m}} e^{-\frac{m-1}{m}\phi^{(m)}_{X/Y,N}}\right) \mathrm{Vol}_{X_y} \\
&\leqslant C \int_{X_y} \left(|F_v|^{\frac{2}{m}} e^{-\phi_N}\right) \mathrm{Vol}_{X_y} = C \cdot \|v\|^{\frac{2}{m}}_{m,y,N} < +\infty,
\end{aligned}$$

where C is a constant given by (2.14). Therefore $v \in \mathrm{H}^0(X_y, K_{X_y} \otimes L|_{X_y} \otimes \mathscr{J}(h_{N_{m-1}}|_{X_y}))$, which proves the lemma. □

By combining Theorem D and Theorem 1.4.8 we immediately get:

Corollary 2.2.10. *Let $f : X \to Y$ and Δ as in Theorem D. Suppose that the determinant of $\mathscr{F}_{m,\Delta}$ is numerically trivial. Then $\left(\mathscr{F}_{m,\Delta}, g^{(m)}_{X/Y,\Delta}\right)$ is a Hermitian flat vector bundle.*

2.2.4 Generalizations

In this subsection we extend the notion of (Griffiths) semipositivity for singular Hermitian metrics introduced in §1.4 to a more flexible one and generalize some of results stated above to this general setting. Throughout this subsection let W be a complex manifold. For a (holomorphic) vector bundle E over W, a singular Hermitian metric h on E is given by a measurable family of semipositive definite Hermitian functions on each fibre of E which is non-singular almost everywhere. Let θ be a smooth $(1,1)$-form on W, then h is called θ-semipositive if for every open subset U of W and for every local holomorphic section $s \in \mathrm{H}^0(W, E^*)$ of the dual bundle of E, the function $\log|s|^2_{h^*}$ is $(-\theta)$-plurisubharmonic ($(-\theta)$-psh), i.e. $\mathrm{d}\mathrm{d}^c \log|s|^2_{h^*} - \theta$ is a positive current on U. Moreover, suppose that (W, ω) is a compact Kähler manifold with Kähler metric ω. Then the vector bundle E is called *θ-weakly semipositively curved* if for every $\epsilon > 0$ small there exists a singular Hermitian metric h_ϵ on E which is $(-\epsilon\omega + \theta)$-semipositive. If θ is a smooth form in the first Chern class of some ($\mathbb{Q}$-)line bundle L, then a θ-semipositive metric is also called *L-semipositive* and a θ-weakly semipositively curved vector bundle is also called *L-weakly semipositively curved*. In particular, E is called *weakly semipositively curved* if it is θ-weakly semipositively curved for $\theta = 0$.

By definition, h is semipositively curved if it is θ-semipositive for $\theta = 0$. Let us remark that if E is a line bundle on W projective, then being (weakly) semipositively curved is equivalent to being pseudoeffective.

As in the semipositivity case, the θ-semipositivity of singular Hermitian metrics is preserved by tensor products (up to multiplying θ), pullback by proper surjective morphisms (up to pulling back θ), and by generically surjective morphisms of vector bundles (thus by symmetric and exterior products, up to multiplying θ). Moreover θ-semipositive singular Hermitian metrics extend (and remaining θ-semipositive) across closed analytic subsets of codimension $\geqslant 2$ and across closed analytic subsets of codimension 1 under the condition that the metric is locally bounded (c.f. [CH19, Proposition 2.4]). In virtue of the aforementioned extension theorem and of [Kob87, Corollary 5.5.15, p. 147] one can naturally extend of the notion of θ-semipositive singular Hermitian metrics to torsion free sheaves.

Remark 2.2.11 (Comparison with the algebro-geometric notion of weak positivity). Suppose that W is projective. For a torsion free sheaf $\mathscr{F}$ on W projective, being (weakly) semipositively curved implies the weak positivity (in the sense of Nakayama [Nak04; Fuj17]), c.f. [PT18; Pău16]; the reciprocal implication is also expected to be true (but still open), and can be regarded as a singular version of Griffiths's conjecture.

Then we can generalize Theorem D to the following:

Theorem 2.2.12. *Let $f : V \to W$ be a fibre space between Kähler manifolds and let (L, h_L) be a line bundle on V equipped with a singular Hermitian metric h_L such that the curvature current $\Theta_{h_L}(L) \geqslant f^*\theta$ for some smooth closed $(1,1)$-form on Y. Suppose that there is an $m \in \mathbb{Z}_{>0}$ such that $\mathscr{J}(h_L^{1/m}|_{V_w}) \simeq \mathscr{O}_{V_w}$ for general $w \in W$. Then the canonical L^2 metric $g^{(m)}_{V/W,L}$ on $f_*\mathscr{O}_V(mK_{V/W} + L)$ is θ-semipositive on Y.*

Proof. This is essentially proved in [CP17, Lemma 5.4], see also [CH19, 2.8.Proposition] and [CCM19, Theorem 2.2(1)]. For the convenience of the readers, we briefly recall the proof. As in the proof of Theorem D, we construct the m-Bergman kernel metric $h^{(m)}_{V/W,h_L}$ on the twisted relative canonical bundle $mK_{V/W}+L$, and equip the line bundle $L_{m-1} := (m-1)K_{V/W}+L$ with the metric

$$h_{L_{m-1}} := \left(h^{(m)}_{V/W,h_L}\right)^{\otimes\frac{m-1}{m}} \otimes h_L^{\otimes\frac{1}{m}};$$

then the metric $g^{(m)}_{V/W,L}$ is constructed as the canonical L^2 metric on the direct image

$$\mathscr{G}_{m,L} := f_*\mathscr{O}_V(K_{V/W}+L_{m-1}) = f_*\mathscr{O}_V(mK_{V/W}+L).$$

Since the construction of the m-Bergman kernel metric and of the L^2 metric is local over W (c.f. §2.2.1 or [BPo08; Pău16; HPS18]), we can assume (by the dd^c-lemma) that θ is given by a weight function ρ, i.e. $\theta = dd^c\rho$. Then $h_{1,L} := h_L\cdot e^{\rho\circ f}$ defines a new singular Hermitian metric on L whose curvature current is positive:

$$\Theta_{h_{1,L}}(L) = \Theta_{h_L}(L) - dd^c(\rho\circ f) = \Theta_{h_L}(L) - f^*\theta \geqslant 0.$$

Now by [Cao17, Theorem 1.2] the m-Bergman kernel metric $h^{(m)}_{V/W,h_{1,L}}$ on the twisted relative canonical bundle $mK_{V/W}+L$ is semipositively curved. Now equip the line bundle L_{m-1} with the singular Hermitian metric

$$h_{1,L_{m-1}} := \left(h^{(m)}_{V/W,h_{1,L}}\right)^{\otimes\frac{m-1}{m}} \otimes h_L^{\otimes\frac{1}{m}},$$

since $\mathscr{J}(h_{1,L}^{1/m}|_{V_w}) \simeq \mathscr{J}(h_L^{1/m}|_{V_w}) \simeq \mathscr{O}_{V_w}$ for general $w\in W$, by Lemma 2.2.9, the natural inclusion

$$f_*(\mathscr{O}_V(mK_{V/W}+L)\otimes\mathscr{J}(h_{1,L_{m-1}})) \hookrightarrow \mathscr{G}_{m,L}$$

is a generic isomorphism. Thus by [DWZZ18, Theorem 9.3] or Theorem 2.2.4 the canonical L^2 metric

$$g_{V/W,h_{1,L_{m-1}}} = g^{(m)}_{V/W,L}\cdot e^{-\rho}$$

is semipositively curved. In other word, for every local section s of the dual sheaf of $\mathscr{G}_{m,L}$, we have

$$0 \leqslant dd^c\log|s|^2_{g^*_{V/W,h_{1,L_{m-1}}}} = dd^c\log|s|^2_{g^{(m)*}_{V/W,L}} - dd^c\rho,$$

which means that the metric $g^{(m)}_{V/W,L}$ is θ-semipositive. □

As a result of the above Theorem 2.2.12, we have:

Corollary 2.2.13 ([CCM19, Theorem 2.2(2)]). *Let $f: V\to W$, (L,h_L) and m as in the Theorem 2.2.12. Assume further that f is projective, V and W are compact and L is f-big. Then for any nef line bundle N on V, the direct image sheaf $f_*\mathscr{O}_V(mK_{V/W}+N+L)$ is θ-weakly positively curved.*

Proof. For the convenience of the readers, we briefly recall the proof. Since L is ϕ-big, there is a singular Hermitian metric h on L such that $\Theta_h(L)+f^*(\omega_W-\theta)\geqslant\omega_V$ as current for some Kähler form ω_W on W (such that ω_W is still a Kähler form) and for some Kähler form ω_V on V. Since N is nef, there are smooth Hermitian metrics $(g_\delta)_{\delta>0}$ on N such that $\Theta_{g_\delta}(N)+\delta\omega_V\geqslant 0$. Now consider the singular Hermitian metric

$$h_\epsilon := h_L^{1-\epsilon}\otimes h^\epsilon\otimes g_\epsilon$$

on the line bundle $L\otimes N$. Then for ϵ sufficiently small (with respect to h) we have

$$\mathscr{J}(h_\epsilon^{1/m}|_{V_w}) = \mathscr{J}(h_L^{(1-\epsilon)/m}\otimes h^{\epsilon/m}|_{V_w}) \simeq \mathscr{O}_{V_w}$$

for general $w\in W$. And by a direct computation we have

$$\Theta_{h_\epsilon}(L\otimes N) \geqslant f^*(-\epsilon\omega_W+\theta).$$

Then Theorem 2.2.12 implies that $f_*\mathscr{O}_V(mK_{V/W}+N+L)$ is θ-weakly semipositively curved. □

2.3 Numerically flat vector bundles and locally constant fibrations

In this section we recall the notion of numerically flat vector bundles as well as its relation to the local constancy of fibre spaces; then we recall a fundamental criterion for numerical flatness. First let us define the numerical flatness for vector bundles on compact Kähler manifolds (c.f. [DPS94, Definition 1.9 & Definition 1.17]):

Definition 2.3.1. Let W be a compact complex manifold. A holomorphic vector bundle E on W is said nef if the line bundle $\mathscr{O}_{\mathbb{P}E}(1)$ on $\mathbb{P}E$ is nef (c.f. [DPS94, Definition 1.2] for the more general definition of nefness of holomorphic line bundles on (non-necessarily algebraic) compact complex manifolds). The vector bundle E is said to be numerically flat if both E and its dual E^* are nef.

As shown in [DPS94, Proposition 1.14 & Proposition 1.15], nefness of vector bundles is preserved by tensor products, by surjection of vector bundles, by pullbacks via surjective morphisms, and thus by symmetric and exterior products. Moreover, by [DPS94, Theorem 1.18], [Sim92, Corollary 3.10] (c.f. also [Den17b, Ch.6, Theorem V]) and [Cao13, Lemma 4.3.3], we have the following structure result on numerically flat vector bundles:

Theorem 2.3.2. *Let W be a compact Kähler manifold and let E be a numerically flat vector bundle on W. Then we have:*

(a) *E admits a filtration*

$$\{0\} = E_0 \subsetneqq E_1 \subsetneqq E_1 \subsetneqq \cdots \subsetneqq E_k = E.$$

where the E_i are vector bundles and the quotients E_{i+1}/E_i are Hermitian flat vector bundles, that is, induced by unitary representations $\pi_1(X)\to \mathrm{U}(r_i)$.

(b) *E is isomorphic to the underlying holomorphic vector bundle of a local system L, such that the natural Gauss-Manin connection ∇ on L is compatible with the filtration in* (a) *and induces flat connections on the quotients E_{i+1}/E_i. In particular, every section of $\mathrm{H}^0(X,E)$ is parallel with respect to ∇.*

Next let us define:

Definition 2.3.3. Let $f: V\to W$ be a fibre space. We call f a *locally constant fibration* if f is a locally trivial fibre bundle with fibre F and there is a representation $\rho:\pi_1(W)\to \mathrm{Aut}(F)$ such that V isomorphic to the quotient of $\widetilde{W}\times F$ by the action of $\pi_1(W)$ given by $\gamma\cdot(w,z) = (\gamma\cdot w, \rho(\gamma)z)$ where $\widetilde{W}$ denotes the universal cover of W.

Remark 2.3.4. In the definition above, we see that $\pi_1(W)$ acts diagonally on $\widetilde{W}\times F$. Hence if we suppose in addition that V is normal, then the natural decomposition $T_{\widetilde{W}\times F}\simeq \mathrm{pr}_1^*T_{\widetilde{W}}\oplus \mathrm{pr}_2^*T_F$ induces a splitting of the tangent sheaf of V into foliations.

As a corollary of Theorem 2.3.2 we have the following proposition which reveals the relation between local constancy of fibre spaces and numerical flatness of direct images (c.f. [Cao19, 2.8.Proposition] and [CCM19, Proposition 2.8]; c.f. also [CH17, 4.1.Proposition] and [Cao13, Proposition 4.3.6]):

Proposition 2.3.5. *Let W be a compact Kähler manifold and let $f : V \to W$ be a flat projective morphism with connected fibres (V is not necessarily smooth). Suppose that there is a f-very ample line bundle L on V such that for every $m > 1$ the direct image $E_m := f_*(mL)$ is a numerically flat vector bundle. Then f is locally constant.*

Proof. We will follow the main line of the argument in the proof of [CCM19, Proposition 2.8]. We nevertheless give some details in order to illustrate how the proof works for V singular. Since L is f-very ample, we have an embedding $i : V \to \mathbb{P}E_1$ over W with $i^*\mathscr{O}_{\mathbb{P}E_1}(1) = L$. Let $\mathscr{I}_V$ be the ideal of V in $\mathbb{P}E_1$, we will show that (up to twisting with some power of $\mathscr{O}_{\mathbb{P}E_1}(1)$) the generating polynomials of $\mathscr{I}_V$ have coefficients being constant functions over W.

By relative Serre vanishing, for m large enough we have a short exact sequence

$$0 \to p_*(\mathscr{I}_V \otimes \mathscr{O}_{\mathbb{P}E_1}(m)) \to p_*(\mathscr{O}_{\mathbb{P}E_1}(m)) \to E_m = f_*(mL) \to 0, \tag{2.15}$$

where p denotes the natural morphism $\mathbb{P}E_1 \to W$. By hypothesis E_1 is a numerically flat vector bundle, then by Theorem 2.3.2 it is a local system, equipped with the Gauss-Manin connection ∇_{E_1}. Take $\gamma : \widetilde{W} \to W$ the universal covering of W, then γ^*E_1 is trivial. And there are $r+1$ global sections $e_0, \cdots, e_r$ in $\mathrm{H}^0(\widetilde{W}, \gamma^*E_1)$ which are parallel with respect to ∇_{E_1} and generate γ^*E_1, where $r := \mathrm{rk}E_1 - 1$.

Now set $F_m := p_*(\mathscr{I}_V \otimes \mathscr{O}_{\mathbb{P}E_1}(m))$. The morphism f being flat, $\mathscr{I}_V$ is flat over W, thus by the same argument as that in [Har77, §III.9, Proof of Theorem 9.9, pp. 261-262], F_m is a vector bundle for m sufficiently large. Then by the short exact sequence (2.15) and by [DPS94, Proposition 1.15], F_m is numerically flat. Then again by Theorem 2.3.2 F_m is a local system, equipped with the Gauss-Manin connection ∇_{F_m}. By the same argument as above, γ^*F_m is a trivial vector bundle and admits generating global sections $f_1, \cdots, f_{s_m}$ which are parallel with respect to ∇_{F_m}, where $s_m := \mathrm{rk}F_m$.

Consider the inclusion

$$\eta : \gamma^*F_m \hookrightarrow \gamma^*p_*\mathscr{O}_{\mathbb{P}E_1}(m) = \gamma^*\mathrm{Sym}^m E_1.$$

By Theorem 2.3.2, the sections $\eta(f_i)$ are all parallel with respect to the connection $\nabla_{\mathrm{Sym}^m E_1}$ induce by ∇_{E_1} on $\mathrm{Sym}^m E_1$. Since $\mathrm{Sym}^m E_1$ is generated by the flat global sections

$$\left(e_0^{\alpha_0} \cdots e_r^{\alpha_r}\right)_{\alpha_j \in \mathbb{Z}_{>0}, \alpha_0 + \cdots + \alpha_r = m}$$

we can write, for every $i = 1, \cdots, s_m$

$$\eta(f_i) = \sum_{\substack{\alpha = (\alpha_0, \cdots, \alpha_r) \in \mathbb{Z}_{>0}^{r+1} \\ |\alpha| = m}} c_{i,\alpha} \cdot e_0^{\alpha_0} \cdots e_r^{\alpha_r},$$

for some constants $c_{i,\alpha} \in \mathbb{C}$. This then implies that the embedding of $\widetilde{V} := V \underset{W}{\times} \widetilde{W}$ into $\widetilde{W} \times \mathbb{P}^r$ over $\widetilde{W}$ is defined by polynomials whose coefficients are independent of $w \in \widetilde{W}$. Hence $\widetilde{V}$ splits into a product $\widetilde{W} \times F$ where F is the general fibre of f. Since E_1 is a flat bundle, it is induced by a representation $\rho_1 : \pi_1(W) \to \mathrm{PGL}(r+1)$. Let $\gamma \in \pi_1(W)$, then $\rho_1(\gamma)$ sends V_w to $V_{\gamma(w)}$ viewed as subvarieties of $\mathbb{P}^r$. But as seen before, the defining polynomial of V_w in $\mathbb{P}^r$ is independent of w hence $\rho_1(\gamma)$ can be seen as an element of $\mathrm{Aut}(F)$, and hence a representation $\rho : \pi_1(W) \to \mathrm{Aut}(F)$. By construction V is isomorphic to the quotient of $\widetilde{V}$ by the action of $\pi_1(W)$, and hence f is locally constant fibration. □

To finish this subsection let us recall the following numerical flatness criterion, which is proved in [CCM19, Proposition 2.7] when W is projective and is extended to Kähler case by [Wu20, §1, Corollary of Main Theorem]:

Proposition 2.3.6. *Let W be a smooth projective variety and let $\mathscr{F}$ be a reflexive sheaf on W. Suppose that $\mathscr{F}$ is weakly positively curved and that $\det\mathscr{F}$ is numerically trivial. Then $\mathscr{F}$ is a numerically flat vector bundle on W.*

2.4 Holomorphic foliations on normal varieties

In this section, we attempt to recollect some results coming from different literatures in order to give a somewhat general account on singular foliations on normal varieties, with some emphasis on the algebro-geometric aspect, for the convenience of the readers. Some results will be used in §4.3.

2.4.1 General results on holomorphic foliations

First recall some definitions:

Definition 2.4.1. Let X be a normal complex variety and let $T_X := (\Omega^1_X)^*$ denotes the tangent sheaf of X (then it is a coherent reflexive sheaf on X). A (singular) foliation on X is a subsheaf $\mathscr{F}$ of T_X satisfying the following two conditions:

(i) $[\mathscr{F},\mathscr{F}]\subseteq\mathscr{F}$, i.e. $\mathscr{F}$ is stable under the Lie bracket (we call such $\mathscr{F}$ *involutive*);

(ii) $\mathscr{F}$ is saturated in T_X, i.e. the quotient $T_X/\mathscr{F}$ is torsion free (which implies that $\mathscr{F}$ is reflexive[1]).

The *codimension* of $\mathscr{F}$ is defined to be $n-\mathrm{rk}\,\mathscr{F}$. The *normal sheaf* of $\mathscr{F}$ is defined to be $N_{\mathscr{F}} := (T_X/\mathscr{F})^{**}$. A *leaf* L of $\mathscr{F}$ is a maximal connected and immersed holomorphic submanifold of X° such that $T_L = \mathscr{F}|_L$ where X° denotes the Zariski open subset of X_{reg} on which $\mathscr{F}|_{X_{\mathrm{reg}}}$ is a subbundle of $T_{X_{\mathrm{reg}}}$ (by [Kob87, Corollary 5.5.15, p. 147], $\mathrm{codim}(X\backslash X^\circ)\geqslant 2$). If X is projective, the *canonical divisor* $K_{\mathscr{F}}$ of $\mathscr{F}$ is defined to be a Weil divisor on X satisfying $\det\mathscr{F}\simeq\mathscr{O}_X(-K_{\mathscr{F}})$ (defined up to linear equivalence).

The following lemma says that the involutivity of a saturated subsheaf of T_X can be checked over any Zariski open of X:

Lemma 2.4.2. *Let X be a normal complex variety and $\mathscr{F}$ a saturated subsheaf of T_X. Then $\mathscr{F}$ is involutive if and only if $\mathscr{F}|_{X_0}$ is involutive for some Zariski open $X_0\subseteq X$.*

Proof. The "only if" part is obvious, we will prove the "if" part as following: notice first that the problem is local, hence we can assume X is a Stein variety, so that every coherent sheaf on X is globally generated (so is $\mathscr{F}$, and $\mathscr{H}$, $\mathscr{H}_1$ below). Consider the hom sheaves

$$\mathscr{H} := \mathscr{H}om_{\mathscr{O}_X}(\bigwedge^2\mathscr{F},\mathscr{F})\subseteq\mathscr{H}_1 := \mathscr{H}om_{\mathscr{O}_X}(\bigwedge^2\mathscr{F},T_X),$$

since $\mathscr{F}$ and T_X are reflexive, by the tensor-hom adjunction and [Har80, Corollary 1.2], so are $\mathscr{H}$ and $\mathscr{H}_1$. Moreover, $\mathscr{H}_1/\mathscr{H}$ is contained in $\mathscr{H}om_{\mathscr{O}_X}(\bigwedge^2\mathscr{F},T_X/\mathscr{F})$, which is torsion free since $T_X/\mathscr{F}$ is torsion free (again by the tensor-hom adjunction and [Har80, Corollary 1.2]), hence $\mathscr{H}_1/\mathscr{H}$ is also torsion free ($\mathscr{H}$ is saturated in $\mathscr{H}_1$). Now consider

[1] The reciprocal is not true, e.g. consider the natural inclusion $\mathscr{O}_{\mathbb{A}^1}(-D)\hookrightarrow\mathscr{O}_{\mathbb{A}^1}\simeq T_{\mathbb{A}^1}$ for D effective divisor, $\mathscr{O}_{\mathbb{A}^1}(-D)$ is reflexive (locally free) but it is not saturated in $\mathscr{O}_{\mathbb{A}^1}$.

$\sigma = [\,\cdot\,,\cdot\,]\big|_{\bigwedge^2\mathscr{F}}$, regarded as a (global) section of $\mathscr{H}_1$. Then the involutivity of $\mathscr{F}$ is equivalent to $\sigma \in \mathscr{H}$. This amounts to show that the image $\bar{\sigma}$ of σ in $\mathscr{H}_1/\mathscr{H}$ is zero. By our assumption, $\bar{\sigma}|_{X_0}$ is zero; but $\mathscr{H}_1/\mathscr{H}$ is torsion free, a fortiori $\bar{\sigma} = 0$, this completes the proof. □

An important observation is that giving a foliation on X is equivalent to giving a meromorphic differential form. In fact we have:

Proposition-Definition 2.4.3 (c.f.[AD14b, §3.1])**.** Let X be a normal complex variety of dimension n. Then we have the following two reciprocal constructions:

- Let $\mathscr{F}$ be a codimension q foliation on X, then the surjection $T_X \to T_X/\mathscr{F}$ induces an inclusion $N^*_{\mathscr{F}} \hookrightarrow \Omega^{[1]}_X := \left(\Omega^1_X\right)^{**}$, which gives rise to a $\det N_{\mathscr{F}}$-valued reflexive q-form $\omega \in \mathrm{H}^0(X, \Omega^q_X[\otimes]\det N_{\mathscr{F}})$ (where '$[\otimes]$' denotes the reflexive hull of the tensor product), which satisfies the following three properties:
  (a) The vanishing locus of ω is of codimension 2;
  (b) ω is locally decomposable (around a general point of X), that is, in a neighbourhood of a general point of X, we can write $\omega = \omega_1 \wedge \cdots \wedge \omega_q$ with ω_i's local 1-forms;
  (c) ω is integrable, that is, for the local decomposition $\omega = \omega_1 \wedge \cdots \wedge \omega_q$ as in (b), one has $d\omega_i \wedge \omega = 0$ for every $i = 1, \cdots, q$.
- Let $\mathscr{L}$ be a reflexive sheaf of rank 1 on X, and let $\omega \in \mathrm{H}^0(X, \Omega^q_X[\otimes]\mathscr{L})$ satisfying the above three conditions (a)(b)(c)[2], consider the morphism $T_X \to \Omega^{q-1}_X[\otimes]\mathscr{L}$ given by the contraction with ω, then the kernel of this morphism is a codimension q foliation on X.

Proof. This is surely well known to experts and is formulated in another way in the literatures when $\mathscr{F}$ is regular and X is smooth (see e.g. [MM03, §1.2, pp. 10-11]). However, due to lack of references treating the singular case, we will give a proof here for the convenience of the readers. The idea of the proof is borrowed from [MM03, §1.2, pp. 10-11].

- Let $\mathscr{F}$ be a foliation on X, then there is a Zariski open subset $X^\circ \subseteq X_{\mathrm{reg}}$ such that $\mathrm{codim}(X \backslash X^\circ) \geqslant 2$ and that $\mathscr{F}|_{X^\circ} \subseteq T_{X^\circ}$ is a subbundle. Then ω is nowhere vanishing on X°, hence the vanishing locus of ω is of codimension $\geqslant 2$. Locally in X°, we can take $v_1, \cdots, v_n$ trivializing sections (local vector fields) of T_{X°, among which $v_1, \cdots, v_{n-q}$ generate $\mathscr{F}|_{X^\circ}$. Let $\alpha_1, \cdots, \alpha_n$ be the dual sections of the v_i's. Then locally $N^*_{\mathscr{F}} = (T_X/\mathscr{F})^*$ is generated by $\alpha_{n-q+1}, \cdots, \alpha_n$, and hence $\det N^*_{\mathscr{F}}$ is generated by $\alpha_{n-q+1} \wedge \cdots \wedge \alpha_n$. Since ω is induced by the inclusion $\det N_{\mathscr{F}} \hookrightarrow \Omega^{[q]}_X$, hence under the local trivialisation of $\mathscr{L}|_{X^\circ}$ given by $\alpha_{n-q+1} \wedge \cdots \wedge \alpha_n$ the differential q-form ω is equal to the tautological section $\alpha_{n-q+1} \wedge \cdots \wedge \alpha_n$, hence ω is locally decomposable. And the kernel of the morphism $T_X \to \Omega^{q-1}_X[\otimes]\det N_{\mathscr{F}}$ can be locally expressed as
$$\left\{ v \text{ local holomorphic vector field} \,\middle|\, \alpha_i(v) = 0, \forall i = n-q+1, \cdots, n \right\},$$
which is then equal to $\mathscr{F}$ on X°; both of them are reflexive, then $\mathscr{F} = \mathrm{Ker}(T_X \to \Omega^{q-1}_X[\otimes]\det N_{\mathscr{F}})$. Finally let us check that ω is integrable. To this end, take any two

[2] The condition (a) is not essential in the construction. In fact, ω can also be regarded as a $\mathscr{L}(D)$-valued q-form for any effective (Weil) divisor D on X, as section of $\Omega^q_X[\otimes]\mathscr{L}(D)$ ω vanishes along D; but this does not change the kernel of the contraction morphism, in fact $T_X \to \Omega^{q-1}_X[\otimes]\mathscr{L}(D)$ is nothing other then composition of $T_X \to \Omega^{q-1}_X[\otimes]\mathscr{L}$ and the inclusion $\mathscr{L} \hookrightarrow \mathscr{L}(D)$. Nevertheless, the condition (a) guarantees that the construction is reciprocal to the first one.

local sections v and w of $\mathscr{F}$, then by the formula (definition) of exterior derivative we get

$$d\alpha_i(v,w) = \frac{1}{2}[v(\alpha_i(w)) - w(\alpha_i(v)) - \alpha_i([v,w])] = 0,\ \forall i = n-q+1,\cdots,n \tag{2.16}$$

since $[v,w]$ is still a local section of $\mathscr{F}$ as a result of the involutivity of $\mathscr{F}$. Therefore, for every $i = n-q+1,\cdots,n$, we can write

$$d\alpha_i = \sum_{j=n-q+1}^{n} \eta_{ij} \wedge \alpha_j,$$

for some local differential 1-forms η_{ij}. Hence $d\alpha_i \wedge \alpha_{n-q+1} \wedge \cdots \wedge \alpha_n = 0$.

- Reciprocally, let $\mathscr{L}$ be a reflexive sheaf of rank 1 on X and $\omega \in \mathrm{H}^0(X, \Omega_X^q[\otimes]\mathscr{L})$ satisfying the conditions (a)(b)(c) as above. Now consider $\mathscr{F} = \mathrm{Ker}(T_X \to \Omega_X^{q-1}[\otimes]\mathscr{L})$. Since $\Omega_X^{q-1}[\otimes]\mathscr{L}$ is torsion free (being reflexive), $\mathscr{F}$ is saturated in T_X (thus reflexive). We check that $\mathscr{F}$ is involutive. In virtue of Lemma 2.4.2, up to replacing X by a Zariski open whose complement is of codimension $\geqslant 2$, we can assume that X is smooth, that ω is nowhere vanishing on X and that ω is locally decomposable around every point of X. Locally we can write $\omega = \omega_1 \wedge \cdots \wedge \omega_q$, since ω is nowhere vanishing, the ω_i's are everywhere linearly independent. Then we can complete $\{\omega_i\}_{i=1,\cdots,q}$ into a family of trivializing local sections $\omega_1,\cdots,\omega_n$ of Ω_X^1. Then locally $\mathscr{F}$ is equal to

$$\Big\{v \text{ local holomorphic vector field} \Big| \omega_i(v) = 0,\ \forall i = 1,\cdots,q\Big\}. \tag{2.17}$$

Since ω is integrable, thus $d\omega_i \wedge \omega_1 \wedge \cdots \wedge \omega_q = 0$ for every $i = 1,\cdots,q$. Write

$$d\omega_i = \sum_{1\leqslant j<k\leqslant n} a_{ijk}\omega_j \wedge \omega_k,$$

since $\omega_j \wedge \omega_k \wedge \omega_1 \wedge \cdots \wedge \omega_q = 0$ for $j \leqslant q$, we get from the integrability condition:

$$\sum_{q+1\leqslant j<k\leqslant n} a_{ijk}\omega_j \wedge \omega_k \wedge \omega_1 \wedge \cdots \wedge \omega_q = 0,$$

which implies that $a_{ijk} = 0$ if $j \leqslant q+1$. Hence we can write

$$d\omega_i = \sum_{j=1}^{q} \eta_{ij} \wedge \omega_j$$

for some local 1-forms η_{ij}; in particular, for every $i = 1,\cdots,q$, $d\omega_i$ annihilates $\mathscr{F}$. Then by the formula (2.16), we see that $[\mathscr{F},\mathscr{F}]$ is annihilated by every ω_i, $i = 1,\cdots,q$, which in turn implies, by the local characterization (2.17) of $\mathscr{F}$ above, that $[\mathscr{F},\mathscr{F}] \subseteq \mathscr{F}$. Hence $\mathscr{F}$ is a foliation on X. Moreover, by the the local expression (2.17) we see that $(\omega_i)_{i=1,\cdots,q}$ is a family of local trivializing sections of $(T_X/\mathscr{F})^*$. In consequence, $\omega \overset{\mathrm{loc}}{=} \omega_1 \wedge \cdots \wedge \omega_q$ is equal to the the rational q-form induced by $\mathscr{F}$.

□

With the help of the construction above, we can define the pullback of a foliation:

Proposition-Definition 2.4.4 (c.f.[Dru18a, §3.4]). Let $\mu : X \dashrightarrow Y$ a dominant meromorphic mapping between normal complex varieties, which restricts to a surjective morphism $\mu^\circ : X^\circ \to Y^\circ$ with X° and Y° smooth Zariski open subsets of X and of Y respectively. Let $\mathscr{G}$ be a foliation on Y, then it induces a foliation $\mu^{-1}\mathscr{G}$ on X as following: as in the Proposition-Definition 2.4.3, $\mathscr{G}$ gives rise to a meromorphic differential q-form $\omega \in \mathrm{H}^0(Y, \Omega^q_Y[\otimes]\det N_{\mathscr{G}})$, then $(\mu^\circ)^*(\omega|_{Y^\circ})$ extends to a meromorphic q-form τ on X_{reg}. By well choosing a rank 1 reflexive sheaf $\mathscr{L}$ on X, τ can be regarded as a section in $\mathrm{H}^0(X, \Omega^q_X[\otimes]\mathscr{L})$ whose vanishing locus has codimension 2 in X_{reg} (thus in X). And by construction it is clear that τ is locally decomposable and integrable, then by Proposition-Definition 2.4.3 τ induces a foliation on X, which we denote by $\mu^{-1}\mathscr{G}$. By construction $\mu^{-1}\mathscr{G}$ is the unique foliation on X whose associated differential q-form coincides with $(\mu^\circ)^*(\omega|_{Y^\circ})$ on X°.

In the proof of our main theorem, we will treat the situation where the tangent sheaf admits a direct sum decomposition into foliations, and we expect that under certain condition this decomposition can be retained via pullback. When the morphism is bimeromorphic, the following lemma provides a criterion to ensure this.

Lemma 2.4.5. *Let $\mu : X \to Y$ be a bimeromorphic morphism between normal complex varieties and let $\mathscr{G}_1$ and $\mathscr{G}_2$ be foliations on Y. Suppose that we have a direct sum decomposition $T_Y \simeq \mathscr{G}_1 \oplus \mathscr{G}_2$, and suppose that the natural morphism $\det(\mu^{-1}\mathscr{G}_1) \to \det\left(T_X/\mu^{-1}\mathscr{G}_1\right)$ is an isomorphism. Then the decomposition of T_Y pulls back to X:*

$$T_X \simeq \mu^{-1}\mathscr{G}_1 \oplus \mu^{-1}\mathscr{G}_2$$

Remark 2.4.6. The lemma does not holds in general without the assumption on the natural morphism between determinant line bundles even for regular foliations on smooth varieties. For example, consider $Y = \mathbb{P}^1 \times \mathbb{P}^1$ and $\mu : X \to Y$ be the blow-up of a general point on $\mathbb{P}^1 \times \mathbb{P}^1$. Then T_Y admits a natural decomposition

$$T_Y \simeq \mathrm{pr}_1^* T_{\mathbb{P}^1} \oplus \mathrm{pr}_2^* T_{\mathbb{P}^1}$$

into regular (algebraically integrable) foliations. This decomposition cannot pullback via μ to X. Otherwise, if it were the case then we have a decomposition

$$T_X \simeq \mathscr{F}_1 \oplus \mathscr{F}_2$$

with $\mathscr{F}_i$ the pullback foliation of $\mathrm{pr}_i^* T_{\mathbb{P}^1}$, (the Zariski closure of) whose general leaf is rationally connected. By semicontinuity $\mathscr{F}_1$ and $\mathscr{F}_2$ are locally free, hence are regular foliations. Therefore, by [Höro7, 2.11.Corollary], $\mathscr{F}_1$ induces a smooth holomorphic submersion, whose fibres are transverse to the leaves of $\mathscr{F}_2$; then by the classical Ehresmann Theorem 2.4.22 (c.f. [CL85, §V.4, Theorem 1 and Theorem 3, pp. 95-99]), X splits into a product of two curves. But X is rationally connected, hence simply connected, then it must be isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, which is absurd

Lemma 2.4.5 follows immediately from the following general fact:

Proposition 2.4.7. *Let X be a normal complex variety and let E be a reflexive sheaf on X and E_1, E_2 saturated subsheaves of E. Suppose that there is a Zariski open X_0 of X such that*

$$E_1|_{X_0} \oplus E_2|_{X_0} \simeq E, \tag{2.18}$$

and suppose that the natural morphism $\det E_1 \to \det(E/E_2)$ is an isomorphism. Then the direct sum decomposition extends globally:

$$E \simeq E_1 \oplus E_2.$$

Proof. Since X is normal, E, E_1 and E_2 are reflexive, up to replace X by a Zariski open whose complementary is of codimension $\geqslant 2$, we can assume that X is smooth, E is a vector bundle and E_1 and E_2 are subbundles of E. Now consider the natural morphism

$$\sigma : E_1 \hookrightarrow E \twoheadrightarrow E/E_2,$$

By (2.18) σ is an isomorphism over X_0, then it must be injective ($\mathrm{Ker}(\sigma)$ is torsion free and generically 0 hence must be 0). Hence E_1 is a locally free (thus reflexive) subsheaf of the vector bundle E/E_2. In addition, the morphism $\det\sigma : \det E_1 \to \det(E/E_2)$ is an isomorphism by the hypothesis. Then by [DPS94, Lemma 1.20] E_1 is a subbundle of E/E_2, hence they must be isomorphic. In particular this means that the short exact sequence

$$0 \to E_2 \to E \to E/E_2 \to 0$$

splits, thus we get the desired direct decomposition. □

Remark 2.4.8. The proposition does not hold in general without the assumption even on the natural morphism between determinant bundles. For example (pointed out by Junyan Cao), consider $X = \mathbb{A}^2$ and $E = T_{\mathbb{A}^2} \simeq \mathscr{O}_{\mathbb{A}^2}^{\oplus 2}$ with E_1 the foliation generated by the global vector field

$$v_1 = z_1 \cdot \frac{\partial}{\partial z_1} + z_2 \cdot \frac{\partial}{\partial z_2},$$

and E_2 the foliation generated by the global vector field

$$v_2 = z_2 \cdot \frac{\partial}{\partial z_1} + z_1 \cdot \frac{\partial}{\partial z_2}.$$

Then E_1 and E_2 are locally free subsheaves of $E = T_{\mathbb{A}^2}$, and generically (out of the line $(z_1 = z_2)$) $E \simeq E_1 \oplus E_2$. But the decomposition cannot extend globally. In fact, the natural morphism $\det E_1 \to \det(E/E_2)$ is zero along the line $(z_1 = z_2)$.

2.4.2 Pfaff fields and invariant subvarieties

Definition 2.4.9. Let X be a normal complex variety. A Pfaff vector field of rank r on X is a non-trivial morphism $\eta : \Omega_X^r \to \mathscr{L}$ where $\mathscr{L}$ is a reflexive sheaf of rank 1. The *singular locus* $\mathrm{Sing}(\eta)$ of η is the closed analytic subspace of X defined by the ideal sheaf $\mathrm{Im}(\Omega_X^r[\otimes]\mathscr{L}^* \xrightarrow{\eta[\otimes]\mathscr{L}^*} \mathscr{O}_X)$. If $\mathscr{L}$ is invertible, then set-theoretically $\mathrm{Sing}(\eta)$ consists of the points at which η is not surjective.

Definition 2.4.10 ([Dru18b, Definition 5.4])**.** Let $\mathscr{F}$ be a foliation on X, then $\mathscr{F}$ induces a Pfaff field of rank $r = \mathrm{rk}\,\mathscr{F}$ on X

$$\eta_{\mathscr{F}} : \Omega_X^r = \bigwedge^r \Omega_X^1 \to \bigwedge^r \mathscr{F}^* \to \det \mathscr{F}^*.$$

The singular locus $\mathrm{Sing}(\mathscr{F})$ of the foliation $\mathscr{F}$ is defined to be the singular locus of the Pfaff field $\eta_{\mathscr{F}}$. And $\mathscr{F}$ is called *weakly regular* if $\mathrm{Sing}(\mathscr{F}) = \varnothing$.

Remark 2.4.11. If X is smooth, then one deduces easily from [DPS94, Lemma 1.20] that (set-theoretically)

$$\begin{aligned}\mathrm{Sing}(\mathscr{F})_{\mathrm{red}} &= \left\{x \in X \,\middle|\, \mathscr{F} \to T_X \text{ is a injective bundle map at } x\right\} \\ &= \left\{x \in X \,\middle|\, \mathscr{F} \text{ is a subbundle of } T_X \text{ at } x\right\}.\end{aligned}$$

Definition 2.4.12. Let X be a normal complex variety and let $\eta : \Omega^r_X \to \mathscr{L}$ be a Pfaff field of rank r on X. Suppose that some reflexive power of $\mathscr{L}$ is invertible. A closed analytic subspace Y of X is called invariant under η if

- none of the irreducible components of Y is contained in $\mathrm{Sing}(\eta)$;
- for some $m \in \mathbb{Z}_{>0}$ such that $\mathscr{L}^{[m]}$ is invertible, the restriction $\eta^{\otimes m} : (\Omega^r_X)^{\otimes m}\big|_Y \to \mathscr{L}^{[m]}\big|_Y$ factors through the natural map $(\Omega^r_X)^{\otimes m}\big|_Y \to (\Omega^r_Y)^{\otimes m}$. Y is said invariant under a $\mathbb{Q}$-Gorenstein foliation $\mathscr{F}$ on X if Y is invariant under its associated Pfaff field $\eta_{\mathscr{F}}$.

Remark 2.4.13. Suppose that Y is a reduced analytic subspace of X and that none of its irreducible components is contained in $\mathrm{Sing}(\eta)$. Then Y is invariant under η if and only if the restriction $\eta|_{Y_{\mathrm{reg}}} : \Omega^r_X\big|_{Y_{\mathrm{reg}}} \to \mathscr{L}|_{Y_{\mathrm{reg}}}$ factors through $\Omega^r_X\big|_{Y_{\mathrm{reg}}} \to \Omega^r_{Y_{\mathrm{reg}}}$. More generally, one can replace Y_{reg} above by any Zariski dense subset of Y_{reg}. This results from the following useful lemma (by taking $Y_0 = Y_{\mathrm{reg}}$ or any Zariski dense subset of Y_{reg}, $\mathscr{M} = (\Omega^r_X)^{\otimes m}\big|_Y$ and $\mathscr{N} = (\Omega^r_Y)^{\otimes m}$ and noting that by [Har77, §II.8, Proposition 8.12, p.] the natural morphism $(\Omega^r_X)^{\otimes m}|_Y \to (\Omega^r_Y)^{\otimes m}$ is surjective):

Lemma 2.4.14 (c.f.[EK03, Proof of Proposition 3.2, p. 10])**.** *Let Y be a reduced complex analytic space, and let $\mathscr{L}$, $\mathscr{M}$ and $\mathscr{N}$ be coherent sheaves on Y with a surjective morphism $\alpha : \mathscr{M} \to \mathscr{N}$. Then a morphism $\beta : \mathscr{M} \to \mathscr{L}$ factors through α if and only if β annihilates $\mathrm{Ker}(\alpha)$. Suppose that $\mathscr{L}$ is torsion free, then β factors through α if and only if there is a Zariski dense subset Y_0 of Y such that $\beta|_{Y_0}$ factors through $\alpha|_{Y_0}$.*

Proof. By arguing components by components we can assume that Y is irreducible, so that Y is a complex variety. Since α is surjective, $\mathscr{N} = \mathrm{Im}(\alpha) = \mathrm{Coker}(\mathrm{Ker}(\alpha) \to \mathscr{M})$, then the first statement results from the universal property of cokernels. Now turn to the second statement: since $\beta|_{Y_0}$ factors through $\alpha|_{Y_0}$, then by the first statement

$$\beta|_{Y_0}(\mathrm{Ker}(\alpha|_{Y_0})) = \beta(\mathrm{Ker}(\alpha))|_{Y_0} = 0;$$

but $\beta(\mathrm{Ker}(\alpha)) \subseteq \mathscr{L}$ is a subsheaf of a torsion free sheaf, hence also torsion free, thus a fortiori $\beta(\mathrm{Ker}(\alpha)) = 0$, which implies, by the first statement, that β factors through α. □

The following lemma gives a characterization of invariant subvarieties which are not contained in the singular locus (other examples of invariant subvarieties can be found in [Dru18b, Lemma 3.5]):

Lemma 2.4.15 (c.f.[AD13, Lemma 2.7])**.** *Let X be a complex manifold and $\mathscr{F}$ a rank r foliation on X with associated Pfaff field $\eta = \eta_{\mathscr{F}} : \Omega^r_X \to \det\mathscr{F}^*$. Set $S := \mathrm{Sing}(\mathscr{F})_{\mathrm{red}}$. Let Y be a closed subvariety of X of dimension r such that Y is not contained in S. Then Y is invariant under η if and only if $Y \backslash S$ is a leaf of $\mathscr{F}$*

Proof. First note that since X is smooth, S is characterized by Remark 2.4.11; it is of codimension $\geqslant 2$ by [Kob87, Corollary 5.5.15, p. 147]. Up to replacing X by $X \backslash S$ we can assume $\mathscr{F}$ is a subbundle of T_X so that $S = \emptyset$ (i.e. $\mathscr{F}$ is a regular foliation). Now take $x \in Y_{\mathrm{reg}}$ and take $v_1, \cdots, v_r$ local holomorphic vector fields around x that generate (locally trivialize) $\mathscr{F}$. By construction η is the dual morphism of the inclusion map $\det\mathscr{F} \hookrightarrow \bigwedge^r T_X$, hence locally it is given by

$$\alpha \longmapsto \alpha(v_1, \cdots, v_r) \cdot \alpha_0 \tag{2.19}$$

where α_0 is a a section of $\det\mathscr{F}^*$ such that $\alpha_0(v_1, \cdots, v_r) = 1$. Since $\Omega^r_X|_Y \to \Omega^r_Y$ is surjective, by Lemma 2.4.14 Y is invariant under η if and only if $\mathrm{Ker}(\Omega^r_X\big|_Y \to \Omega^r_Y)$ is annihilated by $\eta|_Y$. Locally around x (Y is smooth around x) $\mathrm{Ker}(\Omega^r_X\big|_Y \to \Omega^r_Y)$ consists of the

r-forms of the form $df \wedge \beta$ with f a local holomorphic function vanishing along Y and β any local differential $(r-1)$-form. Combined with (2.19) we see easily that locally around x, Y is invariant under η if and only if

$$df \wedge \beta(v_1, \cdots, v_r)\big|_Y = 0.$$

Since $df \wedge \beta|_Y = d(f\beta)|_Y - f\,d\beta|_Y = d(f\beta)|_Y$ since $f|_Y = 0$, hence by the formula of exterior derivative we get

$$\begin{aligned}
df \wedge \beta(v_1, \cdots, v_r)\big|_Y &= d(f\beta)(v_1, \cdots, v_r)\big|_Y \\
&= \frac{1}{r}\sum_{i=1}^{r}(-1)^i v_i(f\beta(v_1 \cdots, \hat{v}_i, \cdots, v_r))\big|_Y \\
&\quad + \frac{1}{r}\sum_{1\leqslant i<j\leqslant r}(-1)^{i+j} f\beta([v_i, v_j], v_1 \cdots, \hat{v}_i, \cdots, \hat{v}_j, \cdots, v_r)\big|_Y \\
&= \frac{1}{r}\sum_{i=1}^{r}(-1)^i v_i(f\beta(v_1 \cdots, \hat{v}_i, \cdots, v_r))\big|_Y \\
&= \frac{1}{r}\sum_{i=1}^{r}(-1)^i \left(v_i(f)\big|_Y \cdot \beta(v_1, \cdots, \hat{v}_i, \cdots, v_r)\big|_Y + f \cdot v_i(\beta(v_1, \cdots, \hat{v}_i, \cdots, v_r))\big|_Y\right) \\
&= \frac{1}{r}\sum_{i=1}^{r}(-1)^i v_i(f)\big|_Y \cdot \beta(v_1, \cdots, \hat{v}_i, \cdots, v_r)\big|_Y
\end{aligned}$$

Hence Y is invariant under η around x if and only if $v_i(f)\big|_Y = 0$ for every local holomorphic function f vanishing along Y and for every $i = 1, \cdots, r$. Since Y is a r-dimensional holomorphic submanifold of X at x, this condition is equivalent to saying that $T_{Y_{\mathrm{reg}}} = \mathscr{F}|_Y$ around x. In consequence, Y is invariant under $\eta \Leftrightarrow Y_{\mathrm{reg}}$ is contained in a leaf of $\mathscr{F} \Leftrightarrow Y = Y_{\mathrm{reg}}$ is a leaf of $\mathscr{F}$ (noting that $\mathscr{F}$ is a regular foliation by our assumption). □

To end this subsection, let us recall the following lemma concerning the extension of Pfaff fields to the normalization:

Lemma 2.4.16 ([Sei66, §4, Theorem C, Corollary, p. 170],[ADK08, Proposition 4.5],[AD14b, Lemma 3.7])**.** *Let X be a normal complex variety and let $\eta : \Omega^r_X \to \mathscr{L}$ be a Pfaff field of rank r on X where $\mathscr{L}$ is reflexive sheaf of rank 1 such that $\mathscr{L}^{[m]}$ is invertible for some $m \in \mathbb{Z}_{>0}$. Let Y be a subvariety of X invariant under η, whose normalization is denoted by $\nu = \nu_Y : \bar{Y} \to Y$. Then the morphism $(\Omega^r_Y)^{\otimes m} \to \mathscr{L}^{[m]}\big|_Y$ extends (uniquely) to a generically surjective morphism $(\Omega^r_{\bar{Y}})^{\otimes m} \to \nu^*\left(\mathscr{L}^{[m]}\big|_Y\right)$.*

2.4.3 Algebraically integrable foliations

Definition 2.4.17. Let X be a normal algebraic variety and let $\mathscr{F}$ be a foliation on X. A leaf of $\mathscr{F}$ is called *algebraic* if it is (Zariski) open in its Zariski closure. $\mathscr{F}$ is called *algebraically integrable* if every leaf of $\mathscr{F}$ is algebraic.

Remark 2.4.18. A typical example of algebraically integrable foliation is one induced by a equidimensional fibre space, i.e. $\mathscr{F} = T_{X/Y} := (\Omega^1_{X/Y})^*$ with $\pi : X \to Y$ a proper equidimensional morphism between normal algebraic varieties with connected fibres. In fact, $\mathscr{F}$ is clearly reflexive, and by virtue of Lemma 2.4.2 one can easily prove that $\mathscr{F}$ is involutive by showing that $\mathscr{F}$ involutive over the smooth locus of π, hence $\mathscr{F}$ is

a foliation on X. In addition, by [CKT16, Lemma 2.31] the canonical divisor of $\mathscr{F}$ is described by the following equality:

$$\mathscr{O}_X(K_{\mathscr{F}}) = \det(\Omega^1_{X/Y}) \simeq \mathscr{O}_X(K_{X/Y} - \mathrm{Ram}(\pi)),$$

where the ramification divisor $\mathrm{Ram}(\pi)$ is defined by:

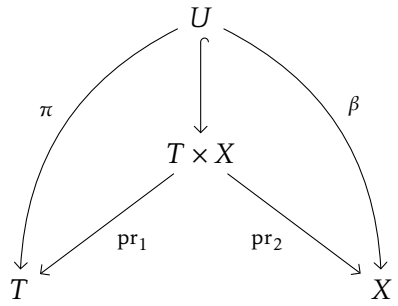
$$\mathrm{Ram}(\pi) = \sum_{D \text{ prime divisor on } X} \max(0, \mathrm{mult}_D(f^* f_* D) - 1) \cdot D.$$

Notice that π is equidimensional, then $\pi^{-1}(Y \setminus Y_{\mathrm{reg}})$ is still of codimension 2 in X, hence pullbacks of Weil divisors are well-defined (c.f. [CKT16, Construction 2.13]).

The following proposition, due to [AD13, Lemma 3.2], says that every algebraically integrable foliation on a normal projective variety is of the form as in Remark 2.4.18 up to pullback by a birational morphism. In particular, one can construct a family whose general fibre parametrizes the closure of a general leaf of $\mathscr{F}$.

Proposition 2.4.19 (c.f.[AD13, Lemma 3.2])**.** *Let X be a normal projective variety and let $\mathscr{F}$ a algebraically integrable foliation. Then there is a unique closed subvariety T of* $\mathrm{Chow}(X)$ *whose general point parametrize the Zariski closure of a general leaf of $\mathscr{F}$. That is, let $U \subseteq T \times X$ be the universal cycle along with morphisms $\pi : U \to T$ and $\beta : U \to X$, then β is birational and for a general point $t \in T$, $\beta(\pi^{-1}(t)) \subseteq X$ is the Zariski closure of a leaf of $\mathscr{F}$.*

Proof. First note that the Zariski closure of any leaf of $\mathscr{F}$ is irreducible and reduced, hence a subvariety of X. Let T_1 be the Zariski closure of the points of $\mathrm{Chow}(X)$ that parametrize leaves of $\mathscr{F}$, then T_1 is a reduced subscheme of $\mathrm{Chow}(X)$; since $\mathrm{Chow}(X)$ has only countably many components (c.f. [Kol96, §I.3, 3.20 Definition, 3.21 Theorem, pp. 51-52]), then so is T_1. Consider the universal cycle U_1 over T_1. Since the leaves are integral, the universal cycle over each component of T_1 is irreducible, hence the irreducible components of U_1 are in one-to-one correspondence with that of T_1, in particular U_1 also has only countably many irreducible components; now the natural map $U_1 \to X$ is surjective, there is a unique component U of U_1 which is dominant over X, and denote by T the component of T_1 corresponding to U. Let $\pi : U \to T$ and $\beta : U \to X$ be the natural morphisms.



Now it remains to show that for $t \in T$ general, $\beta(\pi^{-1}(t)) \subseteq X$ is the Zariski closure of a general leaf of $\mathscr{F}$. To this end, first note that: up to replace X by X° the Zariski open of X_{reg} where $\mathscr{F}$ is a subbundle of $T_{X_{\mathrm{reg}}}$, T by T° where T° is a Zariski open of T_{reg} whose points correspond to the cycles that are not contained in $X \setminus X^\circ$, and U by $U \cap \mathrm{pr}_1^{-1}(T^\circ) \cap \mathrm{pr}_2^{-1}(X^\circ)$ we can assume that X and T are smooth and $\mathscr{F}$ regular (by definition, a leaf is always contained in X°, c.f. Definition 2.4.1). In particular $K_{\mathscr{F}}$ is a Cartier divisor, and $\mathscr{F}$ induces a Pfaff field $\eta = \eta_{\mathscr{F}} : \Omega^r_X \to \mathscr{O}_X(K_{\mathscr{F}})$ where $r = \mathrm{rg}\,\mathscr{F}$. In the sequel we will use Lemma 2.4.15 to conclude; to this end, we will show that η induces

a Pfaff field on $T \times X$ whose restriction to U factors through $\Omega^r_{U/T}$. In fact, η induces a Pfaff field on $T \times X$

$$\mathrm{pr}_2^{-1}\eta : \Omega^r_{T\times X} \simeq \bigwedge^r (\mathrm{pr}_1^*\,\Omega^1_T \oplus \mathrm{pr}_2^*\,\Omega^1_X) \xrightarrow{\text{projection}} \bigwedge^r \mathrm{pr}_2^*\,\Omega^1_X \simeq \mathrm{pr}_2^*\,\Omega^r_X \xrightarrow{\mathrm{pr}_2^*\eta} \mathscr{O}_{T\times X}(\mathrm{pr}_2^* K_{\mathscr{F}}).$$

Then we will show that the restriction morphism $\mathrm{pr}_2^{-1}\eta\big|_U : \Omega^r_{T\times X}\big|_U \to \mathscr{O}_U(\beta^* K_{\mathscr{F}})$ factors through the composition map $\Omega^r_{T\times X}\big|_U \to \Omega^r_U \twoheadrightarrow \Omega^r_{U/T}$ (c.f. [Har77, §II.8, Proposition 8.11, p. 176]). By construction there is a Zariski dense subset of T whose points parametrize the leaves of $\mathscr{F}$; then by the proof of Lemma 2.4.15 $\mathrm{Ker}(\Omega^r_{T\times X}\big|_U \to \Omega^r_{U/T})$ is annihilated by $\mathrm{pr}_2^{-1}\eta\big|_U$ over a Zariski dense of U, thus is annihilated by $\mathrm{pr}_2^{-1}\eta\big|_U$ everywhere on U since $\mathscr{O}_U(\beta^* K_{\mathscr{F}})$ is torsion free. By Lemma 2.4.14 we see that $\mathrm{pr}_2^{-1}\eta\big|_U$ factors through $\Omega_{T\times X}|_U \to \Omega^r_{U/T}$. By the base change for Kähler differentials ([Har77, §II.8, Proposition 8.10, p. 175]) for every $t \in T$ we have $\Omega^r_{U/T}\big|_{U_t} \simeq \Omega^r_{U_t}$ and thus every U_t is invariant under $\mathrm{pr}_2^{-1}\,\eta$, which amounts to say that every $\beta(U_t)$ is invariant under η. By the generic flatness (c.f. [Mat70, §22, pp. 156-159]) and [EGAIV-3, Théorème 12.2.1(x), pp. 179-180], for general $t \in T$, U_t is irreducible and reduced (then so is $\beta(U_t)$), hence by Lemma 2.4.15, $\beta(U_t)$ is the closure of a (general) leaf of $\mathscr{F}$.

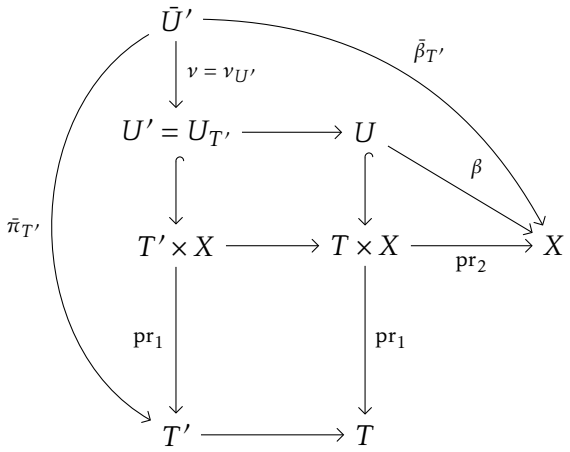
□

The morphism $\pi : U \to T$ constructed above is called the *family of leaves* of $\mathscr{F}$. In the following proposition we study the relation between the canonical divisor of $\mathscr{F}$ and that of the pullback of $\mathscr{F}$ to the family of leaves.

Proposition 2.4.20 ([AD14b, Remark 3.12], [AD14a], [AD16, §3.10]). *Let X be a projective normal variety and $\mathscr{F}$ be a algebraically integrable foliation on X. Let $\pi : U \to T$ the family of leaves of $\mathscr{F}$ as constructed in Proposition 2.4.19. Let $T' \to T$ be any surjective morphism with T' normal and let $U' = U_{T'}$ be the pullback of the universal family U, whose normalization is denoted by $\nu = \nu_{U'} : \bar{U}' \to U'$. Let $\beta_{T'}$ (resp. $\pi_{T'}$, resp. $\bar{\beta}_{T'}$, resp. $\bar{\pi}_{T'}$) be the induced morphism $U' \to X$ (resp. $U' \to T'$, resp. $\bar{U}' \to X$, resp. $\bar{U}' \to T'$). Then*

(a) *The pullback foliation $\bar{\beta}_{T'}^{-1}\mathscr{F}$ is equal to $T_{\bar{U}'/T'} := (\Omega^1_{\bar{U}'/T'})^*$;*

(b) *Assume that $\mathscr{F}$ is $\mathbb{Q}$-Gorenstein, then there is a canonical effective Weil $\mathbb{Q}$-divisor $\Delta_{T'}$ on $\bar{U}'$ such that $K_{\bar{\beta}_{T'}^{-1}\mathscr{F}} + \Delta_{T'} \sim_{\mathbb{Q}} \bar{\beta}_{T'}^* K_{\mathscr{F}}$. If $T' \to T$ is birational then $\Delta_{T'}$ is $\bar{\beta}_{T'}$-exceptional.*



Proof. First notice that since ν is a finite morphism, $\bar{\pi}_{T'} : \bar{U}' \to T'$ is still equidimensional and hence $T_{\bar{U}'/T'}$ is a foliation on $\bar{U}'$. Then (a) is clear : in fact, since $T' \to T$ is surjective, by Proposition 2.4.19 there is a Zariski open of T' over which the fibres of $\pi_{T'} : U' \to T'$

are leaves of $\mathscr{F}$, hence $\bar{\beta}_{T'}^{-1}\mathscr{F}$ and $T_{\bar{U}'/T'}$ coincide over a Zariski open of $\bar{U}'$, then by the uniqueness in Proposition-Definition 2.4.4 a fortiori $\bar{\beta}_{T'}^{-1}\mathscr{F} = T_{\bar{U}'/T'}$. Now turn to the proof of (b). Consider the Pfaff field associated to $\mathscr{F}$

$$\eta := \eta_{\mathscr{F}} : \Omega_X^r \to \mathscr{O}_X(K_{\mathscr{F}}),$$

as in the proof of Proposition 2.4.19 η induces a Pfaff field on $T' \times X$ ($T' \times X$ is normal)

$$\mathrm{pr}_2^{-1}\eta : \Omega_{T'\times X}^r \to \mathscr{O}_{T'\times X}(\mathrm{pr}_2^* K_{\mathscr{F}}),$$

where pr_2^* above denotes the pullback of Weil divisors (or algebraic cycles) by equidimensional (or flat) morphisms (c.f. [CKT16, Construction 2.13] or [Ful84, §1.7, pp. 18-21]); moreover, the restriction morphism

$$(\mathrm{pr}_2^{-1}\eta)^{\otimes m}\big|_{U'} : (\Omega_{T'\times X}^r)^{\otimes m}\big|_{U'} \to \mathscr{O}_{U'}(m\beta_{T'}^* K_{\mathscr{F}})$$

factors through $(\Omega_{T'\times X}^r)^{\otimes m}\big|_{U'} \to (\Omega_{U'/T'}^r)^{\otimes m}$ where m is a positive integer such that $mK_{\mathscr{F}}$ is Cartier, in particular U' is invariant under $\mathrm{pr}_2^{-1}\eta$. Now by Lemma 2.4.16 we get a generically surjective morphism

$$(\Omega_{\bar{U}'}^r)^{\otimes m} \to \mathscr{O}_{\bar{U}'}(m\bar{\beta}_{T'}^* K_{\mathscr{F}})$$

which factors through the natural surjection $\Omega_{\bar{U}'} \twoheadrightarrow \Omega_{\bar{U}'/T'}$. Then we get an injection of rank 1 reflexive sheaves

$$\det(\Omega_{\bar{U}'/T'}^1)^{\otimes m} \hookrightarrow \mathscr{O}_{\bar{U}'}(m\bar{\beta}_{T'}^* K_{\mathscr{F}}),$$

hence there is a unique effective Weil $\mathbb{Q}$-divisor $\Delta_{T'}$ ($m\Delta_{T'}$ is the Weil divisor defined by this injection) such that

$$K_{\bar{\beta}_{T'}^{-1}\mathscr{F}} + \Delta_{T'} \sim_{\mathbb{Q}} \bar{\beta}_{T'}^* K_{\mathscr{F}}.$$

Moreover, combining this with Remark 2.4.18 we get

$$K_{\bar{U}'/T'} - \mathrm{Ram}(\bar{\pi}_{T'}) + \Delta_{T'} \sim_{\mathbb{Q}} \bar{\beta}_{T'}^* K_{\mathscr{F}}.$$

If $T' \to T$ is birational, then by Proposition 2.4.19 $\Delta_{T'}$ is $\bar{\beta}_{T'}$-exceptional. □

We close this subsection by considering algebraically integrable foliations that are weakly regular (c.f. Definition 2.4.10). It is clear that a foliation induced by a equidimensional fibre space (c.f. Remark 2.4.18) is weakly regular, the following result says that the converse is true for (weakly regular) foliations with canonical singularities over $\mathbb{Q}$-factorial klt projective varieties.

Theorem 2.4.21 ([Dru18b, Theorem 6.1]). *Let X be a (normal) $\mathbb{Q}$-factorial projective variety with klt singularities, and let $\mathscr{G}$ be a weakly regular algebraically integrable foliation on X. Suppose in addition that $\mathscr{G}$ has canonical singularities. Then $\mathscr{G}$ is induced by an equidimensional fibre space $\psi : X \to Y$ onto a normal projective variety Y. Moreover, there exists an open subset Y° with complement of codimension $\geqslant 2$ in Y such that $\psi^{-1}(y)$ is irreducible for any $y \in Y^\circ$.*

In the study of algebraically integrable foliations, the family of leaves is a quite useful tool which permits the enter of algebro-geometric methods; nonetheless, by passing to the family of leaves, one loses the control of the singularities. The above Theorem 2.4.21 implies that weakly regular foliations with canonical singularities on a projective variety X with mild singularities have the advantage that there is no need to pass to the family of leaves (since it is isomorphic to X itself), c.f. also Remark 4.4.5.

2.4.4 Foliations transverse to holomorphic submersions

In this subsection we consider regular foliations which are transverse to a smooth fibration and we recall the important (analytic version of) classical Ehresmann theorem. Let $f : V \to W$ be a smooth morphism (holomorphic submersion) between complex manifolds and let $\mathscr{F}$ be a regular foliation on V. Then $\mathscr{F}$ is said to be *transverse to* f if the following two conditions are verified:

(i) The tangent bundle sequence of f gives rise to a direct decomposition $T_V \simeq T_{V/W} \oplus \mathscr{F}$.

(ii) The restriction of f to any leaf of $\mathscr{F}$ is an étale (not necessarily finite) cover.

By [CL85, §V.2, Proposition 1, pp. 91-92] (by [Voi02, §9.1, Proposition 9.5, pp. 209-210] f can be viewed as a $\mathscr{C}^\infty$ fibre bundle), if f is proper then the condition (i) implies (ii). The most important result for these foliations is the following analytic version of the classical Ehresmann theorem :

Theorem 2.4.22 ([Höro7, 3.17.Theorem]). *Let $f : V \to W$ be a holomorphic submersion between complex manifolds and let $\mathscr{F}$ be a regular foliation on V transverse to f. Suppose that W and the general fibre F of f are connected. Then f is an analytic fibre bundle. Moreover, there is a representation $\rho : \pi_1(Y) \to \mathrm{Aut}(F)$ such that X is biholomorphic to $(\tilde{Y} \times F)/\pi_1(Y)$ where $\pi_1(Y)$ acts on $\tilde{Y} \times F$ via $\alpha : (y,s) \to (\alpha(y), \rho(\alpha)(s))$, and $\tilde{Y} \to Y$ denotes the universal cover of Y; in particular, f is a locally constant fibration.*

See [CL85, §V.3, Theorem 1 and Theorem 3, pp. 91-95] for the proof. The above statement is taken from [Höro7, 3.17.Theorem].

Chapter 3

On the Iitaka conjecture $C_{n,m}$ for Kähler fibre spaces

In this chapter we consider the Iitaka conjecture $C_{n,m}$ for Kähler fibre spaces, which is the main content of the article [Wan19].

3.1 Log Kähler version of results of Kawamata and of Viehweg

In this section we will apply the Ohsawa-Takegoshi type extension Theorem 2.1.2 to prove the Theorem A(I). Along the way we also give a proof of the conjecture $C_{n,m}$ over general type bases (c.f. Theorem 3.1.2 below) which is a main ingredient in the proof of Theorem E in §3.2.

Classically the proof of Theorem A(I) and Theorem 3.1.2 is based on Viehweg's weak positivity theorem on the direct image; here we will take a new argument which only depends on the Ohsawa-Takegoshi type extension Theorem 2.1.2. Precisely, Theorem 2.1.2 is used to ensure the effectivity of the twisted relative canonical bundle up to adding an ample line bundle from the base, in virtue of the following auxiliary result:

Lemma 3.1.1. *Let $f : X \to Y$ be an analytic fibre space with X a normal complex variety and Y a projective variety. Let L be a holomorphic line bundles on X such that $\kappa(L) \geqslant 0$ and let A be a ample line bundle on Y. Then*

$$\kappa(X, L \otimes f^*A) = \kappa(F, L|_F) + \dim Y$$

where F denotes the general fibre of f.

Before giving the proof, let us remark that this simple but useful result has been implicitly used in the works on $C_{n,m}$, e.g. [Esn81; Vie83]; it is explicitly formulated in [Cam04, Lemma 4.9] but without proof. For the convenience of the readers, we will give the detailed proof.

Proof of Lemma 3.1.1. Up to multiplying L and A_Y by a sufficiently large and divisible integer, we can assume that $\mathrm{H}^0(X, L) \neq 0$ and A is very ample; we can further assume that the closure of the image of the meromorphic mapping

$$\Phi := \Phi_{|L \otimes f^*A|} : X \dashrightarrow \mathbb{P}V$$

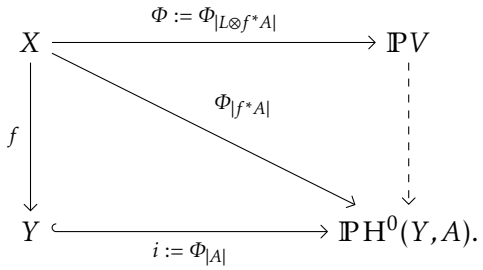
with $V := \mathrm{H}^0(X, L \otimes f^*A)$ is of dimension $\kappa(X, L \otimes f^*A)$. Up to blowing up X we can assume that Φ is an analytic fibre space (c.f. [Uen75, Lemma 5.3, pp. 51-52, and Corollary 5.8, p. 57]). Then consider the sub-linear series defined by the inclusion

$$\mathrm{H}^0(Y, A) \simeq \mathrm{H}^0(X, f^*A) \hookrightarrow \mathrm{H}^0(X, L \otimes f^*A) \simeq \mathrm{H}^0(\mathbb{P}V, \mathscr{O}_{\mathbb{P}V}(1)),$$

this gives rise to a meromorphic mapping

$$\mathbb{P}V \dashrightarrow \mathbb{P}\mathrm{H}^0(Y, A).$$

On the other hand, since A is very ample, the linear series $|A|$ defines an closed embedding $i := \Phi_{|A|} : Y \hookrightarrow \mathbb{P}\mathrm{H}^0(Y, A)$, thus we have the following "commutative" diagram:



In particular, the general fibre G of Φ is contracted by f, hence we get an analytic fibre space

$$\Phi|_F : F \to \mathrm{Im}(\Phi|_F),$$

whose general fibre is isomorphic to G. $\Phi|_F$ is defined by the linear series $|L \otimes f^*A|$ restricted to F, which is a sub-linear series of $|(L \otimes f^*A)|_F| \simeq |L|_F|$, hence we have

$$\kappa(F, L|_F) \geqslant \dim \mathrm{Im}(\Phi|_F) = \dim \mathrm{Im}\, \Phi - \dim Y = \kappa(X, L \otimes f^*A) - \dim Y.$$

In addition, by applying the easy inequality [Uen75, Theorem 5.11, pp. 59-60] to $\Phi|_F$ and $(L \otimes f^*A_Y)|_F$ we get

$$\kappa(F, L|_F) = \kappa(F, (L \otimes f^*A)|_F) \leqslant \kappa(G, (L \otimes f^*A)|_G) + \dim \mathrm{Im}(\Phi|_F) = \dim \mathrm{Im}(\Phi|_F),$$

therefore $\kappa(X, L \otimes f^*A) = \kappa(F, L|_F) + \dim Y$. □

3.1.1 Kähler version of $C^{\log}_{n,m}$ over general type bases

In this subsection we will apply the Ohsawa-Takegoshi type extension Theorem 2.1.2 to recover the result that $C^{\log}_{n,m}$ holds for fibre spaces over general type bases, i.e. to give a new proof of the following theorem:

Theorem 3.1.2 (Kähler version of [Kaw81, Theorem 3], [Vie83, Theorem III]). *Let $f : X \to Y$ be a fibre space between compact complex varieties in Fujiki class $\mathcal{C}$ and let Δ be an $\mathbb{Q}$-effective divisor on X such that (X, Δ) is klt. Suppose that Y of general type (thus Moishezon). Then*

$$\kappa(X, K_X + \Delta) \geqslant \kappa(F, K_F + \Delta_F) + \dim Y,$$

where F denotes the general fibre of f and $\Delta_F := \Delta|_F$.

Let us remark that by virtue of the easy inequality [Uen75, Theorem 5.11, pp. 59-60], the inequality in the theorem is in fact an equality. In order to establish Theorem 3.1.2, we first prove the following lemma, which can be regarded as a (log) Kähler version of [Vie83, Corollary 7.1]:

Lemma 3.1.3. *Let $f : X \to Y$ be an analytic fibre space with X a (compact) Kähler manifold and Y a smooth projective variety. Let Δ be an effective $\mathbb{Q}$-divisor on X such that the pair (X, Δ) is klt. Then for any ample $\mathbb{Q}$-line bundle A_Y on Y, we have*

$$\kappa(X, K_{X/Y} + \Delta + f^*A_Y) = \kappa(F, K_F + \Delta_F) + \dim Y. \tag{3.1}$$

where F denotes the general fibre of f, and $\Delta_F := \Delta|_F$.

Proof. If $\kappa(F, K_F + \Delta_F) = -\infty$, then for any integer $\mu > 0$ sufficiently large and divisible (so that $A_Y^{\otimes\mu}$ is a line bundle and $\mu\Delta$ is an integral divisor) we have

$$\mathscr{F}_{\mu,\Delta} := f_*\left(K_{X/Y}^{\otimes\mu} \otimes \mathscr{O}_X(\mu\Delta)\right) = 0,$$

thus $\mathscr{F}_{\mu,\Delta} \otimes A_Y^{\otimes\mu} = 0$, and in particular

$$\mathrm{H}^0(X, K_{X/Y}^{\otimes\mu} \otimes \mathscr{O}_X(\mu\Delta) \otimes f^*A_Y^{\otimes\mu}) = \mathrm{H}^0(Y, \mathscr{F}_{\mu,\Delta} \otimes A_Y^{\otimes\mu}) = 0,$$

therefore $\kappa(X, K_{X/Y} + \Delta + f^*A_Y) = -\infty$, hence the equality (3.1).

Suppose in the sequel that $\kappa(F, K_F + \Delta_F) \geqslant 0$. Let m be a sufficiently large and divisible positive integer, so that $A_Y^{\otimes m}$ is a line bundle, $m\Delta$ is an integral divisor, $\mathscr{F}_{m,\Delta} \neq 0$ and that there is a very ample line bundle A'_Y on Y satisfying $(A'_Y)^{\otimes 2} \simeq A_Y^{\otimes m}$ such that $A'_Y \otimes K_Y^{-1}$ is ample ant that the following inequality for Seshadri constant holds:

$$\epsilon(A'_Y \otimes K_Y^{-1}, y) > \dim Y, \quad \text{for general } y \in Y.$$

Such an m exists by [Laz04, §5.1, Example 5.1.4, p. 270 and Example 5.1.18, p. 274, Vol.I]. By Theorem 2.2.1 the relative m-Bergman kernel metric $h^{(m)}_{X/Y,m\Delta}$ on $K_{X/Y}^{\otimes m} \otimes \mathscr{O}_X(m\Delta)$ is semi-positively curved. Then as in the proof of Theorem D we consider the line bundle

$$L_{m-1} := K_{X/Y}^{\otimes(m-1)} \otimes \mathscr{O}_X(m\Delta)$$

equipped with the semi-positively curved metric

$$h_{L_{m-1}} := (h^{(m)}_{X/Y,m\Delta})^{\otimes\frac{m-1}{m}} \otimes h_\Delta,$$

where h_Δ denotes the singular Hermitian metric whose local weight is defined by the local equation of Δ. Then apply Theorem 2.1.2 to $L = L_{m-1}$ (by virtue of Lemma 2.2.9) and we get a surjection

$$\mathrm{H}^0(X, K_X \otimes L_{m-1} \otimes f^*(A'_Y \otimes K_Y^{-1})) \twoheadrightarrow \mathrm{H}^0(F, K_F \otimes L_{m-1}|_F),$$

i.e.

$$\mathrm{H}^0(X, K_{X/Y}^{\otimes m} \otimes \mathscr{O}_X(m\Delta) \otimes f^*A'_Y) \twoheadrightarrow \mathrm{H}^0(F, K_F^{\otimes m} \otimes \mathscr{O}_F(m\Delta_F)),$$

which implies that

$$\mathrm{H}^0(X, K_{X/Y}^{\otimes m} \otimes \mathscr{O}_X(m\Delta) \otimes f^*A'_Y) \neq 0. \tag{3.2}$$

By (3.2) we can apply Lemma 3.1.1 to $L = K_{X/Y}^{\otimes m} \otimes \mathscr{O}_X(m\Delta) \otimes f^*A'_Y$ and $A = A'_Y$ and we get

$$\begin{aligned}\kappa(X, K_{X/Y} + \Delta + f^*A_Y) &= \kappa(X, (mK_{X/Y} + m\Delta + f^*A'_Y) + f^*A'_Y) \\ &= \kappa(F, (mK_{X/Y} + m\Delta + f^*A'_Y)|_F) + \dim Y \\ &= \kappa(F, K_F + \Delta_F) + \dim Y.\end{aligned}$$

□

By virtue of Lemma 3.1.3, one easily deduces Theorem 3.1.2:

Proof of Theorem 3.1.2. By Lemma 1.1.4, up to replacing Y by a higher smooth model and up to taking a desingularization of the fibre product, we can assume that X and Y are smooth. Since Y is of general type, it is projective. Then fix an ample line bundle H on Y; its canonical bundle K_Y being big, the Kodaira Lemma (c.f. [KM98, Lemma 2.60, pp. 67-68]) implies that there exists an integer $b > 0$ such that $K_Y^{\otimes b} \otimes H^{-1}$ is effective. Now by applying Lemma 3.1.3 to $A_Y = \frac{1}{b}H$ we obtain

$$\begin{aligned}\kappa(X, K_X + \Delta) &\geqslant \kappa(X, bK_{X/Y} + b\Delta + f^*H) \\ &= \kappa(F, K_F + \Delta_F) + \dim Y,\end{aligned}$$

and it ends the proof of Theorem 3.1.2. □

3.1.2 Iitaka conjecture for Kähler fibre spaces with big determinant bundle of the direct image of some relative pluricanonical bundle

The proof of Theorem A(I) is obtained by combining Lemma 3.1.1 and Theorem 2.1.2 plus the following result:

Theorem 3.1.4 (Kähler version of [CP17, Theorem 3.4]). *Let $f : X \to Y$ be a fibre space with X a compact Kähler manifold and Y a smooth projective variety andd let F be the general fibre of f. Let L be a holomorphic $\mathbb{Q}$-line bundle on X equipped with a singular Hermitian metric h_L such that its curvature current $\Theta_{h_L}(L) \geqslant 0$ and that $\mathscr{J}(h_L) \simeq \mathscr{O}_X$. Suppose that there is an integer $m > 0$ such that $L^{\otimes m}$ is a line bundle and that*

$$f_*\left(K_{X/Y}^{\otimes m} \otimes L^{\otimes m}\right) \neq 0. \tag{3.3}$$

Such m exists if and only if $\kappa(F, K_F + L|_F) \geqslant 0$. Suppose that there is a SNC divisor Σ_Y containing $Y \backslash Y_0$ where Y_0 is the (analytic) Zariski open subset over which f is smooth, such that $f^*\Sigma_Y$ has SNC support (in other word, f is prepared in the sense of [Cam04]). Then there exists a constant $\epsilon_0 > 0$ and an f-exceptional effective $\mathbb{Q}$-divisor E such that the $\mathbb{Q}$-line bundle

$$K_{X/Y} + L + E - \epsilon_0 f^* \det f_*\left(K_{X/Y}^{\otimes m} \otimes L^{\otimes m}\right) \tag{3.4}$$

is pseudoeffective.

Before giving the proof, let us remark that:

Remark 3.1.5. The condition (3.3) concerning the positivity of the Kodaira dimension of the general fibre does not appear in the original statement of [CP17, Theorem 3.14], but is indispensable. In fact, consider for example the case where $Y = \mathrm{pt}$, X is a smooth Fano variety (or more generally a smooth uniruled projective variety) with $\Delta = 0$, f is the structural morphism $X \to \mathrm{pt}$ and $L = \mathscr{O}_X$; f being a smooth morphism, there is no f-exceptional divisors, and the direct image (space of global sections) of $K_X^{\otimes m}$ is always 0, then the $\mathbb{Q}$-line bundle (3.4) is equal to K_X, which can never be pseudoeffective for X Fano (or uniruled projective, by [BDPP13]).

Proof of Theorem 3.1.4. The proof follows the same idea as that of [CP17, Theorem 3.4]; in fact, the algebraicity of f (or equivalently, the algebraicity of X) is not essential in the original proof: it is only used in [CP17] to apply the Ohsawa-Takegoshi extension theorem and [Nak04, III.5.10.Lemma, pp. 107-108]; as have been seen in §2.1 and §1.3 respectively, both of them can be generalized to the Kähler case. Nevertheless, the proof being highly technical, we will give more details for the convenience of the readers. Let us summarize the central idea of the proof as follows: from the natural inclusion of the determinant into the tensor product, we can construct, by the diagonal method of Viehweg, a non-zero section on $X^{(r)}$ (where $X^{(r)}$ denotes the resolution of some fibre product X^r of X over Y) of a line bundle of the form (3.4) (with X replaced by X^r and $\epsilon_0 = 1$); and then we "restrict" this section to the diagonal so that we get a section of the line bundle (3.4) on X. However one cannot deduce the effectivity of the line bundle (3.4), since the section constructed as above can vanish along the diagonal. To overcome this difficulty, we have to take a twisted approach: at the cost of tensoring by an ample divisor coming from Y, we can use the Ohsawa-Takegoshi extension Theorem 2.1.2 to extend pluricanonical forms on the general fibre F (by virtue of the condition (3.3)) to sections of the line bundle of the form (3.4) on $X^{(r)}$, then one can restrict them to the diagonal and get non-zero sections. However, these sections usually have poles, due to the singularities of f; in order to get rid of them, one has to carefully analyse these singularities (this analysis takes up a technical part of the proof), then it turns out that the poles are supported on the non-reduced fibres in codimension 1 and hence one can use

Proposition 2.2.3 to control them. Finally one use an approximation argument to conclude the pseudoeffectivity of the line bundle (3.4). The proof of the theorem proceeds in six steps:

(A) Analysis of singular fibres of f.

In this step, we will use a standard argument to show that the (analytic Zariski) open subset of $y \in Y$ such that X_y is Gorenstein is of codimension $\geqslant 2$ (whilst the generic smoothness only ensure this to be analytic Zariski open). To this end, note

$$Y_{\mathrm{f}} := Y_{\mathrm{flat}} \cap Y_{\mathscr{F}_{m,L}}$$

the (analytic) Zariski open subset over which f is flat and $\mathscr{F}_{m,L} := f_*\left(K_{X/Y}^{\otimes m} \otimes L^{\otimes m}\right)$ is locally free; and denote $X_{\mathrm{f}} := f^{-1}Y_{\mathrm{f}}$. since X and Y are reduced, $\operatorname{codim}_Y(Y \setminus Y_{\mathrm{f}}) \geqslant 2$ (c.f. [Kob87, Corollary 5.5.15, p. 147] and [Ful84, Example A.5.4, p. 416]). By [Mat89, Theorem 23.4, p. 181], for every $y \in Y_{\mathrm{f}}$, the fibre X_y is Gorenstein.

(B) Construction of the fibre product X^r and the canonical section.

Over Y_{f} one has a natural morphism (injection of vector bundles)

$$\det f_*\left(K_{X/Y}^{\otimes m} \otimes L^{\otimes m}\right) \hookrightarrow \bigotimes^r f_*\left(K_{X/Y}^{\otimes m} \otimes L^{\otimes m}\right), \tag{3.5}$$

where $r := \operatorname{rk} \mathscr{F}_{m,L}$, which gives rise to a non-trivial section of

$$\left(\bigotimes^r f_*\left(K_{X/Y}^{\otimes m} \otimes L^{\otimes m}\right)\right) \otimes \left(\det f_*\left(K_{X/Y}^{\otimes m} \otimes L^{\otimes m}\right)\right)^{-1}. \tag{3.6}$$

over Y_{f}. In order to get a section of a line bundle of the form (3.4), we will apply the diagonal method of Viehweg (c.f. for example [Vie95, §6.5, pp. 192-196]). Let

$$X^r := \underbrace{X \underset{Y}{\times} X \underset{Y}{\times} \cdots \underset{Y}{\times} X}_{r \text{ times}}$$

be the r-fold fibre product of X over Y, equipped with a morphism (a Kähler fibration) $f^r : X^r \to Y$ as well as the natural projections $\mathrm{pr}_i : X^r \to X$ to the i-th factor. Denote $X_{\mathrm{f}}^r := (f^r)^{-1}Y_{\mathrm{f}}$, then $f^r|_{X_{\mathrm{f}}}$ is flat; moreover, since Y and $X_y^r = X_y \times \cdots \times X_y$ are Cohen-Macaulay for every $y \in Y_{\mathrm{f}}$, X_{f}^r is also Cohen-Macaulay (by [Mat70, (21.C) Corollary 2, p. 154]). By the base change formula for relative canonical sheaves we see that X_{f}^r is Gorenstein and

$$\omega_{X^r} \otimes f^{r*}K_Y^{-1} = \omega_{X^r/Y} \simeq \bigotimes_{i=1}^r \mathrm{pr}_i^* K_{X/Y} \tag{3.7}$$

Note

$$L_r := \bigotimes_{i=1}^r \mathrm{pr}_i^* L,$$

then by an induction argument, the projection formula together with the base change formula imply that (c.f. [Hör10, Lemma 3.15])

$$\bigotimes^r f_*\left(K_{X/Y}^{\otimes m} \otimes L^{\otimes m}\right) \simeq (f^r)_*\left(\omega_{X^r/Y}^{\otimes m} \otimes L_r^{\otimes m}\right) \quad \text{over } Y_{\mathrm{f}}.$$

In consequence, the morphism (3.5) gives rise to a non-zero section

$$\begin{aligned} s_0 \in \mathrm{H}^0(X_{\mathrm{f}}^r, \omega_{X^r/Y}^{\otimes m} \otimes L_r^{\otimes m} \otimes (f^r)^*\left(\det f_*\left(K_{X/Y}^{\otimes m} \otimes L^{\otimes m}\right)\right)^{-1}) \\ = \mathrm{H}^0(Y_{\mathrm{f}}, \left(\bigotimes^r f_*\left(K_{X/Y}^{\otimes m} \otimes L^{\otimes m}\right)\right) \otimes \left(\det f_*\left(K_{X/Y}^{\otimes m} \otimes L^{\otimes m}\right)\right)^{-1}). \end{aligned} \tag{3.8}$$

(C) Analysis of the singularities of X^r.

Take a desingularization $\mu : X^{(r)} \to X^r$ which is an isomorphism over the smooth locus of X^r. Note $f^{(r)} := f^r \circ \mu$ and $X_{\mathrm{f}}^{(r)} := \mu^{-1} X_{\mathrm{f}}^r$. The natural morphism

$$\mu_* K_{X^{(r)}} \to \omega_{X^r}, \tag{3.9}$$

which is an isomorphism over X^r_{rat} where X^r denotes the (analytic Zariski) open subset of point with rational singularities on X^r, gives rise to a meromorphic section of the line bundle (by virtue of (3.7))

$$K^{-1}_{X^{(r)}/Y} \otimes \mu^* \left(\bigotimes_{i=1}^{r} \mathrm{pr}_i^* K_{X/Y} \right),$$

whose zeros and poles are contained in $X^{(r)} \backslash \mu^{-1} X^r_{\mathrm{rat}}$. In consequence, there are two effective divisors D_1 and D_2 over $X^{(r)}$ such that $\mathrm{Supp}(D_1), \mathrm{Supp}(D_2) \subseteq X^{(r)} \backslash \mu^{-1} X^r_{\mathrm{rat}}$ and that

$$K_{X^{(r)}/Y} \otimes \mathscr{O}_{X^{(r)}}(D_1) = \mu^* \left(\bigotimes_{i=1}^{r} \mathrm{pr}_i K_{X/Y} \right) \otimes \mathscr{O}_{X^{(r)}}(D_2). \tag{3.10}$$

Now let us further analyse the rational singularities locus X^r_{rat} by virtue of our hypothesis on Σ_Y and $f^*\Sigma_Y$. Write

$$f^* \Sigma_Y = \sum_i W_i + \sum_j a_j V_j \tag{3.11}$$

with the W_i's and V_j's prime divisors over X and $a_i \geqslant 2$; by hypothesis,

$$W := \sum_i W_i \quad \text{et} \quad V := \sum_j V_j$$

are (reduced) SNC divisors. As is explained in Remark 1.1.2, the fibre product

$$(X_{\mathrm{f}} \backslash (V \cup f^{-1} \mathrm{Sing}(\Sigma_Y)))^r := \underbrace{(X_{\mathrm{f}} \backslash (V \cup f^{-1} \mathrm{Sing}(\Sigma_Y))) \underset{Y_{\mathrm{f}} \backslash \mathrm{Sing}(\Sigma_Y)}{\times} \cdots \underset{Y_{\mathrm{f}} \backslash \mathrm{Sing}(\Sigma_Y)}{\times} (X_{\mathrm{f}} \backslash (V \cup f^{-1} \mathrm{Sing}(\Sigma_Y)))}_{r \text{ times}}$$

is contained in X^r_{rat}.

In consequence, both D_1 and D_2 are contained in the set $\mathscr{D}$ where $\mathscr{D}$ denotes the set of divisors D on $X^{(r)}$ such that every component Γ of D satisfies (at least) one of the following three conditions:

($\mathscr{D}1$) $f^{(r)}(\Gamma) \subseteq Y \backslash Y_{\mathrm{f}}$ (in particular, Γ is $f^{(r)}$-exceptional);

($\mathscr{D}2$) Γ is $\mathrm{pr}_i \circ \mu$-exceptional for some i;

($\mathscr{D}3$) $\mathrm{pr}_i \circ \mu(\Gamma) = V_j$ for some i and j.

(D) Extension of pluricanonical forms on $X_y^{(r)}$ by Ohsawa-Takegoshi.

The section s_0 (c.f. (3.8)) gives rise the section

$$\mu^* s_0 \in \mathrm{H}^0(X_{\mathrm{f}}^{(r)}, K^{\otimes m}_{X^{(r)}/Y} \otimes \mu^* L_r^{\otimes m} \otimes \mathscr{O}_{X^{(r)}}(mD_1) \otimes f^{(r)*} \left(\det f_* \left(K^{\otimes m}_{X/Y} \otimes L^{\otimes m} \right) \right)^{-1}).$$

Since $\operatorname{codim}_Y Y_{\mathrm{f}} \geqslant 2$, the section $\mu^* s_0$, regarded as a section of the torsion free sheaf (3.6) over Y_{f}, extends to a global section $\bar{s}_0$ of the reflexive hull

$$\begin{aligned}&\left[\left(\bigotimes^r f_*\left(K_{X/Y}^{\otimes m}\otimes L^{\otimes m}\right)\right)\otimes\left(\det f_*\left(K_{X/Y}^{\otimes m}\otimes L^{\otimes m}\right)\right)^{-1}\right]^{\wedge}\\ =&\left[f_*^{(r)}\left(K_{X^{(r)}/Y}^{\otimes m}\otimes\mu^* L_r^{\otimes m}\otimes\mathscr{O}_{X^{(r)}}(mD_1)\otimes f^{(r)*}\left(\det f_*\left(K_{X/Y}^{\otimes m}\otimes L^{\otimes m}\right)\right)^{-1}\right)\right]^{\wedge}.\end{aligned}$$

By Theorem 1.3.1, there is an $f^{(r)}$-exceptional effective divisor D_3 such that

$$\begin{aligned}&\left[f_*^{(r)}\left(K_{X^{(r)}/Y}^{\otimes m}\otimes\mu^* L_r^{\otimes m}\otimes\mathscr{O}_{X^{(r)}}(mD_1)\otimes f^{(r)*}\left(\det f_*\left(K_{X/Y}^{\otimes m}\otimes L^{\otimes m}\right)\right)^{-1}\right)\right]^{\wedge}\\ =&\left[f_*^{(r)}\left(K_{X^{(r)}/Y}^{\otimes m}\otimes\mu^* L_r^{\otimes m}\otimes\mathscr{O}_{X^{(r)}}(mD_1)\right)\right]^{\wedge}\otimes f^{(r)*}\left(\det f_*\left(K_{X/Y}^{\otimes m}\otimes L^{\otimes m}\right)\right)^{-1}\\ =&f_*^{(r)}\left(K_{X^{(r)}/Y}^{\otimes m}\otimes\mu^* L_r^{\otimes m}\otimes\mathscr{O}_{X^{(r)}}(mD_1+D_3)\right)\otimes f^{(r)*}\left(\det f_*\left(K_{X/Y}^{\otimes m}\otimes L^{\otimes m}\right)\right)^{-1}\\ =&f_*^{(r)}\left(K_{X^{(r)}/Y}^{\otimes m}\otimes\mu^* L_r^{\otimes m}\otimes\mathscr{O}_{X^{(r)}}(mD_1+D_3)\otimes f^{(r)*}\left(\det f_*\left(K_{X/Y}^{\otimes m}\otimes L^{\otimes m}\right)\right)^{-1}\right),\end{aligned}$$

hence $\bar{s}_0$ can be regarded as a (global) section of the line bundle

$$K_{X^{(r)}/Y}^{\otimes m}\otimes\mu^* L_r^{\otimes m}\otimes\mathscr{O}_{X^{(r)}}(mD_1+D_3)\otimes f^{(r)*}\det f_*\left(K_{X/Y}^{\otimes m}\otimes L^{\otimes m}\right)^{-1}.$$

Moreover, since the torsion free sheaf (3.6) is locally free on Y_{f}, hence

$$f^{(r)}(\operatorname{Supp}(D_3))\subseteq Y\backslash Y_{\mathrm{f}},$$

in particular, $D_3\in\mathscr{D}$. Now choose $\epsilon\in\mathbb{Q}_{>0}$ small enough such that $\Delta_0:=\epsilon\operatorname{div}(\bar{s}_0)$ is klt on $X^{(r)}$.The $\mathbb{Q}$-line bundle $\mathscr{O}_{X^{(r)}}(\Delta_0)$ is equipped with a canonical singular Hermitian metric h_{Δ_0} whose local weight is given by

$$\phi_{\Delta_0}=\frac{\epsilon}{2}\log|g_{\bar{s}_0}|^2,$$

where $g_{\bar{s}_0}$ denotes a local equation of $\operatorname{div}(\bar{s}_0)$. Denote $L_0:=\mu^* L_r\otimes\mathscr{O}_{X^{(r)}}(\Delta_0)$, this $\mathbb{Q}$-line bundle is equipped with the singular Hermitian metric

$$h_{L_0}:=h_{\Delta_0}\otimes\bigotimes_{i=1}^r\mu^*\operatorname{pr}_i^* h_L.$$

whose curvature current is positive. By strong openness [GZ15a, Theorem 1.1] for ϵ sufficiently small we have

$$\mathscr{J}(h_{L_0})=\mathscr{J}(\bigotimes_{i=1}^r\mu^*\operatorname{pr}_i^* h_L).\tag{3.12}$$

Since μ is supposed to be an isomorphism over Y_0, we have $X_y^{(r)}\simeq X_y\times\cdots\times X_y$ for $y\in Y_0$ (c.f. Step (E1) below), then by Lemma 2.2.5 and [DELoo, Theorem 2.6(i)] we have $\mathscr{J}(h_{L_0}|_{X_y^{(r)}})=\mathscr{O}_{X_y^{(r)}}$ for a.e. $y\in Y_0$.

Let A_Y be an ample line bundle over Y such that the line bundle $A_Y\otimes K_Y^{-1}$ is ample and that the Seshadri constant $\epsilon(A_Y\otimes K_Y^{-1},y)>d:=\dim Y$ for general $y\in Y_0$ (such A_Y exists by [Laz04, §5.1, Example 5.1.4, p. 270 and Example 5.1.18, p. 274, Vol.I]). We claim that the restriction map

$$\mathrm{H}^0(X^{(r)},K_{X^{(r)}/Y}^{\otimes k}\otimes L_0^{\otimes k}\otimes f^{(r)*}A_Y)\longrightarrow\mathrm{H}^0(X_y^{(r)},K_{X_y^{(r)}}^{\otimes k}\otimes L_0^{\otimes k}|_{X_y^{(r)}})\tag{3.13}$$

is surjective for any k sufficiently large and divisible and for every $y \in Y_0$ such that $\mathscr{J}(h_{L_0}|_{X_y^{(r)}}) = \mathscr{O}_{X_y^{(r)}}$. In fact, Δ_0 being effective, the hypothesis (3.3) implies that

$$\begin{aligned} f_*^{(r)}\left(K_{X^{(r)}/Y}^{\otimes k} \otimes L_0^{\otimes k}\right) &= f_*^{(r)}\left(K_{X^{(r)}/Y}^{\otimes k} \otimes \mu^* L_r^{\otimes k} \otimes \mathscr{O}_{X^{(r)}}(k\Delta_0)\right) \\ &\supseteq f_*^{(r)}\left(K_{X^{(r)}/Y}^{\otimes k} \otimes \mu^* L_r^{\otimes k}\right) \neq 0 \end{aligned}$$

for k sufficiently large and divisible (e.g. such that $\epsilon k \in \mathbb{Z}_{>0}$ and k divisible by m) hence the integrability condition in Theorem 2.2.1 is satisfied (c.f. Remark 2.2.2). Moreover, since $\Theta_{h_{L_0}}(L_0) \geqslant 0$, Theorem 2.2.1 implies that the k-Bergman kernel metric $h^{(k)}_{X^{(r)}/Y, kL_0}$ is semi-positively curved. Set $M_k := K_{X^{(r)}/Y}^{\otimes(k-1)} \otimes L_0^{\otimes k}$, equipped with a singular Hermitian metric

$$h_{M_k} := \left(h^{(k)}_{X^{(r)}/Y, kL_0}\right)^{\frac{k-1}{k}} \otimes h_{L_0}$$

whose curvature current is positive. Then by Lemma 2.2.9 one has

$$\mathrm{H}^0(X_y^{(r)}, K_{X_y^{(r)}}^{\otimes k} \otimes L_0^{\otimes k}|_{X^{(r)}} \otimes \mathscr{J}(h_{M_k}|_{X_y^{(r)}})) = \mathrm{H}^0(X_y^{(r)}, K_{X_y^{(r)}}^{\otimes k} \otimes L_0^{\otimes k}|_{X^{(r)}}) \tag{3.14}$$

for a.e. $y \in Y_0$. Hence we can apply Theorem 2.1.2 to

$$K_{X^{(r)}} \otimes M_k \otimes f^{(r)*}(A_Y \otimes K_Y^{-1}) = K_{X^{(r)}/Y}^{\otimes k} \otimes L_0^{\otimes k} \otimes f^{(r)*} A_Y$$

to obtain the surjectivity of the restriction morphism (3.13) for a.e. $y \in Y_0$. Moreover, set $H_k := A_Y \otimes \det f_*\left(K_{X/Y}^{\otimes m} \otimes L^{\otimes m}\right)^{\otimes(-\epsilon k)}$, then we can rewrite (3.13) as

$$\begin{aligned} &\mathrm{H}^0(X^{(r)}, \left(K_{X^{(r)}/Y} \otimes \mu^* L_r\right)^{\otimes(1+\epsilon m)k} \otimes \mathscr{O}_{X^{(r)}}(\epsilon k m D_1 + \epsilon k D_3) \otimes f^{(r)*} H_k) \\ &\xrightarrow{\text{restriction}} \mathrm{H}^0(X_y^{(r)}, \left(K_{X_y^{(r)}} \otimes \mu^* L_r|_{X_y^{(r)}}\right)^{\otimes(1+\epsilon m)k}) \end{aligned} \tag{3.15}$$

for a.e. $y \in Y_0$ and for k sufficiently large and divisible.

(E) Extension of pluricanonical forms from X_y via restriction to the diagonal.
For general $y \in Y_0$ take a section

$$u \in \mathrm{H}^0(X_y, \left(K_{X_y} \otimes L|_{X_y}\right)^{\otimes(1+\epsilon m)k})$$

with k sufficiently large and divisible, we will construct a section s in

$$\mathrm{H}^0(X, (K_{X/Y} \otimes L)^{\otimes(1+\epsilon m)kr} \otimes \mathscr{O}_X(CkV + kE_0) \otimes f^* H_k^{\otimes r}),$$

for $C > 0$ a constant and E_0 an f-exceptional effective divisor, both independent of k, such that $s|_{X_y} = u^{\otimes r}$.

(E1) Extending the section u to a section over $X^{(r)}$ by Step (D)
Set

$$X_0^r := X_0 \underset{Y_0}{\times} X_0 \underset{Y_0}{\times} \cdots \underset{Y_0}{\times} X_0 \subseteq X^r,$$

where $X_0 := f^{-1}(Y_0)$, then X_0^r is smooth, hence $\mu^{-1}(X_0^r) \xrightarrow{\mu}{\sim} X_0^r$ is an isomorphism. In particular, we have

$$X_y^{(r)} \xrightarrow{\mu}{\sim} X_y^r = \underbrace{X_y \times X_y \times \cdots \times X_y}_{r \text{ times}}. \tag{3.16}$$

Hence u gives rise to a section

$$u^{(r)} := \mu^*\left(\bigotimes_{i=1}^{r} \mathrm{pr}_i^* u\right) \in \mathrm{H}^0\Big(X_y^{(r)}, \Big(K_{X_y^{(r)}} \otimes \mu^* L_r|_{X_y^{(r)}}\Big)^{\otimes(1+\epsilon m)k}\Big), \tag{3.17}$$

such that the restriction of $u^{(r)}$ to the diagonal is equal to $u^{\otimes r}$. Using the surjection (3.15) we obtain a section $\sigma^{(r)}$ of the line bundle

$$\Big(K_{X^{(r)}/Y} \otimes \mu^* L_r\Big)^{\otimes(1+\epsilon m)k} \otimes \mathscr{O}_{X^{(r)}}(\epsilon k m D_1 + \epsilon k D_3) \otimes f^{(r)*} H_k, \tag{3.18}$$

such that $\sigma^{(r)}|_{X_y^{(r)}} = u^{(r)}$.

(E2) Restricting the section $\sigma^{(r)}|_{\mu^{-1}X_0^r}$ to the diagonal

In order to restrict $\sigma^{(r)}|_{\mu^{-1}X_0^r}$ to the diagonal, use (3.10) to rewrite the line bundle (3.18) as follows:

$$\begin{aligned}
&\Big(K_{X^{(r)}/Y} \otimes \mu^* L_r\Big)^{\otimes(1+\epsilon m)k} \otimes \mathscr{O}_{X^{(r)}}(\epsilon k m D_1 + \epsilon k D_3) \otimes f^{(r)*} H_k \\
&= \mu^*\left(\bigotimes_{i=1}^{r} \mathrm{pr}_i^*(K_{X/Y} \otimes L)\right)^{\otimes(1+\epsilon m)k} \otimes \mathscr{O}_{X^{(r)}}(-k D_1 + (1+\epsilon m) k D_2 + \epsilon k D_3) \otimes f^{(r)*} H_k .
\end{aligned} \tag{3.19}$$

In consequence, $\sigma^{(r)}$ can be regarded as a meromorphic section of the line bundle

$$\mu^*\left(\bigotimes_{i=1}^{r} \mathrm{pr}_i^*(K_{X/Y} \otimes L)\right)^{\otimes(1+\epsilon m)k} \otimes f^{(r)*} H_k \tag{3.20}$$

whose poles are contained $\mathrm{Supp}(D_2) \cup \mathrm{Supp}(D_3)$. Locally, by choosing a trivialization of the line bundle (3.20), the section $\sigma^{(r)}$ can be written as a meromorphic function $F^{(r)}$ such that

$$g_{D_1}^{-k} g_{D_2}^{(1+\epsilon m)k} g_{D_3}^{\epsilon k} \cdot F^{(r)} \tag{3.21}$$

is holomorphic, where g_{D_l} is a local equation of the divisor D_l $(l = 1, 2, 3)$.

By construction, $D_1, D_2, D_3 \in \mathscr{D}$ (in particular, D_3 is $f^{(r)}$-exceptional), hence there exist constants C_1 et C_2 such that

$$D_l \leqslant C_l \cdot \mu^* \sum_{i=1}^{r} \mathrm{pr}_i^* V, \quad \text{pour } l = 1, 2 \tag{3.22}$$

over $X_{\mathrm{f}}^{(r)} \backslash S$ where $S \subseteq X^{(r)}$ denotes the union of the components in $D_1 + D_2$ which are $\mathrm{pr}_i \circ \mu$-exceptional for every $i = 1, \cdots, r$. By Step (D) we have

$$f^{(r)}(\mathrm{Supp}(D_3)) \subseteq Y \backslash Y_{\mathrm{f}},$$

hence locally over $X_{\mathrm{f}}^{(r)} \backslash S$ the meromorphic function

$$F^{(r)} \cdot \prod_{i=1}^{r} \Big((\mathrm{pr}_i \circ \mu)^* g_V\Big)^{C_2(1+\epsilon m)k} = F^{(r)} \cdot \prod_{i=1}^{r} (\mathrm{pr}_i \circ \mu)^* \Big(g_V^{C_2(1+\epsilon m)k}\Big)$$

is holomorphic where $g_V = \prod_j g_{V_j}$ is a local equation of V.

Note $\delta_{X,r} : X \to X^r$ the inclusion of the diagonal. Then $\mathrm{pr}_i \circ \delta_{X,r} = \mathrm{id}_X$ for $\forall i = 1, \cdots, r$. Since the D_l's $(l = 1, 2, 3)$ are disjoint from

$$\mu^{-1} X_{\mathrm{rat}}^r \supseteq \mu^{-1} X_0^r \supseteq \mu^{-1}(\delta_{X,r}(X_0)),$$

then locally the meromorphic function $F^{(r)}$ is holomorphic over $\mu^{-1}X_0^r$. Therefore we can restrict $\sigma^{(r)}|_{\mu^{-1}X_0^r}$ to the diagonal and obtain a section

$$s_1 := (\mu|_{X_0^r}^{-1} \circ \delta_{X,r}|_{X_0})^* \left(\sigma^{(r)}|_{\mu^{-1}X_0^r}\right)$$

over X_0 of the line bundle

$$(K_{X/Y} \otimes L)^{\otimes(1+\epsilon m)kr} \otimes f^* H_k^{\otimes r}. \tag{3.23}$$

Locally over an open subset of X_0 trivializing the line bundle (3.23) the section s_1 is given by a holomorphic function

$$F_1 := (\mu|_{X_0^r}^{-1} \circ \delta_{X,r}|_{X_0})^* \left(F^{(r)}|_{\mu^{-1}X_0^r}\right).$$

(E3) Extending the section s_1 across the singular fibres of X

In order to extend s_1 across $f^{-1}\Sigma_Y$, one needs to know its behaviour around the W_i's and the V_j's; this can be done by analysing the poles along the D_l's of $\sigma^{(r)}$, regarded as a meromorphic section of the line bundle (3.20), as we explain in the sequel:

(E3-i) By Step (C) $(X_f \backslash (V \cup \mathrm{Sing}(W)))^r$ is contained in X_{rat}^r, thus disjoint to the D_l's ($l = 1, 2, 3$); considering F_1 as a holomorphic function on $\delta_{X,r}(X_0)$), one has

$$\mu^* F_1 = F^{(r)}|_{\mu^{-1}(\delta_{X,r}(X_0))},$$

but the poles of $F^{(r)}$ are contained in $\mathrm{Supp}(D_2) \cup \mathrm{Supp}(D_3)$, hence the function F_1 is bounded near $X_f \backslash (V \cup f^{-1}\,\mathrm{Sing}(\Sigma_Y))$, and thus F_1 can be extended to $X_f \backslash (V \cup f^{-1}\,\mathrm{Sing}(\Sigma_Y))$ by Riemann extension theorem; moreover, by Hartogs extension theorem, F_1 extends to a holomorphic function over $X_f \backslash V$.

(E3-ii) In general, F_1 is not bounded around V. Nevertheless, by Step (E2) the meromorphic function

$$F^{(r)} \cdot \mu^* \prod_{i=1}^{r} \mathrm{pr}_i^* \left(g_V^{C_2(1+\epsilon m)k}\right)$$

is holomorphic over $X_f^r \backslash S$. And the restriction of S to the diagonal is an analytic subset of codimension $\geqslant 2$ (c.f. (E2) for the definition of S), hence the function

$$F_1 \cdot g_V^{C_2(1+\epsilon m)kr}$$

is bounded around a general point of $V \cap X_f$. By Riemann extension theorem (as well as Hartogs extension theorem) F_1 extends across $V \cap X_f$ as a holomorphic local section of the line bundle

$$(K_{X/Y} \otimes L)^{\otimes(1+\epsilon m)kr} \otimes \mathscr{O}_X(CkV) \otimes f^* H_k^{\otimes r},$$

where $C := C_2(1 + \epsilon m)r$ is a constant independent of k. Combining this with (E3-i) we obtain an extension of s_1 to a section over X_f:

$$\bar{s}_1 \in \mathrm{H}^0(X_f, (K_{X/Y} \otimes L)^{\otimes(1+\epsilon m)kr} \otimes \mathscr{O}_X(CkV) \otimes f^* H_k^{\otimes r}).$$

(E3-iii) At last, we will extend $\bar{s}_1$ to a global section, which provides us with the sought section s. In fact, $\bar{s}_1$ can be regarded as a section of the direct image sheaf

$$f_* \left((K_{X/Y} \otimes L)^{\otimes(1+\epsilon m)kr} \otimes \mathscr{O}_X(CkV) \otimes f^* H_k^{\otimes r}\right) \tag{3.24}$$

over Y_{f}. But $\mathrm{codim}_Y(Y\setminus Y_{\mathrm{f}})\geqslant 2$, hence $\bar{s}_1$ extends to a global section s of the reflexive hull of the (torsion free) sheaf (3.24). By Theorem 1.3.1, there is an f-exceptional effective divisor E_0, independent of k, such that

$$\begin{aligned}&f_*\Big((K_{X/Y}\otimes L)^{\otimes(1+\epsilon m)kr}\otimes\mathscr{O}_X(CkV)\otimes f^*H_k^{\otimes r}\Big)^{\wedge}\\&=f_*\Big((K_{X/Y}\otimes L)^{\otimes(1+\epsilon m)kr}\otimes\mathscr{O}_X(CkV+kE_0)\otimes f^*H_k^{\otimes r}\Big),\end{aligned}$$

hence

$$s\in \mathrm{H}^0(X,(K_{X/Y}\otimes L)^{\otimes(1+\epsilon m)kr}\otimes\mathscr{O}_X(CkV+kE_0)\otimes f^*H_k^{\otimes r}).$$

Moreover, by (3.16) as well as the construction of the section $u^{(r)}$ (c.f. (3.17)) we have

$$s|_{X_y}=s_1|_{X_y}=(\delta\circ\mu)^*u^{(r)}=u^{\otimes r}.$$

This finishes (E3) and thus the Step (E).

(F) Conclusion.

By the hypothesis (3.3), for any general $y\in Y$ and for any integer k sufficiently large and divisible (e.g. such that $\epsilon k\in\mathbb{Z}_{>0}$ and that k divisible par m), we have a non-zero section

$$u\in \mathrm{H}^0(X_y,\Big(K_{X_y}\otimes L|_{X_y}\Big)^{\otimes(1+\epsilon m)k}).$$

Assume further that $y\in Y_0$ and $\mathscr{J}(h_{L_0}|_{X_y^{(r)}})=\mathscr{O}_{X_y^{(r)}}$, then by Step (E) above, we can construct a section

$$s\in \mathrm{H}^0(X,(K_{X/Y}\otimes L)^{\otimes(1+\epsilon m)kr}\otimes\mathscr{O}_X(CkV+kE_0)\otimes f^*H_k),$$

for C and E_0 independent of k such that $s|_{X_y}=u^{\otimes r}$. In particular $s\neq 0$, implying that the line bundle

$$(K_{X/Y}\otimes L)^{\otimes(1+\epsilon m)kr}\otimes\mathscr{O}_X(CkV+kE_0)\otimes f^*H_k^{\otimes r} \tag{3.25}$$

is effective. By writing $V=V_{\mathrm{div}}+V_{\mathrm{exc}}$ with V_{div} (resp. V_{exc}) the non-exceptional (resp. exceptional) part of V with respect to f, one can rewrite the line bundle (3.25) as follows:

$$\begin{aligned}&(K_{X/Y}\otimes L)^{\otimes(1+\epsilon m)kr}\otimes\mathscr{O}_X(CkV+kE_0)\otimes f^*H_k^{\otimes r}\\&=(K_{X/Y}\otimes L)^{\otimes(1+\epsilon m)kr}\otimes\mathscr{O}_X(CkV_{\mathrm{div}}+kE_1)\otimes f^*H_k^{\otimes r}\end{aligned}$$

where $E_1=CV_{\mathrm{exc}}+E_0$ is f-exceptional.

In addition, the hypothesis (3.3) implies that the relative m-Bergman kernel metric $h^{(m)}_{X/Y,L}$ on $K_{X/Y}^{\otimes m}\otimes L^{\otimes m}$ is semi-positively curved, hence by Proposition 2.2.3 and (3.11) the line bundle

$$K_{X/Y}\otimes L\otimes\mathscr{O}_X(-bV_{\mathrm{div}})$$

is pseudoeffective, where $b:=\min_j\{a_j-1\}$. There the $\mathbb{Q}$-line bundle

$$\Big((1+\epsilon m)kr+\frac{Ck}{b}\Big)(K_{X/Y}+L)+kE_1+rf^*A_Y-\epsilon krf^*\det f_*\Big(K_{X/Y}^{\otimes m}\otimes L^{\otimes m}\Big)$$

is pseudoeffective. By letting $k\to+\infty$ and by putting

$$E:=\frac{b}{(1+\epsilon m)br+C}E_1\quad\text{et}\quad\epsilon_0:=\frac{\epsilon br}{(1+\epsilon m)br+C}$$

we obtain the pseudoeffectivity of the $\mathbb{Q}$-line bundle (3.4), thus ending the proof of Theorem 3.1.4. □

Now let us turn to the proof of Theorem A(I). In fact one can prove a stronger result as following, whose proof is quite similar to [CP17, Corollary 4.1]:

Theorem 3.1.6. *Let $f : X \to Y$ be a fibre space between compact Kähler manifolds. Let Δ be an effective $\mathbb{Q}$-divisor on X such that (X, Δ) is klt. Suppose that there exists an integer $m > 0$ such that $m\Delta$ is an integral divisor and the determinant line bundle $\det f_*(K_{X/Y}^{\otimes m} \otimes \mathscr{O}_X(m\Delta))$ is big. Then*

$$\kappa(X, K_X + \Delta) \geqslant \kappa(Y) + \kappa(F, K_F + \Delta_F). \tag{3.26}$$

where F denotes the general fibre of f and $\Delta_F := \Delta|_F$. Moreover, if $\kappa(Y) \geqslant 0$ then we have

$$\kappa(X, K_X + \Delta) \geqslant \kappa(F, K_F + \Delta_F) + \dim Y.$$

Proof. The key point of the proof has already been proved in Theorem 3.1.4, the rest is quite similar to that of Theorem 3.1.2. Nevertheless, in order to apply Theorem 3.1.4, one should be able to add an "exceptional" positivity to the pluricanonical bundle; therefore we take a diagram as in Lemma 1.1.3 :

$$\begin{array}{ccc} X' & \xrightarrow{\pi_X} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[\pi_Y]{} & Y, \end{array}$$

and take Δ' an effective $\mathbb{Q}$-divisor on X' as in Lemma 1.1.4, so that every f'-exceptional divisor is also π_X-exceptional and that (X', Δ') is klt. By construction, the morphism f' is smooth over $Y_0' := \pi_Y^{-1}(Y_0)$ where Y_0 denotes the (analytic) Zariski open subset of Y over which f is smooth; $\pi_X|_{X_0'} : X_0' \to X_0$ with $X_0' := (f')^{-1}(Y_0')$ and $X_0 := f^{-1}(Y_0)$ is an isomorphism. In particular, for $y' \in Y_0'$, we have an isomorphism $X'_{y'} \simeq X_y$ (with $y := \pi_Y(y')$) between complex manifolds, implying that $F' \simeq F$ where F' denotes the general fibre of f'; moreover this isomorphism identifies $\Delta'_{F'} := \Delta'|_{F'}$ to Δ_F.

In addition, we have the following (non-trivial) morphism of base change

$$\pi_Y^* f_* \left(K_{X/Y}^{\otimes m} \otimes \mathscr{O}_X(m\Delta)\right) \to f'_* \left(\pi_X^* \left(K_{X/Y}^{\otimes m} \otimes \mathscr{O}_X(m\Delta)\right)\right) \to f'_* \left(K_{X'/Y'}^{\otimes m} \otimes \mathscr{O}_{X'}(m\Delta')\right), \tag{3.27}$$

which is an isomorphism over Y_0'. But π_Y being birational, the line bundle

$$\pi_Y^* \det f_* \left(K_{X/Y}^{\otimes m} \otimes \mathscr{O}_X(m\Delta)\right)$$

is big over Y', therefore the morphism (3.27) implies that the determinant line bundle $\det f'_* \left(K_{X'/Y'}^{\otimes m} \otimes \mathscr{O}_{X'}(m\Delta')\right)$ is also big over Y'. In particular

$$f'_* \left(K_{X'/Y'}^{\otimes m} \otimes \mathscr{O}_{X'}(m\Delta')\right) \neq 0. \tag{3.28}$$

Hence we can apply Theorem 3.1.4 to f', and we get an f'-exceptional $\mathbb{Q}$-divisor E' and $\epsilon_0 \in \mathbb{Q}_{>0}$ such that the $\mathbb{Q}$-line bundle

$$K_{X'/Y'} + \Delta' + E' - \epsilon_0 (f')^* \det f'_* \left(K_{X'/Y'}^{\otimes m} \otimes \mathscr{O}_{X'}(m\Delta')\right)$$

is pseudoeffective. Let us fix a very ample line bundle $A_{Y'}$ on Y' such that $A_{Y'} \otimes K_{Y'}^{-1}$ is ample and that the Seshadri constant $\epsilon(A_{Y'} \otimes K_{Y'}^{-1}, y) > \dim Y$ for general $y \in Y'$ (such $A_{Y'}$ exists by [Laz04, §5.1, Example 5.1.4, p. 270 and Example 5.1.18, p. 274, Vol.I]). Since $\det f'_* \left(K_{X'/Y'}^{\otimes m} \otimes \mathscr{O}_{X'}(m\Delta')\right)$ is big, Kodaira's Lemma (c.f. [KM98, Lemma 2.60, pp. 67-68])

implies that there exists a integer $m_1 > 0$ sufficiently large and divisible and a pseudoeffective line bundle L_0 on X such that $m_1\Delta'$ and m_1E' are integral divisors and that

$$K_{X'/Y'}^{\otimes m_1} \otimes \mathscr{O}_{X'}(m_1(\Delta' + E')) = (f')^* A_{Y'}^{\otimes 2} \otimes L_0.$$

And we have $L_0|_{F'} = K_{F'}^{\otimes m_1} \otimes \mathscr{O}_{F'}(m_1\Delta'_{F'})$. Now L_0 being pseudoeffective, we can equip it with a singular Hermitian metric h_{L_0} whose curvature current is positive. Since Δ' is klt, by strong openness [GZ15a, Theorem 1.1] (or [Ber13, Theorem 1.1]) we can find $m_2 \in \mathbb{Z}_{>0}$ sufficiently large and divisible such that

$$\mathscr{J}\left(h_{\Delta'} \otimes h_{L_0}^{\otimes \frac{1}{m_2}}\right) = \mathscr{O}_{X'}.$$

Now we can endow $K_{X'/Y'}^{\otimes m_2} \otimes \mathscr{O}_{X'}(m_2\Delta') \otimes L_0$ with the relative m_2-Bergman kernel metric $h^{(m_2)}_{X'/Y', m_2\Delta'+L_0}$, then by applying Lemma 2.2.9 to the $\mathbb{Q}$-line bundle $N = \Delta' + \frac{1}{m_2}L_0$ we have

$$\mathrm{H}^0(F', K_{F'} \otimes N_{m_2-1}\big|_{F'} \otimes \mathscr{J}(h_{N_{m_2-1}}\big|_{F'})) = \mathrm{H}^0(F', K_{F'} \otimes N_{m_2-1}\big|_F),$$

where $N_{m_2-1} := K_{X'/Y'}^{\otimes(m_2-1)} \otimes \mathscr{O}_{X'}(m_2\Delta') \otimes L_0$ equipped with the singular Hermitian metric

$$h_{N_{m_2-1}} := \left(h^{(m_2)}_{X'/Y', m_2\Delta'+L_0}\right)^{\otimes \frac{m_2-1}{m_2}} \otimes h_{\Delta'} \otimes h_{L_0}^{\otimes \frac{1}{m_2}}.$$

Now by Theorem 2.1.2 we have a surjection

$$\mathrm{H}^0(X', K_{X'} \otimes N_{m_2-1} \otimes (f')^*(A_{Y'} \otimes K_{Y'}^{-1})) \twoheadrightarrow \mathrm{H}^0(F', K_{F'} \otimes N_{m_2-1}\big|_{F'})$$

which amounts to:

$$\mathrm{H}^0(X', K_{X'/Y'}^{\otimes m_2} \otimes \mathscr{O}_{X'}(m_2\Delta') \otimes L_0 \otimes (f')^* A_{Y'}) \twoheadrightarrow \mathrm{H}^0(F', K_{F'}^{\otimes(m_1+m_2)} \otimes \mathscr{O}_{F'}((m_1+m_2)\Delta'_{F'}))$$

where the space on the right hand side is non-vanishing by (3.28).

Then by applying Lemma 3.1.1 to $L = K_{X'/Y'}^{\otimes m_2} \otimes \mathscr{O}_{X'}(m_2\Delta') \otimes L_0 \otimes (f')^* A_{Y'}$ (noting that $L|_{F'} = K_{F'}^{\otimes(m_1+m_2)} \otimes \mathscr{O}_{F'}((m_1+m_2)\Delta'_{F'})$) and by [Deb01, Lemma 7.11, p. 175] we obtain the following inequality:

$$\begin{aligned}
\kappa(X, K_{X/Y} + \Delta) &= \kappa(X', (m_1+m_2)(K_{X'} - \pi_X^* f^* K_Y + \Delta') + m_1 E') \text{ since } E' \text{ is } \pi_X\text{-exceptional} \\
&\geqslant \kappa(X', (m_1+m_2)(K_{X'/Y'} + \Delta') + m_1 E') \text{ since } K_{Y'/Y} \text{ is } \pi_Y\text{-exceptional effective} \\
&= \kappa(X', m_2 K_{X'/Y'} + m_2\Delta' + L_0 + 2(f')^* A_{Y'}) \\
&= \kappa(X', L + (f')^* A_{Y'}) \\
&= \kappa(F', K_{F'} + \Delta'_{F'}) + \dim Y' \\
&= \kappa(F, K_F + \Delta_F) + \dim Y. \qquad (3.29)
\end{aligned}$$

If $\kappa(Y) = -\infty$ then the inequality (3.26) is automatically established; otherwise, there is an integer $k > 0$ such that $K_Y^{\otimes k}$ is effective, then by (3.29) we get

$$\begin{aligned}
\kappa(X, K_X + \Delta) &= \kappa(X, kK_{X/Y} + k\Delta + kf^* K_Y) \\
&\geqslant \kappa(X, kK_{X/Y} + k\Delta) \\
&\geqslant \kappa(F, K_F + \Delta_F) + \dim Y.
\end{aligned}$$

□

3.2 Albanese maps of compact Kähler manifolds of log Calabi-Yau type

Having proved Theorem 3.1.2, one can follow the same argument as that in [Kaw81] to deduce Theorem E. Let us remark that in [Kaw81] a result equivalent to Theorem E with $\Delta = 0$ is also stated ([Kaw81, Theorem 25]). Similar to [Kaw81] the first step of the proof of Theorem E is to obtain the following proposition, which generalize [Uen75, Theorem 10.9, pp.120-123] and can be regarded as an analytic version of [Kaw81, Theorem 13]:

Proposition 3.2.1. *Let $p : V \to T$ be a finite morphism with V a compact normal complex variety and T a complex torus. Then $\kappa(V) \geqslant 0$, and there is a subtorus S of T and a (projective) normal variety of general type W, which is finite over T/S, such that*

(a) *there is an analytic fibre space $\phi_p : V \to W$ whose general fibre is equal to $\tilde{S}$, a complex torus which admits a finite étale cover $\tilde{S} \to S$ over S.*

(b) *$\kappa(W) = \kappa(V) = \dim W$;*

Before showing the proposition, let us recall the following lemma:

Lemma 3.2.2 ([Uen75, Lemma 9.11, p. 107])**.** *A meromorphic mapping from a complex manifold to a complex torus is always defined everywhere, thus gives rise to a morphism.*

Proof of Proposition 3.2.1. By [Uen75, Lemma 6.3, pp. 66-67], we have $\kappa(V) \geqslant \kappa(T) = 0$. Let $\Phi_V : V' \to W'$ be (a model of) the Iitaka fibration of V where V' is smooth model lying over V and W' a complex manifold. For a general point w' in W', $V_{w'}$ and $V'_{w'}$ are bimeromorphic and thus $\kappa(V_{w'}) = \kappa(V'_{w'}) = 0$, where $V_{w'}$ is the image of $V'_{w'}$ in V. Denote $S_{w'} = p(V_{w'})$ for $w' \in W'$, then by [Uen75, Theorem 10.9, pp.120-123] we have $\kappa(S_{w'}) \geqslant 0$; on the other hand, p being a finite morphism, [Uen75, Lemma 6.3, pp. 66-67] implies that $\kappa(S_{w'}) \leqslant \kappa(V_{w'}) = 0$ for $w' \in W'$ general, hence $\kappa(S_{w'}) = 0$ pour w' general. Again by [Uen75, Theorem 10.9, pp.120-123], $S_{w'}$ is a translate of a subtorus of T for w' general (in particular, $S_{w'}$ is isomorphic to a complex torus for w' general). Therefore $\{S_{w'}\}_{w' \in W'} \subseteq T \times W'$ forms an analytic family of complex varieties over W' whose general fibre is isomorphic to a complex torus; but T has only countably many subtori, hence there exists a subtorus S of T such that for very general w' we have $S_{w'} \simeq S$. Now by (the analytic version of) [Kaw81, Lemma 14] (applied to $f = (V' \to V \to T/S)$ and $g = \Phi_V$), this implies that we have a meromorphic mapping $q' : W' \dashrightarrow T/S$; but W' is smooth, then by Lemma 3.2.2 the meromorphic mapping q' is everywhere defined, hence $q' \circ \Phi_V$ is equal to the composition morphism $V' \to V \xrightarrow{p} T \xrightarrow{\text{quotient}} T/S$.

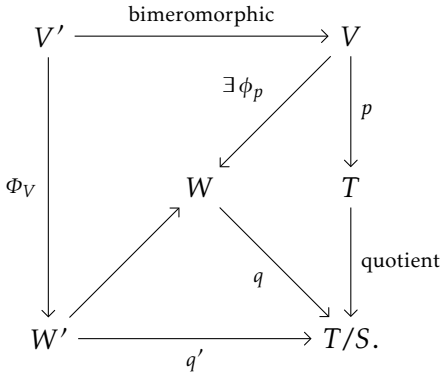
Note $W'_0 = q'(W') =$ image of V in T/S. Since we have

$$\dim W' = \dim V' - \dim V'_w = \dim p(V) - \dim S_w = \dim W'_0,$$

q' is generically finite. Let us consider a Stein factorization of q' given by $q : W \to T/S$ a finite morphism and $W' \to W$ an analytic fibre space; in addition, W is normal by our construction. Since q' is generically finite, $W' \to W$ is a fortiori bimeromorphic, in particular we have

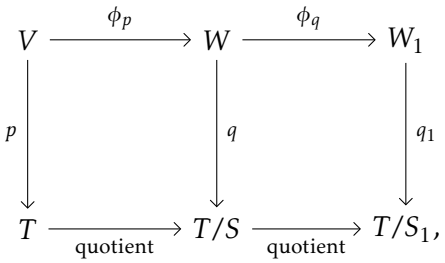
$$\dim W = \dim W' = \kappa(V). \tag{3.30}$$

By construction $q : W \to T/S$ also gives the connected part of the Stein factorization of the proper morphism $V' \xrightarrow{\Phi_V} W' \xrightarrow{q'} T/S$ since $\Phi_{V*}\mathscr{O}_{V'} = \mathscr{O}_{W'}$; $V' \to V$ being a bimeromorphic morphism, the fibres of the morphism $V' \to V$ are connected, hence they are contracted by $V' \xrightarrow{\Phi_V} W' \to W$. By [Debo1, §1.3, Lemma 1.15, pp.12-13] there is a morphism $\phi_p : V \to W$ such that $q \circ \phi_p$ is equal to the morphism $V \xrightarrow{p} T \to T/S$, i.e. the following diagram is commutative:



Moreover, since $V' \to V$ is bimeromorphic, Zariski's Main Theorem (c.f. [Uen75, Corollary 1.14, p. 12]) implies that $\phi_{p*}\mathscr{O}_V = \mathscr{O}_W$, hence ϕ_p is an analytic fibre space; by our construction ϕ_p and q provide us with the Stein factorization of the proper morphism $V \to T \to T/S$. In order to prove (a) it suffices to apply [Kaw81, Theorem 22], which is an analytic version of [KV80, Main Theorem]. In fact, since $\kappa(V_w) = 0$ for $w \in W$ general ($W' \to W$ bimeromorphic), [Kaw81, Theorem 22] implies that the finite surjective morphism $p|_{V_w} : V_w \to p(V_w) = S_w \simeq S$ is a finite étale cover, hence V_w is isomorphic to a (disjoint) union of copies of $\tilde{S}$ with $\tilde{S}$ a complex torus admitting a finite étale cover over S; V_w being connected, we must have $V_w \simeq \tilde{S}$. In other words, ϕ_p is an analytic fibre space whose general fibre equals to $\tilde{S}$. Let us remark that one can further prove that ϕ_p a principle $\tilde{S}$-bundle, for this it suffices to apply [AS60, Theorem 8] which ensures that the deformation of a complex torus is still a complex torus.

In order to establish (b), it remains, by virtue of (3.30), to show that that W is of general type, i.e. $\kappa(W) = \dim W$. To see this, we will follow the same argument as in [Uen75, Proof of Theorem 10.9, p. 122]. Assume by contradiction that $\kappa(W) < \dim W$, then one can apply the above argument to the finite morphism $q : W \to T/S$ and get the following commutative diagram

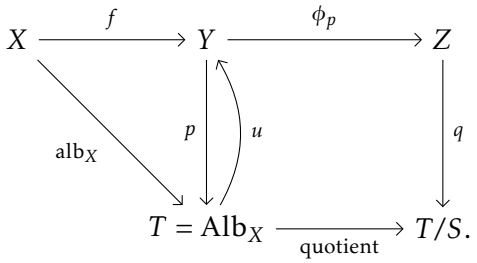


where $\dim W_1 = \kappa(W) < \dim W$, S_1 is a subtorus of T containing S, ϕ_q is an analytic fibre space whose general fibre is equal to $\tilde{S}_1$, a complex torus admitting a finite étale cover over S_1/S, and q_1 is a finite morphism. Then $\phi_q \circ \phi_p : V \to W_1$ is an analytic fibre space whose general fibre is denoted by F. By construction F admits a finite morphism $F \to S_1$, thus F is Kähler and by (a) we have $\kappa(F) \geqslant 0$. Moreover, we have an analytic fibre space $\phi_p|_F : F \to \tilde{S}_1$ whose general fibre is equal to $\tilde{S}$. The canonical bundle $K_{\tilde{S}_1}$ being trivial, consider the relative Bergman kernel metric $h_{F/\tilde{S}_1}$ on $K_F \simeq K_{F/\tilde{S}_1}$ (c.f. §2.2.1). Since $K_{F_t} \simeq K_{\tilde{S}} \simeq \mathscr{O}_{\tilde{S}}$ is trivial for general $t \in \tilde{S}_1$, then by (2.2) and by the Riemann extension theorem, the local weight of $h_{F/\tilde{S}_1}$ is a constant psh function, hence $(K_F, h_{F/\tilde{S}_1})$ is an Hermitian flat line bundle. Consequently we have $\kappa(F) \leqslant 0$ by [Uen75, Example 5.4.3, p. 52], hence $\kappa(F) = 0$. By the easy inequality [Uen75, Theorem 5.11, pp. 59-60] we have

$$\kappa(V) \leqslant \kappa(F) + \dim W_1 = \dim W_1 < \dim W = \kappa(V),$$

which is absurd. Therefore we must have $\kappa(W) = \dim W = \kappa(V)$. □

Proof of Theorem E. Let us consider the Stein factorization of the Albanese map of X given by $f : X \to Y$ an analytic fibre space and $p : Y \to T := \mathrm{Alb}_X$ a finite morphism. Then by Proposition 3.2.1, one can find a subtorus S of T and a projective variety Z of general type which admits a finite morphism $q : Z \to T/S$ such that there is an Kähler fibre space $\phi_p : Y \to Z$ whose general fibre $\tilde{S}$ is a complex torus, which is a finite étale cover over S.



Since Z is of general type, apply Theorem 3.1.2 as well as the easy inequality [Uen75, Theorem 5.11, pp. 59-60] to the Kähler fibre space $f \circ \phi_p : X \to Z$ and we get:

$$0 = \kappa(X, K_X + \Delta) = \kappa(X_z, K_{X_z} + \Delta_z) + \dim Z \geqslant \dim Z,$$

where $z \in Z$ is a general point and $\Delta_z := \Delta|_{X_z}$. Hence Z must be a singleton. In consequence $Y = \tilde{S}$ is a complex torus. By the universal property of the Albanese map, we obtain a unique morphism $u : T \to Y$ of complex tori, such that $u \circ \mathrm{alb}_X = f$ (up to change the base point of alb_X); in particular, the fibres of alb_X are connected, hence alb_X is also an analytic fibre space, hence a Kähler fibre space, thus proves Theorem E. Let us remark that alb_X being an analytic fibre space, then so is p (all its fibres are connected); p is thus a fortiori an isomorphism by Zariski's Main Theorem (c.f. [Uen75, Theorem 1.11, pp. 9-10]). □

3.3 Pluricanonical version of the structure theorem for cohomology jumping loci

In this section we will prove Theorem F by combining the Covering Lemma 1.1.1 and the main result in [Wan16a]. First let us recall some notions: let V be a complex manifold, and let $\mathscr{F}$ be a coherent sheaf on V, for every $k > 0$ denote

$$V^i_k(\mathscr{F}) := \left\{ \rho \in \mathrm{Pic}^0(V) \,\middle|\, \mathrm{h}^i(V, \mathscr{F} \otimes \rho) \geqslant k \right\},$$

the "k-th jumping locus of the i-th cohomology". With the help of the Poincaré line bundle on $V \times \mathrm{Pic}^0(V)$, one can express this as the locus where a certain coherent sheaf (in fact, some higher direct image sheaf) of $\mathrm{Pic}^0(V)$ has rank $\geqslant k$, hence $V^i_k(\mathscr{F})$ is a closed analytic subspace of $\mathrm{Pic}^0(V)$. The study of the cohomology jumping loci was initiated in the works of Green-Lazarsfeld [GL87; GL91] where they treat the case $\mathscr{F} = \mathscr{O}_V$. When $\mathscr{F} = \Omega^p_V$ for V a smooth projective variety (resp. a compact Kähler manifold) these cohomology jumping loci are described by the result of Simpson [Sim93] (resp. of Wang [Wan16a]). Now let everything be as in Theorem F, then as mentioned above, the case $g = \mathrm{id}_X$, $m = 1$ and $\Delta = 0$ has been proved in [Wan16a]. In the sequel we will follow the ideas in [CKP12; HPS18] to deduce Theorem F from this special case. First let us reduce to the proof of Theorem F to a "key lemma".

Reduction to the Key Lemma 3.3.1. The idea of the proof is similar to that of [HPS18, Theorem 10.1]. In fact, when $\Delta = 0$, Theorem F is nothing but the Kähler version of [HPS18, Theorem 10.1]; moreover, as in [HPS18] the theorem is proved by a Baire category theorem argument combined with the following "key lemma":

Lemma 3.3.1 (Key Lemma). *Every irreducible component of* $V_k^0\left(g_*\left(K_X^{\otimes m}\otimes\mathscr{O}_X(m\Delta)\right)\right)$ *is a union of torsion translates of subtori in* $\mathrm{Pic}^0(Y)$.

Assuming that Key Lemma 3.3.1 is true, let us prove Theorem F. Since $\mathrm{Pic}^0(Y)$ is compact, the jumping locus

$$V_k^0\left(g_*\left(K_X^{\otimes m}\otimes\mathscr{O}_X(m\Delta)\right)\right),$$

as a closed analytic subspace of $\mathrm{Pic}^0(Y)$, has only finite many irreducible components, thus it suffices to prove that every irreducible component of

$$V_k^0\left(g_*\left(K_X^{\otimes m}\otimes\mathscr{O}_X(m\Delta)\right)\right)$$

is a torsion translate of a subtorus. Let Z be a irreducible component of

$$V_k^0\left(g_*\left(K_X^{\otimes m}\otimes\mathscr{O}_X(m\Delta)\right)\right).$$

By the Key Lemma 3.3.1, Z is a union of torsion translates of subtori. Then by the following Lemma 3.3.2, Theorem F is proved. □

Lemma 3.3.2. *Let Z be a analytic subvariety of a complex torus T. Suppose that Z is a union of torsion translates of subtori of T. Then Z itself is a torsion translate of a subtorus of T.*

Proof. Since T has only countably many subtori (c.f. [BL04, Chapter 1, Exercise (1-b), p. 20]) and countably many torsion points, hence the set of torsion translates of subtori is countable, then by hypothesis we can write $Z=\bigcup_{n\in\mathbb{N}}E_n$ with each E_n being a torsion translate of a subtorus of T. By the Baire category theorem (Z is (locally) compact, hence it is a Baire space: every countable union of closed subsets of empty interior is of empty interior), there is one E_n, say E_1, which dominates Z, a fortiori $Z=E_1$. □

The following two subsections will be dedicated to the proof of the "key lemma".

Remark 3.3.3. Remark that in order to prove Key Lemma 3.3.1 it suffices to show that every point of

$$V_k^0\left(g_*\left(K_X^{\otimes m}\otimes\mathscr{O}_X(m\Delta)\right)\right)$$

is in a torsion translate of a subtorus contained in $V_k^0\left(g_*\left(K_X^{\otimes m}\otimes\mathscr{O}_X(m\Delta)\right)\right)$. In fact, assume this to be true, and let Z be an irreducible component of

$$V_k^0\left(g_*\left(K_X^{\otimes m}\otimes\mathscr{O}_X(m\Delta)\right)\right),$$

with Z_0 be the dense (analytic Zariski) open subset of Z complementary to the other irreducible components of

$$V_k^0\left(g_*\left(K_X^{\otimes m}\otimes\mathscr{O}_X(m\Delta)\right)\right);$$

then Z_0 is contained in a union of torsion translates of subtori: $Z_0\subseteq\bigcup_\lambda E_\lambda$, with each $E_\lambda\subseteq Z$ being a torsion translate of a subtorus. Hence $Z=\bigcup_\lambda E_\lambda$ by the density of Z_0. By Lemma 3.3.2 we get Key Lemma 3.3.1.

3.3.1 Result of Wang and reduction to the case $g=\mathrm{id}$

In this subsection we consider the case where $m=1$ and $\Delta=0$, this is also the case considered by Simpson and Wang. In particular, if $g=\mathrm{id}$, Theorem F is proved by Botong Wang in [Wan16a]; effectively, he proves the more general:

Proposition 3.3.4 ([Wan16a, Corollary 1.4]). *Let V a compact Kähler manifold, then each $V_k^i(\Omega_V^p)$ is a finite union of torsion translates of subtori in* $\mathrm{Pic}^0(V)$.

In the sequel we shall concentrate on the case $i = 0$, as in Theorem F. For every integer $k > 0$ and for every coherent sheaf $\mathscr{F}$ on X, by the projection formula we have:

$$\begin{aligned} V_k^0(g_*\mathscr{F}) &= \left\{\rho \in \mathrm{Pic}^0(Y) \,\middle|\, \mathrm{h}^0(Y, g_*\mathscr{F} \otimes \rho) \geqslant k\right\} = \left\{\rho \in \mathrm{Pic}^0(Y) \,\middle|\, \mathrm{h}^0(X, \mathscr{F} \otimes g^*\rho) \geqslant k\right\} \\ &= (g^*)^{-1}\left(V_k^0(\mathscr{F}) \cap \mathrm{Im}\, g^*\right) \end{aligned} \tag{3.31}$$

where $g^* : \mathrm{Pic}^0(Y) \to \mathrm{Pic}^0(X)$ is the morphism of complex tori given by $L \mapsto g^*L$. Then the following lemma permit us to reduce to the case $g = \mathrm{id}$:

Lemma 3.3.5. *Let $\alpha : T_1 \to T_2$ a morphism of complex tori. Let $t \in T_2$ a torsion point and $S \subseteq T_2$ a subtorus. Then $\alpha^{-1}(t + S)$ is also a torsion translate of a subtorus in T_1.*

Proof. By [Deb99, §1.2, Théorème 2.3, p. 7] α can be factorized as

$$T_1 \xrightarrow{\text{quotient}} T_1/(\mathrm{Ker}\,\alpha)^0 \xrightarrow[\text{isogeny}]{\bar{\alpha}} T_1/\mathrm{Ker}\,\alpha = \mathrm{Im}\,\alpha \xhookrightarrow{\text{inclusion}} T_2\,.$$

Thus it suffices to prove the lemma in the following three cases:

- α is the quotient by a subtorus,
- α is an isogeny,
- α is the inclusion of a subtorus.

Each of theses cases can be done by elementary linear algebra. We nevertheless give the details for the convenience of the readers.

Case 1: α is the quotient $T \to T/T'$ with $T' \subseteq T$ being a subtorus.
Let $\bar{t} \in T/T'$ be a torsion point such that $m\bar{t} = 0$ in T/T' with $m \in \mathbb{Z}_{>0}$, and let $S \subseteq T/T'$ be a subtorus. Then

$$\alpha^{-1}(\bar{t} + S) = \alpha^{-1}(\bar{t}) + \alpha^{-1}S = t + T' + \alpha^{-1}S\,.$$

In addition, $m\bar{t} = 0$ in T/T' signifies that $mt \in T'$. Since complex tori are divisible, there is $t' \in T'$ such that $mt' = mt$. Hence $t + T' = (t - t') + T'$ with $m(t - t') = 0$ in T, i.e. $t - t'$ is a torsion point in T. In consequence we have

$$\alpha^{-1}(\bar{t} + S) = (t - t') + T' + \alpha^{-1}S = (t - t') + \alpha^{-1}S\,,$$

where $\alpha^{-1}S$ is a subtorus of T and $t - t'$ a torsion point.

Case 2: α is the isogeny $T_1 \to T_2$ of degree n.
Let t be a torsion pint of T_2 and S a subtorus of T_2, then we have $\alpha^{-1}(t + S) = \alpha^{-1}(t) + \alpha^{-1}S$ with

$$\begin{aligned} \alpha^{-1}S &= \text{finite union of subtori} = \bigcup_{i=1}^{n'} S_i \\ \alpha^{-1}(t) &= n \text{ distinct points } = \{t_1, \cdots, t_n\} \end{aligned}$$

where $n' | n$. Since t is a torsion point, there is $m \in \mathbb{Z}_{>0}$ such that $mt = 0$ in T_2, then for $j = 1, \cdots, n$ we have $m\alpha(t_j) = \alpha(mt_j) = 0$, i.e. $mt_j \in \mathrm{Ker}\,\alpha$. But $\#(\mathrm{Ker}\,\alpha) = n$, we must have $n \cdot \mathrm{Ker}\,\alpha = 0$, in particular $nmt_j = 0$, $\forall j$, hence the t_j's are all torsion points. In consequence,

$$\alpha^{-1}(t + S) = \bigcup_{i,j} (t_j + S_i)$$

is a finite union of torsion translates of subtori.

Case 3: α is the inclusion of a subtorus $S_1 \hookrightarrow T$.

Let S_2 be a subtorus of T and we will show that $S_1 \cap (t + S_2)$ is a torsion translate of a subtorus in S_1. Write $T = \mathbb{C}^g/\Gamma$ with $g = \dim T$ and $\Gamma \subseteq \mathbb{C}^g$ a lattice. By [Deb99, Exercice 1.1, pp.9] or [BL04, §1.5, Exercise(1)], there are subgroups $\Gamma_1, \Gamma_2 \subseteq \Gamma$, such that for $i = 1, 2$, Γ_i is stable under multiplication by $\sqrt{-1}$ and generates S_i with $\operatorname{rg}\Gamma_i = 2h_i$ ($h_i = \dim S_i$). For $i = 1, 2$, let W_i be the vector subspace of $\mathbb{C}^g$ generated by Γ_i, then $W_1 \cap W_2$ is generated by $\Gamma_1 \cap \Gamma_2$. Set $k := \dim_{\mathbb{C}}(W_1 \cap W_2)$, then $\operatorname{rg}(\Gamma_1 \cap \Gamma_2) = 2k$. We are then reduced to show that $W_1 \cap (u + W_2)$ (where u is a representative of t in $\mathbb{C}^g$) is of the form $u' + W'$ with $u' + \Gamma_1$ a torsion point in $S_1 = W_1/\Gamma_1$ and W' a vector subspace of W_1.

Now let us choose a basis $\{v_i \,|\, i = 1, \cdots, 2g\}$ of Γ such that $v_1, \cdots, v_{2k}$ form a basis of $\Gamma_1 \cap \Gamma_2$, $v_{2k+1}, \cdots, v_{2h_1}$ generate a supplementary of $\Gamma_1 \cap \Gamma_2$ in Γ_1, $v_{2h_1+1}, \cdots, v_{2(h_1+h_2-k)}$ generate a supplementary of $\Gamma_1 \cap \Gamma_2$ in Γ_2, and $v_{2(h_1+h_2-k)+1}, \cdots, v_{2g}$ generate a supplementary of $\Gamma_1 + \Gamma_2$ in Γ. Note that the v_i's form a $\mathbb{R}$-basis of $\mathbb{C}^g$. If $W_1 \cap (u + W_2) = \varnothing$, then there is nothing to prove, hence we suppose that $W_1 \cap (u + W_2) \neq \varnothing$, i.e. $\exists w \in W_2$ such that $u + w \in W_1$, then $W_1 \cap (u + W_2) = (u + w) + W_1 \cap W_2$; in addition, we have a fortiori $u \in W_1 + W_2$. Since $w \in W_2$ is determined up to $W_1 \cap W_2$, we can assume that the projection of w in $W_1 \cap W_2$ equals 0, i.e. we can write

$$w = \sum_{i=2h_1+1}^{2(h_1+h_2-k)} w_i v_i\,, \quad w_i \in \mathbb{R}.$$

And let us write (noting that $u \in W_1 + W_2$):

$$u = \sum_{i=1}^{2(h_1+h_2-k)} u_i v_i\,.$$

But $u + w \in W_1$, a fortiori we have $u_i + w_i = 0$ for $i = 2h_1 + 1, \cdots, 2(h_1 + h_2 - k)$. Since $t \in T = \mathbb{C}^g/\Gamma$ is a torsion point, there is $m \in \mathbb{Z}_{>0}$ such that $mt = 0$ in T, meaning that $mu \in \Gamma$, hence $mu_i \in \mathbb{Z}$ for $i = 1, \cdots, 2(h_1 + h_2 - k)$, and hence $mw_i \in \mathbb{Z}$ for $i = 2h_1, \cdots, 2(h_1 + h_2 - k)$. In consequence $mw \in \Gamma_2$, $m(u + w) \in \Gamma_1$, therefore $(u + w) + \Gamma_1$ is a trosion point in $S_1 = W_1/\Gamma_1$. □

In particular we obtain immediately:

Proposition 3.3.6. *Let $g : X \to Y$ a morphism between compact Kähler manifolds. Then for every $k > 0$ and for every $0 \leqslant p \leqslant n$, $V^0_k(g_*\Omega^p_X)$ is a finite union of torsion translates of subtori in $\operatorname{Pic}^0(Y)$.*

3.3.2 Proof of the "Key Lemma"

Let us now turn to the proof of Key Lemma 3.3.1. It proceeds in four steps:

(A) Reduction to the case $g = \mathrm{id}$.

First apply the formula (3.31) to $\mathscr{F} = K_X^{\otimes m} \otimes \mathscr{O}_X(m\Delta)$ and then by Lemma 3.3.5 we see that Key Lemma 3.3.1 is true for $V^0_k\left(g_*\left(K_X^{\otimes m} \otimes \mathscr{O}_X(m\Delta)\right)\right)$ as soon as it holds for $V^0_k\left(K_X^{\otimes m} \otimes \mathscr{O}_X(m\Delta)\right)$. In consequence we can suppose that $g = \mathrm{id}$ (and $X = Y$).

(B) Case $m = 1$ and $\Delta = 0$.

This is nothing but Proposition 3.3.6 for $p = n$.

(C) Case $m = 1$ and Δ is of SNC support.

In this step, we consider the case where $m = 1$ and Δ is an effective $\mathbb{Q}$-divisor of SNC support; in addition, we do not require Δ to be an integral divisor, but only assume that it is given by a line bundle L^+, i.e. there is a line bundle L^+, $(L^+)^{\otimes N} \simeq \mathscr{O}_X(N\Delta)$ for any $N \in \mathbb{Z}_{>0}$ which makes $N\Delta$ an integral divisor. In this case, Key Lemma 3.3.1 can be deduced from Covering Lemma 1.1.1 combined with the following auxiliary result (c.f. also [Wan16a, Lemma 3.1]):

Lemma 3.3.7 (analytic version of [HPS18, Lemma 10.3]). *Let $\mathscr{F}$ and $\mathscr{G}$ be coherent sheaves on X such that $\mathscr{F}$ is a direct summand of $\mathscr{G}$. Then for $\forall i \in \mathbb{N}$ and $\forall k \in \mathbb{Z}_{>0}$, each irreducible component of $V^i_k(\mathscr{F})$ is also an irreducible component of $V^i_l(\mathscr{G})$ for some $l \geqslant k$.*

Proof. This is simply a result of Grauert's semi-continuity theorem (c.f. [BS76, §III.4, Theorem 4.12(i), p. 134]) □

Now let L^+ be the line bundle given by Δ. Since (X, Δ) is a klt pair, then $\lfloor \Delta \rfloor = 0$. Moreover, Δ being a $\mathbb{Q}$-divisor of SNC support, then for any N making $N\Delta$ an integral divisor, we can construct, by Covering Lemma 1.1.1, a generically finite morphism $f : V \to X$ of compact Kähler manifolds such that

$$f_* K_V \simeq \bigoplus_{i=0}^{N-1} K_X \otimes (L^+)^{\otimes i} \otimes \mathscr{O}_X(-\lfloor i\Delta \rfloor).$$

By Lemma 3.3.7 each irreducible component of $V^0_k(K_X \otimes L^+)$ is also a irreducible component of a certain $V^0_l(f_* K_V)$ for some $l > 0$. Then by Step (B) (or Proposition 3.3.6), every irreducible component of $V^0_k(K_X \otimes L^+)$ is a torsion translate of a subtorus in $\mathrm{Pic}^0(X)$.

(D) General case.

In order to prove the general case we use a reduction to the case of Step (C). This reduction process is inspired by[CKP12, §1.A-1.C], whose idea has already appeared in [Bud09]. Let L be a point in

$$V^0_k\left(K_X^{\otimes m} \otimes \mathscr{O}_X(m\Delta)\right) \subseteq \mathrm{Pic}^0(X),$$

we will prove in the sequel that there exists a torsion translate of a subtorus contained in

$$V^0_k\left(K_X^{\otimes m} \otimes \mathscr{O}_X(m\Delta)\right)$$

which contains L. $\mathrm{Pic}^0(X)$ being complex torus, thus divisible, we can then write $L = mL_0 = L_0^{\otimes m}$ for some $L_0 \in \mathrm{Pic}^0(Y)$. Then we have $h^0(X, L_{m,\Delta}) \geqslant k$, where

$$L_{m,\Delta} := K_X^{\otimes m} \otimes \mathscr{O}_X(m\Delta) \otimes L_0^{\otimes m}.$$

Now take a log resolution $\mu : X' \to X$ for both Δ and the linear series $\left|L_{m,\Delta}\right|$. Then we can write

$$K_{X'}^{\otimes m} \otimes \mathscr{O}_{X'}(m\Delta') \simeq \mu^*(K_X^{\otimes m} \otimes \mathscr{O}_X(m\Delta)) \otimes \mathscr{O}_{X'}(\sum_{i \in I^+} m a_i E_i), \tag{3.32}$$
$$\mu^* \left|L_{m,\Delta}\right| = \left|\mu^* L_{m,\Delta}\right| = F_{m,\Delta} + \left|M_{m,\Delta}\right|,$$

where:

- $\left\{E_i \,\middle|\, i \in I\right\}$ denotes the set of μ-exceptional prime divisors, and

$$a_i := a(E_i, X, \Delta)$$

denotes the discrepancy of E_i with respect to the pair (X, Δ); I^+ (resp. I^-) is the set of indices i such that $a_i > 0$ (resp. $a_i < 0$).

- Δ' is the effective $\mathbb{Q}$-divisor on X' as in the proof of Lemma 1.1.4, i.e.

$$\Delta' := \mu_*^{-1}\Delta - \sum_{i \in I^-} a_i E_i\,,$$

by Lemma 1.1.4 the pair (X', Δ') is also klt.

- $F_{m,\Delta}$ (resp. $M_{m,\Delta}$) is the fixed part (resp. mobile part) of the linear series $\mu^*\left|L_{m,\Delta}\right|$; by construction, $\left|M_{m,\Delta}\right|$ is base point free.

By construction (μ being a log resolution of Δ and of $\left|L_{m,\Delta}\right|$), $m\Delta' + \sum_{i\in I} E_i$ and $F_{m,\Delta} + \sum_{i\in I} E_i$ are (integral) divisors of SNC support. Let H be a general member in $\left|M_{m,\Delta}\right|$, then H has no common component either with $F_{m,\Delta}$ or with $\sum_{i\in I} E_i$ or with Δ'; by Bertini's theorem, H is smooth (in particular H is reduced), $H + F_{m,\Delta} + \sum_{i\in I} E_i$ is of SNC support. Set

$$F'_{m,\Delta} := F_{m,\Delta} + \sum_{i\in I^+} m a_i E_i\,,$$
$$L'_{m,\Delta} := K_{X'}^{\otimes m} \otimes \mathscr{O}_{X'}(m\Delta') \otimes \mu^* L_0^{\otimes m}\,.$$

Then we have

$$L'_{m,\Delta} \simeq \mu^* L_{m,\Delta} \otimes \mathscr{O}_X(\sum_{i\in I^+} m a_i E_i),$$

thus

$$\left|L'_{m,\Delta}\right| = \left|M_{m,\Delta}\right| + F'_{m,\Delta}.$$

By [Debo1, Lemma 7.11, p. 175] we have

$$\mathrm{H}^0(X, L'_{m,\Delta}) \simeq \mathrm{H}^0(X, L_{m,\Delta}) \simeq \mathrm{H}^0(X', \mathscr{O}_X(M_{m,\Delta})),$$

hence $F'_{m,\Delta}$ is equal to the fixed part of the linear series $\left|L'_{m,\Delta}\right|$ and by construction it has SNC support.

Put

$$\mu_*^{-1}\Delta := \sum_{j\in J} d_j D_j\,, \quad d_j \in \mathbb{Q}_{>0},$$
$$b_j := \text{coefficient of } D_j \text{ in } F_{m,\Delta}\,, \quad j \in J\,,$$
$$b_i := \text{coefficient of } E_i \text{ in } F_{m,\Delta}\,, \quad i \in I^-\,,$$

and take

$$\bar{\Delta} := \Delta' - \sum_{j\in J} \min(d_j, \frac{b_j}{m}) D_j - \sum_{i\in I^-} \min(-a_i, \frac{b_i}{m}) E_i\,,$$
$$\bar{F}_{m,\Delta} := F'_{m,\Delta} - \sum_{j\in J} \min(m d_j, b_j) D_j - \sum_{i\in I^-} \min(-m a_i, b_i) E_i\,,$$

so that $\bar{\Delta}$ and $\bar{F}_{m,\Delta}$ have no common component. We see that $\bar{\Delta} \leqslant \Delta'$, $\bar{F}_{m,\Delta} \leqslant F'_{m,\Delta}$. Now consider the line bundle

$$\bar{L}_{m,\Delta} := K_{X'}^{\otimes m} \otimes \mathscr{O}_{X'}(m\bar{\Delta}) \otimes \mu^* L_0^{\otimes m},$$

then the same argument as above shows that $\bar{F}_{m,\Delta}$ equals the fixed part of the linear series $\left|\bar{L}_{m,\Delta}\right|$, hence we have

$$\left|\bar{L}_{m,\Delta}\right| = \bar{F}_{m,\Delta} + \left|M_{m,\Delta}\right|.$$

In addition we have

$$\begin{aligned}\bar{L}_{m,\Delta} \otimes \mathscr{O}_{X'}(-\Big\lfloor \frac{m-1}{m} \bar{F}_{m,\Delta} \Big\rfloor) &= K_{X'}^{\otimes m} \otimes \mathscr{O}_{X'}(m\bar{\Delta}) \otimes \mu^* L_0^{\otimes m} \otimes \mathscr{O}_{X'}(-\Big\lfloor \frac{m-1}{m} \Big(\bar{F}_{m,\Delta} + H\Big) \Big\rfloor) \\ &\simeq K_{X'} \otimes \mathscr{O}_{X'}(\Delta^+) \otimes \mu^* L_0\end{aligned}$$

where the $\mathbb{Q}$-divisor Δ^+ is defined by

$$\Delta^+ := \bar{\Delta} + \Big\{ \frac{m-1}{m} \Big(\bar{F}_{m,\Delta} + H\Big) \Big\}.$$

Since H has no common component with either $\bar{\Delta}$ or $\bar{F}_{m,\Delta}$, hence

$$\Delta^+ = \bar{\Delta} + \Big\{ \frac{m-1}{m} \bar{F}_{m,\Delta} \Big\} + \frac{m-1}{m} H;$$

but H is reduced, $\bar{\Delta}$ and $\bar{F}_{m,\Delta}$ have no common components, then the coefficients of the irreducible components in Δ^+ are all < 1; since Δ^+ is of SNC support, then [KM98, Corollary 2.31(3), p. 53] implies that the pair (X', Δ^+) is klt. A priori $\mathscr{O}_{X'}(\Delta^+)$ is only a $\mathbb{Q}$-line bundle, but by our construction Δ^+ is given by a line bundle

$$L^+ := \mathscr{O}_{X'}(\Delta^+) = \bar{L}_{m,\Delta} \otimes \mathscr{O}_{X'}(-\Big\lfloor \frac{m-1}{m} \bar{F}_{m,\Delta} \Big\rfloor) \otimes K_{X'}^{-1} \otimes \mu^* L_0^{-1}.$$

Moreover, we have

$$\mathrm{h}^0(X', K_{X'} \otimes L^+ \otimes \mu^* L_0) = \mathrm{h}^0(X', \bar{L}_{m,\Delta} \otimes \mathscr{O}_{X'}(-\Big\lfloor \frac{m-1}{m} \bar{F}_{m,\Delta} \Big\rfloor)) \geqslant \mathrm{h}^0(X', M_{m,\Delta}) \geqslant k,$$

which means that $\mu^* L_0 \in V_k^0(K_{X'} \otimes L^+)$. Let W' be an irreducible component $V_k^0(K_{X'} \otimes L^+)$ containing $\mu^* L_0$. By Step (C) W' is a torsion translate of subtorus, then we can write $W' = \beta_{\mathrm{tor}} + T'_0$ with β_{tor} a torsion point in $\mathrm{Pic}^0(X')$ and T'_0 a subtorus, in particular $\mu^* L_0$ can be written as the sum of β_{tor} and an element of T'_0, thus

$$(m-1)\mu^* L_0 + W' = m\beta_{\mathrm{tor}} + T'_0$$

is also a torsion translate of a subtorus as $m\beta_{\mathrm{tor}}$ is also a torsion point of $\mathrm{Pic}^0(X')$. In addition, $(m-1)\mu^* L_0 + W'$ contains $\mu^* L = m\mu^* L_0$ as $\mu^* L_0 \in W'$. It remains to see that $(m-1)\mu^* L_0 + W'$ is contained in $V_k^0\Big(K_{X'}^{\otimes m} \otimes \mathscr{O}_{X'}(m\Delta')\Big)$. In fact, for every $\alpha \in W'$, we have (since $W' \subseteq V_k^0(K_{X'} \otimes L^+)$):

$$\begin{aligned}\mathrm{h}^0(X', K_{X'}^{\otimes m} \otimes \mathscr{O}_{X'}(m\Delta') \otimes \mu^* L_0^{\otimes(m-1)} \otimes \alpha) &\geqslant \mathrm{h}^0(X', K_{X'}^{\otimes m} \otimes \mathscr{O}_{X'}(m\bar{\Delta}) \otimes \mu^* L_0^{\otimes(m-1)} \otimes \alpha) \\ &= \mathrm{h}^0(X', \bar{L}_{m,\Delta} \otimes \mu^* L_0^{-1} \otimes \alpha) \\ &\geqslant \mathrm{h}^0(X', \bar{L}_{m,\Delta} \otimes \mathscr{O}_{X'}(-\Big\lfloor \frac{m-1}{m} \bar{F}_{m,\Delta} \Big\rfloor) \otimes \mu^* L_0^{-1} \otimes \alpha) \\ &= \mathrm{h}^0(X', K_{X'} \otimes L^+ \otimes \alpha) \geqslant k.\end{aligned}$$

Therefore $(m-1)\mu^*L_0 + W' \subseteq V_k^0\left(K_{X'}^{\otimes m} \otimes \mathscr{O}_{X'}(m\Delta')\right)$.

In virtue of the isomorphism (3.32) we have

$$\begin{aligned} V_k^0\left(K_X^{\otimes m} \otimes \mathscr{O}_X(m\Delta)\right) &= \left\{\rho \in \mathrm{Pic}^0(X) \,\middle|\, \mathrm{h}^0(X, K_X^{\otimes m} \otimes \mathscr{O}_X(m\Delta) \otimes \rho) \geqslant k\right\} \\ &= \left\{\rho \in \mathrm{Pic}^0(X) \,|\, \mathrm{h}^0(X', \mu^*\left(K_X^{\otimes m} \otimes \mathscr{O}_X(m\Delta) \otimes \rho\right)) \geqslant k\right\} \\ &= \left\{\rho \in \mathrm{Pic}^0(X) \,|\, \mathrm{h}^0(X', K_{X'}^{\otimes m} \otimes \mathscr{O}_{X'}(m\Delta') \otimes \mu^*\rho) \geqslant k\right\} \\ &= (\mu^*)^{-1}\left(V_k^0\left(K_{X'}^{\otimes m} \otimes \mathscr{O}_{X'}(m\Delta')\right) \cap \mathrm{Im}\,\mu^*\right), \end{aligned}$$

where the third equality is a consequence of [Debo1, Lemma 7.11, p. 175]. Hence by Lemma 3.3.5,

$$W := (\mu^*)^{-1}\left(((m-1)\mu^*L_0 + W') \cap \mathrm{Im}\,\mu^*\right)$$

is a torsion translate of a subtorus contained in $V_k^0\left(K_X^{\otimes m} \otimes \mathscr{O}_X(m\Delta)\right)$ and $L = mL_0 \in W$. This proves the Key Lemma 3.3.1.

Remark 3.3.8. If X is a smooth projective variety, then one can prove Theorem F for log canonical pair (X, Δ) as follows:

- First apply [BW15, Theorem 1.1] along with [Voi02, Théorème 8.35(ii), p. 201] to prove the Key Lemma 3.3.1 (thus also Theorem F) for $m = 1$ and Δ a reduced SNC divisor (c.f. also [Kaw13]);
- Then by [CKP12, Lemma 2.1] and Lemma 1.1.1 one can deduce further the Key Lemma 3.3.1 for the case of $m = 1$ and Δ a log canonical $\mathbb{Q}$-divisor of SNC support, which is given by a line bundle, but is not necessarily an integral divisor;
- Finally one can follow the same argument as in Step (D) above to prove the Key Lemma and thus Theorem F.

As for the Kähler case, as soon as [BW17, Conjecture 1.2] is solved, one can prove Theorem F for log canonical pair (X, Δ).

3.3.3 Kähler version of a result of Campana-Koziarz-Păun

Before ending this section, let us prove the following significant corollary of Theorem F, which generalizes a result of Campana, Koziarz and Păun to the Kähler case, and will be used in the proof of the Theorem A(II). In the algebraic case, it is proved in [CP11, Theorem 3.1] for $\Delta = 0$, and in [CKP12, Theorem 0.1] for Δ log canonical.

Corollary 3.3.9. *Let (X, Δ) a klt pair with X a Kähler manifold, and let L_0 a numerically trivial line bundle on X, i.e. $L_0 \in \mathrm{Pic}^0(X)$. Then*

(a) $\kappa(X, K_X + \Delta) \geqslant \kappa(X, mK_X + m\Delta + L_0)$, $\forall m \in \mathbb{Z}_{>0}$. *Namely, for any $\mathbb{Q}$-line bundle[1] L on X such that $c_1(L) = c_1(K_X + \Delta)$, we have $\kappa(X, K_X + \Delta) \geqslant \kappa(X, L)$.*

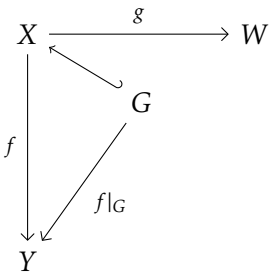
(b) *If there is an integer $m > 0$ such that $\kappa(X, K_X + \Delta) = \kappa(X, mK_X + m\Delta + L_0) = 0$, then L_0 is a torsion point in $\mathrm{Pic}^0(X)$.*

Remark 3.3.10. Before entering into the proof, let us remark that one cannot omit the condition "$\kappa(X, K_X + \Delta) = 0$" in the point (b) above. For example, if (X, Δ) is of log general type, then for any $L_0 \in \mathrm{Pic}^0(X)$ we always have $\kappa(X, K_X + \Delta) = \kappa(X, mK_X + m\Delta + L_0) = \dim X$.

[1] In fact, since $\mathrm{Pic}^0(X)$ is divisible, this a priori $\mathbb{Q}$-line bundle L is an "authentic" line bundle.

Proof of Corollary 3.3.9. We will follow the argument in [CP11] with a small simplification. First prove the point (a), the proof proceeds in three steps:

Step 1: Reduction to the case $\kappa(X, K_X \otimes \mathscr{O}_X(\Delta)) \leqslant 0$. Assuming (a) for any klt pair (X,Δ) with $\kappa(X, K_X+\Delta) \leqslant 0$, we will prove it for any klt pair (X,Δ) with $\kappa(X, K_X+\Delta) > 0$. Let $g : X \dashrightarrow W$ the Iitaka fibration (c.f. [Uen75, §5, Theorem 5.10, p. 58]) of the $\mathbb{Q}$-line bundle $K_X+\Delta$ and $f : X \dashrightarrow Y$ that of $mK_X+m\Delta+L_0$. By Lemma 1.1.4 Point (a) is preserved by log resolutions of (X,Δ), we can thus suppose that f and g are morphisms (instead of meromorphic mappings).



By construction we have $\dim Y = \kappa(X, mK_X + m\Delta + L_0)$, $\dim W = \kappa(X, K_X+\Delta)$. Denoting by F (resp. by G) the general fibre of f (resp. of g), we have

$$\kappa(X, K_X+\Delta) \geqslant \kappa(X, mK_X + m\Delta + L_0) \Leftrightarrow \dim W \geqslant \dim Y \Leftrightarrow \dim G \leqslant \dim F,$$

then it suffices to prove that G is contracted by f (i.e. $f(G) = \mathrm{pt}$). By adjunction formula the $\mathbb{Q}$-line bundle

$$K_G + \Delta_G \simeq (K_X+\Delta)|_G$$

where $\Delta_G := \Delta|_G$, hence $f|_G$ is bimeromorphically equivalent to a meromorphic mapping defined by a sub-linear series of $\left|K_G^{\otimes km} \otimes \mathscr{O}_G(km\Delta) \otimes L_0|_G^{\otimes k}\right|$ for some k sufficiently large and divisible[2]. Therefore it suffices to show

$$\kappa(G, mK_G + m\Delta + L_0|_G) = 0.$$

But by our construction

$$\kappa(G, K_G + \Delta_G) = \kappa(G, (K_X+\Delta)|_G) = 0,$$

hence our assumption implies that (a) holds for the klt pair (G, Δ_G). Since $L_0|_G \in \mathrm{Pic}^0(G)$ we have

$$\kappa(G, mK_G + m\Delta_G + L_0|_G) \leqslant \kappa(G, K_G + \Delta_G) = 0.$$

Step 2: By the previous step, we can assume that $\kappa(X, K_X+\Delta) \leqslant 0$. If $\kappa(X, mK_X + m\Delta + L_0) = -\infty$, then the inequality is automatically established, hence we can assume that $\kappa(X, mK_X + m\Delta + L_0) \geqslant 0$; in addition, up to replacing m and L_0 with some multiples, we can assume that $m\Delta$ is an integral divisor and

$$\mathrm{H}^0(X, K_X^{\otimes m} \otimes \mathscr{O}_X(m\Delta) \otimes L_0) \neq 0.$$

For every integer $k > 0$ denote

$$r_k := \mathrm{h}^0(X, K_X^{\otimes km} \otimes \mathscr{O}_X(km\Delta) \otimes L_0^{\otimes k}) > 0.$$

Then $L_0^{\otimes k} \in V^0_{r_k}\left(K_X^{\otimes km} \otimes \mathscr{O}_X(km\Delta)\right) \subseteq V^0_1\left(K_X^{\otimes km} \otimes \mathscr{O}_X(km\Delta)\right)$, thus by Theorem F, $L_0^{\otimes k} \in \beta_{\mathrm{tor}} + T_0 \subseteq V^0_{r_k}\left(K_X^{\otimes km} \otimes \mathscr{O}_X(km\Delta)\right)$ for β_{tor} a torsion point in $\mathrm{Pic}^0(X)$ and T_0 a subtorus;

[2] In the proof of [CP11, Theorem 3.1], it is said that $f|_G$ is equal to the Iitaka fibration of $mK_G+m\Delta_G+L_0|_G$; but it is not true in general.

in particular, $\beta_{\mathrm{tor}} \in V^0_{r_k}\left(K_X^{\otimes km} \otimes \mathscr{O}_X(km\Delta)\right)$. Let $m_0 > 0$ an integer such that $\beta_{\mathrm{tor}}^{\otimes m_0} \simeq \mathscr{O}_X$. Then

$$\mathrm{h}^0(X, K_X^{\otimes kmm_0} \otimes \mathscr{O}_X(kmm_0\Delta)) \geqslant \mathrm{h}^0(X, K_X^{\otimes km} \otimes \mathscr{O}_X(km\Delta) \otimes \beta_{\mathrm{tor}}) \geqslant r_k. \tag{3.33}$$

Step 3: By hypothesis we have $\kappa(X, K_X+\Delta) \leqslant 0$, hence (3.33) implies that $\kappa(X, K_X+\Delta) = 0$, which means that $r_k \leqslant 1$ for every $k \in \mathbb{Z}_{k>0}$. Therefore $\kappa(X, mK_X + m\Delta + L_0) = 0$. This proves (a).

Now turn to the proof of (b): assume by contradiction that there is a line bundle $L \in \mathrm{Pic}^0(X)$ with L non-torsion such that $\kappa(X, mK_X + m\Delta + L) = \kappa(X, K_X + \Delta) = 0$ for some $m > 0$. Up to replacing m and L with some multiples, we can assume that $m\Delta$ is an integral divisor and that $\mathrm{h}^0(X, K_X^{\otimes m} \otimes \mathscr{O}_X(m\Delta) \otimes L) = 1$, then $L \in V^0_1\left(K_X^{\otimes m} \otimes \mathscr{O}_X(m\Delta)\right)$. By Theorem F there exists $\beta_{\mathrm{tor}} \in \mathrm{Pic}^0(X)_{\mathrm{tor}}$ and T_0 a subtorus in $\mathrm{Pic}^0(X)$ such that $L \in \beta_{\mathrm{tor}} + T_0 \subseteq V^0_1\left(K_X^{\otimes m} \otimes \mathscr{O}_X(m\Delta)\right)$, then we can write $L = \beta_{\mathrm{tor}} \otimes F$ with $F \in T_0$. By our assumption L is not a torsion point in $\mathrm{Pic}^0(X)$, hence F cannot be trivial and thus T_0 is not reduced to a singleton. In consequence there is a (non-trivial) one-parameter subgroup $(F_t)_{t\in\mathbb{R}}$ in T_0 passing through F (by choosing an isomorphism $T_0 \simeq \mathbb{C}^q/\Gamma$, we can take $F_t = t \cdot F$), then for every $t \in \mathbb{R}$, $\beta_{\mathrm{tor}} \otimes F_t \in \beta_{\mathrm{tor}} + T_0 \subseteq V^0_1\left(K_X^{\otimes m} \otimes \mathscr{O}_X(m\Delta)\right)$ hence there is a non-zero section s_t in

$$\mathrm{H}^0(X, K_X^{\otimes m} \otimes \mathscr{O}_X(m\Delta) \otimes \beta_{\mathrm{tor}} \otimes F_t).$$

We claim that:

Claim $(*)$**.** There is a $t \in \mathbb{R}_{>0}$ such that the sections $s_t \otimes s_{-t}$ and $s_0^{\otimes 2}$ are not linearly independent in $\mathrm{H}^0(X, K_X^{\otimes 2m} \otimes \mathscr{O}_X(2m\Delta) \otimes \beta_{\mathrm{tor}}^{\otimes 2})$.

In fact, this leads to a contradiction: we have immediately

$$\mathrm{h}^0(X, K_X^{\otimes 2m} \otimes \mathscr{O}_X(2m\Delta) \otimes \beta_{\mathrm{tor}}^{\otimes 2}) \geqslant 2,$$

which implies that

$$\kappa(X, K_X + \Delta) = \kappa(X, K_X^{\otimes 2m} \otimes \mathscr{O}_X(2m\Delta) \otimes \beta_{\mathrm{tor}}^{\otimes 2}) \geqslant 1,$$

and this contradicts the hypothesis that $\kappa(X, K_X + \Delta) = 0$. Therefore (b) is proved.

Let us prove the Claim $(*)$. Assume by contradiction that $s_t \otimes s_{-t}$ are $s_0^{\otimes 2}$ are linearly dependent for every $t \in \mathbb{R}$. Then $\forall t \in \mathbb{R}$, $\mathrm{div}(s_t) + \mathrm{div}(s_{-t}) = 2\,\mathrm{div}(s_0)$; in particular, $\mathrm{div}(s_t) \leqslant 2\,\mathrm{div}(s_0)$ for every $t \in \mathbb{R}_{>0}$. Take ϵ sufficiently small such that $t \mapsto F_t$ is injective for $t \in]-\epsilon, \epsilon[$. By Dirichlet's drawer principle, there are $t_1, t_2 \in]0, \epsilon[$ such that $\mathrm{div}(s_{t_1}) = \mathrm{div}(s_{t_2})$, hence the divisor

$$0 = \mathrm{div}(s_{t_2}) - \mathrm{div}(s_{t_1}) \in \left|F_{t_2} \otimes F_{t_1}^{-1}\right|,$$

which implies that $F_{t_1} = F_{t_2}$ in $\mathrm{Pic}^0(X)$ with $t_1, t_2 \in]0, \epsilon[$; but this contradicts our hypothesis on ϵ. This proves Claim $(*)$. □

As a by-product of Corollary 3.3.9(a) we obtain the following special case of the Kähler version of the (generalized) log Abundance Conjecture by using the divisorial Zariski decomposition (c.f.[Bou04, Definition 3.7]):

Theorem 3.3.11. *Let (X, Δ) be a klt pair with X a compact Kähler manifold whose numerical dimension $\nu(X, K_X + \Delta) = 0$, then $\kappa(X, K_X + \Delta) = 0$.*

Proof. For the definition of the numerical dimension of (non necessarily nef) $\mathbb{Q}$-line bundles (or cohomology classes in $\mathrm{H}^{1,1}(X, \mathbb{R})$) over a compact Kähler manifold, c.f. [Dem10, §18.13, p. 198]. Since $\nu(K_X + \Delta) = 0$, the $\mathbb{Q}$-line bundle $K_X + \Delta$ is pseudoeffective, hence

we can consider the divisorial Zariski decomposition (c.f. [Bou04, Definition 3.7] and [Dem10, §18.12(d), p. 195]) of its first Chern class:

$$c_1(K_X+\Delta) = \Big\{N\Big(c_1(K_X+\Delta)\Big)\Big\} + \langle c_1(K_X+\Delta)\rangle.$$

By hypothesis $\nu(c_1(K_X+\Delta)) = 0$, which means that $\langle c_1(K_X+\Delta)\rangle = 0$; in other word, the $\mathbb{Q}$-line bundle $K_X+\Delta$ is numerically equivalent to the effective $\mathbb{R}$-divisor $N = N\Big(c_1(K_X+\Delta)\Big)$, a fortiori N is a $\mathbb{Q}$-divisor. Therefore by Corollary 3.3.9(a), we have

$$\kappa(K_X+\Delta) \geqslant \kappa(N) \geqslant 0.$$

Finally by [Dem10, §18.15, p. 199] we get $\kappa(K_X+\Delta) = 0$. □

3.4 Kähler version of $C^{\log}_{n,m}$ for fibre spaces over complex tori

In this section, we will finish the proof of Theorem A. To this end, we do some reductions by an induction on the dimension of T and by applying Theorem 2.2.4, Theorem 3.1.6 and Theorem E; at last, we deduce Theorem A from Corollary 3.3.9.

3.4.1 Reduction to the case T is a simple torus

By an induction on $\dim T$ we can assume that T is a simple torus, i.e. admitting no non-trivial subtori. In fact, if T is not simple, take a non-trivial subtorus $S \subseteq T$ and denote by $q : T \to T/S$ the canonical morphism (of complex analytic Lie groups), this is a Kähler fibre space (more precisely a principle S-bundle). We obtain thus a Kähler fibre space $f' = q \circ f : X \to T/S$, and then by induction hypothesis we have

$$\kappa(X, K_X+\Delta) \geqslant \kappa(F', K_{F'}+\Delta_{F'}),$$

where $\Delta_{F'} := \Delta|_{F'}$ with F' the general fibre f'. In addition, $f|_{F'} : F' \to S$ is also a Kähler fibre space of general fibre F over a complex torus S of dimension $< \dim T$, hence by induction hypothesis we have

$$\kappa(F', K_{F'}+\Delta_{F'}) \geqslant \kappa(F, K_F+\Delta_F),$$

thus we get

$$\kappa(X, K_X+\Delta) \geqslant \kappa(F, K_F+\Delta_F).$$

3.4.2 Dichotomy according to the determinant bundle and reduction to the case of Hermitian flat direct images

For positive integer m such that $m\Delta$ is an integral divisor, consider the direct image

$$\mathscr{F}_{m,\Delta} := f_*\Big(K_X^{\otimes m} \otimes \mathscr{O}_X(m\Delta)\Big) = f_*\Big(K_{X/T}^{\otimes m} \otimes \mathscr{O}_X(m\Delta)\Big).$$

If $\kappa(F, K_F+\Delta_F) = -\infty$ then Part (II) of the Theorem A is automatically established; hence we can assume that $\kappa(F, K_F+\Delta_F) \geqslant 0$. In consequence for m sufficiently divisible $\mathscr{F}_{m,\Delta} \neq 0$. Let us denote by $\mathscr{M}$ the set of positive integers m such that $m\Delta$ is an integral divisor and that $\mathscr{F}_{m,\Delta} \neq 0$, then we can suppose that $\mathscr{M} \neq \varnothing$, this is moreover an additive subset of $\mathbb{N}$. By Theorem D, for $\forall m \in \mathscr{M}$ the torsion free sheaf $\mathscr{F}_{m,\Delta}$ admits a semi-positively curved metric $g^{(m)}_{X/T,\Delta}$; in addition, the induced metric $\det g^{(m)}_{X/T,\Delta}$ on its determinant bundle $\det \mathscr{F}_{m,\Delta}$ is of curvature current

$$\theta_{m,\Delta} := \Theta_{\det g^{(m)}_{X/T,\Delta}}(\det \mathscr{F}_{m,\Delta}) \geqslant 0.$$

In particular, the line bundle $\det \mathscr{F}_{m,\Delta}$ is pseudoeffective on T for every $m \in \mathscr{M}$. By §3.4.1 we can assume that T is a simple torus, hence [Cao15, Proposition 2.2] (c.f. also [CP17, Theorem 3.3]) implies that we fall into the following two cases:

- Either $\theta_{m,\Delta} \not\equiv 0$, in this case T is an Abelian variety equipped with $\det \mathscr{F}_{m,\Delta}$ an ample line bundle;
- Or $\theta_{m,\Delta} \equiv 0$, in this case $\det \mathscr{F}_{m,\Delta}$ is a numerically trivial line bundle, and thus Corollary 2.2.10 implies that $(\mathscr{F}_{m,\Delta}, g^{(m)}_{X/T,\Delta})$ is a Hermitian flat vector bundle.

If there is an integer $m \in \mathscr{M}$ such that the determinant bundle $\det \mathscr{F}_{m,\Delta}$ is ample, then Theorem A(II) can be deduced by Theorem A(I) (which is proved in §3.1.2, c.f. Theorem 3.1.6). Hence in order to finish the proof of Theorem A(II), one only need to tackle the case that that the determinant bundle $\det \mathscr{F}_{m,\Delta}$ is numerically trivial for every $m \in \mathscr{M}$, which implies that $(\mathscr{F}_{m,\Delta}, g^{(m)}_{X/T,\Delta})$ is a Hermitian flat vector bundle for every $m \in \mathscr{M}$.

3.4.3 Reduction to the case $\kappa \leqslant 0$

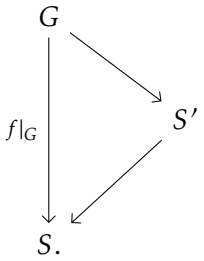
In this subsection we will demonstrate that we can reduce to the case $\kappa(X, K_X + \Delta) \leqslant 0$, which is an observation dating back to Kawamata, c.f. [Kaw81, §3, Proof of Claim 2, pp. 256-266]. Suppose that Theorem A (II) holds true for klt pair (X, Δ) with $\kappa(X, K_X + \Delta) \leqslant 0$. Now take a klt pair (X, Δ) such that $\kappa(X, K_X + \Delta) \geqslant 1$. By Lemma 1.1.4, we can freely replace X by a higher bimeromorphic model (the Kodaira dimension remains unchanged), and in consequence we can suppose that the Iitaka fibration of $K_X + \Delta$ is a morphism, denoted by

$$\phi : X \to Y,$$

whose general fibre is G. Then $\dim Y = \kappa(X, K_X + \Delta) > 0$ and $\kappa(G, K_G + \Delta_G) = 0$ where $\Delta_G := \Delta|_G$. Consider

$$f|_G : G \to f(G) =: S \subseteq T,$$

and take the Stein factorization of $f|_G$:



Case 1: $S \neq T$.

T being a simple torus, [Uen75, Theorem 10.9, pp. 120-123] implies that S is of general type, then so is S' by [Uen75, Lemma 6.3, p. 66-67]. By Theorem 3.1.2, for general $s \in S'$ we have

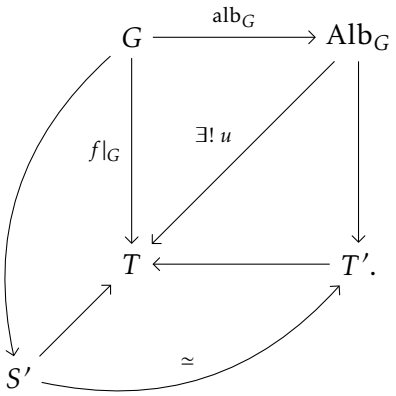
$$0 = \kappa(G, K_G + \Delta_G) = \kappa(G_s, K_{G_s} + \Delta_{G_s}) + \dim S' = \kappa(G_s, K_{G_s} + \Delta_{G_s}) + \dim S,$$

where $\Delta_{G_s} := \Delta|_{G_s} = \Delta_G|_{G_s}$. This forces $\dim S = \dim S' = 0$, hence $f(G) = \mathrm{pt}$, and in consequence G is contained in F. Therefore $\phi|_F : F \to \phi(F) \subseteq Y$ is a Kähler fibre space of general fibre G, and thus by the easy inequality [Uen75, Lemma 5.11, pp. 59-60] we obtain (noting that $\Delta_G = \Delta_F|_G$):

$$\kappa(F, K_F + \Delta_F) \leqslant \kappa(G, K_G + \Delta_G) + \dim h(F) = \dim h(F) \leqslant \dim Y = \kappa(X, K_X + \Delta).$$

Case 2: $S = T$.

First we prove that $S' \to S$ is a finite étale cover (thus S' is also a complex torus) with the help of Theorem E. In fact, let $\mathrm{alb}_G : G \to \mathrm{Alb}_G$ the Albanese map of (G, y) with base point y such that $f(y) = e \in T$. By the universal property of the Albanese map we get a (unique) morphism $u : \mathrm{Alb}_G \to T$ of complex tori (a morphism of complex analytic Lie groups) making the following diagram commutative:

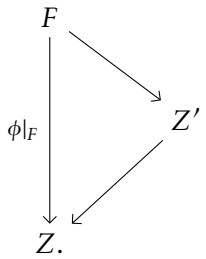


Since $f|_G$ is surjective, then so is u. By [Deb99, Théorème 2.3, p. 7] u can be factorized as $\mathrm{Alb}_G \to T' \to T$ with $\mathrm{Alb}_G \to T'$ the quotient by $\mathrm{Ker}(u)^0$ and $T' \to T$ a finite étale cover. As $\kappa(G, K_G + \Delta_G) = 0$, then by Theorem E the morphism alb_G is an analytic (Kähler) fibre space, thus so is $G \to T'$. Therefore the construction of Stein factorization implies that S' and T' are isomorphic. In particular, $S' \to T$ is a finite étale cover and thus S' is a complex torus.

Put F' to be the general fibre of $G \to S'$, then for general $t \in T$, we have $G_t \simeq F \cap G$ is finite union of copies of F'. Now apply our assumption to $G \to T$ ($\kappa(G, K_G + \Delta_G) = 0$) and we get

$$0 = \kappa(G, K_G + \Delta_G) \geqslant \kappa(F', K_{F'} + \Delta_{F'}).$$

where $\Delta_{F'} := \Delta|_{F'} = \Delta_G|_{F'}$. Furthermore, consider the Stein factorization of $\phi|_F : F \to \phi(F) =: Z \subseteq Y$:



For $z \in Z$ general $F_z \simeq F \cap G$, hence the general fibre of the analytic fibre space $F \to Z'$ is isomorphic à F'. Then by the easy inequality [Uen75, Lemma 5.11, pp. 59-60] we obtain:

$$\kappa(F, K_F + \Delta_F) \leqslant \kappa(F', \Delta_{F'} + \Delta_{F'}) + \dim Z' \leqslant \dim Z' = \dim Z \leqslant \dim Y = \kappa(X, K_X + \Delta).$$

3.4.4 End of the proof of Theorem A

By §3.4.2 we have that $(\mathscr{F}_{m,\Delta}, g^{(m)}_{X/T,\Delta})$ is a Hermitian flat vector bundle for every $m \in \mathscr{M}$. In other words $\mathscr{F}_{m,\Delta}$ is built from a unitary representation of the fundamental group (c.f. for example [Kob87, Proposition 1.4.21, p. 13] or [Dem12, §6, pp. 260-261])

$$\rho_m : \pi_1(T, t_0) \to \mathrm{U}(r_m)$$

where

$$r_m := \operatorname{rk} \mathscr{F}_{m,\Delta} = \mathrm{h}^0(F, K_F^{\otimes m} \otimes \mathscr{O}_F(m\Delta_F)).$$

Since $\pi_1(T, t_0)$ is an Abelian group, every representation of $\pi_1(T)$ can be decomposed into (irreducible) sub-representations of rank 1, hence a decomposition of $\mathscr{F}_{m,\Delta}$ into (numerically trivial) line bundles:

$$\mathscr{F}_{m,\Delta} = L_1 \oplus L_2 \oplus \cdots \oplus L_{r_m}, \quad \text{with } L_i \in \mathrm{Pic}^0(T),\ \forall i = 1, \cdots, r_m. \tag{3.34}$$

Step 1: First prove that $\mathrm{Im}(\rho_m)$ is finite for every $m \in \mathscr{M}$. In fact, suppose by contradiction that there exists $m \in \mathscr{M}$ such that $\mathrm{Im}(\rho_m)$ is infinite, hence there exists $j \in \{1, 2 \cdots, r_m\}$, say $j = 1$, such that L_j is not a torsion point in $\mathrm{Pic}^0(T)$. Consider the natural inclusion $L_1 \hookrightarrow \mathscr{F}_{m,\Delta}$, which induces a non-zero section

$$\mathrm{H}^0(T, \mathscr{F}_{m,\Delta} \otimes L_1^{-1}) = \mathrm{H}^0(X, K_X^{\otimes m} \otimes \mathscr{O}_X(m\Delta) \otimes f^* L_1^{-1}).$$

This implies that $\kappa(X, mK_X + m\Delta + f^*L_1) \geqslant 0$. As $f^*L_1 \in \mathrm{Pic}^0(X)$, by Corollary 3.3.9(a) and §3.4.3 we have

$$\kappa(X, mK_X + m\Delta + f^*L_1) \leqslant \kappa(X, K_X + \Delta) \leqslant 0,$$

hence a fortiori

$$\kappa(X, mK_X + m\Delta + f^*L_1) = \kappa(X, K_X + \Delta) = 0. \tag{3.35}$$

By Corollary 3.3.9(b), the equality (3.35) implies that f^*L_1 is a torsion point in $\mathrm{Pic}^0(X)$, i.e. there is an $e > 0$ such that $f^*L_1^{\otimes e} \simeq \mathscr{O}_X$, meaning that $L_1^{\otimes e} \simeq \mathscr{O}_T$ since the morphism

$$f^* : \mathrm{Pic}^0(T) \to \mathrm{Pic}^0(X)$$

is injective (f being an analytic fibre space). This contradicts our supposition that L_1 is not a torsion element in $\mathrm{Pic}^0(T)$. Hence $\mathrm{Im}(\rho_m)$ is finite for each $m \in \mathscr{M}$.

Step 2: By the precedent step we see that $\mathrm{Im}(\rho_m)$ is a finite group. Set $H_m := \mathrm{Ker}(\rho_m)$, then H_m is normal subgroup of $\pi_1(T)$ of finite index. Hence H_m induces a finite étale cover of T. Up to passing to this finite étale cover (the Kodaira dimension is invariant under finite étale covers) we can assume that the representation ρ_m is trivial, and consequently $\mathscr{F}_{m,\Delta}$ is a trivial vector bundle, then we have

$$\mathrm{h}^0(X, K_X^{\otimes m} \otimes \mathscr{O}_X(m\Delta)) = \mathrm{h}^0(T, \mathscr{F}_{m,\Delta}) = r_m = \mathrm{h}^0(F, K_F^{\otimes m} \otimes \mathscr{O}_F(m\Delta_F)),$$

which implies that $\kappa(X, K_X + \Delta) = \kappa(F, K_F + \Delta_F)$.

3.5 Geometric orbifold version of the $C_{n,m}$-conjecture for Kähler fibre spaces over complex tori

In this last section, we will prove Theorem G, in other word, generalize Part (II) of Theorem A, established in §3.4, to the geometric orbifold setting. Along the way, we also show that $C^{\mathrm{orb}}_{n,m}$ holds when $(Y, B_{f,\Delta})$ is of log general type. Before entering into the proof of theses results, let us first clarify some definitions. Remind that for $f : X \to Y$ analytic fibre space between compact complex manifolds and for Δ effective $\mathbb{Q}$-divisor on X, the branching divisor $B_{f,\Delta}$ is defined as the most effective $\mathbb{Q}$-divisor on Y such that $f^*B_{f,\Delta} \leqslant R_{f,\Delta}$ modulo exceptional divisors (see below, c.f. also Introduction); on the other hand, in [Cam04, Definition 1.29] Frédéric Campana defines a divisor on Y with respect to f and Δ in the setting of geometric orbifolds, named "orbifold base". We will see in the sequel that these two definitions coincide when (X, Δ) is lc. Let us first recall the definition of Campana:

Definition 3.5.1. Let $f : X \to Y$ and Δ as above such that (X, Δ) is lc. For any prime divisor G on Y, write

$$f^*G = \sum_{j \in J(f,G)} \mathrm{Ram}_{G_j}(f) G_j + (f\text{-exceptional divisor}),$$

where $J(f, G)$ is the index set of all prime divisors mapped onto G. Then the orbifold base with respect to f and Δ is defined to be the $\mathbb{Q}$-divisor

$$B_{f,\Delta} := \sum_G \left(1 - \frac{1}{m(f, \Delta; G)}\right) G$$

where the multiplicity $m(f, \Delta; G)$ of G with respect to f and Δ is defined to be

$$m(f, \Delta; G) := \inf\left\{\mathrm{Ram}_{G_j}(f) m(\Delta; G_j) \,\middle|\, j \in J(f, G)\right\}$$

with $m(\Delta; G_j) \in \mathbb{Q}_{\geqslant 1} \cup \{+\infty\}$ satisfying

$$\mathrm{ord}_{G_j}(\Delta) = 1 - \frac{1}{m(\Delta; G_j)}.$$

Now we have:

Lemma 3.5.2. *Let $f : X \to Y$ and Δ as above such that (X, Δ) is lc.Let $B_{f,\Delta}$ be the orbifold base respect to f and Δ in the sense of Campana, as defined in Definition 3.5.1 above. Then there is an f-exceptional effective $\mathbb{Q}$-divisor E such that the $\mathbb{Q}$-divisor $R_{f,\Delta} + E - f^*B_{f,\Delta}$ is effective; and $B_{f,\Delta}$ is the most effective $\mathbb{Q}$-divisor on Y satisfying this property.*

Proof. The second assertion is evident by construction of $B_{f,\Delta}$. In fact, if B is a divisor on Y such that $f^*B \leqslant R_{f,\Delta}$, then for every prime divisor G on Y we have

$$\begin{aligned}\mathrm{ord}_{G_j}(f^*B) = \mathrm{Ram}_{G_j}(f)\,\mathrm{ord}_G(B) \leqslant \mathrm{ord}_{G_j}(R_{f,\Delta}) &= \mathrm{Ram}_{G_j}(f) - 1 + \mathrm{ord}_{G_j}(\Delta) \\ &= \mathrm{Ram}_{G_j}(f) - \frac{1}{m(\Delta; G_j)},\ \forall j \in J(f, G),\end{aligned}$$

where

$$f^*G = \sum_{j \in J(f,G)} \mathrm{Ram}_{G_j} G_j + (f - \text{exceptional divisor});$$

this implies that

$$\mathrm{ord}_G(B) \leqslant 1 - \frac{1}{\mathrm{Ram}_{G_j}(f) m(\Delta; G_j)},\ \forall j \in J(f, G),$$

and hence

$$\begin{aligned}\mathrm{ord}_G(B) \leqslant \inf_{j \in J(f,G)} \left(1 - \frac{1}{\mathrm{Ram}_{G_j}(f) m(\Delta; G_j)}\right) &= 1 - \frac{1}{\inf\left\{\mathrm{Ram}_{G_j}(f) m(\Delta; G_j) \,\middle|\, j \in J(f, G)\right\}} \\ &= \mathrm{ord}_G(B_{f,\Delta}).\end{aligned}$$

Now turn to the proof of the first assertion. To this end, it suffices to show that for any prime divisor D on X such that $f(D)$ is a divisor on Y we have

$$\mathrm{ord}_D(R_{f,\Delta}) = \mathrm{ord}_D(\Sigma_f) + \mathrm{ord}_D(\Delta) \geqslant \mathrm{ord}_D(f^*B_{f,\Delta}). \tag{3.36}$$

Let Σ_Y be a (reduced) divisor containing $Y \setminus Y_0$ with $Y_0 \subset Y$ the smooth locus of f and write

$$f^*\Sigma_Y = \sum_{i \in I} b_i W_i,$$

then

$$\Sigma_f := \sum_{i \in I^{\mathrm{div}}} (b_i - 1) W_i.$$

where I^{div} denotes the set of indices in I such that $f(W_i)$ is a divisor on Y. Now we consider separately the two cases:

Case 1 : $D \not\subset \mathrm{Supp}(\Sigma_f)$. Then $\mathrm{ord}_D(\Sigma_f) = 0$ and a general point of $f(D)$ is contained in Y_0 , thus

$$f^* f(D) = D + (f\text{-exceptional divisor}).$$

In consequence $\mathrm{Ram}_D(f) = 1$ and $J(f, f(D)) = \{D\}$, which implies that $m(f, \Delta; f(D)) = m(\Delta; D)$. Hence

$$\mathrm{ord}_D(f^* B_{f,\Delta}) = \mathrm{ord}_{f(D)}(B_{f,\Delta}) = 1 - \frac{1}{m(\Delta; D)} = \mathrm{ord}_D(\Delta) = \mathrm{ord}_D(\Sigma_f) + \mathrm{ord}_D(\Delta).$$

Case 2 : $D \subset \mathrm{Supp}(\Sigma_f)$. Then $D = W_i$ for some $i \in I^{\mathrm{div}}$. In consequence, $f(W_i) \subset \mathrm{Supp}(\Sigma_Y)$ and

$$f^* f(W_i) = \sum_{j \in J(f, f(W_i))} b_j W_j + (f\text{-exceptional divisor}),$$

with $J(f, f(W_i)) = \left\{ j \in I^{\mathrm{div}} \,\middle|\, f(W_j) = f(W_i) \right\}$ and $\mathrm{Ram}_{W_j}(f) = b_j$. By definition we have

$$m(f, \Delta; f(W_i)) = \inf \left\{ b_j m(\Delta; W_j) \,\middle|\, j \in I^{\mathrm{div}} \text{ and } f(W_j) = f(W_i) \right\} \leqslant b_i m(\Delta; W_i).$$

Hence

$$\begin{aligned} \mathrm{ord}_{W_i}(f^* B_{f,\Delta}) = b_i \cdot \mathrm{ord}_{f(W_i)}(B_{f,\Delta}) &= b_i \left(1 - \frac{1}{m(f, \Delta; f(W_i))} \right) \\ &\leqslant 1 - \frac{1}{b_i m(\Delta; W_i)} = (b_i - 1) + (1 - \frac{1}{m(\Delta; W_i)}) \\ &= \mathrm{ord}_{W_i}(\Sigma_f) + \mathrm{ord}_{W_i}(\Delta). \end{aligned}$$

In both cases, the inequality (3.36) is established for prime divisor D vertical w.r.t. f, hence it end the proof. □

Remark 3.5.3. As a corollary of the above lemma, one sees clearly:

- $f^* B_{f,\Delta}$ being a vertical divisor w.r.t. f (i.e. not dominating Y), it is in fact the most effective divisor on Y such that $f^* B_{f,\Delta} \leqslant R_{f,\Delta^{\mathrm{vert}}} = \Sigma_f + \Delta^{\mathrm{vert}}$ where Δ^{vert} denotes the vertical part of Δ.
- If (X, Δ) is klt and $\mathscr{F}_{m,\Delta} := f_* \left(K^{\otimes m}_{(X,\Delta)/Y} \right) \neq 0$ for some m sufficiently large and divisible, one can easily deduce from Proposition 2.2.3 (applied to $L = \mathscr{O}_X(m\Delta^{\mathrm{horiz}})$ with Δ^{horiz} the horizontal part of Δ) that there is an f-exceptional effective $\mathbb{Q}$-divisor E such that the $\mathbb{Q}$-line bundle $K^{\mathrm{orb}}_{f,\Delta} + E$ is pseudoeffective, where the orbifold relative canonical bundle is defined (as a $\mathbb{Q}$-line bundle) by the formula:

$$K^{\mathrm{orb}}_{f,\Delta} := K_{(X,\Delta)/(Y,B_{f,\Delta})} = K_{X/Y} + \Delta - f^* B_{f,\Delta}.$$

Before proving the Theorem G, let us first prove that the klt version of $C^{\mathrm{orb}}_{n,m}$ holds for fibre spaces over bases of general type in the sense of geometric orbifolds:

Theorem 3.5.4. *Let $f : X \to Y$ be a surjective morphism between compact Kähler manifolds whose general fibre F is connected. Let Δ be an effective $\mathbb{Q}$-divisor on X such that (X, Δ) is klt. Suppose that $(Y, B_{f,\Delta})$ is of log general type. Then*

$$\kappa(X, K_X + \Delta) \geqslant \kappa(F, K_F + \Delta_F) + \dim Y,$$

where $\Delta_F := \Delta|_F$.

Notice that a stronger (log canonical) version of the above theorem is proved in [Cam04] (for X projective) based on the a weak positivity theorem for direct images of twisted pluricanonical bundles. We will give here a new argument depending on the Ohsawa-Takegoshi extension theorem:

Proof of Theorem 3.5.4. First, as in the proof of Theorem 3.1.6, by passing to a higher bimeromorphic model of f, we can assume that f is neat and prepared (in virtue of Lemma 1.1.3 and Lemma 1.1.4), that is, every f-exceptional divisor is also exceptional with respect to some bimeromorphic morphism $X \to X'$ and the the singular locus of f is a (reduced) SNC divisor; in particular, for every effective f-exceptional divisor E_0 on X, we have $\kappa(X, K_X + \Delta) = \kappa(X, K_X + \Delta + E_0)$.

If $\kappa(F, K_F + \Delta_F) = -\infty$ then there is nothing to prove, hence suppose that $\kappa(F, K_F + \Delta_F) \geqslant 0$, this implies that there is $m > 0$ sufficiently large and divisible such that $\mathscr{F}_{m,\Delta} := f_*\left(K^{\otimes m}_{(X,\Delta)/Y}\right) \neq 0$. By Remark 3.5.3, there is an effective f-exceptional $\mathbb{Q}$-divisor E such that the $\mathbb{Q}$-line bundle $K^{\mathrm{orb}}_{f,\Delta} + E$ is pseudoeffective. Since $(Y, B_{f,\Delta})$ is of log general type, Y is projective, one can fix a very ample line bundle A_Y on Y such that the $\mathbb{Q}$-line bundle $A_Y - K_Y - B_{f,\Delta}$ is ample and that the Seshadri constant $\epsilon(A_Y - K_Y - B_{f,\Delta}, y) > \dim Y$ for general y (such an A_Y exists by [Laz04, §5.1, Example 5.1.4, p. 270 and Example 5.1.18, p. 274, Vol.I]). Now by our hypothesis $K_Y + B_{f,\Delta}$ is a big $\mathbb{Q}$-line bundle, then (up to replacing m by a multiple) we can assume that $m(K_Y + B_{f,\Delta}) - 2A_Y$ is effective. Then we have

$$\kappa(X, K_X + \Delta) = \kappa(X, K_X + \Delta + E) \geqslant \kappa(X, mK^{\mathrm{orb}}_{f,\Delta} + mE + 2f^*A_Y).$$

In virtue of Lemma 3.1.1 it suffices to show that

$$\mathrm{H}^0(X, (K^{\mathrm{orb}}_{f,\Delta})^{\otimes m} \otimes \mathscr{O}_X(mE) \otimes f^*A_Y) \neq 0,$$

which is a direct consequence of the Ohsawa-Takegoshi type extension Theorem 2.1.2, as we precise below:

Since Δ is klt, by Theorem 2.2.1 the relative m-Bergman kernel metric $h_{X/Y,m\Delta^{\mathrm{horiz}}}$ on $K^{\otimes m}_{X/Y} \otimes \mathscr{O}_X(m\Delta^{\mathrm{horiz}})$ is semipositive (noting that $\Delta^{\mathrm{horiz}}\big|_F = \Delta_F$). Set

$$\begin{aligned} L_{m-1} &:= K^{\otimes(m-1)}_{X/Y} \otimes \mathscr{O}_X(m\Delta^{\mathrm{horiz}}), \\ L'_{m-1} &:= L_{m-1} \otimes \mathscr{O}_X(mE + m\Delta^{\mathrm{vert}} - (m-1)f^*B_{f,\Delta}), \end{aligned}$$

respectively equipped with the singular Hermitian metrics:

$$\begin{aligned} h_{L_{m-1}} &:= \left(h^{(m)}_{X/Y,m\Delta^{\mathrm{horiz}}}\right)^{\otimes\frac{m-1}{m}} \otimes h_{\Delta^{\mathrm{horiz}}}, \\ h_{L'_{m-1}} &:= h_{L_{m-1}} \otimes h^{\otimes m}_E \otimes h^{\otimes m}_{\Delta^{\mathrm{vert}}} \otimes f^*h^{\otimes(1-m)}_{B_{f,\Delta}}. \end{aligned}$$

where $h_{\Delta^{\mathrm{horiz}}}$, $h_{\Delta^{\mathrm{vert}}}$, h_E and $h_{B_{f,\Delta}}$ denote the canonical singular metrics defined by the divisors. Then by Proposition 2.2.3 and Lemma 3.5.2 the curvature current of $h_{L'_{m-1}}$ sat-

isfies

$$\begin{aligned}\Theta_{h_{L'_{m-1}}}(L'_{m-1}) &= \frac{m-1}{m}\Theta_{h_{X/Y,m\Delta^{\mathrm{horiz}}}}(K_{X/Y}^{\otimes m}\otimes\mathscr{O}_X(m\Delta^{\mathrm{horiz}})+[\Delta]+(m-1)[\Delta^{\mathrm{vert}}]\\ &\quad +m[E]-(m-1)[f^*B_{f,\Delta}]\\ &\geqslant (m-1)\big([\Sigma_f]+[E]+[\Delta^{\mathrm{vert}}]-[f^*B_{f,\Delta}]\big)+[\Delta]+[E]\\ &\geqslant [\Delta]+[E]\geqslant 0.\end{aligned}$$

Moreover, since $L'_{m-1}|_F = L_{m-1}|_F$ and $h_{L'_{m-1}}|_F = h_{L_{m-1}}|_F$, by Lemma 2.2.9 the natural inclusion

$$\begin{aligned}\mathrm{H}^0(F,K_F\otimes L'_{m-1}|_F\otimes\mathscr{J}(h_{L'_{m-1}}|_F)) &= \mathrm{H}^0(F,K_F\otimes L_{m-1}|_F\otimes\mathscr{J}(h_{L_{m-1}}|_F))\\ &\hookrightarrow \mathrm{H}^0(F,K_F\otimes L_{m-1}|_F)=\mathrm{H}^0(F,K_F^{\otimes m}\otimes\mathscr{O}_F(m\Delta_F))\end{aligned}$$

is an isomorphism. Hence by Theorem 2.1.2 we get a surjection

$$\mathrm{H}^0(X,K_X\otimes L'_{m-1}\otimes f^*(A_Y\otimes K^{-1}_{(Y,B_{f,\Delta})}))\twoheadrightarrow \mathrm{H}^0(F,K_F^{\otimes m}\otimes\mathscr{O}_F(m\Delta_F)).$$

Since

$$K_X\otimes L'_{m-1}\otimes f^*(A_Y\otimes K^{-1}_{(Y,B_{f,\Delta})}) = (K^{\mathrm{orb}}_{f,\Delta})^{\otimes m}\otimes\mathscr{O}_X(mE)\otimes f^*A_Y,$$

this proves the non-vanishing of $\mathrm{H}^0(X,(K^{\mathrm{orb}}_{f,\Delta})^{\otimes m}\otimes\mathscr{O}_X(mE)\otimes f^*A_Y)$. □

Finally, let us turn to the proof of Theorem G:

Proof of Theorem G. Let us proceed by induction on $\dim T$. If $B_{f,\Delta}=0$, then Theorem G is reduced to Part (II) of Theorem A. Hence we assume that $B_{f,\Delta}\neq 0$. Then by [Cao15, Proposition 2.2], there is a subtorus S of T of dimension $<\dim T$ and an ample $\mathbb{Q}$-divisor H on $A:=T/S$ such that $\pi^*H=B_{f,\Delta}$ with $\pi: T\to A=T/S$ the quotient map.

Now let $f'=\pi\circ f: X\to A$, which is a fibre space with general fibre F'. Then $f|_{F'}: F'\to S$ is a fibre space with general fibre F. We have $B_{f|_{F'},\Delta_{F'}}\geqslant (B_{f,\Delta})|_S$, as one can easily check: for every component G of $(B_{f,\Delta})|_S$, it arises from a prime divisor of X, hence $B_{f|_{F'},\Delta_{F'}}$ has the same vanishing order over G. This is enough for our use; we nevertheless remark that we have in fact the equality $B_{f|_{F'},\Delta_{F'}}=(B_{f,\Delta})|_S$ since every component of $B_{f|_{F'},\Delta_{F'}}$ must arise from a divisor on X: in fact, every component of $B_{f|_{F'},\Delta_{F'}}$ is either the image of a component of $\Delta_{F'}=\Delta|_{F'}$ or the image of a component of $\Sigma_{f|_{F'}}=(\Sigma_f)|_{F'}$ (we have the equality if we choose S to be a general translate). Now the induction hypothesis gives:

$$\kappa(F',K_{F'}+\Delta_{F'})\geqslant\kappa(F,K_F+\Delta_F)+\kappa(S,(B_{f,\Delta})|_S).$$

Furthermore, since $\kappa(S,(B_{f,\Delta})|_S)\geqslant 0$, we have

$$\kappa(F',K_{F'}+\Delta_{F'})\geqslant\kappa(F,K_F+\Delta_F). \tag{3.37}$$

We claim that

$$\kappa(X,K_X+\Delta)\geqslant\kappa(F',K_{F'}+\Delta_{F'})+\dim A. \tag{3.38}$$

If $\kappa(F',K_{F'}+\Delta_{F'})=-\infty$, then (3.38) evidently holds. Hence we can assume that $\kappa(F',K_{F'}+\Delta_{F'})\geqslant 0$. In this case, for m sufficiently large and divisible,

$$\mathrm{H}^0(F,K_{F'}^{\otimes m}\otimes\mathscr{O}_{F'}(m\Delta_{F'}))\neq 0.$$

Since (X,Δ) is klt, $(F',\Delta_{F'})$ is klt, then by Theorem 2.2.1 we can construct the relative $2m$-Bergman kernel metric $h^{(2m)}_{X/A,2m\Delta^{\mathrm{horiz}}}$ on $K_{X/A}^{\otimes 2m}\otimes\mathscr{O}_X(2m\Delta^{\mathrm{horiz}})\simeq K_X^{\otimes 2m}\otimes\mathscr{O}_X(2m\Delta^{\mathrm{horiz}})$. Now put

$$L:=K_X^{\otimes(2m-1)}\otimes\mathscr{O}_X(2m\Delta+2mE-m(f')^*H)$$

equipped with the singular Hermitian metric

$$h_L := \left(h^{(2m)}_{X/A,2m\Delta^{\mathrm{horiz}}}\right)^{\otimes\frac{2m-1}{2m}} \otimes h_{\Delta^{\mathrm{horiz}}} \otimes h^{\otimes 2m}_{\Delta^{\mathrm{vert}}} \otimes h^{\otimes 2m}_E \otimes (f')^* h^{\otimes(-m)}_H,$$

where E is an f-exceptional effective divisor as in Lemma 3.5.2 and $h_{\Delta^{\mathrm{horiz}}}$, $h_{\Delta^{\mathrm{vert}}}$, h_E and h_H are the canonical singular metrics defined by the divisors. Then by Proposition 2.2.3 and Lemma 3.5.2 the curvature current of h_L satisfies

$$\begin{aligned}\Theta_{h_L}(L) &= \frac{2m-1}{2m}\Theta_{h^{(2m)}_{X/A,\Delta^{\mathrm{horiz}}}}\left(K^{\otimes 2m}_{X/A}\otimes\mathscr{O}_X(2m\Delta^{\mathrm{horiz}})\right) + [\Delta^{\mathrm{horiz}}] + 2m[\Delta^{\mathrm{vert}}] + 2m[E] - m[(f')^*H]\\ &\geqslant (2m-1)[\Sigma_f] + [\Delta^{\mathrm{horiz}}] + 2m[\Delta^{\mathrm{vert}}] + 2m[E] - m[f^*B_{f,\Delta}]\\ &= [\Delta] + [E] + (m-1)([\Sigma_f] + [\Delta^{\mathrm{vert}}] + [E]) + m([\Sigma_f] + [\Delta^{\mathrm{vert}}] + [E] - [f^*B_{f,\Delta}])\\ &\geqslant [\Delta] + [E] + (m-1)([\Sigma_f] + [\Delta^{\mathrm{vert}}] + [E]) \geqslant 0.\end{aligned}$$

Since $h_L|_F = h_{L_{2m-1}}|_F$, where $L_{2m-1} := K^{\otimes(2m-1)}_X \otimes \mathscr{O}_X(2m\Delta^{\mathrm{horiz}})$ equipped with the singular metric

$$h_{L_{2m-1}} := \left(h^{(2m)}_{X/A,\Delta^{\mathrm{horiz}}}\right)^{\otimes\frac{2m-1}{2m}} \otimes h_{\Delta^{\mathrm{horiz}}},$$

then by Lemma 2.2.9 we see that the natural inclusion

$$f'_*(K_{X/A}\otimes L\otimes\mathscr{J}(h_L)) \hookrightarrow f'_*(K_{X/A}\otimes L)$$

is generically an isomorphism, hence by Theorem 2.2.4 the canonical L^2 metric on

$$f'_*(K_{X/A}\otimes L) = f'_*(K^{\otimes 2m}_{X/A}\otimes\mathscr{O}_X(2m\Delta + 2mE))\otimes H^{\otimes(-m)}$$

is semi-positively curved. In particular its determinant is pseudoeffective, which implies that $\det f'_*(K^{\otimes 2m}_{X/A}\otimes\mathscr{O}_X(2m\Delta+2mE))$ is big on A. Since $f'_*(K^{\otimes 2m}_{X/A}\otimes\mathscr{O}_X(2m\Delta+2mE))$ and $f'_*(K^{\otimes 2m}_{X/A}\otimes\mathscr{O}_X(2m\Delta))$ are equal in codimension 1, hence

$$\det f'_*(K^{\otimes 2m}_{X/A}\otimes\mathscr{O}_X(2m\Delta+2mE)) = \det f'_*(K^{\otimes 2m}_{X/A}\otimes\mathscr{O}_X(2m\Delta)),$$

implying that $\det f'_*(K^{\otimes 2m}_{X/A}\otimes\mathscr{O}_X(2m\Delta))$ is big on A. Since $\kappa(A) = 0$, (3.38) results from Theorem 3.1.6.

At last, by combining (3.37) and (3.38) with the easy inequality [Uen75, Theorem 5.11, pp. 59-60] (applied to $\pi : T \to A$) we obtain:

$$\begin{aligned}\kappa(X,\Delta+X) &\geqslant \kappa(F',K_{F'}+\Delta_{F'}) + \dim A \geqslant \kappa(F,K_F+\Delta_F) + \kappa(S,(B_{f,\Delta})|_S) + \dim A\\ &\geqslant \kappa(F,K_F+\Delta_F) + \kappa(T,B_{f,\Delta}).\end{aligned}$$

□

Chapter 4

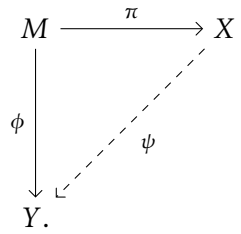
Structure of klt projective varieties with nef anticanonical divisors

In this chapter we study the structure of klt projective varieties with nef anticanonical divisors, which grows from the article [Wan20].

4.1 Positivity and flatness of the direct images

Let X be a klt projective variety with nef anticanonical divisor. In order to give a uniform treatment of the Albanese map and of the MRC fibration of X, we prove in this section some general results on the dominant rational mapping from X to any smooth non-uniruled variety Y; in particular, by virtue of Proposition 2.3.5 we study the direct images of powers of a relatively very ample line bundle on X. Before stating these results, let us set up some general notations (see also [CCM19, Setting 3.1]):

General Settings 4.1.1. *Let $\psi : X \dashrightarrow Y$ be a dominant rational map between projective varieties with Y smooth. Suppose that there is an effective divisor Δ on X such that the pair (X, Δ) is klt and $-(K_X + \Delta)$ is nef. Let $\phi : M \to Y$ be an elimination of indeterminacy of ψ with M smooth and let $\pi : M \to X$ be the induced (birational) morphism. For convenience, we further assume that the branch locus of ϕ is a SNC divisor on Y and that its inverse image on M has SNC support. Let Y_0 be the maximal Zariski open of Y such that ϕ is flat over Y_0 and that for every prime divisor D on Y_0 the pullback $\phi^* D$ is not contained in the exceptional locus of π.*



Write $\mathrm{Exc}(\pi) = \sum_{i \in I} E_i =: E$. Since (X, Δ) is klt, $K_X + \Delta$ is $\mathbb{Q}$-Cartier and we can write:

$$K_M + \pi_*^{-1}\Delta \sim_{\mathbb{Q}} \pi^*(K_X + \Delta) + \sum_i a_i E_i \tag{4.1}$$

with $a_i > -1$, where $\pi_*^{-1}\Delta$ denotes the strict transform of Δ via π. We rewrite the formula above by:

$$K_M + \Delta_M \sim_{\mathbb{Q}} \pi^*(K_X + \Delta) + \sum_{i \in I_{>0}} a_i E_i \tag{4.2}$$

where $I_{>0}$ (resp. $I_{>0}$) is the set of indices i such that $a_i > 0$ (resp. $a_i < 0$) and

$$\Delta_M := \pi_*^{-1}\Delta + \sum_{i \in I_{<0}} (-a_i)E_i.$$

By the klt condition we see that the coefficients of the components in Δ_M are all < 1 thus (M, Δ_M) is klt.

4.1.1 Birational geometry of ψ

Let everything be as in the General Setting 4.1.1. In this subsection we recall some general results on the birational geometry of ψ. They are essentially proved by Qi Zhang in [Zha05, Main Theorem]. The following result is explicitly formulated in [CCM19, Theorem 3.2] for X is smooth.

Proposition 4.1.2. *Let everything be as in the General Setting 4.1.1 except that we only assume that the pair (X, Δ) is log canonical (abbr. lc). Suppose further that Y is not uniruled. Then we have:*

(a) $\kappa(Y) = 0$. *Moreover, if N_Y is an effective $\mathbb{Q}$-divisor $\mathbb{Q}$-linearly equivalent to K_Y, then $\phi^* N_Y$ is π-exceptional; in particular, N_Y is contained in $Y \setminus Y_0$.*

(b) *Δ is horizontal with respect to ψ.*

(c) *$\pi(\phi^{-1}(Y \setminus Y_0))$ is of codimension $\geqslant 2$ in X. In particular, every ϕ-exceptional divisor on M is also π-exceptional.*

(d) *Y_0 has the following Liouville property: every global holomorphic function on Y_0 is constant.*

(e) *ψ is semistable in codimension 1 (c.f. [Zha05, Definition 1]), i.e. for every prime divisor P on Y_0, write $\phi^* P = \sum_i c_i P_i$ with P_i being prime divisor on $\phi^{-1}(Y_0)$ for every i, then $c_i > 1$ implies that P_i is π-exceptional.*

Proof. When X is smooth, the proposition is established in [CH19, Lemma 3.1, Proposition 3.2]. In the singular case, the proof becomes a little subtle. For the convenience of the readers, we will briefly present the proof below following ideas from [Zha05] and [CH19]. The same ideas are also used in the proof of Lemma 4.2.2 below.

Up to further blowing-up M and Y, we can assume that ϕ is smooth outside a SNC divisor $D_Y := \sum_j D_{Y,j}$ (called the *branching divisor* of ϕ) and that $\mathrm{Supp}(\phi^* D_Y + E)$ is SNC. In addition, let us fix a very ample line bundle L on X.

Now take A_M an ample divisor on M, then for any $\epsilon \in \mathbb{Q}_{>0}$ the $\mathbb{Q}$-divisor $-\pi^*(K_X + \Delta) + \epsilon A_M$ is ample since $-(K_X + \Delta)$ is nef; choose an ample $\mathbb{Q}$-divisor H_ϵ on Y such that $-\pi^*(K_X + \Delta) + \epsilon A_M - \phi^* H_\epsilon$ remains ample. Take

$$\Delta_{M,\epsilon} := \Delta_M + \frac{1}{k} \cdot \text{general member of the linear series } \left| k\left(-\pi^*(K_X + \Delta) + \epsilon A_M - \phi^* H_\epsilon\right) \right|,$$

for k sufficiently large and divisible. Then $\Delta_{M,\epsilon}$ is a $\mathbb{Q}$-divisor with coefficients $\leqslant 1$ and has SNC support. By [KM98, Corollary 2.31, pp. 53-55] the pair $(M, \Delta_{M,\epsilon})$ is lc, thus by the weak positivity result [Fuj17, Theorem 1.1], the direct image $\phi_* \mathscr{O}_M(k(K_{M/Y} + \Delta_{M,\epsilon}))$ is weakly positive; moreover, since $K_{M/Y} + \Delta_{M,\epsilon}$ is linearly equivalent to $\epsilon A + \sum_{a_i > 0} a_i E_i$ over the general fibre of ϕ, hence $K_{M/Y} + \Delta_{M,\epsilon}$ is relatively big, in particular we have

$$\phi_* \mathscr{O}_M(k(K_{M/Y} + \Delta_{M,\epsilon})) \neq 0.$$

In consequence, the $\mathbb{Q}$-divisor

$$K_{M/Y} + \Delta_{M,\epsilon} + \phi^* H_\epsilon \sim_{\mathbb{Q}} -\phi^* K_Y + \sum_{i \in I_{>0}} a_i E_i + \epsilon A_M$$

is $\mathbb{Q}$-linearly equivalent to an effective $\mathbb{Q}$-divisor (for details, see the proof of Lemma 4.2.2); by letting $\epsilon \to 0$, we see that $-\phi^* K_Y + \sum_{a_i>0} a_i E_i$ is pseudoeffective.

Finally take $H_1, \cdots, H_{\dim X - 1}$ be general members of the linear series $|\pi^* L|$, and let

$$C := H_1 \cap \cdots \cap H_{\dim X - 1},$$

then C is a movable curve on M, thus

$$(-\phi^* K_Y + \sum_{i \in I_{>0}} a_i E_i) \cdot C \geqslant 0.$$

When $a_i > 0$ the divisor E_i is π-exceptional, then the projection formula implies that $E_i \cdot C = 0$ for every i. Hence we have $\phi^* K_Y \cdot C \leqslant 0$. By our hypothesis Y is not uniruled, then by [BDPP13, Corollary 0.3] K_Y is pseudoeffective, since $C_Y := \phi_* C$ moves in a strongly connecting family (c.f. [BDPP13, §0]), in particular it is movable, thus by [BDPP13, Theorem 0.2] $K_Y \cdot C_Y \geqslant 0$. But on the other hand, we have seen that $K_Y \cdot C_Y = \phi^* K_Y \cdot C \leqslant 0$, hence $K_Y \cdot C_Y = 0$; then by [BDPP13, 9.8 Theorem] we have $\kappa(Y) = 0$. If N_Y is an effective $\mathbb{Q}$-divisor $\mathbb{Q}$-linear equivalent to K_Y, then by the projection formula we have

$$\pi_* \phi^* N_Y \cdot L^{\dim X - 1} = \phi^* N_Y \cdot C = 0;$$

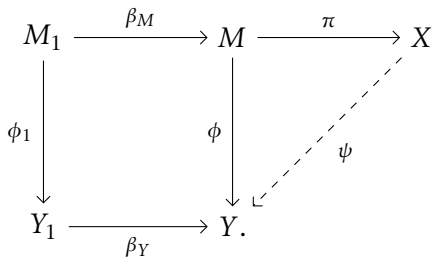
but L being very ample, a fortiori $\pi_* \phi^* N_Y = 0$, meaning that $\phi^* N_Y$ is π-exceptional. This proves (a).

For the point (b), note that in the proof of (a), if we set

$$\Delta_{M,\epsilon} := \pi_*^{-1} \Delta^{\text{horiz}} + \sum_{i \in I_{<0}} a_i E_i + \frac{1}{k} \cdot \text{general member of } \left| k(-\pi^*(K_X + \Delta) + \epsilon A_M - \phi^* H_\epsilon) \right|,$$

with k sufficiently large and divisible, then the same argument as in (a) plus the equality $\phi^* K_Y \cdot C = 0$ shows that $\Delta^{\text{vert}} \cdot C \leqslant 0$, but Δ^{vert} is effective, then a fortiori $\Delta^{\text{vert}} = 0$, which implies that Δ is horizontal. Thus we proved (b).

Now let us prove (c). Take a prime divisor V on M such that $\phi(V) \subseteq Y \setminus Y_0$. By definition of Y_0, if $\phi(V)$ is of codimension 1, then V is automatically π-exceptional; hence we can suppose that $\phi(V)$ is of codimension $\geqslant 2$, i.e. V is ϕ-exceptional. Let $\beta_Y : Y_1 \to Y$ be a desingularization of the blow-up of Y at $\phi(V)$, then $\phi(V) \subseteq \beta_Y(\text{Exc}(\beta_Y))$. Since Y is smooth, we have $K_{Y_1} \sim K_Y + F_Y$ with F_Y effective and β_Y-exceptional, moreover we have $\text{Supp}(F_Y) = \text{Exc}(\beta_Y)$. Take M_1 be a desingularization of the fibre product $M \underset{Y}{\times} Y_1$, with the induced morphisms $\beta_M : M_1 \to M$ and $\phi_1 : M_1 \to Y_1$. And let V_1 be the strict transform of V in M_1. Then $\phi_1(V_1) \subseteq \text{Exc}(\beta_Y)$.



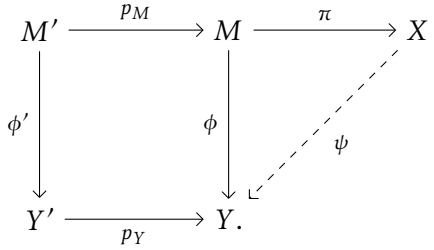
By (b) there exists an effective $\mathbb{Q}$-divisor N_Y which is $\mathbb{Q}$-linearly equivalent to K_Y. Then $\beta_Y^* N_Y + F_Y$ is an effective $\mathbb{Q}$-divisor $\mathbb{Q}$-linearly equivalent to K_{Y_1}. Apply (a) to the dominant rational map $X \dashrightarrow Y_1$ one sees that $\phi_1^*(\beta_Y^* N_Y + F_Y)$ is $(\pi \circ \beta_M)$-exceptional. But

$$\phi_1(V_1) \subseteq \mathrm{Exc}(\beta_Y) = \mathrm{Supp}(F_Y) \subseteq \mathrm{Supp}(\beta_Y^* N_Y + F_Y),$$

therefore $V_1 \subseteq \mathrm{Supp}(\phi^*(\beta_Y^* N_Y + F_Y))$ and thus V_1 is also $(\pi \circ \beta_Y)$-exceptional. This implies that $V = \beta_M(V_1)$ is π-exceptional. Thus we proved (c).

Point (d) is a simple consequence of (c) by the same argument as [CH19, §3.A, Remark 3]. For convenience of the readers let us briefly recall the proof: let $h : Y_0 \to \mathbb{C}$ is a holomorphic function, then its pullback $\phi^* h$ induces a holomorphic function h_1 on $\pi(\phi^{-1}(Y_0) \backslash E)$. By (c) the complement of $\pi(\phi^{-1}(Y_0) \backslash E)$ in X has codimension $\geqslant 2$. Then h_1 extends to a holomorphic function on X, which is constant by Liouville's Theorem. Hence h is constant.

It remains to prove (e). To this end it suffices to show the following statement: for every j write $\phi^* D_{Y,j} = \sum_l m_{j,l} D_{j,l}$, if $m_{j,l} > 1$ then $D_{j,l}$ is π-exceptional. By Kawamata's covering techniques (a Block-Gieseker cover followed by cyclic cover, c.f. [Laz04, Proposition 4.1.6, Theorem 4.1.10, Theorem 4.1.12, pp. 243-247]) we can construct a flat finite cover $p_Y : Y' \to Y$ such that $p_Y^* D_{Y,j} = m_{j,l} D_{Y',j}$ for some smooth prime divisor $D_{Y',j}$ on Y' and that Y' is smooth with $\sum_i p_Y^* E_i + \sum_{k \neq j} p_Y^* D_{Y,k} + D_{Y',j}$ being a reduced SNC divisor. By [Laz04, Proposition 4.1.6] the fibre product $M \underset{Y}{\times} Y'$ is singular along the singular locus of the divisor $\phi^* D_{Y,j}$, in particular, it is singular along the preimage of $D_{j,l}$ since $m_{j,l} > 1$. Take M' a strong desingularization of $M \underset{Y}{\times} Y'$ with induced morphisms $p_M : M' \to M$ and $\phi' : M' \to Y'$.



By [Kle80, Proposition (9), Remark (26)(vii)], $M \underset{Y}{\times} Y'$ is Gorenstein and

$$K_{M \underset{Y}{\times} Y'/Y'} \sim \text{pullback of } K_{M/Y} \text{ to } M \underset{Y}{\times} Y'.$$

over $p_Y^{-1}(Y_{\mathrm{flat}})$ where $Y_{\mathrm{flat}} \subseteq Y$ denotes the flat locus of ϕ. By generic flatness and [Ful84, Example A.5.4, p. 416], $Y \backslash Y_{\mathrm{flat}}$ is of codimension $\geqslant 2$, then so is $Y' \backslash p_Y^{-1}(Y_{\mathrm{flat}})$. By [Rei94, 2.3 Proposition] we can write (for details, see the proof of Lemma 4.2.2)

$$K_{M'/Y'} \sim_{\mathbb{Q}} p_M^* K_{M/Y} + E_{M \underset{Y}{\times} Y'} + E_M - G$$

where $E_{M \underset{Y}{\times} Y'}$ is a (non-necessarily effective) divisor which exceptional for $M' \to M \underset{Y}{\times} Y'$, E_M is a (non-necessarily effective) divisor such that $\phi'(E_M) \subseteq Y' \backslash p_Y^{-1}(Y_{\mathrm{flat}})$ (in particular E_M is ϕ'-exceptional), and G is an effective divisor supported on the preimage of the prime divisors with multiplicity > 1 in $\phi^* D_{Y,j}$. In particular, $p_M(G)$ contains $D_{j,l}$. Combine this with the formula (4.1) we get

$$K_{M'/Y'} \sim_{\mathbb{Q}} p_M^* \pi^*(K_X + \Delta) - p_M^* \phi^* K_Y + \sum_{\lambda \in \Lambda} b_\lambda E'_\lambda + E'_{M \underset{Y}{\times} Y'} + E_M - G.$$

where $E'_{M\underset{Y}{\times}Y'}$ is exceptional for $M' \to M \underset{Y}{\times} Y'$ and, for every $\lambda \in \Lambda$, E'_λ is prime divisor on M' supported on the strict transform via $M' \to M \underset{Y}{\times} Y'$ of the pullback of $\sum_i E_i$ on $M \underset{Y}{\times} Y'$ with $b_\lambda := a_{i_\lambda} \cdot \mathrm{mult}_\lambda$ where i_λ is the index such that $E_{i_\lambda} = p_M(E_\lambda)$ and

$$\mathrm{mult}_\lambda := \text{multiplicity of the image of } E_\lambda \text{ in the pullback of } E_i \text{ on } M \underset{Y}{\times} Y'.$$

By construction of p_Y we see that $\mathrm{mult}_\lambda > 1$ if and only if E_{i_λ} coincide with a divisor contained in the non-reduced part of $\phi^* D_{Y,j}$. In particular, for $\lambda \in \Lambda$ such that $\phi(E_{i_\lambda}) \not\subset \mathrm{Supp}(D_Y)$ we have $\mathrm{mult}_\lambda = 1$ and thus $b_\lambda = a_{i_\lambda} \geqslant -1$.

Now take A' an ample divisor on M'. Since $-K_X$ is nef, for any $\epsilon \in \mathbb{Q}_{>0}$ the $\mathbb{Q}$-divisor $-p_M^* \pi^* K_X + \epsilon A'$ is ample; then choose an ample $\mathbb{Q}$-divisor H'_ϵ on Y' such that $-p_M^* \pi^* K_X + \epsilon A - (\phi')^* H_\epsilon$ remains ample. Take

$$\begin{aligned}\Delta_{M',\epsilon} := & \sum_{\substack{\phi(E_{i_\lambda}) \not\subset \mathrm{Supp}(D_Y) \\ b_\lambda < 0}} (-b_\lambda) E'_\lambda \\ & + \frac{1}{k} \cdot \text{general member of the linear series } \left| k\left(-p_M^* \pi^* K_X + \epsilon A - (\phi')^* H_\epsilon\right) \right|,\end{aligned}$$

for k sufficiently large and divisible. Then $(M', \Delta_{M',\epsilon})$ is a lc pair. Moreover, since the general fibre of ϕ and thus of ϕ' is smooth, $E'_{M\underset{Y}{\times}Y'}$ is ϕ'-vertical; E_M and G are ϕ'-vertical by construction. Therefore $K_{M'/Y'} + \Delta_{M',\epsilon}$ is big on the general fibre of ϕ'. Hence by the same argument as in the proof of (a) we obtain that the $\mathbb{Q}$-divisor

$$K_{M'/Y'} + \Delta_{M',\epsilon} + (\phi')^* H'_\epsilon \sim_{\mathbb{Q}} -p_M^* \phi^* K_Y + \sum_{b_\lambda > 0} b_\lambda E'_\lambda + E'_{M\underset{Y}{\times}Y'} + E_M - G - \sum_{\substack{b_\lambda \leqslant 0 \\ \phi(E_{i_\lambda}) \subset \mathrm{Supp}(D_Y)}} (-b_\lambda) E'_\lambda + \epsilon A_{M'}$$

is $\mathbb{Q}$-linearly equivalent to an effective $\mathbb{Q}$-divisor; by letting $\epsilon \to 0$, we see that

$$-p_M^* \phi^* K_Y + E'_{M\underset{Y}{\times}Y'} + \sum_{b_\lambda > 0} b_\lambda E'_\lambda + E_M - G - \sum_{\substack{b_\lambda \leqslant 0 \\ \phi(E_{i_\lambda}) \subset \mathrm{Supp}(D_Y)}} (-b_\lambda) E'_\lambda$$

is pseudoeffective.

Finally take $H'_1, \cdots, H'_{\dim X - 1}$ be general members of the linear series $\left| p_M^* \pi^* L \right|$, and let

$$C' := H'_1 \cap \cdots \cap H'_{\dim X - 1},$$

then C' is a movable curve on M', thus

$$\left(-p_M^* \phi^* K_Y + \sum_{b_\lambda > 0} b_\lambda E'_\lambda + E'_{M\underset{Y}{\times}Y'} + E_M - G - \sum_{\substack{b_\lambda \leqslant 0 \\ \phi(E_{i_\lambda}) \subset \mathrm{Supp}(D_Y)}} -(b_\lambda) E'_\lambda \right) \cdot C' \geqslant 0.$$

By construction, E'_λ is $(\pi \circ p_M)$-exceptional for λ such that $b_\lambda > 0$, so is $E'_{M\underset{Y}{\times}Y'}$, hence $E_\lambda \cdot C' = E'_{M\underset{Y}{\times}Y'} \cdot C' = 0$ for λ such that $b_\lambda > 0$. Furthermore, by construction E_M is ϕ'-exceptional, hence $p_{M*} E_M$ is ϕ-exceptional, then by (c) E_M is π-exceptional and $E_M \cdot C' = 0$. Therefore we have

$$\left(\sum_{\substack{b_\lambda \leqslant 0 \\ \phi(E_{i_\lambda}) \subset \mathrm{Supp}(D_Y)}} (-b_\lambda) E'_\lambda + G \right) \cdot C' \leqslant -p_M^* \phi^* K_Y \cdot C' = -K_Y \cdot (\phi \circ p_M)_* C'.$$

Since K_Y is pseudoeffective (by assumption Y is not uniruled), $(\phi\circ p_M)_*C'$ is movable, we have

$$\left(\sum_{\substack{b_\lambda\leqslant 0\\ \phi(E_{i_\lambda})\subset \mathrm{Supp}(D_Y)}}(-b_\lambda)E'_\lambda+G\right)\cdot C'\leqslant -K_Y\cdot(\phi\circ p_M)_*C'\leqslant 0.$$

But G is effective, a fortiori $G\cdot C'=0$ and $b_\lambda=0$ or $E'_\lambda\cdot C'=0$ for every $\lambda\in\Lambda_{\mathrm{vert}}$ such that $\phi(E_{i\lambda})\subset\mathrm{Supp}(D_Y)$. By the projection formula this implies that

$$(\pi\circ p_M)_*G\cdot L^{\dim X-1}=0.$$

Since L is very ample, we have $(\pi\circ p_M)_*G=0$. In particular, since $p_{M*}G\supseteq D_{j,l}$, this implies that $D_{j,l}$ is π-exceptional, which proves (d). By the way, the same argument shows that $b_\lambda=0$ for every $\lambda\in\Lambda_{\mathrm{vert}}$ such that $\phi(E_{i_\lambda})\subset\mathrm{Supp}(D_Y)$. □

4.1.2 Positivity and numerical flatness of the direct images

Throughout this subsection, let everything be as in the General Setting 4.1.1, and suppose further that Y is not uniruled. The main purpose of this subsection is to study the positivity of the ϕ-direct images of a sufficiently ample line bundle on M. Before stating these results, let us fix some notations: by (4.2) the $\mathbb{Q}$-divisor

$$-(K_{M/Y}+\Delta_M)+E'\sim_{\mathbb{Q}}-\pi^*(K_X+\Delta) \tag{4.3}$$

is nef, where $E':=\sum_{i\in I_{>0}}a_iE_i-\phi^*N_Y$ with N_Y being an effective $\mathbb{Q}$-divisor $\mathbb{Q}$-linearly equivalent to K_Y (by Proposition 4.1.2(b) such an N_Y exists). By Proposition 4.1.2(a) E' is π-exceptional and the restriction of E' to a general fibre of ϕ is effective. The basic result in this subsection is the following (c.f. [CCM19, Lemma 3.4]):

Proposition 4.1.3. *Let everything be as in the General Setting 4.1.1 with Y non-uniruled and E' as above. Let θ be a smooth $(1,1)$-form on Y and let G be a ϕ-big divisor on M such that $\mathscr{O}_M(G)$ admits a singular Hermitian metric h_G such that $\Theta_{h_G}(\mathscr{O}_M(G))\geqslant\phi^*\theta$. Then for any $q\in\mathbb{Z}_{>0}$ the direct image sheaf $\phi_*\mathscr{O}_M(q(K_{M/Y}+\Delta_M)+G+pE')$ is θ-weakly semipositively curved for any p sufficiently large with respect to q.*

Proof. This can be deduced immediately from the Corollary 2.2.13. Let us briefly recall how the proof goes. Let $p\in\mathbb{Z}_{>0}$ such that pE' is an integral divisor on M, and write

$$q(K_{M/Y}+\Delta_M)+G+pE'=(p+q)(K_{M/Y}+\Delta_M)+G+\underbrace{(-p(K_{M/Y}+\Delta_M)+pE')}_{\text{nef}}.$$

Let h_{Δ_M} be the canonical metric on Δ_M. Since (M,Δ_M) is still klt, $\mathscr{J}(h_{\Delta_M})\simeq\mathscr{O}_M$, thus by [Laz04, §9.5.D, Theorem 9.5.35, pp. 210-211] $\mathscr{J}(h_{\Delta_M}|_{M_y})\simeq\mathscr{O}_{M_y}$ for general y. Hence for p sufficiently large

$$\mathscr{J}((h_G^{1/(p+q)}\otimes h_{\Delta_M})|_{M_y})\simeq\mathscr{O}_{M_y}$$

for general $y\in Y$. For such a p Corollary 2.2.13 implies that $\phi_*\mathscr{O}_M(q(K_{M/Y}+\Delta_M)+G+pE')$ is θ-weakly semipositively curved. □

Then let us recall in the sequel some results in [CH19]. We first remark that:

Remark 4.1.4. Let us remark that most of the results below have been essentially contained in [CCM19]. We carefully state and prove them for the following reason: since X is not necessarily smooth (nor $\mathbb{Q}$-factorial), the pushforward of a (Cartier) $\mathbb{Q}$-divisor on M via π is not necessarily $\mathbb{Q}$-Cartier, thus in general it does not make sense to talk about pseudoeffectivity of them ([CCM19] does not take care of this point). However, since the effectivity of a Weil divisor still makes sense, we will use this to overcome this difficulty.

Proposition 4.1.5 ([CCM19, Lemma 3.5]). *Let everything be as in the General Settinng 4.1.1 with Y non-uniruled and let G be a ϕ-big divisor on M, then for any ample divisor A_Y on Y and for any integers $c, s \in \mathbb{Z}_{>0}$ the $\mathbb{Q}$-divisor*

$$\pi_*\left(G - \phi^* D_{G,c,1} + \frac{1}{s}\phi^* A_Y\right)$$

is $\mathbb{Q}$-linearly equivalent to an effective divisor on X, where $D_{G,c,1}$ is the $\mathbb{Q}$-divisor on Y defined by

$$D_{G,c,1} := \frac{1}{r} \cdot \textit{the Cartier divisor on } Y \textit{ associated to the line bundle } \det\phi_*\mathscr{O}_M(G + cE)$$

with $r = \mathrm{rk}\phi_*\mathscr{O}_M(G + cE)$. If moreover $\pi_*(G - \phi^* D_{G,c,1})$ is $\mathbb{Q}$-Cartier on X, then it is pseudo-effective; in particular, for $k \in \mathbb{Z}_{>0}$ sufficiently large $G - \phi^* D_{G,c,1} + kE$ is pseudoeffective on M.

Proof. The proof is essentially the same as that of [CP17, Theorem 3.4] (c.f. [Wan19, Theorem 3.4] for more details), see also [Cao19, Proposition 3.15], [CH19, Lemma 3.3] and [CCM19, Lemma 3.5]. We give the detailed proof in order to clarify the problems pointed out in Remark 4.1.4

(A) Construction of the fibre product and of the canonical section. Let Y_f be the Zariski open subset of Y over which ϕ is flat and $\phi_*\mathscr{O}_M(G + cE)$ is locally free. Then $\mathrm{codim}_Y(Y \setminus Y_f) \geqslant 2$ and for every $y \in Y_f$ the fibre M_y is Gorenstein (c.f. [Mat89, §23, Theorem 23.4, p. 181]). Over Y_f we have a natural inclusion

$$\det\phi_*\mathscr{O}_M(G + cE)|_{Y_f} \simeq \mathscr{O}_{Y_f}(rD_{G,c,1}|_{Y_f}) \hookrightarrow \bigotimes^r \phi_*\mathscr{O}_M(G + cE)|_{Y_f}. \tag{4.4}$$

Now we take the r-fold fibre product

$$M^r := \underbrace{M \underset{Y}{\times} M \underset{Y}{\times} \cdots \underset{Y}{\times} M}_{r \text{ times}},$$

equipped with natural projections $\mathrm{pr}_i : M^r \to M$ and the natural morphism $\phi^r : M^r \to Y$ such that $\phi \circ \mathrm{pr}_i = \phi^r$ for every i. Set

$$G^r := \sum_{i=1}^r \mathrm{pr}_i^* G,$$
$$E^r := \sum_{i=1}^r \mathrm{pr}_i^* E,$$
$$\Delta_{M^r} := \sum_{i=1}^r \mathrm{pr}_i^* \Delta_M.$$

Let $\mu : M^{(r)} \to M^r$ be a strong desingularization of M^r such that $\mu|_{\mu^{-1}(M^r_{\mathrm{reg}})}$ is an isomorphism, and set $p_i := \mathrm{pr}_i \circ \mu$, $\phi^{(r)} := \phi^r \circ \mu$, $G^{(r)} := \mu^* G^r$, $E^{(r)} := \mu^* E^r$, $\Delta_{M^{(r)}} := \mu^* \Delta_{M^r}$. By the projection formula and by induction we have

$$\phi^{(r)}_*\mathscr{O}_{M^{(r)}}(G^{(r)} + cE^{(r)})|_{Y_f} \simeq \phi^r_*\mathscr{O}_{M^r}(G^r + cE^r)|_{Y_f} \simeq \bigotimes^r \phi_*\mathscr{O}_M(G + cE)|_{Y_f},$$

Then (4.4) induces a non-zero section

$$s_0 \in \mathrm{H}^0(Y_f, \phi^{(r)}_*\mathscr{O}_{M^{(r)}}(G^{(r)} + cE^{(r)}) \otimes (\det\phi_*\mathscr{O}_M(G + cE))^{-1}),$$

By [Nako4, §III.5, 5.10.Lemma, pp. 107-108] (c.f. [Wan19, Theorem 1.13] for more details) there is an effective divisor B_1 supported in $M^{(r)}\backslash(\phi^{(r)})^{-1}(Y_{\mathrm{f}})$ such that s_0 extends to a non-zero section

$$\bar{s}_0 \in \mathrm{H}^0(M^{(r)}, \mathscr{O}_{M^{(r)}}(G^{(r)} + cE^{(r)} + B_1 - r(\phi^{(r)})^* D_{G,c,1})),$$

in particular $\Delta_0 := G^{(r)} + cE^{(r)} + B_1 - r(\phi^{(r)})^* D_{G,c,1}$ is (linearly equivalent to) an effective divisor on $M^{(r)}$.

(B) Comparison of the relative canonical divisors. By induction and the base change formula of the relative canonical sheaf [Kle80, Proposition (9)] we see that M^r_{f} is Gorenstein and the relative dualizing sheaf

$$\omega_{M^r_{\mathrm{f}}/Y} \simeq \mathscr{O}_{M^r_{\mathrm{f}}}(\sum_{i=1}^{r} \mathrm{pr}_i^* K_{M/Y}).$$

The natural morphism $\omega_{M^r_{\mathrm{f}}/Y} \to \mu_* \mathscr{O}_{M^{(r)}}(K_{M^{(r)}/Y})|_{M^r_{\mathrm{f}}}$ (from [Har77, §II.8, Proposition 8.3, p. 176]) is an isomorphism over M^r_{rat}, the rational singularities locus of M^r. By assumption (c.f. General Setting 4.1.1), the branch locus $\mathrm{Branch}(\phi)$ of ϕ is a SNC divisor on Y and $f^*\mathrm{Branch}(\phi)$ has SNC support. Write

$$f^*\mathrm{Branch}(\phi) := \sum_{\lambda} W_\lambda + \sum_{\mu} a_\mu V_\mu$$

with $a_\mu > 1$ for every μ and set

$$W := \sum_{\lambda} W_\lambda, \qquad V := \sum_{\mu} V_\mu,$$

then by [Hör10, 3.13.Lemma] M^r_{f} has rational singularities along

$$\underbrace{(M_{\mathrm{f}}\backslash(V \cup \phi^{-1}\,\mathrm{Sing}(\mathrm{Branch}(\phi)))) \underset{Y_{\mathrm{f}}\backslash \mathrm{Sing}(\mathrm{Branch}(\phi))}{\times} \cdots \underset{Y_{\mathrm{f}}\backslash \mathrm{Sing}(\mathrm{Branch}(\phi))}{\times} (M_{\mathrm{f}}\backslash(V \cup \phi^{-1}\,\mathrm{Sing}(\mathrm{Branch}(\phi))))}_{r \text{ times}}.$$

Hence there is a divisor B_2 on $M^{(r)}$ supported on

$$E^{(r)} \cup (M^{(r)}\backslash\mu^{-1}(M^r_{\mathrm{f}})) \cup \mathrm{Supp}(\sum_{i=1}^{r} \mathrm{pr}_i^* V)$$

such that

$$-(K_{M^{(r)}/Y} + \Delta_{M^{(r)}}) + B_2 \sim \sum_{i=1}^{r} \mathrm{pr}_i^*(-(K_{M/Y} + \Delta_M) + E').$$

(C) Ohsawa-Takegoshi type Extension. For $y \in Y$ general, the general fibre

$$M^r_y := \underbrace{M_y \times \cdots \times M_y}_{r \text{ times}}$$

of ϕ^r is smooth; since μ is an isomorphism over M^r_{reg}, M^r_y is also the general fibre of $\phi^{(r)}$. Now fix a sufficiently ample divisor A_Y on Y divisible by 2, such that $\frac{1}{2}A_Y - K_Y$ separates

all the $(2\dim Y)$-jets. For $s \in \mathbb{Z}_{>0}$, since $\Delta_0 = G^{(r)} + cE^{(r)} + B_1 - r(\phi^{(r)})^* D_{G,c,1}$ is $\phi^{(r)}$-big, there is $\epsilon \in \mathbb{Q}_{>0}$ sufficiently small such that $\epsilon s\Delta_0 + A_Y$ is big. Then we can write

$$\epsilon s\Delta_0 + \frac{1}{2}A_Y \sim_{\mathbb{Q}} H_{s,\epsilon} + \Delta_{s,\epsilon}.$$

with $H_{p,\epsilon}$ an ample $\mathbb{Q}$-divisor and $\Delta_{p,\epsilon}$ an effective $\mathbb{Q}$-divisor. Now let $t \in \mathbb{Z}_{>0}$ sufficiently large such that

$$\left(\Delta_{M^{(r)}} + \frac{1}{st}\Delta_{s,\epsilon} + \frac{1-\epsilon}{t}\Delta_0\right)\Big|_{M^r_y} = \left(\Delta_{M^r} + \frac{1}{st}\Delta_{s,\epsilon} + \frac{1-\epsilon}{t}\Delta_0\right)\Big|_{M^r_y}$$

is klt. Since $st(-K_{M^{(r)}/Y} - \Delta_{M^{(r)}} + B_2) + H_{s,\epsilon}$ is ample, we can apply [Cao19, Theorem 2.10] (c.f. also [Den17a, Theorem 2.11]) to the divisor

$$\begin{aligned} L &:= st(-K_{M^{(r)}/Y} + B_2) + s\Delta_0 + \frac{1}{2}A_Y \\ &\sim_{\mathbb{Q}} [st(-K_{M^{(r)}/Y} - \Delta_{M^{(r)}} + B_2) + H_{s,\epsilon}] + (st\Delta_{M^{(r)}} + \Delta_{s,\epsilon} + (1-\epsilon)s\Delta_0) \end{aligned}$$

to obtain the surjectivity of the restriction morphism

$$\mathrm{H}^0(M^{(r)}, \mathscr{O}_{M^{(r)}}(stK_{M^{(r)}/Y} + L + \frac{1}{2}A_Y)) \to \mathrm{H}^0(M^r_y, \mathscr{O}_{M^r_y}(stK_{M^{(r)}/Y} + L + \frac{1}{2}A_Y)),$$

which can be rewritten as

$$\mathrm{H}^0(M^{(r)}, \mathscr{O}_{M^{(r)}}(sG^{(r)} + scE^{(r)} + sB_1 + stB_2 + (\phi^{(r)})^*(A_Y - srD_{G,c,1}))) \twoheadrightarrow \mathrm{H}^0(M^r_y, \mathscr{O}_{M^r_y}(sG^r + scE^r))$$

(D) Restriction to the diagonal and conclusion. Now take a non-zero section (since $G + cE$ is ϕ-big, such a section exists)

$$u \in \mathrm{H}^0(M_y, \mathscr{O}_{M_y}(pG + pcE))$$

for $y \in Y$ general, then

$$u^{(r)} := \sum_{i=1}^{r} \mathrm{pr}_i^* u \in \mathrm{H}^0(M^r_y, \mathscr{O}_{M^r_y}(sG^r + scE^r)).$$

By **Step (C)** we get a section

$$\sigma^{(r)} \in \mathrm{H}^0(M^{(r)}, \mathscr{O}_{M^{(r)}}(sG^{(r)} + scE^{(r)} + sB_1 + stB_2 + (\phi^{(r)})^*(A_Y - prD_{G,c,1})))$$

such that $\sigma^{(r)}|_{M^r_y} = u^{(r)}$. Since μ is an isomorphism over $(M \setminus \mathrm{Supp}(V+W))^r \subseteq (M^r)_{\mathrm{reg}}$, then $\sigma^{(r)}|_{(M\setminus \mathrm{Supp}(V+W))^r}$ can be restricted to the diagonal and gives rise to a section

$$\sigma' \in \mathrm{H}^0(M \setminus \mathrm{Supp}(V+W), \mathscr{O}_M(srG + srcE + F'_{s,t} + \phi^*(A_Y - srD_{G,c,1})))$$

for some $F'_{s,t}$ supported in $\mathrm{Supp}(E)$ (by Proposition 4.1.2(c) any ϕ-exceptional divisor is also π-exceptional thus contained in $\mathrm{Supp}(E)$). By construction of B_1 and B_2 we know that σ' is bounded around a general point of W; moreover, by Proposition 4.1.2(d) V is contained in $\mathrm{Supp}(E)$, hence there is a π-exceptional divisor $F_{s,t}$ such that σ' extends to a section

$$\sigma \in \mathrm{H}^0(M, \mathscr{O}_M(srG + srcE + F_{s,t} + \phi^*(A_Y - srD_{G,c,1}))).$$

By construction $\sigma|_{M_y} = u^{\otimes r}$, hence $\sigma \neq 0$, which implies that $srG - +srcE + F_{s,t} + \phi^*(A_Y - srD_{G,c,1})$ is linearly equivalent to an effective divisor on M. But E and $F_{s,t}$ are π-exceptional, hence

$$\pi_*\left(G - \phi^* D_{G,c,1} + \frac{1}{s}\phi^* A_Y\right)$$

is $\mathbb{Q}$-linearly equivalent to an effective (Weil) $\mathbb{Q}$-divisor on X. Since this holds for any $s \in \mathbb{Z}_{>0}$, we can take A_Y to be any ample divisor on Y.

If we assume moreover that $\pi_*(G - \phi^* D_{G,c,1})$ is $\mathbb{Q}$-Cartier, then by taking a sufficiently ample divisor A on X containing $\pi_*\phi^* A_Y$, we see that $\pi_*(G - \phi^* D_{G,c,1}) + \frac{1}{s}A$ is $\mathbb{Q}$-linearly equivalent to an effective (Cartier) $\mathbb{Q}$-divisor, hence $\pi_*(G - \phi^* D_{G,c,1})$ is pseudoeffective. In particular, for $k \in \mathbb{Z}_{>0}$ sufficiently large $G - \phi^* D_{G,c,1} + kE$ is pseudoeffective on M. □

Proposition 4.1.6 ([CH19, Lemma 3.5],[CCM19, Proposition 3.6]). *Let everything be as in the General Settinng 4.1.1 with Y non-uniruled and let G be a ϕ-big divisor. Then there is an $c_0 \in \mathbb{Z}_{>0}$ such that for every $c \geqslant c_0$ the natural inclusion $\det \phi_* \mathscr{O}_M(G + cE) \to \det \phi_* \mathscr{O}_M(G + (c+1)E)$ is an isomorphism over Y_0.*

Proof. If X is smooth, then E cannot dominate Y and the proposition is proved in [CH19, Lemma 3.5]. In our case, X is not necessarily smooth and it takes more effort to prove the proposition. We will follow the same argument of [CCM19, Proposition 3.6] with some clarifications (c.f. Remark 4.1.4). The proof can be divided into two steps:

Step 1: Constancy of the rank of the direct images with respect to c. Since $\mathrm{rk}\, \phi_* \mathscr{O}_M(G + cE) = \mathrm{h}^0(M_y, \mathscr{O}_{M_y}(G + cE))$ for $y \in Y$ general, and since E is effective and π-exceptional, it suffices to prove that $\mathrm{h}^0(M_y, \mathscr{O}_{M_y}(G + cE))$ is bounded by a constant for all $c \in \mathbb{Z}_{>0}$. By [Cao19, Theorem 2.10] and by the argument as in **Step (C)** of the proof of Proposition 4.1.5, for p sufficiently large and for A_Y sufficiently ample on Y divisible by 2 and such that $\frac{1}{2}A_Y - K_Y$ separates all the $(2 \dim Y)$-jets, we have a surjection

$$\begin{aligned}\mathrm{H}^0(M, \mathscr{O}_M(G + cE + pE' + \phi^* A_Y)) = \mathrm{H}^0(M, \mathscr{O}_M(p(K_M + \Delta_M) + p(-K_M - \Delta_M + E') + G + cE + \phi^* A_Y)) \\ \twoheadrightarrow \mathrm{H}^0(M_y, \mathscr{O}_{M_y}(G + cE + pE')),\end{aligned}$$

for $y \in Y$ general. Since $E'|_{M_y}$ is effective, we have

$$\mathrm{h}^0(M_y, \mathscr{O}_{M_y}(G + cE)) \leqslant \mathrm{h}^0(M_y, \mathscr{O}_{M_y}(G + cE + pE')) \leqslant \mathrm{h}^0(M, \mathscr{O}_M(G + cE + pE' + \phi^* A_Y)).$$

It remains to see the boundedness of $\mathrm{h}^0(M, \mathscr{O}_M(G + cE + pE' + \phi^* A_Y))$. By [Nak04, §III.5, 5.10.Lemma, pp. 107-108], for c and p sufficiently large,

$$\pi_* \mathscr{O}_M(G + (pk + c)E + \phi^* A_Y) \simeq (\pi_* \mathscr{O}_M(G + \phi^* A_Y))^{**}$$

for any $k \in \mathbb{Z}_{>0}$. Hence for sufficiently large c and for p sufficiently large with respect to c and G we have

$$\mathrm{h}^0(M, \mathscr{O}_M(G + cE + pE' + \phi^* A_Y)) \leqslant \mathrm{h}^0(M, \mathscr{O}_M(G + (c + pk)E + \phi^* A_Y)) = \mathrm{h}^0(X, (\phi_* \mathscr{O}_M(G + \phi^* A_Y))^{**}),$$

where k is a positive integer such that $E' \leqslant kE$. In consequence $\mathrm{h}^0(M, \mathscr{O}_M(G + cE + pE' + \phi^* A_Y))$ is bounded by a constant independent of c and p, and so is $\mathrm{rk}\, \phi_* \mathscr{O}_M(G + cE)$. In other word, there is $c_0 \in \mathbb{Z}_{>0}$ such that for any $c \geqslant c_0$, the rank of $\phi_* \mathscr{O}_M(G + cE)$ is independent of c.

Step 2: Stability of the determinant sheaf over Y_0. By contradiction, let us assume that there is an increasing sequence $(c_k)_{k \in \mathbb{Z}_{>0}}$ such that $c_1 \geqslant c_0$, $c_k \nearrow +\infty$ and that there is some effective divisor B_k on Y such that $B_k \cap Y_0 \neq \varnothing$ (in particular $B_k \neq 0$) and

$$rD_{G,c_{k+1},1} - (rD_{G,c_k,1} + B_k)$$

is linearly equivalent to an effective divisor on M for every k, where $r := \mathrm{rk}\phi_*\mathscr{O}_M(G+c_0E)$ and

$$D_{G,c,1} := \frac{1}{r}\cdot \text{ the Cartier divisor on } Y \text{ associated to } \det\phi_*\mathscr{O}_M(G+cE).$$

By **Step 1** for any $c \geqslant c_0$, $\mathrm{rk}\phi_*\mathscr{O}_M(G+cE) = r$. Then by Proposition 4.1.5 for any ample divisor A_Y on Y and for any $p \in \mathbb{Z}_{>0}$ the $\mathbb{Q}$-divisor

$$\pi_*\left(G - \phi^*D_{G,c_k,1} + \frac{1}{s}\phi^*A_Y\right)$$

is $\mathbb{Q}$-linearly equivalent to an effective divisor. In particular, take $s = r$, we see that for every $N > 0$

$$r\pi_*G + \pi_*\phi^*A_Y - \sum_{k=1}^{N}\pi_*\phi^*B_k$$

is linear equivalent to a Weil divisor on X. But since $B_k \cap Y_0 \neq \emptyset$, ϕ^*B_k is not π-exceptional, hence $\pi_*\phi^*B_k$ is non-zero effective for every k. By letting $N \to +\infty$ we see that this is impossible. □

As an immediate corollary of Proposition 4.1.6 we have

Corollary 4.1.7. *Let everything be as in the General Settinng 4.1.1 with Y non-uniruled and let G be a ϕ-big divisor. Let c_0 be the integer given by the Proposition 4.1.6 and let $c \geqslant c_0$. For every $a \in \mathbb{Z}_{>0}$ set*

$$D_{G,c,a} := \frac{1}{r_a}\cdot \textit{the Cartier divisor on } Y \textit{ associated to the line bundle } \det\phi_*\mathscr{O}_M(aG+acE)$$

where $r_a := \mathrm{rk}\phi_*\mathscr{O}_M(aG+acE)$. Then

(a) *$\phi_*\mathscr{O}_M(G+cE)$ is isomorphic to $\phi_*\mathscr{O}_M(G+kE+pE')$ over Y_0 for any $k \geqslant c$ and for any $p \in \mathbb{Z}_{\geqslant 0}$ rendering pE' integral;*

(b) *Suppose that π_*G and $\pi_*\phi^*D_{G,c,b}$ are $\mathbb{Q}$-Cartier on X for some $b \in \mathbb{Z}_{>0}$. Then $\phi_*\mathscr{O}_M(G+cE)$ is $\frac{1}{b}D_{G,c,b}$-weakly semipositively curved over Y_0.*

Proof. By construction $E' = \sum_{i\in I^+} a_iE_i - \phi^*N_Y$ with N_Y an effective $\mathbb{Q}$-divisor on Y supported out of Y_0, hence Proposition 4.1.6 implies that $\mathrm{rk}\phi_*\mathscr{O}_M(G+kE+pE') = r_1$ and that the natural injection

$$\mathscr{O}_Y(rD_{G,c,1}) \simeq \det\phi_*\mathscr{O}_M(G+cE) \hookrightarrow \det\phi_*\mathscr{O}_M(G+kE+pE')$$

is an isomorphism over Y_0. By [DPS94, Lemma 1.20] this means that the natural inclusion

$$\phi_*\mathscr{O}_M(G+cE) \hookrightarrow \phi_*\mathscr{O}_M(G+kE+pE') \tag{4.5}$$

is an isomorphism over the locally free locus of $\phi_*\mathscr{O}_M(G+kE+pE')\big|_{Y_0}$. Since ϕ is flat over Y_0, both $\phi_*\mathscr{O}_M(G+cE)$ and $\phi_*\mathscr{O}_M(G+kE+pE')$ are reflexive over Y_0, hence (4.5) must be an isomorphism over Y_0. Thus (a) is proved.

As for (b), since by hypothesis π_*G and $\pi_*\phi^*D_{G,c,b}$ are $\mathbb{Q}$-Cartier on X, then by Proposition 4.1.5 we see that $b\pi_*G - \pi_*\phi^*D_{G,c,b}$ is a pseudoeffective ($\mathbb{Q}$-Cartier) $\mathbb{Q}$-divisor on X. In consequence there is an integer $k \in \mathbb{Z}_{>0}$ such that $\pi_*G + kE - \frac{1}{b}\phi^*D_{G,c,b}$ is pseudoeffective on M. Then by Proposition 4.1.3, for p_k sufficiently large $\phi_*\mathscr{O}_M(G+kE+p_kE')$ is $\frac{1}{b}D_{G,c,b}$-weakly semipositively curved. Combine this with (a) we see that $\phi_*\mathscr{O}_M(G+cE)$ is $\frac{1}{b}D_{G,c,b}$-weakly semipositively curved over Y_0, which proves (b). □

Proposition 4.1.8 ([CH19, Proposition 3.6]). *Let everything be as in the General Settinng 4.1.1 with Y non-uniruled. Suppose that ψ is almost holomorphic and let A be a sufficiently ample divisor on X such that for general $y \in Y$ the natural morphism*

$$\mathrm{Sym}^k \mathrm{H}^0(X_y, \mathscr{O}_{X_y}(A)) \to \mathrm{H}^0(X_y, \mathscr{O}_{X_y}(kA)) \tag{4.6}$$

is surjective for every $k \in \mathbb{Z}_{>0}$. Let c_0 be the positive integer given by Proposition 4.1.6 and let c be any integer $\geqslant c_0$. For every $a \in \mathbb{Z}_{>0}$ set

$$D_{A,c,a} := \frac{1}{r_a} \cdot \textit{the Cartier divisor on } Y \textit{ associated to } \det \phi_* \mathscr{O}_M(a\pi^* A + acE)$$

where $r_a := \mathrm{rk}\phi_* \mathscr{O}_M(a\pi^* A + acE)$, and suppose that $\pi_* \phi^* D_{A,c,1}$ is $\mathbb{Q}$-Cartier on X (e.g. when X is $\mathbb{Q}$-factorial). Then for any $m \in \mathbb{Z}_{>0}$ divisible by $r := r_1$ such that $\pi_* \phi^* D_{A,c,m}$ is $\mathbb{Q}$-Cartier, we have

$$\pi_* \phi^* D_{A,c,m} \equiv m \pi_* \phi^* D_{A,c,1}$$

where $\equiv$ denotes the numerical equivalence.

Before proving the proposition, let us first prove the following auxiliary lemma:

Lemma 4.1.9. *Let everything be as in the General Settinng 4.1.1 with Y non-uniruled. Suppose that ψ is almost holomorphic and let A as in Proposition 4.1.8. For every m divisible by r set*

$$\mathscr{U}_{c,m} := \mathrm{Sym}^m \phi_* \mathscr{O}_M(\pi^* A + cE) \otimes \mathscr{O}_Y(-mD_{A,c,1}),$$
$$\mathscr{V}_{c,m} := \phi_* \mathscr{O}_M(m\pi^* A + mcE) \otimes \mathscr{O}_Y(-mD_{A,c,1}),$$

then $\mathscr{U}_{c,m}$ and $\mathscr{V}_{c,m}$ are both weakly semipositively curved on Y_0.

Proof. By hypothesis $\pi_* \phi^* D_{A,c,1}$ is $\mathbb{Q}$-Cartier on X, hence by Corollary 4.1.7(b) we see that $\phi_* \mathscr{O}_M(\pi^* A + cE)$ is $D_{A,c,1}$-weakly semipositively curved on Y_0, which implies that $\mathscr{U}_{c,m}$ is weakly semipositively curved on Y_0.

By (4.6) and by [Debo1, Lemma 7.11] we have a surjection

$$\mathrm{Sym}^m \mathrm{H}^0(M_y, \mathscr{O}_{M_y}(\pi^* A + cE)) \twoheadrightarrow \mathrm{H}^0(M_y, \mathscr{O}_{M_y}(m\pi^* A + mcE))$$

for $y \in Y$ general, from which we see that the natural morphism $\mathscr{U}_{c,m} \to \mathscr{V}_{c,m}$ is generically surjective. Hence $\mathscr{V}_{c,m}$ is also weakly semipositively curved on Y_0. □

Moreover, in the statement of Proposition 4.1.8 we presume the existence of a very ample divisor A on X satisfying the condition (4.6). We will next show that such divisor really exists. More generally we have:

Lemma 4.1.10. *Let V be a normal projective variety and let H be a semiample divisor. Then up to multiplying H the natural morphism*

$$\mathrm{Sym}^k \mathrm{H}^0(V, \mathscr{O}_V(H)) \to \mathrm{H}^0(V, \mathscr{O}_V(kH))$$

is surjective for every $k \in \mathbb{Z}_{>0}$.

Proof. The proof is quite similar to that of [Debo1, §7.1, Proposition 7.6, pp. 172-173]. First by [Debo1, §7.1, Proposition 7.6(b), p. 172], up to replacing V by the image of $|rH|$ for some r sufficiently large and replacing H by the hyperplane divisor, we can assume that H is a very ample and V is embedded into $\mathbb{P}E$ so that $\mathscr{O}_V(H)$ is equal to the pullback of $\mathscr{O}(1)$ where $E := \mathrm{H}^0(V, \mathscr{O}_V(H))$. Then by the Serre vanishing, for s sufficiently large, we have

$$\mathrm{H}^1(\mathbb{P}E, \mathscr{I}_V(ks)) = 0.$$

for any $k \in \mathbb{Z}_{>0}$,where $\mathscr{I}_X$ is the ideal of X in $\mathbb{P}E$. Hence the surjectivity of

$$\mathrm{H}^0(\mathbb{P}E, \mathscr{O}(ks)) \to \mathrm{H}^0(V, \mathscr{O}_V(ksH)).$$

But

$$\mathrm{H}^0(\mathbb{P}E, \mathscr{O}(ks)) \simeq \mathrm{Sym}^{ks}\mathrm{H}^0(\mathbb{P}E, \mathscr{O}(1)) \simeq \mathrm{Sym}^{ks}\mathrm{H}^0(V, \mathscr{O}_V(H)), \tag{4.7}$$

thus

$$\mathrm{Sym}^{ks}\mathrm{H}^0(V, \mathscr{O}_V(H)) \twoheadrightarrow \mathrm{H}^0(V, \mathscr{O}_V(ksH)).$$

in particular by taking $k = 1$ we get

$$\mathrm{Sym}^{ks}\mathrm{H}^0(V, \mathscr{O}_V(H)) \twoheadrightarrow \mathrm{Sym}^{k}\mathrm{H}^0(V, \mathscr{O}_V(sH)).$$

Now the map (4.7) factorizes through

$$\mathrm{Sym}^{ks}\mathrm{H}^0(V, \mathscr{O}_V(H)) \to \mathrm{Sym}^{k}\mathrm{H}^0(V, \mathscr{O}_V(sH)),$$

hence sH satisfies the condition, and the lemma is proved. □

Now let us turn to the proof of Proposition 4.1.8:

Proof of Proposition 4.1.8. By Proposition 4.1.6, as soon as $c \geqslant c_0$, for any $a \in \mathbb{Z}_{>0}$ the divisor $\pi_*\phi^*D_{A,c,a}$ over X is independent of c. Hence it suffices to prove the proposition for a particular choice of $c \geqslant c_0$. By Kleiman's criterion for numerical triviality [GKP16, Lemma 4.1], it suffices to show that for any $(\dim X - 1)$-tuple of ample line bundles $L_1, \cdots, L_{\dim X-1}$ on X the intersection number

$$L_1 \cdot \cdots \cdot L_{\dim X-1} \cdot (\pi_*\phi^*D_{A,c,m} - m\pi_*\phi^*D_{A,c,1}) = 0.$$

To this end, let H_i be a general member of $\left|L_i^{\otimes k}\right|$ for k sufficiently large and set $C = H_1 \cap \cdots \cap H_{\dim X-1}$. By the projection formula it suffices to show that

$$(\phi^*D_{A,c,m} - m\phi^*D_{A,c,1}) \cdot (\pi^{-1}C) = 0.$$

Since $\pi(\mathrm{Exc}(\pi))$ is of codimension 2 in X, C is disjoint from $\pi(\mathrm{Exc}(\pi))$, then $\pi^{-1}C$ is disjoint from E and thus $C_Y := \phi(\pi^{-1}C)$ is contained in Y_0. Let $\bar{C}_Y$ be the normalization of C_Y and let $\bar{i}_{C_Y} : \bar{C}_Y \to Y$ be the natural morphism. Again by the projection formula, we are reduced to show that

$$\bar{i}_{C_Y}^*(D_{A,c,m} - mD_{A,c,1}) = 0.$$

As in Lemma 4.1.9, we set for any m divisible by r

$$\mathscr{U}_{c,m} := \mathrm{Sym}^m \phi_*\mathscr{O}_M(\pi^*A + cE) \otimes \mathscr{O}_Y(-mD_{A,c,1}),$$
$$\mathscr{V}_{c,m} := \phi_*\mathscr{O}_M(m\pi^*A + mcE) \otimes \mathscr{O}_Y(-mD_{A,c,1}),$$

Since $\mathscr{U}_{c,m}$ and $\mathscr{V}_{c,m}$ are torsion free, we can assume that C_Y is contained in the locally free locus of them.

By Lemma 4.1.9 $\mathscr{V}_{c,m}$ is weakly semipositively curved, since C_Y is a general complete intersection curve, C_Y is not contained in the singular locus of the $-\epsilon\omega$-semipositive metric of $\mathscr{V}_{c,m}$, $\mathscr{V}_{c,m}|_{Y_0}$ is semipositively curved on C_Y, in particular $\bar{i}_{C_Y}^* \det \mathscr{V}_{c,m} \geqslant 0$. But $\det \mathscr{V}_{c,m} \simeq \mathscr{O}_Y(r_m D_{A,c,m} - mr_m D_{A,c,1})$, hence we have

$$\bar{i}_{C_Y}^*(D_{A,c,m} - mD_{A,c,1}) \geqslant 0.$$

On the other hand, $\pi_*\phi^*D_{A,c,m}$ is $\mathbb{Q}$-Cartier on X, then by Corollary 4.1.7(b) we see that $\phi_*\mathscr{O}_M(\pi^*A+cE)$ is $\frac{1}{m}D_{A,c,m}$-weakly semipositively curved over Y_0, in consequence

$$\bar{i}^*_{C_Y}\phi_*\mathscr{O}_M(\pi^*A+cE)$$

is $(\frac{1}{m}\bar{i}^*_{C_Y}D_{A,c,m})$-weakly semipositively curved. Hence

$$\bar{i}^*_{C_Y}\det\phi_*\mathscr{O}_M(\pi^*A+cE)-\frac{r}{m}\bar{i}^*_{C_Y}D_{A,c,m}\geqslant 0,$$

implying that

$$\bar{i}^*_{C_Y}(D_{A,c,m}-mD_{A,c,1})\leqslant 0.$$

□

4.2 Albanese map of X

In this section, we take ψ in the General Setting 4.1.1 to be the Albanese map alb_X of X which admits an effective $\mathbb{Q}$-divisor such that (X,Δ) is klt and that the twisted anticanonical divisor $-(K_X+\Delta)$ is nef. In this case, we can take M any smooth model of X, $\phi=\mathrm{alb}_M$ and $Y=\mathrm{Im}(\mathrm{alb}_M)\subseteq\mathrm{Alb}_M$, by [Uen75, Proposition 9.12, pp. 107-108] ψ is independent of the choice of the smooth model M.

First recall the basic properties of the Albanese map (c.f. [Uen75, §9, pp. 94-115]):

Proposition 4.2.1. *Let V a compact Kähler manifold and let $\mathrm{alb}_V:V\to\mathrm{Alb}_V$ be its Albanese map. Then we have:*

(a) *alb_V satisfies the following universal property: every morphism $V\to T$ with T a complex torus factorizes via $\mathrm{alb}_V:V\to\mathrm{Alb}_V$; in addition $\mathrm{Alb}_V\to T$ is a morphism of analytic Lie groups up to a translation. C.f.* [Uen75, Defintion 9.6, pp. 101-102].

(b) *$W:=\mathrm{Im}(\mathrm{alb}_V)$ generates Alb_V, i.e. there is an integer $k>0$ such that the morphism*

$$\begin{aligned}\underbrace{W\times\cdots\times W}_{k\text{ times}}&\longrightarrow\mathrm{Alb}_V,\\(w_1,\cdots,w_k)&\longmapsto w_1+\cdots+w_k,\end{aligned}$$

is surjective. C.f. [Uen75, Lemma 9.14, pp. 108-110].

More generally, for V a compact complex variety in the Fujiki class $\mathscr{C}$ (not necessarily smooth), the Albanese map alb_V of V is defined to be the meromorphic map induced by the Albanese map of a smooth model of V (this definition is independent of the choice of the smooth model by [Uen75, Proposition 9.12, pp. 107-108]). In this case, alb_V has the universal property that every meromorphic map from V to a complex torus factorizes via alb_V (analogous to Proposition 4.2.1(a)), c.f. [Wan16b, Theorem-Definition 2.1].

4.2.1 Everywhere-definedness, surjectivity and connectedness of fibres of alb_X

In this subsection, we briefly recall how one proves that $\psi=\mathrm{alb}_X$ is everywhere defined, surjective and with connected fibres:

- Since (X,Δ) is a klt pair, in particular X has rational singularities (c.f. [KM98, Theorem 5.22, pp. 161-162]) hence by [Kaw85, Lemma 8.1], ψ is a(n) (everywhere defined) morphism $X\to\mathrm{Alb}_X$.

- By Proposition 4.1.2(b), the Kodaira dimension of $\mathrm{Im}(\psi)$ is equal to 0, then [Uen75, Theorem 10.3] implies that $\mathrm{Im}(\psi)$ is a translate of a subtorus of Y; in virtue of Proposition 4.2.1(b), a fortiori $\mathrm{Im}(\psi) = Y$, i.e. ψ is surjective.
- To see that ψ has connected fibres, let us take $\pi : Y' \to Y$ a Stein factorization of ψ with Y' a normal projective variety, then by Proposition 4.1.2(b) we have $\kappa(Y') = 0$, which implies, in virtue of [KV80, Main Theorem], that π is a finite étale cover. Then the theorem of Serre-Lang [Mum70, §18, pp. 167-168] implies that Y' is an abelian variety with π an isogeny. By Proposition 4.2.1(a) π is a fortiori an isomorphism.

4.2.2 Flatness of alb_X

In order to apply Proposition 2.3.5 to ψ one needs to prove first that it is flat. In this subsection we will settle this by following the argument of [LTZZ10]. Recall that X is a normal projective variety which admits an effective $\mathbb{Q}$-divisor Δ such that (X, Δ) is a klt pair and that the twisted anticanonical divisor $-(K_X + \Delta)$ is nef. The flatness of ψ can be deduced from the following lemma. Let us remark that under the additional assumption that V is smooth and $D = 0$, a stronger result is obtained in [EIM20, Proposition 4.1] (they also prove the semistability of the fibre space).

Lemma 4.2.2. *Let $f : V \to W$ a surjective morphism with connected fibres with V a projective Gorenstein variety and W a smooth projective variety. Suppose that there is an effective $\mathbb{Q}$-divisor D on V such that (V, D) is a log canonical pair and that the twisted relative anticanonical divisor $-(K_{V/W} + D)$ is nef on V. Then f is flat.*

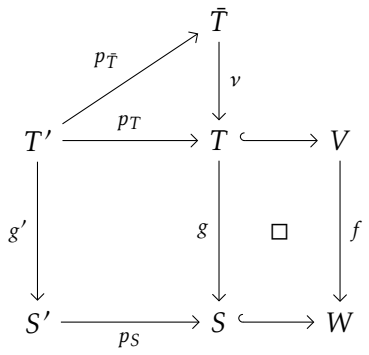
Proof. By the miracle flatness, it suffices to show that f is equi-dimensional. Suppose by contradiction that f is not so, then there is a (closed) point $w_0 \in W$ such that $\dim V_{w_0} > \dim F$ where F denotes the general fibre of f. Now take S to be the complete intersection of $\dim V - \dim V_{w_0} + 1$ general very ample divisors passing through w_0. Then by Bertini S is a smooth projective variety containing w_0 of dimension $\dim S = \dim V_{w_0} - \dim F + 1$. Set $T = S \underset{W}{\times} V$ with $g : T \to S$ the induced morphism, then $\dim T = \dim V_{w_0} + 1$.

Let us remark that in [LTZZ10] it is claimed that T is smooth in codimension 1; but this cannot be true in general since a priori V_{w_0} can be a non-reduced fibre of f which is a codimension 1 subvariety contained in T. We will present below a proof avoiding the use of this claim.

By construction T is a complete intersection in X, thus T is Gorenstein by [Mat89, Exercise 18.1, p. 152]. By adjunction formula [CDGPR94, §II.5, Proposition 5.26, pp. 139-140] one finds that

$$K_{T/S} = K_T - g^* K_S \sim K_{V/W}\big|_T$$

Now take a flattening morphism $p_S : S' \to S$ of g (c.f. [Hir75, Flatenning Theorem]) and take T' a desginularization of the principal component of $T \underset{S}{\times} S$ with $g' : T' \to S'$ and $p_T : T' \to T$ the induced morphisms. Then every g'-exceptional divisor must be p_T-exceptional.



Take the normalization $\nu : \bar{T} \to T$ of T, then p_T factors through ν, and denote by $p_{\bar{T}}$ the induced morphism $T' \to \bar{T}$. By [Rei94, 2.3 Proposition] (noting that T is Cohen-Macaulay thus S_2) we have

$$\nu^* K_T \sim K_{\bar{T}} + \mathrm{Cond}_{\bar{T}}$$

where $\mathrm{Cond}_{\bar{T}}$ is the effective Weil divisor defined by the conductor ideal on $\bar{T}$. Now we can write

$$K_{T'} + (p_T)_*^{-1} D_T \sim_{\mathbb{Q}} p_T^*(K_T + D_T) + E_T - G,$$
$$K_{S'} \sim p_S^* K_S + E_S,$$

where $D_T := D|_T$, E_T is a (non necessarily effective) p_T-exceptional (thus $p_{\bar{T}}$-exceptional) divisor, G is an effective divisor consisting of the non-exceptional components of the pullback of $\mathrm{Cond}_{\bar{T}}$, and E_S is an effective p_S-exceptional divisor (noting that S is smooth). Hence

$$K_{T'/S'} + (p_T)_*^{-1} D_T \sim_{\mathbb{Q}} p_T^*(K_{V/W} + D)\big|_T + E_T - G - (g')^* E_S.$$

Moreover, let F be the general fibre of g, then by construction it is also the general fibre of f, by [KM98, Lemma 5.7, pp. 158-159] (F, D_F) is a lc pair where $D_F := D|_F$, hence the horizontal part of $E_T - G$ has coefficients $\geqslant -1$. Write

$$(E_T - G)^{\mathrm{horiz}} := \sum_{j \in J} b_j B_j$$

with the B_j's being prime divisors, and set

$$\Delta_0 := \sum_{j \in J, b_j < 0} (-b_j) B_j,$$

then every coefficient in Δ_0 is $\leqslant 1$. By the construction of Δ_0, we can rewrite $E_T + \Delta_0 - G$ as $E'_T - G'$ with E'_T being p_T-exceptional and G' being effective whose components come from the conductor divisor of the normalization of T. Clearly the support of E'_T (resp. G') is contained in that of E_T (resp. G).

Since $\dim T = \dim V_{w_0} + 1$, $p_T^{-1}(V_{w_0})$ is a non-p_T-exceptional divisor in T', hence is not g'-exceptional, consequently $g'(p_T^{-1}(V_{w_0}))$ contains a codimension 1 component, which we denote by E. Then $p_S(E) = \{w_0\}$ hence $E \subseteq \mathrm{Supp}(E_S)$ (by assumption g is not flat, hence $E_S \neq 0$ and $\mathrm{Supp}(E_S) \neq \varnothing$).

Take an ample divisor A on T', since $-(K_{V/W} + D)$ is nef then for any $\epsilon \in \mathbb{Q}_{>0}$ the $\mathbb{Q}$-divisor $-p_T^*(K_{V/W} + D)\big|_T + \epsilon A$ is ample. Choose an ample $\mathbb{Q}$-divisor H_ϵ on S' such that $-p_T^*(K_{V/W} + D)\big|_T + \epsilon A - (g')^* H_\epsilon$ is still ample. Let

$$\Delta_\epsilon := (p_T)_*^{-1} D_T + \Delta_0 + \frac{1}{k} \cdot \text{general member of } \left| k\left(- p_T^*(K_{V/W} + D)\big|_T + \epsilon A - (g')^* H_\epsilon \right) \right|,$$

where k is a positive integer sufficiently large and divisible (so that $\epsilon \cdot k \in \mathbb{Z}$ and that kH_ϵ is an integral divisor). Then the coefficients in Δ_ϵ are $\leqslant 1$, thus the pair (T', Δ_ϵ) is lc. By [Fuj17, Theorem 1.1] the direct image sheaf $g'_*\mathscr{O}_{T'}(k(K_{T'/S'} + \Delta_\epsilon))$ is weakly positive; $(E_T - G)^{\mathrm{vert}} - (g')^* E_S$ being g'-vertical and $(E_T - G)^{\mathrm{horiz}} + \Delta_0$ being effective, $K_{T'/S'} + \Delta_\epsilon$ is big on the general fibre of g', in particular we have

$$g'_*\mathscr{O}_{T'}(k(K_{T'/S'} + \Delta_\epsilon)) \neq 0.$$

Hence there is $p \in \mathbb{Z}_{>0}$ such that

$$\hat{\mathrm{S}}^p g'_*\mathscr{O}_{T'}(k(K_{T'/S'} + \Delta_\epsilon)) \otimes \mathscr{O}_{S'}(kpH_\epsilon) \simeq \hat{\mathrm{S}}^p g'_*\mathscr{O}_{T'}(k(\epsilon A + E_T + \Delta_0 - G - (g')^* E_S))$$

is generically globally generated, that is, there is a generically surjective morphism

$$\mathscr{O}_{S'}^{\oplus d} \to \hat{\mathrm{S}}^p g'_*\mathscr{O}_{T'}(k(\epsilon A + E_T + \Delta_0 - G - (g')^* E_S)),$$

where

$$d := \dim \mathrm{H}^0(S', \hat{\mathrm{S}}^p g'_*\mathscr{O}_{T'}(k(\epsilon A + E_T + \Delta_0 - G - (g')^* E_S))) \in \mathbb{Z}_{>0}.$$

Pull it back to T' and combined with the natural (non-trivial) morphism

$$(g')^* \hat{\mathrm{S}}^p g'_*\mathscr{O}_{T'}(k(\epsilon A + E_T + \Delta_0 - G - (g')^* E_S)) \to \mathscr{O}_{T'}(kp(\epsilon A + E_T + \Delta_0 - G - (g')^* E_S))$$

one finds that $\epsilon A + E_T + \Delta_0 - G - (g')^* E_S$ is $\mathbb{Q}$-linearly equivalent to an effective $\mathbb{Q}$-divisor. Letting $\epsilon \to 0$, we obtain that $E_T + \Delta_0 - G - (g')^* E_S = E'_T - G' - (g')^* E_S$ is pseudoeffective. If (V, D) is klt, the pseudoeffectivity result can also be obtained by the semipositivity of the curvature current of the relative m-Bergman kernel metric on the twisted relative canonical bundle (c.f. [Cao17, Theorem 1.2]).

Finally, let L be a very ample line bundle on $\bar{T}$, and let $H_1, \cdots, H_{\dim V_{w_0}}$ be general members of the linear series $\left|p_{\bar{T}}^* L\right|$. Set

$$C := H_1 \cap \cdots \cap H_{\dim V_{w_0}},$$

then C is a movable curve on T', hence $(E'_T - G' - (g')^* E_S) \cdot C \geqslant 0$ by [BDPP13, 0.2 Theorem] (c.f. also [Laz04, vol.II, Theorem 11.4.19, p. 308]). The divisor E'_T being $p_{\bar{T}}$-exceptional, we have $E'_T \cdot C = 0$ by the projection formula. Thus we get

$$(g')^* E \cdot C \leqslant (g')^* E_S \cdot C \leqslant -G' \cdot C \leqslant 0$$

where the last inequality results from the effectivity of G'. On the other hand, $(g')^* E$ is not $p_{\bar{T}}$-exceptional, hence $(p_{\bar{T}})_*(g')^* E$ is an effective (Weil) divisor on $\bar{T}$ (e.g. it contains $\nu^{-1}(V_{w_0})$), thus again by the projection formula one gets

$$(g')^* E \cdot C = (p_{\bar{T}})_*(g')^* E \cdot L^{\dim V_{w_0}} > 0,$$

which is a contradiction. Hence f is flat. □

4.2.3 Reduction to $\mathbb{Q}$-factorial case

In this subsection we prove that in order to prove Theorem B, we can assume that X is $\mathbb{Q}$-factorial. The key ingredient in the proof of this reduction is the following lemma:

Lemma 4.2.3. *Let $p : S \to B$ and $f : S' \to S$ be projective surjective morphisms between normal complex varieties such that $f_*\mathscr{O}_{S'} \simeq \mathscr{O}_S$. Suppose that $p \circ f$ induces a decomposition of S' into a product $B \times Y'$ with $q(Y') = 0$. Then there is a normal projective variety Y along with a projective morphism $g : Y' \to Y$ such that p induces a decomposition of S into a product $B \times Y$ and that under the decompositions $S' \simeq B \times Y'$ and $S \simeq B \times Y$ we have $f = \mathrm{id}_B \times g$.*

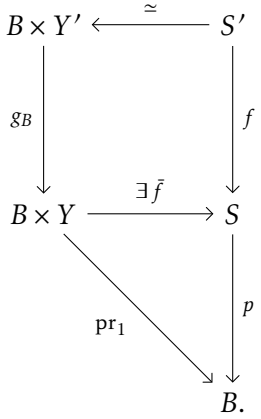
Proof. This is the relative version of [Dru18a, Lemma 4.6]. In fact, when B is a projective variety, it is just a simple corollary of [Dru18a, Lemma 4.6]; in order to apply to our situation we need to treat the case that B is a (non-necessarily compact) complex manifold. The proof can be divided into four parts.

(A) Construction of g. Since $p : S \to B$ is a projective morphism, there is a p-very ample line bundle L on S; since $q(Y') = 0$ by (the analytic version of) [Har77, §III.12, Exercise 12.6, p. 292] there are line bundles $L_B \in \mathrm{Pic}(B)$ and $L_{Y'} \in \mathrm{Pic}(Y')$ such that

$$f^*L \simeq \mathrm{pr}_1^* L_B \otimes \mathrm{pr}_2^* L_{Y'}$$

with $\mathrm{pr}_1 := p \circ f$ and pr_2 being natural projections of $S' \simeq B \times Y'$. Up to replacing L by $L \otimes p^* L_B^{-1}$ we can assume that $f^*L \simeq \mathrm{pr}_2^* L_{Y'}$ for some line bundle $L_{Y'}$. Since f^*L is $(p \circ f)$-relatively generated, hence $L_{Y'}$ is globally generated over Y'. Then by [Laz04, §2.1.B, Theorem 2.1.27, pp. 129-130, Vol.I] for m sufficiently large, $L_{Y'}^{\otimes m}$ defines a morphism $g : Y' \to Y$ with connected fibres. In addition, by construction there is a very ample divisor H on Y such that $g^*\mathscr{O}_Y(H) \simeq L_{Y'}^{\otimes m}$.

(B) Contraction of the fibres of $\mathrm{id}_B \times g$ by f. Set (by identifying S' with $B \times Y'$) $g_B = \mathrm{id}_B \times g : S' \to B \times Y$. Then we have the following commutative diagram:



In this part we will prove that every fibre of g_B is contracted by f. Let $g_B^{-1}(b,z)$ be a positive dimensional fibre of g_B (with $(b,z) \in B \times Y$), since $g_B^{-1}(b,z) \simeq g^{-1}(z) =: Y'_z$, it can be regarded as a subvariety of Y' contracted by g. Let C any curve contained in $g_B^{-1}(b,z)$, then $C \subseteq (p \circ f)^{-1}(b) =: S'_b$ and since C is contracted by g we have

$$(f^*L)|_{S'_b} \cdot C = L_{Y'} \cdot C = \frac{1}{m} g^*H \cdot C = 0,$$

which means that C is contracted by f. Hence every fibre of g_B is contracted by f.

(C) Factorization of f through g_B. In this step we prove that f factorizes through g_B. This can be deduced from the following rigidity lemma, which is nothing but an analytic version of [Deb01, Lemma 1.15, pp. 12-13]:

Lemma 4.2.4. *Let $f_1 : S' \to S_1$ and $f_2 : S' \to S_2$ be proper surjective morphisms between normal complex varieties such that $f_{1*}\mathscr{O}_{S'} \simeq \mathscr{O}_{S_1}$. If f_2 contracts every fibre of f_1, then f_2 factorizes through f_1.*

Proof of the Lemma 4.2.4. The proof is the same as the one of [Deb01, Lemma 1.15, pp. 12-13]. For the convenience of the readers we give the details below to illustrate that the argument in [Deb01, Proof of Lemma 1.15, pp. 12-13] fits into the analytic case. Consider the morphism

$$\phi := (f_1, f_2) : S' \to S_1 \times S_2.$$

Let Γ be the image of ϕ and let $p_1 : \Gamma \to S_1$ and $p_2 : \Gamma \to S_2$ be the natural projections restricted to Γ, then $p_i \circ \phi = f_i$ for $i = 1, 2$. For any $s \in S_1$, f_2 contracts $f_1^{-1}(s) = (\phi \circ p_1)^{-1}(s)$, hence

$$p_1^{-1}(s) = \phi(\phi^{-1}(p_1^{-1}(s))) = \phi(f_1^{-1}(s))$$

is a singleton, hence the proper surjective morphism $p_1 : \Gamma \to S_1$ is a finite morphism. But f_1 has connected fibres, then so is p_1, thus by Stein factorization [Uen75, §1, Theorem 1.9, pp. 8-9] p_1 is an isomorphism. Then we have $\phi = p_1^{-1} \circ f_1$ and

$$f_2 = p_2 \circ \phi = p_2 \circ p_1^{-1} \circ f_1.$$

□

(D) Conclusion. By **(C°)** there is a morphism $\bar{f} : B \times Y \to S$ such that $f = \bar{f} \circ g_B$. Hence

$$g_B^*\left(\bar{f}^* L^{\otimes m}\right) = f^* L^{\otimes m} \simeq \mathrm{pr}_2^* L_{Y'}^{\otimes m} \simeq \mathrm{pr}_2^* g^* \mathscr{O}_Y(H) = g_B^*(\mathrm{pr}_2^* \mathscr{O}_Y(H)).$$

But g has connected fibres, hence so is g_B, in consequence g_B^* is an injective morphism between Picard groups, thus $\bar{f}^* L^{\otimes m} \simeq \mathrm{pr}_2^* \mathscr{O}_Y(H)$. Since H is very ample and L is p-relatively very ample, by looking at every fibre of p, we see that $\bar{f}$ is a finite morphism; but

$$\bar{f}_* \mathscr{O}_{B \times Y} \simeq \bar{f}_* g_{B*} \mathscr{O}_{S'} \simeq f_* \mathscr{O}_{S'} \simeq \mathscr{O}_S,$$

hence $\bar{f}$ is an isomorphism. □

Now let us return to the proof of the reduction of Theorem B, whose idea comes from the author's personal communications with Stéphane Druel (of course, any mistake is the author's):

Reduction to the $\mathbb{Q}$*-factorial case.* Suppose that Theorem B holds for X $\mathbb{Q}$-factorial, let us prove it for general X. Let $g : X^{\mathrm{qf}} \to X$ be a $\mathbb{Q}$-factorialization of X, whose existence is proved in [Kol13, Corollary 1.37, pp. 29-30]. By construction, g is a small birational morphism, hence

$$K_{X^{\mathrm{qf}}} + g_*^{-1}\Delta \sim_{\mathbb{Q}} g^*(K_X + \Delta)$$

then $(X^{\mathrm{qf}}, g_*^{-1}\Delta)$ is a klt pair with the twisted anticanonical divisor $-(K_{X^{\mathrm{qf}}} + g_*^{-1}\Delta)$ nef. In particular $\mathrm{alb}_{X^{\mathrm{qf}}}$ is an everywhere defined morphism; and since the Albanese map is independent of the choice of the birational model, we have $\mathrm{Alb}_{X^{\mathrm{qf}}} = \mathrm{Alb}_X$ and

$$\mathrm{alb}_{X^{\mathrm{qf}}} = g \circ \mathrm{alb}_X.$$

Now by our assumption $\mathrm{alb}_{X^{\mathrm{qf}}}$ is a locally constant fibration whose fibre has vanishing irregularity, then by passing to the universal cover of Alb_X and by Lemma 4.2.3 we see that alb_X is also a locally constant fibration. □

In the sequel of the section, we always assume that X is $\mathbb{Q}$-factorial (so that X itself has klt singularities).

4.2.4 Local constancy of alb_X as fibration

In this subsection let us prove that alb_X is a locally constant fibration (c.f. Definition 2.3.3). In virtue of Proposition 2.3.5, it suffices to find a ψ-very ample divisor A on X such that $\psi_* \mathscr{O}_X(mA)$ is numerically flat for every m where $\psi = \mathrm{alb}_X$.

Recall that we set $\psi = \mathrm{alb}_X$, $\pi : M \to X$ a smooth model of X and $\phi = \mathrm{alb}_M$, $Y = \mathrm{Alb}_M = \mathrm{Alb}_X$, as mentioned at the beginning of §4.2. By §4.2.3 we can assume that X is

$\mathbb{Q}$-factorial. Let A be a very ample divisor on X. Up to multiplying A we can assume that for general $y \in Y$ the natural morphism

$$\mathrm{Sym}^k \mathrm{H}^0(X_y, \mathscr{O}_{X_y}(A)) \to \mathrm{H}^0(X_y, \mathscr{O}_{X_y}(kA))$$

is surjective for every k. As π is birational, π^*A is big and for every $k \in \mathbb{Z}_{>0}$ and for general $y \in Y$ we have a surjection:

$$\mathrm{Sym}^k \mathrm{H}^0(M_y, \mathscr{O}_{M_y}(\pi^*A)) \twoheadrightarrow \mathrm{H}^0(M_y, \mathscr{O}_{M_y}(k\pi^*A));$$

in addition, for any $m, c \in \mathbb{Z}_{>0}$ we have (c.f. [Debo1, Lemma 7.11])

$$\pi_* \mathscr{O}_M(m\pi^*A + mcE) \simeq \mathscr{O}_X(mA),$$

hence

$$\phi_* \mathscr{O}_M(m\pi^*A + mcE) \simeq \psi_* \mathscr{O}_X(mA).$$

Set $D_{A,m}$ be the (unique up to linear equivalence) Cartier divisor on Y associated to the line bundle $\det \psi_* \mathscr{O}_X(mA)$, then by [Ful84, §2.3, Proposition 2.3(c), pp. 33-34] we have

$$\pi_* \phi^* D_{A,m} \sim \psi^* D_{A,m}$$

Since X is $\mathbb{Q}$-factorial, by Proposition 4.1.5, the ($\mathbb{Q}$-Cartier) $\mathbb{Q}$-divisor

$$A - \psi^* D_{A,1}$$

is pseudoeffective. By Proposition 4.1.8, up to multiplying A by a integer divisible by r, we can assume that $\psi^* D_{A,1}$ is an integral Cartier divisor (noting that $\mathrm{Pic}^0(X)$ is an Abelian variety, thus divisible). In consequence, by replacing A by $A - \psi^* D_{A,1}$, we get an integral Cartier divisor A on X such that:

- A is pseudoeffective on X;
- A is ψ-very ample;
- for general $y \in Y$ and for any $k \in \mathbb{Z}_{>0}$ the natural morphism

  $$\mathrm{Sym}^k \mathrm{H}^0(X_y, \mathscr{O}_{X_y}(A)) \to \mathrm{H}^0(X_y, \mathscr{O}_{X_y}(kA))$$

  is surjective;
- $D_{A,1}$ is trivial.

In the sequel we will show that $\psi_* \mathscr{O}_X(mA)$ is numerically flat for every $m \in \mathbb{Z}_{>0}$.

First, since π is birational, π^*A is ϕ-big and the natural morphism

$$\mathrm{Sym}^k \mathrm{H}^0(M_y, \mathscr{O}_{M_y}(\pi^*A)) \to \mathrm{H}^0(M_y, \mathscr{O}_{M_y}(k\pi^*A))$$

is surjective for all $k \in \mathbb{Z}_{>0}$. Since $Y = \mathrm{Alb}_X$ is a complex torus, E' is an effective divisor, hence by Proposition 4.1.3 $\psi_* \mathscr{O}_X(mA) \simeq \phi_* \mathscr{O}_M(m\pi^*A + pE')$ is weakly semipositively curved for every $m \in \mathbb{Z}_{>0}$. By §4.2.2 ψ is flat and thus $\psi_* \mathscr{O}_X(mA)$ is reflextive. Moreover, by Proposition 4.1.8 $D_{A,m} \equiv mD_{A,1} = 0$, i.e. $\det \psi_* \mathscr{O}_X(mA)$ is numerically trivial, then Proposition 2.3.6 implies that $\psi_* \mathscr{O}_X(mA)$ is numerically flat for every $m \in \mathbb{Z}_{>0}$. In virtue of Proposition 2.3.5 we see that ψ is a locally constant fibration. The proof of Theorem B is thus finished.

4.3 MRC fibration for X with simply connected smooth locus

Throughout the section, let X be a projective variety equipped with an effective $\mathbb{Q}$-divisor Δ such that the pair (X,Δ) is klt and that the twisted anticanonical divisor $-(K_X+\Delta)$ is nef, and suppose that $\pi_1(X_{\mathrm{reg}})=\{1\}$. Take the ψ in the General Setting 4.1.1 to be the maximally rationally connected (MRC) fibration of X (c.f. [Deb01, §5.4, Theorem 5.13, pp. 128-129]), we will prove in this section that ψ induces a product structure on X.

4.3.1 Splitting of the tangent sheaf

In this subsection we will prove that following decomposition theorem for the tangent sheaf of X:

Theorem 4.3.1. *Let X be a projective variety whose smooth locus X_{reg} is simply connected. Suppose that there is an effective divisor Δ on X such that the pair (X,Δ) is klt and that the twisted anticanonical divisor $-(K_X+\Delta)$ is nef. Then the tangent sheaf of X admits a splitting*

$$T_X \simeq \mathscr{F}\oplus\mathscr{G}$$

with $\mathscr{F}$ and $\mathscr{G}$ being algebraically integrable foliations. Moreover, the closure of the general leaf of $\mathscr{F}$ is rationally connected and $\det\mathscr{G}\simeq\mathscr{O}_X$.

The proof of this result can be divided into four steps:

Step 1: Reduction to the terminal case. To prove the theorem, we can assume that the pair (X,Δ) is terminal and $\mathbb{Q}$-factorial. In fact, by [BCHM10, Corollary 1.4.3] we can take a ($\mathbb{Q}$-factorial) terminal model $g: X^{\mathrm{term}}\to X$ of X, with an effective $\mathbb{Q}$-divisor Δ^{term} on X^{term} such that

$$K_{X^{\mathrm{term}}}+\Delta^{\mathrm{term}}\sim_{\mathbb{Q}} g^*(K_X+\Delta).$$

Then $-(K_{X^{\mathrm{term}}}+\Delta^{\mathrm{term}})$ is nef. Suppose that $T_{X^{\mathrm{term}}}$ admits a decomposition into algebraically integrable foliations

$$T_{X^{\mathrm{term}}}\simeq\mathscr{F}^{\mathrm{term}}\oplus\mathscr{G}^{\mathrm{term}}$$

with $\det\mathscr{G}^{\mathrm{term}}\simeq\mathscr{O}_{X^{\mathrm{term}}}$ and the closure of the general leaf of $\mathscr{F}^{\mathrm{term}}$ is rationally connected. Then we get a decomposition $T_X\simeq\mathscr{F}\oplus\mathscr{G}$ on X with

$$\mathscr{F}:=(g_*\mathscr{F}^{\mathrm{term}})^{**}\quad\text{and}\quad\mathscr{G}:=(g_*\mathscr{G}^{\mathrm{term}})^{**}.$$

By [Kol96, Proposition 3.3.(3), p. 200], the closure of the general leaf of $\mathscr{F}$ is also rationally connected. Since g is an isomorphism out of a codimension 2 subscheme of X, then $\det\mathscr{G}^{\mathrm{term}}\simeq\mathscr{O}_{X^{\mathrm{term}}}$ implies that $\det\mathscr{G}\simeq\mathscr{O}_X$. It remains to prove that $(X^{\mathrm{term}})_{\mathrm{reg}}$ is simply connected. Since $X_{\mathrm{reg}}\backslash g(\mathrm{Exc}(g))$ can be regarded as an Zariski open in $(X^{\mathrm{term}})_{\mathrm{reg}}$, by [FL81, §0.7 (B)] it suffices to show that $X_{\mathrm{reg}}\backslash g(\mathrm{Exc}(g))$ is simply connected. This can be obtained easily by the following topological result:

Lemma 4.3.2. *Let W be a complex manifold and let Z be an analytic subspace of V of codimension $\geqslant 2$. Then the natural morphism $\pi_1(W\backslash Z)\to\pi_1(W)$ induced by the embedding $W\backslash Z\hookrightarrow W$ is an isomorphism.*

Proof. This result is of course well known to experts, we nevertheless give the proof for the convenience of the readers. The argument is taken from [Pol16]. Let us argue by induction on $\dim Z$. If $\dim Z=0$, then $\dim W\geqslant 2$, and the lemma results from [God81, §X.2, Theorem 2.3, p. 146]. In general, by the induction hypothesis, $\pi_1(W\backslash Z_{\mathrm{sing}})\to\pi_1(W)$ is an isomorphism; then we apply [God81, §X.2, Theorem 2.3, p. 146] to $Z_{\mathrm{reg}}\subset W\backslash Z_{\mathrm{sing}}$ to obtain an isomorphism $\pi_1(W\backslash Z)\to\pi_1(W\backslash Z_{\mathrm{sing}})$, hence we have $\pi_1(W\backslash Z)\xrightarrow{\simeq}\pi_1(W)$. □

Step 2: Triviality of the direct image sheaves. We will prove in this step the following lemma:

Lemma 4.3.3. *Let X be a $\mathbb{Q}$-factorial projective variety and suppose that there is an effective $\mathbb{Q}$-divisor Δ on X such that the pair (X,Δ) is terminal and $-(K_X+\Delta)$ is nef and let everything as in the General Setting 4.1.1 with ψ being the MRC fibration of X. Let A be a sufficiently ample divisor on X such that for every $k\in\mathbb{Z}_{>0}$ and for general $y\in Y$ the natural morphism*

$$\mathrm{Sym}^k \mathrm{H}^0(X_y,\mathscr{O}_{X_y}(A)) \to \mathrm{H}^0(X_y,\mathscr{O}_{X_y}(kA))$$

is surjective. Then the following two torsion free sheaves

$$\mathscr{U}_{c,m} := \mathrm{Sym}^m \phi_*\mathscr{O}_M(\pi^*A+cE)\otimes \det\phi_*\mathscr{O}_M(\pi^*A+cE)^{\otimes -\frac{m}{r}}$$
$$\mathscr{V}_{c,m} := \phi_*\mathscr{O}_M(m\pi^*A+mcE)\otimes \det\phi_*\mathscr{O}_M(\pi^*A+cE)^{\otimes -\frac{m}{r}}$$

are trivial on Y_0 for every $m\in\mathbb{Z}_{>0}$ divisible by r.

Proof. When X is smooth, the theorem is proved in [CH19, Proposition 3.7]; for the singular case, the proof is much more subtle but the main idea remains the same: take a general complete intersection surface in X and prove the triviality of $\mathscr{U}_{c,m}$ and $\mathscr{V}_{c,m}$ on this surface, then try to extend the trivializing sections to Y_0. For the convenience of the readers, we give the details below. Furthermore, for sake of clarity we divide the proof into five parts:

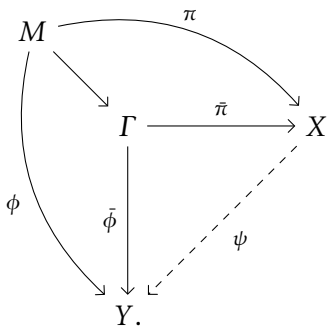
1. General settings: If $\dim X = 1$ then everything is clear, so in the sequel we assume that $\dim X \geqslant 2$. We will only give the proof of triviality on Y_0 for $\mathscr{V}_{c,m}$, for $\mathscr{U}_{c,m}$ the argument is exactly the same (and simpler since $\det\mathscr{U}_{c,m}\simeq\mathscr{O}_Y$). Since ϕ is flat over Y_0, $\mathscr{V}_{c,m}$ is reflexive on Y_0, hence in order to prove the triviality of $\mathscr{V}_{c,m}$ on Y_0, it suffices to show that $\mathscr{V}_{c,m}$ is trivial on $Y_0\cap Y_{\mathscr{V}_{c,m}}$ where $Y_{\mathscr{V}_{c,m}}$ is the locally free locus of $\mathscr{V}_{c,m}$. For every $a\in\mathbb{Z}_{>0}$ set

$$D_{A,c,a} := \frac{1}{r_a}\cdot \text{the Cartier divisor on } Y \text{ associated to the line bundle } \det\mathscr{O}_M(a\pi^*A+acE)$$

where $r_a := \mathrm{rk}\phi_*\mathscr{O}_M(a\pi^*A+acE)$. Then we have

$$\det\mathscr{V}_{c,m}\simeq\mathscr{O}_Y(r_mD_{A,c,m}-mr_mD_{G,c,1}).$$

Since X is not necessarily smooth, the exceptional divisor $E=\mathrm{Exc}(\pi)$ can dominate Y, which will render the arguments in [CH19] invalid. In order to overcome this difficulty, we set Γ to be the normalization of the graph of the rational mapping ψ, up to further blow up M we can assume that $\phi: M\to Y$ and $\pi: M\to X$ both factorize through Γ and denote by $\bar\phi:\Gamma\to Y$ and $\bar\pi:\Gamma\to X$ the corresponding morphisms. By construction, ψ is almost holomorphic (c.f. [Deb01, §5.4, Definition 5.12, p. 128] and [BCEKPRSW02, Definition 2.3]), hence $\mathrm{Exc}(\bar\pi)$ does not dominate Y.



2. Simple connectdeness of a general complete intersection surface in X: Let A be a very ample divisor on X and take $H_1, \cdots, H_{n-1}$ be general hypersurfaces in $|A|$. Set $n := \dim X$ and let $S = H_1 \cap \cdots \cap H_{n-2}$ be the complete intersection surface cut out by $H_1, \cdots, H_{n-2}$ (if $n = 2$ then we simply take $S = X$). Since terminal singularities are smooth in codimension 2 (c.f. [KM98, Corollary 5.18, p. 159]), S is smooth (see also [KM98, Theorem 4.5, p. 113]). Since X is normal, by [FL81, §0.7 (B), p. 33] we have a surjection between fundamental groups $\pi_1(X_{\mathrm{reg}}) \twoheadrightarrow \pi_1(X)$, then $\pi_1(X_{\mathrm{reg}}) = \{1\}$ implies that $\pi_1(X) = \{1\}$. We claim that S is also simply connected:

- If $n = 2$, then $S = X$ is simply connected.
- If $n \geqslant 3$, then by [HL85, Theorem 1.1.3] X_{reg} has the same homotopy type of the space obtained from $H_1 \cap X_{\mathrm{reg}}$ by attaching cells of dimension $\geqslant \dim X$, but the fundamental group of a CW complex only depends on its 2-skeleton, so that we get an isomorphism
$$\pi_1(H_1 \cap X_{\mathrm{reg}}) \xrightarrow{\simeq} \pi_1(X_{\mathrm{reg}}),$$
hence $\pi_1(H_1 \cap X_{\mathrm{reg}}) = \{1\}$. By iterating the argument, we see that $\pi_1(S \cap X_{\mathrm{reg}}) = \{1\}$; but since X is smooth in codimension 2 we have $S \cap X_{\mathrm{reg}} = S$, hence S is simply connected.

3. Triviality of the pullback of $\mathscr{U}_{c,m}$ and $\mathscr{V}_{c,m}$ to a general complete intersection surface in X: Now set $\bar{E} := \mathrm{Exc}(\bar{\pi})$, then $\bar{\pi}|_{\Gamma\backslash\bar{E}} : \Gamma\backslash\bar{E} \to X\backslash\bar{\pi}(\bar{E})$ is an isomorphism and $S \cap \bar{\pi}(\bar{E})$ is of dimension 0. In particular, $\bar{\pi}^{-1}(S\backslash\bar{\pi}(\bar{E})) = \bar{\pi}^{-1}(S)\backslash\bar{E}$ is smooth. By Lemma 4.1.9 $\mathscr{V}_{c,m}$ is weakly positively curved on Y_0 , in consequence $\bar{\phi}^*\mathscr{V}_{c,m}\big|_{\bar{\pi}^{-1}(S)\backslash\bar{E}}$ is also weakly positively curved by §2.2.4. By viewing $\Gamma\backslash\bar{E}$ as a Zariski open of X via the isomorphism $\bar{\pi}|_{\Gamma\backslash\bar{E}} : \Gamma\backslash\bar{E} \to X\backslash\bar{\pi}(\bar{E})$, $\bar{\phi}^*\mathscr{V}_{c,m}\big|_{\Gamma\backslash\bar{E}}$ extends to a reflexive sheaf on X, denoted by $\mathscr{W}_{c,m}$. By the projection formula we have:
$$\det\mathscr{W}_{c,m} \simeq (\bar{\pi}_*\bar{\phi}^*\det\mathscr{V}_{c,m})^{**} \simeq (\pi_*\phi^*\det\mathscr{V}_{c,m})^{**},$$
hence the (unique up to linear equivalence) Weil divisor associated to $\det\mathscr{W}_{c,m}$ is equal to
$$\pi_*\phi^*(r_m D_{A,c,m} - m r_m D_{A,c,1}).$$
Since X is $\mathbb{Q}$-factorial, Proposition 4.1.8 implies that $\det\mathscr{W}_{c,m}$ is a numerically trivial $\mathbb{Q}$-line bundle. Hence by Proposition 2.3.6 $\mathscr{W}_{c,m}|_S$ is a numerically flat vector bundle on S; but S is simply connected, then $\mathscr{W}_{c,m}|_S$ is a trivial vector bundle.

4. Surjectivity of the restriction morphism: Since $\bar{\pi}(\bar{E})$ is of codimension $\geqslant 2$ in X, then we have an isomorphism
$$\mathrm{H}^0(\Gamma\backslash\bar{E}, \bar{\phi}^*\mathscr{V}_{c,m}) \xrightarrow{\simeq} \mathrm{H}^0(X, \mathscr{W}_{c,m}).$$
Since (X, Δ) is terminal, X has rational singularities and in particular X is Cohen-Macaulay. For A sufficiently ample we have
$$\mathrm{H}^1(X, \mathscr{W}_{c,m} \otimes \mathscr{O}_X(-H_1)) \simeq \mathrm{H}^{n-1}(X, \omega_X \otimes \mathscr{W}_m^* \otimes \mathscr{O}_X(H_1)) = 0,$$
where ω_X denotes the dualizing sheaf of X. Then the canonical exact sequence $0 \to \mathscr{O}_X(-H_1) \to \mathscr{O}_X \to \mathscr{O}_{H_1} \to 0$ induces a surjection
$$\mathrm{H}^0(\Gamma\backslash\bar{E}, \bar{\phi}^*\mathscr{V}_{c,m}) \simeq \mathrm{H}^0(X, \mathscr{W}_{c,m}) \twoheadrightarrow \mathrm{H}^0(H_1, \mathscr{W}_{c,m}|_{H_1}).$$
By iterating this argument we see that for $C := S \cap H_{n-1}$ the restriction morphism (since C is disjoint from $\bar{\pi}(\bar{E})$, we can identify $\bar{\pi}^{-1}(C)$ and C)
$$\mathrm{H}^0(\Gamma\backslash\bar{E}, \bar{\phi}^*\mathscr{V}_{c,m}) \to \mathrm{H}^0(C, \mathscr{W}_{c,m}|_C)$$
is surjective.

5. Construction of the trivializing sections and Conclusion: But $\mathscr{W}_{c,m}|_S$ is a trivial vector bundle of rank r_m, we get r_m sections $\sigma_1, \cdots, \sigma_{r_m}$ in $\mathrm{H}^0(\Gamma\backslash\bar{E}, \bar{\phi}^*\mathscr{V}_{c,m})$ whose restrictions to C are everywhere linearly independent. Then $\sigma_1 \wedge \cdots \wedge \sigma_{r_m}$ is a non-zero section in $\mathrm{H}^0(\Gamma\backslash\bar{E}, \bar{\phi}^*\det\mathscr{V}_{c,m})$, which extends, via the isomorphism $\pi|_{\Gamma\backslash\bar{E}} : \Gamma\backslash\bar{E} \to X\backslash\bar{\pi}(\bar{E})$, to a non-zero section of $\mathrm{H}^0(X, \det\mathscr{W}_{c,m})$; but $\det\mathscr{W}_{c,m}$ is a numerically trivial $\mathbb{Q}$-line bundle, then this section must be constant, which implies that $\sigma_1 \wedge \cdots \wedge \sigma_{r_m}$ is a non-zero constant.

We claim that for every i there is a section $\tau_i \in \mathrm{H}^0(Y_0, \mathscr{V}_{c,m})$ such that $\phi^*\tau_i = \sigma_i|_{\bar{\phi}^{-1}(Y_0)\backslash\bar{E}}$. The argument is the same as in [CH19, Proof of Proposition 3.7]. In fact, since ψ is the MRC fibration of X, $\bar{E}$ does not dominate Y, then σ_i induces a section $\bar{\tau}_i \in \mathrm{H}^0(Y_0\backslash\bar{\phi}(\bar{E}), \mathscr{V}_{c,m})$. It remains to show that $\bar{\tau}_i$ extends to Y_0. Since $\mathscr{V}_{c,m}$ is reflexive on Y_0, it suffices to show that $\bar{\tau}_i$ extends to a general point of any divisor P in Y_0. By Proposition 4.1.2(d) $\bar{\phi}^*P$ contains at least a reduced component, hence locally around a general point of P, $\bar{\phi}|_{\Gamma\backslash\bar{E}}\Gamma\backslash\bar{E} \to Y$ admits a local section, which implies that $\bar{\tau}_i$ is locally bounded (with respect to any Hermitian metric) around a general point of P. Hence by Riemann extension $\bar{\tau}_i$ extends to $Y_{\mathscr{V}_{c,m}} \cap Y_0$ and thus to Y_0 by the reflexivity of $\mathscr{V}_{c,m}|_{Y_0}$, in this way for every i we obtain a section $\tau_i \in \mathrm{H}^0(Y_0, \mathscr{V}_{c,m})$ such that $\sigma_i|_{\bar{\phi}^{-1}(Y_0)\backslash\bar{E}} = \bar{\phi}^*\tau_i$.

Now

$$\tau_1 \wedge \cdots \wedge \tau_{r_m} = \bar{\phi}^*(\sigma_1 \wedge \cdots \wedge \sigma_{r_m})\big|_{\bar{\phi}^{-1}(Y_0)\backslash\bar{E}}$$

is a non-zero constant, this implies that the sections $\tau_1, \cdots, \tau_{r_m}$ are everywhere linearly independent on Y_0. Hence the τ_i's give a trivialization of $\mathscr{V}_{c,m}|_{Y_0}$. □

Step 3: Birational version of the decomposition. In the sequel of the proof of Theorem 4.3.1, let us fix a very ample divisor A on X, such that

$$\mathrm{Sym}^k\mathrm{H}^0(X, \mathscr{O}_X(A)) \to \mathrm{H}^0(X, \mathscr{O}_X(kA)) \tag{4.8}$$

is surjective for every $k \in \mathbb{Z}_{>0}$. In this step we will prove that $\phi^{-1}(Y_0)$ is birational to a product, which can be seen as a birational version of the decomposition theorem for X. Let c_0 be as in the Proposition 4.1.6 and let c be any integer $\geqslant c_0$. Set $G := \pi^*A + cE$, and for every $a \in \mathbb{Z}_{>0}$ set

$$D_{A,c,a} := \frac{1}{r_a} \cdot \text{the Cartier divisor on } Y \text{ associated to the line bundle } \det\phi_*\mathscr{O}_M(aG).$$

where $r_a := \mathrm{rk}\phi_*\mathscr{O}_M(aG)$. Then by the Lemma 4.3.3 for every $m \in \mathbb{Z}_{>0}$ divisible by $r := r_1$ the torsion free sheaves

$$\begin{aligned}\mathscr{U}_{c,m} &:= \mathrm{Sym}^m\phi_*\mathscr{O}_M(G) \otimes \mathscr{O}_Y(-mD_{A,c,1}) \\ \mathscr{V}_{c,m} &:= \phi_*\mathscr{O}_M(mG) \otimes \mathscr{O}_Y(-mD_{A,c,1})\end{aligned}$$

are trivial on Y_0. Up to blow up M, we can assume that, the ϕ-relative base locus of G, i.e. the subscheme of M defined by the ideal sheaf $\mathrm{Im}(\phi^*\phi_*\mathscr{O}_M(G) \otimes \mathscr{O}_M(-G) \to \mathscr{O}_M)$, is a divisor. Then we can write

$$G = G_{\mathrm{b}} + G_{\mathrm{f}}$$

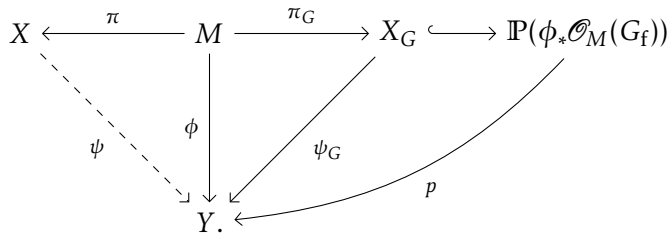
where G_{b} is the ϕ-relative fixed part of the linear series $|G|$ and $G_{\mathrm{f}} := G - G_{\mathrm{b}}$ is ϕ-relatively generated. Now the adjunction morphism admits a factorization

$$\phi^*\phi_*\mathscr{O}_M(G) \twoheadrightarrow \mathscr{O}_M(G_{\mathrm{f}}) \hookrightarrow \mathscr{O}_M(G),$$

that can be pushed down to Y and give morphisms

$$\phi_*\mathscr{O}_M(G) \to \phi_*\mathscr{O}_M(G_{\mathrm{f}}) \hookrightarrow \phi_*\mathscr{O}_M(G).$$

By construction the composition morphism is the identity, hence the inclusion $\phi_*\mathscr{O}_M(G_\mathrm{f}) \hookrightarrow \phi_*\mathscr{O}_M(G)$ is an isomorphism. Then the surjection $\phi^*\phi_*\mathscr{O}_M(G_\mathrm{f}) \twoheadrightarrow \mathscr{O}_M(G_\mathrm{f})$ induces a morphism $\pi_G : M \to \mathbb{P}(\phi_*\mathscr{O}_M(G_\mathrm{f}))$ such that $\mathscr{O}_M(G_\mathrm{f}) = \pi_G^*\mathscr{O}_{\mathbb{P}(\phi_*\mathscr{O}_M(G_\mathrm{f}))}(1)$. Set X_G be the image of π_G with induced morphism $\psi_G : X_G \to Y$, then we have the following commutative diagram:



The main purpose of this step is to prove the following lemma

Lemma 4.3.4. *In the above setting, we have $\psi_G^{-1}(Y_0) \simeq Y_0 \times F$, where F denotes the general fibre of ψ (the MRC fibration ψ is almost holomorphic, hence it makes sense to talk about its general fibre).*

Before entering into the proof of the above lemma let us first prove the following auxiliary result:

Lemma 4.3.5. *Let everything be as above. Then for general $y \in Y$ we have $G_\mathrm{f}|_{M_y} \sim \pi^*A|_{M_y}$ and $G_\mathrm{b}|_{M_y} \sim cE|_{M_y}$. In particular, the general fibre of ψ_G is isomorphic to F.*

Proof. Let us first point out that a major difference between the singular case that we consider in this article and the smooth case treated in [CH19; CCM19] is that if X is singular the exceptional divisor E can dominate Y, in particular $E|_{M_y} \nsim 0$. For general $y \in Y$ consider the morphism $\pi|_{M_y} : M_y \to X_y$, it is a birational morphism with the exceptional divisor being $E|_{M_y}$. By the projection formula (c.f. [Deb01, Lemma 7.11]) we have

$$\mathrm{H}^0(M_y, \mathscr{O}_{M_y}(G)) \simeq \mathrm{H}^0(X_y, \mathscr{O}_{X_y}(A)) \tag{4.9}$$

but $\pi^*A|_{M_y}$ is globally generated, hence $\pi^*A|_{M_y}$ is a fortiori the mobile part of $G|_{M_y}$, that is, $G_\mathrm{f}|_{M_y} = \pi^*A|_{M_y}$; then $G_\mathrm{b}|_{M_y} = cE|_{M_y}$. Consequently the morphism $\pi_G|_{M_y} : M_y \to (X_G)_y$ is given by the linear series $|\pi^*A|_{M_y}|$. But A is very ample on X, hence for general $y \in Y$ the morphism $\pi_G|_{M_y}$ factors through X_y, and its image is isomorphic to $X_y \simeq F$. □

Now let us turn to the proof of Lemma 4.3.4:

Proof of Lemma 4.3.4. The idea of the proof is the same as that of [CH19, §3.C. Proof of Theorem 1.2, Step 1], we nevertheless give the proof for the convenience of the readers. By (4.8) and (4.9) the morphism

$$\mathrm{Sym}^m \phi_*\mathscr{O}_M(G) \to \phi_*\mathscr{O}_M(mG) \tag{4.10}$$

is generically surjective. Twisting with $\mathscr{O}_M(-mD_{A,c,1})$ we get a generically surjective morphism $\mathscr{U}_{c,m} \to \mathscr{V}_{c,m}$, which gives rise to a global section $s \in \mathrm{H}^0(Y, \mathscr{U}_{c,m}^* \otimes \mathscr{V}_{c,m})$. By Lemma 4.3.3, $\mathscr{U}_{c,m}|_{Y_0}$ and $\mathscr{V}_{c,m}|_{Y_0}$ are trivial vector bundles, hence $s|_{Y_0}$ is constant by Proposition 4.1.2(d), in particular the morphism $\mathscr{U}_{c,m} \to \mathscr{V}_{c,m}$ has constant rank over Y_0. Consequently the morphism (4.10) is surjective over Y_0. Now consider the inclusion $\phi_*\mathscr{O}_M(mG_\mathrm{f}) \hookrightarrow \phi_*\mathscr{O}_M(mG)$ we get the following commutative diagram

$$\begin{array}{ccc} \mathrm{Sym}^m \phi_* \mathscr{O}_M(G_{\mathrm{f}}) & \xrightarrow{\simeq} & \mathrm{Sym}^m \phi_* \mathscr{O}_M(G) \\ \downarrow & & \downarrow \\ \phi_* \mathscr{O}_M(mG_{\mathrm{f}}) & \hookrightarrow & \phi_* \mathscr{O}_M(mG) \end{array}$$

Since right column is the morphism (4.10), which is shown to be surjective over Y_0, hence by the Five Lemma the left column is also surjective over Y_0. Again apply the Five Lemma but exchange the role of rows and of columns, then we find that the bottom row is an isomorphism over Y_0. In particular, $\phi_* \mathscr{O}_M(mG_{\mathrm{f}}) \otimes \mathscr{O}_Y(-mD_{A,c,1})$ is trivial over Y_0.

Let $\mathscr{I}_{X_G}$ be the ideal sheaf of X_G in $\mathbb{P}(\phi_* \mathscr{O}_M(G_{\mathrm{f}}))$. Twisting the exact sequence

$$0 \to \mathscr{I}_{X_G} \to \mathscr{O}_{\mathbb{P}(\phi_* \mathscr{O}_M(G_{\mathrm{f}}))} \to \mathscr{O}_{X_G} \to 0$$

with $\mathscr{O}_{\mathbb{P}(\phi_* \mathscr{O}_M(G_{\mathrm{f}}))}(m)$ for m sufficiently large and divisible by r and pushing down to Y we get (by relative Serre vanishing):

$$0 \to p_* \mathscr{I}_{X_G}(m) \to p_* \mathscr{O}_{\mathbb{P}(\phi_* \mathscr{O}_M(G_{\mathrm{f}}))}(m) \simeq \mathrm{Sym}^m \phi_* \mathscr{O}_M(G_{\mathrm{f}}) \to \psi_{G*} \mathscr{O}_{X_G}(m) \to 0. \tag{4.11}$$

where we adapt the notation that for any coherent sheaf $\mathscr{F}$ on $\mathbb{P}(\phi_* \mathscr{O}_M(G_{\mathrm{f}}))$ and for any integer k we set $\mathscr{F}(k) := \mathscr{F} \otimes \mathscr{O}_{\mathbb{P}(\phi_* \mathscr{O}_M(G_{\mathrm{f}}))}(k)$. Since $\pi_G : M \to X_G$ is birational (because it birational on the general fibre of ϕ), the natural morphism $\mathscr{O}_{X_G} \to \pi_{G*} \mathscr{O}_M$ is injective, hence by the projection formula we have an injection

$$\psi_{G*} \mathscr{O}_{X_G}(m) \hookrightarrow \psi_* \mathscr{O}_M(mG_{\mathrm{f}}).$$

Now we consider the composition morphism

$$\mathrm{Sym}^m \phi_* \mathscr{O}_M(G_{\mathrm{f}}) \twoheadrightarrow \psi_{G*} \mathscr{O}_{X_G}(m) \hookrightarrow \phi_* \mathscr{O}_M(mG_{\mathrm{f}}),$$

which is shown to be surjective over Y_0 (the left column of the diagram above), hence the inclusion $\psi_{G*} \mathscr{O}_{X_G}(m) \hookrightarrow \phi_* \mathscr{O}_M(mG_{\mathrm{f}})$ is an isomorphism over Y_0, and in consequence its twisting

$$\psi_{G*} \mathscr{O}_{X_G}(m) \otimes \mathscr{O}_Y(-mD_{A,c,1})$$

is trivial over Y_0. By the exact sequence (4.11) we see that $p_* \mathscr{I}_{X_G}(m) \otimes \mathscr{O}_Y(-mD_{A,c,1})$ is also trivial over Y_0. By Proposition 4.1.2(d) this means that the defining equations of $\psi_G^{-1}(Y_0)$ in $\mathbb{P}(\phi_* \mathscr{O}_M(G_{\mathrm{f}})|_{Y_0}) \simeq Y_0 \times \mathbb{P}^{r-1}$ are constant over Y_0, hence $\psi_G^{-1}(Y_0)$ is isomorphic to the product $Y_0 \times F$ by Lemma 4.3.5. □

Step 4: Proof of the splitting theorem. In this step we will apply Lemma 4.3.4 to conclude. The proof relies on the following auxiliary result:

Lemma 4.3.6. *Let everything be as in Step 3, then every codimension* 1 *component of the exceptional locus of* $\psi_G|_{\phi^{-1}(Y_0)} : \phi^{-1}(Y_0) \to \psi_G^{-1}(Y_0)$ *is contained in* E.

Proof. The proof is similar to [CH19, §3.C. Proof of Theorem 1.2, Step 2], nevertheless in our case X is possibly singular, then E can dominate Y and this renders the argument a little subtle. For the convenience of the readers, we give the proof below. First notice that we have the following observation:

> Since $\pi^* A$ is ϕ-relatively generated, hence $G_{\mathrm{b}} \leqslant cE$. Let Γ be a component of any fibre of ϕ not contained in E, then every component of E restricts to an effective divisor on Γ, hence
>
> $$G_{\mathrm{f}}|_\Gamma = \pi^* A|_\Gamma + (cE - G_{\mathrm{b}})|_\Gamma$$
>
> is big, and thus Γ is not contracted by ψ_G.

Now let us turn to the proof of the lemma. Let $D \subset \phi^{-1}(Y_0)$ be an irreducible Weil divisor contained in the exceptional locus of $\psi_G|_{\phi^{-1}(Y_0)}$. Consider the two cases separately:

- If D is ϕ-horizontal. Then for general $y \in Y_0$, $D|_{M_y}$ is $\psi_G|_{M_y}$-exceptional. But $\psi_G|_{M_y} : M_y \to (X_G)_y \simeq F = X_y$ is induced by the divisor $\pi^*A|_{M_y}$, hence $D|_{M_y}$ is contained in $E|_{M_y}$ and thus D is contained in E.
- If D is ϕ-vertical. Since ϕ is flat over Y_0, $\phi(D)$ is also a divisor. For the general fibre of $\phi|_D : D \to \phi(D)$, it is contracted by ψ_G, then by the observation above it is contained in E. Therefore D is contained in E.

□

By Lemma 4.3.4 we have $\psi_G^{-1}(Y_0) \simeq Y_0 \times F$, then

$$T_{\psi_G^{-1}(Y_0)} \simeq \mathrm{pr}_1^* T_{Y_0} \oplus \mathrm{pr}_2^* T_F. \tag{4.12}$$

Set $X_0 := \phi^{-1}(Y_0)\backslash E$, which can be regarded as a Zariski open of X via the embedding $\pi|_{M\backslash E} : M\backslash E \hookrightarrow X$. By Lemma 4.3.6, $\psi_G|_{X_0} : X_0 \to \psi_G^{-1}(Y_0) \simeq Y_0 \times F$ is an embedding out of a codimension $\geqslant 2$ subscheme. Hence the decomposition (4.12) induces a decomposition

$$T_{X_0} \simeq \mathscr{F}^\circ \oplus \mathscr{G}^\circ, \tag{4.13}$$

with $\mathscr{F}^\circ$ (resp. $\mathscr{G}^\circ$) corresponding to $\mathrm{pr}_2^* T_F$ (resp. $\mathrm{pr}_1^* T_{Y_0}$). By construction, $\mathscr{F}^\circ$ and $\mathscr{G}^\circ$ are algebraically integrable foliations over X_0, with the closure of a general leaf of $\mathscr{F}^\circ$ equal to a Zariski open of F and

$$K_{\mathscr{G}^\circ} \sim \mathrm{pr}_1^* K_{Y_0} = \mathrm{pr}_1^* K_Y|_{Y_0} \sim_{\mathbb{Q}} 0.$$

since by Proposition 4.1.2(a) any effective $\mathbb{Q}$-divisor $\mathbb{Q}$-linearly equivalent to K_Y is supported out of Y_0. By Proposition 4.1.2(c), $X\backslash X_0$ has codimension $\geqslant 2$, hence (4.13) gives rise to a decomposition

$$T_X \simeq \mathscr{F} \oplus \mathscr{G}.$$

with $\mathscr{F}$ (resp. $\mathscr{G}$) being the reflexive hull of the extension of $\mathscr{F}^\circ$ (resp. of $\mathscr{G}^\circ$) to X. By Lemma 2.4.2 $\mathscr{F}$ and $\mathscr{G}$ are algebraically integrable foliations; moreover, the Zariski closure of a general leaf of $\mathscr{F}$ is rationally connected (in fact equal to F) and $K_{\mathscr{G}} \sim_{\mathbb{Q}} 0$. This means that $\det\mathscr{G}|_{X_{\mathrm{reg}}}$ is a torsion line bundle on X_{reg}, but $\pi_1(X_{\mathrm{reg}}) = \{1\}$, then $\det\mathscr{G}|_{X_{\mathrm{reg}}}$ and thus $\det\mathscr{G}$ must be trivial. As a byproduct we get additional information on the splitting:

Lemma 4.3.7. *Let everything be as in the General Setting 4.1.1 with ψ being the MRC fibration of X and suppose that the smooth locus X_{reg} of X is simply connected. Then there is a Zariski open subset X_0 of X such that X_0 is embedded into the product space $Y_0 \times F$.*

Proof. We have proved this for (X, Δ) terminal. For the klt case, let us take a terminal model $g : (X^{\mathrm{term}}, \Delta^{\mathrm{term}}) \to (X, \Delta)$. Then there is a Zariski open $(X^{\mathrm{term}})_0$ such that $(X^{\mathrm{term}})_0$ can be embedded into $Y_0 \times F$. Then $(X^{\mathrm{term}})_0\backslash \mathrm{Exc}(g)$ can be regarded as a Zariski open X_0 of X, whose complement is of codimension $\geqslant 2$ in X. Clearly X_0 can be embedded into $Y_0 \times F$. □

Remark 4.3.8. To end this subsection let us make a remark about how to show that $\phi_*\mathscr{O}_M(mG_{\mathrm{f}}) \hookrightarrow \phi_*\mathscr{O}_M(mG)$ is an isomorphism over Y_0 in the proof of Lemma 4.3.4 in Step 3 above. By taking

$$\mathscr{A}_1 := \mathrm{Sym}^m \phi_*\mathscr{O}_M(G_{\mathrm{f}})\big|_{Y_0}, \qquad \mathscr{B}_1 := \mathrm{Sym}^m \phi_*\mathscr{O}_M(G)\big|_{Y_0},$$
$$\mathscr{A}_2 := \phi_*\mathscr{O}_M(mG_{\mathrm{f}})|_{Y_0}, \qquad \mathscr{B}_2 := \phi_*\mathscr{O}_M(mG)|_{Y_0},$$

we have the following commutative square:

$$\begin{array}{ccc} \mathscr{A}_1 & \xrightarrow[\simeq]{c_1} & \mathscr{B}_1 \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ \mathscr{A}_2 & \xrightarrow[c_2]{} & \mathscr{B}_2, \end{array}$$

with b being surjective. By completing the two row into short exact sequences we get

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathscr{A}_1 & \xrightarrow{c_1} & \mathscr{B}_1 & \longrightarrow & 0 & \longrightarrow & 0 \\ & & {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} & & \downarrow & & \\ 0 & \longrightarrow & \mathscr{A}_2 & \xrightarrow[c_2]{} & \mathscr{B}_2 & \longrightarrow & \mathrm{Coker}(c_2) & \longrightarrow & 0. \end{array}$$

Since b is surjective, the Snake Lemma implies that $\mathrm{Coker}(a) \simeq \mathrm{Coker}(b) = 0$, hence a is surjective. Then exchange the role of rows and of columns we get

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Ker}(a) & \longrightarrow & \mathscr{A}_1 & \xrightarrow{a} & \mathscr{A}_2 & \longrightarrow & 0 \\ & & \downarrow & & {\scriptstyle c_1}\downarrow{\scriptstyle \simeq} & & \downarrow{\scriptstyle c_2} & & \\ 0 & \longrightarrow & \mathrm{Ker}(b) & \longrightarrow & \mathscr{B}_1 & \xrightarrow[b]{} & \mathscr{B}_2 & \longrightarrow & 0. \end{array}$$

Again by the Snake Lemma we have $\mathrm{Coker}(c_2) \simeq \mathrm{Coker}(c_1) = 0$, hence c_2 is also surjective. Clearly this argument works in any Abelian category.

4.3.2 Decomposition theorem for X

In this subsection, let us prove Theorem C. Let X be a projective variety of semi-Fano type with simply connected smooth locus X_{reg}. Then there is an effective $\mathbb{Q}$-divisor on X such that (X, Δ) is klt and that the twisted anticanonical divisor $-(K_X + \Delta)$ is nef. By §4.3.1 we have a direct decomposition of the tangent sheaf into reflexive subsheaves:

$$T_X \simeq \mathscr{F} \oplus \mathscr{G}.$$

with $\mathscr{F}$ and $\mathscr{G}$ algebraically integrable foliations. Moreover, the Zariski closure of a general leaf of $\mathscr{F}$ is rationally connected and $\det\mathscr{G} \simeq \mathscr{O}_X$. Set F (resp. Z) the Zariski closure of the general leaf of $\mathscr{F}$ (resp. of $\mathscr{G}$) and we will prove in the sequel that $X \simeq Z \times F$. In fact, if $\Delta = 0$, this can be immediately deduced from the more general result of Stéphane Druel [Dru18b, Theorem 1.5] on the foliations with numerically trivial canonical class, as will be discussed in §4.4. Nevertheless, we will present here a more elementary proof of the decomposition Theorem C, since the argument can be also be applied to the more general case without assumption on the fundamental group, and we hope that it can be used to give a proof of Conjecture 2 without proving Conjecture 3 or at least reducing it to a much weaker result on the fundamental group than Conjecture 3. The key observation is that the decomposition $T_X \simeq \mathscr{F} \oplus \mathscr{G}$ implies that $\mathscr{F}$ and $\mathscr{G}$ are weakly regular foliations by [Dru18b, Lemma 5.8] (c.f. Definition 2.4.10 or [Dru18b, Definition 5.4] for the definition of the weak regularity).

Step 1: Simple connectedness of the general leaf. In this first step, let us prove the following preparatory result on the topology of the general leaves of the foliations $\mathscr{F}$ and $\mathscr{G}$:

Lemma 4.3.9. *As above let F (resp. Z) be the Zariski closure of a general leaf of $\mathscr{F}$ (resp. of $\mathscr{G}$). Then both F_{reg} and Z_{reg} are simply connected.*

Proof. This follows easily from Lemma 4.3.7. In fact, by Lemma 4.3.7, there is a Zariski open X_0 of X which can be embedded into $Y_0 \times F$ such that $\mathrm{codim}_X(X \backslash X_0) \geqslant 2$. Up to shrinking Y we can assume that $X_0 \subseteq X_{\mathrm{reg}}$, then we have $\mathrm{codim}_{X_{\mathrm{reg}}}(X_{\mathrm{reg}} \backslash X_0) \geqslant 2$. But $\pi_1(X_{\mathrm{reg}}) \simeq \{1\}$, then Lemma 4.3.2 implies that $\pi_1(X_0) \simeq \{1\}$. Since X_0 is smooth, it can be regarded as a Zariski open in $Y_0 \times F_{\mathrm{reg}}$. Then by [FL81, §o.7 (B), p. 33], we have $\pi_1(Y_0 \times F_{\mathrm{reg}}) \simeq \{1\}$, which implies that $\pi_1(Y_0) \simeq \pi_1(F_{\mathrm{reg}}) \simeq \{1\}$. Again by Lemma 4.3.7, we see that Y_0 can be regarded as a Zariski open of Z (and thus of Z_{reg} since Y_0 is smooth). Then by [FL81, §o.7 (B), p. 33] $\pi_1(Z_{\mathrm{reg}}) \simeq \{1\}$. □

Step 2: Reduction to the $\mathbb{Q}$-factorial terminal case. As in the §4.2.3, in this step we will reduce the proof of Theorem C to the $\mathbb{Q}$-factorial case. Assume that Theorem C for X with terminal $\mathbb{Q}$-factorial singularities, let us prove that it holds for general X. To this end, we take a ($\mathbb{Q}$-factorial) terminal model $g : X^{\mathrm{term}} \to X$ of X (by [BCHM10, Corollary 1.4.3]). By construction X^{term} is equipped with an effective $\mathbb{Q}$-divisor Δ^{term} on X^{term} such that

$$K_{X^{\mathrm{term}}} + \Delta^{\mathrm{term}} \sim_{\mathbb{Q}} g^*(K_X + \Delta).$$

hence the twisted anticanonical $-(K_{X^{\mathrm{term}}} + \Delta^{\mathrm{term}})$ is nef. By our assumption, the MRC fibration of X^{term} induces a decomposition $X^{\mathrm{term}} \simeq Z^{\mathrm{term}} \times F^{\mathrm{term}}$ with $K_{Z^{\mathrm{term}}} \sim 0$ and F^{term} rationally connected. But by Lemma 4.3.9 the irregularity of F^{term} is zero, hence by [Dru18a, Lemma 4.6] we get a decomposition $X \simeq Z \times F$, and we have $K_Z \sim 0$ and F rationally connected.

Step 3: Weak Regularity of the foliations and everywhere-definedness of the MRC fibration. In the sequel we always assume that X has $\mathbb{Q}$-factorial terminal singularities. As pointed above, $\mathscr{F}$ and $\mathscr{G}$ are weakly regular foliations. By construction $\mathscr{F}$ is an algebraically integrable foliation, we intend to apply Theorem 2.4.21 ([Dru18b, Theorem 6.1]) to prove that $\mathscr{F}$ is induced by an equidimensional fibre space. To this end, we need to show:

Lemma 4.3.10. *Let everything as above, then the foliation $\mathscr{F}$ has canonical singularities (c.f. [Dru18b, Definition 4.1] or Definition 4.4.2 below).*

Proof. If $K_{\mathscr{F}}$ is Cartier, then the lemma follows immediately from Lemma 4.4.3 below ([Dru18b, Lemma 5.9]). In the general case, $K_{\mathscr{F}} \sim K_X$ is only $\mathbb{Q}$-Cartier, in order to prove the lemma we will make use of the fact that $-(K_X + \Delta)$ is nef and apply [Dru17, Proposition 5.5]; in fact, we will prove more generally that $(\mathscr{F}, \Delta)$ is canonical (c.f. [Dru17, §5.1] or [Spi20, Definition 2.9]; by Proposition 4.1.2, Δ is horizontal with respect to the MRC fibration, hence any component of Δ is not invariant by $\mathscr{F}$). Let $f : V \to W$ be the family of leaves of $\mathscr{F}$, with the natural morphism $\beta : V \to X$. Then by Proposition 2.4.20 and Remark 2.4.18, there is an effective β-exceptional divisor B on V such that

$$K_{\beta^{-1}\mathscr{F}} + B \sim K_{V/W} - \mathrm{Ram}(f) + B \sim_{\mathbb{Q}} \beta^* K_{\mathscr{F}}, \tag{4.14}$$

But since $K_{\mathscr{F}} \sim K_X$ and since (X, Δ) is terminal (thus X is terminal by $\mathbb{Q}$-factoriality), we must have $f^*K_W + \mathrm{Ram}(f) - B \geqslant 0$. In particular, B is f-vertical. By [AD14b, Remark 3.12] or [Dru17, p. 2.13] the general log leaf of $\mathscr{F}$ is $(V_w, B|_{V_w})$ for $w \in W$ general; since B is f-vertical, $B|_{V_w} = 0$. Moreover, by (4.14) $(V, \beta^*\Delta + B - f^*K_W - \mathrm{Ram}(f))$ is terminal (c.f.

[Kol97, 3.5 Definition]), then so is $(V_w, \beta^*\Delta|_{V_w})$ for general $w \in W$ by [KM98, Lemma 5.17, pp. 158-159]. Finally, by writing

$$-K_{\mathscr{F}} \sim -K_X = -(K_X + \Delta) + \Delta$$

with $-(K_X + \Delta)$ nef and Δ effective, we see that the foliated pair $(X, \Delta, \mathscr{F})$ satisfies the condition of [Dru17, Proposition 5.5] and hence $(\mathscr{F}, \Delta)$ is canonical.

□

By virtue of Lemma 4.3.10 above, we can apply Theorem 2.4.21 ([Dru18b, Theorem 6.1]) to conclude that $\mathscr{F}$ is induced by a surjective equidimensional fibre space $f : X \to W$ onto a normal projective variety W. By construction, W is not uniruled. Moreover we have:

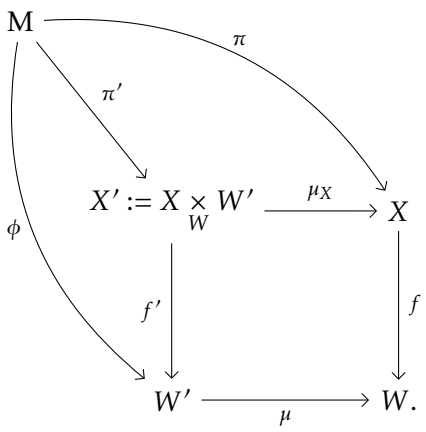
Lemma 4.3.11. *Let everything be as above, then W_{reg} is simply connected.*

Proof. Since X has terminal singularities, by [KM98, Theorem 5.22, pp. 161-162] or [Elk81, Théorème 1] it has rational singularities and in particular X is Cohen-Macaulay, hence by the miracle flatness [Mat89, Theorem 23.1, p. 179] the projective morphism $f|_{f^{-1}W_{\mathrm{reg}}} : f^{-1}W_{\mathrm{reg}} \to W_{\mathrm{reg}}$ is flat. By [Mat89, Theorem 23.7, p. 182] we see that $X_{\mathrm{reg}} \subseteq f^{-1}W_{\mathrm{reg}}$ and X is smooth at $x \in f^{-1}W_{\mathrm{reg}}$ if and only if the fibre $X_{f(x)}$ is smooth at x. Hence $\mathscr{F}$ is locally free over X_{reg} and consequently $\mathscr{F}|_{X_{\mathrm{reg}}}$ and $\mathscr{G}|_{X_{\mathrm{reg}}}$ are both regular foliations on X_{reg}. Then the tangent bundle sequence of the smooth morphism $f|_{X_{\mathrm{reg}}} : X_{\mathrm{reg}} \to W_{\mathrm{reg}}$ gives rise to an isomorphism $\mathscr{G}|_{X_{\mathrm{reg}}} \simeq f^*T_{W_{\mathrm{reg}}}$; and this means that the restricted morphism $f|_{Z_{\mathrm{reg}}} : Z_{\mathrm{reg}} \to W_{\mathrm{reg}}$ is an étale cover, but $f|_{Z_{\mathrm{reg}}}$ is also projective, hence it is a finite étale cover. By Lemma 4.3.9 Z_{reg} is simply connected, hence $f|_{Z_{\mathrm{reg}}}$ is the universal cover of W_{reg} and thus $\pi_1(W_{\mathrm{reg}}) \simeq \pi_1^{\text{ét}}(W_{\mathrm{reg}})$ is finite. Since f is a fibre space, by [SGA1, §X.4, Corollary 1.4, p. 263] we have an exact sequence of étale fundamental groups

$$\pi_1^{\text{ét}}(F) \to \pi_1^{\text{ét}}(f^{-1}W_{\mathrm{reg}}) \to \pi_1^{\text{ét}}(W_{\mathrm{reg}}) \to 1$$

But since X_{reg} is simply connected, by [FL81, §0.7 (B), p. 33] we have $\pi_1^{\text{ét}}(f^{-1}W_{\mathrm{reg}}) = \{1\}$ and thus $\pi_1(W_{\mathrm{reg}}) \simeq \pi_1^{\text{ét}}(W_{\mathrm{reg}}) \simeq \{1\}$. □

Step 4: Decomposition of X. As shown in the preceding step, the MRC fibration is everywhere defined, then the sequel of the proof is quite similar to the argument in §4.2.4. Take a desingularization $\mu : W' \to W$ of W, and let $X' := X \underset{W}{\times} W'$ be fibre product, equipped with the natural morphisms $\mu_X : X' \to X$ and $f' : X' \to W'$. Up to further blowing up M and Y in the General Setting 4.1.1, we can assume that π factorizes through μ_X and $W' = Y$, and let $\pi' : M \to X'$ be the induced morphism. Since W is not uniruled, then so is $Y = W'$.



By Proposition 4.1.2 f is semistable in codimension 1, hence the ramification divisor of f is 0 (c.f. [CKT16, Definition 2.16]), then by [CKT16, Lemma 2.31] we have $K_{X/W} \sim K_{\mathscr{F}} \sim K_X$, which implies in particular that $K_W \sim 0$. Since f is equidimensional and since W' is smooth, by [Kle80, Proposition (9)] we have $K_{X'/W'} \sim \mu_X^* K_{X/W} \sim \mu_X^* K_X$. Since Δ is horizontal with respect to f by Proposition 4.1.2(b), the pullback $\mu_X^*\Delta$ is horizontal with respect to f' by the Proposition 1.6.1, hence a fortiori we have $\mu_X^*\Delta = (\mu_X)_*^{-1}\Delta$ (noting that every μ_X-exceptional divisor is f'-vertical) and thus we can rewrite (4.3) as

$$-(K_{M/W'} + \Delta_M) + E' \sim_{\mathbb{Q}} -(\pi')^*(K_{X'/W'} + (\mu_X)_*^{-1}\Delta) \sim_{\mathbb{Q}} -\pi^*(K_X + \Delta),$$

with E' being π'-exceptional.

Take a very ample divisor A on X, such that for general $w \in W$ the natural morphism

$$\mathrm{Sym}^k \mathrm{H}^0(X_w, \mathscr{O}_{X_w}(A)) \to \mathrm{H}^0(X_w, \mathscr{O}_{X_w}(kA))$$

is surjective for every k. For every integer b set

$$D_{A,b} := \frac{1}{r_b} \cdot \text{the Weil divisor on } W \text{ associated to the rank 1 reflexive sheaf } \det f_*\mathscr{O}_X(bA),$$
$$D'_{A,b} := \frac{1}{r_b} \cdot \text{the Cartier divisor on } W' \text{ associated to the line bundle } \det f'_*\mathscr{O}_{X'}(b\mu_X^*A),$$

where $r_b := \mathrm{rk} f_*\mathscr{O}_X(bA)$. Then by construction we have $\mu_* D'_{A,b} = D_{A,b}$ and

$$\pi_*\phi^* D'_{A,b} \sim \mu_{X*}(f')^* D'_{A,b} \sim f^*\mu_* D'_{A,b} = f^* D_{A,b}.$$

Notice that since f is equidimensional and W is normal, the pullback of Weil divisors via f is defined, c.f. [CKT16, Construction 2.13]. Since X is $\mathbb{Q}$-factorial and since π^*A is big, by Proposition 4.1.5 the ($\mathbb{Q}$-Cartier) $\mathbb{Q}$-divisor

$$A - f^* D_{A,1} \sim \pi_*(\pi^*A - \phi^* D'_{A,1})$$

is pseudoeffective.

By Proposition 4.1.8, up to multiplying A by a integer divisible by r, we can assume that $f^*D_{A,1}$ is an integral Cartier divisor (noting that $\mathrm{Pic}^0(X)$ is an Abelian variety, thus divisible). In consequence, by replacing A by $A - f^*D_{A,1}$, we get an integral Cartier divisor A on X such that:

- A is pseudoeffective on X;
- A is f-very ample;
- for general $w \in W$ and for any $k \in \mathbb{Z}_{>0}$ the natural morphism

  $$\mathrm{Sym}^k \mathrm{H}^0(X_w, \mathscr{O}_{X_w}(A)) \to \mathrm{H}^0(X_w, \mathscr{O}_{X_w}(kA))$$

  is surjective;
- $D_{A,1}$ is trivial.

Since π is birational, π^*A is ϕ-big and by [Debo1, Lemma 7.11] the natural morphism

$$\mathrm{Sym}^k \mathrm{H}^0(M_y, \mathscr{O}_{M_y}(\pi^*A)) \to \mathrm{H}^0(M_y, \mathscr{O}_{M_y}(k\pi^*A))$$

is surjective for all $k \in \mathbb{Z}_{>0}$. Then by Proposition 4.1.3 we have that (noting that E' is π'-exceptional)

$$\mu^* f_*\mathscr{O}_X(mA) \simeq f'_*\mathscr{O}_{X'}(m\mu_X^*A) \simeq \phi_*\mathscr{O}_M(m\pi^*A) \simeq \phi_*\mathscr{O}_M(m\pi^*A + pE')$$

is weakly semipositively curved for every $m \in \mathbb{Z}_{>0}$. Moreover, by Proposition 4.1.8 $f^*D_{A,m} \equiv mf^*D_{A,1} = 0$, i.e. $\det f_*\mathscr{O}_X(mA)$ is numerically trivial, and so is $\det f'_*\mathscr{O}_{X'}(m\mu_X^*A)$. Since f' is equidimensional, $f'_*\mathscr{O}_{X'}(m\mu_X^*A)$ is reflexive for every m, then Proposition 2.3.6 implies that $f'_*\mathscr{O}_{X'}(m\mu_X^*A)$ is numerically flat for every $m \in \mathbb{Z}_{>0}$. By [Har77, §III.9, Proof of Proposition 9.9, pp. 261-262] (c.f. also [ACG11, §IX.2, Proposition (2.5), p. 8]) the local freeness of $f'_*\mathscr{O}_{X'}(m\mu_X^*A)$ implies that f' is flat. Then by virtue of Proposition 2.3.5 we see that f' is a locally constant fibration. Since W_{reg} is simply connected by Lemma 4.3.11, then by [FL81, §0.7 (B), p. 33] so is $Y = W'$. Hence f' induces a decomposition $X' \simeq F \times W'$. The decomposition of X then follows from [Dru18a, Lemma 4.6]. In addition, the decomposition is induced by f, hence a fortiori $W \simeq Z$ and hence $X \simeq F \times Z$. Thus we have just proved Theorem C.

4.4 Foliations with numerically trivial canonical class

As mentioned at the beginning of §4.3.2, Theorem C can be deduced directly by combining Theorem 4.3.1 and the following theorem, which is a variant of [Dru18b, Theorem 1.5]:

Theorem 4.4.1. *Let X be a normal projective variety admitting an effective $\mathbb{Q}$-divisor Δ on X such that (X, Δ) is klt and let $\mathscr{G}$ be an algebraically integrable foliation with canonical singularities. Suppose that the canonical class of $\mathscr{G}$ is numerically trivial. Then there are projective varieties Z and F and a finite quasi-étale cover $f : Z \times F \to X$, such that $f^{-1}\mathscr{G} \simeq \mathrm{pr}_1^* T_Z$.*

Before entering into the proof of Theorem 4.4.1, let us first recall the notion of singularities of foliations:

Definition 4.4.2 ([Dru18b, Defintion 4.1]; see also [McQ08, §I.5],[LPT18, Section 3],)**.** Let $\mathscr{G}$ be a $\mathbb{Q}$-Gorenstein foliation on a normal complex variety X. For any projective bimeromorphic morphism $\beta : V \to X$ with V smooth, there are uniquely determined (c.f. [LPT18, Remark 3.2]) rational numbers $a(E, X, \mathscr{G})$ such that

$$\beta^* \det\mathscr{G} \simeq \det\beta^{-1}\mathscr{G} + \sum_E a(E, X, \mathscr{G})E\,,$$

as $\mathbb{Q}$-line bundles. where E runs over all the exceptional prime divisors of β. The number $a(E, X, \mathscr{G})$ does not depend on β but only depends on the valuation defined by E on the function filed of X. We say that $\mathscr{G}$ has *canonical* (resp. *terminal*) singularities if for every E exceptional over X, $a(E, X, \mathscr{G}) \geqslant 0$ (resp. $a(E, X, \mathscr{G}) > 0$).

In particular, weakly regular foliations (c.f. Definition 2.4.10) on klt varieties have canonical singularities. Indeed we have:

Lemma 4.4.3 ([Dru18b, Lemma 5.9])**.** *Let X be a normal complex variety admitting an effective $\mathbb{Q}$-divisor Δ such that (X, Δ) is klt, and let $\mathscr{G}$ be a foliation on X such that $\det\mathscr{G}$ is a line bundle. Suppose that $\mathscr{G}$ is weakly regular. Then $\mathscr{G}$ has canonical singularities.*

For foliations with numerically trivial canonical class, the converse of Lemma 4.4.3 also holds:

Lemma 4.4.4 ([Dru18b, Corollary 5.23])**.** *Let X be a normal complex variety admitting an effective $\mathbb{Q}$-divisor Δ such that (X, Δ) is klt, and let $\mathscr{G}$ be a foliation on X with canonical singularities. Suppose that $\det\mathscr{G}$ is a line bundle and is numerically trivial, then $\mathscr{G}$ is weakly regular and there is a decomposition $T_X \simeq \mathscr{G} \oplus \mathscr{E}$ of T_X into involutive subsheaves.*

Remark 4.4.5. Let us remark that Lemma 4.4.4 is a key ingredient in the proof of [Dru18b, Theorem 1.5]. In fact, let X be a klt projective variety and let $\mathcal{G}$ be an algebraically integrable foliation on X with numerically trivial canonical class, let us briefly explain the strategy of the proof of [Dru18b, Theorem 1.5]: First by Lemma 4.4.4 $\mathcal{G}$ is weakly regular, hence by Theorem 2.4.21 ([Dru18b, Theorem 6.1]), up to replacing X by a $\mathbb{Q}$-factorialization one can assume that $\mathcal{G}$ is induced by an equidimensional fibre space. Then by separating the Abelian variety factor we can reduce the proof to the case that the leaf of $\mathcal{G}$ has vanishing irregularity and then [Dru18a, Lemma 4.6] permits to conclude.

Remark 4.4.6. In [Dru18b] the above two lemmas are stated for normal variety X with klt singularities. But since the control on the singularities of X is only used to ensure the existence and the universal property of the pullback maps of reflexive differentials ([Dru18b, §2.6]) and since this in fact holds for any "klt space" in the sense of Kebekus (that is, a normal complex variety X admitting an effective $\mathbb{Q}$-divisor Δ such that the pair (X, Δ) is klt) by [Keb13, Theorem 3.1, Proposition 6.1], we see immediately that the two lemmas holds for klt spaces.

Now let us recall the following important characterization of having canonical singularities for foliations with numerically trivial canonical class over *projective* varieties in terms of uniruledness, which first appears in [LPT18, Corollary 3.8] for X smooth and is generalized to singular case in [Dru18b, Proposition 4.22]:

Proposition 4.4.7 ([Dru18b, Proposition 4.22])**.** *Let X be a normal projective variety and let $\mathcal{G}$ be a $\mathbb{Q}$-Gorenstein foliation on X such that $K_{\mathcal{G}} \equiv 0$. Then $\mathcal{G}$ has canonical singularities if and only if $\mathcal{G}$ is not uniruled.*

Recall that a foliation $\mathcal{G}$ on the normal variety X is called *uniruled* if through a general point of X there is a rational curve which is everywhere tangent to $\mathcal{G}$.

Let us turn to the proof of Theorem 4.4.1. The proof is suggested to the author by Stéphane Druel through personal communications (of course, any mistake is the author's), and is very similar to Step 2 of Proof of [Dru18b, Theorem 8.1]. The main idea is to take a $\mathbb{Q}$-factorialization of X, which enables us to apply [Dru18b, Theorem 1.5]. In order to descend the splitting to X we intend to use [Dru18a, Lemma 4.6], to this end we need the following:

Lemma 4.4.8 ([Dru18b, Proposition 8.2])**.** *Let X be a normal projective variety and let $\mathscr{E}$ be an algebraically integrable foliation with canonical singularities on X. Suppose that $\mathscr{E} \simeq \mathcal{O}_X^{\oplus \mathrm{rk}\,\mathscr{E}}$. Then there exist an Abelian variety A, a normal projective variety V and a finite étale cover $f : A \times V \to X$ such that $f^{-1}\mathscr{E} \simeq \mathrm{pr}_1^* T_A$.*

Now we can prove Theorem 4.4.1:

Proof of Theorem 4.4.1. If $\Delta = 0$ this is nothing but [Dru18b, Theorem 1.5]. For the general case, let $\beta : X^{\mathrm{qf}} \to X$ be a $\mathbb{Q}$-factorialization of X, whose existence is proved by [Kol13, Corollary 1.37, pp. 29-30], and let $\mathcal{G}^{\mathrm{qf}} := \beta^{-1}\mathcal{G}$. By construction, β is a small birational morphism, then

$$K_{X^{\mathrm{qf}}} + \beta_*^{-1}\Delta \sim_{\mathbb{Q}} \beta^*(K_X + \Delta),$$

so that $(X^{\mathrm{qf}}, \beta_*^{-1}\Delta)$ remains a klt pair, but X^{qf} is $\mathbb{Q}$-factorial hence X^{qf} itself is klt by [KM98, Corollary 2.35(3), pp. 57-58]. Moreover, since β is small birational, we have

$$K_{\mathcal{G}^{\mathrm{qf}}} \sim_{\mathbb{Q}} \beta^* K_{\mathcal{G}} \equiv 0,$$

hence by [Dru18b, Lemma 4.2(2)] $\mathcal{G}^{\mathrm{qf}}$ also has canonical singularities. Then we can apply [Dru18b, Theorem 1.5] to $(X^{\mathrm{qf}}, \mathcal{G}^{\mathrm{qf}})$ to obtain projective varieties Z^{qf} and F^{qf} with klt singularities and a quasi-étale cover $g^{\mathrm{qf}} : Z^{\mathrm{qf}} \times F^{\mathrm{qf}} \to X^{\mathrm{qf}}$ such that $(g^{\mathrm{qf}})^{-1}\mathcal{G}^{\mathrm{qf}} \simeq \mathrm{pr}_1^* T_{Z^{\mathrm{qf}}}$. And we have

$$\mathrm{pr}_1^* K_{Z^{\mathrm{qf}}} \sim (g^{\mathrm{qf}})^* K_{\mathcal{G}^{\mathrm{qf}}} \equiv 0,$$

implying that $K_{Z^{\mathrm{qf}}} \equiv 0$ (pr_1^* is an injective morphism between Picard groups). By [HP19, 1.5.Theorem], up to a quasi-étale cover, we can assume that $Z^{\mathrm{qf}} \simeq A^{\mathrm{qf}} \times B^{\mathrm{qf}}$ with A^{qf} being an Abelian variety and B^{qf} a normal projective variety with vanishing augmented irregularity. Now let X_1 be the normalization of X in the function field of $Z^{\mathrm{qf}} \times F^{\mathrm{qf}}$, and let $\beta_1 : Z^{\mathrm{qf}} \times F^{\mathrm{qf}} \to X_1$ and $g : X_1 \to X$ be the induced morphism. Set $\Delta_1 := g^*\Delta$ be the pull-back of Δ as Weil divisor (c.f. [CKT16, Construction 2.13]), then (X_1, Δ_1) is klt by [KM98, Proposition 5.20, p. 160]. We have the following commutative diagram

$$\begin{array}{ccc} Z^{\mathrm{qf}} \times F^{\mathrm{qf}} & \xrightarrow{\beta_1,\ \text{small birational}} & X_1 \\ \Big\downarrow{\scriptstyle g^{\mathrm{qf}},\ \text{quasi-étale}} & & \Big\downarrow{\scriptstyle g,\ \text{quasi-étale}} \\ X^{\mathrm{qf}} & \xrightarrow[\beta,\ \text{small birational}]{} & X \end{array}$$

Then $\mathrm{pr}_1^* T_{A^{\mathrm{qf}}}$ is a direct summand of $T_{A^{\mathrm{qf}} \times B^{\mathrm{qf}} \times F^{\mathrm{qf}}} \simeq T_{Z^{\mathrm{qf}} \times F^{\mathrm{qf}}}$, and pushes down via β_1 to an algebraically integrable foliation $\mathscr{E}_1$ on X_1. Similarly, $\mathrm{pr}_2^* T_{B^{\mathrm{qf}}}$ induces an algebraically integrable foliation $\mathscr{G}_1$ on X_1. By construction $\mathscr{E}_1 \oplus \mathscr{G}_1 \simeq g^{-1}\mathscr{G}$ and $\mathscr{E}_1 \simeq \mathscr{O}_{X_1}^{\oplus \mathrm{rk}\mathscr{E}_1}$. Since $\mathscr{E}_1$ is a direct summand of T_{X_1}, $\mathscr{E}_1$ is weakly regular (c.f. [Dru18b, Lemma 5.8]), and thus has canonical singularities by Lemma 4.4.3. By applying Lemma 4.4.8 to $\mathscr{E}_1$ we see that there exist an Abelian variety A_1, a normal projective variety X_2 and a finite étale cover $g_1 : A_1 \times X_2 \to X_1$ such that $g_1^{-1}\mathscr{E}_1 \simeq \mathrm{pr}_1^* T_{A_1}$. Since g_1 is a finite étale cover, $(A_1 \times X_2, g_1^*\Delta_1)$ is klt, and hence for general $a \in A_1$, the pair $(X_2, (g_1^*\Delta_1)|_{\mathrm{pr}_1^{-1}(a)})$ is klt (by identifying X_2 with $\mathrm{pr}_1^{-1}(a)$) by [KM98, Lemma 5.17, p. 158-159]. Since g_1 is a finite étale cover, we have

$$g_1^{-1}\mathscr{E}_1 \oplus g_1^{-1}\mathscr{G}_1 \simeq g_1^{-1}g^{-1}\mathscr{G},$$

hence $g_1^{-1}\mathscr{G}_1$ is a direct summand of $\mathrm{pr}_2^* T_{X_2}$. In consequence, $g_1^{-1}\mathscr{G}_1$ descends to a a(n) (algebraically integrable) foliation $\mathscr{G}_2$ on X_2 via pr_2, i.e. there is a foliation $\mathscr{G}_2$ on X_2 such that $\mathrm{pr}_2^{-1}\mathscr{G}_2 \simeq g_1^{-1}\mathscr{G}_1$. Moreover, by construction $\mathscr{G}_2$ is a direct summand of T_{X_2}, hence $\mathscr{G}_2$ is weakly regular.

By construction we have

$$\beta_1^* K_{\mathscr{G}_1} \sim \beta_1(K_{\mathscr{E}_1} + K_{\mathscr{G}_1}) \sim \beta_1^* K_{g^{-1}\mathscr{G}} \sim K_{Z^{\mathrm{qf}}} \sim 0,$$

hence $K_{\mathscr{G}_1} \sim 0$, which implies that $K_{\mathscr{G}_2} \sim 0$ and in particular $K_{\mathscr{G}_2}$ is a Cartier divisor. By Lemma 4.4.3, $\mathscr{G}_2$ has canonical singularities. Clearly, in order to prove the theorem for X and $\mathscr{G}$, it suffices to prove this for X_2 and $\mathscr{G}_2$. If $\dim A^{\mathrm{qf}} = 0$, then $Z^{\mathrm{qf}} \simeq B^{\mathrm{qf}}$ has vanishing augmented irregularity, in this case [Dru18a, Lemma 4.6] permits us to conclude; otherwise, we have

$$\dim X_2 = \dim X - \dim A_1 = \dim X - \mathrm{rk}\mathscr{E}_1 = \dim X - \dim A^{\mathrm{qf}} < \dim X,$$

then since X_2 admits an effective divisor Δ_2 such that (X_2, Δ_2) is klt, the proof is done by an induction on the dimension. □

Next let us give an alternative proof of Theorem C by using Theorem 4.4.1:

Alternative Proof of Theorem C. Let everything as in the General Setting 4.1.1 with $\psi : X \dashrightarrow Y$ being the MRC fibration of X. By Theorem 4.3.1 the tangent sheaf admits a splitting $T_X \simeq \mathscr{F} \oplus \mathscr{G}$ into algebraically integrable foliations with $K_{\mathscr{G}} \sim 0$. Set F (resp. Z) to be the Zariski closure of the general leaf of $\mathscr{F}$ (resp. $\mathscr{G}$), then F is rationally connected. By Lemma 4.3.7, Y_0 can be regarded as a Zariski open of Z, hence Z is birational to Y; but ψ is the MRC fibration of X, Y is not uniruled, then so is Z. This means that $\mathscr{G}$ is

not uniruled, and by Proposition 4.4.7, $\mathscr{G}$ has canonical singularities. By Theorem 4.4.1 there are projective varieties Z_1 and F_1 and a quasi-étale cover $f : Z_1 \times F_1 \to X$ such that $f^{-1}\mathscr{G} \simeq \mathrm{pr}_1^* T_{Z_1}$. Since $\pi_1(X_{\mathrm{reg}}) \simeq \{1\}$, f must be an isomorphism, then we have $\mathrm{pr}_1^* T_{Z_1} \simeq \mathscr{G}$ and $\mathrm{pr}_1^* T_{F_1} \simeq \mathscr{F}$. In particular, we have $Z_1 \simeq Z$ and $F \simeq F_1$, hence $X \simeq Z \times F$ with $K_Z \sim 0$ and F rationally connected. □

Hopefully we expect that, by proving a more general splitting theorem for tangent sheaves (with no condition on the fundamental group), one is able to use Theorem 4.4.1 to prove the full Conjecture 2.

4.5 Fundamental group of X_{reg}

Let X be a klt projective variety with nef anticanonical divisor $-K_X$. In this section we study the fundamental group of X_{reg}, especially the relation of $\pi_1(X_{\mathrm{reg}})$ to the decomposition theorem and to other folklore conjectures (c.f. Conjecture 4).

4.5.1 Albanese map of X_{reg} and torsion-free nilpotent completion of $\pi_1(X_{\mathrm{reg}})$

In this subsection we will study the Albanese map of X_{reg} and deduce from this the nilpotent completion of $\pi_1(X_{\mathrm{reg}})$ by using the same argument as in [Cam95, §2]. The principal result of this subsection is the following:

Theorem 4.5.1. *Let X be a normal projective variety of semi-Fano type, i.e. there is an effective $\mathbb{Q}$-divisor Δ on X such that (X, Δ) is klt and that the twisted anticanonical divisor $-(K_X + \Delta)$ is nef. Then*

(a) *The Albanese map $\widetilde{\mathrm{alb}}_{X_{\mathrm{reg}}} : X_{\mathrm{reg}} \to \widetilde{\mathrm{Alb}}_{X_{\mathrm{reg}}}$ of X_{reg} is dominant.*

(b) *Let $j : X_{\mathrm{reg}} \hookrightarrow X$ be the open immersion. Then the morphism between fundamental groups induced by $\mathrm{alb}_X \circ j$ gives rise to an isomorphism*

$$\pi_1(X_{\mathrm{reg}})^{\mathrm{nilp}} \xrightarrow{\simeq} \pi_1(\mathrm{Alb}_X).$$

Before turning to the proof of the theorem, let us first recall the definition of the nilpotent completion of a group (c.f. [Cam95, Appendice A]). Let G be a group, define the descending central series of G by $G_1 := G$ and $G_{k+1} = [G, G_k]$ for any $k \in \mathbb{Z}_{>0}$ and set

$$G_\infty := \bigcap_{k \in \mathbb{Z}_{>0}} G_k.$$

Put

$$G'_k = \sqrt{G_k} := \left\{ g \in G \,\middle|\, g^m \in G_k \text{ for some } m \in \mathbb{Z}_{>0} \right\}.$$

for $1 \leqslant k \leqslant \infty$. Then the *torsion-free nilpotent completion* of G is defined to be

$$G^{\mathrm{nilp}} := G/G'_\infty.$$

Let $f : G \to H$ be a group morphism, [Sta65, 3.4.Theorem] gives the following criterion for the induced morphism between nilpotent completion to be injective or isomorphism (c.f. also [Cam95, A.2.Théorème]):

Proposition 4.5.2 ([Sta65, 3.4.Theorem])**.** *Let $f : G \to H$ be a group morphism, and for $1 \leqslant k \leqslant \infty$ let G'_k (resp. H'_k) be the radical of the k-th member in the descending central series of G (resp. of H), as defined above. Suppose that the induced morphism $\mathrm{H}_i(f) : \mathrm{H}_i(G, \mathbb{Q}) \to \mathrm{H}_i(H, \mathbb{Q})$ is an isomorphism for $i = 1$ and surjective for $i = 2$. Then the morphism $f'_k : G'_k \to H'_k$ induced by f is injective for every $1 \leqslant k \leqslant \infty$, and is of finite index if $k < \infty$. Moreover, if f is surjective then f'_k is an isomorphism for every $1 \leqslant k \leqslant \infty$.*

Now let us turn to the proof of Theorem 4.5.1:

Proof of Theorem 4.5.1. Let us first prove (a), i.e. the Albanese map of X_{reg} is dominant. Let Y° be the Zariski closure in $\widetilde{\mathrm{Alb}}_{X_{\mathrm{reg}}}$ of the image of $\widetilde{\mathrm{alb}}_{X_{\mathrm{reg}}}$ and let Y be a smooth compactification of Y° such that $D_Y := Y \backslash Y^\circ$ is a SNC divisor, then we get a dominant rational map $\psi : X \dashrightarrow Y$. Take M to be a strong desingularization of the graph of ψ, then the induced morphism $\pi : M \to X$ is a birational morphism which is an isomorphism over X_{reg}. Let $E = \mathrm{Exc}(\pi)$ be the exceptional divisor of π and let $\phi : M \to Y$ be the natural morphism, then by construction $M \backslash E \simeq X_{\mathrm{reg}}$ and thus $\mathrm{Supp}(\phi^* D_Y) \subseteq E$. Now we are in the same situation as in General Setting 4.1.1, hence by the proof of Proposition 4.1.2(a), for a very ample line bundle L on X and for general members $H_1, \cdots, H_{\dim X - 1}$ in the linear series $|\pi^* L|$, we have

$$\phi^* K_Y \cdot C \leqslant 0$$

where $C := H_1 \cap \cdots \cap H_{\dim X - 1}$. Since $\phi^* D_Y$ is π-exceptional, we have $\phi^* D_Y \cdot C = 0$ hence by the projection formula we get

$$(K_Y + D_Y) \cdot C_Y \leqslant 0$$

where $C_Y := \phi_* C$. By Proposition 1.5.3 we know that $\bar{\kappa}(Y^\circ) := \kappa(Y, K_Y + D_Y) \geqslant 0$ (c.f. Definition 1.5.4 for the definition of logarithmic Kodaira dimension), hence we must have

$$(K_Y + D_Y) \cdot C_Y = 0,$$

but by construction C_Y is moves in a strong connecting family of curves (c.f. [BDPP13, §0]) on Y, hence by [BDPP13, 0.5.Theorem] the numerical dimension $\nu(Y, K_Y + D_Y) = 0$, this implies that $\kappa(Y, K_Y + D_Y) \leqslant \nu(Y, K_Y + D_Y) = 0$. Therefore we must have $\bar{\kappa}(Y^\circ) = \kappa(Y, K_Y + D_Y) = 0$. Again by Proposition 1.5.3 we have that Y° is a semi-Abelian subvariety of $\widetilde{\mathrm{Alb}}_{X_{\mathrm{reg}}}$; but by Propositionn 1.5.7 Y° generates $\widetilde{\mathrm{Alb}}_{X_{\mathrm{reg}}}$, hence we must have $Y^\circ = \widetilde{\mathrm{Alb}}_{X_{\mathrm{reg}}}$, and this proves (a).

Now let us prove (b). It can be deduced by [Cam95, 2.2.Théorème] and by the more general Theorem 4.5.3 below. This theorem, as well as its proof, is pointed out to the author by Benoît Claudon (any mistake, is of course, the author's). □

Theorem 4.5.3. *Let X be a normal projective variety which admits an effective $\mathbb{Q}$-divisor Δ such that the pair (X, Δ) is klt and let $j : X_{\mathrm{reg}} \hookrightarrow X$ be the open immersion. Then*

$$\mathrm{H}^1(j, \mathbb{C}) : \mathrm{H}^1(X, \mathbb{C}) \to \mathrm{H}^1(X_{\mathrm{reg}}, \mathbb{C})$$

is an isomorphism and

$$\mathrm{H}^2(j, \mathbb{C}) : \mathrm{H}^2(X, \mathbb{C}) \to \mathrm{H}^2(X_{\mathrm{reg}}, \mathbb{C})$$

is injective. In particular, j induces an isomorphism between the nilpotent completion of fundamental groups

$$\pi_1(X_{\mathrm{reg}})^{\mathrm{nilp}} \xrightarrow{\simeq} \pi_1(X)^{\mathrm{nilp}}.$$

Proof. The klt condition is used to guarantee the vanishing of $\mathrm{R}^1 j_* \mathbb{C}$. In fact, by [Bra20, Theorem 1], for any point $x \in X$, there is an open neighbourhood U of x such that $\pi_1(U_{\mathrm{reg}})$ is finite, in particular $\mathrm{H}^1(U_{\mathrm{reg}}, \mathbb{C}) = 0$, hence we get $\mathrm{R}^1 j_* \mathbb{C} = 0$. Then consider the Leray spectral sequence associated to j which gives the exact sequence

$$0 \to \mathrm{H}^1(X, \mathbb{C}) \to \mathrm{H}^1(X_{\mathrm{reg}}, \mathbb{C}) \to \mathrm{H}^0(X, \mathrm{R}^1 j_* \mathbb{C}) \to \mathrm{H}^2(X, \mathbb{C}) \to \mathrm{H}^2(X_{\mathrm{reg}}, \mathbb{C}),$$

by the vanishing of $\mathrm{R}^1 j_* \mathbb{C}$ we get the isomorphism $\mathrm{H}^1(X, \mathbb{C}) \simeq \mathrm{H}^1(X_{\mathrm{reg}}, \mathbb{C})$ and the injectivity of $\mathrm{H}^2(X, \mathbb{C}) \to \mathrm{H}^2(X_{\mathrm{reg}}, \mathbb{C})$.

It remains to show that $\pi_1(X_{\mathrm{reg}})^{\mathrm{nilp}} \simeq \pi_1(X)^{\mathrm{nilp}}$. By [FL81, §0.7 (B), p. 33] the fundamental group morphism $\pi_1(j) : \pi_1(X_{\mathrm{reg}}) \to \pi_1(X)$ is surjective, hence by Proposition 4.5.2 it suffices to show that

$$\mathrm{H}_1(\pi_1(j), \mathbb{Q}) : \mathrm{H}_1(\pi_1(X_{\mathrm{reg}}), \mathbb{Q}) \to \mathrm{H}_1(\pi_1(X), \mathbb{Q})$$

is an isomorphism and

$$\mathrm{H}_2(\pi_1(j), \mathbb{Q}) : \mathrm{H}_2(\pi_1(X_{\mathrm{reg}}), \mathbb{Q}) \to \mathrm{H}_2(\pi_1(X), \mathbb{Q})$$

is surjective. We have shown that $\mathrm{H}^1(X, \mathbb{C}) \simeq \mathrm{H}^1(X_{\mathrm{reg}}, \mathbb{C})$, hence $\mathrm{H}_1(\pi_1(j), \mathbb{Q})$ is an isomorphism by [Sta65, §5]. On the other hand, the surjectivity $\mathrm{H}_2(\pi_1(j), \mathbb{Q})$ can be deduced from [Cam95, 2.3.Lemma] and from the injectivity of $\mathrm{H}^2(X, \mathbb{C}) \to \mathrm{H}^2(X_{\mathrm{reg}}, \mathbb{C})$. □

4.5.2 From fundamental group to decomposition theorem

In this subsection, we show that with the help of Theorem B and Theorem C the proof of Conjecture 2 can be reduced to the study of the fundamental group of X. Precisely speaking, we will prove that Conjecture 3 implies Conjecture 2. Let us remark that when X is smooth, the Conjecture 3 is proved by M.Păun in [Pău97] by improving the arguments in the previous work of [DPS93, §1] and by applying the famous theorem of Cheeger-Colding [CC96, Theorem 8.7].

Theorem 4.5.4. *Let X be a normal projective variety of semi-Fano type. Suppose that Conjecture 3 holds for X, i.e. $\pi_1(X_{\mathrm{reg}})$ is of polynomial growth, then (the log version of) Conjecture 2 holds for X, i.e. up to replacing X by a finite quasi-étale cover, the universal cover $\tilde{X}$ of X can be decomposed into a product*

$$\tilde{X} \simeq \mathbb{C}^q \times Z \times F, \tag{4.15}$$

with q being the augmented irregularity of X, Z being a klt projective variety with trivial canonical divisor and F being rationally connected.

Proof. By [KM98, Proposition 5.20, pp. 160-161], any quasi-étale cover of X is still of semi-Fano type. By hypothesis $\pi_1(X_{\mathrm{reg}})$ is of polynomial growth (by [FL81, §0.7 (B), p. 33] so is $\pi_1(X)$), hence by [Gro81, Main Theorem] $\pi_1(X_{\mathrm{reg}})$ is virtually nilpotent, therefore, up to replacing X by a finite étale cover we can assume that $\pi_1(X_{\mathrm{reg}})$ is torsion-free nilpotent.

By Theorem B, the Albanese map $\mathrm{alb}_X : X \to \mathrm{Alb}_X$ is a locally constant fibration. Let V denotes the fibre of alb_X, then $\mathrm{alb}_X|_{X_{\mathrm{reg}}} : X_{\mathrm{reg}} \to \mathrm{Alb}_X$ is a locally trivial fibration whose fibre is isomorphic to V_{reg}. Apply [BT82, §17, p. 209] to $\mathrm{alb}_X|_{X_{\mathrm{reg}}}$ (viewed as a topological fibre bundle) we get a homotopy sequence

$$\cdots \to \pi_2(\mathrm{Alb}_X) \to \pi_1(V_{\mathrm{reg}}) \to \pi_1(X_{\mathrm{reg}}) \to \pi_1(\mathrm{Alb}_X) \to 1.$$

But $\pi_1(X_{\mathrm{reg}})$ is torsion free nilpotent, by Theorem 4.5.1 the morphism $\pi_1(X_{\mathrm{reg}}) \to \pi_1(\mathrm{Alb}_X)$ is an isomorphism. Moreover, since Alb_X is an Abelian variety, we have $\pi_2(\mathrm{Alb}_X) \simeq \{0\}$, hence $\pi_1(V_{\mathrm{reg}}) \simeq \{1\}$. Then the conclusion follows from Theorem C. □

4.5.3 From Conjecture 3 to Conjecture 4

In this subsection we will show that Conjecture 3 implies the Gurjar-Zhang conjecture on the finiteness of the fundamental group of the smooth locus of varieties of Fano type and the Conjecture 4. In fact, we can prove the following more general result:

Proposition 4.5.5. *Let X be a normal projective variety of semi-Fano type with vanishing augmented irregularity. Suppose that $\pi_1(X_{\mathrm{reg}})$ is of polynomial growth, then $\pi_1(X_{\mathrm{reg}})$ is finite.*

Proof. First note that, as in the proof of Theorem 4.5.4, in order to prove the finiteness of $\pi_1(X_{\mathrm{reg}})$ we can replace X by any finite quasi-étale cover; in particular we can assume that $\pi_1(X_{\mathrm{reg}})$ is a torsion-free nilpotent group (by [Gro81, Main Theorem]), so that we have $\pi_1(X_{\mathrm{reg}}) \simeq \pi_1(\mathrm{Alb}_X)$. But the augmented irregularity of X is zero, its Albanese variety Alb_X is trivial, then a fortiori $\pi_1(X_{\mathrm{reg}}) \simeq \{1\}$, in particular $\pi_1(X_{\mathrm{reg}})$ is finite. Thus we proved the proposition. □

By the proposition above, we see that Conjecture 3 implies Conjecture 4; moreover, since varieties of Fano type have vanishing augmented irregularity (every quasi-étale cover of a projective variety of Fano type remains Fano type; by [Zha06, Corollary 1.1] and [Tak00] varieties of Fano type are simply connected), Conjecture 3 implies the Gurjar-Zhang conjecture which states that for any projective variety of Fano type X the fundamental group of X_{reg} is finite and which has recently been confirmed in [Bra20].

Finally, let us make some remarks on the history of the Gurjar-Zhang conjecture and Conjecture 4:

Remark 4.5.6. The Gurjar-Zhang conjecture is first proved for del Pezzo surfaces in [GZ94; GZ95] (c.f. [GZ95, last Remark] for weak Fano surfaces) and the question is explicitly raised in [Zha95, Introduction] for log Fano varieties (c.f. also [Sch07, Question 0.11]) and in [Zha95] the conjecture is proved for canonical (klt) Fano threefolds under some additional assumption that X has isolated singularities ([Zha95, Theorem 1]) or that the index of X is $\geqslant \dim X - 2$ ([Zha95, Theorem 2]). The three-dimensional Fano case is fully confirmed by [TX17, Theorem 1.6]. Then it is proved in [Xu14, Theorem 2] and [GKP16, Theorem 1.13] that the profinite completion of $\pi_1(X_{\mathrm{reg}})$ (which is, isomorphic to the étale fundamental group of X_{reg}) is finite for X weak log Fano. Recently this conjecture has been settled in [Bra20].

As for Conjecture 4, the question is raised in [GGK19] and it is proved therein that for X klt projective with trivial canonical divisor and vanishing augmented irregularity the fundamental group of X_{reg} has only finitely many k-dimensional complex representations for every $k \in \mathbb{Z}_{>0}$, and that the image of each finite dimensional representation of $\pi_1(X_{\mathrm{reg}})$ is finite. It is also proved that the étale fundamental group of X_{reg} is finite for X an irreducible holomorphic symplectic variety or an even-dimensional Calabi-Yau varieties, c.f. [GGK19, §13.1].

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Titre : Positivité des images directes et variétés projectives à courbure non-négative

Mots clés : Conjecture $C_{n,m}$ d'Iitaka ; Programme des modèles minimaux, diviseur anticanonique, application d'Albanese, fibration MRC, feuilletages rationnellement connexes

Résumé : La classification birationnelle des variétés algébrique est un problématique central en géoémtrie algébrique. Récemment grand progrès a été fait vers l'établissement du MMP et l'abondance, et par ces travaux, les variété projectives lisse (ou légèrement singulières) sont birationnellement divisées en deux catégories : 1. les variétés à diviseur canonique pseudoeffectif, qui sont montré d'aboutir à un modèle minimal par le MMP ; 2. les variétés uniréglées, qui sont recouvertes par des courbes rationnelles. Dans cette thèse, des étude raffinées de ces deux catégories de variétés est sont effectuées respectivement, by en suivant la philosophie d'étudier les fibrations canoniques y associées.
Pour une variété X dans la première catégorie, la fibration la plus importante y associée est la fibration d'Iitaka-Kodaira, dont la base est de dimension égale à la dimension de Kodaira de X. Cette thèse traite un corollaire important de l'abondance, à savoir, la conjecture $C_{n,m}$ d'Iitaka, qui énonce la sup-additivité de la dimension de Kodaira dimension par rapport aux fibration algébrique. Dans cette thèse la version kählérienne de $C_{n,m}$ est montré sous l'hypothèse que la base est un tore complexe en développant davantage la positivité des images directes et la version pluricanonique du théorème à la Green-Lazarsfeld-Simpson sur les lieux de sauts de cohomologie. Ceci généralise le résultat principal de Cao-Păun (2017).
Pour les variétés dans la seconde catégorie, l'on étude l'application d'Albanese et la fibration MRC fibration, au lieu de la fibration d'Iitaka-Kodaira. La philosophie dans cette enquête est que si le fibré tangent ou anticanonique admet une certaine positivité, les deux fibrations susmentionnées doivent avoir une structure rigide. Dans cette thèse j'étudie la structure des variétés projective (légèrement singulières) à diviseur anticanonique nef. En appliquant la positivité des images directe et des résultats de la théorie des feuilletages, j'arrive à démontrer que l'application d'Albanese map est une fibration localement constante et que si le lieu lisse est simplement connexe la fibration MRC induit une décomposition en un produit. Ceci généralise les résultats correspondants pour les variétés lisses dans Cao (2019) and Cao-Höring (2019).

Title : Positivity of Direct Images and Projective Varieties with Nonnegative Curvature

Keywords : Iitaka Conjecture $C_{n,m}$; Minimal Model Program ; anticanonical divisor ; Albanese map ; MRC fibration ; rationally connected foliations

Abstract : The birational classification of algebraic varieties is a central problem in algebraic geometry. Recently great progress has been made towards the establishment of the MMP and the Abundance and by these works, smooth (or mildly singular) projective varieties can be birationally divided into two categories : 1. varieties with pseudoeffective canonical divisor, which are shown to reach a minimal model under the MMP ; 2. uniruled varieties, which are covered by rational curves. In this thesis refined studies of these two categories of varieties are carried out respectively, by following the philosophy of studying the canonical fibrations associated to them.
For any variety X in the first category, the most important canonical fibration associated to X is the Iitaka-Kodaira fibration whose base variety is of dimension equal to the Kodaira dimension of X. This thesis tacles an important corollary of the Abundance conjecture, namely, the Iitaka conjecture $C_{n,m}$, which states the supadditivity of the Kodaira dimension with respect to algebraic fibre spaces. In this thesis the Kähler version of $C_{n,m}$ is proved under the assumption that the base variety of the fibre space is a complex torus by further developping the positivity theorem of direct images and the pluricanonical version of the Green-Lazarsfeld-Simpson type theorem on cohomology jumping loci. This generalizes the main result of Cao-Păun (2017).
As for varieties in the second category, one studies the Albanese map and the MRC fibration, instead of the Iitaka-Kodaira fibration. A philosophy in this investigation is that when the tangent bundle or the anticanonical divisor admits certain positivity, the aforementioned two fibrations of the variety should have a rigid structure. In this thesis I study in this thesis the structure of (mildly singular) projective varieties with nef anticanonical divisor. By again applying the positivity of direct images and by applying results from the foliation theory, I manage to prove that the Albanese map of such variety is a locally constant fibration and that if its smooth locus is simply connected then the MRC fibration induces a splitting into a product. These generalize the corresponding results for smooth projective varieties in Cao (2019) and Cao-Höring (2019).

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