Feedback exponential stabilization of open quantum systems undergoing continuous-time measurements

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Résumé

Dans cette thèse, nous nous intéressons à la stabilisation par rétroaction des systèmes quantiques ouverts soumis à des mesures imparfaites en temps continu. Tout d'abord, nous introduisons la théorie du filtrage quantique pour décrire l'évolution temporelle de l'opérateur de densité conditionnelle représentant un état quantique en interaction avec un environnement. Ceci est décrit par une équation différentielle stochastique à valeurs matricielles. Deuxièmement, nous étudions le comportement asymptotique des trajectoires quantiques associées à des systèmes de spin à N niveaux pour des états initiaux donnés, pour les cas avec et sans loi de rétroaction. Dans le cas sans loi de rétroaction, nous montrons la propriété de réduction de l'état quantique à vitesse exponentielle. Ensuite, nous fournissons des conditions suffisantes sur la loi de contrôle assurant une convergence presque sûre vers un état pur prédéterminé correspondant à un vecteur propre de l'opérateur de mesure. Troisièmement, nous étudions le comportement asymptotique des trajectoires de systèmes ouverts à plusieurs qubits pour des états initiaux donnés. Dans le cas sans loi de rétroaction, nous montrons la réduction exponentielle de l'état quantique pour les systèmes N-qubit avec deux canaux quantiques. Dans le cas particulier des systèmes à deux qubits, nous donnons des conditions suffisantes sur la loi de contrôle assurant la convergence asymptotique vers un état cible de Bell avec un canal quantique, et la convergence exponentielle presque sûre vers un état cible de Bell avec deux canaux quantiques. Ensuite, nous étudions le comportement asymptotique des trajectoires des systèmes quantiques ouverts de spin-$\frac{1}{2}$ avec les états initiaux inconnus soumis à des mesures imparfaites en temps continu, et nous fournissons des conditions suffisantes au contrôleur pour garantir la convergence de l'état estimé vers l'état quantique réel lorsque le temps tends vers l'infini. En conclusion, nous discutons de manière heuristique du problème de stabilisation exponentielle des systèmes de spin à N niveaux avec les états initiaux inconnus et nous proposons des lois de rétroaction candidates afin de stabiliser le système de manière exponentielle.

Abstract

In this thesis, we focus on the feedback stabilization of open quantum systems undergoing imperfect continuous-time measurements. First, we introduce the quantum filtering theory to obtain the time evolution of the conditional density operator representing a quantum state in interaction with an environment. This is described by a matrix-valued stochastic differential equation. Second, we study the asymptotic behavior of quantum trajectories associated with N-level quantum spin systems for given initial states, for the cases with and without feedback law. For the case without feedback, we show the exponential quantum state reduction. Then, we provide sufficient conditions on the feedback control law ensuring almost sure exponential convergence to a predetermined pure state corresponding to an eigenvector of the measurement operator. Third, we study the asymptotic behavior of trajectories of open multi-qubit systems for given initial states. For the case without feedback, we show the exponential quantum state reduction for N-qubit systems with two quantum channels. Then, we focus on the two-qubit systems, and provide sufficient conditions on the feedback control law ensuring asymptotic convergence to a target Bell state with one quantum channel, and almost sure exponential convergence to a target Bell state with two quantum channels. Next, we investigate the asymptotic behavior of trajectories of open quantum spin-$\frac{1}{2}$ systems with unknown initial states undergoing imperfect continuous-time measurements, and provide sufficient conditions on the controller to guarantee the convergence of the estimated state towards the actual quantum state when time goes to infinity. Finally, we discuss heuristically the exponential stabilization problem for N-level quantum spin systems with unknown initial states and propose candidate feedback laws to stabilize exponentially the system.

Contents

1 Introduction (version française)

1.1 Motivation et contexte

La théorie du contrôle quantique est un domaine de recherche en plein essor, cependant de nombreux efforts doivent encore être effectués pour rendre cette spécialité plus pratique dans l'ingénierie des dispositifs quantiques. La capacité de contrôler les systèmes quantiques, par exemple, de préparer et de protéger un état quantique souhaité, joue un rôle essentiel dans le développement de technologies quantiques avancées. Les technologies quantiques [NC02] sont censées dépasser les technologies conventionnelles: l'informatique quantique peut être beaucoup plus rapide que l'informatique classique pour résoudre certains problèmes, et la métrologie quantique sera beaucoup plus précise que les techniques conventionnelles dans l'estimation des paramètres.

La dynamique de systèmes quantiques fermés simples peut être décrite par les équations de Schrödinger (dépendantes du temps). Pour de tels systèmes, nous pouvons définir des tâches de contrôle simples pouvant être réalisées à l'aide de techniques de contrôle en boucle ouverte. Cela signifie qu'un signal de commande classique prédéterminé est appliqué à un système quantique fermé et qu'aucune rétroaction n'est impliquée ; vous pouvez trouver un traitement complet dans [d'A07]. La stratégie de contrôle en boucle ouverte a été appliquée dans différents contextes, comme l'approche du contrôle optimal [TR04, KRK+05, BCS09], problèmes de contrôlabilité [BBR10, BCR10], etc. En raison de problèmes de robustesse, une approche de contrôle en boucle ouverte ne peut pas s'appliquer aux tâches de contrôle quantique complexes, telles que la suppression de la décohérence (c'est-à-dire la perte d'informations en raison du couplage inévitable avec l'environnement) et le bruit quantique. Il est donc important d'introduire l'analogue quantique de la théorie de contrôle de la rétroaction classique afin de compenser certaines perturbations, telles que les incertitudes et les dérives des paramètres décrivant le système dynamique, les perturbations externes et les bruits de mesure, pour accroître la robustesse du système. Grossièrement, il existe deux types courants de contrôle de rétroaction quantique en fonction de la nature du contrôleur

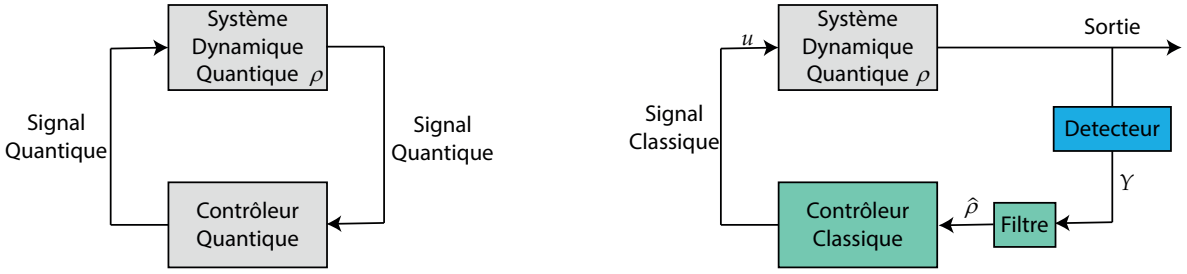
de rétroaction: la rétroaction cohérente et la rétroaction basée sur les mesures. Pour la rétroaction cohérente, le contrôleur est un autre système quantique communiquant avec le système quantique dynamique à contrôler par un signal quantique, c'est-à-dire un faisceau de lumière [Llo00], voir Fig 1.1. Ce type de retour a récemment été appliqué à différentes tâches telles que l'amélioration de la compression [GW09, WM94], les mémoires quantiques et la correction d'erreur [KNPM10]. Pour la rétroaction basée sur les mesures, le contrôleur est un système classique qui traite les informations classiques obtenues en mesurant un observable du système dynamique quantique à contrôler et met en œuvre l'entrée de contrôle appropriée, voir Fig 1.1. Ce type de rétroaction a été envisagé pour la préparation des états pressés [TMW02], la réduction de l'état quantique [vHSM05a, MDR09], etc.

Dans cette thèse, nous nous concentrons sur la stabilisation par retour basée sur la mesure des systèmes quantiques ouverts, c'est-à-dire des systèmes en interaction avec un environnement. L'évolution d'un système quantique ouvert soumis à des mesures indirectes en temps continu est décrite par l'équation maîtresse stochastique quantique, qui a été dérivée par Belavkin en utilisant la théorie du filtrage quantique [Bel89]. La théorie du filtrage quantique, reposant sur le calcul quantique stochastique et la théorie des probabilités quantiques (développée par Hudson et Parthasarathy [HP84]) joue un rôle important en optique quantique et calcul quantique.

Le contrôle de rétroaction basé sur des mesures quantiques peut être considéré comme une branche du contrôle stochastique, qui a été développé pour la première fois par Belavkin dans [Bel83]. Ce domaine a suscité l'intérêt de nombreux chercheurs théoriques et expérimentaux, principalement à partir du début des années 2000, donnant des résultats fondamentaux [vHSM05a, AAS+02, MvH07, Tsu08, ADL02, YTH07, MK05]. En particulier, les études théoriques effectuées dans [MvH07, DMB+09, MDR09, ARM11, ASD+13] ont abouti à la première mise en oeuvre expérimentale de la rétroaction quantique en temps réel basée sur la mesure dans [SDZ+11].

Dans [BvH08], les auteurs ont établi un principe de séparation quantique. De manière similaire au principe de séparation classique, ce résultat permet d'interpréter le problème de contrôle comme un problème de contrôle de retour basé sur l'état pour le filtre (la meilleure estimation, c'est-à-dire l'état conditionnel), sans se soucier de l'état quantique actuel. Cela motive la conception de rétroaction basée sur l'état pour l'équation de filtrage quantique basée sur la connaissance de l'état initial. Dans ce contexte, la stabilisation des filtres quantiques vers les états purs (c'est-à-dire la préparation d'états purs) a un impact majeur sur le développement de nouvelles technologies quantiques.

Dans ce qui suit, nous introduisons d'abord très brièvement quelques postulats importants issus de la mécanique quantique, des systèmes à spin quantique et de l'oscillateur harmonique quantique. Ensuite, nous présentons rapidement les systèmes quantiques ouverts et le filtrage quantique. Enfin, nous évoquons le plan de cette thèse avec les principales contributions.



(a) Contrôle de retour cohérent

(b) Contrôle du retour basé sur les mesures

FIGURE 1.1 – (a) Le système dynamique et le contrôleur sont des systèmes quantiques interagissant l'un avec l'autre. Le contrôleur quantique obtient le signal quantique d'un système dynamique et le traite en utilisant la logique quantique, puis le renvoie de manière cohérente dans le système dynamique quantique. (b) Le système dynamique est un système quantique, tandis que le contrôleur de rétroaction est un système classique, qui ne peut traiter que le signal classique. Le filtre est utilisé pour former un estimateur de l'état actuel du système dynamique en fonction des résultats du détecteur. Ensuite, le contrôleur de rétroaction fournit l'entrée de contrôle, c'est-à-dire une fonction de l'estimateur, qui est appliquée pour contrôler le système dynamique.

1.2 Mécanique quantique

1.2.1 Postulats de la mécanique quantique

En mécanique classique, le mouvement de tout système physique est déterminé si la position et la vitesse de chacun de ses points sont connues en tant que fonctions du temps. Toutes les grandeurs physiques associées au système, à savoir la position, l'énergie, le moment cinétique, etc., peuvent être mesurées avec précision, et ces mesures n'affectent pas le système en général. La mécanique quantique diffère beaucoup de la mécanique classique. Dans ce qui suit, nous donnons les principaux postulats de la mécanique quantique [CTDL18, SN14].

Premier postulat : *A tout moment fixe t_0, l'état d'un système physique est défini en spécifiant un vecteur d'état $\psi(t_0)$ appartenant à un espace de Hilbert complexe séparable $\mathcal{H}$.* Dans la notation de Dirac, nous désignons ce vecteur d'état par $|\psi(t_0)\rangle$ et l'appelons ket. Le vecteur d'état correspondant appartenant au double espace $\mathcal{H}$ est noté $\langle\psi(t_0)|$ et appelé bra. Alors le produit intérieur de deux kets $|\psi\rangle$ et $|\phi\rangle$ appartenant à $\mathcal{H}$ est défini par $\langle\psi|\phi\rangle = \langle\phi|\psi\rangle^*$. Ce premier postulat implique un principe de superposition : une combinaison linéaire de vecteurs d'état est également un vecteur d'état.

Deuxième postulat : *Chaque grandeur physique mesurable $\mathscr{X}$ est décrite par un opérateur hermitien X agissant dans $\mathcal{H}$, cet opérateur est un observable.*

Troisième postulat : *Le seul résultat possible de la mesure d'une quantité physique $\mathscr{X}$ est l'une des valeurs propres de l'observable correspondant X.* Une mesure de $\mathscr{X}$ donne toujours une valeur réelle, puisque X est hermitien. Si le spectre de X est discret, les résultats pouvant être obtenus en mesurant $\mathscr{X}$ sont quantifiés.

Quatrième postulat (cas de dimension finie) : *Lorsque la quantité physique $\mathscr{X}$ est mesurée sur un système à l'état normalisé $|\psi\rangle$, la probabilité $\mathbb{P}(x_n)$ d'obtenir la valeur*

propre x_n de la valeur correspondante observable X est : $\mathbb{P}(x_n) = \sum_{i=1}^{g_n} |\langle u_n^i|\psi\rangle|^2$, *où g_n est le degré de dégénérescence de x_n et $|u_n^i\rangle$ avec $i = 1, 2, \ldots, g_n$ est un ensemble orthonormal de vecteurs qui constitue une base dans l'espace électronique $\mathcal{H}_n$ associé à la valeur propre x_n.* Pour la valeur propre x_n, nous avons $X|u_n^i\rangle = x_n|u_n^i\rangle$ avec $i = 1, \ldots, g_n$, puis nous développons le vecteur d'état $|\psi\rangle$ dans la base orthonormale $\{|u_n^i\rangle\}$ et obtenons $|\psi\rangle = \sum_n \sum_{i=1}^{g_n} c_n^i |u_n^i\rangle$ où $c_n^i = \langle u_n^i|\psi\rangle$. Alors la probabilité $\mathbb{P}(x_n)$ d'obtenir la valeur propre non dégénérée x_n de l'observable correspondant X est: $\mathbb{P}(x_n) = \sum_{i=1}^{g_n} |\langle u_n^i|\psi\rangle|^2 = \sum_{i=1}^{g_n} |c_n^i|^2$. Ensuite, nous définissons le projecteur sur $\mathcal{H}_n$: $P_n = \sum_{i=1}^{g_n} |u_n^i\rangle\langle u_n^i|$, où $P_n^* = P_n$ et $P_n^2 = P_n$, la probabilité mentionnée ci-dessus peut être écrite: $\mathbb{P}(x_n) = \langle\psi|P_n|\psi\rangle$. Puisque nous pouvons écrire l'observable sous la forme $X = \sum_n x_n P_n$, l'attente correspondante de X est définie par $\langle X\rangle := \sum_n x_n \mathbb{P}(x_n) = \langle\psi|X|\psi\rangle$. Une conséquence importante de ce postulat est la suivante : les probabilités prédites pour une mesure arbitraire sont les mêmes pour deux vecteurs d'états proportionnels, elles représentent donc le même état physique. Ainsi, les états d'un système quantique sont des rayons dans un espace de Hilbert $\mathcal{H}$.

Cinquième postulat : *Si une mesure de la quantité physique $\mathscr{X}$ sur un système de l'état $|\psi\rangle$ donne le résultat x_n, l'état du système immédiatement après la mesure est la projection normalisée $\frac{P_n|\psi\rangle}{\sqrt{\langle\psi|P_n|\psi\rangle}}$ de $|\psi\rangle$ sur l'espace propre associé à x_n.* Ce postulat est ce qu'on appelle l'effondrement de l'état ou la réduction du paquet d'onde.

Sixième postulat : *L'évolution temporelle du vecteur d'état $|\psi(t)\rangle$ est régie par l'équation de Schrödinger*:

$$i\hbar\frac{d}{dt}|\psi(t)\rangle = H(t)|\psi(t)\rangle, \tag{1.1}$$

où $H(t)$ est l'observable associé à l'énergie totale du système, appelé opérateur hamiltonien. La solution de l'équation de Schrödinger ci-dessus est donnée par $|\psi(t)\rangle = U(t, t_0)|\psi(t_0)\rangle$, où $U(t, t_0) = exp\left(-\frac{i}{\hbar}\int_{t_0}^{t} H(\tau)d\tau\right)$ et il s'agit d'un opérateur unitaire agissant sur $\mathcal{H}$. En particulier, si H ne dépend pas du temps (cas des systèmes conservateurs), nous avons $U(t, t_0) = e^{-iH(t-t_0)/\hbar}$. Il est également instructif d'étudier la dynamique de l'attente d'un X observable dans un état donné $|\psi(t)\rangle$,

$$\langle X\rangle(t) = \langle\psi(t)|X(t_0)|\psi(t)\rangle = \langle\psi(t_0)|U^*(t, t_0)X(t_0)U(t, t_0)|\psi(t_0)\rangle, \tag{1.2}$$

où notons l'observable X à t_0 comme $X(t_0)$. Étant donné que la nature dépendant du temps du système doit être assurée par une combinaison des vecteurs d'état et des opérateurs, il existe deux approches pour traiter les systèmes évoluant dans le temps en général:

1. représentation de Schrödinger: $|\psi(t_0)\rangle \to |\psi(t)\rangle = U(t, t_0)|\psi(t_0)\rangle$, où les opérateurs sont inchangés.
2. représentation de Heisenberg: $X_H(t_0) = X(t_0) \to X_H(t) = U^*(t, t_0)X_H(t_0)U(t, t_0)$, où les vecteurs d'état sont inchangés. Ici, l'indice H est utilisé pour souligner l'image de Heisenberg.

De (1.2), nous pouvons voir intuitivement que ces deux approches sont équivalentes. En raison de (1.2) et de l'équation de Schrödinger, la dynamique du système dans la représentation de Heisenberg est donnée par $i\hbar\frac{d}{dt}X_H(t) = [X_H(t), H(t)]$, cette équation s'appelle l'équation de Heisenberg.

Jusqu'à présent, nous avons considéré des systèmes dont les états sont parfaitement connus et nous pouvons étudier leurs évolutions dans le temps et prévoir les résultats

de diverses mesures effectuées. Cependant, dans la pratique, l'état du système n'est souvent pas parfaitement déterminé. Ensuite, nous introduisons l'opérateur de densité pour représenter l'état d'un système quantique, qui code une situation avec des informations incomplètes d'une manière plus générale qu'un vecteur d'état. Les informations incomplètes sur le système en mécanique quantique signifient que son état est un mélange statistique d'états $|\psi_1\rangle, |\psi_1\rangle, \ldots$ avec des probabilités $p_1, p_2, \ldots$. Les états $|\psi_1\rangle, |\psi_2\rangle, \ldots$ ne sont pas nécessairement orthogonaux.

Nous utilisons ensuite l'opérateur de densité $\rho(t)$ pour décrire le mélange statistique d'états. On dit que le système est à l'état pur si son état est parfaitement connu, c'est-à-dire que $\rho(t) = |\psi(t)\rangle\langle\psi(t)|$ où $|\psi(t)\rangle$ est le vecteur d'état du système à l'instant t. Notez que, pour les états purs, nous avons $\rho^2 = \rho$ et $\mathrm{Tr}(\rho^2) = 1$. Pour les états mixtes, nous avons $\rho = \sum_k p_k \rho_k$ avec $\rho_k = |\psi_k\rangle\langle\psi_k|$ l'opérateur de densité de l'état pur correspondant à $|\psi\rangle$. Ainsi, nous concluons que l'opérateur de densité possède trois propriétés principales: $\rho^* = \rho$, $\mathrm{Tr}(\rho) = 1$ et $\rho \geq 0$. Ensuite, nous généralisons les quatrième, cinquième et sixième postulats ci-dessus au cas en ce qui concerne les opérateurs de densité.

Quatrième postulat (cas de dimension finie) : *Lorsque la quantité physique $\mathscr{X}$ est mesurée sur un système dans ρ, la probabilité $\mathbb{P}(x_n)$ de l'obtention de la valeur propre x_n de l'observable correspondant X est*: $\mathbb{P}(x_n) = \mathrm{Tr}(\rho P_n)$. En conséquence, l'attente de X est donnée par $\langle X\rangle = \mathrm{Tr}(\rho X)$.

Cinquième postulat : *Si une mesure de la quantité physique $\mathscr{X}$ sur un système dans ρ donne le résultat x_n, l'état du système immédiatement après la mesure est $\frac{P_n \rho P_n}{\mathrm{Tr}(\rho P_n)}$.*

Sixième postulat : *L'évolution temporelle de l'opérateur de densité $\rho(t)$ est régie par l'équation de Liouville-von Neumann*: $i\hbar \frac{d}{dt}\rho(t) = [H(t), \rho(t)]$.

1.2.2 Systèmes de spin quantiques

La fameuse expérience de Stern-Gerlach [CTDL18] a démontré que le moment angulaire du spin est quantifié, c'est-à-dire le moment angulaire du spin $\mathscr{S}_{x,y,z}$ mesuré le long de l'axe-x, y, z n'a que deux valeurs possibles $\frac{\hbar}{2}$ et $-\frac{\hbar}{2}$. Nous associons d'abord un observable S_z à $\mathscr{S}_z$ qui a deux valeurs propres $\frac{\hbar}{2}$ et $-\frac{\hbar}{2}$, et notons les vecteurs propres orthonormés par $|+\rangle$ et $|-\rangle$, c'est-à-dire que $S_z|+\rangle = \frac{\hbar}{2}|+\rangle$ et $S_z|-\rangle = -\frac{\hbar}{2}|-\rangle$. Ensuite, notez l'espace d'état de spin par $\mathcal{H}_2$, qui est bidimensionnel et étalé par $|+\rangle$ et $|-\rangle$. Le vecteur d'état $|\psi\rangle \in \mathcal{H}_2$ peut être représenté par une superposition linéaire de $|+\rangle$ et $|-\rangle$, c'est-à-dire, $|\psi\rangle = \psi_+|+\rangle + \psi_-|-\rangle$ avec $|\psi_+|^2 + |\psi_-|^2 = 1$. Ainsi, dans la base $\{|+\rangle, |-\rangle\}$, l'observable S_z peut être représenté sous forme de matrice $S_z = \frac{\hbar}{2}\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. Les relations de commutation des moments cinétiques le long des trois axes dans le cas de spin-$\frac{1}{2}$ sont données par

$$[S_x, S_y] = i\hbar S_z, \qquad [S_y, S_z] = i\hbar S_x, \qquad [S_z, S_x] = i\hbar S_y.$$

Dans la base $\{|+\rangle, |-\rangle\}$, les variables observables S_x et S_y peuvent être représentées sous forme de matrice sous la forme $S_x = \frac{\hbar}{2}\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ et $S_y = \frac{\hbar}{2}\left(\begin{smallmatrix} 0 & -i \\ i & 0 \end{smallmatrix}\right)$. Ici, on note $\sigma_{x,y,z} = \frac{2}{\hbar} S_{x,y,z}$, les matrices σ_x, σ_y et σ_z sont appelées matrices de Pauli. De toute évidence, toutes les matrices de Pauli sont hermitiennes et ont deux valeurs propres ± 1. Notez que, avec l'identité $\mathbb{1}$, la matrice de Pauli constitue une base pour l'espace vectoriel des matrices appartenant à $\mathbb{C}^{2\times 2}$, donc toute matrice $M = \left(\begin{smallmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{smallmatrix}\right)$ peut être décomposé comme

$$M = \frac{m_{11} + m_{22}}{2}\mathbb{1} + \frac{m_{11} - m_{22}}{2}\sigma_z + \frac{m_{12} + m_{21}}{2}\sigma_x + i\frac{m_{12} + m_{21}}{2}\sigma_y.$$

Par conséquent, toute matrice dans $\mathbb{C}^{2\times 2}$ peut être exprimée sous la forme $M = a_0\mathbb{1} + a_x\sigma_x + a_y\sigma_y + a_z\sigma_z$ avec $a_0, a_x, a_y, a_z \in \mathbb{C}$.

Considérons maintenant spin-$\frac{1}{2}$ particules. En plaçant l'appareil le long d'un angle θ, nous pouvons préparer un état $|\psi\rangle$ tel que $|\psi\rangle = \psi_+|+\rangle + \psi_-|-\rangle$ avec $|\psi_+|^2 + |\psi_-|^2 = 1$ et ces deux coefficients sont déterminés par l'angle θ. Si nous mesurons $\mathscr{S}_z$, nous aurons deux résultats possibles $\frac{\hbar}{2}$ et $-\frac{\hbar}{2}$ avec les probabilités $\mathbb{P}(+) = \langle\psi|(|+\rangle\langle+|)|\psi\rangle = |\psi_+|^2$ et $\mathbb{P}(-) = \langle\psi|(|-\rangle\langle-|)|\psi\rangle = |\psi_-|^2$ respectivement. L'espérance correspondante est $\langle S_z\rangle = \langle\psi|S_z|\psi\rangle = \frac{\hbar}{2}(|\psi_+|^2 - |\psi_-|^2)$. Considérons un atome dans un champ magnétique uniforme : l'hamiltonien indépendant du temps décrivant l'énergie totale du système donné par $H = \omega_0 S_z$, où $\omega > 0$ représente la vitesse angulaire. Ensuite, nous avons $H|+\rangle = E_+|+\rangle$ et $H|-\rangle = E_-|-\rangle$ avec $E_+ = \frac{\hbar\omega_0}{2}$ et $E_- = -\frac{\hbar\omega_0}{2}$, la "fréquence de Bohr" est donnée par $\frac{E_+ - E_-}{\hbar}$. Par l'équation de Schrödinger, nous obtenons $|\psi(t)\rangle = e^{-iH(t-t_0)/\hbar}|\psi(t_0)\rangle$. Utilisons l'opérateur de densité ρ pour décrire l'état d'un système quantique à deux niveaux, où $\rho \in \mathcal{S}_2 := \{\rho \in \mathbb{C}^{2\times 2} | \rho^* = \rho, \mathrm{Tr}(\rho) = 1, \rho \geq 0\}$. En raison des propriétés des matrices de Pauli, nous pouvons écrire ρ sous la forme $\rho = \frac{\mathbb{1} + x\sigma_x + y\sigma_y + z\sigma_z}{2}$ avec $(x, y, z) \in \mathcal{B} := \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 + z^2 \leq 1\}$. Ainsi, $\mathcal{S}_2$ est isomorphe à $\mathcal{B}$, ce qui signifie que tout état d'un système quantique à deux niveaux peut être représenté par un vecteur tridimensionnel (x, y, z) appartenant à la balle d'unité. Un tel vecteur s'appelle le vecteur de Bloch.

Considérons un système quantique qui a deux états dont les énergies sont proches mais très différentes des énergies des autres états du système. Sous cette hypothèse, nous pouvons ignorer tous les autres niveaux d'énergie du système et analyser le système dans un espace à deux dimensions $\mathcal{H}_2$. Prenons les deux vecteurs propres $|e\rangle$ et $|g\rangle$ de l'hamiltonien H, dont les valeurs propres sont respectivement E_e et E_g ($E_g < E_e$), comme base orthonormale de $\mathcal{H}_2$. Ensuite, nous pouvons appliquer la représentation ci-dessus d'un modèle spin-$\frac{1}{2}$ aux systèmes quantiques généraux à deux niveaux.

Nous considérons maintenant les systèmes de spin quantiques de niveau supérieur et dénotons l'espace d'état de spin de dimension N par $\mathcal{H}_N$, avec $2 \leq N < \infty$. Associez le moment cinétique le long de l'axe-x, y, z aux opérateurs hermitiens $J_{x,y,z}$. Les relations de commutation des moments cinétiques le long des trois axes sont données par

$$[J_x, J_y] = i\hbar J_z, \qquad [J_y, J_z] = i\hbar J_x, \qquad [J_z, J_x] = i\hbar J_y.$$

En se référant à [CTDL18, SN14], pour tous les $n \in \{0, \ldots, N-1\}$, les valeurs propres et les vecteurs propres des trois opérateurs de moment angulaire sont donnés par

$$J_x|e_n\rangle = c_n|e_n\rangle + c_{n+1}|e_{n+1}\rangle, \quad J_y|e_n\rangle = -ic_n|e_{n-1}\rangle + ic_{n+1}|e_{n+1}\rangle, \quad J_z|e_n\rangle = (J-n)|e_n\rangle,$$

où $J = \frac{N-1}{2}$, $c_n = \frac{\sqrt{(N-n)n}}{2}$ et $\langle e_n|e_m\rangle = \delta_{m,n}$. L'ensemble des vecteurs d'état orthonormés $\{|e_1\rangle, \ldots, |e_N\rangle\}$ forme une base de l'espace d'état $\mathcal{H}_N$. Ainsi, dans cette base, les opérateurs de moment cinétique peuvent être représentés dans les formes matricielles suivantes

$$J_x = \begin{bmatrix} 0 & c_1 & & & & \\ c_1 & 0 & c_2 & & & \\ & \ddots & \ddots & \ddots & & \\ & & c_{2J-1} & 0 & c_{2J} & \\ & & & c_{2J} & 0 & \end{bmatrix}, \quad J_y = \begin{bmatrix} 0 & -ic_1 & & & & \\ ic_1 & 0 & -ic_2 & & & \\ & \ddots & \ddots & \ddots & & \\ & & ic_{2J-1} & 0 & -ic_{2J} & \\ & & & ic_{2J} & 0 & \end{bmatrix}, \quad J_z = \begin{bmatrix} J & & & & \\ & J-1 & & & \\ & & \ddots & & \\ & & & -J+1 & \\ & & & & -J \end{bmatrix}.$$

Dans ce cas, l'opérateur de densité ρ agissant sur $\mathcal{H}_N$ appartient à l'espace $\mathcal{S}_N := \{\rho \in \mathbb{C}^{N\times N} | \rho^* = \rho, \mathrm{Tr}(\rho) = 1, \rho \geq 0\}$, le vecteur de Bloch correspondant au niveau de N est

donné dans [KK05]. Semblable au cas à deux niveaux, les systèmes quantiques généraux à N, avec $2 \leq N < \infty$, dont l'Hamiltonien a N valeurs propres non dégénérées, peuvent être analysés par le modèle à moment angulaire à N niveaux.

1.2.3 Oscillateur harmonique quantique

L'oscillateur harmonique unidimensionnel est un système important en mécanique classique et quantique. De nombreux systèmes peuvent être approximés par le modèle d'oscillateur harmonique. Considérons un oscillateur harmonique conservateur unidimensionnel, selon la deuxième loi de Newton et la loi de Hooke, nous avons $\frac{d^2x}{dx^2} = -\frac{k}{m}x$ où $m > 0$ représente la masse de la particule et $k > 0$ décrit la rigidité du ressort et notons $\omega^2 = \frac{k}{m}$. L'énergie totale de ce système est alors la somme de l'énergie potentielle et de l'énergie cinétique, $E = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2}$.

En mécanique quantique, nous définissons deux observables Q et P, qui sont des opérateurs auto-adjoints sur l'espace de Hilbert $\mathcal{H}$ associés à la position physique des quantités x et à la quantité de mouvement p, la relation de commutation canonique de Heisenberg $[Q, P] = i\hbar\mathbb{1}$. Conformément au cas classique, nous définissons l'opérateur hamiltonien du système comme suit $H = \frac{P^2}{2m} + \frac{m\omega^2 Q^2}{2}$, il n'est pas évident de résoudre l'équation de valeur propre $H|\psi\rangle = E|\psi\rangle$. Ensuite, nous définissons deux opérateurs non hermitiens $a = \sqrt{\frac{m\omega}{2\hbar}}\left(Q + i\frac{P}{m\omega}\right)$ et son conjugué $a^* = \sqrt{\frac{m\omega}{2\hbar}}\left(Q - i\frac{P}{m\omega}\right)$, connus sous le nom d'opérateur d'annihilation et d'opérateur de création, et ils peuvent nous aider à étudier les valeurs propres et les vecteurs propres de H. Notons $N := a^*a$, par un calcul simple, nous avons

$$N = \frac{m\omega}{2\hbar}\left(Q^2 + \frac{P^2}{m^2\omega^2}\right) + \frac{i}{2\hbar}[Q, P] = \frac{H}{\hbar\omega} - \frac{\mathbb{1}}{2},$$

ce qui implique que $H = \hbar\omega\left(N + \frac{\mathbb{1}}{2}\right)$, où N est appelé opérateur numérique. Ensuite, un calcul direct montre $[N, a] = -a$ et $[N, a^*] = a^*$. Notons une valeur propre et un vecteur propre associé de N par n et $|n\rangle$, c'est-à-dire $N|n\rangle = n|n\rangle$, où $|n\rangle$ sont appelés états de Fock et forment une base orthogonale de l'espace de Hilbert $\mathcal{H}$, $\langle n|m\rangle = \delta_{n,m}$ et $\sum_n |n\rangle\langle n| = \mathbb{1}$. Par conséquent, nous pouvons facilement obtenir $a|0\rangle = 0$. Puisque $n = \langle n|N|n\rangle = \|a|n\rangle\|^2 \geq 0$, alors n n'est pas négatif. En conséquence, nous avons

$$\begin{aligned} Na|n\rangle &= ([N, a] + aN)|n\rangle = (n-1)a|n\rangle, \\ Na^*|n\rangle &= ([N, a^*] + a^*N)|n\rangle = (n+1)a^*|n\rangle, \end{aligned} \tag{1.3}$$

ce qui implique que $a|n\rangle$ et $a^*|n\rangle$ sont également des vecteurs propres de N avec des valeurs propres diminuées et augmentées de un respectivement. De la relation $H|n\rangle = \hbar\omega\left(n + \frac{1}{2}\right)|n\rangle$, on remarque que la diminution ou l'augmentation d'une unité à n représente l'annihilation ou la création d'une unité quantique d'énergie $\hbar\omega$. De plus, comme conséquence de (1.3), $a|n\rangle$ est parallèle à $|n-1\rangle$ et $a^*|n\rangle$ est parallèle to $|n+1\rangle$, ce qui signifie $a|n\rangle = \alpha_1|n-1\rangle$, $a^*|n\rangle = \alpha_2|n+1\rangle$. En raison de la relation de commutation canonique, nous avons $[a, a^*] = \mathbb{1}$, donc $aa^* = N + \mathbb{1}$. Puisque $n = \langle n|a^*a|n\rangle = |\alpha_1|^2$ et $n+1 = \langle n|(N+\mathbb{1})|n\rangle = \langle n|aa^*|n\rangle = |\alpha_2|^2$, et en supposant que α_1 et α_2 sont réels et non négatifs, nous avons $\alpha_1 = \sqrt{n}$ et $\alpha_2 = \sqrt{n+1}$. Enfin, nous obtenons

$$a|n\rangle = \sqrt{n}|n-1\rangle, \quad a^*|n\rangle = \sqrt{n+1}|n+1\rangle.$$

Par conséquent, dans la base $\{|n\rangle\}$, le hamiltonien H, l'opérateur de création a^* et l'opérateur d'annihilation a peuvent être représentés par les matrices suivantes

$$H = \frac{\hbar\omega}{2}\begin{bmatrix} 1 & 0 & 0 & \cdot & \cdot \\ 0 & 3 & 0 & \cdot & \cdot \\ 0 & 0 & 5 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}, \quad a^* = \begin{bmatrix} 0 & 0 & 0 & \cdot & \cdot \\ 1 & 0 & 0 & \cdot & \cdot \\ 0 & \sqrt{2} & 0 & \cdot & \cdot \\ 0 & 0 & \sqrt{3} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}, \quad a = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdot & \cdot \\ 0 & 0 & \sqrt{2} & 0 & \cdot & \cdot \\ 0 & 0 & 0 & \sqrt{3} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}.$$

De plus, l'état général d'un oscillateur harmonique peut être exprimé par une superposition des états de Fock $|n\rangle$. Ces états sont appelés états cohérents et sont définis par

$$a|\alpha\rangle = \alpha|\alpha\rangle,$$

où a est l'opérateur d'annihilation et une solution triviale de l'équation ci-dessus est l'état de vide $|0\rangle$ pour $\alpha = 0$.

Maintenant, nous analysons l'oscillateur harmonique dans l'image de Heisenberg et discutons de l'évolution temporelle des opérateurs. Par souci de simplicité, nous n'ajoutons pas l'indice H pour mettre en valeur l'image de Heisenberg. L'équation de Heisenberg de a est donnée par

$$\frac{da}{dt} = \frac{1}{i\hbar}[a, H] = -i\omega[a, N] = -i\omega a,$$

ce qui implique que $a(t) = a(t_0)e^{-i\omega(t-t_0)}$.

1.3 Systèmes quantiques ouverts

Dans la section précédente, nous avons brièvement discuté du formalisme des systèmes quantiques fermés. Cependant, tout système quantique interagit inévitablement avec un système quantique externe, comme un grand environnement ou un bain de chaleur quantique, etc. Ce type de système est appelé système quantique ouvert.

Systèmes quantiques bipartites Afin de développer un cadre théorique pour traiter ces interactions, considérons un cas simple, où le système quantique d'intérêt consiste en deux sous-systèmes quantiques S_1 et S_2. Ceci est appelé système quantique bipartite et est noté $S_1 \otimes S_2$. En raison du premier postulat de la mécanique quantique, on suppose que le sous-système quantique S_1 est défini par l'état $\psi_1 \in \mathcal{H}_1$ et que le sous-système quantique S_2 est défini par l'état $\psi_2 \in \mathcal{H}_2$. Le système couplé $S_1 \otimes S_2$ est défini par l'état $\psi_1 \otimes \psi_2$ appartenant au produit tenseur $\mathcal{H}_1 \otimes \mathcal{H}_2$. Si X_1 et X_2 sont observables des sous-systèmes quantiques S_1 et S_2, respectivement, ils peuvent être étendus comme observables du système quantique $S_1 \otimes S_2$ agissant sur $\mathcal{H}_1 \otimes \mathcal{H}_2$ en considérant les opérateurs $X_1 \otimes \mathbb{1}$ et $\mathbb{1} \otimes X_2$, respectivement. De même, si ρ_1 et ρ_2 sont des opérateurs de densité de S_1 et S_2, respectivement, alors $\rho_1 \otimes \rho_2$ est un opérateur de densité du système $S_1 \otimes S_2$. Par conséquent, nous pouvons étendre les six postulats discutés dans la section précédente au cas des systèmes quantiques bipartites. Ensuite, afin de déterminer les états marginaux des systèmes quantiques bipartites, nous devons introduire la trace partielle (voir par

exemple, [Att, Chapter 2]), ce qui nous aide à moyenner le système complémentaire. Étant donné un opérateur de densité $\boldsymbol{\rho}$ sur $\mathcal{H}_1 \otimes \mathcal{H}_2$, l'état marginal ρ_1 sur $\mathcal{H}_1$ peut être calculé par

$$\rho_1 = \mathrm{Tr}_{\mathcal{H}_2}(\boldsymbol{\rho}),$$

où la trace partielle est définie par

$$\mathrm{Tr}\big(\mathrm{Tr}_{\mathcal{H}_2}(\boldsymbol{\rho})X_1\big) = \mathrm{Tr}\big(\boldsymbol{\rho}(X_1 \otimes \mathbb{1})\big), \tag{1.4}$$

pour tous les observables X_1 sur $\mathcal{H}_1$.

Systèmes quantiques ouverts Les systèmes quantiques ouverts peuvent être considérés comme la famille de tous les systèmes quantiques bipartites. Définissons le système d'intérêt quantique dynamique S sur un espace de Hilbert $\mathcal{H}_S$, et décrivons l'environnement par un système quantique W sur un espace de Hilbert $\mathcal{H}_W$. Le système-environnement $S \otimes W$ peut être considéré comme un système quantique plus vaste défini sur l'espace de Hilbert $\mathcal{H}_S \otimes \mathcal{H}_W$. Notons $\rho_S(t)$ et $\rho_W(t)$ en tant qu'opérateurs de densité du système et de l'environnement, dont les états initiaux sont donnés par $\rho_S(t_0)$ et $\rho_W(t_0)$. Ensuite, l'évolution temporelle de l'opérateur de densité $\boldsymbol{\rho}(t)$ de l'environnement système est donnée par

$$\boldsymbol{\rho}(t) = U(t,t_0)\big(\rho_S(t_0) \otimes \rho_W(t_0)\big)U^*(t,t_0) \Rightarrow \rho_S(t) = \mathrm{Tr}_{\mathcal{H}_W}\big(\boldsymbol{\rho}(t)\big),$$

où $U(t,t_0)$ est l'opérateur unitaire agissant sur $\mathcal{H}_S \otimes \mathcal{H}_W$. Nous mesurons les $X_1 \otimes \mathbb{1}$ observables. Indiquez l'ensemble des valeurs propres de X_1 par $\{x_k\}$ et l'ensemble des projecteurs associés par $\{P_k\}$. Après la mesure, l'état devient immédiatement

$$\boldsymbol{\rho}'(t) = \frac{P_n U(t,t_0)\big(\rho_S(t_0) \otimes \rho_W(t_0)\big)U^*(t,t_0)P_n}{\mathrm{Tr}\big(P_n U(t,t_0)\big(\rho_S(t_0) \otimes \rho_W(t_0)\big)U^*(t,t_0)P_n\big)},$$

où le dénominateur donne la probabilité d'obtenir x_n. Par conséquent, l'état marginal de post-mesure $\rho'_S(t)$ est donné par

$$\rho'_S(t) = \mathrm{Tr}_{\mathcal{H}_W}\big(\boldsymbol{\rho}'(t)\big) = \frac{\mathrm{Tr}_{\mathcal{H}_W}\big(P_n U(t,t_0)\big(\rho_S(t_0) \otimes \rho_W(t_0)\big)U^*(t,t_0)P_n\big)}{\mathrm{Tr}\big(P_n U(t,t_0)\big(\rho_S(t_0) \otimes \rho_W(t_0)\big)U^*(t,t_0)P_n\big)}, \tag{1.5}$$

qui décrit l'évolution temporelle du système quantique ouvert en cours de mesure.

Équation de filtrage quantique Considérons maintenant un système quantique ouvert défini sur $\mathcal{H}_S$ en interaction avec un champ électromagnétique à l'état de vide défini sur $\mathcal{H}_W$ en cours de mesure en temps continu. Ne considérer qu'un seul canal de mesure de détection homodyne à la fois. Heuristiquement, le champ électromagnétique peut être considéré comme une collection d'oscillateurs harmoniques quantiques, décrits par les opérateurs de champ A_t (processus d'annihilation) et $A_t^\dagger$ (processus de création), définis sur $\mathcal{H}_W$ et qui ne commutent pas entre eux. Ensuite, la dynamique conjointe de l'opérateur unitaire U_t de l'ensemble du système, c'est-à-dire du système quantique ouvert et du champ électromagnétique, défini sur l'espace de Hilbert $\mathcal{H}_S \otimes \mathcal{H}_W$, est donnée par l'équation différentielle stochastique quantique suivante (EDSQ)

$$dU_t = \Big(L \otimes dA_t^\dagger - L^* \otimes dA_t - \Big(\big(L^*L/2 + iH\big) \otimes \mathbb{1}\Big)dt\Big)\,U_t, \qquad U_0 = \mathbb{1}.$$

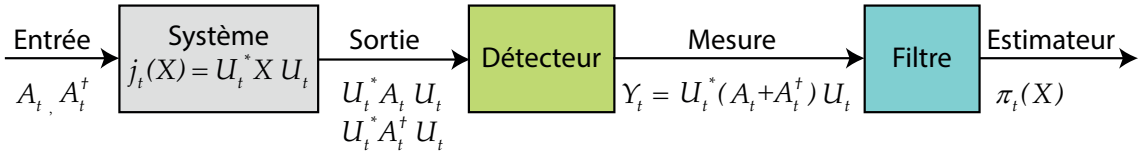


FIGURE 1.2 – Procédure de filtrage quantique. En fonction du résultat du détecteur Y_t, le filtre fournit un estimateur de l'observable X à l'instant t. Notez que nous identifions X et $X \otimes \mathbb{1}$, A_t et $\mathbb{1} \otimes A_t$, $A_t^\dagger$ et $\mathbb{1} \otimes A_t^\dagger$ dans le diagramme ci-dessus.

Alors l'évolution temporelle de l'observable X sur $\mathcal{H}_S$ est $j_t(X) = U_t^*(X \otimes \mathbb{1})U_t$, et l'observation de la détection homodyne à temps t est $Y_t = U_t^*(\mathbb{1} \otimes (A_t + A_t^\dagger))U_t$, voir Fig 1.2.

Par le calcul quantique Itô [HP84], nous avons

$$\begin{aligned} dj_t(X) &= \mathcal{L}\big(j_t(X)\big)dt + dA_t^\dagger[j_t(X), j_t(L)] + [j_t(L^*), j_t(X)]dA_t, \\ dY_t &= (j_t(L) + j_t(L^*))dt + dA_t + dA_t^\dagger, \end{aligned}$$

où $\mathcal{L}(X) := i[H, X] + L^*XL - L^*LX/2 - XL^*L/2$ est appelé générateur de Lindblad. Comme la théorie classique du filtrage stochastique [Kal13, Xio08], selon la formule de Kallianpur-Stribel et la formule Itô, la dynamique d'attente de $j_t(X)$ conditionnée par les mesures enregistrées Y_t jusqu'à temps t, noté $\pi_t(X)$, voir Fig 1.3, peut être décrit par une équation de filtrage normalisée. Cette équation est guidée par le processus d'innovation correspondant au processus d'observation.

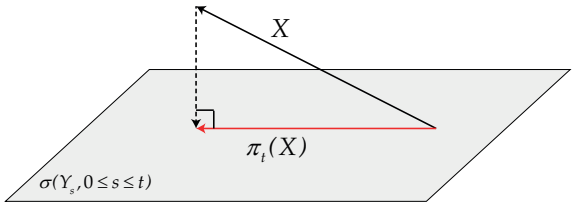


FIGURE 1.3 – L'espérance conditionnelle $\pi_t(X)$ peut être considérée comme la projection sur l'espace couvert par l'enregistrement de mesures $\sigma(Y_s, 0 \le s \le t)$.

Dans la représentation de Schrödinger, nous avons $\pi_t(X) = \mathrm{Tr}(\rho_t X)$, où ρ_t est un opérateur de densité conditionné par les observations jusqu'au temps t. Ainsi, on peut obtenir une équation différentielle stochastique à valeur matricielle pour l'évolution de l'opérateur de densité du système sous des mesures parfaites en temps continu (homodyne), appelée équation principale stochastique. Il s'agit de l'analogue quantique de l'équation de Kushner-Stratonovich ou FKK,

$$\begin{aligned} d\rho_t &= \mathcal{L}^*(\rho_t)dt + \big(L\rho_t + \rho_t L^* - \mathrm{Tr}\big((L + L^*)\rho_t\big)\rho_t\big)dW_t, \\ dY_t &= dW_t + \mathrm{Tr}\big((L + L^*)\rho_t\big)dt, \end{aligned}$$

où W_t est un processus de Wiener unidimensionnel. Des arguments similaires permettent de décrire la dynamique du système sous des mesures imparfaites.

$$d\rho_t = \mathcal{L}^*(\rho_t)dt + \sqrt{\eta}\big(L\rho_t + \rho_t L^* - \mathrm{Tr}\big((L + L^*)\rho_t\big)\rho_t\big)dW_t. \tag{1.6}$$

L'efficacité de la mesure est donnée par $\eta \in [0, 1)$. Notez que, si H et L sont invariants dans le temps, l'espérance classique de l'équation maîtresse stochastique (1.6) s'appelle

l'équation maîtresse de Lindblad

$$\frac{d}{dt}\mathbb{E}(\rho_t) = \mathcal{L}^*\big(\mathbb{E}(\rho_t)\big).$$

De plus, $\mathbb{E}(\rho_t)$ peut également être dérivé en calculant la trace partielle,

$$\mathbb{E}(\rho_t) = \mathrm{Tr}_{\mathcal{H}_W}\big(U_t(\rho_0 \otimes |0\rangle\langle 0|)U_t^*\big).$$

Nous observons donc que l'équation maîtresse de Lindblad joue le rôle d'équation de Fokker-Planck associée à l'équation maîtresse stochastique (1.6).

1.4 Plan de la thèse

Dans cette thèse, nous étudions la stabilisation exponentielle à rétroaction de systèmes quantiques ouverts subissant des mesures imparfaites en temps continu, vers un état cible prédéterminé qui est un état pur correspondant à un vecteur propre des opérateurs de mesure.

Le chapitre 3 fournit une introduction à la théorie du filtrage quantique. Nous introduisons d'abord la théorie des probabilités quantiques comme une extension de la théorie des probabilités classique, qui permet de décrire les phénomènes quantiques. Ensuite, nous présentons les systèmes de spin quantiques et l'oscillateur harmonique quantique dans le cadre de la probabilité quantique. Ensuite, nous discutons des processus stochastiques quantiques sur l'espace de Fock, qui sont appliqués pour caractériser les champs électromagnétiques quantiques libres. Après cela, nous décrivons le calcul quantique de Hudson-Parthasarathy et obtenons des équations différentielles quantiques stochastiques. Enfin, par la formule non commutative de Kallianpur-Striebel, nous obtenons une équation différentielle stochastique matricielle appelée équation de filtrage quantique ou équation de maître stochastique et décrivant l'évolution dans le temps de l'opérateur de densité de systèmes quantiques ouverts en interaction avec un champ électromagnétique subissant des mesures imparfaites en temps continu.

Le chapitre 4 est consacré à la stabilisation exponentielle à rétroaction des systèmes de spin quantique de niveau N soumis à des mesures en temps continu avec des états initiaux connus. Ce chapitre est basé sur nos publications [LAM18, LAM19a]. Nous étudions d'abord le cas où l'entrée de contrôle est désactivée et montrons la réduction exponentielle de l'état quantique en moyenne et presque sûrement. En utilisant des outils de contrôle stochastiques et géométriques, nous fournissons des conditions suffisantes sur la loi de contrôle par rétroaction assurant une convergence exponentielle presque sûre vers un état pur prédéterminé correspondant à un vecteur propre de l'opérateur de mesure J_z. Afin d'atteindre ces résultats, nous établissons des caractéristiques générales des trajectoires quantiques qui présentent un intérêt en elles-mêmes. Nous illustrons les résultats en concevant une classe de lois de commande à rétroaction satisfaisant les conditions susmentionnées et, enfin, nous démontrons l'efficacité de notre méthodologie par le biais de simulations numériques pour les systèmes de spin quantique à trois niveaux.

Le chapitre 5 est consacré à la stabilisation en retour des systèmes multi-qubits soumis à des mesures en temps continu avec des états initiaux connus. Ce chapitre est basé sur les publications [LAM19b] et [LAMa]. Nous étudions d'abord les systèmes multi-bits

avec deux canaux quantiques lorsque l'entrée de contrôle est désactivée, et montrons la réduction exponentielle de l'état quantique en moyenne et presque sûrement. Nous considérons ensuite un système à deux qubits soumis à des mesures en temps continu. En présence d'un canal, nous établissons une convergence asymptotique vers un état de Bell prédéterminé. Avec deux canaux, nous fournissons des conditions suffisantes à la loi de commande de rétroaction continue assurant une convergence exponentielle presque sûre vers un état de Bell prédéterminé. Ceci est obtenu en appliquant des outils stochastiques, des méthodes de Lyapunov et des outils de contrôle géométrique. Dans les deux cas, nous fournissons des expressions explicites de lois de commande de rétroaction satisfaisant les conditions susmentionnées. Enfin, nous démontrons l'efficacité de notre méthodologie par des simulations numériques.

Le chapitre 6 est consacré à la stabilisation exponentielle à rétroaction des systèmes de spin quantiques ouverts avec des états initiaux inconnus. Ce chapitre est basé sur [LAMb]. Nous considérons d'abord les systèmes de spin-$\frac{1}{2}$ quantique ouverts avec des états initiaux inconnus et fournissons des conditions suffisantes sur le contrôleur de rétroaction du filtre quantique associé, qui assurent la convergence de la fidélité de l'état réel et de l'état estimé vers un, quand t tend vers l'infini. Ensuite, nous montrons l'efficacité de notre méthodologie par des simulations numériques. Par la suite, de manière heuristique, nous discutons de la stabilisation exponentielle à rétroaction des systèmes de spin quantiques ouverts de niveau N avec des états initiaux inconnus. Nous terminons ce chapitre en proposant des simulations numériques pour les systèmes de spin quantiques à trois niveaux.

Le chapitre 7 propose des extensions naturelles aux résultats décrits dans les chapitres mentionnés ci-dessus.

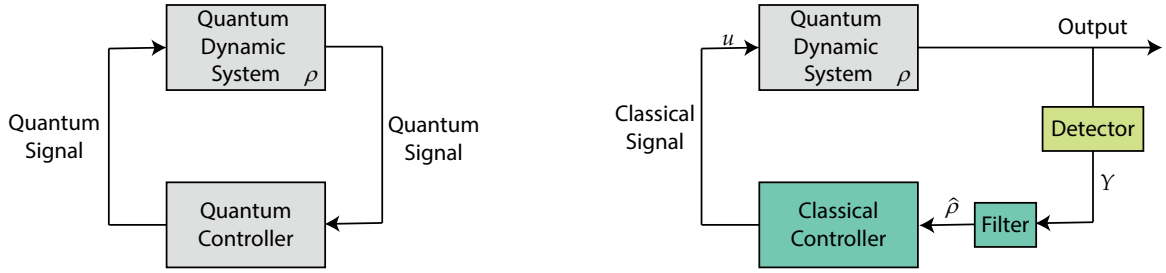
Annexe A fournit des notions de base et des théorèmes issus du calcul stochastique et de la théorie du contrôle stochastique.

2 Introduction

2.1 Motivation and context

Quantum control theory is a rapidly developing research domain, however there are still many efforts that should take place to make this field more practical in engineering of quantum devices. The ability to control quantum systems, e.g., preparing and protecting a desired quantum state, plays an essential role to develop further quantum technologies. Quantum technologies [NC02] are supposed to outperform the conventional technologies. For example, quantum computing can be much faster than conventional computing when solving certain problems, and quantum metrology will be much more precise in parameter estimation than conventional techniques.

The dynamics of simple closed quantum systems can be described by (time-dependent) Schrödinger equations. For such systems, we can define some simple control tasks which can be achieved by using open-loop control techniques. This means that a predetermined classical control signal is applied to a closed quantum system and no feedback is involved, a comprehensive treatment can be found in [d'A07]. Open-loop control strategy has been applied in different contexts like optimal control approach [TR04, KRK+05, BCS09], controllability issues [BBR10, BCR10], etc. Because of robustness issues, an open-loop control approach may fail to apply for complex quantum control tasks, such as suppressing decoherence (i.e., the loss of information due to unavoidable coupling to the environment) and quantum noise. Thus, it is important to introduce quantum analogue of classical feedback control theory, in order to compensate some disturbances, i.e., uncertainties and drifts in the parameters describing the dynamical system, external perturbations and measurement noises, in order to increase the robustness of the system. Roughly speaking, there exist two common types of quantum feedback control depending on the nature of the feedback controller: coherent feedback and measurement-based feedback. For coherent feedback, the controller is another quantum system communicating with the quantum dynamical system to be controlled by a quantum signal, i.e., a beam of light [Llo00], see



(a) Coherent feedback control (b) Measurement-based feedback control

FIGURE 2.1 – (a) Both dynamical system and controller are quantum systems interacting with each other. The quantum controller obtains quantum signal from a dynamical system and processes such signal by using quantum logic, and then feeds the signal coherently back into the quantum dynamical system. (b) The dynamical system is a quantum system, while the feedback controller is a classical system, which can only process classical signal. The filter is used to form an estimator of the actual state of the dynamical system based on the outcomes of the detector. Then the feedback controller provides the control input, i.e., a function of the estimator, which is applied to control the dynamical system.

Fig 2.1. This kind of feedback has been recently applied for different tasks such as squeezing enhancement [GW09, WM94], quantum memories and error correction [KNPM10]. For measurement-based feedback, the controller is a classical system that processes classical information obtained by measuring an observable of the quantum dynamical system to be controlled, and implement the suitable control input, see Fig 2.1. This type of feedback has been considered for squeezed states preparation [TMW02], quantum state reduction [vHSM05a, MDR09], etc.

In this thesis, we focus on measurement-based feedback stabilization of open quantum systems, i.e., systems which are in interaction with an environment. The evolution of an open quantum system undergoing indirect continuous-time measurements is described by the so-called quantum stochastic master equation, which has been derived by Belavkin in quantum filtering theory [Bel89]. The quantum filtering theory, relying on quantum stochastic calculus and quantum probability theory (developed by Hudson and Parthasarathy [HP84]) plays an important role in quantum optics and computation.

Quantum measurement-based feedback control, can be considered as a branch of stochastic control which has been first developed by Belavkin in [Bel83]. This field has attracted the interest of many theoretical and experimental researchers mainly starting from the early 2000s, yielding fundamental results [vHSM05a, AAS+02, MvH07, Tsu08, ADL02, YTH07, MK05]. In particular, theoretical studies carried out in [MvH07, DMB+09, MDR09, ARM11, ASD+13] lead to the first experimental implementation of real-time quantum measurement-based feedback control in [SDZ+11].

In [BvH08], the authors established a quantum separation principle. Similar to the classical separation principle. This result allows to interpret the control problem as a state-based feedback control problem for the filter (the best estimate, i.e., the conditional state), without caring of the actual quantum state. This motivates the state-based feedback design for the quantum filtering equation based on the knowledge of the initial state. In this context, stabilization of quantum filters towards pure states (i.e., the preparation of pure states) has a major impact in developing new quantum technologies.

In the following, we first introduce very briefly some important postulates from quantum mechanics, quantum spin systems and the quantum harmonic oscillator. Next, we present shortly open quantum systems and quantum filtering. Finally, we give the main contributions of this thesis with the outline.

2.2 Quantum mechanics

2.2.1 Postulates of quantum mechanics

In classical mechanics, the motion of any physical system is determined if the position and velocity of each of its points are known as functions of time. All the physical quantities associated with the system, e.g., position, energy, angular momentum, etc. can be measured precisely, and such measurements do not affect the system in general. The quantum mechanics differs a lot from the classical mechanics. In the following, we give the main postulates of quantum mechanics [CTDL18, SN14].

First postulate: *At any fixed time t_0, the state of a physical system is defined by specifying a state vector $\psi(t_0)$ belonging to a separable complex Hilbert space $\mathcal{H}$.* In Dirac's notation, we denote such state vector by $|\psi(t_0)\rangle$ and call it ket. The corresponding state vector belonging to the dual space of $\mathcal{H}$ is denoted $\langle\psi(t_0)|$ and called bra. Then the inner product of two kets $|\psi\rangle$ and $|\phi\rangle$ belonging to $\mathcal{H}$ is defined by $\langle\psi|\phi\rangle = \langle\phi|\psi\rangle^*$. This first postulate implies a superposition principle: a linear combination of state vectors is also a state vector.

Second postulate: *Every measurable physical quantity $\mathscr{X}$ is described by an Hermitian operator X acting in $\mathcal{H}$, this operator is an observable.*

Third postulate: *The only possible result of the measurement of a physical quantity $\mathscr{X}$ is one of the eigenvalues of the corresponding observable X.* A measurement of $\mathscr{X}$ always gives a real value, since X is Hermitian. If the spectrum of X is discrete, the results that can be obtained by measuring $\mathscr{X}$ are quantized.

Fourth postulate (finite dimensional case): *When the physical quantity $\mathscr{X}$ is measured on a system in the normalized state $|\psi\rangle$, the probability $\mathbb{P}(x_n)$ of obtaining the eigenvalue x_n of the corresponding observable X is: $\mathbb{P}(x_n) = \sum_{i=1}^{g_n} |\langle u_n^i|\psi\rangle|^2$, where g_n is the degree of degeneracy of x_n and $|u_n^i\rangle$ with $i = 1, 2, \ldots, g_n$ is an orthonormal set of vectors which forms a basis in the eigensubspace $\mathcal{H}_n$ associated with the eigenvalue x_n.* For the eigenvalue x_n, we have $X|u_n^i\rangle = x_n|u_n^i\rangle$ with $i = 1, \ldots, g_n$, then we expand the state vector $|\psi\rangle$ in the orthonormal basis $\{|u_n^i\rangle\}$ and obtain $|\psi\rangle = \sum_n \sum_{i=1}^{g_n} c_n^i |u_n^i\rangle$ where $c_n^i = \langle u_n^i|\psi\rangle$. Then the probability $\mathbb{P}(x_n)$ of obtaining the non-degenerate eigenvalue x_n of the corresponding observable X is: $\mathbb{P}(x_n) = \sum_{i=1}^{g_n} |\langle u_n^i|\psi\rangle|^2 = \sum_{i=1}^{g_n} |c_n^i|^2$. Next we define the projector onto $\mathcal{H}_n$: $P_n = \sum_{i=1}^{g_n} |u_n^i\rangle\langle u_n^i|$, where $P_n^* = P_n$ and $P_n^2 = P_n$, the probability mentioned above can be written: $\mathbb{P}(x_n) = \langle\psi|P_n|\psi\rangle$. Since we can write the observable as $X = \sum_n x_n P_n$, the corresponding expectation of X is defined by $\langle X\rangle := \sum_n x_n \mathbb{P}(x_n) = \langle\psi|X|\psi\rangle$. An important consequence of this postulate is the following: the probabilities predicted for an arbitrary measurement are the same for two proportional state vectors, thus they represent the same physical state. Thus the states of a quantum system are rays in a Hilbert space $\mathcal{H}$.

Fifth postulate: *If a measurement of the physical quantity $\mathscr{X}$ on a system in the state $|\psi\rangle$ gives the result x_n, the state of the system immediately after measurement is the normalized projection $\frac{P_n|\psi\rangle}{\sqrt{\langle\psi|P_n|\psi\rangle}}$ of $|\psi\rangle$ onto the eigensubspace associated with x_n.* This postulate is so-called collapse of the state or reduction of the wave packet.

Sixth postulate: *The time evolution of the state vector $|\psi(t)\rangle$ is governed by the Schrödinger equation*:

$$i\hbar\frac{d}{dt}|\psi(t)\rangle = H(t)|\psi(t)\rangle, \tag{2.1}$$

where $H(t)$ is the observable associated with the total energy of the system, which is called Hamiltonian operator. The solution of the above Schrödinger equation is given by $|\psi(t)\rangle = U(t,t_0)|\psi(t_0)\rangle$, where $U(t,t_0) = exp\left(-\frac{i}{\hbar}\int_{t_0}^{t} H(\tau)d\tau\right)$ and it is an unitary operator acting on $\mathcal{H}$. In particular, if H does not depend on time (case of conservative systems), we have $U(t,t_0) = e^{-iH(t-t_0)/\hbar}$. It is also instructive to study the dynamics of the expectation of an observable X in a given state $|\psi(t)\rangle$,

$$\langle X\rangle(t) = \langle\psi(t)|X(t_0)|\psi(t)\rangle = \langle\psi(t_0)|U^*(t,t_0)X(t_0)U(t,t_0)|\psi(t_0)\rangle, \tag{2.2}$$

where we denote the observable X at t_0 as $X(t_0)$. Since the time-dependent nature of the system must be carried by some combination of the state vectors and the operators, there are two approaches to deal with systems evolving in time in general:

1. Schrödinger picture: $|\psi(t_0)\rangle \to |\psi(t)\rangle = U(t,t_0)|\psi(t_0)\rangle$, where operators are unchanged.
2. Heisenberg picture: $X_H(t_0) = X(t_0) \to X_H(t) = U^*(t,t_0)X_H(t_0)U(t,t_0)$, where state vectors are unchanged. Here, the subscript H is used to emphasize the Heisenberg picture.

From (2.2), we can see intuitively that these two approaches are equivalent. Due to (2.2) and Schrödinger equation, the dynamics of the system in Heisenberg picture is given by $i\hbar\frac{d}{dt}X_H(t) = [X_H(t), H(t)]$. This equation is called Heisenberg equation.

Until now, we have considered systems whose states are perfectly well known and we can study their time evolutions and predict the results of various measurements performed on them. However, in practice, the state of the system is often not perfectly determined. Then, we introduce the density operator to represent the state of a quantum system, which encodes a situation with incomplete information in a more general way than a state vector. The incomplete information about the system in quantum mechanics means the state of the system is a statistical mixture of states $|\psi_1\rangle, |\psi_1\rangle, \dots$ with probabilities $p_1, p_2, \dots$ The states $|\psi_1\rangle, |\psi_2\rangle, \dots$ are not necessarily orthogonal. Then we use density operator $\rho(t)$ to describe the statistical mixture of states. The system is said to be in a pure state if the state of the system is perfectly known, i.e., $\rho(t) = |\psi(t)\rangle\langle\psi(t)|$ where $|\psi(t)\rangle$ is the state vector of the system at time t. Note that, for the pure states, we have $\rho^2 = \rho$ and $\mathrm{Tr}(\rho^2) = 1$. For the mixed states, we have $\rho = \sum_k p_k\rho_k$ with $\rho_k = |\psi_k\rangle\langle\psi_k|$ the density operator of the pure state corresponding to $|\psi\rangle$. Thus, we conclude that the density operator possesses three main properties: $\rho^* = \rho$, $\mathrm{Tr}(\rho) = 1$ and $\rho \geq 0$.

Next, we generalize the above fourth, fifth and sixth postulates to the case with respect to density operators.

Fourth postulate (finite dimensional case): *When the physical quantity $\mathscr{X}$ is measured on a system in ρ, the probability $\mathbb{P}(x_n)$ of obtaining the eigenvalue x_n of the*

corresponding observable X *is*: $\mathbb{P}(x_n) = \mathrm{Tr}(\rho P_n)$. As a consequence, the expectation of X is given by $\langle X\rangle = \mathrm{Tr}(\rho X)$.

Fifth postulate: *If a measurement of the physical quantity* $\mathscr{X}$ *on a system in* ρ *gives the result* x_n*, the state of the system immediately after measurement is* $\frac{P_n\rho P_n}{\mathrm{Tr}(\rho P_n)}$.

Sixth postulate: *The time evolution of the density operator* $\rho(t)$ *is governed by the Liouville-von Neumann equation*: $i\hbar\frac{d}{dt}\rho(t) = [H(t), \rho(t)]$.

2.2.2 Quantum spin systems

The famous Stern-Gerlach experiment [CTDL18] demonstrated that the spin angular momentum is quantized, that is the spin angular momentum $\mathscr{S}_{x,y,z}$ measured along the axis-x, y, z has only two possible values $\frac{\hbar}{2}$ and $-\frac{\hbar}{2}$. We firstly associate an observable S_z with $\mathscr{S}_z$ which has two eigenvalues $\frac{\hbar}{2}$ and $-\frac{\hbar}{2}$, and denote the corresponding orthonormal eigenvectors by $|+\rangle$ and $|-\rangle$, i.e., $S_z|+\rangle = \frac{\hbar}{2}|+\rangle$ and $S_z|-\rangle = -\frac{\hbar}{2}|-\rangle$. Then denote the spin state space by $\mathcal{H}_2$ which is 2-dimensional and spanned by $|+\rangle$ and $|-\rangle$. The state vector $|\psi\rangle \in \mathcal{H}_2$ can be represented by a linear superposition of $|+\rangle$ and $|-\rangle$, i.e., $|\psi\rangle = \psi_+|+\rangle + \psi_-|-\rangle$ with $|\psi_+|^2 + |\psi_-|^2 = 1$. Thus, in the $\{|+\rangle, |-\rangle\}$ basis, the observable S_z can be represented in matrix form $S_z = \frac{\hbar}{2}\left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)$. The commutation relations of the angular momenta along the three axis in the case of spin-$\frac{1}{2}$ are given by

$$[S_x, S_y] = i\hbar S_z, \qquad [S_y, S_z] = i\hbar S_x, \qquad [S_z, S_x] = i\hbar S_y.$$

In the $\{|+\rangle, |-\rangle\}$ basis, the observables S_x and S_y can be represented in matrix forms as $S_x = \frac{\hbar}{2}\left(\begin{smallmatrix}0 & 1\\ 1 & 0\end{smallmatrix}\right)$ and $S_y = \frac{\hbar}{2}\left(\begin{smallmatrix}0 & -i\\ i & 0\end{smallmatrix}\right)$. Here, we denote $\sigma_{x,y,z} = \frac{2}{\hbar}S_{x,y,z}$, the matrices σ_x, σ_y and σ_z are called Pauli matrices. Obviously, all Pauli matrices are Hermitian and have two eigenvalues ± 1. Note that, together with the identity $\mathbb{1}$, the Pauli matrix form a basis for the vector space of matrices belonging in $\mathbb{C}^{2\times 2}$, thus, any matrix $M = \left(\begin{smallmatrix}m_{11} & m_{12}\\ m_{21} & m_{22}\end{smallmatrix}\right)$ can be decomposed as

$$M = \frac{m_{11}+m_{22}}{2}\mathbb{1} + \frac{m_{11}-m_{22}}{2}\sigma_z + \frac{m_{12}+m_{21}}{2}\sigma_x + i\frac{m_{12}+m_{21}}{2}\sigma_y.$$

Therefore, any matrix in $\mathbb{C}^{2\times 2}$ can be expressed in the form $M = a_0\mathbb{1} + a_x\sigma_x + a_y\sigma_y + a_z\sigma_z$ with $a_0, a_x, a_y, a_z \in \mathbb{C}$.

Now, let us consider spin-$\frac{1}{2}$ particles. By placing the apparatus along an angle θ, we can prepare a state $|\psi\rangle$ such that $|\psi\rangle = \psi_+|+\rangle + \psi_-|-\rangle$ with $|\psi_+|^2 + |\psi_-|^2 = 1$ and these two coefficients are determined by the angle θ. If we measure $\mathscr{S}_z$, then we have two possible outcomes $\frac{\hbar}{2}$ and $-\frac{\hbar}{2}$ with the probabilities $\mathbb{P}(+) = \langle\psi|(|+\rangle\langle+|)|\psi\rangle = |\psi_+|^2$ and $\mathbb{P}(-) = \langle\psi|(|-\rangle\langle-|)|\psi\rangle = |\psi_-|^2$ respectively. The corresponding expectation is $\langle S_z\rangle = \langle\psi|S_z|\psi\rangle = \frac{\hbar}{2}(|\psi_+|^2 - |\psi_-|^2)$. Consider an atom in a uniform magnetic field: the time-independent Hamiltonian describing the total energy of the system given by $H = \omega_0 S_z$, where $\omega > 0$ represents the angular velocity. Then we have $H|+\rangle = E_+|+\rangle$ and $H|-\rangle = E_-|-\rangle$ with $E_+ = \frac{\hbar\omega_0}{2}$ and $E_- = -\frac{\hbar\omega_0}{2}$, the "Bohr frequency" is given by $\frac{E_+ - E_-}{\hbar}$. By the Schrödinger equation, we get $|\psi(t)\rangle = e^{-iH(t-t_0)/\hbar}|\psi(t_0)\rangle$. Let us use the density operator ρ to describe the state of a two-level quantum system, where $\rho \in \mathcal{S}_2 := \{\rho \in \mathbb{C}^{2\times 2} | \rho^* = \rho, \mathrm{Tr}(\rho) = 1, \rho \geq 0\}$. Due to the properties of the Pauli matrices, we can write ρ in the form $\rho = \frac{\mathbb{1} + x\sigma_x + y\sigma_y + z\sigma_z}{2}$ with $(x, y, z) \in \mathcal{B} := \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 + z^2 \leq 1\}$. Thus

$\mathcal{S}_2$ is isomorphic to $\mathcal{B}$, that is any state of two-level quantum system can be represented by a three-dimensional vector (x, y, z) belonging to the unit ball. Such a vector is called the Bloch vector.

Consider a quantum system which has two states whose energies are close to each other and very different from the energies of other states of the system. Under this assumption, we can ignore all the other energy levels of the system, and analyze the system in a two-dimensional space $\mathcal{H}_2$. Take the two eigenvectors $|e\rangle$ and $|g\rangle$ of the Hamiltonian H, whose eigenvalues are E_e and E_g ($E_g < E_e$) respectively, as an orthonormal basis of $\mathcal{H}_2$. Then we can apply the above representation of a spin-$\frac{1}{2}$ model to general two-level quantum systems.

Now, we consider higher level quantum spin systems, and denote the N-dimensional spin state space by $\mathcal{H}_N$, with $2 \leq N < \infty$. Associate the angular momentum along the axis-x, y, z with Hermitian operators $J_{x,y,z}$. The commutation relations of the angular momenta along the three axis are given by

$$[J_x, J_y] = i\hbar J_z, \qquad [J_y, J_z] = i\hbar J_x, \qquad [J_z, J_x] = i\hbar J_y.$$

Referring to [CTDL18, SN14], for all $n \in \{0, \ldots, N-1\}$, the eigenvalues and the eigenvectors of the three angular momentum operators are given by

$$J_x|e_n\rangle = c_n|e_n\rangle + c_{n+1}|e_{n+1}\rangle, \quad J_y|e_n\rangle = -ic_n|e_{n-1}\rangle + ic_{n+1}|e_{n+1}\rangle, \quad J_z|e_n\rangle = (J-n)|e_n\rangle,$$

where $J = \frac{N-1}{2}$, $c_n = \frac{\sqrt{(N-n)n}}{2}$ and $\langle e_n|e_m\rangle = \delta_{m,n}$. The set of the orthonormal state vectors $\{|e_1\rangle, \ldots, |e_N\rangle\}$ form a basis of the state space $\mathcal{H}_N$. Thus, in this basis, the angular momentum operators can be represented in the following matrix forms

$$J_x = \begin{bmatrix} 0 & c_1 & & & & \\ c_1 & 0 & c_2 & & & \\ & \ddots & \ddots & \ddots & & \\ & & c_{2J-1} & 0 & c_{2J} & \\ & & & c_{2J} & 0 & \end{bmatrix}, \quad J_y = \begin{bmatrix} 0 & -ic_1 & & & & \\ ic_1 & 0 & -ic_2 & & & \\ & \ddots & \ddots & \ddots & & \\ & & ic_{2J-1} & 0 & -ic_{2J} & \\ & & & ic_{2J} & 0 & \end{bmatrix}, \quad J_z = \begin{bmatrix} J & & & & \\ & J-1 & & & \\ & & \ddots & & \\ & & & -J+1 & \\ & & & & -J \end{bmatrix}.$$

In this case, the density operator ρ acting on $\mathcal{H}_N$ belongs to the space $\mathcal{S}_N := \{\rho \in \mathbb{C}^{N\times N} | \rho^* = \rho, \mathrm{Tr}(\rho) = 1, \rho \geq 0\}$, the corresponding N-level Bloch vector is given in [KK05]. Similar to the two-level case, the general N-level quantum systems, with $2 \leq N < \infty$, whose Hamiltonian has N non-degenerate eigenvalues can be analyzed by the N-level angular momentum model.

2.2.3 Quantum harmonic oscillator

The one-dimensional harmonic oscillator is an important system in both classical and quantum mechanics. A lot of systems can be approximated by the harmonic oscillator model. Consider a one-dimensional conservative harmonic oscillator, by Newton's second law and Hooke's law, we have $\frac{d^2x}{dx^2} = -\frac{k}{m}x$ where $m > 0$ represents the mass of the particle and $k > 0$ describes the stiffness of the spring, and denote $\omega^2 = \frac{k}{m}$. Then the total energy of this system is the sum of the potential energy and the kinetic energy, $E = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2}$.

In quantum mechanics, we define two observables Q and P, which are self-adjoint operators on the Hilbert space $\mathcal{H}$ associated with the physical quantities position x and

momentum p, satisfying the Heisenberg's canonical commutation relation $[Q, P] = i\hbar\mathbb{1}$. In accordance with the classical case, we define the Hamiltonian operator of the system as $H = \frac{P^2}{2m} + \frac{m\omega^2 Q^2}{2}$, however, it is not obvious to resolve the eigenvalue equation $H|\psi\rangle = E|\psi\rangle$. Then we define two non-Hermitian operators $a = \sqrt{\frac{m\omega}{2\hbar}}\left(Q + i\frac{P}{m\omega}\right)$ and its conjugate $a^* = \sqrt{\frac{m\omega}{2\hbar}}\left(Q - i\frac{P}{m\omega}\right)$, the two non-Hermitian operators are known as the annihilation operator and the creation operator, and they can help us to study the eigenvalues and eigenvectors of H. Denote $N := a^*a$, by a straightforward calculation, we have

$$N = \frac{m\omega}{2\hbar}\left(Q^2 + \frac{P^2}{m^2\omega^2}\right) + \frac{i}{2\hbar}[Q, P] = \frac{H}{\hbar\omega} - \frac{\mathbb{1}}{2},$$

which implies that $H = \hbar\omega\left(N + \frac{\mathbb{1}}{2}\right)$, where N is called number operator. Then a direct computation shows $[N, a] = -a$ and $[N, a^*] = a^*$. Denote an eigenvalue and an associated eigenvector of N by n and $|n\rangle$, i.e., $N|n\rangle = n|n\rangle$, where $|n\rangle$ are called Fock states and form an orthogonal basis of the Hilbert space $\mathcal{H}$, $\langle n|m\rangle = \delta_{n,m}$ and $\sum_n |n\rangle\langle n| = \mathbb{1}$. Consequently, we can obtain easily $a|0\rangle = 0$. Since $n = \langle n|N|n\rangle = \|a|n\rangle\|^2 \geq 0$, then n is non-negative. As a result, we have

$$\begin{aligned} Na|n\rangle &= ([N, a] + aN)|n\rangle = (n-1)a|n\rangle, \\ Na^*|n\rangle &= ([N, a^*] + a^*N)|n\rangle = (n+1)a^*|n\rangle, \end{aligned} \tag{2.3}$$

which implies that $a|n\rangle$ and $a^*|n\rangle$ are also eigenvectors of N with eigenvalues decreased and increased by one respectively. From the relation $H|n\rangle = \hbar\omega\left(n + \frac{1}{2}\right)|n\rangle$, we note that the decrease or increase of n by one amounts to the annihilation or creation of one quantum unit of energy $\hbar\omega$, as the consequence of (2.3), $a|n\rangle$ is parallel to $|n-1\rangle$ and $a^*|n\rangle$ is parallel to $|n+1\rangle$, which means $a|n\rangle = \alpha_1|n-1\rangle$, $a^*|n\rangle = \alpha_2|n+1\rangle$. Due to the canonical commutation relation, we have $[a, a^*] = \mathbb{1}$, thus $aa^* = N + \mathbb{1}$. Since $n = \langle n|a^*a|n\rangle = |\alpha_1|^2$ and $n+1 = \langle n|(N+\mathbb{1})|n\rangle = \langle n|aa^*|n\rangle = |\alpha_2|^2$, and by assuming α_1 and α_2 to be real and non-negative, we have $\alpha_1 = \sqrt{n}$ and $\alpha_2 = \sqrt{n+1}$. Finally, we obtain

$$a|n\rangle = \sqrt{n}|n-1\rangle, \quad a^*|n\rangle = \sqrt{n+1}|n+1\rangle.$$

Therefore, in the $\{|n\rangle\}$ basis, the Hamiltonian H, the creation operator a^* and the annihilation operator a can be represented by the following matrices

$$H = \frac{\hbar\omega}{2}\begin{bmatrix} 1 & 0 & 0 & \cdot & \cdot \\ 0 & 3 & 0 & \cdot & \cdot \\ 0 & 0 & 5 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}, \quad a^* = \begin{bmatrix} 0 & 0 & 0 & \cdot & \cdot \\ 1 & 0 & 0 & \cdot & \cdot \\ 0 & \sqrt{2} & 0 & \cdot & \cdot \\ 0 & 0 & \sqrt{3} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}, \quad a = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdot & \cdot \\ 0 & 0 & \sqrt{2} & 0 & \cdot & \cdot \\ 0 & 0 & 0 & \sqrt{3} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}.$$

Furthermore, the general state of an harmonic oscillator can be expressed as a superposition of Fock states $|n\rangle$. Such states are called coherent states and are defined by

$$a|\alpha\rangle = \alpha|\alpha\rangle,$$

where a is the annihilation operator and a trivial solution of the above equation is the vacuum state $|0\rangle$ for $\alpha = 0$.

Now, we analyze the harmonic oscillator in the Heisenberg picture and discuss the time evolution of the operators. For the purpose of simplicity, we do not add the subscript H to emphasize the Heisenberg picture. The Heisenberg equation of a is given by

$$\frac{da}{dt} = \frac{1}{i\hbar}[a, H] = -i\omega[a, N] = -i\omega a,$$

which implies that $a(t) = a(t_0)e^{-i\omega(t-t_0)}$.

2.3 Open quantum systems

In the previous section, we have briefly discussed the formalism of the closed quantum systems. However, any quantum system interacts unavoidably with an external quantum system, like a large environment or a quantum heat bath. This type of system is called open quantum system.

Bipartite quantum systems In order to develop a theoretical framework for treating these interactions, let us consider a simple case, where the quantum system of interest consists of two quantum subsystems S_1 and S_2. This is called bipartite quantum system and is denoted by $S_1 \otimes S_2$. Due to the first postulate of quantum mechanics, we suppose that the quantum subsystem S_1 is defined by the state $\psi_1 \in \mathcal{H}_1$ and the quantum subsystem S_2 is defined by the state $\psi_2 \in \mathcal{H}_2$. Then the coupled system $S_1 \otimes S_2$ is defined by the state $\psi_1 \otimes \psi_2$ belonging to the tensor product $\mathcal{H}_1 \otimes \mathcal{H}_2$. If X_1 and X_2 are observables of the quantum subsystems S_1 and S_2, respectively, then they can be extended as the observables of the quantum system $S_1 \otimes S_2$ acting on $\mathcal{H}_1 \otimes \mathcal{H}_2$ by considering the operators $X_1 \otimes \mathbb{1}$ and $\mathbb{1} \otimes X_2$, respectively. Similarly, if ρ_1 and ρ_2 are density operators of S_1 and S_2, respectively, then $\rho_1 \otimes \rho_2$ is a density operator of the system $S_1 \otimes S_2$. Therefore, we can extend all six postulates discussed in the previous section to the case of bipartite quantum systems. Next, in order to determine the marginal states for bipartite quantum systems, we need to introduce the partial trace (see e.g., [Att, Chapter 2]), which helps us to average the complementary system. Given a density operator $\boldsymbol{\rho}$ on $\mathcal{H}_1 \otimes \mathcal{H}_2$, the marginal state ρ_1 on $\mathcal{H}_1$ can be calculated by

$$\rho_1 = \mathrm{Tr}_{\mathcal{H}_2}(\boldsymbol{\rho}),$$

where the partial trace is defined by

$$\mathrm{Tr}\big(\mathrm{Tr}_{\mathcal{H}_2}(\boldsymbol{\rho})X_1\big) = \mathrm{Tr}\big(\boldsymbol{\rho}(X_1 \otimes \mathbb{1})\big), \tag{2.4}$$

for all observables X_1 on $\mathcal{H}_1$.

Open quantum systems Open quantum systems can be considered as the family of all bipartite quantum systems. Let us define the quantum dynamical system of interest S on a Hilbert space $\mathcal{H}_S$, and describe the environment by a quantum system W on a Hilbert space $\mathcal{H}_W$. The system-environment $S \otimes W$ can be considered as a larger quantum system defined on the Hilbert space $\mathcal{H}_S \otimes \mathcal{H}_W$. Denote $\rho_S(t)$ and $\rho_W(t)$ as the density operators of the system and the environment, whose initial states are given by $\rho_S(t_0)$ and $\rho_W(t_0)$. Then the time evolution of the density operator $\boldsymbol{\rho}(t)$ of system-environment is given by

$$\boldsymbol{\rho}(t) = U(t, t_0)\big(\rho_S(t_0) \otimes \rho_W(t_0)\big)U^*(t, t_0) \Rightarrow \rho_S(t) = \mathrm{Tr}_{\mathcal{H}_W}\big(\boldsymbol{\rho}(t)\big),$$

where $U(t,t_0)$ is the unitary operator acting on $\mathcal{H}_S \otimes \mathcal{H}_W$. We measure the observable $X_1 \otimes \mathbb{1}$. Denote the set of eigenvalues of X_1 by $\{x_k\}$, and the set of associated projectors by $\{P_k\}$. After the measurement, the state immediately becomes

$$\boldsymbol{\rho}'(t) = \frac{P_n U(t,t_0)\big(\rho_S(t_0)\otimes\rho_W(t_0)\big)U^*(t,t_0)P_n}{\mathrm{Tr}\big(P_n U(t,t_0)\big(\rho_S(t_0)\otimes\rho_W(t_0)\big)U^*(t,t_0)P_n\big)},$$

where the denominator gives the probability of obtaining x_n. Therefore, the post measurement marginal state $\rho'_S(t)$ is given by

$$\rho'_S(t) = \mathrm{Tr}_{\mathcal{H}_W}\big(\boldsymbol{\rho}'(t)\big) = \frac{\mathrm{Tr}_{\mathcal{H}_W}\big(P_n U(t,t_0)\big(\rho_S(t_0)\otimes\rho_W(t_0)\big)U^*(t,t_0)P_n\big)}{\mathrm{Tr}\big(P_n U(t,t_0)\big(\rho_S(t_0)\otimes\rho_W(t_0)\big)U^*(t,t_0)P_n\big)}, \tag{2.5}$$

which describes the time evolution of the open quantum system undergoing measurements.

Quantum filtering equation Now, let us consider an open quantum system defined on $\mathcal{H}_S$ in interaction with an electromagnetic field in the vacuum state defined on $\mathcal{H}_W$ undergoing continuous-time measurements. Consider only one homodyne detection measurement channel at a time. Heuristically, the electromagnetic field can be considered as a collection of quantum harmonic oscillators, described by the field operators A_t (annihilation process) and $A_t^\dagger$ (creation process), which are defined on $\mathcal{H}_W$ and do not commute with each other. Then the joint dynamics of the unitary operator U_t of the whole system, i.e., open quantum system and the electromagnetic field, defined on the Hilbert space $\mathcal{H}_S \otimes \mathcal{H}_W$, is given by the following quantum stochastic differential equation (QSDE)

$$dU_t = \Big(L\otimes dA_t^\dagger - L^*\otimes dA_t - \Big(\big(L^*L/2 + iH\big)\otimes\mathbb{1}\Big)dt\Big)\,U_t, \qquad U_0 = \mathbb{1}.$$

Then the time evolution of the observable X on $\mathcal{H}_S$ is $j_t(X) = U_t^*(X\otimes\mathbb{1})U_t$, and the observation of homodyne detection at time t is $Y_t = U_t^*(\mathbb{1}\otimes(A_t + A_t^\dagger))U_t$, see Fig 2.2.

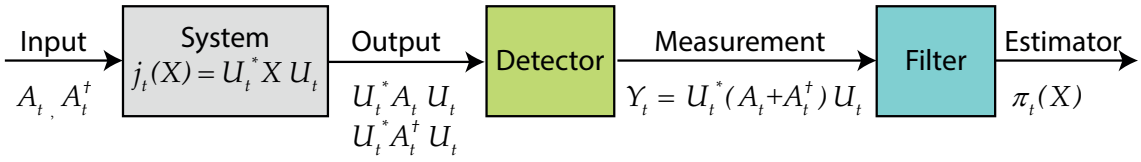


FIGURE 2.2 – Quantum filtering procedure. Based on the outcome of the detector Y_t, the filter provides an estimator of the observable X at time t. Note that, we identify X and $X\otimes\mathbb{1}$, A_t and $\mathbb{1}\otimes A_t$, $A_t^\dagger$ and $\mathbb{1}\otimes A_t^\dagger$ in the above diagram.

By the quantum Itô calculus [HP84], we have

$$\begin{aligned} dj_t(X) &= \mathcal{L}\big(j_t(X)\big)dt + dA_t^\dagger[j_t(X), j_t(L)] + [j_t(L^*), j_t(X)]dA_t, \\ dY_t &= (j_t(L) + j_t(L^*))dt + dA_t + dA_t^\dagger, \end{aligned}$$

where $\mathcal{L}(X) := i[H,X] + L^*XL - L^*LX/2 - XL^*L/2$ is called Lindblad generator. Like classical stochastic filtering theory [Kal13, Xio08], by the Kallianpur-Stribel formula and Itô formula, the dynamics of expectation of $j_t(X)$ conditioned on the recorded measurements Y_t up to time t, denoted by $\pi_t(X)$, see Fig 2.3, can be described by a normalized filtering equation. This equation is driven by the innovation process corresponding to the observation process.

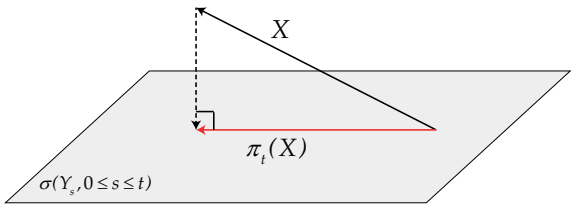


FIGURE 2.3 – The conditional expectation $\pi_t(X)$ can be considered as the projection onto the space spanned by the measurements record $\sigma(Y_s, 0 \le s \le t)$.

In the Schrödinger picture, we have $\pi_t(X) = \mathrm{Tr}(\rho_t X)$, where ρ_t is a density operator conditioned on the observations up to time t. Thus we can obtain a matrix-valued stochastic differential equation for the evolution of the density operator of the system under perfect continuous-time (homodyne) measurements, which is called stochastic master equation, and it is the quantum analogue of the Kushner-Stratonovich or FKK equation,

$$\begin{aligned} d\rho_t &= \mathcal{L}^*(\rho_t)dt + \big(L\rho_t + \rho_t L^* - \mathrm{Tr}\big((L+L^*)\rho_t\big)\rho_t\big)dW_t, \\ dY_t &= dW_t + \mathrm{Tr}\big((L+L^*)\rho_t\big)dt, \end{aligned}$$

where W_t is a one-dimensional Wiener process. By similar arguments, the dynamics of the system under imperfect measurements can be described by

$$d\rho_t = \mathcal{L}^*(\rho_t)dt + \sqrt{\eta}\big(L\rho_t + \rho_t L^* - \mathrm{Tr}\big((L+L^*)\rho_t\big)\rho_t\big)dW_t. \tag{2.6}$$

The measurement efficiency is given by $\eta \in (0,1)$. Note that, if H and L are time-invariant, the classical expectation of the stochastic master equation (2.6) is called Lindblad master equation

$$\frac{d}{dt}\mathbb{E}(\rho_t) = \mathcal{L}^*\big(\mathbb{E}(\rho_t)\big).$$

Moreover $\mathbb{E}(\rho_t)$ can also be derived by calculating the partial trace,

$$\mathbb{E}(\rho_t) = \mathrm{Tr}_{\mathcal{H}_W}\big(U_t(\rho_0 \otimes |0\rangle\langle 0|)U_t^*\big).$$

Thus we observe that Lindblad master equation plays the role of Fokker-Plank equation associated with the stochastic master equation (2.6).

2.4 Contribution and Outline of dissertation

In this thesis, we study feedback exponential stabilization of open quantum systems undergoing imperfect continuous-time measurements, towards a predetermined target state which is a pure state corresponding to an eigenvector of measurement operators.

Chapter 3 provides an introduction to the quantum filtering theory. We first introduce quantum probability theory as an extension of the classical probability theory, which allows to describe quantum phenomena. Then, we present quantum spin systems and the quantum harmonic oscillator in the framework of quantum probability. Next, we discuss quantum stochastic processes on Fock space, which are applied to characterize free quantum electromagnetic fields. After that, we describe the Hudson-Parthasarathy quantum Itô calculus and obtain quantum stochastic differential equations. Finally, by

the noncommutative Kallianpur-Striebel formula, we obtain a matrix-valued stochastic differential equation which is called quantum filtering equation or stochastic master equation. This equation describes the time evolution of the density operator of open quantum systems in interaction with an electromagnetic field undergoing imperfect continuous-time measurements.

Chapter 4 is devoted to feedback exponential stabilization of N-level quantum spin systems undergoing continuous-time measurements with known initial states. This chapter is based on our publications [LAM18, LAM19a]. We first study the case where the control input is turned off, and show the exponential quantum state reduction in mean and almost surely. By using stochastic and geometric control tools, we provide sufficient conditions on the feedback control law ensuring almost sure exponential convergence to a predetermined pure state corresponding to an eigenvector of the measurement operator J_z. In order to achieve these results, we establish general features of quantum trajectories which are of interest by themselves. We illustrate the results by designing a class of feedback control laws satisfying the above-mentioned conditions and finally we demonstrate the effectiveness of our methodology through numerical simulations for three-level quantum spin systems.

Chapter 5 is devoted to feedback stabilization of multi-qubit systems undergoing continuous-time measurements with known initial states. This chapter is based on the publication [LAM19b] and [LAMa]. We first study multi-qubit systems with two quantum channels when the control input is turned off, and show the exponential quantum state reduction in mean and almost surely. Then we consider a two-qubit system undergoing continuous-time measurements. In presence of one channel, we establish asymptotic convergence towards a predetermined Bell state. With two channels, we provide sufficient conditions on the continuous feedback control law ensuring almost sure exponential convergence to a predetermined Bell state. This is obtained by applying stochastic tools, Lyapunov methods and geometric control tools. In both cases, we provide explicit expressions of feedback control laws satisfying the above-mentioned conditions. Finally, we demonstrate the effectiveness of our methodology through numerical simulations.

Chapter 6 is devoted to feedback exponential stabilization of open quantum spin systems with unknown initial states. This chapter is based on [LAMb]. We first consider open quantum spin-$\frac{1}{2}$ systems with unknown initial states, and provide sufficient conditions on the feedback controller of the associated quantum filter, which ensure the convergence of the fidelity of the actual state and the estimated state towards one at infinity. Then we show the effectiveness of our methodology through numerical simulations. Then, heuristically, we discuss feedback exponential stabilization of N-level open quantum spin systems with unknown initial states. We finish this chapter by providing numerical simulations for three-level quantum spin systems.

Chapter 7 proposes some natural extensions to the results described in the chapters mentioned above.

Appendix A provides some basic notions and theorems from stochastic calculus and stochastic control theory.

3

Quantum filtering theory

In this chapter, we introduce the general framework of quantum probability and we focus mainly on quantum filtering theory.

Inspired by the classical filtering problem [Kal13, Xio08], whose purpose is to determine the best estimation of the state of a classical system from noisy observations, the quantum analogue was developed in the 1960s by Davies [Dav69, Dav76] and extended by Belavkin in the 1980s [Bel83, Bel89, Bel95, Bel92], relying on the quantum probability theory and the quantum stochastic calculus [HP84, Par12, Mey06]. For a modern treatment of quantum filtering, we refer to [BvHJ07].

Roughly speaking, using quantum filtering theory, we can derive a matrix-valued stochastic differential equation called stochastic master equation, to describe the time evolution of the state of an open quantum system interacting with an electromagnetic field under homodyne detection.

3.1 From classical to quantum probability

The Russian mathematician A. Kolmogorov provided a rigorous mathematical foundation for the classical probability theory in 1930s, firstly introduced the notion of the probability space, and considered random variables as functions from the probability space to $\mathbb{R}$. Quantum probablity theory provides an approach generalizing the Kolmogorov's probability theory in the framework of quantum mechanics. In this approach, the algebra of classical random variables is replaced by a more general non-commutative algebra appropriate to represent observables of the system (analogue of classical random variables), which helps us to discuss the statistical inference of quantum states.

As a preliminary step towards an introduction of quantum probability, we recall below the main definitions and properties of conditional expectation from classical probability theory.

3.1.1 Classical probability

An important concept in classical probability theory is the notion of probability space, which models a real-world "experiment" containing more than one possible outcome. A probability space $(\Omega, \mathcal{F}, \mathbb{P})$ consists of three parts:

1. a sample space Ω, which is the set of all possible outcomes, each outcome is denoted by ω;
2. a set of events $\mathcal{F}$, which is a σ-field of subsets of Ω, each event in $\mathcal{F}$ is a set containing zeros or more outcomes;
3. $\mathbb{P}$ is a probability measure on $\mathcal{F}$, i.e., $\mathbb{P} : \mathcal{F} \to [0, 1]$, which assigns the probabilities to the events. Since the probability measure contains all the information available on the outcome of any observation, we can consider $\mathbb{P}$ as the "state" of such random system.

A real-valued random variable X on Ω is represented as a real-valued measurable map $X : \Omega \to \mathbb{R}$, i.e., $X^{-1}(E) \in \mathcal{F}$ ($X^{-1}(E)$ is an event) for all $E \in \mathcal{R}$, where $\mathcal{R}$ is the σ-field of Borel subsets of $\mathbb{R}$. The law or distribution of a random variable X is the pushforward measure $\mu_X := X \circ \mathbb{P}$ of X, which is also the probability measure on the Borel subsets of $\mathbb{R}$ given by

$$\mu_X(E) = \mathbb{P}\big(X^{-1}(E)\big) = \mathbb{P}(\omega \in \Omega |\, X(\omega) \in E), \quad \forall E \in \mathcal{R}.$$

Random variables are interpreted as real-valued observable of a system by physicists. In particular, the mean value or expectation can be obtained by repeatedly measuring an observable and averaging. Mathematically, the expectation of an integrable random variable X is given by the following integral

$$\mathbb{E}(X) = \int_{\omega \in \Omega} X(\omega) d\mathbb{P}(\omega) = \int_{x \in \mathbb{R}} x d\mu_X(x),$$

where the second equality can be shown by applying Radon–Nikodym theorem. Suppose f is a Borel measurable real-valued function and μ_X-integrable, then $f(X)$ is still a random variable given by $\big(f(X)\big)(\omega) = f\big(X(\omega)\big)$, we have

$$\mathbb{E}\big(f(X)\big) = \int_{\Omega} \big(f(X)\big)(\omega) d\mathbb{P}(\omega) = \int_{\Omega} f\big(X(\omega)\big) d\mathbb{P}(\omega) = \int_{\mathbb{R}} f(x) d\mu_X(x).$$

By taking f as the exponential function, we can obtain the so-called characteristic function

$$\Phi_X(t) := \mathbb{E}\left(e^{itX}\right) = \int_{\mathbb{R}} e^{itx} d\mu_X(x), \quad \forall t \in \mathbb{R}.$$

Because of the invertibility of Fourier transform, the characteristic function of a random variable completely determines its law or distribution.

Now, let us consider the random vector X, which is a mapping from Ω to $\mathbb{R}^k$, i.e., $\omega \mapsto X(\omega) = \big(X_1(\omega), \ldots, X_n(\omega)\big)$. Then X is simply a n-tuple random variables, and X is measurable if and only if each X_i is. The joint distribution of a n-tuple random variables $(X_1, \ldots, X_n)$ is given by

$$\mu_X(E_1 \times \cdots \times E_n) = \mathbb{P}\left(\bigcap_{i=1}^{n} X_i^{-1}(E_i)\right) = \mathbb{P}\big(\omega \in \Omega |\, X_1(\omega) \in E_1, \ldots, X_n(\omega) \in E_n\big),$$

where each $E_i \in \mathcal{R}$. For the Borel measurable function f, which is μ_X-integrable, we have

$$\mathbb{E}\big(f(X)\big) = \int_\Omega f\big(X(\omega)\big) d\mathbb{P}(\omega) = \int_{\mathbb{R}^n} f(x_1, \ldots, x_n) d\mu_X(x_1, \ldots, x_n).$$

Then the characteristic function of the random vector X is given by

$$\Phi_X(t) := \mathbb{E}\left(e^{it_1X_1 + \cdots + it_nX_n}\right),$$

which completely determines the distribution of the random vector X and the one of its marginals X_i. In this case, the random variables $X_1, \ldots, X_n$ are said to be (totally) independent if and only if for all $E_i \in \mathcal{R}$, we have

$$\mathbb{P}\left(\bigcap_{i=1}^n X_i^{-1}(E_i)\right) = \prod_{i=1}^n \mathbb{P}\big(X_i^{-1}(E_i)\big).$$

There are two important properties concerning the independence of the random variables,

1. if $X_1, \ldots, X_n$ are independent random variables and $f_1, \ldots, f_n$ are Borel measurable functions, then $f_1(X_1), \ldots, f_n(X_n)$ are independent random variables;
2. $X_1, \ldots, X_n$ are independent if and only if

$$\mathbb{E}\left(\prod_{i=1}^n f(X_i)\right) = \prod_{i=1}^n \mathbb{E}\big(f(X_i)\big),$$

for all bounded Borel measurable real-valued functions f.

When an observer possesses only the partial information, it is important to introduce the concept of conditional probability. Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and two events $E, \Lambda \in \mathcal{F}$, the probability of E occurring knowing Λ occurred is given by

$$\mathbb{P}(E|\Lambda) = \frac{\mathbb{P}(E \cap \Lambda)}{\mathbb{P}(\Lambda)}, \quad \mathbb{P}(\Lambda) > 0. \tag{3.1}$$

If $\mathbb{P}(\Lambda) = 0$, we set $P(E|\Lambda) = 0$ for all $E \in \mathcal{F}$. Clearly, $\mathbb{P}(\cdot|\Lambda)$ is a probability measure on $\mathcal{F}$, which is called the "conditional probability relative to Λ". The integral of an integrable random variable X with respect to this probability measure is called the "conditional expectation relative to Λ":

$$\mathbb{E}(X|\Lambda) := \int_\Omega X(\omega) d\mathbb{P}(\omega|\Lambda) = \frac{1}{\mathbb{P}(\Lambda)} \int_\Lambda X(\omega) d\mathbb{P}(\omega). \tag{3.2}$$

Let $\{\Lambda_i | \, i \geq 1\}$ be a countable measurable partition of Ω, i.e.,

$$\Omega = \bigcup_{i=1}^\infty \Lambda_i, \quad \Lambda_i \in \mathcal{F}, \quad \Lambda_i \cap \Lambda_j = \emptyset, \text{ if } i \neq j.$$

Then, due to Equations (3.1) and (3.2), we have the following properties, which are called law of total probability and law of total expectation respectively,

$$\mathbb{P}(E) = \sum_{i=1}^\infty \mathbb{P}(\Lambda_i \cap E) = \sum_{i=1}^\infty \mathbb{P}(\Lambda_i)\mathbb{P}(E|\Lambda_i),$$

$$\mathbb{E}(X) = \sum_{i=1}^\infty \int_{\Lambda_i} X(\omega) d\mathbb{P}(\omega) = \sum_{i=1}^\infty \mathbb{P}(\Lambda_i)\mathbb{E}(X|\Lambda_i),$$

provided that X is integrable. Let us now turn to the conditional expectation with respect to the σ-Borel field $\mathcal{G}$ generated by a countable partition $\{\Lambda_i | \, i \geq 1\}$, which plays a fundamental role in classical estimation and filtering theory. The definition of the conditional expectation with respect to $\mathcal{G}$ is given by the following theorem.

Theorem 3.1.1. *Suppose X is an integrable random variable on $(\Omega, \mathcal{F}, \mathbb{P})$, and $\mathcal{G} \subset \mathcal{F}$ is a sub σ-field of $\mathcal{F}$. Then there exists a random variable $\mathbb{E}(X|\mathcal{G})$ called the conditional expectation of X given $\mathcal{G}$, which has the following two properties*

1. *$\mathbb{E}(X|\mathcal{G})$ is $\mathcal{G}$-measurable ;*
2. *for all $G \in \mathcal{G}$, $\mathbb{E}(X\mathbb{1}_G) = \mathbb{E}\big(\mathbb{E}(X|\mathcal{G})\mathbb{1}_G\big)$, where*

$$\mathbb{E}(X\mathbb{1}_G) = \int_G X(\omega) d\mathbb{P}(\omega), \quad \mathbb{E}\big(\mathbb{E}(X|\mathcal{G})\mathbb{1}_G\big) = \int_G \mathbb{E}(X|\mathcal{G})(\omega) d\mathbb{P}(\omega).$$

Let X and Y be integrable random variables on $(\Omega, \mathcal{F}, \mathbb{P})$, and let $\mathcal{G}$ and $\mathcal{E}$ be sub σ-fields of $\mathcal{F}$, then we state the following important properties of the conditional expectation,

1. irrelevance of independent information : if X is independent of $\mathcal{G}$, then $\mathbb{E}(X|\mathcal{G}) = \mathbb{E}(X)$;
2. linearity: for all $\alpha, \beta \in \mathbb{R}$, $\mathbb{E}(\alpha X + \beta Y|\mathcal{G}) = \alpha\mathbb{E}(X|\mathcal{G}) + \beta\mathbb{E}(Y|\mathcal{G})$;
3. stability: if X is $\mathcal{G}$-measurable, then $\mathbb{E}(X|\mathcal{G}) = X$;
4. module property: if X is $\mathcal{G}$-measurable, then $\mathbb{E}(XY|\mathcal{G}) = X\mathbb{E}(Y|\mathcal{G})$;
5. tower property: if $\mathcal{E} \subset \mathcal{G} \subset \mathcal{F}$, then $\mathbb{E}\big(\mathbb{E}(X|\mathcal{G})|\mathcal{E}\big) = \mathbb{E}(X|\mathcal{E})$;
6. law of total expectation: $\mathbb{E}\big(\mathbb{E}(X|\mathcal{G})\big) = \mathbb{E}(X)$.

The next lemma, called L^2-projection property, concerns the estimation of the exact value of a random variable X based on the information represented by $\mathcal{G}$.

Lemma 3.1.2. *Let X be an integrable random variable on $(\Omega, \mathcal{F}, \mathbb{P})$, and let $\mathcal{G} \subset \mathcal{F}$. Then*

$$\mathbb{E}\left(\big(X - \mathbb{E}(X|\mathcal{G})\big)^2\right) = \min_{Y \in L^2(\Omega, \mathcal{G}, \mathbb{P})} \mathbb{E}\left((X - Y)^2\right).$$

The above lemma implies that the conditional expectation $\mathbb{E}(X|\mathcal{G})$ is the "best" estimation for X among all $\mathcal{G}$-measurable random variable. The following result is an abstract version of the Bayes formula [Xio08], which corresponds to the key concept in the non-linear filtering theory, Kallianpur-Striebel formula.

Theorem 3.1.3. *Suppose that X is an integrable random variable on $(\Omega, \mathcal{F}, \mathbb{P})$ and $\mathcal{G}$ is a sub-σ-field of $\mathcal{F}$. Let $\mathbb{Q} \gg \mathbb{P}$ (i.e., $\mathbb{P}$ is absolutely continuous with respect to the measure $\mathbb{Q}$) be another probability measure such that $M = \frac{d\mathbb{P}}{d\mathbb{Q}}$ (Radon–Nikodym derivative). Then*

$$\mathbb{E}(X|\mathcal{G}) = \frac{\mathbb{E}^{\mathbb{Q}}(XM|\mathcal{G})}{\mathbb{E}^{\mathbb{Q}}(M|\mathcal{G})}.$$

3.1.2 Quantum probability

Below, we introduce the framework of quantum probability and in particular we provide the analogue of the definitions and properties stated in the previous subsection.

Quantum random variable Here, we want to interpret the quantum random variable, in the probabilistic view, as a spectral measure on a real line $\mathbb{R}$ by the famous von Neumann spectral theorem (see [RS80, Att]). In physicist's language, observables are given by self-adjoint operators on a Hilbert space $\mathcal{H}$ and also called quantum random variables (see the fourth postulate of quantum mechanics). Then, we first introduce the powerful tools to study self-adjoint operators: the spectral measure (or projection-valued measure) and the corresponding spectral integration.

Given a Hilbert space $\mathcal{H}$, denote $\mathcal{P}(\mathcal{H})$ the set of orthogonal projectors on $\mathcal{H}$, i.e., the bounded operator P on $\mathcal{H}$ satisfying $P^2 = P^* = P$. Let us consider a measurable space $(\mathbb{R}, \mathcal{R})$, where $\mathcal{R}$ is the σ-field of Borel subsets of $\mathbb{R}$. A $\mathcal{H}$-valued spectral measure on $(\mathbb{R}, \mathcal{R})$ is a mapping $\xi : \mathcal{R} \to \mathcal{P}(\mathcal{H})$, which satisfies $\xi(\emptyset) = 0$, $\xi(\mathbb{R}) = \mathbb{1}$ and $\xi\big(\bigcup_i E_i\big) = \sum_i \xi(E_i)$ for every sequence of disjoint set $\{E_i\}$ belonging to $\mathcal{R}$, and $E_i \downarrow \emptyset$ implies $\xi(E_i) \to 0$ in the strong topology. We fix $\psi \in \mathcal{H}$, and consider the mapping $E \in \mathcal{R} \mapsto \langle \psi, \xi(E)\psi \rangle = \|\xi(E)\psi\|^2 \in \mathbb{R}_+$. Due to the Riesz-Markov theorem, the above mapping uniquely defines a measure $\mu_\psi(\cdot) := \|\xi(\cdot)\psi\|^2$, such that $\mu_\psi(\mathbb{R}) = \|\psi\|^2$. Note that, if ψ is a unit vector on $\mathcal{H}$, then $\mu_\psi(\cdot)$ is a probability measure on $(\mathbb{R}, \mathcal{R})$.

Then, we define the spectral integral for any Borel function. We begin by considering the case of a simple function f on $\mathbb{R}$, that is $f(x) = \sum_{i=1}^n \alpha_i \mathbb{1}_{E_i}(x)$ where the sets $E_i \in \mathcal{R}$ are two by two disjoint and $\mathbb{1}_{E_i}$ denotes the indicator of E_i. We define the spectral integral of such f with respect to ξ,

$$\int_{\mathbb{R}} f(x) d\xi(x) := \sum_{i=1}^n \alpha_i \xi(E_i),$$

which implies that, for any $\psi \in \mathcal{H}$,

$$\langle \psi, \int_{\mathbb{R}} f(x) d\xi(x)\, \psi \rangle = \int_{\mathbb{R}} f(x) d\mu_\psi(x).$$

By introducing the essential supremum and showing the convergence of the integral, we can extend the notation from the simple function to any bounded Borel function. Then, for any Borel function f, we define the set

$$D_f := \left\{ \psi \in \mathcal{H} \,\middle|\, \int_{\mathbb{R}} |f(x)|^2 d\mu_\psi(x) < \infty \right\},$$

which is a dense subspace of $\mathcal{H}$. Finally, given a real bounded Borel measurable function f, we can define the spectral integral $\int_{\mathbb{R}} f(x) d\xi(x)$ on D_f, which is a bounded and self-adjoint operator on $\mathcal{H}$ (The proof can be found in [Att, Proposition 1.98]).

Next, we state the following important theorem: von Neumann's spectral theorem, which connects the physicist's and probabilist's interpretations of random variable (for more details, we refer to [Att, Chapter 7], [Mey06, Chapter 1] and [Par12, Chapter 1]).

Theorem 3.1.4 (von Neumann's spectral theorem [Mey06])**.** *For every self-adjoint operator X, there exists a unique spectral measure $\xi : \mathcal{R} \to \mathcal{P}(\mathcal{H})$, such that*

$$X = \int_{\mathbb{R}} x \, d\xi(x).$$

Then, given any (real or complex) Borel function f on the line, the spectral integral $\int_{\mathbb{R}} f(x) d\xi(x)$ is denoted by $f(X)$.

Due to the one-to-one correspondence between the self-adjoint operator and the associated spectral measure on $\mathbb{R}$ stated by above theorem, we may define the law of a given self-adjoint opertaor X on $\mathcal{H}$ under the state $\psi \in \mathcal{H}$ as follows

$$\mu_\psi(E) = \langle \psi, \xi(E)\psi \rangle = \|\xi(E)\psi\|^2 \in [0,1],$$

where $E \in \mathcal{R}$ and ξ is spectral measure corresponding to X. Then we can define its associated expectation, if it exists, by the following integral

$$\mathbb{E}(X) = \langle \psi, X\psi \rangle = \int_{\mathbb{R}} x \, d\mu_\psi(x).$$

For any bounded Borel function f, we have

$$\mathbb{E}\big(f(X)\big) = \langle \psi, f(X)\psi \rangle = \int_{\mathbb{R}} f(x) d\mu_\psi(x).$$

Hence the corresponding characteristic function is given by

$$\Phi_X(t) = \mathbb{E}\big(e^{itX}\big) = \langle \psi, e^{itX}\psi \rangle = \int_{\mathbb{R}} e^{itx} d\mu_\psi(x).$$

Note that we can consider X as a random variable in the classical probability space $(\mathbb{R}, \mathcal{R}, \mu_\psi)$, the associated spectral measure $\xi(E)$ describes the event that the random variable X takes a value in $E \in \mathcal{R}$.

Furthermore, by [Par12, Theorem 1.8], we can generalize the above discussion on spectral measure and spectral integral for one random variable to the ones for multiple random variables. Next, let us consider the case of two commuting real-valued random variables A and B. By the definition in [RS80, Chapter VIII], two (possibly unbounded) self-adjoint operators A and B are said to be commute if and only if all the projections in their associated spectral measures commute. Then we have the following two equivalent statements:

1. A and B commute.
2. For all $s, t \in \mathbb{R}$, $e^{itA}e^{isB} = e^{isB}e^{itA}$.

Due to [Par12, Corollary 10.9], we can view the two commuting real-valued random variables A and B as a single $\mathbb{R}^2$-valued random variable. Then, by [Par12, Proposition 10.10], we can define the joint probability distribution of the commuting random variables A and B as the probability measure μ_ψ on $\mathbb{R}^2$ for which

$$\int_{\mathbb{R}^2} e^{ita+isb} d\mu_\psi(a,b) = \langle \psi, e^{itA}e^{isB}\psi \rangle = \langle \psi, e^{isB}e^{itA}\psi \rangle, \quad \forall s,t \in \mathbb{R}^2. \tag{3.3}$$

Moreover, the random variables A and B can be realized by the function $f_a(a,b) = a$ and $f_b(a,b) = b$ on the larger probability space $(\mathbb{R}^2, \mathcal{R}^2, \mu_\psi)$.

If the two random variables A and B do not commute, then such two random variables belong to two different probability spaces, we cannot define the joint probability distribution and they cannot be simultaneously realized. It corresponds to the fact that it is impossible to measure two non-commuting observables simultaneously.

Quantum probability space By what precedes, for a given unit vector $\psi \in \mathcal{H}$, we may associate a common probability measure depending on ψ for all commuting quantum observables (self-adjoint operators). However, this does not extend to non-commuting observables.

However, in order to generalize the classical (Kolmogorov) probability theory to allow the quantum mechanical models, following the approach of [BvHJ07, Maa03], we consider the algebra of random variables as the fundamental concept and encode the information contained in the classical structure into an appropriate algebra (von Neumann algebra), then introduce the notion of quantum probability space.

Let us first introduce the following important notions [Maa03]

Definition 3.1.5. A von Neumann algebra is a collection $\mathcal{A}$ of bounded linear operator on a Hilbert space $\mathcal{H}$ containing the identity $\mathbb{1}$ with the following properties,

1. $\mathcal{A}$ is a linear space: $A, B \in \mathcal{A}$ and $\alpha, \beta \in \mathbb{C}$ implies $\alpha A + \beta B \in \mathcal{A}$.
2. $\mathcal{A}$ is $*$-algebra: $A, B \in \mathcal{A}$ implies $AB \in \mathcal{A}$ and $A^* \in \mathcal{A}$.
3. $\mathcal{A}$ is strongly closed: $A_i \in \mathcal{A}$ and for all $\psi \in \mathcal{H}$, $\lim_{i\to\infty} A_i\psi = A\psi$ implies $A \in \mathcal{A}$.

Definition 3.1.6. A state $\boldsymbol{\varphi}$ on von Neumann algebra $\mathcal{A}$ is a functional $\boldsymbol{\varphi} : \mathcal{A} \to \mathbb{C}$ with the following properties,

1. linearity: $A, B \in \mathcal{A}$ and $\alpha, \beta \in \mathbb{C}$ implies $\boldsymbol{\varphi}(\alpha A + \beta B) = \alpha\boldsymbol{\varphi}(A) + \beta\boldsymbol{\varphi}(B)$.
2. positivity: for all $A \in \mathcal{A}$, then $\|A\|_{\boldsymbol{\varphi}} := \boldsymbol{\varphi}(A^*A) \geq 0$.
3. normalization: $\boldsymbol{\varphi}(\mathbb{1}) = 1$.

Moreover, a state $\boldsymbol{\varphi}$ is called faithful if $\boldsymbol{\varphi}(A^*A) = 0$ implies $A = 0$. It is called normal for every sequence $A_1, A_2, \ldots$ in $\mathcal{A}$ with strong limit A, we have $\lim_{n\to\infty} \boldsymbol{\varphi}(A_n) = \boldsymbol{\varphi}(A)$.

Let us illustrate the above notions by a simple example. Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, two function spaces $L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ and $L^2(\Omega, \mathcal{F}, \mathbb{P})$ are well defined. Here $L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ is a Banach space and $L^2(\Omega, \mathcal{F}, \mathbb{P})$ is a Hilbert space. For any $f \in L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ determines an multiplication operator M_f on the Hilbert space $L^2(\Omega, \mathcal{F}, \mathbb{P})$ by $(M_f\psi)(\omega) = f(\omega)\psi(\omega)$. Then, the following result provides a method to construct the von Neumann algebra and the associated state.

Proposition 3.1.7 ([Maa03, Proposition 1.1])**.** *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Then the algebra*

$$\mathcal{A} := \{M_f |\, f \in L^\infty(\Omega, \mathcal{F}, \mathbb{P})\}$$

is a commutative von Neumann algebra of the operators on $\mathcal{H} := L^2(\Omega, \mathcal{F}, \mathbb{P})$, and $\boldsymbol{\varphi} : M_f \mapsto \int f d\mathbb{P}$ is a faithful normal state on $\mathcal{A}$.

The fundamental result in theory of operator algebra known as Gelfrand's theorem [Maa03, Theorem 1.2], states that the commutative von Neumann algebra is equivalent to the algebra of bounded functions acting by multiplication on the Hilbert space $L^2(\Omega, \mathcal{F}, \mathbb{P})$ for some probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

In the following, we consider the non-commutative case. We define a quantum probability space as a pair $(\mathcal{A}, \boldsymbol{\varphi})$, which consists of a von Neumann algebra $\mathcal{A}$ on a Hilbert space $\mathcal{H}$, which is the quantum counterpart of σ-algebra in the classical probability theory, and a faithful normal state $\boldsymbol{\varphi}$ on $\mathcal{A}$, which plays a role of a probability measure. Next, let us show the following three most commonly used types of von Neumann algebra $\mathcal{A}$ [BvHJ07],

1. the set of all bounded linear operators on $\mathcal{H}$, denoted by $\mathscr{B}(\mathcal{H})$, is a von Neumann algebra;
2. $L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ acting on $L^2(\Omega, \mathcal{F}, \mathbb{P})$ by pointwise multiplication is a commutative von Neumann algebra;
3. denote the set called commutant of $\mathcal{B} \subset \mathscr{B}(\mathcal{H})$ by $\mathcal{B}' := \{A \in \mathscr{B}(\mathcal{H}) |\, AB = BA,\ \forall B \in \mathcal{B}\}$, then the double commutant of any self adjoint set [1] $\mathcal{B}$, denoted by $\mathcal{B}''$, is a smallest von Neumann subalgebra of $\mathscr{B}(\mathcal{H})$ that contains $\mathcal{B}$. In particular, $\mathcal{B}$ is a von Neumann algebra if and only if $\mathcal{B} = \mathcal{B}''$. [2]

Let $\mathcal{A}$ be a von Neumann algebra acting on a Hilbert space $\mathcal{H}$. An operator X is said to be affiliated to $\mathcal{A}$, if $\mathcal{H}$ is stable under every operator in $\mathcal{A}'$, the commutant of $\mathcal{A}$, and $XA\psi = AX\psi$ for all $A \in \mathcal{A}'$ and $\psi \in \mathcal{H}$ (see [Mey06, Chapter A.4.3]). Thus, if a bounded and self-adjoint operator X is affiliated to a von Neumann algebra $\mathcal{A}$, then $X \in \mathcal{A}''$ which is equal to $\mathcal{A}$. Due to the spectral decomposition of X which is unique, its spectral measure $\xi(E) \in \mathcal{A}$ for all $E \in \mathcal{R}$. Thus, for a bounded and self-adjoint operator X affiliated to $\mathcal{A}$, the probability of the event that X on $\mathcal{H}$ takes a value in the Borel set $E \in \mathcal{R}$ is given by

$$\mathbb{P}\big(\omega \in \Omega |\, X(\omega) \in E\big) = \boldsymbol{\varphi}\big(\xi(E)\big),$$

which is called the law or distribution of X. The associated expectation is given by

$$\mathbb{E}(X) = \int_{\mathbb{R}} x\, d\boldsymbol{\varphi}_\psi\big(\xi(x)\big).$$

Next, for the final purpose of the quantum filtering theory, we give the definition of conditional expectation in a quantum probability space. The existence and uniqueness of the quantum conditional expectation have been shown in [BvHJ07, Section 3.3].

Definition 3.1.8. Let $(\mathcal{A}, \boldsymbol{\varphi})$ be a quantum probability space and let $\mathcal{B} \subset \mathcal{A}$ be a commutative von Neumann subalgebra. Denote the set called commutant of $\mathcal{B} \subset \mathscr{B}(\mathcal{H})$ in $\mathcal{A}$ by $\mathcal{B}' := \{A \in \mathcal{A} |\, AB = BA,\ \forall B \in \mathcal{B}\}$. Then the map $\boldsymbol{\varphi}(\cdot|\mathcal{B}) : \mathcal{B}' \to \mathcal{B}$ is called (a version of) the conditional expectation from $\mathcal{B}'$ onto $\mathcal{B}$, if for all $X \in \mathcal{B}'$ and $S \in \mathcal{B}$,

$$\boldsymbol{\varphi}\big(\boldsymbol{\varphi}(X|\mathcal{B})S\big) = \boldsymbol{\varphi}(XS).$$

Given a quantum probability space $(\mathcal{A}, \boldsymbol{\varphi})$, let X be the operators affiliated to the $\mathcal{B}'$, the commutant of $\mathcal{B} \subset \mathcal{A}$, then we have the following properties of the quantum condition expectation, which are analogous to the classical case,

1. linearity: for all $\alpha, \beta \in \mathbb{C}$, $\boldsymbol{\varphi}(\alpha X + \beta Y|\mathcal{B}) = \alpha\boldsymbol{\varphi}(X|\mathcal{B}) + \beta\boldsymbol{\varphi}(Y|\mathcal{B})$;
2. positivity: if $X \geq 0$, then $\boldsymbol{\varphi}(X|\mathcal{B}) \geq 0$;
3. module property: if $Y_1, Y_2 \in \mathcal{B}$, then $\boldsymbol{\varphi}(Y_1 X Y_2|\mathcal{B}) = Y_1\boldsymbol{\varphi}(X|\mathcal{B})Y_2$;
4. tower property: if $\mathcal{C} \subset \mathcal{B} \subset \mathcal{A}$, then $\boldsymbol{\varphi}\big(\boldsymbol{\varphi}(X|\mathcal{B})|\mathcal{C}\big) = \boldsymbol{\varphi}(X|\mathcal{C})$;
5. law of total expectation: $\boldsymbol{\varphi}\big(\boldsymbol{\varphi}(X|\mathcal{B})\big) = \boldsymbol{\varphi}(X)$.

The next result [BvHJ07, Theorem 3.16] is the quantum analogue of L^2-projection property of the conditional expectation.

1. If $S \in \mathcal{B}$, then $S^* \in \mathcal{B}$.
2. This is von Neumann bicommutant theorem [Mey06, Chapter A.4.3].

Lemma 3.1.9. *Given a commutative quantum probability space $(\mathcal{A}, \boldsymbol{\varphi})$, let X be the operators affiliated to the $\mathcal{B}'$, the commutant of commutative von Neumann subalgebra $\mathcal{B} \subset \mathcal{A}$. Then*

$$\|X - \boldsymbol{\varphi}(X|\mathcal{B})\|_{\boldsymbol{\varphi}} = \min_{Y \in \mathcal{B}} \|X - Y\|_{\boldsymbol{\varphi}}.$$

The above lemma implies that the quantum conditional expectation $\boldsymbol{\varphi}(X|\mathcal{B})$ is the "best" estimation for X among all operators in $\mathcal{B}$. A crucial tool for quantum filtering theory, quantum analogue of Bayes formula, is given by the following lemma.

Theorem 3.1.10 (Quantum Bayes formula)**.** *Given a quantum probability space $(\mathcal{A}, \boldsymbol{\varphi})$, denote $\mathcal{B}'$ as the commutant of a commutative von Neumann subalgebra $\mathcal{B} \subset \mathcal{A}$. Choose $V \in \mathcal{B}'$ such that $V^*V > 0$ and $\boldsymbol{\varphi}(V^*V) = 1$. Then we can define a new state ϖ on $\mathcal{B}'$ by $\boldsymbol{\varpi}(X) := \boldsymbol{\varphi}(V^*XV)$ and*

$$\boldsymbol{\varpi}(X|\mathcal{B}) = \frac{\boldsymbol{\varphi}(V^*XV|\mathcal{B})}{\boldsymbol{\varphi}(V^*V|\mathcal{B})}, \quad \forall X \in \mathcal{B}'.$$

3.1.3 Revision of quantum spin systems and quantum harmonic oscillators

Here, we discuss two important quantum system models, quantum spin systems and the quantum harmonic oscillator, in the framework of quantum probability. For the purpose of simplicity, we consider only the case of pure states.

Quantum spin systems The observables of the quantum spin systems, which we have discussed in Section 2, are bounded self-adjoint operators acting on a finite dimensional Hilbert space $\mathcal{H}$. Thus it is easy to construct the associated finite dimensional quantum probability space, the commutative von Neumann algebra $\mathcal{A}$ can be obtained as the span of a set of orthogonal projections $\{P_n\}$ such that $\sum_n P_n = \mathbb{1}$. Moreover, in the finite dimensional case, every observable affiliated to $\mathcal{A}$ belongs to $\mathcal{A}$ and all elements of $\mathcal{A}$ commute with each other. The corresponding state is given by the map $\boldsymbol{\varphi} : X \in \mathcal{A} \mapsto \langle \psi, X\psi \rangle \in \mathbb{C}$ with a fixed unit vector $\psi \in \mathcal{H}$. It is easy to verify that the finite dimensional quantum probability space $(\mathcal{A}, \boldsymbol{\varphi})$ is isomorphic to a classical probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with $\mathrm{card}(\Omega) = \dim(\mathcal{H})$. The probability of the event that the random variable X takes its eigenvalue x_n is given by $\mathbb{P}(x_n) = \boldsymbol{\varphi}(P_n) = \langle \psi, P_n\psi \rangle$, where P_n is the eigenprojector corresponding to x_n. The more details and examples in finite dimensional quantum probability space are referred to [BvHJ09].

Quantum harmonic oscillators Two common observables are the position operator Q and the momentum operator P on a Hilbert space $\mathcal{H}$ satisfying the canonical commutation relation $[P, Q] = -i\hbar\mathbb{1}$, which have been already discussed in Section 2 in a heuristic way. Now, we want to reformulate this canonical pair (P, Q) more rigorously in the framework of quantum probability. P. A. Meyer stated the following phrase to emphasize the importance of the canonical pair in his lecture notes [Mey86], "les couples canoniques jouent en probabilités quantique le rôle des variables aléatiores classiques." [3]

3. The canonical pair in quantum probability plays the role of the classical random variables.

Note that, the canonical commutation relation was impossible to be satisfied for the linear operators P and Q acting on a finite dimensional Hilbert space unless $\hbar = 0$, since in this case we have $\mathrm{Tr}\big([P,Q]\big) = 0$ and $\mathrm{Tr}(-i\mathbb{1}) \neq 0$. Furthermore, the two operators satisfying the canonical commutation relation cannot be both bounded (see [RS80, Chapter VIII.5]), which can be shown by contradiction. If they were both bounded, then

$$\begin{aligned}[Q^n, P] &= (Q^n P - Q^{n-1}PQ) + (Q^{n-1}PQ - Q^{n-2}PQ^2) + \cdots + (QPQ^{n-1} - PQ^n) \\ &= Q^{n-1}[Q,P] + Q^{n-2}[Q,P]Q + \cdots + Q[Q,P]Q^{n-1} = i\hbar n Q^{n-1},\end{aligned}$$

which implies that $2\|P\|\|Q\|^n \geq n\hbar\|Q\|^{n-1}$. Then, for all n, we have $2\|P\|\|Q\| \geq n\hbar$.

Here is specific example of Q, P satisfying the canonical commutation relation so-called Schrödinger representation (the physical introduction can be found in [CTDL18, Chapter 2]). We can represent the canonical pair as two unbounded operators on $\mathcal{H} = L^2(\mathbb{R})$ such that, for all $\psi \in \mathcal{H}$,

$$(Q\psi)(x) = x\psi(x), \quad (P\psi)(x) = -i\hbar\frac{d}{dx}\psi(x), \tag{3.4}$$

where the two unbounded operators Q and P are self-adjoint (see [RS80, Chapter VIII]). Furthermore, Stone–von Neumann theorem [Mey06, Chapter III.6] shows the uniqueness of Schrödinger representation of the canonical commutation relation.

Stone's theorem on one-parameter unitary groups [RS80, Chapter VIII.4] can help us to get rid of the unboundedness of the operators P and Q. This provides the mathematically rigorous Weyl commutation relation equivalent to the canonical commutation relation. Firstly, let us introduce one parameter unitary groups. Suppose that A is a self-adjoint operator on a Hilbert space $\mathcal{H}$. By von Neumann spectral theorem (Theorem 3.1.4), define an operator-valued function U_t such that, for all $t \in \mathbb{R}$,

$$U_t := e^{itA} = \int_{\mathbb{R}} e^{itx} d\xi(x).$$

Then $\{U_t\}_{t\in\mathbb{R}}$ are unitary operators that satisfy

1. the group property, i.e., for all $s, t \in \mathbb{R}$, $U_{t+s} = U_t U_s$; and
2. strong continuity property, i.e., for all $t_0 \in \mathcal{R}$ and $\psi \in \mathcal{H}$, $\lim_{t\to t_0} U_t\psi = U_{t_0}\psi$.

Theorem 3.1.11 (Stone's theorem)**.** *Let $\{U_t\}_{t\in\mathbb{R}}$ be a strongly continuous one-parameter group on a Hilbert space $\mathcal{H}$. Then there is a self-adjoint operator A on $\mathcal{H}$ such that $U_t = e^{itA}$.*

Then, by Stone's theorem, we can obtain the following two one-parameter unitary groups $U_t = e^{itP}$ and $V_s = e^{isQ}$ determined by the self-adjoint operators Q and P, whose action on $\mathcal{H}$ is given by

$$(U_t\psi)(x) = \psi(x + \hbar t), \quad (V_s\psi)(x) = e^{isx}\psi(x),$$

which deduces the Weyl commutation relation,

$$U_t V_s = e^{i\hbar ts} V_s U_t. \tag{3.5}$$

Based on the above argument, we can take the von Neumann algebra $\mathcal{A}$ as the set of all bounded linear operator on $\mathcal{H} = L^2(\mathbb{R})$. As an example, we consider a unit vector $\psi \in \mathcal{H}$ given by

$$\psi(x) = (2\pi)^{-1/4}\sigma^{-1/2}e^{-\frac{(x-\mu)^2}{4\sigma^2}},$$

where $\psi(x)$ is the probability density function of a normal distribution $\mathcal{N}(\mu, \sigma)$, thus we have

$$\langle \psi, \psi \rangle = \int_{\mathbb{R}} \psi^2(x) dx = 1.$$

Therefore, the characteristic functions of the Q and P are given by

$$\Phi_Q(t) = \mathbb{E}(V_t) = \langle \psi, V_t \psi \rangle = \int_{\mathbb{R}} \psi(x) e^{itx} \psi(x) dx = e^{it\mu - \frac{t^2\sigma^2}{2}},$$
$$\Phi_P(t) = \mathbb{E}(U_t) = \langle \psi, U_t \psi \rangle = \int_{\mathbb{R}} \psi(x) \psi(x + \hbar t) dx = e^{-\frac{\hbar^2 t^2}{8\sigma^2}}.$$

Therefore, Q and P are Gaussian random variables such that $Q \sim \mathcal{N}(\mu, \sigma^2)$ and $P \sim \mathcal{N}(0, \hbar^2/4\sigma^2)$. Note that, the variance of Q is inversely proportional to the one of P, which implies

$$\mathrm{Var}(Q)\mathrm{Var}(P) = \frac{\hbar^2}{4},$$

which attains the minimal uncertainty of the Heisenberg uncertainty relation, i.e., $\mathrm{Var}(Q)\mathrm{Var}(P) \geq \frac{\hbar^2}{4}$. Furthermore, the minimal uncertainty states of the canonical pair are Gaussian [Mey06, Chapter III].

3.2 Quantum stochastic calculus

3.2.1 Stochastic processes on Fock space

Fock space In the previous section, we have discussed the relation between the single classical probability space and the single quantum probability space. In classical probability theory, N different statistical systems, described in N different classical probability spaces $(\Omega_i, \mathcal{F}_i, \mathbb{P}_i)$, can be represented in a "big" single classical probability space $(\Omega, \mathcal{F}, \mathbb{P})$ by taking the Cartesian product, i.e., $\Omega = \Omega_1 \times \cdots \times \Omega_N$, $\mathcal{F} = \mathcal{F}_1 \times \cdots \times \mathcal{F}_N$ which is the smallest σ-algebra containing all rectangle of $E_1 \times \cdots \times E_N$ with $E_i \in \mathcal{F}_i$, and $\mathbb{P} = \mathbb{P}_1 \times \cdots \times \mathbb{P}_N$ satisfies the following property, for all $E_i \in \mathcal{F}_i$,

$$\mathbb{P}(E_1 \times \cdots \times E_N) = \mathbb{P}_1(E_1) \ldots \mathbb{P}_N(E_N).$$

Now, we present the quantum analogue of the above description, merging indefinite different quantum systems into one picture, by introducing the tensor products of Hilbert spaces [RS80, Att], which are key concepts on open quantum systems. Here, we first describe the tensor product of two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$. For each $\psi_1 \in \mathcal{H}_1$ and $\psi_2 \in \mathcal{H}_2$, let $\psi_1 \otimes \psi_2$ denote the conjugate bilinear form which acts on $\mathcal{H}_1 \times \mathcal{H}_2$ by

$$(\psi_1 \otimes \psi_2)(\phi_1, \phi_2) = \langle \phi_1, \psi_1 \rangle \langle \phi_2, \psi_2 \rangle.$$

Let $\mathcal{E}$ be the set of all finite linear combinations of such forms, we define an inner product $\langle \cdot, \cdot \rangle$ on $\mathcal{E}$ by setting

$$\langle \phi_1 \otimes \phi_2, \psi_1 \otimes \psi_2 \rangle = \langle \phi_1, \psi_1 \rangle \langle \phi_2, \psi_2 \rangle.$$

and extending by linearity to $\mathcal{E}$. Let $(\mathcal{H}_n)_{1 \leq n \leq N}$ be a sequence of Hilbert spaces, we can generalize the above construction to define the tensor products of finitely many Hilbert

space $(\mathcal{H}_n)_{1\leq n\leq N}$, denoted by $\bigotimes_{n=1}^{N}\mathcal{H}_n$. In particular, for a Hilbert spaces $\mathcal{H}$ and for any finite integer N, $\mathcal{H}^{\otimes N} := \mathcal{H}\otimes\cdots\otimes\mathcal{H}$ is called N-fold tensor product of $\mathcal{H}$. Then, we can define the countable tensor product $\bigotimes_{n=1}^{+\infty}\mathcal{H}_n$ as the inductive limit of the space $\bigotimes_{n=1}^{N}\mathcal{H}_n$, when N tends to $+\infty$ (see [Att, Chapter 2.2]). Based on the above construction of the tensor products of Hilbert spaces, let us introduce the important notions of Fock spaces [Mey06, Par12, Att, RS80], to realize the combination of indefinite number of quantum systems. They are fundamental for modelling the typical states (i.e., zero particle states, one particle states and so on) in quantum fields (i.e., electromagnetic fields, etc.). Let $\mathcal{H}$ be a complex Hilbert space, consider its N-fold tensor product $\mathcal{H}^{\otimes N}$ with $N\geq 1$. For $u_1,\ldots,u_N\in\mathcal{H}$, we define the symmetric tensor product [4]

$$u_1\circ\cdots\circ u_N := \frac{1}{N!}\sum_{\sigma\in\mathscr{P}_N} u_{\sigma(1)}\otimes\cdots\otimes u_{\sigma(N)}, \tag{3.6}$$

where $\mathscr{P}_N$ is permutation group on N elements. For $u\in\mathcal{H}$, we denote $u^{\circ N} := u\circ\cdots\circ u$ and $u^{\otimes N} := u\otimes\cdots\otimes u$. By the definition (3.6), we can easily deduce $u^{\circ N} = u^{\otimes N}$. The closed subspace of $\mathcal{H}^{\otimes N}$ generated by all vectors (3.6) is called the N-fold symmetric tensor product of $\mathcal{H}$, denoted by $\mathcal{H}^{\circ N}$. Moreover, we have the following scalar products defined on $\mathcal{H}^{\otimes N}$ and $\mathcal{H}^{\circ N}$ respectively,

$$\langle u_1\otimes\cdots\otimes u_N, v_1\otimes\cdots\otimes v_N\rangle_{\otimes} = \langle u_1,v_1\rangle\ldots\langle u_N,v_N\rangle; \tag{3.7a}$$

$$\langle u_1\circ\cdots\circ u_N, v_1\circ\cdots\circ v_N\rangle_{\circ} = \mathrm{Per}\big(\langle u_i,v_j\rangle\big)_{0\leq i,j\leq N}, \tag{3.7b}$$

where Per denotes the permanent of a matrix, i.e., the determinant without the minus signs. Note that, for the case $N=0$, we make the convention $u^{\otimes 0} = u^{\circ 0} = 1\in\mathbb{C}$, $\mathcal{H}^{\otimes 0} = \mathcal{H}^{\circ 0} = \mathbb{C}$ and call it vacuum subspace.

Definition 3.2.1 (Fock space [Att, Chapter 8.2])**.** We call free (or full) Fock space over $\mathcal{H}$ the Hilbert space

$$\Gamma_f(\mathcal{H}) := \bigoplus_{n=0}^{+\infty}\mathcal{H}^{\otimes n}.$$

We call symmetric (or bosonic) Fock space over $\mathcal{H}$ the Hilbert space

$$\Gamma_s(\mathcal{H}) := \bigoplus_{n=0}^{+\infty}\mathcal{H}^{\circ n}.$$

In quantum probability, the symmetric (bosonic) Fock space $\Gamma_s\big(L^2(\mathbb{R}_+)\big)$ is very important for quantum stochastic calculus, where $\mathbb{R}_+$ corresponds to the time set. It can help us to describe the particle states of the quantum fields in time representation. Then we will see that the Wiener process emerges within this model. Next, let us focus on the symmetric Fock space $\Gamma_s(\mathcal{H})$ for a given Hilbert space $\mathcal{H}$. There exists a particular useful set of vectors $e(u)\in\Gamma_s(\mathcal{H})$ associated with $u\in\mathcal{H}$, called exponential vectors or coherent vectors, given by

$$e(u) = \bigoplus_{n=0}^{+\infty}\frac{u^{\otimes n}}{\sqrt{n!}},\quad u\neq 0, \tag{3.8}$$

4. In this thesis, we focus on the symmetric case. For the definition of antisymmetric tensor product, we refer to [Mey06, Par12, Att, RS80]

with the convention that

$$e(0) = 1 \oplus 0 \oplus 0 \oplus \dots,$$

which is called vacuum vector. It implies that

$$\langle e(u), e(v)\rangle = \left\langle \bigoplus_{n=0}^{+\infty} \frac{u^{\otimes n}}{\sqrt{n!}}, \bigoplus_{n=0}^{+\infty} \frac{v^{\otimes n}}{\sqrt{n!}} \right\rangle = \sum_{n=0}^{+\infty} \frac{1}{n!} \langle u^{\otimes n}, v^{\otimes n}\rangle = \sum_{n=0}^{+\infty} \frac{1}{n!} \big(\langle u, v\rangle\big)^n = e^{\langle u,v\rangle}, \quad (3.9)$$

where the second equality is due to the direct sum of sequence of Hilbert spaces, the third equality comes from (3.7a). Note that, for the purpose of simplicity, we omitted the label for the different types of scalar product. We denote by $\mathcal{E}(\mathcal{H})$ the space of finite linear combinations of exponential vectors, i.e., $\mathcal{E}(\mathcal{H}) := \mathrm{span}\{e(u) | \, u \in \mathcal{H}\}$. This space is called exponential domain and is dense in $\Gamma_s(\mathcal{H})$ (see [Att, Chapter 8.2]). Moreover, the generators $e(u)$ of $\mathcal{E}(\mathcal{H})$ are linearly independent and $\mathcal{E}(\mathcal{H})$ is total [5] in $\Gamma_s(\mathcal{H})$. Then we show an important property of the symmetric Fock space carried by the exponential vectors (3.8) satisfying the relation (3.9).

Theorem 3.2.2 (Exponential property [Par12, Proposition 19.6])**.** *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces. Then there exists a unique unitary isomorphism $U : \Gamma_s(\mathcal{H}_1 \oplus \mathcal{H}_2) \to \Gamma_s(\mathcal{H}_1) \otimes \Gamma_s(\mathcal{H}_2)$ satisfying the relation*

$$Ue(u \oplus v) = e(u) \otimes e(v), \quad \forall u \in \mathcal{H}_1, \forall v \in \mathcal{H}_2.$$

Because of this exponential property, the symmetric Fock spaces are often considered as "exponentials of Hilbert spaces".

Creation and annihilation operators Now, let us introduce two important operators on symmetric Fock spaces by following the approach of Parthasarathy [Par12, Chapter II.20] creation and annihilation operators, which are discussed heuristically in Chapter 2.

Given a group of rigid motions of a given Hilbert space $\mathcal{H}$, any element of such group can be described by a pair (u, U), where $u \in \mathcal{H}$ and U is a unitary operator on $\mathcal{H}$. The action of the pair (u, U) on $v \in \mathcal{H}$ is given by

$$(u, U)v = Uv + u,$$

then v is "rotated" by U and "translated" by u. We define a family of operators, so-called Weyl operators $W(u, U)$, by their action on the exponential vectors $e(v)$ for all $v \in \mathcal{H}$,

$$W(u, U)e(v) = e^{-\langle u,Uv\rangle - \|u\|^2/2} e(Uv + u). \quad (3.10)$$

By the linear independence of the different exponential vectors, we can specify the action of $W(u, U)$ on the exponential domain $\mathcal{E}(\mathcal{H})$. Moreover, for $v, h \in \mathcal{H}$, we have

$$\begin{aligned} &\langle W(u, U)e(h), W(u, U)e(v)\rangle = e^{-\langle u,Uv\rangle - \langle Uh,u\rangle - \|u\|^2} \langle e(Uh + u), e(Uv + u)\rangle \\ &= e^{-\langle u,Uv\rangle - \langle Uh,u\rangle - \|u\|^2} e^{\langle Uh+u,Uv+u\rangle} = e^{\langle Uh,Uv\rangle} = e^{\langle h,v\rangle} = \langle e(h), e(v)\rangle \end{aligned}$$

where the second and the last equality are due to (3.9). It implies that a Weyl operator $W(u, U)$ is an isometry of $\mathcal{E}$ onto itself. Since $\mathcal{E}(\mathcal{H})$ is total in $\Gamma_s(\mathcal{H})$, then by [Par12,

5. A subset $\mathcal{E} \subset \mathcal{X}$ is total in $\mathcal{X}$ if the smallest closed subspace containing $\mathcal{E}$ is $\mathcal{X}$.

Proposition 7.2], $W(u,U)$ can be uniquely extend to a unitary operator on the entire $\Gamma_s(\mathcal{H})$. From (3.10), we can obtain Weyl commutation relation, which is the more general form than (3.5),

$$W(u_1,U_1)W(u_2,U_2) = e^{-i\mathrm{Im}\langle u_1,U_1u_2\rangle}W\big((u_1,U_1)(u_2,U_2)\big). \tag{3.11}$$

Let us consider two special cases, take $(u_1,U_1) = (u,\mathbb{1})$ and $(u_2,U_2) = (0,U)$, then we have

$$W(u,U) = W(u,\mathbb{1})W(0,U).$$

The operator $\Lambda(U) := W(0,U)$ is called the second quantization of U and it can lead to the concept of quantum Poisson process (gauge process). More details can be found in [Par12, Mey06]. We denote $W_u := W(u,\mathbb{1})$, which corresponds to the translation by the vector u. Since $W_{su}W_{tu} = W_{(s+t)u}$, we may define a one-parameter unitary group W_{tu} for $t\in\mathbb{R}$, which is strongly continuous. By Stone's theorem (Theorem 3.1.11), for any $u\in\mathcal{H}$, there exists a self-adjoint operator (random variable or observable) $B(u)$ such that

$$W_{tu} = e^{-itB(u)}, \quad \forall t\in\mathbb{R}, \tag{3.12}$$

where $B(u)$ is called the field operator. As we mentioned above [RS80, Chapter VIII], for $u,v\in\mathcal{H}$, $B(v)$ and $B(u)$ commute if and only if $W(tu)$ and $W(tv)$ commute. Due to Weyl commutation relation (3.11), if $\langle u,v\rangle$ is real, then $W(tu)$ and $W(tv)$ commute. Moreover, for all $u,v,h\in\mathcal{H}$, we have the following relation [Par12, Proposition 20.4],

$$[B(u),B(v)]e(h) = 2i\mathrm{Im}\langle u,v\rangle e(h), \tag{3.13}$$

which is a consequence of applying Weyl commutation relation (3.11). Based on the above discussion, we define the annihilation operator associated with $u\in\mathcal{H}$ by

$$a(u) := \frac{-B(iu)+iB(u)}{2}, \tag{3.14}$$

and define the creation operator associated with $u\in\mathcal{H}$ by

$$a^\dagger(u) := \frac{-B(iu)-iB(u)}{2}. \tag{3.15}$$

The creation and annihilation have the following properties

$$a(u)e(v) = \langle u,v\rangle e(v), \tag{3.16a}$$
$$a^\dagger(u)e(v) = \big(de(v+tu)/dt\big)|_{t=0}, \tag{3.16b}$$
$$\langle e(h),a(u)e(v)\rangle = \langle u,v\rangle\langle e(h),e(v)\rangle, \tag{3.16c}$$
$$\langle e(h),a^\dagger(u)e(v)\rangle = \langle h,u\rangle\langle e(h),e(v)\rangle, \tag{3.16d}$$
$$\langle e(h),a(u)e(v)\rangle = \langle a^\dagger(u)e(h),e(v)\rangle, \tag{3.16e}$$
$$[a(u),a(v)]e(h) = [a^\dagger(u),a^\dagger(v)]e(h) = 0, \tag{3.16f}$$
$$[a(u),a^\dagger(v)]e(h) = \langle u,v\rangle e(h). \tag{3.16g}$$

The proof of above properties can be found in [Par12, Chapter II.20]. Then the next properties explain why $a(u)$ and $a^\dagger(u)$ are called the annihilation and creation operators associated with u,

$$a(u)e(0) = 0,\ a(u)v^{\otimes N} = \sqrt{N}\langle u,v\rangle v^{\otimes N-1}, \tag{3.17a}$$
$$a^\dagger(u)v^{\otimes N} = \frac{1}{\sqrt{N+1}}\sum_{n=0}^{N}\left(v^{\otimes n}\otimes u\otimes v^{\otimes(N-n)}\right). \tag{3.17b}$$

Equation (3.17) shows that $a(u)$ is a mapping from $\mathcal{H}^{\circ N}$ to $\mathcal{H}^{\circ N-1}$ and $a^\dagger(u)$ is a mapping from $\mathcal{H}^{\circ N}$ to $\mathcal{H}^{\circ N+1}$.

Quantum stochastic processes Now, we focus on the Hilbert space $L^2(\mathbb{R}_+)$. For $0 \leq s \leq t < \infty$ and $u \in L^2(\mathbb{R}_+)$, we suppose that there are no jump points and adopt the notations used in [BvHJ07] such that

$$u_{s]} := u\mathbb{1}_{[0,s]}, \quad u_{[s,t]} := u\mathbb{1}_{[s,t]}, \quad u_{[t} := u\mathbb{1}_{[t,\infty)}.$$

Moreover, we have the following decomposition of the Hilbert space $L^2(\mathbb{R}_+)$,

$$L^2(\mathbb{R}_+) = L^2([0,s]) \oplus L^2([s,t]) \oplus L^2([t,\infty)).$$

Thus, for every $u \in L^2(\mathbb{R}_+)$ can be uniquely written as a sum

$$u = u_{s]} + u_{[s,t]} + u_{[t},$$

where $u_{s]} \in L^2([0,s])$, $u_{[s,t]} \in L^2([s,t])$ and $u_{[t} \in L^2([t,\infty))$. Due to the exponential property of the Fock space, Theorem 3.2.2,

$$\Gamma_s\big(L^2(\mathbb{R}_+)\big) = \Gamma_s\big(L^2([0,s])\big) \otimes \Gamma_s\big(L^2([s,t])\big) \otimes \Gamma_s\big(L^2([t,\infty))\big),$$

where the identification between the left and the right sides is established through a unique unitary isomorphism. Next, we define the following von Neumann algebras [Par12, Chapter III.24],

$$\mathscr{B}_{s]} := \mathscr{B}\Big(\Gamma_s\big(L^2([0,s])\big)\Big), \quad \mathscr{B}_{[s,t]} := \mathscr{B}\Big(\Gamma_s\big(L^2([s,t])\big)\Big), \quad \mathscr{B}_{[t} := \mathscr{B}\Big(\Gamma_s\big(L^2([t,\infty))\big)\Big),$$

which realize a decomposition of the von Neumann algebra $\mathscr{B}_{\mathbb{R}_+} := \mathscr{B}\Big(\Gamma_s\big(L^2(\mathbb{R}_+)\big)\Big)$, the set of all bounded operators on $\Gamma_s\big(L^2(\mathbb{R}_+)\big)$,

$$\mathscr{B}_{\mathbb{R}_+} = \mathscr{B}_{s]} \otimes \mathscr{B}_{[s,t]} \otimes \mathscr{B}_{[t}. \tag{3.18}$$

A quantum process $\{X_t | \, t \in \mathbb{R}_+\}$, i.e., a one-parameter family of self-adjoint operators, is called adapted if X_t is affiliated to $\mathscr{B}_{s]}$ for every $t \in \mathbb{R}_+$, equivalently it is of the form $X_{t]} \otimes \mathbb{1}_{[t}$ as an operator on $\Gamma_s\big(L^2([0,t])\big) \otimes \Gamma_s\big(L^2([t,\infty))\big)$. This concept plays the same role as, a stochastic process adapted to a filtration, in the classical probability theory.

Let us revisit the field operator $B(u)$. We define the following processes, for $t \in \mathbb{R}_+$,

$$Q_t := -B(i\mathbb{1}_{[0,t]}), \quad P_t := B(\mathbb{1}_{[0,t]}).$$

Since $P_{t_4} - P_{t_3}$ commute with $P_{t_2} - P_{t_1}$, due to (3.13) for $0 \leq t_1 \leq t_2 \leq t_3 \leq t_4 < \infty$, we can define the joint probability distribution similar to (3.3) of these two commuting random variables. For $x, y \in \mathbb{R}$, the corresponding characteristic function is given by characteristic function is given by the following formula. For

$$\begin{aligned}
&\big\langle e(0), e^{ix(P_{t_4}-P_{t_3})} e^{iy(P_{t_2}-P_{t_1})} e(0)\big\rangle \\
&= \big\langle e(0), e^{ixB(\mathbb{1}_{[t_3,t_4]})} e^{iyB(\mathbb{1}_{[t_1,t_2]})} e(0)\big\rangle = \big\langle e(0), W(-x\mathbb{1}_{[t_3,t_4]}) W(-y\mathbb{1}_{[t_1,t_2]}) e(0)\big\rangle \\
&= \big\langle e(0), W(-x\mathbb{1}_{[t_3,t_4]} - y\mathbb{1}_{[t_1,t_2]}) e(0)\big\rangle = e^{-\left\|x\mathbb{1}_{[t_3,t_4]} + y\mathbb{1}_{[t_1,t_2]}\right\|^2/2} \big\langle e(0), e(-x\mathbb{1}_{[t_3,t_4]} - y\mathbb{1}_{[t_1,t_2]})\big\rangle \\
&= e^{-\left\|x\mathbb{1}_{[t_3,t_4]} + y\mathbb{1}_{[t_1,t_2]}\right\|^2/2} = e^{-x^2(t_4-t_3)/2} e^{-y^2(t_2-t_1)/2},
\end{aligned}$$

where we have used the relations (3.18), (3.12), (3.11), (3.10) and (3.9). The above formula implies that the process P_t has independent increments and $P_t - P_s \sim \mathcal{N}(0, t-s)$ for $0 \leq s \leq t < \infty$ under vacuum vector. Moreover, we have $P_0 = 0$, and by double commutant technique we can construct a commutative von Neumann algebra, so that we can represent the process B_t on a single probability space. Therefore, P_t defines a Wiener process (Brownian motion) under the vacuum vector. By the same argument, we can show that Q_t also defines a Wiener process under the vacuum vector, but in a different probability space with respect to the one defined by P_t, since Q_t do not commute with P_t. Hence, the quantum probability space $\left(\mathscr{B}_{\mathbb{R}_+}, \boldsymbol{\varphi}_f\right)$ with $\boldsymbol{\varphi}_f : X \in \mathscr{B}_{\mathbb{R}_+} \mapsto \langle e(0), Xe(0)\rangle$, admits the above-mentioned two quantum Wiener processes. In quantum optics, these two non-commuting processes can be observed by measuring the vacuum via the homodyne detection [Jac14].

Based on the two quantum Wiener processes, the creation operator (3.15) and the annihilation operator (3.14), let us introduce two fundamental quantum noises

$$A_t := \frac{Q_t + iP_t}{2}, \quad A_t^\dagger := \frac{Q_t - iP_t}{2},$$

where A_t is called annihilation process and $A_t^\dagger$ is called creation process, which play the important role in the quantum stochastic calculus and the quantum stochastic integral.

3.2.2 Classical and quantum stochastic calculus

Classical stochastic calculus Let us first briefly review some important concept of classical stochastic calculus [Øks03, CW90, LG16, RY13]. Consider the one dimensional Wiener process W_t, which can be characterized by the following four facts

1. $W_0 = 0$;
2. W_t is almost surely continuous;
3. W_t has independent increments;
4. $(W_t - W_s) \sim \mathcal{N}(0, t-s)$, for $0 \leq s \leq t$.

Moreover, a celebrated theorem by Paley, Wiener and Zygmund states that, for every $\omega \in \Omega$, the Wiener sample path $W_\cdot(\omega)$ is nowhere differentiable. Thus instead of considering the differential equation formulation, it is more reasonable to describe some physical behavior by means of an integral equation

$$X_T = X_0 + \int_0^T f(t,\omega)dt + \int_0^T g(t,\omega)dW_t.$$

Hence, we need to define a stochastic integral "$\int_0^T g(t,\omega)dW_t$". This type of integral is called Itô stochastic integral, and it is a stochastic generalization of the Riemann-Stieltjes integral [RS80] in analysis. Let $\mathbb{V}$ denote the class of all simple predictable processes (see [CW90, Chapter 2.4]), $\phi(t,\omega) : \mathbb{R}_+ \times \Omega \to \mathbb{R}$, of the following form

$$\phi(t,\omega) = \sum_{n=1}^{N-1} \alpha_n(\omega) \mathbb{1}_{(t_n, t_{n+1}]}(t),$$

where $0 = t_0 \leq \cdots \leq t_N = T$ is a partition of $[0, T]$ and $\alpha_n(\omega)$ is a bounded $\mathcal{F}_{t_n}$-measurable random variable. For $\phi \in \mathbb{V}$, we define the Itô stochastic integral as

$$\int_0^T \phi(t,\omega)dW_t = \sum_{n=1}^{N-1} \alpha_n(\omega)\big(W_{t_{n+1}} - W_{t_n}\big)(\omega).$$

By a straightforward calculation, for all $\phi \in \mathbb{V}$, we can obtain the following properties, which are called Itô isometry,

$$\mathbb{E}\left(\int_0^T \phi(t,\omega)dW_t\right) = 0, \quad \mathbb{E}\left[\left(\int_0^T \phi(t,\omega)dW_t\right)^2\right] = \mathbb{E}\left(\int_0^T \phi^2(t,\omega)dt\right). \tag{3.19}$$

Furthermore, for each predictable process $g(t,\omega)$ such that $\mathbb{E}\left(\int_0^T g^2(t,\omega)dt\right) < \infty$, there exists a sequence $\phi_N(t,\omega) \in \mathbb{V}$ such that

$$\lim_{N\to\infty} \mathbb{E}\left(\int_0^T |g(t,\omega) - \phi_N(t,\omega)|^2 dt\right) = 0.$$

Then we define Itô integral of $g(t,\omega)$ as below

$$\int_0^T g(t,\omega)dW_t := \lim_{N\to\infty} \int_0^T \phi_N(t,\omega)dW_t,$$

where the limit exists in L^2 due to Itô isometry (3.19). In fact, the domain of the integrand g can be extended to a larger class, more details referred to [Øks03, CW90, LG16, RY13]. Finally, the above-mentioned integral equation of X_t is well defined, and its equivalent differential form is given by

$$dX_t = f(t,\omega)dt + g(t,\omega)dW_t.$$

Then let us introduce the most famous theorem in classical stochastic calculus, Itô formula [Øks03, Theorem 4.1.2].

Theorem 3.2.3 (Itô formula)**.** *Let X_t be an Itô process (Itô stochastic integral), which is given by*

$$dX_t = f(t,X_t)dt + g(t,X_t)dW_t.$$

Let $h(t,x)$ be twice continuously differentiable in x and once in t, then $Y_t = h(t,X_t)$ is also an Itô process and

$$\begin{aligned} dY_t &= \mathscr{L}h(t,X_t)dt + \frac{\partial h}{\partial x}g(t,X_t)dW_t, \\ \mathscr{L}h(t,X_t) &:= \frac{\partial h}{\partial t} + \frac{\partial h}{\partial x}f(t,X_t) + \frac{1}{2}\frac{\partial^2 h}{\partial x^2}(t,X_t)g^2(t,X_t), \end{aligned} \tag{3.20}$$

which is computed according to the following Itô rules

$$dtdt = dtdW_t = dW_t dt = 0, \quad dW_t dW_t = dt.$$

Quantum stochastic calculus Let us return to the quantum case and focus on the complete model, the main system (e.g., atoms) interacting with a quantum electromagnetic field. We have already discussed the two fundamental noises A_t and $A_t^\dagger$ on the von Neumann algebra $\mathscr{B}_{\mathbb{R}_+}$, which can characterize the quantum electromagnetic field. Similar to the classical case, we want to define the quantum stochastic integral (stochastic integration of adapted operator-valued process) with respect to these noises, in view of constructing quantum stochastic differential equations (QSDEs) to describe the behavior of the entire model. Denote $\mathcal{H}_s$ as the associated Hilbert space with the main system. In order to couple the fundamental noises to the main system interacting with the field, we need a "larger" Hilbert space $\mathcal{H} := \mathcal{H}_s \otimes \Gamma_s\big(L^2(\mathbb{R}_+)\big)$ together with the von Neumann algebra $\mathscr{A} := \mathscr{B}(\mathcal{H}_s) \otimes \mathscr{B}_{\mathbb{R}_+}$.

As in the classical case, let us firstly define a quantum stochastic integral of an adapted simple operator-valued process L_t in $\mathscr{A}$ with respect to M_t, where M_t denotes A_t or $A_t^\dagger$. For $0 \leq s < t < \infty$, by the factorizability property [Par12, Chapter II.25], $M_t - M_s$ is affiliated to $\mathscr{B}_{[s,t]}$, and acts on exponetial vectors as follows

$$(M_t - M_s)e(u) = e(u_{s]}) \otimes \big((M_t - M_s)e(u_{[s,t]})\big) \otimes e(u_{[t}),$$

where $u \in L^2(\mathbb{R}_+)$ and $(M_t - M_s)e(u_{[s,t]}) \in \Gamma_s\big(L^2([s,t])\big)$. This property is related to the notation of classical process with independent increments. The adapted simple operator-valued process L_t means that L_t is affiliated to $\mathscr{B}(\mathcal{H}_s) \otimes \mathscr{B}_{t]}$ for every $t \in \mathbb{R}_+$, and can be written in the following form

$$L_t = \sum_{n=1}^{N-1} L_{t_n} \mathbb{1}_{(t_n, t_{n+1}]}(t),$$

where $0 = t_0 \leq \cdots \leq t_N = T$ is a partition of $[0,T]$. Now, we extend A_t and $A_t^\dagger$ in $\mathcal{H}$ by ampliating, i.e., tensor them with identity on $\mathcal{H}_s$. For the purpose of simplicity, we identify these operators with their ampliations in $\mathcal{H}$. Following [Par12, Chapter II.25] and [HP84], the quantum stochastic integral of L_t with respect to M_t on $\widetilde{\mathcal{H}} := \mathcal{H}_s \otimes \mathcal{E}\big(L^2(\mathbb{R}_+)\big)$ is given by

$$\left(\int_0^T L_t dM_t\right) f \otimes e(u) := \sum_{n=1}^{N-1} \big(L_{t_n} f \otimes e(u_{t_n]})\big) \otimes \big((M_{t_{n+1}} - M_{t_n})e(u_{[t_n, t_{n+1}]})\big) \otimes e(u_{[t_{n+1}}),$$

where $f \in \mathcal{H}_s$ and $e(u) \in \mathcal{E}\big(L^2(\mathbb{R}_+)\big)$.

Following the classical approach, we want to extend the quantum stochastic integral to a large class of adapted integrands, by approximating with simple processes and taking the limit. However, as stated in [RS80, Chapter VIII.7], the abscence of a common domain is especially troublesome for the convergence of unbounded operators. The solution of Hudson and Parthasarathy [HP84] is to define all relative operators (integrands, integrators and integrals) on $\widetilde{\mathcal{H}}$ [6], which can ensure the existence of the limit in a certain sense.

6. In fact, in [HP84], the authors work on $\mathcal{H}_s \otimes \mathcal{D}$ with $\mathcal{D} := \{e(u) | \, u \in L^2(\mathbb{R}_+) \cap L^\infty_{loc}(\mathbb{R}_+)\} \subset \mathcal{E}\big(L^2(\mathbb{R}_+)\big)$, rather than $\widetilde{\mathcal{H}}$ directly, due to the difficulty of defining the quantum stochastic integral with respect to the third fundamental noise Λ_t, which we do not discuss in this thesis. For our case that integrator is A_t or $A_t^\dagger$, it is sufficient to choose the exponential domain $\mathcal{E}\big(L^2(\mathbb{R}_+)\big)$ [Mey06, Chapter VI.1.6].

Now, we briefly introduce the idea of Hudson-Parthasarathy to define a quantum stochastic integral

$$I_T = \int_0^T (F_t dA_t + G_t dA_t^\dagger + H_t dt) \tag{3.21}$$

as a limit of a sequence of integrals

$$I_T^{(N)} = \int_0^T \left(F_t^{(N)} dA_t + G_t^{(N)} dA_t^\dagger + H_t^{(N)} dt\right)$$

for adapted simple processes $F_t^{(N)}$, $G_t^{(N)}$ and $H_t^{(N)}$. Firstly, [HP84, Corollary 1] which plays a role as Itô isometry (3.19) in the classical case provides an estimation of the quantum stochastic integral, for all $f \in \mathcal{H}_s$ and $e(u) \in \mathcal{E}\big(L^2(\mathbb{R}_+)\big)$

$$\|I_T f \otimes e(u)\|^2 \leq C(T,u) \int_0^T \left(\|F_t f \otimes e(u)\|^2 + \|G_t f \otimes e(u)\|^2 + \|H_t f \otimes e(u)\|^2\right) dt, \tag{3.22}$$

where $C(T,u) < \infty$ for $T > 0$. Similar to the classical case, we can define the quantum stochastic integral I_T for more general integrands as a limit of a sequence $I_T^{(N)}$ in the following sense: for all $f \in \mathcal{H}_s$ and $e(u) \in \mathcal{E}\big(L^2(\mathbb{R}_+)\big)$,

$$\lim_{N\to\infty} \left\langle \left(I_T - I_T^{(N)}\right) f \otimes e(u), \left(I_T - I_T^{(N)}\right) f \otimes e(u) \right\rangle = 0. \tag{3.23}$$

By (3.22), the above limit (3.23) exists if there exists sequences $F_t^{(N)}$, $G_t^{(N)}$ and $H_t^{(N)}$, such that for all $f \in \mathcal{H}_s$ and $e(u) \in \mathcal{E}\big(L^2(\mathbb{R}_+)\big)$,

$$\lim_{N\to\infty} \int_0^T \left(\left\| \left(F_t - F_t^{(N)}\right) f \otimes e(u) \right\|^2 + \left\| \left(G_t - G_t^{(N)}\right) f \otimes e(u) \right\|^2 + \left\| \left(H_t - H_t^{(N)}\right) f \otimes e(u) \right\|^2 \right) dt = 0.$$

Finally, [HP84, Proposition 3.2] shows that every adapted square-integrable process such that for any $T < \infty$,

$$\int_0^T \|L_t f \otimes e(u)\|^2 dt < \infty, \quad \forall f \in \mathcal{H}_s \text{ and } e(u) \in \mathcal{E}\big(L^2(\mathbb{R}_+)\big),$$

admits a suitable approximation by simple processes. The quantum stochastic integral for the adapted square-integrable processes on $\widetilde{\mathcal{H}}$ is well-defined.

Now let us continue to follow the approach of Hudson-Parthasarathy to obtain the quantum analogue of Itô formula on $\widetilde{\mathcal{H}}$, by introducing the first fundamental lemma and the second fundamental lemma [Par12, Chapter II.25] and [HP84, Hud03]. By considering a quantum stochastic integral as a "matrix" intuitively, the first fundamental lemma displays its matrix elements. It can be easily proved with applying (3.16) and above-mentioned approximating procedure by simple processes.

Theorem 3.2.4 (First fundamental lemma)**.** *Let*

$$I_T = \int_0^T (F_t dA_t + G_t dA_t^\dagger + H_t dt)$$

where F_t, G_t and H_t are adapted square-integrable processes. Then, for all $f, g \in \mathcal{H}_s$, $e(u), e(v) \in \mathcal{E}\big(L^2(\mathbb{R}_+)\big)$ and $v, u \in L^2(\mathbb{R}_+)$,

$$\langle f \otimes e(u), I_T g \otimes e(v)\rangle = \int_0^T \big\langle f \otimes e(u), \big(F_t v(t) + u^*(t)G_t + H_t\big) g \otimes e(v)\big\rangle \, dt \tag{3.24a}$$

$$= \int_0^T \left\langle f \otimes e(u), \begin{bmatrix}1, u^*(t)\end{bmatrix} \begin{bmatrix}H_t & F_t \\ G_t & 0\end{bmatrix} \begin{bmatrix}1 \\ v(t)\end{bmatrix} g \otimes e(v)\right\rangle dt. \tag{3.24b}$$

In order to avoid the domain problem of the "multiplication" of two integrals I_T and I'_T, Hudson and Parthasarathy study the scalar product

$$\langle I'_T f \otimes e(u), I_T g \otimes e(v)\rangle, \qquad \forall f, g \in \mathcal{H}_s, \quad e(u), e(u) \in \mathcal{E}\big(L^2(\mathbb{R}_+)\big) \text{ and } u, v \in L^2(\mathbb{R}_+),$$

which is always well-defined, and provides the second fundamental lemma, which can also be easily proved by similar approach as the first one.

Theorem 3.2.5 (Second fundamental lemma)**.** *Let*

$$I_T = \int_0^T (F_t dA_t + G_t dA_t^\dagger + H_t dt), \quad I'_T = \int_0^T (F'_t dA_t + G'_t dA_t^\dagger + H'_t dt),$$

where F_t, G_t, H_t, F'_t, G'_t and H'_t are adapted square-integrable processes. Then, for all $f, g \in \mathcal{H}_s$, $e(u), e(v) \in \mathcal{E}\big(L^2(\mathbb{R}_+)\big)$ and $u, v \in L^2(\mathbb{R}_+)$,

$$\langle I'_T f \otimes e(u), I_T g \otimes e(v)\rangle$$

$$= \int_0^T \big\langle I'_T f \otimes e(u), \big(F_t v(t) + u^*(t)G_t + H_t\big) g \otimes e(v)\big\rangle \, dt \tag{3.25a}$$

$$+ \int_0^T \big\langle \big(F'_t u(t) + v^*(t)G'_t + H'_t\big) f \otimes e(u), I_T g \otimes e(v)\big\rangle \, dt \tag{3.25b}$$

$$+ \int_0^T \langle G'_t f \otimes e(u), G_t g \otimes e(v)\rangle \, dt. \tag{3.25c}$$

Now, let us translate the second fundamental lemma to the more explicit quantum Itô rule. We need to introduce the concept of adjoint-pair: two operators L and $L^\dagger$ are said to be an adjoint-pair, if for all $f, g \in \mathcal{H}_s$, $e(u), e(v) \in \mathcal{E}\big(L^2(\mathbb{R}_+)\big)$ and $u, v \in L^2(\mathbb{R}_+)$,

$$\langle f \otimes e(u), Lg \otimes e(v)\rangle = \langle L^\dagger f \otimes e(u), g \otimes e(v)\rangle.$$

Suppose (F_t, G_t, H_t) and $(F_t^\dagger, G_t^\dagger, H_t^\dagger)$ are three adjoint-pairs, if (F_t, G_t, H_t) are adapted square-integrable, then $(F_t^\dagger, G_t^\dagger, H_t^\dagger)$ are also adapted square-integrable. Define

$$I_T^\dagger = \int_0^T (G_t^\dagger dA_t + F_t^\dagger dA_t^\dagger + H_t^\dagger dt).$$

By the properties (3.16) applied to A_t and $A_t^\dagger$, we can easily show that I_t and $I_t^\dagger$ are adjoint-pairs as well. Under the setting of Theorem 3.2.5, we use the following notation

$$dI_t = F_t dA_t + G_t dA_t^\dagger + H_t dt, \quad dI'_t = F'_t dA_t + G'_t dA_t^\dagger + H'_t dt,$$

to describe the two quantum stochastic integrals

$$I_T - I_0 = \int_0^T (F_t dA_t + G_t dA_t^\dagger + H_t dt), \quad I'_T - I'_0 = \int_0^T (F'_t dA_t + G'_t dA_t^\dagger + H'_t dt).$$

Thus, we have

$$\langle I'_T f \otimes e(u), I_T g \otimes e(v)\rangle = \langle f \otimes e(u), (I'_T)^\dagger I_T g \otimes e(v)\rangle. \tag{3.26}$$

Let us exchange the role of I'_T and $(I'_T)^\dagger$ in (3.26), by Theorem 3.2.5, we can write (3.26) in the following matrix form,

$$\begin{aligned}&\langle f \otimes e(u), I'_T I_T g \otimes e(v)\rangle \\ &= \int_0^T \left\langle f \otimes e(u), [1, u^*(t)] \begin{bmatrix} I'_t H_t + H'_t I_t + F'_t G_t & I'_t F_t + F'_t I_t \\ I'_t G_t + G'_t I_t & 0 \end{bmatrix} \begin{bmatrix} 1 \\ v(t) \end{bmatrix} g \otimes e(v) \right\rangle dt.\end{aligned} \tag{3.27}$$

By Theorem 3.2.4 and Theorem 3.2.5, we can show that $I'_T I_T$ is still a quantum stochastic integral of the form (3.21). Then, compare the matrix in (3.27) to the one in (3.24b), we can obtain the coefficients of the integral $(I'_T I_T)$ with respect to the three integrators dA_t, $dA_t^\dagger$ and dt, such that

$$d(I'_t I_t) = (I'_t H_t + H'_t I_t + F'_t G_t)dt + (I'_t F_t + F'_t I_t)dA_t + (I'_t G_t + G'_t I_t)dA_t^\dagger.$$

Finally, we can obtain the following explicit form of the quantum Itô rule [HP84, Theorem 4.5].

Theorem 3.2.6 (Quantum Itô rule)**.** *Let I_t and I'_t be quantum stochastic integrals of the form*

$$dI_t = F_t dA_t + G_t dA_t^\dagger + H_t dt, \quad dI'_t = F'_t dA_t + G'_t dA_t^\dagger + H'_t dt,$$

where F_t, G_t, H_t, F'_t, G'_t and H'_t are adapted square-integrable processes, and are all bounded in the sense that $\sup_{s\le t} \|\cdot\| < \infty$ for all finite t. Then

$$d(I'_t I_t) = I'_t dI_t + (dI'_t) I_t + dI'_t dI_t,$$

where $I'_t dI_t = I'_t F_t dA_t + I'_t G_t dA_t^\dagger + I'_t H_t dt$, $(dI'_t) I_t = F'_t I_t dA_t + G'_t I_t dA_t^\dagger + H'_t I_t dt$ and $dI'_t dI_t = F'_t G_t dt$ are evaluated according to the following rules

$dI'_t \setminus dI_t$	dA_t	$dA_t^\dagger$	dt
dA_t	0	dt	0
$dA_t^\dagger$	0	0	0
dt	0	0	0

.

3.2.3 Quantum stochastic differential equations

Now, we are ready to discuss the evolution of the main system (e.g., a collection of atoms) interacting with an electromagnetic field. This evolution can be described by a unitary operator on the composite space $\mathcal{H}_s \otimes \Gamma_s\big(L^2(\mathbb{R}_+)\big)$. From physics point of view, by the suitable assumptions and approximations (see [WM09] for more details, and [vHSM05b]

for a brief introduction), the free electromagnetic field along the z-axis is described by a stationary Gaussian wide-band noise $\tilde{a}(t,z)$. If we assume that the atoms are localized at $z=0$, then the dynamics of the open system on $\mathcal{H}_s \otimes \Gamma_s\big(L^2(\mathbb{R}_+)\big)$ can be described by the following differential equation

$$\frac{d}{dt}\widetilde{U}(t) = \big(-iH(t) + L\tilde{a}^*(t,0) - L^*\tilde{a}(t,0)\big)\widetilde{U}(t), \quad \widetilde{U}(0) = \mathbb{1}, \tag{3.28}$$

where we omit the symbol of tensor product and set $\hbar = 1$. The bounded self-adjoint operator $L \in \mathscr{B}(\mathcal{H}_s)$ is the atomic operator specifying the interaction between the atoms and the field, and the bounded self-adjoint operator $H(t) \in \mathscr{B}(\mathcal{H}_s)$ for $t \in \mathbb{R}_+$ represents the time-dependent Hamiltonian of the main system. $H(t)$ can be taken of the form $H_f + u(t)H_c$, where $H_f = H_f^*$ represents the free atomic Hamiltonian and $H_c = H_c^*$ represents the control atomic Hamiltonian modulated by a deterministic (open-loop) scalar control input $u_t \in \mathbb{R}$. By an approximation procedure (see [vHSM05b] and the references therein), Equation (3.28) may be replaced with the following quantum stochastic differential equation (QSDE) driven by the two fundamental quantum noises A_t and $A_t^\dagger$,

$$dU_t = \left(LdA_t^\dagger - L^*dA_t - \frac{1}{2}L^*Ldt - iH(t)dt\right)U_t, \quad U_0 = \mathbb{1}, \tag{3.29}$$

where we identify the operators L, H_f and H_c with their ampliations on $\mathcal{H}_s \otimes \Gamma_s\big(L^2(\mathbb{R}_+)\big)$. The existence and uniqueness of this QSDE can be ensured by a Picard iteration argument [HP84, Par12, Mey06]. Define

$$dU_t^* = U_t^*\left(L^*dA_t - LdA_t^\dagger - \frac{1}{2}L^*Ldt + iH(t)dt\right), \quad U_0^* = \mathbb{1},$$

by the quantum Itô rule, Theorem 3.2.6, we can show that

$$d(U_t^*U_t) = d(U_tU_t^*) = 0, \quad U_0^*U_0 = U_0U_0^* = \mathbb{1},$$

thus the solution of (3.29) U_t is unitary for all t, which is consistent with the case of the solution of the Schrödinger equation. Then we use such U_t to describe the evolution of the open quantum system. For every atomic observable X on $\mathcal{H}_s$, in Heisenberg picture, the time evolution of this observable X is given by a flow

$$j_t : X \otimes \mathbb{1}_\Gamma \mapsto U_t^*(X \otimes \mathbb{1}_\Gamma)U_t \tag{3.30}$$

where $\mathbb{1}_\Gamma$ denotes the identity on $\Gamma_s\big(L^2(\mathbb{R}_+)\big)$. By applying quantum Itô rule and identifying X and $X \otimes \mathbb{1}_\Gamma$, we have

$$dj_t(X) = j_t\big(\mathcal{L}^*(X)\big)dt + j_t\big([L^*, X]\big)dA_t + j_t\big([X, L]\big)dA_t^\dagger, \tag{3.31}$$

where

$$\mathcal{L}(X) := i[H(t), X] + L^*XL - \frac{1}{2}(L^*LX + XL^*L) \tag{3.32}$$

is called Lindblad generator [Lin76].

Equation (3.31) can describe the time evolution of the observable of the main system (atoms) interaction with the field, in the Heisenberg picture. Such interaction can be viewed as a noisy driving force. Now, let us focus on the influence of the main systems on

the quantum field, which is measured by homodyne detection. Note that, in real experiments [Bel92, vHSM05b, vHSM05a], the (homodyne) detection cannot be implemented perfectly, which means that there exists another additional noise $\tilde{A}_t$ independent of A_t that does not interact with the main system. Such noise $\tilde{A}_t$ is called corrupting noise. Then we extend the operators L, H_0, H_1, U_t, U_t^*, A_t, $A_t^\dagger$, $\tilde{A}_t$ and $\tilde{A}_t$ onto $\mathcal{H}_s \otimes \Gamma_s\big(L^2(\mathbb{R}_+)\big) \otimes \Gamma_s\big(L^2(\mathbb{R}_+)\big)$ by tensoring them with identities. Following [Bar06], which deals with the case of quantum optical measurements, the output noise Z_t on $\mathcal{H}_s \otimes \Gamma_s\big(L^2(\mathbb{R}_+)\big) \otimes \Gamma_s\big(L^2(\mathbb{R}_+)\big)$ is given by

$$Z_t = A_t + A_t^\dagger + \kappa(\tilde{A}_t + \tilde{A}_t^\dagger), \quad \kappa \geq 0, \tag{3.33}$$

and therefore the associated observation process Y_t is given by

$$Y_t = U_t^* Z_t U_t = U_t^*(A_t + A_t^\dagger)U_t + \kappa(\tilde{A}_t + \tilde{A}_t^\dagger). \tag{3.34}$$

From the physics point of view, we measure the field observable $A_t + A_t^\dagger$ after interaction with the main system, corrupted by the uncorrelated noises $\tilde{A}_t$ and $\tilde{A}_t^\dagger$. Due to quantum Itô formula (Theorem 3.2.6) we have

$$dY_t = j_t(L + L^*)dt + dA_t + dA_t^\dagger + \kappa(d\tilde{A}_t + d\tilde{A}_t^\dagger), \tag{3.35}$$

where j_t is ampliated in such a way that $j_t : X \otimes \mathbb{1}_\Gamma \otimes \mathbb{1}_\Gamma \mapsto U_t^*(X \otimes \mathbb{1}_\Gamma \otimes \mathbb{1}_\Gamma)U_t$, and Y_t may be interpreted as a noisy observation process of $j_t(L + L^*)$. By using the double commutant technique, we construct a commutative von Neumann algebra $\mathcal{Y}_t$ generated by the observation process Y_s for $s \leq t$, given by

$$\mathcal{Y}_t = \mathrm{vN}\{Y_s |\, s \leq t\} := \big(\{Y_s |\, s \leq t\} \cup \{Y_s^* |\, s \leq t\}\big)''.$$

In the following, we state two fundamental properties of Y_t and $\mathcal{Y}_t$.

Proposition 3.2.7 (Nondemolition property [BvH08, Proposition 2.1])**.** *The observation process Y_t satisfies the self-nondemolition condition, i.e., $\mathcal{Y}_t$ is commutative for all $t \in [0, T]$ with $T < \infty$, and is nondemolition with respect to the flow, i.e., $j_t(X) \in \mathcal{Y}_t'$ for all $X \in \mathscr{B}(\mathcal{H}_s)$ and $t \in [0, T]$.*

The self-nondemolition of the observation process implies that Y_t can be considered as a classical stochastic process. Moreover, the non-demolition property ensures the existence of the conditional expectation $\boldsymbol{\varphi}\big(j_t(X)|\mathcal{Y}_t\big)$ on the quantum probability space $(\mathcal{A}, \boldsymbol{\varphi})$ with $\mathcal{A} = \mathscr{B}(\mathcal{H}_s) \otimes \mathscr{B}_{\mathbb{R}_+} \otimes \mathscr{B}_{\mathbb{R}_+}$ and $\boldsymbol{\varphi} = \boldsymbol{\varphi}_s \otimes \boldsymbol{\varphi}_f \otimes \boldsymbol{\varphi}_f$, where $\boldsymbol{\varphi}_s(\cdot) = \mathrm{Tr}(\rho\,\cdot)$ for a fixed density operator of the main system ρ on $\mathcal{H}_s$, and $\boldsymbol{\varphi}_f(\cdot) = \langle e(0), \cdot\, e(0)\rangle$. Due to Lemma 3.1.9, such conditional expectation $\boldsymbol{\varphi}\big(j_t(X)|\mathcal{Y}_t\big)$ provides the "best" estimation for $j_t(X)$ on $\mathcal{Y}_t$.

We have already discussed the dynamics of an open quantum system where the atmoic Hamiltonian $H(t) = H_f + u(t)H_c$ is modulated by an open-loop scalar control input. Then we follow the approach of [BvH08] to introduce the dynamics of an open quantum system in a feedback control scenario. Denote $\mathcal{Z}_t = \mathrm{vN}\{Z_s |\, s \leq t\}$ as the commutative von Neumann algebra generated by the output noise Z_t defined by (3.33). Next, we define the atomic Hamiltonian as $H_t := H_f + u(Z_{s\leq t})H_c$, where $u(Z_{s\leq t})$ is a bounded real scalar function of the output noise up to time t, and H_t is affiliated to $\mathcal{Z}_t$. Then the controlled quantum stochastic differential equation is given by

$$dU_t = \left(LdA_t^\dagger - L^* dA_t - \frac{1}{2}L^*L dt - iH_t dt\right)U_t, \quad U_0 = \mathbb{1}.$$

By similar calculation to those in the open-loop case (3.31), for an atomic observable $X \in \mathscr{B}(\mathcal{H}_s)$, which is identified with $X \otimes \mathbb{1}_\Gamma \otimes \mathbb{1}_\Gamma$, we have

$$\begin{aligned} dj_t(X) =& j_t\big([L^*, X]\big)dA_t + j_t\big([X, L]\big)dA_t^\dagger + \frac{1}{2}j_t\big(2L^*XL - (L^*LX + XL^*L)\big)dt \\ &+ j_t\big(i[H_f, X]\big)dt + u(Y_{s\leq t})j_t\big(i[H_c, X]\big)dt, \end{aligned} \tag{3.36}$$

where we used the fact that $j_t\big(i[u(Z_{s\leq t})H_c, X]\big) = u(Y_{s\leq t})j_t\big(i[H_c, X]\big)$ with Y_t defined by (3.34), and the atomic Hamiltonian is modulated by a bounded real scalar function of the observation history.

3.3 From classical to quantum filtering theory

In this section, we want to use the information obtained from the observation process to estimate the dynamics of open quantum system. By applying the quantum filtering theory proposed by Belavkin [Bel92], which is close to the classical stochastic filtering theory [Xio08, Kal13], we can obtain the quantum filtering equation of Belavkin, which is also called stochastic master equation.

3.3.1 Classical stochastic filtering theory

Here, we focus on a simple signal-observation model : the real one-dimensional signal process x_t, which is what we want to estimate, and the real one-dimensional observation process y_t satisfies

$$\begin{aligned} dx_t &= b(x_t)dt + c(x_t)dB_t, \\ dy_t &= h(x_t)dt + dB_t, \end{aligned}$$

where B_t is the Brownian motion and the mappings b, c and h are bounded and Lipschitz continuous. We want to estimate the signal x_t based on the available information up to time t, $\mathcal{F}_t^y$. The "best" estimator of the signal $\hat{x}_t$ is given by Lemma 3.1.2,

$$\hat{x}_t = \mathbb{E}\big(x_t|\mathcal{F}_t^y\big).$$

Note that, $\mathbb{E}\big(x_t|\mathcal{F}_t^y\big)$ is $\mathcal{F}_t^y$-measurable. Normally, a function of the signal $f(x_t)$ may be more interesting than the signal x_t itself to estimate. Unless $f(\cdot)$ is linear, $\widehat{f(x_t)} = f(\hat{x}_t)$, in the other cases, we have to find the best estimator of $f(x_t)$. For all $\omega \in \Omega$, consider $\mathbb{P}(x_t \in \cdot|\mathcal{F}_t^y)(\omega)$ as a probability measure. Then, by the monotone convergence theorem, the "best" estimator of $f(x_t)$ is given by $\mathbb{E}\big(f(x_t)|\mathcal{F}_t^y\big)$. Denote $L_t := -\int_0^t h(x_t)dB_t$ which a continuous local martingale, since $h(\cdot)$ is bounded. By [LG16, Theorem 5.23], the stochastic process (see Definition A.1.1), $\mathfrak{E}(L_t)$ given by

$$\mathfrak{E}(L_t) = \exp\left(-\int_0^t h(x_t)dB_t - \frac{1}{2}\int_0^t h^2(x_s)ds\right)$$

is a uniformly integrable martingale. Set $\mathbb{Q}$ as the measure on Ω that is absolutely continuous with respect to $\mathbb{P}$, such that its Radon-Nickodym derivative on $(\Omega, \mathcal{F}_t^B)$ is

$$\frac{d\mathbb{Q}}{d\mathbb{P}}\bigg|_{\mathcal{F}_t^B} = \mathfrak{E}(L_t).$$

In virtue of Girsanov's theorem (Theorem A.1.2) and Levy's characterization,

$$B_t - [B, L]_t = B_t + \int_0^t h(x_t) dB_t = y_t$$

is a $\mathcal{F}_t^B$ Brownian motion under the probability measure $\mathbb{Q}$, where $[\cdot, \cdot]_t$ denotes the quadratic (co)variation defined in (A.1). Set

$$M_t := \mathfrak{E}^{-1}(L_t) = \left.\frac{d\mathbb{P}}{d\mathbb{Q}}\right|_{\mathcal{F}_t^B},$$

by applying Bayes' formula (3.1.3), we can obtain the formula that calculates the conditional expectation under the $\mathbb{P}$ by calculating the ones under the $\mathbb{Q}$, which is called Kallianpur-Striebel formula,

$$\pi_t(f) := \mathbb{E}\big(f(x_t)|\mathcal{F}_t^y\big) = \frac{\mathbb{E}^{\mathbb{Q}}\big(f(x_t)M_t|\mathcal{F}_t^y\big)}{\mathbb{E}^{\mathbb{Q}}\big(M_t|\mathcal{F}_t^y\big)} =: \frac{\sigma_t(f)}{\sigma_t(1)},$$

where σ_t is called the unnormalized filter and π_t is called the optimal filter. By using the stochastic Fubini theorem [Xio08, Lemma 5.4] and Itô formula 3.2.3, we can obtain the following linear equation for the unnormalized filter $\sigma_t(f)$, which is called Zakai's equation,

$$\sigma_t(f) = \sigma_0(f) + \int_0^t \sigma_s(\mathscr{L}f)ds + \int_0^t \sigma_s(hf)dy_s,$$

where $\mathscr{L}f$ is the infinitesimal generator of $f(X_t)$ defined in (3.20). Define the innovation process $dW_t = dy_t - \pi_t(h)dt$, which is a $\mathcal{F}_t^y$-Brownian motion under the original probability measure $\mathbb{P}$. Finally, by Kallianpur-Striebel formula, we can obtain the stochastic differential equation for the optimal filter $\pi_t(f)$, which is called Kushner-Stratonovich equation or FKK equation,

$$\pi_t(f) = \pi_0(f) + \int_0^t \pi_s(\mathscr{L}f)ds + \int_0^t \big(\pi_s(hf) - \pi_s(h)\pi_s(f)\big)dW_s.$$

3.3.2 Quantum filtering theory

Based on the discussion of previous sections, two system-observation models may be considered in the quantum probability space $(\mathcal{A}, \boldsymbol{\varphi})$, one corresponds to the open-loop Hamiltonian control given by,

$$\begin{aligned} dj_t(X) =& j_t\big([L^*, X]\big)dA_t + j_t\big([X, L]\big)dA_t^\dagger + \frac{1}{2}j_t\big(2L^*XL - (L^*LX + XL^*L)\big)dt \\ &+ j_t\big(i[H_f, X]\big)dt + u(t)j_t\big(i[H_c, X]\big)dt, \\ dY_t =& j_t(L + L^*)dt + dZ_t, \end{aligned}$$

while the other one corresponds to the feedback Hamiltonian control given by,

$$\begin{aligned} dj_t(X) =& j_t\big([L^*, X]\big)dA_t + j_t\big([X, L]\big)dA_t^\dagger + \frac{1}{2}j_t\big(2L^*XL - (L^*LX + XL^*L)\big)dt \\ &+ j_t\big(i[H_f, X]\big)dt + u(Y_{s\le t})j_t\big(i[H_c, X]\big)dt, \\ dY_t =& j_t(L + L^*)dt + dZ_t, \end{aligned}$$

where $dZ_t = dA_t + dA_t^\dagger + \kappa(d\tilde{A}_t + d\tilde{A}_t^\dagger)$ with $\kappa \geq 0$ denotes the output noise. In order to make the theory allow the above two cases, let us introduce a new state on the quantum probability space

$$\varpi(X) := \varphi(U_t^* X U_t),$$

where the unitary operator U_t is the solution of the quantum stochastic differential equation defined in the previous section. Due to the definition of quantum conditional expectation (3.1.8), we can easily deduce that, given a commutative von Neumann subalgebra $\mathcal{B} \subset \mathcal{A}$,

$$\varpi(X) = \varphi(U_t^* X U_t) \Rightarrow U_t^* \varpi(X|\mathcal{B}) U_t = \varphi(U_t^* X U_t | U_t^* \mathcal{B} U_t^*), \tag{3.37}$$

where $X \in \mathcal{B}'$ and $U_t^* \mathcal{B} U_t = \mathrm{vN}\{U_t^* X U_t^* | X \in \mathcal{B}\}$. Denote $\mathcal{Z}_t := \mathrm{vN}\{Z_s | s \leq t\}$ and $\mathcal{Y}_t := \mathrm{vN}\{Y_s | s \leq t\}$, then we have $\mathcal{Y}_t = U_t^* \mathcal{Z}_t U_t$ since $Y_t = U_t^* Z_t U_t$. Thus, by the property (3.37), for all ampliate atomic observable X, we have

$$U_t^* \varpi(X|\mathcal{Z}_t) U_t = \varphi\big(j_t(X)|\mathcal{Y}_t\big).$$

Due to the nondemolition property 3.2.7 of Z_t and Y_t, the existence of the above conditional expectations are ensured. In the classical stochastic filtering theory, Girsanov's theorem appears as a powerful tool where we are able to change the measure, for example, we can consider the observation process as a Wiener process in a new probability space. The next lemma can be considered as the quantum analogue of Girsanov's theorem. This lemma is necessary to present the quantum filtering equation. In our presentation, we follow the approaches applied in [BvHJ07, BvH08].

Lemma 3.3.1. *Given the quantum probability space $(\mathcal{A}, \varphi)$, let D_t, F_t, $\tilde{F}_t$ and G_t be bounded processes, and let*

$$dV_t = \big(D_t dA_t^\dagger + F_t dA_t + G_t dt\big) V_t, \quad d\tilde{V}_t = \big(D_t dA_t^\dagger + \tilde{F}_t dA_t + G_t dt\big) \tilde{V}_t.$$

Then, $\varphi(V_t^* X V_t) = \varphi(\tilde{V}_t^* X \tilde{V}_t)$ for all $X \in \mathcal{A}$.

Under the notations of Section 3.2.3, we define

$$dV_t = \left(L dA_t^\dagger - L dA_t - \frac{1}{2} L^* L dt - iH(t) dt\right) V_t, \quad V_0 = \mathbb{1}, \tag{3.38}$$

the only difference between the expressions of dV_t and dU_t defined in (3.29) is the coefficient of dA_t. Due to Lemma 3.3.1, for all ampliate atomic observable X, we have

$$\varpi(X) := \varphi(U_t^* X U_t) = \varphi(V_t^* X V_t).$$

By the quantum Itô formula 3.2.6, we have

$$V_t^* X V_t = X + \int_0^t \big(V_s^* \mathcal{L}(X) V_s\big) ds + \int_0^t \big(V_s^* (L^* X + X L) V_s\big)(dA_t + dA_t^\dagger), \tag{3.39}$$

where $\mathcal{L}(X)$ denotes the Lindblad generator (3.32), for the open-loop case $H(t) = H_f + u(t) H_c$ and for the feedback case $H_t = H_f + u(Z_{s \leq t}) H_c$. Due to the properties of the quantum conditional expectation, the following exchange formulas for quantum conditional expectation and integral, play the same role as the stochastic Fubini theorem [Xio08, Lemma 5.4] in the classical case,

$$\varphi\left(\int_0^t K_s ds \middle| \mathcal{Z}_s\right) = \int_0^t \varphi(K_s|\mathcal{Z}_s) ds, \quad \varphi\left(\int_0^t K_s dZ_s \middle| \mathcal{Z}_s\right) = \int_0^t \varphi(K_s|\mathcal{Z}_s) dZ_s. \tag{3.40}$$

where K_t is adapted and K_s with $s \leq t$ is affiliated to $\mathcal{Z}_s'$. Recall that $Z_t = A_t + A_t^\dagger + \kappa(\tilde{A}_t + \tilde{A}_t^\dagger)$ with $\kappa \geq 0$, where $\tilde{A}_t$ and $\tilde{A}_t^\dagger$ are independent of A_t and $A_t^\dagger$. Denote $M_t := (1-\eta)(A_t + A_t^\dagger) - \eta\kappa(\tilde{A}_t + \tilde{A}_t^\dagger)$, for some $\eta \in \mathbb{R}$ then we have

$$A_t + A_t^\dagger = \eta Z_t + M_t, \quad \forall \eta \in \mathbb{R}.$$

By quantum Itô formula, we have

$$dZ_t dM_t = (1 - \eta - \eta\kappa^2)dt.$$

Note that, if we choose $\eta = 1/(1+\kappa^2) \in (0,1]$, we have $dZ_t dM_t = 0$, which can intuitively be interpreted as the independendence between Z_t and M_t.

$$\boldsymbol{\varphi}\left(\int_0^t K_s dM_s \middle| \mathcal{Z}_s\right) = 0,$$

which can be easily show by elementary arguments and the property (3.40), we have

$$\boldsymbol{\varphi}\left(\int_0^t K_s d(A_s + A_s^\dagger) \middle| \mathcal{Z}_s\right) = \int_0^t \boldsymbol{\varphi}(K_s|\mathcal{Z}_s) d(A_s + A_s^\dagger).$$

Taking the conditional expectation on (3.39) and setting $\eta = 1/(1+\kappa^2)$. Then by the above-mentioned exchange formulas, we have

$$\boldsymbol{\varphi}(V_t^* X V_t|\mathcal{Z}_t) = \boldsymbol{\varphi}(X) + \int_0^t \boldsymbol{\varphi}\big(V_s^*\mathcal{L}(X)V_s|\mathcal{Z}_s\big)ds + \int_0^t \boldsymbol{\varphi}\big(V_s^*(L^*X + XL)V_s|\mathcal{Z}_s\big)(dA_s + dA_s^\dagger).$$

Denote the unnormalized filter $\sigma_t(X) = U_t\boldsymbol{\varphi}(V_t^* X V_t|\mathcal{Z}_t)U_t \in \mathcal{Y}_t$, by quantum Itô formula, we can obtain Belavkin-Zakai equation,

$$d\sigma_t(X) = \sigma_t\big(\mathcal{L}(X)\big)dt + \eta\sigma_t(L^*X + XL)dY_t. \tag{3.41}$$

Due to the self-nondemolition property of Y_t, Y_t can be consider as a classical stochastic process, so that the above differential equation can be considered as classical one. Due to quantum Bayes formula (3.1.10), we define the optimal filter, which is called noncommutative Kallianpur-Striebel formula,

$$\pi_t(X) = \frac{\sigma_t(X)}{\sigma_t(\mathbb{1})}, \tag{3.42}$$

and then by classical Itô formula (3.2.3), we have

$$d\pi_t(X) = \pi_t\big(\mathcal{L}(X)\big)dt + \sqrt{\eta}\big(\pi_t(L^*X + XL) - \pi_t(L^* + L)\pi_t(X)\big)\big(\sqrt{\eta}dY_t + \sqrt{\eta}\pi_t(L^* + L)dt\big).$$

Following the physical convention, we normalize the observation process by replacing dY_t by $\sqrt{\eta}dY_t$, i.e. $(\sqrt{\eta}dY_t)^2 = dt$. Moreover, as we have already discussed, all the states of an open quantum system on a finite-dimensional Hilbert space can be expressed as $\mathrm{Tr}(\rho X)$ for some density operator ρ. Hence, in our atom-field model with finite dimensional $\mathcal{H}_s$, we can always write $\pi_t(X) = \mathrm{Tr}(\rho_t X)$, where the conditional density matrix ρ_t is a random function of the observations up to t. Finally, we obtain a matrix-valued stochastic differential equation, which is called the quantum stochastic master equation,

$$\begin{aligned} d\rho_t = &-[H_f + u_t H_c, \rho_t]dt + \big(L\rho_t L^* - L^*L\rho_t/2 - \rho_t L^* L/2\big)dt \\ &+ \sqrt{\eta}\big(L\rho_t + \rho_t L^* - \mathrm{Tr}((L + L^*)\rho_t)\rho_t\big)\big(dY_t - \sqrt{\eta}\mathrm{Tr}((L + L^*)\rho_t)dt\big), \end{aligned} \tag{3.43}$$

where $u_t = u(t) \in \mathbb{R}$ for the open-loop case, $u_t \in \mathcal{Y}_t$ for the feedback case and $\eta \in (0, 1]$. By applying the innovations method [BvHJ07, Section 7], we have

$$dW_t = dY_t - \sqrt{\eta}\mathrm{Tr}((L + L^*)\rho_t)dt,$$

where W_t is a one-dimensional standard Wiener process.

Remark 3.3.2. In this chapter, we only considered a simplified case, where the main system is finite dimensional, the operators describing the interaction between the main system and the field are time-independent. There is only one quantum channel and the control input affects only on the atomic Hamiltonian. Based on the preliminary discussion on the quantum filtering theory of this chapter, we can adapt the results to more general cases by using more sophisticated arguments [Mey06, Par12, Bel92].

4

Feedback stabilization of open quantum spin systems

In this chapter, we consider the quantum stochastic master equation (3.43) as the starting point and assume that the initial state of the quantum filter (the conditional density operator) ρ_0 matches the actual quantum initial state perfectly, then Equation (3.43) precisely describes the time evolution of the state (density operator) of the main system (atom) interacting with the electromagnetic field. The "wrongly initialized" case will be discussed in Chapter 6.

A quantum stochastic master equation (3.43) is composed of a deterministic part and a stochastic part. The deterministic part, which corresponds to the average dynamics, is given by Lindblad generator (3.32). The stochastic part represents the back-action effect of continuous-time measurements. The solutions of this equation are called quantum trajectories and their properties have been studied in [MvH07, Pel08].

In this chapter, we focus on N-level quantum spin systems (quantum angular momentum systems) interacting with an electromagnetic field, whose stochastic master equation is given by

$$d\rho_t = F(\rho_t)dt + \sqrt{\eta}G(\rho_t)dW_t, \tag{4.1}$$

where

- W_t is a one-dimensional standard Wiener process on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$, where $\mathcal{F}_t$ is the natural filtration of the process W_t,
- the quantum state is described by the density operator ρ, which belongs to the compact space $\mathcal{S}_N := \{\rho \in \mathbb{C}^{N\times N} | \, \rho = \rho^*, \mathrm{Tr}(\rho) = 1, \rho \geq 0\}$,
- the drift term is given by

$$F(\rho) := -i[\omega J_z + u_t J_y, \rho] + M\left(J_z\rho J_z - J_z^2\rho/2 - \rho J_z^2/2\right)$$

  and the diffusion term is given by $G(\rho) := \sqrt{M}(J_z\rho + \rho J_z - 2\mathrm{Tr}(J_z\rho)\rho)$,
- $u := u(\rho)$ denotes the feedback law,

- J_z is the (self-adjoint) angular momentum along the axis z, and it is defined by

$$J_z e_n = (J-n)e_n, \quad n \in \{0,\dots,2J\},$$

where $J = \frac{N-1}{2}$ represents the fixed angular momentum and $\{e_0,\dots,e_{2J}\}$ corresponds to an orthonormal basis of $\mathbb{C}^N$. With respect to this basis, the matrix form of J_z is given by

$$J_z = \begin{bmatrix} J & & & & \\ & J-1 & & & \\ & & \ddots & & \\ & & & -J+1 & \\ & & & & -J \end{bmatrix}, \tag{4.2}$$

- J_y is the (self-adjoint) angular momentum along the axis y, and it is defined by

$$J_y e_n = -ic_n e_{n-1} + ic_{n+1}e_{n+1}, \quad n \in \{0,\dots,2J\}, \tag{4.3}$$

where $c_m = \frac{1}{2}\sqrt{(2J+1-m)m}$. The matrix form of J_y is given by

$$J_y = \begin{bmatrix} 0 & -ic_1 & & & \\ ic_1 & 0 & -ic_2 & & \\ & \ddots & \ddots & \ddots & \\ & & ic_{2J-1} & 0 & -ic_{2J} \\ & & & ic_{2J} & 0 \end{bmatrix},$$

- $\eta \in (0,1]$ measures the efficiency of the detectors, $M > 0$ is the strength of the interaction between the light and the atoms, and $\omega \geq 0$ is a parameter characterizing the free Hamiltonian.

If the feedback $u(\rho)$ is in $\mathcal{C}^1(\mathcal{S}_N, \mathbb{R})$, the existence and uniqueness of the solution of (4.1) as well as the strong Markov property of the solution are ensured by the results established in [MvH07].

In the following sections, we first analyze the behavior of the quantum stochastic differential equation (4.1) for open quantum spin systems, when we turn off the Hamiltonian control input (i.e., $u \equiv 0$), which leads to quantum state reduction phenomena. Based on the benefits and disadvantages of such phenomenon for our control goal, we will review three state-feedback stabilization methods proposed in [vHSM05a, MvH07, Tsu08]. Then we introduce our feedback approach which guarantees the exponential stabilization [LAM19a, LAM18].

4.1 Quantum state reduction

Purification Let us first discuss which roles the drift term $F(\rho)$ and the diffusion term $G(\rho)$ of Equation (4.1) play on the preparation of pure states, and why it is interesting to analyze the behavior of an open quantum system when we turn off the Hamiltonian controller (i.e., $u \equiv 0$). As $\mathrm{Tr}(\rho^2) = 1$ if and only if ρ is pure, we define the following formula to measure the "distance" between the actual state and the set of all pure states, denoted by $\mathcal{P} := \{\rho \in \mathcal{S}_N | \, \mathrm{Tr}(\rho^2) = 1\}$,

$$S(\rho) := 1 - \mathrm{Tr}(\rho^2), \tag{4.4}$$

which is called impurity. The infinitesimal generator of $S(\rho)$ is given by

$$\begin{aligned}\mathscr{L}S(\rho) &= -2\mathrm{Tr}\big(F(\rho)\rho\big) - \eta\mathrm{Tr}\big(G^2(\rho)\big) \\ &= 2M\big(\mathrm{Tr}(\rho^2 J_z^2) - \mathrm{Tr}(\rho J_z \rho J_z)\big) - \eta\mathrm{Tr}\big(G^2(\rho)\big),\end{aligned} \tag{4.5}$$

where $G(\rho) = G^*(\rho)$. By Cholesky decomposition [Bha13], we can write the density operator as $\rho = pp^*$ with $p \in \mathbb{C}^{N\times N}$, then we have

$$\mathrm{Tr}(\rho J_z \rho J_z) = \mathrm{Tr}(p^* J_z p\, p^* J_z p) = \|p^* J_z p\|_{HS}^2,\ \mathrm{Tr}(\rho^2 J_z^2) = \|\rho J_z\|_{HS}^2,\ \mathrm{Tr}\big(G^2(\rho)\big) = \|G(\rho)\|_{HS}^2,$$

where $\|X\|_{HS}^2 := \mathrm{Tr}(X^*X)$ for a finite dimensional matrix X is called Hilbert-Schmidt norm. By Cauchy–Schwarz inequality, we have

$$\mathrm{Tr}^2(\rho J_z \rho J_z) \leq \mathrm{Tr}^2(\rho^2 J_z^2) \Rightarrow \mathrm{Tr}(\rho J_z \rho J_z) \leq \mathrm{Tr}(\rho^2 J_z^2)$$

and the equality holds if and only if ρJ_z and $J_z\rho$ are parallel, which implies that ρ should satisfy $[\rho, J_z] = 0$ if the equality holds. Note that $\mathscr{L}S(\rho)$ does not depend on the control input $u(\rho)$. Moreover, the first term of the right hand side of (4.5) is positive and the second term is negative, which implies that the diffusion term $G(\rho)$ of the quantum stochastic master equation (4.1), unlike the drift term, contributes to increase the purity of the quantum state.

Quantum state reduction Now, we then analyze Equation (4.1) with $u \equiv 0$ and discuss how the diffusion term increases the purity of the quantum state. Such behavior known as quantum state reduction has been already discussed [ABBH01, vHSM05a, MvH07], by stochastic Lyapunov-type approach [Kha11, Mao07]. Denote by $\boldsymbol{\rho}_n := e_n e_n^*$ with $n \in \{0, \ldots, 2J\}$, where e_n is an eigenvector of J_z, then the set of all equilibria of Equation (4.1) with $u \equiv 0$ is given by

$$\bar{E} := \{\boldsymbol{\rho}_0, \ldots, \boldsymbol{\rho}_{2J}\}.$$

Consider the "variance function" $\mathscr{V}(\rho) := \mathrm{Tr}(J_z^2\rho) - \mathrm{Tr}^2(J_z\rho) \geq 0$ of J_z as a candidate Lyapunov function. Then we have,

$$\mathscr{L}\mathscr{V}(\rho) = -\mathrm{Tr}^2\big(J_z G(\rho)\big) = -4M\eta\mathscr{V}^2(\rho) \leq 0,$$

where $\mathscr{V}(\rho) = 0$ if and only if $\rho \in \bar{E}$. By means of the stochastic LaSalle-type theorem (Theorem A.2.3), we can show that $\mathscr{V}(\rho_t)$ converges to 0 almost surely, when t goes to infinity. Then we can conclude that, if $u \equiv 0$, the diffusion term of Equation (4.1) induces a collapse of the quantum state ρ_t towards a pure state corresponding to one of the eigenvectors of the measurement operator J_z almost surely. This suggests that combining the continuous measurement with the feedback control may provide an effective strategy for preparing a selected target state $\boldsymbol{\rho}_{\bar{n}}$ in practice. Some relevant results concerning the construction of such a feedback controller [vHSM05a, MvH07, Tsu08] will be briefly discussed in Section 4.2.

Exponential quantum state reduction Here, we show exponential convergence towards the set $\bar{E}$. This is essential to achieve our main control goal which concerns exponential stabilization of a target state $\boldsymbol{\rho}_{\bar{n}}$. For this purpose, we have estimated the convergence rate of ρ_t towards $\bar{E}$ by stochastic Lyapunov-type method in [LAM19a, Theorem 5.1]. Note

that a similar problem has also been discussed in [BP14]. The authors show the quantum state reduction of (jump-diffusion) stochastic master equation, undergoing continuous-time non-demolition measurement with an exponential convergence rate. In order to prove such a result, they applied martingale techniques and Girsanov theorem, rather than the Lyapunov-type method.

Firstly, we provide an invariant property of ρ_t in this case inspired by analogous results established in [Kha11, Mao07]. This lemma can also make the condition of Itô formula be verified, which will be used in the proof of Theorem 4.1.2. Denote the projection of ρ onto the state $\boldsymbol{\rho}_k$ as $\rho_{k,k} := \mathrm{Tr}(\rho\boldsymbol{\rho}_k)$.

Lemma 4.1.1 ([LAM19a, Lemma 4.1]). *Assume $u \equiv 0$. If $\rho_{k,k}(0) = 0$ for some $k \in \{0, \ldots, 2J\}$, then $\mathbb{P}\big(\rho_{k,k}(t) = 0, \forall\, t \geq 0\big) = 1$, i.e., the set $\{\rho \in \mathcal{S}_N | \, \rho_{k,k} = 0\}$ is a.s. invariant for Equation (4.1). Otherwise, if the initial state satisfies $\rho_{k,k}(0) \neq 0$, then $\mathbb{P}\big(\rho_{k,k}(t) \neq 0, \forall\, t \geq 0\big) = 1$.*

Proof. For $u \equiv 0$, the dynamics of $\rho_{k,k}$ is given by

$$d\rho_{k,k}(t) = \sqrt{\eta}\big(G(\rho_t)\big)_{k,k} dW_t = 2\sqrt{\eta M}\big(J - k - \mathrm{Tr}(J_z\rho_t)\big)\rho_{k,k}(t)dW_t.$$

In particular $|\sqrt{\eta}\big(G(\rho_t)\big)_{k,k}| \leq R\rho_{k,k}(t)$, for some $R > 0$, yielding the first part of the lemma.

Let us now prove the second part of the lemma. Assume that $\rho_{k,k}(0) > 0$ and $\mathbb{P}(\rho_{k,k}(t) \neq 0, \forall\, t \geq 0) < 1$. In particular $\mathbb{P}(\tau < \infty) > 0$, where $\tau := \inf\{t \geq 0 | \, \rho_{k,k}(t) = 0\}$. Let T be sufficiently large so that $\mathbb{P}(\tau \leq T) > 0$. Now, let $\varepsilon \in \big(0, \rho_{k,k}(0)\big)$, and consider any $\mathcal{C}^2$ function V defined on $\mathcal{S}$ such that

$$V(\rho) = \frac{1}{\rho_{k,k}}, \qquad \text{if } \rho_{k,k} > \varepsilon.$$

Then we have $\mathscr{L}V(\rho) = \rho_{k,k}^{-3}\big(\sqrt{\eta}G(\rho)\big)_{k,k}^2 \leq R^2V(\rho)$ if $\rho_{k,k} > \varepsilon$. We further define the time-dependent function $f(\rho, t) = e^{-R^2t}V(\rho)$, whose infinitesimal generator is given by $\mathscr{L}f(\rho, t) = e^{-R^2t}\big(-R^2V(\rho) + \mathscr{L}V(\rho)\big) \leq 0$ if $\rho_{k,k} > \varepsilon$. Now, define the stopping time $\tau_\varepsilon := \inf\{t \geq 0 | \, \rho_{k,k}(t) \notin (\varepsilon, 1)\}$. By Itô formula, we have

$$\mathbb{E}(f(\rho_{\tau_\varepsilon \wedge T}, \tau_\varepsilon \wedge T)) = V_0 + \mathbb{E}\left(\int_0^{\tau_\varepsilon \wedge T} \mathscr{L}f(\rho_s, s)ds\right) \leq V_0 = \frac{1}{\rho_{k,k}(0)}.$$

Since $\tau \geq \tau_\varepsilon$ we deduce that, conditioning to the event $\{\tau \leq T\}$, $f(\rho_{\tau_\varepsilon \wedge T}, \tau_\varepsilon \wedge T) = f(\rho_{\tau_\varepsilon}, \tau_\varepsilon) = e^{-R^2T}\varepsilon^{-1}$, which implies

$$\mathbb{E}\left(e^{-R^2T}\varepsilon^{-1}\mathbb{1}_{\{\tau \leq T\}}\right) = \mathbb{E}\left(f(\rho_{\tau_\varepsilon}, \tau_\varepsilon)\mathbb{1}_{\{\tau \leq T\}}\right) \leq \mathbb{E}\big(f(\rho_{\tau_\varepsilon \wedge T}, \tau_\varepsilon \wedge T)\big) \leq \frac{1}{\rho_{k,k}(0)}.$$

Thus, $\mathbb{P}(\tau \leq T) = \mathbb{E}\left(\mathbb{1}_{\{\tau \leq T\}}\right) \leq \varepsilon e^{R^2T}/\rho_{k,k}(0)$. Letting ε tend to 0, we get $\mathbb{P}(\tau \leq T) = 0$ which gives a contradiction. The proof is then complete. □

Then we show that the quantum state reduction for the system (4.1) towards the invariant set $\bar{E} = \{\boldsymbol{\rho}_0, \ldots, \boldsymbol{\rho}_{2J}\}$ occurs with exponential velocity with respect to the Bures distance (see Definition A.2.4).

Theorem 4.1.2 (Exponential quantum state reduction)**.** *For system* (4.1)*, with* $u \equiv 0$ *and* $\rho_0 \in \mathcal{S}_N$*, the set* $\bar{E}$ *is exponentially stable in mean and a.s. with average and sample Lyapunov exponent less than or equal to* $-\eta M/2$*. Moreover, the probability of convergence to* $\boldsymbol{\rho}_n \in \bar{E}$ *is* $\mathrm{Tr}(\rho_0 \boldsymbol{\rho}_n)$ *for* $n \in \{0, \ldots, 2J\}$*.*

Proof. Let $I := \{k | \, \rho_{k,k}(0) = 0\}$ and $\mathcal{S}_I := \{\rho \in \mathcal{S}_N | \, \rho_{k,k} = 0 \text{ if and only if } k \in I\}$. Then by Lemma 4.1.1, $\mathcal{S}_I$ is a.s. invariant for (4.1). Consider the function

$$V(\rho) = \frac{1}{2} \sum_{\substack{n,m=0 \\ n \neq m}}^{2J} \sqrt{\mathrm{Tr}(\rho \boldsymbol{\rho}_n)\mathrm{Tr}(\rho \boldsymbol{\rho}_m)} = \frac{1}{2} \sum_{\substack{n,m=0 \\ n \neq m}}^{2J} \sqrt{\rho_{n,n}\rho_{m,m}} \geq 0 \tag{4.6}$$

as a candidate Lyapunov function. Note that $V(\rho) = 0$ if and only if $\rho \in \bar{E}$. As $\mathcal{S}_I$ is invariant for (4.1) with $u \equiv 0$ and V is twice continuously differentiable when restricted to $\mathcal{S}_I$, we can compute $\mathscr{L}V(\rho) \leq -\frac{\eta M}{2} V(\rho)$. By Itô formula, for all $\rho_0 \in \mathcal{S}$, we have

$$\mathbb{E}\big(V(\rho_t)\big) = V(\rho_0) + \int_0^t \mathbb{E}\big(\mathscr{L}V(\rho_s)\big) ds \leq V(\rho_0) - \frac{\eta M}{2} \int_0^t \mathbb{E}\big(V(\rho_s)\big) ds.$$

In virtue of Grönwall inequality, we have $\mathbb{E}\big(V(\rho_t)\big) \leq V(\rho_0) e^{-\frac{\eta M}{2} t}$. Next, we show that the candidate Lyapunov function is bounded by the Bures distance from $\bar{E}$. Firstly, we have

$$V(\rho) = \frac{1}{2} \sum_{n=0}^{2J} \left(\sqrt{\rho_{n,n}} \sum_{m \neq n} \sqrt{\rho_{m,m}} \right) \geq \frac{1}{2} \sum_{n=0}^{2J} \sqrt{\rho_{n,n}(1 - \rho_{n,n})} \geq \frac{d_B(\rho, \bar{E})}{2} \sum_{n=0}^{2J} \sqrt{\rho_{n,n}}.$$

Combining with $\sum_{n=0}^{2J} \sqrt{\rho_{n,n}} \geq \sum_{n=0}^{2J} \rho_{n,n} = 1$, we have $\frac{1}{2} d_B(\rho, \bar{E}) \leq V(\rho)$. Let us now prove the converse inequality. Assume that $d_B(\rho, \bar{E}) = \sqrt{2 - 2\sqrt{\rho_{\bar{n},\bar{n}}}}$ for some index $\bar{n}$, then $\sqrt{\rho_{m,m}} \leq \sqrt{1 - \rho_{\bar{n},\bar{n}}} \leq d_B(\rho, \bar{E})$ for $m \neq \bar{n}$. In particular, each addend in $V(\rho)$ is less than or equal to $d_B(\rho, \bar{E})$, and $V(\rho) \leq J(2J+1) d_B(\rho, \bar{E})$. Thus, we have

$$C_1 d_B(\rho, \bar{E}) \leq V(\rho) \leq C_2 d_B(\rho, \bar{E}), \tag{4.7}$$

where $C_1 = 1/2$, $C_2 = J(2J+1)$. It implies,

$$\mathbb{E}\big(d_B(\rho_t, \bar{E})\big) \leq \frac{C_2}{C_1} d_B(\rho_0, \bar{E}) e^{-\frac{\eta M}{2} t}, \quad \forall \rho_0 \in \mathcal{S}_N.$$

which means that the set $\bar{E}$ is exponentially stable in mean with average Lyapunov exponent less than or equal to $-\eta M/2$.

Now we consider the stochastic process $Q(\rho_t, t) = e^{\frac{\eta M}{2} t} V(\rho_t) \geq 0$ whose infinitesimal generator is given by $\mathscr{L}Q(\rho, t) = e^{\frac{\eta M}{2} t} \big(\eta M/2 \, V(\rho) + \mathscr{L}V(\rho)\big) \leq 0$. Hence, the process $Q(\rho_t, t)$ is a positive supermartingale. Due to Doob's martingale convergence theorem [RY13], the process $Q(\rho_t, t)$ converges almost surely to a finite limit as t tends to infinity. Consequently, $Q(\rho_t, t)$ is almost surely bounded, that is $\sup_{t \geq 0} Q(\rho_t, t) = A$, for some a.s. finite random variable A. This implies $\sup_{t \geq 0} V(\rho_t) = A e^{-\frac{\eta M}{2} t}$ a.s. Letting t goes to infinity, we obtain $\limsup_{t \to \infty} \frac{1}{t} \log V(\rho_t) \leq -\frac{\eta M}{2}$ a.s. By the inequality (4.7),

$$\limsup_{t \to \infty} \frac{1}{t} \log d_B(\rho_t, \bar{E}) \leq -\frac{\eta M}{2}, \qquad a.s.$$

which means that the set $\bar{E}$ is a.s. exponentially stable with sample Lyapunov exponent less than or equal to $-\eta M/2$.

In order to calculate the probability of convergence towards $\boldsymbol{\rho}_n \in \bar{E}$, we follow an approach inspired by [ASD+13, ABBH01]. According to the first part of the theorem, the process $\mathrm{Tr}(\rho_t \boldsymbol{\rho}_n)$ converges a.s. to $\mathbb{1}_{\{\rho_t \to \boldsymbol{\rho}_n\}}$. Therefore, by applying the dominated convergence theorem, $\mathrm{Tr}(\rho_t \boldsymbol{\rho}_n)$ converges to $\mathbb{1}_{\{\rho_t \to \boldsymbol{\rho}_n\}}$ in mean. As $\mathscr{L}\mathrm{Tr}(\rho_t \boldsymbol{\rho}_n) = 0$, then $\mathrm{Tr}(\rho_t \boldsymbol{\rho}_n)$ is a positive martingale. Hence,

$$\mathbb{P}(\rho_t \to \boldsymbol{\rho}_n) = \lim_{t\to\infty} \mathbb{E}\big(\mathrm{Tr}(\rho_t \boldsymbol{\rho}_n)\big) = \mathrm{Tr}(\rho_0 \boldsymbol{\rho}_n),$$

and the proof is complete. □

Simulations confirming the quantum state reduction phenomena are shown in Fig. 4.1. In particular, we observe that the expectation of the Lyapunov function $\mathbb{E}\big(V(\rho_t)\big)$ is bounded by the exponential function $V(\rho_0)e^{-\frac{\eta M}{2}t}$, and the expectation of the Bures distance $\mathbb{E}\big(d_B(\rho_t, \bar{E})\big)$ is always below the exponential function $C_2/C_1\, d_B(\rho_0, \bar{E})e^{-\frac{\eta M}{2}t}$, with $C_1 = 1/2$ and $C_2 = 3$ (see Equation (4.7)) in accordance with Theorem 4.1.2.

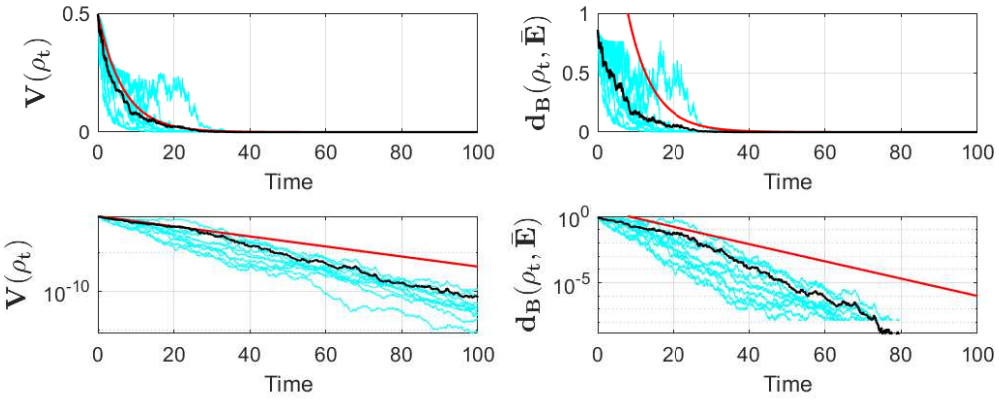


FIGURE 4.1 – Quantum state reduction of a three-level quantum angular momentum system with $u \equiv 0$ starting at $\mathrm{diag}(0.3, 0.4, 0.3)$ when $\omega = 0$, $\eta = 0.3$ and $M = 1$: the black curve represents the mean value of 10 arbitrary sample trajectories, and the red curve represents the exponential reference with exponent $-\eta M/2$. The figures at the bottom are the semi-log versions of the ones at the top.

The results of this section provide the preliminary step to study the exponential stabilization towards the target state $\boldsymbol{\rho}_{\bar{n}}$. This will be discussed in Section 4.3.

4.2 Asymptotic stabilization of open quantum spin systems

Because of the topological structure of the state space $\mathcal{S}_N$ and the quantum state reduction phenomena, many natural feedback controllers cannot stabilize the system (4.1), see [AT08] for examples. In this section, we discuss three main approaches to construct feedback controllers. In [vHSM05a], the authors design for the first time a continuous

quantum feedback controller that globally stabilizes a quantum spin-$\frac{1}{2}$ system (which is a special case of quantum spin systems) towards a pure state corresponding to an eigenvector of σ_z (two-level version of J_z) in the presence of imperfect measurements (i.e., $\eta \in (0,1)$). This feedback controller has been designed by looking numerically for an appropriate global Lyapunov function. Then, in [MvH07], by analyzing the stochastic flow and by using stochastic Lyapunov techniques, the authors constructed a switching feedback controller which globally stabilizes the N-level quantum spin system to a target state, in the presence of imperfect measurements. A continuous version of this feedback controller has been proposed in [Tsu08]. The essential ideas in [vHSM05a, Tsu08] for constructing the continuous feedback controller remain the same: the controllers consist of two parts, the first one contributing to the local convergence to the target state, and the second one driving the system away from the antipodal states.

4.2.1 Continuous feedback laws for open quantum spin-$\frac{1}{2}$ systems

Here, we want to discuss the main ideas of [vHSM05a] to construct a continuous feedback controller. Consider the two-level case of (4.1), with the Hamiltonian given by $H = u_t\sigma_y$ and the measurement operator given by σ_z, where σ_y and σ_z are the two-level versions of J_y and J_z. Recall that σ_x, σ_y and σ_z are called Pauli matrices given by

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

where $\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = \mathbb{1}$. Then, Equation (4.1) becomes

$$d\rho_t = \big(-iu_t[\sigma_y, \rho_t] + M(\sigma_z\rho_t\sigma_z - \rho_t)\big)dt + \sqrt{\eta M}\big(\sigma_z\rho_t + \rho_t\sigma_z - 2\mathrm{Tr}(\sigma_z\rho_t)\rho_t\big)dW_t. \quad (4.8)$$

As we introduced in Chapter 2, for a two-level quantum system, ρ can be uniquely characterized by the Bloch sphere coordinates (x, y, z) as

$$\rho = \frac{\mathbb{1} + x\sigma_x + y\sigma_y + z\sigma_z}{2} = \frac{1}{2}\begin{bmatrix} 1+z & x-iy \\ x+iy & 1-z \end{bmatrix}. \quad (4.9)$$

The vector (x, y, z) belongs to the ball

$$\mathcal{B} := \{(x, y, z) \in \mathbb{R}^3 |\, x^2 + y^2 + z^2 \leq 1\}.$$

The stochastic differential equation (4.8) expressed in the Bloch sphere coordinates takes the following form

$$dx_t = (-2Mx_t + u_tz_t)dt - 2\sqrt{\eta M}x_tz_tdW_t, \quad (4.10a)$$
$$dy_t = -2My_tdt - 2\sqrt{\eta M}y_tz_tdW_t, \quad (4.10b)$$
$$dz_t = -u_tx_tdt + 2\sqrt{\eta M}(1 - z_t^2)dW_t. \quad (4.10c)$$

Since the equations of x_t and z_t do not depend on y_t, we can just focus on Equation (4.10a) and (4.10c) on a disc $\{(x, z) \in \mathbb{R}^2 |\, x^2 + z^2 \leq 1\}$. Let us choose $\boldsymbol{\rho}_g := \mathrm{diag}(0, 1)$ as the target state, which corresponds to $(0, 0, -1)$ in Bloch sphere coordinate. In order to ensure

that the feedback controller is non-zero at the antipodal state $\boldsymbol{\rho}_e := \mathrm{diag}(1,0)$ and vanishes at the target state, the authors construct the feedback laws in [vHSM05a] as below,

$$u_t = \alpha(1+z_t) + \beta x_t, \tag{4.11}$$

where α and β are non-zero and belong to $\mathbb{R}$. By the heuristic discussion for the case $\eta = 1$ in [vHSM05a], we can see that, when ρ_t is around the antipodal state $\boldsymbol{\rho}_e$, the first term of (4.11) dominates the control input, which holds non-zero and contributes to drive the system away the antipodal eigenstate; when ρ_t is around the target state $\boldsymbol{\rho}_g$, the second term of (4.11) dominates the control input, which contributes to the local convergence to the target state. However, it is not easy to find an analytical way to construct the suitable Lyapunov function, corresponding to the chosen feedback controller (4.11), to show the almost sure global asymptotic stability of the target state.

Then, given α, β, M and η, by applying a semidefinite programming technique (e.g., Matlab toolbox SOSTOOLS) based on the semialgebraic geometry, we may look for a global Lyapunov function $V(x,z) \geq 0$ such that $\mathscr{L}V(x,z) \leq 0$. The following example was given in [vHSM05a], suppose $M = 2$, $\eta = 1/2$, $\alpha = -1$ and $\beta = 4$, the Lyapunov function found by SOSTOOLS is given by,

$$V(x,z) = 21.8(1+z) - 5.73x^2 + 10.4x(1+z) - 5.63(1+z)^2,$$

whose infinitesimal generator satisfies $\mathscr{L}V(x,z) \leq 0$. Thus the target state $\boldsymbol{\rho}_g$ can be shown almost surely globally asymptotically stable by applying the stochastic LaSalle-type theorem (Theorem A.2.3).

Therefore, the procedure of asymptotically stabilizing a target state $\boldsymbol{\rho}_{\bar{n}}$ of (4.1) in [vHSM05a] can be resumed through the following steps:

1. fix the parameters appearing in (4.1), i.e., ω, M and η;
2. construct a suitable feedback controller $u(\rho)$ with the fixed parameters ensuring that the target state $\boldsymbol{\rho}_{\bar{n}}$ be the only equilibrium of Equation (4.1);
3. apply the semidefinite programming technique to find a global Lyapunov function guaranteeing the effectiveness of the feedback laws.

By the above-mentioned approach, we can construct a continuous feedback controller for some low-level open quantum spin systems. However, it is difficult to show the stability of target state for higher-level systems or in case of unknown parameters due to the computational capacity of the computer.

4.2.2 Switching feedback laws for open quantum spin systems

To bypass the limitation of the approach in [vHSM05a], the authors in [MvH07] proposed an entirely analytical method to globally stabilize a N-level open quantum spin system towards the target state $\boldsymbol{\rho}_{\bar{n}}$ by implementing a switching feedback laws. The contribution in [MvH07] can be considered as a foundation work for many of the later papers. In particular, the exponential stabilzation results obtained in our works [LAM19a, LAM18] combine some of the arguments in [MvH07] together with further stochastic and geometric control tools, which will be discussed later.

As we mentioned above, the main challenge is due to the geometric symmetry hidden in the state space $\mathcal{S}_N$. The approach proposed in [MvH07] consists in analyzing the quantum

trajectory ρ_t in two subsets of $\mathcal{S}_N$, which are defined based on the value of the function $V(\rho) = 1 - \mathrm{Tr}(\rho\boldsymbol{\rho}_{\bar{n}})$ as below,

$$\mathcal{S}_{\geq 1-\gamma} := \{\rho \in \mathcal{S}_N | \, 1-\gamma \leq V(\rho) \leq 1\}, \quad \mathcal{S}_{\leq 1-\gamma/2} := \{\rho \in \mathcal{S}_N | \, 0 \leq V(\rho) \leq 1-\gamma/2\},$$

where $\gamma \in (0,1)$. Note that there are two obvious but important properties on the above two subsets,

- all antipodal eigenstates belongs to $\mathcal{S}_{\geq 1-\gamma}$, i.e., $\bar{E} \setminus \boldsymbol{\rho}_{\bar{n}} \subset \mathcal{S}_{\geq 1-\gamma}$ and $\boldsymbol{\rho}_{\bar{n}} \notin \mathcal{S}_{\geq 1-\gamma}$;
- the only eigenstate belonging to $\mathcal{S}_{\leq 1-\gamma/2}$ is the target state $\boldsymbol{\rho}_{\bar{n}}$.

Due to this important fact, one can solve the control problem in the following four steps,

Step 1. there exists a $\gamma \in (0,1)$ such that $\mathcal{S}_{\geq 1-\gamma}$ is not invariant when we take the feedback controller as a non-zero constant, e.g., $u(\rho) = 1$;

Step 2. for almost every sample path $\omega \in \Omega$, the quantum trajectory $\rho_t(\omega)$ can exit $\mathcal{S}_{\geq 1-\gamma}$ under the constant control input in finite time $T_1(\omega)$;

Step 3. for almost every sample path $\omega \in \Omega$, there exists a finite time $T_2(\omega)$, for all $t > T_2(\omega)$, the quantum trajectory $\rho_t(\omega)$ under a suitable (switching) feedback controller stays in $\mathcal{S}_{\leq 1-\gamma/2}$ and never exits;

Step 4. for all $\rho_0 \in \mathcal{S}_N$, ρ_t converges to the target state $\boldsymbol{\rho}_{\bar{n}}$ under a suitable feedback controller almost surely.

Then let us resume the proof and clarify the significance of each step briefly.

1. Step 1 can be proved by showing that the same property holds for the corresponding deterministic control system [1], and applying the support theorem (Theorem A.2.1). It means that, there exists a $\gamma \in (0,1)$ such that, for all $\rho_0 \in \mathcal{S}_{\geq 1-\gamma}$, the probability that ρ_t exits this domain in finite time is non-zero. Thus, the constant control input can break the symmetry of the state space in probability.
2. Step 2 can be shown by the so-called Dynkin estimation [Dyn65] based on the compactness of $\mathcal{S}_{\geq 1-\gamma}$ and Feller continuity [MvH07, Lemma 4.5], such technique is widely used in studying one-dimensional diffusion processes [RY13, Chapter VII.3]. Thus, Step 2 is a strengthened version of Step 1, which allows to conclude that the probability of exiting $\mathcal{S}_{\geq 1-\gamma}$ in finite time T_1 is one. We represent this result heuristically by
$$\mathbb{P}\left(\mathcal{S}_{\leq 1-\gamma/2} \xrightarrow{u(\rho)=1} \mathcal{S}_{\geq 1-\gamma}\right) = 1. \tag{4.12}$$
Note that the above mentioned finite time T_1 is a random variable.
3. The infinitesimal generator of $V(\rho)$ is given by $\mathscr{L}V(\rho) = u(\rho)U(\rho, \boldsymbol{\rho}_{\bar{n}})$, where $u(\rho)$ denotes the feedback controller, $U(\rho, \boldsymbol{\rho}_{\bar{n}}) := \mathrm{Tr}\big(i[J_y, \rho]\boldsymbol{\rho}_{\bar{n}}\big)$ and $U(\rho, \boldsymbol{\rho}_{\bar{n}}) = 0$ for all $\rho \in \bar{E}$. By the stochastic Lyapunov-type argument [MvH07, Theorem 2.2], if we choose $u(\rho) = -U(\rho, \boldsymbol{\rho}_{\bar{n}})$ then $\mathscr{L}V(\rho) = -U^2(\rho, \boldsymbol{\rho}_{\bar{n}}) \leq 0$ and the probability of ρ_t exiting $\mathcal{S}_{\geq 1-\gamma/2}$ from $\mathcal{S}_{\leq 1-\gamma}$ is less than one strictly, which can be represented heuristically by,
$$\mathbb{P}\left(\mathcal{S}_{\leq 1-\gamma} \xrightarrow{u(\rho)=-U} \mathcal{S}_{\geq 1-\gamma/2}\right) < 1, \tag{4.13}$$
where the mentioned two domains are defined as
$$\mathcal{S}_{\leq 1-\gamma} := \{\rho \in \mathcal{S} | \, 0 \leq V(\rho) \leq 1-\gamma\} \subset \mathcal{S}_{\leq 1-\gamma/2},$$
$$\mathcal{S}_{\geq 1-\gamma/2} := \{\rho \in \mathcal{S} | \, 1-\gamma/2 \leq V(\rho) \leq 1\} \subset \mathcal{S}_{\geq 1-\gamma}.$$

1. See Theorem A.2.1 to determine the corresponding deterministic control system of the stochastic differential equation (4.1)

Because of the two properties Equation (4.12) and Equation (4.13), by the strong Markov property of ρ_t [MvH07, Proposition 3.7] and Borel-Cantelli lemma [Chu01], for all $\rho_0 \in \mathcal{S}_N$, after a finite time T_2, ρ_t stays in $\mathcal{S}_{\leq 1-\gamma/2}$ and never exits almost surely. Moreover, the only eigenstate in $\mathcal{S}_{\leq 1-\gamma/2}$ is the target state $\boldsymbol{\rho}_{\bar{n}}$. In particular, the above-mentioned switching feedback control laws break the symmetry of the system.

4. Due to the stochastic LaSalle theorem [MvH07, Theorem 2.3], restricted in $\mathcal{S}_{\leq 1-\gamma/2}$, we can show that ρ_t converges to the target state $\boldsymbol{\rho}_{\bar{n}}$ almost surely under the feedback controller $u(\rho) = -U(\rho, \boldsymbol{\rho}_{\bar{n}})$, when t goes to infinity. By using Step 3, we can show that $\boldsymbol{\rho}_{\bar{n}}$ is almost surely globally asymptotically stable under the switching feedback control laws.

Then we conclude the above analysis by the following theorem [MvH07, Theorem 4.2].

Theorem 4.2.1. *Consider the system* (4.1) *evolving in the set* $\mathcal{S}_N$. *Let* $\boldsymbol{\rho}_{\bar{n}}$ *be the target state and let* $\gamma > 0$. *Consider the following control law :*

1. $u(\rho) = -U(\rho, \boldsymbol{\rho}_{\bar{n}})$ *for* $\rho \in \mathcal{S}_{\leq 1-\gamma}$;
2. $u(\rho) = 1$ *for* $\rho \in \mathcal{S}_{\geq 1-\gamma/2}$;
3. *for* $\rho \in \mathfrak{B} := \{\rho \in \mathcal{S} | \, \gamma/2 < \mathrm{Tr}(\rho\boldsymbol{\rho}_{\bar{n}}) < \gamma\}$, *then* $u(\rho) = -U(\rho, \boldsymbol{\rho}_{\bar{n}})$ *if* ρ_t *entered* $\mathfrak{B}$ *through the boundary* $V(\rho) = 1 - \gamma$, *and* $u(\rho) = 1$ *otherwise.*

Then, there exists $\gamma > 0$, *such that* $u(\rho)$ *almost surely globally stabilizes the system* (4.1) *towards* $\boldsymbol{\rho}_{\bar{n}}$.

Fig 4.2 presents a heuristic illustration of the application of the above theorem.

4.2.3 Continuous feedback laws for open quantum spin systems

As we discussed in the previous subsection, [MvH07] for the first time rigorously demonstrated the existence of a feedback controller which almost surely globally stabilizes the system (4.1). However, the switching nature of the feedback law in [MvH07, Theorem 4.2] makes difficult its implementation. Then, based on the analysis of [MvH07], the author in [Tsu08] proposed a continuous version of this switching feedback controller given by,

$$u(\rho) = -\alpha U(\rho, \boldsymbol{\rho}_{\bar{n}}) + \beta P_{\bar{n}}(\rho), \tag{4.14}$$

where

- $\alpha, \beta > 0$ and $\beta^2/8\alpha\eta < 1$, where $\eta \in (0, 1]$ describes the efficiency of the detectors ;
- $P_{\bar{n}}(\rho) := J - \bar{n} - \mathrm{Tr}(J_z\rho)$, and restricted on $\bar{E}$, $P_{\bar{n}}(\rho) = 0$ if and only if $\rho = \boldsymbol{\rho}_{\bar{n}}$.

The esssential ideas to construct such a feedback remain the same as the one given in (4.14) is same as the one of (4.11): $-\alpha U(\rho, \boldsymbol{\rho}_{\bar{n}})$ contributes to the local convergence to the target state, and $\beta P_{\bar{n}}(\rho)$ driving the system away from the antipodal states. The effectiveness of the continuous feedback controller (4.14) can be shown in the following two steps:

1. provide a domain of attraction for a fixed feedback controller (4.14), such that every trajectory staying in this domain and never exiting, ρ_t converges almost surely to the target state $\boldsymbol{\rho}_{\bar{n}}$;
2. show that, under the effect of the continuous feedback controller, for almost every sample path $\omega \in \Omega$, there exists a finite time $T(\omega)$, such that for all $t > T(\omega)$, $\rho_t(\omega)$ stays in the above-mentioned domain of attraction and never exits.

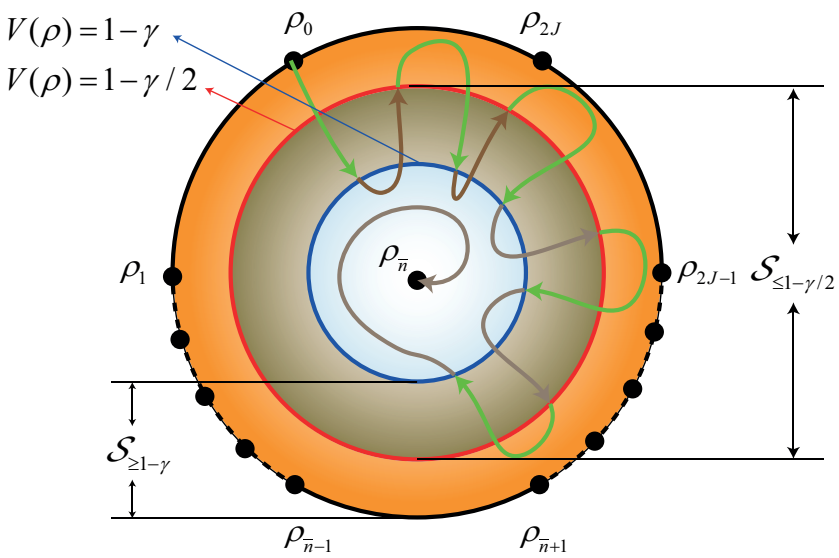


FIGURE 4.2 – A sample path of quantum trajectory starting at an antipodal eigenstate, after four times entering-exiting, ρ_t enters in $\mathcal{S}_{\leq 1-\gamma/2}$ and never exits, then converges to the target state $\boldsymbol{\rho}_{\bar{n}}$ when t goes to infinity. The whole disc corresponds to the state space $\mathcal{S}_N$, we divide it into several parts based on the value of $V(\rho)$. The largest circle represents the set $\{\rho \in \mathcal{S}_N | \, V(\rho) = 1\}$, which contains all antipodal eigenstates ; the two other circles describe the set $\{\rho \in \mathcal{S}_N | \, V(\rho) = 1 - \gamma/2\}$ and $\{\rho \in \mathcal{S}_N | \, V(\rho) = 1 - \gamma\}$ respectively ; the center of the disc represents the set $\{\rho \in \mathcal{S}_N | \, V(\rho) = 0\}$, i.e., the target state $\boldsymbol{\rho}_{\bar{n}}$. The curve with arrows describes the quantum trajectory, the green parts represent the ones with $u(\rho) = 1$, and the brown parts represent the ones with $u(\rho) = -U(\rho, \boldsymbol{\rho}_{\bar{n}})$.

For Step 1, consider a new Lyapunov function,

$$\bar{V}(\rho) = 1 - \mathrm{Tr}^2(\rho\boldsymbol{\rho}_{\bar{n}}),$$

whose infinitesimal generator is given by

$$\begin{aligned}\mathscr{L}\bar{V}(\rho) &= 2\mathrm{Tr}(\rho\boldsymbol{\rho}_{\bar{n}})\mathrm{Tr}\big(F(\rho)\boldsymbol{\rho}_{\bar{n}}\big) - \mathrm{Tr}^2\big(G(\rho)\boldsymbol{\rho}_{\bar{n}}\big) \\ &= -2\mathrm{Tr}(\rho\boldsymbol{\rho}_{\bar{n}})\Big(\alpha\big(-\beta P_{\bar{n}}(\rho)/2\alpha - U(\rho,\boldsymbol{\rho}_{\bar{n}})\big)^2 + 2\eta P_{\bar{n}}^2(\rho)\big(\mathrm{Tr}(\rho\boldsymbol{\rho}_{\bar{n}}) - \gamma_0\big)\Big),\end{aligned}$$

where $\gamma_0 := \beta^2/8\alpha\eta$ belongs to $(0,1)$. Thus, for all $\rho \in \{\rho \in \mathcal{S} | \, \mathrm{Tr}(\rho\boldsymbol{\rho}_{\bar{n}}) \geq \gamma_0\}$, which can be denoted as $\mathcal{S}_{\leq 1-\gamma_0}$ under the setting of the previous subsection, we have $\mathscr{L}V(\rho) \leq 0$. Moreover, the only eigenstate of J_z contained in $\mathcal{S}_{\leq 1-\gamma_0}$ is the target state $\boldsymbol{\rho}_{\bar{n}}$. By the stochastic LaSalle theorem [MvH07, Theorem 2.3], the proof of Step 1 is complete. We note that the role of the diffusion term is crucial in the above argument.

The proof of Step 2 is more difficult than the one with constant control (Steps 1, 2 and 3 of the previous subsection). Unfortunately some parts of the proof of Step 2 have not been given in [Tsu08]. We refer to [LAM19a, Lemma 6.1] and the first two parts of the proof of [LAM19a, Theorem 6.2] for the rigorous and complete demonstration.

4.3 Exponential stabilization of open quantum spin systems

Motivated by exponential convergence given in Theorem 4.1, and inspired by the analysis in [vHSM05a, MvH07, Tsu08], we have proposed a more general method in [LAM19a] to analyze the open quantum spin systems [2], and provided some general conditions on the feedback law enforcing the exponential convergence towards the target state $\boldsymbol{\rho}_{\bar{n}}$. These conditions are obtained mainly by studying the asymptotic behavior of quantum trajectories. Roughly speaking, under such conditions, and making use of the support theorem and other classical stochastic tools, we show that any neighborhood of the target state may be approached with probability one starting from any initial state (Lemma 4.3.6). This result shows the exponential convergence towards the target state by applying local Lyapunov-type arguments (Theorem 4.3.8). To show the convergence towards the target state, previous works applied stochastic LaSalle theorem (see e.g., [vHSM05a, MvH07, Tsu08]) which, unlike Lyapunov-type arguments, do not guarantee exponential stability and does not provide any information on the convergence rate. As demonstration of the general result, explicit parametrized stabilizing feedback laws are exhibited (Theorem 4.3.8 and Theorem 4.3.10). Note that to obtain our main results, some preliminary results on the asymptotic behavior of quantum trajectories associated with the considered system were needed, see Section 4.3.1. We believe that these results are significants by themselves. We point out that the exponential stabilization problem for open quantum spin-$\frac{1}{2}$ system (4.8) has been discussed in [LAM18].

4.3.1 Properties of the quantum trajectories

Our aim here is to establish some basic properties of the quantum trajectories corresponding to Equation (4.1). This subsection is instrumental in order to prove our main results.

Recall that we denote the projection of ρ onto the state $\boldsymbol{\rho}_k$ as $\rho_{k,k} := \mathrm{Tr}(\rho\boldsymbol{\rho}_k)$. In the following we state a lemma inspired by analogous results established in [Kha11, Mao07], which is the version with the feedback control of Lemma 4.1.1. This lemma can be served to overcome the problem of singularity of Lyapunov function in application of Ito formula, which is a necessary condition to show the exponential stability in Theorem 4.3.8.

Lemma 4.3.1. *Let $n \in \{0, \ldots, 2J\}$. Assume that $u \in \mathcal{C}^1(\mathcal{S}_N, \mathbb{R})$ and $u(\boldsymbol{\rho}_n) = 0$. If the initial state satisfies $\rho_0 \neq \boldsymbol{\rho}_n$, then $\mathbb{P}(\rho_t \neq \boldsymbol{\rho}_n, \forall\, t \geq 0) = 1$.*

Proof. Given $\varepsilon > 0$, we consider any $\mathcal{C}^2$ function on $\mathcal{S}_N$ such that

$$V(\rho) = \frac{1}{1-\rho_{n,n}}, \qquad \text{if } \rho_{n,n} < 1-\varepsilon.$$

We find

$$\mathscr{L}V(\rho) = -\frac{u(\rho)\mathrm{Tr}(i[J_y,\rho]\boldsymbol{\rho}_n)}{(1-\rho_{n,n})^2} + \frac{4\eta M[(J-n-\mathrm{Tr}(J_z\rho))\rho_{n,n}]^2}{(1-\rho_{n,n})^3},$$

2. This method is not limited to just analyzing quantum spin systems, it can be use to study the other control problem, which will be discussed in the following chapters.

whenever $\rho_{n,n} < 1 - \varepsilon$.

By applying the assumptions $u \in \mathcal{C}^1(\mathcal{S}_N, \mathbb{R})$ and $u(\boldsymbol{\rho}_n) = 0$, we deduce that $|u(\rho)| = |u(\rho) - u(\boldsymbol{\rho}_n)| \leq C\|\rho - \boldsymbol{\rho}_n\|_{HS} \leq \sqrt{2}C\sqrt{1-\rho_{n,n}}$, where as matrix norm we have used the Hilbert-Schmidt norm. Then by

$$\begin{aligned}\mathrm{Tr}(i[J_y, \rho]\boldsymbol{\rho}_n) &= 2c_{n+1}\mathbf{Re}\{\rho_{n,n+1}\} - 2c_n\mathbf{Re}\{\rho_{n,n-1}\} \\ &\leq 2(c_{n+1} + c_n)\sqrt{\rho_{n,n}(1-\rho_{n,n})},\end{aligned}$$

we have $|u(\rho)\mathrm{Tr}(i[J_y, \rho]\boldsymbol{\rho}_n)| \leq 2C(c_{n+1}+c_n)(1-\rho_{n,n})$. Also, as we have $|J - n - \mathrm{Tr}(J_z\rho)| \leq 2J(1-\rho_{n,n})$, we get $\mathscr{L}V(\rho) \leq KV(\rho)$, with $K = 2C(c_{n+1}+c_n) + 16J^2\eta M$. To conclude the proof, one just applies the same arguments as in the previous lemma. □

Consider the observation process of the system Y_t, whose dynamics satisfies

$$dY_t = dW_t + 2\sqrt{\eta M}\mathrm{Tr}(J_z\rho_t)dt.$$

By Girsanov's theorem (Theorem A.1.2), the process Y_t is a standard Wiener process under a new probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$. Denote by $\mathcal{F}_t^Y := \sigma(Y_s, 0 \leq s \leq t)$ the σ-field generated by the observation process up to time t. Due to Belavkin-Zakai equation (3.41) and noncommutative Kallianpur-Striebel formula (3.42), the Zakai equation associated with Equation (4.1) takes the following linear form

$$d\tilde{\rho}_t = F(\tilde{\rho}_t)dt + \sqrt{\eta}\widetilde{G}(\tilde{\rho}_t)dY_t, \tag{4.15}$$

where $\tilde{\rho}_t = \tilde{\rho}_t^* \geq 0$, $F(\tilde{\rho})$ is defined as in (4.1), and $\widetilde{G}(\tilde{\rho}) := \sqrt{M}(J_z\tilde{\rho} + \tilde{\rho}J_z)$. The equation (4.15) has a unique strong solution [Xio08, Pro04], and the solutions of the equations (4.1) and (4.15) satisfy the relation

$$\rho_t = \tilde{\rho}_t/\mathrm{Tr}(\tilde{\rho}_t), \tag{4.16}$$

which can be verified easily by applying Itô formula and corresponds to the noncommutative Kallianpur-Striebel formula (3.42). In the following lemma, we adapt [MvH07, Lemma 3.2] to the case of positive-definite matrices.

Lemma 4.3.2. *The set of positive-definite matrices is a.s. invariant for* (4.1). *More in general, the rank of ρ_t is a.s. non-decreasing.*

Proof. The initial state of (4.15) with respect to the basis of its eigenstates is given by $\tilde{\rho}_0 = \sum_i \tilde{\lambda}_i\tilde{\psi}_i\tilde{\psi}_i^*$, where $\tilde{\rho}_0\tilde{\psi}_i = \tilde{\lambda}_i\tilde{\psi}_i$ for $i \in \{0, \ldots, 2J\}$. If $\rho_0 > 0$, due to the relation (4.16), we have $\tilde{\rho}_0 > 0$, thus $\tilde{\lambda}_i > 0$ for all i. Extend the probability space by defining $\mathcal{F}_t^{Y,\widetilde{W}} := \sigma(y_s, \widetilde{W}_s, 0 \leq s \leq t)$, where $\widetilde{W}_t$ is a Brownian motion independent of Y_t. Set $B_t := \sqrt{\eta}Y_t + \sqrt{1-\eta}\widetilde{W}_t$, whose quadratic variation satisfies $[B, B]_t = t$. Following [MvH07, Lemma 3.2], we consider the equations

$$\begin{aligned}d\rho_t^i &= F(\rho_t^i)dt + \widetilde{G}(\rho_t^i)\sqrt{\eta}dY_t + \widetilde{G}(\rho_t^i)\sqrt{1-\eta}\,d\widetilde{W}_t, \quad \rho_0^i = \tilde{\psi}_i\tilde{\psi}_i^*, \\ d\tilde{\psi}_i(t) &= (i\omega J_z - iu_tJ_y - M/2J_z^2)\tilde{\psi}_i(t)dt + \sqrt{M}J_z\tilde{\psi}_i(t)dB_t, \quad \tilde{\psi}_i(0) = \tilde{\psi}_i,\end{aligned}$$

where $\tilde{\psi}_i(t) \in \mathbb{C}^N$. The solutions of the equations above satisfy $\rho_t^i = \tilde{\psi}_i(t)\tilde{\psi}_i^*(t)$ by Itô formula. In virtue of [Pro04, Theorem 5.48], for all $t \geq 0$, there exists an almost surely invertible random matrix U_t such that $\tilde{\psi}_i(t) = U_t\tilde{\psi}_i$.

Let $\rho'_t = \sum_i \tilde{\lambda}_i \rho^i_t$, so that in particular $\rho'_0 = \tilde{\rho}_0$ and $\rho'_t = U_t \tilde{\rho}_0 U_t^*$. Due to the linearity of $F(\cdot)$ and $\widetilde{G}(\cdot)$, the stochastic Fubini theorem [Xio08, Lemma 5.4] and the Itô isometry,

$$\mathbb{E}(\rho'_t|\mathcal{F}_t^Y) = \rho'_0 + \int_0^t F\big(\mathbb{E}(\rho'_s|\mathcal{F}_t^Y)\big)ds + \int_0^t \widetilde{G}\big(\mathbb{E}(\rho'_s|\mathcal{F}_t^Y)\big)\sqrt{\eta}dY_s.$$

By the uniqueness in law [RY13, Proposition 9.1.4] of the solution of the equation (4.15), the laws of $\tilde{\rho}_t$ and $\mathbb{E}(\rho'_t|\mathcal{F}_t^Y) = \mathbb{E}(U_t\tilde{\rho}_0 U_t^*|\mathcal{F}_t^Y)$ are equal for all $t \geq 0$. By what precedes $\rho_0 > 0$ implies $\rho'_t > 0$ a.s. which in turn yields $\rho_t = \tilde{\rho}_t/\mathrm{Tr}(\tilde{\rho}_t) > 0$ a.s. We have thus proved that the set of positive-definite matrices is a.s. invariant for (4.1).

Let us now consider the general case in which ρ_0 is not necessarily full rank. We have

$$\mathrm{rank}(\rho'_t) = \mathrm{rank}(U_t\tilde{\rho}_0 U_t^*) = \mathrm{rank}(\tilde{\rho}_0) = \mathrm{rank}(\rho_0), \quad a.s. \tag{4.17}$$

Note that the kernel of any positive semi-definite matrix $\hat{\rho} \in \mathbb{C}^{N\times N}$ coincides with the space $\{\psi \in \mathbb{C}^N | \psi^*\hat{\rho}\psi = 0\}$, and that for almost every path $\rho'_t(\omega)$

$$\{\psi \in \mathbb{C}^N | \mathbb{E}(\psi^*\rho'_t\psi|\mathcal{F}_t^Y) = 0\} \subseteq \{\psi \in \mathbb{C}^N | \psi^*\rho'_t(\omega)\psi = 0\}.$$

This implies $\mathrm{rank}(\tilde{\rho}_t) \geq \mathrm{rank}(\rho'_t) = \mathrm{rank}(\rho_0)$ for any $t \geq 0$ almost surely, which concludes the proof. □

Lemma 4.3.3. *If $\eta = 1$, then the boundary of the state space*

$$\partial\mathcal{S}_N := \{\rho \in \mathcal{S}_N | \ \det(\rho) = 0\}$$

is a.s. invariant for (4.1).

Proof. Based on the proof of Lemma 4.3.2, if $\eta = 1$, we have $B_t = Y_t$ which implies $\tilde{\rho}_t = \rho'_t$. Then by applying the relation (4.17), we get the conclusion. □

The Stratonovich form of Equation (4.1) is given by

$$d\rho_t = \widehat{F}(\rho_t)dt + \sqrt{\eta}G(\rho_t) \circ dW_t, \tag{4.18}$$

where

$$\begin{aligned}\widehat{F}(\rho) :=& -i[\omega J_z + u(\rho)J_y, \rho] + 2\eta M\mathrm{Tr}(J_z\rho)(J_z\rho + \rho J_z - 2\mathrm{Tr}[J_z\rho]\rho) \\ &+ M\left((1-\eta)J_z\rho J_z - \frac{1+\eta}{2}(J_z^2\rho + \rho J_z^2) + 2\eta\mathrm{Tr}(J_z^2\rho)\rho\right),\end{aligned}$$

and G is defined as in (4.1). The corresponding deterministic control system is given by

$$\dot{\rho}_v(t) = \widehat{F}\big(\rho_v(t)\big) + \sqrt{\eta}G\big(\rho_v(t)\big)v(t), \quad \rho_v(0) = \rho_0, \tag{4.19}$$

where $v(t) \in \mathcal{V}$, where $\mathcal{V}$ is the set of all piecewise constant functions from $\mathbb{R}_+$ to $\mathbb{R}$. By the support theorem (Theorem A.2.1), the set $\mathcal{S}_N$ is positively invariant for Equation (4.19).

In the following, we state two important preliminary results, which provide the sufficient conditions on the continuous feedback controller to overcome the challenge of breaking the symmetry of the state space, and will be applied to our stabilization problem in the following subsection. For this purpose, we fix a target state $\boldsymbol{\rho}_{\bar{n}}$ for some $\bar{n} \in \{0, \ldots, 2J\}$.

Proposition 4.3.4. *Suppose $\eta \in (0,1)$ and the feedback controller satisfying the assumptions of Lemma 4.3.1. Assume that $\nabla u \cdot G(\rho_0) \neq 0$ or $\nabla u \cdot \widehat{F}(\rho_0) \neq 0$ for any $\rho_0 \in \{\rho \in \mathcal{S}_N \setminus \boldsymbol{\rho}_{\bar{n}} |\ \rho_{k,k} = 0$ for some k, and $u(\rho) = 0\}$. Then for any initial condition $\rho_0 \in \{\rho \in \mathcal{S}_N \setminus \boldsymbol{\rho}_{\bar{n}} |\ \rho_{k,k} = 0$ for some $k\}$ and $\varepsilon > 0$, there exists at most one trajectory $\rho_v(t)$ of (4.19) starting from ρ_0 which lies in $\partial\mathcal{S}_N$ for t in $[0,\varepsilon]$. For any other initial state $\rho_0 \in \partial\mathcal{S}_N \setminus \boldsymbol{\rho}_{\bar{n}}$ and $v \in \mathcal{V}$, $\rho_v(t) > 0$ for $t > 0$.*

Proof. Define $Z_1(t) := \mathrm{Span}\{e_k |\ \big(\rho_v(t)\big)_{k,k} = 0\}$ and $Z_2(t)$ the eigenspace corresponding to the eigenvalue 0 of $\rho_v(t)$. By definition, $Z_1(t) \subseteq Z_2(t)$ for all $t \geq 0$. Since all the subspaces which are invariant by J_z take the form $\mathrm{Span}\{e_{k_1}, \ldots, e_{k_h}\}$ for $\{k_1, \ldots, k_h\} \subseteq \{0, \ldots, 2J\}$, we deduce that $Z_1(t)$ is the largest subspace of $Z_2(t)$ invariant by J_z.

Denote by $\lambda_k(t)$ and $\psi_k(t)$ for $k \in \{0, \ldots, 2J\}$ the eigenvalues and eigenvectors of $\rho_v(t)$, where, without loss of generality, we assume $\lambda_k(t) \in \mathcal{C}^1$ since $\rho_v(t) \in \mathcal{C}^1$ ([Kat76, Theorem 2.6.8]). In addition, we suppose that the eigenvectors $\psi_k(t)$ form an orthonormal basis of $\mathbb{C}^N$.

Let $\psi_k(t) \in Z_2(t)$ for $t \in [0,\varepsilon]$. In order to provide an expression of the derivative for the eigenvalue λ_k along the path, we observe that

$$\frac{1}{t}(\lambda_k(t+\delta) - \lambda_k(t)) = \frac{1}{\psi_k^*(t+\delta)\psi_k(t)} \left(\psi_k^*(t+\delta) \frac{\rho_v(t+\delta) - \rho_v(t)}{t} \psi_k(t) \right). \tag{4.20}$$

Since ψ_k is a unit vector, then by compactness, we can extract a sequence $\delta_n \searrow 0$ such that $\psi_k(t+\delta_n)$ converges to an eigenvector $\psi_k(t)$ of $\rho_v(t)$. By passing to the limit on the left-hand and right-hand sides of Equation (4.20), we get $\dot{\lambda}_k(t) = \psi_k^*(t)\dot{\rho}_v(t)\psi_k(t) = M(1-\eta)\psi_k^*(t)J_z\rho_v(t)J_z\psi_k(t)$.

If $\psi_k(t) \notin Z_1(t)$ then $J_z\psi_k(t) \notin Z_2(t)$, since otherwise $Z_1(t)$ would not be the largest subspace invariant by J_z contained in $Z_2(t)$. Thus $\dot{\lambda}_k(t) > 0$, which implies $\lambda_k(s) > 0$ for any $s - t > 0$ sufficiently small, then we can deduce that $\dim Z_2(s) \leq \dim Z_1(t)$. Moreover, by continuity of $\rho_v(t)$, we have $Z_1(s) \subseteq Z_1(t)$, for any $s - t > 0$ sufficiently small. Now we consider the case where $Z_1(t) \neq 0$ for $t \geq 0$. In this case, we have two possibilities: either $u\big(\rho_v(\cdot)\big) \equiv 0$ on $[0,\varepsilon]$ for some $\varepsilon > 0$; or $u\big(\rho_v(t)\big) \neq 0$ for arbitrarily small $t > 0$. Note that under the assumptions of the proposition there exists at most one v such that $u\big(\rho_v(\cdot)\big) \equiv 0$. It is therefore sufficient to show that, for the second possibility, $\rho_v(t)$ belongs to the interior of $\mathcal{S}_N$ for all $t > 0$. For this purpose, we first show that for all $t > 0$ such that $u\big(\rho_v(t)\big) \neq 0$ and $Z_1(t) \neq 0$, there exists $s - t > 0$ arbitrarily small such that $u\big(\rho_v(s)\big) \neq 0$ and $Z_1(s) \subsetneqq Z_1(t)$.

Let us pick k such that $e_k \in Z_1(t)$, and at least one between e_{k-1} and e_{k+1} is not contained in $Z_1(t)$ [3]. We now show by contradiction that $e_k \notin Z_1(s)$ for some $s - t > 0$ arbitrarily small. We assume that $e_k \in Z_1(\tau)$ for $\tau \in [t, t+\varepsilon]$, with $\varepsilon > 0$. By setting $q^n(\tau) := \rho_v(\tau)e_n$, for $n \in \{0, \ldots, 2J\}$ and $\tau \geq 0$, the condition $\big(\rho_v(\tau)\big)_{n,n} = 0$ is equivalent to $q^n(\tau) = 0$. In particular, by assumption, $q^k(\tau) = 0$ for $\tau \in [t, t+\varepsilon]$. On this interval we have

$$\dot{q}^k(\tau) = iu\big(\rho_v(\tau)\big)\rho_v(\tau)J_y e_k = u\big(\rho_v(\tau)\big)\rho_v(\tau)\psi = 0,$$

where $\psi := c_k e_{k-1} - c_{k+1}e_{k+1}$. By taking ε small enough we may assume $u\big(\rho_v(\tau)\big) \neq 0$ and therefore the previous equality implies $\rho_v(\tau)\psi = 0$. This means that $\psi \in Z_2(\tau)$ and, since

3. If $k = 0$, the condition is replaced by $e_1 \notin Z_1(t)$ while if $k = 2J$, we assume $e_{2J-1} \notin Z_1(t)$.

$\psi \notin Z_1(\tau)$, by the above argument we have $J_z\psi \notin Z_2(\tau)$ and

$$\psi^* \dot{\rho}_v(\tau)\psi = M(1-\eta)\psi^* J_z \rho_v(\tau) J_z \psi > 0,$$

leading to a contradiction.

Hence, there exists $s-t>0$ arbitrarily small such that $Z_1(s) \subsetneqq Z_1(t)$ and, by continuity of u, $u\big(\rho_v(s)\big) \neq 0$. Thus, by repeating the arguments for a finite number of steps, we can show that there exists $s-t>0$ arbitrary small such that $Z_1(s)=0$. As t may also be chosen arbitrarily small, this means that there exists an arbitrarily small $s>0$ such that $\rho_v(s)>0$.

To conclude the proof, we show that if $\rho_v(t_0)>0$ for some $t_0 \geq 0$, then $\rho_v(t)>0$ for all $t>t_0$. This can be done by considering the flow $\Phi_{t,v} : \mathcal{S}_N \to \mathcal{S}_N$ of Equation (4.19) which associates with each ρ_0, the value $\rho_v(t)$. Since $\Phi_{t,v}$ is a diffeomorphism, if $\rho \in \mathcal{S}_N \setminus \partial\mathcal{S}_N$, there is an open neighborhood U of the state ρ such that $\Phi_{t,v}U \subset \mathcal{S}_N$ is also an open neighborhood of $\Phi_{t,v}\rho$. Thus, $\Phi_{t,v}\rho \in \mathcal{S}_N \setminus \partial\mathcal{S}_N$. The proof is then complete. □

Corollary 4.3.5. *Suppose that the assumptions of Proposition 4.3.4 are satisfied. Then for all $\rho_0 \in \partial\mathcal{S}_N \setminus \boldsymbol{\rho}_{\bar{n}}$, either ρ_t stays on the boundary of $\partial\mathcal{S}_N$ and converges to $\boldsymbol{\rho}_{\bar{n}}$ as t goes to infinity or it exits the boundary in finite time and stays in the interior of $\mathcal{S}_N$ afterwards, almost surely.*

Proof. By the support theorem (Theorem A.2.1) and Proposition 4.3.4, we have $\mathbb{P}(\rho_\nu > 0) > 0$ for all $\nu > 0$ independently of the initial state $\rho_0 \in \mathcal{S}_N \setminus \boldsymbol{\rho}_{\bar{n}}$. Define the set $\mathcal{S}_{\leq\zeta} := \{\rho \in \mathcal{S}_N | \det(\rho) \leq \zeta\} \setminus B_r(\boldsymbol{\rho}_{\bar{n}})$ for any r arbitrary small and the stopping time $\tau_\zeta := \inf\{t \geq 0 | \rho_t \notin \mathcal{S}_{\leq\zeta}\}$. Now by compactness of $\mathcal{S}_{\leq\zeta}$ and the Feller continuity of ρ_t ([MvH07, Lemma 4.5]), it is easy to see that for any $\nu>0$ and $\zeta>0$ small enough, there exists $\varepsilon>0$ such that $\mathbb{P}_{\rho_0}(\tau_\zeta < \nu) > \varepsilon$,[4] independently of $\rho_0 \in \mathcal{S}_{\leq\zeta}$. Then we can conclude that $\sup_{\rho_0 \in \mathcal{S}_{\leq\zeta}} \mathbb{P}_{\rho_0}(\tau_\zeta \geq \nu) \leq 1-\varepsilon$. By Dynkin inequality [Dyn65],

$$\sup_{\rho_0 \in \mathcal{S}_{\leq\zeta}} \mathbb{E}_{\rho_0}(\tau_\zeta) \leq \frac{\nu}{1 - \sup_{\rho_0 \in \mathcal{S}_{\leq\zeta}} \mathbb{P}_{\rho_0}(\tau_\zeta \geq \nu)} \leq \frac{\nu}{\varepsilon} < \infty.$$

By Markov inequality, for all $\rho_0 \in \mathcal{S}_{\leq\zeta}$, we have

$$\mathbb{P}_{\rho_0}(\tau_\zeta = \infty) = \lim_{n\to\infty} \mathbb{P}_{\rho_0}(\tau_\zeta \geq n) \leq \lim_{n\to\infty} \mathbb{E}_{\rho_0}(\tau_\zeta)/n = 0.$$

By arbitrariness of r we deduce that, either $\rho_t > 0$ for some positive time t or ρ_t converges to $\boldsymbol{\rho}_{\bar{n}}$ as t tends to infinity while staying in $\partial\mathcal{S}_N$, almost surely. In addition, by the strong Markov property of ρ_t and Lemma 4.3.2, once ρ_t exits the boundary and enters the interior of $\mathcal{S}$, it stays in the interior afterwards. The proof is hence complete. □

4.3.2 Exponential stabilization by continuous feedback

In this subsection, we study the exponential stabilization of system (4.1) towards a selected target state $\boldsymbol{\rho}_{\bar{n}}$ with $\bar{n} \in \{0, \ldots, 2J\}$. Firstly, we establish a general result ensuring the exponential convergence towards $\boldsymbol{\rho}_{\bar{n}}$ under some assumptions on the feedback control law and an additional local Lyapunov type condition. Next, we design a parametrized family of feedback control laws satisfying such conditions.

4. Recall that $\mathbb{P}_{\rho_0}$ corresponds to the probability law of ρ_t starting at ρ_0; the associated expectation is denoted by $\mathbb{E}_{\rho_0}$.

Almost sure global exponential stabilization Inspired by [Tsu08, Lemma 3.4] and [RY13, Proposition 3.1], in the following lemma we show that, wherever the initial state is, the trajectory ρ_t enters in $B_r(\boldsymbol{\rho}_{\bar{n}})$ with $r > 0$ in finite time almost surely.

Before stating the result, we define $\mathbf{P}_{\bar{n}} := \{\rho \in \mathcal{S}_N | \, J - \bar{n} - \mathrm{Tr}(J_z\rho) = 0\}$ and the "variance function" $\mathscr{V}(\rho) := \mathrm{Tr}(J_z^2\rho) - \mathrm{Tr}^2(J_z\rho)$ of J_z.

Lemma 4.3.6. *Assume the feedback controller satisfying the assumptions of Lemma 4.3.1. Suppose that for any $\rho_0 \in \{\rho \in \mathcal{S}_N | \, \rho_{\bar{n},\bar{n}} = 0\}$, there exists a control $v(t) \in \mathcal{V}$ such that for all $t \in (0, \varepsilon)$, with ε sufficiently small, $u\big(\rho_v(t)\big) \neq 0$, for some solution $\rho_v(t)$ of Equation (4.19). Assume moreover that*

$$\forall \rho \in \mathbf{P}_{\bar{n}} \setminus \boldsymbol{\rho}_{\bar{n}}, \quad 2\eta M \mathscr{V}(\rho)\rho_{\bar{n},\bar{n}} > u(\rho)\mathrm{Tr}(i[J_y, \rho]\boldsymbol{\rho}_{\bar{n}}). \tag{4.21}$$

Then for all $r > 0$ and any given initial state $\rho_0 \in \mathcal{S}_N$, $\mathbb{P}(\tau_r < \infty) = 1$, where $\tau_r := \inf\{t \geq 0 | \, \rho_t \in B_r(\boldsymbol{\rho}_{\bar{n}})\}$ and ρ_t corresponds to the solution of system (4.1).

Proof. The lemma holds trivially for $\rho_0 \in B_r(\boldsymbol{\rho}_{\bar{n}})$, as in that case $\tau_r = 0$. Let us thus suppose that $\rho_0 \in \mathcal{S}_N \setminus B_r(\boldsymbol{\rho}_{\bar{n}})$. We show that there exists $T \in (0, \infty)$ and $\zeta \in (0, 1)$ such that $\mathbb{P}_{\rho_0}(\tau_r < T) > \zeta$. For this purpose, we make use of the support theorem. Therefore, we consider the differential equation

$$\big(\dot{\rho}_v(t)\big)_{\bar{n},\bar{n}} = \Delta_{\bar{n}}\big(\rho_v(t)\big) + 2\sqrt{\eta M} P_{\bar{n}}\big(\rho_v(t)\big)\big(\rho_v(t)\big)_{\bar{n},\bar{n}} v(t),$$

where $v(t) \in \mathcal{V}$ is the control input, and

$$\begin{aligned} \Delta_{\bar{n}}(\rho) :=\ & 2\eta M\big(\mathrm{Tr}(J_z^2\rho) - (J - \bar{n})^2\big)\rho_{\bar{n},\bar{n}} - u(\rho)\mathrm{Tr}\big(i[J_y, \rho]\boldsymbol{\rho}_{\bar{n}}\big) \\ & + 4\eta M P_{\bar{n}}(\rho)\mathrm{Tr}(J_z\rho)\rho_{\bar{n},\bar{n}}, \\ P_{\bar{n}}(\rho) :=\ & J - \bar{n} - \mathrm{Tr}(J_z\rho). \end{aligned}$$

Consider the special case in which $\rho_{\bar{n},\bar{n}}(0) = 0$. By applying similar arguments as in the proof of Proposition 4.3.4, there exists a control input $v \in \mathcal{V}$ such that $\big(\rho_v(t)\big)_{\bar{n},\bar{n}} > 0$ for all $t > 0$. Thus, without loss the generality, we suppose $\rho_{\bar{n},\bar{n}}(0) > 0$. Then we show that there exist a control input v and a time $T \in (0, \infty)$ such that $\rho_v(t) \in B_r(\boldsymbol{\rho}_{\bar{n}})$ for $t \leq T$ in the two following separate cases.

1. Let $\bar{n} \in \{0, 2J\}$. We have $\mathbf{P}_{\bar{n}} = \boldsymbol{\rho}_{\bar{n}}$. Since $\mathcal{S} \setminus B_r(\boldsymbol{\rho}_{\bar{n}})$ is compact, $\Delta_{\bar{n}}(\rho)$ is bounded from above in this domain and $|P_{\bar{n}}(\rho)|$ is bounded from below. Then by choosing the control input $v = KP_{\bar{n}}(\rho)/\rho_{\bar{n},\bar{n}}$, with $K > 0$ sufficiently large, we can guarantee that $\rho_v(t) \in B_r(\boldsymbol{\rho}_{\bar{n}})$ for $t \leq T$ with $T < \infty$ if $\rho_{\bar{n},\bar{n}}(0) > 0$.
2. Now suppose $\bar{n} \in \{1, \cdots, 2J - 1\}$. Due to the compactness of $\mathbf{P}_{\bar{n}} \setminus B_r(\boldsymbol{\rho}_{\bar{n}})$ and the condition (4.21), we have

$$\begin{aligned} m :=\ & \min_{\rho \in \mathbf{P}_{\bar{n}} \setminus B_r(\boldsymbol{\rho}_{\bar{n}})} \Delta_{\bar{n}}(\rho) \\ =\ & \min_{\rho \in \mathbf{P}_{\bar{n}} \setminus B_r(\boldsymbol{\rho}_{\bar{n}})} \Big(2\eta M \mathscr{V}(\rho)\rho_{\bar{n},\bar{n}} - u(\rho)\mathrm{Tr}\big(i[J_y, \rho]\boldsymbol{\rho}_{\bar{n}}\big)\Big) > 0. \end{aligned}$$

Then we define an open set containing $\mathbf{P}_{\bar{n}} \setminus B_r(\boldsymbol{\rho}_{\bar{n}})$,

$$\mathbf{P}_{\bar{n}} \setminus B_r(\boldsymbol{\rho}_{\bar{n}}) \subseteq \mathbf{U} := \{\rho \in \mathcal{S} | \, \Delta_{\bar{n}}(\rho) > m/2\} \subseteq \mathcal{S}_N.$$

Thus, setting $v(t) = 0$ whenever $\rho_v(t) \in \mathbf{U}$, we have

$$\big(\dot{\rho}_v(t)\big)_{\bar{n},\bar{n}} = \Delta_{\bar{n}}\big(\rho_v(t)\big) > m/2 \quad \text{on } \mathbf{U}.$$

Moreover, $\big(\mathcal{S}_N \setminus B_r(\boldsymbol{\rho}_{\bar{n}})\big) \setminus \mathbf{U}$ is compact, then $\Delta_{\bar{n}}(\rho)$ is bounded from above and $|P_{\bar{n}}(\rho)|$ is bounded from below in this domain. For all $\rho_v(t) \in \{\rho \in \mathcal{S}_N | \, \rho_{\bar{n},\bar{n}} > 0\}$, we can take the feedback $v = KP_{\bar{n}}(\rho)/\rho_{\bar{n},\bar{n}}$ with $K > 0$ sufficiently large, so that $\big(\dot{\rho}_v(t)\big)_{\bar{n},\bar{n}}$ is bounded from below on $\big(\mathcal{S}\setminus B_r(\boldsymbol{\rho}_{\bar{n}})\big)\setminus\mathbf{U}$. The proposed input v guarantees that $\rho_v(t) \in B_r(\boldsymbol{\rho}_{\bar{n}})$ for $t \leq T$ with $T < \infty$ if $\rho_{\bar{n},\bar{n}}(0) > 0$.

Therefore, there exists $T \in (0, \infty)$ such that, for all $\rho_0 \in \mathcal{S}_N \setminus B_r(\boldsymbol{\rho}_{\bar{n}})$, there exists $v(t)$ steering the system from ρ_0 to $B_r(\boldsymbol{\rho}_{\bar{n}})$ by time T. By compactness of $\mathcal{S}_N \setminus B_r(\boldsymbol{\rho}_{\bar{n}})$ and the Feller continuity of ρ_t, we have $\sup_{\rho_0 \in \mathcal{S}_N \setminus B_r(\boldsymbol{\rho}_{\bar{n}})} \mathbb{P}_{\rho_0}(\tau_r \geq T) \leq 1 - \zeta < 1$, for some $\zeta > 0$. By Dynkin inequality [Dyn65],

$$\sup_{\rho_0 \in \mathcal{S}_N \setminus B_r(\boldsymbol{\rho}_{\bar{n}})} \mathbb{E}_{\rho_0}(\tau_r) \leq \frac{T}{1 - \sup_{\rho_0 \in \mathcal{S}_N \setminus B_r(\boldsymbol{\rho}_{\bar{n}})} \mathbb{P}_{\rho_0}(\tau_r \geq T)} \leq \frac{T}{\zeta} < \infty.$$

Then by Markov inequality, for all $\rho_0 \in \mathcal{S}_N \setminus B_r(\boldsymbol{\rho}_{\bar{n}})$, we have

$$\mathbb{P}_{\rho_0}(\tau_r = \infty) = \lim_{n\to\infty} \mathbb{P}_{\rho_0}(\tau_r \geq n) \leq \lim_{n\to\infty} \mathbb{E}_{\rho_0}(\tau_r)/n = 0,$$

which implies $\mathbb{P}_{\rho_0}(\tau_r < \infty) = 1$. The proof is complete. □

Remark 4.3.7. The above lemma provides a sufficient condition on the feedback to ensure that ρ_t enters in an arbitrary neighborhood of the target state in finite time almost surely. This result is stronger than [MvH07, Lemma 4.5 and 4.6] which only obtains the existence of a neighborhood where the previous property holds, for a constant control input.

In the following, we state our general result concerning the exponential stabilization of N-level open quantum spin systems (4.1).

Theorem 4.3.8. *Assume that the feedback control law satisfies the assumptions of Lemma 4.3.6. Additionally, suppose that there exists a positive-definite function $V(\rho)$ such that $V(\rho) = 0$ if and only if $\rho = \boldsymbol{\rho}_{\bar{n}}$, and V is continuous on $\mathcal{S}_N$ and twice continuously differentiable on the set $\mathcal{S}_N \setminus \boldsymbol{\rho}_{\bar{n}}$. Moreover, suppose that there exist positive constants C, C_1 and C_2 such that*

(i) $C_1\, d_B(\rho, \boldsymbol{\rho}_{\bar{n}}) \leq V(\rho) \leq C_2\, d_B(\rho, \boldsymbol{\rho}_{\bar{n}})$, *for all* $\rho \in \mathcal{S}_N$, *and*

(ii) $\limsup_{\rho\to\boldsymbol{\rho}_{\bar{n}}} \frac{\mathscr{L}V(\rho)}{V(\rho)} \leq -C$.

Then, $\boldsymbol{\rho}_{\bar{n}}$ is a.s. exponentially stable for the system (4.1) with sample Lyapunov exponent less than or equal to $-C - \frac{K}{2}$, where $K := \liminf_{\rho\to\boldsymbol{\rho}_{\bar{n}}} g^2(\rho)$ and $g(\rho) := \sqrt{\eta}\frac{\partial V(\rho)}{\partial \rho}\frac{G(\rho)}{V(\rho)}$.

Proof. The proof proceeds in three steps:

1. first we show that $\boldsymbol{\rho}_{\bar{n}}$ is locally stable in probability ;
2. next we show that for any fixed $r > 0$ and almost all sample paths, there exists $T < \infty$ such that for all $t \geq T$, $\rho_t \in B_r(\boldsymbol{\rho}_{\bar{n}})$; and
3. finally, we prove that $\boldsymbol{\rho}_{\bar{n}}$ is a.s. exponentially stable with sample Lyapunov exponent less than or equal to $-C - \frac{K}{2}$.

Step 1 : By the condition *(ii)*, we can choose $r > 0$ sufficiently small such that $\mathscr{L}V(\rho) \leq -C(r)V(\rho)$ for $\rho \in B_r(\boldsymbol{\rho}_{\bar{n}}) \setminus \boldsymbol{\rho}_{\bar{n}}$, for some $C(r) > 0$. Let $\varepsilon \in (0,1)$ be arbitrary. By the continuity of $V(\rho)$ and the fact that $V(\rho) = 0$ if and only if $d_B(\rho, \boldsymbol{\rho}_{\bar{n}}) = 0$, we can find $\delta = \delta(\varepsilon, r) > 0$ such that

$$1/\varepsilon \sup_{\rho_0 \in B_\delta(\boldsymbol{\rho}_{\bar{n}})} V(\rho_0) \leq C_1 r. \tag{4.22}$$

Assume that $\rho_0 \in B_\delta(\boldsymbol{\rho}_{\bar{n}})$ and let τ be the first exit time of ρ_t from $B_r(\boldsymbol{\rho}_{\bar{n}})$. By Itô formula, we have

$$\mathbb{E}\big(V(\rho_{t\wedge\tau})\big) \leq V(\rho_0) - C(r)\,\mathbb{E}\left(\int_0^{t\wedge\tau} V(\rho_s)ds\right) \leq V(\rho_0).$$

For all $t \geq \tau$, $d_B(\rho_{t\wedge\tau}, \boldsymbol{\rho}_{\bar{n}}) = d_B(\rho_\tau, \boldsymbol{\rho}_{\bar{n}}) = r$. Hence, by the condition *(i)*,

$$\mathbb{E}\big(V(\rho_{t\wedge\tau})\big) \geq \mathbb{E}\big(\mathbb{1}_{\{\tau\leq t\}}V(\rho_\tau)\big) \geq \mathbb{E}\big(\mathbb{1}_{\{\tau\leq t\}}C_1 d_B(\rho_\tau, \boldsymbol{\rho}_{\bar{n}})\big) = C_1 r\,\mathbb{P}(\tau \leq t).$$

Combining with the inequality (4.22), we have

$$\mathbb{P}(\tau \leq t) \leq \frac{\mathbb{E}\big(V(\rho_{t\wedge\tau})\big)}{C_1 r} \leq \frac{V(\rho_0)}{C_1 r} \leq \varepsilon.$$

Letting t tend to infinity, we get $\mathbb{P}(\tau < \infty) \leq \varepsilon$ which implies

$$\mathbb{P}\big(d_B(\rho_t, \boldsymbol{\rho}_{\bar{n}}) < r \text{ for } t \geq 0\big) \geq 1 - \varepsilon.$$

Step 2 : Since $u = 0$ in $\bar{E}$ if and only if $\rho = \boldsymbol{\rho}_{\bar{n}}$ by Lemma 4.3.6 we obtain, for all $\rho_0 \in \mathcal{S}_N$, $\mathbb{P}(\tau_\delta < \infty) = 1$, where $\tau_\delta := \inf\{t \geq 0 |\, \rho_t \in B_\delta(\boldsymbol{\rho}_{\bar{n}})\}$. It implies that ρ_t enters $B_\delta(\boldsymbol{\rho}_{\bar{n}})$ in a finite time almost surely. Due to Step 1, for all $\rho_0 \in B_\delta(\boldsymbol{\rho}_{\bar{n}})$, $\mathbb{P}(\sigma_r < \infty) \leq \varepsilon$, where $\sigma_r := \inf\{t \geq 0 |\, \rho_t \notin B_r(\boldsymbol{\rho}_{\bar{n}})\}$.

We define two sequences of stopping times $\{\sigma_r^k\}_{k\geq 0}$ and $\{\tau_\delta^k\}_{k\geq 1}$ such that $\sigma_r^0 = 0$, $\tau_\delta^{k+1} = \inf\{t \geq \sigma_r^k |\, \rho_t \in B_\delta(\boldsymbol{\rho}_{\bar{n}})\}$ and $\sigma_r^{k+1} = \inf\{t \geq \tau_\delta^{k+1} |\, \rho_t \notin B_r(\boldsymbol{\rho}_{\bar{n}})\}$. By the strong Markov property, we find

$$\begin{aligned}\mathbb{P}_{\rho_0}(\sigma_r^m < \infty) &= \mathbb{P}_{\rho_0}(\tau_\delta^1 < \infty, \sigma_r^1 < \infty, \ldots, \sigma_r^m < \infty)\\ &= \mathbb{P}_{\rho_{\tau_\delta^1}}(\sigma_r < \infty) \cdots \mathbb{P}_{\rho_{\tau_\delta^m}}(\sigma_r < \infty) \leq \varepsilon^m.\end{aligned}$$

Thus, for all $\rho_0 \in \mathcal{S}_N$, we have $\mathbb{P}(\sigma_r^m < \infty,\ \forall m > 0) = 0$. We deduce that, for almost all sample paths, there exists $T < \infty$ such that, for all $t \geq T$, $\rho_t \in B_r(\boldsymbol{\rho}_{\bar{n}})$, which concludes Step 2.

Step 3 : In this step, we obtain an upper bound of the sample Lyapunov exponent by employing an argument inspired by [Mao07, Theorem 4.3.3]. For $\rho \neq \boldsymbol{\rho}_{\bar{n}}$,

$$\mathscr{L}\log V(\rho) = \frac{\mathscr{L}V(\rho)}{V(\rho)} - \frac{g^2(\rho)}{2}.$$

Due to Lemma 4.3.1, $\boldsymbol{\rho}_{\bar{n}}$ cannot be attained in finite time almost surely, then by Itô formula, we have

$$\log V(\rho_t) = \log V(\rho_0) + \int_0^t \frac{\mathscr{L}V(\rho_s)}{V(\rho_s)}ds + \int_0^t g(\rho_s)dW_s - \frac{1}{2}\int_0^t g^2(\rho_s)ds.$$

Let $m \in \mathbb{Z}_{>0}$ and take arbitrarily $\varepsilon \in (0,1)$. By the exponential martingale inequality (see e.g. [Mao07, Theorem 1.7.4]), we have

$$\mathbb{P}\left(\sup_{0\le t\le m}\left[\int_0^t g(\rho_s)dW_s - \frac{\varepsilon}{2}\int_0^t g^2(\rho_s)ds\right] > \frac{2}{\varepsilon}\log m\right) \le \frac{1}{m^2}.$$

Since $\sum_{m=1}^{\infty}\frac{1}{m^2} < \infty$, by Borel-Cantelli lemma we have that for almost all sample paths there exists m_0 such that, if $m > m_0$, then

$$\sup_{0\le t\le m}\left(\int_0^t g(\rho_s)dW_s - \frac{\varepsilon}{2}\int_0^t g^2(\rho_s)ds\right) \le \frac{2}{\varepsilon}\log m.$$

Thus, for $0 \le t \le m$ and $m > m_0$,

$$\int_0^t g(\rho_s)dW_s \le \frac{2}{\varepsilon}\log m + \frac{\varepsilon}{2}\int_0^t g^2(\rho_s)ds, \quad a.s.$$

We have

$$\log V(\rho_t) \le \log V(\rho_0) + \int_0^t \frac{\mathscr{L}V(\rho_s)}{V(\rho_s)}ds + \frac{2}{\varepsilon}\log m - \frac{1-\varepsilon}{2}\int_0^t g^2(\rho_s)ds, \quad a.s.$$

It gives

$$\limsup_{t\to\infty}\frac{1}{t}\log V(\rho_t) \le \limsup_{t\to\infty}\frac{1}{t}\left(\int_0^t \frac{\mathscr{L}V(\rho_s)}{V(\rho_s)}ds - \frac{1-\varepsilon}{2}\int_0^t g^2(\rho_s)ds\right) \quad a.s.$$

Letting ε tend to zero, we have

$$\limsup_{t\to\infty}\frac{1}{t}\log V(\rho_t) \le \limsup_{t\to\infty}\frac{1}{t}\left(\int_0^t \frac{\mathscr{L}V(\rho_s)}{V(\rho_s)}ds - \frac{1}{2}\int_0^t g^2(\rho_s)ds\right) \quad a.s.$$

For every fixed $T > 0$ consider the event

$$\Omega_T = \{\rho_t \in B_r(\boldsymbol{\rho}_{\bar{n}}) \text{ for all } t \ge T\}.$$

Due to the condition *(ii)*, for almost all $\omega \in \Omega_T$,

$$\begin{aligned}\limsup_{t\to\infty}\frac{1}{t}&\left(\int_0^t \frac{\mathscr{L}V(\rho_s)}{V(\rho_s)}ds - \frac{1}{2}\int_0^t g^2(\rho_s)ds\right)\\ &\le \limsup_{t\to\infty}\frac{1}{t}\left(\int_T^t \frac{\mathscr{L}V(\rho_s)}{V(\rho_s)}ds - \frac{1}{2}\int_T^t g^2(\rho_s)ds\right)\\ &\le -C(r) - \inf_{\rho\in B_r(\boldsymbol{\rho}_{\bar{n}})\setminus\boldsymbol{\rho}_{\bar{n}}}\frac{g^2(\rho)}{2}.\end{aligned}$$

Since T can be taken arbitrarily large and Step 2 implies that $\lim_{T\to\infty}\mathbb{P}(\Omega_T) = 1$, we can conclude that

$$\limsup_{t\to\infty}\frac{1}{t}\log V(\rho_t) \le -C(r) - \inf_{\rho\in B_r(\boldsymbol{\rho}_{\bar{n}})\setminus\boldsymbol{\rho}_{\bar{n}}}\frac{g^2(\rho)}{2}, \quad a.s.$$

Finally, due to the condition *(i)* and since r can be taken arbitrarily small, we have

$$\limsup_{t\to\infty}\frac{1}{t}\log d_B(\rho_t, \boldsymbol{\rho}_{\bar{n}}) \le -C - \frac{K}{2}, \quad a.s.$$

which yields the result. □

Feedback controller design The purpose of this part is to design parametrized feedback laws which stabilize exponentially the system (4.1) almost surely towards some predetermined target state. For the choice of target state, we consider first the particular case $\bar{n} \in \{0, 2J\}$ and then the general case $\bar{n} \in \{0, \cdots, 2J\}$.

In the following theorem, we consider the case $\bar{n} \in \{0, 2J\}$. Before stating the result, we note that we can describe the set $B_{r(\lambda)}(\boldsymbol{\rho}_{\bar{n}}) \setminus \boldsymbol{\rho}_{\bar{n}}$ as follows

$$D_\lambda(\boldsymbol{\rho}_{\bar{n}}) := \{\rho \in \mathcal{S}_N | \, 0 < \lambda < \rho_{\bar{n},\bar{n}} < 1\} = B_{r(\lambda)}(\boldsymbol{\rho}_{\bar{n}}) \setminus \boldsymbol{\rho}_{\bar{n}},$$

where $r(\lambda) = \sqrt{2 - 2\sqrt{\lambda}}$.

Theorem 4.3.9. *Consider system* (4.1) *with* $\rho_0 \in \mathcal{S}_N$ *and assume* $\eta \in (0, 1)$. *Let* $\boldsymbol{\rho}_{\bar{n}} \in \{\boldsymbol{\rho}_0, \boldsymbol{\rho}_{2J}\}$ *be the target state and define the feedback controller*

$$u_{\bar{n}}(\rho) := \alpha\big(1 - \mathrm{Tr}(\rho\boldsymbol{\rho}_{\bar{n}})\big)^\beta - \gamma \,\mathrm{Tr}\big(i[J_y, \rho]\boldsymbol{\rho}_{\bar{n}}\big), \tag{4.23}$$

where $\gamma \geq 0$, $\beta \geq 1$ *and* $\alpha > 0$. *Then the feedback controller* (4.23) *exponentially stabilizes System* (4.1) *almost surely to the equilibrium* $\boldsymbol{\rho}_{\bar{n}}$ *with sample Lyapunov exponent less than or equal to* $-\eta M$.

Proof. To prove the theorem, we show that we can apply Theorem 4.3.8 with the Lyapunov function $V_{\bar{n}}(\rho) = \sqrt{1 - \mathrm{Tr}(\rho\boldsymbol{\rho}_{\bar{n}})}$ for $\bar{n} = 0$ and $\bar{n} = 2J$. First, it is easy to see that $u_{\bar{n}}$ satisfies the assumptions of Lemma 4.3.6 and Lemma 4.3.1. Then, we need to show that the conditions *(i)* and *(ii)* of Theorem 4.3.8 hold true. Note that $\frac{\sqrt{2}}{2} d_B(\rho, \boldsymbol{\rho}_{\bar{n}}) \leq V_{\bar{n}}(\rho) \leq d_B(\rho, \boldsymbol{\rho}_{\bar{n}})$, so that the condition *(i)* is shown. We are left to check the condition *(ii)*. The infinitesimal generator $\mathscr{L}V_{\bar{n}}$ takes the following form

$$\mathscr{L}V_{\bar{n}}(\rho) = \frac{u_{\bar{n}}}{2} \frac{\mathrm{Tr}\big(i[J_y, \rho]\boldsymbol{\rho}_{\bar{n}}\big)}{V_{\bar{n}}(\rho)} - \frac{\eta M}{2} \frac{\big(J - \bar{n} - \mathrm{Tr}(J_z\rho)\big)^2 \mathrm{Tr}^2(\rho\boldsymbol{\rho}_{\bar{n}})}{V_{\bar{n}}^3(\rho)}.$$

If $\bar{\mathbf{n}} = \mathbf{0}$, and $\rho \in D_\lambda(\boldsymbol{\rho}_0)$, we find

$$\frac{u_0}{2} \frac{\mathrm{Tr}\big(i[J_y, \rho]\boldsymbol{\rho}_0\big)}{V_0(\rho)} \leq \alpha c_1 \big(V_0(\rho)\big)^\beta \leq \alpha c_1 (1 - \lambda)^{\frac{\beta-1}{2}} V_0(\rho),$$

since $|\mathrm{Tr}\big(i[J_y, \rho]\boldsymbol{\rho}_0\big)| = 2c_1 |\mathbf{Re}\{\rho_{0,1}\}| \leq 2c_1 |\rho_{0,1}| \leq 2c_1 V_0(\rho)$. Moreover, we have

$$J - \mathrm{Tr}(J_z\rho) = \sum_{k=1}^{2J} k\rho_{k,k} \geq \sum_{k=1}^{2J} \rho_{k,k} = 1 - \rho_{0,0} = \big(V_0(\rho)\big)^2.$$

Thus, for all $\rho \in D_\lambda(\boldsymbol{\rho}_0)$, $\mathscr{L}V_0(\rho) \leq -C_{0,\lambda} V_0(\rho)$, where $C_{0,\lambda} = \frac{\eta M \lambda^2}{2} - \alpha c_1 (1 - \lambda)^{\frac{\beta-1}{2}}$. The case $\bar{\mathbf{n}} = \mathbf{2J}$ may be treated similarly. In particular, for all $\rho \in D_\lambda(\boldsymbol{\rho}_{2J})$, one gets $\mathscr{L}V_{2J}(\rho) \leq -C_{2J,\lambda} V_{2J}(\rho)$, where $C_{2J,\lambda} = \frac{\eta M \lambda^2}{2} - \alpha c_{2J} (1 - \lambda)^{\frac{\beta-1}{2}} = C_{0,\lambda}$.

Furthermore, for $\bar{n} \in \{0, 2J\}$, we have $g^2(\rho) \geq \eta M \lambda^2$, for all $\rho \in D_\lambda(\boldsymbol{\rho}_{\bar{n}})$. Hence, we can apply Theorem 4.3.8 for $\bar{n} \in \{0, 2J\}$, with $C = \frac{\eta M}{2}$ and $K = \eta M$. The proof is complete. □

In the following theorem, we consider the general case $\bar{n} \in \{0, \ldots, 2J\}$.

Theorem 4.3.10. *Consider system* (4.1) *with* $\rho_0 \in \mathcal{S}_N \setminus \partial\mathcal{S}_N$. *Let* $\boldsymbol{\rho}_{\bar{n}} \in \bar{E}$ *be the target state and define the feedback controller*

$$u_{\bar{n}}(\rho) := \alpha\big(P_{\bar{n}}(\rho)\big)^{\beta} = \alpha\big(J - \bar{n} - \mathrm{Tr}(J_z\rho)\big)^{\beta}, \tag{4.24}$$

where $\beta \geq 1$ *and* $\alpha > 0$. *Then the feedback controller* (4.24) *exponentially stabilizes system* (4.1) *almost surely to the equilibrium* $\boldsymbol{\rho}_{\bar{n}}$ *with sample Lyapunov exponent less than or equal to* $-\eta M$ *if* $\bar{n} \in \{0, 2J\}$ *and* $-\eta M/2$ *if* $\bar{n} \in \{1, \ldots, 2J-1\}$.

Proof. Consider the following candidate Lyapunov function

$$V_{\bar{n}}(\rho) = \sum_{k\neq\bar{n}} \sqrt{\mathrm{Tr}(\rho\boldsymbol{\rho}_k)}. \tag{4.25}$$

Due to Lemma 4.3.2, all diagonal elements of ρ_t remain strictly positive for all $t \geq 0$ almost surely. Since $V_{\bar{n}}(\rho)$ is $\mathcal{C}^2$ in $\mathcal{S}_N \setminus \partial\mathcal{S}_N$, we can make use of similar arguments as those in Theorem 4.3.8. First, we show that the following conditions are satisfied.

C.1. $2\eta M\mathscr{V}(\rho)\rho_{\bar{n},\bar{n}} > u_{\bar{n}}\mathrm{Tr}\big(i[J_y,\rho]\boldsymbol{\rho}_{\bar{n}}\big)$, $\forall\,\rho \in \mathbf{P}_{\bar{n}} \setminus \boldsymbol{\rho}_{\bar{n}}$,

C.2. $u_{\bar{n}}(\rho) \leq \overline{C}V_{\bar{n}}(\rho)$ with $\overline{C} > 0$, $\forall\,\rho \in D_\lambda(\boldsymbol{\rho}_{\bar{n}})$.

Roughly speaking, by Lemma 4.3.6, C.1 provides a sufficient condition guaranteeing the accessibility of any arbitrary small neighborhood of $\boldsymbol{\rho}_{\bar{n}}$. C.2 is helpful to obtain a bound of the type $\mathscr{L}V_{\bar{n}} \leq -CV_{\bar{n}}$ on $D_\lambda(\boldsymbol{\rho}_{\bar{n}})$.

We now show that these conditions are satisfied. The property C.1 follows from the fact that, for all $\rho \in \mathbf{P}_{\bar{n}} \setminus \boldsymbol{\rho}_{\bar{n}}$, we have $u_{\bar{n}}(\rho) = 0$ and $\mathscr{V}(\rho) > 0$.

Next, we can show that the property C.2 holds true, because

$$|P_{\bar{n}}(\rho)| = \Big|\sum_{k\neq\bar{n}} k\rho_{k,k} - \bar{n}(1-\rho_{\bar{n},\bar{n}})\Big| \leq \Upsilon(1-\rho_{\bar{n},\bar{n}}),$$

where $\Upsilon := \max\{\bar{n}, 2J-\bar{n}\}$. Then, for all $\rho \in D_\lambda(\boldsymbol{\rho}_{\bar{n}})$,

$$u_{\bar{n}}(\rho) \leq \alpha\Upsilon^{\beta}(1-\rho_{\bar{n},\bar{n}})^{\beta-1/2}\sqrt{1-\rho_{\bar{n},\bar{n}}} \leq \alpha\Upsilon^{\beta}(1-\lambda)^{\beta-1/2}V_{\bar{n}}(\rho).$$

Consider the Lyapunov function (4.25). In the following, we verify the conditions *(i)* and *(ii)* of Theorem 4.3.8. First note that by Jensen inequality, we have $V_{\bar{n}}(\rho) \leq \sqrt{2J}\sqrt{1-\rho_{\bar{n},\bar{n}}}$. Then we get $\frac{\sqrt{2}}{2}d_B(\rho,\boldsymbol{\rho}_{\bar{n}}) \leq V_{\bar{n}}(\rho) \leq \sqrt{2J}d_B(\rho,\boldsymbol{\rho}_{\bar{n}})$, hence the condition *(i)* is shown. In order to verify the condition *(ii)*, we write the infinitesimal generator of the Lyapunov function which has the following form

$$\mathscr{L}V_{\bar{n}}(\rho) = -\frac{u_{\bar{n}}}{2}\sum_{k\neq\bar{n}}\frac{\mathrm{Tr}\big(i[J_y,\rho]\boldsymbol{\rho}_k\big)}{\sqrt{\rho_{k,k}}} - \frac{\eta M}{2}\sum_{k\neq\bar{n}}\big(P_k(\rho)\big)^2\sqrt{\rho_{k,k}}.$$

We find

$$\begin{aligned}\frac{|\mathrm{Tr}\big(i[J_y,\rho]\boldsymbol{\rho}_k\big)|}{\sqrt{\rho_{k,k}}} &= \frac{|c_k\mathbf{Re}\{\rho_{k,k-1}\} - c_{k+1}\mathbf{Re}\{\rho_{k,k+1}\}|}{\sqrt{\rho_{k,k}}} \leq \frac{c_k|\rho_{k,k-1}| + c_{k+1}|\rho_{k,k+1}|}{\sqrt{\rho_{k,k}}}\\ &\leq c_k\sqrt{\rho_{k-1,k-1}} + c_{k+1}\sqrt{\rho_{k+1,k+1}} \leq c_k + c_{k+1}.\end{aligned}$$

For $k \neq \bar{n}$ and for all $\rho \in D_\lambda(\boldsymbol{\rho}_{\bar{n}})$ with $\lambda > 1 - 1/\Upsilon$, we have

$$|J - k - \mathrm{Tr}(J_z\rho)| \geq |\bar{n} - k| - |P_{\bar{n}}(\rho)| \geq 1 - \Upsilon(1 - \rho_{\bar{n},\bar{n}}) \geq 1 - \Upsilon(1 - \lambda) > 0.$$

Thus, for all $\rho \in D_\lambda(\boldsymbol{\rho}_{\bar{n}})$,

$$\mathscr{L}V_{\bar{n}}(\rho) \leq -\left(\frac{\eta M\big(1 - \Upsilon(1-\lambda)\big)^2}{2} - \alpha\Gamma\Upsilon^\beta(1-\lambda)^{\beta-1/2}\right) V_{\bar{n}}(\rho) \leq -C_{\bar{n},\lambda}V_{\bar{n}}(\rho),$$

where $\Gamma := \sum_{k\neq\bar{n}}(c_k + c_{k+1})$ and $C_{\bar{n},\lambda} := \frac{\eta M\left(1-\Upsilon(1-\lambda)\right)^2}{2} - \alpha\Gamma\Upsilon^\beta(1-\lambda)^{\beta-1/2}$.

Furthermore, for $\bar{n} \in \{0, 2J\}$, we have $g^2(\rho) \geq \eta M\lambda^2$, for all $\rho \in D_\lambda(\boldsymbol{\rho}_{\bar{n}})$. Since $C_{\bar{n},\lambda}$ and $\eta M\lambda^2$ converge respectively to $\frac{\eta M}{2}$ and ηM as λ tends to one, by employing the same arguments used earlier in the proof of Theorem 4.3.8, we find that the sample Lyapunov exponent is less than or equal to $-C - K/2$ where $C = \frac{\eta M}{2}$ for $\bar{n} \in \{0, \ldots, 2J\}$, $K = \eta M$ for $\bar{n} \in \{0, 2J\}$ and $K = 0$ for $\bar{n} \in \{1, \ldots, 2J-1\}$.

Remark 4.3.11. Locally around the target state $\boldsymbol{\rho}_{\bar{n}}$, the asymptotic behavior of the Lyapunov function (4.25) is the same as the one of the Lyapunov function (4.6). This is related to the fact that, under the assumptions on $u_{\bar{n}}$, the behavior of the system around the target state is similar to the case $u \equiv 0$. In particular, without feedback and conditioning to the event $\{\exists t' \geq 0 |\, \rho_t \in B_r(\boldsymbol{\rho}_{\bar{n}}),\ \forall t \geq t'\}$, one can show that the trajectories converge a.s. to $\boldsymbol{\rho}_{\bar{n}}$ with sample Lyapunov exponent equal to the one in Theorem 4.3.10.

Remark 4.3.12. Note that the feedback controller satisfies the assumptions of Proposition 4.3.4, that is $\sum_k \frac{\partial P_{\bar{n}}(\rho)}{\partial\rho_{k,k}}\big(G(\rho)\big)_{k,k} \neq 0$ when $u_{\bar{n}}(\rho) = 0$ and $\rho \neq \boldsymbol{\rho}_{\bar{n}}$ (this can be easily shown by applying Cauchy-Schwarz inequality). If $\eta \in (0, 1)$, Theorem 4.3.10 and Corollary 4.3.5 guarantee the convergence of almost all trajectories to the target state even if the initial state ρ_0 lies in the boundary of $\mathcal{S}_N$ (the argument is no more valid if $\eta = 1$ because of Lemma 4.3.3). Unfortunately, these results do not ensure the almost sure exponential convergence towards the target state whenever ρ_0 lies in $\partial\mathcal{S}_N \setminus \boldsymbol{\rho}_{\bar{n}}$. However, we believe that under the assumptions imposed on the feedback, we can still guarantee such convergence property. This is suggested by the following arguments.

Set the event $\Omega_{>0} = \bigcap_{t>0}\{\rho_t > 0\}$ which is $\mathcal{F}_{0+}$-measurable. By the strong Markov property of ρ_t, and by applying Blumenthal zero–one law [RW00a], we have that either $\mathbb{P}(\Omega_{>0}) = 0$ or $\mathbb{P}(\Omega_{>0}) = 1$. In order to conclude that $\mathbb{P}(\Omega_{>0}) = 1$, it would be enough to show that $\mathbb{P}(\Omega_{>0}) > 0$, i.e., ρ_t exits the boundary and enters the interior of $\mathcal{S}_N$ immediately with non-zero probability. Proposition 4.3.4 provides some intuitions about the validity of this property, as it proves that the majority of the trajectories of the associated deterministic equation (4.19) enter the interior of $\mathcal{S}_N$ immediately. It is then tempting to conjecture that under the assumption of Proposition 4.3.4, for all $\rho_0 \in \partial\mathcal{S}_N \setminus \boldsymbol{\rho}_{\bar{n}}$, $\rho_t > 0$ for all $t > 0$ almost surely. If this conjecture is correct, we can generalize Theorem 4.3.10 to the case $\rho_0 \in \mathcal{S}_N$.

Simulations We illustrate here our results by numerical simulations in the case of a three-level quantum angular momentum system. We illustrate the convergence towards the target states $\boldsymbol{\rho}_0$ and $\boldsymbol{\rho}_1$ by applying feedback laws of the form (4.23) and (4.24), respectively.

First, we set $\boldsymbol{\rho}_0$ as the target state ; the corresponding simulations with a feedback law of the form (4.23) and initial condition $\boldsymbol{\rho}_2$ are shown in Fig. 4.3. For this case, we note that a larger α can speed up the exit of the trajectories from a neighborhood of the state $\boldsymbol{\rho}_2$. Similarly, a larger γ may speed up the accessibility of a neighborhood of the target state $\boldsymbol{\rho}_0$. Finally, a larger β can weaken the role of the first term in the feedback law (4.23) on neighborhoods of the target state (a more detailed discussion for the two-level case may be found in [LAM18]).

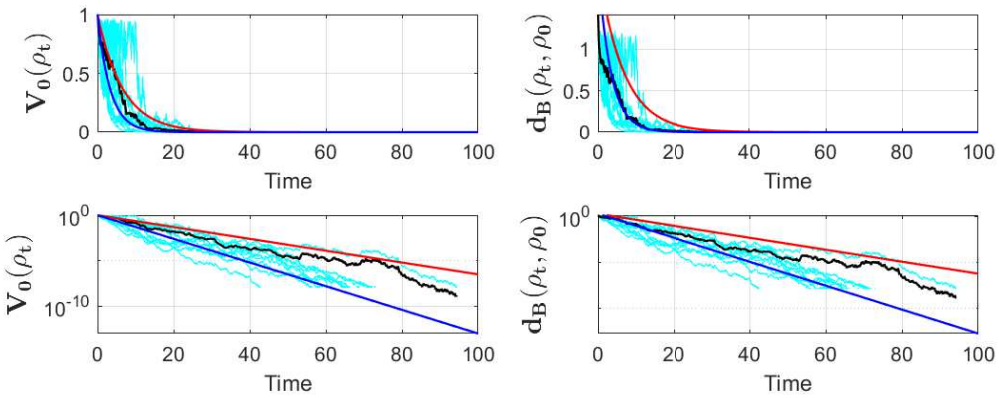


FIGURE 4.3 – Exponential stabilization of a three-level quantum angular momentum system towards $\boldsymbol{\rho}_0$ with the feedback law (4.23)starting at $\boldsymbol{\rho}_2$ with $\omega = 0$, $\eta = 0.3$, $M = 1$, $\alpha = 10$, $\beta = 5$ and $\gamma = 10$: the black curve represents the mean value of 10 arbitrary sample trajectories, the red and blue curves represent the exponential references with exponents $-\eta M/2$ and $-\eta M$ respectively. The figures at the bottom are the semi-log versions of the ones at the top.

Then, we set $\boldsymbol{\rho}_1$ as the target state ; the simulations with a feedback law of the form (4.24) and initial condition $\mathrm{diag}(0.3, 0.4, 0.3) \in \mathrm{int}(\mathcal{S}_N)$) are shown in Fig. 4.4. Finally, we repeat

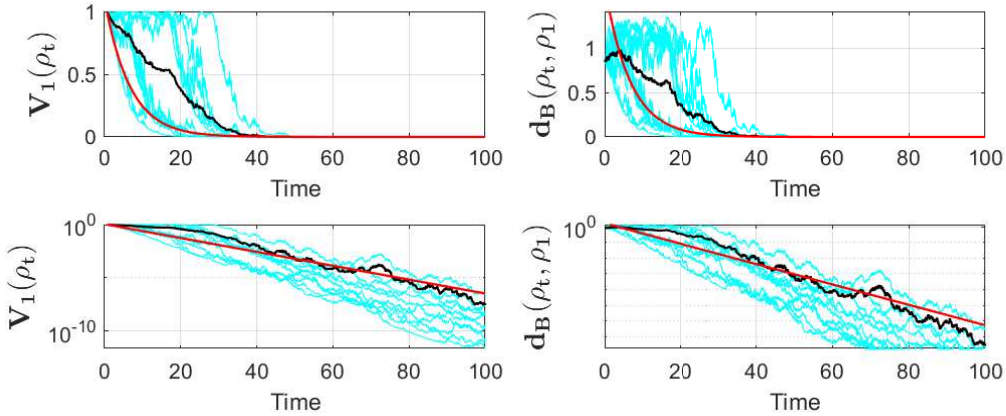


FIGURE 4.4 – Exponential stabilization of a three-level quantum angular momentum system towards $\boldsymbol{\rho}_1$ with the feedback law (4.24) starting at $\mathrm{diag}(0.3, 0.4, 0.3)$ with $\omega = 0$, $\eta = 0.3$, $M = 1$, $\alpha = 0.3$ and $\beta = 10$: the black curve represents the mean value of 10 arbitrary sample trajectories, and the red curve represents the exponential reference with exponent $-\eta M/2$. The figures at the bottom are the semi-log versions of the ones at the top.

the last simulations for the case where the initial condition is $\boldsymbol{\rho}_2$. Simulations show that the trajectories enter immediately in the interior of $\mathcal{S}_N$ and converge exponentially towards the target state.

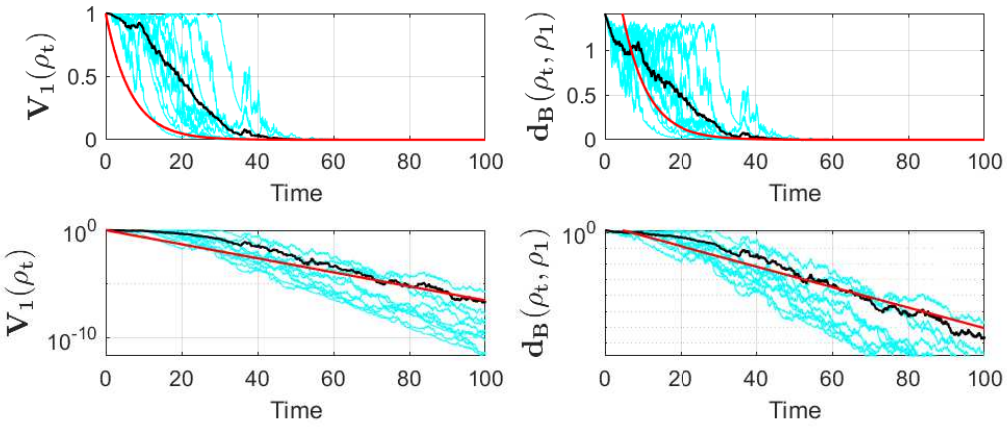


FIGURE 4.5 – Exponential stabilization of a three-level quantum angular momentum system towards $\boldsymbol{\rho}_1$ with the feedback law (4.24) starting at $\boldsymbol{\rho}_2$ with $\omega = 0$, $\eta = 0.3$, $M = 1$, $\alpha = 0.3$ and $\beta = 10$: the black curve represents the mean value of 10 arbitrary sample trajectories, and the red curve represents the exponential reference with exponent $-\eta M/2$. The figures at the bottom are the semi-log versions of the ones at the top.

5

Feedback stabilization of multi-qubit systems

In view of the rapid development of quantum information science [NC02], the generation of quantum entangled states [BŻ17] has become essential in a variety of applications such as quantum teleportation, quantum cryptography and quantum computation. The simplest entangled states are the Bell states, which are pure states corresponding to maximal quantum entanglement of two spin-$\frac{1}{2}$ systems (i.e., two-qubit systems). The four Bell states are given by $\mathbf{\Psi}_\pm = \Psi_\pm \Psi_\pm^*$ and $\mathbf{\Phi}_\pm = \Phi_\pm \Phi_\pm^*$ where

$$\Psi_\pm = \frac{1}{\sqrt{2}} \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} \pm \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ \pm 1 \end{bmatrix},$$
$$\Phi_\pm = \frac{1}{\sqrt{2}} \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} \pm \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ \pm 1 \\ 0 \end{bmatrix}.$$

The extension of Bell states to $\mathbf{n}$-qubit are called GHZ states, which are a class of $N = 2^{\mathbf{n}}$ entangled states given by $\mathbf{GHZ}_k^\pm = |\mathrm{ghz}_k^\pm\rangle\langle \mathrm{ghz}_k^\pm|$ for $k \in \{1, \ldots, N/2\}$ where

$$|\mathrm{ghz}_k^\pm\rangle = \frac{1}{\sqrt{2}} \left(\bigotimes_{m=1}^{\mathbf{n}} |k_m\rangle \pm \bigotimes_{m=1}^{\mathbf{n}} |1 - k_m\rangle \right), \quad \lambda_{k_m} \in \{0, 1\}.$$

Moreover, we can easily verify that $\{|\mathrm{ghz}_k^\pm\rangle\}$ forms an orthogonal basis of $\mathbb{C}^N$. Note that the above mentioned entangled states, Bell states and GHZ states, are all pure states.

Multi-qubit systems undergoing continuous-time measurements represent a particular example of open quantum systems whose evolution can also be described by quantum filtering equations. Here, we consider the following stochastic master equation, more general compared to (3.43), which contains n quantum channels [Bel92, BG09],

$$d\rho_t = F_0(\rho_t)dt + \sum_{k=1}^{n} F_k(\rho_t)dt + \sum_{k=1}^{n} \sqrt{\eta_k} G_k(\rho_t) dW_k(t), \tag{5.1}$$

where the quantum state is described by the density operator ρ_t, which belongs to the compact space $\mathcal{S}_N = \{\rho \in \mathbb{C}^{N \times N} | \, \rho = \rho^*, \mathrm{Tr}(\rho) = 1, \rho \geq 0\}$ with $N = 2^{\mathbf{n}}$ and $\mathbf{n}$ the number

of entangled qubits. Here $W_t = (W_k(t))_{1\leq k\leq n}$ is a n-dimensional standard Wiener process with the natural filtration $\mathcal{F}_t$, and W_k are independent, i.e., for $i, j = 1, \ldots, n$, one has $\langle W_i(t), W_j(t)\rangle = \delta_{i,j}t$. The filtered probability space associated with the above evolution is $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$. The measurement efficiency for the k-th channel is given by $\eta_k \in (0, 1]$. The functions F_0, F_k and G_k are given by the following expressions

$$\begin{aligned} F_0(\rho) &:= -i[H_0, \rho] - i\textstyle\sum_{j=1}^m u_j(\rho)[H_j, \rho], \\ F_k(\rho) &:= L_k\rho L_k - L_k^2\rho/2 - \rho L_k^2/2, \\ G_k(\rho) &:= L_k\rho + \rho L_k - 2\mathrm{Tr}(L_k\rho)\rho. \end{aligned} \tag{5.2}$$

The function u appearing in F_0 denotes the feedback law taking values in $\mathbb{R}^m$, while $H_0 = H_0^* \in \mathbb{C}^{N\times N}$ is the free Hamiltonian, $H_j = H_j^* \in \mathbb{C}^{N\times N}$ are the control Hamiltonians and $L_k = L_k^* \in \mathbb{C}^{N\times N}$ are the measurement operators which may be degenerate. If the feedback $u \in \mathcal{C}^1(\mathcal{S}_N, \mathbb{R}^m)$, the existence and uniqueness of the solution of (5.1) as well as the strong Markov property of the solution can be ensured by the results established in [MvH07].

Concerning stabilization of two-qubit systems with only one quantum channel, some interesting results have been derived in [YTH07] and [MvH07]. In [YTH07], the methods in [vHSM05a] are adopted in order to construct a continuous feedback controller stabilizing the target Bell state starting from almost any initial pure state when the measurement is perfect. Then, in [MvH07], the authors design a switching quantum feedback controller that asymptotically stabilizes the system towards two specific Bell states. On the other hand we are not aware of any result on feedback stabilization of multi-qubit systems, with $\mathbf{n} > 2$, towards an arbitrary GHZ state.

In the following sections, we will first analyze the behavior of $\mathbf{n}$-qubit systems (5.1) for $u \equiv 0$. Starting from the phenomenon of quantum state reduction of two-qubit systems, we will review the switching state-feedback stabilization method in [MvH07] and our continuous version [LAM19b] for two-qubit systems with only one quantum channel. Then, we introduce our feedback approach which ensures the exponential stabilization of two-qubit systems with two quantum channels towards the target Bell state (see [LAM19b]).

5.1 Quantum state reduction for n-qubit systems

As in the paragraph "Purification" of Section 4.1, we can conclude that only the diffusion terms of (5.1) contribute to the increase the purity of the quantum state. Then we discuss how the diffusion terms of (5.1) increase the purity of $\mathbf{n}$-qubit systems with $\mathbf{n} \geq 2$, when we turn off the control input.

Let us consider the case of $\mathbf{n}$-qubit systems with n quantum channels. Assume that the measurement operator of each quantum channel denoted by L_i has K_i different real eigenvalues $\lambda_1^i, \ldots \lambda_{K_i}^i$, and the algebraic multiplicity of each eigenvalue λ_k^i is M_k^i, where $\sum_{k=1}^{K_i} M_k^i = N$. Consider the "variance function" of L_i

$$\mathscr{V}_i(\rho) := \mathrm{Tr}(L_i^2\rho) - \mathrm{Tr}^2(L_i\rho).$$

By the eigendecomposition, we write L_i with respect to a basis of eigenstates as $L_i = \sum_{k=1}^{K_i} \lambda_k^i \big(\sum_{j=1}^{M_k^i} u_{k_j}^i (u_{k_j}^i)^*\big)$ with $L_i u_{k_j}^i = \lambda_k^i u_{k_j}^i$. Denote $\alpha_k^i := \sum_{j=1}^{M_k^i} (u_{k_j}^i)^* \rho u_{k_j}^i \geq 0$ then

$\sum_{k=1}^{K_i} \alpha_k^i = 1$, due to Jensen inequality, we have

$$\mathrm{Tr}(L_i^2\rho) - \mathrm{Tr}^2(L_i\rho) = \sum_{k=1}^{K_i}(\lambda_k^i)^2\alpha_k^i - \left(\sum_{k=1}^{K_i}\lambda_k^i\alpha_k^i\right)^2 \geq 0,$$

the last equality holds if and only if there exists a k such that $\alpha_k^i = 1$, which is equivalent to say that ρ is an eigenstate of L_i associated with λ_k^i, i.e., $L_i\rho = \lambda_k^i\rho$. Thus, $\{\rho \in \mathcal{S}_N | \sum_{i=1}^n \mathscr{V}_i(\rho) = 0\}$ is the set of common spaces of the pure states corresponding to eigenvectors of $L_1, \ldots, L_n$. Suppose $H_0, L_1, \ldots, L_n$ commute with each other, then we have

$$\mathscr{L}\mathscr{V}_i(\rho) = -4\eta_i\big(\mathscr{V}_i(\rho)\big)^2 - \sum_{k\neq i}\mathrm{Tr}^2\big(L_iG_k(\rho)\big) \leq -4\eta_i\big(\mathscr{V}_i(\rho)\big)^2;$$

which implies

$$\mathscr{L}\left(\sum_{i=1}^n \mathscr{V}_i(\rho)\right) \leq -4\left(\sum_{i=1}^n \eta_i\big(\mathscr{V}_i(\rho)\big)^2\right) \leq -4\frac{\boldsymbol{\eta}}{n}\left(\sum_{i=1}^n \mathscr{V}_i(\rho)\right)^2,$$

where $\boldsymbol{\eta} := \min\{\eta_1, \ldots, \eta_n\} > 0$. By using the stochastic LaSalle-type theorem (Theorem A.2.3), we can show that $\sum_{i=1}^n \mathscr{V}_i(\rho_t)$ converges to 0 almost surely, when t goes to infinity.

In order to observe the phenomenon of quantum state reduction of **n**-qubit systems to realize our objective, and in view of stabilizing the systems towards an arbitrary GHZ state, it is sufficient to implement two quantum channels such that the common states of the corresponding measurement operators coincide with the Bell states or GHZ states. For the purpose of simplicity, we take $L_1 = \sqrt{M_1}L_z$ and $L_2 = \sqrt{M_2}L_x$ with $M_1, M_2 > 0$ describing the strength of measurement and

$$L_z = \mathrm{diag}(l_1, \ldots, l_{N/2}, l_{N/2}, \ldots, l_1), \quad L_x = \sigma_x^{\otimes\mathbf{n}}, \tag{5.3}$$

where $l_i \neq l_j$ if $i \neq j$, and we denote $l > 0$ the minimum absolute difference between any two different l_i and l_j, i.e., $l := \min_{i\neq j}\{|l_i - l_j|\}$. Denote $\Lambda_k := \rho_{k,k} + \rho_{\bar{k},\bar{k}}$ with $\bar{k} := N + 1 - k$, then we have

$$\mathscr{V}_1(\rho) = M_1\big(\mathrm{Tr}(L_z^2\rho) - \mathrm{Tr}^2(L_z\rho)\big) = M_1\left(\sum_{k=1}^{N/2} l_k^2\Lambda_k - \left(\sum_{k=1}^{N/2} l_k\Lambda_k\right)^2\right) \geq 0,$$

$$\mathscr{V}_2(\rho) = M_2\big(\mathrm{Tr}(L_x^2\rho) - \mathrm{Tr}^2(L_x\rho)\big) = M_2\left(1 - \left(\sum_{k=1}^{N/2} 2\mathbf{Re}\{\rho_{k,\bar{k}}\}\right)^2\right) \geq 0.$$

For **n**-qubit systems, we denote the set of all GHZ states by

$$\bar{E}_{\mathbf{n}} := \{\mathbf{GHZ}_1^{\pm}, \ldots, \mathbf{GHZ}_{N/2}^{\pm}\}.$$

Then, we deduce that

$$\{\rho \in \mathcal{S}_N | \, \mathscr{V}_1(\rho) + \mathscr{V}_2(\rho) = 0\} = \bar{E}_{\mathbf{n}}.$$

By the above argument and if H_0 commutes with L_x, L_z, we can show that a **n**-qubit system converges to one GHZ state almost surely, when t goes to infinity. However, if

we implement only one quantum channel, and the corresponding measurement operator is chosen to be L_1, then the system converges to the subspace $\{\rho \in \mathcal{S}_N | \, \Lambda_k = 1\}$ for $k \in \{1, \ldots, N/2\}$, which contains two GHZ states $\mathbf{GHZ}_k^{\pm}$.

Now, we focus on **n**-qubit systems with two quantum channels, assuming again that the two measurement operators are $L_1 = \sqrt{M_1}L_z$ and $L_2 = \sqrt{M_2}L_x$, which are defined in (5.3), and with H_0 satisfying $[H_0, L_1] = [H_0, L_2] = 0$. It is well known that, we cannot obtain any information on the rate of convergence by LaSalle theorem. Then, inspired by the approach adopted in the case of quantum spin systems in Theorem 4.1.2, we introduce our method to study **n**-qubit systems with two quantum channels.

Firstly, we provide the invariant properties of ρ_t in this case by the following two lemmas, which are analogous to the first part of Lemma 4.1.1.

Lemma 5.1.1. *Assume $u \equiv 0$. If $\Lambda_k(0) = 0$ for some $k \in \{1, \ldots, N/2\}$, then $\mathbb{P}\big(\Lambda_k(t) = 0, \forall\, t \geq 0\big) = 1$. If the initial state satisfies $\Lambda_k(0) > 0$, then $\mathbb{P}\big(\Lambda_k(t) > 0, \forall\, t \geq 0\big) = 1$.*

Proof. For $u \equiv 0$, the dynamics of $\Lambda_k(t)$ is given by

$$d\Lambda_k(t) = 2\sqrt{\eta_1 M_1}\Lambda_k(t)\big(l_k - \mathrm{Tr}(L_z\rho_t)\big)dW_1(t) + 2\sqrt{\eta_2 M_2}\big(2\mathbf{Re}\{\rho_{k,\bar{k}}\} - \mathrm{Tr}(L_x\rho_t)\Lambda_k(t)\big)dW_2(t).$$

Since $\rho \geq 0$ one has, for all $k \in \{1, \ldots, N/2\}$, $2\mathbf{Re}\{\rho_{k,\bar{k}}\} \leq \Lambda_k$. In particular we have

$$\begin{aligned}
&\Big|2\sqrt{\eta_1 M_1}\Lambda_k(t)\big(l_k - \mathrm{Tr}(L_z\rho_t)\big)\Big| \leq R\Lambda_k(t),\\
&\Big|2\sqrt{\eta_2 M_2}\big(2\mathbf{Re}\{\rho_{k,\bar{k}}\} - \mathrm{Tr}(L_x\rho_t)\Lambda_k(t)\big)\Big| \leq R\Lambda_k(t),
\end{aligned}$$

for some $R > 0$, which yields the first part of the lemma.

Let us now prove the second part of the lemma. Given $\varepsilon > 0$, consider any $\mathcal{C}^2$ function on $\mathcal{S}$ such that

$$V_k(\rho) = \frac{1}{\Lambda_k}, \quad \text{if } \Lambda_k > \varepsilon.$$

A simple computation shows that $\mathscr{L}V_k(\rho) \leq KV_k(\rho)$ if $\Lambda_k > \varepsilon$ for some positive constant K. To conclude the proof, one just applies the same arguments as in Lemma 4.1.1. Roughly speaking, by setting $f(\rho, t) = e^{-Kt}V_k(\rho)$, one has $\mathscr{L}f \leq 0$ whenever $\Lambda_k > \varepsilon$. From this fact one proves that the probability of Λ_k becoming zero in a finite fixed time T is proportional to ε and, being the latter arbitrary, it must be 0. This concludes the proof. □

We denote $V_x(\rho) := 1 - \mathrm{Tr}^2(L_x\rho)$.

Lemma 5.1.2. *Assume $u \equiv 0$. If $V_x(\rho_0) = 0$ then $\mathbb{P}\big(V_x(\rho_t) = 0, \forall\, t \geq 0\big) = 1$. If the initial state satisfies $V_x(\rho_0) > 0$, then $\mathbb{P}\big(V_x(\rho_t) > 0, \forall\, t \geq 0\big) = 1$.*

Proof. For $u \equiv 0$, the dynamics of $V_x(\rho_t)$ is given by

$$\begin{aligned}
dV_x(\rho_t) = &-4\eta_1 M_1 \Gamma^2(\rho_t)dt - 4\eta_2 M_2 V_x^2(\rho_t)dt\\
&- 4\sqrt{\eta_1 M_1}\mathrm{Tr}(L_x\rho_t)\Gamma(\rho_t)dW_1(t) - 4\sqrt{\eta_2 M_2}\mathrm{Tr}(L_x\rho_t)V_x(\rho_t)dW_2(t),
\end{aligned}$$

where $\Gamma(\rho) := \mathrm{Tr}(L_xL_z\rho) - \mathrm{Tr}(L_x\rho)\mathrm{Tr}(L_z\rho)$. Moreover, we can write $\Gamma(\rho)$ in following form,

$$\Gamma(\rho) = \begin{cases} \mathrm{Tr}(L_xL_z\rho) - \mathrm{Tr}(L_z\rho) + \frac{\mathrm{Tr}(L_z\rho)V_x(\rho)}{1+\mathrm{Tr}(L_x\rho)} \leq CV_x(\rho), & \text{if } \mathrm{Tr}(L_x\rho) \geq 0;\\ \mathrm{Tr}(L_xL_z\rho) + \mathrm{Tr}(L_z\rho) + \frac{\mathrm{Tr}(L_z\rho)V_x(\rho)}{1-\mathrm{Tr}(L_x\rho)} \leq CV_x(\rho), & \text{if } \mathrm{Tr}(L_x\rho) < 0, \end{cases}$$

for some $C > 0$. It implies that

$$|4\eta_1 M_1 \Gamma^2(\rho) + 4\eta_2 M_2 V_x^2(\rho)| \leq R V_x(\rho),$$
$$|4\sqrt{\eta_1 M_1}\mathrm{Tr}(L_x\rho)\Gamma(\rho)| \leq R V_x(\rho),$$
$$|4\sqrt{\eta_2 M_2}\mathrm{Tr}(L_x\rho) V_x(\rho)| \leq R V_x(\rho),$$

for some $R > 0$, which yields the first part of the lemma.

Let us now prove the second part of the lemma. Given $\varepsilon > 0$, consider any $\mathcal{C}^2$ function on $\mathcal{S}_N$ such that

$$\mathbf{V}_x(\rho) = \frac{1}{V_x(\rho)}, \quad \text{if } V_x(\rho) > \varepsilon.$$

A simple computation shows that $\mathscr{L}\mathbf{V}_x(\rho) \leq K\mathbf{V}_x(\rho)$ if $\mathbf{V}_x(\rho) > \varepsilon$ for some positive constant K. To conclude the proof, one just applies the same arguments as in the previous lemma and Lemma 4.1.1. □

We now show the exponential convergence towards $\bar{E}_{\mathbf{n}}$ in mean and almost surely for **n**-qubit systems when $u \equiv 0$.

Theorem 5.1.3 (Exponential quantum state reduction)**.** *For system* (5.1) *with* $L_1 = \sqrt{M_1}L_z$, $L_2 = \sqrt{M_2}L_x$, $u \equiv 0$ *and* $\rho_0 \in \mathcal{S}_N$, *the set* $\bar{E}_{\mathbf{n}}$ *is exponentially stable in mean and a.s. with average and sample Lyapunov exponent less than or equal to* $-\min\{\eta_1 M_1 l^2/2\,,\, 2\eta_2 M_2\}$. *Moreover, the probability of convergence to* $\bar{\boldsymbol{\rho}} \in \bar{E}_{\mathbf{n}}$ *is* $\mathrm{Tr}(\rho_0\bar{\boldsymbol{\rho}})$.

Proof. Let $I := \{k|\,\Lambda_k(0) = 0\}$ and $\mathcal{S}_I := \{\rho \in \mathcal{S}_N|\,\Lambda_k = 0 \text{ if and only if } k \in I\}$. Then by Lemma 5.1.1, $\mathcal{S}_I$ is a.s. invariant for (5.1) in this case. Consider the function

$$V(\rho) = U(\rho) + \sqrt{V_x(\rho)} \geq 0 \tag{5.4}$$

as a candidate Lyapunov function, where

$$U(\rho) = \sum_{k\neq h}\sqrt{\Lambda_k\Lambda_h} \geq 0, \quad V_x(\rho) = 1 - \mathrm{Tr}^2(L_x\rho) \geq 0.$$

Note that $V(\rho) = 0$ if and only if $\rho \in \bar{E}_{\mathbf{n}}$. Since the candidate Lyapunov function $V(\rho)$ is twice continuously differentiable when restricted to the invariant sets $\mathcal{S}_I \cap \{\rho \in \mathcal{S}_N|\,V_x(\rho) > 0\}$ and $\mathcal{S}_I \cap \{\rho \in \mathcal{S}_N|\,V_x(\rho) = 0\}$, we can compute the infinitesimal generator of $V(\rho)$,

$$\mathscr{L}V(\rho) \leq -\frac{\eta_1 M_1 l^2}{2}U(\rho) - 2\eta_2 M_2\sqrt{V_x(\rho)} \leq -\bar{C}V(\rho).$$

where $\bar{C} := \min\{\eta_1 M_1 l^2/2, 2\eta_2 M_2\}$. For all $\rho_0 \in \mathcal{S}$, we have

$$\mathbb{E}\big(V(\rho_t)\big) = V(\rho_0) - \bar{C}\int_0^t \mathbb{E}\big(V(\rho_s)\big)ds.$$

In virtue of Grönwall inequality, we have $\mathbb{E}\big(V(\rho_t)\big) \leq V(\rho_0)e^{-\bar{C}t}$. By a straightforward calculation, we can show that the candidate Lyapunov function is bounded from below and above by the Bures distance from $\bar{E}_{\mathbf{n}}$,

$$C_1 d_B(\rho, \bar{E}_{\mathbf{n}}) \leq V(\rho) \leq C_2 d_B(\rho, \bar{E}_{\mathbf{n}}), \tag{5.5}$$

where $C_1 = 1/8$ and $C_2 = N(N/2-1)+4$. It implies

$$\mathbb{E}\big(d_B(\rho_t, \bar{E}_{\mathbf{n}})\big) \leq \frac{C_2}{C_1} d_B(\rho_0, \bar{E}_{\mathbf{n}}) e^{-\bar{C}t}, \quad \forall \rho_0 \in \mathcal{S}_N,$$

which means that the set $\bar{E}_{\mathbf{n}}$ is exponentially stable in mean with average Lyapunov exponent less than or equal to $-\bar{C}$.

Now we consider the stochastic process $Q(\rho_t, t) = e^{\bar{C}t} V(\rho_t) \geq 0$ whose infinitesimal generator is given by $\mathscr{L}Q(\rho, t) = e^{\bar{C}t}\big(\bar{C}\, V(\rho) + \mathscr{L}V(\rho)\big) \leq 0$. Hence, the process $Q(\rho_t, t)$ is a positive supermartingale. Due to Doob's martingale convergence theorem [RY13], the process $Q(\rho_t, t)$ converges almost surely to a finite limit as t tends to infinity. Consequently, $Q(\rho_t, t)$ is almost surely bounded, that is $\sup_{t\geq 0} Q(\rho_t, t) = A$, for some a.s. finite random variable A. This implies $\sup_{t\geq 0} V(\rho_t) = Ae^{-\bar{C}t}$ a.s. Letting t goes to infinity, we obtain $\limsup_{t\to\infty} \frac{1}{t} \log V(\rho_t) \leq -\bar{C}$ a.s. By the inequality (5.5),

$$\limsup_{t\to\infty} \frac{1}{t} \log d_B(\rho_t, \bar{E}_{\mathbf{n}}) \leq -\bar{C}, \qquad a.s. \tag{5.6}$$

which means that the set $\bar{E}_{\mathbf{n}}$ is a.s. exponentially stable with sample Lyapunov exponent less than or equal to $-\bar{C}$.

Finally, the fact that the probability of convergence to $\bar{\boldsymbol{\rho}} \in \bar{E}_{\mathbf{n}}$ is $\mathrm{Tr}(\rho_0 \bar{\boldsymbol{\rho}})$ may be proved by standard arguments (see e.g., Theorem 4.1.2). The proof is complete. □

The simulations for a two-qubit system with two quantum channels, where $L_z = \sigma_z \otimes \sigma_z$ and $L_x = \sigma_x \otimes \sigma_x$, in the case $u \equiv 0$ are shown in Fig. 5.1, we observe that the expectation of the Lyapunov function $\mathbb{E}\big(V(\rho_t)\big)$ is bounded by the exponential function $V(\rho_0)e^{-\bar{C}t}$ with $\bar{C} = \min\{2\eta_1 M_1, 2\eta_2 M_2\}$. Denote the set of four Bell states by

$$\bar{E}_2 := \{\boldsymbol{\Psi}_\pm, \boldsymbol{\Phi}_\pm\}.$$

We can also observe that the expectation of the Bures distance $\mathbb{E}\big(d_B(\rho_t, \bar{E}_2)\big)$ is always bounded by $C_2/C_1 d_B(\rho_0, \bar{E}) e^{-\bar{C}t}$, which confirms the results of Theorem 5.1.3.

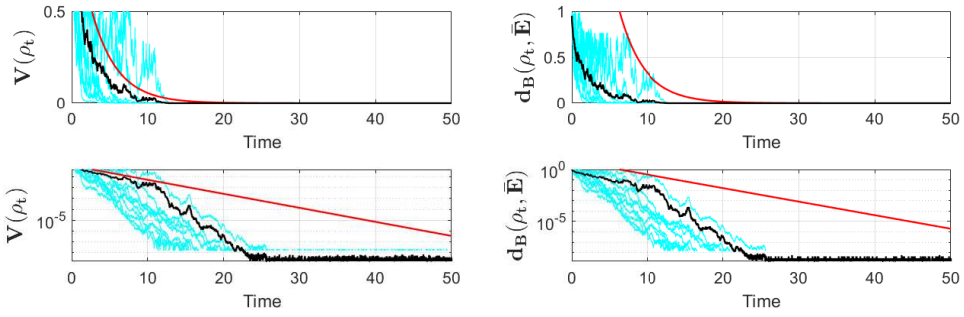


FIGURE 5.1 – Quantum state reduction with two quantum channels and $u \equiv 0$ starting at $\mathrm{diag}(0.2, 0.3, 0.1, 0.4)$, when $\omega = 0.3$, $\eta_1 = 0.3$, $M_1 = 1$, $\eta_2 = 0.4$ and $M_2 = 0.9$: the black curve represents the mean value of the 10 arbitrary samples, and the red curve represents the exponential reference with exponent $-\bar{C}$. The figures at the bottom are the semi-log versions of the ones at the top.

5.2 Asymptotic stabilization of two-qubit systems

In this section, we consider two-qubit systems with only one quantum channel. As we discussed above, the diffusion term of (5.1) contributes to the convergence of the system towards the set of pure states corresponding to vectors belonging to the one of the eigenspaces of the measurement operator L_1, but not necessarily towards a Bell state. Thus, in this case the feedback controller plays two roles,

1. it breaks the attraction of the antipodal eigenspace [1], and
2. it stabilizes the system towards the target Bell state.

We first introduce the switching feedback control law proposed in [MvH07]. Then, inspired by this method and based on the analysis of each sample path of quantum trajectory in [LAM19a], we propose a continuous feedback controller [LAM19b].

Switching feedback control law The method for stabilizing quantum spin systems, discussed in subsection 4.2.2, can be extended to two-qubit systems. For stabilizing quantum spin systems, we need to construct a suitable feedback controller to break the attraction of the antipodal states. However, the obstacle here is more difficult since, in the presence of a single measurement operator, the quantum state reduction phenomenon only predicts the convergence to a subset, and not to an isolated state (unlike Theorem 5.1.3).

In [MvH07], the control goal is to stabilize the system towards $\bar{\boldsymbol{\rho}} \in \{\boldsymbol{\Phi}_+, \boldsymbol{\Phi}_-\}$, then the author choose $L_1 = \sigma_z \otimes \mathbb{1} + \mathbb{1} \otimes \sigma_z = \mathrm{diag}(2, 0, 0, -2)$ as the measurement operator, which contains the eigenvector associated with the target Bell state. The advantage of this choice of L_1 is that each antipodal eigenspace corresponds to only one pure state. Now, let us consider $\boldsymbol{\Phi}_-$ as the target Bell state, the case $\boldsymbol{\Phi}_+$ as the target Bell state can be done in the same manner. In order to avoid the attraction of the antipodal eigenspaces, which are in fact only two pure states given by $\boldsymbol{\rho}_1 := \mathrm{diag}(1, 0, 0, 0)$ and $\boldsymbol{\rho}_4 := \mathrm{diag}(0, 0, 0, 1)$, the authors analyze the quantum trajectory in the following two subspaces of $\mathcal{S}_4$, which are defined based on the value of the function $V(\rho) = 1 - \mathrm{Tr}(\rho \boldsymbol{\Phi}_-)$ as below,

$$\mathcal{S}_{\geq 1-\gamma} := \{\rho \in \mathcal{S}_4 |\, 1 - \gamma \leq V(\rho) \leq 1\}, \quad \mathcal{S}_{\leq 1-\gamma/2} := \{\rho \in \mathcal{S}_4 |\, 0 \leq V(\rho) \leq 1 - \gamma/2\},$$

where $\gamma \in (0, 1)$. Note that, the above two subspaces have two obvious but important properties,

- $\{\boldsymbol{\rho}_1, \boldsymbol{\rho}_4, \boldsymbol{\Phi}_+\} \subset \mathcal{S}_{\geq 1-\gamma}$; and
- the only Bell state belonging to $\mathcal{S}_{\leq 1-\gamma/2}$ is the target state $\boldsymbol{\Phi}_-$.

Due to these facts, we can solve the control problem in the following three steps,

1. by the support theorem, Feller continuity and Dynkin estimation, we can show that, there exists a $\gamma \in (0, 1)$ such that, under the effect of the control Hamiltonian $H_1 = \sigma_y \otimes \mathbb{1}$ and a constant feedback controller $u_1 = 1$, ρ_t exits $\mathcal{S}_{\geq 1-\gamma}$ in finite time almost surely ;
2. by the strong Markov property, stochastic Lyapunov-type argument and Borel-Cantelli lemma, we can show that, for almost every sample path $\omega \in \Omega$, there exists

1. The antipodal eigenspaces the sets of pure states corresponding to vectors belonging to the eigenspaces of the measurement operator, which do not contain the vector associated with the target state.

a finite time $T(\omega)$, such that for all $t > T(\omega)$, the quantum trajectory $\rho_t(\omega)$ stays in $\mathcal{S}_{\leq 1-\gamma/2}$ and never exits, under the (switching) feedback controllers u_1 and u_2 ;

3. by stochastic LaSalle theorem [MvH07, Theorem 2.3], for all $\rho_0 \in \mathcal{S}_4$, ρ_t converges to the target state $\mathbf{\Phi}_-$ under suitable (switching) feedback controllers u_1 and u_2 almost surely.

Note that, in order to apply a stochastic Lyapunov-type argument or stochastic LaSalle argument, we need to find a Lyapunov function whose infinitesimal generator is not positive. In fact, it is never obvious to construct such a Lyapunov function for the system with only one control Hamiltonian, and this issue would be much easier to treat if one adds a second control Hamiltonian. Then we sum up the above analysis by the following theorem [MvH07, Theorem 5.1].

Theorem 5.2.1. *Consider the system* (5.1) *evolving in the set* $\mathcal{S}_4$*, which contains only one quantum channel* $L_1 = \sigma_z \otimes \mathbb{1} + \mathbb{1} \otimes \sigma_z$ *and two control Hamiltonians* $H_1 = \sigma_y \otimes \mathbb{1}$, $H_2 = \mathbb{1} \otimes \sigma_y$*. Let* $\mathbf{\Phi}_-$ *be the target state and let* $\gamma > 0$*. Consider the following control law :*

1. $u_1(\rho) = 1 - \mathrm{Tr}\big(i[H_1, \rho]\mathbf{\Phi}_-\big)$ *and* $u_2(\rho) = 1 - \mathrm{Tr}\big(i[H_2, \rho]\mathbf{\Phi}_-\big)$ *for* $\rho \in \mathcal{S}_{\leq 1-\gamma}$ *;*
2. $u_1(\rho) = 1$ *and* $u_0(\rho) = 0$ *for* $\rho \in \mathcal{S}_{\geq 1-\gamma/2}$ *;*
3. *for* $\rho \in \mathfrak{B} := \{\rho \in \mathcal{S} | \, \gamma/2 < \mathrm{Tr}(\rho \boldsymbol{\rho}_{\bar{n}}) < \gamma\}$*, then* $u_1(\rho) = 1 - \mathrm{Tr}\big(i[H_1, \rho]\mathbf{\Phi}_-\big)$ *and* $u_2(\rho) = 1 - \mathrm{Tr}\big(i[H_2, \rho]\mathbf{\Phi}_-\big)$ *if* ρ_t *entered* $\mathfrak{B}$ *through the boundary* $V(\rho) = 1 - \gamma$*, and* $u_1(\rho) = 1$ *and* $u_0(\rho) = 0$ *otherwise.*

Then, there exists $\gamma > 0$*, such that* $u(\rho)$ *almost surely globally stabilizes the system towards* $\mathbf{\Phi}_-$*.*

Continuous feedback control law Our purpose here is to stabilize the system (5.1) towards an arbitrary Bell state $\bar{\boldsymbol{\rho}} \in \bar{E}_2 = \{\mathbf{\Psi}_\pm, \mathbf{\Phi}_\pm\}$ with only one quantum channel under a continuous feedback controller. We choose

$$L_1 = \sqrt{M_1} L_z, \quad L_z := \sigma_z \otimes \sigma_z = \mathrm{diag}(1, -1, -1, 1)$$

as the measurement operators, and the set of pure states corresponding to the eigenspaces of L_1 and L_2 contain all Bell states. In this case, the diffusion term strengthens the convergence [2] towards $\mathcal{S}_{\Psi_\pm} \cup \mathcal{S}_{\Phi_\pm}$ where

$$\mathcal{S}_{\Psi_\pm} := \{\rho \in \mathcal{S}_4 | \, \rho_{2,2} = \rho_{3,3} = 0\}, \quad \mathcal{S}_{\Phi_\pm} := \{\rho \in \mathcal{S}_4 | \, \rho_{1,1} = \rho_{4,4} = 0\}.$$

Here, we take $H_0 = \omega L_z$ with $\omega > 0$.

Generally speaking, based on the support theorem, trajectories of Equation (5.1) may be interpreted as limits of solutions of the following deterministic equation

$$\begin{aligned} \dot{\rho}_v(t) = -i[H_0 + \textstyle\sum_{k=1}^m u_k H_k \, , \rho_v(t)] + (1 - \eta_1)\big(L_1 \rho_v(t) L_1 - \rho_v(t)\big) \\ + 2\eta_1 \mathrm{Tr}(L_1 \rho) G_1\big(\rho_v(t)\big) + \sqrt{\eta_1} G_1\big(\rho_v(t)\big) v_1(t), \end{aligned} \tag{5.7}$$

with $v_1(t) \in \mathcal{V}$, where $\mathcal{V}$ is the set of all piecewise constant functions from $\mathbb{R}_+$ to $\mathbb{R}$, and G_1 is defined as in (5.2). In particular, the set $\mathcal{S}_4$ is positively invariant for Equation (5.7). Then we provide sufficient conditions on the feedback controller, such that ρ_t can enter an arbitrary neighborhood of the target Bell state. Denote $X_{\bar{\boldsymbol{\rho}}}(\rho) := \mathrm{Tr}(\rho \bar{\boldsymbol{\rho}})$.

2. In fact, we can show this convergence is exponential in mean and almost surely, by the similar approach as the proof of Theorem 4.1.2.

Lemma 5.2.2. *Let $\bar{\boldsymbol{\rho}} = \xi\xi^*$ with $\xi \in \{\Psi_\pm, \Phi_\pm\}$ and suppose $\eta_1 \in (0,1)$. Assume that the feedback controllers and control Hamiltonians satisfy the following conditions*:

C.1. $u \in \mathcal{C}^1(\mathcal{S}_4, \mathbb{R}^m)$, $u_1 \neq 0$ and $u_k \equiv 0$ for $k > 1$ on the set $\{\rho \in \mathcal{S}_4 |\, X_{\bar{\boldsymbol{\rho}}}(\rho) = 0\}$;

C.2. ξ, $H_1\xi$ and $L_zH_1\xi$ are linearly independent and H_1 does not admit an eigenvector orthogonal to the space spanned by them ;

C.3. On $\mathcal{S}_\xi \setminus \bar{\boldsymbol{\rho}}$, $\sum_{k=1}^m u_k[H_k, \rho]$ is not tangent to $\mathcal{S}_\xi$.

Then for all $r > 0$ and any given initial state $\rho_0 \in \mathcal{S}_4$, $\mathbb{P}(\tau_r < \infty) = 1$, where $\tau_r := \inf\{t \geq 0 |\, \rho_t \in B_r(\bar{\boldsymbol{\rho}})\}$ and ρ_t corresponds to the solution of System (5.1) in this case.

Proof. The lemma holds trivially for $\rho_0 \in B_r(\bar{\boldsymbol{\rho}})$, as in this case $\tau_r = 0$. Let us thus suppose that $\rho_0 \in \mathcal{S}_4 \setminus B_r(\bar{\boldsymbol{\rho}})$. We show that there exists $T \in (0, \infty)$ and $\zeta \in (0,1)$ such that $\mathbb{P}_{\rho_0}(\tau_r < T) > \zeta$. For this purpose, we make use of the support theorem. Consider the following differential equation derived from (5.7),

$$\dot{X}_{\bar{\boldsymbol{\rho}}}\big(\rho_v(t)\big) = -\Theta_u\big(\rho_v(t)\big) + \Theta_1\big(\rho_v(t)\big) + 2X_{\bar{\boldsymbol{\rho}}}\big(\rho_v(t)\big)\sqrt{\eta_1 M_1}P_1\big(\rho_v(t)\big)v_1(t),$$

where $v_1(t) \in \mathcal{V}$ is the control input, and

$$\Theta_u(\rho) := \sum_{k=1}^m u_k \mathrm{Tr}\big(i[H_k, \rho]\bar{\boldsymbol{\rho}}\big), \quad \Theta_1(\rho) := 4\eta_1 M_1 \mathrm{Tr}(L_z\rho)P_1(\rho)X_{\bar{\boldsymbol{\rho}}}(\rho),$$
$$P_1(\rho) := \bar{\lambda}_z - \mathrm{Tr}(L_z\rho) \text{ where } L_z\bar{\boldsymbol{\rho}} = \bar{\lambda}_z\bar{\boldsymbol{\rho}}.$$

Firstly, we show by contradiction that, if $X_{\bar{\boldsymbol{\rho}}}\big(\rho_v(0)\big) = 0$ then $X_{\bar{\boldsymbol{\rho}}}\big(\rho_v(t)\big) > 0$ immediately. Suppose $X_{\bar{\boldsymbol{\rho}}}\big(\rho_v(s)\big) = 0$ for $s \in [0, \epsilon]$ with $\epsilon > 0$, which is equivalent to $\rho_v(s)\xi = 0$, take this assumption into account in Equation (5.7), then combining with the condition C.1, we have

$$\dot{\rho}_v(s)\xi = -iu_1\rho_v(s)H_1\xi = 0,$$

which implies $\rho_v(s)H_1\xi = 0$ since $u_1 \neq 0$. After a straightforward calculation, for $s \in [0, \epsilon]$, we have

$$\xi^*H_1\dot{\rho}_v(s)H_1\xi = M_1(1-\eta_1)\xi^*H_1L_z\rho_v(s)L_zH_1\xi = 0,$$

which implies $\rho_v(s)L_zH_1\xi = 0$. Thus, for $s \in [0, \epsilon]$, due to the condition C.2, ϕ is not an eigenvector of H_1 then $[H_1, \rho_v(s)] \neq 0$ which leads to a contradiction. Hence, we can conclude that, under the assumptions of the lemma on the feedback controller and control Hamiltonians, i.e. C.1 and C.2, if $X_{\bar{\boldsymbol{\rho}}}\big(\rho_v(0)\big) = 0$, there exits an arbitrarily small $t_0 > 0$ such that $X_{\bar{\boldsymbol{\rho}}}\big(\rho_v(t_0)\big) > 0$.

Next, we show that, if $P_1\big(\rho_v(0)\big) = 0$ then $|P_1\big(\rho_v(t)\big)| > 0$ immediately. One easily checks that $P_1(\rho) = 0$ implies $\rho \in \mathcal{S}_\xi$, so that $\hat{F}_1(\rho) = G_1(\rho) = 0$. In particular, if $P_1(\rho_v(0)) = 0$ then the right-hand side of Equation (5.7) at time 0 becomes $-i\sum_{k=1}^m u_k[H_k\,,\, \rho_v(0)]$. Due to the condition C.3, if $\rho_v(0) \in \mathcal{S}_\xi \setminus \bar{\boldsymbol{\rho}}$, there exists an arbitrarily small $t_0 > 0$ such that $\rho_v(t_0) \notin \mathcal{S}_\xi$, which implies $|P_1\big(\rho_v(t_0)\big)| > 0$.

Therefore, without loss of generality, we suppose $P_1\big(\rho_v(0)\big)X_{\bar{\boldsymbol{\rho}}}\big(\rho_v(0)\big) > 0$. For $t > 0$, we can thus take the feedback $v_1 = KP_1(\rho)X_{\bar{\boldsymbol{\rho}}}(\rho)$ with $K > 0$ sufficiently large. The proposed control input v guarantees that $\rho_v(t) \in B_r(\bar{\boldsymbol{\rho}})$ for $t \leq T$ with $T < \infty$. Now, considering the stochastic solution of (5.1) in this case, we deduce that $\mathbb{P}(\rho_t \in B_r(\bar{\boldsymbol{\rho}})) > 0$ for $t \leq T$ from the support theorem.

By compactness of $\mathcal{S}_4 \setminus B_r(\bar{\boldsymbol{\rho}})$ and the Feller continuity of ρ_t, we have $\sup_{\rho_0 \in \mathcal{S}_4 \setminus B_r(\bar{\boldsymbol{\rho}})} \mathbb{P}_{\rho_0}(\tau_r \geq T) \leq 1 - \zeta < 1$, for some $\zeta > 0$. By Dynkin inequality [Dyn65],

$$\sup_{\rho_0 \in \mathcal{S}_4 \setminus B_r(\bar{\boldsymbol{\rho}})} \mathbb{E}_{\rho_0}(\tau_r) \leq \frac{T}{1 - \sup_{\rho_0 \in \mathcal{S}_4 \setminus B_r(\bar{\boldsymbol{\rho}})} \mathbb{P}_{\rho_0}(\tau_r \geq T)} \leq \frac{T}{\zeta}.$$

Then by Markov inequality, for all $\rho_0 \in \mathcal{S}_4 \setminus B_r(\bar{\boldsymbol{\rho}})$, we have

$$\mathbb{P}_{\rho_0}(\tau_r = \infty) = \lim_{n \to \infty} \mathbb{P}_{\rho_0}(\tau_r \geq n) \leq \lim_{n \to \infty} \mathbb{E}_{\rho_0}(\tau_r)/n = 0,$$

which implies $\mathbb{P}_{\rho_0}(\tau_r < \infty) = 1$. The proof is complete. □

By employing the first two steps of the proof of Theorem 4.3.8, we can obtain a general result concerning the asymptotic stabilization of System (5.1) with only one quantum channel towards the target Bell state.

Theorem 5.2.3. *Assume that the feedback law u satisfies the assumptions of Lemma 5.2.2 and $u(\bar{\boldsymbol{\rho}}) = 0$. Additionally, suppose that there exists a twice continuously differentiable positive function $V(\rho)$ such that $V(\rho) = 0$ if and only if $\rho = \bar{\boldsymbol{\rho}}$. Moreover, suppose that there exist positive constants C, C_1 and C_2 such that*

(i) $C_1\, d_B^p(\rho, \bar{\boldsymbol{\rho}}) \leq V(\rho) \leq C_2\, d_B^p(\rho, \bar{\boldsymbol{\rho}})$ with $p > 0$, for all $\rho \in \mathcal{S}_4$, and

(ii) $\mathscr{L}V(\rho) \leq 0$, for all $\rho \in B_r(\bar{\boldsymbol{\rho}})$ with $r > 0$.

Then, $\bar{\boldsymbol{\rho}}$ is a.s. asymptotically stable for System (5.1) in this case.

Remark 5.2.4. Theorem 5.2.3 ensures the global asymptotic stabilization of the system only providing local Lyapunov type condition. The additional assumptions on u_k and H_k are used to avoid the presence of invariant subsets of $\mathcal{S}_4$. These conditions are not optimal and may be easily weakened. We believe that by applying Proposition 4.3.4, we can relax these conditions for the case $\eta < 1$. We note that we do not need to find a global Lyapunov condition or apply the LaSalle theorem as in [MvH07, YTH07].

We next discuss an example of application of the previous result. We define the following continuously differentiable function on $[0, 1]$,

$$f(x) = \begin{cases} 0, & \text{if } x \in [0, \epsilon_1); \\ \frac{1}{2} \sin\left(\frac{\pi(2x - \epsilon_1 - \epsilon_2)}{2(\epsilon_2 - \epsilon_1)}\right) + \frac{1}{2}, & \text{if } x \in [\epsilon_1, \epsilon_2); \\ 1, & \text{if } x \in (\epsilon_2, 1], \end{cases}$$

where $0 < \epsilon_1 < \epsilon_2 < 1$. As an example of application of the previous result, we propose the following continuous feedback law and control Hamiltonians inspired by Theorem 5.2.1.

Proposition 5.2.5. *Consider the system (5.1) with $\rho_0 \in \mathcal{S}_4$, $\eta_1 \in (0, 1)$ and $m = 2$. Let $\bar{\boldsymbol{\rho}} \in \bar{E}_2$ be the target state. Define $H_1 = \sigma_z \otimes \sigma_y + \sigma_z \otimes \sigma_x + \mathbb{1} \otimes \sigma_z$ and the feedback laws in the following form*

$$\begin{aligned} u_1(\rho) &= \gamma - \operatorname{Tr}(i[H_1, \rho]\bar{\boldsymbol{\rho}}), \\ u_2(\rho) &= f(X_{\bar{\boldsymbol{\rho}}}(\rho))\big(\gamma - \operatorname{Tr}(i[H_2, \rho]\bar{\boldsymbol{\rho}})\big), \end{aligned} \tag{5.8}$$

where $|\gamma|$ is sufficient large. If

- *$\bar{\boldsymbol{\rho}} = \boldsymbol{\Psi}_\pm$, take $H_2 = \mp\sigma_y \otimes \sigma_z \pm \sigma_x \otimes \sigma_z - \sigma_z \otimes \mathbb{1}$;*

- $\bar{\boldsymbol{\rho}} = \boldsymbol{\Phi}_{\pm}$, *take* $H_2 = \mp\sigma_y \otimes \sigma_z \pm \sigma_x \otimes \sigma_z + \sigma_z \otimes \mathbb{1}$.

Then $\bar{\boldsymbol{\rho}}$ *is a.s. asymptotically stable.*

Proof. We apply Theorem 5.2.3 with the Lyapunov function $V(\rho) = 1 - X_{\bar{\boldsymbol{\rho}}}(\rho)$. We can easily verify that the feedback law and control Hamiltonians satisfy the assumptions of Lemma 5.2.2, $d_B^2(\rho, \bar{\boldsymbol{\rho}}) \leq V(\rho) \leq 2d_B^2(\rho, \bar{\boldsymbol{\rho}})$ in $\mathcal{S}_4$ and $\mathscr{L}V(\rho) \leq 0$ in a neighborhood of $\bar{\boldsymbol{\rho}}$. Hence, the proof is complete. □

The simulations in the case with only one quantum channel are shown in Fig. 5.2 and Fig. 5.3 for $\boldsymbol{\Psi}_+$ as the target state and $\boldsymbol{\Phi}_-$ as the target state respectively. Such simulations confirm the validity of Proposition 5.2.5.

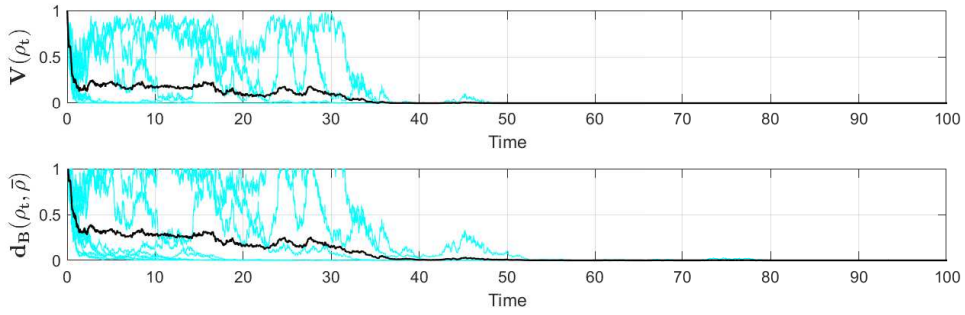


FIGURE 5.2 – Asymptotic stabilization of system (5.1) in this case towards $\boldsymbol{\Psi}_+$ with the feedback laws (5.8) starting at $\boldsymbol{\Phi}_-$, when $\omega = 0.3$, $\eta_1 = 0.3$, $M_1 = 1$, $\epsilon_1 = 0.1$, $\epsilon_2 = 0.4$ and $\gamma = 8$: the black curve represents the mean value of the 10 arbitrary samples.

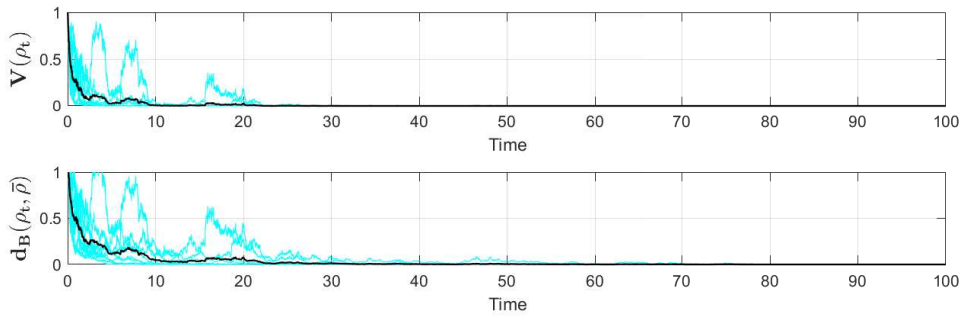


FIGURE 5.3 – Asymptotic stabilization of system (5.1) in this case towards $\boldsymbol{\Phi}_-$ with the feedback laws (5.8) starting at $\boldsymbol{\Psi}_+$, when $\omega = 0.3$, $\eta_1 = 0.3$, $M_1 = 1$, $\epsilon_1 = 0.1$, $\epsilon_2 = 0.4$ and $\gamma = 8$: the black curve represents the mean value of the 10 arbitrary samples.

5.3 Exponential stabilization of two-qubit systems

In this section, we consider two-qubit systems with only one control Hamiltonian H_1 and two quantum channels. The associated measurement operators are given by $L_1 = \sqrt{M_1}L_z$ with $M_1 > 0$ and $L_2 = \sqrt{M_2}L_x$ with $M_2 > 0$, where $L_z := \sigma_z \otimes \sigma_z$ and $L_x := \sigma_x \otimes \sigma_x$. Here $M_1, M_2 > 0$ are the strengths of the interaction between the light and the atoms. We also take $H_0 = \omega L_z$ with $\omega \geq 0$ and use only one control Hamiltonian H_1. Note that the

four Bell states coincide with the common eigenstates of the chosen operators L_1 and L_2. Due to the quantum state reduction in this case, Theorem 5.1.3, if we turn off the control input $u \equiv 0$, the system (5.1) converges exponentially towards one Bell state belonging to $\bar{E}_2 = \{\mathbf{\Psi}_\pm, \mathbf{\Phi}_\pm\}$. Based on this powerful result, we study the exponential stabilization of system (5.1) towards a target state $\bar{\boldsymbol{\rho}} \in \bar{E}_2$. We first establish a general result ensuring the exponential convergence towards $\bar{\boldsymbol{\rho}}$ under some assumptions on the feedback law and an additional local Lyapunov type condition. Next, we design a parametrized family of feedback control laws satisfying such conditions for some choice of the control Hamiltonian. Denote $X_{\bar{\boldsymbol{\rho}}}(\rho) := \mathrm{Tr}(\rho\bar{\boldsymbol{\rho}})$ and $\Theta_u(\rho) := u(\rho)\mathrm{Tr}(i[H_1, \rho]\bar{\boldsymbol{\rho}})$.

Lemma 5.3.1. *Assume that the initial state satisfies $\rho_0 \neq \bar{\boldsymbol{\rho}}$, $u \in \mathcal{C}^1(\mathcal{S}_4, \mathbb{R})$ and $u(\bar{\boldsymbol{\rho}}) = 0$. Then $\mathbb{P}(\rho_t \neq \bar{\boldsymbol{\rho}}, \forall\, t \geq 0) = 1$.*

Proof. Given $\varepsilon > 0$, we consider any $\mathcal{C}^2$ function on $\mathcal{S}$ such that

$$V(\rho) = \frac{1}{1 - X_{\bar{\boldsymbol{\rho}}}(\rho)}, \qquad \text{if } X_{\bar{\boldsymbol{\rho}}}(\rho) < 1 - \varepsilon.$$

We find

$$\mathscr{L}V(\rho) = -\frac{\Theta_u(\rho)}{\big(1 - X_{\bar{\boldsymbol{\rho}}}(\rho)\big)^2} + \frac{4\eta_1 M_1\big(\bar{\lambda}_z - \mathrm{Tr}(L_z\rho)\big)^2\mathrm{Tr}^2(\rho\bar{\boldsymbol{\rho}})}{\big(1 - X_{\bar{\boldsymbol{\rho}}}(\rho)\big)^3} + \frac{4\eta_2 M_2\big(\bar{\lambda}_x - \mathrm{Tr}(L_x\rho)\big)^2\mathrm{Tr}^2(\rho\bar{\boldsymbol{\rho}})}{\big(1 - X_{\bar{\boldsymbol{\rho}}}(\rho)\big)^3},$$

whenever $X(\rho) < 1 - \varepsilon$, where $L_z\bar{\boldsymbol{\rho}} = \bar{\lambda}_z\bar{\boldsymbol{\rho}}$ and $L_x\bar{\boldsymbol{\rho}} = \bar{\lambda}_x\bar{\boldsymbol{\rho}}$.

By applying the assumptions $u \in \mathcal{C}^1(\mathcal{S}_4, \mathbb{R})$ and $u(\bar{\boldsymbol{\rho}}) = 0$, we deduce that

$$|u(\rho)| = |u(\rho) - u(\bar{\boldsymbol{\rho}})| \leq C_1\|\rho - \bar{\boldsymbol{\rho}}\|_{HS} \leq \sqrt{2}C_1\sqrt{1 - X_{\bar{\boldsymbol{\rho}}}(\rho)}$$

for some positive constant C_1, where $\|\cdot\|_{HS}$ denotes the Hilbert-Schmidt norm. By the similar argument, we have $|\mathrm{Tr}(i[H_1, \rho]\bar{\boldsymbol{\rho}})| \leq \sqrt{2}C_2\sqrt{1 - X_{\bar{\boldsymbol{\rho}}}(\rho)}$ for some positive constant C_2. Then we have $|\Theta_u(\rho)| \leq C\big(1 - X_{\bar{\boldsymbol{\rho}}}(\rho)\big)$ for some positive constant C. Moreover,

$$|\bar{\lambda}_z - \mathrm{Tr}(L_z\rho)| \leq 2\big(1 - X_{\bar{\boldsymbol{\rho}}}(\rho)\big), \quad |\bar{\lambda}_x - \mathrm{Tr}(L_x\rho)| \leq \frac{1}{2}\big(1 - X_{\bar{\boldsymbol{\rho}}}(\rho)\big),$$

it is easy to check that $\mathscr{L}V(\rho) \leq KV(\rho)$ for some $K > 0$. To conclude the proof, one just applies the same arguments as in Lemma 4.1.1 . □

Based on the support theorem, the corresponding deterministic control system corresponding to Equation (5.1) in this case is given by

$$\dot{\rho}_v(t) = F_0\big(\rho_v(t)\big) + \sum_{j=1}^{2}\widehat{F}_j\big(\rho_v(t)\big) + \sum_{j=1}^{2}\sqrt{\eta_j}G_j\big(\rho_v(t)\big)v_j(t), \tag{5.9}$$

with $v_1(t)$ and $v_2(t)$ belonging to $\mathcal{V}$, where $\mathcal{V}$ is the set of all piecewise constant functions from $\mathbb{R}_+$ to $\mathbb{R}$, and

$$\begin{aligned}\widehat{F}_1(\rho) :=& (1 - \eta_1)(L_1\rho L_1 - \rho) + 2\eta_1\mathrm{Tr}(L_1\rho)G_1(\rho),\\ \widehat{F}_2(\rho) :=& (1 - \eta_2)(L_2\rho L_2 - \rho) + 2\eta_2\mathrm{Tr}(L_2\rho)G_2(\rho),\end{aligned}$$

with F_0, G_1 and G_2 defined as in (5.2). In particular, the set $\mathcal{S}_4$ is positively invariant for Equation (5.7).

Lemma 5.3.2. *Let $\bar{\boldsymbol{\rho}} = \xi\xi^*$ with $\xi \in \{\Psi_\pm, \Phi_\pm\}$. Suppose $\eta_1, \eta_2 \in (0,1)$ and the feedback controller satisfying the assumptions of Lemma 5.3.1. Assume that the feedback controller and control Hamiltonian satisfy the following conditions*:

1. *$u \neq 0$ on the set $\{\rho \in \mathcal{S}_4 |\, X_{\bar{\boldsymbol{\rho}}}(\rho) = 0\}$;*
2. *ξ, $H_1\xi$, $L_zH_1\xi$ and $L_xH_1\xi$ are linearly independent.*

Then for all $r > 0$ and any given initial state $\rho_0 \in \mathcal{S}_4$, $\mathbb{P}(\tau_r < \infty) = 1$, where $\tau_r := \inf\{t \geq 0 |\, \rho_t \in B_r(\bar{\boldsymbol{\rho}})\}$ and ρ_t corresponds to the solution of System (5.1) in this case.

Proof. The lemma holds trivially for $\rho_0 \in B_r(\bar{\boldsymbol{\rho}})$, as in this case $\tau_r = 0$. Let us thus suppose that $\rho_0 \in \mathcal{S}_4 \setminus B_r(\bar{\boldsymbol{\rho}})$. We show that there exists $T \in (0,\infty)$ and $\zeta \in (0,1)$ such that $\mathbb{P}_{\rho_0}(\tau_r < T) > \zeta$. For this purpose, we make use of the support theorem. Consider the following differential equation derived from (5.9),

$$\dot{X}_{\bar{\boldsymbol{\rho}}}\big(\rho_v(t)\big) = -\Theta_u\big(\rho_v(t)\big) + \sum_{j=1}^{2} \Theta_j\big(\rho_v(t)\big) + 2X_{\bar{\boldsymbol{\rho}}}\big(\rho_v(t)\big) \sum_{j=1}^{2} \sqrt{\eta_j M_j} P_j\big(\rho_v(t)\big) v_j(t),$$

where $v_1(t)$ and $v_2(t)$ belonging to $\mathcal{V}$ are the control inputs, $\Theta_u(\rho) := u_1 \mathrm{Tr}(i[H_1,\rho]\bar{\boldsymbol{\rho}})$ and

$$\begin{aligned} \Theta_1(\rho) &:= 4\eta_1 M_1 \mathrm{Tr}(L_z\rho)P_1(\rho)X_{\bar{\boldsymbol{\rho}}}(\rho), \quad P_1(\rho) := \bar{\lambda}_z - \mathrm{Tr}(L_z\rho) \text{ where } L_z\bar{\boldsymbol{\rho}} = \bar{\lambda}_z\bar{\boldsymbol{\rho}}; \\ \Theta_2(\rho) &:= 4\eta_2 M_2 \mathrm{Tr}(L_x\rho)P_2(\rho)X_{\bar{\boldsymbol{\rho}}}(\rho), \quad P_2(\rho) := \bar{\lambda}_x - \mathrm{Tr}(L_x\rho) \text{ where } L_x\bar{\boldsymbol{\rho}} = \bar{\lambda}_x\bar{\boldsymbol{\rho}}. \end{aligned}$$

Firstly, we show by contradiction that $X_{\bar{\boldsymbol{\rho}}}(\rho_v(t)) > 0$ for $t > 0$. Suppose $X_{\bar{\boldsymbol{\rho}}}\big(\rho_v(s)\big) = 0$ for $s \in [0,\epsilon]$ with $\epsilon > 0$, which is equivalent to $\rho_v(s)\xi = 0$. Take this assumption into account in Equation (5.9), for $s \in [0,\epsilon]$, we have

$$\dot{\rho}_v(s)\xi = -iu\rho_v(s)H_1\xi = 0,$$

which implies $\rho_v(s)H_1\xi = 0$ since $u \neq 0$. After a straightforward calculation, for $s \in [0,\epsilon]$, we have

$$\xi^*H_1\dot{\rho}_v(s)H_1\xi = M_1(1-\eta_1)\xi^*H_1L_z\rho_v(s)L_zH_1\xi + M_2(1-\eta_2)\xi^*H_1L_x\rho_v(s)L_xH_1\xi = 0,$$

which implies that $\rho_v(s)L_zH_1\xi = 0$ and $\rho_v(s)L_xH_1\xi = 0$. Thus, for $s \in [0,\epsilon]$, under the assumption of the lemma, $\rho_v(s) \in \mathbb{C}^{4\times4}$ has four linearly independent eigenvectors corresponding to the eigenvalue zero, which leads to a contradiction. Hence, we can conclude that, under the assumption of the lemma on the feedback law and H_1, if $X_{\bar{\boldsymbol{\rho}}}\big(\rho_v(0)\big) = 0$, there exists an arbitrarily small $t_0 > 0$ such that $X_{\bar{\boldsymbol{\rho}}}\big(\rho_v(t_0)\big) > 0$.

Set $\mathbf{P}_1 := \{\rho \in \mathcal{S}_4 |\, P_1(\rho) = 0\}$ and $\mathbf{P}_2 := \{\rho \in \mathcal{S}_4 |\, P_2(\rho) = 0\}$. We note that $\mathbf{P}_1 \cap \mathbf{P}_2 = \bar{\boldsymbol{\rho}}$. For $t > t_0$, we can thus take the feedbacks $v_1 = KP_1(\rho)X_{\bar{\boldsymbol{\rho}}}(\rho)$ and $v_2 = KP_2(\rho)X_{\bar{\boldsymbol{\rho}}}(\rho)$ with $K > 0$. The proposed control input v guarantees that $\rho_v(t) \in B_r(\bar{\boldsymbol{\rho}})$ for $t \leq T$ with $t_0 < T < \infty$, if K is sufficiently large. Now, considering the stochastic solution of Equation (5.1) in this case, we deduce that $\mathbb{P}\big(\rho_t \in B_r(\bar{\boldsymbol{\rho}})\big) > 0$ for $t \leq T$ from the support theorem (Theorem A.2.1).

By compactness of $\mathcal{S}_4 \setminus B_r(\bar{\boldsymbol{\rho}})$ and the Feller continuity of ρ_t, we have $\sup_{\rho_0 \in \mathcal{S}_4 \setminus B_r(\bar{\boldsymbol{\rho}})} \mathbb{P}_{\rho_0}(\tau_r \geq T) \leq 1 - \zeta < 1$, for some $\zeta > 0$. By Dynkin inequality [Dyn65],

$$\sup_{\rho_0 \in \mathcal{S}_4 \setminus B_r(\bar{\boldsymbol{\rho}})} \mathbb{E}_{\rho_0}(\tau_r) \leq \frac{T}{1 - \sup_{\rho_0 \in \mathcal{S}_4 \setminus B_r(\bar{\boldsymbol{\rho}})} \mathbb{P}_{\rho_0}(\tau_r \geq T)} \leq \frac{T}{\zeta}.$$

Then by Markov inequality, for all $\rho_0 \in \mathcal{S}_4 \setminus B_r(\bar{\boldsymbol{\rho}})$, we have

$$\mathbb{P}_{\rho_0}(\tau_r = \infty) = \lim_{n\to\infty} \mathbb{P}_{\rho_0}(\tau_r \geq n) \leq \lim_{n\to\infty} \mathbb{E}_{\rho_0}(\tau_r)/n = 0,$$

which implies $\mathbb{P}_{\rho_0}(\tau_r < \infty) = 1$. The proof is complete. □

By combining Lemma 5.3.2 and following arguments similar to Theorem 4.3.8, we get the following general result concerning the exponential stabilization towards Bell states.

Theorem 5.3.3. *Assume that $\rho_0 \in \mathcal{S}_4$ and the assumptions of Lemma 5.3.2 are satisfied. Additionally, suppose that there exists a positive-definite function $V(\rho)$ such that $V(\rho) = 0$ if and only if $\rho = \bar{\boldsymbol{\rho}} \in \bar{E}_2$, and V is continuous on $\mathcal{S}_4$ and twice continuously differentiable on the set $\mathcal{S}_4 \setminus \bar{\boldsymbol{\rho}}$. Moreover, suppose that there exist positive constants C, C_1 and C_2 such that*

(i) $C_1\, d_B(\rho, \bar{\boldsymbol{\rho}}) \leq V(\rho) \leq C_2\, d_B(\rho, \bar{\boldsymbol{\rho}})$, $\forall\, \rho \in \mathcal{S}_4$, *and*

(ii) $\limsup_{\rho\to\bar{\boldsymbol{\rho}}} \frac{\mathscr{L}V(\rho)}{V(\rho)} \leq -C$.

Then, $\bar{\boldsymbol{\rho}}$ is a.s. exponentially stable for the system (5.1) with sample Lyapunov exponent less than or equal to $-C - \frac{K}{2}$, where $K := \liminf_{\rho\to\bar{\boldsymbol{\rho}}} \big(g_1^2(\rho) + g_2^2(\rho)\big)$ with $g_j(\rho) := \sqrt{\eta_j} \frac{\partial V(\rho)}{\partial \rho} \frac{G_j(\rho)}{V(\rho)}$ for $j = 1, 2$.

Next, we derive general conditions on the feedback law and the control Hamiltonian which allows us to apply the previous theorem.

Theorem 5.3.4. *Let $\rho_0 \in \mathcal{S}_4$ and $\bar{\boldsymbol{\rho}} \in \bar{E}_2$ be the target state. Suppose that the assumptions of Lemma 5.3.2 and the following relation*

$$\limsup_{\rho\to\bar{\boldsymbol{\rho}}} \Theta_u(\rho)/d_B^2(\rho, \bar{\boldsymbol{\rho}}) = 0 \tag{5.10}$$

are satisfied. Then $\bar{\boldsymbol{\rho}}$ is almost surely exponentially stable with sample Lyapunov exponent less than or equal to $-\bar{C}$ where $\bar{C} = \min\{\eta_1 M_1, \eta_2 M_2\}$.

Proof. To prove the theorem, we show that we can apply Theorem 5.3.3 with the Lyapunov function $V(\rho) = \sqrt{1 - X_{\bar{\boldsymbol{\rho}}}(\rho)}$ with $\bar{\boldsymbol{\rho}} \in \bar{E}_2$. Note that $d_B(\rho, \bar{\boldsymbol{\rho}}) \leq V(\rho) \leq \sqrt{2} d_B(\rho, \bar{\boldsymbol{\rho}})$, we are then left to show the condition (ii). The infinitesimal generator of the Lyapunov function satisfies,

$$\mathscr{L}V(\rho) \leq \frac{\Theta_u(\rho)}{2V(\rho)} - \frac{X_{\bar{\boldsymbol{\rho}}}^2(\rho)\bar{C}}{2V^3(\rho)}\big(P_1^2(\rho) + P_2^2(\rho)\big).$$

Since $\rho \geq 0$, by estimating the right hand side of the above inequality, we obtain the following for all $\rho \in \mathcal{S}_4 \setminus \bar{\boldsymbol{\rho}}$,

$$\mathscr{L}V(\rho) \leq -\frac{\bar{C}}{2} V(\rho) \left(X_{\bar{\boldsymbol{\rho}}}^2(\rho) - \frac{\Theta_u(\rho)}{\bar{C}V^2(\rho)} \right).$$

As $g_1^2(\rho) + g_2^2(\rho) \geq \bar{C} X_{\bar{\boldsymbol{\rho}}}^2(\rho)$ and by using the relation (5.10), we can apply Theorem 5.3.3 with $C = \bar{C}/2$ and $K = \bar{C}$. The proof is hence complete. □

An application of the previous results is given below.

Proposition 5.3.5. *Consider system* (5.1) *with* $\rho_0 \in \mathcal{S}_4$ *and* $\eta_1, \eta_2 \in (0,1)$. *Let* $\bar{\boldsymbol{\rho}} \in \bar{E}_2$ *be the target state. Define the control Hamiltonian as* $H_1 = \sigma_z \otimes \sigma_y + \sigma_z \otimes \sigma_x + \mathbb{1} \otimes \sigma_z$ *and the feedback law as*

$$u(\rho) = \alpha\big(1 - X_{\bar{\boldsymbol{\rho}}}(\rho)\big)^{\beta} - \gamma \mathrm{Tr}(i[H_1, \rho]\bar{\boldsymbol{\rho}}), \tag{5.11}$$

where $\gamma \geq 0$, $\beta \geq 1$ *and* $\alpha > 0$ *sufficiently large. Then* $\bar{\boldsymbol{\rho}}$ *is almost surely exponentially stable with sample Lyapunov exponent less than or equal to* $-\bar{C}$ *where* $\bar{C} = \min\{\eta_1 M_1, \eta_2 M_2\}$.

Proof. By simple computations one can show that the feedback law and the control Hamiltonian satisfy the relation (5.10), the assumptions of Lemma 5.3.1 and Lemma 5.3.2. The result then follows Theorem 5.3.4. □

The simulations for the case with two quantum channels are shown in Fig. 5.4 and Fig. 5.5.

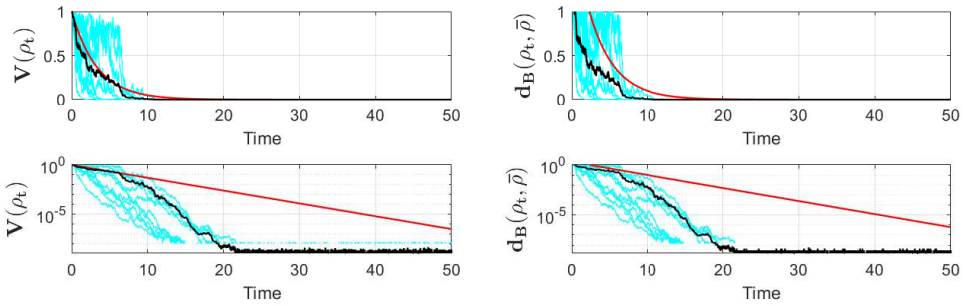


FIGURE 5.4 – Exponential stabilization of system (5.1) towards $\boldsymbol{\Psi}_+$ with the feedback law (5.11) starting at $\boldsymbol{\Phi}_-$, when $\omega = 0.3$, $\eta_1 = 0.3$, $M_1 = 1$, $\eta_2 = 0.4$, $M_2 = 0.9$, $\alpha = 10$, $\beta = 12$ and $\gamma = 1$: the black curve represents the mean value of 10 arbitrary samples, and the red curve represents the exponential reference with exponent $-\bar{C}$. The figures at the bottom are the semi-log versions of the ones at the top.

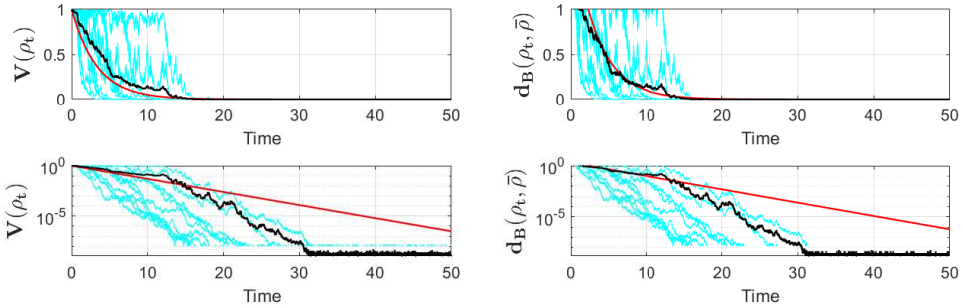


FIGURE 5.5 – Exponential stabilization of system (5.1) towards $\boldsymbol{\Phi}_-$ with the feedback law (5.11) starting at $\boldsymbol{\Psi}_+$, when $\omega = 0.3$, $\eta_1 = 0.3$, $M_1 = 1$, $\eta_2 = 0.4$, $M_2 = 0.9$, $\alpha = 10$, $\beta = 12$ and $\gamma = 1$: the black curve represents the mean value of 10 arbitrary samples, and the red curve represents the exponential reference with exponent $-\bar{C}$. The figures at the bottom are the semi-log versions of the ones at the top.

6

Feedback stabilization of open quantum spin systems with unknown initial states

In Chapter 3, we have already introduced the quantum filtering theory and obtained the stochastic master equation (3.43) whose solution $\hat{\rho}_t$, called quantum filter, is the optimal estimator in L^2 sense of the actual quantum state ρ_t. If the initial state ρ_0 is known, then we can set $\hat{\rho}_0 = \rho_0$, and the stochastic master equation (3.43) describes the time evolution of the quantum state ρ_t the feedback stabilization in this case has been discussed in Chapter 4 and Chapter 5. However, if the quantum filter is "wrongly initialized", i.e., $\hat{\rho}_0 \neq \rho_0$, how can we ensure that the distance between the estimator $\hat{\rho}_t$ and the true quantum state ρ_t converges to zero when t goes to infinity ?

This problem has been investigated in some papers. In a series of papers by van Handel [vH06, vH09b, vH09a, vH10], a sufficient observability condition has been established so that such convergence is guaranteed. However, such condition is not easy to verify even if the system is finite dimensional. In [DKSA06], by showing that the Hilbert-Schmidt inner product of $\hat{\rho}_t$ and ρ_t, i.e., $\mathrm{Tr}(\hat{\rho}_t\rho_t)$, is a sub-martingale, the authors proved the convergence to one of the distance between $\hat{\rho}_t$ and ρ_t when at least one of them is always pure. Then, in [Rou11], by applying Uhlmann's technique [NC02, Theorem 9.4], the author showed that the fidelity between the state of the discrete-time quantum filter [SDZ+11] and the state of the open quantum system is a sub-martingale via a Kraus map. However, this sub-martingale property of the fidelity cannot ensure the convergence of the filter state towards the actual one. In [AMR11] the authors show that the fidelity is a sub-martingale for continuous-time quantum filters with the perfect measurement for arbitrary mixed states. By quantum repeated interaction approach, such result has been extended to the continuous-time case with general measurement imperfections in [APR14]. Then, in [BP14] , the authors showed that, for a more general quantum filter dynamics, which is described by a jump-diffusion stochastic differential equations, when the control input is turned off under non-demolition perfect measurements, the convergence is ensured.

In this chapter, we will firstly consider the convergence problem for the quantum

state of actual quantum spin-$\frac{1}{2}$ systems with imperfect measurements and the estimated state [LAM20]. Next, we discuss the exponential stabilization problem of N-level quantum spin systems (6.8) undergoing imperfect measurements with the unknown initial data towards the predetermined pure state associated with an eigenvector of the measurement operator J_z [LAMb].

6.1 Convergence property of quantum spin-$\frac{1}{2}$ systems

Here, we consider quantum spin-$\frac{1}{2}$ systems. The stochastic master equations of the actual system and its corresponding filter follow the following dynamics,

$$\begin{aligned} d\rho_t &= F_u(\rho_t)dt + L(\rho_t)dt + G(\rho_t)\big(dY_t - 2\sqrt{\eta M}\mathrm{Tr}(\sigma_z\rho_t)dt\big),\\ d\hat{\rho}_t &= F_u(\hat{\rho}_t)dt + L(\hat{\rho}_t)dt + G(\hat{\rho}_t)\big(dY_t - 2\sqrt{\eta M}\mathrm{Tr}(\sigma_z\hat{\rho}_t)dt\big), \end{aligned}$$

where

- the actual quantum state of spin-$\frac{1}{2}$ system is described by ρ, which belongs to the space $\mathcal{S}_2 := \{\rho \in \mathbb{C}^{2\times 2} |\, \rho = \rho^*, \mathrm{Tr}(\rho) = 1, \rho \geq 0\}$. The associated estimated state is described by $\hat{\rho} \in \mathcal{S}_2$,
- $F_u(\rho) := -i/2[\omega\sigma_z + u\sigma_y, \rho]$, $L(\rho) := M/4(\sigma_z\rho\sigma_z - \rho)$ and $G(\rho) := \sqrt{\eta M}/2\big(\sigma_z\rho + \rho\sigma_z - 2\mathrm{Tr}(\sigma_z\rho)\rho\big)$,
- Y_t denotes the observation process of the actual quantum spin-$\frac{1}{2}$ system, which is a continuous semi-martingale whose quadratic variation is given by $[Y, Y]_t = t$. Its dynamics satisfies $dY_t = dW_t + \sqrt{\eta M}\mathrm{Tr}(\sigma_z\rho_t)dt$, where W_t is a one-dimensional standard Wiener process,
- $u := u(\hat{\rho})$ denotes the feedback controller as a function of the estimated state $\hat{\rho}$,
- ω is the difference between the energies of the excited state and the ground state, $\eta \in [0, 1]$ is determined by the efficiency of the detectors, and $M > 0$ is the strength of the interaction between the system and the probe. The matrices σ_x, σ_y and σ_z are the Pauli matrices.

By setting $dY_t = dW_t + 2\sqrt{\eta M}\mathrm{Tr}(\sigma_z\rho_t)dt$, we obtain the following matrix-valued stochastic differential equations describing the time evolution of the pair $(\rho_t, \hat{\rho}_t) \in \mathcal{S}_2 \times \mathcal{S}_2$,

$$d\rho_t = F_u(\rho_t)dt + L(\rho_t)dt + G(\rho_t)dW_t, \tag{6.1}$$

$$d\hat{\rho}_t = F_u(\hat{\rho}_t)dt + L(\hat{\rho}_t)dt + \sqrt{\eta M}G(\hat{\rho}_t)\mathrm{Tr}\big(\sigma_z(\rho_t - \hat{\rho}_t)\big)dt + G(\hat{\rho}_t)dW_t. \tag{6.2}$$

If $u \in \mathcal{C}^1(\mathcal{S}_2, \mathbb{R})$, the existence and uniqueness of the solution of (6.1) and (6.2) can be shown by similar arguments as in [MvH07, Proposition 3.5].

We focus on the fidelity $\mathcal{F}(\rho, \hat{\rho})$ which defines a "distance" between the real state ρ and the estimated state $\hat{\rho}$. In the two-level case, the fidelity can be written in the following form

$$\mathcal{F}(\hat{\rho}, \rho) = \mathrm{Tr}(\hat{\rho}\rho) + 2\sqrt{\det(\hat{\rho})\det(\rho)}.$$

Recall that a density operator can be uniquely characterized by the Bloch sphere coordinates (x, y, z) as

$$\rho = \frac{\mathbb{1} + x\sigma_x + y\sigma_y + z\sigma_z}{2} = \frac{1}{2}\begin{bmatrix} 1+z & x-iy \\ x+iy & 1-z \end{bmatrix}.$$

The vector (x, y, z) belongs to the ball

$$\mathcal{B} := \{(x, y, z) \in \mathbb{R}^3 |\, x^2 + y^2 + z^2 \leq 1\}.$$

Thus the fidelity in the Bloch sphere coordinates is given by

$$\mathcal{F}(\hat{\rho}, \rho) = \mathcal{F}(\hat{\mathbf{v}}, \mathbf{v}) = \frac{1}{2}\left(1 + \mathbf{v}^\top \hat{\mathbf{v}} + \sqrt{(1 - \|\mathbf{v}\|^2)(1 - \|\hat{\mathbf{v}}\|^2)}\right),$$

where $\mathbf{v} := (x, y, z)$ denotes the real state and $\hat{\mathbf{v}} := (\hat{x}, \hat{y}, \hat{z})$ denotes the estimated state in Bloch sphere coordinates. Thus, for the two special cases $\mathcal{F}(\hat{\rho}, \rho) = 1$ and $\mathcal{F}(\hat{\rho}, \rho) = 0$,

1. if $\mathcal{F}(\hat{\rho}, \rho) = 1$, we have $\mathbf{v} = \hat{\mathbf{v}}$;
2. if $\mathcal{F}(\hat{\rho}, \rho) = 0$, we have $\mathbf{v} + \hat{\mathbf{v}} = 0$ and $\|\mathbf{v}\|^2 = \|\hat{\mathbf{v}}\|^2 = 1$.

The stochastic differential equation (6.1) expressed in the Bloch sphere coordinates takes the following form

$$dx_t = \left(-\omega_{eg} y_t - \frac{M}{2} x_t + \hat{u}_t z_t\right) dt - \sqrt{\eta M} x_t z_t dW_t, \tag{6.3a}$$

$$dy_t = \left(\omega_{eg} x_t - \frac{M}{2} y_t\right) dt - \sqrt{\eta M} y_t z_t dW_t, \tag{6.3b}$$

$$dz_t = -\hat{u}_t x_t dt + \sqrt{\eta M}(1 - z_t^2) dW_t. \tag{6.3c}$$

The stochastic differential equation (6.2) in the Bloch sphere coordinates is given by,

$$d\hat{x}_t = \left(-\omega_{eg} \hat{y}_t - \frac{M}{2} \hat{x}_t + \hat{u}_t \hat{z}_t + \eta M \hat{x}_t \hat{z}_t (\hat{z}_t - z_t)\right) dt - \sqrt{\eta M} \hat{x}_t \hat{z}_t dW_t, \tag{6.4a}$$

$$d\hat{y}_t = \left(\omega_{eg} \hat{x}_t - \frac{M}{2} \hat{y}_t + \eta M \hat{y}_t \hat{z}_t (\hat{z}_t - z_t)\right) dt - \sqrt{\eta M} \hat{y}_t \hat{z}_t dW_t, \tag{6.4b}$$

$$d\hat{z}_t = \big(-\hat{u}_t \hat{x}_t - \eta M (1 - \hat{z}_t^2)(\hat{z}_t - z_t)\big) dt + \sqrt{\eta M}(1 - \hat{z}_t^2) dW_t. \tag{6.4c}$$

In order to apply the Itô formula on the fidelity $\mathcal{F}(\rho, \hat{\rho})$, we need the unattainability of the boundary for ρ and $\hat{\rho}$. By straightforward calculations, we can show that

$$\{\rho \in \mathcal{S}_2 |\, \det(\rho) = 0\} = \{\rho \in \mathcal{S}_2 |\, \mathrm{Tr}(\rho^2) = 1\}, \tag{6.5}$$

which means that the boundary $\partial \mathcal{S}_2$ is equal to the set of all pure states $\mathcal{P}$. By a similar argument as in Lemma 4.1.1, we can obtain the following lemma, which states some invariance properties for Equation (6.1) and Equation (6.2). Note that the following result can also be shown by applying Lemma 4.3.2 and Lemma 4.3.3.

Lemma 6.1.1. *If $\rho_0 > 0$, then $\mathbb{P}(\rho_t > 0, \forall t \geq 0) = 1$. The same results hold true for $\hat{\rho}_t$. Also, if $\eta = 1$, $\partial \mathcal{S}_2 \times \mathcal{S}_2$ and $\mathcal{S}_2 \times \partial \mathcal{S}_2$ are a.s. invariant for Equation* (6.1) *and Equation* (6.2).

Proof. The dynamics of the purification function $S(\rho_t) := 1 - \mathrm{Tr}(\rho_t^2)$ is given by

$$dS(\rho_t) = M\big(\frac{(1 - \eta)(1 - z_t^2)}{2} - (1 - \eta z_t^2) S(\rho_t)\big) dt - 2\sqrt{\eta M} z_t S(\rho_t) dW_t.$$

Then, if $\eta = 1$, it is obvious that the set of all pure states $\mathcal{P}$ for Equation (6.1) is a.s. invariant.

Next, let us prove the first part of the lemma. Given $\varepsilon > 0$, consider any $\mathcal{C}^2$ function on $\mathcal{S}$ such that

$$V(\rho) = \frac{1}{S(\rho)}, \quad \text{if } S(\rho) > \varepsilon.$$

We find

$$\mathscr{L}V(\rho) = \left(4M - (1-\eta)\frac{1-z^2}{S(\rho)}\right) V(\rho), \quad \text{if } S(\rho) > \varepsilon.$$

Since $1 - z^2 \geq 2S(\rho)$, then we have $\mathscr{L}V(\rho) \leq KV(\rho)$ if $S(\rho) > \varepsilon$ for some positive constant K. To conclude the proof, one just applies the same arguments as in Lemma 4.1.1. Roughly speaking, by setting $f(\rho, t) = e^{-Kt}V(\rho)$, one has $\mathscr{L}f \leq 0$ whenever $S(\rho) > \varepsilon$. From this fact one proves that the probability of $S(\rho)$ becoming zero in a finite fixed time T is proportional to ε and, being the latter arbitrary, it must be 0. Due to the equality (6.5), $\mathbb{P}(\rho_t > 0, \forall t \geq 0) = 1$ when $\rho_0 > 0$. Concerning $\hat{\rho}_t$ and Equation (6.2), the result can be proved in the same manner. □

We then propose a new method to analyze the behavior of ρ_t and $\hat{\rho}_t$ at the boundary. Denote $\boldsymbol{\rho}_g := \mathrm{diag}(1, 0)$ and $\boldsymbol{\rho}_e := \mathrm{diag}(0, 1)$, which are the the pure states corresponding to the eigenvectors of σ_z.

Lemma 6.1.2. *Assume $\eta \in (0, 1)$ and $u \in \mathcal{C}^1(\mathcal{S}_2, \mathbb{R})$. Suppose that $\hat{\rho}_0$ lies in $\partial\mathcal{S}_2 \setminus \{\boldsymbol{\rho}_g, \boldsymbol{\rho}_e\}$, then $\hat{\rho}_t > 0$ for all $t > 0$ almost surely. In particular, if $u(\boldsymbol{\rho}_g)u(\boldsymbol{\rho}_e) \neq 0$ then, for all $\hat{\rho}_0 \in \partial\mathcal{S}_2$, $\hat{\rho}_t > 0$ for all $t > 0$ almost surely. Moreover, under the assumption $\rho_0 \in \partial\mathcal{S}_2 \setminus \{\boldsymbol{\rho}_g, \boldsymbol{\rho}_e\}$, then $\rho_t > 0$ for all $t > 0$ almost surely. In particular, if $u(\boldsymbol{\rho}_g)u(\boldsymbol{\rho}_e) \neq 0$ then, for all $\rho_0 \in \partial\mathcal{S}_2$, ρ_t exits the boundary in finite time and stays in the interior of $\mathcal{S}_2$ almost surely.*

Proof. Firstly, consider the purification function $S(\hat{\rho}) := 1 - \mathrm{Tr}(\hat{\rho}^2)$ for Equation (6.2), whose dynamics is given by

$$dS(\hat{\rho}_t) = M\Big(\frac{(1-\eta)(1-\hat{z}_t^2)}{2} - (1-\eta\hat{z}_t^2)S(\hat{\rho}_t) - 4\eta(z_t - \hat{z}_t)\hat{z}_t S(\hat{\rho}_t)\Big)dt - 2\sqrt{\eta M}\hat{z}_t S(\hat{\rho}_t)dW_t.$$

Now, since $\eta \in [0, 1)$, for all $\hat{\rho} \in \partial\mathcal{S}_2 \setminus \{B_\epsilon(\boldsymbol{\rho}_g) \cap B_\epsilon(\boldsymbol{\rho}_e)\}\}$ with $\epsilon > 0$ arbitrarily small. By compactness, there exists a $\zeta > 0$ such that $2M(1-\eta)(1-\hat{z}^2) \geq \zeta$. Define $\tau := \inf\{t > 0 | \, \hat{\rho}_t \notin \partial\mathcal{S}_2 \setminus \{B_\epsilon(\boldsymbol{\rho}_g) \cap B_\epsilon(\boldsymbol{\rho}_e)\}$, for all $\hat{\rho}_0 \in \partial\mathcal{S}_2 \setminus \{B_\epsilon(\boldsymbol{\rho}_g) \cap B_\epsilon(\boldsymbol{\rho}_e)\}$ and $t > 0$, by Itô's formula, we have

$$\mathbb{E}(S(\hat{\rho}_{t\wedge\tau})) = \mathbb{E}\left(\int_0^{t\wedge\tau} \frac{1}{2}M(1-\eta)(1-\hat{z}_s^2)ds\right) \geq \zeta\mathbb{E}(t \wedge \tau).$$

By continuity and the definition of τ, $S(\hat{\rho}_{t\wedge\tau}) = 0$ almost surely. This implies that $\mathbb{E}(t \wedge \tau) = 0$. Since we have $\mathbb{E}(t \wedge \tau) \geq t\mathbb{P}(\tau \geq t)$ we deduce that $\mathbb{P}(\tau \geq t) = 0$ for all $t > 0$. Due to the arbitrariness of ϵ, if $\hat{\rho}_0 \in \partial\mathcal{S}_2 \setminus \{\boldsymbol{\rho}_g, \boldsymbol{\rho}_e\}$ then $\hat{\rho}_t$ exits the boundary immediately. Combining with the strong Markov property and Lemma 6.1.1, $\hat{\rho}_t > 0$ for all $t > 0$, almost surely. Moreover, if $\hat{\rho}_0 \in \{\boldsymbol{\rho}_g, \boldsymbol{\rho}_e\}$ then by the condition $u(\hat{\rho}_0) \neq 0$ we deduce the same result.

By repeating the previous lines for the purification function $S(\rho)$, we can show that, if $\rho_0 \in \partial\mathcal{S}_2 \setminus \{\boldsymbol{\rho}_g, \boldsymbol{\rho}_e\}$, then $\rho_t > 0$ for all $t > 0$ almost surely. Moreover, if $u(\boldsymbol{\rho}_g)u(\boldsymbol{\rho}_e) \neq 0$

and $u \in \mathcal{C}^1(\mathcal{S}_2, \mathbb{R})$, then there exist a neighborhood of $\boldsymbol{\rho}_e$ denoted by $B_{r_e}(\boldsymbol{\rho}_e)$ and a neighborhood of $\boldsymbol{\rho}_g$ denoted by $B_{r_g}(\boldsymbol{\rho}_g)$ such that, $u(\hat{\rho}) \neq 0$ for all $\rho \in B_{r_e}(\boldsymbol{\rho}_e) \cup B_{r_g}(\boldsymbol{\rho}_g)$. By applying the similar argument as in the last part of the proof of Lemma 4.3.6, we can show that $\hat{\rho}_t$ can enter in $B_{r_e}(\boldsymbol{\rho}_e) \cup B_{r_g}(\boldsymbol{\rho}_g)$ in finite time almost surely, which means $u(\hat{\rho}_t)$ can become non-zero in finite time almost surely. At once $u \neq 0$, ρ_t can exits the boundary and stay in the interior of $\mathcal{S}$ almost surely. The proof is complete. □

Proposition 6.1.3. *Assume $u(\boldsymbol{\rho}_e)u(\boldsymbol{\rho}_g) \neq 0$ and $u \in \mathcal{C}^1(\mathcal{S}_2, \mathbb{R})$, then for all $(\rho_0, \hat{\rho}_0) \in \mathcal{S}_2 \times \mathcal{S}_2$, $\mathcal{F}(\rho_t, \hat{\rho}_t)$ converges to one almost surely.*

Proof. By Lemma 6.1.1, if $\eta = 1$, then $\partial\mathcal{S}_2 \times \partial\mathcal{S}_2$ and $\mathrm{int}(\mathcal{S}_2) \times \mathrm{int}(\mathcal{S}_2)$ are invariant for the coupled system (6.1)-(6.2) almost surely. Moreover, if ρ_t or $\hat{\rho}_t$ stay in boundary of $\mathcal{S}_2$, then $\mathcal{F}(\rho, \hat{\rho}) = \mathrm{Tr}(\rho\hat{\rho}) \in \mathcal{C}^2$. For the case $\eta \in (0, 1)$, under the assumptions of feedback controller: $u(\boldsymbol{\rho}_e)u(\boldsymbol{\rho}_g) \neq 0$ and $u \in \mathcal{C}^1(\mathcal{S}_2, \mathbb{R})$, by Lemma 6.1.2, $(\rho_t, \hat{\rho}_t)$ can exit the boundary and stay in $\mathrm{int}(\mathcal{S}_2) \times \mathrm{int}(\mathcal{S}_2)$ in finite time almost surely. Note that the fidelity function is $\mathcal{C}^2$ in $\mathrm{int}(\mathcal{S}_2) \times \mathrm{int}(\mathcal{S}_2)$.

Consider the Lyapunov function $\mathcal{V}(\rho, \hat{\rho}) := 1 - \mathcal{F}(\rho, \hat{\rho})$. Denote $\Xi := \sqrt{(1 - \|\mathbf{v}\|^2)(1 - \|\hat{\mathbf{v}}\|^2)}$. For any $u \in \mathcal{C}^1(\mathcal{S}_2, \mathbb{R})$, the infinitesimal generator of $\mathcal{F}(\rho, \hat{\rho})$ is given by

$$\begin{aligned}\mathscr{L}\mathcal{F}(\rho, \hat{\rho}) = \frac{M}{2}(1 - \hat{z}^2)(1 - \mathbf{v}^\top\hat{\mathbf{v}} - \Xi) + \frac{M(1-\eta)}{4\Xi}\Big((1 - \hat{z}^2)(1 - \|\mathbf{v}\|^2) \\ + (1 - z^2)(1 - \|\hat{\mathbf{v}}\|^2) + 2\hat{z}^2(1 - \mathbf{v}^\top\hat{\mathbf{v}} - \Xi)\Xi - 2(1 - z\hat{z})\Xi\Big).\end{aligned} \tag{6.6}$$

In particular, if $\eta = 1$, we have

$$\mathscr{L}\mathcal{F}(\rho, \hat{\rho}) = \frac{M}{2}(1 - \hat{z}^2)(1 - \mathbf{v}^\top\hat{\mathbf{v}} - \Xi) = M(1 - \hat{z}^2)\big(1 - \mathcal{F}(\rho, \hat{\rho})\big). \tag{6.7}$$

For $\eta = 0$, we have

$$\begin{aligned}\mathscr{L}\mathcal{F}(\rho, \hat{\rho}) =& \frac{M}{2}\left(\frac{(1 - \hat{z}^2)(1 - \|\mathbf{v}\|^2) + (1 - z^2)(1 - \|\hat{\mathbf{v}}\|^2)}{2\Xi} + z\hat{z} - \mathbf{v}^\top\hat{\mathbf{v}} - \Xi\right) \\ \geq& \frac{M}{2}\left(\frac{(1 - \hat{z}^2)(1 - \|\mathbf{v}\|^2) + (1 - z^2)(1 - \|\hat{\mathbf{v}}\|^2)}{2\Xi} - \sqrt{(\|\mathbf{v}\|^2 - z^2)(\|\hat{\mathbf{v}}\|^2 - \hat{z}^2)} - \Xi\right) \\ =& \frac{M}{4\Xi}\left(\sqrt{(\|\hat{\mathbf{v}}\|^2 - \hat{z}^2)(1 - \|\mathbf{v}\|^2)} - \sqrt{(\|\mathbf{v}\|^2 - z^2)(1 - \|\hat{\mathbf{v}}\|^2)}\right)^2.\end{aligned}$$

Therefore, for all $\eta \in [0, 1]$ and $(\rho, \hat{\rho}) \in \mathrm{int}(\mathcal{S}_2) \times \mathrm{int}(\mathcal{S}_2)$, we have $\mathscr{L}\mathcal{F}(\rho, \hat{\rho}) \geq 0$ which implies that $\mathscr{L}\mathcal{V}(\rho, \hat{\rho}) \leq 0$. By the stochastic LaSalle-type theorem in [Mao99], we deduce that $\lim_{t\to\infty} \mathscr{L}\mathcal{F}(\rho_t, \hat{\rho}_t) = 0$ almost surely. Since $\mathscr{L}\mathcal{F}(\rho, \hat{\rho})$ for any $\eta \in (0, 1]$ can be written as a convex combination of the expressions (6.7) and (6.8), we have that either $|\hat{z}|$ converges to one or $\mathcal{F}(\rho, \hat{\rho})$ converges to one almost surely. This concludes the proof of the first part of the proposition. The additional assumption $u(\boldsymbol{\rho}_e)u(\boldsymbol{\rho}_g) \neq 0$ rules out the first possibility, completing the proof of the proposition. □

The simulations of the fidelity $\mathcal{F}(\rho_t, \hat{\rho}_t)$ are shown in Fig. 6.1 and the one of the trajectories of ρ_t and $\hat{\rho}_t$ is shown Fig. 6.2. We set $\boldsymbol{\rho}_e$ as the initial state of the real quantum system and $\boldsymbol{\rho}_g$ as the initial state of the quantum filter.

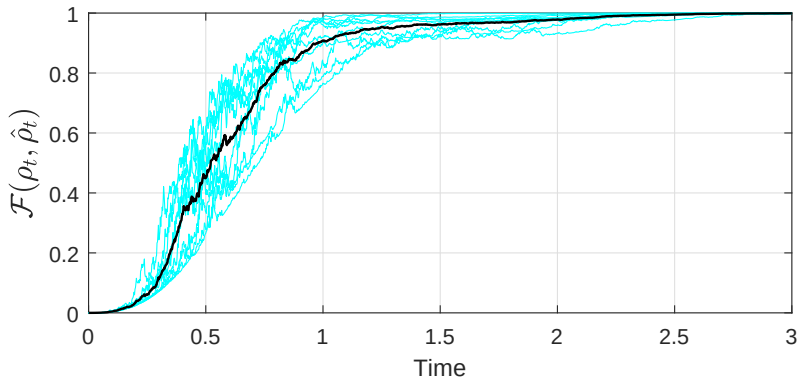


FIGURE 6.1 – Convergence of the fidelity $\mathcal{F}(\rho_t, \hat{\rho}_t)$ towards one with the feedback law $u(\hat{\rho}) \equiv 1$ starting at $(\rho_0, \hat{\rho}_0) = (\boldsymbol{\rho}_e, \boldsymbol{\rho}_g)$, when $\omega = 0.3$, $\eta = 0.3$ and $M = 1$: the black curve represents the mean value of the 10 arbitrary samples.

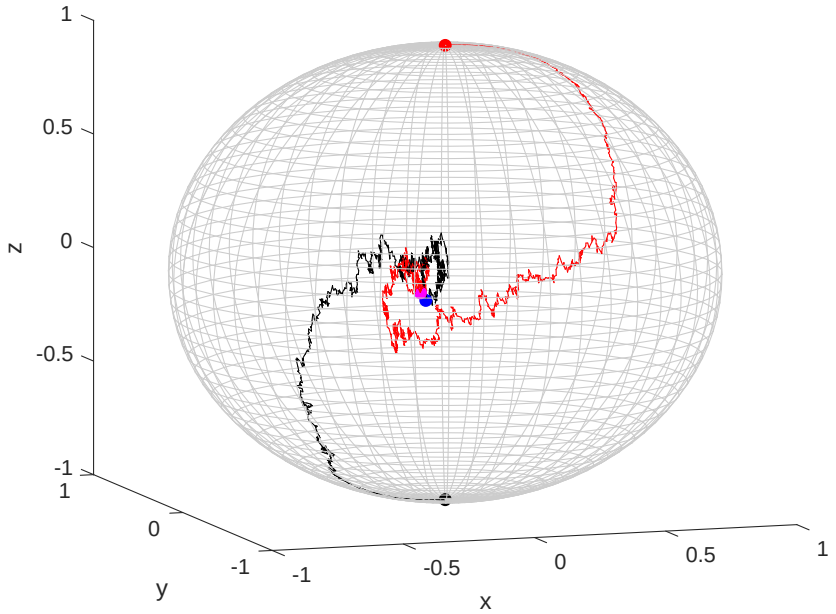


FIGURE 6.2 – Behavior of the trajectories ρ_t and $\hat{\rho}_t$ with the feedback law $u(\hat{\rho}) \equiv 1$ during three seconds, starting at $(\boldsymbol{\rho}_e, \boldsymbol{\rho}_g)$ described by the black and the red point respectively, when $\omega = 0.3$, $\eta_1 = 0.3$ and $M = 1$. The black curve represents the mean value of the 10 arbitrary samples of ρ_t ; the blue point describes the means value of end point of ρ_t ; the red curve represents the mean value of the 10 arbitrary samples of $\hat{\rho}_t$; and the magenta point describes the means value of end point of $\hat{\rho}_t$.

6.2 Exponential stabilization of the coupled N-level quantum spin system

In this section, we consider N-level quantum spin systems with unknown initial states. The stochastic master equations are given by

$$d\rho_t = F_u(\rho_t)dt + L(\rho_t)dt + G(\rho_t)\big(dY_t - 2\sqrt{\eta M}\mathrm{Tr}(J_z\rho_t)dt\big),$$
$$d\hat{\rho}_t = F_u(\hat{\rho}_t)dt + L(\hat{\rho}_t)dt + G(\hat{\rho}_t)\big(dY_t - 2\sqrt{\eta M}\mathrm{Tr}(J_z\hat{\rho}_t)dt\big),$$

where

- the actual quantum state of the quantum spin systems is described by ρ, which belongs to the space $\mathcal{S}_N := \{\rho \in \mathbb{C}^{N\times N} | \, \rho = \rho^*, \mathrm{Tr}(\rho) = 1, \rho \geq 0\}$. The associated estimated state is described by $\hat{\rho} \in \mathcal{S}_N$,
- $F_u(\rho) := -i[\omega J_z + u\, J_y, \rho]$, $L(\rho) := \frac{M}{2}(2J_z\rho J_z - J_z^2\rho - \rho J_z^2)$ and the diffusion term is given by $G(\rho) := \sqrt{\eta M}\big(J_z\rho + \rho J_z - 2\mathrm{Tr}(J_z\rho)\rho\big)$,
- Y_t denotes the observation process of the actual quantum spin system, which is a continuous semi-martingale whose quadratic variation $[Y,Y]_t = t$. Its dynamics satisfy $dY_t = dW_t + 2\sqrt{\eta M}\mathrm{Tr}(J_z\rho_t)dt$, where W_t is a one-dimensional standard Wiener process,
- $u := u(\hat{\rho})$ denotes the feedback controller as a function of the estimated state $\hat{\rho}$,
- J_z and J_y are the (self-adjoint) angular momenta along the axis z and y respectively, $J = \frac{N-1}{2}$ represents the fixed angular momentum, they are defined by Equation (4.2) and Equation (4.3),
- $\eta \in (0,1]$ measures the efficiency of the detectors, $M > 0$ is the strength of the interaction between the system and the probe, and $\omega \geq 0$ is a parameter characterizing the free Hamiltonian.

By setting $dY_t = dW_t + 2\sqrt{\eta M}\mathrm{Tr}(J_z\rho_t)dt$, we obtain the following matrix-valued stochastic differential equations describing the time evolution of the pair $(\rho_t, \hat{\rho}_t) \in \mathcal{S}_N \times \mathcal{S}_N$,

$$d\rho_t = F_u(\rho_t)dt + L(\rho_t)dt + G(\rho_t)dW_t, \tag{6.8}$$
$$d\hat{\rho}_t = F_u(\hat{\rho}_t)dt + L(\hat{\rho}_t)dt + 2\sqrt{\eta M}G(\hat{\rho}_t)\mathrm{Tr}\big(J_z(\rho_t - \hat{\rho}_t)\big)dt + G(\hat{\rho}_t)dW_t. \tag{6.9}$$

If $u \in \mathcal{C}^1(\mathcal{S}_N, \mathbb{R})$, the existence and uniqueness of solutions of (6.8) and (6.9) can be shown by similar arguments as in [MvH07, Proposition 3.5].

Our purpose is to provide sufficient conditions on the feedback controller $u(\hat{\rho})$, which stabilizes exponentially almost surely the coupled system (6.8) and (6.9) towards the target state $(\boldsymbol{\rho}_{\bar{n}}, \boldsymbol{\rho}_{\bar{n}})$ with $\bar{n} \in \{0, \ldots, 2J\}$. Note that, if we turn off the feedback controller, there are N^2 equilibria $(\boldsymbol{\rho}_n, \boldsymbol{\rho}_m)$ with $n, m \in \{0, \ldots, 2J\}$ for the coupled system. However, since the system (6.8) satisfies the non-demolition condition [BP14, Definition 2] and the measurement operator J_z satisfies the non-degeneracy condition [BP14, Assumption (**ND**)], based on [BP14, Proposition 3], we find that the pair $(\rho_t, \hat{\rho}_t)$ exponentially converges towards the following set

$$\bar{\mathbf{E}} := \{(\boldsymbol{\rho}_0, \boldsymbol{\rho}_0), \ldots, (\boldsymbol{\rho}_{2J}, \boldsymbol{\rho}_{2J})\},$$

when $u \equiv 0$ and $\eta = 1$, which is similar to the exponential quantum state reduction phenomenon established in Theorem 4.1.2. Here, we aim to generalize to the coupled

system the methodology developed in Section 3.3 for the quantum trajectories. This is the goal of our work in preparation [LAMb].

Now let us discuss heuristically our method for the feedback exponential stabilization of the coupled system (6.8) and (6.9) towards the target state $(\boldsymbol{\rho}_{\bar{n}}, \boldsymbol{\rho}_{\bar{n}})$. Firstly, we note that the two subsystems share the same feedback controller $u(\hat{\rho})$, which is only a function of the estimated state $\hat{\rho}$. Hence, if we assume $u(\boldsymbol{\rho}_{\bar{n}}) = 0$ and $u(\boldsymbol{\rho}_k) \neq 0$ for all $k \neq \bar{n}$, then $(\boldsymbol{\rho}_k, \boldsymbol{\rho}_{\bar{n}})$ with $k \in \{0, \ldots, 2J\}$ are the N equilibria of the coupled system (6.8) and (6.9). This is one of the major obstacles for our control goal.

Next, similar to Lemma 4.3.6, we need to provide sufficient conditions on the feedback controller to guarantee that $(\rho_t, \hat{\rho}_t)$ can enter in any neighborhood of the target state in finite time almost surely. The first step is to apply the support theorem (Theorem A.2.1) to analyze the corresponding deterministic control systems given by

$$\dot{\rho}_v(t) = F_u\big(\rho_v(t)\big) + \tilde{L}\big(\rho_v(t)\big) + 2\sqrt{\eta M}\mathrm{Tr}\big(J_z\rho_v(t)\big)G\big(\rho_v(t)\big) + G\big(\rho_v(t)\big)v(t), \tag{6.10}$$

$$\dot{\hat{\rho}}_v(t) = F_u\big(\hat{\rho}_v(t)\big) + \tilde{L}\big(\hat{\rho}_v(t)\big) + 2\sqrt{\eta M}\mathrm{Tr}\big(J_z\rho_v(t)\big)G\big(\hat{\rho}_v(t)\big) + G\big(\hat{\rho}_v(t)\big)v(t). \tag{6.11}$$

with $\rho_v(0) = \rho_0$, $\hat{\rho}_v(0) = \hat{\rho}_0$ and $v(t) \in \mathcal{V}$, where $\mathcal{V}$ is the set of all piecewise constant functions from $\mathbb{R}_+$ to $\mathbb{R}$ and $\tilde{L}(\rho) := M\left((1-\eta)J_z\rho J_z - \frac{1+\eta}{2}(J_z^2\rho + \rho J_z^2) + 2\eta\mathrm{Tr}(J_z^2\rho)\rho\right)$. Then we are left to show that there exist trajectories $(\rho_v(t), \hat{\rho}_v(t))$ which can enter in any neighborhood of the target state in finite time under some suitable feedback controller u and control input v. In fact, it is not easy to find appropriate u and v, which achieve the control purpose. The second step would be to show that the probability of the above-mentioned event is one. This could be proved by means of a Dynkin estimation, as in Lemma 4.3.6, if one establishes the existence of a compact "domain of attraction" for the equilibrium $(\boldsymbol{\rho}_{\bar{n}}, \boldsymbol{\rho}_{\bar{n}})$. This seems to be difficult to be shown due to the presence of further $N-1$ equilibria.

Finally, by applying the local stochastic Lyapunov technique provided in the proof of Theorem 4.3.8, we can establish a general result ensuring the exponential convergence under some assumptions on the feedback control law $u(\hat{\rho})$ and an additional local Lyapunov type condition.

Because of the similarity in the above-mentioned method for the coupled system and the analysis of the exponential stabilization of System (4.1) in Section 4.3, based on Theorem 4.3.9 and Theorem 4.3.10 for System (4.1), it is then tempting to conjecture the following.

Conjecture 6.2.1. Consider the coupled system (6.8) and (6.9) with $(\rho_0, \hat{\rho}_0) \in \mathcal{S}_N \times \mathcal{S}_N \setminus (\boldsymbol{\rho}_{\bar{n}}, \boldsymbol{\rho}_{\bar{n}})$ and assume $\eta \in (0, 1)$. Then, the feedback controller

$$u_{\bar{n}}(\hat{\rho}) = \alpha\big(1 - \mathrm{Tr}(\hat{\rho}\boldsymbol{\rho}_{\bar{n}})\big)^{\beta}, \qquad \alpha > 0, \quad \beta \geq 1, \tag{6.12}$$

exponentially stabilizes $(\rho_t, \hat{\rho}_t)$ to $(\boldsymbol{\rho}_{\bar{n}}, \boldsymbol{\rho}_{\bar{n}})$ almost surely for the special case $\bar{n} \in \{0, 2J\}$ with sample Lyapunov exponent less than or equal to $-\eta M$. Moreover, the feedback controller

$$u_{\bar{n}}(\hat{\rho}) = \alpha\big(J - \bar{n} - \mathrm{Tr}(J_z\hat{\rho})\big)^{\beta}, \qquad \alpha > 0, \quad \beta \geq 1, \tag{6.13}$$

exponentially stabilizes $(\rho_t, \hat{\rho}_t)$ to $(\boldsymbol{\rho}_{\bar{n}}, \boldsymbol{\rho}_{\bar{n}})$ almost surely for the general case $\bar{n} \in \{0, \ldots, 2J\}$ with sample Lyapunov exponent less than or equal to $-\eta M/2$ for $\bar{n} \in \{1, \ldots, 2J-1\}$ and $-\eta M$ for $\bar{n} \in \{0, 2J\}$.

In the following, we illustrate the above conjecture through simulations for a three-level quantum spin system. First, we set $(\boldsymbol{\rho}_0, \boldsymbol{\rho}_0)$ as the target state. The corresponding simulations with a feedback law of the form (6.12) and initial condition $(\rho_0, \hat{\rho}_0) = (\boldsymbol{\rho}_2, \boldsymbol{\rho}_1)$ are shown in Fig. 6.3. Then we set $(\boldsymbol{\rho}_1, \boldsymbol{\rho}_1)$ as the target state. The simulations with a feedback law of the form (6.13) and initial condition $(\rho_0, \hat{\rho}_0) = \big(\mathrm{diag}(0.2, 0.2, 0.6), \mathrm{diag}(0.8, 0.1, 0.1)\big)$ belonging to $\mathrm{int}(\mathcal{S}_3) \times \mathrm{int}(\mathcal{S}_3)$ are shown in Fig. 6.4.

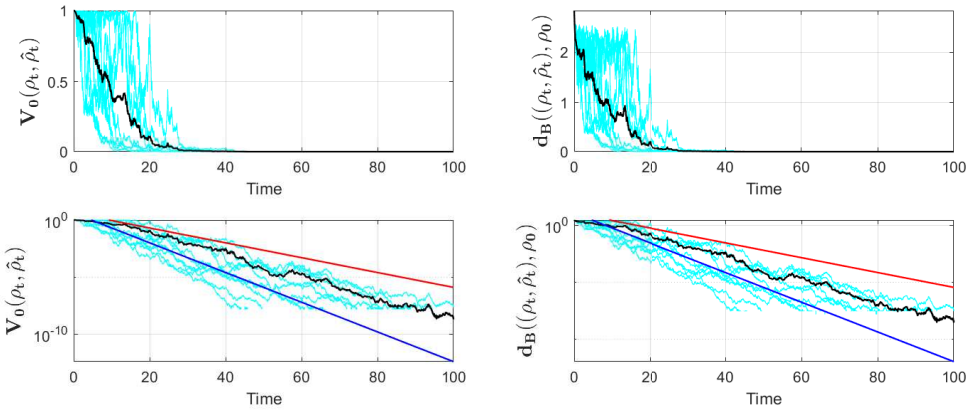


FIGURE 6.3 – Exponential stabilization of a three-level quantum spin system towards $(\boldsymbol{\rho}_0, \boldsymbol{\rho}_0)$ with the feedback law (6.12) starting at $(\rho_0, \hat{\rho}_0) = (\boldsymbol{\rho}_2, \boldsymbol{\rho}_1)$ with $\omega = 0.3$, $\eta = 0.3$, $M = 1$, $\alpha = 5$ and $\beta = 2$. $V_0(\rho, \hat{\rho}) = \sqrt{1 - \rho_{0,0}\hat{\rho}_{0,0}}$ denotes the Lyapunov function, $\mathbf{d_B}((\rho, \hat{\rho}), \boldsymbol{\rho}_0) := d_B(\rho, \boldsymbol{\rho}_0) + d_B(\hat{\rho}, \boldsymbol{\rho}_0)$ and $\frac{\sqrt{2}}{4} V_0(\rho, \hat{\rho}) \leq \mathbf{d_B}((\rho, \hat{\rho}), \boldsymbol{\rho}_0) \leq \sqrt{2} V_0(\rho, \hat{\rho})$. The black curve represents the mean value of 10 arbitrary sample trajectories, and the red and blue curves represent the exponential references with exponents $-\eta M/2$ and $-\eta M$ respectively. The figures at the bottom are the semi-log versions of the ones at the top.

Then, we repeat the last simulations for the case where the initial condition is $(\rho_0, \hat{\rho}_0) = (\boldsymbol{\rho}_2, \boldsymbol{\rho}_0)$. As simulations show, the trajectories ρ_t and $\hat{\rho}_t$ enter immediately in the interior of $\mathcal{S}_3 \times \mathcal{S}_3$ and converge exponentially towards the target state $(\boldsymbol{\rho}_1, \boldsymbol{\rho}_1)$.

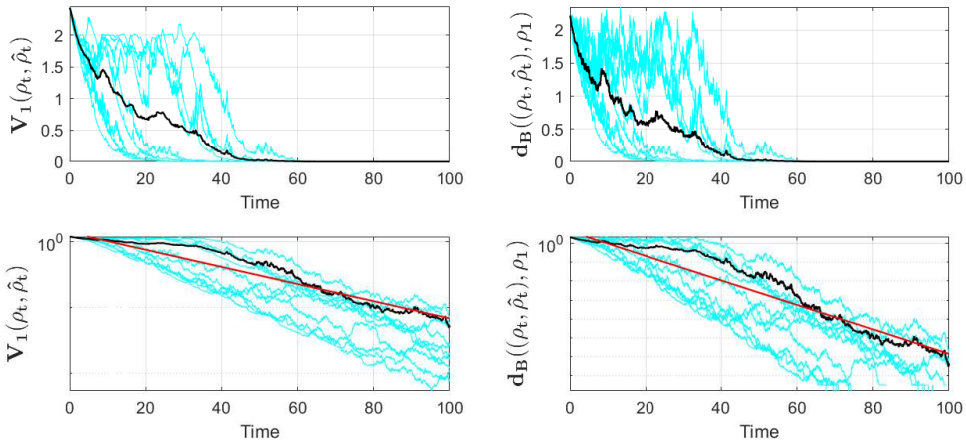


FIGURE 6.4 – Exponential stabilization of a three-level quantum spin system towards $(\boldsymbol{\rho}_1, \boldsymbol{\rho}_1)$ with the feedback law (6.13) starting at $(\rho_0, \hat{\rho}_0) = \big(\mathrm{diag}(0.2, 0.2, 0.6), \mathrm{diag}(0.8, 0.1, 0.1)\big)$ with $\omega = 0.3$, $\eta = 0.3$, $M = 1$, $\alpha = 2$ and $\beta = 2$. $V_1(\rho, \hat{\rho}) = \sum_{k \neq 1} \left(\sqrt{\rho_{k,k}} + \sqrt{\hat{\rho}_{k,k}}\right)$ denotes the Lyapunov function, $\mathbf{d_B}((\rho, \hat{\rho}), \boldsymbol{\rho}_1) := d_B(\rho, \boldsymbol{\rho}_1) + d_B(\hat{\rho}, \boldsymbol{\rho}_1)$ and $\frac{\sqrt{2}}{2} V_1(\rho, \hat{\rho}) \leq \mathbf{d_B}((\rho, \hat{\rho}), \boldsymbol{\rho}_1) \leq \sqrt{2} V_1(\rho, \hat{\rho})$. The black curve represents the mean value of 10 arbitrary sample trajectories, and the red curve represents the exponential reference with exponent $-\eta M/2$. The figures at the bottom are the semi-log versions of the ones at the top.

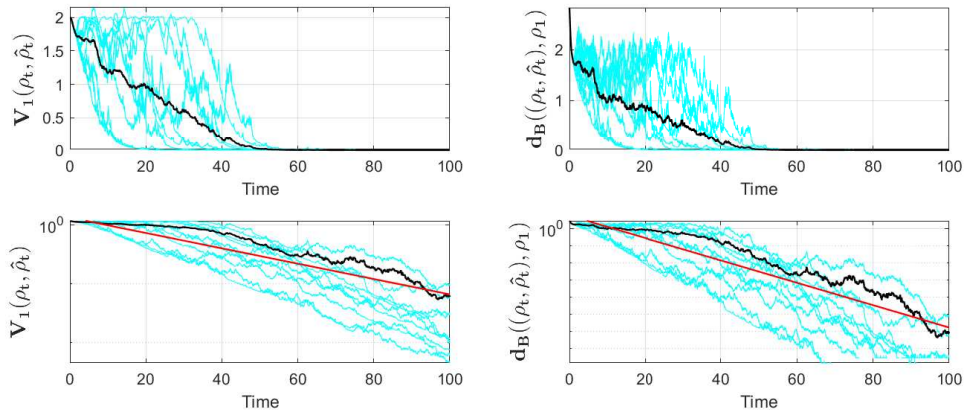


FIGURE 6.5 – Exponential stabilization of a three-level quantum spin system towards $(\boldsymbol{\rho}_1, \boldsymbol{\rho}_1)$ with the feedback law (6.13) starting at $(\rho_0, \hat{\rho}_0) = (\boldsymbol{\rho}_2, \boldsymbol{\rho}_0)$ with $\omega = 0.3$, $\eta = 0.3$, $M = 1$, $\alpha = 2$ and $\beta = 2$. $V_1(\rho, \hat{\rho}) = \sum_{k \neq 1} \left(\sqrt{\rho_{k,k}} + \sqrt{\hat{\rho}_{k,k}}\right)$ denotes the Lyapunov function, $\mathbf{d_B}((\rho, \hat{\rho}), \boldsymbol{\rho}_1) := d_B(\rho, \boldsymbol{\rho}_1) + d_B(\hat{\rho}, \boldsymbol{\rho}_1)$ and $\frac{\sqrt{2}}{2} V_1(\rho, \hat{\rho}) \leq \mathbf{d_B}((\rho, \hat{\rho}), \boldsymbol{\rho}_0) \leq \sqrt{2} V_1(\rho, \hat{\rho})$. The black curve represents the mean value of 10 arbitrary sample trajectories, and the red curve represents the exponential reference with exponent $-\eta M/2$. The figures at the bottom are the semi-log versions of the ones at the top.

7

Conclusion and perspective

In this thesis, we have studied the asymptotic behavior of quantum trajectories associated with different open quantum systems undergoing imperfect continuous-time measurements for the cases with and without feedback law. In Chapter 3, we have introduced the quantum probability theory and the quantum filtering theory. In particular, we have derived stochastic master equations which describe the time evolution of conditional density operators. In Chapter 4, we have discussed the asymptotic behavior of trajectories associated with N-level quantum spin systems for the cases with and without feedback law. Firstly, we have reviewed some previous works on this topic. Then, for the system without feedback, we have shown the exponential convergence towards the set of pure states associated with eigenvectors of the measurement operator J_z (quantum state reduction with exponential rate $\eta M/2$). We have next proved the exponential convergence of N-level quantum angular momentum systems towards an arbitrary predetermined target eigenstate under some general conditions on the feedback law. This was obtained by applying stochastic Lyapunov techniques and analyzing the asymptotic behavior of quantum trajectories. For illustration, we have provided a parametrized feedback law satisfying our general conditions, which stabilizes exponentially the system towards the target state. In Chapter 5, we have studied the asymptotic behavior of trajectories of open multi-qubit systems. Firstly, for the case of two quantum channels and without feedback, we have shown the exponential convergence towards the set of GHZ states. Then we focused on two-qubit systems, for the system with one quantum channel, we have briefly reviewed a previous work and proposed a continuous feedback law stabilizing the system asymptotically towards the target Bell state. In particular, for the case of two quantum channels, we have provided a general result concerning the feedback exponential stabilization towards the target Bell state by applying local stochastic Lyapunov techniques and analyzing the asymptotic behavior of quantum trajectories. Furthermore, we have constructed a parameterized continuous feedback law satisfying the conditions of our general results. In Chapter 6, we have first studied the asymptotic behavior of trajectories of actual open quantum spin-$\frac{1}{2}$ systems with unknown initial states and its associated quantum filter undergoing imperfect continuous-time measurements. Then we have provided sufficient

conditions on the feedback controller, which is a function of quantum filter states, to guarantee that the actual quantum state and the estimated state converge to the same target state when time goes to infinity. Finally, we have discussed feedback exponential stabilization of N-level quantum spin systems with unknown initial states, and heuristically introduced our stabilization method and precised the difficulties that we need to overcome.

Below, we resume some possible future research directions.

- The proof of the conjecture proposed in Remark 4.3.12 is a natural future research line.
- One short term objective is to generalize the results in Chapter 5 concerning the exponential stabilization of multi-qubit systems towards an arbitrary GHZ state [LAMa].
- A further short term goal is to generalize the results in Chapter 4 concerning the exponential stabilization of N-level quantum spin systems towards a predetermined state to the case with unknown initial state [LAMb], which is the topic discussed in Section 6.2.
- We also want to extend our results to the case where there are some delays in the feedback loops. We believe that our results can be naturally adapted for this case.
- Further, we want to extend our results to general open quantum systems. This means that measurement operators have general forms and we have both homodyne and photon counting detections, i.e., the stochastic master equations are driven by both Wiener and Poisson processes.
- For general open quantum systems, we look for exponential stabilization towards a chosen pure subspace by applying feedbacks, similar to the work done in [BPT17], with an open-loop control strategy.
- In addition, for general open quantum systems, we would like to study feedback stabilization when the target states can be non-classical states like Schrödinger cat states and/or the feedback strategy is not based on measurements, for example, when the feedback is coherent.
- Moreover, the stabilization methods of [MvH07, LAM18, LAM19a] are based on the real-time simulation of a quantum filter equation to obtain an estimate of the quantum state. However, this quantum filter equation is in general high-dimensional. It is difficult to achieve real-time simulation of the filter equation as the time scales of quantum systems are short. One possible direction is to look for a reduced filter equation as in [NHM09, TAM17] and provide feedback laws stabilizing such reduced filters.

A Appendix

In this appendix, we firstly recall some notions and theorems of stochastic calculus. Then we provide a brief introduction of stochastic control theory, notions of different types of stability and some basic theorems needed in this thesis.

A.1 Stochastic calculus

Definition A.1.1 (Stochastic exponential [Pro04])**.** For a semimartingale x_t with $x_0 = 0$, the stochastic exponential (Doléans-Dade exponential) of x_t is denoted by $\mathfrak{E}(x_t)$,

$$\mathfrak{E}(x_t) = \exp\left(x_t - \frac{1}{2}[x_t, x_t]\right).$$

It is the semimartingale z_t which is the solution of the following SDE,

$$z_t = 1 + \int_0^t z_s dx_s \Leftrightarrow dz_t = z_t dx_t, \quad z_0 = 1.$$

In the above definition, the bracket process $[\cdot,\cdot]_t$ means the quadratic (co)variation process. For all $T > 0$, if $0 = t_0^n < \cdots < t_{p_n}^n = T$ is a sequence of subdivisions of $[0, T]$ whose mesh tends to 0, and for two semi-martingales x_t and y_t,

$$[x, y]_t := \lim_{n\to\infty} \sum_{i=1}^{p_n} (x_{t_i^n} - x_{t_{i-1}^n})(y_{t_i^n} - y_{t_{i-1}^n}) = x_t y_t - \int x_t dy_t - \int y_t dx_t. \tag{A.1}$$

Theorem A.1.2 (Girsanov's theorem [LG16])**.** *Assume that the probability measures $\mathbb{P}$ and $\mathbb{Q}$ are mutually absolutely continuous on $\mathcal{F}_\infty$. Let L_t be the unique continuous local martingale, $\mathcal{E}(L_t)$ be the martingale with càdlàg sample continuous paths such that, for every $t \geq 0$, $\frac{d\mathbb{Q}}{d\mathbb{P}}\big|_{\mathcal{F}_t^W} = \mathcal{E}(L_t)$. Then, if M_t is a continuous local martingale under $\mathbb{P}$, the process $\widetilde{M}_t = M_t - [M, L]_t$ is a continuous local martingale under $\mathbb{Q}$.*

Theorem A.1.3 (Itô formula). *Given a stochastic differential equation $dq_t = f(q_t)dt + g(q_t)dW_t$, where q_t takes values in $Q \subset \mathbb{R}^p$, the infinitesimal generator is the operator $\mathscr{L}$ acting on twice continuously differentiable functions $V : Q \times \mathbb{R}_+ \to \mathbb{R}$ in the following way*

$$\mathscr{L}V(q,t) := \frac{\partial V(q,t)}{\partial t} + \sum_{i=1}^{p} \frac{\partial V(q,t)}{\partial q_i} f_i(q) + \frac{1}{2}\mathrm{Tr}\big(g^*(q)D^2V(q,t)g(q)\big),$$

where $D^2V(q,t)$ is the Hessian of the function $V(\cdot,t)$. Itô formula describes the variation of the function V along solutions of the stochastic differential equation and is given as follows

$$dV(q,t) = \mathscr{L}V(q,t)dt + \sum_{i=1}^{p} \frac{\partial V(q,t)}{\partial q_i} g_i(q)dW_t.$$

A.2 Stochastic control theory

Consider a stochastic differential equation in Itô form in $\mathbb{R}^K$

$$dx_t = \widehat{X}_0(x_t)dt + \sum_{k=1}^{n} \widehat{X}_k(x_t)dW_k(t), \quad x_0 = x, \tag{A.2}$$

where $W(t) = (W_k(t))_{1\le k\le n}$ is a n-dimensional standard Wiener process on the canonical Wiener space $(\Omega, \mathcal{F}, \mathbb{P})$, and X_k for $0 \le n$ satisfy the usual linear growth and local Lipschitz continuous conditions for existence and uniqueness of solutions [Pro04]. Equation (A.2) can be written in the following Stratonovich form [RW00b]

$$dx_t = X_0(x_t)dt + \sum_{k=1}^{n} X_k(x_t) \circ dW_t^k, \quad x_0 = x,$$

where $X_0(x) = \widehat{X}_0(x) - \frac{1}{2}\sum_{l=1}^{K}\sum_{k=1}^{n} \frac{\partial \widehat{X}_k}{\partial x_l}(x)(\widehat{X}_k)_l(x)$, $(\widehat{X}_k)_l$ denoting the component l of the vector $\widehat{X}_k$, and $X_k(x) = \widehat{X}_k(x)$ for $k \neq 0$.

Theorem A.2.1 (Support theorem [SV72]). *Let $X_0(t,x)$ be a bounded measurable function, uniformly Lipschitz continuous in x and $X_k(t,x)$ be continuously differentiable in t and twice continuously differentiable in x, with bounded derivatives, for $k \neq 0$. Consider the Stratonovich equation*

$$dx_t = X_0(t,x_t)dt + \sum_{k=1}^{n} X_k(t,x_t) \circ dW_t^k, \quad x_0 = x.$$

Let $\mathbb{P}_x$ be the probability law of the solution x_t starting at x. Consider in addition the associated deterministic control system

$$\frac{d}{dt}x_v(t) = X_0(t,x_v(t)) + \sum_{k=1}^{n} X_k(t,x_v(t))v^k(t), \quad x_v(0) = x.$$

with $v^k \in \mathcal{V}$, where $\mathcal{V}$ is the set of all piecewise constant functions from $\mathbb{R}_+$ to $\mathbb{R}$. Now we define $\mathcal{W}_x$ as the set of all continuous paths from $\mathbb{R}_+$ to $\mathbb{R}^K$ starting at x, equipped with

the topology of uniform convergence on compact sets, and $\mathcal{I}_x$ *as the smallest closed subset of* $\mathcal{W}_x$ *such that* $\mathbb{P}_x(x_\cdot \in \mathcal{I}_x) = 1$. *Then,*

$$\mathcal{I}_x = \overline{\{x_v(\cdot) \in \mathcal{W}_x | \, v \in \mathcal{V}^n\}} \subset \mathcal{W}_x.$$

Suppose that the system (A.2) is equipped with the metric $d(x, y)$ for $x, y \in \mathbb{R}^K$, then the distance between a state x and a set $E \subset \mathbb{R}^K$ is defined by $d(x, E) = \min_{y \in E} d(x, y)$. Given $E \subseteq \mathbb{R}^K$ and $r > 0$, we define the neighborhood $B_r(E)$ of E as

$$B_r(E) = \{x \in \mathbb{R}^K | \, d(x, E) < r\}.$$

Definition A.2.2 ([Mao07, Kha11])**.** Let $\bar{E}$ be an invariant set of system (A.2), then $\bar{E}$ is said to be

1. *locally stable in probability*, if for every $\varepsilon \in (0, 1)$ and for every $r > 0$, there exists $\delta = \delta(\varepsilon, r)$ such that,

$$\mathbb{P}\left(x_t \in B_r(\bar{E}) \text{ for } t \geq 0\right) \geq 1 - \varepsilon,$$

   whenever $x_0 \in B_\delta(\bar{E})$.

2. *almost surely asymptotically stable*, if it is locally stable in probability and,

$$\mathbb{P}\left(\lim_{t \to \infty} d(x_t, \bar{E}) = 0\right) = 1,$$

   whenever $x_0 \in \mathbb{R}^K$.

3. *exponentially stable in mean*, if for some positive constants α and β,

$$\mathbb{E}(d(x_t, \bar{E})) \leq \alpha \, d(x_0, \bar{E}) e^{-\beta t},$$

   whenever $x_0 \in \mathbb{R}^K$. The smallest value $-\beta$ for which the above inequality is satisfied is called the *average Lyapunov exponent*.

4. *almost surely exponentially stable*, if

$$\limsup_{t \to \infty} \frac{1}{t} \log d(x_t, \bar{E}) < 0, \quad a.s.$$

   whenever $x_0 \in \mathbb{R}^K$. The left-hand side of the above inequality is called the *sample Lyapunov exponent* of the solution.

Note that any equilibrium $\bar{x}$ of (A.2), that is any state satisfying

$$\widehat{X}_0(\bar{x}) = \widehat{X}_1(\bar{x}) = \cdots = \widehat{X}_n(\bar{x}) = 0,$$

which is a special case of invariant set.

Theorem A.2.3 (stochastic LaSalle-type theorem [Mao99])**.** *Let* E *be a bounded invariant set with respect to the solutions of* (A.2) *and* $x_0 \in E$. *Suppose there exists a continuous, twice differentiable function* $V : E \to \mathbb{R}_+$ *such that* $\mathscr{L}V(x) \leq 0$ *for all* $x \in E$. *Then* $\lim_{t \to \infty} \mathscr{L}V(x_t) = 0$ *almost surely.*

Now we consider N-dimensional quantum systems with $N < \infty$ whose state space is given by

$$\mathcal{S}_N := \{\rho \in \mathbb{C}^{N\times N} | \, \rho = \rho^*, \mathrm{Tr}(\rho) = 1, \rho \geq 0\}.$$

The Bures distance [BŻ17] is defined as follows.

Definition A.2.4. The Bures distance between two quantum states ρ_a and ρ_b in $\mathcal{S}_N$ is defined as

$$d_B(\rho_a, \rho_b) := \sqrt{2 - 2\mathrm{Tr}\left(\sqrt{\sqrt{\rho_b}\rho_a\sqrt{\rho_b}}\right)}.$$

In particular, the Bures distance between a quantum state $\rho_a \in \mathcal{S}_N$ and a pure state $\boldsymbol{\rho}$ is given by

$$d_B(\rho_a, \boldsymbol{\rho}) = \sqrt{2 - 2\sqrt{\mathrm{Tr}(\rho_a\boldsymbol{\rho})}}.$$

Also, the Bures distance between a quantum state ρ_a and a set $E \subseteq \mathcal{S}_N$ is defined as

$$d_B(\rho_a, E) = \min_{\rho\in E} d_B(\rho_a, \rho).$$

Given $E \subseteq \mathcal{S}_N$ and $r > 0$, we define the neighborhood $B_r(E)$ of E as

$$B_r(E) = \{\rho \in \mathcal{S}_N | \, d_B(\rho, E) < r\}.$$

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ÉCOLE DOCTORALE
Sciences et technologies de l'information et de la communication (STIC)

Titre : Stabilisation exponentielle par rétroaction de systèmes quantiques ouverts soumis à des mesures en temps continu

Mots clés : Stabilité stochastique, Stabilité exponentielle, Contrôle quantique, Systèmes quantiques ouverts, Filtrage quantique, Techniques de Lyapunov

Résumé : Dans cette thèse, nous nous intéressons à la stabilisation par rétroaction des systèmes quantiques ouverts soumis à des mesures imparfaites en temps continu. Tout d'abord, nous introduisons la théorie du filtrage quantique pour décrire l'évolution temporelle de l'opérateur de densité conditionnelle représentant un état quantique en interaction avec un environnement. Ceci est décrit par une équation différentielle stochastique à valeurs matricielles. Deuxièmement, nous étudions le comportement asymptotique des trajectoires quantiques associées à des systèmes de spin à N niveaux pour des états initiaux donnés, pour les cas avec et sans loi de rétroaction. Dans le cas sans loi de rétroaction, nous montrons la propriété de réduction de l'état quantique à vitesse exponentielle. Ensuite, nous fournissons des conditions suffisantes sur la loi de contrôle assurant une convergence presque sûre vers un état pur prédéterminé correspondant à un vecteur propre de l'opérateur de mesure. Troisièmement, nous étudions le comportement asymptotique des trajectoires de systèmes ouverts à plusieurs qubits pour des états initiaux donnés. Dans le cas sans loi de rétroaction, nous montrons la réduction exponentielle de l'état quantique pour les systèmes N-qubit avec deux canaux quantiques. Dans le cas particulier des systèmes à deux qubits, nous donnons des conditions suffisantes sur la loi de contrôle assurant la convergence asymptotique vers un état cible de Bell avec un canal quantique, et la convergence exponentielle presque sûre vers un état cible de Bell avec deux canaux quantiques. Ensuite, nous étudions le comportement asymptotique des trajectoires des systèmes quantiques ouverts de spin-$\frac{1}{2}$ avec les états initiaux inconnus soumis à des mesures imparfaites en temps continu, et nous fournissons des conditions suffisantes au contrôleur pour garantir la convergence de l'état estimé vers l'état quantique réel lorsque le temps tends vers l'infini. En conclusion, nous discutons de manière heuristique du problème de stabilisation exponentielle des systèmes de spin à N niveaux avec les états initiaux inconnus et nous proposons des lois de rétroaction candidates afin de stabiliser le système de manière exponentielle.

Title : Feedback exponential stabilization of open quantum systems undergoing continuous-time measurements

Keywords : Stochastic stability, Exponential stability, Quantum control, Open quantum systems, Quantum filtering, Lyapunov techniques

Abstract : In this thesis, we focus on the feedback stabilization of open quantum systems undergoing imperfect continuous-time measurements. First, we introduce the quantum filtering theory to obtain the time evolution of the conditional density operator representing a quantum state in interaction with an environment. This is described by a matrix-valued stochastic differential equation. Second, we study the asymptotic behavior of quantum trajectories associated with N-level quantum spin systems for given initial states, for the cases with and without feedback law. For the case without feedback, we show the exponential quantum state reduction. Then, we provide sufficient conditions on the feedback control law ensuring almost sure exponential convergence to a predetermined pure state corresponding to an eigenvector of the measurement operator. Third, we study the asymptotic behavior of trajectories of open multi-qubit systems for given initial states. For the case without feedback, we show the exponential quantum state reduction for N-qubit systems with two quantum channels. Then, we focus on the two-qubit systems, and provide sufficient conditions on the feedback control law ensuring asymptotic convergence to a target Bell state with one quantum channel, and almost sure exponential convergence to a target Bell state with two quantum channels. Next, we investigate the asymptotic behavior of trajectories of open quantum spin-$\frac{1}{2}$ systems with unknown initial states undergoing imperfect continuous-time measurements, and provide sufficient conditions on the controller to guarantee the convergence of the estimated state towards the actual quantum state when time goes to infinity. Finally, we discuss heuristically the exponential stabilization problem for N-level quantum spin systems with unknown initial states and propose candidate feedback laws to stabilize exponentially the system.

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