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Yisheng Tian

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Arithmétique des groupes algébriques au-dessus du corps des fonctions d'une courbe sur un corps p -adique

Thèse de doctorat de l'Université Paris-Saclay

École doctorale de mathématiques Hadamard n° 574
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**Thèse présentée et soutenue à Orsay,
le 1 octobre 2020, par**

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Introduction

Cette thèse porte sur les points rationnels des groupes algébriques linéaires sur les corps de fonctions p -adiques. En particulier, on s'intéresse aux théorèmes de dualité arithmétique et à l'approximation faible pour les groupes réductifs. On fixe d'abord quelques notations et conventions. Puis pour illustrer la motivation de cette thèse, on rappelle le cas classique des corps de nombres. Enfin, on introduit de nouveaux résultats sur les corps de fonctions p -adiques.

0.1 Notations et conventions

Sauf indication contraire, tous les groupes de cohomologie et d'hypercohomologie seront considérés pour la topologie étale. En particulier, les groupes (hyper)cohomologiques sur les corps sont identifiés avec des groupes (hyper)cohomologiques galoisiens. Tout au long de cette section, L est un corps de caractéristique zéro. On fixe une clôture algébrique \bar{L} de L . Une variété sur L est un schéma séparé de type fini sur L .

Groupes abéliens

Soit A un groupe topologique abélien. Soient $n \geq 1$ un entier et ℓ un nombre premier. On notera:

- ${}_n A$ le sous-groupe de n -torsion de A .
- A_{div} le sous-groupe divisible maximal de A .
- $A_{\text{tors}} = \varinjlim_n {}_n A$ le sous-groupe de torsion de A .
- $A\{\ell\}$ le sous-groupe de torsion ℓ -primaire de A .
- A^\wedge la limite projective de ses quotients finis.
- $A^{(\ell)} := \varprojlim_n A/\ell^n A$ le complété ℓ -adique de A .
- $A_\wedge := \varprojlim_n A/nA$ la limite projective des A/nA .
- $T_\ell(A) := \varprojlim_n \ell^n A$ le module de Tate ℓ -adique de A .
- $A^D := \text{Hom}_{\text{cont}}(A, \mathbb{Q}/\mathbb{Z})$ le groupe des morphismes continus de A dans \mathbb{Q}/\mathbb{Z} , où A est discret si la topologie sur A n'est pas précisée.

Définition 0.1.1. Soit A un groupe abélien de torsion.

- On dit que A est d'**exposant fini** s'il existe un entier $n \geq 1$ tel que $nA = 0$.
- On dit qu'un groupe de torsion A est de **type cofini** si nA est fini pour tout entier $n \geq 1$. Si A est un groupe de torsion ℓ -primaire de type cofini, alors il existe un groupe fini F et un entier r tels que $A \simeq F \oplus (\mathbb{Q}_\ell/\mathbb{Z}_\ell)^{\oplus r}$ (voir [Fuc70, Theorem 25.1]). En particulier, on a $A/A_{\text{div}} \simeq A^{(\ell)}$.

Tores algébriques

Soit A un groupe abélien de type fini. Soit $L[A]$ l'algèbre du groupe A sur L . Alors on note $D(A) := \text{Spec}(L[A])$. C'est un groupe algébrique commutatif (voir [Mil17, 12.3]).

Définition 0.1.2. Soit G un groupe algébrique linéaire sur L .

- (1) On dit que G est diagonalisable si $G \simeq D(A)$ pour un groupe abélien A de type fini.
- (2) On dit que G est de type multiplicatif si $G_{L'}$ est diagonalisable pour une extension algébrique $L'|L$.
- (3) On dit qu'un groupe G de type multiplicatif est un tore si G est lisse et connexe.

Soit T un tore sur L . On note $\mathbf{X}^*(T)$ le module des caractères du tore T et $\mathbf{X}_*(T)$ le module des cocaractères du tore T . Alors $\mathbf{X}^*(T)$ et $\mathbf{X}_*(T)$ sont des groupes abéliens libres de type fini muni d'une action continue galoisienne. De plus, on a un accouplement parfait $\mathbf{X}^*(T) \times \mathbf{X}_*(T) \rightarrow \mathbb{Z}$ entre groupes abéliens libres. On désigne par T' le tore dual de T . C'est l'unique tore sur L tel que $\mathbf{X}^*(T') = \mathbf{X}_*(T)$ comme modules galoisiens.

Définition 0.1.3. Soit T un tore sur L .

- (1) T est dit **quasi-trivial** si $\mathbf{X}^*(T)$ possède une base sur \mathbb{Z} stable sous l'action de $\text{Gal}(\bar{L}|L)$. Un tel T est de la forme $T = \prod_{1 \leq i \leq r} R_{L_i|L}(\mathbb{G}_m)$ où r est un entier, $L_i|L$ est une sous-extension finie de $\bar{L}|L$ et $R_{L_i|L}$ est la restriction de Weil.
- (2) T est dit **flasque** si $H^1(U, \mathbf{X}_*(T)) = 0$ pour tout sous-groupe ouvert U de $\text{Gal}(\bar{L}|L)$. Autrement dit, T est flasque si et seulement si $H^1(L', \mathbf{X}_*(T)) = 0$ pour toute sous-extension finie $L'|L$ de $\bar{L}|L$.

Groupes algébriques linéaires

Soit G un groupe linéaire connexe sur L . On notera:

- $\text{rad}^u(G)$ le radical unipotent de G . C'est un sous-groupe unipotent distingué de G .
- $G^{\text{red}} := G/\text{rad}^u(G)$ le plus grand quotient réductif de G .
- G^{ss} le sous-groupe dérivé de G^{red} . C'est un groupe semi-simple sur L .
- $G^{\text{tor}} := G^{\text{red}}/G^{\text{ss}}$ le plus grand quotient torique de G^{red} . Donc on a $\mathbf{X}^*(G^{\text{tor}}) = \mathbf{X}^*(G)$.

0.1. NOTATIONS ET CONVENTIONS

- G^{sc} le revêtement simplement connexe de G^{ss} . Alors $G^{\text{sc}} \rightarrow G^{\text{ss}}$ est une isogenie centrale.

Soit G un groupe réductif connexe sur L . Par [CT08], on a une suite exacte de groupes connexes réductifs:

$$1 \rightarrow R \rightarrow H \rightarrow G \rightarrow 1$$

dite une **résolution flasque** de G , où H est un groupe quasi-trivial (c'est-à-dire qu'il est une extension d'un tore quasi-trivial par G^{sc}), et R est un tore flasque qui est central dans H . Enfin par [CT08, 0.3], H est une extension de H^{tor} par G^{sc} où H^{tor} est un tore quasi-trivial et $H^{\text{ss}} \simeq G^{\text{sc}}$.

Par [CT08, pp. 94], on a un diagramme commutatif associé à une résolution flasque de G :

$$\begin{array}{ccccccc}
 & & 1 & & 1 & & 1 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 1 & \longrightarrow & F & \longrightarrow & G^{\text{sc}} & \longrightarrow & G^{\text{ss}} \longrightarrow 1 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 1 & \longrightarrow & R & \longrightarrow & H & \longrightarrow & G \longrightarrow 1 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 1 & \longrightarrow & M & \longrightarrow & H^{\text{tor}} & \longrightarrow & G^{\text{tor}} \longrightarrow 1 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 1 & & 1 & & 1
 \end{array} \tag{1}$$

avec des lignes et des colonnes exactes, où $M := \text{Ker}(H^{\text{tor}} \rightarrow G^{\text{tor}})$ est un groupe de type multiplicatif. Enfin, le schéma en groupes fini $F := \text{Ker}(G^{\text{sc}} \rightarrow G^{\text{ss}})$ est aussi le noyau de $R \rightarrow H^{\text{tor}}$. Donc $M \simeq R/F$ est un tore sur L .

Soit $\rho : G^{\text{sc}} \rightarrow G^{\text{ss}} \rightarrow G$ la composition. Soit T un tore maximal de G sur L . Alors $T^{\text{sc}} := \rho^{-1}(T)$ est un tore maximal de G^{sc} . On applique [CT08, Appendice A] à T et on obtient un diagramme commutatif

$$\begin{array}{ccccccc}
 & & 1 & & 1 & & 1 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 1 & \longrightarrow & F & \longrightarrow & T^{\text{sc}} & \longrightarrow & T \cap G^{\text{ss}} \longrightarrow 1 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 1 & \longrightarrow & R & \longrightarrow & T_H & \longrightarrow & T \longrightarrow 1 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 1 & \longrightarrow & M & \longrightarrow & H^{\text{tor}} & \longrightarrow & G^{\text{tor}} \longrightarrow 1 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 1 & & 1 & & 1
 \end{array} \tag{2}$$

avec des lignes et des colonnes exactes, où $T_H \subset H$ est un tore maximal de H .

Revêtements spéciaux

Soit $G_0 \rightarrow G$ une isogénie de groupes réductifs connexes sur L .

Définition 0.1.4 ([San81]). On dit que $G_0 \rightarrow G$ est un **revêtement spécial** si G_0 est le produit d'un groupe semi-simple simplement connexe et d'un tore quasi-trivial.

Pour chaque groupe réductif connexe G sur L , par [San81, Lemme 1.10] il existe un entier $m \geq 1$ et un tore Q quasi-trivial tel que $G \times Q^m$ possède un revêtement spécial.

Espaces homogènes

Soit H un L -groupe linéaire. Soit Y un schéma (non-vide) lisse muni d'une action de H .

Définition 0.1.5.

- (1) On dit que Y est un **espace homogène** sur L sous H si l'action de $H(\bar{L})$ sur $Y(\bar{L})$ est transitive.
- (2) On dit que Y est un **espace homogène principal** (ou un **torseur**) sous H si l'action de $H(\bar{L})$ sur $Y(\bar{L})$ est simplement transitive.

Complexes motiviques

Soit Y une variété lisse sur L . Bloch a défini un complexe de cycles $z^i(Y, \bullet)$ dans [Blo86]. On désigne par $\mathbb{Z}(i) := z^i(-, \bullet)[-2i]$ le complexe motivique étale sur le petit site étale de Y . Par exemple, on a des quasi-isomorphismes $\mathbb{Z}(0) \simeq \mathbb{Z}$ et $\mathbb{Z}(1) \simeq \mathbb{G}_m[-1]$ (voir [Blo86, Corollary 6.4]). Pour un groupe abélien A , on note $A(i) := A \otimes^{\mathbb{L}} \mathbb{Z}(i)$ dans la catégorie dérivée bornée. Enfin, on a des quasi-isomorphismes $\mathbb{Z}/n\mathbb{Z}(i) \simeq \mu_n^{\otimes i}$ où $\mu_n^{\otimes i}$ est en degré 0 et alors on note $\mathbb{Q}/\mathbb{Z}(i) := \varinjlim_{n \geq 1} \mu_n^{\otimes i}$.

Corps de fonctions

Soit k un corps p -adique, c'est-à-dire une extension finie de \mathbb{Q}_p . Soient X une courbe projective lisse géométriquement intègre sur k et K le corps de fonctions de X . Soit $X^{(1)}$ l'ensemble des points fermés sur X . L'anneau local $\mathcal{O}_{X,v}$ est régulier de dimension 1. Autrement dit, $\mathcal{O}_{X,v}$ est un anneau de valuation discrète et on dit que v est une place de K . Alors on peut prendre le complété K_v de K par rapport à la place v . On note \mathcal{O}_v l'anneau des entiers de K_v . De même, on note K_v^h le hensélisé de K par rapport à v et \mathcal{O}_v^h son anneau des entiers. Enfin, on a $\text{cd } K = 3$ et $\text{cd } K_v = 3$ où cd désigne la dimension cohomologique (voir [Ser65]).

Soit Y une variété lisse géométriquement intègre sur K . Donc on peut trouver un schéma \mathcal{Y} lisse géométriquement intègre sur X_0 pour un ouvert de Zariski non-vide $X_0 \subset X$. Par la suite, on peut définir les points adéliques sur Y par

$$Y(\mathbb{A}_K) := \varinjlim_{U \subset X_0} \left(\prod_{v \notin U} Y(K_v) \times \prod_{v \in U} \mathcal{Y}(\mathcal{O}_v) \right).$$

Groupe de Tate–Shafarevich

Soit $C = [T_1 \rightarrow T_2]$ un complexe de tores en degrés -1 et 0 . On note

$$\text{III}^i(C) := \text{Ker} \left(\mathbb{H}^i(K, C) \rightarrow \prod_{v \in X^{(1)}} \mathbb{H}^i(K_v, C) \right).$$

Soit $S \subset X^{(1)}$ un sous-ensemble fini. On note

$$\text{III}_S^i(C) := \text{Ker} \left(\mathbb{H}^i(K, C) \rightarrow \prod_{v \notin S} \mathbb{H}^i(K_v, C) \right).$$

Enfin, on désigne par $\text{III}_\omega^i(C)$ le sous-groupe des éléments de $\mathbb{H}^i(K, C)$ qui sont localement triviaux pour presque tout $v \in X^{(1)}$.

Complexes de cochaînes

Soit (A^\bullet, ∂) un complexe de cochaînes d'une catégorie abélienne. On définit les troncatures respectives de A^\bullet comme suit:

$$\tau_{\leq i} A^\bullet := [\dots \rightarrow A^{i-2} \rightarrow A^{i-1} \rightarrow \text{Ker } \partial^i \rightarrow 0 \rightarrow \dots]$$

en degrés $\leq i$ et

$$\tilde{\tau}_{\leq i} A^\bullet := [\dots \rightarrow A^{i-1} \rightarrow A^i \rightarrow \text{Im } \partial^i \rightarrow 0 \rightarrow \dots]$$

en degrés $\leq i + 1$. Il est clair que $\tau_{\leq i} A^\bullet \rightarrow \tilde{\tau}_{\leq i} A^\bullet$ est un quasi-isomorphisme. Donc on a une suite exacte de complexes

$$0 \rightarrow \tilde{\tau}_{\leq i-1} A^\bullet \rightarrow \tau_{\leq i} A^\bullet \rightarrow H^i(A^\bullet)[-i] \rightarrow 0 \quad (3)$$

et un triangle distingué

$$\tau_{\leq i-1} A^\bullet \rightarrow \tau_{\leq i} A^\bullet \rightarrow H^i(A^\bullet)[-1] \rightarrow \tau_{\leq i-1} A^\bullet[1].$$

Enfin, par calcul direct on a pour un complexe A^\bullet quelconque:

$$\tau_{\leq 1}(A^\bullet[-1]) = (\tau_{\leq 0} A^\bullet)[-1]. \quad (4)$$

0.2 Rappels de résultats sur les corps de nombres

On rappelle quelques résultats classiques sur des corps de nombres. Ensuite, on génère ces résultats sur des corps de fonction p -adiques. Soit L un corps de nombres. Soit $\Gamma = \text{Gal}(\bar{L}|L)$ le groupe de Galois absolu de L . Soit Ω_L l'ensemble des places de L , c'est-à-dire l'ensemble des valeurs absolues non triviales sur L à équivalence près. Pour chaque place $v \in \Omega_L$, on note L_v le complété de L par rapport à v . Soit A un Γ -module discret. On note

$$\text{III}^i(M) := \text{Ker} \left(H^i(L, M) \rightarrow \prod_{v \in \Omega_L} H^i(L_v, M) \right)$$

le groupe de Tate–Shafarevich.

0.2.1 Dualités arithmétiques

Dualité pour les modules finis

On commence avec la dualité locale. Ce fut Tate qui obtint le premier résultat de dualité entre modules finis galoisiens.

Théorème 0.2.1 (Tate). *Soient $v \in \Omega_f$ une place finie et Γ_v le groupe de Galois absolu de L_v . Soit F un Γ_v -module discret fini. On note $\widehat{F} := \text{Hom}(F, \overline{L}_v^\times)$ le dual de Cartier de F . Pour $0 \leq i \leq 2$, on a un accouplement parfait entre groupes finis*

$$H^i(L_v, F) \times H^{2-i}(L_v, \widehat{F}) \rightarrow \mathbb{Q}/\mathbb{Z}.$$

Si on considère la dualité sur L , on n'obtient qu'une dualité entre certains sous-groupes de $H^i(L, F)$ et $H^{3-i}(L, \widehat{F})$:

Théorème 0.2.2 (Poitou–Tate). *Soit F un Γ -module discret fini. On note le dual de Cartier de F par $\widehat{F} := \text{Hom}(F, \overline{L}^\times)$. On a un accouplement parfait entre groupes finis*

$$\text{III}^1(F) \times \text{III}^2(\widehat{F}) \rightarrow \mathbb{Q}/\mathbb{Z}.$$

Dualité pour les tores algébriques

En utilisant la dualité des modules finis, on peut alors déduire la dualité de tores suivante :

Théorème 0.2.3 (Tate–Nakayama). *Soient $v \in \Omega_f$ une place finie et T un tore sur L_v . Soit $\mathbf{X}^*(T)$ le module des caractères de T . Pour $0 \leq i \leq 2$, le cup-produit induit un accouplement*

$$H^i(L_v, T) \times H^{2-i}(L_v, \mathbf{X}^*(T)) \rightarrow \mathbb{Q}/\mathbb{Z}$$

lequel est parfait entre groupes finis pour $i = 1$. Il induit un accouplement parfait entre

- le groupe profini $H^0(L_v, T)^\wedge$ et le groupe discret de torsion $H^2(L_v, \mathbf{X}^*(T))$.
- le groupe discret de torsion $H^2(L_v, T)$ et le groupe profini $H^0(L_v, \mathbf{X}^*(T))^\wedge$.

Notons que ce théorème généralise l'isomorphisme de réciprocity de la théorie du corps de classes local, qui est le cas $i = 0$ et $T = \mathbb{G}_m$. Maintenant on passe à la dualité globale :

Théorème 0.2.4 (Poitou–Tate). *Soient T un tore sur L et $\mathbf{X}^*(T)$ son module des caractères. Pour $i = 1, 2$, on a un accouplement parfait entre groupes finis*

$$\text{III}^i(T) \times \text{III}^{3-i}(\mathbf{X}^*(T)) \rightarrow \mathbb{Q}/\mathbb{Z}.$$

Dualité pour les complexes de tores

On considère les deux cas dans la suite qui généralisent le cas des modules finis et tores, respectivement.

0.2. RAPPELS DE RÉSULTATS SUR LES CORPS DE NOMBRES

- Soit G un groupe connexe réductif sur L . Soit $\rho : G^{\text{sc}} \rightarrow G^{\text{ss}} \rightarrow G$ la composition. Soient $T \subset G$ un tore maximal et $T^{\text{sc}} := \rho^{-1}(T)$ (c'est un tore maximal de G^{sc}). Donc on peut associer à G un complexe de tore $[T^{\text{sc}} \rightarrow T]$. Notez que dans ce cas $T^{\text{sc}} \rightarrow T$ a noyau fini égal à $\text{Ker}(G^{\text{sc}} \rightarrow G^{\text{ss}})$. Si G est semi-simple, alors on a $C \simeq (\text{Ker } \rho)[1]$. C'est-à-dire qu'on généralise le cas des modules finis.
- Soit M un groupe de type multiplicatif sur L . Donc on peut trouver deux tores T_1, T_2 sur L tel qu'il y a une suite exacte $0 \rightarrow M \rightarrow T_1 \rightarrow T_2 \rightarrow 0$ de groupes commutatifs. C'est-à-dire qu'on a un quasi-isomorphisme de complexes $M[1] \simeq [T_1 \rightarrow T_2]$. Notez que dans ce cas $T_1 \rightarrow T_2$ est surjectif.

Soit $C = [T_1 \rightarrow T_2]$ un complexe de tores sur L en degrés -1 et 0 . Soit $\mathbf{X}^*(C) = [\mathbf{X}^*(T_2) \rightarrow \mathbf{X}^*(T_1)]$ le dual de C . On peut étendre T_i à un \mathcal{O}_S -tore \mathcal{T}_i pour un sous-ensemble fini approprié S de places de L , où \mathcal{O}_S est l'anneau des S -entiers dans L . Soit $\mathcal{C} = [\mathcal{T}_1 \rightarrow \mathcal{T}_2]$ un complexe de \mathcal{O}_S -tores en degrés -1 et 0 . On note $\mathbb{H}^i(L, C)$ le groupe d'hypercohomologie galoisienne. Maintenant on note $\mathbb{P}^i(L, C)$ le produit restreint des groupes $\mathbb{H}^i(L_v, C)$ par rapport aux groupes $\text{Im}(\mathbb{H}^i(\mathcal{O}_v, \mathcal{C}) \rightarrow \mathbb{H}^i(L_v, C))$, où \mathcal{O}_v est l'anneau des entiers de L_v . Comme ci-dessus, on définit pareillement

$$\begin{aligned} \mathbb{H}^i(C) &:= \text{Ker}(\mathbb{H}^i(L, C) \rightarrow \mathbb{P}^i(L, C)) \\ \mathbb{H}^i(\mathbf{X}^*(C)) &:= \text{Ker}(\mathbb{H}^i(L, \mathbf{X}^*(C)) \rightarrow \mathbb{P}^i(L, \mathbf{X}^*(C))) \end{aligned}$$

Théorème 0.2.5 (Demarche [Dem11a]). *Soit $v \in \Omega_f$ une place finie. Le cup-produit*

$$\mathbb{H}^i(L_v, C) \times \mathbb{H}^{1-i}(L_v, \mathbf{X}^*(C)) \rightarrow \mathbb{Q}/\mathbb{Z}$$

réalise des dualités parfaites entre les groupes suivants

- le groupe profini $\mathbb{H}^{-1}(L_v, C)^\wedge$ et le groupe discret $\mathbb{H}^2(L_v, \mathbf{X}^*(C))$.
- le groupe profini $\mathbb{H}^0(L_v, C)^\wedge$ et le groupe discret $\mathbb{H}^1(L_v, \mathbf{X}^*(C))$.
- le groupe discret $\mathbb{H}^1(L_v, C)$ et le groupe profini $\mathbb{H}^0(L_v, \mathbf{X}^*(C))^\wedge$.
- le groupe discret $\mathbb{H}^2(L_v, C)$ et le groupe profini $\mathbb{H}^{-1}(L_v, \mathbf{X}^*(C))^\wedge$.

Pour la dualité globale, on a

Théorème 0.2.6 (Demarche [Dem11a]). *Soit $C = [T_1 \xrightarrow{\rho} T_2]$. Alors on a deux accouplements parfaits entre groupes finis*

$$\begin{aligned} \mathbb{H}^0(C) \times \mathbb{H}^2(\mathbf{X}^*(C)) &\rightarrow \mathbb{Q}/\mathbb{Z}, \\ \mathbb{H}^1(C) \times \mathbb{H}^1(\mathbf{X}^*(C)) &\rightarrow \mathbb{Q}/\mathbb{Z}, \\ \mathbb{H}^2(C) \times \mathbb{H}^0_\wedge(\mathbf{X}^*(C)) &\rightarrow \mathbb{Q}/\mathbb{Z} \end{aligned}$$

où $\mathbb{H}^0_\wedge(\mathbf{X}^*(C)) = \text{Ker}(\mathbb{H}^0(L, \mathbf{X}^*(C))_\wedge \rightarrow \mathbb{P}^0(L, \mathbf{X}^*(C))_\wedge)$.

Remarque 0.2.7. La version originale requiert que $\text{Ker } \rho$ soit fini ou que $\text{Coker } \rho$ soit trivial, mais nous pouvons en fait supprimer ces hypothèses.

Enfin, on peut résumer les théorèmes de dualité en une suite exacte. Plus précisément, on a quatre suites de Poitou–Tate pour les complexes de tores.

Theorem 0.2.8 (Demarche [Dem11a]). *Soit $C = [T_1 \xrightarrow{\rho} T_2]$ un complexe de tores.*

(1) *Supposons $\text{Ker } \rho$ fini. On a alors deux suites exactes de groupes topologiques*

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \mathbb{H}^{-1}(L, C) & \longrightarrow & \mathbb{P}^{-1}(L, C) & \longrightarrow & \mathbb{H}^2(L, \mathbf{X}^*(C))^D \\
 & & \searrow & & \searrow & & \searrow \\
 & & \mathbb{H}^0(L, C)_\wedge & \longrightarrow & \mathbb{P}^0(L, C)_\wedge & \longrightarrow & \mathbb{H}^1(L, \mathbf{X}^*(C))^D \\
 & & \searrow & & \searrow & & \searrow \\
 & & \mathbb{H}^1(L, C) & \longrightarrow & \mathbb{P}^1(L, C) & \longrightarrow & \mathbb{H}^0(L, \mathbf{X}^*(C))^D \\
 & & \searrow & & \searrow & & \searrow \\
 & & \mathbb{H}^2(L, C) & \longrightarrow & \mathbb{P}^2(L, C) & \longrightarrow & \mathbb{H}^{-1}(L, \mathbf{X}^*(C))^D \longrightarrow 0
 \end{array}$$

et

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \mathbb{H}^{-1}(L, \mathbf{X}^*(C))_\wedge & \longrightarrow & \mathbb{P}^{-1}(L, \mathbf{X}^*(C))_\wedge & \longrightarrow & \mathbb{H}^2(L, C)^D \\
 & & \searrow & & \searrow & & \searrow \\
 & & \mathbb{H}^0(L, \mathbf{X}^*(C)) & \longrightarrow & \mathbb{P}^0(L, \mathbf{X}^*(C)) & \longrightarrow & \mathbb{H}^1(L, C)^D \\
 & & \searrow & & \searrow & & \searrow \\
 & & \mathbb{H}^1(L, \mathbf{X}^*(C)) & \longrightarrow & \mathbb{P}^1(L, \mathbf{X}^*(C))_{\text{tors}} & \longrightarrow & (\mathbb{H}^0(L, C)^D)_{\text{tors}} \\
 & & \searrow & & \searrow & & \searrow \\
 & & \mathbb{H}^2(L, \mathbf{X}^*(C)) & \longrightarrow & \mathbb{P}^2(L, \mathbf{X}^*(C)) & \longrightarrow & \mathbb{H}^{-1}(L, C)^D \longrightarrow 0.
 \end{array}$$

(2) *Supposons $\text{Coker } \rho$ trivial. On a alors un quasi-isomorphisme $M := \text{Ker } \rho[1] \simeq C$. On a alors deux suites exacte de groupes topologiques*

$$\begin{array}{ccccccc}
 0 & \longrightarrow & H^0(L, M)_\wedge & \longrightarrow & \mathbb{P}^0(L, M)_\wedge & \longrightarrow & H^2(L, \mathbf{X}^*(M))^D \\
 & & \searrow & & \searrow & & \searrow \\
 & & H^1(L, M) & \longrightarrow & \mathbb{P}^1(L, M) & \longrightarrow & H^1(L, \mathbf{X}^*(M))^D \\
 & & \searrow & & \searrow & & \searrow \\
 & & H^2(L, M) & \longrightarrow & \mathbb{P}^2(L, M) & \longrightarrow & H^0(L, \mathbf{X}^*(M))^D \longrightarrow 0
 \end{array}$$

et

$$\begin{array}{ccccccc}
 0 & \longrightarrow & H^0(L, \mathbf{X}^*(M))_\wedge & \longrightarrow & \mathbb{P}^0(L, \mathbf{X}^*(M))_\wedge & \longrightarrow & H^2(L, M)^D \\
 & & \searrow & & \searrow & & \searrow \\
 & & H^1(L, \mathbf{X}^*(M)) & \longrightarrow & \mathbb{P}^1(L, \mathbf{X}^*(M)) & \longrightarrow & H^1(L, M)^D \\
 & & \searrow & & \searrow & & \searrow \\
 & & H^2(L, \mathbf{X}^*(M)) & \longrightarrow & \mathbb{P}^2(L, \mathbf{X}^*(M))_{\text{tors}} & \longrightarrow & (H^0(L, M)^D)_{\text{tors}} \longrightarrow 0.
 \end{array}$$

On peut utiliser ces suites exactes pour étudier le défaut à l'approximation forte pour des groupes linéaires connexes.

0.2.2 Approximation faible

Soient L un corps de nombres et X une variété sur L . Si X possède un point rationnel, alors X possède un L_v -point pour chaque $v \in \Omega_L$.

Définition 0.2.9. Soit X une variété sur L telle que $X(L) \neq \emptyset$. On dit que X vérifie **l'approximation faible** si l'ensemble $X(L)$ est dense dans $\prod_{v \in \Omega_L} X(L_v)$ par rapport à la topologie produit des topologies p -adiques. C'est-à-dire pour chaque sous-ensemble fini $S \subset \Omega_L$, $X(L)$ est dense dans $\prod_{v \in S} X(L_v)$.

Dans cette thèse, on ne considère que l'approximation faible dans les groupes algébriques, car il est intéressant de connaître ces cas en premier. Il ressort clairement de l'approximation faible classique que \mathbb{G}_m satisfait l'approximation faible. Plus généralement, les tores quasi-triviaux satisfont l'approximation faible. En général, on a :

Théorème 0.2.10 ([PR94, Proposition 7.8]). *Soit T un tore déployé par l'extension galoisienne $L'|L$. Soit $S \subset \Omega_L$ une partie finie. Si les groupes de décomposition dans $L'|L$ des places de S sont cycliques, alors T vérifie l'approximation faible.*

Si G est un groupe semi-simple, on peut supposer que G possède un revêtement spécial $G_0 \rightarrow G$ par [San81, Lemme 1.10]. Notez que G_0 vérifie l'approximation faible par le théorème suivant :

Théorème 0.2.11 ([PR94, Theorem 7.8]). *Soit G un groupe semi-simple sur L . Si G est simplement connexe ou adjoint, alors G vérifie l'approximation faible.*

On peut obtenir un défaut à l'approximation faible pour G via $G_0 \rightarrow G$ (voir [San81]). On remarque ici qu'un groupe semi-simple arbitraire peut ne pas vérifier l'approximation faible (voir [PR94, Section 7.3]).

0.2.3 Théorème de Borel–Serre

Dans cette sous-section, on s'intéresse à un résultat de finitude de la cohomologie galoisienne. Soit G un groupe algébrique sur L . On note

$$\text{III}^1(G) := \text{Ker} \left(H^1(L, G) \rightarrow \prod_{v \in \Omega_L} H^1(L_v, G) \right).$$

C'est un ensemble pointé (voir [Ser65]).

Théorème 0.2.12 (Borel–Serre). *Si G est linéaire, alors $\text{III}^1(G)$ est un ensemble fini. De plus, les fibres de $H^1(L, G) \rightarrow \prod_{v \in \Omega_L} H^1(L_v, G)$ sont finies.*

Il est intéressant de savoir quand $\text{III}^1(G)$ est trivial.

Définition 0.2.13. On dit qu'un groupe algébrique G vérifie le **principe local-global** si $\text{III}^1(G) = 1$.

Pour certains groupes semi-simples, le principe local-global est connu.

Théorème 0.2.14. *Soit G un groupe semi-simple sur L .*

(1) *Si G est adjoint, alors $\mathbb{H}^1(G) = 1$ ([PR94, Theorem 6.22]).*

(2) *Si G est simplement connexe, alors on a par [PR94, Theorem 6.4, 6.6]*

- $H^1(L_v, G) = 1$ pour chaque $v \in \Omega_L$ place finie, et
- $H^1(L, G) \rightarrow \prod_{v \in \Omega_\infty} H^1(L_v, G)$ est bijective. En particulier, $\mathbb{H}^1(G) = 1$.

Avec l'aide du théorème ci-dessus, on peut montrer par un argument de torsion que les fibres de l'application diagonale

$$H^1(L, G) \rightarrow \prod_{v \in \Omega_L} H^1(L_v, G)$$

sont finies pour les groupes algébriques linéaires.

0.3 Résultats sur les corps de fonctions p -adiques

Au cours des dernières années, on s'intéresse aux principes locaux-globaux pour les groupes algébriques sur un corps L de dimension cohomologique $\text{cd } L \geq 3$. Par exemple,

- Colliot-Thélène, Parimala et Suresh [CTPS12] ont obtenu des principes locaux-globaux sur l'invariant de Rost pour des groupes semi-simples simplement connexes sur un corps de fonctions p -adique.
- Harari et Szamuely [HS16] ont obtenu des résultats sur des obstructions aux principes locaux-globaux pour des espaces homogènes sous des tores et des groupes réductifs connexes quasi-déployés sur un corps de fonctions p -adique.
- D'autre part, Harari, Scheiderer et Szamuely [HSS15] ont obtenu des résultats sur des obstructions à l'approximation faible pour les tores sur un corps de fonctions p -adique.
- Izquierdo [Izq16] a obtenu des théorèmes de dualités et principes locaux-globaux pour les corps de fonctions sur des corps locaux supérieurs.

Il est donc intéressant de connaître l'arithmétique des groupes réductifs connexes sur un corps de fonctions p -adiques.

0.3.1 Résultats pour les tores

Soit X une courbe projective lisse géométriquement intègre sur un corps p -adique. Soit K le corps de fonctions de X . Soient T un tore sur K et Y un K -torseur sous T . On désigne

$$H_{\text{lc}}^3(Y, \mathbb{Q}/\mathbb{Z}(2)) := \text{Ker} \left(\frac{H^3(Y, \mathbb{Q}/\mathbb{Z}(2))}{\text{Im } H^3(K, \mathbb{Q}/\mathbb{Z}(2))} \rightarrow \prod_{v \in X^{(1)}} \frac{H^3(Y_v, \mathbb{Q}/\mathbb{Z}(2))}{\text{Im } H^3(K_v, \mathbb{Q}/\mathbb{Z}(2))} \right)$$

où $Y_v := Y \times_K K_v$ est le changement de corps de base. On a un analogue de l'accouplement de Brauer–Manin (voir [HS16, pp. 15]):

$$(-, -) : Y(\mathbb{A}_K) \times H_{\text{lc}}^3(Y, \mathbb{Q}/\mathbb{Z}(2)) \rightarrow \mathbb{Q}/\mathbb{Z}.$$

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De plus, pour chaque $\alpha \in H_{\text{lc}}^3(Y, \mathbb{Q}/\mathbb{Z}(2))$, l'application $Y(\mathbb{A}_K) \rightarrow \mathbb{Q}/\mathbb{Z}$, $(y_v) \mapsto ((y_v), \alpha)$ est constante. On note l'image commune par $\rho_Y(\alpha)$ et on obtient une application

$$\rho_Y : H_{\text{lc}}^3(Y, \mathbb{Q}/\mathbb{Z}(2)) \rightarrow \mathbb{Q}/\mathbb{Z}.$$

Théorème 0.3.1 (Harari–Szamuely). *Soit Y un K -torseur sous T tel que $Y(\mathbb{A}_K) \neq \emptyset$. Si $\rho_Y \equiv 0$, alors $Y(K) \neq \emptyset$.*

Donc l'obstruction au principe local-global est décrite par le groupe $H_{\text{lc}}^3(Y, \mathbb{Q}/\mathbb{Z}(2))$ qui est défini par des conditions arithmétiques. Lesdites conditions rendent difficile le calcul de $H_{\text{lc}}^3(Y, \mathbb{Q}/\mathbb{Z}(2))$. Une question naturelle est de montrer que l'obstruction peut être décrite par le groupe $H_{\text{nr}}^3(K(Y), \mathbb{Q}/\mathbb{Z}(2))/H^3(K, \mathbb{Q}/\mathbb{Z}(2))$ de cohomologie non ramifiée, lequel est un invariant birationnel important défini par voie algébrique.

Par la prouver du théorème de Harari–Szamuely, il existe un homomorphisme de groupes $\tau : \text{III}^2(T') \rightarrow H_{\text{lc}}^3(Y, \mathbb{Q}/\mathbb{Z}(2))$ tel que $\rho_Y \circ \tau \equiv 0$ implique $Y(K) \neq \emptyset$ (voir [HS16, pp. 15-16]). Donc il suffit de prouver que l'image de $\text{III}^2(T')$ dans $H^3(K(Y), \mathbb{Q}/\mathbb{Z}(2))/H^3(K, \mathbb{Q}/\mathbb{Z}(2))$ est non ramifiée, c'est-à-dire que cette image est contenu dans $H_{\text{nr}}^3(K(Y), \mathbb{Q}/\mathbb{Z}(2))/H^3(K, \mathbb{Q}/\mathbb{Z}(2))$. En fait, on peut faire mieux en utilisant le théorème de pureté en cohomologie étale:

Théorème 0.3.2 (Corollaire 1.3.2 et 1.4.5, Tian, 2019). *Soit Y un K -torseur sous T tel que $Y(\mathbb{A}_K) \neq \emptyset$. Alors l'image de $\text{III}_{\omega}^2(T')$ dans $H^3(Y, \mathbb{Q}/\mathbb{Z}(2))/\text{Im } H^3(K, \mathbb{Q}/\mathbb{Z}(2))$ est non ramifiée. C'est-à-dire que l'image canonique de $\text{III}_{\omega}^2(T')$ dans $H^3(K(Y), \mathbb{Q}/\mathbb{Z}(2))/H^3(K, \mathbb{Q}/\mathbb{Z}(2))$ via $H^3(Y, \mathbb{Q}/\mathbb{Z}(2))/\text{Im } H^3(K, \mathbb{Q}/\mathbb{Z}(2))$ est contenu dans $H_{\text{nr}}^3(K(Y), \mathbb{Q}/\mathbb{Z}(2))/H^3(K, \mathbb{Q}/\mathbb{Z}(2))$. En particulier, l'image de $\text{III}^2(T')$ dans $H_{\text{lc}}^3(Y, \mathbb{Q}/\mathbb{Z}(2))$ est non ramifiée.*

On fournit deux preuves différentes du théorème ci-dessus. La première méthode marche lorsque $Y(K) \neq \emptyset$. D'autre part, la deuxième approche marche en général si $Y(K) = \emptyset$.

- (1) Le cas facile est lorsque $Y = T$ est un K -torseur trivial (si $Y(K)$ est vide, seule la deuxième preuve marche). C'est-à-dire que Y a un point rationnel. Alors l'application $H^3(K, \mathbb{Q}/\mathbb{Z}(2)) \rightarrow H^3(T, \mathbb{Q}/\mathbb{Z}(2))$ induite par le morphisme structural $T \rightarrow \text{Spec } K$ est injective. Maintenant on prend une résolution flasque $1 \rightarrow R \rightarrow Q \rightarrow T \rightarrow 1$ de T , où R est un tore flasque et Q est un tore quasi-trivial. On considère le diagramme ci-dessous (voir Proposition 1.3.1)

$$\begin{array}{ccc} H^1(K, R') & \longrightarrow & H^4(T, \mathbb{Z}(2))/H^4(K, \mathbb{Z}(2)) \\ \downarrow & & \uparrow \\ H^2(K, T') & \longrightarrow & H^3(T, \mathbb{Q}/\mathbb{Z}(2))/H^3(K, \mathbb{Q}/\mathbb{Z}(2)). \end{array}$$

En fait, la flèche horizontale supérieure est d'image non-ramifiée par construction. Parce que Q est un tore quasi-trivial, on a $H^1(K, Q') = 0$, $\text{III}_{\omega}^2(Q') = 0$, donc $\text{III}_{\omega}^1(R') \simeq \text{III}_{\omega}^2(T')$. Si le diagramme est commutatif, alors l'image de $\text{III}_{\omega}^2(T')$ est non-ramifiée. On montrera la commutativité en section 1.3.

- (2) En général, on ne suppose pas que $Y(K) \neq \emptyset$. Maintenant, on utilise une méthode plus géométrique pour montrer que l'image désirée est non ramifiée. Soit T^c une compactification projective lisse T -équivariante sur K . Soit $Y^c := T^c \times^T Y$ le produit contracté. La stratégie est de montrer

$$\text{Im} \left(\text{III}_{\omega}^2(T') \rightarrow \frac{H^3(Y, \mathbb{Q}/\mathbb{Z}(2))}{\text{Im } H^3(K, \mathbb{Q}/\mathbb{Z}(2))} \right) \subset \text{Im} \left(\frac{H^3(Y^c, \mathbb{Q}/\mathbb{Z}(2))}{\text{Im } H^3(K, \mathbb{Q}/\mathbb{Z}(2))} \rightarrow \frac{H^3(Y, \mathbb{Q}/\mathbb{Z}(2))}{\text{Im } H^3(K, \mathbb{Q}/\mathbb{Z}(2))} \right)$$

or, cette dernière image est non ramifiée. En particulier, on sait que l'image de $\text{III}^2(T')$ dans $H_{\text{lc}}^3(Y, \mathbb{Q}/\mathbb{Z}(2))$ est non-ramifiée.

0.3.2 Dualités arithmétiques

Dans cette thèse, on s'intéresse à l'approximation faible pour des groupes réductifs connexes quasi-déployés sur un corps K de fonctions p -adiques. Comme dans le cas des corps de nombres, on introduira un complexe de tores attaché à un groupe réductif connexe. Par conséquent, la première chose à faire est de développer des théorèmes de dualité arithmétique pour les complexes de tores sur K . Soit $C = [T_1 \rightarrow T_2]$ un complexe de tores sur K en degrés -1 et 0 . Soit $C' = [T_2' \rightarrow T_1']$ le dual de C sur K . On a donc un accouplement $C \otimes^{\mathbf{L}} C' \rightarrow \mathbb{Z}(2)[3]$ (voir [Izq16, pp. 69, Lemme 4.3]) dans la catégorie dérivée des faisceaux étales sur K_v pour toute place $v \in X^{(1)}$. Par la suite on obtient un accouplement

$$\mathbb{H}^i(K_v, C) \times \mathbb{H}^{1-i}(K_v, C') \rightarrow \mathbb{H}^1(K_v, \mathbb{Z}(2)[3]) = \mathbb{H}^4(K_v, \mathbb{Z}(2)).$$

De plus, on a un isomorphisme $\mathbb{H}^4(K_v, \mathbb{Z}(2)) \simeq H^3(K_v, \mathbb{Q}/\mathbb{Z}(2))$ parce que $H^n(K_v, \mathbb{Q}(2)) = 0$ pour $n \geq 3$. Enfin, grâce à la théorie du corps de classes de Kato, on a $H^3(K_v, \mathbb{Q}/\mathbb{Z}(2)) \simeq \mathbb{Q}/\mathbb{Z}$.

Théorème 0.3.3 (Voir 2.3.5, 2.3.7, Tian [Tia19a, Tia19b]). *Soit ℓ un nombre premier. Le cup-produit*

$$\mathbb{H}^i(K_v, C) \times \mathbb{H}^{1-i}(K_v, C') \rightarrow \mathbb{Q}/\mathbb{Z}$$

réalise des dualités parfaites entre les groupes suivants

- le groupe profini $\mathbb{H}^0(K_v, C)_{\wedge}$ et le groupe discret $\mathbb{H}^1(K_v, C')$.
- le groupe discret $\mathbb{H}^1(K_v, C)$ et le groupe profini $\mathbb{H}^0(K_v, C')_{\wedge}$.
- le groupe profini $\mathbb{H}^0(K_v, C)^{(\ell)}$ et le groupe discret $\mathbb{H}^1(K_v, C')\{\ell\}$.
- le groupe discret $\mathbb{H}^1(K_v, C)\{\ell\}$ et le groupe profini $\mathbb{H}^0(K_v, C')^{(\ell)}$.

Le théorème ci-dessus est une synthèse des Proposition 2.3.5 et Corollary 2.3.7. Les preuves sont basées sur des arguments de dévissage pour les triangles distingués $T_1 \rightarrow T_2 \rightarrow C \rightarrow T_1[1]$ et $C \rightarrow C \rightarrow C \otimes^{\mathbf{L}} \mathbb{Z}/n \rightarrow C[1]$.

Pour la dualité globale des groupes d'hypercohomologie galoisienne de C et C' , on a les résultats suivants.

Théorème 0.3.4 (Voir 2.3.14, 2.3.15, 2.3.19, 2.3.20, Tian [Tia19a, Tia19b]).

- (1) *Pour chaque $i \in \mathbb{Z}$, $\text{III}^i(C)$ est un groupe abélien fini. De plus, $\text{III}^i(C) = 0$ pour $i \leq -1$ et $i \geq 3$.*
- (2) *On a un accouplement parfait entre les groupes finis pour $0 \leq i \leq 2$*

$$\text{III}^i(C) \times \text{III}^{2-i}(C') \rightarrow \mathbb{Q}/\mathbb{Z}.$$

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(3) *Supposons $\text{Ker } \rho$ fini. On a un accouplement parfait entre les groupes finis*

$$\mathbb{III}_\lambda^0(C) \times \mathbb{III}^2(C') \rightarrow \mathbb{Q}/\mathbb{Z}$$

$$\text{où } \mathbb{III}_\lambda^0(C) := \text{Ker} \left(\mathbb{H}^0(K, C)_\lambda \rightarrow \mathbb{P}^0(K, C)_\lambda \right).$$

Ce théorème est une synthèse des propositions 2.3.14, 2.3.15, théorèmes 2.3.19 et 2.3.20. Soit X_0 un ouvert de Zariski non-vide assez petit de X tel que T_i s'étende à un X_0 -tore \mathcal{T}_i . Soit $\mathcal{C} = [\mathcal{T}_1 \rightarrow \mathcal{T}_2]$ un complexe de X_0 -tores en degrés -1 et 0 . On explique d'abord les résultats de finitude de $\mathbb{III}^i(C)$.

- On considère le triangle distingué $T_1 \rightarrow T_2 \rightarrow C \rightarrow T_1[1]$. Parce que $H^i(K, P) = 0$ pour tout tore P et tout entier $i \leq -1$ ou $i \geq 3$, on sait que $\mathbb{III}^i(C) = 0$ pour $i \leq -1$ ou $i \geq 3$ par dévissage.
- Pour la finitude de $\mathbb{III}^0(C)$ et $\mathbb{III}^2(C)$, on considère le triangle distingué $M[1] \rightarrow C \rightarrow T \rightarrow M[2]$ où $C = [T_1 \xrightarrow{\rho} T_2]$, $M = \text{Ker } \rho$ et $T = \text{Coker } \rho$. Alors on peut déduire les résultats par dévissage. Plus précisément, $\mathbb{III}^0(C) \simeq \mathbb{III}^1(M)$ est contenu dans un groupe fini. D'autre part, $\mathbb{III}^2(C)$ est de type cofini par dévissage et il est un sous-quotient d'un groupe de torsion de type cofini, donc $\mathbb{III}^2(C)$ est fini.
- Pour $\mathbb{III}^1(C)$, on considère l'image $\mathbb{D}_K^1(U, \mathcal{C}) = \text{Im} \left(\mathbb{H}_c^1(U, \mathcal{C}) \rightarrow \mathbb{H}^1(K, C) \right)$ dans $\mathbb{H}^1(K, C)$. On peut montrer qu'il existe un ouvert $U_0 \subset X$ tel que $\mathbb{D}_K^1(U, \mathcal{C}) = \mathbb{D}_K^1(U_0, \mathcal{C}) = \mathbb{III}^1(C)$ pour tout $U \subset U_0$. De plus, les groupes $\mathbb{D}_K^1(U, \mathcal{C})$ sont de type cofini et d'exposant fini, donc les groupes $\mathbb{D}_K^1(U, \mathcal{C})$ sont finis pour tout U . En particulier, $\mathbb{III}^1(C)$ est fini.

Pour la dualité globale $\mathbb{III}^i(C) \times \mathbb{III}^{2-i}(C') \rightarrow \mathbb{Q}/\mathbb{Z}$, la méthode est la suivante. On a un diagramme commutatif avec lignes exactes

$$\begin{array}{ccccccc} 0 & \longrightarrow & (\text{Ker } \Delta_U)\{\ell\} & \longrightarrow & \mathbb{H}^1(U, \mathcal{C})\{\ell\} & \xrightarrow{\Delta_U} & \left(\prod_{v \in X^{(1)}} \mathbb{H}^1(K_v, C) \right)\{\ell\} \\ & & \downarrow \Phi_U & & \downarrow \text{AV}_U & & \downarrow \text{Loc} \\ 0 & \longrightarrow & (\mathbb{D}_K^1(U, \mathcal{C}')^{(\ell)})^D & \longrightarrow & (\mathbb{H}_c^1(U, \mathcal{C}')^{(\ell)})^D & \longrightarrow & \left(\left(\bigoplus_{v \in X^{(1)}} \mathbb{H}^0(K_v^h, C') \right)^{(\ell)} \right)^D \end{array}$$

où

- Δ_U est le composé $\mathbb{H}^1(U, \mathcal{C}) \rightarrow \mathbb{H}^1(K, C) \rightarrow \prod \mathbb{H}^1(K_v, C)$,
- Loc est induit par les dualités locales et on peut montrer que Loc est un isomorphisme,
- AV_U est induit par une variante de l'accouplement d'Artin–Verdier, et l'application AV_U est surjective avec noyau divisible,
- Φ_U est obtenu par le carré de droite, donc Φ_U est surjective avec $\text{Ker } \Phi_U$ divisible.

Mais le groupe $\varinjlim_U (\text{Ker } \Delta_U)\{\ell\} = \mathbb{III}^i(C)\{\ell\}$ est fini, donc le groupe divisible $\varinjlim_U \text{Ker } \Phi_U$ est trivial. Enfin, on montre que $\mathbb{D}_K^i(U, \mathcal{C}')^{(\ell)} \simeq \mathbb{III}^i(C')\{\ell\}$. On a alors un accouplement parfait entre groupes finis $\mathbb{III}^i(C)\{\ell\} \times \mathbb{III}^{2-i}(C')\{\ell\} \rightarrow \mathbb{Q}_\ell/\mathbb{Z}_\ell$ pour chaque nombre premier ℓ .

Pour la dernière dualité $\mathbb{III}_\wedge^0(C) \times \mathbb{III}^2(C') \rightarrow \mathbb{Q}/\mathbb{Z}$, la stratégie est la suivante. On peut établir un accouplement parfait entre groupes finis

$$\varprojlim_n \mathbb{D}^0(U, \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n) \times \mathbb{D}_{\text{sh}}^1(U, \varinjlim_n \mathcal{C}' \otimes^{\mathbf{L}} \mathbb{Z}/n) \rightarrow \mathbb{Q}/\mathbb{Z} \quad (5)$$

où

$$\mathbb{D}^0(U, \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n) := \text{Im} \left(\mathbb{H}_c^0(U, \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n) \rightarrow \mathbb{H}^0(U, \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n) \right),$$

et

$$\mathbb{D}_{\text{sh}}^1(U, \varinjlim_n \mathcal{C}' \otimes^{\mathbf{L}} \mathbb{Z}/n) := \text{Ker} \left(\mathbb{H}^1(U, \varinjlim_n \mathcal{C}' \otimes^{\mathbf{L}} \mathbb{Z}/n) \rightarrow \prod_{v \in X^{(1)}} H^1(K_v, \varinjlim_n \mathcal{C}' \otimes^{\mathbf{L}} \mathbb{Z}/n) \right).$$

En fait, on a des isomorphismes

$$\begin{aligned} \mathbb{III}_\wedge^0(C) &\simeq \varprojlim_U \varprojlim_n \mathbb{D}^0(U, \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n), \\ \mathbb{III}^2(C') &\simeq \varinjlim_U \mathbb{D}_{\text{sh}}^1(U, \varinjlim_n \mathcal{C}' \otimes^{\mathbf{L}} \mathbb{Z}/n). \end{aligned}$$

Donc on obtient un accouplement parfait $\mathbb{III}_\wedge^0(C) \times \mathbb{III}^2(C') \rightarrow \mathbb{Q}/\mathbb{Z}$ entre groupes finis par (5) après avoir pris la limite inductive sur tout U .

Maintenant, on peut résumer tous les résultats ci-dessus en une suite exacte de type Poitou-Tate.

Théorème 0.3.5 (Theorem 2.4.2, Tian [Tia19b]). *Soit $C = [T_1 \xrightarrow{\rho} T_2]$ un complexe de tores. Supposons que $\text{Ker } \rho$ est fini ou $\text{Coker } \rho$ est trivial. Alors on a une suite exacte de groupes topologiques abéliens*

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{H}^{-1}(K, C)_\wedge & \longrightarrow & \mathbb{P}^{-1}(K, C)_\wedge & \longrightarrow & \mathbb{H}^2(K, C')^D \\ & & \searrow & & \searrow & & \searrow \\ & & \mathbb{H}^0(K, C)_\wedge & \longrightarrow & \mathbb{P}^0(K, C)_\wedge & \longrightarrow & \mathbb{H}^1(K, C')^D \\ & & \searrow & & \searrow & & \searrow \\ & & \mathbb{H}^1(K, C) & \longrightarrow & \mathbb{P}^1(K, C)_{\text{tors}} & \longrightarrow & (\mathbb{H}^0(K, C')_\wedge)^D \\ & & \searrow & & \searrow & & \searrow \\ & & \mathbb{H}^2(K, C) & \longrightarrow & \mathbb{P}^2(K, C)_{\text{tors}} & \longrightarrow & (\mathbb{H}^{-1}(K, C')_\wedge)^D \longrightarrow 0. \end{array}$$

Remarque 0.3.6.

- (1) Les première et dernière ligne sont exactes sans aucune hypothèse sur C .
- (2) Les applications de $\mathbb{P}^i(K, C)$ à $\mathbb{H}^{1-i}(K, C')^D$ sont induites par les dualités locales.
- (3) Les applications de $\mathbb{H}^i(K, C')^D$ à $\mathbb{H}^{2-i}(K, C)$ sont induites par les dualités globales.

0.3.3 Approximation faible

Maintenant on peut utiliser la suite de Poitou–Tate pour étudier l’approximation faible pour des groupes réductifs connexes. Soit K un corps de fonctions p -adiques. Soit G un groupe réductif connexe sur K . D’après Deligne et Borovoi, on considère le composé $\rho : G^{\text{sc}} \rightarrow G^{\text{ss}} \rightarrow G$. Soit T un tore maximal de G . Soit $T^{\text{sc}} := \rho^{-1}(T)$. C’est un tore maximal de G^{sc} . On associe à G un complexe de tores $C := [T^{\text{sc}} \rightarrow T]$ en degrés -1 et 0 . Le résultat suivant indique qu’en général il y a une obstruction à l’approximation faible pour G qui est contrôlée par une sorte de groupe de Tate–Shafarevich de C' .

Théorème 0.3.7 (Voir 3.1.4, Tian [Tia19a]). *Soit G un groupe réductif connexe sur K . Supposons que G^{sc} satisfait l’approximation faible et contient un tore maximal quasi-trivial.¹ On a une suite exacte de groupes*

$$1 \rightarrow \overline{G(K)} \rightarrow \prod_{v \in X^{(1)}} G(K_v) \rightarrow \mathbb{H}_\omega^1(C')^D \rightarrow \mathbb{H}^1(C) \rightarrow 1$$

où $\overline{G(K)}$ désigne l’adhérence de l’image diagonale de $G(K)$ dans $\prod_{v \in X^{(1)}} G(K_v)$ par rapport aux topologies produits v -adiques, et $\mathbb{H}_\omega^1(C')$ désigne le sous-groupes des éléments de $\mathbb{H}^1(K, C')$ localement triviaux pour presque tout $v \in X^{(1)}$.

Notons que contrairement au cas du corps de nombres, actuellement on ne sait pas si tout les groupes semi-simples simplement connexes vérifient l’approximation faible sur des corps de fonctions p -adiques. Cependant, les groupes semi-simples simplement connexes quasi-déployés sont rationels (voir [Har67, Satz 2.2.2]), donc l’approximation faible est vérifiée. En particulier, ces groupes vérifient l’approximation faible. Maintenant, on indique brièvement d’autres raisons pour lesquelles nous n’avons pas abandonné l’hypothèse que G^{sc} est quasi-déployé.

- Soit H un groupe réductif connexe quasi-trivial sur L , c’est-à-dire H est une extension d’un tore quasi-trivial par un groupe simplement connexe. Donc on a une suite exacte $1 \rightarrow H^{\text{sc}} \rightarrow H \rightarrow H^{\text{tor}} \rightarrow 1$ avec $H^{\text{sc}} = H^{\text{ss}}$ simplement connexe et H^{tor} un tore quasi-trivial. Si L est un corps de nombres, alors on a $H^1(L_v, H^{\text{sc}}) = 1$ pour chaque place v finie et $H^1(L, H) \simeq \prod_{v|\infty} H^1(L_v, H^{\text{sc}})$. On a un diagramme commutatif avec les lignes exactes

$$\begin{array}{ccccccc} H^{\text{sc}}(L) & \longrightarrow & H(L) & \longrightarrow & H^{\text{tor}}(L) & \longrightarrow & H^1(L, H^{\text{sc}}) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \simeq \\ \prod_{v \in S} H^{\text{sc}}(L_v) & \longrightarrow & \prod_{v \in S} H(L_v) & \longrightarrow & \prod_{v \in S} H^{\text{tor}}(L_v) & \longrightarrow & \prod_{v|\infty} H^1(L, H^{\text{sc}}) \end{array}$$

où $S \subset \Omega_L$ est un sous-ensemble fini contenant toutes les places archimédiennes. Par la suite la méthode de Sansuc [San81] implique que H vérifie également l’approximation faible. Si L est un corps de fonctions p -adique, on ne sait pas que $H^1(L_v, H^{\text{sc}}) = 1$ pour presque toutes les places v . En particulier, il n’est pas clair que H satisfait l’approximation faible même si H^{sc} satisfait l’approximation faible.

¹Si G est quasi-déployé, alors G^{sc} satisfait l’approximation faible et contient un tore maximal quasi-trivial. Voir Proposition 3.1.1 pour plus de détails.

- Soit H un groupe semi-simple. On a une suite exacte $1 \rightarrow F \rightarrow H^{\text{sc}} \rightarrow H \rightarrow 1$ avec F un schéma en groupe commutatif fini étale. Dans ce cas, la méthode de Sansuc ne donne pas le défaut d'approximation faible pour H pour la même raison.

On rappelle les principales étapes de la preuve du Théorème 0.3.7 comme suit. La première étape consiste à réduire la suite exacte à l'exactitude de

$$1 \rightarrow \overline{G(K)}_S \rightarrow \prod_{v \in S} G(K_v) \rightarrow \mathbb{H}_S^1(C')^D \rightarrow \mathbb{H}^1(C) \rightarrow 1 \quad (6)$$

pour tous les sous-ensembles finis S de $X^{(1)}$, où $\overline{G(K)}_S$ est l'adhérence de l'image diagonale de $G(K)$ dans $\prod_{v \in S} G(K_v)$.

La deuxième étape consiste à montrer que si la suite (6) est exacte pour $G^m \times Q$ où m est un entier positif et Q est un tore quasi-trivial, alors la suite (6) est exacte pour G . En particulier, on peut supposer que G admet un revêtement spécial $G_0 \rightarrow G$ par le lemme de Ono. Notez que G_0 vérifie l'approximation faible parce que G^{sc} et les tores quasi-triviaux satisfont l'approximation faible.

On peut alors déduire l'exactitude de (6). Soit F_0 le noyau de $G_0 \rightarrow G$. Donc F_0 est un groupe commutatif étale fini lequel est central dans G_0 . On a alors un diagramme commutatif

$$\begin{array}{ccccccc} G_0(K) & \longrightarrow & G(K) & \xrightarrow{\partial} & H^1(K, F_0) & \longrightarrow & H^1(K, G_0) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \prod G_0(K_v) & \longrightarrow & \prod G(K_v) & \xrightarrow{\partial_v} & \prod H^1(K_v, F_0) & \longrightarrow & \prod H^1(K_v, G_0) \\ & & \downarrow & & \downarrow & & \\ & & \mathbb{H}_S^1(C')^D & \longrightarrow & \mathbb{H}_S^2(F_0)^D & & \end{array}$$

où tous les produits directs sont sur S . On a

- La troisième colonne est exacte par la suite de Poitou–Tate [HSS15, Theorem 2.3] pour les modules finis.
- Les flèches $H^1(L, F_0) \rightarrow H^1(L, G_0)$ sont surjectives où $L = K$ ou K_v parce que G_0 contient un tore quasi-trivial construit à partir d'un tel tore de G^{sc} .

Ensuite, on obtient l'exactitude des trois premiers termes de (6). Pour les trois derniers termes, on conclut en dualisant la suite exacte

$$1 \rightarrow \mathbb{H}^1(C') \rightarrow \mathbb{H}_S^1(C') \rightarrow \bigoplus_{v \in S} \mathbb{H}^1(K_v, C').$$

La suite exacte $1 \rightarrow \overline{G(K)} \rightarrow \prod_{v \in X^{(1)}} G(K_v) \rightarrow \mathbb{H}_\omega^1(C')^D \rightarrow \mathbb{H}^1(C) \rightarrow 1$ dit que le groupe $\mathbb{H}_\omega^1(C')$ peut être considéré comme un défaut d'approximation faible pour le groupe G .

Remarque 0.3.8. Soient $C' = [T' \rightarrow (T^{\text{sc}})']$ et $\pi_1^{\text{alg}}(\overline{G}) := \mathbf{X}_*(T)/\rho_*\mathbf{X}_*(T^{\text{sc}})$ le groupe fondamental algébrique de G (voir [Bor98] ou [CT08]). Soit G^* l'unique groupe de type multiplicatif tel que $\mathbf{X}^*(G^*) = \pi_1^{\text{alg}}(\overline{G})$. Alors on peut montrer qu'il y a un isomorphisme $\mathbb{H}^i(K, C') \simeq H^{i+1}(K, G^*)$ entre groupes abéliens. En particulier, on a $\mathbb{H}_\omega^1(C') \simeq \mathbb{H}_\omega^2(G^*)$. C'est-à-dire que le défaut à l'approximation faible peut être décrit par $\mathbb{H}_\omega^2(G^*)$.

0.3. RÉSULTATS SUR LES CORPS DE FONCTIONS P -ADIQUES

En fait, on peut reformuler la suite exacte ci-dessus en termes d'obstruction de réciprocité à l'approximation faible. Plus précisément, il existe un accouplement qui annule l'adhérence de l'image diagonale de $G(K)$ à gauche:

$$(-, -) : \prod_{v \in X^{(1)}} G(K_v) \times H_{\text{nr}}^3(K(G), \mathbb{Q}/\mathbb{Z}(2)) \rightarrow \mathbb{Q}/\mathbb{Z}.$$

Voir [CT95, §4.1] pour les définitions et propriétés générales de la cohomologie non-ramifiée. Voir [HSS15, pp. 18, pairing (17)] pour la construction de l'accouplement ci-dessus.

Théorème 0.3.9 (Voir 3.2.1, Tian [Tia19a]). *Soit G un groupe réductif connexe sur K . Supposons que G^{sc} satisfait l'approximation faible et contient un tore maximal quasi-trivial. Il existe un morphisme*

$$u : \text{III}_{\omega}^1(C') \rightarrow H_{\text{nr}}^3(K(G), \mathbb{Q}/\mathbb{Z}(2))$$

tel que tout $(g_v) \in \prod_{v \in X^{(1)}} G(K_v)$ satisfaisant $((g_v), \text{Im } u) = 0$ sous $(-, -)$ ci-dessus est dans l'adhérence $\overline{G(K)}$ de $G(K)$ par rapport à la topologie produit.

On conclut cette section en rappelant la construction de $u : \text{III}_{\omega}^1(C') \rightarrow H_{\text{nr}}^3(K(G), \mathbb{Q}/\mathbb{Z}(2))$. Soit $1 \rightarrow R \rightarrow H \rightarrow G \rightarrow 1$ une résolution flasque de G . On peut montrer qu'il y a un isomorphisme $\text{III}_{\omega}^1(C') \simeq \text{III}_{\omega}^1(R')$ entre groupes abéliens. Donc il suffit de trouver une flèche $H^1(K, R') \rightarrow H_{\text{nr}}^3(K(G), \mathbb{Q}/\mathbb{Z}(2))$ qui est un composé des deux homomorphismes suivants. L'accouplement $R \otimes^{\mathbb{L}} R' \rightarrow \mathbb{Z}(2)[2]$ induit un homomorphisme $H^1(K, R') \rightarrow H^1(G^c, R') \rightarrow H^4(G^c, \mathbb{Z}(2))$. Enfin, on a un homomorphisme $H^4(G^c, \mathbb{Z}(2)) \rightarrow H_{\text{nr}}^3(K(G), \mathbb{Q}/\mathbb{Z}(2))$ par la résolution de Gersten.

0.3.4 Théorème de Borel–Serre

Soit G un groupe linéaire sur K . On peut considérer l'application diagonale

$$\Delta : H^1(K, G) \rightarrow \prod_{v \in X^{(1)}} H^1(K_v, G)$$

où $H^1(-, G)$ est un ensemble de cohomologie galoisienne. On dit que G vérifie le **principe de Hasse** si le noyau $\text{III}^1(G)$ de Δ est trivial. Bien sûr il est très optimiste de conjecturer que tous les groupes linéaires vérifient le principe de Hasse, donc on espère montrer que $\text{III}^1(G)$ est fini pour tous les groupes linéaires. Le premier résultat est donné par Harari et Szamuely:

Theorem 0.3.10 (Voir Section 4.1, Theorem 4.1.9). *Soit G un groupe linéaire sur K . Soit X_0 un ouvert de Zariski non-vide assez petit de X tel que G s'étende à un schéma en groupe \mathcal{G} sur X_0 . Soit $\sigma \in Z^1(X_0, \mathcal{G})$ un cocycle. On note $\sigma_K \in H^1(K, G)$ l'image de σ sous l'application de restriction.*

- *Soit G un groupe linéaire sur K . Soit G° le composant neutre de G . Si $H^1(X_0, \mathcal{G}^{\circ, \sigma}) \rightarrow H^1(K, G^{\circ, \sigma_K})$ d'image fini, alors $\text{III}^1(G^{\sigma_K})$ est fini pour chaque $\sigma \in Z^1(X_0, \mathcal{G})$.*
- *Soit G un groupe réductif connexe sur K . Si $H^1(X_0, \mathcal{G}^{\text{sc}, \sigma}) \rightarrow H^1(K, G^{\text{sc}, \sigma_K})$ d'image fini pour chaque $\sigma \in Z^1(X_0, \mathcal{G})$, alors $\text{III}^1(G^{\sigma_K})$ est fini pour chaque σ .*

Pour la finitude de $\text{III}^1(G)$, la première étape consiste à réduire aux groupes linéaires connexes. Donc on peut alors passer aux groupes réductifs connexes (pour un groupe linéaire connexe, on a $H^1(K, H) \simeq H^1(K, G)$ où H est un sous-groupe maximal connexe réductif de G [PR94, Proposition 2.9]). Enfin, on se réduit au cas des groupes semi-simples simplement connexes en considérant des revêtements spéciaux (voir Proposition 4.1.8).

Par le lemme de Shapiro, on peut supposer que les groupes sont absolument simples simplement connexes. On a le résultat suivant:

Theorem 0.3.11 (Voir Section 4.2). *Soit G un groupe semi-simple simplement connexe. Si les facteurs absolument simples simplement connexes de G sont de type A_n^* , B_n , C_n^* , D_n^* , F_4^{red} ou G_2 , alors $\text{III}^1(G)$ est trivial.*

Soit G un groupe absolument simple simplement connexe sur K . On dit que G est de type

- ${}^1A_n^*$, si $G = \mathbf{SL}_1(D)$ pour une algèbre simple centrale D avec $\text{ind}(D)$ sans facteurs carrés.
- ${}^2A_n^*$, si $G = \mathbf{SU}(h)$ pour une forme hermitienne non singulière h sur (D, τ) , où D est une algèbre à division central avec $\text{ind}(D)$ sans facteurs carrés sur une extension quadratique de K et τ est une involution de deuxième type.
- B_n , si $G = \mathbf{Spin}(q)$ pour une forme quadratique non singulière q de dimension $2n + 1$ sur K .
- C_n^* , si $G = \mathbf{U}(h)$ pour une forme hermitienne non singulière h sur (D, τ) , où D est une algèbre de quaternions sur K et τ est une involution symplectique sur D .
- D_n^* , si $G = \mathbf{Spin}(h)$ pour une forme hermitienne non singulière h sur (D, τ) , où D est une algèbre de quaternions sur K et τ est une involution orthogonale sur D .
- F_4^{red} , si $G = \text{Aut}_{\text{alg}}(J)$ pour une algèbre de Jordan exceptionnelle réduite J sur K de dimension 27.
- G_2 , si $G = \text{Aut}_{\text{alg}}(C)$ pour une algèbre de Cayley C sur K .

En fait, ce théorème découle exactement du même argument que [Hu14] où Hu considère toutes les places de K alors qu'on ne considère que celles provenant de la courbe X . Avec une hypothèse de bonne réduction, on peut montrer que l'application

$$H^3(K, \mathbb{Q}/\mathbb{Z}(2)) \rightarrow \prod_{v \in X^{(1)}} H^3(K_v, \mathbb{Q}/\mathbb{Z}(2))$$

est injective. Par la suite, on obtient un diagramme commutatif

$$\begin{array}{ccc} H^1(K, G) & \longrightarrow & H^3(K, \mathbb{Q}/\mathbb{Z}(2)) \\ \downarrow & & \downarrow \\ \prod_{v \in X^{(1)}} H^1(K_v, G) & \longrightarrow & \prod_{v \in X^{(1)}} H^3(K_v, \mathbb{Q}/\mathbb{Z}(2)) \end{array}$$

où $H^1(-, G) \rightarrow H^3(-, \mathbb{Q}/\mathbb{Z}(2))$ est donné par l'invariant de Rost. De ce point de vue, on peut étudier l'invariant de Rost pour en déduire la finitude de $\text{III}^1(G)$. Par la suite, on le fera au cas par cas pour montrer que $\text{III}^1(G)$ est trivial.

Chapter 1

An obstruction to the Hasse principle for tori

Abstract: We first recall some cohomological obstructions to the Hasse principle for torsors under tori [HS16] and to weak approximation for tori [HSS15]. Subsequently we show that the two obstructions are compatible in the sense of a commutative diagram. Finally we deduce a more general result saying that certain Tate–Shafarevich groups are unramified. Some stated results can be found in [Tia19a, Appendix].

Keywords: unramified cohomology, cohomological obstruction, purity exact sequence.

Let X be a smooth projective geometrically integral curve over a p -adic field. Let K be the function field of X . Let T be a torus over K . For a K -torsor Y under T , we want to study the Hasse principle for Y , i.e. whether $Y(\mathbb{A}_K) \neq \emptyset$ will imply $Y(K) \neq \emptyset$.

Throughout, we will keep the same notations and conventions as in the introduction. Let \bar{K} be an algebraic closure of K . We denote by $\bar{Y} := Y \times_K \bar{K}$ the base change of a K -variety Y to \bar{K} . For a place $v \in X^{(1)}$, we write $Y_v := Y \times_K K_v$.

This chapter is organized as follows. In the first two sections, we recall the constructions of cohomological obstructions to the Hasse principle and to weak approximation for tori over K . In the third section, we compare these two obstructions in an algebraic manner. Finally, we use purity statements to show the obstruction to the Hasse principle comes from some unramified cohomology group.

1.1 An obstruction to the Hasse principle

Let Y be a smooth geometrically integral K -variety. Let \mathcal{Y} be a smooth integral separable X_0 -scheme such that $\mathcal{Y} \times_{X_0} \text{Spec } K \simeq Y$ for some sufficiently small non-empty open subset $X_0 \subset X$. We define

$$Y(\mathbb{A}_K) := \varinjlim_{U \subset X_0} \left(\prod_{v \notin U} Y(K_v) \times \prod_{v \in U} \mathcal{Y}(\mathcal{O}_v) \right).$$

We would like to construct a pairing

$$Y(\mathbb{A}_K) \times H^3(Y, \mathbb{Q}/\mathbb{Z}(2)) \rightarrow \mathbb{Q}/\mathbb{Z} \quad (1.1)$$

analogous to the classical Brauer–Manin pairing as follows. Any K_v -point $y_v \in Y(K_v)$ induces an evaluation map

$$Y(K_v) \times H^3(Y, \mathbb{Q}/\mathbb{Z}(2)) \rightarrow H^3(K_v, \mathbb{Q}/\mathbb{Z}(2)) \simeq \mathbb{Q}/\mathbb{Z}, \quad (y_v, \alpha) \mapsto y_v^*(\alpha) \quad (1.2)$$

where the isomorphism $H^3(K_v, \mathbb{Q}/\mathbb{Z}(2)) \simeq \mathbb{Q}/\mathbb{Z}$ follows from Kato's class field theory for higher local fields [Kat80]. Similarly, a point on $\mathcal{Y}(\mathcal{O}_v)$ will induce an evaluation map

$$\mathcal{Y}(\mathcal{O}_v) \times H^3(\mathcal{Y}, \mathbb{Q}/\mathbb{Z}(2)) \rightarrow H^3(\mathcal{O}_v, \mathbb{Q}/\mathbb{Z}(2))$$

which is actually trivial because (by [Mil06, Chapter II, Proposition 1.1(b)])

$$H^3(\mathcal{O}_v, \mathbb{Q}/\mathbb{Z}(2)) \simeq H^3(\kappa(v), \mathbb{Q}/\mathbb{Z}(2)) = 0$$

(where the last vanishing follows from the fact that $\text{cd}(\kappa(v)) = 2$). Summing up, taking sums (which is actually a finite sum) of (1.2) over all v yields the desired pairing (1.1).

By the generalized Weil reciprocity law ([Ser65, II, Annexe, (3.3) and (2.2)]), the sequence

$$H^3(K, \mathbb{Q}/\mathbb{Z}(2)) \rightarrow \bigoplus_{v \in X^{(1)}} H^3(K_v, \mathbb{Q}/\mathbb{Z}(2)) \rightarrow \mathbb{Q}/\mathbb{Z} \quad (1.3)$$

is a complex. Therefore the pairing (1.1) annihilates both the diagonal image of $Y(K)$ in $Y(\mathbb{A}_K)$ on the left and the image of $H^3(K, \mathbb{Q}/\mathbb{Z}(2)) \rightarrow H^3(Y, \mathbb{Q}/\mathbb{Z}(2))$ induced by the structural morphism $Y \rightarrow \text{Spec } K$ on the right. We put¹

$$H_{\text{lc}}^3(Y, \mathbb{Q}/\mathbb{Z}(2)) := \text{Ker} \left(\frac{H^3(Y, \mathbb{Q}/\mathbb{Z}(2))}{\text{Im } H^3(K, \mathbb{Q}/\mathbb{Z}(2))} \rightarrow \prod_{v \in X^{(1)}} \frac{H^3(Y_v, \mathbb{Q}/\mathbb{Z}(2))}{\text{Im } H^3(K_v, \mathbb{Q}/\mathbb{Z}(2))} \right). \quad (1.4)$$

¹Here the subscript "lc" stands for locally constant elements in $H^3(Y, \mathbb{Q}/\mathbb{Z}(2))$.

Therefore (1.1) induces a pairing

$$(-, -)_{\text{HP}} : Y(\mathbb{A}_K) \times H_{\text{lc}}^3(Y, \mathbb{Q}/\mathbb{Z}(2)) \rightarrow \mathbb{Q}/\mathbb{Z} \quad (1.5)$$

annihilating the diagonal image of $Y(K)$ in $Y(\mathbb{A}_K)$ on the left.

Next, we observe that the pairing $(-, \alpha)_{\text{HP}} : Y(\mathbb{A}_K) \rightarrow \mathbb{Q}/\mathbb{Z}$ is constant for each element $\alpha \in H_{\text{lc}}^3(Y, \mathbb{Q}/\mathbb{Z}(2))$. Take any adelic point $(y_v) \in Y(\mathbb{A}_K)$. Note that each y_v determines a K_v -point on Y_v which enables one to identify $H^3(K_v, \mathbb{Q}/\mathbb{Z}(2))$ as a subgroup of $H^3(Y_v, \mathbb{Q}/\mathbb{Z}(2))$. Thus $y_v^*(\alpha) \in H^3(K_v, \mathbb{Q}/\mathbb{Z}(2))$ coincides with the image of α under $q_v^* : H^3(Y, \mathbb{Q}/\mathbb{Z}(2)) \rightarrow H^3(Y_v, \mathbb{Q}/\mathbb{Z}(2))$ where $q_v : Y_v \rightarrow Y$ denotes the canonical projection. In particular, we obtain a homomorphism provided that $Y(\mathbb{A}_K) \neq \emptyset$:

$$\rho_Y : H_{\text{lc}}^3(Y, \mathbb{Q}/\mathbb{Z}(2)) \rightarrow \mathbb{Q}/\mathbb{Z}, \quad \alpha \mapsto \sum_{v \in X^{(1)}} y_v^*(\alpha).$$

Theorem 1.1.1 (Harari–Szamuely [HS16, Theorem 5.1]). *Let Y be a K -torsor under a torus T such that $Y(\mathbb{A}_K) \neq \emptyset$. If ρ_Y is trivial, then $Y(K) \neq \emptyset$.*

In other words, the cohomological obstruction to the Hasse principle given by $H_{\text{lc}}^3(Y, \mathbb{Q}/\mathbb{Z}(2))$ is the only one. But the group $H_{\text{lc}}^3(Y, \mathbb{Q}/\mathbb{Z}(2))$ is defined by an "arithmetic" condition, i.e. it consists of elements that are trivial everywhere locally, so it is difficult to compute this group. To understand the obstruction given by the group $H_{\text{lc}}^3(Y, \mathbb{Q}/\mathbb{Z}(2))$ better, we would like to relate it with the unramified Galois cohomology group $H_{\text{nr}}^3(K(Y), \mathbb{Q}/\mathbb{Z}(2))$. To this end, we shall

- define a finer obstruction to the Hasse principle by a subgroup of $H_{\text{lc}}^3(Y, \mathbb{Q}/\mathbb{Z}(2))$,
- recall an obstruction to weak approximation for tori using $H_{\text{nr}}^3(K(T), \mathbb{Q}/\mathbb{Z}(2))$,
- and compare the above two obstructions in two different settings.

More precisely, we shall see soon that the above two obstructions form a commutative diagram and we can compare them in this manner. Now let us construct an obstruction finer than (1.5).

Lemma 1.1.2. *Let Y be a K -torsor under a K -torus T . Let T' be the dual torus of T . There is an isomorphism of Galois modules*

$$H^1(\bar{Y}, \mathbb{Q}/\mathbb{Z}(2)) \simeq T'(\bar{K})_{\text{tors}}.$$

Proof. See [HS16, Lemma 5.2]. □

Since the Galois module $T'(\bar{K})/T'(\bar{K})_{\text{tors}}$ is uniquely divisible, it has trivial Galois cohomology groups in positive degrees. Therefore we obtain isomorphisms

$$H^2(K, H^1(\bar{Y}, \mathbb{Q}/\mathbb{Z}(2))) \simeq H^2(K, T'(\bar{K})_{\text{tors}}) \simeq H^2(K, T').$$

On the other hand, the Hochschild–Serre spectral sequence

$$E_2^{p,q} := H^p(K, H^q(\bar{Y}, \mathbb{Q}/\mathbb{Z}(2))) \Rightarrow H^{p+q} := H^{p+q}(Y, \mathbb{Q}/\mathbb{Z}(2))$$

yields a map

$$H^2(K, H^1(\bar{Y}, \mathbb{Q}/\mathbb{Z}(2))) \rightarrow H^3(Y, \mathbb{Q}/\mathbb{Z}(2)) / \text{Im } H^3(K, \mathbb{Q}/\mathbb{Z}(2)).$$

Indeed, since $\text{cd } K = 3$, the differential $d_2^{2,1} : E_2^{2,1} \rightarrow E_2^{4,0}$ is trivial and hence we obtain a map $E_2^{2,1} \rightarrow E_\infty^{2,1} \simeq F^2 H^3 / F^3 H^3 \rightarrow H^3 / F^3 H^3$ where $0 = F^4 H^3 \subset \dots \subset F^1 H^3 \subset F^0 H^3 = H^3$ is a filtration of H^3 . Note that Y is geometrically integral, so $H^0(\bar{Y}, \mathbb{Q}/\mathbb{Z}(2)) = \mathbb{Q}/\mathbb{Z}(2)$. Thus there is a surjective map

$$H^3(K, \mathbb{Q}/\mathbb{Z}(2)) \simeq E_2^{3,0} \rightarrow E_\infty^{3,0} \simeq F^3 H^3.$$

Summing up, these computation yields a map $E_2^{2,1} \rightarrow H^3 / \text{Im } E_2^{3,0}$, as desired. If we restrict ourself to the subgroup $\text{III}^2(T')$ of $H^2(K, T')$, then we obtain a map

$$\tau : \text{III}^2(T') \rightarrow H_{\text{lc}}^3(Y, \mathbb{Q}/\mathbb{Z}(2)).$$

Note that $Y(\mathbb{A}_K) \neq \emptyset$ is equivalent to $Y(K_v) \neq \emptyset$ for all $v \in X^{(1)}$. So $Y(\mathbb{A}_K) \neq \emptyset$ will imply that the class $[Y] \in H^1(K, T)$ actually lies in $\text{III}^1(T)$. Now we arrive at:

Proposition 1.1.3. *Let Y be a K -torsor under a K -torus T such that $Y(\mathbb{A}_K) \neq \emptyset$. Then*

$$\rho_Y \circ \tau(\alpha) = \langle [Y], \alpha \rangle$$

holds up to sign, where $\langle -, - \rangle$ denotes the global duality $\text{III}^1(T) \times \text{III}^2(T') \rightarrow \mathbb{Q}/\mathbb{Z}$ (see [HS16, Theorem 4.1]).

This is [HS16, Proposition 5.3] which is the crucial part of establishing the obstruction to the Hasse principle. Indeed, Theorem 1.1.1 follows immediately from the precedent proposition together with the fact that the global duality pairing is perfect. In this way, we obtain a cohomological obstruction to the Hasse principle by the image of $\text{III}^2(T')$ in $H_{\text{lc}}^3(Y, \mathbb{Q}/\mathbb{Z}(2))$.

1.2 An obstruction to weak approximation

Let Y be a smooth integral variety over K with function field $K(Y)$. The unramified part $H_{\text{nr}}^3(K(Y), \mu_n^{\otimes 2})$ of $H^3(K(Y), \mu_n^{\otimes 2})$ (see [CT95] for more details) is defined as the group of cohomology classes coming from $H^3(A, \mu_n^{\otimes 2})$ for every discrete valuation ring A containing K with fraction field $K(Y)$. Take $y_v \in Y(K_v)$ and $\alpha \in H_{\text{nr}}^3(K(Y), \mu_n^{\otimes 2})$. Lift α uniquely² to $\alpha_v \in H^3(\mathcal{O}_{Y_v, y_v}, \mu_n^{\otimes 2})$. Now α_v goes to $H^3(K_v, \mu_n^{\otimes 2})$ via $H^3(\mathcal{O}_{Y_v, y_v}, \mu_n^{\otimes 2}) \rightarrow H^3(K_v, \mu_n^{\otimes 2})$. Summing up, we obtain an evaluation pairing

$$Y(K_v) \times H_{\text{nr}}^3(K(Y), \mu_n^{\otimes 2}) \rightarrow H^3(K_v, \mu_n^{\otimes 2}).$$

Taking the isomorphism $H^3(K_v, \mathbb{Q}/\mathbb{Z}(2)) \simeq \mathbb{Q}/\mathbb{Z}$ for each $v \in X^{(1)}$ into account, we can construct a pairing

$$(-, -)_{\text{WA}} : \prod_{v \in X^{(1)}} Y(K_v) \times H_{\text{nr}}^3(K(Y), \mathbb{Q}/\mathbb{Z}(2)) \rightarrow \mathbb{Q}/\mathbb{Z} \quad (1.6)$$

Again by the generalized Weil reciprocity law (1.3), the pairing $(-, -)_{\text{WA}}$ annihilates the diagonal image of $Y(K)$. Moreover, $(-, -)_{\text{WA}}$ annihilates the closure of $Y(K)$ in $\prod_{v \in X^{(1)}} Y(K_v)$ under the product of v -adic topologies by a continuity argument (see [Duc97, Part II, Proposition 0.31 and 0.33]).

²Here the uniqueness follows from the injective property for $\mu_n^{\otimes j}$ over discrete valuation rings, see [CT95, §3.6].

Theorem 1.2.1 (Harari–Scheiderer–Szamuely). *There is a homomorphism*

$$u : \mathbb{III}_\omega^2(T') \rightarrow H_{\text{nr}}^3(K(T), \mathbb{Q}/\mathbb{Z}(2))$$

such that each family $(t_v) \in T(K_v)$ annihilated by $(-, \text{Im } u)_{\text{WA}}$ lies in the closure $\overline{T(K)}$ of $T(K)$ with respect to the product topology.

We shall recall the construction of $u : \mathbb{III}_\omega^2(T') \rightarrow H_{\text{nr}}^3(K(T), \mathbb{Q}/\mathbb{Z}(2))$ for later use. Let $1 \rightarrow R \rightarrow Q \rightarrow T \rightarrow 1$ be a flasque resolution of T (recall that R is a flasque torus and Q is a quasi-trivial torus). Dualizing it yields an exact sequence $1 \rightarrow T' \rightarrow Q' \rightarrow R' \rightarrow 1$ of tori which induces an exact sequence

$$0 \rightarrow H^1(K, R') \rightarrow H^2(K, T') \rightarrow H^2(K, Q').$$

Since Q' is quasi-trivial, $\mathbb{III}_\omega^2(Q') = 0$ by [HSS15, Lemma 3.2]. Consequently, we have an isomorphism $\mathbb{III}_\omega^1(R') \simeq \mathbb{III}_\omega^2(T')$ and we obtain a map

$$\mathbb{III}_\omega^2(T') \simeq \mathbb{III}_\omega^1(R') \rightarrow H^1(K, R').$$

On the other hand, Q is endowed with a T -torsor structure under R by the flasque resolution $1 \rightarrow R \rightarrow Q \rightarrow T \rightarrow 1$. Since R is flasque, the T -torsor Q extends to a T^c -torsor Y under R where T^c is a smooth compactification of T . In this way we obtain a class $[Y] \in H^1(T^c, R)$. Note that the pairing $R \otimes^{\mathbb{L}} R' \rightarrow \mathbb{Z}(2)[2]$ (we have constructed in subsection 0.3.2) induces a homomorphism

$$H^1(K, R') \rightarrow H^4(T^c, \mathbb{Z}(2)), \alpha \mapsto \alpha_{T^c} \cup [Y],$$

where α_{T^c} is the image of α in $H^1(T^c, R')$. Moreover, there exist a natural map

$$H^4(T^c, \mathbb{Z}(2)) \rightarrow H_{\text{nr}}^3(K(T), \mathbb{Q}/\mathbb{Z}(2))$$

by [Kah12, Proposition 2.9]. Summing up, we construct u by the composition

$$\mathbb{III}_\omega^2(T') \rightarrow H^1(K, R') \rightarrow H^4(T^c, \mathbb{Z}(2)) \rightarrow H_{\text{nr}}^3(K(T), \mathbb{Q}/\mathbb{Z}(2)).$$

For latter use, we remark that the map $H^4(T^c, \mathbb{Z}(2)) \rightarrow H_{\text{nr}}^3(K(T), \mathbb{Q}/\mathbb{Z}(2))$ together with an isomorphism

$$H^3(K(T), \mathbb{Q}/\mathbb{Z}(2)) \rightarrow H^4(K(T), \mathbb{Z}(2))$$

fits into a commutative diagram

$$\begin{array}{ccc} H^4(T^c, \mathbb{Z}(2)) & \longrightarrow & H_{\text{nr}}^3(K(T), \mathbb{Q}/\mathbb{Z}(2)) \\ \downarrow & & \downarrow \\ H^4(K(T), \mathbb{Z}(2)) & \longrightarrow & H^3(K(T), \mathbb{Q}/\mathbb{Z}(2)). \end{array}$$

1.3 Comparison of two obstructions

In this section, we show that the obstruction to the Hasse principle is compatible with that of weak approximation. More precisely, we prove in the special case $Y = T$ that the image of $\mathbb{III}_\omega^2(T')$ in $H^3(T, \mathbb{Q}/\mathbb{Z}(2))/H^3(K, \mathbb{Q}/\mathbb{Z}(2))$ is unramified. In particular, the image of $\mathbb{III}^2(T')$ in $H_{\text{lc}}^3(T, \mathbb{Q}/\mathbb{Z}(2))$ is unramified.

Proposition 1.3.1. *Let $1 \rightarrow R \rightarrow Q \rightarrow T \rightarrow 1$ be a flasque resolution of T (in particular, we obtain a class $[Q] \in H^1(T, R)$). There is a commutative diagram (up to sign)*

$$\begin{array}{ccc} H^1(K, R') & \longrightarrow & H^4(T, \mathbb{Z}(2))/H^4(K, \mathbb{Z}(2)) \\ \downarrow & & \uparrow \\ H^2(K, T') & \longrightarrow & H^3(T, \mathbb{Q}/\mathbb{Z}(2))/H^3(K, \mathbb{Q}/\mathbb{Z}(2)) \end{array} \quad (1.7)$$

where the right vertical map is induced by the exact sequence

$$0 \rightarrow \mathbb{Z}(2) \rightarrow \mathbb{Q}(2) \rightarrow \mathbb{Q}/\mathbb{Z}(2) \rightarrow 0,$$

the upper horizontal map is defined by the cup-product

$$H^1(K, R') \times H^1(T^c, R) \rightarrow H^4(T^c, \mathbb{Z}(2)),$$

and the lower horizontal map comes from the Hochschild–Serre spectral sequence (see Section 1.1).

Corollary 1.3.2. *The image of $\mathbb{III}_\omega^2(T')$ in $H^3(T, \mathbb{Q}/\mathbb{Z}(2))/H^3(K, \mathbb{Q}/\mathbb{Z}(2))$ has unramified image. In particular, the image of $\mathbb{III}^2(T')$ in $H_{\text{lc}}^3(T, \mathbb{Q}/\mathbb{Z}(2))$ lies in the unramified part under the further map $H_{\text{lc}}^3(T, \mathbb{Q}/\mathbb{Z}(2)) \rightarrow H^3(K(T), \mathbb{Q}/\mathbb{Z}(2))/H^3(K, \mathbb{Q}/\mathbb{Z}(2))$.*

Proof. By construction the map $H^1(K, R') \rightarrow H^4(T, \mathbb{Z}(2))$ induced by the cup-product factors through $H^4(T^c, \mathbb{Z}(2)) \rightarrow H^4(T, \mathbb{Z}(2))$, so the image of $\mathbb{III}_\omega^1(R') \simeq \mathbb{III}_\omega^2(T')$ lies in the unramified part $H_{\text{nr}}^3(K(T), \mathbb{Q}/\mathbb{Z}(2))/H^3(K, \mathbb{Q}/\mathbb{Z}(2))$. In particular, the image of $\mathbb{III}^2(T')$ in $H_{\text{lc}}^3(T, \mathbb{Q}/\mathbb{Z}(2))$ is unramified. \square

The rest of this section is devoted to the proof of Proposition 1.3.1. We begin with some observations on torsion groups under consideration. Let $L|K$ be a finite Galois extension splitting both R and T . The vanishing of $H^1(T_L, R_L) = 0^3$ implies that the class $[Q] \in H^1(T, R)$ is torsion by a restriction-corestriction argument. The spectral sequence

$$H^p(K, \text{Ext}_K^q(\overline{R}', \overline{T}')) \Rightarrow \text{Ext}_K^{p+q}(R', T')$$

together with the vanishing $\text{Ext}_K^1(\overline{R}', \overline{T}')) = 0$ (since split tori are flasque, see [CTS87]) implies that

$$H^1(K, \text{Hom}_K(\overline{R}', \overline{T}')) \rightarrow \text{Ext}_K^1(R', T')$$

is an isomorphism. Thus the group $\text{Ext}_K^1(R', T')$ is torsion. We choose a suitable integer n such that the classes $[Q] \in H^1(T, R)$ and $[Q'] \in \text{Ext}_K^1(R', T')$ are both n -torsion.

Step 1: We verify the commutativity of diagram (1.7) with a different construction of the left vertical arrow as in diagram (1.10).

The Kummer sequence $1 \rightarrow {}_nR \rightarrow R \rightarrow R \rightarrow 1$ yields a surjection $H^1(T, {}_nR) \rightarrow {}_nH^1(T, R)$, i.e. $[Q] = \iota_n([Q_n])$ for some class $[Q_n] \in H^1(T, {}_nR)$ with $\iota_n : H^1(T, {}_nR) \rightarrow H^1(T, R)$ induced by the Kummer sequence.

³Recall that the Picard group of a split torus is trivial. Indeed, let P be a split L -torus. Subsequently, we have $\text{Pic}(P) \simeq H^1(L, \mathbf{X}^*(P))$ which is a sum of copies of $H^1(L, \mathbb{Z}) = 0$.

1.3. COMPARISON OF TWO OBSTRUCTIONS

Let $p : T \rightarrow \text{Spec } K$ be the structural morphism. Let $\mathbf{D}(K)$ be the derived category of bounded complexes of Galois modules. We consider the object $ND(T) = (\tau_{\leq 1} \mathbf{R}p_* \mu_n^{\otimes 2})[1]$ in $\mathbf{D}(K)$ which fits into a distinguished triangle (see [HS16, (17)])

$$\mu_n^{\otimes 2}[1] \rightarrow ND(T) \rightarrow H^1(\overline{T}, \mu_n^{\otimes 2}) \rightarrow \mu_n^{\otimes 2}[2]. \quad (1.8)$$

We will follow [HS13, Proposition 1.1] to construct a map

$$\chi : H^1(T, {}_n R) \rightarrow \text{Hom}_K({}_n R', ND(T)).$$

The pairing ${}_n R \otimes^{\mathbf{L}} {}_n R' \rightarrow \mu_n^{\otimes 2}$ yields a map $H^1(T, {}_n R) \rightarrow H^1(T, \underline{\text{Hom}}({}_n R', \mu_n^{\otimes 2}))$. Moreover, we obtain a map $H^1(T, \underline{\text{Hom}}({}_n R', \mu_n^{\otimes 2})) \rightarrow \text{Ext}_T^1({}_n R', \mu_n^{\otimes 2})$ from the exact sequence in low degrees associated to the local-to-global spectral sequence

$$H^p(T, \underline{\text{Ext}}_T^q({}_n R', \mu_n^{\otimes 2})) \Rightarrow \text{Ext}_T^{p+q}({}_n R', \mu_n^{\otimes 2}).$$

Since $\mathbf{R} \text{Hom}_T({}_n S', -) = \mathbf{R} \text{Hom}_K({}_n S', -) \circ \mathbf{R}p_*$ is a composition, formally there is a canonical isomorphism

$$\text{Ext}_T^1({}_n R', \mu_n^{\otimes 2}) \simeq R^1 \text{Hom}_K({}_n R', \mathbf{R}p_* \mu_n^{\otimes 2}).$$

Because $\tau_{\geq 2} \mathbf{R}p_* \mu_n^{\otimes 2}$ is acyclic in degrees 0 and 1, we obtain an isomorphism

$$R^1 \text{Hom}_K({}_n R', \tau_{\leq 1} \mathbf{R}p_* \mu_n^{\otimes 2}) \simeq R^1 \text{Hom}_K({}_n R', \mathbf{R}p_* \mu_n^{\otimes 2})$$

from the distinguished triangle $\tau_{\leq 1} \mathbf{R}p_* \mu_n^{\otimes 2} \rightarrow \mathbf{R}p_* \mu_n^{\otimes 2} \rightarrow \tau_{\geq 2} \mathbf{R}p_* \mu_n^{\otimes 2} \rightarrow ND(T)$. Now χ is just the composition

$$H^1(T, {}_n R) \rightarrow \text{Ext}_T^1({}_n R', \mu_n^{\otimes 2}) \simeq R^1 \text{Hom}_K({}_n R', \tau_{\leq 1} \mathbf{R}p_* \mu_n^{\otimes 2}) = \text{Hom}_K({}_n R', ND(T)).$$

All the above constructions yield a diagram of cup-products

$$\begin{array}{ccccc} H^1(K, R') & \times & H^1(T, R) & \longrightarrow & H^4(T, \mathbb{Z}(2)) \\ \partial_n \downarrow & & \uparrow \iota_n & & \uparrow \partial \\ H^2(K, {}_n R') & \times & H^1(T, {}_n R) & \longrightarrow & H^3(T, \mu_n^{\otimes 2}) \\ \parallel & & \downarrow & & \parallel \\ H^2(K, {}_n R') & \times & \text{Hom}_K({}_n R', \mathbf{R}p_* \mu_n^{\otimes 2}[1]) & \longrightarrow & H^2(K, \mathbf{R}p_* \mu_n^{\otimes 2}[1]) \\ \parallel & & \parallel & & \uparrow \\ H^2(K, {}_n R') & \times & \text{Hom}_K({}_n R', ND(T)) & \longrightarrow & H^2(K, ND(T)) \end{array} \quad (1.9)$$

where the upper diagram commutes by functoriality of the cup product pairing (see [HS16, diagram (26)] for more details), the middle diagram commutes by [Mil80, Proposition V.1.20], and the commutativity of the lower diagram is evident. Diagram (1.9) gives the commutativity of the left two squares of the following diagram (where the second square comes from the lower three rows):

$$\begin{array}{ccccccccc} H^1(K, R') & \longrightarrow & H^2(K, {}_n R') & \longrightarrow & H^2(K, ND(T)) & \longrightarrow & H^2(K, {}_n T') & \longrightarrow & H^2(K, T') \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H^4(T, \mathbb{Z}(2)) & \longleftarrow & H^3(T, \mu_n^{\otimes 2}) & \longleftarrow & H^3(T, \mu_n^{\otimes 2}) & \longrightarrow & \frac{H^3(T, \mu_n^{\otimes 2})}{H^3(K, \mu_n^{\otimes 2})} & \longrightarrow & \frac{H^3(T, \mathbb{Q}/\mathbb{Z}(2))}{H^3(K, \mathbb{Q}/\mathbb{Z}(2))}. \end{array} \quad (1.10)$$

The right two squares in diagram (1.10) commute by construction of the Hochschild–Serre spectral sequences. Passing to the quotient by respective subgroup of constants and taking limits over all n imply the commutativity of diagram (1.7). Consequently, we are done if the upper row of diagram (1.10) gives the coboundary map $H^1(K, R') \rightarrow H^2(K, T')$ induced by the short exact sequence $1 \rightarrow T' \rightarrow Q' \rightarrow R' \rightarrow 1$ of tori.

Step 2: We check that composite of arrows in the upper row of diagram (1.10) is just the desired coboundary map in diagram (1.7).

The exact sequence $1 \rightarrow {}_n T' \rightarrow T' \rightarrow T' \rightarrow 1$ induces a surjection $\text{Ext}_K^1(R', {}_n T') \rightarrow {}_n \text{Ext}_K^1(R', T')$, so the class $[Q']$ lifts to a class $[M_n] \in \text{Ext}_K^1(R', {}_n T')$. Similarly, the Kummer sequence $1 \rightarrow {}_n R' \rightarrow R' \rightarrow R' \rightarrow 1$ induces an isomorphism

$$\text{Hom}_K({}_n R', {}_n T') \rightarrow {}_n \text{Ext}_K^1(R', {}_n T') = \text{Ext}_K^1(R', {}_n T')$$

by the vanishing of $\text{Hom}_K(R', {}_n T') = 0$ (because $R'(\bar{K})$ is divisible). Hence there is a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & {}_n R' & \longrightarrow & R' & \longrightarrow & R' \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \parallel \\ 0 & \longrightarrow & {}_n T' & \longrightarrow & M_n & \longrightarrow & R' \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \parallel \\ 0 & \longrightarrow & T' & \longrightarrow & Q' & \longrightarrow & R' \longrightarrow 0. \end{array} \quad (1.11)$$

Applying the functor $H^1(K, -)$ to diagram (1.11) yields a commutative diagram

$$\begin{array}{ccc} H^1(K, R') & \longrightarrow & H^2(K, {}_n R') \\ \downarrow & & \downarrow \\ H^2(K, T') & \longleftarrow & H^2(K, {}_n T') \end{array}$$

which tells us the composite $H^1(K, R') \rightarrow H^2(K, {}_n R') \rightarrow H^2(K, {}_n T') \rightarrow H^2(K, T')$ is exact the coboundary map $H^1(K, R') \rightarrow H^2(K, T')$ induced by the bottom row of diagram (1.11). It remains to show the map $H^2(K, {}_n R') \rightarrow H^2(K, {}_n T')$ obtained from $\text{Ext}_K^1(R', {}_n T') \simeq \text{Hom}_K({}_n R', {}_n T')$ coincides with the composition

$$H^2(K, {}_n R') \rightarrow H^2(K, ND(T)) \rightarrow H^2(K, {}_n T').$$

The cup-product pairing

$$\Phi(-, -) : H^1(\bar{T}, {}_n R) \times H^0(\bar{K}, {}_n R') \rightarrow H^1(\bar{T}, \mu_n^{\otimes 2}),$$

defines a map $H^1(\bar{T}, {}_n R) \rightarrow \text{Hom}_{\bar{K}}({}_n R'(\bar{K}), H^1(\bar{T}, \mu_n^{\otimes 2}))$. Again there is a commutative dia-

gram by [Mil80, Proposition V.1.20]:

$$\begin{array}{ccccc}
 H^1(\bar{T}, {}_nR) & \times & {}_nR'(\bar{K}) & \longrightarrow & H^1(\bar{T}, \mu_n^{\otimes 2}) \\
 \downarrow & & \parallel & & \parallel \\
 \mathrm{Hom}_{\bar{K}}({}_nR'(\bar{K}), \mathbf{R}\bar{p}_*\mu_n^{\otimes 2}[1]) & \times & {}_nR'(\bar{K}) & \longrightarrow & H^0(\bar{K}, \mathbf{R}\bar{p}_*\mu_n^{\otimes 2}[1]) \\
 \simeq \uparrow & & \parallel & & \uparrow \\
 \mathrm{Hom}_{\bar{K}}({}_nR'(\bar{K}), ND(\bar{T})) & \times & {}_nR'(\bar{K}) & \longrightarrow & H^0(\bar{K}, ND(\bar{T}))
 \end{array}$$

which may be rewritten into the following commutative diagram

$$\begin{array}{ccc}
 H^1(\bar{T}, {}_nR) & \longrightarrow & \mathrm{Hom}_{\bar{K}}({}_nR'(\bar{K}), \mathbf{R}\bar{p}_*\mu_n^{\otimes 2}[1]) \\
 \Phi_* \downarrow & \searrow \chi & \uparrow \simeq \\
 \mathrm{Hom}_{\bar{K}}({}_nR'(\bar{K}), H^1(\bar{T}, \mu_n^{\otimes 2})) & \xleftarrow{\alpha} & \mathrm{Hom}_{\bar{K}}({}_nR'(\bar{K}), ND(\bar{T}))
 \end{array}$$

where the arrow Φ_* is induced by $\Phi(-, -)$ is the obvious way and the arrow α is induced by the distinguished triangle (1.8). Now $\alpha \circ \chi = \Phi_*$ says that ${}_nR'(\bar{K}) \rightarrow ND(\bar{T}) \rightarrow H^1(\bar{T}, \mu_n^{\otimes 2})$ is the same as $\Phi([Q_n], -)$. In particular, $H^2(K, {}_nR') \rightarrow H^2(K, ND(T)) \rightarrow H^2(K, {}_nT')$ is the same as $H^2(K, {}_nR') \rightarrow H^2(K, {}_nT')$ obtained from the identification $\mathrm{Ext}_K^1(T, {}_nR) \simeq \mathrm{Hom}_K({}_nR', {}_nT')$. \square

1.4 Purity of étale cohomology

Let T be a K -torus and let Y be a K -torsor under T . In this section, we show that the image of $\mathrm{III}_\omega^2(T')$ in $H^3(Y, \mathbb{Q}/\mathbb{Z}(2))/\mathrm{Im} H^3(K, \mathbb{Q}/\mathbb{Z}(2))$ has unramified image in the quotient $H^3(K(Y), \mathbb{Q}/\mathbb{Z}(2))/H^3(K, \mathbb{Q}/\mathbb{Z}(2))$. In particular, the image of $\mathrm{III}^2(T')$ in $H_{\mathrm{lc}}^3(Y, \mathbb{Q}/\mathbb{Z}(2))$ is unramified.

Let T^c be a T -equivariant smooth projective compactification⁴ of T over K . Let $T \subset V_i \subset T^c$ be the open subset of T^c consisting of T -orbits such that $\mathrm{codim}(T^c \setminus V_i, T^c) \geq i$. Let $Y^c = Y \times^T T^c$ and let $U_i = Y \times^T V_i \subset Y^c$. So $U_0 = Y$ by construction and Y^c is cellular by [Cao18, Proposition 2.2(3)]. For $i \geq 1$, $Z_i := U_i \setminus U_{i-1}$ is smooth of codimension i in U_i .

We begin with the computation of some cohomology groups via purity and then deduce a commutative diagram which tells us $\mathrm{III}_\omega^2(T')$ is unramified. Throughout this section, we shall simply write $\mathcal{Q}(i) := \mathbb{Q}/\mathbb{Z}(i)$ for $i \in \mathbb{Z}$.

Lemma 1.4.1. *Suppose*

- $0 \leq r \leq 4$ and $i \geq 2$, or
- $0 \leq r \leq 2$ and $i \geq 1$.

There are isomorphisms

$$H^r(U_i, \mathcal{Q}(2)) \simeq H^r(Y^c, \mathcal{Q}(2)) \quad \text{and} \quad H^r(\bar{U}_i, \mathcal{Q}(2)) \simeq H^r(\bar{Y}^c, \mathcal{Q}(2)).$$

⁴Such compactification exists, see [CTHS05] or [Kol07, Proposition 3.9.1].

Proof. The purity of $U_{i-1} \subset U_i \supset Z_i$ for $i \geq 1$ yields exact sequences (and similar sequences over \bar{K})

$$H^{r-2i}(Z_i, \mathcal{Q}(3-i)) \rightarrow H^r(U_i, \mathcal{Q}(2)) \rightarrow H^r(U_{i-1}, \mathcal{Q}(2)) \rightarrow H^{r+1-2i}(Z_i, \mathcal{Q}(3-i)). \quad (1.12)$$

Thus for $r \leq 4$ and $i \geq 3$, there are isomorphisms

$$H^r(U_i, \mathcal{Q}(2)) \rightarrow H^r(U_{i-1}, \mathcal{Q}(2)) \quad (1.13)$$

by the exact sequence (1.12). In particular, applying (1.13) inductively yields isomorphisms for $r \leq 4$

$$H^r(U_2, \mathcal{Q}(2)) \simeq H^r(Y^c, \mathcal{Q}(2)) \quad \text{and} \quad H^r(\bar{U}_2, \mathcal{Q}(2)) \simeq H^r(\bar{Y}^c, \mathcal{Q}(2)).$$

Since $\text{codim}(Z_2, U_2) = 2$, the exact sequence (1.12) for $U_1 \subset U_2 \supset Z_2$ and $0 \leq r \leq 2$ implies

$$H^r(U_2, \mathcal{Q}(2)) \simeq H^r(U_1, \mathcal{Q}(2)) \quad \text{and} \quad H^r(\bar{U}_2, \mathcal{Q}(2)) \simeq H^r(\bar{U}_1, \mathcal{Q}(2)).$$

Therefore we conclude $H^r(U_1, \mathcal{Q}(2)) \simeq H^r(U_2, \mathcal{Q}(2)) \simeq H^r(Y^c, \mathcal{Q}(2))$ and similar over \bar{K} . \square

Remark 1.4.2. Since \bar{Y}^c is a projective cellular variety, $H^i(\bar{Y}^c, \mathcal{Q}(2)) = 0$ for $i = 2n - 1$ with $n \geq 1$ an integer (see [Ful84, Example 19.1.11] or [Cao18, Théorème 2.6]). Moreover, thanks to $\text{Br}(\bar{Y}^c) = 0$ (since \bar{Y}^c is smooth projective rational) and the Kummer sequence

$$0 \rightarrow H^1(\bar{Y}^c, \mathbb{G}_m)/n \rightarrow H^2(\bar{Y}^c, \mu_n) \rightarrow {}_n H^2(\bar{Y}^c, \mathbb{G}_m) \rightarrow 0,$$

we obtain an isomorphism of Galois modules $H^2(\bar{Y}^c, \mathcal{Q}(1)) \simeq \text{Pic}(\bar{Y}^c) \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z}$ after taking direct limit over all n . So there is an isomorphism $H^2(\bar{Y}^c, \mathcal{Q}(2)) \simeq \text{Pic}(\bar{Y}^c) \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z}$ of abelian groups.

Lemma 1.4.3.

- (1) We have $H^1(\bar{U}_1, \mathcal{Q}(2)) = 0$. The map $H^2(\bar{U}_1, \mathcal{Q}(2)) \rightarrow H^2(\bar{U}_0, \mathcal{Q}(2))$ induced by $U_0 \subset U_1$ is identically zero. In particular, there is an exact sequence

$$0 \rightarrow H^1(\bar{U}_0, \mathcal{Q}(2)) \rightarrow H^0(\bar{Z}_1, \mathcal{Q}(1)) \rightarrow H^2(\bar{U}_1, \mathcal{Q}(2)) \rightarrow 0 \quad (1.14)$$

of Galois modules.

- (2) We have

$$\text{Im}(H^3(Y^c, \mathcal{Q}(2)) \rightarrow H^3(U_1, \mathcal{Q}(2))) = \text{Ker}(H^3(U_1, \mathcal{Q}(2)) \rightarrow H^3(\bar{U}_1, \mathcal{Q}(2))).$$

Therefore

$$\text{Im}\left(\frac{H^3(Y^c, \mathcal{Q}(2))}{H^3(K, \mathcal{Q}(2))} \rightarrow \frac{H^3(U_1, \mathcal{Q}(2))}{H^3(K, \mathcal{Q}(2))}\right) = \text{Ker}\left(\frac{H^3(U_1, \mathcal{Q}(2))}{H^3(K, \mathcal{Q}(2))} \rightarrow H^3(\bar{U}_1, \mathcal{Q}(2))\right). \quad (1.15)$$

- (3) Consider the diagonal map $\Delta : H^2(K, H^0(\bar{Z}_1, \mathcal{Q}(1))) \rightarrow \prod_v H^2(K_v, H^0(\bar{Z}_1, \mathcal{Q}(1)))$ and write the image $(\alpha_v) := \Delta(\alpha)$ of $\alpha \in H^2(K, H^0(\bar{Z}_1, \mathcal{Q}(1)))$ under Δ into a family of local elements. Put

$$\text{III}_{\omega}^2(H^0(\bar{Z}_1, \mathcal{Q}(1))) := \{\alpha \mid \alpha_v = 0 \text{ for all but finitely many } v \in X^{(1)}\}.$$

Then $\text{III}_{\omega}^2(H^0(\bar{Z}_1, \mathcal{Q}(1))) = 0$ and in particular Δ is injective.

Proof.

- (1) By Lemma 1.4.1 and Remark 1.4.2 we conclude $H^1(\overline{U}_1, \mathcal{Q}(2)) \simeq H^1(\overline{Y}^c, \mathcal{Q}(2)) = 0$. Since there are isomorphisms of varieties $\overline{U}_0 = \overline{Y} \simeq \overline{T}$ and $\overline{Y}^c \simeq \overline{T}^c$, there is a commutative diagram

$$\begin{array}{ccc} \text{Pic}(\overline{T}^c) \otimes \mathbb{Q}/\mathbb{Z} & \longrightarrow & \text{Pic}(\overline{T}) \otimes \mathbb{Q}/\mathbb{Z} \\ \downarrow & & \downarrow \\ H^2(\overline{T}^c, \mathcal{Q}(2)) & \longrightarrow & H^2(\overline{T}, \mathcal{Q}(2)). \end{array}$$

Here the vertical arrows are induced by the Kummer sequence $0 \rightarrow \mu_n \rightarrow \mathbb{G}_m \rightarrow \mathbb{G}_m \rightarrow 0$. Note that $\text{Br} \overline{T}^c = 0$ and $\text{Pic} \overline{T} = 0$. Since the left vertical arrow is an isomorphism by the vanishing of $\text{Br} \overline{T}^c$, we conclude that $H^2(\overline{Y}^c, \mathcal{Q}(2)) \rightarrow H^2(\overline{U}_0, \mathcal{Q}(2))$ is identically zero. Finally, the purity sequence (1.12) for $U_0 \subset U_1 \supset Z_1$ together with the above vanishing results imply the desired short exact sequence of Galois modules.

- (2) The purity of $U_1 \subset U_2 \supset Z_2$ induces a commutative diagram with exact rows

$$\begin{array}{ccccc} H^3(U_2, \mathcal{Q}(2)) & \longrightarrow & H^3(U_1, \mathcal{Q}(2)) & \longrightarrow & H^0(Z_2, \mathcal{Q}(1)) \\ \downarrow & & \downarrow & & \downarrow \\ H^3(\overline{U}_2, \mathcal{Q}(2)) & \longrightarrow & H^3(\overline{U}_1, \mathcal{Q}(2)) & \longrightarrow & H^0(\overline{Z}_2, \mathcal{Q}(1)). \end{array}$$

Note that the map $H^0(Z_2, \mathcal{Q}(1)) \rightarrow H^0(\overline{Z}_2, \mathcal{Q}(1))$ is injective. According to Lemma 1.4.1 and Remark 1.4.2, we have

$$H^3(\overline{U}_2, \mathcal{Q}(2)) \simeq H^3(\overline{Y}^c, \mathcal{Q}(2)) = 0.$$

Thus a diagram chasing yields

$$\text{Im} (H^3(U_2, \mathcal{Q}(2)) \rightarrow H^3(U_1, \mathcal{Q}(2))) = \text{Ker} (H^3(U_1, \mathcal{Q}(2)) \rightarrow H^3(\overline{U}_1, \mathcal{Q}(2))).$$

Recall that $H^3(U_2, \mathcal{Q}(2)) \simeq H^3(Y^c, \mathcal{Q}(2))$ by Lemma 1.4.1, we are done.

- (3) Note first that $H^0(\overline{Z}_1, \mathcal{Q}(1))$ is isomorphic to a direct sum of copies of $\mathcal{Q}(1)$ as Galois modules and hence $\text{III}_\omega^2(H^0(\overline{Z}_1, \mathcal{Q}(1)))$ is a direct sum of copies of $\text{III}_\omega^2(\mathcal{Q}(1))$. Recall that $\text{III}_\omega^2(\mathbb{G}_m) = 0$ by [HSS15, Lemma 3.2(a)]. Thanks to the short exact sequence $0 \rightarrow \mathcal{Q}(1) \rightarrow \overline{K}^\times \rightarrow Q \rightarrow 0$ of Galois modules (where the quotient Q is uniquely divisible), there is an isomorphism $H^2(K, \mathcal{Q}(1)) \simeq \text{Br} K$ and hence $\text{III}_\omega^2(\mathcal{Q}(1)) \simeq \text{III}_\omega^2(\mathbb{G}_m) = 0$. \square

In the diagram below, we denote by HU/HL, VF/VB/VL/VM/VR for the horizontal upper/lower, vertical front/back/left/middle/right face, respectively.

Lemma 1.4.4. *There is an exact commutative diagram with surjective vertical arrows*

$$\begin{array}{ccccc} & H^3(K, \tau_{\leq 2} \mathbf{R}\overline{f}_{U_1^*} \mathcal{Q}(2)) & \rightarrow & H^3(K, \tau_{\leq 1} \mathbf{R}\overline{f}_{U_0^*} \mathcal{Q}(2)) & \rightarrow & H^2(K, \tau_{\leq 0} \mathbf{R}\overline{f}_{Z_1^*} \mathcal{Q}(1)) \\ & \swarrow & \downarrow & \swarrow & \downarrow & \downarrow \\ H^3(U_1, \mathcal{Q}(2)) & \longrightarrow & H^3(U_0, \mathcal{Q}(2)) & \longrightarrow & H^2(Z_1, \mathcal{Q}(1)) & \\ \downarrow & & \downarrow & & \downarrow & \\ & H^1(K, H^2(\overline{U}_1, \mathcal{Q}(2))) & \rightarrow & H^2(K, H^1(\overline{U}_0, \mathcal{Q}(2))) & \rightarrow & H^2(K, H^0(\overline{Z}_1, \mathcal{Q}(1))) \\ & \swarrow & \downarrow & \swarrow & \downarrow & \downarrow \\ \frac{H^3(U_1, \mathcal{Q}(2))}{\text{Im } H^3(K, \mathcal{Q}(2))} & \longrightarrow & \frac{H^3(U_0, \mathcal{Q}(2))}{\text{Im } H^3(K, \mathcal{Q}(2))} & \longrightarrow & H^2(Z_1, \mathcal{Q}(1)). & \end{array}$$

Proof. To construct the arrows in the diagram, we shall need the distinguished triangle giving the purity exact sequence. Let $j : \bar{U}_0 \rightarrow \bar{U}_1$ and $i : \bar{Z}_1 \rightarrow \bar{U}_1$ be open and closed immersions, respectively. We consider the exact sequence $0 \rightarrow i_*i^!\mathcal{Q}(2) \rightarrow \mathcal{Q}(2) \rightarrow j_*j^*\mathcal{Q}(2) \rightarrow i_*R^1i^!\mathcal{Q}(2) \rightarrow 0$ of étale sheaves over U_1 (see [Fu11, Corollary 8.5.6]). The isomorphism $R^qj_*\mathcal{Q}(2) \simeq R^{q+1}i^!\mathcal{Q}(2)$ for $q \geq 1$ yields a distinguished triangle

$$i_*Ri^!\mathcal{Q}(2) \rightarrow \mathcal{Q}(2) \rightarrow Rj_*j^*\mathcal{Q}(2) \rightarrow i_*Ri^!\mathcal{Q}(2)[1]. \quad (1.16)$$

Taking $R^2i^!\mathcal{Q}(2) \simeq i^*\mathcal{Q}(1)[-2]$ into account (see [Fu11, Corollary 8.5.6]) and applying the functor $\mathbf{R}f_{U_1*}$ to (1.16) yield a distinguished triangle

$$\mathbf{R}f_{Z_1*}\mathcal{Q}(1)[-2] \rightarrow \mathbf{R}f_{U_1*}\mathcal{Q}(2) \rightarrow \mathbf{R}f_{U_0*}\mathcal{Q}(2) \rightarrow \mathbf{R}f_{Z_1*}\mathcal{Q}(1)[-1] \quad (1.17)$$

where $f_{\mathfrak{X}}$ denotes the structural morphism of a K -scheme \mathfrak{X} .

- The lower row of VB is obtained by taking Galois cohomology of (1.14).
- We construct the upper row of VB. The first dashed arrow is constructed as follows. Let $f : \mathfrak{X} \rightarrow \text{Spec } K$ be a scheme over K with structural morphism f and let \mathcal{F} be an étale sheaf over \mathfrak{X} . Then $H^2(\mathbf{R}\bar{f}_*\mathcal{F}) = R^2\bar{f}_*\mathcal{F} \simeq H^2(\bar{\mathfrak{X}}, \mathcal{F})$. Consider the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \tilde{\tau}_{\leq 1}\mathbf{R}f_{U_1*}\mathcal{Q}(2) & \longrightarrow & \tau_{\leq 2}\mathbf{R}f_{U_1*}\mathcal{Q}(2) & \longrightarrow & H^2(\bar{U}_1, \mathcal{Q}(2))[-2] \longrightarrow 0 \\ & & \downarrow & \swarrow \text{---} & \downarrow & & \downarrow \\ 0 & \longrightarrow & \tilde{\tau}_{\leq 1}\mathbf{R}f_{U_0*}\mathcal{Q}(2) & \longrightarrow & \tau_{\leq 2}\mathbf{R}f_{U_0*}\mathcal{Q}(2) & \longrightarrow & H^2(\bar{U}_0, \mathcal{Q}(2))[-2] \longrightarrow 0 \end{array} \quad (1.18)$$

where the rows are given by (3) and the vertical arrows are induced by (1.17). The arrow $\tau_{\leq 2}\mathbf{R}f_{U_1*}\mathcal{Q}(2) \rightarrow \tau_{\leq 2}\mathbf{R}f_{U_0*}\mathcal{Q}(2)$ factors through $\tilde{\tau}_{\leq 1}\mathbf{R}f_{U_0*}\mathcal{Q}(2)$, since the right vertical map is zero by Lemma 1.4.3(1).

The second dashed arrow is obtained by $\mathbf{R}f_{U_0*}\mathcal{Q}(2) \rightarrow \mathbf{R}f_{Z_1*}\mathcal{Q}(1)[-1]$ from (1.17). Indeed, there is a map

$$\tau_{\leq 1}\mathbf{R}f_{U_0*}\mathcal{Q}(2) \rightarrow \tau_{\leq 1}(\mathbf{R}f_{Z_1*}\mathcal{Q}(1)[-1]) = (\tau_{\leq 0}\mathbf{R}f_{Z_1*}\mathcal{Q}(1))[-1].$$

- The vertical arrows of VB are induced by the distinguished triangle

$$\mathcal{Q}(2) \rightarrow \tau_{\leq j}\mathbf{R}\bar{\pi}_*\mathcal{Q}(2) \rightarrow H^j(\bar{\mathfrak{X}}, \mathcal{Q}(2))[-j] \rightarrow \mathcal{Q}(2)[1]$$

with $\bar{\pi} : \bar{\mathfrak{X}} \rightarrow \bar{K}$ the structural morphism. The map

$$H^3(K, \tau_{\leq 2}\mathbf{R}f_{U_1*}\mathcal{Q}(2)) \rightarrow H^1(K, H^2(\bar{U}_1, \mathcal{Q}(2)))$$

is *surjective* since $H^4(K, \mathbb{Q}/\mathbb{Z}(2)) = 0$. Similarly, the other vertical arrows are also surjective.

- Taking Galois cohomology of the triangle (1.17) yields an exact sequence

$$H^3(K, \mathbf{R}f_{U_1*}\mathcal{Q}(2)) \rightarrow H^3(K, \mathbf{R}f_{U_0*}\mathcal{Q}(2)) \rightarrow H^2(K, \mathbf{R}f_{Z_1*}\mathcal{Q}(1)),$$

i.e. $H^3(U_1, \mathcal{Q}(2)) \rightarrow H^3(U_0, \mathcal{Q}(2)) \rightarrow H^2(Z_1, \mathcal{Q}(1))$ is exact. The horizontal rows of VF are constructed.

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- All the vertical arrows of VF are canonical projections.
- Vertical arrows of HU are induced by the canonical map $\tau_{\leq i} A^* \rightarrow A^*$ of complexes, and that of HL are obtained by the Hochschild–Serre spectral sequence $H^p(K, H^q(\overline{\mathfrak{X}}, \mathcal{Q}(i))) \Rightarrow H^{p+q}(\overline{\mathfrak{X}}, \mathcal{Q}(i))$ where \mathfrak{X} is a variety defined over K .

Now we check the commutativity of the diagram.

- VL, VM and VR commute by construction of the Hochschild–Serre spectral sequence (see [HS16, pp. 17]).
- VF obviously commutes. VB commutes because they are induced by the same distinguished triangle thanks to the commutative diagram (1.18).
- HU commutes by construction of truncation and by functoriality of Galois cohomology.

Thus the horizontal lower face HL commutes by diagram chasing. \square

Corollary 1.4.5. *The image of $\mathbb{III}_\omega^2(T')$ in $H^3(Y, \mathcal{Q}(2))/\text{Im } H^3(K, \mathcal{Q}(2))$ is unramified.*

Proof. By Lemma 1.4.4 and functoriality, the following diagram commutes:

$$\begin{array}{ccccc}
 & & \mathbb{III}_\omega^2(T') & \longrightarrow & \mathbb{III}_\omega^2(H^0(\overline{Z}_1, \mathcal{Q}(1))) \\
 & & \downarrow \iota_T & & \downarrow \iota_Z \\
 H^1(K, H^2(\overline{U}_1, \mathcal{Q}(2))) & \xrightarrow{\Phi} & H^2(K, H^1(\overline{U}_0, \mathcal{Q}(2))) & \longrightarrow & H^2(K, H^0(\overline{Z}_1, \mathcal{Q}(1))) \\
 \text{HS}_1 \downarrow & & \text{HS}_0 \downarrow & & \downarrow \\
 \frac{H^3(U_1, \mathcal{Q}(2))}{\text{Im } H^3(K, \mathcal{Q}(2))} & \xrightarrow{\Psi} & \frac{H^3(Y, \mathcal{Q}(2))}{\text{Im } H^3(K, \mathcal{Q}(2))} & \longrightarrow & H^2(Z_1, \mathcal{Q}(1)).
 \end{array}$$

Here ι_T and ι_Z are respective inclusions.

- According to Lemma 1.4.3, the second row is exact and $\mathbb{III}_\omega^2(H^0(\overline{Z}_1, \mathcal{Q}(1))) = 0$. Subsequently a diagram chasing shows that $\text{Im}(\iota_T) \subset \text{Im } \Phi$.
- By (i) and commutativity of the left lower square, $\text{Im } \text{HS}_0 \circ \iota_T \subset \text{Im } \text{HS}_0 \circ \Phi = \text{Im } \Psi \circ \text{HS}_1$.
- We show $\text{Im } \text{HS}_1 \subset \text{Ker} \left(\frac{H^3(U_1, \mathcal{Q}(2))}{H^3(K, \mathcal{Q}(2))} \rightarrow H^3(\overline{U}_1, \mathcal{Q}(2)) \right)$. Indeed, the Hochschild–Serre spectral sequence $E_2^{p,q} = H^p(K, H^q(\overline{U}_1, \mathcal{Q}(2))) \Rightarrow H^{p+q}(U_1, \mathcal{Q}(2))$ and its standard filtration $0 = F^{r+1}H^r \subset \dots \subset F^0H^r = H^r$ yield $H^3/F^1H^3 \simeq E_\infty^{0,3} = E_3^{0,3} \subset E_2^{0,3} \subset H^3(\overline{U}_1, \mathcal{Q}(2))$, and $E_\infty^{1,2} = E_2^{1,2} \simeq F^1H^3/F^2H^3$ (note that these computations work thanks to $\text{cd } K = 3$). So $E_2^{1,2}$ has trivial image in $H^3(\overline{U}_1, \mathcal{Q}(2))$. Finally, $\text{Im } \text{HS}_1 \subset \text{Im} \left(\frac{H^3(Y^c, \mathcal{Q}(2))}{H^3(K, \mathcal{Q}(2))} \rightarrow \frac{H^3(U_1, \mathcal{Q}(2))}{H^3(K, \mathcal{Q}(2))} \right)$ by Lemma 1.4.3(2).

Thus the image of $\mathbb{III}_\omega^2(T')$ in $H^3(Y, \mathcal{Q}(2))/\text{Im } H^3(K, \mathcal{Q}(2))$ comes from $H^3(Y^c, \mathcal{Q}(2))$, i.e. $\mathbb{III}_\omega^2(T')$ is unramified. In particular, the image of $\mathbb{III}^2(T')$ in $H_{\text{lc}}^3(Y, \mathcal{Q}(2))$ is unramified. \square

If we restrict ourselves to the subgroup $\mathbb{III}^2(T')$ of $\mathbb{III}_\omega^2(T')$, then the image lies in the subgroup $H_{\text{lc}}^3(Y, \mathcal{Q}/\mathbb{Z}(2))$ of $H^3(Y, \mathcal{Q}/\mathbb{Z}(2))/\text{Im } H^3(K, \mathcal{Q}/\mathbb{Z}(2))$. Now Corollary 1.4.5 implies that the image of $\mathbb{III}^2(T')$ is unramified in $H^3(K(Y), \mathcal{Q}/\mathbb{Z}(2))/H^3(K, \mathcal{Q}/\mathbb{Z}(2))$.

Chapter 2

Arithmetic duality theorems

Abstract: We develop some (Artin–Verdier style, local and global) arithmetic duality theorems for a short complex of tori paralleled to the work of Demarche [Dem11a] over number fields and Izquierdo [Izq16] over higher dimensional local fields. Later on, we construct a 12-term Poitou–Tate style exact sequence for a short complex of tori. The stated results are based on the preprints [Tia19a, Tia19b].

Keywords: arithmetic duality theorems, Poitou–Tate sequences.

We shall keep the following notation throughout this chapter. Let $C = [T_1 \xrightarrow{\rho} T_2]$ be a short complex of tori. Let $C' = [T_2' \xrightarrow{\rho'} T_1']$ be the dual of C where T_i' is the respective dual torus of T_i . Let $M = \text{Ker } \rho$ and $T = \text{Coker } \rho$. Thus M is a group of multiplicative type and T is a torus.

We fix some sufficiently small non-empty open subset X_0 of X such that T_1 and T_2 extend to X_0 -tori (in the sense of [SGA3II]) \mathcal{T}_1 and \mathcal{T}_2 , respectively. The complexes $\mathcal{C} = [\mathcal{T}_1 \rightarrow \mathcal{T}_2]$ and $\mathcal{C}' = [\mathcal{T}_2' \rightarrow \mathcal{T}_1']$ over X_0 are defined analogously (these short complexes are concentrated in degree -1 and 0). Moreover, we denote by \mathcal{M} and \mathcal{T} the X_0 -group schemes extending M and T , respectively. By [Izq16, pp. 69, Lemme 4.3] there are respective natural pairings of complexes over K and X_0 :

$$C \otimes^{\mathbf{L}} C' \rightarrow \mathbb{Z}(2)[3] \quad \text{and} \quad \mathcal{C} \otimes^{\mathbf{L}} \mathcal{C}' \rightarrow \mathbb{Z}(2)[3].$$

If $C = [0 \rightarrow T]$ consists of a single torus, then $C \otimes^{\mathbf{L}} C' \simeq T \otimes^{\mathbf{L}} T'[1] \rightarrow \mathbb{Z}(2)[3]$ is constructed in [HS16, pp. 4]. Finally, we write $\mathbb{H}_c^i(X_0, \mathcal{C}) := \mathbb{H}^i(X, j_{0*}\mathcal{C})$ for the compact support cohomology where $j_0 : X_0 \rightarrow X$ denotes the open immersion.

By construction of C , there is a distinguished triangle $T_1 \rightarrow T_2 \rightarrow C \rightarrow T_1[1]$ over K . If we pass to the finite level, there is a distinguished triangle ${}_n T_1[1] \rightarrow {}_n T_2[1] \rightarrow C \otimes^{\mathbf{L}} \mathbb{Z}/n \rightarrow {}_n T_1[2]$ (note that there is a quasi-isomorphism $P \otimes^{\mathbf{L}} \mathbb{Z}/n \simeq {}_n P[1]$ for any K -torus P). Finally, according to the construction of M and T , we obtain a distinguished triangle $M[1] \rightarrow C \rightarrow T \rightarrow M[2]$ over K .

2.1 Preliminaries on injectivity properties

In this section, we show various canonical maps $\mathbb{H}^i(\mathcal{O}_v, -) \rightarrow \mathbb{H}^i(K_v, -)$ and $\mathbb{H}^i(\mathcal{O}_v^h, -) \rightarrow \mathbb{H}^i(K_v^h, -)$ are injective. Here K_v (resp. K_v^h) denotes the completion (resp. the Henselization) of K with respect to a place $v \in X^{(1)}$ and \mathcal{O}_v (resp. \mathcal{O}_v^h) denotes the ring of integers in K_v (resp. K_v^h). We begin with the following comparison result:

Lemma 2.1.1. *The canonical maps $\mathbb{H}^i(K_v^h, C \otimes^{\mathbf{L}} \mathbb{Z}/n) \rightarrow \mathbb{H}^i(K_v, C \otimes^{\mathbf{L}} \mathbb{Z}/n)$ are isomorphisms for all $i \geq -1$.*

Proof. Let F be a finite étale commutative group scheme over K . Note that F is locally constant in the étale topology and that K_v^h and K_v have the same absolute Galois group, therefore $H^i(K_v^h, F) \simeq H^i(K_v, F)$ for any $i \in \mathbb{Z}$. Now the result follows thanks to the distinguished triangle ${}_n T_1 \rightarrow {}_n T_2 \rightarrow C \otimes^{\mathbf{L}} \mathbb{Z}/n[-1] \rightarrow {}_n T_1[1]$ by dévissage. \square

Lemma 2.1.2. *For $-2 \leq i \leq 1$, the map $\mathbb{H}^i(\mathcal{O}_v, C \otimes^{\mathbf{L}} \mathbb{Z}/n) \rightarrow \mathbb{H}^i(K_v, C \otimes^{\mathbf{L}} \mathbb{Z}/n)$ induced by the inclusion $\mathcal{O}_v \subset K_v$ is injective. Thus we may view $\mathbb{H}^i(\mathcal{O}_v, C \otimes^{\mathbf{L}} \mathbb{Z}/n)$ as a subgroup of $\mathbb{H}^i(K_v, C \otimes^{\mathbf{L}} \mathbb{Z}/n)$.*

Proof. For $i = -2$, we observe that $\mathbb{H}^{-2}(\mathcal{O}_v, \mathcal{C}) = 0$ from the distinguished triangle $\mathcal{T}_1 \rightarrow \mathcal{T}_2 \rightarrow \mathcal{C} \rightarrow \mathcal{T}_1[1]$. Since \mathcal{T}_1 is affine (hence separated), $\mathcal{T}_1(\mathcal{O}_v) \rightarrow \mathcal{T}_1(K_v)$ is injective. It follows that the homomorphism $\mathbb{H}^{-1}(\mathcal{O}_v, \mathcal{C}) \rightarrow \mathbb{H}^{-1}(K_v, \mathcal{C})$ is injective by dévissage thanks to the distinguished triangle $\mathcal{T}_1 \rightarrow \mathcal{T}_2 \rightarrow \mathcal{C} \rightarrow \mathcal{T}_1[1]$.

2.1. PRELIMINARIES ON INJECTIVITY PROPERTIES

Now we suppose $i \geq -1$. Let \overline{K}_v be an algebraic closure of K_v and let K_v^{nr} be the maximal unramified extension of K_v . According to [Mil06, II, Proposition 1.1(b)], we obtain $H^i(\mathcal{O}_v, {}_n\mathcal{P}) \simeq H^i(\kappa(v), {}_n\mathcal{P})$ for $i \geq 0$ and $\mathcal{P} = \mathcal{T}_1, \mathcal{T}_2$. It follows that for $i \geq -1$, the canonical map $\mathbb{H}^i(\mathcal{O}_v, \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n) \simeq \mathbb{H}^i(\kappa(v), \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n)$ is an isomorphism thanks to the distinguished triangle ${}_n\mathcal{T}_1[1] \rightarrow {}_n\mathcal{T}_2[1] \rightarrow \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n \rightarrow {}_n\mathcal{T}_1[2]$. Note that $\mathbb{H}^i(\kappa(v), \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n)$ is isomorphic to $\mathbb{H}^i(\text{Gal}(K_v^{\text{nr}}|K_v), \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n)$ by ramification theory. Choose an extension of v to \overline{K}_v and let I_v be the corresponding inertia group. But the short exact sequence $1 \rightarrow I_v \rightarrow \text{Gal}(\overline{K}_v|K_v) \rightarrow \text{Gal}(K_v^{\text{nr}}|K_v) \rightarrow 1$ admits a section [Ser65, II, Appendix §2], consequently $H^i(K_v^{\text{nr}}|K_v, \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n) \rightarrow H^i(K_v, \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n)$ admits a retraction, hence injective. \square

The next lemma states similar results for the complex C . Although it concerns Henselizations $\mathbb{H}^i(\mathcal{O}_v^h, \mathcal{C}) \rightarrow \mathbb{H}^i(K_v^h, \mathcal{C})$, the same argument shows that the injectivity also holds for the completions $\mathbb{H}^i(\mathcal{O}_v, \mathcal{C}) \rightarrow \mathbb{H}^i(K_v, \mathcal{C})$. More generally, we may state and prove the result for an arbitrary local Henselian integral domain.

Lemma 2.1.3. *The homomorphism $\mathbb{H}^i(\mathcal{O}_v^h, \mathcal{C}) \rightarrow \mathbb{H}^i(K_v^h, \mathcal{C})$ induced by the canonical morphism $\text{Spec } K_v^h \rightarrow \text{Spec } \mathcal{O}_v^h$ is injective for $-1 \leq i \leq 2$.*

Proof.

- (1) $i = -1$. This is already proved in the first paragraph of the previous proof.
- (2) $i = 0$. We consider the distinguished triangle $M[1] \rightarrow C \rightarrow T \rightarrow M[2]$. By dévissage, it will be sufficient to show $H^1(\mathcal{O}_v^h, \mathcal{M}) \rightarrow H^1(K_v^h, M)$ is injective. We may realize \mathcal{M} as an extension $1 \rightarrow \mathcal{P} \rightarrow \mathcal{M} \rightarrow \mathcal{F} \rightarrow 1$ of a finite group scheme \mathcal{F} by a torus \mathcal{P} over \mathcal{O}_v (since \mathcal{M} is isotrivial by [SGA3II, Chapitre X, Proposition 5.16]). Recall that $H^1(\mathcal{O}_v^h, \mathcal{P}) \rightarrow H^1(K_v^h, P)$ and $H^1(\mathcal{O}_v^h, \mathcal{F}) \rightarrow H^1(K_v^h, F)$ (see [CTS87] and [HSS15, Proposition 1.2 and 1.3]) are injective, and that $H^0(\mathcal{O}_v^h, \mathcal{F}) = H^0(K_v^h, F)$ since \mathcal{F} is a finite group scheme. It follows that $H^1(\mathcal{O}_v^h, \mathcal{M}) \rightarrow H^1(K_v^h, M)$ is injective by dévissage.
- (3) $i = 1$. Let $q : \mathcal{Q}_2 \rightarrow \mathcal{T}_2$ be an epimorphism of \mathcal{O}_v^h -tori with \mathcal{Q}_2 being quasi-trivial (for example, we may take a flasque resolution of \mathcal{T}_2 , see [CTS87, (1.3.3)]). Let $\mathcal{Q}_1 := \mathcal{Q}_2 \times_{\mathcal{T}_2} \mathcal{T}_1$ and let $\mathcal{Q}_1 \times_{\mathcal{T}_2} \mathcal{T}_1 \rightarrow \mathcal{T}_2$ be the map $(r, t_1) \mapsto q(r)\rho(t_1)^{-1}$. Let $\text{pr}_1 : \mathcal{Q}_1 \rightarrow \mathcal{T}_1$ and $\text{pr}_2 : \mathcal{Q}_1 \rightarrow \mathcal{Q}_2$ be the respective canonical projections. By construction of \mathcal{Q}_1 , we have $q \circ \text{pr}_2 = \rho \circ \text{pr}_1$. A direct verification yields isomorphisms $\text{Ker } \text{pr}_2 \simeq \text{Ker } \rho$ and $\text{Coker } \text{pr}_1 \simeq \text{Coker } \rho$, that is, $\mathcal{J}_0 = [\mathcal{Q}_1 \rightarrow \mathcal{Q}_2]$ is quasi-isomorphic to $\mathcal{J} = [\mathcal{T}_1 \rightarrow \mathcal{T}_2]$. Note that $\mathcal{Q}_2 = \mathcal{Q}_2 \times_{\mathcal{O}_v^h} K_v^h$ is a quasi-trivial K_v^h -torus and being faithfully flat is stable under base change, the same argument as above yields that $J_0 = [Q_1 \rightarrow Q_2]$ with $Q_1 = \mathcal{Q}_1 \otimes_{\mathcal{O}_v^h} K_v^h$ is quasi-isomorphic to the complex $J = [T_1 \rightarrow T_2]$. Thus it suffices to show $\mathbb{H}^1(\mathcal{O}_v^h, \mathcal{J}_0) \rightarrow \mathbb{H}^1(K_v^h, J_0)$ is injective. By construction \mathcal{Q}_2 is a quasi-trivial \mathcal{O}_v^h -torus, thus $H^1(\mathcal{O}_v^h, \mathcal{Q}_2) = H^1(\kappa(v), \mathcal{Q}_2) = 0$ and $H^1(K_v^h, Q_2) = 0$ by Shapiro's lemma and Hilbert's theorem 90. Consequently it will be sufficient to show $H^2(\mathcal{O}_v^h, \mathcal{Q}_1) \rightarrow H^2(K_v^h, Q_1)$ is injective by applying dévissage to the distinguished triangle $\mathcal{Q}_1 \rightarrow \mathcal{Q}_2 \rightarrow \mathcal{J}_0 \rightarrow \mathcal{Q}_1[1]$. Take an exact sequence $1 \rightarrow \mathcal{Q}_1 \rightarrow \mathcal{P}_1 \rightarrow \mathcal{P}_2 \rightarrow 1$ over \mathcal{O}_v^h with \mathcal{P}_1 a quasi-trivial \mathcal{O}_v^h -torus and \mathcal{P}_2 an \mathcal{O}_v^h -torus (for example, see [CTS87, pp. 158, (1.3.1)]). It induces the

commutative diagram below with exact rows

$$\begin{array}{ccccccc}
 0 & \longrightarrow & H^1(\mathcal{O}_v^h, \mathcal{P}_2) & \longrightarrow & H^2(\mathcal{O}_v^h, \mathcal{Q}_1) & \longrightarrow & H^2(\mathcal{O}_v^h, \mathcal{P}_1) \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & H^1(K_v^h, \mathcal{P}_2) & \longrightarrow & H^2(K_v^h, \mathcal{Q}_1) & \longrightarrow & H^2(K_v^h, \mathcal{P}_1)
 \end{array}$$

where $P_i = \mathcal{P}_i \times_{\mathcal{O}_v^h} K_v^h$ for $i = 1, 2$. The left vertical arrow is injective by [CTS87, Theorem 4.1] and the right one is injective by Shapiro's lemma and the injectivity for Brauer groups by [Mil80, IV, Corollary 2.6], therefore the middle one is also injective.

(4) $i = 2$. There is a commutative diagram obtained from the respective Kummer sequences

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \mathbb{H}^1(\mathcal{O}_v^h, \mathcal{C}) \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z} & \longrightarrow & \varinjlim_n \mathbb{H}^1(\mathcal{O}_v^h, \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n) & \longrightarrow & \mathbb{H}^2(\mathcal{O}_v^h, \mathcal{C}) \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & \mathbb{H}^1(K_v^h, \mathcal{C}) \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z} & \longrightarrow & \varinjlim_n \mathbb{H}^1(K_v^h, \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n) & \longrightarrow & H^2(K_v^h, \mathcal{C}) \longrightarrow 0.
 \end{array}$$

Since the groups $\mathbb{H}^1(K_v^h, \mathcal{C})$ and $\mathbb{H}^1(\mathcal{O}_v^h, \mathcal{C}) \simeq \mathbb{H}^1(\kappa(v), \mathcal{C})$ are torsion, we observe that the last arrow in both rows are isomorphisms. Since the middle vertical arrow is injective (see Lemma 2.1.2 and 2.1.1), so is the right one by diagram chasing. \square

Remark 2.1.4. To proceed, let us first briefly explain the degrees under consideration.

- (1) Let P be a K -torus. Then the groups $H^i(K, P)$ and $H^i(K_v, P)$ vanish for $i \geq 3$. See Lemma 2.3.1(3) below for details (see also [SvH03, Corollary 4.10] for the former group and see [HS16, Remark 2.3] for the latter one). Subsequently, $\mathbb{H}^i(K, \mathcal{C}) = 0$ for $i \geq 3$ thanks to the distinguished triangle $T_1 \rightarrow T_2 \rightarrow \mathcal{C} \rightarrow T_1[1]$.
- (2) The group $H^i(K_v, \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n)$ vanishes for $i \leq -3$ or $i \geq 3$. This is a direct consequence of dévissage thanks to the distinguished triangle $\mathcal{C} \rightarrow \mathcal{C} \rightarrow \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n \rightarrow \mathcal{C}[1]$.
- (3) For $v \in X_0^{(1)}$, the group $\mathbb{H}^i(\mathcal{O}_v, \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n)$ vanishes for $i \geq 2$ or $i \leq -3$. Indeed, we consider over \mathcal{O}_v the distinguished triangle ${}_n\mathcal{T}_1[1] \rightarrow {}_n\mathcal{T}_2[1] \rightarrow \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n \rightarrow {}_n\mathcal{T}_1[2]$. Therefore it will be sufficient to show $H^i(\mathcal{O}_v, {}_n\mathcal{P}) = 0$ for $i \geq 3$ and $i \leq -1$, and for any \mathcal{O}_v -torus \mathcal{P} by dévissage. Finally, we have $H^i(\mathcal{O}_v, {}_n\mathcal{P}) = H^i(\kappa(v), {}_n\mathcal{P})$ by [Mil06, II, Proposition 1.1(b)] and the latter group vanishes for $i \geq 3$ for cohomological dimension reasons (see [Ser65, Chapitre II, §5.3]), and $H^i(\mathcal{O}_v, {}_n\mathcal{P}) = 0$ for $i \leq -1$ by construction.

We denote by $\mathbb{P}^i(K, \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n) := \prod' \mathbb{H}^i(K_v, \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n)$ the restricted topological product of the finite discrete groups $\mathbb{H}^i(K_v, \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n)$ with respect to $\mathbb{H}^i(\mathcal{O}_v, \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n)$. Note that the only non-trivial degrees are $-2 \leq i \leq 2$ by Remark 2.1.4. Since $\mathbb{P}^i(K, \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n)$ is a direct limit of discrete groups, it is locally compact. Moreover, we have by Remark 2.1.4

$$\begin{aligned}
 \mathbb{P}^{-2}(K, \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n) &= \prod_{v \in X^{(1)}} \mathbb{H}^{-2}(K_v, \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n) \\
 \mathbb{P}^2(K, \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n) &= \bigoplus_{v \in X^{(1)}} \mathbb{H}^2(K_v, \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n),
 \end{aligned}$$

so $\mathbb{P}^{-2}(K, C \otimes^{\mathbf{L}} \mathbb{Z}/n)$ is profinite, and $\mathbb{P}^2(K, C \otimes^{\mathbf{L}} \mathbb{Z}/n)$ is discrete (it is a direct sum of finite groups).

Similarly, we let $\mathbb{P}^i(K, C)$ be the restricted topological product of the groups $\mathbb{H}^i(K_v, C)$ with respect to the subgroups $\mathbb{H}^i(\mathcal{O}_v, C)$ (see Lemma 2.1.3 above) for $v \in X_0^{(1)}$ and $-1 \leq i \leq 2$.

Lemma 2.1.5. *For $n \geq 1$ and $i \geq -1$, the canonical map $\mathbb{H}^i(\mathcal{O}_v, C)/n \rightarrow \mathbb{H}^i(K_v, C)/n$ induced by the inclusion $\mathbb{H}^i(\mathcal{O}_v, C) \rightarrow \mathbb{H}^i(K_v, C)$ is injective as well for $v \in X_0^{(1)}$. In particular, the map $\mathbb{P}^i(K, C)/n \rightarrow \prod_{v \in X^{(1)}} \mathbb{H}^i(K_v, C)/n$ induced by the inclusion $\mathbb{P}^i(K, C) \subset \prod_{v \in X^{(1)}} \mathbb{H}^i(K_v, C)$ is injective for $i \geq -1$. Moreover, the image is the restricted topological product of $\mathbb{H}^i(K_v, C)/n$ with respect to the subgroups $\mathbb{H}^i(\mathcal{O}_v, C)/n$, i.e. we have*

$$\mathbb{P}^i(K, C)/n \simeq \varinjlim_U \prod_{v \notin U} \mathbb{H}^i(K_v, C)/n \times \prod_{v \in U} \mathbb{H}^i(\mathcal{O}_v, C)/n.$$

Proof. Thanks to the distinguished triangle $C \rightarrow C \rightarrow C \otimes^{\mathbf{L}} \mathbb{Z}/n \rightarrow C[1]$, the injectivity of $\mathbb{H}^i(\mathcal{O}_v, C)/n \rightarrow \mathbb{H}^i(K_v, C)/n$ follows from that of $\mathbb{H}^i(\mathcal{O}_v, C \otimes^{\mathbf{L}} \mathbb{Z}/n) \rightarrow \mathbb{H}^i(K_v, C \otimes^{\mathbf{L}} \mathbb{Z}/n)$ for each $v \in X_0^{(1)}$ (the latter injectivity is ensured by Lemma 2.1.2). Now we consider the following commutative diagram

$$\begin{array}{ccc} \mathbb{P}^i(K, C) & \xlongequal{\quad\quad\quad} & \mathbb{P}^i(K, C) \\ \downarrow & & \downarrow \\ \left(\varinjlim_U \prod_{v \notin U} \mathbb{H}^i(K_v, C) \times \prod_{v \in U} \mathbb{H}^i(\mathcal{O}_v, C) \right) / n & \longrightarrow & \varinjlim_U \prod_{v \notin U} \mathbb{H}^i(K_v, C)/n \times \prod_{v \in U} \mathbb{H}^i(\mathcal{O}_v, C)/n \end{array}$$

where the lower arrow is given by $(x_v) \mapsto (\text{Im } \tilde{x}_v)$ with $(\tilde{x}_v) \in \mathbb{P}^i(K, C)$ being a lift of (x_v) and $\text{Im } \tilde{x}_v$ the image of \tilde{x}_v in $\mathbb{H}^i(K_v, C)/n$. Clearly the lower arrow is surjective. If (x_v) goes to zero, then $\tilde{x}_v \in n\mathbb{H}^i(K_v, C)$ for each v , i.e. $(x_v) = 0$. \square

2.2 Arithmetic dualities in finite level

We first develop some arithmetic duality results and a 15-term Poitou–Tate exact sequence concerning the complexes $C \otimes^{\mathbf{L}} \mathbb{Z}/n$ and $C' \otimes^{\mathbf{L}} \mathbb{Z}/n$ for any $n \geq 1$.

2.2.1 Local dualities

The following local arithmetic duality is a special case of [Izq16, pp. 73, Proposition 4.7]. We quote it here and we briefly recall the idea of the proof.

Proposition 2.2.1. *The following pairing is a functorial perfect pairing of finite groups for $i \in \mathbb{Z}$*

$$\mathbb{H}^i(K_v, C \otimes^{\mathbf{L}} \mathbb{Z}/n) \times \mathbb{H}^{-i}(K_v, C' \otimes^{\mathbf{L}} \mathbb{Z}/n) \rightarrow \mathbb{Q}/\mathbb{Z}. \quad (2.1)$$

Proof. Recall [HS16, pp. 6, pairing (10)] that there is a perfect pairing of finite groups for $j \in \mathbb{Z}$

$$H^j(K_v, {}_nT) \times H^{3-j}(K_v, {}_nT') \rightarrow \mathbb{Q}/\mathbb{Z}.$$

Therefore the distinguished triangles ${}_nT_1[1] \rightarrow {}_nT_2[1] \rightarrow C \otimes^{\mathbf{L}} \mathbb{Z}/n \rightarrow {}_nT_1[2]$ and ${}_nT'_2[1] \rightarrow {}_nT'_1[1] \rightarrow C' \otimes^{\mathbf{L}} \mathbb{Z}/n \rightarrow {}_nT'_2[2]$ yield an isomorphism $\mathbb{H}^i(K_v, C \otimes^{\mathbf{L}} \mathbb{Z}/n) \simeq \mathbb{H}^{-i}(K_v, C' \otimes^{\mathbf{L}} \mathbb{Z}/n)^D$ by dévissage (see [Izq16, pp. 73, Proposition 4.7] for details). \square

We shall need the following additional result on respective annihilators of local dualities.

Proposition 2.2.2. *The annihilator of the group $\mathbb{H}^i(\mathcal{O}_v, \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n)$ is $\mathbb{H}^{-i}(\mathcal{O}_v, \mathcal{C}' \otimes^{\mathbf{L}} \mathbb{Z}/n)$ under the perfect pairing for $-2 \leq i \leq 2$*

$$\mathbb{H}^i(K_v, C \otimes^{\mathbf{L}} \mathbb{Z}/n) \times \mathbb{H}^{-i}(K_v, C' \otimes^{\mathbf{L}} \mathbb{Z}/n) \rightarrow \mathbb{Q}/\mathbb{Z}.$$

Proof. The distinguished triangle ${}_n\mathcal{T}_1[1] \rightarrow {}_n\mathcal{T}_2[1] \rightarrow \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n \rightarrow {}_n\mathcal{T}_1[2]$ over \mathcal{O}_v yields a commutative diagram of finite groups with exact rows:

$$\begin{array}{ccccccccc} H^{i+1}(\mathcal{O}_{v,n}\mathcal{T}_1) & \longrightarrow & H^{i+1}(\mathcal{O}_{v,n}\mathcal{T}_2) & \longrightarrow & \mathbb{H}^i(\mathcal{O}_v, \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n) & \longrightarrow & H^{i+2}(\mathcal{O}_{v,n}\mathcal{T}_1) & \longrightarrow & H^{i+2}(\mathcal{O}_{v,n}\mathcal{T}_2) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow^* \\ H^{i+1}(K_{v,n}T_1) & \longrightarrow & H^{i+1}(K_{v,n}T_2) & \longrightarrow & \mathbb{H}^i(K_v, C \otimes^{\mathbf{L}} \mathbb{Z}/n) & \longrightarrow & H^{i+2}(K_{v,n}T_1) & \longrightarrow & H^{i+2}(K_{v,n}T_2) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H^{2-i}(\mathcal{O}_{v,n}\mathcal{T}'_1)^D & \longrightarrow & H^{2-i}(\mathcal{O}_{v,n}\mathcal{T}'_2)^D & \longrightarrow & \mathbb{H}^{-i}(\mathcal{O}_v, \mathcal{C}' \otimes^{\mathbf{L}} \mathbb{Z}/n)^D & \longrightarrow & H^{1-i}(\mathcal{O}_{v,n}\mathcal{T}'_1)^D & \longrightarrow & H^{1-i}(\mathcal{O}_{v,n}\mathcal{T}'_2)^D \end{array}$$

Recall that F^D denotes its dual $\text{Hom}(F, \mathbb{Q}/\mathbb{Z}(2))$ for a finite (discrete) abelian group F . For cohomological dimension reasons, the following pairing

$$\mathbb{H}^i(\mathcal{O}_{v,n}\mathcal{P}) \times \mathbb{H}^{3-i}(\mathcal{O}_{v,n}\mathcal{P}') \rightarrow \mathbb{H}^3(\mathcal{O}_v, \mathbb{Q}/\mathbb{Z}(2)) \simeq \mathbb{H}^3(\kappa(v), \mathbb{Q}/\mathbb{Z}(2))$$

is trivial for any X_0 -torus \mathcal{P} , and similarly $\mathbb{H}^i(\mathcal{O}_v, \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n) \times \mathbb{H}^{-i}(\mathcal{O}_v, \mathcal{C}' \otimes^{\mathbf{L}} \mathbb{Z}/n) \rightarrow \mathbb{Q}/\mathbb{Z}$ is also trivial. Thus the columns in the diagram are complexes. Note that it will be sufficient to consider $-2 \leq i \leq 0$ by symmetry and that the arrow $*$ is an isomorphism for $i = -2$ and is injective for $i = -1, 0$ (see [HSS15, Proposition 1.2] and its proof). In the sequel, we show the exactness of the middle column case by case.

- (1) $i = -2$. In this case, we have $H^{i+2}(\mathcal{O}_{v,n}\mathcal{T}_j) \simeq H^{i+2}(K_{v,n}T_j)$ and $H^{i+1}(\mathcal{O}_{v,n}\mathcal{T}_j) = H^{i+1}(K_{v,n}T_j) = 0$ for $j = 1, 2$. Thus exactness of the middle column fulfills after a diagram chase.
- (2) $i = -1$. The right two columns of the diagram above are exact by [HSS15, Proposition 1.2]. Moreover, we have an isomorphism $H^{i+1}(\mathcal{O}_{v,n}\mathcal{T}_2) \simeq H^{i+1}(K_{v,n}T_2)$. Now a diagram chase yields the exactness of the middle column.
- (3) $i = 0$. Note that $H^{i+1}(K_{v,n}T_1) \rightarrow H^{2-i}(\mathcal{O}_{v,n}\mathcal{T}'_1)^D$ is surjective because $H^{2-i}(\mathcal{O}_{v,n}\mathcal{T}'_1) \rightarrow H^{2-i}(K_{v,n}\mathcal{T}'_1)$ is an inclusion (see [HSS15, Proposition 1.2]) of finite groups. The exactness of the middle column follows from a diagram chase. \square

Corollary 2.2.3. *For each $i \in \mathbb{Z}$, the following pairing of locally compact topological groups induced by the local dualities is perfect*

$$\mathbb{P}^i(K, C \otimes^{\mathbf{L}} \mathbb{Z}/n) \times \mathbb{P}^{-i}(K, C' \otimes^{\mathbf{L}} \mathbb{Z}/n) \rightarrow \mathbb{Q}/\mathbb{Z}.$$

Proof. This is an immediate consequence of Proposition 2.2.1 and Proposition 2.2.2. \square

2.2.2 Global dualities

We begin with an Artin–Verdier style duality result which plays a role in the proof of the global duality

$$\mathbb{H}^i(\mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n) \times \mathbb{H}^{1-i}(\mathcal{C}' \otimes^{\mathbf{L}} \mathbb{Z}/n) \rightarrow \mathbb{Q}/\mathbb{Z}$$

for $-1 \leq i \leq 2$. We quote the following proposition [Izq16, pp. 70, I.4.4] for convenience and completeness.

Proposition 2.2.4 (Artin–Verdier duality). *Let $U \subset X_0$ be a non-empty open subset. For $i \in \mathbb{Z}$, the following is a perfect pairing between finite groups*

$$\mathbb{H}^i(U, \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n) \times \mathbb{H}_c^{1-i}(U, \mathcal{C}' \otimes^{\mathbf{L}} \mathbb{Z}/n) \rightarrow \mathbb{Q}/\mathbb{Z}.$$

Another input for the proof of global duality is the key exact sequence (2.2) below. We need the following results to assure its exactness.

Proposition 2.2.5. *Let $U \subset X_0$ be a non-empty open subset. Let \mathcal{A} be either \mathcal{C} or $\mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n$.*

(1) *Let $V \subset U$ be a further non-empty open subset. There is an exact sequence*

$$\cdots \rightarrow \mathbb{H}_c^i(V, \mathcal{A}) \rightarrow \mathbb{H}_c^i(U, \mathcal{A}) \rightarrow \bigoplus_{v \in U \setminus V} \mathbb{H}^i(\kappa(v), i_v^* \mathcal{A}) \rightarrow \mathbb{H}_c^{i+1}(V, \mathcal{A}) \rightarrow \cdots$$

where $i_v : \text{Spec } \kappa(v) \rightarrow U$ is the closed immersion.

(2) *There is an exact sequence of hypercohomology groups for $i \geq 1$ if $\mathcal{A} = \mathcal{C}$, and for $i \geq -1$ if $\mathcal{A} = \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n$:*

$$\cdots \rightarrow \mathbb{H}_c^i(U, \mathcal{A}) \rightarrow \mathbb{H}^i(U, \mathcal{A}) \rightarrow \bigoplus_{v \notin U} \mathbb{H}^i(K_v^h, \mathcal{A}) \rightarrow \mathbb{H}_c^{i+1}(U, \mathcal{A}) \rightarrow \cdots$$

where K_v^h is the Henselization of K with respect to the place v and by abuse of notation we write \mathcal{A} for the pull-back of \mathcal{A} by the natural morphism $\text{Spec } K_v^h \rightarrow U$.

(3) *There is an exact sequence for $i \geq 1$ if $\mathcal{A} = \mathcal{C}$, and for $i \geq -1$ if $\mathcal{A} = \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n$:*

$$\cdots \rightarrow \mathbb{H}_c^i(U, \mathcal{A}) \rightarrow \mathbb{H}^i(U, \mathcal{A}) \rightarrow \bigoplus_{v \notin U} \mathbb{H}^i(K_v, \mathcal{A}) \rightarrow \mathbb{H}_c^{i+1}(U, \mathcal{A}) \rightarrow \cdots$$

(4) (Three Arrows Lemma). *Let $V \subset U$ be a further non-empty open subset. We have a commutative diagram*

$$\begin{array}{ccc} \mathbb{H}_c^i(V, \mathcal{A}) & \longrightarrow & \mathbb{H}_c^i(U, \mathcal{A}) \\ \downarrow & & \downarrow \\ \mathbb{H}^i(V, \mathcal{A}) & \longleftarrow & \mathbb{H}^i(U, \mathcal{A}). \end{array}$$

Proof. Actually the proofs follow from [HS16, Proposition 3.1] after replacing cohomology by hypercohomology.

- (1) Applying [Mil80, III. Remark 1.30] to the open immersion $V \rightarrow U$ and the closed immersion $U \setminus V \rightarrow U$ yields the required long exact sequence for hypercohomology.
- (2) The long exact sequence for hypercohomology associated to the open immersion $j : U \rightarrow X$ reads as

$$\cdots \rightarrow \mathbb{H}_{X \setminus U}^i(X, j_! \mathcal{A}) \rightarrow \mathbb{H}^i(X, j_! \mathcal{A}) \rightarrow \mathbb{H}^i(U, \mathcal{A}) \rightarrow \mathbb{H}_{X \setminus U}^{i+1}(X, j_! \mathcal{A}) \rightarrow \cdots$$

Now the same argument as [Mil06, II. Lemma 2.4] implies the required long exact sequence.

- (3) By [HS16, Corollary 3.2], we know that $H^i(K_v^h, P) \simeq H^i(K_v, P)$ for any $i \geq 1$ and for any K -torus P . Thus the isomorphism $\mathbb{H}^i(K_v^h, C) \simeq \mathbb{H}^i(K_v, C)$ for each $i \geq 1$ follows after applying dévissage to the distinguished triangle $T_1 \rightarrow T_2 \rightarrow C \rightarrow T_1[1]$. Applying this to the previous long exact sequence yields the desired long exact sequence for $A = C$. For $A = C \otimes^{\mathbf{L}} \mathbb{Z}/n$, we have $\mathbb{H}^i(K_v^h, C \otimes^{\mathbf{L}} \mathbb{Z}/n) \simeq \mathbb{H}^i(K_v, C \otimes^{\mathbf{L}} \mathbb{Z}/n)$ for $i \geq -1$ by Lemma 2.1.1.
- (4) There is an isomorphism of hypercohomologies $\mathbb{H}_c^i(U, \mathcal{A}) = \mathbb{H}^i(X, j_! \mathcal{A}) \simeq \mathbf{Ext}_X^i(\mathbb{Z}, j_! \mathcal{A})$ by [Mil80, III. Remark 1.6(e)] (where $j : U \rightarrow X$ denotes the open immersion), thus the same argument as [HS16, Proposition 3.1(3)] completes the proof. \square

Put $\mathbb{D}_K^i(U, C \otimes^{\mathbf{L}} \mathbb{Z}/n) := \text{Im}(\mathbb{H}_c^i(U, C \otimes^{\mathbf{L}} \mathbb{Z}/n) \rightarrow \mathbb{H}^i(K, C \otimes^{\mathbf{L}} \mathbb{Z}/n))$. Now we arrive at the key exact sequence proving the global duality between the respective Tate–Shafarevich groups of $C \otimes^{\mathbf{L}} \mathbb{Z}/n$ and $C' \otimes^{\mathbf{L}} \mathbb{Z}/n$.

Proposition 2.2.6. *The following is an exact sequence for $-1 \leq i \leq 1$*

$$\bigoplus_{v \in X^{(1)}} \mathbb{H}^i(K_v, C \otimes^{\mathbf{L}} \mathbb{Z}/n) \rightarrow \mathbb{H}_c^{i+1}(U, C \otimes^{\mathbf{L}} \mathbb{Z}/n) \rightarrow \mathbb{D}_K^{i+1}(U, C \otimes^{\mathbf{L}} \mathbb{Z}/n) \rightarrow 0. \quad (2.2)$$

Proof. We can construct a map $\bigoplus_{v \in X^{(1)}} \mathbb{H}^i(K_v, C \otimes^{\mathbf{L}} \mathbb{Z}/n) \rightarrow \mathbb{H}_c^{i+1}(U, C \otimes^{\mathbf{L}} \mathbb{Z}/n)$ in a similar way as [HS16, pp. 11]. Let us recall the construction for the convenience of the readers. Suppose $\alpha \in \bigoplus_{v \in X^{(1)}} \mathbb{H}^i(K_v, C \otimes^{\mathbf{L}} \mathbb{Z}/n)$ lies in $\bigoplus_{v \notin V} \mathbb{H}^i(K_v, C \otimes^{\mathbf{L}} \mathbb{Z}/n)$ for some non-empty open subset V of U . By Proposition 2.2.5(3), we can send α to $\mathbb{H}_c^{i+1}(V, C \otimes^{\mathbf{L}} \mathbb{Z}/n)$ and hence to $\mathbb{H}_c^{i+1}(U, C \otimes^{\mathbf{L}} \mathbb{Z}/n)$ by covariant functoriality of $\mathbb{H}_c^{i+1}(-, C \otimes^{\mathbf{L}} \mathbb{Z}/n)$. The following commutative diagram (see [HS16, Proposition 4.2] for its commutativity) for $W \subset V$

$$\begin{array}{ccc} \bigoplus_{v \notin W} \mathbb{H}^i(K_v, C \otimes^{\mathbf{L}} \mathbb{Z}/n) & \longrightarrow & \mathbb{H}_c^{i+1}(W, C \otimes^{\mathbf{L}} \mathbb{Z}/n) \\ \uparrow & & \downarrow \\ \bigoplus_{v \notin V} \mathbb{H}^i(K_v, C \otimes^{\mathbf{L}} \mathbb{Z}/n) & \longrightarrow & \mathbb{H}_c^{i+1}(V, C \otimes^{\mathbf{L}} \mathbb{Z}/n) \end{array}$$

shows that the construction does not depend on the choice of V . Finally, the sequence (2.2) is a complex and the square in diagram (2.3) below commutes by the same argument as in the proof of [HS16, Proposition 4.2].

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Conversely, take $\alpha \in \text{Ker}(\mathbb{H}_c^{i+1}(U, \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n) \rightarrow \mathbb{D}_K^{i+1}(U, \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n))$. Let $V \subset U$ be a non-empty open subset. We consider the following diagram for $-1 \leq i \leq 1$:

$$\begin{array}{ccccc} \mathbb{H}_c^{i+1}(V, \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n) & \longrightarrow & \mathbb{H}_c^{i+1}(U, \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n) & \longrightarrow & \bigoplus_{v \in U \setminus V} \mathbb{H}^{i+1}(\kappa(v), \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n) \\ & & \downarrow & & \downarrow \\ & & \mathbb{H}^{i+1}(K, \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n) & \longrightarrow & \bigoplus_{v \in U \setminus V} \mathbb{H}^{i+1}(K_v, \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n). \end{array} \quad (2.3)$$

The upper row is exact by Proposition 2.2.5(1). The left vertical arrow is just the composition

$$\mathbb{H}_c^{i+1}(U, \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n) \rightarrow \mathbb{H}^{i+1}(U, \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n) \rightarrow \mathbb{H}^{i+1}(K, \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n).$$

The right vertical arrow is given by the composition

$$\mathbb{H}^{i+1}(\kappa(v), \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n) \simeq \mathbb{H}^{i+1}(\mathcal{O}_v, \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n) \rightarrow \mathbb{H}^{i+1}(K_v, \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n).$$

By Lemma 2.1.2, the right vertical arrow in diagram (2.3) is injective.

Finally, thanks to the exactness of the upper row in diagram (2.3), α comes from an element $\beta \in \mathbb{H}_c^{i+1}(V, \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n)$ by diagram chasing. Since β goes to zero in $\mathbb{H}^{i+1}(K, \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n)$, we may choose V sufficiently small such that β already maps to zero in $\mathbb{H}^{i+1}(V, \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n)$. Now the proof is completed by Proposition 2.2.5(3). \square

Let $\text{III}^i(\mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n) := \text{Ker}(\mathbb{H}^i(K, \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n) \rightarrow \prod_{v \in X^{(1)}} \mathbb{H}^i(K_v, \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n))$. Now we construct a perfect pairing $\text{III}^i(\mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n) \times \text{III}^{1-i}(\mathcal{C}' \otimes^{\mathbf{L}} \mathbb{Z}/n) \rightarrow \mathbb{Q}/\mathbb{Z}$ of finite groups for $i = -1, 0$.

Theorem 2.2.7. *The following is a perfect pairing of finite groups for each $i \in \mathbb{Z}$:*

$$\text{III}^i(\mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n) \times \text{III}^{1-i}(\mathcal{C}' \otimes^{\mathbf{L}} \mathbb{Z}/n) \rightarrow \mathbb{Q}/\mathbb{Z}.$$

Proof. Thanks to the distinguished triangle ${}_n T_1[1] \rightarrow {}_n T_2[1] \rightarrow \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n \rightarrow {}_n T_1[2]$, we see that $\text{III}^{-2}(\mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n) = 0$ by the injectivity of ${}_n T_1(K) \rightarrow {}_n T_1(K_v)$. For $i \leq -3$ and $i \geq 3$, we have $H^i(K, {}_n T_j) = 0$ for $i \geq 4$ and $j = 1, 2$ by Remark 2.1.4(1) and the Kummer sequences. Now it follows that $\mathbb{H}^i(K, \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n) = 0$ for $i \leq -3$ and $i \geq 3$ by the above distinguished triangle and dévissage. In particular, $\text{III}^i(\mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n) = 0$ for $i \leq -3$ and $i \geq 3$. Thus it will be sufficient to consider the cases $-1 \leq i \leq 2$.

We define¹ $D_{\text{sh}}^i(U, \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n)$ to be the kernel of the last arrow of the upper row in the following diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & D_{\text{sh}}^i(U, \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n) & \longrightarrow & \mathbb{H}^i(U, \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n) & \longrightarrow & \prod_{v \in X^{(1)}} \mathbb{H}^i(K_v, \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n) \\ & & \downarrow \text{---} & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathbb{D}_K^{1-i}(U, \mathcal{C}' \otimes^{\mathbf{L}} \mathbb{Z}/n)^D & \longrightarrow & \mathbb{H}_c^{1-i}(U, \mathcal{C}' \otimes^{\mathbf{L}} \mathbb{Z}/n)^D & \longrightarrow & \left(\bigoplus_{v \in X^{(1)}} \mathbb{H}^{-i}(K_v, \mathcal{C}' \otimes^{\mathbf{L}} \mathbb{Z}/n) \right)^D. \end{array}$$

The middle vertical arrow is an isomorphism by Proposition 2.2.4 and the same holds for the right one by Proposition 2.2.1. It follows that the left vertical arrow is an isomorphism as well.

¹Since $\mathbb{H}^i(K, \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n) \simeq \varinjlim_U \mathbb{H}^i(U, \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n)$, we have $\varinjlim_U D_{\text{sh}}^i(U, \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n) \simeq \text{III}^i(\mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n)$.

Since the group $\mathbb{H}_c^{1-i}(U, \mathcal{C}' \otimes^{\mathbf{L}} \mathbb{Z}/n)$ is finite and the functor $\mathbb{H}_c^{1-i}(-, \mathcal{C}' \otimes^{\mathbf{L}} \mathbb{Z}/n)$ is covariant, $\{\mathbb{D}_K^{1-i}(U, \mathcal{C}' \otimes^{\mathbf{L}} \mathbb{Z}/n)\}_{U \subset X_0}$ forms a decreasing family of finite abelian groups. Thus there exists a non-empty open subset $U_0 \subset X_0$ such that $\mathbb{D}_K^{1-i}(U, \mathcal{C}' \otimes^{\mathbf{L}} \mathbb{Z}/n) = \mathbb{D}_K^{1-i}(U_0, \mathcal{C}' \otimes^{\mathbf{L}} \mathbb{Z}/n)$ for all non-empty open subset $U \subset U_0$ (here we used Proposition 2.2.5(3) implicitly), i.e. we deduce that $\mathbb{D}_K^{1-i}(U, \mathcal{C}' \otimes^{\mathbf{L}} \mathbb{Z}/n) \simeq \mathbb{H}^{1-i}(C' \otimes^{\mathbf{L}} \mathbb{Z}/n)$ for all $U \subset U_0$. Now passing to the direct limit over all U of the isomorphism $\mathbb{D}_{\text{sh}}^i(U, \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n) \simeq \mathbb{H}^{1-i}(C' \otimes^{\mathbf{L}} \mathbb{Z}/n)^D$ yields $\mathbb{H}^i(C \otimes^{\mathbf{L}} \mathbb{Z}/n) \simeq \varinjlim_n \mathbb{D}_{\text{sh}}^i(U, \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n) \simeq \mathbb{H}^{1-i}(C' \otimes^{\mathbf{L}} \mathbb{Z}/n)^D$. \square

2.2.3 The Poitou–Tate sequence

Lemma 2.2.8. *Let \mathcal{A} be either \mathcal{C} or $\mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n$ over $U \subset X_0$ and let A be its generic fibre. For $V \subset U$, if $\alpha \in \mathbb{H}^i(V, \mathcal{A})$ is such that $\alpha_v \in \mathbb{H}^i(K_v, A)$ belongs to $\mathbb{H}^i(\mathcal{O}_v, \mathcal{A})$ for all $v \in U \setminus V$, then $\alpha \in \text{Im}(\mathbb{H}^i(U, \mathcal{A}) \rightarrow \mathbb{H}^i(V, \mathcal{A}))$ for $i \in \mathbb{Z}$.*

Proof. The localization sequences [Fu11, Proposition 5.6.11] for the respective pairs of open immersions $V \subset U$ and $\text{Spec } K_v \subset \text{Spec } \mathcal{O}_v$ (actually here we do the same argument as loc. cit. by replacing injective resolutions by injective Cartan–Eilenberg resolutions) together with [Mil80, pp. 93, 1.28] induce the following commutative diagram with exact rows²

$$\begin{array}{ccccc} \mathbb{H}^i(U, \mathcal{A}) & \longrightarrow & \mathbb{H}^i(V, \mathcal{A}) & \longrightarrow & \bigoplus_{v \in U \setminus V} \mathbb{H}_v^{i+1}(\mathcal{O}_v^h, \mathcal{A}) \\ \downarrow & & \downarrow & & \downarrow \\ \bigoplus_{v \in U \setminus V} \mathbb{H}^i(\mathcal{O}_v, \mathcal{A}) & \longrightarrow & \bigoplus_{v \in U \setminus V} \mathbb{H}^i(K_v, A) & \longrightarrow & \bigoplus_{v \in U \setminus V} \mathbb{H}_v^{i+1}(\mathcal{O}_v, \mathcal{A}). \end{array}$$

By [DH18, Lemma 2.6]³ the right vertical map is an isomorphism, so a diagram chasing yields the desired result. \square

Lemma 2.2.9. *There are exact sequences for $n \geq 1$ and $-2 \leq i \leq 2$:*

$$\mathbb{H}^i(K, C \otimes^{\mathbf{L}} \mathbb{Z}/n) \rightarrow \mathbb{P}^i(K, C \otimes^{\mathbf{L}} \mathbb{Z}/n) \rightarrow \mathbb{H}^{-i}(K, C' \otimes^{\mathbf{L}} \mathbb{Z}/n)^D.$$

Proof. For $V \subset U \subset X_0$ and $-1 \leq i \leq 2$, we obtain an exact sequence

$$\mathbb{H}^i(V, \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n) \rightarrow \prod_{v \notin V} \mathbb{H}^i(K_v, C \otimes^{\mathbf{L}} \mathbb{Z}/n) \rightarrow \mathbb{H}_c^{i+1}(V, \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n)$$

by Proposition 2.2.5(3). Subsequently the following is an exact sequence by Lemma 2.2.8

$$\mathbb{H}^i(U, \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n) \rightarrow \prod_{v \notin U} \mathbb{H}^i(K_v, C \otimes^{\mathbf{L}} \mathbb{Z}/n) \times \prod_{v \in U \setminus V} \mathbb{H}^i(\mathcal{O}_v, \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n) \rightarrow \mathbb{H}_c^{i+1}(V, \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n).$$

²Here we have used the fact that $\text{Cone}(\mathbf{R}\Gamma_{U \setminus V} \rightarrow \mathbf{R}\Gamma_U)$ agrees with $\mathbf{R}\Gamma_V$. Indeed, it suffices to show that Cone preserves triangles and satisfies the desired universal property. It preserves triangles since it forms a functor on the derived category and it satisfies the desired universal property because it agrees with $\mathbf{R}\Gamma_V$ for sheaves by [Mil80, pp. 93, 1.28]. Therefore we can pass from a single sheaf to a short complex.

³Actually here the commutativity is simpler because we can work in the étale cohomology rather than the fppf cohomology.

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By Artin–Verdier duality 2.2.4, we obtain an isomorphism

$$\mathbb{H}_c^{i+1}(V, \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n) \simeq \mathbb{H}^{-i}(V, \mathcal{C}' \otimes^{\mathbf{L}} \mathbb{Z}/n)^D.$$

Taking inverse limit⁴ of the above exact sequence over V yields an exact sequence

$$\mathbb{H}^i(U, \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n) \rightarrow \prod_{v \notin U} \mathbb{H}^i(K_v, \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n) \times \prod_{v \in U} \mathbb{H}^i(\mathcal{O}_v, \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n) \rightarrow \mathbb{H}^{-i}(K, \mathcal{C}' \otimes^{\mathbf{L}} \mathbb{Z}/n)^D.$$

Now we conclude the desired exact sequence by taking direct limit over U .

In particular, we obtain an exact sequence $\mathbb{H}^2(K, \mathcal{C}' \otimes^{\mathbf{L}} \mathbb{Z}/n) \rightarrow \mathbb{P}^2(K, \mathcal{C}' \otimes^{\mathbf{L}} \mathbb{Z}/n) \rightarrow \mathbb{H}^{-2}(K, \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n)^D$ by applying the case $i = 2$ to \mathcal{C}' . It follows that there are exact sequences

$$\mathbb{H}^{-2}(K, \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n) \rightarrow \mathbb{P}^{-2}(K, \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n) \rightarrow \mathbb{H}^2(K, \mathcal{C}' \otimes^{\mathbf{L}} \mathbb{Z}/n)^D$$

by dualizing the above sequence of discrete abelian groups (recall that double dual of a finite abelian group is itself). \square

To close this section, we summarize all the above arithmetic dualities into a 15-term exact sequence as follows.

Theorem 2.2.10. *The following is a 15-term exact sequence for $n \geq 1$*

$$\begin{array}{l} 0 \longrightarrow \mathbb{H}^{-2}(K, \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n) \longrightarrow \mathbb{P}^{-2}(K, \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n) \longrightarrow \mathbb{H}^2(K, \mathcal{C}' \otimes^{\mathbf{L}} \mathbb{Z}/n)^D \\ \left(\longrightarrow \mathbb{H}^{-1}(K, \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n) \longrightarrow \mathbb{P}^{-1}(K, \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n) \longrightarrow \mathbb{H}^1(K, \mathcal{C}' \otimes^{\mathbf{L}} \mathbb{Z}/n)^D \right) \\ \left(\longrightarrow \mathbb{H}^0(K, \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n) \longrightarrow \mathbb{P}^0(K, \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n) \longrightarrow \mathbb{H}^0(K, \mathcal{C}' \otimes^{\mathbf{L}} \mathbb{Z}/n)^D \right) \\ \left(\longrightarrow \mathbb{H}^1(K, \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n) \longrightarrow \mathbb{P}^1(K, \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n) \longrightarrow \mathbb{H}^{-1}(K, \mathcal{C}' \otimes^{\mathbf{L}} \mathbb{Z}/n)^D \right) \\ \left(\longrightarrow \mathbb{H}^2(K, \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n) \longrightarrow \mathbb{P}^2(K, \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n) \longrightarrow \mathbb{H}^{-2}(K, \mathcal{C}' \otimes^{\mathbf{L}} \mathbb{Z}/n)^D \longrightarrow 0 \right) \end{array} \quad (2.4)$$

Proof. The injectivity of the first arrow is a direct consequence of the injectivity of ${}_nT_1(K) \rightarrow {}_nT_1(K_v)$ in view of the distinguished triangle ${}_nT_1[1] \rightarrow {}_nT_2[1] \rightarrow \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n \rightarrow {}_nT_1[2]$. The surjectivity of the last arrow follows from dualizing the injective map $\mathbb{H}^{-2}(K, \mathcal{C}' \otimes^{\mathbf{L}} \mathbb{Z}/n) \rightarrow \mathbb{P}^{-2}(K, \mathcal{C}' \otimes^{\mathbf{L}} \mathbb{Z}/n)$.

Next, we show that the map $\mathbb{H}^i(K, \mathcal{C}' \otimes^{\mathbf{L}} \mathbb{Z}/n) \rightarrow \mathbb{P}^i(K, \mathcal{C}' \otimes^{\mathbf{L}} \mathbb{Z}/n)$ has discrete image for $-1 \leq i \leq 2$. Since $\mathbb{P}^2(K, \mathcal{C}' \otimes^{\mathbf{L}} \mathbb{Z}/n)$ itself is discrete, there is nothing to do. For $i = 0, \pm 1$, suppose $\alpha \in \text{Im}(\mathbb{H}^i(K, \mathcal{C}' \otimes^{\mathbf{L}} \mathbb{Z}/n) \rightarrow \mathbb{P}^i(K, \mathcal{C}' \otimes^{\mathbf{L}} \mathbb{Z}/n))$ lies in $\prod_{v \notin U} \mathbb{H}^i(K_v, \mathcal{C}' \otimes^{\mathbf{L}} \mathbb{Z}/n) \times \prod_{v \in U} \mathbb{H}^i(\mathcal{O}_v, \mathcal{C}' \otimes^{\mathbf{L}} \mathbb{Z}/n)$ for some $U \subset X_0$. Then α comes from the finite group $\mathbb{H}^i(U, \mathcal{C}' \otimes^{\mathbf{L}} \mathbb{Z}/n)$ by Lemma 2.2.8. Therefore the intersection of $\text{Im}(\mathbb{H}^i(K, \mathcal{C}' \otimes^{\mathbf{L}} \mathbb{Z}/n) \rightarrow \mathbb{P}^i(K, \mathcal{C}' \otimes^{\mathbf{L}} \mathbb{Z}/n))$ with any open subset of $\mathbb{P}^i(K, \mathcal{C}' \otimes^{\mathbf{L}} \mathbb{Z}/n)$ is finite, i.e. $\mathbb{H}^i(K, \mathcal{C}' \otimes^{\mathbf{L}} \mathbb{Z}/n)$ has discrete image. Now dualizing the exact sequences

$$0 \rightarrow \mathbb{H}^i(\mathcal{C}' \otimes^{\mathbf{L}} \mathbb{Z}/n) \rightarrow \mathbb{H}^i(K, \mathcal{C}' \otimes^{\mathbf{L}} \mathbb{Z}/n) \rightarrow \mathbb{P}^i(K, \mathcal{C}' \otimes^{\mathbf{L}} \mathbb{Z}/n)$$

yields the exactness at all the remaining terms. \square

⁴Note that $\mathbb{H}^i(\mathcal{O}_v, \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n)$ and $\mathbb{H}_c^{i+1}(V, \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n)$ are finite groups, so taking inverse limit keeps the exactness.

2.3 Results for complexes of tori

We begin with a list of properties of abelian groups under consideration.

Lemma 2.3.1. *Let P be a K -torus extending to a U_0 -torus \mathcal{P} for some sufficiently small non-empty open subset U_0 of X . Let U be a non-empty open subset of U_0 . Let L be either K or K_v .*

- (1) *The torsion groups $\mathbb{H}^1(U, \mathcal{C})_{\text{tors}}$ and $\mathbb{H}_c^1(U, \mathcal{C})_{\text{tors}}$ are of cofinite type.*
- (2) *For $i \geq 2$, the groups $\mathbb{H}^i(U, \mathcal{C})$ and $\mathbb{H}_c^i(U, \mathcal{C})$ are torsion of cofinite type.*
- (3) *The group $H^1(K, P)$ has finite exponent and the group $H^1(K_v, P)$ is finite. Moreover, $H^i(L, P) = 0$ for $i \geq 3$. Finally, the groups $\mathbb{H}^i(P)$ are finite for each $i \geq 0$.*
- (4) *Let Φ be a group of multiplicative type over K . Then the groups $H^1(L, \Phi)$ and $H^3(L, \Phi)$ have finite exponents.*
- (5) *Suppose $M := \text{Ker } \rho$ is finite. Then the groups $\mathbb{H}^{-1}(K_v, C)$ have a common finite exponent for all $v \in X^{(1)}$. Moreover, the groups $\mathbb{H}^{-1}(K, C)$ and $\mathbb{H}^1(K, C)$ are torsion of finite exponent.*
- (6) *Suppose $T := \text{coker } \rho$ is trivial. The groups $\mathbb{H}^0(K, C)$ and $\mathbb{H}^2(K, C)$ are torsion of finite exponent.*

Proof.

- (1) The first statement is a consequence of the exact sequence

$$0 \rightarrow \mathbb{H}^0(U, \mathcal{C})/n \rightarrow \mathbb{H}^0(U, \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n) \rightarrow {}_n\mathbb{H}^1(U, \mathcal{C}) \rightarrow 0$$

induced by the distinguished triangle $\mathcal{C} \rightarrow \mathcal{C} \rightarrow \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n \rightarrow \mathcal{C}[1]$. The same argument works for $\mathbb{H}_c^1(U, \mathcal{C})_{\text{tors}}$.

- (2) By [HS16, Corollary 3.3 and Proposition 3.4(1)], the groups $H^i(U, \mathcal{T}_2)$ and $H^{i+1}(U, \mathcal{T}_1)$ are torsion of cofinite type for $i \geq 2$. Now we deduce that $\mathbb{H}^i(U, \mathcal{C})$ is torsion by the exactness of $H^i(U, \mathcal{T}_2) \rightarrow \mathbb{H}^i(U, \mathcal{C}) \rightarrow H^{i+1}(U, \mathcal{T}_1)$. The group $\mathbb{H}^i(U, \mathcal{C})$ is of cofinite type thanks to the short exact sequence $0 \rightarrow \mathbb{H}^{i-1}(U, \mathcal{C})/n \rightarrow \mathbb{H}^{i-1}(U, \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n) \rightarrow {}_n\mathbb{H}^i(U, \mathcal{C}) \rightarrow 0$. The same argument works for $\mathbb{H}_c^i(U, \mathcal{C})$.
- (3) For $L = K$ or K_v , the group $H^1(L, P)$ has finite exponent is a direct consequence of Hilbert's theorem 90. Moreover, the group $H^1(K_v, P)$ is of cofinite type because there is a surjective map $H^1(K_v, {}_nP) \rightarrow {}_nH^1(K_v, P)$ induced by the Kummer sequence.

The group $H^3(K, P)$ is the direct limit of the groups $H^3(V, \mathcal{P})$ for $V \subset U_0$, but by [SvH03, Corollary 4.10] $H^3(V, \mathcal{P}) = 0$ for V sufficiently small and so $H^3(K, P) = 0$. For the group $H^3(K_v, P)$, we deduce that $H^3(K_v, P) \simeq \varinjlim_n H^3(K_v, {}_nP)$ from the Kummer sequence. Thus it suffices to show that $\varinjlim_n H^0(K_v, {}_nP') = 0$ by [HS16, (10)]. Note that $(K_v^\times)_{\text{tors}} = (\kappa(v)^\times)_{\text{tors}}$ is finite, so $H^0(K_v, P')_{\text{tors}}$ is finite as well by a restriction-corestriction argument. As a consequence, $H^0(K_v, {}_nP')$ has a common finite exponent

for each n and it follows that the limit $\varprojlim_n H^0(K_v, {}_n P') = 0$ vanishes. For cohomological dimension reasons, we have $H^i(L, {}_n P) = 0$ for $i \geq 4$. Subsequently, we see that $H^i(L, P) \simeq \varinjlim_n H^i(L, {}_n P) = 0$ for $i \geq 4$ where the first isomorphism follows from the Kummer sequence $0 \rightarrow H^{i-1}(L, P)/n \rightarrow H^i(L, {}_n P) \rightarrow {}_n H^i(L, P) \rightarrow 0$.

Finally, the finiteness of $\mathbb{H}^i(P)$ follows from [HS16, Proposition 3.4(2)].

- (4) Embed Φ into a short exact sequence $0 \rightarrow P \rightarrow \Phi \rightarrow F \rightarrow 0$ where P is an L -torus and F is a finite étale commutative group scheme. Thus there is an exact sequence $H^i(L, P) \rightarrow H^i(L, \Phi) \rightarrow H^i(L, F)$ for $i \geq 1$. By dévissage, it follows that $H^1(L, \Phi)$ has finite exponent by Hilbert's Theorem 90 and so does $H^3(L, \Phi)$ by $H^3(L, P) = 0$.
- (5) Note that there is an isomorphism $\mathbb{H}^{-1}(K_v, C) \simeq \mathbb{H}^0(K_v, M)$ thanks to the distinguished triangle $M[1] \rightarrow C \rightarrow T \rightarrow M[2]$. Since M is finite by assumption, $\mathbb{H}^{-1}(K_v, C)$ has a common finite exponent for each $v \in X^{(1)}$. The group $\mathbb{H}^{-1}(K, C)$ has finite exponent for the same reason. Thanks to the exact sequence $H^2(K, M) \rightarrow \mathbb{H}^1(K, C) \rightarrow H^1(K, T)$, we deduce that $\mathbb{H}^1(K, C)$ have finite exponent by dévissage.
- (6) In this case, the short complex C is quasi-isomorphic to $M[1]$. Thus the desired results follow from (4). \square

Remark 2.3.2. Note that the finiteness of $\text{Ker } \rho$ is equivalent to the finiteness of the cokernel $\text{Coker}(\mathbf{X}^*(T_2) \rightarrow \mathbf{X}^*(T_1))$, and hence it is equivalent to the injectivity of $\mathbf{X}_*(T_1) \rightarrow \mathbf{X}_*(T_2)$. Therefore the finiteness of $\text{Ker } \rho$ amounts to saying that $\rho' : T_2' \rightarrow T_1'$ is surjective, and vice versa. By Lemma 2.3.1(5,6), we see that

- If $\text{Ker } \rho$ is finite, then $\mathbb{H}^{-1}(K, C)$, $\mathbb{H}^0(K, C')$, $\mathbb{H}^1(K, C)$ and $\mathbb{H}^2(K, C')$ are torsion of finite exponent.
- If $\text{Coker } \rho$ is trivial, then $\mathbb{H}^{-1}(K, C')$, $\mathbb{H}^0(K, C)$, $\mathbb{H}^1(K, C')$ and $\mathbb{H}^2(K, C)$ are torsion of finite exponent.

2.3.1 An Artin–Verdier style duality

The following result is some sort of variation of the classical Artin–Verdier duality theorem, which provides a more precise statement concerning the ℓ -primary part.

Proposition 2.3.3. *Let $U \subset X_0$ be any non-empty open subset. For $0 \leq i \leq 2$, there is a pairing with divisible left kernel for each prime number ℓ ,*

$$\mathbb{H}^i(U, \mathcal{C})\{\ell\} \times \mathbb{H}_c^{2-i}(U, \mathcal{C}')^{(\ell)} \rightarrow \mathbb{Q}/\mathbb{Z}.$$

Proof. First, recall [Izq16, Proposition 1.4.4] that there is a perfect pairing of finite groups

$$\mathbb{H}^i(U, \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n) \times \mathbb{H}_c^{1-i}(U, \mathcal{C}' \otimes^{\mathbf{L}} \mathbb{Z}/n) \rightarrow \mathbb{Q}/\mathbb{Z} \quad (2.5)$$

for $i \in \mathbb{Z}$. The pairing $\mathcal{C} \otimes^{\mathbf{L}} \mathcal{C}' \rightarrow \mathbb{Z}(2)[3]$ induces a pairing $\mathbb{H}^i(U, \mathcal{C}) \times \mathbb{H}_c^{2-i}(U, \mathcal{C}') \rightarrow \mathbb{Q}/\mathbb{Z}$ by [HS16, Lemma 1.1]. In particular, we obtain pairings

$${}_{\ell^n} \mathbb{H}^i(U, \mathcal{C}) \times \mathbb{H}_c^{2-i}(U, \mathcal{C}')/\ell^n \rightarrow \mathbb{Q}/\mathbb{Z} \quad \text{and} \quad \mathbb{H}^i(U, \mathcal{C})/\ell^n \times {}_{\ell^n} \mathbb{H}^{2-i}(U, \mathcal{C}') \rightarrow \mathbb{Q}/\mathbb{Z}$$

which fit into the following commutative diagram with exact rows (it commutes by functoriality of the cup-product analogous to [HS16, diagram (26)]):

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \mathbb{H}^{i-1}(U, \mathcal{C})/\ell^n & \longrightarrow & \mathbb{H}^{i-1}(U, \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/\ell^n) & \longrightarrow & \ell^n \mathbb{H}^i(U, \mathcal{C}) \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & (\ell^n \mathbb{H}_c^{3-i}(U, \mathcal{C}'))^D & \longrightarrow & \mathbb{H}_c^{2-i}(U, \mathcal{C}' \otimes^{\mathbf{L}} \mathbb{Z}/\ell^n)^D & \longrightarrow & (\mathbb{H}_c^{2-i}(U, \mathcal{C}')/\ell^n)^D \longrightarrow 0.
 \end{array}$$

Since the middle vertical arrow is an isomorphism by the pairing (2.5), we obtain an isomorphism by snake lemma

$$\mathbb{K}_n^i(U) \simeq \text{Coker} \left(\mathbb{H}^{i-1}(U, \mathcal{C})/\ell^n \rightarrow (\ell^n \mathbb{H}_c^{3-i}(U, \mathcal{C}'))^D \right),$$

where $\mathbb{K}_n^i(U) = \text{Ker} \left(\ell^n \mathbb{H}^i(U, \mathcal{C}) \rightarrow (\mathbb{H}_c^{2-i}(U, \mathcal{C}')/\ell^n)^D \right)$. Taking direct limit over all n yields an isomorphism

$$\varinjlim_n \mathbb{K}_n^i(U) \simeq \varinjlim_n \text{Coker} \left(\mathbb{H}^{i-1}(U, \mathcal{C})/\ell^n \rightarrow (\ell^n \mathbb{H}_c^{3-i}(U, \mathcal{C}'))^D \right).$$

The latter limit is a quotient of the divisible group $\varinjlim_n (\ell^n \mathbb{H}_c^{3-i}(U, \mathcal{C}'))^D \simeq \left(\varprojlim_n \ell^n \mathbb{H}_c^{3-i}(U, \mathcal{C}') \right)^D$, so it is also divisible. Indeed, since $\mathbb{H}_c^{3-i}(U, \mathcal{C}')\{\ell\}$ is a torsion group of cofinite type, so it is of the form $(\mathbb{Q}_\ell/\mathbb{Z}_\ell)^{\oplus r} \oplus F_\ell$ where F_ℓ is a finite ℓ -group. Thus the dual of its Tate module $\varprojlim_n \ell^n \mathbb{H}_c^{3-i}(U, \mathcal{C}')$ is a direct sum of copies of $\mathbb{Q}_\ell/\mathbb{Z}_\ell$, i.e. $\left(\varprojlim_n \ell^n \mathbb{H}_c^{3-i}(U, \mathcal{C}') \right)^D$ is divisible. Being isomorphic to a quotient of the divisible group $\left(\varprojlim_n \ell^n \mathbb{H}_c^{2-i}(U, \mathcal{C}') \right)^D$, we see that $\varinjlim_n \mathbb{K}_n^i(U)$ is divisible as well. Passing to the direct limit over all n yield an exact sequence (by definition of $\mathbb{K}_n^i(U)$ and exactness of direct limit) of abelian groups

$$0 \rightarrow \varinjlim_n \mathbb{K}_n^i(U) \rightarrow \mathbb{H}^i(U, \mathcal{C})\{\ell\} \rightarrow (\mathbb{H}_c^{2-i}(U, \mathcal{C}')^{(\ell)})^D$$

which guarantees the required pairing having divisible left kernel. \square

Remark 2.3.4. We shall see later in Theorem 2.3.19 that there exists a non-empty open subset U_0 of X_0 such that the induced map $\mathbb{H}^1(U, \mathcal{C})\{\ell\} \rightarrow (\mathbb{H}_c^1(U, \mathcal{C}')^{(\ell)})^D$ is an isomorphism for each $U \subset U_0$, because the direct limit $\varinjlim_n \mathbb{K}_n^i(U)$ is contained in a finite group (so it vanishes as it a finite divisible group).

2.3.2 Local dualities

In this subsection, we prove local dualities for the completion K_v and the Henselization K_v^h with respect to v .

Proposition 2.3.5 (Local dualities). *Let ℓ be a prime number.*

- (1) *There is a perfect pairing functorial in C between discrete and profinite groups:*

$$\mathbb{H}^1(K_v, C) \times \mathbb{H}^0(K_v, C')^\wedge \rightarrow \mathbb{Q}/\mathbb{Z}.$$

- (2) *There is a perfect pairing functorial in C between finite groups:*

$$\ell^n \mathbb{H}^1(K_v, C) \times \mathbb{H}^0(K_v, C')/\ell^n \rightarrow \mathbb{Q}/\mathbb{Z}.$$

Proof.

(1) The distinguished triangle $T'_2 \rightarrow T'_1 \rightarrow C' \rightarrow T'_2[1]$ induces an exact sequence

$$H^0(K_v, T'_2) \rightarrow H^0(K_v, T'_1) \rightarrow \mathbb{H}^0(K_v, C') \rightarrow H^1(K_v, T'_2) \rightarrow H^1(K_v, T'_1).$$

Since $H^1(K_v, T'_2)$ is finite by Lemma 2.3.1(3), by [HS05, Appendix, Proposition] there is an exact sequence

$$H^0(K_v, T'_1)^\wedge \rightarrow \mathbb{H}^0(K_v, C')^\wedge \rightarrow H^1(K_v, T'_2) \rightarrow H^1(K_v, T'_1)$$

and a complex

$$H^0(K_v, T'_2)^\wedge \rightarrow H^0(K_v, T'_1)^\wedge \rightarrow \mathbb{H}^0(K_v, C')^\wedge.$$

Now the statement follows from the same argument as [Izq16, Proposition 1.4.9(ii)].

(2) Consider the following exact commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{H}^0(K_v, C)/\ell^n & \longrightarrow & \mathbb{H}^0(K_v, C \otimes^{\mathbf{L}} \mathbb{Z}/\ell^n) & \longrightarrow & \ell^n \mathbb{H}^1(K_v, C) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & (\ell^n \mathbb{H}^1(K_v, C'))^D & \longrightarrow & \mathbb{H}^0(K_v, C' \otimes^{\mathbf{L}} \mathbb{Z}/\ell^n)^D & \longrightarrow & (\mathbb{H}^0(K_v, C')/\ell^n)^D \longrightarrow 0 \end{array}$$

with the middle vertical arrow being an isomorphism of finite groups (see the proof of [Izq16, Proposition 1.4.9(i)]). It follows that the right vertical arrow is surjective. Moreover, we have a commutative diagram

$$\begin{array}{ccc} \ell^n \mathbb{H}^1(K_v, C) & \longrightarrow & (\mathbb{H}^0(K_v, C')/\ell^n)^D \\ \downarrow & & \downarrow \\ \mathbb{H}^1(K_v, C) & \longrightarrow & (\mathbb{H}^0(K_v, C')^\wedge)^D \end{array}$$

where the lower horizontal arrow is injective by (1). Therefore the upper horizontal arrow is also injective and hence it is an isomorphism. \square

Remark 2.3.6. By a similar argument as Proposition 2.3.5(2), we obtain a perfect pairing between profinite and discrete groups

$$\mathbb{H}^0(K_v, C)_\wedge \times \mathbb{H}^1(K_v, C') \rightarrow \mathbb{Q}/\mathbb{Z}.$$

So we can identify $\mathbb{H}^0(K_v, C)_\wedge$ with the profinite completion $\mathbb{H}^0(K_v, C)^\wedge$ by Proposition 2.3.5(1).

Corollary 2.3.7. *Let ℓ be a prime number.*

(1) *There is a perfect pairing between discrete and profinite groups:*

$$\mathbb{H}^1(K_v, C)\{\ell\} \times \mathbb{H}^0(K_v, C')^{(\ell)} \rightarrow \mathbb{Q}/\mathbb{Z}.$$

(2) *There is a perfect pairing between locally compact groups:*

$$\left(\prod_{v \in X^{(1)}} \mathbb{H}^1(K_v, C) \right) \{\ell\} \times \left(\bigoplus_{v \in X^{(1)}} \mathbb{H}^0(K_v, C') \right)^{(\ell)} \rightarrow \mathbb{Q}/\mathbb{Z}.$$

More precisely, the former group is a direct limit of profinite groups and the latter is a projective limit of discrete torsion groups.

Proof. We apply the local duality Proposition 2.3.5(2), i.e. the isomorphism ${}_{\ell^n} \mathbb{H}^1(K_v, C) \simeq (\mathbb{H}^0(K_v, C')/\ell^n)^D$.

(1) Passing to the direct limit over all n yields $\mathbb{H}^1(K_v, C)\{\ell\} \simeq (\mathbb{H}^0(K_v, C')^{(\ell)})^D$.

(2) Taking product over all places gives isomorphisms

$${}_{\ell^n} \left(\prod_v \mathbb{H}^1(K_v, C) \right) \simeq \prod_v (\mathbb{H}^0(K_v, C')/\ell^n)^D \simeq \left(\left(\bigoplus_v \mathbb{H}^0(K_v, C') \right) / \ell^n \right)^D.$$

Thus the desired perfect pairing follows by passing to the direct limit over all $n \geq 1$. \square

Remark 2.3.8. Analogously, there is a perfect pairing between locally compact groups

$$\left(\prod_{v \in X^{(1)}} \mathbb{H}^0(K_v, C) \right) \{\ell\} \times \left(\bigoplus_{v \in X^{(1)}} \mathbb{H}^1(K_v, C') \right)^{(\ell)} \rightarrow \mathbb{Q}/\mathbb{Z}.$$

More precisely, the former group is a direct limit of profinite groups and the latter is a projective limit of discrete torsion groups.

The next lemma is probably well-known:

Lemma 2.3.9. *Let $A_1 \rightarrow A_2 \rightarrow A_3 \rightarrow 0$ be an exact sequence of abelian groups. If ${}_{\ell} A_3$ is finite, then $A_1^{(\ell)} \rightarrow A_2^{(\ell)} \rightarrow A_3^{(\ell)} \rightarrow 0$ is exact for each prime number ℓ .*

Proof. Let's say $f : A_1 \rightarrow A_2$, $g : A_2 \rightarrow A_3$ and $g_n : A_2/\ell^n \rightarrow A_3/\ell^n$. Thus there is a short exact sequence $0 \rightarrow \text{Ker } g_n \rightarrow A_2/\ell^n \rightarrow A_3/\ell^n \rightarrow 0$. Since $\text{Ker } g_n$ is a quotient of A_1/ℓ^n , $\{\text{Ker } g_n\}$ forms a surjective system in the sense of [AM69, Proposition 10.2] and it follows that $0 \rightarrow \varprojlim \text{Ker } g_n \rightarrow A_2^{(\ell)} \rightarrow A_3^{(\ell)} \rightarrow 0$ is exact. By the snake lemma, there is an exact sequence $0 \rightarrow {}_{\ell^n} \text{Ker } g \rightarrow {}_{\ell^n} A_2 \rightarrow {}_{\ell^n} A_3 \rightarrow (\text{Ker } g)/\ell^n \rightarrow \text{Ker } g_n \rightarrow 0$. But ${}_{\ell^n} A_3$ is finite by assumption, we conclude that $(\text{Ker } g)^{(\ell)} \rightarrow \varprojlim \text{Ker } g_n$ is surjective by Mittag-Leffler condition. Finally, let $\text{Ker } f_n := \text{Ker} (A_1/\ell^n \rightarrow (\text{Ker } g)/\ell^n)$. Then $\{\text{Ker } f_n\}$ is a surjective system (because $\text{Ker } f/\ell^n \rightarrow \text{Ker } f_n$ is surjective), and hence $A_1^{(\ell)} \rightarrow (\text{Ker } g)^{(\ell)}$ is surjective. Summing up, the sequence $A_1^{(\ell)} \rightarrow A_2^{(\ell)} \rightarrow A_3^{(\ell)} \rightarrow 0$ is exact. \square

Lemma 2.3.10. *Let ℓ be a prime number. Let T be a K -torus and let $C = [T_1 \rightarrow T_2]$ be as above.*

(1) *The natural map $H^0(K_v^h, T) \rightarrow H^0(K_v, T)$ induces an isomorphism $H^0(K_v^h, T)^{(\ell)} \simeq H^0(K_v, T)^{(\ell)}$. Moreover, there is an isomorphism $\mathbb{H}^0(K_v^h, C)^{(\ell)} \simeq \mathbb{H}^0(K_v, C)^{(\ell)}$.*

(2) *For $i \geq 1$, there is an isomorphism $\mathbb{H}^i(K_v^h, C) \rightarrow \mathbb{H}^i(K_v, C)$.*

Proof.

- (1) The same argument as [Dem11a, Lemme 3.7] yields an isomorphism $H^0(K_v^h, T)/\ell^n \simeq H^0(K_v, T)/\ell^n$. Therefore the first assertion follows by passing to the inverse limit over all n . For the second statement, since $H^1(K_v^h, T_i) \simeq H^1(K_v, T_i)$ is finite for $i = 1, 2$, there is a commutative diagram of complexes with rows exact at the last four terms by Lemma 2.3.9:

$$\begin{array}{ccccccccc} H^0(K_v^h, T_1)^{(\ell)} & \longrightarrow & H^0(K_v^h, T_2)^{(\ell)} & \longrightarrow & \mathbb{H}^0(K_v^h, C)^{(\ell)} & \longrightarrow & H^1(K_v^h, T_1) & \longrightarrow & H^1(K_v^h, T_2) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H^0(K_v, T_1)^{(\ell)} & \longrightarrow & H^0(K_v, T_2)^{(\ell)} & \longrightarrow & \mathbb{H}^0(K_v, C)^{(\ell)} & \longrightarrow & H^1(K_v, T_1) & \longrightarrow & H^1(K_v, T_2). \end{array}$$

Now all the vertical arrows except the middle one are isomorphisms, and hence the middle one is also an isomorphism by the 5-lemma.

- (2) By [HS16, Corollary 3.2], we know that $H^i(K_v^h, T) \simeq H^i(K_v, T)$ for each $i \geq 1$ and for each K -torus T . Thus the isomorphism $\mathbb{H}^i(K_v^h, C) \simeq \mathbb{H}^i(K_v, C)$ for each $i \geq 1$ follows after applying dévissage to the distinguished triangle $T_1 \rightarrow T_2 \rightarrow C \rightarrow T_1[1]$. \square

Corollary 2.3.11. *There is a perfect pairing between direct limit of profinite groups and projective limit of discrete torsion groups:*

$$\left(\prod_{v \in X^{(1)}} \mathbb{H}^1(K_v, C) \right) \{\ell\} \times \left(\bigoplus_{v \in X^{(1)}} \mathbb{H}^0(K_v^h, C') \right)^{(\ell)} \rightarrow \mathbb{Q}/\mathbb{Z}.$$

Proof. The same argument as Proposition 2.3.5 yields a perfect pairing

$$\ell^n \mathbb{H}^1(K_v^h, C) \times \mathbb{H}^0(K_v^h, C')/\ell^n \rightarrow \mathbb{Q}/\mathbb{Z}$$

of finite groups. So Lemma 2.3.10(2) implies that $\mathbb{H}^0(K_v^h, C')/\ell^n \simeq \mathbb{H}^0(K_v, C')/\ell^n$. The desired perfect pairing is an immediate consequence by the same argument as Corollary 2.3.7(2). \square

2.3.3 Global dualities: finiteness results

The next goal is to establish a perfect pairing $\mathbb{III}^i(C) \times \mathbb{III}^{2-i}(C') \rightarrow \mathbb{Q}/\mathbb{Z}$ between finite groups. We first prove the finiteness of $\mathbb{III}^1(C)$. Recall that $\mathbb{III}_\omega^i(C)$ denotes the subgroup of $\mathbb{H}^i(K, C)$ consisting of locally trivial elements for all but finitely many $v \in X^{(1)}$.

Lemma 2.3.12. *Let $C = [T_1 \rightarrow T_2]$ be a short complex of tori. The group $\mathbb{III}_\omega^1(C)$ is of finite exponent.*

Proof. Let $L|K$ be a finite Galois extension that splits both T_1 and T_2 . Then for $i = 1, 2$, the L -tori $T_{i,L} = T_i \times_K L$ are products of \mathbb{G}_m , and $H^1(L, T_{2,L}) = 0$ by Hilbert's theorem 90. The distinguished triangle $T_1 \rightarrow T_2 \rightarrow C \rightarrow T_1[1]$ induces a commutative diagram

$$\begin{array}{ccc} \mathbb{H}^1(K, C) & \xrightarrow{\partial} & H^2(K, T_1) \\ \text{res} \downarrow & & \downarrow \text{res} \\ 0 & \longrightarrow & \mathbb{H}^1(L, C_L) \longrightarrow H^2(L, T_{1,L}) \end{array}$$

where $C_L = [T_{1,L} \rightarrow T_{2,L}]$. Recall [HSS15, Lemma 3.2(2)] that $\text{III}_\omega^2(T_1)$ is of finite exponent, and hence a restriction-corestriction argument shows that $\text{III}_\omega^1(C)$ is of finite exponent. \square

The following result provides both the finiteness of $\text{III}^1(C)$ and a crucial point in the proof of global duality.

Lemma 2.3.13. *For any non-empty open subset $U \subset X_0$ and any U -torus \mathcal{T} , put $D_K^2(U, \mathcal{T}) = \text{Im}(H_c^2(U, \mathcal{T}) \rightarrow H^2(K, \mathcal{T}))$. Then there exists a non-empty open subset $U_2 \subset X_0$ such that $D_K^2(U, \mathcal{T})$ is of finite exponent for any non-empty open subset $U \subset U_2$.*

Proof. By a restriction-corestriction argument, it will be sufficient to show that $D_K^2(U, \mathbb{G}_m)$ is of finite exponent. By [Gro68, pp. 96, (2.9)], there is an exact sequence

$$H^0(k, \underline{\text{Pic}}_{X/k}) \rightarrow \text{Br } k \rightarrow \text{Br } X \rightarrow H^1(k, \underline{\text{Pic}}_{X/k}) \rightarrow H^3(k, \mathbb{G}_m)$$

where $\underline{\text{Pic}}_{X/k}$ denotes the relative Picard functor. Since k is a p -adic local field, we conclude $\text{Br } X/\text{Br}_0 X \simeq H^1(k, \underline{\text{Pic}}_{X/k})$ for cohomological dimension reasons. Recall that there is a canonical short exact sequence $0 \rightarrow \underline{\text{Pic}}_{X/k}^\circ \rightarrow \underline{\text{Pic}}_{X/k} \rightarrow \mathbb{Z} \rightarrow 0$, thus $H^1(k, \underline{\text{Pic}}_{X/k})$ is a quotient of $H^1(k, \underline{\text{Pic}}_{X/k}^\circ)$ where $\underline{\text{Pic}}_{X/k}^\circ$ is an abelian variety. But $H^1(k, \underline{\text{Pic}}_{X/k}^\circ)$ is dual to $\underline{\text{Pic}}_{X/k}^\circ(k)^\vee$ by Tate duality over local fields [Mil06, Chapter I, Corollary 3.4], we deduce that $H^1(k, \underline{\text{Pic}}_{X/k}^\circ) \simeq F_0 \oplus (\mathbb{Q}_p/\mathbb{Z}_p)^{\oplus r}$ with F_0 a finite abelian group by Mattuck's theorem (see [Mat55] and [Mil06, pp. 41]).

Suppose first there is a rational point $e \in X(k)$ on X . Let $e^* : \text{Br } X \rightarrow \text{Br } k \simeq \mathbb{Q}/\mathbb{Z}$ be the induced map and put $\text{Br}_e X = \{\alpha \in \text{Br } X \mid e^*(\alpha) = 0\}$. Note that in this case the map $\text{Br } k \rightarrow \text{Br } X$ induced by the structural morphism $X \rightarrow \text{Spec } k$ is injective and there is an isomorphism $H^1(k, \underline{\text{Pic}}_{X/k}^\circ) \simeq H^1(k, \underline{\text{Pic}}_{X/k})$. Thus there is a split short exact sequence $0 \rightarrow \text{Br}_e X \rightarrow \text{Br } X \rightarrow \text{Br } k \rightarrow 0$ and consequently $\text{Br}_e X \simeq \text{Br } X/\text{Br } k$. Moreover, if $e \notin U$, then $D_K^2(U, \mathbb{G}_m) \subset \text{Ker}(\text{Br } U \rightarrow \bigoplus_{v \notin U} \text{Br } K_v) \subset \text{Br}_e X$. Indeed, applying [HS16, Proposition 3.1(2)] to $U \subset X$ and the étale sheaf \mathbb{G}_m yields an exact sequence

$$\bigoplus_{v \notin U} H^1(\kappa(v), i_v^* \mathbb{G}_m) \rightarrow H_c^2(U, \mathbb{G}_m) \rightarrow H_c^2(X, \mathbb{G}_m) = \text{Br}(X).$$

By Hilbert's Theorem 90, the first term vanishes and we conclude that $H_c^2(U, \mathbb{G}_m) \subset \text{Br}(X)$. Similarly, by applying [HS16, Proposition 3.1(1)] to the étale sheaf \mathbb{G}_m , we obtain an exact sequence

$$0 = \bigoplus_{v \notin U} H^1(K_v, \mathbb{G}_m) \rightarrow H_c^2(U, \mathbb{G}_m) \rightarrow H^2(U, \mathbb{G}_m) \rightarrow \bigoplus_{v \notin U} H^2(K_v, \mathbb{G}_m),$$

i.e. we have $H_c^2(U, \mathbb{G}_m) = \text{Im}(H_c^2(U, \mathbb{G}_m) \rightarrow H^2(U, \mathbb{G}_m)) = \text{Ker}(\text{Br } U \rightarrow \bigoplus_{v \notin U} \text{Br } K_v)$. In particular, the kernel is contained in $\text{Br } X$. Now any α in the kernel comes from $H_c^2(U, \mathbb{G}_m) = H^2(X, j_{U!} \mathbb{G}_m)$. Thus $e^*(\alpha) = 0$ if $e \notin U$. It follows that there is an injective map $D_K^2(U, \mathbb{G}_m) \rightarrow F_0 \oplus (\mathbb{Q}_p/\mathbb{Z}_p)^{\oplus r}$. Next, we show that there is a non-empty open subset $U_2 \subset X_0$ such that the decreasing sequence $\{D_K^2(U, \mathbb{G}_m)\{\ell\}\}$ is stable for $U \subset U_2$. By [HS16, Proposition 3.4], the group $H_c^2(U, \mathbb{G}_m)$ is of cofinite type and hence so is $D_K^2(U, \mathbb{G}_m)$. Since F_0 is finite, there exists *only* finitely many $\ell \neq p$ such that ℓ divides the order of F_0 . As a consequence, there exists a non-empty open subset $U_1 \subset X_0$ (which is independent of ℓ) such that $D_K^2(U, \mathbb{G}_m)\{\ell\} =$

⁵Note that $\underline{\text{Pic}}_{X/k}^\circ$ is the Jacobian of a curve, so it is isomorphic to its dual.

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$D_K^2(U_1, \mathbb{G}_m)\{\ell\}$ holds for any non-empty open subset $U \subset U_1$ and for each $\ell \neq p$ by [HS16, Lemma 3.7]. Again the decreasing sequence $\{D_K^2(U, \mathbb{G}_m)\{p\}\}_{U \subset U_1}$ stabilizes, so there exists some $U_2 \subset U_1$ such that $D_K^2(U, \mathbb{G}_m) = D_K^2(U_2, \mathbb{G}_m)$ for all $U \subset U_2$. Note that $\text{III}^2(\mathbb{G}_m)$ is the direct limit of $D^2(U, \mathbb{G}_m)$. Letting U run through all non-empty open subsets of U_2 yields $D_K^2(U_2, \mathbb{G}_m) = \text{III}^2(\mathbb{G}_m) = 0$, where the vanishing of $\text{III}^2(\mathbb{G}_m)$ is a consequence of [HSS15, Lemma 3.2].

In general, there exists a finite Galois extension $k'|k$ such that $X(k') \neq \emptyset$. Put $X_{k'} := X \times_k k'$ and $U_{k'} := U \times_k k'$. Thus $U_{k'}$ is open in $X_{k'}$. By Galois descent, for a non-empty open subset $V' \subset X_{k'}$, we can find a non-empty open subset $V \subset X$ such that $V_{k'} \subset V'$. Therefore we may choose a non-empty open subset $U \subset X$ such that both U and $U_{k'}$ are sufficiently small in X and $X_{k'}$ respectively, i.e. we may choose such a U such that $D_{K'}^2(U_{k'}, \mathbb{G}_m) \subset \text{Br } K'$ vanishes. Let K' be the function field of $X_{k'}$. Therefore a restriction-corestriction argument implies that $D_K^2(U, \mathbb{G}_m) \subset \text{Br } K$ has finite exponent. \square

We put for $i \geq 0$

$$\mathbb{D}_K^i(U, \mathcal{C}) := \text{Im}(\mathbb{H}_c^i(U, \mathcal{C}) \rightarrow \mathbb{H}^i(K, C)).$$

Proposition 2.3.14. *There exists a non-empty open subset U_0 of X_0 such that*

$$\mathbb{D}_K^1(U, \mathcal{C}) = \mathbb{D}_K^1(U_0, \mathcal{C}) = \text{III}^1(C). \quad (2.6)$$

for each non-empty open subset $U \subset U_0$. Moreover, the group $\text{III}^1(C)$ is finite.

Proof. By Lemma 2.3.13, the group $D_K^2(U, \mathcal{T}_1)$ is of finite exponent for U sufficiently small. Since $H^1(K, T_2)$ is of finite exponent, it follows that $\mathbb{D}_K^1(U, \mathcal{C})$ is of finite exponent (say N) by dévissage. In particular, the epimorphism $\mathbb{H}_c^1(U, \mathcal{C}) \rightarrow \mathbb{D}_K^1(U, \mathcal{C})$ factors through $\mathbb{H}_c^1(U, \mathcal{C}) \rightarrow \mathbb{H}_c^1(U, \mathcal{C})/N$. Recall that $\mathbb{H}_c^1(U, \mathcal{C})/N$ is a subgroup of the finite group $\mathbb{H}_c^1(U, \mathcal{C} \otimes^{\mathbb{L}} \mathbb{Z}/N)$, hence its quotient $\mathbb{D}_K^1(U, \mathcal{C})$ is finite.

For non-empty open subsets $V \subset U \subset X_0$ of X_0 , we have $\mathbb{D}_K^1(V, \mathcal{C}) \subset \mathbb{D}_K^1(U, \mathcal{C})$ by covariant functoriality of $\mathbb{H}_c^1(-, \mathcal{C})$. The decreasing sequence $\{\mathbb{D}_K^1(U, \mathcal{C})\}_{U \subset X_0}$ of finite abelian groups must be stable, hence there exists a non-empty open subset U_0 of X_0 such that $\mathbb{D}_K^1(U, \mathcal{C}) = \mathbb{D}_K^1(U_0, \mathcal{C})$ for each non-empty open subset $U \subset U_0$. Note that $\mathbb{D}_K^1(U, \mathcal{C}) \subset \text{Ker}(\mathbb{H}^1(K, C) \rightarrow \prod_{v \notin U} \mathbb{H}^1(K_v, C))$ by Proposition 2.2.5(3). Letting U run through all non-empty open subset of U_0 implies that $\mathbb{D}_K^1(U, \mathcal{C}) = \mathbb{D}_K^1(U_0, \mathcal{C}) = \text{III}^1(C)$. Since the former two groups are finite, so is $\text{III}^1(C)$. \square

Proposition 2.3.15. *The groups $\text{III}^0(C)$ and $\text{III}^2(C)$ are finite.*

Proof. Let L be K or K_v for $v \in X^{(1)}$. We consider the distinguished triangle

$$M[1] \rightarrow C \rightarrow T \rightarrow M[2] \quad (2.7)$$

over L . By Lemma 2.3.1(4), the groups $H^1(L, M)$ and $H^3(L, M)$ have finite exponents.

- We consider the exact sequences $H^3(L, M) \rightarrow \mathbb{H}^2(L, C) \rightarrow H^2(L, T) \rightarrow H^4(L, M)$ obtained from the distinguished triangle (2.7). Note that $\text{III}^2(T)$ is finite by Lemma 2.3.1(3). In particular, $\text{III}^2(C)$ has finite exponent by dévissage and it remains to show that $\text{III}^2(C)$ is of cofinite type. Since $\mathbb{H}^2(K, C)$ is the direct limit of $\mathbb{H}^2(U, \mathcal{C})$, each $\alpha \in \text{III}^2(C)$ comes from some $\mathbb{H}^2(U, \mathcal{C})$ with U a non-empty open subset of X_0 . In particular, α lies in the

image of $\mathbb{H}_c^2(U, \mathcal{C})$ by Proposition 2.2.5(3). We conclude that α comes from $\mathbb{H}_c^2(X_0, \mathcal{C})$ by the Three Arrows Lemma and hence $\mathbb{III}^2(C)$ is a subquotient of $\mathbb{H}_c^2(X_0, \mathcal{C})$ (which is torsion of cofinite type by Lemma 2.3.1(2)). As a consequence, $\mathbb{III}^2(C)$ is of cofinite type. Therefore $\mathbb{III}^2(C)$ is finite.

- The exact sequence $0 \rightarrow H^1(K, M) \rightarrow \mathbb{H}^0(K, C) \rightarrow H^0(K, T)$ obtained from (2.7) yields an isomorphism $\mathbb{III}^1(M) \simeq \mathbb{III}^0(C)$ as $\mathbb{III}^0(T) = 0$. Since we may embed M into a short exact sequence $0 \rightarrow M \rightarrow P_1 \rightarrow P_2 \rightarrow 0$ with P_1 and P_2 being K -tori, there is a quasi-isomorphism $M[1] \simeq [P_1 \rightarrow P_2]$. Subsequently, an analogous argument as above implies that $\mathbb{III}^1(M) \subset \text{Im}(H_c^1(X_0, \mathcal{M}) \rightarrow H^1(K, M))$. But this map factors through $H_c^1(X_0, \mathcal{M})/N \rightarrow H^1(K, M)$ because $H^1(K, M)$ has finite exponent for some positive integer N by Lemma 2.3.1(4). Finally, $H_c^1(X_0, \mathcal{M})/N$ injects into the finite group $H_c^1(X_0, \mathcal{M} \otimes^{\mathbf{L}} \mathbb{Z}/N)$ (we have seen its finiteness in Proposition 2.2.4) thanks to the distinguished triangle $M \rightarrow M \rightarrow M \otimes^{\mathbf{L}} \mathbb{Z}/n \rightarrow M[1]$, so it is finite as well. Hence $\mathbb{III}^1(M) \simeq \mathbb{III}^0(C)$ is contained in this finite image which completes the proof. \square

Remark 2.3.16. Thus all non-trivial Tate–Shafarevich groups of the complex C are finite:

- $\mathbb{III}^i(C)$ is a finite group for $0 \leq i \leq 2$ by Proposition 2.3.14 and Proposition 2.3.15.
- $\mathbb{III}^i(C) = 0$ for $i \leq -1$ and $i \geq 3$ by dévissage thanks to the distinguished triangle $T_1 \rightarrow T_2 \rightarrow C \rightarrow T_1[1]$.

2.3.4 Global dualities: perfect pairings

The goal of this section is to establish a perfect pairing of finite abelian groups:

$$\mathbb{III}^i(C) \times \mathbb{III}^{2-i}(C') \rightarrow \mathbb{Q}/\mathbb{Z}.$$

To state a key step, we first construct a map

$$\bigoplus_{v \in X^{(1)}} \mathbb{H}^i(K_v^h, C) \rightarrow \mathbb{H}_c^{i+1}(U, \mathcal{C})$$

for some non-empty open subset U of X_0 and $i = 0, 1$. Take $\alpha \in \bigoplus_{v \in X^{(1)}} \mathbb{H}^i(K_v^h, C)$ supported outside some non-empty open subset V of U , i.e. $\alpha_v = 0$ for $v \in V$. Applying Proposition 2.2.5(1) to V sends α to $\mathbb{H}_c^{i+1}(V, \mathcal{C})$, and so α is sent to $\mathbb{H}_c^{i+1}(U, \mathcal{C})$ by the covariant functoriality of $\mathbb{H}_c^{i+1}(-, \mathcal{C})$. The construction is independent of the choice of V by the same argument as [HS16, pp. 11, (12)].

Proposition 2.3.17. *For $i = 0, 1$, there is an exact sequence*

$$\bigoplus_{v \in X^{(1)}} \mathbb{H}^i(K_v^h, C) \rightarrow \mathbb{H}_c^{i+1}(U, \mathcal{C}) \rightarrow \mathbb{D}_K^{i+1}(U, \mathcal{C}) \rightarrow 0.$$

Proof. The sequence is a complex by exactly the same argument of [HS16, Proposition 4.2]. The surjectivity of the last arrow is just the definition of $\mathbb{D}_K^{i+1}(U, \mathcal{C})$. Take $\alpha \in \text{Ker}(\mathbb{H}_c^{i+1}(U, \mathcal{C}) \rightarrow$

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$\mathbb{D}_K^{i+1}(U, \mathcal{C})$) and a non-empty open subset $V \subset U$. Consider the diagram

$$\begin{array}{ccccc} \mathbb{H}_c^{i+1}(V, \mathcal{C}) & \longrightarrow & \mathbb{H}_c^{i+1}(U, \mathcal{C}) & \longrightarrow & \bigoplus_{v \in U \setminus V} \mathbb{H}^{i+1}(\kappa(v), i_v^* \mathcal{C}) \\ & & \downarrow & & \downarrow \\ & & \mathbb{H}^{i+1}(K, \mathcal{C}) & \longrightarrow & \bigoplus_{v \in U \setminus V} \mathbb{H}^{i+1}(K_v, \mathcal{C}) \end{array}$$

where the right vertical arrow is constructed as the composite

$$\mathbb{H}^{i+1}(\kappa(v), i_v^* \mathcal{C}) \simeq \mathbb{H}^{i+1}(\mathcal{O}_{U,v}^h, \mathcal{C}) \rightarrow \mathbb{H}^{i+1}(K_v^h, \mathcal{C}) \simeq \mathbb{H}^{i+1}(K_v, \mathcal{C}).$$

The first isomorphism is a consequence of $H^{i+1}(\kappa(v), i_v^* \mathcal{P}) \simeq H^{i+1}(\mathcal{O}_{U,v}^h, \mathcal{P})$ for any U_0 -torus \mathcal{P} (see [Mil06, Chapter II, Proposition 1.1(b)]) and a dévissage argument. The diagram commutes for the same reason as in the proof of [HS16, Proposition 4.2]. Since the right vertical arrow is injective by Lemma 2.1.3, a diagram chase shows that α comes from $\mathbb{H}_c^{i+1}(V, \mathcal{C})$. According to Three Arrows Lemma, we may take V sufficiently small such that α already goes to zero in $\mathbb{H}^{i+1}(V, \mathcal{C})$. Therefore α comes from $\bigoplus_{v \notin V} \mathbb{H}^i(K_v^h, \mathcal{C})$ by Proposition 2.2.5(1) and hence the desired sequence is indeed exact. \square

Lemma 2.3.18. *There exists a non-empty open subset U_0 of X_0 such that*

$$\mathbb{D}_K^2(U, \mathcal{C}) = \mathbb{D}_K^2(U_0, \mathcal{C}) = \mathbb{I}^2(\mathcal{C}). \quad (2.8)$$

for each non-empty open subset $U \subset U_0$.

Proof. Since $\mathbb{H}_c^2(U, \mathcal{C})$ is torsion of cofinite type by Lemma 2.3.1, so is $\mathbb{D}_K^2(U, \mathcal{C})$. Hence the decreasing family $\{\mathbb{D}_K^2(U, \mathcal{C})\{\ell\}\}_{U \subset X_0}$ of ℓ -primary torsion groups must be stable by [HS16, Lemma 3.7]. Let us say $\mathbb{D}_K^2(U, \mathcal{C})\{\ell\} = \mathbb{D}_K^2(U_0, \mathcal{C})\{\ell\}$ for some open subset $U_0 \subset X_0$ and for each non-empty open subset $U \subset U_0$. Letting U run through all non-empty open subsets of U_0 , we conclude $\mathbb{D}_K^2(U_0, \mathcal{C})\{\ell\} = \mathbb{I}^2(\mathcal{C})\{\ell\}$ by Proposition 2.2.5(3). \square

Now we know for $i = 1, 2$ that $\mathbb{D}_K^i(U, \mathcal{C})$ is finite, by Lemma 2.3.9 there is an exact sequence

$$\left(\bigoplus_{v \in X^{(1)}} \mathbb{H}^i(K_v^h, \mathcal{C}) \right)^{(\ell)} \rightarrow \mathbb{H}_c^{i+1}(U, \mathcal{C})^{(\ell)} \rightarrow \mathbb{D}_K^{i+1}(U, \mathcal{C})^{(\ell)} \rightarrow 0 \quad (2.9)$$

for $i = 0, 1$. We arrive at the global duality of the short complex \mathcal{C} .

Theorem 2.3.19. *There is a perfect, functorial in \mathcal{C} , pairing of finite groups:*

$$\mathbb{I}^i(\mathcal{C}) \times \mathbb{I}^{2-i}(\mathcal{C}') \rightarrow \mathbb{Q}/\mathbb{Z}.$$

Proof. We proceed by constructing a perfect pairing of finite groups

$$\mathbb{I}^i(\mathcal{C})\{\ell\} \times \mathbb{I}^{2-i}(\mathcal{C}')\{\ell\} \rightarrow \mathbb{Q}/\mathbb{Z}$$

for a fixed prime ℓ and for $i = 0, 1$ (the case $i = 2$ follows by symmetry). Define $\mathbb{D}_{\text{sh}}^i(U, \mathcal{C})$ by the exact sequence

$$0 \rightarrow \mathbb{D}_{\text{sh}}^i(U, \mathcal{C}) \rightarrow \mathbb{H}^i(U, \mathcal{C}) \rightarrow \prod_{v \in X^{(1)}} \mathbb{H}^i(K_v, \mathcal{C})$$

for each $U \subset U_0$. Dualizing the exact sequence (2.9) yields the following commutative diagram (by a similar argument as [CTH15, Proposition 4.3(f)], replacing the pairing $\mathcal{F} \otimes \mathcal{F}' \rightarrow \mathbb{G}_m$ there with $\mathcal{C} \otimes^{\mathbf{L}} \mathcal{C}' \rightarrow \mathbb{Z}(2)[3]$) with exact rows

$$\begin{array}{ccccc}
 0 & \longrightarrow & \mathbb{D}_{\text{sh}}^i(U, \mathcal{C})\{\ell\} & \longrightarrow & \mathbb{H}^i(U, \mathcal{C})\{\ell\} & \longrightarrow & \left(\prod_{v \in X^{(1)}} \mathbb{H}^i(K_v, C) \right) \{\ell\} \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & (\mathbb{D}_K^{2-i}(U, \mathcal{C}')^{(\ell)})^D & \longrightarrow & (\mathbb{H}_c^{2-i}(U, \mathcal{C}')^{(\ell)})^D & \longrightarrow & \left(\bigoplus_{v \in X^{(1)}} \mathbb{H}^{1-i}(K_v^h, C') \right)^{(\ell)D}
 \end{array}$$

where the middle vertical arrow is induced by the Artin–Verdier duality (see Proposition 2.3.3) and the right vertical arrow is induced by the local duality (see Corollary 2.3.11). The first vertical arrow is induced by $\mathbb{H}^i(U, \mathcal{C})\{\ell\} \rightarrow (\mathbb{H}_c^{2-i}(U, \mathcal{C}')^{(\ell)})^D$ in view of the commutativity of the right square. By local duality Corollary 2.3.11, the right vertical arrow is an isomorphism, and it follows that the kernels of the first two vertical arrows are identified. Moreover, the middle vertical arrow is surjective, thus so is the left one by diagram chasing. Passing to the direct limit of the dashed arrow induces an exact sequence of abelian groups

$$0 \rightarrow \varinjlim_U \varinjlim_n \mathbb{K}_n^i(U) \rightarrow \varinjlim_U \mathbb{D}_{\text{sh}}^i(U, \mathcal{C})\{\ell\} \rightarrow \varinjlim_U (\mathbb{D}_K^{2-i}(U, \mathcal{C}')^{(\ell)})^D \rightarrow 0.$$

Recall that $\varinjlim_n \mathbb{K}_n^i(U)$ is the divisible kernel of the middle vertical arrow introduced in Proposition 2.3.3. Note that the second limit is just $\mathbb{H}^i(C)\{\ell\}$ by definition of $\mathbb{D}_{\text{sh}}^i(U, \mathcal{C})$. In particular, the first limit is trivial being a divisible subgroup of a finite abelian group. On the other hand, there are isomorphisms of finite abelian groups for $i = 0, 1$:

$$\mathbb{D}_K^{2-i}(U, \mathcal{C}')^{(\ell)} = \mathbb{D}_K^{2-i}(U, \mathcal{C}')\{\ell\}^{(\ell)} = \mathbb{H}^{2-i}(C')\{\ell\}^{(\ell)} \simeq \mathbb{H}^{2-i}(C')\{\ell\}$$

where the central equality holds by Proposition 2.3.14 and Lemma 2.3.18. Summing up, we acquire an isomorphism of finite abelian groups

$$\mathbb{H}^i(C)\{\ell\} \rightarrow \mathbb{H}^{2-i}(C')\{\ell\}^D$$

for each prime number ℓ and $i = 0, 1$, as required. \square

2.3.5 Global dualities: additional results

We shall need an additional global duality concerning inverse limits to connect the first two rows in the Poitou–Tate sequence.

Theorem 2.3.20. *Put $\mathbb{H}_\wedge^0(C) := \text{Ker}(\mathbb{H}^0(K, C)_\wedge \rightarrow \mathbb{P}^0(K, C)_\wedge)$. If $\text{Ker} \rho$ is finite, then we can construct a perfect pairing of finite groups*

$$\mathbb{H}_\wedge^0(C) \times \mathbb{H}^2(C') \rightarrow \mathbb{Q}/\mathbb{Z}.$$

The rest of this section is devoted to the proof of Theorem 2.3.20 which is analogous to that of [Dem09, pp. 86–88]. We proceed by reducing the question into various limits in finite level.

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Lemma 2.3.21. *Let $C = [T_1 \xrightarrow{\rho} T_2]$ (here $\text{Ker } \rho$ is not necessarily finite). The natural map is an isomorphism*

$$\mathbb{H}_\wedge^0(C) \rightarrow \varprojlim_n \mathbb{H}^0(C \otimes^{\mathbf{L}} \mathbb{Z}/n).$$

Proof. Consider the Kummer exact sequences $0 \rightarrow \mathbb{H}^0(K_v, C)/n \rightarrow \mathbb{H}^0(K_v, C \otimes^{\mathbf{L}} \mathbb{Z}/n) \rightarrow {}_n\mathbb{H}^1(K_v, C) \rightarrow 0$ for all $v \in X^{(1)}$, and $0 \rightarrow \mathbb{H}^0(\mathcal{O}_v, \mathcal{C})/n \rightarrow \mathbb{H}^0(\mathcal{O}_v, \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n) \rightarrow {}_n\mathbb{H}^1(\mathcal{O}_v, \mathcal{C}) \rightarrow 0$ for all $v \in X_0^{(1)}$. Moreover, the complex $0 \rightarrow \mathbb{P}^0(K, C)/n \rightarrow \mathbb{P}^0(K, C \otimes^{\mathbf{L}} \mathbb{Z}/n) \rightarrow {}_n\mathbb{P}^1(K, C) \rightarrow 0$ is an exact sequence by Lemma 2.1.5. Therefore there is a commutative diagram with exact rows by taking inverse limit over all n in respective Kummer sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{H}^0(K, C)_\wedge & \longrightarrow & \varprojlim_n \mathbb{H}^0(K, C \otimes^{\mathbf{L}} \mathbb{Z}/n) & \longrightarrow & \Phi_K \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathbb{P}^0(K, C)_\wedge & \longrightarrow & \varprojlim_n \mathbb{P}^0(K, C \otimes^{\mathbf{L}} \mathbb{Z}/n) & \longrightarrow & \Phi_\Pi \longrightarrow 0. \end{array} \quad (2.10)$$

where $\Phi_K \subset \varprojlim_n {}_n\mathbb{H}^1(K, C)$ and $\Phi_\Pi \subset \varprojlim_n {}_n\mathbb{P}^1(K, C)$ (here the inverse limit may not be right exact because the involved groups are infinite). Recall that $\mathbb{H}^1(C)$ is finite (Remark 2.3.16). As a consequence, the kernel of the right vertical arrow is contained in $\varprojlim_n \mathbb{H}^1(C) = 0$, hence $\Phi_K \rightarrow \Phi_\Pi$ is injective. Therefore there are isomorphisms

$$\mathbb{H}_\wedge^0(C) \simeq \text{Ker} \left(\varprojlim_n \mathbb{H}^0(K, C \otimes^{\mathbf{L}} \mathbb{Z}/n) \rightarrow \varprojlim_n \mathbb{P}^0(K, C \otimes^{\mathbf{L}} \mathbb{Z}/n) \right) \simeq \varprojlim_n \mathbb{H}^0(C \otimes^{\mathbf{L}} \mathbb{Z}/n)$$

by the snake lemma, as required. \square

The following lemmas tell us that $\varprojlim_n \mathbb{H}^0(C \otimes^{\mathbf{L}} \mathbb{Z}/n)$ is an inverse limit of subgroups of $\mathbb{H}^0(U, \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n)$.

Lemma 2.3.22. *Let \mathcal{F} be a finite étale commutative group scheme over X_0 . For $i \geq 0$, we have $\varprojlim_n H^i(X_0, {}_n\mathcal{F}) = 0$.*

Proof. Since \mathcal{F} is a finite group scheme, $\mathcal{F} = {}_N\mathcal{F}$ for some positive integer N . In particular, we see that $H^i(X_0, {}_n\mathcal{F})$ is a torsion group having exponent N . Take $(x_n) \in \varprojlim_n H^i(X_0, {}_n\mathcal{F})$. For each $n \geq 1$, consider the following commutative diagram

$$\begin{array}{ccc} H^i(X_0, {}_{nN^2}\mathcal{F}) & \longrightarrow & H^i(X_0, \mathcal{F}) \\ \downarrow & & \downarrow \\ H^i(X_0, {}_{nN}\mathcal{F}) & \longrightarrow & H^i(X_0, \mathcal{F}) \end{array}$$

where the horizontal arrows are isomorphisms (because $\mathcal{F} = {}_N\mathcal{F}$) and the right vertical arrow is given by multiplication by N . Then $x_{nN} = Nx_{nN^2}$ for each positive integer n and it follows that $x_{nN} = 0$. In particular, the image x_n of x_{nN} in $H^i(X_0, {}_n\mathcal{F})$ is zero, i.e. $\varprojlim_n H^i(X_0, {}_n\mathcal{F}) = 0$. \square

Recall that we denote by $M := \text{Ker } \rho$ the kernel of $\rho : T_1 \rightarrow T_2$. Moreover, we denote by \mathcal{M} the kernel of $\mathcal{T}_1 \rightarrow \mathcal{T}_2$.

Lemma 2.3.23. *Suppose $\text{Ker } \rho$ is finite. Then $\varprojlim_n \mathbb{H}^0(X_0, \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n) \rightarrow \varprojlim_n \mathbb{H}^0(K, C \otimes^{\mathbf{L}} \mathbb{Z}/n)$ is injective.*

Proof. According to Lemma 2.3.22, we obtain that $\varprojlim_n H^i(X_0, {}_n\mathcal{M}) = 0$. We put $T_{\mathbb{Z}/n}(C) := H^0(C[-1] \otimes^{\mathbf{L}} \mathbb{Z}/n)$ and consider the distinguished triangle ${}_nM[2] \rightarrow C \otimes^{\mathbf{L}} \mathbb{Z}/n \rightarrow T_{\mathbb{Z}/n}(C)[1] \rightarrow {}_nM[3]$ (see [Dem11a, Lemme 2.3]). Thus $\varprojlim_n \mathbb{H}^0(X_0, C \otimes^{\mathbf{L}} \mathbb{Z}/n) \rightarrow \varprojlim_n \mathbb{H}^0(K, C \otimes^{\mathbf{L}} \mathbb{Z}/n)$ is injective by the following commutative diagram

$$\begin{array}{ccc} \varprojlim_n \mathbb{H}^0(X_0, C \otimes^{\mathbf{L}} \mathbb{Z}/n) & \longrightarrow & \varprojlim_n H^1(X_0, T_{\mathbb{Z}/n}(C)) \\ \downarrow & & \downarrow \\ \varprojlim_n \mathbb{H}^0(K, C \otimes^{\mathbf{L}} \mathbb{Z}/n) & \longrightarrow & \varprojlim_n H^1(K, T_{\mathbb{Z}/n}(C)) \end{array}$$

(the right vertical arrow is injective by the same argument as [Dem11a, Proposition 5.3(2)]). \square

We put $\mathbb{D}^i(U, C \otimes^{\mathbf{L}} \mathbb{Z}/n) := \text{Im}(\mathbb{H}_c^i(U, C \otimes^{\mathbf{L}} \mathbb{Z}/n) \rightarrow \mathbb{H}^i(U, C \otimes^{\mathbf{L}} \mathbb{Z}/n))$ for $U \subset X_0$. So there are inclusions

$$\varprojlim_n \mathbb{D}^i(U, C \otimes^{\mathbf{L}} \mathbb{Z}/n) \subset \varprojlim_n \mathbb{H}^i(U, C \otimes^{\mathbf{L}} \mathbb{Z}/n) \subset \varprojlim_n \mathbb{H}^i(K, C \otimes^{\mathbf{L}} \mathbb{Z}/n).$$

If $V \subset U$ is an open subset, then $\varprojlim_n \mathbb{D}^i(V, C \otimes^{\mathbf{L}} \mathbb{Z}/n) \subset \varprojlim_n \mathbb{D}^i(U, C \otimes^{\mathbf{L}} \mathbb{Z}/n)$ viewed as subgroups of $\varprojlim_n \mathbb{H}^i(K, C \otimes^{\mathbf{L}} \mathbb{Z}/n)$. We can then take the inverse limit $\varprojlim_U \varprojlim_n \mathbb{D}^0(U, C \otimes^{\mathbf{L}} \mathbb{Z}/n)$ over all $U \subset X_0$.

Lemma 2.3.24. *Suppose $\text{Ker } \rho$ is finite. The following map is an isomorphism*

$$\varprojlim_U \varprojlim_n \mathbb{D}^0(U, C \otimes^{\mathbf{L}} \mathbb{Z}/n) \rightarrow \varprojlim_n \mathbb{III}^0(C \otimes^{\mathbf{L}} \mathbb{Z}/n)$$

with transition maps given by covariant functoriality of the functor $\mathbb{H}_c^0(-, C \otimes^{\mathbf{L}} \mathbb{Z}/n)$.

Proof. By the injectivity of Lemma 2.3.23, we can take $\bigcap_{U \subset X_0} \varprojlim_n \mathbb{D}^0(U, C \otimes^{\mathbf{L}} \mathbb{Z}/n)$ in the inverse limit $\varprojlim_n \mathbb{H}^0(K, C \otimes^{\mathbf{L}} \mathbb{Z}/n)$. It follows from the definition of \varprojlim_U that the intersection over $U \subset X_0$ coincides with inverse limit. By Proposition 2.2.5(1) and (3), we conclude that $\bigcap_{U \subset X_0} \varprojlim_n \mathbb{D}^0(U, C \otimes^{\mathbf{L}} \mathbb{Z}/n) = \varprojlim_n \mathbb{III}^0(C \otimes^{\mathbf{L}} \mathbb{Z}/n)$. \square

Next we describe $\mathbb{III}^2(C')$. Again we write L for K or K_v and consider the Kummer sequence $0 \rightarrow \mathbb{H}^1(L, C')/n \rightarrow \mathbb{H}^1(L, C' \otimes^{\mathbf{L}} \mathbb{Z}/n) \rightarrow {}_n\mathbb{H}^2(L, C') \rightarrow 0$. Since $\mathbb{H}^i(L, C')$ is torsion for $i \geq 1$, taking the direct limit over all n yields an isomorphism $\mathbb{H}^1(L, \varinjlim_n C' \otimes^{\mathbf{L}} \mathbb{Z}/n) \simeq \mathbb{H}^2(L, C')$ (because $\mathbb{H}^1(L, C') \otimes \mathbb{Q}/\mathbb{Z} = 0$). In particular, we obtain an isomorphism

$$\mathbb{III}^1(\varinjlim_n C' \otimes^{\mathbf{L}} \mathbb{Z}/n) \simeq \mathbb{III}^2(C')$$

of finite abelian groups. Put

$$\mathbb{D}_{\text{sh}}^1(U, \varinjlim_n C' \otimes^{\mathbf{L}} \mathbb{Z}/n) := \text{Ker}(\mathbb{H}^1(U, \varinjlim_n C' \otimes^{\mathbf{L}} \mathbb{Z}/n) \rightarrow \prod_{v \in X^{(1)}} \varinjlim_n H^1(K_v, C' \otimes^{\mathbf{L}} \mathbb{Z}/n)).$$

If $V \subset U$ is a smaller open subset, then there is a homomorphism

$$\mathbb{D}_{\text{sh}}^1(U, \varinjlim_n C' \otimes^{\mathbf{L}} \mathbb{Z}/n) \rightarrow \mathbb{D}_{\text{sh}}^1(V, \varinjlim_n C' \otimes^{\mathbf{L}} \mathbb{Z}/n)$$

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induced by the restriction maps $\mathbb{H}^1(U, \varinjlim_n \mathcal{C}' \otimes^{\mathbf{L}} \mathbb{Z}/n) \rightarrow \mathbb{H}^1(V, \varinjlim_n \mathcal{C}' \otimes^{\mathbf{L}} \mathbb{Z}/n)$. In particular, we can take the following direct limit over all $U \subset X_0$ and we obtain an isomorphism

$$\varinjlim_U \mathbb{D}_{\text{sh}}^1(U, \varinjlim_n \mathcal{C}' \otimes^{\mathbf{L}} \mathbb{Z}/n) \simeq \mathbb{H}^1(\varinjlim_n \mathcal{C}' \otimes^{\mathbf{L}} \mathbb{Z}/n)$$

by construction. Consequently, we reduce Theorem 2.3.20 to showing that

$$\varprojlim_n \mathbb{D}^0(U, \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n) \times \mathbb{D}_{\text{sh}}^1(U, \varinjlim_n \mathcal{C}' \otimes^{\mathbf{L}} \mathbb{Z}/n) \rightarrow \mathbb{Q}/\mathbb{Z}$$

is a perfect pairing.

We shall need the following compatibility between local duality and Artin–Verdier duality.

Lemma 2.3.25. *Let U be a sufficiently small non-empty open subset of X_0 . There is a commutative diagram*

$$\begin{array}{ccc} \bigoplus_{v \in X^{(1)}} \mathbb{H}^{-1}(K_v, \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n) & \times & \prod_{v \in X^{(1)}} \mathbb{H}^1(K_v, \mathcal{C}' \otimes^{\mathbf{L}} \mathbb{Z}/n) \longrightarrow \mathbb{Q}/\mathbb{Z} \\ \downarrow & & \uparrow \\ \mathbb{H}_c^0(U, \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n) & \times & \mathbb{H}^1(U, \mathcal{C}' \otimes^{\mathbf{L}} \mathbb{Z}/n) \longrightarrow \mathbb{Q}/\mathbb{Z} \end{array} \quad (2.11)$$

where the left vertical arrow is constructed analogous to the first arrow of (2.2), and the middle one is the composition $\mathbb{H}^1(U, \mathcal{C}' \otimes^{\mathbf{L}} \mathbb{Z}/n) \rightarrow \mathbb{H}^1(K, \mathcal{C}' \otimes^{\mathbf{L}} \mathbb{Z}/n) \rightarrow \prod \mathbb{H}^1(K_v, \mathcal{C}' \otimes^{\mathbf{L}} \mathbb{Z}/n)$.

Proof. The proof is essentially the same as that of [CTH15, Proposition 4.3(f)]. We first observe by the same argument as *loc. cit.* that it will be sufficient to show the commutativity of diagram (2.11) when $v \notin U$. Recall there are isomorphisms $H^1(U, \mathcal{C}' \otimes^{\mathbf{L}} \mathbb{Z}/n) \simeq \text{Ext}_U^1(\mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n, \mu_n^{\otimes 2})$ and $H^1(K_v, \mathcal{C}' \otimes^{\mathbf{L}} \mathbb{Z}/n) \simeq \text{Ext}_{K_v}^1(\mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n, \mu_n^{\otimes 2})$. Hence it suffices to show the commutativity of the following diagram

$$\begin{array}{ccc} \text{Hom}_{K_v}(\mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n, \mu_n^{\otimes 2}[1]) & \times & \mathbb{H}^{-1}(K_v, \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n) \longrightarrow \mathbb{Q}/\mathbb{Z} \\ \uparrow & & \downarrow \\ \text{Hom}_U(\mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n, \mu_n^{\otimes 2}[1]) & \times & \mathbb{H}_c^0(U, \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n) \longrightarrow \mathbb{Q}/\mathbb{Z}, \end{array}$$

where the Hom are taken in respective derived categories. Take $\alpha_U \in \text{Hom}_U(\mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n, \mu_n^{\otimes 2}[1])$ and let α_v be its image in $\text{Hom}_{K_v}(\mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n, \mu_n^{\otimes 2}[1])$. Let $j_U : U \rightarrow X$ and $j_v : \text{Spec } K_v \rightarrow \text{Spec } \mathcal{O}_v$ be the respective open immersions. Recall that there is an isomorphism

$$\mathbb{H}^{-1}(K_v, \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n) \simeq \mathbb{H}_v^0(\mathcal{O}_v, j_{v!}(\mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n)).$$

In view of the commutative diagram

$$\begin{array}{ccccc} \mathbb{H}_v^0(\mathcal{O}_v, j_{v!}(\mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n)) & \xleftarrow{\simeq} & \mathbb{H}_v^0(X, j_{U!}(\mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n)) & \longrightarrow & \mathbb{H}^0(X, j_{U!}(\mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n)) \\ (\alpha_v)_* \downarrow & & (\alpha_U)_* \downarrow & & (\alpha_U)_* \downarrow \\ \mathbb{H}_v^1(\mathcal{O}_v, j_{v!}(\mu_n^{\otimes 2})) & \xleftarrow{\simeq} & \mathbb{H}_v^1(X, j_{U!}(\mu_n^{\otimes 2})) & \longrightarrow & \mathbb{H}^1(X, j_{U!}(\mu_n^{\otimes 2})), \end{array}$$

we conclude that the desired commutativity of diagram (2.11). \square

Lemma 2.3.26. *Suppose $M := \text{Ker } \rho$ is finite. Then the canonical map*

$$\mathbb{H}^1(U, \varinjlim_n \mathcal{C}' \otimes^{\mathbf{L}} \mathbb{Z}/n) \rightarrow \prod \varinjlim_n H^1(K_v, \mathcal{C}' \otimes^{\mathbf{L}} \mathbb{Z}/n)$$

factors through $\bigoplus \varinjlim_n H^1(K_v, \mathcal{C}' \otimes^{\mathbf{L}} \mathbb{Z}/n) \subset \prod \varinjlim_n H^1(K_v, \mathcal{C}' \otimes^{\mathbf{L}} \mathbb{Z}/n)$.

Proof. Indeed, the finiteness of $\text{Ker } \rho$ is equivalent to the surjectivity of $\rho' : T'_2 \rightarrow T'_1$. Therefore we obtain a quasi-isomorphism $(\text{Ker } \rho')[1] \simeq \mathcal{C}'$. But $\text{Ker } \rho'$ is a group of multiplicative type, we may extend it to X_0 and embed it into a short exact sequence $0 \rightarrow \mathcal{P} \rightarrow \text{Ker } \rho' \rightarrow \mathcal{F} \rightarrow 0$ over \mathcal{O}_v for $v \in X_0^{(1)}$ where \mathcal{P} is a torus and \mathcal{F} is a finite group scheme. Note that $H^3(\mathcal{O}_v, \mathcal{P}) = 0$, we conclude that $H^3(\mathcal{O}_v, \text{Ker } \rho') \simeq H^3(\mathcal{O}_v, \mathcal{F}) \simeq H^3(\kappa(v), \mathcal{F}) = 0$. According to the distinguished triangle $\mathcal{C}' \rightarrow \mathcal{C}' \rightarrow \mathcal{C}' \otimes^{\mathbf{L}} \mathbb{Z}/n \rightarrow \mathcal{C}'[1]$, we have an identification $\mathbb{H}^2(\mathcal{O}_v, \mathcal{C}') \simeq \varinjlim_n \mathbb{H}^1(\mathcal{O}_v, \mathcal{C}' \otimes^{\mathbf{L}} \mathbb{Z}/n)$. But $\mathbb{H}^2(\mathcal{O}_v, \mathcal{C}') \simeq H^3(\mathcal{O}_v, \text{Ker } \rho') = 0$, it follows that $\varinjlim_n \mathbb{H}^1(\mathcal{O}_v, \mathcal{C}' \otimes^{\mathbf{L}} \mathbb{Z}/n) = 0$. So the image of $\varinjlim_n \mathbb{H}^1(U, \mathcal{C}' \otimes^{\mathbf{L}} \mathbb{Z}/n)$ in $\prod_v \varinjlim_n \mathbb{H}^1(K_v, \mathcal{C}' \otimes^{\mathbf{L}} \mathbb{Z}/n)$ lies in the subgroup $\bigoplus_v \varinjlim_n \mathbb{H}^1(K_v, \mathcal{C}' \otimes^{\mathbf{L}} \mathbb{Z}/n)$. \square

Since direct limits commute with direct sums, we obtain an exact sequence

$$0 \rightarrow \mathbb{D}_{\text{sh}}^1(U, \varinjlim_n \mathcal{C}' \otimes^{\mathbf{L}} \mathbb{Z}/n) \rightarrow \mathbb{H}^1(U, \varinjlim_n \mathcal{C}' \otimes^{\mathbf{L}} \mathbb{Z}/n) \rightarrow \varinjlim_n \prod_v H^1(K_v, \mathcal{C}' \otimes^{\mathbf{L}} \mathbb{Z}/n)$$

via

$$\bigoplus_v \varinjlim_n H^1(K_v, \mathcal{C}' \otimes^{\mathbf{L}} \mathbb{Z}/n) \simeq \varinjlim_n \bigoplus_v H^1(K_v, \mathcal{C}' \otimes^{\mathbf{L}} \mathbb{Z}/n) \subset \varinjlim_n \prod_v H^1(K_v, \mathcal{C}' \otimes^{\mathbf{L}} \mathbb{Z}/n).$$

Lemma 2.3.27. *The following is a perfect pairing of abelian groups*

$$\varinjlim_n \mathbb{D}^0(U, \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n) \times \mathbb{D}_{\text{sh}}^1(U, \varinjlim_n \mathcal{C}' \otimes^{\mathbf{L}} \mathbb{Z}/n) \rightarrow \mathbb{Q}/\mathbb{Z}.$$

Proof. We consider the following diagram with exact rows

$$\begin{array}{ccccc} 0 \longrightarrow \mathbb{D}_{\text{sh}}^1(U, \varinjlim_n \mathcal{C}'/n) & \longrightarrow & \mathbb{H}^1(U, \varinjlim_n \mathcal{C}'/n) & \longrightarrow & \varinjlim_n \prod H^1(K_v, \mathcal{C}'/n) \\ & \uparrow & \uparrow \simeq & & \uparrow * \\ & \mathbb{D}^0(U, \mathcal{C}/n) & \mathbb{H}_c^0(U, \mathcal{C}/n) & & \mathbb{H}^{-1}(K_v, \mathcal{C}/n) \\ 0 \longrightarrow \varinjlim_n (\mathbb{D}^0(U, \mathcal{C}/n))^D & \longrightarrow & \varinjlim_n (\mathbb{H}_c^0(U, \mathcal{C}/n))^D & \longrightarrow & \varinjlim_n (\bigoplus \mathbb{H}^{-1}(K_v, \mathcal{C}/n))^D \end{array} \quad (2.12)$$

where $A/n := A \otimes^{\mathbf{L}} \mathbb{Z}/n$ for a short complex A of tori and the lower row is exact by Proposition 2.2.5. Note that the arrow $*$ is constructed as the composition

$$\varinjlim_n \left(\bigoplus_v \mathbb{H}^{-1}(K_v, \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n) \right)^D \simeq \varinjlim_n \prod_v \left(\mathbb{H}^{-1}(K_v, \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n)^D \right) \simeq \varinjlim_n \prod_v \mathbb{H}^1(K_v, \mathcal{C}' \otimes^{\mathbf{L}} \mathbb{Z}/n)$$

where the last isomorphism follows from local dualities. The square in diagram (2.12) commutes by Lemma 2.3.25. Now a diagram chase shows that the left vertical arrow is an isomorphism. \square

2.4 Poitou–Tate sequences

In this section, we prove the following

Theorem. *Suppose either $\text{Ker } \rho$ is finite or $\text{Coker } \rho$ is trivial. Then there is an exact sequence of topological abelian groups*

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \mathbb{H}^{-1}(K, C)_\wedge & \longrightarrow & \mathbb{P}^{-1}(K, C)_\wedge & \longrightarrow & \mathbb{H}^2(K, C')^D \\
 & & \underbrace{\hspace{10em}} & & & & \\
 & & \longrightarrow & \mathbb{H}^0(K, C)_\wedge & \longrightarrow & \mathbb{P}^0(K, C)_\wedge & \longrightarrow & \mathbb{H}^1(K, C')^D \\
 & & \underbrace{\hspace{10em}} & & & & \\
 & & \longrightarrow & \mathbb{H}^1(K, C) & \longrightarrow & \mathbb{P}^1(K, C)_{\text{tors}} & \longrightarrow & (\mathbb{H}^0(K, C')_\wedge)^D \\
 & & \underbrace{\hspace{10em}} & & & & \\
 & & \longrightarrow & \mathbb{H}^2(K, C) & \longrightarrow & \mathbb{P}^2(K, C)_{\text{tors}} & \longrightarrow & (\mathbb{H}^{-1}(K, C')_\wedge)^D \longrightarrow 0
 \end{array} \tag{2.13}$$

where $\mathbb{P}^i(K, C)_{\text{tors}}$ denotes the torsion subgroup of the group $\mathbb{P}^i(K, C)$ for $i = 1, 2$.

Remark 2.4.1.

(1) Actually in the diagram above, the first row

$$0 \longrightarrow \mathbb{H}^{-1}(K, C)_\wedge \longrightarrow \mathbb{P}^{-1}(K, C)_\wedge \longrightarrow \mathbb{H}^2(K, C')^D$$

and the last row

$$\mathbb{H}^2(K, C) \longrightarrow \mathbb{P}^2(K, C)_{\text{tors}} \longrightarrow (\mathbb{H}^{-1}(K, C')_\wedge)^D \longrightarrow 0$$

are exact for an arbitrary short complex C .

(2) To prove the exactness of the middle rows, the assumptions on ρ really play a role. Moreover, we shall need global dualities between $\mathbb{H}_\wedge^0(C)$ and $\mathbb{H}^2(C')$ to connect the first two rows and the last two rows. As we have seen during the proof of Theorem 2.3.20, it is necessary to assume $\text{Ker } \rho$ is finite.

(3) Finally, according to Lemma 2.3.1, the subscripts in the first and the third rows are superfluous when $\text{Ker } \rho$ is finite, and the subscripts in the second and the last rows are superfluous when $\text{Coker } \rho$ is trivial. \square

We begin with the topologies on $\mathbb{H}^i(K, C)$ and $\mathbb{P}^i(K, C)$.

- For each i , the groups $\mathbb{H}^i(K, C)$ are endowed with the discrete topology and $\mathbb{H}^i(K, C)_\wedge$ are endowed with the subspace topology of the product $\prod_n \mathbb{H}^i(K, C)/n$. Its topology is not profinite since each component $\mathbb{H}^i(K, C)/n$ is not necessarily a finite group in general.
- For $i = -1, 0$, we give $\mathbb{P}^i(K, C)$ the restricted product topology. Moreover, the group $\mathbb{P}^i(K, C)_\wedge$ is equipped with the subspace topology of the product $\prod_n \mathbb{P}^i(K, C)/n$.
- For $i = 1, 2$, The group $\mathbb{P}^i(K, C)_{\text{tors}}$ is endowed with the direct limit topology. More precisely, ${}_n\mathbb{P}^i(K, C)$ is equipped with the restricted product topology with respect to the discrete topology on each ${}_n\mathbb{H}^i(K_v, C)$, and their direct limit $\mathbb{P}^i(K, C)_{\text{tors}}$ is equipped with the corresponding direct limit topology.

Now we give a more concrete statement of the main result of this section.

Theorem 2.4.2.

- Suppose $\text{Ker } \rho$ is finite. There is a 12-term exact sequence of topological abelian groups

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \mathbb{H}^{-1}(K, C) & \longrightarrow & \mathbb{P}^{-1}(K, C) & \longrightarrow & \mathbb{H}^2(K, C')^D \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & \mathbb{H}^0(K, C)_\wedge & \longrightarrow & \mathbb{P}^0(K, C)_\wedge & \longrightarrow & \mathbb{H}^1(K, C')^D \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & \mathbb{H}^1(K, C) & \longrightarrow & \mathbb{P}^1(K, C) & \longrightarrow & \mathbb{H}^0(K, C')^D \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & \mathbb{H}^2(K, C) & \longrightarrow & \mathbb{P}^2(K, C)_{\text{tors}} & \longrightarrow & (\mathbb{H}^{-1}(K, C')_\wedge)^D \longrightarrow 0
 \end{array}$$

- Suppose $\text{Coker } \rho$ is trivial. There is a 12-term exact sequence of topological abelian groups

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \mathbb{H}^{-1}(K, C)_\wedge & \longrightarrow & \mathbb{P}^{-1}(K, C)_\wedge & \longrightarrow & \mathbb{H}^2(K, C')^D \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & \mathbb{H}^0(K, C) & \longrightarrow & \mathbb{P}^0(K, C) & \longrightarrow & \mathbb{H}^1(K, C')^D \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & \mathbb{H}^1(K, C) & \longrightarrow & \mathbb{P}^1(K, C)_{\text{tors}} & \longrightarrow & (\mathbb{H}^0(K, C')_\wedge)^D \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & \mathbb{H}^2(K, C) & \longrightarrow & \mathbb{P}^2(K, C) & \longrightarrow & \mathbb{H}^{-1}(K, C')^D \longrightarrow 0
 \end{array}$$

The proof of the theorem consists of several steps. We first establish perfect pairings between the restricted topological products for any short complex C . Subsequently, we deduce the exactness of the first and the last rows again for any C . Finally, we deal with the more complicated exact sequence in the middle of the diagram with either $\text{Ker } \rho$ being finite or $\text{Coker } \rho$ being trivial.

2.4.1 Step 1: Dualities of restricted topological products

We proceed as in the finite level to obtain pairings between $\mathbb{P}^i(K, C)_\wedge$ and $\mathbb{P}^{1-i}(K, C')_{\text{tors}}$ for $i = -1, 0$. Recall Lemma 2.1.5 that $\mathbb{H}^i(\mathcal{O}_v, \mathcal{C})/n \rightarrow \mathbb{H}^i(K_v, C)/n$ is injective for each $v \in X_0^{(1)}$. Therefore we are allowed to identify $\mathbb{H}^i(\mathcal{O}_v, \mathcal{C})_\wedge$ with a subgroup of $\mathbb{H}^i(K_v, C)_\wedge$ for $v \in X_0^{(1)}$ by the left exactness of inverse limits. In this step, all the conclusions are valid without any assumption on the short complex C .

Proposition 2.4.3. *For $i = -1, 0$, the annihilator of $\mathbb{H}^{1-i}(\mathcal{O}_v, C')$ is $\mathbb{H}^i(\mathcal{O}_v, \mathcal{C})_\wedge$ under the perfect pairing*

$$\mathbb{H}^i(K_v, C)_\wedge \times \mathbb{H}^{1-i}(K_v, C') \rightarrow \mathbb{Q}/\mathbb{Z}.$$

Proof. We consider the following commutative diagram with exact rows for $i = -1, 0$

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \mathbb{H}^i(\mathcal{O}_v, \mathcal{C})/n & \longrightarrow & \mathbb{H}^i(\mathcal{O}_v, \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n) & \longrightarrow & \mathbb{H}^{i+1}(\mathcal{O}_v, \mathcal{C}) \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & \mathbb{H}^i(K_v, C)/n & \longrightarrow & \mathbb{H}^i(K_v, C \otimes^{\mathbf{L}} \mathbb{Z}/n) & \longrightarrow & \mathbb{H}^{i+1}(K_v, C).
 \end{array}$$

Take $t \in \mathbb{H}^i(K_v, C)/n$ such that t is orthogonal to ${}_n\mathbb{H}^{1-i}(\mathcal{O}_v, C')$. Then the image s of t in $\mathbb{H}^i(K_v, C \otimes^{\mathbf{L}} \mathbb{Z}/n)$ is orthogonal to $\mathbb{H}^{-i}(\mathcal{O}_v, C' \otimes^{\mathbf{L}} \mathbb{Z}/n)$ and it follows that $s \in \mathbb{H}^i(\mathcal{O}_v, C \otimes^{\mathbf{L}} \mathbb{Z}/n)$ by Proposition 2.2.2. The right vertical arrow is injective by Lemma 2.1.3, thus a diagram chasing shows that t lies in $\mathbb{H}^i(\mathcal{O}_v, C)/n$. \square

Corollary 2.4.4. *For $i = -1, 0$, there are isomorphisms $\mathbb{P}^i(K, C)_{\wedge} \simeq (\mathbb{P}^{1-i}(K, C')_{\text{tors}})^D$ of locally compact groups.*

Proof. This is an immediate consequence of Lemma 2.1.5 and Proposition 2.4.3. \square

2.4.2 Step 2: Exactness of the first and the last rows

In this step, all the conclusions are valid without any assumption on the short complex C .

Proposition 2.4.5. *There is an exact sequence of locally compact groups*

$$0 \rightarrow \mathbb{H}^2(C) \rightarrow \mathbb{H}^2(K, C) \rightarrow \mathbb{P}^2(K, C)_{\text{tors}} \rightarrow (\mathbb{H}^{-1}(K, C')_{\wedge})^D \rightarrow 0. \quad (2.14)$$

Proof. We consider the following commutative diagram with exact rows (by respective Kummer sequences) and middle column (by Theorem 2.2.10):

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{H}^1(K, C)/n & \longrightarrow & \mathbb{H}^1(K, C \otimes^{\mathbf{L}} \mathbb{Z}/n) & \longrightarrow & {}_n\mathbb{H}^2(K, C) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathbb{P}^1(K, C)/n & \longrightarrow & \mathbb{P}^1(K, C \otimes^{\mathbf{L}} \mathbb{Z}/n) & \longrightarrow & {}_n\mathbb{P}^2(K, C) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & ({}_n\mathbb{H}^0(K, C'))^D & \longrightarrow & \mathbb{H}^{-1}(K, C' \otimes^{\mathbf{L}} \mathbb{Z}/n)^D & \longrightarrow & (\mathbb{H}^{-1}(K, C')/n)^D & \longrightarrow & 0. \end{array} \quad (2.15)$$

Taking direct limit over all n of the last two columns in diagram (2.15) yields the commutative diagram

$$\begin{array}{ccccccc} \varinjlim_n \mathbb{H}^1(K, C \otimes^{\mathbf{L}} \mathbb{Z}/n) & \longrightarrow & \varinjlim_n \mathbb{P}^1(K, C \otimes^{\mathbf{L}} \mathbb{Z}/n) & \longrightarrow & (\varinjlim_n \mathbb{H}^{-1}(K, C' \otimes^{\mathbf{L}} \mathbb{Z}/n))^D & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ \mathbb{H}^2(K, C) & \longrightarrow & \mathbb{P}^2(K, C)_{\text{tors}} & \longrightarrow & (\mathbb{H}^{-1}(K, C')_{\wedge})^D & & \end{array} \quad (2.16)$$

Since $\mathbb{H}^2(K, C)$ is torsion and $\mathbb{H}^3(K, C) = 0$, taking direct limit of the Kummer sequence

$$0 \rightarrow \mathbb{H}^2(K, C)/n \rightarrow \mathbb{H}^2(K, C \otimes^{\mathbf{L}} \mathbb{Z}/n) \rightarrow {}_n\mathbb{H}^3(K, C) \rightarrow 0.$$

yields $\varinjlim_n \mathbb{H}^2(K, C \otimes^{\mathbf{L}} \mathbb{Z}/n) = 0$. Taking direct limit in Theorem 2.2.10 yields the exactness of the upper row. The left vertical arrow in (2.16) is an isomorphism since $\mathbb{H}^1(K, C) \otimes \mathbb{Q}/\mathbb{Z} = 0$ and the middle one is surjective by the exactness of \varinjlim_n . If the right vertical arrow is an isomorphism, then a diagram chasing yields the exact sequence (2.14).

So it remains to show $\varprojlim_n \mathbb{H}^{-1}(K, C' \otimes^{\mathbb{L}} \mathbb{Z}/n) \simeq \mathbb{H}^{-1}(K, C')_{\wedge}$. We see that the vanishing $\varprojlim_n {}_n\mathbb{H}^0(K, C') = 0$ is enough because of the Kummer sequence

$$0 \rightarrow \mathbb{H}^{-1}(K, C')/n \rightarrow \mathbb{H}^{-1}(K, C' \otimes^{\mathbb{L}} \mathbb{Z}/n) \rightarrow {}_n\mathbb{H}^0(K, C') \rightarrow 0,$$

and therefore we reduce to show $\mathbb{H}^0(K, C')_{\text{tors}}$ has finite exponent. Indeed, as $H^0(K, \mathbb{G}_m)_{\text{tors}} = (K^{\times})_{\text{tors}} = (k^{\times})_{\text{tors}}$ is finite, we see that $H^0(K, P)_{\text{tors}}$ has finite exponent for a K -torus P by a restriction-corestriction argument. The distinguished triangle $\text{Ker } \rho'[1] \rightarrow C' \rightarrow \text{Coker } \rho' \rightarrow \text{Ker } \rho'[2]$ yields an exact sequence $0 \rightarrow H^1(K, \text{Ker } \rho')_{\text{tors}} \rightarrow \mathbb{H}^0(K, C')_{\text{tors}} \rightarrow \mathbb{H}^0(K, \text{Coker } \rho')_{\text{tors}}$. By Lemma 2.3.1(4) $H^1(K, \text{Ker } \rho')$ has finite exponent, so is $\mathbb{H}^0(K, C')_{\text{tors}}$ by dévissage. \square

Lemma 2.4.6. *Let $A_1 \rightarrow A_2 \rightarrow A_3 \rightarrow A_4$ be an exact sequence of Hausdorff, second countable and locally compact topological abelian groups with continuous maps. Then the map $A_2 \rightarrow A_3$ is strict. In particular, we have an exact sequence $A_3^D \rightarrow A_2^D \rightarrow A_1^D$.*

Proof. We show that $A_2/\text{Im } A_1 \rightarrow A_3$ induces a homeomorphism onto a closed subgroup. Its image is closed since it equals the closed subgroup $\text{Ker}(A_3 \rightarrow A_4)$ of A_3 . It induces a homeomorphism onto its image by [Bou74, Chapitre IX, §5, Proposition 6]. Now if an element of A_2^D goes to zero in A_1^D , then it becomes an element of $(A_2/\text{Im } A_1)^D$. But $A_2/\text{Im } A_1 \rightarrow A_3$ is a homeomorphism onto a closed subgroup of A_3 , we conclude that $A_3^D \rightarrow A_2^D \rightarrow A_1^D$ is exact. \square

Corollary 2.4.7. *There is an exact sequence of locally compact groups*

$$0 \rightarrow \mathbb{H}^{-1}(K, C)_{\wedge} \rightarrow \mathbb{P}^{-1}(K, C)_{\wedge} \rightarrow \mathbb{H}^2(K, C')^D \rightarrow \mathbb{H}^2(C')^D \rightarrow 0.$$

Proof. Applying Proposition 2.4.5 to C' yields an exact sequence

$$\mathbb{H}^2(K, C') \rightarrow \mathbb{P}^2(K, C')_{\text{tors}} \rightarrow (\mathbb{H}^{-1}(K, C)_{\wedge})^D \rightarrow 0. \quad (2.17)$$

It follows that the desired sequence is exact at the first three terms by dualizing the sequence (2.17) and applying Corollary 2.4.4 and Lemma 2.4.6. Applying Lemma 2.4.6 to the exact sequence $0 \rightarrow \mathbb{H}^2(C') \rightarrow \mathbb{H}^2(K, C') \rightarrow \mathbb{P}^2(K, C')_{\text{tors}}$ yields the desired exactness at the last three terms. \square

2.4.3 Step 3: Exactness of middle rows: finite kernel case

We will systematically assume that $M := \text{Ker } \rho$ is finite from Proposition 2.4.8 to Proposition 2.4.12.

Proposition 2.4.8. *There is an exact sequence of locally compact groups*

$$\varprojlim_n \mathbb{H}^0(K, C \otimes^{\mathbb{L}} \mathbb{Z}/n) \rightarrow \varprojlim_n \mathbb{P}^0(K, C \otimes^{\mathbb{L}} \mathbb{Z}/n) \rightarrow \mathbb{H}^0(K, \varinjlim_n C' \otimes^{\mathbb{L}} \mathbb{Z}/n)^D.$$

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Proof. The proof is similar to [Dem11a, Lemme 6.2]. We consider the following commutative diagram

$$\begin{array}{ccccccccc}
H^0(K, T_{\mathbb{Z}/n}(C)) & \xrightarrow{\Phi_n^K} & H^2(K, {}_nM) & \longrightarrow & \mathbb{H}^0(K, C/n) & \longrightarrow & H^1(K, T_{\mathbb{Z}/n}(C)) & \xrightarrow{\Psi_n^K} & H^3(K, {}_nM) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathbb{P}^0(K, T_{\mathbb{Z}/n}(C)) & \xrightarrow{\Phi_n^\Pi} & \mathbb{P}^2(K, {}_nM) & \longrightarrow & \mathbb{P}^0(K, C/n) & \longrightarrow & \mathbb{P}^1(K, T_{\mathbb{Z}/n}(C)) & \xrightarrow{\Psi_n^\Pi} & \mathbb{P}^3(K, {}_nM) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
H^3(K, T_{\mathbb{Z}/n}(C)')^D & \rightarrow & H^1(K, ({}_nM)')^D & \rightarrow & \mathbb{H}^0(K, C'/n)^D & \rightarrow & H^2(K, T_{\mathbb{Z}/n}(C)')^D & \rightarrow & H^0(K, ({}_nM)')^D
\end{array}$$

where $A/n := A \otimes^{\mathbf{L}} \mathbb{Z}/n$ for $A = C, C'$, the upper two rows are exact since they are induced by the distinguished triangle ${}_nM[2] \rightarrow C \otimes^{\mathbf{L}} \mathbb{Z}/n \rightarrow T_{\mathbb{Z}/n}(C)[1] \rightarrow {}_nM[3]$, and the columns except the middle one are exact by [HSS15, Theorem 2.3]. The product $\mathbb{P}^0(K, T_{\mathbb{Z}/n}(C)) = \prod_{v \in X^{(1)}} \mathbb{H}^0(K_v, T_{\mathbb{Z}/n}(C))$ is compact since $H^0(K_v, T_{\mathbb{Z}/n}(C))$ is finite. Now taking inverse limit of the middle three columns of the above diagram over all n yields the following commutative diagram

$$\begin{array}{ccccccc}
\varprojlim_n \text{Coker } \Phi_n^K & \longrightarrow & \varprojlim_n H^0(K, C/n) & \longrightarrow & \varprojlim_n \text{Ker } \Psi_n^K & \longrightarrow & \varprojlim_n^1 \text{Coker } \Phi_n^K \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\varprojlim_n \text{Coker } \Phi_n^\Pi & \longrightarrow & \varprojlim_n \mathbb{P}^0(K, C/n) & \longrightarrow & \varprojlim_n \text{Ker } \Psi_n^\Pi & \longrightarrow & \varprojlim_n^1 \text{Coker } \Phi_n^\Pi \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
& & \varprojlim_n H^0(K, C'/n)^D & \rightarrow & \varprojlim_n H^2(K, T_{\mathbb{Z}/n}(C)')^D & \rightarrow & \varprojlim_n^1 H^0(K, ({}_nM)')^D.
\end{array}$$

Moreover, the third column of the above diagram also fits into the commutative diagram

$$\begin{array}{ccccc}
0 & \longrightarrow & \varprojlim_n \text{Ker } \Psi_n^K & \longrightarrow & \varprojlim_n H^1(K, T_{\mathbb{Z}/n}(C)) & \longrightarrow & \varprojlim_n \text{Im } \Psi_n^K \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \varprojlim_n \text{Ker } \Psi_n^\Pi & \longrightarrow & \varprojlim_n \mathbb{P}^1(K, T_{\mathbb{Z}/n}(C)) & \longrightarrow & \varprojlim_n \text{Im } \Psi_n^\Pi \\
& & & & \downarrow & & \\
& & & & \varprojlim_n H^2(K, T_{\mathbb{Z}/n}(C)')^D & &
\end{array} \tag{2.18}$$

with $\varprojlim_n \text{Im } \Psi_n^K \subset \varprojlim_n H^3(K, {}_nM) = 0$ being zero (because M is finite). We conclude that $\varprojlim_n \text{Ker } \Psi_n^K \rightarrow \varprojlim_n H^1(K, T_{\mathbb{Z}/n}(C))$ is surjective. Let $\Delta_n : H^1(K, T_{\mathbb{Z}/n}(C)) \rightarrow \mathbb{P}^1(K, T_{\mathbb{Z}/n}(C))$ denote the diagonal map. Thanks to [Jen72, Théorème 7.3], \varprojlim_n^1 of the finite groups $\text{Ker } \Delta_n$ vanishes, therefore $\varprojlim_n H^1(K, T_{\mathbb{Z}/n}(C)) \rightarrow \varprojlim_n H^1(K, T_{\mathbb{Z}/n}(C))/\text{Ker } \Delta_n$ is surjective. Now it follows that the middle column is exact.

Take $\alpha \in \varprojlim_n \mathbb{P}^0(K, C \otimes^{\mathbf{L}} \mathbb{Z}/n)$ such that it goes to zero in $\varprojlim_n \mathbb{H}^0(K, C' \otimes^{\mathbf{L}} \mathbb{Z}/n)^D$. By functoriality, $\beta := \text{Im } \alpha \in \varprojlim_n \mathbb{P}^1(K, T_{\mathbb{Z}/n}(C))$ goes to zero in $\varprojlim_n H^2(K, T_{\mathbb{Z}/n}(C)')^D$. Thus β

comes from $\gamma \in \varprojlim_n H^1(K, T_{\mathbb{Z}/n}(C))$ by the exactness of the middle column in diagram (2.18). Since $\varprojlim_n \text{Ker } \Psi_n^K \rightarrow \varprojlim_n H^1(K, T_{\mathbb{Z}/n}(C))$ is surjective, γ comes from some $\gamma' \in \varprojlim_n \text{Ker } \Psi_n^K$. But $\varprojlim_n^1 \text{Coker } \Phi_n^K \rightarrow \varprojlim_n^1 \text{Coker } \Phi_n^\Pi$ is injective by Lemma 2.4.9 below, so γ' comes from $\tau \in \varprojlim_n H^1(K, C/n)$. By construction we see that $\alpha, \text{Im } \tau \in \varprojlim_n \mathbb{P}^0(K, C \otimes^{\mathbf{L}} \mathbb{Z}/n)$ have the same image in $\varprojlim_n \text{Ker } \Psi_n^\Pi$, hence α and $\text{Im } \tau$ differs from an element in $\varprojlim_n \mathbb{P}^2(K, {}_nM)$. Recall that M is finite, thus $\varprojlim_n \text{Coker } \Phi_n^\Pi \subset \varprojlim_n \mathbb{P}^2(K, {}_nM) = 0$ and α comes from $\varprojlim_n H^1(K, C/n)$. \square

Lemma 2.4.9. *The homomorphism $\varprojlim_n^1 \text{Coker } \Phi_n^K \rightarrow \varprojlim_n^1 \text{Coker } \Phi_n^\Pi$ is an isomorphism.*

Proof. As $H^0(L, T_{\mathbb{Z}/n}(C))$ is finite for $L = K, K_v$, $\mathbb{P}^0(K, T_{\mathbb{Z}/n}(C)) = \prod_{v \in X^{(1)}} H^0(K_v, T_{\mathbb{Z}/n}(C))$ is compact. Thus we obtain $\varprojlim_n^1 H^0(K, T_{\mathbb{Z}/n}(C)) = 0$ and $\varprojlim_n^1 \mathbb{P}^0(K, T_{\mathbb{Z}/n}(C)) = 0$ by [Jen72, Théorème 7.3]. Moreover, the image $\text{Im } \Phi_n^K$ of $H^0(K, T_{\mathbb{Z}/n}(C))$ in $H^2(K, {}_nM)$ is finite, so $\varprojlim_n^1 H^2(K, {}_nM) \simeq \varprojlim_n^1 \text{Coker } \Phi_n^K$ by the short exact sequence $0 \rightarrow \text{Im } \Phi_n^K \rightarrow H^2(K, {}_nM) \rightarrow \text{Coker } \Phi_n^K \rightarrow 0$. Similarly, we obtain $\varprojlim_n^1 \mathbb{P}^2(K, {}_nM) \simeq \varprojlim_n^1 \text{Coker } \Phi_n^\Pi$.

Let I_n denote the image of $H^2(K, {}_nM) \rightarrow \mathbb{P}^2(K, {}_nM)$. So

$$0 \rightarrow \mathbb{I}^2({}_nM) \rightarrow H^2(K, {}_nM) \rightarrow I_n \rightarrow 0$$

is an exact sequence. The finiteness of $\mathbb{I}^2({}_nM)$ yields an isomorphism $\varprojlim_n^1 H^2(K, {}_nM) \simeq \varprojlim_n^1 I_n$. Moreover, the cokernel of $I_n \rightarrow \mathbb{P}^2(K, {}_nM)$ is a subgroup of the group $H^1(K, ({}_nM)')^D$ (see the proof of Proposition 2.4.8). The finiteness of M yields⁶

$$\varprojlim_n (H^1(K, ({}_nM)')^D) \simeq H^1(K, \varinjlim_n ({}_nM)')^D = 0$$

and thus $\varprojlim_n \text{Coker}(I_n \rightarrow \mathbb{P}^2(K, {}_nM)) = 0$ by the left exactness of inverse limits. Moreover, the finiteness of M implies that there is an integer (say e) such that ${}_{ne}M \rightarrow {}_nM$ the multiplication by e is trivial for each integer n . In particular, the system $\{\text{Coker}(I_n \rightarrow \mathbb{P}^2(K, {}_nM))\}$ satisfies the Mittag-Leffler condition, hence $\varprojlim_n^1 \text{Coker}(I_n \rightarrow \mathbb{P}^2(K, {}_nM)) = 0$ by [Wei94, Proposition 3.5.7]. We conclude that $\varprojlim_n^1 I_n \rightarrow \varprojlim_n^1 \mathbb{P}^2(K, {}_nM)$ is an isomorphism and therefore we have $\varprojlim_n^1 \text{Coker } \Phi_n^K \simeq \varprojlim_n^1 I_n \simeq \varprojlim_n^1 \text{Coker } \Phi_n^\Pi$. \square

Corollary 2.4.10. *There is an exact sequence of locally compact groups*

$$0 \rightarrow \mathbb{I}^0_\wedge(C) \rightarrow \mathbb{H}^0(K, C)_\wedge \rightarrow \mathbb{P}^0(K, C)_\wedge \rightarrow \mathbb{H}^1(K, C')^D \rightarrow \mathbb{I}^1(C')^D \rightarrow 0.$$

Proof. We consider again the diagram (2.10) with exact rows and middle column (by Proposition 2.4.8):

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{H}^0(K, C)_\wedge & \longrightarrow & \varprojlim_n \mathbb{H}^0(K, C \otimes^{\mathbf{L}} \mathbb{Z}/n) & \longrightarrow & \Phi_K \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathbb{P}^0(K, C)_\wedge & \longrightarrow & \varprojlim_n \mathbb{P}^0(K, C \otimes^{\mathbf{L}} \mathbb{Z}/n) & \longrightarrow & \Phi_\Pi \longrightarrow 0. \\ & & \downarrow & & \downarrow & & \\ & & \mathbb{H}^1(K, C')^D & \longrightarrow & \mathbb{H}^0(K, \varinjlim_n C' \otimes^{\mathbf{L}} \mathbb{Z}/n)^D & & \end{array}$$

⁶Note that the transition maps are not given by the inclusion $({}_nM)' \rightarrow ({}_{nm}M)'$ in which case we obtain $\varinjlim_n H^1(K, ({}_nM)') \simeq H^1(K, M')$.

2.4. POITOU–TATE SEQUENCES

Since $\mathbb{H}^0(K, C' \otimes^{\mathbf{L}} \mathbb{Z}/n) \rightarrow {}_n\mathbb{H}^1(K, C')$ is a surjective map between finite discrete groups, there is an injective map $({}_n\mathbb{H}^1(K, C'))^D \rightarrow (\mathbb{H}^0(K, C' \otimes^{\mathbf{L}} \mathbb{Z}/n))^D$ and hence an injection $\mathbb{H}^1(K, C')^D \rightarrow \mathbb{H}^0(K, \varprojlim_n C' \otimes^{\mathbf{L}} \mathbb{Z}/n)^D$. It follows that the left vertical column is a complex by diagram chasing. But the map $\Phi_K \rightarrow \Phi_{\Pi}$ is injective (see the proof of Lemma 2.3.21), another diagram chasing then tells us the left column is exact.

To show the exactness of the last three terms, we consider the exact sequence $0 \rightarrow \mathbb{I}\mathbb{I}^1(C') \rightarrow \mathbb{H}^1(K, C') \rightarrow \mathbb{P}^1(K, C')_{\text{tors}}$. Now we obtain the desired exactness by Lemma 2.4.6. \square

Remark 2.4.11. By Lemma 2.3.1, the groups $\mathbb{H}^{-1}(K, C)$ and $\mathbb{P}^{-1}(K, C)$ are torsion of finite exponent, therefore they are isomorphic to respective completions $\mathbb{H}^{-1}(K, C)_{\wedge}$ and $\mathbb{P}^{-1}(K, C)_{\wedge}$. Again by Lemma 2.3.1, $\mathbb{H}^1(K, C)$ and $\mathbb{P}^1(K, C)$ have finite exponents. Consequently, we obtain $\mathbb{H}^1(K, C) \simeq \mathbb{H}^1(K, C)_{\wedge}$ and $\mathbb{P}^1(K, C)_{\text{tors}} = \mathbb{P}^1(K, C) \simeq \mathbb{P}^1(K, C)_{\wedge}$. Moreover, the finiteness of M implies that $\rho' : T'_2 \rightarrow T'_1$ is surjective and hence $\mathbb{H}^0(K, C')$ has finite exponent by Lemma 2.3.1. Summing up, the subscripts "tors" and " \wedge " in the first and the third rows of (2.13) are superfluous.

Proposition 2.4.12. *There is an exact sequence of locally compact groups*

$$0 \rightarrow \mathbb{I}\mathbb{I}^1(C) \rightarrow \mathbb{H}^1(K, C) \rightarrow \mathbb{P}^1(K, C) \rightarrow \mathbb{H}^0(K, C')^D \rightarrow \mathbb{H}^2(K, C).$$

Proof. Taking inverse limit in diagram (2.15) yields a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{H}^1(K, C) & \longrightarrow & \varprojlim_n \mathbb{H}^1(K, C \otimes^{\mathbf{L}} \mathbb{Z}/n) & \longrightarrow & \Psi_K \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathbb{P}^1(K, C) & \longrightarrow & \varprojlim_n \mathbb{P}^1(K, C \otimes^{\mathbf{L}} \mathbb{Z}/n) & \longrightarrow & \Psi_{\Pi} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathbb{H}^0(K, C')^D & \longrightarrow & \varprojlim_n \mathbb{H}^{-1}(K, C' \otimes^{\mathbf{L}} \mathbb{Z}/n)^D & & \end{array}$$

We observe that $\Psi_K \rightarrow \Psi_{\Pi}$ is injective because its kernel is contained in $\varprojlim_n {}_n\mathbb{I}\mathbb{I}^2(C) = 0$ (recall that $\mathbb{I}\mathbb{I}^2(C)$ is finite). The exactness of the middle column follows from the vanishing $\varprojlim_n^1 \mathbb{I}\mathbb{I}^1(C \otimes^{\mathbf{L}} \mathbb{Z}/n) = 0$ (by [Jen72, Théorème 7.3]). Thus a diagram chasing yields the exactness of the left column.

Now we verify the exactness of $\mathbb{P}^1(K, C) \rightarrow \mathbb{H}^0(K, C')^D \rightarrow \mathbb{H}^2(K, C)$. Consider the commutative diagram with vertical arrows obtained from respective Kummer sequences:

$$\begin{array}{ccccc} \mathbb{P}^0(K, C \otimes^{\mathbf{L}} \mathbb{Z}/n) & \longrightarrow & \mathbb{H}^0(K, C' \otimes^{\mathbf{L}} \mathbb{Z}/n)^D & \longrightarrow & \mathbb{H}^1(K, C \otimes^{\mathbf{L}} \mathbb{Z}/n) \\ \downarrow & & \downarrow & & \downarrow \\ {}_n\mathbb{P}^1(K, C) & \longrightarrow & (\mathbb{H}^0(K, C')/n)^D & \xrightarrow{*} & {}_n\mathbb{H}^2(K, C), \end{array} \tag{2.19}$$

where the upper row is exact by Theorem 2.2.10 and the arrow $*$ is the composite

$$(\mathbb{H}^0(K, C')/n)^D \rightarrow (\mathbb{I}\mathbb{I}^0(C')/n)^D \simeq {}_n\mathbb{I}\mathbb{I}^2(C) \rightarrow {}_n\mathbb{H}^2(K, C).$$

The left square in (2.19) commutes by Proposition 2.2.1 and the right one commutes by construction and Theorem 2.3.19. Finally, the middle vertical arrow is surjective because

$\mathbb{H}^0(K, C' \otimes^{\mathbf{L}} \mathbb{Z}/n)$ and $\mathbb{H}^0(K, C')/n$ are discrete groups. Passing to the direct limit over all n , the right vertical arrow in diagram (2.19) becomes an isomorphism as $\mathbb{H}^1(K, C) \otimes \mathbb{Q}/\mathbb{Z} = 0$. Now a diagram chasing implies the exactness of $\mathbb{P}^1(K, C) \rightarrow \mathbb{H}^0(K, C')^D \rightarrow \mathbb{H}^2(K, C)$ (here we used the identification $\mathbb{H}^0(K, C') \simeq \mathbb{H}^0(K, C')_{\wedge}$). \square

2.4.4 Step 4: Exactness of middle rows: surjective case

Suppose $\rho : T_1 \rightarrow T_2$ is surjective. Thus $\mathbf{X}^*(T_2) \rightarrow \mathbf{X}^*(T_1)$ is injective and $\mathbf{X}_*(T_1) \rightarrow \mathbf{X}_*(T_2)$ has finite cokernel. Moreover, the morphism $\rho' : T'_2 \rightarrow T'_1$ has finite kernel. Therefore $\mathbb{H}^{-1}(K, C')$, $\mathbb{H}^0(K, C)$, $\mathbb{H}^1(K, C')$ and $\mathbb{H}^2(K, C)$ are torsion groups having finite exponent by Lemma 2.3.1. Subsequently, we see that $\mathbb{H}^0(K, C)_{\wedge} = \mathbb{H}^0(K, C)$, $\mathbb{P}^0(K, C)_{\wedge} = \mathbb{P}^0(K, C)$, $\mathbb{P}^2(K, C)_{\text{tors}} = \mathbb{P}^2(K, C)$ and $\mathbb{H}^{-1}(K, C')_{\wedge} = \mathbb{H}^{-1}(K, C')$, i.e. the subscripts in diagram (2.13) are superfluous. After Corollary 2.4.7 and Proposition 2.4.5, it remains to show the following proposition.

Proposition 2.4.13. *Suppose $\rho : T_1 \rightarrow T_2$ is surjective. Then there is an exact sequence*

$$\begin{aligned} 0 \rightarrow \mathbb{H}^0(C) \rightarrow \mathbb{H}^0(K, C) \rightarrow \mathbb{P}^0(K, C) \rightarrow \mathbb{H}^1(K, C')^D \\ \rightarrow \mathbb{H}^1(K, C) \rightarrow \mathbb{P}^1(K, C)_{\text{tors}} \rightarrow (\mathbb{H}^0(K, C')_{\wedge})^D \rightarrow \mathbb{H}^2(K, C). \end{aligned}$$

Proof.

- We show the exactness of $\mathbb{H}^1(K, C) \rightarrow \mathbb{P}^1(K, C)_{\text{tors}} \rightarrow (\mathbb{H}^0(K, C')_{\wedge})^D$. Since $\mathbb{H}^1(K, C')$ has finite exponent, $\varprojlim_n \mathbb{H}^1(K, C') = 0$ and hence $\mathbb{H}^0(K, C')_{\wedge} \simeq \varprojlim_n \mathbb{H}^0(K, C' \otimes^{\mathbf{L}} \mathbb{Z}/n)$ by the Kummer sequence $0 \rightarrow \mathbb{H}^0(K, C')/n \rightarrow \mathbb{H}^0(K, C' \otimes^{\mathbf{L}} \mathbb{Z}/n) \rightarrow {}_n\mathbb{H}^1(K, C') \rightarrow 0$. Consider the commutative diagram with exact upper row by Theorem 2.2.10:

$$\begin{array}{ccccc} \varinjlim_n \mathbb{H}^0(K, C \otimes^{\mathbf{L}} \mathbb{Z}/n) & \longrightarrow & \varinjlim_n \mathbb{P}^0(K, C \otimes^{\mathbf{L}} \mathbb{Z}/n) & \longrightarrow & (\varprojlim_n \mathbb{H}^0(K, C' \otimes^{\mathbf{L}} \mathbb{Z}/n))^D \\ \downarrow & & \downarrow & & \downarrow \simeq \\ \mathbb{H}^1(K, C) & \longrightarrow & \mathbb{P}^1(K, C)_{\text{tors}} & \longrightarrow & (\mathbb{H}^0(K, C')_{\wedge})^D. \end{array}$$

Now the exactness of the lower row follows by diagram chasing.

- We consider the following commutative diagram (see (2.19)) for $i = -1, 0$:

$$\begin{array}{ccccc} \mathbb{P}^i(K, C \otimes^{\mathbf{L}} \mathbb{Z}/n) & \longrightarrow & \mathbb{H}^{-i}(K, C' \otimes^{\mathbf{L}} \mathbb{Z}/n)^D & \longrightarrow & \mathbb{H}^{i+1}(K, C \otimes^{\mathbf{L}} \mathbb{Z}/n) \\ \downarrow & & \downarrow & & \downarrow \\ {}_n\mathbb{P}^{i+1}(K, C) & \longrightarrow & (\mathbb{H}^{-i}(K, C')/n)^D & \longrightarrow & {}_n\mathbb{H}^{i+2}(K, C). \end{array} \quad (2.20)$$

Note that $\mathbb{H}^0(K, C) \simeq H^1(K, M)$ (since we have a quasi-isomorphism $C \simeq M[1]$) and $\mathbb{H}^1(K, C)$ are torsion. Thus $\mathbb{H}^{i+1}(K, C) \otimes \mathbb{Q}/\mathbb{Z} = 0$ for $i = -1, 0$. It follows that the right vertical arrow in diagram (2.20) becomes an isomorphism after taking direct limit. Therefore we get the exactness of $\mathbb{P}^0(K, C) \rightarrow \mathbb{H}^1(K, C')^D \rightarrow \mathbb{H}^1(K, C)$ and $\mathbb{P}^1(K, C)_{\text{tors}} \rightarrow (\mathbb{H}^0(K, C')_{\wedge})^D \rightarrow \mathbb{H}^2(K, C)$ by diagram chasing.

- Now we have an exact sequence $\mathbb{H}^1(K, C') \rightarrow \mathbb{P}^1(K, C')_{\text{tors}} \rightarrow (\mathbb{H}^0(K, C)_{\wedge})^D$ by the previous point. Dualizing it yields an exact sequence $\mathbb{H}^0(K, C)_{\wedge} \rightarrow \mathbb{P}^0(K, C)_{\wedge} \rightarrow \mathbb{H}^1(K, C')^D$. But the groups $\mathbb{H}^0(L, C)$ have finite exponents for $L = K, K_v$, thus we obtain an exact sequence $\mathbb{H}^0(K, C) \rightarrow \mathbb{P}^0(K, C) \rightarrow \mathbb{H}^1(K, C')^D$. \square

Example 2.4.14.

- (1) Let P be a K -torus that extends to an X_0 -torus \mathcal{P} . We consider the special case that $C = [0 \rightarrow P]$ and $C' = P'[1]$. By definition, $\mathbb{H}^{-1}(L, P) = 0$ for $L = K$ or K_v and hence the first two terms in diagram (2.13) vanish automatically. The third term vanishes by Lemma 2.3.1. Moreover, we see that the groups $H^1(K, P')$ and $\mathbb{P}^1(K, P)$ are torsion having finite exponents by Hilbert's Theorem 90. Thus $\mathbb{P}^1(K, P)_{\text{tors}} = \mathbb{P}^1(K, P)$ and the canonical map $H^1(K, P') \rightarrow H^1(K, P')_{\wedge}$ is an isomorphism. The remaining 9 terms in diagram (2.13) read as

$$\begin{array}{ccccccc}
 0 & \longrightarrow & H^0(K, P)_{\wedge} & \longrightarrow & \mathbb{P}^0(K, P)_{\wedge} & \longrightarrow & H^2(K, P')^D \\
 & & \longleftarrow & & \longleftarrow & & \longleftarrow \\
 & & H^1(K, P) & \longrightarrow & \mathbb{P}^1(K, P) & \longrightarrow & H^1(K, P')^D \\
 & & \longleftarrow & & \longleftarrow & & \longleftarrow \\
 & & H^2(K, P) & \longrightarrow & \mathbb{P}^2(K, P)_{\text{tors}} & \longrightarrow & (H^0(K, P')_{\wedge})^D \longrightarrow 0
 \end{array}$$

which is the Poitou–Tate exact sequence for tori [HSS15, Theorem 2.9].

- (2) Let M be a group of multiplicative type over K . We may embed it into a short exact sequence $0 \rightarrow M \rightarrow T_1 \rightarrow T_2 \rightarrow 0$ with T_1 and T_2 being K -tori. In particular, there is a quasi-isomorphism $M[1] \simeq C$ where $C := [T_1 \rightarrow T_2]$. In this case, the cokernel of $T_1 \rightarrow T_2$ is trivial and we obtain a Poitou–Tate sequence for short complexes C_M and thus for groups of multiplicative type:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & H^0(K, M)_{\wedge} & \longrightarrow & \mathbb{P}^0(K, M)_{\wedge} & \longrightarrow & \mathbb{H}^2(K, C')^D \\
 & & \longleftarrow & & \longleftarrow & & \longleftarrow \\
 & & H^1(K, M) & \longrightarrow & \mathbb{P}^1(K, M) & \longrightarrow & \mathbb{H}^1(K, C')^D \\
 & & \longleftarrow & & \longleftarrow & & \longleftarrow \\
 & & H^2(K, M) & \longrightarrow & \mathbb{P}^2(K, M)_{\text{tors}} & \longrightarrow & (\mathbb{H}^0(K, C')_{\wedge})^D \\
 & & \longleftarrow & & \longleftarrow & & \longleftarrow \\
 & & H^3(K, M) & \longrightarrow & \mathbb{P}^3(K, M) & \longrightarrow & \mathbb{H}^{-1}(K, C')^D \longrightarrow 0
 \end{array}$$

- (3) Let G be a connected reductive group over K . Let G^{sc} be the universal covering of the derived subgroup G^{ss} of G . Let $\rho : G^{\text{sc}} \rightarrow G^{\text{ss}} \rightarrow G$ be the composite. Let T be a maximal torus of G and let $T^{\text{sc}} := \rho^{-1}(T)$ be the inverse image of T in G^{sc} . Thus T^{sc} is a maximal torus of G^{sc} . Following [Bor98], we write $H_{\text{ab}}^i(K, G) = \mathbb{H}^i(K, C)$ and $\mathbb{P}_{\text{ab}}^i(K, G) = \mathbb{P}^i(K, C)$ with $C = [T^{\text{sc}} \rightarrow T]$ for the abelianized Galois cohomologies. So the Poitou–Tate sequence (2.13) yields an exact sequence for the abelianization of Galois

cohomology of G as follows

$$\begin{array}{ccccccc}
 0 & \longrightarrow & H_{\text{ab}}^{-1}(K, G) & \longrightarrow & \mathbb{P}_{\text{ab}}^{-1}(K, G) & \longrightarrow & \mathbb{H}^2(K, C')^D \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & H_{\text{ab}}^0(K, G)_{\wedge} & \longrightarrow & \mathbb{P}_{\text{ab}}^0(K, G)_{\wedge} & \longrightarrow & \mathbb{H}^1(K, C')^D \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & H_{\text{ab}}^1(K, G) & \longrightarrow & \mathbb{P}_{\text{ab}}^1(K, G) & \longrightarrow & \mathbb{H}^0(K, C')^D \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & H_{\text{ab}}^2(K, G) & \longrightarrow & \mathbb{P}_{\text{ab}}^2(K, G)_{\text{tors}} & \longrightarrow & (\mathbb{H}^{-1}(K, C')_{\wedge})^D \longrightarrow 0.
 \end{array}$$

Hopefully it will give a defect to strong approximation, which is analogous to the number field case [Dem11b].

Finally, we relate $\mathbb{H}^i(K, C')$ in Example 2.4.14(3) with a simpler cohomology group. Recall that the algebraic fundamental group $\pi_1^{\text{alg}}(\overline{G})$ (see [Bor98, §1] and [CT08, §6] for more information) of a connected reductive group G is $\pi_1^{\text{alg}}(\overline{G}) := \mathbf{X}_*(T)/\rho_*\mathbf{X}_*(T^{\text{sc}})$ where $\rho_* : \mathbf{X}_*(T^{\text{sc}}) \rightarrow \mathbf{X}_*(T)$ is induced by $\rho : T^{\text{sc}} \rightarrow T$.

Corollary 2.4.15. *Let G be a connected reductive group. Let $C = [T^{\text{sc}} \xrightarrow{\rho} T]$ be as above. Let G^* be the group of multiplicative type such that $X^*(G^*) = \pi_1^{\text{alg}}(\overline{G})$. There is an isomorphism $H^{i+1}(K, G^*) \simeq \mathbb{H}^i(K, C')$. In particular, we have an exact sequence*

$$0 \rightarrow \mathbb{H}^1(C) \rightarrow \mathbb{H}^1(K, C) \rightarrow \mathbb{P}^1(K, C) \rightarrow H^1(K, G^*)^D.$$

Proof. Let $T^{\text{ss}} := T \cap G^{\text{ss}}$ and let $G^{\text{tor}} := T/T^{\text{ss}}$. Thus there is a short exact sequence of short complexes

$$0 \rightarrow [(G^{\text{tor}})' \rightarrow 0] \rightarrow [T' \rightarrow (T^{\text{sc}})'] \rightarrow [(T^{\text{ss}})' \rightarrow (T^{\text{sc}})'] \rightarrow 0. \quad (2.21)$$

By [CT08, Proposition 6.4], there is a short exact sequence of abelian groups

$$0 \rightarrow (\text{Ker } \rho)(-1) \rightarrow \pi_1^{\text{alg}}(\overline{G}) \rightarrow \mathbf{X}_*(G^{\text{tor}}) \rightarrow 0.$$

Here $(\text{Ker } \rho)(-1) := \text{Hom}_{\mathbb{Z}}(X^*(\text{Ker } \rho), \mathbb{Q}/\mathbb{Z})$. Note that $(\text{Ker } \rho)(-1)$ is the module of characters of $(\text{Ker } \rho)' := \text{Hom}(\text{Ker } \rho, \mathbb{Q}/\mathbb{Z}(2))$. Thus there is an exact sequence of groups of multiplicative type

$$0 \rightarrow (G^{\text{tor}})' \rightarrow G^* \rightarrow (\text{Ker } \rho)' \rightarrow 0. \quad (2.22)$$

Since $T^{\text{sc}} \rightarrow T^{\text{ss}}$ is an isogeny with kernel $\text{Ker } \rho$, its dual isogeny $(T^{\text{ss}})' \rightarrow (T^{\text{sc}})'$ has kernel $(\text{Ker } \rho)'$, i.e. there is a quasi-isomorphic $[(T^{\text{ss}})' \rightarrow (T^{\text{sc}})'] \simeq (\text{Ker } \rho)'[1]$. By definition there is an exact sequence $X_*(T^{\text{sc}}) \xrightarrow{\rho_*} X_*(T) \rightarrow \pi_1^{\text{alg}}(\overline{G}) \rightarrow 0$, so there is a corresponding exact sequence $0 \rightarrow G^* \rightarrow T' \rightarrow (T^{\text{sc}})'$ of groups of multiplicative type. In particular, we obtain a morphism of short complexes $G^*[1] \rightarrow C'$. Summing up, there is a commutative diagram of short complexes with exact rows obtained from (2.22) and (2.21):

$$\begin{array}{ccccccc}
 0 & \longrightarrow & (G^{\text{tor}})'[1] & \longrightarrow & G^*[1] & \longrightarrow & (\text{Ker } \rho)'[1] \longrightarrow 0 \\
 & & \parallel & & \downarrow & & \downarrow \\
 0 & \longrightarrow & (G^{\text{tor}})'[1] & \longrightarrow & C' & \longrightarrow & [(T^{\text{ss}})' \rightarrow (T^{\text{sc}})'] \longrightarrow 0.
 \end{array}$$

Since the right vertical arrow is a quasi-isomorphism, so is the middle one as is seen by taking cohomology and applying the 5-lemma. Thus $H^{i+1}(K, G^*) \simeq \mathbb{H}^i(K, C')$ and the desired sequence follows from Example 2.4.14(3). \square

Remark 2.4.16. Corollary 2.4.15 gives an abelianized version of the Kottwitz–Borovoi sequence [Bor98, Theorem 5.16] over p -adic function fields K . Hopefully over such K there is an exact sequence $1 \rightarrow \text{III}^1(G) \rightarrow H^1(K, G) \rightarrow \mathbb{P}^1(K, G) \rightarrow H^1(K, G^*)^D$ of pointed sets for connected linear groups which can be used to give an obstruction to weak approximation for homogenous spaces under some linear group with connected reductive stabilizer G .

Chapter 3

Obstructions to weak approximation

Abstract: Using global arithmetic duality theorems, we give an explicit obstruction to weak approximation for certain connected reductive groups over p -adic function fields generalizing the results of Harari–Scheiderer–Szamuely for tori over such fields. We shall also relate this cohomological obstruction to some unramified cohomology group. The stated results are based on the preprint [[Tia19a](#)].

Keywords: connected reductive groups, cohomological obstruction, weak approximation.

3.1 Defects to weak approximation

Let G be a connected reductive group over K . Let G^{ss} be the derived subgroup of G and let $G^{\text{sc}} \rightarrow G^{\text{ss}}$ be the universal covering of G^{ss} . Thus G^{ss} is semi-simple and G^{sc} is simply connected. We consider the composition $\rho : G^{\text{sc}} \rightarrow G^{\text{ss}} \rightarrow G$. Let $T \subset G$ be a maximal torus over K and let $T^{\text{sc}} = \rho^{-1}(T)$. Recall that T^{sc} is a maximal torus of G^{sc} . We apply the arithmetic duality theorems developed in the previous chapter to the morphism $\rho : T^{\text{sc}} \rightarrow T$, i.e. to the complexes $C = [T^{\text{sc}} \rightarrow T]$ and $C' = [T' \rightarrow (T^{\text{sc}})']$ concentrated in degree -1 and 0 . We say that G^{sc} satisfies the condition $(*)$ if it has weak approximation and contains a quasi-trivial maximal torus. We shall see soon that G^{sc} satisfies $(*)$ if G is quasi-split.

Proposition 3.1.1. *Let H be a quasi-split semi-simple simply connected group over K . Then H satisfies weak approximation with respect to any finite set $S \subset X^{(1)}$ of places.*

Proof. Let B be a Borel subgroup of H defined over K and let P be a maximal K -torus contained in B . Applying [HS16, Lemma 6.7 and its proof] implies that $\mathbf{X}^*(P) \simeq \mathbf{X}^*(B) \simeq \text{Pic}(\overline{H/B})$ is a permutation module, i.e. P is a quasi-trivial torus. Moreover, P is a Levi subgroup of B by [BT65, Corollaire 3.14]. Now [Thä96, Corollary 1.5] yields a bijection from the defect of weak approximation $\prod_{v \in S} H(K_v)/\overline{H(K)}_S$ for H to that $\prod_{v \in S} P(K_v)/\overline{P(K)}_S$ for P with respect to any finite set S of places. Here $\overline{H(K)}_S$ (resp. $\overline{P(K)}_S$) denotes the closure of $H(K)$ (resp. $P(K)$) in $\prod_{v \in S} H(K_v)$ (resp. $\prod_{v \in S} P(K_v)$) with respect to the product of v -adic topologies. Since P is a quasi-trivial torus, by construction of Weil restriction it is an open subscheme of some affine space. Hence P satisfies weak approximation and so is H . \square

Actually quasi-split semi-simple simply connected groups are rational (see [Har67, Satz 2.2.2]), thus they satisfy weak approximation. However, the above proposition shows how to describe the defect to weak approximation for such groups.

Proposition 3.1.2. *Let H be a connected reductive group over K . The following are equivalent: (1) H is quasi-split; (2) H^{ss} is quasi-split; (3) H^{sc} is quasi-split.*

Proof.

- By [SGA3III, Proposition 6.2.8(ii)], there is a one-to-one correspondence between Borel subgroups of H and that of H^{ss} , so H is quasi-split if and only if H^{ss} is quasi-split.
- Suppose H^{sc} is quasi-split. Since the universal covering $q : H^{\text{sc}} \rightarrow H^{\text{ss}}$ is faithfully flat, [Mil17, Proposition 17.68] implies that q sends Borel subgroups of H^{sc} to Borel subgroups of H^{ss} . In particular, H^{ss} is quasi-split.
- Suppose H^{ss} is quasi-split and let B^{ss} be a Borel subgroup of H^{ss} . According to [Mil17, Proposition 17.20], there exists a Borel subgroup B^{sc} of H^{sc} such that $q(B^{\text{sc}}) = B^{\text{ss}}$. In particular, H^{sc} is quasi-split. \square

The following corollary says that our technical assumption $(*)$ holds if G is quasi-split.

Corollary 3.1.3. *If H is a quasi-split connected reductive group over K , then H^{sc} satisfies $(*)$.*

Proof. Indeed, H^{sc} is quasi-split by Proposition 3.1.2 and so it satisfies weak approximation by Proposition 3.1.1. Moreover, H^{sc} contains a quasi-trivial maximal torus by [HS16, Lemma 6.7]. \square

3.1. DEFECTS TO WEAK APPROXIMATION

Suppose G^{sc} contains a quasi-trivial maximal torus $T^{\text{sc}} \subset G^{\text{sc}}$. Because $G^{\text{sc}} \rightarrow G^{\text{ss}}$ is faithfully flat, the image T^{ss} of T^{sc} in G^{ss} is again a maximal torus by [Hum75, §21.3, Corollary C]. Therefore we may choose a maximal torus $T \subset G$ of G such that $T \cap G^{\text{ss}} = T^{\text{ss}}$, i.e. $T^{\text{sc}} = \rho^{-1}(T)$. By [Bor98, Section 2.4], different choices of $[T^{\text{sc}} \rightarrow T]$ give rise to the same hypercohomology group¹ and thus we are allowed to fix a quasi-trivial maximal torus T^{sc} . Recall that for any finite set $S \subset X^{(1)}$ of places, we denote by $\mathbb{I}_S^1(C)$ the subgroup of $\mathbb{H}^1(K, C)$ consisting of elements that are trivial in $\mathbb{H}^1(K_v, C)$ for each $v \notin S$. Moreover, we denote by $\mathbb{I}_\omega^1(C) := \bigcup_S \mathbb{I}_S^1(C)$ where S runs through all finite sets of places of K .

Theorem 3.1.4. *Let G be a connected reductive group such that G^{sc} satisfies $(*)$.*

(1) *Let $S \subset X^{(1)}$ be a finite set of places. There is an exact sequence of groups*

$$1 \rightarrow \overline{G(K)}_S \rightarrow \prod_{v \in S} G(K_v) \rightarrow \mathbb{I}_S^1(C')^D \rightarrow \mathbb{I}^1(C) \rightarrow 1. \quad (3.1)$$

Here $\overline{G(K)}_S$ denotes the closure of the diagonal image of $G(K)$ in $\prod_{v \in S} G(K_v)$ for the product topology.

(2) *There is an exact sequence of groups*

$$1 \rightarrow \overline{G(K)} \rightarrow \prod_{v \in X^{(1)}} G(K_v) \rightarrow \mathbb{I}_\omega^1(C')^D \rightarrow \mathbb{I}^1(C) \rightarrow 1. \quad (3.2)$$

Here $\overline{G(K)}$ denotes the closure of the diagonal image of $G(K)$ in $\prod_{v \in X^{(1)}} G(K_v)$ for the product topology.

Example 3.1.5. Let us first look at two special cases of the sequence (3.1).

(1) If G is semi-simple, then there is an exact sequence $1 \rightarrow F \rightarrow G^{\text{sc}} \rightarrow G \rightarrow 1$ of algebraic groups with F finite and central in G^{sc} . In particular, there are exact sequences of commutative algebraic groups

$$1 \rightarrow F \rightarrow T^{\text{sc}} \rightarrow T \rightarrow 1 \quad \text{and} \quad 1 \rightarrow F' \rightarrow T' \rightarrow (T^{\text{sc}})' \rightarrow 1.$$

Here we denote by $F' := \text{Hom}(F, \mathbb{Q}/\mathbb{Z}(2))$ which plays a similar role as the Cartier dual of F and the latter sequence is obtained from the dual isogeny of $T^{\text{sc}} \rightarrow T$. Consequently there are quasi-isomorphisms $C \simeq F[1]$ and $C' \simeq F'[1]$, and hence we obtain isomorphisms of abelian groups $\mathbb{I}_S^1(C) \simeq \mathbb{I}_S^2(F)$ and $\mathbb{I}^1(C') \simeq \mathbb{I}^2(F')$. Therefore the exact sequence (3.1) reads as

$$1 \rightarrow \overline{G(K)}_S \rightarrow \prod_{v \in S} G(K_v) \rightarrow \mathbb{I}_S^2(F')^D \rightarrow \mathbb{I}^2(F) \rightarrow 1.$$

Here the second arrow is given by the composite of the coboundary map $G(K_v) \rightarrow H^1(K_v, F)$ and the local duality $H^1(K_v, F) \times H^2(K_v, F') \rightarrow \mathbb{Q}/\mathbb{Z}$, and the last one is given by the global duality $\mathbb{I}^2(F) \times \mathbb{I}^2(F') \rightarrow \mathbb{Q}/\mathbb{Z}$ for finite Galois modules (see [HS16, (10) and Theorem 4.4] for details).

¹Indeed, let Z^{sc} (resp. Z) be the centre of G^{sc} (resp. G). Then there is a quasi-isomorphism $[Z^{\text{sc}} \rightarrow Z] \rightarrow [T^{\text{sc}} \rightarrow T]$ of short complexes. In particular, different choices of $[T^{\text{sc}} \rightarrow T]$ yield isomorphic hypercohomology groups.

- (2) If $G = T$ is a torus, then $C = [T^{\text{sc}} \rightarrow T]$ is quasi-isomorphic to the complex $[0 \rightarrow T] \simeq T$ and its dual $C' = [T' \rightarrow (T^{\text{sc}})']$ is quasi-isomorphic to the complex $[T' \rightarrow 0] \simeq T'[1]$. So we obtain isomorphisms of abelian groups $\mathbb{H}^1(C) \simeq \mathbb{H}^1(T)$ and $\mathbb{H}_S^1(C') \simeq \mathbb{H}_S^2(T')$. Now the exact sequence (3.1) is of the following form

$$1 \rightarrow \overline{T(K)}_S \rightarrow \prod_{v \in S} T(K_v) \rightarrow \mathbb{H}_S^2(T')^D \rightarrow \mathbb{H}^1(T) \rightarrow 1.$$

This is the obstruction to weak approximation for tori given by Harari, Scheiderer and Szamuely in [HSS15].

The rest of this section is devoted to the proof of Theorem 3.1.4. For a field L and $i = 0, 1$, we shall denote by $\text{ab}^i : H^i(L, G) \rightarrow \mathbb{H}^i(L, C)$ the abelianization map in the sequel. The construction of the abelianization map is set forth in [Bor98, Section 3].

Lemma 3.1.6. *Let L be a field of characteristic zero and let G be a connected reductive group over L such that G^{sc} satisfies (*). Then the canonical map $\text{ab}^0 : H^0(L, G) \rightarrow \mathbb{H}^0(L, C)$ is surjective.*

Proof. Let $A \rightarrow B$ be a crossed module (see [Bor98, Section 3]) of (not necessarily abelian) $\text{Gal}(\overline{L}|L)$ -groups concentrated in degree -1 and 0 . We write $\mathbb{H}_{\text{rel}}^i(L, [A \rightarrow B])$ for the non-abelian hypercohomology in the sense of [Bor98, Section 3] for $-1 \leq i \leq 1$. Now we view $\rho : T^{\text{sc}} \rightarrow T$ and $\rho : G^{\text{sc}} \rightarrow G$ as crossed modules of $\text{Gal}(\overline{L}|L)$ -groups concentrated in degree -1 and 0 . Recall that T^{sc} is chosen to be quasi-trivial. We have a commutative diagram of crossed modules of $\text{Gal}(\overline{L}|L)$ -groups

$$\begin{array}{ccc} [T^{\text{sc}} \rightarrow T] & \longrightarrow & [T^{\text{sc}} \rightarrow 1] \\ \downarrow & & \downarrow \\ [G^{\text{sc}} \rightarrow G] & \longrightarrow & [G^{\text{sc}} \rightarrow 1]. \end{array}$$

Applying the functor $\mathbb{H}_{\text{rel}}^0(\text{Gal}(\overline{L}|L), -)$ with values in the category of pointed sets and taking into account the identification $\mathbb{H}_{\text{rel}}^0(\text{Gal}(\overline{L}|L), [A \rightarrow 1]) \simeq H^1(\text{Gal}(\overline{L}|L), A)$ (see [Bor98, Example 3.1.2(2)]), we obtain the following commutative diagram of pointed sets

$$\begin{array}{ccc} \mathbb{H}_{\text{rel}}^0(L, [T^{\text{sc}} \rightarrow T]) & \longrightarrow & H^1(L, T^{\text{sc}}) = 0 \\ \downarrow & & \downarrow \\ \mathbb{H}_{\text{rel}}^0(L, [G^{\text{sc}} \rightarrow G]) & \longrightarrow & H^1(L, G^{\text{sc}}). \end{array}$$

By [Bor98, Lemma 3.8.1], there are isomorphisms of pointed sets

$$\mathbb{H}^0(L, C) \simeq \mathbb{H}_{\text{rel}}^0(L, [T^{\text{sc}} \rightarrow T]) \simeq \mathbb{H}_{\text{rel}}^0(L, [G^{\text{sc}} \rightarrow G])$$

Since T^{sc} is quasi-trivial, we conclude that $\mathbb{H}^0(L, C) \rightarrow H^1(L, G^{\text{sc}})$ is trivial. According to [Bor98, 3.10], there is an exact sequence of pointed sets

$$G(L) \rightarrow \mathbb{H}^0(L, C) \rightarrow H^1(L, G^{\text{sc}}).$$

In particular, the abelianization map $\text{ab}^0 : G(L) \rightarrow \mathbb{H}^0(L, C)$ is surjective. \square

3.1. DEFECTS TO WEAK APPROXIMATION

We now proceed as in [San81]. The first step is to show the following:

Lemma 3.1.7. *Let $m \geq 1$ be an integer and let Q be a quasi-trivial K -torus. If the sequence (3.1) is exact for $G^m \times_K Q$, then it is also exact for G .*

Proof. Evidently the exactness of (3.1) for the direct product G^m will imply the exactness of (3.1) for G . We claim if (3.1) is exact for the product $G \times_K Q$ of G by some quasi-trivial K -torus Q , then (3.1) is also exact for G . Since $T \subset G$ is a maximal torus of G , $T \times_K Q$ is a maximal torus of $G \times_K Q$. Moreover, the derived subgroup of $G \times_K Q$ is just $\mathcal{D}(G \times_K Q) = G^{\text{ss}}$, so we have a composite $\rho_Q : G^{\text{sc}} \rightarrow G^{\text{ss}} \rightarrow G \times_K Q$. We introduce the complex $C_Q = [T^{\text{sc}} \rightarrow T \times_K Q]$ which is concentrated in degree -1 and 0 . Consider the following commutative diagram

$$\begin{array}{ccccccccc} H^1(K, T^{\text{sc}}) & \longrightarrow & H^1(K, T \times_K Q) & \longrightarrow & \mathbb{H}^1(K, C_Q) & \longrightarrow & H^2(K, T^{\text{sc}}) & \longrightarrow & H^2(K, T \times_K Q) \\ & & \parallel & & \downarrow & & \parallel & & \downarrow \\ H^1(K, T^{\text{sc}}) & \longrightarrow & H^1(K, T) & \longrightarrow & \mathbb{H}^1(K, C) & \longrightarrow & H^2(K, T^{\text{sc}}) & \longrightarrow & H^2(K, T) \end{array}$$

where the second vertical map is an isomorphism since $T \times_K Q \rightarrow T$ admits a section and Q is a quasi-trivial K -torus. Since $\mathbb{H}^2(T^{\text{sc}}) = 0$ by [HSS15, Lemma 3.2(a)], we obtain isomorphisms $\mathbb{H}^1(C_Q) \simeq \mathbb{H}^1(T) \simeq \mathbb{H}^1(C)$ of abelian groups by diagram chasing. Similarly, $\mathbb{H}_S^1(C'_Q)^D \simeq \mathbb{H}_S^1(C')^D$ holds for any finite subset S of places. Indeed, we have the following commutative diagram (and similar diagrams over K_v)

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{H}^1(K, C') & \longrightarrow & H^2(K, T') & \longrightarrow & H^2(K, (T^{\text{sc}})') \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathbb{H}^1(K, C'_Q) & \longrightarrow & H^2(K, T' \times Q') & \longrightarrow & H^2(K, (T^{\text{sc}})') \end{array}$$

with exact rows, where $C'_Q := [T' \times Q' \rightarrow (T^{\text{sc}})']$ is the dual of C_Q . According to [HSS15, Lemma 3.2], we see that $\mathbb{H}_S^2(Q') = 0$. Now a diagram chase yields $\mathbb{H}_S^1(C') \simeq \mathbb{H}_S^1(C'_Q)$. Finally, recall that quasi-trivial tori are K -rational, hence in particular Q satisfies weak approximation. It follows that the cokernel of the first map in (3.1) is stable under multiplying G by a quasi-trivial torus. Subsequently the exactness of (3.1) for $G \times_K Q$ yields the exactness of (3.1) for G . \square

Therefore to prove the exactness of (3.1), we are free to replace G by $G^m \times_K Q$ for some integer m and some quasi-trivial K -torus Q . Suppose now that G^{sc} satisfies (*). By [BT65, Proposition 2.2] and Ono's lemma [San81, Lemme 1.7], there exist an integer $m \geq 1$, quasi-trivial K -tori Q and Q_0 such that $G^{\text{sc}, m} \times_K Q \rightarrow G^m \times_K Q_0$ is a central K -isogeny. By Lemma 3.1.7, we may therefore assume that G has a special covering² $1 \rightarrow F_0 \rightarrow G_0 \rightarrow G \rightarrow 1$ where G_0 satisfies weak approximation having derived subgroup $\mathcal{D}G_0 = G^{\text{sc}}$. Moreover, G_0 contains a quasi-trivial maximal torus T_0 over K such that $T_0 \cap G^{\text{sc}} = T^{\text{sc}}$ and that the sequence $1 \rightarrow F_0 \rightarrow T_0 \rightarrow T \rightarrow 1$ is exact by construction. Therefore we may assume G admits a special covering in the sequel.

The second step is to show the exactness at the first three terms:

²Recall that an isogeny $G_0 \rightarrow G$ of connected reductive groups is called a special covering, if G_0 is the product of a semi-simple simply connected group with a quasi-trivial torus.

Lemma 3.1.8. *There is an exact sequence of groups*

$$1 \rightarrow \overline{G(K)}_S \rightarrow \prod_{v \in S} G(K_v) \rightarrow \mathbb{III}_S^1(C')^D.$$

Proof. After passing to the dual isogeny of the exact sequence $1 \rightarrow F_0 \rightarrow T_0 \rightarrow T \rightarrow 1$, we obtain an isomorphism of abelian groups $\mathbb{III}_S^2(F'_0) \simeq \mathbb{III}_S^2(T')$ since $\mathbb{III}_\omega^2(T'_0) = 0$ by [HSS15, Lemma 3.2(a)] (note that T'_0 is a quasi-trivial torus). Moreover, the distinguished triangle $T' \rightarrow (T^{\text{sc}})' \rightarrow C' \rightarrow T'[1]$ induces an isomorphism $\mathbb{III}_S^1(C') \simeq \mathbb{III}_S^2(T')$ for the same reason. In particular, we obtain an isomorphism $\mathbb{III}_S^2(F'_0) \simeq \mathbb{III}_S^1(C')$ which fits into the commutative diagram with $F'_0 = \text{Hom}(F_0, \mathbb{Q}/\mathbb{Z}(2))$:

$$\begin{array}{ccccccc} G_0(K) & \longrightarrow & G(K) & \xrightarrow{\partial} & H^1(K, F_0) & \longrightarrow & H^1(K, G_0) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \prod_{v \in S} G_0(K_v) & \longrightarrow & \prod_{v \in S} G(K_v) & \xrightarrow{\partial_v} & \prod_{v \in S} H^1(K_v, F_0) & \longrightarrow & \prod_{v \in S} H^1(K_v, G_0) \\ & & \downarrow & & \downarrow & & \\ & & \mathbb{III}_S^1(C')^D & \xrightarrow{\simeq} & \mathbb{III}_S^2(F'_0)^D & & \end{array} \quad (3.3)$$

We show that the lower square in diagram (3.3) commutes. Let $C_0 := [T_0 \rightarrow T]$ be the short complex associated to the isogeny $G_0 \rightarrow G$. Thus we have a quasi-isomorphism $F_0[1] \simeq C_0$ of complexes. Note that we have a map $C \rightarrow C_0$ of complexes induced by the inclusion $G^{\text{sc}} \rightarrow G_0$ (recall that $G^{\text{sc}} = \mathcal{D}G_0$). Moreover, both $H^0(K_v, G) \rightarrow H^1(K_v, F_0) \simeq \mathbb{H}^0(K_v, C_0)$ and $H^0(K_v, G) \rightarrow \mathbb{H}^0(K_v, C) \rightarrow \mathbb{H}^0(K_v, C_0)$ are induced by the short exact sequence $1 \rightarrow F_0 \rightarrow G_0 \rightarrow G \rightarrow 1$. It follows that the left square in the following diagram commutes:

$$\begin{array}{ccccc} H^0(K_v, G) & \longrightarrow & \mathbb{H}^0(K_v, C) & \longrightarrow & \mathbb{III}_S^1(C')^D \\ \downarrow & & \downarrow & & \downarrow \\ H^1(K_v, F_0) & \longrightarrow & \mathbb{H}^0(K_v, C_0) & \longrightarrow & \mathbb{III}_S^1(C'_0)^D. \end{array}$$

The right-hand side square commutes as a consequence of the functoriality of cup-product over K_v . Finally, taking the quasi-isomorphisms $F_0[1] \simeq C_0$ and $F'_0[1] \simeq C'_0$ into account yields the commutativity of the lower square in diagram (3.3). Now let us go back to diagram (3.3). Recall that the third column is exact by [HSS15, Lemma 3.1]. We claim the coboundary map ∂ is surjective. Since F_0 is contained in T_0 , the inclusion $F_0 \rightarrow G_0$ factors through $T_0 \rightarrow G_0$. It follows that $H^1(K, F_0) \rightarrow H^1(K, G_0)$ factors through the trivial map $0 = H^1(K, T_0) \rightarrow H^1(K, G_0)$. By construction T_0 is quasi-trivial, so the vanishing $H^1(K, T_0) = 0$ implies that $H^1(K, F_0) \rightarrow H^1(K, G_0)$ is the trivial map, i.e. ∂ is surjective. Similarly, the coboundary map $\partial_v : G(K_v) \rightarrow H^1(K_v, F_0)$ is surjective for each place $v \in X^{(1)}$. Since G_0 satisfies weak approximation by assumption, we see that $G_0(K)$ has dense image in $\prod_{v \in S} G_0(K_v)$. Finally, a diagram chasing yields the desired exact sequence. \square

In order to prove Theorem 3.1.4(1), the last step is to show the exactness of the last three terms. By definition there is an exact sequence of discrete abelian groups:

$$1 \rightarrow \mathbb{III}^1(C') \rightarrow \mathbb{III}_S^1(C') \rightarrow \bigoplus_{v \in S} \mathbb{H}^1(K_v, C').$$

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Dualizing the sequence yields an exact sequence of profinite groups:

$$\prod_{v \in S} \mathbb{H}^0(K_v, C)^\wedge \rightarrow \mathbb{III}_S^1(C')^D \rightarrow \mathbb{III}^1(C) \rightarrow 1$$

by Proposition 2.3.5 and Theorem 2.3.19. Since $\mathbb{III}_S^1(C') \simeq \mathbb{III}_S^2(T')$ is a finite group by [HSS15, Lemma 3.2], the groups $\prod_{v \in S} \mathbb{H}^0(K_v, C)$ and $\prod_{v \in S} \mathbb{H}^0(K_v, C)^\wedge$ have the same image in $\mathbb{III}_S^1(C')^D$. By Lemma 3.1.6, the canonical abelianization map $\prod_{v \in S} G(K_v) \rightarrow \prod_{v \in S} \mathbb{H}^0(K_v, C)$ is surjective which guarantees the desired exactness.

Proof of Theorem 3.1.4(2). Passing to the projective limit of (3.1) over all finite subset $S \subset X^{(1)}$ yields an exact sequence of groups

$$1 \rightarrow \overline{G(K)} \rightarrow \prod_{v \in X^{(1)}} G(K_v) \rightarrow \mathbb{III}_\omega^1(C')^D.$$

Dualizing the exact sequence of discrete groups

$$1 \rightarrow \mathbb{III}^1(C') \rightarrow \mathbb{III}_\omega^1(C') \rightarrow \bigoplus_{v \in X^{(1)}} \mathbb{H}^1(K_v, C')$$

yields an exact sequence of profinite groups

$$\prod_{v \in X^{(1)}} \mathbb{H}^0(K_v, C)^\wedge \rightarrow \mathbb{III}_\omega^1(C')^D \rightarrow \mathbb{III}^1(C) \rightarrow 0.$$

Since $G(K_v) \rightarrow \mathbb{H}^0(K_v, C)$ is surjective and $\prod_{v \in X^{(1)}} \mathbb{H}^0(K_v, C)$ is dense in $\prod_{v \in X^{(1)}} \mathbb{H}^0(K_v, C)^\wedge$, it will be sufficient to show the image of $\prod_{v \in X^{(1)}} G(K_v)$ is closed in $\mathbb{III}_\omega^1(C')^D$. In view of diagram (3.3), the quotient of $\prod_{v \in X^{(1)}} G(K_v)$ by $\overline{G(K)}$ is isomorphic to the quotient of the profinite group $\prod_{v \in X^{(1)}} H^1(K_v, F_0)$ by the closure of the image of $H^1(K, F_0)$. Consequently, the quotient of $\prod_{v \in X^{(1)}} G(K_v)$ by $\overline{G(K)}$ is compact and hence the image of $\prod_{v \in X^{(1)}} G(K_v)$ in $\mathbb{III}_\omega^1(C')^D$ is closed, as required. \square

Actually the defect of weak approximation for G can also be given by a simpler group $\mathbb{III}_\omega^2(G^*)$ where G^* is the group of multiplicative type whose character module is $\mathbf{X}^*(G^*) = \pi_1^{\text{alg}}(\overline{G})$. Here π_1^{alg} (see [Bor98, §1] or [CT08, §6] for more details) denotes the algebraic fundamental group of a connected reductive group G defined as $\pi_1^{\text{alg}}(\overline{G}) := \mathbf{X}_*(T)/\rho_*\mathbf{X}_*(T^{\text{sc}})$ where $\rho_* : \mathbf{X}_*(T^{\text{sc}}) \rightarrow \mathbf{X}_*(T)$ is induced by $\rho : G^{\text{sc}} \rightarrow G$.

Proposition 3.1.9. *Let G be a connected reductive group such that G^{sc} satisfies (*). Let G^* be the group of multiplicative type whose character module is $\pi_1^{\text{alg}}(\overline{G})$. There is an isomorphism of abelian groups $\mathbb{III}_\omega^2(G^*) \rightarrow \mathbb{III}_\omega^1(C')$.*

Proof. Recall that there is an isomorphism $H^{i+1}(K, G^*) \simeq \mathbb{H}^i(K, C')$ of abelian groups by Corollary 2.4.15. In particular, we have $\mathbb{III}^2(G^*) \simeq \mathbb{III}^1(C')$. \square

Thus there is an exact sequence of groups by Theorem 3.1.4(2)

$$1 \rightarrow \overline{G(K)} \rightarrow \prod_{v \in X^{(1)}} G(K_v) \rightarrow \mathbb{III}_\omega^2(G^*)^D.$$

Consequently, the defect of weak approximation may also be given by the group $\mathbb{III}_\omega^2(G^*)$.

3.2 Reciprocity obstructions

The next theorem is the promised generalization of [HSS15, Theorem 4.2] to the non-commutative case. Let Y be a smooth integral variety over K . Recall that we have constructed a pairing (see page 30)

$$\prod_{v \in X^{(1)}} Y(K_v) \times H_{\text{nr}}^3(K(Y), \mathbb{Q}/\mathbb{Z}(2)) \rightarrow \mathbb{Q}/\mathbb{Z}. \quad (3.4)$$

Theorem 3.2.1. *Let G be a connected reductive group such that G^{sc} satisfies (*). There exists a homomorphism*

$$u : \mathbb{III}_{\omega}^1(C') \rightarrow H_{\text{nr}}^3(K(G), \mathbb{Q}/\mathbb{Z}(2))$$

such that each family $(g_v) \in \prod_{v \in X^{(1)}} G(K_v)$ satisfying $((g_v), \text{Im } u) = 0$ under the pairing (3.4) lies in the closure $\overline{G(K)}$ with respect to the product topology.

More precisely, the obstruction is given by $\text{Im}(H^3(G^c, \mu_n^{\otimes 2}) \rightarrow H_{\text{nr}}^3(K(G), \mathbb{Q}/\mathbb{Z}(2)))$ for some sufficiently large n where G^c is a smooth compactification of G over K .

Proof. Let us first construct the desired homomorphism $u : \mathbb{III}_{\omega}^1(C') \rightarrow H_{\text{nr}}^3(K(G), \mathbb{Q}/\mathbb{Z}(2))$. Fix a quasi-trivial maximal torus $T^{\text{sc}} \subset G^{\text{sc}}$ and let $T \subset G$ be a maximal torus such that $T^{\text{sc}} = \rho^{-1}(T)$. Recall the fundamental diagrams (1) and (2) associated with a flasque resolution $1 \rightarrow R \rightarrow H \rightarrow G \rightarrow 1$. Recall also that H is a connected reductive group such that H^{tor} is a quasi-trivial torus. Since T^{sc} and H^{tor} are quasi-trivial tori, we obtain $H^1(K, (T_H)') = 0$ and $\mathbb{III}_{\omega}^2((T_H)') = 0$ by the associated long exact sequence of $1 \rightarrow (H^{\text{tor}})' \rightarrow (T_H)' \rightarrow (T^{\text{sc}})' \rightarrow 1$. Dualizing the middle row $1 \rightarrow R \rightarrow T_H \rightarrow T \rightarrow 1$ of the diagram (2) yields an exact sequence

$$0 \rightarrow H^1(K, R') \rightarrow H^2(K, T') \rightarrow H^2(K, (T_H)')$$

of abelian groups. The distinguished triangle $T' \rightarrow (T^{\text{sc}})' \rightarrow C' \rightarrow T'[1]$ yields an isomorphism $\mathbb{III}_{\omega}^2(T') \simeq \mathbb{III}_{\omega}^1(C')$ of abelian groups, and hence we obtain

$$\mathbb{III}_{\omega}^1(R') \simeq \mathbb{III}_{\omega}^2(T') \simeq \mathbb{III}_{\omega}^1(C'). \quad (3.5)$$

Subsequently, we find a homomorphism $\mathbb{III}_{\omega}^1(C') \rightarrow H^1(K, R')$ via the inclusion $\mathbb{III}_{\omega}^1(R') \rightarrow H^1(K, R')$ in view of (3.5).

Because R is a flasque K -torus, applying [CTS87, Theorem 2.2(i)] implies that the class $[H] \in H^1(G, R)$ comes from a class $[Y] \in H^1(G^c, R)$, where G^c is a smooth compactification of G . The pairing $R \otimes^{\mathbf{L}} R' \rightarrow \mathbb{Z}(2)[2]$ now induces a homomorphism

$$H^1(K, R') \rightarrow H^4(G^c, \mathbb{Z}(2)), \quad a \mapsto a_{G^c} \cup [Y]$$

with a_{G^c} denoting the image of a under $H^1(K, R') \rightarrow H^1(G^c, R')$. The same argument as in [HSS15, Theorem 4.2] shows that there is a natural map

$$H^4(G^c, \mathbb{Z}(2)) \rightarrow H_{\text{nr}}^3(K(G), \mathbb{Q}/\mathbb{Z}(2))$$

fitting into a commutative diagram

$$\begin{array}{ccc} H^4(G^c, \mathbb{Z}(2)) & \longrightarrow & H_{\text{nr}}^3(K(G), \mathbb{Q}/\mathbb{Z}(2)) \\ \downarrow & & \downarrow \\ H^4(K(G), \mathbb{Z}(2)) & \xleftarrow{\simeq} & H^3(K(G), \mathbb{Q}/\mathbb{Z}(2)). \end{array}$$

3.2. RECIPROCITY OBSTRUCTIONS

Now take a family $(g_v) \in \prod_{v \in X^{(1)}} G(K_v)$ of local points. By Theorem 3.1.4(2), the family (g_v) lies in the closure $\overline{G(K)}$ if and only if (g_v) is orthogonal to $\mathbb{III}_\omega^1(C')$. We consider the commutative diagram (up to sign) of various cup-products

$$\begin{array}{ccc}
 \mathbb{H}^0(K_v, C) & \times & \mathbb{H}^1(K_v, C') \longrightarrow \mathbb{Q}/\mathbb{Z} \\
 \uparrow & & \downarrow \\
 H^0(K_v, T) & \times & H^2(K_v, T') \longrightarrow \mathbb{Q}/\mathbb{Z} \\
 \delta_v \downarrow & & \uparrow \\
 H^1(K_v, R) & \times & H^1(K_v, R') \longrightarrow \mathbb{Q}/\mathbb{Z}.
 \end{array}$$

Recall that the map $\prod G(K_v) \rightarrow \mathbb{III}_\omega^1(C')^D$ is constructed as the composition $\prod H^0(K_v, G) \rightarrow \prod \mathbb{H}^0(K_v, C) \rightarrow \mathbb{III}_\omega^1(C')^D$. Thus $(g_v) \in \prod G(K_v)$ is orthogonal to $\mathbb{III}_\omega^1(C')$ if and only if its image $(\text{ab}^0(g_v)) \in \prod \mathbb{H}^0(K_v, C)$ is orthogonal to $\mathbb{III}_\omega^1(C')$. Since $H^1(K_v, T^{\text{sc}}) = 0$ by the quasi-trivialness of T^{sc} , the map $H^0(K_v, T) \rightarrow \mathbb{H}^0(K_v, C)$ is surjective. In particular, there exists $t_v \in H^0(K_v, T)$ such that its image in $\mathbb{H}^0(K_v, C)$ equals $\text{ab}_v^0(g_v)$. The diagram together with Theorem 3.1.4 imply that $(g_v) \in \overline{G(K)}$ if and only if $(\delta_v t_v)$ is orthogonal to $\mathbb{III}_\omega^1(R')$. Recall we have isomorphisms (3.5). More explicitly, it means that

$$0 = \sum_{v \in X^{(1)}} \langle a_v, \text{ab}_v^0(g_v) \rangle_v = \sum_{v \in X^{(1)}} \text{ab}_v^0(g_v) \cup a_v = \sum_{v \in X^{(1)}} \delta_v t_v \cup a_v$$

for each $a \in \mathbb{III}_\omega^1(C') \simeq \mathbb{III}_\omega^1(R')$ (here the first two a_v lie in $\mathbb{H}^1(K_v, C')$ while the last lies in $H^1(K_v, R')$). Note that $\delta_v t_v$ is given by $t_v^* : H^1(T, R) \rightarrow H^1(K_v, R)$, $[T_H] \mapsto [T_H](t_v) = [Y](g_v)$. Let a_T be the image of $a \in H^1(K, R')$ in $H^2(K, T')$ and let a_{G^c} be the image of $a \in H^1(K, R')$ in $H^1(G^c, R')$. It follows that

$$\sum_{v \in X^{(1)}} \delta_v t_v \cup a_v = \sum_{v \in X^{(1)}} ([T_H] \cup a_T)(t_v) = \sum_{v \in X^{(1)}} ([Y] \cup a_{G^c})(g_v) \quad (3.6)$$

holds thanks to the commutative diagram

$$\begin{array}{ccc}
 H^1(G^c, R) & \times & H^1(G^c, R') \longrightarrow \mathbb{Q}/\mathbb{Z} \\
 \downarrow & & \downarrow \\
 H^1(G, R) & \times & H^1(G, R') \longrightarrow \mathbb{Q}/\mathbb{Z} \\
 \downarrow & & \downarrow \\
 H^1(T, R) & \times & H^1(T, R') \longrightarrow \mathbb{Q}/\mathbb{Z}.
 \end{array}$$

Note that the vanishing of the last term in (3.6) means that (g_v) is orthogonal to the image of u under the pairing $(-, -)$ which completes the proof of the first statement.

Recall that $H^1(K, R')$ has finite exponent (say N). Then $[Y] \in H^1(G^c, R)$ goes to $\partial_N([Y]) \in H^2(G^c, {}_N R)$ induced by a Kummer sequence, and we can lift $a \in H^1(K, R')$ to $a_N \in H^1(K, {}_N R')$. Now we obtain a class $(a_N)_{G^c} \cup \partial_N([Y]) \in H^3(G^c, \mu_N^{\otimes 2})$ induced by ${}_N R \otimes^{\mathbf{L}} {}_N R' \rightarrow \mu_N^{\otimes 2}$ and it restricts to a class in $H_{\text{nr}}^3(K(G), \mu_N^{\otimes 2})$. \square

Remark 3.2.2. The following argument was pointed out to the author by Colliot-Thélène. Let G be a quasi-split reductive group over K . Let B be a Borel subgroup of G containing a maximal torus T of G and let B^- be the unique Borel subgroup of G such that $B \cap B^- = T$. Let $U^+ = \text{rad}^u(B)$ and $U^- = \text{rad}^u(B^-)$ be respective unipotent radicals of B and B^- . Then the big cell of G is $U^- \times T \times U^+$ by [Con, Proposition 1.4.11] (which is dense in G). But U^\pm are isomorphic to some affine spaces as varieties, the big cell of G is thus isomorphic to $T \times \mathbb{A}^N$ for suitable N , i.e. G is stably birational to its maximal torus T . In this point of view, one sees that $H_{\text{nr}}^3(K(G), \mathbb{Q}/\mathbb{Z}(2)) \simeq H_{\text{nr}}^3(K(T), \mathbb{Q}/\mathbb{Z}(2))$, and that weak approximation for G is equivalent to that for T . Subsequently, the previous theorem gives a more precise description of the cohomological obstruction (namely, the obstruction is given by the image of $H^3(G^c, \mu_n^{\otimes 2})$ in $H_{\text{nr}}^3(K(G), \mathbb{Q}/\mathbb{Z}(2))$).

Chapter 4

A Theorem of Borel–Serre

Abstract: This chapter is a report on a Borel–Serre style theorem announcing the finiteness of the Tate–Shafarevich sets of linear groups defined over p -adic function fields. We first reduce the problem to the finiteness of the Tate–Shafarevich sets of absolutely simple simply connected groups. Subsequently, we use the method of Hu [Hu14] to give a list of groups having trivial Tate–Shafarevich sets.

Keywords: Galois cohomology, Borel–Serre theorem, semi-simple simply connected groups.

Throughout, linear groups over a field means an affine group scheme of finite type. Let G be a linear group over K . As usual, we define the Tate–Shafarevich set

$$\text{III}^1(G) := \text{Ker} \left(H^1(K, G) \rightarrow \prod_{v \in X^{(1)}} H^1(K_v, G) \right)$$

of G to be the set of locally trivial elements of $H^1(K, G)$. We shall ask the following:

Question. Is $\text{III}^1(G)$ a finite set for every linear group G ?

If the question has a positive answer, then a twisting argument shows that the fibres of the global-to-local map are finite. This chapter is organized as follows. In the first section, we show that the finiteness of $\text{III}^1(G)$ for all linear groups G follows from that for all semi-simple simply connected groups. In the second section, we use Shapiro’s lemma to pass to absolutely simple simply connected groups. At this stage, we are able to show that $\text{III}^1(G)$ is trivial (hence finite) for a list of absolutely simple simply connected groups.

4.1 Reduction to semi-simple simply connected groups

This section consists of unpublished results of Harari and Szamuely. The author thanks them for their permission to include these results here.

Recall [Ser65, III.4.1, Proposition 8] that a perfect field L is of type (\mathbf{F}) if the set $H^1(L, A)$ is finite for every finite $\text{Gal}(\bar{L}|L)$ -group A . Moreover, if L is a perfect field of type (\mathbf{F}) , then $H^1(L, G)$ is finite for any linear group G by [Ser65, III.4.3, Theorem 4].

Lemma 4.1.1. *For each $v \in X^{(1)}$, the field K_v is of type (\mathbf{F}) .*

Proof. Let $\Gamma_v := \text{Gal}(\bar{K}_v|K_v)$ be the absolute Galois group of K_v . Let A be a finite Γ_v -group and set $I_v = \text{Gal}(\bar{K}_v|K_v\bar{k})$. Consider the restriction-inflation sequence of pointed sets

$$1 \rightarrow H^1(k, A^{I_v}) \rightarrow H^1(K_v, A) \rightarrow H^1(I_v, A).$$

Since k is of type (\mathbf{F}) (see [Ser65, III.4.2, Example(d)] or [PR94, Proposition 6.13]), we conclude that $H^1(k, A^{I_v})$ is a finite set. Analogously, all sets $H^1(k, (A^\sigma)^{I_v})$ are finite where A^σ is the twist of A by some cocycle $\sigma \in Z^1(K_v, A)$. Thus the fibres of $H^1(K_v, A) \rightarrow H^1(I_v, A)$ are finite. Finally, since $K_v\bar{k}^1$ is of type (\mathbf{F}) by [Ser65, III.4.2, Example(c)], we see that $H^1(I_v, A)$ is finite as well. In particular, $H^1(K_v, A)$ is finite. \square

Notation 4.1.2. Now let G be a linear algebraic group over K . Let X_0 be a sufficiently small non-empty open subset of X such that G extends to a smooth group scheme \mathcal{G} over X_0 . Let $H^1(X_0, \mathcal{G})$ be the Čech cohomology set in the étale topology (see [Sko01, §2.2]). For each Čech cocycle $\sigma \in Z^1(X_0, \mathcal{G})$, we denote by \mathcal{G}^σ the twisted group scheme over X_0 .

The next proposition tells us that the finiteness of the Tate–Shafarevich set plays a key role in proving the finiteness of the fibres of the global-to-local map.

Proposition 4.1.3. *Let X_0 be as above. The following are equivalent:*

¹Since we have an identification $K_v \simeq \kappa(v)((t))$, we conclude that $K_v\bar{k} \simeq \bar{k}((t))$.

- (a) The set $\text{III}^1(G^\sigma)$ is finite for each $\sigma \in Z^1(K, G)$.
- (b) The fibres of $\Delta : H^1(K, G) \rightarrow \prod_{v \in X_0^{(1)}} H^1(K_v, G)$ are finite.

Proof. Note that $H^1(K_v, G)$ is finite for each $v \in X^{(1)}$ since K_v is of type **(F)** by Lemma 4.1.1.

Assume (a). It will be sufficient to show $\text{Ker } \Delta$ is finite by replacing G by its twist. Observe that the last term of the exact sequence

$$1 \longrightarrow \text{III}^1(G) \longrightarrow \text{Ker}(\Delta) \xrightarrow{j} \prod_{v \notin X_0} H^1(K_v, G)$$

is finite. Now the fibres of j are finite by (a), so is $\text{Ker } \Delta$ by the finiteness of the last term.

Assume (b). Since $\text{Ker } \Delta^\sigma$ is finite by assumption, we conclude that $\text{III}^1(G^\sigma)$ is finite as well being a subset of $\text{Ker } \Delta^\sigma$. \square

According to the previous proposition, it will be sufficient to show the finiteness of the Tate–Shafarevich sets for all linear groups. We shall see soon that such finiteness is a consequence of the finiteness of the image of $H^1(X_0, \mathcal{G}) \rightarrow H^1(K, G)$.

Lemma 4.1.4. *Let X_0 be as above. Consider the following statements.*

- (a) The image of the restriction map $H^1(X_0, \mathcal{G}) \rightarrow H^1(K, G)$ is finite.
- (b) The set $\text{III}^1(G^{\sigma_K})$ is finite for every $\sigma \in Z^1(X_0, \mathcal{G})$ where $\sigma_K \in H^1(K, G)$ is the image of the class of σ in $H^1(X_0, \mathcal{G})$.

Then (a) implies (b) for any linear algebraic group G , and (a) is equivalent to (b) for any semi-simple simply connected group G .

Proof. First observe that (a) implies the same assertion with G replaced by G^σ thanks to the following commutative diagram (with bijective horizontal arrows)

$$\begin{array}{ccc} H^1(X_0, \mathcal{G}) & \longleftarrow & H^1(X_0, \mathcal{G}^\sigma) \\ \downarrow & & \downarrow \\ H^1(K, G) & \longleftarrow & H^1(K, G^{\sigma_K}). \end{array}$$

By Harder’s lemma ([Har67, Lemma 4.1.3]), the image of $H^1(U, \mathcal{G})$ in $H^1(K, G)$ contains $\text{III}^1(G)$ for all $U \subset X_0$. Thus (a) implies (b) for any linear algebraic group G .

Now let G be a semi-simple simply connected group and assume (b). We claim that there is an exact sequence of pointed sets

$$1 \rightarrow \text{III}^1(G) \rightarrow \text{Im} (H^1(X_0, \mathcal{G}) \rightarrow H^1(K, G)) \rightarrow \prod_{v \notin X_0} H^1(K_v, G).$$

Indeed, for $v \in X_0$ the composite map $H^1(X_0, \mathcal{G}) \rightarrow H^1(K, G) \rightarrow H^1(K_v, G)$ factors through $H^1(\mathcal{O}_v, \mathcal{G}) = H^1(\kappa(v), \mathcal{G})$ which is trivial by Serre’s Conjecture II (proven by Kneser over p -adic fields). Therefore if $x \in \text{Im} (H^1(X_0, \mathcal{G}) \rightarrow H^1(K, G))$ goes to the distinguished element in the rightmost set, then it goes to the distinguished element in $H^1(K_v, G)$ for each $v \in X^{(1)}$. This shows the exactness in the middle. The finiteness of $H^1(K_v, G)$ shows that the finiteness of the middle term follows from the finiteness of the fibres of the last map, which in turn implied by assumption (b). \square

From now on, we shall work with the restriction map $\text{Res} : H^1(X_0, \mathcal{G}) \rightarrow H^1(K, G)$. We claim that if Res has finite image for all inner forms of G^{sc} , then the image of Res is finite for all linear groups G .

Lemma 4.1.5.

- The set $H^1(X_0, \mathcal{F})$ is finite for each finite X_0 -group scheme \mathcal{F} .
- The group $H^2(X_0, \mathcal{F})$ is finite for each finite commutative X_0 -group scheme \mathcal{F} .

Proof. Actually we have $H^i(X_0, \mathcal{F}) = H^i(\pi_1(X_0), \mathcal{F}(\overline{K}))$ where the latter set is finite (for $i = 2$, the latter set is actually a group since \mathcal{F} is commutative). Indeed, the profinite group $\pi_1(X_0 \times_k \overline{k})$ is topologically finitely generated (see [SGA1, X.2.9]), hence it is of type **(F)** by [Ser65, III.4.1, Proposition 9]. Now $\pi_1(X_0)$ is a group of type **(F)** because it is an extension of $\text{Gal}(\overline{k}|k)$ by $\pi_1(X_0 \times_k \overline{k})$. \square

Subsequently, we can pass from linear groups to connected linear groups as follows.

Corollary 4.1.6. *Let G be a linear algebraic group over K with neutral component G° . If the map $H^1(X_0, \mathcal{G}^{\circ, \sigma}) \rightarrow H^1(K, G^{\circ, \sigma_K})$ has finite image for each twist G°, σ_K} of G° by $\sigma \in Z^1(X_0, \mathcal{G})$, then so does $H^1(X_0, \mathcal{G}) \rightarrow H^1(K, G)$. Here σ_K denotes the image of σ in $H^1(K, G)$.*

Proof. The connected-étale sequence [Mil17, Chapter 5, §H] yields a commutative diagram

$$\begin{array}{ccccc} H^1(X_0, \mathcal{G}^\circ) & \longrightarrow & H^1(X_0, \mathcal{G}) & \longrightarrow & H^1(X_0, \mathcal{G}/\mathcal{G}^\circ) \\ \downarrow & & \downarrow & & \downarrow \\ H^1(K, G^\circ) & \longrightarrow & H^1(K, G) & \longrightarrow & H^1(K, G/G^\circ) \end{array}$$

with exact rows of pointed sets. Now a diagram chase together with Lemma 4.1.5 yield the desired result. \square

Remark 4.1.7. Recall that for a connected linear group G over K (which is of characteristic zero), we have $H^1(K, H) \simeq H^1(K, G)$ where H is a maximal connected reductive subgroup of G (see [PR94, Proposition 2.9]). Thus we can pass from connected linear groups to connected reductive groups. Recall that we denote by G^{sc} the simply connected covering of the derived subgroup G^{ss} of a connected reductive group G .

Proposition 4.1.8. *Let G be a connected reductive group. If $H^1(X_0, \mathcal{G}^{\text{sc}, \sigma}) \rightarrow H^1(K, G^{\text{sc}, \sigma_K})$ has finite image for each $\sigma \in Z^1(X_0, \mathcal{G})$, then so does $H^1(X_0, \mathcal{G}) \rightarrow H^1(K, G)$.*

Proof. The assertion holds for G if and only if it holds for some finite direct power G^r . Moreover, since $H^1(K, Q) = 0$ for any quasi-trivial K -torus Q , we are free to multiply G by a quasi-trivial torus. Now [San81, Lemme 1.10] implies that we can assume G admits a special covering by replacing G with $G^r \times Q$ if necessary. More precisely, there is a short exact sequence of algebraic groups $1 \rightarrow F \rightarrow G_0 \rightarrow G \rightarrow 1$ where $G_0 \rightarrow G$ is a central isogeny with $G_0 = G^{\text{sc}} \times Q_0$ for some

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quasi-trivial torus Q_0 . We fix an extension $1 \rightarrow \mathcal{F} \rightarrow \mathcal{G}_0 \rightarrow \mathcal{G} \rightarrow 1$ of $1 \rightarrow F \rightarrow G_0 \rightarrow G \rightarrow 1$ to X_0 . Consider the following commutative diagram with exact rows

$$\begin{array}{ccccc} H^1(X_0, \mathcal{G}_0) & \longrightarrow & H^1(X_0, \mathcal{G}) & \xrightarrow{\partial} & H^2(X_0, \mathcal{F}) \\ \downarrow & & \downarrow & & \downarrow \\ H^1(K, G_0) & \longrightarrow & H^1(K, G) & \longrightarrow & H^2(K, F) \end{array}$$

(see [Ser65, §I.5.7, Proposition 43]). Since $H^2(X_0, \mathcal{F})$ is finite by Lemma 4.1.5, it suffices to show the finiteness of each fibre of

$$\mathrm{Im} \left(H^1(X_0, \mathcal{G}) \rightarrow H^1(K, G) \right) \rightarrow \mathrm{Im} \left(H^2(X_0, \mathcal{F}) \rightarrow H^2(K, F) \right).$$

By assumption and construction, the image of $H^1(X_0, \mathcal{G}_0^\sigma) \rightarrow H^1(K, G_0^{\sigma_K})$ is finite for each $\sigma \in Z^1(X_0, \mathcal{G})$. Now a diagram chase yields the desired finiteness.² \square

We summarize the results of this section into the following theorem. As above, we denote by $\sigma_K \in H^1(K, G)$ its image under the restriction map for a cocycle $\sigma \in Z^1(X_0, \mathcal{G})$.

Theorem 4.1.9.

- Let G be a linear K -group with neutral component G° . If $H^1(X_0, \mathcal{G}^{\circ, \sigma}) \rightarrow H^1(K, G^{\circ, \sigma_K})$ has finite image, then $\mathrm{III}^1(G^{\sigma_K})$ is finite for each $\sigma \in Z^1(X_0, \mathcal{G})$.
- Let G be a connected reductive K -group. If $H^1(X_0, \mathcal{G}^{\mathrm{sc}, \sigma}) \rightarrow H^1(K, G^{\mathrm{sc}, \sigma_K})$ has finite image for each $\sigma \in Z^1(X_0, \mathcal{G})$, then $\mathrm{III}^1(G^{\sigma_K})$ is finite for each such σ .

Proof. According to Lemma 4.1.4, the finiteness of $\mathrm{III}^1(G^{\sigma_K})$ will follow from the finiteness of the image under the restriction map $H^1(X_0, \mathcal{G}) \rightarrow H^1(K, G)$. Corollary 4.1.6 yields the first point. Finally, we deduce the last point from Proposition 4.1.8. \square

Remark 4.1.10. Suppose G is an absolutely almost simple simply connected group over K . The finiteness of $\mathrm{III}^1(G)$ (and even of $\mathrm{III}^1(G^\sigma)$ for $\sigma \in Z^1(K, G)$ by a twisting argument) would follow from the finiteness of the fibres of the Rost invariant $H^1(K, G) \rightarrow H^3(K, \mu_N^{\otimes 2})$ (here N is a sufficiently large positive integer depending on G , see [KMRT98, Proposition 31.40]). More precisely, let us consider the following commutative diagram

$$\begin{array}{ccccccc} 1 & \longrightarrow & \mathrm{III}^1(G) & \longrightarrow & H^1(K, G) & \longrightarrow & \prod H^1(K_v, G) \\ & & \downarrow & & \downarrow & & \downarrow \\ 1 & \longrightarrow & \mathrm{III}^3(\mu_N^{\otimes 2}) & \longrightarrow & H^3(K, \mu_N^{\otimes 2}) & \longrightarrow & \prod H^3(K_v, \mu_N^{\otimes 2}) \end{array}$$

where the rows are exact and the vertical arrows are given by respective Rost invariants. Now it is clear that each element of $\mathrm{III}^1(G)$ lies in some fibre over $\mathrm{III}^3(\mu_N^{\otimes 2})$. Thus it will be sufficient to show that the group $\mathrm{III}^3(\mu_N^{\otimes 2})$ is finite. Indeed, elements of $\mathrm{III}^3(\mu_N^{\otimes 2})$ are all contained in the image of the finite group $H^3(U, \mu_N^{\otimes 2})$ for a suitable open subset $U \subset X$ (see the proof of [HS16, Proposition 3.6] or the proof of Proposition 2.3.15).

²Here we are twisting with cocycles of $Z^1(X_0, \mathcal{G})$ as in [Gir71, §IV, Proposition 4.3.4]. In particular these twists are X_0 -inner forms of G^{sc} .

4.2 Semi-simple simply connected case

In this section, we show that the set $\text{III}^1(G)$ is trivial for certain semi-simple simply connected groups. All the arguments are essentially the same as [Hu14]. However, Hu considered the set Ω of all discrete valuations of K while we consider only those coming from the curve X . More precisely, in [Hu14] the main theorem asserts that the set

$$\text{III}_{\Omega}^1(G) := \text{Ker} \left(H^1(K, G) \rightarrow \prod_{v \in \Omega} H^1(K_v, G) \right)$$

is trivial for a list of groups. In our situation, we consider only a subset $X^{(1)} \subset \Omega$ of places of K , and hence our $\text{III}^1(G)$ contains $\text{III}_{\Omega}^1(G)$ as a subset. In particular, our result is slightly better than Hu's original one.

Passing to absolutely simple groups

The first step is passing to absolutely simple simply connected groups. Let G be a semi-simple simply connected algebraic group over K . According to [KMRT98, Theorem 26.8], one can then write $G = \prod_i R_{L_i|K}(G_i)$ where G_i is an absolutely simple simply connected algebraic group over some finite (separable) extension L_i of K , and $R_{L_i|K}$ denotes the corresponding Weil restriction. Thus it suffices to show the finiteness of $\text{III}^1(G)$ for absolutely simple simply connected groups (here we have implicitly used the non-commutative version of Shapiro's lemma, see [KMRT98, Lemma 29.6]). From now on, G will always be an absolutely simple simply connected group over K .

Preliminaries on injectivity properties

Let us recall two injectivity properties (under good reduction assumption) of some global-to-local maps before going further. We shall frequently consider the fundamental diagram

$$\begin{array}{ccc} H^1(K, G) & \xrightarrow{R_G} & H^3(K, \mu_n^{\otimes 2}) \\ \downarrow & & \downarrow \\ \prod_{v \in X^{(1)}} H^1(K_v, G) & \xrightarrow{(R_{G,v})} & \prod_{v \in X^{(1)}} H^3(K_v, \mu_n^{\otimes 2}) \end{array} \quad (4.1)$$

where $R_G, R_{G,v}$ denote the Rost invariants (see [KMRT98, §31.B]) and $n = n_G$ is an integer depending on G (see [KMRT98, Proposition 31.40]). The injectivity of the right vertical arrow is a step in the proof of [HS16, Proposition 6.2]. We quote it as a lemma for further reference.

Lemma 4.2.1. *Suppose X extends to a smooth proper relative curve \mathcal{X} over \mathcal{O}_k . The following map is injective*

$$H^3(K, \mathbb{Q}/\mathbb{Z}(2)) \rightarrow \prod_{v \in X^{(1)}} H^3(K_v, \mathbb{Q}/\mathbb{Z}(2)).$$

Proof. By [Kat86, Proposition 5.2], the kernel of the above map can be identified with the kernel of

$$\text{loc}_{\mathcal{X}} : H^2(L, \mathbb{Q}/\mathbb{Z}(1)) \rightarrow \prod_{w \in \mathcal{X}_{\text{sp}}^{(1)}} H^2(L_w, \mathbb{Q}/\mathbb{Z}(1))$$

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where \mathcal{X}_{sp} is the special fibre of $\mathcal{X} \rightarrow \text{Spec } \mathcal{O}_k$, L denotes the function field of \mathcal{X}_{sp} , and L_w stands for the completion of L with respect to $w \in \mathcal{X}_{\text{sp}}^{(1)}$. But \mathcal{X}_{sp} is smooth by good reduction assumption, thus the injectivity of $\text{loc}_{\mathcal{X}}$ follows from the Brauer–Hasse–Noether theorem over global function fields. \square

Let $W(K)$ be the Witt group related to quadratic forms (see [Lam05, §II.1]). Then $W(K)$ is endowed with a ring structure where the multiplication is induced by the tensor product of quadratic forms. Let $I(K)$ be the class of even dimensional quadratic forms in $W(K)$. Moreover, $I(K)$ carries an ideal structure in $W(K)$. For any integer $n \geq 1$, we denote by $I^n(K)$ the n -th power of the ideal $I(K)$ in $W(K)$. Note that $I^n(K)$ is generated by the classes of n -fold Pfister forms as an abelian group (see [Lam05, VII, pp. 202]). The next lemma plays a role when we consider absolutely simple simply connected groups of type C_n .

Lemma 4.2.2. *Suppose X extends to a smooth proper relative curve \mathcal{X} over \mathcal{O}_k . The diagonal map below is injective:*

$$I^3(K) \rightarrow \prod_{v \in X^{(1)}} I^3(K_v).$$

Proof. There is a commutative diagram with injective right vertical map by Lemma 4.2.1:

$$\begin{array}{ccc} I^3(K) & \xrightarrow{e^3} & H^3(K, \mathbb{Z}/2) \\ \downarrow & & \downarrow \\ \prod_{v \in X^{(1)}} I^3(K_v) & \xrightarrow{(e_v^3)} & \prod_{v \in X^{(1)}} H^3(K_v, \mathbb{Z}/2) \end{array}$$

with e^3 and (e_v^3) induced by the Arason invariants (see [Ara75]). Since $\text{cd}_2(K) \leq 3$, [AEJ86, pp. 654, Corollary 2] implies $I^4(K) = 0$. So e^3 is injective by [AEJ86, Theorem 4] and we conclude by diagram chasing. \square

Remark 4.2.3. Suppose that X extends to a (not necessarily smooth) flat proper relative curve \mathcal{X} over \mathcal{O}_k . Since the kernel of $\text{loc}_{\mathcal{X}}$ is always finite, the proof of Lemma 4.2.1 shows that the map $H^3(K, \mathbb{Q}/\mathbb{Z}(2)) \rightarrow \prod_{v \in X^{(1)}} H^3(K_v, \mathbb{Q}/\mathbb{Z}(2))$ always has a finite kernel. Similarly, we conclude without good reduction assumption that $I^3(K) \rightarrow \prod_{v \in X^{(1)}} I^3(K_v)$ has a finite kernel.

Statement of the main theorem

Recall that we have defined a finer classification of absolutely simple simply connected groups (see page 26). In the sequel, we shall show the following theorem case by case.

Theorem 4.2.4. *Suppose X extends to a smooth proper relative curve \mathcal{X} over \mathcal{O}_k . Let G be a semi-simple simply connected group over K . If every absolutely simple simply connected factor of G is of type A_n^* , B_n , C_n^* , D_n^* , F_4^{red} or G_2 , then $\text{III}^1(G)$ is trivial.³*

Subsequently we assume systematically that X extends to a smooth relative curve over \mathcal{O}_k .

³If we consider the set Ω of all discrete valuations of K rather than the places in $X^{(1)}$ coming from the curve X , then there are other known cases [CTPS12, Theorem 4.3]. For unitary groups, see also the recent preprint by Parimala and Suresh [PS20].

Remark 4.2.5. Let us consider the diagram (4.1). Under the good reduction assumption, we know that the map $H^3(K, \mathbb{Q}/\mathbb{Z}(2)) \rightarrow \prod_{v \in X^{(1)}} H^3(K_v, \mathbb{Q}/\mathbb{Z}(2))$ is injective. Moreover, if the kernel of the Rost invariant map is trivial, then it follows immediately from diagram (4.1) that III^1 is trivial. We thank Parimala Raman for pointing out to us that the kernel of the Rost invariant map is trivial in the following cases:

- For groups of type B_n, C_n, D_n and certain unitary groups, see [Pre13].
- For groups of the form $\mathbf{SL}_1(A)$ with $\text{ind}(A)$ coprime to p , see [PPS18].

In particular, the Hasse principle holds in these cases and we get a supplement to Theorem 4.2.4.

4.2.1 Type A: inner type

Proposition 4.2.6. *Let A be a central simple algebra of square-free index over K and let $G = \mathbf{SL}_1(A)$. Then the diagonal map $H^1(K, G) \rightarrow \prod_{v \in X^{(1)}} H^1(K_v, G)$ is injective. In particular, $\text{III}^1(G)$ is trivial.*

Proof. Suslin's theorem [Sus84, Theorem 24.4] yields for $L = K$ or K_v the injectivity of the Rost invariant $H^1(L, \mathbf{SL}_1(A)) \rightarrow H^3(L, \mu_n^{\otimes 2})$. Now we conclude by diagram chasing in (4.1). \square

Remark 4.2.7. In this case, we can drop the good reduction assumption to get a weaker result. Indeed, a diagram chase in (4.1) together with Remark 4.2.3 shows that $H^1(K, G) \rightarrow \prod_{v \in X^{(1)}} H^1(K_v, G)$ has a finite kernel.

4.2.2 Type B

Let L be either K or K_v . Let q be a non-singular quadratic form of rank $(2n + 1) \geq 3$ over L . We denote by $\text{Sn}(q_L)$ the image of the spinor norm map $\mathbf{SO}(q)(L) \rightarrow L^\times/L^{\times 2}$ arising as the connecting map associated to the short exact sequence $1 \rightarrow \mu_2 \rightarrow \mathbf{Spin}(q) \rightarrow \mathbf{SO}(q) \rightarrow 1$ of algebraic groups over L .

Proposition 4.2.8. *Let q be a non-singular quadratic form over K .*

- (1) *If q is of rank 3 over K , then the diagonal map below is injective*

$$\frac{K^\times/K^{\times 2}}{\text{Sn}(q_K)} \rightarrow \prod_{v \in X^{(1)}} \frac{K_v^\times/K_v^{\times 2}}{\text{Sn}(q_{K_v})}.$$

- (2) *If $\text{rank } q \geq 5$, then $\mathbf{SO}(q)(K) \rightarrow K^\times/K^{\times 2}$ is surjective, i.e. $\text{Sn}(q_K) = K^\times/K^{\times 2}$.*

Proof.

- If $\text{rank}(q) = 3$, we may assume $q = \langle 1, a, b \rangle$ after scaling. Consider the quaternion K -algebra $D = (-a, -b)_K$. So $\text{Sn}(q_K) = \text{Nrd}(D^\times)$ modulo squares and the result follows from Proposition 4.2.6.
- If $\text{rank } q \geq 5$, then the surjectivity of spinor norm holds by [Hu14, Corollary 4.4]. \square

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Proposition 4.2.9. *Let q be a non-singular quadratic form of rank $2n + 1$ over K with $n \geq 1$ and let $G = \mathbf{Spin}(q)$. Then $\mathbb{III}^1(G)$ is trivial.*

Proof. Take $x \in \mathbb{III}^1(G)$. Let $\pi_* : H^1(K, G) \rightarrow H^1(K, \mathbf{SO}(q))$ be the map induced by $\pi : G \rightarrow \mathbf{SO}(q)$. Then $\pi_*(x)$ corresponds to a non-singular quadratic form q' with the same rank as q (recall that $H^1(K, \mathbf{SO}(q))$ classifies non-singular quadratic forms having the same rank as q). By [Sch85, pp. 89, Merkurjev's Theorem], the class $q \perp (-q') \in W(K)$ lies in $I^3(K)$. But $x \in \mathbb{III}^1(G)$ implies that its canonical image in $H^1(K_v, G)$ is trivial for each v , so the image $(q \perp (-q'))_v$ of $q \perp (-q')$ in $I^3(K_v)$ vanishes. According to Lemma 4.2.2, we have $q \simeq q'$ over K , i.e. $x \in \text{Ker } \pi_*$. Thus x comes from some $\alpha \in \frac{K^\times/K^{\times 2}}{\text{Sn}(q_K)}$ by [Ser65, I.5.5, Corollary 1]. Now we can conclude by the following commutative diagram

$$\begin{array}{ccc} 1 & \longrightarrow & \frac{K^\times/K^{\times 2}}{\text{Sn}(q_K)} \longrightarrow H^1(K, G) \\ & & \downarrow \\ 1 & \longrightarrow & \prod \frac{K_v^\times/K_v^{\times 2}}{\text{Sn}(q_{K_v})} \longrightarrow \prod H^1(K_v, G). \end{array}$$

Indeed, $x \in \mathbb{III}^1(G)$ implies that the image of α in $\prod \frac{K_v^\times/K_v^{\times 2}}{\text{Sn}(q_{K_v})}$ is trivial, hence $\alpha = 1$ by Proposition 4.2.8. \square

4.2.3 Type C

Let D be a quaternion division algebra over K with standard involution $*$. Let $h : V \times V \rightarrow D$ be a Hermitian form over $(D, *)$. Let $q_h : V \rightarrow L$ be the quadratic form given by $q_h(v) = h(v, v)$. Then sending h to q_h defines an injective group homomorphism $W(D, *) \rightarrow W(K)$ by [Sch85, Theorem 10.1.7].

Proposition 4.2.10. *Let $G = \mathbf{U}(h)$. Then $H^1(K, G) \rightarrow \prod_{v \in X^{(1)}} H^1(K_v, G)$ is injective.*

Proof. The pointed set $H^1(K, G)$ classifies up to isomorphism Hermitian forms over $(D, *)$ of the same rank as h . Let h_1 and h_2 be Hermitian forms over $(D, *)$ of the same rank as h . Put $h' = h_1 \perp (-h_2)$. Thus h' has even rank and it follows that the class of $q_{h'}$ in the Witt group $W(K)$ lies in the subgroup $I^3(K) = I(K) \cdot I^2(K)$. We conclude that $[q_{h_1}] - [q_{h_2}] = [q_{h'}] \in I^3(K)$. Suppose $(h_1)_v \simeq (h_2)_v$ for all $v \in X^{(1)}$. According to Lemma 4.2.2, we obtain $[q_{h'}] = 0$ in $I^3(K)$. This implies that $q_{h_1} \simeq q_{h_2}$ over K , i.e. $h_1 \simeq h_2$. \square

4.2.4 Type D

Lemma 4.2.11. *Let $(D, *)$ be a quaternion division algebra over K . Let h be a non-singular Hermitian form of rank $n \geq 2$ over $(D, *)$. Then the canonical map $\frac{K^\times/K^{\times 2}}{\text{Sn}(h_K)} \rightarrow \prod_{v \in X^{(1)}} \frac{K_v^\times/K_v^{\times 2}}{\text{Sn}(h_{K_v})}$ is injective.*

Proof. This follows from exactly the same proof as [Hu14, Proposition 5.1] except that we use Proposition 4.2.6 instead of [Hu14, Theorem 3.7] there. \square

Proposition 4.2.12. *Let $(D, *)$ be a quaternion division algebra over K . Let h be a non-singular Hermitian form of rank $n \geq 2$ over $(D, *)$. If $G = \mathbf{Spin}(h)$, then $\mathbb{III}^1(G) = 1$.*

Proof. Take $\xi \in \text{III}^1(G)$. Its image under $H^1(K, G) \rightarrow H^1(K, \mathbf{SU}(h)) \rightarrow H^1(K, \mathbf{U}(h))$ corresponds to the class of a Hermitian form h' having the same rank and discriminant as h such that the Clifford invariant $\text{Cl}(h \perp (-h'))$ is trivial in ${}_2\text{Br}(K)/\langle D \rangle$. By the same argument as [Hu14, Theorem 5.4], we deduce that $h \perp (-h')$ has trivial Rost invariant. Now [Hu14, Corollary 5.3] yields an isomorphism $h \simeq h'$ of Hermitian forms. Consequently, the image of ξ in $H^1(K, \mathbf{U}(h))$ is trivial and we can deduce further that its image in $H^1(K, \mathbf{SU}(h))$ is already trivial by [BFP98, Lemma 7.11]. Consider the exact commutative diagram

$$\begin{array}{ccccccc} 1 & \longrightarrow & \frac{K^\times/K^{\times 2}}{\text{Sn}(h_K)} & \longrightarrow & H^1(K, G) & \longrightarrow & H^1(K, \mathbf{SU}(h)) \\ & & \downarrow & & \downarrow & & \downarrow \\ 1 & \longrightarrow & \prod_{v \in X^{(1)}} \frac{K_v^\times/K_v^{\times 2}}{\text{Sn}(h_{K_v})} & \longrightarrow & \prod_{v \in X^{(1)}} H^1(K_v, G) & \longrightarrow & \prod_{v \in X^{(1)}} H^1(K_v, \mathbf{SU}(h)). \end{array}$$

We have seen ξ has trivial image in $H^1(K, \mathbf{SU}(h))$, so ξ comes from some $\alpha \in \frac{K^\times/K^{\times 2}}{\text{Sn}(h_K)}$. Thus α is trivial everywhere locally by diagram chasing. But the first vertical map is injective, so $\alpha = 1$ and hence $\xi = 1$. \square

4.2.5 Type A: outer case

Throughout this subsection, $L|K$ is a quadratic field extension, (D, τ) is a central division algebra over L with a unitary involution τ such that $L^\tau = K$, and h, h_1, h_2 are non-singular Hermitian forms over (D, τ) . We will denote by $\text{ind}(D)$ the index of D over L .

Proposition 4.2.13 (See [Hu14, Proposition 6.1 and 6.12]).

- Suppose $\text{ind}(D)$ is odd. Suppose h_1, h_2 have the same rank and discriminant. If $[h_1] = [h_2]$ in $W(D \otimes_K K_v, \tau)$ for every $v \in X^{(1)}$, then $h_1 \simeq h_2$ over (D, τ) .
- Suppose $\text{ind}(D)$ is even and $4 \nmid \text{ind}(D)$. Suppose h has even rank, trivial discriminant and trivial Rost invariant (see [Hu14, 2.12] for the definition of Rost invariants of Hermitian forms). Then $[h] = 0$ in $W(D, \tau)$ if and only if $[h \otimes_K K_v] = 0$ in $W(D \otimes_K K_v, \tau)$ for every $v \in X^{(1)}$.

Proof.

- Suppose $\text{ind}(D)$ is odd. Using Lemma 4.2.2 instead of [Hu14, Lemma 3.8] in Hu's argument yields the desired result.
- Suppose $\text{ind}(D)$ is even. In this case, we may write $D = Q \otimes_L D_{\text{odd}}$ where Q is a quaternion division algebra over L and D_{odd} is a central division algebra of odd index over L . Subsequently, we can copy verbatim the proof of [Hu14, Proposition 6.12]. \square

Lemma 4.2.14. Suppose $\text{ind}(D)$ is square-free. The following map is injective:

$$\frac{(R_{L|K}^1 \mathbb{G}_m)(K)}{\text{Nrd}(\mathbf{U}(h)(K))} \rightarrow \prod_{v \in X^{(1)}} \frac{(R_{L|K}^1 \mathbb{G}_m)(K_v)}{\text{Nrd}(\mathbf{U}(h)(K_v))}.$$

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Proof. Let us briefly explain how to deduce the result from Hu's argument.

- Suppose $\text{ind}(D) = 2$. In this case, replacing [Hu14, Theorem 4.5] by Proposition 4.2.12 yields the desired injectivity.
- Suppose $\text{ind}(D)$ is odd and square-free. Then replacing [Hu14, Theorem 3.7] by Proposition 4.2.6 implies that the desired map is injective.
- Suppose $\text{ind}(D)$ is even and square-free. The injectivity follows from exactly the same argument as [Hu14, Lemma 6.2]. \square

Proposition 4.2.15. *Suppose $\text{ind}(D)$ is square-free and $p \neq 2$ if $\text{ind}(D)$ is even. Let $G = \mathbf{SU}(h)$. Then the following map has trivial kernel*

$$H^1(K, G) \rightarrow \prod_{v \in X^{(1)}} H^1(K_v, G).$$

Proof. Take $\xi \in \mathbb{I}^1(G)$. Let h' be a Hermitian form over (D, τ) such that its class $[h'] \in H^1(K, \mathbf{U}(h))$ equals the image of ξ under the canonical map $H^1(K, \mathbf{SU}(h)) \rightarrow H^1(K, \mathbf{U}(h))$.

- Suppose $\text{ind}(D)$ is odd. Note that h and h' have the same rank and discriminant and they are locally isomorphic since ξ is locally trivial. It follows from Proposition 4.2.13 that there is an isomorphism $h \simeq h'$ of Hermitian forms over (D, τ) . In particular, $\xi \in H^1(K, \mathbf{SU}(h))$ goes to the trivial element $[h] \in H^1(K, \mathbf{U}(h))$. Now consider the following exact commutative diagram

$$\begin{array}{ccccc} 1 & \longrightarrow & \frac{(R_{L|K}^1 \mathbb{G}_m)(K)}{\text{Nrd}(\mathbf{U}(h)(K))} & \longrightarrow & H^1(K, \mathbf{SU}(h)) & \longrightarrow & H^1(K, \mathbf{U}(h)) \\ & & \downarrow & & \downarrow & & \downarrow \\ 1 & \longrightarrow & \prod_{v \in X^{(1)}} \frac{(R_{L|K}^1 \mathbb{G}_m)(K_v)}{\text{Nrd}(\mathbf{U}(h)(K_v))} & \longrightarrow & \prod_{v \in X^{(1)}} H^1(K_v, \mathbf{SU}(h)) & \longrightarrow & \prod_{v \in X^{(1)}} H^1(K_v, \mathbf{U}(h)). \end{array}$$

According to Lemma 4.2.14, the left vertical arrow is injective. Finally, a diagram chasing shows that ξ is trivial.

- Suppose $\text{ind}(D)$ is even. The form $h' \perp (-h)$ has even rank, trivial discriminant and is locally hyperbolic. We can show that $h' \perp (-h)$ has trivial Rost invariant (see [Hu14, Theorem 6.13]). Now Proposition 4.2.13 implies that $h' \simeq h$ over (D, τ) . Now the desired result follows from the same argument as the previous case. \square

4.2.6 Type F

For each exceptional Jordan algebra J of dimension 27 over L , there are three associated invariants⁴:

$$f_3(J) \in H^3(L, \mathbb{Z}/2), \quad f_5(J) \in H^5(L, \mathbb{Z}/2) \quad \text{and} \quad g_3(J) \in H^3(L, \mathbb{Z}/3).$$

⁴See [Ser95, §9] for more information. Here f_3 and g_3 are given by the Rost invariant.

Moreover, $g_3(J) = 0$ if and only if J is reduced, and two reduced exceptional Jordan algebras are isomorphic if and only if they have the same f_3 and f_5 invariants.

We say an absolutely simple simply connected group G is of type F_4^{red} , if $G = \text{Aut}_{\text{alg}}(J)$ for some reduced exceptional Jordan algebra J of dimension 27.

Proposition 4.2.16. *Let G be an absolutely simple simply connected group of type F_4^{red} over K . Then $\text{III}^1(G) = 1$.*

Proof. Suppose $G = \text{Aut}_{\text{alg}}(J)$ for some reduced exceptional Jordan algebra J of dimension 27. For cohomological dimension reasons, we have $f_5(J) = 0$. Take $\xi \in \text{III}^1(G)$ and let J' be an exceptional Jordan algebra representing ξ . By Lemma 4.2.1, we conclude that $f_3(J) = f_3(J')$ and $0 = g_3(J) = g_3(J')$. Thus J' is also reduced with $f_3(J') = f_3(J)$ and $f_5(J') = f_5(J) = 0$, i.e. $J' \simeq J$. □

4.2.7 Type G

Proposition 4.2.17. *Let G be an absolutely simple simply connected group of type G_2 over K . Then $\text{III}^1(G) = 1$.*

Proof. We repeat verbatim the proof of [Hu14, Theorem 3.10]. By assumption, there is an isomorphism $G \simeq \mathbf{Aut}_{\text{alg}}(C)$ for some Cayley K -algebra C . Take $\xi \in \text{III}^1(G)$ and let C' be a Cayley algebra representing ξ . So $C_{K_v} \simeq C'_{K_v}$ for each $v \in X^{(1)}$. Recall two Cayley algebras are isomorphic if and only if their norm forms are isomorphic. Since the norm form of a Cayley algebra is a 3-fold Pfister form (see [KMRT98, pp. 460]), it follows from Lemma 4.2.1 that $C \simeq C'$. □

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Titre: Arithm tique des groupes alg briques au-dessus du corps des fonctions d'une courbe sur un corps p -adique

Mots cl s: points rationnels, obstruction cohomologique, principe de Hasse, approximation faible, groupes r ductifs, espaces homog nes, toseurs, cohomologie galoisienne, th or mes de dualit  arithm tique

R sum : Dans cette th se, on consid re l'arithm tique des groupes lin aires sur les corps de fonctions p -adiques. On divise la th se en plusieurs parties.

Dans la premi re partie, on rappelle une obstruction cohomologique au principe de Hasse pour les toseurs sous un tore [HS16] et une obstruction   l'approximation faible pour les tores [HSS15]. Par la suite, on compare les obstructions ci-dessus de deux mani res diff rentes. En particulier, on montre que l'obstruction au principe de Hasse pour les toseurs sous un tore peut  tre d crite par un groupe de cohomologie non ramifi e.

Dans la deuxi me partie, on  tablit quelques th or mes de dualit  arithm tique et on d -

duit une suite exacte de type Poitou–Tate pour les complexes courts de tores. Plus tard, on parvient   trouver un d faut d'approximation faible pour certains groupes r ductifs connexes en utilisant un morceau de la suite de Poitou–Tate.

Dans la derni re partie, on consid re un th or me de Borel–Serre de finitude en cohomologie galoisienne. Le premier ingr dient est que la finitude du noyau de l'application locale-globale pour les groupes lin aires d coulera de celle des groupes g om triquement simples simplement connexes. Par la suite, on montre que ce noyau est un ensemble fini pour une liste de groupes g om triquement simples simplement connexes.

Title: Arithmetic of algebraic groups over the function field of a curve defined over a p -adic field

Keywords: rational points, cohomological obstruction, the Hasse principle, weak approximation, reductive groups, homogeneous spaces, torsors, Galois cohomology, arithmetic duality

Abstract: This thesis deals with the arithmetic of linear groups over p -adic function fields. We divide the thesis into several parts.

In the first part, we recall a cohomological obstruction to the Hasse principle for torsors under tori [HS16] and another obstruction to weak approximation for tori [HSS15]. Subsequently we compare the two obstructions in two different manners. In particular, we show that the obstruction to the Hasse principle for torsors under tori can be described by an unramified cohomology group.

In the second part, we establish some arith-

metic duality theorems and deduce a Poitou–Tate style exact sequence for a short complex of tori. Later on, we manage to find a defect to weak approximation for certain connected reductive groups using a piece of the Poitou–Tate sequence.

In the last part, we consider a Borel–Serre style finiteness theorem in Galois cohomology. The first ingredient is that the finiteness of the kernel of the global-to-local map for linear groups will follow from that of absolutely simple simply connected groups. Subsequently, we show the kernel is a finite set for a list of absolutely simple simply connected groups.