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François Généræu

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## THÈSE

Pour obtenir le grade de

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Présentée par

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préparée au sein du **Laboratoire Jean Kuntzmann**  
dans l'**École Doctorale Mathématiques, Sciences et**  
**technologies de l'information, Informatique**

## **Sur une approximation variationnelle stable du cut locus, et un problème isopérimétrique non local**

## **On a stable variational approximation of the cut locus, and a non-local isoperimetric problem**

Thèse soutenue le **3 juin 2020**,  
devant le jury composé de :

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**Abstract.** This thesis is composed of two parts. In the first part, we study a generalization of the variational problem of elastic-plastic torsion problem to manifolds. We show that in the case of manifolds, the problem is not equivalent to an obstacle type problem, contrary to the euclidean case, but we establish the equivalence when the parameter of the problem goes to infinity. We show, as in the euclidean case, that the non-contact set contains the cut locus of the manifold, and converges to the latter in the Hausdorff sense. Furthermore, we show that the minimizers of the problem are uniformly semiconcave. We deduce a stable approximation of the cut locus, in the spirit of the lambda medial axis of Chazal and Lieutier. We then use this result to compute numerically the cut locus of some surfaces of varied geometries.

In the second part, we study an extension of a nonlocal isoperimetric problem. More precisely, we add a confinement potential to Gamow's liquid drop model for the nucleus. We then study large volume minimizers. We show that for certain sets of parameters, large volume minimizers converge to the ball, or may even exactly be the ball. Moreover, we develop a numerical method for this variational problem. Our results confirm numerically a conjecture of Choksi and Peletier, in dimension 2: it seems that minimizers of Gamow's liquid drop model are balls as long as they exist.

**Résumé.** Cette thèse comporte deux parties. Dans la première partie, nous étudions une généralisation du problème variationnel de torsion élastique-plastique à des variétés. Nous montrons que dans le cas des variétés, le problème n'est pas équivalent à un problème d'obstacle, contrairement au cas euclidien, mais nous établissons l'équivalence lorsque le paramètre du problème tend vers l'infini. Nous montrons, comme dans le cas euclidien, que l'ensemble de non contact contient le cut locus de la variété, et converge vers ce dernier au sens de Hausdorff. Nous montrons de plus que les minimiseurs du problème sont uniformément semiconcaves. Nous en déduisons une approximation stable de cut locus, dans l'esprit du lambda axe médian de Chazal et Lieutier. Nous utilisons ensuite ce résultat pour calculer numériquement le cut locus de surfaces de géométries variées.

Dans la seconde partie, nous étudions une extension d'un problème isopérimétrique non local. Précisément, on adjoint un potentiel de confinement au modèle de goutte liquide du noyau de Gamow. Nous étudions alors les minimiseurs de grand volume. Nous montrons que pour certains jeux de paramètres, les minimiseurs de grand volume convergent vers des boules, voire sont exactement des boules. Nous développons ensuite une méthode numérique pour ce problème variationnel. Cela permet de confirmer numériquement une conjecture de Choksi et Peletier en dimension 2 : dans ce cas les minimiseurs du modèle de Gamow semble être des boules si ils existent.

---

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# Introduction

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In this thesis, we present some contributions to the field of calculus of variations. Our results are both theoretical and numerical.

In Part I, we study an extension of the elastic-plastic torsion problem to manifolds and its relation with the cut locus, and derive a numerical method to compute the cut locus of a surface. Chapter 2 is based on the paper [41], to appear in the journal 'Nonlinearity'. The work in Chapter 3 and Chapter 4 was conducted in collaboration with Édouard Oudet and Bozhidar Velichkov, and are based on two papers to be submitted. For a synthetic view of this part, see the Outline of Part I.

In Part II, we study an isoperimetric problem that is a variation of Gamow's liquid drop model. It is based on the published paper [42]. The work in Chapter 2 was conducted in collaboration with Edouard Oudet. For a synthetic view of this part, please see the Outline of Part II.

We tried to make each Chapter relatively self-contained.

## a Obstacle problems

In part I, our focus will be on a variation of the elastic-plastic torsion problem, which is introduced in Section b. It is a particular example of an obstacle type variational problem. The most classical obstacle problem is the following: given a smooth open set  $\Omega \subset \mathbb{R}^n$  and some continuous functions  $\phi : \Omega \rightarrow \mathbb{R}$  (the obstacle) and  $g : \partial\Omega \rightarrow \mathbb{R}$  (the boundary condition), one wants to minimize the following:

$$\inf \left\{ \frac{1}{2} \int_{\Omega} |\nabla v|^2 : v \in H^1(\Omega), v \geq \phi \text{ and } v|_{\partial\Omega} = g \right\}. \quad (\text{a.1})$$

With reasonable assumptions on  $g$  and  $\phi$  (basically to guarantee that there is at least one candidate to the variational problem), one can show that there is a minimizer  $u \in H_0^1(\Omega)$  by the direct method of calculus of variations, and that this minimizer is unique by the convexity of the Dirichlet energy. The graph of  $u$  typically models the resting position of an elastic membrane above  $\Omega$ , attached at the height given by  $g$  on  $\partial\Omega$ , and constrained to stay above an obstacle given by the graph of  $\phi$ . There are many generalizations or variants of this famous problem, and many applications, such as fluid filtration in a porous media, elastic-plastic torsion, optimal control and financial mathematics, see for instance [71], [52], [37]. The literature on the subject is vast, and it is still a major area of research in calculus of variations



and PDEs. We will mention some generalizations later on. But let us first explain a few key properties on this simple model.

For any positive test function  $\psi \in C_c^\infty(\Omega)$ , writing the minimality of  $u$  against a competitor  $u + t\psi$  and taking the limit as  $t \downarrow 0$ , one gets, in the sense of distributions,

$$\Delta u \leq 0.$$

Moreover, assuming that  $u$  is continuous, this last inequality becomes an equality (the Euler-Lagrange equation of the Dirichlet energy) in the open set  $\{u > \phi\} \cap \Omega$ . The set  $\mathcal{N} := \{u > \phi\} \cap \Omega$ , is called the *non-contact set*, while the set  $\mathcal{C} := \{u = \phi\} \cap \Omega$  is called the *contact set*. The set  $\Gamma := \partial\mathcal{C} \cap \Omega$  is the interface between the contact and non-contact sets. It is called the *free boundary*.

Let us denote by  $C^{1,1}(\Omega)$  the space of  $C^1$  functions with Lipschitz gradients on  $\Omega$ . With the right regularity assumptions on  $\Omega$  and  $g$  (for instance, smooth), if the obstacle  $\phi$  is in  $C^{1,1}(\Omega)$ , then  $u$  is in  $C^{1,1}(\Omega)$  (see for instance [81]). This regularity is optimal. Indeed, in this case, we have almost everywhere in  $\Omega$ ,

$$\Delta u = \chi_{\mathcal{C}} \Delta \phi.$$

In particular, the Laplacian of  $u$  cannot be expected to be continuous, even if  $\phi$  is smooth.

Even though we will not be interested in the question of the regularity of the free boundary in this work, let us describe this important aspect briefly. The development of the regularity theory for free boundaries started in the 70's with the seminal paper of Caffarelli [16], and has been an active area of research since then. The free boundary  $\Gamma$  has finite  $(n - 1)$ -Hausdorff measure. It can be divided into a regular part and a singular part. The singular part is relatively open in  $\Gamma$ , and is locally a smooth manifold of dimension  $n - 1$ . The singular part is therefore closed, and is contained in a  $C^1$  manifold of dimension  $n - 1$ . The singular part consists in points at which the density of the contact set  $\mathcal{C}$  is null, see [17]. In a recent paper [32], Figalli, Ros-Oton and Serra proved that in dimension 2 and 3, the free boundary of the classical obstacle problem is 'generically regular', *i.e.* for a 'generic' obstacle  $\phi$ , the singular part of  $\Gamma$  is empty, and so  $\Gamma$  is a smooth  $(n - 1)$ -manifold. For the elastic-plastic torsion problem introduced below, the free boundary is always regular ([37]).

Let  $K := \{v \in H^1(\Omega) : v \geq \phi \text{ and } v|_{\partial\Omega} = g\}$ . Note that  $K$  is a convex subset of  $H^1(\Omega)$ . Let  $v \in K$ . By writing the minimality of  $u$  against the competitor  $u + t(v - u)$  and taking the limit as  $t \downarrow 0$ , one can see that the classical obstacle problem (a.1) can be rephrased as follows: find  $u \in K$  such that

$$\forall v \in K, \quad (Au - f, v - u) \leq 0,$$

where  $A = \Delta$ ,  $f = 0$  and  $(\cdot, \cdot)$  is the duality product on  $H^{-1}(\Omega) \times H_0^1(\Omega)$ . A problem of this form is usually referred to as a *variational inequality*. A direct generalization is to consider more general elliptic operators  $A$  and functions  $f$ . This includes the case of the elastic-plastic torsion problem, introduced in the next section. The optimal  $C^{1,1}$  regularity of  $u$  and the regularity of the free boundary has been studied by many authors, including for some non-linear and degenerate elliptic operator  $A$ , see for instance [34].

Let us end this section by mentioning two other famous obstacle type problems. In the thin obstacle problem, one minimizes the Dirichlet energy with Dirichlet boundary conditions as in

(a.1), but candidates are only constrained to stay above an obstacle  $\phi$  on a  $(n-1)$ -hypersurface  $S \subset \Omega$ , see [37, Chapter 1, Section 11].

In parabolic obstacle problems, a time dependency is added. This is the case for instance of the Stephan problem, which models the evolution of the temperature of some ice immersed in water, see [71]. A typical formulation would be

$$\begin{cases} u \geq 0 & \text{in } \Omega, \\ \partial_t - \Delta u = -1 & \text{in } \{u > 0\} \cap \Omega, \\ \partial_t - \Delta u \geq -1 & \text{in } \Omega. \end{cases}$$

## b Elastic-plastic torsion problem and medial axis

Let  $\Omega$  be a smooth bounded open set of  $\mathbb{R}^n$ . The *elastic-plastic torsion problem* is the following variational problem, defined for any  $m > 0$ .

$$\inf \left\{ \int_{\Omega} |\nabla v|^2 - mv : v \in H_0^1(\Omega), |\nabla v| \leq 1 \right\} \quad (\text{b.1})$$

One can show that this functional has a minimizer by the direct method of calculus of variations. The minimizer is unique because we are minimizing a strictly convex functional over a convex subset of  $H_0^1(\Omega)$ . Let  $v_m$  be the unique minimizer of (b.1). Physically speaking,  $v_m$  represents the stress function of a long bar of cross section  $\Omega$ , twisted with an angle  $m$ .

The elastic-plastic torsion problem and the properties of its minimizer  $v_m$  have been intensively studied in the 60s and 70s by various authors (such as Ting, Brézis, Sibony, Gerhardt, Caffarelli, Rivière, Friedman, Pozzi...), see for instance [78], [14], [13], [20], [79], [21], [38] and [19].

An important tool in the study of this problem is the equivalence with an obstacle problem. A consequence of the gradient constraint  $|\nabla v_m| \leq 1$ , is the following bound on  $v_m$ :

$$v_m \leq d_{\partial\Omega},$$

where  $d_{\partial\Omega}$  is the distance function to the boundary of  $\Omega$ :

$$d_{\partial\Omega}(x) = \min_{y \in \partial\Omega} |x - y|.$$

Indeed, as  $\Omega$  is smooth and  $v_m \in H_0^1(\Omega)$ , we may extend  $v_m$  by 0 outside  $\Omega$ , to get a function  $\widetilde{v}_m \in H^1(\mathbb{R}^n)$ , such that  $|\nabla \widetilde{v}_m| = \mathbb{1}_{\Omega} |\nabla v_m| \leq 1$ . So  $\widetilde{v}_m$  is 1-Lipschitz on  $\mathbb{R}^n$ , and so  $v_m$  is 1-Lipschitz on  $\Omega$  for the euclidean metric. In particular, for any  $x \in \Omega$  and  $y \in \partial\Omega$  such that  $d_{\partial\Omega}(x) = |x - y|$ , we have

$$v_m(x) \leq v_m(y) + |x - y| = |x - y| = d_{\partial\Omega}(x).$$

So  $v_m$  is also a competitor in the obstacle problem

$$\inf \left\{ \int_{\Omega} |\nabla v|^2 - mv : v \in H_0^1(\Omega), v \leq d_{\partial\Omega} \right\}. \quad (\text{b.2})$$

Note that the above problem also admits a unique minimizer, by the convexity of the functional and the constraint. Let us call it  $v_m^d$  ('d' is for the 'distance' obstacle). Brézis and Sibony showed in [14] that  $v_m$  is actually also the minimizer of this problem, *i.e.*  $v_m^d = v_m$ .

**Theorem b.1.** *Let us define*

$$\begin{aligned}\mathcal{F}(v) &:= \int_{\Omega} |\nabla v|^2 - mv, \\ K_1 &:= \{v \in H_0^1(\Omega) : |\nabla v| \leq 1\}, \\ K_2 &:= \{v \in H_0^1(\Omega) : v \leq d_{\partial\Omega}\}.\end{aligned}$$

*Then, the minimizer  $v_m$  of  $\mathcal{F}$  under the constraint  $v_m \in K_1$  is also a minimizer of  $\mathcal{F}$  under the constraint  $v_m \in K_2$ .*

To prove this, one only needs to prove that  $|\nabla v_m^d| \leq 1$ . Let us give a sketch of their proof. For  $\lambda > 0$  small, let  $v_\lambda$  be the unique minimizer of

$$\inf \left\{ \int_{\Omega} |\nabla v|^2 - mv - \lambda v^2 : v \in H_0^1(\Omega), v \leq d_{\partial\Omega} \right\}. \quad (\text{b.3})$$

The idea is to show that  $|\nabla v_\lambda| \leq 1$ , and take the limit as  $\lambda \rightarrow 0$  to conclude. Let  $\tilde{v} : \mathbb{R}^n \rightarrow \mathbb{R}$  be the extension of  $v_\lambda$  by 0 outside  $\Omega$ . For  $h \in \mathbb{R}^n$ , define

$$\begin{aligned}v_h^+(x) &= \max\{\tilde{v}(x-h) - |h|, \tilde{v}(x)\}, & E_h^+ &= \{x \in \mathbb{R}^n : \tilde{v}(x-h) - |h| > \tilde{v}(x)\}, \\ v_h^-(x) &= \max\{\tilde{v}(x+h) + |h|, \tilde{v}(x)\}, & E_h^- &= \{x \in \mathbb{R}^n : \tilde{v}(x+h) + |h| < \tilde{v}(x)\}.\end{aligned}$$

Using the fact  $v_h^+$  and  $v_h^-$  are also competitors in (b.3), one can write the minimality of  $v_\lambda$  in (b.3) against these competitors. After some computations, one gets

$$\lambda \int_{E_h^+} (\tilde{v}(x-h) - \tilde{v}(x) - |h|)^2 dx \leq - \int_{E_h^+} |h| (\tilde{v}(x-h) - \tilde{v}(x) - |h|) dx,$$

which implies that the measure of  $E_h^+$  is null, and conclude the proof. Generalizing these techniques, Treu, Mironescu, Mariconda and Sifdari generalized this result to a broader class of variational problems with convex constraints on the gradient, see [80], [62] and [72]. However, none of these will apply to our variant of the problem on manifolds, for which the equivalence of constraints fail in general (see Proposition 3.3.1).

The optimal regularity  $v_m \in C^{1,1}(\Omega)$  was first obtained in [21], using the equivalent formulation (b.2). The *elastic* and *plastic* sets are defined as follows:

$$\begin{aligned}E_m &:= \{x \in \Omega : |\nabla v_m(x)| < 1\}, \\ P_m &:= \{x \in \Omega : |\nabla v_m(x)| = 1\},\end{aligned}$$

It is then easy to prove that these are actually the non-contact and contact sets (see [18, §3]):

$$\begin{aligned}E_m &= \{x \in \Omega : v_m(x) < d_{\partial\Omega}(x)\}, \\ P_m &= \{x \in \Omega : v_m(x) = d_{\partial\Omega}(x)\}.\end{aligned}$$

Caffarelli and Rivière proved that the free boundary  $\partial E_m \cap \Omega$  has no singular point, and is therefore smooth, in [20].

The *medial axis* of  $\Omega$ , denoted by  $\mathcal{M}(\Omega)$ , is defined as the set of points of  $\Omega$  that have at least two closest points on the boundary  $\partial\Omega$  of  $\Omega$ :

$$\mathcal{M}(\Omega) := \{x \in \Omega : \exists y, z \in \partial\Omega, y \neq z \text{ and } d_{\partial\Omega}(x) = |x-y| = |x-z|\}.$$

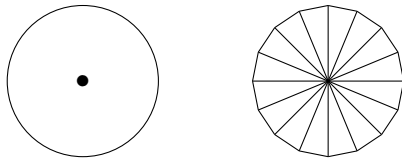


Figure 1: A circle with its medial axis, and a polygonal approximation of a circle, with its medial axis.

Equivalently,  $\mathcal{M}(\Omega)$  is the set of points of  $\Omega$  at which the distance function  $d_{\partial\Omega}$  is not differentiable. In [18], Caffarelli and Friedman used the notion of *ridge* of  $\Omega$  introduced by Ting, which is precisely the closure of  $\mathcal{M}(\Omega)$ . Equivalently, it is the complement of the largest open set where the distance function is smooth.

They proved the following.

**Theorem b.2** (Theorems 3.2 and 4.1 in [18]). *For any  $m > 0$ , we have*

$$\overline{\mathcal{M}(\Omega)} \subset E_m.$$

*Furthermore, there exists a constant  $C > 0$  depending on  $\Omega$  only such that*

$$\forall m > 0, \quad d^{\mathcal{H}}(\mathcal{M}(\Omega), E_m) \leq \frac{C}{m},$$

*where  $d^{\mathcal{H}}$  denotes the Hausdorff distance.*

**The  $\lambda$ -medial axis.** The medial axis, also called *skeleton*, is used in a wide variety of applications in image processing and computer vision. Many researchers have proposed a method to compute it, and are still trying to improve its computation. We refer to the book [73] from 2017 for a review about the medial axis, its computation and applications.

One important geometric property of the medial axis  $\mathcal{M}(\Omega)$  is that it is unstable with respect to small perturbations of the boundary of  $\Omega$  - see Figure 1. This instability makes computing numerically  $\mathcal{M}(\Omega)$  quite tricky. Indeed, any numerical approximation of  $\Omega$  (for instance, with polygons) might introduce an artificial (and large) medial set. In order to give a theoretical framework to tackle the problem of stability, Chazal and Lieutier defined in [23] the so called  $\lambda$ -medial axis of  $\Omega$ , by setting for any  $\lambda > 0$ ,

$$\mathcal{M}_\lambda(\Omega) := \{x \in \Omega : r(x) \geq \lambda\},$$

where  $r(x)$  is the radius of the smallest ball containing the set of all closest points to  $x$  on  $\partial\Omega$ , *i.e.* the set  $\{z \in \partial\Omega : |x - z| = d_{\partial\Omega}(x)\}$ . Note that the map  $\lambda \mapsto \mathcal{M}_\lambda(\Omega)$  is non-increasing, and we have

$$\mathcal{M}(\Omega) = \bigcup_{\lambda > 0} \mathcal{M}_\lambda(\Omega).$$

It is further proved in [23, Section 3, Theorem 2] that  $\mathcal{M}_\lambda(\Omega)$  has the same homotopy type as  $\mathcal{M}(\Omega)$ , for  $\lambda$  small enough. These facts justify that, for fixed  $\Omega$ ,  $\mathcal{M}_\lambda(\Omega)$  is a good approximation of  $\mathcal{M}(\Omega)$ , for  $\lambda$  small enough. The crucial difference though is that  $\mathcal{M}_\lambda(\Omega)$  is stable with respect to small variations of  $\Omega$ , whereas  $\mathcal{M}(\Omega)$  is not. Indeed we have the following theorem.

**Theorem b.3** (Consequence of Theorem 3 in [23]). *For any  $0 < \lambda' < \lambda$ , there exists  $\varepsilon > 0$  such that, if  $\Omega$  and  $\tilde{\Omega}$  are two open bounded subset of  $\mathbb{R}^n$  such that  $d^H(\Omega^c, \tilde{\Omega}^c) < \varepsilon$ , then*

$$\sup_{x \in \mathcal{M}_\lambda(\tilde{\Omega})} d(x, \mathcal{M}_{\lambda'}(\Omega)) \leq 2\sqrt{\text{diam}(\Omega)\varepsilon},$$

where  $\text{diam}(\Omega)$  is the diameter of  $\Omega$ .

This means that although it is difficult to compute numerically  $\mathcal{M}(\Omega)$  directly, one can compute  $\mathcal{M}_\lambda(\Omega)$  instead. In the same paper [23], the authors use Voronoi diagrams to compute an approximation of  $\mathcal{M}_\lambda(\Omega)$  from a noisy sample of points of the boundary of  $\Omega$ .

We will need the following definition.

**Definition b.4.** *A function  $u : \Omega \rightarrow \mathbb{R}$  is said to be  $C$ -semiconcave (or just semiconcave) if the function  $x \mapsto C|x|^2 - u(x)$  is convex.*

As  $\Omega$  is smooth, we know that  $d_{\partial\Omega}$  is semiconcave (see [22, (iii) in Proposition 2.2.2]). As explained with more details in the Appendix A.1, this implies that, as concave functions,  $d_{\partial\Omega}$  has a well defined generalized gradient whose norm is given by the formula:

$$|\nabla d_{\partial\Omega}|(x) = \max(0, \sup_{v \in S^{n-1}} \partial_v^+ d_{\partial\Omega}(x)),$$

where  $\partial_v^+ d_{\partial\Omega}(x)$  denotes the directional derivative of  $d_{\partial\Omega}$  in the direction  $v$ . What is more, as it can be seen from [23, Section 2.1], we have the following formula for the  $\lambda$ -medial axis.

$$\mathcal{M}_\lambda(\Omega) = \left\{ x \in \Omega : |\nabla d_{\partial\Omega}(x)|^2 \leq 1 - \frac{\lambda^2}{d_{\partial\Omega}^2(x)} \right\}.$$

Motivated by this formula, we will prove the following.

**Theorem b.5** (Theorem 1.3.1). *Let  $\Omega$  be a smooth bounded open subset of  $\mathbb{R}^n$ . For  $m > 0$ , let  $v_m$  be the (unique) solution of the variational problem:*

$$\inf \left\{ \int_{\Omega} |\nabla v|^2 - mv : v \in H_0^1(\Omega), |\nabla v| \leq 1 \right\}.$$

Let us define the set  $E_{m,\lambda}$  by:

$$E_{m,\lambda} := \left\{ x \in \Omega : |\nabla v_m(x)|^2 \leq 1 - \frac{\lambda^2}{v_m^2(x)} \right\}.$$

Then, for any  $\lambda > 0$  and  $\varepsilon > 0$ , we have

$$\sup_{x \in E_{m,\lambda}} d(x, \mathcal{M}_\lambda(\Omega)) \xrightarrow{m \rightarrow +\infty} 0 \quad \text{and} \quad \sup_{x \in \mathcal{M}_{\lambda+\varepsilon}(\Omega)} d(x, E_{m,\lambda}) \xrightarrow{m \rightarrow +\infty} 0.$$

This theorem basically shows that the set  $E_{m,\lambda}$  is a good approximation of the  $\lambda$ -medial axis. More importantly, in Chapter 3 we will also use this strategy to approximate the cut locus of a manifold, which is introduced below.

The most important step in the proof of Theorem b.5, is to show that functions  $(v_m)_{m>0}$  are uniformly semiconcave.

**Theorem b.6** (Theorem 1.3.2). *Let  $C_d > 0$  be such that  $d_{\partial\Omega}$  is  $C_d$ -semiconcave. There exists  $m_0 > 0$  such that for any  $m > m_0$ , the solution  $v_m$  of the elasto-plastic torsion problem (b.1) is  $C_d$ -semiconcave.*

The proof of Theorem 1.3.2 relies on the two following facts:

- inside  $E_m$ ,  $v_m$  has constant Laplacian,
- outside  $E_m$ ,  $v_m$  is equal to  $d_{\partial\Omega}$ , which is  $C_d$ -semiconcave, for some constant  $C_d > 0$  that depends only on  $\Omega$ .

The main tool we use is Korevaar's convexity maximum principle [54, Theorem 1.3]. Let us present this result. Let  $u : \Omega \rightarrow \mathbb{R}$  be a  $C^2$  function satisfying an equation of the form

$$P(x, u(x), \nabla u(x), D^2u(x)) = 0$$

in  $\Omega$ , where  $P$  is a certain elliptic operator, and  $D^2u$  is the Hessian of  $u$ . The convexity function of  $u$  is the function  $c$  defined for any  $[x, y] \subset \bar{\Omega}$  and  $\lambda \in [0, 1]$  by

$$c(x, y, \lambda) := (1 - \lambda)u(x) + \lambda u(y) - u((1 - \lambda)x + \lambda y).$$

Under suitable conditions on the elliptic operator  $P$ , Korevaar's convexity maximum principle states that the minimum of  $c$  is attained at a triple  $(x, y, \lambda)$  such that one of the points  $x$ ,  $y$  or  $(1 - \lambda)x + \lambda y$  belongs to the boundary of  $\Omega$ . In particular, this result applies to the case where  $P$  is the euclidean Laplacian.

## c Distance function and cut locus

Let  $M$  be a smooth Riemannian manifold without boundary of dimension  $n \geq 2$ ,  $b$  a point of  $M$ ,  $d$  the distance function of  $M$  and  $d_b := d(\cdot, b)$  the distance function to the point  $b$ . We will now introduce the notion of *cut locus*, which is closely related to the function  $d_b$ . Let us first recall some notions of Riemannian geometry.

**Definition c.1.** *Let  $\gamma : [a, b] \rightarrow M$  be a geodesic, and  $J : [a, b] \rightarrow TM$  a vector field along  $\gamma$ . Then,  $J$  is called a Jacobi field if there exists a variation of  $\gamma$  through geodesics whose  $J$  is the variation field, i.e. a smooth map  $\Gamma : (-1, 1) \times [a, b] \rightarrow M$  such that  $\Gamma(0, \cdot) = \gamma$  and for any  $s \in (-1, 1)$ ,  $\Gamma(s, \cdot)$  is a geodesic, and*

$$\partial_1 \Gamma(0, t) = J(t).$$

*Furthermore, if there exists a Jacobi field  $J$  along  $\gamma$  such that  $J(a) = 0$  and  $J(b) = 0$ , then we say that  $\gamma(a)$  and  $\gamma(b)$  are conjugate points along  $\gamma$ .*

**Definition c.2.** *Let  $T > 0$  and  $\gamma : [0, T] \rightarrow M$  be a unit speed geodesic such that  $\gamma(0) = b$ ,  $t_0 \in (0, T)$  and  $p = \gamma(t_0)$ . We say that  $p$  is a cut point of  $b$  along  $\gamma$  if  $\gamma$  is length minimizing between  $b$  and  $p$ , but not after  $p$ , i.e.  $d_b(\gamma(t)) = t$  for  $t \leq t_0$ , and  $d_b(\gamma(t)) < t$  for  $t > t_0$ . A point  $q \in M$  is said to be a cut point of  $b$  if there exists a geodesic  $\gamma$  such that  $q$  is a cut point of  $b$  along  $\gamma$ .*

**Definition c.3.** *The cut locus of  $b$  in  $M$ , denoted by  $Cut_b(M)$ , is defined as the set of all cut points of  $b$ .*

We recall the following.

**Definition c.4.** *Let  $q \in M$  and  $V \in T_qM$  be such that there exists a well defined geodesic  $\gamma : [0, 1] \rightarrow M$  such that  $\gamma(0) = q$  and  $\dot{\gamma}(0) = V$ . Then, the (global) exponential map at  $V$  is defined by  $\text{Exp}(V) = \gamma(1)$ . The restriction of the exponential map  $\text{Exp}$  on  $T_qM$  is denoted by  $\text{exp}_q$  and called the exponential map at  $q$ .*

The following facts can all be found in [74, Chapter III, Section 4]:

**Proposition c.5.** *Some well known facts related to the cut locus include:*

- *$Cut_b(M)$  is the set of points  $p$  in  $M$ , such that either there exist at least two minimizing geodesics from  $b$  to  $p$ , or the points  $b$  and  $p$  are conjugate,*
- *$Cut_b(M)$  is the closure of the set of points  $p$  in  $M$ , such that there exist at least two minimizing geodesics from  $b$  to  $p$ ,*
- *the function  $d_b$  is smooth outside  $Cut_b(M) \cup \{b\}$ , and it is differentiable at a point  $p$  if and only if there exists a unique minimizing geodesics between  $b$  and  $p$ ,*
- *in particular,  $Cut_b(M) \cup \{b\}$  is the closure of the set of points of non-differentiability of  $d_b$ ,*
- *the exponential map  $\text{exp}_b$  is a diffeomorphism from an open set of  $T_bM$  onto  $M \setminus Cut_b(M)$ ,*
- *$Cut_b(M)$  is a deformation retract of  $M \setminus \{b\}$ . In particular, these two sets have the same homotopy type, and so  $Cut_b(M)$  inherits much of the topology of  $M$  (like homology groups, for instance). See [74, Chapter III, Section 4, proposition 4.5] for a precise statement.*

We know quite a bit about the structure of the cut locus. Let us give some of its properties, to get an idea of what the cut locus looks like. First, it is the union of smooth hypersurfaces consisting of points with exactly two minimizing geodesics, plus a set of Hausdorff dimension at most  $n - 2$ . A reader-friendly reference for this is [49]. Furthermore, the cut locus has locally finite  $n - 1$ -dimensional Hausdorff measure, see for instance [50, Theorem B]. To our knowledge, the most precise results in this direction are contained in [9], where points of the cut locus are classified, up to a set of Hausdorff dimension at most  $n - 3$ . We refer to [8, Section 1.3] for additional structural properties of the cut locus.

An important property of the distance function is *semiconcavity*. Let us give a definition of semiconcavity on manifolds.

**Definition c.6.** *Given a constant  $C > 0$ , a function  $u$  is said to be  $C$ -semiconcave on  $M$  if and only if for any unit speed geodesic  $\gamma : [a, b] \rightarrow M$ , the function  $t \mapsto Ct^2 - u(\gamma(t))$  is convex. We say that  $u$  is semiconcave if  $u$  is  $C$ -semiconcave for some constant  $C > 0$ . We say that  $u$  is locally semiconcave if for any  $p \in M$ ,  $u$  is semiconcave in a neighborhood of  $p$ .*

*Remark c.7.* This definition is compatible with the previous definition of semiconcavity for functions on  $\mathbb{R}^n$ .

In [60, Proposition 3.4], C.Mantegazza and A.Mennucci showed that the distance function is a viscosity solution of the Hamilton-Jacobi equation  $|\nabla d_b| = 1$ . Using some regularity result ([25]) for Hamilton-Jacobi equations, they deduced that for any local chart  $\psi$  of  $M$ , the map  $d_b \circ \psi^{-1}$  is locally semiconcave as a function on  $\mathbb{R}^n$  (see also [22] for a more reader-friendly reference about semiconcave functions and Hamilton-Jacobi equations). In the Appendix A.1 about semiconcave functions, we show that it is equivalent to saying that  $d_b$  is locally semiconcave. A different proof of the local semiconcavity of distance functions can also be found in [5]. In [61] (and in particular Theorem 2.10), C.Mantegazza, G.Mascellani and G.Uraltsev used this fact along with geometric measure theory tools to give a complete description of the distributional Hessian and Laplacian of  $d_b$ . They made use of the fact that the gradient of semiconcave functions has bounded variations, in particular. They proved the following.

**Theorem c.8.** *The distributional Hessian of  $d_b$  is given by:*

$$D^2 d_b = \widetilde{D^2 d_b} \text{Vol} - (\nu \otimes \nu) |\nabla d_b^+ - \nabla d_b^-| \mathcal{H}_{\text{Cut}_b(M)}^{n-1},$$

where  $\widetilde{D^2 d_b}$  is the classical Hessian of  $d_b$  wherever  $d_b$  is smooth (i.e. almost everywhere),  $\text{Vol}$  is the Riemannian volume measure,  $\nu$  is a unit normal vector to  $\text{Cut}_b(M)$  wherever  $\text{Cut}_b(M)$  is smooth (i.e.  $\mathcal{H}^{n-1}$ -almost everywhere),  $\nabla d_b^+$  and  $\nabla d_b^-$  are limits of  $\nabla d_b$  to either side of  $\text{Cut}_b(M)$ , and  $\mathcal{H}_{\text{Cut}_b(M)}^{n-1}$  is the  $n-1$ -dimensional Hausdorff measure restricted to  $\text{Cut}_b(M)$ . The distributional Laplacian of  $d_b$  is given by:

$$\Delta d_b = \widetilde{\Delta d_b} \text{Vol} - |\nabla d_b^+ - \nabla d_b^-| \mathcal{H}_{\text{Cut}_b(M)}^{n-1},$$

where  $\widetilde{\Delta d_b}$  is the classical Laplacian of  $d_b$ , wherever  $d_b$  is smooth.

Our main result in Chapter 2 is a complement to the description of the non-smooth behavior of the distance function on the cut locus. It deals with the Laplacian of  $d_b$  in the sense of barriers.

**Definition c.9.** *Let  $\psi : M \rightarrow \mathbb{R}$  be a continuous function, and  $p \in M$ . We say that the Laplacian of  $\psi$  at  $p$  is  $-\infty$  in the sense of barriers if for any  $A > 0$ , there exists a smooth function  $\phi : M \rightarrow \mathbb{R}$  defined on a neighborhood of  $p$  such that*

$$\phi \geq \psi, \quad \phi(p) = \psi(p) \quad \text{and} \quad \Delta \phi(p) \leq -A.$$

We will prove the following.

**Theorem c.10** (Theorem 2.1.11). *Let  $M$  be a smooth manifold without boundary of dimension  $n \geq 2$ , and  $b \in M$ . Let  $d_b$  be the distance function to the point  $b$ , and  $\text{Cut}_b(M)$  the cut locus with respect to  $b$ . The Laplacian of  $d_b$  is  $-\infty$  at every point of  $\text{Cut}_b(M)$ , in the sense of barriers.*

This result proves useful when one wants to apply the maximum principle to a function involving the distance function  $d_b$ . It will be applied to the study of a variation of the elastic-plastic torsion problem on manifolds, that we will now introduce.



## d A variation of the elastic-plastic torsion problem on manifolds

Let us now assume that the manifold  $M$  is compact. The cut locus is a fundamental object in Riemannian geometry, and it is a natural problem to try and find ways to compute it numerically. While the medial axis of  $\Omega$  is unstable with respect to non-smooth variation of the boundary of  $\Omega$ , the cut locus is unstable with respect to non-smooth variations of the metric of  $M$ . Indeed, it is proved in [4] that the cut locus is unstable for some  $C^1$  variations of the metric of  $M$  (but stable for  $C^2$  variations of the metric of  $M$ ). In particular, the cut locus of a surface  $S$  cannot simply be approximated with the cut locus of a triangulation of  $S$ . Our strategy will be to extend the results presented above linking the  $\lambda$ -medial axis and the elastic-plastic torsion problem to the case of the cut locus, and to derive a numerical approximation of the cut locus of triangulated surfaces. We will study the following variational problem in Chapter 3:

$$\inf \left\{ \int_M |\nabla u|^2 - mu : u \in H^1(M), |\nabla u| \leq 1, u(b) = 0 \right\}, \quad (\text{d.1})$$

where  $m > 0$  is a given constant. We will extend the results for the elastic-plastic torsion problem presented earlier in this introduction.

*Remark d.1.* We could have replaced the point  $b$  with a smooth hypersurface  $S \subset M$  in the above problem. However, our interest here is in the instability of the cut locus with respect to the metric of  $M$ , and this aspect is already present with a point  $b$  instead of  $S$ . So for simplicity, we focus on the case of the cut locus of a point.

Note that by convexity of the functional, this problem has a unique minimizer  $u_m$ . One may ask if an analog of Theorem b.1 is also true for  $u_m$ , with  $d_{\partial\Omega}$  replaced with  $d_b$  and  $\Omega$  with  $M$ . Interestingly, it is not.

**Theorem d.2** (Theorem 3.3.1). *There exists a surface of revolution  $M$  and a parameter  $m > 0$  such that, if  $u_m^d$  denotes the unique minimizer of the following problem:*

$$\inf \left\{ \int_M |\nabla u|^2 - mu : u \in H^1(M), u \leq d_b \right\},$$

*then we have  $u_m \neq u_m^d$ .*

However, we prove that the equality holds if  $m$  is large enough, depending on the Ricci curvature of  $M$ .

**Theorem d.3** (Theorem 3.3.2). *Let  $K \geq 0$  be such that*

$$\text{Ric} \geq -K,$$

*where Ric denotes the Ricci curvature tensor of the manifold  $M$ . If*

$$m \geq \max(\sqrt{nK(1 + K \text{diam}(M)^2)}, nK \text{diam}(M)),$$

*then  $|\nabla u_m^d| \leq 1$ , or equivalently*

$$u_m^d = u_m.$$

The technique of the proof is different from the one in [14], which was specific to the flat case. Our proof relies on the maximum principle, applied to a function involving  $|\nabla u_m^d|^2$ , and the Euler-Lagrange equation  $\Delta u_m^d = -2m$ .

For the elastic-plastic torsion problem, as mentioned earlier, we know that the elastic set contains the medial axis, and the former converges to the latter in the Hausdorff sense. For problem (d.1), we have the analogous result:

**Theorem d.4** (Theorem 3.1.2). *We have  $Cut_b(M) \subset \{|\nabla u_m| < 1\}$ , and the following convergence holds in the Hausdorff sense:*

$$\{|\nabla u_m| < 1\} \xrightarrow{m \rightarrow +\infty} Cut_b(M).$$

Once we have the inclusion, the proof of the convergence is simple. It is based on the maximum principle, applied to the right function. Note to this end that  $u_m$  verifies on the whole manifold  $M$ ,  $\Delta u_m \geq -2m$ , in the sense of distributions. To prove the inclusion, we will use the maximum principle and the Theorem c.10 about the Laplacian of the distance function on the cut locus.

We will now state our analogous result to Theorem b.5 for problem (d.1). This result will be used in Chapter 4 to compute a stable approximation of the cut locus. We recall that the function  $d_b$  is locally semiconcave on  $M \setminus \{b\}$ . This implies that, as convex functions,  $d_b$  has a well defined generalized gradient at every point – see Appendix A.1, Proposition A.1.6. At a point  $p$  where  $d_b$  is not differentiable, the norm of its gradient is given by

$$|\nabla d_b|(p) := \max(0, \sup_{v \in T_x M, |v|=1} \partial_v^+ d_b(p)).$$

As shown in Appendix A.1, Proposition A.1.10, if there exist two minimizing geodesics from  $p$  to  $b$ , then  $|\nabla d_b(p)| < 1$ . In particular, we have

$$Cut_b(M) = \overline{\{|\nabla d_b| < 1\}}.$$

Following the analogous formula for the  $\lambda$ -medial axis, we define a  $\lambda$ -cut locus by

$$Cut_b(M)_\lambda := \left\{ x \in M : |\nabla d_b(x)|^2 \leq 1 - \frac{\lambda^2}{d_b^2(x)} \right\}, \quad \text{for } \lambda > 0.$$

It is clear that the map  $\lambda \mapsto Cut_b(M)_\lambda$  is non-increasing, and we have

$$Cut_b(M) = \bigcup_{\lambda > 0} \overline{Cut_b(M)_\lambda}.$$

Recall that  $u_m$  denotes the minimizer of (d.1). We set

$$E_m := \left\{ x \in M : |\nabla u_m(x)| < 1 \right\},$$

and  $E_{m,\lambda} := \left\{ x \in M : |\nabla u_m(x)|^2 \leq 1 - \frac{\lambda^2}{u_m^2(x)} \right\}.$

We will prove the following.

**Theorem d.5** (Theorem 3.1.2). *There exists  $m_0 > 0$ , depending only on the manifold  $M$ , such that for any  $\varepsilon > 0$ , there exists a constant  $C > 0$  such that, for any  $m > m_0$ ,*

$$u_m \text{ is } C\text{-semiconcave on } M \setminus B(b, \varepsilon).$$

What is more,

$$\sup_{x \in E_{m,\lambda}} d(x, \text{Cut}_b(M)_\lambda) \xrightarrow{m \rightarrow +\infty} 0, \quad \text{and} \quad \sup_{x \in \text{Cut}_b(M)_{\lambda+\varepsilon}} d(x, E_{m,\lambda}) \xrightarrow{m \rightarrow +\infty} 0.$$

Once the uniform semiconcavity is proved, the rest of the theorem is strictly similar to the corresponding one for the classical elastic-plastic torsion problem. The proof of semiconcavity, however, is different. Indeed, in this case, we cannot apply Korevaar's maximum principle to prove the semiconcavity in  $E_m$ . It can be seen that proving the semiconcavity is equivalent to finding a constant  $C$  such that for any  $p \in E_m$  and  $X \in T_p M$  of norm 1, we have  $D^2 u_m(X, X) \leq C$ . Our strategy will be to assume that there exists  $p \in E_m$  and  $X \in T_p M$  that maximizes the quantity  $D^2 u_m(X, X)$ . Then, we extend  $X$  by a well chosen vector field  $\tilde{X}$ , and find a suitable subharmonic function involving  $D^2 u_m(\tilde{X}, \tilde{X})$ . This technique is inspired from the proof of second order estimate for hessian equations in Riemannian manifolds by Guan in [45].

**Numerical approximation.** Now let us consider the case  $M = S$ , a compact real analytic surface without boundary embedded in  $\mathbb{R}^3$ . In Chapter 4, we will use the above results to compute a numerically stable approximation of the cut locus of  $S$ . A few authors have done it before, with different techniques. These techniques may be classified into two categories.

*Geodesic approximation on parametrized surfaces.* This approach was used in [76] and [63]. In [76], on genus 1 parametrized surfaces, the authors computed a degree 4 polynomial approximation of the exponential map using the geodesic equation, and deduced an approximation of the cut locus from there. In [63], the authors used the deformable simplicial complexes (DSC) method and finite differences techniques for geodesic computations, to compute geodesic circles of increasing radius and their self-intersection, *i.e.* the cut locus. They apply the method to genus 1 surfaces. These papers contain no proof of convergence of the computed cut locus.

*Exact geodesic computation on discretized surfaces.* This approach was used in [48] and [28]. In [48], the authors computed the geodesics on a convex triangulated surface. They deduced an approximation of the cut locus of the triangulated surface, and filtered it according to the angle formed by the geodesics meeting at a point of the approximated cut locus, to make their approximation stable. They applied the method to ellipsoids. There is no proof of convergence. In [28], the authors computed shortest curves on a graph obtained from a sufficiently dense sample of points of a surface. From there they deduced an approximation of the cut locus, and filtered it according to the maximal distance (called *spread*) between the geodesics meeting at a point of the approximated cut locus. They proved that the set they compute converges to the cut locus (see [28, Theorem 4.1]).

We also mention [11], where the authors use some more geometric tools to compute (numerically) the cut locus of an ellipsoid, or a sphere with some particular metric with singularities.

The strategy we use is quite different. By the above results, for  $\lambda > 0$  to be 'chosen small', and  $m$  'large enough', the set  $E_{m,\lambda} := \left\{ |\nabla u_m|^2 \leq 1 - \frac{\lambda^2}{u_m^2} \right\}$  can be used as an approximation of  $\text{Cut}_b(S)_\lambda$ . In turn, we can see that  $\text{Cut}_b(S)_\lambda$  is a good approximation of  $\text{Cut}_b(S)$ . First,

we have the following convergence in the Hausdorff sense:

$$Cut_b(S)_\lambda \xrightarrow{\lambda \rightarrow 0} Cut_b(S).$$

Furthermore, we will show (see Proposition 4.2.2) that for  $\lambda$  small enough,  $Cut_b(S)_\lambda$  has a connected component that is homotopy equivalent to  $Cut_b(S)$ . On a triangulation  $S_h$  of the surface  $S$ , the minimizer  $u_m$  can be approximated with a function  $u_h$  by the finite element methods. We will prove the following convergence in measure (see Proposition 4.4.4):

$$\left\{ |\nabla u_h|^2 \leq 1 - \frac{\lambda^2}{u_h^2} \right\} \xrightarrow{h \rightarrow 0} E_{m,\lambda}.$$

The efficiency of the method is illustrated on complex geometries in Section 4.5.

## e A non-local isoperimetric problem

In part II, our focus will be on a non-local isoperimetric problem.

**The classical isoperimetric problem.** Let  $n \in \mathbb{N}^*$  be an integer. For any measurable set  $E \subset \mathbb{R}^n$ , the perimeter of  $E$  is defined as the total variation of the indicator function of  $E$ :

$$P(E) := \int_{\mathbb{R}^n} |D\chi_E| = \sup \left\{ \int_E \operatorname{div} f : f \in C_c^1(\mathbb{R}^n), \quad |f|_{L^\infty} \leq 1 \right\}.$$

The quantity  $P(E)$  may be infinite, and when it is finite, we say that  $E$  is a set of finite perimeter. If  $E$  is a  $C^1$  open set, then as one would expect, we have  $P(E) = \mathcal{H}^{n-1}(\partial E)$ , where  $\mathcal{H}^{n-1}$  is the  $(n-1)$ -dimensional Hausdorff measure. This definition is due to De Giorgi. It allowed him, using Steiner symmetrization, to give a complete proof of the isoperimetric inequality. The classical isoperimetric problem is the following variational problem:

$$\min_{E \subset \mathbb{R}^n, |E|=m} P(E),$$

where  $|E|$  is the volume of  $E$ , and  $m > 0$  is a constant. It is now well known that for any  $m > 0$ , this problem has a unique minimizer up to translation, namely  $B[m]$ , the ball of volume  $m$ . Moreover, a sharp quantitative estimate was proved in [40] by Fusco, Maggi and Pratelli. Before stating it, we need a definition. Given a set  $E$  of volume  $m$ , we call *Fraenkel asymmetry* the quantity

$$\lambda(E) := \min_{x \in \mathbb{R}^n} \frac{|E\Delta(x + B[m])|}{|E|},$$

where  $E\Delta F = (E \cup F) \setminus (E \cap F)$  is the symmetric difference of  $E$  and  $F$ . There exists a constant  $C(n)$  depending only on the dimension of the ambient space such that for any set  $E$  of volume  $m$ ,

$$P(E) - P(B[m]) \geq C(n)P(B[m])\lambda(E)^2.$$

See also [39] for a refinement of this inequality, where the difference  $P(E) - P(B[m])$  also controls the oscillation of the normal to the (reduced) boundary of  $E$ . The quantitative isoperimetric inequality plays an instrumental role in the study of minimizers of small masses

of Gamow's liquid drop model. Our book of reference for the perimeter is Giusti's book, [43]. Maggi's book [59] is probably more reader-friendly though (but does not include the study of  $BV$  functions).

**Gamow's liquid drop problem.** Gamow's liquid drop model for the atomic nucleus consists in:

$$\inf_{E \subset \mathbb{R}^3, |E|=m} P(E) + \int_{E \times E} \frac{dx dy}{|x - y|},$$

where

- $P(E)$  is the perimeter of  $E$ ,
- $m$ , called the *mass*, is a positive constant.

In this model, the nucleus is thought of as an incompressible charged liquid. The perimeter term models the short-range attractive nuclear force, whereas the other term, similar to an electrostatic energy, models the repulsive forces between the protons of the nucleus. Gamow invented this model in the 1930's, in an attempt to predict the shape of nuclei, the nonexistence of large nuclei, and the existence of a nucleus with the least energy per nucleon. These three goals have only been attained relatively recently (in the 2010's, see below).

**Some variations of the liquid drop model.** The most studied variant of Gamow's model is the following:

$$\inf_{E \subset \mathbb{R}^2, |E|=m} P(E) + \mathcal{V}_\alpha(E) \tag{e.1}$$

where

- $P(E)$  is the perimeter of  $E$ ,
- $\mathcal{V}_\alpha(E) := \int_{E \times E} \frac{dx dy}{|x - y|^{n - \alpha}}$  is the Riesz energy of  $E$ ,
- $n \geq 2$  (the dimension of the ambient space),  $m > 0$  and  $\alpha \in (0, n)$  are constants.

The physical case corresponds to  $n = 3$  and  $\alpha = 2$ . In this case, the Riesz energy boils down to the electrostatic energy  $\int_{E \times E} \frac{dx dy}{|x - y|}$ .

It was shown in [33] by Figalli, Fusco, Maggi, Millot, and Morini, that if the mass  $m$  is small enough, then the problem (e.1) admits a unique minimizer (up to translation), namely the ball of volume  $m$ . See also the papers of Knüpfer, Muratov, Julin, Bonacini and Cristoferi [53], [64], [51] and [10], for anterior and slightly less general results.

On the other hand, for  $\alpha \in (n - 2, n)$ , it was shown in [64] that for  $m$  large enough there is no minimizer of problem (e.1). This result was simultaneously proved in [58] by Lu and Otto in the physical case. See also [35] for a short proof with a quantitative bound, by Frank, Killip and Nam.

In [36], Frank showed in the physical case that there exists a set which minimizes the quantity  $(P(E) + \mathcal{V}_\alpha(E))/|E|$ , thus proving that in Gamow's model, there is a nucleus with the least energy per nucleon.

In [33], the authors studied a generalization of problem (e.1), where the perimeter term is replaced with a non-local perimeter.

Another interesting variation consists in replacing the repulsive Riesz energy with a repulsive energy involving another kernel. Let  $K \in L^1(\mathbb{R}^n)$ . Problem (e.1) can be rephrased as

$$\inf_{E \subset \mathbb{R}^n, |E|=m} P(E) + \int_{E \times E} K(x-y) dx dy,$$

with  $K(x) = |x|^{\alpha-n}$ . In [70], Rigot considered the above problem with  $K$  an  $L^1$  compactly supported function, and proved that it has a minimizer for any mass  $m$ . The following papers are posterior to the present work. In [67], Pegon proved the same result in the case of radially symmetric non-increasing kernels  $K$  with some regularity and integrability conditions at infinity. He also showed that, under the same conditions, after rescaling, the problem  $\Gamma$ -converges to the classical isoperimetric problem as the mass  $m$  goes to  $+\infty$ . In [65], Muratov and Simon considered the kernel  $|x|^{-3}$  in dimension 2, and also studied the minimizers of large mass.

In [3], Alama, Bronsard, Choksi, and Topaloglu studied problem (e.1) with the addition of a confining background potential of the form  $\frac{Z}{|x|^\beta}$ . They proved that it admits a minimizer for any mass  $m$ , and study the minimizers as  $Z \rightarrow 0$ .

In [44], Goldman, Novaga and Roger studied the small mass minimizers of problem (e.1) with the addition of the Willmore energy.

As shown in [24], Gamow's liquid drop model is also related to diblock copolymers, and the Ohta–Kawasaki functional. This is also an active field of research, see for instance [2] and [82].

**Our problem.** To restore the existence of a minimizer for large masses in (e.1), we will add the energy associated to the potential  $A|x|^\beta$  to our functional, as we expect it to counter the spreading effect of the  $\mathcal{V}_\alpha$  term. Thus we will be interested in the following modification of the original problem (e.1):

$$\inf_{E \subset \mathbb{R}^n, |E|=m} \mathcal{E}_{\alpha,\beta,A}(E) := P(E) + \mathcal{V}_\alpha(E) + \mathcal{U}_{\beta,A}(E), \quad (\text{e.2})$$

where

- $\mathcal{U}_{\beta,A}(E) := \int_E A|x|^\beta dx$ ,
- $A \geq 0$  and  $\beta > 0$  are constants.

Indeed, an elementary proof will show that this extra confinement potential (no pun intended) restores the compactness of a minimizing sequence for any mass  $m$ . Therefore, this problem always has a minimizer. We will then be interested in the behavior of these minimizers when the mass  $m$  goes to  $+\infty$ , with particular emphasis in the case  $\beta > \alpha$ . We will prove the following.

**Theorem e.1** (Theorem 1.1.1). *Given  $\alpha \in (0, n)$ ,  $\beta > 0$  and  $A > 0$ , assume  $\alpha < \beta$ . Let  $(E_m)_{m>0}$  be a family of minimizers in (1.1.3), such that  $|E_m| = m$ , and let  $E_m^*$  be the rescaling of  $E_m$  of the same mass as the unit ball  $B$  centered at 0 (ie  $E_m^* = \left(\frac{|B|}{m}\right)^{\frac{1}{n}} E_m$ ). Then, up to modifying the sets  $(E_m)$  by some sets of measure 0, the boundaries of the sets  $(E_m^*)$  Hausdorff-converge to the boundary of  $B$  as  $m \rightarrow +\infty$ .*

Using the techniques of [33], one can see that in this case, if  $\beta > 1$  then balls of large masses are locally minimizing. On the contrary if  $\beta < 1$  then balls of large masses are not

locally minimizing. By locally minimizing, we mean that a set  $E$  is a local minimizer if a small  $L^1$  perturbation of  $E$  always has greater energy than  $E$ . With the additional assumption that  $\beta > 1$  we will actually prove that minimizers of large masses are exactly balls.

**Theorem e.2** (Theorem 1.1.2). *Given  $\alpha \in (0, n)$ ,  $\beta > 0$  and  $A > 0$ , assume  $1 < \alpha < \beta$ . There exists a mass  $m_1 = m_1(n, \alpha, \beta, A) > 0$  such that for any  $m > m_1$  the ball of volume  $m$  centered at 0 is the unique minimizer (1.1.3), up to a set of measure 0.*

We conjecture that this last theorem holds also for any  $\alpha \in (0, 1]$  as long as  $\beta > 1$ . However, this seems to require different techniques.

We started working on some aspects of this functional under the supervision of Vincent Millot for our Master 2 thesis. In part II, the new results start with Section 1.4. In particular, the results presented in this introduction are new.

**Numerical minimization** In Chapter 2, we will develop a method to compute numerically the minimizers of (e.2). In particular, we will apply this method to the original problem (e.1). Indeed, the theoretical knowledge we have so far on problem (e.1) raises two natural questions. Is it always the case (i.e. for any value of  $\alpha \in (0, n)$ ) that there is no minimizer for  $m$  large enough? Is there a set of parameters  $n$ ,  $\alpha$  and  $m$ , such that there exists a minimizer that is different from a ball? Our numerical results indicate that in dimension 2, the answers are positive and negative respectively, as conjectured by Choksi and Peletier in [24, Conjecture 6.1]

**$\Gamma$ -convergence.** To justify our numerical method, we will prove some  $\Gamma$ -convergence results. This is the notion of convergence of functionals one generally wants to use when studying variational problems.

**Definition e.3.** *Let  $X$  be a topological space,  $\mathcal{F}$  a real functional on  $X$ , and  $(\mathcal{F}_\varepsilon)_{\varepsilon>0}$  a family of real functionals on  $X$ . We say that  $\mathcal{F}_\varepsilon$   $\Gamma$ -converges to  $\mathcal{F}$  as  $\varepsilon$  goes to 0, and we write  $\mathcal{F}_\varepsilon \xrightarrow[\varepsilon \rightarrow 0]{\Gamma} \mathcal{F}$ , if the two following assertions hold:*

- for any  $u \in X$  and any family  $(u_\varepsilon)_{\varepsilon>0}$  of  $X$  such that  $u_\varepsilon \xrightarrow[\varepsilon \rightarrow 0]{} u$ , we have

$$\liminf_{\varepsilon \rightarrow 0} \mathcal{F}_\varepsilon(u_\varepsilon) \geq \mathcal{F}(u),$$

- for any  $u \in X$ , there exists a family  $(u_\varepsilon)_{\varepsilon>0}$  of  $X$  such that  $u_\varepsilon \xrightarrow[\varepsilon \rightarrow 0]{} u$  and

$$\lim_{\varepsilon \rightarrow 0} \mathcal{F}_\varepsilon(u_\varepsilon) = \mathcal{F}(u).$$

*Remark e.4.* We also say that a sequence of functionals  $(\mathcal{F}_N)$   $\Gamma$ -converges to a function  $\mathcal{F}$  if the assertions above hold with  $\varepsilon$  and 0 replaced with  $N$  and  $+\infty$ .

This definition guarantees that if, for any  $\varepsilon > 0$ ,  $u_\varepsilon$  is a minimizer of  $\mathcal{F}_\varepsilon$  in  $X$ , and the family  $(u_\varepsilon)$  converges to  $u \in X$  as  $\varepsilon \rightarrow 0$ , then  $u$  is also a minimizer of  $\mathcal{F}$  in  $X$ . To minimize numerically a given functional  $\mathcal{F}$ , the general strategy is the following:

1. find a sequence of discrete functionals  $(\mathcal{F}_N)_{N \geq 0}$  that  $\Gamma$ -converges to  $\mathcal{F}$ ,
2. find numerically a minimizer  $u_N$  of  $\mathcal{F}_N$  for some  $N$  considered 'large',
3. extrapolate a limit point  $u$  of the sequence  $(u_N)_{N \geq 0}$  from the results. This  $u$  should be a minimizer of  $\mathcal{F}$ .

For point 3 to even have a chance to work, we need the sequence  $(u_N)_{N \geq 0}$  to be precompact:

*Terminology* e.5. Following [6], we say that a family of functionals  $(\mathcal{F}_\varepsilon)_{\varepsilon > 0}$  defined on a metric space  $X$  enjoys property (C) (for compactness) if for any family  $(u_\varepsilon)_{\varepsilon > 0}$  of elements of  $X$  such that  $(\mathcal{F}_\varepsilon(u_\varepsilon))_{\varepsilon > 0}$  is bounded, there is a subsequence of  $(u_\varepsilon)_{\varepsilon > 0}$  that converges in  $X$ .

If a family of functionals  $(\mathcal{F}_\varepsilon)_{\varepsilon > 0}$  enjoys property (C) and  $\Gamma$ -converges to a limit functional  $\mathcal{F}$  when  $\varepsilon$  goes to 0, then we know that for  $\varepsilon$  small enough, minimizers of  $\mathcal{F}_\varepsilon$  are close to minimizers of  $\mathcal{F}$ . When discretizing our functional  $\mathcal{F}$ , along with the  $\Gamma$ -convergence, we will prove that the property (C) is verified.

We refer to the book [12] for more details about  $\Gamma$ -convergence. The lecture [6] also contains a brief introduction to  $\Gamma$ -convergence that is sufficient to our needs.

**Modica-Mortola theorem.** The numerical treatment of the perimeter is classical. To deal with the perimeter term, we will use the well known Modica-Mortola theorem. This theorem allows us to relax the perimeter functional on sets, i.e. characteristic functions, into a functional on functions taking values in  $[0, 1]$ . Therefore we will be able to use the vector space structure of functions and, after discretization (*step three* below), usual optimization tools for functionals on  $\mathbb{R}^d$ .

**Theorem e.6** (Modica-Mortola theorem.). *Let  $\Omega$  be a Lipschitz open bounded subset of  $\mathbb{R}^n$  and  $\varepsilon > 0$ . Let  $W$  be the double well potential defined on  $[0, 1]$  by  $W(x) = x(1-x)$ . Let us set  $\sigma := 2 \int_0^1 \sqrt{W(u)} du$ . We define the set  $X$ , and the functionals  $\mathcal{F}_\varepsilon : X \rightarrow \overline{\mathbb{R}}$  by*

$$X := \left\{ u \in L^1(\Omega, [0, 1]) : \int u = m \right\}, \quad \text{endowed with the strong } L^1\text{-topology,}$$

and

$$\mathcal{F}_\varepsilon(u) = \begin{cases} \sigma^{-1} \int_{\mathbb{R}^n} \left( \varepsilon |\nabla u|^2 + \varepsilon^{-1} W(u) \right) & \text{if } u \in H_0^1(\Omega, [0, 1]), \\ +\infty & \text{otherwise,} \end{cases} \quad (\text{e.3})$$

If  $(u_\varepsilon)_{\varepsilon > 0}$  is a family of functions in  $H_0^1(\Omega, [0, 1])$  and  $E \subset \Omega$  is any measurable subset of  $\Omega$ , such that  $u_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} \mathbf{1}_E$  in  $L^1(\Omega)$ , then we have

$$P(E) \leq \liminf_{\varepsilon \rightarrow 0} \mathcal{F}_\varepsilon(u_\varepsilon).$$

Moreover, if  $E \subset\subset \Omega$  is a compact subset of  $\Omega$  of finite perimeter, then there exists a family of functions  $(u_\varepsilon)_{\varepsilon > 0}$  in  $X$ , such that  $u_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} \mathbf{1}_E$ , and

$$P(E) = \lim_{\varepsilon \rightarrow 0} \mathcal{F}_\varepsilon(u_\varepsilon).$$

Finally, the family of functionals  $(\mathcal{F}_\varepsilon)$  enjoys property (C).



One can find a reader-friendly proof of these facts in [6]. Here, we will only give the following heuristic. If  $\varepsilon > 0$  is very small, then  $\varepsilon^{-1}W(u)$  is very large, except if  $u$  is 0 or 1. Therefore the minimizers of  $\mathcal{F}_\varepsilon$  are functions that are equal to 0 or 1 except on a set of small measure.

**Method of the numerical minimization.** Here are the modifications of the variational problem (1.1.3) that we will use in Chapter 2 to arrive at a finite dimensional variational problem that can be easily numerically solved. All steps will be justified by a  $\Gamma$ -convergence and compactness result.

*Step one* is standard when dealing with the perimeter. We use the classical Modica-Mortola theorem to relax the functional on sets, i.e. characteristic functions, into a functional on functions taking values in  $[0, 1]$ . This allows us to use the vector space structure of functions and, after discretization (*step three*), usual optimization tools for functionals on  $\mathbb{R}^d$ .

*Step two* is the key step for dealing with the non-local term  $\mathcal{V}_\alpha$ . We will replace the ambient space  $\mathbb{R}^n$  with a large square with periodic boundary conditions, whose size is a new relaxation parameter. Then, we can approximate the non-local term  $\mathcal{V}_\alpha$  by a simple expression in Fourier variable.

In *step three*, we discretize the problem by considering only trigonometric functions with frequencies lower than some integer  $N$ , and by computing the integral terms with riemann sums.

The numerical results are presented in section 2.3.

## Part I

Elastic-plastic torsion problem,  
distance function and cut locus



## Outline of part I

This part is independent of part II. We will try and give a global view of the content of this part. Then, each Chapter will start with a more detailed specific introduction, giving also more context about their respective topics.

We are interested in the approximation of the cut locus of a Riemannian manifold through a variational problem. Given a smooth Riemannian manifold  $M$  and a point  $b$  in  $M$ , the cut locus of  $b$  in  $M$ , denoted by  $Cut_b(M)$ , is the set of points at which geodesics starting from  $b$  stop being minimizing. Alternatively, it is also the set of points around which the distance function to  $b$ , denoted by  $d_b$ , is not smooth. Or the closure of the set of points  $p$  such that there exist at least two minimizing geodesics from  $b$  to  $p$ . The cut locus is quite a fundamental object in Riemannian geometry, and it has been studied by many mathematicians from varied backgrounds. See Section 2.1 for more details. It is a difficult object to compute numerically, because it is unstable with respect to  $C^1$  variations of the metric of  $M$ . In particular, the cut locus of the polyhedral approximation of a given surface  $S$  may be quite larger than the cut locus of  $S$  itself.

In Chapter 1, we first study the elastic-plastic torsion problem, in relation with the medial axis. Given a smooth bounded open set of  $\mathbb{R}^n$ , the medial axis of  $\Omega$ , denoted by  $\mathcal{M}(\Omega)$ , is the set of points  $x \in \Omega$  that have at least two closest points on the boundary of  $\Omega$ ,  $\partial\Omega$ . Alternatively, it is the set of points at which the distance function to  $\partial\Omega$  is not differentiable. The elastic-plastic torsion problem is the following variational problem:

$$\inf \left\{ \int_{\Omega} |\nabla v|^2 - mv : v \in H_0^1(\Omega), |\nabla v| \leq 1 \right\}. \quad (0.1)$$

It has been intensively studied in the 60's and 70's by various mathematicians. See Section 1.1 for more details. Let  $v_m$  be its unique minimizer. In [18], in particular, it was proved that the set  $\{|\nabla v_m| < 1\}$  contains the set  $\overline{\mathcal{M}(\Omega)}$ , and that the former converges in the Hausdorff sense to the latter as  $m$  goes to  $+\infty$ . Like the cut locus, the medial axis has some kind of instability. Indeed,  $\mathcal{M}(\Omega)$  is unstable with respect to non-smooth variations of the boundary of  $\Omega$  (see Figure 1.1 in Chapter 1). It makes it non-trivial to compute numerically as well. In 2005, Chazal and Lieutier introduced the notion of  $\lambda$ -medial axis to give a theoretical framework to tackle the problem of instability. It is defined, for any  $\lambda > 0$ , as

$$\mathcal{M}_{\lambda}(\Omega) := \{x \in \Omega : r(x) \geq \lambda\},$$

where  $r(x)$  is the radius of the smallest ball containing the set of all closest points to  $x$  on  $\partial\Omega$ , *i.e.* the set  $\{z \in \partial\Omega : |x - z| = d_{\partial\Omega}(x)\}$ . Basically, they showed that for  $\lambda$  small enough,  $\mathcal{M}_{\lambda}(\Omega)$  is a good approximation of  $\mathcal{M}(\Omega)$  and it is stable with respect to variations of the boundary of  $\Omega$ . Therefore, one can try and compute  $\mathcal{M}_{\lambda}(\Omega)$  instead of  $\mathcal{M}(\Omega)$ , and that is what they did in their paper [23]. See [73] for many more ways of computing the medial axis, and applications. Another useful formula for  $\mathcal{M}_{\lambda}(\Omega)$  can be derived using the generalized gradient of  $d_{\partial\Omega}$ . Indeed,  $d_{\partial\Omega}$  is locally semiconcave. This means that, locally, there exists a constant  $C > 0$  such that the function  $x \mapsto C|x|^2 - d_{\partial\Omega}(x)$  is convex. In particular, as convex functions, it has a well defined generalized gradient. See Appendix A.1 for more details. Now, one can see that the  $\lambda$ -medial axis is also given by the formula

$$\mathcal{M}_{\lambda}(\Omega) = \left\{ x \in \Omega : |\nabla d_{\partial\Omega}(x)|^2 \leq 1 - \frac{\lambda^2}{d_{\partial\Omega}^2(x)} \right\}.$$

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Motivated by these facts, we prove that the set  $E_{m,\lambda}$  defined by

$$E_{m,\lambda} := \left\{ x \in \Omega : |\nabla v_m(x)|^2 \leq 1 - \frac{\lambda^2}{v_m^2(x)} \right\}$$

is basically a good approximation of  $\mathcal{M}_\lambda(\Omega)$  if  $m$  is large enough.

**Theorem** (Theorem 1.3.1). *Let  $v_m$  be the unique solution of the elastic-plastic torsion problem (0.1). Then, for any  $\lambda > 0$  and  $\varepsilon > 0$ , we have*

$$\sup_{x \in E_{m,\lambda}} d(x, \mathcal{M}_\lambda(\Omega)) \xrightarrow{m \rightarrow +\infty} 0 \quad \text{and} \quad \sup_{x \in \mathcal{M}_{\lambda+\varepsilon}(\Omega)} d(x, E_{m,\lambda}) \xrightarrow{m \rightarrow +\infty} 0.$$

This means that a stable version of the cut locus can be approximated from the minimizer of a convex variational problem.

The strategy of the proof is to show that the minimizers  $(v_m)_{m>0}$  of the elastic-plastic torsion problem are uniformly semiconcave when  $m$  is large enough.

**Theorem** (Theorem 1.3.2). *Let  $C_d > 0$  be such that  $d_{\partial\Omega}$  is  $C_d$ -semiconcave. There exists  $m_0 > 0$  such that for any  $m > m_0$ , the solution  $v_m$  of the elasto-plastic torsion problem (0.1) is  $C_d$ -semiconcave.*

This is done by using Korevaar's convexity maximum principle [54].

The idea is then to apply the same strategy to the study of the cut locus and the problem of instability with respect to the metric, and to derive a method for computing the cut locus of a surface. In Chapter 3, we study the following variational problem:

$$\inf \left\{ \int_M |\nabla u|^2 - mu : u \in H^1(M), |\nabla u| \leq 1, u(b) = 0 \right\}. \quad (0.2)$$

It is a kind of generalization of the elastic-torsion problem, where the ambient euclidean space is replaced with a smooth compact Riemannian manifold without boundary  $\overline{M}$ , and the boundary of  $\partial\Omega$  is replaced with the point  $b$ . In this framework, the analog of  $\mathcal{M}(\Omega)$  is  $Cut_b(M)$ . We extend some results known for the elastic-plastic torsion problem to this framework. Let  $Cut_b(M)_\lambda \subset M$  be defined by

$$Cut_b(M)_\lambda := \left\{ p \in M : |\nabla d_b(x)|^2 \leq 1 - \frac{\lambda^2}{d_b^2(x)} \right\},$$

where  $\nabla d_b(x)$  is the generalized gradient of  $d_b$  (see Propositions A.1.9 and A.1.6). We will prove the following theorem.

**Theorem** (Theorem 3.1.2). *There exists  $m_0 > 0$ , depending only on the manifold  $M$ , such that for any  $m > m' > m_0$ , the minimizer  $u_m$  of (0.2) is locally  $C^{1,1}$  on  $M \setminus \{b\}$ , and*

$$Cut_b(M) \subset \{|\nabla u_m| < 1\} \subset \{|\nabla u_{m'}| < 1\}.$$

Moreover,

$$E_m \xrightarrow{m \rightarrow +\infty} Cut_b(M) \quad \text{in the Hausdorff sense.}$$

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Given any  $\varepsilon > 0$ , there exists a constant  $C > 0$  such that, for any  $m > m_0$ ,

$$u_m \text{ is } C\text{-semiconcave on } M \setminus B(b, \varepsilon).$$

Finally,

$$\sup_{x \in E_{m,\lambda}} d(x, \text{Cut}_b(M)_\lambda) \xrightarrow{m \rightarrow +\infty} 0, \quad \text{and} \quad \sup_{x \in \text{Cut}_b(M)_{\lambda+\varepsilon}} d(x, E_{m,\lambda}) \xrightarrow{m \rightarrow +\infty} 0.$$

This result will be used in chapter 4 to build a new method for the numerical approximation of the cut locus of a surface.

In our study, we will need to establish that the gradient constraint  $|\nabla u| \leq 1$  in (0.2) can be replaced with the obstacle constraint  $u \leq d_b$ . Contrary to the case of the classical obstacle problem (0.1), this is not true in general. However, it is true if  $m$  is large enough. Indeed, let  $u_m^d$  be the minimizer of the following variational problem:

$$\inf \left\{ \int_M |\nabla u|^2 - mu : u \in H^1(M), u \leq d_b \right\}.$$

We will prove the following.

**Theorem** (Theorem 3.3.1). *There exists a surface of revolution  $M$  and a parameter  $m > 0$  such that  $u_m \neq u_m^d$ .*

**Theorem** (Theorem 3.3.2). *Let  $K \geq 0$  be such that*

$$\text{Ric} \geq -K, \tag{0.3}$$

where  $\text{Ric}$  denotes the Ricci curvature tensor of the manifold  $M$ . If

$$m \geq \max(\sqrt{nK(1 + K \text{diam}(M)^2)}, nK \text{diam}(M)),$$

then

$$u_m^d = u_m.$$

In Chapter 4, we address the question of the numerical approximation of the cut locus based on previous modeling. We introduce a finite element method based on triangulated surfaces, prove the convergence of our scheme and illustrate its efficiency on complex geometries.

In Chapter 2, after introducing the notion of cut locus, and some important properties of  $d_b$  and  $\text{Cut}_b(M)$ , we prove a proposition about the behavior of  $d_b$  on  $\text{Cut}_b(M)$ , that is later used in Chapter 3. More specifically, we prove:

**Theorem** (Theorem 2.1.11). *Let  $M$  be a smooth manifold without boundary of dimension  $n \geq 2$ , and  $b \in M$ . Let  $d_b$  be the distance function to the point  $b$ , and  $\text{Cut}_b(M)$  the cut locus with respect to  $b$ . The Laplacian of  $d_b$  is  $-\infty$  at every point of  $\text{Cut}_b(M)$ , in the sense of barriers. This means that for any  $p \in \text{Cut}_b(M)$ , for any  $A > 0$ , there exists a smooth function  $\phi$  defined in a neighborhood of  $p$  such that*

$$\phi \geq d_b, \quad \phi(p) = d_b(p) \quad \text{and} \quad \Delta \phi(p) \leq -A.$$

*Remark 0.1.* We could have replaced the point  $b$  with a smooth hypersurface  $S \subset M$  in all our considerations. However, our interest here is in the instability of the cut locus with respect to the metric of  $M$ , and this aspect is already present with a point  $b$  instead of  $S$ . So for simplicity, we focus on the case of the cut locus of a point.



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# Medial axis of a euclidean domain and elastic-plastic torsion problem

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## 1.1 The elastic-plastic torsion problem.

In this chapter,  $\Omega$  is a smooth bounded open set of  $\mathbb{R}^n$ . The *elastic-plastic torsion problem* is the following variational problem, defined for any  $m > 0$ .

$$\inf \left\{ \int_{\Omega} |\nabla v|^2 - mv : v \in H_0^1(\Omega), |\nabla v| \leq 1 \right\}. \quad (1.1.1)$$

One can show that this functional has a minimizer by the direct method of calculus of variations. The minimizer is unique because we are minimizing a strictly convex functional over a convex subset of  $H_0^1(\Omega)$ . Let  $v_m$  be the unique minimizer of (1.1.1). (See also [46] for general existence and uniqueness results for more general functionals of that type.) Physically speaking,  $v_m$  represents the stress function of a long bar of cross section  $\Omega$ , twisted with an angle  $m$ . The elastic-plastic torsion problem and the properties of its minimizer  $v_m$  have been intensively studied in the 60s and 70s by various authors (such as Ting, Brézis, Sibony, Gerhardt, Caffarelli, Rivière, Friedman, Pozzi...), see for instance [78], [14], [13], [20], [79], [21], [38] and [19].

In particular, we know from [21] that  $v_m$  is locally  $C^{1,1}$  in  $\Omega$ . What is more, we know from [14] that the gradient constraint in (1.1.1) can be replaced with an obstacle type constraint involving the distance function  $d_{\partial\Omega}$  to the boundary of  $\Omega$ . Indeed, we have

$$0 \leq v_m \leq d_{\partial\Omega} \quad \text{and} \quad v_m = \arg \min \left\{ \int_{\Omega} |\nabla v|^2 - mv : v \in H_0^1(\Omega), v \leq d_{\partial\Omega} \right\}. \quad (1.1.2)$$

where

$$d_{\partial\Omega}(x) = \min_{y \in \partial\Omega} |x - y|.$$

Following [18], we define the *elastic* and *plastic* sets as follows:

$$\begin{aligned} E_m &= \{x \in \Omega : |\nabla v_m(x)| < 1\}, \\ P_m &= \{x \in \Omega : |\nabla v_m(x)| = 1\}, \end{aligned}$$



It is then easy to prove (see [18, §3]) that

$$\begin{aligned} E_m &= \{x \in \Omega : v_m(x) < d_{\partial\Omega}(x)\}, \\ P_m &= \{x \in \Omega : v_m(x) = d_{\partial\Omega}(x)\}. \end{aligned} \tag{1.1.3}$$

In the open set  $E_m$ , we have the Euler-Lagrange equation of the variational problem:

$$\Delta v_m = -2m \quad \text{in } E_m. \tag{1.1.4}$$

At a point  $x \in P_m$ , we have  $v_m(x) = d_{\partial\Omega}(x)$  and  $v_m \leq d_{\partial\Omega}$  around  $x$ , so

$$\nabla v_m(x) = \nabla d_{\partial\Omega}(x). \tag{1.1.5}$$

The *medial axis* of  $\Omega$ , denoted by  $\mathcal{M}(\Omega)$ , is defined as the set of points of  $\Omega$  that have at least two closest points on the boundary  $\partial\Omega$  of  $\Omega$ :

$$\mathcal{M}(\Omega) := \{x \in \Omega : \exists y, z \in \partial\Omega, y \neq z \text{ and } d_{\partial\Omega}(x) = |x - y| = |x - z|\}.$$

Equivalently,  $\mathcal{M}(\Omega)$  is the set of points of  $\Omega$  at which the distance function  $d_{\partial\Omega}$  is not differentiable. In [18], Caffarelli and Friedman used the notion of *ridge* of  $\Omega$ , which is precisely the closure of  $\mathcal{M}(\Omega)$ . Equivalently, it is the set of points around which the distance function is not smooth.

They proved the following.

**Theorem 1.1.1** (Theorems 3.2 and 4.1 in [18].). *For any  $m > 0$ , we have*

$$\overline{\mathcal{M}(\Omega)} \subset E_m.$$

*Furthermore, there exists a constant  $C > 0$  depending on  $\Omega$  only such that*

$$\forall m > 0, \quad d^{\mathcal{H}}(\mathcal{M}(\Omega), E_m) \leq \frac{C}{m},$$

*where  $d^{\mathcal{H}}$  denotes the Hausdorff distance.*

We will also use the following fact. Equation (4.2) in [18] gives a constant  $C > 0$  depending only on  $\Omega$  such that

$$\forall x \in \Omega, \quad d(x, \{v_m = d_{\partial\Omega}\}) \leq \frac{C}{m}.$$

As  $d_{\partial\Omega}$  and  $v_m$  are both 1-Lipschitz, we get that for a (different) constant  $C' > 0$ , we have

$$\forall m > 0, \quad \|d_{\partial\Omega} - v_m\|_{L^\infty(\Omega)} \leq \frac{C'}{m}. \tag{1.1.6}$$

## 1.2 $\mathcal{M}_\lambda(\Omega)$ : the $\lambda$ -medial axis

The medial axis, also called *skeleton*, is used in a wide variety of applications in image processing and computer vision. Many researchers have proposed more or less efficient methods to compute it, and are still trying to improve its computation. We refer to the book [73] from 2017 for a review about the medial axis, its computation and applications. In this section,

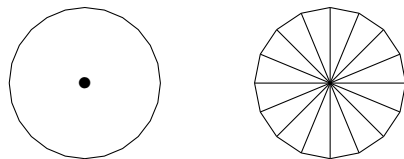


Figure 1.1: A circle with its medial axis, and a polygonal approximation of a circle, with its medial axis.

we introduce the notion of  $\lambda$ -medial axis that was invented in 2005 by Chazal and Lieutier to give a theoretical framework to tackle the problem of stability.

Indeed, one important geometric property of the medial axis  $\mathcal{M}(\Omega)$  is that it is unstable with respect to small perturbations of the boundary of  $\Omega$  - see Figure 1.1. This instability makes computing numerically  $\mathcal{M}(\Omega)$  quite tricky. Indeed, any numerical approximation of  $\Omega$  (for instance, with polygons) might introduce an artificial (and large) medial set. In order to deal with this problem, in [23], Chazal and Lieutier defined the so called  $\lambda$ -medial axis of  $\Omega$  by setting, for any  $\lambda > 0$ ,

$$\mathcal{M}_\lambda(\Omega) := \{x \in \Omega : r(x) \geq \lambda\}, \quad (1.2.1)$$

where  $r(x)$  is the radius of the smallest ball containing the set of all closest points to  $x$  on  $\partial\Omega$ , i.e. the set  $\{z \in \partial\Omega : |x - z| = d_{\partial\Omega}(x)\}$ . Note that the map  $\lambda \mapsto \mathcal{M}_\lambda(\Omega)$  is non-increasing, and we have

$$\mathcal{M}(\Omega) = \bigcup_{\lambda > 0} \mathcal{M}_\lambda(\Omega).$$

It is further proved in [23, Section 3, Theorem 2] that  $\mathcal{M}_\lambda(\Omega)$  has the same homotopy type as  $\mathcal{M}(\Omega)$ , for  $\lambda$  small enough. These facts justify that, for fixed  $\Omega$ ,  $\mathcal{M}_\lambda(\Omega)$  is a good approximation of  $\mathcal{M}(\Omega)$ , for  $\lambda$  small enough. The crucial difference though is that  $\mathcal{M}_\lambda(\Omega)$  is stable with respect to small variations of  $\Omega$ , whereas  $\mathcal{M}(\Omega)$  is not. Indeed we have the following theorem.

**Theorem 1.2.1** (Consequence of Theorem 3 in [23]). *For any  $0 < \lambda' < \lambda$ , there exists  $\varepsilon > 0$  such that, if  $\Omega$  and  $\tilde{\Omega}$  are two open bounded subset of  $\mathbb{R}^n$  such that  $d^H(\Omega^c, \tilde{\Omega}^c) < \varepsilon$ , then*

$$\sup_{x \in \mathcal{M}_\lambda(\tilde{\Omega})} d(x, \mathcal{M}_{\lambda'}(\Omega)) \leq 2\sqrt{\text{diam}(\Omega)\varepsilon},$$

where  $\text{diam}(\Omega)$  is the diameter of  $\Omega$ .

This means that although it is difficult to compute numerically  $\mathcal{M}(\Omega)$  directly, one can compute  $\mathcal{M}_\lambda(\Omega)$  instead. In the same paper [23], the authors use voronoi diagrams to compute an approximation of  $\mathcal{M}_\lambda(\Omega)$  from a noisy sample of points of the boundary of  $\Omega$ .

We will need the following definition.

**Definition 1.2.2.** *A function  $u : \Omega \rightarrow \mathbb{R}$  is said to be  $C$ -semiconcave (or just semiconcave) if the function  $x \mapsto C|x|^2 - u(x)$  is convex. This is equivalent to having, for any  $\lambda \in [0, 1]$  and  $x, y \in \Omega$  such that  $[x, y] \in \Omega$ ,*

$$(1 - \lambda)u(x) + \lambda u(y) - u((1 - \lambda)x + \lambda y) \leq C\lambda(1 - \lambda)|y - x|^2$$

Then, we have:

**Lemma 1.2.3.** *Let  $\Omega$  be a  $C^2$  open bounded subset of  $\mathbb{R}^n$ . There is a constant  $C_d > 0$  such that  $d_{\partial\Omega}$  is  $C_d$ -semiconcave on  $\Omega$ .*

This lemma is proved in [22, (iii) of Proposition 2.2.2], under slightly more general assumptions. Here we only give the idea of the proof.

*Idea of the proof.* As  $\Omega$  is of class  $C^2$ , you can write the complement of  $\Omega$  as a union of balls of radius  $r > 0$  for some fixed  $r$ . Then,  $d_{\partial\Omega}$  is the infimum of the distance functions to these balls of radius  $r$ . One can verify that the distance function to a ball is  $C$ -semiconcave for some  $C$ , and the supremum of a family of convex functions is convex, so  $d_{\partial\Omega}$  is also  $C$ -semiconcave.  $\square$

As explained with more details in the Appendix A.1, this implies that, as concave functions,  $d_{\partial\Omega}$  has a well defined generalized gradient whose norm is given by the formula:

$$|\nabla d_{\partial\Omega}|(x) = \max(0, \sup_{v \in S^{n-1}} \partial_v^+ d_{\partial\Omega}(x)),$$

where  $\partial_v^+ d_{\partial\Omega}(x)$  denotes the directional derivative of  $d_{\partial\Omega}$  in the direction  $v$ . What is more, as it can be seen from [23, Section 2.1], we have

$$|\nabla d_{\partial\Omega}(x)|^2 = 1 - \frac{r(x)^2}{d_{\partial\Omega}(x)^2},$$

and so the following formula for the  $\lambda$ -medial axis holds.

$$\mathcal{M}_\lambda(\Omega) = \left\{ x \in \Omega : |\nabla d_{\partial\Omega}(x)|^2 \leq 1 - \frac{\lambda^2}{d_{\partial\Omega}^2(x)} \right\}. \quad (1.2.2)$$

In the next section, we will use this formula to approximate  $\mathcal{M}(\Omega)$  with the minimizer of the elastic-torsion problem presented in Section 1.1.

### 1.3 Elastic-plastic torsion problem and $\lambda$ -medial axis

In this section, motivated by the two preceding sections, we prove the following refinement of Theorem 1.1.1. It is the main result of this chapter.

**Theorem 1.3.1.** *Let  $\Omega$  be a smooth bounded open subset of  $\mathbb{R}^n$ . For  $m > 0$ , let  $v_m$  be the unique solution of the variational problem:*

$$\inf \left\{ \int_{\Omega} |\nabla v|^2 - mv : v \in H_0^1(\Omega), |\nabla v| \leq 1 \right\}. \quad (1.3.1)$$

Let us define the set  $E_{m,\lambda}$  by:

$$E_{m,\lambda} := \left\{ x \in \Omega : |\nabla v_m(x)|^2 \leq 1 - \frac{\lambda^2}{v_m^2(x)} \right\}. \quad (1.3.2)$$

Then, for any  $\lambda > 0$  and  $\varepsilon > 0$ , we have

$$\sup_{x \in E_{m,\lambda}} d(x, \mathcal{M}_\lambda(\Omega)) \xrightarrow{m \rightarrow +\infty} 0 \quad \text{and} \quad \sup_{x \in \mathcal{M}_{\lambda+\varepsilon}(\Omega)} d(x, E_{m,\lambda}) \xrightarrow{m \rightarrow +\infty} 0.$$

This theorem basically shows that the set  $E_{m,\lambda}$  is a good approximation of the  $\lambda$ -medial axis. In particular, it yields a new method for the numerical computation of the medial axis, based on the solution of a convex variational problem. In Chapter 3, we will use this strategy to approximate the cut locus of a manifold. The numerical results will be presented in Chapter 4.

The most important step in the proof of Theorem 1.3.1, is to show that the functions  $(v_m)_{m>0}$  are uniformly semiconcave.

**Theorem 1.3.2.** *Let  $C_d > 0$  be such that  $d_{\partial\Omega}$  is  $C_d$ -semiconcave. There exists  $m_0 > 0$  such that for any  $m > m_0$ , the solution  $v_m$  of the elasto-plastic torsion problem (1.1.1) is  $C_d$ -semiconcave.*

With this uniform semiconcavity result, Theorem 1.3.1 follows from the uniform convergence of  $v_m$  to  $d_{\partial\Omega}$  when  $m \rightarrow +\infty$ . The proof of 1.3.1 is almost identical to the proof of Proposition 3.5.6. We still give it here in full details.

The proof of Theorem 1.3.2 relies on the following two facts:

- inside  $E_m$ ,  $v_m$  has constant Laplacian,
- outside  $E_m$ ,  $v_m$  is equal to  $d_{\partial\Omega}$ , which is  $C_d$ -semiconcave, for some constant  $C_d > 0$  the depends only on  $\Omega$ .

The main tool is Korevaar's convexity maximum principle [54, Theorem 1.3]. Let us present this result. Let  $f : \Omega \rightarrow \mathbb{R}$  be a  $C^2$  function satisfying an equation of the form

$$P(x, f(x), \nabla f(x), Hf(x)) = 0$$

in  $\Omega$ , for a certain elliptic operator  $P$ . The convexity function of  $f$  is the function  $c$  defined for any  $[x, y] \subset \overline{\Omega}$  and  $\lambda \in [0, 1]$  by

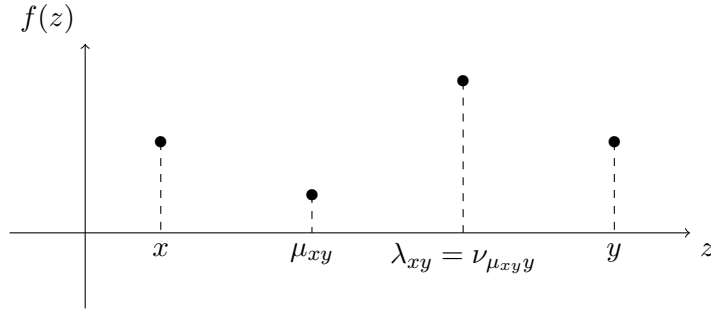
$$c(x, y, \lambda) := (1 - \lambda)f(x) + \lambda f(y) - f((1 - \lambda)x + \lambda y).$$

Under suitable conditions on the elliptic operator  $P$ , Korevaar's convexity maximum principle states that the minimum of  $c$  is attained at a triple  $(x, y, \lambda)$  such that one of the points  $x$ ,  $y$  or  $(1 - \lambda)x + \lambda y$  belongs to the boundary of  $\Omega$ . In particular, this result applies to the case where  $P$  is the euclidean Laplacian.

We are ready to prove Theorem 1.3.2. The first part of the proof is identical to the first part of the proof of Proposition 3.5.5. We still give it in full detail, for the reader's convenience.

*Proof of Theorem 1.3.2.* Let  $C_d$  be such that the distance function  $d_{\partial\Omega}$  is  $C_d$ -semiconcave. As  $\Omega$  is smooth, every point of  $\Omega$  in a neighborhood of  $\partial\Omega$  has a unique projection on  $\partial\Omega$ . Therefore the distance between  $\mathcal{M}(\Omega)$  and  $\partial\Omega$  is positive. From Theorem 1.1.1, we know that the elastic set  $E_m$  converges to  $\mathcal{M}(\Omega)$  in the Hausdorff sense when  $m$  goes to  $+\infty$ . So there exists  $m_0$  such that, for any  $m > m_0$ ,  $\overline{E_m} \subset \Omega$ . Let  $\varepsilon > 0$ . We will show that for any  $m > m_0$ ,  $v_m$  is  $(C_d + \varepsilon)$ -semiconcave, which is sufficient to conclude the proof. Let us introduce some notations. For  $a, b \in \mathbb{R}$  and  $\lambda \in (0, 1)$ , we set

$$\lambda_{ab} := (1 - \lambda)a + \lambda b.$$


 Figure 1.2: Construction of  $\tilde{\gamma}$  and  $\nu$ .

For any  $x, y \in \bar{\Omega}$  such that  $[x, y] \in \bar{\Omega}$ ,  $\lambda \in [0, 1]$  and  $v$  a function on  $\bar{\Omega}$ , let us define

$$\begin{aligned} f_v(z) &:= (C_d + \varepsilon)|z|^2 - v(z), \\ c(x, y, \lambda, v) &:= \lambda f_v(x) f_v(y) - f_v(\lambda_{xy}). \end{aligned} \quad (1.3.3)$$

In order to prove that  $f_{v_m}$  is convex (i.e. that  $v_m$  is  $(C_d + \varepsilon)$  semiconcave), we need to show the following:

$$\inf_{x, y, \lambda} c(x, y, \lambda, v_m) \geq 0 \quad (1.3.4)$$

where the infimum is taken over  $x, y \in \bar{\Omega}$  such that  $[x, y] \subset \bar{\Omega}$  and  $\lambda \in [0, 1]$ . Let us argue by contradiction and assume that (1.3.4) does not hold. By continuity of  $c$ , the infimum in (1.3.4) is a minimum, attained at a triple  $(x, y, \lambda)$ , that verifies

$$c(x, y, \lambda, v_m) < 0.$$

Now we show that

$$(x, y) \subset E_m \quad (= \{v_m < d_{\partial\Omega}\}). \quad (1.3.5)$$

If not, then there exists  $\mu \in (0, 1)$  such that  $\mu_{xy} \notin E_m$ , i.e.  $v_m(\mu_{xy}) = d_{\partial\Omega}(\mu_{xy})$ . Recalling that  $v_m \leq d_{\partial\Omega}$ , we get

$$c(x, y, \mu, v_m) \geq c(x, y, \mu, d_{\partial\Omega}) > 0, \quad (1.3.6)$$

where the last inequality comes from the  $C_d$ -semiconcavity of  $d_{\partial\Omega}$ . In particular,  $\mu \neq \lambda$ . Let us assume for instance that  $\mu < \lambda$  (the case  $\mu > \lambda$  being similar) and show that  $c(x, y, \lambda, v_m)$  is not minimal in (1.3.4). Figure 1.2 may help justify intuitively the following construction. Let  $\nu \in (0, 1)$  be such that

$$\nu_{\mu_{xy}y} = \lambda_{xy}. \quad (1.3.7)$$

We have

$$\begin{aligned} c(\mu_{xy}, y, \nu, v_m) &= (1 - \nu)f_{v_m}(\mu_{xy}) + \nu f_{v_m}(y) - f_{v_m}(\nu_{\mu_{xy}y}) \\ &= (1 - \nu)f_{v_m}(\mu_{xy}) + \nu f_{v_m}(y) - f_{v_m}(\lambda_{xy}) \\ &= c(x, y, \lambda, v_m) - (1 - \lambda)f_{v_m}(x) + (\nu - \lambda)f_{v_m}(y) + (1 - \nu)f_{v_m}(\mu_{xy}). \end{aligned} \quad (1.3.8)$$

Now after some elementary calculations, (1.3.7) translates into

$$\begin{cases} 1 - \lambda &= (1 - \nu)(1 - \mu), \\ \nu - \lambda &= -(1 - \nu)\mu, \end{cases}$$

so (1.3.8) becomes

$$\begin{aligned}
 c(\mu_{xy}, y, \nu, v_m) &= c(x, y, \lambda, v_m) - (1 - \nu)((1 - \mu)f_{v_m}(x) + \mu f_{v_m}(y) - f_{v_m}(\mu_{xy})) \\
 &= c(x, y, \lambda, v_m) - (1 - \nu)c(x, y, \mu, v_m) \\
 &< c(x, y, \lambda, v_m),
 \end{aligned} \tag{1.3.9}$$

where in the last inequality we used the fact that  $c(x, y, \mu, v_m) > 0$  (equation (1.3.6)). This contradicts the minimality of  $c(x, y, \lambda, v_m)$ . Thus (1.3.5) is true, and we are left to show that  $v_m$  is  $(C_d + \varepsilon)$ -semiconcave inside  $E_m$ .

The function

$$f_{v_m} : z \mapsto (C_d + \varepsilon)|z|^2 - v_m(z)$$

has constant Laplacian on  $E_m$ . So we may apply Korevaar's convexity maximum principle (see [54, Theorem 1.3]) to get that the minimum of the function  $c(\cdot, \cdot, \cdot, v_m)$  on  $\overline{E_m} \times \overline{E_m} \times [0, 1]$  is attained at a triple  $(x, y, \lambda)$  such that  $x, y$ , or  $\lambda_{xy}$  is in  $\partial E_m$ . As proved above,  $\lambda_{xy} \in \partial E_m$  is not an option, so  $x \in \partial E_m$  or  $y \in \partial E_m$ . Let us treat the case  $x \in \partial E_m$ , for instance. We have chosen  $m_0$  so that for  $m > m_0$ ,  $\overline{E_m} \subset \Omega$ . In particular  $x \in \Omega$ . We get that  $x \in P_m$ , and in particular (recalling (1.1.3) and (1.1.5))

$$v_m(x) = d_{\partial\Omega}(x) \quad \text{and} \quad \nabla v_m(x) = d_{\partial\Omega}(x). \tag{1.3.10}$$

As  $d_{\partial\Omega}$  is  $C_d$ -semiconcave, we have for any  $\mu \in (0, 1)$ ,

$$\begin{aligned}
 (1 - \mu)(C_d|x|^2 - d_{\partial\Omega}(x)) + \mu(C_d|y|^2 - d_{\partial\Omega}(y)) \\
 \geq C_d|(1 - \mu)x + \mu y|^2 - d_{\partial\Omega}((1 - \mu)x + \mu y).
 \end{aligned}$$

With (1.3.10), it yields as  $\mu \rightarrow 0$ ,

$$\begin{aligned}
 (1 - \mu)(C_d|x|^2 - d_{\partial\Omega}(x)) + \mu(C_d|y|^2 - d_{\partial\Omega}(y)) \\
 \geq C_d|(1 - \mu)x + \mu y|^2 - v_m((1 - \mu)x + \mu y) + o(\mu).
 \end{aligned}$$

Since  $v_m \leq d_{\partial\Omega}$ ,

$$\begin{aligned}
 (1 - \mu)(C_d|x|^2 - v_m(x)) + \mu(C_d|y|^2 - v_m(y)) \\
 \geq C_d|(1 - \mu)x + \mu y|^2 - v_m((1 - \mu)x + \mu y) + o(\mu).
 \end{aligned} \tag{1.3.11}$$

Multiplying the following equality by  $\varepsilon$

$$(1 - \mu)|x|^2 + \mu|y|^2 = |(1 - \mu)x + \mu y|^2 + \mu(1 - \mu)|x - y|^2,$$

and adding it to (1.3.11), we get

$$\begin{aligned}
 (1 - \mu)((C_d + \varepsilon)|x|^2 - v_m(x)) + \mu((C_d + \varepsilon)|y|^2 - v_m(y)) \\
 \geq (C_d + \varepsilon)|(1 - \mu)x + \mu y|^2 - v_m((1 - \mu)x + \mu y) + \varepsilon\mu(1 - \mu)|x - y|^2 + o(\mu),
 \end{aligned}$$

and so for  $\mu$  small enough,

$$\begin{aligned}
 (1 - \mu)((C_d + \varepsilon)|x|^2 - v_m(x)) + \mu((C_d + \varepsilon)|y|^2 - v_m(y)) \\
 > (C_d + \varepsilon)|(1 - \mu)x + \mu y|^2 - v_m((1 - \mu)x + \mu y),
 \end{aligned}$$

i.e.  $c(x, y, \mu, v_m) > 0$ . As before, we show that it implies

$$c(\mu_{xy}, y, \nu, v_m) < c(x, y, \lambda, v_m),$$

where  $\nu$  is such that  $\nu_{\mu_{xy}y} = \lambda_{xy}$ . This contradicts the minimality of  $c(x, y, \lambda)$ . Therefore hypothesis (1.3.4) holds, which concludes the proof.  $\square$

We can now prove Theorem 1.3.1.

*Proof of Theorem 1.3.1.* Let us argue by contradiction, and assume that the following convergence does not hold.

$$\sup_{x \in E_{m, \lambda}} d(x, \mathcal{M}_\lambda(\Omega)) \xrightarrow{m \rightarrow +\infty} 0 \quad (1.3.12)$$

Let  $(m_k)_{k \geq 0}$  and  $(x_k)_{k \geq 0}$  be sequences in  $(0, \infty)$  and  $\Omega$  respectively, such that  $m_k \xrightarrow{k \rightarrow +\infty} +\infty$  and for some  $\eta > 0$ , for any  $k \geq 0$ ,  $x_k \in E_{m_k, \lambda}$  and

$$d(x_k, \mathcal{M}_\lambda(\Omega)) > \eta. \quad (1.3.13)$$

Up to taking subsequences, we may assume that  $(x_k)$  converges to a point  $x_\infty \in \bar{\Omega}$ . As  $x_k \in E_{m_k, \lambda}$  for any  $k$ , we actually have  $x_\infty \in \Omega$ . As the functions  $(v_m)_{m > 0}$  are uniformly semiconcave and converge to  $d_{\partial\Omega}$  as  $m$  goes to  $\infty$  (Theorem 1.3.2 and (1.1.6)), we get from Proposition A.1.8:

$$\liminf_{k \rightarrow \infty} |\nabla v_{m_k}(x_k)| \geq |\nabla d_{\partial\Omega}(x_\infty)|.$$

Using the facts that  $x_k \in E_{m_k, \lambda}$  and  $v_{m_k}$  converges uniformly to  $d_{\partial\Omega}$ , this implies

$$|\nabla d_{\partial\Omega}(x_\infty)|^2 \leq 1 - \frac{\lambda^2}{d_{\partial\Omega}(x_\infty)^2},$$

and so  $x_\infty \in \mathcal{M}_\lambda(\Omega)$ . But this contradicts (1.3.13). Thus (1.3.12) is proved.

We are left to prove the following convergence:

$$\sup_{x \in \mathcal{M}_{\lambda+\varepsilon}(\Omega)} d(x, E_{m, \lambda}) \xrightarrow{m \rightarrow +\infty} 0. \quad (1.3.14)$$

Once again, we argue by contradiction and assume that it does not hold. Let  $(m_k)_{k \geq 0}$  and  $(x_k)_{k \geq 0}$  be sequences of  $(0, \infty)$  and  $\Omega$  respectively, such that  $m_k \xrightarrow{k \rightarrow +\infty} +\infty$  and for some  $\eta > 0$ , for any  $k \geq 0$ ,  $x_k \in \mathcal{M}_{\lambda+\varepsilon}(\Omega)$  and

$$d(x_k, E_{m_k, \lambda}) > 2\eta. \quad (1.3.15)$$

Up to taking subsequences, we can assume that  $(x_k)$  converges to a point  $x_\infty$  in  $\mathcal{M}_{\lambda+\varepsilon}(\Omega)$ . The last inequality implies that for any  $k$  large enough, we have

$$d(x_\infty, E_{m_k, \lambda}) > \eta.$$

According to (1.1.6), there exists a constant  $C_1 > 0$  such that for any  $y \in \Omega$  and  $k \geq 0$ ,  $v_{m_k}(y)^2 \geq d_{\partial\Omega}(y)^2 - C_1/m_k$ . As  $d_{\partial\Omega}$  is 1-Lipschitz and bounded, we deduce that there exists

a constant  $C_2 > 0$  such that for any  $y \in B(x_\infty, \eta)$  and  $k \geq 0$ ,  $v_{m_k}(y)^2 \geq d_{\partial\Omega}(x_\infty)^2 - C_2\eta - C_1/m_k$ . Let  $\eta_0 > 0$ ,  $k_0 > 0$  and  $\delta_0 > 0$  be constants such that for any  $\eta \leq \eta_0$  and  $k \geq k_0$ ,

$$\delta := \frac{(\lambda + \varepsilon)^2}{d_{\partial\Omega}(x_\infty)^2} - \frac{\lambda^2}{d_{\partial\Omega}(x_\infty)^2 - C_2\eta - C_1/m_k} \geq \delta_0 > 0.$$

For any  $y \in B(x_\infty, \eta)$ , we have

$$\begin{aligned} 1 - \frac{\lambda^2}{v_{m_k}(y)^2} &\geq \left(1 - \frac{\lambda^2}{d_{\partial\Omega}(x_\infty)^2 - C_2\eta - C_1/m_k}\right) \\ &= 1 - \frac{(\lambda + \varepsilon)^2}{d_{\partial\Omega}(x_\infty)^2} + \delta. \end{aligned} \quad (1.3.16)$$

As  $x_\infty \in \mathcal{M}_{\lambda+\varepsilon}(\Omega)$ , we have  $\left(1 - \frac{(\lambda+\varepsilon)^2}{d_{\partial\Omega}(x_\infty)^2}\right) \geq 0$ , and so for any  $\eta \leq \eta_0$ ,  $k \geq k_0$  and  $y \in B(x_\infty, \eta)$ ,

$$1 - \frac{\lambda^2}{v_{m_k}(y)^2} \geq \delta \geq \delta_0 > 0. \quad (1.3.17)$$

Let  $(y_t)_{t \geq 0}$  be the curve defined by

$$y_0 = x_\infty \quad \text{and} \quad \frac{dy_t}{dt} = \nabla v_{m_k}(y_t).$$

Let  $T > 0$  be such that for any  $t \in [0, T]$ ,  $d(y_t, x_\infty) \leq \eta$ , and in particular  $y_t \notin E_{m_k, \lambda}$ . We have

$$\begin{aligned} v_{m_k}(y_T) - v_{m_k}(x_\infty) &= \int_{(0, T)} |\nabla v_{m_k}(y_t)|^2 dt \quad \text{by definition of } (y_t), \\ &\geq \int_{(0, T)} \left(1 - \frac{\lambda^2}{v_{m_k}(y_t)^2}\right) dt \quad \text{because } y_t \notin E_{m_k, \lambda}, \\ &\geq T\delta \quad \text{because of (1.3.17)}. \end{aligned}$$

As  $v_{m_k}$  is bounded by the diameter of  $\Omega$ , this estimate implies that there exists a finite largest time  $T > 0$  such that for any  $t \in [0, T]$ ,  $d(y_t, x_\infty) \leq \eta$ . In particular,  $d(x_\infty, y_T) = \eta$ . From Lemma 1.2.3, we know that there exists a constant  $C_d > 0$  such that  $d_{\partial\Omega}$  is semiconcave. In particular, we have (see Proposition A.1.7)

$$\begin{aligned} d_{\partial\Omega}(y_T) - d_{\partial\Omega}(x_\infty) &\leq |\nabla d_{\partial\Omega}(x_\infty)| d(x_\infty, y_T) + C_d(d(x_\infty, y_T))^2 \\ &= |\nabla d_{\partial\Omega}(x_\infty)| \eta + C_d \eta^2, \\ &\leq \left(\sqrt{1 - \frac{(\lambda + \varepsilon)^2}{d_{\partial\Omega}(x_\infty)^2}}\right) \eta + C_d \eta^2 \quad \text{because } x_\infty \in \mathcal{M}_{\lambda+\varepsilon}(\Omega). \end{aligned} \quad (1.3.18)$$

In addition,

$$\begin{aligned} v_{m_k}(y_T) - v_{m_k}(x_\infty) &= \int_{(0, T)} |\nabla v_{m_k}(y_t)| \left| \frac{dy_t}{dt} \right| dt \quad \text{by definition of } (y_t), \\ &\geq \int_{(0, T)} \sqrt{1 - \frac{\lambda^2}{v_{m_k}(y_t)^2}} \left| \frac{dy_t}{dt} \right| dt \quad \text{because } y_t \notin E_{m_k, \lambda}, \\ &\geq \left( \inf_{t \in (0, T)} \sqrt{1 - \frac{\lambda^2}{v_{m_k}(y_t)^2}} \right) \eta. \end{aligned} \quad (1.3.19)$$



Using lemma 3.4.1 again, we know that there exists a constant  $C_3 > 0$  such that  $d_{\partial\Omega}(y_T) - d_{\partial\Omega}(x_\infty) \geq v_{m_k}(y_T) - v_{m_k}(x_\infty) - \frac{C_3}{m_k}$ . Therefore, estimates (1.3.18) and (1.3.19) yield

$$C_d \eta^2 + \left( \sqrt{1 - \frac{(\lambda + \varepsilon)^2}{d_{\partial\Omega}(x_\infty)^2}} - \inf_{t \in (0, T)} \sqrt{1 - \frac{\lambda^2}{v_{m_k}(y_t)^2}} \right) \eta + \frac{C_3}{m_k} \geq 0.$$

Using (1.3.16), we find that there exists a constant  $C_4 > 0$  such that for any  $\eta \leq \eta_0$  and  $k \geq k_0$ ,

$$\left( \sqrt{1 - \frac{(\lambda + \varepsilon)^2}{d_{\partial\Omega}(x_\infty)^2}} - \inf_{t \in (0, T)} \sqrt{1 - \frac{\lambda^2}{v_{m_k}(y_t)^2}} \right) \leq -C_4.$$

In particular, for any  $\eta \leq \eta_0$  and  $k \geq k_0$ ,

$$C_d \eta^2 - C_4 \eta + \frac{C_3}{m_k} \geq 0.$$

This gives a contradiction if  $k$  is large enough. Thus, (1.3.14) is true. This concludes the proof.  $\square$

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## Distance function and cut locus

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### 2.1 Introduction

Before stating and proving the result of this chapter, let us introduce and give some context about the notions of distance function and cut locus.

In this chapter,  $M$  is a smooth Riemannian manifold without boundary of dimension  $n \geq 2$ ,  $b$  a point of  $M$ ,  $d$  the distance function of  $M$  and  $d_b := d(\cdot, b)$  the distance function to the point  $b$ . We are interested in the properties of the function  $d_b$ . Before giving an account of these properties, let us recall some notions of Riemannian geometry.

**Definition 2.1.1.** *Let  $\gamma : [a, b] \rightarrow M$  be a geodesic, and  $J : [a, b] \rightarrow TM$  a vector field along  $\gamma$ . Then,  $J$  is called a Jacobi field if there exists a variation of  $\gamma$  through geodesics whose  $J$  is the variation field, i.e. a smooth map  $\Gamma : (-1, 1) \times [a, b] \rightarrow M$  such that  $\Gamma(0, \cdot) = \gamma$  and for any  $s \in (-1, 1)$ ,  $\Gamma(s, \cdot)$  is a geodesic, and*

$$\partial_1 \Gamma(0, t) = J(t).$$

*Moreover, if there exists a Jacobi field  $J$  along  $\gamma$  such that  $J(a) = 0$  and  $J(b) = 0$ , then we say that  $\gamma(a)$  and  $\gamma(b)$  are conjugate points along  $\gamma$ .*

**Definition 2.1.2.** *Let  $T > 0$  and  $\gamma : [0, T] \rightarrow M$  be a unit speed geodesic such that  $\gamma(0) = b$ ,  $t_0 \in (0, T)$  and  $p = \gamma(t_0)$ . We say that  $p$  is a cut point of  $b$  along  $\gamma$  if  $\gamma$  is length minimizing between  $b$  and  $p$ , but not after  $p$ , i.e  $d_b(\gamma(t)) = t$  for  $t \leq t_0$ , and  $d_b(\gamma(t)) < t$  for  $t > t_0$ .*

**Definition 2.1.3.** *The cut locus of  $b$  in  $M$ , denoted by  $Cut_b(M)$ , is defined as the set of all cut points of  $b$ .*

We recall the following.

**Definition 2.1.4.** *Let  $q \in M$  and  $V \in T_q M$  be such that there exists a well defined geodesic  $\gamma : [0, 1] \rightarrow M$  such that  $\gamma(0) = q$  and  $\dot{\gamma}(0) = V$ . Then, the (global) exponential map at  $V$  is defined by  $\text{Exp}(V) = \gamma(1)$ . The restriction of the exponential map  $\text{Exp}$  on  $T_q M$  is denoted by  $\text{exp}_q$  and called the exponential map at  $q$ .*

The following facts can all be found in [74, Chapter III, Section 4]:

**Proposition 2.1.5.** *Some well known facts related to the cut locus include:*

- $Cut_b(M)$  is the set of points  $p$  in  $M$ , such that either there exist at least two minimizing geodesics from  $b$  to  $p$ , or the points  $b$  and  $p$  are conjugate,
- $Cut_b(M)$  is the closure of the set of points  $p$  in  $M$ , such that there exist at least two minimizing geodesics from  $b$  to  $p$ ,
- the function  $d_b$  is smooth outside  $Cut_b(M) \cup \{b\}$ , and it is differentiable at a point  $p$  if and only if there exist a unique minimizing geodesics between  $b$  and  $p$ ,
- in particular,  $Cut_b(M) \cup \{b\}$  is the closure of the set of points of non-differentiability of  $d_b$ ,
- the exponential map  $\exp_b$  is a diffeomorphism from an open set of  $T_bM$  onto  $M \setminus Cut_b(M)$ ,
- $Cut_b(M)$  is a deformation retract of  $M \setminus \{b\}$ . In particular, these two sets have the same homotopy type, and so  $Cut_b(M)$  inherits much of the topology of  $M$  (like homology groups, for instance). See [74, Chapter III, Section 4, proposition 4.5] for a precise statement.

**Proposition 2.1.6.** *The function  $d_b$  is 1-Lipschitz. At every point  $p$  where  $d_b$  is differentiable,  $\nabla d_b$  is given by the speed vector of the minimizing unit speed geodesic from  $b$  to  $p$ . In particular, we have  $|\nabla d_b| = 1$  at every point where  $d_b$  is differentiable.*

We know quite a bit about the structure of the cut locus. Let us give some of its properties, to get an idea of what the cut locus looks like. First, it is the union of smooth hypersurfaces consisting of points with exactly two minimizing geodesics, plus a set of Hausdorff dimension at most  $n - 2$ . A reader-friendly reference for this is [49]. Moreover, the cut locus has locally finite  $n - 1$ -dimensional Hausdorff measure, see for instance [50, Theorem B]. To our knowledge, the most precise results in this direction are contained in [9], where points of the cut locus are classified, up to a set of Hausdorff dimension at most  $n - 3$ . We refer to [8, Section 1.3] for additional structural properties of the cut locus.

An important property of the distance function, which we will be using in the rest of the thesis, is *semiconcavity*. Let us give a definition.

**Definition 2.1.7.** *Given a constant  $C > 0$ , a function  $u$  is said to be  $C$ -semiconcave on  $M$  if and only if for any unit speed geodesic  $\gamma : [a, b] \rightarrow M$ , the function  $t \mapsto Ct^2 - u(\gamma(t))$  is convex. We say that  $u$  is semiconcave if  $u$  is  $C$ -semiconcave for some constant  $C > 0$ . We say that  $u$  is locally semiconcave if for any  $p \in M$ ,  $u$  is semiconcave in a neighborhood of  $p$ .*

*Remark 2.1.8.* For a function  $u$  defined on an open set of  $\mathbb{R}^n$ ,  $u$  is  $C$ -semiconcave if and only if the map  $x \mapsto C|x|^2 - u(x)$  is convex.

In [60, Proposition 3.4], C.Mantegazza and A.Mennucci showed that the distance function is a viscosity solution of the Hamilton-Jacobi equation  $|\nabla d_b| = 1$ . Using some regularity result ([25]) for Hamilton-Jacobi equations, they deduced that for any local chart  $\psi$  of  $M$ , the map  $d_b \circ \psi^{-1}$  is locally semiconcave as a function on  $\mathbb{R}^n$  (see also [22] for a more reader-friendly reference about semiconcave functions and Hamilton-Jacobi equations). In the Appendix A.1 about

semiconcave functions, we show that it is equivalent to saying that  $d_b$  is locally semiconcave. We realized afterwards that a different proof of the local semiconcavity of distance functions can also be found in [5]. In [61] (and in particular Theorem 2.10), C.Mantegazza, G.Mascellani and G.Uraltsev used this fact along with geometric measure theory tools to give a complete description of the distributional Hessian and Laplacian of  $d_b$ . They proved the following.

**Theorem 2.1.9.** *The distributional Hessian of  $d_b$  is given by:*

$$D^2d_b = \widetilde{D^2d_b}\text{Vol} - (\nu \otimes \nu) |\nabla d_b^+ - \nabla d_b^-| \mathcal{H}_{\text{Cut}_b(M)}^{n-1},$$

where  $\widetilde{D^2d_b}$  is the classical Hessian of  $d_b$  where  $d_b$  is smooth (i.e. almost everywhere),  $\text{Vol}$  is the Riemannian volume measure,  $\nu$  is a unit normal vector to  $\text{Cut}_b(M)$  where  $\text{Cut}_b(M)$  is smooth (i.e.  $\mathcal{H}^{n-1}$ -almost everywhere),  $\nabla d_b^+$  and  $\nabla d_b^-$  are limits of  $\nabla d_b$  to either side of  $\text{Cut}_b(M)$ , and  $\mathcal{H}_{\text{Cut}_b(M)}^{n-1}$  is the  $n - 1$ -dimensional Hausdorff measure restricted to  $\text{Cut}_b(M)$ . The distributional Laplacian of  $d_b$  is given by:

$$\Delta d_b = \widetilde{\Delta d_b}\text{Vol} - |\nabla d_b^+ - \nabla d_b^-| \mathcal{H}_{\text{Cut}_b(M)}^{n-1},$$

where  $\widetilde{\Delta d_b}$  is the classical Laplacian of  $d_b$  where  $d_b$  is smooth.

The main result of this Chapter is a complement to the description of the non-smooth behavior of the distance function on the cut locus. We are interested in the Laplacian of  $d_b$  in the sense of barriers.

**Definition 2.1.10.** *Let  $\psi : M \rightarrow \mathbb{R}$  be a continuous function, and  $p \in M$ . We say that the Laplacian of  $\psi$  at  $p$  is  $-\infty$  in the sense of barriers if for any  $A > 0$ , there exists a smooth function  $\phi : M \rightarrow \mathbb{R}$  defined on a neighborhood of  $p$  such that*

$$\phi \geq \psi, \quad \phi(p) = \psi(p) \quad \text{and} \quad \Delta\phi(p) \leq -A.$$

We prove the following.

**Theorem 2.1.11.** *Let  $M$  be a smooth manifold without boundary of dimension  $n \geq 2$ , and  $b \in M$ . Let  $d_b$  be the distance function to the point  $b$ , and  $\text{Cut}_b(M)$  the cut locus with respect to  $b$ . The Laplacian of  $d_b$  is  $-\infty$  at every point of  $\text{Cut}_b(M)$ , in the sense of barriers.*

This result proves useful when one wants to apply the maximum principle to a function involving the distance function  $d_b$ . It will be applied to the study of a variational problem in Chapter 3.

## 2.2 Proof of Theorem 2.1.11

**Ideas of the proof.** The proof might seem very technical if the reader is not used to the elementary tools of Riemannian geometry used in the proof, such as normal coordinates, Jacobi fields and covariant derivatives. We will explain the idea of the proof in dimension 2. We will refer to constructions that appear on Figure 2.1. Let  $p \in \text{Cut}_b(M)$ . The case where there exist two minimizing geodesics between  $b$  and  $p$  is simple— see below. So let us assume that there exists a unique minimizing geodesic between  $b$  and  $p$ , along which the two points are

conjugate. Let  $r, R > 0$  be such that  $d_b(p) = R + r$ . For technical reasons,  $r$  is to be chosen small. Let  $p_\varepsilon$  a point that is just  $\varepsilon$  further away than  $p$  on the geodesic  $\gamma$  from  $b$  to  $p$ . Let  $r_\varepsilon := r + \varepsilon$ . The barrier we will use is  $\phi_\varepsilon = R + d(\cdot, \partial B(p_\varepsilon, r_\varepsilon))$ . Let  $q$  be the point of  $\gamma$  such that  $d_b(q) = R$ , so that the circles  $\partial B(b, R)$ ,  $\partial B(p, r)$  and  $\partial B(p_\varepsilon, r_\varepsilon)$  are all tangent at  $q$ . We first prove that the circles  $\partial B(b, R)$  and  $\partial B(p, r)$  have the same curvature at  $q$ . This is where we use the fact that the points  $b$  and  $p$  are conjugate. Then, we show that the curvature of  $\partial B(p_\varepsilon, r_\varepsilon)$  is larger than the curvature of  $\partial B(p, r)$  at  $q$ , from which we deduce that the curvature of  $\partial B(p_\varepsilon, r_\varepsilon)$  is larger than the curvature of  $\partial B(b, R)$  at  $q$ . We use this fact to show that we have  $\phi_\varepsilon \geq d_b$  in a neighborhood of  $p$ . Finally, an easy computation will show that  $\Delta\phi_\varepsilon(p) \xrightarrow{\varepsilon \rightarrow 0} -\infty$ , which will conclude the proof.

We will make heavy use of the notion of exponential map of a hypersurface in the proof. Let us recall it and fix some notations.

**Definition 2.2.1.** *Let  $S$  be a smooth oriented hypersurface  $S$  of  $M$ . Let  $\nu_S$  be the unit normal vector to  $S$ . The exponential map of  $S$  is defined by*

$$\begin{aligned} \exp_S : S \times [0, \infty] &\rightarrow M \\ (\theta, t) &\mapsto \text{Exp}(t\nu_S(\theta)) \end{aligned}$$

We will also make heavy use of the notion of normal coordinates. Let us recall this notion and its fundamental properties.

**Proposition and definition 2.2.2.** *For any point  $p \in M$ , the map  $\exp_p : T_pM \rightarrow M$  is a local diffeomorphism from a neighborhood of  $0 \in T_pM$  to a neighborhood of  $p \in M$ . Therefore, after identifying  $T_pM$  with  $\mathbb{R}^n$  by the choice of an orthonormal basis for the metric of  $M$  on  $T_pM$ , the map  $\psi = \exp_p^{-1}$  is a local chart of  $M$  around  $p$ , and so it defines some coordinates  $(x^i)$ . These coordinates are called normal coordinates at  $p$ .*

**Proposition 2.2.3.** *In normal coordinates at  $p$ , the tangent vectors  $\partial_i$  have vanishing covariant derivatives at  $p$ . Moreover, the metric of  $M$  is euclidean at  $p$ , up to order 1. More precisely, the coordinates of the metric  $g$  of  $M$  verify*

$$g^{ij}(p) = \delta^{ij} \quad \text{and} \quad \partial_i g^{ij}(p) = 0.$$

*In particular, recalling the following expression for the Laplacian in coordinates*

$$\Delta u = \frac{1}{\sqrt{\det g}} \partial_i (\sqrt{\det g} g^{ij} \partial_j u),$$

*one can see that in normal coordinates at  $p$  we have  $\Delta u(p) = \sum_i \partial_i^2 u(p)$ . (Please note that the equality holds at  $p$  only.)*

We will use the following lemma in the proof of the theorem. It is a well known fact when the hypersurface  $S$  considered is replaced with a point. We give a proof here, as we could not find it in the literature.

**Lemma 2.2.4.** *Let  $S$  be a smooth hypersurface of  $M$  such that  $\bar{S}$  is compact. Let  $\mu : [0, r] \rightarrow M$  be a geodesic such that  $q := \mu(0) \in S$ . We assume that  $\mu$  is length minimizing between  $p := \mu(r)$  and  $\bar{S}$ , and that  $p$  is not a cut point of  $q$  along  $\mu$ . Then, the exponential map  $\exp_S$*

is a diffeomorphism from a neighborhood  $U(q, r)$  of  $(q, r)$  to a neighborhood  $U(p)$  of  $p$ , and on  $U(p)$  we have

$$d(\cdot, S) = \pi_2 \circ (\exp_{S|_{U(q,r)}})^{-1},$$

where  $\pi_2 : S \times \mathbb{R} \rightarrow \mathbb{R}$  is the projection on the second coordinate. In particular,  $d(\cdot, S)$  is smooth around  $p$ .

*Proof.* As  $p$  is not a cut point of  $q$  along  $\gamma$ , it is not a focal point of  $S$  (see [74, Chapter III, lemma 2.11]). In particular,  $\exp_S$  is a local diffeomorphism from  $(q, r)$  to  $p$ . As  $q$  is the unique closest point to  $p$  on the compact set  $\bar{S}$ , for any neighborhood  $U_S(q)$  of  $q$  in  $S$ , there exists a neighborhood  $U(p)$  of  $p$  such that for any point  $p' \in U(p)$  the closest points to  $p'$  on  $\bar{S}$  are inside  $U_S(q)$ . We choose such  $U(p)$  and  $U_S(q)$  so that, for some  $\varepsilon > 0$ ,  $\exp_S$  is a diffeomorphism from  $U(q, r) := U_S(q) \times (r - \varepsilon, r + \varepsilon)$  to  $U(p)$ . Let  $p' \in U(p)$  and  $r' := d(p', S)$ . A closest point to  $p'$  on  $S$  is a point  $q' \in U_S(q)$  such that  $p' = \exp_S(q', r')$ , and so  $d(p', S) = r' = \pi_2 \circ (\exp_{S|_{U(q,r)}})^{-1}(p')$ . This concludes the proof.  $\square$

*Proof of Theorem 2.1.11.* Let  $p \in \text{Cut}_b(M)$  and  $A > 0$ . We know that either there exist two minimizing geodesics from  $b$  to  $p$ , or there exists a unique minimizing geodesic from  $b$  to  $p$  along which the two points are conjugate.

*Case one.* There exist two minimizing geodesics from  $b$  to  $p$ . This implies that  $d_b$  is not differentiable at  $p$ . Let  $\delta > 0$  be smaller than the distance between  $p$  and  $\text{Cut}_p(M)$ , and  $\psi : B(p, \delta) \rightarrow \mathbb{R}^n$  be a normal coordinate chart around  $p$ . According to [60, Proposition 3.4], there exists  $C > 0$  such that the function  $x \mapsto C|x|^2 - d_b \circ \psi^{-1}(x)$  is convex. As it is not differentiable at 0, it has at least two subgradients  $v$  and  $w$ . We have

$$C|x|^2 - d_b \circ \psi^{-1}(x) \geq \max(v \cdot x, w \cdot x) - d_b(p), \quad (2.2.1)$$

with equality for  $x = 0$ . For  $B > 0$  to be chosen large enough later, let us define the function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  by

$$f(x) := \frac{1}{2}(v \cdot x + w \cdot x) + B(v \cdot x - w \cdot x)^2 - d_b(p).$$

Then, for  $x$  in a neighborhood of 0 we have

$$\begin{aligned} f(x) &= \max(v \cdot x, w \cdot x) - \frac{1}{2}|v \cdot x - w \cdot x| + B(v \cdot x - w \cdot x)^2 - d_b(p) \\ &\leq \max(v \cdot x, w \cdot x) - d_b(p) \end{aligned} \quad (2.2.2)$$

with equality at  $x = 0$ . Setting  $\phi = C|\psi|^2 - f \circ \psi$ , we get from (2.2.1) and (2.2.2) that for  $q$  in a neighborhood of  $p$ ,

$$d_b(q) \leq \phi(q),$$

with equality at  $q = p$ . Moreover, as  $\psi$  is a normal coordinate chart, we have

$$\Delta\phi(p) = \Delta(\phi \circ \psi^{-1})(0) = 2nC - 2B|v - w|^2.$$

As  $B$  is independent of  $v, w$  and  $A$ , we may now chose it large enough to have  $\Delta\phi(p) \leq -A$ , which concludes case one.

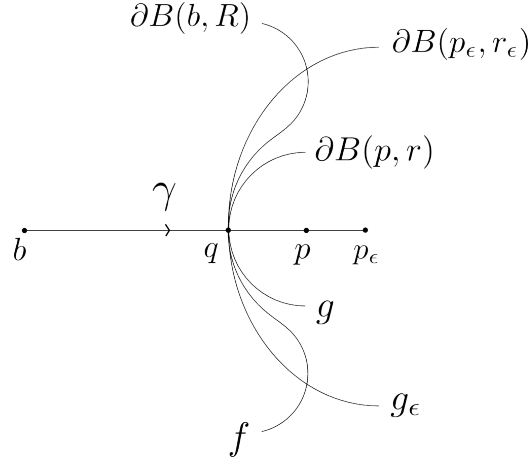


Figure 2.1: Construction of  $r, R, q, p_\varepsilon, r_\varepsilon, f, g, g_\varepsilon$ .

*Case two.* There exists a unique minimizing geodesic  $\gamma$  such that  $\gamma(0) = b$  and  $\gamma(1) = p$ . Let  $r > 0$ , and let  $q$  be the intersection point of  $\gamma$  with the sphere  $\partial B(p, r)$ , and  $R = d_b(q)$ . For  $\varepsilon > 0$ , let us define

$$\begin{aligned} p_\varepsilon &:= \gamma(1 + \varepsilon), \\ r_\varepsilon &:= r + d(p, p_\varepsilon). \end{aligned}$$

We may choose  $r$  and  $\varepsilon$  small enough so that  $p$  and  $p_\varepsilon$  are contained in the totally normal geodesic ball  $B(q, \delta)$  for some  $\delta > 0$  (see [29, Chapter 3, Remark 3.8]). This implies in particular that for any points  $p'$  and  $q'$  in neighborhoods of  $p$  and  $q$  respectively, there exists a unique minimizing geodesic  $\mu_{(p', q')}$  between  $p'$  and  $q'$ , and  $\mu_{(p', q')}$  depends continuously on  $(p', q')$ . Let us fix some normal coordinates at  $q$  on  $B(q, \delta)$  such that the unit normal vector to  $\partial B(b, R)$  at  $q$  is  $\partial_n$ . We will use these normal coordinates throughout the rest of the proof. In these coordinates, around  $q$ ,  $\partial B(b, R)$  is the graph of a smooth function  $f$  such that  $f(0) = 0$  and  $\nabla f(0) = 0$ . Likewise,  $\partial B(p, r)$  and  $\partial B(p_\varepsilon, r_\varepsilon)$  can be seen around  $q$  as the graphs of some smooth functions  $g$  and  $g_\varepsilon$  respectively, such that  $g(0) = g_\varepsilon(0) = 0$  and  $\nabla g(0) = \nabla g_\varepsilon(0) = 0$ . *Step one.* We prove that

$$\exists v \in \mathbb{R}^{n-1}, v \neq 0, \quad v \cdot Hf(0)v = v \cdot Hg(0)v, \quad (2.2.3)$$

where  $Hf$  and  $Hg$  are the Hessian matrix of the functions  $f$  and  $g$ . As the points  $b$  and  $p$  are conjugate, there exists a Jacobi field  $J$  along  $\gamma$  such that  $J(0) = 0$ ,  $J(1) = 0$  and  $J(t) \neq 0$  for  $t \in (0, 1)$ . We may extend it to a global smooth field  $\tilde{J}$  on  $M$ . Let  $(\Phi_{\tilde{J}}^s)_{s \in \mathbb{R}}$  be the flow of this vector field, defined by the following equations:

$$\forall x \in M, \Phi_{\tilde{J}}^0(x) = x \quad \text{and} \quad \forall s \in \mathbb{R}, \frac{d}{ds} \Phi_{\tilde{J}}^s(x) = \tilde{J}(\Phi_{\tilde{J}}^s(x)).$$

We define a variation  $\Gamma$  of the curve  $\gamma$  as follows:

$$\forall s \in \mathbb{R}, \forall t \in [0, 1], \Gamma(s, t) := \Phi_{\tilde{J}}^s(\gamma(t)).$$

For any  $s \in \mathbb{R}$ ,  $\Gamma(s, \cdot)$  is still a curve from  $b$  to  $p$ , as  $\tilde{J}(0) = 0$  and  $\tilde{J}(1) = 0$ . Therefore  $\Gamma$  is a fixed endpoints variation of  $\gamma$ . As  $\gamma$  is a geodesic, we have

$$\frac{d}{ds}[\text{length}(\Gamma(s, \cdot))]_{s=0} = 0.$$

As the variation field  $J$  of  $\Gamma$  is a Jacobi field, we also have (see [56, Proposition 10.14] for instance)

$$\frac{d^2}{ds^2}[\text{length}(\Gamma(s, \cdot))]_{s=0} = 0. \quad (2.2.4)$$

By the implicit function theorem, for any  $s$  close enough to 0, there exist some unique times  $t_R(s)$  and  $t_r(s)$  such that

$$d(\Gamma(s, t_R(s)), b) = R \quad \text{and} \quad d(\Gamma(s, t_r(s)), p) = r.$$

Moreover, the functions  $s \mapsto t_R(s)$  and  $s \mapsto t_r(s)$  are smooth. Now we set

$$P(s) := \Gamma(s, t_R(s)) \quad \text{and} \quad Q(s) := \Gamma(s, t_r(s)).$$

We have

$$\begin{aligned} \text{length}(\Gamma(s, \cdot)) &= \text{length}(\Gamma(s, \cdot)|_{[0, t_R(s)]}) + \text{length}(\Gamma(s, \cdot)|_{[t_R(s), t_r(s)]}) \\ &\quad + \text{length}(\Gamma(s, \cdot)|_{[t_r(s), 1]}) \\ &\geq d(\Gamma(s, 0), \Gamma(s, t_R(s))) + d(\Gamma(s, t_R(s)), \Gamma(s, t_r(s))) \\ &\quad + d(\Gamma(s, t_r(s)), \Gamma(s, 1)) \\ &= R + d(P(s), Q(s)) + r. \end{aligned}$$

Meanwhile, as  $s$  goes to 0, thanks to (2.2.4), we also have

$$\text{length}(\Gamma(s, \cdot)) = \text{length}(\gamma) + o(s^2) = R + r + o(s^2),$$

so  $d(P(s), Q(s)) \leq o(s^2)$ . In our normal coordinates around  $q$ , we have  $d(x, y) \geq c|x - y|$  where  $c > 0$  is a constant and  $|\cdot|$  is the euclidean norm. This comes from the fact that the metric  $g$  of  $M$  and the euclidean metric are locally equivalent in any coordinates system. With this remark, we get

$$P(s) = Q(s) + o(s^2), \quad (2.2.5)$$

where the equality is to be understood in coordinates. For any  $s$  small, let  $x_P(s), x_Q(s) \in \mathbb{R}^{n-1}$  be such that

$$P(s) = (x_P(s), f(x_P(s))) \quad \text{and} \quad Q(s) = (x_Q(s), g(x_Q(s))).$$

The functions  $x_P$  and  $x_Q$  are smooth because  $P$  and  $Q$  are. Using the fact that  $\nabla f(0) = \nabla g(0) = 0$ , (2.2.5) gives

$$x'_P(0) \cdot Hf(0)x'_P(0) = x'_Q(0) \cdot Hg(0)x'_Q(0). \quad (2.2.6)$$

Let us show that  $x'_Q(0) = x'_P(0)$ . We have

$$\begin{aligned} P'(0) &= \partial_s \Gamma(0, t_R(0)) + \partial_t \Gamma(0, t_R(0))t'_R(0) \\ &= J(t_R(0)) + \dot{\gamma}(t_R(0))t'_R(0). \end{aligned} \quad (2.2.7)$$



By definition the curve  $P$  is on the sphere  $\partial B(b, R)$ , so  $P'(0)$  is tangent to this sphere and orthogonal to  $\dot{\gamma}(t_R(0))$  by the Gauss lemma. Also, as  $J(0) = 0$  and  $J(1) = 0$ , the Jacobi field  $J$  is normal to  $\gamma$  (see [56, lemma 10.6]), so  $J(t_R(0))$  is orthogonal to  $\dot{\gamma}(t_R(0))$ . Combined with these facts, (2.2.7) yields  $P'(0) = J(t_R(0))$ . Likewise, we have  $Q'(0) = J(t_r(0)) = J(t_R(0))$ . So  $P'(0) = Q'(0)$ , and consequently  $x'_P(0) = x'_Q(0)$ . Moreover, as  $J(t) \neq 0$  for  $t \in (0, 1)$ , we have  $x'_P(0) \neq 0$ . So setting  $v = x'_P(0)$ , with (2.2.6) we get (2.2.3).

*Step two.* Now we want to show that

$$v \cdot Hg(0)v > v \cdot Hg_\varepsilon(0)v. \quad (2.2.8)$$

Let us first show that  $g \geq g_\varepsilon$  in a neighborhood of 0. Let us argue by contradiction and assume that for some  $x \in \mathbb{R}^{n-1}$ , we have  $g(x) < g_\varepsilon(x)$ . Let  $\mu : [0, 1] \rightarrow M$  be the shortest geodesic from the point of coordinate  $(x, g(x))$  to the point of coordinate  $(0, r)$ , *i.e.*  $p$ . If  $x$  has been taken close enough to 0, then for  $t \in [0, 1]$ ,  $\mu(t)$  stays inside the normal neighborhood of  $q$ ,  $B(q, \delta)$ , on which our normal coordinates are defined. For  $t \in [0, 1]$ , let  $x(t) \in \mathbb{R}^{n-1}$  and  $z(t) \in \mathbb{R}$  be such that in coordinates,  $\mu(t) = (x(t), z(t))$ . We have  $z(0) = g(x) < g_\varepsilon(x) = g_\varepsilon(x(0))$ . And  $z(1) = r > 0 = g_\varepsilon(0) = g_\varepsilon(x(1))$ . As  $\mu$  is continuous, there exists  $t \in (0, 1)$  such that  $z(t) = g_\varepsilon(x(t))$ , *i.e.*  $\mu(t) \in \partial B(p_\varepsilon, r_\varepsilon)$ . This implies

$$\begin{aligned} r_\varepsilon &= d(\mu(t), p_\varepsilon) \\ &\leq d(\mu(t), p) + d(p, p_\varepsilon) \\ &< d(\mu(0), p) + d(p, p_\varepsilon) \\ &= r + d(p, p_\varepsilon) \\ &= r_\varepsilon, \end{aligned}$$

which gives a contradiction. We conclude that  $g \geq g_\varepsilon$ . In particular, for any  $w \in \mathbb{R}^{n-1}$ , we have  $w \cdot Hg(0)w \geq w \cdot Hg_\varepsilon(0)w$ . Thus, the matrix  $Hg(0) - Hg_\varepsilon(0)$  is symmetric non-negative. To show that it is positive definite and conclude that (2.2.8) holds, we only need to show that

$$\forall w \in \mathbb{R}^{n-1}, \quad Hg(0)w \neq Hg_\varepsilon(0)w. \quad (2.2.9)$$

The geodesic  $\gamma$  is minimizing between  $p_\varepsilon$  and  $\partial B(p_\varepsilon, r_\varepsilon)$ . In particular, the point  $p$  is not a cut point of  $q$  along  $\gamma$ , so lemma 2.2.4 above shows that the differential of the map  $\exp_{\partial B(p_\varepsilon, r_\varepsilon)}$  is invertible at  $(q, r)$ . On the contrary, the map  $\exp_{\partial B(p, r)}(\cdot, r)$  is constant and so its differential is null at  $q$ . In particular, for any  $w \in T_q \partial B(p, r)$ , we have

$$D_w \nu_{\partial B(p_\varepsilon, r_\varepsilon)}(q) \neq D_w \nu_{\partial B(p, r)}(q), \quad (2.2.10)$$

where  $D_w$  denotes the covariant derivative in the direction  $w$ . Let us define the families of vectors  $(u_i^g)_{1 \leq i \leq n-1}$  and  $(u_i^{g_\varepsilon})_{1 \leq i \leq n-1}$  at the point of coordinates  $(x, z) \in \mathbb{R}^{n-1} \times \mathbb{R}$ , by  $u_i^g(x, z) := \partial_i + \partial_i g(x) \partial_n$  and  $u_i^{g_\varepsilon}(x, z) := \partial_i + \partial_i g_\varepsilon(x) \partial_n$ . These vectors form a basis of the tangent spaces of  $\partial B(p, r)$  and  $\partial B(p_\varepsilon, r_\varepsilon)$  respectively, as we recall that these surfaces are the graphs of the functions  $g$  and  $g_\varepsilon$ . Furthermore, the two basis are identical at  $(0, 0)$ . Thus, (2.2.10) is equivalent to

$$\exists i \in \{1, \dots, n-1\}, \quad u_i^{g_\varepsilon} \cdot D_w \nu_{\partial B(p_\varepsilon, r_\varepsilon)}(q) \neq u_i^g \cdot D_w \nu_{\partial B(p, r)}(q).$$

Note that if some vector fields  $X$  and  $Y$  are orthogonal, then we have  $X \cdot Y = 0$ , and so  $D_w X \cdot Y + X \cdot D_w Y = 0$ . So we get

$$\begin{aligned} \exists i \in \{1, \dots, n-1\}, D_w u_i^{g_\varepsilon} \cdot \nu_{\partial B(p_\varepsilon, r_\varepsilon)}(q) &\neq D_w u_i^g \cdot \nu_{\partial B(p, r)}(q), \\ \text{i.e. } \exists i \in \{1, \dots, n-1\}, D_w u_i^{g_\varepsilon} \cdot \partial_n &\neq D_w u_i^g \cdot \partial_n. \end{aligned}$$

Recalling that in normal coordinates, the tangent vectors  $(\partial_j)_{1 \leq j \leq n}$  have vanishing covariant derivatives at  $0$ , we get

$$\exists i \in \{1, \dots, n-1\}, w^j \partial_{ji} g_\varepsilon(0) \neq w^j \partial_{ji} g(0),$$

which is (2.2.9). So (2.2.8) holds.

*Step three.* Putting steps one and two together, we get

$$\exists v \neq 0, \quad v \cdot (H g_\varepsilon(0) - H f(0))v < 0.$$

In particular, there exists a basis  $(v_i)_{1 \leq i \leq n-1}$  of  $\mathbb{R}^{n-1}$  with  $v_1 = v$ , that is orthogonal for the quadratic form  $q_\varepsilon(w) = w \cdot (H g_\varepsilon(0) - H f(0))w$ . Let  $k_\varepsilon$  be a quadratic form of  $\mathbb{R}^{n-1}$  such that the  $(v_i)$  are orthogonal for  $k_\varepsilon$ ,  $k_\varepsilon(v) = 0$  and for  $i \geq 2$ ,  $k_\varepsilon(v_i) = \max(0, q_\varepsilon(v_i) + 1)$ . This way we have  $k_\varepsilon > q_\varepsilon$ , and  $k_\varepsilon \geq 0$ . Now let us set  $h_\varepsilon = g_\varepsilon - k_\varepsilon$ . Note that in dimension 2, we actually have  $h_\varepsilon = g_\varepsilon$ , and the rest of the proof is a bit less technical. By construction we have

$$h_\varepsilon \leq g_\varepsilon, \tag{2.2.11}$$

$$h_\varepsilon = g_\varepsilon \quad \text{on } \mathbb{R}v, \tag{2.2.12}$$

$$h_\varepsilon \leq f \quad \text{on } B(0, \rho_\varepsilon), \text{ for some } \rho_\varepsilon > 0, \tag{2.2.13}$$

where (2.2.13) comes from the fact that the Hessian of  $h_\varepsilon$  at  $0$  verifies  $H h_\varepsilon(0) = H g_\varepsilon(0) - k_\varepsilon < H g_\varepsilon(0) - q_\varepsilon = H f(0)$ , and  $h_\varepsilon(0) = f(0) = 0$ ,  $\nabla h_\varepsilon(0) = \nabla f(0) = 0$ . It will be convenient, at the end of the proof, to have a smooth function  $h : \mathbb{R}^{n-1} \rightarrow \mathbb{R}$  such that

$$h(0) = 0, \quad \nabla h(0) = 0, \quad \text{and for all } \varepsilon > 0 \text{ sufficiently small, } \quad h_\varepsilon \geq h. \tag{2.2.14}$$

To see that such function exists, we note that for  $\varepsilon < \varepsilon'$ , we have  $g_\varepsilon \geq g_{\varepsilon'}$ . This can be proven the same way we proved  $g_\varepsilon \leq g$  at the beginning of step two. We then fix  $\varepsilon' > 0$ . Therefore, for any  $\varepsilon$  sufficiently small, we have  $g_{\varepsilon'} \leq g_\varepsilon \leq g$ . This allows us to see that the quadratic forms  $q_\varepsilon$ , and then  $k_\varepsilon$ , are bounded independently of  $\varepsilon$ , by a quadratic form  $k$ . In turn, this implies that the function  $h = g_{\varepsilon'} - k$  verifies (2.2.14). Let  $S_{h_\varepsilon}$  be the hypersurface of  $M$  defined in our normal coordinates around  $q$  as the graph of the function  $h_\varepsilon$  on  $B(0, \rho_\varepsilon)$ . Let us set

$$\phi_\varepsilon = d(\cdot, S_{h_\varepsilon}) + R.$$

We will show that for  $\varepsilon$  sufficiently small,  $\phi_\varepsilon$  is the function  $\phi$  we are looking for to prove the theorem.

Let us first show that  $\phi_\varepsilon \geq d_b$  in a neighborhood of  $p$ . Let  $p' \in B(q, \delta)$  be a point near  $p$ . Let  $\mu : [0, 1] \rightarrow M$  be a length minimizing geodesic between  $S_{h_\varepsilon}$  and  $p'$ . For  $t \in [0, 1]$ , let  $(x(t), z(t)) \in \mathbb{R}^{n-1} \times \mathbb{R}$  be the coordinates of  $\mu(t)$ . As  $\mu(0) \in S_{h_\varepsilon}$ , we have  $z(0) = h_\varepsilon(x(0)) \leq f(x(0))$ . Moreover, the coordinates  $(x, z) = (0, r)$  of  $p$  verify  $z > f(x)$ . Therefore, provided

$p'$  is close enough to  $p$ , we have  $z(1) \geq f(x(1))$ . Thus, there exists  $t \in [0, 1]$  such that  $z(t) = f(x(t))$ , *i.e.*  $\mu(t) \in \partial B(b, R)$ . This implies

$$d(\cdot, S_{h_\varepsilon}) \geq d(\cdot, \partial B(b, R)) \quad \text{on a neighborhood of } p, \quad (2.2.15)$$

and so  $\phi_\varepsilon \geq d_b$  on a neighborhood of  $p$ .

As  $q \in S_{h_\varepsilon}$ , we also have  $\phi_\varepsilon(p) \leq d(p, q) + R = d_b(p)$ . So we get  $\phi_\varepsilon(p) = d_b(p)$  as well.

We are left to show that  $\phi_\varepsilon$  is smooth around  $p$ , and that given  $A > 0$ , we have  $\Delta\phi_\varepsilon(p) \leq -A$ , provided  $\varepsilon$  has been taken small enough. Let us first show that  $\phi_\varepsilon$  is smooth around  $p$ . Between  $p_\varepsilon$  and  $q$ , the geodesic  $\gamma$  is minimizing among all geodesics from  $p_\varepsilon$  to  $\partial B(p_\varepsilon, r_\varepsilon)$ . The same way we have shown that  $h_\varepsilon \leq f$  implies  $d(\cdot, S_{h_\varepsilon}) \geq d(\cdot, \partial B(b, R))$ , one can show that  $h_\varepsilon \leq g_\varepsilon$  implies

$$d(\cdot, S_{h_\varepsilon}) \geq d(\cdot, \partial B(p_\varepsilon, r_\varepsilon)) \quad \text{on a neighborhood of } p. \quad (2.2.16)$$

Therefore, between  $q$  and  $p_\varepsilon$ ,  $\gamma$  is still minimizing among geodesics from  $p_\varepsilon$  to  $S_{h_\varepsilon}$ . We may apply lemma 2.2.4 to conclude that  $\phi_\varepsilon$  is smooth at  $p$ . Now for  $t$  in a neighborhood of 0, let us define  $c(t) := \exp_{\partial B(p_\varepsilon, r_\varepsilon)}(r, \exp_q(tv, g_\varepsilon(tv)))$ . We have,

$$\begin{aligned} d(c(t), S_{h_\varepsilon}) &\geq d(c(t), \partial B(p_\varepsilon, r_\varepsilon)) \quad \text{because of (2.2.16),} \\ &= d(c(t), \exp_q(tv, g_\varepsilon(tv))) \quad \text{because of lemma 2.2.4,} \\ &= d(c(t), \exp_q(tv, h_\varepsilon(tv))) \quad \text{because of (2.2.12),} \\ &\geq d(c(t), S_{h_\varepsilon}). \end{aligned}$$

Therefore, the first inequality is actually an equality:

$$d(c(t), S_{h_\varepsilon}) = d(c(t), \partial B(p_\varepsilon, r_\varepsilon)).$$

As  $d(\cdot, S_{h_\varepsilon})$  and  $d(\cdot, \partial B(p_\varepsilon, r_\varepsilon))$  have the same gradient at  $c(0) = p$ , we deduce that

$$D_{\dot{c}(0), \dot{c}(0)}^2[d(\cdot, S_{h_\varepsilon})]_p = D_{\dot{c}(0), \dot{c}(0)}^2[d(\cdot, \partial B(p_\varepsilon, r_\varepsilon))]_p.$$

We have  $\dot{c}(0) \neq 0$  because of lemma 2.2.4 again. Setting  $w_1 := \dot{c}(0)/|\dot{c}(0)|$ , we get

$$D_{w_1, w_1}^2[d(\cdot, S_{h_\varepsilon})]_p = D_{w_1, w_1}^2[d(\cdot, \partial B(p_\varepsilon, r_\varepsilon))]_p. \quad (2.2.17)$$

Let us complete  $w_1$  into an orthonormal basis  $(w_1, \dots, w_n)$  of  $T_pM$ . Let  $S_h$  be the hypersurface of  $M$  defined in our normal coordinates at  $q$  as the graph of the function  $h$  from (2.2.14). The same way we have proved (2.2.15), one can prove that  $d(\cdot, S_{h_\varepsilon}) \leq d(\cdot, S_h)$ . As these two functions are equal up to order 1 at  $p$ , we deduce that

$$D_{w_i, w_i}^2[d(\cdot, S_{h_\varepsilon})]_p \leq D_{w_i, w_i}^2[d(\cdot, S_h)]_p.$$

Moreover, we know from [60, Proposition 3.4] that there exists a constant  $C > 0$  such that for all  $i$ ,  $D_{w_i, w_i}^2[d(\cdot, S_h)]_p \leq C$ , so we get

$$\forall i \geq 2, D_{w_i, w_i}^2[d(\cdot, S_{h_\varepsilon})]_p \leq C,$$

where  $C$  is independent of  $\varepsilon$ . Combining this inequality with (2.2.17), we get

$$\Delta\phi_\varepsilon(p) \leq (n-1)C + D_{w_1, w_1}^2 [d(\cdot, \partial B(p_\varepsilon, r_\varepsilon))]_p. \quad (2.2.18)$$

Let us pick some normal coordinates  $(x^i)_{1 \leq i \leq n}$  at  $p_\varepsilon$ , such that we have  $\partial_n = \dot{\gamma}$  on the curve  $\gamma$ . By the Gauss lemma, the set of points that are equidistant to  $\partial B(p_\varepsilon, r_\varepsilon)$  are orthogonal to the geodesics starting out orthogonally from  $\partial B(p_\varepsilon, r_\varepsilon)$ . Therefore we have  $\dot{c}(0) \cdot \dot{\gamma}(1) = 0$ , or equivalently  $w_1 \cdot \partial_n = 0$ . In particular, if  $(w_1^i)$  are the coordinates of  $w$  relatively to the normal coordinates  $(x^i)$  at  $p_\varepsilon$ , we have  $w_1^n = 0$ . In these coordinates we have  $d(x, \partial B(p_\varepsilon, r_\varepsilon)) = r_\varepsilon - |x|$ , so

$$\begin{aligned} D_{w_1, w_1}^2 [d(\cdot, \partial B(p_\varepsilon, r_\varepsilon))](x) &= -D_{w_1, w_1}^2 |x| \\ &= -w_1^i w_1^j \left( \partial_{ij} |x| - \Gamma_{ij}^k \partial_k |x| \right) \\ &= -w_1^i w_1^j \left( -\frac{\delta_{ij}}{|x|} + \frac{x_i x_j}{|x|^3} + \Gamma_{ij}^k(x) \frac{x_k}{|x|} \right), \end{aligned}$$

where the  $(\Gamma_{ij}^k)$  are the Christoffel symbols. We apply this formula at the point  $p$  of coordinates  $(0, \dots, 0, -d(p, p_\varepsilon))$ , and obtain

$$D_{w_1, w_1}^2 [d(\cdot, \partial B(p_\varepsilon, r_\varepsilon))](p) = \frac{w_1^i w_1^j \delta_{ij}}{d(p, p_\varepsilon)} - w_1^i w_1^j \Gamma_{ij}^n(p). \quad (2.2.19)$$

As we are in normal coordinates, there exist a continuous function  $\rho : [0, +\infty) \rightarrow [0, +\infty)$  such that  $\rho(0) = 0$  and

$$\left| \Gamma_{ij}^k(x) \right| \leq \rho(|x|) \quad \text{and} \quad |g^{ij}(x) - \delta_{ij}| \leq |x| \rho(|x|).$$

Moreover, note that we only consider coordinates centered at some points  $p_\varepsilon$  contained in a bounded neighborhood of  $p$ , so the function  $\rho$  is independent of  $\varepsilon$ . With these remarks, (2.2.19) yields

$$\left| D_{w_1, w_1}^2 [d(\cdot, \partial B(p_\varepsilon, r_\varepsilon))](p) - \frac{|w|^2}{d(p, p_\varepsilon)} \right| \leq 2 \left( \sum_i w_1^i \right)^2 \rho(d(p, p_\varepsilon)).$$

As  $\varepsilon \rightarrow 0$ , the right hand side goes to 0, and  $\frac{|w|^2}{d(p, p_\varepsilon)} = \frac{1}{d((p, p_\varepsilon))} \rightarrow +\infty$ , so

$$D_{w_1, w_1}^2 [d(\cdot, \partial B(p_\varepsilon, r_\varepsilon))](p) \rightarrow -\infty,$$

which, together with (2.2.18), concludes the proof.  $\square$



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## Cut locus and variational problem

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### 3.1 Introduction

In this chapter, we will use the same notations as in Chapter 2, but our ambient manifold  $M$  is assumed to be compact. We recall that  $n \geq 2$  is an integer,  $M$  a smooth  $n$ -dimensional compact Riemannian manifold without boundary,  $b \in M$  any point of  $M$  (that can be thought of as a base point), and  $d_b$  the distance function to the point  $b$  in  $M$ .

**Definition 3.1.1.** *The cut locus of  $b$  in  $M$  can be defined as the closure of the set of points  $p \in M$  such that there exists at least two minimizing geodesics between  $p$  and  $b$ . We will denote it by  $Cut_b(M)$ . Equivalently, it is also the set of points around which the distance function to the point  $b$  - denoted by  $d_b$  - is not smooth.*

We refer to the introduction of Chapter 2 for a more detailed introduction to the notion of cut locus.

The goal of this Chapter is to study an extension of the elastic plastic torsion problem to manifolds, and its relation with the cut locus of  $b$  in  $M$ . We are interested in the following variational problem.

$$\inf \left\{ \int_M |\nabla u|^2 - mu : u \in H^1(M), |\nabla u| \leq 1, u(b) = 0 \right\}, \quad (3.1.1)$$

where  $m > 0$  is a given constant. We extend the results for the elastic-plastic torsion problem presented in Chapter 1. Our main results are summed up in Theorem 3.1.2 below. For the reader's convenience and in order to make this Chapter relatively self-contained, we will recall some notions and results from the preceding chapters. The elastic-plastic torsion problem is defined as follows:

$$\inf \left\{ \int_{\Omega} |\nabla v|^2 - mv : v \in H_0^1(\Omega), |\nabla v| \leq 1 \right\}, \quad (1.1.1)$$

where  $\Omega$  is a smooth open subset of  $\mathbb{R}^n$ .

Notice that both the functionals and the constraints in (3.1.1) and (1.1.1) are strictly convex. As a consequence, the two problems (3.1.1) and (1.1.1) have a unique minimizer. Let us call them  $u_m$  and  $v_m$  respectively. The elastic-plastic torsion problem and the properties

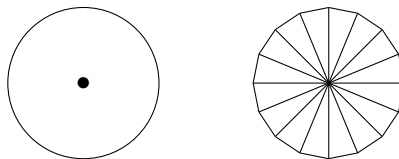


Figure 3.1: A circle with its medial axis, and a polygonal approximation of a circle, with its medial axis.

of its minimizer  $v_m$  have been intensively studied by various authors in the 60's and 70's (see Chapter 1 for some references). In particular, it was proved in [21] that  $v_m$  is locally  $C^{1,1}$ . What is more, it was proved in [14] that the gradient constraint in (1.1.1) can be replaced with an obstacle-type constraint where the obstacle is the distance function to the boundary of  $\Omega$ , *i.e.* the function  $d_{\partial\Omega}$  defined by

$$d_{\partial\Omega}(x) := \min \{|x - y| : y \in \partial\Omega\}.$$

Indeed, we have

$$v_m \leq d_{\partial\Omega} \quad \text{and} \quad v_m = \arg \min \left\{ \int_{\Omega} |\nabla v|^2 - mv : v \in H_0^1(\Omega), v \leq d_{\partial\Omega} \right\}. \quad (3.1.2)$$

One may ask if an analog of (3.1.2) is also true for  $u_m$ , with  $d_{\partial\Omega}$  replaced with  $d_b$ . Interestingly, it is not. We build a counterexample in Theorem 3.3.1. However, we prove that the equality holds if  $m$  is large enough, depending on the Ricci curvature of  $M$ . This is Theorem 3.3.2. The technique of the proof is different from the one in [14], which was specific to the flat case.

The *medial axis* of  $\Omega$ , denoted by  $\mathcal{M}(\Omega)$ , is defined as the set of points of  $\Omega$  that have at least two closest points on the boundary  $\partial\Omega$  of  $\Omega$ :

$$\mathcal{M}(\Omega) := \{x \in \Omega : \exists y, z \in \partial\Omega, y \neq z \text{ and } d_{\partial\Omega}(x) = |x - y| = |x - z|\}.$$

Equivalently,  $\mathcal{M}(\Omega)$  is the set of points of  $\Omega$  at which the distance function  $d_{\partial\Omega}$  is not differentiable. In [18], the authors proved that for any  $m > 0$ ,  $\overline{\mathcal{M}(\Omega)}$  is contained in the set  $\{|\nabla v_m| < 1\}$ , and that the latter converges to the former in the Hausdorff sense as  $m$  goes to  $+\infty$ . We prove an analogous result for problem (3.1.1), where the cut locus plays the role of  $\overline{\mathcal{M}(\Omega)}$ , see Theorem 3.1.2 below.

One important geometric property of the medial axis  $\mathcal{M}(\Omega)$  is that it is unstable with respect to small perturbations of the boundary of  $\Omega$  - see Figure 3.1. This instability makes computing numerically  $\mathcal{M}(\Omega)$  quite tricky. Indeed, any numerical approximation of  $\Omega$  (for instance, with polygons) might introduce an artificial (and large) medial set. In order to deal with this problem, in [23], Chazal and Lieutier defined the so called  $\lambda$ -*medial axis* of  $\Omega$ , denoted by  $\mathcal{M}_\lambda(\Omega)$ . The latter can be seen to verify

$$\mathcal{M}_\lambda(\Omega) = \left\{ x \in \Omega : |\nabla d_{\partial\Omega}(x)|^2 \leq 1 - \frac{\lambda^2}{d_{\partial\Omega}^2(x)} \right\}, \quad (3.1.3)$$

where  $\nabla d_{\partial\Omega}$  denotes the generalized gradient wherever  $d_{\partial\Omega}$  is not differentiable. We have the formula

$$|\nabla d_{\partial\Omega}|(x) = \max(0, \sup_{v \in S^{n-1}} \partial_v^+ d_{\partial\Omega}(x)).$$

It is proved in [23] that  $\mathcal{M}_\lambda(\Omega)$  is a good approximation of  $\mathcal{M}(\Omega)$ , for  $\lambda > 0$  small enough. The crucial difference though is that  $\mathcal{M}_\lambda(\Omega)$  is stable with respect to small variations of the boundary of  $\Omega$ , whereas  $\mathcal{M}(\Omega)$  is not. We refer the reader to [23, Section 4] for precise statements and proofs (see also Section 1.2 for more details). This means that although it is difficult to compute numerically  $\mathcal{M}(\Omega)$  directly, one can compute  $\mathcal{M}_\lambda(\Omega)$  instead.

Motivated by these facts, we proved the following in Chapter 1 – see Theorem 1.3.1. For  $\lambda > 0$ , let us define the set  $E_{m,\lambda}$  by

$$E_{m,\lambda} := \left\{ x \in \Omega : |\nabla v_m(x)|^2 \leq 1 - \frac{\lambda^2}{v_m^2(x)} \right\}.$$

Then, we have

$$\sup_{x \in E_{m,\lambda}} d(x, \mathcal{M}_\lambda(\Omega)) \xrightarrow{m \rightarrow +\infty} 0 \quad \text{and} \quad \sup_{x \in \mathcal{M}_{\lambda+\varepsilon}(\Omega)} d(x, E_{m,\lambda}) \xrightarrow{m \rightarrow +\infty} 0, \quad \text{for any } \varepsilon > 0.$$

While the medial axis of  $\Omega$  is unstable with respect to non-smooth variation of the boundary of  $\Omega$ , the cut locus is unstable with respect to non-smooth variations of the metric of  $M$ . Indeed, it is proved in [4] that the cut locus is unstable for some  $C^1$  variations of the metric of  $M$  (but stable for  $C^2$  variations of the metric of  $M$ ). So it is a natural question to try and compute a stable approximation of  $Cut_b(M)$ . In the following, we prove an analogous result to Theorem 1.3.1 for problem (3.1.1). This result will then be used in Chapter 4 to compute a stable approximation of the cut locus. Before stating it, let us first define semiconcavity for functions on a manifold. A function  $u : M \rightarrow \mathbb{R}$  is said to be  $C$ -semiconcave if for any unit speed geodesic  $\gamma$ , the function  $t \mapsto Ct^2 - u(\gamma(t))$  is convex. We know from [60, Proposition 3.4] that the function  $d_b$  is locally semiconcave on  $M \setminus \{b\}$ . This implies that, as convex functions,  $d_b$  has a well defined generalized gradient at every point – see Appendix A.1, Proposition A.1.6. At a point  $p$  where  $d_b$  is not differentiable, the norm of its gradient is given by

$$|\nabla d_b|(p) := \max(0, \sup_{v \in T_x M, |v|=1} \partial_v^+ d_b(p)). \quad (3.1.4)$$

As shown in Appendix A.1, Proposition A.1.10, if there exist two minimizing geodesics from  $p$  to  $b$ , then  $|\nabla d_b(p)| < 1$ . In particular, we have

$$Cut_b(M) = \overline{\{|\nabla d_b| < 1\}}. \quad (3.1.5)$$

Following what has been done for the medial axis, we define a  $\lambda$ -cut locus by

$$Cut_b(M)_\lambda := \left\{ x \in M : |\nabla d_b(x)|^2 \leq 1 - \frac{\lambda^2}{d_b^2(x)} \right\}, \quad \text{for } \lambda > 0. \quad (3.1.6)$$

Recall that  $u_m$  denotes the minimizer of (3.1.1). We set

$$E_m := \{x \in M : |\nabla u_m(x)| < 1\},$$

$$\text{and } E_{m,\lambda} := \left\{ x \in M : |\nabla u_m(x)|^2 \leq 1 - \frac{\lambda^2}{u_m^2(x)} \right\}. \quad (3.1.7)$$

Our main results are summed up in the following theorem.



**Theorem 3.1.2.** *There exists  $m_0 > 0$ , depending only on the manifold  $M$ , such that for any  $m > m' > m_0$ , the minimizer  $u_m$  of (3.1.1) is locally  $C^{1,1}$  on  $M \setminus \{b\}$ , and*

$$\text{Cut}_b(M) \subset \{|\nabla u_m| < 1\} \subset \{|\nabla u_{m'}| < 1\}. \quad (3.1.8)$$

Moreover,

$$E_m \xrightarrow{m \rightarrow +\infty} \text{Cut}_b(M) \quad \text{in the Hausdorff sense.} \quad (3.1.9)$$

Given any  $\varepsilon > 0$ , there exists a constant  $C > 0$  such that, for any  $m > m_0$ ,

$$u_m \text{ is } C\text{-semiconcave on } M \setminus B(b, \varepsilon). \quad (3.1.10)$$

Finally,

$$\sup_{x \in E_{m,\lambda}} d(x, \text{Cut}_b(M)_\lambda) \xrightarrow{m \rightarrow +\infty} 0, \quad \text{and} \quad \sup_{x \in \text{Cut}_b(M)_{\lambda+\varepsilon}} d(x, E_{m,\lambda}) \xrightarrow{m \rightarrow +\infty} 0. \quad (3.1.11)$$

In addition to its own theoretical interest, this theorem is used in Chapter 4 to compute numerically a stable approximation of the cut locus of a closed surface.

*Remark 3.1.3.* We could have replaced the point  $b$  with a smooth hypersurface  $S \subset M$  in the above problem. However, our interest here is in the instability of the cut locus with respect to the metric of  $M$ , and this aspect is already present with a point  $b$  instead of  $S$ . So for simplicity, we focus on the case of the cut locus of a point.

The rest of the Chapter is organized as follows. In Section 3.2, we prove the  $C^{1,1}$  regularity of  $u_m$ . In Section 3.3, we prove that the analogous identity to (3.1.2) does not hold in general for problem (3.1.1), but it holds for  $m$  large enough. In Section 3.4, we prove the assertions (3.1.8) and (3.1.9) of Theorem 3.1.2. In Section 3.5, we prove the assertions (3.1.10) and (3.1.11) of Theorem 3.1.2.

## 3.2 Regularity

Let  $u_m^d$  ('d' is for the 'distance' constraint) be the minimizer of the following strictly convex variational problem:

$$\inf \left\{ \int_M |\nabla u|^2 - mu : u \in H^1(M), u \leq d_b \right\}. \quad (3.2.1)$$

It is the same problem as (3.1.1), where the constraint on the gradient has been replaced with an obstacle-type constraint where the obstacle is the distance function  $d_b$ . In this section, we prove the  $C^{1,1}$  regularity of  $u_m^d$ . In the next section, we will prove that for  $m$  large enough we have  $u_m^d = u_m$ . Thanks to Theorem 2.1.11, we can prove the following.

**Lemma 3.2.1.** *For any  $m > 0$ , there exists a function  $\tilde{d}_b$  that is smooth on  $M \setminus \{b\}$ , such that  $u_m^d \leq \tilde{d}_b \leq d_b$  on  $M$  and  $\tilde{d}_b < d_b$  on  $\text{Cut}_b(M)$ . In particular,  $u_m^d$  is also the solution of*

$$\inf \left\{ \int_M |\nabla u|^2 - mu : u \in H^1(M), u \leq \tilde{d}_b \right\}. \quad (3.2.2)$$

*Proof.* It is a consequence of the minimality of  $u_m$  in (3.2.1) that we have in the sense of distributions:

$$\Delta u_m \geq -m.$$

This implies, from the theory of subharmonic distributions, that  $u_m$  has an upper semi-continuous representative. It is proved in [47, section 9], in the present setting of manifolds. For a reference about subharmonic functions in the euclidean space, see for instance [26].

Let  $x \in \text{Cut}_b(M)$ . According to Theorem 2.1.11 we can find a smooth function  $\phi$  defined on an open neighborhood  $N$  of  $x$  such that:

$$\phi \geq d_b, \quad \phi(x) = d_b(x) \quad \text{and} \quad \Delta \phi \leq -m - 1.$$

Then the function  $u_m - \phi$  is non-positive strictly subharmonic on  $N$  and so by the maximum principle for subharmonic functions, we have

$$(u_m - \phi)(x) < 0, \quad \text{i.e.} \quad u_m(x) < d_b(x).$$

Thus, the function  $u_m - d_b$  is upper semi-continuous and negative on the compact set  $\text{Cut}_b(M)$ , so there exists  $\varepsilon > 0$  such that the set  $\{u_m < d_b - \varepsilon\}$  is an open set containing  $\text{Cut}_b(M)$ . Now, choose any smooth bump function  $\rho : M \rightarrow [0, 1]$  that is compactly supported in  $\{u_m < d_b - \varepsilon\} \setminus \{b\}$  and such that  $\rho = 1$  on  $\text{Cut}_b(M)$ , and a smooth function  $d_{b,\varepsilon} : M \rightarrow \mathbb{R}$  such that

$$d_b - \varepsilon \leq d_{b,\varepsilon} < d_b. \tag{3.2.3}$$

We set

$$\tilde{d}_b := (1 - \rho)d_b + \rho d_{b,\varepsilon}. \tag{3.2.4}$$

This function is smooth on  $M \setminus b$  because  $d_b$  is smooth on the complement of  $\text{Cut}_b(M) \cup \{b\}$ . The condition  $\tilde{d}_b \leq d_b$  is a consequence of  $d_{b,\varepsilon} < d_b$ . The inequality  $\tilde{d}_b < d_b$  on  $\text{Cut}_b(M)$  is also a consequence of  $d_{b,\varepsilon} < d_b$ , plus the fact that  $\rho = 1$  on  $\text{Cut}_b(M)$ . We are left to show that  $u_m \leq \tilde{d}_b$ . On the complement of the support of  $\rho$ , we have  $\tilde{d}_b = d_b$ , so  $u_m \leq \tilde{d}_b$ . On the support of  $\rho$ , (3.2.4) and (3.2.3) imply

$$u_m \leq d_b - \varepsilon \leq d_{b,\varepsilon} = (1 - \rho)d_{b,\varepsilon} + \rho d_{b,\varepsilon} \leq (1 - \rho)d_b + \rho d_{b,\varepsilon} = \tilde{d}_b,$$

This concludes the proof. □

One could adapt to the manifold framework the regularity theorems for the classical obstacle problem on a euclidean domain and, with the preceding lemma, deduce the regularity of  $u_m^d$ . Rather than doing that, we will use lemma 3.2.1 to reduce our problem to a classical obstacle problem on a euclidean domain. Let us start with the following regularity lemma.

**Lemma 3.2.2.** *For any  $m > 0$ , the function  $u_m^d$  is continuous on  $M$ .*

*Proof.* We will reduce our problem to a classical obstacle type variational problem on an open subset of  $\mathbb{R}^n$ , by a series of elementary modifications, and apply a classical  $W^{2,p}$  regularity theorem.

From lemma 3.2.1, we know that there exists an open set  $U \subset M$  and  $\varepsilon > 0$  such that  $\text{Cut}_b(M) \subset U$  and  $u_m^d \leq d_b - \varepsilon$  on  $U$ . On  $U$ ,  $u_m^d$  verifies the Euler-Lagrange equation of

(3.2.1), i.e.  $\Delta u_m^d = -2m$ . In particular, it is smooth on  $U$ . Let  $\Omega \subset M$  be a smooth open set such that

$$U^c \subset \Omega, \quad \partial\Omega \subset U \quad \text{and} \quad \text{Cut}_b(M) \cap \bar{\Omega} = \emptyset.$$

As  $U^c \subset \Omega$ , it suffices to show that  $u_m^d$  is continuous on  $\Omega$ . As  $\partial\Omega \subset U$ ,  $u_m^d$  is smooth on  $\partial\Omega$ , so there exists a smooth function  $v_m$  on  $\bar{\Omega}$  such that  $v_m = u_m^d$  on  $\partial\Omega$ . Then, one can check that  $u_m^d$  is a solution of the following variational problem:

$$\inf \left\{ \int_{\Omega} |\nabla u|^2 - mu : u \in H^1(\Omega), u \leq d_b, u|_{\partial\Omega} = v_m|_{\partial\Omega} \right\}.$$

As a consequence,  $u_m^d - v_m$  is a solution of the following variational problem:

$$\inf \left\{ \int_{\Omega} |\nabla v|^2 - (m + \Delta v_m)v : v \in H_0^1(\Omega), v \leq d_b - v_m \right\}. \quad (3.2.5)$$

Because we have  $\text{Cut}_b(M) \cap \bar{\Omega} = \emptyset$ , the exponential map at  $b$  is a diffeomorphism onto  $\Omega$ . Let  $\phi : \Omega \rightarrow \tilde{\Omega} \subset \mathbb{R}^n$  be a normal coordinates chart centered at  $b$ . Let  $g = (g^{ij})$  denotes the metric of  $M$  in the coordinates defined by  $\phi$ , and  $\det g$  its determinant. We recall that the Riemannian volume measure is given in coordinates by  $\sqrt{\det g} dx$ . So we have

$$\begin{aligned} \int_{\Omega} (|\nabla v|^2 - (m + \Delta v_m)v) &= \int_{\tilde{\Omega}} \left( g^{ij} \partial_i (v \circ \phi^{-1}) \partial_j (v \circ \phi^{-1}) \sqrt{\det g} \right. \\ &\quad \left. - (m + \Delta v_m) \circ \phi^{-1} v \circ \phi^{-1} \sqrt{\det g} \right), \end{aligned}$$

so  $(u_m^d - v_m) \circ \phi^{-1}$  is a minimizer of

$$\inf \left\{ \int_{\tilde{\Omega}} g^{ij} \sqrt{\det g} \partial_i w \partial_j w - Fw : w \in H_0^1(\tilde{\Omega}), w \leq \psi \right\}, \quad (3.2.6)$$

where we have set  $\psi := (d_b - v_m) \circ \phi^{-1}$  and  $F := (m + \Delta v_m) \circ \phi^{-1} \sqrt{\det g}$ . We want to apply [81, Theorem 4.32]. For this we need to write the above variational problem into a variational inequality. Let  $w$  be a competitor in (3.2.6). Writing down the minimality of  $w_m := (u_m^d - v_m) \circ \phi^{-1}$  against the competitor  $w_m + t(w - w_m)$ , for  $t \in (0, 1)$  small, we find that

$$\langle Aw_m, w_m - w \rangle \geq \langle F, w_m - w \rangle,$$

where  $A$  is the elliptic operator defined on  $H_0^1(\tilde{\Omega})$  by  $Aw := -\partial_j (g^{ij} \sqrt{\det g} \partial_i w)$ . From there, we can apply [81, Theorem 4.32] to deduce that, for any  $p < n$ , if  $\min(A\psi, F) \in L^p(\tilde{\Omega})$ , then  $Aw_m \in L^p(\tilde{\Omega})$ . To check that  $\min(A\psi, F) \in L^p(\tilde{\Omega})$ , it is enough to check that  $A(d_b \circ \phi^{-1}) \in L^p(\tilde{\Omega})$ . As  $d_b$  is smooth except at  $b$ , it is enough to check that  $(A(d_b \circ \phi^{-1}))^p$  is integrable at 0. But this is a consequence of the fact that  $-\Delta d_b \circ \phi^{-1} = \frac{1}{\sqrt{\det g}} A(d_b \circ \phi^{-1})$ , and lemma 3.2.5 below, from which we deduce that  $A(d_b \circ \phi^{-1})(x)$  is equivalent to  $\frac{n-1}{|x|}$  when  $x$  goes to 0. Therefore, for  $p < n$ ,  $(A(d_b \circ \phi^{-1}))^p$  is integrable at 0, and so  $Aw_m \in L^p(\tilde{\Omega})$ . By elliptic regularity, this implies  $w_m \in W^{2,p}(\tilde{\Omega})$ , for any  $p < n$ . By the Sobolev embeddings,  $w_m$  is then continuous on  $\tilde{\Omega}$ , and so  $u_m^d$  is continuous on  $\Omega$ . This concludes the proof.  $\square$

We can now define the set  $E_m^d := \{u_m^d < d_b\}$ , for any  $m > 0$ . It is an open subset of  $M$ , on which  $u_m^d$  verifies the Euler-Lagrange equation of problem (3.1.1):  $\Delta u_m^d = -2m$ . We can now prove the following lemma.

**Lemma 3.2.3.** *For any  $m > 0$ , we have  $u_m^d = d_b$  in a neighborhood of  $b$ .*

*Proof.* Let us assume that we have constructed a  $C^1$  function  $v$  on  $\overline{B}(b, R)$  for some  $R > 0$ , such that

$$\begin{cases} v \leq d_b & \text{in } B(b, R), & (3.2.7) \\ v = d_b & \text{in } B(b, \varepsilon) \text{ for some } \varepsilon \in (0, R), & (3.2.8) \\ v < 0 & \text{in } \partial B(b, R), & (3.2.9) \\ \Delta v \geq -m & \text{in } B(b, R) \text{ in the distributional sense.} & (3.2.10) \end{cases}$$

We will then show that we have  $u_m^d \geq v$ . The construction of  $v$  is postponed to the end of the proof. From Lemma 3.2.2, we know that the function  $v - u_m^d$  is continuous. Let us first assume that  $v - u_m^d$  attains a positive maximum at a point  $x \in \overline{B}(b, R)$ . We have

$$0 < v(x) - u_m^d(x) \leq d_b(x) - u_m^d(x),$$

so  $x \in E_m^d$ . Moreover, we have  $u_m^d \geq 0$  since  $\max(u_m^d, 0)$  is a better competitor than  $u_m^d$  in (3.2.1), so

$$v - u_m^d \leq v < 0 \quad \text{on} \quad \partial B(b, R),$$

and so  $x \in B(b, R)$ . Hence the function  $v - u_m^d$  attains a positive maximum inside the open set  $E_m^d \cap B(b, R)$ , but its Laplacian verifies in the distributional sense:

$$\Delta(v - u_m^d) = \Delta v + m \geq 0, \quad (3.2.11)$$

which yields a contradiction by the maximum principle. Then, the maximum of  $v - u_m^d$  on  $\overline{B}(b, R)$  is non-positive, and we get

$$u_m^d \geq v = d_b \quad \text{in} \quad B(b, \varepsilon),$$

which concludes the proof.

Let us now construct the function  $v$  that was used above. Let  $R > 0$  be small enough so that  $\overline{B}(b, R)$  is contained in a normal neighborhood of  $b$ . In polar coordinates around  $b$ , we define  $v$  as a radial function. For  $\varepsilon > 0$  to be chosen small enough later, let  $f : [0, R] \rightarrow [0, \infty)$  be the  $C^1$  function such that

$$\begin{cases} f(r) = r & \text{if } r \leq \varepsilon, \\ f''(r) + \frac{n-1}{r} f'(r) = -\frac{m}{2} & \text{if } r > \varepsilon. \end{cases} \quad (3.2.12)$$

If  $n = 2$ , the unique  $C^1$  solution to this system is given by:

$$\begin{cases} f(r) = r & \text{if } r \leq \varepsilon, \\ f(r) = \varepsilon + \frac{m}{8}(\varepsilon^2 - r^2) + (\varepsilon + \frac{m}{4}\varepsilon^2) \ln\left(\frac{r}{\varepsilon}\right) & \text{if } r > \varepsilon. \end{cases} \quad (3.2.13)$$

If  $n = 3$ , then the solution is

$$\begin{cases} f(r) = r & \text{if } r \leq \varepsilon, \\ f(r) = \varepsilon + \frac{m}{4n}(\varepsilon^2 - r^2) \\ \quad + (\varepsilon^{n-1} + \frac{m}{2n}\varepsilon^n) \frac{1}{n-2} \left( \frac{1}{\varepsilon^{n-2}} - \frac{1}{r^{n-2}} \right) & \text{if } r > \varepsilon. \end{cases} \quad (3.2.14)$$

Then, we set in standard polar coordinates  $v(x) = f(r)$  for  $x \in B(b, R)$ . For  $r \leq \varepsilon$ , the constraint (3.2.8) is verified by definition. (For  $r > \varepsilon$ , we chose  $f$  so that  $\Delta v$  is small, but still larger than  $-m$ .)

Let us show that (3.2.7) holds. Let us set  $g(r) := f(r) - r$  and prove that  $g \leq 0$ . We have  $g(r) = 0$  for  $r \leq \varepsilon$  so it is sufficient to prove that  $g'(r) \leq 0$  for  $r \geq \varepsilon$ . But, as  $f$  verifies (3.2.12),  $g$  verifies

$$g'' + \frac{n-1}{r}g' = -m - \frac{n-1}{r} \quad \text{for } r \geq \varepsilon.$$

In particular, whenever  $g'(r) = 0$ , we have  $g''(r) < 0$ . This implies  $g'(r) \leq 0$  for  $r \geq \varepsilon$ , and so (3.2.7) is verified.

Now let us show that (3.2.10) holds if  $R$  has been taken small enough. We use the following expression of the Laplacian in coordinates:

$$\Delta v = \frac{1}{\sqrt{\det g}} \partial_i (\sqrt{\det g} g^{ij} \partial_j v),$$

where  $g = (g^{ij})$  is the metric of the manifold  $M$ , and  $\det g$  its determinant. We apply this formula to polar coordinates to find that, on  $B(b, R) \setminus \overline{B}(b, \varepsilon)$ , we have in the classical sense

$$\begin{aligned} \Delta v &= \frac{1}{\sqrt{\det g}} \partial_r (\sqrt{\det g} f'(r)) \\ &= f'' + \frac{\partial_r \det g}{2 \det g} f' \\ &= f'' + \frac{n-1}{r} f' + \left( \frac{\partial_r \det g}{2 \det g} - \frac{n-1}{r} \right) f' \\ &= -\frac{m}{2} + \left( \frac{\partial_r \det g}{2 \det g} - \frac{n-1}{r} \right) f'. \end{aligned} \quad (3.2.15)$$

Note that by applying the Laplacian formula in polar coordinates to the distance function  $d_b(x) = r$ , we find that

$$\Delta d_b = \frac{\partial_r \det g}{2 \det g}. \quad (3.2.16)$$

Because of lemma 3.2.5, we also have

$$\Delta d_b(x) = \frac{n-1}{r} + o(1).$$

With (3.2.15) and (3.2.16), this last equation yields in the classical sense

$$\Delta v = -\frac{m}{2} + o(1)f'(r) \quad \text{on } B(b, R) \setminus \overline{B}(b, \varepsilon). \quad (3.2.17)$$

Furthermore, it is clear from the following expression that  $f'$  is bounded on  $[\varepsilon, R]$ , by a constant independent of  $R$ , as long as we choose  $R \leq 1$ :

$$f'(r) = -\frac{m}{n}r + (\varepsilon^{n-1} + \frac{m}{n}\varepsilon^n)\frac{1}{r^{n-1}} \quad \text{for } \varepsilon \leq r \leq R.$$

Hence from (3.2.17) we see that by taking  $R$  small enough (independently of  $\varepsilon$ ), we can ensure that

$$\Delta v \geq -m \quad \text{on } B(b, R) \setminus \overline{B}(b, \varepsilon).$$

But from (3.2.17), we see that the above is also true on  $B(b, \varepsilon)$  if  $\varepsilon$  is small enough. Thus the function  $v$  is  $C^1$  on  $B(b, R)$  and verifies  $\Delta v(x) \geq -m$  when  $x \notin \partial B(b, \varepsilon)$ , hence (3.2.10) holds. It is also clear from (3.2.13) and (3.2.14) that the constraint (3.2.9) is verified if  $\varepsilon$  is taken small enough. This concludes the proof.  $\square$

We can now prove a stronger regularity proposition.

**Proposition 3.2.4.** *For any  $\varepsilon > 0$ , the function  $u_m^d$  belongs to  $C^{1,1}(M \setminus B(b, \varepsilon))$ .*

*Proof.* We reproduce the proof of lemma 3.2.2, but we replace the open set  $\Omega$  with  $\widehat{\Omega} := \Omega \setminus B(b, \varepsilon)$ , and the function  $v_m$  with a function  $\widehat{v}_m$  that is smooth and such that  $w_m = u_m$  on  $\partial\widehat{\Omega}$ . We know that such a function exists because  $u_m$  is smooth on  $\partial B(b, \varepsilon)$  for  $\varepsilon$  small enough, as it can be seen from lemma 3.2.3. This way, we can apply the stronger  $W^{2,\infty}$  regularity result for the obstacle problem [81, Theorem 4.38], since  $d_b$  is smooth on  $\widehat{\Omega}$ . We get that  $u_m^d$  belongs to  $W^{2,\infty} = C^{1,1}(\widehat{\Omega})$ . As  $u_m^d$  is smooth on  $E_m^d$  and  $\partial\widehat{\Omega} \subset E_m^d$ , then  $u_m^d$  is  $C^{1,1}$  on  $\widehat{\Omega} \cup E_m = M \setminus B(b, \varepsilon)$ .  $\square$

We end this section with the following elementary computational lemma.

**Lemma 3.2.5.** *We have*

$$\Delta d_b(p) \underset{p \rightarrow b}{=} \frac{n-1}{d_b(p)} + o(1).$$

*Proof.* We compute  $\Delta d_b$  in normal coordinates centered at  $b$ . Let  $g = (g^{ij})$  be the metric of  $M$  in these coordinates. We have

$$\Delta d_b(x) = \frac{1}{\sqrt{\det g}} \partial_i (\sqrt{\det g} g^{ij} \partial_j d_b)(x).$$

In normal coordinates, the metric is euclidean up to order 1 as  $x$  goes to 0. So we have

$$g^{ij}(x) = \delta^{ij} + o(x), \quad \partial_i (\sqrt{\det g} g^{ij})(x) = o(1) \quad \text{and} \quad \frac{1}{\sqrt{\det g}} = 1 + o(x).$$

Furthermore, in normal coordinates, we have  $d_b(x) = |x|$ , and so

$$\delta^{ij} \partial_{ij} d_b(x) = \frac{n-1}{|x|}.$$

Putting all the above together, as  $x$  goes to 0, we get

$$\Delta d_b(x) = \frac{n-1}{|x|} + o(1),$$

which concludes the proof.  $\square$

### 3.3 Equivalence of the two constraints

The constraints  $|\nabla u| \leq 1$  and  $u(b) = 0$ , in problem (3.1.1), imply  $u \leq d_b$ . However, contrary to the case of the elastic-plastic torsion problem, these two constraints may not be replaced with the constraint  $u \leq d_b$  in general. Indeed, we have the following proposition. We remind the reader that  $u_m$  and  $u_m^d$  denotes the minimizer of problems (3.1.1) and (3.2.1) respectively.

**Theorem 3.3.1.** *There exists a surface of revolution  $M$  and a parameter  $m > 0$  such that  $u_m \neq u_m^d$ .*

However, when  $m$  is large enough, we do have  $u_m^d = u_m$ :

**Theorem 3.3.2.** *Let  $K \geq 0$  be such that*

$$\text{Ric} \geq -K, \tag{3.3.1}$$

where  $\text{Ric}$  denotes the Ricci curvature tensor of the manifold  $M$ . If

$$m \geq \frac{1}{2} \max(\sqrt{nK(1 + K \text{diam}(M)^2)}, nK \text{diam}(M)),$$

then  $|\nabla u_m^d| \leq 1$ , or equivalently

$$u_m^d = u_m.$$

Let us now prove these two theorems.

*Proof of Theorem 3.3.1.* Let  $r_\theta$  denote the rotation of  $\mathbb{R}^3$  of angle  $\theta \in [0, 2\pi)$  around the  $z$ -axis. Let  $T := 10^{10}$  and  $r, h : [0, T] \rightarrow \mathbb{R}$  be two smooth functions such that

$$\begin{aligned} \gamma : t \mapsto (r(t), 0, h(t)) \text{ is a unit speed curve.} \\ M := \{r_\theta(\gamma(t)) : (t, \theta) \in [0, T] \times [0, 2\pi]\} \text{ is a smooth surface,} \\ r(0) = 0, \\ r([1, 2]) \subset [1, +\infty), \\ r([3, 4]) \subset (0, 10^{-10}), \\ r([5, T-1]) \subset [1, +\infty). \end{aligned}$$

This information is pictured in Figure 3.2. We chose  $b = (0, 0, 0)$  as the base point on  $M$ , and  $m = 10^{-10}$ . Let us assume that  $u_m^d = u_m$  and build a better competitor in (3.2.1) to contradict the minimality of  $u_m^d$ . We will first reduce (3.2.1) to a one-dimensional problem. Note that the functional we are minimizing is rotation-invariant. More precisely, for any  $\theta \in (0, 2\pi)$  and  $u \in H^1(M)$ , we have

$$\int_M |\nabla(u \circ r_\theta)|^2 - m(u \circ r_\theta) = \int_M |\nabla u|^2 - mu. \tag{3.3.2}$$

By uniqueness of the minimizer  $u_m^d$ , we deduce that  $u_m^d$  is rotation-invariant, i.e. there exists a function  $\rho_m : [0, T] \rightarrow \mathbb{R}$  such that for any  $\theta \in [0, 2\pi)$  and  $t \in [0, T]$ ,  $u_m^d(r_\theta(\gamma(t))) = \rho_m(t)$ . Thus  $u_m^d$  is a minimizer of (3.2.1) among rotation-invariant functions. Let  $u : M \rightarrow \mathbb{R}$  be any rotation-invariant function, and  $\rho : [0, T] \rightarrow \mathbb{R}$  be such that for any  $\theta \in [0, 2\pi)$ ,

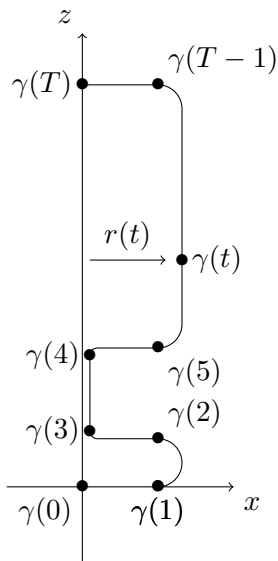


Figure 3.2: Curve  $\gamma$  generating  $M$  by rotating around the  $z$ -axis.

$u(r_\theta(\gamma(t))) = \rho(t)$ . We will translate the minimization problem (3.2.1) on  $u$  into a problem on  $\rho$ .

First, because  $M$  is a surface of revolution, all the geodesics starting from  $b = (0, 0, 0)$  have a constant angle  $\theta$  (this is an exercise in Riemannian geometry). Thus, they are of the form  $t \mapsto r_\theta(\gamma(t))$  for some  $\theta \in [0, 2\pi)$ . These are actually unit speed geodesics as  $\gamma$  is unit speed. Hence,  $d_b(r_\theta(\gamma(t))) = t$ , and the constraint  $u \leq d_b$  in (3.2.1) is equivalent to  $\rho(t) \leq t$ .

Secondly, we translate the  $H^1$  constraint. To this end, let us define some coordinates  $(t, \theta)$  on  $M$  via the map

$$\begin{aligned} \phi : (0, T) \times (0, 2\pi) &\rightarrow M \\ (t, \theta) &\mapsto r_\theta(\gamma(t)) \end{aligned} .$$

We have

$$\begin{aligned} \int_M |\nabla u|^2 &= \int_{(0, T) \times (0, 2\pi)} (|\nabla u|^2 \circ \phi) J\phi dt d\theta \\ &= \int_{(0, T) \times (0, 2\pi)} |\nabla u|^2 (r_\theta(\gamma(t))) r(t) dt d\theta \\ &= 2\pi \int_{(0, T)} |\nabla u|^2 (\gamma(t)) r(t) dt, \end{aligned} \tag{3.3.3}$$

because  $u$  is rotation-invariant. Furthermore, as  $u$  is rotation-invariant, its gradient at the point  $\gamma(t)$  is parallel to  $\gamma'(t)$ , and so

$$|\rho'(t)| = |\nabla u(\gamma(t)) \cdot \gamma'(t)| = |\nabla u(\gamma(t))| |\gamma'(t)| = |\nabla u(\gamma(t))| .$$

Hence (3.3.3) gives

$$\int_M |\nabla u|^2 = 2\pi \int_{(0, T)} \rho'(t)^2 r(t) dt .$$



Thus, the constraint  $u \in H^1(M)$  in (3.2.1) is equivalent to  $v \in H^1((0, T), r(t)dt)$ .

Thirdly, we may compute the functional likewise:

$$\int_M |\nabla u|^2 - mu = 2\pi \int_{(0, T)} (\rho'(t)^2 - m\rho(t)) r(t)dt.$$

Putting these facts together, as  $u_m^d$  is a minimizer in (3.2.1), we deduce that  $\rho_m$  is a minimizer of :

$$\inf \left\{ \int_{(0, T)} (\rho'(t)^2 - m\rho(t)) r(t)dt : \rho \in H^1((0, T), r(t)dt), \rho(t) \leq t \right\}. \quad (3.3.4)$$

The idea of the rest of the proof is the following. First, we recall the assumption  $u_m^d = u_m$ , which means that  $|\nabla u_m^d| \leq 1$ , and so  $|\rho_m'| \leq 1$ . Now, if  $\rho_m(4)$  is close to 4, then  $\rho_m'(t)$  is close to 1 for  $t \leq 4$ , so a competitor  $v$  such that  $\rho'(t)$  is small for  $t \leq 4$  will contradict the minimality of  $\rho_m$  in (3.3.4). If on the contrary  $\rho_m(4)$  is significantly smaller than 4, then for  $t \geq 4$ ,  $\rho_m(t)$  will be significantly smaller than  $t$ , so a competitor  $\rho$  such that  $\rho(t)$  is closer to  $t$  for  $t \geq 4$  will contradict the minimality of  $\rho_m$  in (3.3.4). Because we chose  $r$  very small on the interval  $[3, 4]$  (see Figure 3.2), we can define a competitor  $\rho$  independently on  $[0, 3]$  and  $[4, T]$ , without paying much for the behavior of  $\rho$  on  $[3, 4]$ .

*Case one:*  $\rho_m(4) \in [3.5, 4]$ . Let us define a competitor  $\rho$  for (3.3.4):

$$\begin{aligned} \rho &: [0, T] \rightarrow \mathbb{R} \\ t &\mapsto \begin{cases} 0 & \text{if } t \in [0, 3], \\ 4(t-3) & \text{if } t \in [3, 4] \\ \rho_m(t) + 4 - \rho_m(4) & \text{if } t \geq 4, \end{cases} \end{aligned}$$

Let us call  $\mathcal{F}(\rho)$  the functional appearing in (3.3.4). We have, from the definition of  $r$  and  $\rho$ ,

$$\begin{aligned} \mathcal{F}(\rho) &= \int_{(3, 4)} (16 - 4m(t-3)) r(t)dt + \int_{(4, T)} (\rho_m'^2(t) - m\rho_m(t)) r(t)dt \\ &\quad - m(4 - \rho_m(4)) \int_{(4, T)} r(t)dt \\ &\leq (16 - 0) \cdot 10^{-10} + \int_{(4, T)} (\rho_m'^2(t) - m\rho_m(t)) r(t)dt - 0 \end{aligned} \quad (3.3.5)$$

so

$$\begin{aligned} \mathcal{F}(\rho) - \mathcal{F}(\rho_m) &\leq 16 \cdot 10^{-10} - \int_{(0, 4)} (\rho_m'^2(t) - m\rho_m(t)) r(t)dt \\ &\leq 16 \cdot 10^{-10} - \int_{(1, 2)} \rho_m'^2(t) r(t)dt + m \int_{(0, 4)} \rho_m(t)dt \\ &\leq 16 \cdot 10^{-10} - \int_{(1, 2)} \rho_m'^2(t) r(t)dt + m \int_{(0, 4)} tdt \\ &= 16 \cdot 10^{-10} - \int_{(1, 2)} \rho_m'^2(t) r(t)dt + \frac{16}{2}m. \end{aligned} \quad (3.3.6)$$

We are left to bound from below the integral term in (3.3.6). By the Hölder inequality we have

$$\int_{(1,2)} \rho'_m \leq \left( \int_{(1,2)} \frac{1}{r} \right)^{\frac{1}{2}} \left( \int_{(1,2)} \rho_m'^2 r \right)^{\frac{1}{2}},$$

and so

$$\int_{(1,2)} \rho_m'^2 r \geq \frac{(\rho_m(2) - \rho_m(1))^2}{\int_{(1,2)} \frac{1}{r}} \geq (\rho_m(2) - \rho_m(1))^2, \quad (3.3.7)$$

by the construction of  $r$ . Now we use the fact  $u_m^d = u_m$ , which means that  $|\nabla u_m^d| \leq 1$ , and so  $|\rho'_m| \leq 1$ . With the running assumption  $\rho_m(4) \geq 3.5$ , this implies  $\rho_m(2) \geq 1.5$ . As  $\rho_m(1) \leq 1$ , we get  $\rho_m(2) - \rho_m(1) \geq 0.5$ . Then, (3.3.7) and (3.3.6) yield

$$\mathcal{F}(\rho) - \mathcal{F}(\rho_m) \leq 16 \cdot 10^{-10} - 0.25 + \frac{9}{2}m. \quad (3.3.8)$$

Recalling that we have chosen  $m = 10^{-10}$ , it contradicts the minimality of  $\rho_m$  in (3.3.4).

*Case two:*  $\rho_m(4) \leq 3.5$ . We use the same competitor  $\rho$  as in case one. We even perform similar estimates, the only difference being that we do not estimate the term  $-m(4 - \rho_m(4)) \int_{(4,T)} r(t)dt$  by 0 as in (3.3.5). Thus (3.3.6) becomes instead:

$$\begin{aligned} \mathcal{F}(\rho) - \mathcal{F}(\rho_m) &\leq 16 \cdot 10^{-10} - \int_{(1,2)} \rho_m'^2(t)r(t)dt + \frac{16}{2}m \\ &\quad - m(4 - \rho_m(4)) \int_{(4,T)} r(t)dt. \\ &\leq 16 \cdot 10^{-10} + \frac{16}{2}m - 0.5m \int_{(4,T)} r(t)dt \\ &\leq 16 \cdot 10^{-10} + \frac{16}{2}m - 0.5m \int_{(5,T-1)} r(t)dt. \end{aligned}$$

Recalling that we have chosen  $m = 10^{-10}$ ,  $T = 10^{10}$  and  $r \geq 1$  between 5 and  $T - 1$ , it contradicts the minimality of  $\rho_m$  in (3.3.4). This concludes the proof.  $\square$

*Proof of Theorem 3.3.2.* Recall that the function  $u_m^d$  is  $C^1$  except at  $b$ , by Proposition 3.2.4. Let  $x \neq b$  be in the contact set  $P_m^d := \{u_m^d = d_b\}$ . By the lemma 3.2.1, we have  $x \notin \text{Cut}_b(M)$ , and so the distance function  $d_b$  is differentiable at  $x$ . It is a simple consequence of the constraint  $u_m^d \leq d_b$  and the equality  $u_m^d(x) = d_b(x)$  that we have  $\nabla u_m^d(x) = \nabla d_b(x)$ . The desired inequality  $|\nabla u_m^d(x)| \leq 1$  follows.

In the non-contact set  $E_m^d = \{u_m^d < d_b\}$ , the function  $u_m^d$  verifies the Euler-Lagrange equation

$$\Delta u_m^d = -2m. \quad (3.3.9)$$

In particular it is smooth, and we may apply the Bochner-Weitzenböck formula:

$$\Delta \left| \nabla u_m^d \right|^2 = 2\text{Ric}(\nabla u_m^d, \nabla u_m^d) + 2 \left| D^2 u_m^d \right|^2 + 2(\nabla \Delta u_m^d, \nabla u_m^d), \quad (3.3.10)$$

where  $\text{Ric}$  denotes the Ricci curvature tensor on the manifold  $M$  and  $D^2u_m^d$  is the second covariant derivative of  $u_m^d$ . The last term is 0 because  $\Delta u_m^d = -2m$  ((3.3.9)). As for the second term, we have:

$$\begin{aligned} |D^2u_m^d|^2 &\geq \frac{1}{n} \left( \text{Trace}(D^2u_m^d) \right)^2 \\ &= \frac{4m^2}{n} \quad \text{by (3.3.9)}. \end{aligned}$$

As the manifold  $M$  is compact, there exists a constant  $K > 0$  (depending on  $M$  only) such that the Ricci curvature is bounded from below by  $-K$ . In the end, formula (3.3.10) yields

$$\Delta |\nabla u_m^d|^2 + 2K |\nabla u_m^d|^2 \geq \frac{8}{n} m^2. \quad (3.3.11)$$

Now notice that by (3.3.9),

$$\begin{aligned} \Delta((u_m^d)^2) &= 2u_m^d \Delta u_m^d + 2 |\nabla u_m^d|^2 \\ &= -4mu_m^d + 2 |\nabla u_m^d|^2, \end{aligned}$$

so (3.3.11) gives

$$\begin{aligned} \Delta(|\nabla u_m^d|^2 + K(u_m^d)^2) &= \frac{8}{n} m^2 - 4Kmu_m^d \\ &\geq \frac{8}{n} m^2 - 4Kmd_b \\ &\geq \frac{8}{n} m^2 - 4Kmdiam(M) \end{aligned}$$

Thus, if  $m \geq \frac{n}{2} K diam(M)$ , the function  $|\nabla u_m^d|^2 + K(u_m^d)^2$  is subharmonic in the non-contact set  $E_m^d$ . From lemma 3.2.3, we have  $\overline{E_m^d} \subset M \setminus \{b\}$ , and with Proposition 3.2.4, we get that the function  $|\nabla u_m^d|^2 + K(u_m^d)^2$  is continuous on  $\overline{E_m^d} \subset M \setminus \{b\}$ . Therefore we may apply the maximum principle to get

$$\begin{aligned} |\nabla u_m^d|^2 &\leq |\nabla u_m^d|^2 + K(u_m^d)^2 \\ &\leq \sup_{\partial E_m^d} \left( |\nabla u_m^d|^2 + K(u_m^d)^2 \right) \\ &= 1 + K \sup_{\partial E_m^d} (u_m^d)^2 \\ &\leq 1 + K \sup_{\partial E_m^d} (d_b)^2 \\ &\leq 1 + K diam(M)^2 \end{aligned} \quad (3.3.12)$$

With (3.3.11), this last inequality gives

$$\Delta |\nabla u_m^d|^2 \geq \frac{8}{n} m^2 - 2K(1 + K diam(M)^2)$$

Thus, whenever the right-hand side is non-negative, the maximum principle applied to the function  $|\nabla u_m^d|^2$  on the open set  $E_m^d$  implies  $|\nabla u_m^d|^2 \leq 1$ . This concludes the proof.  $\square$

### 3.4 Convergence of the non-contact set

In this section we show that the non-contact set  $E_m^d = \{u_m^d < d_b\}$  Hausdorff-converges to  $Cut_b(M)$ .

**Lemma 3.4.1.** *We have  $\|d_b - u_m^d\|_{L^\infty(M)} \leq C/m$ , for some positive constant  $C$  depending on  $M$  only.*

*Proof.* We only need to prove the proposition for  $m$  large enough. Therefore, thanks to Theorem 3.3.2, we will assume that  $m$  is large enough so that  $|\nabla u_m^d| \leq 1$ . We only need to show the estimate on  $E_m^d$  since outside this set,  $u_m^d$  and  $d_b$  are the same. We will show that for  $m$  large enough, we have

$$\forall \bar{p} \in E_m^d, \exists p \in (E_m^d)^c \quad \text{such that} \quad d(p, \bar{p}) < 5n/m. \quad (3.4.1)$$

This will conclude the proof since by the 1-Lipschitzianity of  $u_m^d$  and  $d_b$ , we then have

$$\begin{aligned} |d_b(\bar{p}) - u_m^d(\bar{p})| &\leq |d_b(p) - u_m^d(p)| + 2d(\bar{p}, p) \\ &= 0 + 2d(\bar{p}, p) \\ &\leq \frac{5n}{m}, \end{aligned}$$

which is what we need. In order to prove (3.4.1), we argue by contradiction and assume that  $B_{5n/m}(\bar{p}) \subset E_m^d$ . We want to apply the maximum principle to the function  $v$  defined on  $B_{5n/m}(\bar{p})$  by the following formula

$$v(p) := u_m^d(p) - \inf_{\partial B_{\frac{5n}{m}}(\bar{p})} u_m^d + \frac{m}{2n} \left( d_{\bar{p}}(p)^2 - \left( \frac{5n}{m} \right)^2 \right).$$

For any  $p \in B_{5n/m}(\bar{p})$ , we have  $\Delta u_m^d(p) = -2m$  because we have assumed  $B_{5n/m}(\bar{p}) \subset E_m^d$ . To estimate the Laplacian of  $d_{\bar{p}}^2$ , we use some normal coordinates  $(x^i)$  centered at  $\bar{p}$ . In these coordinates, the metric is euclidean up to order 1, uniformly in  $\bar{p}$  since  $M$  is compact, and  $d_{\bar{p}}(x) = |x|$ . This argument has been detailed in the proof of lemma 3.2.5, so we do not reproduce it here. We get that for  $m$  large enough, independently of  $\bar{p}$ ,

$$\forall p \in B_{5n/m}(\bar{p}), \quad \Delta d_{\bar{p}}^2(p) \leq 2(2n).$$

Therefore, we obtain on  $B_{5n/m}(\bar{p}) \subset E_m^d$ ,

$$\Delta v \leq -2m + \frac{m}{2n} 2(2n) = 0.$$

So we can apply the maximum principle to  $v$  to get

$$v(\bar{p}) \geq \inf_{\partial B_{\frac{5n}{m}}(\bar{p})} v,$$

*i.e.*

$$u_m^d(\bar{p}) - \inf_{\partial B_{\frac{5n}{m}}(\bar{p})} u_m^d - \frac{m}{4n} \left( \frac{5n}{m} \right)^2 \geq 0. \quad (3.4.2)$$

As we have taken  $m$  large enough so that  $|\nabla u_m^d| \leq 1$ , we also have

$$u_m^d(\bar{p}) - \inf_{\partial B_{\frac{5n}{m}}(\bar{p})} u_m^d \leq \frac{5n}{m} < \frac{m}{4n} \left( \frac{5n}{m} \right)^2,$$

which contradicts the estimate (3.4.2). This concludes the proof.  $\square$

**Theorem 3.4.2.** *For any  $m > m' > 0$ , we have  $Cut_b(M) \subset E_m^d \subset E_{m'}^d$ , and*

$$E_m^d \xrightarrow{m \rightarrow \infty} Cut_b(M) \quad \text{in the Hausdorff sense.}$$

*Proof.* The fact that, for any  $m > 0$ ,  $Cut_b(M) \subset E_m^d$ , is a direct consequence of lemma 3.2.1. Let us prove the second inclusion. For  $m > m' > 0$ , note that by the respective minimality of  $u_m^d$  and  $u_{m'}^d$ , we have

$$\begin{aligned} & \int_M |\nabla \max(u_{m'}^d, u_m^d)|^2 - m \int_M \max(u_{m'}^d, u_m^d) \geq \int_M |\nabla u_m^d|^2 - m \int_M u_m^d, \\ \text{and} \quad & \int_M |\nabla \min(u_{m'}^d, u_m^d)|^2 - m' \int_M \min(u_{m'}^d, u_m^d) \geq \int_M |\nabla u_{m'}^d|^2 - m' \int_M u_{m'}^d. \end{aligned}$$

Using the formulas

$$\begin{aligned} \nabla \max(u_{m'}^d, u_m^d) &= \nabla u_{m'}^d \mathbf{1}_{\{u_{m'}^d > u_m^d\}} + \nabla u_m^d \mathbf{1}_{\{u_{m'}^d \leq u_m^d\}}, \\ \nabla \min(u_{m'}^d, u_m^d) &= \nabla u_m^d \mathbf{1}_{\{u_{m'}^d > u_m^d\}} + \nabla u_{m'}^d \mathbf{1}_{\{u_{m'}^d \leq u_m^d\}}, \end{aligned}$$

we obtain

$$\begin{aligned} & \int_{\{u_{m'}^d > u_m^d\}} \left( |\nabla u_{m'}^d|^2 - |\nabla u_m^d|^2 \right) \geq -m \int_{\{u_{m'}^d > u_m^d\}} \left( u_m^d - u_{m'}^d \right), \\ \text{and} \quad & \int_{\{u_{m'}^d > u_m^d\}} \left( |\nabla u_m^d|^2 - |\nabla u_{m'}^d|^2 \right) \geq -m' \int_{\{u_{m'}^d > u_m^d\}} \left( u_{m'}^d - u_m^d \right). \end{aligned}$$

Summing these two inequalities, we get

$$0 \geq (m - m') \int_{\{u_{m'}^d > u_m^d\}} \left( u_{m'}^d - u_m^d \right),$$

and so  $u_m^d \geq u_{m'}^d$ . In particular,  $E_m^d \subset E_{m'}^d$ .

We are left to show the Hausdorff convergence in  $E_m^d$  to  $Cut_b(M)$ . Given  $\varepsilon > 0$ , let us set

$$\Omega_\varepsilon := \{x \in M : d(x, Cut_b(M)) > \varepsilon\}.$$

We will show that for  $m$  large enough we have  $E_m^d \subset (\Omega_{2\varepsilon})^c$ , which will conclude the proof. Let  $\phi : M \rightarrow \mathbb{R}$  be a function such that  $\phi \leq d_b$  on  $M$ ,  $\phi = d_b$  on  $\Omega_{2\varepsilon}$ ,  $\phi < d_b$  on  $\partial\Omega_\varepsilon$ , and  $\phi$  is smooth on  $M$  except at  $b$ . We want to apply the maximum principle to the function  $\phi - u_m^d$  on  $E_m^d \cap \Omega_\varepsilon$ . We have

$$\Delta(\phi - u_m^d) = \Delta\phi + m \quad \text{on} \quad E_m^d \cap \Omega_\varepsilon,$$

so for  $m$  large enough the function  $\phi - u_m^d$  is subharmonic on  $E_m^d \cap \Omega_\varepsilon$ . On  $\partial\Omega_\varepsilon$ , we have  $\phi < d_b$  and  $u_m^d$  converges uniformly to  $d_b$  as  $m$  tends to  $+\infty$  (lemma 3.4.1) so  $\phi - u_m^d \leq 0$ , for  $m$  large enough. On  $\partial E_m^d$ , we have  $\phi - u_m^d = \phi - d_b \leq 0$ . Thus the maximum principle implies that for  $m$  large enough, we have  $\phi - u_m^d \leq 0$  on  $E_m^d \cap \Omega_\varepsilon$ . As  $\phi = d_b$  on  $\Omega_{2\varepsilon}$ , we get  $u_m^d \geq d_b$  on  $E_m^d \cap \Omega_{2\varepsilon}$ . Since by definition we have  $u_m^d < d_b$  on  $E_m^d$ , we get  $E_m^d \subset (\Omega_{2\varepsilon})^c$ , which concludes the proof.  $\square$

### 3.5 Semiconcavity

In this section, we prove (3.1.10) and (3.1.11) of Theorem 3.1.2. We begin with a definition. For  $a, b, \lambda \in \mathbb{R}$ , we set  $\lambda_{ab} = (1 - \lambda)a - \lambda b$ .

**Definition 3.5.1.** *Given a constant  $C > 0$ , a function  $u$  is said to be  $C$ -semiconcave on  $M$  if and only if for any unit speed geodesic  $\gamma : [a, b] \rightarrow M$ , the function  $t \mapsto Ct^2 - u(\gamma(t))$  is convex. This is equivalent to having, for any  $\lambda \in [0, 1]$ ,*

$$(1 - \lambda)u(\gamma(a)) + \lambda u(\gamma(b)) - u(\gamma(\lambda_{ab})) \leq C\lambda(1 - \lambda)(b - a)^2.$$

We say that  $u$  is semiconcave if  $u$  is  $C$ -semiconcave for some constant  $C > 0$ . We say that  $u$  is locally semiconcave if for any  $p \in M$ ,  $u$  is semiconcave in a neighborhood of  $p$ .

*Remark 3.5.2.* On the compact manifold  $M$ , semiconcavity and local semiconcavity are the same.

In [60, Definition 2.5], the authors define local semiconcavity using charts as follows: a function  $u$  is said to be locally semiconcave if for any chart  $\psi$ , the function  $u \circ \psi^{-1}$  is locally semiconcave as a function on  $\mathbb{R}^n$ . In the appendix, Proposition A.1.4, we show that this notion of local semiconcavity is equivalent to the definition of local semiconcavity we gave.

In [60, Proposition 3.4], the authors proved that the distance function is locally semiconcave on  $M \setminus \{b\}$ . As a consequence, recalling remark 3.5.2, we have the following proposition:

**Proposition 3.5.3.** *Given  $\varepsilon > 0$ , the distance function  $d_b$  is  $C$ -semiconcave on  $M \setminus B(b, \varepsilon)$ , for some  $C > 0$ .*

We also have the following obvious proposition, obtained from the characterization of twice differentiable convex functions on the real line:

**Proposition 3.5.4.** *A  $C^2$  function  $u : M \rightarrow \mathbb{R}$  is  $C$ -semiconcave if and only if its second order covariant differential verifies  $D^2u \leq 2C$ .*

We can now prove the following:

**Proposition 3.5.5.** *Given  $\varepsilon > 0$ , there exists a constant  $C > 0$  such that for any  $m$  large enough, the function  $u_m$  is  $C$ -semiconcave on  $M \setminus B(b, \varepsilon)$ .*

We start with a proof that is specific to the dimension 2. It uses the determinant of symmetric matrices of size 2, to estimate their eigenvalues. We will give a proof that works in any dimension afterwards. The first part of the proof is almost identical to the first part of the proof of Theorem 1.3.2.

*Proof of Proposition 3.5.5 in dimension 2.* The reader may find the recalls of appendix A.1 useful to follow this proof. For  $m$  large enough, we recall that  $u_m = u_m^d$  (Theorem 3.3.2). This will be used throughout the proof. Let  $C_d$  be such that the distance function  $d_b$  is  $C_d$ -semiconcave on  $M \setminus B(b, \varepsilon)$ . We will show that  $u_m$  is  $(C_d + 1)$ -semiconcave for  $m$  large enough. Let us introduce some notations. For  $a, b \in \mathbb{R}$  and  $\lambda \in (0, 1)$ ,

$$\lambda_{ab} := (1 - \lambda)a + \lambda b.$$

For any unit speed geodesic  $\gamma : [a, b] \rightarrow M$ ,  $\lambda \in (0, 1)$  and  $v$  a function on  $M$ , let us define

$$c(\gamma, \lambda, v) := \lambda(1 - \lambda)(C_d + 1)(b - a)^2 - ((1 - \lambda)v(\gamma(a)) + \lambda v(\gamma(b)) - v(\gamma(\lambda_{ab}))). \quad (3.5.1)$$

We need to show the following:

$$\inf_{\gamma, \lambda} c(\gamma, \lambda, u_m) \geq 0 \quad (3.5.2)$$

where the infimum is taken over unit speed geodesics defined over finite intervals. Let us argue by contradiction and assume that (3.5.2) does not hold. As  $u_m$  verifies  $|\nabla u_m| \leq 1$ , it is 1-Lipschitz. As any unit speed geodesic  $\gamma : [a, b] \rightarrow M$  is also 1-Lipschitz, the function  $u_m \circ \gamma$  is 1-Lipschitz. By definition of  $c$ , this implies

$$c(\gamma, \lambda, u_m) \geq \lambda(1 - \lambda)(C_d + 1)(b - a)^2 + 2\lambda(1 - \lambda)(b - a).$$

In particular it is clear that if the geodesic  $\gamma$  is long enough (*i.e.*  $(b - a)$  is large enough), then we have  $c(\gamma, \lambda, u_m) \geq 0$ . Hence we can assume that the infimum in (3.5.2) is taken over unit speed geodesics whose length is bounded by a given constant. By continuity of  $c$  for the uniform convergence of curves, and compactness (for the uniform convergence of curves) of geodesics with bounded length in the compact manifold  $M$ , the infimum in (3.5.2) is a minimum, attained at a point  $(\gamma, \lambda)$ , with  $\gamma$  defined on a segment  $[a, b]$ . Now we show that

$$\gamma((a, b)) \subset E_m = \{u_m < d_b\}. \quad (3.5.3)$$

If not, then there exists  $\mu \in (0, 1)$  such that  $\gamma(\mu_{ab}) \notin E_m$ , *i.e.*  $u_m(\gamma(\mu_{ab})) = d_b(\gamma(\mu_{ab}))$ . Recalling that  $u_m \leq d_b$ , we get

$$c(\gamma, \mu, u_m) \geq c(\gamma, \mu, d_b) > 0, \quad (3.5.4)$$

where the last inequality comes from the  $C_d$ -semiconcavity of  $d_b$ . In particular,  $\mu \neq \lambda$ . Let us assume for instance that  $\mu < \lambda$  (the case  $\mu > \lambda$  being similar) and show that  $c(\gamma, \lambda, u_m)$  is not minimal in (3.5.2). Figure 3.3 may help justify intuitively the following construction. Let  $\nu \in (0, 1)$  be such that

$$\nu_{\mu_{abb}} = \lambda_{ab}. \quad (3.5.5)$$

Let  $\tilde{\gamma}$  be the unit speed geodesic defined by  $\tilde{\gamma} := \gamma|_{[\mu_{ab}, b]}$ . Let us set  $f(t) := (C_d + 1)t^2 - u_m(\gamma(t))$ . Then, we have

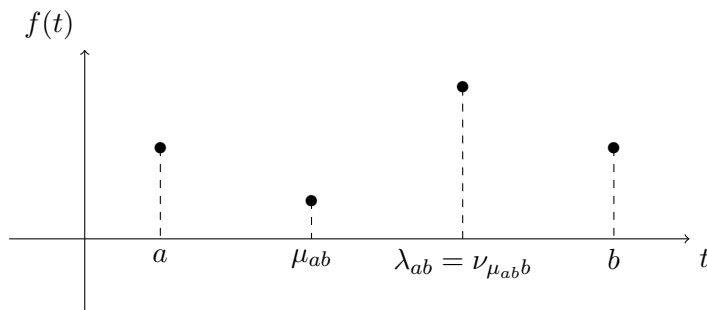
$$\begin{aligned} c(\tilde{\gamma}, \nu, u_m) &= (1 - \nu)f(\mu_{ab}) + \nu f(b) - f(\nu_{\mu_{abb}}) \\ &= (1 - \nu)f(\mu_{ab}) + \nu f(b) - f(\lambda_{ab}) \\ &= c(\gamma, \lambda, u_m) - (1 - \lambda)f(a) + (\nu - \lambda)f(b) + (1 - \nu)f(\mu_{ab}). \end{aligned} \quad (3.5.6)$$

Now after some elementary calculations, (3.5.5) translates into

$$\begin{cases} 1 - \lambda &= (1 - \nu)(1 - \mu), \\ \nu - \lambda &= -(1 - \nu)\mu, \end{cases}$$

so (3.5.6) becomes

$$\begin{aligned} c(\tilde{\gamma}, \nu, u_m) &= c(\gamma, \lambda, u_m) - (1 - \nu)((1 - \mu)f(a) + \mu f(b) - f(\mu_{ab})) \\ &= c(\gamma, \lambda, u_m) - (1 - \nu)c(\gamma, \mu, u_m). \end{aligned} \quad (3.5.7)$$


 Figure 3.3: Construction of  $\tilde{\gamma}$  and  $\nu$ .

With (3.5.4), this contradicts the minimality of  $c(\gamma, \lambda, u_m)$ . Thus (3.5.3) is true, and we are left to show that  $u_m$  is  $(C_d + 1)$ -semiconcave inside  $E_m$ .

As  $u_m$  has constant Laplacian in  $E_m$ , it is smooth, and by Proposition 3.5.4, its  $(C_d + 1)$ -semiconcavity boils down to the pointwise condition

$$D^2 u_m \leq (C_d + 1)Id. \quad (3.5.8)$$

To prove (3.5.8), it is enough to show that

$$\det D^2 u_m \geq -2(C_d + 1)m. \quad (3.5.9)$$

Indeed, let  $\lambda_m \leq \mu_m$  be the eigenvalues of  $D^2 u_m$ . Recalling that  $\lambda_m + \mu_m = \Delta u_m = -2m$ , we have

$$\begin{aligned} \det D^2 u_m \geq -2(C_d + 1)m &\Leftrightarrow \lambda_m \mu_m \geq -2(C_d + 1)m \\ &\Leftrightarrow -\mu_m^2 - 2m\mu_m \geq -2(C_d + 1)m \\ &\Rightarrow \mu_m \leq (C_d + 1) \\ &\Leftrightarrow D^2 u_m \leq (C_d + 1), \end{aligned}$$

and so (3.5.9) implies (3.5.8). Now to prove (3.5.9), the strategy is to show that it holds on the boundary of  $E_m$ , and to apply the maximum principle to a suitable superharmonic function. If the surface  $M$  was flat, as we will see in the following, we could apply this strategy directly to  $\det D^2 u_m$ . Let us first compute its Laplacian. Given any local orthonormal basis  $(e_1, e_2)$  of the tangent plane to  $M$ , we set  $D_{ij}^2 u_m := D^2 u_m(e_i, e_j)$ . Then, we have

$$\begin{aligned} \Delta(\det D^2 u_m) &= \Delta((D_{11}^2 u_m)(D_{22}^2 u_m) - (D_{12}^2 u_m)^2) \\ &= \Delta(D_{11}^2 u_m)D_{22}^2 u_m + \Delta(D_{22}^2 u_m)D_{11}^2 u_m \\ &\quad + 2\nabla(D_{11}^2 u_m) \cdot \nabla(D_{22}^2 u_m) \\ &\quad - \Delta((D_{12}^2 u_m)^2). \end{aligned}$$



Using the equation  $-2m = \Delta u_m = D_{11}^2 u_m + D_{22}^2 u_m$ , we get

$$\begin{aligned}
\Delta(\det D^2 u_m) &= -\Delta(D_{22}^2 u_m)D_{22}^2 u_m - \Delta(D_{11}^2 u_m)D_{11}^2 u_m \\
&\quad - \nabla(D_{22}^2 u_m) \cdot \nabla(D_{22}^2 u_m) - \nabla(D_{11}^2 u_m) \cdot \nabla(D_{11}^2 u_m) \\
&\quad - \Delta((D_{12}^2 u_m)^2) \\
&= -\frac{1}{2}\Delta((D_{11}^2 u_m)^2 + (D_{22}^2 u_m)^2 + 2(D_{12}^2 u_m)^2) \\
&= -\frac{1}{2}\Delta(|D^2 u_m|^2). \tag{3.5.10}
\end{aligned}$$

Now this right hand side can be conveniently computed in a coordinate-free way using classical tensorial differential calculus on Riemannian manifolds with abstract index notation. We write down the explicit computations here as it does not seem to be a classical identity. The metric tensor of  $M$  is denoted by  $g_{ab}$  and the Riemann curvature tensor by  $R_{abcd}$ . We will denote by  $|T|$  the norm of any tensor  $T$ . We have

$$\begin{aligned}
\Delta(|D^2 u_m|^2) &= g^{ab} D_a D_b ((D_c D_d u_m)(D^c D^d u_m)) \\
&= g^{ab} (2(D_a D_b D_c D_d u_m)(D^c D^d u_m) + 2(D_a D_c D_d u_m)(D_b D^c D^d u_m)) \\
&= 2g^{ab} (D_a D_b D_c D_d u_m)(D^c D^d u_m) + 2(D_a D_c D_d u_m)(D^a D^c D^d u_m) \\
&= 2g^{ab} (D_a D_b D_c D_d u_m)(D^c D^d u_m) + 2|D^3 u_m|^2 \\
&\geq 2g^{ab} (D_a D_b D_c D_d u_m)(D^c D^d u_m), \tag{3.5.11}
\end{aligned}$$

We will use the notation  $D_{[ab]} := D_a D_b - D_b D_a$ . We have

$$D_a D_b D_c D_d u_m = D_a D_{[bc]} D_d u_m + D_{[ac]} D_b D_d u_m + D_c D_a D_{[bd]} u_m + D_c D_{[ad]} D_b u_m + D_c D_d D_a D_b u_m.$$

In the following, any constant that depends on  $M$  only will be denoted by  $C^M$ . By definition of the Riemann tensor we have

$$D_a D_{[bc]} D_d u_m = D_a (R_{bced} D^e u_m) = (D_a R_{bced}) D^e u_m + R_{bced} D_a D^e u_m,$$

and so

$$|D_a D_{[bc]} D_d u_m| \geq -C^M |\nabla u_m| - C^M |D^2 u_m|. \tag{3.5.12}$$

Likewise,

$$|D_c D_{[ad]} D_b u_m| \geq -C^M |\nabla u_m| - C^M |D^2 u_m|. \tag{3.5.13}$$

To compute the term  $D_{[ac]} D_b D_d u_m$ , let us pick some coordinates  $(x^i)$  and write  $D_b D_d u_m = D_{ij}^2 u_m dx_b^i dx_d^j$ . Then, we have

$$\begin{aligned}
D_{[ac]} D_b D_d u_m &= D_{[ac]} (D_{ij}^2 u_m dx_b^i dx_d^j) \\
&= (D_{[ac]} D_{ij}^2 u_m) dx_b^i dx_d^j + D_{ij}^2 u_m (D_{[ac]} dx_b^i) dx_d^j \\
&\quad + D_{ij}^2 u_m dx_b^i (D_{[ac]} dx_d^j) \\
&= 0 + D_{ij}^2 u_m R_{aceb} (dx^i)^e dx_d^j + D_{ij}^2 u_m dx_b^i R_{aced} (dx^j)^e \\
&= R_{aceb} D^e D_d u_m + R_{aced} D_b D^e u_m,
\end{aligned}$$

and so

$$|D_{[ac]}D_bD_du_m| \geq -C^M |D^2u_m|. \quad (3.5.14)$$

By symmetry of the tensor  $D^2u_m$ , we have

$$D_cD_aD_{[bd]}u_m = 0. \quad (3.5.15)$$

Putting (3.5.12), (3.5.13), (3.5.14) and (3.5.15) together, we find

$$|g^{ab}D_aD_bD_cD_du_m| \geq -C^M |\nabla u_m| - C^M |D^2u_m| - |g^{ab}D_cD_dD_aD_bu_m|.$$

In  $E_m$ ,  $u_m$  has constant Laplacian, so

$$g^{ab}D_cD_dD_aD_bu_m = D_cD_dg^{ab}D_aD_bu_m = D_cD_d\Delta u_m = 0.$$

So we get

$$|g^{ab}D_aD_bD_cD_du_m| \geq -C^M |\nabla u_m| - C^M |D^2u_m|. \quad (3.5.16)$$

With this estimate, the estimate (3.5.11) yields

$$\Delta(|D^2u_m|^2) \geq (-C^M |\nabla u_m| - C^M |D^2u_m|) |D^2u_m| \geq -C^M |\nabla u_m|^2 - C^M |D^2u_m|^2.$$

Coming back to (3.5.10), we get:

$$\Delta(\det D^2u_m) \leq C^M |\nabla u_m|^2 + C^M |D^2u_m|^2. \quad (3.5.17)$$

To get a function with non-positive Laplacian, we need to add some terms to  $\det D^2u_m$  to get a non-positive right hand side in (3.5.17). First we use the Bochner-Weitzenböck formula (3.3.10) to get

$$\Delta(|\nabla u_m|^2) \geq |D^2u_m|^2 - C^M |\nabla u_m|^2. \quad (3.5.18)$$

With (3.5.17), we obtain

$$\Delta(\det D^2u_m - C^M |\nabla u_m|^2) \leq C^M |\nabla u_m|^2. \quad (3.5.19)$$

Secondly, we use the equality

$$\Delta(u_m^2) = 2|\nabla u_m|^2 + 2u_m\Delta u_m = 2|\nabla u_m|^2 - 2mu_m,$$

to get

$$\Delta(u_m^2) \geq 2|\nabla u_m|^2 - C^M m, \quad (3.5.20)$$

where we have used that  $u_m$  is bounded by the diameter of  $M$  since  $u_m \leq d_b$ . So we may transform (3.5.19) into

$$\Delta(\det D^2u_m - C^M |\nabla u_m|^2 - C^M u_m^2) \leq C^M m,$$

Finally, we use the equation  $\Delta u_m = -2m$  to get that for  $m$  large enough,

$$\Delta(\det D^2u_m - C^M |\nabla u_m|^2 - C^M u_m^2 + C^M u_m) \leq 0. \quad (3.5.21)$$

Now that we have a superharmonic function on  $E_m$ , we need to look at its boundary values to apply the maximum principle. Thus we need to control the second order derivatives of  $u_m$  near the boundary of  $E_m$ . As  $u_m$  is not  $C^2$  across  $\partial E_m$ , we cannot directly use the fact that the hessian of  $d_b$  is bounded on  $\partial E_m$ . We will use a theorem for obstacle problems on  $\mathbb{R}^n$ . Let us show that  $u_m$  is the solution of an obstacle problem on an open set of  $\mathbb{R}^2$ . Then, we will apply [37, Theorem 3.8] to conclude that

$$\forall x \in \partial E_m, \quad \lim_{\substack{y \rightarrow x \\ y \in E_m}} D^2 u_m(y) \leq C_d Id. \quad (3.5.22)$$

The minimality of  $u_m$  in (3.2.1) implies

$$-\Delta u_m - 2m \geq 0, \quad u_m \leq d_b \quad \text{and} \quad (-\Delta u_m - 2m)(u_m - d_b) = 0. \quad (3.5.23)$$

Let  $\tilde{\Omega}$  be defined as in the proof of 3.2.4. Let  $\phi : \tilde{\Omega} \rightarrow \tilde{U}$  be a normal coordinates chart. Writing down (3.5.23) in these coordinates, we find

$$A\tilde{u}_m - 2m \geq 0, \quad \tilde{u}_m \leq \psi \quad \text{and} \quad (A\tilde{u}_m - 2m)(\tilde{u}_m - \psi) = 0,$$

where  $A$  is the Laplacian of  $M$  in the coordinates defined by  $\phi$ ,  $\tilde{u}_m = u_m \circ \phi^{-1}$  and  $\psi = d_b \circ \phi^{-1}$ . This is the form of [37, Chapter 2, equation (3.16)], so we can apply [37, Chapter 2, Theorem 3.8], to deduce that

$$\forall p \in \partial E_m, \forall X \in \mathbb{R}^2 \quad \lim_{\substack{q \rightarrow p \\ q \in E_m}} D^2 \tilde{u}_m(\phi(q))(X, X) \leq D^2 \psi(\phi(p))(X, X). \quad (3.5.24)$$

Moreover, we have

$$\begin{aligned} D^2 \tilde{u}_m &= D^2 u_m \circ (D\phi^{-1}, D\phi^{-1}) + Du_m \circ D^2 \phi^{-1}, \\ D^2 \psi &= D^2 d_b \circ (D\phi^{-1}, D\phi^{-1}) + Dd_b \circ D^2 \phi^{-1}, \end{aligned}$$

and  $Du_m = Dd_b$  on  $\partial E_m$  because  $u_m$  is  $C^1$ . Thus, (3.5.24) yields:

$$\forall p \in \partial E_m, \forall X \in \mathbb{R}^2 \quad \lim_{\substack{q \rightarrow p \\ q \in E_m}} D^2 u_m(q)(X_q, X_q) \leq D^2 d_b(p)(X_p, X_p),$$

where we have set  $X_q := D\phi^{-1}(\phi(q))X$ . As  $d_b$  is  $C_d$ -semiconcave, with Proposition 3.5.4, we get

$$\forall p \in \partial E_m, \forall X \in \mathbb{R}^2 \quad \lim_{\substack{q \rightarrow p \\ q \in E_m}} D^2 u_m(q)(X_q, X_q) \leq C_d |X_p|^2. \quad (3.5.25)$$

From there, we deduce that for  $q \in E_m$  close enough to  $\partial E_m$ , we have

$$D^2 u_m(q) \leq C_d + 1/4. \quad (3.5.26)$$

Indeed, if not, there exist a sequence  $(q_k)$  of points of  $E_m$  whose distance to  $\partial E_m$  goes to 0, and a sequence  $(X_k)$  of unit vectors of  $\mathbb{R}^2$  such that for any  $k \in \mathbb{N}$ ,

$$D^2 u_m(q_k)((X_k)_{q_k}, (X_k)_{q_k}) > C_d + \frac{1}{4}. \quad (3.5.27)$$

As  $E_m$  is precompact, up to extracting a subsequence, we can assume that  $(q_k)$  converges to a point  $p \in \partial E_m$ , and  $(X_k)$  converges to a vector  $Y \in \mathbb{R}^2$ . Because of (3.5.25), we have

$$\lim_{k \rightarrow \infty} D^2 u_m(q_k)(Y_{q_k}, Y_{q_k}) \leq C_d. \quad (3.5.28)$$

Furthermore, we know from proposition 3.2.4 that  $D^2 u_m$  is locally bounded. As  $(X_k)_{q_k} - Y_{q_k}$  converges to 0 when  $k$  goes to  $\infty$ , this implies

$$\lim_{k \rightarrow \infty} D^2 u_m(q_k)((X_k)_{q_k}, (X_k)_{q_k}) - D^2 u_m(q_k)(Y_{q_k}, Y_{q_k}) = 0. \quad (3.5.29)$$

Inequalities (3.5.27), (3.5.28) and (3.5.29) yield a contradiction. So (3.5.26) is true.

Let  $q$  be such that (3.5.26) holds. Let  $\lambda \leq \mu$  be the eigenvalues of  $D^2 u_m(q)$ . The inequality (3.5.26) translates into

$$\mu \leq C_d + \frac{1}{4}. \quad (3.5.30)$$

Furthermore, inside  $E_m$  we have  $\Delta u_m = -2m$ , so

$$\lambda + \mu = -2m. \quad (3.5.31)$$

For  $m$  large enough, we see that (3.5.30) and (3.5.31) imply

$$\det D^2 u_m(q) = \lambda \mu = -\mu^2 - 2m\mu \geq -2(C_d + \frac{1}{2})m.$$

Recalling that  $|\nabla u_m| \leq 1$  (Theorem 3.3.2) and that  $u_m$  is bounded by the diameter of  $M$ , we deduce that for  $m$  large enough (depending on  $M$  only), at any point  $q \in E_m$  sufficiently close to  $\partial E_m$ ,

$$\det D^2 u_m - C^M |\nabla u_m|^2 - C^M u_m^2 + C^M u_m \geq -2(C_d + \frac{1}{2})m - C^M \geq -2(C_d + \frac{3}{4})m.$$

Recalling (3.5.21), by the maximum principle on  $E_m$ , we get

$$\det D^2 u_m - C^M |\nabla u_m|^2 - C^M u_m^2 + C^M u_m \geq -(C_d + \frac{3}{4})m.$$

Using the boundedness of the terms  $\nabla u_m$  and  $u_m$  once more, we find that inside  $E_m$  and for  $m$  large enough, (3.5.9) is valid. This concludes the proof.  $\square$

Now let us give a proof of Proposition 3.5.5 that works in any dimension. This technique is inspired from the proof of second order estimate for hessian equations in Riemannian manifolds by Guan in [45]. The reader may find the recalls of appendix A.2 useful to follow this proof. We will use many arguments and estimates from the proof of the 2-dimensional case.

*Proof of Proposition 3.5.5.* As in the preceding proof, we show that it is enough to prove that for  $m$  large enough,  $u_m$  is  $C$ -semiconcave in  $E_m$ , where  $C$  is a constant that depends only on  $M$ . Let  $C_1, C_2, C_3 > 0$  be some constants to be taken large enough later. For  $p \in E_m$  and  $X \in S^{n-1}(T_p M)$ , let us define

$$f(p, X) := D^2 u_m(X, X) + C_1 |\nabla u_m|^2(p) + C_2 u_m^2(p) - C_3 u_m(p). \quad (3.5.32)$$

We will show that for a good choice of constant  $C_i$  depending only on  $M$ ,  $f$  is bounded by a constant that depends only on  $M$ . This will conclude the proof since  $|\nabla u_m|^2$ ,  $u_m^2$  and  $u_m$  are also bounded by a constant depending only on  $M$ . Let us assume that there exist  $q \in E_m$  and  $Y \in S^{n-1}(T_q M)$  such that

$$f(q, Y) = \sup_{\substack{p \in E_m \\ X \in S^{n-1}(T_p M)}} f(p, X), \quad (3.5.33)$$

and show that it gives a contradiction. Let us pick some normal coordinates at  $q$  such that  $\partial_1(q) = Y$ . We then extend the vector  $Y$  into a vector field (still denoted by  $Y$ ) in a neighborhood of  $q$ , by setting  $Y := \partial_1 / |\partial_1|$ . As  $D\partial_1(q) = 0$ , we also have  $DY(q) = 0$ . Moreover, as the manifold is compact,  $D^2Y(q)$  is bounded by a constant that depends on  $M$  only. In the following, any constant that depends on  $M$  only will be denoted by  $C^M$ . We have

$$|D^2Y(q)| \leq C^M. \quad (3.5.34)$$

We will show that the Laplacian of  $p \mapsto f(p, Y(p))$  is positive at  $q$ , which contradicts the maximality of  $(q, Y(q))$  in (3.5.33). Let us estimate  $\Delta(D^2u_m(Y, Y))$  at the point  $q$ .

$$\begin{aligned} \Delta(D^2u_m(Y, Y)) &= g^{ab} D_a D_b (D_{cd}^2 u_m Y^c Y^d) \\ &= g^{ab} \left( D_{abcd}^4 u_m Y^c Y^d + D_{acd}^3 u_m D_b (Y^c Y^d) \right. \\ &\quad \left. + D_{bcd}^3 u_m D_a (Y^c Y^d) + D_{cd}^2 u_m D_{ab}^2 (Y^c Y^d) \right). \end{aligned}$$

The second term is null because it contains  $D_b(Y^c Y^d) = (D_b Y^c) Y^d + Y^c D_b Y^d$ , and  $DY = 0$ . The third term is also null. By our preceding remark (3.5.34) about  $D^2Y$ , the fourth term is bounded from below by  $-C^M - |D^2u_m|^2$ . Finally, the first term is treated just like in the preceding proof. Recall the estimate (3.5.16):

$$\left| g^{ab} D_a D_b D_c D_d u_m \right| \geq -C^M |\nabla u_m| - C^M |D^2u_m|.$$

From this and the fact  $Y$  has norm 1, we deduce

$$\left| g^{ab} D_a D_b D_c D_d u_m Y^c Y^d \right| \geq -C^M - |\nabla u_m|^2 - |D^2u_m|^2.$$

We obtain, at the point  $q$ ,

$$\Delta(D^2u_m(Y, Y)) \geq -C^M - |D^2u_m|^2 - |\nabla u_m|^2. \quad (3.5.35)$$

We recall the following from the preceding proof:

$$\begin{aligned} \Delta |\nabla u_m|^2 &\geq |D^2u_m|^2 - C^M |\nabla u_m|^2, \quad (\text{estimate (3.5.18)}) \\ \Delta u_m^2 &\geq |\nabla u_m|^2 - C^M m, \quad (\text{estimate (3.5.20)}) \\ \Delta u_m &= -2m. \end{aligned}$$

Using these three estimates and (3.5.35), we find some constants  $C_1^M, C_2^M, C_3^M > 0$  such that

$$\Delta(D^2u_m(Y, Y) + C_1^M |\nabla u_m|^2 + C_2^M u_m^2 - C_3^M u_m) > 0.$$

So, in the definition of  $f$  (3.5.32), if we take  $C_i = C_i^M$  for  $i = 1, 2, 3$ , we get

$$\Delta(f(p, Y(p)))_{p=q} > 0,$$

which contradicts the maximality of  $(q, Y(q))$ . Therefore the supremum in (3.5.33) is not attained in  $E_m$ . We conclude as in the preceding proof by showing that, by the semiconcavity of  $d_b$ ,  $f(p, X)$  is bounded by a constant independent of  $m$  when  $p$  gets close to  $\partial E_m$  and  $X \in S^{n-1}(T_p M)$ .  $\square$

Now, putting all sections together, only (3.1.11) remains to be proven in Theorem 3.1.2. This is the object of the following proposition.

**Proposition 3.5.6.** *Let  $\lambda > 0$ . Let  $Cut_b(M)_\lambda$  and  $E_{m,\lambda}$  be defined as in (3.1.6) and (3.1.7). We have*

$$\sup_{p \in E_{m,\lambda}} d(p, Cut_b(M)_\lambda) \xrightarrow{m \rightarrow +\infty} 0, \quad (3.5.36)$$

$$\text{and } \sup_{p \in Cut_b(M)_{\lambda+\varepsilon}} d(p, E_{m,\lambda}) \xrightarrow{m \rightarrow +\infty} 0. \quad (3.5.37)$$

*Proof.* Let us assume by contradiction that (3.5.36) does not hold. Let  $(m_k)_{k \geq 0}$  and  $(p_k)_{k \geq 0}$  be sequences in  $(0, \infty)$  and  $M$  respectively, such that  $m_k \xrightarrow{k \rightarrow +\infty} +\infty$  and for some  $\eta > 0$ , for any  $k \geq 0$ ,  $p_k \in E_{m_k,\lambda}$  and

$$d(p_k, Cut_b(M)_\lambda) > \eta. \quad (3.5.38)$$

Up to taking subsequences, we may assume that  $(p_k)$  converges to a point  $p_\infty$  in  $M$ . As the functions  $(u_m)_{m > 0}$  are uniformly semiconcave and converge to  $d_b$  as  $m$  goes to  $\infty$  (Proposition 3.5.5 and lemma 3.4.1), we get from Proposition A.1.8:

$$\liminf_{k \rightarrow \infty} |\nabla u_{m_k}(p_k)| \geq |\nabla d_b(p_\infty)|.$$

Using the facts that  $p_k \in E_{m_k,\lambda}$  and  $u_{m_k}$  converges uniformly to  $d_b$ , this implies

$$|\nabla d_b(p_\infty)|^2 \leq 1 - \frac{\lambda^2}{d_b(p_\infty)^2},$$

and so  $p_\infty \in Cut_b(M)_\lambda$ . But this contradicts (3.5.38). Thus (3.5.36) is proved.

For the proof of (3.5.37), once again we argue by contradiction and assume that it does not hold. Let  $(m_k)_{k \geq 0}$  and  $(p_k)_{k \geq 0}$  be sequences of  $(0, \infty)$  and  $M$  respectively, such that  $m_k \xrightarrow{k \rightarrow +\infty} +\infty$  and for some  $\eta > 0$ , for any  $k \geq 0$ ,  $p_k \in Cut_b(M)_{\lambda+\varepsilon}$  and

$$d(p_k, E_{m_k,\lambda}) > 2\eta. \quad (3.5.39)$$

Up to taking subsequences, we can assume that  $(p_k)$  converges to a point  $p_\infty$  in  $Cut_b(M)_{\lambda+\varepsilon}$ . The last inequality implies that for any  $k$  large enough, we have

$$d(p_\infty, E_{m_k,\lambda}) > \eta.$$

According to lemma 3.4.1, there exists a constant  $C_1 > 0$  such that for any  $p \in M$  and  $k \geq 0$ ,  $u_{m_k}(p)^2 \geq d_b(p)^2 - C_1/m_k$ . As  $d_b$  is 1-Lipschitz and bounded, we deduce that there exists a

constant  $C_2 > 0$  such that for any  $p \in B(p_\infty, \eta)$  and  $k \geq 0$ ,  $u_{m_k}(p)^2 \geq d_b(p_\infty)^2 - C_2\eta - C_1/m_k$ . Let  $\eta_0 > 0$ ,  $k_0 > 0$  and  $\delta_0 > 0$  be constants such that for any  $\eta \leq \eta_0$  and  $k \geq k_0$ ,

$$\delta := \frac{(\lambda + \varepsilon)^2}{d_b(p_\infty)^2} - \frac{\lambda^2}{d_b(p_\infty)^2 - C_2\eta - C_1/m_k} \geq \delta_0 > 0.$$

For any  $p \in B(p_\infty, \eta)$ , we have

$$\begin{aligned} 1 - \frac{\lambda^2}{u_{m_k}(p)^2} &\geq \left(1 - \frac{\lambda^2}{d_b(p_\infty)^2 - C_2\eta - C_1/m_k}\right) \\ &= 1 - \frac{(\lambda + \varepsilon)^2}{d_b(p_\infty)^2} + \delta. \end{aligned} \quad (3.5.40)$$

As  $p_\infty \in \text{Cut}_b(M)_{\lambda+\varepsilon}$ , we have  $\left(1 - \frac{(\lambda+\varepsilon)^2}{d_b(p_\infty)^2}\right) \geq 0$ , and so for any  $\eta \leq \eta_0$ ,  $k \geq k_0$  and  $p \in B(p_\infty, \eta)$ ,

$$1 - \frac{\lambda^2}{u_{m_k}(p)^2} \geq \delta \geq \delta_0 > 0. \quad (3.5.41)$$

Let  $(q_t)_{t \geq 0}$  be the curve defined by

$$q_0 = p_\infty \quad \text{and} \quad \frac{dq_t}{dt} = \nabla u_{m_k}(q_t).$$

Let  $T > 0$  be such that for any  $t \in [0, T]$ ,  $d(q_t, p_\infty) \leq \eta$ , and in particular  $q_t \notin E_{m_k, \lambda}$ . We have

$$\begin{aligned} u_{m_k}(q_T) - u_{m_k}(p_\infty) &= \int_{(0, T)} |\nabla u_{m_k}(q_t)|^2 dt \quad \text{by definition of } (q_t), \\ &\geq \int_{(0, T)} \left(1 - \frac{\lambda^2}{u_{m_k}(q_t)^2}\right) dt \quad \text{because } q_t \notin E_{m_k, \lambda}, \\ &\geq T\delta \quad \text{because of (3.5.41)}. \end{aligned}$$

As  $u_{m_k}$  is bounded by the diameter of  $M$ , this estimate implies that there exists a finite largest time  $T > 0$  such that for any  $t \in [0, T]$ ,  $d(q_t, p_\infty) \leq \eta$ . In particular,  $d(p_\infty, q_T) = \eta$ . Let  $\gamma$  be a unit speed minimizing geodesic between  $p_\infty$  and  $q_T$ . As  $d_b$  is semiconcave (Proposition 3.5.3), there exists a constant  $C_d > 0$  such that the function  $f : t \mapsto C_d t^2 - d_b(\gamma(t))$  is convex. In particular, we have  $f(0) - f(d(p_\infty, q_T)) \leq -f'(0)d(p_\infty, q_T)$ , which yields

$$\begin{aligned} d_b(q_T) - d_b(p_\infty) &\leq |\nabla d_b(p_\infty)| d(p_\infty, q_T) + C_d(d(p_\infty, q_T))^2 \\ &= |\nabla d_b(p_\infty)| \eta + C_d \eta^2, \\ &\leq \left(\sqrt{1 - \frac{(\lambda + \varepsilon)^2}{d_b(p_\infty)^2}}\right) \eta + C_d \eta^2 \quad \text{because } p_\infty \in \text{Cut}_b(M)_{\lambda+\varepsilon}. \end{aligned} \quad (3.5.42)$$

In addition,

$$\begin{aligned}
 u_{m_k}(q_T) - u_{m_k}(p_\infty) &= \int_{(0,T)} |\nabla u_{m_k}(q_t)| \left| \frac{dq_t}{dt} \right| dt \quad \text{by definition of } (q_t), \\
 &\geq \int_{(0,T)} \sqrt{1 - \frac{\lambda^2}{u_{m_k}(q_t)^2}} \left| \frac{dq_t}{dt} \right| dt \quad \text{because } q_t \notin E_{m_k, \lambda}, \\
 &\geq \left( \inf_{t \in (0,T)} \sqrt{1 - \frac{\lambda^2}{u_{m_k}(q_t)^2}} \right) \eta.
 \end{aligned} \tag{3.5.43}$$

Using lemma 3.4.1 again, we know that there exists a constant  $C_3 > 0$  such that  $d_b(q_T) - d_b(p_\infty) \geq u_{m_k}(q_T) - u_{m_k}(p_\infty) - \frac{C_3}{m_k}$ . Therefore, estimates (3.5.42) and (3.5.43) yield

$$C_d \eta^2 + \left( \sqrt{1 - \frac{(\lambda + \varepsilon)^2}{d_b(p_\infty)^2}} - \inf_{t \in (0,T)} \sqrt{1 - \frac{\lambda^2}{u_{m_k}(q_t)^2}} \right) \eta + \frac{C_3}{m_k} \geq 0.$$

Using (3.5.40), we find that there exists a constant  $C_4 > 0$  such that for any  $\eta \leq \eta_0$  and  $k \geq k_0$ ,

$$\left( \sqrt{1 - \frac{(\lambda + \varepsilon)^2}{d_b(p_\infty)^2}} - \inf_{t \in (0,T)} \sqrt{1 - \frac{\lambda^2}{u_{m_k}(q_t)^2}} \right) \leq -C_4.$$

In particular, for any  $\eta \leq \eta_0$  and  $k \geq k_0$ ,

$$C_d \eta^2 - C_4 \eta + \frac{C_3}{m_k} \geq 0.$$

This gives a contradiction if  $k$  is large enough. This concludes the proof.  $\square$





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## Numerical computation of the cut locus

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### 4.1 Introduction

Let  $S$  be a compact real analytic surface without boundary embedded in  $\mathbb{R}^3$ , and  $b \in S$  any point of  $S$  (that can be thought of as a base point).

**Definition 4.1.1.** *The cut locus of  $b$  in  $S$  can be defined as the closure of the set of points  $p \in S$  such that there exists at least two minimizing geodesics between  $p$  and  $b$ . We will denote it by  $Cut_b(S)$ . Equivalently, it is also the set of points around which the distance function to the point  $b$  - denoted by  $d_b$  - is not smooth.*

We refer to Chapter 2 for a more detailed introduction to the notion of cut locus. The cut locus is a fundamental object in Riemannian geometry, and it is a natural problem to try and find ways to compute it numerically. In this chapter, we propose a numerical approximation of  $Cut_b(S)$ , based on the convex variational problem on  $S$  (3.1.1), with proven convergence – see Section 4.5 for some numerical results. It is not trivial to compute  $Cut_b(S)$  because it is not stable with respect to  $C^1$ -small variations of  $S$ , see for instance [4, Example 2]. In particular, one cannot approximate the cut locus of  $S$  with the cut locus of a piecewise linear approximation of  $S$ .

**Related works.** Let us review the techniques used in the past by different authors to approximate the cut locus. We may divide them into two categories.

*Geodesic approximation on parametrized surfaces.* This approach was used in [76] and [63]. In [76], on genus 1 parametrized surfaces, the authors computed a degree 4 polynomial approximation of the exponential map using the geodesic equation, and deduced an approximation of the cut locus from there. In [63], the authors used the deformable simplicial complexes (DSC) method and finite differences techniques for geodesic computations, to compute geodesic circles of increasing radius and their self-intersection, *i.e.* the cut locus. They apply the method to genus 1 surfaces. These papers contain no proof of convergence of the computed cut locus.

*Exact geodesic computation on discretized surfaces.* This approach was used in [48] and [28]. In [48], the authors computed the geodesics on a convex triangulated surface. They deduced an approximation of the cut locus of the triangulated surface, and filtered it according to the angle formed by the geodesics meeting at a point of the approximated cut locus, to make their approximation stable. They applied the method to ellipsoids. There is no proof of convergence.

In [28], the authors computed shortest curves on a graph obtained from a sufficiently dense sample of points of a surface. From there they deduced an approximation of the cut locus, and filtered it according to the maximal distance (called *spread*) between the geodesics meeting at a point of the approximated cut locus. They proved that the set they compute converges to the cut locus (see [28, Theorem 4.1]).

We also mention [11], where the authors use some more geometric tools to compute (numerically) the cut locus of an ellipsoid, or a sphere with some particular metric with singularities.

**Our method.** The strategy we use is quite different. Given  $m > 0$  a constant, let  $u_m$  be the minimizer of the following variational problem

$$\inf \left\{ \int_S |\nabla_S u|^2 - mu : u \in H^1(M), |\nabla_S u| \leq 1, u(b) = 0 \right\}, \quad (4.1.1)$$

where  $\nabla_S$  denotes the gradient operator on the surface  $S$ . For  $\lambda > 0$  to be chosen small, we will use the set  $E_{m,\lambda} := \left\{ |\nabla_S u_m|^2 \leq 1 - \frac{\lambda^2}{u_m^2} \right\}$  as an approximation of  $Cut_b(S)$ . This is justified by Section 4.2, plus the theoretical results regarding problem (4.1.1) obtained in Chapter 3, which will be summarized in Section 4.3. Now the set  $E_{m,\lambda}$  can be well approximated using finite elements on a triangulation of the surface  $S$ .

The rest of the Chapter is organized as follows. In Section 4.2, we recall the definition of " $\lambda$ -cut locus"  $Cut_b(S)_\lambda$  and show that it can be used as an approximation of the compute cut locus. In Section 4.3, we recall the results from Chapter 3 which states that the set  $E_{m,\lambda}$  defined above is a good approximation of  $Cut_b(S)_\lambda$  if  $m$  is large enough. In Section 4.4, we discretize problem (4.1.1) using finite elements and prove that the " $\lambda$ -discrete  $E_{m,\lambda}$ " converges to  $E_{m,\lambda}$ . In Section 4.5, we present the results of some numerical experiments.

## 4.2 $\lambda$ -Cut locus

We recall that, according to [60, Proposition 3.4], the function  $d_b$  is locally semiconcave on  $S \setminus \{b\}$ , so it has a generalized gradient everywhere on  $S \setminus \{b\}$ , whose norm is given by the following formula (see propositions A.1.9 and A.1.6):

$$|\nabla_S d_b|(x) = \max(0, \sup_{v \in T_x S, |v|=1} \partial_v^+ d_b(x)).$$

We recall the identity (3.1.5):

$$Cut_b(M) = \overline{\{|\nabla d_b| < 1\}}.$$

In Section 3.1, following the definition of the  $\lambda$ -medial axis by Chazal and Lieutier, we define the  $\lambda$ -cut locus as

$$Cut_b(S)_\lambda := \left\{ x \in S \setminus \{b\} : |\nabla_S d_b(x)|^2 \leq 1 - \frac{\lambda^2}{d_b^2(x)} \right\},$$

where  $\nabla d_b$  denotes the generalized gradient of  $d_b$  wherever it is not differentiable. The following obvious proposition holds.

**Proposition 4.2.1.** *The map  $\lambda \mapsto Cut_b(S)_\lambda$  is non-increasing, and*

$$Cut_b(S) = \bigcup_{\lambda > 0} Cut_b(S)_\lambda.$$

In addition, we have the following proposition.

**Proposition 4.2.2.** *If  $S$  is a real analytic surface, then for  $\lambda > 0$  small enough, one of the connected component of  $Cut_b(S)_\lambda$  has the same homotopy type as  $Cut_b(S)$ , while the other connected components, if any, are contractible.*

These two propositions justify that  $Cut_b(S)_\lambda$  is a good approximation of  $Cut_b(S)$ , for  $\lambda > 0$  small enough.

Using Proposition A.1.10, Proposition 4.2.2 will mainly be a consequence of [28, Proposition 3.4] and the proof of [28, Proposition 3.5]. Following [28], we will use the following terminology. A point  $x$  of a finite graph  $G$  is called a tree point if  $G \setminus \{x\}$  has a connected component whose closure is a tree. Otherwise,  $x$  is called a cycle point. It is a consequence of the proof of [28, Proposition 3.5] that any closed connected subset of  $G$  that contains all cycle points is a deformation retract of  $G$ .

*Proof of Proposition 4.2.2.* As  $S$  is real analytic, the cut locus  $Cut_b(S)$  is a finite graph (see [66] in dimension 2, and [15] for the generalization to arbitrary dimensions). According to the Proposition A.1.10, given any  $\theta > 0$ , if  $\lambda$  has been taken small enough, then for any point  $x \in Cut_b(S) \setminus Cut_b(S)_\lambda$ , the angle between the minimizing geodesics from  $b$  to  $x$ , at the point  $x$ , is smaller than  $\theta$ . Given two unit speed minimizing geodesics  $\gamma_1$  and  $\gamma_2$ , following [28], the *spread* between  $\gamma_1$  and  $\gamma_2$  is defined as

$$spd(\gamma_1, \gamma_2) = \sup_t d(\gamma_1(t), \gamma_2(t)).$$

The geodesics  $\gamma_1$  and  $\gamma_2$  verify the geodesic equation:

$$\ddot{\gamma}^l + \Gamma_{ij}^l \dot{\gamma}^i \dot{\gamma}^j = 0,$$

where the  $\Gamma_{ij}^l$  are the Christoffel symbols of the metric of  $S$ . In particular, by continuity of the solution of a second order differential equation with respect to initial conditions, if the angle between  $\gamma_1$  and  $\gamma_2$  at their common starting point is small, then their spread is also small. Therefore, applying [28, Proposition 3.4], we deduce that if  $\lambda$  has been taken small enough, then any point  $x \in Cut_b(S) \setminus Cut_b(S)_\lambda$  is a *tree point* of  $Cut_b(S)$ . It remains to show that  $Cut_b(S)_\lambda$  is closed to conclude that it is a deformation retract of  $Cut_b(S)$  and conclude the proof. But this is a consequence of the fact that  $d_b$  is semiconcave, and the upper semicontinuity of the generalized gradient of convex functions (see also Proposition A.1.8 for instance).  $\square$

Therefore, we will use  $Cut_b(S)_\lambda$  as an approximation of  $Cut_b(S)$  for  $\lambda$  small enough.

### 4.3 Approximation with a variational problem

For  $m > 0$ , recall that  $u_m$  is the minimizer in (4.1.1). For  $\lambda > 0$ , let us define the set  $E_{m,\lambda}$  by

$$E_{m,\lambda} := \left\{ x \in S \setminus \{b\} : |\nabla_S u_m(x)|^2 \leq 1 - \frac{\lambda^2}{u_m^2(x)} \right\}.$$

We recall the following theorem from Chapter 3.

**Theorem 4.3.1** (Theorem 3.1.2). *There exists  $m_0 > 0$  such that for any  $m > m_0$ , the function  $u_m$  is locally  $C^{1,1}$  on  $S \setminus \{b\}$ . For any  $m > m' > m_0$ ,*

$$\text{Cut}_b(S) \subset \{|\nabla_S u_m| < 1\} \subset \{|\nabla_S u_{m'}| < 1\}. \quad (4.3.1)$$

Moreover,

$$\{|\nabla_S u_m| < 1\} \xrightarrow{m \rightarrow +\infty} \text{Cut}_b(S) \quad \text{in the Hausdorff sense.} \quad (4.3.2)$$

Finally, for any  $\varepsilon > 0$ ,

$$\sup_{x \in E_{m,\lambda}} d(x, \text{Cut}_b(S)_\lambda) \xrightarrow{m \rightarrow +\infty} 0, \quad \text{and} \quad \sup_{x \in \text{Cut}_b(S)_{\lambda+\varepsilon}} d(x, E_{m,\lambda}) \xrightarrow{m \rightarrow +\infty} 0. \quad (4.3.3)$$

Therefore, we can use  $E_{m,\lambda}$  as an approximation of  $\text{Cut}_b(S)_\lambda$ . In conclusion, we will use  $E_{m,\lambda}$  as an approximation of  $\text{Cut}_b(S)$ .

## 4.4 Discretization

### 4.4.1 Finite elements of order $r$ on a surface approximation of order $k$

In this section we introduce a discretization framework adapted to the variational problem (4.1.1) based on finite elements. We follow the notations of [27, 31].

Let  $S$  be a compact oriented smooth two-dimensional surface embedded in  $\mathbb{R}^3$ . For  $x \in S$ , we denote by  $\nu(x)$  the oriented normal vector field on  $S$ . Let  $d : \mathbb{R}^3 \rightarrow \mathbb{R}$  be the signed distance associated to  $S$  and  $U_\eta = \{x \in \mathbb{R}^3, |d(x)| < \eta\}$  the tubular neighborhood of  $S$  of width  $\eta > 0$ . It is well known that if  $\eta$  is small enough (for instance  $0 < \eta < \min_{i=1,2} \frac{1}{|\kappa_i|} L_\infty(S)$  where the  $(\kappa_i)$  stand for the extremal sectional curvatures of  $S$ ), then for every  $x \in U_\eta$  it exists a unique  $a(x) \in S$  such that

$$x = a(x) + d(x)\nu(a(x)) = a(x) + d(x)\nabla d(x). \quad (4.4.1)$$

We consider  $S_h^1$  a triangular approximation of  $S$  whose vertices lie on  $S$  and whose faces are quasi-uniform and shape regular of diameter at most  $h > 0$ . Moreover, we will assume that  $\mathcal{T}_h$ , the set of triangular faces of  $S_h$ , are contained in some tubular neighborhood  $U_\eta$  such that the map  $a$  defined by (4.4.1) is unique.

For  $k \geq 2$  and for a triangle  $T \in \mathcal{T}_h$ , we consider the  $n_k$  Lagrange basis functions  $\Phi_1^k, \dots, \Phi_{n_k}^k$  of degree  $k$  and define the discrete projection on  $S_h$  by:

$$a_k(x) = \sum_{j=1}^{n_k} a(x_j)\Phi_j^k(x) \quad (4.4.2)$$

where  $x_1, \dots, x_{n_k}$  are the nodal points associated to the basis functions. Now we can define  $S_h^k$  a polynomial approximation of order  $k$  of  $S$  associated to  $\mathcal{T}_h$

$$S_h^k = \{a_k(x), x \in S_h\}. \quad (4.4.3)$$

Observe that by definition the image by  $a$  of the nodal points are both on  $S$  and on  $S_h^k$ . Let us now introduce the finite element spaces on  $S_h = S_h^1$  and  $S_h^k$  for  $k \geq 2$ . For every integer  $r \geq 1$ , let

$$L_h^r = \{\chi \in C^0(S_h), \chi|_T \in \mathbb{P}_r, \forall T \in \mathcal{T}_h\} \quad (4.4.4)$$

where  $\mathbb{P}_r$  is the family of polynomials of degree at most  $r$ . Analogously, for  $k \geq 2$  let

$$L_h^{r,k} = \{\hat{\chi} \in C^0(S_h^k), \hat{\chi} = \chi \circ a_k^{-1}, \text{ for some } \chi \in L_h^r\}. \quad (4.4.5)$$

Analogously to (4.1.1), we define

$$\min_{\substack{u \in L_h^{r,k} \\ \left| \nabla_{S_h^k} u \right| \leq 1 \\ u(b)=0}} F_h^k(u) \quad (4.4.6)$$

where  $F_h^k(u) = \int_{S_h^k} \left( \left| \nabla_{S_h^k} u \right|^2 - mu \right)$  and  $b$  some fixed nodal points of the mesh  $\mathcal{T}_h$ .

#### 4.4.2 Convergence of the lifted minimizers

In order to prove the convergence of our numerical approach, let us first establish that our discrete problem converges in values in the sense of Proposition 4.4.2. For a function  $u$  defined on  $S_h^k$ , we introduce its lifted function  $u^l$  onto  $S$  defined by the relation  $u^l(b) = u(x)$  for  $b \in S$  where  $x$  is the unique point of  $S_h^k$  which satisfies  $a(x) = b$ .

Below, we focus our analysis in the piecewise linear case  $r = k = 1$  which contains all the main ingredients of a proof for the general  $(r, k)$  case. For every  $h > 0$ , the convex optimization problem (4.4.6) has a unique solution.

**Lemma 4.4.1.** *The differential of the projection  $a$  onto  $S$ , when restricted to the tangent space of  $S_h$ , is the identity, up to order 2 in  $h$ :*

$$Da|_{TS_h} = Id + \mathcal{O}(h^2).$$

*The second differential of  $a$ , when restricted to the tangent space of  $S_h$ , is null, up to order 1 in  $h$ :*

$$D^2a|_{TS_h} = \mathcal{O}(h).$$

*Proof.* The identity estimate on  $Da$  is a direct consequence of [31, equations (4.12), (4.13) and (4.11)], and the fact that, following the notations of [31, lemma 4.1], we have  $\nu_{n+1}^2 = 1 - \sum_{j \leq n} \nu_j^2$ . The estimate on  $D^2a$  follows from the same equations, plus the identity  $D^2a(x) = -2\nabla d(x)D^2d(x)$ . □

Defining  $F(u) = \int_S \left( |\nabla_S u|^2 - mu \right)$ , we have

**Proposition 4.4.2.** *Let  $u_{m,h}$  be the solution of problem (4.4.6) for  $k = r = 1$ . Let  $Lu_{m,h}^l := \frac{u_{m,h}^l}{|\nabla_S u_{m,h}^l|_{L^\infty(S)}}$  be the 1-Lipschitz normalization of  $u_{m,h}^l$ . Then,  $Lu_{m,h}^l \in H^1(S)$  and*

$$F(Lu_{m,h}^l) = \min_{\substack{u \in H^1(S) \\ \left| \nabla_S u \right| \leq 1 \\ u(b)=0}} F(u) + \mathcal{O}(h^{\frac{1}{2}}).$$

*Proof. step 1.* Let  $u_m$  be the solution of problem (4.1.1). For  $\varepsilon > 0$ , let  $u_{m,\varepsilon} : S \rightarrow \mathbb{R}$  be defined by:

$$u_{m,\varepsilon} = \begin{cases} \frac{d_b(x)^2}{2\varepsilon} & \text{if } d_b(x) \leq \varepsilon \\ u_m(x) - \frac{\varepsilon}{2} & \text{if } d_b(x) \geq \varepsilon. \end{cases}$$

According to lemma 3.2.3, we have  $u_m = d_b$  in a neighborhood of  $b$ . Therefore, for  $\varepsilon > 0$  small enough, we have  $u_m = d_b$  on  $B(b, 2\varepsilon)$ . In particular, we deduce that  $u_{m,\varepsilon}$  is  $C^1$  on  $S$ . As  $d_b^2$  is smooth in a neighborhood of  $b$ , the gradient of  $d_b^2/2\varepsilon$  is  $\mathcal{O}(\varepsilon^{-1})$ -Lipschitz on  $B(b, \varepsilon)$ . Moreover, as  $u_m = d_b$  on  $B(b, 2\varepsilon)$ , the gradient of  $u_m$  is  $\mathcal{O}(\varepsilon^{-1})$ -Lipschitz on  $B(b, 2\varepsilon) \setminus B(b, \varepsilon)$ . According to Proposition 3.2.4,  $u_m$  is also locally  $C^{1,1}$  on  $S \setminus \{b\}$ . Therefore its gradient is  $\mathcal{O}(\varepsilon^{-1})$ -Lipschitz on  $S \setminus B(b, \varepsilon)$ . Putting ths facts together, we obtain that  $u_{m,\varepsilon}$  is  $C^{1,1}$  on  $S$ , and the Lipschitz constant of its gradient is  $\mathcal{O}(\varepsilon^{-1})$ . Furthermore, as  $d_b$  and  $u_m$  are both 1-Lipschitz, we have  $|\nabla u_{m,\varepsilon}| \leq 1$ . Now for  $\varepsilon > 0$ , consider

$$v_{h,\varepsilon} := \frac{I_h u_{m,\varepsilon}}{|\nabla_{S_h} I_h u_{m,\varepsilon}|_{L^\infty(S_h)}},$$

where  $I_h u_{m,\varepsilon}$  is the  $\mathbb{P}^1$  Lagrange interpolation of  $u_{m,\varepsilon}$  on  $S_h$ . For  $x \in S_h$ , observe that we have the relation  $I_h u_{m,\varepsilon}(x) = I_h(u_{m,\varepsilon} \circ a)(x)$  which says that  $I_h u_{m,\varepsilon}$  is the standard (flat) interpolation of the composed function  $u_{m,\varepsilon} \circ a$ . From lemma 4.4.1, we know that on every triangle, the differential of  $a$  is  $\mathcal{O}(h)$ -Lipschitz, and  $a$  is  $\mathcal{O}(1)$ -Lipschitz. As the gradient of  $u_{m,\varepsilon}$  is  $\mathcal{O}(\varepsilon^{-1})$ -Lipschitz, we deduce that on every triangle, the gradient of  $u_m \circ a$  is  $\mathcal{O}(\varepsilon^{-1})$ -Lipschitz. By the quasi uniformity of the mesh, we obtain the uniform interpolation estimates on  $S_h$ :

$$I_h u_{m,\varepsilon}(x) = (u_{m,\varepsilon} \circ a)(x) + \mathcal{O}(\varepsilon^{-1}h^2) \quad (4.4.7)$$

and

$$\nabla_{S_h} I_h u_{m,\varepsilon}(x) = \nabla_{S_h} (u_{m,\varepsilon} \circ a)(x) + \mathcal{O}(\varepsilon^{-1}h).$$

With lemma 4.4.1, we deduce for all  $x \in S_h$ ,

$$\nabla_{S_h} I_h u_{m,\varepsilon}(x) = \nabla_S u_{m,\varepsilon}(a(x)) + \mathcal{O}(\varepsilon^{-1}h). \quad (4.4.8)$$

Recall that we have  $|\nabla_S u_{m,\varepsilon}|_{L^\infty(S)} = 1$ . Therefore the last identity yields

$$|\nabla_{S_h} I_h u_{m,\varepsilon}|_{L^\infty(S_h,\varepsilon)} = 1 + \mathcal{O}(\varepsilon^{-1}h).$$

Thus,  $v_{h,\varepsilon} = I_h u_{m,\varepsilon}(1 + \mathcal{O}(\varepsilon^{-1}h))$ , and so

$$F_h(v_{h,\varepsilon}) = F_h(I_h u_{m,\varepsilon}) + \mathcal{O}(\varepsilon^{-1}h). \quad (4.4.9)$$

Applying lemma 4.4.1 again, with a simple change of variable, we find that for any function  $f : S_h \rightarrow \mathbb{R}$ ,

$$\int_{S_h} f \circ a = \int_S f + \mathcal{O}(h^2).$$

Recalling (4.4.7) and (4.4.8), we obtain

$$F_h(I_h u_{m,\varepsilon}) = F(u_{m,\varepsilon}) + \mathcal{O}(\varepsilon^{-1}h). \quad (4.4.10)$$

Furthermore, we have

$$\int_S |u_{m,\varepsilon} - u_m| \leq \mathcal{O}(\varepsilon) \quad \text{and} \quad \int_S |\nabla u_{m,\varepsilon} - \nabla u_m|^2 \leq \mathcal{O}(\varepsilon^2),$$

so

$$F(u_{m,\varepsilon}) = F(u_m) + \mathcal{O}(\varepsilon).$$

Combining this with (4.4.9) and (4.4.10), we find

$$F_h(v_{h,\varepsilon}) = F(u_m) + \mathcal{O}(\varepsilon^{-1}h) + \mathcal{O}(\varepsilon).$$

Choosing  $\varepsilon = h^{\frac{1}{2}}$ , this yields

$$\min_{\substack{u \in H^1(S_h) \\ |\nabla_{S_h} u| \leq 1 \\ u(b)=0}} F_h \leq \min_{\substack{u \in H^1(S) \\ |\nabla_S u| \leq 1 \\ u(b)=0}} F + \mathcal{O}(h^{\frac{1}{2}}). \quad (4.4.11)$$

*step 2.* Symmetrically, let  $u_{m,h}$  the solution of the discrete problem (4.4.6),  $u_h^l := u_{m,h} \circ (a|_{S_h})^{-1}$  its lifted version on  $S$ , and  $Lu_{m,h}^l := \frac{u_h^l}{|\nabla_{S_h} u_h^l|_{L^\infty(S_h)}}$ . We show as before, using the equation  $u_{m,h} = u_h^l \circ a$ , that  $F(Lu_{m,h}^l) = F_h(u_{m,h}) + \mathcal{O}(h)$ . With (4.4.11), this implies

$$\min_{\substack{u \in H^1(S) \\ |\nabla_S u| \leq 1 \\ u(b)=0}} F \leq F(Lu_{m,h}^l) \leq \min_{\substack{u \in H^1(S) \\ |\nabla_S u| \leq 1 \\ u(b)=0}} F + \mathcal{O}(h^{\frac{1}{2}}),$$

which concludes the proof of the proposition.  $\square$

We can now establish the convergence of the minimizers:

**Proposition 4.4.3.**

$$\left| \nabla u_{m,h}^l - \nabla u_m \right|_{L^2(S)}^2 = \mathcal{O}(h^{\frac{1}{2}}) \quad \text{and} \quad \left| u_{m,h}^l - u_m \right|_{L^1(S)} = \mathcal{O}(h^{\frac{1}{2}}).$$

*Proof.* Consider  $v = \frac{1}{2}(Lu_{m,h}^l + u_m)$ . Then,  $v$  is admissible for problem (4.1.1), so  $F(v) \geq F(u_m)$ . Moreover, the following algebraic identity holds

$$F(v) = \frac{1}{2}F(Lu_{m,h}^l) + \frac{1}{2}F(u_m) - \frac{1}{4} \int_S |\nabla_S u_m - \nabla_S Lu_{m,h}^l|^2.$$

Therefore, we have

$$\frac{1}{2}F(Lu_{m,h}^l) - \frac{1}{2}F(u_m) \geq \frac{1}{4} \int_S |\nabla_S u_m - \nabla_S Lu_{m,h}^l|^2,$$

which proves, with Proposition 4.4.2, that

$$\left| \nabla Lu_{m,h}^l - \nabla u_m \right|_{L^2(S)}^2 = \mathcal{O}(h^{\frac{1}{2}}). \quad (4.4.12)$$



Furthermore, we have

$$F(Lu_{m,h}^l) - F(u_m) = \int_S \left( \left| \nabla Lu_{m,h}^l \right|^2 - |\nabla u_m|^2 \right) - m \int_S (Lu_{m,h}^l - u_m).$$

The last two equations imply

$$\left| Lu_{m,h}^l - u_m \right|_{L^1(S)} = \mathcal{O}(h^{\frac{1}{2}}). \quad (4.4.13)$$

As in the proof of Proposition 4.4.2, using the relation  $u_{m,h} = u_{m,h}^l \circ h$ , we show that  $Lu_{m,h}^l = u_{m,h}^l(1 + \mathcal{O}(h^2))$ . Together with (4.4.12) and (4.4.13), this concludes the proof.  $\square$

We just proved that the sequence of the lifted minimizers converges with an order at least  $1/2$  to the minimizer of problem (4.1.1). By analogy with the more standard variational context [27, 31], we expect a convergence of order  $\mathcal{O}((h^r + h^{k+1})^{\frac{1}{2}})$  using an approximation of orders  $(r, k)$ .

### 4.4.3 Convergence in measure

Let us recall that the set  $E_{m,\lambda}$  is defined by

$$E_{m,\lambda} = \left\{ x \in S \setminus \{b\} : |\nabla_S u_m(x)|^2 \leq 1 - \frac{\lambda^2}{u_m^2(x)} \right\}.$$

**Proposition 4.4.4.** *For any  $\lambda > 0$  and  $\varepsilon > 0$  with  $\varepsilon < \lambda/2$ , let us define*

$$E_{m,\lambda,h} := \left\{ x \in S \setminus \{b\} : \left| \nabla_S u_{m,h}^l(x) \right|^2 \leq 1 - \frac{\lambda^2}{(u_{m,h}^l)^2(x)} \right\}.$$

Then, we have

$$|E_{m,\lambda+\varepsilon} \setminus E_{m,\lambda,h}| = \mathcal{O}(h^{\frac{1}{2}}) \quad \text{and} \quad |E_{m,\lambda,h} \setminus E_{m,\lambda-\varepsilon}| = \mathcal{O}(h^{\frac{1}{2}}).$$

*Proof.* By definition of  $E_{m,\lambda}$  and  $E_{m,\lambda,h}$ , we have

$$E_{m,\lambda+\varepsilon} \setminus E_{m,\lambda,h} \subset \left\{ \left| \nabla u_{m,h}^l \right|^2 - |\nabla u_m|^2 > \frac{(\lambda + \varepsilon)^2}{u_m^2} - \frac{\lambda^2}{(u_{m,h}^l)^2} \right\}.$$

Therefore, on  $E_{m,\lambda+\varepsilon} \setminus E_{m,\lambda,h}$ , we have

$$\begin{aligned} \left| \nabla u_{m,h}^l \right|^2 - |\nabla u_m|^2 &> \frac{(\lambda + \varepsilon)^2 - \lambda^2}{u_m^2} + \lambda^2 \left( \frac{1}{u_m^2} - \frac{1}{(u_{m,h}^l)^2} \right) \\ &\geq \frac{2\varepsilon\lambda + \varepsilon^2}{u_m^2} - \lambda^2 \frac{2}{\min(u_m, u_{m,h}^l)^3} |u_m - u_{m,h}^l| \\ &= \frac{2\varepsilon\lambda + \varepsilon^2}{(\text{diam } S)^2} - \lambda^2 \frac{2}{(u_m + \mathcal{O}(h^{\frac{1}{2}}))^3} |u_m - u_{m,h}^l|, \end{aligned}$$

where  $\text{diam } S$  is the diameter of  $S$ . By definition of  $E_{m,\lambda}$ , we also have  $E_{m,\lambda+\varepsilon} \subset \{u_m \geq (\lambda + \varepsilon)\}$ , so on  $E_{m,\lambda+\varepsilon} \setminus E_{m,\lambda,h}$ ,

$$\left| \nabla u_{m,h}^l \right|^2 - |\nabla u_m|^2 > \frac{2\varepsilon\lambda + \varepsilon^2}{(\text{diam } S)^2} - \lambda^2 \frac{2}{(\lambda + \varepsilon + \mathcal{O}(h^{\frac{1}{2}}))^3} |u_m - u_{m,h}^l|.$$

So for  $h$  large enough, we have

$$\begin{aligned} & E_{m,\lambda+\varepsilon} \setminus E_{m,\lambda,h} \\ & \subset \left\{ \left| \nabla u_{m,h}^l \right|^2 - |\nabla u_m|^2 + 2\lambda |u_m - u_{m,h}^l| > \frac{2\varepsilon\lambda + \varepsilon^2}{(\text{diam } S)^2} \right\} \\ & \subset \left\{ \left| \nabla u_{m,h}^l \right|^2 - |\nabla u_m|^2 > \frac{2\varepsilon\lambda + \varepsilon^2}{2(\text{diam } S)^2} \right\} \cup \left\{ |u_m - u_{m,h}^l| > \frac{2\varepsilon\lambda + \varepsilon^2}{4\lambda(\text{diam } S)^2} \right\}. \end{aligned}$$

Now from Proposition 4.4.3, we know that for any  $\eta > 0$ , we have the following estimates

$$\eta \left| \left\{ \left| \nabla u_{m,h}^l \right|^2 - |\nabla u_m|^2 > \eta \right\} \right| \leq \int_S \left| \left| \nabla u_{m,h}^l \right|^2 - |\nabla u_m|^2 \right| = \mathcal{O}(h^{\frac{1}{2}}),$$

and

$$\eta \left| \left\{ |u_m - u_{m,h}^l| > \eta \right\} \right| \leq \int_S |u_m - u_{m,h}^l| = \mathcal{O}(h^{\frac{1}{2}}).$$

This gives the estimate  $|E_{m,\lambda+\varepsilon} \setminus E_{m,\lambda,h}| = \mathcal{O}(h^{\frac{1}{2}})$ . The other estimate is proved by the same method.  $\square$

*Remark 4.4.5.* We expect a convergence of order  $\mathcal{O}((h^r + h^{k+1})^{\frac{1}{2}})$  using an approximation of orders  $(r, k)$ .

All the preceding sections together justify that the set  $E_{m,\lambda,h}$  is a good approximation of the cut locus of  $b$  in  $M$ , if  $m$  is large enough, and  $\lambda$  and  $h$  are small enough.

## 4.5 Numerical illustrations

### 4.5.1 Cut locus approximation

We established the convergence of the minimizers of solutions of problems (4.1.1) when  $h$  tends to 0. For a fixed  $h > 0$ , this convex discrete problems is of quadratic type with an infinite number of conic pointwise constraints. By the way, it is important to observe that for  $k = r = 1$ , the gradient pointwise bounds for a function of  $\mathbb{P}^1$  is equivalent to a single discrete conic constraint on every triangle with respect to the degrees of freedom of  $\mathbb{P}^1(\mathcal{T}_h)$ .

Nevertheless, we observed in our experiments that using  $\mathbb{P}^1$  elements may lead to approximated cut loci with some tiny artificial connected components. Motivated by this lack of precision, we use in all following illustrations elements of order  $r > 1$ .

For the general case  $r > 1$ , the bound constraint on the gradient can not be easily reduced to a finite set of discrete constraints. In our experiments, we approximated the constraint  $|\nabla_{S_h^k} u|_{L^\infty(S_h^k)} \leq 1$  by forcing this constraint only on a finite number of points of the mesh. In practice, we imposed these constraints on the Gauss quadrature points of order  $g$  on every triangle of  $\mathcal{T}_h$ .

We illustrate in Figures 4.1, 4.2, 4.3 and 4.4 the approximation of the cut locus provided by our approach. These computations have been carried out on meshes of approximated  $10^5$  triangles for  $k = 2$  and  $r = 3$  using high precision quadrature formula associated to 17 Gauss points on every element of the mesh. Moreover, for  $r = 3$ , we imposed the conic gradient constraints on the  $g = 9$  Gauss points of every triangle. In order to solve the resulting linear conic constrained quadratic optimization problem, we used the *JuMP* modeling language and the finite elements library *Getfem++* [30, 69] combined with *Mosek* optimization solver [7]. For such a precision, the optimization solver identified a solution in less than one hour on a standard computer.



Figure 4.1: Three different views of the approximation of a cut locus on a standard torus



Figure 4.2: Three different views of the approximation of a cut locus on a standard torus, without representing the surface

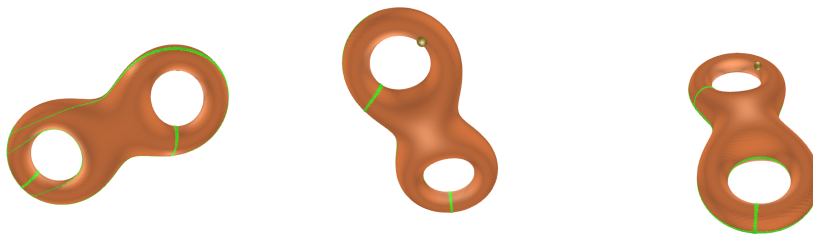


Figure 4.3: Three different views of the approximation of a cut locus on a torus of genus 2

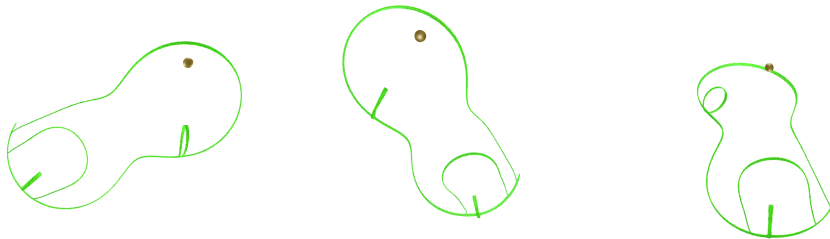


Figure 4.4: Three different views of the approximation of a cut locus on a torus of genus 2, without representing the surface

### 4.5.2 Approximation of the boundary of Voronoi cells

All previous theoretical results still hold if we replace the source point  $b$  by any compact subset of the surface  $S$ . For instance, if  $b$  is replaced by a set of points, the singular set of the distance function can be decomposed as the union of the boundary of voronoi cells and the cut loci of every point intersected with its voronoi cell. As a consequence, if the distribution of source points is homogeneous enough, that is every voronoi cell is small enough, the singular part of the distance function will be exactly equal to the boundary of the voronoi cells. We illustrate this remark in the following experiments. We used exactly the same framework as in previous sections and just replaced the pointwise condition at  $b$  with the analogous pointwise Dirichlet conditions at every source point. Figure 4.5 and 4.6 represent the voronoi diagrams obtained with 10, 30 and 100 points for surfaces of genus 2 and 3. The expected computational complexity is exactly of the same order as with a single source point.

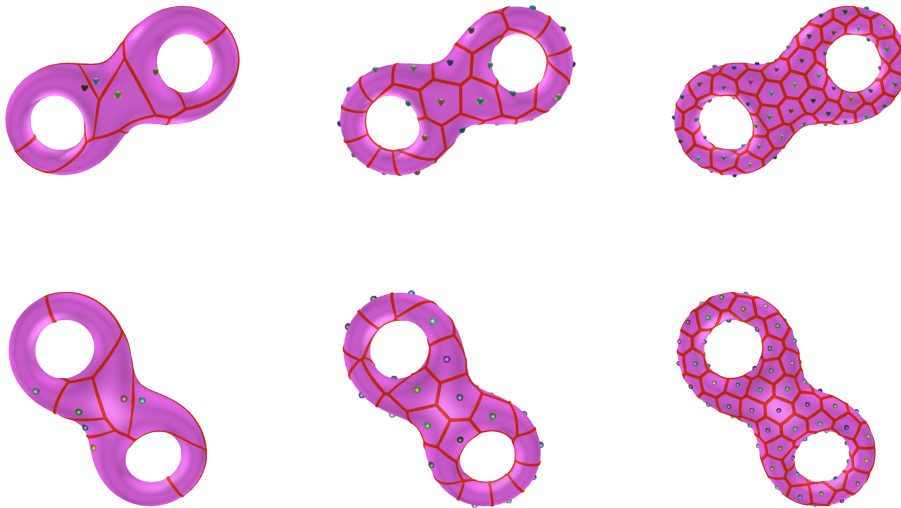


Figure 4.5: Approximation of the voronoi cells on a torus of genus 2 of 10, 30 and 100 points. Every column represent two different views

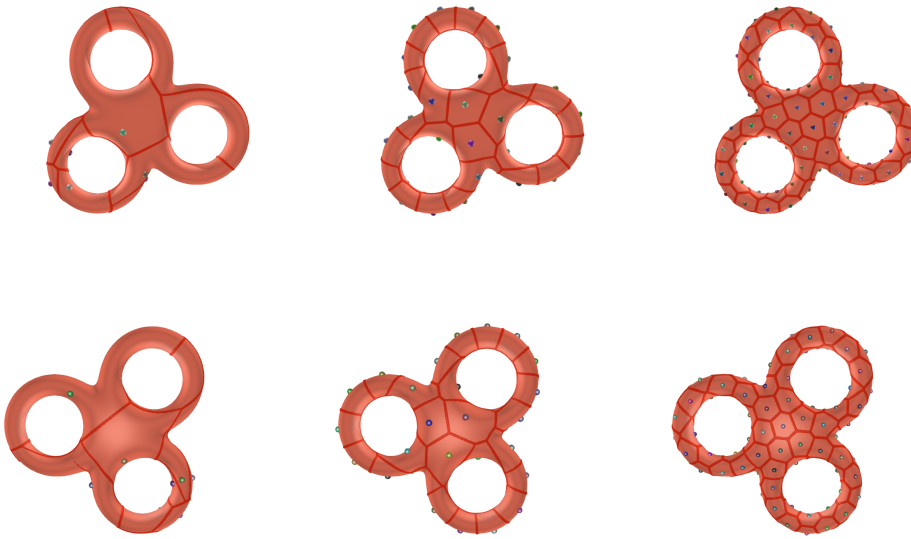


Figure 4.6: Approximation of the voronoi cells on a torus of genus 3 of 10, 30 and 100 points. Every column represent two different views

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## Appendices to part I

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### A.1 Semiconcavity

Some of the properties of semiconcave functions we need are given in [68], in the framework of Alexandrov spaces, and sometimes without proof. A more detailed reference for semiconcave functions in the framework of euclidean spaces is [22]. In this appendix, we give complete proofs of the properties of semiconcave functions that we need in the framework of smooth Riemannian manifolds. To our knowledge, Proposition A.1.4 is not proved (or even actually stated) anywhere in the literature. We could not find a proof of the formula for the directional derivatives of distance functions (Proposition A.1.10) in the Riemannian context either. Here,  $M$  is any smooth Riemannian manifold without boundary, and  $g$  its metric.

For any  $a, b \in \mathbb{R}$  and  $\lambda \in [0, 1]$ , we define  $\lambda_{ab} := (1 - \lambda)a + \lambda b$ . Let us recall the definition of semiconcavity.

**Definition A.1.1.** *Given a constant  $C > 0$ , a function  $u : M \rightarrow \mathbb{R}$  is said to be  $C$ -semiconcave on  $M$  if and only if for any unit speed geodesic  $\gamma : [a, b] \rightarrow M$ , the function  $t \mapsto Ct^2 - u(\gamma(t))$  is convex. This is equivalent to having, for any  $\lambda \in [0, 1]$ ,*

$$\lambda_{u(\gamma(a))u(\gamma(b))} - u(\gamma(\lambda_{ab})) \leq C\lambda(1 - \lambda)(b - a)^2.$$

*We say that  $u$  is semiconcave if  $u$  is  $C$ -semiconcave for some constant  $C > 0$ . We say that  $u$  is locally semiconcave if for any  $p \in M$ ,  $u$  is semiconcave in a neighborhood of  $p$ .*

*Remark A.1.2.* On  $\mathbb{R}^n$ , a function  $u$  is  $C$ -semiconcave if and only if the map  $x \mapsto C|x|^2 - u(x)$  is convex.

We will need the following lemma to estimate the difference between two geodesics linking a pair of given points, for two different metrics.

**Lemma A.1.3.** *Let  $g$  be a metric on the unit ball  $B(0, 1) \subset \mathbb{R}^n$ . There exists a constant  $B > 0$  such that for any unit speed geodesic  $\gamma : [a, b] \rightarrow (B(0, 1), g)$  and  $\lambda \in [0, 1]$ , we have*

$$|\gamma(\lambda_{ab}) - \lambda_{\gamma(a)\gamma(b)}| \leq B\lambda(1 - \lambda)(b - a)^2.$$

*Proof.* It suffices to prove that the estimate holds for  $\lambda \leq \frac{1}{2}$ , as the case  $\lambda \geq \frac{1}{2}$  can be deduced by considering  $\tilde{\gamma} : t \mapsto \gamma(b-t)$  instead of  $\gamma$ . A unit speed geodesic  $\gamma : [a, b] \rightarrow (B(0, 1), g)$  verifies the geodesic equation

$$\ddot{\gamma}^l + \Gamma_{ij}^l \dot{\gamma}^i \dot{\gamma}^j = 0,$$

where the  $\Gamma_{ij}^l$  are the Christoffel symbols of the metric  $g$ . As  $\gamma$  is unit speed, the  $(\dot{\gamma}^i)$  are bounded, uniformly in  $\gamma$ . Therefore, there exists a constant  $\alpha > 0$  independent of  $\gamma$  such that  $|\ddot{\gamma}| \leq \alpha$ . By integration, we find  $|\dot{\gamma}(t) - \dot{\gamma}(a) - \ddot{\gamma}(a)(t-a)| \leq \alpha(t-a)^2$ . Evaluating this expression at  $b$  yields  $|\dot{\gamma}(b) - \dot{\gamma}(a) - \ddot{\gamma}(a)(b-a)| \leq \alpha(b-a)^2$ . From these two estimates, we deduce

$$\left| \dot{\gamma}(t) - \dot{\gamma}(a) - \frac{\dot{\gamma}(b) - \dot{\gamma}(a)}{b-a}(t-a) \right| \leq \alpha(t-a)^2 + \alpha(b-a)(t-a).$$

Taking  $t = (1-\lambda)a + \lambda b$  in this estimate yields

$$\begin{aligned} |\dot{\gamma}((1-\lambda)a + \lambda b) - ((1-\lambda)\dot{\gamma}(a) + \lambda\dot{\gamma}(b))| &\leq \alpha\lambda(1+\lambda)(b-a)^2 \\ &= \frac{\alpha(1+\lambda)}{1-\lambda}\lambda(1-\lambda)(b-a)^2. \end{aligned}$$

Taking  $B := \frac{\alpha(1+1/2)}{1-1/2}$ , this proves the desired estimate when  $\lambda \leq \frac{1}{2}$ . This concludes the proof.  $\square$

We can now prove that defining local semiconcavity through charts (as in [60]), or through geodesics, is the same.

**Proposition A.1.4.** *Let  $u : M \rightarrow \mathbb{R}$  be a locally Lipschitz function. Then,  $u$  is locally semiconcave if and only if for any chart  $\psi$  of  $M$ ,  $u \circ \psi^{-1}$  is locally semiconcave as a function on  $\mathbb{R}^n$ .*

*Proof.* Let us assume that  $u$  is locally semiconcave. Let  $\psi : U \rightarrow V$  be a chart from an open set  $U$  of  $M$  to an open set  $V$  of  $\mathbb{R}^n$ , and  $y \in V$ . Let  $f := u \circ \psi^{-1}$ . We want to show that  $f$  is semiconcave in a neighborhood of  $y$ , as a function of  $\mathbb{R}^n$ . We first observe that  $f$  is locally semiconcave on the manifold  $(V, \psi_*g)$ . Let  $V' \subset V$  be a neighborhood of  $y$  that is geodesically convex for the metric  $\psi_*g$ , and such that there exists a constant  $C > 0$  such that  $f$  is  $C$ -semiconcave on  $(V', \psi_*g)$ . Let  $d$  denote the distance function on  $(V', \psi_*g)$ . Up to taking  $V'$  smaller, we may assume that the metric  $\psi_*g$  is bounded on  $V'$ , and so there exists a constant  $\beta > 0$  such that

$$\forall x, y \in V', \quad d(x, y) \leq \beta |x - y|.$$

Let  $x, y \in V'$  be such that  $[x, y] \subset V'$ , and  $\lambda \in [0, 1]$ . Let  $\gamma : [a, b] \rightarrow V'$  be a unit speed geodesic of  $(V', \psi_*g)$  from  $x$  to  $y$ . By the  $C$ -semiconcavity of  $f$  on  $(V', \psi_*g)$ , we have

$$\begin{aligned} \lambda_{f(x)f(y)} - f(\lambda_{xy}) &= \lambda_{f(\gamma(a))f(\gamma(b))} - f(\lambda_{\gamma(a)\gamma(b)}) \\ &\leq C\lambda(1-\lambda)(b-a)^2 + f(\gamma(\lambda_{ab})) - f(\lambda_{\gamma(a)\gamma(b)}) \\ &\leq C\lambda(1-\lambda)(b-a)^2 + \text{Lip}(f) |\gamma(\lambda_{ab}) - \lambda_{\gamma(a)\gamma(b)}|. \end{aligned}$$

Applying lemma A.1.3 above, we get a constant  $B > 0$  such that

$$\begin{aligned} \lambda_{f(x)f(y)} - f(\lambda_{xy}) &\leq (C + \text{Lip}(f)B)\lambda(1-\lambda)(b-a)^2 \\ &= (C + \text{Lip}(f)B)\lambda(1-\lambda)(d(x, y))^2 \\ &\leq (C + \text{Lip}(f)B)\beta^2\lambda(1-\lambda) |x - y|^2, \end{aligned}$$

and so  $f$  is semiconcave on  $V'$ , as a function of  $\mathbb{R}^n$ .

Reciprocally, let us assume that  $u \circ \psi^{-1}$  is locally semiconcave as a function of  $\mathbb{R}^n$  for any chart  $\psi$ . Then, we can show that  $u \circ \psi^{-1}$  is locally semiconcave for the metric  $\psi_*g$ , for any chart  $\psi$ , by using the same technique. From there we deduce that  $u$  is locally semiconcave. This concludes the proof.  $\square$

*Remark A.1.5.* One can show that any semiconcave function on  $M$  is also Lipschitz, using the analogous property for functions on the real line. Therefore, the Lipschitzianity assumption is not actually needed.

By composition with charts, semiconcave functions inherits some properties of concave functions. More precisely, let  $\psi : B(p, r) \rightarrow B(0, r) \subset \mathbb{R}^n$  be a normal coordinate chart at  $p$ , for some  $r > 0$ . The function  $u \circ \psi^{-1}$  is semiconcave as a function of  $\mathbb{R}^n$ . Furthermore, the differential of  $\psi$  at  $p$  is an isometry. The following proposition then follows from the properties of concave functions on  $\mathbb{R}^n$ .

**Proposition and definition A.1.6.** *Let  $u : M \rightarrow \mathbb{R}$  be a locally Lipschitz and semiconcave function. At every point  $p \in M$ ,  $u$  admits a directional derivative  $\partial_v^+ u(p)$  in any direction  $v \in T_p M \setminus \{0\}$ , defined by*

$$\partial_v^+ u(p) = \frac{d}{dt}[u(\gamma(t))]_{t=0},$$

where  $\gamma : [0, 1] \rightarrow M$  is any curve such that  $\gamma(0) = p$  and  $\dot{\gamma}(0) = v$ .

Moreover, the map  $v \mapsto \partial_v^+ u(p)$  is concave and positively homogeneous on  $T_p M$ . In particular, if it attains a positive maximum in the closed unit ball of  $T_p M$ , then this maximum is attained at a unique vector  $v_p$ . Otherwise, we set  $v_p = 0$ . We define the generalized gradient of  $u$  at  $p$  by

$$\nabla u(p) := \partial_{v_p}^+ u(p)v_p.$$

The norm of  $\nabla u(p)$  is given by the following formula:

$$|\nabla u(p)| = \max(0, \max_{v \in T_p M, |v|=1} \partial_v^+ u(p)).$$

**Proposition A.1.7.** *Let  $u : M \rightarrow \mathbb{R}$  be a locally Lipschitz and locally semiconcave function. Let  $x, y \in M$  be such that there exists a geodesic from  $x$  to  $y$  (which is always the case if  $M$  is complete). Then,*

$$u(y) \leq u(x) + |\nabla u(x)| d(x, y) + Cd(x, y)^2$$

*Proof.* Let  $\gamma : [0, d(x, y)] \rightarrow M$  be a geodesic from  $x$  to  $y$ . As the function  $f : t \mapsto Ct^2 - u \circ \gamma(t)$  is convex, we have

$$f(d(x, y)) \geq f(0) + f'(0)d(x, y).$$

The estimate follows.  $\square$

**Proposition A.1.8.** *Let  $C > 0$ . If  $(u_k)$  is a sequence of locally Lipschitz  $C$ -semiconcave functions that converges pointwise to a function  $u_\infty$ , then  $u_\infty$  is also semiconcave, and for any sequence of points  $(p_k)$  that converges to a point  $p_\infty$ , we have*

$$\liminf_{k \rightarrow \infty} |\nabla u_{m_k}(p_k)| \geq |\nabla u_\infty(p_\infty)|.$$



*Proof.* Let  $\varepsilon > 0$ . Let  $q \in M$  be such that

$$|\nabla u_\infty(p_\infty)| \leq \frac{u_\infty(q) - u_\infty(p_\infty)}{d(q, p_\infty)} - \varepsilon.$$

Then, we have

$$\begin{aligned} |\nabla u_\infty(p_\infty)| &\leq \liminf_{k \rightarrow \infty} \frac{u_k(q) - u_k(p_k)}{d(q, p_k)} - \varepsilon \\ &\leq \liminf_{k \rightarrow \infty} |\nabla u_k(p_k)| + 2Cd(q, p_k) - \varepsilon \\ &= \liminf_{k \rightarrow \infty} |\nabla u_k(p_k)| - \varepsilon, \end{aligned}$$

where we have used the Proposition A.1.7 above in the second estimate. As it is valid for any  $\varepsilon$ , this concludes the proof.  $\square$

We recall the following proposition.

**Proposition A.1.9** (Proposition 3.4 in [60]). *For any compact subset  $K$  of a smooth manifold without boundary  $M$ , the distance function to the set  $K$  is locally semiconcave on  $M \setminus K$ .*

We end this appendix with the following proposition about the generalized gradient of the distance function to a general compact set  $K$ .

**Proposition A.1.10.** *Let  $M$  be a smooth Riemannian manifold without boundary,  $K$  a compact subset of  $M$ , and  $d_K$  the distance function to  $K$ . Let  $p$  be a point of  $M$  such that there exist several minimizing geodesics from  $p$  to  $K$ . We denote the set of unit speed geodesics from  $p$  to  $K$  that are minimizing between  $p$  and  $K$  by  $\text{geod}(p, K)$ . For any  $v \in T_p M$ , we have*

$$\partial_v^+ d_K(p) = \min_{\gamma \in \text{geod}(p, K)} -\dot{\gamma}(0) \cdot v.$$

In particular,

$$|\nabla d_K|(p) = \max\{0, \max_{v \in T_p M, |v|=1} \min_{\gamma \in \text{geod}(p, K)} -\dot{\gamma}(0) \cdot v\}.$$

A similar result for the distance function to a point is proved in [1, Theorem 4.5.6], in the more general framework of Alexandrov spaces, but with some additional restrictions on the curvature of the ambient space. A similar result for the directional derivatives of the function  $d(\cdot, \cdot)$  is also stated in [1, Lemma 3.2], but the proof given there mainly consists in saying that the proof in [1, Theorem 4.5.6] also works in the case of Riemannian manifolds without the restriction on the curvature. For any reader that is not familiar with Alexandrov spaces, that is not very satisfying. So here is a complete proof in the Riemannian context.

*Proof.* Let  $\gamma : [0, d_K(p)] \rightarrow M$  be a geodesic of  $\text{geod}(p, K)$ . Let  $a = \gamma(d_K(p)/2)$ . As  $\gamma$  is minimizing between  $p$  and  $\gamma(d_K(p))$ , we have  $a \notin \text{Cut}_p(M)$ , and so  $p \notin \text{Cut}_a(M)$ . In particular, the function  $d_a$  is differentiable at  $p$ , and  $\nabla d_a(p) = -\dot{\gamma}(0)$ . Thus, we have

$$\begin{aligned} \frac{d_K(\exp_p(tv)) - d_K(p)}{t} &\leq \frac{d_a(\exp_p(tv)) + d_K(a) - d_K(p)}{t} \\ &= \frac{d_a(\exp_p(tv)) - d_a(p)}{t} \\ &\rightarrow -\dot{\gamma}(0) \cdot v \quad \text{as } t \rightarrow 0. \end{aligned}$$

This proves

$$\partial_v^+ d_K(p) \leq \min_{\gamma \in \text{geod}(p, K)} -\dot{\gamma}(0) \cdot v. \quad (\text{A.1.1})$$

Now, for every  $t > 0$ , let  $\gamma_t \in \text{geod}(\exp_p(tv), K)$ . For  $t$  small enough, the length of  $\gamma_t$  is bounded by  $d_K(p) + 1$ . By compactness of the set of geodesics of length bounded by a given constant, there exists a sequence of positive numbers  $(t_n)_{n \geq 0}$  that converges to 0, such that  $\gamma_n := \gamma_{t_n}$  converges to a unit speed geodesic  $\gamma$  as  $n \rightarrow +\infty$ . As  $K$  is closed,  $\gamma$  is a geodesic from  $p$  to  $K$ . Moreover, we have

$$\text{length}(\gamma) = \lim_{n \rightarrow \infty} \text{length}(\gamma_n) = \lim_{n \rightarrow \infty} d_K(\exp_p(t_n v)) = d_K(p),$$

so  $\gamma \in \text{geod}(p, K)$ . Let  $R = \min\{\text{inj}(M), d_K(p)/2\}$ , where  $\text{inj}(M)$  is the injectivity radius of  $M$ . In particular for any  $(x, y)$  such that  $d(x, y) < R$  and  $x \neq y$ , the distance function  $d(\cdot, \cdot)$  is smooth in a neighborhood of  $(x, y)$  in  $M \times M$ . For  $n \in \mathbb{N}$ , let  $b_n := \gamma_n(R)$ , and  $b_\infty = \gamma(R)$ . Let  $U, V \subset M$  be precompact neighborhoods of  $p$  and  $b_\infty$  respectively such that  $d(\cdot, \cdot)$  is smooth on  $\bar{U} \times \bar{V}$ . For  $n$  large enough, we have  $\exp_p(t_n v) \in U$  and  $b_n \in V$ , and so

$$\begin{aligned} d_K(p) &\leq d_K(b_n) + d(b_n, p) \\ &= d_K(\exp_p(t_n v)) - d(b_n, \exp_p(t_n v)) + d(b_n, p) \\ &= d_K(\exp_p(t_n v)) - \nabla_2 d(b_n, p) \cdot v + o(t_n), \end{aligned} \quad (\text{A.1.2})$$

where  $\nabla_2$  is the gradient with respect to the second coordinate. We have

$$\nabla_2 d(b_n, p) \xrightarrow{n \rightarrow \infty} \nabla_2 d(b_\infty, p) = -\dot{\gamma}(0)$$

because  $d(\cdot, \cdot)$  is smooth on  $U \times V$ . So (A.1.2) yields

$$\liminf_{n \rightarrow \infty} \frac{d_K(\exp_p(t_n v)) - d_K(p)}{t_n} \geq -\dot{\gamma}(0) \cdot v.$$

In particular,

$$\partial_v^+ d_K(p) \geq \min_{\gamma \in \text{geod}(p, K)} -\dot{\gamma}(0) \cdot v.$$

With (A.1.1), this concludes the proof. □

## A.2 Tensor calculus and covariant derivatives

In this appendix, we recall some notions of tensor calculus and covariant derivation on Riemannian manifolds. These notions are used in the proof of Proposition 3.5.5 in particular. If the reader has studied these notions before, we hope that these reminders will be sufficient to understand the computations of the proof of Proposition 3.5.5. We try and give a definition of every notion needed, along with the computation rules we need. We give a list of formulae at the end of the section. If the reader is totally unfamiliar with these notions, those reminders might be too brief for an introduction.

Let us give some specific references for each notion. A brief introduction to the notions of tensor, tensor product, trace of a tensor, tensor field over a manifold, can be found in [55, Chapter 3]]. For the notions of raising and lowering indices, inner product on tensors, see [55, Chapter 3]]. For covariant derivatives, see [55, Chapter 4]).

**Tensors.** Let  $V$  be a  $n$ -dimensional vector space (you may think of  $V$  as the tangent space of a point  $p \in M, T_pM$ ). Let  $k, l \in \mathbb{N}$ . For any  $i \in \{1, \dots, k+l\}$ , let  $V_i$  denote either  $V$  or its dual  $V^*$ . A  $(k, l)$ -tensor on  $V$  is a multilinear map

$$T : V_1 \times \dots \times V_{k+l} \rightarrow \mathbb{R},$$

where  $V^*$  appears  $k$  times in the product  $V_1 \times \dots \times V_{k+l}$  and  $V$  appears  $l$  times. In this case we say that  $T$  is a tensor of rank  $k+l$ .

In particular, a  $(0, 1)$  tensor is just a linear form on  $V$ , and a  $(1, 0)$  tensor is an element of  $V^*$ , so it can be seen as a vector of  $V$ . Reciprocally, any vector  $X \in V$  will automatically be seen as a  $(1, 0)$  tensor that will still be denoted by  $X$ . More generally a tensor  $T : \underbrace{V^* \times \dots \times V^*}_k \times \underbrace{V \times \dots \times V}_l \rightarrow \mathbb{R}$  can be seen as an element of the tensorial product  $\underbrace{V \otimes \dots \otimes V}_k \otimes \underbrace{V^* \otimes \dots \otimes V^*}_l$ .

Let  $(e_i)_{1 \leq i \leq n}$  be a basis of  $V$ , whose dual basis is  $(e^j)_{1 \leq j \leq n}$ . The coordinates of a  $(k, l)$  tensor  $T : \underbrace{V^* \times \dots \times V^*}_k \times \underbrace{V \times \dots \times V}_l \rightarrow \mathbb{R}$  in the basis  $(e_i)_{1 \leq i \leq n}$  are defined by

$$T^{j^1 \dots j^k}_{i_1 \dots i_l} := T(e^{i_1}, \dots, e^{i_l}, e^{j^1}, \dots, e^{j^k}).$$

Note that the indices to corresponding arguments in  $V$  are subscripts, while the indices corresponding to arguments in  $V^*$  are superscripts. If  $T$  were to take arguments in  $V$  and  $V^*$  in a different order, then we would write the indices in the coordinates of  $T$  in the corresponding order, still writing subscripts for indices corresponding to arguments in  $V$  and superscripts for indices corresponding to arguments in  $V^*$ . For instance, if  $S : V \times V^* \times V \rightarrow \mathbb{R}$  is a  $(1, 2)$  tensor we will write

$$S_i^j{}_k := S(e_i, e^j, e_k).$$

**Abstract index notation.** We may call a tensor by its coordinates, *i.e.* we may write 'the tensor  $S_a^b{}_c$ ' instead of 'the tensor  $S$ '. This way we can encode the type of a tensor directly in its name. This will also prove useful for computations. However, when doing so, we will

use letters  $a, b, c, d, \dots$  while we keep the letters  $i, j, k, l, \dots$  for actual coordinates in a basis. Therefore,  $T_{a_1 \dots a_l}^{b_1 \dots b_k}$  is non-other than the tensor  $T$ , while the  $T_{i_1 \dots i_l}^{j_1 \dots j_k}$  are the coordinates of the tensor  $T$  in the basis we have chosen. One may think of the  $a_i$  and  $b_i$  as mere labelling of the slots of  $T$ . This convention is called *abstract index notation*.

**Tensor product.** Given two tensors  $S$  and  $T$  of rank  $p$  and  $q$  respectively, their tensor product is a tensor of rank  $p + q$  defined by

$$S \otimes T(u_1, \dots, u_p, v_1, \dots, v_q) = S(u_1, \dots, u_p)T(v_1, \dots, v_q).$$

The coordinates of  $S \otimes T$  are the products of the coordinates of  $S$  and  $T$ . For instance, the coordinates of the tensor product of  $S_a^b$  and  $T_e^d$  are

$$(S \otimes T)_{i k}^{j l} = S_i^j T_k^l.$$

Therefore, using the convention of abstract index notation, the tensor product of  $S_a^b$  and  $T_{de}$  will be denoted by  $S_a^b T_{de}$ .

**Trace of a tensor.** Let  $T_a^b$  be a  $(1, 1)$  tensor. For any  $X \in V$ ,  $T(X, \cdot) \in V^{**} \simeq V$ , so  $T$  can also be seen as an endomorphism of  $V$ . Its trace as a tensor, denoted by  $tr(T)$  is defined as the trace of the associated endomorphism of  $V$ . The trace of  $T$  is also called *contraction* of  $T$ . In any basis  $(e_i)$  of  $V$ , we have the formula

$$tr(T) = \sum_{i=1}^n T_i^i.$$

We will use the Einstein summation convention, where an index repeated twice means we are summing over that index. Therefore, we have

$$tr(T) = T_i^i.$$

In abstract index notation, we will denote by  $T_a^a$  the trace of  $T$ . We can extend this definition to any  $(k + 1, l + 1)$  tensor  $T$ . All we need is a lower index and an upper index. The result is then a  $(k, l)$  tensor. However, one must specify with respect to which indices the trace is taken. This can be done when writing the coordinates of the tensor. For instance, we can take the trace of the tensor  $S_{ab}^{cd}$  with respect to the first and last indices, to get a  $(1, 1)$  tensor whose coordinates are  $S_{ij}^{ki}$ . We will denote this tensor by  $S_{ab}^{ca}$ . The tensor  $S_{ab}^{ba}$  would denote the trace of the tensor  $S$  with respect to the first and fourth coordinates, and the second and third coordinates, taken successively (the order does not matter.)

As mentioned before, any vector  $X$  of  $V$  can be seen as a  $(1, 0)$  tensor. We still denote by  $X$  the associated  $(1, 0)$  tensor. Now let  $T^{a_1 \dots a_k}_{b_1 \dots b_{l+1}}$  be a  $(k, l + 1)$  tensor. Then,  $T(\cdot, \dots, \cdot, X)$  is a  $(k, l)$  tensor, and one can see (using coordinates) that it is the trace of the tensor product of  $T$  and  $X$ , *i.e.*

$$T(\cdot, \dots, \cdot, X)^{a_1 \dots a_k}_{b_1 \dots b_l} = T^{a_1 \dots a_k}_{b_1 \dots b_l c} X^c.$$

The tensor  $T^{a_1 \dots a_k}_{b_1 \dots b_l c} X^c$  is also called *contraction* of  $T$  and  $X$ . Therefore, the result of the action of a tensor on a vector is the contraction of this tensor with the vector (seen as a  $(1, 0)$

tensor). Likewise, the result of the action of any tensor  $T^{a_1 \dots a_k}_{b_1 \dots b_l}$  on the arguments  $\phi^1, \dots, \phi^k, X_1, \dots, X_l$ , is

$$T(\phi_1, \dots, \phi_k, X_1, \dots, X_l) = T^{a_1 \dots a_k}_{b_1 \dots b_l} \phi_{a_1}^1 \dots \phi_{a_k}^k X_1^{b_1} \dots X_l^{b_l}.$$

**Raising and lowering indices.** Now we assume that  $V$  is equipped with a scalar product  $g$ . (If  $V$  is  $T_p M$ ,  $g$  can be the metric of the manifold  $M$ .) The scalar product induces an isomorphism between  $V$  and its dual  $V^*$ . We will also denote this isomorphism by  $g$ . Its inverse is denoted by  $g^{-1}$ . By composition with this isomorphism, we can transform a  $(0, 1)$  tensor  $T$  into a  $(1, 0)$  tensor  $\tilde{T}$  and *vice-versa* by setting

$$\tilde{T}(X) = T(g^{-1}(X)).$$

We denote by  $g_{ij}$  and  $g^{ij}$  the components of the maps  $g$  and  $g^{-1}$  respectively. In coordinates, we have the relations

$$\tilde{T}^i = g^{ij} T_j \quad \text{and} \quad T_j = g_{ij} \tilde{T}^i$$

We will denote the tensor  $\tilde{T}$  by  $T^a$ . We have

$$T^a = g^{ab} T_b,$$

which means that  $T^a$  is the contraction of  $g^{ab}$  and  $T_b$ , or equivalently the trace of the tensorial product of  $g^{ab}$  and  $T_b$ . We call this operation 'raising' and index. We may also lower an index thanks to the relation

$$T_a = g_{ab} T^a.$$

Likewise, we can transform any  $(k, l + 1)$  tensor into a  $(k + 1, l)$  tensor, and *vice-versa*, by composition with the isomorphism  $g$ , or equivalently by taking the trace of the tensorial product with  $g$ . We may also raise several indices successively. The order does not matter. For instance, for a tensor  $S_{ab}$ , we have  $S^{ab} = g^{ac} g^{bd} S_{cd}$ .

**More about the trace.** With the operation of raising and lowering indices, we can define the trace of a tensor with respect to any two indices, even if they are both low or both up: if this two indices are low, raise one of them and take the trace, otherwise lower one of them and take the trace. For instance, the trace of the tensor  $S_{ab}$  is  $S_a^a (= S_{ab} g^{ba} = tr(S_{ab} g^{bc}))$ .

**Inner product of tensors.** We can extend the scalar product  $g$  on  $V$  to a scalar product on tensors of the same type, by requiring that if  $(e_i)$  is an orthonormal basis and  $(e^i)$  its dual, then the family composed of tensor products of elements of  $(e_i)$  and  $(e^i)$  of a given type is orthonormal. It can also be described in terms of raising and lowering indices and taking the trace. Indeed, to take the scalar product of two tensors  $T$  and  $S$  of the same type, raise all the lower indices and lower all the upper indices of  $T$ , and take the trace of the tensor product of  $T$  and  $S$  with respect to all pair of corresponding indices. For instance, for some tensors  $T_{ab}$  and  $S_{ab}$ , we have

$$g(T, S) = T^{ab} S_{ab} \quad \left( = g^{ac} g^{bd} T_{cd} S_{ab} \right).$$

We also define the norm of a tensor  $T$  by

$$|T|^2 = g(T, T). \tag{A.2.1}$$

If  $T = T_{ab}$ , we get  $|T^2| = T_{ab}T^{ab}$ .

**Tensor fields.** A tensor field  $T$  on  $M$  is the collection of tensors of  $T_pM$  for every point  $p \in M$ . A tensor field  $T$  is said to be smooth if its coordinates in a smooth basis of  $TM$  are smooth.

**Covariant derivative.** Let  $\mathfrak{X}(M)$  denote the set of smooth vector fields over  $M$ . The covariant derivative  $D$  of  $M$  (or *Levi-Civita connection*) is the unique map

$$D : \mathfrak{X}(M) \times \mathfrak{X}(M) \rightarrow \mathfrak{X}(M),$$

that is  $C^\infty(M)$ -linear in the first variable,  $\mathbb{R}$ -linear in the second variable and such that for any  $f \in C^\infty(M)$  and  $X, Y, Z \in \mathfrak{X}(M)$ ,

$$\begin{aligned} D_Y(fX) &= D_Y(f)X + fD_Y(X), && \text{(product rule)} \\ D_Y(g(X, Z)) &= g(D_YX, Z) + g(X, D_YZ), && \text{(compatibility with the metric)} \end{aligned}$$

where  $D_Y(f)$  is the differential of  $f$  at  $Y$ . Because of the  $C^\infty(M)$ -linearity, the map  $Y \mapsto D_YX$  is a well defined endomorphism on every tangent space  $T_pM$ , i.e a  $(1, 1)$  tensor field. Therefore we can define the (total) covariant derivative tensor  $DX$  of  $X$  by the relation

$$DX(\phi, Y) = D_YX(\phi).$$

In abstract index notation, the tensor  $DX$  will be denoted by  $D_aX^b$ , so that  $D_YX = (D_aX^b)Y^a$ .

The covariant derivative can be extended to any kind of tensor, by requiring only that it satisfies some product rules. Therefore, for any tensor  $(k, l)$  tensor  $T$ , there is a well defined  $(k, l + 1)$  tensor  $DT$ . The second covariant derivative (or *Hessian*) of  $T$  is  $D^2T = D(DT)$ .

In abstract index notation, the covariant derivative of a tensor field  $T^{a_1 \dots a_k}_{b_1 \dots b_l}$  is denoted by  $D_c T^{a_1 \dots a_k}_{b_1 \dots b_l}$ . The second covariant derivative is denoted by  $D_d D_c T^{a_1 \dots a_k}_{b_1 \dots b_l}$ , etc. We may also use the notation  $D_{a_1 \dots a_k}^k$  in place of  $D_{a_1} \dots D_{a_k}$ .

The covariant derivative satisfies the following product rule :

$$D(S \otimes T) = DS \otimes T + S \otimes DT$$

The covariant derivative commutes with the trace. In particular, an expression of the type  $D_a T_{bc}{}^{bc}$  makes sense, as we do not need to specify if it denotes the derivative of the trace of the tensor  $T$ , or the trace of the derivative of the tensor  $T$ .

Let us give an example of computation with the covariant derivative on tensors. Using the facts that the covariant derivatives commute with the trace and that it satisfies a product rule on tensor products, we deduce for instance that for any tensor  $H_{ab}$ ,

$$D_c(|H|^2) = D_c(H_{ab}H^{ab}) = (D_c H_{ab})H^{ab} + H_{ab}(D^c H^{ab}).$$

By this example, we hope to show that computations with covariant derivatives in abstract index notation are fairly intuitive.

**Warning.** Let  $T$  be any tensor field, and  $X$  and  $Y$  some vector fields. The tensor  $D_a D_b T X^a Y^b$  is not the same as  $D_X(D_Y T)$ . Stated otherwise, evaluating the second covariant derivative of

$T$  on  $X$  and  $Y$  is not the same as derivating  $T$  in the directions  $Y$  and  $X$  successively. Indeed, we have  $D_Y T = D_b T Y^b$ , so some derivatives of  $Y^b$  will appear in  $D_X(D_Y T)$ .

**Laplacian.** The Laplacian of a function  $u$  is the trace of the second covariant derivative of  $u$ , i.e.

$$\Delta u := g^{ab} D_a D_b u.$$

**Riemann curvature tensor.** The Riemann curvature tensor is a  $(0, 4)$  tensor  $R_{abcd}$  defined by

$$\forall X, Y, Z, W \in \mathfrak{X}(M), \quad R(X, Y, Z, W) = g(D^2 Z(X, Y) - D^2 Z(Y, X), W).$$

Though  $Z$  needs to be a vector field (and not just a vector) for this definition to make sense a priori, it only depends on the pointwise values of  $Z$ , and so  $R$  is actually a tensor.

We will denote the commutator of  $D_a$  and  $D_b$  by  $D_{[ab]}^2$ :

$$D_{[ab]}^2 := D_a D_b - D_b D_a.$$

With this notation, we have for any vector  $Z$ :

$$D_{[ab]}^2 Z^c = R_{abd}{}^c Z^d.$$

**Summary of useful formulae.** Here we specifically recall the formulas we use in the proof of Proposition 3.5.5, in abstract index notation.

- Trace: we recall that when two (abstract) indices are repeated, it means we are taking the trace with respect to those indices.

- Tensor applied to vectors:

$$T(X, Y) = T_{ab} X^a Y^b.$$

- Norm of a tensor:

$$\left| T_{b_1 \dots b_l}{}^{a_1 \dots a_k} \right|^2 = T_{b_1 \dots b_l}{}^{a_1 \dots a_k} T^{b_1 \dots b_l}{}_{a_1 \dots a_k}.$$

- Laplacian:

$$\Delta v = g^{ab} D_a D_b v.$$

- Product rule:

$$D_a (T_{b_1 \dots b_l}{}^{a_1 \dots a_k} S_{b_1 \dots b_j}{}^{c_1 \dots c_i}) = (D_a T_{b_1 \dots b_l}{}^{a_1 \dots a_k}) S_{b_1 \dots b_j}{}^{c_1 \dots c_i} + T_{b_1 \dots b_l}{}^{a_1 \dots a_k} (D_a S_{b_1 \dots b_j}{}^{c_1 \dots c_i}).$$

- Compatibility with the metric: symbols  $D$  and  $g$  can be inverted.

$$D_a (g^{bc} T_{b_1 \dots b_l}{}^{a_1 \dots a_k}) = g^{bc} D_a T_{b_1 \dots b_l}{}^{a_1 \dots a_k}.$$

- Riemann curvature tensor:

$$D_{[ab]}^2 Z^c = R_{abd}{}^c Z^d.$$

## Part II

# A non-local isoperimetric problem





## Outline of part II

In this part, we study a non-local isoperimetric problem derived from Gamow's liquid drop model for the nucleus. More precisely, we are interested in

$$\min_{E \subset \mathbb{R}^n, |E|=m} P(E) + \mathcal{V}_\alpha(E) + \mathcal{U}_{\beta,A}(E), \quad (1.1.3)$$

where

- $P(E)$  is the perimeter of  $E$ ,
- $\mathcal{V}_\alpha(E) := \int_{E \times E} \frac{dx dy}{|x-y|^{n-\alpha}}$ ,
- $\mathcal{U}_{\beta,A}(E) := \int_E A |x|^\beta dx$ ,
- $n \in \mathbb{N}^*$  (the dimension of the ambient space),  $m > 0$  (called the *mass*),  $\alpha \in (0, n)$  and  $A \geq 0$  are constants.

If  $A = 0$ , we recover Gamow's liquid drop model, which does not admit any minimizer for  $m$  sufficiently large. If  $A > 0$ , this problem admits a minimizer for any mass  $m$ .

In the first chapter, we study minimizers of large mass (*i.e* when  $m \rightarrow +\infty$ ). We show the following.

**Theorem** (Theorem 1.1.1). *Given  $\alpha \in (0, n)$ ,  $\beta > 0$  and  $A > 0$ , assume  $\alpha < \beta$ . Let  $(E_m)_{m>0}$  be a family of minimizers in (1.1.3), such that  $|E_m| = m$ , and let  $E_m^*$  be the rescaling of  $E_m$  of the same mass as the unit ball  $B$  centered at 0 (*i.e*  $E_m^* = \left(\frac{|B|}{m}\right)^{\frac{1}{n}} E_m$ ). Then, up to modifying the sets  $(E_m)$  by some sets of measure 0, the boundaries of the sets  $(E_m^*)$  Hausdorff-converge to the boundary of  $B$  as  $m \rightarrow +\infty$ .*

However, from Proposition 1.3.3 we know that if  $\beta < 1$ , then large volume minimizers are not exactly balls. Indeed, in this case large balls are not even locally minimizing. On the contrary, in the case  $\beta > 1$ , if we assume in addition that  $\alpha > 1$ , then we have:

**Theorem** (Theorem 1.1.2). *Given  $\alpha \in (0, n)$ ,  $\beta > 0$  and  $A > 0$ , assume  $1 < \alpha < \beta$ . There exists a mass  $m_1 = m_1(n, \alpha, \beta, A) > 0$  such that for any  $m > m_1$  the ball of volume  $m$  centered at 0 is the unique minimizer (1.1.3), up to a set of measure 0.*

In the second chapter, we develop a numerical method to compute the minimizers. Our results seem to confirm that the following conjecture of R. Choksi and M. Peletier regarding Gamow's model is true in dimension 2: for  $A = 0$ , if problem (1.1.3) admits a minimizer, then it is a ball.

We started working on some aspects of the first Chapter under the supervision of Vincent Millot for our Master 2 thesis. The new results start with Section 1.4. In particular, the results presented in this introduction are new.



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# Large volume minimizers

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We started working on some aspects of this Chapter under the supervision of Vincent Millot for our Master 2 thesis. The new results start with Section 1.4.

## 1.1 Introduction

### 1.1.1 The classical isoperimetric problem

Let  $n \in \mathbb{N}^*$  be an integer. For any measurable set  $E \subset \mathbb{R}^n$ , the perimeter of  $E$  is defined as the total variation of the indicator function of  $E$ :

$$P(E) := \int_{\mathbb{R}^n} |D\mathbf{1}_E| = \sup \left\{ \int_E \operatorname{div} f : f \in C_c^1(\mathbb{R}^n), \quad |f|_{L^\infty} \leq 1 \right\}.$$

Our book of reference for the perimeter is Giusti's book, [43]. Maggi's book [59] is probably more reader-friendly though (but does not include the study of  $BV$  functions). The quantity  $P(E)$  may be infinite, and when it is finite, we say that  $E$  is a set of finite perimeter. If  $E$  is a  $C^1$  open set, then as one would expect, we have  $P(E) = \mathcal{H}^{n-1}(\partial E)$ , where  $\mathcal{H}^{n-1}$  is the  $(n-1)$ -dimensional Hausdorff measure. The properties of the perimeter we need will be recalled along the way, when needed. The classical isoperimetric problem is the following variational problem:

$$\min_{E \subset \mathbb{R}^n, |E|=m} P(E),$$

where  $|E|$  is the volume of  $E$ , and  $m > 0$  is a constant. It is well known that for any  $m > 0$ , this problem has a unique minimizer up to translation, namely  $B[m]$ , the ball of volume  $m$ . Furthermore, a sharp quantitative estimate was proved in [40]. Before stating it, we need a definition. Given a set  $E$  of volume  $m$ , we call *Fraenkel asymmetry* the quantity

$$\lambda(E) := \min_{x \in \mathbb{R}^n} \frac{|E \Delta (x + B[m])|}{|E|},$$

where  $E \Delta F = (E \cup F) \setminus (E \cap F)$  is the symmetric difference of  $E$  and  $F$ . There exists a constant  $C(n)$  depending only on the dimension of the ambient space such that for any set  $E$  of volume  $m$ ,

$$P(E) - P(B[m]) \geq C(n)P(B[m])\lambda(E)^2.$$

See also [39] for a refinement of this inequality, where the difference  $P(E) - P(B[m])$  also controls the oscillation of the normal to the (reduced) boundary of  $E$ .

### 1.1.2 Gamow's liquid drop problem

Gamow's liquid drop model for the atomic nucleus consists in:

$$\inf_{E \subset \mathbb{R}^2, |E|=m} P(E) + \int_{E \times E} \frac{dx dy}{|x - y|} \quad (1.1.1)$$

where

- $P(E)$  is the perimeter of  $E$ ,
- $m$ , called the *mass*, is a positive constant.

In this model, the nucleus is thought of as an incompressible charged liquid. The perimeter term models the short-range attractive nuclear force, whereas the other term, similar to an electrostatic energy, models the repulsive forces between the protons of the nucleus. Gamow invented this model in the 1930's, in an attempt to predict the shape of nuclei, the nonexistence of large nuclei, and the existence of a nucleus with the least energy per nucleon. These three goals have only been attained relatively recently (in the 2010's, see next subsection). Problem (1.1.2) has been studied by mathematicians as an interesting extension of the classical isoperimetric problem. Indeed, two terms are competing: the perimeter tends to round things up and is minimized by balls, whereas the non-local term tends to spread the mass, and is maximized by balls. This last fact is a consequence of Riesz rearrangement inequalities (see [57]).

### 1.1.3 Some variations of the liquid drop model

The most studied variant of Gamow's model is the following:

$$\inf_{E \subset \mathbb{R}^2, |E|=m} P(E) + \mathcal{V}_\alpha(E) \quad (1.1.2)$$

where

- $P(E)$  is the perimeter of  $E$ ,
- $\mathcal{V}_\alpha(E) := \int_{E \times E} \frac{dx dy}{|x - y|^{n - \alpha}}$  is the Riesz energy of  $E$ ,
- $n \geq 2$  (the dimension of the ambient space),  $m > 0$  and  $\alpha \in (0, n)$  are constants.

The physical case corresponds to  $n = 3$  and  $\alpha = 2$ . In this case, the Riesz energy boils down to the electrostatic energy  $\int_{E \times E} \frac{dx dy}{|x - y|}$ .

It was shown in [33] by Figalli, Fusco, Maggi, Millot, and Morini, that if the mass  $m$  is small enough, then the problem (1.1.2) admits a unique minimizer (up to translation), namely the ball of volume  $m$ . See also the papers of Knüpfer, Muratov, Julin, Bonacini and Cristoferi [53], [64], [51] and [10], for anterior and slightly less general results.

On the other hand, for  $\alpha \in (n - 2, n)$ , it was shown in [64] that for  $m$  large enough there is no minimizer of problem (1.1.2). This result was simultaneously proved in [58] by Lu and

Otto in the physical case. See also [35] for a short proof with a quantitative bound, by Frank, Killip and Nam.

In [36], Frank showed in the physical case that there exists a set which minimizes the quantity  $(P(E) + \mathcal{V}_\alpha(E))/|E|$ , thus proving that in Gamow's model, there is a nucleus with the least energy per nucleon.

In [33], the authors studied a generalization of problem (1.1.2), where the perimeter term is replaced with a non-local perimeter.

Another interesting variation consists in replacing the repulsive Riesz energy with a repulsive energy involving another kernel. Let  $K \in L^1(\mathbb{R}^n)$ . Problem (1.1.2) can be rephrased as

$$\inf_{E \subset \mathbb{R}^n, |E|=m} P(E) + \int_{E \times E} K(x-y) dx dy,$$

with  $K(x) = |x|^{\alpha-n}$ . In [70], Rigot considered the above problem with  $K$  an  $L^1$  compactly supported function, and proved that it has a minimizer for any mass  $m$ . The following papers are posterior to the present work. In [67], Pegon proved the same result in the case of radially symmetric non-increasing kernels  $K$  with some regularity and integrability conditions at infinity. He also showed that, under the same conditions, after rescaling, the problem  $\Gamma$ -converges to the classical isoperimetric problem as the mass  $m$  goes to  $+\infty$ . In [65], Muratov and Simon considered the kernel  $|x|^{-3}$  in dimension 2, and also studied the minimizers of large mass.

In [3], Alama, Bronsard, Choksi, and Topaloglu studied problem (1.1.2) with the addition of a confining background potential of the form  $\frac{Z}{|x|^\beta}$ . They proved that it admits a minimizer for any mass  $m$ , and study the minimizers as  $Z \rightarrow 0$ .

In [44], Goldman, Novaga and Roger studied the small mass minimizers of problem (1.1.2) with the addition of the Willmore energy.

As shown in [24], Gamow's liquid drop model is also related to diblock copolymers, and the Ohta–Kawasaki functional. This is also an active field of research, see for instance [2] and [82].

### 1.1.4 Our problem

To restore the existence of a minimizer for large masses in (1.1.2), we add the energy associated to the potential  $A|x|^\beta$  to our functional, as we expect it to counter the spreading effect of the  $\mathcal{V}_\alpha$  term. Thus we are interested in the following modification of the original problem (1.1.2):

$$\inf_{E \subset \mathbb{R}^n, |E|=m} \mathcal{E}_{\alpha,\beta,A}(E) := P(E) + \mathcal{V}_\alpha(E) + \mathcal{U}_{\beta,A}(E), \quad (1.1.3)$$

where

- $\mathcal{U}_{\beta,A}(E) := \int_E A|x|^\beta dx$ ,
- $A \geq 0$  and  $\beta > 0$  are constants.

As easily proved in Section 1.2, we indeed have the existence of a minimizer in (1.1.3) for any mass  $m$ .

In section 1.3, we extend some known results about minimality of small balls, and the domain (of masses  $m$ ) of local minimality of balls (depending on the parameters  $\alpha$  and  $\beta$ ). We do not give complete proofs, but recall briefly the techniques used in [33] to get these results.

In section 1.4, we study large volume minimizers (*i.e.* when  $m$  is large) of (1.1.3) when  $\alpha < \beta$ . More precisely we prove the following theorems:

**Theorem 1.1.1.** *Given  $\alpha \in (0, n)$ ,  $\beta > 0$  and  $A > 0$ , assume  $\alpha < \beta$ . Let  $(E_m)_{m>0}$  be a family of minimizers in (1.1.3), such that  $|E_m| = m$ , and let  $E_m^*$  be the rescaling of  $E_m$  of the same mass as the unit ball centered at the origin,  $B$  (ie  $E_m^* = \left(\frac{|B|}{m}\right)^{\frac{1}{n}} E_m$ ). Then, up to modifying the sets  $(E_m)$  by some sets of measure 0, the boundaries of the sets  $(E_m^*)$  Hausdorff-converge to the boundary of  $B$  as  $m \rightarrow +\infty$ .*

However, from Proposition 1.3.3 we know that if  $\beta < 1$ , then large volume minimizers are not exactly balls. Indeed, in this case large balls are not even locally minimizing. On the contrary, in the case  $\beta > 1$ , if we assume in addition that  $\alpha > 1$ , then we have:

**Theorem 1.1.2.** *Given  $\alpha \in (0, n)$ ,  $\beta > 0$  and  $A > 0$ , assume  $1 < \alpha < \beta$ . There exists a mass  $m_1 = m_1(n, \alpha, \beta, A) > 0$  such that for any  $m > m_1$  the ball of volume  $m$  centered at 0 is the unique minimizer (1.1.3), up to a set of measure 0.*

We conjecture that this last theorem holds also for any  $\alpha \in (0, 1]$  as long as  $\beta > 1$ . However, this seems to require different techniques.

## 1.2 Existence of a minimizer in (1.1.3)

In this section, we prove the following easy proposition:

**Proposition 1.2.1.** *As long as  $A > 0$ , problem (1.1.3) admits a minimizer for any mass  $m > 0$ .*

*Notation.* We denote by  $B$  the unit ball of  $\mathbb{R}^n$ , and by  $B[m]$  the ball of volume  $m$  centered at 0.

*Proof.* Let  $(E_k)$  be a minimizing sequence for the variational problem (1.1.3). By replacing  $E_k$  with the ball  $B[m]$  if necessary, we can assume

$$\mathcal{E}_{\alpha, \beta, A}(E_k) \leq \mathcal{E}_{\alpha, \beta, A}(B[m]), \quad \text{for all } k \in \mathbb{N}. \quad (1.2.1)$$

Set  $g(x) = A|x|^\beta$ . As  $g(x) \xrightarrow{|x| \rightarrow +\infty} +\infty$ , we can take a sequence of positive radius  $(R_k)_{k \in \mathbb{N}}$  and a sequence of positive constants  $(A_j)_{j \in \mathbb{N}}$  such that  $A_j \xrightarrow{j \rightarrow \infty} +\infty$  and for all  $x \notin B_{R_j}$ ,  $g(x) > A_j$ .

For any  $j \in \mathbb{N}$ , the sequence  $(E_k \cap B_{R_j})_{k \in \mathbb{N}}$  is a sequence of uniformly bounded Borel sets, with uniformly bounded perimeters. Indeed, the inequalities  $P(E_k \cap B_{R_j}) \leq P(E_k)$ ,  $P \leq \mathcal{E}_{\alpha, \beta, A}$  and (1.2.1) together give  $P(E_k \cap B_{R_j}) \leq \mathcal{E}_{\alpha, \beta, A}(B[m])$  for all  $k$ .

Therefore we can extract a  $L^1$ -converging subsequence of  $(E_k \cap B_{R_j})_{k \in \mathbb{N}}$ . Doing that for all  $j \in \mathbb{N}$  and using a diagonal argument, we get a subsequence of  $(E_k)_{k \in \mathbb{N}}$  that converges locally in  $L^1$  to a Borel set  $E \subset \mathbb{R}^n$ . Using the lower semi-continuity of the perimeter and Fatou's lemma in  $\mathcal{V}_\alpha(E_k)$  and  $\mathcal{U}_{\beta, A}(E_k)$ , we get that

$$\mathcal{E}_{\alpha, \beta, A}(E) \leq \liminf_{k \rightarrow \infty} \mathcal{E}_{\alpha, \beta, A}(E_k). \quad (1.2.2)$$

Now we show that  $|E| = m$ . By Fatou's lemma, from  $|E_k| = m$ , we get  $|E| \leq m$ . Also, for any  $j \in \mathbb{N}$ , from the inequalities  $\mathcal{U}_{\beta,A}(E_k \setminus B_{R_j}) \leq \mathcal{U}_{\beta,A}(E_k)$ ,  $\mathcal{U}_{\beta,A} \leq \mathcal{E}_{\alpha,\beta,A}$  and (1.2.1) we get

$$\mathcal{U}_{\beta,A}(E_k \setminus B_{R_j}) \leq \mathcal{E}_{\alpha,\beta,A}(B[m]). \quad (1.2.3)$$

But

$$\mathcal{U}_{\beta,A}(E_k \setminus B_{R_j}) = \int_{E_k \setminus B_{R_j}} g(x) dx \geq A_j |E_k \setminus B_{R_j}|. \quad (1.2.4)$$

Thus (1.2.3) and (1.2.4) together give

$$|E_k \cap B_{R_j}| = m - |E_k \setminus B_{R_j}| \geq m - \frac{\mathcal{E}_{\alpha,\beta,A} B[m]}{A_j}.$$

Taking the limit  $k \rightarrow \infty$ , then  $j \rightarrow \infty$ , we obtain

$$|E \cap B_{R_j}| \geq m - \frac{\mathcal{E}_{\alpha,\beta,A} B[m]}{A_j}, \quad \text{then} \quad |E| \geq m.$$

Thus  $|E| = m$ . With (1.2.2), it means that  $E$  is a minimizer of the variational problem (1.1.3).  $\square$

*Remark 1.2.2.* It is clear from the proof that Proposition 1.2.1 is true if we replace the potential  $A|x|^\beta$  by any  $L^1_{loc}$  non-negative function  $g$  such that  $g(x) \xrightarrow{|x| \rightarrow +\infty} +\infty$ .

### 1.3 Extension of some known results

In this section we recall two known results about the variational problem (1.1.2), and extend it to (1.1.3), recalling only the techniques used in [33]. The first result states that if the mass  $m$  is small enough, then problem (1.1.2) admits a unique (up to translation) minimizer, namely the ball of volume  $m$ . The same holds for problem (1.1.3):

**Proposition 1.3.1.** *Given  $\alpha \in (0, n)$ ,  $\beta > 0$ ,  $A > 0$ , there exists a constant  $m_0(n, \alpha, \beta, A) > 0$  such that for any  $m \in (0, m_0)$ , problem (1.1.3) admits the ball of volume  $m$  centered at 0 as its unique minimizer.*

It is a direct consequence of the same theorem for problem (1.1.2) (see [33, Theorem 1.3]), as balls centered at 0 are also volume-constrained minimizers of  $\mathcal{U}_{\beta,A}$ . This last fact is a consequence of Riesz inequality regarding symmetric decreasing rearrangements (see [57] for rearrangement inequalities). Note that Proposition 1.3.1 is true if we replace the potential  $A|x|^\beta$  with a radially symmetric non-decreasing function  $g$ .

The second result deals with local minimality of balls.

*Terminology 1.3.2.* We say that a set  $E \subset \mathbb{R}^n$  of volume  $m$  is a  $L^1$ -local minimizer in (1.1.3) if there exists  $\varepsilon > 0$  such that for any set  $F \subset \mathbb{R}^n$  such that  $|F| = |E|$  and  $|E \Delta F| < \varepsilon$ ,  $\mathcal{E}_{\alpha,\beta,A}(E) \leq \mathcal{E}_{\alpha,\beta,A}(F)$ .

In the case of problem (1.1.2), we know from [33, Theorem 1.5] that there exists a  $m_* > 0$  such that if  $m < m_*$ , then  $B[m]$  is a  $L^1$ -local minimizer in (1.1.2), and if  $m > m_*$  then  $B[m]$  is *not* a  $L^1$ -local minimizer in (1.1.2). As stated in the next theorem, the addition of the  $\mathcal{U}_{\beta,A}$  term may modify this situation, but we can still apply the techniques used in [33] to get a similar result.



**Proposition 1.3.3.** *Given  $\alpha \in (0, n)$ ,  $\beta > 0$  and  $A > 0$ ,*

1. *if  $\alpha > \beta$ , then there exists a mass  $m_*(n, \alpha, \beta, A) > 0$  such that if  $m < m_*$ , then  $B[m]$  is a  $L^1$ -local minimizer in (1.1.3), and if  $m > m_*$  then  $B[m]$  is not a  $L^1$ -local minimizer in (1.1.3),*
2. *if  $\alpha = \beta$ , then either the same holds, or (if  $\alpha > 1$  and  $A$  is small enough)  $B[m]$  is a  $L^1$ -local minimizer in (1.1.3) for any  $m > 0$ ,*
3. *if  $\alpha < \beta$  and  $\beta > 1$ , then there exists a mass  $m_*(n, \alpha, \beta, A) > 0$  such that if  $m > m_*$  then  $B[m]$  is a  $L^1$ -local minimizer in (1.1.3),*
4. *if  $\alpha < \beta$  and  $\beta < 1$ , then there exists a mass  $m_*(n, \alpha, \beta, A) > 0$  such that if  $m > m_*$  then  $B[m]$  is not a  $L^1$ -local minimizer in (1.1.3),*
5. *if  $\alpha < \beta$  and  $\beta = 1$ , then the conclusion of either 3 or 4 holds (depending on the value of  $A$ ).*

*Remark 1.3.4.* The conclusions in points 3, 4 and 5 are less precise than in points 1 and 2.

*Ideas of the proof.* The method used in [33] still applies to our functional  $\mathcal{E}_{\alpha, \beta, A} = P + \mathcal{V}_\alpha + \mathcal{U}_{\beta, A}$ . Given  $m > 0$ , let us proceed to a rescaling of the functional and set

$$\gamma = \left( \frac{m}{|B|} \right)^{1/n} \quad \text{and} \quad \mathcal{E}_{\alpha, \beta, A, \gamma} := P + \gamma^{1+\alpha} \mathcal{V}_\alpha + \gamma^{1+\beta} \mathcal{U}_{\beta, A},$$

so that for any set  $E$  of volume  $m$ , the set  $E^* = \frac{1}{\gamma} E$  has volume  $|B|$  and

$$\mathcal{E}_{\alpha, \beta, A}(E) = \gamma^{n-1} \mathcal{E}_{\alpha, \beta, A, \gamma}(E^*).$$

Thus we are reduced to finding the values of  $\gamma > 0$  such that the unit ball  $B$  centered at 0 is a local minimizer of  $\mathcal{E}_{\alpha, \beta, A, \gamma}$ .

Following [33, Section 6] we can compute the second variation of  $\mathcal{E}_{\alpha, \beta, A, \gamma}$  at  $B$ . The terms  $P$  and  $\mathcal{V}_\alpha$  are treated in [33] and the term  $\mathcal{U}_{\beta, A}$  adds no further difficulty. We find that given any smooth compactly supported vector field  $X$ , such that the volume of  $B$  is preserved under the flow  $(\Phi_t^X)_{t>0}$  of  $X$ , we have:

$$\begin{aligned} \delta^2 \mathcal{E}_{\alpha, \beta, A, \gamma}(B)[X] &= \sum_{k \geq 2} (\lambda_k - \lambda_1) \left( 1 - \gamma^{1+\alpha} \frac{\mu_k^\alpha - \mu_1^\alpha}{\lambda_k - \lambda_1} + \gamma^{1+\beta} \frac{A\beta}{\lambda_k - \lambda_1} \right) a_k (X \cdot \nu_B)^2 \\ &\quad + \gamma^{1+\beta} A\beta a_1 (X \cdot \nu_B)^2, \end{aligned}$$

where

- $\delta^2 \mathcal{E}_{\alpha, \beta, A, \gamma}(B)[X] := \frac{d^2}{dt^2} [\mathcal{E}_{\alpha, \beta, A, \gamma}(\Phi_t^X(B))]_{t=0}$ ,
- $\nu_B$  is the unit outer normal vector to  $\partial B$ ,
- $a_k(X \cdot \nu_B)$  are the coefficient of the function  $X \cdot \nu_B : \partial B \rightarrow \mathbb{R}$  with respect to an orthonormal basis of spherical harmonics,

- $\lambda_k = k(n+k-2)$  is the  $k$ -th eigenvalue of the Laplacian on the sphere  $\partial B$ ,
- $\mu_k^\alpha$  is the  $k$ -th eigenvalue of the operator  $\mathcal{R}_\alpha$  defined by

$$\mathcal{R}_\alpha u(x) := 2 \int_{\partial B} \frac{u(x) - u(y)}{|x-y|^{n-\alpha}} d\mathcal{H}^{n-1}(y), \quad \forall u \in C^1(\partial B).$$

From there we deduce that, defining

$$S_* = \{\gamma > 0 : 1 - \gamma^{1+\alpha} \frac{\mu_k^\alpha - \mu_1^\alpha}{\lambda_k - \lambda_1} + \gamma^{1+\beta} \frac{A\beta}{\lambda_k - \lambda_1} \geq 0, \forall k \geq 2\}, \quad (1.3.1)$$

if  $\gamma \notin S_*$ , then there exists a vector field  $X$  such that

$$|\phi_t^X(B)| = |B| \quad \text{and} \quad \mathcal{E}_{\alpha,\beta,A,\gamma}(\phi_t^X(B)) < \mathcal{E}_{\alpha,\beta,A,\gamma}(B) \quad \text{for } t \text{ small enough.}$$

Thus  $B$  is not a  $L^1$ -local minimizer of  $\mathcal{E}_{\alpha,\beta,A,\gamma}$  if  $\gamma \notin S_*$ .

Now let us set

$$\tilde{S}_* = \{\gamma > 0 : 1 - \gamma^{1+\alpha} \frac{\mu_k^\alpha - \mu_1^\alpha}{\lambda_k - \lambda_1} + \gamma^{1+\beta} \frac{A\beta}{\lambda_k - \lambda_1} > 0, \forall k \geq 2\}. \quad (1.3.2)$$

We assume  $\gamma \in \tilde{S}_*$  and explain how to show that  $B$  is a  $L^1$ -local minimizer of  $\mathcal{E}_{\alpha,\beta,A,\gamma}$ . First, we note that it is true in a certain class of nearly spherical sets. More precisely, let  $E$  be a nearly spherical set associated to a  $C^1$  function  $u : \partial B \rightarrow \mathbb{R}$ :

$$E := \{s(1+u(x))x, x \in \partial B, 0 \leq s \leq 1\}.$$

Assume that  $|E| = |B|$  and  $\int_E |x|^{\beta-2} x dx = 0$ . Then, using some explicit computations and Taylor expansions, we can show that there exist some constants  $\varepsilon(n, \alpha, \beta, A, \gamma) > 0$  and  $C(n, \alpha, \beta, A, \gamma) > 0$  such that if  $\|u\|_{C^1(\partial B)} \leq \varepsilon(n, \alpha, \beta, A, \gamma)$ , then

$$\mathcal{E}_{\alpha,\beta,A,\gamma}(E) - \mathcal{E}_{\alpha,\beta,A,\gamma}(B) \geq C(n, \alpha, \beta, A, \gamma) \left( \|u\|_{L^2(\partial B)}^2 + \|\nabla(u)\|_{L^2(\partial B)}^2 \right). \quad (1.3.3)$$

Next we argue by contradiction and assume that we have a sequence of Borel sets  $(E_k)$  such that for any  $k$ ,  $|E_k| = |B|$ ,  $\mathcal{E}_{\alpha,\beta,A,\gamma}(E_k) < \mathcal{E}_{\alpha,\beta,A,\gamma}(B[m])$  and  $|E_k \Delta B| \xrightarrow[k \rightarrow \infty]{} 0$ . We consider a set  $F_k$  solution of the penalized problem:

$$\inf\{\mathcal{E}_{\alpha,\beta,A,\gamma}(E) + M |E \Delta E_k|, E \subset \mathbb{R}^n\},$$

with  $M > 0$  to be taken large enough. The role of the set  $F_k$  is to be "close to  $E_k$ ", and to be a  $\Lambda$ -minimizer in the sense that

$$P(F_k) \leq P(E) + \Lambda |E \Delta F_k|, \quad \text{for any Borel set } E.$$

Thus we show that  $F_k$  is a  $\Lambda$ -minimizer for some  $\Lambda$  uniform in  $k$ , and that  $|F_k \Delta B| \xrightarrow[k \rightarrow \infty]{} 0$ , which implies by classical regularity theory that  $F_k$  is an almost spherical set. Up to translating and rescaling  $F_k$  we can apply inequality (1.3.3). Only simple manipulations are left to get a contradiction.

At this stage we have two sets  $S_*$  and  $\tilde{S}_*$  defined by (1.3.1) and (1.3.2), such that if  $\left(\frac{m}{|B|}\right)^{1/n} \in \tilde{S}_*$  then  $B[m]$  is a  $L^1$ -local minimizer in (1.1.3), and if  $\left(\frac{m}{|B|}\right)^{1/n} \notin S_*$  then  $B[m]$  is *not* a  $L^1$ -local minimizer in (1.1.3). We are left to study the variations of the functions

$$\gamma \mapsto 1 - \gamma^{1+\alpha} \frac{\mu_k^\alpha - \mu_1^\alpha}{\lambda_k - \lambda_1} + \gamma^{1+\beta} \frac{A\beta}{\lambda_k - \lambda_1}, \quad k \geq 2$$

to get the conclusions of the theorem. This is done in Appendix 1.A.  $\square$

## 1.4 Large volume minimizers for $\alpha < \beta$

### 1.4.1 Hausdorff convergence of large volume minimizers for $\alpha < \beta$

Here we prove Theorem 1.1.1, *i.e.* that large volume minimizers of (1.1.3) are almost balls if  $\alpha < \beta$ . Note that if  $\beta < 1$ , we know that large volume minimizers are not exactly balls. Indeed, in virtue of Proposition 1.3.3, balls are not even local minimizers in this case.

The idea behind the proof is that if  $\alpha < \beta$ , then for a Borel set  $E \subset \mathbb{R}^n$  of volume  $m > 0$  with  $m$  large, the predominant term in  $\mathcal{E}_{\alpha,\beta,A}(E)$  is  $\mathcal{U}_{\beta,A}(E)$ . This can be seen by rescaling:

$$\mathcal{E}_{\alpha,\beta,A}(E) = \gamma^{n-1} \left( P(E^*) + \gamma^{1+\alpha} \mathcal{V}_\alpha(E^*) + \gamma^{1+\beta} \mathcal{U}_{\beta,A}(E^*) \right), \quad (1.4.1)$$

where we have set  $\gamma := \left(\frac{m}{|B|}\right)^{\frac{1}{n}}$  and  $E^* := \frac{1}{\gamma}E$ . As the unique volume constrained minimizer of  $\mathcal{U}_{\beta,A}$  is the ball  $B[m]$ , this implies that if  $E$  is a minimizer of  $\mathcal{E}_{\alpha,\beta,A}$  at mass  $m$  for  $m$  large,  $\mathcal{U}_{\beta,A}(E)$  must be close to  $\mathcal{U}_{\beta,A}(B[m])$ . This in turn will imply that  $E$  is close to  $B[m]$ . Note that according to the rescaling (1.4.1), proving Theorem 1.1.1 is equivalent to proving that if  $(E_\gamma)_{\gamma>0}$  is a family of Borel sets such that  $|E_\gamma| = |B|$ , and each set  $E_\gamma$  is a volume-constrained minimizer of the functional

$$\mathcal{E}_{\alpha,\beta,A,\gamma} := P + \gamma^{1+\alpha} \mathcal{V}_\alpha + \gamma^{1+\beta} \mathcal{U}_{\beta,A}, \quad (1.4.2)$$

then, up to modifying each set  $E_\gamma$  by a set of measure 0, the boundaries of the sets  $(E_\gamma)$  Hausdorff-converge to the boundary of the unit ball  $B$  centered at 0 as  $\gamma \rightarrow +\infty$ . First we will show the following convergence in measure:

**Lemma 1.4.1.** *We have*

$$|E_\gamma \Delta B| \xrightarrow{\gamma \rightarrow \infty} 0.$$

We will need the following stability lemma for the potential energy  $\mathcal{U}_{\beta,A}$ .

**Lemma 1.4.2.** *For any Borel set  $E \subset \mathbb{R}^n$  of volume  $|B|$ , we have*

$$\mathcal{U}_{\beta,A}(E) - \mathcal{U}_{\beta,A}(B) \geq \frac{A\beta}{8P(B)} |E \Delta B|^2.$$

*Proof.* Let  $E \subset \mathbb{R}^n$  be a Borel set of volume  $|B|$ . Define  $r_1 \geq 0$  and  $r_2 > 0$  to be such that  $|\{x \in \mathbb{R}^n : r_1 \leq |x| \leq 1\}| = |\{x \in \mathbb{R}^n : 1 \leq |x| \leq r_2\}| = |E \setminus B| = |B \setminus E|$ . Explicitely,

$r_1 = \left(1 - n \frac{|E \setminus B|}{P(B)}\right)^{\frac{1}{n}}$  and  $r_2 = \left(1 + n \frac{|E \setminus B|}{P(B)}\right)^{\frac{1}{n}}$ . We then have

$$\begin{aligned} \mathcal{U}_{\beta,A}(E) - \mathcal{U}_{\beta,A}(B) &= \int_{E \setminus B} A |x|^\beta dx - \int_{B \setminus E} A |x|^\beta dx \\ &\geq \int_{\{x \in \mathbb{R}^n : 1 \leq |x| \leq r_2\}} A |x|^\beta dx - \int_{\{x \in \mathbb{R}^n : r_1 \leq |x| \leq 1\}} A |x|^\beta dx \end{aligned}$$

(for  $x \rightarrow |x|^\beta$  is radially symmetric non-decreasing),

$$\begin{aligned} &= AP(B) \left( \int_1^{r_2} r^\beta r^{n-1} dr - \int_{r_1}^1 r^\beta r^{n-1} dr \right) \\ &= \frac{AP(B)}{n+\beta} \left( r_2^{n+\beta} - 1 - \left(1 - r_1^{n+\beta}\right) \right) \\ &= \frac{AP(B)}{n+\beta} \left( \left(1 + n \frac{|E \setminus B|}{P(B)}\right)^{\frac{n+\beta}{n}} - 1 \right. \\ &\quad \left. - \left(1 - \left(1 - n \frac{|E \setminus B|}{P(B)}\right)^{\frac{n+\beta}{n}}\right) \right). \end{aligned} \tag{1.4.3}$$

Now, setting  $\lambda := \frac{n+\beta}{n}$  and  $f(r) := \left((1+r)^\lambda - 1 - \left(1 - (1-r)^\lambda\right)\right)$ , we have

$$f''(r) = \lambda(\lambda-1) \left( (1+r)^{\lambda-2} + (1-r)^{\lambda-2} \right) \geq \lambda(\lambda-1)$$

As  $f'(0) = f(0) = 0$ , we get  $f(r) \geq \lambda(\lambda-1) \frac{r^2}{2}$ , which yields the result with (1.4.3).  $\square$

Lemma 1.4.1 is then easily deduced from Lemma 1.4.2 :

*Proof of Lemma 1.4.1.* We have

$$\begin{aligned} \gamma^{1+\beta} \mathcal{U}_{\beta,A}(E_\gamma) &\leq \mathcal{E}_{\alpha,\beta,A,\gamma}(E_\gamma) \\ &\leq \mathcal{E}_{\alpha,\beta,A,\gamma}(B) \\ &= P(B) + \gamma^{1+\alpha} \mathcal{V}_\alpha(B) + \gamma^{1+\beta} \mathcal{U}_{\beta,A}(B), \end{aligned}$$

so

$$\mathcal{U}_{\beta,A}(E_\gamma) - \mathcal{U}_{\beta,A}(B) \leq \frac{1}{\gamma^{1+\beta}} \left( P(B) + \gamma^{1+\alpha} \mathcal{V}_\alpha(B) \right).$$

This implies  $\mathcal{U}_{\beta,A}(E_\gamma) - \mathcal{U}_{\beta,A}(B) \xrightarrow{\gamma \rightarrow \infty} 0$ , which concludes the proof thanks to Lemma 1.4.2.  $\square$

We are now in position to prove Theorem 1.1.1.

*Proof of Theorem 1.1.1.* Let  $(E_\gamma)_{\gamma>0}$  be a family of Borel sets such that  $|E_\gamma| = |B|$ , and each set  $E_\gamma$  is a volume-constrained minimizer of the functional

$$\mathcal{E}_{\alpha,\beta,A,\gamma} := P + \gamma^{1+\alpha} \mathcal{V}_\alpha + \gamma^{1+\beta} \mathcal{U}_{\beta,A}.$$

We need to show that, up to modifying each set  $E_\gamma$  by a set of measure 0, the boundaries of the sets  $(E_\gamma)$  Hausdorff-converge to the boundary of the unit ball  $B$  centered at 0 as  $\gamma \rightarrow +\infty$ .

*Step one.* We show that given  $R > 1$ , for  $\gamma$  large enough we have  $E_\gamma \subset B_R$ , up to a set of measure 0.

Given  $R > 1$ , set  $F = \mu(E_\gamma \cap B_R)$ , with  $\mu > 0$  such that  $|F| = |B|$ , ie  $\mu = \left(\frac{|E_\gamma|}{|E_\gamma \cap B_R|}\right)^{\frac{1}{n}} = \left(\frac{1}{1-u}\right)^{\frac{1}{n}}$ , where  $u = \frac{|E_\gamma \setminus B_R|}{|E_\gamma|}$ . We have

$$\begin{aligned} \mathcal{E}_{\alpha,\beta,A,\gamma}(F) &= \mu^{n-1}P(E_\gamma \cap B_R) + \mu^{n+\alpha}\gamma^{1+\alpha}\mathcal{V}_\alpha(E_\gamma \cap B_R) \\ &\quad + \mu^{n+\beta}\gamma^{1+\beta}\mathcal{U}_{\beta,A}(E_\gamma \cap B_R) \\ &\leq \mu^{n+\beta}\mathcal{E}_{\alpha,\beta,A,\gamma}(E_\gamma \cap B_R). \end{aligned} \quad (1.4.4)$$

Take  $\eta > 0$  to be chosen later, and then  $\varepsilon > 0$  such that for all  $v \in [0, \varepsilon)$ ,  $\left(\frac{1}{1-v}\right)^{\frac{n+\beta}{n}} \leq 1 + \left(\frac{n+\beta}{n} + \eta\right)v$ . According to Lemma 1.4.1, if  $\gamma$  has been taken large enough, we can assume that  $u \leq \varepsilon$ , and so  $\mu^{n+\beta} \leq 1 + \left(\frac{n+\beta}{n} + \eta\right)u$ . Then, using  $P(E_\gamma \cap B_R) \leq P(E_\gamma)$ ,  $\mathcal{V}_\alpha(E_\gamma \cap B_R) \leq \mathcal{V}_\alpha(E_\gamma)$  and

$$\mathcal{U}_{\beta,A}(E_\gamma) - \mathcal{U}_{\beta,A}(E_\gamma \cap B_R) \geq A|E_\gamma \setminus B_R|R^\beta,$$

we find

$$\begin{aligned} \mathcal{E}_{\alpha,\beta,A,\gamma}(F) &\leq \left(1 + \left(\frac{n+\beta}{n} + \eta\right)u\right)\mathcal{E}_{\alpha,\beta,A,\gamma}(E_\gamma \cap B_R) \\ &\leq \mathcal{E}_{\alpha,\beta,A,\gamma}(E_\gamma) - \gamma^{1+\beta}A|E_\gamma \setminus B_R|R^\beta + \left(\frac{n+\beta}{n} + \eta\right)u\mathcal{E}_{\alpha,\beta,A,\gamma}(E_\gamma) \\ &= \mathcal{E}_{\alpha,\beta,A,\gamma}(E_\gamma) + \left(\left(\frac{n+\beta}{n} + \eta\right)\mathcal{E}_{\alpha,\beta,A,\gamma}(E_\gamma) - \gamma^{1+\beta}AR^\beta|B|\right)u \\ &\leq \mathcal{E}_{\alpha,\beta,A,\gamma}(E_\gamma) + \left(\left(\frac{n+\beta}{n} + \eta\right)\mathcal{E}_{\alpha,\beta,A,\gamma}(B) - \gamma^{1+\beta}AR^\beta|B|\right)u. \end{aligned} \quad (1.4.5)$$

But as  $\gamma \rightarrow \infty$ ,

$$\begin{aligned} \mathcal{E}_{\alpha,\beta,A,\gamma}(B) &= \gamma^{1+\beta}\mathcal{U}_{\beta,A}(B) + o(\gamma^{1+\beta}) \\ &= \gamma^{1+\beta}A\frac{1}{n+\beta}P(B) + o(\gamma^{1+\beta}) \\ &= \gamma^{1+\beta}A\frac{n}{n+\beta}|B| + o(\gamma^{1+\beta}). \end{aligned}$$

So with (1.4.5),

$$\mathcal{E}_{\alpha,\beta,A,\gamma}(F) \leq \mathcal{E}_{\alpha,\beta,A,\gamma}(E_\gamma) + \left(\gamma^{1+\beta}A|B|\left(1 + \frac{n}{n+\beta}\eta - R^\beta\right) + o(\gamma^{1+\beta})\right)u.$$

Recall that  $R > 1$ , so that if  $\eta$  has been taken small enough, we get that for  $\gamma$  large enough,  $\mathcal{E}_{\alpha,\beta,A,\gamma}(F) \leq \mathcal{E}_{\alpha,\beta,A,\gamma}(E_\gamma)$ , with equality if and only if  $u=0$ , i.e.  $E_\gamma \subset B_R$  up to a set of measure 0.

*Step two.* We show that given  $\delta > 0$ , for  $\gamma$  large enough we have  $B_{1-\delta} \subset E_\gamma$ , up to a set of measure 0. This is done by taking some mass of  $E_\gamma$  outside a certain ball  $B_R$  and putting it in  $E_\gamma \cap B_r$  for a well chosen  $r$ . In the proof we use Lemma 1.4.4 below to show that the perimeter decreases under such a transformation for a well chosen  $r \in (1 - \delta, 1)$ . On the other hand, the increase of  $\mathcal{V}_\alpha$  is compensated by the decrease of  $\mathcal{U}_{\beta,A}$  if  $\gamma$  has been taken large enough.

Let us set  $F = B \setminus E_\gamma$  and  $\varepsilon = \delta/2$ . From lemma 1.4.1 we know that if  $\gamma$  has been taken large enough we have  $|F| < |B| \left(\frac{\varepsilon}{2}\right)^n$ . Thus we can apply lemma 1.4.4 below with  $r_0 = 1 - \varepsilon$ , to get a  $r \in (1 - \delta, 1 - \varepsilon)$  such that

$$P(F, B_r) \geq \mathcal{H}^{n-1}(F \cap \partial B_r). \quad (1.4.6)$$

As  $|E_\gamma| = |B|$ , we have  $|E_\gamma \setminus B| = |B \setminus E_\gamma| \geq |B_r \setminus E_\gamma|$ , so there exists  $R \geq 1$  such that  $|E_\gamma \setminus B_R| = |B_r \setminus E_\gamma|$ . Now let us set

$$E'_\gamma = (E_\gamma \cap B_R) \cup B_r,$$

and compare  $\mathcal{E}_{\alpha,\beta,A,\gamma}(E'_\gamma)$  and  $\mathcal{E}_{\alpha,\beta,A,\gamma}(E_\gamma)$ . Using classical formulae for the perimeter of the union or the intersection of a set with a ball (see [43, remark 2.14]), we have

$$\begin{aligned} P(E'_\gamma) &= \mathcal{H}^{n-1}(E_\gamma \cap \partial B_R) + P(E_\gamma, \overline{B_r}^c \cap B_R) + \mathcal{H}^{n-1}(E_\gamma^c \cap \partial B_r), \\ P(E_\gamma) &= P(E_\gamma, B_R^c) + P(E_\gamma, \overline{B_r}^c \cap B_R) + P(E_\gamma, \overline{B_r}), \end{aligned}$$

so that

$$\begin{aligned} P(E'_\gamma) - P(E_\gamma) &= \mathcal{H}^{n-1}(E_\gamma \cap \partial B_R) - P(E_\gamma, B_R^c) \\ &\quad + \mathcal{H}^{n-1}(E_\gamma^c \cap \partial B_r) - P(E_\gamma, \overline{B_r}). \end{aligned} \quad (1.4.7)$$

From the classical inequality  $P(E_\gamma \cap B_R) \leq P(E_\gamma)$ , we get that  $\mathcal{H}^{n-1}(E_\gamma \cap \partial B_R) \leq P(E_\gamma, B_R^c)$ , so (1.4.7) gives

$$P(E'_\gamma) - P(E_\gamma) \leq \mathcal{H}^{n-1}(E_\gamma^c \cap \partial B_r) - P(E_\gamma, \overline{B_r}).$$

But

$$\mathcal{H}^{n-1}(E_\gamma^c \cap \partial B_r) = \mathcal{H}^{n-1}(E_\gamma^c \cap B \cap \partial B_r) = \mathcal{H}^{n-1}((B \setminus E_\gamma) \cap \partial B_r),$$

and

$$P(E_\gamma, \overline{B_r}) = P(E_\gamma^c, \overline{B_r}) = P(E_\gamma^c \cap B, \overline{B_r}) = P(B \setminus E_\gamma, \overline{B_r}),$$

So, recalling that  $F = B \setminus E_\gamma$ , we obtain

$$\begin{aligned} P(E'_\gamma) - P(E_\gamma) &\leq \mathcal{H}^{n-1}(F \cap \partial B_r) - P(F, \overline{B_r}) \\ &\leq \mathcal{H}^{n-1}(F \cap \partial B_r) - P(F, B_r), \end{aligned}$$

so by the choice of  $r$ ,

$$P(E'_\gamma) \leq P(E_\gamma). \quad (1.4.8)$$

Now we estimate the variation of  $\mathcal{V}_\alpha$ . Let us define the non-local potential:

$$\Phi_E^\alpha(x) = \int_E \frac{dy}{|x - y|^{n-\alpha}}.$$

With this notation, we have

$$\begin{aligned}
 \mathcal{V}_\alpha(E'_\gamma) - \mathcal{V}_\alpha(E_\gamma) &= \int_{E'_\gamma} \Phi_{E'_\gamma}^\alpha - \int_{E_\gamma} \Phi_{E_\gamma}^\alpha \\
 &= \int_{E'_\gamma} \Phi_{E'_\gamma}^\alpha - \int_{E_\gamma} \Phi_{E'_\gamma}^\alpha + \int_{E'_\gamma} \Phi_{E_\gamma}^\alpha - \int_{E_\gamma} \Phi_{E_\gamma}^\alpha \\
 &= \int_{E'_\gamma \setminus E_\gamma} \Phi_{E'_\gamma}^\alpha - \int_{E_\gamma \setminus E'_\gamma} \Phi_{E'_\gamma}^\alpha + \int_{E'_\gamma \setminus E_\gamma} \Phi_{E_\gamma}^\alpha - \int_{E_\gamma \setminus E'_\gamma} \Phi_{E_\gamma}^\alpha \\
 &\leq 4 \sup_{|F|=|B|} \|\Phi_F^\alpha\|_\infty |E_\gamma \setminus E'_\gamma|. \tag{1.4.9}
 \end{aligned}$$

By the Hardy-Littlewood rearrangement inequality, we have, for any  $F \subset \mathbb{R}^n$  such that  $|F| = |B|$  and  $x \in \mathbb{R}^n$ ,

$$\Phi_F^\alpha(x) = \int_F \frac{dy}{|x-y|^{n-\alpha}} = \int_{\mathbb{R}^n} \frac{\mathbb{1}_{\{-x+F\}}}{|y|^{n-\alpha}} dy \leq \int_{\mathbb{R}^n} \frac{\mathbb{1}_B}{|y|^{n-\alpha}} dy = C(n, \alpha).$$

So (1.4.9) yields

$$\mathcal{V}_\alpha(E'_\gamma) - \mathcal{V}_\alpha(E_\gamma) \leq C(n, \alpha) |E_\gamma \setminus B_r|. \tag{1.4.10}$$

As for  $\mathcal{U}_{\beta,A}$ , we have

$$\begin{aligned}
 \mathcal{U}_{\beta,A}(E'_\gamma) - \mathcal{U}_{\beta,A}(E_\gamma) &= \int_{B_r \setminus E_\gamma} A|x|^\beta dx - \int_{E_\gamma \setminus B_R} A|x|^\beta dx \\
 &\leq \int_{B_r \setminus E_\gamma} Ar^\beta dx - \int_{E_\gamma \setminus B_R} AR^\beta dx \\
 &\leq \int_{B_r \setminus E_\gamma} A(1-\varepsilon)^\beta dx - \int_{E_\gamma \setminus B_R} Adx \\
 &= A \left( (1-\varepsilon)^\beta |B_r \setminus E_\gamma| - |E_\gamma \setminus B_R| \right) \\
 &= A \left( (1-\varepsilon)^\beta - 1 \right) |E_\gamma \setminus B_r|.
 \end{aligned}$$

This last estimate with (1.4.8) and (1.4.10) gives

$$\mathcal{E}_{\alpha,\beta,A,\gamma}(E'_\gamma) - \mathcal{E}_{\alpha,\beta,A,\gamma}(E_\gamma) \leq \left( \gamma^{1+\alpha} C(n, \alpha) + \gamma^{1+\beta} A \left( (1-\varepsilon)^\beta - 1 \right) \right) |E_\gamma \setminus B_r|.$$

As  $E_\gamma$  is a minimizer, we have  $\mathcal{E}_{\alpha,\beta,A,\gamma}(E'_\gamma) - \mathcal{E}_{\alpha,\beta,A,\gamma}(E_\gamma) \geq 0$ , so for  $\alpha < \beta$  and  $\gamma$  large enough (depending only of  $n, \alpha, \beta, A, \delta$ ), this last inequality implies

$$|B_r \setminus E_\gamma| = 0, \quad i.e. \quad B_r \subset E_\gamma,$$

up to a set of measure 0. This concludes *Step two*.

*Step three.* The first and second steps show that for any  $k \in \mathbb{N}^*$ , there exists  $\gamma_k > 0$  such that for any  $\gamma > \gamma_k$ , there exist some sets  $P_{\gamma,k} \subset B$  and  $Q_{\gamma,k} \subset B^c$  of measure 0, such that

$$B_{1-1/k} \subset (E_\gamma \cup P_{\gamma,k}) \setminus Q_{\gamma,k} \subset B_{1+1/k}.$$

For any  $\gamma > 0$ , let us define the sets

$$P_\gamma := \bigcup_{\{k \in \mathbb{N}^* : \gamma > \gamma_k\}} P_{\gamma,k} \quad \text{and} \quad Q_\gamma := \bigcup_{\{k \in \mathbb{N}^* : \gamma > \gamma_k\}} Q_{\gamma,k},$$

and

$$\widetilde{E}_\gamma = (E_\gamma \cup P_\gamma) \setminus Q_\gamma.$$

Then, for any  $\gamma > 0$ ,  $E_\gamma$  and  $\widetilde{E}_\gamma$  are equal up to a set of measure 0, and for any  $k \geq 1$ , for any  $\gamma > \gamma_k$ ,

$$B_{1-1/k} \subset \widetilde{E}_\gamma \subset B_{1+1/k}.$$

In particular, we have

$$\sup_{x \in \partial \widetilde{E}_\gamma} d(x, \partial B) \xrightarrow{\gamma \rightarrow +\infty} 0. \quad (1.4.11)$$

Reciprocally, let  $y \in \partial B$ . For any  $k \geq 1$ , for any  $\gamma > \gamma_k$ , we have

$$(1 - 2/k)y \in \widetilde{E}_\gamma \quad \text{and} \quad (1 + 1/k)y \notin \widetilde{E}_\gamma,$$

so there exists  $t \in [1 - 2/k, 1 + 1/k]$  such that  $ty \in \partial \widetilde{E}_\gamma$ . In particular,

$$d(y, \partial \widetilde{E}_\gamma) \leq 2/k.$$

This shows that

$$\sup_{y \in \partial B} d(y, \partial \widetilde{E}_\gamma) \xrightarrow{\gamma \rightarrow +\infty} 0.$$

With (1.4.11), we get that that boundary of  $\widetilde{E}_\gamma$  converges to the boundary of  $B$  in the Hausdorff sense when  $\gamma$  goes to  $+\infty$ . This concludes the proof.  $\square$

*Remark 1.4.3.* With this proof, we see that the result of Theorem 1.1.1 is also valid for any  $\alpha \in (0, n)$  and  $\beta > 0$  if, instead of letting the mass  $m$  go to  $+\infty$ , we let the quantity  $A\gamma^{\beta-\alpha}$  go to  $+\infty$  (with  $\gamma = \left(\frac{m}{|B|}\right)^{\frac{1}{n}}$ ).

**Lemma 1.4.4.** *Given  $F \subset \mathbb{R}^n$  a set of finite perimeter,  $r_0 > 0$ , and  $\varepsilon > 0$ , assume that*

$$|F| \leq |B| \left(\frac{\varepsilon}{2}\right)^n. \quad (1.4.12)$$

*Then, there exists  $r \in (r_0 - \varepsilon, r_0)$  such that,*

$$P(F, B_r) \geq \mathcal{H}^{n-1}(F \cap \partial B_r). \quad (1.4.13)$$

*Proof.* We argue by contradiction and assume that (1.4.12) holds, and

$$\forall r \in (r_0 - \varepsilon, r_0), P(F, B_r) < \mathcal{H}^{n-1}(F \cap \partial B_r).$$

Adding  $\mathcal{H}^{n-1}(F \cap \partial B_r)$  to both sides, this is equivalent to

$$P(F \cap B_r) < 2\mathcal{H}^{n-1}(F \cap \partial B_r).$$



Using the isoperimetric inequality we get

$$\left(\frac{|F \cap B_r|}{|B|}\right)^{\frac{n-1}{n}} P(B) < 2\mathcal{H}^{n-1}(F \cap \partial B_r). \quad (1.4.14)$$

Now set for  $r \geq 0$ ,  $f(r) = |F \cap B_r|$ . We can assume  $f(r) \neq 0$  for all  $r \in (r_0 - \varepsilon, r_0)$  otherwise the lemma is trivially true. We have for almost all  $r \in (0, \infty)$ ,

$$f'(r) = \mathcal{H}^{n-1}(F \cap \partial B_r).$$

Thus (1.4.14) gives for almost all  $r \in (r_0 - \varepsilon, r_0)$ ,

$$\frac{1}{n} f'(r) f(r)^{\frac{1}{n}-1} > \frac{P(B)}{2n|B|^{\frac{n-1}{n}}} = \frac{|B|^{\frac{1}{n}}}{2}.$$

Integrating on the interval  $(r_0 - \varepsilon, r_0)$ , we get

$$f(r_0)^{\frac{1}{n}} - f(r_0 - \varepsilon)^{\frac{1}{n}} > \frac{\varepsilon |B|^{\frac{1}{n}}}{2},$$

so

$$f(r_0)^{\frac{1}{n}} > \frac{\varepsilon |B|^{\frac{1}{n}}}{2},$$

which contradicts (1.4.12).  $\square$

### 1.4.2 Large volume minimizers = balls for $\alpha < \beta$ and $\beta > 1$

Here we prove Theorem 1.1.2, *i.e.* that if we assume in addition that  $\alpha > 1$ , then large volume minimizers are exactly balls. We conjecture that the theorem is also true when  $\alpha \in (0, 1]$ , as long as  $\beta > 1$ . For  $\beta < 1$ , it cannot be true as we know from Proposition 1.3.3 that for  $m$  large the ball  $B[m]$  is not even a local minimizer. Note that in dimension 1, using Theorem 1.1.1, one can perform some computations to show that the theorem is indeed true under the more general assumption  $\beta > \max(1, \alpha)$ .

The proof relies heavily on the following simple lemma:

**Lemma 1.4.5.** *If  $\alpha > 1$ , then there exists a constant  $C(n, \alpha) > 0$  such that for any set  $E \subset \mathbb{R}^n$  of volume  $|E| = |B|$ , we have*

$$\|\Phi_E^\alpha\|_{C^1(\mathbb{R}^n)} \leq C(n, \alpha), \quad \text{where} \quad \Phi_E^\alpha(x) = \int_E \frac{dx}{|x-y|^{n-\alpha}}.$$

This lemma is not true as soon as  $\alpha \leq 1$ , where we just get  $\alpha$ -Hölder continuity instead of Lipschitz continuity. We refer to [10, Proposition 2.1] for a proof.

*Proof of Theorem 1.1.2.* Rescaling the functional as usual, we need to show that for  $\gamma$  large enough, if  $E \subset \mathbb{R}^n$  is such that  $|E| = |B|$ , and  $E$  is a volume-constrained minimizer of  $\mathcal{E}_{\alpha, \beta, A, \gamma}$  (see (1.4.2)), then  $E=B$ . Let us show that for  $\gamma > 0$  large enough, we have

$$\gamma^{1+\alpha} \mathcal{V}_\alpha(E) + \gamma^{1+\beta} \mathcal{U}_{\beta, A}(E) \geq \gamma^{1+\alpha} \mathcal{V}_\alpha(B) + \gamma^{1+\beta} \mathcal{U}_{\beta, A}(B). \quad (1.4.15)$$

The theorem will then result from the isoperimetric inequality:  $P(E) > P(B)$  if  $E \neq B$ . We divide the proof of (1.4.15) into two steps. In *step one* we compare  $E$  to the subgraph of a function over the sphere, by concentrating the mass of  $E$  on each half line through the origin. In *step two*, we show that (1.4.15) holds for subgraphs of sufficiently small functions over the sphere.

*Step one.* For any  $x \in \partial B$ , define  $u(x) \in \mathbb{R}$  by the equation

$$\int_0^{1+u(x)} r^{n-1} dr = \int_{\mathbb{R}_+} \mathbf{1}_{rx \in E} r^{n-1} dr. \quad (1.4.16)$$

Then, set

$$E_u = \{t(1 + u(x)), t \in [0, 1], x \in \partial B\}. \quad (1.4.17)$$

We have

$$|E_u| = \int_{\partial B} \int_0^{1+u(x)} r^{n-1} dr d\mathcal{H}^{n-1}(x) \quad (1.4.18)$$

$$= \int_{\partial B} \int_{\mathbb{R}_+} \mathbf{1}_{rx \in E} r^{n-1} dr d\mathcal{H}^{n-1}(x) \quad (1.4.19)$$

$$= |E|, \quad (1.4.20)$$

thus  $E_u$  satisfies the volume constraint. Now we estimate the variation of  $\mathcal{U}_{\beta,A}$ . From Theorem 1.1.1 we know that, taking  $\gamma$  large enough, we can assume  $B_{\frac{1}{2}} \subset E$ . Thus we have

$$\begin{aligned} \mathcal{U}_{\beta,A}(E_u) - \mathcal{U}_{\beta,A}(E) &= \int_{E_u \setminus B_{\frac{1}{2}}} A|x|^\beta dx - \int_{E \setminus B_{\frac{1}{2}}} A|x|^\beta dx \\ &= \int_{\partial B} \int_{(\frac{1}{2}, u(x))} A r^\beta r^{n-1} d\mathcal{H}^{n-1}(x) \\ &\quad - \int_{\partial B} \int_{(\frac{1}{2}, \infty)} \mathbf{1}_{rx \in E} A r^\beta r^{n-1} d\mathcal{H}^{n-1}(x) \\ &= \int_{\partial B} \left( \int_{(\frac{1}{2}, u(x))} A r^\beta r^{n-1} dr - \int_{E_x} A r^\beta r^{n-1} dr \right) d\mathcal{H}^{n-1}(x), \end{aligned} \quad (1.4.21)$$

where we have set

$$E_x := \{r \geq \frac{1}{2} : rx \in E\}.$$

Here we need a simple lemma from optimal transportation on the real line.

**Lemma 1.4.6.** *Given a measurable set  $S \subset (1/2, \infty)$  such that  $\int_S r^{n-1} dr < \infty$ , let  $u > 0$  be such that*

$$\int_S r^{n-1} dr = \int_{(\frac{1}{2}, u)} r^{n-1} dr.$$

*Then, there exists a unique non-decreasing measurable map  $T : (1/2, u) \rightarrow (1/2, \infty)$  such that*

$$\mathbf{1}_S r^{n-1} dr = T\#(\mathbf{1}_{(1/2, u)} r^{n-1} dr),$$

i.e. for any non-negative measurable function  $f$ ,

$$\int_S f(r)r^{n-1}dr = \int_{(1/2,u)} f(T(r))r^{n-1}dr. \quad (1.4.22)$$

Moreover, we have

$$\forall r \in (1/2, u), T(r) \geq r.$$

*Proof.* The existence and uniqueness of the map  $T$  is given by the result [75, Theorem 2.5]. Only the fact that  $T(r) \geq r$  needs to be checked. For  $r \in (1/2, u)$ , taking  $f = \mathbb{1}_{(1/2, T(r))}$  in (1.4.22) yields

$$\int_{S \cap (1/2, T(r))} t^{n-1} dt = \int_{(1/2, u)} \mathbb{1}_{T(t) \leq T(r)} t^{n-1} dt.$$

Let us argue by contradiction and assume that  $T(r) < r$ . As  $T$  is non-decreasing, the previous equality yields

$$\begin{aligned} \int_{S \cap (1/2, T(r))} t^{n-1} dt &\geq \int_{(1/2, r)} t^{n-1} dt \\ &> \int_{(1/2, T(r))} t^{n-1} dt \\ &\geq \int_{S \cap (1/2, T(r))} t^{n-1} dt, \end{aligned}$$

which gives a contradiction. This concludes the proof.  $\square$

For each  $x \in \partial B$ , we apply this lemma to  $S = E_x$ , to get a corresponding map  $T_x$ . Then, (1.4.21) becomes

$$\mathcal{U}_{\beta, A}(E_u) - \mathcal{U}_{\beta, A}(E) = \int_{\partial B} \left( \int_{(\frac{1}{2}, u(x))} (Ar^\beta - AT_x(r)^\beta) r^{n-1} dr \right) d\mathcal{H}^{n-1}(x). \quad (1.4.23)$$

Now let us compute the variation of the Riesz energy  $\mathcal{V}_\alpha$  in a similar fashion :

$$\begin{aligned} \mathcal{V}_\alpha(E_u) - \mathcal{V}_\alpha(E) &= \int_{E_u} \Phi_{E_u}^\alpha - \int_E \Phi_E^\alpha \\ &= \int_{E_u} \Phi_{E_u}^\alpha + \int_{E_u} \Phi_E^\alpha - \int_E \Phi_{E_u}^\alpha - \int_E \Phi_E^\alpha \\ &= \int_{E_u} (\Phi_{E_u}^\alpha + \Phi_E^\alpha) - \int_E (\Phi_{E_u}^\alpha + \Phi_E^\alpha) \\ &= \int_{\partial B} \int_{(1/2, u(x))} [(\Phi_{E_u}^\alpha + \Phi_E^\alpha)(rx) \\ &\quad - (\Phi_{E_u}^\alpha + \Phi_E^\alpha)(T_x(r)x)] r^{n-1} dr d\mathcal{H}^{n-1}(x). \end{aligned} \quad (1.4.24)$$

To estimate (1.4.23) and (1.4.24), we use the two following inequalities:

$$\forall x \in \partial B, \forall s \geq r > \frac{1}{2}, r^\beta - s^\beta \leq -C(\beta) |r - s|,$$

$$(\Phi_{E_u}^\alpha + \Phi_E^\alpha)(rx) - (\Phi_{E_u}^\alpha + \Phi_E^\alpha)(sx) \leq C(n, \alpha) |r - s|,$$

where the second inequality comes from Lemma 1.4.5. With these and (1.4.23) and (1.4.24), we get

$$\begin{aligned} & \left( \gamma^{1+\alpha} \mathcal{V}_\alpha(E_u) + \gamma^{1+\beta} \mathcal{U}_{\beta,A}(E_u) \right) - \left( \gamma^{1+\alpha} \mathcal{V}_\alpha(E) + \gamma^{1+\beta} \mathcal{U}_{\beta,A}(E) \right) \\ & \leq \int_{\partial B} \int_{(1/2, u(x))} \left( \gamma^{1+\alpha} C(n, \alpha) - \gamma^{1+\beta} AC(\beta) \right) |r - T_x(r)| r^{n-1} dr d\mathcal{H}^{n-1}(x). \end{aligned}$$

From this inequality we get that if  $\gamma$  is large enough (depending only on  $n, \alpha, \beta, A$ ), then

$$\gamma^{1+\alpha} \mathcal{V}_\alpha(E_u) + \gamma^{1+\beta} \mathcal{U}_{\beta,A}(E_u) \leq \gamma^{1+\alpha} \mathcal{V}_\alpha(E) + \gamma^{1+\beta} \mathcal{U}_{\beta,A}(E). \quad (1.4.25)$$

*Step two.* We show that there exists  $\varepsilon = \varepsilon(n, \alpha, \beta, A) > 0$ , such that for any  $\gamma$  large enough, if  $\|u\|_{L^\infty(\partial B)} < \varepsilon$ , then

$$\gamma^{1+\alpha} \mathcal{V}_\alpha(B) + \gamma^{1+\beta} \mathcal{U}_{\beta,A}(B) \leq \gamma^{1+\alpha} \mathcal{V}_\alpha(E_u) + \gamma^{1+\beta} \mathcal{U}_{\beta,A}(E_u). \quad (1.4.26)$$

Remark that by Theorem 1.1.1, the condition  $\|u\|_{L^\infty(\partial B)} < \varepsilon$  is satisfied if  $\gamma$  has been taken large enough. The inequality (1.4.26) will result from this computational lemma, whose proof is postponed :

**Lemma 1.4.7.** *Given a measurable function  $u : \partial B \rightarrow \mathbb{R}$  with  $\|u\|_{L^\infty(\partial B)} < 1$ , set for  $t \geq 0$*

$$E_t := \{s(1 + tu(x))x, x \in \partial B, s \in [0, 1)\}.$$

*Assume that  $|E_t| = |B|$ . Then, for  $t$  small enough, depending only on the dimension  $n$ , we have*

$$\mathcal{U}_{\beta,A}(E_t) \geq \mathcal{U}_{\beta,A}(B) + A\beta \frac{t^2}{2} \|u\|_{L^2(\partial B)}^2 - C(n, \beta)t^3 \|u\|_{L^2(\partial B)}^2, \quad (1.4.27)$$

and

$$\begin{aligned} \mathcal{V}_\alpha(E_t) & \geq \mathcal{V}_\alpha(B) - \frac{t^2}{2} \left( [u]_{\frac{1-\alpha}{2}}^2 - \alpha(n + \alpha) \|u\|_{L^2(\partial B)}^2 \right) \\ & \quad - C(n)t^3 \left( [u]_{\frac{1-\alpha}{2}}^2 + \alpha \mathcal{V}_\alpha(B) \|u\|_{L^2(\partial B)}^2 \right), \end{aligned} \quad (1.4.28)$$

where

$$[u]_{\frac{1-\alpha}{2}}^2 = \int_{\partial B \times \partial B} \frac{|u(x) - u(y)|^2}{|x - y|^{n-\alpha}} d\mathcal{H}^{n-1}(x) d\mathcal{H}^{n-1}(y).$$

Indeed for  $\alpha > 1$ , we have

$$\begin{aligned} [u]_{\frac{1-\alpha}{2}}^2 & \leq \int_{\partial B \times \partial B} \frac{2(|u(x)|^2 + |u(y)|^2)}{|x - y|^{n-\alpha}} d\mathcal{H}^{n-1}(x) d\mathcal{H}^{n-1}(y) \\ & = 4 \int_{\partial B \times \partial B} \frac{|u(x)|^2}{|x - y|^{n-\alpha}} d\mathcal{H}^{n-1}(x) d\mathcal{H}^{n-1}(y) \\ & = 4 \int_{\partial B} \left( \int_{\partial B} \frac{d\mathcal{H}^{n-1}(y)}{|x - y|^{n-\alpha}} \right) |u(x)|^2 d\mathcal{H}^{n-1}(x) \\ & = C(n, \alpha) \int_{\partial B} |u(x)|^2 d\mathcal{H}^{n-1}(x) \\ & = C(n, \alpha) \|u\|_{L^2(\partial B)}^2, \end{aligned}$$

so that (1.4.28) gives

$$\mathcal{V}_\alpha(E_t) \geq \mathcal{V}_\alpha(B) - \frac{t^2}{2}C(n, \alpha)\|u\|_{L^2(\partial B)}^2 - C(n, \alpha)t^3\|u\|_{L^2(\partial B)}^2.$$

This implies that for  $t$  small enough, depending only on  $n$  and  $\alpha$ , we have

$$\mathcal{V}_\alpha(E_t) \geq \mathcal{V}_\alpha(B) - t^2C(n, \alpha)\|u\|_{L^2(\partial B)}^2.$$

Likewise, we get from (1.4.27) that for  $t$  small enough, depending only on  $n$ ,  $\beta$  and  $A$ , we have

$$\mathcal{U}_{\beta, A}(E_t) \geq \mathcal{U}_{\beta, A}(B) + t^2C(n, \beta, A)\|u\|_{L^2(\partial B)}^2.$$

These last two inequalities imply that there exists  $\varepsilon = \varepsilon(n, \alpha, \beta, A) > 0$ , such that if  $\|u\|_{L^\infty(\partial B)} < \varepsilon$ , then

$$\begin{aligned} \gamma^{1+\alpha}\mathcal{V}_\alpha(B) + \gamma^{1+\beta}\mathcal{U}_{\beta, A}(B) &\leq \gamma^{1+\alpha}\mathcal{V}_\alpha(E_u) + \gamma^{1+\beta}\mathcal{U}_{\beta, A}(E_u) \\ &\quad + \left( \gamma^{1+\alpha}C(n, \alpha, \beta, A)\|u\|_{L^2(\partial B)}^2 - \gamma^{1+\beta}C(n, \alpha, \beta, A)\|u\|_{L^2(\partial B)}^2 \right), \end{aligned}$$

which in turn implies (1.4.26) for  $\gamma$  large enough.

The estimate (1.4.15) is now a consequence of (1.4.25) and (1.4.26). The theorem results from (1.4.15) and the isoperimetric inequality.  $\square$

*Proof of Lemma 1.4.7.* The proof of (1.4.28) is given in [33, equation (5.20)], under the hypothesis  $\|u\|_{C^1(\partial B)} \leq 1$  instead of  $\|u\|_{L^\infty(\partial B)} \leq 1$ . However it is clear from the proof that it holds also for  $\|u\|_{L^\infty(\partial B)} \leq 1$  only. (The reason why it was stated with the stronger hypothesis  $\|u\|_{C^1(\partial B)} \leq 1$  is because it is needed to get the corresponding estimate for the perimeter.)

Let us prove (1.4.27). Using spherical coordinates, we can compute

$$\begin{aligned} \mathcal{U}_{\beta, A}(E_t) &= \int_{\partial B} \int_0^{1+tu(x)} A |rx|^\beta r^{n-1} dr d\mathcal{H}^{n-1}(x) \\ &= \int_{\partial B} \int_0^1 A (1+tu(x))^{n+\beta} r^{n+\beta-1} dr d\mathcal{H}^{n-1}(x) \\ &= \int_{\partial B} A \frac{(1+tu)^{n+\beta}}{n+\beta} d\mathcal{H}^{n-1}. \end{aligned}$$

Setting  $h(t) := \int_{\partial B} (1+tu)^{n+\beta} d\mathcal{H}^{n-1}$ , we then have  $\mathcal{U}_{\beta, A}(E_t) - \mathcal{U}_{\beta, A}(B) = \frac{A}{n+\beta}(h(t) - h(0))$ . The Taylor-Lagrange inequality gives

$$(1+tu)^{n+\beta} \geq 1 + (n+\beta)tu + (n+\beta)(n+\beta-1)\frac{(tu)^2}{2} - C(n, \beta)\frac{(tu)^3}{3},$$

So

$$\begin{aligned} \frac{1}{n+\beta}(h(t) - h(0)) &\geq \int_{\partial B} tud\mathcal{H}^{n-1} + (n+\beta-1) \int_{\partial B} \frac{(tu)^2}{2} d\mathcal{H}^{n-1} - C(n, \beta)t^3 \int_{\partial B} u^3 d\mathcal{H}^{n-1} \\ &\geq \int_{\partial B} tud\mathcal{H}^{n-1} + (n+\beta-1) \int_{\partial B} \frac{(tu)^2}{2} d\mathcal{H}^{n-1} - C(n, \beta)t^3\|u\|_{L^2(\partial B)}^2. \end{aligned} \tag{1.4.29}$$

Now we use the volume constraint  $|E_t| = |B|$  to estimate  $\int tu$ . The volume constraint can be expressed as

$$\int_{\partial B} (1 + tu)^n d\mathcal{H}^{n-1} = \int_{\partial B} 1 d\mathcal{H}^{n-1},$$

and so

$$\begin{aligned} \int_{\partial B} tud\mathcal{H}^{n-1} &= \int_{\partial B} \left( tu - \frac{1}{n} ((1 + tu)^n - 1) \right) d\mathcal{H}^{n-1} \\ &= - \sum_{k=2}^n \frac{1}{n} \binom{n}{k} \int_{\partial B} (tu)^k d\mathcal{H}^{n-1} \\ &\geq - \frac{n-1}{2} \int_{\partial B} (tu)^2 d\mathcal{H}^{n-1} - C(n)t^3 \|u\|_{L^2(\partial B)}^2 \end{aligned} \quad (1.4.30)$$

This with (1.4.29) gives (1.4.27).  $\square$

## 1.A Appendix: study of the sets $\tilde{S}_*$ and $S_*$ from the proof of Proposition 1.3.3

In this appendix, we give explicit forms of the sets  $\tilde{S}_*$  and  $S_*$ , needed in the proof of Proposition 1.3.3. Let us define, for all  $k \geq 2$ , a function  $f_k : (0, \infty) \rightarrow \mathbb{R}$  by

$$f_k(\gamma) = 1 - \gamma^{1+\alpha} \frac{\mu_k^\alpha - \mu_1^\alpha}{\lambda_k - \lambda_1} + \gamma^{1+\beta} \frac{A\beta}{\lambda_k - \lambda_1}.$$

Then, the sets  $\tilde{S}_*$  and  $S_*$  from the proof of Proposition 1.3.3 are defined by

$$\tilde{S}_* = \{\forall k \geq 2, f_k > 0\} \quad \text{and} \quad S_* = \{\forall k \geq 2, f_k \geq 0\}.$$

As stated in [33, equations (7.4), (7.5) and (7.6)], we have

$$\mu_k^\alpha = \begin{cases} \frac{2^{1+\alpha} \pi^{\frac{n-1}{2}} \Gamma(\frac{1+\alpha}{2})}{1-\alpha} \left( \frac{\Gamma(k+\frac{n-\alpha}{2})}{\Gamma(k+\frac{n-2+\alpha}{2})} - \frac{\Gamma(\frac{n-\alpha}{2})}{\Gamma(\frac{n-2+\alpha}{2})} \right) & \text{if } \alpha \in (0, 1), \\ 2^\alpha \pi^{\frac{n-1}{2}} \frac{\Gamma(\frac{\alpha-1}{2})}{\Gamma(\frac{n-\alpha}{2})} \left( \frac{\Gamma(\frac{n-\alpha}{2})}{\Gamma(\frac{n-2+\alpha}{2})} - \frac{\Gamma(k+\frac{n-\alpha}{2})}{\Gamma(k+\frac{n-2+\alpha}{2})} \right) & \text{if } \alpha \in (1, n), \\ \frac{4\pi^{\frac{n-1}{2}}}{\Gamma(\frac{n-1}{2})} \left( \frac{\Gamma'(k+\frac{n-1}{2})}{\Gamma(k+\frac{n-1}{2})} - \frac{\Gamma'(\frac{n-1}{2})}{\Gamma(\frac{n-1}{2})} \right) & \text{if } \alpha = 1. \end{cases} \quad (1.A.1)$$

Recall also that for any  $k \geq 0$ ,  $\lambda_k = k(n+k-2)$ . Now let us treat each case enumerated in Proposition 1.3.3 separately.

*Case 1:  $\alpha > \beta$ .* A simple study of the sign of  $f'_k$  shows that each  $f_k$  is increasing from 0 to a point  $\gamma_k$ , then decreasing from  $\gamma_k$  to  $+\infty$ . Furthermore,  $f_k(0) = 1$  and  $\lim_{+\infty} f_k = -\infty$ , so each  $f_k$  has exactly one zero and is positive left of this zero and negative right of it. At last, for any constant  $K > 0$ ,  $f_k(\gamma) \xrightarrow[k \rightarrow \infty]{} 1$  uniformly in  $\gamma \leq K$ . Putting these facts together shows that the sets  $\tilde{S}_*$  and  $S_*$  have the forms:

$$\tilde{S}_* = (0, m_*) \quad \text{and} \quad S_* = (0, m_*],$$

for some critical mass  $m_* > 0$  (depending on  $n, \alpha, \beta$  and  $A$ ).

*Case 2:*  $\alpha = \beta$ . For any  $k$ ,  $f_k$  is either decreasing or increasing or constant (depending on the size of  $A$ ). If none of them is decreasing, then for any  $k$ ,  $f_k \geq f_k(0) = 1$ , so

$$\tilde{S}_* = S_* = (0, +\infty).$$

Otherwise the same arguments as in *case one* shows again that

$$\tilde{S}_* = (0, m_*) \quad \text{and} \quad S_* = (0, m_*],$$

for some critical mass  $m_* > 0$  (depending on  $n, \alpha, \beta$  and  $A$ ).

*Cases 3, 4 and 5:*  $\alpha < \beta$ . A simple study of the sign of  $f'_k$  shows that each  $f_k$  is decreasing from 0 to a point  $\gamma_k$ , then increasing from  $\gamma_k$  to  $+\infty$ , and we have:

$$\gamma_k = \left( \frac{1 + \alpha}{A\beta(1 + \beta)} (\mu_k^\alpha - \mu_1^\alpha) \right)^{\frac{1}{\beta - \alpha}}. \quad (1.A.2)$$

Another simple computation shows that

$$\min f_k = f_k(\gamma_k) = 1 - \left( \frac{1 + \alpha}{A\beta(1 + \beta)} \right)^{\frac{1 + \alpha}{\beta - \alpha}} \frac{\beta - \alpha}{1 + \beta} \frac{(\mu_k^\alpha - \mu_1^\alpha)^{\frac{1 + \beta}{\beta - \alpha}}}{\lambda_k - \lambda_1}. \quad (1.A.3)$$

We must treat the subcases  $\alpha > 1$ ,  $\alpha = 1$  and  $\alpha < 1$  separately.

*Subcase one:*  $\alpha > 1$ . We use the following classical Stirling formula:

$$\Gamma(x) \underset{x \rightarrow +\infty}{\sim} \sqrt{\frac{2\pi}{x}} \left(\frac{x}{e}\right)^x,$$

to find that

$$\frac{\Gamma(k + \frac{n - \alpha}{2})}{\Gamma(k + \frac{n - 2 + \alpha}{2})} \underset{k \rightarrow \infty}{\sim} k^{1 - \alpha}.$$

With (1.A.1), this means that the sequence  $(\mu_k^\alpha)$  is bounded. As  $\lambda_k \xrightarrow[k \rightarrow \infty]{} \infty$ , we get from (1.A.3)

$$\min f_k \xrightarrow[k \rightarrow \infty]{} 1.$$

Thus there exists an index  $k_0$  such that

$$\tilde{S}_* = \bigcap_{k=2}^{k_0} \{f_k > 0\} \quad \text{and} \quad S_* = \bigcap_{k=2}^{k_0} \{f_k \geq 0\}. \quad (1.A.4)$$

As for any  $k$ ,  $\lim_{+\infty} f_k = +\infty$ , we get that  $\tilde{S}_*$  and  $S_*$  both contain an unbounded interval, which is what we wanted.

*Subcase two:*  $\alpha = 1$ . We use the classical asymptotics of the digamma function  $\frac{\Gamma'}{\Gamma}$ :

$$\frac{\Gamma'}{\Gamma}(x) \underset{x \rightarrow +\infty}{\sim} \ln(x),$$

to find that, according to (1.A.3),

$$\min f_k \xrightarrow[k \rightarrow +\infty]{} 1.$$

We conclude as above.

*Subcase three:*  $\alpha < 1$ . Once again we use the asymptotics

$$\frac{\Gamma(k + \frac{n-\alpha}{2})}{\Gamma(k + \frac{n-2+\alpha}{2})} \underset{k \rightarrow \infty}{\sim} k^{1-\alpha},$$

to find that

$$\frac{(\mu_k^\alpha - \mu_1^\alpha)^{\frac{1+\beta}{\beta-\alpha}}}{\lambda_k - \lambda_1} \underset{k \rightarrow \infty}{\sim} \frac{(k^{1-\alpha})^{\frac{1+\beta}{\beta-\alpha}}}{k^2} = k^{\frac{(1-\beta)(1+\alpha)}{\beta-\alpha}}. \quad (1.A.5)$$

If  $\beta > 1$ , once again we have

$$\min f_k \underset{k \rightarrow +\infty}{\longrightarrow} 1,$$

and we conclude as above. If  $\beta < 1$ , then we have

$$\min f_k \underset{k \rightarrow +\infty}{\longrightarrow} -\infty \quad (1.A.6)$$

Also, by definition we have

$$f_{k+1}(\gamma_k) - 1 = -\gamma_k^{1+\alpha} \frac{\mu_{k+1}^\alpha - \mu_1^\alpha}{\lambda_{k+1} - \lambda_1} + \gamma_k^{1+\beta} \frac{A\beta}{\lambda_{k+1} - \lambda_1}.$$

As the sequence  $(\mu_k^\alpha)$  is increasing we get

$$\begin{aligned} f_{k+1}(\gamma_k) - 1 &\leq -\gamma_k^{1+\alpha} \frac{\mu_k^\alpha - \mu_1^\alpha}{\lambda_{k+1} - \lambda_1} + \gamma_k^{1+\beta} \frac{A\beta}{\lambda_{k+1} - \lambda_1} \\ &= (f_k(\gamma_k) - 1) \frac{\lambda_k - \lambda_1}{\lambda_{k+1} - \lambda_1} \\ &\underset{k \rightarrow \infty}{\longrightarrow} -\infty. \end{aligned} \quad (1.A.7)$$

With (1.A.6), this means that

$$[\gamma_k, \gamma_{k+1}] \subset (\tilde{S}_*)^c \cap (S_*)^c.$$

Moreover, from (1.A.2), we have  $\gamma_k \underset{k \rightarrow +\infty}{\longrightarrow} +\infty$ , so  $\tilde{S}_*$  and  $S_*$  are both bounded, which is what we wanted. At last, if  $\beta = 1$  we find using (1.A.5) that there exists a constant  $C_\alpha$  such that

$$\min f_k \underset{k \rightarrow \infty}{\longrightarrow} 1 - \frac{C_\alpha}{A^{\frac{1+\alpha}{\beta-\alpha}}}.$$

For  $A^{\frac{1+\alpha}{\beta-\alpha}} > C_\alpha$ , we conclude as in *case one*. For  $A^{\frac{1+\alpha}{\beta-\alpha}} < C_\alpha$ , we conclude as above that  $\tilde{S}_*$  and  $S_*$  are both bounded. For  $A^{\frac{1+\alpha}{\beta-\alpha}} < C_\alpha$ , we have to use the following more precise form of Stirling's approximation:

$$\Gamma(x) \underset{x \rightarrow +\infty}{\sim} \sqrt{\frac{2\pi}{x}} \left(\frac{x}{e}\right)^x \left(1 + O\left(\frac{1}{x}\right)\right).$$

Proceeding to simple asymptotic expansions, we find that for  $k$  large enough we have

$$\min f_k > 0.$$

We conclude as in *case one*.





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## Numerical minimization

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In this Chapter we present our method and results for the numerical minimization of the variational problem (1.1.3). We apply this method to the original problem (1.1.2). Indeed, the theoretical knowledge we have so far on problem (1.1.2) raises two natural questions. Is it always the case (i.e. for any value of  $\alpha \in (0, n)$ ) that there is no minimizer for  $m$  large enough? Is there a set of parameters  $n$ ,  $\alpha$  and  $m$ , such that there exists a minimizer that is different from a ball? Our numerical results indicate that in dimension 2, the answers are positive and negative respectively.

### 2.1 Preliminaries

#### 2.1.1 $\Gamma$ -convergence

In this section, we introduce the notion of  $\Gamma$ -convergence, which we will be using to justify our numerical method. This is the notion of convergence of functionals one generally wants to use when studying variational problems. We refer to the book [12] for more details about  $\Gamma$ -convergence. The lecture [6] also contains a brief introduction to  $\Gamma$ -convergence that is sufficient to our needs.

**Definition 2.1.1.** *Let  $X$  be a topological space,  $\mathcal{F}$  a real functional on  $X$ , and  $(\mathcal{F}_\varepsilon)_{\varepsilon>0}$  a family of real functionals on  $X$ . We say that  $\mathcal{F}_\varepsilon$   $\Gamma$ -converges to  $\mathcal{F}$  as  $\varepsilon$  goes to 0, and we write  $\mathcal{F}_\varepsilon \xrightarrow[\varepsilon \rightarrow 0]{\Gamma} \mathcal{F}$ , if the two following assertions hold:*

1. *for any  $u \in X$  and any family  $(u_\varepsilon)_{\varepsilon>0}$  of  $X$  such that  $u_\varepsilon \xrightarrow[\varepsilon \rightarrow 0]{} u$ , we have*

$$\liminf_{\varepsilon \rightarrow 0} \mathcal{F}_\varepsilon(u_\varepsilon) \geq \mathcal{F}(u),$$

2. *for any  $u \in X$ , there exists a family  $(u_\varepsilon)_{\varepsilon>0}$  of  $X$  such that  $u_\varepsilon \xrightarrow[\varepsilon \rightarrow 0]{} u$  and*

$$\lim_{\varepsilon \rightarrow 0} \mathcal{F}_\varepsilon(u_\varepsilon) = \mathcal{F}(u).$$

When dealing with the perimeter term, we will also use the following relaxation of the  $\Gamma$ -convergence definition. This definition and notation is not standard.

**Definition 2.1.2.** *With the same notations as in the previous definition, we say that  $\mathcal{F}_\varepsilon$   $\tilde{\Gamma}$ -converges to  $\mathcal{F}$  as  $\varepsilon$  goes to 0, and we write  $\mathcal{F}_\varepsilon \xrightarrow[\varepsilon \rightarrow 0]{\tilde{\Gamma}} \mathcal{F}$ , if the assertion 1 of the previous definition holds for any  $u \in X$ , and the assertion 2 holds for at least one  $u \in X$  that minimizes the functional  $\mathcal{F}$ .*

*Remark 2.1.3.* We will actually use the following equivalent two assertions

- for any  $u \in X$  and any family  $(u_\varepsilon)_{\varepsilon>0}$  of  $X$  such that  $u_\varepsilon \xrightarrow[\varepsilon \rightarrow 0]{} u$ , we have

$$\liminf_{\varepsilon \rightarrow 0} \mathcal{F}_\varepsilon(u_\varepsilon) \geq \mathcal{F}(u),$$

- for any  $u \in X$  (respectively, in the case of  $\tilde{\Gamma}$ -convergence, there exists a minimizer  $u \in X$  of  $\mathcal{F}$  such that), there exists a family  $(u_\varepsilon)_{\varepsilon>0}$  of  $X$  such that  $u_\varepsilon \xrightarrow[\varepsilon \rightarrow 0]{} u$  and

$$\limsup_{\varepsilon \rightarrow 0} \mathcal{F}_\varepsilon(u_\varepsilon) \leq \mathcal{F}(u).$$

*Remark 2.1.4.* We also say that a sequence of functionals  $(\mathcal{F}_N)$   $\Gamma$ -converges to a function  $\mathcal{F}$  if the assertions above hold with  $\varepsilon$  and 0 replaced with  $N$  and  $+\infty$ .

Both definitions guarantee that if, for any  $\varepsilon > 0$ ,  $u_\varepsilon$  is a minimizer of  $\mathcal{F}_\varepsilon$  in  $X$ , and the family  $(u_\varepsilon)$  converges to  $u \in X$  as  $\varepsilon \rightarrow 0$ , then  $u$  is also a minimizer of  $\mathcal{F}$  in  $X$ . To minimize numerically a given functional  $\mathcal{F}$ , our strategy is the following:

1. find a sequence of discrete functionals  $(\mathcal{F}_N)_{N \in \mathbb{N}}$  that  $\tilde{\Gamma}$ -converges to  $\mathcal{F}$ ,
2. find numerically a minimizer  $u_N$  of  $\mathcal{F}_N$  for some  $N$  considered 'large',
3. extrapolate a limit point  $u$  of the sequence  $(u_N)_{N \in \mathbb{N}}$  from the results. This  $u$  should be a minimizer of  $\mathcal{F}$ .

For point 3 to even have a chance to work, we need the sequence  $(u_N)_{N \in \mathbb{N}}$  to be precompact:

*Terminology 2.1.5.* Following [6], we say that a family of functionals  $(\mathcal{F}_\varepsilon)_{\varepsilon>0}$  defined on a metric space  $X$  enjoys property (C) (for compactness) if for any family  $(u_\varepsilon)_{\varepsilon>0}$  of elements of  $X$  such that  $(\mathcal{F}_\varepsilon(u_\varepsilon))_{\varepsilon>0}$  is bounded, there is a subsequence of  $(u_\varepsilon)_{\varepsilon>0}$  that converges in  $X$ .

If a family of functionals  $(\mathcal{F}_\varepsilon)_{\varepsilon>0}$  enjoys property (C) and  $\tilde{\Gamma}$ -converges to a limit functional  $\mathcal{F}$  when  $\varepsilon$  goes to 0, then we know that for  $\varepsilon$  small enough, minimizers of  $\mathcal{F}_\varepsilon$  are close to minimizers of  $\mathcal{F}$ .

To conclude this section, we have the following elementary proposition.

**Proposition 2.1.6.** *If  $\mathcal{G} : X \rightarrow \mathbb{R}$  is continuous and  $\mathcal{F}_\varepsilon \xrightarrow[\varepsilon \rightarrow 0]{\Gamma} \mathcal{F}$  on  $X$ , then  $\mathcal{F}_\varepsilon + \mathcal{G} \xrightarrow[\varepsilon \rightarrow 0]{\Gamma} \mathcal{F} + \mathcal{G}$  on  $X$ .*

### 2.1.2 Modica-Mortola theorem

The numerical treatment of the perimeter is classical. To deal with the perimeter term, we will use the well-known Modica-Mortola theorem. This theorem allows us to relax the perimeter functional on sets, i.e. characteristic functions, into a functional on functions taking values in  $[0, 1]$ . Therefore we will be able to use the vector space structure of functions and, after discretization (*step three*), usual optimization tools for functionals on  $\mathbb{R}^d$ .

**Theorem 2.1.7** (Modica-Mortola theorem.). *Let  $\Omega$  be a Lipschitz open bounded subset of  $\mathbb{R}^n$  and  $\varepsilon > 0$ . Let  $W$  be the double well potential defined on  $[0, 1]$  by  $W(x) = x(1-x)$ . Let us set  $\sigma := 2 \int_0^1 \sqrt{W(u)} du$ . We define the set  $X$ , and the functionals  $\mathcal{F}_\varepsilon : X \rightarrow \overline{\mathbb{R}}$  by*

$$X := \left\{ u \in L^1(\Omega, [0, 1]) : \int u = m \right\}, \quad \text{endowed with the strong } L^1\text{-topology,}$$

and

$$\mathcal{F}_\varepsilon(u) = \begin{cases} \sigma^{-1} \int_{\mathbb{R}^n} \left( \varepsilon |\nabla u|^2 + \varepsilon^{-1} W(u) \right) & \text{if } u \in H_0^1(\Omega, [0, 1]), \\ +\infty & \text{otherwise,} \end{cases} \quad (2.1.1)$$

If  $(u_\varepsilon)_{\varepsilon>0}$  is a family of functions in  $H_0^1(\Omega, [0, 1])$  and  $E \subset \Omega$  is any measurable subset of  $\Omega$ , such that  $u_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} \mathbf{1}_E$  in  $L^1(\Omega)$ , then we have

$$P(E) \leq \liminf_{\varepsilon \rightarrow 0} \mathcal{F}_\varepsilon(u_\varepsilon).$$

Moreover, if  $E \subset\subset \Omega$  is a compact subset of  $\Omega$  of finite perimeter, then there exists a family of functions  $(u_\varepsilon)_{\varepsilon>0}$  in  $X$ , such that  $u_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} \mathbf{1}_E$ , and

$$P(E) = \lim_{\varepsilon \rightarrow 0} \mathcal{F}_\varepsilon(u_\varepsilon).$$

Finally, the family of functionals  $(\mathcal{F}_\varepsilon)$  enjoys property (C).

Reader-friendly proofs of these assertions can be found in [6]. Here, we will only give the following heuristic. If  $\varepsilon > 0$  is very small, then  $\varepsilon^{-1}W(u)$  is very large, except if  $u$  is 0 or 1. Therefore the minimizers of  $\mathcal{F}_\varepsilon$  are functions that are equal to 0 or 1 except on a set of small measure.

## 2.2 Method of the numerical minimization

We present a series of three modifications of the variational problem (1.1.3) to arrive at a finite dimensional variational problem that can be easily numerically solved. All steps are justified by a  $\Gamma$ -convergence and compactness result.

*Step one* is standard when dealing with the perimeter. We use the classical Modica-Mortola theorem to relax the functional on sets, i.e. characteristic functions, into a functional on functions taking values in  $[0, 1]$ . This allows us to use the vector space structure of functions and, after discretization (*step three*), usual optimization tools for functionals on  $\mathbb{R}^d$ .

*Step two* is the key step for dealing with the non-local term  $\mathcal{V}_\alpha$ . We replace the ambient space  $\mathbb{R}^n$  with a large square with periodic boundary conditions, whose size is a new relaxation

parameter. Then, we can approximate the non-local term  $\mathcal{V}_\alpha$  by a simple expression in Fourier variable.

In *step three*, we discretize the problem by considering only trigonometric functions with frequencies lower than some integer  $N$ , and by computing the integral terms with riemann sums.

Let us now describe and justify each step precisely.

**Step one.** We use the classical Modica-Mortola theorem to replace this problem on subsets of  $\mathbb{R}^n$ , i.e. functions taking only values 0 or 1, with a problem on functions taking any value between 0 and 1. Let  $m > 0$ . First, remember that we proved in the proof of Proposition 1.2.1 that minimizers of our problem (1.1.3) are uniformly bounded. Let  $\Omega$  be a Lipschitz bounded open set such that one minimizer of (1.1.3) is compactly included in  $\Omega$ . Given a (small)  $\varepsilon > 0$ , we define the set  $X$ , and the functionals  $\mathcal{F}_\varepsilon : X \rightarrow \overline{\mathbb{R}}$  and  $\mathcal{F} : X \rightarrow \overline{\mathbb{R}}$  by

$$X := \left\{ u \in L^1(\mathbb{R}^n, [0, 1]) : u|_{\Omega^c} = 0 \quad \text{and} \quad \int u = m \right\}, \quad \text{endowed with the strong } L^1\text{-topology,}$$

and

$$\mathcal{F}_\varepsilon(u) = \begin{cases} \sigma^{-1} \int_{\mathbb{R}^n} \left( \varepsilon |\nabla u|^2 + \varepsilon^{-1} W(u) \right) + \mathcal{V}_\alpha(u) + A \int_{\mathbb{R}^n} u(x) |x|^\beta dx & \text{if } u|_\Omega \in H_0^1(\Omega, [0, 1]) \\ +\infty & \text{otherwise,} \end{cases} \quad (2.2.1)$$

$$\mathcal{F}(u) = \begin{cases} P(E) + \mathcal{V}_\alpha(E) + A \int_E |x|^\beta dx & \text{if } u = \mathbf{1}_E, \text{ with } E \subset \Omega, \\ +\infty & \text{otherwise,} \end{cases} \quad (2.2.2)$$

where we have used the natural notation  $\mathcal{V}_\alpha(u) = \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{u(x)u(y)}{|x-y|^{n-\alpha}} dx dy$ , and  $W$  is the following double well potential on  $[0, 1]$ :  $W(x) = x(1-x)$ . Then, from the Modica-Mortola theorem and the fact that the two last terms of the functionals  $\mathcal{F}_\varepsilon$  and  $\mathcal{F}$  are continuous on  $X$ , we have the following proposition.

**Proposition 2.2.1.** *The following  $\tilde{\Gamma}$ -convergence holds on  $X$ , endowed with the strong  $L^1(\mathbb{R}^n)$ -topology.*

$$\mathcal{F}_\varepsilon \xrightarrow[\varepsilon \rightarrow 0]{\tilde{\Gamma}} \mathcal{F}.$$

Moreover,  $(\mathcal{F}_\varepsilon)$  enjoys property (C).

**Step two.** For simplicity, we will first reduce the domain of definition of  $\mathcal{F}_\varepsilon$ , and use a stronger topology in this step. Let us set

$$X' := \left\{ u \in L^1(\mathbb{R}^n, [0, 1]) : u|_\Omega \in H_0^1(\Omega, [0, 1]), \quad u|_{\Omega^c} = 0 \quad \text{and} \quad \int u = m \right\},$$

endowed with the weak  $H^1(\mathbb{R}^n)$ -topology. The non-local repulsive term has a simple expression in Fourier variable :

$$\mathcal{V}_\alpha(u) = \frac{C(n, \alpha)}{(2\pi)^n} \int_{\mathbb{R}^n} |\xi|^{-\alpha} |\hat{u}(\xi)|^2 d\xi, \quad (2.2.3)$$

with  $\hat{u}$  the Fourier transform of  $u$  and  $C(n, \alpha) := \frac{2^\alpha \pi^{\frac{n}{2}} \Gamma(\frac{\alpha}{2})}{\Gamma(\frac{n-\alpha}{2})}$ , and  $\Gamma$  the usual gamma function. This can be seen by noting that  $\mathcal{V}_\alpha(u) = \int u \mathcal{I}_\alpha(u)$  with  $\mathcal{I}_\alpha(u)$  the Riesz potential of  $u$ , and using the Fourier expression of the Riesz potential (see [77, Part V]). To exploit this formula, we will replace the ambient space  $\mathbb{R}^n$  with a large torus of size  $T > 0$ . Thus we will approximate  $\mathcal{V}_\alpha(u)$  by

$$\mathcal{V}_{\alpha,T}(u) := \frac{C(n, \alpha)}{T^n} \sum_{k \in \mathbb{Z}^n \setminus \{0\}} \left| \frac{2k\pi}{T} \right|^{-\alpha} |c_{k,T}(u)|^2, \quad (2.2.4)$$

where  $c_{k,T}(u) := \int_{(-\frac{T}{2}, \frac{T}{2})^n} u(x) e^{-\frac{2ik\pi x}{T}} dx$  is the  $k$ -th Fourier coefficient of  $u$  on  $[-T/2, T/2]^n$ , for some (large)  $T > 0$ . More precisely, let us define the functional  $\mathcal{F}_{\varepsilon,T} : X' \rightarrow \overline{\mathbb{R}}$  by

$$\mathcal{F}_{\varepsilon,T}(u) = \sigma^{-1} \int_{\mathbb{R}^n} \left( \varepsilon |\nabla u|^2 + \varepsilon^{-1} W(u) \right) + \mathcal{V}_{\alpha,T}(u) + A \int_{\mathbb{R}^n} u(x) |x|^\beta dx$$

Then, we have the following proposition.

**Proposition 2.2.2.** *The following  $\Gamma$ -convergence holds on  $X'$ , endowed with the weak  $H^1(\mathbb{R}^n)$ -topology:*

$$\mathcal{F}_{\varepsilon,T} \xrightarrow[T \rightarrow \infty]{\Gamma} \mathcal{F}_\varepsilon. \quad (2.2.5)$$

Moreover,  $(\mathcal{F}_{\varepsilon,T})_{T>0}$  enjoys property (C).

We emphasize the following remark:

*Remark 2.2.3.* To prove property (C) in this step, the assumption that all functions are supported in a given bounded set  $\Omega$  is important. Numerically, we also observe that the method fails without this assumption. See Section 2.3 for further comments.

Let us prove Proposition 2.2.2.

*Proof of Proposition 2.2.2.*

*Proof of property (C).* Let  $(u_T)_{T>0}$  be a sequence of  $X'$  such that  $(\mathcal{F}_{\varepsilon,T}(u_T))_{T>0}$  is bounded. Then,  $(u_T)_{T>0}$  is bounded in  $H^1(\mathbb{R}^n)$ , so up to taking a subsequence, it converges weakly to a function  $u \in H^1(\mathbb{R}^n)$ . Up to taking another subsequence, we may also assume that the convergence hold almost everywhere. So,  $u$  takes its values in  $[0, 1]$ , and we have  $u|_{\Omega^c} = 0$ . In particular, we also have  $u|_\Omega \in H_0^1(\Omega)$ . The condition  $\int u = m$  is verified as well by dominated convergence. Hence property (C) is verified.

*Proof of the lim inf inequality for  $\Gamma$ -convergence.* Here we assume that  $(u_T)_{T>0}$  is a sequence of  $X'$  that converges to a limit  $u \in X'$  with respect to the weak  $H^1(\mathbb{R}^n)$ -topology. We need to show that

$$\liminf_{T \rightarrow \infty} \mathcal{F}_{\varepsilon,T}(u_T) \geq \mathcal{F}_\varepsilon(u). \quad (2.2.6)$$

Because of the weak  $H^1(\mathbb{R}^n)$ -convergence, we have

$$\liminf_{T \rightarrow \infty} \int_{\mathbb{R}^n} |\nabla u_T|^2 \geq \int_{\mathbb{R}^n} |\nabla u|^2. \quad (2.2.7)$$

By Fatou's lemma, we also have

$$\liminf_{T \rightarrow \infty} \int_{\mathbb{R}^n} W(u_T) \geq \int_{\mathbb{R}^n} W(u) \quad \text{and} \quad \liminf_{T \rightarrow \infty} \int_{\mathbb{R}^n} u_T |x|^\beta \geq \int_{\mathbb{R}^n} u |x|^\beta. \quad (2.2.8)$$

We are left to show that

$$\liminf_{T \rightarrow \infty} \mathcal{V}_{\alpha, T}(u_T) \geq \mathcal{V}_{\alpha}(u). \quad (2.2.9)$$

Setting  $\delta_T = \frac{(2\pi)^n}{C(n, \alpha)} (\mathcal{V}_{\alpha, T}(u_T) - \mathcal{V}_{\alpha}(u))$ , (2.2.9) is equivalent to

$$\liminf_{T \rightarrow \infty} \delta_T \geq 0.$$

Recalling equation (2.2.3) and definition (2.2.4), we have

$$\begin{aligned} \delta_T &= \left(\frac{2\pi}{T}\right)^n \sum_{k \in \mathbb{Z}^n \setminus \{0\}} \left| \frac{2k\pi}{T} \right|^{-\alpha} |c_{k, T}(u_T)|^2 - \int |\xi|^{-\alpha} |\hat{u}(\xi)|^2 d\xi \\ &= \left(\frac{2\pi}{T}\right)^n \sum_{k \in \mathbb{Z}^n \setminus \{0\}} \left| \frac{2k\pi}{T} \right|^{-\alpha} \left| \hat{u}_T\left(\frac{2k\pi}{T}\right) \right|^2 - \int |\xi|^{-\alpha} |\hat{u}(\xi)|^2 d\xi \\ &= \left(\frac{2\pi}{T}\right)^n \sum_{\substack{k \in \mathbb{Z}^n \setminus \{0\} \\ \left| \frac{2k\pi}{T} \right| \leq R}} \left( \left| \frac{2k\pi}{T} \right|^{-\alpha} \left| \hat{u}_T\left(\frac{2k\pi}{T}\right) \right|^2 - \left| \frac{2k\pi}{T} \right|^{-\alpha} \left| \hat{u}\left(\frac{2k\pi}{T}\right) \right|^2 \right) \\ &\quad + \left(\frac{2\pi}{T}\right)^n \sum_{\substack{k \in \mathbb{Z}^n \setminus \{0\} \\ \left| \frac{2k\pi}{T} \right| > R}} \left| \frac{2k\pi}{T} \right|^{-\alpha} \left| \hat{u}_T\left(\frac{2k\pi}{T}\right) \right|^2 \\ &\quad + \left(\frac{2\pi}{T}\right)^n \sum_{\substack{k \in \mathbb{Z}^n \setminus \{0\} \\ \left| \frac{2k\pi}{T} \right| \leq R}} \left| \frac{2k\pi}{T} \right|^{-\alpha} \left| \hat{u}\left(\frac{2k\pi}{T}\right) \right|^2 \\ &\quad - \int_{|\xi| \leq R} |\xi|^{-\alpha} |\hat{u}(\xi)|^2 d\xi - \int_{|\xi| > R} |\xi|^{-\alpha} |\hat{u}(\xi)|^2 d\xi \\ &=: S_1(T, R) + S_2(T, R) + S_3(T, R) - I_1(R) - I_2(R), \end{aligned}$$

with obvious notations. We have

$$S_2 \geq 0, \quad (2.2.10)$$

and

$$I_2 \leq R^{-\alpha} \int |\hat{u}(\xi)|^2 d\xi = (2\pi)^n R^{-\alpha} \int u^2 \leq (2\pi)^n R^{-\alpha} \int u = (2\pi)^n R^{-\alpha} m, \quad (2.2.11)$$

where we used the isometry property of the Fourier transform and the fact that  $u$  takes its values in  $[0, 1]$ . From (2.2.10) and (2.2.11) we deduce

$$\liminf_{T \rightarrow \infty} \delta_T \geq \liminf_{T \rightarrow \infty} S_1(T, R) + \liminf_{T \rightarrow \infty} (S_3(T, R) - I_1(R)) - (2\pi)^n R^{-\alpha} m. \quad (2.2.12)$$

As the Fourier transform is 1-Lipschitz from  $L^1(\mathbb{R}^n)$  to  $L^\infty(\mathbb{R}^n)$ , and  $u_T \rightarrow u$  in  $L^1(\mathbb{R}^n)$ , we have

$$\forall \xi \in \mathbb{R}^n, \left| |\hat{u}_T(\xi)|^2 - |\hat{u}(\xi)|^2 \right| \leq \varepsilon(T),$$

with  $\varepsilon(T) \xrightarrow{T \rightarrow \infty} 0$ . Thus,

$$0 \leq S_1(T, R) \leq \left(\frac{2\pi}{T}\right)^n \sum_{\substack{k \in \mathbb{Z}^n \setminus \{0\} \\ \left|\frac{2k\pi}{T}\right| \leq R}} \left|\frac{2k\pi}{T}\right|^{-\alpha} \varepsilon(T). \quad (2.2.13)$$

We are left to show that

$$\limsup_{T \rightarrow \infty} \left(\frac{2\pi}{T}\right)^n \sum_{\substack{k \in \mathbb{Z}^n \setminus \{0\} \\ \left|\frac{2k\pi}{T}\right| \leq R}} \left|\frac{2k\pi}{T}\right|^{-\alpha} < \infty,$$

to conclude that  $S_1(T, R) \xrightarrow{T \rightarrow \infty} 0$ . But this comes from the fact that

$$\left(\frac{2\pi}{T}\right)^n \sum_{\substack{k \in \mathbb{Z}^n \setminus \{0\} \\ \left|\frac{2k\pi}{T}\right| \leq R}} \left|\frac{2k\pi}{T}\right|^{-\alpha} \xrightarrow{T \rightarrow \infty} \int_{|\xi| \leq R} |\xi|^{-\alpha} d\xi. \quad (2.2.14)$$

Indeed using the monotonicity of  $x \mapsto x^{-\alpha}$ , we have in dimension 1,

$$\frac{2\pi}{T} \sum_{0 < \frac{2k\pi}{T} \leq R} \left(\frac{2(k+1)\pi}{T}\right)^{-\alpha} \leq \int_{\frac{2\pi}{T} < x \leq R} x^{-\alpha} dx \leq \frac{2\pi}{T} \sum_{0 < \frac{2k\pi}{T} \leq R} \left(\frac{2k\pi}{T}\right)^{-\alpha},$$

From this we deduce convergence (2.2.14) in dimension 1, and the convergence in any dimension can be shown with the same idea. In conclusion, we have  $S_1(T, R) \xrightarrow{T \rightarrow \infty} 0$  and so (2.2.12) becomes

$$\liminf_{T \rightarrow \infty} \delta_T \geq \liminf_{T \rightarrow \infty} (S_3(T, R) - I_1(R)) - (2\pi)^n R^{-\alpha} m. \quad (2.2.15)$$

Given  $\eta > 0$  small, we have

$$\begin{aligned} |S_3(T, R) - I_1(R)| &\leq \\ &\left| \left(\frac{2\pi}{T}\right)^n \sum_{\substack{k \in \mathbb{Z}^n \setminus \{0\} \\ \eta \leq \left|\frac{2k\pi}{T}\right| \leq R}} \left|\frac{2k\pi}{T}\right|^{-\alpha} \left|\hat{u}\left(\frac{2k\pi}{T}\right)\right|^2 - \int_{\eta \leq |\xi| \leq R} |\xi|^{-\alpha} |\hat{u}(\xi)|^2 d\xi \right| \\ &\quad + \left| \left(\frac{2\pi}{T}\right)^n \sum_{\substack{k \in \mathbb{Z}^n \setminus \{0\} \\ 0 < \left|\frac{2k\pi}{T}\right| \leq \eta}} \left|\frac{2k\pi}{T}\right|^{-\alpha} \left|\hat{u}\left(\frac{2k\pi}{T}\right)\right|^2 \right| + \left| \int_{|\xi| \leq \eta} |\xi|^{-\alpha} |\hat{u}(\xi)|^2 d\xi \right|. \end{aligned} \quad (2.2.16)$$

In the right-hand side of this last inequality, the first term goes to 0 as  $T \rightarrow \infty$  by classical convergence of Riemann sums for continuous functions on a nice domain. For the second term,



we use the estimate  $\|\hat{u}\|_{L^\infty} \leq \|u\|_{L^1}$ , and convergence (2.2.14), to get

$$\begin{aligned} \limsup_{T \rightarrow \infty} |S_3(T, R) - I_1(R)| &\leq \left| \int_{|\xi| \leq \eta} |\xi|^{-\alpha} d\xi \right| \|u\|_{L^1}^2 \\ &\quad + \left| \int_{|\xi| \leq \eta} |\xi|^{-\alpha} |\hat{u}(\xi)|^2 d\xi \right| \|u\|_{L^1}^2, \end{aligned}$$

and so letting  $\eta$  go to 0,

$$\limsup_{T \rightarrow \infty} |S_3(T, R) - I_1(R)| = 0.$$

From this and (2.2.15) we deduce

$$\liminf_{T \rightarrow \infty} \delta_T \geq (2\pi)^n R^{-\alpha} m,$$

and so letting  $R$  go to  $\infty$  we get  $\liminf_{T \rightarrow \infty} \delta_T = 0$ , and so (2.2.9) is proved. Putting (2.2.7), (2.2.8) and (2.2.9) together concludes the proof of (2.2.6).

*Proof of the limsup inequality for  $\Gamma$ -convergence.* Let  $u \in X'$ . We need to find a sequence  $(u_T)_{T>0}$  of  $X'$  that converges to  $u$  weakly in  $H^1(\mathbb{R}^n)$ , such that

$$\limsup_{T \rightarrow \infty} \mathcal{F}_{\varepsilon, T}(u_T) \leq \mathcal{F}_\varepsilon(u). \quad (2.2.17)$$

We simply set  $u_T := u$ , for every  $T > 0$ . We only need to verify that  $\limsup_{T \rightarrow \infty} \mathcal{V}_{\alpha, T}(u_T) \leq \mathcal{V}_\alpha(u)$ . We proceed to the same computations as for the liminf inequality, using  $S_1, S_2, S_3, I_1$  and  $I_2$  as above. We deal with the terms  $S_3, I_1$  and  $I_2$  exactly as above. The term  $S_1$  is 0 here. For  $S_2$ , we use the isometry property of the Fourier transform to write

$$\begin{aligned} S_2(T, R) &= \left( \frac{2\pi}{T} \right)^n \sum_{\left| \frac{2k\pi}{T} \right| > R} \left| \frac{2k\pi}{T} \right|^{-\alpha} \left| \hat{u}_T \left( \frac{2k\pi}{T} \right) \right|^2 \\ &\leq \left( \frac{2\pi}{T} \right)^n \sum_{\left| \frac{2k\pi}{T} \right| > R} R^{-\alpha} \left| \hat{u}_T \left( \frac{2k\pi}{T} \right) \right|^2 \\ &\leq \frac{(2\pi)^n}{R^\alpha} \int_{(-\frac{T}{2}, \frac{T}{2})^n} u_T^2 \\ &= \frac{(2\pi)^n}{R^\alpha} \int_{\mathbb{R}^n} u^2 \quad \text{for } T \text{ large enough.} \end{aligned}$$

This is enough, with the previous estimates, to conclude that (2.2.17) holds. This concludes the proof.  $\square$

**Step three.** As the final step, we discretize the variational problem. In this step, we will use the functional space

$$X'' := \left\{ u \in H^1((-T/2, T/2)^n) : \int_{(-\frac{T}{2}, \frac{T}{2})^n} u = m \right\},$$

endowed with the weak  $H^1((-T/2, T/2)^n)$ -topology. Let us assume that  $T$  is large enough so that  $\Omega \subset\subset (-T/2, T/2)^n$ . If  $u \in X''$  takes its values in  $[0, 1]$  and verifies  $u \in H_0^1(\Omega)$  and  $u|_{\Omega^c} = 0$ , we may extend it by 0 outside of  $(-T/2, T/2)^n$ , to get a function  $\tilde{u} \in X'$ , and define  $\mathcal{F}_{\varepsilon, T}(u) := \mathcal{F}_{\varepsilon, T}(\tilde{u})$ . Otherwise, we set  $\mathcal{F}_{\varepsilon, T}(u) = +\infty$ . This way, the functional  $\mathcal{F}_{\varepsilon, T}$  is well defined on  $X''$ . For  $N \in 2\mathbb{N}^*$  large, we consider the space

$$E_N := \left\{ u \in \text{Vect}(e^{\frac{2i\pi}{T}k \cdot x})_{k \in \{-\frac{N}{2}+1, \dots, \frac{N}{2}\}^n} : \forall j \in \{-\frac{N}{2}+1, \dots, \frac{N}{2}\}^n, \quad u(j\frac{T}{N}) \in [0, 1], \right. \\ \left. u(j\frac{T}{N}) = 0 \quad \text{if} \quad j\frac{T}{N} \notin \Omega, \quad \text{and} \quad \int_{(-\frac{T}{2}, \frac{T}{2})^n} u = m \right\}.$$

For  $u \in E_N$ , we set

$$\mathcal{W}_N(u) := \left(\frac{T}{N}\right)^n \sum_{j \in \{-\frac{N}{2}+1, \dots, \frac{N}{2}\}^n} W(u(j\frac{T}{N})),$$

and

$$\mathcal{U}_{\beta, A, N}(u) := A \left(\frac{T}{N}\right)^n \sum_{j \in \{-\frac{N}{2}+1, \dots, \frac{N}{2}\}^n} u(j\frac{T}{N}) \left| j\frac{T}{N} \right|^\beta.$$

Then, we define the functional  $\mathcal{F}_{\varepsilon, T, N} : X'' \rightarrow \overline{\mathbb{R}}$  by

$$\mathcal{F}_{\varepsilon, T, N}(u) = \begin{cases} \sigma^{-1}\varepsilon \int_{(-\frac{T}{2}, \frac{T}{2})^n} |\nabla u|^2 + \sigma^{-1}\varepsilon^{-1}\mathcal{W}_N(u) + \mathcal{V}_{\alpha, T}(u) + \mathcal{U}_{\beta, A, N} & \text{if } u \in E_N, \\ +\infty & \text{otherwise.} \end{cases}$$

Then, we have the following proposition.

**Proposition 2.2.4.** *The following  $\Gamma$ -convergence holds on  $X''$ , endowed with the weak  $H^1((-T/2, T/2)^n)$ -topology:*

$$\mathcal{F}_{\varepsilon, T, N} \xrightarrow[N \rightarrow \infty]{\Gamma} \mathcal{F}_{\varepsilon, T}. \quad (2.2.18)$$

Moreover,  $(\mathcal{F}_{\varepsilon, T, N})_{N > 0}$  enjoys property (C).

In the proof of 2.2.4, we will use the following technical lemma, which shows that a trigonometric function whose frequencies are lower than  $N$  is well represented by its values on a grid with step size  $1/N$ .

**Lemma 2.2.5.** *Let  $(u_N)$  be a converging sequence in  $L^2([0, 1]^n)$ , such that for every  $N \in 2\mathbb{N}^*$ ,  $u_N \in \text{Vect}(e^{2i\pi k \cdot x})_{k \in \{-\frac{N}{2}+1, \dots, \frac{N}{2}\}^n}$ . Then, for any bounded uniformly continuous functions  $\phi : \mathbb{R} \rightarrow \mathbb{R}$  and  $\psi : [0, 1]^n \rightarrow \mathbb{R}$ , we have*

$$\left| \frac{1}{N^n} \sum_{j \in \frac{1}{N}\mathbb{Z}^n \cap [0, 1]^n} \psi(j)\phi(u_N(j)) - \int_{[0, 1]^n} \psi(x)\phi(u_N(x))dx \right| \xrightarrow[N \rightarrow \infty]{} 0.$$

Equivalently, for any  $T > 0$ , let  $(u_N)$  be a converging sequence in  $L^2([-T/2, T/2]^n)$ , such that for every  $N \in 2\mathbb{N}^*$ ,  $u_N \in \text{Vect}(e^{\frac{2i\pi}{T}k \cdot x})_{k \in \{-\frac{N}{2}+1, \dots, \frac{N}{2}\}^n}$ . Then, for any bounded uniformly

continuous functions  $\phi : \mathbb{R} \rightarrow \mathbb{R}$  and  $\psi : [-T/2, T/2]^n \rightarrow \mathbb{R}$ , we have

$$\left| \left( \frac{T}{N} \right)^n \sum_{j \in \{-\frac{N}{2}+1, \dots, \frac{N}{2}\}^n} \psi(j \frac{T}{N}) \phi(u_N(j \frac{T}{N})) - \int_{(-\frac{T}{2}, \frac{T}{2})^n} \psi(x) \phi(u_N(x)) dx \right| \xrightarrow{N \rightarrow \infty} 0.$$

*Proof.* We only need to prove the lemma in the case of functions on  $[0, 1]^n$ . For  $N \in 2\mathbb{N}^*$ ,  $k \in \{-\frac{N}{2} + 1, \dots, \frac{N}{2}\}^n$ , let us define the function  $e_k^N : [0, 1]^n \rightarrow \mathbb{R}$  by setting,

$$\forall j \in \frac{1}{N}\mathbb{Z}^n \cap [0, 1]^n, \quad \forall x \in j + [0, 1/N]^n, \quad e_k^N(x) = e^{2ik\pi \cdot j}.$$

We also define

$$\tilde{u}_N = \sum_{k \in \{-\frac{N}{2}+1, \dots, \frac{N}{2}\}^n} c_k(u_N) e_k^N,$$

where the  $(c_k(u_N))_{k \in \{-\frac{N}{2}+1, \dots, \frac{N}{2}\}^n}$  are the Fourier coefficients of  $u_N$ . Then, we can rewrite (2.2.19) as

$$\left| \int_{[0,1]^n} \phi(\tilde{u}_N) - \int_{[0,1]^n} \phi(u_N) \right| \xrightarrow{N \rightarrow \infty} 0. \quad (2.2.19)$$

Let  $(K_N)$  be a sequence of positive integers, diverging slowly enough (to be precised later) to  $+\infty$ . We define some functions  $v_N$  and  $\tilde{v}_N$  by

$$\begin{aligned} v_N &= \sum_{k \in \{-\frac{K_N}{2}+1, \dots, \frac{K_N}{2}\}^n} c_k(u_N) e_k, \\ \tilde{v}_N &= \sum_{k \in \{-\frac{K_N}{2}+1, \dots, \frac{K_N}{2}\}^n} c_k(u_N) e_k^N. \end{aligned}$$

As  $(u_N)$  converges in  $L^2([0, 1]^n)$ , we have

$$\|u_N - v_N\|_{L^2([0,1]^n)} \xrightarrow{N \rightarrow \infty} 0.$$

Also, noting that the family  $(e_k^N)_{k \in \{-\frac{N}{2}+1, \dots, \frac{N}{2}\}^n}$  is orthonormal in  $L^2([0, 1]^n)$ , we have

$$\|\tilde{u}_N - \tilde{v}_N\|_{L^2([0,1]^n)} \xrightarrow{N \rightarrow \infty} 0.$$

The function  $\phi$  being uniformly continuous and bounded, it is easy to show that (2.2.19) is equivalent to

$$\left| \int_{[0,1]^n} \phi(\tilde{v}_N) - \int_{[0,1]^n} \phi(v_N) \right| \xrightarrow{N \rightarrow \infty} 0,$$

which is in turn equivalent to

$$\left| \frac{1}{N^n} \sum_{j \in \frac{1}{N}\mathbb{Z}^n \cap [0,1]^n} \phi(v_N(j)) - \int_{[0,1]^n} \phi(v_N) \right| \xrightarrow{N \rightarrow \infty} 0. \quad (2.2.20)$$

Now, we have

$$\begin{aligned}
 \|\nabla v_N\|_{L^\infty} &\leq \sum_{k \in \{-\frac{K_N}{2}+1, \dots, \frac{K_N}{2}\}^n} |2\pi k| |c_k(u_N)| \\
 &\leq C(n)K_N \sum_{k \in \{-\frac{K_N}{2}+1, \dots, \frac{K_N}{2}\}^n} |c_k(u_N)| \\
 &\leq C(n)K_N (K_N)^{n/2} \left( \sum_{k \in \{-\frac{K_N}{2}+1, \dots, \frac{K_N}{2}\}^n} |c_k(u_N)|^2 \right)^{1/2} \\
 &= C(n)K_N^{n/2+1} \|v_N\|_{L^2([0,1]^n)}.
 \end{aligned} \tag{2.2.21}$$

where we have used the Hölder inequality in (2.2.21), and  $C(n)$  denotes a constant depending on  $n$  only. From this last estimate, we get

$$\begin{aligned}
 \forall j \in \frac{1}{N}\mathbb{Z}^n \cap [0,1]^n, \forall x \in j + [0,1/N]^n, \\
 |v_N(j) - v_N(x)| \leq C(n) \frac{K_N^{n/2+1}}{N} \|v_N\|_{L^2([0,1]^n)}.
 \end{aligned}$$

Denoting by  $\omega_\phi$  the modulus of continuity of  $\phi$ , this yields

$$\begin{aligned}
 &\left| \frac{1}{N^n} \sum_{j \in \frac{1}{N}\mathbb{Z}^n \cap [0,1]^n} \phi(v_N(j)) - \int_{[0,1]^n} \phi(v_N) \right| \\
 &\leq \sum_{j \in \frac{1}{N}\mathbb{Z}^n \cap [0,1]^n} \int_{j+[0,1/N]^n} |\phi(v_N(j)) - \phi(v_N)| \\
 &\leq \sum_{j \in \frac{1}{N}\mathbb{Z}^n \cap [0,1]^n} \int_{j+[0,1/N]^n} \omega_\phi \left( C(n) \frac{K_N^{n/2+1}}{N} \|v_N\|_{L^2([0,1]^n)} \right) \\
 &= \omega_\phi \left( C(n) \frac{K_N^{n/2+1}}{N} \|v_N\|_{L^2([0,1]^n)} \right).
 \end{aligned}$$

Choosing  $(K_N)$  such that  $K_N^{n/2+1}/N \xrightarrow{N \rightarrow \infty} 0$ , and recalling that the sequence  $(v_N)$  is bounded in  $L^2([0,1]^n)$  (because  $(u_N)$  is), the last inequality concludes the proof.  $\square$

*Proof of Proposition 2.2.4.*

*Proof of property (C).* Let  $(u_N)$  be a sequence of  $X''$  such that for any  $N$ ,  $u_N \in E_N$ , and  $(\mathcal{F}_{\varepsilon,T,N}(u_N))$  is bounded. Then,  $(u_N)$  is bounded in  $H^1([-T/2, T/2]^n)$  so, up to taking a subsequence, it converges weakly in  $H^1([-T/2, T/2]^n)$ , strongly in  $L^2([-T/2, T/2]^n)$ , such that  $\int u = m$ .

*Proof of the lim inf inequality.* Let  $(u_N)$  be a sequence of  $X''$  that converges to a function  $u \in X''$ , and such that for every  $N \in 2\mathbb{N}^*$ ,  $u_N \in E_N$ . We need to show that

$$\liminf_{N \rightarrow \infty} \mathcal{F}_{\varepsilon,T,N}(u_N) \geq \mathcal{F}_{\varepsilon,T}(u).$$

We may assume that the left hand-side is finite. Up to taking a subsequence, we may then assume that  $(\mathcal{F}_{\varepsilon, T, N}(u_N))_{N \in 2\mathbb{N}^*}$  is bounded. Because of property (C), up to taking another subsequence, we may assume that  $(u_N)$  converges strongly in  $L^2((-T/2, T/2)^n)$  and almost everywhere to  $u$ .

Let us show that  $u$  takes its values in the interval  $[0, 1]$ , and that  $u|_{\Omega^c} = 0$ .

To prove that  $u$  takes its values in  $[0, 1]$ , let us argue by contradiction and assume it is not the case. Then, there exists a continuous function  $\phi : \mathbb{R} \rightarrow \mathbb{R}_+$  that is compactly supported in the complement of  $[0, 1]$ , and such that

$$\int_{(-\frac{T}{2}, \frac{T}{2})^n} \phi(u) > 0.$$

Applying Lemma 2.2.5 with  $\psi = 1$ , we find

$$\int_{(-\frac{T}{2}, \frac{T}{2})^n} \phi(u_N) \xrightarrow{N \rightarrow +\infty} 0.$$

By the dominated convergence theorem, this can be rewritten as

$$\int_{(-\frac{T}{2}, \frac{T}{2})^n} \phi(u) = 0,$$

which gives a contradiction. So  $u$  takes its values in  $[0, 1]$ .

To prove that  $u|_{\Omega^c} = 0$ , we also argue by contradiction, and assume it is not the case. Then, there exists a continuous function  $\psi : [-T/2, T/2]^n \rightarrow \mathbb{R}$  that is supported in  $\Omega^c$ , and such that

$$\int_{(-\frac{T}{2}, \frac{T}{2})^n} \psi u > 0.$$

Applying Lemma 2.2.5 with  $\phi(x) = x$ , we find

$$\int_{(-\frac{T}{2}, \frac{T}{2})^n} \psi u_N \xrightarrow{N \rightarrow +\infty} 0.$$

By the dominated convergence theorem, this can be rewritten as

$$\int_{(-\frac{T}{2}, \frac{T}{2})^n} \psi u = 0,$$

which gives a contradiction. So  $u|_{\Omega^c} = 0$ .

By the weak  $H^1((-T/2, T/2)^n)$  convergence, we have

$$\liminf_{N \rightarrow \infty} \int_{(-\frac{T}{2}, \frac{T}{2})^n} |\nabla u_N|^2 \geq \int_{(-\frac{T}{2}, \frac{T}{2})^n} |\nabla u|^2.$$

By the strong  $L^2((-T/2, T/2)^n)$  convergence, we have

$$\lim_{N \rightarrow \infty} \mathcal{V}_{\alpha, T}(u_N) = \mathcal{V}_{\alpha, T}(u).$$

Thus, the only problematic terms are  $\mathcal{W}_N$  and  $\mathcal{U}_{\beta,A,N}$ . Let us apply Lemma 2.2.5 to  $(u_N)$ , with  $\psi = 1$  and  $\phi = \widetilde{W}$ , where  $\widetilde{W}$  is any uniformly continuous bounded extension of  $W$ . We get

$$\left| \mathcal{W}_N(u_N) - \int_{(-\frac{T}{2}, \frac{T}{2})^n} \widetilde{W}(u_N) \right| \xrightarrow{N \rightarrow +\infty} 0. \quad (2.2.22)$$

By the dominated convergence theorem, we have

$$\lim_{N \rightarrow +\infty} \int_{(-\frac{T}{2}, \frac{T}{2})^n} \widetilde{W}(u_N) = \int_{(-\frac{T}{2}, \frac{T}{2})^n} W(u).$$

With (2.2.22), this implies

$$\lim_{N \rightarrow +\infty} \mathcal{W}_N(u_N) = \int_{(-\frac{T}{2}, \frac{T}{2})^n} W(u).$$

Using the same strategy, we may also show that

$$\lim_{N \rightarrow +\infty} \mathcal{U}_{\beta,A,N}(u_N) = A \int_{(-\frac{T}{2}, \frac{T}{2})^n} u(x) |x|^\beta dx.$$

*Proof of the limsup inequality.* Let  $u \in X''$ . We need to find a recovery sequence  $(u_N)$  such that for any  $N \in 2\mathbb{N}^*$ ,  $u_N \in E_N$ , and

$$\limsup_{N \rightarrow \infty} \mathcal{F}_{\varepsilon,T,N}(u_N) \leq \mathcal{F}_{\varepsilon,T}(u)$$

The domain of finiteness of  $\mathcal{F}_{\varepsilon,T}$  is

$$\mathcal{D}_{\mathcal{F}_{\varepsilon,T}} = \{u \in X'' : u \in H_0^1(\Omega, [0, 1]) \text{ and } u|_{\Omega^c} = 0\}$$

The set  $C^\infty \cap \mathcal{D}_{\mathcal{F}_{\varepsilon,T}}$  of smooth functions of  $\mathcal{D}_{\mathcal{F}_{\varepsilon,T}}$  is a dense subset of  $\mathcal{D}_{\mathcal{F}_{\varepsilon,T}}$  for the strong  $H^1((-\frac{T}{2}, \frac{T}{2})^n)$ -topology. Moreover, the functional  $\mathcal{F}_{\varepsilon,T}$  is continuous over  $\mathcal{D}_{\mathcal{F}_{\varepsilon,T}}$  for the strong  $H^1((-\frac{T}{2}, \frac{T}{2})^n)$ -topology. Therefore, we may assume that  $u \in C^\infty \cap \mathcal{D}_{\mathcal{F}_{\varepsilon,T}}$ . Let  $(c_k)_{k \in \mathbb{Z}^n}$  be the Fourier coefficients of  $u$ , so that

$$u = \sum_{k \in \mathbb{Z}^n} c_k e_k,$$

where  $e_k(x) = \exp(\frac{2i\pi}{T} k \cdot x)$ . For  $N \in 2\mathbb{N}^*$ , we define

$$v_N = \sum_{k \in \{-\frac{N}{2}+1, \dots, \frac{N}{2}\}^n} c_k e_k.$$

The sequence  $(v_N)$  cannot be used directly as a recovery sequence because it does not verify

$$\forall j \in \{-\frac{N}{2} + 1, \dots, \frac{N}{2}\}^n, \quad v_N(j \frac{T}{N}) \in [0, 1], \quad \text{and} \quad v_N(j \frac{T}{N}) = 0 \quad \text{if} \quad j \frac{T}{N} \notin \Omega. \quad (2.2.23)$$

However, because  $u$  is smooth, by classical uniform estimates for the Fourier serie of a  $C^{n+4}$  function, there exists a constant  $C > 0$  such that for any  $N \in 2\mathbb{N}^*$ ,

$$|u - v_N|_{L^\infty} \leq \frac{C}{N^{n+3}}. \quad (2.2.24)$$

We recall that the discrete Fourier transform is a bijection: for any two families of complex numbers  $(w_j)_{j \in \{-\frac{N}{2}+1, \dots, \frac{N}{2}\}^n}$  and  $(d_k)_{k \in \{-\frac{N}{2}+1, \dots, \frac{N}{2}\}^n}$ , we have

$$\begin{aligned} & \left( \forall k \in \left\{-\frac{N}{2} + 1, \dots, \frac{N}{2}\right\}^n, \quad d_k = \frac{1}{N^n} \sum_{j \in \left\{-\frac{N}{2} + 1, \dots, \frac{N}{2}\right\}^n} w_j e_k\left(-\frac{jT}{N}\right) \right) \\ & \iff \left( \forall j \in \left\{-\frac{N}{2} + 1, \dots, \frac{N}{2}\right\}^n, \quad w_j = \sum_{k \in \left\{-\frac{N}{2} + 1, \dots, \frac{N}{2}\right\}^n} d_k e_k\left(\frac{jT}{N}\right) \right) \end{aligned} \quad (2.2.25)$$

Let  $p : \mathbb{R} \rightarrow [0, 1]$  be the projection onto  $[0, 1]$ . Let us set

$$\forall j \in \left\{-\frac{N}{2} + 1, \dots, \frac{N}{2}\right\}^n, \quad w_j = \begin{cases} p(v_N(\frac{jT}{N})) & \text{if } \frac{jT}{N} \in \Omega, \\ 0 & \text{if } \frac{jT}{N} \notin \Omega. \end{cases}$$

Note that, because of (2.2.24), we have the estimate

$$\left| w_j - v_N\left(\frac{jT}{N}\right) \right| \leq \frac{C}{N^{n+3}}. \quad (2.2.26)$$

Then we define

$$\begin{aligned} c_k^N &:= \frac{1}{N^n} \sum_{j \in \left\{-\frac{N}{2} + 1, \dots, \frac{N}{2}\right\}^n} w_j e_k\left(-\frac{jT}{N}\right), \\ w_N &:= \sum_{k \in \left\{-\frac{N}{2} + 1, \dots, \frac{N}{2}\right\}^n} c_k^N e_k. \end{aligned}$$

Then, by the bijectivity of the discrete Fourier transform, we have  $w_N(\frac{jT}{N}) = w_j$ . Now,  $w_N$  verifies the constraints (2.2.23), but it may not verify  $\int w_N = m$ . However, we have

$$\begin{aligned} \left| \int w_N - m \right| &= T^n |c_0^N - c_0| \\ &= T^n \left| \frac{1}{N^n} \sum_{j \in \left\{-\frac{N}{2} + 1, \dots, \frac{N}{2}\right\}^n} \left( w_N\left(\frac{jT}{N}\right) - v_N\left(\frac{jT}{N}\right) \right) \right| \\ &\leq \frac{T^n}{N^n} \sum_{j \in \left\{-\frac{N}{2} + 1, \dots, \frac{N}{2}\right\}^n} \left| w_N\left(\frac{jT}{N}\right) - v_N\left(\frac{jT}{N}\right) \right| \\ &= \frac{T^n}{N^n} \sum_{j \in \left\{-\frac{N}{2} + 1, \dots, \frac{N}{2}\right\}^n} \left| w_j - v_N\left(\frac{jT}{N}\right) \right| \\ &\leq \frac{C}{N^{n+3}} \quad \text{because of (2.2.26),} \end{aligned} \quad (2.2.27)$$

We will modify the values  $w_N(\frac{jT}{N})$  on a small subset of  $\Omega$ , so that the new values still verify (2.2.23), but the mean of the new function is  $m$ . Recall that  $u$  is continuous, null outside  $\Omega$ ,

and not identically null. In particular, there exists  $\eta > 0$  and  $x \in \Omega$  such that  $\eta \leq u \leq 1 - \eta$  on  $B(x, \eta)$ . Because of (2.2.24) and (2.2.26), we may also require, up to taking  $\eta$  smaller, that for any  $N$  large enough,

$$\forall j \in \left\{-\frac{N}{2} + 1, \dots, \frac{N}{2}\right\}^n, \quad \frac{jT}{N} \in B(x, \eta) \implies \eta \leq w_N\left(\frac{jT}{N}\right) \leq 1 - \eta. \quad (2.2.28)$$

Let  $K$  be the cardinal of the set

$$\left\{j \in \left\{-\frac{N}{2} + 1, \dots, \frac{N}{2}\right\}^n : \frac{jT}{N} \in B(x, \eta)\right\}.$$

For  $N$  large enough, there exists  $j \in \left\{-\frac{N}{2} + 1, \dots, \frac{N}{2}\right\}^n$  such that  $\frac{jT}{N} \in B(x, \eta)$ , so that  $K \neq 0$ . So we can set

$$\delta_j^N = \begin{cases} \frac{N^n}{T^n K} (m - \int w_N) & \text{if } \frac{jT}{N} \in B(x, \eta), \\ 0 & \text{otherwise.} \end{cases}$$

Note that, because of (2.2.27) and (2.2.28), for some constant  $C > 0$ , we have for any  $N$  large enough,

$$\forall j \in \left\{-\frac{N}{2} + 1, \dots, \frac{N}{2}\right\}^n, \quad |\delta_j^N| \leq \frac{C}{N^{n+3}}. \quad (2.2.29)$$

Finally, let us define

$$\begin{aligned} \tilde{c}_k^N &:= \frac{1}{N^n} \sum_{j \in \left\{-\frac{N}{2} + 1, \dots, \frac{N}{2}\right\}^n} (w_N\left(\frac{jT}{N}\right) + \delta_j^N) e_k\left(-\frac{jT}{N}\right), \\ u_N &:= \sum_{k \in \left\{-\frac{N}{2} + 1, \dots, \frac{N}{2}\right\}^n} \tilde{c}_k^N e_k. \end{aligned}$$

Then, by the bijectivity of the discrete Fourier transform, we have

$$\forall j \in \left\{-\frac{N}{2} + 1, \dots, \frac{N}{2}\right\}^n, \quad u_N\left(\frac{jT}{N}\right) = w_N\left(\frac{jT}{N}\right) + \delta_j^N.$$

and so by construction  $u_N$  verifies the constraints (2.2.23). Moreover, we have

$$\begin{aligned} \int u_N &= T^n \tilde{c}_0^N \\ &= \frac{T^n}{N^n} \sum_{j \in \left\{-\frac{N}{2} + 1, \dots, \frac{N}{2}\right\}^n} (w_N\left(\frac{jT}{N}\right) + \delta_j^N) \\ &= T^n \tilde{c}_0^N + \frac{T^n}{N^n} \sum_{j \in \left\{-\frac{N}{2} + 1, \dots, \frac{N}{2}\right\}^n} \delta_j^N \\ &= \int w_N + \left(m - \int w_N\right) \\ &= m. \end{aligned}$$



Thus we have proved that  $u_N \in E_N$ . By construction, for any  $j \in \{-\frac{N}{2} + 1, \dots, \frac{N}{2}\}^n$ , we have

$$\begin{aligned} \left| u_N\left(\frac{jT}{N}\right) - u\left(\frac{jT}{N}\right) \right| &= \left| w_N\left(\frac{jT}{N}\right) + \delta_j^N - u\left(\frac{jT}{N}\right) \right| \\ &= \left| w_j + \delta_j^N - u\left(\frac{jT}{N}\right) \right| \\ &\leq \left| w_j - v_N\left(\frac{jT}{N}\right) \right| + \left| v_N\left(\frac{jT}{N}\right) - u\left(\frac{jT}{N}\right) \right| + |\delta_j^N| \\ &\leq \frac{C}{N^{n+3}} \quad \text{because of (2.2.26) (2.2.24) and (2.2.29),} \end{aligned}$$

for some constant  $C > 0$  independent of  $N$ . This implies

$$\forall k \in \left\{-\frac{N}{2} + 1, \dots, \frac{N}{2}\right\}^n, \quad |\tilde{c}_k^N - c_k| \leq \frac{C}{N^{n+3}}.$$

From this estimate, we deduce

$$\begin{aligned} |u - u_N|_{H^1((-T/2, T/2)^n)} &\leq C \sum_{k \in \mathbb{Z}^n} |k|^2 |\tilde{c}_k^N - c_k|^2 \\ &\leq C \sum_{k \in \{-\frac{N}{2} + 1, \dots, \frac{N}{2}\}^n} |k|^2 \frac{C}{N^{n+3}} + \sum_{k \notin \{-\frac{N}{2} + 1, \dots, \frac{N}{2}\}^n} |k|^2 |c_k|^2 \\ &\leq \frac{C}{N} + \sum_{k \notin \{-\frac{N}{2} + 1, \dots, \frac{N}{2}\}^n} |k|^2 |c_k|^2 \\ &\xrightarrow{N \rightarrow +\infty} 0. \end{aligned}$$

So  $(u_N)$  converges strongly to  $u$  in  $H^1((-T/2, T/2)^n)$ . We can now conclude, using Lemma 2.2.5 as in the proof of the lim inf inequality, that

$$\lim_{N \rightarrow +\infty} \mathcal{F}_{\varepsilon, T, N}(u_N) = \mathcal{F}_{\varepsilon, T}(u).$$

This concludes the proof. □

## 2.3 Numerical results

In the physical case, *i.e.* in dimension  $n = 3$  and for  $\alpha = 2$ , R. Choksi and M. Peletier conjectured the following (see [24, Conjecture 6.1]):

**Conjecture 2.3.1.** *For  $n = 3$  and  $\alpha = 2$ , as long as there is a minimizer in Gamow's liquid drop problem (1.1.2), it is a ball. Also, when there is no minimizer, the infimum of the energy is attained by a finite number of balls of the same volume, infinitely far away from each other.*

As far as it is known, this conjecture may hold for any  $n$  and  $\alpha$ . In any dimension  $n \geq 2$ , for  $\alpha$  close enough to  $n$ , M. Bonacini and R. Cristoferi proved that as long as a minimizer exists for problem (1.1.2), it is a ball. Moreover, when there is no minimizer, they showed that

the infimum of the energy is attained by a finite number of balls (of possibly different volumes) infinitely far away from each other, see [10, Theorem 2.12]. Our numerical results suggest that in dimension 2, the conjecture holds for any  $\alpha \in (0, 2)$  (*i.e.* the whole admissible range). Note that if the conjecture holds, we can compute explicitly the mass  $m_1(n, \alpha) > 0$  such that there is a minimizer in (1.1.2) if and only if  $m < m_1$ . Indeed, given  $m > 0$ , let us set

$$f(m) = P(B[m]) + \mathcal{V}_\alpha(B[m]).$$

Then, define  $m_k$  as the unique solution of

$$kf\left(\frac{m}{k}\right) = (k+1)f\left(\frac{m}{k+1}\right).$$

Note that  $kf\left(\frac{m}{k}\right)$  is the energy of  $k$  balls of volume  $m/k$ , infinitely far away from each other. Using the homogeneity of  $P$  and  $\mathcal{V}_\alpha$  we find that

$$m_k = |B| \left( \frac{(k+1)^{\frac{1}{n}} - k^{\frac{1}{n}}}{(k)^{-\frac{\alpha}{n}} - (k+1)^{-\frac{\alpha}{n}}} \frac{P(B)}{\mathcal{V}_\alpha(B)} \right)^{\frac{n}{1+\alpha}}. \quad (2.3.1)$$

We also set  $m_0 = 0$ . The sequence  $(m_k)$  is increasing. Then, an equivalent formulation of conjecture 2.3.1, generalized to any dimension  $n$  and parameter  $\alpha \in (0, n)$ , is:

**Conjecture 2.3.2.** *If  $m \in [0, m_1]$ , the ball of volume  $m$  is the unique minimizer of (1.1.2). For  $m \in (m_1, \infty)$ , there is no minimizer in (1.1.2). Moreover, for any  $k \in \mathbb{N}^*$  and  $m \in [m_{k-1}, m_k]$ , we have*

$$\inf_{E \subset \mathbb{R}^n, |E|=m} P(E) + \mathcal{V}_\alpha(E) = kf\left(\frac{m}{k}\right).$$

*In particular, as long as there is a minimizer in (1.1.2), it is a ball. When there is no minimizer, the infimum of the energy is attained by a finite number of balls of the same volume, infinitely far away from each other.*

To get minimizers of (1.1.2) for different volume constraint, we set the volume constraint to 1 and add a constant  $c_m$  to the term  $\mathcal{V}_\alpha$ . Indeed, minimizing

$$\inf_{E \subset \mathbb{R}^n, |E|=1} P(E) + c_m \mathcal{V}_\alpha(E)$$

is equivalent to minimizing (1.1.2) provided

$$c_m = m^{\frac{1+\alpha}{n}}. \quad (2.3.2)$$

The choice of  $T$  is made so that, if  $\mathbb{1}_{B[1]}^N$  is the discretization of the ball of volume 1 with side step  $\frac{T}{N}$ , we have

$$\frac{\mathcal{V}_{\alpha, T}(\mathbb{1}_{B[1]}^N) - \mathcal{V}_\alpha(B[1])}{\mathcal{V}_\alpha(B[1])} \leq 1\%.$$

Meanwhile, given the number of discretization points  $N = 2^{11}$ , we cannot increase  $T$  too much, otherwise the discretization of candidate minimizers is less and less precise.

For instance, for  $\alpha = 1$  and  $n = 2$ , we have

- for  $T = 5\pi$ :  $\frac{\mathcal{V}_{\alpha,T}(\mathbb{1}_B^N) - \mathcal{V}_{\alpha}(B[1])}{\mathcal{V}_{\alpha}(B[1])} \simeq 0.08$ ,
- for  $T = 10\pi$ :  $\frac{\mathcal{V}_{\alpha,T}(\mathbb{1}_B^N) - \mathcal{V}_{\alpha}(B[1])}{\mathcal{V}_{\alpha}(B[1])} \simeq 0.04$ ,
- for  $T = 20\pi$ :  $\frac{\mathcal{V}_{\alpha,T}(\mathbb{1}_B^N) - \mathcal{V}_{\alpha}(B[1])}{\mathcal{V}_{\alpha}(B[1])} \simeq 0.01$ .

These numerical estimates lead us to chose  $T = 20\pi$ . See Appendix 2.A for the method used to compute  $\mathcal{V}_{\alpha}(B[1])$ .

We display the results obtained for  $\alpha = 1$ , and  $c_m = 1.5, 1.6$  in Figure 2.1.

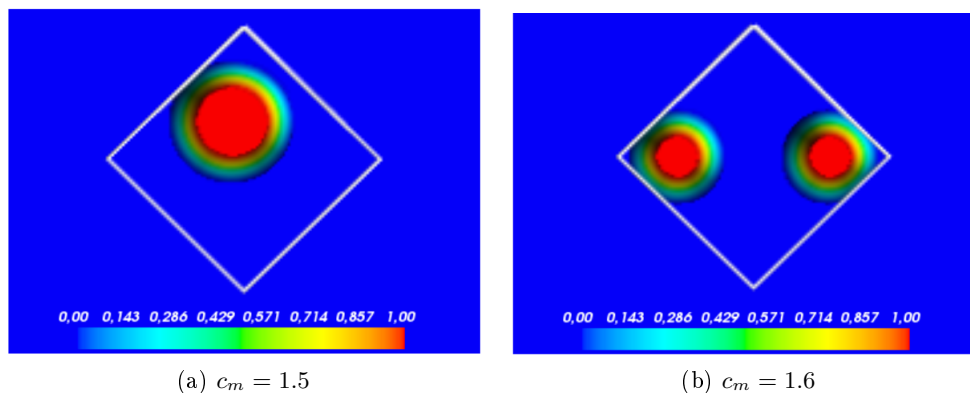


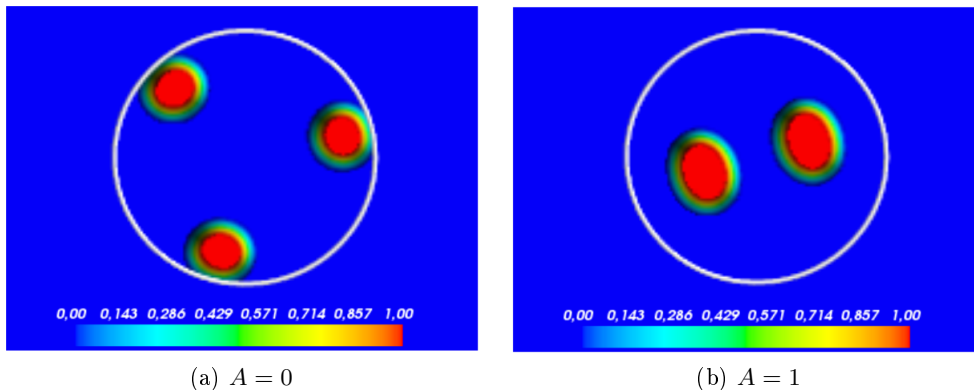
Figure 2.1:  $\alpha = 1$ ,  $A = 0$ ,  $\Omega$  is a square.

Here the box  $\Omega$  in which all functions are supported (see Section 2.2) has been chosen to be a square of diagonal length  $\pi$  (and is represented by white lines). We emphasize that this box is needed to get the right minimizers, both theoretically and numerically. Theoretically, the condition that functions are supported in a fixed bounded set is needed for the compactness property (C) (again see Section 2.2) to be satisfied, both in *step one* and in *step two*, as we let the size of the domain  $T$  go to infinity. Numerically, without this box, for  $c_m = 1.5$ , simulations yield two balls (instead of one as shown on Figure 2.1a) that get further and further away from each other as  $T$  increases. But this configuration *does not* converge to an admissible candidate, so it definitely does not converge to a minimizer.

We observe that for  $c_m = 1.6$ , we get two balls in opposite corners of the square  $\Omega$ : it is consistent with the expected repulsive behaviour of the non-local term  $\mathcal{V}_{\alpha}$ . Moreover, using (2.3.1) and (2.3.2), we find that, if conjecture 2.3.2 is true, there must be a minimizer up to  $c_m \simeq 1.67$ . Numerically, we find that there is a minimizer up to a constant  $c_m \in (1.5, 1.6)$ , which is relatively close to 1.67. We also observe similar results for different values of  $\alpha$ , including in the near field-dominated regime  $\alpha < 1$ .

For  $\Omega$  a disk of diameter  $\pi$ , if one increases further  $c_m$ , we get three balls located near the boundary of  $\Omega$ , as shown in Figure 2.2a for  $c_m = 3.0$ . This is consistent with the conjecture that the energy is minimized by balls of the same volume. To illustrate the effect of the confining potential, we display in Figure 2.2b the minimizer for  $c_m = 3.0$ ,  $A = 1$  and  $\beta = 16$ .

Finally, let us mention that the number of discretization points is  $N = 2^{11}$  in each direction. Numerical minimization is made using the solver IPOPT [83]. The computation time on a standard computer is about an hour.

Figure 2.2:  $\alpha = 1$ ,  $c_m = 3.0$ ,  $\beta = 16$ ,  $\Omega$  is a disk.

## 2.A Appendix: computation of $\mathcal{V}_\alpha(B[1])$

Here we explain how we compute  $\mathcal{V}_\alpha(B[1])$  numerically, as needed in Section 2.3 to choose the value of  $T$ . In order to compute numerically the improper integral

$$\mathcal{V}_\alpha(B[1]) = \int_{B[1] \times B[1]} \frac{dx dy}{|x - y|^{2-\alpha}},$$

we add a small term  $\varepsilon > 0$  to the denominator of the integrand. So we compute

$$\mathcal{V}_{\alpha,\varepsilon}(B[1]) = \int_{B[1] \times B[1]} \frac{dx dy}{|x - y|^{2-\alpha} + \varepsilon}.$$

To control the error introduced by the parameter  $\varepsilon$ , we need to estimate the difference  $\Delta_\varepsilon := \mathcal{V}_\alpha(B[1]) - \mathcal{V}_{\alpha,\varepsilon}(B[1])$ . We have

$$\begin{aligned} \Delta_\varepsilon &= \int_{B[1] \times B[1]} \frac{dx dy}{|x - y|^{2-\alpha}} - \int_{B[1] \times B[1]} \frac{dx dy}{|x - y|^{2-\alpha} + \varepsilon} \\ &= \int_{B[1] \times B[1]} \frac{\varepsilon dx dy}{|x - y|^{2-\alpha} (|x - y|^{2-\alpha} + \varepsilon)}. \end{aligned}$$

So

$$\begin{aligned}
 \Delta_\varepsilon &\leq \int_{B[1] \times B[1]} \mathbb{1}_{|x-y| < r} \frac{dx dy}{|x-y|^{2-\alpha}} + \int_{B[1] \times B[1]} \mathbb{1}_{|x-y| \geq r} \frac{\varepsilon dx dy}{|x-y|^{2-\alpha} r^{2-\alpha}} \\
 &\leq \int_{B[1] \times \mathbb{R}^2} \mathbb{1}_{|x-y| < r} \frac{dx dy}{|x-y|^{2-\alpha}} + \frac{\varepsilon}{r^{2-\alpha}} \int_{B[1] \times B[1]} \frac{dx dy}{|x-y|^{2-\alpha}} \\
 &\leq \int_{B[1] \times \mathbb{R}^2} \mathbb{1}_{|y| < r} \frac{dx dy}{|y|^{2-\alpha}} + \frac{\varepsilon}{r^{2-\alpha}} \int_{B[1]} \int_{B[1]} \frac{dx dy}{|y|^{2-\alpha}} \\
 &= \int_{|y| < r} \frac{dy}{|y|^{2-\alpha}} + \frac{\varepsilon}{r^{2-\alpha}} \int_{B[1]} \frac{dy}{|y|^{2-\alpha}} \\
 &= 2\pi \int_0^r \frac{\rho^{2-1} d\rho}{\rho^{2-\alpha}} + \frac{\varepsilon}{\rho^{2-\alpha}} 2\pi \int_0^{\frac{1}{\sqrt{\pi}}} \frac{\rho^{2-1} d\rho}{\rho^{2-\alpha}} \\
 &= \frac{2\pi}{\alpha} \left( r^\alpha + \frac{\varepsilon}{r^{2-\alpha} \pi^{\frac{\alpha}{2}}} \right),
 \end{aligned}$$

for some  $r > 0$ . This last bound attains its minimum for  $r = \left( \frac{(2-\alpha)\varepsilon}{\alpha \pi^{\frac{\alpha}{2}}} \right)^{\frac{1}{2}}$ . From there we deduce

$$\Delta_\varepsilon \leq \frac{2\pi}{\alpha} \frac{2}{2-\alpha} \left( \frac{2-\alpha}{\alpha} \right)^{\frac{\alpha}{2}} \left( \frac{\varepsilon}{\pi^{\frac{\alpha}{2}}} \right)^{\frac{\alpha}{2}}.$$

With  $\alpha = 1$ , we get

$$\Delta_\varepsilon \leq 4\pi^{\frac{3}{4}} \sqrt{\varepsilon}.$$

Now the proper integral  $\mathcal{V}_{\alpha,\varepsilon}(B[1])$  can be expressed in polar coordinates, and computed with arbitrary precision in the Julia language, using the *HCubature* package.

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