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THÈSE

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DOCTORAT DE L'UNIVERSITÉ DE TOULOUSE
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Présentée et soutenue par
Van Tu LE

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Dynamique des endomorphismes post-critiquement algébriques

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Résumé

Dans cette thèse, j'étudie la dynamique des endomorphismes de l'espace projectif complexe. Je m'intéresse aux endomorphismes post-critiquement algébriques, une notion qui généralise celle de fractions rationnelles post-critiquement finies en dimension 1. En particulier, j'étudie les valeurs propres d'un endomorphisme post-critiquement algébrique le long de l'orbite d'un point périodique. En dimension 1, un résultat bien connu, qui remonte aux travaux de Pierre Fatou, dit que ces valeurs sont soit nulles soit de module strictement plus supérieurs à 1. Dans cette thèse, j'étudie une conjecture qui généralise ce résultat en dimension au moins 2.

Dans la première partie de cette thèse, j'étudie une famille des endomorphismes post-critiquement algébriques introduite dans la thèse de Sarah Koch. En utilisant la caractérisation topologique des fractions rationnelles de William Thurston, sous certaines conditions, Sarah Koch a associé à une fraction rationnelle post-critiquement finie g un endomorphisme post-critiquement algébrique f . Lorsque g est un polynôme quadratique, je donne une caractérisation détaillée des valeurs propres de l'endomorphisme associé f en ses points fixes. En particulier, je montre que celles-ci sont soit nulles soit de modules strictement supérieurs à 1. Ce résultat suggère la validité de la conjecture.

Dans la deuxième partie, je montre que la conjecture est vraie dans le cas de dimension 2 sans hypothèse supplémentaire et en toute dimension lorsque les points périodiques sont en dehors de l'ensemble post-critique et sans autre hypothèse.

Mots clés: dynamique holomorphe, post-critiquement finies, post-critiquement algébriques, valeurs propres, points périodiques.

Abstract

In this thesis, I study the dynamics of endomorphisms of the complex projective space. I am interested in post-critically algebraic endomorphisms, a notion which generalizes that of post-critically finite rational maps in dimension 1. In particular, I study the eigenvalues of a post-critically algebraic endomorphism along the orbit of a periodic point. In dimension 1, a well-known result, which is due to Pierre Fatou, states that these values are either zero or of modules strictly greater than 1. In this thesis, I study a conjecture which generalizes this result in dimension at least 2.

In the first part of this thesis, I study a family of post-critically algebraic endomorphisms introduced in Sarah Koch's thesis. Using the topological characterization of rational maps of William Thurston, under certain conditions, Sarah Koch associated with a post-critically finite rational map g a post-critically algebraic endomorphism f . When g is a quadratic polynomial, I give a detailed characterization of the eigenvalues of the endomorphism f at its fixed points. In particular, I show that these values are either zero or of modules strictly greater than 1. This result provides evidence of the validity of the conjecture.

In the second part, I show that the conjecture is true in the case of dimension 2 without additional hypotheses and in any dimension when the periodic points are outside the post-critical set and without other hypotheses.

Keywords. holomorphic dynamics, post-critically finite, post-critically algebraic, eigenvalues, periodic points.

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Introduction

In this thesis, we discuss the discrete dynamics of holomorphic endomorphisms of $\mathbb{C}\mathbb{P}^n$. Without any further indication, every endomorphism considered in this thesis is holomorphic. Given an endomorphism $f : \mathbb{C}\mathbb{P}^n \rightarrow \mathbb{C}\mathbb{P}^n$, one of the main dynamical questions is to understand the asymptotic behavior of the sequence

$$z_0, z_1 = f(z_0), \dots, z_n = f^{\circ n}(z_0), \dots$$

as n tends to ∞ and as z_0 varies in $\mathbb{C}\mathbb{P}^n$. The theory has been studied since the beginning of the XIX century (see [Ale94]) in the case of one complex variable, and the case of several variables has thrived with a lot of beautiful results in the late 40 years.

Let $f : \mathbb{C}\mathbb{P}^n \rightarrow \mathbb{C}\mathbb{P}^n$ be an endomorphism of degree $d \geq 2$. The critical locus C_f of f is the set of points z such that the derivative $D_z f : T_z \mathbb{C}\mathbb{P}^n \rightarrow T_{f(z)} \mathbb{C}\mathbb{P}^n$ is not surjective. The endomorphism f is called *post-critically algebraic* if the post-critical set of f , that is

$$PC(f) = \bigcup_{j \geq 1} f^{\circ j}(C_f),$$

is an algebraic set of codimension one. If $n = 1$, such endomorphisms are called *post-critically finite*, since proper algebraic sets in $\mathbb{C}\mathbb{P}^1$ are finite sets.

A point $z \in \mathbb{C}\mathbb{P}^n$ is called a *periodic point* of f if there exists an integer $p \geq 1$ such that $f^{\circ p}(z) = z$. The smallest integer p satisfying such a condition is called the *period* of z and the orbit of z is called a *periodic cycle of period p of f* . A periodic point z is called a *fixed point* if its period is 1. An *eigenvalue of f along the cycle of a periodic point z of period p* is an eigenvalue of $D_z f^{\circ p}$. When $n = 1$, such a value is called the *multiplier of f along the cycle of z* .

In what follows, we focus on the notion of post-critically algebraic endomorphisms and study the most elementary yet intriguing dynamical object: their periodic points. Our goal is to prove that, for such endomorphisms, eigenvalues along a periodic cycle are either 0 or of modulus strictly bigger than 1. This has been done in the unidimensional

case and the higher dimensional case has been received a lot of attention lately. In this thesis, we are able to close the case when $n = 2$ and we obtain several partial results in higher dimension. Let us get into details.

When $n = 1$, the multipliers of a rational map along its periodic cycles and the dynamics of its critical points are closely related. We have the following fundamental result, which is essentially due to Fatou.

Theorem 1. Let $f : \mathbb{C}\mathbb{P}^1 \rightarrow \mathbb{C}\mathbb{P}^1$ be a post-critically finite rational map and λ be the multiplier of f along a periodic cycle. Then either $\lambda = 0$ or $|\lambda| > 1$.

This thesis seeks for a generalization of this theorem in higher dimension. More precisely, we propose the following conjecture.

Conjecture A. Let f be a post-critically algebraic endomorphism of $\mathbb{C}\mathbb{P}^n$ with $n \geq 2$ and λ be an eigenvalue of f along a periodic cycle. Then either $\lambda = 0$ or $|\lambda| > 1$.

When $n = 1$, the proof of Theorem 1 relies on the study of the relation between critical orbits and periodic cycles of endomorphisms of $\mathbb{C}\mathbb{P}^1$. When $n \geq 2$, most of the tools used in the unidimensional proof are no longer available. Some weaker form of Conjecture A has been studied by several authors. For example, it was studied in

- [FS94, Theorem 6.1] under the assumption that the complement of $PC(f)$ is Kobayashi hyperbolic and hyperbolically embedded,
- [Jon98, Proposition 2.9] in the case of dimension $n = 2$ and under the assumption that C_f has no periodic component,
- [Ast20, Theorem B] under a mild geometric assumption on the irreducible components of $PC(f)$.

However, no complete result has been obtained. This thesis, with three chapters, is devoted to the study of Conjecture A in higher dimension.

In Chapter 1

After a brief review of some important notions, we study several examples of post-critically algebraic endomorphisms in dimension one and higher. In dimension one, we focus on those classes of post-critically finite rational maps whose multipliers along periodic cycles can be computed explicitly. Then we study the natural generalizations of these examples in higher dimension. We also consider some other classes of post-critically

algebraic endomorphisms which are defined as product maps or are semiconjugate to another post-critically algebraic endomorphism via a ramified covering. In some cases, we can compute the eigenvalues along periodic cycles by the same technique used with examples in dimension one. In some other cases, we can only control the eigenvalues along periodic cycles outside the post-critical set. In all cases, the eigenvalues under control are either 0 or of modulus strictly bigger than 1.

In Chapter 2

We reinforce the evidences of Conjecture A by studying the eigenvalues at fixed points of a class of post-critically algebraic endomorphisms constructed by Sarah Koch. In [Koc13], Koch introduced a systematic way to produce infinitely many post-critically algebraic endomorphisms of $\mathbb{C}\mathbb{P}^n$ with $n \geq 2$. In this chapter, we study the eigenvalues of the following class of such post-critically algebraic maps at their fixed points. We refer to [Koc13] for the general definitions and construction.

Definition 0.0.1. Let $(k, m) \in \mathbb{N} \times \mathbb{N}^*$ be such that $k + m - 1 \geq 2$. The map

$$G_{k,m} : \mathbb{C}^{k+m-1} \rightarrow \mathbb{C}^{k+m-1}$$

is defined as,

- if $k = 0$, $G_{0,m} : \mathbb{C}^{m-1} \rightarrow \mathbb{C}^{m-1}$ is of the form

$$G_{0,m} : \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{m-1} \end{pmatrix} \mapsto \begin{pmatrix} -x_{m-1}^2 \\ x_1^2 - x_{m-1}^2 \\ \vdots \\ x_{m-2}^2 - x_{m-1}^2 \end{pmatrix}.$$

- if $k \neq 0$, $G_{k,m} : \mathbb{C}^{k+m-1} \rightarrow \mathbb{C}^{k+m-1}$ is of the form

$$G_{k,m} : \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{k+m-1} \end{pmatrix} \mapsto \begin{pmatrix} \left(-\frac{x_{k+m-1} + x_{k-1}}{2}\right)^2 \\ \left(x_1 - \frac{x_{k+m-1} + x_{k-1}}{2}\right)^2 \\ \vdots \\ \left(x_{k+m-2} - \frac{x_{k+m-1} + x_{k-1}}{2}\right)^2 \end{pmatrix}.$$

Given two integers $(k, m) \in \mathbb{N} \times \mathbb{N}^*$ such that $k + m - 1 \geq 2$, it follows from [Koc13, Theorem 5.17] and [Koc13, Theorem 5.18] that $G_{k,m}$ induces a post-critically algebraic

endomorphism of $\mathbb{C}\mathbb{P}^{k+m-2}$ of degree 2. In addition, according to [Koc13, Corollary 7.2], if z is a periodic point of $G_{k,m}$ outside the post-critical set, then every eigenvalue of $G_{k,m}$ along the orbit of z has modulus strictly bigger than 1. In this chapter, we show that all eigenvalues of $G_{k,m}$ at a fixed point are either 0 or of modulus strictly bigger than 1, i.e. we give an affirmative answer to Conjecture A for the cases of fixed points of $G_{k,m}$.

Theorem A. Let $(k, m) \in \mathbb{N} \times \mathbb{N}^*$ be such that $k + m - 1 \geq 2$ and μ be an eigenvalue of $G_{k,m}$ at a fixed point. Then either $\mu = 0$ or $|\mu| > 1$.

Moreover, we obtain a complete characterization of the eigenvalues of $D_z G_{k,m}$ when z is a fixed point of $G_{k,m}$. More precisely, let $z = (z_1, \dots, z_{k+m-1}) \in \mathbb{C}^{k+m-1}$ be a fixed point of $G_{k,m}$. An important observation, which is a consequence of the construction of $G_{k,m}$, is that the polynomial $P(t) = t^2 + z_1 \in \mathbb{C}[t]$ is post-critically finite.

When $z \notin PC(G_{k,m})$, Theorem A follows from [Koc13, Corollary 7.2], which is based on the work of Thurston. When $z \in PC(G_{k,m})$, we show that there exist two integers $(k', m') \in \mathbb{N} \times \mathbb{N}^*$ and a vector subspace E of \mathbb{C}^{k+m-1} of dimension $k' + m' - 1$ such that E is invariant under $G_{k,m}$ and the restriction of $G_{k,m}$ to E is conjugate to $G_{k',m'}$. Moreover, $z \notin PC(G_{k',m'})$. Thus, by applying the results due to Koch, we can understand the eigenvalues of $D_z G_{k,m}$ when their associated eigenvectors are tangent to E .

Our main contribution is the characterization of the eigenvalues of $D_z G_{k,m}$ with associated eigenvectors which are not tangent to E . It turns out that there is a relation between such eigenvalues and the multiplier of P along the periodic cycle in its critical orbit. More precisely, the critical value of P , which is z_1 , is mapped under the k' -th iterate of P to a periodic cycle of P of period m' and of multiplier λ . We show that the set of the eigenvalues of $D_z G_{k,m}$ whose associated eigenvectors are not tangent to E is

$$\begin{cases} \{0\} & \text{if } k' = 0 \\ \{\mu \mid \mu^m = \lambda^{\frac{m}{m'}}, \mu^{m'} \neq \lambda\} & \text{if } k' \neq 0. \end{cases}$$

Since P is post-critically finite, it follows from Theorem 1 that, in any case, every eigenvalue of $D_z G_{k,m}$ is either 0 or of modulus strictly bigger than 1.

In Chapter 3

We study Conjecture A in the general setting. Let $f : \mathbb{C}\mathbb{P}^n \rightarrow \mathbb{C}\mathbb{P}^n$ be a post-critically algebraic endomorphism of $\mathbb{C}\mathbb{P}^n$ and z be a periodic point of f . Since every iterate of f is also post-critically algebraic, by passing to an iterate, it is enough to assume that z is a fixed point of f . We have to deal with two main cases: the fixed point is either outside the post-critical set or inside the post-critical set.

The case when the fixed point is outside the post-critical set was first studied in [FS94]. Under the assumption that the complement of the post-critically set is Kobayashi hyperbolic and hyperbolically embedded, they showed that every eigenvalue of $D_z f$ has modulus at least 1 (see [FS94, Theorem 6.1]). Note that the Koch maps satisfy this geometric condition. In [FS94, Theorem 5.3], the author proved that the Kobayashi hyperbolicity of the complement of the post-critical set is a generic condition in the space of holomorphic endomorphisms of $\mathbb{C}\mathbb{P}^n$ of a given degree. However, the map

$$\mathbf{p}_d : \begin{array}{ccc} \mathbb{C}\mathbb{P}^n & \rightarrow & \mathbb{C}\mathbb{P}^n \\ [z_0 : \dots : z_n] & \rightarrow & [z_0^d : \dots : z_n^d] \end{array} \quad \text{with } d \geq 2,$$

which is the simplest example of a post-critically algebraic endomorphism, does not satisfy this condition. Indeed, the post-critical set $PC(\mathbf{p}_d)$ of \mathbf{p}_d is the union of $n + 1$ hyperplanes $\{x_i = 0 \mid 0 \leq i \leq n\}$. The map $t \mapsto [e^t : 1 : \dots : 1]$ is a non-constant holomorphic function avoiding $PC(\mathbf{p}_d)$. Thus, according to [Kob13, Proposition 3.6.1], the complement of $PC(\mathbf{p}_d)$ in $\mathbb{C}\mathbb{P}^n$ is not Kobayashi hyperbolic.

We improve the method of [FS94, Theorem 6.1] to get rid of the Kobayashi hyperbolic assumption and we are able to exclude the possibility of eigenvalues of modulus 1. The first main result of this chapter is the following.

Theorem B. Let f be a post-critically algebraic endomorphism of $\mathbb{C}\mathbb{P}^n$ and λ be an eigenvalue of f along a periodic cycle outside the post-critical set. Then $|\lambda| > 1$.

Let us give a sketch of the proof. We prove the result by contradiction. More precisely, without loss of generality, assume that $z \notin PC(f)$ is a fixed point of f and that $D_z f$ has at least one eigenvalue of modulus 1. Denote by E_n the sum of generalized eigenspaces associated to such eigenvalues. We build an immersed center manifold of f at z , i.e. an immersed complex submanifold M of $\mathbb{C}\mathbb{P}^n$ containing z such that $f(M) \subseteq M$ and, in a neighborhood of z , $f|_M$ is conjugate to the linear map $D_z f : E_n \rightarrow E_n$. Let λ be an eigenvalue of $D_z f$ of modulus 1 and $v \in E_n$ be an eigenvector associated to λ . By using the local conjugacy between $f|_M$ and $D_z f|_{E_n}$, we show the existence of a holomorphic map $\tau : \mathbb{D}(0, r) \rightarrow \mathbb{C}\mathbb{P}^n$ such that $\tau(\mathbb{D}(0, r)) \subset \mathbb{C}\mathbb{P}^n \setminus PC(f)$,

$$\tau(0) = z \text{ and } \tau'(0) = v.$$

Moreover, for all $t \in \mathbb{D}(0, r)$,

$$f \circ \tau(t) = \tau(\lambda t).$$

Using the latter property, we show that τ can be extended holomorphically to $\mathbb{D}(0, R)$

for a maximal $R < +\infty$ and $\tau(se^{i\theta})$ tends to a limit value in $PC(f)$ as s tends to R^- for almost every value $\theta \in [0, 2\pi)$. By applying a lemma due to Fatou, we show that the defining polynomial of $PC(f)$ vanishes identically on $\tau(\mathbb{D}(0, R))$, and hence, vanishes at $\tau(0) = z$. This contradicts the assumption that $z \notin PC(f)$.

When the fixed point is inside the post-critical set, we restrict our study to dimension $n = 2$. The second main result of this chapter is the proof of Conjecture [A](#) when $n = 2$ without any additional hypothesis.

Theorem C. Let f be a post-critically algebraic endomorphism of $\mathbb{C}\mathbb{P}^2$ and λ be an eigenvalue of f along a periodic cycle. Then either $\lambda = 0$ or $|\lambda| > 1$.

More precisely, let f be a post-critically algebraic endomorphism of $\mathbb{C}\mathbb{P}^2$ and z be a periodic point of f . Thanks to Theorem [B](#), we can assume that z is a fixed point of f in $PC(f)$. The derivative $D_z f$ has two eigenvalues λ_1 and λ_2 , counting multiplicities. We consider two subcases: either z is a regular point of $PC(f)$, or z is a singular point of $PC(f)$.

If the fixed point z is a regular point of $PC(f)$, the tangent space $T_z PC(f)$ is invariant under $D_z f$. Then $D_z f$ admits an eigenvalue λ_1 with associated eigenvectors in $T_z PC(f)$. The other eigenvalue λ_2 arises as the eigenvalue of the linear endomorphism $\overline{D_z f} : T_z \mathbb{C}\mathbb{P}^2 / T_z PC(f) \rightarrow T_z \mathbb{C}\mathbb{P}^2 / T_z PC(f)$ induced by $D_z f$. On the one hand, we prove that the eigenvalue λ_1 has modulus strictly bigger than 1 by using the normalization of irreducible algebraic curves and Theorem [1](#). On the other hand, we show that either $\lambda_2 = 0$ or $|\lambda_2| > 1$ by following the same idea used to prove Theorem [B](#).

If the fixed point z is a singular point of $PC(f)$, in most of the cases, there exists a relation between λ_1 and λ_2 . Then by using Theorem [1](#), we deduce that for $j = 1, 2$, either $\lambda_j = 0$ or $|\lambda_j| > 1$. This was observed in [[Jon98](#)]. For the sake of completeness, we will recall the detailed statements and include the proof. This allows us to conclude the proof of Conjecture [A](#) for post-critically algebraic endomorphisms of $\mathbb{C}\mathbb{P}^2$.

Chapter 1

Preliminaries

We expect the readers to be familiar with the basic notions of algebraic geometry, complex manifolds, holomorphic maps between complex manifolds and the basic notions of complex dynamics in higher dimension. We will quickly recall these topics in section 1.1 and we refer to [Gun90, Huy06, Har13],[SSC⁺10] for more details. Experienced readers can skip directly to 1.3, where we shall discuss the main object of this thesis: *the eigenvalues of post-critically algebraic endomorphisms at its periodic points*. The convention throughout the rest of this thesis is that all complex manifolds are Hausdorff, second countable and paracompact. Without further notice, every endomorphism of a complex manifold in this thesis will be holomorphic.

1.1 Complex manifolds and holomorphic maps

We refer to [Huy06] for an introduction of complex manifolds and holomorphic maps between complex manifolds. In this thesis, we are interested in endomorphisms of complex projective spaces. Given an integer $n \geq 1$, recall that the complex projective space $\mathbb{C}\mathbb{P}^n$ is a compact connected complex manifold of dimension n , defined as the quotient of $\mathbb{C}^{n+1} \setminus \{0\}$ by the equivalence relation

$$(x_0, \dots, x_n) \sim (y_0, \dots, y_n) \Leftrightarrow \exists \lambda \in \mathbb{C}^* \text{ such that } x_i = \lambda y_i \quad \text{for } 0 \leq i \leq n.$$

Denote by $[x_0 : \dots : x_n] \in \mathbb{C}\mathbb{P}^n$ the equivalence class of $(x_0, \dots, x_n) \in \mathbb{C}^{n+1} \setminus \{0\}$. We denote by $\pi : \mathbb{C}^{n+1} \setminus \{0\} \rightarrow \mathbb{C}\mathbb{P}^n$ the canonical projection, i.e. the map defined by

$$\pi(x_0, \dots, x_n) = [x_0 : \dots : x_n].$$

The space $\mathbb{C}\mathbb{P}^n$ is equipped with the complex structure defined by the $n + 1$ holomorphic charts $\{(U_i, \varphi_i : U_i \rightarrow \mathbb{C}^n), 0 \leq i \leq n + 1\}$, where

$$U_i = \{[x_0 : \dots : x_n] \in \mathbb{C}\mathbb{P}^n \mid x_i \neq 0\}, \quad \varphi_i([x_0 : \dots : x_n]) = \left(\frac{x_0}{x_i}, \dots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \dots, \frac{x_n}{x_i} \right).$$

We will identify \mathbb{C}^n with the open set U_0 via the inverse of φ_0 , i.e. the map defined by

$$(x_1, \dots, x_n) \mapsto [1 : x_1 : \dots : x_n].$$

Every holomorphic map $f : \mathbb{C}\mathbb{P}^n \rightarrow \mathbb{C}\mathbb{P}^n$, i.e. an endomorphism of $\mathbb{C}\mathbb{P}^n$, is rational (see [FS94, Theorem 2.1]). This means that there exist $n + 1$ homogeneous polynomials $P_i \in \mathbb{C}[x_0, \dots, x_n], 0 \leq i \leq n$ of the same degree $d \geq 1$ such that $\bigcap_{0 \leq i \leq n} P_i^{-1}(0) = \{0\}$ and

$$f \circ \pi = \pi \circ (P_0, \dots, P_n).$$

Such a polynomial map $(P_0, \dots, P_n) : \mathbb{C}^{n+1} \rightarrow \mathbb{C}^{n+1}$ is called a *lift* of f to \mathbb{C}^{n+1} . The integer $d \geq 1$ is called the *algebraic degree*, or the *degree*, of f . Conversely, a map $F = (P_0, \dots, P_n) : \mathbb{C}^{n+1} \rightarrow \mathbb{C}^{n+1}$, where P_0, \dots, P_n are homogeneous polynomials of degree d satisfying $\bigcap_{0 \leq i \leq n} P_i^{-1}(0) = \{0\}$, induces an endomorphism of $\mathbb{C}\mathbb{P}^n$ of degree d .

Analytic sets and algebraic sets

We are also interested in analytic subsets of complex manifolds, and particularly, algebraic subsets of \mathbb{C}^n or $\mathbb{C}\mathbb{P}^n$. We shall recall some important definitions and properties of such objects. For further details and proofs of the following statements, the readers are invited to see [Chi12, CDK12] for the case of analytic sets, and [RS13] for the case of algebraic subsets.

Let M be a complex manifold. A set $X \subseteq M$ is an *analytic set* if, for every $z \in X$, there exist an open neighborhood U of z and a finite set of holomorphic functions $f_1, \dots, f_k : U \rightarrow \mathbb{C}$ such that

$$U \cap X = \{x \in U \mid f_1(x) = \dots = f_k(x) = 0\}.$$

Let X be a non-empty analytic set of a complex manifold M . A point $z \in X$ is called a *regular point* (or *smooth point*) if X is a submanifold in some open neighborhood of z , and it is called a *singular point* otherwise. Denote by $\text{Reg } X$ the set of regular points and by $\text{Sing } X$ the set of singular points of X . Note that $\text{Sing } X$ is again an analytic set. An

analytic set X is called *irreducible* if X is not the union of two proper analytic subsets or equivalently, $\text{Reg } X$ is a connected manifold. An analytic set $X' \subset X$ is said to be an irreducible component of X , if X' is irreducible and maximal, i.e. is contained in no other irreducible analytic set $X'' \subset X$. An irreducible component of an analytic set X is the closure of a connected component of $\text{Reg } X$. The regular part $\text{Reg } X$ of X can have several connected components which are complex manifolds of different dimensions. The *dimension of X* , $\dim X$, is the maximum of the dimensions of connected components of $\text{Reg } X$. The *codimension of X in M* is $\dim M - \dim X$. An analytic set is called *purely k -dimensional* if every connected component of $\text{Reg } X$ has the same dimension k . A pure dimensional analytic set of codimension one in a complex manifold M can be defined locally as the zero locus of a single holomorphic function, i.e. as a *hypersurface* (see [CDK12, Chapter III]).

Assume $n \geq 1$. A set $\Gamma \subset \mathbb{C}^n$ is called *an algebraic set of \mathbb{C}^n* if its elements are common zeros of finite polynomials P_1, \dots, P_k of n complex variables. The set $\{P_1, \dots, P_k\}$ is called *a set of defining polynomials of Γ* . A set $X \subset \mathbb{C}\mathbb{P}^n$ is called a *projective algebraic set* (or *an algebraic set of $\mathbb{C}\mathbb{P}^n$*) if $\pi^{-1}(X)$ is an algebraic set of \mathbb{C}^{n+1} which is defined by a set of homogeneous polynomials. Such a set of polynomials is also said to be a set of defining polynomials of X . The family of algebraic sets in $\mathbb{C}\mathbb{P}^n$ is stable under intersections and finite union. *The Zariski topology* on $\mathbb{C}\mathbb{P}^n$ is the topology whose closed sets are the algebraic subsets of $\mathbb{C}\mathbb{P}^n$. By definition, algebraic subsets of $\mathbb{C}\mathbb{P}^n$ (resp. of \mathbb{C}^n) are analytic subsets of $\mathbb{C}\mathbb{P}^n$ (resp. \mathbb{C}^n). Conversely, Chow's theorem asserts that every analytic set of $\mathbb{C}\mathbb{P}^n$ which is closed under the usual topology is algebraic (see [Gun90, Volume II, Theorem M.3]). We can define regular points, singular points, dimension, codimension for an algebraic set, of $\mathbb{C}\mathbb{P}^n$ (resp. of \mathbb{C}^n) as for an analytic set of $\mathbb{C}\mathbb{P}^n$ (resp. of \mathbb{C}^n). It is known that an irreducible algebraic set of codimension one can be defined by only one polynomial (see [RS13, Theorem 1.21]).

Given an algebraic set $X \subset \mathbb{C}\mathbb{P}^n$, the set $\widehat{X} := \pi^{-1}(X) \cup \{0\} \subset \mathbb{C}^{n+1}$ is called the *affine cone* over X . The geometries of X and \widehat{X} are closely related. In particular, $\dim \widehat{X} = \dim X + 1$ and

$$\text{Reg } X = \pi(\text{Reg } \widehat{X}), \text{Sing } X = \pi(\text{Sing } \widehat{X} \setminus \{0\})$$

(see [Har13, Exercise 2.10] or [Chi12, Proposition 1.2.5]). Moreover, X has codimension one in $\mathbb{C}\mathbb{P}^n$ if and only if the affine cone over X is also of codimension one in \mathbb{C}^{n+1} .

Let Γ be an algebraic subset of \mathbb{C}^n , which is defined by k polynomials $P_1, \dots, P_k \in \mathbb{C}[x_1, \dots, x_n]$. Recall that we can identify \mathbb{C}^n with the open subset U_0 of $\mathbb{C}\mathbb{P}^n$ by the homeomorphism φ_0 . Hence, Γ is identified with a subset of $\mathbb{C}\mathbb{P}^n$. The closure $\overline{\Gamma}^{\text{proj}}$ of

Γ with respect to the Zariski topology in $\mathbb{C}\mathbb{P}^n$ is called the *projective closure* of Γ . A set of defining polynomials of $\bar{\Gamma}^{proj}$ is $\{Q_1, \dots, Q_n\} \subset \mathbb{C}[x_0, x_1, \dots, x_n]$, where Q_i is the *homogenization* of P_i , i.e.

$$Q_i = x_0^{\deg P_i} P_i \left(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0} \right) \quad \forall i \in \{1, \dots, n\}.$$

In particular, Γ is of codimension one if and only if $\bar{\Gamma}^{proj}$ is of codimension one.

Critical points, critical values

In this paragraph, we will discuss holomorphic maps between complex manifolds. We will mainly work with maps between complex manifolds of the same dimension, in the following, all considered complex manifolds will have the same dimension.

Definition 1.1.1. Let M and N be complex manifolds of the same dimension and $f : M \rightarrow N$ be a holomorphic map. The *critical set* C_f is the set of points $z \in M$ such that the derivative $D_z f : T_z M \rightarrow T_{f(z)} N$ is not surjective. The *critical value set* $V_f := f(C_f)$ is the image of C_f under f .

Let M and N be complex manifolds of the same dimension and $f : M \rightarrow N$ be a holomorphic map. Assume $z \in M$. Let (U, φ) of M be a chart containing z and (V, ψ) of N be a chart containing $f(z)$ such that $f(U) \subset V$. The point z is a *critical point* of f if and only if the (complex) Jacobian of $\psi \circ f \circ \varphi^{-1}$ at $\varphi(z)$ is not invertible. In particular, the critical set of f , if it is neither empty nor M , is a pure dimensional analytic set of codimension one, which is locally defined by the determinant of the Jacobian of f in local charts (see [Huy06, Proposition 1.1.13]).

Ramified coverings

Recall that, given two connected topological spaces X and Y and a covering map $f : X \rightarrow Y$, the cardinal of $f^{-1}(y)$, which can be finite or infinite, does not depend on $y \in Y$ and is called *the order of the covering*.

Definition 1.1.2. Let M and N be complex manifolds of the same dimension. A holomorphic map $f : M \rightarrow N$ is called a *ramified covering* (or *branched covering*) if there exists an analytic set D of codimension one in N such that the map

$$f : M \setminus f^{-1}(D) \rightarrow N \setminus D$$

is a covering of finite order (see [Gun90, Volume II, Definition C.3]). We will say that f is *ramified over* D .

Let $f : M \rightarrow N$ be a ramified covering ramifying over $D \subset N$. If N is connected, the order of the covering $f : M \setminus f^{-1}(D) \rightarrow N \setminus D$ does not depend on D and is called *the order of f* . Assume $z \in N$ and $w \in f^{-1}(z)$. For every connected open neighborhood $V_z \subset N$ of z , denote by V_w the connected component $f^{-1}(V_z)$ containing w . The restricted map $f|_{V_w} : V_w \rightarrow V_z$ is a ramified covering. Note that when V_z varies in a local basis of z , V_w forms a local basis of w . Moreover, the order of the ramified covering $f|_{V_w}$ is constant when we consider V_z small enough. This common order is called the *branching order* or the *multiplicity of f at w* . The set of points in M of branching order at least 2 is called *the branch locus*. In fact, the branch locus of f is precisely the critical set C_f . In particular, f is ramified over the critical value set. The following result will be important.

Theorem 1.1.3 (Chapter 3. Proposition 1.2 [SSC+10]). *Let $n \geq 1$ and $f : \mathbb{C}\mathbb{P}^n \rightarrow \mathbb{C}\mathbb{P}^n$ be an endomorphism of $\mathbb{C}\mathbb{P}^n$ of degree $d \geq 1$. Then f is a ramified covering of order d^n .*

In fact, every lift to \mathbb{C}^{n+1} of an endomorphism of degree d of $\mathbb{C}\mathbb{P}^n$ is also a ramified covering of order d^{n+1} .

Let $f : \mathbb{C}\mathbb{P}^n \rightarrow \mathbb{C}\mathbb{P}^n$ be an endomorphism of degree $d \geq 1$ and let $F : \mathbb{C}^{n+1} \rightarrow \mathbb{C}^{n+1}$ be a lift of f . If $d = 1$, then f is unramified, i.e. is a covering, and hence f is a biholomorphism. If $d \geq 2$, since $\mathbb{C}\mathbb{P}^n$ is simply connected, f cannot be a covering. Hence, f is ramifying over the critical value set V_f which is an algebraic set of codimension one of $\mathbb{C}\mathbb{P}^n$. In this case, the map F is also a ramified covering ramifying over V_F which is the affine cone over V_f . In particular, V_F is an algebraic set of codimension one of \mathbb{C}^{n+1} .

Normal families of holomorphic maps

We will deal with families of holomorphic maps of complex manifolds. An important notion in this setting is that of a normal family of holomorphic maps. We shortly recall here the essential tools of the theory of normal families of holomorphic maps.

Let M, N be connected complex manifolds. Denote by $C^0(M, N)$ the set of continuous maps from M to N and by $\text{Hol}(M, N) \subset C^0(M, N)$ the set of holomorphic maps. The standard topology on $C^0(M, N)$ is the *compact-open topology* (see [Wu67, Part I. Section 1]) By convention, N is paracompact, hence it is metrizable. Once a distance is chosen on N , a sequence $(f_n)_n \subset C^0(M, N)$ converges to $f \in C^0(M, N)$ if and only if $(f_n)_n$ converges to f uniformly on compact sets of M , i.e. *locally uniformly*.

Definition 1.1.4. A subset $\mathcal{F} \subset \text{Hol}(M, N)$ is called a *normal family* if every sequence of elements of \mathcal{F} contains a subsequence converging locally uniformly.

One of the most important characterizations of normal families is Montel's theorem, whose proof can be found for instance in [Gum90, Volume I, Theorem F.3].

Theorem 1.1.5 (Montel's theorem). *Let M be a complex manifold. A family $\mathcal{F} \subseteq \text{Hol}(M, \mathbb{C}^n)$ is normal if and only if it is locally uniformly bounded, i.e. uniformly bounded on compact sets of M .*

We conclude this section by the following result, which is due to Cazacu [AC03]. It asserts that normality is preserved after lifting by a covering.

Theorem 1.1.6 (Theorem 4 [AC03]). *Let M, N, P be connected complex manifold. Let $\varpi : N \rightarrow P$ be a covering and let $\{f_n : M \rightarrow P\}_{n \geq 0}$, $\{g_n : M \rightarrow N\}_{n \geq 0}$ be sequences of holomorphic maps such that, for every $n \geq 0$, the following diagram commutes.*

$$\begin{array}{ccc} N & \xleftarrow{g_n} & M \\ \varpi \downarrow & \swarrow f_n & \\ P & & \end{array}$$

If f_n converges locally uniformly on M to f_0 and there exists a point $x \in M$ such that $g_n(x)$ converges to $g_0(x)$, then g_n converges locally uniformly to g_0 .

1.2 Dynamics in complex projective spaces

Dynamics of endomorphisms of a complex manifold

We will be interested in the family defined by the iterates of a self-map of a complex manifold. More precisely, let M be a complex manifold and $f : M \rightarrow M$ be an endomorphism. Given an integer $j \geq 0$, denote by

$$f^{\circ j} = \underbrace{f \circ \dots \circ f}_{j \text{ times}}$$

the j -th iterate of f . Given a point $z \in M$, the orbit of z under f is the sequence $(f^{\circ j}(z))_{j \geq 0}$. Complex dynamics study the asymptotic behavior of the family of iterates $\{f^{\circ j}\}_{j \geq 0}$ of f as n tends to infinity. We are concerned about the asymptotic behavior of the orbit of z when z varies in M and n tends to infinity. In this thesis, we shall focus on the set of points with finite orbit.

Definition 1.2.1. Let M be a complex manifold and let $f : M \rightarrow M$ be an endomorphism. A point $z \in M$ is called a *periodic point* of f if there exists an integer $p \geq 1$ such

that $f^{\circ p}(z) = z$. The smallest integer satisfying such a property is called the *period* of z . When $p = 1$, z is called a *fixed point*. A point z is a *preperiodic point* of f if $f^{\circ k}(z)$ is periodic for some integer $k \geq 0$. The smallest integer k such that $f^{\circ k}(z)$ is periodic is called the *preperiod*. The orbit of z is called a *cycle* of f .

In particular, a point $z \in M$ is preperiodic if and only if the orbit of z under f is finite. Furthermore, a preperiodic point of preperiod 0 is periodic. Given a periodic point z of period p of f , every point in the orbit of z is a fixed point of $f^{\circ p}$. The derivative $D_z f^{\circ p}$ of $f^{\circ p}$ at z is a linear endomorphism of $T_z M$. The eigenvalues of $D_z f^{\circ p}$ play an important role in the study of the dynamics of f in a neighborhood of the orbit of z .

Definition 1.2.2. Let M be a complex manifold of dimension one and let $f : M \rightarrow M$ be an endomorphism. Let z be a periodic point of period p . The unique eigenvalue λ of $D_z f^{\circ p}$ is called the *multiplier of f along the orbit of z* . The cycle of z is called

- *superattracting* if $\lambda = 0$,
- *attracting* if $0 < |\lambda| < 1$,
- *neutral* or *indifferent* if $|\lambda| = 1$, additionally
 - *parabolic* if λ is a root of unity,
 - *elliptic* otherwise,
- *repelling* if $|\lambda| > 1$.

When M has dimension at least 2, in general, $D_z f^{\circ p}$ has several eigenvalues. With a slight abuse of notation, we will use the following definition to characterize such values.

Definition 1.2.3. Let M be a complex manifold let $f : M \rightarrow M$ be an endomorphism. Let $z \in M$ be a periodic point of period p of f . An eigenvalue λ of $D_z f^{\circ p}$ is called an *eigenvalue of f along the cycle of z* (or *at the cycle of z*). We say that λ is

- *superattracting* if $\lambda = 0$,
- *attracting* if $0 < |\lambda| < 1$,
- *neutral* or *indifferent* if $|\lambda| = 1$, additionally
 - *parabolic* if λ is a root of unity,
 - *elliptic* otherwise,

- *repelling* if $|\lambda| > 1$.

Definition 1.2.4. Let M be a complex manifold and $f : M \rightarrow M$ be an endomorphism. The cycle of a periodic point z is called *superattracting* (resp. *attracting*, *indifferent or neutral*, *repelling*) *cycle* if all eigenvalue of f along the orbit of z are superattracting (resp. attracting, indifferent or neutral, repelling).

Endomorphisms of \mathbb{CP}^n

Let $f : \mathbb{CP}^n \rightarrow \mathbb{CP}^n$ be an endomorphism of degree $d \geq 2$ with $n \geq 1$. Since we are mainly interested in periodic points, the following result is important.

Theorem 1.2.5 (Corollary 3.2, [FS94]). *Let $f : \mathbb{CP}^n \rightarrow \mathbb{CP}^n$ be an endomorphism of degree $d \geq 2$. Given an integer $p \geq 1$, f has finitely many periodic points of period p .*

We refer to [Mil11] for further details on the dynamics of $f : \mathbb{CP}^n \rightarrow \mathbb{CP}^n$ when $n = 1$ and to [SSC⁺10, Chapter 3] when $n \geq 2$.

Recall that f is a ramified covering, ramifying over the critical value set V_f . Similarly, for every $j \geq 1$, $f^{\circ j}$ is a ramified covering ramifying over $V_{f^{\circ j}}$. The *post-critical set* $PC(f)$ is the union of the critical values set of all iterates of f , i.e.

$$PC(f) = \bigcup_{j \geq 1} V_{f^{\circ j}}.$$

Note that, for every $j \geq 2$, $V_{f^{\circ j}} = f(C_f) \cup f^{\circ 2}(C_f) \cup \dots \cup f^{\circ j}(C_f)$. In particular,

$$PC(f) = \bigcup_{j \geq 1} f^{\circ j}(C_f).$$

Since f maps an algebraic set onto an algebraic set of the same dimension, $PC(f)$ is a countable union of codimension one algebraic sets of \mathbb{CP}^n . We will focus on the case when this union is finite.

Definition 1.2.6. Let f be an endomorphism of \mathbb{CP}^n of degree $d \geq 2$. The map f is called *post-critically algebraic* if its post-critical set $PC(f)$ is an algebraic set of codimension one of \mathbb{CP}^n .

Remark 1.2.7. Let f be an endomorphism of \mathbb{CP}^n and F be a lift of f . Since C_F is the algebraic cone over C_f , the map f is post-critically algebraic if and only if the $PC(F)$ is an algebraic set of codimension one in \mathbb{C}^{n+1} . We shall discuss more about the notion of being post-critically algebraic for polynomial maps of \mathbb{C}^n in Section 1.3.2.

In dimension one, post-critically algebraic endomorphisms are called *post-critically finite* rational maps and have been one of most important topic in the theory of complex dynamics. Some typical post-critically finite examples are $x \mapsto x^2, x \mapsto \frac{1}{x^2}, x \mapsto x^2 - 2$. To the best of my knowledge, post-critically algebraic endomorphisms of \mathbb{CP}^n , for $n \geq 2$, were first mentioned in [BDM⁺90]. In this problem list, Curtis McMullen asked for the generalizations of post-critically finite rational maps in \mathbb{CP}^n which preserve the Kobayashi hyperbolicity of the complement of the post-critical set. In [FS92], John Erik Fornæss and Nessim Sibony answered this question by introducing a class of maps on \mathbb{CP}^n , for $n \geq 2$, of the form

$$\begin{aligned} h_\lambda : \mathbb{CP}^n &\rightarrow \mathbb{CP}^n \\ [x_0 : x_1 : \dots : x_n] &\rightarrow [x_0^d : \lambda(x_0 - 2x_1)^d : \dots : \lambda(x_0 - 2x_n)^d] \end{aligned}$$

with $d \geq 2$ and $(1 - 2\lambda)^d = 1$. This class generalizes the class of post-critically finite rational maps $\{\lambda \left(\frac{x-2}{x}\right)^d \mid \left(\frac{\lambda-2}{\lambda}\right)^d = 1\}$. Moreover, for every permutation of coordinates $\sigma : \{x_0, x_1, \dots, x_n\} \rightarrow \{x_0, x_1, \dots, x_n\}$, they observed that the map $\sigma \circ h_\lambda$ is post-critically algebraic. This thesis is devoted to the study of the dynamics of post-critically algebraic endomorphisms of \mathbb{CP}^n .

1.3 Examples of post-critically algebraic endomorphisms

Let f be an endomorphism of \mathbb{CP}^n and let $z \in \mathbb{CP}^n$ be a periodic point of f of period p . Recall that z is a fixed point of f^{op} and the derivative $D_z f^{op}$ of f^{op} at z is a linear endomorphism of $T_z \mathbb{CP}^n$. We want to study the eigenvalues of $D_z f^{op} : T_z \mathbb{CP}^n \rightarrow T_z \mathbb{CP}^n$.

When $n = 1$, such a value, which is called the *multiplier* along the cycle of z , plays a crucial role in the study of rational maps in a neighborhood of a periodic point. In most of the cases, given a periodic point and its multiplier, one can understand the behavior of the family of iterates of f in a full neighborhood of this periodic point. Moreover, the behavior of critical points controls the existence of certain types of periodic points. We refer to [Mil11, §8, 9, 10, 11] for more details. In particular, when a rational map is post-critically finite, we have the following well-known result, which is essentially due to Fatou.

Theorem 1.3.1. *Let f be a post-critically finite rational map and λ be the multiplier of f along a periodic cycle. Then either $\lambda = 0$ or $|\lambda| > 1$.*

The classical proof of this result is based on the study of the local dynamics of rational maps in a neighborhood of a periodic point. More precisely, let z be a periodic point of a rational map f of degree at least 2 (not necessarily post-critically finite). Then,

- if the cycle of z is either attracting or parabolic, then the cycle of z is the limit of the infinite orbit of some critical point (see [Mil11, Lemma 8.5] for the attracting case and [Mil11, Theorem 10.15] for the parabolic case),
- if the cycle of z is elliptic indifferent then,
 - either f is linearizable¹ in a neighborhood of the cycle of z , i.e. the cycle of z is contained in a cycle of Siegel discs², and the boundary of the Siegel disk containing the cycle of z is accumulated by the infinite orbit of some critical point ([Mil11, Theorem 11.7]),
 - or f is not linearizable in a neighborhood of the cycle of z and the cycle of z is accumulated by the infinite orbit of some critical point ([Mil11, Theorem 11.17] or [Mil11, Theorem 14.4]).

In higher dimension, we may have several eigenvalues along a given periodic cycle and the dynamics in a neighborhood of a periodic cycle is much more complicated. Since post-critically algebraic endomorphisms of $\mathbb{C}P^n$ with $n \geq 2$ are the generalizations of post-critically finite rational maps, it is natural to ask whether Theorem 1.3.1 remains true in higher dimension. The results used in the classical proof are generally not available when $n \geq 2$. In the next two paragraphs, we shall analyse some examples in dimension one and their natural generalizations in higher dimension. For these examples, we can either compute explicitly the eigenvalues at periodic cycles or control the eigenvalues at the cycles outside the post-critical set. These examples give evidences of a possible generalization of Theorem 1.3.1.

1.3.1 Examples in dimension one

We review three families of rational maps, whose construction can be generalized to higher dimension: the families of *finite quotients of affine maps*. We refer to [Mil06] for more details about the construction and the dynamics of such maps.

¹Let p be the period of z . The map f is linearizable in a neighborhood of the cycle of z if there exist an open neighborhood U of z , a holomorphic injective map $\varphi : U \rightarrow \mathbb{C}$ such that $\varphi(z) = 0$ and, for every $x \in U \cap \varphi(U)$, $\varphi \circ f^{\circ p}(x) = (f^{\circ p})'(z)\varphi(x)$.

²The maximal domain on which f is linearizable (see [Mil11, Definition 11.2]).

Note that \mathbb{CP}^1 is isomorphic to the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Every rational map of degree d can be written in the form $\frac{P(x)}{Q(x)}$ where P, Q are polynomials without common irreducible divisor and $d = \max\{\deg P, \deg Q\}$.

Example 1.3.2. Let f be a post-critically finite rational map of degree $d \geq 2$ such that $PC(f)$ has exactly two points. Using Riemann-Hurwitz's formula, one can show that, in such a case, $PC(f)$ is totally invariant, i.e. $f^{-1}(PC(f)) = PC(f)$. It is classical that f is conjugate to either $x \mapsto x^d$ or $x \mapsto \frac{1}{x^d}$ (see [Mil11, Lemma 4.9]). In particular, periodic points of f are either in $PC(f)$ or roots of unity and the multiplier of f at periodic cycles is either 0 or of modulus $d^p > 2$.

The maps in Example 1.3.2 are the simplest ones of post-critically finite rational maps. We shall see in Example 1.3.11 that we can nevertheless use them to construct post-critically algebraic endomorphisms of \mathbb{CP}^n , $n \geq 2$ with interesting properties.

Example 1.3.3. Given an integer $d \geq 2$, a polynomial T_d of degree d is called the d -th *Chebyshev polynomial* if it satisfies that

$$T_d \circ \varphi = \varphi \circ p_d.$$

where $\varphi(x) := x + \frac{1}{x}$ and $p_d(x) = x^d$. In other words, we have the following commutative diagram.

$$\begin{array}{ccc} \hat{\mathbb{C}} & \xrightarrow{x \mapsto x^d} & \hat{\mathbb{C}} \\ x \mapsto x + \frac{1}{x} \downarrow & & \downarrow x \mapsto x + \frac{1}{x} \\ \hat{\mathbb{C}} & \xrightarrow{T_d} & \hat{\mathbb{C}} \end{array}$$

For examples, we have:

$$T_2(x) = x^2 - 2, \quad T_3(x) = x^3 - 3x, \quad T_4(x) = x^4 - 4x^2 + 2, \dots$$

One can show that that for every $d \geq 2$, $PC(T_d) = \{2, -2, \infty\}$, and hence T_d is post-critically finite.

Example 1.3.4. Given any lattice $\Lambda = \mathbb{Z} + \tau\mathbb{Z}$ with $\tau \in \mathbb{C} \setminus \mathbb{R}$, the quotient $\mathbb{T} := \mathbb{C}/\Lambda$ is a compact Riemann surface, which is called *a torus*. The automorphism $x \mapsto -x$ of \mathbb{T} has four fixed points. The quotient S of \mathbb{T} by identifying each $x \in \mathbb{T}$ with $-x$ is a Riemann surface. By Riemann-Hurwitz's formula, one can show that S is isomorphic to \mathbb{CP}^1 . Denote by $\varphi : \mathbb{T} \rightarrow \mathbb{CP}^1$ a quotient map. The map φ is a ramified covering of order 2, with four critical points of branching order 2 which are fixed points of the automorphism

$x \mapsto -x$. The doubling map $D_2 : x \mapsto 2x$ commutes with the map $x \mapsto -x$, and hence induces a map $f : \mathbb{C}\mathbb{P}^1 \rightarrow \mathbb{C}\mathbb{P}^1$ such that

$$f \circ \wp = \wp \circ D_2.$$

Such a rational map f is called a *Lattès map*. We refer to [Mil11, §7] for the details of the construction. The map f is post-critically finite with the post-critical set is precisely four critical values of \wp (see [Mil06, Lemma 3.4]).

We will take a closer look at periodic cycles and the multipliers at periodic cycles of the latter two classes of maps. Let f be either a Chebyshev polynomial or a Lattès map. By definition, f is semiconjugate to a holomorphic self-map, say g , of a complex manifold Γ , which is either $\hat{\mathbb{C}}$ or a torus. The map $\varphi : \Gamma \rightarrow \mathbb{C}\mathbb{P}^1$, which satisfies the semiconjugacy relation

$$\varphi \circ g = f \circ \varphi, \tag{1.3.1}$$

is a branched covering of finite order. The semiconjugacy relation provides a relation between periodic points (and hence the multipliers) of f and those of g . More precisely, if $z \in \mathbb{C}\mathbb{P}^1$ is a fixed point of f , then equality (1.3.1) implies that $g(\varphi^{-1}(z)) \subseteq \varphi^{-1}(z)$. Since φ is of finite order, $\varphi^{-1}(z)$ is a finite set which is invariant by g . Note that every iterate of f is still of the same type of f , either a Chebyshev polynomial or a Lattès map. Hence, by passing to an iterate, we can assume that there exists a point $w \in \Gamma$ such that $g(w) = w$ and $\varphi(w) = z$. The following result shows that we can compute the multiplier of f at z once the multiplier of g at w is known.

Lemma 1.3.5. *Let $f, g, \varphi : (\mathbb{C}, 0) \rightarrow (\mathbb{C}, 0)$ be holomorphic germs such that*

$$\varphi \circ g = f \circ \varphi.$$

Assume that f, g and φ are not constantly vanishing. Then there exists an integer $k \geq 1$ such that $f'(0) = g'(0)^k$.

Proof. Since f, g and φ are not constantly vanishing, there exist three integers $r, s, k \in \mathbb{N}^*$ and three constants $a_r, b_s, c_k \in \mathbb{C} \setminus \{0\}$ such that, in a neighborhood of 0, we can write

$$f(x) = a_r x^r + O(x^{r+1}),$$

$$g(x) = b_s x^s + O(x^{s+1}).$$

and

$$\varphi(x) = c_k x^k + O(x^{k+1})$$

The hypothesis $\varphi \circ g = f \circ \varphi$ implies that

$$c_k (b_s x^s + O(x^{s+1}))^k + O((b_s x^s + O(x^{s+1}))^{k+1}) = a_r (c_k x^k + O(x^{k+1}))^r + O((c_k x^k + O(x^{k+1}))^{r+1}),$$

and hence,

$$c_k b_s^k x^{sk} + O(x^{sk+1}) = a_r c_k^r x^{rk} + O(x^{rk+1}).$$

Comparing the coefficient of the first non-trivial terms in both sides, we deduce that

$$\begin{cases} r = s \\ c_k b_s^k = a_r c_k^r \end{cases}$$

If $r > 1$, and hence $s > 1$, we have $f'(0) = 0 = g'(0)$. If $r = s = 1$, then $b_1 = g'(0)$, $a_1 = f'(0)$ and

$$c_k g'(0)^k = f'(0) c_k$$

Since $c_k \neq 0$, we deduce that $g'(0)^k = f'(0)$. \square

Applying Lemma 1.3.5, we deduce that a multiplier of f at a fixed point is up to some powers of a multiplier of g at its fixed points. Note that g is either a power map (as in Example 1.3.2) or an affine map, which means that we can compute explicitly the multipliers of g at periodic points. Therefore, we can also compute explicitly the multipliers of f at its fixed points, and hence also at periodic points. In particular, every multiplier of either a Chebyshev map or a Lattès map is either superattracting or repelling.

1.3.2 Examples in higher dimension

Given an endomorphism of $\mathbb{C}\mathbb{P}^n$ with $n \geq 2$, we want to study the eigenvalues of its derivative along a periodic cycle. It is natural to expect that the algebraicity of the post-critical set will allow us to control the eigenvalues at periodic points and we can generalize Theorem 1.3.1. As stated in the Introduction, to study the eigenvalues of post-critically algebraic endomorphisms, one can make use of some geometric properties on the post-critical set or on its complement, like the Kobayashi hyperbolicity of the complement (see [FS94, Theorem 6.1]) or the regularity of the intersection of irreducible components of the post-critical set (see [Ast20, Theorem B]). We can also rely on some dynamical assumptions on the post-critical set, like the non-periodicity of critical components (see [Jon98, Theorem 2.9]). However, there exist several examples of post-critically algebraic endomorphisms in the literature which do not satisfy these assumptions. We shall review

some of them in the following. We will focus on the possible type of eigenvalues of such maps at periodic points. In some cases, we can give explicitly the values of the eigenvalues. However, in other cases, we only prove that the periodic cycles outside the post-critical set has only repelling eigenvalues. The question of characterizing the possible values of the eigenvalues of a post-critically algebraic endomorphism at all its periodic cycles remains open in general.

Post-critically algebraic regular polynomial maps

We shall extend the notion of being post-critically algebraic to some classes of polynomial endomorphisms of \mathbb{C}^n with $n \geq 2$. In particular, we can see how to generalize the class of post-critically finite polynomials to higher dimension by considering product maps. In this case, we can control the eigenvalues of the product thanks to Theorem 1.3.1.

Assume $n \geq 2$. A polynomial map $F = (P_1, \dots, P_n), P_i \in \mathbb{C}[x_1, \dots, x_n]$ is called a *polynomial map of degree d* if $\deg P_i = d$ for every $i \in \{1, \dots, n\}$. The *homogenization of a polynomial map F of degree d* is a homogeneous polynomial map $G = (Q_0, \dots, Q_n) : \mathbb{C}^{n+1} \rightarrow \mathbb{C}^{n+1}$ such that $Q_0 = x_0^d$ and Q_i is the homogenization of P_i , i.e.

$$Q_i = x_0^d P_i \left(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0} \right) \quad \forall i \in \{1, \dots, n\}.$$

Definition 1.3.6. A polynomial map $F : \mathbb{C}^n \rightarrow \mathbb{C}^n$ of degree $d \geq 2$ is *regular* if the homogenization of F induces an endomorphism of $\mathbb{C}\mathbb{P}^n$.

This notion follows the notions of regular polynomial maps given by Bedford and Jonsson in [BJ00] and it is different from the notion of regular polynomial automorphism used in [Sib99]. One can show that every lift of an endomorphism of $\mathbb{C}\mathbb{P}^n$ is regular. A general polynomial endomorphism of \mathbb{C}^2 does not have some important properties which are crucial for our work later, like the properness or the finiteness,. Therefore, we will restrict our attention to only regular polynomial maps. The following lemma provides a way to characterize when a polynomial map is regular.

Lemma 1.3.7. *Let $F = (P_1, \dots, P_n) : \mathbb{C}^n \rightarrow \mathbb{C}^n$ be a polynomial map of degree d . For every $i \in \{1, \dots, n\}$, let $P_{i,d}$ be the sum of monomials of degree d of P_i . Set $H := (P_{1,d}, \dots, P_{n,d}) : \mathbb{C}^n \rightarrow \mathbb{C}^n$. Then, F is regular if and only if $H^{-1}(0) = \{0\}$.*

Proof. For every $i \in \{1, \dots, n\}$, we can write $P_i \in \mathbb{C}[x_1, \dots, x_n]$ in the form

$$P_i = P_{i,d} + P_{i,d-1} + \dots + P_{i,1} + P_{i,0}$$

where $P_{i,j}$ is the sum of monomials of degree j of P_i for every $i \in \{1, \dots, n\}$ and $j \in \{0, \dots, d\}$. The homogenization $G = (Q_0, Q_1, \dots, Q_n)$ of F is of the form

$$Q_0 = x_0^d \quad \text{and} \quad Q_i = P_{i,d} + x_0 P_{i,d-1} + \dots + x_0^{d-1} P_{i,1} + x_0^d P_{i,0} \quad \forall i \in \{1, \dots, n\}.$$

By definition, F is regular if only if G induces an endomorphism of $\mathbb{C}\mathbb{P}^n$, i.e. $G^{-1}(0) = \{0\}$.

Assume that $(z_0, z) \in \mathbb{C} \times \mathbb{C}^n$ such that $G(z_0, z) = 0$. Note that $Q_0 = 0$ if and only if that $z_0 = 0$. Moreover, for every $i \in \{1, \dots, n\}$, $Q_i(0, z) = P_{i,d}(z)$. Consequently, $G(z_0, z) = 0$ if and only if $z_0 = 0$ and

$$H(z) = (P_{1,d}(z), \dots, P_{n,d}(z)) = 0.$$

Hence, $G^{-1}(0) = \{0\}$ if and only if $H^{-1}(0) = \{0\}$. □

Definition 1.3.8. A regular polynomial map $F : \mathbb{C}^n \rightarrow \mathbb{C}^n$ of degree $d \geq 2$ is called *post-critically algebraic* if the homogenization of F induces a post-critically algebraic endomorphism of $\mathbb{C}\mathbb{P}^n$.

Definition 1.3.9. Let $F : \mathbb{C}^n \rightarrow \mathbb{C}^n$ be a regular polynomial map. The *post-critical set* of F is the union of the images of C_F under iterations of F ,

$$PC(F) := \bigcup_{j \geq 1} F^{\circ j}(C_F).$$

We can characterize, as we show in the following result, the post-critically algebraic regular polynomial maps by the algebraicity of its post-critical set.

Lemma 1.3.10. *Let $F : \mathbb{C}^n \rightarrow \mathbb{C}^n$ be a regular polynomial map of degree $d \geq 2$. The map F is post-critically algebraic if and only if the post-critical set $PC(F)$ of F is a codimension one algebraic set of \mathbb{C}^n .*

Proof. Set $F = (P_1, \dots, P_n)$ with $P_i \in \mathbb{C}[x_1, \dots, x_n]$. Denote by $G = (Q_0, \dots, Q_n) : \mathbb{C}^{n+1} \rightarrow \mathbb{C}^{n+1}$ the homogenization of F , i.e. $Q_0 = x_0^d$ and

$$Q_i = x_0^d P_i \left(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0} \right) \quad \forall i \in \{1, \dots, n\}.$$

Denote by g the endomorphism of $\mathbb{C}\mathbb{P}^n$ induced by G . We first show that the critical set C_g of g is the union of the projective closure of C_F with the hyperplane $\{x_0 = 0\}$.

For every $i, j \in \{1, \dots, n\}$, we have

$$\frac{\partial Q_i}{\partial x_j} = x_0^{d-1} \frac{\partial P_i}{\partial x_i} \left(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0} \right).$$

In other words, $\frac{\partial Q_i}{\partial x_j}$ is the homogenization of $\frac{\partial P_i}{\partial x_j}$ for every $i, j \in \{1, \dots, n\}$. Let J_F be the Jacobian determinant of F at (x_1, \dots, x_n) and J_Q be the Jacobian determinant of Q at (x_0, x_1, \dots, x_n) . Let H be the homogenization of J_F . Then,

$$J_Q = dx_0^{d-1} \cdot H.$$

Consequently,

$$C_g = \{[x_0 : x_1 : \dots : x_n] \in \mathbb{C}\mathbb{P}^n \mid x_0 = 0 \text{ or } H(x_0, x_1, \dots, x_n) = 0\}.$$

Since J_F is a defining polynomial of C_F , the polynomial H , which is the homogenization of J_F , is a defining polynomial of the projective closure of C_F . Therefore,

$$C_g = \{x_0 = 0\} \cup \overline{C_F}^{proj}.$$

If we identify \mathbb{C}^n with the open set $U_0 = \{x_0 \neq 0\} \subset \mathbb{C}\mathbb{P}^n$, the set U_0 is invariant under g , i.e. $g(U_0) \subseteq U_0$, and the restriction of g to U_0 is precisely F . Thus

$$PC(g) = \{x_0 = 0\} \cup \overline{PC(F)}^{proj}$$

where $\overline{PC(F)}^{proj}$ is the closure with respect to Zariski topology of $PC(F)$ in $\mathbb{C}\mathbb{P}^n$. By definition, F is post-critically algebraic if and only if $PC(g)$ is an algebraic set of codimension one in $\mathbb{C}\mathbb{P}^n$. Hence, F is post-critically algebraic if and only if $PC(F)$ is an algebraic set of codimension one in \mathbb{C}^n . \square

The following example is the generalization of Example 1.3.2.

Example 1.3.11. The simplest example of post-critically algebraic regular polynomial map is the map $\mathfrak{p}_d : \mathbb{C}^n \rightarrow \mathbb{C}^n$,

$$\mathfrak{p}_d(x_1, \dots, x_n) := (x_1^d, \dots, x_n^d) \quad \text{with } d \geq 2.$$

According to Lemma 1.3.10, for every $d \geq 2$ and $n \geq 1$, \mathfrak{p}_d is post-critically algebraic regular polynomial map. The post-critical set of \mathfrak{p}_d is the union of n coordinates hyperplanes $\{x_i = 0\}$ of \mathbb{C}^n . In fact, when $n \geq 2$, the map \mathfrak{p}_d induces an endomorphism

$\mathbf{p}_d : \mathbb{C}\mathbb{P}^{n-1} \rightarrow \mathbb{C}\mathbb{P}^{n-1}$, which is also a post-critically algebraic endomorphism.

Every periodic point of period p of \mathbf{p}_d is of the form $(\zeta_1, \dots, \zeta_n)$ where ζ_i is either 0 or a $(d^p - 1)$ -th root of unity. Hence, every eigenvalue of \mathbf{p}_d at a periodic point of period p is either 0 or of modulus d^p , i.e. either superattracting or repelling.

Such an example is particularly interesting since it induces a post-critically algebraic endomorphism of $\mathbb{C}\mathbb{P}^{n-1}$ whose the complement of the post-critical set is not hyperbolic. More precisely, the post-critical set of $\mathbf{p}_d : \mathbb{C}\mathbb{P}^{n-1} \rightarrow \mathbb{C}\mathbb{P}^{n-1}$ is the union of n projective hyperplanes $\Gamma_i := \{[x_1 : \dots : x_n] \in \mathbb{C}\mathbb{P}^{n-1} \mid x_i = 0\}$, $1 \leq i \leq n$. The following non-trivial holomorphic map $\varphi : \mathbb{C} \rightarrow \mathbb{C}\mathbb{P}^{n-1}$,

$$\varphi(t) = [e^t : 1 : \dots : 1].$$

satisfies that $\varphi(\mathbb{C}) \cap \Gamma_i = \emptyset$ for every $i \in \{1, \dots, n\}$, which implies that $\varphi(\mathbb{C}) \cap PC(\mathbf{p}_d) = \emptyset$. According to [Kob13, Proposition 3.6.1], $\mathbb{C}\mathbb{P}^{n-1} \setminus PC(\mathbf{p}_d)$ is not hyperbolic. In particular, we cannot apply [FS94, Theorem 6.1]. However, we can still compute explicitly the eigenvalues of \mathbf{p}_d at a periodic cycle. It follows from a similar computation as the one with \mathbf{p}_d that each eigenvalues of \mathbf{p}_d at a periodic cycle of period p is either 0 or d^p .

The previous example can be generalized to the class of polynomial maps which are products of post-critically finite polynomials. More precisely, assume $n \geq 2$ and let $f_i \in \mathbb{C}[x_i]$, $i \in \{1, \dots, n\}$ be n polynomials of the same degree $d \geq 2$. Set $F := (f_1, \dots, f_n) : \mathbb{C}^n \rightarrow \mathbb{C}^n$,

$$F(x_1, \dots, x_n) = (f_1(x_1), \dots, f_n(x_n)).$$

Since f_i , $1 \leq i \leq n$ are of the same degree $d \geq 2$, we can write

$$f_i(x_i) = a_i x_i^d + b_{d-1} x_i^{d-1} + b_{d-2} x_i^{d-2} + \dots, a_i \neq 0 \quad \forall i \in \{1, \dots, n\}.$$

The map $H = (a_1 x_1^d, \dots, a_n x_n^d) : \mathbb{C}^n \rightarrow \mathbb{C}^n$ satisfies that $H^{-1}(0) = \{0\}$. According to Lemma 1.3.7, $F = (f_1, \dots, f_n)$ is regular. The following result shows F is post-critically algebraic when f_i are post-critically finite. Moreover, we can control eigenvalues of such maps thanks to one-dimensional results.

Proposition 1.3.12. *Assume $n \geq 2$. Let $f_i \in \mathbb{C}[x_i]$, $i \in \{1, \dots, n\}$ be n post-critically finite polynomials of the same degree $d \geq 2$ and set $F := (f_1, \dots, f_n) : \mathbb{C}^n \rightarrow \mathbb{C}^n$,*

$$F(x_1, \dots, x_n) = (f_1(x_1), \dots, f_n(x_n)).$$

Let g be the endomorphism of $\mathbb{C}\mathbb{P}^n$ induced by the homogenization of F . Then g is post-critically algebraic. Moreover, if λ is an eigenvalue of g at a periodic cycle, then either $\lambda = 0$ or $|\lambda| > 1$.

Proof. First, we show that F is post-critically algebraic. The Jacobian matrix of F at a point $z \in \mathbb{C}^n$ is of the form

$$\begin{pmatrix} f'_1(z_1) & 0 & \dots & 0 \\ 0 & f'_2(z_2) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & f'_n(z_n) \end{pmatrix}$$

Hence, $C_F = \bigcup_{1 \leq i \leq n} \mathbb{C}^{i-1} \times C_{f_i} \times \mathbb{C}^{n-i}$, whence

$$PC(F) = \bigcup_{i \in \{1, \dots, n\}} \mathbb{C}^{i-1} \times PC(f_i) \times \mathbb{C}^{n-i}.$$

Since f_i is post-critically finite for every $i \in \{1, \dots, n\}$, i.e. $PC(f_i)$ is finite, $PC(F)$ is a finite union of hypersurfaces. According to Lemma 1.3.10, F is post-critically algebraic, hence by definition, so is g .

Second, we show that each eigenvalue of g at a periodic cycle is either superattracting or repelling. Denote by $[x_0 : x_1 : \dots : x_n]$ the homogeneous coordinates of $\mathbb{C}\mathbb{P}^n$. Observe that if we identify \mathbb{C}^n with the open set $\{x_0 \neq 0\} \subset \mathbb{C}\mathbb{P}^n$, then \mathbb{C}^n and $\Gamma = \{x_0 = 0\} \subset \mathbb{C}\mathbb{P}^n$ are invariant under g and the restriction of g to \mathbb{C}^n is F .

Let $z = (z_1, \dots, z_n) \in \mathbb{C}^n$ be a periodic point of period p of g . Since the restriction of g to \mathbb{C}^n is F , it is enough to consider z as a periodic point of F . Since F acts coordinates-wise, for every $i \in \{1, \dots, n\}$, z_i is a periodic point of f_i of period p_i of multiplier λ_i . Moreover, p is the lowest common multiple of $\{p_i, 1 \leq i \leq n\}$. Let λ be an eigenvalue of F at z , i.e. λ is an eigenvalue of $D_z F^{op}$. Since the Jacobian matrix of F is diagonal, there exists $i \in \{1, \dots, n\}$ such that

$$\lambda = \lambda_i^{\frac{p}{p_i}}.$$

Since f_i is post-critically finite, λ_i is either superattracting or repelling. Whence, so is λ .

To conclude, let $z \in \Gamma = \{x_0 = 0\}$ be a periodic point of g . Observe that $T_z \Gamma$ is an

invariant subspace of $D_z g$ and that

$$\text{Spec } D_z g = \{0\} \cup \text{Spec } D_z g|_{T_z \Gamma}.$$

It is enough to consider $\text{Spec } D_z g|_{T_z \Gamma}$. The map restricted to Γ by g is of the form

$$[0 : x_1 : \dots : x_n] \mapsto [0 : a_1 x_1^d : \dots : a_n x_n^d].$$

In other words, the restriction to Γ of g is conjugate to the endomorphism \mathbf{p}_d of $\mathbb{C}\mathbb{P}^{n-1}$, which is induced by the map \mathbf{p}_d studied in Example 1.3.11. Consequently, each eigenvalue of $\text{Spec } D_z g|_{T_z \Gamma}$ is either superattracting or repelling. \square

We will conclude this paragraph by computing the eigenvalues at periodic cycles of an example of post-critically algebraic regular polynomial map which is not a product.

Example 1.3.13. Given an integer $d \geq 2$, consider the map $f : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ defined by

$$f : (x_1, x_2) \mapsto (-x_1^d + x_2^{d-1}, x_2^d).$$

The critical set C_f has two irreducible components, $\{x_2 = 0\}$ and $\{x_1 = 0\}$. The critical curve $\{x_2 = 0\}$ is invariant. The critical curve $\{x_1 = 0\}$ is mapped to the singular curve $\{x_1^d - x_2^{d-1} = 0\}$ and the map f maps $\{x_1^d - x_2^{d-1} = 0\}$ back to $\{x_1 = 0\}$. This class of post-critically algebraic maps was introduced in [Ron08], as example of post-critically algebraic maps whose post-critical sets have a periodic singular irreducible components.

We can show that, for every $d \geq 2$, every eigenvalue at periodic points of f is either superattracting or repelling. Let (z, w) be a periodic point of period m of f . Note that w is a periodic point of the polynomial $p_d(t) = t^d \in \mathbb{C}[t]$. The line $L := \mathbb{C} \times \{x_2 = w\}$ is invariant by $f^{\circ m}$. Moreover, the map induced by f on L is conjugate to a post-critically finite polynomial. Hence, the eigenvalue λ_1 of $D_q f^{\circ m}$ with associated eigenvectors tangent to L is either superattracting or repelling. If $D_q f^{\circ m}$ is not diagonalizable, then we are done.

Assume that $D_q f^{\circ m}$ is diagonalizable with two distinct eigenvalues $\lambda_1 \neq \lambda_2$, i.e. $\text{Spec } D_q f^{\circ m} = \{\lambda_1, \lambda_2\}$. In particular, there exists an eigenvector (u, v) of $D_q f^{\circ m}$ associated to λ_2 such that $(u, v) \notin T_w L$, i.e. $v \neq 0$. The Jacobian matrix of $f^{\circ m}$ at (z, w) is of the form

$$\begin{pmatrix} \lambda_1 & \alpha \\ 0 & (p_d^{\circ m})'(w) \end{pmatrix} \quad \alpha \in \mathbb{C}.$$

Since $v \neq 0$, $\lambda_2 = (p_d^{\circ m})'(w)$, i.e. λ_2 is the multiplier of p_d at a periodic point w .

According to Example 1.3.2, λ_2 is either superattracting to repelling.

Maps defined by a semiconjugacy

We discuss post-critically algebraic maps (endomorphisms of $\mathbb{C}\mathbb{P}^n$ or regular polynomial maps) which are semiconjugate to post-critically algebraic regular polynomials. These maps usually have post-critical sets with singular irreducible components. In general, the eigenvalues at periodic cycles of such post-critically algebraic endomorphisms are not completely characterized. Except for the class of symmetric products for which we can compute explicitly the eigenvalues, we shall see that we can only control the eigenvalues of the cycles outside the post-critical set.

Example 1.3.14 (Symmetric products). Symmetric products have been used to produce examples of endomorphisms of $\mathbb{C}\mathbb{P}^n$ with certain dynamical properties when $n \geq 2$. Their dynamics is rather simple and quite well understood (see [SSC⁺10, Ued93, GHK]). In particular, we shall see that symmetric products of post-critically finite rational maps are post-critically algebraic and every eigenvalue of a symmetric product at a periodic point is either superattracting or repelling.

Given $n \geq 2$, the symmetric group \mathfrak{S}_n defines an action on $(\mathbb{C}\mathbb{P}^1)^n$ by permuting the coordinates. The quotient space is isomorphic to $\mathbb{C}\mathbb{P}^n$ (see [Maa05, Corollary 2.6]). Denote by $\eta_n : (\mathbb{C}\mathbb{P}^1)^n \rightarrow \mathbb{C}\mathbb{P}^n$ the quotient map. Let $f : \mathbb{C}\mathbb{P}^1 \rightarrow \mathbb{C}\mathbb{P}^1$ be a rational map. The n -symmetric product of f is an endomorphism $S_f : \mathbb{C}\mathbb{P}^n \rightarrow \mathbb{C}\mathbb{P}^n$ such that the following diagram commutes.

$$\begin{array}{ccc} (\mathbb{C}\mathbb{P}^1)^n & \xrightarrow{(f, \dots, f)} & (\mathbb{C}\mathbb{P}^1)^n \\ \downarrow \eta_n & & \downarrow \eta_n \\ \mathbb{C}\mathbb{P}^n & \xrightarrow{S_f} & \mathbb{C}\mathbb{P}^n \end{array}$$

The following lemma shows that the n -symmetric product of a post-critically finite rational map is post-critically algebraic.

Lemma 1.3.15 (Lemma 3.1, [GHK]). *Let f be a rational map and let S_f be the n -*

symmetric product of f . The post-critical set of S_f is

$$PC(S_f) = \eta_n \left(\bigcup_{1 \leq j \leq n} (\mathbb{CP}^1)^{j-1} \times PC(f_j) \times (\mathbb{CP}^1)^{n-j} \cup \bigcup_{1 \leq i < j \leq n} \Omega_{i,j} \right)$$

where $\Omega_{i,j} = \{(x_1, \dots, x_n) \in (\mathbb{CP}^1)^n \mid f(x_i) = f(x_j)\}$. In particular, f is post-critically finite if and only if the n -symmetric product of f is post-critically algebraic.

The periodic points of S_f are closely related to those of f . In fact, let $z = \eta_n(z_1, \dots, z_n)$ be a periodic point of period p of S_f , where $z_i \in \mathbb{CP}^1$ for $i \in \{1, \dots, n\}$. Then,

$$\eta_n(z_1, \dots, z_n) = S_f^{\circ p}(\eta_n(z_1, \dots, z_n)) = \eta_n(f^{\circ p}(z_1), \dots, f^{\circ p}(z_n)).$$

This means that there exists a permutation $\sigma \in \mathfrak{S}_n$ such that for every $i \in \{1, \dots, n\}$, $z_{\sigma(i)} = f^{\circ p}(z_i)$. Since every element of \mathfrak{S}_n has finite order, there exists an integer $m \geq 1$ such that for every $i \in \{1, \dots, n\}$, $z_i = f^{\circ mp}(z_i)$. In other words, for every $i \in \{1, \dots, n\}$, z_i is a periodic point of f . Conversely, assume that z_1, \dots, z_n are periodic points of f of period p_1, \dots, p_n respectively. Let $p = \text{lcm}(p_1, \dots, p_n)$ be the lowest common multiplier of p_1, \dots, p_n . Then

$$S_f^{\circ p} \circ \eta_n(z_1, \dots, z_n) = \eta_n(f^{\circ p}(z_1), \dots, f^{\circ p}(z_n)) = \eta_n(z_1, \dots, z_n),$$

i.e. $\eta_n(z_1, \dots, z_n)$ is a periodic point of S_f . Hence, the set periodic points of S_f is precisely the set of points of the form $\eta_n(z_1, \dots, z_n)$ where z_i are periodic points of f .

The following result show the eigenvalues of S_f at a periodic point, which can be computed in terms of the multipliers of f at its periodic points.

Proposition 1.3.16 (Proposition 2.3 [GHK]). *Let $f : \mathbb{CP}^1 \rightarrow \mathbb{CP}^1$ be a rational map and S_f be the n -symmetric product of f . Assume that z_1, \dots, z_n are n periodic points of f and that $\eta_n(z_1, \dots, z_n)$ is a periodic point of period p of S_f . Let $P \in \mathbb{C}[t]$ be the characteristic polynomial of $D_{\eta_n(z_1, \dots, z_n)} S_f^{\circ p}$.*

- If a periodic point of multiplier λ of f occurs m times in (z_1, \dots, z_n) , then P is divisible by

$$(t - \lambda)(t - \lambda^2) \dots (t - \lambda^m).$$

- If applying η_n to (z_1, \dots, z_n) results in a periodic orbit of length ml of multiplier λ for f collapsing to a periodic orbit of length l for S_f (this can occur when say the

whole orbit is contained in $\{z_1, \dots, z_n\}$), then P is divisible by

$$t^m - \lambda.$$

In particular, combining with Theorem 1.3.1, this result implies that every eigenvalue at a periodic point of a n -symmetric product of a post-critically finite rational map is either superattracting or repelling. We refer to [GHK] for further discussion about the arithmetic dynamics of symmetric products.

We will now discuss the generalization in higher dimension of Lattès maps and Chebyshev polynomials.

Example 1.3.17. [Generalized Lattès maps] An endomorphism $f : \mathbb{C}\mathbb{P}^n \rightarrow \mathbb{C}\mathbb{P}^n$ of degree $d \geq 2$ is called a *Lattès map* if there exist a ramified covering $\varphi : \mathbb{T}^n \rightarrow \mathbb{C}\mathbb{P}^n$ and an endomorphism $A : \mathbb{T}^n \rightarrow \mathbb{T}^n$ such that the following diagram commutes.

$$\begin{array}{ccc} \mathbb{T}^n & \xrightarrow{A} & \mathbb{T}^n \\ \varphi \downarrow & & \downarrow \varphi \\ \mathbb{C}\mathbb{P}^n & \xrightarrow{f} & \mathbb{C}\mathbb{P}^n \end{array},$$

i.e.

$$\varphi \circ A = f \circ \varphi.$$

where \mathbb{T}^n is a complex torus of dimension n and A is induced by an affine self-map of \mathbb{C}^n whose linear part is of the form $\sqrt{d} \cdot U$, where U is an isometry with respect a norm on \mathbb{C}^n (see [Dup03]). The dynamics of Lattès maps has been deeply studied in several papers (see [BL98, BL01, Din01, Dup03, BD05, GHK, Bie19]). In particular, in [GHK], the authors showed that the n -symmetric products of Lattès rational maps are higher dimensional Lattès maps.

Regarding eigenvalues at periodic points, it is open whether Conjecture A is true for Lattès maps, that is, whether every eigenvalue of a Lattès map along a periodic cycle is either superattracting or repelling? When $n = 2$, in [Ron10], Rong gave the classification of Lattès maps and they satisfy the assumption in [Jon98, Proposition 2.9]. This means that the critical set of a Lattès map does not have periodic irreducible component. In particular, every periodic cycles of a Lattès map of $\mathbb{C}\mathbb{P}^2$ is repelling. We shall see in Chapter 3 that we can conclude the case $n = 2$ for every post-critically algebraic maps. In general, it is also an open question that whether the complement of post-critical set of a Lattès map is Kobayashi hyperbolic, hence we can not apply [FS94, Theorem 6.1]

to control the eigenvalues of a Lattès map at a periodic cycle outside the post-critical set. However, thanks to the semiconjugacy with an affine map, we can show that every eigenvalue of a Lattès map at a periodic point outside an algebraic set containing the ramification locus of φ is repelling. For the convenience of the readers, we also recall the proof that Lattès maps are post-critically algebraic.

Proposition 1.3.18. *Let $f : \mathbb{C}\mathbb{P}^n \rightarrow \mathbb{C}\mathbb{P}^n$ be a Lattès map of degree $d \geq 2$. Then f is post-critically algebraic. Moreover, there exists an algebraic set V of codimension one containing $PC(f)$ such that every eigenvalue of f at a periodic point outside V is repelling.*

Proof. Let $\varphi : \mathbb{T}^n \rightarrow \mathbb{C}\mathbb{P}^n$ be a ramified covering and $A : \mathbb{T}^n \rightarrow \mathbb{T}^n$ be an affine map, whose linear part is $\sqrt{d} \cdot U$ with U is an isometry, such that

$$\varphi \circ A = f \circ \varphi. \quad (1.3.1)$$

Denote by V_φ the set of critical values of φ . Since φ is ramified covering which is not a covering, the critical value set V_φ is non-empty. Hence, V_φ is an algebraic set of codimension one of $\mathbb{C}\mathbb{P}^n$. Observe that

$$V_f \cup f(V_\varphi) \subset V_\varphi.$$

Indeed, assume $x \in \mathbb{C}\mathbb{P}^n$ and $y \in \varphi^{-1}(x)$. Deriving both sides of (1.3.1) at y , we have $D_{A(y)}\varphi \circ (\sqrt{d} \cdot U) = D_x f \circ D_y \varphi$ since the linear part of A is $\sqrt{d} \cdot U$. Note that $\text{rank}(\sqrt{d} \cdot U) = n$. Hence,

$$\text{rank } D_{A(y)}\varphi = \text{rank}(D_x f \circ D_y \varphi).$$

If $x \in C_f$, i.e. $\text{rank } D_x f < n$, then $\text{rank } D_{A(y)}\varphi < n$. This means that $A(y)$ is a critical point of φ . In particular, $\varphi \circ A(y) \in V_\varphi$. Thus, $f \circ \varphi(y) = \varphi \circ A(y) \in V_\varphi$. Hence, $f(x) \in V_\varphi$ and

$$V_f = f(C_f) \subset V_\varphi.$$

If $x \in V_\varphi$, then we can choose a point $y \in \varphi^{-1}(x)$ such that $\text{rank } D_y \varphi < n$. Hence, $\text{rank } D_{A(y)}\varphi = \text{rank}(D_x f \circ D_y \varphi) < n$. Similarly, we deduce that $f(x) \in V_\varphi$, i.e.

$$f(V_\varphi) \subset V_\varphi.$$

Therefore, we deduce that $V_f \cup f(V_\varphi) \subset V_\varphi$. Consequently, $f(V_f) \subset f(V_\varphi) \subset V_\varphi$.

Inductively, we deduce that, for every $j \geq 0$, $f^{\circ j}(V_f) \subset V_\varphi$. Hence

$$PC(f) = \bigcup_{j \geq 0} f^{\circ j}(V_f) \subset V_\varphi.$$

Since V_φ is an algebraic set of codimension one, f is post-critically algebraic.

We shall now show that each eigenvalue of f at a periodic point outside V_φ is repelling. Since $f^{\circ j}$ is also a Lattès map for every $j \geq 1$, it is enough to consider eigenvalues at fixed points. Assume z is a fixed point of f and let λ be an eigenvalue of $D_z f$. Then (1.3.1) implies that $A(\varphi^{-1}(z)) \subset \varphi^{-1}(z)$. Since φ has finite order, there exist a point $w \in \varphi^{-1}(z)$ and an integer $p \geq 1$ such that $A^{\circ p}(w) = w$. Deriving the equality $\varphi \circ A^{\circ p} = f^{\circ p} \circ \varphi$ at w , we have

$$D_w \varphi \circ (\sqrt{d} \cdot U)^p = D_z f^{\circ p} \circ D_w \varphi$$

If $z \notin V_\varphi$, $D_w \varphi$ is invertible. Hence, $(\sqrt{d} \cdot U)^p$ and $D_z f^{\circ p}$ are conjugate. Consequently, λ^p is an eigenvalue of $(\sqrt{d} \cdot U)^p$. Since U is unitary,

$$|\lambda^p| = d^{\frac{p}{2}},$$

i.e. $\lambda = \sqrt{d^p} > 1$. Hence, λ is repelling. □

Example 1.3.19. [Generalized Chebyshev polynomials]³ Set $p_2 : \mathbb{C}^2 \rightarrow \mathbb{C}^2$,

$$p_2(t_1, t_2) = (t_1^2, t_2^2)$$

and $\psi : (\mathbb{C}^*)^2 \rightarrow (\mathbb{C}^*)^2$,

$$(x, y) = \psi(t_1, t_2) = \left(t_1 + t_2 + \frac{1}{t_1 t_2}, \frac{1}{t_1} + \frac{1}{t_2} + t_1 t_2 \right).$$

The polynomial map $P : \mathbb{C}^2 \rightarrow \mathbb{C}^2$,

$$P(x, y) = (x^4 - 4x^2 y + 2y^2 + 4x, y^4 - 4xy^2 + 2x^2 + 4y)$$

is called a *generalized Chebyshev polynomial*. The polynomial P satisfies the following relation

$$\psi \circ p_2 = P \circ \psi, \tag{1.3.2}$$

³It is worth noting that some Chebyshev maps and Lattès maps belong to another class of maps which preserve an algebraic web. In dimension 2, this class of maps was studied in [DJ10]

i.e. it makes the following diagram commuting.

$$\begin{array}{ccc} (\mathbb{C}^*)^2 & \xrightarrow{P^2} & (\mathbb{C}^*)^2 \\ \psi \downarrow & & \downarrow \psi \\ \mathbb{C}^2 & \xrightarrow{P} & \mathbb{C}^2 \end{array}$$

The generalization of Chebyshev polynomials to higher dimension was considered in [Ves87, Ves91, Wit88, HW88]. Roughly speaking, given two integers k and n , a generalized Chebyshev polynomial is a polynomial map $P_k : \mathbb{C}^n \rightarrow \mathbb{C}^n$ of \mathbb{C}^n such that $P_k(\Phi(x)) = \Phi(kx)$, where Φ is a map from some complex Lie group to \mathbb{C}^n . In some cases, like the map P above, the map P_k is a post-critically algebraic regular polynomial. Dynamics of the map P have been studied in [Uch09]. The analogue map defined on \mathbb{C}^3 is studied in [Uch17]. We refer to [Wit88] for an exposition of the construction.

The critical set of P has two irreducible components, $\{xy - 1 = 0\}$ and $\{x^2y^2 - 2x^3 - 2y^3 + 4xy - 1 = 0\}$. They are all mapped onto a singular curve $V = \{x^2y^2 - 4x^3 - 4y^3 + 18xy - 27 = 0\}$, which is invariant by P . Thus, P is post-critically algebraic and the post-critical set of P has a unique irreducible singular component V . Moreover, $\psi : (\mathbb{C}^*)^2 \setminus \psi^{-1}(V) \rightarrow \mathbb{C}^2 \setminus V$ is a covering of order 6 (see [Uch09, p.997]).

We now show that every periodic cycle of P which is outside its post-critical set V is repelling. Let us first consider a fixed point $z \notin V$ of P . The equality (1.3.2) implies that $p_2(\psi^{-1}(z)) \subset \psi^{-1}(z)$. Since $\psi : (\mathbb{C}^*)^2 \setminus V \rightarrow \mathbb{C}^2 \setminus V$ is a covering of order 6, $\psi^{-1}(z)$ is a finite set of cardinality 6, and hence $p_2^{\circ 6}$ fixes every point in $\psi^{-1}(z)$. Consequently, we can choose $w \in \psi^{-1}(z)$ such that $p_2^{\circ 6}(w) = (w)$, $\psi(w) = z$ and ψ is an isomorphism in a neighborhood of w . Differentiating both sides of the equality $\psi \circ p_2^{\circ 6} = P^{\circ 6} \circ \psi$ at w , we have

$$D_w\psi \circ D_w p_2^{\circ 6} = D_z P^{\circ 6} \circ D_w\psi.$$

Thus, $D_w p_2^{\circ 6}$ and $D_z P^{\circ 6}$ are conjugate. Let λ be an eigenvalue of $D_z P$. Then λ^6 is an eigenvalue of $D_z P^{\circ 6}$. Therefore, λ^6 is an eigenvalue of $D_w p_2^{\circ 6}$. Note that w does not belong to the post-critical set of p_2 . Thus, according to Example 1.3.11, λ^6 is repelling, hence so is λ . If $z \notin V$ is a periodic point of period p of P , then z is a fixed point of $P^{\circ p}$. Since $P^{\circ p}$ is semiconjugate to $p_2^{\circ p}$ via ψ , by similar arguments as the above, every eigenvalue of $D_z P^{\circ p}$ is repelling.

Chapter 2

Eigenvalues at fixed points of Koch maps

2.1 A family of post-critically algebraic endomorphisms

In Chapter 1, we saw that one can produce examples of post-critically algebraic endomorphisms of $\mathbb{C}\mathbb{P}^n$ with $n \geq 2$ by using products of post-critically finite rational maps or by using semiconjugacy. In [Koc13], Sarah Koch introduced a systematic way to produce post-critically algebraic endomorphisms of $\mathbb{C}\mathbb{P}^n$, $n \geq 2$. In this chapter, we study the eigenvalues of the following class of such post-critically algebraic maps at their fixed points. We refer to [Koc13] for the general definitions and construction.

Definition 2.1.1. Let $(k, m) \in \mathbb{N} \times \mathbb{N}^*$ be such that $k + m - 1 \geq 2$. The map

$$G_{k,m} : \mathbb{C}^{k+m-1} \rightarrow \mathbb{C}^{k+m-1}$$

is defined as,

- if $k = 0$, $G_{0,m} : \mathbb{C}^{m-1} \rightarrow \mathbb{C}^{m-1}$ is of the form

$$G_{0,m} : \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{m-1} \end{pmatrix} \mapsto \begin{pmatrix} -x_{m-1}^2 \\ x_1^2 - x_{m-1}^2 \\ \vdots \\ x_{m-2}^2 - x_{m-1}^2 \end{pmatrix}.$$

- if $k \neq 0$, $G_{k,m} : \mathbb{C}^{k+m-1} \rightarrow \mathbb{C}^{k+m-1}$ is of the form

$$G_{k,m} : \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{k+m-1} \end{pmatrix} \mapsto \begin{pmatrix} \left(-\frac{x_{k+m-1}+x_{k-1}}{2}\right)^2 \\ \left(x_1 - \frac{x_{k+m-1}+x_{k-1}}{2}\right)^2 \\ \vdots \\ \left(x_{k+m-2} - \frac{x_{k+m-1}+x_{k-1}}{2}\right)^2 \end{pmatrix}.$$

Given two integers $(k, m) \in \mathbb{N} \times \mathbb{N}^*$ such that $k + m - 1 \geq 2$, it follows from [Koc13, Theorem 5.17] and [Koc13, Theorem 5.18] that $G_{k,m}$ induces a post-critically algebraic endomorphism of \mathbb{CP}^{k+m-2} of degree 2. In addition, according to [Koc13, Corollary 7.2], every periodic cycles of $G_{k,m}$ which is outside the post-critical set is repelling. This means that if z is a periodic point of $G_{k,m}$ outside the post-critical set, then every eigenvalue of $G_{k,m}$ along the orbit of z is repelling. The question of what is the possible values of the eigenvalues of $G_{k,m}$ along a periodic cycle inside the post-critical set is left open.

In this chapter, we will study the eigenvalues of $G_{k,m}$ for $(k, m) \in \mathbb{N} \times \mathbb{N}^*$, $k + m - 1 \geq 2$ at its fixed point. We will check Conjecture A for this cases, i.e. we will show that each eigenvalue of $G_{k,m}$ at a fixed point either 0 or has modulus strictly bigger than 1. More precisely, we will show the following result.

Theorem A. Let $(k, m) \in \mathbb{N} \times \mathbb{N}^*$ be such that $k + m - 1 \geq 2$ and μ be an eigenvalue of $G_{k,m}$ at a fixed point. Then either $\mu = 0$ or $|\mu| > 1$.

Let z be a fixed point of $G_{k,m}$. When $z \notin PC(G_{k,m})$, this was done by [Koc13, Corollary 7.2]. In general, it turns out that there is a relation between the eigenvalues of $D_z G_{k,m}$ and the polynomial $P(t) = t^2 + z_1 \in \mathbb{C}[t]$. This relation depends on the relative position of z and $PC(G_{k,m})$ and on how $G_{k,m}$ and $G_{k',m'}$ are related for different pairs $(k, m), (k', m')$. To study more easily these objects, we introduce a partial order \preceq on $\mathbb{N} \times \mathbb{N}^*$, a dynamically equivalent family of maps, that we denote by

$$\{F_{k,m} : \mathcal{M}_{k,m} \rightarrow \mathcal{M}_{k,m}, (k, m) \in \mathbb{N} \times \mathbb{N}^*\},$$

where $\mathcal{M}_{k,m}$ is a subspace of the vector spaces of complex sequences $\mathbb{C}^{\mathbb{N}^*}$. We shall use these objects to show in Theorem A' a complete characterization of the eigenvalues of $G_{k,m}$ at its fixed points. The rest of this chapter is devoted to the constructions of $F_{k,m}$ and the proof of Theorem A.

2.2 An alternative construction

2.2.1 Construction of $F_{k,m}$ and $\mathcal{M}_{k,m}$

Assume $(k, m) \in \mathbb{N} \times \mathbb{N}^*$. Denote by $\mathcal{E} = \mathbb{C}^{\mathbb{N}^*}$ the vector space of complex sequences $\mathbf{x} = (x_i)_{i \geq 1}$. Set $\mathbf{0} = (0, 0, \dots)$. Let $\mathcal{L} \subset \mathcal{E}$ be the one-dimensional subspace consisting of constant sequences and $\mathcal{H}_{k,m} \subset \mathcal{E}$ be the hyperspace defined by

$$\mathcal{H}_{k,m} := \{\mathbf{x} \in \mathcal{E} \mid x_k + x_{k+m} = 0\}$$

with the convention $x_0 := 0$.

Lemma 2.2.1. *For every $(k, m) \in \mathbb{N} \times \mathbb{N}^*$, we have $\mathcal{E} = \mathcal{H}_{k,m} \oplus \mathcal{L}$.*

Proof. On one hand, given $\mathbf{x} \in \mathcal{E}$, define $\mathbf{y} \in \mathcal{E}$ by

$$y_i := x_i - \kappa \text{ with } \kappa = \begin{cases} x_m & \text{if } k = 0 \\ \frac{x_k + x_{k+m}}{2} & \text{if } k \geq 1 \end{cases}.$$

Then $y_k + y_{k+m} = 0$ hence $\mathbf{y} \in \mathcal{H}_{k,m}$. Note that $\mathbf{x} - \mathbf{y} \in \mathcal{L}$ hence

$$\mathcal{E} = \mathcal{H}_{k,m} + \mathcal{L}.$$

On the other hand, assume $\mathbf{x} \in \mathcal{H}_{k,m} \cap \mathcal{L}$. Then $x_{k+m} + x_k = 0$ and $x_{k+m} = x_k$, whence $x_k = x_{k+m} = 0$. Since \mathbf{x} is a constant sequence, it vanishes identically, that is

$$\mathcal{H}_{k,m} \cap \mathcal{L} = \{\mathbf{0}\}. \quad \square$$

Denote by

$$\pi_{k,m} : \mathcal{E} \rightarrow \mathcal{E}$$

the projection to $\mathcal{H}_{k,m}$ parallel to \mathcal{L} . Consider the map $\mathcal{Q} : \mathcal{E} \rightarrow \mathcal{E}$ defined by

$$\mathcal{Q}(\mathbf{x}) = \mathbf{y} \quad \text{with} \quad y_1 := 0 \text{ and } y_i = x_{i-1}^2, i \geq 1.$$

Definition 2.2.2. Given $(k, m) \in \mathbb{N} \times \mathbb{N}^*$, the *moduli map of type (k, m)* is the map

$$F_{k,m} := \pi_{k,m} \circ \mathcal{Q} : \mathcal{E} \rightarrow \mathcal{E}.$$

Characterizations of $F_{k,m}$

Lemma 2.2.3. *Given $(k, m) \in \mathbb{N} \times \mathbb{N}^*$, for every $\mathbf{x}, \mathbf{y} \in \mathcal{E}$, we have*

$$F_{k,m}(\mathbf{x}) = \mathbf{y} \Leftrightarrow \begin{cases} y_{k+m} + y_k = 0 \\ y_i = x_{i-1}^2 + y_1 \quad \text{for all } i \geq 2. \end{cases}$$

In particular,

$$y_1 = \begin{cases} -x_{m-1}^2 & \text{if } k = 0, \\ -\frac{x_{k-1}^2 + x_{k+m-1}^2}{2} & \text{if } k \geq 1. \end{cases}$$

Proof. Assume $\mathbf{x} \in \mathcal{E}$ and $\mathbf{y} = \pi_{k,m}(\mathcal{Q}(\mathbf{x})) = F_{k,m}(\mathbf{x})$. On the one hand, $\mathbf{y} \in \pi_{k,m}(\mathcal{E}) = \mathcal{H}_{k,m}$, i.e. $y_k + y_{k+m} = 0$. On the other hand, set $\mathbf{z} = \mathcal{Q}(\mathbf{x})$, i.e. $z_1 = 0$ and $z_i = x_{i-1}^2$ for all $i \geq 1$. Then for all $i \geq 1$,

$$y_i = z_i - \kappa \text{ with } \kappa := \begin{cases} z_m & \text{if } k = 0 \\ \frac{z_k + z_{k+m}}{2} & \text{if } k \geq 1 \end{cases}.$$

In particular,

$$y_1 = z_1 - \kappa = -\kappa = \begin{cases} -x_{m-1}^2 & \text{if } k = 0, \\ -\frac{x_{k-1}^2 + x_{k+m-1}^2}{2} & \text{if } k \geq 1. \end{cases}$$

whence for all $i \geq 2$, $y_i = z_i + y_1$, i.e. $y_i = x_{i-1}^2 + y_1$.

Conversely, assume $\mathbf{x}, \mathbf{y} \in \mathcal{E}$ such that $y_k + y_{k+m} = 0$ and for all $i \geq 2$, $y_i = x_{i-1}^2 + y_1$. In particular, $\mathbf{y} \in \mathcal{H}_{k,m}$. Set $\mathbf{z} = \mathbf{y} - \mathcal{Q}(\mathbf{x})$. Then for all $i \geq 2$,

$$z_i = y_i - x_{i-1}^2 = y_1$$

hence $\mathbf{z} \in \mathcal{L}$. In other words, $\mathbf{y} = \pi_{k,m}(\mathcal{Q}(\mathbf{x})) = F_{k,m}(\mathbf{x})$. □

Although $F_{k,m}$ is defined on a vector space of infinite dimension, we will now see that the dynamics of $F_{k,m}$ is captured entirely by some finite dimensional vector space. Given $(k, m) \in \mathbb{N} \times \mathbb{N}^*$, let $\mathcal{P}_{k,m} \subset \mathcal{E}$ be the subspace of preperiodic sequences of preperiod at most k to a cycle of period dividing m , i.e.

$$\mathcal{P}_{k,m} := \{\mathbf{x} \in \mathcal{E} \mid x_{i+m} = x_i \text{ for } i \geq k+1\}.$$

Since sequences in $\mathcal{P}_{k,m}$ are uniquely determined by the first $k+m$ entries, the vector space $\mathcal{P}_{k,m}$ has finite dimension $k+m$.

Definition 2.2.4. Given $(k, m) \in \mathbb{N} \times \mathbb{N}^*$, the space $\mathcal{M}_{k,m} := \mathcal{P}_{k,m} \cap \mathcal{H}_{k,m}$ is called *the moduli space of type (k, m)* .

Note that the constant sequence $(1, 1, \dots)$ is in $\mathcal{P}_{k,m} \setminus \mathcal{H}_{k,m}$ and that $\mathcal{H}_{k,m}$ has codimension one in \mathcal{E} . Hence, $\mathcal{M}_{k,m}$ is a vector space of dimension $k + m - 1$. The following two lemmas show the importance of $\mathcal{M}_{k,m}$.

Lemma 2.2.5. *We have $F_{k,m}(\mathcal{M}_{k,m}) = \mathcal{M}_{k,m}$ and $F_{k,m} : \mathcal{M}_{k,m} \rightarrow \mathcal{M}_{k,m}$ is a nondegenerate homogeneous map of degree 2.*

Proof. Let us first prove that $\mathcal{Q}(\mathcal{M}_{k,m}) \subseteq \mathcal{P}_{k,m}$. Assume $\mathbf{x} \in \mathcal{M}_{k,m}$. Since $\mathbf{x} \in \mathcal{P}_{k,m} \cap \mathcal{H}_{k,m}$, \mathbf{x} has preperiod k and period dividing m and $x_{k+m} = -x_m$. In particular, $x_{k+m}^2 = x_m^2$. Consequently, setting $\mathbf{y} := \mathcal{Q}(\mathbf{x})$,

$$\begin{cases} y_{k+1} = x_k^2 + y_1 = x_{k+m}^2 + y_1 = y_{k+m+1} \\ y_i = x_{i-1}^2 + y_1 = x_{m+i-1}^2 + y_1 = y_{i+m} \quad \text{for all } i \geq k+2. \end{cases}$$

Thus $\mathcal{Q}(\mathbf{x}) \in \mathcal{P}_{k,m}$

Since $\pi_{k,m}(\mathcal{P}_{k,m}) \subset \mathcal{P}_{k,m}$, we deduce that $F_{k,m}(\mathcal{M}_{k,m}) \subset \mathcal{P}_{k,m}$. Since $F_{k,m}(\mathcal{E}) \subset \mathcal{H}_{k,m}$,

$$F_{k,m}(\mathcal{M}_{k,m}) \subset \mathcal{P}_{k,m} \cap \mathcal{H}_{k,m} = \mathcal{M}_{k,m}.$$

Clearly, the map \mathcal{Q} is homogeneous of degree 2 and the map $\pi_{k,m}$ is homogeneous of degree 1, thus $F_{k,m}$ is homogeneous of degree 2.

Let us now prove that $F_{k,m}$ is nondegenerate. Assume $\mathbf{x} \in \mathcal{M}_{k,m}$ and

$$\pi_{k,m} \circ \mathcal{Q}(\mathbf{x}) = F_{k,m}(\mathbf{x}) = \mathbf{0}.$$

Then $\mathbf{y} := \mathcal{Q}(\mathbf{x}) \in \text{Ker}(\pi_{k,m}) = \mathcal{L}$. By definition of \mathcal{Q} , $y_1 = 0$. Since $\mathbf{y} \in \mathcal{L}$, \mathbf{y} is a constant sequence thus $\mathbf{y} = \mathbf{0}$. This implies that $x_i = 0$ for all $i \geq 1$, i.e. $\mathbf{x} = \mathbf{0}$.

Since $\mathcal{M}_{k,m}$ has finite dimension and $F_{k,m} : \mathcal{M}_{k,m} \rightarrow \mathcal{M}_{k,m}$ is homogeneous and nondegenerate, it is surjective. \square

Lemma 2.2.6. *We have that $\bigcap_{n \geq 1} F_{k,m}^{\circ n}(\mathcal{E}) = \mathcal{M}_{k,m}$.*

Proof. According to the previous lemma, $F_{k,m}(\mathcal{M}_{k,m}) = \mathcal{M}_{k,m}$. In addition, $F_{k,m}(\mathcal{E}) \subset \mathcal{H}_{k,m}$. Since $\mathcal{M}_{k,m} = \mathcal{H}_{k,m} \cap \mathcal{P}_{k,m}$, it is enough to prove that

$$\bigcap_{n \geq 1} F_{k,m}^{\circ n}(\mathcal{E}) \subset \mathcal{P}_{k,m}.$$

This follows from the following claim: for all $n \geq 1$, if $\mathbf{x} \in F_{k,m}^{\circ n}(\mathcal{E})$, then $x_{i+m} = x_i$ for all $i \in \{k+1, \dots, k+n-1\}$.

Let us prove this claim by induction. If $n = 1$, the claim is empty, so there is nothing to prove. Assume it holds for some $n \geq 1$. Assume $\mathbf{y} \in F_{k,m}^{\circ(n+1)}(\mathcal{E})$, i.e. $\mathbf{y} = F_{k,m}(\mathbf{x})$ with $\mathbf{x} \in F_{k,m}^{\circ n}(\mathcal{E})$. The induction hypothesis implies that for all $i \in \{k+2, \dots, k+n\}$, $x_{i+m-1} = x_{i-1}$, so that $y_{i+m} = y_i$. In addition, since $\mathbf{x} \in \mathcal{H}_{k,m}$, $x_{k+m} = x_k$, so that $y_{k+m+1} = y_{k+1}$. Thus the claim is true for $n+1$. \square

2.2.2 The main result about $F_{k,m}$

Lemma 2.2.5 and Lemma 2.2.6 allow us to restrict our study to the dynamics of $F_{k,m}$ on $\mathcal{M}_{k,m}$. With a slight abuse of notations, from now on, we shall denote by $F_{k,m}$ the restriction of $F_{k,m} : \mathcal{E} \rightarrow \mathcal{E}$ to $\mathcal{M}_{k,m}$. The following result sums up the properties of $F_{k,m} : \mathcal{M}_{k,m} \rightarrow \mathcal{M}_{k,m}$ which are important for us.

Theorem A'. Given $(k, m) \in \mathbb{N} \times \mathbb{N}^*$, let $\mathbf{z} \in \mathcal{M}_{k,m}$ be a fixed point of the map $F_{k,m} : \mathcal{M}_{k,m} \rightarrow \mathcal{M}_{k,m}$. Let k' be the preperiod of the sequence \mathbf{z} and m' be its period. Then,

1. the polynomial $P(t) = t^2 + z_1 \in \mathbb{C}[t]$ is post-critically finite and $\mathbf{z} = (P^{\circ j}(0))_{j \geq 1}$; in particular, the critical value z_1 of P is preperiodic of preperiod k' to a cycle of period m' of multiplier λ ,
2. there exists a partial order \preceq on $\mathbb{N} \times \mathbb{N}^*$ such that $(k', m') \preceq (k, m)$ if and only if $\mathcal{M}_{k',m'} \subseteq \mathcal{M}_{k,m}$ is a $F_{k,m}$ -invariant subspace and the restriction of $F_{k,m}$ to $\mathcal{M}_{k',m'}$ is $F_{k',m'}$,
3. if $k+m-1 \geq 2$, $F_{k,m} : \mathcal{M}_{k,m} \rightarrow \mathcal{M}_{k,m}$ and $G_{k,m} : \mathbb{C}^{k+m-1} \rightarrow \mathbb{C}^{k+m-1}$ are conjugate,
4. $\text{Spec } D_{\mathbf{z}} F_{k',m'} \subset \mathbb{C} \setminus \overline{\mathbb{D}}$,
5. If $(k, m) \neq (k', m')$,

$$\text{Spec } (D_{\mathbf{z}} F_{k,m})^* |_{(\mathcal{M}_{k',m'})^0} = \begin{cases} \{0\} & \text{if } k' = 0 \\ \{\mu \mid \mu^m = \lambda^{\frac{m}{m'}}, \mu^{m'} \neq \lambda\} & \text{if } k' \neq 0 \end{cases} .$$

where $\mathcal{M}_{k',m'}^0 = \{\omega \in \mathcal{M}_{k,m}^* \mid \omega|_{\mathcal{M}_{k',m'}} \equiv 0\}$ is the annihilator of $\mathcal{M}_{k',m'}$ in $\mathcal{M}_{k,m}$

In particular, we shall see that Theorem A is a direct consequence of Theorem A'. The rest of this chapter is devoted to the proof of Theorem A'. Item 1 is proved in

Proposition 2.4.4. The partial order \preceq will be introduced in Definition 2.2.8 and item 2 will be proved in Proposition 2.2.9. Item 3 is proved in Proposition 2.2.7. Item 4 is due to Koch, and we recall its proof in Proposition 2.4.8 for the sake of completeness. Our main contribution is the proof of item 5 which will be proved in Proposition 2.4.10. Finally, we prove Theorem A by using the results above.

2.2.3 $F_{k,m}$ is conjugate to $G_{k,m}$

The following lemma assures that the class of maps $F_{k,m}$ for $(k, m) \in \mathbb{N} \times \mathbb{N}^*$ is a good alternative when one wants to study $G_{k,m}$. Recall that when $k = 0, m \geq 3$, we have

$$G_{0,m} : \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{m-1} \end{pmatrix} \mapsto \begin{pmatrix} -x_{m-1}^2 \\ x_1 - x_{m-1}^2 \\ \vdots \\ x_{m-2}^2 - x_{m-1}^2 \end{pmatrix}.$$

and when $k \geq 1, m \geq 2$, we have

$$G_{k,m} : \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{k+m-1} \end{pmatrix} \mapsto \begin{pmatrix} \left(-\frac{x_{k+m-1}+x_{k-1}}{2}\right)^2 \\ \left(x_1 - \frac{x_{k+m-1}+x_{k-1}}{2}\right)^2 \\ \vdots \\ \left(x_{k+m-2} - \frac{x_{k+m-1}+x_{k-1}}{2}\right)^2 \end{pmatrix}.$$

Proposition 2.2.7. *Let $(k, m) \in \mathbb{N} \times \mathbb{N}^*$ be such that $k + m - 1 \geq 2$, the maps $F_{k,m}$ and $G_{k,m}$ are holomorphically conjugate.*

Proof. When $k = 0$, let $\tau_0 : \mathcal{M}_{0,m} \rightarrow \mathbb{C}^{m-1}$ be the linear map defined by

$$\tau_0(\mathbf{x}) = (x_1, \dots, x_{m-1})$$

Observe that τ_0 is an isomorphism. In fact, since it is a linear map of two vector spaces of the same dimension, it is enough to prove that τ_0 is injective. Assume $\mathbf{x} \in \mathcal{M}_{0,m}$ and $\tau_0(\mathbf{x}) = 0$. In particular, $\mathbf{x} \in \mathcal{M}_{0,m}$ implies that $x_m = 0$, and $\tau_0(\mathbf{x}) = 0$ implies that $x_1 = \dots = x_{m-1} = 0$. For every $i \geq m + 1$, there exists $i' \in \{1, \dots, m\}$ such that $i \equiv i' \pmod{m}$. Since $\mathbf{x} \in \mathcal{M}_{0,m}$, $x_i = x_{i'} = 0$. In other words, $x_i = 0$ for every $i \geq 1$, whence τ_0 is an isomorphism.

We have

$$\tau_0 \circ F_{0,m} = G_{0,m} \circ \tau_0.$$

In fact, assume $\mathbf{x} \in \mathcal{M}_{0,m}$ and $\mathbf{y} = F_{0,m}(\mathbf{x})$. According to Lemma 2.2.3, for every $i \geq 2$ $y_i = x_{i-1}^2 + y_1$ with $y_1 = -x_{m-1}^2$. Hence

$$\tau_0 \circ F_{0,m}(\mathbf{x}) = (y_1, \dots, y_{m-1}) = (-x_{m-1}^2, x_1^2 - x_{m-1}^2, \dots, x_{m-2}^2 - x_{m-1}^2).$$

Since

$$G_{0,m} \circ \tau_0(\mathbf{x}) = G_{0,m}(x_1, \dots, x_{m-1}) = (-x_{m-1}^2, x_1^2 - x_{m-1}^2, \dots, x_{m-2}^2 - x_{m-1}^2),$$

we have $\tau_0 \circ F_{0,m} = G_{0,m} \circ \tau_0$. This proves the statement when $k = 0$.

When $k \geq 1$, let $\tau : \mathcal{M}_{k,m} \rightarrow \mathbb{C}^{k+m-1}$ be the linear map defined by

$$\tau(\mathbf{x}) = (x_2 - x_1, x_3 - x_1, \dots, x_{k+m-1} - x_1, x_{k+m} - x_1).$$

Observe that τ is an isomorphism. Indeed, since it is clearly a linear map of two vector spaces of the same dimension, it is enough to prove that τ is injective. Assume $\mathbf{x} \in \mathcal{M}_{k,m}$ and $\tau(\mathbf{x}) = (0, \dots, 0)$. Then $x_1 = x_2 = \dots = x_{k+m-1} = x_{k+m}$. Since $x_{k+m} + x_k = 0$, we deduce that $x_k = x_{k+m} = -x_k$, and hence $x_k = 0$. Whence, $x_i = 0$ for all $i \in \{1, \dots, k+m\}$. And since $x_{i+m} = x_i$ for all $i \geq k+1$, we have $x_i = 0$ for all $i \geq k+m+1$.

Then, to conclude, it is enough to show that

$$\tau \circ F_{k,m} = G_{k,m} \circ \tau.$$

Assume $\mathbf{x} \in \mathcal{M}_{k,m}$. First, we compute $\tau \circ F_{k,m}(\mathbf{x})$. Set $\mathbf{y} = F_{k,m}(\mathbf{x})$. According to Lemma 2.2.3, we have for all $i \geq 2$, $y_i = x_{i-1}^2 + y_1$. In addition, $\tau(\mathbf{y}) = (y_2 - y_1, y_3 - y_1, \dots, y_{k+m} - y_1)$. Hence

$$\tau \circ F_{k,m}(\mathbf{x}) = \tau(\mathbf{y}) = (x_1^2, x_2^2, \dots, x_{k+m-1}^2). \quad (2.2.1)$$

Second, we compute $G_{k,m} \circ \tau(\mathbf{x})$. Set $(w_1, \dots, w_{k+m-1}) = \tau(\mathbf{x})$. Then, for all $i \in \{1, \dots, k+m-1\}$, $w_i = x_{i+1} - x_1$, or equivalently,

$$w_i + x_1 = x_{i+1}.$$

Since $\mathbf{x} \in \mathcal{M}_{k,m}$ with $k \geq 1$, $x_k + x_{k+m} = 0$. Hence

$$-\frac{w_{k+m-1} + w_{k-1}}{2} = -\frac{1}{2}(x_{k+m} - x_1 + x_k - x_1) = x_1$$

and

$$G_{k,m} \circ \tau(\mathbf{x}) = G_{k,m} \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_{k+m-1} \end{pmatrix} = \begin{pmatrix} \left(-\frac{w_{k+m-1} + w_{k-1}}{2}\right)^2 \\ \left(w_1 - \frac{w_{k+m-1} + w_{k-1}}{2}\right)^2 \\ \vdots \\ \left(w_{k+m-2} - \frac{w_{k+m-1} + w_{k-1}}{2}\right)^2 \end{pmatrix} = \begin{pmatrix} x_1^2 \\ x_2^2 \\ \vdots \\ x_{k+m-1}^2 \end{pmatrix}.$$

Combining with (2.2.1), we have

$$G_{k,m} \circ \tau(\mathbf{x}) = (x_1^2, x_2^2, \dots, x_{k+m-1}^2) = \tau \circ F_{k,m}(\mathbf{x}). \quad \square$$

2.2.4 Comparing moduli maps

Our initial expectation was that for arbitrary pairs (k_1, m_1) and (k_2, m_2) in $\mathbb{N} \times \mathbb{N}^*$, the maps F_{k_1, m_1} and F_{k_2, m_2} would agree on the intersection $\mathcal{M}_{k_1, m_1} \cap \mathcal{M}_{k_2, m_2}$. However this is not true as shown in the following example. Consider the sequence

$$\mathbf{x} := \{2, 0, 0, \dots, 0, \dots\}$$

Then $\mathbf{x} \in \mathcal{M}_{2,1} \cap \mathcal{M}_{3,1}$ and

$$\mathcal{Q}(\mathbf{x}) = \{0, 4, 0, 0, 0, 0, \dots\}$$

However,

$$F_{2,1}(\mathbf{x}) = \{-2, 2, -2, -2, -2, -2, \dots\} \text{ and } F_{3,1}(\mathbf{x}) = \{0, 4, 0, 0, 0, 0, \dots\}.$$

We will now see that if some order $(k', m') \preceq (k, m)$ is satisfied, then $\mathcal{M}_{k', m'} \subseteq \mathcal{M}_{k, m}$ and $F_{k', m'}$ is the restriction of $F_{k, m}$ to $\mathcal{M}_{k', m'}$.

Definition 2.2.8. Let \preceq be the partial order on $\mathbb{N} \times \mathbb{N}^*$ defined by

$$(k', m') \preceq (k, m) \Leftrightarrow \begin{cases} m' \text{ divides } m, \\ \text{either } k' = k \text{ or } (k' = 0 \text{ and } m' \text{ divides } k). \end{cases}$$

The strict order \prec is defined by

$$(k', m') \prec (k, m) \Leftrightarrow (k', m') \preceq (k, m) \text{ and } (k', m') \neq (k, m).$$

Proposition 2.2.9. *For two pairs of integers $(k, m), (k', m') \in \mathbb{N} \times \mathbb{N}^*$, we have that*

$$(k', m') \preceq (k, m) \Leftrightarrow \begin{cases} \mathcal{M}_{k', m'} \subseteq \mathcal{M}_{k, m}, \\ F_{k, m}|_{\mathcal{M}_{k', m'}} = F_{k', m'}. \end{cases}$$

Proof. Let us first assume that $(k', m') \preceq (k, m)$ and prove that $\mathcal{M}_{k', m'} \subseteq \mathcal{M}_{k, m}$. Assume $\mathbf{x} \in \mathcal{M}_{k', m'}$. Then \mathbf{x} is preperiodic of period less than k' to a cycle of period dividing m' . Since $k' \leq k$ and $m' \mid m$, we deduce that $\mathbf{x} \in \mathcal{P}_{k, m}$. We need to show that $x_k + x_{k+m} = 0$. Indeed,

- if $k' = k$, since $m' \mid m$, $x_{k+m} + x_k = x_{k'+m'} + x_{k'} = 0$.
- if $k' = 0$; in that case $m' \mid k$; since $\mathbf{x} \in \mathcal{M}_{0, m'}$, we have that $x_{k+m} = x_k = 0$.

Let us now assume that $\mathcal{M}_{k', m'} \subseteq \mathcal{M}_{k, m}$. We claim that $(k', m') \preceq (k, m)$. Indeed,

- either $k' = 0$; in this case, consider $\mathbf{x} \in \mathcal{M}_{0, m'}$ given by $x_i = 0$ if $m' \mid i$ and 1 otherwise. If $x_{k+m} + x_k = 0$ then necessarily, $x_{k+m} = x_k = 0$ thus m' divides k and m .
- or $k' \geq 1$; in this case, consider $\mathbf{x} \in \mathcal{M}_{k', m'}$ given by $x_{k'} = -1, x_{k'+jm'} = 1$ for $j \geq 1$ and $x_i = 2$ otherwise. If $x_{k+m} + x_k = 0$ then $x_k = -1$ and $x_{k+m} = 1$. Hence $k = k'$ and $m' \mid m$.

Under the assumption that $(k', m') \preceq (k, m)$, let us prove that $F_{k, m}|_{\mathcal{M}_{k', m'}} = F_{k', m'}$. Assume $\mathbf{x} \in \mathcal{M}_{k', m'}$. Set $\mathbf{y} = F_{k', m'}(\mathbf{x})$ and $\mathbf{z} := F_{k, m}(\mathbf{x})$. According to Lemma 2.2.3, for all $i \geq 2$, $y_i = x_{i-1}^2 + y_1, z_i = x_{i-1}^2 + z_1$ where

$$y_1 = \begin{cases} -x_{m'-1}^2 & \text{if } k' = 0 \\ -\frac{x_{k'-1}^2 + x_{k'+m'-1}^2}{2} & \text{if } k' \geq 1 \end{cases} \quad \text{and} \quad z_1 = \begin{cases} -x_{m-1}^2 & \text{if } k = 0 \\ -\frac{x_{k-1}^2 + x_{k+m-1}^2}{2} & \text{if } k \geq 1. \end{cases}$$

It is enough to prove that $y_1 = z_1$.

- Case $k' = 0$. In that case, k and m are multiples of m' . If $k = 0$ then $y_1 = -x_{m'-1}^2 = -x_{m-1}^2 = z_1$. If $k \neq 0$, $x_{k+m-1} = x_{k-1} = x_{m'-1}$ hence

$$y_1 = -x_{m'-1}^2 = -\frac{x_{k-1}^2 + x_{k+m-1}^2}{2} = z_1.$$

- Case $k' \neq 0$. In that case, $k' = k$ and m is a multiple of m' . Then

$$x_{k+m-1} = \begin{cases} -x_{k'+m'-1} & \text{if } m' - 1 = 0 \text{ and } m - 1 \geq 1 \\ x_{k'+m'-1} & \text{otherwise.} \end{cases}$$

Then

$$y_1 = -\frac{x_{k'-1}^2 + x_{k'+m'-1}^2}{2} = -\frac{x_{k-1}^2 + x_{k+m-1}^2}{2} = z_1. \quad \square$$

2.3 Dynamics of moduli maps on their post-critical sets

In the following, we fix a pair of integers $(k, m) \in \mathbb{N} \times \mathbb{N}^*$. Define

$$C_{k,m} := \text{the critical set of } F_{k,m} : \mathcal{M}_{k,m} \rightarrow \mathcal{M}_{k,m}$$

$$V_{k,m} := \text{the critical value set of } F_{k,m} : \mathcal{M}_{k,m} \rightarrow \mathcal{M}_{k,m}$$

and

$$PC_{k,m} := \text{the post-critical set of } F_{k,m} : \mathcal{M}_{k,m} \rightarrow \mathcal{M}_{k,m}.$$

We shall now study the dynamics of $F_{k,m}$ on $PC_{k,m}$. If $k + m = 1$, $\dim \mathcal{M}_{k,m} = 0$ and $F_{k,m} : \mathcal{M}_{k,m} \rightarrow \mathcal{M}_{k,m}$ is trivial. From now on, we fix $(k, m) \in \mathbb{N} \times \mathbb{N}^*$ such that $k + m \geq 2$.

In [Koc13], Koch proved that the maps $G_{k,m}$ is post-critically algebraic for every $(k, m) \in \mathbb{N} \times \mathbb{N}^*$. Since $G_{k,m}$ and $F_{k,m}$ are conjugate, the moduli maps $F_{k,m}$ are also post-critically algebraic. Since we need to understand better the dynamics of $F_{k,m}$ on its post-critical set, so we will reprove here that $F_{k,m}$ is post-critically algebraic.

Lemma 2.3.1. *We have that $C_{k,m} = \{\mathbf{x} \in \mathcal{M}_{k,m} \mid x_i = 0 \text{ for some } 1 \leq i \leq k + m - 1\}$*

Proof. Recall that $F_{k,m} = \pi_{k,m} \circ \mathcal{Q}$. Differentiating both sides, we see that for any $\mathbf{x} \in \mathcal{M}_{k,m}$ and for any $\mathbf{v} \in T_{\mathbf{x}}\mathcal{E} = \mathcal{E}$,

$$D_{\mathbf{x}}F_{k,m}(\mathbf{v}) = \pi_{k,m} \circ D_{\mathbf{x}}\mathcal{Q}(\mathbf{v}) = \pi_{k,m}(0, 2x_1v_1, 2x_2v_2, \dots)$$

On the one hand, assume $\mathbf{x} \in C_{k,m}$. Then there exists $\mathbf{v} \in T_{\mathbf{x}}\mathcal{M}_{k,m} \setminus \{\mathbf{0}\}$ such that $x_iv_i = 0$ for all $i \geq 1$. Observe that there exists $i \in \{1, \dots, k + m - 1\}$ such that $v_i \neq 0$ whence $x_i = 0$. Indeed otherwise, $v_{k+m} = -v_k = 0$ and by preperiodicity, $v_i = 0$ for all $i \geq 1$.

On the other hand, given $\mathbf{x} \in \mathcal{E}$, define $\mathbf{v} \in \mathcal{E}$ by

$$v_j = \begin{cases} -1 & \text{if } x_j = 0 \text{ and } j = k \\ 1 & \text{if } x_j = 0 \text{ and } j \neq k \\ 0 & \text{if } x_j \neq 0 \end{cases}$$

Then $x_j v_j = 0$ for all $j \geq 1$ so that $D_{\mathbf{x}} F_{k,m}(\mathbf{v}) = \mathbf{0}$. Moreover, if $\mathbf{x} \in \mathcal{M}_{k,m}$ then $\mathbf{v} \in \mathcal{M}_{k,m}$. Finally, if there exists $i \in \{1, \dots, k+m-1\}$ such that $x_i = 0$ then $\mathbf{v} \neq \mathbf{0}$ whence $\mathbf{x} \in C_{k,m}$. \square

Definition 2.3.2. Denote by

$$\Delta_{k,m} = \{\mathbf{x} \in \mathcal{M}_{k,m} \mid \text{there exists } 1 \leq i < j \leq k+m \text{ such that } x_i = x_j\}$$

The set $\Delta_{k,m}$ consists of $\binom{k+m}{2}$ hyperplanes.

Proposition 2.3.3. We have that $V_{k,m} \subseteq \Delta_{k,m}$ and $F_{k,m}(\Delta_{k,m}) \subseteq \Delta_{k,m}$. Consequently, $PC_{k,m} \subseteq \Delta_{k,m}$.

Proof. Let $\mathbf{x} \in C_{k,m}$ and set $\mathbf{y} = F_{k,m}(\mathbf{x})$. Then by Lemma 2.3.1, there exists $i \in \{1, \dots, k+m-1\}$ such that $x_i = 0$. By Lemma 2.2.3, we have

$$y_{i+1} = x_i^2 + y_1 = y_1$$

Thus $\mathbf{y} \in \Delta_{k,m}$, whence $V_{k,m} \subseteq \Delta_{k,m}$.

Now we prove that $\Delta_{k,m}$ is invariant under $F_{k,m}$. Assume $\mathbf{x} \in \Delta_{k,m}$ and set $\mathbf{y} = F_{k,m}(\mathbf{x})$. Then there exist $1 \leq i < j \leq k+m$ such that $x_i = x_j$. By Lemma 2.2.3, for every $l \geq 2$, $y_l = x_{l-1}^2 + y_1$. Note that since $\mathbf{x} \in \mathcal{M}_{k,m}$, $x_k + x_{k+m} = 0$ with the convention $x_0 := 0$.

- If $j \leq k+m-1$, we have $y_{i+1} = x_i^2 + y_1 = x_j^2 + y_1 = y_{j+1}$.
- If $j = k+m$, then
 - either $i = k$ so that $x_k = x_i = x_j = x_{k+m}$; since $x_k + x_{k+m} = 0$, $x_k = 0$ whence $\mathbf{x} \in C_{k,m}$ and $\mathbf{y} \in \Delta_{k,m}$;
 - or $i \neq k$ so that $i+1 \neq k+1$; since $x_k = -x_{k+m} = -x_i$, we have

$$y_{i+1} = x_i^2 + y_1 = x_k^2 + y_1 = y_{k+1}.$$

Hence, in any case, we have $\mathbf{y} \in \Delta_{k,m}$, i.e. $F_{k,m}(\Delta_{k,m}) \subset \Delta_{k,m}$ and the lemma is proved. \square

2.4 Fixed points of moduli maps

In this section, we shall study the eigenvalues of the derivative of $F_{k,m}$ at its fixed points.

2.4.1 Relation with post-critically finite polynomials

There is a close connection between fixed points of moduli maps and post-critically finite quadratic polynomials. More precisely, we will consider monic centered quadratic polynomials,

$$P(t) = t^2 + c \in \mathbb{C}[t], c \in \mathbb{C}$$

The critical orbit of such a polynomial is the sequence $\mathbf{c}_P \in \mathcal{E}$ defined by

$$\mathbf{c}_P = (c_i)_{i \geq 1} \in \mathcal{E} \text{ where } c_i = P^{\circ i}(0).$$

Since the preperiod and the period of a preperiodic sequence will be extensively discussed in this chapter, we introduce the following notions.

Definition 2.4.1. Given integers $k \geq 0, m \geq 1$, a sequence $\mathbf{x} \in \mathcal{E}$ is called *preperiodic of type (k, m)* if for every $i \geq k + 1$, $x_{i+m} = x_i$, *preperiodic of exact type (k, m)* if, additionally, k and m are the smallest integers satisfying such conditions.

For a sequence of exact type (k, m) , the pair (k, m) consists of the preperiod k and the period m . The vector space $\mathcal{P}_{k,m}$ is the space of preperiodic sequences of type (k, m) .

Definition 2.4.2. A *polynomial of (exact) type (k, m)* is a monic centered quadratic polynomial P whose critical orbit \mathbf{c}_P is of (exact) type (k, m) .

In other words, a polynomial is of type (k, m) if and only if its critical orbit belongs to $\mathcal{P}_{k,m}$. Note that a polynomial of type (k, m) is post-critically finite.

Remark 2.4.3. Let P be a polynomial of type (k, m) . If $k = 0$, then the critical value c of P is a periodic point of period dividing m , i.e. $P^{\circ m}(c) = c$. In other words, $P^{\circ(m-1)}(c) \in P^{-1}(c)$. However, since P is a unicritical polynomial, $P^{-1}(c)$ consists of exactly one point which is the critical point of P . This means that $P^{\circ(m-1)}(c)$ is in fact the critical point of P . This is the case if and only if the critical point of P is also a periodic point of type $(0, m)$.

Proposition 2.4.4. *Given $(k, m) \in \mathbb{N} \times \mathbb{N}^*$, assume $\mathbf{z} \in \mathcal{M}_{k,m}$ and set $P(t) = t^2 + z_1$. Then*

$$F_{k,m}(\mathbf{z}) = \mathbf{z} \Leftrightarrow \begin{cases} \mathbf{z} = \mathbf{c}_P \\ P \text{ is of exact type } (k', m') \preceq (k, m). \end{cases}$$

Proof. Assume P is of exact type $(k', m') \preceq (k, m)$ and $\mathbf{z} = \mathbf{c}_P$. We have to prove that $F_{k,m}(\mathbf{z}) = \mathbf{z}$. Since $\mathcal{M}_{k',m'} \subseteq \mathcal{M}_{k,m}$ and $F_{k,m}|_{\mathcal{M}_{k',m'}} = F_{k',m'}$, it is enough to prove that $F_{k',m'}(\mathbf{z}) = \mathbf{z}$. In other words, we only need to consider the case $(k', m') = (k, m)$. In that case, the critical orbit $\mathbf{z} = (P^{oi}(0))_{i \geq 1}$ of P is a preperiodic sequence of exact type (k, m) . In particular,

$$\begin{cases} z_{k+m+1} = z_{k+1}, \\ z_{k+m} \neq z_k. \end{cases}$$

Note that for every $i \geq 2$, $z_i = P^{oi}(0) = z_{i-1}^2 + z_1$. According to Lemma 2.2.3, in order to prove that $F_{k,m}(\mathbf{z}) = \mathbf{z}$, it is enough to prove $z_k + z_{k+m} = 0$.

- If $k = 0$, by Remark 2.4.3, the critical point 0 of P is periodic of period m , so that

$$z_m = P^{om}(0) = 0$$

hence $z_0 + z_m = 0$.

- If $k \neq 0$, the fact that $z_{k+m+1} = z_{k+1}$ implies that $z_{k+m}^2 + z_1 = z_k^2 + z_1$ whence

$$(z_{k+m} - z_k)(z_{k+m} + z_k) = 0.$$

Since $z_{k+m} \neq z_k$, it implies that $z_{k+m} + z_k = 0$.

In either cases, $z_k + z_{k+m} = 0$ whence $F_{k,m}(\mathbf{z}) = \mathbf{z}$.

Assume now $F_{k,m}(\mathbf{z}) = \mathbf{z}$. First, let us prove that P is of type (k, m) . According to Lemma 2.2.3, for every $i \geq 2$, we have that $z_i = z_{i-1}^2 + z_1$ hence

$$z_i = P(z_{i-1})$$

In other words, \mathbf{z} is the sequence of iterates of z_1 under P . Recall that by Lemma 2.2.6, $\mathcal{M}_{k,m}$ contains every fixed point of $F_{k,m}$ hence $\mathbf{z} \in \mathcal{M}_{k,m} \subset \mathcal{P}_{k,m}$. Therefore, the polynomial P is a polynomial of type (k, m) . Second, let (k', m') be the exact type of P . We need to prove that $(k', m') \preceq (k, m)$, i.e.,

$$\begin{cases} m' \mid m \\ \text{either } k' = k \text{ or } (k' = 0 \text{ and } m' \mid k) \end{cases}$$

Since (k', m') is the exact type of the orbit of z_1 , $k' \leq k$ and $m' \mid m$. If $k' = k$, we are done. If $k' \neq k$, we need to prove that $k' = 0$ and $m' \mid k$. Since $\mathbf{z} \in \mathcal{M}_{k',m'}$, $k' + 1 \leq k$ and $m' \mid m$, we have that $z_k = z_{k+m}$. Since $\mathbf{z} \in \mathcal{M}_{k,m}$, we also have $z_{k+m} + z_k = 0$.

Therefore, $P^{\circ k}(0) = z_k = 0$. In other words, 0 is a periodic point of P , i.e. $k' = 0$, and the period of 0 is m' . Moreover, $P^{\circ k}(0) = 0$ also implies that k is a multiple of m' . Thus, we can conclude that $(k', m') \preceq (k, m)$. \square

The partial order \preceq enables us to study the relative positions of the fixed points of $F_{k,m}$ and $\Delta_{k,m}$.

Lemma 2.4.5. *Let \mathbf{z} be a fixed point of $F_{k,m}$ and let (k', m') be the exact type of \mathbf{z} . Then, $\mathbf{z} \in \Delta_{k,m}$ if and only if $(k', m') \prec (k, m)$.*

Proof. Assume $\mathbf{z} \in \Delta_{k,m}$, i.e. there exists $1 \leq i < j \leq k + m$ such that $z_i = z_j$. In particular, \mathbf{z} is a preperiodic sequence of preperiod at most $i - 1$ and of period dividing $j - i$. Whence, since (k', m') is the exact type of \mathbf{z} , we have $k' \leq i - 1$ and m' divides $j - i$.

- If $i \leq k$ then $k' \leq i - 1 < k$.
- If $i \geq k + 1$ then $j - i \leq k + m - (k + 1) < m$. Since $m' \mid j - i$, we have $m' < m$.

In both cases, we have $(k', m') \neq (k, m)$. Note that, according to Proposition 2.4.4, $(k', m') \preceq (k, m)$. Hence $(k', m') \prec (k, m)$.

Conversely, assume $(k', m') \prec (k, m)$. In particular, $k' \leq k$, $m' \leq m$ and $(k', m') \neq (k, m)$. Note that \mathbf{z} is of exact type (k', m') . Hence,

$$z_{k'+1} = z_{k'+m'+1}.$$

If $k' \neq k$ then $k' < k$. Whence $k' + 1$ and $k' + m' + 1$ are integers in $\{1, \dots, k + m\}$. If $k' = k$ then $m' < m$. In this case, $k + 1, k + m' + 1$ are also in $\{1, \dots, k + m\}$. Therefore, in both cases, we deduce by that $\mathbf{z} \in \Delta_{k,m}$. \square

2.4.2 Eigenvalues of moduli maps at fixed points

In order to study the eigenvalues of the derivative of moduli maps at one of its fixed point, we will in fact study its transpose. Note that when $k + m = 1$, $\mathcal{M}_{k,m} = \{\mathbf{0}\}$ and $F_{k,m}$ is trivial. Let us fix $(k, m) \in \mathbb{N} \times \mathbb{N}^*$ such that $k + m \geq 2$. Assume $\mathbf{z} \in \mathcal{M}_{k,m}$ is a fixed point of $F_{k,m}$. We will describe the transpose of the derivative $D_{\mathbf{z}}F_{k,m} : T_{\mathbf{z}}\mathcal{M}_{k,m} \rightarrow T_{\mathbf{z}}\mathcal{M}_{k,m}$. Since $\mathcal{M}_{k,m}$ is a vector space, there is a canonical identification of $T_{\mathbf{z}}\mathcal{M}_{k,m}$ with $\mathcal{M}_{k,m}$, the derivative $D_{\mathbf{z}}F_{k,m} : T_{\mathbf{z}}\mathcal{M}_{k,m} \rightarrow T_{\mathbf{z}}\mathcal{M}_{k,m}$ identifies with a linear map

$$L : \mathcal{M}_{k,m} \rightarrow \mathcal{M}_{k,m},$$

and the transpose identifies with the pull-back map

$$L^* : \mathcal{M}_{k,m}^* \rightarrow \mathcal{M}_{k,m}^*.$$

The dual space $\mathcal{M}_{k,m}^*$

For $i \geq 1$, let $\omega_i \in \mathcal{M}_{k,m}^*$ be the linear form defined by for all $\mathbf{v} \in \mathcal{M}_{k,m}$,

$$\omega_i(\mathbf{v}) := v_i.$$

Lemma 2.4.6. *The family $\{\omega_i, 1 \leq i \leq k+m-1\}$ is a basis of $\mathcal{M}_{k,m}^*$.*

Proof. Note that $\dim \mathcal{M}_{k,m} = k+m-1$ hence it is enough to prove that $\{\omega_1, \dots, \omega_{k+m-1}\}$ are linearly independent. Assume that

$$\sum_{1 \leq i \leq k+m-1} \lambda_i \omega_i = 0 \text{ with } \lambda_i \in \mathbb{C}.$$

Let $i \geq 1$. To prove that $\lambda_i = 0$, consider the vector $\mathbf{v} \in \mathcal{M}_{k,m}$ defined by

- if $i < k$, $v_j = \begin{cases} 1 & \text{if } j = i \\ 0 & \text{otherwise,} \end{cases}$
- if $i = k$, $v_j = \begin{cases} 1 & \text{if } j = i = k \\ -1 & \text{if } j > k \text{ and } j \equiv k \pmod{m} \\ 0 & \text{otherwise,} \end{cases}$
- if $i > k$, $v_j = \begin{cases} 1 & \text{if } j \geq i \text{ and } j \equiv i \pmod{m} \\ 0 & \text{otherwise.} \end{cases}$

In any case, we have

$$0 = \sum_{1 \leq i \leq k+m-1} \lambda_i \omega_i(\mathbf{v}) = \lambda_i v_i = \lambda_i. \quad \square$$

The transpose of the derivative

Observe that the transpose $L^* : \mathcal{M}_{k,m}^* \rightarrow \mathcal{M}_{k,m}^*$ is the pull-back of forms, i.e. for all $\omega \in \mathcal{M}_{k,m}^*$,

$$L^* \omega = \omega \circ L.$$

For all $i \geq 1$, set

$$\delta_i = 2z_i$$

where $\mathbf{z} = (z_1, z_2, \dots) \in \mathcal{M}_{k,m}$ is a fixed point of $F_{k,m}$.

Lemma 2.4.7. *We have that*

$$L^*\omega_1 = \begin{cases} -\delta_{m-1}\omega_{m-1} & \text{if } k = 0 \\ -\frac{\delta_{k-1}\omega_{k-1} + \delta_{k+m-1}\omega_{k+m-1}}{2} & \text{otherwise,} \end{cases}$$

and for all $i \geq 2$,

$$L^*\omega_i = \delta_{i-1}\omega_{i-1} + L^*\omega_1.$$

Proof. Recall that for all $\mathbf{v} \in \mathcal{M}_{k,m}$,

$$L(\mathbf{v}) = \pi_{k,m} \circ D_{\mathbf{z}}\mathcal{Q}(\mathbf{v}).$$

Set $\mathbf{u} = D_{\mathbf{z}}\mathcal{Q}(\mathbf{v})$ then

$$u_1 = 0 \quad \text{and for all } i \geq 2, \quad u_i = 2z_{i-1}v_{i-1} = \delta_{i-1}v_{i-1}.$$

In addition, if $\mathbf{w} := \pi_{k,m}(\mathbf{u})$ then

$$\text{for all } i \geq 2, \quad w_i = u_i + w_1 \text{ with } w_1 = \begin{cases} -u_m & \text{if } k = 0 \\ -\frac{u_k + u_{k+m}}{2} & \text{if } k \neq 0. \end{cases}$$

Combining those formulas, we obtain that $\mathbf{w} = L(\mathbf{v})$ and for all $i \geq 2$,

$$w_i = \delta_{i-1}v_{i-1} + w_1 \text{ with } w_1 = \begin{cases} -\delta_{m-1}v_{m-1} & \text{if } k = 0 \\ -\frac{\delta_{k-1}v_{k-1} + \delta_{k+m-1}v_{k+m-1}}{2} & \text{otherwise.} \end{cases} \quad (2.4.1)$$

We deduce that for all $\mathbf{v} \in \mathcal{M}_{k,m}$,

$$L^*\omega_1(\mathbf{v}) = \omega_1 \circ L(\mathbf{v}) = w_1 = \begin{cases} -\delta_{m-1}v_{m-1} & \text{if } k = 0 \\ -\frac{\delta_{k-1}v_{k-1} + \delta_{k+m-1}v_{k+m-1}}{2} & \text{otherwise,} \end{cases}$$

hence

$$L^*\omega_1 = \begin{cases} -\delta_{m-1}\omega_{m-1} & \text{if } k = 0 \\ -\frac{\delta_{k-1}\omega_{k-1} + \delta_{k+m-1}\omega_{k+m-1}}{2} & \text{otherwise.} \end{cases}$$

In addition, for all $i \geq 2$ and for all $\mathbf{v} \in \mathcal{M}_{k,m}$, we have

$$L^* \omega_i(\mathbf{v}) = \omega_i \circ L(\mathbf{v}) = \delta_{i-1} v_{i-1} + \omega_1 \circ L(\mathbf{v})$$

hence

$$L^* \omega_i = \delta_{i-1} \omega_{i-1} + L^* \omega_1. \quad \square$$

Fixed points outside the post-critical set

According to Section 2.2.3, the map $F_{k,m}$ is conjugate to the map $G_{k,m}$ constructed by Koch [Koc13]. By [Koc13, Corollary 7.2], the derivative of $G_{k,m}$ at its fixed points outside the post-critical set has only repelling eigenvalues, whence so does $F_{k,m}$. For the sake of completeness, we give here the proof of this property. For further discussion about the arithmetics of such eigenvalues, we refer to [BEK20]. The main content of this paragraph is the following result.

Proposition 2.4.8. *Given $(k, m) \in \mathbb{N} \times \mathbb{N}^*$, let $\mathbf{z} \notin PC_{k,m}$ be a fixed point of the moduli map $F_{k,m}$. Then $D_{\mathbf{z}} F_{k,m}$ has only repelling eigenvalues.*

Proof. Since $\mathcal{M}_{k,m}$ has finite dimension, it is suffice to prove that the transpose L^* of $D_{\mathbf{z}} F_{k,m}$ has only repelling eigenvalues.

Recall that, by Lemma 2.4.6, the family $\{\omega_i : \mathcal{M}_{k,m} \rightarrow \mathbb{C}\}_{i \in \{1, \dots, k+m-1\}}$ is a basis of $\mathcal{M}_{k,m}^*$. According to Lemma 2.4.7, setting $\delta_i = 2z_i$, we have

$$L^* \omega_1 = \begin{cases} -\delta_{m-1} \omega_{m-1} & \text{if } k = 0 \\ -\frac{\delta_{k-1} \omega_{k-1} + \delta_{k+m-1} \omega_{k+m-1}}{2} & \text{if } k \geq 1, \end{cases}$$

and for all $i \geq 2$,

$$L^* \omega_i = \delta_{i-1} \omega_{i-1} + L^* \omega_1.$$

Since \mathbf{z} is a fixed point of $F_{k,m}$ of exact type (k, m) , we have that for all $i \in \{1, \dots, k+m-1\}$, $\delta_i \neq 0$. Indeed, note that according to Proposition 2.4.4, \mathbf{z} is the critical orbit of $P(t) = t^2 + z_1$. Assume that $\delta_i = 0$ for some $i \in \{1, \dots, k+m-1\}$. Then $P^{\circ i}(0) = z_i = 0$. This implies that $k = 0$ and m divides i . However $i \leq k+m-1 < m$, hence contradiction.

We may therefore define a linear map $L_* : \mathcal{M}_{k,m}^* \rightarrow \mathcal{M}_{k,m}^*$ by

$$\forall i \in \{1, \dots, k+m-1\} \quad L_*(\omega_i) = \frac{\omega_{i+1} - \omega_1}{\delta_i}. \quad (2.4.2)$$

Lemma 2.4.9. *The linear map L^* is invertible and its inverse is L_* .*

Proof. We need to prove that $L_* \circ L^* = L^* \circ L_* = \text{id}$. First, observe that for all $i \in \{1, \dots, k+m-1\}$,

$$L^* \circ L_*(\omega_i) = L^* \left(\frac{\omega_{i+1} - \omega_1}{\delta_i} \right) = \frac{1}{\delta_i} (L^*(\omega_{i+1}) - L^*(\omega_1)) = \frac{1}{\delta_i} (\delta_i \omega_i + L^*(\omega_1) - L^*(\omega_1)) = \omega_i.$$

Second, we prove $L_* \circ L^* = \text{Id}_{\mathcal{M}_{k,m}^*}$. Note that, by the definition of $\mathcal{M}_{k,m}$, we have that

$$\omega_{k+m} = \begin{cases} 0 & \text{if } k = 0 \\ -\omega_k & \text{if } k \geq 1 \end{cases} \quad \text{and} \quad \forall i \geq k+m+1 \quad \omega_{i+m} = \omega_i.$$

To compute $L_* \circ L^*(\omega_1)$, observe that if $k = 0$, then

$$L_* \circ L^*(\omega_1) = L_*(-\delta_{m-1}\omega_{m-1}) = -\delta_{m-1}L_*(\omega_{m-1}) = -(\omega_m - \omega_1) = \omega_1$$

and if $k \geq 1$, then

$$\begin{aligned} L_* \circ L^*(\omega_1) &= L_* \left(-\frac{\delta_{k-1}\omega_{k-1} + \delta_{k+m-1}\omega_{k+m-1}}{2} \right) \\ &= -\frac{1}{2}((\omega_k - \omega_1) + (\omega_{k+m} - \omega_1)) = \omega_1. \end{aligned}$$

In both cases, $L_* \circ L^*(\omega_1) = \omega_1$. For $L_* \circ L^*(\omega_i)$ with $i \in \{2, \dots, k+m-1\}$,

$$L_* \circ L^*(\omega_i) = L_*(\delta_{i-1}\omega_{i-1} + L^*(\omega_1)) = \delta_{i-1}\frac{\omega_i - \omega_1}{\delta_{i-1}} + L_*L^*(\omega_1) = \omega_i.$$

Thus, the linear map $L_* : \mathcal{M}_{k,m}^* \rightarrow \mathcal{M}_{k,m}^*$ is indeed the inverse of L^* □

In order to prove Proposition 2.4.8, it is therefore enough to prove that every eigenvalue of $L_* : \mathcal{M}_{k,m}^* \rightarrow \mathcal{M}_{k,m}^*$ is contained in the open unit disc \mathbb{D} . Inspired by the proof of [Koc13, Corollary 7.2], we will show that L_* is conjugate to a linear transformation on a space of meromorphic quadratic differentials on \mathbb{C} , whose eigenvalues are all contained in \mathbb{D} .

Consider the quadratic polynomial $P(t) := t^2 + z_1$, so that $z_i = P^{\circ i}(0)$ for all $i \geq 1$. Following Milnor [Mil14], denote by $\mathfrak{Q}(\mathbb{C})$ the space of meromorphic quadratic differentials on \mathbb{C} which have at worst simple poles and let us use the notation $Q \in \mathfrak{Q}(\mathbb{C})$ with

$$Q = q(t)dt^2.$$

and $q(t)$ is a meromorphic function. Let $U \subset \mathbb{C}$ be a sufficiently large disk so that $P^{-1}(U)$

is compactly contained in U and for $Q \in \mathfrak{Q}(\mathbb{C})$, consider the norm

$$\|Q\|_U := \iint_U |q(t)dt^2|.$$

The pushforward of Q by P is the quadratic differential $P_*Q \in \mathfrak{Q}(\mathbb{C})$ defined by

$$P_*Q := \sum_{P(u)=t} \frac{q(u)}{(P'(u))^2} dt^2.$$

It follows from the triangle inequality that

$$\|P_*Q\|_U \leq \|Q\|_{P^{-1}(U)} < \|Q\|_U.$$

For $i \geq 1$, let $Q_i \in \mathfrak{Q}(\mathbb{C})$ be the quadratic differential defined by

$$Q_i := \frac{dt^2}{t - z_i}.$$

According to [Mil14, Lemma 1]

$$\forall i \in \{1, \dots, k + m - 1\} \quad P_*Q_i = \frac{Q_{i+1} - Q_1}{\delta_i}. \quad (2.4.3)$$

The quadratic differentials $(Q_i)_{i \in [1, k+m-1]}$ span a vector space $\mathcal{Q}_P \subset \mathfrak{Q}(\mathbb{C})$ of dimension $k + m - 1$. According to Equation (2.4.3), this subspace is invariant by P_* . According to Equations (2.4.2) and (2.4.3), the linear map $\iota : \mathcal{Q}_P \rightarrow \mathcal{M}_{k,m}$ which sends $Q_i \in \mathfrak{Q}(\mathbb{C})$ to $\omega_i \in \mathcal{M}_{k,m}$ is an isomorphism which conjugates $P_* : \mathcal{Q}_P \rightarrow \mathcal{Q}_P$ to $L_* : \mathcal{M}_{k,m}^* \rightarrow \mathcal{M}_{k,m}^*$.

Since $\|P_*Q\|_U < \|Q\|_U$ for all $Q \in \mathcal{Q}_P$, the spectrum of $P_* : \mathcal{Q}_P \rightarrow \mathcal{Q}_P$ is contained in the unit disk. It follows that the spectrum of $L_* : \mathcal{M}_{k,m}^* \rightarrow \mathcal{M}_{k,m}^*$ is contained in the unit disk as required. □

Fixed points inside the post-critical set

We will now study the derivatives of moduli maps at fixed points which are inside the post-critical set. Let $z \in PC_{k,m}$ be a fixed point of $F_{k,m}$ and let (k', m') be the exact type of z .

According to Lemma 2.4.5, $(k', m') \prec (k, m)$ and, by Proposition 2.2.9, $\mathcal{M}_{k',m'} \subsetneq \mathcal{M}_{k,m}$

is invariant under $F_{k,m}$. Since $\mathcal{M}_{k',m'}$ is invariant under $D_{\mathbf{z}}F_{k,m}$, the vector space

$$\mathcal{M}_{k',m'}^0 = \{\omega \in \mathcal{M}_{k,m}^* \mid \omega|_{\mathcal{M}_{k',m'}} \equiv 0\},$$

which is called *the annihilator of $\mathcal{M}_{k',m'}$* in $\mathcal{M}_{k,m}$, is invariant under the transpose L^* and we have the following decomposition

$$\text{Spec } L = \text{Spec}(L|_{\mathcal{M}_{k',m'}}) \cup \text{Spec}\left(L^*|_{\mathcal{M}_{k',m'}^0}\right). \quad (2.4.4)$$

Moreover, according to Proposition 2.2.9, we have

$$L|_{\mathcal{M}_{k',m'}} = D_{\mathbf{z}}F_{k',m'}.$$

Whence, by Proposition 2.4.8, $L|_{\mathcal{M}_{k',m'}}$ has only repelling eigenvalues. In order to describe $\text{Spec } L$, we need to study $\text{Spec}\left(L^*|_{\mathcal{M}_{k',m'}^0}\right)$. We will prove the following result.

Proposition 2.4.10. *Given $(k, m) \in \mathbb{N} \times \mathbb{N}^*$, let $\mathbf{z} \in PC_{k,m}$ be a fixed point of $F_{k,m}$ of exact type $(k', m') \prec (k, m)$. Let λ be the multiplier of the polynomial $P(t) = t^2 + z_1 \in \mathbb{C}[t]$ along the cycle of $P^{\circ k'}$ (z_1). Then*

$$\text{Spec}\left((D_{\mathbf{z}}F_{k,m})^*|_{\mathcal{M}_{k',m'}^0}\right) = \begin{cases} \{0\} & \text{if } k' = 0 \\ \{\mu \mid \mu^m = \lambda^{\frac{m}{m'}}, \mu^{m'} \neq \lambda\} & \text{if } k' \neq 0. \end{cases}$$

The rest of this section is devoted to the proof of this proposition. To simplify the notation, we denote by L^* the restriction of $(D_{\mathbf{z}}F_{k,m})^*$ to $\mathcal{M}_{k,m}^0$. The study of the transpose $L^* : \mathcal{M}_{k',m'}^0 \rightarrow \mathcal{M}_{k',m'}^0$ is divided into two cases, $k' = 0$ and $k' \neq 0$, and each case will be treated separately.

Proof of Proposition 2.4.10 when $k' = 0$. Since $(0, m') \preceq (k, m)$, m' divides k and m . It is enough to prove that $L^* : \mathcal{M}_{0,m'}^0 \rightarrow \mathcal{M}_{0,m'}^0$ is nilpotent. Recall that for $i \geq 1$, the form $\omega_i : \mathcal{M}_{k,m} \rightarrow \mathbb{C}$ is defined by $\omega_i(\mathbf{v}) = v_i$. For $i \geq 1$, set $\alpha_i : \mathcal{M}_{k,m} \rightarrow \mathbb{C}$ defined by

$$\alpha_i = \omega_i - \omega_{i+m'}.$$

Recall that $\mathcal{M}_{0,m'}^0 = \{\omega : \mathcal{M}_{k,m} \rightarrow \mathbb{C} \text{ such that } \omega|_{\mathcal{M}_{0,m'}} \equiv 0\}$.

Lemma 2.4.11. *We have $\mathcal{M}_{0,m'}^0 = \text{Span}\{\alpha_i, 1 \leq i \leq k+m\}$.*

Proof. By duality, it is equivalent to show that

$$\mathcal{M}_{0,m'} = \bigcap_{1 \leq i \leq k+m} \text{Ker } \alpha_i.$$

Assume $\mathbf{v} \in \mathcal{M}_{0,m'}$. Then for all $j \geq 1$, $v_j = v_{j+m'}$. Given $i \in \{1, \dots, k+m\}$, we have

$$\alpha_i(\mathbf{v}) = \omega_i(\mathbf{v}) - \omega_{i+m'}(\mathbf{v}) = v_i - v_{i+m'} = 0.$$

Hence $\mathcal{M}_{0,m'} \subseteq \bigcap_{1 \leq i \leq k+m} \text{Ker } \alpha_i$.

Conversely, assume $\mathbf{v} \in \bigcap_{1 \leq i \leq k+m} \text{Ker } \alpha_i$, i.e. for all $i \in \{1, \dots, k+m\}$, $v_i = v_{i+m'}$. In order to prove that $\mathbf{v} \in \mathcal{M}_{0,m'}$, we will prove that for all $j \geq k+m+1$, $v_j = v_{j+m'}$ and that $v_{m'} = 0$. Given $j \geq k+m+1$, there exists an integer $j' \in \{k+1, \dots, k+m\}$ such that $j \equiv j' \pmod{m}$. Since $\mathbf{v} \in \mathcal{M}_{k,m}$, we have $v_j = v_{j'}$ and $v_{j+m'} = v_{j'+m'}$. Moreover, the fact that $\mathbf{v} \in \text{Ker } \alpha_{j'}$ implies that $v_{j'} = v_{j'+m'}$. Thus

$$v_j = v_{j'} = v_{j'+m'} = v_{j+m'}.$$

In order to conclude, we need to show that $v_{m'} = 0$. Note that the previous argument shows that \mathbf{v} is a periodic sequence of period dividing m' . Since m' divides k and m ,

$$v_{m'} = v_k = v_{k+m}.$$

Since $\mathbf{v} \in \mathcal{M}_{k,m}$, we have $v_k + v_{k+m} = 0$, whence $v_{m'} = v_k = v_{k+m} = 0$. □

Lemma 2.4.12. *We have $L^* \alpha_1 = 0$ and for $i \geq 2$, $L^* \alpha_i = \delta_{i-1} \alpha_{i-1}$.*

Proof. According to Lemma 2.4.7, for all $i \geq 2$,

$$L^* \omega_i = \delta_{i-1} \omega_{i-1} + L^* \omega_1.$$

Hence, if $i \geq 2$,

$$L^* \alpha_i = L^*(\omega_i - \omega_{i+m'}) = \delta_{i-1} \omega_{i-1} - \delta_{i+m'-1} \omega_{i+m'-1}.$$

Since $\mathbf{z} \in \mathcal{M}_{0,m'}$, we have $\delta_{i-1} = 2z_{i-1} = 2z_{i+m'-1} = \delta_{i+m'-1}$. Hence

$$L^* \alpha_{i-1} = \delta_{i-1}(\omega_{i-1} - \omega_{i+m'-1}) = \delta_{i-1} \alpha_{i-1}.$$

If $i = 1$, since $m' \geq 1$, we have $1 + m' \geq 2$ so that

$$L^* \omega_{1+m'} = \delta_{m'} \omega_{m'} + L^* \omega_1.$$

Hence

$$L^* \alpha_1 = L^*(\omega_1 - \omega_{1+m'}) = -\delta_{m'} \omega_{m'}.$$

Since $\mathbf{z} \in \mathcal{M}_{0,m'}$, we have $z_{m'} = 0$. Therefore, $\delta_{m'} = 2z_{m'} = 0$ and $L^* \alpha_1 = 0$. \square

It follows from Lemma 2.4.11 and Lemma 2.4.12 that $L^* : \mathcal{M}_{0,m'}^0 \rightarrow \mathcal{M}_{0,m'}^0$ is nilpotent. \square

Proof of Proposition 2.4.10 when $k' \neq 0$. In this case, since $(k', m') \prec (k, m)$, we have

$$k' = k \text{ and } m = pm' \text{ with } p \geq 2.$$

Let λ be the multiplier of $P(t) = t^2 + z_1$ at $P^{\circ k}(z_1)$. Note that, according to Proposition 2.4.4, \mathbf{z} is the critical orbit of P . Since \mathbf{z} is preperiodic of preperiod $k > 0$, the critical point 0 of P is preperiodic, i.e. $\lambda \neq 0$. We will show that

$$\text{Spec}(L^* : \mathcal{M}_{k,m'}^0 \rightarrow \mathcal{M}_{k,m'}^0) = \{\mu \mid \mu^m = \lambda^p, \mu^{m'} \neq \lambda\}.$$

Given $j \in \mathbb{Z}/m\mathbb{Z}$, denote by \underline{j} the representative of j in $\{k+1, \dots, k+m\}$, define a linear form $\beta_j : \mathcal{M}_{k,m} \rightarrow \mathbb{C}$ by

$$\beta_j := \omega_{\underline{j}} - \omega_{\underline{j+m'}}.$$

Note that for all $j \in \mathbb{Z}/m\mathbb{Z}$, $\beta_j : \mathcal{M}_{k,m} \rightarrow \mathbb{C}$ is non trivial. In fact, for any $j \in \mathbb{Z}/m\mathbb{Z}$, define $\mathbf{u} \in \mathcal{M}_{k,m}$ by

$$u_i = \begin{cases} 1 & \text{if } i \geq k+1 \text{ and } i \equiv \underline{j} \pmod{m} \\ -1 & \text{if } i = k \text{ and } \underline{j} = k+m \\ 0 & \text{otherwise.} \end{cases}$$

Since $m' < m$, $\beta_j(\mathbf{u}) = u_{\underline{j}} = 1 \neq 0$.

We will show that these forms span $\mathcal{M}_{k,m'}^0 \subsetneq \mathcal{M}_{k,m}$ and use them to study the linear map $L^* : \mathcal{M}_{k,m'}^0 \rightarrow \mathcal{M}_{k,m'}^0$. The properties we need are provided by the following lemmas.

Lemma 2.4.13. *We have $\mathcal{M}_{k,m'}^0 = \text{Span}\{\beta_j, j \in \mathbb{Z}/m\mathbb{Z}\}$.*

Proof. By duality, it is equivalent to show that

$$\mathcal{M}_{k,m'} = \bigcap_{j \in \mathbb{Z}/m\mathbb{Z}} \text{Ker } \beta_j.$$

Assume $\mathbf{v} \in \mathcal{M}_{k,m'}$. Then for all $i \geq k+1$, $v_i = v_{i+m'}$. Since $\underline{j} \in j$ and m' divides m , we have $\underline{j} \equiv \underline{j+m'} \pmod{m'}$. Moreover, $\underline{j} \geq k+1$ and $\underline{j+m'} \geq k+1$, whence

$$\beta_j(\mathbf{v}) = v_{\underline{j}} - v_{\underline{j+m'}} = 0.$$

This shows that $\mathcal{M}_{k,m'} \subseteq \bigcap_{j \in \mathbb{Z}/m\mathbb{Z}} \text{Ker } \beta_j$.

Conversely, assume $\mathbf{v} \in \bigcap_{j \in \mathbb{Z}/m\mathbb{Z}} \text{Ker } \beta_j$. We want to prove that for all integer $i \geq k+1$, $v_i = v_{i+m'}$ and that $v_k + v_{k+m'} = 0$. First, assume $i \geq k+1$ and let $j \in \mathbb{Z}/m\mathbb{Z}$ be the congruence class of i . Since $\underline{j} \in j$, we have $i \equiv \underline{j} \pmod{m}$. Moreover $\mathbf{v} \in \text{Ker } \beta_j$, and so

$$v_{\underline{j}} - v_{\underline{j+m'}} = \beta_j(\mathbf{v}) = 0.$$

From the fact that $\mathbf{v} \in \mathcal{M}_{k,m}$, we therefore deduce that $v_i = v_{\underline{j}}$ and $v_{\underline{j+m'}} = v_{i+m'}$. Thus,

$$v_i = v_{i+m'}.$$

Second, let us show $v_k + v_{k+m'} = 0$. The previous argument shows that \mathbf{v} is preperiodic of preperiod at most k and of period dividing m' . Since m' divides m , we deduce that $v_{k+m'} = v_{k+m}$. Since $\mathbf{v} \in \mathcal{M}_{k,m}$, we have $v_k + v_{k+m} = 0$. Thus $v_k + v_{k+m'} = 0$. \square

Thus, it is now important to understand the how L^* acts on $\{\beta_j, j \in \mathbb{Z}/m\mathbb{Z}\}$. Given $j \in \mathbb{Z}/m\mathbb{Z}$, set

$$\sigma_j = \delta_{\underline{j}} = 2z_{\underline{j}}.$$

Lemma 2.4.14. *For $j \in \mathbb{Z}/m\mathbb{Z}$, we have $L^*\beta_j = \sigma_{j-1}\beta_{j-1}$.*

Proof. For $j \in \mathbb{Z}/m\mathbb{Z}$, recall that \underline{j} is the representative of j in $\llbracket k+1, k+m \rrbracket$. Let us first prove that for all $j \in \mathbb{Z}/m\mathbb{Z}$,

$$L^*\omega_{\underline{j}} = \sigma_{j-1}\omega_{\underline{j-1}} + L^*\omega_1. \tag{2.4.5}$$

Indeed, if $\underline{j} = k+1$ then $\underline{j-1} = k+m$, whence $\sigma_{j-1}\omega_{\underline{j-1}} = \delta_{k+m}\omega_{k+m}$. According to Lemma 2.4.7, we have $L^*\omega_{k+1} = \delta_k\omega_k + L^*\omega_1$. Since $\mathbf{z} \in \mathcal{M}_{k,m}$, we have $\delta_k = 2z_k =$

$-2z_{k+m} = -\delta_{k+m}$. Moreover, $\omega_k = -\omega_{k+m}$, whence $\delta_k\omega_k = \delta_{k+m}\omega_{k+m}$. Thus

$$L^*\omega_{k+1} = \delta_{k+m}\omega_{k+m} + L^*\omega_1.$$

If $\underline{j} \neq k+1$ then $\underline{j-1} = \underline{j} - 1$. According to Lemma 2.4.7, we have

$$L^*\omega_{\underline{j}} = \delta_{\underline{j-1}}\omega_{\underline{j-1}} + L^*\omega_1 = \delta_{\underline{j-1}}\omega_{\underline{j-1}} + L^*\omega_1 = \sigma_{j-1}\omega_{\underline{j-1}} + L^*\omega_1.$$

In any case, we have the equality (2.4.5). Hence

$$L^*\beta_j = L^*(\omega_{\underline{j}} - \omega_{\underline{j+m'}}) = \sigma_{j-1}\omega_{\underline{j-1}} - \sigma_{j+m'-1}\omega_{\underline{j+m'-1}}.$$

Note that $\underline{j-1}$ and $\underline{j+m'-1}$, which are congruence modulo m' , are two integers at least $k+1$. Since $\mathbf{z} \in \mathcal{M}_{k,m'}$, we have $\sigma_{j-1} = \delta_{\underline{j-1}} = \delta_{\underline{j+m'-1}} = \sigma_{j+m'-1}$. Thus

$$L^*\beta_j = \sigma_{j-1}(\omega_{\underline{j-1}} - \omega_{\underline{j+m'-1}}) = \sigma_{j-1}\beta_{j-1}. \quad \square$$

Recall that λ is the multiplier of $P(t) = t^2 + z_1$ at the periodic point z_{k+1} of period $m' = \frac{m}{p}$.

Lemma 2.4.15. *For all $j \in \mathbb{Z}/m\mathbb{Z}$,*

$$(L^*)^{m'}(\beta_j) = \lambda\beta_{j-m'} \quad \text{and} \quad (L^*)^m(\beta_j) = \lambda^p\beta_j.$$

Proof. The second equality is the straightforward consequence of the first one. Hence, it is enough to prove the first equality. According to Proposition 2.4.4, \mathbf{z} is the critical orbit of the polynomial $P(t) = t^2 + z_1$, i.e. $z_i = P^i(0)$. In particular, for any $j \in \mathbb{Z}/m\mathbb{Z}$, we have $P(z_{\underline{j-1}}) = z_{\underline{j}}$. Since \mathbf{z} is of type (k, m') , the multiplier λ of the cycle of $z_{\underline{j}}$ is

$$\lambda = \prod_{i \in \llbracket 1, m' \rrbracket} P'(z_{\underline{j-i}}) = \prod_{i \in \llbracket 1, m' \rrbracket} \delta_{\underline{j-i}} = \prod_{i \in \llbracket 1, m' \rrbracket} \sigma_{j-i}.$$

Hence, by Lemma 2.4.14, we have

$$(L^*)^{m'}(\beta_j) = \left(\prod_{i \in \llbracket 1, m' \rrbracket} \sigma_{j-i} \right) \beta_{j-m'} = \lambda\beta_{j-m'},$$

and

$$(L^*)^m(\beta_j) = (L^*)^{m'p}(\beta_j) = \lambda^p\beta_j. \quad \square$$

Let ν be a m' -th root of λ . Set

$$T = \frac{L^*}{\nu} : \mathcal{M}_{k,m'}^0 \rightarrow \mathcal{M}_{k,m'}^0.$$

We shall prove that T is diagonalizable with simple eigenvalues and the eigenvalues of T are m -th roots of unity except 1. According to Lemma 2.4.15, for all $j \in \mathbb{Z}/m\mathbb{Z}$, $T^{m'}(\beta_j) = \beta_{j-m'}$. In addition,

$$\begin{aligned} \sum_{n \in m'\mathbb{Z}/m\mathbb{Z}} \beta_n &= \sum_{n \in m'\mathbb{Z}/m\mathbb{Z}} (\omega_n - \omega_{n+m'}) = 0 \\ &= \sum_{n \in m'\mathbb{Z}/m\mathbb{Z}} \omega_n - \sum_{n \in m'\mathbb{Z}/m\mathbb{Z}} \omega_{n+m'} = 0. \end{aligned}$$

Recall that $p = \frac{m}{m'}$. Hence

$$\beta_0 + T^{m'}(\beta_0) + \dots + T^{m'(p-1)}(\beta_0) = 0$$

Applying $m' - 1$ times T and adding the results, we deduce that

$$\beta_0 + T(\beta_0) + T^2(\beta_0) + \dots + T^{m'p-1}(\beta_0) = 0$$

According to Lemma 2.4.13 and Lemma 2.4.14, the set $\{\beta_0, L^*(\beta_0), (L^*)^2(\beta_0), \dots\}$ generates $\mathcal{M}_{k,m'}^0$. Hence $\{\beta_0, T(\beta_0), T^2(\beta_0), \dots\}$ also generates $\mathcal{M}_{k,m'}^0$. Therefore,

$$\text{Id} + T + T^2 + \dots + T^{m-1} = 0 \tag{2.4.6}$$

This means that the minimal polynomial of T divides the polynomial $1 + X + X^2 + \dots + X^{m-1}$. Consequently, T is diagonalizable and the eigenvalues of T are roots of unity which are not 1. We now show that T has only simple eigenvalues. Assume $\zeta \in \text{Spec } T$. Let $v \in \mathcal{M}_{k,m'}$ be an eigenvector associated to ζ . Set

$$H_\zeta = \frac{1}{m} \left(\text{Id} + \frac{T}{\zeta} + \dots + \frac{T^{m-1}}{\zeta^{m-1}} \right).$$

The equality (2.4.6) implies that $T^m = \text{Id}$. Additionally, $\zeta^m = 1$. Hence

$$H_\zeta \circ \frac{T}{\zeta} = H_\zeta \quad \text{so that} \quad H_\zeta \circ T^j = \zeta^j H_\zeta \quad \forall j \geq 1.$$

In addition, $\{\beta_0, T(\beta_0), \dots\}$ generates $\mathcal{M}_{k,m'}^0$, hence

$$\text{Im } H_\zeta \subseteq \text{Span } H_\zeta(\beta_0).$$

Note that $H_\zeta(v) = v$. Hence

$$v \subseteq \text{Im } H_\zeta \subseteq \text{Span } H_\zeta(\beta_0).$$

Thus the eigenspace associated to the eigenvalue ζ of T has dimension 1, i.e. T has only simple eigenvalues.

Since $T = \frac{L^*}{\nu}$, L^* is diagonalizable with simple eigenvalues which are m -th roots of $\nu^m = \lambda^p$. In addition, 1 is not an eigenvalue of T hence ν is not an eigenvalue of L^* . Since ν is an arbitrary m' -th root of λ , we deduce that

$$\text{Spec } L^* \subseteq \{\mu^m = \lambda^p, \mu^{m'} \neq \lambda\}.$$

Since L^* has only simple eigenvalues, $\#\text{Spec } L^* = \dim \mathcal{M}_{k,m'}^0 = m - m'$. Hence

$$\text{Spec}(L^* : \mathcal{M}_{k,m'}^0 \rightarrow \mathcal{M}_{k,m'}^0) = \{\mu^m = \lambda^p, \mu^{m'} \neq \lambda\} \quad \square$$

We conclude this chapter by showing that Conjecture A is true for the eigenvalues of $G_{k,m}$ at a fixed point.

Proof of Theorem A. According to Proposition 2.2.7, $F_{k,m}$ and $G_{k,m}$ are conjugate. Hence it is enough to consider an eigenvalue μ of $F_{k,m}$ at a fixed point $z \in \mathcal{M}_{k,m}$ and show that either $\mu = 0$ or $|\mu| > 1$. When $z \notin PC(F_{k,m})$, the result follows from Proposition 2.4.8. When $z \in PC(F_{k,m})$, according to Proposition 2.4.4 and Lemma 2.4.5, the polynomial $P(t) = t^2 + z_1 \in \mathbb{C}[t]$ is post-critically finite and that the orbit of z_1 under P is preperiodic of preperiod k' to a cycle of length m' and of multiplier λ such that $(k', m') \prec (k, m)$. Moreover, by Proposition 2.2.9, we have that $\mathcal{M}_{k',m'} \subsetneq \mathcal{M}_{k,m}$, $F_{k,m}|_{\mathcal{M}_{k',m'}} = F_{k',m'}$, and hence, we have the following decomposition

$$\text{Spec } D_z F_{k,m} = \text{Spec}(D_z F_{k,m}|_{\mathcal{M}_{k',m'}}) \cup \text{Spec} \left((D_z F_{k,m})^*|_{\mathcal{M}_{k',m'}^0} \right).$$

If $\mu \in \text{Spec } D_z F_{k,m}|_{\mathcal{M}_{k',m'}} = \text{Spec } D_z F_{k',m'}$, it follows from Item 4 of Theorem A' that $|\mu| > 1$, and we are done. If $\mu \in \text{Spec} \left((D_z F_{k,m})^*|_{\mathcal{M}_{k',m'}^0} \right)$, it follows from Item 5 of Theorem A' that either $\mu = 0$ if $k' = 0$ or $\mu^m = \lambda^{\frac{m}{m'}}$ if $k' \neq 0$. If $k' = 0$, we are done. If $k' \neq 0$, since P is post-critically finite, λ is repelling, i.e. $|\lambda| > 1$, and so

$$|\mu| = |\lambda|^{\frac{1}{m'}} > 1.$$

□

Chapter 3

Eigenvalues along periodic cycles of post-critically algebraic endomorphisms

3.1 Statements of the main results

As we have seen in the previous chapters, in many cases, the eigenvalues of a post-critically algebraic endomorphism of $\mathbb{C}\mathbb{P}^n$ along a periodic cycle are either 0 or of modulus strictly bigger than 1. In this chapter, we shall focus on the following conjecture.

Conjecture A. *Let f be a post-critically algebraic endomorphism of $\mathbb{C}\mathbb{P}^n$ and λ be an eigenvalue of f along a periodic cycle. Then either $\lambda = 0$ or $|\lambda| > 1$.*

This conjecture was studied, without being stated explicitly in the form above, in:

- [FS94, Theorem 6.1] under the assumption that the complement of $PC(f)$ is Kobayashi hyperbolic and hyperbolically embedded,
- [Jon98, Proposition 2.9] in the case of dimension $n = 2$ and under the assumption that C_f has no periodic component,
- [Ast20, Theorem B] under a mild geometric assumption on the irreducible components of $PC(f)$.

In this chapter, we improve the above results: we prove the conjecture when $n = 2$ without any additional hypothesis, and also in any dimension for $z \notin PC(f)$.

More precisely, let $f : \mathbb{C}\mathbb{P}^n \rightarrow \mathbb{C}\mathbb{P}^n$ be a post-critically algebraic endomorphism of $\mathbb{C}\mathbb{P}^n$ and let z be a periodic point of f . Since every iterate of f is also post-critically finite, by passing to an iterate, it is enough to assume that z is a fixed point of f . We have to deal with two main cases: the fixed point is either outside the post-critical set or inside the post-critical set.

When the fixed point is outside the post-critical set, we improve the method of [FS94, Theorem 6.1] to get rid of the Kobayashi hyperbolic assumption and exclude the possibility of eigenvalues of modulus 1. We obtain the following general result.

Theorem B. Let f be a post-critically algebraic endomorphism of $\mathbb{C}\mathbb{P}^n$ and λ be an eigenvalue of f along a periodic cycle outside the post-critical set. Then $|\lambda| > 1$.

When the fixed point is inside the post-critical set, we restrict our study to dimension $n = 2$. The second main result of this chapter is the proof of Conjecture A when $n = 2$ without any additional hypothesis.

Theorem C. Let f be a post-critically algebraic endomorphism of $\mathbb{C}\mathbb{P}^2$ and λ be an eigenvalue of f along a periodic cycle. Then either $\lambda = 0$ or $|\lambda| > 1$.

More precisely, let f be a post-critically algebraic endomorphism of $\mathbb{C}\mathbb{P}^2$ and let z be a periodic point of f . Thanks to Theorem B, we can assume that z is a fixed point of f in $PC(f)$. The derivative $D_z f$ has two eigenvalues λ_1 and λ_2 , counting multiplicities. We consider two subcases: either z is a regular point of $PC(f)$, or z is a singular point of $PC(f)$.

If the fixed point z is a regular point of $PC(f)$, the tangent space $T_z PC(f)$ is invariant by $D_z f$. Then $D_z f$ admits an eigenvalue λ_1 with associated eigenvectors in $T_z PC(f)$. The other eigenvalue λ_2 arises as the eigenvalue of the linear endomorphism $\overline{D_z f} : T_z \mathbb{C}\mathbb{P}^2 / T_z PC(f) \rightarrow T_z \mathbb{C}\mathbb{P}^2 / T_z PC(f)$ induced by $D_z f$. On one hand, we prove that the eigenvalue λ_1 has modulus strictly bigger than 1 by using the normalization of irreducible algebraic curves and Theorem 1, which is a classical theorem about the eigenvalue of post-critically finite rational map. On the other hand, we show that either $\lambda_2 = 0$ or $|\lambda_2| > 1$ by following the same idea used to prove Theorem B,

If the fixed point z is a singular point of $PC(f)$, in most of the cases, there exists a relation between λ_1 and λ_2 . Then by using Theorem 1, we deduce that for $j = 1, 2$, either $\lambda_j = 0$ or $|\lambda_j| > 1$. This was observed in [Jon98]. For the sake of completeness, we will recall the detailed statements and include the proof. This allows us to conclude the proof of Conjecture A for post-critically algebraic endomorphisms of $\mathbb{C}\mathbb{P}^2$.

The rest of this chapter is devoted to the proof of Theorem B and Theorem C. In Section 3.2, we will recall the results of Ueda and prove that when a fixed point is not

a critical point then every eigenvalue has modulus at least 1. In Section 3.3, we will present the strategy and the proof of Theorem B. In Section 3.4, we will give the proof that for an eigenvalue λ of a post-critically algebraic map at a fixed point which is a regular point of the post-critical set and the associated eigenvectors are not tangent to the post-critical set, then λ is either zero or of modulus strictly bigger than 1. In Section 3.5, we will study the dynamics of post-critically algebraic endomorphisms of $\mathbb{C}\mathbb{P}^2$ and prove Theorem C.

3.2 Periodic cycles outside the critical set

In this section, we will prove that the eigenvalues of a post-critically algebraic endomorphism of $\mathbb{C}\mathbb{P}^n$ at a fixed point, which is not a critical point, have modulus at least or equal to 1. The proof relies on the existence of an open subset on which we can find a family of inverse branches and the fact that the family of inverse branches of endomorphisms of $\mathbb{C}\mathbb{P}^n$ is normal. These results are due to Ueda, [Ued98].

Recall that endomorphisms f of $\mathbb{C}\mathbb{P}^n$ are finite branched covering ramifying over $f(C_f)$. If f is post-critically algebraic, for every $j \geq 1$, $f^{\circ j}$ is ramified over $PC(f)$.

Let $z \notin C_f$ be a fixed point of f . Then $f^{\circ j}$ is locally invertible in a neighborhood of z . The following result which are due to Ueda ensures that we can find a common open neighborhood on which inverse branches of $f^{\circ j}$ fixing z are well-defined for every $j \geq 1$.

Lemma 3.2.1 ([Ued98], Lemma 3.8). *Let X be a complex manifold and D be an analytic subset of X of codimension 1. For every point $x \in X$, if W is a simply connected open neighborhood of x such that $(W, W \cap D)$ is homeomorphic to a cone over $(\partial W, \partial W \cap D)$ with vertex at x , then for every branched covering $\eta : Y \rightarrow W$ ramifying over $D \cap W$, the set $\eta^{-1}(x)$ consists of only one point.*

Proposition 3.2.2. *Let f be a post-critically algebraic endomorphism of $\mathbb{C}\mathbb{P}^n$ of degree $d \geq 2$ and let $z \notin C_f$ be a fixed point of f . Let W be a simply connected open neighborhood of z such that $(W, W \cap PC(f))$ is homeomorphic to a cone over $(\partial W, \partial W \cap PC(f))$ with vertex at z . Then there exists a family of holomorphic inverse branches $h_j : W \rightarrow \mathbb{C}\mathbb{P}^n$ of $f^{\circ j}$ fixing z , that is,*

$$h_j(z) = z, f^{\circ j} \circ h_j = \text{Id}|_W.$$

Note that for a fixed point z of a post-critically algebraic endomorphism f of $\mathbb{C}\mathbb{P}^n$, since $PC(f)$ is an algebraic set, there always exists a simply connected neighborhood W of z such that $(W, W \cap PC(f))$ is homeomorphic to a cone over $(\partial W, \partial W \cap PC(f))$ with vertex at z . Indeed, if $z \notin PC(f)$ then we can take any simply connected neighborhood

W of z in $\mathbb{C}\mathbb{P}^n \setminus PC(f)$. If $z \in PC(f)$ then it follows from [Sea19, Theorem 3.2] that such a neighborhood always exists. We refer to [Mil16] to an approach when z is an isolated singularity of $PC(f)$, see also [Sea05, Remark 2.3].

Proof of Proposition 3.2.2. For every $j \geq 1$, denote by W_j the connected component of $f^{-j}(W)$ containing z . Since $f^{\circ j}$ are branched covering ramifying over $PC(f)$, $f^{\circ j}$ induces a branched covering

$$f_j := f^{\circ j}|_{W_j} : W_j \rightarrow W$$

ramifying over $W \cap PC(f)$. By Lemma 3.2.1, we deduce that $f_j^{-1}(z)$ consists of only one point, which is in fact z . Since W is simply connected hence connected, the order of the branched covering f_j coincides with the branching order of $f^{\circ j}$ at z . Note that z is not a critical point of $f^{\circ j}$ therefore the branching order of $f^{\circ j}$ at z is 1. This means that f_j is branched covering of order one of complex manifolds, thus f_j is homeomorphism hence biholomorphism (see [Gun90, Corollary 11Q]). The holomorphic map $h_j : W \rightarrow W_j$ defined as the inverse of f_j is the map we are looking for. \square

Once we obtain a family of inverse branches, the following theorem, which is due to Ueda, implies that this family is normal.

Theorem 3.2.3 ([Ued98], Theorem 2.1). *Let f be an endomorphism of $\mathbb{C}\mathbb{P}^n$. Let X be a complex manifold with a holomorphic map $\pi : X \rightarrow \mathbb{C}\mathbb{P}^n$. Let $\{h_j : X \rightarrow \mathbb{C}\mathbb{P}^n\}_j$ be a family of holomorphic lifts of $f^{\circ j}$ by π , that is $f^{\circ j} \circ h_j = \pi$. Then $\{h_j\}_j$ is a normal family.*

Thus, for a fixed point z of a post-critically algebraic endomorphism f , if z is not a critical point (or equivalently, $D_z f$ is invertible), we can obtain an open neighborhood W of z in $\mathbb{C}\mathbb{P}^n$ and a normal family of holomorphic maps $\{h_j : W \rightarrow \mathbb{C}\mathbb{P}^n\}_j$ such that

$$f^{\circ j} \circ h_j = \text{Id}_W, h_j(z) = z.$$

The normality of $\{h_j\}_j$ implies that $\{D_z h_j\}_j$ is a uniformly bounded sequence (with respect to a fixed norm $\|\cdot\|$ on $T_z \mathbb{C}\mathbb{P}^n$). Since $D_z h_j = (D_z f)^{-j}$, we can deduce that every eigenvalue of $D_z f$ has modulus at least 1. Consequently, we can find a $D_z f$ -invariant decomposition of $T_z \mathbb{C}\mathbb{P}^n$ as the following direct sum

$$T_z \mathbb{C}\mathbb{P}^n = \left(\bigoplus_{\lambda \in \text{Spec } D_z f, |\lambda|=1} E_\lambda \right) \oplus \left(\bigoplus_{\lambda \in \text{Spec } D_z f, |\lambda|>1} E_\lambda \right) = E_n \oplus E_r$$

where E_λ is the generalized eigenspace of the eigenvalue λ . We call E_n the neutral eigenspace and E_r the repelling eigenspace of $D_z f$ (see also 3.3.2). If E_λ is not generated by eigenvectors (or equivalently, $D_z f|_{E_\lambda}$ is not diagonalizable), we can find at least two generalized eigenvectors e_1, e_2 of $D_z f$ corresponding to λ such that

$$D_z f(e_1) = \lambda e_1, D_z f(e_2) = \lambda e_2 + e_1.$$

Then $D_z h_j(e_2) = \lambda^{-j} e_2 - j \lambda^{-(j+1)} e_1$. If $|\lambda| = 1$, then $\|D_z h_j(e_2)\|$ tends to ∞ as j tends to ∞ . This contradicts the uniform boundedness of $\{D_z h_j\}_j$. Hence $D_z f|_{E_\lambda}$ is diagonalizable. Thus, we have proved the following result.

Corollary 3.2.4. *Let f be a post-critically algebraic endomorphism of $\mathbb{C}\mathbb{P}^n$ of degree $d \geq 2$ and let λ be an eigenvalue of f at a fixed point $z \notin C_f$. Then $|\lambda| \geq 1$. Moreover, if $|\lambda| = 1$, $D_z f|_{E_\lambda}$ is diagonalizable.*

3.3 Periodic cycles outside the post-critical set

In this section, we prove Theorem B.

Theorem B. *Let f be a post-critically algebraic endomorphism of $\mathbb{C}\mathbb{P}^n$ and λ be an eigenvalue of f along a periodic cycle outside the post-critical set. Then $|\lambda| > 1$.*

In fact, we will consider an equivalent statement and prove this equivalent statement.

3.3.1 Equivalent problem in the affine case.

Recall that for an endomorphism $f : \mathbb{C}\mathbb{P}^n \rightarrow \mathbb{C}\mathbb{P}^n$ of degree d , there exists a polynomial endomorphism

$$F = (P_1, \dots, P_{n+1}) : \mathbb{C}^{n+1} \rightarrow \mathbb{C}^{n+1}$$

where P_i are homogeneous polynomials of the same degree $d \geq 1$ and $F^{-1}(0) = \{0\}$ such that

$$f \circ \pi = \pi \circ F$$

where $\pi : \mathbb{C}^{n+1} \setminus \{0\} \rightarrow \mathbb{C}\mathbb{P}^n$ is the canonical projection. The integer d is called the *algebraic degree* (or *degree*) of f . Such a map F is called a *lift* of f to \mathbb{C}^{n+1} . Further details about holomorphic endomorphisms of $\mathbb{C}\mathbb{P}^n$ and their dynamics can be found in [Sib99], [Gun90], [SSC⁺10], [FS94],[FS95].

Lifts to \mathbb{C}^{n+1} of an endomorphism of $\mathbb{C}\mathbb{P}^n$ belong to a class of *non-degenerate homogeneous polynomial endomorphisms* of \mathbb{C}^{n+1} . More precisely, a non-degenerate homogeneous polynomial endomorphism of \mathbb{C}^n of algebraic degree d is a polynomial map $F : \mathbb{C}^n \rightarrow \mathbb{C}^n$ such that $F(\lambda z) = \lambda^d z$ for every $z \in \mathbb{C}^n, \lambda \in \mathbb{C}$ and $F^{-1}(0) = \{0\}$. Conversely, such a map induces an endomorphism of $\mathbb{C}\mathbb{P}^n$. This kind of maps has been studied extensively in [HP94]¹. If we consider \mathbb{C}^{n+1} as a dense open set of $\mathbb{C}\mathbb{P}^{n+1}$ by the inclusion $(\zeta_0, \zeta_1, \dots, \zeta_n) \mapsto [\zeta_0 : \zeta_1 : \dots : \zeta_n : 1]$ then F can be extended to an endomorphism of $\mathbb{C}\mathbb{P}^{n+1}$. Moreover, this extension fixes the hypersurface at infinity $\mathbb{C}\mathbb{P}^{n+1} \setminus \mathbb{C}^{n+1} \cong \mathbb{C}\mathbb{P}^n$ and the restriction to this hypersurface is the endomorphism of $\mathbb{C}\mathbb{P}^n$ induced by F .

Thus, if f is a post-critically algebraic endomorphism of $\mathbb{C}\mathbb{P}^n$, every lift F of f to \mathbb{C}^{n+1} is the restriction of a post-critically algebraic endomorphism of $\mathbb{C}\mathbb{P}^{n+1}$ to \mathbb{C}^{n+1} . Post-critically algebraic non-degenerate homogeneous polynomial endomorphisms of \mathbb{C}^{n+1} have similar properties, which are proved in Section 2, as post-critically algebraic endomorphisms of $\mathbb{C}\mathbb{P}^n$. More precisely, we can sum up in the following proposition.

Proposition 3.3.1. *Let F be a post-critically algebraic non-degenerate homogeneous polynomial endomorphism of \mathbb{C}^{n+1} of degree $d \geq 2$ and let $z \notin C_F$ be a fixed point of F . Then, we have the following assertions.*

- a. *Let X be a complex manifold and let $\pi : X \rightarrow \mathbb{C}^{n+1}$ be a holomorphic map. Then every family of holomorphic maps $\{h_j : X \rightarrow \mathbb{C}^{n+1}\}_j$, which satisfies that $F^{\circ j} \circ h_j = \pi$ for every $j \geq 1$, is normal.*
- b. *There exist a simply connected open neighborhood W of z in \mathbb{C}^{n+1} and a family $\{h_j : W \rightarrow \mathbb{C}^{n+1}\}_j$ of inverse branches of iterates of F , i.e. $F^{\circ j} \circ h_j = \text{Id}_W$, fixing z .*
- c. *Every eigenvalue λ of $\text{Spec}(D_z F)$ has modulus at least 1. The tangent space $T_z \mathbb{C}^{n+1}$ admits a $D_z F$ -invariant decomposition $T_z \mathbb{C}^{n+1} = E_n \oplus E_r$ where the neutral eigenspace E_n is the sum of generalized eigenspaces corresponding to eigenvalues of modulus 1 and the repelling E_r is the sum of generalized eigenspaces corresponding to eigenvalues of modulus strictly bigger than 1.*
- d. *If $|\lambda| = 1$ then $D_z f|_{E_\lambda}$ is diagonalizable.*

Remark 3.3.2. Regarding eigenvalues at fixed points, studying eigenvalues at fixed points of post-critically algebraic endomorphisms of $\mathbb{C}\mathbb{P}^n$ is equivalent to studying eigenvalues

¹In fact, these are regular polynomial self-maps of \mathbb{C}^n (see 1.3.2) which are homogeneous

at fixed points of post-critically algebraic non-degenerate homogeneous polynomial self-map of \mathbb{C}^{n+1} . More precisely, let $f : \mathbb{C}\mathbb{P}^n \rightarrow \mathbb{C}\mathbb{P}^n$ be an endomorphism of degree $d \geq 2$ and let F be a lift of f . Assume z is a fixed point of f . Then, the complex line L containing $\pi^{-1}(z)$ is invariant under F and the map induced by F on L is conjugate to $x \mapsto x^d$. In particular, there exists a fixed point $w \in L \setminus \{0\}$ of F such that $\pi(w) = z$ and $D_w F$ preserves $T_w L \subset T_w \mathbb{C}^{n+1}$ with an eigenvalue d . Then, $D_w F$ descends to a linear endomorphism of the quotient space $T_w \mathbb{C}^{n+1}/T_w(\mathbb{C}w)$ which is conjugate to $D_z f : T_z \mathbb{C}\mathbb{P}^n \rightarrow T_z \mathbb{C}\mathbb{P}^n$. Hence a value λ is an eigenvalue of $D_w F$ if and only if either λ is an eigenvalue of $D_z f$ or $\lambda = d$.

Conversely, if w is a fixed point of F then either $w = 0$ (and the eigenvalues of $D_0 F$ are all equal to 0) or w induces a fixed point $\pi(w)$ of f . Since we consider only post-critically algebraic endomorphisms of degree $d \geq 2$, Conjecture A is equivalent to the following conjecture.

Conjecture B. *Let f be a non-degenerate homogeneous polynomial post-critically algebraic endomorphism of \mathbb{C}^n and let λ be an eigenvalue of f along a periodic cycle. Then either $\lambda = 0$ or $|\lambda| > 1$.*

The advantage of this observation is that we can make use of some nice properties of the affine space \mathbb{C}^n . More precisely, we will use the fact that the tangent bundle of \mathbb{C}^n is trivial

3.3.2 Strategy of the proof of Theorem B

Recall that an eigenvalues λ of $D_z f : T_z \mathbb{C}^n \rightarrow T_z \mathbb{C}^n$ is called

- superattracting if $\lambda = 0$,
- attracting if $0 < |\lambda| < 1$,
- neutral if $|\lambda| = 1$,
 - parabolic or rational if λ is a root of unity,
 - elliptic or irrational if λ is not a root of unity,
- repelling if $|\lambda| > 1$,

(see Definition 1.2.3). By Remark 3.3.2, in order to prove Theorem B, it is enough to prove the following result.

Proposition 3.3.1. *Let f be a post-critically algebraic non-degenerate homogeneous polynomial endomorphism of \mathbb{C}^n of degree $d \geq 2$. Let λ be an eigenvalue of f at a fixed point $z \notin PC(f)$. Then λ is repelling.*

The strategy of the proof is as follows.

Step 1. Set $X = \mathbb{C}^n \setminus PC(f)$ and let $\pi : \tilde{X} \rightarrow X^2$ be its universal covering. We construct a holomorphic map $g : \tilde{X} \rightarrow \tilde{X}$ such that

$$f \circ \pi \circ g = \pi$$

and g fixes a point $[z]$ such that $\pi([z]) = z$.

Step 2. We prove that the family $\{g^{om}\}_m$ is normal. Then there exists a closed complex submanifold M of \tilde{X} passing through $[z]$ such that $g|_M$ is an automorphism and $\dim M$ is the number of eigenvalues of $D_z f$ of modulus 1, i.e. neutral eigenvalues, counted with multiplicities. Due to Corollary 3.2.4, it is enough to prove that $\dim M = 0$.

Step 3. In order to prove that $\dim M = 0$, we proceed by contradiction. Assuming that $\dim M > 0$. We then construct a holomorphic mapping $\Phi : M \rightarrow T_{[z]}M$ such that $\Phi([z]) = 0$, $D_{[z]}\Phi = \text{Id}$ and

$$\Phi \circ g = D_{[z]}g \circ \Phi.$$

We deduce that $D_z f$ has no parabolic eigenvalue.

Step 4. Assume that λ is a neutral irrational eigenvalue and v is an associated eigenvector. We prove that the irreducible component Γ of $\Phi^{-1}(\mathbb{C}v)$ containing $[z]$ is smooth and $\Phi|_\Gamma$ maps Γ biholomorphically onto a disc $\mathbb{D}(0, R)$, $0 < R < +\infty$.

Step 5. Denote by $\kappa : \mathbb{D}(0, R) \rightarrow M$ the inverse of $\Phi|_\Gamma$. We prove that $\pi \circ \kappa$ has radial limits almost everywhere on $\partial\mathbb{D}(0, R)$ and these radial limits land on ∂X . We deduce from this a contradiction and Proposition 3.3.1 is proved.

3.3.3 Lifting the backward dynamics via the universal covering

Denote by $X = \mathbb{C}^n \setminus PC(f)$ the complement of $PC(f)$ in \mathbb{C}^n . Since $PC(f)$ is an algebraic set, the set X is a connected open subset of \mathbb{C}^n then the universal covering of X is well

²Since we are now only work on \mathbb{C}^n , we won't use π as the canonical projection from \mathbb{C}^{n+1} to $\mathbb{C}\mathbb{P}^n$

defined. Denote by $\pi : \tilde{X} \rightarrow X$ the universal covering of X defined by

$$\tilde{X} = \{[\gamma] \mid \gamma \text{ is a path in } X \text{ starting at } z_0\} \quad \text{and} \quad \pi([\gamma]) = \gamma(1),$$

where $[\gamma]$ denotes the homotopy class of γ in X , fixing the endpoints $\gamma(0)$ and $\gamma(1)$. Denote by $[z]$ the element in \tilde{X} representing the homotopy class of the constant path at z . We endow \tilde{X} with a complex structure such that $\pi : \tilde{X} \rightarrow X$ is a holomorphic covering map.

Set $Y = f^{-1}(X) \subset X$ and $\tilde{Y} = \pi^{-1}(Y) \subset \tilde{X}$. Since $f : Y \rightarrow X$ is a covering map, every path γ in X starting at z_0 lifts to a path $f^*\gamma \subset Y$ starting at z_0 . In addition, if γ_1 and γ_2 are homotopic in X , then $f^*\gamma_1$ and $f^*\gamma_2$ are homotopic in $Y \subset X$, in particular in X . Thus, this pullback map f^* induces a map $g : \tilde{X} \rightarrow \tilde{X}$ such that the following diagram commutes:

$$\begin{array}{ccc} \tilde{X} & \xleftarrow{g} & \tilde{X} \\ \pi \downarrow & & \downarrow \pi \\ X & \xrightarrow{f} & X \end{array}$$

Note that $g([z]) = [z]$. In addition, g is holomorphic since in local charts given by π , it coincides with inverse branches of f .

3.3.4 Normality of maps on the universal covering

We will prove that the family $\{g^{oj}\}_j$ is a normal family. For every integer $j \geq 1$, define $k_j = \pi \circ g^{oj}$ so that $f^{oj} \circ k_j = \pi$.

Lemma 3.3.2. *The family $\{k_j : \tilde{X} \rightarrow X\}_j$ is normal and any limit takes values in X .*

Proof. By Proposition 3.3.1, the family $\{k_j : \tilde{X} \rightarrow \mathbb{C}^n\}_j$ is normal. Denote by $Q : \mathbb{C}^n \rightarrow \mathbb{C}$ a polynomial such that $PC(f)$ is the zeros locus of Q . Consider the family

$$Q_j = Q \circ k_j : \tilde{X} \rightarrow \mathbb{C}.$$

The family $\{Q_j\}$ is a normal family of nonvanishing function. Then by Hurwitz's theorem, every limit map is either a nonvanishing function or a constant function. But $Q_j([z]) = Q(z) \neq 0$ hence every limit map is a nonvanishing function, i.e. every limit map of $\{k_j\}$ is valued in X . Thus $\{k_j : \tilde{X} \rightarrow X\}$ is normal. \square

We can deduce the normality of $\{g^{oj}\}_j$.

Proposition 3.3.3. *The family $\{g^{\circ j}\}_j$ is normal.*

Proof. Let $\{g^{\circ j_s}\}_s$ be a sequence of iterates of g . Extracting subsequences if necessary, we can assume that k_{j_s} converges to a holomorphic map $k : \tilde{X} \rightarrow X$.

Since \tilde{X} is simply connected and $\pi : \tilde{X} \rightarrow X$ is a holomorphic covering map, there exists a holomorphic map $g_0 : \tilde{X} \rightarrow \tilde{X}$ such that $\pi \circ g_0 = k$ and $g_0([z]) = [z]$. Note that for every $j \geq 1$, $g^{\circ j}([z]) = [z]$, thus the sequence $\{g^{\circ j}([z])\}_j$ converges to $g_0([z])$. According to [AC03, Theorem 4], the sequence $\{g^{\circ j_s}\}_s$ converges locally uniformly to g_0 . This shows that $\{g^{\circ j}\}_j$ is normal. \square

Remark 3.3.4. The proof of Lemma 3.3.2 relies on the post-critically algebraic hypothesis. Without the post-critically algebraic assumption, for a fixed point z which is not accumulated by the critical set, we can still consider the connected component U of $\mathbb{C}^n \setminus \overline{PC(f)}$ containing z and the construction follows. Then we will need some control on the geometry of U to prove that the family $\{g^{\circ j}\}$ is normal. For example, if U is a pseudoconvex open subset of \mathbb{C}^n or in general, if U is a taut manifold, then $\{g^{\circ j}\}_j$ is normal.

3.3.5 Consequences of normality

The normality of the family of iterates of g implies many useful information. In particular, following Abate [Aba89, Corollary 2.1.30-2.1.31], we derive the existence of a center manifold of g on \tilde{X} .

Theorem 3.3.5. *Let X be a connected complex manifold and let g be an endomorphism of X . Assume that g has a fixed point z . If the family of iterates of g is normal, then*

1. *Every eigenvalue of $D_z g$ is contained in the closed unit disc.*
2. *The tangent space $T_z X$ admits a $D_z g$ -invariant decomposition $T_z X = \tilde{E}_n \oplus \tilde{E}_a$ such that $D_z g|_{\tilde{E}_n}$ has only neutral eigenvalues and $D_z g|_{\tilde{E}_a}$ has only attracting or superattracting eigenvalues.*
3. *The linear map $D_z g|_{\tilde{E}_n}$ is diagonalizable.*
4. *There exists a limit map ρ of iterates of g such that $\rho \circ \rho = \rho$*
5. *The set of fixed points of ρ , which is $\rho(X)$, is a closed submanifold of X . Set $M = \rho(X)$.*
6. *The submanifold M is invariant by g . In fact, $g|_M$ is an automorphism.*

7. The submanifold M contains z and $T_z M = \widetilde{E}_n$.

Applying this theorem to \widetilde{X} and $g : \widetilde{X} \rightarrow \widetilde{X}$ fixing $[z]$, we deduce that $D_{[z]}g$ has only eigenvalues of modulus at most 1 and $T_{[z]}\widetilde{X}$ admits a $D_{[z]}g$ -invariant decomposition as $T_{[z]}\widetilde{X} = \widetilde{E}_n \oplus \widetilde{E}_a$. Differentiating both sides of $f \circ \pi \circ g = \pi$ at $[z]$, we have

$$D_z f \cdot D_{[z]}\pi \cdot D_{[z]}g = D_{[z]}\pi.$$

Hence λ is a neutral eigenvalue of $D_z f$ if and only if λ^{-1} is a neutral eigenvalue of $D_{[z]}g$. Consequently, $D_{[z]}\pi$ maps \widetilde{E}_n to the neutral eigenspace E_n of $D_z f$, \widetilde{E}_a onto the repelling eigenspace E_r of $D_z f$ (see Proposition 3.3.1).

We also obtain a closed center manifold M of g at $[z]$, i.e. if λ is a neutral eigenvalue of $D_{[z]}g$ of eigenvector v , then $v \in T_{[z]}M$. So in order to prove Theorem 3.3.1, it is enough to prove that $\dim M = 0$. The first remarkable property of M is that $\pi(M)$ is a bounded set in \mathbb{C}^n .

Proposition 3.3.6. *The image $\pi(M)$ is bounded.*

Proof. For every $[\gamma] \in \widetilde{X}$, note that $\{\pi(g^{oj}([\gamma]))\}_j$ is in fact a sequence of backward iterations of $\gamma(1)$ by f and the ω -limit set of backward images of \mathbb{C}^n by f is bounded. More precisely, since f is a homogeneous polynomial endomorphism of \mathbb{C}^n of degree $d \geq 2$, the origin 0 is superattracting and the basin of attraction \mathcal{B} is bounded with the boundary is $\partial\mathcal{B} = H_f^{-1}(0)$ where

$$H_f(w) = \lim_{j \rightarrow \infty} \frac{1}{d^j} \log \|f^{oj}(w)\|$$

for $w \in \mathbb{C}^n \setminus \{0\}$. The function H_f is called the *potential function* of f (see [HP94]). It is straight forward by computation that $H_f(\pi(m)) = 0$ for every $m \in M$. Hence $\pi(M) \subset \partial\mathcal{B}$ is bounded. \square

3.3.6 Semiconjugacy on the center manifold

Assume that $\dim M > 0$. Denote by Λ the restriction of $D_{[z]}g$ on $T_{[z]}M$. The following proposition assures that we can semiconjugate $g|_M$ to Λ .

Proposition 3.3.7. *Let M be a complex manifold and let g be an endomorphism of M such that the family of iterates of g is normal. Assume that g has a fixed point z such that $D_z g$ is diagonalizable with only neutral eigenvalues and that there exists a holomorphic*

map $\varphi : M \rightarrow T_z M$ such that $\varphi(z) = 0, D_z \varphi = \text{Id}$. Then there exists a holomorphic map $\Phi : M \rightarrow T_z M$ such that $\Phi(z) = 0, D_z \Phi = \text{Id}$ and

$$D_z g \circ \Phi = \Phi \circ g.$$

Proof. Consider the family $\{(D_z g)^{-n} \circ \varphi \circ g^{on} : M \rightarrow T_{[z]} M\}_n$, we know that $\{g^{on}\}_n$ is normal, thus $\{\varphi \circ g^{oj}\}_j$ is locally uniformly bounded. The linear map $D_z g$ is diagonalizable with neutral eigenvalues, so $\{D_z g^{-n}\}_n$ is uniformly bounded on any bounded set. Then $\{(D_z g)^{-n} \circ \varphi \circ g^{on}\}_n$ is a normal family. Denote by Φ_N the Cesaro average of $\{(D_z g)^{-n} \circ \varphi \circ g^{on}\}_n$.

$$\Phi_N = \frac{1}{N} \sum_{n=0}^{N-1} (D_z g)^{-n} \circ \varphi \circ g^{on}.$$

The family $\{\Phi_N\}_N$ is also locally uniformly bounded, thus normal. Observe that

$$\begin{aligned} \Phi_N \circ g &= \frac{1}{N} \sum_{n=0}^{N-1} (D_z g)^{-n} \circ \varphi \circ g^{o(n+1)} \\ &= D_z g \Phi_N + D_z g \left(-\frac{1}{N} \varphi + \frac{1}{N} ((D_z g)^{-(N+1)} \circ \varphi \circ g^{o(N+1)}) \right). \end{aligned}$$

For every subsequence $\{N_k\}$, the second term on the right hand side converges locally uniformly to 0. So for every limit map Φ of $\{\Phi_N\}_N$, Φ satisfies that

$$\Phi \circ g = D_z g \circ \Phi.$$

Since g fixes z we have that for every $N \geq 1$,

$$D_z \Phi_N = \frac{1}{N} \sum_{n=0}^{N-1} (D_z g)^{-n} \circ D_z \varphi \circ D_z g^{on} = \text{Id}.$$

So $D_z \Phi = \text{Id}$ for every limit map Φ of $\{\Phi_N\}_N$. □

Now we consider the complex manifold M obtained in Step 3.3.5 and the restriction of g on M which is an automorphism with a fixed point $[z]$. Since Λ has only neutral eigenvalues, in order to apply Proposition 3.3.7, we need to construct a holomorphic map $\varphi : M \rightarrow T_{[z]} M$ such that $D_{[z]} \varphi = \text{Id}$. The map φ is constructed as the following

composition:

$$M \xrightarrow{\mathbf{i}} \tilde{X} \xrightarrow{\pi} X \xrightarrow{\delta} T_z X \xrightarrow{(D_{[z]}\pi)^{-1}} T_{[z]}\tilde{X} \xrightarrow{\pi_{\tilde{E}_a}} T_{[z]}M.$$

where $\delta : X \rightarrow T_z X$ is a holomorphic map tangent to identity, $\pi_{\tilde{E}_a} : T_{[z]}\tilde{X} \rightarrow T_{[z]}M$ is the projection parallel to \tilde{E}_a , $\mathbf{i} : M \rightarrow \tilde{X}$ is the canonical inclusion and its derivative $D_{[z]}\mathbf{i} : T_{[z]}M \rightarrow T_{[z]}\tilde{X}$ is again the canonical inclusion. Then $D_{[z]}\varphi : T_{[z]}M \rightarrow T_{[z]}M$ is

$$\begin{aligned} D_{[z]}\varphi &= D_{[z]}(\pi_{\tilde{E}_a} \circ (D_{[z]}\pi)^{-1} \circ \delta \circ \pi \circ \mathbf{i}) \\ &= \pi_{\tilde{E}_a} \circ (D_{[z]}\pi)^{-1} \circ D_z \delta \circ D_{[z]}\pi \circ D_{[z]}\mathbf{i} = \text{Id}. \end{aligned}$$

Remark 3.3.8. The existence of a holomorphic map $\delta : X \rightarrow T_z X$ tangent to identity is one of the advantages we mentioned in Remark 3.3.2. It comes from the intrinsic nature of the tangent space of affine spaces. In this case, X is an open subset of \mathbb{C}^n which is an affine space directed by \mathbb{C}^n .

Corollary 3.3.9. *Let z be a fixed point of a non-degenerate homogeneous polynomial post-critically algebraic endomorphism f of \mathbb{C}^n . Assume that $z \notin PC(f)$. If λ is a neutral eigenvalue of $D_z f$ then λ is an irrational eigenvalue.*

Proof. It is equivalent to consider a neutral eigenvalue λ of $D_{[z]}g$ and assume that $\lambda = e^{2\pi i \frac{p}{q}}$. Hence $(D_{[z]}g)^q$ fixes pointwise the line $\mathbb{C}v$ in $T_{[z]}M$. This means that locally near $[z]$, g^{oq} fixes $\Phi^{-1}(\mathbb{C}v)$ hence f^{oq} fixes $\pi(\Phi^{-1}(\mathbb{C}v))$ near z . Note that Φ is locally invertible near $[z]$ hence $\Phi^{-1}(\mathbb{C}v)$ is a complex manifold of dimension one near $[z]$. Then $\pi(\Phi^{-1}(\mathbb{C}v))$ is a complex manifold near z because π is locally biholomorphic. In particular, $\pi(\Phi^{-1}(\mathbb{C}v))$ contains uncountably many fixed points of f^{oq} . This is a contradiction because f^{oq} has only finitely many fixed points (see [SSC⁺10, Proposition 1.3]). Hence λ is an irrational eigenvalue. □

3.3.7 Linearization along the neutral direction

We obtained a holomorphic map $\Phi : M \rightarrow T_{[z]}M$, $\Phi([z]) = 0$, $D_{[z]}\Phi = \text{Id}$ and

$$\Phi \circ g|_M = \Lambda \circ \Phi \tag{3.3.1}$$

where $\Lambda = D_{[z]}g|_{T_{[z]}M}$. Let $\lambda = e^{2\pi i \theta}$, $\theta \in \mathbb{R} \setminus \mathbb{Q}$ be an irrational eigenvalue of Λ and $\mathbb{C}v$ be a complex line of direction v in $T_{[z]}M$. The line $\mathbb{C}v$ is invariant by Λ , i.e. $\Lambda(\mathbb{C}v) = \mathbb{C}v$,

hence $\Sigma := \Phi^{-1}(\mathbb{C}v)$ is invariant by g . Denote by Γ the irreducible component of $\Phi^{-1}(\mathbb{C}v)$ containing $[z]$.

Lemma 3.3.10. *Set $\Gamma_0 = \Gamma \setminus \text{Sing } \Sigma$. Then $g(\Gamma) = \Gamma$ and $g(\Gamma_0) = \Gamma_0$.*

Proof. On one hand, since g is automorphism, it maps irreducible analytic sets to irreducible analytic sets. On the other hand, $D_{[z]}\Phi = \text{Id}$ then by Inverse function theorem, $\Phi^{-1}(\mathbb{C}v)$ is smooth near $[z]$, hence Γ is the only irreducible component of $\Phi^{-1}(\mathbb{C}v)$ near $[z]$. Then $g(\Gamma) = \Gamma$.

Concerning Γ_0 , we observe that $\text{Sing } \Sigma = \Sigma \cap C_\Phi$ where $C_\Phi = \{x \in M \mid \text{rank } D_x \Phi < \dim M\}$ the set of critical points of Φ . By differentiating (3.3.1) at $x \in M$, we have

$$D_{g(x)}\Phi \circ D_x g|_M = \Lambda \circ D_x \Phi.$$

We deduce that $g(C_\Phi) = C_\Phi$. Moreover, $g(\Sigma) = \Sigma$. Thus $g(\text{Sing } \Sigma) = \text{Sing } \Sigma$ and hence $g(\Gamma_0) = \Gamma_0$. \square

Note that $\Gamma_0 \subset \text{Reg } \Gamma$ is smooth since $\text{Sing } \Gamma \subset \Gamma \cap \text{Sing } \Sigma$ and in fact Γ_0 is a Riemann surface. In particular, $[z] \in \Gamma_0$.

We will prove that Γ_0 is biholomorphic to a disc and $\Phi|_{\Gamma_0}$ is conjugate to an irrational rotation. Then we can deduce from that $\Gamma = \Gamma_0$ and $\Phi|_\Gamma$ conjugates $g|_\Gamma$ to an irrational rotation. Let us first recall an important theorem in the theory of dynamics in one complex dimension.

Theorem 3.3.11 (see [Mil11, Theorem 5.2]). *Let S be a hyperbolic Riemann surface and let $g : S \rightarrow S$ be a holomorphic map with a fixed point z . If z is an irrational fixed point with multiplier λ then S is biholomorphic to the unit disc and g is conjugate to the irrational rotation $\zeta \mapsto \lambda\zeta$.*

Lemma 3.3.12. *The Riemann surface Γ_0 is hyperbolic, $\Phi(\Gamma_0) = \mathbb{D}(0, R)$ with $R \in (0, +\infty)$ and $\Phi|_{\Gamma_0} : \Gamma_0 \rightarrow \mathbb{D}(0, R)$ is a biholomorphism conjugating $g|_{\Gamma_0}$ to the irrational rotation $\zeta \mapsto \lambda\zeta$, i.e.*

$$\Phi \circ g|_{\Gamma_0} = \lambda \cdot \Phi|_{\Gamma_0}.$$

Proof. Recall that $\pi(M)$ is bounded in \mathbb{C}^n . Thus π induces a non constant bounded holomorphic function from Γ_0 to \mathbb{C}^n . Therefore, Γ_0 is a hyperbolic Riemann surface. Note that $[z]$ is a fixed point of the holomorphic map $g|_{\Gamma_0}$ with the irrational multiplier λ . Then we can apply Theorem 3.3.11 to obtain a conjugacy $\psi : \Gamma_0 \rightarrow \mathbb{D}(0, 1)$ such that $\psi \circ g|_{\Gamma_0} = \lambda \cdot \psi$.

Denote by $\Psi := \Phi \circ \psi^{-1} : \mathbb{D}(0, 1) \rightarrow S := \Phi(\Gamma_0)$ (see the diagram below). Then we have $\Psi(\lambda z) = \lambda \Psi(z)$ for every $z \in \mathbb{D}$

$$\begin{array}{ccc}
 \mathbb{D}(0, 1) & \xrightarrow{\lambda \cdot} & \mathbb{D}(0, 1) \\
 \psi \uparrow & & \psi \uparrow \\
 \Gamma_0 & \xrightarrow{g} & \Gamma_0 \\
 \Phi \downarrow & & \Phi \downarrow \\
 S := \Phi(\Gamma_0) & \xrightarrow{\lambda \cdot} & S
 \end{array}
 \begin{array}{c}
 \Psi \left(\begin{array}{c} \uparrow \\ \downarrow \end{array} \right) \Psi
 \end{array}$$

It follows that $\Psi(z) = \Psi'(0)z$ for every $z \in \mathbb{D}$. Therefore $\Phi|_{\Gamma_0} = \Psi'(0) \cdot \psi$ is a conjugacy conjugating $g|_{\Gamma_0}$ to $z \mapsto \lambda z$. In particular, $\Phi(\Gamma_0) = \mathbb{D}(0, R)$, $R = |\Psi'(0)| \in (0, +\infty)$ and $\Phi|_{\Gamma_0}$ is a biholomorphism. \square

Proposition 3.3.13. *The analytic set Γ is smooth and and the map*

$$\Phi|_{\Gamma} : \Gamma \rightarrow \Phi(\Gamma) = \mathbb{D}(0, R)$$

is a biholomorphic with $R \in (0, +\infty)$.

Proof. It is enough to prove that $\Gamma = \Gamma_0$. From Lemma 3.3.12, we deduce that Γ_0 is simply connected. Note that $\Gamma_0 \subset \text{Reg } \Gamma$ is the complement of a discrete set $\Gamma \cap \text{Sing } \Sigma$ in Γ . We denote $\hat{\Gamma}$ the normalization of Γ , a Riemann surface (see [Chi12]), and by $\tilde{\Gamma}$ the universal covering of $\hat{\Gamma}$. Since $\Gamma_0 \subset \text{Reg } \Gamma$, the preimage $\hat{\Gamma}_0$ of Γ_0 by the normalization, which is isomorphic to Γ_0 , is simply connected. Hence the preimage of $\hat{\Gamma}_0$ by the universal covering in $\tilde{\Gamma}$ is a simply connected open subset in $\tilde{\Gamma}$ with discrete complement. Then either $\tilde{\Gamma}$ is biholomorphic to the unit disc (or \mathbb{C}) and $\Gamma_0 = \Gamma$ or $\tilde{\Gamma}$ is biholomorphic to $\mathbb{C}\mathbb{P}^1$ and $\Gamma \setminus \Gamma_0$ is only one point. Since $\pi|_{\Gamma}$ is a non constant bounded holomorphic function valued in \mathbb{C}^n , the only case possible is that $\tilde{\Gamma}$ is biholomorphic to a disc and $\Gamma_0 = \Gamma$. \square

Thus, we obtain a biholomorphic map $\kappa := (\Phi|_{\Gamma})^{-1} : \mathbb{D}(0, R) \rightarrow \Gamma \hookrightarrow M$, $\kappa(0) = [z]$ with $R \in (0, +\infty)$.

3.3.8 End of the proof

Denote by $\tau = \pi \circ \kappa$. Note that $\tau(0) = z$. Since $\tau(\mathbb{D}(0, R)) \subset \pi(M)$ is bounded hence by Fatou-Riesz's theorem (see [Mil11, Theorem A.3]), the radial limit

$$\tau_{\theta} = \lim_{r \rightarrow R^-} \pi \circ \kappa(re^{i\theta})$$

exists for almost every $\theta \in [0, 2\pi)$.

Remark 3.3.14. This is another advantage we mentioned in Remark 3.3.2.

Proposition 3.3.15. *If τ_θ exists, $\tau_\theta \in PC(f)$.*

Proof. Consider θ such that τ_θ exists and $\tau_\theta \notin PC(f)$, i.e. $\tau_\theta \in X$. Note that

$$\gamma_R : [0, 1] \rightarrow X$$

where $\gamma_R(t) = \tau(tRe^{i\theta})$, $\gamma_R(1) = \tau_\theta$ is a well-defined path in X starting at z hence it defines an element in \tilde{X} . Moreover, in \tilde{X} , the family of paths $\{[\gamma_r]\}_{0 \leq r \leq R}$

$$\gamma_r : [0, 1] \rightarrow X$$

where $\gamma_r(t) = \tau(tr e^{i\theta})$ converges to $[\gamma_R]$ as $r \rightarrow R^-$. A quick observation is that in \tilde{X} , we have $[\gamma_r] = \kappa(re^{i\theta}) \subset M$ for every $r \in [0, 1)$. Since M is a closed submanifold of \tilde{X} , then $[\gamma_R] \in M$ or in fact $[\gamma_R] \in \Gamma$. Recall that $\Phi : \Gamma \rightarrow \mathbb{D}(0, R)$ is a biholomorphic mapping hence

$$re^{i\theta} = \Phi([\gamma_r]) \xrightarrow{r \rightarrow R^-} \Phi([\gamma_R]) \in \mathbb{D}(0, R).$$

But $re^{i\theta} \xrightarrow{r \rightarrow R^-} Re^{i\theta} \notin \mathbb{D}(0, R)$ which yields a contradiction. Thus $\tau_\theta \in PC(f)$. \square

Now, denote by Q a defining polynomial of $PC(f)$ then

$$Q \circ \tau : \mathbb{D}(0, R) \rightarrow \mathbb{C}$$

has vanishing radial limit $\lim_{r \rightarrow R^-} Q \circ \tau(re^{i\theta})$ for almost every $\theta \in [0, 2\pi)$. Then, $Q \circ \tau$ vanishes identically on $\mathbb{D}(0, R)$ (see [Mil11, Theorem A.3]). In particular, $Q \circ \tau(0) = Q(z) = 0$ hence $z \in PC(F)$. It is a contradiction and our proof of Proposition 3.3.15 and Theorem 3.3.1 is complete.

3.4 Periodic cycles in the regular locus: the transversal eigenvalue

Now we consider a fixed point z of a post-critically algebraic endomorphism f of $\mathbb{C}\mathbb{P}^n$ which is a regular point of $PC(f)$. Then $T_z PC(f)$ is well-defined and it is a $D_z f$ -invariant subspace of $T_z \mathbb{C}\mathbb{P}^n$. On one hand, it is natural to expect that our method of the previous case can be extended to prove that $D_z f|_{T_z PC(f)}$ has only repelling eigenvalues (it cannot

have superattracting eigenvalues, see Remark 3.4.5 below). Unfortunately, there are some difficulties due to the existence of singularities of codimension higher than 1 that we cannot overcome easily. On the other hand, we are able to adapt our method to prove that the transversal eigenvalue with respect to $T_z PC(f)$, i.e. the eigenvalue of $\overline{D_z f} : T_z \mathbb{C}P^n / T_z PC(f) \rightarrow T_z \mathbb{C}P^n / T_z PC(f)$, is repelling. More precisely, we will prove the following result.

Proposition 3.4.1. *Let f be a post-critically algebraic endomorphism of $\mathbb{C}P^n$ of degree $d \geq 2$ and let z be a point of $\text{Reg } PC(f)$. Then the eigenvalue of the linear map $\overline{D_z f} : T_z \mathbb{C}P^n / T_z PC(f) \rightarrow T_z \mathbb{C}P^n / T_z PC(f)$ is either repelling or superattracting.*

By Remark 3.3.2, it is equivalent to prove the following result.

Proposition 3.4.2. *Let f be a post-critically algebraic non-degenerate homogeneous polynomial endomorphism of \mathbb{C}^n of degree $d \geq 2$ and let z be a point of $\text{Reg } PC(f)$. Then the eigenvalue of the linear map $\overline{D_z f} : T_z \mathbb{C}^n / T_z PC(f) \rightarrow T_z \mathbb{C}^n / T_z PC(f)$ is either repelling or superattracting.*

Since $PC(f)$ has codimension one, $\overline{D_z f}$ has exactly one eigenvalue and we denote it by λ . The value λ is also an eigenvalue of $D_z f$. The proof of Proposition 3.4.2 will occupy the rest of this section.

3.4.1 Strategy of the proof

Denote by $X = \mathbb{C}^n \setminus PC(f)$.

Step 1. We first prove that if $\lambda \neq 0$ then $|\lambda| \geq 1$ and z is not a critical point. Then we prove that $|\lambda| = 1$ will lead to a contradiction. By assuming $|\lambda| = 1$, following from the discussion in Section 3.2, there exists an eigenvector v of $D_z f$ corresponding to λ such that $v \notin T_z PC(f)$.

Our goal is to build a holomorphic map $\tau : \mathbb{D}(0, R)^* \rightarrow X$ such that $\tau(0) = z$, $\tau'(0) = v$ and τ has radial limit almost everywhere and these radial limits land on $PC(f)$ whenever they exist. The construction of τ occupies Step 2 to Step 6 and the contradiction will be deduced in Step 7.

Step 2. We construct a connected complex manifold \tilde{X} of dimension n with two holomorphic maps $\pi : \tilde{X} \rightarrow X$, $g : \tilde{X} \rightarrow \tilde{X}$ such that

$$f \circ \pi \circ g = \pi.$$

- Step 3. We prove that $\{g^{\circ j}\}_j$ is a normal family. Then we extract a subsequence $\{g^{\circ j_k}\}_k$ converging to a retraction $\rho : \tilde{X} \rightarrow \tilde{X}$, i.e. $\rho \circ \rho = \rho$.
- Step 4. We will study $M = \rho(\tilde{X})$. More precisely, we will prove that M is closely related to a limit manifold of inverse branches of f at z .
- Step 5. We construct a holomorphic map $\Phi : M \rightarrow E_n$ which semi-conjugates g to the restriction of $(D_z f)^{-1}$ to the neutral eigenspace E_n (see Proposition 3.3.1.c).
- Step 6. We prove that there exists an irreducible component Γ of $\Phi^{-1}(\mathbb{C}v)$ which is smooth and biholomorphic to the punctured disc. More precisely, we prove that $\Phi(\Gamma) = \mathbb{D}(0, R)^*$ with $R \in (0, +\infty)$ and the map $\tau := \pi \circ (\Phi|_{\Gamma})^{-1}$ extends to a holomorphic map from $\mathbb{D}(0, R)$ to \mathbb{C}^n so that $\tau(0) = z, \tau'(0) = v$.
- Step 7. We prove that the map τ has radial limit almost everywhere and the limit belong to $PC(f)$ if it exists. It implies that $\pi \circ \tau \subset PC(f)$ which yields a contradiction to the fact that $v \notin T_z PC(f)$. This means that the assumption $|\lambda| = 1$ is false thus Proposition 3.4.2 is proved.

3.4.2 Existence of the transversal eigenvector

Let us recall the following result due to Grauert.

Proposition 3.4.3 ([GR58], Satz 10). *Let U, V be open neighborhood of 0 in \mathbb{C}^n and let $f : U \rightarrow V$ be a holomorphic branched covering of order k ramifying over $V_f = \{\zeta_n = 0\} \cap V$. Then there exists a biholomorphism $\Phi : U \rightarrow W$ such that the following diagram commutes*

$$\begin{array}{ccc} & & W \\ & \nearrow \Phi & \downarrow (\zeta_1, \dots, \zeta_n) \mapsto (\zeta_1, \dots, \zeta_{n-1}, \zeta_n^k) \\ U & \xrightarrow{f} & V \end{array}$$

In particular, the branched locus $B_f = \Phi^{-1}(\{\zeta_n = 0\} \cap W)$ is smooth and $f|_{B_f} : B_f \rightarrow V_f$ is a biholomorphism.

This is in fact a local statement and we can apply it to a post-critically algebraic non-degenerate homogeneous polynomial endomorphism of \mathbb{C}^n to obtain the following result.

Proposition 3.4.4 ([Ued98], Lemma 3.5). *Let f be a post-critically algebraic non-degenerate homogeneous polynomial endomorphism of \mathbb{C}^n of degree $d \geq 2$. Then*

$$f^{-1}(\text{Reg } PC(f)) \subset \text{Reg } PC(f)$$

and

$$f : f^{-1}(\text{Reg } PC(f)) \rightarrow \text{Reg } PC(f)$$

is locally a biholomorphism.

Remark 3.4.5. In particular, Proposition 3.4.4 implies that if f has a fixed point $z \in \text{Reg } PC(f)$ then $D_z f|_{T_z PC(f)}$ is invertible. Hence $D_z f|_{T_z PC(f)}$ does not have any super-attracting eigenvalue.

If $\lambda \neq 0$ then z is not a critical point. By Proposition 3.3.1, the modulus of λ is at least 1. Then we will prove Proposition 3.4.1 by contradiction by assuming that $|\lambda| = 1$. If $|\lambda| = 1$, there exists an associated eigenvector v of $D_z f$ such that $v \notin T_z PC(f)$. Indeed, note that

$$\text{Spec}(D_z f) = \text{Spec}(D_z f|_{T_z PC(f)}) \cup \text{Spec}(\overline{D_z f})$$

where $\text{Spec}(\overline{D_z f})$ has only one eigenvalue λ of modulus one. Then the repelling eigenspace E_r is included in $T_z PC(f)$. The diagonalizability of $D_z f|_{E_n}$ implies that E_n is generated by a basis of eigenvectors. The vector v is such an eigenvector which is not in $T_z PC(f)$.

3.4.3 (X, z) -homotopy and related constructions

Denote by $X = \mathbb{C}^n \setminus PC(f)$.

Construction of \tilde{X}

We construct a complex manifold \tilde{X} , a covering map $\pi : \tilde{X} \rightarrow X$ and a holomorphic map $g : \tilde{X} \rightarrow \tilde{X}$ such that

$$f \circ \pi \circ g = \pi.$$

Denote by

$$\Xi = \{\gamma : [0, 1] \rightarrow \mathbb{C}^n \text{ continuous map such that } \gamma(0) = z, \gamma((0, 1]) \subset X\}$$

the space of paths starting at z and varying in X . Let $\gamma_0, \gamma_1 \in \Xi$. We say that γ_0 and γ_1 are (X, z) -homotopic if there exists a continuous map $H : [0, 1] \times [0, 1] \rightarrow \mathbb{C}^n$ such that

$$H(0, s) = z, H(1, s) = \gamma_0(1) = \gamma_1(1),$$

$$H(t, 0) = \gamma_0(t), H(t, 1) = \gamma_1(t),$$

$$H(t, s) \subset X \forall t \neq 0.$$

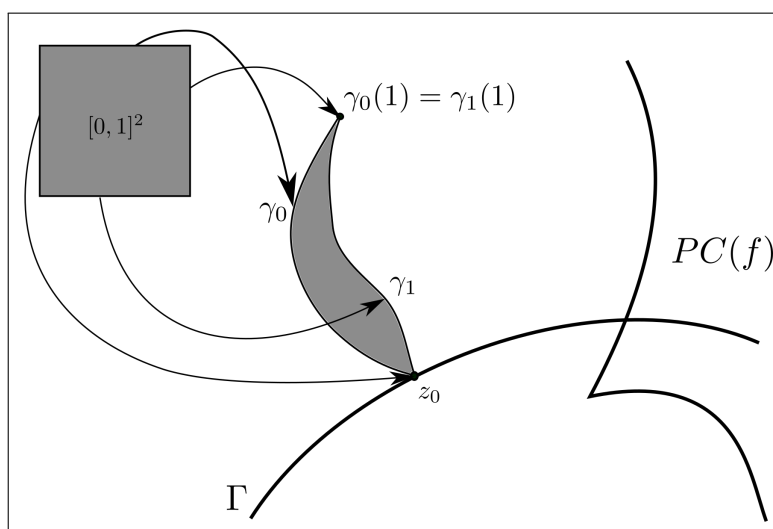


Figure 3.4-1: Two paths which are homotopic relative to X

Denote by $\gamma_0 \sim_X \gamma_1$ if γ_0 and γ_1 are (X, z) -homotopic. In other words, γ_0 and γ_1 are homotopic by a homotopy of paths $\{\gamma_t, t \in [0, 1]\}$ such that $\gamma_t \in \Xi$ for every t . It is easy to see that (X, z) -homotopy is an equivalence relation on Ξ . Denote by \tilde{X} the quotient space of Ξ by this relation and by $[\gamma]$ the equivalent class of $\gamma \in \Xi$. Denote by

$$\pi : \tilde{X} \rightarrow X, \pi([\gamma]) = \gamma(1)$$

the projection. We endow \tilde{X} with a topology constructed in the same way as the topology of a universal covering. More precisely, let \mathfrak{B} be the collection of simply connected open subsets of X . Note that \mathfrak{B} is a basis for the usual topology of X . We consider the topology on \tilde{X} which is defined by a basis of open subsets $\{U_{[\gamma]}\}_{U \in \mathfrak{B}, [\gamma] \in X}$ where $\gamma(1) \in U$ and

$$U_{[\gamma]} = \{[\gamma * \alpha] \mid \alpha \text{ is a path in } U \text{ starting at } \gamma(1)\}.$$

We can transport the complex structure of X to \tilde{X} and this will make \tilde{X} a complex

manifold of dimension n . Note that π is also a holomorphic covering map.

Lifts of inverse branches of f

We will construct a holomorphic mapping

$$g : \tilde{X} \rightarrow \tilde{X}$$

which is induced by the pullback action of f on paths in Ξ .

Lemma 3.4.6. *Let γ be a path in Ξ . Then there exists a unique path $f^*\gamma \in \Xi$ such that $f \circ f^*\gamma = \gamma$.*

Proof. Since f is locally invertible at z and since $f^{-1}(X) \subset X$, there exists $t_0 \in [0, 1]$ such that $\gamma|_{[0, t_0]} \in \Xi$ and $f^{-1} \circ \gamma|_{[0, t_0]}$ is a well-defined element in Ξ . Then the path $f^*\gamma$ is the concatenation of $f^{-1} \circ \gamma|_{[0, t_0]}$ with the lifting $f^*\gamma|_{[t_0, 1]}$ of the path $\gamma|_{[t_0, 1]}$ by the covering $f : f^{-1}(X) \rightarrow X$. This construction does not depend on the choice of t_0 . \square

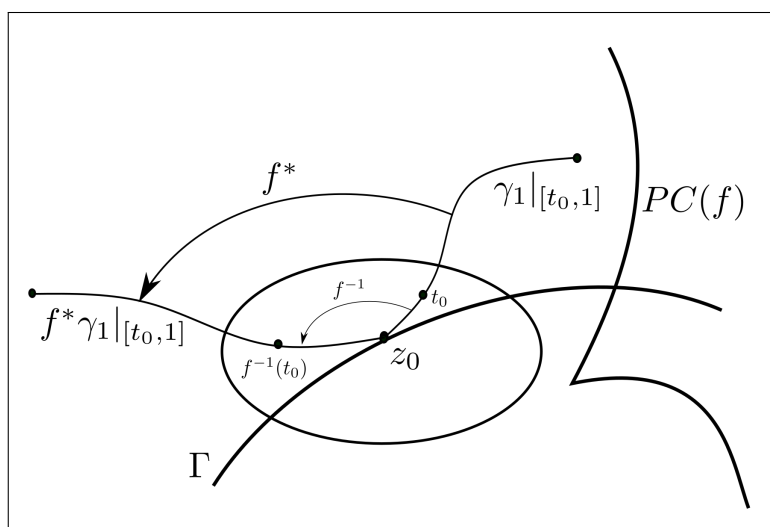


Figure 3.4-2: Pulling back of an element in Ξ .

Lemma 3.4.7. *Let $\gamma_0, \gamma_1 \in \Xi$. If $[\gamma_0] = [\gamma_1]$ then $[f^*\gamma_0] = [f^*\gamma_1]$.*

Proof. Lemma 3.4.6 implies that the pull-back of a homotopy of paths in Ξ between γ_0 and γ_1 is a homotopy of paths in Ξ between $f^*\gamma_0$ and $f^*\gamma_1$. See also Figure 3.4.3. \square

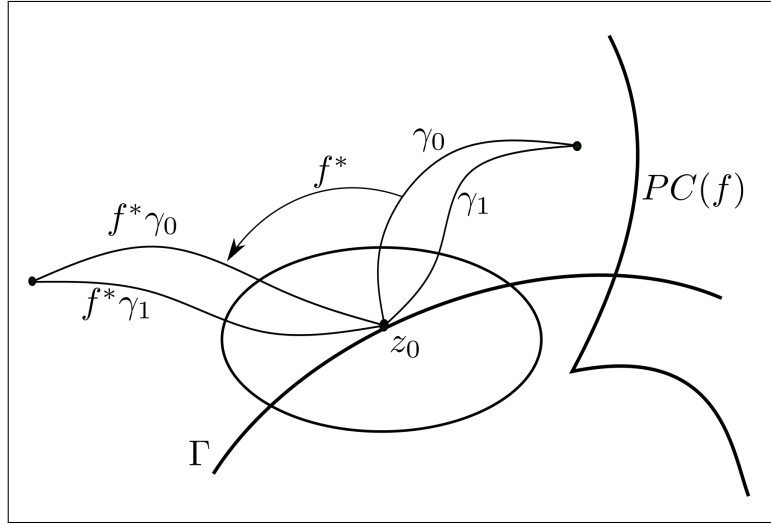


Figure 3.4-3: Pull-back preserves the (X, z) -homotopic paths

The two previous lemmas allow us to define a map $g : \tilde{X} \rightarrow \tilde{X}$ as follow

$$g([\gamma]) = [f^* \gamma].$$

Then $f \circ \pi \circ g = \pi$.

The connectedness of \tilde{X} .

The connectedness of \tilde{X} is not obvious from the construction. We will introduce a notion of *regular neighborhood* which is not only useful in proving \tilde{X} is connected but also very important later.

Definition 3.4.8. A bounded open subset W of \mathbb{C}^n containing z is called a *regular neighborhood* of z if:

1. $(W, W \cap PC(f))$ is homeomorphic to a cone over $(\partial W, \partial W \cap PC(f))$ with a vertex at z .
2. For every path $\gamma_0, \gamma_1 \in \Xi$ such that $\gamma_0([0, 1]), \gamma_1([0, 1]) \subset W$ and $\gamma_0(1) = \gamma_1(1)$. Then $\gamma_0 \sim_X \gamma_1$.

Let W be an open subset of \mathbb{C}^n containing z . Set

$$\tilde{W} = \{[\gamma] | \gamma \in \Xi, \gamma((0, 1]) \subset W\}.$$

Lemma 3.4.9. *If W is a regular neighborhood of z , then $\pi : \widetilde{W} \rightarrow W \setminus PC(f)$ is a biholomorphism.*

Proof. We can observe that \widetilde{W} is open. Indeed, for an element $[\gamma]$ in \widetilde{W} , let U be an open set in $W \setminus PC(f)$ containing $\gamma(1)$, then $U_{[\gamma]} \subset \widetilde{W}$. The projection $\pi|_{\widetilde{W}} : \widetilde{W} \rightarrow W \setminus PC(f)$ is surjective since $W \setminus PC(f)$ is path-connected. So we have to prove that $\pi : \widetilde{W} \rightarrow W \setminus PC(f)$ is injective. Indeed, let z be a point in $W \setminus PC(f)$ and let $[\gamma_0], [\gamma_1]$ be two elements in \widetilde{W} such that $\pi([\gamma_0]) = \pi([\gamma_1]) = z$, i.e. $\gamma_0(1) = \gamma_1(1)$. Since W is regular, we have $\gamma_0 \sim_X \gamma_1$ or $[\gamma_0] = [\gamma_1]$. So $\pi|_{\widetilde{W}}$ is injective hence biholomorphic. \square

In particular, \widetilde{W} is path-connected. If $\gamma : [0, 1] \rightarrow X$ is a path in Ξ then the path $\gamma_s : [0, 1] \rightarrow \mathbb{C}^n$ defined by $\gamma_s(t) = \gamma(t(1-s))$ also belongs to Ξ and $\gamma_s([0, 1]) = \gamma([0, 1-s])$. It follows that every element in \widetilde{X} can be joined by paths to an element in \widetilde{W} thus \widetilde{X} is path-connected hence connected.

Now we will prove that we can indeed find a regular neighborhood when z is a regular point of $PC(f)$. Let $(\zeta_1, \dots, \zeta_n)$ be a local coordinates vanishing at z in which $PC(f)$ is given by $\{\zeta_1 = 0\}$. Let U be the unit polydisk centered at z .

Proposition 3.4.10. *Any polydisc centered at z in U is a regular neighborhood.*

Proof. Let γ_0 and γ_1 be two elements of Ξ such that $\gamma_0([0, 1]), \gamma_1([0, 1]) \in U$ and $\gamma_0(1) = \gamma_1(1)$. Consider the loop $\eta = \gamma_0 * (-\gamma_1)$ and the continuous map $H : [0, 1] \times [0, 1] \rightarrow \mathbb{C}^n$ defined by

$$H(t, s) = s \cdot \eta(t).$$

The loop η bounds $H([0, 1] \times [0, 1] \setminus \{(0, 0)\}) \subset X$, which implies that $\gamma_0 \sim_X \gamma_1$. \square

Remark 3.4.11. The construction above also implies that z admits a basis of neighborhoods consisting of regular neighborhoods.

Dynamics of g on regular neighborhoods.

Let W be a regular neighborhood of z and

$$\sigma : W \setminus PC(f) \rightarrow \widetilde{W}$$

be the inverse of $\pi|_{\widetilde{W}}$. Note that if W is constructed as above, then by Proposition 3.3.1.b, there exists a family of holomorphic maps $h_j : W \rightarrow \mathbb{C}^n, j \geq 1$ such that

$$h_j(z) = z, f^{\circ j} \circ h_j = \text{Id}_W.$$

We can deduce from the definition of g that for every $j \geq 1$, $\pi \circ g^{oj} \circ \sigma = h_j|_{W \setminus PC(f)}$. More precisely, let $[\gamma] \in \widetilde{W}$, i.e. $\gamma((0, 1]) \subset W \setminus PC(f)$, then by definition, we have for every $j \geq 1$,

$$g^{oj}([\gamma]) = [h_j \circ \gamma]. \quad (3.4.1)$$

Recall that the family $\{h_j : W \rightarrow \mathbb{C}^n\}_j$ is normal. The assumption $|\lambda| = 1$ allows us to control the value taken by any limit maps of this family. Note that $f^{-1}(X) \subset X$ hence $h_j(W \setminus PC(f)) \subset X$ for every $j \geq 1$.

Lemma 3.4.12. *Let $h = \lim_{s \rightarrow \infty} h_{j_s}$ be a limit map of $\{h_j : W \rightarrow \mathbb{C}^n\}_j$. Then*

$$h(W \setminus PC(f)) \subset X.$$

Proof. Recall that $PC(f)$ is the zero locus of a polynomial $Q : \mathbb{C}^n \rightarrow \mathbb{C}$. Then

$$Q \circ h = \lim_{s \rightarrow +\infty} Q \circ h_{j_s}.$$

Since $h_{j_s}(W \setminus PC(f)) \subset X$, the map $Q \circ h|_{W \setminus PC(f)}$ is the limit of a sequence of nonvanishing holomorphic functions. By Hurwitz's theorem, $Q \circ h|_{W \setminus PC(f)}$ is either a nonvanishing function or identically 0, i.e. either $h(W \setminus PC(f)) \subset X$ or $h(W \setminus PC(f)) \subset PC(f)$.

Let v be an eigenvector of $D_z f$ associated to the eigenvalue λ . Then we have $D_z h_j(v) = \frac{1}{\lambda^j} v$. Hence

$$D_z h(v) = \frac{1}{\lambda'} v$$

for some limit value λ' of $\{\lambda^j\}_j$. Since we assumed that $|\lambda| = 1$ hence $|\lambda'| = 1$. The fact that $v \notin T_z PC(f)$ implies that $D_z h(v) \notin T_z PC(f)$. Consequently, we have

$$h(W \setminus PC(f)) \cap X \neq \emptyset$$

thus $h(W \setminus PC(f)) \subset X$. □

3.4.4 Normality of family of maps lifted via the relative homotopy

We will prove that $\{g^{oj} : \widetilde{X} \rightarrow \widetilde{X}\}_j$ is normal. Following 3.3.4, it is enough to prove two following lemmas.

Lemma 3.4.13. *The family $\{k_j = \pi \circ g^{oj} : \widetilde{X} \rightarrow X\}_j$ is normal and any limit map can be lifted by π to a holomorphic endomorphism of \widetilde{X} .*

Proof. Note that $\{k_j : \tilde{X} \rightarrow \mathbb{C}^n\}_j$ is locally uniformly bounded hence normal (see Proposition 3.3.6). Consider a limit map k of this family, by using Hurwitz's theorem, we deduce that either $k(\tilde{X}) \subset X$ or $k(\tilde{X}) \subset PC(f)$.

Let W be a regular neighborhood of z then there exists a family $\{h_j : W \rightarrow \mathbb{C}^n\}_j$ of $f^{\circ j}$ fixing z (see 3.4.3). We have

$$k_j|_{\tilde{W}} \circ \sigma = h_j|_{W \cap X}$$

where $\sigma : W \setminus PC(f) \rightarrow \tilde{W}$ is the section of $\pi|_{\tilde{W}}$. Therefore $k|_{\tilde{W}} \circ \sigma$ is a limit map of $\{h_j|_{W \setminus PC(f)}\}$. Lemma 3.4.12 implies that $k(\tilde{W}) \subset X$, and hence, $k(\tilde{X}) \subset X$. Thus $\{k_j : \tilde{X} \rightarrow X\}_j$ is normal.

We shall now show that the map $k : \tilde{X} \rightarrow X$ can be lifted to a map from \tilde{X} to \tilde{X} . Set

$$h := k|_{\tilde{W}} \circ \sigma$$

then h is a limit map of $\{h_j|_{W \setminus PC(f)}\}$. For each element $[\gamma] \in \tilde{X}$, we denote by η the image of γ under the analytic continuation of h along γ . Note that h fixes z . Thus, Lemma 3.4.12 implies that $\eta \in \Xi$. This construction does not depend on the choice of γ in the equivalence class $[\gamma]$. Thus, the map

$$\begin{aligned} \tilde{k} : \tilde{X} &\rightarrow \tilde{X} \\ [\gamma] &\mapsto [\eta] \end{aligned}$$

is well-defined. The map $\pi \circ \tilde{k}$ coincides with k on an open set \tilde{W} , and hence on \tilde{X} . In other words, \tilde{k} is a lifted map of k by π . \square

Hence we deduce that:

Proposition 3.4.14. *The family $\{g^{\circ j} : \tilde{X} \rightarrow \tilde{X}\}_j$ is normal.*

Proof. Let $\{g^{\circ j_s}\}_s$ be sequence of iterates of g . Extracting subsequences if necessary, we can assume that $\{k_{j_s}\}_s$ converges locally uniformly to a holomorphic map $k : \tilde{X} \rightarrow X$. By Lemma 3.4.13, there exists a holomorphic map $\tilde{k} : \tilde{X} \rightarrow \tilde{X}$ so that $\pi \circ \tilde{k} = k$. We will prove that $\{g^{\circ j_s}\}_s$ converges locally uniformly to \tilde{k} . Applying [AC03, Theorem 4], it is enough to prove that there exists an element $[\gamma] \in \tilde{X}$ such that $g^{\circ j_s}([\gamma])$ converges to $\tilde{k}([\gamma])$.

We consider a regular neighborhood W of z and the family $\{h_j : W \rightarrow \mathbb{C}^n\}_j$ of $f^{\circ j}$ fixing z (see 3.4.3). Let $[\gamma]$ be an element in \tilde{W}^3 and an associated path $\tilde{\gamma} : [0, 1] \rightarrow$

³In Section 3.3.4, we choose an element representing z . Such an element does not exist in this case

$\tilde{X}, \tilde{\gamma}(t) = [\gamma|_{[0,t]}$ in \tilde{X} . Since $[\gamma] \in \tilde{W}$, $\tilde{\gamma} = \sigma \circ \gamma$. Then

$$\begin{aligned} \tilde{k}([\gamma]) = [k \circ \tilde{\gamma}] &= \lim_{s \rightarrow \infty} [k_{j_s} \circ \tilde{\gamma}] \\ &= \lim_{s \rightarrow \infty} [h_{j_s} \circ \gamma] = \lim_{s \rightarrow \infty} g^{\circ j_s}([\gamma]). \end{aligned}$$

Thus we conclude the proof the proposition. \square

Following [Aba89, Corollary 2.1.29], the normality of $\{g^{\circ j}\}_j$ implies that

- there exists a subsequence $\{g^{\circ j_k}\}_k$ converging to a holomorphic retraction

$$\rho : \tilde{X} \rightarrow \tilde{X}$$

of \tilde{X} , i.e. $\rho \circ \rho = \rho$,

- by [Car86], the image $M = \rho(\tilde{X})$ is a closed submanifold of \tilde{X} ,
- by [Aba89, Corollary 2.1.31], M is invariant by g and $g|_M$ is an automorphism.

3.4.5 Existence of the center manifold

We will study the dynamics of g restricted on M . The difference between the construction of universal covering used in the first case (the fixed point is outside $PC(f)$) and the construction of \tilde{X} in this case is that \tilde{X} does not contain a point representing z . Hence it is not straight forward that we can relate the dynamics of g on M with the dynamics of f near z .

We consider the objects introduced in Section 3.4.3. In particular, we consider a regular neighborhood W of z in \mathbb{C}^n and the family $\{h_j : W \rightarrow \mathbb{C}^n\}_j$ of inverse branches fixing z of $f^{\circ j}$ on W . Recall that

$$\sigma : W \setminus PC(f) \rightarrow \tilde{W}$$

is the inverse of the biholomorphism $\pi : \tilde{W} \rightarrow W \setminus PC(f)$ and that $\lim_{k \rightarrow \infty} g^{\circ j_k} = \rho$ is a holomorphic retraction on \tilde{X} .

Define a holomorphic map $\tilde{H} : W \setminus PC(f) \rightarrow \mathbb{C}^n$ as follows:

$$\tilde{H} = \pi \circ \rho \circ \sigma = \lim_{k \rightarrow +\infty} \pi \circ g^{\circ j_k} \circ \sigma.$$

but it is enough to consider an element representing a point near z

By (3.4.1), we have $\tilde{H} = \lim_{k \rightarrow +\infty} h_{j_k}|_{W \setminus PC(f)}$. Since $\{h_j : W \rightarrow \mathbb{C}^n\}_j$ is normal, by passing to subsequences, we can extend \tilde{H} to a holomorphic map $H : W \rightarrow \mathbb{C}^n$ such that $H = \lim_{k \rightarrow +\infty} h_{j_k} : W \rightarrow \mathbb{C}^n$.

Note that $h_j(z) = z$ for every $j \geq 1$. Then $H(z) = z$. By continuity of H , there exists an open neighborhood U of z in W such that $H(U) \subset W$. Note that we choose U to be a regular neighborhood of z (see Remark 3.4.11) and we can shrink U whenever we need to. Recall that for every $[\gamma] \in \tilde{W}$, we have $g^{oj}([\gamma]) = [h_j \circ \gamma]$. Then for $[\gamma] \in \tilde{U} := \sigma(U \setminus PC(f))$, we have

$$\rho([\gamma]) = \lim_{k \rightarrow \infty} g^{oj_k}([\gamma]) = \lim_{k \rightarrow \infty} [h_{j_k} \circ \gamma] = [H \circ \gamma] \subset \tilde{W}$$

In other words, $\rho(\tilde{U}) \subset \tilde{W}$. Hence

$$H(U \setminus PC(f)) = \pi \circ \rho \circ \sigma(U \setminus PC(f)) \subset W \setminus PC(f). \quad (3.4.1)$$

Moreover, since $\sigma \circ \pi|_{\tilde{W}} = \text{Id}_{\tilde{W}}$, the composition

$$\tilde{H} \circ \tilde{H} = \pi \circ \rho \circ \sigma \circ \pi \circ \rho \circ \sigma.$$

is well defined on $U \setminus PC(f)$ and equals to $\tilde{H}|_{U \setminus PC(f)}$. Since H is the extension of \tilde{H} , we deduce that

$$H \circ H(U) = H(U).$$

Proposition 3.4.15. *The set $H(U)$ is a submanifold of W containing z whose dimension is the number of neutral eigenvalues of $D_z f$ counted with multiplicities. Moreover, $T_z H(U) = E_n$ and $D_z f|_{T_z H(U)}$ is diagonalizable.*

Proof. The first assertion is due to [Car86] since $H \circ H = H$ on U . The rest are consequences of the fact that H is a limit map of the family $\{h_j : W \rightarrow \mathbb{C}^n\}_j$ of inverse branches fixing z of f (see Corollary 3.2.4). \square

Lemma 3.4.16. $\dim M = \dim H(U)$.

Proof. Since \tilde{X} is connected, M is also a connected complex manifold. Thus $\dim M = \text{rank } D_x \rho$ for every $x \in \tilde{M}$. In particular, if we choose $x = \sigma(z)$ with $w \in U \setminus PC(f)$ then

$$\text{rank } D_x \rho = \text{rank}_w H = \dim H(U)$$

thus $\dim M = \dim H(U)$. □

In other words,

$$M_X := \pi(M) \cup H(U)$$

is a submanifold of \mathbb{C}^n in a neighborhood of z . Moreover, following Proposition 3.3.6, we can deduce that $\pi(M)$ is a bounded set in \mathbb{C}^n .

3.4.6 Semiconjugacy on the center manifold

Denote by $\Lambda = (D_z f|_{E_n})^{-1}$. We will construct a holomorphic $\Phi : M \rightarrow E_n$ such that

$$\Phi \circ g|_M = \Lambda \circ \Phi.$$

The construction follows the idea in Section 3.3.6 and the connection established in Step 3 between g and inverse branches of f at z .

Lemma 3.4.17. *There exists a holomorphic map $\Phi : M \rightarrow E_n$ such that $\Phi \circ g = \Lambda \circ \Phi$.*

Proof. We consider a holomorphic map $\varphi : M \rightarrow E_n$ constructed as the following composition

$$M \hookrightarrow \tilde{X} \xrightarrow{\pi} \mathbb{C}^n \xrightarrow{\delta} T_z \mathbb{C}^n \xrightarrow{\pi_{E_r}} E_n$$

where $\delta : \mathbb{C}^n \rightarrow T_z \mathbb{C}^n$ is a holomorphic map such that $\delta(z) = 0$, $D_z \delta = \text{Id}$, $\pi_{E_r} : T_z \mathbb{C}^n \rightarrow E_n$ is a projection on E_n parallel to E_r . Note that since $\pi(M)$ is bounded, $\varphi(M)$ is also a bounded set in E_n .

We consider the family

$$\Lambda^{-j} \circ \varphi \circ g^{oj} : M \rightarrow E_n, j \geq 0.$$

Since Λ is diagonalizable with only neutral eigenvalues, this family is uniformly bounded hence so is the family of its Cesàro averages $\{\Phi_N = \frac{1}{N} \sum_{j=0}^{N-1} \Lambda^{-j} \circ \varphi \circ g^{oj}\}_N$. Therefore, $\{\Phi_N\}_N$ is normal and every limit map Φ of $\{\Phi_N\}_N$ satisfies that

$$\Phi \circ g = \Lambda \circ \Phi.$$

Note that $\Phi(M)$ is also a bounded set in E_n . □

Let us fix such a limit map $\Phi = \lim_{k \rightarrow \infty} \Phi_{N_k}$. We will prove that Φ restricted to $\tilde{U} \cap M$ is a biholomorphism. In order to do so, we consider the following holomorphic function

$$\Phi_1 := \Phi \circ \sigma|_{H(U)} : H(U \setminus PC(f)) \rightarrow \mathbb{C}^n.$$

Since $H(U \setminus PC(f)) \subset \pi(M)$ is a bounded set in \mathbb{C}^n , the map Φ_1 is bounded hence we can extend it to a holomorphic function on $H(U)$. By an abuse of notations, we denote the extension by Φ_1 . We will prove that Φ_1 is invertible in a neighborhood of z in $H(U)$.

More precisely, on $H(U) \setminus PC(f)$, we have

$$\Phi_1 = \lim_{k \rightarrow \infty} \frac{1}{N_k} \sum_{j=0}^{N_k-1} \Lambda^{-j} \circ \varphi \circ g^{oj} \circ \sigma|_{H(U) \setminus PC(f)}$$

Consider the map $\varphi_1 : \pi(M) \cup H(U) \rightarrow E_n, \varphi_1 = \pi_{E_r} \circ \delta|_{M_X}$. Then $\varphi_1(z) = 0, D_z \varphi_1 = \text{Id}$ and $\varphi = \varphi_1|_{\pi(M)} \circ \pi$. It follows that

$$\begin{aligned} \Phi_1 &= \lim_{k \rightarrow \infty} \frac{1}{N_k} \sum_{j=0}^{N_k-1} \Lambda^{-j} \circ \varphi_1|_{\pi(M)} \circ \pi \circ g^{oj} \circ \sigma|_{H(U) \setminus PC(f)} \\ &= \lim_{k \rightarrow \infty} \frac{1}{N_k} \sum_{j=0}^{N_k-1} \Lambda^{-j} \circ \varphi_1|_{\pi(M)} \circ h_j|_{H(U) \setminus PC(f)} \end{aligned}$$

Note that $D_z h_j|_{H(U)} = \Lambda^j$, then we can deduce that $\Phi_1(z) = 0$ and $D_z \Phi_1 = \text{Id}$. Therefore, there exists a regular neighborhood V of z in W such that Φ_1 is biholomorphic on $V \cap H(U)$. Consequently, since σ is a biholomorphism, the neighborhood V induces an open neighborhood $\tilde{V} = \sigma(V \setminus PC(f))$ in \tilde{W} such that Φ is biholomorphic on $\tilde{V} \cap \mathcal{M}$. By shrinking U , we can assume that $V = U$ hence $\Phi|_{\tilde{U} \cap M}$ is a biholomorphism.

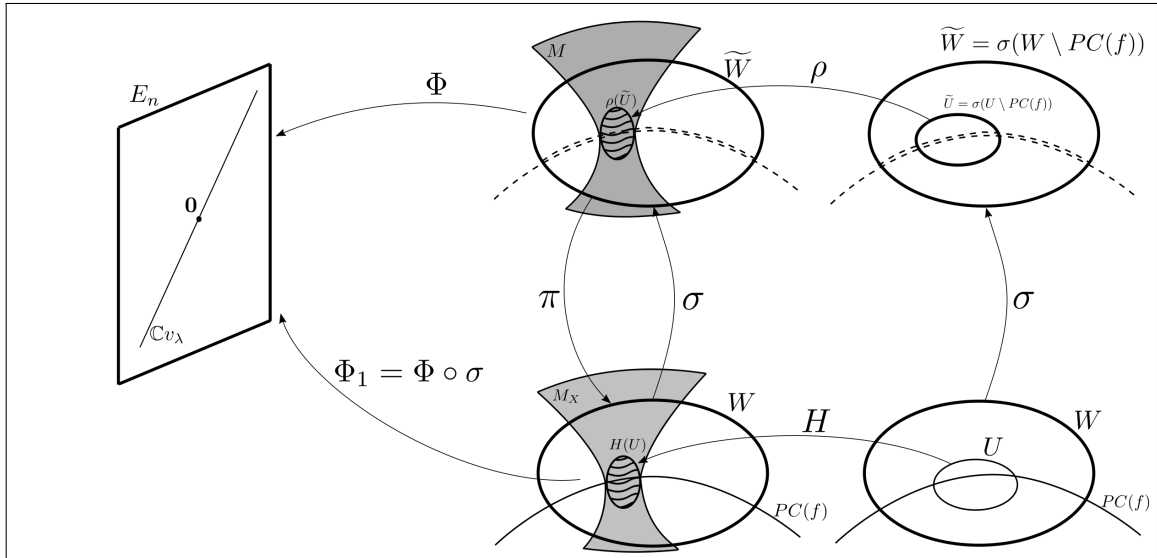


Figure 3.4-4: Constructions on M

3.4.7 Linearization along the neutral direction

The map Φ_1 extends the image of Φ in the sense that $\Phi(M) \cup \Phi_1(H(U))$ contains a full neighborhood of 0 in E_n . Let $v \in E_n$ be an eigenvector of $D_z f$ associated to λ . We will study $\Phi^{-1}(\mathbb{C}v)$ by studying $\Phi_1^{-1}(\mathbb{C}v)$.

Denote by Γ_1 the irreducible component of $\Phi_1^{-1}(\mathbb{C}v)$ containing z . Since $D_z \Phi_1 = \text{Id}$, Γ_1 is a submanifold of dimension one of $H(U)$ near z and $T_z \Gamma_1 = \mathbb{C}v$. Note that $v \notin T_z PC(f)$, then by shrinking U if necessary, we can assume that $\Gamma_1 \cap PC(f) = \{z\}$. In other words, $\Gamma_1 \setminus \{z\}$ is a smooth component of $\Phi_1^{-1}(\mathbb{C}v)$ in $H(U) \setminus PC(f)$.

Since $\Phi_1 = \Phi \circ \sigma|_{H(U) \setminus PC(f)}$ and σ is a biholomorphism, there exists a unique irreducible component Γ of $\Phi^{-1}(\mathbb{C}v)$ such that Γ contains $\sigma(\Gamma_1 \setminus \{z\})$. Moreover, $\Phi(\Gamma)$ is a punctured neighborhood of 0 in $\mathbb{C}v$. This means that $0 \notin \Phi(\Gamma)$ but $\Phi(\Gamma) \cup \{0\}$ contains an open neighborhood of 0 in $\mathbb{C}v$. We will prove that Γ is in fact biholomorphic to a punctured disc and $\Phi|_\Gamma$ is a biholomorphism conjugating $g|_\Gamma$ to the irrational rotation $\zeta \mapsto \lambda\zeta$.

Following Section 3.3.7, we consider $\Gamma_0 = \Gamma \setminus C_\Phi$ where C_Φ the set of critical points of Φ . Then Γ_0 is a hyperbolic Riemann surface which is invariant by g . The map g induces an automorphism $g|_{\Gamma_0}$ on Γ_0 such that $g|_{\Gamma_0}^{\circ j_k}$ converges to $\rho = \text{Id}_M$ which is identity on Γ_0 .

On one hand, Γ_0 contains $\sigma(\Gamma_1 \setminus PC(f))$ hence $\Phi(\Gamma_0)$ is also a punctured neighborhood of 0 in $\mathbb{C}v$. On another hand, g restricted on $\sigma(\Gamma_1 \setminus PC(f))$ is conjugate to h restricted on $\Gamma_1 \setminus PC(f)$. Note that h fixes $z = \Gamma_1 \cap PC(f)$. Hence we can consider an abstract Riemann surface $\Gamma_0^* = \Gamma_0 \cup \{z\}$ and two holomorphic maps $\iota : \Gamma_0 \rightarrow \Gamma_0^*$, $\Phi^* : \Gamma_0^* \rightarrow \mathbb{C}v \subset E_n$ so that ι is an injective holomorphic map, $\Gamma_0^* \setminus \iota(\Gamma_0) = \{z\}$, $\Phi^*(z) = 0$ and the following diagram commutes.

$$\begin{array}{ccc} & & \Gamma_0^* \\ & \swarrow \Phi^* & \uparrow \iota \\ E_n & \xleftarrow{\Phi} & \Gamma_0 \end{array}$$

Moreover, Γ_0^* admits an automorphism g^* fixing the point z with multiplier λ and extends $g|_{\Gamma_0}$ in the sense that $g^* \circ \iota = \iota \circ g$. Note that $\Phi^*(\Gamma_0^*) = \Phi(\Gamma_0) \cup \{0\}$ is bounded in E_n . Then by arguing as in Lemma 3.3.12, we deduce that.

Lemma 3.4.18. *The Riemann surface Γ_0^* is biholomorphic to a disc $\mathbb{D}(0, R)$, $R \in (0, +\infty)$ and $\Phi^* : \Gamma_0^* \rightarrow \Phi^*(\Gamma_0^*) = \mathbb{D}(0, R)$, $\Phi^*(z) = 0$ is a biholomorphism conjugating g^* to the irrational rotation $\zeta \rightarrow \lambda\zeta$.*

Consequently, Γ_0 is biholomorphic to $\mathbb{D}(0, R)$ and $\Phi|_\Gamma$ is a biholomorphism.

Proposition 3.4.19. *The set Γ is smooth and the map*

$$\Phi|_{\Gamma} : \Gamma \rightarrow \Phi(\Gamma) = \mathbb{D}(0, R)^*$$

is a biholomorphism with $R \in (0, +\infty)$.

Proof. It is enough to prove that $\Gamma = \Gamma_0$. The idea is similar to the proof of Proposition 3.3.13

Note that Γ_0 is the complement of a discrete set $\Gamma \cap \text{Sing } \Phi^{-1}(\mathbb{C}v)$ in Γ and Γ_0 is biholomorphic to a punctured disc $\mathbb{D}(0, R)^*$. Moreover, $\Phi(\Gamma_0) \subset \Phi(\Gamma)$ is also a punctured neighborhood of 0 and $\Gamma_0 \subset \text{Reg } \Gamma$ has discrete complement.

Then we can consider an abstract one dimensional analytic space $\Gamma^* = \Gamma \cup \{z\}$ such that $\Gamma_0^* \subset \Gamma^*$ and $\Gamma^* \setminus \Gamma_0^*$ is a discrete set containing singular points of Γ^* (which is exactly $\Gamma \setminus \Gamma_0$). Then by arguing similarly to 3.3.13, we can deduce that Γ^* is biholomorphic to $\mathbb{D}(0, R)$ hence the proposition is proved. \square

3.4.8 End of the proof

Denote by $\tau_1 := \pi \circ (\Phi|_{\Gamma})^{-1} : \mathbb{D}(0, R)^* \rightarrow \mathbb{C}^n$. The map τ_1 has a holomorphic extension to the map $\tau : \mathbb{D}(0, R) \rightarrow \mathbb{C}^n$ such that $\tau(0) = z, \tau'(0) = v$. The map τ takes values in $\pi(M)$, which is bounded, hence the radial limit

$$\tau_{\theta} = \lim_{r \rightarrow R^-} \tau(re^{i\theta})$$

exists for almost every $\theta \in [0, 2\pi)$.

Proposition 3.4.20. *$\tau_{\theta} \in PC(f)$ if it exists.*

Proof. Note that $\tau(\mathbb{D}(0, R)) \cap PC(f) = (\pi(\Gamma) \cup \Gamma_1) \cap PC(f) = \{z\}$. Hence $\tau(\mathbb{D}(0, R) \setminus \{0\}) \subset M$. It implies that $\tau(\mathbb{D}(0, R) \setminus \{0\}) \subset X$. Then by arguing similarly to Proposition 3.3.15, we deduce that the radial limit $\tau_{\theta} \in PC(f)$ if this limit exists. \square

Recall that Q is the defining polynomial of $PC(f)$ then $Q \circ \tau$ has vanishing radial limit for almost every $\theta \in [0, 2\pi)$. This means that $Q \circ \tau$ is identically 0. Hence $\tau(\mathbb{D}(0, 1)) \subset PC(f)$. This is a contradiction since $\tau'(0) = v \notin T_z PC(f)$. The proof the Proposition 3.4.2 is complete.

3.5 Periodic cycles of post-critically algebraic endomorphisms of \mathbb{CP}^2

This section is devoted to the proof of Theorem C.

Theorem C. Let f be a post-critically algebraic endomorphism of \mathbb{CP}^2 and λ be an eigenvalue of f along a periodic cycle. Then either $\lambda = 0$ or $|\lambda| > 1$.

If the periodic cycle is not in $PC(f)$, then $\lambda \neq 0$ and the result follows from Theorem B. Therefore, without loss of generality, we may assume that z is a fixed point of f in $PC(f)$. Note that if λ is an eigenvalue of f at a fixed point z , then λ^j is an eigenvalue of $f^{\circ j}$ at the fixed point z . If we can prove that λ^j is either superattracting or repelling, so is λ . Thus, in order to prove Theorem C, we can always consider f up to some iterates if necessary.

After passing to an iterate, we may assume that the fixed point belongs to an invariant irreducible component Γ of $PC(f)$. The reason why we have to restrict to dimension $n = 2$ is that, in this case, Γ is an algebraic curve. There is a normalization $\mathbf{n} : \hat{\Gamma} \rightarrow \Gamma$ where $\hat{\Gamma}$ is a smooth compact Riemann surface and \mathbf{n} is a biholomorphism outside a finite set (see [RS13], [Gun90] or [Chi12]). And there is a holomorphic endomorphism $\hat{f} : \hat{\Gamma} \rightarrow \hat{\Gamma}$ such that $\mathbf{n} \circ \hat{f} = f \circ \mathbf{n}$.

In section 3.5.1, we analyse the dynamics of $\hat{f} : \hat{\Gamma} \rightarrow \hat{\Gamma}$ and in particular, we show that when f is post-critically algebraic, then \hat{f} is post-critically finite. In Section 3.5.2, we complete the proof in the case where the fixed point belongs to the regular part of $PC(f)$ and in Section 3.5.3, we complete the proof in the case where the fixed point belongs to the singular part of $PC(f)$.

3.5.1 Dynamics on an invariant curve

Assume that $f : \mathbb{CP}^2 \rightarrow \mathbb{CP}^2$ is an endomorphism of degree $d \geq 2$ (not necessarily post-critically algebraic) and $\Gamma \subset \mathbb{CP}^2$ is an irreducible algebraic curve such that $f(\Gamma) = \Gamma$. Let $\mathbf{n} : \hat{\Gamma} \rightarrow \Gamma$ be a normalization of Γ and $\hat{f} : \hat{\Gamma} \rightarrow \hat{\Gamma}$ be an endomorphism such that $\mathbf{n} \circ \hat{f} = f \circ \mathbf{n}$.

According to [FS94, Theorem 7.4], the endomorphism $\hat{f} : \hat{\Gamma} \rightarrow \hat{\Gamma}$ has degree $d \geq 2$. It follows from the Riemann-Hurwitz Formula that the compact Riemann surface $\hat{\Gamma}$ has genus 0 or 1. In addition, if the genus is 1, then \hat{f} has no critical point and all fixed points of \hat{f} are repelling with common repelling eigenvalue λ satisfying $|\lambda| = \sqrt{d}$. If the genus is 0, then the following lemma implies that the postcritical set of \hat{f} and f are closely related.

Lemma 3.5.1. *Denote by $V_{\hat{f}}$ and V_f the set of critical values of \hat{f} and f respectively. Then*

$$V_{\hat{f}} \subset \begin{cases} \mathbf{n}^{-1}(V_f) & \text{if } \Gamma \not\subset V_f \\ \mathbf{n}^{-1}(\text{Sing } V_f) & \text{if } \Gamma \subset V_f \end{cases}.$$

Proof. The set of critical values of \hat{f} is characterized by the following property: $x \notin V_{\hat{f}}$ if and only if for every $y \in \hat{f}^{-1}(x)$, \hat{f} is injective near y . Note that $\mathbf{n} : \hat{\Gamma} \rightarrow \Gamma$ induces a parametrization of the germ (Γ, x) such that for every $x \in \Gamma$ and for every $y \in \mathbf{n}^{-1}(x)$, \mathbf{n} is injective near y (see also 2.3, [Wal04]).

- If $\Gamma \not\subset V_f$, let $x \notin \mathbf{n}^{-1}(V_f)$ and let $y \in \hat{f}^{-1}(x)$. Then $\mathbf{n}(y) \in f^{-1}(\mathbf{n}(x))$. Since $\mathbf{n}(x) \notin V_f$ then f is injective near $\mathbf{n}(y)$. Combining with the fact that \mathbf{n} is locally injective, we deduce that \hat{f} is injective near y . Thus $x \notin V_{\hat{f}}$.
- If $\Gamma \subset V_f$, let $x \notin \mathbf{n}^{-1}(\text{Sing } V_f)$ and let $y \in \hat{f}^{-1}(x)$. Then $\mathbf{n}(y) \in f^{-1}(\mathbf{n}(x))$ and $\mathbf{n}(x) \in \text{Reg } V_f$. By Proposition 3.4.4, we can deduce that $\mathbf{n}(y) \in f^{-1}(\text{Reg } V_f)$ and $f|_{f^{-1}(\text{Reg } V_f)}$ is locally injective. It implies that \hat{f} is also injective near y . Hence $x \notin V_{\hat{f}}$.

Thus we obtain the conclusion of the lemma. \square

Proposition 3.5.2. *If $\hat{\Gamma}$ has genus 0 and f is a post-critically algebraic endomorphism then \hat{f} is a PCF endomorphism.*

Proof. We have that

$$PC(f) = \bigcup_{j \geq 1} V_{f^{oj}} \quad \text{and} \quad PC(\hat{f}) = \bigcup_{j \geq 1} V_{\hat{f}^{oj}}.$$

Since $\mathbf{n} \circ \hat{f}^{oj} = f^{oj} \circ \mathbf{n}$ for all $j \geq 1$, applying the previous lemma to f^{oj} and \hat{f}^{oj} yields

$$PC(\hat{f}) \subset \begin{cases} \mathbf{n}^{-1}(PC(f)) & \text{if } \Gamma \not\subset PC(f) \\ \mathbf{n}^{-1}(\text{Sing } PC(f)) & \text{if } \Gamma \subset PC(f). \end{cases}$$

In both cases, $PC(\hat{f})$ is contained in the preimage by \mathbf{n} of a proper algebraic subset of Γ , which therefore is finite. Since \mathbf{n} is proper, $PC(\hat{f})$ is finite and so, \hat{f} is PCF. \square

Assume that f has a fixed point z which is a regular point of Γ (which is not necessarily an irreducible component of $PC(f)$). Since \mathbf{n} is a biholomorphism outside the preimage

of singular points of Γ , the point $\mathbf{n}^{-1}(z)$ is a fixed point of \hat{f} and \mathbf{n} will conjugate $D_z f|_{T_z \Gamma}$ and $D_{\mathbf{n}^{-1}(z)} \hat{f}$. Denote by λ the eigenvalue of $D_z f|_{T_z \Gamma}$. Then λ is also the eigenvalue of $D_{\mathbf{n}^{-1}(z)} \hat{f}$. The previous discussion allows us to conclude that either $\lambda = 0$ or $|\lambda| > 1$. Thus we can deduce the following result.

Lemma 3.5.3. *Let f be a post-critically algebraic endomorphism of $\mathbb{C}\mathbb{P}^2$ of degree $d \geq 2$, let $\Gamma \subset \mathbb{C}\mathbb{P}^2$ be an invariant irreducible algebraic curve, let $z \in \text{Reg}\Gamma$ be a fixed point of f and let λ be the eigenvalue of $D_z f|_{T_z \Gamma}$. Then, either $\lambda = 0$ or $|\lambda| > 1$.*

3.5.2 Periodic cycles in the regular locus of the post-critical set

Proof of Theorem C – first part. Let f be a post-critically algebraic endomorphism of $\mathbb{C}\mathbb{P}^2$ with a fixed point z which is a regular point of $PC(f)$. Denote by Γ the irreducible component of $PC(f)$ containing z . Then Γ is invariant by f . Denote by $\overline{D_z f} : T_z \mathbb{C}\mathbb{P}^2 / T_z \Gamma \rightarrow T_z \mathbb{C}\mathbb{P}^2 / T_z \Gamma$ the linear endomorphism induced by $D_z f$. Note that

$$\text{Spec}(D_z f) = \text{Spec}(D_z f|_{T_z \Gamma}) \cup \text{Spec}(\overline{D_z f})$$

By Proposition 3.4.4, the eigenvalue of $D_z f|_{T_z \Gamma}$ is not 0 hence repelling by Lemma 3.5.3. By Proposition 3.4.1, the eigenvalue of $\overline{D_z f}$ is either superattracting or repelling. Thus Theorem C is proved when the fixed point is a regular point of the post-critical set. \square

3.5.3 Periodic cycles in the singular locus of the post-critical set

When the fixed point z is a singular point of $PC(f)$, by passing to some iterates of f , we can assume that f induces a holomorphic germ at z which fixes a singular germ of curve at z which is induced by some irreducible components of $PC(f)$. On the one hand, from the local point of view, there exists (in most of cases) a relation between two eigenvalues of $D_z f$ as a holomorphic germ fixing a singular germ of curve. On the other hand, from the global point of view, these eigenvalues can be identified with the eigenvalue the germ at a fixed point of the lifts of f via the normalization of $PC(f)$. Then by Proposition 3.5.2, we can conclude Theorem C.

Holomorphic germ of $(\mathbb{C}^2, 0)$ fixing a singular germ of curve

Let $(\Sigma, 0)$ be an irreducible germ of curve at 0 in $(\mathbb{C}^2, 0)$ defined by a holomorphic germ $g : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}, 0)$. In local coordinates (x, y) of \mathbb{C}^2 , if $g(0, y) \not\equiv 0$, i.e. g does not

identically vanish on $\{x = 0\}$, it is well known that there exists an injective holomorphic germ $\gamma : (\mathbb{C}, 0) \rightarrow (\mathbb{C}^2, 0)$ of the form

$$\gamma(t) = (t^m, \alpha t^n + O(t^{n+1}))$$

parameterizing Σ , i.e. $\gamma((\mathbb{C}, 0)) = (\Sigma, 0)$ (see [Wal04, Theorem 2.2.6]). If Σ is singular, after a change of coordinates, α can be 1 and m and n satisfy that $1 < m < n, m \nmid n$. The germ γ is called a Puiseux parametrization. In fact, if Σ is a germ induced by an algebraic curve Γ in $\mathbb{C}\mathbb{P}^2$, then γ coincides with the germ induced by the normalization morphism. When Σ is singular, the integers m and n are called the first two Puiseux characteristics of Γ and they are invariants of the equisingularity class of Γ . In particular, m and n do not depend the choice of local coordinates. We refer to [Wal04] for further discussion about Puiseux characteristics. We refer also to [ZT73] and references therein for discussion about equisingular invariants.

Now, consider a proper⁴ holomorphic germ $g : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, 0)$ and a singular germ of curve $(\Sigma, 0)$. If Σ is invariant by g , i.e. $g(\Sigma) = \Sigma$, g acts as a permutation on irreducible branches of Σ . Then by passing to some iterates of g , we assume that there exists an invariant branch. The following propositions show that there exists a relation between two eigenvalues of D_0g .

When g has an invariant singular branch, the following result was observed by Jons-son, [Jon98].

Proposition 3.5.4. *Let Σ be an irreducible singular germ of curve parametrized by $\gamma : (\mathbb{C}, 0) \rightarrow (\mathbb{C}^2, 0)$ of the form*

$$\gamma(t) = (t^m, t^n + O(t^{n+1})), 1 < m < n, m \nmid n.$$

Let $g : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, 0)$ and $\hat{g} : (\mathbb{C}, 0) \rightarrow (\mathbb{C}, 0), \hat{g}(t) = \lambda t + O(t^2)$ be holomorphic germs such that

$$g \circ \gamma = \gamma \circ \hat{g}.$$

Then the eigenvalues of D_0g are λ^m and λ^n .

Proof. The germ g has an expansion of the form

$$g(x, y) = (ax + by + h_1(x, y), cx + dy + h_2(x, y))$$

⁴Proper germ means that $g^{-1}(0) = 0$. In particular, an endomorphism of $\mathbb{C}\mathbb{P}^2$ induces a proper germ at its fixed points.

where $h_1(x, y) = O(\|(x, y)\|^2)$, $h_2(x, y) = O(\|(x, y)\|^2)$. Replacing those expansions in the equation $\gamma \circ \hat{g} = g \circ \gamma$, we have

$$\begin{aligned} (\lambda^m t^m + O(t^{m+1}), \lambda^n t^n + O(t^{n+1})) &= (at^m + bt^n + h_1(t^m, t^n + O(t^{n+1})), \\ &\quad ct^m + dt^n + h_2(t^m, t^n + O(t^{n+1}))). \end{aligned}$$

Comparing coefficients of the term t^m in each coordinate, we deduce that $a = \lambda^m$ and $c = 0$. Comparing coefficients of the term t^n in the second coordinate, since $m \nmid n$, the expansion of h_2 can not contribute any term of order t^n , hence $d = \lambda^n$. The linear part of g has the form $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ hence a, d are eigenvalues of D_0g . In other words, λ^m and λ^n are eigenvalues of D_0g . \square

When g has an invariant smooth branch which is the image of another branch, g is not an injective germ hence 0 is an eigenvalue of D_0g . This case was not considered in [Jon98] since Jonsson assumed that there is no periodic critical point.

Proposition 3.5.5. *Let $g : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, 0)$ be a proper holomorphic germ and let Σ_1, Σ_2 be irreducible germs of curves at 0 such that $\Sigma_1 \neq \Sigma_2, g(\Sigma_1) = \Sigma_2, g(\Sigma_2) = \Sigma_2$. If Σ_2 is smooth then the eigenvalues of D_zg are 0 and λ where λ is the eigenvalue of $D_0g|_{T_0\Sigma_2}$.*

Proof. Since Σ_2 is smooth, we choose a local coordinates (x, y) of $(\mathbb{C}^2, 0)$ such that $\Sigma_2 = \{x = 0\}$. Since Σ_1 and Σ_2 are distinct irreducible germs, the defining function of Σ_1 does not identically vanish on Σ_2 . Then we can find a Puiseux parametrization of Σ_1 of the form

$$\gamma(t) = (t^m, \alpha t^n + O(t^{n+1})), \alpha \in \mathbb{C} \setminus \{0\}$$

where m, n are positive integers (see [Wal04, Theorem 2.2.6]). The germ g has an expansion of the form

$$g(x, y) = (ax + by + h_1(x, y), cx + dy + h_2(x, y))$$

where $h_1(x, y) = O(\|(x, y)\|^2)$, $h_2(x, y) = O(\|(x, y)\|^2)$. The invariance of Σ_2 implies that $b = 0$ and g has the form

$$g(x, y) = (x(a + h_3(x, y)), cx + dy + h_2(x, y))$$

where $h_3(x, y) = O(\|(x, y)\|)$, $h_2(x, y) = O(\|(x, y)\|^2)$. Replacing γ and f in the equation

$g(\Sigma_1) = \Sigma_2$, we have

$$t^m(a + h_3(t^m, \alpha t^n + O(t^{n+1}))) = 0$$

hence $a = 0$. Then the linear part of D_0g is $(0, cx + dy)$ hence 0 and d are the eigenvalues of D_0g where d is the eigenvalue of the restriction of D_0g to $T_0\Sigma_2$. \square

And finally, if g has two invariant smooth branches which are tangent, we have the following result (see also [Jon98]).

Proposition 3.5.6. *Let $g : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, 0)$ be a proper holomorphic germ and let Σ_1, Σ_2 be irreducible invariant germs of smooth curves at 0 . If Σ_1 and Σ_2 intersect tangentially, i.e. $\Sigma_1 \neq \Sigma_2$ and $T_0\Sigma_1 = T_0\Sigma_2$ then there exists a positive integer m such that the eigenvalues of D_0g are λ and λ^m where λ is the eigenvalue of $D_0g|_{T_0\Sigma_1}$.*

Proof. Since Σ_1 is smooth, we can choose a local coordinates (x, y) such that $\Sigma_1 = \{y = 0\}$. The defining function of Σ_2 can not identically vanish on $\{x = 0\}$ since otherwise, Σ_2 and Σ_1 would not be tangent. Then Σ_2 has a parametrization of the form

$$\gamma(t) = (t, t^m + O(t^{m+1})).$$

Since $\Sigma_1 = \{y = 0\}$ is invariant, g has an expansion in the coordinates (x, y) of the form:

$$g(x, y) = (\lambda x + by + h_1(x, y), y(d + h_2(x, y)))$$

where $\lambda, b, d \in \mathbb{C}$, $h_1(x, y) = O(\|(x, y)\|^2)$, $h_2(x, y) = O(\|(x, y)\|)$. The linear part of D_0g is $(\lambda x + by, dy)$ thus λ, d are eigenvalues of $D_z f$. Letting $x = t, y = t^m + O(t^{m+1})$, we have

$$f(t, t^m + O(t^{m+1})) = (\lambda t + O(t^2), dt^m + O(t^{m+1}))$$

Since $f(\Sigma_2) = \Sigma_2$, we deduce that $d = \lambda^m$. Note that λ is the eigenvalue of $D_0g|_{T_0\Sigma_1}$. \square

Using these observations, we can conclude the proof of Theorem C.

Proof of Theorem C – final. Let f be a post-critically algebraic endomorphism of $\mathbb{C}\mathbb{P}^2$ and let z be a fixed point such that z is a singular point of $PC(f)$. We will look at the germ $(PC(f), z)$ induced by $PC(f)$ at z and prove Theorem C depending on how f acts on irreducible branches of $(PC(f), z)$. Passing to an iterate of f if necessary, we assume that there exists a branch Σ of $(PC(f), z)$ such that $f(\Sigma) = \Sigma$. Denote by Γ the irreducible component of $PC(f)$ inducing Σ . Then Γ is also invariant by f . Denote by $\mathfrak{n} : \hat{\Gamma} \rightarrow \Gamma$ the normalization of Γ and by \hat{f} the lifting of f by the normalization $\mathfrak{n} : \hat{\Gamma} \rightarrow \Gamma$.

If Σ is singular then $\mathbf{n}^{-1}(z)$ is a finite set and $\hat{f}(\mathbf{n}^{-1}(z)) \subset \mathbf{n}^{-1}(z)$ since z is a fixed point of f . Then by passing up to some iterations, we can assume that \hat{f} fixes a point $w_0 \in \mathbf{n}^{-1}(z)$. By Proposition 3.5.4, the eigenvalues of $D_z f$ are λ^m, λ^n where λ is the eigenvalue of $D_{w_0} \hat{f}$ and m and n are the first two Puiseux characteristics of Σ . By Proposition 3.5.2, λ is either superattracting or repelling. Hence so are λ^m and λ^n .

If Σ is smooth, the tangent space $T_z \Sigma$ is well-defined and invariant by $D_z f$. Denote by λ the eigenvalue of $D_z f|_{T_z \Sigma}$. Even if Γ can be singular (for example, z can be a self-intersection point of Γ), there exists a point $w \in \hat{\Gamma}$, $\mathbf{n}(w) = z$ such that w is a fixed point of \hat{f} and \mathbf{n} induces an invertible germ $\mathbf{n} : (\hat{\Gamma}, w) \rightarrow (\Sigma, z)$. Arguing similarly to Lemma 3.5.3, we can deduce that either $\lambda = 0$ or $|\lambda| > 1$. To deal with the other eigenvalue, since $(PC(f), z)$ is singular, we have one of the following cases:

1. There exists a branch Σ_1 such that $f(\Sigma_1) = \Sigma$. By Proposition 3.5.5, the eigenvalues of $D_z f$ are 0 and λ where λ is the eigenvalue of $D_z f|_{T_z \Sigma}$. Hence we are done by previous discussion.
2. There exists a smooth invariant branch Σ_1 such that Σ and Σ_1 intersect transversally. Denote by Γ_1 the irreducible component of $PC(f)$ containing Σ_1 (which can be Γ) thus Γ_1 are invariant by f . Since Σ and Σ_1 are transversal, the eigenvalues of $D_z f$ are λ and λ_1 where λ_1 is the eigenvalue of $D_z f|_{T_z \Sigma_1}$. Arguing similarly to the case of λ , we deduce λ_1 is also either superattracting or repelling.
3. There exists a smooth invariant branch Σ_1 such that Σ and Σ_1 intersect tangentially. Denote by Γ_1 the irreducible component of $PC(f)$ inducing Σ_1 (which is possibly equal to Γ) thus Γ_1 are invariant by f . By Proposition 3.5.6, the eigenvalues of $D_z f$ are λ and λ^m . Note that λ is either superattracting or repelling hence so does λ^m .

□

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