



Relations between entropies, exponents and dimensions

Yuntao Zang

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Relations between entropies, exponents and dimensions

**Thèse de doctorat de
Soochow University et de l'université Paris-Saclay**

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Titre: Relations entre entropies, exposants et dimensions

Mots cl  s: entropies; entropie-hyperbolique; exposants; Formule d'entropie de Ledrappier-Young; Th  orie de Pesin

R  sum  : Nous   tudions les relations entre entropies, exposants et dimensions pour des dynamiques diff  rentiables. Nous proposons une preuve simplifi  e de la formule d'entropie de Ledrappier-Young. Ensuite, en combinant cette formule et la th  orie de Pesin, nous majorons l'entropie d'une mesure ergodique par une combinaison d'exposants de Lyapunov et de divers invariants d  finis par la dynamique sur les vari  t  s instables telles que croissance du volume ou entropies.

Nous analysons   galement de fa  on syst  matique les entropies dites dimensionnelles et les syst  mes entropie-hyperboliques qu'elles permettent de d  finir.

Title: Relations between entropies, exponents and dimensions

Keywords: entropies; entropy-hyperbolic; exponents; Ledrappier-Young entropy formula; Pesin theory

Abstract: We focus on the relations between entropies, exponents and dimensions for differentiable dynamics. We propose a simplified proof of Ledrappier-Young entropy formula. Then by using Ledrappier-Young entropy formula and Pesin theory, we bound the entropy of an ergodic measure by a combination of Lyapunov exponents and various invariants on the unstable manifolds such as volume growth, topological entropies and fibered entropies.

We also systemically investigate the so called dimensional entropies and their applications to entropy-hyperbolic systems.

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Chapter 1

Introduction

1.1 Introduction (Français)

1.1.1 Ergodicité et entropie

La théorie des systèmes dynamiques concentre les changements dans le temps d'un point donné dans un espace. Les exemples incluent le système solaire, le flux de liquide et de gaz, le stock de marché, le nombre de poissons dans un étang, le temps de la terre et ainsi de suite. Il existe des applications vastes et profondes de la théorie des systèmes dynamiques dans de nombreux domaines avancés des sciences comme la biologie, la physique, la théorie de l'information, etc. Il y a généralement deux parties d'un système dynamique: l'espace des phases et la dynamique. L'espace de phase fournit tous les états possibles en question. Il s'agit généralement d'un ensemble de points avec des structures mesurables, topologiques ou différentiables. La dynamique est un ensemble de règles qui régit le mouvement d'un point dans l'espace des phases. Soyons clairs à travers une abstraction mathématique. En un mot, un système dynamique est défini X avec une auto-carte f . Ce n'est généralement pas très intéressant sans ajouter plus de structure sur X et plus de restrictions sur f . La façon la plus courante et la plus simple de le faire est de supposer que X est un espace mesurable pour certaines σ algèbre $\mathcal{B}$ et f préserve la structure mesurable dans le sens où la pré-image d'un ensemble mesurable est également mesurable. Nous avons maintenant la définition formelle des systèmes dynamiques, mais quel genre de question qui intéresse les dynamiques?

Commençons par la fonction d'expansion linéaire sur l'intervalle $[0, 1)$:

$$f(x) = 2x \mod 1.$$

Considérons une fonction continue ϕ sur $[0, 1)$. Nous définissons la moyenne spa-

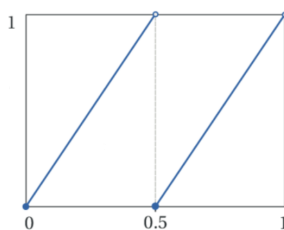


FIGURE 1.1 – La fonction $f(x) = 2x \mod 1$

tiale de ϕ comme

$$\int \phi d \text{Leb}$$

et la moyenne temporelle de ϕ comme la limite quand n tend vers l'infini de

$$\frac{1}{n} \sum_{i=0}^{n-1} \phi(f^i(x)).$$

Une question fondamentale en dynamique est le problème de comprendre quand les moyennes spatiales sont égales aux moyennes temporelles. La réponse à cette question conduit au concept d'ergodicité qui joue un rôle crucial dans les systèmes dynamiques, en particulier dans la théorie ergodique. Une mesure de probabilité μ sur X est appelée ergodique si un sous-ensemble mesurable A avec $f^{-1}(A) = A$ a une mesure nulle ou une mesure complète. Le célèbre travail de Birkhoff [1] montre que pour une mesure ergodique μ , les moyennes spatiales sont égales aux moyennes temporelles. De retour à l'exemple ci-dessus, on peut montrer que la mesure de Lebesgue est ergodique. Par conséquent, pour Lebesgue, il faut presque pointer x ,

$$\frac{1}{n} \sum_{i=0}^{n-1} \phi(f^i(x)) \longrightarrow \int \phi d \text{Leb}.$$

Introduisons maintenant une autre quantité qui décrit la complexité d'un système dynamique donné. Imaginez que nous prenions une boîte de milliers de pièces de puzzle et que nous les jetions accidentellement sur le sol. Il est raisonnable de s'attendre à ce que les pièces aient une chance de se placer exactement en place et de créer un puzzle terminé. Mais pourquoi ne voyons-nous pas cela se produire en réalité? La réponse non mathématique est que les chances sont écrasantes contre elle. Mais que faire s'il n'y a que deux pièces de puzzle dans la boîte? Ensuite, on peut imaginer qu'il y a une grande chance d'obtenir un puzzle terminé par une goutte aléatoire. Cet exemple reflète l'essence de l'entropie. Nous utilisons l'entropie comme mesure du désordre. Plus l'entropie est grande, plus nous obtenons de variations désordonnées que d'ordonnées. L'entropie peut être définie dans les catégories topologiques, ergodiques ou différentiables. Nous nous concentrons principalement sur deux types d'entropie, l'entropie métrique d'une mesure ergodique notée $h(f, \mu)$ et l'entropie topologique du système notée $h_{\text{top}}(f)$.

1.1.2 Formule d'entropie de Pesin

Puisque l'entropie mesure la complexité, il est tout à fait naturel d'en considérer la borne supérieure. Le résultat classique dans cette direction est l'inégalité de Margulis-Ruelle [28] qui délimite l'entropie par les soi-disant exposants de Lyapunov.

Theorem 1.1.2.1. [28] Soit f une carte C^1 sur une variété compacte M et μ une mesure de probabilité Borel ergodique.

$$h(f, \mu) \leq \sum_{i=1}^u \lambda_i(f, \mu) \cdot \dim E^i \quad (1.1)$$

où $\lambda_i(f, \mu)$, $1 \leq i \leq u$ sont les exposants positifs de Lyapunov et E^i , $1 \leq i \leq u$ sont les faisceaux de vecteurs Oseledets correspondants [23].

En considérant une mesure ergodique appuyée sur un point fixe hyperbolique, on peut voir que l'inégalité ci-dessus peut être nette. La question est alors de savoir dans quelle condition nous pouvons obtenir une égalité. Ceci est partiellement répondu par Pesin [24].

Theorem 1.1.2.2. [24] *Soit f un difféomorphisme $C^{1+\alpha}$ ($\alpha > 0$) sur une variété compacte M et μ une mesure de probabilité Borel ergodique qui est absolument continue w.r.t. la mesure de Lebesgue sur M . Ensuite nous avons*

$$h(f, \mu) = \sum_{i=1}^u \lambda_i(f, \mu) \cdot \dim E^i. \quad (1.2)$$

L'égalité (1.8) ci-dessus est souvent appelée 'formule d'entropie de Pesin'.

Plus tard, Ledrappier et Young ont renforcé ce résultat (mais en C^2) en montrant que

Theorem 1.1.2.3 (Formule d'entropie de Ledrappier et Young [17]). *Soit f un difféomorphisme C^2 sur une variété compacte M et μ une mesure ergodique de probabilité de Borel. Alors μ a des mesures conditionnelles absolument continues sur les variétés instables si et seulement si*

$$h(f, \mu) = \sum_{i=1}^u \lambda_i(f, \mu) \cdot \dim E^i. \quad (1.3)$$

Dans un article consécutif [18], Ledrappier et Young se sont concentrés sur la recherche d'une formule d'entropie pour $h(f, \mu)$ valide pour toute mesure de probabilité ergodique μ . Le résultat suivant fournit une description géométrique profonde de l'entropie métrique. Nous écrivons l'exposant de Lyapunov λ_i pour faire court.

Theorem 1.1.2.4. [18] *Soit f un difféomorphisme C^2 sur une variété compacte M et μ une mesure ergodique de probabilité de Borel.*

$$h(f, \mu) = \sum_{j=1}^u r_j \cdot \lambda_j \quad (1.4)$$

où $r_1, \dots, r_u$ sont des dimensions transversales satisfaisant $r_j \leq \dim E^j$.

En effet, Ledrappier et Young ont prouvé un résultat plus général. Ils ont défini l'entropie le long de i -th foliation instable dénotée par $h_i(f, \mu)$ (voir Proposition 2.3.1). Ils ont montré (Theorem C' in [18]): pour tout $1 \leq i \leq u$,

$$h(f, \mu) = h_i(f, \mu) + \sum_{j=i+1}^u r_j \cdot \lambda_j. \quad (1.5)$$

Pour $1 \leq i \leq u$, que $\mu_x^{\xi^i}$ soit une famille de mesures conditionnelles w.r.t. une partition subordonnée donnée ξ^i (voir la Définition 2.2.1). Nous définissons les valeurs μ presque:

$$\overline{\delta}_i \triangleq \limsup_{\rho \rightarrow 0} \frac{\log \mu_x^{\xi^i}(B^i(x, \rho))}{\log \rho}, \quad \underline{\delta}_i \triangleq \liminf_{\rho \rightarrow 0} \frac{\log \mu_x^{\xi^i}(B^i(x, \rho))}{\log \rho}.$$

Le principal obstacle pour prouver la formule (1.5) est d'afficher $\overline{\delta}_i = \underline{\delta}_i$ pour $1 \leq i \leq u$ et les relier aux entropies le long des feuillages instables. Remarquez

que la dimension transversale r_i est définie comme $r_i = \delta_i - \delta_{i-1}$. Pour prouver $\bar{\delta}_i = \delta_i$, Ledrappier et Young se sont concentrés sur les propriétés clés suivantes: pour $2 \leq i \leq u$,

1. $h_i - h_{i-1} \geq \lambda_i(\bar{\delta}_i - \bar{\delta}_{i-1})$,
2. $h_i - h_{i-1} \leq \lambda_i(\underline{\delta}_i - \underline{\delta}_{i-1})$.

Mais nous notons que les preuves de ces deux propriétés sont basées sur des méthodes très différentes. La première inégalité ci-dessus se trouve dans la section 10 dans [18] tandis que la seconde se trouve dans la section 5 dans [17]. Dans la Section 2, nous réorganisons leurs arguments pour obtenir une preuve unifiée de ces deux propriétés.

1.1.3 Limite supérieure de l'entropie métrique

Comme mentionné précédemment, l'inégalité de Margulis-Ruelle (1.1) montre que les exposants de Lyapunov positifs se sont liés au-dessus de l'entropie métrique. Comme peut-être observé pour la première fois par Katok, cette inégalité implique que les mesures avec entropie positive des difféomorphismes de surface sont hyperboliques, c'est-à-dire sans zéro exposant de Lyapunov. Katok [13] a ensuite pu analyser une telle dynamique en utilisant la théorie de Pesin dans le paramètre $C^{1+\alpha}$.

En utilisant également la théorie de Pesin, Newhouse [21] a prouvé une autre limite pour l'entropie d'une mesure ergodique cette fois par la croissance en volume de sous-variétés qui sont transversales à ses variétés stables. Dans le paramètre C^1 avec fractionnement dominé, sans utiliser la théorie de Pesin, Saghin [29] et Guo-Liao-Sun-Yang [12] délimités au-dessus de l'entropie métrique par un mélange entre le Lyapunov positif exposants et la croissance en volume de certains sous-sous-variétés. En utilisant le résultat de Ledrappier-Young [18], Cogswell [10] a prouvé que la croissance en volume des variétés locales instables est plus grande que l'entropie métrique. La preuve de Cogswell suppose une régularité de C^2 puisque cela est requis dans le travail de Ledrappier-Young. Du côté topologique, pour le difféomorphisme $C^{1+\alpha}$, Przytycki [25] a prouvé que l'entropie topologique est limitée au-dessus par le taux de croissance de certaines formes différentielles. Plus tard, Kozlovski [14] a montré que c'est une égalité si le système est C^∞ .

Dans cette thèse, nous avons plusieurs résultats sur la borne supérieure de l'entropie métrique.

Croissance volumique, entropie topologique et entropie fibrée

Nous généralisons l'idée de Cogswell dans [10] pour établir une limite supérieure plus générale pour les systèmes C^2 sans supposer un fractionnement dominé. Nous avons lié l'entropie d'une mesure par une combinaison d'exposants de Lyapunov (comme dans l'inégalité de Ruelle) et de diverses croissances de variétés instables telles que la croissance en volume (comme dans l'inégalité de Newhouse [21]). Nous utiliserons cette nouvelle limite pour étendre l'argument d'hyperbolicité de Katok mentionné ci-dessus au-delà de la dimension deux. Pour plus de détails, voir la section 1.2.4.

Vague Statements of main results. Soit f un difféomorphisme C^2 sur une variété compacte M et μ toute mesure de probabilité ergodique et invariante. Considérez ses exposants Lyapunov positifs $\lambda_1 > \dots > \lambda_u$ et les i -th variétés locales instables correspondantes $W_{\text{loc}}^i(x)$ pour presque chaque $x \in M$ et $i = 1, \dots, u$.

Alors l'entropie $h(f, \mu)$ est bornée, pour tout indice $1 \leq i \leq u$, par la somme de la croissance en volume presque partout de $W_{\text{loc}}^i(x)$ et les exposants transversaux de Lyapunov $\lambda_{i+1}, \dots, \lambda_u$, répétés selon la multiplicité.

Dans cette inégalité, la croissance en volume peut être remplacée par une entropie fibrée ou une entropie topologique de $W_{\text{loc}}^i(x)$.

Nous donnons des déclarations complètes et précises dans la section suivante après avoir introduit les notions requises. Voir en particulier le théorème C.

Entropie dimensionnelle uniforme sur les sous-variétés

Dans cette section, pour les difféomorphismes $C^{1+\alpha}$, nous avons lié l'entropie par une somme des plus petits exposants de Lyapunov et une certaine entropie (appelée entropie dimensionnelle, voir 5.1.3) des sous-variétés transversales. Voici une liste des principales différences par rapport aux résultats précédents présentés dans la section 1.1.3.

- Nous utilisons principalement la théorie de Pesin, donc nous ne supposons pas une structure uniforme comme le fractionnement dominé.
- Nous n'avons pas à considérer la dynamique de certains sous-variétés lisses spéciales comme les variétés instables. Au lieu de cela, nous nous concentrons sur une famille dense de sous-variétés C^0 appelée famille ample (voir Définition 5.1.1).
- Nous n'utilisons pas le résultat de Ledrappier et Young comme [10] et Theorem C. Cela nous permet de travailler avec $C^{1+\alpha}$ et fournit une observation géométrique claire (voir la section 5).

Nous définissons d'abord **entropie dimensionnelle uniforme w.r.t. une famille $\mathcal{A}$ de sous-variétés** par

$$H(f, \mathcal{A}) \triangleq \lim_{\tau \rightarrow 0} \limsup_{n \rightarrow +\infty} \frac{1}{n} \sup_{\gamma \in \mathcal{A}} \log \#E(n, \tau, \gamma)$$

où $E(n, \varepsilon, \gamma)$ est un sous-ensemble maximal (n, τ) séparé de γ .

Nous considérons un type spécial de famille de sous-variétés C^0 dans la définition 5.6.1 appelée ample family. En gros, une ample family est une très grande famille de sous-variétés i -dimensionnelles qui sont denses en M dans un certain sens. Nous définissons ensuite uniform i -dimensional entropy on ample families par

$$H^i(f) \triangleq \inf \left\{ H(f, \mathcal{A}) \mid \mathcal{A} \text{ est une ample family de sous-variétés } i\text{-dimensionnelles} \right\}.$$

Nous présentons maintenant notre résultat sur l'entropie dimensionnelle.

Theorem A. Soit f un difféomorphisme $C^{1+\alpha}$ ($\alpha > 0$) sur une variété compacte et que μ soit une mesure ergodique. Répertoriez les exposants Lyapunov de μ comme $\lambda_1 < \lambda_2 < \dots < \lambda_l$. Pour tout $0 \leq k \leq l$, soit $i = \dim(E^{k+1} \oplus E^{k+2} \oplus \dots \oplus E^l)$. On a alors

$$h(f, \mu) \leq H^i(f) + \sum_{j=1}^k \lambda_j^+ \dim(E^j)$$

où $\lambda_j^+ = \max\{0, \lambda_j\}$.

Remark 1.1.1.

- Si $k = l$, nous obtenons l'inégalité Ruelle [28] et si $k = 0$, nous obtenons le résultat trivial $h(f, \mu) \leq h_{\text{top}}(f)$.
- Nous pouvons remplacer f par la carte temporelle d'un flux $C^{1+\alpha}$, puis nous obtenons une inégalité similaire pour le flux.

1.1.4 Applications au difféomorphisme entropie-hyperbolique

Dans cette section, nous appliquons nos résultats à une classe de difféomorphismes appelés entropie-hyperbolique (d'après [6]). Les épreuves sont dans la Section 4.

Nous fixons un atlas fini $\mathcal{A} = \{(U, \phi)\}$ de M . Étant donné $r \geq 1$, pour une sous-variété C^r $\gamma : (0, 1)^k \rightarrow M$, définissez la norme C^r - de γ as

$$\|\gamma\|_r = \max_{1 \leq i \leq r} \sup_{t \in (0, 1)^k} \max\{ \|D^i(\phi \circ \gamma)|_t\| \mid (U, \phi) \in \mathcal{A} \}$$

où D^i est le i -dérivée.

Definition 1.1.2. L'entropie k -dimensionnelle sur les sous-variétés C^r est

$$h_{\text{top}}^{k,r}(f) \triangleq \sup\{h_{\text{top}}(f, \gamma) \mid \gamma \text{ est } C^r \text{ sous-variété de dimension } k \text{ avec } \|\gamma\|_r < 1\}.$$

Un difféomorphisme f est entropie-hyperbolique si

$$d_{C^r}(f) + d_{C^r}(f^{-1}) = \dim M$$

où $d_{C^r}(f) = \min\{0 \leq k \leq \dim M : h_{\text{top}}^{k,r}(f) = h_{\text{top}}(f)\}$.

Remark 1.1.3. Rappelons que nous avons défini l'entropie dimensionnelle uniforme sur de ample family $(H^k(f))$ dans la section précédente. Ici $h_{\text{top}}^{k,r}(f)$ est défini w.r.t. Sous-variétés C^r .

A partir de maintenant, nous considérons r comme la douceur supposée du difféomorphisme et pour simplifier, nous le supprimons des notations et de la terminologie. Nous utilisons également la notation $d(f)$ au lieu de $d_{C^r}(f)$.

Notez qu'un difféomorphisme entropie-hyperbolique sur une variété compacte M ($\dim M \geq 1$) a toujours une entropie positive. Ici, nous utilisons des définitions analogues de l'entropie dimensionnelle comme dans [6] où Buzzi a systématiquement discuté et conjecturé les propriétés de l'entropie dimensionnelle et de l'hyperbolicité entropique.

Pour $\lambda > 0$, nous disons qu'une mesure ergodique est λ -hyperbolic si pour un exposant de Lyapunov λ_i de μ , la valeur absolue de λ_i est supérieur ou égal à λ . Nous disons qu'un point périodique hyperbolique est λ -hyperbolique si la mesure ergodique supportée sur les orbites de ce point périodique est λ -hyperbolique.

L'indice instable d'une mesure ergodique est défini comme la dimension de ses variétés instables.

Nous déduisons du Theorem C:

Theorem 1.1.4. Soit f un difféomorphisme C^2 sur une variété compacte M qui est entropie-hyperbolique. Il y a une constante $h < h_{\text{top}}(f)$ telle que toute mesure ergodique μ avec $h(f, \mu) > h$ est $\frac{h(f, \mu) - h}{\dim M}$ hyperbolique et l'indice instable de μ est $d(f)$.

Remark 1.1.5. On peut facilement obtenir de la preuve du Theorem 1.2.4 que $d(f) + d(f^{-1}) \leq \dim M$ pour le C^2 difféomorphisme arbitraire.

En utilisant le Katok's horseshoe theorem, cela implique:

Corollary 1.1.6. Tout C^2 difféomorphisme entropie-hyperbolique f sur une variété compacte satisfait:

- $h_{\text{top}}(f) = \sup h_{\text{top}}(f, K)$ où K s'étend sur les ensembles invariants hyperboliques compacts;
- f est un point de semi-continuité inférieure de l'entropie topologique dans la topologie C^1 .

En particulier, f est un point de continuité de l'entropie topologique dans la topologie C^∞ .

Une mesure de probabilité invariante ν maximise l'entropie si l'entropie métrique de ν est égale à l'entropie topologique du système. Ensuite, nous montrons des exemples de difféomorphisme entropie-hyperbolique.

Ben Ovadia (Theorem 1.5 dans [31]) montre que tout difféomorphisme $C^{1+\alpha}$ sur une variété compacte a au plus dénombrable de nombreuses mesures hyperboliques ergodiques maximisant l'entropie. Par conséquent:

Corollary 1.1.7. Soit f un difféomorphisme C^2 sur une variété compacte M . Si f est entropie-hyperbolique, il a au plus de nombreuses mesures ergodiques maximisant l'entropie.

En utilisant le codage de Sarig [30] dans la dimension deux, Buzzi (Theorem 1.2 dans [7]) a montré une borne inférieure multiplicative sur le nombre de points périodiques. Nous utilisons la généralisation dimensionnelle supérieure de Ben Ovadia [31] du codage de Sarig pour étendre le résultat de Buzzi à n'importe quelle dimension lorsque le système est entropie-hyperbolique.

En suivant Buzzi dans [7], nous définissons

$$\text{per}_\lambda(f, n) \triangleq \{x \in M \mid \{f^k(x), k \in \mathbb{N}\} \text{ a la cardinalité } n \text{ et est } \lambda\text{-hyperbolic}\}.$$

Rappelons que la constante h provient du Theorem 1.2.4.

Theorem 1.1.8. Soit f un difféomorphisme C^∞ sur une variété compacte M . Si f est entropie-hyperbolique, alors il y a un entier $p \geq 1$ tel que pour tout $\lambda < \frac{h_{\text{top}}(f) - h}{\dim M}$,

$$\liminf_{\substack{n \rightarrow \infty \\ p|n}} e^{-nh_{\text{top}}(f)} \cdot \#\text{per}_\lambda(f, n) \geq p.$$

Ensuite, nous montrons quelques exemples de systèmes entropie-hyperboliques.

Soit $\pi : \mathbb{R}^2 \rightarrow \mathbb{T}^2 := \mathbb{R}^2/\mathbb{Z}^2$ la projection naturelle de l'avion au tore. Alors π est une carte bijective entre $[0, 1]^2$ et $\mathbb{T}^2$. Notons $\Delta \triangleq \{0\} \times [0, 1) \cup (0, 1) \times \{0\}$.

Nous introduisons la propriété suivante:

$$f(p) = p \text{ pour tous } p \text{ dans un voisinage } \mathcal{U} \text{ de } \pi(\Delta), \quad (1.6)$$

Example 1.1.9. Soit f_1 et f_2 deux difféomorphismes lisses C^∞ sur $\mathbb{T}^2$ avec entropie topologique positive. S'ils satisfont la condition (1.6) ci-dessus, alors $f_1 \times f_2$ est entropie-hyperbolique sur $\mathbb{T}^4$.

Remark 1.1.10. *Nous nous attendons à ce que cela puisse être généralisé à toute surface fermée et que la condition (1.6) puisse être supprimée. Plus important encore, nous nous attendons à ce que l'hyperbolicité d'entropie soit C^∞ ouverte de sorte que les perturbations de $f_1 \times f_2$ soient également entropie-hyperboliques. Voir chapitre 6. Voir Chapter 6 pour plus de détails.*

1.2 Introduction (English)

1.2.1 Ergodicity and entropy

The theory of dynamical systems focuses the changes over time of a given point in a space. Examples include the solar system, the flow of liquid and gas, the stock of market, the number of fish in a pond, the weather of earth and so on. There are wide and profound applications of dynamical systems theory in many advanced areas of sciences like biology, physics, information theory, etc. There are usually two parts of a dynamical system: the phase space and the dynamics. The phase space provides all the possible states in question. Usually it is a set of points with measurable, topological or differentiable structures. The dynamics is collection of rules which governs the movement of a point in the phase space. Let us make things clear through a mathematical abstraction. In a nutshell, a dynamical system is a set X with a self-map f . It is usually not very interesting without adding more structures on X and more restrictions on f . The most common and simple way to do this is to assume that X is a measurable space for some σ algebra $\mathcal{B}$ and f preserves the measurable structure in the sense that the pre-image of a measurable set is also measurable. Now we have the formal definition of dynamical systems, but what kind of question that interests dynamicists ?

Let us start with the linear expanding function on the interval $[0, 1]$:

$$f(x) = 2x \mod 1.$$

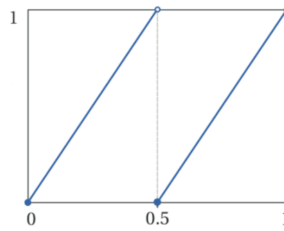


FIGURE 1.2 – The function $f(x) = 2x \mod 1$

Consider a continuous function ϕ on $[0, 1]$. We define the space average of ϕ as

$$\int \phi d\text{Leb}$$

and the time average of ϕ as

$$\frac{1}{n} \sum_{i=0}^{n-1} \phi(f^i(x)).$$

A basic question in dynamics is the problem of understanding when the space averages are equal to time averages. The answer to this question leads to the concept of

ergodicity which plays a crucial role in dynamical systems, especially in ergodic theory. A probability measure μ on X is called ergodic if any measurable subset A with $f^{-1}(A) = A$ has zero measure or full measure. The celebrated work of Birkhoff [1] shows that for an ergodic measure μ , the space averages are equal to time averages. Back to the above example, one can show that the Lebesgue measure is ergodic. As a consequence, for Lebesgue almost point x ,

$$\frac{1}{n} \sum_{i=0}^{n-1} \phi(f^i(x)) \longrightarrow \int \phi d \text{Leb}.$$

Now let us introduce another quantity that describes the complexity of a given dynamical system. Imagine we take a box of thousands of puzzle pieces and we accidentally dump them out on the ground. It is reasonable to expect that there is a chance for the pieces to fall exactly into place and create a completed puzzle. But why don't we see this happens in reality? The non-mathematical answer is that the odds are overwhelmingly against it. But what if there are only two puzzle pieces in the box? Then one can image that there is a big chance to get a completed puzzle by a random drop. This example reflects the essence of entropy. We use entropy as a measure of disorder. The larger the entropy is, the more disorderly variations we get than orderly ones. The entropy can be defined in the topological, ergodic or differentiable categories. We mostly focus on two kinds of entropy, the metric entropy of an ergodic measure denoted by $h(f, \mu)$ and the topological entropy of the system denoted by $h_{\text{top}}(f)$.

1.2.2 Pesin's entropy formula

Since the entropy measures the complexity, it is quite natural to consider the upper bound of it. The classical result on this direction is the Margulis-Ruelle Inequality [28] which bounds above the entropy by the so called Lyapunov exponents.

Theorem 1.2.2.1. [28] *Let f be a C^1 map on a compact manifold M and μ an ergodic Borel probability measure.*

$$h(f, \mu) \leq \sum_{i=1}^u \lambda_i(f, \mu) \cdot \dim E^i \quad (1.7)$$

where $\lambda_i(f, \mu), 1 \leq i \leq u$ are the positive Lyapunov exponents and $E^i, 1 \leq i \leq u$ are the corresponding Oseledets' vector bundles [23].

By considering an ergodic measure supported on a hyperbolic fixed point, one can see that the inequality above can be sharp. Then the question is under what condition we can get an equality. This is partially answered by Pesin [24].

Theorem 1.2.2.2. [24] *Let f be a $C^{1+\alpha}$ ($\alpha > 0$) diffeomorphism on a compact manifold M and μ an ergodic Borel probability measure which is absolutely continuous w.r.t. the Lebesgue measure on M . Then we have*

$$h(f, \mu) = \sum_{i=1}^u \lambda_i(f, \mu) \cdot \dim E^i. \quad (1.8)$$

The above equality (1.8) is often called 'Pesin's entropy formula'.

Later, Ledrappier and Young strengthened this result (but in C^2 setting) by showing that

Theorem 1.2.2.3. [17] *Let f be a C^2 diffeomorphism on a compact manifold M and μ an ergodic Borel probability measure. Then μ has absolutely continuous conditional measures on unstable manifolds if and only if*

$$h(f, \mu) = \sum_{i=1}^u \lambda_i(f, \mu) \cdot \dim E^i. \quad (1.9)$$

In a consecutive paper [18], Ledrappier and Young focused on finding an entropy formula for $h(f, \mu)$ valid for any ergodic probability measure μ . The following result provides a profound geometrical description of the metric entropy. We write the Lyapunov exponent λ_i for short.

Theorem 1.2.2.4 (Ledrappier-Young entropy formula [18]). *Let f be a C^2 diffeomorphism on a compact manifold M and μ an ergodic Borel probability measure.*

$$h(f, \mu) = \sum_{j=1}^u r_j \cdot \lambda_j \quad (1.10)$$

where $r_1, \dots, r_u$ are some transverse dimensions satisfying $r_j \leq \dim E^j$.

Indeed, Ledrappier and Young proved a more general result. They defined the entropy along i -th unstable foliation denoted by $h_i(f, \mu)$ (see Proposition 2.3.1). They showed (Theorem C' in [18]): for any $1 \leq i \leq u$,

$$h(f, \mu) = h_i(f, \mu) + \sum_{j=i+1}^u r_j \cdot \lambda_j. \quad (1.11)$$

For $1 \leq i \leq u$, let $\mu_x^{\tilde{\zeta}^i}$ be a family of conditional measures w.r.t. a given subordinate partition $\tilde{\zeta}^i$ (see definition 2.2.1). We define the μ almost values:

$$\overline{\delta}_i \triangleq \limsup_{\rho \rightarrow 0} \frac{\log \mu_x^{\tilde{\zeta}^i}(B^i(x, \rho))}{\log \rho}, \quad \underline{\delta}_i \triangleq \liminf_{\rho \rightarrow 0} \frac{\log \mu_x^{\tilde{\zeta}^i}(B^i(x, \rho))}{\log \rho}.$$

The key obstacle to prove formula (1.11) is to show $\overline{\delta}_i = \underline{\delta}_i$ for $1 \leq i \leq u$ and relate them to the entropies along unstable foliations. Remark that the transverse dimension r_i is define as $r_i = \delta_i - \delta_{i-1}$. To prove $\overline{\delta}_i = \underline{\delta}_i$, Ledrappier and Young focused on the following key properties: for $2 \leq i \leq u$,

1. $h_i - h_{i-1} \geq \lambda_i(\overline{\delta}_i - \overline{\delta}_{i-1})$,
2. $h_i - h_{i-1} \leq \lambda_i(\underline{\delta}_i - \underline{\delta}_{i-1})$.

But we note the proofs of these two properties are based on very different methods. The first inequality above is in Section 10 in [18] while the second one is in Section 5 in [17]. In Section 2, we reorganize their arguments to get a unified proof of these two properties.

1.2.3 Upper bound of metric entropy

As mentioned before, Margulis-Ruelle Inequality (1.7) shows that the positive Lyapunov exponents bound above the metric entropy. As perhaps first observed by Katok, this inequality implies that measures with positive entropy of surface diffeomorphisms are hyperbolic, i.e., without zero Lyapunov exponents. Katok [13] was then able to analyze such dynamics using the Pesin theory in the $C^{1+\alpha}$ setting.

Also using Pesin theory, Newhouse [21] proved another bound for the entropy of an ergodic measure this time by the volume growth of sub-manifolds which are transverse to its stable manifolds. In the C^1 setting with dominated splitting, without using Pesin theory, Saghin [29] and Guo-Liao-Sun-Yang [12] bounded above the metric entropy by a mixture between the positive Lyapunov exponents and the volume growth of some sub-manifold. By using Ledrappier-Young's result [18], Cogswell [10] proved that the volume growth of local unstable manifolds is larger than the metric entropy. Cogswell's proof assumes C^2 smoothness since this is required in Ledrappier-Young's work. On the topological side, for $C^{1+\alpha}$ diffeomorphism, Przytycki [25] proved that the topological entropy is bounded above by the growth rate of some differential forms. Later Kozlovski [14] showed it is an equality if the system is C^∞ .

In this thesis, we have several results on the upper bound of metric entropy.

Volume growth, topological entropy and fibered entropy

We generalize Cogswell's idea in [10] to establish a more general upper bound for C^2 systems without assuming dominated splitting. We bound the entropy of a measure by a combination of Lyapunov exponents (as in Ruelle's inequality) and various growths of unstable manifolds such as volume growth (as in Newhouse's inequality [21]). We will use this new bound to extend the above mentioned Katok's hyperbolicity argument beyond dimension two. For more detail, see Section 1.2.4.

Our proof is a combination of Ledrappier-Young's entropy formula [18] and Pesin theory. We also discuss some extensions for hyperbolic measures in $C^{1+\alpha}$ case.

Vague Statements of main results. *Let f be a C^2 diffeomorphism on a compact manifold M and let μ be any ergodic, invariant probability measure. Consider its positive Lyapunov exponents $\lambda_1 > \dots > \lambda_u$ and the corresponding i -th local unstable manifolds $W_{\text{loc}}^i(x)$ for almost every $x \in M$ and $i = 1, \dots, u$.*

Then the entropy $h(f, \mu)$ is bounded, for any index $1 \leq i \leq u$, by the sum of the almost everywhere volume growth of $W_{\text{loc}}^i(x)$ and the transverse Lyapunov exponents $\lambda_{i+1}, \dots, \lambda_u$, repeated according to multiplicity.

In this inequality, the volume growth can be replaced by fibered entropy or topological entropy of $W_{\text{loc}}^i(x)$.

We give complete and precise statements in the next section after introducing the required notions. See in particular Theorem C in Chapter 1.2.3.

Uniform dimensional entropy on submanifolds

In this section, for $C^{1+\alpha}$ diffeomorphisms, we bound the entropy by a sum of the smallest Lyapunov exponents and some entropy (called uniform dimensional entropy, see 5.1.3) of transverse submanifolds. Here is a list of the main differences from the previous results introduced in Section 1.2.3.

- We mainly use Pesin's theory, so we do not assume a uniform structure like dominated splitting.
- We do not have to consider the dynamics on some special smooth submanifolds like unstable manifolds. Instead, we focus on a dense family of C^0 submanifolds called ample family (see Definition 5.1.1).

- We do not use Ledrappier and Young's result like [10] and Theorem C. This allows us to work in $C^{1+\alpha}$ setting and it provides a clear geometric observation (see Section 5).

We first define the uniform dimensional entropy w.r.t. a family $\mathcal{A}$ of submanifolds by

$$H(f, \mathcal{A}) \triangleq \lim_{\tau \rightarrow 0} \limsup_{n \rightarrow +\infty} \frac{1}{n} \sup_{\gamma \in \mathcal{A}} \log \#E(n, \tau, \gamma)$$

where $E(n, \tau, \gamma)$ is a maximal (n, τ) separated subset of γ .

We consider a special kind of family of C^0 submanifolds in Definition 5.6.1 called ample family. Roughly speaking, an ample family is a very large family of i -dimensional submanifolds which are dense in M in some sense. We then define the uniform i -dimensional entropy on ample families by

$$H^i(f) \triangleq \inf \left\{ H(f, \mathcal{A}) \mid \mathcal{A} \text{ is an ample family of } i\text{-dimensional submanifolds} \right\}.$$

We now introduce our result on dimensional entropy.

Theorem A. *Let f be a $C^{1+\alpha}$ ($\alpha > 0$) diffeomorphism on a compact manifold and let μ be an ergodic measure. List the Lyapunov exponents of μ as $\lambda_1 < \lambda_2 < \dots < \lambda_l$. For any $0 \leq k \leq l$, let $i = \dim(E^{k+1} \oplus E^{k+2} \oplus \dots \oplus E^l)$. Then we have*

$$h(f, \mu) \leq H^i(f) + \sum_{j=1}^k \lambda_j^+ \dim(E^j)$$

where $\lambda_j^+ = \max\{0, \lambda_j\}$.

Remark 1.2.1.

- If $k = l$, we get the Ruelle inequality [28] and if $k = 0$, we get the variational principle $h(f, \mu) \leq h_{\text{top}}(f)$.
- We can replace f by the time-one map of a $C^{1+\alpha}$ flow, then we get a similar inequality for flow.

1.2.4 Applications to entropy-hyperbolic diffeomorphism

In this section, we apply our results to a class of diffeomorphisms called entropy-hyperbolic (after [6]). The proofs are in Section 4.

We fix a finite atlas $\mathcal{A} = \{(U, \phi)\}$ of M . Given $r \geq 1$, for a C^r sub-manifold $\gamma : (0, 1)^k \rightarrow M$, define the C^r -norm of γ as

$$\|\gamma\|_r = \max_{1 \leq i \leq r} \sup_{t \in (0, 1)^k} \max\{\|D^i(\phi \circ \gamma)|_t\| \mid (U, \phi) \in \mathcal{A}\}$$

where D^i is the i -th derivative.

Definition 1.2.2. *The k -dimensional entropy on C^r sub-manifolds is*

$$h_{\text{top}}^{k,r}(f) \triangleq \sup\{h_{\text{top}}(f, \gamma) \mid \gamma \text{ is } C^r \text{ sub-manifold of dimension } k \text{ with } \|\gamma\|_r < 1\}.$$

A diffeomorphism f is C^r entropy-hyperbolic if

$$d_{C^r}(f) + d_{C^r}(f^{-1}) = \dim M$$

where $d_{C^r}(f) = \min\{0 \leq k \leq \dim M : h_{\text{top}}^{k,r}(f) = h_{\text{top}}(f)\}$.

Remark 1.2.3. Recall that we have defined uniform dimensional entropy on ample families $(H^k(f))$ in the previous section. Here $h_{\text{top}}^{k,r}(f)$ is defined w.r.t. C^r sub-manifolds.

From now on, we take r to be the assumed smoothness of the diffeomorphism and for simplicity, we drop it from the notations and terminology. We also use the notation $d(f)$ instead of $d_{C^r}(f)$.

Note that an entropy-hyperbolic diffeomorphism on compact manifold M ($\dim M \geq 1$) always has positive entropy. Here we use analogous definitions of dimensional entropy as in [6] where Buzzi systematically discussed and conjectured the properties of dimensional entropy and entropy-hyperbolicity.

For $\lambda > 0$, we say that an ergodic measure is λ -hyperbolic if for any Lyapunov exponent λ_i of μ , the absolute value of λ_i is strictly larger than λ . We say that a hyperbolic periodic point is λ -hyperbolic if the ergodic measure supported on the orbits of this periodic point is λ -hyperbolic.

The unstable index of an ergodic measure is defined as the dimension of its unstable manifolds.

We will deduce from Theorem C:

Theorem 1.2.4. Let f be a C^2 diffeomorphism on a compact manifold M which is entropy-hyperbolic. There is a constant $h < h_{\text{top}}(f)$ such that any ergodic measure μ with $h(f, \mu) > h$ is $\frac{h(f, \mu) - h}{\dim M}$ -hyperbolic.

Remark 1.2.5. One can easily get from the proof of Theorem 1.2.4 that $d(f) + d(f^{-1}) \leq \dim M$ for arbitrary C^2 diffeomorphism f .

Using Katok's horseshoe theorem, this implies:

Corollary 1.2.6. Any C^2 entropy-hyperbolic diffeomorphism f on a compact manifold satisfies:

- $h_{\text{top}}(f) = \sup h_{\text{top}}(f, K)$ where K ranges over the compact hyperbolic invariant sets;
- f is a point of lower semi-continuity of the topological entropy in the C^1 -topology.

In particular, f is a point of continuity of the topological entropy in the C^∞ topology.

An invariant probability measure ν maximizes entropy if the metric entropy of ν is equal to the topological entropy of the system.

Ben Ovadia (Theorem 1.5 in [31]) shows that any $C^{1+\alpha}$ diffeomorphism on a compact manifold has at most countably many ergodic hyperbolic measures maximizing entropy. Therefore:

Corollary 1.2.7. Let f be a C^2 diffeomorphism on a compact manifold M . If f is entropy-hyperbolic, then it has at most countably many ergodic measures maximizing entropy.

By using Sarig's coding [30] in dimension two, Buzzi (Theorem 1.2 in [7]) showed a multiplicative lower bound on the number of periodic points. We use Ben Ovadia's higher-dimensional generalization [31] of Sarig's coding to extend Buzzi's result to any dimension when the system is entropy-hyperbolic.

Following Buzzi in [7], we define

$$\text{per}_\lambda(f, n) \triangleq \{x \in M \mid \{f^k(x), k \in \mathbb{N}\} \text{ has cardinality } n \text{ and is } \lambda\text{-hyperbolic}\}.$$

Recall that constant h is from Theorem 1.2.4.

Theorem 1.2.8. *Let f be a C^∞ diffeomorphism on a compact manifold M . If f is entropy-hyperbolic, then there is an integer $p \geq 1$ such that for any $\lambda < \frac{h_{\text{top}}(f) - h}{\dim M}$,*

$$\liminf_{\substack{n \rightarrow \infty \\ p|n}} e^{-nh_{\text{top}}(f)} \cdot \#\text{per}_\lambda(f, n) \geq p.$$

Next we show some examples of entropy-hyperbolic systems.

Let $\pi : \mathbb{R}^2 \rightarrow \mathbb{T}^2 := \mathbb{R}^2 / \mathbb{Z}^2$ be the natural projection from the plane to the torus. Then π is a bijective map between $[0, 1)^2$ and $\mathbb{T}^2$. Denote $\Delta \triangleq \{0\} \times [0, 1) \cup (0, 1) \times \{0\}$.

We introduce the following property:

$$f(p) = p \text{ for all } p \text{ in some neighbourhood } \mathcal{U} \text{ of } \pi(\Delta), \quad (1.12)$$

Example 1.2.9. *Let f_1 and f_2 be two C^∞ -smooth diffeomorphisms on $\mathbb{T}^2$ with positive topological entropy. If they satisfy condition (1.12) above, then $f_1 \times f_2$ is entropy-hyperbolic on $\mathbb{T}^4$.*

Remark 1.2.10. *We expect that this can be generalized to any closed surface and that condition (1.12) can be removed. More importantly, we expect that entropy hyperbolicity is C^∞ open so that perturbations of $f_1 \times f_2$ are also entropy-hyperbolic. See Chapter 6 for more detail.*

Part I

The Ledrappier-Young theory

Chapter 2

Dimensions along unstable foliations: Ledrappier-Young entropy formula

2.1 Introduction

This chapter is devoted to an exposition of Ledrappier-Young entropy formula (Theorem 1.2.2.4) and its proof, for which we offer a simplification.

Let f be a $C^{1+\alpha}$ diffeomorphism on a compact manifold M and μ an ergodic measure. We list the positive Lyapunov exponents as $\lambda_1 > \lambda_2 > \cdots > \lambda_u$. By Pesin theory, for $1 \leq i \leq u$ and for any $x \in R_\mu$ (the set of regular points, see Section 2.2 below), the i -th global unstable manifold

$$W^i(x) \triangleq \{y \in M \mid \limsup_{n \rightarrow +\infty} \frac{1}{n} \log d(f^{-n}(x), f^{-n}(y)) \leq -\lambda_i\}$$

is a $C^{1+\alpha}$ immersed sub-manifold. Given $1 \leq i \leq u$, for a measurable partition ξ^i subordinate to W^i (See Definition 2.2.1), let $\{\mu_x^{\xi^i}\}$ be the disintegration of μ w.r.t. ξ^i . Ledrappier and Young [18] have defined the entropy along i -th unstable foliation h_i (for more detail, see Proposition 2.3.1):

$$h_i \triangleq \lim_{\tau \rightarrow 0} \liminf_{n \rightarrow +\infty} -\frac{1}{n} \log \mu_x^{\xi^i}(V^i(x, n, \tau)) = \lim_{\tau \rightarrow 0} \limsup_{n \rightarrow +\infty} -\frac{1}{n} \log \mu_x^{\xi^i}(V^i(x, n, \tau))$$

where $V^i(x, n, \tau)$ is the dynamical ball (See Section 2.2).

They also define dimensions δ_i w.r.t. each ξ^i . The Ledrappier-Young entropy formula relates entropies, dimensions and exponents. More precisely, for any $1 \leq i \leq u$ (assuming $h_0 = 0$), they proved

$$h_i = h_{i-1} + (\delta_i - \delta_{i-1}) \cdot \lambda_i.$$

Moreover $h_u = h(f, \mu)$. Hence we get the Ledrappier-Young entropy formula:

$$h(f, \mu) = \sum_{j=1}^u r_j \cdot \lambda_j$$

where $r_1, \dots, r_u$ are called the transverse dimensions defined by $r_i = \delta_i - \delta_{i-1}$.

As in Ledrappier and Young's original approach, the basic tools are Pesin theory and the construction of good metrics on the quotients $\zeta^i(x)/\zeta^{i-1}$. These good metrics have been constructed precisely by Ledrappier and Young for the case $i = u + 1$, but we will use them in all intermediate exponents.

The core of the proof of Ledrappier-Young entropy formula is to show that these dimensions $\{\delta_i\}$ along i -th unstable manifold are well defined for μ -a.e. x :

$$\bar{\delta}_i := \limsup_{\rho \rightarrow 0} \frac{\log \mu_x^{\zeta^i}(B^i(x, \rho))}{\log \rho} = \liminf_{\rho \rightarrow 0} \frac{\log \mu_x^{\zeta^i}(B^i(x, \rho))}{\log \rho} =: \underline{\delta}_i.$$

For the direction with the largest Lyapunov exponent λ_1 , one can easily compare the size of dynamical ball and the natural ball (see Lemma 2.2.5 for a precise statement). As a direct consequence, the above formula holds for $i = 1$. But for the case $i \geq 2$, the situation becomes complicated. Since the expansion rates on different directions are related to different Lyapunov exponents, one can not compare the size of dynamical ball and the natural ball by a direct argument as in the case $i = 1$. To conquer this problem, Ledrappier and Young divided the proof into two parts (see also Section 1.2.2:

1. $h_i - h_{i-1} \geq \lambda_i(\bar{\delta}_i - \bar{\delta}_{i-1})$,
2. $h_i - h_{i-1} \leq \lambda_i(\underline{\delta}_i - \underline{\delta}_{i-1})$.

The strategies of the proofs are quite different. For the proof of the first inequality above (Section 10 in [18]), by fully using the advantage of 'lim sup', they analyzed the cardinality of the dynamical balls in the λ_{i-1} direction which are contained in a given natural ball in the λ_i direction. Thanks to the uniform dynamical behavior for the points in a Pesin block, they related the cardinality of these dynamical balls with these upper dimensions $\bar{\delta}_i, \bar{\delta}_{i-1}$. Due to some technical reason, this kind of argument can not easily be applied to prove the second inequality. Therefore Ledrappier and Young considered the quotient space $\zeta^i(x)/\zeta^{i-1}$ where the dynamics on ζ^{i-1} is "somehow ignored". Then they only focus on the dynamics on the direction with the Lyapunov exponent λ_i on which they get the desired transverse dimension.

We propose a simplified proof of Ledrappier-Young entropy formula. The advantage of our new proof is that these two parts are generated by the same idea and are formulated in a parallel way. Recall that by Pesin theory, on the λ_i -direction, the expansion rate (w.r.t the Lyapunov norm) is bounded above and below w.r.t. λ_i up to some small error. The key point that inspires us to unify the proofs is that we observe in Proposition 5.1 (the key result) in [17], they only used the upper bound of the expansion rate on the λ_i -direction. We extend the idea of Proposition 5.1 in [17] by considering the lower bound of the expansion rate. Together with their arguments, we bound above and below the dynamics on $\zeta^i(x)/\zeta^{i-1}$. This leads us to prove $\bar{\delta}_i = \underline{\delta}_i$ which are related to the entropies along unstable manifolds.

Let us remark that even though the C^2 smoothness is assumed in Ledrappier and Young's theory ([17], [18]), most of the arguments, especially these involving Pesin theory, are only based on $C^{1+\alpha}$ setting. The only place where we essentially use C^2 smoothness is Lemma 2.2.7.

2.2 Definitions and basic properties

Let f be a $C^{1+\alpha}$ ($\alpha > 0$) diffeomorphism on a compact manifold M , that is, f is differentiable and its differential is Hölder-continuous with some positive exponent α . Let μ be an ergodic probability measure. Oseledets' Theorem [23] states that there are an invariant measurable subset R_μ with full measure, an invariant measurable decomposition $T_{R_\mu}M = E^1 \oplus E^2 \oplus \cdots \oplus E^l$ and finitely many numbers $\lambda_1 > \lambda_2 > \cdots > \lambda_l$ such that for any $x \in R_\mu$ and any nonzero vector $v \in E_x^j$, we have

$$\lim_{n \rightarrow \pm\infty} \frac{1}{n} \log \|Df_x^n(v)\| = \lambda_j.$$

We define the i -th local unstable manifold,

$$W_\rho^i(x) \triangleq \text{connected part of } W^i(x) \cap B(x, \rho) \text{ containing } x$$

where $B(x, \rho)$ is the ball center at x with radius ρ . At each $x \in R_\mu$, we fix a positive number $r(x)$ such that $W_{r(x)}^i(x)$ is an embedded sub-manifold.

For an ergodic measure μ , we always assume its positive Lyapunov exponents are listed by $\lambda_1 > \lambda_2 > \cdots > \lambda_u$.

Definition 2.2.1. Let f be a $C^{1+\alpha}$ diffeomorphism on a compact manifold M and let μ be an invariant measure. For $1 \leq i \leq u$, we say a measurable partition ξ^i is subordinate to W^i if for μ -a.e. x ,

- $\xi^i(x) \subset W^i(x)$,
- $\xi^i(x)$ contains an open neighborhood of x w.r.t. the intrinsic topology on $W^i(x)$.

Remark 2.2.2.

- We refer to [26] for background on measurable partitions and associated systems of conditional measures.
- Lemma 9.1.1 in [18] shows the existence of increasing subordinate measurable partitions. Here a partition η is called increasing if $\eta(x) \subset f(\eta(f^{-1}(x)))$ for μ -a.e. x .

Recall that $r(x) > 0$, $x \in R_\mu$ is such that $W_{r(x)}^i(x)$ is an embedded sub-manifold. Indeed, by Pesin theory, we can assume that for any $x \in R_\mu$, $r(x)$ is such that

$$\lim_{n \rightarrow +\infty} \frac{1}{n} \log r(f^n(x)) = 0.$$

In light of this, we introduce in the following a collection of results in classical Pesin theory. For more detail, see section 8 in [18] and Proposition 3.3 in [16] (which originates from Part I in [15]).

Lemma 2.2.3. Let f be a $C^{1+\alpha}$ diffeomorphism on a compact manifold M and let μ be an ergodic measure. List the positive Lyapunov exponents of μ as $\lambda_1 > \lambda_2 > \cdots > \lambda_u$. For any $\varepsilon > 0$, we can find an increasing sequence of measurable sets $\Lambda_1^\varepsilon \subset \Lambda_2^\varepsilon \subset \cdots \subset \Lambda_k^\varepsilon \subset \cdots \subset R_\mu$ and a sequence of decreasing numbers $\{r_k\}_{k \geq 1}$ with $0 < r_k < 1$ and $r_k \rightarrow 0$ such that

- $\cup_k \Lambda_k^\varepsilon = R_\mu$.

- $f^n(\Lambda_k^\varepsilon) \subset \Lambda_{k+n}^\varepsilon, \forall k, n \geq 1$.
- For any $x \in \Lambda_k^\varepsilon, r_k \leq r(x)$ and any $y \in W_{r_k}^i(x), 1 \leq i \leq u$,

$$r_k e^{-n(\lambda_i + \varepsilon)} d(x, y) \leq d(f^{-n}(x), f^{-n}(y)) \leq r_k^{-1} e^{-n(\lambda_i - \varepsilon)} d(x, y), \quad \forall n \geq 0.$$

- There is a constant K such that for $k \geq 1, x \in \Lambda_k^\varepsilon, \rho \leq r_k$ and $1 \leq i \leq u$,

$$\text{Vol}(W_\rho^i(x)) \leq K \cdot \rho^{\sum_{j=1}^i \dim E^j}.$$

- $e^{-\varepsilon} \leq r_{k+1}/r_k \leq e^\varepsilon, \forall k \geq 1$.

Remark 2.2.4.

- Here, for example, one can choose $r_k = e^{-\varepsilon k}$.
- Note that $W^i(x)$ is tangent to $\sum_{l=1}^i \dim E_x^l$ at x . These small numbers $\{r_k\}_{k \geq 1}$ indicate the size of Pesin charts. When $W_\rho^i(x)$ is in the Pesin chart of x , we can assume that it is contained in a small cone around x and therefore its volume is determined by its radius up to a uniform constant K .

For an ergodic measure μ and for $1 \leq i \leq u$, we write $B^i(x, r)$ as $W_r^i(x)$ for simplicity.

We define the dynamical ball in the i -th unstable manifold:

$$V^i(x, n, \tau) \triangleq \{y \in W_{r(x)}^i(x) \mid d(f^j(x), f^j(y)) \leq \tau, 0 \leq j \leq n-1\}.$$

For the largest Lyapunov exponent λ_1 , we have the following observation.

Lemma 2.2.5. *Let f be a $C^{1+\alpha}$ ($\alpha > 0$) diffeomorphism on a compact manifold and μ an ergodic measure. For every $\varepsilon > 0$ and almost every x , we choose some large $k \in \mathbb{N}$ such that $x \in \Lambda_k^\varepsilon$ (See Lemma 2.2.3). Then there is $\tau_0(x) > 0$ such that for $n \geq 0$,*

$$B^1(x, r_k^2 e^{-n(\lambda_1 + 2\varepsilon)}) \subset V^1(x, n, r_k e^{-n\varepsilon}), \quad V^1(x, n, \tau_0(x)) \subset B^1(x, e^{-n(\lambda_1 - 2\varepsilon)}).$$

Proof. Given $m, k \in \mathbb{N}$ and $x \in \Lambda_k$, for any $\delta \leq r_{m+k}$, by Lemma 2.2.3,

$$B^1(x, \delta \cdot r_{m+k} \cdot e^{-m(\lambda_1 + \varepsilon)}) \subset f^{-m}(B^1(f^m(x), \delta)).$$

Given $n \geq 0$, note that $r_k e^{-n\varepsilon} \cdot r_{m+k} \geq e^{-2n\varepsilon} \cdot r_k^2$ for any $0 \leq m \leq n$. Hence the formula above implies

$$f^m(B^1(x, r_k^2 e^{-n(\lambda_1 + 2\varepsilon)})) \subset B^1(f^m(x), r_k e^{-n\varepsilon}), \quad \forall 0 \leq m \leq n.$$

We then get the first inclusion.

For the second property, note that for any δ small, there is some large k such that $\mu(\Lambda_k) > 1 - \delta$. For μ -a.e. $x \in \Lambda_k$, choose N_x large enough such that for any $n \geq N_x$, there is some $l_n \in [(1 - \delta)n, n]$ such that $f^{l_n}(x) \in \Lambda_k$. Note that this is a consequence of Birkhoff ergodic theorem. By Lemma 2.2.3, for $n \geq N_x$ and for any $\tau \leq r_k$,

$$f^{-l_n}(B^1(f^{l_n}(x), \tau)) \subset B^1(x, \tau \cdot r_k^{-1} \cdot e^{-l_n(\lambda_1 - \varepsilon)}) \subset B^1(x, \tau \cdot r_k^{-1} \cdot e^{n\delta(\lambda_1 - \varepsilon)} e^{-n(\lambda_1 - \varepsilon)}).$$

We can assume that δ and $\tau_0(x)$ are small enough such that

- for $n \leq N_x$, $V^1(x, n, \tau_0(x)) \subset B^1(x, e^{-n(\lambda_1 - \varepsilon)})$,
- for $n \geq 0$, $e^{n\delta(\lambda_1 - \varepsilon)} \leq e^{n\varepsilon}$,
- $\tau_0(x) \cdot r_k^{-1} \leq 1$.

We then get the second property by noting that

$$V^1(x, n, \tau_0(x)) \subset f^{-l_n}(B(f^{l_n}(x), \tau_0(x))), \quad n \geq 0.$$

□

Given $\varepsilon > 0$ and $x \in R_\mu$, we fix some k such that $x \in \Lambda_k^\varepsilon$. We define

$$S^{cu}(x) \triangleq \{y \in M \mid d(f^{-n}(x), f^{-n}(y)) \leq r_k e^{-n\varepsilon}, \forall n \geq 0\}.$$

Next we introduce a measurable partition ξ^{u+1} similar to these subordinate partitions $\{\xi^i\}_{1 \leq i \leq u}$ above (see Definition 2.2.1). For a measurable partition ξ , we define where $\xi^+ = \bigvee_{n=0}^{+\infty} f^n \xi$. By Lemma 2.4.2 in [17], we have the following result.

Lemma 2.2.6. *Let f be a $C^{1+\alpha}$ diffeomorphism on a compact manifold M and let μ be an ergodic measure. Given $\varepsilon > 0$, there is a measurable partition ξ with $H_\mu(\xi) < +\infty$ such that $\xi^+(x) \subset S^{cu}(x)$, $x \in R_\mu$.*

From now To simply the statements in the following Lemma, we define $\lambda_{u+1} = 0$ for temporary use.

The following is a collection of the results in Section 3 and Section 4 in [17]. We list them here to emphasize the key properties that will be used in the reorganization of Ledrappier-Young result in Section 2.3. Note that here is the only part that we use C^2 smoothness.

Lemma 2.2.7. *Let f be a C^2 diffeomorphism on a compact manifold and μ an ergodic measure. For any $\varepsilon > 0$, there are a subset Λ with positive measure, a constant C and some increasing partitions $\{\xi^i\}_{1 \leq i \leq u}$ (subordinate to $\{W^i\}_{1 \leq i \leq u}$) and an increasing measurable partition ξ^{u+1} such that for μ -a.e. x and for any $1 \leq i \leq u+1$, there are another metrics on $\xi^i(x)/\xi^{i-1}$ denoted by $\hat{d}_x^i$ and $\tilde{d}_x^i$ with the following properties:*

1. If $y \in \xi^i(x)$, $\tilde{d}_x^i = \tilde{d}_y^i$ and $\tilde{d}_x^i$ does not depend on ε .
2. $|h(f, \mu) - h_\mu(f, \xi^{u+1})| \leq \varepsilon$.
3. There is an injective map $\pi_i : \xi^i(x) \rightarrow \mathbb{R}^{\sum_{j=1}^i \dim E^j} = \mathbb{R}^{\dim E^i} \times \mathbb{R}^{\sum_{j=1}^{i-1} \dim E^j}$ such that π_i maps each $\xi^{i-1}(y) \subset \xi^i(x)$ into some plane $t \times \mathbb{R}^{\sum_{j=1}^{i-1} \dim E^j}$ and the induced projection map $\tilde{\pi}_i : (\xi^i(x)/\xi^{i-1}, \tilde{d}_x^i) \rightarrow \mathbb{R}^{\dim E^i}$ is a bi-Lipschitz homeomorphism on its image. Moreover the bi-Lipschitz constant is C for all $x \in \Lambda$.
4. For any n , any $y \in (f^{-n}\xi^i)(x)$ and $1 \leq m \leq n$,

$$e^{m(\lambda_i - \varepsilon)} \hat{d}_x^i(x, y) \leq \hat{d}_{f^m(x)}^i(f^m(x), f^m(y)) \leq e^{m(\lambda_i + \varepsilon)} \hat{d}_x^i(x, y).$$

5. If $x \in \Lambda$,

$$C^{-1} \hat{d}_x^i(\cdot, \cdot) \leq \tilde{d}_x^i(\cdot, \cdot) \leq C \hat{d}_x^i(\cdot, \cdot).$$

6. If $x \notin \Lambda$,

$$\tilde{d}_x^i(\cdot, \cdot) = \tilde{d}_{f^{-n_0}(x)}^i(f^{-n_0}(\cdot), f^{-n_0}(\cdot))$$

where n_0 is the smallest positive number such that $f^{-n_0}(x) \in \Lambda$.

Proof. In [17], these properties are showed only for the case $i = u + 1$ (i.e., the case of zero Lyapunov exponent $\lambda_{u+1} = 0$). But we can get these properties for general i in a parallel way. For example, one can refer to Chapter IX in the book [32] for a detailed argument. Here we only give the detailed reference of these properties for $i = u + 1$ in Ledrappier and Young's work:

1. See the properties of the metric d^T (which is our $\tilde{d}^i$) defined in Section 3.4 in [17].
2. See Lemma 2.4.2 in [17] and Part A in the proof of Proposition 5.1 in [17].
3. See Section 4.2 in [17].
4. See Lemma 2.3.2 in [17] for the upper bound. One can get lower bound in a parallel way.
5. See Lemma 4.2.3 in [17].
6. See the definition of d^T in Section 3.4 in [17].

□

Remark 2.2.8.

- The metric $\hat{d}_x^i$ acts like the distance induced by the Lyapunov norms (see Lemma 5.5.1).
- Given $\varepsilon > 0$, let $\mathcal{P}$ be a finite partition such that $h_\mu(f, \mathcal{P}) \geq h(f, \mu) - \varepsilon$. We then define ξ^{u+1} in Lemma 2.2.7 as $\xi^{u+1} = \mathcal{P} \vee \xi^+$ where ξ is from Lemma 2.2.6. The partition ξ^{u+1} is the partition η_2 in Section 3.2 in [17].
- The third property is the precise place where Ledrappier and Young used C^2 smoothness.
- We can assume that $\mu(\Lambda)$ is arbitrarily small. To see this, note that the first 5 properties hold trivially for any subset $\Lambda' \subset \Lambda$ with positive measure. For any point $x \in \Lambda \setminus \Lambda'$, if we replace $\tilde{d}_x^i(\cdot, \cdot)$ by $\tilde{d}_{f^{-m_0}(x)}^i(f^{-m_0}(\cdot), f^{-m_0}(\cdot))$ where m_0 is the smallest positive number such that $f^{-m_0}(x) \in \Lambda'$, then we get another two families of metrics which make all properties above hold on the subset Λ' .

The following is a general result in measure theory. For more details, see Lemma 4.14 in [17].

Lemma 2.2.9. *Let μ be a Borel probability measure on a manifold M . Then for μ -a.e. x , we have*

$$\limsup_{\rho \rightarrow 0} \frac{\log \mu(B(x, \rho))}{\log \rho} \leq \dim M.$$

We use $B^k(x, \rho)$ to denote the ball in $\mathbb{R}^k$ for temporary use (in the following Lemma).

We next show a result similar with Lemma 11.31 in [18]. The difference is that we consider the lower and upper bound of the point-wise dimension while Lemma 11.31 in [18] only considers the lower bound.

Lemma 2.2.10. Let μ be a Borel probability measure on $\mathbb{R}^p \times \mathbb{R}^q$ and let $\pi : \mathbb{R}^p \times \mathbb{R}^q \rightarrow \mathbb{R}^p$ be the projection. Let $\{\mu_t\}_{t \in \mathbb{R}^q}$ be the conditional measures of μ w.r.t. the measurable partition $\{t \times \mathbb{R}^q \mid t \in \mathbb{R}^p\}$. Assume that there are numbers $\bar{\delta}, \underline{\delta} \geq 0$ such that for μ -a.e. (t, x) ,

$$\bar{\delta} \geq \limsup_{\rho \rightarrow 0} \frac{\log \mu_t(B^q(x, \rho))}{\log \rho}, \quad \underline{\delta} \leq \liminf_{\rho \rightarrow 0} \frac{\log \mu_t(B^q(x, \rho))}{\log \rho}.$$

Then for μ -a.e. (t, x) ,

$$\bar{r}(t) + \bar{\delta} \geq \limsup_{\rho \rightarrow 0} \frac{\log \mu(B^{p+q}((t, x), \rho))}{\log \rho}, \quad \underline{r}(t) + \underline{\delta} \leq \liminf_{\rho \rightarrow 0} \frac{\log \mu(B^{p+q}((t, x), \rho))}{\log \rho}$$

where

$$\bar{r}(t) \triangleq \limsup_{\rho \rightarrow 0} \frac{\log \mu \circ \pi^{-1}(B^p(t, \rho))}{\log \rho}, \quad \underline{r}(t) \triangleq \liminf_{\rho \rightarrow 0} \frac{\log \mu \circ \pi^{-1}(B^p(t, \rho))}{\log \rho}.$$

Remark 2.2.11. The statements and the proofs of the two parts are parallel. For the ‘lim inf’ part, see Lemma 11.31 in [18]. For completeness, we next prove the ‘lim sup’ part.

Proof of Lemma 2.2.10. We next prove for μ -a.e. (t, x) ,

$$\bar{r}(t) + \bar{\delta} \geq \limsup_{\rho \rightarrow 0} \frac{\log \mu(B^{p+q}((t, x), \rho))}{\log \rho}.$$

For any $\varepsilon > 0$, choose a subset $\Lambda \subset \mathbb{R}^p \times \mathbb{R}^q$ with $\mu(\Lambda) \geq 1 - \varepsilon$ and a large integer $N \in \mathbb{N}$ such that for any $(t, x) \in \Lambda$ and any $n \geq N$,

$$\mu_t(B^q(x, \frac{\sqrt{2}}{2}e^{-n})) \geq e^{-(\bar{\delta}+\varepsilon)n}.$$

Note for any $(t_0, x_0) \in \Lambda$ and any $n \geq N$,

$$\begin{aligned} \mu(B^{p+q}((t_0, x_0), e^{-n})) &\geq \mu(\Lambda \cap B^{p+q}((t_0, x_0), e^{-n})) \\ &\geq \int_{\Lambda \cap B^p(t_0, \frac{\sqrt{2}}{2}e^{-n})} \mu_t(\Lambda \cap B^q(x_0, \frac{\sqrt{2}}{2}e^{-n})) \mu \circ \pi^{-1}(dt) \\ &\geq \mu \circ \pi^{-1}(B^p(t_0, \frac{\sqrt{2}}{2}e^{-n})) \times e^{-(\bar{\delta}+\varepsilon)n}. \end{aligned}$$

Also note

$$\limsup_{\rho \rightarrow 0} \frac{\log \mu(B^{p+q}((t_0, x_0), \rho))}{\log \rho} = \limsup_{n \rightarrow +\infty} \frac{\log(\mu(B^{p+q}((t_0, x_0), e^{-n})))}{-n}$$

and

$$\bar{r}(t_0) = \limsup_{n \rightarrow +\infty} \frac{\log(\mu \circ \pi^{-1}(B^p(t_0, \frac{\sqrt{2}}{2}e^{-n})))}{-n}.$$

Then by the arbitrariness of n and ε , we get the conclusion. $\square$

The next result is a trick that reveals the relation between continuous parameters and discrete parameters.

Lemma 2.2.12. *Let $\phi : (0, 1) \rightarrow (0, 1)$ be a non-decreasing function. Let $\{a_n\}_{n \in \mathbb{N}} \subset (0, 1)$ be a sequence such that*

$$\lim_{n \rightarrow +\infty} \frac{\log a_{n+1}}{\log a_n} = 1 \text{ and } a_n \rightarrow 0.$$

Then

$$\liminf_{x \rightarrow 0} \frac{\log \phi(x)}{\log x} = \liminf_{n \rightarrow +\infty} \frac{\log \phi(a_n)}{\log a_n}, \quad \limsup_{x \rightarrow 0} \frac{\log \phi(x)}{\log x} = \limsup_{n \rightarrow +\infty} \frac{\log \phi(a_n)}{\log a_n}.$$

Proof. The proofs of the two formulas are similar. We just show the first one.

Note the trivial direction:

$$\liminf_{x \rightarrow 0} \frac{\log \phi(x)}{\log x} \leq \liminf_{n \rightarrow +\infty} \frac{\log \phi(a_n)}{\log a_n}.$$

For each $x \in (0, 1)$, let $n(x) = \max\{n \in \mathbb{N} \mid a_n \leq x\}$. Since ϕ is non-decreasing,

$$\liminf_{x \rightarrow 0} \frac{\log \phi(x)}{\log x} \geq \liminf_{x \rightarrow 0} \frac{\log \phi(a_{n(x)+1})}{\log a_{n(x)}} \geq \liminf_{n \rightarrow +\infty} \frac{\log \phi(a_{n+1})}{\log a_{n+1}} \cdot \frac{\log a_{n+1}}{\log a_n}.$$

We then conclude. $\square$

As a consequence of ergodicity, the following shows that the upper dimension and the lower dimension are almost constant.

Lemma 2.2.13. *Let f be a $C^{1+\alpha}$ diffeomorphism on a compact manifold M and let μ be an ergodic measure. For $1 \leq i \leq u$ and for any increasing partition ξ^i subordinate to W^i , the measurable functions $\bar{\delta}_i(\cdot)$ and $\underline{\delta}_i(\cdot)$ are constant almost everywhere where*

$$\bar{\delta}_i(x) \triangleq \limsup_{\rho \rightarrow 0} \frac{\log \mu_x^{\xi^i}(B^i(x, \rho))}{\log \rho}, \quad \underline{\delta}_i(x) \triangleq \liminf_{\rho \rightarrow 0} \frac{\log \mu_x^{\xi^i}(B^i(x, \rho))}{\log \rho}.$$

Proof. For μ almost every point x , note the following two facts:

- there is a constant C_x such that $B^i(f(x), C_x \cdot \rho) \subset f(B^i(x, \rho))$ for all sufficiently small ρ ,
- since μ is invariant and ξ^i is increasing, for all sufficiently small ρ ,

$$\mu_x^{\xi^i}(A) = \mu_{f(x)}^{\xi^i}(f(A)) \cdot \mu_x^{\xi^i}((f^{-1}\xi^i)(x)), \quad \forall A \subset B^i(x, \rho).$$

For μ almost every point x , let $\{\rho_n\}$ be a sequence with $\rho_n \rightarrow 0$ such that

$$\bar{\delta}_i(x) = \lim_{n \rightarrow +\infty} \frac{\log \mu_x^{\xi^i}(B^i(x, \rho_n))}{\log \rho_n}.$$

By the two facts above,

$$\bar{\delta}_i(x) \leq \limsup_{n \rightarrow +\infty} \frac{\log \mu_{f(x)}^{\xi^i}(B^i(f(x), C_x \cdot \rho_n)) + \log \mu_x^{\xi^i}((f^{-1}\xi^i)(x))}{\log \rho_n} \leq \bar{\delta}_i(f(x)).$$

Then by ergodicity, $\bar{\delta}_i(\cdot)$ is constant almost everywhere. With similar arguments, we can also get the result for $\underline{\delta}_i(\cdot)$. $\square$

2.3 Proof of Ledrappier-Young's entropy formula

In this section, we prove Ledrappier-Young's entropy formula by a relatively simple argument.

2.3.1 Entropies along unstable foliations

Before we give the proof of Ledrappier-Young's entropy formula, we give a proof of Proposition 7.2.1 in [18]. This proposition justifies the definition of the entropies along unstable foliations under $C^{1+\alpha}$ setting. Note that Ledrappier and Young [18] assumed C^2 smoothness to prove the entropy formula. But their proof of Proposition 7.2.1 in [18] only uses Pesin Theory and $C^{1+\alpha}$ smoothness. Even though the general idea is from [18], our proof is self-contained and is not intensely involved in Pesin theory.

Recall that for two measurable partitions ξ and η , $\xi \vee \eta$ denotes the partition $\{\xi(x) \cap \eta(x)\}_{x \in R_\mu}$ and $\xi^+ = \bigvee_{n=0}^{+\infty} f^n \xi$. Let $h_\mu(f, \xi)$ denote the entropy of ξ w.r.t. f and let $H_\mu(\xi|\eta)$ denote the mean conditional entropy.

Recall that $r(x) > 0$, $x \in R_\mu$ (R_μ is the Lyapunov regular set of μ) is such that $W_{r(x)}^i(x)$ is an embedded sub-manifold. Recall the definition of the dynamical ball in the i -th unstable manifold:

$$V^i(x, n, \tau) \triangleq \{y \in W_{r(x)}^i(x) \mid d(f^j(x), f^j(y)) \leq \tau, 0 \leq j \leq n-1\}.$$

Remark that here in the definition of the dynamical ball $V^i(x, n, \tau)$, we use the global metric d on M , unlike the definition in [18] that uses the intrinsic metric on the sub-manifold $W^i(x)$. But since we only consider the case when $\tau \rightarrow 0$, our definition of $h_i(f, \mu)$ coincides with theirs. For simplicity, sometimes we write h_i instead of $h_i(f, \mu)$.

Proposition 2.3.1 (Proposition 7.2.1 in [18]). *Let f be a $C^{1+\alpha}$ diffeomorphism on a compact manifold M and let μ be an ergodic measure. For $1 \leq i \leq u$ and for any increasing partition ξ^i subordinate to W^i and for μ almost every point x ,*

$$\lim_{\tau \rightarrow 0} \liminf_{n \rightarrow +\infty} -\frac{1}{n} \log \mu_x^{\xi^i}(V^i(x, n, \tau)) = \lim_{\tau \rightarrow 0} \limsup_{n \rightarrow +\infty} -\frac{1}{n} \log \mu_x^{\xi^i}(V^i(x, n, \tau)) = H_\mu(\xi^i|f\xi^i).$$

Remark 2.3.2.

- Note that the quantities above do not depend on the choice of the subordinate partition ξ^i (See Lemma 3.5.7). So it is proper to denote it by $h_i(f, \mu)$ or h_i for simplicity. See Lemma 3.12 in [17] for more detail.
- Since ξ^i is increasing, i.e., $\xi^i(x) \subset f(\xi^i(f^{-1}(x)))$ for μ -a.e. x , we always have

$$h_\mu(f, \xi^i) \triangleq H_\mu(\xi^i|f(\xi^i)) = H_\mu(\xi^i|f\xi^i).$$

Before the proof, we first give some lemmas that will be used.

The following lemma is a general result in ergodic theory.

Lemma 2.3.3 (Lemma 9.3.1 [18]). *Let $(X, \mathcal{B}, \mu)$ be a Lebesgue probability space and let f be a measurable map such that μ is ergodic. Assume that ξ is an increasing measurable*

partition which generates $\mathcal{B}$ and η is an partition with finite entropy, then for μ -a.e. x ,

$$\lim_{n \rightarrow +\infty} -\frac{1}{n} \log \mu_x^{\tilde{\zeta}} \left(\left(\bigvee_{j=0}^{n-1} f^{-j}(\eta \vee \tilde{\zeta}) \right)(x) \right) = H(f^{-1}\tilde{\zeta}|\tilde{\zeta}).$$

The following is a result that compares the size of subordinate partitions and dynamical balls up to some small error (a partition with finite entropy). Recall that Λ_k^ε is the Pesin block in Lemma 2.2.3. The difference between the original statement (Lemma 9.3.3 in [18]) and ours is that here we only consider the dynamics on a Pesin block to simplify the arguments.

Lemma 2.3.4. *Let f be a $C^{1+\alpha}$ diffeomorphism on a compact manifold and μ an ergodic measure. For any $\varepsilon, \tau \in (0, 1)$, any $1 \leq i \leq u$, any $k \in \mathbb{N}$ and any measurable partition ξ^i subordinate to W^i with $\xi^i(x) \subset W_{r_k}^i(x)$, $x \in \Lambda_k^\varepsilon$, there is a partition η with finite entropy ($H_\mu(\eta) < +\infty$) such that for μ -a.e. $x \in \Lambda_k^\varepsilon$,*

$$\left(\bigvee_{j=0}^{n-1} f^{-j}\eta \right)(x) \cap \xi^i(x) \subset V^i(x, n, \tau), \quad \forall n \geq 1.$$

Remark 2.3.5. *The special measurable partition ξ^i above always exists. In fact, this is a direct consequence of the way how subordinate partitions are constructed. See precisely Lemma 9.11 in [18].*

Proof. For μ -a.e. x , define

$$C \triangleq \sup_{z \in M} \|Df_z\|, \quad n_0(x) \triangleq \min\{n \geq 1 \mid f^n(x) \in \Lambda_k^\varepsilon\}$$

and

$$\rho(x) = \begin{cases} C^{-n_0(x)} \cdot r_k e^{-n_0(x)\varepsilon} \cdot \tau, & x \in \Lambda_k^\varepsilon \\ 1, & \text{else} \end{cases}.$$

Note that ρ is well defined almost everywhere and by Kac's Lemma, $\log \rho$ is μ -integrable. By Lemma 13.3 in [20], we can find a partition η with finite entropy such that

$$\eta(x) \subset B(x, \rho(x)), \quad \mu - a.e. \ x \in M.$$

For μ -a.e. $x \in \Lambda_k^\varepsilon$, for $0 \leq j \leq n_0(x)$ and for any $y \in (\bigvee_{j=0}^{n-1} f^{-j}\eta)(x) \cap \xi^i(x)$, since $y \in \eta(x)$, by the definition of ρ , we have $d(f^j(x), f^j(y)) \leq \tau$. And also note that the term $r_k e^{-n_0(x)\varepsilon}$ guarantees that $f^j(y) \in W_{r_{k+j}}^i(f^j(x))$. Since $f^{n_0(x)}(x) \in \Lambda_k^\varepsilon$ and $f^{n_0(x)}(y) \in \eta(f^{n_0(x)}(x))$, we can replace x, y by $f^{n_0(x)}(x), f^{n_0(x)}(y)$. Then an inductive argument completes the proof. $\square$

Now we give the proof of Proposition 2.3.1.

Proof of Proposition 2.3.1. Let's first show that

$$\lim_{\tau \rightarrow 0} \liminf_{n \rightarrow +\infty} -\frac{1}{n} \log \mu_x^{\xi^i} (V^i(x, n, \tau)) \geq H(f^{-1}\xi^i|\xi^i).$$

To see this, for $\tau > 0$, define

$$\Lambda_\tau \triangleq \{x \mid B^i(x, \tau) \subset \xi^i(x)\}, \quad N_x^n \triangleq \max\{0 \leq j \leq n-1 \mid f^j(x) \in \Lambda_\tau\}.$$

Note

$$V^i(x, n, \tau) = \bigcap_{j=0}^{n-1} f^{-j}(B^i(f^j(x), \tau)).$$

For μ -a.e. x ,

$$\begin{aligned} \lim_{\tau \rightarrow 0} \liminf_{n \rightarrow +\infty} -\frac{1}{n} \log \mu_x^{\xi^i}(V^i(x, n, \tau)) &\geq \lim_{\tau \rightarrow 0} \liminf_{n \rightarrow +\infty} -\frac{1}{n} \log \mu_x^{\xi^i}(f^{-N_x^n}(\xi^i(f^{N_x^n}(x)))) \\ &= \lim_{\tau \rightarrow 0} \liminf_{n \rightarrow +\infty} -\frac{1}{n} \log \mu_x^{\xi^i}((f^{-N_x^n} \xi^i)(x)) \\ &= \lim_{\tau \rightarrow 0} \liminf_{n \rightarrow +\infty} -\frac{1}{n} \sum_{j=0}^{N_x^n-1} \log \mu_x^{f^{-j} \xi^i}((f^{-j-1} \xi^i)(x)) \\ &= \lim_{\tau \rightarrow 0} \liminf_{n \rightarrow +\infty} -\frac{1}{n} \sum_{j=0}^{N_x^n-1} \log \mu_{f^j(x)}^{\xi^i}((f^{-1} \xi^i)(f^j(x))) \\ &\geq \lim_{\tau \rightarrow 0} \liminf_{n \rightarrow +\infty} -\frac{1}{n} \sum_{j=0}^{N_x^n-1} \chi_{\Lambda_\tau}(f^j(x)) \cdot \log \mu_{f^j(x)}^{\xi^i}((f^{-1} \xi^i)(f^j(x))) \\ &= \lim_{\tau \rightarrow 0} \liminf_{n \rightarrow +\infty} -\frac{1}{n} \sum_{j=0}^{n-1} \chi_{\Lambda_\tau}(f^j(x)) \cdot \log \mu_{f^j(x)}^{\xi^i}((f^{-1} \xi^i)(f^j(x))) \\ &= \lim_{\tau \rightarrow 0} \int \chi_{\Lambda_\tau} \cdot \log \mu_x^{\xi^i}((f^{-1} \xi^i)(x)) d\mu \\ &= \int \log \mu_x^{\xi^i}((f^{-1} \xi^i)(x)) d\mu \\ &= H(f^{-1} \xi^i | \xi^i). \end{aligned}$$

For the other direction, we next show for μ -a.e. x ,

$$\lim_{\tau \rightarrow 0} \limsup_{n \rightarrow +\infty} -\frac{1}{n} \log \mu_x^{\xi^i}(V^i(x, n, \tau)) \leq H(f^{-1} \xi^i | \xi^i).$$

Given $k \in \mathbb{N}$, by Lemma 2.3.4, for any $\tau > 0$, we have a partition η with finite entropy such that for μ -a.e. $x \in \Lambda_k^\varepsilon$,

$$\left(\bigvee_{j=0}^{n-1} f^{-j} \eta \right)(x) \cap \xi^i(x) \subset V^i(x, n, \tau), \quad \forall n \geq 1.$$

Hence by Lemma 2.3.3, for μ -a.e. $x \in \Lambda_k^\varepsilon$,

$$\begin{aligned} \limsup_{n \rightarrow +\infty} -\frac{1}{n} \log \mu_x^{\xi^i}(V^i(x, n, \tau)) &\leq \limsup_{n \rightarrow +\infty} -\frac{1}{n} \log \mu_x^{\xi^i} \left(\left(\bigvee_{j=0}^{n-1} f^{-j} \eta \right)(x) \cap \xi^i(x) \right) \\ &\leq \limsup_{n \rightarrow +\infty} -\frac{1}{n} \log \mu_x^{\xi^i} \left(\left(\bigvee_{j=0}^{n-1} f^{-j} (\eta \vee \xi^i) \right)(x) \right) \\ &= H(f^{-1} \xi^i | \xi^i). \end{aligned}$$

We recall that $\mu(\cup_k \Lambda_k^\varepsilon) = 1$. Then we get the result by the arbitrariness of k .

□

2.3.2 Proof of Ledrappier-Young entropy formula

Now we give the proof of Ledrappier-Young's entropy formula.

Theorem 2.3.6. *Let f be a C^2 diffeomorphism on a compact manifold and μ an ergodic measure. For $1 \leq i \leq u$, μ -a.e. x and any measurable partition ξ^i subordinate to W^i , we have*

$$\limsup_{\rho \rightarrow 0} \frac{\log \mu_x^{\xi^i}(B^i(x, \rho))}{\log \rho} = \liminf_{\rho \rightarrow 0} \frac{\log \mu_x^{\xi^i}(B^i(x, \rho))}{\log \rho} := \delta_i.$$

Furthermore, we have $\delta_i - \delta_{i-1} = \frac{h_i - h_{i-1}}{\lambda_i} \leq \dim E^i$ and $h(f, \mu) = h_u$. Here we define $\delta_0 = h_0 = 0$ for convenience.

Once they had defined the quantities $\{\delta_i\}$, Ledrappier and Young called the difference

$$r_i \triangleq \delta_i - \delta_{i-1}$$

the transverse dimension.

By Theorem 2.3.6, we immediately get the following Ledrappier-Young entropy formula.

Corollary 2.3.7 (Ledrappier-Young entropy formula, Theorem 1.2.2.4). *Let f be a C^2 diffeomorphism on a compact manifold and μ an ergodic measure.*

$$h(f, \mu) = \sum_{i=1}^u \lambda_i r_i,$$

Proof of Theorem 2.3.6. For $1 \leq i \leq u$, by Lemma 2.2.13, we denote the μ -a.e. value of

$$\bar{\delta}_i \triangleq \limsup_{\rho \rightarrow 0} \frac{\log \mu_x^{\xi^i}(B^i(x, \rho))}{\log \rho}, \quad \underline{\delta}_i \triangleq \liminf_{\rho \rightarrow 0} \frac{\log \mu_x^{\xi^i}(B^i(x, \rho))}{\log \rho}.$$

We prove the result by an inductive process. Let us first show the case $i = 1$:

$$\bar{\delta}_1 = \underline{\delta}_1 = \frac{h_1}{\lambda_1}.$$

To see this, given $\varepsilon > 0$, for μ -a.e. $x \in \Lambda_k^\varepsilon$, by Lemma 2.2.5, we have

$$\begin{aligned} \liminf_{\rho \rightarrow 0} \frac{\log \mu_x^{\xi^1}(B^1(x, \rho))}{\log \rho} &= \liminf_{n \rightarrow +\infty} \frac{\log \mu_x^{\xi^1}(B^1(x, r_k^2 e^{-n(\lambda_1 + 2\varepsilon)}))}{-n(\lambda_1 + 2\varepsilon) + 2 \log r_k} \\ &\geq \liminf_{n \rightarrow +\infty} \frac{\log \mu_x^{\xi^1}(V^1(x, n, r_k e^{-n\varepsilon}))}{-n(\lambda_1 + 2\varepsilon) + 2 \log r_k} \\ &= \frac{1}{\lambda_1 + 2\varepsilon} \liminf_{n \rightarrow +\infty} -\frac{1}{n} \log \mu_x^{\xi^1}(V^1(x, n, r_k e^{-n\varepsilon})) \\ &\geq \frac{h_1}{\lambda_1 + 2\varepsilon}. \end{aligned}$$

On the other direction, for μ -a.e. $x \in \Lambda_k^\varepsilon$, by Lemma 2.2.5, we have

$$\begin{aligned}
\limsup_{\rho \rightarrow 0} \frac{\log \mu_x^{\tilde{\zeta}^1}(B^1(x, \rho))}{\log \rho} &= \limsup_{n \rightarrow +\infty} \frac{\log \mu_x^{\tilde{\zeta}^1}(B^1(x, e^{-n(\lambda_1 - 2\varepsilon)}))}{-n(\lambda_1 - 2\varepsilon)} \\
&\leq \limsup_{n \rightarrow +\infty} \frac{\log \mu_x^{\tilde{\zeta}^1}(V^1(x, n, \tau_0(x)))}{-n(\lambda_1 - 2\varepsilon)} \\
&= \frac{1}{\lambda_1 - 2\varepsilon} \limsup_{n \rightarrow +\infty} -\frac{1}{n} \log \mu_x^{\tilde{\zeta}^1}(V^1(x, n, \tau_0(x))) \\
&\leq \frac{h_1}{\lambda_1 - 2\varepsilon}.
\end{aligned}$$

Then the result of the case $i = 1$ follows by the arbitrariness of ε and k .

Assume that we have $\bar{\delta}_{i-1} = \underline{\delta}_{i-1}$. Let us proceed.

For $\varepsilon > 0$, by Lemma 2.2.7, we can assume that there are a constant C , a subset Λ and two metrics $\hat{d}_x^i, \tilde{d}_x^i$ on $\tilde{\zeta}^i(x)/\tilde{\zeta}^{i-1}$ with properties described in Lemma 2.2.7. By the remark below Lemma 2.2.7, we can assume that $\mu(\Lambda)$ is small enough such that

$$C^{-4\mu(\Lambda)} e^{-2\varepsilon(\lambda_i - \varepsilon)} \geq e^{-10\varepsilon(\lambda_i - \varepsilon)}. \quad (2.1)$$

For μ -a.e. x , define

$$\begin{aligned}
\tilde{B}^i(x, \delta) &\triangleq \{y \in \tilde{\zeta}^i(x) \mid \tilde{d}_x^i(x, y) \leq \delta\}; \\
g(x) &\triangleq \mu_x^{\tilde{\zeta}^{i-1}}((f^{-1}\tilde{\zeta}^{i-1})(x)); \\
g_\delta(x) &\triangleq \frac{\mu_x^{\tilde{\zeta}^i}(\tilde{B}^i(x, \delta) \cap (f^{-1}\tilde{\zeta}^i)(x))}{\mu_x^{\tilde{\zeta}^i}(\tilde{B}^i(x, \delta))}; \\
g^-(x) &\triangleq \inf_{\delta > 0} g_\delta(x).
\end{aligned}$$

One can check (see Part B in Proposition 5.1 in [17] or the proof of Proposition IX.4.6 in [32]) that by Lemma 4.1.3 in [17], $g_\delta \rightarrow g$ almost everywhere and

$$\int -\log g^- d\mu < +\infty.$$

For μ -a.e. x , let $t_0 \leq 0 < t_1 < t_2 < t_3 < \dots$ be the successive times with $f^{t_i}(x) \in \Lambda$.

For large n and $j < n$, define

$$\begin{aligned}
a(j) &\triangleq \sup\{i \mid t_i \leq j, f^{t_i}(x) \in \Lambda\}, \\
\alpha(x, n, j) &\triangleq \tilde{B}^i(f^j(x), C^{2a(j)} e^{-(n-t_{a(j)})(\lambda_i + \varepsilon)}), \\
\beta(x, n, j) &\triangleq \tilde{B}^i(f^j(x), C^{-2a(j)} e^{-(n-t_{a(j)})(\lambda_i - \varepsilon)}).
\end{aligned}$$

Claim 2.3.7.1.

$$\begin{aligned}
\alpha(x, n, j) \cap (f^{-1}\tilde{\zeta}^i)(f^j(x)) &\subset f^{-1}\alpha(x, n, j+1), \\
f^{-1}\beta(x, n, j+1) &\subset \beta(x, n, j) \cap (f^{-1}\tilde{\zeta}^i)(f^j(x)).
\end{aligned}$$

Remark 2.3.8. Proposition 5.1 in [17] only considers these $\alpha(x, n, j)$. We add these $\beta(x, n, j)$ to get an upper bound on the dimension.

Proof. The proof of these two formulas are similar, so let us just show the second one.

If $j \neq t_i - 1$ for any t_i which implies $a(j) = a(j+1)$, then by definition and Property 6 in Lemma 2.2.7, we get

$$f^{-1}\beta(x, n, j+1) = \beta(x, n, j) \cap (f^{-1}\xi^i)(f^j(x)).$$

Else if $j = t_i - 1$ for some i , i.e., $f^{j+1}(x) = f^{t_i}(x) \in \Lambda$. Note that $a(j) = i - 1$. Then for any $y \in f^{-1}\beta(x, n, j+1)$, by Lemma 2.2.7,

$$\begin{aligned} \tilde{d}_{f^j(x)}^i(f^j(x), y) &= \tilde{d}_{f^{t_{i-1}}(x)}^i(f^{t_{i-1}}(x), f^{-(t_i-t_{i-1}-1)}(y)) \\ &\leq C\tilde{d}_{f^{t_{i-1}}(x)}^i(f^{t_{i-1}}(x), f^{-(t_i-t_{i-1}-1)}(y)) \\ &\leq Ce^{-(t_i-t_{i-1})(\lambda_i-\varepsilon)}\tilde{d}_{f^{j+1}(x)}^i(f^{j+1}(x), f(y)) \\ &\leq C^2e^{-(t_i-t_{i-1})(\lambda_i-\varepsilon)}\tilde{d}_{f^{j+1}(x)}^i(f^{j+1}(x), f(y)) \\ &\leq C^2e^{-(t_i-t_{i-1})(\lambda_i-\varepsilon)}C^{-2i}e^{-(n-t_i)(\lambda_i-\varepsilon)} \\ &= C^{-2a(j)}e^{-(n-t_{a(j)})(\lambda_i-\varepsilon)} \end{aligned}$$

Hence $y \in \beta(x, n, j)$. □

The third property in Lemma 2.2.7 tells us that it is sufficient to work in the standard Euclidean space. Combining this with Lemma 2.2.9, one can show that for μ -a.e. x , there is some N_x such that

$$-\frac{1}{n} \log \mu_x^i(\tilde{B}^i(x, e^{-10\varepsilon n(\lambda_i-\varepsilon)})) \leq 20\varepsilon(\lambda_i - \varepsilon) \dim M, \quad \forall n \geq N_x. \quad (2.2)$$

Choose N large enough such that $\mu(\Delta_\varepsilon) \geq 1 - \frac{\varepsilon}{10}$ where $\Delta_\varepsilon = \{x \mid N_x \leq N\}$.

Denote

$$T_n \triangleq \max\{n(1-2\varepsilon) \leq k \leq n(1-\varepsilon) \mid f^k(x) \in \Delta_\varepsilon\}. \quad (2.3)$$

Note that for μ -a.e. x and for any large n , T_n is well defined. Hence we have

$$\begin{aligned} &\mu_x^{\xi^i}(\alpha(x, n, 0)) \\ &= \prod_{j=0}^{T_n-1} \frac{\mu_{f^j(x)}^{\xi^i}(\alpha(x, n, j))}{\mu_{f^{j+1}(x)}^{\xi^i}(\alpha(x, n, j+1))} \times \mu_{f^{T_n}(x)}^{\xi^i}(\alpha(x, n, T_n)) \\ &= \prod_{j=0}^{T_n-1} \frac{\mu_{f^j(x)}^{\xi^i}(\alpha(x, n, j))}{\mu_{f^j(x)}^{\xi^i}(f^{-1}(\alpha(x, n, j+1)))} \times \mu_{f^{T_n}(x)}^{\xi^i}(\alpha(x, n, T_n)) \\ &= \prod_{j=0}^{T_n-1} \left[\frac{\mu_{f^j(x)}^{\xi^i}(f^{-1}(\alpha(x, n, j+1)))}{\mu_{f^j(x)}^{\xi^i}(\alpha(x, n, j))} \right]^{-1} \times \prod_{j=0}^{T_n-1} \mu_{f^j(x)}^{\xi^i}(f^{-1}(\xi^i)(f^j(x))) \times \mu_{f^{T_n}(x)}^{\xi^i}(\alpha(x, n, T_n)) \\ &\leq \prod_{j=0}^{T_n-1} \left[\frac{\mu_{f^j(x)}^{\xi^i}(\alpha(x, n, j) \cap (f^{-1}\xi^i)(f^j(x)))}{\mu_{f^j(x)}^{\xi^i}(\alpha(x, n, j))} \right]^{-1} \times \prod_{j=0}^{T_n-1} \mu_{f^j(x)}^{\xi^i}(f^{-1}(\xi^i)(f^j(x))) \times \mu_{f^{T_n}(x)}^{\xi^i}(\alpha(x, n, T_n)) \\ &= \prod_{j=0}^{T_n-1} (g_{\delta(x, n, j)}(f^j(x)))^{-1} \times \prod_{j=0}^{T_n-1} \mu_{f^j(x)}^{\xi^i}(f^{-1}(\xi^i)(f^j(x))) \times \mu_{f^{T_n}(x)}^{\xi^i}(\alpha(x, n, T_n)) \end{aligned}$$

where we write $\delta(x, n, j) \triangleq C^{2a(j)} e^{-(n-t_{a(j)})(\lambda_i + \varepsilon)}$ for simplicity.

Therefore we have

$$\begin{aligned}
& (\lambda_i + \varepsilon) \liminf_{\rho \rightarrow 0} \frac{\log \mu_x^{\xi^i}(\tilde{B}^i(x, \rho))}{\log \rho} \\
&= \liminf_{n \rightarrow +\infty} -\frac{1}{n} \log \mu_x^{\xi^i}(\alpha(x, n, 0)) \\
&\geq \liminf_{n \rightarrow +\infty} \frac{1}{n} \sum_{j=0}^{T_n-1} \log g_{\delta(x, n, j)}(f^j(x)) + \liminf_{n \rightarrow +\infty} \sum_{j=0}^{T_n-1} -\frac{1}{n} \log \mu_{f^j(x)}^{\xi^i}(f^{-1}(\xi^i)(f^j(x))) \\
&\geq \liminf_{n \rightarrow +\infty} \frac{1}{n} \sum_{j=0}^{T_n-1} \log g_{\delta(x, n, j)}(f^j(x)) + (1 - 2\varepsilon)h_i.
\end{aligned} \tag{2.4}$$

Note that in the inequality above, the term

$$\liminf_{n \rightarrow +\infty} -\frac{1}{n} \log \mu_{f^{T_n}(x)}^{\xi^i}(\alpha(x, n, T_n))$$

is ignored (because it is non-negative).

Similarly, we can also get

$$\begin{aligned}
& (\lambda_i - \varepsilon) \limsup_{\rho \rightarrow 0} \frac{\log \mu_x^{\xi^i}(\tilde{B}^i(x, \rho))}{\log \rho} \\
&\leq \limsup_{n \rightarrow +\infty} \frac{1}{n} \sum_{j=0}^{T_n-1} \log g_{\delta'(x, n, j)}(f^j(x)) + (1 - \varepsilon)h_i + \limsup_{n \rightarrow +\infty} -\frac{1}{n} \log \mu_{f^{T_n}(x)}^{\xi^i}(\beta(x, n, T_n))
\end{aligned} \tag{2.5}$$

where $\delta'(x, n, j) \triangleq C^{-2a(j)} e^{-(n-t_{a(j)})(\lambda_i - \varepsilon)}$.

Claim 2.3.8.1.

$$\begin{aligned}
& \liminf_{n \rightarrow +\infty} \frac{1}{n} \sum_{j=0}^{T_n-1} \log g_{\delta(x, n, j)}(f^j(x)) \geq -(1 - \varepsilon)(h_{i-1} + 3\varepsilon); \\
& \limsup_{n \rightarrow +\infty} \frac{1}{n} \sum_{j=0}^{T_n-1} \log g_{\delta'(x, n, j)}(f^j(x)) \leq -(1 - 2\varepsilon)(h_{i-1} - \varepsilon).
\end{aligned}$$

Proof. Since $g_\delta \rightarrow g$ and $\int \log g d\mu = -h_{i-1}$, choose δ_1 small enough such that

$$\int_{M \setminus E_{\delta_1}} \log g^- d\mu \geq -\varepsilon, \quad \int_{E_{\delta_1}} \log g d\mu \in [-h_{i-1} - \varepsilon, -h_{i-1} + \varepsilon]$$

where $E_{\delta_1} = \{x \mid \log g(x) - \varepsilon \leq \log g_\delta(x) \leq \log g(x) + \varepsilon, \forall \delta \leq \delta_1\}$.

We first note that if n is sufficiently large, $\delta(x, n, j) \leq \delta_1$ and $\delta'(x, n, j) \leq \delta_1$ for any $j \leq T_n \leq n(1 - \varepsilon)$. Indeed, for any sufficiently large n ,

$$\frac{1}{2}n\mu(\Lambda) \leq \#\{j \leq n \mid f^j(x) \in \Lambda\} \leq 2n\mu(\Lambda).$$

Hence

$$\delta(x, n, j) = C^{2a(j)} e^{-(n-t_{a(j)})(\lambda_i + \varepsilon)} \leq C^{4n\mu(\Lambda)} e^{-\varepsilon n(\lambda_i + \varepsilon)} \rightarrow 0.$$

And also,

$$\delta'(x, n, j) = C^{-2a(j)} e^{-(n-t_{a(j)})(\lambda_i - \varepsilon)} \leq C^{-n\mu(\Lambda)} e^{-\varepsilon n(\lambda_i - \varepsilon)} \rightarrow 0.$$

Therefore for any large n ,

$$\frac{1}{n} \sum_{j=0}^{T_n-1} \log g_{\delta(x, n, j)}(f^j(x)) \geq \frac{1}{n} \sum_{\substack{j=0, \\ f^j(x) \in E_{\delta_1}}}^{T_n-1} [\log g(f^j(x)) - \varepsilon] + \frac{1}{n} \sum_{\substack{j=0, \\ f^j(x) \notin E_{\delta_1}}}^{T_n-1} \log g^-(f^j(x)).$$

Similarly,

$$\frac{1}{n} \sum_{j=0}^{T_n-1} \log g_{\delta'(x, n, j)}(f^j(x)) \leq \frac{1}{n} \sum_{\substack{j=0, \\ f^j(x) \in E_{\delta_1}}}^{T_n-1} [\log g(f^j(x)) + \varepsilon].$$

Then by Birkhoff ergodic theorem, we can get the conclusion. $\square$

Claim 2.3.8.2.

$$\limsup_{n \rightarrow +\infty} -\frac{1}{n} \log \mu_{f^{T_n}(x)}^{\xi^i}(\beta(x, n, T_n)) \leq 20\varepsilon(\lambda_i - \varepsilon) \dim M.$$

Proof. By (2.1), (2.2) and (2.3), we have

$$\begin{aligned} & \limsup_{n \rightarrow +\infty} -\frac{1}{n} \log \mu_{f^{T_n}(x)}^{\xi^i}(\beta(x, n, T_n)) \\ & \leq \limsup_{n \rightarrow +\infty} -\frac{1}{n} \log \mu_{f^{T_n}(x)}^{\xi^i}(\tilde{B}^i(f^{T_n}(x), C^{-4n\mu(\Lambda)} e^{-2\varepsilon n(\lambda_i - \varepsilon)})) \\ & \leq \limsup_{n \rightarrow +\infty} -\frac{1}{n} \log \mu_{f^{T_n}(x)}^{\xi^i}(\tilde{B}^i(f^{T_n}(x), e^{-10\varepsilon n(\lambda_i - \varepsilon)})) \\ & \leq 20\varepsilon(\lambda_i - \varepsilon) \dim M. \end{aligned}$$

$\square$

In summary, by (2.4) and Claim 2.3.8.1, we have

$$\liminf_{\rho \rightarrow 0} \frac{\log \mu_x^{\tilde{\xi}^i}(\tilde{B}^i(x, \rho))}{\log \rho} \geq \frac{(1 - 2\varepsilon)h_i - (1 - \varepsilon)(h_{i-1} + 3\varepsilon)}{\lambda_i + \varepsilon}. \quad (2.6)$$

By (2.5), Claim 2.3.8.1 and Claim 2.3.8.2,

$$\limsup_{\rho \rightarrow 0} \frac{\log \mu_x^{\tilde{\xi}^i}(\tilde{B}^i(x, \rho))}{\log \rho} \leq \frac{(1 - \varepsilon)h_i - (1 - 2\varepsilon)(h_{i-1} - \varepsilon) + 20\varepsilon(\lambda_i - \varepsilon) \dim M}{\lambda_i - \varepsilon}.$$

Then by the arbitrariness of ε , we get

$$\limsup_{\rho \rightarrow 0} \frac{\log \mu_x^{\tilde{\xi}^i}(\tilde{B}^i(x, \rho))}{\log \rho} = \liminf_{\rho \rightarrow 0} \frac{\log \mu_x^{\tilde{\xi}^i}(\tilde{B}^i(x, \rho))}{\log \rho} = \frac{h_i - h_{i-1}}{\lambda_i}. \quad (2.7)$$

Recall that in Lemma 2.2.7, we identify $\xi^i(x)$ with $\mathbb{R}^{\sum_{j=1}^i \dim E^j} = \mathbb{R}^{\dim E^i} \times \mathbb{R}^{\sum_{j=1}^{i-1} \dim E^j}$ through an injective map π_i whose induced map $\tilde{\pi}_i$ is bi-Lipschitz.

Let $p = \dim E^i$ and $q = \sum_{j=1}^{i-1} \dim E^j$. Consider the push-forward Borel probability measure $(\pi_i)_*(\mu_x^{\zeta^i})$ on $\mathbb{R}^p \times \mathbb{R}^q$.

For μ -a.e. $y \in \zeta^i(x)$, define

$$\bar{r}_i(y) \triangleq \limsup_{\rho \rightarrow 0} \frac{\log \mu_y^{\zeta^i}(\tilde{B}^i(y, \rho))}{\log \rho}, \quad \underline{r}_i(y) \triangleq \liminf_{\rho \rightarrow 0} \frac{\log \mu_y^{\zeta^i}(\tilde{B}^i(y, \rho))}{\log \rho}.$$

Note that by (2.7), $\bar{r}_i(y) = \underline{r}_i(y)$.

Recall that by Lemma 2.2.13, $\bar{\delta}_i$ and $\underline{\delta}_i$ are constant almost everywhere. For $x \in \Lambda$, by the uniform bi-Lipschitz continuity of $\tilde{\pi}_i$ and Lemma 2.2.10, we have

$$\bar{\delta}_{i-1} + \bar{r}_i(y) \geq \bar{\delta}_i, \quad \underline{\delta}_{i-1} + \underline{r}_i(y) \leq \underline{\delta}_i.$$

By our assumption $\bar{\delta}_{i-1} = \underline{\delta}_{i-1}$, we then get that

$$\bar{\delta}_i = \underline{\delta}_i = \delta_i, \quad \delta_i - \delta_{i-1} = \frac{h_i - h_{i-1}}{\lambda_i}.$$

Since $(\zeta^i(x)/\zeta^{i-1}, \tilde{d}_x^i)$ is lipeomomorphic to $\mathbb{R}^{\dim E^i}$, by Lemma 2.2.9, we get

$$\frac{h_i - h_{i-1}}{\lambda_i} \leq \bar{r}_i(y) \leq \dim E^i.$$

For the part $h(f, \mu) = h_u$, it looks like the case of $i = u + 1$. But the problem is that we haven't defined the quantity h_{u+1} (note $\lambda_{u+1} = 0$ is already defined). Now in order to apply the arguments above, we can define $h_{u+1} = h_\mu(f, \zeta^{u+1}) = H(\zeta^{u+1}|f\zeta^{u+1})$ where ζ^{u+1} is from Lemma 2.2.7 with $|h(f, \mu) - h_\mu(f, \zeta^{u+1})| \leq \varepsilon$. Be aware that here the number h_{u+1} might depend on ε and the special partition ζ^{u+1} . This is unlike the case for $1 \leq i \leq u$ where ζ^i is subordinate to W^i . But all the arguments above can be directly applied for the case $i = u + 1$ once h_{u+1} is assigned.

By Lemma 4.1, when $i = u + 1$, for the left side of formula (2.6), we have

$$\liminf_{\rho \rightarrow 0} \frac{\log \mu_x^{\zeta^{u+1}}(\tilde{B}^{u+1}(x, \rho))}{\log \rho} \leq \dim M.$$

This is independent of ε . As a consequence, we then get

$$(1 - \varepsilon)(h_{u+1} - h_u - 3\varepsilon) \leq \varepsilon \dim M.$$

Hence $h_u = h(f, \mu)$ by the arbitrariness of ε .

□

Chapter 3

Entropy bounds from unstable dynamics

3.1 Introduction

In this section, we will relate the various invariants to the metric entropy of an ergodic measure. To achieve this, the key intermediate step is to compare them with the entropies along unstable foliations (introduced by Ledrappier and Young, see Proposition 2.3.1). The idea of all these comparisons are similar. Note that by definition, roughly speaking, the entropies along unstable foliations measure, in a probabilistic view, the shrinking speed of the dynamical balls in the unstable manifolds. On the other hand, by their definitions, the topological and fibered entropies we are considering are formulated by counting the cardinality of maximal separated subsets of a given set. The key point is to relate the probabilistic information (these h_i) with the topological information (cardinality). The idea is pretty simple. Let us first give the readers a quick taste of what kind of problems we are handling in this section.

Consider a probability measure μ on the interval $[0, 1]$. Assume that given any radius r , the measure of the ball $B(x, r)$ is uniformly bounded above and below for all $x \in [0, 1]$. For a subset $E \subset [0, 1]$, we may ask: how many disjoint balls of radius r at least and at most that we need to cover E almost everywhere? Of course, the upper cardinality and lower cardinality of these disjoint balls are determined by the upper bound and lower bound of the measures of these balls. In our situation, we have to consider the measures of dynamical balls instead of these natural balls. But the idea is the same.

3.2 Definitions of quantities from unstable dynamics

There are various invariants other than entropies that can also measure the complexity of dynamics. For example, these popular invariants are the rate of growth of periodic points, dimensional entropies, the volume growth of submanifolds and some other invariants related to the growth rate of exterior powers of derivatives. For a systematic survey on this topic, see [19]. A natural question that interests many dynamicists is the relation between these invariants and entropies. In this thesis, we mainly focus on the upper bounds of the metric entropy of an ergodic measure. We prepare the precise definitions of these invariants.

In this section, we always let f be a $C^{1+\alpha}$ ($\alpha > 0$) diffeomorphism on a compact manifold M and let μ be an ergodic measure.

The volume of a sub-manifold $\gamma \subset M$ of constant dimension is denoted by $\text{Vol}(\gamma)$. The lower volume growth of such a sub-manifold $\gamma \subset M$ with $\text{Vol}(\gamma) < \infty$ is:

$$\underline{v}(f, \gamma) \triangleq \liminf_{n \rightarrow \infty} \frac{1}{n} \log^+ \text{Vol}(f^n(\gamma))$$

where $\log^+ a = \max\{0, \log a\}$.

We now introduce the key concepts of our results. They are well defined by Lemma 3.5.1 in Section 3.5.

Recall the definitions of i -th local unstable manifolds $W_\rho^i(x)$ in Section 2.2.

Definition 3.2.1. Given $1 \leq i \leq u$, the μ -a.e. lower volume growth rate of W^i is the μ -a.e. value of

$$\underline{v}_i(f, \mu) \triangleq \inf_{\rho} \underline{v}(f, W_\rho^i(x)).$$

Let $E(n, \varepsilon, \gamma)$ denote a maximal (n, ε) separated subset of a C^1 sub-manifold γ . For the definitions of separated subset and some other basic concepts in ergodic theory, see the book [33].

Definition 3.2.2. Given $1 \leq i \leq u$, the μ -a.e. lower topological entropy of W^i is the μ -a.e. value of

$$h_{\text{top}}^i(f, \mu) \triangleq \inf_{\rho} \lim_{\varepsilon \rightarrow 0} \liminf_{n \rightarrow +\infty} \frac{1}{n} \log \#E(n, \varepsilon, W_\rho^i(x)).$$

Remark 3.2.3. Recall that the topological entropy of $W_\rho^i(x)$ is

$$h_{\text{top}}(f, W_\rho^i(x)) \triangleq \lim_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow +\infty} \frac{1}{n} \log \#E(n, \varepsilon, W_\rho^i(x)).$$

Hence we have $h_{\text{top}}^i(f, \mu) \leq \inf_{\rho} h_{\text{top}}(f, W_\rho^i(x))$ for any $x \in R_\mu$. Note $\inf_{\rho} h_{\text{top}}(f, W_\rho^i(x))$ is also μ -a.e. constant.

Recall that $\{\mu_x^i\}$ is the disintegration of μ w.r.t. a given measurable partition subordinate to W^i . For $1 \leq i \leq u$, $x \in R_\mu$ and $\lambda \in (0, 1)$, define

$$N_\lambda(\mu_x^i, n, \varepsilon) \triangleq \min\{\#C \subset R_\mu : \mu_x^i(\bigcup_{y \in C} V^i(y, n, \varepsilon)) \geq \lambda\}.$$

Definition 3.2.4. Given $1 \leq i \leq u$, the upper fibered Katok entropy of W^i is the μ -a.e. value of

$$\bar{h}_i^K(f, \mu) \triangleq \inf_{\lambda} \lim_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow +\infty} \frac{1}{n} \log N_\lambda(\mu_x^i, n, \varepsilon).$$

Similarly, the lower fibered Katok entropy of W^i is the μ -a.e. value of

$$\underline{h}_i^K(f, \mu) \triangleq \inf_{\lambda} \lim_{\varepsilon \rightarrow 0} \liminf_{n \rightarrow +\infty} \frac{1}{n} \log N_\lambda(\mu_x^i, n, \varepsilon).$$

Remark 3.2.5. The above definition is analogous to the formula of Katok in [13], expressing the metric entropy as the growth rate of the cardinality of maximal separated sets. To be precise, we first note that the definition of $N_\lambda(\mu_x^i, n, \varepsilon)$ above can be applied to any probability measure ν , i.e., we can define

$$N_\lambda(\nu, n, \varepsilon) \triangleq \min\{\#C \subset R_\nu : \nu(\bigcup_{x \in C} B(x, n, \varepsilon)) \geq \lambda\}$$

where $B(x, n, \varepsilon)$ is the dynamical ball. Katok in [13] showed that for any $\lambda > 0$,

$$h(f, \nu) = \lim_{\varepsilon \rightarrow 0} \liminf_{n \rightarrow +\infty} \frac{1}{n} \log N_\lambda(\nu, n, \varepsilon) = \lim_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow +\infty} \frac{1}{n} \log N_\lambda(\nu, n, \varepsilon).$$

We will show similar result in next section (See Theorem B).

3.3 Statements of main results

From now on, when we say $C^{1+\alpha}$ diffeomorphism, we always assume that $\alpha > 0$. Recall the definitions of the entropies (denoted by $h_i(f, \mu)$) along unstable foliations (see Proposition 2.3.1).

Theorem B. *Let f be a $C^{1+\alpha}$ diffeomorphism on a compact manifold M . Let μ be an ergodic measure. List the positive Lyapunov exponents of μ as $\lambda_1 > \lambda_2 > \dots > \lambda_u > 0$. Then for $1 \leq i \leq u$, the entropy along the i -th unstable foliation satisfies:*

1. $h_i(f, \mu) = \underline{h}_i^K(f, \mu) = \bar{h}_i^K(f, \mu);$
2. $h_i(f, \mu) \leq \underline{h}_{\text{top}}^i(f, \mu);$
3. $h_i(f, \mu) \leq \underline{v}_i(f, \mu).$

Remark 3.3.1. *By considering the Dirac measure of a hyperbolic fix point, we note that the second and the third inequalities above can be sharp.*

Recall that when f is C^2 , Ledrappier and Young (Theorem C' in [18]) have shown the following entropy formula (see previous chapter): for any $1 \leq i \leq u$,

$$h(f, \mu) = h_i(f, \mu) + \sum_{j=i+1}^u r_j \cdot \lambda_j.$$

where $r_1, \dots, r_u$ are some transverse dimensions satisfying $r_j \leq \dim E^j$. Therefore, Theorem B immediately implies:

Theorem C. *Let f be a C^2 diffeomorphism on a compact manifold M . Let μ be an ergodic measure. List the positive Lyapunov exponents of μ as $\lambda_1 > \lambda_2 > \dots > \lambda_u$. Then for $1 \leq i \leq u$,*

$$\begin{aligned} h(f, \mu) &= \underline{h}_i^K(f, \mu) + \sum_{j=i+1}^u r_j \cdot \lambda_j, \\ h(f, \mu) &\leq \underline{h}_{\text{top}}^i(f, \mu) + \sum_{j=i+1}^u r_j \cdot \lambda_j, \\ h(f, \mu) &\leq \underline{v}_i(f, \mu) + \sum_{j=i+1}^u r_j \cdot \lambda_j. \end{aligned}$$

Theorem C above is the formal statement of the vague result in page 11.

When the measure μ is hyperbolic, i.e., when μ has no zero Lyapunov exponents, the result in Theorem C is true for $i = u$ without the C^2 assumption.

Theorem D. *Let f be a $C^{1+\alpha}$ diffeomorphism on a compact manifold M . Let μ be an ergodic measure. If μ is hyperbolic, then*

$$h_u(f, \mu) = h(f, \mu).$$

Remark 3.3.2. As a consequence of Theorems **B** and **D**,

$$h(f, \mu) = \underline{h}_u^K(f, \mu) = \bar{h}_u^K(f, \mu).$$

Moreover, this quantity is bounded above both by $\underline{h}_{\text{top}}^\mu(f, \mu)$ and $\underline{v}_u(f, \mu)$.

3.4 Remarks

Let us explain our motivation beyond the desire to prove natural inequalities.

Theorem **C** is to be used to study some entropy-hyperbolic diffeomorphisms (as suggested by Buzzi [6]). We will investigate some large class non-uniformly hyperbolic systems whose ergodic measures of entropy close to the topological entropy are nevertheless hyperbolic and of given index. For more detail, see Section 1.2.4.

Theorem **D** extends by a simple argument the Ledrappier-Young entropy formula in the $C^{1+\alpha}$ setting assuming hyperbolicity. This is used in some ongoing work by other authors (J. Buzzi, S. Crovisier, O. Sarig).

Note that A. Brown [3] gives this $C^{1+\alpha}$ generalization without the hyperbolicity assumption. More precisely, he gives a proof of a uniform bi-Lipschitz property of the stable holonomies inside center-unstable manifolds. However, his argument is technical and only a preprint at the time we are writing this. Hence we believe that our simple, half-page argument has some interest.

3.5 Basic properties

In this section, we list some basic results that will be used later.

Lemma 3.5.1. *Let f be a $C^{1+\alpha}$ diffeomorphism on a compact manifold M . Let μ be an ergodic measure. Then the following four functions are constant almost everywhere.*

$$\begin{aligned} & \inf_{\rho} \underline{v}(f, W_{\rho}^i(x)), \\ & \inf_{\rho} \lim_{\varepsilon \rightarrow 0} \liminf_{n \rightarrow +\infty} \frac{1}{n} \log \#E(n, \varepsilon, W_{\rho}^i(x)), \\ & \inf_{\lambda} \lim_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow +\infty} \frac{1}{n} \log N_{\lambda}(\mu_x^i, n, \varepsilon), \\ & \inf_{\lambda} \lim_{\varepsilon \rightarrow 0} \liminf_{n \rightarrow +\infty} \frac{1}{n} \log N_{\lambda}(\mu_x^i, n, \varepsilon). \end{aligned}$$

Proof. We first explain the measurability of these functions.

Since the infimum or the limit of a sequence of measurable functions is also measurable, it is enough to check that $\text{Vol}(f^n(W_{\rho}^i(x))), E(n, \varepsilon, W_{\rho}^i(x)), N_{\lambda}(\mu_x^i, n, \varepsilon)$ are measurable w.r.t. x .

Recall that a result of Pesin theory gives that the i -th local unstable manifold $W_{\rho}^i(x)$ varies measurably w.r.t. x (see Theorem 16, page 195 in [11]). This gives the measurability of $\text{Vol}(f^n(W_{\rho}^i(x)))$ and $E(n, \varepsilon, W_{\rho}^i(x))$ by noting that the composition of measurable functions is still measurable. Since the family of conditional measures μ_x^i of μ (w.r.t. measurable subordinate partitions, see Definition 2.2.1) varies measurably w.r.t. x , one can get that $N_{\lambda}(\mu_x^i, n, \varepsilon)$ is also measurable.

Once we get the measurability, one can check that these functions are f -invariant. Hence by ergodicity, they are constant almost everywhere. $\square$

Recall the notations in Lemma 2.2.3. A result of standard Pesin Theory (e.g. remarks below Lemma 2.2.3 in [17]) shows that:

Lemma 3.5.2. *Let f be a $C^{1+\alpha}$ diffeomorphism on a compact manifold M and let μ be an hyperbolic ergodic measure. Given $\varepsilon > 0$ and $x \in R_\mu$, assume that $x \in \Lambda_k^\varepsilon$ for some k . Then*

$$S^{cu}(x) \subset W^u(x).$$

Here recall that

$$S^{cu}(x) \triangleq \{y \in M \mid d(f^{-n}(x), f^{-n}(y)) \leq r_k e^{-n\varepsilon}, \forall n \geq 0\}.$$

Remark 3.5.3. *Roughly speaking, $S^{cu}(x)$ above is just the set of points whose backward trajectory always stays in the same Pesin chart of the backward trajectory of x . Hence in general, $S^{cu}(x)$ is the local center unstable manifold of x . But when the measure is hyperbolic, the above lemma says that $S^{cu}(x)$ reduces to the local unstable manifold.*

Recall that for two measurable partitions ξ and η , $\xi \vee \eta$ denotes the partition $\{\xi(x) \cap \eta(x)\}_{x \in R_\mu}$ and $\xi^+ = \bigvee_{n=0}^{+\infty} f^n \xi$. Recall that $h_\mu(f, \xi)$ denotes the entropy of ξ w.r.t. f and $H_\mu(\xi | \eta)$ denotes the mean conditional entropy.

We say η is finer than ξ denoted by $\xi \leq \eta$ if $\eta(x) \subset \xi(x)$ for μ -a.e. x . For partitions with finite mean entropy, the finer partition has larger entropy. The following is an extension of this property to non-finite partitions.

Lemma 3.5.4 (Property 8.7 in [27]). *Let f be a homeomorphism on a compact metric space X . Assume that μ is an f -invariant probability measure. Let ξ, η be two measurable partitions (possibly with infinite mean entropy) with η being finer than ξ . If the mean conditional entropy $H_\mu(\eta | f(\xi^+))$ is finite, then $h_\mu(f, \xi) \leq h_\mu(f, \eta)$.*

Remark 3.5.5. *Rokhlin's article [27] mainly discusses entropy theory for endomorphisms where most definitions and properties are stated by using f^{-1} . Since here we assume that f is a homeomorphism, our statement of Lemma 3.5.4 is equivalent to the original statement of Property 8.7 in [27].*

Recall that a measurable partition ξ is called increasing if $\xi \leq f^{-1}\xi$.

Lemma 3.5.6. *Let f be a homeomorphism on a compact metric space X . Assume that μ is an f -invariant probability measure. Let ξ, η be two increasing measurable partitions with $h_\mu(f, \xi) < +\infty, h_\mu(f, \eta) < +\infty$. Then for any integer $n \geq 1$,*

$$h_\mu(f, \xi \vee \eta) = h_\mu(f, \xi \vee f^n \eta).$$

Proof. Since ξ, η are increasing, for any $n \geq 1$, we have

$$f^n \xi \vee f^n \eta \leq \xi \vee f^n \eta \leq \xi \vee \eta.$$

In order to apply Lemma 3.5.4, we note that

$$\begin{aligned} H_\mu(\xi \vee \eta | f((\xi \vee f^n \eta)^+)) &= H_\mu(\xi \vee \eta | f\xi \vee f^{n+1}\eta) \\ &= H_\mu(\xi | f\xi \vee f^{n+1}\eta) + H_\mu(\eta | \xi \vee f^{n+1}\eta) \\ &\leq H_\mu(\xi | f\xi) + H_\mu(\eta | f^{n+1}\eta) \\ &= h_\mu(f, \xi) + nh_\mu(f, \eta) \\ &< +\infty. \end{aligned}$$

By Lemma 3.5.4, we have

$$h_\mu(f, \xi \vee f^n \eta) \leq h_\mu(f, \xi \vee \eta).$$

To conclude, we prove the converse inequality by applying the previous one to $\xi^1 = f^n \eta$ and $\eta^1 = \xi$, obtaining:

$$h_\mu(f, \xi \vee \eta) = h_\mu(f, f^n \xi \vee f^n \eta) \leq h_\mu(f, \xi \vee f^n \eta).$$

□

The following is an extension of Lemma 3.1.2 in [17]. The main difference is that here we only assume one of the two partitions is subordinate. The proof is essentially identical to that of Lemma 3.1.2 in [17]. For completeness, we present it.

Lemma 3.5.7. *Let f be a $C^{1+\alpha}$ diffeomorphism on a compact manifold M and let μ be an ergodic measure. Let ξ^u be an increasing partition subordinate to W^u . Assume that β is a measurable partition satisfying*

1. β is increasing,
2. for μ -a.e. x , $\beta(x) \subset W^u(x)$,
3. for μ -a.e. x , $\text{diam}((f^{-n}(\beta))(x)) \rightarrow 0$.

Then

$$h_\mu(f, \xi^u \vee \beta) = h_\mu(f, \beta).$$

Proof. Since both ξ^u and β are increasing and their entropies w.r.t. f and μ are finite, by Lemma 3.5.6, for any $n \geq 1$, we have

$$\begin{aligned} h_\mu(f, \xi^u \vee \beta) &= h_\mu(f, (f^n \xi^u) \vee \beta) \\ &= H_\mu((f^n \xi^u) \vee \beta \mid (f^{n+1} \xi^u) \vee f\beta) \\ &= H_\mu(\beta \mid (f^{n+1} \xi^u) \vee f\beta) + H_\mu(f^n \xi^u \mid (f^{n+1} \xi^u) \vee \beta) \\ &= H_\mu(\beta \mid (f^{n+1} \xi^u) \vee f\beta) + H_\mu(\xi^u \mid (f \xi^u) \vee f^{-n} \beta). \end{aligned}$$

By the third assumption on β , $f^{-n} \beta$ tends increasingly to the partition ε into points. Note that $H_\mu(\xi^u \mid (f \xi^u) \vee f^{-n} \beta) \leq H_\mu(\xi^u \mid f \xi^u) < +\infty$, by Property 5.11 in [27], the second term $H_\mu(\xi^u \mid (f \xi^u) \vee f^{-n} \beta)$ above goes to $H_\mu(\xi^u \mid \varepsilon) = 0$ as $n \rightarrow +\infty$. So it is sufficient to prove $H_\mu(\beta \mid (f^{n+1} \xi^u) \vee f\beta) \rightarrow H(\beta \mid f\beta) = h_\mu(f, \beta)$. First note that

$$H_\mu(\beta \mid (f^{n+1} \xi^u) \vee f\beta) \leq H(\beta \mid f\beta).$$

Write the conditional measures of μ w.r.t. $(f^{n+1} \xi^u) \vee f\beta$ as $\{\mu_x^n\}_{x \in M}$ and the conditional measures w.r.t. $f\beta$ as $\{\mu_x\}_{x \in M}$. By definition,

$$H_\mu(\beta \mid (f^{n+1} \xi^u) \vee f\beta) = \int -\log \mu_x^n(\beta(x)) d\mu(x), \quad H(\beta \mid f\beta) = \int -\log \mu_x(\beta(x)) d\mu(x).$$

Let

$$\Omega_n = \{x \mid f\beta(x) \subset f^{n+1} \xi^u(x)\}.$$

Since $\xi^u(x)$ contains an open neighborhood of x in $W^u(x)$ w.r.t. the sub-manifold topology, by assumptions 2 and 3 on β , $\{\Omega_n\}$ is a non-decreasing sequence and

$\mu(\Omega_n) \rightarrow 1$ as $n \rightarrow +\infty$. For $x \in \Omega_n$, by definition, $(f^{n+1}\xi^u)(x) \cap (f\beta)(x) = (f\beta)(x)$. Then one can show that this implies

$$-\log \mu_x^n(\beta(x)) = -\log \mu_x(\beta(x)), \quad \mu \text{ a.e. } x \in \Omega_n.$$

Hence the non-negative functions $\{-\log \mu_{(\cdot)}^n(\beta(\cdot))\}$ tend pointwise to $-\log \mu_{(\cdot)}(\beta(\cdot))$. By Fatou's Lemma,

$$\lim_{n \rightarrow +\infty} H_\mu(\beta | (f^{n+1}\xi^u) \vee f\beta) \geq H(\beta | f\beta).$$

□

3.6 Proof of Theorem B

We prove the assertions in Theorem B one by one.

Proof of Property 1

We first prove $h_i(f, \mu) \leq \underline{h}_i^K(f, \mu)$.

By Proposition 2.3.1 and by removing a set of zero measure from R_μ if necessary, we can assume that

$$\lim_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow +\infty} -\frac{1}{n} \log \mu_x^i(V^i(x, n, \varepsilon)) = \lim_{\varepsilon \rightarrow 0} \liminf_{n \rightarrow +\infty} -\frac{1}{n} \log \mu_x^i(V^i(x, n, \varepsilon)) = h_i(f, \mu), \quad \forall x \in R_\mu.$$

We write $h_i(f, \mu)$ as h_i for short.

For any $\eta, \varepsilon > 0$, define

$$\Delta_\eta^\varepsilon \triangleq \{x \in R_\mu \mid \liminf_{n \rightarrow +\infty} -\frac{1}{n} \log \mu_x^i(V^i(x, n, 2\varepsilon)) > h_i - \eta\}.$$

Then $\cup_{\varepsilon > 0} \Delta_\eta^\varepsilon = R_\mu$.

For $j \in \mathbb{N}$ and $p \in \Delta_\eta^\varepsilon$, define

$$\Delta_\eta^\varepsilon(p, j) \triangleq \{x \in \Delta_\eta^\varepsilon \mid \mu_p^i(V^i(x, n, 2\varepsilon)) \leq e^{-n(h_i - \eta)}, \forall n \geq j\}.$$

By definition,

$$\Delta_\eta^\varepsilon(p, j) \subset \Delta_\eta^\varepsilon(p, j+1), \quad \mu_p^i(\cup_j \Delta_\eta^\varepsilon(p, j)) = \mu_p^i(\Delta_\eta^\varepsilon).$$

Fix any $\lambda > 0$ and $p \in R_\mu$ and choose ε small enough and N large enough such that

$$\mu_p^i(\Delta_\eta^\varepsilon(p, j)) \geq 1 - \frac{\lambda}{2}, \quad \forall j \geq N.$$

For $n \in \mathbb{N}$, let $C_n \subset R_\mu$ be a subset such that $\#C_n = N_\lambda(\mu_p^i, n, \varepsilon)$ and $\mu_p^i(\cup_{y \in C_n} V^i(y, n, \varepsilon)) \geq \lambda$. Hence we have

$$\mu_p^i(\Delta_\eta^\varepsilon(p, n) \cap (\cup_{y \in C_n} V^i(y, n, \varepsilon))) \geq \frac{\lambda}{2}, \quad \forall n \geq N.$$

Let $A_n \subset C_n$ be such that for each $y \in A_n$, we have $V^i(y, n, \varepsilon) \cap \Delta_\eta^\varepsilon(p, n) \neq \emptyset$. For $y \in A_n$, we fix any $\tilde{y} \in V^i(y, n, \varepsilon) \cap \Delta_\eta^\varepsilon(p, n)$. Then we have

$$V^i(y, n, \varepsilon) \subset V^i(\tilde{y}, n, 2\varepsilon).$$

Hence for $n \geq N$,

$$\frac{\lambda}{2} \leq \mu_p^i \left(\bigcup_{y \in A_n} V^i(\tilde{y}, n, 2\varepsilon) \right) \leq N_\lambda(\mu_p^i, n, \varepsilon) \times \sup_{x \in \Delta_\eta^\varepsilon(p, n)} \mu_p^i(V^i(x, n, 2\varepsilon)) \leq N_\lambda(\mu_p^i, n, \varepsilon) \times e^{-n(h_i - \eta)}.$$

Therefore for any ε small enough (depending on η) and n large enough,

$$N_\lambda(\mu_p^i, n, \varepsilon) \geq \frac{\lambda}{2} \cdot e^{n(h_i - \eta)}.$$

Then by the arbitrariness of η and λ , we get

$$h_i(f, \mu) \leq \inf_{\lambda} \lim_{\varepsilon \rightarrow 0} \liminf_{n \rightarrow +\infty} \frac{1}{n} \log N_\lambda(\mu_p^i, n, \varepsilon) = \underline{h}_i^K(f, \mu).$$

Next we prove $h_i(f, \mu) \geq \bar{h}_i^K(f, \mu)$. The arguments are similar as above.

For any $\eta, \varepsilon > 0$, define

$$\Omega_\eta^\varepsilon \triangleq \{x \in R_\mu \mid \limsup_{n \rightarrow +\infty} -\frac{1}{n} \log \mu_x^i(V^i(x, n, \frac{\varepsilon}{2})) < h_i + \eta\}.$$

Then $\bigcup_{\varepsilon > 0} \Omega_\eta^\varepsilon = R_\mu$.

For $j \in \mathbb{N}$ and $p \in \Delta_\eta^\varepsilon$, define

$$\Omega_\eta^\varepsilon(p, j) \triangleq \{x \in \Delta_\eta^\varepsilon \mid \mu_p^i(V^i(x, n, \frac{\varepsilon}{2})) \geq e^{-n(h_i + \eta)}, \forall n \geq j\}.$$

By definition,

$$\Omega_\eta^\varepsilon(p, j) \subset \Omega_\eta^\varepsilon(p, j+1), \quad \mu_p^i(\bigcup_j \Omega_\eta^\varepsilon(p, j)) = \mu_p^i(\Omega_\eta^\varepsilon).$$

Fix any $\lambda > 0$ and $p \in R_\mu$, choose ε small enough and N large enough such that

$$\mu_p^i(\Omega_\eta^\varepsilon(p, j)) \geq \lambda, \quad \forall j \geq N.$$

For $n \in \mathbb{N}$, let $F_n \subset \Omega_\eta^\varepsilon(p, n)$ be a maximal (n, ε) separated set of $\Omega_\eta^\varepsilon(p, n) \cap \xi^i(p)$. Then $\{V^i(y, n, \varepsilon)\}_{y \in F_n}$ covers $\Omega_\eta^\varepsilon(p, n) \cap \xi^i(p)$. Hence $\#F_n \geq N_\lambda(\mu_p^i, n, \varepsilon)$. And we also have

$$y_1, y_2 \in F_n, y_1 \neq y_2 \implies V^i(y_1, n, \frac{\varepsilon}{2}) \cap V^i(y_2, n, \frac{\varepsilon}{2}) = \emptyset.$$

Hence for $n \geq N$,

$$N_\lambda(\mu_p^i, n, \varepsilon) \leq \#F_n \leq \frac{1}{\sup_{x \in \Omega_\eta^\varepsilon(p, n)} \mu_p^i(V^i(x, n, \frac{\varepsilon}{2}))} \leq e^{n(h_i + \eta)}.$$

Then by the arbitrariness of η and λ , we get

$$h_i(f, \mu) \geq \inf_{\lambda} \lim_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow +\infty} \frac{1}{n} \log N_{\lambda}(\mu_p^i, n, \varepsilon) = \bar{h}_i^K(f, \mu).$$

Proof of Property 2

We next prove $h_i(f, \mu) \leq \underline{h}_{\text{top}}^i(f, \mu)$.

Since ξ^i is a partition subordinate to W^i , for any $\rho > 0$, we assume that for any $x \in R_{\mu}$, $\mu_x^i(W_{\rho}^i(x)) > 0$.

For $\rho, \varepsilon, \eta > 0, n, j \in \mathbb{N}$ and $p \in R_{\mu}$, let $F_{\eta}^{\varepsilon}(p, j, n)$ be a (n, ε) -separated subset of $\Delta_{\eta}^{\varepsilon}(p, j) \cap W_{\rho}^i(p)$ with maximum cardinality. It is a cover, hence we have

$$\begin{aligned} \mu_p^i(\Delta_{\eta}^{\varepsilon}(p, j) \cap W_{\rho}^i(p)) &\leq \mu_p^i\left(\bigcup_{x \in F_{\eta}^{\varepsilon}(p, j, n)} V^i(x, n, \varepsilon)\right) \\ &\leq \#F_{\eta}^{\varepsilon}(p, j, n) \times \sup_{x \in \Delta_{\eta}^{\varepsilon}(p, j)} \mu_p^i(V^i(x, n, \varepsilon)) \\ &\leq \#F_{\eta}^{\varepsilon}(p, j, n) \times e^{-n(h_i - \eta)}, \quad \forall j, \forall n \geq j. \end{aligned}$$

Hence

$$\#F_{\eta}^{\varepsilon}(p, j, n) \geq \frac{\mu_p^i(\Delta_{\eta}^{\varepsilon}(p, j) \cap W_{\rho}^i(p))}{e^{n(h_i - \eta)}}, \quad \forall j, \forall n \geq j.$$

Choose ε small enough (depending on η) and j large enough such that $\mu_p^i(\Delta_{\eta}^{\varepsilon}(p, j) \cap W_{\rho}^i(x)) > 0$. Then taking $\liminf_{n \rightarrow +\infty} \frac{1}{n} \log$ on both sides, we have

$$\liminf_{n \rightarrow +\infty} \frac{1}{n} \log \#E(n, \varepsilon, W_{\rho}^i(x)) \geq \liminf_{n \rightarrow +\infty} \frac{1}{n} \log \#F_{\eta}^{\varepsilon}(p, j, n) \geq h_i - \eta.$$

Since η and ρ are arbitrary, we get $h_i(f, \mu) \leq \underline{h}_{\text{top}}^i(f, \mu)$.

Proof of Property 3

We next prove $h_i(f, \mu) \leq \underline{v}_i(f, \mu)$.

Applying Lemma 2.2.3 for any $\varepsilon > 0$, we obtain an increasing sequence of measurable sets $\{\Lambda_k^{\varepsilon} \subset R_{\mu}\}$.

Let us first note that, for any $k, n \in \mathbb{N}$, any $x \in \Lambda_k^{\varepsilon}$, any $\tau \leq r_k e^{-n\varepsilon}$ and any y with $f^n(y) \in W_{\tau}^i(f^n(x))$,

$$d(f^{n-j}(x), f^{n-j}(y)) \leq r_{k+n}^{-1} e^{-j(\lambda_i - \varepsilon)} d(f^n(x), f^n(y)), \quad \forall 0 \leq j \leq n.$$

Hence for any $k, n \in \mathbb{N}$, any $x \in \Lambda_k^{\varepsilon}$ and any $\tau \leq r_k e^{-n\varepsilon}$, $f^n(V^i(x, n, \tau))$ contains an i -th local sub-manifold $W_{r_{k+n}\tau}^i(f^n(x))$.

Since the function $\mu_p^i(V^i(p, n, \tau))$ is non-decreasing w.r.t. τ , for any sequence $\{\tau_n\}$ with $\tau_n \rightarrow 0$, we have for $p \in R_{\mu}$,

$$\liminf_{n \rightarrow +\infty} -\frac{1}{n} \log \mu_p^i(V^i(p, n, \tau_n)) \geq \liminf_{n \rightarrow +\infty} -\frac{1}{n} \log \mu_p^i(V^i(p, n, \tau)), \quad \forall \tau > 0.$$

Hence in particular, for $k \in \mathbb{N}$, $p \in \Lambda_k^\varepsilon$ and $x \in \Lambda_k^\varepsilon \cap \zeta^i(p)$,

$$\liminf_{n \rightarrow +\infty} -\frac{1}{n} \log \mu_p^i(V^i(x, n, r_k e^{-n\varepsilon})) \geq h_i.$$

For any $j \in \mathbb{N}$, $p \in \Lambda_k^\varepsilon$ and $\rho > 0$ with $W_\rho^i(p) \subset \zeta^i(p)$, define

$$\Lambda_{k,j}^{\varepsilon,p} = \{x \in \Lambda_k^\varepsilon \cap W_\rho^i(p) \mid \mu_p^i(V^i(x, n, r_k e^{-n\varepsilon})) \leq e^{-n(h_i - \varepsilon)}, \forall n \geq j\}.$$

By definition,

$$\Lambda_{k,j}^{\varepsilon,p} \subset \Lambda_{k,j+1}^{\varepsilon,p}, \quad \mu_p^i\left(\bigcup_j \Lambda_{k,j}^{\varepsilon,p}\right) = \mu_p^i(\Lambda_k^\varepsilon \cap W_\rho^i(p)).$$

Let $F_{n,j,k}^{\varepsilon,p}$ be a $(n, r_k e^{-n\varepsilon})$ -separated subset of $\Lambda_{k,j}^{\varepsilon,p}$ with maximum cardinality. Then we have

$$\begin{aligned} \mu_p^i(\Lambda_{k,j}^{\varepsilon,p}) &\leq \mu_p^i\left(\bigcup_{x \in F_{n,j,k}^{\varepsilon,p}} V^i(x, n, r_k e^{-n\varepsilon})\right) \\ &\leq \#F_{n,j,k}^{\varepsilon,p} \times \sup_{x \in \Lambda_{k,j}^{\varepsilon,p}} \mu_p^i(V^i(x, n, r_k e^{-n\varepsilon})). \end{aligned} \quad (3.1)$$

Note that for any $x \in F_{n,j,k}^{\varepsilon,p} \subset W_\rho^i(p)$, $V^i(x, n, r_k e^{-n\varepsilon}) \subset W_{2\rho}^i(p)$ for all n such that $r_k e^{-n\varepsilon} < \rho$. Since the sets $\{f^n(V^i(x, n, \frac{1}{2}r_k e^{-n\varepsilon}))\}_{x \in F_{n,j,k}^{\varepsilon,p}}$ are mutually disjoint, for all large n , we have

$$\text{Vol}(f^n(W_{2\rho}^i(x))) \geq \sum_{x \in F_{n,j,k}^{\varepsilon,p}} \text{Vol}(f^n(V^i(x, n, \frac{1}{2}r_k e^{-n\varepsilon}))).$$

Recall that each $f^n(V^i(x, n, \frac{1}{2}r_k e^{-n\varepsilon}))$ contains a piece of i -th local unstable manifold $W_{\frac{1}{2}r_{k+n}r_k e^{-n\varepsilon}}^i(f^n(x))$. Thus

$$\text{Vol}(f^n(W_{2\rho}^i(x))) \geq \#F_{n,j,k}^{\varepsilon,p} \times K \times \left(\frac{1}{2}r_{k+n}r_k e^{-n\varepsilon}\right)^{\sum_{l=1}^i \dim E^l} \quad (3.2)$$

where K is the constant from Lemma 2.2.3.

Combining (3.1) and (3.2), for all large n ,

$$\begin{aligned} \text{Vol}(f^n(W_{2\rho}^i(x))) &\geq \frac{\mu_p^i(\Lambda_{k,j}^{\varepsilon,p})}{\sup_{x \in \Lambda_{k,j}^{\varepsilon,p}} \mu_p^i(V^i(x, n, r_k e^{-n\varepsilon}))} \times K \times \left(\frac{1}{2}r_{k+n}r_k e^{-n\varepsilon}\right)^{\sum_{l=1}^i \dim E^l} \\ &\geq \frac{\mu_p^i(\Lambda_{k,j}^{\varepsilon,p})}{e^{-n(h_i - \varepsilon)}} \times K \times \left(\frac{1}{2}r_{k+n}r_k e^{-n\varepsilon}\right)^{\sum_{l=1}^i \dim E^l}, \quad \forall k \in \mathbb{N}, \forall p \in \Lambda_k^\varepsilon, \forall j \in \mathbb{N}. \end{aligned}$$

Since $\mu_p^i(W_\rho^i(p)) > 0$, we choose k, j large enough such that $\mu_p^i(\Lambda_{k,j}^{\varepsilon,p}) > 0$. Taking $\liminf_{n \rightarrow +\infty} \frac{1}{n} \log$ on both sides, we have

$$\liminf_{n \rightarrow +\infty} \frac{1}{n} \log \text{Vol}(f^n(W_{2\rho}^i(p))) \geq h_i - \varepsilon - 2\varepsilon \sum_{l=1}^i \dim E^l, \quad \forall k \in \mathbb{N}, \forall p \in \Lambda_k^\varepsilon.$$

Hence we have

$$\liminf_{n \rightarrow +\infty} \frac{1}{n} \log \text{Vol}(f^n(W_{2\rho}^i(p))) \geq h_i - \varepsilon - 2\varepsilon \sum_{l=1}^i \dim E^l, \quad \forall p \in R_\mu.$$

By the arbitrariness of ε and ρ , we get the conclusion.

3.7 Proof of Theorem D

We now prove Theorem D, i.e. $h_u(f, \mu) = h(f, \mu)$ using the arguments of Ledrappier and Young in [17].

Proof. Let ξ^u be any increasing measurable partition subordinate to W^u . By Proposition 2.3.1 (see Remark 2.3.2), $h_u(f, \mu) = H_\mu(\xi^u | f\xi^u) = h_\mu(f, \xi^u)$. Hence $h_u(f, \mu) \leq h(f, \mu)$. So it is sufficient to prove $h_u(f, \mu) \geq h(f, \mu)$.

In the following argument, some properties only hold for μ -a.e. x . But without loss of generality, we assume that these properties hold for any $x \in R_\mu$.

For $\varepsilon > 0$ and $x \in \Lambda_k^\varepsilon$, let $S^{cu}(x)$ be the set in Lemma 3.5.2, i.e.:

$$S^{cu}(x) \triangleq \{y \in M \mid d(f^{-n}(x), f^{-n}(y)) \leq r_k e^{-n\varepsilon}, \forall n \geq 0\}.$$

By Lemma 2.2.6, there is a measurable partition ξ with $H_\mu(\xi) < +\infty$ such that $\xi^+(x) \subset S^{cu}(x)$, $x \in R_\mu$ where $\xi^+ = \bigvee_{n=0}^{+\infty} f^n \xi$. Since $H_\mu(\xi) < +\infty$, we can assume that $h_\mu(f, \xi) \geq h(f, \mu) - \varepsilon$.

By Lemma 3.5.2, $\xi^+(x) \subset W^u(x)$, $x \in R_\mu$.

We note the following facts:

- $h_u(f, \mu) = H_\mu(\xi^u | f\xi^u) \geq h_\mu(f, \xi^u \vee \xi^+)$.

The proof is similar to Lemma 3.2.1 in [17]. To see this, similar to the proof of Lemma 3.5.7, we note that for any $n \geq 1$,

$$h_\mu(f, \xi^u \vee \xi^+) = H_\mu(\xi^u | (f^{n+1}\xi^+) \vee f\xi^u) + H_\mu(\xi^+ | (f\xi^+) \vee f^{-n}\xi^u).$$

Since the second term goes to zero as $n \rightarrow +\infty$, we get $h_\mu(f, \xi^u \vee \xi^+) \leq H_\mu(\xi^u | f\xi^u)$.

- Since ξ^+ is increasing, $\xi^+(x) \subset W^u(x)$ and $\text{diam}(f^{-n}(\xi^+(x))) \rightarrow 0$ for μ -a.e. x , Lemma 3.5.7 yields

$$h_\mu(f, \xi^u \vee \xi^+) = h_\mu(f, \xi^+).$$

- Since ξ^+ is increasing,

$$h_\mu(f, \xi^+) = H_\mu(\xi^+ | f(\xi^+)) = H_\mu(\xi \vee f(\xi^+) | f(\xi^+)) = H_\mu(\xi | f(\xi^+)) = h_\mu(f, \xi).$$

Hence we have

$$h_u(f, \mu) \geq h_\mu(f, \xi) \geq h(f, \mu) - \varepsilon.$$

Since ε is arbitrarily small, we get the conclusion. □

Chapter 4

Proofs of applications to entropy-hyperbolicity

4.1 Introduction

In this chapter, we prove the results stated in Section 1.2.4. The first observation of the condition ‘entropy-hyperbolic’ is Theorem 1.2.4: measures with large entropy are hyperbolic with same index. Other applications to entropy-hyperbolic diffeomorphisms, the continuity property of entropy function (Corollary 1.2.6), cardinality of measures maximizing entropy (Corollary 1.2.7) and the growth rate of periodic points (Corollary 1.2.8) are essentially based on Theorem 1.2.4.

Note that if $\dim M \geq 1$, f is entropy-hyperbolic implies that $h_{\text{top}}(f) > 0$.

4.2 Proofs

Proof of Theorem 1.2.4. Note $h_{\text{top}}(f) > 0$ by entropy-hyperbolicity. So $d(f)$ and $d(f^{-1})$ are positive.

Define

$$h \triangleq \max\{h_{\text{top}}^{d(f)-1}(f), h_{\text{top}}^{d(f^{-1})-1}(f^{-1})\}.$$

By definition, $h < h_{\text{top}}(f)$. Let μ be an ergodic measure with $h(f, \mu) > h$. Let λ_0 be the smallest non-negative Lyapunov exponent and denote its multiplicity by d_0 . We need to show $\lambda_0 > \frac{h(f, \mu) - h}{\dim M}$. Similar argument can be applied to the non-positive Lyapunov exponent, hence we omit the proof of that part. Let s (resp. u) be the stable (resp. unstable) index of μ , i.e., the sum of the dimensions of the Oseledec’s subspaces on which the Lyapunov exponents are negative (resp. positive). Going by contradiction, we assume that $\lambda_0 \leq \frac{h(f, \mu) - h}{\dim M}$. By Theorem C (where we use C^2 smoothness), we have

$$h(f, \mu) \leq \lambda_0 \cdot d_0 + h_{\text{top}}^{u-d_0}(f) < h(f, \mu) - h + h_{\text{top}}^{u-1}(f), \quad h(f^{-1}, \mu) \leq h_{\text{top}}^s(f^{-1}).$$

Together with the fact that $h < h(f^{-1}, \mu)$, we have

$$h < \min\{h_{\text{top}}^{u-1}(f), h_{\text{top}}^s(f^{-1})\}.$$

Then by the definition of h ,

$$\begin{cases} d(f) - 1 \leq u - 2 \\ d(f^{-1}) - 1 \leq s - 1. \end{cases} \quad (4.1)$$

Hence $d(f) + d(f^{-1}) < u + s$ which is a contradiction. Therefore $\lambda_0 > \frac{h(f, \mu) - h}{\dim M}$ which means that μ is $\frac{h(f, \mu) - h}{\dim M}$ -hyperbolic. Moreover, one can also get that the index of μ is $d(f)$. $\square$

Remark 4.2.1. The proof of Theorem 1.2.4 shows $d(f) + d(f^{-1}) \leq s + u$ for any ergodic measure μ with large entropy. Hence by the variational principle [33, Theorem 8.6], we always have $d(f) + d(f^{-1}) \leq \dim M$.

Recall that a compact f -invariant subset Λ is a horseshoe if there are positive integers m, n and subsets $\Lambda_1, \dots, \Lambda_m$ such that

- $\Lambda = \bigcup_{i=1}^m \Lambda_i$.
- $f^m(\Lambda_i) = \Lambda_i$, $f(\Lambda_i) = \Lambda_{i+1}$ for $1 \leq i \leq m$ (assume $\Lambda_{m+1} = \Lambda_1$).
- $f^m|_{\Lambda_1}$ is conjugate to the full shift on the space of n symbols.

Proof of Corollary 1.2.6. By Theorem 1.2.4, there is some constant $h < h_{\text{top}}(f)$ such that any ergodic measure with entropy larger than h is hyperbolic. By Katok's horseshoe theorem [13], for any $\varepsilon > 0$, there is a hyperbolic horseshoe Λ such that $h_{\text{top}}(f|_{\Lambda}) \geq h(f, \mu) - \varepsilon$. By the variational principle [33, Theorem 8.6] and the arbitrariness of ε , we get the first conclusion.

Recall that hyperbolic horseshoe is stable under C^1 perturbations. If $f_n \xrightarrow{C^1} f$, then

$$\liminf_{n \rightarrow +\infty} h_{\text{top}}(f_n) \geq h_{\text{top}}(f|_{\Lambda}) \geq h(f, \mu) - \varepsilon.$$

Hence we get that $h_{\text{top}}(\cdot)$ is lower semi-continuous at f w.r.t. the C^1 topology.

Furthermore, by Newhouse's result [22], $h_{\text{top}}(\cdot)$ is always upper continuous w.r.t. the C^∞ topology, hence it is continuous at f w.r.t. the C^∞ topology in our setting. $\square$

Theorem 1.2.8 is proved analogously to Buzzi's result (Theorem 1.2 in [7]) up to the following changes:

1. We use Ben Ovadia's coding [31] instead of Sarig's coding [29] which is only for surface diffeomorphisms.
2. For surface diffeomorphism with positive entropy, the Ruelle-Margulis inequality suffices to show that measures maximizing the entropy are hyperbolic. In higher dimension, we use entropy-hyperbolicity to get this property by Theorem 1.2.4.

Let $A = (a_{i,j})$ be a transition matrix (a finite or countable matrix of zeroes and ones with no columns or rows which are all zeroes) and let Σ be the symbolic space induced by A , i.e.,

$$\Sigma = \{(x_i)_{i \in \mathbb{Z}} \mid a_{x_i, x_{i+1}} = 1, i \in \mathbb{Z}\}.$$

A topological Markov shift is the symbolic space (Σ, σ) where σ is the left shift on Σ .

The first step to prove Theorem 1.2.8 is to generalize the injective coding, Theorem 1.1 in [7] to any dimension. The key improvement over Ben Ovadia's result is the injectivity, whereas he only established finiteness. We also improve the hyperbolicity estimate (see item 4).

Theorem 4.2.2. *Let f be a $C^{1+\alpha}$ diffeomorphism on a compact manifold M . For all numbers $0 < \chi' < \chi$, there are a topological Markov shift Σ_χ and a Hölder continuous map $\pi_\chi : \Sigma_\chi \rightarrow M$ such that*

1. $\pi_\chi \circ \sigma = f \circ \pi_\chi$,
2. π_χ is injective,
3. $\pi_\chi(\Sigma_\chi)$ has full measure for all χ -hyperbolic measures,
4. For any periodic point p in Σ_χ , the periodic point $\pi_\chi(p)$ is χ' -hyperbolic.

Outline of the proof. The proof is essentially the same as Theorem 1.1 in [7].

For a topological Markov shift (Σ, σ) , let $\Sigma^\#$ denote the set of points $(x_i)_{i \in \mathbb{Z}} \in \Sigma$ such that there exist u, v in the alphabet such that u occurs infinitely many times in $(x_i)_{i \leq 0}$ and v occurs infinitely many times in $(x_i)_{i \geq 0}$.

Ben Ovadia [31] shows for any $\chi > 0$, there are a locally compact topological Markov shift Σ_χ and a Hölder continuous map $\pi_\chi : \Sigma_\chi \rightarrow M$ such that

1. $\pi_\chi \circ \sigma = f \circ \pi_\chi$ and π_χ has the Bowen property (see the definition in [7]) w.r.t. some locally finite relation on the alphabet,
2. $\pi_\chi|_{\Sigma_\chi^\#}$ is finite to one,
3. $\pi_\chi(\Sigma_\chi^\#)$ has full measure for all χ -hyperbolic measures,
4. For any periodic point p in Σ_χ , the periodic point $\pi_\chi(p)$ is $\frac{2}{3}\chi$ -hyperbolic (Proposition 4.4 in [31]).

The number $\frac{2}{3}\chi$ is not essential and can be replaced by any number $\chi' < \chi$. See Lemma 6.1 in [7] for the 2-dimensional case.

By the Main Theorem of [7] (page 3), we can replace the coding (π_χ, Σ_χ) by a new coding which also satisfies Property 1,3 and 4 above and this new coding can be injective. Hence we can assume that (π_χ, Σ_χ) itself is the desired injective coding. $\square$

Now we explain the proof of Theorem 1.2.8.

Proof of Theorem 1.2.8. Since f is C^∞ , Newhouse [22] shows the existence of an ergodic measure μ maximizing the entropy. By Theorem 1.2.4, μ is $\frac{h_{\text{top}}(f) - h}{\dim M}$ -hyperbolic. Let $\chi = \frac{h_{\text{top}}(f) - h}{\dim M}$ and $\chi' = \lambda$. By Theorem 4.2.2, we get an injective coding $\pi_\chi : \Sigma_\chi \rightarrow M$. We conclude as in Section 6 of [7]. $\square$

Consider a C^∞ diffeomorphism f on $\mathbb{T}^2$. We define the induced map $\hat{f} : [0, 1]^2 \rightarrow [0, 1]^2$ as the unique continuous extension of $\pi^{-1}|_{[0,1]^2} \circ f \circ \pi$ on $[0, 1]^2$. In general, $\hat{f}$ is just a piece-wise C^∞ map. But if f satisfies condition (1.12), then one can show that $\hat{f}$ is C^∞ on $[0, 1]^2$.

Proof of Example 1.2.9. First notice that f_1, f_2 are entropy-hyperbolic with $d(f_1) = d(f_2) = 1$. Let $\hat{f}_1$ and $\hat{f}_2$ be the induced C^∞ maps on $[0, 1]^2$ as described before Example 1.2.9. Recall that π is the natural bijective map between $[0, 1]^2$ and $\mathbb{T}^2$.

Under the condition (1.12), one can show that,

- $h_{\text{top}}(f_i) = h_{\text{top}}(\hat{f}_i)$, $i = 1, 2$,

- $h_{\text{top}}^1(f_1 \times f_2) = h_{\text{top}}^1(\hat{f}_1 \times \hat{f}_2)$.

The first property is straightforward by condition (1.12). For the second property, we first note by definition that $h_{\text{top}}^1(f_1 \times f_2) \geq h_{\text{top}}^1(\hat{f}_1 \times \hat{f}_2)$. To see the other direction, consider any C^∞ curve γ on $\mathbb{T}^4 = \mathbb{T}^2 \times \mathbb{T}^2$. It can be written as a finite union of curves $\{\gamma_i\}$ with small diameter and any such curve γ_i is either contained in $\mathcal{U} \times \mathcal{U}$ ($\mathcal{U}$ is from condition (1.12)) or $(\pi \times \pi)^{-1} \circ \gamma_i$ is a C^∞ curve in $(0, 1)^2 \times (0, 1)^2$. Then either $h_{\text{top}}(f_1 \times f_2, \gamma_i) = 0$ or $h_{\text{top}}(f_1 \times f_2, \gamma_i) = h_{\text{top}}(\hat{f}_1 \times \hat{f}_2, (\pi \times \pi)^{-1} \circ \gamma_i)$. Thus $h_{\text{top}}(f_1 \times f_2, \gamma) \leq h_{\text{top}}(\hat{f}_1 \times \hat{f}_2, (\pi \times \pi)^{-1} \circ \gamma)$. As a consequence, we have $h_{\text{top}}^1(f_1 \times f_2) \leq h_{\text{top}}^1(\hat{f}_1 \times \hat{f}_2)$.

By a result of Burguet (Theorem 4.35 in [4]),

$$h_{\text{top}}^1(\hat{f}_1 \times \hat{f}_2) \leq \max\{h_{\text{top}}(\hat{f}_1), h_{\text{top}}(\hat{f}_2)\}.$$

Hence

$$h_{\text{top}}^1(f_1 \times f_2) \leq \max\{h_{\text{top}}(f_1), h_{\text{top}}(f_2)\} < h_{\text{top}}(f_1 \times f_2).$$

Therefore $d(f_1 \times f_2) \geq 2$. Applying the same argument to $f_1^{-1} \times f_2^{-1}$, we have $d(f_1^{-1} \times f_2^{-1}) \geq 2$. Thus by Remark 4.2.1, $d(f_1 \times f_2) + d(f_1^{-1} \times f_2^{-1}) = 4$. □

Part II

Dimensional Entropies

Chapter 5

Upper bound: uniform dimensional entropy

5.1 Precise definition of uniform dimensional entropy

The uniform dimensional entropy is a concept related to topological entropy which was introduced by Buzzi (see [6]). We next give precise definitions.

Let M be a compact Riemannian manifold. Let δ_0 be a fixed number such that for any $x \in M$, the exponential map $\exp_x : \mathcal{B}(x, \delta_0) \rightarrow M$ is an embedding where

$$\mathcal{B}(x, \delta_0) \triangleq \{v \in T_x M \mid \|v\| \leq \delta_0\}.$$

Let D^i be the open unit disk in $\mathbb{R}^i$. An (i -dimensional) C^r ($r \geq 0$) submanifold γ is defined as a C^r embedding $\gamma : D^i \rightarrow M$. We sometimes identify the embedding with its image for convenience. Given $\delta \leq \frac{\delta_0}{100}$ and $1 \leq i \leq \dim M$, we define a standard family $\mathcal{A}^{\text{std}}(i, \delta)$ of i -dimensional C^0 submanifolds of M ,

$$\mathcal{A}^{\text{std}}(i, \delta) \triangleq \{\exp_x \circ (\delta \cdot \mathcal{L}|_{D^i} + v) \mid \|v\| \leq \delta, v \in T_x M, \mathcal{L} : \mathbb{R}^i \rightarrow T_x M \text{ linear isometry}, x \in M\}.$$

Let $\mathcal{A}$ be a collection of C^0 submanifolds (not necessarily have fixed dimension). For a C^0 submanifold γ , we write $\gamma \prec \mathcal{A}$ if γ (the image) is a subset of some submanifold in $\mathcal{A}$. Let $\mathcal{A}'$ be another collection of C^0 submanifolds. We write $\mathcal{A}' \prec \mathcal{A}$ if for any $\gamma' \in \mathcal{A}'$, $\gamma' \prec \mathcal{A}$.

Definition 5.1.1. Given $0 < \delta < \frac{\delta_0}{100}$ and $1 \leq i \leq \dim M$, we say a collection of C^0 submanifolds $\mathcal{A}(i, \delta)$ is an (i, δ) ample family if for any $\gamma_{\text{std}} \in \mathcal{A}^{\text{std}}(i, \delta)$ and $\varepsilon > 0$, there is $\gamma \prec \mathcal{A}(i, \delta)$ such that $d_{C^0}(\gamma_{\text{std}}, \gamma) \leq \varepsilon$ where d_{C^0} is the C^0 distance.

Remark 5.1.2. Note that the ample family $\mathcal{A}(i, \delta)$ might contain submanifolds with different dimensions.

Associated with an ample family, we can define its dimensional entropy.

Definition 5.1.3. Define the uniform dimensional entropy w.r.t. a family $\mathcal{A}$ of submanifolds by

$$H(f, \mathcal{A}) \triangleq \lim_{\tau \rightarrow 0} \limsup_{n \rightarrow +\infty} \frac{1}{n} \sup_{\gamma \in \mathcal{A}} \log \#E(n, \tau, \gamma)$$

where $E(n, \varepsilon, \gamma)$ is a maximal (n, τ) separated subset of γ .

Define the uniform i -dimensional entropy by

$$H^i(f) \triangleq \inf \left\{ H(f, \mathcal{A}(i, \delta)) \mid \mathcal{A}(i, \delta) \text{ is an } (i, \delta) \text{ ample family} \right\}.$$

Remark 5.1.4.

- Similar ideas were introduced by Buzzi in [6] where he systematically discussed the properties and applications of dimensional entropy.
- We assume that $H^0(f) = 0$ and note that $H^{\dim M}(f) = h_{\text{top}}(f)$.
- $H^i(f) \leq H^{i+1}(f)$. This is a consequence of the second property in Lemma 5.4.4.
- The reason why we add the word “uniform” is that here we put “sup” inside the formula (instead of putting it before $\lim_{\tau \rightarrow 0}$).

5.2 Idea of the proof of Theorem A

The goal of this chapter is to prove Theorem A stated in the introduction of this thesis. For completeness, we first recall it.

Theorem A. Let f be a $C^{1+\alpha}$ ($\alpha > 0$) diffeomorphism on a compact manifold and let μ be an ergodic measure. List the Lyapunov exponents of μ as $\lambda_1 < \lambda_2 < \dots < \lambda_l$. For any $0 \leq k \leq l$, let $i = \dim(E^{k+1} \oplus E^{k+2} \oplus \dots \oplus E^l)$. Then we have

$$h(f, \mu) \leq H^i(f) + \sum_{j=1}^k \lambda_j^+ \dim(E^j)$$

where $\lambda_j^+ = \max\{0, \lambda_j\}$.

In this section, we introduce the idea of the proof of Theorem A.

Let f be a $C^{1+\alpha}$ ($\alpha > 0$) diffeomorphism on a compact manifold M and μ an ergodic probability measure. Oseledets' Theorem [23] states that there are an invariant measurable subset R_μ with full measure, an invariant measurable decomposition $T_{R_\mu} M = E^1 \oplus E^2 \oplus \dots \oplus E^l$ and finitely many numbers $\lambda_1 < \lambda_2 < \dots < \lambda_l$ such that for any $x \in R_\mu$ and any nonzero vector $v \in E_x^j$, we have

$$\lim_{n \rightarrow \pm\infty} \frac{1}{n} \log \|Df_x^n(v)\| = \lambda_j.$$

Given $1 \leq k \leq l$, in the classical Pesin theory, one bounds above the expansion rate of relatively k -th stable manifold (tangent to $\oplus_{j=1}^i E^j$) by the largest exponent λ_k . A lot of information is lost because on different directions the expansion rates are different (even though they are all smaller than λ_k). The key obstacle of proving Theorem A is to estimate precisely the size of relatively stable manifolds. Different directions should be taken into careful consideration. We give a version of relatively stable manifold. See Theorem E where we bound the expansion rate of the relatively stable manifold by the corresponding Lyapunov exponents up to some small errors.

We next show the outline of the proof of Theorem A by using our relatively stable manifolds.

By Katok's result (see Proposition 5.4.3), to bound the metric entropy, we only have to estimate the cardinality of a maximal separated subset of some set Λ with

positive measure. Given $\tau > 0$ and $n \in \mathbb{N}$, let $\alpha_j = \tau e^{-n(\lambda_j^+ + \epsilon)}$. By Theorem E, we get that the stable manifolds $\{\mathcal{W}_{\Gamma_x^k}(\alpha_1, \dots, \alpha_k)\}_{x \in \Lambda}$ whose diameter keeps less than τ when we iterate. We then choose a family F_n of finitely many disks in the ample family (see Definition 5.1.1) such that each $\mathcal{W}_{\Gamma_x^k}(\alpha_1, \dots, \alpha_k)$ has non-empty intersection with some $\gamma \in F_n$. But how many disks do we need to have this non-intersection property? It turns out by a geometric argument (see Proposition 5.6.4), the cardinality of F_n can be bounded by $\prod_{j=1}^k \alpha_j^{-\dim(E^j)}$ up to some constant.

For $\gamma \in F_n$, let $E(n, \tau, \gamma)$ be a maximal (n, τ) separate subset of γ . Then we get that $\cup_{\gamma \in F_n} E(n, \tau, \gamma)$ contains a maximal $(n, 2\tau)$ separated subset of Λ . Then

$$\#E(n, 2\tau, \Lambda) \leq \prod_{j=1}^k \alpha_j^{-\dim(E^j)} \times \sup_{\gamma \in F_n} \#E(n, \tau, \gamma).$$

This leads us to the formula in Theorem A.

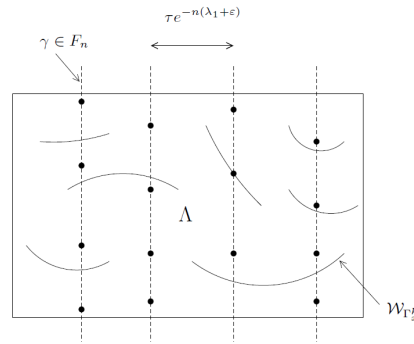


FIGURE 5.1 – Maximal separated subset of Λ

5.3 A version of stable manifold theorem

We first give a precise statement of our stable manifold theorem.

For $1 \leq j \leq l$, let π_j be the projection to E^j w.r.t. the Oseledets' decomposition $T_{R_\mu} M = E^1 \oplus E^2 \oplus \dots \oplus E^l$.

Given $1 \leq k \leq l$ and $x \in R_\mu$, define

$$\Delta_k(x) \triangleq \{\psi_x : \oplus_{j=1}^k E_x^j \rightarrow T_x M \mid \psi_x \text{ is continuous}\},$$

$$\tilde{\Delta}_k(x) \triangleq \{\phi_x : \oplus_{j=1}^k E_x^j \rightarrow \oplus_{j=1}^k E_x^j \mid \exists K \text{ s.t. } \|\pi_i(\phi_x(v))\| \leq K \|\pi_i(v)\|, v \in \oplus_{j=1}^k E_x^j, 1 \leq i \leq k\},$$

$$\Delta_k \triangleq \bigcup_{x \in R_\mu} \{\Delta_k(x)\}, \quad \tilde{\Delta}_k \triangleq \bigcup_{x \in R_\mu} \{\tilde{\Delta}_k(x)\}$$

The set Δ_k can be viewed as a collection of submanifolds in the tangent space in which we will choose some good submanifolds as our relatively stable manifolds (see Theorem E below). The element ϕ in $\tilde{\Delta}_k$ is some kind of change of coordinate where the constant K indicates that ϕ can not rotate too much. Note that we do not assume any topology on Δ_k and $\tilde{\Delta}_k$.

We say a map $\Gamma^k : R_\mu \rightarrow \Delta_k$ (resp. $\tilde{\Delta}_k$) is a section map, if at each $x \in R_\mu$, the image, denoted by Γ_x^k , is contained in $\Delta_k(x)$ (resp. $\tilde{\Delta}_k(x)$). For any section map $\Gamma^k :$

$R_\mu \rightarrow \Delta_k$, any $x \in R_\mu$ and any non-negative numbers $\eta_1, \dots, \eta_k$, define

$$\mathcal{W}_{\Gamma_x^k}(\eta_1, \dots, \eta_k) \triangleq \exp_x(\{\Gamma_x^k(v_1, \dots, v_k) \in T_x M \mid v_i \in E_x^i, \|v_i\| \leq \eta_i, i \leq k\}).$$

We can view $\mathcal{W}_{\Gamma_x^k}(\eta_1, \dots, \eta_k)$ as some local submanifold generated by the section map Γ^k .

Theorem E. *Let f be a $C^{1+\alpha}$ diffeomorphism on a compact manifold M . Then for any ergodic measure μ , there is a constant ε_0 such that for $\varepsilon \in (0, \varepsilon_0)$ and any integer $1 \leq k \leq l$, there are two section maps $\Gamma^k : R_\mu \rightarrow \Delta_k$ and $\phi : R_\mu \rightarrow \Delta_k$ such that*

- For any $x \in R_\mu$, any $n \in \mathbb{N}$ and any $\tau \in (0, \frac{1}{k})$,

$$\text{diam}(f^i(\mathcal{W}_{\Gamma_x^k}(\tau e^{-n(\lambda_1^+ + \varepsilon)}, \dots, \tau e^{-n(\lambda_k^+ + \varepsilon)})) \leq \tau, \quad \forall 0 \leq i \leq n$$

where $\lambda_j^+ = \max\{0, \lambda_j\}$.

- There is a subset $\Omega_\varepsilon \subset R_\mu$ with $\mu(\Omega_\varepsilon) \geq 1 - \varepsilon$ such that

$$\text{Lip}(\hat{\Gamma}_x^k \circ \phi_x - \text{Id}) \leq \varepsilon, \quad \forall x \in \Omega_\varepsilon$$

where $\hat{\Gamma}_x^k = (\pi_1 \circ \Gamma_x^k, \dots, \pi_k \circ \Gamma_x^k)$ and Id is the identity map on $\oplus_{j=1}^k E_x^j$.

Remark 5.3.1. $\mathcal{W}_{\Gamma_x^k}$ above can be viewed as a piece of well reparameterized stable manifold. The first property says that its expansion on each direction is well controlled by the corresponding Lyapunov exponent.

By the classical Pesin theory, we know that stable manifold at x is tangent to the corresponding Oseledec's space at x . As a consequence, locally, the projection of the stable manifold to the corresponding Oseledec's space should be close to the identity map. The second property then says that after some change of coordinate by ϕ_x , our reparameterization also keeps this property.

Note that here we do not need the measurability of Γ^k and ϕ .

5.4 Basic definitions and properties

In the following, we will recall some classical definitions of entropy. For more detail, see Walters' book [33].

Definition 5.4.1. Let f be a homeomorphism on a compact metric space X . Let $\bar{B}(x, \tau)$ be the closed ball centered at x with radius τ . For a compact subset Γ , a subset E of Γ is called a (n, τ) separated set of Γ if for any $y \in E$, the closed dynamical ball

$$\bigcap_{i=0}^{n-1} f^{-i} \bar{B}(f^i y, \tau)$$

contains no other point of E . A (n, τ) separated set E is called maximal if for any $p \in \Gamma$, there is $q \in E$ and $0 \leq k \leq n-1$ such that $d(f^k(p), f^k(q)) \leq \tau$.

The topological entropy of Γ denoted by $h_{\text{top}}(f, \Gamma)$ is defined as follows:

$$h_{\text{top}}(f, \Gamma) = \lim_{\tau \rightarrow 0} \limsup_{n \rightarrow \infty} \frac{1}{n} \log \#E(n, \tau, \Gamma)$$

where $E(n, \tau, \Gamma)$ is a maximal (n, τ) separated set of Γ .

We use $h_{\text{top}}(f)$ to denote the topological entropy of the whole space X .

Definition 5.4.2. For an f -invariant Borel probability measure μ , the entropy of a finite partition $\mathcal{P} = \{P_1, \dots, P_k\}$ is defined by

$$H_\mu(\mathcal{P}) = \sum_{i=1}^k -\mu(P_i) \log \mu(P_i).$$

The entropy of f with respect to a finite partition $\mathcal{P}$ is defined by

$$h_\mu(f, \mathcal{P}) = \lim_{n \rightarrow +\infty} \frac{1}{n} H_\mu\left(\bigvee_{i=0}^{n-1} f^{-i}(\mathcal{P})\right),$$

where

$$\bigvee_{i=0}^{n-1} f^{-i}(\mathcal{P}) \triangleq \{B \mid B = \bigcap_{i=0}^{n-1} f^{-i}(P_{i_j}), P_{i_j} \in \mathcal{P}\}.$$

The metric entropy of f is defined by

$$h(f, \mu) \triangleq \sup\{h_\mu(f, \mathcal{P}) \mid \mathcal{P} \text{ is a finite partition}\}.$$

The following is a result of Katok [13].

Proposition 5.4.3. Let f be a homeomorphism on a compact metric space X and μ an ergodic measure. For any compact subset $\Lambda \subset X$ with positive measure, we have

$$h(f, \mu) \leq \lim_{\tau \rightarrow 0} \limsup_{n \rightarrow +\infty} \frac{1}{n} \log \#E(n, \tau, \Lambda)$$

where $E(n, \tau, \Lambda)$ is a maximal (n, τ) separated subset of Λ .

In the following, we show some basic properties of ample family.

Lemma 5.4.4. For the ample family, we have the following properties:

- An (i, δ) ample family is also an (i, δ') ample family for any $\delta' \leq \delta$.
- An $(i+1, \delta)$ ample family is also an (i, δ) ample family for $1 \leq i \leq \dim M - 1$.

Proof. Given $\varepsilon > 0$, consider a submanifold $\mathcal{A}^{\text{std}}(i, \delta') \ni \gamma'_{\text{std}} = \delta' \cdot \mathcal{L}|_{D^i} + v$ for some $x \in M$, some $v \in T_x M$ with $\|v\| \leq \delta'$ and some linear isometry $\mathcal{L} : \mathbb{R}^i \rightarrow T_x M$. Let $\gamma_{\text{std}} = \delta \cdot \mathcal{L}|_{D^i} + v$, then $\gamma_{\text{std}} \in \mathcal{A}^{\text{std}}(i, \delta)$. Let $\mathcal{A}(i, \delta)$ be an (i, δ) ample family, then there is some $\gamma \prec \mathcal{A}(i, \delta)$ such that $d_{C^0}(\gamma_{\text{std}}, \gamma) \leq \varepsilon$. Define $\gamma' : D^i \rightarrow M$ as $\gamma'(p) = \gamma(\frac{\delta'}{\delta} \cdot p)$. Then by definition, $\gamma' \prec \mathcal{A}^{\text{std}}(i, \delta)$ and for any $p \in D^i$,

$$d(\gamma'_{\text{std}}(p), \gamma'(p)) = d(\gamma_{\text{std}}(\frac{\delta'}{\delta} \cdot p), \gamma(\frac{\delta'}{\delta} \cdot p)) \leq \varepsilon.$$

Hence $\mathcal{A}^{\text{std}}(i, \delta)$ is also an (i, δ') ample family.

Given $\varepsilon > 0$, consider a submanifold $\mathcal{A}^{\text{std}}(i, \delta) \ni \gamma_{\text{std}}^i = \delta \cdot \mathcal{L}|_{D^i} + v$ for some $x \in M$, some $v \in T_x M$ with $\|v\| \leq \delta$ and some linear isometry $\mathcal{L} : \mathbb{R}^i \rightarrow T_x M$. Let $\tilde{\mathcal{L}} : D^{i+1} \rightarrow T_x M$ be a linear isometry such that for any $p \in D^i$, $\tilde{\mathcal{L}}((p, 0)) = \mathcal{L}(p)$ where $(p, 0) \in \mathbb{R}^{i+1} = \mathbb{R}^i \times \mathbb{R}$. Let $\gamma_{\text{std}}^{i+1} = \delta \cdot \tilde{\mathcal{L}}|_{D^{i+1}} + v$, then $\gamma_{\text{std}}^{i+1} \in \mathcal{A}^{\text{std}}(i+1, \delta)$. Let $\mathcal{A}(i+1, \delta)$ be an $(i+1, \delta)$ ample family, then there is some $\gamma^{i+1} \prec \mathcal{A}(i+1, \delta)$

such that $d_{C^0}(\gamma_{\text{std}}^{i+1}, \gamma^{i+1}) \leq \varepsilon$. Define $\gamma^i : D^i \rightarrow M$ as

$$\gamma^i = \gamma^{i+1} \circ (\gamma_{\text{std}}^{i+1})^{-1} \circ \gamma_{\text{std}}^i.$$

By definition, $\gamma^i \prec \mathcal{A}(i+1, \delta)$ and for any $p \in D^i$,

$$d(\gamma_{\text{std}}^i(p), \gamma^i(p)) = d(\gamma_{\text{std}}^{i+1} \circ (\gamma_{\text{std}}^{i+1})^{-1} \circ \gamma_{\text{std}}^i(p), \gamma^{i+1} \circ (\gamma_{\text{std}}^{i+1})^{-1} \circ \gamma_{\text{std}}^i(p)) \leq \varepsilon.$$

We then get the second property. □

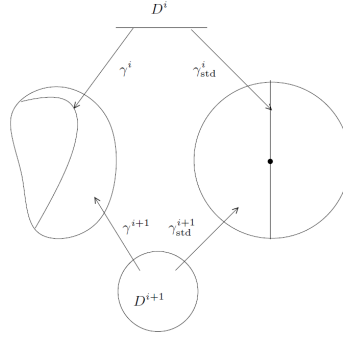


FIGURE 5.2 – Second property

5.5 Proof of Theorem E

In this section, we prove Theorem E. First we will introduce some lemmas.

5.5.1 Lyapunov norm and extension lemma

Given $\varepsilon > 0$ and an f -invariant subset Ω , a function $\phi : \Omega \rightarrow \mathbb{R}^+$ is called ε -tempered if for any $x \in \Omega$, $e^{-\varepsilon} \leq \phi(f(x))/\phi(x) \leq e^\varepsilon$.

We introduce the classical results about Lyapunov norm which can be found in [11] Proposition 4.

Lemma 5.5.1. *Let f be a $C^{1+\alpha}$ diffeomorphism on a compact manifold M . Given an ergodic measure μ and $\varepsilon > 0$, there are a measurable family of norms $\{||| \cdot |||_x\}_{x \in R_\mu}$ and an ε -tempered measurable function $A_\varepsilon \geq 1$ on R_μ such that for any $x \in R_\mu$ and any $1 \leq i \leq l$,*

- $e^{\lambda_i - \varepsilon} < |||Df|_{E_x^i}||| < e^{\lambda_i + \varepsilon}$;
- $||v|| \leq |||v||| \leq A_\varepsilon ||v||, \forall v \in T_x M$;
- $|||v_1 + \dots + v_l||| = \max\{|||v_1|||, \dots, |||v_l|||\}, \forall v_j \in E_x^j, 1 \leq j \leq l$.

The family of norms in Lemma 5.5.1 is called Lyapunov norm w.r.t. μ . From now on, when an ergodic measure μ is involved, for each ε , we always fix one of its Lyapunov norms and denote it by $\{||| \cdot |||_x\}_{x \in R_\mu}$.

By the continuity of f and the compactness, there is a small number $r > 0$ such that for any $x \in M$,

- $F_x \triangleq \exp_{f(x)}^{-1} \circ f \circ \exp_x|_{B(0,r)}$ is a diffeomorphism onto its image as smooth as f ;

- for any $y, z \in \mathcal{B}(x, r)$, we have

$$\frac{1}{2} \|\exp_x^{-1}(y) - \exp_x^{-1}(z)\| \leq d(y, z) \leq 2 \|\exp_x^{-1}(y) - \exp_x^{-1}(z)\|.$$

We use F to denote the induced (by these F_x) bundle map from TM to itself.

The next is an extension lemma from [11] Proposition 7.

Lemma 5.5.2. *Let f be a $C^{1+\alpha}$ diffeomorphism on a compact manifold M . Given an ergodic measure μ and $\varepsilon, \eta > 0$, there is a measurable map $\tilde{F} : T_{R_\mu} M \rightarrow T_{R_\mu} M$ with $\tilde{F}(T_x M) = T_{f(x)} M$ and an ε -tempered measurable function $C_\varepsilon \leq r$ on R_μ such that for any $x \in R_\mu$,*

- $\text{Lip}_{||\cdot||}(\tilde{F}_x - Df_x) < \eta$;
- $\tilde{F}_x(v) = F_x(v)$ for any $v \in T_x M$ with $||v|| \leq C_\varepsilon(x)$.

5.5.2 Stable manifold theorem in Banach spaces

Next we will introduce a version of stable (relatively) manifold theorem in Banach spaces.

For a vector space with a decomposition $E = E^1 \oplus E^2 \oplus \dots \oplus E^{k+1}$ in a Banach space E , we denote by π_j the respect projection w.r.t. that decomposition.

Definition 5.5.3. *Let f be a map on a Banach space E , a ρ -stable manifold is defined by*

$$\mathcal{W}(f, \rho) \triangleq \{v \in E \mid \sup_{n \in \mathbb{N}} \rho^{-n} \|f^n v\| < +\infty\}.$$

In this subsection, we prove the following key proposition which is a special version of stable manifold theorem in Banach spaces. It is the main ingredient to prove Theorem E.

Proposition 5.5.4. *Given $0 < \theta_1, \theta_2 < 1$ and an integer $k \geq 0$, there are two constants $K > 1$ and $0 < \varepsilon_0 < 1$ with the following property:*

Let E be a Banach space with a decomposition $E = E^1 \oplus E^2 \oplus \dots \oplus E^k \oplus E^{k+1}$ and a maximum norm defined by $|(v_1, v_2, \dots, v_{k+1})| = \max\{|v_1|, |v_2|, \dots, |v_{k+1}|\}$. Let T be a continuous invertible linear map on E with

- $T(E^j) = E^j, \forall 1 \leq j \leq k+1$;
- $\theta_1 \leq ||T_1|| \leq ||T|| \leq \theta_1^{-1}$ where $T_j \triangleq T|_{E^j}$;
- $\sup_j ||T_j|| / ||T_{j+1}^{-1}||^{-1} < \theta_2$.

For any $\varepsilon < \varepsilon_0$, let f be an Lipschitz map on E to itself with $f(0) = 0$ and

$$\text{Lip}(f - T) \leq \frac{\varepsilon}{2},$$

then there is a Lipschitz map $\Phi_k : \bigoplus_{j=1}^k E^j \rightarrow E$ with $\Phi_k(0) = 0$ such that

1. *For any $1 \leq j \leq k$,*

$$\mathcal{W}(f, ||T_j|| + \varepsilon) = \{\Phi_k(v_1, \dots, v_j, 0, \dots, 0) \mid v_i \in E^i, 1 \leq i \leq j\};$$

2. *$\text{Lip}(\Phi_k) \leq k$ and $\text{Lip}(\hat{\Phi}_k - \text{Id}) \leq K \cdot \text{Lip}(f - T)$ where $\hat{\Phi}_k = (\pi_1 \circ \Phi_k, \dots, \pi_k \circ \Phi_k)$ and Id is the identity map on $\bigoplus_{j=1}^k E^j$;*

3. For any $(v_1, \dots, v_k) \in \oplus_{j=1}^k E^j$,

$$\pi_j(\Phi_k(v_1, \dots, v_j, 0, \dots, 0) - \Phi_k(v_1, \dots, v_{j-1}, 0, \dots, 0)) = v_j, \forall 1 \leq j \leq k;$$

4. For any $(v_1, \dots, v_k) \in \oplus_{j=1}^k E^j$ and any $n \in \mathbb{N}$, there is some $(u_1, \dots, u_k) \in \oplus_{j=1}^k E^j$ with

$$\|u_j\| \leq (\|T_j\| + \varepsilon)^n \|v_j\| + K\varepsilon \sum_{i=j+1}^k (\|T_i\| + \varepsilon)^n \|v_i\|, 1 \leq j \leq k$$

such that

$$f^n(\Phi_k(v_1, \dots, v_k)) = \Phi_k(u_1, \dots, u_k).$$

Remark 5.5.5.

- The reason why we introduce these global parameters θ_1, θ_2 and K, ε_0 is that we want these constants to be independent of different choices of norms for Banach spaces. This origins from the fact that Lyapunov norm is usually not unique.
- Instead of considering only one stable manifold like Proposition 5.5.7 (in the following), Proposition 5.5.4 considers a family of nested stable manifolds

$$\mathcal{W}(f, \|T_1\| + \varepsilon) \subset \mathcal{W}(f, \|T_2\| + \varepsilon) \subset \dots \subset \mathcal{W}(f, \|T_k\| + \varepsilon).$$

- Note that here we use the images of maps (these Φ_k) to denote the stable manifolds, unlike the usual way (e.g., Lemma 5.5.6) where we use graphs of maps to denote the stable manifolds.

Let E be a Banach space with a decomposition $E = E^1 \oplus E^2$ and let $g : E^1 \rightarrow E^2$ be a map. The graph of g , denoted by $\text{Graph}(g)$, is defined as the set $\{(v, g(v)) \mid v \in E^1\}$.

The following is the classical stable manifold theorem in Banach space. It is from [11] Theorem A5.

Lemma 5.5.6. [11] Let E be a Banach space with a decomposition $E = E^1 \oplus E^2$ with a maximum norm defined by $\|(v_1, v_2)\| = \max\{\|v_1\|, \|v_2\|\}$. Let T be a continuous linear map on E with

- $T(E^1) \subset E^1, T(E^2) = E^2$ and T is invertible on E^2 ;
- $\|T_1\| < \rho < \|T_2^{-1}\|^{-1}$ for some number ρ where $T_i \triangleq T|_{E^i}$.

Then for any Lipschitz map f on E with $f(0) = 0$ and $\text{Lip}(f - T) < \min\{\rho - \|T_1\|, \|T_2^{-1}\|^{-1} - \rho\}$, there is a Lipschitz map $g : E^1 \rightarrow E^2$ such that

- $\text{Graph}(g) = \mathcal{W}(f, \rho)$;
- $\text{Lip}(f|_{\text{Graph}(g)}) \leq \|T_1\| + \text{Lip}(f - T)$;
- $\text{Lip}(g) < 1$.

The following is an extension of Lemma 5.5.6 above.

Lemma 5.5.7. Given $0 < \theta_1, \theta_2 < 1$, there is $0 < \varepsilon_0 < 1$ satisfying the following property:

Let E be a Banach space with a decomposition $E = E^1 \oplus E^2$ with a maximum norm defined by $\|(v_1, v_2)\| = \max\{\|v_1\|, \|v_2\|\}$. Let T be a continuous linear map on E with

- $T(E^1) \subset E^1, T(E^2) = E^2$ and T is invertible on E^2 ;
- $\theta_1 \leq \|T_1\| \leq \|T\| \leq \theta_1^{-1}$ where $T_i \triangleq T|_{E^i}$;
- $\|T_1\|/\|T_2^{-1}\|^{-1} \leq \theta_2$.

Then for any Lipschitz map f on E with $f(0) = 0$ and $\text{lip}(f - T) < \varepsilon_0$, there is a map $g : E^1 \rightarrow E^2$ such that

- $\text{Graph}(g) = \mathcal{W}(f, \|T_1\| + \varepsilon_0)$;
- $\text{Lip}(f|_{\text{Graph}(g)}) \leq \|T_1\| + \text{Lip}(f - T)$;
- $\text{Lip}(g) \leq \frac{\text{Lip}(f-T)}{\varepsilon_0} < 1$.

Proof. Let $\varepsilon_0 = \min\{\frac{1}{2}\theta_1(\theta_2^{-1} - 1), \frac{1}{2}\}$. One can show that this implies

$$\varepsilon_0 < \rho - \|T_1\| = \|T_2^{-1}\|^{-1} - \rho, \quad \text{where } \rho \triangleq \frac{\|T_1\| + \|T_2^{-1}\|^{-1}}{2}.$$

By Lemma 5.5.6, there is a Lipschitz map $g : E^1 \rightarrow E^2$ such that

- $\text{Graph}(g) = \mathcal{W}(f, \rho)$;
- $\text{Lip}(f|_{\text{Graph}(g)}) \leq \|T_1\| + \text{Lip}(f - T)$;
- $\text{Lip}(g) < 1$.

Note that the first two properties imply

$$\text{Graph}(g) = \mathcal{W}(f, \rho) = \mathcal{W}(f, \|T_1\| + \varepsilon_0).$$

Now it remains to show $\text{Lip}(g) \leq \alpha \triangleq \frac{\text{Lip}(f-T)}{\varepsilon_0}$.

Define a linear map $S : E \rightarrow E$ by $S(v_1, v_2) = (v_1, \alpha^{-1}v_2)$ and define $\tilde{f} = S^{-1} \circ f \circ S$. Note that $T = S^{-1} \circ T \circ S$ by definition. Let π_1, π_2 be the projections w.r.t. the decomposition $E = E^1 \oplus E^2$. By a direct computation, one can show that

$$\text{Lip}(\pi_1 \circ \tilde{f} - \pi_1 \circ T) \leq \alpha^{-1} \text{Lip}(\pi_1 \circ f - \pi_1 \circ T),$$

$$\text{Lip}(\pi_2 \circ \tilde{f} - \pi_2 \circ T) \leq \text{Lip}(\pi_2 \circ f - \pi_2 \circ T).$$

As a consequence,

$$\text{Lip}(\tilde{f} - T) \leq \alpha^{-1} \text{Lip}(f - T) < \varepsilon_0.$$

Hence applying Lemma 5.5.6 again for $\tilde{f}$, we get the corresponding $\tilde{g}$, one can show that

$$g = \alpha \cdot \tilde{g}.$$

We then have

$$\text{Lip}(g) = \alpha \text{Lip}(\tilde{g}) \leq \frac{\text{Lip}(f - T)}{\varepsilon_0}.$$

□

Remark 5.5.8.

- If $E^2 = 0$, then we can just define $g : E \rightarrow 0$ by $g(v) = 0$ for any $v \in E$. This makes Lemma 5.5.7 trivially true.

- The main difference between Lemma 5.5.7 and Lemma 5.5.6 is that Lemma 5.5.7 shows better Lipschitz constant for g (will be used later) while Lemma 5.5.6 only shows that $\text{Lip}(g) < 1$.

Now we give the proof of Proposition 5.5.4 by mainly using Proposition 5.5.7.

Proof of Proposition 5.5.4. For θ_1, θ_2 and $k \in \mathbb{N}$, by Lemma 5.5.7, we get a constant $0 < \varepsilon_0 < 1$. By shrinking ε_0 , we can also assume that $\varepsilon_0 < \theta_1 - \theta_1\theta_2$ which implies $\theta_2 + \frac{\varepsilon_0}{\theta_1} < 1$.

For the parameters θ_1 and $\theta_2 + \frac{\varepsilon_0}{\theta_1}$, by Lemma 5.8.1, we get a constant $K_2 > 1$. Let $K = \max\{(k-1)\varepsilon_0^{-1}, K_2\}$.

For any $\varepsilon < \varepsilon_0$, assume that f is a Lipschitz map on E with $f(0) = 0$ and

$$\text{Lip}(f - T) \leq \frac{\varepsilon}{2}.$$

For each $j \leq l$, by Lemma 5.5.7, there is a Lipschitz map $g_j : \oplus_{i=1}^j E^i \rightarrow \oplus_{i=j+1}^{k+1} E^i$ with $g_j(0) = 0$ such that

- $\text{Graph}(g_j) = \mathcal{W}(f, \|T_j\| + \varepsilon_0) = \mathcal{W}(f, \|T_j\| + \varepsilon)$.
This is a consequence of $\text{Lip}(f|_{\text{Graph}(g_j)}) \leq \|T_j\| + \frac{\varepsilon}{2}$.
- $\text{Lip}(g_j) \leq \frac{\text{Lip}(f-T)}{\varepsilon_0} < 1$.

Define $\phi_j : \oplus_{i=1}^j E^i \rightarrow E^{j+1}$ as

$$\phi_j(v_1, v_2, \dots, v_j) \triangleq \pi_{j+1} \circ g_j(v_1, v_2, \dots, v_j).$$

By definition, we have $\text{Lip}(\phi_j) < \frac{\text{Lip}(f-T)}{\varepsilon_0} < 1$. Now define $\Phi_k : \oplus_{j=1}^k E^j \rightarrow E$ as

$$\Phi_k(v_1, v_2, \dots, v_k) \triangleq (\alpha_1(v_1), \alpha_2(v_1, v_2), \alpha_3(v_1, v_2, v_3), \dots, \alpha_k(v_1, v_2, \dots, v_k), g_k(\alpha_1, \alpha_2, \dots, \alpha_k))$$

where $\{\alpha_i\}_{i \leq k}$ are defined inductively:

- $\alpha_1(v_1) = v_1$;
- For $i \leq k-1$, if $\alpha_i(v_1, \dots, v_i)$ is defined, then define $\alpha_{i+1}(v_1, \dots, v_i, v_{i+1}) = v_{i+1} + \phi_i(\alpha_1(v_1), \dots, \alpha_i(v_1, \dots, v_i))$.

One can check that Φ_k is continuous and $\Phi_k(0) = 0$. Moreover, for any $1 \leq j \leq k$, we have the following properties of Φ_k :

1. $\{\Phi_k(v_1, \dots, v_j, 0, \dots, 0) \mid v_i \in E^i, 1 \leq i \leq j\} = \text{Graph}(g_j) = \mathcal{W}(f, \|T_j\| + \varepsilon)$;

This is a direct consequence of the definition. We also have that for any $v_1, v_2, \dots, v_j$ with $v_i \in E^i, 1 \leq i \leq j$, there are $u_1, u_2, \dots, u_j$ with $u_i \in E^i, 1 \leq i \leq j$ such that

$$f(\Phi_k(v_1, v_2, \dots, v_j, 0, \dots, 0)) = \Phi_k(u_1, u_2, \dots, u_j, 0, \dots, 0).$$

2. $\text{Lip}(\Phi_k) \leq k$ and $\text{Lip}(\hat{\Phi}_k - Id) \leq K \cdot \text{Lip}(f - T)$;

The later inequality is from the following estimation:

$$\begin{aligned}
& \|(\hat{\Phi}_k - Id)(v_1, v_2, \dots, v_k) - (\hat{\Phi}_k - Id)(v'_1, v'_2, \dots, v'_k)\| \\
&= \max_{1 \leq j \leq k-1} \{ \|\alpha_j(v_1, \dots, v_j) - \alpha_j(v'_1, \dots, v'_j)\| \} \\
&\leq \varepsilon_0^{-1} \text{Lip}(f - T) \max_{1 \leq j \leq k-1} \left\{ \sum_{i=1}^j \|v_i - v'_i\| \right\} \\
&= \varepsilon_0^{-1} \text{Lip}(f - T) \sum_{j=1}^{k-1} \|v_j - v'_j\| \\
&\leq (k-1) \varepsilon_0^{-1} \text{Lip}(f - T) \|(v_1, v_2, \dots, v_k) - (v'_1, v'_2, \dots, v'_k)\| \\
&\leq K \cdot \text{Lip}(f - T) \|(v_1, v_2, \dots, v_k) - (v'_1, v'_2, \dots, v'_k)\|.
\end{aligned}$$

Combined with the fact that $\text{Lip}(g_k) < 1$ and $\varepsilon_0^{-1} \text{Lip}(f - T) < 1$, we then also get $\text{Lip}(\Phi_k) \leq k$.

$$3. \pi_j(\Phi_k(v_1, \dots, v_j, 0, \dots, 0) - \Phi_k(v_1, \dots, v_{j-1}, 0, \dots, 0)) = v_j.$$

$$\|\Phi_k(v_1, \dots, v_k) - \Phi_k(v_1, \dots, v_{j-1}, 0, \dots, 0)\| \leq \sum_{i=j}^k \|v_i\|,$$

Indeed, let $\beta_i \in \oplus_{n=1}^i E^n$ denote the first i components of $\Phi_k(v_1, v_2, \dots, v_k)$ and $\beta'_i \in \oplus_{n=1}^i E^n$ denote the first i components of $\Phi_k(v_1, \dots, v_{j-1}, 0, 0, \dots, 0)$, then for $1 \leq k$, we have

$$\begin{aligned}
& \|\Phi_k(v_1, v_2, \dots, v_k) - \Phi_k(v_1, \dots, v_{j-1}, 0, 0, \dots, 0)\| \\
&= \|(0, \dots, 0, v_j, v_{j+1} + \phi_j(\beta_j) - \phi_j(\beta'_j), \dots, \\
&\quad v_k + \phi_{k-1}(\beta_{k-1}) - \phi_{k-1}(\beta'_{k-1}), g_k(\beta_k) - g_k(\beta'_k))\| \\
&= \|(0, \dots, 0, v_j, v_{j+1} + \phi_j(\beta_j) - \phi_j(\beta'_j), \dots, \\
&\quad v_k + \phi_{k-1}(\beta_{k-1}) - \phi_{k-1}(\beta'_{k-1}), 0)\| \text{ (since } \text{Lip}(g_k) < 1) \\
&\leq \sum_{i=j}^k \|v_i\| \text{ (since } \text{Lip}(\phi_i) < 1).
\end{aligned}$$

It remains to show the last(4th) property. To see this, we first show the formula w.r.t. one-step iteration:

Claim 5.5.8.1. For any $(v_1, \dots, v_k) \in \oplus_{j=1}^k E^j$, there is some $(u_1, \dots, u_k) \in \oplus_{j=1}^k E^j$ with

$$\|u_j\| \leq (\|T_j\| + \varepsilon) \|v_j\| + \varepsilon \sum_{i=j+1}^k \|v_i\|, \quad 1 \leq j \leq k$$

such that

$$f(\Phi_k(v_1, \dots, v_k)) = \Phi_k(u_1, \dots, u_k).$$

proof of Claim. Write f as

$$f = (f_1, f_2, \dots, f_l) = (h_1 + T_1, h_2 + T_2, \dots, h_l + T_l)$$

where $h_j : E \rightarrow E^j$ satisfies $\text{Lip}(h_j) \leq \varepsilon/2$ (since $\text{Lip}(f - T) < \varepsilon/2$).

We first show that for any $(v_1, \dots, v_k) \in \bigoplus_{j=1}^k E^j$ and $1 \leq j \leq k$,

$$\|f_j \circ \Phi_k(v_1, v_2, \dots, v_k) - f_j \circ \Phi_k(v_1, \dots, v_{j-1}, 0, \dots, 0)\| \leq \|T_j\| \cdot \|v_j\| + \frac{\varepsilon}{2} \sum_{i=j}^k \|v_i\|. \quad (5.1)$$

To see this, recall,

$$\begin{aligned} & \Phi_k(v_1, v_2, \dots, v_k) - \Phi_k(v_1, \dots, v_{j-1}, 0, \dots, 0) \\ &= (0, \dots, 0, v_j, v_{j+1} + \phi_j(\beta_j) - \phi_j(\beta'_j), \dots, v_k + \phi_{k-1}(\beta_{k-1}) - \phi_{k-1}(\beta'_{k-1}), g_k(\beta_k) - g_k(\beta'_k)) \end{aligned}$$

where these β_i and β'_i are defined as before. Then (5.1) follows from the computation:

$$\begin{aligned} & \|f_j \circ \Phi_k(v_1, v_2, \dots, v_k) - f_j \circ \Phi_k(v_1, \dots, v_{j-1}, 0, \dots, 0)\| \\ & \leq \|h_j \circ \Phi_k(v_1, v_2, \dots, v_k) - h_j \circ \Phi_k(v_1, \dots, v_{j-1}, 0, \dots, 0)\| \\ & \quad + \|T_j \circ (\Phi_k(v_1, v_2, \dots, v_k) - \Phi_k(v_1, \dots, v_{j-1}, 0, \dots, 0))\| \\ & \leq \|T_j\| \cdot \|v_j\| + \frac{\varepsilon}{2} \sum_{i=j}^k \|v_i\| \text{ (By the third property of } \Phi_k \text{)}. \end{aligned}$$

Similarly as above, let $\gamma_i \in \bigoplus_{n=1}^i E^n$ and $\gamma'_i \in \bigoplus_{n=1}^i E^n$ denote respectively the first i components of $f(\Phi_k(v_1, v_2, \dots, v_k)) \in \text{Image}(\Phi_k)$ and $f(\Phi_k(v_1, \dots, v_{j-1}, 0, \dots, 0)) \in \text{Image}(\Phi_k)$. First note that by the first property of Φ_k , there is $(w_1, \dots, w_{j-1}, 0, \dots, 0)$ such that

$$(\gamma'_1, \dots, \gamma'_{k+1}) = f(\Phi_k(v_1, \dots, v_{j-1}, 0, \dots, 0)) = \Phi_k(w_1, \dots, w_{j-1}, 0, \dots, 0).$$

Then by the definition of Φ_k , we have

$$\gamma'_j = f_j \circ \Phi_k(v_1, \dots, v_{j-1}, 0, \dots, 0) = 0 + \phi_{j-1}(\gamma'_{j-1}) = \phi_{j-1}(\gamma'_{j-1}).$$

For any $1 \leq j \leq k$, similarly with the proof of (5.1), one can show

$$\begin{aligned} \|\gamma_{j-1} - \gamma'_{j-1}\| &= \max_{1 \leq i \leq j-1} \{ \|(h_i + T_i)(\Phi_k(v_1, v_2, \dots, v_k)) \\ & \quad - (h_i + T_i)(\Phi_k(v_1, \dots, v_{j-1}, 0, \dots, 0))\| \} \\ &\leq \frac{\varepsilon}{2} \sum_{i=j}^k \|v_i\|. \end{aligned}$$

Since $\text{Lip}(\phi_{j-1}) < 1$, the above implies $\|\phi_{j-1}(\gamma_{j-1}) - \phi_{j-1}(\gamma'_{j-1})\| \leq \frac{\varepsilon}{2} \sum_{i=j}^k \|v_i\|$.

Then we have

$$\begin{aligned}
& \|f_j \circ \Phi_k(v_1, v_2, \dots, v_k) - \phi_{j-1}(\gamma_{j-1})\| \\
& \leq \|f_j \circ \Phi_k(v_1, v_2, \dots, v_k) - f_j \circ \Phi_k(v_1, \dots, v_{j-1}, 0, 0, \dots, 0)\| \\
& \quad + \|f_j \circ \Phi_k(v_1, \dots, v_{j-1}, 0, \dots, 0) - \phi_{j-1}(\gamma_{j-1})\| \\
& = \|f_j \circ \Phi_k(v_1, v_2, \dots, v_k) - f_j \circ \Phi_k(v_1, \dots, v_{j-1}, 0, \dots, 0)\| \\
& \quad + \|\phi_{j-1}(\gamma'_{j-1}) - \phi_{j-1}(\gamma_{j-1})\| \\
& \leq (\|T_j\| + \varepsilon)\|v_j\| + \varepsilon \sum_{i=j+1}^k \|v_i\|.
\end{aligned}$$

For each $1 \leq j \leq k$, define $u_j \triangleq f_j \circ \Phi_k(v_1, v_2, \dots, v_k) - \phi_{j-1}(\gamma_{j-1})$, then

$$\|u_j\| \leq (\|T_j\| + \varepsilon)\|v_j\| + \varepsilon \sum_{i=j+1}^k \|v_i\|$$

and by the definition of Φ_k , we have

$$f(\Phi_k(v_1, v_2, \dots, v_k)) = \Phi_k(u_1, u_2, \dots, u_k).$$

□

The 4th property is then a direct application of Lemma 5.8.1. To see this, set $\rho_j = \|T_j\| + \varepsilon$, $1 \leq j \leq k$. Since $\varepsilon_0 < \theta_1 - \theta_1\theta_2$, we have $\rho_1 \geq \theta_1$ and

$$\sup_j \frac{\rho_j}{\rho_{j+1}} < \theta_2 + \frac{\varepsilon_0}{\theta_1} < 1.$$

For $n = 0, 1, 2, \dots$, define $(u_n^1, u_n^2, \dots, u_n^k)$ to be such that

$$f^n(\Phi_k(v_1, \dots, v_k)) = \Phi_k(u_n^1, u_n^2, \dots, u_n^k).$$

By Claim 5.5.8.1, we have

$$\|u_n^j\| \leq \rho_j \|u_{n-1}^j\| + \varepsilon \sum_{i=j+1}^k \|u_{n-1}^i\|, \quad 1 \leq j \leq k.$$

We then get the conclusion by using 5.8.1.

□

5.5.3 Proof of Theorem E

Next we show the proof of Theorem E. We first show a key proposition.

Proposition 5.5.9. *Let f be a $C^{1+\alpha}$ diffeomorphism on a compact manifold M , then for any ergodic measure μ , there is a constant ε_0 such that for any positive number $\varepsilon < \varepsilon_0$ and any $k \leq l$, there are two section maps $\Gamma^k : R_\mu \rightarrow \Delta_k$ and $\phi : R_\mu \rightarrow \tilde{\Delta}_k$ such that*

- For any $x \in R_\mu$, any non-negative numbers $\eta_1, \dots, \eta_k < 1$ and $n \in \mathbb{N}$, if

$$\max_{1 \leq j \leq k} \{e^{i(\lambda_j + \varepsilon)} \eta_j + \varepsilon \sum_{m=j+1}^k e^{i(\lambda_m + \varepsilon)} \eta_m\} \leq 1, \quad \forall 0 \leq i \leq n,$$

then $f^n(\mathcal{W}_{\Gamma_x^k}(\eta_1, \dots, \eta_k))$ is contained in

$$\mathcal{W}_{\Gamma_{f^n(x)}^k}(e^{n(\lambda_1+\varepsilon)}\eta_1 + \varepsilon \sum_{j=2}^k e^{n(\lambda_j+\varepsilon)}\eta_j, e^{n(\lambda_2+\varepsilon)}\eta_2 + \varepsilon \sum_{j=3}^k e^{n(\lambda_j+\varepsilon)}\eta_j, \dots, e^{n(\lambda_k+\varepsilon)}\eta_k).$$

- For any $x \in R_\mu$ and any non-negative numbers $\eta_1, \dots, \eta_k < 1$,

$$\text{diam}(\mathcal{W}_{\Gamma_x^k}(\eta_1, \dots, \eta_k)) \leq \frac{1}{k} \max_{1 \leq j \leq k} \eta_j.$$

- There is a subset $\Omega_\varepsilon \subset R_\mu$ with $\mu(\Omega_\varepsilon) \geq 1 - \varepsilon$ such that

$$\text{Lip}(\hat{\Gamma}_x^k \circ \phi_x - \text{Id}) \leq \varepsilon, \quad \forall x \in \Omega_\varepsilon$$

where $\hat{\Gamma}_x^k = (\pi_1 \circ \Gamma_x^k, \dots, \pi_k \circ \Gamma_x^k)$ and Id is the identity map on $\oplus_{j=1}^k E_x^j$.

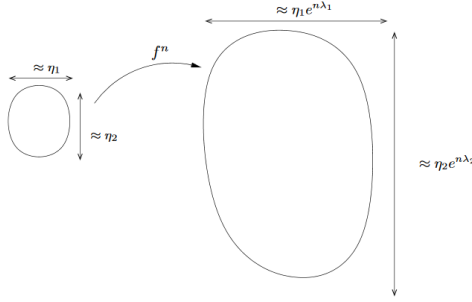


FIGURE 5.3 – Expansion rate

Let us first show that the proof of Theorem E by using Proposition 5.5.9.

Proof of Theorem E. The last properties in Theorem E and Proposition 5.5.9 are the same. We only have to show the first property in Theorem E.

For any $\tau \leq \frac{1}{k}$, any $n \in \mathbb{N}$ and $1 \leq j \leq k$, set $\alpha_j = \tau e^{-n(\lambda_j^+ + \varepsilon)}$, then

$$\max_{1 \leq j \leq k} \{e^{i(\lambda_j + \varepsilon)}\alpha_j + \varepsilon \sum_{m=j+1}^k e^{i(\lambda_m + \varepsilon)}\alpha_m\} \leq k\tau \leq 1, \quad \forall 0 \leq i \leq n.$$

Thus by Proposition 5.5.9, for any $0 \leq i \leq n$, we have

$$\begin{aligned} & \text{diam}(f^i(\mathcal{W}_{\Gamma_x^k}(\alpha_1, \dots, \alpha_k))) \\ & \leq \text{diam}(\mathcal{W}_{\Gamma_{f^i(x)}^k}(e^{i(\lambda_1 + \varepsilon)}\alpha_1 + \varepsilon \sum_{j=2}^k e^{i(\lambda_j + \varepsilon)}\alpha_j, e^{i(\lambda_2 + \varepsilon)}\alpha_2 + \varepsilon \sum_{j=3}^k e^{i(\lambda_j + \varepsilon)}\alpha_j, \dots, e^{i(\lambda_k + \varepsilon)}\alpha_k)) \\ & \leq \frac{1}{k} \max_{1 \leq j \leq k} \{e^{i(\lambda_j + \varepsilon)}\alpha_j + \varepsilon \sum_{m=j+1}^k e^{i(\lambda_m + \varepsilon)}\alpha_m\} \\ & \leq \tau. \end{aligned}$$

□

Now Let us show the proof of Proposition 5.5.9. The main technique is to apply our stable manifold theorem in Banach space (Proposition 5.5.4) to the compact manifold M .

Proof of Proposition 5.5.9. For the ergodic measure μ and any $k \leq l$, let

$$\theta_1 = e^{-\lambda_l - 2\kappa}, \quad \theta_2 = \sup_{j \leq l-1} \frac{e^{\lambda_j + 2\kappa}}{e^{\lambda_{j+1} - 2\kappa}} < 1$$

where $\kappa = \frac{1}{10} \min_{j \leq l-1} \{\lambda_{j+1} - \lambda_j\}$. Thus by Lemma 5.5.4, we get two constants $K > 1$ and $0 < \varepsilon_0 < 1$, we may assume that $\varepsilon_0 < \kappa$.

For any $\varepsilon < \varepsilon_0$, by Lemma 5.5.1, we have the Lyapunov norm $||| \cdot |||$ w.r.t. ε . let B be the Banach space of bounded Borel sections of $T_{R_\mu} M$ w.r.t. the norm defined by $|||\sigma||| \triangleq \sup_{x \in R_\mu} \{|||\sigma(x)|||\}$. B can be decomposed as $B = B^1 \oplus B^2 \cdots \oplus B^l$ where $B^i \triangleq \{\sigma \in B \mid \sigma(x) \in E_x^i\}$. Define the induced map

$$G : B \rightarrow B, \quad G(\sigma)(x) = Df_{f^{-1}(x)}(\sigma(f^{-1}(x))).$$

We Note

- G is an invertible linear map preserving each B^j , i.e. $G(B^j) = B^j$, $\forall 1 \leq j \leq l$;
- $\theta_1 \leq |||G_j||| \leq |||G||| \leq \theta_1^{-1}$ since $\varepsilon < \kappa$;
- $\sup_j |||G_j||| / |||G_{j+1}||| < \theta_2$. Indeed, this is because by the first property in Lemma 5.5.1, we have

$$e^{\lambda_j - \kappa} < e^{\lambda_j - \varepsilon} \leq |||G_j||| \leq e^{\lambda_j + \varepsilon} < e^{\lambda_j + \kappa}.$$

Let $K_3 > 1$ be a number large enough such that $\mu(\Omega'_\varepsilon) \geq 1 - \varepsilon$ where

$$\Omega_\varepsilon \triangleq \{x \in R_\mu \mid A_\varepsilon(x) \leq K_3\}.$$

Let

$$\eta = \min\left\{\frac{\varepsilon}{2}, \frac{\varepsilon}{KK_3}\right\},$$

by Lemma 5.5.2, we get an extended map $\tilde{F} : T_{R_\mu} \rightarrow T_{R_\mu}$. Define

$$H : B \rightarrow B, \quad H(\sigma)(x) = \tilde{F}_{f^{-1}(x)}(\sigma(f^{-1}(x))).$$

By Lemma 5.5.2, $H(0) = 0$ and $\text{Lip}_{|||\cdot|||}(H - G) < \eta \leq \varepsilon/2$.

With these configuration, we can apply Proposition 5.5.4 to G and H to get a map $\Phi_k : \oplus_{j=1}^k B^j \rightarrow B$ with $\Phi_k(0) = 0$ such that

- For any $j \leq k$,

$$\{\sigma \in B \mid \sup_{n \in \mathbb{N}} (|||G_j||| + \varepsilon)^{-n} |||H^n \sigma||| < +\infty\} = \{\Phi_k(v_1, \dots, v_j, 0, \dots, 0) \mid v_i \in B^i, 1 \leq i \leq j\};$$

- $\text{Lip}_{|||\cdot|||}(\Phi_k) \leq k$ and $\text{Lip}_{|||\cdot|||}(\hat{\Phi}_k - Id) \leq K\eta \leq \frac{\varepsilon}{K_3}$ where $\hat{\Phi}_k = (\pi_1 \circ \Phi_k, \dots, \pi_k \circ \Phi_k)$ and Id is the identity map on $\oplus_{j=1}^k B^j$;

- For any $(\sigma_1, \dots, \sigma_k) \in \oplus_{j=1}^k B^j$,

$$\pi_j(\Phi_k(\sigma_1, \dots, \sigma_j, 0, \dots, 0) - \Phi_k(\sigma_1, \dots, \sigma_{j-1}, 0, \dots, 0)) = \sigma_j, \forall 1 \leq j \leq k; \quad (2)$$

- For any $(\sigma_1, \dots, \sigma_k) \in \oplus_{j=1}^k B^j$ and any $n \in \mathbb{N}$, there is some $(\tilde{\sigma}_1, \dots, \tilde{\sigma}_k) \in \oplus_{j=1}^k B^j$ with

$$|||\tilde{\sigma}_j||| \leq (|||G_j||| + \varepsilon)^n |||\sigma_j||| + K\varepsilon \sum_{i=j+1}^k (|||G_i||| + \varepsilon)^n |||\sigma_i|||, \quad 1 \leq j \leq k$$

such that

$$H^n(\Phi_k(\sigma_1, \dots, \sigma_k)) = \Phi_k(\tilde{\sigma}_1, \dots, \tilde{\sigma}_k). \quad (3)$$

Define a map $\tilde{\Gamma}^k : R_\mu \rightarrow \Delta_k$ as following: for any $\sigma \in \oplus_{j=1}^k B^j$ and any $x \in R_\mu$, $\tilde{\Gamma}^k(x) \triangleq \Phi_k(\sigma)(x)$. We use $\tilde{\Gamma}_x^k(\sigma(x))$ to denote $\tilde{\Gamma}^k(x)(\sigma(x))$. We first Note, by the continuity of Φ_k , $\tilde{\Gamma}_x^k$ is a continuous map from $\oplus_{j=1}^k E_x^j$ to $T_x M$.

Claim 5.5.9.1. $\tilde{\Gamma}^k$ is well defined, i.e., for any $\sigma, \hat{\sigma} \in \oplus_{j=1}^k B^j$ and any $x \in R_\mu$, if $\sigma(x) = \hat{\sigma}(x)$, then $\Phi_k(\sigma)(x) = \Phi_k(\hat{\sigma})(x)$.

Proof. Recall by Lemma 5.5.7, for any $j \leq k$, there is $g_j : \oplus_{i=1}^j B^i \rightarrow \oplus_{i=j+1}^l B^i$ such that $\text{Graph}(g_j)$ equals to the following set:

$$\{\sigma \in B \mid \sup_{n \in \mathbb{N}} (|||G_j||| + \varepsilon)^{-n} |||H^n \sigma||| < +\infty\} = \{\Phi_k(v_1, \dots, v_j, 0, \dots, 0) \mid v_i \in B^i, 1 \leq i \leq j\}.$$

As a consequence, for any $(\sigma_1, \dots, \sigma_{j-1}) \in \oplus_{i=1}^{j-1} B^i$,

$$\pi_j(\Phi_k(\sigma_1, \dots, \sigma_{j-1}, 0, \dots, 0)) = \pi_j(g_{j-1}(\alpha_1, \dots, \alpha_{j-1}))$$

where $\alpha_i = \pi_i(\Phi_k(\sigma_1, \dots, \sigma_{j-1}, 0, \dots, 0))$, $\forall 1 \leq i \leq j-1$. Hence by formula (2), we get that for any $(\sigma_1, \dots, \sigma_k) \in \oplus_{j=1}^k B^j$ and any $1 \leq j \leq k$,

$$\pi_j(\Phi_k(\sigma_1, \dots, \sigma_k)) = \sigma_j + \pi_j(\Phi_k(\sigma_1, \dots, \sigma_{j-1}, 0, \dots, 0)) = \sigma_j + \pi_j(g_{j-1}(\alpha_1, \dots, \alpha_{j-1})).$$

In particular, $\pi_1(\Phi_k(\sigma_1, 0, \dots, 0)) = \sigma_1$. Then by a direct inductive argument, to prove $\tilde{\Gamma}^k$ is well defined, it is sufficient to show that for any $j \leq k$, any $\sigma, \hat{\sigma} \in \oplus_{i=1}^j B^i$ and any $x \in R_\mu$, if $\sigma(x) = \hat{\sigma}(x)$, then $g_j(\sigma)(x) = g_j(\hat{\sigma})(x)$. To see this, define

$$\sigma_0(y) = \begin{cases} g_j(\hat{\sigma})(x) & y = x, \\ g_j(\sigma)(y) & y \neq x. \end{cases}$$

By definition, $\sigma_0 \in \oplus_{i=j+1}^l B^i$ and one can show that for any $n \in \mathbb{N}$,

$$|||H^n(\sigma, \sigma_0)||| \leq \max\{|||H^n(\sigma, g_j(\sigma))|||, |||H^n(\hat{\sigma}, g_j(\hat{\sigma}))|||\}.$$

Thus $(\sigma, \sigma_0) \in \text{Graph}(g_j)$ which implies $(\sigma, \sigma_0) = (\sigma, g_j(\sigma))$. Therefore $g_j(\sigma)(x) = g_j(\hat{\sigma})(x)$. □

Recall $F_x = \exp_{f(x)}^{-1} \circ f \circ \exp_x|_{B(0,r)}$ and $\tilde{F}_x$ is an extension of F_x according to Lemma 5.5.2. By Claim (1) above and the invariance of Φ_k under H , we have

$$\tilde{F}_x(\text{Image}(\tilde{\Gamma}_x^k)) \subset \text{Image}(\tilde{\Gamma}_{f(x)}^k), \forall x \in R_\mu. \quad (4)$$

By the formulas (3) and (4), for any $x \in R_\mu$, any $n \in \mathbb{N}$ and any $(v_1, \dots, v_k) \in \oplus_{j=1}^k E_x^j$, there is some $(u_1, \dots, u_k) \in \oplus_{j=1}^k E_{f^n(x)}^j$ with

$$|||u_j||| \leq (|||G_j||| + \varepsilon)^n |||v_j||| + K\varepsilon \sum_{m=j+1}^k (|||G_m||| + \varepsilon)^n |||v_m|||, 1 \leq j \leq k$$

such that

$$\tilde{F}_x^n(\tilde{\Gamma}_x^k(v_1, \dots, v_k)) = \tilde{\Gamma}_{f^n(x)}^k(u_1, \dots, u_k). \quad (5)$$

Claim 5.5.9.2. Let $C_1 = 2^{-1}k^{-2}C_\varepsilon$ where C_ε is from Lemma 5.5.2. For any $x \in R_\mu$, any $(v_1, \dots, v_k) \in \oplus_{j=1}^k E_x^j$ and any $n \in \mathbb{N}$, if

$$\max_{1 \leq j \leq k} \{ (|||G_j||| + \varepsilon)^i |||v_j||| + K\varepsilon \sum_{l=j+1}^k (|||G_l||| + \varepsilon)^i |||v_l||| \} \leq e^{-i\varepsilon}, \forall 0 \leq i \leq n,$$

then

$$F_x^n(\Gamma_x^k(v_1, \dots, v_k)) = \tilde{F}_x^n(\Gamma_x^k(v_1, \dots, v_k))$$

where $F_x^n = F_{f^{n-1}(x)} \circ \dots \circ F_x$ (similar to $\tilde{F}_x^n$) and Γ_x^k is defined as

$$\Gamma_x^k(v_1, \dots, v_k) \triangleq \tilde{\Gamma}_x^k(\Theta(x, v_1)v_1, \dots, \Theta(x, v_k)v_k) \quad \text{where} \quad \Theta(x, v) \triangleq C_1(x) \frac{|||v|||_x}{|||v|||_x}.$$

Proof. We will check this property by an induction. We first check the case when $i = 0$. For any $x \in R_\mu$, note $\text{Lip}_{|||\cdot|||}(\tilde{\Gamma}_x^k) \leq k$ (since $\text{Lip}_{|||\cdot|||}(\Phi_k) \leq k$), for any $(v_1, v_2, \dots, v_k) \in \oplus_{j=1}^k E_x^j$ with each $|||v_j||| \leq 1$, we have

$$\begin{aligned} |||\Gamma_x^k(v_1, v_2, \dots, v_k)||| &= |||\tilde{\Gamma}_x^k(\Theta(x, v_1)v_1, \dots, \Theta(x, v_k)v_k)||| \\ &\leq k \max_{1 \leq i \leq k} |||\Theta(x, v_i)v_i||| \quad (\text{Since } \tilde{\Gamma}_x^k(0) = 0) \\ &= kC_1(x) \max_{1 \leq i \leq k} |||v_i||| \\ &\leq C_\varepsilon(x). \end{aligned}$$

Thus by Lemma 5.5.2,

$$F_x(\Gamma_x^k(v_1, v_2, \dots, v_k)) = \tilde{F}_x(\Gamma_x^k(v_1, v_2, \dots, v_k)).$$

Now assume that for $i - 1$, we have

$$F_x^{i-1}(\Gamma_x^k(v_1, v_2, \dots, v_k)) = \tilde{F}_x^{i-1}(\Gamma_x^k(v_1, v_2, \dots, v_k)).$$

By formula (5), for the vector $(\Theta(x, v_1)v_1, \dots, \Theta(x, v_k)v_k)$, there is some $(u_1, \dots, u_k) \in \oplus_{j=1}^k E_{f^{i-1}(x)}^j$ with

$$|||u_j||| \leq (|||G_j||| + \varepsilon)^{i-1} \Theta(x, v_j) |||v_j||| + K\varepsilon \sum_{m=i+1}^k (|||G_m||| + \varepsilon)^{i-1} \Theta(x, v_m) |||v_m|||, \quad 1 \leq j \leq k$$

such that

$$\begin{aligned} F_x^{i-1}(\Gamma_x^k(v_1, \dots, v_k)) &= \tilde{F}_x^{i-1}(\Gamma_x^k(v_1, \dots, v_k)) \\ &= \tilde{F}_x^{i-1}(\tilde{\Gamma}_x^k(\Theta(x, v_1)v_1, \dots, \Theta(x, v_k)v_k)) \\ &= \tilde{\Gamma}_{f^{i-1}(x)}^k(u_1, \dots, u_k). \end{aligned}$$

Thus we have

$$F_x^i(\Gamma_x^k(v_1, \dots, v_k)) = \tilde{F}_x^i(\Gamma_x^k(v_1, \dots, v_k)).$$

Now we have

$$\begin{aligned} &|||\tilde{\Gamma}_{f^{i-1}(x)}^k(u_1, \dots, u_k)|||_{f^{i-1}(x)} \\ &\leq k \max_{1 \leq j \leq k} |||u_j||| \\ &\leq k \max_{1 \leq j \leq k} \{ (|||G_j||| + \varepsilon)^{i-1} \Theta(x, v_j) |||v_j||| + K\varepsilon \sum_{m=j+1}^k (|||G_m||| + \varepsilon)^{i-1} \Theta(x, v_m) |||v_m||| \} \\ &= kC_1(x) \max_{1 \leq j \leq k} \{ (|||G_j||| + \varepsilon)^{i-1} |||v_j||| + K\varepsilon \sum_{m=j+1}^k (|||G_m||| + \varepsilon)^{i-1} |||v_m||| \} \\ &\leq C_\varepsilon(x) e^{-(i-1)\varepsilon} \\ &\leq C_\varepsilon(f^{i-1}(x)). \end{aligned}$$

Thus we have

$$F_x^i(\Gamma_x^k(v_1, \dots, v_k)) = \tilde{F}_x^i(\Gamma_x^k(v_1, \dots, v_k)).$$

□

Now for any $x \in R_\mu$, any $\eta_1, \dots, \eta_k$ and any $n \in \mathbb{N}$, if

$$\max_{1 \leq j \leq k} \{ (|||G_j||| + \varepsilon)^i \eta_j + K\varepsilon \sum_{m=j+1}^k (|||G_m||| + \varepsilon)^i \eta_m \} \leq e^{-i\varepsilon}, \quad \forall 0 \leq i \leq n,$$

then for any $(v_1, v_2, \dots, v_k) \in \oplus_{j=1}^k E_x^j$ with each $||v_j|| \leq \eta_j$, by Claim (2), we have

$$\begin{aligned} &F_x^n(\Gamma_x^k(v_1, \dots, v_k)) \\ &= \tilde{F}_x^n(\Gamma_x^k(v_1, \dots, v_k)) \\ &= \tilde{F}_x^n(\tilde{\Gamma}_x^k(\Theta(x, v_1)v_1, \dots, \Theta(x, v_k)v_k)) \\ &= \tilde{\Gamma}_{f^n(x)}^k(u_1, \dots, u_k) \\ &= \Gamma_{f^n(x)}^k\left(\frac{u_1}{\Theta(f^n(x), u_1)}, \dots, \frac{u_k}{\Theta(f^n(x), u_k)}\right) \end{aligned}$$

where, by (5), $(u_1, \dots, u_k)$ is some vector in $\oplus_{j=1}^k E_{f^n(x)}^j$ satisfying

$$|||u_j||| \leq (|||G_j||| + \varepsilon)^n \Theta(x, v_j) |||v_j||| + K\varepsilon \sum_{m=j+1}^k (|||G_m||| + \varepsilon)^n \Theta(x, v_m) |||v_m|||, \quad 1 \leq j \leq k.$$

Thus for any $1 \leq j \leq k$, we have

$$\begin{aligned} \left\| \frac{u_j}{\Theta(f^n(x), u_j)} \right\| &= \frac{1}{C_1(f^n(x))} |||u_j||| \\ &\leq \frac{1}{C_1(f^n(x))} ((|||G_j||| + \varepsilon)^n \Theta(x, v_j) |||v_j||| \\ &\quad + K\varepsilon \sum_{m=j+1}^k (|||G_m||| + \varepsilon)^n \Theta(x, v_m) |||v_m|||) \\ &= \frac{1}{C_1(f^n(x))} ((|||G_j||| + \varepsilon)^n C_1(x) |||v_j||| \\ &\quad + K\varepsilon \sum_{m=j+1}^k (|||G_m||| + \varepsilon)^n C_1(x) |||v_m|||) \\ &\leq e^{n\varepsilon} ((|||G_j||| + \varepsilon)^n |||v_j||| + K\varepsilon \sum_{m=j+1}^k (|||G_m||| + \varepsilon)^n |||v_m|||). \end{aligned}$$

Recall by definition,

$$\mathcal{W}_{\Gamma_x^k}(\eta_1, \dots, \eta_k) \triangleq \exp_x(\{\Gamma_x^k(v_1, \dots, v_k) \in T_x M \mid v_j \in E_x^j, |||v_j||| < \eta_j, j \leq k\}),$$

thus we have proved that for any $x \in R_\mu$, any $\eta_1, \dots, \eta_k$ and any $n \in \mathbb{N}$, if

$$\max_{1 \leq j \leq k} \{ (|||G_j||| + \varepsilon)^i \eta_j + K\varepsilon \sum_{m=j+1}^k (|||G_m||| + \varepsilon)^i \eta_m \} \leq e^{-i\varepsilon}, \quad \forall 0 \leq i \leq n,$$

then

$$f^n(\mathcal{W}_{\Gamma_x^k}(\eta_1, \dots, \eta_k)) \subset \mathcal{W}_{\Gamma_{f^n(x)}^k}(c_1, c_2, \dots, c_k)$$

where

$$c_j = (e^\varepsilon (|||G_j||| + \varepsilon))^n \eta_1 + K\varepsilon \sum_{m=j+1}^k (e^\varepsilon (|||G_m||| + \varepsilon))^n \eta_m, \quad 1 \leq j \leq k.$$

This is almost the first property of Theorem E, we will next show the conclusion above actually implies the final formula.

Let $\tilde{K}$ be a number such that for any $1 \leq j \leq k$,

$$e^\varepsilon (|||G_j||| + \varepsilon) \leq e^{\lambda_j + \tilde{K}\varepsilon}, \quad \forall \varepsilon < \varepsilon_0.$$

Note this is possible because $|||G_j||| \leq e^{\lambda_j + \varepsilon}$ by Lemma 5.5.1. Replacing ε by $\hat{\varepsilon} = \varepsilon / (\tilde{K} + K)$, by the above result, we get:

for any $x \in R_\mu$, any $\eta_1, \dots, \eta_k$ and any $n \in \mathbb{N}$, if

$$\max_{1 \leq j \leq k} \{ (|||G_j||| + \hat{\varepsilon})^i \eta_j + K\hat{\varepsilon} \sum_{m=j+1}^k (|||G_m||| + \hat{\varepsilon})^i \eta_m \} \leq e^{-i\hat{\varepsilon}}, \forall 0 \leq i \leq n,$$

then

$$f^n(\mathcal{W}_{\Gamma_x^k}(\eta_1, \dots, \eta_k)) \subset \mathcal{W}_{\Gamma_{f^n(x)}^k}(c_1, c_2, \dots, c_k)$$

where

$$c_j = (e^{\hat{\varepsilon}}(|||G_j||| + \hat{\varepsilon}))^n \eta_j + K\hat{\varepsilon} \sum_{m=j+1}^k (e^{\hat{\varepsilon}}(|||G_m||| + \hat{\varepsilon}))^n \eta_m, 1 \leq j \leq k.$$

For any $0 \leq i \leq n$ and any $1 \leq j \leq k$, a direct computation shows

$$e^{i(\lambda_j + \varepsilon)} \eta_j + \varepsilon \sum_{m=j+1}^k e^{i(\lambda_m + \varepsilon)} \eta_m \leq 1 \implies (|||G_j||| + \hat{\varepsilon})^i \eta_j + K\hat{\varepsilon} \sum_{m=j+1}^k (|||G_m||| + \hat{\varepsilon})^i \eta_m \leq e^{-i\hat{\varepsilon}}$$

and

$$(e^{\hat{\varepsilon}}(|||G_j||| + \hat{\varepsilon}))^i \eta_j + K\hat{\varepsilon} \sum_{m=j+1}^k (e^{\hat{\varepsilon}}(|||G_m||| + \hat{\varepsilon}))^i \eta_m \leq e^{i(\lambda_j + \varepsilon)} \eta_j + \varepsilon \sum_{m=j+1}^k e^{i(\lambda_m + \varepsilon)} \eta_m.$$

Thus for any $x \in R_\mu$, any $\eta_1, \dots, \eta_k$ and any $n \in \mathbb{N}$, if

$$\max_{1 \leq j \leq k} \{ e^{i(\lambda_j + \varepsilon)} \eta_j + \varepsilon \sum_{m=j+1}^k e^{i(\lambda_m + \varepsilon)} \eta_m \} \leq 1, \forall 0 \leq i \leq n,$$

then $f^n(\mathcal{W}_{\Gamma_x^k}(\eta_1, \dots, \eta_k))$ is contained in

$$\mathcal{W}_{\Gamma_{f^n(x)}^k}(e^{n(\lambda_1 + \varepsilon)} \eta_1 + \varepsilon \sum_{j=2}^k e^{n(\lambda_j + \varepsilon)} \eta_j, e^{n(\lambda_2 + \varepsilon)} \eta_2 + \varepsilon \sum_{j=3}^k e^{n(\lambda_j + \varepsilon)} \eta_j, \dots, e^{n(\lambda_k + \varepsilon)} \eta_k).$$

For the second property, for any $v_1, v_2, \dots, v_k$ with each $v_j \in E_x^j$ and $\|v_j\| \leq \eta_j \leq 1$,

$$\begin{aligned} \|\Gamma_x^k(v_1, \dots, v_k)\| &\leq \|\Gamma_x^k(v_1, \dots, v_k)\| \\ &= \|\tilde{\Gamma}_x^k(\Theta(x, v_1)v_1, \dots, \Theta(x, v_k)v_k)\| \\ &\leq k \max_{1 \leq j \leq k} \|\Theta(x, v_j)\| \|v_j\| \\ &= kC_1(x) \max_{1 \leq j \leq k} \|v_j\| \\ &\leq \frac{C_\varepsilon(x)}{2k} \max_{1 \leq j \leq k} \eta_j. \end{aligned}$$

Thus

$$\text{diam}(\mathcal{W}_{\Gamma_x^k}(\eta_1, \dots, \eta_k)) \leq \frac{C_\varepsilon(x)}{k} \max_{1 \leq j \leq k} \eta_j \leq \frac{1}{k} \max_{1 \leq j \leq k} \eta_j.$$

To prove the third property, for $x \in R_\mu$ and any $v = (v_1, \dots, v_k)$, define

$$\phi_x(v_1, \dots, v_k) \triangleq (\Theta^{-1}(x, v_1)v_1, \dots, \Theta^{-1}(x, v_k)v_k).$$

By definition, $\phi_x \in \tilde{\Delta}_k(x)$, hence we get the section map $\phi : R_\mu \rightarrow \tilde{\Delta}_k$.

Recall $\text{Lip}_{||\cdot||,||\cdot||}(\hat{\Phi}_k - Id) \leq \varepsilon$. Let $\hat{\Gamma}_x^k = (\pi_1 \circ \Gamma_x^k, \dots, \pi_k \circ \Gamma_x^k)$ and Id be the identity map on $\bigoplus_{j=1}^k E_x^j$. Note $\hat{\Phi}_k = \hat{\Gamma}_x^k \circ \phi$, hence for any $x \in \Omega_\varepsilon$, under the original Riemannian norm, we have

$$\text{Lip}(\hat{\Gamma}_x^k \circ \phi_x - Id) \leq \varepsilon.$$

□

5.6 An estimation on cardinality

In this section, we will construct a very general result that will be used to prove Theorem A.

Let $E = E^1 \oplus E^2 \cdots \oplus E^k \oplus E^{k+1}$ be a finite-dimensional Hilbert space. Given $1 \leq j \leq k$, for positive numbers $\eta_j, \eta_{j+1} \cdots, \eta_k$, define

$$[\eta_j, \eta_{j+1}, \dots, \eta_k] \triangleq \{(v_j, v_{j+1}, \dots, v_k) \in E^j \oplus E^{j+1} \cdots \oplus E^k \mid \|v_i\| \leq \eta_i, j \leq i \leq k\}.$$

Given $\delta > 0$, let $B(v, \delta)$ be the open ball at $v \in E$ with radius r . For a subspace H , let $B^H(v, \delta) = \{v + w \mid w \in H, \|w\| \leq \delta\}$. We abbreviate $B^{E^j}(v, \delta)$ to $B^j(v, \delta)$.

We first give the definition of ample family in Hilbert spaces, similar to the definition on manifold.

Recall that D^i is the open unite disk in $\mathbb{R}^i$. An $(i\text{-dimensional}) C^r$ ($r \geq 0$) submanifold γ is defined as a C^r embedding $\gamma : D^i \rightarrow E$. Given $\delta > 0$ and $1 \leq i \leq \dim E$, we define a standard family $\mathcal{A}^{\text{std}}(i, \delta)$ (For simplicity, we use the same notation as in the case of manifold) of i -dimensional C^0 submanifolds of E ,

$$\mathcal{A}^{\text{std}}(i, \delta) \triangleq \{\delta \cdot \mathcal{L}|_{D^i} + v \mid \|v\| \leq \delta, v \in E, \mathcal{L} : \mathbb{R}^i \rightarrow E \text{ is an linear isometry}\}.$$

We can similarly define the notation $\prec$ as in the paragraph above Definition 5.1.1.

Definition 5.6.1. Given $\delta > 0$ and $1 \leq i \leq \dim E$, we say a collection of C^0 submanifolds $\mathcal{A}(i, \delta)$ is an (i, δ) ample family if for any $\gamma_{\text{std}} \in \mathcal{A}^{\text{std}}(i, \delta)$ and $\varepsilon > 0$, there is $\gamma \prec \mathcal{A}(i, \delta)$ such that $d_{C^0}(\gamma_{\text{std}}, \gamma) \leq \varepsilon$ where d_{C^0} is the C^0 distance induced by the inner product in E .

Definition 5.6.2. Let $E = E^1 \oplus E^2 \cdots \oplus E^k \oplus E^{k+1}$ be a finite-dimensional Hilbert space. Given $\delta > 0, 0 < \theta < 1$ and some positive numbers $\eta_1, \eta_2, \dots, \eta_k \ll \delta$, a nearly rectangular box is a continuous map $\gamma : [\eta_1, \eta_2, \dots, \eta_k] \rightarrow B(0, \delta)$ such that for any $v, w \in [\eta_1, \dots, \eta_k]$ and any $1 \leq j \leq k$,

$$|\|\pi_j(\gamma(v) - \gamma(w))\| - \|\pi_j(v - w)\|| \leq \theta \sum_{i=1}^k \|\pi_i(v - w)\|.$$

The set of all such γ is denoted by $\mathcal{B}(\theta, \delta, \eta_1, \eta_2, \dots, \eta_k)$.

Remark 5.6.3. Denote $\hat{\gamma} = \pi_{\leq k} \circ \gamma$ where $\pi_{\leq k} = \pi_1 + \dots + \pi_k$ is the projection on $E^1 \oplus E^2 \cdots \oplus E^k$. Then $\mathcal{B}(\theta, \delta, \eta_1, \eta_2, \dots, \eta_k)$ is the collection of γ such that $\hat{\gamma}$ is close (indicated by θ) to the identity map on $E^1 \oplus E^2 \cdots \oplus E^k$.

Next is the main result in this section. We will use it to find how many submanifolds in an ample family that are necessary to intersect all nearly rectangular boxes.

Proposition 5.6.4 (Geometric estimate). Let $E = E^1 \oplus E^2 \dots \oplus E^k \oplus E^{k+1}$ be a finite-dimensional Hilbert space and let $i = \dim(E^{k+1})$. Given $\delta < 1$ and $0 < \theta < 2^{-k}(k+1)^{-1}$, there is a constant L such that for any positive numbers $\alpha_1, \alpha_2, \dots, \alpha_k < 1$ and any $(i, 10\delta)$ -ample family $\mathcal{A}(i, 10\delta)$, there is a subset $\Omega \subset \mathcal{A}(i, 10\delta)$ with the following properties:

- $\#\Omega \leq L \prod_{j=1}^k \alpha_j^{-\dim(E^j)}$;
- For any $\gamma \in \mathcal{B}(\theta, \delta, \alpha_1, \alpha_2, \dots, \alpha_k)$, there is $\gamma' \in \Omega$ such that $\text{Image}(\gamma) \cap \text{Image}(\gamma') \neq \emptyset$.

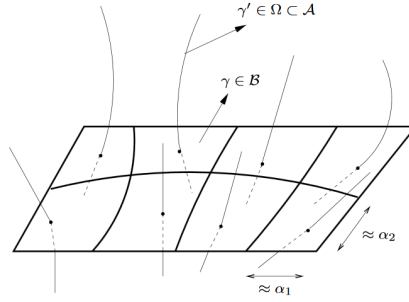


FIGURE 5.4 – Geometric estimate

We first make some preparations before the proof.

Lemma 5.6.5. Let E be a finite-dimensional Hilbert space. Given two positive numbers $r, \theta > 0$, assume that $\gamma : B^E(0, r) \rightarrow E$ is an injective continuous map satisfying

$$\|\gamma(v) - \gamma(0)\| \geq \theta \|v\|, \forall v \in B(0, r).$$

Then $B(\gamma(0), \frac{\theta r}{2}) \subset \text{Image}(\gamma)$.

Proof. Since γ is an injective continuous map, it is a homomorphism onto its image by Brouwer Invariance of Domain Theorem. As a consequence, $\gamma(\partial(B(0, \frac{2r}{3}))) = \partial(\gamma(B(0, \frac{2r}{3})))$ where $\partial(A)$ is boundary of a subset A w.r.t. the subspace topology. Thus

$$\|v - \gamma(0)\| \geq \frac{2\theta}{3} \|v\| = \frac{2\theta r}{3}, \forall v \in \partial(B(0, r)).$$

Next we show the subset $\gamma(B(0, r)) \cap B(\gamma(0), \frac{\theta r}{2})$ is a closed subset in $B(\gamma(0), \frac{\theta r}{2})$ w.r.t. induced topology of $B(\gamma(0), \frac{\theta r}{2})$. Let $\{v_n \in \gamma(B(0, r)) \cap B(\gamma(0), \frac{\theta r}{2})\}_{n \in \mathbb{N}}$ be a sequence of vectors with $v_n \rightarrow v$ for some $v \in B(\gamma(0), \frac{\theta r}{2})$. If $v \notin \gamma(B(0, r))$, then $v \in \partial(B(0, r))$ which contradicts the above inequality. Thus $v \in \gamma(B(0, r)) \cap B(\gamma(0), \frac{\theta r}{2})$. Since γ is an open map, $\gamma(B(0, r)) \cap B(\gamma(0), \frac{\theta r}{2})$ is also open w.r.t. induced topology of $B(\gamma(0), \frac{\theta r}{2})$. Since $B(\gamma(0), \frac{\theta r}{2})$ is connected, $\gamma(B(0, r)) \cap B(\gamma(0), \frac{\theta r}{2}) = B(\gamma(0), \frac{\theta r}{2})$. $\square$

The following result shows that the intersection of a nearly rectangular box with a plane defined by the first coordinate is again a nearly rectangular box. This can be viewed as some version of Implicit Function Theorem.

Lemma 5.6.6 (Implicit Function Theorem). Let $E = E^1 \oplus E^2 \dots \oplus E^k \oplus E^{k+1}$ be a finite-dimensional Hilbert space. For any $\theta < \frac{1}{k+1}$, any $1 > \eta_1 \geq \eta_2 \geq \dots \geq \eta_k > 0$

and any continuous map $\gamma : [\eta_1, \eta_2, \dots, \eta_k] \rightarrow E$ with the property that for any $v, w \in [\eta_1, \dots, \eta_k]$ and any $1 \leq j \leq k$,

$$| \|\pi_j(\gamma(v) - \gamma(w))\| - \|\pi_j(v - w)\| | \leq \theta \sum_{i=1}^k \|\pi_i(v - w)\|.$$

Then there is $\omega \in E^1$ with $B^1(\omega, \frac{\theta\eta_1}{10}) \subset \pi_1(\gamma([\eta_1, \eta_2, \dots, \eta_k]))$ such that for any $y \in B^1(\omega, \frac{\theta\eta_1}{10})$, there is a unique map $\phi : [\frac{\theta\eta_2}{10}, \frac{\theta\eta_3}{10}, \dots, \frac{\theta\eta_k}{10}] \rightarrow B^1(0, \eta_1)$ such that

1. for any $w \in [\frac{\theta\eta_2}{10}, \frac{\theta\eta_3}{10}, \dots, \frac{\theta\eta_k}{10}]$,

$$\pi_1(\gamma(\phi(w), w)) = y;$$

2. define $\tilde{\gamma} : [\frac{\theta\eta_2}{10}, \frac{\theta\eta_3}{10}, \dots, \frac{\theta\eta_k}{10}] \rightarrow E^2 \oplus E^3 \oplus \dots \oplus E^k \oplus E^{k+1}$ by

$$\tilde{\gamma}(v) \triangleq (\pi_2(\gamma(\phi(v), v)), \pi_3(\gamma(\phi(v), v)), \dots, \pi_k(\gamma(\phi(v), v)), \pi_{k+1}(\gamma(\phi(v), v))).$$

Then $\tilde{\gamma}$ is a continuous map such that for any $w, v \in [\frac{\theta\eta_2}{10}, \frac{\theta\eta_3}{10}, \dots, \frac{\theta\eta_k}{10}]$ and any $2 \leq j \leq k$,

$$| \|\pi_j(\gamma(v) - \gamma(w))\| - \|\pi_j(v - w)\| | \leq 2\theta \sum_{i=2}^k \|\pi_i(v - w)\|.$$

Proof. For any $(v_2, \dots, v_k), (w_2, \dots, w_k) \in [\frac{\theta\eta_2}{10}, \frac{\theta\eta_3}{10}, \dots, \frac{\theta\eta_k}{10}]$, we first note the following two properties:

- $B^1(\pi_1(\gamma(0, w_2, w_3, \dots, w_k)), \frac{k\theta\eta_1}{2}) \subset \{\pi_1(\gamma(w_1, w_2, w_3, \dots, w_k)) \mid w_1 \in B^1(0, \eta_1)\}.$

To see this, note that the map

$$\gamma_1 \triangleq \pi_1 \circ \gamma(\cdot, w_2, \dots, w_k) : B^1(0, \eta_1) \rightarrow E^1$$

is an continuous injective map and we have

$$\|\gamma_1(u) - \gamma_1(0)\| \geq (1 - \theta)\|u\| \geq k\theta\|u\|, \forall u \in B^1(0, \eta_1).$$

Thus by Lemma 5.6.5, we have

$$B^1(\pi_1(\gamma(0, w_2, \dots, w_k)), \frac{k\theta\eta_1}{2}) \subset \{\pi_1(\gamma(w_1, w_2, w_3, \dots, w_k)) \mid w_1 \in B^1(0, \eta_1)\}.$$

•

$$\|\pi_1(\gamma(0, v_2, \dots, v_k)) - \pi_1(\gamma(0, w_2, \dots, w_k))\| \leq \theta \sum_{j=2}^k \|v_j - w_j\| \leq 2\theta^2 \sum_{j=2}^k \frac{\eta_j}{10} \leq \frac{k\theta\eta_2}{5}.$$

Hence, there is $\omega \in \pi_1(\gamma([\eta_1, \eta_2, \dots, \eta_k]))$ such that for any $(w_2, w_3, \dots, w_k) \in [\frac{\theta\eta_2}{10}, \frac{\theta\eta_3}{10}, \dots, \frac{\theta\eta_k}{10}]$,

$$B^1(\omega, \frac{\theta\eta_1}{10}) \subset B^1(\omega, \frac{k\theta\eta_1}{10}) \subset B^1(\pi_1(\gamma(0, w_2, \dots, w_k)), \frac{k\theta\eta_1}{2}).$$

And the last subset above is contained in $\{\pi_1(\gamma(w_1, w_2, \dots, w_k)) \mid w_1 \in B^1(0, \eta_1)\}.$

In other words, for any $y \in B^1(\omega, \frac{\theta\eta_1}{10})$ and any $(w_2, w_3, \dots, w_k) \in [\frac{\theta\eta_2}{10}, \frac{\theta\eta_3}{10}, \dots, \frac{\theta\eta_k}{10}]$, there is a $w_1 \in B^1(0, \eta_1)$ such that

$$\pi_1(\gamma(w_1, w_2, w_3, \dots, w_k)) = y.$$

Furthermore, this w_1 is unique, denoted by $\phi(w_2, w_3, \dots, w_k)$. Hence we got property 1.

We next show $\tilde{\gamma}$ satisfies Property 2. Let's first show

$$\|\phi(v_2, \dots, v_k) - \phi(w_2, \dots, w_k)\| \leq \sum_{j=2}^k \|v_j - w_j\|.$$

It is from the following fact:

$$\begin{aligned} & \|\phi(v_2, \dots, v_k) - \phi(w_2, \dots, w_k)\| - \theta \|\phi(v_2, \dots, v_k) - \phi(w_2, \dots, w_k)\| - \theta \sum_{j=2}^k \|v_j - w_j\| \\ & \leq \|\pi_1(\gamma(\phi(v_2, \dots, v_k), v_2, \dots, v_k)) - \pi_1(\gamma(\phi(w_2, \dots, w_k), w_2, \dots, w_k))\| \\ & = 0. \end{aligned}$$

By its definition, $\tilde{\gamma}$ is a continuous map. For any $2 \leq j \leq k$, a direct estimation shows

$$\begin{aligned} & \|\pi_j(\tilde{\gamma}(v_2, \dots, v_k)) - \pi_j(\tilde{\gamma}(w_2, \dots, w_k))\| \\ & = \|\pi_j(\gamma(\phi(v_2, \dots, v_k), v_2, \dots, v_k)) - \pi_j(\gamma(\phi(w_2, \dots, w_k), w_2, \dots, w_k))\| \\ & \geq \|v_j - w_j\| - \theta \|\phi(v_2, \dots, v_k) - \phi(w_2, \dots, w_k)\| - \theta \sum_{i=2}^k \|v_i - w_i\| \\ & \geq \|v_j - w_j\| - 2\theta \sum_{i=2}^k \|v_i - w_i\|. \end{aligned}$$

Similarly,

$$\begin{aligned} & \|\pi_j(\tilde{\gamma}(v_2, \dots, v_k)) - \pi_j(\tilde{\gamma}(w_2, \dots, w_k))\| \\ & = \|\pi_j(\gamma(\phi(v_2, \dots, v_k), v_2, \dots, v_k)) - \pi_j(\gamma(\phi(w_2, \dots, w_k), w_2, \dots, w_k))\| \\ & \leq \|v_j - w_j\| + \theta \|\phi(v_2, \dots, v_k) - \phi(w_2, \dots, w_k)\| + \theta \sum_{i=2}^k \|v_i - w_i\| \\ & \leq \|v_j - w_j\| + 2\theta \sum_{i=2}^k \|v_i - w_i\|. \end{aligned}$$

□

Remark 5.6.7. For the case $k = 1$, the map ϕ stated in Lemma 5.6.6 is defined on an empty set, so we always assume that Property 1 and Property 2 remains trivially true. In this situation, Lemma 5.6.6 only produces the existence of some $\omega \in E^1$ with $B^1(\omega, \frac{\theta\eta_1}{10}) \subset \pi_1(\gamma([\eta_1]))$ which is a direct consequence of Lemma 5.6.5.

Lemma 5.6.8 (Poincaré-Miranda Theorem [34]). If $f = (f_1, f_2, \dots, f_n) : [0, 1]^n \rightarrow \mathbb{R}^n$ is a continuous map such that for any $1 \leq j \leq n$ and any $v \triangleq (x_1, \dots, x_{j-1}, 1, x_{j+1}, \dots, x_n)$, $w \triangleq (x_1, \dots, x_{j-1}, -1, x_{j+1}, \dots, x_n) \in [0, 1]^n$,

$$f_j(v) > 0, \quad f_j(w) < 0,$$

then there is $x \in [0, 1]^n$ with $f(x) = 0$.

Lemma 5.6.9. *Let $E = F \oplus G$ be a finite-dimensional Hilbert space and let $i = \dim(G)$. There is $C_1 > 0$ satisfying the following property:*

Given any $0 < \varepsilon \leq \delta < 1$, any $y \in F$ and any continuous map $f : B^F(y, \varepsilon) \rightarrow G$ with $\text{Graph}(f) \subset B(0, \delta)$. Let $g : D^i \rightarrow E$ be a continuous map such that there is some linear isometry $\mathcal{L} : \mathbb{R}^i \rightarrow G$ such that

$$d_{C^0}(\chi, g) < C_1 \varepsilon$$

where $\chi : D^i \rightarrow E$ is defined by $\chi(v) = y + 10\delta\mathcal{L}(v)$. Then we have $\text{Graph}(f) \cap \text{Image}(g) \neq \emptyset$.

Remark 5.6.10. *Note that by definition, $\text{Graph}(f) \cap \text{Image}(\chi) = (y, f(y))$, so Lemma 5.6.9 just says that a submanifold C^0 -close to χ also has non-empty intersection with the image of f .*

Proof. Let C_1 be a constant such that for any $v = v_F + v_G$, we have $\|v_F\|, \|v_G\| \leq C_1^{-1}\|v\|$.

We fix any linear isometry $A : \mathbb{R}^{\dim E} \rightarrow E$ such that $A|_{\mathbb{R}^i} = \mathcal{L}$. Hence A can be viewed as an extension of $\mathcal{L}$. Note $\mathbb{R}^{\dim E} = A^{-1}(F) \oplus \mathbb{R}^i$.

Define

$$\tilde{f} \triangleq A^{-1} \circ f \circ A|_{A^{-1}(B^F(y, \varepsilon))} = A^{-1} \circ f \circ A|_{B^{A^{-1}(F)}(A^{-1}(y), \varepsilon)}, \quad \tilde{g} \triangleq A^{-1} \circ g, \quad \tilde{\chi} \triangleq A^{-1} \circ \chi.$$

To prove $\text{Graph}(f) \cap \text{Image}(g) \neq \emptyset$, it is sufficient to prove $\text{Graph}(\tilde{f}) \cap \text{Image}(\tilde{g}) \neq \emptyset$. We next show there are some $u \in B^{A^{-1}(F)}(A^{-1}(y), \varepsilon)$ and $v \in D^i$ such that

$$(u, \tilde{f}(u)) = \tilde{g}(v) = (\pi_{A^{-1}(F)}(\tilde{g}(v)), \pi_{\mathbb{R}^i}(\tilde{g}(v))). \quad (5.2)$$

Define a continuous map $\Psi : D^i \rightarrow \mathbb{R}^i$ by

$$\Psi(v) \triangleq \pi_{\mathbb{R}^i}(\tilde{g}(v)) - \tilde{f}(\pi_{A(F)}(\tilde{g}(v))).$$

Since $d_{C^0}(\chi, g) < C_1 \varepsilon$ and A is isometric, for any $v \in D^i$, we have

$$\begin{aligned} \|A^{-1}(y) - \pi_{A^{-1}(F)}(\tilde{g}(v))\| &= \|\pi_{A^{-1}(F)}(\tilde{\chi}(v)) - \pi_{A^{-1}(F)}(\tilde{g}(v))\| \\ &= \|\pi_F(\chi(v)) - \pi_F(g(v))\| \\ &\leq C_1^{-1}\|\chi(v) - g(v)\| < \varepsilon. \end{aligned}$$

This shows that Ψ is well defined on D^i . Hence to prove (5.2), it is sufficient to find some $v \in D^i$ such that $\Psi(v) = 0$. To see this, we first note that for any $v \in D^i$ and any $j \leq i$, since we are using the standard norm in $\mathbb{R}^i$,

$$\begin{aligned} |10r \cdot \pi_j(v) - \pi_j(\pi_{\mathbb{R}^i}(\tilde{g}(v)))| &= |\pi_j(\pi_{\mathbb{R}^i}(\tilde{\chi}(v))) - \pi_j(\pi_{\mathbb{R}^i}(\tilde{g}(v)))| \\ &\leq \|\pi_{\mathbb{R}^i}(\tilde{\chi}(v)) - \pi_{\mathbb{R}^i}(\tilde{g}(v))\| \\ &= \|\pi_G(\chi(v)) - \pi_G(g(v))\| \\ &\leq C_1^{-1}\|\chi(v) - g(v)\| \\ &< \varepsilon. \end{aligned}$$

Now consider any $j \leq i$ and any

$$v = (x_1, \dots, x_{j-1}, \frac{1}{2}, x_{j+1}, \dots, x_i), w = (x_1, \dots, x_{j-1}, -\frac{1}{2}, x_{j+1}, \dots, x_i) \in [-\frac{1}{2}, \frac{1}{2}]^i \subset D^i,$$

we have

$$\begin{aligned} \pi_j(\Psi(v)) &= \pi_j(\pi_{\mathbb{R}^i}(\tilde{g}(v))) - \pi_j(\tilde{f}(\pi_{A^{-1}(F)}(\tilde{g}(v)))) \\ &\geq 10\delta \cdot \pi_j(v) - \varepsilon - \pi_j(\tilde{f}(\pi_{A^{-1}(F)}(\tilde{g}(v)))) \\ &\geq 10\delta \cdot \pi_j(v) - \varepsilon - \varepsilon \\ &\geq 3\delta > 0. \end{aligned}$$

Similarly, $\pi_j(\Psi(w)) \leq -3\delta < 0$. By Poincaré-Miranda Theorem 5.6.8, there is some $v \in D^i$ such that $\Psi(v) = 0$ which completes the proof. $\square$

Proof of Proposition 5.6.4. Denote $[\eta_1, \eta_2, \dots, \eta_k] \triangleq [\frac{\alpha_1}{10}, \frac{\alpha_2}{10}, \dots, \frac{\alpha_k}{10}]$ and

$$C \triangleq \frac{1}{10} \times \frac{2^{\frac{k(k-1)}{2}} \theta^k}{10^k}.$$

We first note that there is a constant L such that for any $\eta_1, \eta_2, \dots, \eta_k < 1$ and $1 \leq j \leq k$, there is a subset $S^j \subset E^j$ with $\#S^j \leq L^{\frac{1}{k}} \eta_j^{-\dim(E^j)}$ which is $C\eta_j$ -dense in $B^j(0, 1)$.

Let $S = S^1 \times S^2 \times \dots \times S^k$. Thus $\#S \leq L \prod_{j=1}^k \eta_j^{-\dim(E^j)}$.

Claim 5.6.10.1. For any $\gamma \in \mathcal{B}(\theta, \delta, \eta_1, \eta_2, \dots, \eta_k)$, there is $y \in S$ such that

$$\gamma\left(\left[\frac{\eta_1}{10}, \frac{\eta_2}{10}, \dots, \frac{\eta_k}{10}\right]\right) \cap (y + E^{k+1}) \neq \emptyset.$$

Proof of the Claim. Without loss of generality, we assume that $\eta_1 \geq \eta_2 \geq \dots \geq \eta_k$. For $X \triangleq [\alpha_j, \alpha_{j+1}, \dots, \alpha_k] \subset E^j \oplus E^{j+1} \oplus \dots \oplus E^k$, we denote $P_i(X) \triangleq \alpha_{i+j-1}$, $1 \leq i \leq k-j+1$. To simplify the notations, for any $1 \leq n \leq k$, we inductively define:

$$Y_0 \triangleq [\eta_1, \eta_2, \dots, \eta_k], \quad Y_n \triangleq \left[\frac{2^{n-1}\theta}{10}P_2(Y_{n-1}), \frac{2^{n-1}\theta}{10}P_3(Y_{n-1}), \dots, \frac{2^{n-1}\theta}{10}P_{k-n+1}(Y_{n-1})\right],$$

Define

$$X_n \triangleq [P_1(Y_0), P_1(Y_1), \dots, P_1(Y_{n-1})].$$

Actually one can show that for $2 \leq i \leq k-n+1$,

$$P_i(Y_n) = \frac{2^{\frac{n(n-1)}{2}} \theta^n}{10^n} \eta_{n+i-1}.$$

We fix $\gamma \in \mathcal{B}(\theta, \eta_1, \eta_2, \dots, \eta_k)$. Next we will define an inductive process. Define the property $P(n)$, $1 \leq n \leq k-1$ in the following:

Property P(n):

For $1 \leq i \leq n$, there are $y_i \in \pi_i(\gamma([\eta_1, \eta_2, \dots, \eta_k])) \cap S^i$ and a map $\Phi_n : Y_n \rightarrow X_n$ such that

- for any $v \in Y_n$,

$$\pi_i(\gamma(\Phi_n(v), v)) = y_i, \quad \forall 1 \leq i \leq n;$$

- define $\gamma_n : Y_n \rightarrow E^{n+1} \oplus E^{n+2} \oplus \dots \oplus E^{k+1}$ by

$$\gamma_n(v) \triangleq (\pi_{n+1}(\gamma(\Phi_n(v), v)), \pi_{n+2}(\gamma(\Phi_n(v), v)), \dots, \pi_k(\gamma(\Phi_n(v), v)), \pi_{k+1}(\gamma(\Phi_n(v), v))).$$

Then γ_n is a continuous map such that for any $(w_{n+1}, w_{n+2}, \dots, w_k), (v_{n+1}, v_{n+2}, \dots, v_k) \in Y_n$ and any $n+1 \leq j \leq k$,

$$| |\pi_j(\gamma_n(v_{n+1}, \dots, v_k) - \gamma_n(w_{n+1}, \dots, w_k)) - \|v_j - w_j\|| | \leq 2^n \theta \sum_{i=n+1}^k \|v_i - w_i\|.$$

Note that Property $P(1)$ is a just consequence of Lemma 5.6.6. Now assuming Property $P(n)$, we next use Property $P(n)$ to prove Property $P(n+1)$.

With the configuration in Property $P(n)$, we apply Lemma 5.6.6 to $\gamma_n : Y_n \rightarrow E^{n+1} \oplus E^{n+2} \oplus \dots \oplus E^{k+1}$ and the parameter $2^n \theta$. Then there is some ball $B^{n+1}(\omega, \frac{2^n \theta}{10} P_1(Y_n)) \subset \pi_{n+1}(\gamma([\eta_1, \eta_2, \dots, \eta_k]))$. by choosing some $y_{n+1} \in B^{n+1}(\omega, \frac{2^n \theta}{10} P_1(Y_n)) \cap S^{n+1}$, we get an associated map $\phi : Y_{n+1} \rightarrow B^{n+1}(0, P_1(Y_n))$ such that for any $v \in Y_{n+1}$,

$$\pi_{n+1}(\gamma_n(\phi(v), v)) = y_{n+1}.$$

We define $\Phi_{n+1} : Y_{n+1} \rightarrow X_{n+1}$ by

$$\Phi_{n+1}(v) \triangleq (\Phi_n(\phi(v), v), \phi(v)).$$

By definition, since $P_{n+1}(X_{n+1}) = P_1(Y_n)$, hence $\Phi_{n+1}(v) \in X_{n+1}$.

For any $v \in Y_{n+1}$ and any $1 \leq i \leq n$, by the induction above (Property $P(n)$), we have

$$\pi_i(\gamma(\Phi_{n+1}(v), v)) = \pi_i(\gamma(\Phi_n(\phi(v), v), \phi(v), v)) = y_i.$$

For the case $i = n+1$, we have

$$\pi_{n+1}(\gamma(\Phi_{n+1}(v), v)) = \pi_{n+1}(\gamma(\Phi_n(\phi(v), v), \phi(v), v)) = \pi_{n+1}(\gamma_n(\phi(v), v)) = y_{n+1}.$$

Then by a direct restatement of Property 2 in Lemma 5.6.6, we get Property $P(n+1)$.

By this induction process, we eventually reach the case when $n = k-1$, by the Remark 5.6.7, we apply Lemma 5.6.6 once again and then we find some $y_k \in \pi_k(\gamma([\eta_1, \eta_2, \dots, \eta_k])) \cap S^k$ and $w_k \in Y_{k-1}$ such that $\pi_j(\gamma(w)) = y_j$, $1 \leq j \leq k$ where $w = (\Phi_{k-1}(w_k), w_k) \in [\eta_1, \eta_2, \dots, \eta_k] = [\frac{\alpha_1}{10}, \frac{\alpha_2}{10}, \dots, \frac{\alpha_k}{10}]$.

Let $y = (y_1, y_2, \dots, y_k)$, we finish the proof of the claim. □

Fix $y \in S$ in Claim 5.6.10.1. Next we will prepare some configuration to apply Lemma 5.6.9.

Let C_0 be a constant such that for any $(v_1, v_2, \dots, v_k) \in E^1 \oplus E^2 \oplus \dots \oplus E^k$,

$$C_0^{-1} \sum_{j=1}^k \|v_j\| \leq \|(v_1, v_2, \dots, v_k)\| \leq C_0 \sum_{j=1}^k \|v_j\|.$$

Denote $\tilde{\gamma} \triangleq (\pi_1 \circ \gamma, \pi_2 \circ \gamma, \dots, \pi_k \circ \gamma)$. Since $\gamma \in \mathcal{B}(\theta, \alpha_1, \alpha_2, \dots, \alpha_k)$ and $\theta < \frac{1}{k+1}$, for any $(v_1, \dots, v_k), (w_1, \dots, w_k) \in [\alpha_1, \alpha_2, \dots, \alpha_k]$, we have

$$\begin{aligned} \|\tilde{\gamma}(v_1, \dots, v_k) - \tilde{\gamma}(w_1, \dots, w_k)\| &\geq C_0^{-1} \sum_{j=1}^k \|\pi_j(\gamma(v_1, \dots, v_k)) - \pi_j(\gamma(w_1, \dots, w_k))\| \\ &\geq C_0^{-1} \theta \sum_{j=1}^k \|v_j - w_j\| \\ &\geq C_0^{-1} \theta \|(v_1, \dots, v_k) - (w_1, \dots, w_k)\|. \end{aligned}$$

Together with the fact that for $w \in [\frac{\alpha_1}{10}, \frac{\alpha_2}{10}, \dots, \frac{\alpha_k}{10}]$, there is some $\varepsilon < \delta$ independent of $\gamma \in \mathcal{B}(\theta, \varepsilon, \alpha_1, \alpha_2, \dots, \alpha_k)$ such that $B^F(w, \varepsilon) \subset \text{Image}(\tilde{\gamma})$ where $F = E^1 \oplus E^2 \dots \oplus E^k$. Let $G = E^{k+1}$ and define $f : B^F(y, \varepsilon) \rightarrow G$ as $f(v) = \pi_{k+1}(\gamma(\tilde{\gamma}^{-1}(v)))$ (note that $\tilde{\gamma}$ is invertible). Since $\mathcal{A}(i, 10\delta)$ is ample, we can find some $\gamma' \in \mathcal{A}(i, 10\delta)$ such that $d_{C^0}(B^G(y, 10\delta), \gamma') \leq C_1 \varepsilon$ where C_1 is from Lemma 5.6.9. Hence by Lemma 5.6.9, we get that $\text{Graph}(f) \cap \text{Image}(\gamma') \neq \emptyset$. Note $\text{Graph}(f) \subset \text{Image}(\gamma)$, we then have $\text{Image}(\gamma) \cap \text{Image}(\gamma') \neq \emptyset$. For each $y \in S$, collect its corresponding $\gamma' \in \mathcal{A}(i, 10\delta)$ into Ω , we complete the proof. $\square$

5.7 Proof of the Theorem A

Next we prove Theorem A:

Proof of Theorem A. For the ergodic measure μ , by Theorem E, we get a constant $\varepsilon_0 < 1$, a section map $\Gamma^k : R_\mu \rightarrow \Delta_k$, a subset Ω_ε with $\mu(\Omega_\varepsilon) \geq 1 - \varepsilon$ and a family of maps $\{\phi_x\}_{x \in \Omega_\varepsilon}$. We choose $C > 1$ large enough and a subset $\Omega'_\varepsilon \subset \Omega_\varepsilon$ with $\mu(\Omega'_\varepsilon) > 0$ such that for any $x \in \Omega'_\varepsilon$,

$$\sup_{v \in \oplus_{j=1}^k E_x^j} \frac{\|\pi_j(\phi_x(v))\|}{\|\pi_j(v)\|} < C, \quad 1 \leq j \leq k.$$

By Lusin's Lemma 5.8.2, we can find a compact subset $\Lambda \subset \Omega'_\varepsilon$ with $\mu(\Lambda) > 0$ such that the Oseledec's splitting $\{T_x M = E_x^1 \oplus E_x^2 \oplus \dots \oplus E_x^l\}_{x \in \Lambda}$ is continuous on Λ .

By Proposition 5.4.3, we have

$$h(f, \mu) \leq \lim_{\tau \rightarrow 0} \limsup_{n \rightarrow +\infty} \frac{1}{n} \log \#E(n, \tau, \Lambda)$$

where $E(n, \tau, \Lambda)$ is a maximal (n, τ) separated subset of Λ . Let $0 < \tau, \delta < \frac{r}{100} \ll \frac{1}{k}$. Since Λ is compact, there are finitely many balls with radius δ that covers Λ . Thus we can choose some point $y \in \Lambda$ such that

$$\limsup_{n \rightarrow +\infty} \frac{1}{n} \log \#E(n, \tau, \Lambda) = \limsup_{n \rightarrow +\infty} \frac{1}{n} \log \#(E(n, \tau, \Lambda) \cap B(y, \delta)).$$

On the other hand, to apply Lemma 5.6.4, we next prepare the configuration. Consider the Banach space $T_y M = E_y^1 \oplus E_y^2 \oplus \dots \oplus E_y^k \oplus (\oplus_{i=k+1}^l E_y^i)$. Since the Oseledec's splitting $\{T_x M = E_x^1 \oplus E_x^2 \oplus \dots \oplus E_x^l\}_{x \in \Lambda}$ is continuous on Λ , by shrinking δ , we assume that for any $x \in B(y, \delta) \cap \Lambda$,

- there is an isomorphism $\{L_x : T_y M \rightarrow T_x M\}$ with $1 - \varepsilon \leq \text{Lip}(L_x) \leq 1 + \varepsilon$ which maps E_y^j to E_x^j for $1 \leq j \leq l$.
- $\exp_y^{-1} \circ \exp_x = L_x^{-1} + \psi_x$ where $\psi_x : T_y M \rightarrow T_x M$ is some map with $\text{Lip}(\psi_x) \leq 2\varepsilon$.

Note for any $\eta_1, \dots, \eta_k \leq \delta$ and any $x \in B(y, \delta) \cap \Lambda$,

$$\mathcal{W}_{\Gamma_x^k}(\eta_1, \dots, \eta_k) \subset B(y, 2\delta).$$

Define $\tilde{\Gamma}_x^k = \exp_y^{-1} \circ \exp_x \circ \Gamma_x^k \circ \phi_x \circ L_x : \oplus_{j=1}^k E_y^j \rightarrow T_y M$ and let

$$[\eta_1, \dots, \eta_k] = \{(v_1, \dots, v_k) \in \oplus_{j=1}^k E_y^j \mid v_j \in E_y^j, \|v_j\| < \eta_j, 1 \leq j \leq k\}.$$

When n is large enough, we assume that

$$\alpha_j \triangleq \tau e^{-n(\lambda_j^+ + \varepsilon)} \leq \delta, \quad 1 \leq j \leq k.$$

Then

- $\text{Image}(\exp_y \circ \tilde{\Gamma}_x^k|_{[\frac{\alpha_1}{2\varepsilon}, \dots, \frac{\alpha_k}{2\varepsilon}]) \subset \mathcal{W}_{\Gamma_x^k}(\alpha_1, \dots, \alpha_k)$. This is a consequence of $\text{Lip}(L_x) \leq 2$ and the property of ϕ_x ;
- By shrinking ε_0 , for any $\varepsilon < \varepsilon_0$ and any $x \leq \delta$, we may assume that for any $(v_1, \dots, v_l), (w_1, \dots, w_k) \in [\eta_1, \dots, \eta_k]$ and any $1 \leq j \leq k$,

$$||\pi_j(\tilde{\Gamma}_x^k(v_1, \dots, v_k) - \tilde{\Gamma}_x^k(w_1, \dots, w_k))|| - ||v_j - w_j|| \leq \theta \sum_{i=1}^k ||v_i - w_i||$$

where $\theta = 10^{-k}(k+1)^{-1}$.

To see this, let K_4 be a number such that for any $(v_1, \dots, v_l) \in T_y M$,

$$||v_j|| \leq K_4 ||(v_1, \dots, v_l)|| \leq K_4^2 \sum_{i=1}^l ||v_i||.$$

Recall the definitions in Theorem E, $\hat{\Gamma}_x^k = (\pi_1 \circ \Gamma_x^k, \dots, \pi_k \circ \Gamma_x^k)$ and Id is the identity map on $\oplus_{j=1}^k E_x^j$. By Theorem E, for $x \in \Lambda$, we write $\hat{\Gamma}_x^k \circ \phi_x = Id + h_x$ where $\text{Lip}(h_x) \leq \varepsilon$. Hence we have

$$\begin{aligned} & ||\pi_j(\tilde{\Gamma}_x^k(v_1, \dots, v_k)) - \pi_j(\tilde{\Gamma}_x^k(w_1, \dots, w_k))|| \\ &= ||v_j - w_j + \pi_j \circ L_x^{-1} \circ h_x \circ L_x(v_1, \dots, v_k) - \pi_j \circ L_x^{-1} \circ h_x \circ L_x(w_1, \dots, w_k) \\ &\quad + \pi_j \circ \psi_x \circ \Gamma_x^k \circ \phi_x \circ L_x(v_1, \dots, v_k) - \pi_j \circ \psi_x \circ \Gamma_x^k \circ \phi_x \circ L_x(w_1, \dots, w_k)|| \\ &\leq ||v_j - w_j|| + 3K_4\varepsilon(1 + \varepsilon)^2 ||(v_1, \dots, v_k) - (w_1, \dots, w_k)|| \\ &\leq ||v_j - w_j|| + 3K_4\varepsilon(1 + \varepsilon)^2 \sum_{i=1}^k ||v_i - w_i||. \end{aligned}$$

Similarly,

$$||v_j - w_j|| - 3K_4\varepsilon(1 + \varepsilon)^2 \sum_{i=1}^k ||v_i - w_i|| \leq ||\pi_j(\tilde{\Gamma}_x^k(v_1, \dots, v_k)) - \pi_j(\tilde{\Gamma}_x^k(w_1, \dots, w_k))||.$$

Therefore if we assume that ε_0 is such that $3K_4^2\varepsilon_0(1+\varepsilon_0)^2 \leq \theta$, we get the conclusion.

Given an (i, δ_0) ample family, recall that an (i, δ_0) ample family $\mathcal{A}(i, \delta_0)$ is also an $(i, 10\delta)$ ample family if we assume that $10\delta \leq \delta_0$. Now apply Proposition 5.6.4 to these $\{\hat{\Gamma}_x\}_{x \in B(y, \delta) \cap \Lambda}$ and the ample family $\mathcal{A}(i, \delta_0)$, we get a constant L and a family of submanifolds $F_n \subset \mathcal{A}(i, \delta_0)$ with $\#F_n \leq L \prod_{j=1}^k \alpha_j^{-\dim(E_j)}$ such that for any $x \in B(y, \delta) \cap \Lambda$, there is some $\gamma \in F_n$ with $\exp_y(\gamma) \cap \mathcal{W}_{\Gamma_x^k}(\alpha_1, \dots, \alpha_k) \neq \emptyset$. Note here the constant L does not depend on n . For each $\gamma \in F_n$, let $E(n, \tau, \gamma)$ be a maximal (n, τ) separate subset of γ . Since for $0 \leq i \leq n$, $\text{diam}(f^i(\mathcal{W}_{\Gamma_x^k}(\alpha_1, \dots, \alpha_k))) \leq \tau$, we have that $\cup_{\gamma \in F_n} E(n, \tau, \gamma)$ contains a maximal $(n, 2\tau)$ separated subset in $B(y, \delta) \cap \Lambda$. Therefore we have

$$\begin{aligned}
h(f, \mu) &\leq \lim_{\tau \rightarrow 0} \limsup_{n \rightarrow +\infty} \frac{1}{n} \log \#E(n, \tau, \Lambda) \\
&= \lim_{\tau \rightarrow 0} \limsup_{n \rightarrow +\infty} \frac{1}{n} \log \#(E(n, 2\tau, \Lambda) \cap B(y, \delta)) \\
&\leq \lim_{\tau \rightarrow 0} \limsup_{n \rightarrow +\infty} \frac{1}{n} \log \#(\cup_{\gamma \in F_n} E(n, \tau, \gamma)) \\
&\leq \lim_{\tau \rightarrow 0} \limsup_{n \rightarrow +\infty} \frac{1}{n} \log \left(\prod_{j=1}^k L(\alpha_j)^{-\dim(E_j)} \sup_{\gamma \in \mathcal{A}(i, \delta_0)} \#E(n, \tau, \gamma) \right) \\
&\leq \lim_{\tau \rightarrow 0} \limsup_{n \rightarrow +\infty} \frac{1}{n} \log \left(\prod_{j=1}^k L(\alpha_j)^{-\dim(E_j)} \sup_{\gamma \in \mathcal{A}(i, \delta_0)} \#E(n, \tau, \gamma) \right) \\
&\leq \sum_{j=1}^k \lambda_j^+ \dim(E_j) + d\varepsilon + \lim_{\tau \rightarrow 0} \limsup_{n \rightarrow +\infty} \frac{1}{n} \sup_{\gamma \in \mathcal{A}(i, \delta_0)} \log \#E(n, \tau, \gamma) \\
&= \sum_{j=1}^k \lambda_j^+ \dim(E_j) + d\varepsilon + H(f, \mathcal{A}(i, \delta_0)).
\end{aligned}$$

By the arbitrariness of ε and ample family, we get the conclusion. $\square$

5.8 Appendix

5.8.1 A recurrence formula

Lemma 5.8.1. *For any numbers $0 < \theta_1, \theta_2 < 1$ and $k \in \mathbb{N}$, there is $K_2 > 1$ such that for any numbers $0 < \rho_1 < \rho_2 < \dots < \rho_k$ with $\rho_1 \geq \theta_1$ and $\sup_{l \leq k-1} \rho_l / \rho_{l+1} < \theta_2$, any $\varepsilon < 1$ and any double indexed positive numbers $\{\{a_n^1\}_{n \geq 0}, \{a_n^2\}_{n \geq 0}, \dots, \{a_n^k\}_{n \geq 0}\}$, if it satisfies the following recurrence formula*

$$a_n^i \leq \rho_i a_{n-1}^i + \varepsilon \sum_{j=i+1}^k a_{n-1}^j, \quad \forall 1 \leq i \leq k, n \geq 1,$$

then we have

$$a_n^i \leq \rho_i^n a_0^i + \varepsilon K \sum_{j=i+1}^k \rho_j^n a_0^j, \quad \forall 1 \leq i \leq k, n \geq 1.$$

Proof. By an inductive computation, one can get

$$a_n^i \leq \rho_i^n a_0^i + \varepsilon \sum_{m=1}^n \sum_{j=i+1}^k \rho_i^{m-1} a_{n-m}^j, \forall 1 \leq i \leq k, n \geq 1.$$

We will perform an inductive argument. Let $M_k = 1$, we note for positive numbers $\{\{a_n^1\}_{n \geq 0}, \{a_n^2\}_{n \geq 0}, \dots, \{a_n^k\}_{n \geq 0}\}$ satisfying the recurrence formula, we have $a_n^k \leq M_k \rho_k^n a_0^k$. Now assume that there is a constant M_{l+1} such that for any positive numbers $\{\{a_n^1\}_{n \geq 0}, \{a_n^2\}_{n \geq 0}, \dots, \{a_n^k\}_{n \geq 0}\}$ satisfying the recurrence formula, we have

$$a_n^i \leq \rho_i^n a_0^i + \varepsilon M_{l+1} \sum_{t=i+1}^k \rho_i^n a_0^t, \forall l+1 \leq i \leq k, n \geq 1.$$

We will use an inductive argument to a constant M_l . Let us compute in the following:

$$\begin{aligned} a_n^l &\leq \rho_l^n a_0^l + \varepsilon \sum_{m=1}^n \sum_{j=l+1}^k \rho_l^{m-1} a_{n-m}^j \\ &\leq \rho_l^n a_0^l + \varepsilon \sum_{m=1}^n \sum_{j=l+1}^k \rho_l^{m-1} (\rho_j^{n-m} a_0^j + \varepsilon M_{l+1} \sum_{t=j+1}^k \rho_t^{n-m} a_0^t) \\ &= \rho_l^n a_0^l + \varepsilon \sum_{m=1}^n \sum_{j=l+1}^k (\rho_j^n \rho_l^{-1} \frac{\rho_l^m}{\rho_j^m} a_0^j + \varepsilon M_{l+1} \sum_{t=j+1}^k \rho_t^n \rho_l^{-1} \frac{\rho_l^m}{\rho_t^m} a_0^t) \\ &= \rho_l^n a_0^l + \varepsilon \sum_{j=l+1}^k (\rho_j^n \rho_l^{-1} \sum_{m=1}^n \frac{\rho_l^m}{\rho_j^m} a_0^j + \varepsilon M_{l+1} \sum_{t=j+1}^k \rho_t^n \rho_l^{-1} \sum_{m=1}^n \frac{\rho_l^m}{\rho_t^m} a_0^t) \\ &= \rho_l^n a_0^l + \varepsilon \sum_{j=l+1}^k \rho_j^n a_0^j (\rho_l^{-1} ((1 + (j-l-1)\varepsilon M_{l+1}) \sum_{m=1}^n \frac{\rho_l^m}{\rho_j^m})) \\ &\leq \rho_l^n a_0^l + (\rho_l^{-1} k(1 + (k-l-1)\varepsilon M_{l+1}) \sum_{m=1}^{+\infty} \frac{\rho_l^m}{\rho_{l+1}^m}) \varepsilon \sum_{j=l+1}^k \rho_j^n a_0^j. \end{aligned}$$

Since $\varepsilon < 1$, by choosing

$$M_l = \theta_1^{-1} k(1 + (k-l-1)M_{l+1}) \sum_{m=1}^{+\infty} \theta_2^m < +\infty,$$

we get

$$a_n^i \leq \rho_i^n a_0^i + \varepsilon M_l \sum_{j=i+1}^k \rho_j^n a_0^j, \forall l \leq i \leq k, n \geq 1.$$

Then eventually, let $K_2 = M_1 + 1$, if the numbers $\{\{a_n^1\}_{n \geq 0}, \{a_n^2\}_{n \geq 0}, \dots, \{a_n^k\}_{n \geq 0}\}$ satisfy the recurrence formula, then

$$a_n^i \leq \rho_i^n a_0^i + \varepsilon K \sum_{j=i+1}^k \rho_j^n a_0^j, \forall 1 \leq i \leq k, n \geq 1.$$

□

5.8.2 Lusin's Lemma

Lemma 5.8.2. *Let μ be a Borel probability measure on a metric space X and Y a second-countable topological space. Assume that $f : X \rightarrow Y$ is a measurable map. Then for any $\varepsilon > 0$, there is a closed set $E \subset X$ with $\mu(E) \geq 1 - \varepsilon$ such that f restricted on E is continuous.*

Proof. Let $\{V_n\}$ be a countable basis of Y , for any $\varepsilon > 0$ and any V_n , let $K_n \subset f^{-1}(V_n)$ and $K'_n \subset X \setminus f^{-1}(V_n)$ be two closed subset such that

$$\mu(K_n) \geq (1 - \frac{\varepsilon}{2^n})\mu(f^{-1}(V_n)), \quad \mu(K'_n) \geq (1 - \frac{\varepsilon}{2^n})\mu(X \setminus f^{-1}(V_n)).$$

Let $K = \bigcap_n (K_n \cup K'_n)$, thus K is closed and $\mu(K) \geq 1 - \varepsilon$.

For any $x \in K$, assume that $f(x) \in V_n$ for some $n \in \mathbb{N}$. By definition, x belongs to $X \setminus K'_n$ which is open in X and by definition, $f(X \setminus K'_n) \subset V_n$. This shows that f is continuous at x . $\square$

Chapter 6

Further questions

6.1 Openness of entropy-hyperbolicity

For specific systems, the entropy-hyperbolicity (see Definition 1.2.2) is a difficult condition to check. For Anosov systems, we believe it is affirmative. The main difficulty is that the definition of the dimensional entropy $h_{\text{top}}^k(f)$ (Section 1.2.4) engages with a very large family of submanifolds and there are not many easy ways to control the entropies of all these submanifolds.

Question 6.1.1. *A C^∞ Anosov diffeomorphism on a compact boundaryless manifold is entropy-hyperbolic.*

Remark 6.1.2.

- *The proof might be an adjustment of Burguet's proof of Theorem 4.35 in [4] and Buzzzi's proof of Theorem 3.2 in [5]. They provided a way of bounding the dimensional entropy for certain C^∞ maps on the cubes in $\mathbb{R}^n$.*
- *For Anosov diffeomorphism, we also believe that the C^∞ smoothness can be replaced by C^1 .*

Beyond Anosov systems, we are interested in finding more complicated systems that are entropy-hyperbolic. We think that some special perturbations of Anosov systems are the best candidates.

Question 6.1.3. *Let f be a C^∞ Anosov diffeomorphism on a compact boundaryless manifold M and let g be another C^1 diffeomorphism. Assume there is an open set $\mathcal{U} \subset M$ such that $f = g$ on $M \setminus \mathcal{U}$ and the topological entropy of g on $\mathcal{U}$ is zero. Then g is entropy-hyperbolic.*

Remark 6.1.4.

- *The condition of the entropy of g on $\mathcal{U}$ being zero might be relaxed. For example, it might be enough to just assume that the entropy of g on $\mathcal{U}$ is strictly less than the entropy of g on M .*
- *In a work of Bonatti and Viana [2], in four dimension, they constructed examples of robustly transitive diffeomorphisms without invariant hyperbolic subbundles. It is also interesting to check whether or not the Bonatti-Viana example is entropy-hyperbolic. This can also be extended to some other examples generated from Bonatti-Viana example, e.g. [8], [9].*

As in the remark of Example 1.2.9, we conjecture that entropy-hyperbolicity is an open property in C^∞ topology. In fact, this relies on the upper continuity of the dimensional entropy.

Question 6.1.5. Assume f is a C^∞ entropy-hyperbolic diffeomorphism on a compact boundaryless manifold. Then the map $h_{\text{top}}^k(\cdot)$ is upper continuous in C^∞ topology at the point f for $1 \leq k \leq \dim M$.

Let us explain how to use the upper continuity of the dimensional entropy above to show the openness of entropy-hyperbolicity. Let

$$\varepsilon = \frac{h_{\text{top}}(f) - h_{\text{top}}^{d(f)-1}(f)}{10}.$$

For g close to f in C^∞ topology, the upper continuity of $h_{\text{top}}^k(\cdot)$ gives

$$h_{\text{top}}^{d(f)-1}(g) \leq h_{\text{top}}^{d(f)-1}(f) + \varepsilon.$$

By Corollary 1.2.6, we can assume

$$h_{\text{top}}(f) \leq h_{\text{top}}(g) + \varepsilon.$$

By the choice of ε , we then get

$$h_{\text{top}}^{d(f)-1}(g) < h_{\text{top}}(g)$$

which implies $d(g) \geq d(f)$. On the other hand, similarly we get $d(g^{-1}) \geq d(f^{-1})$. Since f is entropy-hyperbolic, by Remark 1.2.5, g is also entropy-hyperbolic.

The motivation to show the upper continuity of the dimensional entropy comes Buzzi's work in [5] where he considered it for some special C^∞ maps on cubes. To be precise, Buzzi showed:

Theorem 6.1.0.1. Let $f_1, \dots, f_d : [0, 1] \rightarrow [0, 1]$ be C^∞ maps with $h_{\text{top}}(f_1) \geq \dots \geq h_{\text{top}}(f_d) > 0$. For $1 \leq k \leq d$, the map $h_{\text{top}}^k(\cdot)$ is upper continuous in C^∞ topology at the point $f_1 \times \dots \times f_d$.

There are two key obstacles to extend Buzzi's result to our situation:

- our dynamics is on a general compact manifold instead of cubes. In Buzzi's proof, as far as we know, many tools he used are currently available for cubes. For example, he used polynomials (which are only reasonably defined on cubes) to approximate general submanifolds on the cubes. A possible solution to avoid tremendous extensions of Buzzi's method to a general manifold is to build a global connection between cubes and a general manifold (e.a., atlas, embedding). But we have to be careful that under this global connection, we might lose (or hard to show) the equivalence of $h_{\text{top}}^k(\cdot)$ on cubes and the general manifold. That is the reason why we come up with the redundant condition (1.12) to simplify the arguments.
- the product $f_1 \times \dots \times f_d$ is quite a special form of dynamics. Many specialties (essential or non-essential) of the product dynamics is used in Buzzi's argument.

6.2 Dimensional entropy on ample families

Next let us introduce another quantity that is similar with $h_{\text{top}}^k(f)$ except here we only consider the dynamics on ample families.

Definition 6.2.1. Define the i -dimensional entropy on ample families by

$$h^i(f) \triangleq \inf_{\mathcal{A}(i,\delta)} \left\{ \sup_{\gamma \in \mathcal{A}(i,\delta)} h_{\text{top}}(f, \gamma) \right\}$$

where the infimum is taken over (i, δ) ample families (see definition 5.1.1) for all $\delta > 0$.

Remark 6.2.2. Recall that $H^i(f)$ is called uniform dimensional entropy (depending on the location of ‘sup’ in the definitions). One can see by the definition, $H^i(f) \geq h^i(f)$.

We expect that Theorem A can be strengthened by the dimensional entropy $h^i(f)$.

Question 6.2.3. Let f be a $C^{1+\alpha}$ diffeomorphism on a compact manifold and let μ be an ergodic measure with Lyapunov exponents $\lambda_1 < \lambda_2 < \dots < \lambda_l$. For any $1 \leq k \leq l$, let $i = \dim(E_{k+1} \oplus E_{k+2} \oplus \dots \oplus E_l)$. Then we have

$$h_\mu(f) \leq h^i(f) + \sum_{j=1}^k \lambda_j^+ \dim(E_j)$$

where $\lambda_j^+ = \max\{0, \lambda_j\}$.

Also inspired by Buzzi’s work, we conjecture:

Question 6.2.4. Assume f is a C^∞ entropy-hyperbolic diffeomorphism on a compact manifold without boundary. Then the two maps $H^k(\cdot), h^k(\cdot)$ are upper continuous in C^∞ topology at the point f for $1 \leq k \leq \dim M$.

Remark 6.2.5. One can also define entropy-hyperbolicity by using $H^k(\cdot)$ or $h^k(\cdot)$ instead of $h_{\text{top}}^k(\cdot)$. If the above conjecture is true, one can similarly get the openness of entropy-hyperbolicity in C^∞ topology.

We note that in our Theorem A, the cut dimension i must be the sum of the dimensions of the corresponding Oseledec’s spaces, i.e., $i = \dim(E_{k+1} \oplus E_{k+2} \oplus \dots \oplus E_l)$. One might wonder what if i is arbitrary. So we have the following conjecture:

Question 6.2.6. Let f be a $C^{1+\alpha}$ diffeomorphism on a compact manifold and let μ be an ergodic measure with Lyapunov exponents $\lambda_1 < \lambda_2 < \dots < \lambda_l$. For any $i \leq \dim M$, we have

$$h_\mu(f) \leq h^i(f) + \sum_{j=1}^{n_i-1} \lambda_j^+ \dim(E_j) + \lambda_{n_i}^+ \left(\left(\sum_{j=n_i}^l \dim E_j \right) - i \right)$$

where n_i is the integer such that $\sum_{j=n_i+1}^l \dim E_j \leq i < \sum_{j=n_i}^l \dim E_j$.

Remark 6.2.7.

- Note that if we don’t choose i to be the sum of the dimensions of Oseledec’s spaces. The only direct consequence we get from Theorem A is the following:

$$h_\mu(f) \leq H^i(f) + \sum_{j=1}^{n_i} \lambda_j^+ \dim(E_j).$$

- *The conjecture above seems very nature. But the proof can not easily follow the idea of Theorem A. The main difficulty is that if i is not the sum of the dimensions of the corresponding Oseledets' spaces, usually we can not expect the existence of the relatively stable manifold with dimension i as in Theorem E.*

Nomenclature

- Λ_k^ε Pesin block, see Lemma 2.2.3
- $\mathcal{A}(i, \delta)$ ample family, see Definition 5.1.1
- $\mathcal{W}(f, \rho)$ ρ -stable manifold, see Definition 5.5.3
- $\hat{d}_x^i, \tilde{d}_x^i$ Two special metrics, see Lemma 2.2.7
- ξ^i a measurable partition subordinate to W^i , see Definition 2.2.1
- ξ^{u+1} a special partition, see the paragraph before Lemma 2.2.7
- $\{\mu_x^i\}, \{\mu_x^{\xi^i}\}$ the family of conditional measures w.r.t. the measurable subordinate partition ξ^i
- $h(f, \mu)$ metric entropy of an invariant measure μ , see Definition 5.4.2
- $H^i(f)$ dimensional entropy, see Definition 5.1.3
- $h_{\text{top}}(f)$ topological entropy, see Definition 5.4.1
- $h_i(f, \mu), h_i$ entropies along unstable foliations, see Proposition 2.3.1
- $V^i(x, n, \tau)$ dynamical ball, see Section 2.2
- $W_\rho^i(x)$ i -th local unstable manifold, see section 2.2

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