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Multi-leader-follower models: theoretical analysis, simulation, and application to eco-industrial parks

van Kien Cao

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THÈSE

Pour obtenir le grade de Docteur

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Multi-Leader-Follower Models: Theoretical Analysis, Simulation and Application to Eco-Industrial Parks

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“THE PURPOSE OF OUR LIVES IS TO BE HAPPY.”
- *Dalai Lama* -

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Perpignan, July 17, 2020
VAN KIEN CAO

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Introduction

This thesis covers two research topics: Topic I, we focused on quasi-variational inequality problems and generalized Nash equilibrium problems; and Topic II, we studied the modeling and optimization of eco-industrial parks.

Both of these two topics share as a common characteristic the concept of multi-leader-follower games. Indeed on one hand multi-leader-follower games are actually particular cases of generalized Nash games which themselves are often reformulated as quasi-variational inequalities. And on the other hand eco-industrial parks optimal design will be obtain in this thesis thanks to a model based on multi-leader-follower games.

Topic I: Quasi-variational inequality problems and generalized Nash equilibrium problems

After their introduction by Stampacchia in the 1960s (see [3, 4]), variational and quasi-variational inequalities have been a rich field of research for the mathematical community, with a lot of applications to physics, mechanics, economics, among others. Nowadays, the modern quasi-variational inequality problem (in the sense of Stampacchia) considers two set-valued operators $K : C \rightrightarrows C$ and $T : C \rightrightarrows X^*$, where C is a nonempty subset of a locally convex space X , and it consists in finding a point $x \in C$ satisfying

1. x is a fixed point of K ; and
2. there exists $x^* \in T(x)$ such that for every $y \in K(x)$, $\langle x^*, y - x \rangle \geq 0$.

One of the most recent existence results in this line can be found in [5], where it is assumed that the operator T is *quasi-monotone* and *locally upper sign-continuous*. On one hand, quasi-monotonicity is known to be one of the weakest monotonicity-type properties. On the other hand, local upper sign-continuity has proved to be one of the most adapted and easily verified continuity-type property.

A particular form of quasi-variational inequalities that has a lot of interest in game theory, transportation problems and economics is given by product sets, that is, when $C = \prod_i C_i$ and so the involved set-valued maps T and K take also a product form (i.e. $T = \prod_i T_i$ and $K = \prod_i K_i$).

The first purpose of this thesis is to address quasi-variational inequality problems over product sets considering the assumptions of quasi-monotonicity and local upper sign-continuity only in the component

operators. One of the most important difficulties, in order to obtain the existence results for quasi-variational inequalities over product sets, is that quasi-monotonicity and local upper sign-continuity are not preserved by the product of set-valued maps.

But the developments on quasi-variational inequalities on product sets have been done to provide efficient answers for our main target problem, that is multi-leader-follower problems which are, as noticed above specific generalized Nash equilibrium problems. Indeed, as it will be precisely explained in the forthcoming subsection 1.1.2 generalized Nash games are complex problems that are often reformulated as quasi-variational inequalities which are, due to the intrinsic structure of generalized Nash games, defined on product sets.

Topic II: The modeling and optimization of eco-industrial park

The concept of Eco-Industrial Park (EIP) aims to reduce the ecological impact of industrial production by diminishing the energy and/or raw material consumption. More precisely it consists in designing a connection network between companies grouped into a park in order to be able to organize some exchange of power/vapor, raw materials ... in such a way that, at the same time:

1. the production level of each company is maintained;
2. the production cost of each company is reduced;
3. the amount of energy/raw material needed on the full park is reduced.

This problem has been addressed since the 90'. Recently, in works of Boix et al. [6] and Kastner et al. [7], it has been pointed out that there is still a lack of systematic methods for designing the optimal configuration of an EIP. In previous studies [6, 8, 9], water integration networks (which is a classical example of EIP) were modeled as a cooperative economy, in the framework of multi-objective optimization (MOO). This approach consists in creating a vector function of $n + 1$ coordinates given by

$$C(F) = (\text{Cost}_1(F), \dots, \text{Cost}_n(F), Z(F))$$

where $\text{Cost}_i(\cdot)$ is the cost function of the enterprise i , $Z(\cdot)$ is the global consumption of natural resources, and F is the flux vector describing the distribution in the exchange network. Then, the aim is to solve the problem of “minimizing” C with respect to F , satisfying the physical constraints of the model. The result of such minimization is called a *Pareto front*, which consists in all vectors F for which none of the coordinates of C can be made better off without making someone worse off. Usually the designer selects one of these solutions considering as criteria the distance to an utopia point.

The main problem of such approach is that points of the Pareto front are not necessarily economically stable: first, a Pareto point requires the enterprises to cooperate and share information, which is rarely the case of an EIP. Second, due to the non-cooperative economy, the different enterprises may deviate from the selection of the designer since they may improve their cost function by unilaterally change their operation. In terms of game theory, the solutions of the MOO approach are *a social optimization which may fail to respect incentives* (see [10, Chapter 1]).

To overcome the difficulty of the MOO approach, we have proposed an innovative approach to find an optimal design of EIP, an approach based on tools of non-cooperative game theory. More precisely the concepts of multi-leader-follower game have been implemented. This kind of model is a mixture between bilevel optimization problems and generalized Nash equilibrium problems (GNEP for short). We refer the

reader to [10, 11] for a primer in non-cooperative games, to [12, 13] for the theory of bilevel optimization, and [14, 15] for a survey on generalized Nash equilibrium problems.

Two approaches, both based on multi-leader-follower concepts, have been developed in this thesis in order to improve the optimal design of EIP. The first one, called *blind-input model* (see subsection 1.3.1), mainly revisits the pioneering work of [2] and makes it more “efficient” by proposing under mild assumptions a reformulation in a mixed integer linear programming problem (MILP). In the second approach, called *control-input model* (see subsection 1.3.2), an alternative adaptation of the previous model is proposed, giving more control to the companies/followers.

1.1 Variational analysis on product spaces

1.1.1 Quasi-variational inequality problems

The quasi-variational inequality is an extension of a variational inequality in which the constraint set is also depending on the variable. This category of variational inequalities has proved to be a very tool for the modeling of many problems in transport, telecommunications and economics, to give few examples. The importance of theory as well as applications of quasi-variational inequality has been developed by several researchers. One of the most recent existence results in this line can be found in [5], where it is assumed that the operator T is *quasi-monotone* and *locally upper sign-continuous*. On one hand, quasi-monotonicity is known to be one of the weakest monotonicity-type properties, and it plays a fundamental role in quasi-convex optimization. On the other hand, upper sign-continuity, introduced by Hadjisavvas in [16], has proved to be one of the most adapted and easily verified continuity-type property, while being really weaker than the classic upper semicontinuity assumption. The strategy of [5] is strongly based on [17], and it relies on stability results for the solutions sets of parametrized variational inequalities, previously developed in [18, 19, 20, 21].

A particular form of variational and quasi-variational inequalities that has a lot of interest in game theory, transportation problems and economics is given by product sets, that is, when $C = \prod_i C_i$ and so the involved set-valued maps T and K take also a product form (i.e. $T = \prod_i T_i$ and $K = \prod_i K_i$). This decomposable structure, which is a particular case of systems of quasi-variational inequalities, has been already studied in the literature, for both variational inequalities (see, e.g., [22, 23, 24, 25, 26, 27, 28, 29]) and quasi-variational inequalities (see, e.g., [30, 31]). However, all these works obtain existence results in the context of *pseudomonotonicity* (or some modifications of this notion), which is known to be too strong for many applications, in particular in economics.

One of the most important difficulties in order to replicate the existence results of [5, 17] for quasi-variational inequalities over product sets is that quasi-monotonicity and local upper sign-continuity are not preserved by the product of set-valued maps. This can be illustrated thanks to the following very simple example:

Let $C_1 = [-2, 2]$, $C_2 = [-2, 2]$ and $C = [-2, 2] \times [-2, 2]$. For any $x_2 \in C_2$, let $T_1(\cdot, x_2) : C_1 \rightrightarrows \mathbb{R}$ be defined by $T_1(x_1, x_2) = \{x_1^2\}$. For $x_1 \in C_1$, let $T_2(x_1, \cdot) : C_2 \rightrightarrows \mathbb{R}$ be defined by $T_2(x_1, x_2) = \{1 + x_2^2\}$. Then, both component operators are quasi-monotone, but the product operator $T : C \rightrightarrows \mathbb{R}^2$ defined by $T(x) = \{x_1^2\} \times \{1 + x_2^2\}$ is not. The quasi-monotonicity of the set-valued maps $T_1(\cdot, x_2)$ and $T_2(x_1, \cdot)$ comes directly from the fact that they are the derivatives of the quasi-convex functions $x_1 \mapsto x_1^3/3$ and $x_2 \mapsto x_2 + x_2^3/3$, respectively. Now the non-quasi-monotonicity of T can be observed by considering the points $x = (0, 1/2)$ and $y = (-2, 1)$.

In the literature mentioned in the precedent paragraph, this difficulty is overcome either by exploiting the stronger regularity of pseudomonotone operators, or assuming directly the hypothesis of generalized monotonicity on the product operator T , rather than on the component operators T_i . But both of these hypothesis are too strong for many applications.

In our work, we address the quasi-variational inequality problem over product sets considering the assumptions of quasi-monotonicity and local upper sign-continuity only on the component operators. In doing so, we present a new stability result, under the new notion of *net-lower sign-continuity*. This new stability result is an improvement with respect to [19], and it is better adapted to the product structure than [21].

In several senses, our result is an improvement of the existence theorems in [5] and [17]. First, it works with quasi-variational inequalities in the infinite dimensional setting. Second, it shows the existence of solutions for quasi-variational inequalities over product sets. Finally, net-lower sign-continuity is a weaker hypothesis compared to the settings followed by [19] and [21]. Moreover importantly our result opens the door to powerful applications to generalized Nash equilibrium problems (GNEP for short), since it is well known that GNEPs can be reformulated as quasi-variational inequalities over product sets, respectively (see, e.g., [32]). This application to game theory will be the main aim of the forthcoming section 1.1.2.

1.1.2 Generalized Nash equilibrium problems

The *generalized Nash equilibrium problem* (GNEP for short) was first introduced by Debreu [33] as early as 1952. The GNEP is an extension of the classical *Nash equilibrium problem* (NEP for short) in which both the payoff function and strategy set of each player depend on the decision of the other players. GNEPs have become an important part of the research and attracted much more attention over the years. In fact, GNEP is at a crossroad of several different disciplines, for example, economics, computer science, engineering, mathematics and operations research. From a mathematical point of view, the GNEP is a fundamental modeling tool for non-cooperative multi-leader-follower games, for example in electricity markets (see [34], [35] also other references) and in eco-industrial parks (see [2]). We refer the reader to [32] for the detailed overview on the historical development of the GNEP as well as the literature overview, solution theory, and algorithms and its many other applications.

Formally, the GNEP consists of p players and each player v controls his strategy variables $x_v \in \mathbb{R}^{n_v}$. Let us denote by x the vector of all strategies

$$x = (x_1, \dots, x_p) \in \mathbb{R}^n \quad \text{with} \quad n = n_1 + n_2 + \dots + n_p,$$

and by x_{-v} the strategy vector of the other players except the player v . To emphasize the v th player's strategies within x , we often write (x_v, x_{-v}) instead of x . The strategy of player v belongs to a strategy set, i.e., $x_v \in K_v(x_{-v})$, that depends on the strategy variables of the other players. Given the strategies x_{-v} of the other players, the aim of player v is to choose a strategy x_v solving

$$\min_{x_v} \theta_v(x_v, x_{-v}), \quad \text{subject to} \quad x_v \in K_v(x_{-v}), \quad (P_v(x_{-v}))$$

where $\theta_v : \mathbb{R}^n \rightarrow \mathbb{R}$ is the objective function of player v . The GNEP is the problem of finding a vector $\bar{x} \in \mathbb{R}^n$ so that, for all v , \bar{x}_v solves $P_v(\bar{x}_{-v})$. Such a vector \bar{x} is called *generalized Nash equilibrium*. A point \bar{x} is, therefore, a generalized Nash equilibrium if no player can unilaterally decrease his objective function by choosing a different strategy. If the feasible sets $K_v(x_{-v})$ of each player does not depend on the rival player's strategies (and are thus constant sets), then \bar{x} is called a (classical) Nash equilibrium.

GNEP is a difficult problem to solve and the best way to find solutions of GNEP is to reformulate it into variational problems for which theoretical and numerical tools have been developed. Indeed, it is

well known that when the objective functions $\theta_v(\cdot, x_{-v})$ are continuously differentiable and convex, and the maps K_v are closed and convex valued for all v , then the GNEP can be equivalently reformulated as a quasi-variational inequality, see e.g. [32]. This reformulation has been extended in paper [36], thanks to the concept of adjusted normal operator (see also [37], [38]), to the case where the objective functions $\theta_v(\cdot, x_{-v})$ are semistrictly quasi-convex without assuming any differentiability and the constraint sets are jointly convex. In [36], authors also proved an existence result for semistrictly quasi-convex GNEP.

Some work has been already studied for GNEP in certain specific structure setting the problems, see [32] [36] [39] [40] [41] [42] [43] [44]. One of the classic existence results for GNEPs in the literature is the results in the article of Debreu [33]. In [33], the authors assume that the objective function θ_v is continuous for any v , for every x_{-v} , the function $\theta_v(\cdot, x_{-v})$ is quasi-convex, and the set-valued map K_v is upper and lower semicontinuous. Our main contribution in this article is to eliminate continuity of θ_v by replacing it by a “continuity” of the sublevel sets of the payoff functions. Our approach is based on the concept of the *adjusted normal operator* [37], the *net-lower sign-continuity* which was recently introduced in our previous paper [45], and the reformulation of the GNEP in term of quasi-variational inequalities.

Moreover, the GNEP provides a mathematical model for noncooperative multi-leader-follower games where each player solves a bilevel optimization problem. In the case where one or more players assume the role of leader(s) in the game, then a multi-leader-follower game arises.

1.2 Some reminders on Stackelberg-type optimization problems

1.2.1 Bilevel optimization problems

Bilevel programming problems (BLPP) are hierarchical optimization problems in which some of the variables are constrained to be optimal solutions to another parametric optimization problem. There are two kinds of decision variables in such this optimization problem. The first decision variable is a so-called leader, the second decision variable is a so-called follower, and the decisions are taken sequentially. The leader makes his strategy first by anticipating the responses of the follower, and then the follower relies on the leader’s strategy by solving an optimization problem with the strategy of the leader as a parameter. Figure 1.2.1 illustrates a general BLPP arising with hierarchical leader-follower structures.

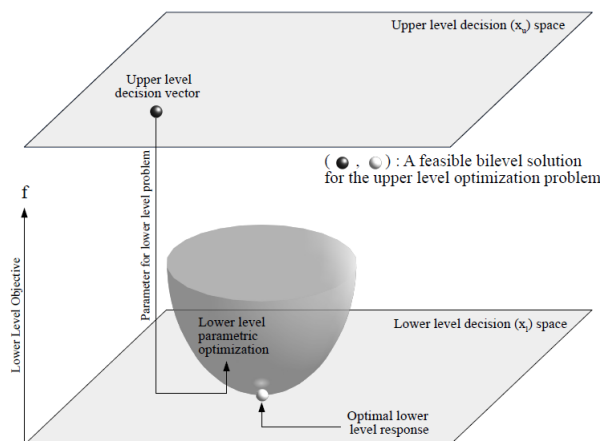


Figure 1.2.1: A general sketch of a bilevel problem. Figure obtained from [1].

BLPP was first realized in the domain of game theory by a German economist Stackelberg in his monograph on market economy in 1934 [46] and in the domain of mathematical programming by Bracken

and McGill in 1972 [47]. As it is well-known, BLPPs have a wide range of applications in real-world problems, such as management science, economics, engineering, transportation, business, environmental economics and others [48, 49]. BLPP is non-smooth and by nature non-convex, so to find optimality conditions for it and to solve it, the problem has to be reformulated as a single-level problem. For a survey on the historical development of BLPP, as well as a literature overview, and algorithms, we refer to [48].

Bilevel programming has two optimization problems where the constraint region of the first-level optimization problem is partly determined through the other second-level optimization problem. The latter problem is given as

$$\min_y \{f(x, y) : g(x, y) \leq 0, y \in Y\}, \quad (1.2.1)$$

where $f : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}$, $g : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^p$, $Y \subseteq \mathbb{R}^m$ is a nonempty closed set. This problem is called the lower level optimization problem or the problem of follower [48].

Let denote

$$\varphi(x) = \min_y \{f(x, y) : g(x, y) \leq 0, y \in Y\} \quad (1.2.2)$$

the *optimal value function*, and $\Psi : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ the *solution set mapping* of problem (1.2.1) for a fixed value of x :

$$\Psi(x) := \{y \in Y : g(x, y) \leq 0, f(x, y) \leq \varphi(x)\}. \quad (1.2.3)$$

Let

$$\text{gph}\Psi = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m : y \in \Psi(x)\}$$

be the graph of the solution set mapping Ψ . Then, the bilevel optimization problem is given as

$$\text{“min”} \{F(x, y) : G(x) \leq 0, (x, y) \in \text{gph}\Psi, x \in X\}, \quad (1.2.4)$$

where $F : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}$, $G : \mathbb{R}^n \rightarrow \mathbb{R}^q$ and $X \subseteq \mathbb{R}^n$ is a closed set. This problem is called the upper level optimization problem or the problem of leader [48].

It is worthwhile noting that the lower level problem may have multiple lower level optimal solutions for any given upper level decision vector. Thus bilevel problem (1.2.4) is ill-posed. This raises an ambiguity in the computation of the upper-level objective function value, which is difficult for the leader to predict which point in $\Psi(x)$ the follower will choose. Obviously, it is hard to determine the leader's solution. Thus, the quotation marks around “min” in (1.2.4) are used to indicate this ambiguity. To overcome this such obstacle, two main strategies have been widely studied are optimistic formulation (weak) and pessimistic formulation (strong) [48]. In the optimistic formulation, the leader may assume that the follower always selects a best optimal solution in $\Psi(x)$ with respect to the leader's objective function. It is formulated as:

$$\min_x \{\Phi_0(x) : G(x) \leq 0, x \in X\},$$

where

$$\Phi_0(x) = \min_y \{F(x, y) : y \in \Psi(x)\}. \quad (1.2.5)$$

Roughly speaking, this problem is almost equivalent to the problem

$$\min_{x, y} \{F(x, y) : G(x) \leq 0, x \in X, (x, y) \in \text{gph}\Psi\} \quad (1.2.6)$$

For the survey on existence of optimistic bilevel optimum and additional results on optimality conditions, we would name [50, 51, 52, 53].

Alternatively, in the pessimistic formulation, the leader has to take into account the follower's ability to select the worst solution with respect to the leader's objective function. Such a worst-case choice function of the follower defined as:

$$\min_x \{ \Phi_p(x) : G(x) \leq 0, x \in X \},$$

where

$$\Phi_p(x) = \max_y \{ F(x, y) : y \in \Psi(x) \}. \quad (1.2.7)$$

For discussion pessimistic on existence of optimistic bilevel optimum and additional results on optimality conditions, we refer to [51, 54, 55].

The bilevel programming model has much more advantages than single-level programming model. They are summarized as follows [56].

1. The bilevel programming can analyze two different and even conflicting objectives at the same time in the decision-making process.
2. The multiple criteria decision-making methods of bilevel programming can more accurately predict real situations.
3. The bilevel programming method can explicitly represent mutual actions between two different levels of the decision-makers.

Because of these advantages, bilevel programming becomes a useful tool for the design and optimization of industrial water networks in eco-industrial parks.

1.2.2 Multi-Leader-Follower games

In a game, when one of the players, called the leader, can decide before the other players, called the followers, make their decisions after observing the decision of the leader, the game is called the Stackelberg game. The Stackelberg game, also known as the single-leader-follower game (SLFG), was proposed by von Stackelberg in the 1930s [57]. Generally, in the SLFG, the leader optimizes the upper-level problem and the followers optimize the lower-level problems jointly. More precisely, the leader makes their decisions first by anticipating the responses of followers. Based on the leaders' decisions, all followers compete with each other in a parametric non-cooperative Nash or generalized Nash game in the lower-level with the strategies of leaders as exogenous parameters. As a bilevel program, the Stackelberg game can be reformulated as a mathematical program with equilibrium constraints (MPEC), which has been studied extensively in recent years, by incorporating the optimality conditions for the followers' problems into the constraints of the leader's problem. Generally, an MPEC is an optimization problem which contains two sets of variables called decision variables and response variables. Some or all of MPEC constraints are represented by a parametric variational inequality or complementarity problem with respect to the response variables, which is parameterized by the decision variables [58]. The MPEC has extensively been studied in the last two decades [59]. Applications of SLFGs are found for example in electricity markets [58, 60] and eco-industrial parks [2].

In the real world, we should have to consider the competition among several firms and a different of agents. Such a problem can be modeled as the multi-leader-follower game (MLFG), such as a deregulated

electricity market [58, 60]. MLFGs are a class of hierarchical games in which several players take the position as leaders and the rest of players who serve as followers. As a bilevel program, all leaders compete with each other in a non-cooperative Nash or generalized Nash game in the upper-level and make their decisions first by anticipating the responses of followers. At the same time, all followers select their own optimal responses by competing with each other in a Nash or generalized Nash game in the lower-level parameterized by the leader's decision. The MLFG has recently been studied by some researchers and used to model several problems in game theory with many applications in economics, operations research and other fields [61]. The MLFG may further be classified into the game which contains only one follower, called the multi-leader single-follower (MLSF) game. An application of MLSF game is used to model the design and optimization of industrial water networks in eco-industrial parks [2]. Moreover, it is very recently the concept of Multi-Leader-Disjoint-Followers (MLDF) game that was first introduced in [34], where several leaders act according to a generalized Nash equilibrium and each of the problem of the leader only depends on the decision variables of the other leaders and of the decision variables of a list of followers that only interact with the problem of the leader. This MLDF game is simpler than classical MLMF game since the sets of feasible solutions of the follower problems does not depend on the choice of the other leaders.

On the other hand, the MLFG can be reformulated as an equilibrium problem with equilibrium constraints (EPEC), one of the major approaches for MLFG. The acceptability of this approach is discussed in [62]. An EPEC is an equilibrium problem consisting of several parametric MPECs which contain the players' strategies as parameters. However, finding an equilibrium point of an EPECs is much more difficult than solving a single MPECs, because the constraints of each leader's problem depend on the other rival leaders' strategies, and all leaders share decision variables of the followers. Thus, the equilibria of an EPEC can be achieved when all MPECs are solved simultaneously. For a survey on this approach, we refer the reader to the references [63, 62]. Recently, in [62] the authors have provided an existence result for optimistic SLMF games.

Now, we are in the position to state the general formulation of MLFG. Let $\mathcal{N} = \{1, 2, \dots, N\}$ and $\mathcal{M} = \{1, 2, \dots, M\}$ denote the set of leaders and follower, respectively. Let $x_v \in \mathbb{R}^{n_v}$ denote the strategy vector, $X_v(x_{-v}) \subseteq \mathbb{R}^{n_v}$ denote the strategy set depending on other leaders' strategies, and $\theta_v : \mathbb{R}^{n+m} \rightarrow \mathbb{R}$ denote the cost function of leader v . Let $y_\omega \in \mathbb{R}^{m_\omega}$ denote the strategy vector, $Y_\omega(x, y_{-\omega}) \subseteq \mathbb{R}^{n+m-m_\omega}$ denote the strategy set depending on other followers' strategies and $\gamma_\omega : \mathbb{R}^{n+m} \rightarrow \mathbb{R}$ denote the cost function of follower ω . Here $n = n_1 + \dots, n_N$ and $m = m_1 + \dots, m_M$. In the conventional formulation of multi-leader multi-follower games, leader $v \in \mathcal{N}$ solves a parametrized optimization problem of the following kind

$$\begin{aligned} \min_{x_v, y} \quad & \theta_v(x_v, x_{-v}, y) \\ \text{s.t.} \quad & \begin{cases} x_v \in X_v(x_{-v}) \\ y \in Y(x) \end{cases} \end{aligned}$$

where, $x_{-v} = (x_1, \dots, x_{v-1}, x_{v+1}, \dots, x_N) \in \mathbb{R}^{n-n_v}$ and $y = (y_1, \dots, y_M) \in \mathbb{R}^m$, and $Y(x) = \prod_{\omega=1}^M Y_\omega(x, y_{-\omega})$.

While the follower ω solves the following optimization problem

$$\begin{aligned} \min_{y_\omega} \quad & \gamma_\omega(x, y_\omega, y_{-\omega}) \\ \text{s.t.} \quad & \{ y_\omega \in Y_\omega(x, y_{-\omega}) \} \end{aligned}$$

where, $x = (x_1, \dots, x_N) \in \mathbb{R}^{n-n_v}$ and $y = (y_1, \dots, y_M) \in \mathbb{R}^m$. Notice that the followers' objective functions and strategy sets depend on the leader's strategy x . For a more general overview of the structure of MLFG, we refer the reader to Figure 1.2.2.

In the forthcoming sections, the design and optimization of industrial water networks in eco-industrial parks are studied by formulating and solving multi-leader-follower game problems.

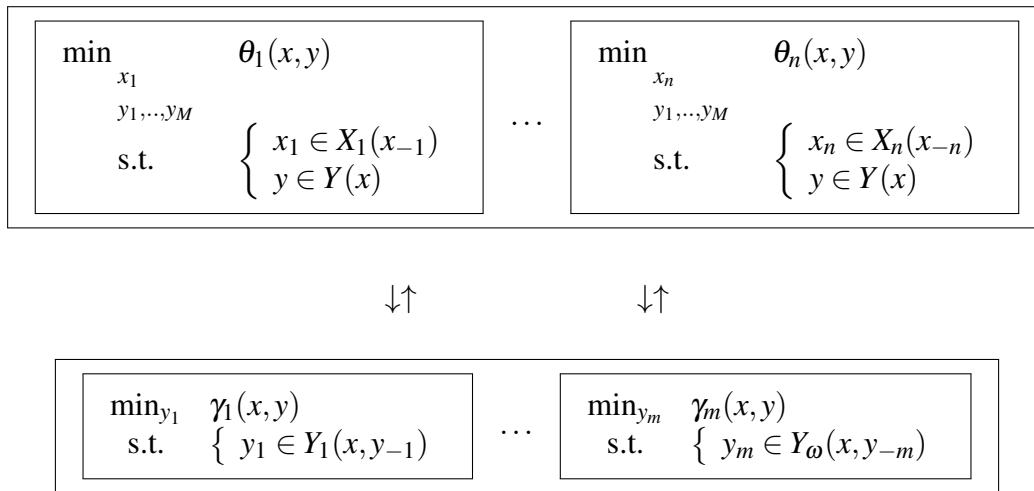


Figure 1.2.2: General structure for MLFG

1.3 Eco-Industrial Parks

1.3.1 Optimal design of exchange networks with blind inputs and its application to eco-industrial parks

In the last few decades, the development of the industrialized countries has led to an increasing depletion of natural resources such as freshwater and energy (see, e.g., [64, 65]). The conservation and sustainable use of such resources play an important role in both, environmental impact and business success within the industry. In response to preserve environment while increasing the utilities of the enterprises, the concept of *industrial ecology* has emerged [6]. Industrial ecology (IE) was first introduced in 1989 by Frosch and Gallopoulos [66]. They wrote “the consumption of energy and materials is optimized, waste generation is minimized and the effluents of one process ... serve as the raw material for another process”. This is an approach to the industrial design of products and processes and the implementation of sustainable manufacturing strategies. The idea is directly related to another concept, *industrial symbiosis*, which involves “separate industries in a collective approach to competitive advantage involving physical exchange of materials, energy, water and/or by-products” (see [67]). One key concept of industrial symbiosis is then the *exchange networks*.

A perfect example of an exchange network which illustrates the notion of industrial symbiosis is the concept of Eco-Industrial Parks (EIP). This notion has several definitions, but one widely accepted is “an industrial system of planned materials and energy exchanges that seeks to minimize energy and raw materials use, minimize waste, and build sustainable economic, ecological and social relationships” [6, 9, 68].

The best-known case of EIP development took place in Kalundborg in Denmark. The primary partners in Kalundborg are firms, local communities, and a lake. The participants exchange water, steam, and electricity, and also exchange a variety of residues that become feedstocks in other processes. Some of the benefits are reduction in carbon dioxide (CO₂) and in sulfur dioxide (SO₂) emissions; transformation of wastes into raw materials; reduction in coal, oil, and water flows.

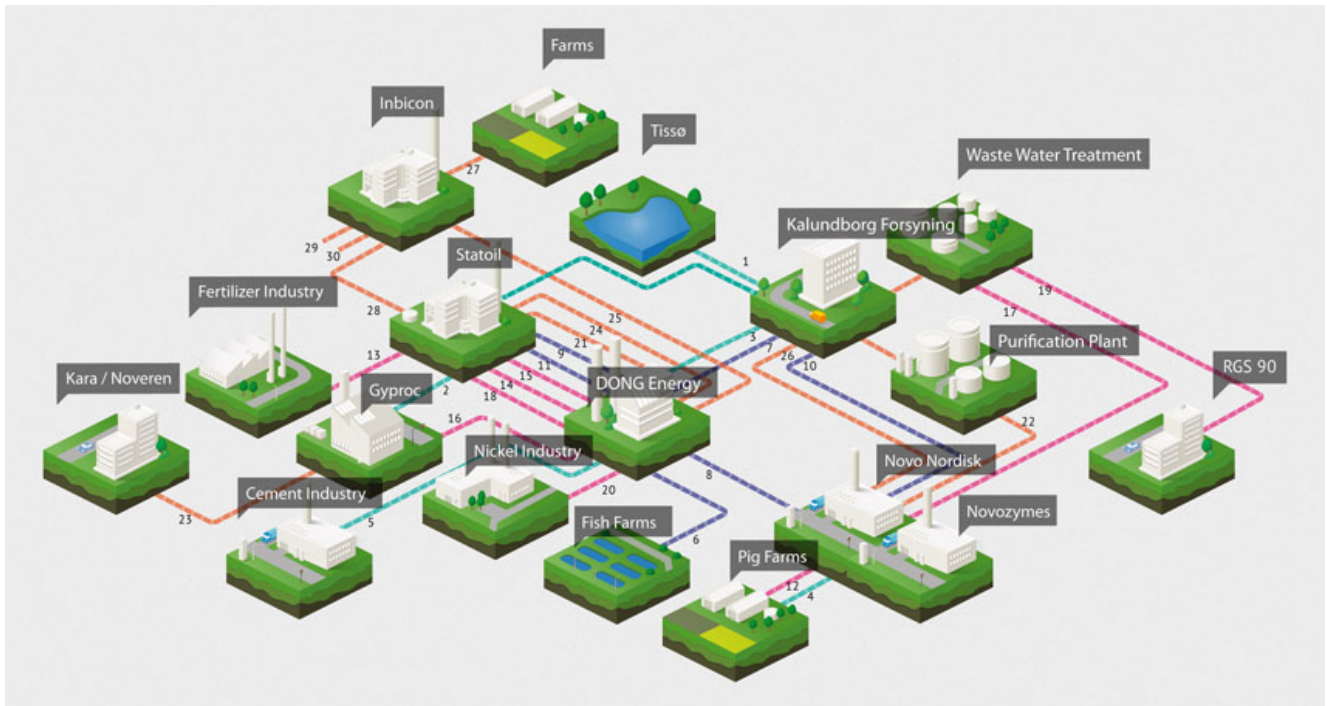


Figure 1.3.1: Eco-Industrial Park in Kalundborg ¹.

In the seminal work of Ramos et al. [2] a novel game theory approach has been proposed, by modeling the EIP design problem as a Single-Leader-Multi-Follower (SLMF) game (see [61, 63]): at the upper level, there is the EIP authority which wants to minimize the consumption of natural resources $Z(F)$, while at the lower level, each enterprise tries to minimize her cost function $Cost_i(F)$, related to her processes, consumption of natural resources and activity within the EIP. The authority of the park must choose the connections of the exchange network as well as the operation of regeneration units, while each enterprise controls their consumption of natural resources and their output distribution. Figure 1.3.2 shows the general scheme of such a model. For Single-Leader-Multi-Follower games, we refer to [63] and the references therein.

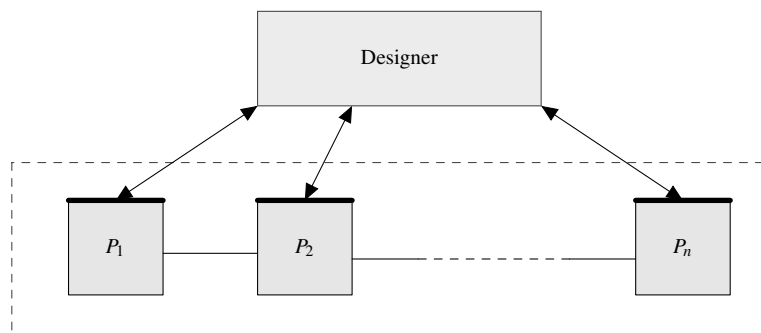


Figure 1.3.2: General scheme of SLMF Game

The main implicit assumption done in [2] is that each enterprise *can only control her outlet distribution and her own fresh water consumption*, but they are forced to accept whatever is sent to them through the

¹<https://hanspeterhonkoop.wordpress.com/2014/10/22/analysis-of-a-regional-network-of-production-avr/>

exchange network. Furthermore, they have no knowledge about the particular actions of the other agents of the network, excepting only the amount and quality of the final inlet flux that they receive. In practice, this situation corresponds to the case where at the entrance each agent of the network has a mixer, and so she is only aware of the total input she is receiving. In other words, when participating to the exchange network, each agent accepts to have a *blind input*.

While this model respect incentive consistency, it has two main drawbacks: the first one is that the rule that the park's authority imposes, that is, the blind input, is too restrictive. Indeed, under this paradigm, an enterprise may be forced to receive too much polluted water which could turn into higher costs than the stand-alone operation outside the park (examples are easy to construct with two enterprises). This violates the economical principle (well known in contract theory and mechanism design) of *individual rationality*: an enterprise will participate in the EIP only if it is convenient to her (see [69, 70, 71]); the second one is the strategy to compute a solution. In [2], the authors implemented the classic general approach to solve bilevel games, that is, to reformulate it as a *mathematical programming with complementarity constraints* (MPCC). Loosely speaking, for a given network, they write the Karush-Kuhn-Tucker conditions of each problem of the lower level game, and put them as constraints in the authority's problem. Then they implemented a Branch-and-Bound heuristic to obtain an approximated optimal exchange network, solving at each iteration the problem described above. However, it is known that the MPCC problems, which is a particular class of *mathematical programming with equilibrium constraints* (MPEC), are hard to solve (see, e.g., [72, 73, 74]) and the heuristic itself doesn't guarantee a real solution of the problem [75, 76]. The literature on theoretical and algorithmic aspects of MPCC and MPEC problems is large and still an active field of research in mathematics.

In this work, we further investigate the model proposed in Ramos et al. [2], but considering its abstract form for general exchange networks. This abstract model is called *Blind-Input model*, since we consider the constraint of full acceptance for each enterprise. To solve the drawback given by the Individual Rationality constraint, we introduce the notion of *Blind-Input contract*, which is an economical contract between the authority and each enterprise in order to participate in the blind-input model. We prove that, under some linear structure of the costs functions $\text{Cost}_i(\cdot)$ of each enterprise, the blind-input model can be reduced from a Single-Leader-Multi-Follower problem to a single mixed-integer optimization problem. Thank to this reformulation, examples of EIP of realistic size are then studied numerically. For details of the data, we refer the reader to [77]. The details per enterprise are given in Table 1.1.

	Fresh Water	Shared Water	Social Cost
Stand-Alone	541.00	0	189.37
Optimal EIP	365.37	278.85	133.79

Table 1.1: Global Results without Reg. Units

The corresponding optimal configuration is the following.

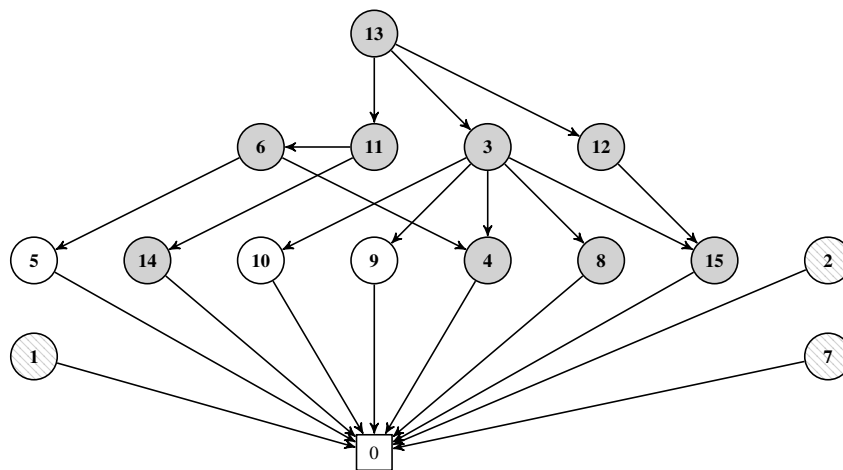


Figure 1.3.3: The configuration in the case without regeneration units, $\alpha = 0.95$ and $\text{Coef} = 1$. Gray nodes are consuming strictly positive fresh water. Dashed nodes are operating in stand-alone mode.

1.3.2 Optimal design of exchange networks with control inputs in eco-industrial parks

Nowadays, increasing industrialization and urbanization are causing environmental problems when proper methods and planning are absent. To remedy this situation, Eco-Industrial Parks (EIP) arise as a new way to design and optimize industrial networks. The problem of EIPs is to reduce the environmental impact of industrial production. This involves reducing the consumption of energy and/or raw materials (water, energy-steam, etc) by a group of companies located in the same industrial park, or designing/creating new industrial parks incorporating these aspects. This is achieved by reusing the waste from one industrial process as a utility from another process, either in raw form if the “contamination” is low enough or via regeneration facilities. However, to convince companies to take part in an EIP, it is essential to make sure that each participant gains in competitiveness (reduction in production costs in most cases). Since these advantages depend on its configuration, proper planning and design are critical. However, the system methods for designing the optimal configuration of EIP are lacking and the optimal configurations in the literature are limited in some respects [6, 67].

The design and optimization of water-exchange networks in EIP are a such complex problem. Recently, the results in [77] developed an abstract Blind-Input model based on game theory approach, modeling the EIP design problem as Single-Leader-Multi-Follower (SLMF) game: at the upper level, there is the EIP authority which wants to minimize the consumption of natural resources, while at the lower level, each enterprise tries to minimize there operating costs. More precisely, the EIP authority makes the decisions first by anticipating the responses of enterprises at lower level. At the same time, all enterprises at lower level select their own optimal responses by competing with each other in a GNEP parameterized by the leader’s decision.

In [77], the authors introduced the notion Blind-Input contract. More precisely, when participating to the exchange network, each enterprise accepts to have a blind input in the sense that enterprises control only their outlet distributions and their own freshwater, and the designer commits to guarantee a minimal relative improvement in comparison with the stand-alone operation of each agent. In [77], the authors implemented that, under some linear structure of the costs functions $\text{Cost}_i(\cdot)$ of each enterprise, the blind-input model can be reduced from a Single-Leader-Multi-Follower problem to a single mixed-integer optimization problem.

In this work, we further investigate the blind-input model proposed in [77]. This model is called

Control-Input model, since, when participating in the exchange network, each enterprise has the ability to control all of his inlet flux. The control-input model will be reduced from a Single-Leader-Multi-Follower problem to a single mixed-integer optimization problem. The approach is validated on a case study of exchange water network in EIP without regeneration units. Obtained results are compared against the blind-input model.

Now, we compare the results of the control-input model with the blind-input model in the case model without regeneration units. For details of the data, we refer the reader to [78].

	Fresh Water	Social Cost
Stand-Alone	418.13	146.34
Optimal EIP with blind-input model	318.97	114.45
Optimal EIP with control-input model	272.97	100.95

Table 1.2: Global Results without Reg. Units

The optimal configuration corresponding to control-input model:

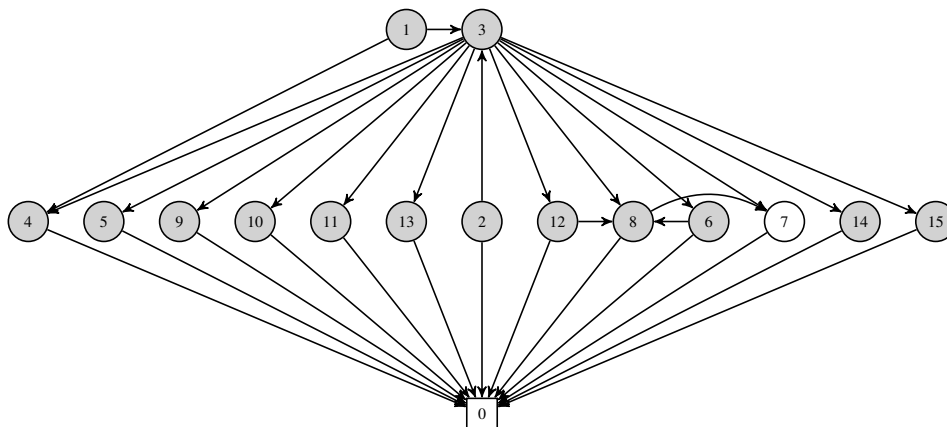


Figure 1.3.4: The optimal configuration in the case $\alpha_i = 0.95$ and $\text{Coef} = 1$. Gray nodes are consuming strictly positive fresh water.

1.4 Our articles

The thesis content presented in this thesis is based on our four articles, namely:

- Article 1: D. Aussel, K. Cao Van, and D. Salas: “Quasi-Variational Inequality Problems over Product Sets with Quasi-monotone Operators”, *SIAM Journal on Optimization*, vol 29, no. 2, pp 1558-1577, 2019.
- Article 2: D. Aussel, K. Cao Van, and D. Salas: “Existence Results for Generalized Nash Equilibrium Problems under Continuity-like Properties of Sublevel Sets”, *submitted to SIAM Journal on Optimization*, 21p, 2020.
- Article 3: D. Salas, K. Cao Van, D. Aussel, and L. Montastruc: “Optimal Design of Exchange Networks with Blind Inputs: Applications to Eco-Industrial Parks”, *Computers & Chemical Engineering*, vol. 143, p. 107053, 2020.

- Article 4: D. Aussel and K. Cao Van: “Optimal Design of Exchange Networks with Control Inputs in Eco-Industrial Parks”, *submitted to Energy Economics*, 23p, 2020.

These articles will be presented in chapters 2,3,4,5, respectively.

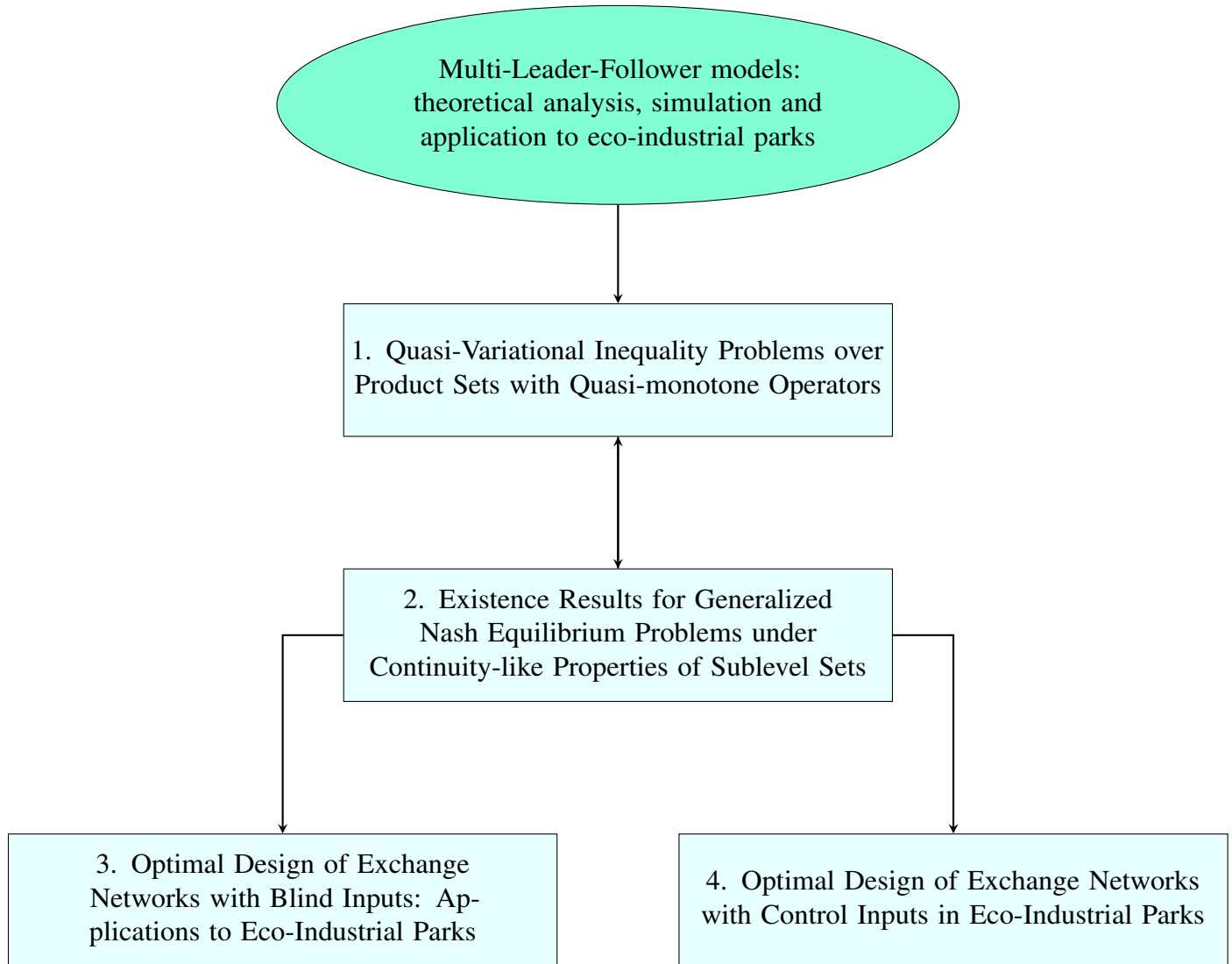


Figure 1.4.1: Organizational diagram of the articles that make up the thesis.

PART I

Quasi-Variational Inequality Problems and Generalized Nash Equilibrium Problems

Quasi-Variational Inequality Problems over Product Sets with Quasi-monotone Operators

This article has been done in the common work with Didier Aussel and David Salas. It was published on *SIAM Journal on Optimization*, vol 29, no. 2, pp 1558-1577, 2019.

Abstract

Quasi-variational inequalities are variational inequalities in which the constraint map depends on the current point. Due to this characteristic, specific proofs have been built to prove adapted existence results. Semicontinuity and generalized monotonicity are assumed and many efforts have been made in the last decades to use the weakest concepts. In the case of quasi-variational inequalities defined on a product of spaces, the existence statements in the literature require pseudomonotonicity of the operator, a hypothesis that is too strong for many applications, in particular in economics. On the other hand, the current minimal hypotheses for existence results for general quasi-variational inequalities are quasi-monotonicity and local upper sign-continuity. But since the product of quasi-monotone (respectively, locally upper sign-continuous) operators is not in general quasi-monotone (respectively locally upper sign-continuous) it is thus quite difficult to use these general-type existence result in the quasi-variational inequalities defined on a product of spaces. In this work we prove, in an infinite-dimensional setting, several existence results for product-type quasi-variational inequalities by only assuming the quasi-monotonicity and local upper sign-continuity of the component operators. Our technique of proof is strongly based on an innovative stability result and on the new concept of *net-lower sign-continuity*.

Keywords: Quasi-variational inequality, Quasi-monotone operator, Product sets, Fixed points, Stability, Net-lower sign-continuity.

AMS subject classifications. 49J53, 47H05, 49J40

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2.1 Introduction

After their introduction by Stampacchia in the 1960s (see [3, 4]), variational and quasi-variational inequalities have been a rich field of research for the mathematical community, with many applications to physics, mechanics, and economics, among others. Nowadays, the modern quasi-variational inequality problem (in the sense of Stampacchia) considers two set-valued operators $K : C \rightrightarrows C$ and $T : C \rightrightarrows X^*$, where C is a nonempty subset of a locally convex space X , and it consists in finding a point $x \in C$ satisfying that

1. x is a fixed point of K ; and
2. there exists $x^* \in T(x)$ such that for every $y \in K(x)$, $\langle x^*, y - x \rangle \geq 0$.

Since the classical existence result of Tan [79], which assumes upper semicontinuity of T and lower semicontinuity of K , a lot of effort has been exerted to obtain existence results with weaker continuity hypotheses, essentially by considering general monotonicity assumptions on the set-valued map T . We refer the reader to [80, 81] for a comprehensive presentation of such developments in the finite-dimensional setting, and to [82] for a survey in the Banach space setting.

One of the most recent existence results in this line can be found in [5], in which it is assumed that the operator T is *quasi-monotone* and *locally upper sign-continuous*. On one hand, quasi-monotonicity is known to be one of the weakest monotonicity-type properties, and it plays a fundamental role in quasi-convex optimization. On the other hand, upper sign-continuity, introduced by Hadjisavvas in [16], has proved to be one of the most adapted and easily verified continuity-type properties, while being really weaker than the classic upper semicontinuity assumption. The strategy of [5] is strongly based on [17], and it relies on stability results for the solutions sets of parametrized variational inequalities, previously developed in [18, 19, 20, 21].

A particular form of variational and quasi-variational inequalities that has received a lot of interest in game theory, transportation problems, and economics is given by product sets, that is, when $C = \prod C_i$ and so the involved set-valued maps T and K also take a product form (i.e., $T = \prod T_i$ and $K = \prod K_i$). This decomposable structure, which is a particular case of systems of quasi-variational inequalities, has been already studied in the literature for both variational inequalities (see, e.g., [22, 23, 24, 25, 26, 27, 28, 29]) and quasi-variational inequalities (see, e.g., [30, 31]). However, all these works obtain existence results in the context of *pseudomonotonicity* (or some modifications of the notion), which is known to be too strong for many applications, and in particular in economics.

One of the biggest difficulties in replicating the existence results of [5, 17] for quasi-variational inequalities over product sets is that quasi-monotonicity and local upper sign-continuity are not preserved

by the product of set-valued maps. In the literature mentioned in the preceding paragraph, this difficulty is overcome by either exploiting the stronger regularity of pseudomonotone operators or assuming directly the hypothesis of generalized monotonicity on the product operator T , rather than on the component operators T_i .

In this work, we address the quasi-variational inequality problem over product sets considering the assumptions of quasi-monotonicity and local upper sign-continuity only on the component operators. In doing so, we present a new stability result, under the new notion of *net-lower sign-continuity*. This new stability result is an improvement with respect to [19], and it is better adapted to the product structure than [21].

The work is organized as follows: in section 3.2 we present some preliminary definitions, notation, and existing results, and formalize the quasi-variational inequalities over product sets. Also, in this section we provide two simple counterexamples showing that quasi-monotonicity and local upper sign-continuity are not preserved in general by the product operations. In Section 2.3 we introduce the notion of net-lower sign-continuity and show our main stability result, Proposition 2.3.9. A comparison is made between our result and the existing literature (specifically with [19, 20, 21]). In section 3.3 we present the main existence results for quasi-variational inequalities over product sets. Finally, in Section 2.5, we close the paper with some final comments.

2.2 Preliminaries and problem formulation

2.2.1 Preliminary notions and notation

In this section, we recall some notation and definitions that will be used later.

In what follows, X and Y will be Banach spaces, and X^* and Y^* their respective topological dual spaces. We always use $\langle \cdot, \cdot \rangle$ to denote the duality product for any Banach space and its dual. For a Banach space X , we denote by w the weak topology on X and by w^* the weak-star topology on X^* . The norm of X is denoted by $\|\cdot\|$. For $x \in X$ and $r > 0$, $B_X(x, r)$ (or simply $B(x, r)$, if there is no ambiguity) stands for the open ball centered on x of radius r . We say that a locally convex topology τ is consistent with the duality $\langle X, X^* \rangle$ if the topological dual of (X, τ) is X^* . For more details on dualities and the associated topologies, we refer the reader to [83] and [84].

For a topological space (U, τ_U) and a point $u \in U$, we write $\mathcal{N}_U(u, \tau_U)$ (or simply $\mathcal{N}(u, \tau_U)$ or $\mathcal{N}(u)$ if there is no confusion) to describe the family of neighbourhoods of u in U , given by the topology τ_U . Recall that the topological space (U, τ_U) is said to be first countable if each point $u \in U$ has a countable basis of neighbourhoods.

For a subset $A \subseteq U$, we write $\text{int}_{\tau_U} A$ and \overline{A}^{τ_U} to denote the interior and closure of A , respectively. If there is no confusion, we may simply write $\text{int}A$ and \overline{A} , omitting the topology. For a Banach space X and a subset A of X we write $\text{conv}A$ and $\overline{\text{conv}A}$ to denote the convex hull and the closed convex hull of A . For any $x, y \in X$, we use the notation $[x, y]$, $]x, y[$, and $]x, y]$ for the segments $[x, y] = \{(1-t)x + ty : t \in [0, 1]\}$, $]x, y[= \{(1-t)x + ty : t \in]0, 1[\}$, and $]x, y] = \{(1-t)x + ty : t \in]0, 1]\}$.

Recall that a pair (\mathcal{A}, \prec) is said to be a *directed set* if \prec is a preorder of \mathcal{A} and for each $\alpha_1, \alpha_2 \in \mathcal{A}$ there exists $\alpha_3 \in \mathcal{A}$ such that $\alpha_1 \prec \alpha_3$ and $\alpha_2 \prec \alpha_3$. In general, we will omit the preorder, saying simply that \mathcal{A} is a directed set. For a set U , a subset $(u_\alpha)_{\alpha \in \mathcal{A}}$ is said to be a net in U if the set of indexes \mathcal{A} is a directed set. If there is no ambiguity, we may simply write $(u_\alpha)_\alpha$ or (u_α) to denote the net. For a net $(u_\alpha)_{\alpha \in \mathcal{A}}$, we say that a net $(u_\beta)_{\beta \in \mathcal{B}}$ is a subnet of it if

1. there exists a function $\varphi : \mathcal{B} \rightarrow \mathcal{A}$ such that, for any $\alpha_0 \in \mathcal{A}$, there exists a $\beta_0 \in \mathcal{B}$ satisfying that

$$\alpha_0 \leq_{\mathcal{A}} \varphi(\beta) \quad \forall \beta \in \mathcal{B} \text{ such that } \beta_0 \leq_{\mathcal{B}} \beta,$$

where $\leq_{\mathcal{A}}$ and $\leq_{\mathcal{B}}$ are the preorders of \mathcal{A} and \mathcal{B} , respectively;

2. for each $\beta \in \mathcal{B}$, $u_{\beta} = u_{\varphi(\beta)}$.

If (U, τ_U) is a topological space, a net $(u_{\alpha})_{\alpha \in \mathcal{A}}$ in U is said to be τ_U -convergent to $u \in U$ if for every neighbourhood $V \in \mathcal{N}(u)$ there exists $\alpha_V \in \mathcal{A}$ such that for every $\alpha \geq \alpha_V$, $u_{\alpha} \in V$. For more details on nets and subnets, we refer the reader to [83], [85] and [86].

For a family $\mathcal{A} := \{A_{\alpha} : \alpha \in \mathcal{A}\}$ of nonempty subsets of X , a family $\{z_{\alpha} : \alpha \in \mathcal{A}\} \subseteq X$ is said to be a *selection* of \mathcal{A} if for every $\alpha \in \mathcal{A}$, $z_{\alpha} \in A_{\alpha}$.

Let A and B be two nonempty sets. For a set-valued map $T : A \rightrightarrows B$ we denote by $\text{Gr}T$ the graph of T , that is,

$$\text{Gr}T := \{(a, b) \in A \times B : b \in T(a)\}.$$

If (B, τ_B) is a topological space, we respectively denote by $\text{int}T$ and by \overline{T} the interior and the closure set-valued maps from A to B , given by

$$\begin{aligned} (\text{int}T)(a) &:= \text{int}T(a) & \forall a \in A, \\ (\overline{T})(a) &:= \overline{T(a)} & \forall a \in A. \end{aligned}$$

We assume the reader is familiar with the theory of set-valued maps and the different notions of semicontinuity involved with them, like upper and lower semicontinuity, closedness (also known as outer semicontinuity), the Painlevé-Kuratowski semilimits, etc. For a survey on such topics, we refer the reader to [87], which presents a comprehensive analysis in the Banach space setting, and to [83] for a more complete presentation involving general topological spaces.

Recall that, for a nonempty subset C of X and a set-valued map $T : C \rightrightarrows X^*$, the *Stampacchia variational inequality* associated with T and C is

$$\text{find } x \in C \text{ such that } \exists x^* \in T(x) \text{ with } \langle x^*, y - x \rangle \geq 0 \quad \forall y \in C. \quad (2.2.1)$$

We denote by $S(T, C)$ its solution set. We also consider the set of nontrivial solutions, $S^*(T, C)$, defined by

$$S^*(T, C) := \{x \in C : \exists x^* \in T(x) \setminus \{0\} \text{ with } \langle x^*, y - x \rangle \geq 0 \quad \forall y \in C\}. \quad (2.2.2)$$

Note that one always has $S^*(T, C) = S(T \setminus \{0\}, C) \subseteq S(T, C)$. Also, recall that the *Minty variational inequality* associated with T and C is

$$\text{find } x \in C \text{ such that } \langle y^*, y - x \rangle \geq 0 \quad \forall y \in C, \forall y^* \in T(y). \quad (2.2.3)$$

The solution set of the Minty variational inequality problem will be denoted by $M(T, C)$. It is not hard to see that $M(T, C)$ is convex and closed, provided that C is convex and closed.

Finally, for C and T as before, and for a set-valued map $K : C \rightrightarrows C$, the *quasi-variational inequality* associated with T and K is

$$\text{find } x \in K(x) \text{ such that } \exists x^* \in T(x) \text{ with } \langle x^*, y - x \rangle \geq 0 \quad \forall y \in K(x). \quad (2.2.4)$$

We denote by $\text{QVI}(T, K)$ its solution set. As before, we also consider the set of non-trivial solutions, $\text{QVI}^*(T, K)$, defined by

$$\text{QVI}^*(T, K) := \left\{ x \in C : \begin{array}{l} x \in K(x) \text{ and } \exists x^* \in T(x) \setminus \{0\} \\ \text{with } \langle x^*, y - x \rangle \geq 0 \quad \forall y \in K(x) \end{array} \right\}. \quad (2.2.5)$$

Again, one always has $\text{QVI}^*(T, K) = \text{QVI}(T \setminus \{0\}, K) \subseteq \text{QVI}(T, K)$.

In what follows, we will use the notation $S(T, C)$, $S^*(T, C)$, $M(T, C)$, $\text{QVI}(T, K)$ and $\text{QVI}^*(T, K)$ to also denote indistinctly both the solution sets and the corresponding variational problems.

In the literature, existence results for $S(T, C)$ and $\text{QVI}(T, K)$ usually have two types of hypotheses on T (and K): continuity-type assumptions and geometrical-type assumptions. One of the most classic existence results for $\text{QVI}(T, K)$ in the infinite-dimensional setting is [79, Theorem 1], which states the following

Theorem 2.2.1 [Tan, 1985] *Let X be a locally convex Hausdorff space, C be a nonempty convex compact subset of X , and $T : C \rightrightarrows X^*$ and $K : C \rightrightarrows C$ be two set-valued maps such that*

- (i) *K is lower semicontinuous with nonempty convex compact values,*
- (ii) *T is upper semicontinuous with nonempty convex compact values.*

Then $\text{QVI}(T, K)$ is nonempty.

The analogous version of the above theorem for $S(T, C)$ can be traced back to [88, Theorem 6]. In Theorem 2.2.1, the continuity-type hypotheses of T and K are upper semicontinuity and lower semicontinuity, respectively, while the geometrical-type hypotheses are that both are convex compact valued, and that C is also convex and compact. In [88, Theorem 6], the same hypotheses on T and C are used.

However, once we need a weaker continuity-type hypothesis on T (that is, upper semicontinuity of T is not verified), the geometrical-type hypothesis must be reinforced. The most classic way to do it is to assume some general monotonicity on T . In this article, we focus only on the weakest one presented in the literature: quasi-monotonicity. For a survey on the different types of general monotonicity of set-valued operators, we refer the reader to [89].

Definition 2.2.2 *Let C be a nonempty subset of X . A set-valued map $T : C \rightrightarrows X^*$ is said to be*

- (i) *quasi-monotone on C if for all $(x, x^*), (y, y^*) \in \text{Gr}T$, the implication*

$$\langle x^*, y - x \rangle > 0 \implies \langle y^*, y - x \rangle \geq 0$$

holds;

- (ii) *properly quasi-monotone on C if for all $x_1, x_2, \dots, x_n \in C$, and all $x \in \text{conv}\{x_1, \dots, x_n\}$, there exists $i \in \{1, \dots, n\}$ such that*

$$\langle x_i^*, x_i - x \rangle \geq 0 \quad \forall x_i^* \in T(x_i).$$

It is known that proper quasi-monotonicity implies quasi-monotonicity (see, for example, [21]).

While studying pseudomonotone operators, Hadjisavvas introduced in [16] the notion of *upper sign-continuity*, which is a weak version of directional upper semicontinuity. After that, the concept was reused in [5, 18, 19, 20, 21, 17] and it has been proved to be well adapted to quasi-monotone operators. It is worth mentioning that this concept plays a fundamental role in the existence results of [5] and [17]. We recall the definition of upper sign-continuity and its local version.

Definition 2.2.3 Let C be a nonempty convex subset of X and let $T : C \rightrightarrows X^*$ be a set-valued map with nonempty values. We say that T is

(i) upper sign-continuous on C if for every $x, y \in C$, the implication

$$\left(\forall t \in]0, 1[, \inf_{x_t^* \in T(x_t)} \langle x_t^*, y - x \rangle \geq 0 \right) \implies \sup_{x^* \in T(x)} \langle x^*, y - x \rangle \geq 0$$

holds, where $x_t := (1 - t)x + ty$;

(ii) locally upper sign-continuous on C if for every $x \in C$, there exists a convex neighbourhood V_x and an upper sign-continuous map $\Phi_x : V_x \cap C \rightrightarrows X^*$ with nonempty convex w^* -compact values satisfying that $\Phi_x(y) \subseteq T(y) \setminus \{0\}$ for all $y \in V_x \cap C$.

Remark 2.2.4 It is important to observe that, due to the condition that 0 is not an element of the submap $\Phi_x(y)$, upper sign-continuity of a set-valued map does not imply in general its local upper sign-continuity. Nevertheless, if $0 \notin T(x)$ for each $x \in C$ and if T has nonempty convex w^* -compact values, then upper sign-continuity implies local upper sign-continuity.

2.2.2 Product-type set-valued maps

Let I be a finite index set, that is, $I = \{1, 2, \dots, n\}$. For each $i \in I$, let X_i be a Banach space with dual X_i^* , and C_i be a nonempty subset of X_i . We write

$$C = \prod_{i \in I} C_i, \quad C_{-i} = \prod_{j \neq i, j \in I} C_j, \quad X = \prod_{i \in I} X_i, \quad X^* = \prod_{i \in I} X_i^*. \quad (2.2.6)$$

For each $x \in X$ and $i \in I$, we write $x = (x_i, x_{-i})$, which is a classical shortcut to denote the vector $x = (x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_n)$, where $x_i \in X_i$.

For each $i \in I$ and each $x_{-i} \in C_{-i}$, let $T_i(\cdot, x_{-i}) : C_i \rightrightarrows X_i^*$ and $K_i(\cdot, x_{-i}) : C_i \rightrightarrows C_i$ be two set-valued maps. We set

$$T(x) = \prod_{i \in I} T_i(x_i, x_{-i}) \quad \text{and} \quad K(x) = \prod_{i \in I} K_i(x_i, x_{-i}). \quad (2.2.7)$$

In what follows, we will refer to the maps $T_i(\cdot, x_{-i})$ and $K_i(\cdot, x_{-i})$ (for all $i \in I$) as the *component operators* and the maps T and K as the *product operators*.

As we stated before, our aim is to extend the results of [5, 17] to quasi-variational inequalities over product sets, only assuming the hypotheses on the component operators. Indeed our motivation comes from the fact that, even for variational inequality problems (that is, $K_i(x_i, x_{-i}) = C_i$ for every $(x_i, x_{-i}) \in C$) the main hypotheses of the existence results in [5, 17], namely quasi-monotonicity and local upper sign-continuity, are not preserved by the product operators.

Example 1 Let $C_1 = [-2, 2]$, $C_2 = [-2, 2]$, and $C = [-2, 2] \times [-2, 2]$. For any $x_2 \in C_2$, let $T_1(\cdot, x_2) : C_1 \rightrightarrows \mathbb{R}$ be defined by $T_1(x_1, x_2) = \{x_1^2\}$. For $x_1 \in C_1$, let $T_2(x_1, \cdot) : C_2 \rightrightarrows \mathbb{R}$ be defined by $T_2(x_1, x_2) = \{1 + x_2^2\}$. Then, both component operators are quasi-monotone, but the product operator $T : C \rightrightarrows \mathbb{R}^2$ defined by $T(x) = \{x_1^2\} \times \{1 + x_2^2\}$ is not.

Proof. First, let us observe that for any $(x_1, x_2) \in C$, the set-valued maps $T_1(\cdot, x_2)$ and $T_2(x_1, \cdot)$ are both quasi-monotone. Indeed, it is enough to note that they are the derivatives of the quasi-convex functions $x_1 \mapsto x_1^3/3$ and $x_2 \mapsto x_2 + x_2^3/3$, respectively (for a survey in quasi-convexity and its relation with quasi-monotone operators, see [82]).

However, the product operator T is not quasimonotone on C . Let us consider the points $x = (0, 1/2)$ and $y = (-2, 1)$. Then, for $x^* \in T(x)$ we have

$$\langle x^*, y - x \rangle = \left\langle \begin{pmatrix} x_1^2 \\ 1 + x_2^2 \end{pmatrix}, \begin{pmatrix} y_1 - x_1 \\ y_2 - x_2 \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} 0 \\ 5/4 \end{pmatrix}, \begin{pmatrix} -2 \\ 1/2 \end{pmatrix} \right\rangle = \frac{5}{8} > 0.$$

But, for $y^* \in T(y)$ we have that

$$\langle y^*, y - x \rangle = \left\langle \begin{pmatrix} y_1^2 \\ 1 + y_2^2 \end{pmatrix}, \begin{pmatrix} y_1 - x_1 \\ y_2 - x_2 \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} 4 \\ 2 \end{pmatrix}, \begin{pmatrix} -2 \\ 1/2 \end{pmatrix} \right\rangle = -7 < 0,$$

which contradicts Definition 2.2.2.(i), finishing the proof. \square

Example 2 Let $C_1 = [-1, 1]$, $C_2 = [-1, 1]$, and $C = [-1, 1] \times [-1, 1]$. For $x_2 \in C_2$, let $T_1(\cdot, x_2) : C_1 \rightrightarrows \mathbb{R}$ be defined by $T_1(x_1, x_2) = \{-1\}$. For $x_1 \in C_1$, let $T_2(x_1, \cdot) : C_2 \rightrightarrows \mathbb{R}$ be defined by

$$T_2(x_1, x_2) = \begin{cases} \{1\} & \text{if } x_2 < 0 \\ \{\frac{1}{2}\} & \text{if } x_2 = 0 \\ \{1\} & \text{if } x_2 > 0. \end{cases}$$

Then, each component operator is upper sign-continuous but the product operator $T : C \rightrightarrows \mathbb{R}^2$ given by $T(x) = T_1(x_1, x_2) \times T_2(x_1, x_2)$ is not even locally upper sign-continuous.

Proof. Note first that, for any $x_2 \in C_2$, $T_1(\cdot, x_2)$ is constant, and thus it is obviously upper sign-continuous on C_1 . Now, for $x_1 \in C_1$, let us show that $T_2(x_1, \cdot)$ is also upper sign-continuous on C_2 . Indeed, choose $v, w \in C_2$ such that

$$\forall t \in]0, 1[, \inf_{v_t^* \in T_2(x_1, v_t)} \langle v_t^*, w - v \rangle \geq 0,$$

with $v_t = (1 - t)v + tw$. Then, it is not hard to realize that $w - v \geq 0$. Thus, since the only element $v^* \in T_2(x_1, v)$ is positive, we get that $\sup_{v^* \in T_2(x_1, v)} \langle v^*, w - v \rangle \geq 0$, thus concluding that $T_2(x_1, \cdot)$ is upper sign-continuous as we claimed.

Now, let us prove that the product operator T is not locally upper sign-continuous on C . Let us consider $x = (0, 0) \in C$ and $r > 0$. Since T is single valued, the only suboperator that one can consider is $\Phi_x = T|_{B(x, r) \cap C}$.

However, considering $y = (r/2, r/2) \in B(x, r) \cap C$ and writing $x_t = (1 - t)x + ty = t(r/2, r/2)$ we have that

$$\inf_{x_t^* \in \Phi_x(x_t)} \langle x_t^*, y - x \rangle = \left\langle \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} r/2 \\ r/2 \end{pmatrix} \right\rangle = 0 \quad \forall t \in]0, 1[,$$

but

$$\sup_{x^* \in \Phi_x(x)} \langle x^*, y - x \rangle = \left\langle \begin{pmatrix} -1 \\ 1/2 \end{pmatrix}, \begin{pmatrix} r/2 \\ r/2 \end{pmatrix} \right\rangle = -\frac{r}{4} < 0,$$

which yields that Φ_x is not upper sign-continuous. Since Φ_x and r are arbitrary, T is not locally upper sign-continuous. \square

Remark 2.2.5 Note that, thanks to Remark 2.2.4, Example 2 shows that both, upper sign-continuity and local upper sign-continuity are not preserved by the product operator.

Our main aim in this work is to state existence results for product-type quasi-variational inequalities. As an example, we present the following main theorem, proved in Section 3.3 as Corollary 2.4.4, which provides some weak sufficient condition for the existence of solutions of such problems.

Theorem 2.2.6 For each $i \in I$, let C_i be a nonempty weakly compact convex subset of X_i , $T_i : C_i \times C_{-i} \rightrightarrows X_i^*$ be a set-valued map with nonempty convex values and $K_i : C_i \times C_{-i} \rightrightarrows C_i$ be a set-valued map with nonempty values. Consider T and K defined as in (2.2.7). Assume that

- (i) for each $i \in I$, the set-valued map $K_i : C_i \times C_{-i} \rightrightarrows C_i$ is weakly closed and its values are convex with nonempty interior;
- (ii) for each $i \in I$ and each $x_{-i} \in C_{-i}$, $T_i(\cdot, x_{-i}) : C_i \rightrightarrows X_i^*$ is quasi-monotone and locally upper sign-continuous;
- (iii) for each $i \in I$, the pair of set-valued maps $(T_i, \text{int}K_i)$ is weakly net-lower sign-continuous with respect to the parameter pair (C_i, C_{-i}) .

Then $\text{QVI}^*(T, K)$ is nonempty.

Note that in this theorem condition (iii) is based on a new concept, called *net-lower sign-continuity*, linking the operators T_i and K_i . It will be introduced and studied in Section 2.3 and is used as a minimal hypothesis in order to obtain some stability results needed in the proof of Theorem 2.2.6. These stability results follow the spirit of [19] and [21].

In several senses, the above result is an improvement of the existence theorems in [5] and [17]. First, it works with quasi-variational inequalities in the infinite-dimensional setting. Second, it shows the existence of solutions for quasi-variational inequalities over product sets, regardless of the obstructions presented in Examples 1 and 2. Finally, net-lower sign-continuity is a weaker hypothesis with respect to the settings followed by [19] and [21].

The proof of Theorem 2.2.6 is based on Kakutani's fixed point theorem (see, e.g., [83, Theorem 6.4.10]) and follows the technique inspired by the proof of [11, Theorem 4.3.1], also used in [5]. This theorem presents the classic existence result for equilibria in abstract economies, and the main idea of the proof is to apply a fixed point theorem to the product of *specific parametrized argmin-sets*. Even though the points in these argmin-sets are not necessarily coherent with the abstract economy, a fixed point of their product becomes an equilibrium. In [5], this technique is applied directly to the parametrized sets $S^*(T, K(x))$, when T and K are not product operators. In our setting, this approach is not possible, since $S^*(T, K(x))$ may not enjoy the necessary properties that we need. Thus, we introduced new suitable parametrized sets, associated with perturbed Minty-type variational inequalities, and we adjust this technique to obtain our main result.

2.3 Stability for perturbed Minty-type problems

In this section we introduce the notions of *net-lower sign-continuity* (subsection 2.3.1) and *extended-Minty variational inequalities* (subsection 2.3.2). As we mentioned before, both notions are needed to prove Theorem 2.2.6.

2.3.1 Net-lower sign-continuity

Definition 2.3.1 Let (U, τ_U) and (Λ, τ_Λ) be two topological spaces, Y be a Banach space, and τ_Y be a locally convex topology consistent with the duality $\langle Y, Y^* \rangle$. Let $T : Y \times \Lambda \rightrightarrows Y^*$ and $K : U \times \Lambda \rightrightarrows Y$ be two set-valued maps. The pair (T, K) is said to be $(\tau_U \times \tau_\Lambda)$ - τ_Y net-lower sign-continuous with respect to the parameter pair (U, Λ) at $(\mu, \lambda) \in U \times \Lambda$ and $y \in K(\mu, \lambda)$ if for every net $(\mu_\alpha, \lambda_\alpha)_\alpha \subseteq U \times \Lambda$ converging to (μ, λ) , every $z \in \overline{K(\mu, \lambda)}^{\tau_Y}$, and every selection $(z_\alpha)_\alpha$ of $(\overline{K(\mu_\alpha, \lambda_\alpha)}^{\tau_Y})_\alpha$ τ_Y -converging to z , the following condition holds:

$$\left\{ \begin{array}{l} \text{If for every subnet } (\mu_\beta, \lambda_\beta)_\beta \text{ of } (\mu_\alpha, \lambda_\alpha)_\alpha \text{ and every selection } (y_\beta)_\beta \text{ of } (K(\mu_\beta, \lambda_\beta))_\beta \\ \tau_Y\text{-converging to } y \text{ one has that} \\ \\ \limsup_{\beta} \sup_{y_\beta^* \in T(y_\beta, \lambda_\beta)} \langle y_\beta^*, z_\beta - y_\beta \rangle \leq 0, \\ \\ \text{then, } \sup_{y^* \in T(y, \lambda)} \langle y^*, z - y \rangle \leq 0, \end{array} \right.$$

where $(z_\beta)_\beta$ is the corresponding subnet of $(z_\alpha)_\alpha$ induced by the index set of $(\mu_\beta, \lambda_\beta)_\beta$.

We simply say that (T, K) is $(\tau_U \times \tau_\Lambda)$ - τ_Y net-lower sign-continuous with respect to the parameter pair (U, Λ) if it is so at each $(\mu, \lambda) \in U \times \Lambda$ and each $y \in K(\mu, \lambda)$.

If there is no ambiguity, we may omit the parameter pair (U, Λ) , and the topologies of U and Λ , saying only that the pair (T, K) is τ_Y net-lower sign-continuous. If τ_Y is the norm topology, we say that (T, K) is norm net-lower sign-continuous, and if τ_Y is the weak topology, we say that (T, K) is weakly net-lower sign-continuous.

If T is fixed and K depends only on U , that is, $K : U \rightrightarrows Y$, we will say that (T, K) is τ_U - τ_Y net-lower sign-continuous with respect to U if, considering the natural extension $\hat{K} : U \times \{0\} \rightrightarrows Y$ and $\hat{T} : Y \times \{0\} \rightrightarrows Y^*$ given by $\hat{K}(\mu, 0) = K(\mu)$ and $\hat{T}(y, 0) = T(y)$, the pair (\hat{T}, \hat{K}) is $(\tau_U \times \{\emptyset, \{0\}\})$ - τ_Y net-lower sign-continuous with respect to the parameter pair $(U, \{0\})$.

Note that if τ_U , τ_Λ , and τ_Y are first countable topologies, then we can replace nets by sequences in Definition 3.3.2.

Proposition 2.3.2 Let (U, τ_U) and (Λ, τ_Λ) be two topological spaces, Y be a Banach space and τ_Y be a locally convex topology consistent with the duality $\langle Y, Y^* \rangle$. Suppose that all three topologies are first countable. Let $T : Y \times \Lambda \rightrightarrows Y^*$ and $K : U \times \Lambda \rightrightarrows Y$ be two set-valued maps. Then, the pair (T, K) is $(\tau_U \times \tau_\Lambda)$ - τ_Y net-lower sign-continuous with respect to the parameter pair (U, Λ) at $(\mu, \lambda) \in U \times \Lambda$ and $y \in K(\mu, \lambda)$ if and only if for every sequence $(\mu_n, \lambda_n)_n \subseteq U \times \Lambda$ converging to (μ, λ) , every $z \in \overline{K(\mu, \lambda)}^{\tau_Y}$, and every selection $(z_n)_n$ of $(\overline{K(\mu_n, \lambda_n)}^{\tau_Y})_n$ τ_Y -converging to z , the following condition holds:

$$\left\{ \begin{array}{l} \text{If for every subsequence } (\mu_{n_k}, \lambda_{n_k})_k \text{ of } (\mu_n, \lambda_n)_n \text{ and every selection } (y_{n_k})_k \text{ of} \\ (K(\mu_{n_k}, \lambda_{n_k}))_k \text{ } \tau_Y\text{-converging to } y \text{ one has that} \\ \\ \limsup_k \sup_{y_{n_k}^* \in T(y_{n_k}, \lambda_{n_k})} \langle y_{n_k}^*, z_{n_k} - y_{n_k} \rangle \leq 0, \\ \\ \text{then, } \sup_{y^* \in T(y, \lambda)} \langle y^*, z - y \rangle \leq 0. \end{array} \right.$$

Proof. To simplify the notation, let us define the support function $\sigma : \Lambda \times Y \times Y \rightarrow \overline{\mathbb{R}}$ given by

$$\sigma(\lambda, y, z) := \sup_{y^* \in T(y, \lambda)} \langle y^*, z - y \rangle.$$

Since there is no ambiguity, we will omit the involved topologies. For the sufficiency, assume that the sequential condition holds for (μ, λ) and y , but that there exist a net $(\mu_\alpha, \lambda_\alpha)_\alpha$ converging to (μ, λ) , an element $z \in \overline{K(\mu, \lambda)}$, and a selection $(z_\alpha)_\alpha$ of $\left(\overline{K(\mu_\alpha, \lambda_\alpha)}\right)_\alpha$ converging to z such that

$$\begin{aligned} \forall (\mu_\beta, \lambda_\beta)_\beta \text{ subnet, } \forall (y_\beta)_\beta \text{ selection of } (K(\mu_\beta, \lambda_\beta))_\beta \text{ converging to } y, \\ \limsup_\beta \sigma(\lambda_\beta, y_\beta, z_\beta) \leq 0, \text{ and } \sigma(\lambda, y, z) > 0. \end{aligned} \quad (2.3.1)$$

Let us denote by A the set of indexes of this net. We claim that for every $\varepsilon > 0$ the following statement holds:

$$\exists \alpha_\varepsilon \in A, \exists V_\varepsilon \in \mathcal{N}(y) \quad \forall \alpha \geq \alpha_\varepsilon, \forall y_\alpha \in K(\mu_\alpha, \lambda_\alpha) \cap V_\varepsilon, \sigma(\lambda_\alpha, y_\alpha, z_\alpha) \leq \varepsilon. \quad (2.3.2)$$

If not, there would exist $\varepsilon > 0$ such that for all $\alpha \in A$ and all neighborhoods $V \in \mathcal{N}(y)$, there exist $\alpha_V \geq \alpha$ and $y_{\alpha_V} \in K(\mu_{\alpha_V}, \lambda_{\alpha_V}) \cap V$ with $\sigma(\lambda_{\alpha_V}, y_{\alpha_V}, z_{\alpha_V}) > \varepsilon$. Now, consider the index set D given by all tuples (α, V, α_V) given as before, with the following preorder:

$$(\alpha, V, \alpha_V) \geq (\alpha', V', \alpha'_{V'}) \iff \alpha \geq \alpha', V \subseteq V' \text{ and } \alpha_V \geq \alpha'_{V'}.$$

Then, considering the function $\varphi : D \rightarrow A$ given by $\varphi(\alpha, V, \alpha_V) = \alpha_V$, and noting that D is a directed set, it is not hard to see that $(\mu_d, \lambda_d)_{d \in D}$ (with the identification $(\mu_d, \lambda_d) = (\mu_{\varphi(d)}, \lambda_{\varphi(d)})$) is a subnet of $(\mu_\alpha, \lambda_\alpha)_{\alpha \in A}$. Now, for each $d = (\alpha, V, \alpha_V) \in D$ we can choose the element $y_d = y_{\varphi(d)} \in K(\mu_d, \lambda_d)$ given by the construction of the index set D , entailing that $y_d \rightarrow y$ and that

$$\limsup_d \sigma(\lambda_d, y_d, z_d) \geq \varepsilon.$$

This is a contradiction with (2.3.1) and so the claim is proved.

Now, let $(O_n)_{n \in \mathbb{N}}$ and $(W_n)_{n \in \mathbb{N}}$ be two decreasing bases of neighbourhoods of $\mathcal{N}(\mu, \lambda)$ and $\mathcal{N}(0)$, respectively. Using condition (2.3.2), we may choose a sequence $(\alpha_n)_n$ in A such that for all $n \in \mathbb{N}$

1. $\alpha_n \leq \alpha_{n+1}$;
2. $z_{\alpha_n} \in z + W_n$ and $(\mu_{\alpha_n}, \lambda_{\alpha_n}) \in O_n$;
3. $\alpha_n \geq \alpha_{1/n}$, where $(\alpha_{1/n}, V_{1/n})$ is the index-neighbourhood pair given by (2.3.2) for $\varepsilon = 1/n$.

Now, clearly $(\mu_{\alpha_n}, \lambda_{\alpha_n}) \rightarrow (\mu, \lambda)$ and $z_{\alpha_n} \rightarrow z$. Let $(\mu_{\alpha_{n_k}}, \lambda_{\alpha_{n_k}})_k$ be a subsequence of $(\mu_{\alpha_n}, \lambda_{\alpha_n})$ and $(y_{n_k})_k$ be a selection of $K(\mu_{\alpha_{n_k}}, \lambda_{\alpha_{n_k}})$ converging to y . For every $m \in \mathbb{N}$ and every k large enough we have that

$$\alpha_{n_k} \geq \alpha_{1/m} \quad \text{and} \quad y_{n_k} \in V_{1/m},$$

and so

$$\limsup_k \sigma(\lambda_{\alpha_{n_k}}, y_{n_k}, z_{\alpha_{n_k}}) \leq 1/m.$$

Since this holds for every $m \in \mathbb{N}$, and the subsequence $(\mu_{\alpha_{n_k}}, \lambda_{\alpha_{n_k}})_k$ and the selection $(y_{n_k})_k$ are arbitrary, we deduce by (??) that $\sigma(\lambda, y, z) \leq 0$, which is a contradiction. We conclude then that (T, K) is net-lower sign-continuous at (μ, λ) and y .

For the necessity, assume that (T, K) is net-lower sign-continuous with respect to the parameter pair (U, Λ) at $(\mu, \lambda) \in U \times \Lambda$ and $y \in K(\mu, \lambda)$, but suppose that there exists a sequence (μ_n, λ_n) in $U \times \Lambda$ converging to (μ, λ) , an element $z \in \overline{K(\mu, \lambda)}$ and a selection $(z_n)_n$ of $\left(\overline{K(\mu_n, \lambda_n)}\right)_n$ converging to z such that for every subsequence $(\mu_{n_k}, \lambda_{n_k})_k$ of $(\mu_n, \lambda_n)_n$ and every selection $(y_{n_k})_k$ of $(K(\mu_{n_k}, \lambda_{n_k}))_k$ converging to y one has that

$$\limsup_k \sigma(\lambda_{n_k}, y_{n_k}, z_{n_k}) \leq 0,$$

but, $\sigma(\lambda, y, z) > 0$. Then, there exists a subnet $(\mu_\beta, \lambda_\beta)_\beta$ converging to (μ, λ) and a selection $(y_\beta)_\beta$ of $(K(\mu_\beta, \lambda_\beta))_\beta$ converging to y such that

$$\limsup_\beta \sigma(\lambda_\beta, y_\beta, z_\beta) > 0.$$

Let B be the directed index set of the subnet $(\mu_\beta, \lambda_\beta)_\beta$ and let $\varphi : B \rightarrow \mathbb{N}$ be the index function given by the definition of subnets (see subsection 2.2.1). This yields that there exists $\varepsilon > 0$ such that

$$\forall \beta \in B \exists \beta' \geq \beta \text{ such that } \sigma(\lambda_{\beta'}, y_{\beta'}, z_{\beta'}) > \varepsilon. \quad (2.3.3)$$

Now, let $(W_k)_{k \in \mathbb{N}}$ be a decreasing base of neighbourhoods of $\mathcal{N}(0)$. Using (2.3.3), we may choose a sequence $(\beta_k)_k$ in B such that, for all $k \in \mathbb{N}$,

1. $y_{\beta_k} \in y + W_k$;
2. $\beta_{k+1} \geq \beta_k$ and $\varphi(\beta_{k+1}) > \varphi(\beta_k)$;
3. $\sigma(\lambda_{\beta_k}, y_{\beta_k}, z_{\beta_k}) > \varepsilon, \forall k \in \mathbb{N}$.

It is not hard to see that $(\mu_{\beta_k}, \lambda_{\beta_k})_k$ is a subsequence of $(\mu_n, \lambda_n)_n$, and $(y_{\beta_k})_k$ is converging to y . However, we have that

$$\limsup_k \sigma(\lambda_{\beta_k}, y_{\beta_k}, z_{\beta_k}) = \inf_k \sup_{l \geq k} \sigma(\lambda_{\beta_l}, y_{\beta_l}, z_{\beta_l}) \geq \varepsilon$$

which is a contradiction, finishing the proof. □

Net-lower sign-continuity seems to be rather technical. Nevertheless, it can be verifiable for a large family of set-valued maps. The following proposition gives a sufficient condition to have norm net-lower sign-continuity.

Proposition 2.3.3 *Let (Λ, τ_Λ) and (U, τ_U) be two first countable topological spaces and Y be a Banach space. Let $T : Y \times \Lambda \rightrightarrows Y^*$ and $K : U \times \Lambda \rightrightarrows Y$ be two set-valued maps with nonempty values. Suppose that for every sequence $(\mu_n, \lambda_n)_n \subseteq U \times \Lambda$ converging to (μ, λ) and every $y \in K(\mu, \lambda)$ we have that*

$$T(y, \lambda) \subseteq \overline{\text{conv}} \left(\bigcup w^*\text{-seq-Limsup}_k T(y_{n_k}, \lambda_{n_k}) \right), \quad (2.3.4)$$

where the union is taken over all subsequences $(\mu_{n_k}, \lambda_{n_k})_k$ of $(\mu_n, \lambda_n)_n$ and all selections $(y_{n_k})_k$ of $(K(\mu_{n_k}, \lambda_{n_k}))_k$ converging to y , and seq-Limsup stands for the Painlevé-Kuratowski sequential upper limit of sets (see, e.g., [87, Definition 1.1.3]). Then, the pair (T, K) is norm net-lower sign-continuous.

Proof. For every $(\mu, \lambda) \in U \times \Lambda$ and every $y \in K(\mu, \lambda)$, let us denote by $A(y, \lambda)$ the set inside the closed convex hull on the right-hand side of the inclusion of (2.3.4).

Let $(\mu_n, \lambda_n)_n$ be a sequence in $U \times \Lambda$ converging to (μ, λ) , $y \in K(\mu, \lambda)$, and let $(z_n)_n$ be a selection of $(K(\mu_n, \lambda_n))_n$ converging to $z \in K(\mu, \lambda)$, and suppose the hypothesis of the implication of (??) holds. Let $y^* \in A(y, \lambda)$. We claim that $\langle y^*, z - y \rangle \leq 0$.

Indeed, since $y^* \in A(y, \lambda)$, there exists a subsequence $(\mu_{n_k}, \lambda_{n_k})_k$ of $(\mu_n, \lambda_n)_n$ and a selection $(y_{n_k})_k$ of $(K(\mu_{n_k}, \lambda_{n_k}))_k$ converging to y such that

$$y^* \in w^*\text{-seq-Limsup}_k T(y_{n_k}, \lambda_{n_k}).$$

Without loss of generality, we may assume that there exists a sequence $(y_{n_k}^*)_k$ w^* -converging to y^* , with $y_{n_k}^* \in T(y_{n_k}, \lambda_{n_k})$ for all $k \in \mathbb{N}$. Since $(y_{n_k}^*)_k$ is bounded thanks to the uniform boundedness principle, we can write

$$\begin{aligned} \langle y^*, z - y \rangle &= \lim_k \langle y_{n_k}^*, z_{n_k} - y_{n_k} \rangle \\ &\leq \limsup_k \sup_{w_{n_k}^* \in T(y_{n_k}, \lambda_{n_k})} \langle w_{n_k}^*, z_{n_k} - y_{n_k} \rangle \leq 0. \end{aligned}$$

Thus, our claim is proven. Then, it is not hard to see that

$$\sup_{y^* \in T(y, \lambda)} \langle y^*, z - y \rangle \leq \sup_{y^* \in \overline{\text{conv}}A(y, \lambda)} \langle y^*, z - y \rangle = \sup_{y^* \in A(y, \lambda)} \langle y^*, z - y \rangle \leq 0,$$

proving, in view of Proposition 2.3.2, that the pair (T, K) is norm net-lower sign-continuous. \square

Remark 2.3.4 In Proposition 2.3.3, seq-Limsup can be replaced by the Painlevé - Kuratowski upper limit, and the local boundedness of T must be assumed. Indeed, this assumption is necessary because when one were to use the usual Limsup , then the sequence $(y_{n_k}^*)_k$ would be replaced by a net, for which one cannot directly apply the uniform boundedness principle.

Note that the inclusion (2.3.4) is quite well known in convex analysis. For example, let us consider two finite dimensional spaces Y_1 and Y_2 , and set $\Lambda = Y_2$, and $K : Y_2 \rightrightarrows Y_1$ given by $K(y_2) := Y_1$. If we consider any function $f : Y_1 \times Y_2 \rightarrow \mathbb{R}$ which is convex in the first variable and jointly continuous, then defining the operator $T : Y_1 \times Y_2 \rightrightarrows Y_1^*$ by

$$T(y_1, y_2) := \partial(f(\cdot, y_2))(y_1),$$

where $\partial f(\cdot, y_2)$ stands for the convex subdifferential of $f(\cdot, y_2)$, we get that inclusion (2.3.4) holds as a direct consequence of [87, Theorem 7.6.4]. Indeed, following the notation of this theorem, for any sequence $(y_2^n) \subseteq Y_2$ converging to $y_2 \in Y_2$ we can identify $V_n := f(\cdot, y_2^n)$ and $V := f(\cdot, y_2)$. Then, since V is the graphical limit (see [87, Definition 7.1.1]) of V_n thanks to the continuity of f , we can conclude that, for every $y_1 \in Y_1$,

$$T(y_1, y_2) = \partial V(y_1) = \text{Limsup}_{y_1^n \rightarrow y_1} \partial V_n(y_1^n) = \bigcup_{y_1^n \rightarrow y_1} \text{Limsup}_n T(y_1^n, y_2^n).$$

In particular, inclusion (2.3.4) is verified, and thus the pair (T, K) is norm net-lower sign-continuous.

We finish this section with the following proposition, which shows that net-lower sign-continuity is weaker as a hypothesis than those assumed in [5], at least when the operator T is locally bounded, which is the case in most of the applications. Recall that a map $T : C \rightrightarrows X^*$ is said to be *dually lower semicontinuous* if for any $x \in C$ and any sequence $(y_n) \subseteq C$ converging to $y \in C$ the following implication holds:

$$\liminf_n \sup_{y_n^* \in T(y_n)} \langle y_n^*, x - y_n \rangle \leq 0 \implies \sup_{y^* \in T(y)} \langle y^*, x - y \rangle \leq 0. \quad (2.3.5)$$

Proposition 2.3.5 *Let C be a convex w -compact subset of X , and let $K : C \rightrightarrows C$ and $T : C \rightrightarrows X^*$ be two set-valued maps with nonempty values. Suppose that*

- (i) \overline{K} is lower semicontinuous with convex values;
- (ii) T is dually lower semicontinuous and locally bounded.

Then, considering $U = C$ (with its induced strong topology), we have that both (T, K) and (T, \overline{K}) are norm net-lower sign-continuous with respect to U .

Proof. We will only prove that (T, K) is norm net-lower sign-continuous with respect to U . The case (T, \overline{K}) is similar. Since all the topologies involved are first countable, it is enough to prove the sequential characterization of net-lower sign-continuity given by Proposition 2.3.2. Thus, let us consider a point $\mu \in U$, a point $y \in K(\mu)$, a sequence (μ_n) converging to μ , and a selection z_n of $(\overline{K}(\mu_n))_n$ converging to some point $z \in \overline{K}(\mu)$, and assume that the hypothesis of (??) holds. Since \overline{K} is lower semicontinuous, there exists a selection (y_n) of $\overline{K}(\mu_n)$ converging to y . Furthermore, without loss of generality, we may take $y_n \in K(\mu_n)$, for each $n \in \mathbb{N}$. Then, we can write

$$\liminf_n \sup_{y_n^* \in T(y_n)} \langle y_n^*, z_n - y_n \rangle \leq \limsup_n \sup_{y_n^* \in T(y_n)} \langle y_n^*, z_n - y_n \rangle \leq 0.$$

Now, since T is locally bounded, for every $\varepsilon \geq 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that, for every $k \geq n_\varepsilon$,

$$\begin{aligned} \sup_{y_k^* \in T(y_k)} \langle y_k^*, z - y_k \rangle &\leq \sup_{y_k^* \in T(y_k)} \langle y_k^*, z_k - y_k \rangle + M \|z - z_k\| \\ &\leq \sup_{y_k^* \in T(y_k)} \langle y_k^*, z_k - y_k \rangle + \varepsilon, \end{aligned}$$

where $M > 0$ is a constant such that $T(y_n) \subseteq B_{X^*}(0, M)$ for every $n \in \mathbb{N}$ large enough. We get that

$$\liminf_n \sup_{y_n^* \in T(y_n)} \langle y_n^*, z - y_n \rangle \leq \liminf_n \sup_{y_n^* \in T(y_n)} \langle y_n^*, z_n - y_n \rangle + \varepsilon \leq \varepsilon,$$

and since ε is arbitrary, we deduce that $\liminf_n \sup_{y_n^* \in T(y_n)} \langle y_n^*, z - y_n \rangle \leq 0$. Since T is dually lower semicontinuous, this yields $\sup_{y^* \in T(y)} \langle y^*, z - y \rangle \leq 0$, and so condition (??) is verified, finishing the proof. \square

2.3.2 Extended Minty variational inequalities

Definition 2.3.6 *Let C be a nonempty subset of a Banach space Y , and let $T : C \rightrightarrows Y^*$ be a set-valued map. We define the extended-Minty variational inequality as follows:*

$$\text{find } y \in \overline{C} \text{ such that } \langle z^*, z - y \rangle \geq 0, \quad \forall z \in C, \forall z^* \in T(z). \quad (2.3.6)$$

We denote by $M^E(T, C)$ both the extended-Minty variational inequality associated with T and C and its set of solutions.

Clearly, one always has $M(T, C) \subseteq M^E(T, C)$. Furthermore, if C is closed, then $M(T, C) = M^E(T, C)$.

A particularly interesting extended-Minty variational inequality is the one we obtain when we consider $\text{int}C$ instead of C . The following lemma shows the relations between $M^E(T, \text{int}C)$ and $S(T, C)$ when C is a nonempty convex closed set.

Lemma 2.3.7 *Let C be a nonempty closed convex subset of a Banach space Y with $\text{int}C \neq \emptyset$ and let $T : C \rightrightarrows Y^*$ be a set-valued map.*

- (i) *If T is upper sign-continuous on C with w^* -compact convex values, then $M^E(T, \text{int}C) \subseteq S(T, C)$.*
- (ii) *If T is locally upper sign-continuous on C , then $M^E(T, \text{int}C) \subseteq S^*(T, C)$.*
- (iii) *If T is quasi-monotone, then $S^*(T, C) \subseteq M^E(T, \text{int}C)$.*

Proof. (i) Let y be an element of $M^E(T, \text{int}C)$. Since C is convex, for any $z \in \text{int}C$ and any $t \in]0, 1[$ we have that $y_t = (1-t)y + tz \in \text{int}C$. Then, for every $t \in]0, 1[$,

$$\inf_{y_t^* \in T(y_t)} \langle y_t^*, z - y \rangle = 1/t \inf_{y_t^* \in T(y_t)} \langle y_t^*, y_t - y \rangle \geq 0.$$

Finally, since T is upper sign-continuous and w^* -compact valued, we have

$$\forall z \in \text{int}C, \max_{y^* \in T(y)} \langle y^*, z - y \rangle \geq 0. \quad (2.3.7)$$

Applying Sion's minimax theorem (see [90]), we get that

$$\inf_{z \in \text{int}C} \max_{y^* \in T(y)} \langle y^*, z - y \rangle = \max_{y^* \in T(y)} \inf_{z \in \text{int}C} \langle y^*, z - y \rangle = \max_{y^* \in T(y)} \inf_{z \in C} \langle y^*, z - y \rangle.$$

where the last equality follows since $C = \overline{\text{int}C}$ and each $y^* \in Y^*$ is continuous. Then, by (2.3.7), we conclude that $y \in S(T, C)$.

(ii) Let y be an element of $M^E(T, \text{int}C)$. Since T is locally upper sign-continuous at y , there exists a convex neighbourhood V_y of y and an upper sign-continuous map $\Phi_y : V_y \cap C \rightrightarrows Y^*$ with nonempty convex w^* -compact values satisfying $\Phi_y(z) \subseteq T(z) \setminus \{0\}, \forall z \in V_y \cap C$.

Now, let $z \in \text{int}C$. There exists z_1 such that $z_1 = (1-t)y + tz \in [y, z] \cap V_y \cap \text{int}C$ (with $0 < t < 1$), and so one has

$$0 \leq \langle v^*, v - y \rangle = t' \langle v^*, z_1 - y \rangle = t't \langle v^*, z - y \rangle,$$

for all $v \in]y, z_1] \subseteq \text{int}C$ and all $v^* \in \Phi_y(v)$ (where $t' \in]0, 1[$) is such that $v = (1-t')y + t'z_1$). Hence $\inf_{v^* \in \Phi_y(v)} \langle v^*, z_1 - y \rangle \geq 0$ and, according to the upper sign-continuity of Φ_y , $\sup_{y^* \in \Phi_y(y)} \langle y^*, z_1 - y \rangle \geq 0$. In addition, since $\Phi_y(y)$ is w^* -compact, there exists $y^* \in \Phi_y(y)$ such that $\langle y^*, z_1 - y \rangle \geq 0$ and therefore $\langle y^*, z - y \rangle \geq 0$. In other words, we have

$$\forall z \in \text{int}C, \max_{y^* \in \Phi_y(y)} \langle y^*, z - y \rangle \geq 0. \quad (2.3.8)$$

At this point, we can do the same as in the proof of (i) and conclude that $y \in S(\Phi_y, C) \subseteq S^*(T, C)$.

(iii) Let y be an element of $S^*(T, C)$ and $y^* \in T(y) \setminus \{0\}$ such that $\langle y^*, z - y \rangle \geq 0$ for all $z \in C$. Then, for all $z \in \text{int}C$, one has $\langle y^*, z - y \rangle > 0$ and thus, by quasi-monotonicity, $\langle z^*, z - y \rangle \geq 0$ for each $z^* \in T(z)$. This yields that $y \in M^E(T, \text{int}C)$, finishing the proof. \square

From [91], it is well known that if T is properly quasi-monotone (see Definition 2.2.2) and C is a weakly compact and convex subset of a Banach space, then the (classical) Minty variational inequality admits at least one solution, that is $M(T, C) \neq \emptyset$. The proposition below describes some sufficient conditions under which the extended-Minty variational inequality with respect to $\text{int}C$ has some solutions, that is, $M^E(T, \text{int}C) \neq \emptyset$.

Proposition 2.3.8 *Let C be a nonempty weakly compact convex subset of X with $\text{int}C \neq \emptyset$ and let $T : C \rightrightarrows X^*$ be quasi-monotone and locally upper sign-continuous. Then $M^E(T, \text{int}C)$ is nonempty.*

Proof. Since the set-valued map T is quasi-monotone and locally upper sign-continuous, then the set-valued map $T \setminus \{0\}$ is also quasi-monotone and locally upper sign-continuous. In addition, since C is a nonempty weakly compact convex set, we can apply [17, Theorem 2.1], obtaining that $S(T \setminus \{0\}, C) \neq \emptyset$. Since $S(T \setminus \{0\}, C) = S^*(T, C)$, the conclusion follows from Lemma 2.3.7(iii). \square

Let us now state the stability result for extended-Minty solution sets.

Proposition 2.3.9 *Let U and Λ be two topological spaces and Y be a Banach space. Let $T : Y \times \Lambda \rightrightarrows Y^*$ and $K : U \times \Lambda \rightrightarrows Y$ be two set-valued maps with nonempty values. Let us suppose that*

- (i) *the set-valued map $\bar{K} : U \times \Lambda \rightrightarrows Y$ given by $\bar{K}(\mu, \lambda) := \overline{K(\mu, \lambda)}$ is $(\tau_U \times \tau_\Lambda)$ - w -closed;*
- (ii) *the pair (T, K) is weakly net-lower sign-continuous with respect to the parameter pair (U, Λ) .*

Then, the set-valued map $\Phi : U \times \Lambda \rightrightarrows Y$ given by

$$\Phi(\mu, \lambda) := M^E(T(\cdot, \lambda), K(\mu, \lambda))$$

is $(\tau_U \times \tau_\Lambda)$ - w -closed.

Proof. Let $(\mu_\alpha, \lambda_\alpha)_\alpha \subseteq U \times \Lambda$ and $(z_\alpha)_\alpha \subseteq Y$ be two nets satisfying that

$$(\mu_\alpha, \lambda_\alpha) \rightarrow (\mu, \lambda), \quad z_\alpha \xrightarrow{w} z \quad \text{and} \quad z_\alpha \in \Phi(\mu_\alpha, \lambda_\alpha).$$

We want to prove that $z \in \Phi(\mu, \lambda)$. Since \bar{K} is $(\tau_U \times \tau_\Lambda)$ - w -closed, we have that $z \in \overline{K(\mu, \lambda)}$. Fix $y \in K(\mu, \lambda)$, let $(\mu_\beta, \lambda_\beta)_\beta$ be a subnet of $(\mu_\alpha, \lambda_\alpha)_\alpha$, and let $(y_\beta)_\beta$ be a selection of $K(\mu_\beta, \lambda_\beta)_\beta$ w -converging to y . Since $z_\alpha \in \Phi(\mu_\alpha, \lambda_\alpha)$, we know that

$$\langle y_\beta^*, z_\beta - y_\beta \rangle \leq 0 \quad \forall y_\beta^* \in T(y_\beta, \lambda_\beta).$$

This yields that

$$\limsup_\beta \sup_{y_\beta^* \in T(y_\beta, \lambda_\beta)} \langle y_\beta^*, z_\beta - y_\beta \rangle \leq 0,$$

and so, since the pair (T, K) is weakly net-lower sign-continuous with respect to the parameter pair (U, Λ) , we conclude that

$$\langle y^*, y - z \rangle \geq 0 \quad \forall y^* \in T(y, \lambda).$$

Since y is arbitrary, $z \in \Phi(\mu, \lambda)$, finishing the proof. \square

Corollary 2.3.10 *Let U and Λ be two topological spaces and Y be a Banach space. Let $T : Y \times \Lambda \rightrightarrows Y^*$ and $K : U \times \Lambda \rightrightarrows Y$ be two set-valued maps with nonempty values. Let us suppose that*

- (i) *the set-valued map K is $(\tau_U \times \tau_\Lambda)$ - w -closed and its values are convex with nonempty interior;*
- (ii) *for every $(\mu, \lambda) \in U \times \Lambda$, $T(\cdot, \lambda)$ is quasi-monotone and locally upper sign-continuous on $K(\mu, \lambda)$;*
- (iii) *the pair $(T, \text{int}K)$ is weakly net-lower sign-continuous with respect to the parameter pair (U, Λ) .*

Then, the set-valued map $\Phi : U \times \Lambda \rightrightarrows Y$ given by

$$\Phi(\mu, \lambda) := S^*(T(\cdot, \lambda), K(\mu, \lambda))$$

is $(\tau_U \times \tau_\Lambda)$ - w -closed.

Proof. Observe that, under hypothesis (ii), Lemma 2.3.7 entails that

$$S^*(T(\cdot, \lambda), K(\mu, \lambda)) = M^E(T(\cdot, \lambda), \text{int}K(\mu, \lambda))$$

for every $(\mu, \lambda) \in U \times \Lambda$. Thus, since $K(\mu, \lambda) = \overline{\text{int}K(\mu, \lambda)}$, we can directly apply Proposition 2.3.9 to obtain the desired conclusion. \square

Remark 2.3.11 Corollary 2.3.10 must be compared with [19, Theorem 4.2] and with [20, Proposition 3.1]. Both results are a direct consequence of Corollary 2.3.10 since the weakly net-lower sign-continuity of the pair (T, K) can be easily derived as a combination of hypotheses (iii) and (iv) of [19, Theorem 4.2], as well as a combination of hypotheses (i) and (iii) of [20, Proposition 3.1]. Furthermore, if the operator T is locally bounded and it is fixed (it doesn't depend on Λ), we can apply Proposition 2.3.5 to derive the net-lower sign-continuity of the pair (T, K) from the lower semicontinuity of K and the dual lower semicontinuity of T . Thus, the above corollary also generalizes [21, Proposition 4.3] for locally bounded operators.

2.4 Existence results for quasi-variational inequality problems

In this section we present our main results, namely, the existence of solutions for quasi-variational inequality problems over product sets, following the hypotheses set out in [17, Theorem 2.1]. Recall that $I, X_i, X_{-i}, C_i, C_{-i}, K_i, T_i, X, K$, and T are defined as in subsection 2.2.2, particularly as in (2.2.6) and (2.2.7).

We divide our results in two cases: (1) we consider the case when $\text{int}K(x) \neq \emptyset$, for any $x \in C$ for which we obtain positive results for both properly quasi-monotone and quasi-monotone operators (see Theorem 2.4.1); and (2) the general case, for which we obtain positive results only for properly quasi-monotone operators (see Theorem 2.4.6).

2.4.1 Existence results for constraints mapping with nonempty interior values and quasi-monotone operators

Theorem 2.4.1 *For each $i \in I$, let C_i be a nonempty weakly compact convex subset of X_i and let $T_i : C_i \times C_{-i} \rightrightarrows X_i^*$ and $K_i : C_i \times C_{-i} \rightrightarrows C_i$ be two set-valued maps with nonempty values. Consider T and K defined as in (2.2.7). Assume that*

- (i) *for each $i \in I$, the set-valued map $K_i : C_i \times C_{-i} \rightrightarrows C_i$ is w -closed and its values are convex with nonempty interior;*
- (ii) *for each $i \in I$, the pair of set-valued maps $(T_i, \text{int}K_i)$ is weakly net-lower sign-continuous with respect to the parameter pair (C_i, C_{-i}) ;*
- (iii) *for each $i \in I$, each $x_i \in C_i$, and each $x_{-i} \in C_{-i}$, one has*

$$M^E(T_i(\cdot, x_{-i}), \text{int}K_i(x_i, x_{-i})) \neq \emptyset.$$

Then,

- (a) if, for each $i \in I$ and each $x_{-i} \in C_{-i}$, the map $T_i(\cdot, x_{-i}) : C_i \rightrightarrows X_i^*$ is upper sign-continuous and w^* -compact convex valued, then $\text{QVI}(T, K)$ is nonempty;
- (b) if, for each $i \in I$ and each $x_{-i} \in C_{-i}$, the map $T_i(\cdot, x_{-i}) : C_i \rightrightarrows X_i^*$ is locally upper sign-continuous, then $\text{QVI}^*(T, K)$ is nonempty.

Before proving Theorem 2.4.1, let us state some useful lemmas that will be needed.

Lemma 2.4.2 *Let K be a nonempty convex subset of X and let $T : K \rightrightarrows X^*$ be a set-valued map. Then $M^E(T, \text{int}K)$ is convex.*

Proof. Let $x_1, x_2 \in M^E(T, \text{int}K)$ and $t \in [0, 1]$. Set $x = tx_1 + (1-t)x_2$ and take $y \in \text{int}K$ and $y^* \in T(y)$. Since $x_1, x_2 \in M^E(T, \text{int}K)$, we have that (for $i = 1, 2$)

$$x_i \in \bar{K} \quad \text{and} \quad \langle y^*, y - x_i \rangle \geq 0.$$

Therefore, $x \in \bar{K}$ due to the convexity of K , and

$$\langle y^*, y - x \rangle = t \langle y^*, y - x_1 \rangle + (1-t) \langle y^*, y - x_2 \rangle \geq 0.$$

Since y and y^* are arbitrary, we conclude that $x \in M^E(T, \text{int}K)$, finishing the proof. \square

Lemma 2.4.3 *For each $i \in I$, let C_i be a nonempty subset of X_i , let $C = \prod_{i \in I} C_i$. Let $\varphi_i : C \rightrightarrows C_i$ be a set-valued map, and let $\varphi : C \rightrightarrows C$ be the product set-valued map defined by $\varphi(x) = \prod_{i \in I} \varphi_i(x)$. Let T and K be defined as in (2.2.7). If, for every $i \in I$ and every $x \in C$, φ_i is given by*

(i) $\varphi_i(x) := S(T_i(\cdot, x_{-i}), K_i(x_i, x_{-i}))$, then

$$\bar{x} \in \varphi(\bar{x}) \iff \bar{x} \in \text{QVI}(T, K);$$

(ii) $\varphi_i(x) := S^*(T_i(\cdot, x_{-i}), K_i(x_i, x_{-i}))$, then

$$\bar{x} \in \varphi(\bar{x}) \implies \bar{x} \in \text{QVI}^*(T, K).$$

Proof. (i) For the necessity, assume $\bar{x} \in \varphi(\bar{x})$. By definition, we can write

$$\begin{aligned} \bar{x} \in \varphi(\bar{x}) &\iff \forall i \in I, \bar{x}_i \in \varphi_i(\bar{x}) \\ &\iff \forall i \in I, \bar{x}_i \in S(T_i(\cdot, \bar{x}_{-i}), K_i(\bar{x}_i, \bar{x}_{-i})). \end{aligned}$$

Thus, for every $i \in I$ we have that $\bar{x}_i \in K_i(\bar{x})$ and that there exists $\bar{x}_i^* \in T_i(\bar{x})$ such that $\langle \bar{x}_i^*, y_i - \bar{x}_i \rangle \geq 0$ for every $y_i \in K_i(\bar{x})$. Now, putting $\bar{x}^* = (\bar{x}_1^*, \dots, \bar{x}_n^*)$ we get that $\bar{x} \in K(\bar{x})$, $\bar{x}^* \in T(\bar{x})$, and

$$\langle \bar{x}^*, y - \bar{x} \rangle = \sum_{i \in I} \langle \bar{x}_i^*, y_i - \bar{x}_i \rangle \geq 0, \forall y = (y_1, \dots, y_n) \in K(\bar{x}).$$

In other words, $\bar{x} \in \text{QVI}(T, K)$.

For the sufficiency, assume now that $\bar{x} \in \text{QVI}(T, K)$, that is $\bar{x} \in K(\bar{x})$ and there exists $\bar{x}^* = (\bar{x}_1^*, \dots, \bar{x}_n^*) \in T(\bar{x})$ such that

$$\langle \bar{x}^*, y - \bar{x} \rangle = \sum_{i \in I} \langle \bar{x}_i^*, y_i - \bar{x}_i \rangle \geq 0, \forall y = (y_1, \dots, y_n) \in K(\bar{x}).$$

Fix $i \in I$, choose $y_i \in K_i(\bar{x})$ and put $z = (\bar{x}_1, \dots, y_i, \dots, \bar{x}_n) \in K(\bar{x})$. By applying the latter inequality, we get that

$$\langle \bar{x}_i^*, y_i - \bar{x}_i \rangle = \sum_{j \in I} \langle \bar{x}_j^*, z_j - \bar{x}_j \rangle = \langle \bar{x}^*, z - \bar{x} \rangle \geq 0.$$

Therefore, for any $i \in I$, $\bar{x}_i \in S(T_i(\cdot, \bar{x}_{-i}), K_i(\bar{x}_i, \bar{x}_{-i}))$ which implies that $\bar{x} \in \varphi(\bar{x})$, finishing the proof.

(ii) Let $\bar{x} \in \varphi(\bar{x})$. Following the same reasoning as that of the necessity proof in part (i), we can deduce that for every $i \in I$, $\bar{x}_i \in K_i(\bar{x})$ and there exists $\bar{x}_i^* \in T_i(\bar{x}) \setminus \{0\}$ such that $\langle \bar{x}_i^*, y_i - \bar{x}_i \rangle \geq 0$ for every $y_i \in K_i(\bar{x})$. This yields that $\bar{x} \in K(\bar{x})$, that $\bar{x}^* = (\bar{x}_1^*, \dots, \bar{x}_n^*) \in T(\bar{x}) \setminus \{0\}$, and that $\langle \bar{x}^*, y - \bar{x} \rangle \geq 0$ for every $y \in K(\bar{x})$. In other words, $\bar{x} \in \text{QVI}^*(T, K)$. \square

Proof of Theorem 2.4.1. For each $i \in I$, let us consider the set-valued map $\Phi_i : C \rightrightarrows C_i$ defined by $\Phi_i(x) := M^E(T_i(\cdot, x_{-i}), \text{int}(K_i(x)))$, and $\Phi : C \rightrightarrows C$ defined by $\Phi(x) = \prod_{i \in I} \Phi_i(x)$.

Hypothesis (iii) implies that, for any $i \in I$, $\Phi_i(x) \neq \emptyset$ and therefore $\Phi(x) \neq \emptyset$ for any $x \in C$.

Since $\text{int}K_i(x) \neq \emptyset$ for all $i \in I$ and all $x \in C$, and since hypotheses (i) and (ii) hold, Proposition 2.3.9 entails that for each $i \in I$, the set-valued map Φ_i is weakly closed. Hence, Φ is weakly closed. Moreover, combining the weak compactness of C and the fact that $\Phi : C \rightrightarrows C$ is weakly closed, we deduce that Φ is weakly upper semicontinuous.

Finally, for each $i \in I$, the set-valued map $K_i : C \rightrightarrows C_i$ is convex valued and $\text{int}K_i(x) \neq \emptyset$ for all $x \in C$. Then Lemma 2.4.2 yields that for any $i \in I$ and any $x \in C$, $\Phi_i(x)$ is a convex set, implying thus that the map Φ is convex valued.

By using Kakutani's fixed-point theorem (see [83, Theorem 6.4.10]), there exists $\bar{x} \in \Phi(\bar{x})$. Conclusion (a) (resp., (b)) follows from Lemma 2.4.3 and Lemma 2.3.7(i) (resp., Lemma 2.3.7(ii)). \square

Assumption (iii) of Theorem 2.4.1, that is, the nonemptiness of the extended Minty variational inequalities $M^E(T_i(\cdot, x_{-i}), \text{int}K_i(x_i, x_{-i}))$, is somehow "artificial" in the sense that it is not a direct assumption on the data of the variational problem, namely, on T_i and K_i . The corollary below describes a complete set of "direct assumptions" on T_i and K_i ensuring the existence of solutions for the quasi-variational inequalities $\text{QVI}(T, K)$ and $\text{QVI}^*(T, K)$.

Corollary 2.4.4 *For each $i \in I$, let C_i be a nonempty weakly compact convex subset of X_i and let $T_i : C_i \times C_{-i} \rightrightarrows X_i^*$ and $K_i : C_i \times C_{-i} \rightrightarrows C_i$ be two set-valued maps with nonempty values. Consider T and K defined as in (2.2.7). Assume that*

- (i) *for each $i \in I$, the set-valued map $K_i : C_i \times C_{-i} \rightrightarrows C_i$ is weakly closed and its values are convex with nonempty interior;*
- (ii) *for each $i \in I$, the pair of set-valued maps $(T_i, \text{int}K_i)$ is weakly net-lower sign-continuous with respect to the parameter pair (C_i, C_{-i}) .*

Then,

- (a) *if for each $i \in I$ and each $x_{-i} \in C_{-i}$, $T_i(\cdot, x_{-i}) : C_i \rightrightarrows X_i^*$ is properly quasi-monotone, w^* -compact convex valued and upper sign-continuous, then $\text{QVI}(T, K)$ is nonempty;*
- (b) *if for each $i \in I$ and each $x_{-i} \in C_{-i}$, $T_i(\cdot, x_{-i}) : C_i \rightrightarrows X_i^*$ is quasi-monotone and locally upper sign-continuous, then $\text{QVI}^*(T, K)$ is nonempty.*

Proof. We will prove each statement separately.

- (a) Since, for each $i \in I$ and $x_{-i} \in C_{-i}$, the set-valued map $K_i(\cdot, x_{-i}) : C_i \rightrightarrows C_i$ is weakly compact convex valued and $T_i(\cdot, x_{-i}) : C_i \rightrightarrows X_i^*$ is properly quasi-monotone, it is known (see [91, Theorem 5.1]) that

$$M(T_i(\cdot, x_{-i}), K_i(x_i, x_{-i})) \neq \emptyset,$$

and thus $M^E(T_i(\cdot, x_{-i}), \text{int}(K_i(x_i, x_{-i}))) \neq \emptyset$. Finally, by Theorem 2.4.1(a), it follows that $\text{QVI}(T, K)$ is nonempty.

- (b) Since, for each $i \in I$ and each $x_{-i} \in C_{-i}$, the set-valued map $K_i(\cdot, x_{-i}) : C_i \rightrightarrows C_i$ is weakly compact convex valued and $T_i(\cdot, x_{-i}) : C_i \rightrightarrows X_i^*$ is quasi-monotone and locally upper sign-continuous, it is known (see [17, Theorem 2.1]) that $S^*(T_i(\cdot, x_{-i}), K(x_i, x_{-i})) \neq \emptyset$ for every $x_i \in C_i$. By Lemma 2.3.7(iii), we deduce that $M^E(T_i(\cdot, x_{-i}), \text{int}K(x_i, x_{-i})) \neq \emptyset$, and so the conclusion follows by Theorem 2.4.1(b).

□

Remark 2.4.5 Note that the main result we presented in section 3.2, namely Theorem 2.2.6, is exactly part (b) of Corollary 2.4.4.

2.4.2 Existence results for the general case with properly quasi-monotone operators

Our aim in this subsection is to state existence results for the quasi-variational inequalities $\text{QVI}(T, K)$ and $\text{QVI}^*(T, K)$ without assuming the nonemptiness of the interior of the constraint sets $K_i(\cdot, x_{-i})$. The price to pay for weakening this hypothesis is that the following theorem needs the nonemptiness of the parametrized Minty solution sets $M(T_i(\cdot, x_{-i}), K_i(x_i, x_{-i}))$, and so the corresponding version of Corollary 2.4.4 will only consider properly quasi-monotone operators.

Theorem 2.4.6 *For each $i \in I$, let C_i be a nonempty weakly compact convex subset of X_i and let $T_i : C_i \times C_{-i} \rightrightarrows X_i^*$ and $K_i : C_i \times C_{-i} \rightrightarrows C_i$ be two set-valued maps with nonempty values. Consider T and K defined as in (2.2.7). Assume that*

- (i) *for each $i \in I$, the set-valued map $K_i(\cdot, x_{-i}) : C_i \times C_{-i} \rightrightarrows C_i$ is weakly closed with convex values;*
- (ii) *for each $i \in I$, the pair of set-valued maps (T_i, K_i) is weakly net-lower sign-continuous with respect to the parameter pair (C_i, C_{-i}) ;*
- (iii) *for each $i \in I$ and each $(x_i, x_{-i}) \in C_i \times C_{-i}$, $M(T_i(\cdot, x_{-i}), K_i(x_i, x_{-i})) \neq \emptyset$.*

Then,

- (a) *if, for each $i \in I$ and each $x_{-i} \in C_{-i}$, $T_i(\cdot, x_{-i}) : C_i \rightrightarrows X_i^*$ is upper sign-continuous and w^* -compact convex valued, then $\text{QVI}(T, K)$ is nonempty;*
- (b) *if, for each $i \in I$ and each $x_{-i} \in C_{-i}$, $T_i(\cdot, x_{-i}) : C_i \rightrightarrows X_i^*$ is locally upper sign-continuous, then $\text{QVI}^*(T, K)$ is nonempty.*

Proof. The proof follows the same arguments as that of Theorem 2.4.1, using directly the Minty solution set $M(T_i(\cdot, x_{-i}), K_i(x_i, x_{-i}))$ and invoking [21, Lemma 3.1] instead of Lemma 2.3.7. □

Corollary 2.4.7 *For each $i \in I$, let C_i be a nonempty weakly compact convex subset of X_i and let $T_i : C_i \times C_{-i} \rightrightarrows X_i^*$ and $K_i : C_i \times C_{-i} \rightrightarrows C_i$ be two set-valued maps with nonempty values. Consider T and K defined as in (2.2.7). Assume that*

- (i) *for each $i \in I$, the set-valued map $K_i(\cdot, x_{-i}) : C_i \rightrightarrows C_i$ is weakly closed with convex values;*
- (ii) *for each $i \in I$, the pair of set-valued maps (T_i, K_i) is weakly net-lower sign-continuous with respect to the parameter pair (C_i, C_{-i}) ;*
- (iii) *for each $i \in I$ and each $x_{-i} \in C_{-i}$, $T_i(\cdot, x_{-i}) : C_i \rightrightarrows X_i^*$ is properly quasi-monotone.*

Then,

- (a) *if, for each $i \in I$ and each $x_{-i} \in C_{-i}$, $T_i(\cdot, x_{-i}) : C_i \rightrightarrows X_i^*$ is upper sign-continuous and w^* -compact convex valued, then $\text{QVI}(T, K)$ is nonempty;*
- (b) *if, for each $i \in I$ and each $x_{-i} \in C_{-i}$, $T_i(\cdot, x_{-i}) : C_i \rightrightarrows X_i^*$ is locally upper sign-continuous, then $\text{QVI}^*(T, K)$ is nonempty.*

Proof. The proof is exactly the same as that of Corollary 2.4.4, invoking Theorem 2.4.6 instead of Theorem 2.4.1, and [91, Theorem 5.1]. □

2.5 Final comments

In this work, we have considered quasi-variational inequality problems over product sets considering the assumptions of quasi-monotonicity and upper sign-continuity only in the component operators. One of the most important difficulties, in obtaining the existence results for quasi-variational inequalities over product sets, is that quasi-monotonicity and upper sign-continuity are not preserved by the product of set-valued maps (see Examples 1 and 2). However, by introducing the new notion of net-lower sign-continuity, which is used as a minimal hypothesis in obtaining the stability result of Proposition 2.3.9, and employing the well known Kakutani fixed point theorem, we have overcome these difficulties and successfully established the existence results for the solution of our problem in the infinite-dimensional setting.

Our existence results extend the approaches of the existing literature (see [22, 23, 24, 25, 26, 27, 28, 29, 30, 31]) to the quasi-monotone setting, but more importantly they open the door to powerful applications to Nash equilibrium problems and generalized Nash equilibrium problems, since it is well known that they can be reformulated as variational and quasi-variational inequalities over product sets, respectively (see, e.g., [32]). This application to game theory will be the main aim of a forthcoming work by the same authors.

Existence results for Generalized Nash Equilibrium Problems under continuity-like properties of sublevel sets

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Abstract

A generalized Nash equilibrium problem (GNEP) corresponds to a non cooperative interaction between a finite set of players in which the cost function and the feasible set of each player depend on the decisions of the others. The classical existence result for generalized equilibria due to Arrow-Debreu requires continuity of the cost functions. In this work, we provide an existence of solutions transferring this hypothesis to a “continuity-like” condition over the sublevel sets of the aforementioned functions. Comparison with Reny’s approach for discontinuous games is also considered.

Keywords: Generalized Nash equilibrium problem, Quasi-variational inequality, Quasi-convexity, Set-valued map, Sublevel set, Normal operator.

AMS subject classifications. 91A10, 49J52, 49J53

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3.1 Introduction

The generalized Nash equilibrium problem (GNEP for short) was first introduced by Debreu [33] as early as 1952. The GNEP is an extension of the classical Nash equilibrium problem (NEP for short) in which the cost function and strategy set of each player depend on the decisions of the other players. GNEPs have become an important part of applied mathematics research and attracted much more attention over the years. In fact, GNEP is at a crossroad of several different disciplines, for example, economics, computer science, engineering, mathematics and operations research. From a mathematical point of view, the GNEP is a fundamental modeling tool for noncooperative multi-leader-follower games, for example in electricity power markets (see [34], [35] and the references therein) and in Eco-Industrial Parks (see [2, 78, 77]). We refer the reader respectively to [32] and [92] for a detailed overview on the history of the GNEP and multi-leader-follower games, and their many other applications.

Formally, a GNEP consists of p players where each player $v \in I = \{1, \dots, p\}$ controls his strategy variables $x_v \in \mathbb{R}^{n_v}$. Let us denote by x the vector of all strategies

$$x = (x_1, \dots, x_p) \in \mathbb{R}^N \quad \text{with} \quad N = n_1 + n_2 + \dots + n_p,$$

and by x_{-v} the strategy vector of the other players except the player v . Following classical notations, we often write (x_v, x_{-v}) instead of x to emphasize the v -th player's strategies within x . The strategy of player v belongs to a strategy set, i.e. $x_v \in K_v(x_{-v})$, that depends on the strategy variables of the other players.

Given the strategies x_{-v} of the other players, the aim of player v is to choose a strategy x_v solving $P_v(x_{-v})$

$$\begin{aligned} \min_{x_v} \quad & \theta_v(x_v, x_{-v}), \\ \text{s. t.} \quad & x_v \in K_v(x_{-v}), \end{aligned} \tag{3.1.1}$$

where $\theta_v : \mathbb{R}^N \rightarrow \mathbb{R}$ is the objective function, also known as cost function, of player v , and the set-valued map K_v , known as the constraints map, takes values in a subset C_v of \mathbb{R}^{n_v} . The GNEP is the problem of finding a vector $\bar{x} \in \mathbb{R}^N$ so that, for all v , \bar{x}_v solves $P_v(\bar{x}_{-v})$. Such a vector \bar{x} is called *generalized Nash equilibrium*. A point \bar{x} is, therefore, a generalized Nash equilibrium if no player can unilaterally decrease his objective function by choosing a different strategy. If the feasible sets $K_v(x_{-v})$ of each player do not depend on the rival player's strategies (and are thus constant sets), then the GNEP reduces to the NEP.

It is well known that when the objective functions $\theta_v(\cdot, x_{-v})$ are continuously differentiable and convex, and the maps K_v are closed and convex valued for all v , then $\text{GNEP}(\theta, K)$ can be reformulated as quasi-variational inequalities (see e.g. [32]). This reformulation has been extended in [36], thanks to the concept of adjusted normal operator (see also [37, 38]), to the case where the objective functions $\theta_v(\cdot, x_{-v})$ are semistrictly quasiconvex without assuming any differentiability, and the constraint sets are jointly convex. In [36], authors also proved an existence result for semistrictly quasiconvex GNEP.

Some work has been already done concerning existence of solutions for GNEP, always under certain specific structure the problems in (3.1.1) [32, 36, 40, 41, 42, 43, 44]. The most classical existence results for GNEPs in the literature is due to Arrow and Debreu in [93] where it is assumed that the objective function θ_v is continuous for any v , for every x_{-v} , the function $\theta_v(\cdot, x_{-v})$ is quasiconvex, and the set-valued map K_v is upper and lower semicontinuous. Latter on, Reny [94] (see also [95]) proved an existence result for GNEP with discontinuous cost functions but assuming the so-called *better-reply secure* property.

Our main contribution in this article is to eliminate the continuity hypothesis of θ_v by replacing it by a "continuity-type hypothesis" of the sublevel sets of the cost functions. Our approach is based on the concepts

of the *adjusted normal operator* [37] and the *net-lower sign-continuity*, which was recently introduced in [45], and on the reformulation of the GNEP in term of quasi-variational inequalities.

The work is organized as follows: Section 3.2 is dedicated to notations, definitions and preliminaries while in Section 3.3 we prove our main existence results for generalized Nash equilibrium problem. Then in Section 3.4 we show that our hypothesis are different, but complementary, with the ones used by Arrow-Debreu and Reny. A short conclusion is given in Section 3.5.

3.2 Notation and preliminaries

In this section we first recall definitions and notations, which will be used. In the sequel \mathbb{R}^p is equipped with the Euclidian norm $\|\cdot\|$ associated with the scalar product $\langle \cdot, \cdot \rangle$. For any $x, y \in \mathbb{R}^p$, we set the notation $[x, y]$, $]x, y[$, and $]x, y]$, respectively, for the segments $[x, y] = \{(1-t)x + ty : t \in [0, 1]\}$, $]x, y[= \{(1-t)x + ty : t \in]0, 1[\}$, and $]x, y] = \{(1-t)x + ty : t \in]0, 1] \}$. For $x \in \mathbb{R}^p$ and $\rho > 0$, we denote by $\mathbb{B}(x, \rho)$ and $\overline{\mathbb{B}}(x, \rho)$, the open ball and the closed ball of center x and radius ρ , respectively. We denote by \mathbb{S}_p the unit sphere of \mathbb{R}^p . We denote the topological interior, the convex hull, and the closure of $A \subseteq \mathbb{R}^p$ by $\text{int}A$, $\text{conv}A$, and $\text{cl}A$, respectively.

Given an extended-real function $f : \mathbb{R}^p \rightarrow \mathbb{R} \cup \{\infty\}$, the domain of f is the set $\text{dom}f = \{x \in \mathbb{R}^p \mid f(x) < +\infty\}$. For any $\lambda \in \mathbb{R}$, we denote by $S_\lambda(f)$ and $S_\lambda^<(f)$ the sublevel set and the strict sublevel set, respectively, associated with f and λ :

$$S_\lambda(f) = \{x \in \mathbb{R}^p : f(x) \leq \lambda\} \quad \text{and} \quad S_\lambda^<(f) = \{x \in \mathbb{R}^p : f(x) < \lambda\}. \quad (3.2.1)$$

If there is no confusion, we may omit the argument function f in the above notation, simply writing S_λ and $S_\lambda^<$. We denote the set of global minimizers of f over a set K as $\text{argmin}_K f$. If $K = \mathbb{R}^p$, we simply write $\text{argmin} f$ instead.

The function $f : \mathbb{R}^p \rightarrow \mathbb{R} \cup \{\infty\}$ is said to be

- *Quasiconvex* on a convex subset of $C \subset \text{dom}f$ if, for any $x, y \in C$ and $\lambda \in [0, 1]$, we have

$$f(\lambda x + (1 - \lambda)y) \leq \max\{f(x), f(y)\}.$$

An equivalent and useful characterization of quasiconvexity is that the function f is quasiconvex on $\text{dom}f$ if and only if its sublevel set $S_\lambda(f)$ is convex for any $\lambda \in \mathbb{R}$.

- *Semistrictly quasiconvex* on a convex subset of $C \subset \text{dom}f$, if f is quasiconvex and for any $x, y \in C$,

$$f(x) < f(y) \text{ implies } f(z) < f(y) \text{ for all } z \in]x, y[.$$

Roughly speaking, a semistrictly quasiconvex function is a quasiconvex function which does not admit “flat part”, except possibly for $\text{argmin} f$.

Let us now recall from [37] the fundamental concept of adjusted sublevel set which provides, for quasiconvex functions, more information than the usual concepts of sublevel sets.

Definition 3.2.1 (Adjusted sublevel set) *Given a quasiconvex function $f : \mathbb{R}^p \rightarrow \mathbb{R} \cup \{\infty\}$, the adjusted sublevel set is defined by*

$$S_f^a(x) = S_{f(x)} \cap \overline{\mathbb{B}}(S_{f(x)}^<, \rho_x),$$

where $\rho_x = \text{dist}(x, S_{f(x)}^<)$, if $x \notin \text{argmin} f$, and $S_f^a(x) = S_{f(x)}$ otherwise. Here

$$\overline{\mathbb{B}}(S_{f(x)}^<, \rho_x) = \text{cl} \left(S_{f(x)}^< + \overline{\mathbb{B}}(0, \rho_x) \right)$$

denotes the closed ρ_x -neighbourhood of the set strict sublevel set $S_{f(x)}^<$.

It is important to notice that one always has the following inclusions, for all $x \in \mathbb{R}^p$:

$$S_f^<(x) \subseteq S_f^a(x) \subseteq S_f(x).$$

and that

$$f \text{ is quasiconvex} \iff S_f^a(x) \text{ is convex, } \forall x \in \text{dom} f.$$

In [37], the concept of adjusted sublevel sets and its normal operator have proved to be useful on the study of quasiconvex programming.

Definition 3.2.2 (Normal operator) *Given a quasiconvex function $f : \mathbb{R}^p \rightarrow \mathbb{R} \cup \{\infty\}$, the normal operator associated with f is a set-valued map $N_f^a : \mathbb{R}^p \rightrightarrows \mathbb{R}^p$ which is given as*

$$N_f^a(x) = \left\{ z \in \mathbb{R}^p : \langle z, y - x \rangle \leq 0, \forall y \in S_f^a(x) \right\} = (S_f^a(x) - \{x\})^\circ,$$

where $(S_f^a(x) - \{x\})^\circ$ is the (negative) polar set of the $S_f^a(x) - \{x\}$.

Observe that, in the case of a semistrictly quasiconvex function, $N_f^a(x)$ is simply the normal cone the sublevel set $S_{f(x)}$ or to the strict sublevel set $S_{f(x)}^<$, that is for any $x \notin \text{argmin}_{\mathbb{R}^p} f$, $N_f^a(x) = N_f(x) = N_f^<(x)$.

It is worth mentioning the following definition of upper sign-continuity which was introduced by Hadjisavvas [96].

Definition 3.2.3 (Upper sign-continuity) *Let C be a nonempty convex subset of \mathbb{R}^p and let $T : C \rightrightarrows \mathbb{R}^p$ be a set-valued map with nonempty values. We say that T is upper sign-continuous on C if for every $x, y \in C$, the following implication holds:*

$$\left(\forall t \in]0, 1[, \inf_{x_t^* \in T(x_t)} \langle x_t^*, y - x \rangle \geq 0 \right) \implies \sup_{x^* \in T(x)} \langle x^*, y - x \rangle \geq 0,$$

where $x_t := (1 - t)x + ty$.

Let us recall from [87] the useful definitions of *Lower limit* and *Upper limit* in the sense of Kuratowski: for any sequence $(S_n)_n$ of subsets of \mathbb{R}^n ,

$$\begin{aligned} x \in \liminf_n S_n &\iff \exists (x_n)_n \subset \mathbb{R}^n \text{ such that } x = \lim_{n \rightarrow \infty} x_n \text{ and } x_n \in S_n, \forall n. \\ x \in \limsup_n S_n &\iff \begin{cases} \exists (S_{n_k})_k \text{ subsequence of } (S_n)_n \text{ and } \exists (x_{n_k})_k \subset \mathbb{R}^n \\ \text{such that } x = \lim_{k \rightarrow \infty} x_{n_k} \text{ and } x_{n_k} \in S_{n_k}, \forall k. \end{cases} \end{aligned}$$

The concept of Mosco-convergence of a sequence of subsets has been introduced in [97] to study the convergence properties of the solutions of variational inequalities. Let us recall the following characterization of the Mosco-convergence of a sequence of sets:

Definition 3.2.4 (Mosco-convergence) *Let S be a subset of \mathbb{R}^p and $(S_n)_n$ be a sequence of subsets of \mathbb{R}^p . We say that the sequence $(S_n)_n$ converges to a subset S in the sense of Mosco, if both of the following equalities hold:*

$$\text{Lim sup}_n S_n \subset S \quad \text{and} \quad S \subset \text{Lim inf}_n S_n.$$

Recall that, for a sequence of sets $(S_n)_n$ and a sequence of set-valued maps $T_n : \mathbb{R}^p \rightrightarrows \mathbb{R}^p$ we define

$$\text{Limsup}_{S_n \ni x_n \rightarrow x} T_n(x_n) := \left\{ \begin{array}{l} \exists (x_n)_n \text{ converging to } x \text{ with } x_n \in S_n \\ y \in \mathbb{R}^p : \text{ and a sequence } (y_n)_n \text{ with } y_n \in T_n(x_n), \\ \text{such that } y \text{ is a cluster point of } (y_n)_n. \end{array} \right\}. \quad (3.2.2)$$

Lemma 3.2.5 *Let $(K_n)_{n \in \mathbb{N}}$ be a sequence of convex closed sets of \mathbb{R}^N such that*

(i) $\text{int}(K_n) \neq \emptyset$, for all $n \in \mathbb{N}$;

(ii) $(K_n)_{n \in \mathbb{N}}$ Mosco-converges to a set K of \mathbb{R}^N with $\text{int}(K) \neq \emptyset$.

Then, for every $y \in \text{int}(K)$ and every sequence $(y_n)_{n \in \mathbb{N}}$ converging to y , there exists $n_0 \in \mathbb{N}$ large enough such that

$$y_n \in \text{int}(K_n), \quad \forall n \geq n_0.$$

Proof. Let $y \in \text{int}(K)$ and $(y_n)_n$ be a sequence converging to y . For any n , let us define $z_n = \text{proj}_{K_n}(y_n)$. Note that

$$\|z_n - y\| \leq \|z_n - \text{proj}_{K_n}(y)\| + \|y - \text{proj}_{K_n}(y)\| \leq \|y_n - y\| + d(y, K_n) \rightarrow 0.$$

Thus, the sequence $(z_n)_n$ converges to y . Assume that the desired condition doesn't hold, that is, there exists a subsequence $(y_{n_k})_k$ of (y_n) such that $y_{n_k} \notin \text{int}(K_{n_k})$, for any k . This yields that $z_{n_k} \in \text{bd}(K_{n_k})$ for every $k \in \mathbb{N}$. By Separating Hyperplane Theorem, we have that for each $k \in \mathbb{N}$, there exists $\zeta_{n_k} \in N_{K_{n_k}}(z_{n_k}) \cap \mathbb{S}_N$. Without loss of generality, we may assume that $(\zeta_{n_k})_k$ converges to $\zeta \in \mathbb{S}_N$.

By [87, Corollary 7.6.5], we deduce that $\zeta \in N_K(y)$, which is a contradiction since $N_K(y) = \{0\}$. \square

For a nonempty subset C of \mathbb{R}^n and a set-valued map $T : C \rightrightarrows \mathbb{R}^n$, the *Stampacchia variational inequality* associated to T and C is

$$\text{find } x \in C \text{ such that } \exists x^* \in T(x) \text{ with } \langle x^*, y - x \rangle \geq 0, \forall y \in C.$$

We denote by $S(T, C)$ its solution set.

Also, recall that the *Minty variational inequality* associated to T and C is

$$\text{find } x \in C \text{ such that } \langle y^*, y - x \rangle \geq 0, \forall y \in C, \forall y^* \in T(y).$$

The solution set of the Minty variational inequality problem will be denoted by $M(T, C)$. It is not hard to see that $M(T, C)$ is convex and closed, provided that C is convex and closed.

Finally, for C and T as before, and for a set-valued map $K : C \rightrightarrows C$, the *quasi-variational inequality* associated to T and K is

$$\text{find } x \in K(x) \text{ such that } \exists x^* \in T(x) \text{ with } \langle x^*, y - x \rangle \geq 0, \forall y \in K(x).$$

The associated solution set will be denoted by $\text{QVI}(T, K)$.

One of the main advantages of the normal operator approach is that it provides the following sufficient optimality condition for quasiconvex optimization.

Proposition 3.2.6 ([98]) *Let $f : \mathbb{R}^p \rightarrow \mathbb{R} \cup \{\infty\}$ be a quasiconvex function, radially continuous on $\text{dom}f$, and C be a nonempty subset of $\text{dom}f$. If $C \subseteq \text{int}(\text{dom}f)$, then any solution of the Stampacchia variational inequality defined by the operator $N_f^a \setminus \{0\}$ on C is a global minimizer of f over C .*

This is a very powerful result since it allows to recover the perfect situation of convex optimization whereas a first order-type condition turns out to be sufficient for global optimality. Nevertheless the fact that radial continuity is asked makes it less useful for the purpose of this work where our aim is to reduce the continuity assumptions which will be made on the cost functions of the players. Therefore we state, in the forthcoming Proposition 3.2.9, the same kind of sufficient optimality conditions but under alternative hypothesis.

Definition 3.2.7 (sub-boundarily constant functions) *A function $f : \mathbb{R}^p \rightarrow \mathbb{R}$ is said to be sub-boundarily constant on a subset C if, for every $x \in C$, one has that*

$$f \text{ is constant over } S_f^a(x) \setminus \text{int}S_f^a(x). \quad (3.2.3)$$

Remark 3.2.8 The reader can easily verify the following observations concerning the above definition:

- i) Note that due to the special structure of the adjusted sublevel sets S_f^a , the subset $S_f^a(x) \setminus \text{int}S_f^a(x)$ has nothing to do in general with the level set $L_f(x) = \{y \in \mathbb{R}^n : f(y) = f(x)\}$, even whenever f is quasiconvex.
- ii) If f is radially continuous, which is one of the hypothesis considered in Proposition 3.2.6, then f is sub-boundarily constant on $\text{dom}f$.
- iii) Also, if $p = 1$, that is f is defined over \mathbb{R} , and if f is quasiconvex, then f must be sub-boundarily constant on its domain.
- iv) Note that, if f is sub-boundarily constant, then for every $\lambda > \inf_{\mathbb{R}^p} f$, the sublevel sets $S_\lambda(f)$ must have nonempty interior.

Proposition 3.2.9 *Let $f : \mathbb{R}^p \rightarrow \mathbb{R}$ be a quasiconvex and sub-boundarily constant function, and $C \subset \mathbb{R}^p$ be a convex set. Then, any solution of the Stampacchia variational inequality defined by the operator $N_f^a \setminus \{0\}$ on C is a global minimizer of f over C , that is*

$$S(N_f^a \setminus \{0\}, C) \subset \underset{C}{\text{argmin}} f.$$

Proof. Let $\bar{x} \in S(N_f^a \setminus \{0\}, C)$ and $x^* \in N_f^a(\bar{x}) \setminus \{0\}$ be such that

$$\langle x^*, y - \bar{x} \rangle \geq 0, \quad \forall y \in C. \quad (3.2.4)$$

Since $S_f^a(\bar{x})$ and C are convex, the above inequality yields that $\bar{x} \in C \setminus \text{int}(S_f^a(\bar{x}))$. Now, assume, for a contradiction, that there exists $y \in C$ such that $f(y) < f(\bar{x})$. Then, $y \in S_f^<(\bar{x}) \subset S_f^a(\bar{x})$. Combining the definition of $N_f^a(x) \setminus \{0\}$ together with (3.2.4), one immediately have $\langle x^*, y - \bar{x} \rangle = 0$.

Now since f is sub-boundarily constant on $\text{dom}f$, y is also an element of $\text{int}S_f^a(\bar{x})$ and thus there exists $\varepsilon > 0$ such that $B(y, \varepsilon)$ is included into $S_f^a(\bar{x})$. Since $x^* \neq 0$ there exists $d \in \mathbb{R}^n$ such that $\langle x^*, d \rangle > 0$. Then for $t > 0$ small enough, $z = y + td$ is an element of $B(y, \varepsilon)$ and thus of $S_f^a(\bar{x})$ and $\langle x^*, z - \bar{x} \rangle \leq 0$. But this is impossible since $\langle x^*, z - \bar{x} \rangle = \langle x^*, y - \bar{x} \rangle + t \langle x^*, d \rangle > 0$. As a conclusion and since y is arbitrary on C , this proves that $f(\bar{x}) = \min_C f$. \square

3.3 Existence results of GNEP

In this section we present our main results, namely, the existence of solutions for the GNEP. Let us set I , C_v , C_{-v} , θ_v and K_v as they are defined in Section 4.1 when we introduce the optimization problems (3.1.1). Moreover let us define the following notations, which will be used in the sequel:

$$C := \prod_{v \in I} C_v; \quad C_{-v} := \prod_{v' \neq v, v' \in I} C_{v'}, \quad v \in I. \quad (3.3.1)$$

and

$$\theta(x) := \prod_{v \in I} \theta_v(x_v, x_{-v}); \quad K(x) := \prod_{v \in I} K_v(x_{-v}). \quad (3.3.2)$$

In what follows, we will use the notation $\text{GNEP}(\theta, K)$ to denote the generalized Nash game induced by the problems (3.1.1), as well as the solution set of this game. We divided our results in two cases: (1) we consider the case of the general constraints set-valued maps K_v , for all $v \in I$; and (2) we consider the particular case with nonemptiness assumption on the interior of the values of the constraints set-valued maps K_v , for all $v \in I$. Under this extra assumption, we are able to deduce Corollary 3.3.8, which establishes the main goal of this work: to deduce the existence of solutions for $\text{GNEP}(\theta, K)$ under continuity-like hypotheses over the sublevel sets of the cost functions θ_v . Proofs are based on the concept of adjusted normal operator [37] and a reformulation of the GNEP in terms of quasi-variational inequalities.

3.3.1 Case of general constraints set-valued maps

We open this subsection by establishing its main result, where existence of solutions of $\text{GNEP}(\theta, K)$ is deduced in terms of a continuity-like property of the normal operators of the cost functions θ_v .

Theorem 3.3.1 *For any $v \in I$, let C_v be a nonempty, compact and convex subset of \mathbb{R}^{n_v} , $\theta_v : \mathbb{R}^N \rightarrow \mathbb{R}$ and $K_v : C_{-v} \rightrightarrows C_v$. Then, the $\text{GNEP}(\theta, K)$ admits a generalized Nash equilibrium if*

- (i) *for every $x_{-v} \in C_{-v}$, the function $\theta_v(\cdot, x_{-v})$ is quasiconvex, lower semicontinuous and sub-boundarily constant on $\text{dom} f$;*
- (ii) *for every $x \in \mathbb{R}^N$, the intersection $C_v \cap \text{argmin}_{\mathbb{R}^{n_v}} \theta_v(\cdot, x_{-v}) = \emptyset$;*
- (iii) *the set-valued map $K_v : C_{-v} \rightrightarrows C_v$ is closed with compact convex values;*
- (iv) *for every sequence $(x_{-v}^n)_{n \in \mathbb{N}} \subseteq C_{-v}$ converging to x_{-v} and every $y_v \in K_v(x_{-v})$ one has*

$$N_{\theta_v(\cdot, x_{-v})}^a(y_v) \subseteq \text{Limsup}_{K_v(x_{-v}^n) \ni y_v^n \rightarrow y_v} N_{\theta_v(\cdot, x_{-v}^n)}^a(y_v^n). \quad (3.3.3)$$

The proof of Theorem 3.3.1 is based on a quasi-variational inequality reformulation of the initial GNEP and the net-lower sign-continuity concept will play a fundamental role into the proof. Let us first recall the definition of the net-lower sign-continuity of a couple of set-valued maps (see [45, Definition 3.1]).

Definition 3.3.2 (net-lower sign-continuity) *Let (U, τ_U) and (Λ, τ_Λ) be two topological spaces, Y be a Banach space and τ_Y be a locally convex topology consistent with the duality $\langle Y, Y^* \rangle$. Suppose that all*

three topologies are first countable. Let $T : Y \times \Lambda \rightrightarrows Y^*$ and $K : U \times \Lambda \rightrightarrows Y$ be two set-valued maps. Then, the pair (T, K) is said to be $(\tau_U \times \tau_\Lambda)$ - τ_Y net-lower-sign continuous with respect to the parameter pair (U, Λ) at $(\mu, \lambda) \in U \times \Lambda$ and $y \in K(\mu, \lambda)$ if for every sequence $(\mu_n, \lambda_n)_n \subseteq U \times \Lambda$ converging to (μ, λ) , every $z \in \text{cl}^{\tau_Y}(K(\mu, \lambda))$ and every selection $(z_n)_n$ of $(\text{cl}^{\tau_Y}(K(\mu_n, \lambda_n)))_n$ τ_Y -converging to z , the following condition holds:

$$\left\{ \begin{array}{l} \text{If for every subsequence } (\mu_{n_k}, \lambda_{n_k})_k \text{ of } (\mu_n, \lambda_n)_n \text{ and every selection } (y_{n_k})_k \text{ of} \\ (K(\mu_{n_k}, \lambda_{n_k}))_k \text{ } \tau_Y\text{-converging to } y \text{ one has that} \\ \\ \limsup_k \sup_{y_{n_k}^* \in T(y_{n_k}, \lambda_{n_k})} \langle y_{n_k}^*, z_{n_k} - y_{n_k} \rangle \leq 0, \\ \\ \text{then, } \sup_{y^* \in T(y, \lambda)} \langle y^*, z - y \rangle \leq 0. \end{array} \right.$$

We simply say that (T, K) is $(\tau_U \times \tau_\Lambda)$ - τ_Y net-lower sign-continuous with respect to the parameter pair (U, Λ) if it is so at each $(\mu, \lambda) \in U \times \Lambda$ and each $y \in K(\mu, \lambda)$.

The definition we present here is in fact a characterization of the original definition given in [45], which is given with generalized sequences without the requirement of first countable topologies. However, in our setting, it is enough to give the sequential characterization as definition.

While the definition of net-lower sign-continuity may be hard to understand at first glance, it should be interpreted as follows. The topological spaces U and Λ must be understood as the parameters set of the variational inequality of the form $S(T(\cdot, \lambda), K(\mu, \lambda))$. For a given parameter pair (λ, μ) , a point $y \in K(\lambda, \mu)$ and a point $z \in \text{cl}(K(\mu, \lambda))$, one wants to estimate the value of $\sup_{y^* \in T(y, \lambda)} \langle y^*, z - y \rangle$, when (λ, μ) and z are being approximated by (generalized) sequences (λ_n, μ_n) and (z_n) , respectively. Then, the definition of net-lower sign-continuity says that, if regardless of the approximating (generalized) subsequences $(\mu_{n_k}, \lambda_{n_k})$ and (z_{n_k}) , and regardless the way of approximating y by a (generalized) sequence (y_{n_k}) , one has that $\sup_{y_{n_k}^* \in T(y_{n_k}, \lambda_{n_k})} \langle y_{n_k}^*, z_{n_k} - y_{n_k} \rangle$ is nonpositive in the limit, then the value we are interested in, that is $\sup_{y^* \in T(y, \lambda)} \langle y^*, z - y \rangle$, must remain nonpositive. This is a very weak notion of continuity, as it is discussed in [45]. In practice, under some mild complementary conditions, when (z_n) is a (generalized) sequence of solutions of $S(T(\cdot, \lambda_n), K(\mu_n, \lambda_n))$, net-lower sign-continuity yields that the limit point z is also a solution of $S(T(\cdot, \lambda), K(\mu, \lambda))$.

Finally, in the forthcoming proof, we will use that in fact inclusion (3.3.3) entails net-lower sign-continuity when $\Lambda = C_{-v}$, $U = \{0\}$ and when, for $\lambda = x_{-v}$, $K(\mu, \lambda)$ is replaced by $K_v(x_{-v})$ and $T(\cdot, \lambda)$ is set as $N_{\theta_v(\cdot, x_{-v})}^a(\cdot) \cap \mathbb{S}_{n_v}$. This is given by [45, Proposition 3.3].

The following lemma will be useful to prove Theorem 3.3.1.

Lemma 3.3.3 *Let (Λ, τ_Λ) and (U, τ_U) be two first countable topological spaces and Y be a Banach space. Let $T : Y \times \Lambda \rightrightarrows Y^*$ and $K : U \times \Lambda \rightrightarrows Y$ be two set-valued maps with nonempty values. Assume that (T, K) is net-lower sign-continuous. Then $(\text{conv}T, K)$ and $(\text{cl}(\text{conv}T), K)$ are net-lower sign-continuous, where $\text{conv}T(y) = \text{conv}(T(y))$ and $\text{cl}(\text{conv}T)(y) = \text{cl}(\text{conv}(T(y)))$.*

Proof. Let $(\mu_n, \lambda_n)_n$ be a sequence in $U \times \Lambda$ converging to (μ, λ) , $y \in K(\mu, \lambda)$ and let $(z_n)_n$ be a selection of $(\text{cl}(K(\mu_n, \lambda_n)))_n$ converging to $z \in \text{cl}(K(\mu, \lambda))$, and suppose that for every subsequence $(\mu_{n_k}, \lambda_{n_k})_k$ of $(\mu_n, \lambda_n)_n$ and every selection $(y_{n_k})_k$ of $(K(\mu_{n_k}, \lambda_{n_k}))_k$ τ_Y -converging to y one has that

$$\limsup_k \sup_{y_{n_k}^* \in \text{conv}T(y_{n_k}, \lambda_{n_k})} \langle y_{n_k}^*, z_{n_k} - y_{n_k} \rangle \leq 0.$$

This implies

$$\limsup_k \sup_{y_{n_k}^* \in T(y_{n_k}, \lambda_{n_k})} \langle y_{n_k}^*, z_{n_k} - y_{n_k} \rangle \leq 0.$$

Since (T, K) is net-lower sign-continuous, we deduce that

$$\sup_{y^* \in T(y, \lambda)} \langle y^*, z - y \rangle \leq 0.$$

By noting that $z - y$ is fixed, we deduce that

$$\sup_{y^* \in \text{conv}T(y, \lambda)} \langle y^*, z - y \rangle = \sup_{y^* \in T(y, \lambda)} \langle y^*, z - y \rangle \leq 0,$$

proving that $(\text{conv}T, K)$ is net-lower sign-continuous. The proof for $(\overline{\text{conv}T}, K)$ is similar. \square

Now we are ready to prove our theorem.

Proof of Theorem 3.3.1. Let us define the following set-valued map $F : C \rightrightarrows \mathbb{R}^N$ which, for any $x = (x_1, \dots, x_p) \in C_1 \times \dots \times C_p$, is given by

$$F(x) := F_1(x) \times \dots \times F_p(x),$$

where

$$F_V : C_V \times C_{-V} \rightrightarrows \mathbb{R}^{n_V} \\ (x_V, x_{-V}) \mapsto \text{conv} \left(N_{\theta_V(\cdot, x_{-V})}^a(x_V) \cap \mathbb{S}_{n_V} \right).$$

Let us observe that, according to [99, Proposition 4.4] and [100, Proposition 2.2], the operator $F_V(\cdot, x_{-V})$ is upper sign-continuous with nonempty convex compact values on C_V .

By hypothesis (iv), for every sequence $(x_{-V}^n)_n \subset C_{-V}$ converging to x_{-V} and every $y_V \in K_{-V}(x_{-V})$ we have that

$$N_{\theta_V(\cdot, x_{-V})}^a(y_V) \subseteq \text{Limsup}_{K_V(x_{-V}^n) \ni y_V^n \rightarrow y_V} N_{\theta_V(\cdot, x_{-V}^n)}^a(y_V^n).$$

Now, we claim that

$$\widehat{F}_V(y_V, x_{-V}) \subseteq \text{Limsup}_{K_V(x_{-V}^n) \ni y_V^n \rightarrow y_V} \widehat{F}_V(y_V^n, x_{-V}^n), \quad (3.3.4)$$

where

$$\widehat{F}_V : C_V \times C_{-V} \rightrightarrows \mathbb{R}^{n_V} \\ (y_V, x_{-V}) \mapsto N_{\theta_V(\cdot, x_{-V})}^a(y_V) \cap \mathbb{S}_{n_V}.$$

Indeed, let $w_V \in \widehat{F}_V(y_V)$. We know then that

$$w_V \in \text{Limsup}_{K_V(x_{-V}^n) \ni y_V^n \rightarrow y_V} N_{\theta_V(\cdot, x_{-V}^n)}^a(y_V^n),$$

and so, there exist a sequence $(y_V^n)_n$ converging to y_V with $y_V^n \in K_V(x_{-V}^n)$ and a sequence $(w_V^n)_n$ with $w_V^n \in N_{\theta_V(\cdot, x_{-V}^n)}^a(y_V^n)$ such that w_V is a cluster point of $(w_V^n)_n$, namely, there exists a subsequence $(w_V^{n_k})_k$ of $(w_V^n)_n$ such that $w_V^{n_k} \rightarrow w_V$.

Let us define the sequence $\widehat{w}_v^n = \frac{1}{\|w_v^n\|} \cdot w_v^n$. Clearly $\widehat{w}_v^n \in \widehat{F}_v(y_v^n, x_{-v}^n)$. Moreover, since $\|w_v\| > 0$,

$$\widehat{w}_v^{n_k} = \frac{1}{\|w_v^{n_k}\|} \cdot w_v^{n_k} \rightarrow \frac{1}{\|w_v\|} \cdot w_v = w_v.$$

Therefore, $w_v \in \text{Limsup}_{K_v(x_{-v}^n) \ni y_v^n \rightarrow y_v} \widehat{F}_v(y_v^n, x_{-v}^n)$, proving our claim.

Now, it is direct that

$$\text{Limsup}_{K_v(x_{-v}^n) \ni y_v^n \rightarrow y_v} \widehat{F}_v(y_v^n, x_{-v}^n) \subseteq \text{conv} \left(\bigcup_k \text{Limsup} \widehat{F}_v(y_v^{n_k}, x_{-v}^{n_k}) \right),$$

where the union is taken over all subsequences $(x_{-v}^{n_k})_k$ of $(x_{-v}^n)_n$ and all selections $(y_v^{n_k})_k$ of $(K_v(x_{-v}^n))_k$ converging to y_v . Hence,

$$\widehat{F}_v(y_v, x_{-v}) \subseteq \text{conv} \left(\bigcup_k \text{Limsup} \widehat{F}_v(y_v^{n_k}, x_{-v}^{n_k}) \right).$$

This last inclusion yields, by [45, Proposition 3.3], that (\widehat{F}_v, K_v) is net-lower sign-continuous. Using Lemma 3.3.3, we get that (F_v, K_v) is net-lower sign-continuous.

Now, fix $x \in C$. We will prove that $M(F_v(\cdot, x_{-v}), K_v(x_{-v}))$ is nonempty. Since $\theta_v(\cdot, x_{-v})$ is lower semicontinuous and $K_v(x_{-v})$ is compact, we get that the set $A_v := \text{argmin}_{K_v(x_{-v})} \theta_v(\cdot, x_{-v})$ is nonempty and closed. Let $\alpha := \min_{K_v(x_{-v})} \theta_v(\cdot, x_{-v})$ and let us denote by $S_\alpha^<$ the strict sublevel set of $\theta_v(\cdot, x_{-v})$ of level α . Since $K_v(x_{-v}) \cap \text{argmin}_{\mathbb{R}^{n_v}} \theta_v(\cdot, x_{-v}) = \emptyset$, we have that $S_\alpha^<$ is nonempty and so, we may choose $\bar{x}_v \in A_v$ such that

$$\text{dist}(\bar{x}_v, S_\alpha^<) = \min_{y_v \in A_v} \text{dist}(y_v, S_\alpha^<)$$

Note that, with this construction, $\bar{x}_v \in S_{\theta_v(\cdot, x_{-v})}^a(y_v)$ for all $y_v \in K_v(x_{-v})$. Indeed, on one hand if $\theta_v(y_v, x_{-v}) > \alpha$, then

$$\bar{x}_v \in A_v = S_\alpha(\theta_v(\cdot, x_{-v})) \subset S_{\theta_v(\cdot, x_{-v})}^<(y_v) \subset S_{\theta_v(\cdot, x_{-v})}^a(y_v).$$

On the other hand, if $\theta_v(y_v, x_{-v}) = \alpha$, then

$$\text{dist}(\bar{x}_v, S_\alpha^<) \leq \text{dist}(y_v, S_\alpha^<) =: \rho_{y_v}$$

and thus, $\bar{x}_v \in S_\alpha \cap \overline{\mathbb{B}}(S_\alpha^<, \rho_{y_v}) = S_{\theta_v(\cdot, x_{-v})}^a(y_v)$.

Now, choose $y_v \in K_v(x_{-v})$ and $y_v^* \in F_v(y_v, x_{-v})$. Since $\bar{x}_v \in S_{\theta_v(\cdot, x_{-v})}^a(y_v)$, we have that

$$\langle y_v^*, y_v - \bar{x}_v \rangle \geq 0.$$

Then, all the hypotheses in [45, Theorem 4.6] are fulfilled, showing that the quasi-variational inequality QVI(F, K) admits at least a solution.

Let \bar{x} be such a solution. We claim that $\bar{x} \in \text{GNEP}(\theta, K)$, or equivalently that for any $v \in I$, \bar{x}_v is an element of A_v (see notation above). Fix $v \in I$. Since $S_{\theta_v(\cdot, \bar{x}_{-v})}^a(\bar{x}_v)$ has nonempty interior thanks to hypotheses (i) and (ii), one has that $N_{\theta_v(\cdot, \bar{x}_{-v})}^a(\bar{x}_v)$ is a pointed cone, and so $F_v(\bar{x}) \subset N_{\theta_v(\cdot, \bar{x}_{-v})}^a(\bar{x}_v) \setminus \{0\}$. (see, e.g., [36, Lemma 3.1]). Therefore

$$\bar{x}_v \in S \left(N_{\theta_v(\cdot, \bar{x}_{-v})}^a \setminus \{0\}, K_v(\bar{x}_{-v}) \right).$$

Then, Proposition 3.2.9 yields that $\bar{x}_v \in A_v$, proving our claim and completing the proof. \square

3.3.2 Existence results of GNEP for constraints maps with nonempty interior values

Our aim in this subsection is to state existence results for the generalized Nash equilibrium problem $\text{GNEP}(\theta, K)$ in the same line that Theorem 3.3.1, but profiting from the extra hypothesis of nonemptiness of the interior of the constraint sets $K_V(x_{-V})$, in order to weaken the continuity-type hypothesis. We start this section by presenting a first result, very similar to Theorem 3.3.1.

Theorem 3.3.4 *For any $v \in I$, let C_v be a nonempty, compact and convex subset of \mathbb{R}^{n_v} , $\theta_v : \mathbb{R}^N \rightarrow \mathbb{R}$ and $K_v : C_{-v} \rightrightarrows C_v$. Then, the $\text{GNEP}(\theta, K)$ admits a generalized Nash Equilibrium if*

- (i) *for every $x_{-v} \in C_{-v}$, the function $\theta_v(\cdot, x_{-v})$ is quasiconvex, lower semicontinuous and sub-boundarily constant on $\text{dom} f$;*
- (ii) *for every $x \in \mathbb{R}^N$, the intersection $C_v \cap \text{argmin } \theta_v(\cdot, x_{-v}) = \emptyset$;*
- (iii) *the set-valued map $K_v : C_{-v} \rightrightarrows C_v$ is closed and its values are compact and convex, with nonempty interior;*
- (iv) *for every sequence $(x_{-v}^n)_{n \in \mathbb{N}} \subseteq C_{-v}$ converging to x_{-v} and every $y_v \in \text{int}(K_v(x_{-v}))$ one has*

$$N_{\theta_v(\cdot, x_{-v})}^a(y_v) \subseteq \underset{\text{int}K_v(x_{-v}^n) \ni y_v^n \rightarrow y_v}{\text{Limsup}} N_{\theta_v(\cdot, x_{-v}^n)}^a(y_v^n).$$

Proof. Let us define the set-valued maps F, F_v, \widehat{F} and \widehat{F}_v as we did in the proof of Theorem 3.3.1. Following the same strategy we did before, we can prove that for every sequence $(x_{-v}^n)_n \subset C_{-v}$ converging to x_{-v} and every $y_v \in \text{int}(K_v(x_{-v}))$ one has that

$$\widehat{F}_v(y_v, x_{-v}) = \underset{\text{int}K_v(x_{-v}^n) \ni y_v^n \rightarrow y_v}{\text{Limsup}} \widehat{F}_v(y_v^n, x_{-v}^n),$$

and so, we can also deduce that $(\widehat{F}_v, \text{int}K_v)$ and $(F_v, \text{int}K_v)$ are both net-lower sign-continuous.

In this case, following the same arguments as in proof of Theorem 3.3.1, we can show that $M^E(F_v(\cdot, x_{-v}), \text{int}K_v(x_{-v}))$ is nonempty, where $M^E(T, K)$ stands for the extended-Minty variational inequality defined by the map T and the subset K (see [45]).

Then, all the hypotheses of in [45, Theorem 4.1] are fulfilled, showing that there exists a solution \bar{x} of $\text{QVI}(F, K)$.

Let $x^* \in F(\bar{x})$ be such that $\langle x^*, y - \bar{x} \rangle \geq 0$, for any $y \in K(\bar{x})$. Since $K(\bar{x}) = \prod_{v \in I} K_v(\bar{x}_{-v})$, it is not hard to realize that for every v ,

$$\langle x_v^*, y_v - \bar{x}_v \rangle \geq 0, \quad \forall y_v \in K_v(\bar{x}_{-v}).$$

In particular, combining again hypotheses (i) and (ii), and an adaptation of [36, Lemma 3.1], we obtain that $x_v^* \neq 0$ and therefore the above inequality is strict for every $y_v \in \text{int}K_v(\bar{x}_{-v})$. Now, assume that \bar{x}_v is not a minimizer of $\theta_v(\cdot, \bar{x}_{-v})$ on $K_v(\bar{x}_{-v})$. Then, there exists $x'_v \in K_v(\bar{x}_{-v})$ such that $\theta_v(x'_v, \bar{x}_{-v}) < \theta_v(\bar{x}_v, \bar{x}_{-v})$ and so

$$x'_v \in \text{int} \left(S_{\theta_v(\cdot, \bar{x}_{-v})}^a(\bar{x}_v) \right).$$

This yields that there exists $y_v \in \text{int} \left(S_{\theta_v(\cdot, \bar{x}_{-v})}^a(\bar{x}_v) \right) \cap \text{int}K_v(\bar{x}_{-v})$ and so

$$\langle x_v^*, y_v - \bar{x}_v \rangle \leq 0,$$

which is a contradiction. We conclude then that $\bar{x}_v \in \operatorname{argmin}_{K_v(\bar{x}_v)} \theta_v(\cdot, \bar{x}_v)$. Since the index v is arbitrary, \bar{x} is a solution of GNEP(θ, K). \square

Our aim now, is to replace the technical assumption (iv) in Theorem 3.3.4 by a more natural one which will use the following convergence concept.

Definition 3.3.5 Let $(f_n)_n$ be a sequence of quasiconvex functions. We will say that:

1. the sequence $(f_n)_n$ sub-converges to a function f at the point y if there is a sequence $(y_n)_{n \in \mathbb{N}} \subseteq \mathbb{R}^N$ converging to y such that $S_{f_n}^a(y_n)$ Mosco-converges to $S_f^a(y)$;
2. the sequence $(f_n)_n$ sub-converges to a function f on the set K if $(f_n)_n$ sub-converge to f at any $y \in K$.

Lemma 3.3.6 Let $(f_n)_n$ be a sequence of quasiconvex, lower semi-continuous functions and let $(K_n)_{n \in \mathbb{N}}$ be a sequence of nonempty closed convex subsets of \mathbb{R}^N such that

- (i) $\operatorname{int}(K_n) \neq \emptyset$, for all $n \in \mathbb{N}$;
- (ii) $(K_n)_{n \in \mathbb{N}}$ Mosco-converges to a subset K of \mathbb{R}^N with $\operatorname{int}(K) \neq \emptyset$;
- (iii) $(f_n)_n$ sub-converges to a function f on $\operatorname{int}(K)$.

Then, for every $y \in \operatorname{int}(K)$,

$$N_f^a(y) \subseteq \operatorname{Limsup}_{\operatorname{int}(K_n) \ni y_n \rightarrow y} N_{f_n}^a(y_n).$$

Proof. Fix $y \in \operatorname{int}(K)$. Let $(y_n)_{n \in \mathbb{N}}$ be the sequence given by hypothesis (iii). Wlog, by Lemma 3.2.5, we can assume that $y_n \in \operatorname{int}(K_n)$ for all $n \in \mathbb{N}$. Let us define the sets

$$\forall n, C_n = S_{f_n}^a(y_n) \cap K_n \quad \text{and} \quad C = S_f^a(y) \cap K.$$

We claim that $(C_n)_{n \in \mathbb{N}}$ Mosco-converges to C .

First, on the one hand, let x be any element of $\operatorname{Limsup} C_n$. Then, there exists a sequence $(x_n)_n$ such that $x_n \in S_{f_n}^a(y_n) \cap K_n$ for every $n \in \mathbb{N}$ and x is a cluster point of this sequence. This yields that $x \in \operatorname{Limsup}_n S_{f_n}^a(y_n) \subset S_f^a(y)$ and $x \in \operatorname{Limsup}_n K_n \subset K$. Thus, $x \in C$, proving that,

$$\operatorname{Limsup}_n C_n \subset C.$$

On the other hand, choose $x \in C' = \operatorname{int}K \cap S_f^a(y)$, which is nonempty since $y \in C'$. Then, we have that $x \in S_f^a(y) \subset \operatorname{Liminf}_n S_{f_n}^a(y_n)$ and so, there exists a sequence $(x_n)_n$ converging to x with $x_n \in S_{f_n}^a(y_n)$. By Lemma 3.2.5, there exists $n_0 \in \mathbb{N}$ large enough such that for all $n \geq n_0$, $x_n \in \operatorname{int}K_n$. Thus, for all $n \geq n_0$, we have that $x_n \in C_n$, which proves that $x \in \operatorname{Liminf}_n C_n$, and so

$$C' \subset \operatorname{Liminf}_n C_n.$$

Note that C' is dense in C , that is, $\operatorname{cl}(C') = C$. Indeed, choose $x \in C$. Then, since $x \in K$ and $y \in \operatorname{int}K$, convexity of K yields that the open segment $]x, y[$ is included in $\operatorname{int}K$. Since $S_f^a(y)$ is also convex, we have that $]x, y[\subset S_f^a(y)$, showing that $]x, y[\subset C'$. This proves that $x \in \operatorname{cl}(C')$ and, since x is arbitrary, that $C = \operatorname{cl}(C')$. Since $(C_n)_n$ is a sequence of closed sets, then $\operatorname{Liminf}_n C_n$ is also closed, and so

$$C = \operatorname{cl}(C') \subset \operatorname{Liminf}_n C_n.$$

This proves our claim. Then, [87, Corollary 7.6.5] entails that

$$N_C(y) = \operatorname{Limsup}_{C_n \ni y'_n \rightarrow y} N_{C_n}(y'_n).$$

Now, since $y \in C \subset S_f^a(y)$ (which are both convex sets) then $N_f^a(y) \subset N_C(y)$. Furthermore, for each $n \in \mathbb{N}$ and each $y'_n \in \operatorname{int}(K_n) \cap S_{f_n}^a(y_n)$ we have that

$$y'_n \in \operatorname{bd}(S_{f_n}^a(y_n)) \implies S_{f_n}^a(y'_n) \subset S_{f_n}^a(y_n). \quad (3.3.5)$$

Indeed, choose $y'_n \in \operatorname{bd}(S_{f_n}^a(y_n))$. First, by Definition 3.2.1, one has that $y'_n \in S_{f_n(y_n)}(f_n)$ and so $f(y'_n) \leq f(y_n)$. If $f_n(y'_n) < f_n(y_n)$, then

$$S_{f_n}^a(y'_n) \subset S_{f_n(y'_n)}(f_n) \subset S_{f_n(y_n)}^<(f_n) \subset S_{f_n}^a(y_n),$$

and thus, the implication (3.3.5) holds. Now, if $f_n(y'_n) = f_n(y_n)$, then, denoting by α the common value, Definition 3.2.1 entails that

$$\rho' := \operatorname{dist}(y'_n, S_\alpha^<(f_n)) \leq \operatorname{dist}(y_n, S_\alpha^<(f_n)) =: \rho,$$

and thus,

$$S_{f_n}^a(y'_n) = S_\alpha(f_n) \cap \overline{\mathbb{B}}(S_\alpha^<(f_n), \rho') \subset S_\alpha(f_n) \cap \overline{\mathbb{B}}(S_\alpha^<(f_n), \rho) = S_{f_n}^a(y_n).$$

This proves that implication (3.3.5) always holds. Therefore, again by locality of normal cones, for any $y'_n \in \operatorname{int}(K_n) \cap S_{f_n}^a(y_n)$, we can write

$$N_{C_n}(y'_n) = (S_{f_n}(y_n) - y'_n)^\circ \subset \begin{cases} N_{f_n}^a(y'_n) & \text{if } y'_n \in \operatorname{bd}(S_{f_n}^a(y_n)) \\ \{0\} & \text{otherwise.} \end{cases}$$

Either way, we always have that $N_{C_n}(y'_n) \subseteq N_{f_n}^a(y'_n)$. Finally, defining the sets $A_n := \operatorname{int}(K_n) \cap S_{f_n}^a(y_n)$ and applying Lemma 3.2.5, we can write

$$\begin{aligned} N_f^a(y) &= N_C(y) \\ &= \operatorname{Limsup}_{C_n \ni y'_n \rightarrow y} N_{C_n}(y'_n) \\ &= \operatorname{Limsup}_{A_n \ni y'_n \rightarrow y} N_{C_n}(y'_n) \subseteq \operatorname{Limsup}_{A_n \ni y'_n \rightarrow y} N_{f_n}^a(y'_n) \\ &\subseteq \operatorname{Limsup}_{\operatorname{int}(K_n) \ni y'_n \rightarrow y} N_{f_n}^a(y'_n). \end{aligned}$$

The conclusion follows. □

Remark 3.3.7 As one can observe, in the above proof, one can use a weaker version of assumption (iii) that is the sub-convergence hypothesis. Indeed Lemma 3.3.6 still hold true if one only assume that for any $y \in \operatorname{int}(K)$, the sequence $(f_n)_n$ sub-converges at y to a certain quasiconvex function f_y . Then of course the conclusion would be that, for every $y \in \operatorname{int}(K)$,

$$N_{f_y}^a(y) \subseteq \operatorname{Limsup}_{\operatorname{int}(K_n) \ni y_n \rightarrow y} N_{f_n}^a(y_n).$$

Now, combining Lemma 3.3.6 and Theorem 3.3.4, we immediately obtain the following existence result of solution for a generalized Nash equilibrium problem, which is the result we announced at the beginning of the article.

Corollary 3.3.8 *For any $v \in I$, let C_v be a nonempty, compact and convex subset of \mathbb{R}^{n_v} , $\theta_v : \mathbb{R}^N \rightarrow \mathbb{R}$ and $K_v : C_{-v} \rightrightarrows C_v$. Then, the GNEP(θ, K) admits a generalized Nash Equilibrium if*

- (i) *for every $x_{-v} \in C_{-v}$, the function $\theta_v(\cdot, x_{-v})$ is quasiconvex, lower semicontinuous and sub-boundarily constant on $\text{dom}f$;*
- (ii) *for every $x \in \mathbb{R}^N$, then the intersection $C_v \cap \text{argmin} \theta_v(\cdot, x_{-v}) = \emptyset$;*
- (iii) *the set-valued map $K_v : C_{-v} \rightrightarrows C_v$ is closed and its values are compact convex with nonempty interior;*
- (iv) *for every sequence $(x_{-v}^n)_{n \in \mathbb{N}}$ converging to x_{-v} , the sets $K_v(x_{-v}^n)$ Mosco-converge to $K_v(x_{-v})$;*
- (v) *for every sequence $(x_{-v}^n) \subseteq C_{-v}$ converging to some x_{-v} , the sequence $\theta_v(\cdot, x_{-v}^n)$ sub-converges to $\theta_v(\cdot, x_{-v})$ on $\text{int}K_v(x_{-v})$.*

3.4 Comparisons with existing results

In the previous section several existence theorems have been proved for quasiconvex generalized Nash equilibrium problems. It is nevertheless important to be able to compare the considered hypotheses to the ones assumed in the classical result of Arrow-Debreu for GNEP or in the discontinuous approach of Reny for NEP. Each of the forthcoming subsections is thus devoted to the comparison between our results and these two baseline existence results.

3.4.1 Continuity vs convergence of adjusted sublevel sets

In their seminal work, Arrow and Debreu [93] proved the existence of generalized Nash equilibrium assuming, for any player v , the quasiconvexity of the cost function $\theta_v(x_v, x_{-v})$ with respect to the player's variable x_v and the continuity of this cost function with respect to both x_v and x_{-v} . Thus to understand the extent and limitation of our contributions, this subsection is devoted to comparison of this continuity assumption to our new hypotheses ((iv) in Theorem 3.3.1 and (v) in Corollary 3.3.8).

In order to simplify the context of the comparison, one can consider problems where, for any v , $K_v(x_{-v}) = C_v$ and C_v has nonempty interior. Namely, we want to now weather for a function $\theta : C_v \times C_{-v} \rightarrow \mathbb{R}$ which is quasiconvex and lower semicontinuous in the first variable, the continuity with respect of its second variable is weaker or stronger than the Mosco-convergence of its adjusted sublevel sets in the sense of hypothesis (v) of Corollary 3.3.8.

The main conclusion is that these two hypotheses are non-comparable, since it is possible to construct examples where one of them is verified, but not the other one.

First it is an immediate to construct discontinuous functions satisfying the Mosco-convergence of the sublevel sets. Just consider, for example, $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ defined by

$$f(x, y) = \begin{cases} x & \text{if } x \leq 0 \\ x + 1 & \text{otherwise} \end{cases}$$

Let us now describe the converse case, that is a continuous function failing the convergence of adjusted sublevel sets.

Example 3 Consider the function $f : [0, 1]^2 \times [-1, 1] \rightarrow \mathbb{R}$ given by

$$f(x, y) = \begin{cases} \|x\|_\infty & \text{if } \|x\|_\infty \in [0, 1/3] \\ \frac{1}{3} + |y| (\|x\|_\infty - \frac{1}{3}) & \text{if } \|x\|_\infty \in (1/3, 2/3) \\ (\|x\|_\infty - \frac{2}{3}) + \frac{|y|+1}{3} & \text{if } \|x\|_\infty \in [2/3, 1]. \end{cases}$$

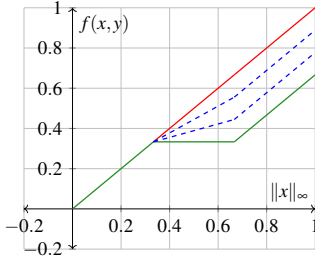


Figure 3.4.1: f with respect to $\|x\|_\infty$. Plotted for $y = 1$ (red), $y = 2/3, 1/3$ (dashed blue), and $y = 0$ (green).

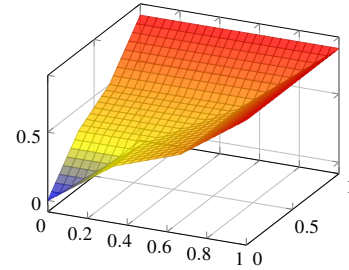


Figure 3.4.2: $f(x, y)$ for $y = 1/2$.

It is not hard to realize that f is continuous with respect to both variables x and y , that it is quasiconvex in the first coordinate $x = (x_1, x_2)$, and that for every $y \neq 0$, the sublevel sets coincide with the sublevel sets of the supremum norm, that is,

$$S_{f(\cdot, y)}(x) = S_{\|\cdot\|_\infty}(x), \quad \forall x \in [0, 1]^2.$$

Now, choose $\bar{x} \in [0, 1]^2$ such that $1/3 < \bar{x}_1 < 1/2$ and $\bar{x}_2 > 1/2$. This yields that $\|\bar{x}\|_\infty \in]1/3, 2/3[$ and so, \bar{x} is in the flat part of $f(\cdot, 0)$. Assuming that $\rho_{\bar{x}} := \text{dist}(\bar{x}; S_{f(\cdot, 0)}^<(\bar{x})) \leq 1/3$, then

$$S_{f(\cdot, 0)}^a(\bar{x}) = \overline{\mathbb{B}}(S_{f(\cdot, 0)}^<(\bar{x}), \rho_{\bar{x}}) = \frac{1}{3}\mathbb{B}_\infty + \rho_{\bar{x}}\mathbb{B},$$

where \mathbb{B}_∞ denotes the unit ball of the supremum norm. Now, fix any sequence $(y_n)_n \subset (0, 1]$ converging to 0, and any sequence $(x_n)_n \subset [0, 1]^2$ converging to \bar{x} . For every $n \in \mathbb{N}$ large enough, one has that

$$x_{2,n} \geq x_{1,n},$$

where $x_{i,n}$ denotes the i th coordinate of x_n . Thus,

$$N_{S_{f(\cdot, y_n)}^a}^a(x_n) = N_{S_{f(\cdot, y_n)}}(x_n) = N_{\|\cdot\|_\infty}(x_n) = \mathbb{R}_+ \cdot (0, 1).$$

This yields that for every sequence $(y_n)_n \subset]0, 1]$ converging to 0,

$$\text{Limsup}_{x_n \rightarrow \bar{x}} N_{f(\cdot, y_n)}^a(x_n) = \mathbb{R}_+ \cdot (0, 1).$$

where the Limsup is taken over all sequences $(x_n)_n$ in $[0, 1]^2$ converging to \bar{x} . However, as Figure 3.4.3 illustrates, the adjusted normal cone of $f(\cdot, 0)$ at \bar{x} is given by

$$N_{f(\cdot, 0)}^a(\bar{x}) = \mathbb{R}_+(\bar{x} - \text{proj}(\bar{x}, \frac{1}{3}\overline{\mathbb{B}}_\infty)) = \mathbb{R}_+(\bar{x} - (1/3, 1/3)) \neq \mathbb{R}_+ \cdot (0, 1).$$

Thus, hypothesis (iv) in Theorem 3.3.4 is not verified and, as a consequence of Lemma 3.3.6, neither is hypothesis (v) of Corollary 3.3.8.

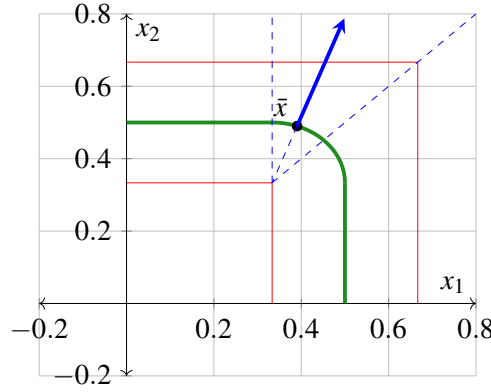


Figure 3.4.3: In red, the sublevel set and strict sublevel set of $f(\cdot, 0)$ at \bar{x} . In green, the adjusted sublevel set of $f(\cdot, 0)$ at \bar{x} . Dashed in blue, the lines $x_1 = x_2$ and $x_1 = 1/3$. Thick in blue, the adjusted normal cone of $f(\cdot, 0)$ at \bar{x} .

The above example shows that convergence of adjusted normal cones or adjusted sublevel sets are not weaker hypotheses than continuity of cost functions in the general case. However, if the cost functions are *semistrictly quasiconvex* in the first variable instead of simply quasiconvex, hypotheses (iv) of Theorem 3.3.4 and (v) of Corollary 3.3.8 are automatically verified by continuous functions.

This result is based on [101], where it is shown that epi-convergence can be characterized by convergence of sublevel sets. Recall that a sequence of lower semicontinuous functions $(f_n : \mathbb{R}^p \rightarrow \overline{\mathbb{R}} : n \in \mathbb{N})$ epi-converges to a lower-semicontinuous function $f : \mathbb{R}^p \rightarrow \overline{\mathbb{R}}$ if the following two conditions hold:

- (a) $f(x) \leq \liminf f_n(x_n)$, whenever a sequence $(x_n) \subset \mathbb{R}^p$ converges to $x \in \mathbb{R}^p$; and
- (b) for all $x \in \mathbb{R}^p$, there exists a sequence $(x_n) \subset \mathbb{R}^p$ such that $f_n(x_n) \rightarrow f(x)$.

Epi-convergence is already a classic tool in nonsmooth variational analysis and it can be found in classic monographs as [102]. In what follows, we will say that a function $f : \mathbb{R}^{n_v} \times \mathbb{R}^{n-v} \rightarrow \mathbb{R}$ is epi-continuous in the second variable if

- (a) for every $x_{-v} \in \mathbb{R}^{n-v}$, $f(\cdot, x_{-v})$ is lower-semicontinuous.
- (b) for every sequence $(x_{-v}^n)_n \subset \mathbb{R}^{n-v}$ converging to a point $\bar{x}_{-v} \in \mathbb{R}^{n-v}$, one has that

$$f(\cdot, x_{-v}^n) \text{ epi-converges to } f(\cdot, \bar{x}_{-v}).$$

It is not hard to verify that if f is lower semicontinuous in the first variable and continuous in the second variable, then it is epi-continuous in the second variable as well. Moreover, it is easy to construct examples where the converse doesn't hold.

Theorem 3.4.1 *Let $f : \mathbb{R}^{n_v} \times \mathbb{R}^{n-v} \rightarrow \mathbb{R}$ be a function such that*

- (i) *for every $x_{-v} \in \mathbb{R}^{n-v}$, the function $f(\cdot, x_{-v})$ is lower semicontinuous and semistrictly quasiconvex.*
- (ii) *f is epi-continuous in the second variable.*

Then, hypothesis (v) of Corollary 3.3.8 holds, that is, for every sequence $(x_{-v}^n)_n \subset \mathbb{R}^{n-v}$ converging to x_{-v} and every $y_v \in \mathbb{R}^{n_v} \setminus \operatorname{argmin} f(\cdot, x_{-v})$, there exists a sequence $(y_v^n)_n \subset \mathbb{R}^{n_v}$ such that

$$S_{f(\cdot, x_{-v}^n)}(y_v^n) \text{ Mosco-converges to } S_{f(\cdot, x_{-v})}(y_v).$$

Proof. By hypothesis (i), we have that for every $(x_v, x_{-v}) \in \mathbb{R}^{n_v} \times \mathbb{R}^{n-v}$, the adjusted level set of $f(\cdot, x_{-v})$ at x_v coincides with the level set of $f(\cdot, x_{-v})$ at x_v , that is,

$$S_{f(\cdot, x_{-v})}^a(x_v) = S_{f(\cdot, x_{-v})}(x_v).$$

Now, fix a sequence $(x_{-v}^n)_n \subset \mathbb{R}^{n-v}$ converging to x_{-v} and a point $y_v \in \mathbb{R}^{n_v}$ which is not a global minimizer of $f(\cdot, x_{-v})$. Let $\alpha = f(y_v, x_{-v})$. Since $f(\cdot, x_{-v}^n)$ epi-converges to $f(\cdot, x_{-v})$ by hypothesis (ii), we have by [101, Theorem 3.1] that there exists a sequence $(\alpha_n)_n \subset \mathbb{R}$ converging to α such that

$$S_{\alpha_n}(f(\cdot, x_{-v}^n)) \text{ Mosco-converges to } S_{\alpha}(f(\cdot, x_{-v})).$$

where $S_{\alpha}(f(\cdot, x_{-v}))$ denotes the sublevel set of the function $x_v \mapsto f(x_v, x_{-v})$ at level α . To simplify notation, let us write $A_n = S_{\alpha_n}(f(\cdot, x_{-v}^n))$ and $A = S_{\alpha}(f(\cdot, x_{-v}))$. Since $A \subset \text{Liminf}_n A_n$, there exists a sequence $(y_v^n)_n \subset \mathbb{R}^{n_v}$ such that $y_v^n \in A_n$ for every $n \in \mathbb{N}$ and converging to y_v . Let $\beta_n = f(y_v^n, x_{-v}^n)$ and let $B_n = S_{\beta_n}(f(\cdot, x_{-v}^n))$.

By epi-continuity of f and the fact that $\beta_n \leq \alpha_n$ for all $n \in \mathbb{N}$, we have that

$$\beta_n \text{ converges to } \alpha \quad \text{and} \quad \text{Limsup}_n B_n \subset A.$$

To finish the proof, it is enough to show that $A \subset \text{Liminf}_n B_n$. To do so, let us write $B = \text{Liminf}_n B_n$ and suppose that there exists $z \in A \setminus B$. On the one hand, since $z \in A$, we have that $f(z, x_{-v}) \leq \alpha$. On the other hand, by epi-continuity, there exists a sequence $(z_n)_n \subset \mathbb{R}^{n_v}$ such that $f(z_n, x_{-v}^n)$ converges to $f(z, x_{-v})$. In particular, since $z \notin B$, there exists a subsequence $(z_{n_k})_k$ of $(z_n)_n$ such that $z_{n_k} \notin B_{n_k}$. Thus,

$$f(z, x_{-v}) = \lim_k f(z_{n_k}, x_{-v}^{n_k}) \geq \lim_k \beta_{n_k} = \alpha.$$

We conclude that $f(z, x_{-v}) = \alpha$.

Choose now $m \in \mathbb{R}^{n_v}$ such that $f(m, x_{-v}) < f(y_v, x_{-v})$. By epi-continuity, there exists a sequence $(m_n)_n \subset \mathbb{R}^{n_v}$ converging to m and such that $f(m_n, x_{-v}^n) \rightarrow f(m, x_{-v})$. Since $\beta_n \rightarrow \alpha > f(m, x_{-v})$, a simple convergence argument tell us that for $n \in \mathbb{N}$ large enough, one has that

$$f(m_n, x_{-v}^n) < \beta_n,$$

and thus, we may assume without loss of generality that $m_n \in B_n$ for all $n \in \mathbb{N}$. We conclude then that $m \in B$. Now, since $z \notin B$ and B is closed, there exists $\varepsilon > 0$ small enough such that $\mathbb{B}(z, \varepsilon) \cap B = \emptyset$. Now, let $u = z + \frac{\varepsilon}{2}(m - z)$. By construction, $u \in A \setminus B$, and so, following the same reasoning we did for z , we conclude that $f(u, x_{-v}) = \alpha$. However, this is a contradiction, since $f(m, x_{-v}) < f(z, x_{-v})$ and so the semistrict quasiconvexity of $f(\cdot, x_{-v})$ yields that $f(u, x_{-v}) < f(z, x_{-v})$. We conclude that $A \subset \text{Liminf}_n B_n$, which completes the proof. \square

3.4.2 Comparison with Reny-type result

In the case of a Nash game (not generalized Nash game) Corollary 3.3.8 reduces to

Corollary 3.4.2 *For any $v \in I$, let C_v be a nonempty compact and convex subset of \mathbb{R}^{n_v} , $\theta_v : \mathbb{R}^N \rightarrow \mathbb{R}$. Then, the NEP(θ, C) admits a Nash Equilibrium if*

- (i) *for every $x_{-v} \in C_{-v}$, the function $\theta_v(\cdot, x_{-v})$ is quasiconvex, lower semicontinuous and sub-boundarily constant on $\text{dom} f$;*

- (ii) for every $x \in \mathbb{R}^N$, the intersection $C_v \cap \operatorname{argmin} \theta_v(\cdot, x_{-v}) = \emptyset$;
- (iii) for each $v \in I$, C_v has nonempty interior.
- (iv) for every sequence $(x_{-v}^n)_n \subseteq C_{-v}$ converging to some x_{-v} , the sequence $\theta_v(\cdot, x_{-v}^n)$ sub-converges to $\theta_v(\cdot, x_{-v})$ on $\operatorname{int}C_v$.

As mentioned for the Generalized Nash case, the advantage of the above result is that only lower semicontinuity of the objective functions θ_v is assumed, instead of the classical continuity hypothesis. To our knowledge, the only work in which existence of equilibria is proved under semicontinuity assumption is due to Reny [94](see also Bich-Laraki [95]).

Proposition 3.4.3 (Reny) *For any $v \in I$, let C_v be a nonempty, compact and convex subset of \mathbb{R}^{n_v} , $\theta_v : \mathbb{R}^N \rightarrow \mathbb{R}$. Then, the NEP(θ, C) admits a Nash Equilibrium if*

- (i) for every $x_{-v} \in C_{-v}$, the function $\theta_v(\cdot, x_{-v})$ is quasiconvex and lower semi-continuous;
- (ii) the game is better-reply secure.

Let us recall, adapting the notations of Laraki-Bich, that the game satisfies the *Better-reply secure* if, for any

$$(x, \lambda) := ((x_v)_v, (\lambda_v)_v) \in \operatorname{cl} \left\{ ((u_v)_v, (\theta_v(u_v))_v) : (u_v)_v \in C = \prod_v C_v \right\}$$

with (x, λ) not a Nash equilibrium of the game, then for at least one of the players v , there exists $d_v \in C_v$ such that

$$\bar{\theta}_v(d_v, x_{-v}) := \limsup_{x'_{-v} \rightarrow x_{-v}} \theta_v(d_v, x'_{-v}) < \lambda_v. \quad (3.4.1)$$

As it will be shown in the following simple example, Corollary 3.4.2 furnishes, in this restricted context of Nash equilibrium problem (instead of Generalized Nash equilibrium case), an alternative set of assumptions compared to the "better-reply secure" hypothesis.

Example 4 Set $N = 2$, $C_1 = C_2 = [0, 1]$ and the players' objective functions θ_1 and θ_2 by

$$\theta_1(x_1, x_2) = \begin{cases} x_1 & \text{if } x_2 \in]0, 1] \\ x_1 - 2 & \text{if } x_2 = 0 \end{cases} \quad \text{and} \quad \theta_2(x_2, x_1) = \begin{cases} 0 & \text{if } x_2 > -1 \\ -1 & \text{if } x_2 \leq -1. \end{cases}$$

We claim that this game is not better reply secure. Indeed, consider the point $x^* = (1/2, 0)$ and $\lambda = \theta(1/2, 0) = (-3/2, 0)$, which is clearly not an equilibrium of the game. Since the cost function of player 2 is constant over C_2 , there exists no $d_2 \in C_2$ satisfying (3.4.1) for $v = 2$.

Now, choose any point $d_1 \in C_1 = [0, 1]$ and a neighbourhood $U \in \mathcal{N}(0)$, and choose any point $x_2 \in U \cap [0, 1]$ with $x_2 \neq 0$. We have then that

$$\theta_1(d_1, x_2) = d_1 \geq 0 > -3/2.$$

It follows that there exist no $d_1 \in C_1$ satisfying (3.4.1) for $v = 1$. The claim is then verified.

In contrast, by construction of the cost functions θ_1 and θ_2 , and recalling Remark 3.2.8.iii), the equilibrium problem $NEP(\theta, C)$ clearly satisfies hypotheses (i) – (iii) of Corollary 3.4.2.

To verify hypothesis (iv), fix $v \in \{1, 2\}$, a sequence $(x_{-v}^k)_k \subset [0, 1]$ converging to any $x_{-v} \in [0, 1]$ and fix any $y_v \in]0, 1[$. Define the sequence $(y_v^k)_k$ by $y_v^k = y_v$. Regardless if $v = 1$ or $v = 2$, it always holds in this example that

$$S_{\theta_v(\cdot, x_{-v}^k)}^a(y_v) =]-\infty, y_v].$$

Thus, $S_{\theta_v(\cdot, x_{-v}^k)}^a(y_v)$ Mosco-converges to $S_{\theta_v(\cdot, x_{-v})}^a(y_v)$ trivially. Hypothesis (iv) holds, and so Corollary 3.4.2 applies.

3.5 Conclusions

In this work, we prove existence results for generalized Nash equilibrium problem in the finite dimensional setting. They provide alternative set of assumptions compared to the Arrow-Debreu pioneering work and the "better-reply secure" approach of Reny. More precisely, we eliminate the continuous assumption of θ_v by replacing it by a "continuity-type" of the sublevel sets of the cost functions. Our approach is based on the concept of the *adjusted normal operator* [37], the *net-lower sign-continuity*[45], and the reformulation of the generalized Nash equilibrium problem in term of quasi-variational inequalities.

Of course some important open research questions and perspectives remain. First of all our existence results are proved in the finite-dimensional setting. Thus, it could be interesting, but out of the scope of this work, to consider possible adaptations of our approach in the infinite-dimensional setting in order to tackle GNEP in which the decision variables of the players are possibly functions. Second, the hypothesis, used all along our paper, that "For every $x \in \mathbb{R}^N$, the intersection $C_v \cap \operatorname{argmin} \theta_v(\cdot, x_{-v}) = \emptyset$ " could appear to be quite technical. It has been introduced since the normal operator has ill behavior at the argmin of the function due to the very definition of the adjusted level sets. A modification of the proof of Theorem 3.3.1 by using an adapted version of the operators F_v (possibly taking inspiration from [36]) could be a possible way to drop this hypothesis.

Finally, our results hold for quasiconvex functions that are also sub-boundarily constant, which represent a large class of nonpathological functions. Morally speaking, these are the functions that don't present any oscillatory behavior along the boundary of adjusted sublevel sets. It is hard to imagine that first order optimality conditions can entail global optimality (in the spirit of Proposition 3.2.9) without a property like sub-boundarily constant condition. However, it is worth to pay some more attention to this definition, to try to better understand what should be the underlying property for quasiconvex programming that can guarantee sufficient first order optimality conditions, well known for the ideal case of convex programming.

PART II

Modeling and Optimization of Eco-Industrial Parks

Optimal Design of Exchange Networks with Blind Inputs: Applications to Eco-Industrial Parks

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Abstract

Motivated by the design and optimization of the water exchange networks in Eco-Industrial Parks (EIP), we investigate the abstract Blind-Input model for general exchange networks. This abstract model is based on a Game Theory approach, formulating it as a Single-Leader-Multi-Follower (SLMF) game: at the upper level, there is an authority (leader) that aims to minimize the consumption of natural resources, while, at the lower level, agents (followers) try to minimize their operating costs. We introduce the notion of Blind-Input contract, which is an economic contract between the authority and the agents in order to ensure the participation of the latter ones in the exchange networks. More precisely, when participating in the exchange network, each agent accepts to have a blind input in the sense that she controls only her output fluxes, and the authority commits to guarantee a minimal relative improvement in comparison with the agent's stand-alone operation. The SLMF game is equivalently transformed into a single mixed-integer optimization problem. Thanks to this reformulation, examples of EIP of realistic size are then studied numerically.

Keywords: Single-Leader-Multi-Follower game, Eco-industrial park, mixed integer programming, water networks.

AMS subject classifications. 90C26, 90C11, 90C90

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4.1 Introduction

In the last few decades, the development of the industrialized countries has led to an increasing depletion of natural resources such as freshwater and energy (see, e.g., [64, 65]). The conservation and sustainable use of such resources play an important role in both, environmental impact and business success within the industry. In response to preserve the environment while increasing the utilities of the enterprises, the concept of *industrial ecology* has emerged [6].

Industrial ecology (IE) was first introduced in 1989 by Frosch and Gallopoulos [66]. They wrote “the consumption of energy and materials is optimized, waste generation is minimized and the effluents of one process . . . serve as the raw material for another process”. This is an approach to the industrial design of products and processes and the implementation of sustainable manufacturing strategies. The idea is directly related to another concept, *industrial symbiosis*, which involves “separate industries in a collective approach to competitive advantage involving physical exchange of materials, energy, water and/or by-products” (see [67]). One key concept of industrial symbiosis is then the *exchange networks*.

A perfect example of an exchange network which illustrates the notion of industrial symbiosis is the concept of Eco-Industrial Parks (EIP). This notion has several definitions, but one widely accepted is “an industrial system of planned materials and energy exchanges that seeks to minimize energy and raw materials use, minimize waste, and build sustainable economic, ecological and social relationships” [6, 9, 68].

Recently, in works of Boix et al. [6] and Kastner et al. [7], it has been pointed out that there is still a lack of systematic methods for designing the optimal configuration of an EIP. In previous studies [6, 9, 8], water integration networks (which is a classical example of EIP) were modeled as a cooperative economy, in the framework of multi-objective optimization (MOO). This approach consist in creating a vector function of $n + 1$ coordinates given by

$$C(F) = (\text{Cost}_1(F), \dots, \text{Cost}_n(F), Z(F))$$

where $\text{Cost}_i(\cdot)$ is the cost function of the enterprise i , $Z(\cdot)$ is the global consumption of natural resources, and F is the flux vector describing the distribution in the exchange network. Then, the aim is to solve the

problem of “minimizing” C with respect to F , satisfying the physical constraints of the model. The result of such minimization is called a *Pareto front*, which consists in all vectors F for which none of the coordinates of C can be improved without worsen another one [103, 104]. Usually an authority, representing the EIP’s designer, selects one of this solutions considering as criteria the distance to an utopia point.

Nomenclature	
Latin symbols:	
n	number of independent agents
m	number of regulated agents
\mathcal{P}	set of independent agents
\mathcal{R}	set of regulated agents
I_P	index set of independent agents
I_R	index set of regulated agents
I	assembly of index sets I_P and I_R
I_0	assembly of index set I and sink node 0
E	network topology
E_{\max}	set of all admissible connections of the network
E^c	set of connections that are not in E
$E_{i,\text{act}}$	set of active arcs of agent i
E_{st}	stand-alone topology
\mathcal{E}	set of all valid topologies
$\text{Cost}_i(\cdot)$	operating cost of agent i
STC_i	stand-alone cost of agent i
$C(i, j)$	arc class of (i, j)
\mathcal{C}_i	family of all arc classes exiting from agent i
D	set of all arc classes of active agents
y	boolean variable
$x_{i,j}$	flux through the connection (i, j)
x_i	outlet flux vector of agent i
x_{-i}	vector of all fluxes not exiting from agent i
x_{-i}^P	vector of all fluxes exiting from an independent agent other than i
x	complete vector of fluxes through the network
z_i	consumption of natural resource of the i th agent
$Z(\cdot)$	total consumption of natural resources
$g_i(\cdot)$	input validation function of agent i
F_i	vector of fluxes exiting from enterprise i (water exchange network)
F_{-i}	vector of all fluxes not exiting from enterprise i (water exchange network)
F^P	vector of fluxes exiting from enterprises (water exchange network)
F_{-i}^P	vector of all fluxes exiting from an enterprise other than i (water exchange network)
F^R	vector of fluxes exiting from regeneration units (water exchange network)
F	flux vector describing the distribution in the water exchange network
M_i	contaminant load of enterprise i [g/h]
$C_{i,\text{in}}, C_{i,\text{out}}$	maximum contaminant concentration allowed in inlet/outlet of processes [ppm]
$C_{r,\text{in}}$	minimum inlet concentration allowed of reg. units [ppm]
$C_{r,\text{out}}$	exact outlet contaminant concentration of reg. units [ppm]
A	the lifetime of the park [h]
Coef	Penalization coefficient of stand-alone agents
Acronyms:	
EIP	Eco-Industrial Park
GNEP	generalized Nash equilibrium problem
Eq	the set of equilibria for the induced GNEP
KKT	Karush-Kuhn-Tucker
MILP	Mixed-integer linear programming
MPEC	mathematical programs with equilibrium constraint
SLMF	Single-Leader-Multi-Follower
STC	stand-alone cost
Greek Symbols:	
α	the minimal relative gain that each agent ask for participating in the network
c	the marginal cost of freshwater consumption [\$/T]
$\beta_{i,0}$	the discharge cost of polluted water of enterprise i [\$/T]
$\delta_{i,j}$	the cost sending polluted water from enterprise i to j [\$/T]
Γ_r	the marginal cost of regenerating water [\$/T]
ψ	power associated to Γ_r

The main problem with such an approach is that points of the Pareto front are not necessarily economi-

cally stable: first, a Pareto point requires the enterprises to cooperate and share information, which is rarely the case of an EIP. Second, due to the noncooperative economy, the different enterprises may deviate from the selection of the authority since they may improve their cost function by unilaterally changing their operation. In terms of game theory, a solution of the MOO approach is *a social optimization which may fail to respect incentives* (see [10, Chapter 1]).

To solve this incompatibility, again in the context of water integration networks, in the seminal work of Ramos et al. [2], further developed in [105], a novel game theory approach has been proposed, by modeling the EIP design problem as a Single-Leader-Multi-Follower (SLMF) game (see [61, 63]): since the agents do not want to exchange information, a confidential centralization through an authority of the park is introduced. Then, at the upper level, there is the EIP authority which wants to minimize the consumption of natural resources $Z(F)$, while at the lower level, each enterprise tries to minimize her cost function $\text{Cost}_i(F)$, related to her processes, consumption of natural resources and activity within the EIP. The authority of the park must choose the connections of the exchange network and the operation of the regeneration units, while each enterprise controls their consumption of natural resources and their output flux distribution. Based on the EIP authority decisions, all enterprises compete with each other in a parametric non-cooperative generalized Nash game with the strategies of the EIP authority as exogenous parameters. Figure 4.1.1 shows the general scheme of such a model, where the enterprises are considered the economic agents of the game. We refer the reader to [10, 11] for a primer in noncooperative games, to [14, 15] for a survey of Generalized Nash Equilibrium problems, and [12, 13] for the theory of bilevel optimization. For Single-Leader-Multi-Follower games, we refer to [63] and the references therein.

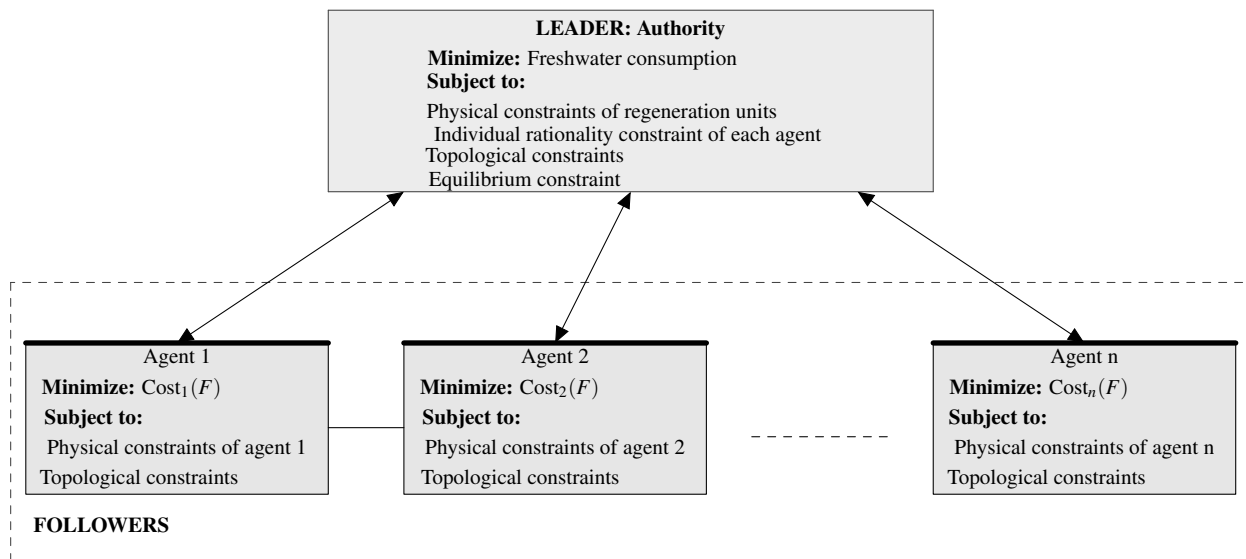


Figure 4.1.1: General scheme of SLMF Game

The main implicit assumption done in [2] is that each enterprise *can only control her outlet distribution and her own fresh water consumption*, but they are forced to accept whatever is sent to them through the exchange network. Furthermore, they have no knowledge about the particular actions of the other agents of the network, excepting only the amount and quality of the final inlet flux. In practice, this situation corresponds to the case when, at the entrance, each agent of the network has a mixer, and so she is only aware of the total input she is receiving, as Figure 4.1.2 illustrates. In other words, when participating in the exchange network, each agent accepts to have a *blind input*.

While this model respects incentive consistency, it has two main drawbacks: the first one is that the rule that the park's authority imposes, that is, the blind input, is too restrictive. Indeed, under this paradigm, an

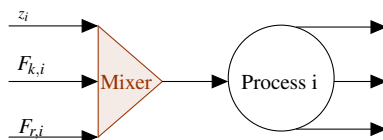


Figure 4.1.2: Blind-Input Schema. z_i , $F_{k,i}$ and $F_{r,i}$ are freshwater consumption, wastewater sent from agent k to i , and regenerated water sent from regeneration unit r to i , respectively.

enterprise may be forced to receive too much polluted water which could turn into higher costs than the stand-alone operation outside the park (examples are easy to construct with two enterprises). This violates the economical principle (well known in contract theory and mechanism design) of *individual rationality*: an enterprise will participate in the EIP only if it is convenient to her (see [69, 70, 71]); the second one is the strategy to compute a solution. In [2], the authors implemented the classic general approach to solve bilevel games, that is, to reformulate it as a *mathematical programming with complementarity constraints* (MPCC): loosely speaking, for a given network, they write the Karush-Kuhn-Tucker (KKT) conditions of each problem of the lower level game, and put them as constraints in the authority's problem. Then they implemented a Branch-and-Bound heuristic to obtain an approximated optimal exchange network, solving at each iteration the problem described above. However, it is known that the MPCC problems, which is a particular class of *mathematical programming with equilibrium constraints* (MPEC), are hard to solve (see, e.g., [72, 73, 74]) and the heuristic itself doesn't guarantee a real solution of the problem [75, 76]. The literature on theoretical and algorithmic aspects of MPCC and MPEC problems is large and still an active field of research in mathematics.

In this work, we further investigate the model proposed in Ramos et al. [2] for water exchange networks, briefly described in Section 4.2 and fully exposed in Section 5.2, but considering its abstract form for general exchange networks in Subsection 4.3.2. This abstract model is called *Blind-Input model*, since we consider the constraint of full acceptance for each enterprise. To solve the drawback given by the Individual Rationality constraint, we introduce in Section 4.3.3 the notion of *Blind-Input contract*, which is an economical contract between the authority and each enterprise in order to participate in the Blind-Input model. We prove that, under some linear structure of the costs functions $\text{Cost}_i(\cdot)$ of each enterprise, the Blind-Input model can be reduced from a Single-Leader-Multi-Follower problem to a single mixed-integer optimization problem. This reduction, which is our main contribution, is presented in Section 4.4.

The proposed reformulation of the Blind-Input model opens the door to a lot of new developments, from the numerical treatment of huge size problems thanks to classical MILP solvers to exhaustive search of equilibria for small/medium size applications. This is illustrated in the second part of the article for water exchange networks in Eco-Industrial Parks: Section 4.6 illustrates a case of study and the obtained results which are then discussed in Section 4.7. Conclusions and perspectives are presented in Section 4.8.

It is worth to mention that, even though this work is motivated by the design problem of water exchange networks, its abstract formulation presented in Section 4.3 allows to apply it to other type of networks, as for example energy networks ([6, 106]). In Section 4.8, we will comment which are the main elements needed to apply the Blind-Input model to other contexts.

To survey our contributions, a comparison between this work and [2] is given in Table 4.1. It is important to mention that the noncooperative approach using SLMF games in EIPs is very recent and, up to our knowledge, there is no other reference in the literature different from [2, 105] to compare our results with.

Comparison criteria	Ramos et al [2].	This work
Number of enterprises	3	15
Number of processes per enterprise	5	1
Regeneration units	Yes	Yes
Several processes per enterprise	Yes	No
Tools to model the EIPs	SLMF game	SLMF game
Presence of Blind-Input model	Implicitly used. Not formalized. Economic drawbacks.	Explicit formalization. Introduction of Blind-Input contract as economic instrument.
Solution Method	MPCC reformulation + Branch-and-Bound Heuristic	Mixed-Integer Linear programming (MILP) reduction.
Properties of the solution	MPCC is hard to solve and existing algorithms are not robust. The solution of the MPCC may fail to be a solution of the SLMF game.	MILP algorithms are robust. Commercial solvers are available. Any global solution of the MILP problem is a global solution of the SLMF game.
The operating cost of each participating enterprise in the EIP is lower than that of stand-alone.	No	Yes

Table 4.1: Comparison between Ramos et al. [2] and the present work. The first two rows are related to the numerical examples used in each article.

4.2 Motivation: EIP model for water exchange

In this section, we briefly describe the model of water exchange network used to describe Eco-Industrial Parks. The model can be found in [6, 2] among others. A detailed version is further exposed in Section 5.2.

In an Eco-Industrial Park (EIP), several enterprises exchange wastes to reduce the global consumption of natural resources. Each time an enterprise uses the natural resource in her industrial process, it comes out degraded, but still can be used as input for other enterprises in the park. One of the most classical examples of EIP (see, e.g., [6, 107]) corresponds to the modeling of water exchange networks: each enterprise needs to consume water for her industrial processes and the outgoing water is partially polluted. Other examples using different natural resources like energy or heat can be found in [8, 108].

In [2], the design of a water exchange network is treated according to the following assumptions: first, the park has a fixed number of n enterprises, each enterprise i has to dilute an amount M_i of contaminant, and the outlet concentration of contaminant must be less than a fixed concentration $C_{i,\text{out}}$. It is usually assumed that each enterprise i has always an *optimal operation*, in the sense that the outlet concentration of contaminant is always equal to $C_{i,\text{out}}$.

Second, each enterprise i can accept partially polluted water, but with a maximal concentration $C_{i,\text{in}}$. This concentration is measured after a mixer (see Figure 4.1.2) in such a way that no enterprise can really know the operation of the other enterprises. However, this measurement, that we will denote g_i and which depends on the actions of the other enterprises, allows enterprise i to perform two fundamental actions: 1) report infeasibilities to the authority of the park, whenever the incoming water after the mixer doesn't fulfill the constraints; and 2) compute how much fresh water she needs to complete its process attaining the outlet concentration $C_{i,\text{out}}$.

Third, each enterprise has a cost function that depends on four factors: 1) the marginal cost of fresh water that she consumes, that we denote c_i ; 2) the marginal cost of polluted water that she discharges to the environment, that we denote $\gamma_{i,0}$; 3) the cost of sending polluted water through a connection of the park;

and 4) the cost of receiving water from other agents of the park (other enterprises but also regeneration units controlled by the authority). The authority transfers the investment cost of the EIP to the enterprises via the last two costs: the first one, via a marginal cost γ which depends on the connections that enterprise i uses to send water; and the second one via a cost function $\text{Cost}_i^{\text{in}}$ that will depend on the actions of the other enterprises.

Moreover, the main assumptions for the pricing instruments are that the prices of fresh water and discharged water are exogenous, and that the authority has no interest of making any profit, and therefore she will fix the prices of using the connections only to recover the investment and maintenance costs. This yields to the following scenario: each enterprise wants to minimize its cost of the use of water while the authority is in charge of the ecological concerns by minimizing the fresh water consumption.

Finally, as we mentioned before, the authority may have regeneration units. Each regeneration unit r receives polluted water and reduces its contaminant concentration up to a certain value $C_{r,\text{out}}$. Then, it sends the water to the enterprises for reuse. The costs associated to the regeneration units are charged to the enterprises through the inlet cost function $\text{Cost}_i^{\text{in}}$.

4.3 Blind-input model

Taking inspiration from the water management model described in Section 4.2, our aim in this section is to define the concept of abstract Blind-Input model for general exchange networks. We divided the model in two parts: the physical model, which gives the constraints that the network must satisfy; and the economical model, which gives the incentives of each agent of the network, as well as the Blind-Input contract between the agents and the authority, which will ensure the participation of the agents.

4.3.1 Network Model

We first consider two main actors: a set of agents participating to an exchange network, and an authority that aims to minimize the consumption of natural resources. Among the agents, we differentiate a set $\mathcal{P} := \{P_1, \dots, P_n\}$ of independent agents, and a set $\mathcal{R} = \{R_1, \dots, R_m\}$ of regulated agents (controlled by the authority). Regulated agents don't have economical motivations, but they act on the exchange network following the indications of the authority. In the context of water exchange in EIP, the independent agents are the enterprises, and the regulated ones model the regeneration units [2].

We identify the independent agents with the index set $I_P = \{1, \dots, n\}$ and the regulated agents with $I_R = \{n+1, \dots, n+m\}$. We set $I = I_P \cup I_R$ and $I_0 = \{0\} \cup I$, where 0 represents the sink node.

We define an *exchange network* as a simple directed graph (I_0, E) , where the edge $e = (i, j) \in E$ means that the agent i can send part of her output to the agent j . The extra node 0 is identified as a sink node, which represent the possibility of discharge of the output. A valid network (I_0, E) must satisfy the following five conditions:

- I. $E \subseteq E_{\text{max}}$, where E_{max} is the set of all admissible connections of the network.
- II. (I_0, E) is a simple graph, that is, there is no multiple edges nor graph loops in E .
- III. Each independent agent $i \in I_P$ is connected with the sink node, that is, $(i, 0) \in E$.
- IV. Each regulated agent $r \in I_R$ is not connected with the sink node, that is, $(r, 0) \notin E$.
- V. The sink node has not exit edges in E that is $(0, i) \notin E_{\text{max}}$, for any $i \in I$.

In what follows, we will call E the topology of the network (I_0, E) , and we will denote by \mathcal{E} the set of all valid topologies. Nevertheless, in order to simplify notations, the network (I_0, E) will be only represented by its topology E . Observe that this representation may lead to ambiguity, since the set E doesn't allow to distinguish possible isolated regulated agents (independent agents are never isolated, given hypothesis III). However, this is not a problem, since any isolated regulated agent will be simply removed from the network.

For each edge $(i, j) \in E_{\max}$, we set the variable $x_{i,j}$ which represents the flux through the connection (i, j) . For each agent $i \in I$, we set $x_i := (x_{i,j} : (i, j) \in E_{\max})$, being thus the outcome vector of agent i . Finally, we set $x = (x_{i,j} : (i, j) \in E_{\max})$, the complete vector of fluxes through the network.

To simplify the mathematical model we use, let us introduce some notation. We put $x^R := (x_r : r \in I_R)$ and $x^P := (x_i : i \in I_P)$. In what follows, for an agent $i \in I$, we will write

$$\begin{aligned} x_{-i} &:= (x_{k,j} : (k, j) \in E_{\max}, k \in I \setminus \{i\}), \\ x_{-i}^P &:= (x_{k,j} : (k, j) \in E_{\max}, k \in I_P \setminus \{i\}). \end{aligned}$$

For a topology subset $A \subseteq E_{\max}$, we write

$$x|_A := (x_{i,j} : (i, j) \in A).$$

Similarly, we define $x_i|_A, x_{-i}|_A, x^P|_A, x_{-i}^P|_A$ and $x^R|_A$. It will be useful also to denote $A^c := E_{\max} \setminus A$.

4.3.2 Physical Model

Let us fix a network topology $E \in \mathcal{E}$. If E is implemented, then for each agent $i \in I$, the physical model of the network is given by the following six operational constraints:

1. **Null fluxes outside the network:** each agent can use only the connections in the topology E . Thus, we set

$$x_i|_{E^c} = 0, \quad (4.3.1)$$

that is, for every edge $(i, j) \notin E$, the flux $x_{i,j}$ is zero.

2. **Consumption of natural resource:** the consumption of natural resource of the i th agent is given by the output fluxes of the other players, that is,

$$z_i = z_i(x_{-i}). \quad (4.3.2)$$

This assumption is derived from an optimal response hypothesis: we assume that, for a given value of x_{-i} , the agent i is capable of compute exactly the minimal amount of natural resource z_i that she has to consume in order to perform her inner processes.

3. **Balance constraint:** the fluxes must satisfy the Kirchoff's law for the agent $i \in I$, that is,

$$z_i(x_{-i}) + \sum_{(k,i) \in E} x_{k,i} = \sum_{(i,j) \in E} x_{i,j}. \quad (4.3.3)$$

Since 0 is the sink node, it is not subject to this balance constraint.

4. **Input consistency:** there exists a real-valued function g_i which allows the agent $i \in I$ to validate the input coming from the other agents. We write this validation as an abstract inequality constraint

$$g_i(x_{-i}) \leq 0. \quad (4.3.4)$$

This constraint may represent maximal inlet fluxes, maximal inlet contaminant concentration, minimal inlet temperature, etc.

5. **Positivity of fluxes:** we assume that the fluxes on the graph, as well as the consumed natural resource are all positive, that is,

$$x_i \geq 0 \quad \text{and} \quad z_i(x_{-i}) \geq 0. \quad (4.3.5)$$

6. **Extra authority constraints:** the exchange network may require additional constraints. We will model them here through an abstract inclusion

$$x \in X,$$

where $X \subset \mathbb{R}^{|E_{\max}|}$ represents the abstract additional feasible set.

Remark 4.3.1 Here, we assume that the degradation of the natural resource is implicit in the connections of the topology E . In this general model, we suppose that agent i can compute the degradation of its inlet flux through the functions g_i and z_i .

An important element of this model is the total lack of direct information among the agents. We suppose that agent i cannot know the actions of other agents, that is, she doesn't have access to the exact value of x_{-i} . However, she counts with indirect observations: even though x_{-i} is unknown, the values of $z_i(x_{-i})$, $g_i(x_{-i})$ and the total inlet flux $\sum_{(k,i) \in E} x_{k,i}$ are available. For water exchange, this could be interpreted as a measurement of the amount of water and contaminant concentration after the mixer of Figure 4.1.2. This is a very important feature of our model, since enterprises want to keep as much private information as possible. The only agent that has all information is the authority, who has access to the full vector x .

4.3.3 Economical Model

In this setting, the network authority has two vectors of decision variables: she must choose the topology of the network $E \in \mathcal{E}$ and she controls the operation of the regulated agents, that is, the output vectors x_r , for every $r \in I_R$. Each independent agent $i \in I_P$ controls her output vector x_i .

We assume that the authority doesn't pay any cost associated to the implementation and operation of the network. Instead, she transfers all these costs through a function $\gamma: E_{\max} \rightarrow \mathbb{R}_+$, where $\gamma((i,j)) = \gamma_{i,j}$ represents the marginal cost for sending one unit of flux through the connection (i,j) . Using this pricing, the independent agents will pay the investment cost of the network and also the operation of the regulated agents. Thus, if there is a connection $(r_1, r_2) \in E_{\max}$ between two regulated agents $r_1, r_2 \in I_R$, we assume that $\gamma_{r_1, r_2} = 0$.

Since all the investment cost is transferred to the independent agents, the authority is only concerned about minimizing the consumption of the natural resources, and so she aims to minimize the function

$$Z(x) := \sum_{i \in I} z_i(x_{-i}). \quad (4.3.6)$$

Remark 4.3.2 It could be argued that the authority must be also concerned about efficiency of the network, by considering the total investment cost of the park. However, we assume that the pricing instrument γ is given exclusively to pay the investment and maintenance cost of the park, and that it will be implemented as efficiently as possible. The discussion over efficiency and right pricing instruments, is out of the scope of this work.

On the other hand any independent agent $i \in I_P$ wants to minimize her global cost Cost_i , which can be separated into three components: the consumption of the natural resource $z_i(x_{-i})$, the cost of discharging (using the connection $(i, 0)$), and the use of the exchange network. Therefore her cost function Cost_i is given as:

$$\text{Cost}_i(x_i, x_{-i}^P, x^R, E) = A \left[c_i \cdot z_i(x_{-i}) + \text{Cost}_i^{\text{in}}(x_{-i}^P, x^R) + \sum_{(i,j) \in E} \gamma_{i,j} \cdot x_{i,j} \right]. \quad (4.3.7)$$

where $A[h]$ is a time constant that measures the lifetime of the park and $\text{Cost}_i^{\text{in}}(x_{-i}^P, x^R)$ is the inlet operating cost of an agent i , and it satisfies that

$$\sum_{(k,i) \in E_{\text{max}}} x_{k,i} = 0 \implies \text{Cost}_i^{\text{in}}(x_{-i}^P, x^R) = 0.$$

Observe that, the cost concerning the exit connections is linear, and so, the cost function is linear in the first component x_i .

Remark 4.3.3 Again, in terms of costs, agent i doesn't have direct access to the actions of the other agents. However, she must pay an operating cost $\text{Cost}_i^{\text{in}}(x_{-i}^P, x^R)$ that is communicated to her by the authority. The choice of this function as pricing instrument could be studied, but this is out of the scope of the work. For now, we will suppose that agent i has enough indirect information (through measurements after the mixer of Figure 4.1.2) to consider the cost $\text{Cost}_i^{\text{in}}(x_{-i}^P, x^R)$ as correct and therefore to accept it.

With this model, the minimization problem of the i th independent agent (parametrized by the topology E , the actions of regulated agents x^R and the actions of the other independent agents x_{-i}^P) leads to problem $P_i(x_{-i}^P, x^R, E)$:

$$\begin{aligned} & \min_{x_i} \text{Cost}_i(x_i, x_{-i}^P, x^R, E) \\ & \text{s.t.} \begin{cases} z_i(x_{-i}) + \sum_{(k,i) \in E} x_{k,i} = \sum_{(i,j) \in E} x_{i,j} \\ g_i(x_{-i}) \leq 0 \\ z_i(x_{-i}) \geq 0 \\ x_i \geq 0 \\ x_i|_{E^c} = 0. \end{cases} \end{aligned} \quad (4.3.8)$$

We denote by $\text{Eq}(x^R, E)$ the set of equilibria for the induced generalized Nash equilibrium problem (GNEP, for short) given by the vector x^R and the topology E , that is

$$x^P \in \text{Eq}(x^R, E) \iff \forall i \in I_P, x_i \text{ solves } P_i(x_{-i}^P, x^R, E). \quad (4.3.9)$$

As we already discussed in Section 4.1, the main problem of this model is that each independent agent only controls her output vector x_i , which is not realistic. She is forced by the authority to fully accept

any inlet fluxes, which may be harmful. Thus, without any extra constraint, agent i may not be willing to participate in the network.

Thus, to solve this problem, the authority must “buy” the participation of agent i . This is modeled by the *Blind-Input contract*: agent i accepts to control only her output fluxes, and the authority commits to guarantee a minimal relative improvement of her cost, with respect to the stand-alone operation of agent i .

To formalize this requirement in the contract, let us denote the stand-alone topology by $E_{\text{st}} \in \mathcal{E}$, that is,

$$E_{\text{st}} := \{(i, 0) : i \in I_P\}.$$

For each independent agent $i \in I_P$ we define the stand-alone cost STC_i , as the optimal value of the problem $P_i(0, 0, E_{\text{st}})$, that is,

$$\text{STC}_i = (c_i + \gamma_{i,0}) \cdot z_i(0)$$

In other words, STC_i is the cost of the i th agent assuming that all other agents (independent and regulated) are inactive, i.e. when agent i only send fluxes to the sink node and doesn't receive any complementary fluxes from other agents. Then, for each independent agent P_i , we can formulate the commitment of minimal improvement in the Blind-Input contract as the following constraint:

$$\text{Cost}_i(x_i, x_{-i}^P, x^R, E) \leq \alpha \cdot \text{STC}_i, \quad (4.3.10)$$

where $\alpha \in]0, 1[$ is the minimal relative gain that each agent ask for participating in the network. We assume that $\alpha > 0$ since, it is impossible to eliminate all costs, and that $\alpha < 1$ since no agent is indifferent concerning her participation in the network. Indeed, if $\text{Cost}_i(x_i, x_{-i}^P, x^R, E) = \text{STC}_i$, then the agent i will prefer not to participate, since she has no gain, entering an exchange network is complicated and she knows she may be “helping the competition”.

Finally, we can write the authority's problem as

$$\begin{aligned} & \min_{E \in \mathcal{E}, x \in \mathbb{R}^{|E_{\text{max}}|}} Z(x) \\ \text{s.t.} & \begin{cases} x \in X, \\ z_r(x_{-r}) + \sum_{(k,r) \in E} x_{k,r} = \sum_{(r,j) \in E} x_{r,j}, & \forall r \in I_R, \\ z_r(x_{-r}) \geq 0, & \forall r \in I_R, \\ g_r(x_{-r}) \leq 0, & \forall r \in I_R, \\ x^R \geq 0, \\ x^R|_{E^c} = 0, \\ x^P \in \text{Eq}(x^R, E), \\ \text{Cost}_i(x_i, x_{-i}^P, x^R, E) \leq \alpha \cdot \text{STC}_i, & \forall i \in I_P. \end{cases} \end{aligned} \quad (4.3.11)$$

The optimization problem (4.3.11) can be interpreted as follows: the authority will propose to the agents a topology E and an operation $x \in \mathbb{R}^{|E_{\text{max}}|}$ which satisfy all the physical constraints and also, such that the operation x respects: 1) the *incentive consistency*, in the sense that no agent will have incentives to unilaterally deviate from the proposal due to the constraint $x^P \in \text{Eq}(x^R, E)$; and 2) the *individual rationality* of each agent, in the sense that all agents will participate in the network since their participation has been bought through the constraint (4.3.10). The first criteria solves the economical inconsistency of MOO approach, and the second criteria solves the participation problem of the Single-Leader-Follower approach.

Remark 4.3.4 In this work, we do not claim novelty in the constraint $x^P \in \text{Eq}(x^R, E)$. This is the main contribution of [2]. However, the constraint (4.3.10) is new. In terms of modeling and in this context, the fact to “attract” the independent agents towards a participation in the general exchange network constitutes one of the important novelties of this work.

Remark 4.3.5 After reading the forthcoming Section 4.4, the reader will observe that all proofs and reductions could be made considering different values of α for each independent agent, that is, putting a value $\alpha_i \in]0, 1[$ for each $i \in I_P$. The value of α_i represents the “cost” of buying the participation of the i th independent agent, which is exactly $(1 - \alpha_i)\text{STC}_i$. However, allowing to have different costs depending on the enterprise rises the natural question of how to decide these values. This problem lies in contract theory (for an introduction to the field, we refer to [70, 71]) and it is out the scope of the article. Thus, we will consider only uniform values of α , which can be interpreted as a public call for participation in the network. Uniform values of α , however, imply that the cost of buying the participation of an agent is proportional to her size, due to the factor STC_i .

Remark 4.3.6 An important factor we do not consider in this work is the rebound effect that costs reductions may have on the operation of agents. For example, in terms of water exchange, a diminution of costs of agent i with respect to STC_i may induce an increment of wastes production, that is, a variation in M_i . Thus, this rebound effect may change the value of $z_i(x_{-i})$. Even though this is a very interesting problem, we suppose that the demand of natural resource is given by a fixed process, on which the costs within the network have no effect. In other words, the consumption of natural resource of each agent is inelastic.

4.4 Mixed-Integer programming reduction

The formulation of the authority’s problem (4.3.11) has the form of a general MPEC problem (see, e.g., [72, 73, 74]). This section is devoted to prove that this MPEC formulation, which is known to be hard to solve, can be reformulated as a single Mixed-Integer programming problem.

This reduction can be interpreted as follows: Blind-Input models are a social optimization problem where, through Blind-Input contracts, the cooperation of each independent agent has been bought. This social optimization is also economically stable, since implicitly it respect an equilibrium constraint ($x^P \in \text{Eq}(x^R, E)$). This reduction/reformulation will be presented in three steps.

4.4.1 Characterization of equilibria

The following theorem characterizes the equilibrium set $\text{Eq}(x^R, E)$ as a system of equations. This allows to reduce the MPEC of problem (4.3.11) to a single optimization problem. The reduction we do here is based on the observation that, once every agent has committed to a Blind-Input contract, her actions become predictable through the cost functions. Thus, the authority can choose the network E such that each action of an independent agent is induced to reach the social optimum.

To formalize this idea, let us introduce the notion of *active arcs*. Given a topology E , for each independent agent $i \in I_P$ we define the set of active arcs of i , denoted by $E_{i,\text{act}}$, as all the arcs $e \in E$ having minimum cost, that is,

$$E_{i,\text{act}} := \left\{ (i, j) \in E : \gamma_{i,j} = \gamma_i^* := \min_{(i,k) \in E} \gamma_{i,k} \right\}. \quad (4.4.1)$$

As convention, for any regulated agent $r \in I_R$, we set $E_{r,\text{act}} = \{(r, j) : (r, j) \in E\}$.

Theorem 4.4.1 For $E \in \mathcal{E}$ and $x^R \geq 0$ fixed, the equilibrium set $\text{Eq}(x^R, E)$ is given by

$$\text{Eq}(x^R, E) = \left\{ x^P : \forall i \in I_P, \begin{cases} z_i(x_{-i}) + \sum_{(k,i) \in E} x_{k,i} = \sum_{(i,j) \in E} x_{i,j} \\ g_i(x_{-i}) \leq 0 \\ z_i(x_{-i}) \geq 0 \\ x_i|_{E_{i,\text{act}}^c} = 0 \\ x_i \geq 0 \end{cases} \right\} \quad (4.4.2)$$

Thus, the authority's problem (4.3.11) is equivalent to the following Mixed-Integer Programming problem:

$$\begin{aligned} & \min_{x \in \mathbb{R}^{|E_{\max}|}, E \in \mathcal{E}} Z(x) \\ & s.t. \begin{cases} x \in X, \\ z_i(x_{-i}) + \sum_{(k,i) \in E} x_{k,i} = \sum_{(i,j) \in E} x_{i,j}, & \forall i \in I \\ x_i|_{E_{i,\text{act}}^c} = 0, & \forall i \in I \\ g_i(x_{-i}) \leq 0, & \forall i \in I \\ z_i(x_{-i}) \geq 0, & \forall i \in I \\ \text{Cost}_i(x_i, x_{-i}^P, x^R, E) \leq \alpha_i \cdot \text{STC}_i, & \forall i \in I_P \\ x \geq 0. \end{cases} \end{aligned} \quad (4.4.3)$$

Proof. The second part of the proof is easily verified by replacing the constraint $x^P \in \text{Eq}(x^R, E)$ by the system of equations in the right hand of equality (4.4.2), and then just reorganizing. Thus, we only need to prove (4.4.2).

To simplify notation, let us denote by $S(x^R, E)$ the right-hand set of (4.4.2). First, let us prove that $S(x^R, E) \subseteq \text{Eq}(x^R, E)$. Fix $x^P \in S(x^R, E)$. Since $E_{i,\text{act}} \subset E$ for each $i \in I_P$, it is not hard to see that x_i is a feasible set of $P_i(x_{-i}^P, x^R, E)$.

Now, fix $i \in I_P$ and let x'_i be another feasible point of $P_i(x_{-i}^P, x^R, E)$. Then, $x'_i \geq 0$ and it satisfies the balance constraint (4.3.3), which yields that

$$\begin{aligned} \Delta \text{Cost}_i &= \sum_{(i,j) \in E} \gamma_{i,j} x'_{i,j} - \gamma_i^* \left(\sum_{(i,j) \in E_{i,\text{act}}} x_{i,j} \right) \\ &\geq \gamma_i^* \left(\sum_{(i,j) \in E} x'_{i,j} - \sum_{(i,j) \in E_{i,\text{act}}} x_{i,j} \right) \\ &\geq 0, \end{aligned}$$

where $\Delta \text{Cost}_i := \text{Cost}_i(x'_i, x_{-i}^P, x^R) - \text{Cost}_i(x_i, x_{-i}^P, x^R)$ and the last inequality is due to the fact that

$$\begin{aligned} \sum_{(i,j) \in E} x'_{i,j} &= z_i(x_{-i}) + \sum_{(k,i) \in E} x_{k,i} \\ &= \sum_{(i,j) \in E} x_{i,j} = \sum_{(i,j) \in E_{i,\text{act}}} x_{i,j}. \end{aligned}$$

Thus, x_i solves $P_i(x_{-i}^P, x^R, E)$, and since this holds for every $i \in I_P$, we deduce that $x^P \in \text{Eq}(x^R, E)$.

Now, let us prove that $\text{Eq}(x^R, E) \subseteq S(x^R, E)$. Let $x^P \in \text{Eq}(x^R, E)$, and suppose that $x^P \notin S(x^R, E)$. Since for each $i \in I_P$ the vector x_i is a feasible point of $P(x_{-i}^P, x^R, E)$, the only way for x^P not to belong to $S(x^R, E)$ is that there exist $i_0 \in I_P$ such that $x_{i_0}|_{E_{i_0, \text{act}}^c} \neq 0$. Thus, there is $(i_0, j_0) \in E \setminus E_{i_0, \text{act}}$ such that $x_{i_0, j_0} > 0$. Let $(i_0, j_1) \in E_{i_0, \text{act}}$ (which is nonempty by definition) and let us consider the vector x'_{i_0} given by

$$x'_{i_0, k} = \begin{cases} x_{i_0, k} & \text{if } k \in I \setminus \{j_0, j_1\}, \\ 0 & \text{if } k = j_0, \\ x_{i_0, j_1} + x_{i_0, j_0} & \text{if } k = j_1. \end{cases}$$

We have that $x'_{i_0} \geq 0$ (since $x_{i_0} \geq 0$) and also

$$z_i(x_{-i_0}) + \sum_{(k, i_0) \in E} x_{k, i_0} = \sum_{(i_0, j) \in E} x_{i_0, j} = \sum_{(i_0, j) \in E} x'_{i_0, j}.$$

Thus, since x_{-i_0} remains the same, x'_{i_0} is a feasible point of $P_i(x_{-i_0}^P, x^R, E)$. Furthermore, denoting by $\Delta \text{Cost}_{i_0} = \text{Cost}_{i_0}(x'_{i_0}, x_{-i_0}^P, x^R, E) - \text{Cost}_{i_0}(x_{i_0}, x_{-i_0}^P, x^R, E)$, we have that

$$\begin{aligned} \Delta \text{Cost}_{i_0} &= \sum_{(i_0, j) \in E} \gamma_{i_0, j} x'_{i_0, j} - \sum_{(i_0, j) \in E} \gamma_{i_0, j} x_{i_0, j} \\ &= (\gamma_{i_0, j_1} - \gamma_{i_0, j_0}) x_{i_0, j_0} \\ &= (\gamma^* - \gamma_{i_0, j_0}) x_{i_0, j_0} < 0, \end{aligned}$$

since, by construction, $\gamma_{i_0, j_0} > \gamma^*$. This yields that x_{i_0} doesn't solve $P_i(x_{-i_0}^P, x^R, E)$, which is a contradiction. Thus, $x^P \in S(x^R, E)$, finishing the proof. \square

Intuitively, the above theorem says that, given a topology E , each independent agent $i \in I_P$ will only use the connections of minimal cost to send the excess of flux, that is, she will use only her active arcs. Furthermore, each independent agent is indifferent to the distribution of fluxes among the active arcs, so any feasible vector x^P satisfying the constraint $x_i|_{E_{i, \text{act}}^c} = 0$ for every $i \in I_P$ must be an equilibrium. This simplification is strongly based on the linearity of the costs functions with respect to the agent's variable x_i .

4.4.2 Mixed-Integer formulation

Theorem 5.3.1 establishes the remarkable fact that the MPEC formulation of the authority's problem can be reformulated as a "classical" programming problem. But actually, a part of the variables of this programming problem lies in the set of topologies of the exchange network and so, it can be considered as difficult to implement numerically. This is why, in this section, we will show how one can finally work with a more classical mixed-integer programming problem.

Let us first introduce the key notion that we will use to arrive to the final formulation, that is, what we call *arc classes*: let $(i, j) \in E_{\max}$. We define the *arc class* of (i, j) as the set

$$C(i, j) := \begin{cases} \{(i, k) \in E_{\max} : \gamma_{i, k} = \gamma_{i, j}\} & \text{if } i \in I_P \\ \{(i, k) \in E_{\max}\} & \text{if } i \in I_R \end{cases}. \quad (4.4.4)$$

We denote by \mathcal{C}_i the family of all arc classes exiting from i , that is, $\mathcal{C}_i = \{C(i, j) : (i, j) \in E_{\max}\}$. Finally, for $C \in \mathcal{C}_i$ we define the utilization cost of the class by

$$\gamma(C) := \gamma_{i,j},$$

where (i, j) is any representative of C .

Observe that, for two arcs $(i, j), (i, k) \in E_{\max}$ such that $\gamma_{i,j} = \gamma_{i,k}$, one has that $C(i, j) = C(i, k)$. Thus, a class $C \in \mathcal{C}_i$ may have many representations of the form $C(i, j)$. Furthermore, the family \mathcal{C}_i induces a partition of the set of arcs “exiting from” agent i , that is

- $\bigcup_{C \in \mathcal{C}_i} C = \{e \in E_{\max} : e = (i, j) \text{ for some } j \in I_0\}$.
- For any two classes $C, C' \in \mathcal{C}_i$, either $C = C'$ or $C \cap C' = \emptyset$.

Moreover, it is not hard to verify that for each topology $E \in \mathcal{E}$ and for each agent $i \in I_P$, there exists one class $C \in \mathcal{C}_i$ such that

$$E_{i,\text{act}} \subseteq C, \tag{4.4.5}$$

and this class must satisfy that

$$\gamma(C) \leq \gamma(C(i, 0)). \tag{4.4.6}$$

This class is then given by $C = C(i, j)$ where (i, j) is any element of $E_{i,\text{act}}$. We will call it the *active class* of E of the agent i , and we will denote it by $C_i(E)$.

Without loss of generality, we will assume that every class $C \in \mathcal{C}_i$ satisfies (4.4.6). If not, any connection in a class violating (4.4.6) would never be used, and therefore, in practice, they can be erased from E_{\max} without changing the problem.

Now, let $D = \bigcup_{i \in I_P} \mathcal{C}_i$, the set of all arc classes of independent agents. We introduce the boolean variable $y = (y_C)_{C \in D} \in \{0, 1\}^{|D|}$ in the following way: for each independent agent $i \in I_P$ and each arc class $C \in \mathcal{C}_i$, we set

$$y_C = \begin{cases} 1 & \text{if } C \text{ is the active class of } i, \\ 0 & \text{otherwise.} \end{cases}$$

From $y \in \{0, 1\}^{|D|}$, we will build the graph associated to y as

$$E(y) = \left(\bigcup \{C : y_C = 1\} \right) \cup \{(i, 0) : i \in I_P\} \cup \{(r, j) \in E_{\max} : r \in I_R\}. \tag{4.4.7}$$

We consider then the following Mixed-Integer optimization problem:

$$\begin{aligned}
& \min_{x \in \mathbb{R}^N, y \in \{0,1\}^{|D|}} Z(x) \\
& \text{s.t.} \left\{ \begin{array}{l} x \in X, \\ z_i(x_{-i}) + \sum_{(k,i) \in E_{\max}} x_{k,i} = \sum_{(i,j) \in E_{\max}} x_{i,j}, \quad \forall i \in I, \\ \sum_{C \in \mathcal{C}_i} y_C = 1, \quad \forall i \in I_P, \\ \sum_{(i,j) \in C} x_{i,j} \leq B \cdot y_C, \quad \forall C \in D, \\ g_i(x_{-i}) \leq 0, \quad \forall i \in I_P, \\ z_i(x_{-i}) \geq 0, \quad \forall i \in I, \\ \text{Cost}_i(x_i, x_{-i}^P, x^R, E(y)) \leq \alpha_i \cdot \text{STC}_i, \quad \forall i \in I_P, \\ x \geq 0, \end{array} \right. \quad (4.4.8)
\end{aligned}$$

where B is a real number chosen arbitrarily, but bigger than the maximum of the total entering flux over all enterprises. A simple option to set B is the value $Z(0)$, which corresponds to the total consumption of the natural resource when there is no exchange network.

Here, the constraint $\sum_{C \in \mathcal{C}_i} y_C = 1$ says that, for the i th agent, only one class is active. Also, the constraint

$$\sum_{(i,j) \in C} x_{i,j} \leq B \cdot y_C, \quad \forall C \in D$$

ensures that, whenever (i, j) doesn't belong to the active class of the i th agent, then $x_{i,j} = 0$.

Theorem 4.4.2 *For every feasible point (x, y) of (4.4.8), the pair $(x, E(y))$ is a feasible point of (4.4.3). Conversely, for every feasible point (x, E) of (4.4.3), the pair (x, y^E) is a feasible point of (4.4.8), where $y^E \in \{0, 1\}^{|D|}$ is given by*

$$y_C^E = \begin{cases} 1 & \text{if } C = C_i(E) \text{ for some } i \in I_P, \\ 0 & \text{otherwise.} \end{cases}$$

Finally, one has that

1. if (x, E) is an optimal solution of (4.4.3), then (x, y^E) is an optimal solution of (4.4.8).

2. if (x, y) is an optimal solution of (4.4.8), then $(x, E(y))$ is an optimal solution of (4.4.3).

Proof. Let (x, y) be a feasible point of (4.4.8). Let us fix an agent $i \in I_P$ and let C_i be the unique class in \mathcal{C}_i such that $y_{C_i} = 1$. Then, by construction, we know that

$$E(y)_{i, \text{act}} = C_i \text{ and } \sum_{(i,j) \in E_{\max} \setminus C_i} x_{i,j} \leq B \cdot \sum_{C \in \mathcal{C}_i \setminus \{C_i\}} y_C = 0.$$

We deduce then that

$$x_i \Big|_{E(y)_{i, \text{act}}^c} = 0.$$

Since this constraint is valid for every active agent $i \in I_P$, and since $E(y)$ contains all exiting arcs for every regulated agent $r \in I_R$, we can rewrite the balance constraint in problem (4.4.8) as

$$z(x_{-i}) + \sum_{(k,i) \in E(y)} x_{k,i} = \sum_{(i,j) \in E(y)} x_{i,j}, \quad \forall i \in I.$$

We deduce then that $(x, E(y))$ is a feasible point of problem (4.4.3).

Now, let (x, E) be a feasible point of problem (4.4.3). By inclusion (5.3.5), for each independent agent $i \in I_P$, there exists a unique active class $C_i(E)$. Let us define $y^E \in \{0, 1\}^{|D|}$ as in the statement of the theorem.

Then, for every $i \in I_P$, $\sum_{C \in \mathcal{C}_i} y_C^E = 1$. Now, fix a class $C \in D$, and let $i \in I_P$ such that $C \in \mathcal{C}_i$. We have that

$$\sum_{(i,j) \in C} x_{i,j} \leq \begin{cases} B = B \cdot y_C^E & \text{if } C = C_i(E), \\ 0 = B \cdot y_C^E & \text{if } C \neq C_i(E), \end{cases}$$

where the second inequality comes from the fact that, whenever $C \neq C_i(E)$, then $C \subseteq E_{i,\text{act}}^c$ and so $x_i|_C = 0$.

For an agent $i \in I_P$, the fact that $E_{i,\text{act}} \subseteq E(y^E)$ lead us to the fact that

$$\text{Cost}_i(x_i, x_{-i}^P, x^R, E(y^E)) = \text{Cost}_i(x_i, x_{-i}^P, x^R, E),$$

and so, the constraint (4.3.10) is satisfied. We deduce that (x, y^E) is a feasible point of (4.4.8), since all other constraints are directly satisfied given that (x, E) is feasible for problem (4.4.3).

Now, let us assume that (x, E) is also optimal for problem (4.4.8). From the development above, for every other feasible point (x', y') of (4.4.8), we know that $(x', E(y'))$ is also a feasible point of problem (4.4.3), and so, $Z(x) \leq Z(x')$. Thus, (x, y^E) is optimal for the problem (4.4.8).

Let now assume that (x, y) is an optimal solution of problem (4.4.8) and suppose, by absurd, that $(x, E(y))$ is not optimal for problem (4.4.3). Then, there exists a feasible point (x', E') of problem (4.4.3) such that $Z(x') < Z(x)$. But, as proved above, $(x', y^{E'})$ is also feasible for problem (4.4.8), showing that (x, y) is not optimal for (4.4.8), which is a contradiction. The proof is then completed. \square

The reader could observe that, a priori, the mixed-integer problem (4.4.8) is smaller than problem (4.4.3) in some sense, since it admits only certain topologies (those ones of the form $E(y)$ for some feasible point $y \in \{0, 1\}^{|D|}$). However the above theorem shows that the set of flux distributions x for which (x, E) is an optimal solution of (4.4.3) for at least one topology E coincides with the set of flux distributions x for which (x, y) is an optimal solution of (4.4.8) for at least one y .

4.4.3 Null Class as exit option

Physically, we know that the network has always a feasible point, which is the stand-alone configuration, that is, the topology E_{st} and the fluxes given by the individual operations of the independent agents and inactivity of the regulated ones. However, when we include the individual rationality constraint (4.3.10), the problem may become infeasible.

Infeasibility of problem (4.4.3) means that the authority is not capable to find a solution that respect the Blind-Input contracts with all the agents. Thus, we need to include the possibility of excluding some agents from the network.

Formally, for each independent agent $i \in I_P$, we include a boolean variable $y_{i,\text{null}} \in \{0, 1\}$ such that

$$y_{i,\text{null}} = \begin{cases} 1 & \text{if } i \text{ breaks the Blind-Input contract,} \\ 0 & \text{otherwise.} \end{cases}$$

With this new variable, we modify problem (4.4.8) as follows:

1. For each agent $i \in I_P$, we put

$$y_{i,\text{null}} + \sum_{C \in \mathcal{C}_i} y_C = 1,$$

meaning that, either one arc class is active or the agent is outside the network.

2. For each agent $i \in I_P$, we put

$$\begin{aligned} \sum_{(i,j) \in C(i,0)} x_{i,j} &\leq B \cdot (y_{C(i,0)} + y_{i,\text{null}}) \\ \sum_{(i,j) \in E_{\max}, j \neq 0} x_{i,j} &\leq B \cdot (1 - y_{i,\text{null}}) \end{aligned}$$

This is to ensure that, if the agent breaks the Blind-Input contract, then she will use the discharge arc $(i, 0)$.

3. For each agent $i \in I_P$, we put

$$\sum_{(k,i) \in E_{\max}} x_{k,i} \leq B \cdot (1 - y_{i,\text{null}}).$$

This constraint establishes that, if the agent breaks the Blind-Input contract, then nobody can send her any flux.

4. For each agent $i \in I_P$, we put

$$\text{Cost}_i(x_i, x_{-i}^P, x^R, E(y)) \leq \alpha_i \text{STC}_i \cdot (1 - y_{i,\text{null}}) + \text{STC}_i \cdot y_{i,\text{null}}. \quad (4.4.9)$$

Here, the individual rationality constraint is active only when $y_{i,\text{null}} = 0$. Otherwise, since the agent is not connected to the network, her cost will coincide with STC_i .

We set $\bar{D} = D \cup \{\text{Null}_i : i \in I_P\}$, where Null_i is the null class, associated to $y_{i,\text{null}}$, and $D_0 = D \setminus \{C(i, 0) : i \in I_P\}$. Denoting

$$\text{STC}_i(y_{i,\text{null}}) := \alpha_i \text{STC}_i \cdot (1 - y_{i,\text{null}}) + \text{STC}_i \cdot y_{i,\text{null}},$$

the new optimization problem becomes

$$\begin{aligned}
 & \min_{x \in \mathbb{R}^N, y \in \{0,1\}^{|\bar{D}|}} Z(x) \\
 & \text{s.t.} \left\{ \begin{array}{l}
 x \in X, \\
 z_i(x_{-i}) + \sum_{(k,i) \in E_{\max}} x_{k,i} = \sum_{(i,j) \in E_{\max}} x_{i,j}, \quad \forall i \in I, \\
 y_{i,\text{null}} + \sum_{C \in \mathcal{C}_i} y_C = 1, \quad \forall i \in I_P, \\
 \sum_{(i,j) \in C} x_{i,j} \leq B \cdot y_C, \quad \forall C \in D_0, \\
 \sum_{(i,j) \in C(i,0)} x_{i,j} \leq B \cdot (y_{C(i,0)} + y_{i,\text{null}}), \quad \forall i \in I_P, \\
 \sum_{(i,j) \in E_{\max}, j \neq 0} x_{i,j} \leq B \cdot (1 - y_{i,\text{null}}), \quad \forall i \in I_P, \\
 \sum_{(k,i) \in E_{\max}} x_{k,i} \leq B \cdot (1 - y_{i,\text{null}}), \quad \forall i \in I_P, \\
 g_i(x_{-i}) \leq 0, \quad \forall i \in I_P, \\
 z_i(x_{-i}) \geq 0, \quad \forall i \in I, \\
 \text{Cost}_i(x_i, x_{-i}^P, x^R, E(y)) \leq \text{STC}_i(y_{i,\text{null}}), \quad \forall i \in I_P, \\
 x \geq 0.
 \end{array} \right. \quad (4.4.10)
 \end{aligned}$$

Observe that, whenever $y_{i,\text{null}} = 0$, then all constraints for the i th agent are the same that those established in problem (4.4.8). Also, if $y_{i,\text{null}} = 1$, the only feasible solution for i is the stand-alone operation. Thus, in this new problem, the authority first choose all the agents that will participate in the network, represented by the set

$$I'_P = \{i \in I_P : y_{i,\text{null}} = 0\},$$

and then it solves problem (4.4.8) replacing I by $I' = I'_P \cup I_R$. Of course, as it is formulated, the authority takes both decisions simultaneously, by solving problem (5.3.9). It is not hard to verify that any optimal solution of problem (5.3.9) is an optimal solution of Problem (4.4.8) for the reduced set of agents I' . We leave this verification to the reader.

4.5 Blind-Input model for water exchange networks

In this section we come back to our original motivation presented in Section 4.2, the water exchange networks in Eco-Industrial Parks. We are now ready to describe in detail the model, and how it fits into the Blind-Input model developed so far.

First, an EIP consists in a set of enterprises $\mathcal{P} := \{P_1, \dots, P_n\}$, that are connected in an exchange network. Each enterprise P_i can be connected either to other enterprises, or to some regeneration units, which we denote $\mathcal{R} := \{R_1, \dots, R_m\}$. The regeneration units are controlled by a central authority. This authority plays the role of designer (when deciding the connections within the network) and regulator of the park's operation. Finally, we include a sink node 0, that represents a wastes' pit to discharge useless polluted water. We identify \mathcal{P} with the index set $I_P := \{1, \dots, n\}$ and \mathcal{R} with $I_R := \{n+1, \dots, n+m\}$. We put $I = I_P \cup I_R$ and $I_0 = \{0\} \cup I$. Finally, for an agent of the park we refer either to an enterprise or to a regeneration unit.

Each enterprise $i \in I_P$ generates a fixed amount of pollutant M_i [g], coming from her internal production process, that needs to be diluted before exiting the enterprise. To do so, enterprise i must buy an amount of fresh water z_i [T/h] such that, after dilution, the pollutant concentration in the exit flux is less than a limit concentration $C_{i,\text{out}}$ [ppm]. If the enterprise discharges this polluted water into the sink node, she has to pay a tax associated to the contamination she is producing. We will assume a *hypothesis of optimal response*: each enterprise $i \in I_P$ consumes exactly the fresh water she needs to attain $C_{i,\text{out}}$, and therefore, her output pollutant concentration is always equal to this constant.

We denote the marginal cost of fresh water as c [\$/T], and the tax of discharged water as β [\$/T]. Observe that, if the enterprise i doesn't participate in the EIP, then her water consumption z_i must be

$$z_i = \frac{M_i}{C_{i,\text{out}}}.$$

Then, the cost of her stand-alone operation, which we denote by STC_i [\$], is given by

$$\text{STC}_i = A \cdot (c + \beta) \frac{M_i}{C_{i,\text{out}}},$$

where A [h] is a time constant that measures the lifetime of the park.

The goal of the authority is to built (and operate) an exchange network so part of this polluted water could be reused by other enterprises, minimizing the global consumption of fresh water within the park. Here, an exchange network for the EIP is a simple directed graph (I_0, E) , where the connection $(i, j) \in E$ means that the agent i can send her output water to the agent j . In this sense, if agent i uses the connection $(i, 0)$, then it means that she is discharging water outside the park, to the environment.

Defining the sets

$$\begin{aligned} E_{\text{st}} &:= \{(i, 0) : i \in I_P\} \\ E_{\text{max}} &:= \{(i, j) : i \in I_P, j \in I_0\} \cup \{(r, j) : r \in I_R, j \in I_P\}, \end{aligned} \tag{4.5.1}$$

a valid exchange network must satisfy that $E_{\text{st}} \subset E \subset E_{\text{max}}$. This definition yields that: 1) for every enterprise there is always the possibility of discharge; 2) the regeneration units can send water only to enterprises; and 3) the sink node doesn't have any exit connections (it is not possible to recover water once it is discharged). We denote by \mathcal{E} the family of valid networks for the EIP. Finally, for any $E \in \mathcal{E}$, we denote by E^c the set of connections that are not in E , that is, $E^c = E_{\text{max}} \setminus E$.

Note that, on the one hand, the set E_{st} is the *stand-alone configuration*, where each enterprise only has access to fresh water and, after using it, she must discharge it to the sink node. On the other hand, E_{max} stands for the complete park, in the sense that all enterprises are connected between them, and all of them have access to the regeneration units.

If an enterprise $i \in I_P$ receives fluxes from other agents within the EIP, then these fluxes pass through a mixer. After the mixing, the inlet flux is then mixed with the purchased fresh water z_i . Then, the contaminant concentration of the total flux cannot surpass a limit inlet concentration $C_{i,\text{in}}$, which is given by technical constraints. We always have that $C_{i,\text{in}} < C_{i,\text{out}}$. This structure is illustrated in Figure 4.5.1.

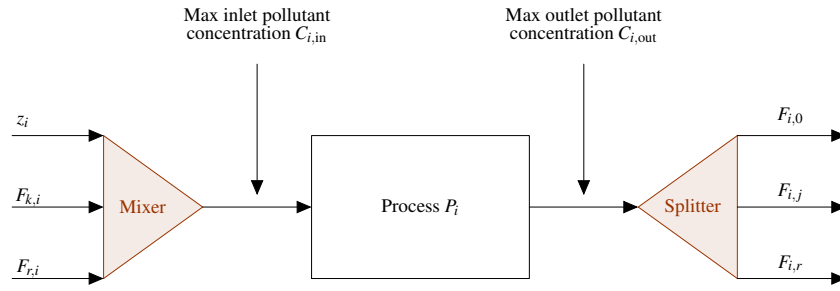


Figure 4.5.1: Water mixture description for a given enterprise. Here $C_{i,in} \leq C_{i,out}$

Every regeneration unit $r \in I_R$, as the enterprises, has limit concentrations as well. The inlet concentration for r must be between a threshold given by a minimal inlet concentration, $C_{r,in}$ [ppm]. The output concentration $C_{r,out}$ [ppm] denotes the concentration of the output flux after the regeneration process. The main difference with the enterprises is that, while enterprises increase the contaminant concentration of fluxes, regeneration units reduce it. Therefore, we always have $C_{r,out} \leq C_{r,in}$. Figure 4.5.2 illustrates the operation of regeneration units.

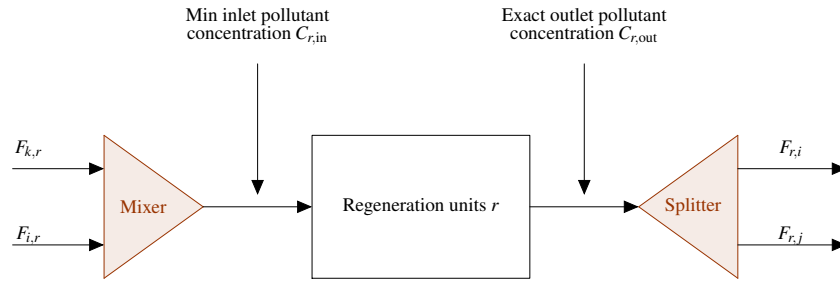


Figure 4.5.2: Water mixture description for a given regeneration unit. Here $C_{r,in} \geq C_{r,out}$. The indexes i, j, k belong to I_P .

For each $(i, j) \in E_{\max}$ we denote by $F_{i,j}$ [T/h] the water flux going from i to j through the connection (i, j) . We consider the following notation:

- $F_i = \{F_{i,j} : j \in I\}$ is the vector of fluxes exiting from agent i .
- $F_{-i} = (F_{k,j} : k \in I \setminus \{i\})$ is the vector of all fluxes not exiting from agent i .
- $F_{-i}^P = (F_{k,j} : k \in I_P \setminus \{i\})$ is the vector of all fluxes exiting from an enterprise different than i .
- $F^R = (F_r : r \in I_R)$ is the vector of fluxes exiting from regeneration units.
- $F^P = (F_i : i \in I_P)$ is the vector of fluxes exiting from enterprises.

Finally, for an agent $i \in I$, we may write $F = (F_i, F_{-i})$, to stress the exiting fluxes of agent i . Moreover, if $i \in I_P$, we may also write

$$F_{-i} = (F_{-i}^P, F^R) \quad \text{and} \quad F = (F_i, F_{-i}^P, F^R),$$

to distinguish the actions of other enterprises and the actions of regeneration units. This is classic notation in game theory (see, e.g., [10, 11, 14]).

Then, for a fixed network E , a valid flux vector $F = (F_{i,j} : (i, j) \in E_{\max})$ must satisfy the following constraints:

1. **Use of connections in E :** Since E represents the available connections, we must put

$$\forall (i, j) \in E^c, F_{i,j} = 0. \quad (4.5.2)$$

2. **Water mass balance:** Around an agent $i \in I$ (different from the sink node), we have that

$$z_i + \sum_{(k,i) \in E} F_{k,i} = \sum_{(i,j) \in E} F_{i,j}. \quad (4.5.3)$$

If $i \in I_R$ then $z_i = 0$, that is, the regeneration units don't consume fresh water. The sink node is not subject to balance constraints.

3. **Contaminant mass balance:** Around an enterprise $i \in I_P$, we have that

$$M_i + \sum_{(k,i) \in E} C_{k,\text{out}} F_{k,i} = C_{i,\text{out}} \sum_{(i,j) \in E} F_{i,j}, \quad (4.5.4)$$

where the right-hand term corresponds to the inlet contaminant mass coming from other agents, and the left-hand corresponds to the outlet contaminant mass. Thanks to the hypothesis of optimal response, the inlet mass of an agent can be expressed in terms of fluxes, since for each agent $k \in I$, the outlet concentration $C_{k,\text{out}}$ is always attained. Observe that regeneration units are not subject to contaminant mass balance, since they clean the water that pass through them.

4. **Inlet/outlet concentration constraints:** For an enterprise $i \in I_P$ we have that:

$$\sum_{(k,i) \in E} C_{k,\text{out}} F_{k,i} \leq C_{i,\text{in}} \left(z_i + \sum_{(k,i) \in E} F_{k,i} \right). \quad (4.5.5)$$

In parallel, for regeneration unit $r \in I_R$ we have that:

$$C_{r,\text{in}} \sum_{(k,r) \in E} F_{k,r} \leq \sum_{(k,r) \in E} C_{k,\text{out}} F_{k,r}. \quad (4.5.6)$$

Both constraints are formulated in terms of contaminant mass, but in practice they represent constraints of concentration.

5. **Positivity of fluxes:** All the fluxes in the park must be positive:

$$\forall (i, j) \in E, F_{i,j} \geq 0 \quad \text{and} \quad \forall i \in I_P, z_i \geq 0. \quad (4.5.7)$$

Observe that, combining equations (5.2.3) and (4.5.4) we obtain:

$$M_i + \sum_{(k,i)} C_{k,\text{out}} F_{k,i} = C_{i,\text{out}} \left(z_i + \sum_{(k,i) \in E} F_{k,i} \right), \quad \forall i \in I_P,$$

and so, the fresh water bought by the enterprise $i \in I_P$ is given by the fluxes of the other agents, that is,

$$z_i(F_{-i}) = \frac{1}{C_{i,\text{out}}} \left(M_i + \sum_{(k,i) \in E} (C_{k,\text{out}} - C_{i,\text{out}}) F_{k,i} \right). \quad (4.5.8)$$

In this model, each enterprise $i \in I_P$ has a cost function that she wants to minimize, defined by

$$\text{Cost}_i(F) = A \left[c \cdot z_i(F_{-i}) + \beta F_{i,0} + 2\delta \sum_{r \in I_R} (F_{i,r} + F_{r,i}) + \delta \sum_{k,j \in I_P} (F_{k,i} + F_{i,j}) + \sum_{r \in I_R} \Gamma_r F_{r,i}^\psi \right], \quad (4.5.9)$$

where Γ_r [\$/T] is the marginal cost of regenerating water depending on the technology of the regeneration unit and δ [\$/T] is the marginal cost of using a sharing connection. Note that the regenerated water cost is non-linear, due to the power $\psi < 1$, which we usually set between 0.6 and 0.8 (see e.g. [2]). In fact the higher the volume of regenerated water, the lesser the operating cost of the regeneration unit. We assume that $\beta \gg 2\delta$, that is, that the cost β is much higher than 2δ .

Observe also that each enterprise pays both for the entering water and the exiting water, for every connection she has. In the case of the connections between two enterprises, this means that the cost of the connection is divided uniformly between the sending enterprise and the receiving enterprise. In the case of regeneration units, the 2δ factor means that the enterprise must pay for the operation (when sending to and receiving from) of the regeneration unit.

On her part, the authority wants to minimize the consumption of the natural resources, and so she tries to minimize the function

$$Z(F) = \sum_{i \in I_P} z_i(F_{-i}). \quad (4.5.10)$$

All investment costs are trespassed to the enterprises by the marginal prices δ and Γ_r and the exponent ψ .

In order to get each enterprise $i \in I_P$ to participate in the EIP, the authority engages in a *Blind-Input contract* with them of constant $\alpha \in]0, 1[$. On the one hand, for each enterprise $i \in I_P$, the authority must ensure a relative improvement of α in the costs, with respect to the stand-alone operation, that is,

$$\text{Cost}_i(F_i, F_{-i}) \leq \alpha \text{STC}_i. \quad (4.5.11)$$

On the other hand, each enterprise commits to accept every inlet flux sent to her through the connections of the park, whenever these fluxes respect her physical constraints (which are given by (5.2.5)).

With all these considerations, for a given network $E \in \mathcal{E}$, the problem of each enterprise is given by problem $P_i(F_{-i}, E)$

$$\begin{aligned} & \min_{F_i} \text{Cost}_i(F_i, F_{-i}) \\ & \text{s.t.} \begin{cases} \text{Equations (5.2.1)-(5.2.3)-(5.2.5)}, \\ z_i(F_{-i}) \geq 0, \\ F_i \geq 0. \end{cases} \end{aligned} \quad (4.5.12)$$

Observe that constraint (4.5.4) is implicit in the expression of $z_i(F_{-i})$ given by (4.5.8). For a network $E \in \mathcal{E}$ and a fixed operation of the regeneration units F^R , we say that a vector F^P is an equilibrium for the enterprises if and only if

$$\forall i \in I_P, F_i \text{ solves the problem } P_i(F_{-i}, E).$$

We denote by $\text{Eq}(F^R, E)$ the set of equilibria for F^R and E . Then, the problem of the authority is

$$\begin{aligned}
& \min_{F \in \mathbb{R}^{|E_{\max}|}, E \in \mathcal{E}} Z(F) \\
& \text{s.t.} \begin{cases} \sum_{(i,r) \in E} F_{i,r} = \sum_{(r,j) \in E} F_{r,j}, & \forall r \in I_R, \\ \sum_{(k,r) \in E} C_{k,\text{out}} F_{k,r} \leq C_{r,\text{in}}^{\max} \sum_{(k,r) \in E} F_{k,r}, & \forall r \in I_R, \\ C_{r,\text{in}}^{\min} \sum_{(k,r) \in E} F_{k,r} \leq \sum_{(k,r) \in E} C_{k,\text{out}} F_{k,r}, & \forall r \in I_R, \\ F_{r,j} = 0, & \forall (r,j) \in E^c, r \in I_R, \\ F^P \in \text{Eq}(F^R, E), \\ F^R \geq 0, \\ \text{Cost}_i(F_i, F_{-i}) \leq \alpha \text{STC}_i, & \forall i \in I_P. \end{cases} \quad (4.5.13)
\end{aligned}$$

It is not hard to verify that Problem (5.2.12) together with the lower level problems (5.2.9) fit into the general Blind-Input model developed in Section 4.3. Indeed we only need to identify the variable x with the flux vector F , to observe that the costs functions have the structure described in Section 4.3.3 and to note that constraint (5.2.5) can be rewritten with the form

$$g_i(F_{-i}) \leq 0.$$

Thus, in the following, we will apply the reformulation described in Section 4.4.

4.5.1 Problem formulation without regeneration units

Now, we can reduce problem (5.2.12) to a Mixed-Integer programming problem. In this section, we analyze the case when there is no regeneration unit, that is, $I_R = \emptyset$.

To do so, we need to identify the pricing function $\gamma : E_{\max} \rightarrow \mathbb{R}_+$ which gives the marginal price of using exiting connections in the park and then, for each $i \in I_P$, we need to find the arc classes

$$C(i, j) = \{(i, k) \in E_{\max} : \gamma(i, j) = \gamma(i, k)\}.$$

Finally, we compute for each enterprise $i \in I_P$ the class set $\mathcal{C}_i = \{C(i, j) : (i, j) \in E_{\max}\}$.

Looking at the cost function (5.2.8) and assuming that there is no regeneration units, the pricing function γ is given by

$$\gamma(i, j) = \begin{cases} \delta & \text{if } j \in I_P, \\ \beta & \text{if } j = 0. \end{cases} \quad (4.5.14)$$

Thus, for each enterprise $i \in I_P$, the set \mathcal{C}_i is equal to $\{C_{i,p}, C_{i,0}\}$ where

$$\begin{aligned}
C_{i,p} &= \{(i, j) \in E_{\max} : j \in I_P\} \\
C_{i,0} &= \{(i, 0)\}.
\end{aligned}$$

Now, for each agent $i \in I_P$, we include three integer variables, $y_{i,p}, y_{i,0}, y_{i,\text{null}} \in \{0, 1\}$ with the following interpretation:

- If $y_{i,p} = 1$, it means that the connections in $C_{i,p}$ are included in the network.

- If $y_{i,0} = 1$, it means that the connection $(i,0)$ is the only exit connection for i , and i participates in the EIP.
- If $y_{i,null} = 1$, it means that the connection $(i,0)$ is the only exit connection for i , and i does not participate in the EIP (she works in stand-alone mode).

The main point is that only one of these variables takes the value 1, and in doing so it determines the network E to be implemented and the operation that each enterprises can do within this network. We denote by $y \in \{0,1\}^{3n}$ the vector of all integer variables of all enterprises. If an enterprise i is considered in the park, then equation (5.2.11) is active, but if not, then her cost coincides with STC_i . Thus, we define the upper bound for the costs of enterprises as a function of the variable $y_{i,null}$ given by

$$\text{STC}_i(y_{i,null}) = \alpha \text{STC}_i \cdot (1 - y_{i,null}) + \text{STC}_i \cdot y_{i,null}. \quad (4.5.15)$$

Then, applying the reformulation of Section 5.3.2 we obtain the following problem:

$$\begin{aligned} \min_{F,y} \quad & Z(F) \\ \text{s.t.} \quad & \left\{ \begin{array}{ll} z_i(F_{-i}) + \sum_{(k,i) \in E_{\max}} F_{k,i} = \sum_{(i,j) \in E_{\max}} F_{i,j}, & \forall i \in I_P, \\ y_{i,null} + y_{i,p} + y_{i,0} = 1, & \forall i \in I_P, \\ \sum_{(i,j) \in C_{i,p}} F_{i,j} \leq B \cdot y_{i,p}, & \forall i \in I_P, \\ F_{i,0} \leq B \cdot (y_{i,0} + y_{i,null}), & \forall i \in I_P, \\ \sum_{(i,j) \in E_{\max}, j \neq 0} F_{i,j} \leq B \cdot (1 - y_{i,null}), & \forall i \in I_P, \\ \sum_{(k,i) \in E_{\max}} F_{k,i} \leq B \cdot (1 - y_{i,null}), & \forall i \in I_P, \\ \sum_{(k,i) \in E_{\max}} C_{k,\text{out}} F_{k,i} \leq C_{i,\text{in}} \left(z_i(F_{-i}) + \sum_{(k,i) \in E_{\max}} F_{k,i} \right), & \forall i \in I_P, \\ z_i(F_{-i}) \geq 0, & \forall i \in I_P, \\ \text{Cost}_i(F) \leq \text{STC}_i(y_{i,null}), & \forall i \in I_P, \\ F \geq 0. \end{array} \right. \end{array} \quad (4.5.16)$$

Here, B is a constant large enough so all fluxes within the park, regardless the connections, are less than B . In practice, we set B as

$$B := \sum_{i \in I_P} z_i(0) = \sum_{i \in I_P} \frac{M_i}{C_{i,\text{out}}}, \quad (4.5.17)$$

that is, the total fresh water consumption assuming that each enterprise works in stand-alone operation.

4.5.2 Problem formulation including regeneration units

Let us now consider a network with regeneration units. Again, we need to identify the pricing function $\gamma: E_{\max} \rightarrow \mathbb{R}_+$ and the class sets \mathcal{C}_i , for each enterprise $i \in I_P$. Recall that we don't need to work with

classes for regeneration units, since they are controlled by the authority and so they don't have economic incentives.

The function γ in this situation is given by

$$\gamma(i, j) = \begin{cases} \delta & \text{if } j \in I_P, \\ 2\delta & \text{if } j \in I_R, \\ \beta & \text{if } j = 0. \end{cases} \quad (4.5.18)$$

Thus, for each enterprise $i \in I_P$, the set $\mathcal{C}_i = \{C_{i,p}, C_{i,r}, C_{i,0}\}$ where

$$\begin{aligned} C_{i,p} &= \{(i, j) \in E_{\max} : j \in I_P\} \\ C_{i,r} &= \{(i, r) \in E_{\max} : r \in I_R\} \\ C_{i,0} &= \{(i, 0)\}. \end{aligned}$$

As in Subsection 4.5.1, for each agent $i \in I_P$, we include four integer variables, $y_{i,p}, y_{i,r}, y_{i,0}, y_{i,\text{null}} \in \{0, 1\}$ with the following interpretation:

- If $y_{i,p} = 1$, it means that the connections in $C_{i,p}$ are included in the network.
- If $y_{i,r} = 1$, it means that the connections in $C_{i,r}$ are included in the network.
- If $y_{i,0} = 1$, it means that the connection $(i, 0)$ is the only exit connection for i , and i participates in the EIP.
- If $y_{i,\text{null}} = 1$, it means that the connection $(i, 0)$ is the only exit connection for i , and i does not participate in the EIP (she works in stand-alone mode).

Again, only one of this variables can take the value 1. Keeping the same notation $y \in \{0, 1\}^{4n}$ for the complete vector of integer variables, setting $\text{STC}_i(y_{i,\text{null}})$ and B as in definitions (4.5.15) and (5.3.8), respectively, and applying the reformulation of Subsection 5.3.2, we obtain the following problem:

$$\begin{aligned}
& \min_{F,y} Z(y) \\
& \text{s.t.} \left\{ \begin{array}{ll}
z_i(F_{-i}) + \sum_{(k,i) \in E_{\max}} F_{k,i} = \sum_{(i,j) \in E_{\max}} F_{i,j}, & \forall i \in I_P, \\
\sum_{(k,r) \in E_{\max}} F_{k,r} = \sum_{(r,j) \in E_{\max}} F_{r,j}, & \forall r \in I_R, \\
y_{i,\text{null}} + y_{i,p} + y_{i,r} + y_{i,0} = 1, & \forall i \in I_P, \\
\sum_{(i,j) \in C_{i,p}} F_{i,j} \leq B \cdot y_{i,p}, & \forall i \in I_P, \\
\sum_{(i,j) \in C_{i,r}} F_{i,j} \leq B \cdot y_{i,r}, & \forall i \in I_P, \\
F_{i,0} \leq B \cdot (y_{i,0} + y_{i,\text{null}}), & \forall i \in I_P, \\
\sum_{(i,j) \in E_{\max}, j \neq 0} F_{i,j} \leq B \cdot (1 - y_{i,\text{null}}), & \forall i \in I_P, \\
\sum_{(k,i) \in E_{\max}} F_{k,i} \leq B \cdot (1 - y_{i,\text{null}}), & \forall i \in I_P, \\
\sum_{(k,i) \in E_{\max}} C_{k,\text{out}} F_{k,i} \leq C_{i,\text{in}} \left(z_i(F_{-i}) + \sum_{(k,i) \in E_{\max}} F_{k,i} \right), & \forall i \in I_P, \\
\sum_{(r,k) \in E_{\max}} C_{r,\text{in}}^{\min} F_{r,k} - \sum_{(k,r) \in E_{\max}} C_{k,\text{out}} F_{k,r} \leq 0, & \forall r \in I_R, \\
\sum_{(k,r) \in E_{\max}} C_{k,\text{out}} F_{k,r} - \sum_{(r,k) \in E_{\max}} C_{r,\text{in}}^{\max} F_{r,k} \leq 0, & \forall r \in I_R, \\
z_i(F_{-i}) \geq 0, & \forall i \in I_P, \\
\text{Cost}_i(F) \leq \text{STC}_i(y_{i,\text{null}}), & \forall i \in I_P, \\
F \geq 0. &
\end{array} \right. \quad (4.5.19)
\end{aligned}$$

4.6 Simulation with some academic examples

In this section we present numerical examples of the Blind-Input model applied to water exchange networks in Eco-Industrial Parks. The detailed model we use is described in Section 5.2. The optimization problems we solve correspond to adaptations of problem (5.3.9) to the EIP: specifically, we solve problem (5.3.7) for parks without regeneration units, and problem (4.5.19) for parks with regeneration units.

The study case we present consists on an EIP made up of 15 enterprises, each one including only one process, and 3 regeneration units. Data is partially inspired from [2, 109]. It is assumed that the EIP operates for one hour, that is, $A = 1$ h. We consider three different regeneration units which differ by their capacity to regenerate water. Concerning the power ψ , we fix $\psi = 0.6$.

The data of 15 enterprises is given in Table 5.2. Prices are shown in Table 5.1. In addition, the operating parameters of regeneration units are illustrated in Table 4.4. In terms of concentration limits, we set $C_{r,\text{out}} = C_{r,\text{in}}$ for each $r \in I_R$, to simplify the problem.

Enterprise i	$C_{i,in}$ (ppm)	$C_{i,out}$ (ppm)	M_i (g/h)
1	0	100	7500
2	0	200	6000
3	50	100	5000
4	80	800	30000
5	400	800	4000
6	20	100	2500
7	50	100	2200
8	80	400	5000
9	100	800	30000
10	400	1000	4000
11	30	60	2000
12	25	50	2000
13	25	75	5000
14	50	800	30000
15	100	200	13000

Table 4.2: Enterprises' Parameters.

Parameter	Value (\$/tonne)
c	0.13
β	0.22
δ	0.01

Table 4.3: Associated costs.

Unit r	$C_{r,out} = C_{r,in}$ (ppm)	Γ_r (\$/tonne)
1	30	0.85
2	40	0.695
3	50	0.54

Table 4.4: Different regeneration units. We suppose $C_{r,in} = C_{r,out}$.

Since the optimization problems we solve (problems (5.3.7) and (4.5.19)) may have several solutions (see Section 4.6.3 for an example), we introduce a penalization term in the objective function in order to obtain the one having more participating enterprises. We replace $Z(F)$ by

$$Z(F) + \text{Coef} \cdot \sum_{i \in I_P} y_{i,null}, \quad (4.6.1)$$

where $\text{Coef} \geq 0$ is a coefficient to penalize those optimal solutions that leave more enterprises outside the park. If Coef is too large, we may sacrifice optimality of fresh water consumption by forcing enterprises into the park. Numerical experiments show that $\text{Coef} = 1$ is a good value. The selection and impact of this coefficient should be further studied but such analysis is out the scope of the article.

All simulations were implemented with Julia v1.2.0 programming language [?], using Gurobi v8.1.1 as solver [?].

4.6.1 Numerical experiment without regeneration units

Here, we present the results of the simulations with the data exposed before. The optimized configuration of the EIP is presented in Figure 4.6.1, and it corresponds to a Blind-Input contract with $\alpha = 0.95$ and $\text{Coef} = 1$. Detailed results are summarized in Table 4.5. Detailed results of fluxes within the network are presented in Table 4.10.

Of course, the optimization results are sensible to the parameter α chosen for the Blind-Input contract. Figure 4.6.2 shows the results when considering $\alpha \in [0.5, 0.99]$, in terms of total fresh water consumption and the number of enterprises left out of the EIP, that is, the number of stand-alone enterprises. For values of α smaller than 0.5, it would mean that the authority is offering an improvement of more than 50% to each participant of the park, which is unrealistic.

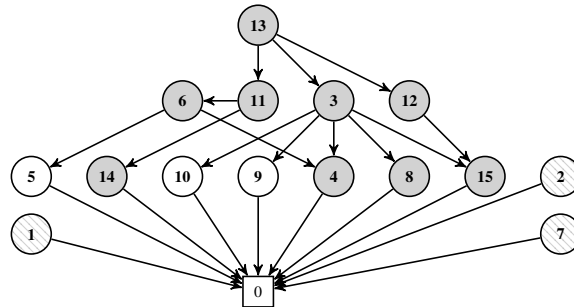


Figure 4.6.1: The configuration in the case without regeneration units, $\alpha = 0.95$ and $\text{Coef} = 1$. Gray nodes are consuming strictly positive fresh water. Dashed nodes are operating in stand-alone mode.

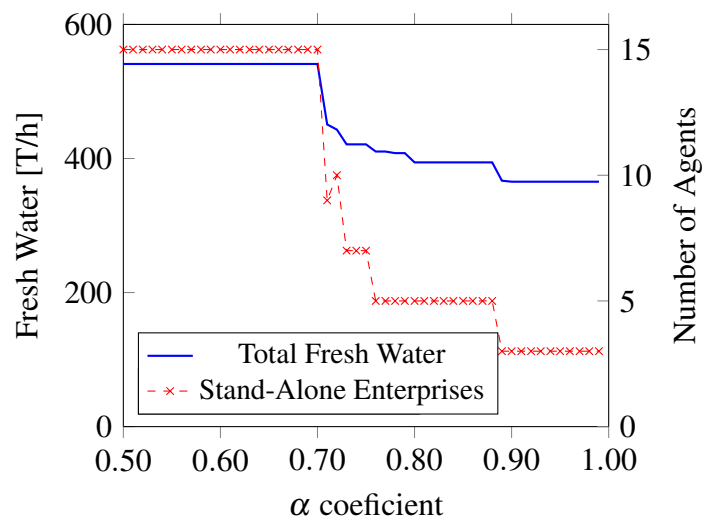


Figure 4.6.2: Sensitivity Analysis for $\alpha \in [0.50, 0.99]$ and $\text{Coef} = 1$. Total fresh water consumption and number of stand-alone enterprises.

Enterprise	Fresh Water Stand-Alone	Fresh Water in EIP	Cost _i Stand-Alone	Cost _i in EIP
1*	75.00	75.00	26.25	26.25
2*	30.00	30.00	10.50	10.50
3	50.00	42.08	17.50	6.52
4	37.50	8.33	13.13	10.58
5	5.00	0.00	1.75	1.31
6	25.00	20.83	8.75	3.13
7*	22.00	22.00	7.70	7.70
8	12.50	3.13	4.38	3.97
9	37.50	0.00	13.13	9.86
10	4.00	0.00	1.40	1.02
11	33.33	35.42	11.67	5.13
12	40.00	53.33	14.00	8.00
13	66.67	66.67	23.33	9.33
14	37.50	6.66	13.13	10.00
15	65.00	1.92	22.75	20.49
Total	541.00	365.37	189.37	133.79

Table 4.5: Summary of results of the EIP without regeneration units. Marked enterprises (*) are left outside the park, operating stand-alone.

4.6.2 Numerical experiment including regeneration units

The main difficulty to solve Problem (4.5.19) is that the function Cost_i described in (5.2.8) is nonlinear under the presence of Regeneration Units. To tackle this obstruction, for each unit $r \in I_R$ we construct a piecewise linear approximation of the functions $f_r : x \mapsto \Gamma_r \cdot x^\psi$, which is the marginal costs of regenerating water. This approximation, that we denote ℓ_r , is constructed in the interval $[0, B]$ (where B is the constant given in (5.3.8)), with the reference points $S = (s_0, s_1, s_2, s_3, s_4, s_5) = (0, 0.1B, 0.2B, 0.4B, 0.6B, B)$. Figure 4.6.3 shows the approximations ℓ_r for the regeneration units described in Table 4.4.

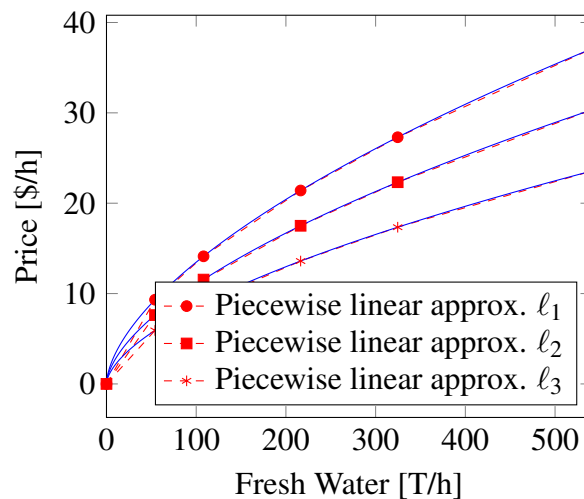


Figure 4.6.3: Piecewise linear approximations (dashed red) of marginal cost of regenerating water. The function ℓ_r corresponds to Reg. Unit $r \in I_R = \{1, 2, 3\}$. The original function f_r is given in blue. The parameters of f_r correspond to the data presented at the beginning of Section 4.6.

The nonuniform partition of the interval $[0, B]$ is due to the fact that B is not a tight bound: each value

$F_{r,j}$ with $r \in I_R$ and $i \in I_P$ should be a lot less than B . Thus, the approximation ℓ_r must be more precise in the first part of the interval.

As we did in Subsection 4.6.1, we present the detailed results of the simulations for $\alpha = 0.95$ and $\text{Coef} = 1$ in Table 4.6. The obtained configuration of the park is given in Figure 4.6.1. Detailed results of fluxes within the network are presented in Table 4.11.

Enterprise	Fresh Water Stand-Alone	Fresh Water in EIP	Cost _i Stand-Alone	Cost _i in EIP
1	75.00	75.00	26.25	10.50
2	30.00	30.00	10.50	4.20
3	50.00	0.00	17.50	11.16
4	37.50	0.00	13.13	5.58
5	5.00	0.00	1.75	1.53
6	25.00	10.42	8.75	5.98
7	22.00	0.00	7.70	6.50
8	12.50	0.00	4.38	12.47
10	4.00	0.00	1.40	0.15
11	33.33	7.15	11.67	11.08
12	40.00	18.94	14.00	13.30
13	66.67	16.67	23.33	17.75
14	37.50	0.00	13.13	6.41
15	65.00	0.00	22.75	21.40
Total	541.00	158.17	189.35	132.17

Table 4.6: Summary of results of the EIP considering regeneration units with $\text{Coef} = 1$.

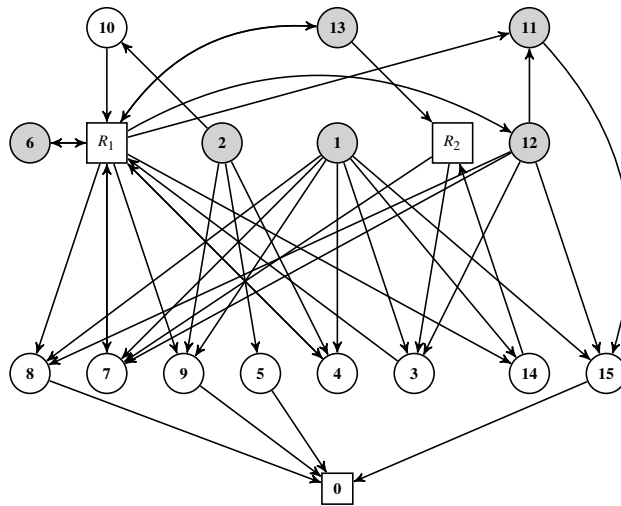


Figure 4.6.4: The configuration considering regeneration units, $\alpha = 0.95$ and $\text{Coef} = 1$. Gray nodes are consuming strictly positive fresh water. Regeneration unit R_3 is not used and therefore removed from the park.

Finally, the sensitivity analysis of the results with respect to $\alpha \in [0.5, 0.99]$ is shown in Figure 4.6.5, again, showing the variation of total fresh water consumption and the number of stand-alone enterprises. As in the case without regeneration units, we neglect values of α smaller than 0.5 due to the impossibility of implement such a contract.

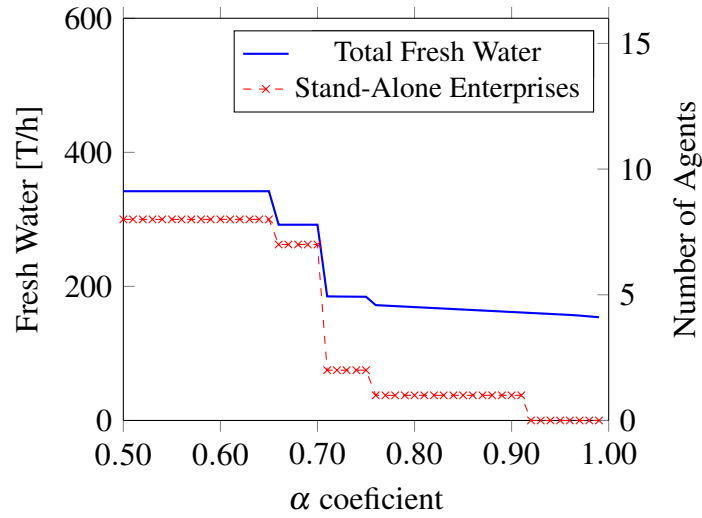


Figure 4.6.5: Sensitivity Analysis for $\alpha \in [0.50, 0.99]$ and Coef = 1. Total fresh water consumption and number of stand-alone enterprises.

4.6.3 Small case with multiple solutions

In this last part, we present an small example to show that problem (5.3.7) may have multiple optimal solutions. This example presents two optimal solutions that have different number of participating enterprises. The case of study consists on an EIP made up of 5 enterprises described in Table 4.7. The prices of the example are the same as before, presented in Table 5.1, and the Blind-Input parameter is fixed as $\alpha = 0.99$.

Enterprise	$C_{i,in}$ (ppm)	$C_{i,out}$ (ppm)	M_p (g/h)
1	175	898	30010
2	90	200	3000
3	30	35	1000
4	150	530	32030
5	400	1095	90000

Table 4.7: Enterprises' Parameters for small size case of study

Given that the example is small, we were able to explore all combinations of participating enterprises: each time, we chose a subset $A \subset I_P = \{1, 2, 3, 4, 5\}$ and we solved problem (5.3.7) with the extra constraint

$$y_{i,null} = \begin{cases} 0 & \text{if } i \in A \\ 1 & \text{otherwise.} \end{cases}$$

This exhaustive exploration can be done only because the set of enterprises is small, since the number of possible subsets $A \subset I_P$ is $2^{|I_P|}$.

After exploring all possible combination, we found two different optimal configurations that are shown in Figures 4.6.6 and 4.6.7.

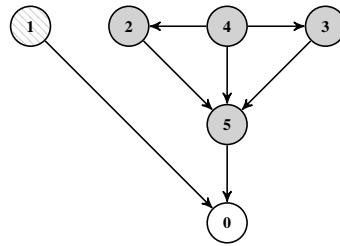


Figure 4.6.6: The configuration in the case without regeneration units, $\alpha = 0.99$. Gray nodes are consuming strictly positive fresh water.

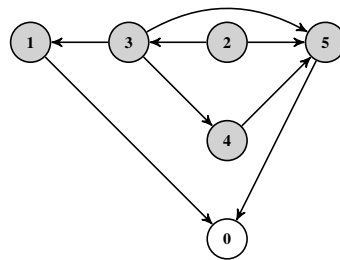


Figure 4.6.7: The configuration in the case without regeneration units, $\alpha = 0.99$. Gray nodes are consuming strictly positive fresh water.

The results of fresh water consumption and economic cost are summarized in tables 4.8 and 4.9.

Enterprise	Stand-Alone	First Solution	Second Solution
1	33.42	33.42*	32.36
2	15	18.65	15
3	28.57	33.98	64.68
4	60.43	60.43	62.26
5	82.19	2.03	0
Total	219.61	148.51	148.51

Table 4.8: **Fresh Water consumption [T/h]**. Stand-alone configuration and optimal solutions. In the first solution, Enterprise 1 is left in stand-alone operation.

Enterprise	Stand-Alone	First Solution	Second Solution
1	11.70	11.70*	11.58
2	5.25	2.65	2.1
3	10	4.76	8.30
4	21.15	8.46	5.39
5	28.77	26.71	27.45
Total	76.87	54.28	54.82

Table 4.9: **Economic cost [\$/h]**. Stand-alone configuration and optimal solutions. In the first solution, Enterprise 1 is left in stand-alone operation.

4.7 Discussion

The results presented for our academic example, in both cases without and with regeneration units, show that optimal networks under Blind-Input contracts provide substantial improvements in terms of reducing the global consumption of water and the cost of each participating enterprise. Relatively speaking, the EIP without regeneration units reduces the amount of consumed water in 32% with respect to the stand-alone configuration, while the EIP with regeneration units reduces it is more than 70%.

Moreover, each participating enterprise has a cost reduction of at least 5%, as promised by the Blind-Input contract. Exploring both, tables 4.5 and 4.6, the reader can observe that the reduction is not uniform among the enterprises. This unbalanced benefit is due to the nature of the problem: the authority is only concerned by the total water consumption, and therefore, the costs of the enterprises is not relevant beyond the Blind-Input contract. This is acceptable recalling that the Blind-Input model maintains the information of each enterprise private.

From the perspective of each enterprise, she only interacts economically with the authority and so she only perceives her gains. It is therefore economically appealing to participate in the EIP under the network proposed by the authority. However, the model presented in this work may not be economically efficient in the sense that the total cost for the enterprises may not be optimal. Figure 4.7.1 shows a sensitivity analysis of total cost for enterprises depending on the Blind-Input parameter $\alpha \in [0.50, 0.99]$.

One would expect that total cost, at least starting from some threshold where enough enterprises participate into the park, should be increasing with respect to α , since larger the α , less exigent are the enterprises. However, this is not the case. The behavior of the total cost is quite chaotic, which is more evident for the case with regeneration units. In terms of total cost of enterprises, the only behavior that is natural is that the influence of regeneration units not only reduces the total water consumption, but also the total cost the enterprises.

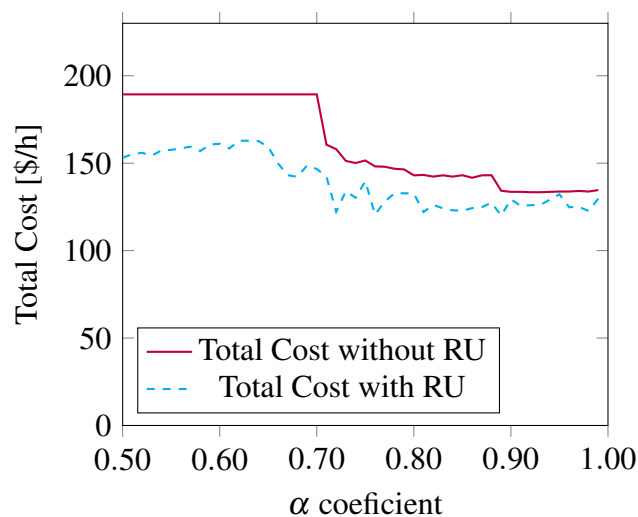


Figure 4.7.1: Sensitivity Analysis for $\alpha \in [0.50, 0.99]$ and Coef = 1. Total cost of the park for both study cases: without and with regeneration units.

This behavior in economic cost can be explained by the multiplicity of optimal configurations. The small size example presented in Section 4.6.3 emphasizes that there may be several optimal configurations in terms of total water consumption, differing in the number of stand-alone agents and total cost of enterprises. An interesting question rising from this example and the behavior of costs showed in Figure 4.7.1 is how to

select the “best” optimal configuration.

In order to minimize the number of stand-alone enterprises we introduce the penalization (5.4.2). However, some other penalization could be considered to try to force economic efficiency. In real-size problems (more than 10 enterprises), it is not possible to explore all configurations, and therefore penalized objective functions seem to be the solution. This could also be regarded in the sense of multi-objective optimization: the authority could consider not only the total water consumption, but also the number of stand-alone enterprises and the economic efficiency as optimization criteria. However, this could be delicate, since it is important to keep the minimization of fresh water consumption as the primary goal of the park. In the case of study without regeneration units, Figure 4.6.2 shows that the minimization of stand-alone enterprises could be relegated for a better solution in terms of water consumption: for $\alpha = 0.71$, 9 enterprises are left in stand-alone operation, while for $\alpha = 0.72$, 10 enterprises are left in stand-alone operation. While this could be interpreted as an ill behavior of the algorithm, it is completely correct. Since $\alpha = 0.72$ is bigger, the authority has more freedom to build the optimal network, and while she could implement the same solution that for $\alpha = 0.71$, she chooses to left one extra enterprise outside of the park, and then reducing the fresh water consumption from 450.79 [T/h] (the optimal value for $\alpha = 0.71$) to 443.08 [T/h] (the optimal value for $\alpha = 0.72$).

This same example shows another limitation for the Blind-Input contract: on the one hand, if enterprises are too exigent (α small), the authority may be incapable to find a economically feasible configuration. In fact, for $\alpha \leq 0.70$, the optimal solution is the stand-alone configuration, which means that the authority cannot build a park that attracts the exigent agents and also that reduces the total fresh water consumption. On the other hand, if enterprises are too polluting, they may not be able to participate in the park, even without economic constraints. Indeed, starting from $\alpha = 0.89$, the authority leaves three enterprises outside the park, since their participation doesn't contribute to the minimization of fresh water. This is solved only by introducing regeneration units, where full participation is attained starting from $\alpha = 0.92$.

4.8 Conclusion and perspectives

Models for general exchange could lead to quite difficult problems like single-leader-multi-follower problems. In this work, our aim is to emphasize that under the Blind-Input paradigm which is economically consistent, one can find some solutions by simply solving a mixed-integer linear problem. This clearly allows to tackle large scale problems efficiently and propose exchange politics that attract enterprises to participate.

In order to apply the Blind-Input model to an exchange network, the following main elements must be present: 1) for each independent agent of the park, Kirchoff's law and positivity of fluxes must be the only constraints of the her optimization problem that include their own variables (i.e., the vector x_i); 2) the quality of each inlet flux must depend only on the sending agent; and 3) the optimal response hypothesis must allow each agent to compute the amount of natural resources needed. These three conditions are present in many other examples. Particularly, energy networks also fit this profile (see, e.g., [6, 106]). Under these conditions, our general model of Section 4.3 (and of course the reductions and methods of Section 4.4) can be adapted and so, the network can be designed/operated following the Blind-Input paradigm. It is important to note that the economic model is a design decision, suitable for Eco-Industrial Parks involving noncooperative enterprises.

Of course, the Blind-Input paradigm can appear to be quite restrictive since followers are forced to accept the incoming fluxes. While this aspect is compensated by the Blind-Input contract, it could be interesting for a future work to consider exchange models in which agents could have more “control” on their exchanges. Taking advantage of the MILP formulation, some other theoretical results could be

deduced on the exchange models; for example, stability results, dual formulations, among others.

In terms of the physical model, several other developments can be considered for future research. In this work, we only treat enterprises with single processes. Multi-process agents should also be explored. Specifically in terms of the water exchange model, we only consider a single-pollutant model. Multi-pollutant exchange networks (where enterprises must dilute different types of pollutants) introduce a particular difficulty with the inlet and outlet concentrations. The optimal response hypothesis changes, since each enterprise will have several outlet concentrations (one for each pollutant) to observe, entailing that these concentrations would become variable, changing the contaminant mass balance constraints as well as the formula for the fresh water consumption. A concrete perspective of this work is to include the multi-contaminant problem coupled with the energy network, following the developments of [110]. Finally, Eco-Industrial Parks with several exchange networks could be explored as well under the Blind-Input paradigm, following the spirit of [105].

4.9 Network fluxes of simulations

The values of the flux corresponding to the case without and with regeneration unit in sections 4.6.1 and 4.6.2 are given in Tables 4.10 and 4.11, respectively. The entrance (i, j) of both tables corresponds to the flux sent from agent i to agent j .

Enterprise	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	Sink
1	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	75.00
2	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	30.00
3	0.00	0.00	0.00	7.79	0.00	0.00	0.00	12.50	42.86	4.44	0.00	0.00	0.00	0.00	6.15	0.00
4	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	41.67
5	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	5.71
6	0.00	0.00	0.00	25.54	5.71	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
7	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	22.00
8	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	15.63
9	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	42.86
10	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	4.44
11	0.00	0.00	0.00	0.00	0.00	10.42	0.00	0.00	0.00	0.00	0.00	0.00	0.00	33.33	0.00	0.00
12	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	80.00	0.00
13	0.00	0.00	31.67	0.00	0.00	0.00	0.00	0.00	0.00	0.00	8.33	26.67	0.00	0.00	0.00	0.00
14	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	40.00
15	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	88.08

Table 4.10: The values of the flux in the case without regeneration units, $\alpha = 0.95$ and Coef = 1.

Enterprise and Regeneration units	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	R_1	R_2	R_3	Sink
1	0.00	0.00	10.82	9.60	0.00	0.00	8.31	10.61	18.50	0.00	0.00	0.00	0.00	11.43	5.74	0.00	0.00	0.00	0.00
2	0.00	0.00	0.00	8.30	6.67	0.00	0.00	0.00	10.03	5.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
3	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	100.00	0.00	0.00	0.00
4	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	41.67	0.00	0.00	0.00
5	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	6.67
6	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	31.25	0.00	0.00	0.00
7	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	44.00	0.00	0.00	0.00
8	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	15.63
9	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	42.86
10	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	5.00	0.00	0.00	0.00
11	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	66.67	0.00	0.00	0.00	0.00
12	0.00	0.00	35.08	0.00	0.00	0.00	3.24	1.90	0.00	0.00	10.72	0.00	0.00	0.00	20.62	0.00	0.00	0.00	0.00
13	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	62.55	37.45	0.00	0.00
14	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	40.00	0.00	0.00
15	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	93.02
R_1	0.00	0.00	0.00	23.76	0.00	20.83	9.10	3.09	14.33	0.00	48.79	52.65	83.33	28.57	0.00	0.00	0.00	0.00	0.00
R_2	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
R_3	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00

Table 4.11: The values of the flux in the case with regeneration units, $\alpha = 0.95$ and Coef = 1.

Optimal Design of Exchange Water Networks with Control Inputs in Eco-Industrial Parks

This article has been done in the common work with Didier Aussel. It has been submitted to Energy Economics, 23p, 2020.

Abstract

Industrial water conservation is an important adaptation to preserve the environment. Eco-Industrial Parks (EIPs) have been designed to encourage the establishment of water exchange networks between enterprises in order to minimize freshwater consumption and wastewater discharge by maximizing wastewater reuse. This control-input model presents a mathematical programming formulation for designing and optimizing industrial water networks in EIPs based on formulating and solving Single-Leader Multi-Follower (SLMF) game problems. Enterprises (followers) aim to minimize their operating costs by reusing wastewater from other enterprises, while the designer (leader) aims to minimize the consumption of natural resources within the ecopark. Moreover, when participating in the ecopark, enterprises can control all their input fluxes and the designer guarantees a minimal relative improvement in comparison with the stand-alone operation of each enterprise. The SLMF game is transformed into a single mixed-integer optimization problem. The obtained results are compared with the results of the blind-input model [77].

Keywords: Single-Leader-Multi-Follower game, Eco-industrial park, mixed integer programming, water networks.

AMS subject classifications. 90C26, 90C11, 90C90

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5.1 Introduction

The development of industrialization and urbanization along with other human activities on the environment in many countries around the world makes the earth heavily polluted. Therefore, decision makers need to have practical policies and actions to prevent these risks. Facing these issues, numerous approaches/concepts have been proposed, in the last few decades, for protecting the global environment while increasing the economic utilities based on the concept of *sustainable development*. Some concepts linked to sustainable development such as *Industrial Ecology* (IE) have emerged [6]. IE was first introduced in 1989 by Frosch and Gallopoulos [66]. They proposed that resource consumption and waste generation are minimized by allowing the waste materials from one industry to serve as raw material for another. This idea is directly related to another concept, namely *Industrial Symbiosis* (IS), which involves “separate industries in a collective approach to competitive advantage involving physical exchange of materials, energy, water and by-products” [67]. Industrial symbiosis can be realized through single-industry dominated complexes including chemicals and petrochemicals, pulp and paper, waste management which includes material reuse and recycle, etc. As a feature of IS, the geographical proximity between participating industries is essential because the transportation of waste materials is expensive. The most widespread manifestations of these kinds of IS are *Eco-Industrial Parks* (EIPs).

The objective of EIPs is to reduce the production cost of each participating enterprise and the environmental impact of industrial production while the production level of each industry is maintained. This involves reducing the consumption of energy and/or raw materials (water, energy-steam, etc) of a group of companies located in the same industrial park, or designing/creating new industrial parks incorporating these aspects. This is achieved by reusing the waste from one industrial process as a utility from another process, either in raw form if the “contamination” is low enough or via regeneration facilities. However, to convince companies to take part in an EIP, it is essential to make sure that each participant gains in competitiveness (reduction in production costs in most cases). Since these advantages depend on the EIP configuration, proper planning and design are critical. However, the system methods for designing the optimal configuration of EIP are lacking and the models of water exchange network used to model Eco-Industrial Parks in the literature are limited in some respects [6, 2, 67, 77].

The design and optimization of water-exchange networks in EIPs are complex problems due to their actual sizes, their range of physical constraints, to be taken into account. In the literature, there are two main approaches for designing and optimizing water-exchange networks in the EIP: multi-objective optimization (MOO) on one hand and non-cooperative game theory on the other hand. The MOO perspective is based on coalition cooperative games in which enterprises make binding agreements to coordinate their strategies and share their information. The solution of such an approach is called *Pareto optimal*. A solution is said to be Pareto optimal if no one can be made better off without making someone worse off [103]. After

solving the multi-objective optimization problem the designer selects one of Pareto optimal solutions considering an auxiliary criterion, usually the distance to a utopia point. The basic obstacle to EIP is the need for cooperation between enterprises with different interests, in particular by sharing data on their production processes. This kind of cooperation between enterprises can only be implemented when there is trust between partners. Note that such an issue usually does not exist in designing and optimizing of exchange networks in EIPs [111]. Therefore, due to the non-cooperative context, the different enterprises may deviate from the selection of the designer since they may improve their benefits by unilaterally change their operation. We refer the reader to [6, 8, 9], for the survey on multi-objective optimization approach.

Another suitable approach for designing and optimizing a water-exchange networks in EIPs is the non-cooperative game theory approach. A game is non-cooperative if the participants do not make binding commitments to coordinate their strategies. The theory of non-cooperative games corresponds to a mathematical analysis of strategy and conflict, in which a player success in making choices depends on the choice of others. In fact, an EIP can be considered as a collection of non-cooperative agents by introducing an EIP impartial authority whose role is to collect all necessary data, in a confidential way, to design the EIP. More precisely, enterprises optimize their operating costs while the EIP authority aims to minimize resource consumption. This kind of problem can be modeled as Single-Leader-Multi-Follower (SLMF) game with leader-follower strategy. The upper-level decision-maker (leader) is the EIP authority and the enterprises are the lower-level decision-makers (followers). The EIP authority makes his decision first by anticipating the responses of enterprises. Based on the EIP authority' decisions, all enterprises compete with each other in a parametric non-cooperative generalized Nash game at the lower-level with the strategies of the EIP authority as exogenous parameters. It's worth mentioning that, at the lower level, enterprises play a generalized Nash equilibrium between them, so enterprises involved would be able to keep confidential data, without the need to share them with the other enterprises of the park. In the context of non-cooperative games, an optimal solution for EIP design can be achieved and proposed by obtaining a generalized Nash equilibrium. Due to the Nash equilibrium concept, no enterprise can unilaterally deviate in order to improve his payoff by choosing a different strategy. We refer the reader to [11] for a primer in non-cooperative games, to [15] for a survey of generalized Nash equilibrium problems. For SLMF games, we refer to [61, 63, 112, 113] and the references therein. Figure ?? shows the general scheme of such a model.

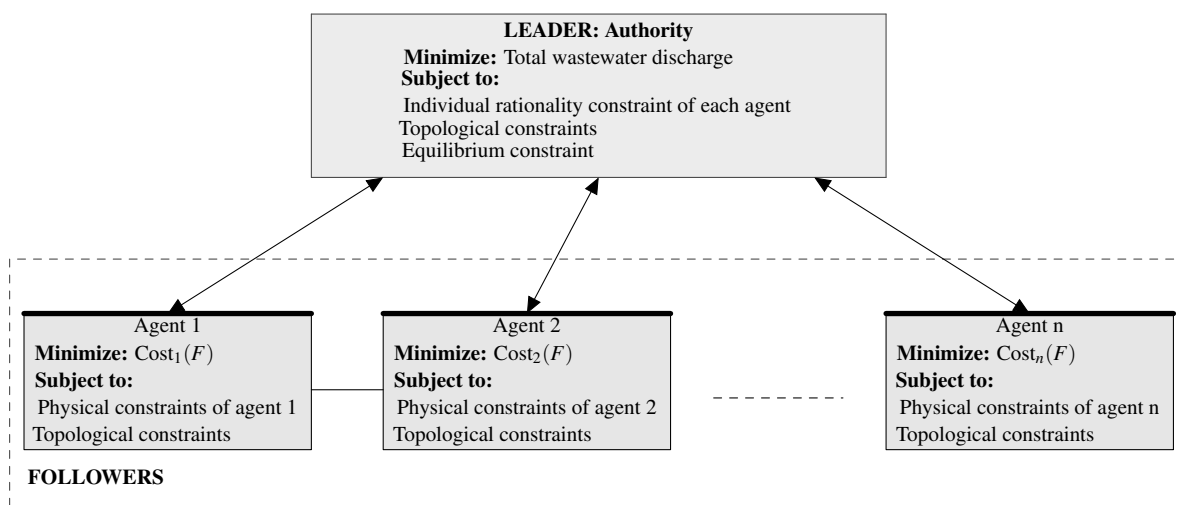


Figure 5.1.1: General scheme of SLMF Game

It is clear that the latter approach is more realistic in designing and optimizing the water exchange

network in the EIP because first, it helps to reduce the overall freshwater consumption; second, it reduces the operating costs of each enterprise; third, enterprises operating in the EIP do not need to share their information with other enterprises of the park, which is clearly a very important issue in the design of an optimal EIP. The SLMF approach of the optimal design of EIP has been introduced for the first time in [2] and then specialized in [77, 108]). In fact in [77], the authors have profoundly revisited the SLMF approach of [2] and have developed an abstract Blind-Input model for water exchange networks in EIPs. The main implicit assumption done in the blind-input model [77] is that each enterprise can only control his outlet distribution. They are thus forced to accept whatever is sent to them through the exchange network. This assumption seems actually quite restrictive since the enterprise may be forced to receive too much polluted water which could turn into higher costs than the stand-alone operation outside the park. However, to overcome this obstacle, the authors came up with the concept of a Blind-Input contract, which guarantees that the designer commits a minimal relative improvement of his operating cost, with respect to the stand-alone operation of the enterprise.

In the present work, we propose another model for designing and optimizing exchange water networks by considering that each enterprise controls his input flux, which seems more realistic in the case of exchange networks. In other words, when participating in the exchange network, each enterprise has the ability to control the amount of water coming from the other enterprises. This model is called *control input model*. Under some linear structure of the costs functions $\text{Cost}_i(\cdot)$ of each enterprise, the control-input model can be reduced from a SLMF problem to a single mixed-integer linear optimization problem. The approach is validated on a case study of exchange water network in EIPs without regeneration units. Obtained results are compared with the blind-input model [77].

The rest of this paper is organized as follows. Section 5.2 provides the general problem statement which briefly describes the problem addressed in this article and present a control input model for water exchange networks in EIPs, based on a single-leader-multiple-follower model. A reformulation of the control input problem as a mixed integer linear programming problem is addressed in Section 5.3. Numerical experiments on reasonably large EIP are presented in Section 5.4 where comparisons with blind input approach are also provided. Finally, Conclusions and perspectives are presented in Section 5.5.

5.2 Control-Input model for water exchange networks in EIPs

5.2.1 Problem statement

Let us consider a set of enterprises $\mathcal{P} := \{P_1, \dots, P_n\}$ that are co-located in the same industrial park and are governed by rules made in the park. Each enterprise has its own pre-defined water input requirement and quality characteristics, as well as the quantity and quality of available output wastewater. After the operation of each enterprise, the discharge wastewater can be used as input for other enterprises in the park. Contrary to the model developed in [77] and [2], our *Control-Input* model is based on the assumption that each enterprise has a control on his polluted inputs, namely, each enterprise has the ability to set the amount of water coming from the other enterprises. The goal of the model is to establish an optimal exchange network so that the total freshwater consumption, the total discharge wastewater, and the operating cost of each participating enterprise in the park are minimized, while satisfying all process and environmental constraints. The problem is structured as a SLMF problem wherein the EIP authority is the upper-level decision-maker and the enterprises interact through a GNEP as lower-level decision-makers. Each enterprise wants to minimize his cost of the use of water while the designer is in charge of the ecological concern by minimizing the fresh water consumption, thus encouraging the recycling or reuse of wastewater streams.

The n enterprises of the considered EIP are connected thanks to an exchange network. A sink node,

represented by the index 0 is of course included. It represents a waste pit to discharge polluted water. Each enterprise P_i can be connected to other enterprises and/or to the sink node. Setting by $I := \{1, \dots, n\}$ the index set of the \mathcal{P} we also define $I_0 = \{0\} \cup I$.

5.2.2 Enterprises' problem

The goal of the designer is to built an exchange network so that part of this polluted water could be reused by other enterprises, reducing the global consumption of fresh water within the park. Here, an exchange network for the EIP is a simple directed graph (I_0, E) , where the connection $(i, j) \in E$ means that the enterprise i can send its output water to the enterprise j . In this sense, if the enterprise i uses the connection $(i, 0)$, then it means that it is discharging water outside the park, to the environment.

Defining the sets

$$\begin{aligned} E^{\text{st}} &:= \{(i, 0) : i \in I\}, \\ E^{\text{max}} &:= \{(i, j) : i \in I, j \in I\}, \\ E_0^{\text{max}} &:= \{(i, j) : i \in I, j \in I_0\} = E^{\text{max}} \cup E^{\text{st}}, \end{aligned}$$

a valid exchange network E_0 is the set of connections satisfying $E^{\text{st}} \subset E_0 \subset E_0^{\text{max}}$. Note that the set E^{st} is the *stand-alone configuration*, where each enterprise only has access to fresh water and, after using it, he must discharge it to the sink node. On the other hand E_0^{max} stands for the complete park, in the sense that all enterprises are connected between them, and each enterprise has a connection with the sink node. This definition yields that: 1) for every enterprise there is always the possibility of discharge; and 2) the sink node doesn't have any exit connections (it is not possible to recover water once it is discharged). We denote by \mathcal{E} the family of valid networks for the EIP. Finally, for any $E_0 \in \mathcal{E}$, the associated networks E and E_0^c stands respectively for $E = E_0 \setminus E^{\text{st}}$ and the set of connections that are not in E_0 , that is, $E_0^c = E_0^{\text{max}} \setminus E_0$.

For each $(k, i) \in E^{\text{max}}$ we denote by $F_{k,i}$ [T/h] the water flux going from k to i through the connection (k, i) . We consider the following notation:

- $F_i = (F_{k,i} : k \in I)$ is the vector of fluxes going to enterprise i ,
- $F_{-i} = (F_{k,j} : j \in I \setminus \{i\})$ is the vector of all fluxes not going to enterprise i and not going to sink node,
- $F = (F_i : i \in I)$ is the vector of fluxes between enterprises,
- for any $i \in I$, z_i stands for the inlet fresh water and $F_{i,0}$ is the flux sent by enterprise i to the sink node.

Finally, following classical notations of game theory, for an enterprise $i \in I$, we may write $F = (F_i, F_{-i})$, to stress the vector of fluxes between enterprise i . Then, for a fixed network E_0 , a valid flux vectors $F = (F_{k,i} : (k, i) \in E^{\text{max}})$ and $F_0 = (F_{i,0} : i \in I)$ must satisfy the following constraints:

1. Use of connections in E_0 : since E_0 represents the available connections, we must put

$$\forall (k, i) \in E_0^c, F_{k,i} = 0. \quad (5.2.1)$$

2. Positivity of fluxes: all the fluxes in the park must be positive:

$$\forall (k, i) \in E_0, F_{k,i} \geq 0 \quad \text{and} \quad \forall i \in I, z_i \geq 0. \quad (5.2.2)$$

3. Water mass balance: since no water losses are considered, for any valid network E_0 and for each enterprise $i \in I$, we have

$$z_i + \sum_{(k,i) \in E} F_{k,i} = \sum_{(i,j) \in E} F_{i,j} + F_{i,0}. \quad (5.2.3)$$

It is important to note that while each enterprise i controls his inlet flux F_i and the outlet fluxes F_{-i} are under the control of the concurrent enterprises, the fluxes $(F_{i,0})_{i \in I}$ corresponding to the amounts of discharged water are directly deduced by the water balance constraint (5.2.3) and will not be variables neither of the enterprises nor of the designer.

Thus for an given couple (E_0, F) of valid network and associated fluxes, an enterprise will be:

- in *semi-stand-alone* situation if, for any $k \in I$, $F_{k,i} = 0$;
- in *stand-alone* situation if, for any $k, j \in E$, $F_{k,i} = 0$ and $F_{i,j} = 0$.

In the first situation the enterprise i doesn't exchange any flux with the other enterprises while in the semi-stand-alone case, the enterprise i send some polluted water to the other enterprises but doesn't receive any polluted water.

Each enterprise $i \in I$ thus receives the fluxes from other enterprises within the EIP. Nevertheless for technical constraints on the process P_i , the pollutant concentration delivered by the other enterprises cannot exceed a certain maximum value denoted here by $C_{i,in}$ [ppm]. We assume here that each enterprises i , in order to use less fresh water, will actually accept a maximum of polluted water. The inlet flux is then mixed with the purchased fresh water z_i generating therefore the inlet flux $z_i + \sum_{(k,i) \in E} F_{k,i}$. On the other hand enterprise i generates a fixed amount of pollutant M_i [g], coming from his internal production process. This pollutant is then diluted into the outlet water flux $\sum_{(i,j) \in E} F_{i,j} + F_{i,0}$, for which is usually assumed, in the design of EIP, that the pollutant concentration is less than a maximum value $C_{i,out}$ [ppm]. Actually considering that enterprise i will optimize his process, we will assume, as it is classically done for EIP the so-called *hypothesis of optimal response* that each enterprise $i \in I$ consumes exactly the fresh water it needs to attain $C_{i,out}$, and therefore, its output pollutant concentration is always equal to this constant. Obviously we have that $0 < C_{i,in} < C_{i,out}$. This functioning is illustrated in Figure 5.2.1.

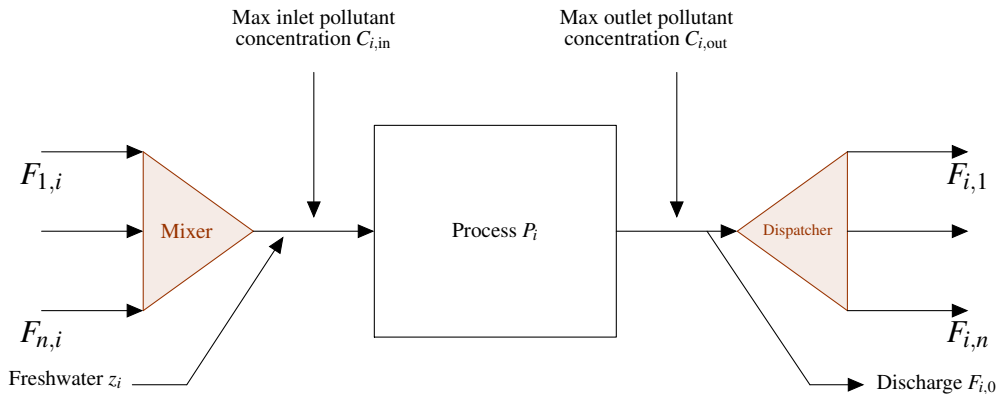


Figure 5.2.1: Water mixture description for a given process P_i .

Taking into account the above explanations, for any valid network E_0 , the water mixture model immediately leads to the following constraints:

1. Contaminant mass balance: each enterprise $i \in I$ wants to receive the maximum of pollutant and thus the contaminant mass balance is

$$M_i + C_{i,\text{in}} \Phi_i(z_i, (F_{k,i})_k) = C_{i,\text{out}} \left(\sum_{(i,j) \in E} F_{i,j} + F_{i,0} \right), \quad (5.2.4)$$

where the auxiliary function $\Phi_i : \mathbb{R} \times \mathbb{R}^{n-1} \rightarrow \mathbb{R}$ is defined as

$$\Phi(z_i, (F_{k,i})_k) = \begin{cases} z_i + \sum_{(k,i) \in E} F_{k,i} & \text{If there exists } k \text{ such that } F_{k,i} > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Note that the auxiliary function Φ_i allows to take into account that when enterprise i doesn't receive any polluted water, thus being in a semi-stand-alone or stand-alone situation, then the pollutant only comes from the internal generation M_i .

2. Inlet/outlet concentration constraints: for an enterprise $i \in I$ we have that:

$$\sum_{(k,i) \in E} C_{k,\text{out}} F_{k,i} \leq C_{i,\text{in}} \left(z_i + \sum_{(k,i) \in E} F_{k,i} \right). \quad (5.2.5)$$

Observe that, in the case of an enterprise i in semi-stand-alone or stand-alone situation, combining equations (5.2.3) and (5.2.4) we obtain:

$$M_i = C_{i,\text{out}} \left(\sum_{(i,j) \in E} F_{i,j} + F_{i,0} \right),$$

and so, the discharge of wastewater by the enterprise $i \in I$ is given by the fluxes controlled by the other enterprises, thanks to the formula

$$F_{i,0}(F_{-i}) = \frac{M_i}{C_{i,\text{out}}} - \sum_{(i,j) \in E} F_{i,j}. \quad (5.2.6)$$

For all the other enterprises, neither in semi-stand-alone nor in stand-alone situation, by combining (5.2.3) and (5.2.4), one gets

$$M_i + C_{i,\text{in}} \left(\sum_{(i,j) \in E} F_{i,j} + F_{i,0} \right) = C_{i,\text{out}} \left(\sum_{(i,j) \in E} F_{i,j} + F_{i,0} \right),$$

and so, the discharge of wastewater by the enterprise $i \in I$ is given by the fluxes controlled by the other enterprises, thanks to the following formula

$$F_{i,0}(F_{-i}) = \frac{M_i}{C_{i,\text{out}} - C_{i,\text{in}}} - \sum_{(i,j) \in E} F_{i,j}. \quad (5.2.7)$$

Now we denote by c [\$/T] the marginal cost of fresh water, and for any enterprise i by $\gamma_{i,0}$ [\$/T] the unit tax for discharged water. Observe that, if the enterprise i doesn't receive polluted water (i.e. is in semi-stand-alone or stand-alone situation), then its fresh water consumption z_i must be

$$z_i = \frac{M_i}{C_{i,\text{out}}}.$$

Then, the cost of stand-alone operation, which we denote by STC_i [\$], is given by

$$STC_i = A \cdot (c + \gamma_{i,0}) \frac{M_i}{C_{i,out}},$$

where A [h] is a time constant that measures the lifetime analysis of the park.

In the SLMF model, given a valid exchange network E_0 , each enterprise $i \in I$ has a cost function that he wants to minimize, defined by

$$\text{Cost}_i(z_i, F_i, F_{-i}, E_0) = A \left[c \cdot z_i + \left(\sum_{(k,i) \in E} \gamma_{k,i} F_{k,i} + \sum_{(i,j) \in E} \gamma_{i,j} F_{i,j} \right) + \gamma_{i,0} F_{i,0}(F_{-i}) \right], \quad (5.2.8)$$

where $\gamma_{l,m}$ [\$/T] is the unit cost for the use of shared connection between node l and m , for any $(l,m) \in E$. Observe that each enterprise pays for each connection he uses, both for the inlet fluxes and outlet fluxes. In the case of the connections between two enterprises, this means that the cost of the connection is divided uniformly between the sending enterprise and the receiving enterprise.

With all these considerations, for a given valid exchange network $E_0 \in \mathcal{E}$, the problem of each enterprise i is given by problem $P_i(z_{-i}, F_{-i}, E_0)$:

$$\begin{aligned} & \min_{z_i, F_i} \text{Cost}_i(z_i, F_i, F_{-i}, E_0) \\ & s.t. \begin{cases} \text{Equations (5.2.1) -(5.2.3) -(5.2.5),} \\ z_i \geq 0, \\ F_i \geq 0 \\ F_{i,0}(F_{-i}) \geq 0. \end{cases} \end{aligned} \quad (5.2.9)$$

Observe that constraint (5.2.4) is implicit in the expressions of $F_{i,0}(F_{-i})$ given by (5.2.6) and (5.2.7) and thus finally by the cost function expression (5.2.8). For a network $E_0 \in \mathcal{E}$, we say that a vector (z, F) is an equilibrium for the enterprises if and only if

$$\forall i \in I, (z_i, F_i) \text{ solves the problem } P_i(z_{-i}, F_{-i}, E_0).$$

We denote by $\text{Eq}(E_0)$ the set of equilibriums for the valide exchange network E_0 .

5.2.3 Designer's problem

The designer being in charge of the reduction of the environmental impact of the park, he wants to minimize the total wastewater discharge, that is function

$$F_0(F) = \sum_{i \in I} F_{i,0}(F_{-i}). \quad (5.2.10)$$

So by selecting an appropriate network E_0 and a compatible operation $(z, F) \in \text{Eq}(E_0)$, the designer will propose an optimal design for the park. Nevertheless there is no a priori guarantee that doing so the resulting cost of each enterprise will be lower than his stand-alone cost STC_i and thus that every enterprise will finally accept to participate to the EIP. Thus in order to have strong arguments to encourage enterprise to be involved into the EIP, the designer engages in a contract with them. This contract will guarantee a

minimal relative gain, with respect to the stand-alone operation, denoted by $\alpha \in]0, 1[$, on the cost of every enterprise.

$$\text{Cost}_i(z_i, F_i, F_{-i}, E_0) \leq \alpha \cdot \text{STC}_i. \quad (5.2.11)$$

This type of minimal relative gain contract was first introduced in [77].

Then, the problem of the designer is

$$\begin{aligned} & \min_{E_0 \in \mathcal{E}_0, z \in \mathbb{R}^n, F \in \mathbb{R}^{|E^{\max}|}} F_0(F) \\ & \text{s.t.} \begin{cases} (z, F) \in \text{Eq}(E_0), \\ \text{Cost}_i(z_i, F_i, F_{-i}, E_0) \leq \alpha \cdot \text{STC}_i, \quad \forall i \in I. \end{cases} \end{aligned} \quad (5.2.12)$$

The optimization problem 5.2.12 can be interpreted as follows: the designer will propose to the enterprises a valid exchange network E_0 , a vector z , and an operation $F \in \mathbb{R}^{|E^{\max}|}$ which satisfy all the physical constraints and also, such that the operation F respects:

- 1) the *incentive consistency*, in the sense that no enterprise will have incentives to unilaterally deviate from the proposal due to the constraint $(z, F) \in \text{Eq}(E_0)$;
- 2) the *individual rationality* of each enterprise, in the sense that all enterprises will participate in the network since their participation has been bought through the constraint (5.2.11).

Let us finally bring to the fore that the designer problem corresponds to the so-called optimistic approach of the SLMF game. Indeed for any given valid exchange network E_0 , the set $\text{Eq}(E_0)$ can contain more than one equilibrium and the designer choose the one which minimizes the function $F_0(F)$. For more details on the different possible approaches for SLMF games the reader can refer to [61].

5.3 Mixed-Integer programming reduction

The formulation of designer's problem (5.2.12) has the form of a *mathematical programming with equilibrium constraints* (MPEC). This section is devoted to prove that this MPEC formulation, which is known to be hard to solve (see, e.g., [72, 73, 74]), can be reformulated as a single Mixed-Integer programming problem.

5.3.1 Characterization of equilibriums

The following theorem characterizes the equilibrium set $\text{Eq}(E_0)$ as a system of equations. This allows to reduce the MPEC of problem (5.2.12) to a single optimization problem.

Let us denote by $I_{i,act} = \{k \in I \setminus \{i\} : \gamma_{k,i} = \min_{l \in I, l \neq i} \gamma_{l,i}\}$. This minimal marginal cost will be denoted by γ_i^* while $E_{i,act}$ stands for the set $\{(k, i) : i \in I_{i,act}\}$.

Theorem 5.3.1 For any valid exchange network $E_0 \in \mathcal{E}_0$ and denoting by $S(E_0)$ the set

$$S(E_0) = \left\{ (z, F) : \forall i \in I, \begin{cases} z_i + \sum_{(k,i) \in E_{i,\text{act}}} F_{k,i} = \sum_{(i,j) \in E_0} F_{i,j} + F_{i,0}(F_{-i}) \\ \sum_{(k,i) \in E_{i,\text{act}}} C_{k,\text{out}} F_{k,i} - C_{i,\text{in}} \left(z_i + \sum_{(k,i) \in E_0} F_{k,i} \right) \leq 0 \\ z_i \geq 0 \\ F_i|_{E_{i,\text{act}}^c} = 0 \\ F_i \geq 0 \\ F_{i,0}(F_{-i}) \geq 0 \end{cases} \right\} \quad (5.3.1)$$

then, one has $S(E_0) \subseteq \text{Eq}(E_0)$. Furthermore, any optimal solution (E_0, z, F) of the auxiliary mathematical programming problem

$$\begin{aligned} \min_{E_0 \in \mathcal{E}_0, z \in \mathbb{R}^n, F \in \mathbb{R}^{|E^{\text{max}}|}} F_0(F) \\ \text{s.t. } \begin{cases} (z, F) \in S(E_0) \\ \text{Cost}_i(z_i, F_i, F_{-i}, E_0) \leq \alpha \cdot \text{STC}_i, \quad \forall i \in I. \end{cases} \end{aligned} \quad (5.3.2)$$

is an optimal solution of the SLMF problem (5.2.12).

Proof. We only need to prove the inclusion $S(E_0) \subseteq \text{Eq}(E_0)$ since the second part of the statement is a direct consequence by replacing, in the formulation of (5.2.12), the constraint “ $(z, F) \in \text{Eq}(E_0)$ ” by “ $(z, F) \in S(E_0)$ ”.

Thus let us fix $(z, F) \in S(E_0)$. It is not hard to see that, for any i , (z_i, F_i) is a feasible point of $P_i(z_{-i}, F_{-i}, E_0)$.

Now, fix $i \in I$ and let (z'_i, F'_i) be another feasible point of $P_i(z_{-i}, F_{-i}, E_0)$. Then, $z'_i \geq 0$, $F'_i \geq 0$ and the water mass balance constraint (5.2.3) is satisfied.

Therefore one has

$$\begin{aligned} \text{Cost}_i(z'_i, F'_i, F_{-i}, E_0) - \text{Cost}_i(z_i, F_i, F_{-i}, E_0) &= A \left[c(z'_i - z_i) + \sum_{(k,i) \in E_0} \gamma_{ki} F'_{k,i} - \gamma_i^* \sum_{k \in I_{i,\text{act}}} F_{k,i} \right] \\ &\geq A \left[c(z'_i - z_i) + \gamma_i^* \left(\sum_{(k,i) \in E_0} F'_{k,i} - \sum_{k \in I_{i,\text{act}}} F_{k,i} \right) \right] \\ &\geq A \left[\min\{c, \gamma_i^*\} \left[\left(z'_i + \sum_{(k,i) \in E_0} F'_{k,i} \right) - \left(z_i + \sum_{k \in I_{i,\text{act}}} F_{k,i} \right) \right] \right] \\ &= 0 \end{aligned} \quad (5.3.3)$$

because, using that $F_i|_{E_{i,\text{act}}^c} = 0$ and the mass balance constraint (5.2.3) for (z'_i, F'_i) and (z_i, F_i) ,

$$z'_i + \sum_{(k,i) \in E_0} F'_{k,i} = \sum_{j \in I} F_{i,j} + F_{i,0}(F_{-i}) = z_i + \sum_{(k,i) \in E_0} F_{k,i} = z_i + \sum_{k \in I_{i,\text{act}}} F_{k,i}. \quad (5.3.4)$$

Note that equality (5.3.4) is still valid if enterprise i is in semi-stand-alone or stand-alone situation with (z_i, F_i) or/and with (z'_i, F'_i) . Thus, (z_i, F_i) solves $P_i(z_{-i}, F_{-i}, E_0)$, and since this holds for every $i \in I$, we deduce that $(z, F) \in \text{Eq}(E_0)$. \square

Even if the resolution of the SLMF game has been drastically simplified thanks to Theorem 5.3.1 and its description of a subset of solutions of the SLMF as solutions of the single optimization problem (5.3.2), an adaptation of this optimization problem is still needed since one of its variables is an exchange network.

Definition 5.3.2 (Arc Classes) *Let $(k, i) \in E^{\max}$. We define the arc class of (k, i) as the set*

$$C(k, i) := \{(l, i) \in E^{\max} : \gamma_{l,i} = \gamma_{k,i}\}.$$

We denote by \mathcal{C}_i the family of all arc classes going to i , that is, $\mathcal{C}_i = \{C(k, i) : (k, i) \in E^{\max}\}$.

Observe that, for two arcs $(k, i), (h, i) \in E^{\max}$ such that $\gamma_{k,i} = \gamma_{h,i}$, one has that $C(k, i) = C(h, i)$. Thus, a class $C \in \mathcal{C}_i$ may have many representations of the form $C(k, i)$. Furthermore, the family \mathcal{C}_i induces a partition of the set of arcs “going to” agent i , that is

- $\bigcup_{C \in \mathcal{C}_i} C = \{e \in E_{\max} : e = (k, i) \text{ for some } k \in I\}$.
- For any two classes $C, C' \in \mathcal{C}_i$, either $C = C'$ or $C \cap C' = \emptyset$.

Moreover, it is not hard to verify that for each valid exchange network $E_0 \in \mathcal{E}_0$ and for each agent $i \in I$, there exists one class $C \in \mathcal{C}_i$ such that

$$E_{i,\text{act}} \subseteq C, \quad (5.3.5)$$

This class is then given by $C = C(k, i)$ where (k, i) is any element of $E_{i,\text{act}}$. We will call it the *active class* of E of the agent i , and we will denote it by $C_i(E)$.

Now, let $D = \bigcup_{i \in I} \mathcal{C}_i$, the set of all arc classes of active agents. We introduce the boolean variable $y = (y_C)_{C \in D} \in \{0, 1\}^{|D|}$ in the following way: for each independent agent $i \in I$ and each arc class $C \in \mathcal{C}_i$, we set

$$y_C = \begin{cases} 1 & \text{if } C \text{ is the active class of } i, \\ 0 & \text{otherwise.} \end{cases}$$

From $y \in \{0, 1\}^{|D|}$, we will build the graph associated to y as

$$E_0(y) = \left(\bigcup \{C : y_C = 1\} \right) \cup \{(i, 0) : i \in I\}. \quad (5.3.6)$$

Then, we consider the following Mixed-Integer optimization problem:

$$\begin{aligned} & \min_{F, z, y} F_0(F) \\ & \text{s.t.} \begin{cases} z_i + \sum_{(k,i) \in E^{\max}} F_{k,i} = \sum_{(i,j) \in E^{\max}} F_{i,j} + F_{i,0}(F_{-i}), & \forall i \in I, \\ \sum_{(k,i) \in E^{\max}} C_{k,\text{out}} F_{k,i} \leq C_{i,\text{in}} \left(z_i + \sum_{(k,i) \in E^{\max}} F_{k,i} \right), & \forall i \in I, \\ \sum_{C \in \mathcal{C}_i} y_C = 1, & \forall i \in I, \\ \sum_{(k,i) \in C} F_{k,i} \leq K \cdot y_C, & \forall C \in D, \\ z_i \geq 0, & \forall i \in I, \\ F \geq 0, \\ F_{i,0}(F_{-i}) \geq 0, & \forall i \in I, \\ \text{Cost}_i(z_i, F_i, F_{-i}, E_0(y)) \leq \alpha \cdot \text{STC}_i, & \forall i \in I. \end{cases} \end{aligned} \quad (5.3.7)$$

Here, K is a constant large enough so all fluxes within the park, regardless the connections, are less than K . In practice, we can set K as

$$K := \sum_{i \in I} F_{i,0}(0) = \sum_{i \in I} \frac{M_i}{C_{i,\text{out}} - C_{i,\text{in}}}, \quad (5.3.8)$$

that is, the total fresh water consumption assuming that none of the enterprises works in stand-alone situation.

Theorem 5.3.3 *For every feasible point (F, z, y) of (5.3.7), the triple $(E_0(y), z, F)$ is a feasible point of (5.3.2). Conversely, for every feasible point (F, z, E_0) of (5.3.2), the triple (F, z, y^E) is a feasible point of (5.3.7), where $y^E \in \{0, 1\}^{|D|}$ is given by*

$$y_C^E = \begin{cases} 1 & \text{if } C = C_i(E_0) \text{ for some } i \in I, \\ 0 & \text{otherwise.} \end{cases}$$

As a consequence,

1. if (E_0, z, F) is an optimal solution of (5.3.2), then (F, z, y_0^E) is an optimal solution of (5.3.7).
2. if (F, z, y) is an optimal solution of (5.3.7), then $(E_0(y), z, F)$ is an optimal solution of (5.3.2).

Proof. Let (F, z, y) be a feasible point of (5.3.7). Let us fix an enterprise $i \in I$ and let C_i be the unique class in \mathcal{C}_i such that $y_{C_i} = 1$. Then, by construction, we know that

$$E(y)_{i,\text{act}} = C_i \quad \text{and} \quad \sum_{(k,i) \in E^{\text{max}} \setminus C_i} F_{k,i} \leq K \cdot \sum_{C \in \mathcal{C}_i \setminus C_i} y_C = 0.$$

We deduce then that

$$F_i|_{E(y)_{i,\text{act}}^c} = 0.$$

Since this constraint is valid for every active enterprise $i \in I$. We deduce then that $(E_0(y), z, F)$ is a feasible point of problem (5.3.2).

Now, let (E_0, z, F) be a feasible point of problem (5.3.2). By inclusion (5.3.5), for each enterprise $i \in I$, there exists a unique active class $C_i(E_0)$. Let us define $y_0^E \in \{0, 1\}^{|D|}$ as in the statement of the theorem.

Then, for every $i \in I$, $\sum_{C \in \mathcal{C}_i} y_C^{E_0} = 1$. Now, fix a class $C \in D$, and let $i \in I$ such that $C \in \mathcal{C}_i$. We have that

$$\sum_{(k,i) \in C} F_{k,i} \leq \begin{cases} K \cdot y_C^{E_0} & \text{if } C = C_i(E_0), \\ 0 = K \cdot y_C^{E_0} & \text{if } C \neq C_i(E_0), \end{cases}$$

where the second inequality comes from the fact that, whenever $C \neq C_i(E_0)$, then $C \subseteq E_{i,\text{act}}^c$ and so $F_i|_C = 0$.

For an enterprise $i \in I$, the fact that $E_{i,\text{act}} \subseteq E(y^{E_0})$ lead us to the fact that

$$\text{Cost}_i(z_i, F_i, F_{-i}, E_0(y^{E_0})) = \text{Cost}_i(z_i, F_i, F_{-i}, E_0),$$

and so, the constraint (5.2.11) is satisfied with $E_0(y^{E_0})$. We deduce that (F, z, y^{E_0}) is a feasible point of (5.3.7).

The two last implications of the theorem follows directly from the above developments. \square

5.3.2 Involved enterprises: an automatic process

Physically, we know that the network has always a feasible point, which is the Stand-Alone configuration, that is, the topology E^{st} . However, when we include the individual rationality constraint (5.2.11), the problem may become infeasible since $\alpha < 1$.

Infeasibility of problem (5.3.7) means that the designer is not capable to find a solution that respect the contracts with all the enterprises. Thus, we need to include the possibility of excluding some enterprises from the network.

Formally, for each enterprise $i \in I$, we define a boolean variable $y_{i,\text{null}} \in \{0, 1\}$ such that

$$y_{i,\text{null}} = \begin{cases} 1 & \text{if } i \text{ breaks the contract (5.2.11),} \\ 0 & \text{otherwise.} \end{cases}$$

and denote by y_{null} the vector $y_{\text{null}} = (y_{i,\text{null}})_{i \in I}$. With this new variable, we modify problem (5.3.7) adding the following constraints

1. For each enterprise $i \in I$, we put

$$y_{i,\text{null}} + \sum_{C \in \mathcal{C}_i} y_{C_i} = 1,$$

meaning that, either one arc class is active or the enterprise is outside the network.

2. For each enterprise $i \in I$, we put

$$\sum_{(i,j) \in E^{\text{max}}} F_{i,j} \leq K \cdot (1 - y_{i,\text{null}}).$$

This is to ensure that, if the enterprise breaks the contract, then he cannot send his polluted water to any enterprises.

3. For each enterprise $i \in I$, we put

$$\sum_{(k,i) \in E^{\text{max}}} F_{k,i} \leq K \cdot (1 - y_{i,\text{null}}).$$

This constraint establishes that, if the enterprise breaks the contract, then nobody can send him any flux.

4. For each enterprise $i \in I$, we put

$$\text{Cost}_i(z_i, F_i, F_{-i}, E_0(y)) \leq \alpha \text{STC}_i \cdot (1 - y_{i,\text{null}}) + \text{STC}_i \cdot y_{i,\text{null}}.$$

Here, the individual rationality constraint is active only when $y_{i,\text{null}} = 0$. Otherwise, since the enterprise is not connected to the network, his cost will coincide with STC_i .

Denoting

$$\text{STC}_i(y_{i,\text{null}}) := \alpha \text{STC}_i \cdot (1 - y_{i,\text{null}}) + \text{STC}_i \cdot y_{i,\text{null}},$$

the new optimization problem becomes

$$\begin{aligned}
& \min_{F, z, y, y_{\text{null}}} F_0(F) \\
& \text{s.t.} \left\{ \begin{array}{l}
z_i + \sum_{(k,i) \in E^{\text{max}}} F_{k,i} = \sum_{(i,j) \in E^{\text{max}}} F_{i,j} + F_{i,0}(F_{-i}), \quad \forall i \in I, \\
\sum_{(k,i) \in E^{\text{max}}} C_{k,\text{out}} F_{k,i} \leq C_{i,\text{in}} \left(z_i + \sum_{(k,i) \in E^{\text{max}}} F_{k,i} \right), \quad \forall i \in I, \\
y_{i,\text{null}} + \sum_{C \in \mathcal{C}_i} y_C = 1, \quad \forall i \in I, \\
\sum_{(k,i) \in C} F_{k,i} \leq K \cdot y_C, \quad \forall C \in D, \\
\sum_{(i,k) \in E^{\text{max}}} F_{i,k} \leq K \cdot (1 - y_{i,\text{null}}), \quad \forall i \in I, \\
\sum_{(k,i) \in E^{\text{max}}} F_{k,i} \leq K \cdot (1 - y_{i,\text{null}}), \quad \forall i \in I, \\
z_i \geq 0, \quad \forall i \in I, \\
F \geq 0, \\
F_{i,0}(F_{-i}) \geq 0, \quad \forall i \in I, \\
\text{Cost}_i(z_i, F_i, F_{-i}, E_0(y)) \leq \text{STC}_i(y_{i,\text{null}}), \quad \forall i \in I.
\end{array} \right. \quad (5.3.9)
\end{aligned}$$

Observe that, whenever $y_{i,\text{null}} = 0$, then all constraints for the i th enterprise are the same that those established in problem (5.3.7). Also, if $y_{i,\text{null}} = 1$, the only feasible solution for i is the stand-alone operation. Thus, in this new problem, the designer first choose all the enterprises that will participate in the network, represented by the set

$$I' = \{i \in I : y_{i,\text{null}} = 0\},$$

and then it solves problem (5.3.7) replacing I by I' . Of course, as it is formulated, the designer takes both decisions simultaneously, by solving problem (5.3.9). It is not hard to verify that any optimal solution of problem (5.3.9) is an optimal solution of Problem (5.3.7) (and thus also of the associated SLMF problem) for the reduced set of enterprises I' . We leave this verification to the reader.

5.4 Numerical experiments

In this section we present numerical examples of the control-input model for water exchange networks in Eco-Industrial parks. To simplify, we assume that the unit cost of using a sharing connection between two nodes is uniform. More precisely,

$$\gamma_{l,m} = \delta, \text{ for all } l, m \in I \quad \text{and} \quad \gamma_{i,0} = \delta, \text{ for all } i \in I. \quad (5.4.1)$$

Thus, for each enterprise $i \in I$, the family of all arc classes going to i defined by $\mathcal{C}_i = \{C_{p,i}\} = \{(k,i) \in E^{\text{max}} : k \in I\}$.

Now, for each agent $i \in I$, we include two integer variables, $y_{p,i}, y_{i,\text{null}} \in \{0, 1\}$ with the following interpretation:

- If $y_{p,i} = 1$, it means that the connections in $C_{p,i}$ are included in the network.

- If $y_{i,\text{null}} = 1$, it means that the connection $(i, 0)$ is the only exit connection for i , and i does not participate in the EIP (he works in stand-alone mode).

Since the optimization problem (5.4.3) may have several optimal solutions, we introduce a penalization term in the objective function in order to obtain the solutions involving more participating enterprises. We replace $F_0(F)$ by

$$F_0(F) + \text{Coef} \cdot \sum_{i \in I} y_{i,\text{null}}, \quad (5.4.2)$$

where $\text{Coef} \geq 0$ is a coefficient to penalize those optimal solutions that leave more enterprises outside the park. The choice of an appropriate value for Coef will be discussed in the forthcoming subsection 5.4.3.

Finally it's worth mentioning that we will add the constraint $F_{i,i} = 0$ for all $i \in I$ to avoid that enterprises send his polluted water to himself. Therefore, the considered problem is the following Mixed Integer Linear Programming problem

$$\begin{aligned} \min_{F, z, y} \quad & F_0(F) + \text{Coef} \cdot \sum_{i \in I} y_{i,\text{null}} \\ \text{s.t.} \quad & \left\{ \begin{array}{ll} z_i + \sum_{(k,i) \in E^{\text{max}}} F_{k,i} = \sum_{(i,j) \in E^{\text{max}}} F_{i,j} + F_{i,0}, & \forall i \in I, \\ \sum_{(k,i) \in E^{\text{max}}} C_{k,\text{out}} F_{k,i} \leq C_{i,\text{in}} \left(z_i + \sum_{(k,i) \in E^{\text{max}}} F_{k,i} \right), & \forall i \in I, \\ y_{p,i} + y_{i,\text{null}} = 1, & \forall i \in I, \\ \sum_{(k,i) \in C_{p,i}} F_{k,i} \leq K \cdot y_{p,i}, & \forall i \in I, \\ \sum_{(i,k) \in E^{\text{max}}} F_{i,k} \leq K \cdot (1 - y_{i,\text{null}}), & \forall i \in I, \\ \sum_{(k,i) \in E^{\text{max}}} F_{k,i} \leq K \cdot (1 - y_{i,\text{null}}), & \forall i \in I, \\ z_i \geq 0, & \forall i \in I, \\ F \geq 0, & \\ F_{i,0}(F_{-i}) \geq 0, & \forall i \in I, \\ F_{i,i} = 0, & \forall i \in I, \\ \text{Cost}_i(z_i, F_i, F_{-i}, E_0(y)) \leq \text{STC}_i(y_{i,\text{null}}), & \forall i \in I. \end{array} \right. \end{aligned} \quad (5.4.3)$$

All simulations have been implemented with Julia v1.3.1 programming language using Gurobi v9.0.1 as solver.

5.4.1 Results and discussion of case study

The study case which we present here consists on an EIP made up of 15 enterprises, each one including only one process. It is assumed that the EIP operates for one hour, that is, $A = 1$ h.

Prices are shown in Table 5.1 and the data of 15 enterprises is given in Table 5.2.

Parameter	Value (\$/tonne)
c	0.13
β	0.22
δ	0.01

Table 5.1: Associated costs.

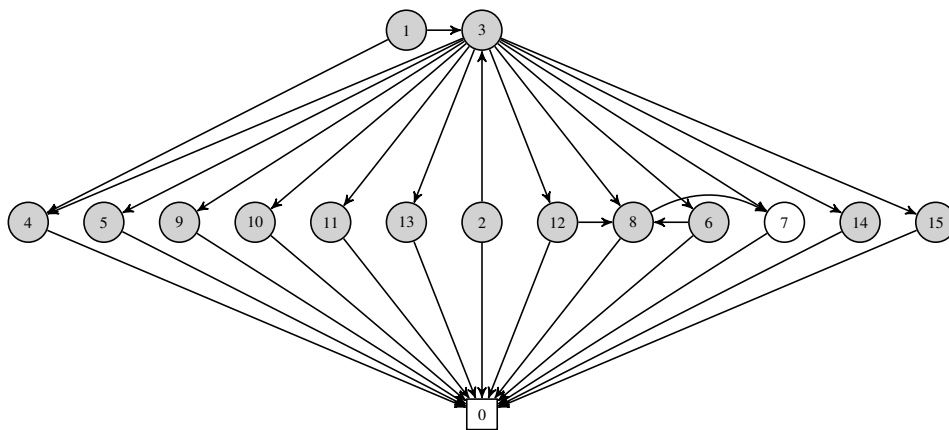
Enterprise	$C_{i,in}$ (ppm)	$C_{i,out}$ (ppm)	M_p (g/h)
1	0	100	7500
2	0	200	6000
3	50	100	8000
4	80	800	30000
5	90	400	4000
6	20	900	8000
7	150	1000	40000
8	80	200	7000
9	70	800	20000
10	90	900	9000
11	30	800	6000
12	25	500	2000
13	25	750	5000
14	50	1000	30000
15	60	700	13000

Table 5.2: Process data for the water allocation problem.

Here, we present the results of the simulations with the above data. The resulting optimized EIP network is shown in Figure 5.4.1, and it corresponds to $\alpha = 0.95$ and $\text{Coef} = 1$. This optimal network provides operating cost of each enterprise and total freshwater consumption that are lower than a stand-alone network as shown in Table 5.3.

Enterprise	Freshwater Stand-Alone (T/h)	Freshwater in EIP (T/h)	Cost _i Stand-Alone (MMUSD/hour)	Cost _i in EIP (MMUSD/hour)	% Reduction in Cost _i
1	75.00	75.00	26.25	10.50	60.00
2	30.00	30.00	10.50	8.04	23.47
3	80.00	91.74	28.00	14.21	49.26
4	37.50	8.33	13.13	10.58	19.37
5	10.00	1.29	3.50	3.12	10.78
6	8.88	7.27	3.11	2.96	5.00
7	40.00	0.00	14.00	10.82	22.69
8	35.00	12.31	12.25	9.95	18.76
9	25.00	8.22	8.75	7.29	16.71
10	10.00	1.11	3.50	2.69	23.17
11	7.50	5.45	2.63	2.45	6.79
12	4.00	3.16	1.40	1.33	5.00
13	6.67	5.17	2.33	2.21	5.42
14	30.00	15.79	10.50	9.16	12.78
15	18.57	8.13	6.50	5.65	13.13
Total	418.13	272.97	146.34	100.95	31.02

Table 5.3: Summary of results of the EIP.

Figure 5.4.1: The optimal configuration in the case $\alpha_i = 0.95$ and $\text{Coef} = 1$. Gray nodes are consuming strictly positive fresh water.

For this case study, when the enterprises operate stand alone, then the entire system consumes a total 418.13 (T/h) of freshwater and, of course, generates 418.13 (T/h) of wastewater. The optimal design obtain with SLMF approach allows to reduce its freshwater requirement to 272.92 (T/h), and thus generates 272.92 (T/h) of wastewater, which is equivalent to a reduction of 34.73 %. Furthermore, the water demand of enterprise 7 is entirely supplied by other enterprises.

Each participating enterprise has a cost reduction of at least 5%, as promised by the contract. Total operating cost is decreased regarding the standalone case, as expected, from 146.34 (\$/h) to 100.95 (\$/h), which means a decrease of 31.02%. Nevertheless, exploring Table 5.3, the reader can observe that the cost reduction is not uniform among the enterprises. In the optimized network, enterprise 1 achieves the highest percentage reduction of operating cost corresponding to 60.00% while enterprises 6 and 12 have

the lowest reduction corresponding to 5.00% with respect to the stand-alone configuration. Moreover this repartition of the relative gain change with the contract, that is with the value of α . Figure 5.4.2 is a box plot to describe the distribution of the cost reduction of participating enterprises in the park with parameters α in the interval $[0.80, 0.99]$.

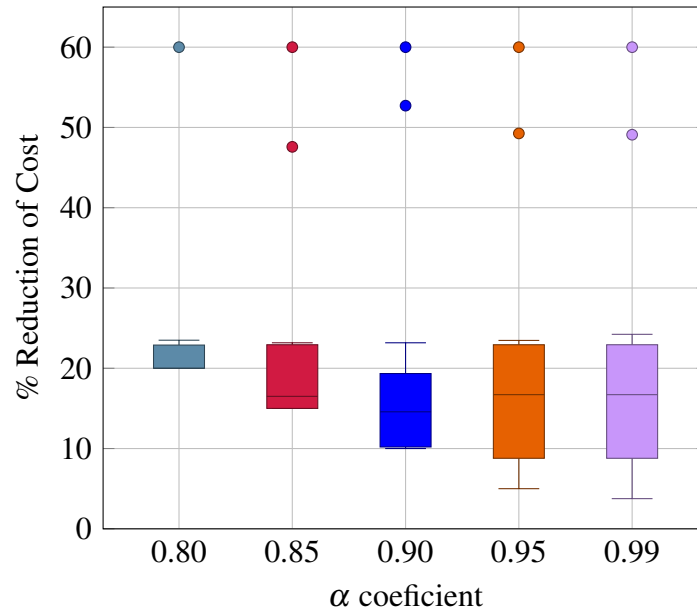


Figure 5.4.2: % Reduction of cost with $\alpha \in [0.80, 0.99]$ and Coef = 1. The small circles are drawn on the chart to indicate outlier points.

A box plot is a standardized way of displaying the distribution of data based on a five number summary: lower limit, lower quartile (Q_1 : 25th Percentile), median (Q_2 : 50th Percentile), upper quartile (Q_3 : 75th Percentile), and upper limit. The difference ($Q_3 - Q_1$) is called the interquartile range or IQR. If an observation falls outside of the interval

$$[\text{lower limit}, \text{upper limit}] = [Q_1 - 1.5 \times \text{IQR}, Q_3 + 1.5 \times \text{IQR}],$$

then it is considered as an outlier. Outliers are usually treated as abnormal values that can affect the overall observation due to its very high or low extreme values. To understand more about the box plot, we refer to [114]. Let us choose $\alpha = 0.95$ to analyze the distribution of the cost reduction of enterprises, and other parameters α can be analyzed analogously. All enterprises will reduce less than or equal to 44.15% (upper limit). All enterprises can reduce operating costs by at least 5% (lower limit). At least 75% of enterprises can reduce costs by 8.79% or higher. The lower (resp. upper) quartile Q_1 (resp. Q_3) is 8.79% (resp. 22.93%) which is equivalent to that there are 25% (resp. 75%) of enterprises can reduced their operating costs less than or equal to 8.79% (resp. 22.93%), while the median reduction Q_2 in this case is 16.71%, so exactly half the enterprises are reduced lesser or higher 16.71%. compared to their stand-alone operating costs. There are two outlier points outside the boxplot: 49.25% and 60.00%, respectively.

The impact of the value of the contract coefficient α on the total cost, total wastewater discharge or the number of stand alone enterprises will be illustrated in the forthcoming subsection 5.4.2. Nevertheless it is interesting to compare the optimal design (network and table of results) obtained with a more ambitious contrat guaranteeing a relative gain of 14%, that is with $\alpha = 0.86$.

Enterprise	Freshwater Stand-Alone (T/h)	Freshwater in EIP (T/h)	Cost _i Stand-Alone (MMUSD/hour)	Cost _i in EIP (MMUSD/hour)	% Reduction in Cost _i
1	75.00	75.00	26.25	10.50	60.00
2	30.00	30.00	10.50	9.03	14.00
3	80.00	82.50	28.00	13.22	52.80
4	37.50	8.33	13.13	10.58	19.37
5	10.00	1.29	3.50	3.01	14.00
6*	8.88	8.89	3.11	3.11	0.00
7	40.00	8.47	14.00	11.84	15.43
8	35.00	11.67	12.25	10.54	14.00
9	25.00	9.83	8.75	7.48	14.51
10	10.00	1.11	3.50	2.69	23.17
11*	7.50	7.50	2.63	2.63	0.00
12*	4.00	4.00	1.40	1.40	0.00
13*	6.67	6.67	2.33	2.33	0.00
14	30.00	15.79	10.50	9.03	14.00
15	18.57	8.13	6.50	5.59	14.00
Total	418.13	279.16	146.34	102.97	29.64

Table 5.4: Summary of optimal results of the EIP with $\alpha = 0.86$. Marked enterprises (*) are left outside the park, that is operating stand-alone.

A first observation that we can do is that the total freshwater consumption (=wastewater discharge) and the total cost obtained with $\alpha = 0.86$ are almost the same as the one with $\alpha = 0.95$. But by guaranteeing an higher relative gain for each involved enterprise, four enterprises (6, 11, 12 and 13) are not included in the EIP network because their characteristics ($M_i, C_{i,in}, C_{i,out}$) would “penalize” the optimal design. Thus the increase of relative gain only benefit to selected subset of enterprises.

The optimal network obtained with $\alpha = 0.86$ is also quite different from the one corresponding to $\alpha = 0.95$. For example the enterprise 7 which was not using fresh water but was reusing water from enterprises 3 and 8 for $\alpha = 0.95$, now received water from five different enterprises and moreover use 40 (T/h) of fresh water.

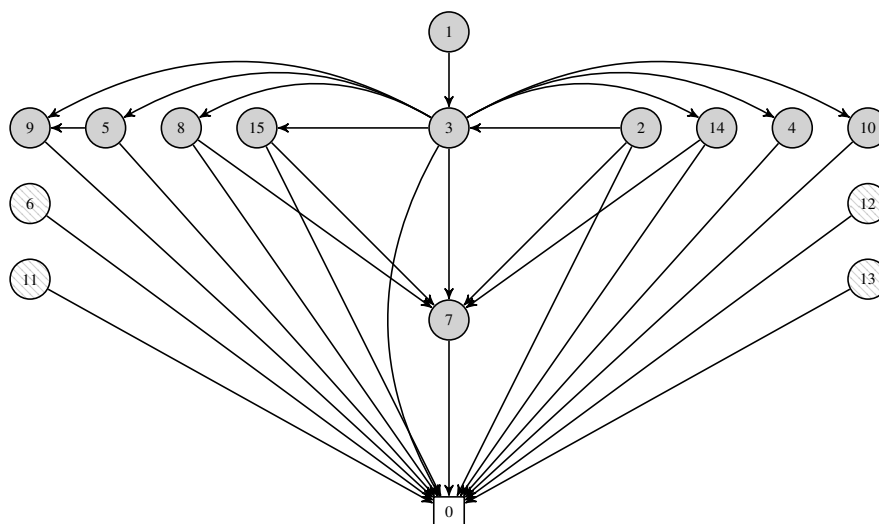


Figure 5.4.3: The optimal configuration in the case $\alpha_i = 0.86$ and Coef = 1. Gray nodes are consuming strictly positive fresh water. Dashed nodes are operating in stand-alone mode.

5.4.2 Comparison between the control input model and the blind input model

One of the goals of the control input model is to propose an alternative approach to the blind input model developed in [77]. It is therefore important to compare the optimal results of both approaches. The criteria we used for comparing the two models are as follows: first, the ability to reduce freshwater consumption and thus wastewater; second, the number of enterprises involved into the optimal EIP and last, the ability to reduce the operating costs of each enterprise. These criteria will be discussed step by step.

First of all and for both models, the total of freshwater consumption and wastewater discharge decreases when α increases as shown in Figure 5.4.4. Note that only the wastewater discharge curve is included below because, of course, both are the same. But the control input model allows a strong reduction of freshwater consumption and wastewater discharge compare to the blind input model. One of the reason why is that, as it will be see in Figure 5.4.4 for the same relative cost reduction α there are always more enterprises participating in the EIP for control input model than with the blind input model. In the optimized networks, the control input model achieves a minimum total of freshwater consumption as well as wastewater discharge is 272.71 (T/h) corresponding to 34.78% with respect to the stand-alone configuration, while the blind input model achieves a minimum total freshwater consumption is 318.67 (T/h) corresponding to 23.78% with respect to the stand-alone configuration.

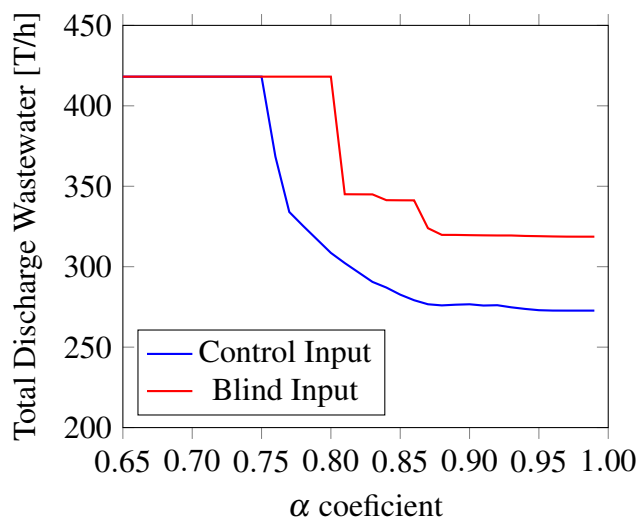


Figure 5.4.4: Sensitivity analysis for $\alpha \in [0.65, 0.99]$ and Coef = 1. Total discharge wastewater.

The number of enterprises which operate stand-alone for both models is shown in Figure 5.4.5. Roughly speaking, for $\alpha \in [0.65, 0.99]$, the number of enterprises operating stand-alone with control inputs is always less than that of enterprises with blind inputs. Moreover, with $\alpha \in [0.76, 0.80]$, the control input model shows its clear superiority on the blind input model. Indeed with the control inputs the designer can build a park for which not only a wastewater discharge reduction is achieved compare to blind input results but also the designer can attracts the exigent enterprises by guaranteeing a relative gain of more than 20% while with such a rate blind input model only propose full stand alone situation. Another interesting feature is that if $\alpha \geq 0.92$ then the control input model can ensure that all enterprises will participate in the park while blind input model reach this full involvement only for $\alpha > 0.96$. Finally, for $\alpha \leq 0.75$, the optimal solution is the stand-alone configuration for both models, thus 0.75 playing the role of a threshold value for the relative gain.

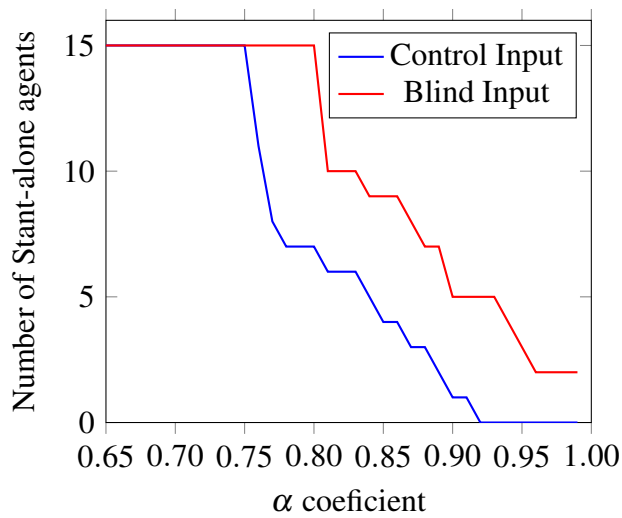


Figure 5.4.5: Sensitivity analysis for $\alpha \in [0.65, 0.99]$ and Coef = 1. Number of stand-alone enterprises in the park..

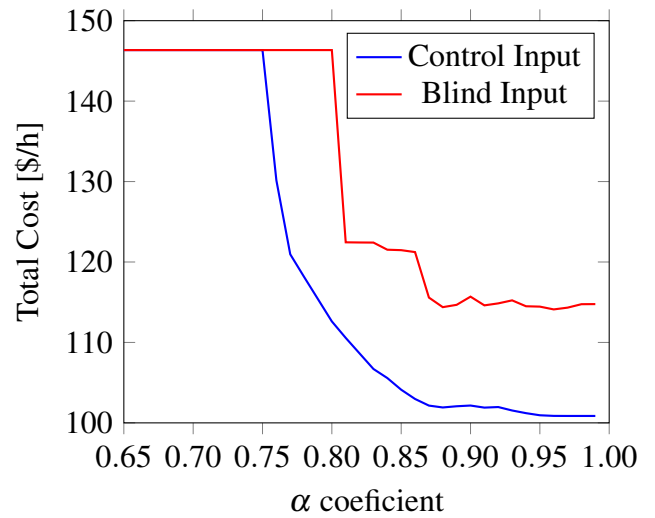


Figure 5.4.6: Sensitivity analysis for $\alpha \in [0.65, 0.99]$ and Coef = 1. Total operating cost in the park.

As observed from Figure 5.4.6, the total operating cost in the EIP with control inputs is always less than the one with blind inputs. The behavior of the total cost with both models is declining evenly. In the optimized networks, the control input model achieves a minimum total operating cost is 100.86 (\$/h) corresponding to 31.08% with respect to the stand-alone configuration, while the blind input model achieves a minimum total operating cost is 114.10 (\$/h).

5.4.3 Sensitivity analysis for coefficient Coef

The aim of the penalization of the objective function of the designer (representing the total wastewater discharge) by the term $\text{Coef} \cdot \sum_{i \in I} y_{i, \text{null}}$ is to force the algorithm to select optimal solution for which the maximum of enterprises are involved into the EIP. Figure 5.4.7 confirms the effect of this penalization since the number of enterprise in stand alone situation decreases as the coefficient Coef increases. But on the other hand figures 5.4.8 and 5.4.9 bring to the fore the somehow intuitive drawback that by encouraging solution with more enterprises, the total discharged water and the total cost of the park naturally increase. Indeed the “additional enterprises” involved thanks to the penalization have not favorable characteristics $(M_i, C_{i, \text{in}}, C_{i, \text{out}})$ and thus they penalize, in a sense, the efficiency of the EIP.

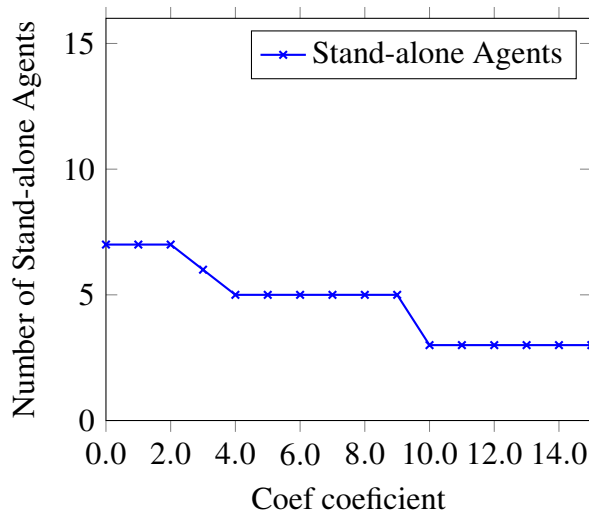


Figure 5.4.7: Sensitivity analysis for $\text{Coef} \in [0, 15]$ and $\alpha = 0.80$. Number of stand-alone enterprises in the park.

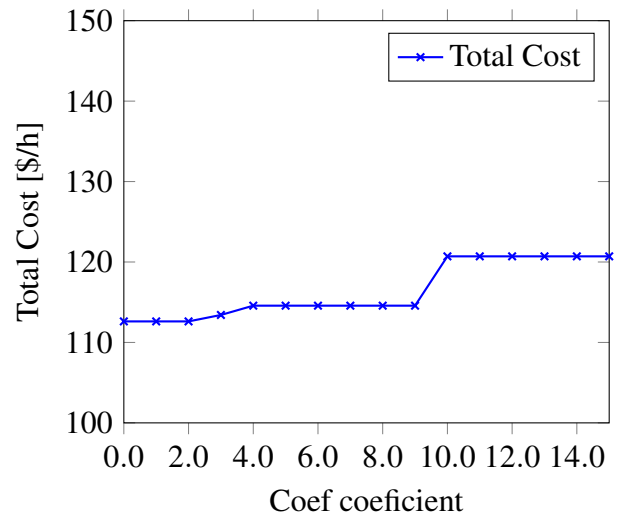


Figure 5.4.8: Sensitivity analysis for $\text{Coef} \in [0, 15]$ and $\alpha = 0.80$. Total operating cost in the park.

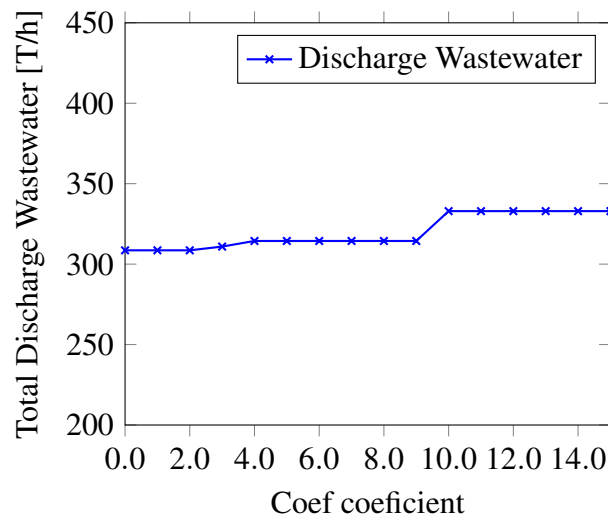


Figure 5.4.9: Sensitivity analysis for $\text{Coef} \in [0, 15]$ and $\alpha = 0.80$. Total discharge wastewater.

5.5 Conclusion and perspectives

In this work, we address the design and optimization of water exchange networks in eco-industrial parks by formulating and solving Single-Leader-Multi-Follower games. Using the particular characteristics of the SLMF model, we show that some solutions can be found by simply solving an auxiliary Mixed-Integer Linear programming problem, allowing thus to tackle larger EIP networks. In our model, we consider that each participating enterprise can control the amount of water coming from other enterprises. In another word, when participating in the network, each enterprise can control his input flux, which is more realistic

than other models in the literature [2, 77]. The results show that the proposed approach is efficient. Indeed, the total discharge wastewater (=the total freshwater consumption) and total network cost in the optimal network have been reduced by 34.73% and 31.02% respectively in which the designer ensures that each participating enterprise has a cost reduction of at least 5%.

Nevertheless our approach use the assumptions that each enterprise only manages one industrial process and that no regeneration units are used. This represents a limitation of our work, in particular concerning the second hypothesis since as it has been emphasized in [2] and [77], the introduction of regeneration units usually allows some significant improvements of the results. It would be thus interesting, but out of the scope of this work, to analyse to which extend it could be possible to use our calculus and reformulation for an EIP in which either enterprises handle more than one processes or regeneration units are involved.

Conclusion and Perspectives

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6.1 Conclusion

Rounding this thesis, we would like to collect the new results presented here. The results presented in this thesis relate to a number of topics of quasi-variational inequality problems, generalized Nash equilibrium problems, and the modeling of eco-industrial parks, respectively. Of course, each topic is related to each other.

In the first part, we have considered quasi-variational inequality problems over product sets considering the assumptions of quasi-monotonicity and upper sign-continuity only in the component operators. One of the most important difficulties, in order to obtain the existence results for quasi-variational inequalities over product sets, is that quasi-monotonicity and upper sign-continuity are not preserved by the product of set-valued maps. However, by introducing the new notion of net-lower sign-continuity, which is used as a minimal hypothesis in order to obtain the stability result of Proposition 2.3.9, and employing the well known Kakutani's fixed point theorem, we have overcome these difficulties and successfully established the existence results for the solution of our problem in the infinite dimensional setting. These classic results open the door to powerful applications to Nash equilibrium problems and generalized Nash equilibrium problems, since it is well known that they can be reformulated as variational and quasi-variational inequalities over product sets, respectively (see, e.g., [32]). Thus it become the main motivation for us to work on generalized Nash equilibrium problem. More precisely, we have considered the existence solutions for generalized Nash equilibrium problem in finite dimensional setting. Our existence results are an extension of the classic result of Debreu [33]. Specifically, we eliminate the continuous assumption of θ_v payoff function in [33] by replacing it by a "continuity" of the sublevel sets of the payoff functions. Our approach is based on the concept of the *adjusted normal operator* [37], the *net-lower sign-continuity*[45], and the reformulation of the generalized Nash equilibrium problem in term of quasi-variational inequality over product sets.

In the second part, we design and optimize the industrial water networks in eco-industrial parks formulating and solving multi-leader-follower game problems. Indeed, multi-leader-follower games are actually particular cases of generalized Nash games. Each follower's objective is to minimize the operating cost at the lower-level problem, while the leader's objective is to minimize the total consumption of freshwater within the ecopark at the upper-level problem. We propose in this thesis two models for designing and optimizing the water exchange networks in eco-industrial parks: the first model, we assume that each enterprise can only control his consumption of freshwater and his output flux, with this type of model we call it is *Blind Input model*. The second model, we assume that each enterprise has control over his polluted input, namely, each enterprise has the ability to set the amount of water coming from the other enterprises, with this type of model we call it is *Control Input model*. Models for exchange networks could lead to quite difficult problems like multi-leader-follower game problems. However, using the particular characteristics of the multi-leader-follower model, we show that some solutions can be found by simply solving an auxiliary mixed-integer linear programming problem. This clearly allows to tackle large scale problems efficiently and propose exchange politics that attract enterprises to participate. The results show that both models are efficient and practical for modeling eco-industrial parks. A comparison between the blind input model and the control input model is considered. And the results show that the control input model has better results than the blind input model.

6.2 Perspectives

There are still multiple aspects which could lead to further developments and research. An important generalization that would be to consider is the type of equilibrium and quasi-equilibrium problem over product sets. The existence results for generalized Nash equilibrium problems in the infinite-dimensional setting are actually challenging for the researcher as well as numerical methods to solve such kinds of problems. So, it would be interesting to consider these developments in future research. Eco-industrial parks are a hot topic for researchers. There are many questions for this topic, especially for the blind input model and the control input model. For example, what would we do if we considered more than one process for each enterprise for both models? or what happens to the model when instead of a single contaminant, multiple contaminant information is to be handled for both models? So far, the control input model only works for the case without regeneration units, thus it could be a possibility to extend the control input model for the case with regeneration units.

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Titre: Modèles Multi-leader-follower: analyse théorique, numérique et application aux parc éco-industriels

Mots clés: optimisation, inégalité quasi-variationnelle, équilibres de Nash, multi-leader-follower game, parc éco-industriel.

Résumé: Les travaux de recherche présentés dans cette thèse portent sur deux sujets: nous nous sommes d'abord concentrés sur les problèmes d'inégalités quasi-variationnelles et sur les problèmes d'équilibre de Nash généralisés; nous avons ensuite étudié la modélisation et l'optimisation des parcs éco-industriels (EIP).

Dans la première partie, nous obtenons des résultats d'existence d'inégalités quasi-variationnelles sur des ensembles produits en dimension infinie en considérant les hypothèses de quasimonotonie et de continuité locale upper-sign uniquement sur les opérateurs de composants. Une des difficultés les plus importantes, pour obtenir les résultats de l'existence d'inégalités quasi-variables sur des ensembles de produits, est que la quasimonotonie et la continuité du signe supérieur local ne sont pas préservées par le produit de multiapplications. En outre, des résultats d'existence pour les problèmes d'équilibre de Nash généralisés sont également obtenus dans cette partie grâce au concept d'opérateur normal ajusté et à la reformulation des problèmes d'équilibre de Nash généralisés en termes d'inégalités quasi-variables.

Dans la deuxième partie, nous concevons et optimisons des réseaux d'eau dans les parcs éco-industriels en les formulant et en les résolvant grâce à des problèmes multi-leader-follower. Ce type de modèle est un mélange entre un problème d'optimisation à deux niveaux et un problème d'équilibre de Nash généralisé. L'objectif de chaque entreprise est de minimiser le coût d'exploitation au problème de niveau inférieur, tandis que l'objectif de l'autorité EIP, au problème de niveau supérieur, est de minimiser la consommation totale d'eau douce au sein de l'écoparc. Nous concevons et optimisons les réseaux d'échange d'eau dans l'EIP selon deux approches : premièrement, dans le modèle *Blind Input*, nous supposons que chaque entreprise ne peut contrôler que sa consommation d'eau douce et son flux de sortie. Dans la seconde approche, appelée *Control Input*, nous supposons que chaque entreprise a le contrôle de ses entrées polluées, c'est-à-dire que chaque entreprise a la possibilité de fixer la quantité d'eau provenant des autres entreprises. Les résultats montrent que les deux approches proposées sont efficaces. Une comparaison entre le modèle *Blind Input* et le modèle *Control Input* est également menée.

Title: Multi-leader-follower models: theoretical analysis, simulation and application to eco-industrial parks

Keywords: optimization, quasi-variational inequality, Nash equilibrium, multi-leader-follower game, eco-industrial park.

Abstract: The research work presented in this thesis covers two topics: first, we focused on quasi-variational inequality problems and generalized Nash equilibrium problems; and second, we studied the modeling and optimization of the eco-industrial park (EIP).

In the first part, we obtain the existence results for quasi-variational inequality problems over product sets in the infinite-dimensional setting considering the assumptions of quasi-monotonicity and local upper sign-continuity only in the component operators. One of the most important difficulties, in order to obtain the existence results for quasi-variational inequalities over product sets, is that quasi-monotonicity and local upper sign-continuity are not preserved by the product of set-valued maps. Furthermore, existence results for generalized Nash equilibrium problems are also obtained in this part through the concept of adjusted normal operator and reformulation of the generalized Nash equilibrium problem in terms of quasi-variational inequalities.

In the second part, we design and optimize the industrial water networks in eco-industrial parks formulating them and solving them thanks to multi-leader-follower game problems. This kind of model is a mixture between the bilevel optimization problem and the generalized Nash equilibrium problem. Each enterprise's objective is to minimize the operating cost at the lower level problem, while the objective of the EIP authority, at the upper-level problem, is to minimize the total consumption of freshwater within the eco-park. We design and optimize the water exchange networks in EIP with two approaches: first, in the *Blind Input model*, we assume that each enterprise can only control his consumption of freshwater and his output flux. In the second approach, called *Control Input model*, we assume that each enterprise has control over his polluted inputs, namely, each enterprise has the ability to set the amount of water coming from the other enterprises. The results show that both proposed approaches are efficient. A comparison between the blind input model and the control input model is also considered.