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Anis Dhifaoui

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Équations de Stokes en domaine extérieur avec des conditions aux limites de type Navier

Thèse de doctorat de
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&
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(mention Mathématiques)

par

Anis DHIFAOUI

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UNIVERSITÉ DE FRANCHE-COMTÉ • CNRS • UMR 6623

Une civilisation sans la Science, ce serait aussi absurde qu'un poisson sans bicyclette.

(Pierre Desproges)

Une démonstration n'est pas autre chose que la résolution d'une vérité en d'autres vérités déjà connues.

(Leibniz)

Running water has within itself an infinite number of movements which are greater or less than its principal course.

(Leonardo da Vinci)

Dédicaces

À mes parents

ma mère Wanissa et mon père Neji
qui ont toujours souhaité ma réussite.

À tous mes proches

Zied, Blegacem, Alya, Wissem, Ahlem, Issa, Rim,
Alaeddine, Nesrin, Syrin, Abir, Amal, Rawen,
Ahmed, Oussema, Soulaymen, Abir, Manar ...

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Contents

Notations **1**

I Introduction général **3**

II Basic Concepts on Weighted Sobolev spaces **11**

1 Notation and Weighted Sobolev Spaces 11
2 Some Properties 13
3 Preliminaries Results 16
4 The Navier boundary conditions and related properties 19
5 Weighted Korn's inequalities 22

III Some results concerning the Laplace and the Stokes problems **27**

1 The Laplace's equation 27
1.1 The Laplace operator in the whole space $\mathbb{R}^3$ 28
1.2 The Laplace problem with Dirichlet boundary condition 28
1.3 The Laplace problem with Neumann boundary condition 30
2 The Stokes problems 32
2.1 The Stokes problem in the whole space $\mathbb{R}^3$ 33
2.2 The Stokes problem with Dirichlet boundary conditions in bounded domain 34
2.3 A mixed Stokes problem 34

IV Weighted Hilbert spaces for the Stationary exterior Stokes problem with Navier slip boundary conditions **37**

1 Introduction 37
2 Variational solution 38
3 Strong solutions 42

4 Very weak solution 53

V Stokes problem with Navier boundary conditions in L^p-theory **63**

1 Introduction 63

2 Strong solution in $W^{2,p}_{k+1}(\Omega) \times W^{1,p}_{k+1}(\Omega)$ 63

3 Very weak solutions in $W^{0,p}_{-2}(\Omega) \times W^{-1,p}_{-2}(\Omega)$ 73

Références bibliographiques **81**

Notations

Notations générales

$\mathbb{N}$	L'ensemble des nombres entiers naturels
$\mathbb{Z}$	L'ensemble des nombres entiers relatifs
$\mathbb{R}^3$	Espace euclidien muni de sa norme usuelle notée $\lvert \cdot \rvert$
Ω	Ouvert de $\mathbb{R}^3$
$\partial\Omega$ ou Γ	Frontière de Ω
$\boldsymbol{n}$	Normale unitaire extérieure de Ω
$\boldsymbol{x} = (x_1, x_2, x_3)$	Elément de Ω
$\boldsymbol{v} = (v_1, v_2, v_3)$	Champ de vecteurs
$p > 1$ ou q	Exposant de Lebesgue
$p' > 1$	Exposant conjugé de p vérifiant $\frac{1}{p} + \frac{1}{p'} = 1$
$\boldsymbol{x} \cdot \boldsymbol{y}$	Produit scalaire de $\boldsymbol{x}$ et $\boldsymbol{y}$
$\boldsymbol{x} \times \boldsymbol{y}$	Produit vectoriel de $\boldsymbol{x}$ et $\boldsymbol{y}$
$\boldsymbol{A} : \boldsymbol{B}$	Produit matriciel de $\boldsymbol{A}$ et $\boldsymbol{B}$
$r = \lvert\boldsymbol{x}\rvert$	Norme de $\boldsymbol{x} \in \mathbb{R}^3$
$\rho = (1 + r^2)^{1/2}$	Poids
${}^T\boldsymbol{u}$	Le transposé du vecteur $\boldsymbol{u}$
$\boldsymbol{v}_\tau = \boldsymbol{v} - (\boldsymbol{v} \cdot \boldsymbol{n})\boldsymbol{n}$	Composante tangentielle de $\boldsymbol{v}$
$\operatorname{supp} \boldsymbol{u}$	Support de $\boldsymbol{u}$

$\nabla \boldsymbol{u}$	Gradient de $\boldsymbol{u}$ défini par $\nabla \boldsymbol{u} \stackrel{\text{def}}{=} \left(\frac{\partial \boldsymbol{u}}{\partial x_1}, \frac{\partial \boldsymbol{u}}{\partial x_2}, \frac{\partial \boldsymbol{u}}{\partial x_3}\right)$
$\operatorname{div} \boldsymbol{u}$	Gradient de $\boldsymbol{u}$ défini par $\operatorname{div}\boldsymbol{u} \stackrel{\text{def}}{=} \sum_{i=1}^{3} \frac{\partial \boldsymbol{u}_i}{\partial x_i}$
$\mathbf{curl}\, \boldsymbol{u}$	Rotationnel de $\boldsymbol{u}$ défini par $\mathbf{curl}\boldsymbol{u} \stackrel{\text{def}}{=} \nabla \times \boldsymbol{u}$
$\mathbf{D}(\boldsymbol{u})$	Tenseur des déformations de $\boldsymbol{u}$ défini par $\mathbf{D}(\boldsymbol{u}) \stackrel{\text{def}}{=} \frac{1}{2}(\nabla \boldsymbol{u} + {}^T\nabla \boldsymbol{u})$
X'	Espace dual de X
$\langle ., . \rangle_{X' \times X}$	Produit de dualité X', X

Espaces fonctionnels

$\mathcal{D}(\Omega)$	Ensemble des fonctions $\mathcal{C}^\infty$ à support compact sur Ω
$\mathcal{D}_\sigma(\Omega)$	Ensemble des fonctions $\mathcal{C}^\infty$ à support compact sur Ω à divergence nulle
$\mathcal{D}'(\Omega)$	Espace des distributions sur Ω
$\mathcal{S}'(\Omega)$	Espace des distributions tempérées sur Ω
$L^p(\Omega)$	Espace des fonctions $\boldsymbol{u}$ mesurables sur Ω telles que $\int_\Omega \lvert\boldsymbol{u}\rvert^p d\mathbf{x} < \infty$
$L^p_{\text{loc}}(\Omega)$	Espace des fonctions $\boldsymbol{u} \in \mathrm{L}^p(\Omega')$ pour tout ouvert borné Ω' avec $\overline{\Omega'} \subset \Omega$
$W^{m,p}(\Omega)$	Espace de Sobolev d'ordre m équipé de la norme $\lVert . \rVert_{W^{m,p}(\Omega)}$
$H^m(\Omega)$	Espace de Sobolev $W^{m,2}(\Omega)$
$\mathring{W}^{m,p}(\Omega)$	Adhérence de $\mathcal{D}(\Omega)$ par rapport à la norme $\lVert . \rVert_{W^{m,p}(\Omega)}$, avec Ω non borné
$\mathring{W}^{m,p}(\Omega')$	Adhérence de $\mathcal{D}(\Omega)$ par rapport à la norme $\lVert . \rVert_{W^{m,p}(\Omega')}$, avec Ω' borné

Chapter I

Introduction général

En mécanique des fluides, les équations de **Navier-Stokes** sont des équations aux dérivées partielles non linéaires qui décrivent le mouvement des écoulements des fluides visqueux incompressibles ou non incompressibles. Elles modélisent un fluide comme un milieu continu, c'est-à-dire caractérisé par des grandeurs physiques définies en tout point de l'espace et à tout instant. Dans l'équation, on verra donc apparaitre un paramètre ν qui représente la viscosité du fluide et ρ représente la densité du fluide. Dans cette thèse, nous nous intéressons à des fluides incompressibles. Le système peut s'écrire alors sous la forme :

$$\begin{aligned} \rho\Big(\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\nabla\,\boldsymbol{u}\Big) - \nu\Delta\,\boldsymbol{u} + \nabla\,\pi &= \rho\boldsymbol{f} \quad \text{dans } \Omega\times(0,T), \\ \operatorname{div}\boldsymbol{u} &= 0 \quad \text{dans } \Omega\times(0,T). \end{aligned}$$

Le problème consiste à trouver le champs de vitesse du fluide $\boldsymbol{u}$ et la pression π. La donnée $\boldsymbol{f}$ représente les forces extérieures appliquées au fluide. Les équations de Navier-Stokes se révèlent cruciales pour décrire de nombreux phénomènes. Elles sont utilisées pour comprendre les mouvements des courants dans les océans, ainsi que ceux des grandes masses d'air dans l'atmosphère. Ces équations entrent également dans l'étude de la circulation du sang dans nos artères, dans la simulation des trajectoires d'air autour d'une aile d'un avion et dans la simulation des tourbillons. Ces équations sont même utilisées dans les jeux vidéos pour améliorer le réalisme de certaines scènes [76].

Les théories mathématiques liées aux équations de Navier-Stockes reposent essentiellement sur

les travaux des années trente de J. Leray [57, 58]. Ses travaux ont fourni des résultats d'existence et d'unicité en dimension deux. Malheureusement jusqu'à présent, on ne sait pas démontrer si pour toute condition initiale, il existe une solution régulière et globale en temps en trois dimensions d'espace. Nous pouvons indiquer que la résolution des équations de **Navier-Stokes** fait partie de l'un des sept **Problèmes du Millénaire** proposés par la Fondation Clay. Pour le mathématicien, le modèle de Navier-Stokes reste source de multiples questions, malgré des progrès conséquents dans les soixante-dix dernières années. Pour un large panorama des résultats classiques et des problèmes ouverts, nous invitons le lecteur à consulter par exemple R. Temam [77] et P.L. Lions [61].

Dans ce travail, nous nous intéressons à l'analyse mathématique d'un écoulement de fluide stationnaire visqueux et incompressible autour d'un obstacle supposé borné. Étant donné un ouvert borné régulier Ω' dont l'adhérence représente un obstacle, et $\Omega = \mathbb{R}^3 \setminus \overline{\Omega'}$, nous considérons plus précisément, le problème de Navier-Stokes stationnaire suivant :

$$\begin{aligned} -\nu\Delta \boldsymbol{u} + \rho \boldsymbol{u}\cdot\nabla\boldsymbol{u} + \nabla\pi = \rho\boldsymbol{f} \quad \text{dans} \quad \Omega, \\ \operatorname{div}\boldsymbol{u} = 0 \quad \text{dans} \quad \Omega. \end{aligned} \tag{I.1}$$

Comme le domaine Ω est non borné, nous devons ajouter une condition à l'infini comme suit :

$$\lim_{|\boldsymbol{x}|\to+\infty} \boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{u}_\infty, \tag{I.2}$$

où $\boldsymbol{u}_\infty \in \mathbb{R}^3$ est la vitesse du fluide à l'infini.

Outre les équations qui doivent être satisfaites en tout point du domaine, il est nécessaire d'imposer également des conditions qui doivent être satisfaites au bord de l'obstacle. À ce jour la plupart des travaux mathématiques portant sur les équations de Navier-Stokes extérieurs considèrent des conditions classiques de non-glissement, dites aussi conditions de Dirichlet, introduites par **G. G. Stokes** en 1845 :

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{sur} \quad \Gamma = \partial\Omega. \tag{I.3}$$

Selon, **Stokes**, un fluide en contact avec une paroi solide ne peut pas glisser le long de la paroi, au contraire sa vitesse au bord doit être nulle. Dans la littérature, plusieurs travaux ont considéré la condition (I.3) dans leurs modèles et il serait impossible de lister toutes les références sur ce sujet. Nous pouvons citer par exemple [3, 32, 33, 37–40, 44, 45, 47, 50–52, 54, 55, 67, 73, 74].

Bien que moins vaste que celles portant sur des conditions aux limites de type Dirichlet, il

existe cependant une littérature importante qui s'intéresse à d'autres types de conditions aux limites. En effet, dans les applications, il est possible de se trouver face à des problèmes où les conditions de type (I.3) ne sont plus valables, tels que l'écoulement d'un fluide visqueux et incompressible autour d'un obstacle ayant une certaine rugosité. Dans une telle application, les conditions de Dirichlet, donc de non-glissement, peuvent conduire à des phénomènes de couche limite près de la surface comme l'a indiqué Serrin dans [71]. Par conséquent, une autre approche a été introduite où il est supposé que, en raison de la rugosité de bord du domaine et de la viscosité du fluide, il y a une couche stagnante de liquide près à la frontière permettant au fluide de glisser (voir par exemple [28]). En 1827, Navier [68] a proposé une condition dite condition de glissement avec friction à la paroi qui permet de prendre en compte l'effet de glissement du fluide près du bord et de mesurer la friction :

$$\boldsymbol{u} \cdot \boldsymbol{n} = 0, \quad 2[\mathbf{D}(\boldsymbol{u})\boldsymbol{n}]_\tau + \alpha \boldsymbol{u}_\tau = \mathbf{0} \quad \text{sur} \quad \Gamma, \tag{I.4}$$

où

$$\mathbf{D}(\boldsymbol{u}) = \frac{1}{2}\left(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T\right) \tag{I.5}$$

est le tenseur de déformations, α est un coefficient de friction, $\boldsymbol{n}$ est le vecteur normal de Γ et la notation $[\cdot]_\tau$ désigne la composante tangentielle d'un vecteur sur Γ. La première condition dans (I.4) est la condition d'imperméabilité de l'obstacle et la deuxième condition exprime le fait que la vitesse tangentielle au lieu d'être nulle comme dans la condition de non-glissement (I.3), est proportionnelle à la composante tangentielle du tenseur des contraintes. Parmi les premiers travaux sur les équations de Navier-Stokes avec les conditions de Navier (I.4) dans le cas où Ω est un ouvert borné est celui de à V.A Solonnikov et V.E Schadilov en 1973 [72]. Dans leurs article, les auteurs ont étudié le problème de Navier-Stokes avec la condition de Navier sur une partie du bord et une condition de Dirichlet sur le reste du bord. Ensuite ils ont obtenu des résultats d'existence de solutions faibles dans $H^1(\Omega) \times L^2(\Omega)$ et aussi de solutions fortes dans $H^2(\Omega) \times H^1(\Omega)$. Depuis lors, plusieurs travaux s'en suivirent sur l'existence, l'unicité, la régularité, etc. de la solution. On va citer ici quelques éléments bibliographiques.

En 1985, G. Mulon et F. Salemi [65, 66] ont donné des résultats d'existence, d'unicité et de régularité pour les problèmes de Navier-Stokes stationnaires et d'évolution avec des conditions aux limites mixtes qui incluent (I.4) sans friction ($\alpha = 0$) et (I.3) dans le cas d'un domaine borné ou d'un domaine extérieur.

H. Beirao da Veiga [17] a prouvé l'existence de solutions faibles pour l'équation de Navier-Stokes avec des conditions aux limites de Navier dans un demi-espace. L'équation de Navier-Stokes dans le cas d'un bord plat a été étudiée par Berselli [20]. Dans ce travail il a prouvé l'existence

de solution généralisée, de solutions fortes et de solutions très faibles dans $L^3(\Omega)$. Nous pouvons également citer H. Beirao da Veiga et F. Crispo [16, 31] qui ont établi des résultats de convergence lorsque la viscosité tend vers zéro.

D'autre part, les conditions de Navier (I.4) sont aussi utilisées dans la simulation numérique comme dans le travail de Berselli et *al.* [21] ainsi que dans celui de Parès [69]. On peut aussi se référer aux travaux de R. Verfürth [78] et A. Liakos [59, 60].

Un autre type de conditions aux limites qui a également été considéré concerne des conditions de glissement sans friction de type **Navier** :

$$\boldsymbol{u} \cdot \boldsymbol{n} = 0, \quad \mathbf{curl}\ \boldsymbol{u} \times \boldsymbol{n} = \mathbf{0} \quad \text{sur} \quad \Gamma. \tag{I.6}$$

Cette condition est aussi appelée conditions aux limites de Hodge. Ceci étant, nous donnons quelques éléments bibliographiques sur l'analyse de modèles qui considèrent ces conditions lorsque les domaines sont bornés ou non (cas d'un domaine extérieur ou demi espace). Dans le cas de domaines bornés, nous pouvons citer par exemple [13–15, 18, 79]. En domaines non bornés, le lecteur pourra consulter [7, 64]. Pour le demi espace, on peut citer [12].

Notons que, dans le cas d'un bord plat et lorsque le coefficient de friction est nul, les deux conditions (I.4) et (I.6) sont équivalentes.

Une étape essentielle pour étudier les équations de Navier-Stokes est de commencer par étudier des problèmes linéaires qui les approchent. Une linéarisation consiste à supposer que $\boldsymbol{u} = \boldsymbol{v} + \boldsymbol{u}_\infty$ et en injectant cela dans les équations de Navier-Stokes on obtient

$$\begin{aligned} -\nu\Delta\boldsymbol{v} + \boldsymbol{u}_\infty \cdot \nabla\, \boldsymbol{v} + \nabla\pi = \boldsymbol{f} - \boldsymbol{v} \cdot \nabla\, \boldsymbol{v} \quad \text{dans} \quad \Omega, \\ \operatorname{div} \boldsymbol{v} = 0 \quad \text{dans} \quad \Omega. \end{aligned}$$

Ici, nous avons supposé que le fluide a une densité $\rho = 1$ afin d'alléger la présentation. Si on suppose que le fluide est au repos à l'infini c'est-à-dire $\boldsymbol{u}_\infty = \mathbf{0}$, nous obtenons le problème de Stokes stationnaire suivant :

$$\begin{aligned} -\nu\Delta\boldsymbol{v} + \nabla\pi = \boldsymbol{f} \quad \text{dans} \quad \Omega, \\ \operatorname{div} \boldsymbol{v} = 0 \quad \text{dans} \quad \Omega. \end{aligned} \tag{I.7}$$

L'objectif de cette thèse est d'étudier le problème de Stokes (I.7) avec des conditions aux limites de Navier (I.4) dans un domaine non borné tridimensionnel. On s'intéresse aux résultats

d'existence et d'unicité de solutions généralisées et de solutions régulières et également de solutions très faibles. Ce problème a été bien étudié lorsqu'il est posé dans un ouvert borné, on peut citer par exemple [9, 10] pour $\alpha = 0$ et [1, 8] pour $\alpha \neq 0$. Dans ces travaux, les espaces de Sobolev classiques fournissent, dans ce cas, un cadre fonctionnel adéquat pour une telle étude. Lorsque le domaine n'est pas borné, cas qui nous intéresse ici, une approche similaire à celle qui a été fait dans le cas d'un domaine borné est défaillante. Dans ce cas, les espaces de Sobolev classiques ne sont pas adaptés car il est nécessaire de décrire le comportement à l'infini des solutions (décroissance ou croissance). C'est pourquoi il est nécessaire d'introduire un cadre fonctionnel adéquat pour contrôler le comportement à l'infini des solutions du problème. Les espaces de Sobolev avec poids constituent un de ces bons cadres. Les espaces de Sobolev à poids considérés ici sont des extensions des espaces de Sobolev classiques, munis de poids de type $\rho(\mathbf{x})^k = \left((1+|\mathbf{x}|^2)^{1/2}\right)^k$ qui permettent de contrôler la croissance ou la décroissance des fonctions à l'infini et cela, en faisant varier le paramètre k. Nous disposons d'une grande liberté de choix quant au comportement à l'infini des fonctions considérées.

Notons que les espaces de Sobolev à poids sont algébriquement et topologiquement identiques à l'espace de Sobolev classique quand le domaine est borné. En effet, les poids utilisés ici n'ont aucun intérêt dans le cas d'un domaine borné. De façon plus générale, quand le domaine est non borné, les fonctions des espaces à poids ont les mêmes propriétés locales que celles des espaces classiques. Ainsi, quand la frontière est bornée (comme pour les domaines extérieurs), les théorèmes de trace dans les espaces de Sobolev à poids résultent directement de ceux des espaces de Sobolev classiques.

Il est à noté que les espaces de Sobolev à poids sont également utilisés comme un cadre fonctionnel pour une méthode numérique. Cette méthode est appelée méthode des éléments finis inversés (IFEM), a été développée par Boulmezaoud dans [23]. Cette dernière est construite pour prendre en compte les conditions aux limites à l'infini, sans avoir recours à l'utilisation de couche limite ou de conditions absorbantes. Le comportement asymptotique de la solution numérique est décrit dans des espaces de Sobolev à poids, et la croissance ou la décroissance des fonctions à l'infini sont exprimées à l'aide de fonctions de poids. Notons que la méthode des éléments finis a été appliquée efficacement à de nombreux problèmes elliptiques définis dans des domaines non bornés (voir [22, 24, 25]). On peut aussi citer [19], dont lequel les auteurs utilisent les espaces de Sobolev à poids comme cadre fonctionnel pour une méthode qui généralise la méthode (IFEM) à la classe des méthodes des éléments finis Galerkin Discontinu (DGFEM).

À notre connaissance, les seuls travaux qui ont étudié le problème de Stokes (I.7) extérieur avec des conditions aux limites de Navier (I.4) sont les travaux de G. Mulon et F. Salemi [65, 66] et

les travaux de A. Russo et A. Tartaglione [70].

Cette thèse s'articule autour des outils suivants :

1- Nous commençons par l'analyse variationnelle du problème. Le fait d'utiliser les conditions de Navier, conduit à écrire la formulation variationnelle, non pas en fonction de gradients, mais en fonction de tenseurs de déformation. Ainsi, à la place des inégalités de Poincaré à poids, nous sommes amenés à établir des inégalités de Korn à poids afin d'obtenir la coercivité de la forme bilinéaire associée à la formulation variationnelle. A partir de l'analyse variationnelle et en combinant les propriétés des équations de Stokes dans un domaine borné ainsi que dans tout l'espace, nous étudions l'existence et l'unicité de solutions faibles ayant des comportements à l'infini différents de celui de la solution variationnelle.

2- Nous poursuivons ensuite nos travaux en étudions les solutions fortes et très faibles, toujours dans un cadre hilbertien.

3- On termine cette thèse par l'extension de certains résultats obtenus dans la théorie L^p.

Cette thèse est organisée de la façon suivante :

Le deuxième chapitre est naturellement dédié aux notations, aux définitions et aux propriétés des espaces de Sobolev avec poids sur lesquels nous nous appuyons dans la suite. Il s'agit principalement de rappeler des résultats de densité et d'injections liées à ces espaces démonter par Hanouzet [49], Cantor [27] et Kudrjavcev [56]. Nous rappelons aussi les inégalités de Hardy qui jouent un rôle clé dans la résolution des problèmes aux limites elliptiques (pour le problème de Laplace et Stokes avec des conditions aux limites de Dirichlet). Néanmoins, comme nous l'avons déjà évoqué précédemment, dans notre analyse variationnelle, nous avons besoin des inégalités de Korn à poids. Nous les établissons dans ce chapitre en suivant l'approche introduite dans [30], où les auteurs ont proposé une autre preuve des inégalités de Korn dans les domaines bornés qui repose sur le fait que les champs de tenseur symétriques qui satisfont à la condition dite de compatibilité de Saint-Venant (voir (II.32)), peuvent s'écrire comme le tenseur de déformation d'un champ de vecteurs.

Dans le troisième chapitre, nous rappelons des résultats concernant le problème de Laplace dans un domaine extérieur de $\mathbb{R}^3$ [5]. Nous présentons une classe complète de résultats d'existence,

d'unicité et de régularité pour ce problème avec des conditions aux limites du type Dirichlet et Neumann. Ces résultats ont été obtenus dans [6] qui traite de la théorie L^p avec le poids particulier $k = 0$, [63] pour l'extension $k \in \mathbb{Z}$ et enfin [48] pour le cadre hilbertien L^2 avec une gamme complète de comportement à l'infini. Dans notre travail, nous avons besoin de résoudre le problème de Laplace avec des conditions de type Neumann et avec une donnée $f \in L^p(\Omega)$. Il est à noter qu'à notre connaissance, ce résultat n'existe pas dans la littérature à l'exception du cas $p = 2$. Ainsi, nous traitons ce problème dans ce chapitre. Ensuite, nous rappelons des résultats concernant le problème de Stokes dans $\mathbb{R}^3$ ainsi que le problème de Stokes dans un domaine borné avec des conditions sur le bord de type Dirichlet (I.3). L'étude de solutions généralisées et de solutions fortes de ce problème ont été faites par un grand nombre d'auteurs, de différents points de vue voir [4, 11, 41, 42]. Enfin, comme nous l'avons évoqué dans la première partie de l'articulation de la thèse, l'étude de solutions généralisées du problème de Stokes (I.7) avec les conditions de Navier (I.4), sera faite en combinant les propriétés d'un problème de Stokes dans un domaine borné et celui de tout l'espace. Ceci sera effectué au moyen d'une partition de l'unité. Ainsi, le domaine borné considéré ici consiste en l'intersection du domaine extérieur Ω avec une boule contenant l'obstacle. Les conditions de Navier seront imposées sur le bord de l'obstacle tandis qu'une condition de Dirichlet sera imposée sur le bord de la boule. Nous serons donc amenés à nous intéresser à un problème mixte.

Le quatrième chapitre est dédié à l'analyse hilbertienne du problème **Stokes** (I.7) dans un domaine extérieur Ω avec des conditions aux limites de type **Navier** (I.4). Nous cherchons l'existence et l'unicité de solutions dans $W^{2,2}_{k+1}(\Omega) \times W^{1,2}_{k+1}(\Omega)$. Pour cela, nous commençons par établir l'existence et l'unicité de la solution variationnelle dans $W^{1,2}_0(\Omega) \times L^2(\Omega)$. Cependant dans certains problèmes de la mécanique de fluides, il est possible de se trouver face à des données qui ne sont pas régulières. C'est pourquoi nous nous intéressons ici à la recherche de solutions dites très faibles de type $(\boldsymbol{u}, \pi) \in W^{0,2}_{-k-1}(\Omega) \times W^{-1,2}_{-k-1}(\Omega)$. A notre connaissance, la notion de solutions très faible a été introduite par Lions et Magenes dans [62]. Récemment, ce concept a été développé par Farwig, Galdi et Sohr dans [34–36]. L'étude des solutions très faibles reposent très fortement sur des arguments de dualité à partir des solutions fortes. En conséquence, dans l'étude de ces solutions très faibles, la régularité du domaine restera inchangée par rapport à celle faite sur l'étude des solutions fortes. Une des difficultés consiste à donner un sens aux traces de fonctions moins régulières et à obtenir, par le biais de lemmes de densité, les formules de Green adéquates.

Dans le dernier chapitre, on s'intéresse à la théorie L^p pour le problème de Stokes (I.7) avec des conditions aux limites de type Navier (I.4) dans un domaine extérieur Ω connexe de $\mathbb{R}^3$ avec $\partial\Omega = \Gamma$ de classe $C^{2,1}$. Nous montrons l'existence et l'unicité de solutions fortes c'est-à-dire

$(\boldsymbol{u}, \pi) \in W^{2,p}_{k+1}(\Omega) \times W^{1,p}_{k+1}(\Omega)$ pour $2 \leqslant p < \infty$ et $k \in \mathbb{Z}$. Nous utilisons des techniques similaires que le chapitre IV. Nous traitons aussi le cas de solutions très faibles dans ce chapitre mais pour un poids particulier. Plus précisément, nous montrons l'existence et l'unicité des solutions $(\boldsymbol{u}, \pi) \in W^{0,p}_{-2}(\Omega) \times W^{-1,p}_{-2}(\Omega)$ pour $1 < p < 2$.

Chapter II

Basic Concepts on Weighted Sobolev spaces

Résumé: Dans ce chapitre, on introduit un cadre fonctionnel adéquat pour la résolution du problème de Stokes et de Navier-Stokes dans des domaines non bornés (espace entier, demi-espace, domaines extérieurs). Dans de tels domaines, il est nécessaire de décrire le comportement à l'infini des solutions (décroissance ou croissance). Un bon cadre fonctionnel adéquat est donné par les espaces de Sobolev avec poids car ils permettent de considérer une grande variété de comportements. Nous présentons ici les propriétés fonctionnelles de ces espaces sur lesquels nous nous appuyons et nous donnons des résultats fondamentaux. Il s'agit principalement à présenter des résultats introduits par Hanouzet [49], Cantor [27], Kudrjavcev [56]. Ensuite, nous présentons quelques propriétés sur la conditions de Navier. Le point-clé ici est la preuve de l'inégalité du type Korn afin d'obtenir la coercivité de la forme bilinéaire associée à la formulation variationnelle.

1 Notation and Weighted Sobolev Spaces

We recall the main Notation and results, concerning the weighted Sobolev spaces, which we shall use later on. In what follows, p is a real number in the interval $]1, \infty[$. The dual exponent of p denoted by p' is given by the following relation:

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

We will use bold characters for vector and matrix fields. A point in $\mathbb{R}^3$ is denoted by $\boldsymbol{x} = (x_1, x_2, x_3)$ and its distance to the origin by

$$r = |\boldsymbol{x}| = \left(x_1^2 + x_2^2 + x_3^2\right)^{1/2}.$$

For any multi-index $\boldsymbol{\lambda} \in \mathbb{N}^3$, we denote by $\partial^{\boldsymbol{\lambda}}$ the differential operator of order $\boldsymbol{\lambda}$,

$$\partial^{\boldsymbol{\lambda}} = \frac{\partial^{|\lambda|}}{\partial_1^{\lambda_1} \partial_2^{\lambda_2} \partial_3^{\lambda_3}}, \quad |\lambda| = \lambda_1 + \lambda_2 + \lambda_3.$$

We denote by $[s]$ the integer part of s. For any $k \in \mathbb{Z}$, $\mathcal{P}_k$ stands for the space of polynomials of degree less than or equal to k and $\mathcal{P}_k^{\Delta}$ the harmonic polynomials of $\mathcal{P}_k$. If k is a negative integer, we set by convention $\mathcal{P}_k = \{0\}$.

Let Ω' be a bounded connected open set in $\mathbb{R}^3$ with boundary $\partial\Omega' = \Gamma$ and let Ω its complement $i.e$ $\Omega = \mathbb{R}^3 \diagdown \overline{\Omega'}$. In this work, we shall also denote by B_R the open ball of radius $R > 0$ centred at the origin with boundary ∂B_R. In particular, since Ω' is bounded, we can find some R_0, such that $\Omega' \subset B_{R_0}$ and we introduce, for any $R \geq R_0$, the set

$$\Omega_R = \Omega \cap B_R.$$

Given a Banach space B, with dual space B' and a closed subspace X of B, we denote by $B' \perp X$ the subspace of B' orthogonal to X, i.e.

$$B' \perp X = \{f \in B'; < f, v >= 0 \ \forall\, v \in X\} = (B/X)'.$$

The space $B' \perp X$ is also called the polar space of X in B'.

Given $\boldsymbol{M}$ and $\boldsymbol{N}$ two matrices fields, such that $\boldsymbol{M} = (m_{ij})_{1\leqslant i,j\leqslant 3}$ and $\boldsymbol{N} = (n_{ij})_{1\leqslant i,j\leqslant 3}$, then we define $\boldsymbol{M} : \boldsymbol{N} = (m_{ij}n_{ij})_{1\leqslant i,j\leqslant 3}$. Finally, as usual, $C > 0$ denotes a generic constant the value of which may change from line to line and even at the same line.

In order to control the behavior at infinity of our functions and distributions we use for basic weights the quantity $\rho(\boldsymbol{x}) = (1 + r^2)^{1/2}$ which is equivalent to r at infinity, and to one on any bounded subset of $\mathbb{R}^3$.
For $k \in \mathbb{Z}$, we introduce

$$W_k^{0,p}(\Omega) = \Big\{u \in \mathcal{D}'(\Omega),\ \rho^k u \in L^p(\Omega)\Big\},$$

which is a Banach space equipped with the norm:

$$\|u\|_{W_k^{0,p}(\Omega)} = \|\rho^k u\|_{L^p(\Omega)}.$$

For any non-negative integers m, real numbers $p > 1$ and $k \in \mathbb{Z}$. We define the weighted Sobolev space for $3/p + k \notin \{1, \cdots, m\}$:

$$W_k^{m,p}(\Omega) = \Big\{u \in \mathcal{D}'(\Omega);\ \forall \boldsymbol{\lambda} \in \mathbb{N}^3 : 0 \leq |\boldsymbol{\lambda}| \leq m,\ \rho^{k-m+|\boldsymbol{\lambda}|}\partial^{\boldsymbol{\lambda}} u \in L^p(\Omega)\Big\}.$$

It is a reflexive Banach space equipped with the norm:

$$\|u\|_{W_k^{m,p}(\Omega)} = \left(\sum_{0 \leqslant |\boldsymbol{\lambda}| \leqslant m} \|\rho^{k-m+|\boldsymbol{\lambda}|} \partial^{\boldsymbol{\lambda}} u\|^p_{L^p(\Omega)} \right)^{1/p}.$$

We define the semi-norm

$$|u|_{W_k^{m,p}(\Omega)} = \left(\sum_{|\boldsymbol{\lambda}|=m} \|\rho^k \partial^{\boldsymbol{\lambda}} u\|_{L^p(\Omega)} \right)^{1/p}.$$

Let us give some examples of such space that will be often used in the remaining of the thesis.

1) For $m = 1$, we have

$$W_k^{1,p}(\Omega) := \{u \in \mathcal{D}'(\Omega);\ \rho^{k-1} u \in L^p(\Omega),\ \rho^k \nabla u \in L^p(\Omega)\}$$

2) For $m = 2$, we have

$$W_{k+1}^{2,p}(\Omega) := \left\{ u \in W_k^{1,p}(\Omega), \rho^{k+1} \nabla^2 u \in L^p(\Omega) \right\},$$

Remark II.1
Note that if $\frac{3}{p} + k \in \{1, ..., m\}$, *we need to add the logarithmic weight in the definition of weight Sobolev spaces introduced above. The logarithmic weight is defined by* $ln(2 + |x|^2)$ *see [48] for more details. For example if* $3/p + k = 1$, *we have*

$$W_k^{1,p}(\Omega) := \{\boldsymbol{u} \in \mathcal{D}'(\Omega);\ \rho^{k-1}(ln(2 + r^2))^{-1} \boldsymbol{u} \in L^p(\Omega),\ \rho^k \nabla \boldsymbol{u} \in L^p(\Omega)\}.$$

2 Some Properties

In this section, we present some basic properties on weighted Sobolev spaces. For more details, the reader can refer to [5, 6, 49].

Properties II.1

- *The space* $\mathcal{D}(\overline{\Omega})$ *is dense in* $W_k^{m,p}(\Omega)$.

- *For any* $m \in \mathbb{N}^*$ *and* $3/p + k \neq 1$, *we have the following continuous embedding:*

$$W_k^{m,p}(\Omega) \hookrightarrow W_{k-1}^{m-1,p}(\Omega). \tag{II.1}$$

- *For any k, $m \in \mathbb{Z}$ and for any $\lambda \in \mathbb{N}^3$, the mapping*

$$u \in W_k^{m,p}(\Omega) \longrightarrow \partial^\lambda u \in W_k^{m-|\lambda|,p}(\Omega) \tag{II.2}$$

is continuous.

- *If $3/p + k \notin \{1, \cdots, m\}$, $3/p + k - \mu \notin \{1, \cdots, m\}$ and $m \in \mathbb{Z}$ the mapping*

$$u \in W_k^{m,p}(\Omega) \longrightarrow \rho^\mu u \in W_{k-\mu}^{m,p}(\Omega)$$

is an isomorphism.

The space $W_k^{m,p}(\Omega)$ sometimes contains some polynomial functions. Let j be defined as follow:

$$j = \begin{cases} [m - (3/p + k)] & \text{if} \quad 3/p + k \notin \mathbb{Z}^-, \\ m - 3/p - k - 1 & \text{otherwise.} \end{cases} \tag{II.3}$$

Then $\mathcal{P}_j$ is the space of all polynomials included in $W_k^{m,p}(\Omega)$.
The norm of the quotient space $W_k^{m,p}(\Omega)/\mathcal{P}_j$ is given by:

$$||u||_{W_k^{m,p}(\Omega)/\mathcal{P}_j} = \inf_{\mu \in \mathcal{P}_j} ||u + \mu||_{W_k^{m,p}(\Omega)}.$$

All the local properties of $W_k^{m,p}(\Omega)$ coincide with those of the corresponding classical Sobolev spaces $W^{m,p}(\Omega)$. Hence, it also satisfies the usual trace theorems on the boundary Γ. Therefore, we can define the space

$$\mathring{W}_k^{m,p}(\Omega) = \{u \in W_\alpha^{m,p}(\Omega), \ \ \gamma_0 u = 0, \, \gamma_1 u = 0, \, \cdots, \gamma_{m-1} u = 0\}.$$

Note that when $\Omega = \mathbb{R}^3$, we have $\mathring{W}_k^{m,p}(\mathbb{R}^3) = W_k^{m,p}(\mathbb{R}^3)$. The space $\mathcal{D}(\Omega)$ is dense in $\mathring{W}_k^{m,p}(\Omega)$. Therefore, the dual space of $\mathring{W}_k^{m,p}(\Omega)$, denoting by $W_{-k}^{-m,p'}(\Omega)$, is a space of distributions with the norm

$$|| \, u \, ||_{W_{-k}^{-m,p'}(\Omega)} = \sup_{v \in \mathring{W}_k^{m,p}(\Omega)} \frac{\langle u, v \rangle_{W_{-k}^{-m,p'}(\Omega) \times \mathring{W}_k^{m,p}(\Omega)}}{|| \, v \, ||_{W_k^{m,p}(\Omega)}}.$$

We state the Hardy's inequalities which play a key role in solving elliptic problems.

Theorem II.1 *Let Ω be an lipschitzian exterior domain. Let $m \geqslant 1$, $k \in \mathbb{Z}$ and $1 < p < \infty$. There exists a constant $C = C(p, k, \Omega) > 0$ such that*

1)

$$\forall u \in W_k^{m,p}(\Omega), \quad ||u||_{W_k^{m,p}(\Omega)/\mathcal{P}_{j'}} \leqslant |u|_{W_k^{m,p}(\Omega)}, \tag{II.4}$$

where $j' = \min(j, 0)$ *and* j *is the highest degree of polynomials belonging to* $W_k^{m,p}(\Omega)$.

2)

$$\forall u \in \mathring{W}_k^{m,p}(\Omega), \quad ||u||_{W_k^{m,p}(\Omega)} \leqslant |u|_{W_k^{m,p}(\Omega)}. \tag{II.5}$$

The inequalities (II.4) and (II.5) are the reason of choosing the weight functions in the definition of $W_k^{m,p}(\Omega)$. The proof of this theorem can be found in the case $\Omega = \mathbb{R}^3$ in [5] and its extension to an exterior domain Ω in [6] and in [48] for $p = 2$. From Theorem II.1 and the Sobolev embeddings, we have the following continuous and dense embedding:

$$W_0^{1,p}(\Omega) \hookrightarrow L^{\frac{3p}{3-p}}(\Omega), \quad \text{if} \quad 1 < p < 3. \tag{II.6}$$

By duality, we have

$$L^{\frac{3p'}{3+p'}}(\Omega) \hookrightarrow W_0^{-1,p'}(\Omega), \quad \text{if} \quad 3/2 < p' < \infty. \tag{II.7}$$

The following result follows from Theorem II.1 (see [5]).

Proposition II.1 *Let* $m \geqslant 1$ *and* $u \in \mathcal{D}'(\mathbb{R}^3)$ *such that*

$$\forall \lambda \in \mathbb{N}^3 : \quad |\lambda| = m, \quad \partial^\lambda u \in L^p(\mathbb{R}^3).$$

i) If $1 < p < 3$, *then there exists a polynomial* $K(u) \in \mathcal{P}_{m-1}$ *such that* $u + K(u)$ *belongs to* $W_0^{m,p}(\mathbb{R}^3)$ *and*

$$\inf_{\mu \in \mathcal{P}_{[m-3/p]}} ||\, u + K(u) + \mu \,||_{W_0^{m,p}(\mathbb{R}^3)} \leq C|u|_{W_0^{m,p}(\mathbb{R}^3)}.$$

ii) If $p \geq 3$, *then* $u \in W_0^{m,p}(\mathbb{R}^3)$ *and we have*

$$\inf_{\mu \in \mathcal{P}_{m-1}} ||\, u + \mu \,||_{W_0^{m,p}(\mathbb{R}^3)} \leq C|u|_{W_0^{m,p}(\mathbb{R}^3)}.$$

The proof of the following theorem can be found in [63, Proposition 2.1].

Theorem II.2
Let k, l *be real numbers. Let* λ *be a polynomial that belongs to* $W_k^{1,p}(\Omega) + W_l^{1,q}(\Omega)$. *Then* λ *belongs to* $\mathcal{P}_\gamma$ *where*

$$\gamma = \max\Big([1 - \frac{3}{p} - k], [1 - \frac{3}{q} - l]\Big).$$

The theorem below states De Rham's Theorem.

Theorem II.3 *Let $\mathcal{O}$ be an open domain in $\mathbb{R}^3$, $\boldsymbol{f} \in \mathcal{D}'(\mathcal{O})$ and for all $\boldsymbol{\varphi} \in \mathcal{D}(\mathcal{O})$ such that* $\operatorname{div} \boldsymbol{\varphi} = 0$, $\langle \boldsymbol{f}, \boldsymbol{\varphi} \rangle_{\mathcal{D}'(\mathcal{O}) \times \mathcal{D}(\mathcal{O})} = 0$. *Then, there exists g in $\mathcal{D}'(\mathcal{O})$, such that $\boldsymbol{f} = \nabla g$.*

The next theorem characterizes distributions that belongs to weighted Sobolev spaces. For the proof, the reader can refer to [44, Theorem 2.7].

Theorem II.4 *Let m and k belong to $\mathbb{Z}$ and assume that Ω is Lipschitz continuous. Let q be a distribution of $\mathcal{D}'(\Omega)$ such that ∇q belongs to $W^{m,2}_{m+k}(\Omega)$. Then, if $k \leqslant -1$, q belongs to $W^{m+1,2}_{m+k}(\Omega)$ and*

$$\|q\|_{W^{m+1,2}_{m+k}(\Omega)/\mathbb{R}} \leqslant C \|\nabla q\|_{W^{m,2}_{m+k}(\Omega)}. \tag{II.8}$$

If $k \geqslant 0$, there exists a unique real constant c and a unique s in $W^{m+1,2}_{m+k}(\Omega)$ such that q has the decomposition:

$$q = c + s$$

and

$$\|s\|_{W^{m+1,2}_{m+k}(\Omega)} \leqslant C \|\nabla q\|_{W^{m,2}_{m+k}(\Omega)}. \tag{II.9}$$

3 Preliminaries Results

The purpose of this section is to introduce some weighted Sobolev spaces that are specific for the study of the Stokes problem (I.7) with the Navier boundary conditions (I.4). Let us first introduce some notations. For any vector fields $\boldsymbol{u}$ and $\boldsymbol{v}$ of $\mathbb{R}^3$, we define

$$\boldsymbol{u} \times \boldsymbol{v} = (u_2 v_3 - v_3 u_2,\, u_3 v_1 - u_1 v_3,\, u_1 v_2 - u_2 v_1)^T$$

and

$$\mathbf{curl}\ \boldsymbol{u} = \nabla \times \boldsymbol{u}.$$

We note that, the vector-valued Laplace operator of a vector field $\boldsymbol{v}$ is equivalently defined by

$$\Delta\, \boldsymbol{v} = \nabla \operatorname{div} \boldsymbol{v} - \mathbf{curl}\, \mathbf{curl}\, \boldsymbol{v}. \tag{II.10}$$

Or by

$$\Delta\, \boldsymbol{v} = 2\mathrm{div}\mathbf{D}(\boldsymbol{v}) - \nabla\, \mathrm{div}\boldsymbol{v}. \tag{II.11}$$

We start by introducing the following spaces for $k \in \mathbb{Z}$ and $1 < p < \infty$.

Definition II.1 *The space $H_k^p(\mathbf{curl},\Omega)$ is defined by:*

$$H_k^p(\mathbf{curl},\Omega) = \left\{ \boldsymbol{v} \in W_k^{0,p}(\Omega); \mathbf{curl}\ \boldsymbol{v} \in W_{k+1}^{0,p}(\Omega) \right\} ,$$

and is provided with the norm:

$$\|\boldsymbol{v}\|_{H_k^p(\mathbf{curl},\Omega)} = \left(\|\boldsymbol{v}\|^p{}_{W_k^{0,p}(\Omega)} + \|\mathbf{curl}\ \boldsymbol{v}\|^p{}_{W_{k+1}^{0,p}(\Omega)} \right)^{\frac{1}{p}} .$$

The space $H_k^p(\mathrm{div},\Omega)$ is defined by:

$$H_k^p(\mathrm{div},\Omega) = \left\{ \boldsymbol{v} \in W_k^{0,p}(\Omega); \mathrm{div}\ \boldsymbol{v} \in W_{k+1}^{0,p}(\Omega) \right\} ,$$

and is provided with the norm:

$$\|\boldsymbol{v}\|_{H_k^p(\mathrm{div},\Omega)} = \left(\|\boldsymbol{v}\|^p_{W_k^{0,p}(\Omega)} + \|\mathrm{div}\ \boldsymbol{v}\|^p_{W_{k+1}^{0,p}(\Omega)} \right)^{\frac{1}{p}} .$$

These definitions will be also used when Ω is replaced by $\mathbb{R}^3$.

Observe that $\mathcal{D}(\overline{\Omega})$ is dense in $H_k^p(\mathrm{div},\Omega)$ and in $H_k^p(\mathbf{curl},\Omega)$. For the proof, one can use the same arguments than for the proof of the density of $\mathcal{D}(\overline{\Omega})$ in $W_k^{m,p}(\Omega)$ (see [48, 49]). Therefore, recalling that $\boldsymbol{n}$ is the unit normal vector to the boundary Γ pointing outside Ω, if $\boldsymbol{v}$ belongs to $H_k^p(\mathrm{div},\Omega)$, then $\boldsymbol{v}$ has normal trace $\boldsymbol{v}\cdot\boldsymbol{n}$ in $W^{-1/p,p}(\Gamma)$, where $W^{-1/p,p}(\Gamma)$ denotes the dual space of $W^{1/p,p'}(\Gamma)$. By the same way, if $\boldsymbol{v}$ belongs to $H_k^p(\mathbf{curl},\Omega)$, then $\boldsymbol{v}$ has a tangential trace $\boldsymbol{v}\times\boldsymbol{n}$ that belongs to $W^{-1/p,p}(\Gamma)$. Similarly as in bounded domain, we have the trace theorems i.e, for $k\in\mathbb{Z}$ there exists a constant $C>0$, such that

$$\forall \boldsymbol{v} \in H_k^p(\mathrm{div},\Omega),\ ||\boldsymbol{v}\cdot\boldsymbol{n}||_{W^{-1/p,p}(\Gamma)} \leq C||\boldsymbol{v}||_{H_k^p(\mathrm{div},\Omega)}, \tag{II.12}$$

$$\forall \boldsymbol{v} \in H_k^p(\mathbf{curl},\Omega),\ ||\boldsymbol{v}\times\boldsymbol{n}||_{W^{-1/p,p}(\Gamma)} \leq C||\boldsymbol{v}||_{H_k^p(\mathbf{curl},\Omega)} \tag{II.13}$$

and the following Green's formulas holds: For any $\boldsymbol{v}\in H_k^p(\mathrm{div},\Omega)$ and $\varphi\in W_{-k}^{1,p'}(\Omega)$, we have

$$\langle \boldsymbol{v}\cdot\boldsymbol{n},\varphi\rangle_\Gamma = \int_\Omega \boldsymbol{v}\cdot\nabla\,\varphi\,dx + \int_\Omega \varphi\,\mathrm{div}\,\boldsymbol{v}\,dx, \tag{II.14}$$

where $\langle .,.\rangle_\Gamma$ denotes the duality pairing between $W^{-1/p,p}(\Gamma)$ and $W^{1/p,p'}(\Gamma)$.
For any $\boldsymbol{v}\in H_k^p(\mathbf{curl},\Omega)$ and $\boldsymbol{\varphi}\in W_{-k}^{1,p'}(\Omega)$, we have

$$\langle \boldsymbol{v}\times\boldsymbol{n},\boldsymbol{\varphi}\rangle_\Gamma = \int_\Omega \boldsymbol{v}\cdot\mathbf{curl}\,\boldsymbol{\varphi}\,dx - \int_\Omega \mathbf{curl}\,\boldsymbol{v}\cdot\boldsymbol{\varphi}dx. \tag{II.15}$$

The closures of $\mathcal{D}(\Omega)$ in $H_k^p(\mathrm{div},\Omega)$ and in $H_k^p(\mathbf{curl},\Omega)$ are denoted respectively by $\mathring{H}_k^p(\mathbf{curl},\Omega)$ and $\mathring{H}_k^p(\mathrm{div},\Omega)$ and can be characterized respectively by:

$$\mathring{H}_k^p(\mathbf{curl},\Omega) = \{\boldsymbol{v} \in H_k^p(\mathbf{curl},\Omega);\ \boldsymbol{v}\times\boldsymbol{n} = \mathbf{0} \text{ on } \Gamma\},$$

$$\mathring{H}_k^p(\mathrm{div},\Omega) = \{\boldsymbol{v} \in H_k^p(\mathrm{div},\Omega);\ \boldsymbol{v}\cdot\boldsymbol{n} = 0 \text{ on } \Gamma\}.$$

For $1 < p < \infty$ and $k \in \mathbb{Z}$, we denote by $[\mathring{H}_k^p(\mathrm{div};\Omega)]'$ and $[\mathring{H}_k^p(\mathbf{curl};\Omega)]'$ the dual spaces of $\mathring{H}_k^p(\mathbf{curl},\Omega)$ and $\mathring{H}_k^p(\mathrm{div},\Omega)$ respectively. We can characterize theses spaces as it is stated in the following proposition.

Proposition II.2

1) *A distribution $\boldsymbol{f}$ belongs to $[\mathring{H}_k^p(\mathrm{div};\Omega)]'$ if and only if there exist functions $\boldsymbol{\psi} \in W_{-k}^{0,p'}(\Omega)$ and $\chi \in W_{-k-1}^{0,p'}(\Omega)$, such that $\boldsymbol{f} = \boldsymbol{\psi} + \nabla\chi$. Moreover*

$$\|\boldsymbol{\psi}\|_{W_{-k}^{0,p'}(\Omega)} + \|\chi\|_{W_{-k-1}^{0,p'}(\Omega)} \leq C\|\boldsymbol{f}\|_{[\mathring{H}_k^p(\mathrm{div};\Omega)]'}. \tag{II.16}$$

2) *A distribution $\boldsymbol{f}$ belongs to $[\mathring{H}_k^p(\mathbf{curl},\Omega)]'$ if and only if there exist functions $\psi \in W_{-k}^{0,p'}(\Omega)$ and $\chi \in W_{-k-1}^{0,p'}(\Omega)$, such that $\boldsymbol{f} = \psi + \mathbf{curl}\,\chi$. Moreover*

$$\|\psi\|_{W_{-k}^{0,p'}(\Omega)} + \|\chi\|_{W_{-k-1}^{0,p'}(\Omega)} \leq C\|\boldsymbol{f}\|_{[\mathring{H}_k^p(\mathbf{curl},\Omega)]'}. \tag{II.17}$$

Proof. The proof of point 1) and 2) are very similar, so we do only the proof of the first result.

($\Leftarrow$) Let $\boldsymbol{\psi} \in W_{-k}^{0,p'}(\Omega)$ and $\chi \in W_{-k-1}^{0,p'}(\Omega)$, we have

$$\forall \boldsymbol{v} \in \mathcal{D}(\Omega), \quad \langle \boldsymbol{\psi} + \nabla\chi,\ \boldsymbol{v}\rangle_{\mathcal{D}'(\Omega)\times\mathcal{D}(\Omega)} = \int_\Omega (\boldsymbol{\psi}\cdot\boldsymbol{v} - \chi\,\mathrm{div}\,\boldsymbol{v})\,\mathrm{d}\boldsymbol{x}.$$

Therefore, the linear mapping $\boldsymbol{\ell} : \boldsymbol{v} \longmapsto \int_\Omega(\boldsymbol{\psi}\cdot\boldsymbol{v} - \chi\,\mathrm{div}\,\boldsymbol{v})\,\mathrm{d}\boldsymbol{x}$ defined on $\mathcal{D}(\Omega)$ is continuous for the norm of $\mathring{H}_k^p(\mathrm{div},\,\Omega)$. Since $\mathcal{D}(\Omega)$ is dense in $\mathring{H}_k^p(\mathrm{div},\,\Omega)$, $\boldsymbol{\ell}$ can be extended by continuity to a mapping still called $\boldsymbol{\ell} \in [\mathring{H}_k^p(\mathrm{div},\,\Omega)]'$. Thus $\boldsymbol{\psi} + \nabla\chi$ is an element of $[\mathring{H}_k^{p'}(\mathrm{div},\,\Omega)]'$.

($\Rightarrow$) Conversely, let $E = W_k^{0,p}(\Omega) \times W_{k+1}^{0,p}(\Omega)$ equipped with the following norm:

$$\|\boldsymbol{v}\|_E = (\|\boldsymbol{v}\|_{W_k^{0,p}(\Omega)}^p + \|\mathrm{div}\,\boldsymbol{v}\|_{W_{k+1}^{0,p}(\Omega)}^p)^{\frac{1}{p}}.$$

The mapping $T : \boldsymbol{v} \in \mathring{H}^p_k(\text{div}, \Omega) \to (\boldsymbol{v}, \text{div}\boldsymbol{v}) \in E$ is an isometry from $\mathring{H}^p_k(\text{div}, \Omega)$ in E. Suppose $G = T(\mathring{H}^p_k(\text{div}, \Omega))$ and let $S = T^{-1} : G \longrightarrow \mathring{H}^p_k(\text{div}, \Omega)$. Thus, we can define the following mapping:

$$\boldsymbol{v} \in G \longmapsto \langle \boldsymbol{f}, S\boldsymbol{v} \rangle_{[\mathring{H}^p_k(\text{div}, \Omega)]' \times \mathring{H}^p_k(\text{div}, \Omega)} \quad \text{for} \quad \boldsymbol{f} \in [\mathring{H}^p_k(\text{div}, \Omega)]'$$

which is a linear continuous form on G. Thanks to Hahn-Banach's Theorem, such form can be extended to a linear continuous form on E, denoted by $\boldsymbol{f}^*$ such that

$$||\boldsymbol{f}^*||_{E'} = ||\boldsymbol{f}||_{[\mathring{H}^p_k(\text{div}, \Omega)]'}. \tag{II.18}$$

From the Riesz's Representation lemma, there exist $\boldsymbol{\psi} \in W^{0,p'}_{-k}(\Omega)$ and $\chi \in W^{0,p'}_{-k-1}(\Omega)$, such that for any $\boldsymbol{v} = (\boldsymbol{v}_1, v_2) \in E$,

$$\langle \boldsymbol{f}^*, \boldsymbol{v} \rangle_{E' \times E} = \int_\Omega \boldsymbol{v}_1 \cdot \boldsymbol{\psi}\, dx + \int_\Omega v_2\, \chi\, dx,$$

with $||\boldsymbol{f}^*||_{E'} = \max\left\{ ||\boldsymbol{\psi}||_{W^{0,p'}_{-k}(\Omega)}, ||\chi||_{W^{0,p'}_{-k-1}(\Omega)} \right\}$. In particular, if $\boldsymbol{v} = T\boldsymbol{\varphi} \in G$, where $\boldsymbol{\varphi} \in \mathcal{D}(\Omega)$, we have:

$$\langle \boldsymbol{f}, \boldsymbol{\varphi} \rangle_{[\mathring{H}^p_k(\text{div}, \Omega)]' \times \mathring{H}^p_k(\text{div}, \Omega)} = \langle \boldsymbol{\psi} - \nabla\, \chi, \boldsymbol{\varphi} \rangle_{[\mathring{H}^p_k(\text{div}, \Omega)]' \times \mathring{H}^p_k(\text{div}, \Omega)},$$

and (II.16) follows immediately from (II.18).

□

As a consequence of Proposition II.2 and the imbedding (II.1) we have, for any $k \in \mathbb{Z}$ and $1 < p < \infty$, the following imbeddings

$$[\mathring{H}^p_k(\text{div}; \Omega)]' \subset W^{-1,p'}_{-k-1}(\Omega) \tag{II.19}$$

and

$$[\mathring{H}^p_k(\text{curl}; \Omega)]' \subset W^{-1,p'}_{-k-1}(\Omega). \tag{II.20}$$

4 The Navier boundary conditions and related properties

Let us introduce some notations related to the boundary. First, for any vector field $\boldsymbol{v}$ on Γ, we can write

$$\boldsymbol{v} = \boldsymbol{v}_\tau + (\boldsymbol{v} \cdot \boldsymbol{n})\boldsymbol{n}, \tag{II.21}$$

where $\boldsymbol{v}_\tau$ is the projection of $\boldsymbol{v}$ on the tangent hyper-plan to Γ. Next, for any point $\boldsymbol{x}$ on Γ, one may choose an open neighbourhood $\mathcal{W}$ of $\boldsymbol{x}$ in Γ small enough to allow the existence of two families of $\mathcal{C}^2$ curves on $\mathcal{W}$ and where the lengths s_1 and s_2 along each family of curves are possible system of coordinates. Denoting by $\boldsymbol{\tau}_1$, $\boldsymbol{\tau}_2$ the unit tangent vectors to each family of curves, we have

$$\boldsymbol{v}_\tau = (\boldsymbol{v}\cdot\boldsymbol{\tau}_1)\boldsymbol{\tau}_1 + (\boldsymbol{v}\cdot\boldsymbol{\tau}_2)\boldsymbol{\tau}_2.$$

As a result for any $\boldsymbol{v}\in\mathcal{D}(\overline{\Omega})$ the following formula holds (see [9, Lemma 2.1])

$$2[\mathbf{D}(\boldsymbol{v})\boldsymbol{n}]_\tau = \nabla_\tau(\boldsymbol{v}\cdot\boldsymbol{n}) + \left(\frac{\partial\boldsymbol{v}}{\partial\boldsymbol{n}}\right)_\tau - \Lambda\boldsymbol{v} \qquad \text{on} \quad \Gamma, \tag{II.22}$$

where

$$\Lambda\,\boldsymbol{v} = \sum_{k=1}^{2}\left(\boldsymbol{v}_\tau\cdot\frac{\partial\boldsymbol{n}}{\partial s_k}\right)\boldsymbol{\tau}_k.$$

For a weight k=0, we want to look for a weak solution for the problem (I.7)-(I.4), so the solution $\boldsymbol{u}$ is in $W_0^{1,2}(\Omega)$, which is why it is necessary to define the tangential trace of the strain tensor. To that end, we first introduce the spaces:

$$V_{\sigma,T}(\Omega) = \left\{\boldsymbol{v}\in W_0^{1,2}(\Omega),\ \operatorname{div}\boldsymbol{v}=0 \text{ in } \Omega \text{ and } \boldsymbol{v}\cdot\boldsymbol{n}=0 \text{ on } \Gamma\right\} \tag{II.23}$$

and

$$E(\Omega) = \left\{\boldsymbol{v}\in W_0^{1,2}(\Omega),\ \Delta\,\boldsymbol{v}\in[\mathring{H}^2_{-1}(\operatorname{div};\Omega)]'\right\}, \tag{II.24}$$

equipped with the norm of $W_0^{1,2}(\Omega)$ and

$$\|\boldsymbol{v}\|_{E(\Omega)} = \|\boldsymbol{v}\|_{W_0^{1,2}(\Omega)} + \|\Delta\boldsymbol{v}\ \|_{[\mathring{H}^2_{-1}(\operatorname{div};\Omega)]'}$$

respectively. We recall that the space $\mathcal{D}(\overline{\Omega})$ is dense in $E(\Omega)$ (see [7, Lemma 5.1]). As a consequence, we obtain the following trace result.

Lemma II.1 *The linear mapping $\Theta\,:\,\boldsymbol{u}\to[\boldsymbol{D}(\boldsymbol{u})\boldsymbol{n}]_\tau$ defined on $\mathcal{D}(\overline{\Omega})$ can be extended to a linear and continuous mapping*

$$\Theta\,:\,E(\Omega)\to H^{-1/2}(\Gamma).$$

Moreover, we have the Green formula: For any $\boldsymbol{u}\in E(\Omega)$ and $\boldsymbol{\varphi}\in V_{\sigma,T}(\Omega)$,

$$-\langle\Delta\,\boldsymbol{u},\boldsymbol{\varphi}\rangle_\Omega = 2\int_\Omega \boldsymbol{D}(\boldsymbol{u}):\boldsymbol{D}(\boldsymbol{\varphi})\,d\boldsymbol{x} - 2\,\langle[\boldsymbol{D}(\boldsymbol{u})\boldsymbol{n}]_\tau,\boldsymbol{\varphi}\rangle_\Gamma\,. \tag{II.25}$$

Where $\langle\cdot,\cdot\rangle_\Omega$ *denotes the dualities between* $[\mathring{H}^2_{-1}(\mathrm{div};\Omega)]$ *and* $\mathring{H}^2_{-1}(\mathrm{div};\Omega)$.

Proof. It is a standard proof that follows the ideas of [62]. The goal is to prove that the mapping Θ defined on $\mathcal{D}(\overline{\Omega})$ is continuous for the norm of $E(\Omega)$. Let us first observe that if $\boldsymbol{u}\in\mathcal{D}(\overline{\Omega})$ and $\boldsymbol{\varphi}\in W^{1,2}_0(\Omega)$ with $\boldsymbol{\varphi}\cdot\boldsymbol{n}=0$ on Γ, then thanks to the identity (II.10) and the Green formula (II.14), we have

$$-\langle\Delta\,\boldsymbol{u},\boldsymbol{\varphi}\rangle_\Omega = 2\int_\Omega \mathbf{D}(\boldsymbol{u}):\nabla\boldsymbol{\varphi}\,d\boldsymbol{x} - 2\langle\mathbf{D}(\boldsymbol{u})\boldsymbol{n},\boldsymbol{\varphi}\rangle_\Gamma - \int_\Omega \mathrm{div}\,\boldsymbol{u}\,\mathrm{div}\,\boldsymbol{\varphi}d\boldsymbol{x}.$$

Next, the fact that $\boldsymbol{\varphi}\cdot\boldsymbol{n}=0$ on Γ also implies that

$$\langle\mathbf{D}(\boldsymbol{u})\boldsymbol{n},\boldsymbol{\varphi}\rangle_\Gamma = \Big\langle\Big(\big[\mathbf{D}(\boldsymbol{u})\boldsymbol{n}\big]\cdot\boldsymbol{n}\Big)\boldsymbol{n}+[\mathbf{D}(\boldsymbol{u})\boldsymbol{n}]_\tau,\boldsymbol{\varphi}\Big\rangle_\Gamma = \langle[\mathbf{D}(\boldsymbol{u})\boldsymbol{n}]_\tau,\boldsymbol{\varphi}\rangle_\Gamma. \tag{II.26}$$

It follows that, for any $\boldsymbol{u}\in\mathcal{D}(\overline{\Omega})$ and $\boldsymbol{\varphi}\in V_{\sigma,T}(\Omega)$, we have

$$-\langle\Delta\,\boldsymbol{u},\boldsymbol{\varphi}\rangle_\Omega = 2\int_\Omega \mathbf{D}(\boldsymbol{u}):\nabla\boldsymbol{\varphi}\,d\boldsymbol{x} - 2\,\langle[\mathbf{D}(\boldsymbol{u})\boldsymbol{n}]_\tau,\boldsymbol{\varphi}\rangle_\Gamma\,.$$

Finally, since we have

$$\int_\Omega \mathbf{D}(\boldsymbol{u}):\nabla\boldsymbol{\varphi}\,d\boldsymbol{x} = \int_\Omega \mathbf{D}(\boldsymbol{u}):\mathbf{D}(\boldsymbol{\varphi})\,d\boldsymbol{x},$$

we arrive at

$$-\langle\Delta\,\boldsymbol{u},\boldsymbol{\varphi}\rangle_\Omega = 2\int_\Omega \mathbf{D}(\boldsymbol{u}):\mathbf{D}\,(\boldsymbol{\varphi})d\boldsymbol{x} - 2\,\langle[\mathbf{D}(\boldsymbol{u})\boldsymbol{n}]_\tau,\boldsymbol{\varphi}\rangle_\Gamma\,. \tag{II.27}$$

Let now $\boldsymbol{\mu}$ be any element of $H^{1/2}(\Gamma)$. Then, there exists an element $\boldsymbol{\varphi}$ in $W^{1,2}_0(\Omega)$, such that $\mathrm{div}\,\boldsymbol{\varphi}=0$ in Ω and $\boldsymbol{\varphi}=\boldsymbol{\mu}_\tau$ on Γ with the estimate

$$\|\boldsymbol{\varphi}\|_{W^{1,2}_0(\Omega)} \leqslant C\|\boldsymbol{\mu}_\tau\|_{H^{1/2}(\Gamma)} \leqslant C\|\boldsymbol{\mu}\|_{H^{1/2}(\Gamma)}. \tag{II.28}$$

Consequently, using (IV.42), we can write

$$\begin{aligned}
\Big|2\,\langle[\mathbf{D}(\boldsymbol{u})\boldsymbol{n}]_\tau,\boldsymbol{\mu}\rangle_\Gamma\Big| &= \Big|2\,\langle[\mathbf{D}(\boldsymbol{u})\boldsymbol{n}]_\tau,\boldsymbol{\mu}_\tau\rangle_\Gamma\Big| = \Big|2\,\langle[\mathbf{D}(\boldsymbol{u})\boldsymbol{n}]_\tau,\boldsymbol{\varphi}\rangle_\Gamma\Big| \\
&\leqslant \|\Delta\,\boldsymbol{u}\|_{[\mathring{H}^2_{-1}(\mathrm{div},\Omega)]}\|\boldsymbol{\varphi}\|_{H_{-1}(\mathrm{div},\Omega)} + 2\|\mathbf{D}(\boldsymbol{u})\|_{L^2(\Omega)}\|\mathbf{D}(\boldsymbol{\varphi})\|_{L^2(\Omega)} \\
&\leqslant \Big(\|\Delta\,\boldsymbol{u}\|^2_{[\mathring{H}^2_{-1}(\mathrm{div},\Omega)]'} + 2\|\mathbf{D}(\boldsymbol{u})\|^2_{L^2(\Omega)}\Big)^{1/2} \times \Big(\|\boldsymbol{\varphi}\|^2_{W^{1,2}_0(\Omega)} + 2\|\mathbf{D}(\boldsymbol{\varphi})\|^2_{L^2(\Omega)}\Big)^{1/2} \\
&\leqslant C\|\boldsymbol{u}\|_{E(\Omega)}\,\|\boldsymbol{\varphi}\|_{W^{1,2}_0(\Omega)}.
\end{aligned}$$

Using inequalities (II.28) we deduce that

$$\Big|2\,\langle[\mathbf{D}(\boldsymbol{u})\boldsymbol{n}]_\tau,\boldsymbol{\mu}\rangle_\Gamma\Big| \leqslant C\|\boldsymbol{u}\|_{E(\Omega)}\|\boldsymbol{\mu}\|_{H^{1/2}(\Gamma)}$$

which implies that

$$\|[\mathbf{D}(\boldsymbol{u})\boldsymbol{n}]_\tau\|_{H^{-1/2}(\Gamma)} \leqslant C\|\boldsymbol{u}\|_{E(\Omega)}.$$

Therefore, the linear mapping $\Theta : \boldsymbol{u} \to [\mathbf{D}(\boldsymbol{u})\boldsymbol{n}]_\tau$ defined in $\mathcal{D}(\overline{\Omega})$ is continuous for the norm of $E(\Omega)$. Since $\mathcal{D}(\overline{\Omega})$ is dense in $E(\Omega)$, Θ can be extended by continuity to a mapping still called Θ defined on $E(\Omega)$ to $H^{-1/2}(\Gamma)$ and formula (II.25) holds for all $\boldsymbol{u} \in E(\Omega)$ and $\varphi \in V_{\sigma,T}(\Omega)$. $\square$

5 Weighted Korn's inequalities

This part is devoted to the proof of weighted Korn's inequalities by following the approach proposed by Ciarlet and Ciarlet in [30], where the authors proposed another proof of Korn's inequalities in bounded domains based on the fact that symmetric tensor fields that satisfy the so-called Saint-Venant compatibility condition (see (II.32)), can be written as the rate-of-strain tensor of a vector field. Let us mention that other variant weighted Korn's inequalities can be found for instance in [53].

We start with a first lemma on a specific existence of vector potential.

Lemma II.2 *Suppose that Ω is of class $\mathcal{C}^{1,1}$ and let $k \in \mathbb{Z}$. Assume that $\boldsymbol{u}$ belongs to $\mathring{W}^{1,2}_k(\Omega)$ and satisfies* $\operatorname{div} \boldsymbol{u} = 0$ *in Ω. Then, there exists $\boldsymbol{\psi} \in \mathring{W}^{2,2}_k(\Omega)$ satisfying*

$$\boldsymbol{u} = \boldsymbol{curl}\ \boldsymbol{\psi} \quad \textit{in} \quad \Omega.$$

Proof. Denote by $\widetilde{\boldsymbol{u}}$ the extension of $\boldsymbol{u}$ by zero in Ω'. Then, $\widetilde{\boldsymbol{u}}$ belongs to $W^{1,2}_k(\mathbb{R}^3)$ and satisfies $\operatorname{div} \widetilde{\boldsymbol{u}} = 0$ in $\mathbb{R}^3$. Thanks to [43, theorems 3.2 and 3.5], there exists $\widetilde{\boldsymbol{w}}$ that belongs to $W^{2,2}_k(\mathbb{R}^3)$, such that

$$\widetilde{\boldsymbol{u}} = \mathbf{curl}\ \widetilde{\boldsymbol{w}} \quad \text{in} \quad \mathbb{R}^3.$$

Denote now by $\boldsymbol{w'}$ the restriction of $\widetilde{\boldsymbol{w}}$ on Ω'. Then $\mathbf{curl}\ \boldsymbol{w'} = \mathbf{0}$ in Ω' and as a consequence $\boldsymbol{w'} = \nabla\Phi'$, where $\Phi' \in H^3(\Omega')$ (see for instance [46]). Let $\widetilde{\Phi} \in W^{3,2}_k(\mathbb{R}^3)$ be an extension of Φ' in the whole space $\mathbb{R}^3$ and let $\widetilde{\boldsymbol{\psi}} \in W^{2,2}_k(\mathbb{R}^3)$ be defined by $\widetilde{\boldsymbol{\psi}} = \widetilde{\boldsymbol{w}} - \nabla\widetilde{\Phi}$. We have

$$\widetilde{\boldsymbol{\psi}}_{|\Omega'} = \widetilde{\boldsymbol{w}}_{|\Omega'} - \nabla\Phi' = \boldsymbol{w'} - \nabla\Phi' = \mathbf{0}. \tag{II.29}$$

As a consequence, denoting by $\boldsymbol{\psi}$ the restriction of $\widetilde{\boldsymbol{\psi}}$ to Ω, then $\boldsymbol{\psi}$ belongs to $\mathring{W}^{2,2}_k(\Omega)$. Furthermore, in view of (II.29), we have

$$\mathbf{curl}\ \widetilde{\boldsymbol{\psi}} = \mathbf{curl}\ \widetilde{\boldsymbol{w}} = \widetilde{\boldsymbol{u}} \quad \text{in} \quad \mathbb{R}^3,$$

which implies that

$$\boldsymbol{u} = \mathbf{curl}\ \boldsymbol{\psi} \quad \text{in} \quad \Omega.$$

□

The second lemma is a result on a variant of de Rham's theorem.

Lemma II.3 *Suppose that Ω is of class $\mathcal{C}^{1,1}$ and let $k \in \mathbb{Z}$. Assume that $\boldsymbol{f}$ belongs to $W_k^{-1,2}(\Omega)$ and satisfies*

$$\boldsymbol{curl}\ \boldsymbol{f} = \mathbf{0} \quad \textit{in} \quad \Omega. \tag{II.30}$$

Then, there exists $p \in W_k^{0,2}(\Omega)$ such that $\boldsymbol{f} = \nabla p$.

Proof. Let us introduce the space

$$\mathcal{D}_\sigma(\Omega) = \Big\{\boldsymbol{\varphi} \in \mathcal{D}(\Omega),\ \operatorname{div} \boldsymbol{\varphi} = 0\Big\}. \tag{II.31}$$

It is clear that if $\boldsymbol{\varphi}$ is in $\mathcal{D}_\sigma(\Omega)$, then it also belongs to $\mathring{W}_{-k}^{1,2}(\Omega)$. Thus, according to Lemma II.2, there exists $\boldsymbol{\psi} \in \mathring{W}_{-k}^{2,2}(\Omega)$ such that $\boldsymbol{\varphi} = \mathbf{curl}\ \boldsymbol{\psi}$. We can now write

$$\langle \boldsymbol{f}\ , \boldsymbol{\varphi}\rangle_{W_k^{-1,2}(\Omega)\times \mathring{W}_{-k}^{1,2}(\Omega)} = \langle \boldsymbol{f}\ , \mathbf{curl}\ \boldsymbol{\psi}\rangle_{W_k^{-1,2}(\Omega)\times \mathring{W}_{-k}^{1,2}(\Omega)} = \langle \mathbf{curl}\ \boldsymbol{f}\ , \boldsymbol{\psi}\rangle_{W_k^{-2,2}(\Omega)\times \mathring{W}_{-k}^{2,2}(\Omega)} = 0.$$

Hence, thanks to de Rham's theorem, there exists $q \in \mathcal{D}'(\Omega)$ such that $\boldsymbol{f} = \nabla q$. It follows from theorem II.4, that there exists a constant c such that $p = q + c \in W_k^{0,2}(\Omega)$ which ends the proof. □

Let us now introduce the spaces

$$W_k(\text{sym};\Omega) = \Big\{\boldsymbol{e} = (e_{ij}) \in W_k^{0,2}(\Omega),\ e_{ij} = e_{ji} \text{ in } \Omega\Big\}$$

and

$$\mathcal{K}(\Omega) = \Big\{\boldsymbol{v} \in \mathcal{P}_1,\ \boldsymbol{v}(\boldsymbol{x}) = \boldsymbol{a} + \boldsymbol{b} \times \boldsymbol{x},\ \boldsymbol{a},\ \boldsymbol{b} \in \mathbb{R}^3\Big\}.$$

The theorem below gives conditions that have to satisfy symmetric matrix fields to be identified as the rate-of-strain tensor of a vector field.

Theorem II.5 *Suppose that Ω is of class $\mathcal{C}^{1,1}$ and let $k \in \mathbb{Z}$. Assume that $\boldsymbol{e}$ belongs to $W_k(sym;\Omega)$ and satisfies the so-called Saint-Venant compatibility conditions*

$$\forall i,j,k,\ell = 1,2,3, \quad \mathcal{R}_{ijk\ell}(\boldsymbol{e}) = \partial_{\ell j} e_{ik} + \partial_{ki} e_{j\ell} - \partial_{\ell i} e_{jk} - \partial_{kj} e_{i\ell} = 0. \tag{II.32}$$

Then, there exists $\boldsymbol{v} \in W_k^{1,2}(\Omega)$ such that $\boldsymbol{e} = \boldsymbol{D}(\boldsymbol{v})$. The vector field $\boldsymbol{v}$ is unique if $k \geq 0$ and unique up to a an element of $\mathcal{K}(\Omega)$ if $k \leq -1$.

Proof.

- Uniqueness : If $\boldsymbol{v}$ belongs to $\mathcal{D}'(\Omega)$ and satisfies $\mathbf{D}(\boldsymbol{v}) = \mathbf{0}$, then $\boldsymbol{v}$ belongs to $\mathcal{K}(\Omega)$ (see for instance [29]). If in addition $\boldsymbol{v}$ belongs to $W_k^{1,2}(\Omega)$ with $k \geq 0$, then necessarily $\boldsymbol{v} = \mathbf{0}$ because in this case, there are no polynomials in the space $W_k^{1,2}(\Omega)$. Observe that if $k = -1$, then the space $\mathcal{K}(\Omega)$ is reduced to $\mathcal{P}_0$.

- Existence : Let $\boldsymbol{e}$ be in $W_k(\mathrm{sym};\Omega)$ and for any $i, j = 1, 2, 3$, let $\boldsymbol{f}_{ij}$ be defined by

$$f_{ijk} = \partial_j e_{ik} - \partial_i e_{jk}, \quad \forall k = 1, 2, 3.$$

  Then clearly $\boldsymbol{f}_{ij}$ belongs to $W_k^{-1,2}(\Omega)$ and one can observe that we have $\mathbf{curl}\ \boldsymbol{f}_{ij} = \mathbf{0}$ in Ω. Thanks to Lemma II.3, there exists $p_{ij} \in W_k^{0,2}(\Omega)$ such that $\boldsymbol{f}_{ij} = \nabla p_{ij}$. Observe also that $\nabla(p_{ij} + p_{ji}) = \mathbf{0}$ with $p_{ij} + p_{ji} \in W_k^{0,2}(\Omega)$. We deduce that, if $k \geq -1$, $p_{ij} + p_{ji} = 0$ and if $k \leq -2$, there exists a unique constant c such that $p_{ij} + p_{ji} = c$ and so we can choose the functions p_{ij} such that $p_{ij} + p_{ji} = 0$. Set now $q_{ij} = e_{ij} + p_{ij} \in W_k^{0,2}(\Omega) \subset W_{k-1}^{-1,2}(\Omega)$. One can verify that $\mathbf{curl}\ \boldsymbol{q}_i = \mathbf{0}$. Therefore, using again Lemma II.3, there exists v_i that belongs to $W_{k-1}^{0,2}(\Omega)$ such that $\boldsymbol{q}_i = \nabla v_i$. This shows that ∇v_i belongs to $W_k^{0,2}(\Omega)$. Consequently v_i belongs to $W_k^{1,2}(\Omega)$. Besides, since $p_{ij} + p_{ji} = 0$, we have

$$\frac{1}{2}(\partial_j v_i + \partial_i v_j) = \frac{1}{2}(q_{ij} + q_{ji}) = e_{ij} + \frac{1}{2}(p_{ij} + p_{ij}) = e_{ij},$$

  which shows that $\boldsymbol{e} = \mathbf{D}(\boldsymbol{v})$.

□

We are now in a position to state variant weighted Korn's inequalities.

Theorem II.6 *Suppose that Ω is of class $\mathcal{C}^{1,1}$ and let $k \geq 0$ be an integer. Then, there exists $C > 0$ such that*

$$\forall \boldsymbol{v} \in W_k^{1,2}(\Omega), \quad \|\boldsymbol{v}\|_{W_k^{1,2}(\Omega)} \leq C \|\boldsymbol{D}(\boldsymbol{v})\|_{W_k^{0,2}(\Omega)}. \tag{II.33}$$

Proof. Let us introduce the space

$$\mathcal{E}_k(\Omega) = \Big\{ \boldsymbol{e} \in W_k(\mathrm{sym};\Omega),\ \mathcal{R}_{ijk\ell}(\boldsymbol{e}) = 0 \Big\}.$$

Theorem II.5 allows to introduce the following linear mapping

$$\mathcal{F} : \mathcal{E}_k(\Omega) \mapsto W_k^{1,2}(\Omega),$$

defined for each $\boldsymbol{e} \in \mathcal{E}_k(\Omega)$ by $\mathcal{F}(\boldsymbol{e}) = \boldsymbol{v}$ the unique element of $W_k^{1,2}(\Omega)$ that satisfies $\boldsymbol{e} = \mathbf{D}(\boldsymbol{v})$.

The mapping $\mathcal{F}$ is injective since $\mathcal{F}(\boldsymbol{e}) = \mathbf{0}$ implies that $\boldsymbol{v} = \mathbf{0}$ which implies that $\boldsymbol{e} = \mathbf{D}(\mathbf{0}) = \mathbf{0}$. Next, for any $\boldsymbol{v} \in W_k^{1,2}(\Omega)$, then clearly $\mathbf{D}(\boldsymbol{v})$ belongs to $\mathcal{E}_k(\Omega)$ and this implies that the mapping $\mathcal{F}$ is onto. Finally the inverse mapping $\mathcal{F}^{-1} : W_k^{1,2}(\Omega) \mapsto \mathcal{E}_k(\Omega)$ defined for each $\boldsymbol{v} \in W_k^{1,2}(\Omega)$ by $\mathcal{F}^{-1}(\boldsymbol{v}) = \mathbf{D}(\boldsymbol{v})$, is clearly continuous. Hence from the closed Range theorem of Banach, the mapping $\mathcal{F}$ is an isomorphism. As a consequence, there exists a constant $C > 0$ such that

$$\forall \boldsymbol{e} \in \mathcal{E}_k(\Omega), \quad \|\mathcal{F}(\boldsymbol{e})\|_{W_k^{1,2}(\Omega)} \leq C\|\boldsymbol{e}\ \|_{W_k^{0,2}(\Omega)},$$

which implies (II.33). □

When $k \leq -1$, we have similar result in a quotient space because the uniqueness of the vector field $\boldsymbol{v}$ stated in theorem II.5 is up to an element of $\mathcal{K}(\Omega)$.

Theorem II.7 *Suppose that Ω is of class $\mathcal{C}^{1,1}$ and let $k \leq -1$ be an integer. Then, there exists $C > 0$ such that*

$$\forall \boldsymbol{v}\ \in W_k^{1,2}(\Omega), \quad \inf_{\boldsymbol{q}\ \in \mathcal{K}(\Omega)} \|\boldsymbol{v} + \boldsymbol{q}\ \|_{W_k^{1,2}(\Omega)} \leq C\|\boldsymbol{D}(\boldsymbol{v})\|_{W_k^{0,2}(\Omega)}. \tag{II.34}$$

Chapter III

Some results concerning the Laplace and the Stokes problems

Résumé: Dans ce chapitre, nous rappelons des résultats concernant le problème de Laplace dans un domaine extérieur de $\mathbb{R}^3$ (voir [5]). Une classe complète de résultats d'existence, d'unicité et de régularité est obtenue pour des conditions aux limites du type Dirichlet et Neumann, nous pouvons citer [6, 48, 63]. Nous étudions le problème de Laplace avec des conditions aux limites de type Neumann pour des données particulières $f \in L^p(\Omega)$. Ensuite, nous rappelons des résultats concernant le problème de Stokes dans $\mathbb{R}^3$ (voir [2]) ainsi que le problème de Stokes dans un domaine borné avec conditions sur le bord de type Dirichlet nous pouvons citer [4, 11, 38, 41, 42]. On termine ce chapitre par l'étude des solutions régulières du problème de Stokes avec des conditions mixte (Navier et Dirichlet) dans un domaine borné.

1 The Laplace's equation

This section is devoted to recall the solution of the Laplace equations in $\mathbb{R}^3$ (see [5]) and Ω, with either Dirichlet and Neumann boundary conditions on Γ (see for instance [6, 48, 63]). J. Giroire in [48], studied the Neumann problem in the Hilbert framework. In [63], the authors investigated the harmonic Neumann problem in L^p theory, for the exterior domain with boundary of class $\mathcal{C}^{1,1}$, note that these results have been also proved by Specovius-Neugebauer [75] with boundary of class at leas $\mathcal{C}^2$. C. Amrouche, V. Girault and J. Giroire in [6], studied the Neumann problem with data f belongs to $W_0^{-1,p}(\Omega) \cap L^p(\Omega)$, they got the existence of solutions in $W_0^{1,p}(\Omega)$. In this work, we give a result concerning the Laplace problem with Neumann boundary conditions with data f belongs to $L^p(\Omega)$, we got the existence of solutions in $W_{-1}^{1,p}(\Omega)$.

1.1 The Laplace operator in the whole space $\mathbb{R}^3$

In this subsection, we recall some results concerning the Laplace problem in the whole space $\mathbb{R}^3$. The proof of the following theorem can be found in [5, Theorems 6.6, 9.5, 9.9].

Theorem III.1 *Let k and p satisfy*

$$k \in \mathbb{Z}, \quad 3/p + k \notin \mathbb{Z}^- \quad \text{and } 3/p' - k \notin \mathbb{Z}^-. \qquad \textbf{\textit{(H)}}$$

Then the Laplace operator defined by

$$\Delta : W_k^{1,p}(\mathbb{R}^3)/P^{\triangle}_{[1-3/p-k]} \longrightarrow W_k^{-1,p}(\mathbb{R}^3) \perp P^{\triangle}_{[1-3/p'+k]} \tag{III.1}$$

is an isomorphism.
Assume moreover that $3/p + k \neq 1$ and $3/p' - k \neq 1$, then for any integer $m \geqslant 1$, the Laplace operators defined by

$$\Delta : W_{k+m}^{1+m,p}(\mathbb{R}^3)/P^{\triangle}_{[1-3/p-k]} \longrightarrow W_{k+m}^{-1+m,p}(\mathbb{R}^3) \perp P^{\triangle}_{[1-3/p'+k]} \tag{III.2}$$

and

$$\Delta : W_{-k-m}^{1-m,p}(\mathbb{R}^3)/P^{\triangle}_{[1-3/p+k]} \longrightarrow W_{-k-m}^{-1-m,p}(\mathbb{R}^3) \perp P^{\triangle}_{[1-3/p'-k]} \tag{III.3}$$

are isomorphisms.

The above isomorphism results are also valid for real values of k satisfying at least **(H)**. But for the sake of simplicity, we restrict ourselves to $k \in \mathbb{Z}$.

1.2 The Laplace problem with Dirichlet boundary condition

In the following subsection, we assume that Ω is an exterior domain of $\mathbb{R}^3$ with Γ of class $\mathcal{C}^{1.1}$ if $p \neq 2$ and Lipschitz-continuous if $p = 2$. We recall some results related to the following Laplace equation with Dirichlet boundary condition:

$$-\Delta\, u = f \quad \text{in} \quad \Omega \quad \text{and} \quad u = g \quad \text{on} \quad \Gamma, \tag{III.4}$$

where f is in $W_k^{-1,p}(\Omega)$ and g is in $W^{1-1/p,p}(\Gamma)$. We start by giving the definition of the kernel of the Laplace operator for any integer $k \in \mathbb{Z}$:

$$\mathcal{A}^{\triangle}_{k,p} = \left\{ \chi \in W^{1,p}_{-k}(\Omega); \quad \Delta\, \chi = 0 \quad \text{in} \quad \Omega \quad \text{and} \quad \chi = 0 \quad \text{on } \Gamma \right\}.$$

In contrast with a bounded domain, the Dirichlet problem for the Laplace operator with zero data can have nontrivial solutions in an exterior domain; it depends upon the exponent of the weight. We recall the characterizations proved in [63] for $k \in \mathbb{Z}$ and $1 < p < \infty$.

Proposition III.1 *The space $\mathcal{A}^{\Delta}_{k,p}$ is a subspace of all functions in $W^{1,p}_k(\Omega)$ of the form $v(\lambda) - \lambda$, where λ runs over all polynomials of $\mathcal{P}^{\Delta}_{[1-3/p-k]}$ and $v(\lambda)$ is the unique solution in $W^{1,2}_0(\Omega) \cap W^{1,p}_k(\Omega)$ of the Dirichlet problem*

$$\Delta\, v(\lambda) = 0 \quad \text{in} \quad \Omega \quad \text{and} \quad v(\lambda) = \lambda \quad \text{on } \Gamma. \tag{III.5}$$

In particular, $\mathcal{A}^{\Delta}_{k,p} = \{0\}$ when $k > 1 - 3/p$.
Moreover, $\mathcal{A}^{\Delta}_{k,p}$ is a finite-dimensional space of the same dimension as $\mathcal{P}^{\Delta}_{[1-3/p-k]}$.

Now, we recall the following theorem of existence and uniqueness for any $k \in \mathbb{Z}$ and $1 < p < \infty$.

Theorem III.2 *Let k and p satisfy **(H)**. Then for any f in $W^{-1,p}_k(\Omega)$ and g in $W^{1-1/p,p}(\Gamma)$ such that*

$$\forall \lambda \in \mathcal{A}^{\Delta}_{-k,p'}, \quad \langle f, \lambda \rangle_{W^{-1,p}_k(\Omega) \times \mathring{W}^{1,p'}_{-k}(\Omega)} = \langle g, \frac{\partial \lambda}{\partial \boldsymbol{n}} \rangle_{W^{1/p',p}(\Gamma) \times W^{-1/p',p'}(\Gamma)}, \tag{III.6}$$

problem (III.4) has a unique solution u in $W^{1,p}_k(\Omega) \big/ \mathcal{A}^{\Delta}_{k,p}$. In addition, there exists a constant C, independent of u, f and g, such that

$$\|u\|_{W^{1,p}_k(\Omega) \big/ \mathcal{A}^{\Delta}_{k,p}} \leqslant C \Big(\|f\|_{W^{-1,p}_k(\Omega)} + \|g\|_{W^{1/p',p}(\Gamma)} \Big) \tag{III.7}$$

The proof of Theorem III.2, can be found for the case $p = 2$ and $k \in \mathbb{Z}$ in [48], the case $k = 0$ and $1 < p < \infty$ is proved by [4] and finally the case $\boldsymbol{f} = \mathbf{0}$, $k \in \mathbb{Z}$ and $1 < p < \infty$ has been studied in [63].

Remark III.1 *Let Ω be an exterior domain of $\mathbb{R}^3$ with boundary Γ of class $\mathcal{C}^{1.1}$, assume that f is in $W^{0,p}_k(\Omega)$ and g in $W^{1+1/p',p}(\Gamma)$. Then, if k and p satisfy **(H)** and the additional assumptions $3/p + k \neq 1$ and $3/p + k \neq 2$, then one can prove that the solution u discussed in Theorem III.2 belongs to $W^{2,p}_k(\Omega) \big/ \mathcal{A}^{\Delta}_{k-1,p}$. In addition, there exists a constant C, independent of u, f and g, such that*

$$\|u\|_{W^{2,p}_k(\Omega) \big/ \mathcal{A}^{\Delta}_{k-1,p}} \leqslant C \Big(\|f\|_{W^{0,p}_k(\Omega)} + \|g\|_{W^{1+1/p',p}(\Gamma)} \Big) \tag{III.8}$$

This regularity result is obtained via an adequate partition of unity that allows to split u into the solution of a Laplace problem in $\mathbb{R}^3$ is solved by the Theorem III.1 (isomorphism (III.2) with

$m = 1$ and k replaced $k-1$ and the solution of a Dirichlet problem for the Laplace operator in a bounded domain.

1.3 The Laplace problem with Neumann boundary condition

We consider the following Laplace problem:

$$\Delta u = f \quad \text{in} \quad \Omega, \quad \frac{\partial u}{\partial \boldsymbol{n}} = g \quad \text{on} \quad \Gamma. \tag{III.9}$$

We define the Kernel of the Laplace with Neumann boundary condition:

$$\mathcal{N}^{\Delta}_{k,p} = \left\{ \chi \in W^{1,p}_k(\Omega); \quad \Delta \chi = 0 \quad \text{in } \Omega \quad \text{and} \quad \frac{\partial \chi}{\partial \boldsymbol{n}} = 0 \quad \text{on } \Gamma \right\}.$$

The characterizations for $k = 0$ and $1 < p < \infty$ is proved by [6] and the case $k \in \mathbb{Z}$, $p = 2$ has been studied in [48]. We recall the characterization for $k \in \mathbb{Z}$ and $1 < p < \infty$(see [63]).

Proposition III.2 *The space $\mathcal{N}^{\Delta}_{k,p}$ is a subspace of all functions in $W^{1,p}_k(\Omega)$ of the form $w(\lambda) - \lambda$, where λ runs over all polynomials of $\mathcal{P}^{\Delta}_{[1-3/p-k]}$ and $w(\lambda)$ is the unique solution in $W^{1,2}_0(\Omega) \cap W^{1,p}_k(\Omega)$ of the Neumann problem*

$$\Delta w(\lambda) = 0 \quad \text{in } \Omega \quad \text{and} \quad \frac{\partial w(\lambda)}{\partial \boldsymbol{n}} = \frac{\partial \lambda}{\partial \boldsymbol{n}} \quad \text{on } \Gamma. \tag{III.10}$$

Here also, we set $\mathcal{N}^{\Delta}_{k,p} = \{0\}$ when $k \leq 1 - 3/p$ and $\mathcal{N}^{\Delta}_{k,p} = \mathbb{R}$ if $-3/p \leqslant k \leqslant 1 - 3/p$. Moreover, $\mathcal{N}^{\Delta}_{k,p}$ is a finite-dimentional space of the same dimension as $\mathcal{P}^{\Delta}_{[1-3/p-k]}$.

We recall now the two theorems blow that solve the harmonic Neumann problem (see [63])

Theorem III.3 *Let k and p satisfy **(H)**. Assume moreover that $k < -1 + 3/p$. Let $f = 0$ and for any g in $W^{-1/p,p}(\Gamma)$, problem (III.9) has a solution u in $W^{1,p}_k(\Omega)$ unique up to an element of $\mathcal{N}^{\Delta}_{k,p}$. In addition, there exists a constant C, independent of u and g, such that*

$$\|u\|_{W^{1,p}_k(\Omega)/\mathcal{N}^{\Delta}_{k,p}} \leqslant C \|g\|_{W^{-1/p,p}(\Gamma)} \tag{III.11}$$

Theorem III.4 *Let k and p satisfy **(H)**. Assume moreover that $k \geq -1 + 3/p$. Let $f = 0$ and for any g in $W^{-1/p,p}(\Gamma)$, problem (III.9) has a solution u in $W^{1,p}_k(\Omega)$ if and only if*

$$\forall v \in \mathcal{N}^{\Delta}_{-k,p'}, \quad \langle g, v \rangle_{W^{-1/p,p}(\Gamma) \times W^{1/p,p'}(\Gamma)} = 0. \tag{III.12}$$

The solution u is unique up to an element of $\mathcal{N}^{\Delta}_{k,p}$. In addition, there exists a constant C,

independent of u and g, such that

$$\|u\|_{W^{1,p}_k(\Omega)/\mathcal{N}^{\Delta}_{k,p}} \leqslant C\|g\|_{W^{-1/p,p}(\Gamma)} \tag{III.13}$$

Remark III.2 *If Ω is an exterior domain with boundary of class $\mathcal{C}^{1,1}$ and if the datum has the additional regularity: g belongs to $W^{1/p',p}(\Gamma)$ and if k and p satisfy the additional assumptions $3/p+k \neq 1$ and $3/p+k \neq 2$, then as in Remark III.1, the solution u of Problem (III.9) discussed in Theorem III.3 and Theorem III.4 belongs to $W^{2,p}_k(\Omega)$.*

The next theorem states an existence, uniqueness and regularity result for problem (III.9), for $k = 0$ and $1 < p < \infty$.

Theorem III.5 *Let $1 < p < \infty$ be any real number and assume that Γ is of class $\mathcal{C}^{1,1}$ if $p \neq 2$. For any f in $L^p(\Omega)$ and g in $W^{-1/p,p}(\Gamma)$. Then, the problem (III.9) has a solution $u \in W^{1,p}_{-1}(\Omega)$ unique up to element of $\mathcal{N}^{\Delta}_{-1,p}(\Omega)$ and we have the following estimate:*

$$\|u\|_{W^{1,p}_{-1}(\Omega)/\mathcal{N}^{\Delta}_{-1,p}(\Omega)} \leqslant C\Big(\|f\|_{L^p(\Omega)} + \|g\|_{W^{-1/p,p}(\Omega)}\Big) \tag{III.14}$$

If in addition, g in $W^{1/p',p}(\Gamma)$, the solution u of problem (III.9) belongs to $W^{2,p}_0(\Omega)$ and satisfies

$$\|u\|_{W^{2,p}_0(\Omega)} \leqslant C\Big(\|f\|_{L^p(\Omega)} + \|g\|_{W^{1/p',p}(\Omega)}\Big) \tag{III.15}$$

Proof. Let us extend f by zero in Ω' and let $\widetilde{f}$ denote the extended function. Then $\widetilde{f}$ belongs to $L^p(\mathbb{R}^3)$. Thanks Theorem III.1, there exists a unique function $\widetilde{v} \in W^{2,p}_0(\mathbb{R}^3)/\mathcal{P}^{\Delta}_{[2-3/p]}$ such that

$$-\Delta\,\widetilde{v} = \widetilde{f} \quad \text{in} \quad \mathbb{R}^3. \tag{III.16}$$

Then $\nabla\,\widetilde{v}\cdot\boldsymbol{n}$ belongs to $W^{1-1/p,p}(\Gamma) \hookrightarrow W^{-1/p,p}(\Gamma)$. It follows from Theorem III.3, that the following problem:

$$\Delta\, w = 0 \quad \text{in} \quad \Omega \quad \text{and} \quad \nabla\, w\cdot\boldsymbol{n} = g - \nabla\,\widetilde{v}\cdot\boldsymbol{n} \quad \text{on} \quad \Gamma, \tag{III.17}$$

has a solution $w \in W^{1,p}_{-1}(\Omega)$ unique up to element of $\mathcal{N}^{\Delta}_{-1,p}(\Omega)$. Thus $u = \widetilde{v}_{|\Omega} + w \in W^{1,p}_{-1}(\Omega)$ is the required solution of $(\mathcal{L}_p)$. The uniqueness follows immediately from Proposition III.2 .
Now, suppose that g belongs to $W^{1/p',p}(\Gamma)$. The aim is to prove that u belongs to $W^{2,p}_0(\Omega)$. To that end, let us introduce the following partition of unity:

$$\begin{aligned} &\varphi,\,\psi \in \mathcal{C}^{\infty}(\mathbb{R}^3), \quad 0 \leq \varphi,\,\psi \leq 1, \quad \varphi + \psi = 1 \quad \text{in} \quad \mathbb{R}^3, \\ &\varphi = 1 \quad \text{in} \quad B_R, \quad \operatorname{supp}\varphi \subset B_{R+1}. \end{aligned} \tag{III.18}$$

Let P be a continuous linear mapping from $W^{1,p}_{-1}(\Omega)$ to $W^{1,p}_{-1}(\mathbb{R}^3)$, such that $P\,u = \widetilde{u}$. Then $\widetilde{u}$ belongs to $W^{1,p}_{-1}(\mathbb{R}^3)$ and can be written as:

$$\widetilde{u} = \varphi\,\widetilde{u} + \psi\,\widetilde{u}.$$

Next, one can easily observe that $\widetilde{u}$ satisfies the following problem:

$$-\Delta\,\psi\,\widetilde{u} = f_1 \quad \text{in} \quad \mathbb{R}^3, \tag{III.19}$$

with

$$f_1 = \widetilde{f}\psi - (2\nabla\widetilde{u}\nabla\psi + \widetilde{u}\Delta\,\psi).$$

Owing to the support of ψ, f_1 has the same regularity as f and so belongs to $L^p(\mathbb{R}^3)$. It follows from theorem III.1, that there exists z in $W^{2,p}_0(\mathbb{R}^3)$ such that $-\Delta\,z = f_1$ in $\mathbb{R}^3$. This implies that $\psi\,\widetilde{u} - z$ is a harmonic tempered distribution and therefore a harmonic polynomial that belongs to $\mathcal{P}^{\Delta}_{[2-3/p]}$. The fact that $\mathcal{P}^{\Delta}_{[2-3/p]} \subset W^{2,p}_0(\mathbb{R}^3)$ yields that $\psi\,\widetilde{u}$ belongs to $W^{2,p}_0(\mathbb{R}^3)$. In particular, we have $\psi\,\widetilde{u} = u$ outside B_{R+1}, so the restriction of u to $\partial\,B_{R+1}$ belongs to $W^{2-1/p,p}(\partial\,B_{R+1})$. Therefore, $\varphi\,u \in W^{1,p}(\Omega_{R+1})$ satisfies the following problem:

$$\begin{cases} -\Delta\,\varphi\,u = f_2 & \text{in} \quad \Omega_{R+1}, \\ \nabla\,\varphi\,u\cdot\boldsymbol{n} = g & \text{on} \quad \Gamma, \\ \varphi\,u = \psi\,\widetilde{u} & \text{on} \quad \partial\,B_{R+1}, \end{cases} \tag{III.20}$$

where $f_2 \in L^p(\Omega_{R+1})$ have similar expression as f_1 with ψ remplaced by φ. According [Remark 3.2, [6]], $\varphi\,u \in W^{2,p}(\Omega_{R+1})$, which in turn shows that $\varphi\,\widetilde{u}$ also belongs to $W^{2,p}(\Omega_{R+1})$. This implies that $u \in W^{2,p}_0(\Omega)$. □

2 The Stokes problems

The purpose of this section is to present some results about the Stokes problem in the whole space $\mathbb{R}^3$ and in a bounded domain Ω' of $\mathbb{R}^3$ with Dirichlet boundary conditions. We end this section by solving the mixed Stokes problem with Navier and Dirichlet boundary conditions.

2.1 The Stokes problem in the whole space $\mathbb{R}^3$

We recall here some basic results concerning the Stokes problem in $\mathbb{R}^3$:

$$-\Delta\, \boldsymbol{u} + \nabla\, \pi = \boldsymbol{f} \quad \text{and} \quad \operatorname{div} \boldsymbol{u} = \chi \quad \text{in} \quad \mathbb{R}^3\,. \tag{III.21}$$

These results can be found in [43] for $p = 2$ and [2] for $1 < p < \infty$. Let us first introduce the kernel of the Stokes operator

$$\mathcal{N}_k^p(\mathbb{R}^3) = \Big\{(\boldsymbol{u}, \pi) \in W_k^{1,p}(\mathbb{R}^3) \times W_k^{0,p}(\mathbb{R}^3),\ \ -\Delta\, \boldsymbol{u} + \nabla\, \pi = \boldsymbol{0} \ \text{ and } \ \operatorname{div} \boldsymbol{u} = 0 \ \text{ in } \ \mathbb{R}^3\Big\}$$

and the space of polynomials

$$N_k = \Big\{(\boldsymbol{\lambda}, \mu) \in \mathcal{P}_k \times \mathcal{P}_{k-1}^{\Delta},\ \ -\Delta\, \boldsymbol{\lambda} + \nabla\, \mu = \boldsymbol{0} \ \text{ and } \ \operatorname{div} \boldsymbol{\lambda} = 0\Big\}.$$

Recall that by agreement on the notation $\mathcal{P}_k$, the space $N_k = \{(\boldsymbol{0}, 0)\}$ when $k < 0$ and $N_0 = \mathcal{P}_0 \times \{0\}$.

The next proposition characterizes the kernel of (III.21).

Proposition III.3 *Let $1 < p < \infty$ and k be integers. Then $\mathcal{N}_k^p(\mathbb{R}^3) = N_{[1-k-3/p]}$. In particular, $\mathcal{N}_k^p(\mathbb{R}^3) = \{(\boldsymbol{0}, 0)\}$ if $k > 1 - 3/p$.*

The next theorem states an existence, uniqueness and regularity result for problem (III.21).

Theorem III.6 *Assume that $k \in \mathbb{Z}$ and $1 < p < \infty$. If $(\boldsymbol{f}, \chi) \in W_k^{-1,p}(\mathbb{R}^3) \times W_k^{0,p}(\mathbb{R}^3)$ satisfies the compatibility condition:*

$$\forall (\boldsymbol{\lambda}, \mu) \in N_{[1-3/p'+k]}, \quad \langle \boldsymbol{f}, \boldsymbol{\lambda}\rangle_{W_k^{-1,p}(\mathbb{R}^3)\times W_{-k}^{1,p'}(\mathbb{R}^3)} - \langle \chi, \mu\rangle_{W_k^{0,p}(\mathbb{R}^3)\times W_{-k}^{0,p'}(\mathbb{R}^3)} = 0, \tag{III.22}$$

then problem (III.21) has a solution $(\boldsymbol{u}, \pi) \in W_k^{1,p}(\mathbb{R}^3) \times W_k^{0,p}(\mathbb{R}^3)$ unique up to an element of $N_{[1-3/p-k]}$ and we have the estimate

$$\inf_{(\boldsymbol{\lambda},\mu)\in N_{[1-3/p-k]}} \Big(\|\boldsymbol{u} + \boldsymbol{\lambda}\|_{W_k^{1,p}(\mathbb{R}^3)} + \|\pi + \mu\|_{W_k^{0,p}(\mathbb{R}^3)}\Big) \leqslant C\Big(\|\boldsymbol{f}\,\|_{W_k^{-1,p}(\mathbb{R}^3)} + \|\chi\|_{W_k^{0,p}(\mathbb{R}^3)}\Big).$$

Furthermore, if $(\boldsymbol{f}, \chi) \in W_{k+1}^{0,p}(\mathbb{R}^3) \times W_{k+1}^{1,p}(\mathbb{R}^3)$, then $(\boldsymbol{u}, \pi) \in W_{k+1}^{2,p}(\mathbb{R}^3) \times W_{k+1}^{1,p}(\mathbb{R}^3)$.

The proofs of Proposition III.3 and Theorem III.6 can be found in [2].

2.2 The Stokes problem with Dirichlet boundary conditions in bounded domain

In this part, we recall some results concerning the Stokes problem with Dirichlet boundary condition in an bounded domain Ω':

$$\begin{cases} -\Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{f} & \text{in } \Omega', \\ \operatorname{div} \boldsymbol{u} = \chi & \text{in } \Omega', \\ \boldsymbol{u} = \boldsymbol{g} & \text{on } \partial\Omega'. \end{cases} \tag{III.23}$$

Basic results on weak and strong solutions of problem (III.23) may be summarized by the following theorem (see [4, 38]).

Theorem III.7 *For every $\boldsymbol{f}$, χ, $\boldsymbol{g}$ with*

$$\boldsymbol{f} \in W^{-1,p}(\Omega'), \quad \chi \in L^p(\Omega'), \quad \boldsymbol{g} \in W^{1-1/p,p}(\partial\Omega'),$$

and satisfying the following compatibility condition:

$$\int_{\Omega'} \chi d\boldsymbol{x} = \langle \boldsymbol{g} \cdot \boldsymbol{n}, 1 \rangle_{W^{-1/p,p}(\partial\Omega') \times W^{1/p,p'}(\partial\Omega')}, \tag{III.24}$$

the Stokes problem (III.23) has a unique solution $\boldsymbol{u} \in W^{1,p}(\Omega')$ and $\pi \in L^p(\Omega')/\mathbb{R}$. Moreover, there exists a constant $C > 0$ depending only on p and Ω' such that

$$\|\boldsymbol{u}\|_{W^{1,p}(\Omega')} + \|\pi\|_{L^p(\Omega')/\mathbb{R}} \leqslant C\Big(\|\boldsymbol{f}\|_{W^{-1,p}(\Omega')} + \|\chi\|_{L^p(\Omega')} + \|\boldsymbol{g}\|_{W^{1-1/p,p}(\partial\Omega')} \Big). \tag{III.25}$$

Moreover, if

$$\boldsymbol{f} \in L^p(\Omega'), \quad \chi \in W^{1,p}(\Omega'), \quad \boldsymbol{g} \in W^{2-1/p,p}(\partial\Omega'),$$

then $\boldsymbol{u} \in W^{2,p}(\Omega')$, $\pi \in W^{1,p}(\Omega')$ and there exists a constant $C > 0$ depending only on p and Ω' such that

$$\|\boldsymbol{u}\|_{W^{2,p}(\Omega')} + \|\pi\|_{W^{1,p}(\Omega')} \leqslant C\Big(\|\boldsymbol{f}\|_{L^p(\Omega')} + \|\chi\|_{W^{1,p}(\Omega')} + \|\boldsymbol{g}\|_{W^{2-1/p,p}(\partial\Omega')} \Big). \tag{III.26}$$

2.3 A mixed Stokes problem

Let $R > 0$ be a real number large enough so that $\overline{\Omega'} \subset B_R$. We recall that $\Omega_R = \Omega \cap B_R$ and $\partial B_R = \Gamma_R$. Now, we study the following mixed boundary value problem: Given $\boldsymbol{f}$, χ, g, $\boldsymbol{h}$ and $\boldsymbol{a}$. We look for $(\boldsymbol{u}, \pi)$ satisfying

$$\begin{cases} -\Delta\, \boldsymbol{u} + \nabla\, \pi = \boldsymbol{f} \quad \textit{and} \quad \operatorname{div} \boldsymbol{u} = \chi & \text{in } \Omega_R, \\ \boldsymbol{u} \cdot \boldsymbol{n} = g \quad \textit{and} \quad 2[\mathbf{D}(\boldsymbol{u})\boldsymbol{n}]_\tau + \alpha \boldsymbol{u}_\tau = \boldsymbol{h} & \text{on } \Gamma, \\ \boldsymbol{u} = \boldsymbol{a} \ \text{ on } \Gamma_R. \end{cases} \tag{III.27}$$

Theorem III.8 *Assume that $1 < p < \infty$. Let the pair $(\boldsymbol{u}, \pi) \in H^2(\Omega_R) \times H^1(\Omega_R)$ be a solution to the problem (III.27) with data $\boldsymbol{f} \in L^p(\Omega_R)$, $\chi \in W^{1,p}(\Omega_R)$, $g \in W^{1+1/p',p}(\Gamma)$, $\boldsymbol{h} \in W^{1-1/p,p}(\Gamma)$, $\boldsymbol{a} \in W^{1+1/p',p}(\partial B_R)$ such that $\boldsymbol{h} \cdot \boldsymbol{n} = 0$ on Γ. Then, we also have $\boldsymbol{u} \in W^{2,p}(\Omega_R)$ and $\pi \in W^{1,p}(\Omega_R)$.*

Proof.

Step 1. The case $1 < p \leqslant 2$.
Since $(\boldsymbol{u}, \pi) \in H^2(\Omega_R) \times H^1(\Omega_R)$ then its is clear that $(\boldsymbol{u}, \pi) \in W^{2,p}(\Omega_R) \times W^{1,p}(\Omega_R)$ for any $1 < p \leqslant 2$.

Step 2. The case $2 < p < \infty$.
Let us introduce the following partition of unity:

$$\begin{aligned} &\theta_1,\, \theta_2 \in \mathcal{C}^\infty(\overline{\Omega}_R), \quad 0 \leq \theta_1,\, \theta_2 \leq 1, \quad \theta_1 + \theta_2 = 1 \quad \text{in} \quad \Omega_R, \\ &\theta_1 = 1 \quad \text{in} \quad B_{R/3}, \quad \operatorname{supp} \theta_1 \subset B_{2R/3}. \end{aligned}$$

Then we can write $\boldsymbol{u} = \theta_1 \boldsymbol{u} + \theta_2 \boldsymbol{u} = \boldsymbol{u}_1 + \boldsymbol{u}_2$ and $\pi = \theta_1 \pi + \theta_2 \pi = \pi_1 + \pi_2$. Now it is clear that

$$-\Delta\, \boldsymbol{u}_1 + \nabla\, \pi_1 = \boldsymbol{f}_1 \quad \text{and} \quad \operatorname{div} \boldsymbol{u}_1 = \chi_1 \quad \text{in } \Omega_R, \tag{III.28}$$

where $\boldsymbol{f}_1 = \theta_1 \boldsymbol{f} - \boldsymbol{u}\, \Delta\, \theta_1 - 2\nabla\, \boldsymbol{u} \cdot \nabla\, \theta_1 + \pi\, \nabla\, \theta_1 \in L^p(\Omega_R)$ and $\chi_1 = \theta\, \chi + \boldsymbol{u}\, \nabla\, \theta_1 \in W^{1,p}(\Omega_R)$ for $2 \leqslant p \leqslant 6$. It is also clear that on the boundaries, we have

$$\begin{cases} \boldsymbol{u}_1 \cdot \boldsymbol{n} = g \quad , \quad 2[\mathbf{D}(\boldsymbol{u}_1)\boldsymbol{n}]_\tau + \alpha \boldsymbol{u}_{1\tau} = \boldsymbol{h} & \text{on } \Gamma, \\ \boldsymbol{u}_1 \cdot \boldsymbol{n} = 0 \quad , \quad 2[\mathbf{D}(\boldsymbol{u}_1)\boldsymbol{n}]_\tau + \alpha \boldsymbol{u}_{1\tau} = \boldsymbol{0} & \text{on } \partial B_R, \end{cases} \tag{III.29}$$

where $\boldsymbol{h} \in W^{1-1/p,p}(\Gamma)$ and $g \in W^{1+1/p',p}(\Gamma)$, we deduce that $(\boldsymbol{u}_1, \pi_1)$ that satisfies (III.28)-(III.29) belongs to $W^{2,p}(\Omega_R) \times W^{1,p}(\Omega_R)$ for $2 < p \leqslant 6$ see [1, Theorem 2.1]. Similar arguments show $(\boldsymbol{u}_2, \pi_2)$ satisfies the following Stokes problem with the Dirichlet

boundary

$$\begin{cases} -\Delta\, \boldsymbol{u}_2 + \nabla\, \pi_2 = \boldsymbol{f}_2 \quad \text{and} \quad \operatorname{div} \boldsymbol{u}_2 = \chi_2 \quad \text{in} \;\; \Omega_R, \\ \boldsymbol{u}_2 = 0 \quad \text{on} \;\; \Gamma \quad \text{and} \quad \boldsymbol{u}_2 = \boldsymbol{a} \quad \text{on} \;\; \partial B_R. \end{cases} \tag{III.30}$$

Where $\boldsymbol{f}_2 \in L^p(\Omega_R)$ and $\chi_2 \in W^{1,p}(\Omega_R)$ have similar expression as $\boldsymbol{f}_1$ and χ_1 with θ_1 replaced by θ_2. Since $\boldsymbol{a} \in W^{1+1/p',p}(\partial B_R)$, then according to Theorem III.7, $(\boldsymbol{u}_2, \pi_2)$ belongs to $W^{2,p}(\Omega_R) \times W^{1,p}(\Omega_R)$.
Now, suppose that $p > 6$. The above argument shows that $(\boldsymbol{u}, \pi)$ belongs to $W^{2,6}(\Omega_R) \times W^{1,6}(\Omega_R)$ and we can repeat the same argument with $p = 6$ instead $p = 2$. We have $W^{1,6}(\mathcal{O}) \hookrightarrow L^q(\mathcal{O})$ for any real number $q > 6$. We know that the embedding

$$W^{2,p}(\Omega') \hookrightarrow W^{1,q}(\Omega'),$$

for any $q \in [1, \infty]$ if $p > 3$. Then we have $\boldsymbol{u}_{|\Gamma} \in W^{2-1/6,6}(\Gamma) \hookrightarrow W^{1-1/p}(\Gamma)$ for all $p > 6$. Consequently, by the some reasoning we deduce that the solution $(\boldsymbol{u}, \pi)$ belongs to $W^{2,p}(\Omega_R) \times W^{1,p}(\Omega_R)$.

$\square$

Chapter IV

Weighted Hilbert spaces for the Stationary exterior Stokes problem with Navier slip boundary conditions

Résumé: L'objet de ce chapitre est la résolution du problème stationnaire de **Stokes** avec des conditions aux limites de types **Navier** dans un domaine extérieur Ω de $\mathbb{R}^3$. Nous montrons ici l'existence et l'unicité de solutions variationnelle et de solutions fortes ainsi que les solutions très faible du problème de Stokes. Nous nous intéressons, dans ce chapitre, par le cadre hilbertien.

1 Introduction

Let Ω' be a bounded connected open set in $\mathbb{R}^3$ with boundary $\partial\Omega' = \Gamma$ of class $C^{1,1}$ representing an obstacle and let Ω its complement *i.e.* $\Omega = \mathbb{R}^3 \diagdown \overline{\Omega'}$. We consider the Stokes problem with Navier slip boundary conditions in Ω: Given vectors fields $\boldsymbol{f}$ and $\boldsymbol{h}$ and a scalars functions χ and g, we assume that α is a positive function.

$$(\mathcal{S}_T)\begin{cases} -\Delta\,\boldsymbol{u} + \nabla\,\pi = \boldsymbol{f} \quad \text{and} \quad \operatorname{div}\boldsymbol{u} = \chi \quad \text{in} \quad \Omega, \\ \boldsymbol{u}\cdot\boldsymbol{n} = g \quad \text{and} \quad 2[\mathbf{D}(\boldsymbol{u})\boldsymbol{n}]_\tau + \alpha\boldsymbol{u}_\tau = \boldsymbol{h} \quad \text{on} \quad \Gamma. \end{cases}$$

Where the unknowns are $\boldsymbol{u}$ the velocity of the fluid and its pressure π.
Thus, instead of the inequalities of Poincaré, we are led to establish inequalities of Korn

Our analysis of problem $(\mathcal{S}_T)$ follows the same lines as in [48] for the study of the exterior Laplace problems or [47] and [44] for the exterior Stokes problem with no-slip boundary conditions: we first look for the variational solution. The fact of using the Navier conditions, leads to writing the variational formulation, not as a function of gradients, but as a function of strain tensors (I.5). Thus, instead of the inequalities of Poincaré, but we need the Korn's inequalities (established in Chapter II) in order to obtain the coercivity of the bilinear form associated with the variational formulation. Next, we look for solutions that have a different behavior at infinity from the variational solution. To that end, we combine results on the Stokes problem set in the whole space to catch the behavior at infinity and results in bounded domains to take into account the boundary conditions. We start by looking for solutions that grow at infinity or decay weaker than the variational solution. But afterwards, unlike the exterior problem of Laplace with Dirichlet boundary or the problem $(\mathcal{S}_T)$, duality arguments can not be applied in our context to find solutions that decay faster than the variational solution. Instead we use arguments that are similar as proving regularity results. Here, the arguments are based on adequate extensions of solutions to $(\mathcal{S}_T)$ that we already have at hand. These techniques were previoulsy applied in [48] (see also [63]) for the exterior Laplace equation with Neumann boundary condition. Next, we want to prove the existence of another class of solutions called very weak solution for less regular data. The idea is based on using the regularity of solutions and a duality argument.

2 Variational solution

As usual the existence and the uniqueness of a variational solution to $(\mathcal{S}_T)$ rely on the Lax-Milgram theorem. We assume that α is a positive function that belongs to $L^\infty(\Gamma)$. The proposition below gives a variational formulation for $(\mathcal{S}_T)$.

Proposition IV.1 *Suppose that Ω is of class $\mathcal{C}^{1,1}$. Assume that $\chi = g = 0$, let $\boldsymbol{f} \in [\mathring{H}^2_{-1}(\mathrm{div};\Omega)]'$ and $\boldsymbol{h} \in H^{-1/2}(\Gamma)$ such that*

$$\boldsymbol{h}\cdot\boldsymbol{n} = 0 \quad \text{on} \quad \Gamma. \tag{IV.1}$$

Then, finding $(\boldsymbol{u},\pi) \in W^{1,2}_0(\Omega)\times L^2(\Omega)$ satisfying $(\mathcal{S}_T)$ in the sense of distributions, is equivalent to the problem:

$$\begin{cases} \text{find } \boldsymbol{u} \in V_{\sigma,T}(\Omega) \text{ such that,} \\ 2\displaystyle\int_\Omega \boldsymbol{D}(\boldsymbol{u}) : \mathbf{D}(\boldsymbol{\varphi})d\boldsymbol{x} + \int_\Gamma \alpha(x)\boldsymbol{u}_\tau\boldsymbol{\varphi}d\sigma = \langle \boldsymbol{f},\boldsymbol{\varphi}\rangle_\Omega + \langle \boldsymbol{h},\boldsymbol{\varphi}\rangle_\Gamma, \quad \forall \boldsymbol{\varphi} \in V_{\sigma,T}(\Omega), \end{cases} \tag{IV.2}$$

where we recall that the space $V_{\sigma,T}(\Omega)$ is defined in (II.23).

Proof. Let $(\boldsymbol{u}, \pi) \in W_0^{1,2}(\Omega) \times L^2(\Omega)$ be a solution of $(\mathcal{S}_T)$. Thanks to the characterization given in Proposition II.2, $-\Delta \boldsymbol{u} = \boldsymbol{f} - \nabla \pi$ belongs to $[\mathring{H}^2_{-1}(\mathrm{div};\Omega)]'$ and thus, $\boldsymbol{u}$ is in the space $E(\Omega)$ defined in (II.24). Then we can write $\forall \boldsymbol{\varphi} \in V_{\sigma,T}(\Omega) \subset \mathring{H}^2_{-1}(\mathrm{div};\Omega)$,

$$\langle -\Delta \boldsymbol{u} + \nabla \pi, \boldsymbol{\varphi}\rangle_\Omega = \langle -\Delta \boldsymbol{u}, \boldsymbol{\varphi}\rangle_\Omega + \langle \nabla \pi, \boldsymbol{\varphi}\rangle_\Omega = \langle \boldsymbol{f}, \boldsymbol{\varphi}\rangle_\Omega, \tag{IV.3}$$

where $\langle \cdot, \cdot \rangle_\Omega$ denotes the dualities between $[\mathring{H}^2_{-1}(\mathrm{div};\Omega)]'$ and $\mathring{H}^2_{-1}(\mathrm{div};\Omega)$. Due to the density of $\mathcal{D}(\Omega)$ in $\mathring{H}^2_{-1}(\mathrm{div};\Omega)$, we have

$$\langle \nabla \pi, \boldsymbol{\varphi}\rangle_\Omega = -\int_\Omega \pi \mathrm{div}\, \boldsymbol{\varphi}\, d\boldsymbol{x}, \quad \forall \boldsymbol{\varphi} \in \mathring{H}^2_{-1}(\mathrm{div};\Omega).$$

Particularly

$$\langle \nabla \pi, \boldsymbol{\varphi}\rangle_\Omega = 0 \quad \text{if} \quad \boldsymbol{\varphi} \in V_{\sigma,T}(\Omega).$$

Now using the Green formula (II.25) in (IV.3), we deduce that every solution of $(\mathcal{S}_T)$ also solves (IV.2).

Conversely, let $\boldsymbol{u} \in V_{\sigma,T}(\Omega)$ be a solution of (IV.2). Then for all $\boldsymbol{\varphi} \in \mathcal{D}_\sigma(\Omega)$ (defined in (II.31)), we have on the one hand

$$2\int_\Omega \mathbf{D}(\boldsymbol{u}) : \mathbf{D}(\boldsymbol{\varphi}) d\boldsymbol{x} = \langle \boldsymbol{f}, \boldsymbol{\varphi}\rangle_{\mathcal{D}'(\Omega)\times\mathcal{D}(\Omega)}$$

and on the other hand

$$2\int_\Omega \mathbf{D}(\boldsymbol{u}) : \mathbf{D}(\boldsymbol{\varphi}) d\boldsymbol{x} = -\langle \Delta\, \boldsymbol{u}, \boldsymbol{\varphi}\rangle_{\mathcal{D}'(\Omega)\times\mathcal{D}(\Omega)}.$$

As a consequence, we obtain

$$\langle -\Delta\, \boldsymbol{u} - \boldsymbol{f}, \boldsymbol{\varphi}\rangle_{\mathcal{D}'(\Omega)\times\mathcal{D}(\Omega)} = 0, \quad \forall \boldsymbol{\varphi} \in \mathcal{D}_\sigma(\Omega).$$

Hence, by de Rham's theorem, there exists $q \in \mathcal{D}'(\Omega)$ such that

$$-\Delta\, \boldsymbol{u} - \boldsymbol{f} = \nabla q \quad \text{in} \quad \Omega. \tag{IV.4}$$

Thanks to (II.19), we have $\boldsymbol{f} \in [\mathring{H}^2_{-1}(\mathrm{div};\Omega)]' \subset W_0^{-1,2}(\Omega)$ and thus $-\Delta\, \boldsymbol{u} - \boldsymbol{f} \in W_0^{-1,2}(\Omega)$. It follows from Theorem II.4, that there exists a constant c such that $\pi = q + c \in L^2(\Omega)$. Because $\boldsymbol{u}$ belongs to $V_{\sigma,T}(\Omega)$, we have $\mathrm{div}\, \boldsymbol{u} = 0$ in Ω and $\boldsymbol{u} \cdot \boldsymbol{n} = 0$ on Γ.

It remains to prove that the second boundary condition $2[\mathbf{D}(\boldsymbol{u})\boldsymbol{n}]_\tau + \alpha \boldsymbol{u}_\tau = \boldsymbol{h}$ on Γ holds. To that end, let us multiply (IV.4) by $\boldsymbol{\varphi} \in V_{\sigma,T}(\Omega) \subset \mathring{H}^2_{-1}(\mathrm{div};\Omega)$. This yields

$$\langle -\Delta\, \boldsymbol{u} + \nabla\, \pi, \boldsymbol{\varphi}\rangle_\Omega = \langle \boldsymbol{f}, \boldsymbol{\varphi}\rangle_\Omega\,.$$

Observe now that $\boldsymbol{u}$ also belongs to $E(\Omega)$. Hence, using the Green formula (II.25), we obtain:

$$2\int_\Omega \mathbf{D}(\boldsymbol{u}) : \mathbf{D}(\boldsymbol{\varphi})d\boldsymbol{x} - \langle 2[\mathbf{D}(\boldsymbol{u})\boldsymbol{n}]_\tau, \boldsymbol{\varphi}\rangle_\Gamma = \langle \boldsymbol{f}, \boldsymbol{\varphi}\rangle_\Omega\,. \tag{IV.5}$$

Therefore, from (IV.2) and (IV.5) we deduce that

$$\forall \boldsymbol{\varphi} \in V_{\sigma,T}(\Omega), \qquad \langle 2[\mathbf{D}(\boldsymbol{u})\boldsymbol{n}]_\tau + \alpha \boldsymbol{u}_\tau, \boldsymbol{\varphi}\rangle_\Gamma = \langle \boldsymbol{h}, \boldsymbol{\varphi}\rangle_\Gamma\,.$$

Let now $\boldsymbol{\mu}$ be any element of the space $H^{1/2}(\Gamma)$. So, there exists $\boldsymbol{\varphi} \in W^{1,2}_0(\Omega)$ such as $\mathrm{div}\,\boldsymbol{\varphi} = 0$ in Ω and $\boldsymbol{\varphi} = \boldsymbol{\mu}_{\boldsymbol{\tau}}$ on Γ. It is clear that $\boldsymbol{\varphi} \in V_{\sigma,T}(\Omega)$ and

$$\begin{aligned}
\langle 2[\mathbf{D}(\boldsymbol{u})\boldsymbol{n}]_\tau + \alpha \boldsymbol{u}_\tau - \boldsymbol{h}, \boldsymbol{\mu}\rangle_\Gamma &= \langle 2[\mathbf{D}(\boldsymbol{u})\boldsymbol{n}]_\tau + \alpha \boldsymbol{u}_\tau - \boldsymbol{h}, \boldsymbol{\mu}_{\boldsymbol{\tau}}\rangle_\Gamma \\
&= \langle 2[\mathbf{D}(\boldsymbol{u})\boldsymbol{n}]_\tau + \alpha \boldsymbol{u}_\tau - \boldsymbol{h}, \boldsymbol{\varphi}\rangle_\Gamma \\
&= 0.
\end{aligned}$$

This implies that

$$2[\mathbf{D}(\boldsymbol{u})\boldsymbol{n}]_\tau + \alpha \boldsymbol{u}_\tau = \boldsymbol{h} \text{ on } \Gamma.$$

□

Proposition IV.2 *Suppose that Ω is of class $\mathcal{C}^{1,1}$. Assume that $\chi = g = 0$ and let $\boldsymbol{f} \in [\mathring{H}^2_{-1}(\mathrm{div};\Omega)]'$ and $\boldsymbol{h} \in H^{-1/2}(\Gamma)$ satisfying (IV.1). Then, the Stokes problem $(\mathcal{S}_T)$ has a unique solution $(\boldsymbol{u}, \pi) \in W^{1,2}_0(\Omega) \times L^2(\Omega)$. In addition, we have the following estimate:*

$$\|\boldsymbol{u}\|_{W^{1,2}_0(\Omega)} + \|\pi\|_{L^2(\Omega)} \leqslant C\Big(\|\boldsymbol{f}\,\|_{[\mathring{H}^2_{-1}(\mathrm{div};\Omega)]'} + \|\boldsymbol{h}\,\|_{H^{-1/2}(\Gamma)}\Big). \tag{IV.6}$$

Proof. Due to the equivalence between problem $(\mathcal{S}_T)$ and the variational formulation (IV.2), it is enough to solve (IV.2). This is done by the use of Lax-Milgram's theorem. Consider the bilinear form

$$\mathcal{A} : V_{\sigma,T}(\Omega) \times V_{\sigma,T}(\Omega) \to \mathbb{R}$$

defined by

$$\mathcal{A}(\boldsymbol{u},\boldsymbol{v}) = \int_\Omega \mathbf{D}(\boldsymbol{u}) : \mathbf{D}(\boldsymbol{v}) d\boldsymbol{x} + \alpha \langle \boldsymbol{u}_\tau, \boldsymbol{v}_\tau \rangle_\Gamma.$$

The mapping $\mathcal{A}(\cdot,\cdot)$ is clearly continuous and thanks to the Korn's inequality (II.33), it is also coercive in $V_{\sigma,T}(\Omega)$. Moreover, the linear form $\ell : V_{\sigma,T}(\Omega) \to \mathbb{R}$, defined by

$$\ell(\boldsymbol{v}) = \langle \boldsymbol{f}, \boldsymbol{u} \rangle_\Omega + \langle \boldsymbol{h}, \boldsymbol{v} \rangle_\Gamma,$$

is continuous on $V_{\sigma,T}(\Omega)$. Hence, using Lax-Milgram's theorem, we deduce that problem (IV.2) has a unique solution $\boldsymbol{u}$ in $V_{\sigma,T}(\Omega)$.
Let us now prove estimate (IV.6). On the one hand, using the variational problem and Korn's inequality (II.33), we obtain

$$\|\boldsymbol{u}\|_{W_0^{1,2}(\Omega)} \leq C\Big(\|\boldsymbol{f}\|_{[\mathring{H}^2_{-1}(\mathrm{div};\Omega)]'} + \|\boldsymbol{h}\|_{H^{-1/2}(\Gamma)} \Big). \tag{IV.7}$$

On the other hand, we have

$$\begin{aligned} \|\nabla \pi\|_{W_0^{-1,2}(\Omega)} &\leq \|\boldsymbol{f}\|_{H_1^{-1}(\mathrm{div};\Omega)} + \|\Delta \boldsymbol{u}\|_{W_0^{-1,2}(\Omega)} \\ &\leq C\Big(\|\boldsymbol{f}\|_{H_1^{-1}(\mathrm{div};\Omega)} + \|\boldsymbol{u}\|_{W_0^{1,2}(\Omega)} \Big) \\ &\leq C\Big(\|\boldsymbol{f}\|_{H_1^{-1}(\mathrm{div};\Omega)} + \|\boldsymbol{h}\|_{H^{-1/2}(\Gamma)} \Big). \end{aligned}$$

Combining this estimate, (IV.7) and the fact that (see [47, Corollary 3.3])

$$\|\pi\|_{L^2(\Omega)} \leq C \|\nabla \pi\|_{W_0^{-1,2}(\Omega)},$$

allows to obtain (IV.6). □

We are now in a position to prove the existence and uniqueness of a variational solution to the complete Stokes problem $(\mathcal{S}_T)$.

Theorem IV.1 *Suppose that Ω is of class $\mathcal{C}^{1,1}$. Assume that $\boldsymbol{f} \in [\mathring{H}^2_{-1}(\mathrm{div};\Omega)]'$, $\chi \in L^2(\Omega)$, $g \in H^{1/2}(\Gamma)$ and $\boldsymbol{h} \in H^{-1/2}(\Gamma)$ satisfies the compatibility condition (IV.1). Then, the Stokes problem $(\mathcal{S}_T)$ has a unique solution $(\boldsymbol{u},\pi) \in W_0^{1,2}(\Omega) \times L^2(\Omega)$. Moreover, the following estimate holds:*

$$\|\boldsymbol{u}\|_{W_0^{1,2}(\Omega)} + \|\pi\|_{L^2(\Omega)} \leq C\Big(\|\boldsymbol{f}\|_{[\mathring{H}^2_{-1}(\mathrm{div};\Omega)]'} + \|\chi\|_{L^2(\Omega)} + \|g\|_{H^{1/2}(\Gamma)} + \|\boldsymbol{h}\|_{H^{-1/2}(\Gamma)} \Big). \tag{IV.8}$$

Proof. Let us first solve the below exterior Neumann problem:

$$\Delta\theta = \chi \quad \text{in} \quad \Omega \quad \text{and} \quad \frac{\partial \theta}{\partial \boldsymbol{n}} = g \quad \text{on} \quad \Gamma. \tag{IV.9}$$

Thanks to [48] (see also [44, Theorem 3.9]), this problem has a unique solution $\theta \in W_0^{2,2}(\Omega)/\mathbb{R}$ satisfying the estimate:

$$\|\theta\|_{W_0^{2,2}(\Omega)/\mathbb{R}} \leqslant C\left(\|\chi\|_{L^2(\Omega)} + \|g\|_{H^{1/2}(\Gamma)}\right).$$

Next we look for a pair $(\boldsymbol{z}, \pi) \in W_0^{1,2}(\Omega) \times L^2(\Omega)$ satisfying

$$\begin{cases} -\Delta\, \boldsymbol{z} + \nabla\, \pi = \boldsymbol{f} + \nabla\, \chi \quad \text{and} \quad \operatorname{div} \boldsymbol{z} = 0 \quad \text{in} \quad \Omega, \\ \boldsymbol{z} \cdot \boldsymbol{n} = 0 \quad \text{and} \quad 2[\mathbf{D}(\boldsymbol{z})\boldsymbol{n}]_\tau + \alpha \boldsymbol{z}_\tau = \boldsymbol{h'} \quad \text{on} \quad \Gamma, \end{cases} \tag{IV.10}$$

where $\boldsymbol{h'} = \boldsymbol{h} - 2[\mathbf{D}(\nabla\, \theta)\boldsymbol{n}]_\tau - \alpha(\nabla\, \theta)_\tau$.
Observe that in view of the assumption on $\boldsymbol{f}$ and Proposition II.2, $\boldsymbol{f}+\nabla\chi$ belongs to $[\mathring{H}^2_{-1}(\operatorname{div};\Omega)]'$. Besides, as θ belongs to $W_0^{2,2}(\Omega)$, then $\nabla\theta$ belongs to $W_0^{1,2}(\Omega)$ (see (II.2)). As a consequence, $\boldsymbol{h'}$ belongs to $H^{-1/2}(\Gamma)$ and satisfies

$$\|\boldsymbol{h'}\|_{H^{-1/2}(\Gamma)} \leq C\left(\|\boldsymbol{h}\|_{H^{-1/2}(\Gamma)} + \|\theta\|_{W_0^{2,2}(\Omega)/\mathbb{R}}\right) \tag{IV.11}$$

and clearly satisfies $\boldsymbol{h'} \cdot \boldsymbol{n} = 0$ on Γ. According to Proposition IV.2, problem (IV.10) has a unique solution $(\boldsymbol{z}, \pi) \in W_0^{1,2}(\Omega) \times L^2(\Omega)$ satisfying

$$\|\boldsymbol{z}\|_{W_0^{1,2}(\Omega)} + \|\pi\|_{L^2(\Omega)} \leq C\left(\|\boldsymbol{f}\|_{H_1^{-1}(\operatorname{div};\Omega)} + \|\chi\|_{L^2(\Omega)} + \|\boldsymbol{h'}\|_{H^{-1/2}(\Omega)}\right). \tag{IV.12}$$

Thus setting $\boldsymbol{u} = \boldsymbol{z} + \nabla\, \theta$, the pair $(\boldsymbol{u}, \pi) \in (W_0^{1,2}(\Omega) \times L^2(\Omega))$ is the unique solution of $(\mathcal{S}_T)$ and estimate (IV.8) follows from (IV.11) and (IV.12). □

Remark IV.1 *Let us mention that due to Proposition II.2, the statement of Theorem IV.1 remains valid if we assume that $\boldsymbol{f}$ belongs to $W_1^{0,2}(\Omega)$. This space is much more convenient for applications and for the numerical approximation of the Stokes problem $(\mathcal{S}_T)$.*

3 Strong solutions

In this part, we investigate the well-posedness of strong solutions in $W_{k+1}^{2,2}(\Omega) \times W_{k+1}^{1,2}(\Omega)$ with $k \in \mathbb{Z}$. In the behavior at infinity point of view, we are looking for solutions $(\boldsymbol{u}, \pi)$ that grow or decay weaker at infinity ($k \leq 0$) than the variational solution or decay faster at infinity ($k > 0$) than this latter solution. In order to deal with the uniqueness issues, we first need to characterize the kernel of problem $(\mathcal{S}_T)$.

For $k \in \mathbb{Z}$, we introduce:

$$\begin{aligned}\mathcal{N}_k^2(\Omega) \;=\; & \Big\{(\boldsymbol{u},\pi) \in W_k^{1,2}(\Omega) \times W_k^{0,2}(\Omega);\\ & -\Delta\,\boldsymbol{u} + \nabla\,\pi = 0,\ \operatorname{div}\boldsymbol{u} = 0 \text{ in } \Omega \text{ and } \boldsymbol{u}\cdot\boldsymbol{n} = 0,\ 2[\mathbf{D}(\boldsymbol{u})\boldsymbol{n}]_\tau + \alpha\boldsymbol{u}_\tau = 0 \text{ on } \Gamma\Big\}.\end{aligned}$$

The characterization of the kernel $\mathcal{N}_k^2(\Omega)$ is given by the following proposition:

Proposition IV.3 *Suppose that Ω is of class $\mathcal{C}^{1,1}$ and assume $k \in \mathbb{Z}$.*

- *If $k \geq 0$, then $\mathcal{N}_k^2(\Omega) = \Big\{(\mathbf{0},0)\Big\}$.*
- *If $k < 0$, then $\mathcal{N}_k^2(\Omega) = \{(\boldsymbol{v}-\boldsymbol{\lambda},\theta-\mu);\quad (\boldsymbol{\lambda},\mu) \in N_{-k-1}\}$, where $(\boldsymbol{v},\theta) \in W_0^{1,2}(\Omega) \times L^2(\Omega)$ is the unique solution of the following problem:*

$$\begin{cases} -\Delta\,\boldsymbol{v} + \nabla\,\theta = \mathbf{0} \quad \textit{and} \quad \operatorname{div}\boldsymbol{v} = 0 \quad \textit{in} \quad \Omega,\\ \boldsymbol{v}\cdot\boldsymbol{n} = \boldsymbol{\lambda}\cdot\boldsymbol{n} \quad \textit{and} \quad 2[\boldsymbol{D}(\boldsymbol{v})\boldsymbol{n}]_\tau + \alpha\boldsymbol{v}_\tau = 2[\boldsymbol{D}(\boldsymbol{\lambda})\boldsymbol{n}]_\tau + \alpha\boldsymbol{\lambda}_\tau \quad \textit{on} \quad \Gamma. \end{cases} \tag{IV.13}$$

Moreover, the space $\mathcal{N}_k^2(\Omega)$ is a finite-dimensional space and $\dim\mathcal{N}_k^2(\Omega) = \dim N_{-k-1} = \dim\,\mathcal{P}_{-k-1}\times\mathcal{P}^{\Delta}_{-k-2}$.

Proof.

- If $k \geq 0$, because $W_k^{1,2}(\Omega) \times W_k^{0,2}(\Omega) \hookrightarrow W_0^{1,2}(\Omega) \times L^2(\Omega)$, we have $\mathcal{N}_k^2(\Omega) \subset \mathcal{N}_0^2(\Omega)$. Now thanks to Theorem IV.2, we have $\mathcal{N}_0^2(\Omega) = \Big\{(\mathbf{0},0)\Big\}$ which ends the proof for this case.
- If $k < 0$, then let us assume that $(\boldsymbol{u},\pi)$ belongs to $\mathcal{N}_k^2(\Omega)$. The pair $(\boldsymbol{u},\pi)$ has an extension $(\widetilde{\boldsymbol{u}},\widetilde{\pi})$ that belongs to $W_k^{1,2}(\mathbb{R}^3) \times W_k^{0,2}(\mathbb{R}^3)$. Set now

$$\boldsymbol{F} = -\Delta\,\widetilde{\boldsymbol{u}} + \nabla\,\widetilde{\pi} \quad \text{and} \quad e = \operatorname{div}\widetilde{\boldsymbol{u}}.$$

  Then the pair $(\boldsymbol{F},e)$ belongs to $W_k^{-1,2}(\mathbb{R}^3)\times W_k^{0,2}(\mathbb{R}^3)$ and has a compact support. Therefore $(\boldsymbol{F},e)$ also belongs to $W_0^{-1,2}(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$. It follows from Theorem III.6, that there exists a solution $(\boldsymbol{v},\theta) \in W_0^{1,2}(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$ such that

$$-\Delta\,\boldsymbol{v} + \nabla\,\theta = -\Delta\,\widetilde{\boldsymbol{u}} + \nabla\,\widetilde{\pi} \quad \text{and} \quad \operatorname{div}\boldsymbol{v} = \operatorname{div}\widetilde{\boldsymbol{u}} \quad \text{in} \quad \mathbb{R}^3.$$

  The fact that $k < 0$ implies that $(\boldsymbol{v}-\widetilde{\boldsymbol{u}},\theta-\widetilde{\pi})$ also belongs to $W_k^{1,2}(\mathbb{R}^3) \times W_k^{0,2}(\mathbb{R}^3)$. It follows from Proposition III.3 that $\boldsymbol{v} = \widetilde{\boldsymbol{u}} + \boldsymbol{\lambda}$ and $\theta = \widetilde{\pi} + \mu$, where $(\boldsymbol{\lambda},\mu) \in N_{-k-1}$. We deduce that the restriction of $\boldsymbol{v}$ and θ to Ω satisfy (IV.13).

Let us now prove that the kernel $\mathcal{N}_k^2(\Omega)$ has a finite dimension. Thanks to Theorem IV.1, we can introduce the linear mapping

$$\mathcal{T} \,:\, N_{-k-1} \mapsto \mathcal{N}_k^2(\Omega)$$

defined for each $(\boldsymbol{\lambda}, \mu) \in N_{-k-1}$, by $\mathcal{T}(\boldsymbol{\lambda}, \mu) = (\boldsymbol{v}(\boldsymbol{\lambda}) - \boldsymbol{\lambda}, \theta(\boldsymbol{\lambda}) - \mu)$, where $(\boldsymbol{v}(\boldsymbol{\lambda}, \theta(\boldsymbol{\lambda})) \in W_0^{1,2}(\Omega) \times L^2(\Omega)$ is the unique solution of (IV.13).
Using the first part of the proof, this mapping is clearly onto. It is also injective, since $\mathcal{T}(\boldsymbol{\lambda}, \mu) = \Big\{(\mathbf{0}, 0)\Big\}$, implies that the pair $\Big(\boldsymbol{v}\,(\boldsymbol{\lambda}), \theta(\boldsymbol{\lambda})\Big) \in W_0^{1,2}(\Omega) \times L^2(\Omega)$ is a pair of polynomials of $\mathcal{P}_{-k-1} \times \mathcal{P}^{\Delta}_{-k-2}$. But the space $W_0^{1,2}(\Omega) \times L^2(\Omega)$ does not contain polynomials, which implies that $\Big(\boldsymbol{v}\,(\boldsymbol{\lambda}), \theta(\boldsymbol{\lambda})\Big) = \Big\{(\mathbf{0}, 0)\Big\}$ which in turn also implies that $(\boldsymbol{\lambda}, \mu) = \Big\{(\mathbf{0}, 0)\Big\}$. Finally, thanks to (IV.8), $\mathcal{T}$ is continuous. Hence the operator $\mathcal{T}$ is an isomorphism from N_{-k-1} to $\mathcal{N}_k^2(\Omega)$. Since the dimension of N_{-k-1} is finite and is equal to the dimension of $\mathcal{P}_{-k-1} \times \mathcal{P}^{\Delta}_{-k-2}$, this ends the proof. $\square$

We start by looking for strong solutions for the case $k \leq 0$.

Theorem IV.2 *Suppose that Ω is of class $\mathcal{C}^{2,1}$ and let $k \leq 0$ be an integer. Assume that $\boldsymbol{f} \in W_{k+1}^{0,2}(\Omega)$, $\chi \in W_{k+1}^{1,2}(\Omega)$, $g \in H^{3/2}(\Gamma)$ and $\boldsymbol{h} \in H^{1/2}(\Gamma)$ satisfying (IV.1). Then the Stokes problem $(\mathcal{S}_T)$ has a solution $(\boldsymbol{u}, \pi) \in W_{k+1}^{2,2}(\Omega) \times W_{k+1}^{1,2}(\Omega)$ unique up to an element of $\mathcal{N}_k^2(\Omega)$ and we have:*

$$\begin{aligned} &\inf_{(\boldsymbol{\lambda},\mu)\in\mathcal{N}_k^2(\Omega)} \Big(\|\boldsymbol{u} + \boldsymbol{\lambda}\|_{W_{k+1}^{2,2}(\Omega)} + \|\pi + \mu\|_{W_{k+1}^{1,2}(\Omega)} \Big) \\ &\leqslant C\Big(\|\boldsymbol{f}\,\|_{W_{k+1}^{0,2}(\Omega)} + \|\chi\|_{W_{k+1}^{1,2}(\Omega)} + \|g\|_{H^{3/2}(\Gamma)} + \|\boldsymbol{h}\|_{H^{1/2}(\Gamma)} \Big). \end{aligned} \tag{IV.14}$$

Proof. The uniqueness result follows immediately from Proposition IV.3. The proof of the existence result is made of two steps.

- We first establish the existence of a solution in $W_k^{1,2}(\Omega) \times W_k^{0,2}(\Omega)$.

  The main idea is to use the results on the Stokes problem in the whole space $\mathbb{R}^3$. To that end, let $\widetilde{\boldsymbol{f}}$ denote the extension by zero of $\boldsymbol{f}$. Then $\widetilde{\boldsymbol{f}}$ belongs to $W_{k+1}^{0,2}(\mathbb{R}^3)$. Besides, by lifting the trace of χ on Γ to Ω', we have an extension $\widetilde{\chi}$ of χ that belongs to $W_{k+1}^{1,2}(\mathbb{R}^3)$. Thanks to Theorem III.6, the following Stokes problem in $\mathbb{R}^3$

  $$-\Delta\, \widetilde{\boldsymbol{u}} + \nabla\, \widetilde{\pi} = \widetilde{\boldsymbol{f}} \quad \text{and} \quad \operatorname{div} \widetilde{\boldsymbol{u}} = \widetilde{\chi} \quad \text{in} \quad \mathbb{R}^3,$$

  has a solution $(\widetilde{\boldsymbol{u}}, \widetilde{\pi})$ that belongs to $W_{k+1}^{2,2}(\mathbb{R}^3) \times W_{k+1}^{0,2}(\mathbb{R}^3)$.

  Let us now consider the following problem:

$$\begin{cases} -\Delta\, \boldsymbol{z} + \nabla\theta = \boldsymbol{0} \quad \text{and} \quad \operatorname{div} \boldsymbol{z} = 0 \quad \text{in} \quad \Omega, \\ \boldsymbol{z}\cdot\boldsymbol{n} = \widetilde{\boldsymbol{u}}\cdot\boldsymbol{n} + g \quad \text{and} \quad 2[\mathbf{D}(\boldsymbol{z})\boldsymbol{n}]_\tau + \alpha\boldsymbol{z}_\tau = 2[\mathbf{D}(\widetilde{\boldsymbol{u}})\boldsymbol{n}]_\tau + \alpha\widetilde{\boldsymbol{u}}_\tau + \boldsymbol{h} \quad \text{on} \quad \Gamma. \end{cases} \tag{IV.15}$$

The fact that $\widetilde{\boldsymbol{u}} \in W^{2,2}_{k+1}(\mathbb{R}^3)$ and $\boldsymbol{h} \in H^{1/2}(\Gamma)$ imply that $2[\mathbf{D}(\widetilde{\boldsymbol{u}})\boldsymbol{n}]_\tau + \alpha\widetilde{\boldsymbol{u}}_\tau + \boldsymbol{h} \in H^{1/2}(\Gamma) \hookrightarrow H^{-1/2}(\Omega)$. Moreover, from (IV.1), we clearly have

$$\Big(2[\mathbf{D}(\boldsymbol{u})\boldsymbol{n}]_\tau + \alpha\boldsymbol{u}_\tau + \boldsymbol{h}\Big)\cdot\boldsymbol{n} = 0 \quad \text{on} \quad \Gamma.$$

Therefore, thanks to Theorem IV.1, Problem (IV.15) has a unique solution $(\boldsymbol{z},\theta) \in W^{1,2}_0(\Omega)\times L^2(\Omega)$. Since $k \leqslant 0$, then the pair $(\boldsymbol{z},\theta)$ also belongs to $W^{1,2}_k(\Omega)\times W^{0,2}_k(\Omega)$. Setting now $\boldsymbol{u} = \boldsymbol{z} - \widetilde{\boldsymbol{u}}_{|\Omega}$ and $\pi = \theta - \widetilde{\pi}_{|\Omega}$, then the pair $(\boldsymbol{u},\pi)$ belongs to $W^{1,2}_k(\Omega)\times W^{0,2}_k(\Omega)$ and satisfies $(\mathcal{S}_T)$.

- We shall now prove that the pair $(\boldsymbol{u},\pi)$ belongs to $W^{2,2}_{k+1}(\Omega)\times W^{1,2}_{k+1}(\Omega)$.

  Here we use regularity arguments on the Stokes problem set in bounded domains and in the whole space $\mathbb{R}^3$. To that end, we introduce the same partition of unity as in Theorem III.5. Let $(\widetilde{\boldsymbol{u}},\widetilde{\pi}) \in W^{1,2}_k(\mathbb{R}^3 \times W^{0,2}_k(\mathbb{R}^3)$ be an extension of $(\boldsymbol{u},\pi)$ to the whole space $\mathbb{R}^3$. We can write:

$$\widetilde{\boldsymbol{u}} = \varphi\widetilde{\boldsymbol{u}} + \psi\widetilde{\boldsymbol{u}} \quad \text{and} \quad \widetilde{\pi} = \varphi\widetilde{\pi} + \psi\widetilde{\pi}.$$

  Then it is enough to show that the pairs $(\varphi\widetilde{\boldsymbol{u}},\varphi\widetilde{\pi})$ and $(\psi\widetilde{\boldsymbol{u}},\psi\widetilde{\pi})$ belong to $W^{2,2}_{k+1}(\mathbb{R}^3)\times W^{1,2}_{k+1}(\mathbb{R}^3)$. To that end, consider first

$$-\Delta\,(\psi\widetilde{\boldsymbol{u}}) + \nabla\,(\psi\widetilde{\pi}) = \boldsymbol{f}_1 \quad \text{and} \quad \operatorname{div}(\psi\widetilde{\boldsymbol{u}}) = \chi_1 \quad \text{in} \quad \mathbb{R}^3, \tag{IV.16}$$

  where

$$\boldsymbol{f}_1 = \boldsymbol{f}\psi - (2\nabla\widetilde{\boldsymbol{u}}\nabla\psi + \boldsymbol{u}\Delta\,\psi) + \widetilde{\pi}\nabla\,\psi \quad \text{and} \quad \chi_1 = \chi\psi + \widetilde{\boldsymbol{u}}\cdot\nabla\,\psi.$$

  We easily see that $\boldsymbol{f}_1$ and χ_1 have bounded supports and belong to $L^2_{loc}(\mathbb{R}^3)\times H^1_{loc}(\mathbb{R}^3)$. As a consequence, $(\boldsymbol{f}_1,\chi_1)$ belongs to $W^{0,2}_{k+1}(\mathbb{R}^3)\times W^{1,2}_{k+1}(\mathbb{R}^3)$. Therefore, thanks to Theorem III.6, there exists $(\boldsymbol{w},q) \in W^{2,2}_{k+1}(\mathbb{R}^3)\times W^{1,2}_{k+1}(\mathbb{R}^3)$ satisfying

$$-\Delta\,\boldsymbol{w} + \nabla\,q = \boldsymbol{f}_1 \quad \text{and} \quad \operatorname{div}\boldsymbol{w} = \chi_1 \quad \text{in} \quad \mathbb{R}^3.$$

Let $\boldsymbol{v} = \psi\widetilde{\boldsymbol{u}} - \boldsymbol{w}$ and $\theta = \psi\widetilde{\pi} - q$, then $(\boldsymbol{v}, \theta)$ belongs to $W^{1,2}_k(\mathbb{R}^3) \times W^{0,2}_k(\mathbb{R}^3)$ and satisfies

$$-\Delta\, \boldsymbol{v} + \nabla\, \theta = \boldsymbol{0} \quad \text{and} \quad \operatorname{div} \boldsymbol{v} = 0 \quad \text{in} \quad \mathbb{R}^3.$$

It follows from Proposition III.3 that $(\boldsymbol{v}, \theta) \in N_{-k-1}$. Since $N_{-k-1} \subset W^{2,2}_{k+1}(\mathbb{R}^3) \times W^{1,2}_{k+1}(\mathbb{R}^3)$, we deduce that the pair $(\psi\widetilde{\boldsymbol{u}}, \psi\widetilde{\pi})$ also belongs to $W^{2,2}_{k+1}(\mathbb{R}^3) \times W^{1,2}_{k+1}(\mathbb{R}^3)$.

Consider now the system

$$-\Delta\,(\varphi\widetilde{\boldsymbol{u}}) + \nabla\,(\varphi\widetilde{\pi}) = \boldsymbol{f}_2 \quad \text{and} \quad \operatorname{div}(\varphi\widetilde{\boldsymbol{u}}) = \chi_2,$$

where $\boldsymbol{f}_2$ and χ_2 have similar expressions as $\boldsymbol{f}_1$ and χ_1 with ψ remplaced by φ. It is easy to check that $(\boldsymbol{f}_2, \chi_2)$ belongs to $L^2(\Omega_{R+1}) \times H^1(\Omega_{R+1})$. In particular, we have $\widetilde{\boldsymbol{u}} = \psi\widetilde{\boldsymbol{u}}$ outside B_{R+1}, so the restriction of $\boldsymbol{u}$ to $\partial\, B_{R+1}$ belongs to $H^{1/2}(\partial\, B_{R+1})$. Its clear that $(\varphi\widetilde{\boldsymbol{u}}, \varphi\widetilde{\pi})$ belongs to $H^2(\Omega_{R+1}) \times H^1(\Omega_{R+1})$ and satisfies the following Stokes problem with mixed boundary conditions

$$\begin{cases} -\Delta(\varphi\widetilde{\boldsymbol{u}}) + \nabla(\varphi\widetilde{\pi}) = \boldsymbol{f}_2 \quad \text{and} \quad \operatorname{div}(\varphi\widetilde{\boldsymbol{u}}) = \chi_2 \quad \text{in} \quad \Omega_{R+1}, \\ (\varphi\widetilde{\boldsymbol{u}}) \cdot \boldsymbol{n} = g \quad \text{and} \quad 2[\mathbf{D}(\varphi\widetilde{\boldsymbol{u}})\boldsymbol{n}]_\tau + \alpha(\varphi\widetilde{\boldsymbol{u}})_\tau = \boldsymbol{h} \quad \text{on} \quad \Gamma, \\ \varphi\widetilde{\boldsymbol{u}} = \psi\widetilde{\boldsymbol{u}} \quad \text{on} \quad \partial B_{R+1} \end{cases} \tag{IV.17}$$

Then thanks to Theorem III.8, we prove that $(\varphi\widetilde{\boldsymbol{u}}, \varphi\widetilde{\pi}) \in H^2(\Omega_{R+1}) \times H^1(\Omega_{R+1})$ solution of problem (IV.17), which also implies that $(\varphi\widetilde{\boldsymbol{u}}, \varphi\widetilde{\pi})$ belongs to $W^{2,2}_{k+1}(\mathbb{R}^3) \times W^{1,2}_{k+1}(\mathbb{R}^3)$.

Consequently, the pair $(\boldsymbol{u}, \pi)$ belongs to $W^{2,2}_{k+1}(\Omega) \times W^{1,2}_{k+1}(\Omega)$ and this ends the proof.

□

We now look for strong solutions for the case $k > 0$. Note that so far, the use of Theorem III.6 was straightforward because when $k \leq 0$ the compatibility condition (III.22) does not appear since N_{k-1} is reduced to $\{\boldsymbol{0}, 0\}$. This is not anymore the case when $k > 0$. Thus, to be able to still apply Theorem III.6, we need to find appropriate extensions of the functions of interest.

Theorem IV.3 *Suppose that Ω is of class $\mathcal{C}^{2,1}$ and let $k > 0$ be an integer. Assume that $\boldsymbol{f} \in W^{0,2}_{k+1}(\Omega)$, $g \in H^{3/2}(\Gamma)$, $\chi \in W^{1,2}_{k+1}(\Omega)$ and $\boldsymbol{h} \in H^{1/2}(\Gamma)$ satisfying (IV.1). Assume moreover that the following compatibility condition is satisfied*

$$\forall(\boldsymbol{\xi}, \eta) \in \mathcal{N}^2_{-k}(\Omega), \quad \int_\Omega \boldsymbol{f} \cdot \boldsymbol{\xi} d\boldsymbol{x} - \int_\Omega \chi\, \eta d\boldsymbol{x} = \langle g, 2[\boldsymbol{D}(\boldsymbol{\xi})\boldsymbol{n}] \cdot \boldsymbol{n} - \eta\rangle_\Gamma - \langle \boldsymbol{h}, \boldsymbol{\xi}\rangle_\Gamma. \tag{IV.18}$$

Then, problem $(\mathcal{S}_T)$ has a unique solution $(\boldsymbol{u}, \pi) \in W^{2,2}_{k+1}(\Omega) \times W^{1,2}_{k+1}(\Omega)$ and we have the following

estimate:

$$\|\boldsymbol{u}\|_{W^{2,2}_{k+1}(\Omega)} + \|\pi\|_{W^{1,2}_{k+1}(\Omega)} \leq C\Big(\|\boldsymbol{f}\|_{W^{0,2}_{k+1}(\Omega)} + \|\boldsymbol{h}\|_{H^{1/2}(\Gamma)} + \|\chi\|_{W^{1,2}_{k+1}(\Omega)} + \|g\|_{H^{3/2}(\Gamma)}\Big). \quad \text{(IV.19)}$$

Proof. Observe first that the uniqueness is a straightforward consequence of Proposition IV.3. We now divide the proof of the theorem into several parts.

- Compatibility condition. In this part, we prove that (IV.18) is a necessary condition. Let $(\boldsymbol{\xi}, \eta)$ be in $\mathcal{N}^2_{-k}(\Omega)$. For any $(\boldsymbol{\varphi}, \psi) \in \mathcal{D}(\overline{\Omega}) \times \mathcal{D}(\overline{\Omega})$, using identity (II.11) and the Green formula (II.14), we have

$$\begin{aligned}\int_\Omega \Big[\Big(-\Delta\,\boldsymbol{\varphi} + \nabla\,\psi\Big)\cdot\boldsymbol{\xi} - \eta\,\mathrm{div}\,\boldsymbol{\varphi}\Big]\,d\boldsymbol{x} \;&=\; 2\int_\Omega \mathbf{D}(\boldsymbol{\varphi}) : \nabla\boldsymbol{\xi}\,d\boldsymbol{x} - 2\langle \mathbf{D}(\boldsymbol{\varphi})\boldsymbol{n}, \boldsymbol{\xi}\rangle_\Gamma \\ &\quad + \int_\Omega \boldsymbol{\varphi}\cdot\nabla\eta\,d\boldsymbol{x} - \langle \boldsymbol{\varphi}\cdot\boldsymbol{n}, \eta\rangle_\Gamma. \qquad \text{(IV.20)}\end{aligned}$$

  Observe that we have

$$\int_\Omega \mathbf{D}(\boldsymbol{\varphi}) : \nabla\boldsymbol{\xi}\,d\boldsymbol{x} = \int_\Omega \nabla\boldsymbol{\varphi} : \mathbf{D}(\boldsymbol{\xi})\,d\boldsymbol{x}$$

  and using again (II.14), we get

$$2\int_\Omega \mathbf{D}(\boldsymbol{\varphi}) : \nabla\boldsymbol{\xi}\,d\boldsymbol{x} = -\int_\Omega \boldsymbol{\varphi}\cdot\Delta\boldsymbol{\xi}\,d\boldsymbol{x} + 2\langle \boldsymbol{D}(\boldsymbol{\xi})\boldsymbol{n}, \boldsymbol{\varphi}\rangle_\Gamma.$$

  Combining this equality with (IV.20) and recalling that the pair $(\boldsymbol{\xi}, \eta)$ belongs to $\mathcal{N}^2_{-k}(\Omega)$, we obtain

$$\int_\Omega \Big[\Big(-\Delta\,\boldsymbol{\varphi}+\nabla\,\psi\Big)\cdot\boldsymbol{\xi}-\eta\,\mathrm{div}\,\boldsymbol{\varphi}\Big]\,d\boldsymbol{x} = -2\langle \mathbf{D}(\boldsymbol{\varphi})\boldsymbol{n}, \boldsymbol{\xi}\rangle_\Gamma + 2\langle \mathbf{D}(\boldsymbol{\xi})\boldsymbol{n}, \boldsymbol{\varphi}\rangle_\Gamma - \langle \boldsymbol{\varphi}\cdot\boldsymbol{n}, \eta\rangle_\Gamma. \quad \text{(IV.21)}$$

  Now, on the one hand, because $\boldsymbol{\xi}\cdot\boldsymbol{n} = 0$ on Γ, we can write

$$2\langle \mathbf{D}(\boldsymbol{\varphi})\boldsymbol{n}, \boldsymbol{\xi}\rangle_\Gamma = 2\langle \Big[\mathbf{D}(\boldsymbol{\varphi})\boldsymbol{n}\Big]_\tau, \boldsymbol{\xi}_\tau\rangle_\Gamma = \langle 2\Big[\mathbf{D}(\boldsymbol{\varphi})\boldsymbol{n}\Big]_\tau + \alpha\boldsymbol{\varphi}_\tau, \boldsymbol{\xi}_\tau\rangle_\Gamma - \langle \alpha\boldsymbol{\varphi}_\tau, \boldsymbol{\xi}_\tau\rangle. \quad \text{(IV.22)}$$

  On the other hand, because $2[\mathbf{D}(\boldsymbol{\xi})\boldsymbol{n}]_\tau + \alpha\boldsymbol{\xi}_\tau = \mathbf{0}$, we can first write

$$\begin{aligned}2\langle \mathbf{D}(\boldsymbol{\xi})\boldsymbol{n}, \boldsymbol{\varphi}\rangle_\Gamma &= 2\langle [\mathbf{D}(\boldsymbol{\xi})\boldsymbol{n}]_\tau, \boldsymbol{\varphi}_\tau\rangle_\Gamma + 2\Big\langle \Big(\Big[\mathbf{D}(\boldsymbol{\xi})\boldsymbol{n}\Big]\cdot\boldsymbol{n}\Big)\boldsymbol{n}, (\boldsymbol{\varphi}\cdot\boldsymbol{n})\boldsymbol{n}\Big\rangle_\Gamma \\ &= -\langle \alpha\boldsymbol{\xi}_\tau, \boldsymbol{\varphi}_\tau\rangle_\Gamma + 2\Big\langle \Big(\Big[\mathbf{D}(\boldsymbol{\xi})\boldsymbol{n}\Big]\cdot\boldsymbol{n}\Big)\boldsymbol{n}, (\boldsymbol{\varphi}\cdot\boldsymbol{n})\boldsymbol{n}\Big\rangle_\Gamma.\end{aligned}$$

Next, because $\boldsymbol{n}$ is normalized and belongs to $W^{1,\infty}(\Gamma)$, we have the relation

$$2\Big\langle\Big(\Big[\mathbf{D}(\boldsymbol{\xi})\boldsymbol{n}\Big]\cdot\boldsymbol{n}\Big)\boldsymbol{n},(\boldsymbol{\varphi}\cdot\boldsymbol{n})\boldsymbol{n}\Big\rangle_\Gamma = 2\Big\langle\Big[\mathbf{D}(\boldsymbol{\xi})\boldsymbol{n}\Big]\cdot\boldsymbol{n},\boldsymbol{\varphi}\cdot\boldsymbol{n}\Big\rangle_\Gamma. \tag{IV.23}$$

As a result, we have

$$2\langle\mathbf{D}(\boldsymbol{\xi})\boldsymbol{n},\boldsymbol{\varphi}\rangle_\Gamma = -\langle\alpha\boldsymbol{\xi}_\tau,\boldsymbol{\varphi}_\tau\rangle_\Gamma + \langle 2\Big[\mathbf{D}(\boldsymbol{\xi})\boldsymbol{n}\Big]\cdot\boldsymbol{n},\boldsymbol{\varphi}\cdot\boldsymbol{n}\rangle_\Gamma. \tag{IV.24}$$

Plugging (IV.22) and (IV.24) in (IV.21) leads to

$$\int_\Omega [(-\Delta\,\boldsymbol{\varphi}+\nabla\,\psi)\cdot\boldsymbol{\xi}-\eta\,\mathrm{div}\,\boldsymbol{\varphi}]\,d\boldsymbol{x} = \langle\boldsymbol{\varphi}\cdot\boldsymbol{n},2[\mathbf{D}(\boldsymbol{\xi})\boldsymbol{n}]\cdot\boldsymbol{n}-\eta\rangle_\Gamma-\langle 2[\mathbf{D}(\boldsymbol{\varphi})\boldsymbol{n}]_\tau+\alpha\,\boldsymbol{\varphi}_\tau,\boldsymbol{\xi}_\tau\rangle_\Gamma. \tag{IV.25}$$

Then, for each $(\boldsymbol{\xi},\eta)\in\mathcal{N}^2_{-k}(\Omega)$, the Green's formula (IV.25) holds for any pair $(\boldsymbol{\varphi},\psi)\in W^{2,2}_{k+1}(\Omega)\times W^{1,2}_{k+1}(\Omega)$ by density. In particular, if $(\boldsymbol{u},\pi)\in W^{2,2}_{k+1}(\Omega)\times W^{1,2}_{k+1}(\Omega)$ is a solution of $\mathcal{S}_T$, then (IV.18) holds.

- Existence. Here we prove that problem $(\mathcal{S}_T)$ has a solution $(\boldsymbol{u},\pi)$ that belongs to $W^{1,2}_k(\Omega)\times W^{0,2}_k(\Omega)$. The proof is made of two steps.

Step 1. The case $g=0$.

Since $k>0$, then $(\boldsymbol{f},\chi)$ belongs to $W^{0,2}_1(\Omega)\times W^{1,2}_1(\Omega)$. Thanks to Theorem IV.2, problem $(\mathcal{S}_T)$ has a solution $(\boldsymbol{u},\pi)\in W^{2,2}_1(\Omega)\times W^{1,2}_1(\Omega)$. It remains now to prove that $(\boldsymbol{u},\pi)$ belongs to $W^{1,2}_k(\Omega)\times W^{0,2}_k(\Omega)$. To that end, we shall use again properties of the Stokes problem in the whole space $\mathbb{R}^3$. As we mention before, we first need appropriate extensions of $\boldsymbol{u}$ and π defined in $\mathbb{R}^3$. So let us consider the following Stokes problem in the bounded domain Ω':

$$\begin{cases}-\Delta\boldsymbol{u}'+\nabla\pi'=\mathbf{0} \quad\text{and}\quad \mathrm{div}\,\boldsymbol{u}'=0 \quad\text{in}\quad \Omega',\\ \boldsymbol{u}'=\boldsymbol{u}\quad\text{on}\quad\Gamma.\end{cases} \tag{IV.26}$$

Since $\boldsymbol{u}\cdot\boldsymbol{n}=0$, then thanks to Theorem III.7, the problem (IV.26) has a solution $(\boldsymbol{u}',\pi')\in H^2(\Omega')\times H^1(\Omega')$ Set now

$$\widetilde{\boldsymbol{u}}=\begin{cases}\boldsymbol{u}\quad\text{in}\quad\Omega,\\ \boldsymbol{u}'\quad\text{in}\quad\Omega',\end{cases}\qquad\text{and}\qquad \widetilde{\pi}=\begin{cases}\pi\quad\text{in}\quad\Omega,\\ \pi'\quad\text{in}\quad\Omega'.\end{cases}$$

Then clearly, the pair $(\widetilde{\boldsymbol{u}},\widetilde{\pi})$ belongs to $W^{1,2}_0(\mathbb{R}^3)\times W^{0,2}_1(\mathbb{R}^3)$. The goal is now to identity $(\widetilde{\boldsymbol{u}},\widetilde{\pi})$ with a solution of the Stokes problem in $\mathbb{R}^3$ that belongs to $W^{1,2}_k(\Omega)\times W^{0,2}_k(\Omega)$. Let us set

$$\boldsymbol{F}=-\Delta\,\widetilde{\boldsymbol{u}}+\nabla\widetilde{\pi}\quad\text{and}\quad e=\mathrm{div}\,\widetilde{\boldsymbol{u}}. \tag{IV.27}$$

In order to apply Theorem III.6 with data $\boldsymbol{F}$ and e, we need to show that $(\boldsymbol{F}, e)$ belongs to $W_k^{-1,2}(\mathbb{R}^3) \times W_k^{0,2}(\mathbb{R}^3)$ and satisfies (III.22) . For any $\boldsymbol{\varphi} \in \mathcal{D}(\mathbb{R}^3)$, we have:

$$\begin{aligned}\langle \boldsymbol{F}, \boldsymbol{\varphi}\rangle_{\mathcal{D}'(\mathbb{R}^3)\times\mathcal{D}(\mathbb{R}^3)} &= \langle -\Delta\, \widetilde{\boldsymbol{u}} + \nabla\, \widetilde{\pi}, \boldsymbol{\varphi}\rangle_{\mathcal{D}'(\mathbb{R}^3)\times\mathcal{D}(\mathbb{R}^3)} = -\int_{\mathbb{R}^3} \widetilde{\boldsymbol{u}} \cdot \Delta\boldsymbol{\varphi}\, d\boldsymbol{x} - \int_{\mathbb{R}^3} \widetilde{\pi}\, \mathrm{div}\ \boldsymbol{\varphi}\, d\boldsymbol{x} \\ &= \underbrace{-\int_\Omega \boldsymbol{u} \cdot \Delta\boldsymbol{\varphi}\, d\boldsymbol{x} - \int_\Omega \pi\, \mathrm{div}\ \boldsymbol{\varphi}\, d\boldsymbol{x}}_{I_1} \underbrace{- \int_{\Omega'} \boldsymbol{u'} \cdot \Delta\boldsymbol{\varphi}\, d\boldsymbol{x} - \int_{\Omega'} \pi'\, \mathrm{div}\ \boldsymbol{\varphi}\, d\boldsymbol{x}}_{I_2}.\end{aligned}$$

On the one hand, in Ω, using identity (II.11) and the Green formula (II.14), we can write:

$$\begin{aligned} I_1 &= \int_\Omega \boldsymbol{u} \cdot \Big(\nabla(\mathrm{div}\ \boldsymbol{\varphi}) - 2\mathrm{div}\ \mathbf{D}(\boldsymbol{\varphi})\Big) d\boldsymbol{x} - \int_\Omega \pi\, \mathrm{div}\ \boldsymbol{\varphi}\, d\boldsymbol{x} \\ &= -\int_\Omega \mathrm{div}\boldsymbol{u}\ \mathrm{div}\, \boldsymbol{\varphi}\, d\boldsymbol{x} + \langle \boldsymbol{u} \cdot \boldsymbol{n}, \mathrm{div}\ \boldsymbol{\varphi}\rangle_\Gamma + 2\int_\Omega \nabla\, \boldsymbol{u} : \mathbf{D}(\boldsymbol{\varphi}) d\boldsymbol{x} - 2\langle \mathbf{D}(\boldsymbol{\varphi})\boldsymbol{n}, \boldsymbol{u}\rangle_\Gamma - \int_\Omega \pi\, \mathrm{div}\ \boldsymbol{\varphi}\, d\boldsymbol{x}. \end{aligned}$$

Using the relation

$$\int_\Omega \nabla \boldsymbol{u} : \mathbf{D}(\boldsymbol{\varphi})\, d\boldsymbol{x} = \int_\Omega \mathbf{D}(\boldsymbol{u}) : \nabla\boldsymbol{\varphi}\, d\boldsymbol{x}$$

and again (II.11), (II.14) and recalling that $g = 0$, we arrive at:

$$I_1 = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{\varphi}\, d\boldsymbol{x} + 2\langle \mathbf{D}(\boldsymbol{u})\boldsymbol{n}, \boldsymbol{\varphi}\rangle_\Gamma - \langle \boldsymbol{\varphi} \cdot \boldsymbol{n}, \chi\rangle_\Gamma - 2\langle \mathbf{D}(\boldsymbol{\varphi})\boldsymbol{n}, \boldsymbol{u}\rangle_\Gamma - \langle \boldsymbol{\varphi} \cdot \boldsymbol{n}, \pi\rangle_\Gamma.$$

On the other hand as previously, in Ω', we can write;

$$\begin{aligned} I_2 &= \int_{\Omega'} \boldsymbol{u'} \cdot \Big(\nabla(\mathrm{div}\ \boldsymbol{\varphi}) - 2\mathrm{div}\ \mathbf{D}(\boldsymbol{\varphi})\Big) d\boldsymbol{x} - \int_{\Omega'} \pi'\, \mathrm{div}\ \boldsymbol{\varphi}\, d\boldsymbol{x} \\ &= -\int_{\Omega'} \mathrm{div}\boldsymbol{u'}\ \mathrm{div}\, \boldsymbol{\varphi}\, d\boldsymbol{x} - \langle \boldsymbol{u'} \cdot \boldsymbol{n}, \mathrm{div}\, \boldsymbol{\varphi}\rangle_\Gamma + 2\int_{\Omega'} \nabla\, \boldsymbol{u'} : \mathbf{D}(\varphi) d\boldsymbol{x} \\ &\quad + 2\langle \boldsymbol{u'}, \mathbf{D}(\boldsymbol{\varphi})\boldsymbol{n}\rangle_\Gamma - \int_{\Omega'} \pi\, \mathrm{div}\ \boldsymbol{\varphi}\, d\boldsymbol{x} \\ &= -2\langle \mathbf{D}(\boldsymbol{u'})\boldsymbol{n}, \boldsymbol{\varphi}\rangle_\Gamma + 2\langle \mathbf{D}(\boldsymbol{\varphi})\boldsymbol{n}, \boldsymbol{u'}\rangle_\Gamma + \langle \boldsymbol{\varphi} \cdot \boldsymbol{n}, \pi'\rangle_\Gamma. \end{aligned}$$

Therefore, denoting by $\widetilde{\boldsymbol{f}} \in W_{k+1}^{0,2}(\mathbb{R}^3)$ the extension of $\boldsymbol{f}$ by zero in Ω' , we deduce that for any $\boldsymbol{\varphi} \in \mathcal{D}(\mathbb{R}^3)$,

$$\begin{aligned} \langle \boldsymbol{F}, \boldsymbol{\varphi}\rangle_{\mathcal{D}'(\mathbb{R}^3)\times\mathcal{D}(\mathbb{R}^3)} &= \int_{\mathbb{R}^3} \widetilde{\boldsymbol{f}} \cdot \boldsymbol{\varphi} d\boldsymbol{x} + 2\langle \mathbf{D}(\boldsymbol{u})\boldsymbol{n} - \mathbf{D}(\boldsymbol{u'})\boldsymbol{n}, \boldsymbol{\varphi}\rangle_\Gamma \\ &\quad + \langle \boldsymbol{\varphi} \cdot \boldsymbol{n}, \pi' - \pi\rangle_\Gamma - \langle \boldsymbol{\varphi} \cdot \boldsymbol{n}, \chi\rangle_\Gamma. \end{aligned} \tag{IV.28}$$

Since $\mathcal{D}(\mathbb{R}^3)$ is dense in $W_{-k}^{1,2}(\mathbb{R}^3)$ and $\widetilde{\boldsymbol{f}}$ belongs to $W_{k+1}^{0,2}(\mathbb{R}^3)$, then (IV.28) is still valid for any $\boldsymbol{\varphi} \in W_{-k}^{1,2}(\mathbb{R}^3)$ which implies that $\boldsymbol{F}$ belongs to $W_k^{-1,2}(\mathbb{R}^3)$.

Now for any $\phi \in \mathcal{D}(\mathbb{R}^3)$, we have

$$\begin{aligned}\langle e, \phi\rangle_{\mathcal{D}'(\mathbb{R}^3)\times\mathcal{D}(\mathbb{R}^3)} &= \langle \operatorname{div} \widetilde{\boldsymbol{u}}, \phi\rangle_{\mathcal{D}'(\mathbb{R}^3)\times\mathcal{D}(\mathbb{R}^3)} = -\int_{\mathbb{R}^3} \widetilde{\boldsymbol{u}} \cdot \nabla \phi d\boldsymbol{x} \\ &= \int_\Omega \operatorname{div} \boldsymbol{u} \ \ \phi d\boldsymbol{x} + \int_{\Omega'} \operatorname{div} \boldsymbol{u'} \ \ \phi d\boldsymbol{x} - \langle \boldsymbol{u}\cdot\boldsymbol{n}, \phi\rangle_\Gamma + \langle \boldsymbol{u'}\cdot\boldsymbol{n}, \phi\rangle_\Gamma \\ &= \int_\Omega \chi\, \phi d\boldsymbol{x}.\end{aligned}$$

Due to the density of $\mathcal{D}(\mathbb{R}^3)$ in $W^{0,2}_{-k}(\mathbb{R}^3)$, we obtain

$$\langle e, \phi\rangle_{\mathcal{D}'(\mathbb{R}^3)\times\mathcal{D}(\mathbb{R}^3)} = \int_\Omega \chi\, \phi d\boldsymbol{x}, \quad \forall \phi \in W^{0,2}_{-k}(\mathbb{R}^3), \tag{IV.29}$$

which implies that e belongs to $W^{0,2}_k(\mathbb{R}^3)$.

As a result, we have proved that $(\boldsymbol{F}, e)$ belongs to $W^{-1,2}_k(\mathbb{R}^3) \times W^{0,2}_k(\mathbb{R}^3)$. Let us now prove that $\boldsymbol{F}$ and e satisfy (III.22), which in view of (IV.28) and (IV.29), amounts to prove that for any $(\boldsymbol{\lambda}, \mu)$ in N_{k-1}

$$\begin{aligned}&\int_{\mathbb{R}^3} \widetilde{\boldsymbol{f}} \cdot \boldsymbol{\lambda} d\boldsymbol{x} - \int_\Omega \chi\, \mu d\boldsymbol{x} + 2\langle \mathbf{D}(\boldsymbol{u})\boldsymbol{n} - \mathbf{D}(\boldsymbol{u'})\boldsymbol{n}, \boldsymbol{\lambda}\rangle_\Gamma \\ &+\langle \boldsymbol{\lambda}\cdot\boldsymbol{n}, \pi' - \pi\rangle_\Gamma - \langle \boldsymbol{\lambda}\cdot\boldsymbol{n}, \chi\rangle_\Gamma = 0.\end{aligned} \tag{IV.30}$$

So let $(\boldsymbol{\lambda}, \mu) \in N_{k-1}$ and let $(\boldsymbol{v}(\boldsymbol{\lambda}), \theta(\boldsymbol{\lambda}))$ be in $W^{1,2}_0(\Omega) \times L^2(\Omega)$ such that the pair $(\boldsymbol{v}(\boldsymbol{\lambda}) - \boldsymbol{\lambda}, \theta(\boldsymbol{\lambda}) - \mu)$ belongs to $\mathcal{N}^2_{-k}(\Omega)$. Note that thanks to Theorem IV.2, the pair $(\boldsymbol{v}(\boldsymbol{\lambda}), \theta(\boldsymbol{\lambda}))$ also belongs to $W^{2,2}_1(\Omega) \times W^{1,2}_1(\Omega)$.

Now, for any $(\boldsymbol{v}, \theta) \in W^{2,2}_1(\Omega) \times W^{1,2}_1(\Omega)$ such that $\operatorname{div} \boldsymbol{v} = 0$, computations in Ω yield

$$\begin{aligned}&\int_\Omega \boldsymbol{f}\cdot \boldsymbol{v} \ \ d\boldsymbol{x} \ = \int_\Omega (-\Delta \boldsymbol{u} \ + \nabla\pi)\cdot \boldsymbol{v} \ d\boldsymbol{x} \\ &= 2\int_\Omega \nabla \boldsymbol{u} \ : \mathbf{D}(\boldsymbol{v} \ )d\boldsymbol{x} \ + \langle \boldsymbol{v} \ \cdot \boldsymbol{n} \ , \chi\rangle_\Gamma - 2\langle \mathbf{D}(\boldsymbol{u} \ )\boldsymbol{n} \ , \boldsymbol{v} \ \rangle_\Gamma + \langle \boldsymbol{v} \ \cdot \boldsymbol{n} \ , \pi\rangle_\Gamma \\ &= -\int_\Omega \Delta \boldsymbol{v} \ \cdot \boldsymbol{u} \ d\boldsymbol{x} \ + \langle \boldsymbol{v} \ \cdot \boldsymbol{n} \ , \chi\rangle_\Gamma + 2\langle \mathbf{D}(\boldsymbol{v} \ )\boldsymbol{n} \ , \boldsymbol{u} \ \rangle_\Gamma - 2\langle \mathbf{D}(\boldsymbol{u} \ )\boldsymbol{n} \ , \boldsymbol{v} \ \rangle_\Gamma + \langle \boldsymbol{v} \ \cdot \boldsymbol{n} \ , \pi\rangle_\Gamma \\ &= \int_\Omega (-\Delta \boldsymbol{v} \ + \nabla\theta)\cdot \boldsymbol{u} \ d\boldsymbol{x} \ + \int_\Omega \chi\theta d\boldsymbol{x} \ + \langle \boldsymbol{v} \ \cdot \boldsymbol{n} \ , \chi\rangle_\Gamma + 2\langle \mathbf{D}(\boldsymbol{v} \ )\boldsymbol{n} \ , \boldsymbol{u} \ \rangle_\Gamma \\ &- 2\langle \mathbf{D}(\boldsymbol{u} \ )\boldsymbol{n} \ , \boldsymbol{v} \ \rangle_\Gamma + \langle \boldsymbol{v} \ \cdot \boldsymbol{n} \ , \pi\rangle_\Gamma.\end{aligned}$$

In particular we have

$$\begin{aligned}&\int_\Omega \boldsymbol{f}\cdot \boldsymbol{v} \ (\boldsymbol{\lambda}) \, d\boldsymbol{x} \ - \int_\Omega \chi\theta(\boldsymbol{\lambda}) d\boldsymbol{x} \ - \langle \boldsymbol{v} \ (\boldsymbol{\lambda})\cdot \boldsymbol{n} \ , \chi\rangle_\Gamma - 2\langle \mathbf{D}(\boldsymbol{v} \ (\boldsymbol{\lambda}))\boldsymbol{n} \ , \boldsymbol{u} \ \rangle_\Gamma \\ &+2\langle \mathbf{D}(\boldsymbol{u} \ )\boldsymbol{n} \ , \boldsymbol{v} \ (\boldsymbol{\lambda})\rangle_\Gamma - \langle \boldsymbol{v} \ (\boldsymbol{\lambda})\cdot \boldsymbol{n} \ , \pi\rangle_\Gamma = 0.\end{aligned} \tag{IV.31}$$

Now making the difference between (IV.31) and (IV.18) with $\boldsymbol{\xi} = \boldsymbol{v}(\boldsymbol{\lambda}) - \boldsymbol{\lambda}$ and $\eta = \theta(\boldsymbol{\lambda}) - \mu$

and recalling that $\boldsymbol{v}\ (\boldsymbol{\lambda}) \cdot \boldsymbol{n}\ = \boldsymbol{\lambda} \cdot \boldsymbol{n}$ on Γ, yields

$$\int_\Omega \boldsymbol{f} \cdot \boldsymbol{\lambda}\, d\boldsymbol{x}\ - \int_\Omega \chi \mu d\boldsymbol{x}\ - \langle \boldsymbol{v}\ (\boldsymbol{\lambda}) - \boldsymbol{\lambda}, \boldsymbol{h}\ \rangle_\Gamma - \langle \boldsymbol{\lambda} \cdot \boldsymbol{n}\ , \chi \rangle_\Gamma - 2\langle \mathbf{D}(\boldsymbol{v}\ (\boldsymbol{\lambda}))\boldsymbol{n}\ , \boldsymbol{u}\ \rangle_\Gamma$$
$$+2\langle \mathbf{D}(\boldsymbol{u}\ )\boldsymbol{n}, \boldsymbol{v}\ (\boldsymbol{\lambda})\rangle_\Gamma - \langle \boldsymbol{\lambda} \cdot \boldsymbol{n}\ , \pi \rangle_\Gamma = 0. \tag{IV.32}$$

Next, computations on Ω' yield

$$\int_{\Omega'} (-\Delta\, \boldsymbol{u'} + \nabla\, \pi') \cdot \boldsymbol{\lambda} d\boldsymbol{x} = 0$$
$$= -\int_{\Omega'} \boldsymbol{u'} \cdot \Delta\, \boldsymbol{\lambda} d\boldsymbol{x} + 2\langle \mathbf{D}(\boldsymbol{u'})\boldsymbol{n}, \boldsymbol{\lambda}\rangle_\Gamma - 2\langle \mathbf{D}(\boldsymbol{\lambda})\boldsymbol{n}, \boldsymbol{u'}\rangle_\Gamma - \langle \boldsymbol{\lambda} \cdot \boldsymbol{n}, \pi' \rangle_\Gamma.$$

The fact that $\boldsymbol{u}\ = \boldsymbol{u'}$ on Γ, implies

$$\int_{\Omega'} \boldsymbol{u'} \cdot \Delta \boldsymbol{\lambda}\, d\boldsymbol{x}\ = \int_{\Omega'} \boldsymbol{u'} \cdot \nabla \mu\, d\boldsymbol{x}\ = -\int_{\Omega'} \mu\, \mathrm{div}\, \boldsymbol{u'} d\boldsymbol{x}\ + \langle \boldsymbol{u'} \cdot \boldsymbol{n}\ , \mu \rangle_\Gamma = 0$$

and we deduce that

$$2\langle \mathbf{D}(\boldsymbol{u'})\boldsymbol{n}, \boldsymbol{\lambda}\rangle_\Gamma - 2\langle \mathbf{D}(\boldsymbol{\lambda})\boldsymbol{n}, \boldsymbol{u}\rangle_\Gamma - \langle \boldsymbol{\lambda} \cdot \boldsymbol{n}, \pi' \rangle_\Gamma = 0. \tag{IV.33}$$

Combining (IV.32) and (IV.33) yields

$$\begin{aligned} &\int_{\mathbb{R}^3} \widetilde{\boldsymbol{f}} \cdot \boldsymbol{\lambda} d\boldsymbol{x} - \int_\Omega \chi\, \mu d\boldsymbol{x} - \langle \boldsymbol{\lambda} \cdot \boldsymbol{n}, \chi \rangle_\Gamma + \langle \boldsymbol{\lambda} \cdot \boldsymbol{n}, \pi' - \pi \rangle_\Gamma - 2\langle \mathbf{D}(\boldsymbol{u'})\boldsymbol{n}, \boldsymbol{\lambda}\rangle_\Gamma \\ &+ 2\langle \mathbf{D}(\boldsymbol{\lambda})\boldsymbol{n} - \mathbf{D}(\boldsymbol{v}(\boldsymbol{\lambda}))\boldsymbol{n}, \boldsymbol{u}\rangle_\Gamma + 2\langle \mathbf{D}(\boldsymbol{u}\ )\boldsymbol{n}, \boldsymbol{v}(\boldsymbol{\lambda})\rangle_\Gamma - \langle \boldsymbol{v}(\boldsymbol{\lambda}) - \boldsymbol{\lambda}, \boldsymbol{h}\ \rangle_\Gamma = 0. \end{aligned} \tag{IV.34}$$

Let us now compute the terms of the second line of (IV.34). Due to the fact that $g = 0$ on Γ and using the Navier boundary condition in (IV.13), we have

$$2\langle \mathbf{D}(\boldsymbol{\lambda})\boldsymbol{n} - \mathbf{D}(\boldsymbol{v}(\boldsymbol{\lambda}))\boldsymbol{n}, \boldsymbol{u}\rangle_\Gamma = 2\big\langle [\mathbf{D}(\boldsymbol{\lambda})\boldsymbol{n}]_\tau - [\mathbf{D}(\boldsymbol{v}(\boldsymbol{\lambda}))\boldsymbol{n}]_\tau, \boldsymbol{u}_\tau \big\rangle_\Gamma$$
$$= \langle \alpha \boldsymbol{v}\ (\boldsymbol{\lambda})_\tau - \boldsymbol{\lambda}_\tau, \boldsymbol{u}\ _\tau \rangle_\Gamma.$$

Next, using the fact that $\boldsymbol{v}\ (\boldsymbol{\lambda}) \cdot \boldsymbol{n}\ = \boldsymbol{\lambda} \cdot \boldsymbol{n}$ on Γ,

$$2\langle \mathbf{D}(\boldsymbol{u}\ )\boldsymbol{n}, \boldsymbol{v}(\boldsymbol{\lambda})\rangle_\Gamma = 2\big\langle [\mathbf{D}(\boldsymbol{u}\ )\boldsymbol{n}]_\tau, \boldsymbol{v}(\boldsymbol{\lambda})_\tau \big\rangle_\Gamma + 2\big\langle ([\mathbf{D}(\boldsymbol{u}\ )\boldsymbol{n}] \cdot \boldsymbol{n}\ )\boldsymbol{n}\ , (\boldsymbol{v}(\boldsymbol{\lambda}) \cdot \boldsymbol{n}\ )\boldsymbol{n}\ \big\rangle_\Gamma$$
$$= 2\big\langle [\mathbf{D}(\boldsymbol{u}\ )\boldsymbol{n}]_\tau, \boldsymbol{v}(\boldsymbol{\lambda})_\tau \big\rangle_\Gamma + 2\langle (\mathbf{D}(\boldsymbol{u}\ )\boldsymbol{n}) \cdot \boldsymbol{n}, \boldsymbol{\lambda} \cdot \boldsymbol{n}\rangle_\Gamma.$$

Finally, from (IV.1), we clearly have

$$\langle \boldsymbol{v}(\boldsymbol{\lambda}) - \boldsymbol{\lambda}, \boldsymbol{h}\ \rangle_\Gamma = \langle \boldsymbol{v}(\boldsymbol{\lambda})_\tau - \boldsymbol{\lambda}_\tau, \boldsymbol{h}\ \rangle_\Gamma.$$

Combining these three expressions, we obtain

$$
\begin{aligned}
&2\langle \mathbf{D}(\boldsymbol{\lambda})\boldsymbol{n} - \mathbf{D}(\boldsymbol{v}(\boldsymbol{\lambda}))\boldsymbol{n}, \boldsymbol{u}\rangle_\Gamma + 2\langle \mathbf{D}(\boldsymbol{u}\ )\boldsymbol{n}, \boldsymbol{v}(\boldsymbol{\lambda})\rangle_\Gamma - \langle \boldsymbol{v}(\boldsymbol{\lambda}) - \boldsymbol{\lambda}, \boldsymbol{h}\ \rangle_\Gamma \\
&= \langle \alpha \boldsymbol{v}(\boldsymbol{\lambda})_\tau - \boldsymbol{\lambda}_\tau, \boldsymbol{u}_\tau\rangle_\Gamma + 2\big\langle \big[\mathbf{D}(\boldsymbol{u})\boldsymbol{n}\big]_\tau, \boldsymbol{v}(\boldsymbol{\lambda})_\tau\big\rangle_\Gamma + 2\langle (\mathbf{D}(\boldsymbol{u})\boldsymbol{n})\cdot\boldsymbol{n}, \boldsymbol{\lambda}\cdot\boldsymbol{n}\rangle_\Gamma - \langle \boldsymbol{v}(\boldsymbol{\lambda})_\tau - \boldsymbol{\lambda}_\tau, \boldsymbol{h}\ \rangle_\Gamma \\
&= \langle \alpha \boldsymbol{v}(\boldsymbol{\lambda})_\tau - \boldsymbol{\lambda}_\tau, \boldsymbol{u}_\tau\rangle_\Gamma + 2\big\langle \big[\mathbf{D}(\boldsymbol{u})\boldsymbol{n}\big]_\tau, \boldsymbol{v}(\boldsymbol{\lambda})_\tau - \boldsymbol{\lambda}_\tau\big\rangle_\Gamma + 2\langle \mathbf{D}(\boldsymbol{u})\boldsymbol{n}, \boldsymbol{\lambda}\rangle_\Gamma - \langle \boldsymbol{v}(\boldsymbol{\lambda})_\tau - \boldsymbol{\lambda}_\tau, \boldsymbol{h}\ \rangle_\Gamma \\
&= \big\langle \boldsymbol{v}(\boldsymbol{\lambda})_\tau - \boldsymbol{\lambda}_\tau, 2\big[\mathbf{D}(\boldsymbol{u})\boldsymbol{n}\big]_\tau + \alpha \boldsymbol{u}_\tau\big\rangle_\Gamma + 2\langle \mathbf{D}(\boldsymbol{u})\boldsymbol{n}, \boldsymbol{\lambda}\rangle_\Gamma - \langle \boldsymbol{v}(\boldsymbol{\lambda})_\tau - \boldsymbol{\lambda}_\tau, \boldsymbol{h}\ \rangle_\Gamma \\
&= \langle \boldsymbol{v}(\boldsymbol{\lambda})_\tau - \boldsymbol{\lambda}_\tau, \boldsymbol{h}\ \rangle_\Gamma + 2\langle \mathbf{D}(\boldsymbol{u})\boldsymbol{n}, \boldsymbol{\lambda}\rangle_\Gamma - \langle \boldsymbol{v}(\boldsymbol{\lambda})_\tau - \boldsymbol{\lambda}_\tau, \boldsymbol{h}\ \rangle_\Gamma \\
&= 2\langle \mathbf{D}(\boldsymbol{u})\boldsymbol{n}, \boldsymbol{\lambda}\rangle_\Gamma.
\end{aligned}
$$

Plugging this in (IV.34) allows to get (IV.30).

Therefore it follows from Theorem III.6, that there exists a unique solution $(\widetilde{\boldsymbol{z}}, \widetilde{q}) \in (\boldsymbol{W}^1_k(\mathbb{R}^3) \times W^{0,2}_k(\mathbb{R}^3))$ satisfying the following Stokes problem:

$$
-\Delta\, \widetilde{\boldsymbol{z}} + \nabla\, \widetilde{q} = \boldsymbol{F} \quad \text{and} \quad \operatorname{div} \widetilde{\boldsymbol{z}} = e \quad \text{in} \quad \mathbb{R}^3.
$$

Using (IV.27), we obtain

$$
-\Delta\, (\widetilde{\boldsymbol{z}} - \widetilde{\boldsymbol{u}}) + \nabla\, (\widetilde{q} - \widetilde{\pi}) = \mathbf{0} \quad \text{and} \quad \operatorname{div} (\widetilde{\boldsymbol{z}} - \widetilde{\boldsymbol{u}}) = 0 \quad \text{in} \quad \mathbb{R}^3.
$$

It follows that $(\widetilde{\boldsymbol{z}} - \widetilde{\boldsymbol{u}}, \widetilde{q} - \widetilde{\pi})$ belongs to $\Big(W^{1,2}_k(\mathbb{R}^3) + W^{1,2}_0(\mathbb{R}^3)\Big) \times \Big(W^{0,2}_k(\mathbb{R}^3) + W^{0,2}_1(\mathbb{R}^3)\Big)$ and since $k > 0$ then $(\widetilde{\boldsymbol{z}} - \widetilde{\boldsymbol{u}}, \widetilde{q} - \widetilde{\pi})$ also belongs to $W^{1,2}_0(\mathbb{R}^3) \times W^{0,2}_1(\mathbb{R}^3)$. Thanks to Proposition III.3, we deduce that $\widetilde{\boldsymbol{z}} - \widetilde{\boldsymbol{u}} = \mathbf{0}$ and $\widetilde{q} - \widetilde{\pi} = 0$ which imply that the solution $(\boldsymbol{u}, \pi)$ belongs indeed to $W^{1,2}_k(\Omega) \times W^{0,2}_k(\Omega)$.

Step 2. The case $g \neq 0$.

The aim is to lift the function g. To that end, consider the following exterior Neumann problem

$$
\Delta\, v = 0 \quad \text{in} \quad \Omega \quad \text{and} \quad \frac{\partial v}{\partial \boldsymbol{n}} = g \quad \text{on} \quad \Gamma. \tag{IV.35}
$$

This problem has a solution v that belongs to $W^{3,2}_0(\Omega)$ (see [48] or [44, Theorem 3.10]). Let R be a positive real number large enough so that $\overline{\Omega'} \subset B_R$ and let $\psi \in \mathcal{D}(\overline{\Omega})$ such that $0 \leq \psi \leq 1$, supp $\psi \subset \overline{\Omega}_{R+1}$ and $\psi = 1$ in $\overline{\Omega}_R$. Set now $w = v\psi$, then w has obviously a compact support and thus w belongs to $W^{3,2}_{k+1}(\Omega)$. Observe that we are not able to look for a solution v of (IV.35) that belongs directly to $W^{3,2}_{k+1}(\Omega)$ because in this case, the datum g must satisfy a compatibility condition (see relation (3.4) of [44, Theorem 3.10]).

Consider now the following problem:

$$
\begin{cases}
-\Delta \boldsymbol{z} + \nabla \pi = \boldsymbol{f} + \Delta(\nabla w) \quad \text{and} \quad \operatorname{div} \boldsymbol{z} = \chi - \Delta w \quad \text{in} \quad \Omega, \\
\boldsymbol{z}\cdot\boldsymbol{n} = 0 \quad \text{and} \quad 2[\mathbf{D}(\boldsymbol{z})\boldsymbol{n}]_\tau + \alpha \boldsymbol{z}_\tau = \boldsymbol{K} \quad \text{on} \quad \Gamma,
\end{cases} \tag{IV.36}
$$

where

$$\boldsymbol{K} = \boldsymbol{h} - 2[\mathbf{D}(\nabla w)\boldsymbol{n}]_\tau - \alpha(\nabla w)_\tau.$$

It is clear that $\boldsymbol{K}\cdot\boldsymbol{n} = 0$ on Γ, $\boldsymbol{K} \in H^{1/2}(\Gamma)$ and $\boldsymbol{f} + \Delta(\nabla w)$ belongs to $W^{0,2}_{k+1}(\Omega)$. According to step 1, problem (IV.36) has a solution in $W^{1,2}_k(\Omega)\times W^{0,2}_k(\Omega)$ if and only if

$$\forall(\boldsymbol{\xi},\eta)\in\mathcal{N}^2_{-k}(\Omega),\quad \int_\Omega(\boldsymbol{f}+\Delta(\nabla w))\cdot\boldsymbol{\xi}d\boldsymbol{x} - \int_\Omega(\chi-\Delta w)\eta d\boldsymbol{x} = -\langle\boldsymbol{K},\boldsymbol{\xi}\rangle_\Gamma. \quad \text{(IV.37)}$$

But if $(\boldsymbol{\xi},\eta)$ is in $\mathcal{N}^2_{-k}(\Omega)$, then since $\nabla w\in W^{2,2}_{k+1}(\Omega)$, we can write (IV.25) for the pair $(-\nabla w, 0)$ and we obtain

$$\begin{aligned}\int_\Omega\Delta(\nabla w)\cdot\boldsymbol{\xi}d\boldsymbol{x} + \int_\Omega\Delta w\ \eta d\boldsymbol{x} \quad &= \quad -\langle g, 2[\mathbf{D}(\boldsymbol{\xi})\boldsymbol{n}]\cdot\boldsymbol{n} - \eta\rangle_\Gamma \\ &\quad +\langle 2[\mathbf{D}(\nabla w)\boldsymbol{n}]_\tau + \alpha(\nabla w)_\tau, \boldsymbol{\xi}\rangle_\Gamma. \qquad \text{(IV.38)}\end{aligned}$$

Combining (IV.18) and (IV.38) allows to obtain (IV.37). Thus setting $\boldsymbol{u} = \boldsymbol{z} + \nabla w \in W^{1,2}_k(\Omega)$, then the pair $(\boldsymbol{u},\pi)\in W^{1,2}_k(\Omega)\times W^{0,2}_k(\Omega)$ is a solution of $(\mathcal{S}_T)$.

- Regularity. Finally, to prove that the solution $(\boldsymbol{u},\pi)\in W^{1,2}_k(\Omega)\times W^{0,2}_k(\Omega)$ of $(\mathcal{S}_T)$ established previously, actually belongs to $W^{2,2}_{k+1}(\Omega)\times W^{1,2}_{k+1}(\Omega)$, we can proceed as in the proof of Theorem IV.2 with the use of the partition of unity III.18.

□

4 Very weak solution

In this section, we study the existence and the uniqueness of very weak solutions in $W^{0,2}_{-k-1}(\Omega)\times W^{-1,2}_{-k-1}(\Omega)$, where $k\in\mathbb{Z}$ for the Stokes problem $(\mathcal{S}_T)$. The main idea consists in the use of a duality argument with the strong solutions. We assume that α is a positive function that belongs to $W^{1,\infty}(\Gamma)$.

To that end, we will introduce the following space:

$$T^2_{k+1}(\Omega) = \{\boldsymbol{v}\in\mathring{H}^2_{k-1}(\mathrm{div},\Omega);\quad \mathrm{div}\,\boldsymbol{v}\in\mathring{W}^{1,2}_{k+1}(\Omega)\},$$

which is a Hilbert space equipped with the following norm

$$\|\boldsymbol{v}\|_{T^2_{k+1}(\Omega)} = \|\boldsymbol{v}\|_{W^{0,2}_{k-1}(\Omega)} + \|\mathrm{div}\,\boldsymbol{v}\|_{W^{1,2}_{k+1}(\Omega)}$$

The following Lemma state some properties related to the space $T^2_{k+1}(\Omega)$.

Lemma IV.1 *Let $k \in \mathbb{Z}$, the following properties hold:*

1- A distribution $\boldsymbol{f}$ belongs to $(T^2_{k+1}(\Omega))'$ if and only if there exist $\boldsymbol{\phi} \in W^{0,2}_{-k+1}(\Omega)$ and $f_0 \in W^{-1,2}_{-k-1}(\Omega)$, such that

$$\boldsymbol{f} = \boldsymbol{\phi} + \nabla f_0$$

2- The space $\mathcal{D}(\Omega)$ is dense in $T^2_{k+1}(\Omega)$.

3- For any $\chi \in W^{-1,2}_{-k-1}(\Omega)$ and $\boldsymbol{v} \in T^2_{k+1}(\Omega)$, we have

$$\langle \nabla \chi, \boldsymbol{v} \rangle_{(T^2_{k+1}(\Omega))' \times T^2_{k+1}(\Omega)} = -\langle \chi, \operatorname{div} \boldsymbol{v} \rangle_{W^{-1,2}_{-k-1}(\Omega) \times \mathring{W}^{1,2}_{k+1}(\Omega)}. \tag{IV.39}$$

Giving a meaning to the boundary condition of the Stokes problem $(\mathcal{S}_T)$ is not easy. For this reason, we need to introduce the Hilbert space, for $k \in \mathbb{Z}$:

$$\mathbf{H}^2_{-k-1}(\Delta;\Omega) = \{\boldsymbol{v} \in W^{0,2}_{-k-1}(\Omega);\ \ \Delta\, \boldsymbol{v} \in (T^2_{k+1}(\Omega))'\},$$

which has the following norm:

$$\|\boldsymbol{v}\|_{\mathbf{H}^2_{-k-1}(\Delta;\Omega)} = \|\boldsymbol{v}\|_{W^{0,2}_{-k-1}(\Omega)} + \|\Delta\, \boldsymbol{v}\|_{(T^2_{k+1}(\Omega))'}$$

Lemma IV.2 *The space $\mathcal{D}(\overline{\Omega})$ is dense in $\boldsymbol{H}^2_{-k-1}(\Delta;\Omega)$.*

The proofs of Lemma IV.1 and Lemma IV.2 can be found in [64]. Finally, in order to write a Green formula, we define for $k \in \mathbb{Z}$

$$\mathbf{S}^2_{k+1}(\Omega) = \{\boldsymbol{v} \in W^{2,2}_{k+1}(\Omega);\quad \operatorname{div} \boldsymbol{v} = 0,\quad \boldsymbol{v} \cdot \boldsymbol{n} = 0,\ 2[\mathbf{D}(\boldsymbol{v})\boldsymbol{n}]_\tau + \alpha \boldsymbol{v}_\tau = 0 \text{ on } \Gamma\}$$

The next lemma will help us to prove a trace result.

Lemma IV.3 *Let α belongs to $W^{1,\infty}(\Gamma)$. The linear mapping $\Theta\ :\ \boldsymbol{u} \mapsto 2[\boldsymbol{D}(\boldsymbol{u})\boldsymbol{n}]_\tau + \alpha \boldsymbol{u}_\tau$ defined on $\mathcal{D}(\overline{\Omega})$ can be extended to a linear and continuous mapping still denoted by Θ, from $\boldsymbol{H}^2_{-k-1}(\Delta;\Omega)$ into $H^{-3/2}(\Gamma)$ and we have the following Green's formula: for any $\boldsymbol{u} \in \boldsymbol{H}^2_{-k-1}(\Delta;\Omega)$ and $\varphi \in \boldsymbol{S}^2_{k+1}(\Omega)$,*

$$\begin{aligned}\langle \Delta\, \boldsymbol{u}, \varphi \rangle_{(T^2_{k+1}(\Omega))' \times T^2_{k+1}(\Omega)} \ &= \ \int_\Omega \boldsymbol{u} \cdot \Delta\, \varphi d\boldsymbol{x} + \langle 2[\boldsymbol{D}(\boldsymbol{u})\boldsymbol{n}]_\tau + \alpha \boldsymbol{u}_\tau, \varphi \rangle_{H^{-3/2}(\Gamma) \times H^{3/2}(\Gamma)} \\ &\quad - \langle 2[\boldsymbol{D}(\varphi)\boldsymbol{n}] \cdot \boldsymbol{n}, \boldsymbol{u} \cdot \boldsymbol{n} \rangle_{H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)}. \end{aligned} \tag{IV.40}$$

Proof. The goal is to prove that the mapping Θ defined on $\mathcal{D}(\overline{\Omega})$ is continuous for the norm of $\mathbf{H}^2_{-k-1}(\Delta;\Omega)$. Let $\boldsymbol{u} \in \mathcal{D}(\overline{\Omega})$ and $\boldsymbol{\varphi} \in \mathbf{S}^2_{k+1}(\Omega)$. Using the identity (II.11) and the Green's formula (II.14), we obtain

$$\begin{aligned}\langle \Delta\, \boldsymbol{u}, \boldsymbol{\varphi}\rangle_{(T^2_{k+1}(\Omega))'\times T^2_{k+1}(\Omega))} &= -2\int_\Omega \mathbf{D}(\boldsymbol{u}) : \nabla\boldsymbol{\varphi}\, d\boldsymbol{x} + \int_\Omega \operatorname{div}\boldsymbol{u}\operatorname{div}\boldsymbol{\varphi} d\boldsymbol{x} + 2\langle \mathbf{D}(\boldsymbol{u})\boldsymbol{n}, \boldsymbol{\varphi}\rangle_\Gamma \\ &= \int_\Omega \boldsymbol{u}\cdot\Delta\,\boldsymbol{\varphi} d\mathbf{x} + 2\langle \mathbf{D}(\boldsymbol{u})\boldsymbol{n}, \boldsymbol{\varphi}\rangle_\Gamma - 2\langle \mathbf{D}(\boldsymbol{\varphi})\boldsymbol{n}, \boldsymbol{u}\rangle_\Gamma. \qquad \text{(IV.41)}\end{aligned}$$

Since $2[\mathbf{D}(\boldsymbol{\varphi})\boldsymbol{n}]_\tau = -\alpha\boldsymbol{\varphi}_\tau$ on Γ, we have

$$-2\langle \mathbf{D}(\boldsymbol{\varphi})\boldsymbol{n}, \boldsymbol{u}\rangle = \langle \alpha\boldsymbol{\varphi}_\tau, \boldsymbol{u}_\tau\rangle_\Gamma - 2\langle [\mathbf{D}(\boldsymbol{\varphi})\boldsymbol{n}]\cdot\boldsymbol{n}, \boldsymbol{u}\cdot\boldsymbol{n}\rangle_\Gamma$$

Plunging this in (IV.41), for any $\boldsymbol{u} \in \mathcal{D}(\overline{\Omega})$ and $\boldsymbol{\varphi} \in \mathbf{S}^2_{k+1}(\Omega)$ we have

$$\begin{aligned}\langle \Delta\, \boldsymbol{u}, \boldsymbol{\varphi}\rangle_{(T^2_{k+1}(\Omega))'\times T^2_{k+1}(\Omega))} &= \int_\Omega \boldsymbol{u}\cdot\Delta\,\boldsymbol{\varphi} d\mathbf{x} + \langle 2[\mathbf{D}(\boldsymbol{u})\boldsymbol{n}]_\tau + \alpha\boldsymbol{u}_\tau, \boldsymbol{\varphi}\rangle_{H^{-3/2}(\Gamma)\times H^{3/2}(\Omega)} \\ &\quad -2\langle [\mathbf{D}(\boldsymbol{\varphi})\boldsymbol{n}]\cdot\boldsymbol{n}, \boldsymbol{u}\cdot\boldsymbol{n}\rangle_{H^{1/2}(\Gamma)\times H^{-1/2}(\Gamma)}.\end{aligned}$$

Let now $\boldsymbol{\mu}$ be any element of $H^{3/2}(\Gamma)$. Since $\alpha \in W^{1,\infty}(\Gamma)$, then $\alpha\boldsymbol{\mu}_\tau \in H^{1/2}(\Gamma)$. So there exists an element $\boldsymbol{\varphi}$ in $W^{2,2}_{k+1}(\Omega)$ such that

$$\boldsymbol{\varphi} = \boldsymbol{\mu}_\tau \quad \text{and} \quad \frac{\partial\boldsymbol{\varphi}}{\partial\boldsymbol{n}} = \Lambda\,\boldsymbol{\mu} - \boldsymbol{n}\operatorname{div}_\Gamma\boldsymbol{\mu}_\tau - \alpha\boldsymbol{\mu}_\tau \quad \text{on } \Gamma.$$

Where

$$\Lambda\,\boldsymbol{\mu} = \sum_{k=1}^{2}\left(\boldsymbol{\mu}_\tau\cdot\frac{\partial\boldsymbol{n}}{\partial s_k}\right)\boldsymbol{\tau}_k.$$

As $\Lambda\,\boldsymbol{\mu}\cdot\boldsymbol{n} = 0$ on Γ, we have

$$\frac{\partial\boldsymbol{\varphi}}{\partial\boldsymbol{n}}\cdot\boldsymbol{n} = -\operatorname{div}_\Gamma\boldsymbol{\mu}_\tau \quad \text{on } \Gamma$$

and we recall the following formula (see [4]):

$$\operatorname{div}\boldsymbol{v} = \operatorname{div}_\Gamma\boldsymbol{v}_\tau + \boldsymbol{\beta}\,\boldsymbol{v}\cdot\boldsymbol{n} + \frac{\partial\,\boldsymbol{v}}{\partial\,\boldsymbol{n}}\cdot\boldsymbol{n} \quad \text{on } \Gamma, \qquad \text{(IV.42)}$$

where $\boldsymbol{\beta}$ denotes the mean curvature of Γ, $\operatorname{div}_\Gamma$ is the surface divergence. We deduce that

$$\operatorname{div}\boldsymbol{\varphi} = 0 \quad \text{on } \Gamma.$$

Using relations (II.22), we have

$$2[\mathbf{D}(\boldsymbol{\varphi})\boldsymbol{n}]_\tau + \alpha\boldsymbol{\varphi}_\tau \;=\; \left(\frac{\partial\boldsymbol{\varphi}}{\partial\boldsymbol{n}}\right)_\tau - \Lambda\boldsymbol{\varphi} + \alpha\boldsymbol{\varphi}_\tau.$$

Since $\dfrac{\partial\boldsymbol{\varphi}}{\partial\boldsymbol{n}} = \Lambda\,\boldsymbol{\mu} - \boldsymbol{n}\,\mathrm{div}_\Gamma\boldsymbol{\mu}_\tau - \alpha\boldsymbol{\mu}_\tau$ on Γ and $\boldsymbol{\varphi} = \boldsymbol{\mu}_\tau$ on Γ, then we have

$$\begin{aligned}\left(\frac{\partial\boldsymbol{\varphi}}{\partial\boldsymbol{n}}\right)_\tau &= (\Lambda\,\boldsymbol{\mu})_\tau - \alpha\boldsymbol{\mu}_\tau\\ &= \Lambda\,\boldsymbol{\varphi} - \alpha\boldsymbol{\varphi}_\tau\end{aligned}$$

We deduce that $2[\mathbf{D}(\boldsymbol{\varphi})\boldsymbol{n}]_\tau + \alpha\boldsymbol{\varphi}_\tau = 0$ on Γ. Which implies that the function $\boldsymbol{\varphi}$ belongs to $\mathbf{S}^2_{k+1}(\Omega)$
In addition, we have the following estimate

$$\|\boldsymbol{\varphi}\|_{W^{2,2}_{k+1}(\Omega)} \leqslant C\|\boldsymbol{\mu}_\tau\|_{H^{3/2}(\Gamma)} \leqslant C\|\boldsymbol{\mu}\|_{H^{3/2}(\Gamma)}. \tag{IV.43}$$

Consequently,

$$\begin{aligned}\Big|\langle 2[\mathbf{D}(\boldsymbol{u})\boldsymbol{n}]_\tau + \alpha\boldsymbol{u}_\tau, \boldsymbol{\mu}\rangle_\Gamma\Big| &= \Big|\langle 2[\mathbf{D}(\boldsymbol{u})\boldsymbol{n}]_\tau + \alpha\boldsymbol{u}_\tau, \boldsymbol{\mu}_\tau\rangle_\Gamma\Big| = \Big|\langle 2[\mathbf{D}(\boldsymbol{u})\boldsymbol{n}]_\tau + \alpha\boldsymbol{u}_\tau, \boldsymbol{\varphi}\rangle_\Gamma\Big|\\ &\leqslant \Big|\langle\Delta\,\boldsymbol{u}, \boldsymbol{\varphi}\rangle_\Omega\Big| + \Big|\int_\Omega \boldsymbol{u}\cdot\Delta\,\boldsymbol{\varphi}d\mathbf{x} - 2\langle[\mathbf{D}(\boldsymbol{\varphi})\boldsymbol{n}]\cdot\boldsymbol{n}, \boldsymbol{u}\cdot\boldsymbol{n}\rangle_{H^{1/2}(\Gamma)\times H^{-1/2}(\Gamma)}\Big|\\ &\leqslant \|\Delta\,\boldsymbol{u}\|_{(T^2_{k+1}(\Omega))'}\|\boldsymbol{\varphi}\|_{T^2_{k+1}(\Omega)} + \|\boldsymbol{u}\|_{W^{0,2}_{-k-1}(\Omega)}\|\boldsymbol{\varphi}\|_{W^{2,2}_{k+1}(\Omega)}\\ &\leqslant C\|\boldsymbol{u}\|_{\mathbf{H}^2_{-k-1}(\Delta;\Omega)}\|\boldsymbol{\varphi}\|_{W^{2,2}_{k+1}(\Omega)}\end{aligned}$$

Thus, using (IV.43), we obtain that for any $\boldsymbol{u}\in\mathcal{D}(\overline{\Omega})$:

$$\|2[\mathbf{D}(\boldsymbol{u})\boldsymbol{n}]_\tau + \alpha\boldsymbol{u}_\tau\|_{H^{-3/2}(\Gamma)} \leqslant C\|\boldsymbol{u}\|_{\mathbf{H}^2_{-k-1}(\Delta;\Omega)}.$$

Therefore, the linear mapping $\Theta : \boldsymbol{u} \to 2[\mathbf{D}(\boldsymbol{u})\boldsymbol{n}]_\tau + \alpha\boldsymbol{u}_\tau$ defined in $\mathcal{D}(\overline{\Omega})$ is continuous for the norm of $\mathbf{H}^2_{-k-1}(\Delta;\Omega)$. Finally, by density of $\mathcal{D}(\overline{\Omega})$ in $\mathbf{H}^2_{-k-1}(\Delta;\Omega)$, we can extend this mapping from $\mathbf{H}^2_{-k-1}(\Delta;\Omega)$ into $H^{-3/2}(\Gamma)$ and formula (IV.40) holds. □

Now, we begin by giving the definition of very weak solutions.

Definition IV.1 *Given $\boldsymbol{f}$, χ, g and $\boldsymbol{h}$ with*

$$\boldsymbol{f}\in(\boldsymbol{T}^2_{k+1}(\Omega))', \quad \chi\in W^{0,2}_{-k}(\Omega), \quad g\in H^{-1/2}(\Gamma), \quad \boldsymbol{h}\in H^{-3/2}(\Gamma)$$

such that $\boldsymbol{h}\cdot\boldsymbol{n}=0$ *on* Γ*, a pair* $(\boldsymbol{u},\pi)\in W^{0,2}_{-k-1}(\Omega)\times W^{-1,2}_{-k-1}(\Omega)$ *is called very weak solution of* $(\mathcal{S}_T)$ *if and only if, for any* $(\boldsymbol{\varphi},q)\in \boldsymbol{S}^2_{k+1}(\Omega)\times W^{1,2}_{k+1}(\Omega)$*, the relations*

$$\begin{aligned}&-\int_\Omega \boldsymbol{u}\cdot\Delta\,\boldsymbol{\varphi}d\boldsymbol{x}-\langle\pi,\operatorname{div}\boldsymbol{\varphi}\rangle_{W^{-1,2}_{-k-1}(\Omega)\times\mathring{W}^{1,2}_{k+1}(\Omega)}\\&=\langle\boldsymbol{f},\boldsymbol{\varphi}\rangle_{(\boldsymbol{T}^2_{k+1}(\Omega))'\times\boldsymbol{T}^2_{k+1}(\Omega)}+\langle\boldsymbol{h},\boldsymbol{\varphi}\rangle_{H^{-3/2}(\Gamma)\times H^{3/2}(\Gamma)}-\langle g,2[\boldsymbol{D}(\boldsymbol{\varphi})\boldsymbol{n}]\cdot\boldsymbol{n}\rangle_{H^{-1/2}(\Gamma)\times H^{1/2}(\Gamma)}\end{aligned}\tag{IV.44}$$

and

$$\int_\Omega \boldsymbol{u}\cdot\nabla\,qd\boldsymbol{x}=-\int_\Omega\chi qd\boldsymbol{x}+\langle g,q\rangle_{H^{-1/2}(\Gamma)\times H^{1/2}(\Gamma)}\tag{IV.45}$$

are satisfied.

Note that $\mathbf{S}^2_{k+1}(\Omega)$ is included in $\boldsymbol{T}^2_{k+1}(\Omega)$, which insures that all brackets are well defined

Proposition IV.4 *Under the assumptions of Definition IV.1, the following two statements are equivalent:*
i) $(\boldsymbol{u},\pi)\in W^{0,2}_{-k-1}(\Omega)\times W^{-1,2}_{-k-1}(\Omega)$ *is a very weak solution of* $(\mathcal{S}_T)$*.*
ii) $(\boldsymbol{u},\pi)\in W^{0,2}_{-k-1}(\Omega)\times W^{-1,2}_{-k-1}(\Omega)$ *satisfies the problem* $(\mathcal{S}_T)$ *in the sense of distributions.*

Proof.

i)⇒ ii) Let $(\boldsymbol{u},\pi)\in W^{0,2}_{-k-1}(\Omega)\times W^{-1,2}_{-k-1}(\Omega)$ a very weak solution of $(\mathcal{S}_T)$. Therefore, it follows immediately from (IV.44) and (IV.45), that for any $\boldsymbol{\varphi}\in\mathcal{D}(\Omega)$, we have

$$-\Delta\,\boldsymbol{u}+\nabla\,\pi=\boldsymbol{f}\quad\text{and}\quad\operatorname{div}\boldsymbol{u}=\chi\quad\text{in}\quad\Omega.$$

It remains now to prove that the Navier boundary conditions are satisfied. Let us first prove that $\boldsymbol{u}\cdot\boldsymbol{n}=g$ on Γ. For this, we consider the equation $\operatorname{div}\boldsymbol{u}=\chi$ in Ω. We multiply this equation by $q\in W^{1,2}_{k+1}(\Omega)$ and compare with (IV.45). We get

$$\langle\boldsymbol{u}\cdot\boldsymbol{n},q\rangle_{H^{-1/2}(\Gamma)\times H^{1/2}(\Gamma)}=\langle g,q\rangle_{H^{-1/2}(\Gamma)\times H^{1/2}(\Gamma)}.$$

Which yields $\boldsymbol{u}\cdot\boldsymbol{n}=g\in H^{-1/2}(\Gamma)$.
Next, writing

$$-\Delta\,\boldsymbol{u}=\boldsymbol{f}-\nabla\,\pi.$$

Using Lemma IV.1, we deduce that $\boldsymbol{u}$ belongs to $\mathbf{H}^2_{-k-1}(\Delta\,;\Omega)$. It follows from Lemma IV.3 that $2[\mathbf{D}(\boldsymbol{u})\boldsymbol{n}]_\tau+\alpha\boldsymbol{u}_\tau\in H^{-3/2}(\Gamma)$. Applying (IV.40) and (IV.39) enables us to write: for

any $\boldsymbol{\varphi} \in \mathbf{S}^2_{k+1}(\Omega)$,

$$\begin{aligned} &-\int_\Omega \boldsymbol{u}\cdot\Delta\,\boldsymbol{\varphi} d\mathbf{x} - \langle 2[\mathbf{D}(\boldsymbol{u})\boldsymbol{n}]_\tau + \alpha\boldsymbol{u}_\tau, \boldsymbol{\varphi}\rangle_{H^{-3/2}(\Gamma)\times H^{3/2}(\Gamma)} \\ &+\langle 2[\mathbf{D}(\varphi)\boldsymbol{n}]\cdot\boldsymbol{n}, \boldsymbol{u}\cdot\boldsymbol{n}\rangle_{H^{1/2}(\Gamma)\times H^{-1/2}(\Gamma} = \langle \boldsymbol{f}, \boldsymbol{\varphi}\rangle_{(\boldsymbol{T}^2_{k+1}(\Omega))'\times \boldsymbol{T}^2_{k+1}(\Omega)} + \langle \pi, \operatorname{div}\boldsymbol{\varphi}\rangle_{W^{-1,2}_{-k-1}(\Omega)\times \mathring{W}^{1,2}_{k+1}(\Omega)}. \end{aligned} \tag{IV.46}$$

Comparing (IV.44) with (IV.46), we get, for any $\boldsymbol{\varphi} \in \mathbf{S}^2_{k+1}(\Omega)$

$$\langle 2[\mathbf{D}(\boldsymbol{u})\boldsymbol{n}]_\tau + \alpha\boldsymbol{u}_\tau, \boldsymbol{\varphi}\rangle_{H^{-3/2}(\Gamma)\times H^{3/2}(\Gamma)} = \langle \boldsymbol{h}, \boldsymbol{\varphi}\rangle_{H^{-3/2}(\Gamma)\times H^{3/2}(\Gamma)}.$$

Let $\boldsymbol{\mu} \in H^{3/2}(\Gamma)$, there exists $\boldsymbol{\varphi} \in \mathbf{S}^2_{k+1}(\Omega)$ such that $\boldsymbol{\varphi} = \boldsymbol{\mu}_\tau$ on Γ (see proof of Lemma IV.3). Consequently,

$$\langle 2[\mathbf{D}(\boldsymbol{u})\boldsymbol{n}]_\tau + \alpha\boldsymbol{u}_\tau, \boldsymbol{\mu}\rangle_{H^{-3/2}(\Gamma)\times H^{3/2}(\Gamma)} = \langle \boldsymbol{h}, \boldsymbol{\mu}\rangle_{H^{-3/2}(\Gamma)\times H^{3/2}(\Gamma)}$$

and we deduce that $2[\mathbf{D}(\boldsymbol{u})\boldsymbol{n}]_\tau + \alpha\boldsymbol{u}_\tau = \boldsymbol{h}$ on Γ.

ii)⇒i) Suppose that $(\boldsymbol{u}, \pi) \in W^{0,2}_{-k-1}(\Omega) \times W^{-1,2}_{-k-1}(\Omega)$ satisfies the problem $(\mathcal{S}_T)$ in the sense of distributions. Then, we have

$$-\Delta\boldsymbol{u} + \nabla\,\pi = \boldsymbol{f} \quad \text{and} \quad \operatorname{div}\boldsymbol{u} = \chi \quad \text{in} \quad \mathcal{D}'(\Omega),$$

which implies that, for any $\boldsymbol{\varphi} \in \mathcal{D}(\Omega)$, we have

$$\langle -\Delta\boldsymbol{u} + \nabla\,\pi, \boldsymbol{\varphi}\rangle_{\mathcal{D}'(\Omega)\times\mathcal{D}(\Omega)} = \langle \boldsymbol{f}, \boldsymbol{\varphi}\rangle_{\mathcal{D}'(\Omega)\times\mathcal{D}(\Omega)}. \tag{IV.47}$$

Since $\mathcal{D}(\Omega)$ is dense in $T^2_{k+1}(\Omega)$, then (IV.47) is still valid for any $\boldsymbol{\varphi} \in T^2_{k+1}(\Omega)$. In particular, (IV.47) is still valid for any $\boldsymbol{\varphi} \in \mathbf{S}^2_{k+1}(\Omega) \subset T^2_{k+1}(\Omega)$. Since $\nabla\,\pi \in (T^2_{k+1}(\Omega))'$ and $-\Delta\,\boldsymbol{u} = \boldsymbol{f} - \nabla\,\pi \in (T^2_{k+1}(\Omega))'$. Then according to (IV.39) and (IV.40), it is clear that (IV.44) holds.
Now from the equation $\operatorname{div}\boldsymbol{u} = \chi$ in Ω, we can deduce that for any $q \in \mathcal{D}(\Omega)$

$$\int_\Omega \operatorname{div}\boldsymbol{u} q d\mathbf{x} = \int_\Omega \chi q d\mathbf{x}.$$

Using the Green's formula (II.14), we can deduce (IV.45).

□

Now, we are in position to prove the existence and uniqueness of a very weak solution for Stokes problem $(\mathcal{S}_T)$.

Theorem IV.4 *Given any $\boldsymbol{f}$, χ, g and $\boldsymbol{h}$ with*

$$\boldsymbol{f} \in (\boldsymbol{T}^2_{k+1}(\Omega))', \quad \chi \in W^{0,2}_{-k}(\Omega), \quad g \in H^{-1/2}(\Gamma), \quad \boldsymbol{h} \in H^{-3/2}(\Gamma),$$

such that $\boldsymbol{h} \cdot \boldsymbol{n} = 0$ on Γ and for any $(\boldsymbol{\xi}, \eta) \in \mathcal{N}^2_{k+1}(\Omega)$,

$$\langle \boldsymbol{f}, \boldsymbol{\xi} \rangle_{(\boldsymbol{T}^2_{k+1}(\Omega))' \times \boldsymbol{T}^2_{k+1}(\Omega)} - \int_\Omega \chi\, \eta d\boldsymbol{x} = \langle g, 2[\boldsymbol{D}(\boldsymbol{\xi})\boldsymbol{n}] \cdot \boldsymbol{n} - \eta \rangle_{H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)} - \langle \boldsymbol{h}, \boldsymbol{\xi} \rangle_{H^{-3/2}(\Gamma) \times H^{3/2}(\Gamma)}, \tag{IV.48}$$

then, problem $(\mathcal{S}_T)$ has a solution $(\boldsymbol{u}, \pi) \in W^{0,2}_{-k-1}(\Omega) \times W^{-1,2}_{-k-1}(\Omega)$ unique up to an element of $\mathcal{N}^2_{-k+1}(\Omega)$.

Proof. Observe that, in view of Proposition IV.4, if the pair $(\boldsymbol{u}, \pi)$ that belongs to $W^{0,2}_{-k-1}(\Omega) \times W^{-1,2}_{-k-1}(\Omega)$ is a solution of $(\mathcal{S}_T)$, then for any $(\boldsymbol{\varphi}, q) \in \mathbf{S}^2_{k+1}(\Omega) \times W^{1,2}_{k+1}(\Omega)$, adding (IV.44) and (IV.45), we have

$$\begin{aligned} &\int_\Omega \boldsymbol{u} \cdot \big(-\Delta \boldsymbol{\varphi} + \nabla q \big) d\mathbf{x} - \langle \pi, \operatorname{div} \boldsymbol{\varphi} \rangle_{W^{-1,2}_{-k-1}(\Omega) \times \mathring{W}^{1,2}_{k+1}(\Omega)} = \langle \boldsymbol{f}, \boldsymbol{\varphi} \rangle_{(\boldsymbol{T}^2_{k+1}(\Omega))' \times \boldsymbol{T}^2_{k+1}(\Omega)} \\ &+ \langle \boldsymbol{h}, \boldsymbol{\varphi} \rangle_{H^{-3/2}(\Gamma) \times H^{3/2}(\Gamma)} - \int_\Omega \chi q d\mathbf{x} + \langle g, q - 2[\mathbf{D}(\boldsymbol{\varphi})\boldsymbol{n}] \cdot \boldsymbol{n} \rangle_{H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)}. \end{aligned} \tag{IV.49}$$

In particular, if $\forall (\boldsymbol{\varphi}, q) \in \mathcal{N}^2_{k+1}(\Omega)$ we obtain (IV.48).

It remains now to look for $(\boldsymbol{u}, \pi)$ belongs to $W^{0,2}_{-k-1}(\Omega) \times W^{-1,2}_{-k-1}(\Omega)$ satisfying (IV.49). To that end, we proceed in two steps depending on the datum g.

Step 1: We assume $g = 0$.

Let $\mathbf{T}$ be the linear form defined from $W^{0,2}_{k+1}(\Omega) \times \mathring{W}^{1,2}_{k+1}(\Omega) \perp \mathcal{N}^2_{-k+1}(\Omega)$ onto $\mathbb{R}$ by

$$\mathbf{T}(\boldsymbol{F}, \theta) = \langle \boldsymbol{f}, \boldsymbol{\varphi} \rangle_{(T^2_{k+1}(\Omega))' \times T^2_{k+1}(\Omega)} + \langle \boldsymbol{h}, \boldsymbol{\varphi} \rangle_{H^{-3/2}(\Gamma) \times H^{3/2}(\Gamma)} - \int_\Omega \chi q d\mathbf{x},$$

where the pair $(\boldsymbol{\varphi}, q) \in W^{2,2}_{k+1}(\Omega) \times W^{1,2}_{k+1}(\Omega)$ is a solution of the following problem (see Theorem IV.2 and Theorem IV.3):

$$\begin{cases} -\Delta \boldsymbol{\varphi} + \nabla q = \boldsymbol{F} \quad \textit{and} \quad \operatorname{div} \boldsymbol{\varphi} = -\theta & \text{in } \Omega, \\ \boldsymbol{\varphi} \cdot \boldsymbol{n} = 0 \quad \textit{and} \quad 2[\mathbf{D}(\boldsymbol{\varphi})\boldsymbol{n}]_\tau + \alpha \boldsymbol{\varphi}_\tau = \boldsymbol{0} & \text{on } \Gamma \end{cases}$$

and satisfying the following estimate:

$$\inf_{(\boldsymbol{\lambda},\mu)\in\mathcal{N}^2_{-k+1}(\Omega)}\Big(\|\boldsymbol{\varphi}+\boldsymbol{\lambda}\|_{\boldsymbol{W}^{2,2}_{k+1}(\Omega)}+\|q+\mu\|_{W^{1,2}_{k+1}(\Omega)}\Big)\leqslant C\Big(\|\boldsymbol{F}\|_{W^{0,2}_{k+1}(\Omega)}+\|\theta\|_{W^{1,2}_{k+1}(\Omega)}\Big). \tag{IV.50}$$

Then for any pair $(\boldsymbol{F},\theta)\in W^{0,2}_{k+1}(\Omega)\times W^{1,2}_{k+1}(\Omega)$ and for any $(\boldsymbol{\lambda},\mu)\in\mathcal{N}^2_{-k+1}(\Omega)$, we can write

$$\begin{aligned}
|\mathbf{T}(\boldsymbol{F},\theta)| &= \Big|\langle\boldsymbol{f},\boldsymbol{\varphi}\rangle_{(T^2_{k+1}(\Omega))'\times T^2_{k+1}(\Omega)}+\langle\boldsymbol{h},\boldsymbol{\varphi}\rangle_{H^{-3/2}(\Gamma)\times H^{3/2}(\Gamma)}-\int_\Omega\chi q d\mathbf{x}\Big|\\
&= \Big|\langle\boldsymbol{f},\boldsymbol{\varphi}+\boldsymbol{\lambda}\rangle_{(T^2_{k+1}(\Omega))'\times T^2_{k+1}(\Omega)}+\langle\boldsymbol{h},\boldsymbol{\varphi}+\boldsymbol{\lambda}\rangle_{H^{-3/2}(\Gamma)\times H^{3/2}(\Gamma)}-\int_\Omega\chi(q+\mu)d\mathbf{x}\Big|\\
&\leqslant C\Big(\|\boldsymbol{f}\|_{(\boldsymbol{T}^2_{k+1}(\Omega))'}+\|\chi\|_{W^{0,2}_{-k}(\Omega)}+\|\boldsymbol{h}\|_{H^{-3/2}(\Gamma)}\Big)\Big(\|\boldsymbol{\varphi}+\boldsymbol{\lambda}\|_{\boldsymbol{W}^{2,2}_{k+1}(\Omega)}+\|q+\mu\|_{W^{1,2}_{k+1}(\Omega)}\Big).
\end{aligned}$$

Using (IV.50), we have

$$\begin{aligned}
|\mathbf{T}(\boldsymbol{F},\theta)| &\leqslant C\inf_{(\boldsymbol{\lambda},\mu)\in\mathcal{N}^2_{-k+1}(\Omega)}\Big(\|\boldsymbol{\varphi}+\boldsymbol{\lambda}\|_{\boldsymbol{W}^{2,2}_{k+1}(\Omega)}+\|q+\mu\|_{W^{1,2}_{k+1}(\Omega)}\Big)\\
&\leqslant C\Big(\|\boldsymbol{F}\|_{W^{0,2}_{k+1}(\Omega)}+\|\theta\|_{W^{1,2}_{k+1}(\Omega)}\Big)
\end{aligned}$$

From this, we can deduce that the linear form $\mathbf{T}$ is continuous on the space $W^{0,2}_{k+1}(\Omega)\times\mathring{W}^{1,2}_{k+1}(\Omega)\perp\mathcal{N}^2_{-k+1}(\Omega)$. Since the dual space of $W^{0,2}_{k+1}(\Omega)\times\mathring{W}^{1,2}_{k+1}(\Omega)\perp\mathcal{N}^2_{-k+1}(\Omega)$ is $W^{0,2}_{-k-1}(\Omega)\times W^{-1,2}_{-k-1}(\Omega)/\mathcal{N}^2_{-k+1}(\Omega)$. From Riesz's Representation theorem, we deduce that there exists a unique $(\boldsymbol{u},\pi)$ belongs to $W^{0,2}_{-k-1}(\Omega)\times W^{-1,2}_{-k-1}(\Omega)/\mathcal{N}^2_{-k+1}(\Omega)$ satisfying (IV.49).

Step 2: Assume $g\neq 0$.

Let $g\in H^{-1/2}(\Gamma)$. Consider the following exterior Neumann problem

$$\Delta\,\Theta=0\quad\text{in }\Omega\quad\text{and}\quad\nabla\,\Theta\cdot\boldsymbol{n}=g\quad\text{on }\Gamma. \tag{IV.51}$$

Then thanks to Theorem III.5, this problem has a solution Θ that belongs to $W^{1,2}_{-1}(\Omega)$. Let R be a positive real number large enough so that $\overline{\Omega'}\subset B_R$ and let $\psi\in\mathcal{D}(\overline{\Omega})$ such that $0\leq\psi\leq 1$, supp $\psi\subset\overline{\Omega}_{R+1}$ and $\psi=1$ in $\overline{\Omega}_R$. Set now $w=\Theta\psi$, then w has obviously a compact support and thus w belongs to $W^{1,2}_{-k-1}(\Omega)$. Next, $\Delta\,w=\Theta\,\Delta\,\psi+2\nabla\,\Theta\,\nabla\,\psi$ has a compact support and thus belongs to $W^{0,2}_{-k}(\Omega)$.

Next we look for a pair $(\boldsymbol{z},\pi)\in W^{0,2}_{-k-1}(\Omega)\times W^{-1,2}_{-k-1}(\Omega)$ satisfying

$$\begin{cases}-\Delta\,\boldsymbol{z}+\nabla\,\pi=\boldsymbol{f}+\Delta(\nabla\,w)\quad\text{and}\quad\operatorname{div}\boldsymbol{z}=\chi-\Delta\,w\quad\text{in}\quad\Omega,\\ \boldsymbol{z}\cdot\boldsymbol{n}=0\quad\text{and}\quad 2[\mathbf{D}(\boldsymbol{z})\boldsymbol{n}]_\tau+\alpha\boldsymbol{z}_\tau=\boldsymbol{h}'\quad\text{on}\quad\Gamma,\end{cases} \tag{IV.52}$$

where $\boldsymbol{h}'=\boldsymbol{h}-2[\mathbf{D}(\nabla\,w)\boldsymbol{n}]_\tau-\alpha(\nabla\,w)_\tau$.

Observe that in view of the assumption on $\boldsymbol{f}$ and Lemma IV.1, $\boldsymbol{f}+\Delta(\nabla w)$ belongs to $(T^2_{k+1}(\Omega))'$. Besides, as w belongs to $W^{1,2}_{-k-1}(\Omega)$, then ∇w belongs to $W^{0,2}_{-k-1}(\Omega)$ and $\Delta(\nabla w)$ belongs to $(T^2_{k+1}(\Omega))'$. As a consequence, $\boldsymbol{h'}$ belongs to $H^{-3/2}(\Gamma)$ and clearly satisfies $\boldsymbol{h'}\cdot\boldsymbol{n}=0$ on Γ. Thus, due to the first step, problem (IV.52) has a solution $(\boldsymbol{z},\pi)\in W^{0,2}_{-k-1}(\Omega)\times W^{-1,2}_{-k-1}(\Omega)$ if the following condition is satisfied $\forall(\boldsymbol{\xi},\eta)\in\mathcal{N}^2_{k+1}(\Omega)$.

$$\begin{aligned}&\langle \boldsymbol{f}\ +\Delta(\nabla w),\boldsymbol{\xi}\rangle_{(T^2_{k+1}(\Omega))'\times T^2_{k+1}(\Omega)}-\int_\Omega(\chi-\Delta w)\,\eta d\boldsymbol{x}\\&+\langle \boldsymbol{h}',\boldsymbol{\xi}\rangle_{H^{-3/2}(\Gamma)\times H^{3/2}(\Gamma)}=0.\end{aligned}\tag{IV.53}$$

Let $(\boldsymbol{\xi},\eta)$ be in $\mathcal{N}^2_{k+1}(\Omega)$. For any $(\boldsymbol{\varphi},\psi)\in\mathcal{D}(\overline{\Omega})\times\mathcal{D}(\overline{\Omega})$, we have

$$\begin{aligned}&\int_\Omega\big[(-\Delta\boldsymbol{\varphi}+\nabla\psi)\cdot\boldsymbol{\xi}-\eta\,\mathrm{div}\,\boldsymbol{\varphi}\big]d\boldsymbol{x}\\&=\langle\boldsymbol{\varphi}\cdot\boldsymbol{n},2[\mathbf{D}(\boldsymbol{\xi})\boldsymbol{n}]\cdot\boldsymbol{n}-\eta\rangle_{H^{-1/2}(\Gamma)\times H^{1/2}(\Gamma)}-\langle 2[\mathbf{D}(\boldsymbol{\varphi})\boldsymbol{n}]_\tau+\alpha\,\boldsymbol{\varphi}_\tau,\xi_\tau\rangle_{H^{-3/2}(\Gamma)\times H^{3/2}(\Gamma)}.\end{aligned}\tag{IV.54}$$

In particular, the Green's formula (IV.54) holds for any $\boldsymbol{\varphi}=\nabla w\in W^{0,2}_{-k-1}(\Omega)$ and $\psi=0$. We obtain

$$\begin{aligned}&\langle-\Delta(\nabla w),\boldsymbol{\xi}\rangle_{(\boldsymbol{T}^2_{k+1}(\Omega))'\times\boldsymbol{T}^2_{k+1}(\Omega)}-\int_\Omega\Delta w\ \eta d\boldsymbol{x}\\&=\langle g,2[\mathbf{D}(\boldsymbol{\xi})\boldsymbol{n}]\cdot\boldsymbol{n}-\eta\rangle_{H^{-1/2}(\Gamma)\times H^{1/2}(\Gamma)}-\langle\boldsymbol{h}-\boldsymbol{h}',\boldsymbol{\xi}\rangle_{H^{-3/2}(\Gamma)\times H^{3/2}(\Gamma)}=0.\end{aligned}\tag{IV.55}$$

Combining (IV.48) and (IV.55) allows to obtain (IV.53). Thus setting $\boldsymbol{u}=\boldsymbol{z}+\nabla w$, the pair $(\boldsymbol{u},\pi)\in W^{0,2}_{-k-1}(\Omega)\times W^{-1,2}_{-k-1}(\Omega)$ is the solution of $(\mathcal{S}_T)$.

□

Chapter V

Stokes problem with Navier boundary conditions in L^p-theory

Résumé: Nous étudions dans ce chapitre le problème de Stokes $(\mathcal{S}_T)$ dans la théorie L^p. Nous montrons ici l'existence et l'unicité de solutions fortes dans $W^{2,p}_{k+1}(\Omega) \times W^{1,p}_{k+1}(\Omega)$ pour $p \geqslant 2$ et $k \in \mathbb{Z}$. Nous traitons aussi le cas de solutions très faibles dans $W^{0,p}_{-2}(\Omega) \times W^{-1,p}_{-2}(\Omega)$ pour $1 < p < 2$.

1 Introduction

In this chapter, we study the problem $(\mathcal{S}_T)$ in L^p-theory. We first look for the strong solutions in $W^{2,p}_{k+1}(\Omega) \times W^{1,p}_{k+1}(\Omega)$ with $k \in \mathbb{Z}$ and $p \geqslant 2$. To that end, we combine results on the Stokes problem set in the whole space to catch the behavior at infinity and results in bounded domains to take into account the boundary conditions. Next, we investigate the very weak solutions in $W^{0,p}_{-2}(\Omega) \times W^{-1,p}_{-2}(\Omega)$ for the Stokes problem $(\mathcal{S}_T)$ with $1 < p < 2$. The main idea here relies on the use of a duality argument using the strong solutions. Note that, this section is very similar to the section 4 in chapter IV.

2 Strong solution in $W^{2,p}_{k+1}(\Omega) \times W^{1,p}_{k+1}(\Omega)$

In this section, we investigate the well-posedness of strong solutions in $W^{2,p}_{k+1}(\Omega) \times W^{1,p}_{k+1}(\Omega)$ with $k \in \mathbb{Z}$ and $p \geqslant 2$. We assume that α is a positive function that belongs to $L^\infty(\Gamma)$. In

order to deal with the uniqueness issues, we first need to characterize the kernel of problem $(\mathcal{S}_T)$. For $k \in \mathbb{Z}$, we introduce:

$$\begin{aligned}\mathcal{N}^p_{k+1}(\Omega) \;=\; & \Big\{(\boldsymbol{u},\pi) \in W^{2,p}_{k+1}(\Omega) \times W^{1,p}_{k+1}(\Omega);\\ & -\Delta\,\boldsymbol{u} + \nabla\,\pi = 0,\ \operatorname{div}\boldsymbol{u} = 0 \text{ in } \Omega \text{ and } \boldsymbol{u}\cdot\boldsymbol{n} = 0,\ 2[\mathbf{D}(\boldsymbol{u})\boldsymbol{n}]_\tau + \alpha\boldsymbol{u}_\tau = 0 \text{ on } \Gamma\Big\}.\end{aligned}$$

The characterization of $\mathcal{N}^p_{k+1}(\Omega)$ is given by the following proposition:

Proposition V.1 *Consider an exterior domain Ω with boundary Γ of class $\mathcal{C}^{2.1}$ and let $p \geqslant 2$. We have*

$$\mathcal{N}^p_{k+1}(\Omega) = \Big\{(\boldsymbol{v}-\boldsymbol{\lambda},\theta-\mu);\quad (\boldsymbol{\lambda},\mu) \in N_{[1-k-3/p]}\Big\},$$

where $(\boldsymbol{v},\theta) \in W^{2,p}_{k+1}(\Omega)\cap W^{2,2}_1(\Omega)\times W^{1,p}_{k+1}(\Omega)\cap W^{1,2}_1(\Omega)$ is the unique solution of problem (IV.13). In particular, $\mathcal{N}^p_{k+1}(\Omega) = \{(\boldsymbol{0},0)\}$ if $k > 1-3/p$. Moreover, the space $\mathcal{N}^p_{k+1}(\Omega)$ is a finite-dimensional space and $\dim \mathcal{N}^p_{k+1}(\Omega) = \dim N_{[1-k-3/p]}$.

Proof. Let us assume that $(\boldsymbol{u},\pi)$ belongs to $\mathcal{N}^p_{k+1}(\Omega)$. The pair $(\boldsymbol{u},\pi)$ has an extension $(\widetilde{\boldsymbol{u}},\widetilde{\pi})$ that belongs to $W^{2,p}_{k+1}(\mathbb{R}^3) \times W^{1,p}_{k+1}(\mathbb{R}^3)$. Set now

$$\boldsymbol{F} = -\Delta\,\widetilde{\boldsymbol{u}} + \nabla\,\widetilde{\pi} \quad \text{and} \quad e = \operatorname{div}\widetilde{\boldsymbol{u}}.$$

Then the pair $(\boldsymbol{F},e)$ belongs to $W^{0,p}_{k+1}(\mathbb{R}^3)\times W^{1,p}_{k+1}(\mathbb{R}^3)$ and has a compact support. Since $p \geqslant 2$, therefore $(\boldsymbol{F},e)$ also belongs to $W^{0,2}_1(\mathbb{R}^3)\times W^{1,2}_1(\mathbb{R}^3)$. It follows from Theorem III.6, that there exists a unique solution $(\boldsymbol{v},\theta) \in W^{2,2}_1(\mathbb{R}^3) \times W^{1,2}_1(\mathbb{R}^3)$ such that

$$-\Delta\,\boldsymbol{v} + \nabla\,\theta = -\Delta\,\widetilde{\boldsymbol{u}} + \nabla\,\widetilde{\pi} \quad \text{and} \quad \operatorname{div}\boldsymbol{v} = \operatorname{div}\widetilde{\boldsymbol{u}} \quad \text{in} \quad \mathbb{R}^3.$$

It follows that $(\boldsymbol{v}-\widetilde{\boldsymbol{u}},\theta-\widetilde{\pi})$ belongs to $\Big(W^{2,2}_1(\mathbb{R}^3)+W^{2,p}_{k+1}(\mathbb{R}^3)\Big)\times\Big(W^{1,2}_1(\mathbb{R}^3)+W^{1,p}_{k+1}(\mathbb{R}^3)\Big)$. Hence, there exits $(\boldsymbol{\lambda},\mu) \in N_{[1-k-3/p]}$ such that $(\boldsymbol{v}-\widetilde{\boldsymbol{u}},\theta-\widetilde{\pi}) = (\boldsymbol{\lambda},\mu)$. Thus, $(\boldsymbol{v},\theta)$ belongs to $\Big(W^{2,2}_1(\mathbb{R}^3)\cap W^{2,p}_{k+1}(\mathbb{R}^3)\Big)\times\Big(W^{1,2}_1(\mathbb{R}^3)\cap W^{1,p}_{k+1}(\mathbb{R}^3)\Big)$, its restriction to Ω belongs to $\Big(W^{2,2}_1(\Omega)\cap W^{2,p}_{k+1}(\Omega)\Big)\times\Big(W^{1,2}_1(\Omega)\cap W^{1,p}_{k+1}(\Omega)\Big)$ and satisfies (IV.13).

Let us now prove that the kernel $\mathcal{N}^p_{k+1}(\Omega)$ has a finite dimension. From the previous part, we can introduce the linear mapping

$$\mathcal{T}\ :\ N_{[1-k-3/p]} \mapsto \mathcal{N}^p_{k+1}(\Omega)$$

defined for each $(\boldsymbol{\lambda},\mu) \in N_{[1-k-3/p]}$, by $\mathcal{T}(\boldsymbol{\lambda},\mu) = (\boldsymbol{v}(\boldsymbol{\lambda})-\boldsymbol{\lambda},\theta(\boldsymbol{\lambda})-\mu)$, where $(\boldsymbol{v}(\boldsymbol{\lambda}),\theta(\boldsymbol{\lambda})) \in \Big(W^{2,2}_1(\Omega)\cap W^{2,p}_{k+1}(\Omega)\Big)\times\Big(W^{1,2}_1(\Omega)\cap W^{1,p}_{k+1}(\Omega)\Big)$ is the unique solution of (IV.13).

Using the first part of the proof, this mapping is clearly onto. It is also injective, since $\mathcal{T}(\boldsymbol{\lambda}, \mu) = \Big\{(\mathbf{0}, 0)\Big\}$, implies that the pair $\Big(\boldsymbol{v}\ (\boldsymbol{\lambda}), \theta(\boldsymbol{\lambda})\Big) \in W^{2,2}_1(\Omega) \times W^{1,2}_1(\Omega)$ is polynomials. But the space $W^{2,2}_1(\Omega) \times W^{1,2}_1(\Omega)$ does not contain polynomials, which implies that $\Big(\boldsymbol{v}\ (\boldsymbol{\lambda}), \theta(\boldsymbol{\lambda})\Big) = \Big\{(\mathbf{0}, 0)\Big\}$ which in turn also implies that $(\boldsymbol{\lambda}, \mu) = \Big\{(\mathbf{0}, 0)\Big\}$. Finally, thanks to (IV.14), $\mathcal{T}$ is continuous. Hence the operator $\mathcal{T}$ is an isomorphism from $N_{[1-k-3/p]}$ to $\mathcal{N}^p_{k+1}(\Omega)$. Then the dimension of $\mathcal{N}^p_{k+1}(\Omega)$ is finite and is equal to the dimension of $N_{[1-k-3/p]}$, this ends the proof. $\square$

The next Theorem solves the problem $(\mathcal{S}_T)$ when $p \geqslant 2$, our study is based on strong solutions in a Hilbertian framework, that's why we will take the data $\boldsymbol{f}$ and χ have a compact support.

Theorem V.1 *Assume that $p \geqslant 2$. Let $\boldsymbol{f} \in W^{0,p}_{k+1}(\Omega)$, $\chi \in W^{1,p}_{k+1}(\Omega)$, $g \in W^{1+1/p',p}(\Gamma)$ and $\boldsymbol{h} \in W^{1-1/p,p}(\Gamma)$ such that $\boldsymbol{f}$ and χ have a compact support, $\boldsymbol{h} \cdot \boldsymbol{n} = 0$ on Γ and the following compatibility condition is satisfied*

$$\forall (\boldsymbol{\xi}, \eta) \in \mathcal{N}^{p'}_{-k+1}(\Omega), \qquad \int_\Omega \boldsymbol{f} \cdot \boldsymbol{\xi} d\boldsymbol{x} - \int_\Omega \chi\, \eta d\boldsymbol{x} = \int_\Gamma g\Big(2[\boldsymbol{D}(\boldsymbol{\xi})\boldsymbol{n}] \cdot \boldsymbol{n} - \eta\Big) d\boldsymbol{\sigma} - \int_\Gamma \boldsymbol{h} \cdot \boldsymbol{\xi} d\boldsymbol{\sigma}. \quad \text{(V.1)}$$

Then, the Stokes problem $(\mathcal{S}_T)$ has a solution $(\boldsymbol{u}, \pi) \in W^{2,p}_{k+1}(\Omega) \times W^{1,p}_{k+1}(\Omega)$ unique up to an element of $\mathcal{N}^p_{k+1}(\Omega)$. In addition, we have the following estimate:

$$\begin{aligned}
&\inf_{(\boldsymbol{\lambda},\mu) \in \mathcal{N}^p_{k+1}(\Omega)} \Big(\|\boldsymbol{u} + \boldsymbol{\lambda}\|_{W^{2,p}_{k+1}(\Omega)} + \|\pi + \mu\|_{W^{1,p}_{k+1}(\Omega)} \Big) \\
&\leqslant C \Big(\|\boldsymbol{f}\|_{W^{0,p}_{k+1}(\Omega)} + \|\chi\|_{W^{1,p}_{k+1}(\Omega)} + \|\boldsymbol{h}\|_{W^{1-1/p,p}(\Gamma)} + \|g\|_{W^{1+1/p',p}(\Gamma)} \Big).
\end{aligned}$$

Proof. Observe first that the uniqueness is a straightforward consequence of Proposition V.1. We now divide the proof of the theorem into several parts.

- Compatibility condition. In this part, we prove that (V.1) is a necessary condition. Let $(\boldsymbol{\xi}, \eta)$ be in $\mathcal{N}^{p'}_{-k+1}(\Omega)$. For any $(\boldsymbol{\varphi}, \psi) \in \mathcal{D}(\overline{\Omega}) \times \mathcal{D}(\overline{\Omega})$. Using the same calculation as in the proof of Theorem IV.3, we have

$$\begin{aligned}
&\int_\Omega \Big[\Big(-\Delta \boldsymbol{\varphi} + \nabla \psi\Big) \cdot \boldsymbol{\xi} - \eta \operatorname{div} \boldsymbol{\varphi}\Big] d\boldsymbol{x} \\
&= \int_\Gamma (\boldsymbol{\varphi} \cdot \boldsymbol{n}) \Big(2[\mathbf{D}(\boldsymbol{\xi})\boldsymbol{n}] \cdot \boldsymbol{n} - \eta\Big) d\boldsymbol{\sigma} - \int_\Gamma \boldsymbol{\xi}_\tau \cdot \Big(2[\mathbf{D}(\boldsymbol{\varphi})\boldsymbol{n}]_\tau + \alpha\, \boldsymbol{\varphi}_\tau\Big) d\boldsymbol{\sigma}. \qquad \text{(V.2)}
\end{aligned}$$

  Then, the last Green's formula holds for any pair $(\boldsymbol{\varphi}, \psi) \in W^{2,p}_{k+1}(\Omega) \times W^{1,p}_{k+1}(\Omega)$ by density. In particular, if $(\boldsymbol{u}, \pi) \in W^{2,p}_{k+1}(\Omega) \times W^{1,p}_{k+1}(\Omega)$ is a solution of $(\mathcal{S}_T)$, then (V.1) holds.

- Existence. Here we prove that problem $(\mathcal{S}_T)$ has a solution $(\boldsymbol{u},\pi)$ that belongs to $W^{1,p}_k(\Omega)\times W^{0,p}_k(\Omega)$. We start with the case $2\leqslant p\leqslant 6$, the proof is made of two steps.

Step 1. The case $g=0$.

Since $p\geqslant 2$ and $(\boldsymbol{f},\chi)$ have a support compact, then we have $(\boldsymbol{f},\chi)$ belongs to $W^{0,2}_1(\Omega)\times W^{1,2}_1(\Omega)$. In addition, its clear that $\boldsymbol{h}$ belongs to $H^{1/2}(\Gamma)$. Thanks to Theorem IV.2, problem $(\mathcal{S}_T)$ has a solution $(\boldsymbol{u},\pi)\in W^{2,2}_1(\Omega)\times W^{1,2}_1(\Omega)$. It remains now to prove that $(\boldsymbol{u},\pi)$ belongs to $W^{1,p}_k(\Omega)\times W^{0,p}_k(\Omega)$. To that end, we shall use again properties of the Stokes problem in the whole space $\mathbb{R}^3$. Now, we first need appropriate extensions of $\boldsymbol{u}$ and π defined in $\mathbb{R}^3$. Since $\boldsymbol{u}\cdot\boldsymbol{n}=0$ on Γ and applying Theorem III.7, then there exists $(\boldsymbol{u'},\pi')\in H^2(\Omega')\times H^1(\Omega')$ satisfying (IV.26). Setting

$$\widetilde{\boldsymbol{u}}=\begin{cases}\boldsymbol{u} & \text{in} \quad \Omega,\\ \boldsymbol{u'} & \text{in} \quad \Omega',\end{cases}\qquad\text{and}\qquad \widetilde{\pi}=\begin{cases}\pi & \text{in} \quad \Omega,\\ \pi' & \text{in} \quad \Omega'.\end{cases}$$

Then clearly, the pair $(\widetilde{\boldsymbol{u}},\widetilde{\pi})$ belongs to $W^{1,2}_1(\mathbb{R}^3)\times W^{0,2}_1(\mathbb{R}^3)$. The goal is now to identify $(\widetilde{\boldsymbol{u}},\widetilde{\pi})$ with a solution of the Stokes problem in $\mathbb{R}^3$ that belongs to $W^{1,p}_k(\Omega)\times W^{0,p}_k(\Omega)$. Let us set

$$\boldsymbol{F}=-\Delta\,\widetilde{\boldsymbol{u}}+\nabla\widetilde{\pi}\quad\text{and}\quad e=\operatorname{div}\widetilde{\boldsymbol{u}}. \tag{V.3}$$

In order to apply Theorem III.6, we need to show that $(\boldsymbol{F},e)$ belongs to $W^{-1,p}_k(\mathbb{R}^3)\times W^{0,p}_k(\mathbb{R}^3)$ and satisfies (III.22). Therefore, denoting by $\widetilde{\boldsymbol{f}}\in W^{0,p}_{k+1}(\mathbb{R}^3)$ the extension of $\boldsymbol{f}$ by zero in Ω', we deduce that for any $\boldsymbol{\varphi}\in\mathcal{D}(\mathbb{R}^3)$

$$\begin{aligned}\langle\boldsymbol{F},\boldsymbol{\varphi}\rangle_{\mathcal{D}'(\mathbb{R}^3)\times\mathcal{D}(\mathbb{R}^3)}&=\int_{\mathbb{R}^3}\widetilde{\boldsymbol{f}}\cdot\boldsymbol{\varphi}d\boldsymbol{x}+2\int_\Gamma\boldsymbol{\varphi}\cdot\Big(\mathbf{D}(\boldsymbol{u})\boldsymbol{n}-\mathbf{D}(\boldsymbol{u'})\boldsymbol{n}\Big)d\boldsymbol{\sigma}\\&+\int_\Gamma\big(\boldsymbol{\varphi}\cdot\boldsymbol{n}\big)\big(\pi'-\pi\big)d\boldsymbol{\sigma}-\int_\Gamma(\boldsymbol{\varphi}\cdot\boldsymbol{n})\chi d\boldsymbol{\sigma}.\end{aligned}\tag{V.4}$$

This calculations are the same as in the previous chapter IV (see (IV.28)).
Since $\Big(2\mathbf{D}(\boldsymbol{u})\boldsymbol{n}-2\mathbf{D}(\boldsymbol{u'})\boldsymbol{n}\Big)\,|_\Gamma$ and $\Big(\pi'-\pi\Big)\,|_\Gamma$ belongs to $H^{1/2}(\Gamma)$.

$$\begin{aligned}&\Big|\langle\boldsymbol{F},\boldsymbol{\varphi}\rangle_{\mathcal{D}'(\mathbb{R}^3)\times\mathcal{D}(\mathbb{R}^3)}\Big|\leqslant\Big|\int_{\mathbb{R}^3}\widetilde{\boldsymbol{f}}\cdot\boldsymbol{\varphi}d\boldsymbol{x}\Big|+\Big|\langle\boldsymbol{\varphi}\cdot\boldsymbol{n},\chi\rangle_{H^{-1/2}(\Gamma)\times H^{1/2}(\Gamma)}\Big|\\&+\Big|2\langle\mathbf{D}(\boldsymbol{u})\boldsymbol{n}-\mathbf{D}(\boldsymbol{u'})\boldsymbol{n},\boldsymbol{\varphi}\rangle_{H^{1/2}(\Gamma)\times H^{-1/2}(\Gamma)}\Big|+\Big|\langle\boldsymbol{\varphi}\cdot\boldsymbol{n},\pi'-\pi\rangle_{H^{-1/2}(\Gamma)\times H^{1/2}(\Gamma)}\Big|\\\leqslant\;&\|\widetilde{\boldsymbol{f}}\|_{W^{0,p}_{k+1}(\mathbb{R}^3)}\|\boldsymbol{\varphi}\|_{W^{0,p'}_{-2}(\mathbb{R}^3)}+\|\chi\|_{W^{1,2}_{k+1}(\Omega)}\|\boldsymbol{\varphi}\|_{H^{-1/2}(\Gamma)}\\&+\|2\mathbf{D}(\boldsymbol{u})\boldsymbol{n}-2\mathbf{D}(\boldsymbol{u'})\boldsymbol{n}\|_{H^{1/2}(\Gamma)}\|\boldsymbol{\varphi}\|_{H^{-1/2}(\Gamma)}+\|\boldsymbol{\varphi}\cdot\boldsymbol{n}\|_{H^{-1/2}(\Gamma)}\|\pi'-\pi\|_{H^{1/2}(\Gamma)}\\\leqslant\;&\|\boldsymbol{f}\|_{W^{0,p}_{k+1}(\Omega)}\|\boldsymbol{\varphi}\|_{W^{1,p'}_{-k}(\mathbb{R}^3)}+C\|\boldsymbol{\varphi}\|_{H^{-1/2}(\Gamma)}\end{aligned}$$

As we have

$$H^{1/2}(\Gamma) \hookrightarrow W^{-1/6,6}(\Gamma) \hookrightarrow W^{-1/p,p}(\Gamma), \text{for any } 2 \leqslant p \leqslant 6$$

We obtain

$$\begin{aligned} \|\boldsymbol{\varphi}\|_{H^{-1/2}(\Gamma)} &\leqslant C\|\boldsymbol{\varphi}\|_{W^{1/p,p'}(\Gamma)} \\ &\leqslant C\|\boldsymbol{\varphi}\|_{W^{1,p'}_{-k}(\Omega)} \\ &\leqslant C\|\boldsymbol{\varphi}\|_{W^{1,p'}_{-k}(\mathbb{R}^3)} \end{aligned}$$

Then, we have

$$\left|\langle \boldsymbol{F}, \boldsymbol{\varphi}\rangle_{\mathcal{D}'(\mathbb{R}^3)\times\mathcal{D}(\mathbb{R}^3)}\right| \leqslant C\|\boldsymbol{\varphi}\|_{W^{1,p'}_{-k}(\mathbb{R}^3)}.$$

Since $\mathcal{D}(\mathbb{R}^3)$ is dense in $W^{1,p'}_{-k}(\mathbb{R}^3)$, then (V.4) is still valid for any $\boldsymbol{\varphi} \in W^{1,p'}_{-k}(\mathbb{R}^3)$ which implies that $\boldsymbol{F}$ belongs to $W^{-1,p}_{k}(\mathbb{R}^3)$.
Now for any $\phi \in \mathcal{D}(\mathbb{R}^3)$, we have

$$\begin{aligned} \langle e, \phi\rangle_{\mathcal{D}'(\mathbb{R}^3)\times\mathcal{D}(\mathbb{R}^3)} &= \langle \operatorname{div} \widetilde{\boldsymbol{u}}, \phi\rangle_{\mathcal{D}'(\mathbb{R}^3)\times\mathcal{D}(\mathbb{R}^3)} = -\int_{\mathbb{R}^3} \widetilde{\boldsymbol{u}} \cdot \nabla \phi d\boldsymbol{x} \\ &= \int_\Omega \operatorname{div} \boldsymbol{u} \ \ \phi d\boldsymbol{x} + \int_{\Omega'} \operatorname{div} \boldsymbol{u'} \ \ \phi d\boldsymbol{x} - \int_\Gamma (\boldsymbol{u}\cdot\boldsymbol{n})\phi d\boldsymbol{\sigma} + \int_\Gamma (\boldsymbol{u'}\cdot\boldsymbol{n})\phi d\boldsymbol{\sigma} \\ &= \int_\Omega \chi\, \phi d\boldsymbol{x}. \end{aligned} \tag{V.5}$$

Since $\chi \in W^{1,p}_{k+1}(\Omega)$. Then, we have

$$\left|\langle e, \phi\rangle_{\mathcal{D}'(\mathbb{R}^3)\times\mathcal{D}(\mathbb{R}^3)}\right| \leqslant \|\chi\|_{W^{0,p}_k(\Omega)}\|\phi\|_{W^{0,p'}_{-k}(\Omega)} \leqslant C\|\phi\|_{W^{0,p'}_{-k}(\Omega)}$$

Due to the density of $\mathcal{D}(\mathbb{R}^3)$ in $W^{0,p'}_{-k}(\mathbb{R}^3)$, then (V.5) is still valid for any $\phi \in W^{0,p'}_{-k}(\mathbb{R}^3)$, which implies that e belongs to $W^{0,p}_k(\mathbb{R}^3)$. As a result, we have proved that $(\boldsymbol{F}, e)$ belongs to $W^{-1,p}_k(\mathbb{R}^3) \times W^{0,p}_k(\mathbb{R}^3)$.
Let us now prove that $\boldsymbol{F}$ and e satisfy (III.22), which in view of (V.4) and (V.5), amounts to prove that for any $(\boldsymbol{\lambda}, \mu)$ in $N_{[1-3/p'+k]}$

$$\begin{aligned} &\int_{\mathbb{R}^3} \widetilde{\boldsymbol{f}} \cdot \boldsymbol{\lambda} d\boldsymbol{x} - \int_\Omega \chi\, \mu d\boldsymbol{x} + 2\int_\Gamma \boldsymbol{\lambda}\cdot\Big(\mathbf{D}(\boldsymbol{u})\boldsymbol{n} - \mathbf{D}(\boldsymbol{u'})\boldsymbol{n}\Big) d\boldsymbol{\sigma} \\ &+ \int_\Gamma \Big(\boldsymbol{\lambda}\cdot\boldsymbol{n}\Big)\Big(\pi' - \pi\Big) d\boldsymbol{\sigma} - \int_\Gamma (\boldsymbol{\lambda}\cdot\boldsymbol{n})\chi d\boldsymbol{\sigma} = 0. \end{aligned} \tag{V.6}$$

So let $(\boldsymbol{\lambda}, \mu) \in N_{[1-3/p'+k]}$ and let $(\boldsymbol{v}(\boldsymbol{\lambda}), \theta(\boldsymbol{\lambda}))$ be in $W^{2,p'}_{-k+1}(\Omega) \cap W^{2,2}_1(\Omega) \times W^{1,p'}_{-k+1}(\Omega) \cap$

$W_1^{1,2}(\Omega)$ such that the pair $(\boldsymbol{v}(\boldsymbol{\lambda})-\boldsymbol{\lambda},\theta(\boldsymbol{\lambda})-\mu)$ belongs to $\mathcal{N}^{p'}_{-k+1}(\Omega)$.
Now, for any $(\boldsymbol{v},\theta)\in W^{2,p'}_{-k+1}(\Omega)\cap W^{2,2}_1(\Omega)\times W^{1,p'}_{-k+1}(\Omega)\cap W^{1,2}_1(\Omega)$ such that $\operatorname{div}\boldsymbol{v}=0$, computations in Ω yields

$$\begin{aligned}&\int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\,(\boldsymbol{\lambda})\,d\boldsymbol{x}\;-\int_\Omega \chi\theta(\boldsymbol{\lambda})d\boldsymbol{x}\;-\int_\Gamma(\boldsymbol{v}\,(\boldsymbol{\lambda})\cdot\boldsymbol{n}\,)\chi d\boldsymbol{\sigma}\\&-2\int_\Gamma \boldsymbol{u}\;\cdot\mathbf{D}(\boldsymbol{v}\,(\boldsymbol{\lambda}))\boldsymbol{n}\;d\boldsymbol{\sigma}+2\int_\Gamma \boldsymbol{v}\,(\boldsymbol{\lambda})\cdot\mathbf{D}(\boldsymbol{u}\,)\boldsymbol{n}\;d\boldsymbol{\sigma}-\int_\Gamma(\boldsymbol{v}\,(\boldsymbol{\lambda})\cdot\boldsymbol{n}\,)\pi d\boldsymbol{\sigma}=0.\end{aligned}\tag{V.7}$$

Now making the difference between (V.7) and (V.1) with $\boldsymbol{\xi}=\boldsymbol{v}(\boldsymbol{\lambda})-\boldsymbol{\lambda}$ and $\eta=\theta(\boldsymbol{\lambda})-\mu$ and recalling that $\boldsymbol{v}\,(\boldsymbol{\lambda})\cdot\boldsymbol{n}\;=\boldsymbol{\lambda}\cdot\boldsymbol{n}$ on Γ, yields

$$\begin{aligned}&\int_\Omega \boldsymbol{f}\cdot\boldsymbol{\lambda}\,d\boldsymbol{x}\;-\int_\Omega \chi\mu d\boldsymbol{x}\;-\int_\Gamma(\boldsymbol{v}\,(\boldsymbol{\lambda})-\boldsymbol{\lambda})\cdot\boldsymbol{h}\;d\boldsymbol{\sigma}-\int_\Gamma(\boldsymbol{\lambda}\cdot\boldsymbol{n}\,)\chi d\boldsymbol{\sigma}\\&-2\int_\Gamma \boldsymbol{u}\;\cdot\mathbf{D}(\boldsymbol{v}\,(\boldsymbol{\lambda}))\boldsymbol{n}\;d\boldsymbol{\sigma}+2\int_\Gamma \boldsymbol{v}\,(\boldsymbol{\lambda})\cdot\mathbf{D}(\boldsymbol{u}\,)\boldsymbol{n}\;d\boldsymbol{\sigma}-\int_\Gamma(\boldsymbol{v}\,(\boldsymbol{\lambda})\cdot\boldsymbol{n}\,)\pi d\boldsymbol{\sigma}=0.\end{aligned}\tag{V.8}$$

Computations on Ω' yield

$$\begin{aligned}&\int_{\Omega'}\big(-\Delta\,\boldsymbol{u'}+\nabla\,\pi'\big)\cdot\boldsymbol{\lambda}d\boldsymbol{x}=0\\&=-\int_{\Omega'}\boldsymbol{u'}\cdot\Delta\,\boldsymbol{\lambda}d\boldsymbol{x}+2\int_\Gamma\boldsymbol{\lambda}\cdot\mathbf{D}(\boldsymbol{u'})\boldsymbol{n}d\boldsymbol{\sigma}-2\int_\Gamma\boldsymbol{u'}\cdot\mathbf{D}(\boldsymbol{\lambda})\boldsymbol{n}d\boldsymbol{\sigma}-\int_\Gamma(\boldsymbol{\lambda}\cdot\boldsymbol{n})\pi' d\boldsymbol{\sigma}.\end{aligned}$$

The fact that $\boldsymbol{u}\;=\boldsymbol{u'}$ on Γ, implies

$$\int_{\Omega'}\boldsymbol{u'}\cdot\Delta\boldsymbol{\lambda}\,d\boldsymbol{x}\;=\int_{\Omega'}\boldsymbol{u'}\cdot\nabla\mu\,d\boldsymbol{x}\;=-\int_{\Omega'}\mu\operatorname{div}\boldsymbol{u'}d\boldsymbol{x}\;+\int_\sigma\mu(\boldsymbol{u'}\cdot\boldsymbol{n}\,)d\boldsymbol{\sigma}=0$$

and we deduce that

$$2\int_\Gamma\boldsymbol{\lambda}\cdot\mathbf{D}(\boldsymbol{u'})\boldsymbol{n}d\boldsymbol{\sigma}-2\int_\Gamma\boldsymbol{u'}\cdot\mathbf{D}(\boldsymbol{\lambda})\boldsymbol{n}d\boldsymbol{\sigma}-\int_\Gamma(\boldsymbol{\lambda}\cdot\boldsymbol{n})\pi' d\boldsymbol{\sigma}=0.\tag{V.9}$$

Combining (V.8) and (V.9) yields

$$\begin{aligned}&\int_{\mathbb{R}^3}\widetilde{\boldsymbol{f}}\cdot\boldsymbol{\lambda}d\boldsymbol{x}-\int_\Omega\chi\,\mu d\boldsymbol{x}-\int_\Gamma\chi(\boldsymbol{\lambda}\cdot\boldsymbol{n})d\boldsymbol{\sigma}+\int_\Gamma(\boldsymbol{\lambda}\cdot\boldsymbol{n})(\pi'-\pi)d\boldsymbol{\sigma}-2\int_\Gamma\boldsymbol{\lambda}\cdot\mathbf{D}(\boldsymbol{u'})\boldsymbol{n}d\boldsymbol{\sigma}\\&+2\int_\Gamma\boldsymbol{u}\cdot\big(\mathbf{D}(\boldsymbol{\lambda})\boldsymbol{n}-\mathbf{D}(\boldsymbol{v}(\boldsymbol{\lambda}))\boldsymbol{n}\big)d\boldsymbol{\sigma}+2\int_\Gamma\boldsymbol{v}(\boldsymbol{\lambda})\cdot\mathbf{D}(\boldsymbol{u}\;)\boldsymbol{n}d\boldsymbol{\sigma}-\int_\Gamma(\boldsymbol{v}(\boldsymbol{\lambda})-\boldsymbol{\lambda})\boldsymbol{h}\;d\boldsymbol{\sigma}=0.\end{aligned}\tag{V.10}$$

Due to the fact that $g=0$ on Γ and using the Navier boundary condition in (IV.13),

we have

$$2\int_\Gamma \boldsymbol{u}\cdot\Big(\mathbf{D}(\boldsymbol{\lambda})\boldsymbol{n} - \mathbf{D}(\boldsymbol{v}(\boldsymbol{\lambda}))\boldsymbol{n}\Big)d\boldsymbol{\sigma} = 2\int_\Gamma \boldsymbol{u}_\tau\cdot\Big(\Big[\mathbf{D}(\boldsymbol{\lambda})\boldsymbol{n}\Big]_\tau - \Big[\mathbf{D}(\boldsymbol{v}(\boldsymbol{\lambda}))\boldsymbol{n}\Big]_\tau\Big)d\boldsymbol{\sigma}$$
$$= \alpha\int_\Gamma \boldsymbol{u}_\tau\cdot\Big(\boldsymbol{v}(\boldsymbol{\lambda})_\tau - \boldsymbol{\lambda}_\tau\Big)d\boldsymbol{\sigma}.$$

Next, using the fact that $\boldsymbol{v}(\boldsymbol{\lambda})\cdot\boldsymbol{n} = \boldsymbol{\lambda}\cdot\boldsymbol{n}$ on Γ,

$$2\int_\Gamma \boldsymbol{v}(\boldsymbol{\lambda})\cdot\mathbf{D}(\boldsymbol{u})\boldsymbol{n}d\boldsymbol{\sigma} = 2\int_\Gamma \boldsymbol{v}(\boldsymbol{\lambda})_\tau\cdot\Big[\mathbf{D}(\boldsymbol{u})\boldsymbol{n}\Big]_\tau d\boldsymbol{\sigma} + 2\int_\Gamma(\boldsymbol{\lambda}\cdot\boldsymbol{n})[\mathbf{D}(\boldsymbol{u})\boldsymbol{n}]\boldsymbol{n}d\boldsymbol{\sigma}.$$

Finally, using that $\boldsymbol{h}\cdot\boldsymbol{n} = 0$ on Γ, we obtain

$$\int_\Gamma\Big(\boldsymbol{v}(\boldsymbol{\lambda}) - \boldsymbol{\lambda}\Big)\boldsymbol{h}\ d\boldsymbol{\sigma} = \int_\Gamma\Big(\boldsymbol{v}(\boldsymbol{\lambda})_\tau - \boldsymbol{\lambda}_\tau\Big)\boldsymbol{h}\ d\boldsymbol{\sigma}.$$

Combining these three expressions and after calculation, we obtain

$$2\int_\Gamma \boldsymbol{\lambda}\cdot\mathbf{D}(\boldsymbol{u})\boldsymbol{n}d\boldsymbol{\sigma} = 2\int_\Gamma \boldsymbol{u}\cdot\Big(\mathbf{D}(\boldsymbol{\lambda})\boldsymbol{n} - \mathbf{D}(\boldsymbol{v}(\boldsymbol{\lambda}))\boldsymbol{n}\Big)d\boldsymbol{\sigma} + 2\int_\Gamma \boldsymbol{v}(\boldsymbol{\lambda})\cdot\mathbf{D}(\boldsymbol{u})\boldsymbol{n}d\boldsymbol{\sigma}$$
$$-\int_\Gamma(\boldsymbol{v}(\boldsymbol{\lambda}) - \boldsymbol{\lambda})\boldsymbol{h}\ d\boldsymbol{\sigma}. \tag{V.11}$$

Plugging this in (V.10) allows to get (V.6).

Therefore it follows from Theorem III.6, that there exists a solution $(\widetilde{\boldsymbol{z}}, \widetilde{q}) \in (\boldsymbol{W}^{1,p}_k(\mathbb{R}^3)\times W^{0,p}_k(\mathbb{R}^3))$ satisfying the following Stokes problem:

$$-\Delta\,\widetilde{\boldsymbol{z}} + \nabla\,\widetilde{q} = \boldsymbol{F} \quad\text{and}\quad \operatorname{div}\widetilde{\boldsymbol{z}} = e \quad\text{in}\quad \mathbb{R}^3.$$

Using (V.3), we obtain

$$-\Delta\,(\widetilde{\boldsymbol{z}} - \widetilde{\boldsymbol{u}}) + \nabla\,(\widetilde{q} - \widetilde{\pi}) = \mathbf{0} \quad\text{and}\quad \operatorname{div}(\widetilde{\boldsymbol{z}} - \widetilde{\boldsymbol{u}}) = 0 \quad\text{in}\quad \mathbb{R}^3.$$

It follows that $(\widetilde{\boldsymbol{z}}-\widetilde{\boldsymbol{u}}, \widetilde{q}-\widetilde{\pi})$ belongs to $\Big(W^{1,p}_k(\mathbb{R}^3) + W^{1,2}_1(\mathbb{R}^3)\Big)\times\Big(W^{0,p}_k(\mathbb{R}^3) + W^{0,2}_1(\mathbb{R}^3)\Big)$, then $(\widetilde{\boldsymbol{z}} - \widetilde{\boldsymbol{u}}, \widetilde{q} - \widetilde{\pi})$ also belongs to $\mathbf{N}_{[1-3/p-k]}$. We deduce that there exist $(\boldsymbol{\lambda}, \mu) \in \mathbf{N}_{[1-3/p-k]}$, then $\widetilde{\boldsymbol{z}} - \widetilde{\boldsymbol{u}} = \boldsymbol{\lambda}$ and $\widetilde{q} - \widetilde{\pi} = \mu$ which imply that the solution $(\boldsymbol{u}, \pi)$ belongs indeed to $W^{1,p}_k(\Omega) \times W^{0,p}_k(\Omega)$.

Regularity. Finally, we prove that the solution $(\boldsymbol{u}, \pi) \in W^{1,p}_k(\Omega) \times W^{0,p}_k(\Omega)$ of $(\mathcal{S}_T)$ established previously, belongs to $W^{2,p}_{k+1}(\Omega) \times W^{1,p}_{k+1}(\Omega)$. Here we use regularity arguments on the Stokes problem set in bounded domains and in the whole space $\mathbb{R}^3$. To that end, we introduce the same partition of unity as in Theorem III.5. Let

$(\widetilde{\boldsymbol{u}}, \widetilde{\pi}) \in W_k^{1,p}(\mathbb{R}^3) \times W_k^{0,p}(\mathbb{R}^3)$ be an extension of $(\boldsymbol{u}, \pi)$ to the whole space $\mathbb{R}^3$. We can write:

$$\widetilde{\boldsymbol{u}} = \varphi\widetilde{\boldsymbol{u}} + \psi\widetilde{\boldsymbol{u}} \quad \text{and} \quad \widetilde{\pi} = \varphi\widetilde{\pi} + \psi\widetilde{\pi}.$$

Then it is enough to show that the pairs $(\varphi\widetilde{\boldsymbol{u}}, \varphi\widetilde{\pi})$ and $(\psi\widetilde{\boldsymbol{u}}, \psi\widetilde{\pi})$ belong to $W_{k+1}^{2,p}(\mathbb{R}^3) \times W_{k+1}^{1,p}(\mathbb{R}^3)$. To that end, consider first

$$-\Delta\,(\psi\widetilde{\boldsymbol{u}}) + \nabla\,(\psi\widetilde{\pi}) = \boldsymbol{f}_1 \quad \text{and} \quad \operatorname{div}(\psi\widetilde{\boldsymbol{u}}) = \chi_1 \quad \text{in} \quad \mathbb{R}^3, \tag{V.12}$$

where

$$\boldsymbol{f}_1 = \boldsymbol{f}\psi - (2\nabla\widetilde{\boldsymbol{u}}\nabla\psi + \boldsymbol{u}\Delta\,\psi) + \widetilde{\pi}\nabla\,\psi \quad \text{and} \quad \chi_1 = \chi\psi + \widetilde{\boldsymbol{u}}\cdot\nabla\,\psi.$$

We easily see that $\boldsymbol{f}_1$ and χ_1 have bounded supports and belong to $L^p_{loc}(\mathbb{R}^3) \times W^{1,p}_{loc}(\mathbb{R}^3)$. As a consequence, $(\boldsymbol{f}_1, \chi_1)$ belongs to $W^{0,p}_{k+1}(\mathbb{R}^3) \times W^{1,p}_{k+1}(\mathbb{R}^3)$. Using the regularity of the Stokes problem (see Theorem III.6), we deduce that the pair $(\psi\widetilde{\boldsymbol{u}}, \psi\widetilde{\pi})$ also belongs to $W^{2,p}_{k+1}(\mathbb{R}^3) \times W^{1,p}_{k+1}(\mathbb{R}^3)$.

Consider now the system

$$-\Delta\,(\varphi\widetilde{\boldsymbol{u}}) + \nabla\,(\varphi\widetilde{\pi}) = \boldsymbol{f}_2 \quad \text{and} \quad \operatorname{div}(\varphi\widetilde{\boldsymbol{u}}) = \chi_2,$$

where $\boldsymbol{f}_2$ and χ_2 have similar expressions as $\boldsymbol{f}_1$ and χ_1 with ψ remplaced by φ. It is easy to check that $(\boldsymbol{f}_2, \chi_2)$ belongs to $L^p(\Omega_{R+1}) \times W^{1,p}(\Omega_{R+1})$. In particular, we have $\widetilde{\boldsymbol{u}} = \psi\widetilde{\boldsymbol{u}}$ outside B_{R+1}, so the restriction of $\boldsymbol{u}$ to $\partial\,B_{R+1}$ belongs to $W^{1+1/p',p}(\partial\,B_{R+1})$. Its clear that $(\varphi\widetilde{\boldsymbol{u}}, \varphi\widetilde{\pi})$ belongs to $H^2(\Omega_{R+1}) \times H^1(\Omega_{R+1})$ and satisfies (IV.17). Then thanks to Theorem III.8, we prove that $(\varphi\widetilde{\boldsymbol{u}}, \varphi\widetilde{\pi}) \in W^{2,p}(\Omega_{R+1}) \times W^{1,p}(\Omega_{R+1})$ solution of problem (IV.17), which also implies that $(\varphi\widetilde{\boldsymbol{u}}, \varphi\widetilde{\pi})$ belongs to $W^{2,p}_{k+1}(\mathbb{R}^3) \times W^{1,p}_{k+1}(\mathbb{R}^3)$.

Consequently, the pair $(\boldsymbol{u}, \pi)$ belongs to $W^{2,p}_{k+1}(\Omega) \times W^{1,p}_{k+1}(\Omega)$ if $2 \leqslant p \leqslant 6$.

Step 2. The case $g \neq 0$.
Let $\boldsymbol{w}$ be in $W^{3,p}_{k+1}(\Omega)$ such that $\dfrac{\partial \boldsymbol{w}}{\partial \boldsymbol{n}} = g$ on Γ. According to step 1, problem (IV.36) has a solution in $W^{2,p}_{k+1}(\Omega) \times W^{1,p}_{k+1}(\Omega)$ if and only if, $\forall(\boldsymbol{\xi}, \eta) \in \mathcal{N}^{p'}_{-k+1}(\Omega)$,

$$\int_\Omega \Big(\boldsymbol{f} + \Delta\,(\nabla\boldsymbol{w})\Big)\cdot\boldsymbol{\xi}d\boldsymbol{x} - \int_\Omega(\chi - \Delta\,\boldsymbol{w})\eta d\boldsymbol{x} = -\int_\Gamma \boldsymbol{K}\cdot\boldsymbol{\xi}d\boldsymbol{\sigma}. \tag{V.13}$$

But if $(\boldsymbol{\xi}, \eta)$ is in $\mathcal{N}^{p'}_{-k+1}(\Omega)$, then since $\nabla\boldsymbol{w} \in W^{2,p}_{k+1}(\Omega)$, we can write (V.2) for the

pair $(-\nabla \boldsymbol{w}, 0)$ and we obtain

$$\int_\Omega \Delta(\nabla \boldsymbol{w}) \cdot \boldsymbol{\xi} d\boldsymbol{x} + \int_\Omega \Delta \boldsymbol{w}\, \eta d\boldsymbol{x} = -\int_\Gamma g\Big(2[\mathbf{D}(\boldsymbol{\xi})\boldsymbol{n}] \cdot \boldsymbol{n} - \eta\Big) d\boldsymbol{\sigma}$$
$$+ \int_\Gamma \Big(2[\mathbf{D}(\nabla \boldsymbol{w})\boldsymbol{n}]_\tau + \alpha(\nabla \boldsymbol{w})_\tau\Big) \cdot \boldsymbol{\xi} d\boldsymbol{\sigma}. \tag{V.14}$$

Combining (V.1) and (V.14) allows to obtain (V.13). Thus setting $\boldsymbol{u} = \boldsymbol{z} + \nabla \boldsymbol{w} \in W^{2,p}_{k+1}(\Omega)$, then the pair $(\boldsymbol{u}, \pi) \in W^{2,p}_{k+1}(\Omega) \times W^{1,p}_{k+1}(\Omega)$ is a solution of $(\mathcal{S}_T)$.

Now, suppose that $p > 6$. The above argument shows that $(\boldsymbol{u}, \pi)$ belongs to $W^{2,6}_{k+1}(\Omega) \times W^{1,6}_{k+1}(\Omega)$ and we can repeat the same argument with $p = 6$ instead of $p = 2$ using the fact if $\mathcal{O}$ is a lipschitzian bounded domain, we have $W^{1,6}(\mathcal{O}) \hookrightarrow L^q(\mathcal{O})$ for any real number $q > 1$. We know that the following embedding holds

$$W^{2,p}(\Omega') \hookrightarrow W^{1,q}(\Omega'),$$

for any $q \in [1, \infty]$ if $p > 3$. Then we have $\boldsymbol{u} \in W^{2-1/6,6}(\Gamma) \hookrightarrow W^{1-1/p}(\Gamma)$ for all $p > 6$. Thus establishes the existence of solution $(\boldsymbol{u}, \pi)$ in $W^{2,p}_{k+1}(\Omega) \times W^{1,p}_{k+1}(\Omega)$ of problem $(\mathbb{S}_T)$ when $p > 6$.

□

We finally close this section by the following theorem.

Theorem V.2 *Assume that $p \geqslant 2$. Let $\boldsymbol{f} \in W^{0,p}_{k+1}(\Omega)$, $\chi \in W^{1,p}_{k+1}(\Omega)$, $g \in W^{1+1/p',p}(\Gamma)$, $\boldsymbol{h} \in W^{1-1/p,p}(\Gamma)$ such that $\boldsymbol{h} \cdot \boldsymbol{n} = 0$ on Γ and that the compatibility condition (V.1) is satisfied. Then, the Stokes problem $(\mathcal{S}_T)$ has a solution $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{2,p}_{k+1}(\Omega) \times W^p_{k+1}(\Omega)$ unique up to an element of $\mathcal{N}^p_{k+1}(\Omega)$. In addition, we have the following estimate:*

$$\inf_{(\boldsymbol{\lambda},\mu) \in \mathcal{N}^p_{k+1}(\Omega)} \Big(\|\boldsymbol{u} + \boldsymbol{\lambda}\|_{W^{2,p}_{k+1}(\Omega)} + \|\pi + \mu\|_{W^{1,p}_{k+1}(\Omega)} \Big)$$
$$\leqslant C\Big(\|\boldsymbol{f}\|_{W^{0,p}_{k+1}(\Omega} + \|\chi\|_{W^{1,p}_{k+1}(\Omega)} + \|\boldsymbol{h}\|_{\boldsymbol{W}^{1-1/p,p}(\Gamma)} + \|g\|_{\boldsymbol{W}^{1+1/p',p}(\Gamma)} \Big).$$

Proof.
Here we prove that problem $(\mathcal{S}_T)$ has a solution $(\boldsymbol{u}, \pi)$ that belongs to $\in W^{2,p}_{k+1}(\Omega) \times W^{1,p}_{k+1}(\Omega)$. To that end, we proceed in two steps.

Step 1. The case $k < 3/p' - 1$.
First, let extend $\boldsymbol{f}$ by zero in Ω' and denote by $\widetilde{\boldsymbol{f}} \in W^{0,p}_{k+1}(\mathbb{R}^3)$ the extended function.

Moreover, let $\widetilde{\chi} \in W^{1,p}_{k+1}(\mathbb{R}^3)$ be an extension of χ. We consider the following problem:

$$-\Delta\, \widetilde{\boldsymbol{w}} + \nabla\, \widetilde{\eta} = \widetilde{\boldsymbol{f}} \quad \text{and} \quad \operatorname{div} \widetilde{\boldsymbol{w}} = \widetilde{\chi} \quad \text{in } \mathbb{R}^3. \tag{V.15}$$

Since $k < 3/p' - 1$, then $\mathbf{N}_{[1-3/p'+k]} = \{(\boldsymbol{0}, 0)\}$ and thus applying Theorem III.6 , we deduce that this problem has a solution $(\widetilde{w}, \widetilde{\eta}) \in W^{2,p}_{k+1}(\mathbb{R}^3) \times W^{1,p}_{k+1}(\mathbb{R}^3)$. Denoting the restriction to Ω by $(\boldsymbol{w}, \eta)$ belongs to $W^{2,p}_{k+1}(\Omega) \times W^{1,p}_{k+1}(\Omega)$. Since $\boldsymbol{w} \in W^{2,p}_{k+1}(\Omega)$, then we have $[\mathbf{D}(\boldsymbol{w})\boldsymbol{n}]_\tau$ belongs to $W^{1-1/p,p}(\Gamma)$. Consider now the following problem:

$$\begin{cases} -\Delta\, \boldsymbol{v} + \nabla\, \theta = 0, \quad \operatorname{div} \boldsymbol{v} = 0 \quad \text{in } \Omega, \\ \boldsymbol{v} \cdot \boldsymbol{n} = G \quad \text{and } 2[\mathbf{D}(\boldsymbol{v})\boldsymbol{n}]_\tau + \alpha \boldsymbol{v}_\tau = \boldsymbol{H} \quad \text{on } \Gamma, \end{cases} \tag{V.16}$$

Where $\boldsymbol{H} = -2[\mathbf{D}(\boldsymbol{w})\boldsymbol{n}]_\tau - \alpha\, \boldsymbol{w}_\tau + \boldsymbol{h}$ and $G = g - \boldsymbol{w} \cdot \boldsymbol{n}$. Its clear that $G \in W^{1+1/p',p}(\Gamma)$ and $\boldsymbol{H} \in W^{1-1/p,p}(\Gamma)$ such that $\boldsymbol{H} \cdot \boldsymbol{n} = 0$ on Γ. According to Theorem V.1, the problem (V.16) has a solution in $W^{2,p}_{k+1}(\Omega) \times W^{1,p}_{k+1}(\Omega)$. Hence, the pair $(\boldsymbol{u}, \pi) = (\boldsymbol{w} + \boldsymbol{v}, \eta + \theta)$ belongs to $W^{2,p}_{k+1}(\Omega) \times W^{1,p}_{k+1}(\Omega)$ and satisfies problem $(\mathcal{S}_T)$.

Step 2. The case $k \geqslant 3/p' - 1$. We split this step in two cases:

The case $g = 0$.
Since $k \geqslant 3/p' - 1 > 0$, then $(\boldsymbol{f}, \chi)$ belongs to $W^{0,p}_1(\Omega) \times W^{1,p}_1(\Omega)$. According to Step 1, the problem $(\mathcal{S}_T)$ has a solution $(\boldsymbol{u}, \pi)$ belongs to $W^{2,p}_1(\Omega) \times W^{1,p}_1(\Omega)$. It remains now to prove that $(\boldsymbol{u}, \pi)$ belongs to $W^{2,p}_{k+1}(\Omega) \times W^{1,p}_{k+1}(\Omega)$. To that end, we shall use again properties of the Stokes problem in the whole space $\mathbb{R}^3$. Since $\boldsymbol{u} \cdot \boldsymbol{n} = 0$ on Γ, problem (IV.26) has a solution $(\boldsymbol{u}', \pi') \in W^{2,p}(\Omega') \times W^{1,p}(\Omega')$ (see Theorem III.7). Set now

$$\widetilde{\boldsymbol{u}} = \begin{cases} \boldsymbol{u} & \text{in } \Omega, \\ \boldsymbol{u}' & \text{in } \Omega', \end{cases} \qquad \text{and} \qquad \widetilde{\pi} = \begin{cases} \pi & \text{in } \Omega, \\ \pi' & \text{in } \Omega'. \end{cases}$$

Then clearly, the pair $(\widetilde{\boldsymbol{u}}, \widetilde{\pi})$ belongs to $W^{1,p}_1(\mathbb{R}^3) \times W^{0,p}_1(\mathbb{R}^3)$. In order to apply Theorem III.6 with data $-\Delta\, \widetilde{\boldsymbol{u}} + \nabla\, \widetilde{\pi}$ and $\operatorname{div} \widetilde{\boldsymbol{u}}$, we need to show that $(-\Delta\, \widetilde{\boldsymbol{u}} + \nabla\, \widetilde{\pi}, \operatorname{div} \widetilde{\boldsymbol{u}})$ belongs to $W^{-1,p}_k(\mathbb{R}^3) \times W^{0,p}_k(\mathbb{R}^3)$ and satisfies (III.22). Therefore, denoting by $\widetilde{\boldsymbol{f}} \in W^{0,p}_{k+1}(\mathbb{R}^3)$ the extension of $\boldsymbol{f}$ by zero in Ω'. For any $\boldsymbol{\varphi} \in \mathcal{D}(\mathbb{R}^3)$, using the same calculation as in the proof of TheoremV.1, we have:

$$\begin{aligned} \langle -\Delta\, \widetilde{\boldsymbol{u}} + \nabla\, \widetilde{\pi}, \boldsymbol{\varphi} \rangle_{\mathcal{D}'(\mathbb{R}^3) \times \mathcal{D}(\mathbb{R}^3)} \;=\; & \int_{\mathbb{R}^3} \widetilde{\boldsymbol{f}} \cdot \boldsymbol{\varphi} d\boldsymbol{x} + 2 \int_\Gamma \boldsymbol{\varphi} \cdot \Big(\mathbf{D}(\boldsymbol{u})\boldsymbol{n} - \mathbf{D}(\boldsymbol{u}')\boldsymbol{n} \Big) d\boldsymbol{\sigma} \\ & + \int_\Gamma \big(\boldsymbol{\varphi} \cdot \boldsymbol{n} \big) \big(\pi' - \pi \big) d\boldsymbol{\sigma} - \int_\Gamma \boldsymbol{\varphi} \cdot \boldsymbol{n}, \chi d\boldsymbol{\sigma}. \end{aligned} \tag{V.17}$$

Since $\mathcal{D}(\mathbb{R}^3)$ is dense in $W^{1,p'}_{-k}(\mathbb{R}^3)$ and $\widetilde{\boldsymbol{f}}$ belongs to $W^{0,p}_{k+1}(\mathbb{R}^3)$, then (V.17) is still valid for any $\boldsymbol{\varphi} \in W^{1,p'}_{-k}(\mathbb{R}^3)$ which implies that $-\Delta\, \widetilde{\boldsymbol{u}} + \nabla\, \widetilde{\pi}$ belongs to $W^{-1,p}_{k}(\mathbb{R}^3)$.
Now for any $\phi \in \mathcal{D}(\mathbb{R}^3)$, we have

$$\langle \operatorname{div} \widetilde{\boldsymbol{u}}, \phi \rangle_{\mathcal{D}'(\mathbb{R}^3)\times\mathcal{D}(\mathbb{R}^3)} = \int_\Omega \chi\, \phi d\boldsymbol{x}. \tag{V.18}$$

Since χ belongs to $W^{1,p}_{k+1}(\Omega)$ and due to the density of $\mathcal{D}(\mathbb{R}^3)$ in $W^{0,p'}_{-k}(\mathbb{R}^3)$, then (V.18) is still valid for any $\phi \in W^{0,p'}_{-k}(\mathbb{R}^3)$, which implies that $\operatorname{div} \widetilde{\boldsymbol{u}}$ belongs to $W^{0,p}_{k}(\mathbb{R}^3)$.
As a consequence, we have proved that $(-\Delta\, \widetilde{\boldsymbol{u}} + \nabla\, \widetilde{\pi}, \operatorname{div} \widetilde{\boldsymbol{u}})$ belongs to $W^{-1,p}_{k}(\mathbb{R}^3) \times W^{0,p}_{k}(\mathbb{R}^3)$. Using the same calculation as in the proof of TheoremV.1, we prove that $(-\Delta\, \widetilde{\boldsymbol{u}} + \nabla\, \widetilde{\pi}, \operatorname{div} \widetilde{\boldsymbol{u}})$ satisfy (III.22). Therefore it follows from Theorem III.6, that there exists a solution $(\widetilde{\boldsymbol{z}}, \widetilde{q}) \in (\boldsymbol{W}^{1,p}_{k}(\mathbb{R}^3) \times W^{0,p}_{k}(\mathbb{R}^3))$ satisfying the following Stokes problem:

$$-\Delta\, (\widetilde{\boldsymbol{z}} - \widetilde{\boldsymbol{u}}) + \nabla\, (\widetilde{q} - \widetilde{\pi}) = \mathbf{0} \quad \text{and} \quad \operatorname{div} (\widetilde{\boldsymbol{z}} - \widetilde{\boldsymbol{u}}) = 0 \quad \text{in} \quad \mathbb{R}^3.$$

It follows that $(\widetilde{\boldsymbol{z}} - \widetilde{\boldsymbol{u}}, \widetilde{q} - \widetilde{\pi})$ belongs to $\left(W^{1,p}_{k}(\mathbb{R}^3) + W^{1,p}_{1}(\mathbb{R}^3)\right) \times \left(W^{0,p}_{k}(\mathbb{R}^3) + W^{0,p}_{1}(\mathbb{R}^3)\right)$, then $(\widetilde{\boldsymbol{z}} - \widetilde{\boldsymbol{u}}, \widetilde{q} - \widetilde{\pi})$ also belongs to $\mathbf{N}_{[1-3/p-k]}$. We deduce that there exist $(\boldsymbol{\lambda}, \mu) \in \mathbf{N}_{[1-3/p-k]}$, such that $\widetilde{\boldsymbol{z}} - \widetilde{\boldsymbol{u}} = \boldsymbol{\lambda}$ and $\widetilde{q} - \widetilde{\pi} = \mu$ which imply that the solution $(\boldsymbol{u}, \pi)$ belongs indeed to $W^{1,p}_{k}(\Omega) \times W^{0,p}_{k}(\Omega)$.

Finally, to prove that the solution $(\boldsymbol{u}, \pi) \in W^{1,p}_{k}(\Omega) \times W^{0,p}_{k}(\Omega)$ of $(\mathcal{S}_T)$ established previously, actually belongs to $W^{2,p}_{k+1}(\Omega) \times W^{1,p}_{k+1}(\Omega)$, we can proceed as in the proof of Theorem V.1 with the use of the partition of unity (III.18).

For the case $g \neq 0$, we proceed as in the same way as in the proof of Theorem V.1, we prove that the pair $(\boldsymbol{u}, \pi)$ belongs to $W^{2,p}_{k+1}(\Omega) \times W^{1,p}_{k+1}(\Omega)$ is a solution of $(\mathcal{S}_T)$.

□

3 Very weak solutions in $W^{0,p}_{-2}(\Omega) \times W^{-1,p}_{-2}(\Omega)$

We are going to study the existence result and the uniqueness of very weak solutions in $W^{0,p}_{-2}(\Omega) \times W^{-1,p}_{-2}(\Omega)$ for the Stokes problem $(\mathcal{S}_T)$ for any $1 < p < 2$. We assume that α is a positive function that belongs to $W^{1,\infty}(\Gamma)$
The set below is a natural extension of that introduced in the previous chapter:

$$\mathbf{T}_2^p(\Omega) = \{\boldsymbol{v} \in \mathring{H}_0^p(\mathrm{div}, \Omega); \quad \mathrm{div}\, \boldsymbol{v} \in \mathring{W}_2^{1,p}(\Omega)\},$$

which is a Banach space equipped with the norm

$$\|\boldsymbol{v}\|_{\mathbf{T}_2^p(\Omega)} = \|\boldsymbol{v}\|_{W_1^{1,p}(\Omega)} + \|\mathrm{div}\, \boldsymbol{v}\|_{W_2^{1,p}(\Omega)}$$

The following Lemmas state some properties related to the space $\mathbf{T}_2^p(\Omega)$, these are extensions of the properties given in the previous chapter. First, we will characterize the dual space of $\mathbf{T}_2^p(\Omega)$.

Lemma V.1 *Let $1 < p < \infty$. A distribution $\boldsymbol{f}$ belongs to $(\boldsymbol{T}_2^p(\Omega))'$ if and only if there exist $\boldsymbol{\phi} \in L^{p'}(\Omega)$ and $f_0 \in W_{-2}^{-1,p'}(\Omega)$, such that*

$$\forall \boldsymbol{\varphi} \in \boldsymbol{T}_2^p(\Omega), \quad \langle \boldsymbol{f}, \boldsymbol{\varphi} \rangle_{(\boldsymbol{T}_2^p(\Omega))' \times \boldsymbol{T}_2^p(\Omega)} = \int_\Omega \boldsymbol{\phi} \cdot \boldsymbol{\varphi} d\boldsymbol{x} + \langle f_0, \mathrm{div}\, \boldsymbol{\varphi} \rangle_{W_{-2}^{-1,p'}(\Omega) \times \mathring{W}_2^{1,p}(\Omega)}. \tag{V.19}$$

Moreover, we have the estimate

$$\|\boldsymbol{f}\|_{(T_2^p(\Omega))'} = \max\{\|\boldsymbol{\phi}\|_{L^{p'}(\Omega)}, \|f_0\|_{W_{-2}^{-1,p'}(\Omega)}\} \tag{V.20}$$

Proof.

Let $\boldsymbol{f} \in (\mathbf{T}_2^p(\Omega))'$ and we set $\mathbf{E} = L^p(\Omega) \times \mathring{W}_2^{1,p}(\Omega)$ equipped with the norm

$$\forall \boldsymbol{v} = (\boldsymbol{v}_0, v_1) \in \mathbf{E}, \quad \|\boldsymbol{v}\|_{\mathbf{E}} = \|\boldsymbol{v}_0\|_{L^p(\Omega)} + \|v_1\|_{W_2^{1,p}(\Omega)}.$$

The mapping $\mathbf{L} \ : \ \boldsymbol{w} \to (\boldsymbol{w}, \mathrm{div}\, \boldsymbol{w})$ is an isometry from $\mathbf{T}_2^p(\Omega)$ onto $\mathbf{E}$. Suppose $\mathbf{G} = \mathbf{L}(\mathbf{T}_2^p(\Omega))$ and $\mathbf{S} = \mathbf{L}^{-1}$: $\mathbf{G} \to \mathbf{T}_2^p(\Omega)$, then the linear mapping defined by

$$\boldsymbol{v} \in \mathbf{G} \mapsto \langle \boldsymbol{f}, \mathbf{S}\boldsymbol{v} \rangle_{(T_2^p(\Omega))' \times T_2^p(\Omega)}$$

is linear and continuous on $\mathbf{G}$. Therefore thanks to Hahn-Banach theorem, that there exists an extension $\boldsymbol{f}^*$ of $\mathbf{L}$ which s linear and continuous on $\mathbf{E}$ satisfying $\|\boldsymbol{f}^*\|_{\mathbf{E}'} = \|\boldsymbol{f}\|_{(T_2^p(\Omega))'}$. From the Riesz's Representation theorem, there exists $\boldsymbol{\phi}$ belongs to $L^{p'}(\Omega)$ and f_0 belongs to $W_{-2}^{-1,p'}(\Omega)$ such that for any $\boldsymbol{v} = (\boldsymbol{v}_0, v_1) \in \mathbf{E}$

$$\langle \boldsymbol{f}^*, \boldsymbol{v} \rangle_{\mathbf{E}' \times \mathbf{E}} \quad = \quad \int_\Omega \boldsymbol{\phi} \cdot \boldsymbol{v}_0 \, d\mathbf{x} + \langle f_0, v_1 \rangle_{W_{-2}^{-1,p'}(\Omega) \times \mathring{W}_2^{1,p}(\Omega)}$$

and it is clear that $\|\boldsymbol{f}^*\|_{\mathbf{E}'} = \|\boldsymbol{f}\|_{(T_2^p(\Omega))'} = \max\{\|\boldsymbol{\phi}\|_{L^{p'}(\Omega)}, \|f_0\|_{W_{-2}^{-1,p'}(\Omega)}\}$.

□

Now, as a consequence of the above Lemma, we present the result of density and the Green's formula.

Lemma V.2 *Let $1 < p < \infty$. The following properties hold:*

i) The space $\mathcal{D}(\Omega)$ is dense in $\boldsymbol{T}^p_2(\Omega)$. As a consequence, relation (V.19) implies that

$$\boldsymbol{f} = \boldsymbol{\phi} + \nabla f_0 \tag{V.21}$$

ii) For any $\chi \in W^{-1,p}_{-2}(\Omega)$ and $\boldsymbol{v} \in \boldsymbol{T}^{p'}_2(\Omega)$, we have

$$\langle \nabla \chi, \boldsymbol{v} \rangle_{(\boldsymbol{T}^{p'}_2(\Omega))' \times \boldsymbol{T}^{p'}_2(\Omega)} = -\langle \chi, \operatorname{div} \boldsymbol{v} \rangle_{W^{-1,p}_{-2}(\Omega) \times \mathring{W}^{1,p'}_2(\Omega)}. \tag{V.22}$$

Proof.

i) Let $\boldsymbol{\ell} \in (\mathbf{T}^p_2(\Omega))'$ such that

$$\langle \boldsymbol{\ell}, \boldsymbol{\varphi} \rangle_{(\mathbf{T}^p_2(\Omega))' \times \mathbf{T}^p_2(\Omega)} = 0, \quad \forall \boldsymbol{\varphi} \in \mathcal{D}(\Omega).$$

Using Lemma V.1, there exist $\boldsymbol{\phi} \in L^{p'}(\Omega)$ and $\boldsymbol{f}_0 \in W^{-1,p'}_{-2}(\Omega)$, such that

$$\langle \boldsymbol{\ell}, \boldsymbol{\varphi} \rangle_{(\mathbf{T}^p_2(\Omega))' \times \mathbf{T}^p_2(\Omega)} = \int_\Omega \boldsymbol{\phi} \cdot \boldsymbol{\varphi} d\mathbf{x} + \langle f_0, \operatorname{div} \boldsymbol{\varphi} \rangle_{W^{-1,p'}_{-2}(\Omega) \times W^{1,p}_2(\Omega)}$$

As a consequence

$$\boldsymbol{\phi} - \nabla f_0 = 0, \quad \text{in } \mathcal{D}'(\Omega).$$

Then, we have $\nabla f_0 = \boldsymbol{\phi}$ belongs to $L^{p'}(\Omega)$, we propose to extend f_0 to $\mathbb{R}^3$. As Ω' is bounded, we can chose once and for all B_{R_0} denote the ball centred at to the origin, with radius $R_0 > 0$ such that $\overline{\Omega'} \subset B_{R_0}$ and let $\Omega_{R_0} = \Omega \cap B_{R_0}$. Then f_0 is a distribution of $\mathcal{D}(\Omega_{R_0})$ and ∇f_0 belongs to $L^{p'}(\Omega_{R_0})$. Since Ω_{R_0} is a bounded domain, f_0 belongs to $W^{1,p'}(\Omega_{R_0})$ (see for instance [4]). By Calderón [26], f_0 has an $W^{1,p'}$ extension $\overline{f}_0$ in Ω' such that the function

$$\widetilde{f}_0 = \begin{cases} \overline{f}_0 & \text{in } \Omega', \\ f_0 & \text{in } \Omega, \end{cases}$$

belongs to $W^{1,p'}(B_{R_0})$ and $\nabla \widetilde{f}_0$ belongs to $L^{p'}(\mathbb{R}^3)$. It follows from [5, Proposition 4.3] that if $3 < p'$, then $\widetilde{f}_0$ belongs to $W_0^{1,p'}(\mathbb{R}^3)$ then f_0 belongs to $W_0^{1,p'}(\Omega)$ and if $1 < p' < 3$, then there exists a constant c, depending on $\widetilde{f}_0$ and s belongs to $W_0^{1,p'}(\mathbb{R}^3)$ such that $\widetilde{f}_0 = s + c$ belongs to $W_0^{1,p'}(\mathbb{R}^3)$. Then, $\widetilde{f}_0 = s_{|\Omega} + c$ belongs to $W_0^{1,p'}(\Omega)$, because we have the imbedding $W_0^{1,p'}(\Omega) \hookrightarrow W_{-2}^{-1,p'}(\Omega)$, then $s_{|\Omega} \in W_{-2}^{-1,p'}(\Omega)$ which in turn implies that $c \in W_{-2}^{-1,p'}(\Omega)$. Since $1 < p' < 3$, the space $W_{-2}^{-1,p'}(\Omega)$ does not contain polynomial functions and thus $c = 0$, it follows that $f_0 = s_{|\Omega}$ belongs to $W_0^{1,p'}(\Omega)$. For any $\boldsymbol{v} \in \mathbf{T}_2^p(\Omega)$, we can write

$$\begin{aligned} \langle \boldsymbol{\ell}, \boldsymbol{v} \rangle_{(\mathbf{T}_2^p(\Omega))' \times \mathbf{T}_2^p(\Omega)} &= \int_\Omega \boldsymbol{\phi} \cdot \boldsymbol{v} d\mathbf{x} + \langle f_0, \operatorname{div} \boldsymbol{v} \rangle_{W_{-2}^{-1,p'}(\Omega) \times \mathring{W}_2^{1,p}(\Omega)} \\ &= \int_\Omega \boldsymbol{\phi} \cdot \boldsymbol{v} d\mathbf{x} + \int_\Omega f_0 \operatorname{div} \boldsymbol{v} d\mathbf{x} \\ &= \int_\Omega \nabla f_0 \cdot \boldsymbol{v} d\mathbf{x} + \int_\Omega f_0 \operatorname{div} \boldsymbol{v} d\mathbf{x} \end{aligned}$$

Since, $\boldsymbol{v}$ belongs $\mathring{H}_0^p(\operatorname{div}, \Omega)$ and $f_0 \in W_0^{1,p'}(\Omega)$, then using (II.14) to the first integral of the last relation, we obtain

$$\langle \boldsymbol{\ell}, \boldsymbol{v} \rangle_{(\mathbf{T}_2^p(\Omega))' \times \mathbf{T}_2^p(\Omega)} = 0, \quad \forall \boldsymbol{v} \in \mathbf{T}_2^p(\Omega). \tag{V.23}$$

ii) Due the density of $\mathcal{D}(\Omega)$ in $\mathbf{T}_2^p(\Omega)$, then the Green formula (V.22) is holds.

□

We extend here the sense given to the boundary condition in Chapter IV. For this reason, we need to introduce the Banach space

$$\mathbf{H}_{-2}^p(\Delta; \Omega) = \{\boldsymbol{v} \in W_{-2}^{0,p}(\Omega); \ \Delta \boldsymbol{v} \in (\mathbf{T}_2^{p'}(\Omega))'\},$$

which has the following norm:

$$\|\boldsymbol{v}\|_{\mathbf{H}_{-2}^p(\Delta;\Omega)} = \|\boldsymbol{v}\|_{W_{-2}^{0,p}(\Omega)} + \|\Delta \boldsymbol{v}\|_{(\mathbf{T}_2^{p'}(\Omega))'}.$$

The following lemma will help us to prove a trace result.

Lemma V.3 *The space $\mathcal{D}(\overline{\Omega})$ is dense in $\boldsymbol{H}_{-2}^p(\Delta; \Omega)$.*

Proof. Let $\boldsymbol{\ell}\in(\mathbf{H}^p_{-2}(\Delta;\Omega))'$ such that for any $\boldsymbol{v}\in\mathcal{D}(\overline{\Omega})$, we have $\langle\boldsymbol{\ell},\boldsymbol{v}\rangle=0$. We want to prove that $\boldsymbol{\ell}=0$. Using the Riesz's Representation lemma, there exists $(\boldsymbol{f}_0,\boldsymbol{f}_1)$ belongs to $W^{0,p'}_2(\Omega)\times\mathbf{T}^{p'}_2(\Omega)$ such that for any $\boldsymbol{v}\in\mathbf{H}^p_{-2}(\Delta;\Omega)$,

$$\langle\boldsymbol{\ell},\boldsymbol{v}\rangle_{(\mathbf{H}^p_{-2}(\Delta;\Omega))'\times\mathbf{H}^p_{-2}(\Delta;\Omega)}=\int_\Omega\boldsymbol{f}_0\cdot\boldsymbol{v}d\mathbf{x}+\langle\Delta\,\boldsymbol{v},\boldsymbol{f}_1\rangle_{(\mathbf{T}^{p'}_2(\Omega))'\times\mathbf{T}^{p'}_2(\Omega)}\tag{V.24}$$

Since ℓ vanishes on $\mathcal{D}(\overline{\Omega})$, we have

$$\boldsymbol{f}_0+\Delta\,\boldsymbol{f}_1=0\quad\text{ on }\Omega.$$

Observe that we can extend by zero the functions $\boldsymbol{f}_0$ and $\boldsymbol{f}_1$, in such a way that $(\widetilde{\boldsymbol{f}_0},\widetilde{\boldsymbol{f}_1})$ belongs to $W^{0,p'}_2(\mathbb{R}^3)\times\mathbf{T}^{p'}_2(\mathbb{R}^3)$. Now, we take $\boldsymbol{\varphi}\in\mathcal{D}(\mathbb{R}^3)$. Then, we have

$$\int_{\mathbb{R}^3}\widetilde{\boldsymbol{f}_0}\cdot\boldsymbol{\varphi}d\mathbf{x}+\int_{\mathbb{R}^3}\widetilde{\boldsymbol{f}_1}\cdot\Delta\,\boldsymbol{\varphi}d\mathbf{x}=\langle\boldsymbol{\ell},\boldsymbol{\varphi}_{|\Omega}\rangle=0,\tag{V.25}$$

and thus $\widetilde{\boldsymbol{f}_0}+\Delta\,\widetilde{\boldsymbol{f}_1}=0$ in $\mathcal{D}'(\mathbb{R}^3)$. Using that $\widetilde{\boldsymbol{f}_0}$ belongs to $W^{0,p'}_2(\mathbb{R}^3)$ then $\Delta\,\widetilde{\boldsymbol{f}_1}=-\widetilde{\boldsymbol{f}_0}\in W^{0,p'}_2(\mathbb{R}^3)$ and due to the density of $\mathcal{D}(\mathbb{R}^3)$ in $W^{2,p}_0(\mathbb{R}^3)$, we have

$$\langle\Delta\,\widetilde{\boldsymbol{f}_1},\varphi\rangle_{W^{0,p'}_2(\mathbb{R}^3)\times W^{0,p}_{-2}(\mathbb{R}^3)}=\langle\widetilde{\boldsymbol{f}_1},\Delta\,\varphi\rangle_{L^{p'}(\mathbb{R}^3)\times L^p(\Omega)},\quad\forall\varphi\in W^{2,p}_0(\mathbb{R}^3).$$

This shows that $\Delta\,\widetilde{\boldsymbol{f}_1}\in W^{0,p'}_2(\mathbb{R}^3)$ is orthogonal to polynomial of $\mathcal{P}^{\Delta}_{[2-3/p]}$. Then, according to Theorem III.1, there exist $\boldsymbol{\lambda}\in W^{2,p'}_2(\mathbb{R}^3)$ such that $-\Delta\,\boldsymbol{\lambda}=\Delta\,\widetilde{\boldsymbol{f}_1}$. Thus the harmonic function $\boldsymbol{\lambda}+\widetilde{\boldsymbol{f}_1}$ belongs to $W^{2,p'}_2(\mathbb{R}^3)+L^{p'}(\mathbb{R}^3)\subset L^{p'}(\mathbb{R}^3)$. Then $\widetilde{\boldsymbol{f}_1}=\boldsymbol{\lambda}\in W^{2,p'}_2(\mathbb{R}^3)$ and therefore $\boldsymbol{f}_1\in\mathring{W}^{2,p'}_2(\Omega)$. As $\mathcal{D}(\Omega)$ is dense in $\mathring{W}^{2,p'}_2(\Omega)$, there exist a sequence $(\boldsymbol{\varphi}_n)_n\subset\mathcal{D}(\Omega)$ such that $\boldsymbol{\varphi}_n\longrightarrow\boldsymbol{f}_1$ in $W^{2,p'}_2(\Omega)$, when $n\longrightarrow\infty$. In particular $\boldsymbol{\varphi}_n\longrightarrow\boldsymbol{f}_1$ in $\mathbf{T}^{p'}_2(\Omega)$. Finally, we consider $\boldsymbol{v}\in\mathbf{H}^p_{-2}(\Delta;\Omega)$ and we have to prove that $\langle\boldsymbol{\ell},\boldsymbol{v}\rangle=0$. Observe that

$$\begin{aligned}
\langle\ell,\boldsymbol{v}\rangle_{(\mathbf{H}^p_{-2}(\Delta;\Omega))'\times\mathbf{H}^p_{-2}(\Delta;\Omega)} &= \int_\Omega\boldsymbol{v}\cdot\boldsymbol{f}_0+\langle\Delta\,\boldsymbol{v},\boldsymbol{f}_1\rangle_{(\mathbf{T}^{p'}_2(\Omega))'\times\mathbf{T}^{p'}_2(\Omega)}\\
&= -\int_\Omega\boldsymbol{v}\cdot\Delta\,\boldsymbol{f}_1+\langle\Delta\,\boldsymbol{v},\boldsymbol{f}_1\rangle_{(\mathbf{T}^{p'}_2(\Omega))'\times\mathbf{T}^{p'}_2(\Omega)}\\
&= \lim_{n\to\infty}\Big(-\int_\Omega\boldsymbol{v}\cdot\Delta\,\boldsymbol{\varphi}_n+\langle\Delta\,\boldsymbol{v},\boldsymbol{\varphi}_n\rangle_{(\mathbf{T}^{p'}_2(\Omega))'\times\mathbf{T}^{p'}_2(\Omega)}\Big)\\
&= \lim_{n\to\infty}\Big(-\int_\Omega\boldsymbol{v}\cdot\Delta\,\boldsymbol{\varphi}_n+\int_\Omega\boldsymbol{v}\cdot\Delta\,\boldsymbol{\varphi}_n\Big)\\
&= 0
\end{aligned}$$

□

The following space is a natural extension of the one introduced in the previous chapter:

$$\mathbf{S}_2^p(\Omega) = \{\boldsymbol{v} \in W_2^{2,p}(\Omega); \quad \operatorname{div} \boldsymbol{v} = 0, \quad \boldsymbol{v} \cdot \boldsymbol{n} = 0, \ [\mathbf{D}(\boldsymbol{v})\boldsymbol{n}]_\tau + \alpha \boldsymbol{v}_\tau = 0 \text{ on } \Gamma\}$$

The following lemma proves that for any $\boldsymbol{v}$ in $\mathbf{H}^p_{-2}(\Delta;\Omega)$, using the fact $\alpha \in W^{1,\infty}(\Gamma)$, we have $2[\mathbf{D}(\boldsymbol{v})\boldsymbol{n}]_\tau + \alpha \boldsymbol{v}$ is well defined and belongs to $W^{-1-1/p,p}(\Gamma)$.

Lemma V.4 *Let $\alpha \in W^{1,\infty}(\Gamma)$. The linear mapping $\Theta : \boldsymbol{u} \mapsto 2[\boldsymbol{D}(\boldsymbol{u})\boldsymbol{n}]_\tau + \alpha \boldsymbol{u}_\tau$ defined on $\mathcal{D}(\overline{\Omega})$ can be extended to a linear and continuous mapping still denoted by Θ, from $\boldsymbol{H}^p_{-2}(\Delta;\Omega)$ into $W^{-1-1/p,p}(\Gamma)$ and we have the following Green formula: for any $\boldsymbol{u} \in \boldsymbol{H}^p_{-2}(\Delta;\Omega)$ and $\boldsymbol{\varphi} \in \boldsymbol{S}_2^{p'}(\Omega)$,*

$$\begin{aligned}\langle \Delta\, \boldsymbol{u}, \boldsymbol{\varphi}\rangle_{(\boldsymbol{T}_2^{p'}(\Omega))' \times \boldsymbol{T}_2^{p'}(\Omega)} \quad = \quad & \int_\Omega \boldsymbol{u} \cdot \Delta\, \boldsymbol{\varphi} d\boldsymbol{x} + \langle 2[\boldsymbol{D}(\boldsymbol{u})\boldsymbol{n}]_\tau + \alpha \boldsymbol{u}_\tau, \boldsymbol{\varphi}\rangle_{W^{-1-1/p,p}(\Gamma)\times W^{1+1/p,p'}(\Gamma)} \\ & -\langle 2[\boldsymbol{D}(\boldsymbol{\varphi})\boldsymbol{n}] \cdot \boldsymbol{n}, \boldsymbol{u} \cdot \boldsymbol{n}\rangle_{W^{1/p,p'}(\Gamma)\times W^{-1/p,p}(\Gamma)}. \end{aligned} \tag{V.26}$$

The proof of this Lemma is very similar to that of Lemma IV.3.

We begin by precising the meaning of very weak variational formulation, this is a extension of the definition given in the previous chapter

Definition V.1 *Given $\boldsymbol{f}$, χ, g and $\boldsymbol{h}$ with*

$$\boldsymbol{f} \in (\boldsymbol{T}_2^{p'}(\Omega))', \quad \chi \in W_1^{0,p}(\Omega), \quad g \in W^{-1/p,p}(\Gamma), \quad \boldsymbol{h} \in W^{-1-1/p,p}(\Gamma),$$

such that $\boldsymbol{h} \cdot \boldsymbol{n} = 0$ on Γ. Then a pair $(\boldsymbol{u}, \pi) \in W_{-2}^{0,p}(\Omega) \times W_{-2}^{-1,p}(\Omega)$ is called very weak solution of $(\mathcal{S}_T)$ if, for any $(\varphi, q) \in \boldsymbol{S}_2^{p'}(\Omega) \times W_2^{1,p'}(\Omega)$, the relations

$$\begin{aligned} & -\int_\Omega \boldsymbol{u} \cdot \Delta\, \boldsymbol{\varphi} d\boldsymbol{x} - \langle \pi, \operatorname{div} \boldsymbol{\varphi}\rangle_{W_{-2}^{-1,p}(\Omega)\times \mathring{W}_2^{1,p'}(\Omega)} = \langle \boldsymbol{f}, \boldsymbol{\varphi}\rangle_{(\boldsymbol{T}_2^{p'}(\Omega))'\times \boldsymbol{T}_2^{p'}(\Omega)} \\ & +\langle \boldsymbol{h}, \boldsymbol{\varphi}\rangle_{W^{-1-1/p,p}(\Gamma)\times W^{1+1/p,p'}(\Gamma)} - \langle g, 2[\boldsymbol{D}(\boldsymbol{\varphi})\boldsymbol{n}] \cdot \boldsymbol{n}\rangle_{W^{-1/p,p}(\Gamma)\times W^{1/p,p'}(\Gamma)} \end{aligned} \tag{V.27}$$

and

$$\int_\Omega \boldsymbol{u} \cdot \nabla\, q d\boldsymbol{x} = -\int_\Omega \chi q d\boldsymbol{x} + \langle g, q\rangle_{W^{-1/p,p}(\Gamma)\times W^{1/p,p'}(\Gamma)}. \tag{V.28}$$

are satisfied.

Proposition V.2 *Under the assumptions of Definition V.1, the following tow statements are equivalent:*

i) $(\boldsymbol{u}, \pi) \in W_{-2}^{0,p}(\Omega) \times W_{-2}^{-1,p}(\Omega)$ *is a very weak solution of* $(\mathcal{S}_T)$.

ii) $(\boldsymbol{u}, \pi) \in W_{-2}^{0,p}(\Omega) \times W_{-2}^{-1,p}(\Omega)$ *satisfies the problem* $(\mathcal{S}_T)$ *in the sense of distributions.*

The proof of this proposition is very similar to that of Proposition IV.4.
Now, we are position to prove the existence and uniqueness of a very weak solution for Stokes problem $(\mathcal{S}_T)$.

Theorem V.3 *Let $1<p<2$, given any $\boldsymbol{f}$, χ, g and $\boldsymbol{h}$ with*

$$\boldsymbol{f}\in(\boldsymbol{T}^{p'}_2(\Omega))',\quad \chi\in W^{0,p}_{-1}(\Omega),\quad g\in W^{-1/p,p}(\Gamma),\quad \boldsymbol{h}\in W^{-1-1/p,p}(\Gamma),$$

such that $\boldsymbol{h}\cdot\boldsymbol{n}=0$ on Γ. Then, the problem $(\mathcal{S}_T)$ has a unique solution $(\boldsymbol{u},\pi)$ belongs to $W^{0,p}_{-2}(\Omega)\times W^{-1,p}_{-2}(\Omega)$. Moreover, we have the estimate:

$$\|\boldsymbol{u}\|_{W^{0,p}_{-2}(\Omega)}+\|\pi\|_{W^{-1,p}_{-2}(\Omega)}\leqslant C\Big(\|\boldsymbol{f}\|_{(\boldsymbol{T}^{p'}_2(\Omega))'}+\|\chi\|_{W^{0,p}_{-1}(\Omega)}+\|g\|_{W^{-1/p,p}(\Gamma)}+\|\boldsymbol{h}\|_{W^{-1-1/p,p}(\Gamma)}\Big).$$

Proof.
The proof is quite similar to the proof of Theorem IV.4. If the pair $(\boldsymbol{u},\pi)$ belongs to $W^{0,p}_{-2}(\Omega)\times W^{-1,p}_{-2}(\Omega)$ is a solution of $(\mathcal{S}_T)$, then for any $(\boldsymbol{\varphi},q)\in\mathbf{S}^{p'}_2(\Omega)\times W^{1,p'}_2(\Omega)$, adding (V.27) and (V.28), we have

$$\begin{aligned}-\int_\Omega \boldsymbol{u}\cdot\Big(\Delta\,\boldsymbol{\varphi}+\nabla\,q\Big)d\mathbf{x}-\langle\pi,\operatorname{div}\boldsymbol{\varphi}\rangle_{W^{-1,p}_{-2}(\Omega)\times\mathring{W}^{1,p'}_2(\Omega)}&=\langle\boldsymbol{f},\boldsymbol{\varphi}\rangle_{(\boldsymbol{T}^{p'}_2(\Omega))'\times\boldsymbol{T}^{p'}_2(\Omega)}\\ +\langle\boldsymbol{h},\boldsymbol{\varphi}\rangle_{W^{-1-1/p,p}(\Gamma)\times W^{1+1/p,p'}(\Gamma)}-\int_\Omega\chi q d\mathbf{x}+\langle g,q-2[\mathbf{D}(\boldsymbol{\varphi})\boldsymbol{n}]\cdot\boldsymbol{n}\rangle_{W^{-1/p,p}(\Gamma)\times W^{1/p,p'}(\Gamma)}&.\end{aligned}\tag{V.29}$$

It remains now to look for $(\boldsymbol{u},\pi)$ belongs to $W^{0,p}_{-2}(\Omega)\times W^{-1,p}_{-2}(\Omega)$ satisfying (V.29).

The proof of the existence result is made of two steps depending on the datum g.

Step 1: We assume $g=0$.
According to Theorem V.2, for any $(\boldsymbol{F},\theta)$ belongs to $W^{0,p'}_2(\Omega)\times\mathring{W}^{1,p'}_2(\Omega)$ there exists a unique solution $(\varphi,q)\in W^{2,p'}_2(\Omega)\times W^{1,p'}_2(\Omega)$ such that

$$\begin{cases}-\Delta\,\varphi+\nabla\,q=\boldsymbol{F}\quad\textit{and}\quad\operatorname{div}\varphi=-\theta & \text{in }\Omega,\\ \varphi\cdot\boldsymbol{n}=0\quad\textit{and}\quad 2[\mathbf{D}(\varphi)\boldsymbol{n}]_\tau+\alpha\varphi_\tau=\boldsymbol{0} & \text{on }\Gamma\end{cases}$$

and

$$\|\boldsymbol{\varphi}\|_{\boldsymbol{W}^{2,p'}_2(\Omega)}+\|q\|_{W^{1,p'}_2(\Omega)}\leqslant C\Big(\|\boldsymbol{F}\|_{W^{0,p'}_2(\Omega)}+\|\theta\|_{W^{1,p'}_2(\Omega)}\Big).$$

Let $\mathbf{T}$ be the linear form defined from $W^{0,p'}_2(\Omega)\times\mathring{W}^{1,p'}_2(\Omega)$ onto $\mathbb{R}$ by

$$\mathbf{T}(\boldsymbol{F},\theta)=\langle\boldsymbol{f},\varphi\rangle_{(\boldsymbol{T}^{p'}_2(\Omega))'\times\boldsymbol{T}^{p'}_2(\Omega)}+\langle\boldsymbol{h},\varphi\rangle_{W^{-1-1/p,p}(\Gamma)\times W^{1+1/p,p'}(\Gamma)}-\int_\Omega\chi q d\mathbf{x}.$$

Note that for any $(\boldsymbol{F},\theta)\in W_2^{0,p'}(\Omega)\times W_2^{1,p'}(\Omega)$

$$\begin{aligned}|\mathbf{T}(\boldsymbol{F},\theta)| &= \left|\langle \boldsymbol{f},\varphi\rangle_{(\boldsymbol{T}_2^{p'}(\Omega))'\times \boldsymbol{T}_2^{p'}(\Omega)} + \langle \boldsymbol{h},\varphi\rangle_{W^{-1-1/p,p}(\Gamma)\times W^{1+1/p,p'}(\Gamma)} - \int_\Omega \chi q d\mathbf{x}\right| \\ &\leqslant C\Big(\|\boldsymbol{f}\|_{(\boldsymbol{T}_2^{p'}(\Omega))'} + \|\chi\|_{W_{-1}^{0,p}(\Omega)} + \|\boldsymbol{h}\|_{W^{-1-1/p,p}(\Gamma)}\Big)\Big(\|\boldsymbol{F}\|_{W_2^{0,p'}(\Omega)} + \|\theta\|_{W_2^{1,p'}(\Omega)}\Big)\end{aligned}$$

From this we can deduce that the linear form $\mathbf{T}$ is continuous on $W_2^{0,p'}(\Omega)\times \mathring{W}_2^{1,p'}(\Omega)$ and we deduce that there exists a unique pair $(\boldsymbol{u},\pi)$ belongs to $W_{-2}^{0,p}(\Omega)\times W_{-2}^{-1,p}(\Omega)$ satisfying (V.29).

Step 2: Assume $g\neq 0$.
We solve the Neumann problem (IV.51) with $g\in W^{-1/p,p}(\Gamma)$, then thanks Theorem III.5, there exists Θ that belongs to $W_{-1}^{1,p}(\Omega)$ solution of (IV.51). Let $\psi\in\mathcal{D}(\overline{\Omega})$ such that $0\leq\psi\leq 1$, supp $\psi\subset\overline{\Omega}_{R+1}$ and $\psi=1$ in $\overline{\Omega}_R$. Set now $w=\Theta\psi$, then w has obviously a compact support and thus w belongs to $W_{-2}^{1,p}(\Omega)$. Setting $\boldsymbol{z}=\boldsymbol{u}+\nabla w$ the rest of the proof runs as in proof of Theorem IV.4.

□

Remark V.1 *Note that, we have studied the very weak solution for a particular case $k=1$, because we have not been able to disassemble the density result of Lemma V.2 for all the weights. The problem that faces is the extension of f_0 into the whole space $\mathbb{R}^3$, if one knows demonstrate that the extension is in $W_{-k-1}^{-1,p'}(\mathbb{R}^3)$, we can prove the density with all weights*

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ECOLE DOCTORALE
CARNOT - PASTEUR

Titre : Équations de Stokes en domaine extérieur avec des conditions aux limites de type Navier

Mots clés : Mécanique de fluide - Equations de Stokes - Conditions de type Navier - Domaine extérieur - Espace de Sobolev à poids - Inégalité de Korn.

Résumé : On s'intéresse aux équations stationnaires de Stokes posées dans un domaine extérieur tridimensionnel décrivant l'écoulement d'un fluide visqueux et incompressible autour d'un obstacle qu'est supposé borné. La particularité ici réside dans les conditions au bord de l'obstacle que nous avons imposées. En effet, nous supposons que l'obstacle a une certaine rugosité et par conséquent, le fluide n'adhère pas au bord de l'obstacle mais, au contraire, il existe une friction dont on suppose décrite par les conditions aux limites de type Navier. Ces dernières modélisent d'une part l'imperméabilité de l'obstacle et d'autre part le fait que la composante tangentielle de la vitesse du fluide sur l'obstacle est proportionnelle au tenseur des déformations. Ce problème a été bien étudié lorsqu'il est posé dans un domaine borné. Les espaces de Sobolev classiques fournissent, dans ce cas, un cadre fonctionnel adéquat pour une étude complète. Cependant lorsque le domaine n'est pas borné, ces espaces ne sont plus adaptés car il est nécessaire de décrire le comportement à l'infini des solutions. On choisit alors de poser le problème dans des espaces de Sobolev avec des poids polynomiaux qui précisent la croissance ou la décroissance des fonctions à l'infini. Dans ce travail, nous commençons par effectuer une analyse hilbertienne du problème. Le point-clé ici est d'établir des inégalités de type Korn avec poids afin d'obtenir la coercivité de la forme bilinéaire associée à la formulation variationnelle. Nous continuons par démontrer des résultats d'existence, d'unicité et de régularité de solutions fortes et très faibles. Enfin, nous étudions l'extension de certains résultats en théorie L^p.

Title : Stokes equations in an exterior domain with Navier boundary conditions

Keywords: Fluide Mechanics - Stokes equations - Navier slip boundary conditions - Exterior domain - Weighted Sobolev spaces - Korn's inequalities.

Abstract: In this manuscript, we investigate the three-dimensional stationary Stokes problem with the Navier slip boundary conditions, describing the flow of a viscous and incompressible fluid past an obstacle where it is assumed that the fluid may slip at the boundary. The particularity here lies in the conditions at the boundary of the obstacle which have imposed. Indeed, it is assumed that the obstacle has a roughness and therefore, the fluid does not adhere to the edge of the obstacle but there is friction that is assumed to be described by the Navier boundary conditions. The latter models is the impermeability of the obstacle on the one hand and the fact that the tangential component of the fluid speed on the obstacle is proportional to the stress tensor on the other. Because the flow domain is unbounded, the data and the solutions are assumed to belong to adequate weighted Sobolev spaces in order to control and to precise the behavior of the functions at infinity. We prove the existence, uniqueness and regularity of solutions. The case of very weak solutions is also treated. The main contribution of this work is that the results are established for a very wide range of behavior at infinity (growth or decay at infinity). Our analysis is made of different steps: we first look for the variational solution. The key point here is the proof of variant weighted Korn's inequalities in order to get the coercivity of the bilinear form associated to the variational formulation. Next, We continue to demonstrate results of existence, uniqueness and regularity of strong and very weak solutions. Finally, we study the extension of certain results in L^p −theory .