Motivic Approach to Enumerating Vector Bundles

Peng Du

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Peng DU

Thèse dirigée par **Jean FASEL**, Professeur des universités, Université Grenoble Alpes

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Une approche motivique de l'énumération de fibrés vectoriels

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devant le jury composé de :

Monsieur JEAN FASEL
PROFESSEUR DES UNIVERSITES, UNIVERSITE GRENOBLE ALPES, Directrice de thèse
Monsieur BAPTISTE CALMES
MAITRE DE CONFERENCES HDR, UNIVERSITE D'ARTOIS, Examinatrice
Monsieur FREDERIC DEGLISE
DIRECTEUR DE RECHERCHE, CNRS DELEGATION CENTRE-EST, Examinatrice
Monsieur LOUIS FUNAR
DIRECTEUR DE RECHERCHE, CNRS DELEGATION ALPES, Examinatrice
Monsieur MATTHIAS WENDT
PROFESSEUR DES UNIVERSITES, UNIV. BERGISCHE DE WUPPERTAL - ALLEMAGNE, Rapportrice
Madame KIRSTEN WICKELGREN
PROFESSEUR DES UNIVERSITES, UNIVERSITE DE DUKE - ETATS-UNIS, Rapportrice

Abstract

In this thesis, we establish, via obstruction theory in motivic homotopy theory, some enumeration results on vector bundles of rank d over a smooth affine k-algebra A of dimension d for a base field k, in analogy with some results of James-Thomas [**47**]. In the rank d case, we recover in particular results of Suslin and Bhatwadekar on cancellation of such vector bundles. Admitting a conjecture of Asok and Fasel, we prove cancellation of such modules of rank $d-1$ if the base field k is algebraically closed. Using similar methods, we also obtain cancellation results for symplectic vector bundles of critical rank.

Key words. Motivic homotopy theory, vector bundles, Moore-Postnikov systems, obstruction theory, algebraic K-theory, Chern classes.

Résumé

Dans ce travail, nous utilisons la théorie de l'obstruction en théorie homotopique des schémas pour obtenir certains résultats d'énumération de fibrés vectoriels sur des algèbres lisses de dimension d sur un corps k fixé. Dans un premier temps, nous énumérons les fibrés vectoriels de rang d sur ces algèbres, obtenant au passage de nouvelles preuves de certains théorèmes de Suslin et Bhatwadekar. Nous étudions ensuite les fibrés de rang $d-1$, prouvant au passage une conjecture de Suslin en admettant une conjecture de Asok et Fasel. Finalement, nous utilisons des méthodes similaires pour prouver des résultats de simplification pour des fibrés symplectiques de rang critique.

Mots-clés. Théorie de l'homotopie motivique, fibrés vectoriels, systèmes Moore-Postnikov, théorie de l'obstruction, K-théorie algébrique, classes de Chern.

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Notations

Below we list some abbreviations and symbols we will use, which we usually don't explain further in the text; other abbreviations and symbols will be introduced along the way with explanations.

Symbol	Meaning
$\mathcal{S}\mathsf{et}$	the category of sets
$\mathbf{\Delta}$	the cosimplicial indexing category
$\mathcal{C}\mathsf{at}$	the category of all (small) categories
$\mathcal{T}\mathsf{op}$	the category of all (nice, e.g. compactly generated weakly Hausdorff) topological spaces
$\mathcal{S}\mathsf{ch}$	the category of schemes
$\mathcal{S}\mathsf{m}_S$	the category of smooth schemes over a fixed base scheme S
$\mathcal{G}\mathsf{r}$	the category of groups
$\mathcal{A}\mathsf{b}$	the category of abelian groups
$\mathcal{M}\mathsf{od}$	the category of modules over a fixed ring
$\mathcal{Q}\mathsf{coh}$	the category of quasi-coherent sheaves over a scheme (or even a locally ringed space)
ker	the kernel of a morphism (in an abelian category)
im	the image of a morphism (in an abelian category)
coker	the cokernel of a morphism (in an abelian category)
$\mathsf{s}\mathcal{S}\mathsf{et}$	the category of simplicial sets
$\mathcal{K}\mathsf{an}$	the category of Kan complexes
$\mathsf{s}\mathcal{G}\mathsf{r}$	the category of simplicial groups
$\mathsf{s}\mathcal{A}\mathsf{b}$	the category of simplicial abelian groups
$\mathsf{c}\mathcal{M} = \mathcal{M}^{\mathbf{\Delta}}$	the category of cosimplicial objects of a category $\mathcal{M}$
$\mathsf{s}\mathcal{M} = \mathcal{M}^{\mathbf{\Delta}^{\mathrm{op}}}$	the category of simplicial objects of a category $\mathcal{M}$
$\mathcal{M}_* = (* \downarrow \mathcal{M})$	the (comma) category of pointed objects in a category $\mathcal{M}$ (when $\mathcal{M}$ admits a final object $*$)
$\mathrm{Ho}(\mathcal{M})$	the homotopy category of a model category $\mathcal{M}$
$\mathcal{H}$	Ho $\mathsf{s}\mathcal{S}\mathsf{et}$, where $\mathsf{s}\mathcal{S}\mathsf{et}$ is equipped with the standard Kan-Quillen model structure
$\mathcal{H}_*$	Ho $\mathsf{s}\mathcal{S}\mathsf{et}_*$
$(\mathcal{C}, \tau)$ or $\mathcal{C}_\tau$	a (small) Grothendieck site with underlying category $\mathcal{C}$ and a Grothendieck topology τ
$\mathcal{P}\mathsf{re}(\mathcal{C})$	the category of presheaves (of sets) on a (small) category $\mathcal{C}$
$\mathcal{S}\mathsf{hv}(\mathcal{C}_\tau)$	the category of sheaves on a (small) site $(\mathcal{C}, \tau)$
$\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})$	the category of simplicial presheaves on a (small) category $\mathcal{C}$
$\mathsf{s}\mathcal{S}\mathsf{hv}(\mathcal{C}_\tau)$	the category of simplicial sheaves on a (small) site $(\mathcal{C}, \tau)$
$\mathcal{P}\mathsf{re}.\mathcal{A}\mathsf{b}(\mathcal{C})$	the category of presheaves of abelian groups on a (small) category $\mathcal{C}$
$\mathcal{S}\mathsf{hv}.\mathcal{A}\mathsf{b}(\mathcal{C}_\tau)$	the category of sheaves of abelian groups on a (small) site $(\mathcal{C}, \tau)$
$\mathsf{s}\mathcal{P}\mathsf{re}.\mathcal{A}\mathsf{b}(\mathcal{C})$	the category of simplicial presheaves of abelian groups on a (small) category $\mathcal{C}$
$\mathsf{s}\mathcal{S}\mathsf{hv}.\mathcal{A}\mathsf{b}(\mathcal{C}_\tau)$	the category of simplicial sheaves of abelian groups on a (small) site $(\mathcal{C}, \tau)$
$\mathcal{S}\mathsf{pt}_T(\mathcal{M})$	the category of T-spectra on a pointed model category $\mathcal{M}$ (with $T \in \mathcal{M}$ a parameter object)
$\mathcal{S}\mathsf{pt}_T(\mathcal{C}_\tau)$	abbreviation for $\mathcal{S}\mathsf{pt}_T(\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})_*)$, for some site $(\mathcal{C}, \tau)$ and a parameter object $T \in \mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})_*$
$\mathrm{Map} = \mathrm{Map}_{\mathcal{M}}$	the simplicial mapping space functor $\mathcal{M}^{\mathrm{op}} \times \mathcal{M} \to \mathsf{s}\mathcal{S}\mathsf{et}$ on a simplicial model category $\mathcal{M}$
$\underline{\mathrm{Hom}}$	the internal-hom functor $\mathcal{M}^{\mathrm{op}} \times \mathcal{M} \to \mathcal{M}$ of a category $\mathcal{M}$ (if it exists)
$\cong$	isomorphism (in a category or with respect to some structure)
$A \simeq B$	exhibiting a weak equivalence between the two objects A and B (in a model category)
$f \simeq g : A \to Z$	exhibiting a homotopy between the two morphisms f and g (in a model category)
H/n	$H/nH = \mathrm{coker}(H \xrightarrow{n} H)$, H an abelian group, $n \in \mathbb{Z}$ (so $k^\times/2 = k^\times/(k^\times)^2$, k a field)
${}_nH$	$\ker(H \xrightarrow{n} H)$, H an abelian group, $n \in \mathbb{Z}$
iff	if and only if

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Introduction (in English)

The goal of this thesis is to study cancellation properties of finitely generated projective modules of critical rank over smooth affine algebras. Here, by critical rank we mean that the rank is equal to the dimension of the base ring. One easily translates this using the language of algebraic geometry to statements about vector bundles over smooth affine varieties.

We begin with describing some classical results in commutative ring and module theory. Let R be a commutative ring, and P a finitely generated projective R-module (of constant rank n). We say that the projective module P is *cancellative* if for any $m > 0$, $P \oplus R^m \cong Q \oplus R^m$ for some R-module Q implies $P \cong Q$. Note that if R is a local ring, then any finitely generated projective R-module is cancellative (since such modules are always free and the isomorphism class of a finitely generated free module is uniquely determined by its rank in that case).

We have the *Serre splitting theorem*, which says that for a finitely generated projective module P over a commutative noetherian ring R of Krull dimension d, if its rank is at least $d+1$, then it has a free summand of rank 1; and *Bass cancellation theorem*: any finitely generated projective module P over R of rank at least $d+1$ is cancellative. The results of Serre and Bass are optimal, in the sense that there are commutative rings R of Krull dimension d and projective R-modules of rank d which are indecomposable and not cancellative.

It's interesting to understand further cancellation properties of projective modules of lower rank. In order to use nice results in algebraic geometry, we restrict to the case when the commutative ring A is a k-algebra, k is a field. Even for k algebraically closed ($k = \bar{k}$), Mohan Kumar **[52]** gives a negative answer in the case when the rank $n = d - 2$ is prime ($d = \dim A$): there exists a rank $n = d - 2$ stably free module which is not free.

We are thus left to deal with $n = d$ and $n = d - 1$, which are the problems we will explore in this thesis. For the case $n = d$, Suslin **[87]** confirms cancellation in the case $k = \bar{k}$. This result was later extended to the case where the base field is a C_1-field by Bhatwadekar **[19]**. For the case $n = d - 1$, Fasel-Rao-Swan **[30]** confirm cancellation for stably free modules in the case $k = \bar{k}$ (with other mild conditions).

We push the problems forward further by dressing with homotopy theoretic clothes, namely, we will use the modern tools of $\mathbb{A}^1$-homotopy theory (or motivic homotopy theory) initiated in Morel-Voevodsky **[69]** and further developped in Morel **[68]**. As optical apparatus can "see" something we can't see by naked eyes, this homotopical apparatus can detect some fascinating phenomenon which is hardly to find otherwise. Precisely, we explore cancellation properties of vector bundles using obstruction theory in $\mathbb{A}^1$-homotopy theory, following some ideas in **[7, 9]**. This reflects the fact that algebraic geometry and topology share a long history of interaction, cross-fertilization and competition.

There are the results of James-Thomas **[47]** on similar problems in the topological situation. The motivic homotopy theory is a suitable framework that allows us to make similar operations in algebraic geometry. We extend their results to the motivic homotopy setting, namely, we identify certain set of homotopy lifting classes with nice source and target with the cokernel of a certain map of abelian groups associated to some $\mathbb{A}^1$-homotopy classes. We then give a formula for that map, following the method of **[47]**. This is essentially a study of the derived mapping space in $\mathbb{A}^1$-homotopy theory (see Corollary 4.1.2), taking advantage of the fact that stable things are abelian group objects even in the *unstable* motivic homotopy category.

Combining with the Suslin matrix construction and some known results on the cohomology of certain motivic spaces, we are able to show that the image of that map contains $n!$ multiple of the target abelian group, which is the cohomology of the variety Spec A with coefficients in some Milnor (or Milnor-Witt) K-theory sheaf, by manipulating some characteristic classes. Hence the set of homotopy lifting classes is a singleton if the target abelian group is $n!$-divisible. It's worth to remark that Suslin's matrix construction plays a very important role here—it provides *sufficiently many elements* in the target abelian group which lie in the image of the map in question.

Further, using recent results—the Rost-Schmid complex and the motivic Bloch-Kato conjecture, one can show that the target cohomology group is indeed $n!$-divisible in many cases (e.g. when all elements in the residue fields at all closed points of Spec A are $n!$-th power).

In the $n = d$ case, we will mostly focus on the odd rank case. For the even rank case, there are some further difficulties. On the one hand, as the $\mathbb{A}^1$-fundamental group of BGL_n is non-trivial (i.e. BGL_n is not $\mathbb{A}^1$-simply connected), the relevant $\mathbb{A}^1$-homotopy fiber sequence we get via $\mathbb{A}^1$-homotopical obstruction theory is not principal hence do not fall into our framework as in **[47]**; so we make the compromise that we restrict to enumeration for *oriented* vector bundles. On the other hand, the first nontrivial homotopy sheaf of the relevant space is more difficult to study, even in the case of oriented bundles. Nevertheless, it's still quite useful to get results using BSL_n etc. instead of BGL_n, hence studying enumeration problem for non-stable *oriented* vector bundles. So we will mostly discuss enumeration problem for oriented vector bundles.

The main $\mathbb{A}^1$-homotopy foundation comes from Morel's work **[68]** where the base fields are all assumed to be perfect (and infinity, but later, Hogadi-Kulkarni **[42]** gave a published version of Gabber's presentation lemma for finite fields,

which confirms that Morel's results are indeed also true for finite fields), but every field k contains a perfect (prime) subfield k_0, and since the k-group schemes $\mathrm{GL}_n/k, \mathrm{SL}_n/k$ are extended from k_0, our arguments in the text *hold for a general base field* k as well; cf. [**11**, Comments on the proof of Theorem 3.1.7]. To make the statements more concise, we just assume everywhere that k is perfect.

Overview of the main results.

Our main result in the $n = d$ case is the following (for more details, see Theorems 4.2.1 and 4.2.3).

Theorem. *Let k be a field, A a smooth affine k-algebra of odd Krull dimension $d \geqslant 3$, and $X = \mathrm{Spec}(A)$, let ξ be a stable vector bundle over X, whose classifying map is still denoted $\xi : X \to \mathrm{BGL}$. Let $\varphi_* : [X, \mathrm{BGL}_d]_{\mathbb{A}^1} \to [X, \mathrm{BGL}]_{\mathbb{A}^1}$ be the stabilizing map. Then there is a bijection*

$$\varphi_*^{-1}(\xi) \cong \mathrm{coker}\big(\mathrm{K}_1(X) \xrightarrow{\Delta(c_{d+1},\xi)} \mathrm{H}^d(X; \mathbf{K}^{\mathrm{M}}_{d+1})\big),$$

where the homomorphism $\Delta(c_{d+1}, \xi)$ is given as follows: for $\beta \in \mathrm{K}_1(X) = [X, \Omega\mathrm{BGL}]_{\mathbb{A}^1} = [X, \mathrm{GL}]_{\mathbb{A}^1}$,

$$\Delta(c_{d+1}, \xi)\beta = (\Omega c_{d+1})(\beta) + \sum_{r=1}^{d}((\Omega c_r)(\beta)) \cdot c_{d+1-r}(\xi).$$

Here, $(\Omega c_i)(\beta) \in \mathrm{H}^{i-1}(X; \mathbf{K}^{\mathrm{M}}_i)$ for $i = 1, \cdots, d+1$ are the Chern classes of β and $c_i(\xi)$ are the ordinary Chern classes of ξ.

In other words, given a vector bundle ξ of rank d over X, then the set of isomorphism classes of vector bundles ν such that $\nu \oplus \mathscr{O}_X^m \cong \xi \oplus \mathscr{O}_X^m$ (for some m) is in bijection with $\mathrm{coker}\Delta(c_{d+1}, \xi)$.

In case A is of even dimension $d \geqslant 2$ and the base field is of cohomological dimension at most 2, the same results as above hold for oriented bundles (see Theorem 4.2.11).

As a corollary, we see that if k is algebraically closed or of cohomological dimension at most 1 then all rank d vector bundles are cancellative (Theorem 4.2.9), hence recovering the cancellation theorems of Suslin and Bhatwadekar. A similar result holds for d even and oriented vector bundles (see Theorems 4.2.11 and 4.2.13).

It's also possible to explore the idea to get further cancellation results in the rank $d-1$ case, which is more difficult. For the moment we only obtain a conditional cancellation result by a study of a 2-stage Moore-Postnikov factorization, in which the first stage is (unconditional and) very similar to the results in the rank d case and requires a condition on the characteristic of the base field k (namely, $\mathrm{char}(k) = 0$ or $\mathrm{char}(k) \geqslant d$; see Theorem 4.3.5) to ensure that the isomorphism class of a stable bundle has at most one lifting to the first stage in the tower; for the second stage, we invoke a conjecture of Asok-Fasel (Conjecture 4.3.6) describing $\pi_d^{\mathbb{A}^1}(\mathbb{A}^d \setminus 0)$: if k be a perfect field with $\mathrm{char}(k) \neq 2$, then there is an exact sequence (at least after d-fold contractions)

$$\mathbf{K}^{\mathrm{M}}_{d+2}/24 \to \pi_d^{\mathbb{A}^1}(\mathbb{A}^d \setminus 0) \to \mathbf{GW}^d_{d+1} \to 0. \tag{4.13}$$

This gives a vanishing result on some cohomology group and ensures that the isomorphism class of the lifting map from the first stage to the second stage in the tower (exists and) is unique, which again involves studying the derived mapping space in $\mathbb{A}^1$-homotopy theory, taking advantage of the abelian group object structures.

The (conditional) cancellation result in the $n = d - 1$ case is the following (see Theorem 4.3.8).

Theorem. *Assume the base field k is algebraically closed. Let A be a smooth k-algebra of Krull dimension $d \geqslant 3$ and assume $\mathrm{char}(k) = 0$ or $\mathrm{char}(k) \geqslant d$. Let $X = \mathrm{Spec}\ A$. If the Asok-Fasel Conjecture 4.3.6 holds, then every oriented rank $d-1$ vector bundle over X is cancellative: In other words, given an oriented vector bundle ξ of rank $d-1$ over X, then any vector bundle ν such that $\nu \oplus \mathscr{O}_X^n \cong \xi \oplus \mathscr{O}_X^n$ is isomorphic to ξ.*

In a similar fashion, one can deduce enumeration results on symplectic vector bundles as follows (Theorems 4.4.2 and 4.4.3).

Theorem. *Let k is a perfect field with $\mathrm{char}(k) \neq 2$ and $\mathrm{c.d.}_2(k) \leqslant 2$, $X = \mathrm{Spec}(A)$ a smooth affine k-scheme of dimension $d = 2n + 1 \geqslant 3$. Let ξ be a stable symplectic vector bundle over X, whose classifying map is still denoted by $\xi : X \to \mathrm{BSp}$.*

(1) *There is an abelian group homomorphism*

$$\Delta(b_{n+1}, \xi) : \mathrm{KSp}_1(X) = [X, \Omega\mathrm{BSp}]_{\mathbb{A}^1} \to [X, \mathrm{K}(\mathbf{K}^{\mathrm{MW}}_{2n+2}, 2n+1)]_{\mathbb{A}^1} = \mathrm{H}^{2n+1}(X; \mathbf{K}^{\mathrm{MW}}_{2n+2})$$

associated to the class $\xi \in [X, \mathrm{BSp}]_{\mathbb{A}^1}$ given by

$$\Delta(b_{n+1}, \xi)(\beta) = (\Omega b_{n+1})(\beta) + \sum_{r=1}^{n}((\Omega b_r)(\beta)) \cdot b_{n+1-r}(\xi),$$

whose cokernel is in bijection with the set of rank $d - 1 = 2n$ symplectic vector bundles over X.

(2) *We have $(2 \cdot d!) \cdot \mathrm{H}^d(X; \mathbf{K}^{\mathrm{M}}_{d+1}) \subset \mathrm{im}(\tau\Delta(b_{n+1}, \xi))$.*

(3) *There is a surjective homomorphism*

$$\mathrm{H}^d(X; \mathbf{K}^{\mathrm{M}}_{d+1})/2 \cdot d! \twoheadrightarrow \mathrm{coker}\Delta(b_{n+1}, \xi).$$

(4) *If* $\mathrm{H}^d(X;\mathbf{K}^{\mathrm{M}}_{d+1})$ *is* $d!$*-divisible, then* $\mathrm{coker}\Delta(b_{n+1},\xi)=0$. *So in this case, any rank* $2n=d-1$ *symplectic vector bundle is cancellative. Moreover, the map*

$$(b_{n+1})_*:\pi_1(\mathrm{RMap}(X,\mathrm{BSp}),\xi)\to\pi_1(\mathrm{RMap}(X,\mathrm{K}(\mathbf{K}^{\mathrm{MW}}_{d+1},d+1)),0)=\mathrm{H}^d(X;\mathbf{K}^{\mathrm{MW}}_{d+1})$$

is surjective for every $\xi\in[X,\mathrm{BSp}]_{\mathbb{A}^1}$.

(5) *If* $\mathrm{c.d.}(k)\leqslant 1$*, then every rank* $2n=d-1$ *symplectic vector bundle over* X *is cancellative.*

In particular, for $d=2n+1\geqslant 3$ *and a given symplectic vector bundle* ξ *of rank* $d-1$ *over* X*, if* $\mathrm{c.d.}(k)\leqslant 1$*, then any symplectic vector bundle* ν *such that* $\nu\oplus H^m\cong\xi\oplus H^m$ *is in fact isomorphic to* ξ*, where* H *is the trivial rank* 2 *symplectic vector bundle over* X.

Structure of the thesis.

In Chapter 1, we discuss some of the categorical results and some abstract homotopy theory as well as some homotopy theory of simplicial sets we need later. In particular, the wonderful results of Quillen (and streamlined by some others) on homotopy (co)limits in model categories and pre-triangulated categories and pointed model categories are discussed in some detail; these basically allow us to perform most things we can do about topological spaces to many abstract setting (hence *abstract* homotopy theory). Some results of Hirschhorn about (left) Bousfield localization of model categories are also presented. This chapter, which we hope to be useful and of interest in its own right, is mainly to set language and to fix notations.

Chapter 2 is a homotopical investigation of simplicial (pre)sheaves, we mostly follow [**50, 48**]. We first give a relatively complete description of Grothendieck topologies (using covering sieves) and Grothendieck sites in great generality, which allows very conceptual understanding of sheaves and sheaf conditions on a Grothendieck site. Then we discuss the projective and injective model structures on the simplicial (pre)sheaf category, which is pure categorical and has nothing to do with the Grothendieck topology of the site. Definitely we want the Grothendieck topology to play a role. Indeed, the Grothendieck topology is essential on defining local weak equivalences which is part of Jardine's local model structure on the simplicial (pre)sheaf category, and local fibrations which are very important in our discussion on homotopical descent of simplicial (pre)sheaves (as in Section 2.7). Section 2.7 is a presentation of the main results of [**26**] on hypercoverings, which in particular give a model structure Quillen equivalent to Jardine's local model structure and we give in Theorem 2.7.13 various equivalent conditions for a simplicial (pre)sheaf to satisfy descent. §2.8 is about torsors and their homotopical classifications and §2.9 gives a homotopical interpretation of sheaf cohomology which is very conceptual and is very natural in Jardine's general homotopical framework. Note that we don't always follow the historical development, rather, we present things in the way which we believe to be the most concise manner.

As a reward of this long odyssey, we are finally able to define the $\mathbb{A}^1$-homotopy theory (or motivic homotopy theory) in a few lines via left Bousfield localizations and easily deduce many important results in Chapter 3, once we understand well the Nisnevich topology that we use. Basic properties and constructions are given in §3.1, with some emphasis on the importance of Morel's notions of strongly/strictly $\mathbb{A}^1$-invariant sheaves of (abelian) groups, mainly following the recent development as in [**68, 8, 11**]. §3.2 is aimed at giving different models of some motivic spaces (e.g. the motivic spheres $\mathbb{A}^n\setminus 0$) as well as some fundamental tools, like the long exact sequence of sheaves of motivic homotopy groups associated to an $\mathbb{A}^1$-homotopy fibre sequence and the Moore-Postnikov systems in motivic homotopy theory; sampling computations of some sheaves of motivic homotopy groups are given in the end. In §3.3, we collect some recent results of Fabien Morel, Marco Schlichting, Aravind Asok, Marc Hoyois and Matthias Wendt about the so-called *affine BG-property* and affine representability results, which in particular say that the functors assigning vector bundles of a fixed rank as well as torsors for certain algebraic groups over a smooth affine variety are representable by some concrete models in the $\mathbb{A}^1$-homotopy category. §3.4 is a summary of Schlichting's results on Hermitian K-theory. In §3.5, we discuss briefly stable motivic homotopy theory, i.e. the theory on motivic $\mathbb{P}^1$-spectra, as developed in [**50**, Chapter 10] and [**49**].

The last chapter can be viewed as a simplified version of the main results proved in [**25**], where we originally follow closely the methods developed by James-Thomas [**47**]. We believe that our new treatment here is more elegant and reveal the essentials, and also [**47**] could be written in the way as we do here. In §4.1, we first identify the first non-trivial $\mathbb{A}^1$-homotopy sheaf of the $\mathbb{A}^1$-homotopy fiber of the map from non-stable to stable classifying spaces, then we give a description of the homotopy type of some stable classifying spaces and some derived mapping spaces, where we use a little stable motivic homotopy theory. In §4.2, we set up the framework of enumeration of lifting $\mathbb{A}^1$-homotopy classes, following some ideas developed in James-Thomas [**47**] in the classical homotopy setting, yielding enumeration results for non-stable (oriented) vector bundles by some computations and identifying some characteristic classes (with mild conditions). §4.3 gives enumeration results for non-stable (oriented) vector bundles below critical rank, in which case we don't have a complete answer. Precisely, we give a (conditional) cancellation result in the $n=d-1$ case, provided the Asok-Fasel Conjecture 4.3.6 holds; the difficulty is that we need a better understanding of certain $\mathbb{A}^1$-homotopy sheaf of the motivic spheres $\mathbb{A}^n\setminus 0$. In §4.4, we also study cancellation properties of symplectic vector bundles along the same line. In the last section (§4.5), we list some questions related to the present work and possible further development, hopefully to get progress in the near future.

Following the main text, there are three appendices. Appendix A is on Kan extensions, which is very effective in creating adjunctions (but much more than that) and we have used it several times in the text. Appendix B summarizes some results about the étale sites and étale cohomology we need. Appendix C is on unimodular elements in projective modules, we present more and in more general form than we need, just for completeness.

Introduction (en français)

Le but de cette thèse est d'étudier les propriétés de simplification des modules projectifs de génération finie sur des algènres lisses sur un corps de base k. Nous nous intéresserons plus particulièrement aux modules projectifs dont le rang est proche de la dimension de la k-algèbre sur laquelle ils sont définis. En termes géométriques, le problème ci-dessus est de comprendre les propriétés de simplification des fibrés vectoriels de rang proche de celui de la variété affine lisse sur laquelles ils sont définis.

Commençons par rappeler quelques résultats classiques concernant la théorie des modules en algèbre commutative. Soit R un anneau commutatif et soit P un R-module projectif de type fini et de rang constant n. Le module projectif P est dit *simplifiable* si tout isomorphisme de la forme $P \oplus R^m \cong Q \oplus R^m$ pour $m > 0$ implique l'existence d'un isomorphisme $P \cong Q$. Notons par exemple que si R est un anneau local, tout module projectif P est simplifiable (ceci vient du fait que tout module projectif sur un anneau local est libre, et sa classe d'isomorphisme est déterminée par son rang).

Un résultat classique dû à Serre dit que tout module projectif P sur un anneau commutatif R de dimension de Krull d admet un facteur libre de rang 1 si son rang est au moins $d+1$. En d'autres termes, tout module projectif P sur un anneau de dimension de Krull d peut s'écrire $P \cong Q \oplus R^m$ pour un certain module projectif Q de rang d. Par ailleurs, un autre résultat classique dû à Bass, le *théorème de simplification de Bass*, dit que tout module projectif de rang supérieur ou égal à $d+1$ est simplifiable. Les résultats de Serre et Bass sont optimaux, dans le sens qu'il existe des anneaux commutatifs R de dimension de Krull d et des R-modules projectifs de rang d qui sont indécomposables et non simplifiables.

Néanmoins, il est possible de démontrer des résultats plus précis que ceux de Serre et Bass dans des cas particuliers importants. Pour être en mesure d'utiliser toute la puissance de la géométrie algébrique, nous nous restreindrons dès maintenant au cas d'une algèbre commutative A de type fini sur un corps k, supposée de dimension de Krull d. Dans le cas où k est algébriquement clos, Suslin a démontré dans **[87]** que tout module projectif de rang d sur A est simplifiable. Il a ensuite posé la question de savoir s'il était possible de démontrer le même résultat pour des modules de rang plus petit que d. Quelques années plus tard, Mohan Kumar **[52]** a construit explicitement pour tout premier p des exemples de modules projectifs de rang p sur des algèbres lisses sur k de dimension de Krull $p+2$ qui étaient stablement libres mais pas libre. La question de Suslin, restée ouverte depuis la fin des annés 70, est donc réduite à savoir si tout module projectif de rang $d-1$ sur une algèbre lisse de dimension d sur un corps k algébriquement clos est simplifiable. Un premier résultat dans cette direction a été obtenu par Fasel-Rao-Swan, qui ont montré dans dans **[30]** que tout module stablement libre de rang $d-1$ sur une telle algèbre est en fait libre.

Dans ce mémoire, nous habillons le problème ci-dessus avec des habits homotopiques pour obtenir des extensions des résultats ci-dessus, et notamment une preuve de la conjecture de Suslin, malheureusement conditionnée à une conjecture de Asok-Fasel. De manière plus précise, nous utilisons les outils de la théorie $\mathbb{A}^1$-homotopique des schémas (aussi connue sous le nom de théorie de l'homotopie motivique) développée par Morel-Voevodsky (**[68]**) pour obtenir un nouvel éclairage du problème. Cet appareil homotopique est capable de détecter des phénomènes qu'il est difficile de cerner en utilisant les outils dits classiques. Pour ce faire, nous utilisons la théorie de l'obstruction en théorie $\mathbb{A}^1$-homotopique des schémas pour prouver quelques propriétés de simplification pour des modules projectifs, suivant en cela les idées de **[7, 9]**. Cette philosophie illustre une fois de plus la riche interaction entre la géométrie algébrique et la topologie algébrique, qui partagent une longue histoire commune.

En topologie, on note certains résultats classiques (notamment dûs à James-Thomas **[47]**) sur des problèmes de stabilisation de fibrés vectoriels. La théorie de l'homotopie motivique fournit le cadre conceptuel nous permettant de démontrer les mêmes résultats en géométrie algébrique. Nous étendons donc les résultats classiques à ce contexte; plus précisément nous montrons dans certaines situations que l'ensemble des fibrés vectoriels stablement isomorphes à un fibré vectoriel donné est en bijection avec les conoyau d'un homomorphisme explicite entre groupes abéliens qu'il est possible de calculer. Notre méthode consiste essentiellement à étudier de manière précise certains ensembles simpliciaux d'applications, utilisant de façon répétée le fait que certains de ces ensembles sont en fait des groupes abéliens simpliciaux.

Une fois cette partie de nature très topologique sous le sapin, nous passons à une étude cohomologique de certains objets pour tirer de nos résultats généraux des corollaires bien plus précis. Nous utilisons en particulier une fameuse construction de matrices due à Suslin pour démontrer que l'ensemble des fibrés vectoriels stablement isomorphes à un fibré vectoriel donné est de torsion. Nous en déduisons que ce groupe est trivial dans des situations assez générales, obtenant en particulier des nouvelles preuves des fameux théorèmes de Suslin **[87]** et Bhatwadekar **[19]**, qui démontrent que les modules projectifs de rang d sur une algèbre lisse de dimension d sur un corps algébriquement clos (Suslin) ou de dimension cohomologique 1 (Bhatwadekar) sont simplifiables. Nous passons ensuite au cas bien plus difficile où les modules projectifs considérés sont cette fois de dimension $d-1$ sur une algèbre lisse de dimension d sur un corps algébriquement clos. Nous obtenons dans ce cas une preuve que ces modules projectifs sont simplifiables, répondant ainsi

de manière affirmative à la conjecture de Suslin. Malheureusement, cette preuve est conditionnée à une conjecture de Asok-Fasel sur la structure d'un certain faisceau d'homotopie motivique d'une sphère algébrique.

Pour terminer cette brève introduction, notons que les fondations motiviques qui ont permis les résultats de cette thèse proviennent quasi-exclusivement du livre de Morel **[68]**. Dans cet ouvrage, l'hypothèse que le corps de base est infini parfait est omniprésente. Cependant, des résultats ultérieurs sur le lemme de présentation de Gabber dûs à Hogadi-Kulkarni **[42]** permettent d'étendre les résultats de Morel au cas des corps parfaits, mais pas nécessairement infinis. Du fait que tout corps contient un sous-corps premier qui est parfait, et que les objets motiviques que nous considérons sont définis sur ces sous-corps, nos arguments peuvent être in fine étendus dans le cas de tout corps de base (voir par exemple les arguments de **[11**, commentaires sur la preuve du théorème 3.1.7]).

Survol des résultats principaux.

Notre résultat principal dans le cas des modules projectifs de rang égal à celui de l'algèbre est le suivant (Theorems 4.2.1 and 4.2.3):

Théorème. *Soit k un corps parfait, A une k-algèbre lisse de dimension de Krull impaire $d \geqslant 3$ et $X = \mathrm{Spec}(A)$. Soit encore ξ un fibré vectoriel stable sur X, classifié par une application $\xi : X \to \mathrm{BGL}$, et $\varphi_* : [X, \mathrm{BGL}_d]_{\mathbb{A}^1} \to [X, \mathrm{BGL}]_{\mathbb{A}^1}$ l'application de stabilisation. Alors, on a une bijection*

$$\varphi_*^{-1}(\xi) \cong \mathrm{coker}\big(\mathrm{K}_1(X) \xrightarrow{\Delta(c_{d+1},\xi)} \mathrm{H}^d(X; \mathbf{K}^{\mathrm{M}}_{d+1})\big),$$

où l'homomorphisme $\Delta(c_{d+1}, \xi)$ est défini comme suit: pour $\beta \in \mathrm{K}_1(X) = [X, \Omega\mathrm{BGL}]_{\mathbb{A}^1} = [X, \mathrm{GL}]_{\mathbb{A}^1}$,

$$\Delta(c_{d+1}, \xi)\beta = (\Omega c_{d+1})(\beta) + \sum_{r=1}^{d}((\Omega c_r)(\beta)) \cdot c_{d+1-r}(\xi).$$

Ici, $(\Omega c_i)(\beta) \in \mathrm{H}^{i-1}(X; \mathbf{K}^{\mathrm{M}}_i)$ pour $i = 1, \cdots, d+1$ sont les classes de Chern de β et $c_i(\xi)$ sont les classes de Chern de ξ. En d'autres termes, étant donné un fibré vectoriel ξ de rang d sur X, l'ensemble des classes d'isomorphisme de fibrés vectoriels ν tels que $\nu \oplus \mathscr{O}_X^m \cong \xi \oplus \mathscr{O}_X^m$ pour un certain entier m est en bijection avec $\mathrm{coker}\Delta(c_{d+1}, \xi)$.

Dans le cas où A est de dimension paire $d \geqslant 2$, et que le corps de base est de dimension cohomologique au plus 2, les mêmes résultats sont valides pour des fibrés vectoriels orientés.

Comme nous l'avons évoqué ci-dessus, un corollaire de ce théorème est que les fibrés vectoriels de rang d (les fibrés sont supposés orientés si d est pair) sont simplifiable si le corps de base est de dimension cohomologique au plus 1. Ceci donne une démonstration alternative aux théorèmes de Suslin et Bhatwadekar (Theorem 4.2.9 dans le cas impair, et Theorems 4.2.11 and 4.2.13 dans le cas pair).

Le cas des modules projectifs de rang inférieur à celle de l'algèbre de base est bien plus difficile. Nous obtenons pour le moment un théorème conditionnel en considérant la factorisation en deux étapes de Moore-Postnikov. La première étape est très similaire au cas précédent (et inconditionnelle) mais requiert certaines hypothèses sur la caractéristique du corps de base ($\mathrm{char}(k) = 0$ ou $\mathrm{char}(k) \geqslant d$; voir Theorem 4.3.5). La deuxième étape est plus délicate et repose sur une conjecture d'Asok-Fasel (Conjecture 4.3.6) décrivant de manière précise $\pi_d^{\mathbb{A}^1}(\mathbb{A}^d \setminus 0)$. Si k est un corps parfait tel que $\mathrm{char}(k) \neq 2$, on a une suite exacte (au moins après d-fold contractions)

$$\mathbf{K}^{\mathrm{M}}_{d+2}/24 \to \pi_d^{\mathbb{A}^1}(\mathbb{A}^d \setminus 0) \to \mathbf{GW}^d_{d+1} \to 0. \tag{4.13}$$

On en déduit que certains groupes de cohomologie sont nuls, ce qui permet de démontrer qu'une application de stabilisation est bien injective.

Théorème. *Soit k un corps algébriquement clos et A une k-algèbre lisse de dimension de Krull $d \geqslant 3$. Supposons que $\mathrm{char}(k) = 0$ ou $\mathrm{char}(k) \geqslant d$ Soit finalement $X = \mathrm{Spec}\ A$. Si la conjecture Conjecture 4.3.6 est vérifiée, tout module projectif orienté de rang $d-1$ est simplifiable. En d'autres termes, deux modules projectifs orientés ξ et ν de rang $d-1$ sont stablement isomorphes si et seulement s'ils sont isomorphes.*

Nous terminons ce survol des principaux résultats de ce mémoire par le cas des fibrés symplectiques (Theorems 4.4.2 and 4.4.3).

Théorème. *Soit k un corps parfait tel que $\mathrm{char}(k) \neq 2$ et $\mathrm{c.d.}_2(k) \leqslant 2$. Soit $X = \mathrm{Spec}(A)$ un schéma affine lisse sur k de dimension $d = 2n+1 \geqslant 3$ et ξ un fibré vectoriel symplectique stable sur X, classifié par une application $\xi : X \to \mathrm{BSp}$.*

(1) *Il existe un homomorphisme de groupes abéliens*

$$\Delta(b_{n+1}, \xi) : \mathrm{KSp}_1(X) = [X, \Omega\mathrm{BSp}]_{\mathbb{A}^1} \to [X, \mathrm{K}(\mathbf{K}^{\mathrm{MW}}_{2n+2}, 2n+1)]_{\mathbb{A}^1} = \mathrm{H}^{2n+1}(X; \mathbf{K}^{\mathrm{MW}}_{2n+2})$$

associé à la classe $\xi \in [X, \mathrm{BSp}]_{\mathbb{A}^1}$. Il est donné par

$$\Delta(b_{n+1}, \xi)(\beta) = (\Omega b_{n+1})(\beta) + \sum_{r=1}^{n}((\Omega b_r)(\beta)) \cdot b_{n+1-r}(\xi)$$

et son conoyau est en bijection avec l'ensemble des fibrés vectoriels symplectiques de rang $d-1 = 2n$ sur X stablement isomorphes à ξ.

(2) *On a $(2 \cdot d!) \cdot \mathrm{H}^d(X; \mathbf{K}^{\mathrm{M}}_{d+1}) \subset \mathrm{im}(\tau\Delta(b_{n+1}, \xi))$.*

(3) *Il existe un homomorphisme surjectif*

$$\mathrm{H}^d(X;\mathbf{K}^{\mathrm{M}}_{d+1})/2\cdot d! \twoheadrightarrow \mathrm{coker}\Delta(b_{n+1},\xi).$$

(4) *Si* $\mathrm{H}^d(X;\mathbf{K}^{\mathrm{M}}_{d+1})$ *est* $d!$*-divisible, alors* $\mathrm{coker}\Delta(b_{n+1},\xi)=0$*. Par conséquent, tout fibré vectoriel symplectique de rang* $2n=d-1$ *sur* X *est simplifiable. De plus, l'application*

$$(b_{n+1})_*:\pi_1(\mathrm{RMap}(X,\mathrm{BSp}),\xi)\to\pi_1(\mathrm{RMap}(X,\mathrm{K}(\mathbf{K}^{\mathrm{MW}}_{d+1},d+1)),0)=\mathrm{H}^d(X;\mathbf{K}^{\mathrm{MW}}_{d+1})$$

est surjective pour tout $\xi\in[X,\mathrm{BSp}]_{\mathbb{A}^1}$.

(5) *Si* $\mathrm{c.d.}(k)\leqslant 1$*, tout fibré vectoriel symplectique de rang* $2n=d-1$ *sur* X *est simplifiable. En d'autres termes deux fibrés symplectiques* ξ *et* ν *de rang* $d-1$ *sont stablement isomorphes si et seulement s'ils sont isomorphes.*

Structure de la thèse.

Dans le premier chapitre, nous discutons certains résultats catégoriels et homotopiques dont nous aurons besoin plus tard. Nous passons également en revue certains aspects de la théorie homotopique des ensembles simpliciaux, et en particulier les magnifiques résultats de Quillen (et autres) sur les (co-)limites homotopiques dans les catégories modèles (pointées) et les catégories prétriangulées. Nous présentons également quelques résultats de Hirschhorn sur la localisation (à gauche) de Bousfield de catégories modèles. Ce chapitre, que nous espérons intéressant et utile, a essentiellement ppour but de fixer les notations et le language utilisé dans le reste de la thèse.

Le second chapitre est dédié à l'étude des (pré)faisceaux simpliciaux, d'après **[50, 48]**. Nous commençons par donner une description relativement complète des topologies et sites de Grothendieck en toute généralité, permettant ainsi une bonne compréhension des faisceaux sur ces sites. Nous discutons ensuite les structures modèle injective et projective sur les préfaisceaux simpliciaux, qui sont purement catégorielles et indépendantes de la topologie de Grothendieck considérée. Néanmoins, la topologie sur le site joue un rôle essentiel dans la définition des équivalences faibles locales, qui sont centrales dans la structure modèle locale de Jardine sur la catégorie des (pré)faisceaux simpliciaux, ainsi que dans la définition des fibrations locales qui sont importantes dans notre discussion de la descente homotopique des (pré)faisceaux simpliciaux. Section 2.7 est une présentation des principaux résultats de **[26]** sur les hyper-recouvrements, qui donnent en particulier une structure modèle Quillen-équivalente à celle de Jardine. Nous donnons dans Theorem 2.7.13 plusieurs conditions équivalentes pour qu'un (pré)faisceau simplicial satisfasse cette notion de descente. Le paragraphe §2.8 est une classification homotopique de certains torseurs, alors que §2.9 donne une interprétation homotopique très conceptuelle de la cohomologie des faisceaux. Notons que nous ne suivons pas toujours les développements historiques des sujets présentés, optant plutôt pour le traitement qui nous semble le plus concis.

La récompense de notre longue odyssée arrive au chapitre 3, où nous définissons enfin la catégorie $\mathbb{A}^1$-homotopique (ou théorie de l'homotopie motivique) en quelques lignes via une localisation de Bousfield. Nous énonçons quelques résultats importants, qui sont conséquence d'une bonne compréhension de la topologie de Nisnevich. Avant cela, nous nous concentrons sur quelques propriétés et constructions de base dans la section §3.1, où nous nous appuyons sur les notions de faisceaux fortement/strictement $\mathbb{A}^1$-invariant dues à Morel, suivant en cela les développements récents de **[68, 8, 11]**. La section §3.2 a pour but de donner différents modèles explicites de certains espaces motiviques, telles que les sphères algébriques $\mathbb{A}^n\setminus 0$, ainsi que de rappeler certains outils tels que les suites exactes longues associées aux suites de fibrations $\mathbb{A}^1$-homotopiques ou les systèmes de Moore-Postnikov. Nous présentons également certains calculs de faisceaux homotopiques motiviques en fin de section. Le paragraphe §3.3 collecte certains résultats récents de Morel, Schlichting, Asok, Hoyois et Wendt sur la propriété BG et la représentabilité affines; montrant en particulier que les foncteurs associant l'ensemble des classes d'isomorphisme de torseurs sous un groupe bien choisi à un schéma affine lisse sont représentables dans la catégorie $\mathbb{A}^1$-homotopique. Nous terminons dans les section §3.4 et §3.5 avec un bref résumé sur les résultats de Schlichting sur la K-théorie hermitienne et un survol de la catégorie homotopique des $\mathbb{P}^1$-spectres, telle que développée dans [**50**, Chapter 10] et [**49**].

Le dernier chapitre peut être vu comme une version simplifiée des résultats principaux de **[25]**, où nous avions suivi étrroitement les méthodes développées par James-Thomas [**47**]. Nous pensons que ce nouvel éclairage est plus élégant et dévoile l'essentiel des arguments. Nous notons que [**47**] pourrait être réécrit de la manière dont nous le faisons. Dans la section §4.1, nous commençons par identifier le premier faisceau d'homotopie motivique non trivial de la fibre $\mathbb{A}^1$-homotopique de l'application entre les espaces classifiants stables et instables, puis nous donnons une description du type d'homotopie de certains espaces classifiants stables, ainsi que de certains espaces dérivés. Nous utilisons pour ce faire un peu de théorie homotopique stable. Dans §4.2, nous construisons le cadre général qui nous permet d'énumérer les relèvements de classes $\mathbb{A}^1$-homotopiques, suivant en cela certaines idées de James-Thomas [**47**] dans le cadre classique. Ceci nous permet d'identifier quelques classes caractéristique (sous quelques conditions supplémentaires) et d'obtenir notre résultat principal sur les fibrés vectoriels de rang égal à celui de l'algèbre sur laquelle ils sont définis. Dans §4.3, nous passons à l'étude des fibrés vectoriels de rang inférieur et nous obtenons une réponse conditionnée à une conjecture d'Asok-Fasel (Conjecture 4.3.6). La principale difficulté est d'obtenir une meilleure compréhension d'un certain faisceau d'homotopie motivique de la sphère $\mathbb{A}^n\setminus 0$, et nous utilisons la conjecture ci-dessus pour parvenir à nos fins. Enfin, nous étudions dans le paragraphe §4.4 les fibrés symplectiques, utilisant les techniques développées dans les sections précédentes. Nous terminons par quelques questions tirées du présent travail, ainsi que par quelques directions de recherche que nous espérons suivre dans le futur.

On trouvera trois appendices à la fin de ce mémoire. Le premier porte sur les extensions de Kan, qui est une méthode particulièrement efficace pour créer des adjonctions (et bien plus) que nous avons utilisée plusieurs fois dans le texte. Le second résume certains résultats sur le site et la cohomologie étales. Finalement, le dernier appendice est à propos des éléments unimodulaires dans les modules projectifs. Nous donnons plus de résultats que strictement nécessaire, ceci en vue d'être le plus exhaustif possible.

Chapter 1

Preliminaries on model categories and simplicial sets

In this chapter, we discuss some of the categorical constructions we will use later, for the convenience and for setting languages. We set down the framework of abstract homotopy theory and simplicial sets we need in the chapters that follow. We are mainly following **[33, 40, 45]**.

1.1. Some notions in category theory

We assume the basics of category theory. We recall some results to fix notations.

Let $\mathcal{I}, \mathcal{E}$ be categories with $\mathcal{I}$ small, $X \in \mathcal{E}^{\mathcal{I}}$ an $\mathcal{I}$-diagram in $\mathcal{E}$, i.e. X is a functor $X : \mathcal{I} \to \mathcal{E}$. Then the *colimit* $\operatorname{colim}_{\mathcal{I}} X$ and the *limit* $\lim_{\mathcal{I}} X$ (provided that they exist) are characterized as follows: there are isomorphisms

$$\mathcal{E}(\operatorname{colim}_{\mathcal{I}} X, Y) \cong \lim_{i \in \mathcal{I}^{\mathrm{op}}} \mathcal{E}(X_i, Y),$$

$$\mathcal{E}(W, \lim_{\mathcal{I}} X) \cong \lim_{i \in \mathcal{I}} \mathcal{E}(W, X_i),$$

natural in $W, Y \in \mathcal{E}$. In other words, for $e \in \mathcal{E}$, denote $h^e = \mathcal{E}(e, -) : \mathcal{E} \to \mathcal{S}\mathsf{et}$ and $h_e = \mathcal{E}(-, e) : \mathcal{E}^{\mathrm{op}} \to \mathcal{S}\mathsf{et}$, then $\operatorname{colim}_{\mathcal{I}} X = e$ in $\mathcal{E}$ exists iff $\lim_{\mathcal{I}} h^X = h^e$ in $\mathcal{S}\mathsf{et}^{\mathcal{E}}$, and $\lim_{\mathcal{I}} X = e$ in $\mathcal{E}$ exists iff $\lim_{\mathcal{I}} h_X = h_e$ in $\mathcal{S}\mathsf{et}^{\mathcal{E}^{\mathrm{op}}}$. [1]

We only consider (co)limits indexed by small categories. A category $\mathcal{E}$ is said to be *complete* resp. *cocomplete* if all (small) limits resp. colimits in $\mathcal{E}$ exist. It's said to be *bicomplete* if it is complete and cocomplete. Mostly, we will ignore set-theoretic problems whenever it may be a problem. Some categorical notions are not well-known, e.g. Reedy categories, in this case, we give precise references about them along the way.

Yoneda lemma says that, there are canonical bijections

$$\mathrm{Fun}(h^e, F) \xrightarrow{\cong} F(e), \mathrm{Fun}(h_e, G) \xrightarrow{\cong} G(e)$$

for functors $F : \mathcal{E} \to \mathcal{S}\mathsf{et}, G : \mathcal{E}^{\mathrm{op}} \to \mathcal{S}\mathsf{et}$.

One can define the *initial object of* $\mathcal{E}$, denoted $\varnothing$, to be the colimit of the empty diagram in $\mathcal{E}$, and dually, the *final object*, or *terminal object*, of $\mathcal{E}$, denoted $*$, the limit of the empty diagram in $\mathcal{E}$, whenever they exist. If the map $\varnothing \to *$ is an isomorphism, the common value is called a *zero object*, or *null object* of $\mathcal{E}$.

For a functor $X : \mathcal{I} \to \mathcal{E}$, if the index category $\mathcal{I}$ has an initial object $\varnothing$, then $\lim_{\mathcal{I}} X = X(\varnothing)$; if $\mathcal{I}$ has a final object $*$, then $\operatorname{colim}_{\mathcal{I}} X = X(*)$.

If a zero object exists, we call $\mathcal{E}$ a *pointed category*. In this case, for any two objects a, b, there is a unique morphism $0 = 0_{a,b} : a \to b$ which factors through a zero object, called the *zero morphism* from a to b (doesn't depend on the choice of the zero object through which it factors). Also, for any morphism $f : a \to b$, we define the *cokernel*, or *cofiber*, of f, to be the coequalizer $g : b \to c$ of the maps $f, 0_{a,b} : a \rightrightarrows b$, and the *kernel*, or *fiber*, of f, to be the equalizer $g : e \to b$ of the maps $f, 0_{a,b} : a \rightrightarrows b$, whenever they exist.

In $\mathcal{S}\mathsf{et}$, *filtered colimits* commute with finite limits, *reflexive coequalizers* commute with finite products.

Let Q be the following commutative diagram in $\mathcal{E}$:

$$\begin{array}{ccc} A & \xrightarrow{p} & B \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle u} \\ C & \xrightarrow{v} & D. \end{array}$$

We have (by checking universal properties)

- If $B \times C$ exists, then Q is a pullback diagram iff $A \longrightarrow B \times C \overset{u\circ p_1}{\underset{v\circ p_2}{\rightrightarrows}} D$ is an equalizer diagram.
- If $B \coprod C$ exists, then Q is a pushout diagram iff $A \overset{\iota_1 \circ p}{\underset{\iota_2 \circ q}{\rightrightarrows}} B \coprod C \longrightarrow D$ is a coequalizer diagram.

An object A of $\mathcal{E}$ is said to be an *(abelian) cogroup object* if $h^A : \mathcal{E} \to \mathcal{S}\mathsf{et}$ factors through the forgetful functor $\mathcal{G}\mathsf{r} \to \mathcal{S}\mathsf{et}$ ($\mathcal{A}\mathsf{b} \to \mathcal{S}\mathsf{et}$). Thus there are natural transformations $h^A \times h^A \xrightarrow{m^A} h^A, h^A \to h^A, * \to h^A$ (and $h^A \times h^A \to h^A \times h^A$) making some diagrams of functors commutative. By Yoneda lemma (and e.g. $h^A \times h^A = h^{A \coprod A}$), these correspond morphisms $A \to A \coprod A$ (comultiplication) etc., whenever these objects exist. We sometimes say the cogroup structure is given by the comultiplication morphism $A \to A \coprod A$ even if we don't know if $A \coprod A$ exists (if it doesn't exist, we understand this expression as an abbreviation for the natural transformation of functors). Then the *left coaction* of the cogroup object A on an object X is given by natural transformations $h^A \times h^X \to h^X$ etc, and we say the coaction is

[1] Note that the limits in the functor category always exist, the point here is that they are (co)representable. We can write this as $\lim_{\mathcal{I}^{\mathrm{op}}} h^X = h^{\operatorname{colim}_{\mathcal{I}} X}, \lim_{\mathcal{I}} h_X = h_{\lim_{\mathcal{I}} X}$, so for example, we have $h^a \times_{h^c} h^b = h^{a \coprod_c b}, h_a \times_{h_c} h_b = h_{a \times_c b}$.

given by the coaction morphism $X \to A \coprod X$ even if we don't know if $A \coprod X$ exists. The notion of *right coaction* is defined similarly. For two morphisms $a, b : A \to X$ viewed as elements of the group $\mathcal{E}(A, X)$, their product in $\mathcal{E}(A, X)$ corresponds to the composition $h^X \xrightarrow{\Delta} h^X \times h^X \xrightarrow{a\times b} h^A \times h^A \xrightarrow{m^A} h^A$.

An object G of $\mathcal{E}$ is said to be an *(abelian) group object* if $h_G : \mathcal{E}^{\mathrm{op}} \to \mathcal{S}\mathsf{et}$ factors through the forgetful functor $\mathcal{G}\mathsf{r} \to \mathcal{S}\mathsf{et}$ ($\mathcal{A}\mathsf{b} \to \mathcal{S}\mathsf{et}$). We say the group structure is given by the multiplication morphism $G \times G \to G$ (in fact $h_G \times h_G \xrightarrow{m_G} h_G$) even if we don't know if $G \times G$ exists. Then a *left action* of the group object G on an object X is given by natural transformations $h_G \times h_X \to h_X$ etc., and we say the action is given by the action morphism $G \times X \to X$ even if we don't know if $G \times X$ exists (if it doesn't exist, we understand this expression as an abbreviation for the natural transformation of functors/map of presheaves). The notion of *right action* is defined similarly. For two morphisms $a, b : X \to G$ viewed as elements of the group $\mathcal{E}(X, G)$, their product in $\mathcal{E}(X, G)$ corresponds to the composition $h_X \xrightarrow{\Delta} h_X \times h_X \xrightarrow{a\times b} h_G \times h_G \xrightarrow{m_G} h_G$.

Proposition 1.1.1 (Eckmann-Hilton principle)**.** *Let A be a cogroup object and G be a group object of $\mathcal{E}$, then the two group operations on the set $\mathcal{E}(A, G)$ induced by A and by G coincide and turn $\mathcal{E}(A, G)$ into an abelian group.*

Proof. The point is to check that the two operations satisfy the condition of the *Eckmann-Hilton argument*, which can be found in [**2**, Proposition 2.2.12] (suitably adapted, at least when finite coproducts and finite products exist). □

A morphism $f : X \to Y$ in $\mathcal{E}$ is a *monomorphism* iff the induced map $f_* : h_X \to h_Y$ is an object-wise injection; a morphism $f : X \to Y$ in $\mathcal{E}$ is a *epimorphism* iff the induced map $f^* : h^Y \to h^X$ is an object-wise injection. It's easy to see that a morphism $f : X \to Y$ is a monomorphism iff the canonical morphism $X \to X \times_Y X$ is an isomorphism. Dually, a morphism $f : X \to Y$ is an epimorphism iff the canonical morphism $Y \coprod_X Y \to Y$ is an isomorphism.

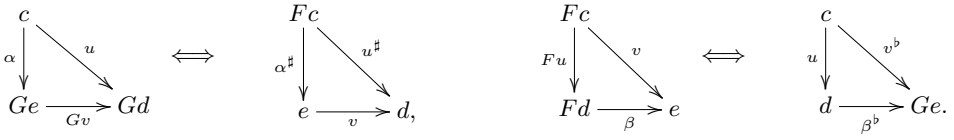
For an adjoint pair

$$F : \mathcal{C} \rightleftarrows \mathcal{D} : G,$$

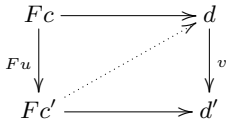
there is the *unit* $\eta : 1_{\mathcal{C}} \Rightarrow GF$ and the *counit* $\varepsilon : FG \Rightarrow 1_{\mathcal{D}}$, which satisfy the *triangle identities* $\varepsilon_F \circ (F\eta) = 1_F : F \Rightarrow F, (G\varepsilon)\circ\eta_G = 1_G : G \Rightarrow G$. The adjunction $\varphi_{c,d} : \mathcal{D}(Fc, d) \to \mathcal{C}(c, Gd)$ is then given by $\varphi_{c,d}(v) = v^\flat = (Gv)\circ\eta_c, \varphi^{-1}_{c,d}(u) = u^\sharp = \varepsilon_d \circ (Fu)$. Sometimes we denote this adjunction by (F, G, φ) or $(F, G) : \mathcal{C} \to \mathcal{D}$. We have,

$$(Gv) \circ \alpha = u \Leftrightarrow v \circ \alpha^\sharp = u^\sharp,$$

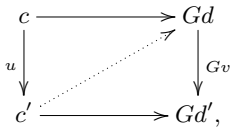
$$\beta \circ (Fu) = v \Leftrightarrow \beta^\flat \circ u = v^\flat.$$



From the above relations, it's also easy to find that the commutativity of a diagram



is the same as the commutativity of a diagram



both with or without the dotted lifting maps (the horizontal maps are related by the adjunction). This is implicitly used in [**40**, Proposition 7.2.17] and to give basic results on Quillen pairs in Proposition 8.5.3 in *loc.cit.*

For two morphisms i, p in a category $\mathcal{M}$, we use $i \boxslash p$ to indicate that i has the left lifting property with respect to p and p has the right lifting property with respect to i.[2] For a class $\mathbf{C}$ of morphisms in a category $\mathcal{M}$, we use $\mathrm{RLP}(\mathbf{C}) = \mathbf{C}^{\boxslash}$ to denote the class of morphisms in $\mathcal{M}$ that have the right lifting property with respect to each element of $\mathbf{C}$, and use $\mathrm{LLP}(\mathbf{C}) = {}^{\boxslash}\mathbf{C}$ to denote the class of morphisms in $\mathcal{M}$ that have the left lifting property with respect to each element of $\mathbf{C}$.

Denote by $c : \mathcal{E} \to \mathcal{E}^{\mathcal{I}}$ the constant diagram functor, then we have the adjunctions

$$\operatorname*{colim}_{\mathcal{I}} : \mathcal{E}^{\mathcal{I}} \rightleftarrows \mathcal{E} : c$$

and

$$c : \mathcal{E} \rightleftarrows \mathcal{E}^{\mathcal{I}} : \lim_{\mathcal{I}},$$

whenever the (co)limits of all diagrams in $\mathcal{E}$ of shape $\mathcal{I}$ exist. The relation between colimits and limits is given by

$$\operatorname*{colim}_{\mathcal{I}^{\mathrm{op}}} X^{\mathrm{op}} = \lim_{\mathcal{I}} X, \ \lim_{\mathcal{I}^{\mathrm{op}}} X^{\mathrm{op}} = \operatorname*{colim}_{\mathcal{I}} X$$

[2] The symbol $\boxslash$ is suggestive of the lifting-extension diagram in homotopy theory, which appears in [**74**, §11.1].

(as objects in $\mathcal{E}$) whenever one side of the (co)limit exists. Here $X^{\mathrm{op}} : \mathcal{I}^{\mathrm{op}} \to \mathcal{E}^{\mathrm{op}}$ is the opposite diagram of X. This relation is in concert with the fact that, given an adjoint pair

$$F : \mathcal{C} \rightleftarrows \mathcal{D} : G,$$

we also have an adjoint pair

$$G^{\mathrm{op}} : \mathcal{D}^{\mathrm{op}} \rightleftarrows \mathcal{C}^{\mathrm{op}} : F^{\mathrm{op}}.$$

Note that if $\mathcal{C}, \mathcal{D}$ are model categories and (F, G) is a Quillen pair (or a Quillen equivalence), then so is $(G^{\mathrm{op}}, F^{\mathrm{op}})$ (giving the opposite categories the dual model structures). And in that case, we have $(\mathbf{L}F)^{\mathrm{op}} = \mathbf{R}(F^{\mathrm{op}})$.

For categories $\mathcal{C}$ and $\mathcal{D}$ with $\mathcal{C}$ small, the assignment $\mathcal{D}^{\mathcal{C}} \to (\mathcal{D}^{\mathrm{op}})^{\mathcal{C}^{\mathrm{op}}}, F \mapsto F^{\mathrm{op}}$ is an anti-equivalence, i.e. it induces an equivalence of categories $(\mathcal{D}^{\mathcal{C}})^{\mathrm{op}} \simeq (\mathcal{D}^{\mathrm{op}})^{\mathcal{C}^{\mathrm{op}}}$. We make the classes of adjunctions from $\mathcal{C}$ to $\mathcal{D}$ a category $\mathrm{Adj}(\mathcal{C}, \mathcal{D})$ by defining morphisms to be natural transformations of the left adjoints in the adjunctions. Then $\mathrm{Adj}(\mathcal{C}, \mathcal{D})$ is a full subcategory of $\mathcal{D}^{\mathcal{C}}$. For $(F, G, \varphi), (F', G', \varphi') \in \mathrm{Adj}(\mathcal{C}, \mathcal{D})$, a natural transformation $\gamma : F \Rightarrow F'$ induces a natural transformation $\tilde{\gamma} : G' \Rightarrow G$ by the formula $\tilde{\gamma}_d = \varphi_{G'd,d}(\varepsilon'_d \circ \gamma_{G'd})$ (i.e. $\tilde{\gamma}_d$ corresponds to the map $FG'd \xrightarrow{\gamma} F'G'd \xrightarrow{\varepsilon'} d$ via the adjunction). If $\mathcal{C}, \mathcal{D}$ are both small, then the assignment $\mathrm{Adj}(\mathcal{C}, \mathcal{D}) \to \mathrm{Adj}(\mathcal{D}^{\mathrm{op}}, \mathcal{C}^{\mathrm{op}}), (F, G) \mapsto (G^{\mathrm{op}}, F^{\mathrm{op}})$ is an anti-equivalence: $\mathrm{Adj}(\mathcal{C}, \mathcal{D})^{\mathrm{op}} \simeq \mathrm{Adj}(\mathcal{D}^{\mathrm{op}}, \mathcal{C}^{\mathrm{op}}), (F, G) \mapsto (G^{\mathrm{op}}, F^{\mathrm{op}}), (\gamma : F \Rightarrow F') \mapsto (\tilde{\gamma}^{\mathrm{op}} : G^{\mathrm{op}} \Rightarrow G'^{\mathrm{op}})$. [3]

A digression on the cosimplicial indexing category

The *cosimplicial indexing category* $\mathbf{\Delta}$ is the category of finite ordinal numbers, it has as objects the sets $[n] = \{0, 1, \cdots, n\}$, for all $n \in \mathbb{N}$. Consider $\mathbb{N}$ as a poset (with its natural order) and each $[n]$ as a subposet. A morphism $f : [n] \to [m]$ is an order-preserving map (not necessarily injective) of posets.

There are distinguished morphisms: for $n \in \mathbb{N}, 0 \leqslant i, j \leqslant n$,

- *coface maps* $d^i = d^i_n : [n-1] \to [n]$, $d^i(0 < 1 < \cdots < n-1) = (0 < 1 < \cdots < i-1 < i+1 < \cdots < n), n \geqslant 1$.
- *codegeneracy maps* $s^j = s^j_n : [n+1] \to [n]$, $s^j(0 < 1 < \cdots < n+1) = (0 < 1 < \cdots < j = j < \cdots < n), n \geqslant 0$.

They satisfy the *cosimplicial identities*

$$\begin{cases} d^j d^i = d^i d^{j-1}, & i < j, \\ s^j d^i = d^i s^{j-1}, & i < j, \\ s^j d^j = 1_{[n-1]} = s^j d^{j+1}, & \\ s^j d^i = d^{i-1} s^j, & i > j+1, \\ s^j s^i = s^i s^{j+1}, & i \leqslant j. \end{cases}$$

In the above, it should be understood that, for example, the first means that $d^j_{n+1} d^i_n = d^i_{n+1} d^{j-1}_n, i < j$. Thus we will have $(s^0)^n = (s^0)^{n+1} d^i, (s^0)^{n+1} = (s^0)^n s^j$. It's easy to verify that there are the following coequalizer diagrams in $\mathbf{\Delta}$:

$$[n-1] \underset{d^{j+1}}{\overset{d^j}{\rightrightarrows}} [n] \xrightarrow{s^j} [n-1], n \geqslant 1, 0 \leqslant j < n.$$

More generally, we have the following coequalizer diagrams in $\mathbf{\Delta}$:

$$[n-1] \underset{d^j}{\overset{d^i}{\rightrightarrows}} [n] \xrightarrow{s^i s^{i+1} \cdots s^{j-1}} [n-(j-i)], n \geqslant 1, 0 \leqslant i < j \leqslant n.$$

(One uses that $s^i s^{i+1} \cdots s^{j-1}$ is given by $\begin{pmatrix} 0 < 1 < \cdots < i < i+1 < \cdots < j < j+1 < \cdots < n \\ 0 < 1 < \cdots < i = \cdots = i < i+1 < \cdots < n-(j-i) \end{pmatrix}$.)

The following result is [**31**, Chapter II, Lemma 2.2].

Proposition 1.1.2. *Any nonidentity morphism $f : [n] \to [m]$ in $\mathbf{\Delta}$ can be factored uniquely as*

$$f = d^{i_a}_m \cdots d^{i_1}_{m-a+1} s^{j_1}_{n-b} \cdots s^{j_b}_{n-1},$$

where $\{i_1 < \cdots < i_a\} = [m] \setminus \mathrm{im}(f), \{j_1 < \cdots < j_b\} = \{j \in [n-1] : f(j) = f(j+1)\}$. We have $n - m = b - a$.

Thus $f = v \circ u : [n] \xrightarrow{u} [n-b] = [m-a] \xrightarrow{v} [m]$, with $u = s^{j_1}_{n-b} \cdots s^{j_b}_{n-1}$ surjective and $v = d^{i_a}_m \cdots d^{i_1}_{m-a+1}$ injective. We call this factorization $f = v \circ u$ the epi-monic factorization *of f.*

Remark 1.1.3. This result implies that

- The cosimplicial indexing category $\mathbf{\Delta}$ is a Reedy category in the sense of [**40**, §15.1].
- In $\mathbf{\Delta}$, monomorphisms=injections, epimorphisms=surjections.[4] Moreover, every epimorphism has a section, and every monomorphism has a cosection: for $u = s^{j_1} \cdots s^{j_b}$, let $r = d^{j_b} \cdots d^{j_1}$, then $ur = 1$; for $v = d^{i_a} \cdots d^{i_1}$, let $l = s^{i_1} \cdots s^{i_a}$, then $lv = 1$.

Given a category $\mathcal{E}$, we can consider the *category of cosimplicial objects of* $\mathcal{E}$, the functor category $\mathsf{c}\mathcal{E} = \mathbf{\Delta}\mathcal{E} = \mathcal{E}^{\mathbf{\Delta}}$, morphisms being natural transformations of functors. For $Y \in \mathsf{c}\mathcal{E}$, we write $Y^n = Y([n])$, there are *coface maps* $d^i = d^i_n : Y^{n-1} \to Y^n$ and *degeneracy maps* $s^j = s^j_n : Y^{n+1} \to Y^n$. They also satisfy the *cosimplicial identities*.

The *simplicial indexing category* is the opposite category $\mathbf{\Delta}^{\mathrm{op}}$.

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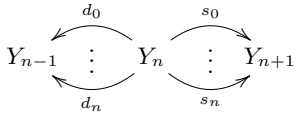
[3] Without smallness assumption, one can still make the stated equivalence into a 1-1 correspondence. If $\mathcal{C}$ (or $\mathcal{D}$) is small, then $\mathrm{Adj}(\mathcal{C}, \mathcal{D})$ and $\mathrm{Adj}(\mathcal{D}^{\mathrm{op}}, \mathcal{C}^{\mathrm{op}})$ are locally small.

[4] If $f(i) = f(j)$, consider the two maps $[0] \to [n]$ with image i and j respectively; if $k \in [m] \setminus \mathrm{im}(f)$, consider the two maps $g, h : [m] \to [1]$ which map the numbers less than k (if any) to 0 and map the numbers bigger than k (if any) to 1, but $g(k) = 0, h(k) = 1$.

The *category of simplicial objects of* $\mathcal{E}$ is the functor category $\mathsf{s}\mathcal{E} = \mathbf{\Delta}^{\mathrm{op}}\mathcal{E} = \mathcal{E}^{\mathbf{\Delta}^{\mathrm{op}}}$, morphisms being natural transformations of functors. For $Y \in \mathsf{s}\mathcal{E}$, we write $Y_n = Y([n])$, there are *face maps* $d_i = d_i^n : Y_n \to Y_{n-1}, n \geqslant 1$ and *degeneracy maps* $s_j = s_j^n : Y_n \to Y_{n+1}, n \geqslant 0$. They satisfy the *simplicial identities*, dual of the cosimplicial identities:

$$\begin{cases} d_i d_j = d_{j-1} d_i, & i < j, \\ d_i s_j = s_{j-1} d_i, & i < j, \\ d_j s_j = 1_{Y_n} = d_{j+1} s_j, & \\ d_i s_j = s_j d_{i-1}, & i > j+1, \\ s_i s_j = s_{j+1} s_i, & i \leqslant j. \end{cases}$$



For $Y \in \mathsf{s}\mathcal{E}$, the maps $(s_0)^n : Y_0 \to Y_n$ define a monomorphism $\mathrm{cs}_* Y_0 \to Y$ in $\mathsf{s}\mathcal{E}$ (for the notation, see Section 1.2).

Remark 1.1.4. If $u : [m] \to [n]$ is surjective, then (since it has a section), the induced map $u^* : Y_n \to Y_m$ is a monomorphism in $\mathcal{E}$. If $u : [m] \to [n]$ is injective, then (since it has a cosection), the induced map $u^* : Y_n \to Y_m$ is an epimorphism in $\mathcal{E}$.

In particular, we get *the category of simplicial sets* $\mathsf{s}\mathcal{S}\mathsf{et}$. This category is bicomplete, (co)limits are obtained in a degree-wise manner. Using the simplicial identities and induction, one can show that, for $i_1 \in [1], \cdots, i_n \in [n], i_{n+1} \in [n+1], m \in [n]$, there exist $i'_1 \in [1], \cdots, i'_n \in [n]$ such that $d^1_{i_1} \cdots d^n_{i_n} d^{n+1}_{i_{n+1}} s^n_m = d^1_{i'_1} \cdots d^n_{i'_n} : Y_n \to Y_0$.

Remark 1.1.5. If $u : [m] \to [n]$ is injective, then the induced map $u_* : \Delta^m \to \Delta^n$ (see Section 1.5) is a monomorphism in $\mathsf{s}\mathcal{S}\mathsf{et}$. If $u : [m] \to [n]$ is surjective, then the induced map $u_* : \Delta^m \to \Delta^n$ is an epimorphism in $\mathsf{s}\mathcal{S}\mathsf{et}$.

For $X \in \mathsf{s}\mathcal{S}\mathsf{et}$, an n-simplex $a \in X_n$ is said to be a *degenerate n-simplex* if $a = s_j(b)$ for some j and $b \in X_{n-1}$; otherwise it's called a *non-degenerate n-simplex*. The set of *degenerate n-simplices* and the set of *non-degenerate n-simplices* of X are denoted respectively $\mathrm{D}X_n$ and $\mathrm{N}X_n$. Clearly, $\mathrm{N}X_0 = X_0$ and $X_n = \mathrm{D}X_n \coprod \mathrm{N}X_n$.

Any functor $F : \mathcal{D} \to \mathcal{E}$ induces functors $\mathsf{c}\mathcal{D} \to \mathsf{c}\mathcal{E}$ and $\mathsf{s}\mathcal{D} \to \mathsf{s}\mathcal{E}$ in the obvious way, which we still denote by F.

1.2. Homotopy (co)limits in model categories

We assume basic knowledge on model categories; nevertheless we recall a few facts to fix notations. When working with model categories, we reserve the symbol $\cong$ for isomorphisms, and to distinguish, we use $\simeq$ to indicate weak equivalences. For a model category $\mathcal{M}$, we use $Q, R : \mathcal{M} \to \mathcal{M}$ denote respectively the functorial cofibrant and fibrant replacement obtained by the functorial factorizations for the model structure of $\mathcal{M}$, we use $\mathcal{M}_c, \mathcal{M}_f, \mathcal{M}_{cf}$ to denote respectively the full subcategories of cofibrant, fibrant, cofibrant-fibrant objects of $\mathcal{M}$ ([**45**, Proposition 1.2.3]). We sometimes use $[A, Y] = [A, Y]_{\mathcal{M}} = [A, Y]_{\mathrm{Ho}\ \mathcal{M}}$ to denote the hom-set $(\mathrm{Ho}\ \mathcal{M})(A, Y)$.

Let $F : \mathcal{M} \to \mathcal{N}$ be a functor between model categories. If F maps weak equivalences between cofibrant objects in $\mathcal{M}$ to weak equivalences in $\mathcal{N}$, then its total left derived functor $\mathbf{L}F : \mathrm{Ho}(\mathcal{M}) \to \mathrm{Ho}(\mathcal{N})$ exists, it is the composite $\mathrm{Ho}(\mathcal{M}) \xrightarrow{\mathrm{Ho}(Q)} \mathrm{Ho}(\mathcal{M}_c) \xrightarrow{\mathrm{Ho}(F)} \mathrm{Ho}(\mathcal{N})$; if F maps weak equivalences between fibrant objects in $\mathcal{M}$ to weak equivalences in $\mathcal{N}$, then its total right derived functor $\mathbf{R}F : \mathrm{Ho}(\mathcal{M}) \to \mathrm{Ho}(\mathcal{N})$ exists, it is the composite $\mathrm{Ho}(\mathcal{M}) \xrightarrow{\mathrm{Ho}(R)} \mathrm{Ho}(\mathcal{M}_f) \xrightarrow{\mathrm{Ho}(F)} \mathrm{Ho}(\mathcal{N})$ (see [**40**, Proposition 8.4.8] or [**45**, Definition 1.3.6 and the discussions thereafter]). Moreover, in the first case, the total right derived functor of $F^{\mathrm{op}} : \mathcal{M}^{\mathrm{op}} \to \mathcal{N}^{\mathrm{op}}$ exists, and $\mathbf{R}(F^{\mathrm{op}}) = (\mathbf{L}F)^{\mathrm{op}} : \mathrm{Ho}(\mathcal{M})^{\mathrm{op}} \to \mathrm{Ho}(\mathcal{N})^{\mathrm{op}}$; in the second case, the total left derived functor of $F^{\mathrm{op}} : \mathcal{M}^{\mathrm{op}} \to \mathcal{N}^{\mathrm{op}}$ exists, and $\mathbf{L}(F^{\mathrm{op}}) = (\mathbf{R}F)^{\mathrm{op}} : \mathrm{Ho}(\mathcal{M})^{\mathrm{op}} \to \mathrm{Ho}(\mathcal{N})^{\mathrm{op}}$.

Proposition 1.2.1 ([**40**, Proposition 14.7.5 and Remark 14.7.6])**.** *Let $\mathcal{C}, \mathcal{D}$ be small categories and $F : \mathcal{C} \to \mathcal{D}$ a functor. Then there are a natural isomorphism*

$$\operatorname*{colim}_{\mathcal{D}^{\mathrm{op}}} \mathrm{B}(- \downarrow F) \cong \mathrm{B}\mathcal{C}$$

and a natural weak equivalence $\operatorname*{hocolim}_{\mathcal{D}^{\mathrm{op}}} \mathrm{B}(- \downarrow F) \to \mathrm{B}\mathcal{C}$.

Let $\mathcal{M}$ be a model category. For an object X in $\mathcal{M}$, denote by $\mathrm{cc}_* X$ the constant diagram in $\mathsf{c}\mathcal{M} = \mathcal{M}^{\mathbf{\Delta}}$ and $\mathrm{cs}_* X$ the constant diagram in $\mathsf{s}\mathcal{M} = \mathcal{M}^{\mathbf{\Delta}^{\mathrm{op}}}$. We give $\mathsf{c}\mathcal{M}, \mathsf{s}\mathcal{M}$ the Reedy model structures (see [**40**, Chapter 15] for Reedy categories and Reedy model structures). A *cosimplicial resolution* of X is a cofibrant approximation $\widetilde{X} \to \mathrm{cc}_* X$ of the constant diagram $\mathrm{cc}_* X$ in $\mathsf{c}\mathcal{M}$ and a *simplicial resolution* of X is a fibrant approximation $\mathrm{cs}_* X \to \widehat{X}$ of the constant diagram $\mathrm{cs}_* X$ in $\mathsf{s}\mathcal{M}$. In these cases, the map $\widetilde{X}^0 \to X$ is a cofibrant approximation of X in $\mathcal{M}$ and the map $X \to \widehat{X}_0$ is a fibrant approximation of X in $\mathcal{M}$; $\widetilde{X}^0 \coprod \widetilde{X}^0 \xrightarrow{d^0 \coprod d^1} \widetilde{X}^1 \xrightarrow{s^0} \widetilde{X}^0$ is a cylinder object for $\widetilde{X}^0$, $\widehat{X}_0 \xrightarrow{s_0} \widehat{X}_1 \xrightarrow{d_0 \times d_1} \widehat{X}_0 \times \widehat{X}_0$ is a path object for $\widehat{X}_0$ ([**40**, Propositions 16.1.5 and 16.1.6]).

A map $\widetilde{X} \to \mathrm{cc}_* X$ is a cosimplicial resolution of X iff $\widetilde{X}$ is Reedy cofibrant in $\mathsf{c}\mathcal{M}$, all the coface and codegeneracy operators of $\widetilde{X}$ are weak equivalences, and the map $\widetilde{X}^0 \to X$ is a cofibrant approximation of X in $\mathcal{M}$; a map $\mathrm{cs}_* X \to \widehat{X}$ is a simplicial resolution of X iff $\widehat{X}$ is Reedy fibrant in $\mathsf{s}\mathcal{M}$, all the face and degeneracy operators of $\widehat{X}$ are weak equivalences, and the map $X \to \widehat{X}_0$ is a fibrant approximation of X in $\mathcal{M}$ ([**40**, Proposition 16.1.27]).

There exist a *functorial cosimplicial resolution* and a *functorial simplicial resolution* on $\mathcal{M}$ (see [**40**, Proposition 16.1.9]).

Left Quillen functors preserve cosimplicial resolutions of cofibrant objects and right Quillen functors preserve simplicial resolutions of fibrant objects (see [**40**, Proposition 16.2.1]).

If $\mathcal{M}$ is a simplicial model category, let W be a cofibrant approximation of X in $\mathcal{M}$ and Z a fibrant approximation of X in $\mathcal{M}$, then $\widetilde{W} = W \otimes \Delta^\bullet$ is a cosimplicial resolution of X and $\widehat{Z} = Z^{\Delta^\bullet}$ is a simplicial resolution of X.

For $K \in \mathsf{s}\mathcal{S}\mathsf{et}, \widetilde{X} \in \mathsf{c}\mathcal{M}$ and $\widehat{Y} \in \mathsf{s}\mathcal{M}$, define (see [**40**, Definition 16.3.1] or [**45**, Proposition 3.1.5 and Remark 3.1.7])

$$\widetilde{X} \otimes K := \operatorname*{colim}_{(\Delta^n \to K) \in \mathbf{\Delta} K} \widetilde{X}^n, \ \widehat{Y}^K := \lim_{(\Delta^n \to K) \in \mathbf{\Delta}^{\mathrm{op}} K} \widehat{Y}_n. \tag{1.1}$$

Then $\widetilde{X}\otimes\Delta^n=\widetilde{X}^n, \widetilde{X}\otimes\partial\Delta^n=\mathrm{L}_n\widetilde{X}, \widehat{Y}^{\Delta^n}=\widehat{Y}_n, \widehat{Y}^{\partial\Delta^n}=\mathrm{M}_n\widehat{Y}$ (see [**40**, Lemma 16.3.6 and Proposition 16.3.7]).

Now let $X,Y\in\mathcal{M}, \widetilde{X}\in\mathsf{c}\mathcal{M}, \widehat{Y}\in\mathsf{s}\mathcal{M}$, we denote

$$\mathcal{M}(\widetilde{X},Y)_n=\mathcal{M}(\widetilde{X}^n,Y), \mathcal{M}(X,\widehat{Y})_n=\mathcal{M}(X,\widehat{Y}_n), \mathcal{M}(\widetilde{X},\widehat{Y})_{n,m}=\mathcal{M}(\widetilde{X}^m,\widehat{Y}_n), \tag{1.2}$$

then we get simplicial sets $\mathcal{M}(\widetilde{X},Y), \mathcal{M}(X,\widehat{Y})$ and a bisimplicial set $\mathcal{M}(\widetilde{X},\widehat{Y})$ with simplicial operators induced from the cosimplicial operators of $\widetilde{X}$ and simplicial operators of $\widehat{Y}$ and hence a simplicial set $\mathrm{diag}\,\mathcal{M}(\widetilde{X},\widehat{Y})$ with $(\mathrm{diag}\,\mathcal{M}(\widetilde{X},\widehat{Y}))_n=\mathcal{M}(\widetilde{X}^n,\widehat{Y}_n)$. There are the following adjoint relations ([**40**], Theorem 16.4.2):

$$\mathcal{M}(\widetilde{X}\otimes K,Y)\cong\mathsf{s}\mathcal{S}\mathsf{et}(K,\mathcal{M}(\widetilde{X},Y)), \mathcal{M}(X,\widehat{Y}^K)\cong\mathsf{s}\mathcal{S}\mathsf{et}(K,\mathcal{M}(X,\widehat{Y})).$$

We will use $\mathrm{map}(X,Y)$ to denote any one of these three simplicial sets for some unspecified cosimplicial resolution $\widetilde{X}$ of X and simplicial resolution $\widehat{Y}$ of Y, and call it a *homotopy function complex from X to Y*. For a small category $\mathcal{C}$ and a functor $K:\mathcal{C}\to\mathsf{s}\mathcal{S}\mathsf{et}$, we have natural isomorphisms ([**40**, Proposition 16.4.3])

$$\operatorname*{colim}_{\mathcal{C}}(\widetilde{X}\otimes K(-))\cong\widetilde{X}\otimes\operatorname*{colim}_{\mathcal{C}}K, \widehat{Y}^{\operatorname*{colim}_{\mathcal{C}}K}\cong\lim_{\mathcal{C}^{\mathrm{op}}}(\widehat{Y}^{K(-)}).$$

We denote by $\mathrm{CRes}(X)$ the category of cosimplicial resolutions of X and $\mathrm{FibAp}(X)$ the category of fibrant replacements of X, similarly for other notations.

For $X,Y\in\mathcal{M}$, there are *the category of left homotopy function complexes* $\mathrm{LHFC}(X,Y)=\mathrm{CRes}(X)^{\mathrm{op}}\times\mathrm{FibAp}(Y)$, *the category of right homotopy function complexes* $\mathrm{RHFC}(X,Y)=\mathrm{CofAp}(X)^{\mathrm{op}}\times\mathrm{SRes}(Y)$, *the category of two-sided homotopy function complexes* $\mathrm{TSHFC}(X,Y)=\mathrm{CRes}(X)^{\mathrm{op}}\times\mathrm{SRes}(Y)$, the change homotopy function complex maps are weak equivalences of Kan complexes ([**40**, Theorem 17.4.8]). All of these are full subcategories of $((\mathrm{cc}_*X,\mathrm{cs}_*Y)\downarrow(\mathsf{c}\mathcal{M})^{\mathrm{op}}\times\mathsf{s}\mathcal{M})$, and have contractible classifying spaces (see [**40**, §17.1-§17.3]). We then have *the category of homotopy function complexes* $\mathrm{HFC}(X,Y)$ ([**40**, Definition 17.4.9]) having the same properties as the above three, and containing these three as full subcategories.

Let $\mathcal{M}$ be a *combinatorial model category*, $\mathcal{C}$ a small category, then the diagram category $\mathcal{M}^{\mathcal{C}}$ has the *projective model structure* where the weak equivalences and fibrations are object-wise, and also the *injective model structure* where the weak equivalences and cofibrations are object-wise, we denote them respectively by $\mathcal{M}^{\mathcal{C}}_{\mathrm{proj}}$ and $\mathcal{M}^{\mathcal{C}}_{\mathrm{inj}}$ (see [**57**, Proposition A.2.8.2]). These two model structures are both combinatorial. In this case, we have Quillen pairs

$$\operatorname*{colim}_{\mathcal{C}}:\mathcal{M}^{\mathcal{C}}_{\mathrm{proj}}\rightleftarrows\mathcal{M}:c:\mathcal{M}\rightleftarrows\mathcal{M}^{\mathcal{C}}_{\mathrm{inj}}:\lim_{\mathcal{C}}\text{ and }1:\mathcal{M}^{\mathcal{C}}_{\mathrm{proj}}\rightleftarrows\mathcal{M}^{\mathcal{C}}_{\mathrm{inj}}:1,$$

the last pair is a Quillen equivalence. We denote some fixed choices of functorial cofibrant and fibrant replacement by $Q_{\mathrm{proj}}, R_{\mathrm{proj}}$ and $Q_{\mathrm{inj}}, R_{\mathrm{inj}}$. Note that R_{proj} and Q_{inj} can be obtained by applying R *resp.* Q of $\mathcal{M}$ to the diagrams, while Q_{proj} and R_{inj} are usually quite difficult to get.

Proposition 1.2.2 ([**40**, Proposition 15.10.12])**.** *Let $\mathcal{M}$ be a model category.*

- *Given a map of sequences in $\mathcal{M}$*

$$\begin{array}{ccccccccc}
X_0 & \xrightarrow{i_0} & X_1 & \xrightarrow{i_1} & X_2 & \xrightarrow{i_2} & \cdots \\
\downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \\
Y_0 & \xrightarrow[j_0]{} & Y_1 & \xrightarrow[j_1]{} & Y_2 & \xrightarrow[j_2]{} & \cdots
\end{array}$$

  in which all the objects are cofibrant, the maps i_n and j_n are cofibrations and f_n are weak equivalences, then the induced map on colimits $\operatorname{colim} f_n:\operatorname{colim}X_n\to\operatorname{colim}Y_n$ *is a weak equivalence of cofibrant objects.*
- *Given a map of towers in $\mathcal{M}$*

$$\begin{array}{ccccccccc}
\cdots & \xrightarrow{p_3} & X_2 & \xrightarrow{p_2} & X_1 & \xrightarrow{p_1} & X_0 \\
 & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} \\
\cdots & \xrightarrow[q_3]{} & Y_2 & \xrightarrow[q_2]{} & Y_1 & \xrightarrow[q_1]{} & Y_0
\end{array}$$

  in which all the objects are fibrant, the maps p_n and q_n are fibrations and f_n are weak equivalences, then the induced map on limits $\lim f_n:\lim X_n\to\lim Y_n$ *is a weak equivalence of fibrant objects.*

Proposition 1.2.3 ([**40**, Proposition 17.9.1])**.** *Let $\mathcal{M}$ be a model category, λ an ordinal. Given a map of λ-sequences in $\mathcal{M}$*

$$\begin{array}{ccccccccc}
X_0 & \xrightarrow{i_0} & X_1 & \xrightarrow{i_1} & X_2 & \xrightarrow{i_2} & \cdots \\
\downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \\
Y_0 & \xrightarrow[j_0]{} & Y_1 & \xrightarrow[j_1]{} & Y_2 & \xrightarrow[j_2]{} & \cdots
\end{array}$$

in which all the objects are cofibrant, the maps i_α and j_α are cofibrations and f_α are weak equivalences for all $\alpha<\lambda$, then the induced map on colimits $\operatorname{colim}f_\alpha:\operatorname{colim}X_\alpha\to\operatorname{colim}Y_\alpha$ *is a weak equivalence.*

Proposition 1.2.4 ([**40**, Proposition 17.9.3]). *Let $\mathcal{M}$ be a left proper model category, λ an ordinal. Given a map of λ-sequences in $\mathcal{M}$*

$$\begin{array}{ccccccccc} X_0 & \xrightarrow{i_0} & X_1 & \xrightarrow{i_1} & X_2 & \xrightarrow{i_2} & \cdots \\ \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \\ Y_0 & \xrightarrow[j_0]{} & Y_1 & \xrightarrow[j_1]{} & Y_2 & \xrightarrow[j_2]{} & \cdots \end{array}$$

in which the maps i_α and j_α are cofibrations and f_α are weak equivalences for all $\alpha < \lambda$, then the induced map on colimits $\operatorname{colim} f_\alpha : \operatorname{colim} X_\alpha \to \operatorname{colim} Y_\alpha$ *is a weak equivalence.*

We will need the notion of *cosimplicial frames* and *simplicial frames* on a model category $\mathcal{M}$ as well as on the diagram category $\mathcal{M}^{\mathcal{C}}$, and also *Reedy frames*, see [**40**, Definition 16.6.1, Definition 16.7.2 and Definition 16.7.8]. These frames exist and can be made functorial, see [**40**, Proposition 16.6.8, Theorem 16.6.9 and Proposition 16.7.11]. If $\mathcal{M}$ is a simplicial model category, we always take the *standard frames* $\widetilde{X} = X \otimes \Delta^\bullet$ and $\widehat{X} = X^{\Delta^\bullet}$ ([**40**, Definition 16.6.5 and Proposition 16.6.23]) and in this case, $\widetilde{X} \otimes K \cong X \otimes K, \widehat{X}^K \cong X^K$ naturally ([**40**, Proposition 16.6.6]). See [**40**, Proposition 16.6.7] for an interesting result between (co)simplicial frames and (co)simplicial resolutions, which roughly says that *a (co)simplicial resolution of an object is the same thing as a (co)simplicial frame of a (co)fibrant replacement of that object.*

Definition 1.2.5. Let $\mathcal{M}$ be a model category, $\mathcal{C}$ a small category, let $X \in \mathcal{M}^{\mathcal{C}}$ and let $\widetilde{X}$ be a cosimplicial frame on X, $\widehat{X}$ a simplicial frame on X.

- For $K \in \mathsf{sSet}^{\mathcal{C}^{\mathrm{op}}}$, define $X \otimes^{\widetilde{X}}_{\mathcal{C}} K \in \mathcal{M}$ via the coequalizer diagram
$$\coprod_{(\theta: c \to c') \in \mathcal{C}} \widetilde{X}_c \otimes K_{c'} \rightrightarrows \coprod_{c \in \mathcal{C}} \widetilde{X}_c \otimes K_c \to X \otimes^{\widetilde{X}}_{\mathcal{C}} K$$
of the obvious maps.
- For $K \in \mathsf{sSet}^{\mathcal{C}}$, define $\hom^{\mathcal{C}}_{\widehat{X}}(K, X) \in \mathcal{M}$ via the equalizer diagram
$$\hom^{\mathcal{C}}_{\widehat{X}}(K, X) \to \prod_{c \in \mathcal{C}} (\widehat{X}_c)^{K_c} \rightrightarrows \prod_{(\theta: c \to c') \in \mathcal{C}} (\widehat{X}_{c'})^{K_c}$$
of the obvious maps.
- The *homotopy colimit* of X is $\operatorname*{hocolim}_{\mathcal{C}} X := X \otimes^{\widetilde{X}}_{\mathcal{C}} \mathrm{B}(- \downarrow \mathcal{C})^{\mathrm{op}}$.
- The *homotopy limit* of X is $\operatorname*{holim}_{\mathcal{C}} X := \hom^{\mathcal{C}}_{\widehat{X}}(\mathrm{B}(\mathcal{C} \downarrow -), X)$.

Remark 1.2.6. A homotopy limit needs not be a limit in the homotopy category, and a limit in the homotopy category needs not be a homotopy limit; homotopy categories do not usually have many limits and colimits at all (aside from products and coproducts—homotopy products and homotopy coproducts are products and coproducts in the homotopy category). Even if a homotopy category does happen to have limits and colimits, these need not be the same as homotopy limits and homotopy colimits.

Remark 1.2.7. If $\mathcal{M}$ is a simplicial model category, we take the standard frames, thus we have the coequalizer diagram

$$\coprod_{(\theta: c \to c') \in \mathcal{C}} X_c \otimes \mathrm{B}(c' \downarrow \mathcal{C})^{\mathrm{op}} \rightrightarrows \coprod_{c \in \mathcal{C}} X_c \otimes \mathrm{B}(c \downarrow \mathcal{C})^{\mathrm{op}} \to \operatorname*{hocolim}_{\mathcal{C}} X$$

and the equalizer diagram

$$\operatorname*{holim}_{\mathcal{C}} X \to \prod_{c \in \mathcal{C}} (X_c)^{\mathrm{B}(\mathcal{C} \downarrow c)} \rightrightarrows \prod_{(\theta: c \to c') \in \mathcal{C}} (X_{c'})^{\mathrm{B}(\mathcal{C} \downarrow c)}.$$

Note that in this case the homotopy colimits and homotopy limits depend only on the simplicial structure of the underlying category of $\mathcal{M}$, having little to do with the model structure, though not completely: the homotopy colimits and homotopy limits depend on the model structure in a secondary fashion, since the simplicial structure has interplay with the model structure. In fact, the homotopy invariance of the homotopy colimits and homotopy limits below will depend on the model structure of $\mathcal{M}$.

Moreover, for $X \in \mathcal{M}^{\mathcal{C}}$ and $Y \in \mathcal{M}$, there is a natural isomorphism $\operatorname{Map}(\operatorname*{hocolim}_{\mathcal{C}} X, Y) \cong \operatorname*{holim}_{\mathcal{C}^{\mathrm{op}}} \operatorname{Map}(X(-), Y)$ (see [**40**, Theorem 18.1.10]).

Let G be a (discrete) group and X be a topological space such that G acts on X. The action can be viewed as a functor $X : G \to \mathsf{Top}$, where we view G as a category with one object as usual. Then the *homotopy fixed points* $X^{\mathsf{h}G}$ can be defined as $X^{\mathsf{h}G} = \operatorname*{holim}_{G} X$. The *homotopy orbit space* $X_{\mathsf{h}G}$ can be defined as $X_{\mathsf{h}G} = \operatorname*{hocolim}_{G} X$[5] (see [**70**, Example 8.2.11, 8.2.23]).

Proposition 1.2.8. *Let $\mathcal{M}$ be a (framed) model category, $\mathcal{C}$ a small category, and $X \in \mathcal{M}^{\mathcal{C}}$.*

- *If $P \in \mathsf{sSet}^{\mathcal{C}^{\mathrm{op}}}$ is a single point at every object, then there is a natural isomorphism $X \otimes^{\widetilde{X}}_{\mathcal{C}} P \cong \operatorname*{colim}_{\mathcal{C}} X$, and the unique map $\mathrm{B}(- \downarrow \mathcal{C})^{\mathrm{op}} \to P$ induces a natural map $\operatorname*{hocolim}_{\mathcal{C}} X \to \operatorname*{colim}_{\mathcal{C}} X$. If moreover $\mathcal{C}$ is a Reedy category and X is Reedy cofibrant in $\mathcal{M}^{\mathcal{C}}$, then this natural map is a weak equivalence of cofibrant objects in $\mathcal{M}$.*
- *If $P \in \mathsf{sSet}^{\mathcal{C}}$ is a single point at every object, then there is a natural isomorphism $\hom^{\mathcal{C}}_{\widehat{X}}(P, X) \cong \lim_{\mathcal{C}} X$, and the unique map $\mathrm{B}(\mathcal{C} \downarrow -) \to P$ induces a natural map $\lim_{\mathcal{C}} X \to \operatorname*{holim}_{\mathcal{C}} X$. If moreover $\mathcal{C}$ is a Reedy category and X is Reedy fibrant in $\mathcal{M}^{\mathcal{C}}$, then this natural map is a weak equivalence of fibrant objects in $\mathcal{M}$.*

[5] By [**33**, Chapter IV, Example 1.10], if X is a simplicial set, then $X_{\mathsf{h}G} = \operatorname*{hocolim}_{G} X \cong \mathrm{E}G \times_G X$.

- *If $P \in \mathsf{sSet}^{\mathcal{C}^{\mathrm{op}}}$ is a single point at every object, then* $\operatorname{hocolim}_{\mathcal{C}^{\mathrm{op}}} P \cong \mathrm{B}\mathcal{C}$ *in* sSet.

Proof. For the first two points, see [**40**, Proposition 19.2.9 and Example 19.2.10]. The third point is proved in [**40**, Proposition 18.1.6]. □

Proposition 1.2.9 ([**40**, Proposition 19.2.13])**.** *Let $\mathcal{M}$ be a model category, $\mathcal{C}$ a small category, let $X \in \mathcal{M}^{\mathcal{C}}$ and let $\widetilde{X}$ be a cosimplicial frame on X, $\widehat{X}$ a simplicial frame on X.*

- *For $K \in \mathsf{sSet}^{\mathcal{C}^{\mathrm{op}}}$ and $Y \in \mathcal{M}$, there is a natural bijection*
$$\mathcal{M}(X \otimes_{\mathcal{C}}^{\widetilde{X}} K, Y) \cong \mathsf{sSet}^{\mathcal{C}^{\mathrm{op}}}(K, \mathcal{M}(\widetilde{X}, Y)).$$
- *For $K \in \mathsf{sSet}^{\mathcal{C}}$ and $Y \in \mathcal{M}$, there is a natural bijection*
$$\mathcal{M}(Y, \hom_{\widehat{X}}^{\mathcal{C}}(K, X)) \cong \mathsf{sSet}^{\mathcal{C}}(K, \mathcal{M}(Y, \widehat{X})).$$

Proposition 1.2.10 (Homotopy invariance, see [**40**, Theorem 19.4.2 and Theorem 19.4.4])**.** *Let $\mathcal{M}$ be a (framed) model category, $\mathcal{C}$ a small category.*

- *If $f : X \to Y$ is a map in $\mathcal{M}^{\mathcal{C}}$ that is an object-wise weak equivalence of cofibrant objects. Then the induced map $f_* : \operatorname{hocolim}_{\mathcal{C}} X \to \operatorname{hocolim}_{\mathcal{C}} Y$ is also an weak equivalence of cofibrant objects.*
- *If $f : X \to Y$ is a map in $\mathcal{M}^{\mathcal{C}}$ that is an object-wise weak equivalence of fibrant objects. Then the induced map $f_* : \operatorname{holim}_{\mathcal{C}} X \to \operatorname{holim}_{\mathcal{C}} Y$ is also an weak equivalence of fibrant objects.*
- *If $X \in \mathcal{M}^{\mathcal{C}}$ is an object-wise cofibrant diagram and $Y \in \mathcal{M}$ is fibrant, then* $\operatorname{map}(\operatorname{hocolim}_{\mathcal{C}} X, Y)$ *is naturally weakly equivalent to* $\operatorname{holim}_{\mathcal{C}} \operatorname{map}(X, Y)$.
- *If $X \in \mathcal{M}$ is cofibrant and $Y \in \mathcal{M}^{\mathcal{C}}$ is an object-wise fibrant diagram, then* $\operatorname{map}(X, \operatorname{holim}_{\mathcal{C}} Y)$ *is naturally weakly equivalent to* $\operatorname{holim}_{\mathcal{C}} \operatorname{map}(X, Y)$.

Proposition 1.2.11 ([**40**, Theorem 19.4.5])**.** *Let $\mathcal{M}, \mathcal{N}$ be (framed) model categories, $\mathcal{C}$ a small category and $F : \mathcal{M} \rightleftarrows \mathcal{N} : U$ a Quillen pair.*

- *There is an essentially unique zig-zag of natural transformations of functors $\mathcal{M}^{\mathcal{C}} \to \mathcal{N}$ (induced by maps of frames) from $F \circ \operatorname{hocolim}_{\mathcal{C}}$ to $\operatorname{hocolim}_{\mathcal{C}} \circ F$, it is a natural zig-zag of weak equivalences on object-wise cofibrant diagrams.*
- *There is an essentially unique zig-zag of natural transformations of functors $\mathcal{N}^{\mathcal{C}} \to \mathcal{M}$ (induced by maps of frames) from $U \circ \operatorname{holim}_{\mathcal{C}}$ to $\operatorname{holim}_{\mathcal{C}} \circ U$, it is a natural zig-zag of weak equivalences on object-wise fibrant diagrams.*

Definition 1.2.12 ([**40**, Definition 19.6.1])**.** Let $\mathcal{C}, \mathcal{D}$ be small categories and $F : \mathcal{C} \to \mathcal{D}$ a functor.

- We call F *homotopy left cofinal* or *homotopy initial* if for every $d \in \mathcal{D}$, $\mathrm{B}(F \downarrow d)$ is contractible.
- We call F *homotopy right cofinal* or *homotopy terminal* if for every $d \in \mathcal{D}$, $\mathrm{B}(d \downarrow F)$ is contractible.

Proposition 1.2.13 ([**40**, Proposition 19.6.6])**.** *Let $\mathcal{C}, \mathcal{D}$ be small categories and $F : \mathcal{C} \to \mathcal{D}$ a functor. Let $\mathcal{M}$ be a (framed) model category and let $X \in \mathcal{M}^{\mathcal{D}}$.*

- *There is a natural isomorphism*
$$\operatorname{hocolim}_{\mathcal{C}} F^* X \cong X \otimes_{\mathcal{D}} \mathrm{B}(- \downarrow F)^{\mathrm{op}}.$$
- *There is a natural isomorphism*
$$\operatorname{holim}_{\mathcal{C}} F^* X \cong \hom^{\mathcal{D}}(\mathrm{B}(F \downarrow -), X).$$

Proposition 1.2.14 ([**40**, Proposition 19.6.12 and Theorem 19.6.13])**.** *Let $\mathcal{C}, \mathcal{D}$ be small categories.*

- *A functor $F : \mathcal{C} \to \mathcal{D}$ is homotopy right cofinal iff for every (framed) model category $\mathcal{M}$ and every object-wise cofibrant diagram $X \in \mathcal{M}^{\mathcal{D}}$, the natural map* $\operatorname{hocolim}_{\mathcal{C}} F^* X \to \operatorname{hocolim}_{\mathcal{D}} X$ *is a weak equivalence.*
- *A functor $F : \mathcal{C} \to \mathcal{D}$ is homotopy left cofinal iff for every (framed) model category $\mathcal{M}$ and every object-wise fibrant diagram $X \in \mathcal{M}^{\mathcal{D}}$, the natural map* $\lim_{\mathcal{D}} X \to \operatorname{holim}_{\mathcal{C}} F^* X$ *is a weak equivalence.*

In the above, it's sufficient to take $\mathcal{M} = \mathsf{sSet}$ for being homotopy right/left cofinal.

Remark 1.2.15. A functor $F : \mathcal{C} \to \mathcal{D}$ is homotopy right cofinal iff the map $\mathrm{B}(- \downarrow F)^{\mathrm{op}} \to \mathrm{B}(- \downarrow \mathcal{D})^{\mathrm{op}}$ is a weak equivalence of cofibrant objects in $\mathsf{sSet}^{\mathcal{D}^{\mathrm{op}}}$ (see the proof of [**40**, Theorem 19.6.7]).

For $c \in \mathcal{C}$, if $K \in \mathsf{sSet}^{\mathcal{C}^{\mathrm{op}}}$, the natural map $K_c \to h^c \otimes_{\mathcal{C}} K$ is an isomorphism; if $K \in \mathsf{sSet}^{\mathcal{C}}$, the natural map $\hom^{\mathcal{C}}(h^c, K) \to K_c$ is an isomorphism ([**40**, Proposition 19.6.9]). Here we regarded h^c as a $\mathcal{C}$-diagram of discrete simplicial sets. Moreover, there are natural isomorphisms ([**40**, Corollary 19.6.10])

$$\operatorname{hocolim}_{\mathcal{C}} h^c \cong \mathrm{B}(c \downarrow \mathcal{C})^{\mathrm{op}},$$
$$\operatorname{holim}_{\mathcal{C}} h^c \cong \mathrm{B}(\mathcal{C} \downarrow c).$$

For a functor $F : \mathcal{C} \to \mathcal{D}$ between small categories, $c \in \mathcal{C}$, there are natural isomorphisms ([**40**, Corollary 19.6.11])

$$h^c \otimes_{\mathcal{D}} \mathrm{B}(- \downarrow F)^{\mathrm{op}} \cong \mathrm{B}(c \downarrow F)^{\mathrm{op}},$$
$$\hom^{\mathcal{D}}(h^c, \mathrm{B}(F \downarrow -)) \cong \mathrm{B}(F \downarrow c).$$

The *Bousfield-Kan map* in $\mathsf{cs}\mathcal{S}\mathsf{et}$ is the map $\phi : \mathrm{B}(\mathbf{\Delta} \downarrow -) \to \Delta^\bullet$ that for $k, n \geqslant 0$ takes the n-simplex

$$((\![i_0] \xrightarrow{\sigma_0} [i_1] \xrightarrow{\sigma_1} \cdots \xrightarrow{\sigma_{n-1}} [i_n]\!], ([i_n] \xrightarrow{\tau} [k]))$$

of $\mathrm{B}(\mathbf{\Delta} \downarrow [k])$ to the n-simplex

$$(\tau\sigma_{n-1} \cdots \sigma_1\sigma_0(i_0), \tau\sigma_{n-1} \cdots \sigma_1(i_1), \cdots, \tau\sigma_{n-1}(i_{n-1}), \tau(i_n))$$

of Δ^k. We also denote $\phi : \mathrm{B}(- \downarrow \mathbf{\Delta}^{\mathrm{op}})^{\mathrm{op}} \to \Delta^\bullet$ the composition of the isomorphism $\mathrm{B}(- \downarrow \mathbf{\Delta}^{\mathrm{op}})^{\mathrm{op}} \cong \mathrm{B}(\mathbf{\Delta} \downarrow -)$ with the map $\phi : \mathrm{B}(\mathbf{\Delta} \downarrow -) \to \Delta^\bullet$ (see [**40**, Definition 18.7.1]).

Let $\mathcal{M}$ be a model category. Endow $\mathsf{s}\mathcal{M}$ with the Reedy framed diagram category structure, let $X \in \mathsf{s}\mathcal{M}$, then the *Bousfield-Kan map* for X is the map $\phi_* : \operatorname{hocolim}_{\mathbf{\Delta}^{\mathrm{op}}} X \to |X|$ that is the composition

$$\operatorname{hocolim}_{\mathbf{\Delta}^{\mathrm{op}}} X \cong X \otimes_{\mathbf{\Delta}^{\mathrm{op}}} \mathrm{B}(- \downarrow \mathbf{\Delta}^{\mathrm{op}})^{\mathrm{op}} \xrightarrow{1_X \otimes_{\mathbf{\Delta}^{\mathrm{op}}} \phi} X \otimes_{\mathbf{\Delta}^{\mathrm{op}}} \Delta^\bullet =: |X|.$$

This map is natural in X.

Endow $\mathsf{c}\mathcal{M}$ with the Reedy framed diagram category structure. Let $X \in \mathsf{c}\mathcal{M}$, then the *Bousfield-Kan map* for X is the map $\phi^* : \mathrm{Tot}(X) \to \operatorname{holim}_{\mathbf{\Delta}} X$ that is the composition

$$\mathrm{Tot}(X) := \hom^{\mathbf{\Delta}}(\Delta^\bullet, X) \xrightarrow{\hom^{\mathbf{\Delta}}(\phi, 1_X)} \hom^{\mathbf{\Delta}}(\mathrm{B}(\mathbf{\Delta} \downarrow -), X) \cong \operatorname{holim}_{\mathbf{\Delta}} X.$$

This map is natural in X. (See [**40**, Definition 19.8.6].) Here, $|-| : \mathsf{s}\mathcal{M} \to \mathcal{M}$ is the *realization functor* and $\mathrm{Tot} : \mathsf{c}\mathcal{M} \to \mathcal{M}$ is the *total object functor* (see [**40**, Definition 19.8.1]).

For a simplicial model category $\mathcal{M}$, if X is Reedy cofibrant in $\mathsf{s}\mathcal{M}$, then the Bousfield-Kan map $\phi_* : \operatorname{hocolim}_{\mathbf{\Delta}^{\mathrm{op}}} X \to |X|$ is a weak equivalence; if X is Reedy fibrant in $\mathsf{c}\mathcal{M}$, then the Bousfield-Kan map $\phi^* : \mathrm{Tot}(X) \to \operatorname{holim}_{\mathbf{\Delta}} X$ is a weak equivalence (see [**40**, Theorem 19.8.7]). If we only assume $\mathcal{M}$ is a model category, then we still have natural weak equivalences $\operatorname{hocolim}_{\mathbf{\Delta}^{\mathrm{op}}} X \simeq |X|, \operatorname{holim}_{\mathbf{\Delta}^{\mathrm{op}}} X \simeq \mathrm{Tot}(X)$ (see [**40**, Theorem 19.8.4]).

By Proposition 1.5.6, for $X \in \mathsf{s}\mathcal{S}\mathsf{et}$, view the set X_n as a discrete simplicial set, there is a coequalizer diagram

$$\coprod_{([m] \to [n]) \in \mathbf{\Delta}} X_n \otimes \Delta^m \rightrightarrows \coprod_{[n] \in \mathbf{\Delta}} X_n \otimes \Delta^n \to X.$$

So let $D_X : \mathbf{\Delta}^{\mathrm{op}} \to \mathsf{s}\mathcal{S}\mathsf{et}, [n] \to X_n$, this means that the realization $|D_X|$ is canonically isomorphic to X. Similarly, if $\mathcal{C}$ is a small category and $X \in \mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})$, then we have a canonical isomorphism $|D_X| \cong X$. Moreover, $D_X : \mathbf{\Delta}^{\mathrm{op}} \to \mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})_{\mathrm{inj}}, [n] \to X_n$ is Reedy cofibrant and the Bousfield-Kan map gives a sectionwise weak equivalence $\operatorname{hocolim}_{\mathbf{\Delta}^{\mathrm{op}}} D_X \xrightarrow{\simeq} X$.

Proposition 1.2.16. *Let $\mathcal{M}$ be a combinatorial simplicial model category, $\mathcal{C}$ a small category, and $X \in \mathcal{M}^{\mathcal{C}}$.*

- *The total left derived functor of $\operatorname{colim}_{\mathcal{C}}$ at X is the homotopy colimit of its (functorial) object-wise cofibrant replacement $Q_{\mathrm{inj}}(X) = Q(X)$, i.e. we have $\mathbf{L}\operatorname{colim}_{\mathcal{C}} X \cong \operatorname{colim}_{\mathcal{C}} Q_{\mathrm{proj}}(X) \cong Q_{\mathrm{inj}}(X) \otimes_{\mathcal{C}} \mathrm{B}(- \downarrow \mathcal{C})^{\mathrm{op}}$ in $\mathrm{Ho}(\mathcal{M})$. If moreover X is object-wise cofibrant, then $\mathbf{L}\operatorname{colim}_{\mathcal{C}} X \cong X \otimes_{\mathcal{C}} \mathrm{B}(- \downarrow \mathcal{C})^{\mathrm{op}}$.*
- *The total right derived functor of $\lim_{\mathcal{C}}$ at X is the homotopy limit of its (functorial) object-wise fibrant replacement $R_{\mathrm{proj}}(X) = R(X)$, i.e. we have $\mathbf{R}\lim_{\mathcal{C}} X \cong \lim_{\mathcal{C}} R_{\mathrm{inj}}(X) \cong \hom^{\mathcal{C}}(\mathrm{B}(\mathcal{C} \downarrow -), R_{\mathrm{proj}}(X))$ in $\mathrm{Ho}(\mathcal{M})$. If moreover X is object-wise fibrant, then $\mathbf{R}\lim_{\mathcal{C}} X \cong \hom^{\mathcal{C}}(\mathrm{B}(\mathcal{C} \downarrow -), X)$.*

Proof. This is the main result of [**32**, §4] and its dual. □

Remark 1.2.17. This result together with the homotopy invariance results above indicate that it's better to *define* the homotopy (co)limits as the derived functor, which would be homotopy invariant in the full generality, and this is done much more often in the literature, and we will do so as well. See a similar result [**33**, Chapter VIII, Lemma 2.11] in another setting (diagrams of simplicial sets). For more on the different notions of "homotopy (co)limits", see [**24**, §2.3.2 (pp.57-61)].

So from now on, we will make the *convention* that when computing homotopy (co)limits for diagrams in combinatorial simplicial model categories, we always use the above completely homotopy invariant formulas:

$$\operatorname{hocolim}_{\mathcal{C}} X = Q(X) \otimes_{\mathcal{C}} \mathrm{B}(- \downarrow \mathcal{C})^{\mathrm{op}}, \operatorname{holim}_{\mathcal{C}} X = \hom^{\mathcal{C}}(\mathrm{B}(\mathcal{C} \downarrow -), R(X)),$$

where $Q, R : \mathcal{M} \to \mathcal{M}$ are some functorial cofibrant and fibrant replacement of $\mathcal{M}$. Of course we then lose the nice property in Remark 1.2.7 since we have to take some replacement before using the computing formulas.

1.3. Pre-triangulated categories and pointed model categories

In this section, we collect some nice features about the homotopy categories of pointed model categories, encapsulated in the notion of pre-triangulated categories (but have some other nice properties).

First recall that a *(symmetric) monoidal category* is a category $\mathcal{C}$ with a bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, a unit object S, a natural associativity isomorphism $a = a_{X,Y,Z} : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$, a natural left unit isomorphism $\ell = \ell_X : S \otimes X \to X$, a natural right unit isomorphism $r = r_X : X \otimes S \to X$ (and a commutativity isomorphism $T = T_{X,Y} : X \otimes Y \to Y \otimes X$ with $T \circ T = 1, r = \ell \circ T$) subject to some coherence conditions (which we omit, see [**45**, Definition 4.1.1 and Definition 4.1.4]). Examples include $\mathcal{S}\mathsf{et}, \mathsf{s}\mathcal{S}\mathsf{et}, \mathsf{s}\mathcal{S}\mathsf{et}_*, \mathrm{Ho}\,\mathsf{s}\mathcal{S}\mathsf{et}, \mathrm{Ho}\,\mathsf{s}\mathcal{S}\mathsf{et}_*$.

Let $\mathcal{C}, \mathcal{D}, \mathcal{E}$ be categories. An *adjunction of two variables from* $\mathcal{C} \times \mathcal{D}$ *to* $\mathcal{E}$ (see [**45**, Definition 4.1.12]) is a quintuple $(\otimes, \mathrm{Hom}_r, \mathrm{Hom}_\ell, \varphi_r, \varphi_\ell)$, where $\otimes : \mathcal{C} \times \mathcal{D} \to \mathcal{E}, \mathrm{Hom}_r : \mathcal{D}^{\mathrm{op}} \times \mathcal{E} \to \mathcal{C}, \mathrm{Hom}_\ell : \mathcal{C}^{\mathrm{op}} \times \mathcal{E} \to \mathcal{D}$ are functors, and φ_r, φ_ℓ are natural isomorphisms of functors

$$\mathcal{C}(C, \mathrm{Hom}_r(D, E)) \xleftarrow{\varphi_r} \mathcal{E}(C \otimes D, E) \xrightarrow{\varphi_\ell} \mathcal{D}(D, \mathrm{Hom}_\ell(C, E)).$$

Sometimes we write this adjunction as $(\otimes, \mathrm{Hom}_r, \mathrm{Hom}_\ell, \varphi_r, \varphi_\ell)$ or $(\otimes, \mathrm{Hom}_r, \mathrm{Hom}_\ell)$ or $\otimes : \mathcal{C} \times \mathcal{D} \to \mathcal{E}$.

A *closed monoidal category* is a category $\mathcal{C}$ with an octuple $(\otimes, a, \ell, r, \mathrm{Hom}_r, \mathrm{Hom}_\ell, \varphi_r, \varphi_\ell)$ making $\mathcal{C}$ a monoidal category and $(\otimes, \mathrm{Hom}_r, \mathrm{Hom}_\ell, \varphi_r, \varphi_\ell) : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is an adjunction of two variables.

Then a *(right) closed* $\mathcal{C}$*-module* is a category $\mathcal{M}$ with functors $\wedge : \mathcal{M} \times \mathcal{C} \to \mathcal{M}, \mathrm{Hom} : \mathcal{C}^{\mathrm{op}} \times \mathcal{M} \to \mathcal{M}, \mathrm{Map} : \mathcal{M}^{\mathrm{op}} \times \mathcal{M} \to \mathcal{C}$ satisfying some coherence conditions and such that $(\wedge, \mathrm{Hom}, \mathrm{Map}) : \mathcal{M} \times \mathcal{C} \to \mathcal{M}$ is an adjunction of two variables. So we have natural isomorphisms $(X \wedge K) \wedge L \cong X \wedge (K \otimes L)$ and natural isomorphisms

$$\mathcal{M}(X, \mathrm{Hom}(K, Y)) \xleftarrow{\varphi_r} \mathcal{M}(X \wedge K, Y) \xrightarrow{\varphi_\ell} \mathcal{C}(K, \mathrm{Map}(X, Y)).$$

For any model category $\mathcal{M}$, its homotopy category $\mathrm{Ho}\,\mathsf{sSet}$ is a closed $\mathrm{Ho}\,\mathsf{sSet}$-module, and if $\mathcal{M}$ is pointed, then $\mathrm{Ho}\,\mathcal{M}$ is a closed $\mathrm{Ho}\,\mathsf{sSet}_*$-module (see the discussion at the end of this section). There are also the notions of (symmetric) monoidal functors between (symmetric) monoidal categories, a (symmetric) $\mathcal{C}$-algebra, monoidal model categories and their modules, etc. (for all these and many examples, see [**45**, Chapter 4]).

Write $\mathcal{H}_* = \mathrm{Ho}\,\mathsf{sSet}_*$. Let $\mathcal{S}$ be a (right) closed $\mathcal{H}$-module, then $\mathcal{S}$ is also pointed. We sometimes write $[A, Y] = [A, Y]_{\mathcal{S}}$ for the hom-set $\mathcal{S}(A, Y)$ and we denote the zero map in $[A, Y]$ by $0 = 0_{A,Y} = [*]$. We then have the *suspension functor* Σ and the *loop functor* Ω given by $\Sigma A = A \wedge S^1, \Omega A = \mathrm{Hom}(S^1, A)$, they form an adjoint pair $\Sigma : \mathcal{S} \rightleftarrows \mathcal{S} : \Omega$. Moreover, for $n \in \mathbb{N}$, we have $\Sigma^n A = A \wedge S^n, \Omega^n A = \mathrm{Hom}(S^n, A)$. Thus for objects $A, Y \in \mathcal{S}$ we have natural isomorphisms

$$[\Sigma^n A, Y] \cong [A, \Omega^n Y] \cong [S^n, \mathrm{Map}(A, Y)]_{\mathsf{sSet}_*} \cong \pi_n \mathrm{Map}(A, Y).$$

In particular, we have natural isomorphisms (cf. [**2**, Lemma 5.5.2])

$$[A, Y] \cong \pi_0 \mathrm{Map}(A, Y).$$

So in $\mathcal{S}$, ΣA is a cogroup object and $\Sigma^n A$ is an abelian cogroup object for $n \geqslant 2$, ΩA is a group object and $\Omega^n A$ is an abelian group object for $n \geqslant 2$.

For any $n \geqslant 1$, there is a (right) action of ΩY on $\Omega^n Y$, being the conjugation for $n = 1$, which for any object W, is given by the action of $\pi_1 \mathrm{Map}(W, Y)$ on $\pi_n \mathrm{Map}(W, Y)$ (see Section 1.6). Dually, there is a (right) coaction of ΣA on $\Sigma^n A$.

We are now ready to introduce the notion of pre-triangulated categories ([**45**, Definition 6.5.1]).

Definition 1.3.1. A **pre-triangulated category** is a nontrivial (right) closed $\mathcal{H}_*$-module $\mathcal{S}$, with a collection **CS** of *cofiber sequences*, or *left triangles*, and a collection **FS** of *fiber sequences*, or *right triangles*, satisfying the following conditions.

(a) A cofiber sequence is of the form $X \xrightarrow{f} Y \xrightarrow{g} Z$ of morphisms in $\mathcal{S}$ with a right coaction of the cogroup object ΣX on Z. A fiber sequence is of the form $X \xrightarrow{f} Y \xrightarrow{g} Z$ of morphisms in $\mathcal{S}$ with a right action of the group object ΩZ on X.

(b) The two collections **CS**, **FS** are closed under isomorphisms.

(c) For any object X in $\mathcal{S}$, we have $(* \to X \xrightarrow{1} X) \in \mathbf{CS}, (X \xrightarrow{1} X \to *) \in \mathbf{FS}$.

(d) For any morphism $f : X \to Y$ in $\mathcal{S}$, there exist morphisms $g : Y \to Z, h : W \to X$ such that $(X \xrightarrow{f} Y \xrightarrow{g} Z) \in \mathbf{CS}, (W \xrightarrow{h} X \xrightarrow{f} Y) \in \mathbf{FS}$.

(e) Cofiber sequences shift to the right: for any $(X \xrightarrow{f} Y \xrightarrow{g} Z) \in \mathbf{CS}$, we have $(Y \xrightarrow{g} Z \xrightarrow{\partial} \Sigma X) \in \mathbf{CS}$, where ∂ is the composite $Z \to Z \coprod \Sigma X \xrightarrow{0 \coprod 1} \Sigma X$, the first map is the coaction; so for an object W in $\mathcal{S}$ and $a \in [\Sigma X, W]$ we have $\partial^* a = [*] \cdot a \in [Z, W]$. The ΣY-coaction on ΣX is given by the composite $\Sigma X \to \Sigma X \coprod \Sigma X \xrightarrow{1 \coprod \Sigma f} \Sigma X \coprod \Sigma Y \xrightarrow{1 \coprod i} \Sigma X \coprod \Sigma Y$, where the first map is the comultiplication morphism of ΣX, i is the coinverse of ΣY.[6] Any cofiber sequence isomorphic to a shifted one $Y \xrightarrow{g} Z \xrightarrow{\partial} \Sigma X$ is called a *principal cofiber sequence*.

Dually, fiber sequences shift to the left: for any $(X \xrightarrow{f} Y \xrightarrow{g} Z) \in \mathbf{FS}$, we have $(\Omega Z \xrightarrow{\partial} X \xrightarrow{f} Y) \in \mathbf{FS}$, where ∂ is the composite $\Omega Z \xrightarrow{(0,1)} X \times \Omega Z \to X$, the second map is the action; so for an object W in $\mathcal{S}$ and $a \in [W, \Omega Z]$ we have $\partial_* a = [*] \cdot a \in [W, X]$. The ΩY-action on ΩZ is given by the composite $\Omega Z \times \Omega Y \xrightarrow{1 \times i} \Omega Z \times \Omega Y \xrightarrow{1 \times \Omega g} \Omega Z \times \Omega Z \to \Omega Z$, where the last map is the multiplication morphism of ΩZ, i is the inverse of ΩY.[7] Any fiber sequence isomorphic to a shifted one $\Omega Z \xrightarrow{\partial} X \xrightarrow{f} Y$ is called a *principal fiber sequence*.

(f) Fill-in maps exist: for any solid-arrow commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} & & \vdots{\scriptstyle\gamma} \\ X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' \end{array}$$

[6] See Section 1.1.

[7] For the classical results of this kind on pointed topological spaces, see [**2**, Theorem 4.3.4 and Theorem 4.3.7].

in $\mathcal{S}$ where the rows are in **CS**, there is a $\Sigma\alpha$-equivariant map $\gamma : Z \to Z'$ making the entire diagram commutative.

Dually, for any solid-arrow commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \\ {\scriptstyle\alpha}\vdots & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} \\ X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' \end{array}$$

in $\mathcal{S}$ where the rows are in **FS**, there is an $\Omega\gamma$-equivariant map $\alpha : X \to X'$ making the entire diagram commutative.

(g) Verdier's octahedral axiom and its dual hold: for maps $X \xrightarrow{u} Y \xrightarrow{v} Z$ in $\mathcal{S}$, there is a commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{u} & Y & \xrightarrow{d} & C_u \curvearrowleft \Sigma X \\ {\scriptstyle 1}\| & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle r} \\ X & \xrightarrow{vu} & Z & \xrightarrow{a} & C_{vu} \curvearrowleft \Sigma X \\ & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle s} \quad \downarrow{\scriptstyle \Sigma u} \\ & & C_v & \xrightarrow{1} & C_v \curvearrowleft \Sigma Y \end{array}$$

where the 3-term rows and columns are in **CS**, r is ΣX-equivariant, s is Σu-equivariant, and the ΣC_u-coaction on C_v is given by the composite $C_v \to C_v \coprod \Sigma Y \xrightarrow{1 \coprod \Sigma d} C_v \coprod \Sigma C_u$.

There is also a commutative diagram

$$\begin{array}{ccccc} \Omega Y & \xrightarrow{\Omega v} & \Omega Z & & \Omega Z \\ \curvearrowleft & & \curvearrowleft & & \curvearrowleft \\ F_u & \xrightarrow{i} & F_{vu} & \xrightarrow{q} & F_v \\ {\scriptstyle 1}\| & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} \\ F_u & \xrightarrow{g} & X & \xrightarrow{u} & Y \\ & & \downarrow{\scriptstyle vu} & & \downarrow{\scriptstyle v} \\ & & Z & \xrightarrow{1} & Z \end{array}$$

where the 3-term rows and columns are in **FS**, q is ΩZ-equivariant, i is Ωv-equivariant, and the ΩF_v-action on F_u is given by the composite $F_u \times \Omega F_v \xrightarrow{1\times\Omega c} F_u \times \Omega Y \to F_u$.[8]

(h) Cofiber and fiber sequences are compatible: let $(X \xrightarrow{f} Y \xrightarrow{g} Z) \in \mathbf{CS}, (X' \xrightarrow{i} Y' \xrightarrow{p} Z') \in \mathbf{FS}$. Suppose we have a solid-arrow commutative diagram

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{\partial} & \Sigma X \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} & & {\scriptstyle\gamma}\vdots & & \downarrow{\scriptstyle(\alpha^\sharp)^{-1}} \\ \Omega Z' & \xrightarrow{\partial} & X' & \xrightarrow{i} & Y' & \xrightarrow{p} & Z' \end{array}$$

in $\mathcal{S}$ where $\alpha^\sharp$ is the element of the group $\mathcal{S}(\Sigma X, Z')$ adjoint of α and -1 means taking inverse in that group, the dotted arrow γ exists making the entire diagram commutative.

Dually, suppose we have a solid-arrow commutative diagram

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{\partial} & \Sigma X \\ {\scriptstyle(\delta^\flat)^{-1}}\downarrow & & {\scriptstyle\beta}\vdots & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\delta} \\ \Omega Z' & \xrightarrow{\partial} & X' & \xrightarrow{i} & Y' & \xrightarrow{p} & Z' \end{array}$$

in $\mathcal{S}$, the dotted arrow β exists making the entire diagram commutative.

(i) The functor $\wedge : \mathcal{S} \times \mathcal{H}_* \to \mathcal{S}$ preserves cofiber sequences in both variables; the functors $\mathrm{Hom} : \mathcal{H}_*^{\mathrm{op}} \times \mathcal{S} \to \mathcal{S}$ and $\mathrm{Map} : \mathcal{S}^{\mathrm{op}} \times \mathcal{S} \to \mathcal{H}_*$ preserve fiber sequences in the second variable and convert cofiber sequences to fiber sequences in the first variable (see [**45**, Corollary 6.4.2 and Proposition 6.4.5] for the related (co)actions).

A large portion of [**45**, §6.1-§6.4] is devoted to prove that *the homotopy categories of pointed model categories are pre-triangulated categories* in the above sense, with structure maps $(\wedge^{\mathrm{L}}, \mathrm{RHom}_*, \mathrm{RMap}_{*r})$ or $(\wedge^{\mathrm{L}}, \mathrm{RHom}_*, \mathrm{RMap}_{*\ell})$ (and these two are equivalent, see below), where in particular the (co)fiber sequences are those isomorphic to the "standard" ones ([**45**, Definition 6.2.6]), the related actions are given by [**45**, Theorem 6.2.1], and the condition (i) above is treated in [**45**, §6.4]. We make the caution that when using the above formalism in specific situations, one should replace $(\wedge, \mathrm{Hom}, \mathrm{Map}, \Sigma, \Omega)$ by their specific incarnations; typically we are dealing with homotopy categories of pointed model categories, then we should use the derived forms of these functors.

The following results generalize Theorem 1.6.8 and Remark 1.6.9.

[8] To be compared with the following facts in abelian categories: $(Z/X)/(Y/X) = Z/Y, \ker u = \ker(\ker vu \xrightarrow{u} \ker v)$.

Proposition 1.3.2. *Let $\mathcal{S}$ be a pre-triangulated category. For any fiber sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{S}$, we have a long sequence*

$$\cdots \to \Omega^{n+1}Z \xrightarrow{\Omega^n\partial} \Omega^n X \xrightarrow{\Omega^n f} \Omega^n Y \xrightarrow{\Omega^n g} \Omega^n Z \xrightarrow{\Omega^{n-1}\partial} \cdots \xrightarrow{\Omega\partial} \Omega X \xrightarrow{\Omega f} \Omega Y \xrightarrow{\Omega g} \Omega Z \xrightarrow{\partial} X \xrightarrow{f} Y \xrightarrow{g} Z,$$

in which, every two consecutive arrows form a fiber sequence.

There are (right) actions of ΩY on every object above: for $n \geqslant 1$, the action on $\Omega^n Z$ is given by the composition $\Omega^n Z \times \Omega Y \xrightarrow{1\times\Omega g} \Omega^n Z \times \Omega Z \xrightarrow{\cdot} \Omega^n Z$, and the action on $\Omega^n X$ fits into the following commutative diagram

$$\begin{array}{ccc}
\Omega^n X \times \Omega X & \xrightarrow{\cdot} & \Omega^n X \\
{\scriptstyle 1\times\Omega f}\downarrow & & \downarrow{\scriptstyle 1} \\
\Omega^n X \times \Omega Y & \xrightarrow{\cdot} & \Omega^n X \\
{\scriptstyle \Omega^n f\times 1}\downarrow & & \downarrow{\scriptstyle \Omega^n f} \\
\Omega^n Y \times \Omega Y & \xrightarrow{\cdot} & \Omega^n Y.
\end{array}$$

If we endow X, Y, Z with the trivial ΩY-action, then the above sequence is ΩY-equivariant.[9]

Dually, for any cofiber sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{S}$, there is a ΣY-equivariant long sequence, in which, every two consecutive arrows form a cofiber sequence.

Theorem 1.3.3 ([**45**, Proposition 6.5.3])**.** *Let $\mathcal{S}$ be a pre-triangulated category.*

(a) *Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be a cofiber sequence in $\mathcal{S}$, then for any object W of $\mathcal{S}$, there is a $[\Sigma Y, W]$-equivariant long exact sequence of pointed sets*

$$\cdots \xrightarrow{(\Sigma\partial)^*} [\Sigma Z, W] \xrightarrow{(\Sigma g)^*} [\Sigma Y, W] \xrightarrow{(\Sigma f)^*} [\Sigma X, W] \xrightarrow{\partial^*} [Z, W] \xrightarrow{g^*} [Y, W] \xrightarrow{f^*} [X, W].$$

This sequence is natural in $(X \xrightarrow{f} Y \xrightarrow{g} Z) \in \mathbf{CS}$. Moreover,

- *for $a, b \in [Z, W]$, we have $g^*a = g^*b$ iff there exists $x \in [\Sigma X, W]$ such that $a \cdot x = b$, i.e. a, b are in the same orbit of the group action;*
- *for $c, d \in [\Sigma X, W]$, we have $\partial^* c = \partial^* d$ iff there exists $y \in [\Sigma Y, W]$ such that $c = d \cdot (\Sigma f)^* y$ in the group $[\Sigma X, W]$.*

(b) *Given a commutative diagram*

$$\begin{array}{ccccc}
X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \\
{\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} \\
X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z'
\end{array}$$

where the rows are cofiber sequences and the map γ is $\Sigma\alpha$-equivariant. If α, β are isomorphisms, then so is γ.

Below is the dual result, for which we give a proof (sketch). See also [**2**, Corollary 4.2.12 and Corollary 4.2.19] and [**2**, Theorem 4.4.5 and Theorem 4.4.8] in the classical setting of pointed topological spaces (and see [**2**, §5.4] for interesting results on special W).

Theorem 1.3.4. *Let $\mathcal{S}$ be a pre-triangulated category.*

(a) *Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be a fiber sequence in $\mathcal{S}$, then for any object W of $\mathcal{S}$, there is a $[W, \Omega Y]$-equivariant long exact sequence of pointed sets*

$$\cdots \xrightarrow{(\Omega\partial)_*} [W, \Omega X] \xrightarrow{(\Omega f)_*} [W, \Omega Y] \xrightarrow{(\Omega g)_*} [W, \Omega Z] \xrightarrow{\partial_*} [W, X] \xrightarrow{f_*} [W, Y] \xrightarrow{g_*} [W, Z].$$

This sequence is natural in $(X \xrightarrow{f} Y \xrightarrow{g} Z) \in \mathbf{FS}$. Moreover,

- *for $a, b \in [W, X]$, we have $f_*a = f_*b$ iff there exists $z \in [W, \Omega Z]$ such that $a \cdot z = b$, i.e. a, b are in the same orbit of the group action;*
- *for $c, d \in [W, \Omega Z]$, we have $\partial_* c = \partial_* d$ iff there exists $y \in [W, \Omega Y]$ such that $c = d \cdot (\Omega g)_* y$ in the group $[W, \Omega Z]$.*

(b) *Given a commutative diagram*

$$\begin{array}{ccccc}
X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \\
{\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} \\
X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z'
\end{array}$$

where the rows are fiber sequences and the map α is $\Omega\gamma$-equivariant. If β, γ are isomorphisms, then so is α.

Proof. For (a), note that by condition (i) in the definition of pre-triangulated categories, there is a fiber sequence

$$\mathrm{Map}(W, X) \xrightarrow{f} \mathrm{Map}(W, Y) \xrightarrow{g} \mathrm{Map}(W, Z)$$

in $\mathcal{H}_* = \mathrm{Ho}\,\mathsf{s}\mathcal{S}\mathsf{et}_*$. Since we have natural isomorphisms

$$[W, \Omega^n X] \cong \pi_n \mathrm{Map}(W, X),$$

[9] See Remark 1.6.9.

the results then follow from the results on simplicial sets as in Theorem 1.6.8 (of course one should check that under the above isomorphisms, all related maps are compatible, i.e. these isomorphisms give a commutative diagram of the homotopy sequences, which is essentially due to the adjunctions and the way we define the maps), or the corresponding results in topological spaces, as Hovey did.

Now we treat (b). By (a), for any object W of $\mathcal{S}$, we have the following commutative diagram:

$$\begin{array}{ccccccccc}
[W,\Omega Y] & \xrightarrow{(\Omega g)_*} & [W,\Omega Z] & \xrightarrow{\partial_*} & [W,X] & \xrightarrow{f_*} & [W,Y] & \xrightarrow{g_*} & [W,Z] \\
\downarrow{\scriptstyle (\Omega\beta)_*} & & \downarrow{\scriptstyle (\Omega\gamma)_*} & & \downarrow{\scriptstyle \alpha_*} & & \downarrow{\scriptstyle \beta_*} & & \downarrow{\scriptstyle \gamma_*} \\
[W,\Omega Y'] & \xrightarrow{(\Omega g')_*} & [W,\Omega Z'] & \xrightarrow{\partial_*} & [W,X'] & \xrightarrow{f'_*} & [W,Y'] & \xrightarrow{g'_*} & [W,Z'].
\end{array}$$

We will show that the middle vertical map $\alpha_* : [W,X] \to [W,X']$ is a bijection, then α is an isomorphism by Yoneda lemma.

For any $x' \in [W,X']$, let $y' := f'_* x' = \beta_* y$, then $\gamma_* g_* y = g'_* \beta_* y = g'_* f'_* x' = 0$, thus $g_* y = 0$. Take $x \in [W,X]$ with $f_* x = y$, then $f'_* \alpha_* x = \beta_* f_* x = \beta_* y = y' = f'_* x'$, so by (a), there exist $z \in [W,\Omega Z], z' = (\Omega\gamma)_* z \in [W,\Omega Z']$ such that $x' = (\alpha_* x) \cdot z' = \alpha_*(x \cdot z) \in \operatorname{im} \alpha_*$ (as α_* is $(\Omega\gamma)_*$-equivariant), so α_* is surjective.

Now we prove that the restriction of α_* to the orbit of the base point $0 = [*]$ of the pointed set $[W,X]$ is injective. Let $h \in [W,\Omega Z], b = [*] \cdot h \in [W,X]$ and suppose $\alpha_*(b) = [*] \in [W,X']$, i.e. $[*] \cdot h' = [*]$, where $h' := (\Omega\gamma)_* h \in [W,\Omega Z']$. Note that $\partial_* h' = [*] \cdot h'$, so we have $\partial_* h' = [*], h' \in \ker \partial_* = \operatorname{im}(\Omega g')_*$, so we can write $h' = (\Omega g')_* u'$ for some $u' \in [W,\Omega Y']$. Thus $u' = (\Omega\beta)_* u$ for some $u \in [W,\Omega Y]$ and $h = (\Omega g)_* u$. Thus $b = [*] \cdot h = [*] \cdot (\Omega g)_* u = \partial_* (\Omega g)_* u = [*]$. This proves the injectivity of the restriction of α_* to the orbit of $0 = [*]$ of the pointed set $[W,X]$.

Next we perform the "*change of base point*" procedure for the pointed set $[W,X]$ to prove the injectivity of the restriction of α_* to general orbits of the pointed set $[W,X]$. Of course we cannot change base points for objects in the abstract category $\mathcal{S}$, but we can indeed do this in our fundamental category $\mathcal{H}_* = \mathrm{Ho}\, \mathsf{s}\mathcal{S}\mathsf{et}_*$. For this, we also need to *change viewpoint* a bit, as follows. We first replace the above fiber sequence $\mathrm{Map}(W,X) \xrightarrow{f} \mathrm{Map}(W,Y) \xrightarrow{g} \mathrm{Map}(W,Z)$ in $\mathcal{H}_* = \mathrm{Ho}\, \mathsf{s}\mathcal{S}\mathsf{et}_*$ by an isomorphic fiber sequence $F \xrightarrow{i} E \xrightarrow{p} B$, with $p : E \to B$ a Kan fibration of Kan complexes in $\mathsf{s}\mathcal{S}\mathsf{et}_* = (* \downarrow \mathsf{s}\mathcal{S}\mathsf{et})$ and $i : F \to E$ is the inclusion of the fiber (over the base point of B, and F shares the same base point as E). We do the same for the second row and get a commutative diagram:[10]

$$\begin{array}{ccccccccc}
\pi_1(E) & \xrightarrow{p_*} & \pi_1(B) & \xrightarrow{\partial_*} & \pi_0(F) & \xrightarrow{i_*} & \pi_0(E) & \xrightarrow{p_*} & \pi_0(B) \\
\downarrow{\scriptstyle (\Omega\beta)_*} & & \downarrow{\scriptstyle (\Omega\gamma)_*} & & \downarrow{\scriptstyle \alpha_*} & & \downarrow{\scriptstyle \beta_*} & & \downarrow{\scriptstyle \gamma_*} \\
\pi_1(E') & \xrightarrow{p'_*} & \pi_1(B') & \xrightarrow{\partial_*} & \pi_0(F') & \xrightarrow{i'_*} & \pi_0(E') & \xrightarrow{p'_*} & \pi_0(B').
\end{array}$$

Now we assume $h \in \pi_1(B), a \in \pi_0(F), b = a \cdot h, \alpha_* a = \alpha_* b$. We change the base point of F and E from the original $*$ to $a \in F$. Note that the map ∂_* will change accordingly (taking values in the orbit of the new base point a) and we get a new diagram as above (by [**40**, Theorem 7.6.5], with new base points, $F \xrightarrow{i} E \xrightarrow{p} B$ is still a fiber sequence in $\mathcal{H}$); the map $(\Omega\beta)_*$ also changes but still is an isomorphism. Now α_* maps the (new) base point a and the point b in F to the new base point $a' = \alpha_*(a) \in \pi_0(F')$. We conclude by the result proven above.[11] □

Now we present some further results on function complexes. By the discussion of Example A.2.6, there are equivalences of categories

$$\mathsf{c}\mathcal{M} = \mathcal{M}^{\mathbf{\Delta}} \xrightarrow{\simeq} \mathrm{Adj}(\mathsf{s}\mathcal{S}\mathsf{et}, \mathcal{M})$$

given by mapping $A^\bullet$ to the adjoint pair $A^\bullet \otimes - : \mathsf{s}\mathcal{S}\mathsf{et} \rightleftarrows \mathcal{M} : \mathcal{M}(A^\bullet, -)$, where $\mathcal{M}(A^\bullet, Y)$ is as defined at the beginning of this section, and

$$\mathsf{s}\mathcal{M} = \mathcal{M}^{\mathbf{\Delta}^{\mathrm{op}}} \xrightarrow{\simeq} \mathrm{Adj}(\mathcal{M}, \mathsf{s}\mathcal{S}\mathsf{et}^{\mathrm{op}})$$

given by mapping $Y_\bullet$ to the pair of functors $\mathcal{M}(-, Y_\bullet) : \mathcal{M}^{\mathrm{op}} \to \mathsf{s}\mathcal{S}\mathsf{et}$ and $\mathrm{Hom}(-, Y_\bullet) : \mathsf{s}\mathcal{S}\mathsf{et}^{\mathrm{op}} \to \mathcal{M}$, where $\mathcal{M}(A, Y_\bullet)$ is as defined at the beginning of Section 1.2 (see [**45**, Proposition 3.1.5 and Remark 3.1.7]).

Any model category $\mathcal{M}$ has a canonical framing ([**45**, Theorem 5.2.8]) in the sense of [**45**, Definition 5.2.7], denoted A° and $A_\circ$. For $A, Y \in \mathcal{M}$, we thus have the factorizations $\ell^\bullet A \to A^\circ \to r^\bullet A$ in $\mathsf{c}\mathcal{M}$ (a cofibration followed by a trivial fibration that is an isomorphism in degree 0), and $\ell_\bullet A \to A_\circ \to r_\bullet A$ in $\mathsf{s}\mathcal{M}$ (a trivial cofibration followed by a fibration that is an isomorphism in degree 0) with $A^{\circ 0} = A = A_{\circ 0}$. Where $\ell^n A = \coprod_{n+1} A, r^n A = A$, $\ell_n A = A, r_n A = \prod_{n+1} A$, and there are adjunctions $\ell^\bullet : \mathcal{M} \rightleftarrows \mathsf{c}\mathcal{M} : \mathrm{Ev}^0 : \mathsf{c}\mathcal{M} \rightleftarrows \mathcal{M} : r^\bullet$ and $\ell_\bullet : \mathcal{M} \rightleftarrows \mathsf{s}\mathcal{M} : \mathrm{Ev}_0 : \mathsf{s}\mathcal{M} \rightleftarrows \mathcal{M} : r_\bullet$ (cf. Example A.2.7 and see [**45**, Example 5.2.4]). So if A is cofibrant, then so is $\ell^\bullet A$; if A is fibrant, then so is $r_\bullet A$. This implies that the framing defined as in [**45**, Definition 5.2.7] is slightly more restrictive than that in [**40**]. However this will have some advantages sometimes, as we have a good choice of framing as A° and $A_\circ$. If A is cofibrant, then A° is a fibrant cosimplicial resolution and if Y is cofibrant, then $Y_\circ$ is a cofibrant simplicial resolution (in the sense of [**40**, Definition 16.1.2]).

Hence for $A, Y \in \mathcal{M}$, we get the adjunctions (canonically associated to the model structure, as functorial factorization of maps is part of the model structure)

$$A \otimes - : \mathsf{s}\mathcal{S}\mathsf{et} \rightleftarrows \mathcal{M} : \mathrm{Map}_\ell(A, -)$$

[10] Of course we could do so from the beginning, once and for all.

[11] We thus get a result for the stabilizers under the same assumption: for any $x \in [W,X]$, $(\Omega\gamma)_*(\mathrm{Stab}(x)) = \mathrm{Stab}(\alpha_*(x))$ as subgroups of $[W,\Omega Z']$.

and

$$(Y^-)^{\mathrm{op}} = \mathrm{Hom}(-,Y)^{\mathrm{op}} : \mathsf{s\mathcal{S}et} \rightleftarrows \mathcal{M}^{\mathrm{op}} : \mathrm{Map}_r(-,Y).$$

We call $\mathrm{Map}_\ell(A,Y)$ the *left function complex* and $\mathrm{Map}_r(A,Y)$ the *right function complex* from A to Y (see [**45**, Remark 5.2.9]). If A is cofibrant, Y is fibrant in $\mathcal{M}$ and equip $\mathcal{M}^{\mathrm{op}}$ with the dual model structure, then the above two adjunctions are Quillen adjunctions (see [**45**, Corollary 5.4.4]).

We thus have the following functors

$$\otimes : \mathcal{M} \times \mathsf{s\mathcal{S}et} \to \mathcal{M}, (A,K) \mapsto A \otimes K := A^\circ \otimes K,$$

$$\mathrm{Map}_\ell : \mathcal{M}^{\mathrm{op}} \times \mathcal{M} \to \mathsf{s\mathcal{S}et}, (A,Y) \mapsto \mathrm{Map}_\ell(A,Y) := \mathcal{M}(A^\circ, Y)$$

satisfying $\mathcal{M}(A \otimes K, Y) \cong \mathsf{s\mathcal{S}et}(K, \mathrm{Map}_\ell(A,Y))$, and

$$\mathrm{Hom} : \mathsf{s\mathcal{S}et}^{\mathrm{op}} \times \mathcal{M} \to \mathcal{M}, (K,Y) \mapsto Y^K = \mathrm{Hom}(K,Y) := \mathrm{Hom}(K, Y_\circ),$$

$$\mathrm{Map}_r : \mathcal{M}^{\mathrm{op}} \times \mathcal{M} \to \mathsf{s\mathcal{S}et}, (A,Y) \mapsto \mathrm{Map}_r(A,Y) := \mathcal{M}(A, Y_\circ)$$

satisfying $\mathcal{M}(A, \mathrm{Hom}(K,Y)) \cong \mathsf{s\mathcal{S}et}(K, \mathrm{Map}_r(A,Y))$ (see [**45**, Remark 5.2.9]), see (1.1) and (1.2) for the related notations, which are the same as those given in [**45**, Proposition 3.1.5 and Remark 3.1.7], see also [**40**, Definition 16.3.1].

Note however that if $\mathcal{M}$ is a simplicial model category, then the above canonical framing A° and $A_\circ$ is in general not the one yields by the simplicial operations with $\Delta^\bullet$ ([**45**, Remark 5.2.10]), i.e. the standard framing.

The total left derived functor of $\otimes : \mathcal{M} \times \mathsf{s\mathcal{S}et} \to \mathcal{M}$ exists, denoted by $\otimes^{\mathrm{L}} = \mathbf{L}\otimes : (\mathrm{Ho}\ \mathcal{M}) \times (\mathrm{Ho}\ \mathsf{s\mathcal{S}et}) \to \mathrm{Ho}\ \mathcal{M}$. There is a natural isomorphism $X \otimes^{\mathrm{L}} \Delta^0 \cong X$ in Ho $\mathcal{M}$; the total right derived functors of $\mathrm{Hom} : \mathsf{s\mathcal{S}et}^{\mathrm{op}} \times \mathcal{M} \to \mathcal{M}$ and $\mathrm{Map}_\ell, \mathrm{Map}_r : \mathcal{M}^{\mathrm{op}} \times \mathcal{M} \to \mathsf{s\mathcal{S}et}$ exist, denoted respectively by $\mathrm{RHom} = \mathbf{R}\mathrm{Hom} : (\mathrm{Ho}\ \mathsf{s\mathcal{S}et})^{\mathrm{op}} \times (\mathrm{Ho}\ \mathcal{M}) \to \mathrm{Ho}\ \mathcal{M}$ and $\mathrm{RMap}_\ell = \mathbf{R}\mathrm{Map}_\ell, \mathrm{RMap}_r = \mathbf{R}\mathrm{Map}_r : (\mathrm{Ho}\ \mathcal{M})^{\mathrm{op}} \times (\mathrm{Ho}\ \mathcal{M}) \to \mathrm{Ho}\ \mathsf{s\mathcal{S}et}$, they are naturally isomorphic: $\mathrm{RMap}_\ell \xrightarrow{\cong} \mathrm{RMap}_r$. These functors are computed as follows: for $K \in \mathsf{s\mathcal{S}et}$ and $X, Y \in \mathcal{M}$,

$$X \otimes^{\mathrm{L}} K = (QX)^\circ \otimes K \in \mathcal{M},$$

$$\mathrm{RHom}(K,Y) = \mathrm{Hom}(K, (RY)_\circ) \in \mathcal{M},$$

$$\mathrm{RMap}_\ell(X,Y) = \mathcal{M}((QX)^\circ, RY) \in \mathsf{s\mathcal{S}et},$$

$$\mathrm{RMap}_r(X,Y) = \mathcal{M}(QX, (RY)_\circ) \in \mathsf{s\mathcal{S}et}.$$

The latter two RMap_ℓ and RMap_r always take values in Kan complexes ([**40**, Proposition 16.6.3 and Corollary 16.5.3]). By [**45**, Proposition 5.4.5] and [**40**, Proposition 16.6.3, 16.1.13, and Corollary 16.5.5], up to isomorphisms in $\mathcal{H} = \mathrm{Ho}\ \mathsf{s\mathcal{S}et}$, in the above computation formulas, we can use any framing other than the canonical framing (A° and $A_\circ$). We use the canonical framing as it's functorial. Thus all the homotopy function complexes from X to Y are weakly equivalent to $\mathrm{RMap}_r(X,Y)$ (use [**40**, Theorem 17.4.14]).

In particular, if $\mathcal{M}$ is a simplicial model category, we can use the standard framing $\widetilde{X} = X \otimes \Delta^\bullet$ and $\widehat{X} = X^{\Delta^\bullet}$, so we have weak equivalences

$$\mathrm{RMap}_\ell(X,Y) \simeq \mathrm{RMap}_r(X,Y) \simeq \mathrm{Map}(QX, RY) = \mathrm{RMap}(X,Y)$$

in $\mathsf{s\mathcal{S}et}$ (use [**40**, Proposition 9.1.9 and Corollary 9.3.3]); similarly, the functors $\otimes^{\mathrm{L}}$ and RHom can indeed be identified (up to equivalences of functors) with the derived functors of the tensor product $\otimes$ and the power map that are part of the simplicial structure of $\mathcal{M}$. When working with simplicial model categories, we will always use RMap, rather than RMap_ℓ or RMap_r.

There are natural isomorphisms

$$[X \otimes^{\mathrm{L}} K, Y] \cong [K, \mathrm{RMap}_\ell(X,Y)] \cong [K, \mathrm{RMap}_r(X,Y)] \cong [X, \mathrm{RHom}(K,Y)]. \tag{1.3}$$

They form an adjunction of two variables $(\mathrm{Ho}\ \mathcal{M}) \times (\mathrm{Ho}\ \mathsf{s\mathcal{S}et}) \to \mathrm{Ho}\ \mathcal{M}$, for which Ho $\mathcal{M}$ is a closed (Ho $\mathsf{s\mathcal{S}et}$)-module (see [**45**, Theorem 5.4.9 and Theorem 5.5.3]).

Assume now that $F : \mathcal{M} \rightleftarrows \mathcal{N} : G$ is a Quillen pair between model categories, then by eq. (1.3), for any $K \in \mathsf{s\mathcal{S}et}, A \in \mathcal{M}, Y \in \mathcal{N}$, we have natural isomorphisms (of sets)

$$\begin{aligned}[K, \mathrm{RMap}_r(FA,Y)] \cong & [K, \mathrm{RMap}_r(FQA, RY)] \cong [(FQA) \otimes^{\mathrm{L}} K, RY] \cong [\mathbf{L}F(QA) \otimes^{\mathrm{L}} K, RY] \cong [\mathbf{L}F((QA) \otimes^{\mathrm{L}} K), RY] \\ \cong & [(QA) \otimes^{\mathrm{L}} K, \mathbf{R}G(RY)] \cong [K, \mathrm{RMap}_r(QA, \mathbf{R}G(RY))] \cong [K, \mathrm{RMap}_r(A, GY)].\end{aligned}$$

Yoneda lemma for $\mathcal{H} = \mathrm{Ho}\ \mathsf{s\mathcal{S}et}$ then yields a natural isomorphism in $\mathcal{H}$:

$$\mathrm{RMap}_r(FA,Y) \cong \mathrm{RMap}_r(A, GY). \tag{1.4}$$

For simplicial model categories, this relation is easier to deduce.

For a model category $\mathcal{M}$, let $\mathcal{M}_* := (* \downarrow \mathcal{M})$, called the *pointed category of* $\mathcal{M}$, it's indeed a pointed category, and its hom-sets can be described via the cartesian square of sets:

$$\begin{array}{ccc} \mathcal{M}_*((X,x),(Y,y)) & \longrightarrow & \mathcal{M}(X,Y) \\ \downarrow & & \downarrow{\scriptstyle x^*} \\ [0] & \xrightarrow[y]{} & \mathcal{M}(*,Y). \end{array}$$

In fact, $\mathcal{M}_*$ is also a model category ([**45**, Proposition 1.1.8]). For an object X of $\mathcal{M}$, we use (X,v) (and sometimes X) to denote the object $* \xrightarrow{v} X$ of $\mathcal{M}_*$. There is a Quillen adjunction $+ : \mathcal{M} \rightleftarrows \mathcal{M}_* : U$, where $X_+ = (* \to X \coprod *), U(* \to X) = X$, thus $[X_+, (Y,y)]_{\mathcal{M}_*} \cong [X,Y]_{\mathcal{M}}$ (if $\mathcal{M}$ is already a pointed category, then $(+,U)$ is a pair of isomorphisms of

categories: note then that $X \coprod * = X \coprod_* * = X$). If $\mathcal{M}$ is cofibrantly generated, or cellular, or left proper, or right proper, or proper, then so is $\mathcal{M}_*$ (see [**41**]).

For a (small) diagram $F : \mathcal{I} \to \mathcal{M}_*$, we have $\lim\limits_{\mathcal{I}} F = (* \to \lim\limits_{\mathcal{I}} UF)$; for the colimit, we let $\mathcal{I}^+$ denote the category $\mathcal{I}$ with an extra initial object $*$ added, and let $F^+ : \mathcal{I}^+ \to \mathcal{M}_*$ be the unique diagram extending F by letting $F^+(*) = *$, then $\operatorname{colim}\limits_{\mathcal{I}} F = (* \to \operatorname{colim}\limits_{\mathcal{I}^+} UF^+)$. The coproduct in $\mathcal{M}_*$ will be denoted by $\vee$, called the *wedge product*, $X \vee Y$ is the quotient of $X \coprod Y$ (in $\mathcal{M}$) by identifying the basepoints (see [**45**, p.4]; if $\mathcal{M}$ is already pointed, then $\vee = \coprod$ in $\mathcal{M} = \mathcal{M}_*$). There is also the *smash product* $\wedge$ in $\mathcal{M}_*$ defined by the pushout diagram

$$\begin{array}{ccc} X \vee Y & \longrightarrow & X \times Y \\ \downarrow & \ulcorner & \downarrow \\ * & \longrightarrow & X \wedge Y. \end{array}$$

Usually the map $X \vee Y \to X \times Y$ is a monomorphism, so one can write $X \wedge Y = X \times Y / X \vee Y$. For $X, Y \in \mathcal{M}$, we have $X_+ \wedge Y_+ = (X \times Y)_+$. (Of course, these operations only depend on the category structure of $\mathcal{M}$.) If $\mathcal{M}$ is cartesian closed, then $(\mathcal{M}_*, \wedge)$ is a monoidal category and the functor $(-)_+ : (\mathcal{M}, \times) \to (\mathcal{M}_*, \wedge)$ adding basepoints is a monoidal functor.

Assume now that $\mathcal{M}$ is a simplicial model category. Then for $(X, x), (Y, y) \in \mathcal{M}_*, K \in \mathsf{sSet}$, we can define the *simplicial mapping space* $\operatorname{Map}_*((X, x), (Y, y)) \in \mathsf{sSet}$ via the cartesian square

$$\begin{array}{ccc} \operatorname{Map}_*((X, x), (Y, y)) & \longrightarrow & \operatorname{Map}(X, Y) \\ \downarrow & & \downarrow{\scriptstyle x^*} \\ \Delta^0 & \xrightarrow[y]{} & \operatorname{Map}(*, Y) \end{array} \tag{1.5}$$

in sSet, the *tensor product* $(X, x) \otimes K := (X, x) \wedge K_+ = X \otimes K / x \otimes K$ (which is naturally pointed, where $x \otimes K$ is the image of the morphism $x \otimes \mathrm{id}_K : * \otimes K \to X \otimes K$), which is given by the pushout diagram (in $\mathcal{M}$)

$$\begin{array}{ccc} * \otimes K & \xrightarrow{x \otimes \mathrm{id}_K} & X \otimes K \\ \downarrow & \ulcorner & \downarrow \\ * & \longrightarrow & (X, x) \wedge K_+, \end{array}$$

and the *power* $(X, x)^K = X^K$ pointed by $* = *^K \xrightarrow{x^K} X^K$ (cf. [**40**, Example 9.1.14]). With these operations, $\mathcal{M}_*$ becomes a simplicial model category. The simplicial mapping space $\operatorname{Map}_*((X, x), (Y, y)) \in \mathsf{sSet}$ is naturally pointed by the composite $X \to * \xrightarrow{y} Y$ and it fits into the equalizer diagram

$$\operatorname{Map}_*((X, x), (Y, y)) \longrightarrow \operatorname{Map}(X, Y) \underset{y_*}{\overset{x^*}{\rightrightarrows}} \operatorname{Map}(*, Y) \tag{1.6}$$

of simplicial sets, where $y_* : \operatorname{Map}(X, Y) \to \Delta^0 \xrightarrow{y} \operatorname{Map}(*, Y)$. We have $\operatorname{Map}(A, Y) \cong \operatorname{Map}_*(A_+, (Y, y))$ for $A \in \mathcal{M}, (Y, y) \in \mathcal{M}_*$.

It's also interesting to note that a morphism $H : (X, x) \otimes K = (X, x) \wedge K_+ \to (Y, y)$ in $\mathcal{M}_*$ is the same as a morphism $X \otimes K \to Y$ in $\mathcal{M}$ which sends $x \otimes K$ to y. In particular, a "pointed homotopy" $H : (X, x) \otimes \Delta^1 \to (Y, y)$ (a "homotopy" that "sends all the stages of the basepoint x of X to the basepoint y of Y") is the same as a (pointed) morphism $H : (X, x) \wedge \Delta^1_+ \to (Y, y)$.

Let $F : \mathcal{M} \rightleftarrows \mathcal{N} : G$ be a Quillen pair between model categories, then it yields functorially another Quillen pair $F_* : \mathcal{M}_* \rightleftarrows \mathcal{N}_* : G_*$. We have $G_*(* \to Y) = (* \to GY)$ (i.e. $UG_* = GU$), and $F_*(X, v)$ fits into the following pushout diagram

$$\begin{array}{ccc} F(*) & \xrightarrow{F(v)} & F(X) \\ \downarrow & \ulcorner & \downarrow \\ * & \longrightarrow & F_*(X, v). \end{array}$$

Moreover, $F_*(X_+)$ is naturally isomorphic to $(FX)_+$ ([**45**, Proposition 1.3.5]).

Now assume $\mathcal{M}$ is a pointed model category. We denote the function complexes by

$$\begin{gathered} \wedge : \mathcal{M} \times \mathsf{sSet}_* \to \mathcal{M}, (A, K) \mapsto A \wedge K := A^\circ \wedge K, \\ \operatorname{Map}_{*\ell} : \mathcal{M}^{\mathrm{op}} \times \mathcal{M} \to \mathsf{sSet}_*, (A, Y) \mapsto \operatorname{Map}_{*\ell}(A, Y) := \mathcal{M}(A^\circ, Y), \\ \operatorname{Hom}_* : \mathsf{sSet}_*^{\mathrm{op}} \times \mathcal{M} \to \mathcal{M}, (K, Y) \mapsto Y^K = \operatorname{Hom}_*(K, Y) := \operatorname{Hom}_*(K, Y_\circ), \\ \operatorname{Map}_{*r} : \mathcal{M}^{\mathrm{op}} \times \mathcal{M} \to \mathsf{sSet}_*, (A, Y) \mapsto \operatorname{Map}_{*r}(A, Y) := \mathcal{M}(A, Y_\circ). \end{gathered}$$

We have ([**45**, p.144]) for $K \in \mathsf{sSet}$ and $A, Y \in \mathcal{M}$,

$$A \wedge K_+ = A \otimes K, \operatorname{Hom}_*(K_+, Y) = \operatorname{Hom}(K, Y).$$

We also have their derived versions

$$\wedge^{\mathrm{L}} = \mathbf{L}\wedge : (\mathrm{Ho}\ \mathcal{M}) \times (\mathrm{Ho}\ \mathsf{sSet}_*) \to \mathrm{Ho}\ \mathcal{M},$$

$$\mathrm{RHom}_* = \mathbf{R}\mathrm{Hom}_* : (\mathrm{Ho}\ \mathsf{s}\mathcal{S}\mathsf{et}_*)^{\mathrm{op}} \times (\mathrm{Ho}\ \mathcal{M}) \to \mathrm{Ho}\ \mathcal{M},$$
$$\mathrm{RMap}_{*\ell} = \mathbf{R}\mathrm{Map}_{*\ell} \cong \mathrm{RMap}_{*r} = \mathbf{R}\mathrm{Map}_{*r} : (\mathrm{Ho}\ \mathcal{M})^{\mathrm{op}} \times (\mathrm{Ho}\ \mathcal{M}) \to \mathrm{Ho}\ \mathsf{s}\mathcal{S}\mathsf{et}_*.$$

They are computed as follows: for $K \in \mathsf{s}\mathcal{S}\mathsf{et}_*$ and $X, Y \in \mathcal{M}$,

$$X \wedge^{\mathrm{L}} K = (QX)^\circ \wedge K \in \mathcal{M},$$
$$\mathrm{RHom}_*(K, Y) = \mathrm{Hom}_*(K, (RY)_\circ) \in \mathcal{M},$$
$$\mathrm{RMap}_{*\ell}(X, Y) = \mathcal{M}((QX)^\circ, RY) \in \mathsf{s}\mathcal{S}\mathsf{et}_*,$$
$$\mathrm{RMap}_{*r}(X, Y) = \mathcal{M}(QX, (RY)_\circ) \in \mathsf{s}\mathcal{S}\mathsf{et}_*.$$

There are natural isomorphisms

$$[X \wedge^{\mathrm{L}} K, Y] \cong [K, \mathrm{RMap}_{*\ell}(X, Y)] \cong [K, \mathrm{RMap}_{*r}(X, Y)] \cong [X, \mathrm{RHom}_*(K, Y)]. \tag{1.7}$$

They form an adjunction of two variables $(\mathrm{Ho}\ \mathcal{M}) \times (\mathrm{Ho}\ \mathsf{s}\mathcal{S}\mathsf{et}_*) \to \mathrm{Ho}\ \mathcal{M}$, for which $\mathrm{Ho}\ \mathcal{M}$ is a closed $(\mathrm{Ho}\ \mathsf{s}\mathcal{S}\mathsf{et}_*)$-module. See [**45**, Theorem 5.7.3]. As eq. (1.4), if $F : \mathcal{M} \rightleftarrows \mathcal{N} : G$ is a Quillen pair between pointed model categories, then there is a natural isomorphism in $\mathcal{H}_* = \mathrm{Ho}\ \mathsf{s}\mathcal{S}\mathsf{et}_*$.

$$\mathrm{RMap}_{*r}(FA, Y) \cong \mathrm{RMap}_{*r}(A, GY). \tag{1.8}$$

We have the following characterization of weak equivalences in a model category, rephrasing [**40**, Theorem 17.7.7] (the last two points follow from the Yoneda lemma applied to the homotopy category $\mathrm{Ho}(\mathcal{M})$, or one can take a cofibrant/fibrant replacement of the given map by using [**40**, Theorem 7.8.6]; cf. [**40**, Theorem 17.8.10]).

Theorem 1.3.5 (Characterizing weak equivalences)**.** *Let $\mathcal{M}$ be a model category and let $g : X \to Y$ be a map in $\mathcal{M}$. Then the following are equivalent:*

- *The map g is a weak equivalence.*
- *For any (cofibrant) object W in $\mathcal{M}$, the induced map $g_* : \mathrm{RMap}_r(W, X) \to \mathrm{RMap}_r(W, Y)$ is a weak equivalence in $\mathsf{s}\mathcal{S}\mathsf{et}$.*
- *For any (fibrant) object Z in $\mathcal{M}$, the induced map $g^* : \mathrm{RMap}_r(Y, Z) \to \mathrm{RMap}_r(X, Z)$ is a weak equivalence in $\mathsf{s}\mathcal{S}\mathsf{et}$.*
- *For any (cofibrant) object W in $\mathcal{M}$, the induced map $g_* : [W, X] \to [W, Y]$ is a bijection.*
- *For any (fibrant) object Z in $\mathcal{M}$, the induced map $g^* : [Y, Z] \to [X, Z]$ is a bijection.*

Remark 1.3.6. In fact, if $\mathcal{M}$ is left proper and cofibrantly generated with I a generating cofibration set, then in the second condition above, we only need W ranges over domains and codomains of maps in I, and no need for left properness assumption if domains and codomains of maps in I are already cofibrant; see [**46**, Proposition 3.2] and the paragraph after its proof.

Using the general abstract formalism in this section, one can codify problems for general model categories to problems about (homotopy theory of) simplicial sets (or topological spaces), and then decode using familiar theory in classical homotopy theory to get useful information about the original model categories. But this is very hard to do and is usually not efficient, in special cases, e.g. motivic homotopy theory, one can take advantage of the interval object $\mathbb{A}^1$, which gives nice cylinder objects in algebro-geometric situations, to obtain useful information.

1.4. Left Bousfield localization of model categories

In this section, we collect some results of Hirschhorn about (left) Bousfield localization of model categories, for complete treatment, see [**40**, Chapter 3 and Chapter 4].

Throughout we fix a model category $\mathcal{M}$ and a class of maps $\mathcal{C}$ in $\mathcal{M}$.

Definition 1.4.1. A *left localization of $\mathcal{M}$ with respect to $\mathcal{C}$* is a model category $\mathrm{L}_{\mathcal{C}}\mathcal{M}$ together with a left Quillen functor $j : \mathcal{M} \to \mathrm{L}_{\mathcal{C}}\mathcal{M}$ such that

- The total left derived functor $\mathbf{L}j : \mathrm{Ho}\ \mathcal{M} \to \mathrm{Ho}\ \mathrm{L}_{\mathcal{C}}\mathcal{M}$ takes the images in $\mathrm{Ho}\ \mathcal{M}$ of $\mathcal{C}$ to isomorphisms in $\mathrm{Ho}\ \mathrm{L}_{\mathcal{C}}\mathcal{M}$.
- If $\mathcal{N}$ is a model category and $\varphi : \mathcal{M} \to \mathcal{N}$ is a left Quillen functor such that the total left derived functor $\mathbf{L}\varphi : \mathrm{Ho}\ \mathcal{M} \to \mathrm{Ho}\ \mathcal{N}$ takes the images in $\mathrm{Ho}\ \mathcal{M}$ of $\mathcal{C}$ to isomorphisms in $\mathrm{Ho}\ \mathcal{N}$, then there is a unique left Quillen functor $\delta : \mathrm{L}_{\mathcal{C}}\mathcal{M} \to \mathcal{N}$ such that $\delta j = \varphi$.

Definition 1.4.2 ([**40**, Definition 3.1.4])**.** An object W of $\mathcal{M}$ is *$\mathcal{C}$-local* if W is fibrant in $\mathcal{M}$ and for every map $f : A \to B$ in $\mathcal{C}$, the map $f^* : \mathrm{RMap}_r(B, W) \to \mathrm{RMap}_r(A, W)$ is an isomorphism in $\mathcal{H} = \mathrm{Ho}\ \mathsf{s}\mathcal{S}\mathsf{et}$. If $\mathcal{C} = \{f\}$, we then say W is an *f-local* object, if moreover f is the unique map $\varnothing \to A$, we then say W is an *A-local* or *A-null* object.

A map $g : X \to Y$ in $\mathcal{M}$ is called a *$\mathcal{C}$-local (weak) equivalence* if for every $\mathcal{C}$-local object W of $\mathcal{M}$, the map $g^* : \mathrm{RMap}_r(Y, W) \to \mathrm{RMap}_r(X, W)$ is an isomorphism in $\mathcal{H} = \mathrm{Ho}\ \mathsf{s}\mathcal{S}\mathsf{et}$. If $\mathcal{C} = \{f\}$, we then say g is an *f-local (weak) equivalence*, if moreover f is the unique map $\varnothing \to A$, we then say g is an *A-local (weak) equivalence*. Tautologically, every map in $\mathcal{C}$ is a $\mathcal{C}$-local equivalence.

Note that if $\mathcal{M}$ is a simplicial model category, we can use RMap instead of RMap_r and we will do so whenever this is the case.

A *$\mathcal{C}$-localization* of an object X is a $\mathcal{C}$-local object $\widehat{X}$ together with a $\mathcal{C}$-local equivalence $j : X \to \widehat{X}$. If moreover j is a cofibration, it is then called a *cofibrant $\mathcal{C}$-localization* of X. The notion of *$\mathcal{C}$-localization* of a map $g : X \to Y$ is defined in the obvious way ([**40**, Definition 3.2.16]).

Let $F : \mathcal{M} \rightleftarrows \mathcal{N} : U$ be a Quillen pair, let X be fibrant in $\mathcal{N}$, then X is $\mathbf{L}F\mathcal{C}$-local iff UX is $\mathcal{C}$-local ([**40**, Proposition 3.1.12]).

Proposition 1.4.3 ([**40**, Lemma 3.2.1])**.** *Let X, Y be fibrant in $\mathcal{M}$, $g : X \to Y$ a weak equivalence, then X is $\mathcal{C}$-local iff Y is $\mathcal{C}$-local.*

Proposition 1.4.4 ([**40**, Lemma 3.2.5])**.** *Let $\mathcal{I}$ be a small category, $g : X \to Y$ a map in $\mathcal{M}^{\mathcal{I}}$. If for each $i \in \mathcal{I}$, $g_i : X_i \to Y_i$ is a $\mathcal{C}$-local equivalence between cofibrants, then the induced map $\operatorname{hocolim}_{\mathcal{I}} X \to \operatorname{hocolim}_{\mathcal{I}} Y$ is also a $\mathcal{C}$-local equivalence.*

Proposition 1.4.5 ([**40**, Theorem 3.2.13])**.** *If $g : X \to Y$ is a $\mathcal{C}$-local equivalence between $\mathcal{C}$-local objects, then g is a weak equivalence.*

The first two points of the following result are [**40**, Theorem 3.2.17, 3.2.18]. The last point follows easily from the formula for RMap_ℓ and the fact $\mathrm{RMap}_\ell \xrightarrow{\cong} \mathrm{RMap}_r$ given in Section 1.3. These results can be regarded as an abstract descent result which we specialize in Theorem 2.7.13.

Proposition 1.4.6 (Abstract descent)**.** *Assumption as before.*

(1) *Let X be an object of $\mathcal{M}$ with a $\mathcal{C}$-localization $j : X \to \widehat{X}$, then j is a weak equivalence iff some (hence any) fibrant approximation of X is $\mathcal{C}$-local.*[12]
(2) *Let $g : X \to Y$ be a map in $\mathcal{M}$ with a $\mathcal{C}$-localization $\widehat{g} : \widehat{X} \to \widehat{Y}$, then g is a $\mathcal{C}$-local equivalence iff $\widehat{g}$ is a weak equivalence.*
(3) *Let X be an object of $\mathcal{M}$ such that for every map $f : A \to B$ in $\mathcal{C}$, the map $f^* : \mathrm{RMap}_r(B, X) \to \mathrm{RMap}_r(A, X)$ is an isomorphism in $\mathcal{H} = \mathrm{Ho}\ \mathtt{sSet}$.*[13] *Then for any fibrant approximation $j : X \to \widehat{X}$ of X, $\widehat{X}$ is $\mathcal{C}$-local and $j : X \to \widehat{X}$ is a $\mathcal{C}$-localization of X.*

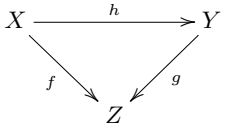
Definition 1.4.7. The *left Bousfield localization of $\mathcal{M}$ with respect to $\mathcal{C}$*, if it exists, is a model structure $\mathrm{L}_{\mathcal{C}}\mathcal{M}$ on the underlying category $\mathcal{M}$ in which,

- weak equivalences $\mathbf{W}$: $\mathcal{C}$-local equivalences in $\mathcal{M}$,
- cofibrations $\mathbf{C}$: cofibrations in $\mathcal{M}$,
- fibrations $\mathbf{F} = \mathrm{RLP}(\mathbf{W} \cap \mathbf{C})$.

Proposition 1.4.8 ([**40**, Proposition 3.3.3, 3.3.4, 3.3.5])**.** *If the left Bousfield localization $\mathrm{L}_{\mathcal{C}}\mathcal{M}$ of $\mathcal{M}$ with respect to $\mathcal{C}$ exists, then*

- *Any weak equivalence in $\mathcal{M}$ is a weak equivalence in $\mathrm{L}_{\mathcal{C}}\mathcal{M}$.*
- *The two model structures $\mathcal{M}$ and $\mathrm{L}_{\mathcal{C}}\mathcal{M}$ have the same trivial fibrations.*
- *Any fibration in $\mathrm{L}_{\mathcal{C}}\mathcal{M}$ is a fibration in $\mathcal{M}$.*
- *Any trivial cofibration in $\mathcal{M}$ is a trivial cofibration in $\mathrm{L}_{\mathcal{C}}\mathcal{M}$.*

Thus $1 : \mathcal{M} \rightleftarrows \mathrm{L}_{\mathcal{C}}\mathcal{M} : 1$ form a Quillen pair.



If $f = gh$, with f, g being fibrations in $\mathrm{L}_{\mathcal{C}}\mathcal{M}$ and h a weak equivalence in $\mathrm{L}_{\mathcal{C}}\mathcal{M}$, then h is a weak equivalence in $\mathcal{M}$.

Proposition 1.4.9 ([**40**, Proposition 3.3.6])**.** *Assume the left Bousfield localization $\mathrm{L}_{\mathcal{C}}\mathcal{M}$ of $\mathcal{M}$ with respect to $\mathcal{C}$ exists. Let X be fibrant in $\mathcal{M}$ and $j : X \to \widehat{X}$ a cofibrant $\mathcal{C}$-localization of X, then the following are equivalent:*

- *The object X is $\mathcal{C}$-local.*
- *The map $j : X \to \widehat{X}$ is a weak equivalence in $\mathcal{M}$.*
- *The map $j : X \to \widehat{X}$ is a homotopy equivalence in $(X \downarrow \mathcal{M})$.*
- *The map $j : X \to \widehat{X}$ is the inclusion of a strong deformation retract.*[14]

Given a map $f : A \to B$ in $\mathcal{M}$, a *horn on f* is a map $\widetilde{A} \otimes \Delta^n \coprod_{\widetilde{A} \otimes \partial\Delta^n} \widetilde{B} \otimes \partial\Delta^n \to \widetilde{B} \otimes \Delta^n$ for some $n \geqslant 0$ and some cosimplicial resolution $\widetilde{f} : \widetilde{A} \to \widetilde{B}$ which is a Reedy cofibration. Define the class $\Lambda(\mathcal{C})$ of *horns on* $\mathcal{C}$ to be the class of horns on some element f of $\mathcal{C}$ (for all $n \geqslant 0$, but fix one $\widetilde{f}$ for each f) ([**40**, Definition 3.3.8, 4.2.1]).

Proposition 1.4.10 ([**40**, Lemma 3.3.11])**.** *A fibrant object W in $\mathcal{M}$ is $\mathcal{C}$-local iff $(W \to *) \in \mathrm{RLP}(\Lambda(\mathcal{C}))$.*

Proposition 1.4.11 ([**40**, Proposition 3.3.14])**.** *If the left Bousfield localization $\mathrm{L}_{\mathcal{C}}\mathcal{M}$ of $\mathcal{M}$ with respect to $\mathcal{C}$ exists, $g : X \to Y$ is a fibration in $\mathrm{L}_{\mathcal{C}}\mathcal{M}$ and Y is $\mathcal{C}$-local, then X is $\mathcal{C}$-local.*

Proposition 1.4.12 ([**40**, Proposition 3.3.15, 3.3.16])**.** *Assume the left Bousfield localization $\mathrm{L}_{\mathcal{C}}\mathcal{M}$ of $\mathcal{M}$ with respect to $\mathcal{C}$ exists.*

- *If $f = gh$, where f is a fibration in $\mathcal{M}$, g is a fibration in $\mathrm{L}_{\mathcal{C}}\mathcal{M}$ and h is a weak equivalence in $\mathcal{M}$, then f is a fibration in $\mathrm{L}_{\mathcal{C}}\mathcal{M}$.*
- *If X, Y are $\mathcal{C}$-local, then a map $f : X \to Y$ is a fibration in $\mathrm{L}_{\mathcal{C}}\mathcal{M}$ iff it is a fibration in $\mathcal{M}$.*

[12] This seemingly stronger result follows easily from the original [**40**, Theorem 3.2.17] (which assumes that X is already fibrant), together with [**40**, Proposition 8.1.7].

[13] Some authors call objects satisfying this condition to be $\mathcal{C}$-local, such as [**69**, Definition 2.3.1], without assuming the objects being fibrant. Our result here says they are not so different, taking a fibrant approximation yields essentially the same notion, and by Proposition 1.4.18 below, if $\mathrm{L}_{\mathcal{C}}\mathcal{M}$ exists, then for W cofibrant, we have $[W, X]_{\mathcal{M}} = [W, \widehat{X}]_{\mathcal{M}} = [W, \widehat{X}]_{\mathrm{L}_{\mathcal{C}}\mathcal{M}} = [W, X]_{\mathrm{L}_{\mathcal{C}}\mathcal{M}}$.

[14] In the sense of [**40**, Definition 7.6.10].

Proposition 1.4.13 ([**40**, Theorem 3.3.19]). *Assume the left Bousfield localization* $\mathrm{L}_{\mathcal{C}}\mathcal{M}$ *of* $\mathcal{M}$ *with respect to* $\mathcal{C}$ *exists, then the functor* $1 : \mathcal{M} \to \mathrm{L}_{\mathcal{C}}\mathcal{M}$ *is a left localization of* $\mathcal{M}$ *with respect to* $\mathcal{C}$.

Theorem 1.4.14 ([**40**, Theorem 3.3.20]). *Let* $F : \mathcal{M} \rightleftarrows \mathcal{N} : U$ *be a Quillen pair. Assume the left Bousfield localizations* $\mathrm{L}_{\mathcal{C}}\mathcal{M}$ *and* $\mathrm{L}_{\mathbf{L}F\mathcal{C}}\mathcal{N}$ *exist, then*

- *The adjunction* $F : \mathrm{L}_{\mathcal{C}}\mathcal{M} \rightleftarrows \mathrm{L}_{\mathbf{L}F\mathcal{C}}\mathcal{N} : U$ *is also a Quillen pair.*
- *If* $F : \mathcal{M} \rightleftarrows \mathcal{N} : U$ *is a Quillen equivalence, then so is* $F : \mathrm{L}_{\mathcal{C}}\mathcal{M} \rightleftarrows \mathrm{L}_{\mathbf{L}F\mathcal{C}}\mathcal{N} : U$.

Proposition 1.4.15 ([**40**, Proposition 3.4.1, 3.4.4, 3.4.6]). *Assume the left Bousfield localization* $\mathrm{L}_{\mathcal{C}}\mathcal{M}$ *of* $\mathcal{M}$ *with respect to* $\mathcal{C}$ *exists and* $\mathcal{M}$ *is left proper. Then*

- $\mathrm{L}_{\mathcal{C}}\mathcal{M}$ *is also left proper.*
- *An object* W *in* $\mathcal{M}$ *is* $\mathcal{C}$*-local iff it is fibrant in* $\mathrm{L}_{\mathcal{C}}\mathcal{M}$.
- *If* $f = gh$, *with* f, g *being fibrations in* $\mathcal{M}$ *and* h *a weak equivalence in* $\mathcal{M}$, *then* f *is a fibration in* $\mathrm{L}_{\mathcal{C}}\mathcal{M}$ *iff* g *is a fibration in* $\mathrm{L}_{\mathcal{C}}\mathcal{M}$.

Proposition 1.4.16 ([**40**, Proposition 3.4.7]). *Assume that the left Bousfield localization* $\mathrm{L}_{\mathcal{C}}\mathcal{M}$ *of* $\mathcal{M}$ *with respect to* $\mathcal{C}$ *exists and* $\mathcal{M}$ *is right proper. Suppose that* f *is a fibration in* $\mathcal{M}$ *and that there is a homotopy cartesian square*

$$\begin{array}{ccc} X & \xrightarrow{j_X} & \widehat{X} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \widehat{f}} \\ Y & \xrightarrow{j_Y} & \widehat{Y} \end{array}$$

in $\mathcal{M}$ *with* j_X, j_Y *being* $\mathcal{C}$*-localizations, then* f *is a fibration in* $\mathrm{L}_{\mathcal{C}}\mathcal{M}$.

Proposition 1.4.17 ([**40**, Proposition 3.4.8]). *Assume that the left Bousfield localization* $\mathrm{L}_{\mathcal{C}}\mathcal{M}$ *of* $\mathcal{M}$ *with respect to* $\mathcal{C}$ *exists, and that both* $\mathcal{M}$ *and* $\mathrm{L}_{\mathcal{C}}\mathcal{M}$ *are right proper. Suppose that* f *is a fibration in* $\mathcal{M}$ *and that* $\widehat{f}$ *is a* $\mathcal{C}$*-localization of* f. *Then* f *is a fibration in* $\mathrm{L}_{\mathcal{C}}\mathcal{M}$ *iff the* $\mathcal{C}$*-localization square*

$$\begin{array}{ccc} X & \xrightarrow{j_X} & \widehat{X} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \widehat{f}} \\ Y & \xrightarrow{j_Y} & \widehat{Y} \end{array}$$

is a homotopy cartesian square in $\mathcal{M}$.

Proposition 1.4.18 ([**40**, Lemma 3.5.1, 3.5.2]). *Assume that the left Bousfield localization* $\mathrm{L}_{\mathcal{C}}\mathcal{M}$ *of* $\mathcal{M}$ *with respect to* $\mathcal{C}$ *exists,* X *is cofibrant in* $\mathcal{M}$ *and* Y *is* $\mathcal{C}$*-local. Then two maps* $X \to Y$ *are homotopic in* $\mathcal{M}$ *iff they are homotopic in* $\mathrm{L}_{\mathcal{C}}\mathcal{M}$. *Consequently the set of homotopy classes* $[X, Y]$ *is independent of whether we consider in* $\mathcal{M}$ *or in* $\mathrm{L}_{\mathcal{C}}\mathcal{M}$: $[X, Y]_{\mathcal{M}} = [X, Y]_{\mathrm{L}_{\mathcal{C}}\mathcal{M}}$.[15]

Thus, the right derived functor $\mathrm{Ho}\,\mathrm{L}_{\mathcal{C}}\mathcal{M} \to \mathrm{Ho}\,\mathcal{M}$ *of the Quillen pair* $1 : \mathcal{M} \rightleftarrows \mathrm{L}_{\mathcal{C}}\mathcal{M} : 1$ *is fully faithful with essential image the full subcategory consisting of* $\mathcal{C}$*-local objects.*

Proposition 1.4.19 ([**40**, Proposition 3.5.3]). *Assume that the left Bousfield localization* $\mathrm{L}_{\mathcal{C}}\mathcal{M}$ *of* $\mathcal{M}$ *with respect to* $\mathcal{C}$ *exists, and that* $\mathcal{M}$ *is left proper. Let* X, Y *be (cofibrant) objects in* $\mathcal{M}$. *Then a map* $g : X \to Y$ *is a* $\mathcal{C}$*-local equivalence iff for every* $\mathcal{C}$*-local object* W, *the induced map* $g^* : [Y, W] \to [X, W]$ *is a bijection.*[16]

Proposition 1.4.20 ([**26**, Lemma 6.3]). *Let* $\mathcal{C}, \mathcal{C}'$ *be two classes of maps in* $\mathcal{M}$ *such that the left Bousfield localizations* $\mathrm{L}_{\mathcal{C}}\mathcal{M}, \mathrm{L}_{\mathcal{C}'}\mathcal{M}$ *exist. The two localizations are the same iff, for any fibration in* $\mathrm{L}_{\mathcal{C}}\mathcal{M}$ *between* $\mathcal{C}$*-local objects* $X \to Y$, *if it is a* $\mathcal{C}'$*-local equivalence, then it is a* $\mathcal{C}$*-local equivalence.*

Theorem 1.4.21 (Existence of left Bousfield localizations [**40**, Theorem 4.1.1]). *Let* $\mathcal{M}$ *be a left proper cellular model category and let* S *be a* set *of maps in* $\mathcal{M}$.[17] *Then the left Bousfield localization* $\mathrm{L}_S\mathcal{M}$ *of* $\mathcal{M}$ *with respect to* S *exists, it is also left proper and cellular. If moreover* $\mathcal{M}$ *is a simplicial model category, then* $\mathrm{L}_S\mathcal{M}$ *is also a simplicial model category with the same simplicial structure.*

Remark 1.4.22. In the above, let f be the coproduct of the maps in S, then the left Bousfield localization $\mathrm{L}_S\mathcal{M}$ is the same as $\mathrm{L}_{\{f\}}\mathcal{M}$.

It's tempting to get more general results on the existence of left Bousfield localizations with respect to a *class* of maps in a model category $\mathcal{M}$, however, there are set-theoretic issues.

Nevertheless, Jardine shows that the left Bousfield localization of the category of simplicial presheaves on a Grothendieck site with respect to the class of local weak equivalences exists, see the discussions in Section 2.6 and Section 2.7. Note that this result doesn't follow from the above theorem.

Remark 1.4.23. There is yet another existence theorem about left Bousfield localizations due to Jeff Smith, which holds for the so called *combinatorial model categories* (which are left proper), for the precise statement, see [**17**, Theorem 4.7]. See also [**57**, Proposition A.3.7.3] (for combinatorial *simplicial* model categories). For the notion of combinatorial model categories, see [**17**, Definition 1.21] or [**35**, Definition 3.13] or [**57**, Definition A.2.6.1].

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[15] Then it's easy to see that this also holds for any object X and any object Y of $\mathcal{M}$ such that for every map $f : A \to B$ in $\mathcal{C}$, the map $f^* : \mathrm{RMap}_r(B, Y) \to \mathrm{RMap}_r(A, Y)$ is an isomorphism in $\mathcal{H} = \mathrm{Ho}\,\mathtt{sSet}$.

[16] As we already assume the left Bousfield localization $\mathrm{L}_{\mathcal{C}}\mathcal{M}$ exists, this is in fact a special case of Theorem 1.3.5, since for a $\mathcal{C}$-local object W (that is, W is fibrant in $\mathrm{L}_{\mathcal{C}}\mathcal{M}$), $[X, W] := [X, W]_{\mathcal{M}} \cong [X, W]_{\mathrm{L}_{\mathcal{C}}\mathcal{M}}$.

[17] For the notion of cellular model categories, see [**40**, Definition 12.1.1].

The procedure of Left Bousfield localization has important applications in various homotopy context, especially in the local model structure on the category of simplicial (pre)sheaves and stable homotopy theory, see e.g. **[26, 46, 49, 50, 64, 65]**.

1.5. Simplicial sets

In this section, we give a brief account on the basics of simplicial sets. The state of the art treatment of this topic is the monograph **[33]** by Goerss-Jardine, from which most of the results presented here are extracted. We already described briefly the notion of simplicial sets in Section 1.1. Recall that the category of simplicial sets $\mathsf{s\mathcal{S}et}$ is bicomplete. A simplicial set is *reduced* if it has only one vertex, such simplicial sets form a full subcategory of $\mathsf{s\mathcal{S}et}$, denoted $\mathsf{s\mathcal{S}et}_0$. A *pointed simplicial set* is a (non-empty) simplicial set X together with a fixed map $\Delta^0 = * \to X$ in $\mathsf{s\mathcal{S}et}$, we thus get the category of pointed simplicial sets $\mathsf{s\mathcal{S}et}_*$, it's canonically identified with the comma category $(* \downarrow \mathsf{s\mathcal{S}et})$ and there is a forgetful functor $\mathsf{s\mathcal{S}et}_* \to \mathsf{s\mathcal{S}et}$. We just use the symbol $*$ to refer to the base point of any pointed simplicial set.

Here are some commonly used simplicial sets and some constructions of new simplicial sets from the old ones:

- Let X be a simplicial set, then a *subsimplicial set* is a simplicial set Y with $Y_n \subset X_n$ for all n and is stable by the simplicial operations of X, we denote $Y \subset X$. In this case, Y is also a simplicial set. There is then a well-defined notion of *quotient* X/Y, formed by degree-wise quotient (as for any (co)limit in $\mathsf{s\mathcal{S}et}$).[18]
- If $X^{(i)}, i \in I$ is a family of subsimplicial sets of X, then so are their *union* and their *intersection* (degree-wise). In fact, they are respectively the *colimit* and *limit* in $\mathsf{s\mathcal{S}et}$ of the diagram formed by the inclusions of the pairwise intersections $X^{(i)} \cap X^{(j)}$ into the $X^{(i)}$'s.
- Let $\mathcal{E}$ be a small category, identify $[n]$ with the category $0 \to 1 \to \cdots \to n$, we get the *classifying space* or *nerve* $\mathrm{B}\mathcal{E}$ of $\mathcal{E}$, with $\mathrm{B}\mathcal{E}_n = \mathcal{C}\mathsf{at}([n], \mathcal{E})$ (strings of n composable arrows in $\mathcal{E}$). For a map $\theta : [m] \to [n]$ in $\mathbf{\Delta}$, the simplicial structure map $\theta^* : \mathrm{B}\mathcal{E}_n \to \mathrm{B}\mathcal{E}_m$ is given by
$$\theta^*(v_0 \to v_1 \to \cdots \to v_n) = (v_{\theta(0)} \to v_{\theta(1)} \to \cdots \to v_{\theta(m)}),$$
on the right hand side, if $\theta(k) = \theta(k+1)$, the arrow $v_{\theta(k)} \to v_{\theta(k+1)}$ is 1; if $\theta(k) < \theta(k+1)$, it is the composite of the arrows on the left hand side from $v_{\theta(k)}$ to $v_{\theta(k+1)}$.
- For a group G, viewed as a category with 1 object, we get the *classifying space* $\mathrm{B}G$ of G.
- Given a map $f : X \to Y$ in $\mathsf{s\mathcal{S}et}$ and $Z \subset Y$ a subsimplicial set, then $\mathrm{im}(f) = f(X) \subset Y$ and $f^{-1}(Z) \subset X$ are subsimplicial sets. It's the pullback of Z along f.

  For a map $f : X \to Y$ in $\mathsf{s\mathcal{S}et}_*$, we call $f^{-1}(*) \subset X$ the *fibre* of f, it's also pointed (with base point the same as that of X).
- The *standard n-simplex* $\Delta^n = \mathbf{\Delta}(-, [n]) \cong \mathrm{B}[n]$ satisfies $\mathsf{s\mathcal{S}et}(\Delta^n, Y) \cong Y_n$ for any $Y \in \mathsf{s\mathcal{S}et}$. Its *boundary* $\partial\Delta^n$ is defined as $\partial\Delta^n = \bigcup_{0 \leqslant i \leqslant n} d^i(\Delta^{n-1}) \subset \Delta^n$ for $n \geqslant 1$. We let $\partial\Delta^0 = \varnothing$. Define the *simplicial n-sphere* to be the quotient $S^n := \Delta^n/\partial\Delta^n$ for $n \geqslant 1$ and $S^0 := \Delta^0 \coprod \Delta^0 = \partial\Delta^1$. We make it pointed in the obvious way. For a subset $J = \{j_0 < \cdots < j_r\} \subset [n]$, we denote $\Delta^J = \Delta^{j_0, \cdots, j_r} \subset \Delta^n$ to be the image of the map $\Delta^r \hookrightarrow \Delta^n$ induced by the inclusion $J \hookrightarrow [n]$.

  The *k-th horn* Λ^n_k $(n \geqslant 1, 0 \leqslant k \leqslant n)$[19] is defined as $\Lambda^n_k = \bigcup_{0 \leqslant i \leqslant n, i \neq k} d^i(\Delta^{n-1}) \subset \partial\Delta^n \subset \Delta^n$.

  An *inner horn* is a horn Λ^n_k with $n \geqslant 2, 0 < k < n$, and the horns $\Lambda^n_0, \Lambda^n_n, n \geqslant 1$, are called *outer horns*.

  For $n \geqslant 2$, there are coequalizer diagrams[20]
$$\coprod_{0 \leqslant i < j \leqslant n} \Delta^{n-2} \rightrightarrows \coprod_{0 \leqslant l \leqslant n} \Delta^{n-1} \xrightarrow{\coprod d^l} \partial\Delta^n$$
and
$$\coprod_{0 \leqslant i < j \leqslant n, i, j \neq k} \Delta^{n-2} \rightrightarrows \coprod_{0 \leqslant l \leqslant n, l \neq k} \Delta^{n-1} \xrightarrow{\coprod d^l} \Lambda^n_k$$
where in both diagrams, the two parallel arrows on the indices $i < j$ are given by d^{j-1} and d^i followed respectively by the inclusions in the i-th and j-th summand.

  To see that the diagrams above are indeed coequalizers, we need the following result, for which the first statement can be proved by hand and the second follows from the first.

  Lemma 1.5.1. *Given in $\mathcal{S}\mathsf{et}$ a family of maps $u_i : A_i \to D$, let $C := \mathrm{im}(\coprod u_i : \coprod A_i \to D) = \bigcup_i u_i(A_i)$, then there is a coequalizer diagram (with obvious maps)*
$$\coprod_{i \neq j} A_i \times_D A_j \rightrightarrows \coprod_l A_l \xrightarrow{\coprod u_l} C$$
in $\mathcal{S}\mathsf{et}$.[21] *Same result holds with $\mathsf{s\mathcal{S}et}$ in place of $\mathcal{S}\mathsf{et}$.*

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[18] But the "difference" $X \setminus Y$, which we want to be a degree-wise difference, is not always well-defined. For a counterexample, let $X = \Delta^1, Y = d^1\Delta^0 \subset X$. Let $u \in X_1$ be represented by the map $1 : \Delta^1 \to \Delta^1$, then $u \in X_1 \setminus Y_1$, but $d_1 u = u d^1 \in Y_0$ is not in the difference $X_0 \setminus Y_0$.

[19] The symbol Λ^0_0 doesn't make sense (and is not $\varnothing$).

[20] In view of the fact that $\mathsf{s\mathcal{S}et} = \mathcal{P}\mathsf{re}(\mathbf{\Delta})$, this characterizes $\partial\Delta^n$ as a sieve of $[n]$, generated by the family $d^0, \cdots, d^n$. See Section 2.2.

[21] Here the first term can be replaced by the coproduct over all pair of indices (i, j), or just half of the pairs (by requiring $i < j$, if there is a total order on the indices), basically because the pairs with $i = j$ give no "relation", while the other half of pairs $(i > j)$ give the same "relation" as those with $i < j$. Note also that $A_i \times_D A_j \cong A_i \times_C A_j \cong u_i(A_i) \cap u_j(A_j) \subset C \subset D$, and after identifying the fibre

By this lemma, we get a coequalizer diagram

$$\coprod_{0\leqslant i<j\leqslant n} \Delta^{n-1}\times_{\Delta^n}\Delta^{n-1} \rightrightarrows \coprod_{0\leqslant l\leqslant n}\Delta^{n-1}\xrightarrow{\coprod d^l}\partial\Delta^n.$$

We have a cartesian square

$$\begin{array}{ccc}[n-2] & \xrightarrow{d^{j-1}} & [n-1]\\ {\scriptstyle d^i}\downarrow & \lrcorner & \downarrow{\scriptstyle d^i}\\ [n-1] & \xrightarrow{d^j} & [n]\end{array}$$

in $\mathbf{\Delta}$, so there is a canonical isomorphism $\Delta^{n-1}\times_{\Delta^n}\Delta^{n-1}\cong\Delta^{n-2}$ (since the Yoneda embedding commutes with fibre products). This proves that the first diagram is a coequalizer, a similar argument yields the second statement.

- For $X,Y\in\mathsf{s\mathcal{S}et}$, we have the *product* $X\times Y=X\otimes Y$ given by $(X\times Y)_n=X_n\times Y_n$, and the *internal hom* or the *function complex* $\underline{\mathrm{Hom}}(X,Y)=Y^X$ given by $\underline{\mathrm{Hom}}(X,Y)_n=\mathsf{s\mathcal{S}et}(X\times\Delta^n,Y)$. We have the *exponential law*
$$\mathsf{s\mathcal{S}et}(X\otimes K,Y)\cong\mathsf{s\mathcal{S}et}(X,Y^K).$$
- For T a topological space, its *singular set* $\mathrm{Sing}(T)\in\mathsf{s\mathcal{S}et}$ is given by $\mathrm{Sing}(T)_n=\mathcal{T}\mathsf{op}(|\Delta^n|,T)$.
- We give a concrete description of (co)skeleta sk_nX and cosk_nX for $X\in\mathsf{s\mathcal{S}et}$: sk_nX is a subsimplicial set of X, and it is the smallest subsimplicial set of X among those $Z\subset X$ with $Z_k=X_k, 0\leqslant k\leqslant n$. Moreover,
$$(\mathrm{sk}_nX)_m=\bigcup_{\alpha\in([m]\downarrow\mathbf{\Delta}_n)}\mathrm{im}(\alpha^*)=\bigcup_{\alpha:[m]\to[k]\text{ surjective, }k\leqslant n}\mathrm{im}(\alpha^*)\subset X_m,$$
where for a map $\alpha:[m]\to[k]$ in $\mathbf{\Delta}$, α^* is the map $X(\alpha):X_k\to X_m$.

  There is the adjunction $\mathrm{sk}_n:\mathsf{s\mathcal{S}et}\rightleftarrows\mathsf{s\mathcal{S}et}:\mathrm{cosk}_n$, so the set of k-simplices of the coskeleton cosk_nX is identified with $\mathsf{s\mathcal{S}et}(\mathrm{sk}_n\Delta^k,X)$. From the above description, it's easy to find that, for a family $X^{(i)},i\in I$ of subsimplicial sets of X, we have
$$\mathrm{sk}_n(\bigcup X^{(i)})=\bigcup\mathrm{sk}_n(X^{(i)})\subset X.$$

To show the result on skeleta, we need the following result

Lemma 1.5.2. *Consider a commutative square*

$$\begin{array}{ccc}[m] & \xrightarrow{\alpha} & [k]\\ {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle\gamma}\\ [l] & \xrightarrow{1} & [l],\end{array}$$

in $\mathbf{\Delta}$, where the map γ is defined by letting $\gamma(s):=\sup\{\beta(i):i\in[m],\alpha(i)\leqslant s\}$. If α is surjective (as a map of the underlying sets), then the above square induces a pushout square in $\mathsf{s\mathcal{S}et}$:

$$\begin{array}{ccc}\Delta^m & \xrightarrow{\alpha_*} & \Delta^k\\ {\scriptstyle\beta_*}\downarrow & & \downarrow{\scriptstyle\gamma_*}\\ \Delta^l & \xrightarrow{1} & \Delta^l.\end{array}$$

Proof. By the result of Example A.2.7 (see also Section 2.1), viewing objects of $\mathsf{s\mathcal{S}et}$ as presheaves on $\mathbf{\Delta}$, it's sufficient to show that the evaluating at all $[n]\in\mathbf{\Delta}$ of this square is a pushout square in $\mathcal{S}\mathsf{et}$:

$$\begin{array}{ccc}\mathbf{\Delta}([n],[m]) & \xrightarrow{\alpha_*} & \mathbf{\Delta}([n],[k])\\ {\scriptstyle\beta_*}\downarrow & & \downarrow{\scriptstyle\gamma_*}\\ \mathbf{\Delta}([n],[l]) & \xrightarrow{1} & \mathbf{\Delta}([n],[l]).\end{array}$$

This is a direct verification of the universal property for being a pushout, once observed that for any $h\in\mathbf{\Delta}([n],[k])$, there exists an element $g\in\mathbf{\Delta}([n],[m])$ such that $h=\alpha\circ g$, where one such map g can be obtained by setting $g(r):=\sup\{j\in[m]:\alpha(j)=h(r)\}$ (here we need the fact that α is surjective). □

We come back to the proof of our main result:

$$(\mathrm{sk}_nX)_m=\bigcup_{\alpha\in([m]\downarrow\mathbf{\Delta}_n)}\mathrm{im}(\alpha^*)=\bigcup_{\alpha:[m]\to[k]\text{ surjective, }k\leqslant n}\mathrm{im}(\alpha^*).$$

By the epi-monic decomposition of maps in $\mathbf{\Delta}$, it's clear that the last two sets are equal, which we denote by Y_m, they form a subsimplicial set Y of X. We need to show that the canonical map

$$(\mathrm{sk}_nX)_m=\operatorname*{colim}_{([k]\to[m])\in(\mathbf{\Delta}_n^{\mathrm{op}}\downarrow[m])}X_k=\operatorname*{colim}_{([m]\xrightarrow{\alpha}[k]),k\leqslant n}X_k\to X_m$$

products with these intersections, the result becomes a trivial fact in $\mathcal{S}\mathsf{et}$: the union of a family of subsets of a fixed set can be identified with the quotient of the disjoint union of these sets by identifying the pairwise intersections.

is injective (its image is clearly the set $Y_m \subset X_m$).

Suppose that $a : \Delta^k \to X, b : \Delta^l \to X$ are two simplices of X with $k \leqslant l \leqslant n$, corresponding respectively to the maps $\alpha : [m] \to [k], \beta : [m] \to [l]$ in $\mathbf{\Delta}$, which have the same image in X_m under the above map, i.e. the diagram

$$\begin{array}{ccc} \Delta^m & \xrightarrow{\alpha_*} & \Delta^k \\ {\scriptstyle \beta_*}\downarrow & & \downarrow{\scriptstyle a} \\ \Delta^l & \xrightarrow[b]{} & X \end{array}$$

commutes. Again by the epi-monic decomposition of maps in $\mathbf{\Delta}$, we can assume that the maps α, β are both surjective, thus the lemma above says that the map a factors through the map b: $a = b \circ \gamma_*$, thus the two simplices a, b represent the same element in the colimit. This proves that $\mathrm{sk}_n X$ is a subsimplicial set of X, the remaining statements are now obvious.

Proposition 1.5.3. *For any $X \in \mathsf{s\mathcal{S}et}$ and $n \geqslant 2$, we have bijections*

$$\mathsf{s\mathcal{S}et}(\partial\Delta^n, X) \cong \{(y_0, \cdots, y_n) \in (X_{n-1})^{n+1} : d_i y_j = d_{j-1} y_i \text{ for } i < j\},$$

$$\mathsf{s\mathcal{S}et}(\Lambda^n_k, X) \cong \{(y_0, \cdots, \hat{y_k}, \cdots, y_n) \in (X_{n-1})^n : d_i y_j = d_{j-1} y_i \text{ for } i < j, i, j \neq k\}.$$

Moreover, we have $(\mathrm{cosk}_0 X)_1 \cong \mathsf{s\mathcal{S}et}(\partial\Delta^1, X) = X_0 \times X_0$ and for $n > 1$ there is (in $\mathcal{S}\mathsf{et}$) the equalizer diagram

$$(\mathrm{cosk}_{n-1} X)_n \cong \mathsf{s\mathcal{S}et}(\partial\Delta^n, X) \to \prod_{0 \leqslant l \leqslant n} X_{n-1} \rightrightarrows \prod_{0 \leqslant i < j \leqslant n} X_{n-2}.$$

Proposition 1.5.4. *For a map $f : X \to Y$ in $\mathsf{s\mathcal{S}et}$, we have $f(\mathrm{sk}_n X) = f(X) \cap (\mathrm{sk}_n Y) = \mathrm{sk}_n f(X)$.*

If $X \subset Y$, then $\mathrm{sk}_n X = X \cap \mathrm{sk}_n Y$.

Proof. It's clear that $f(\mathrm{sk}_n X) \subset f(X) \cap (\mathrm{sk}_n Y)$. Conversely, let $d \in (f(X) \cap (\mathrm{sk}_n Y))_m$, then there exists $b \in X_m, c \in Y_k$ and a surjective map $\alpha : [m] \to [k]$ in $\mathbf{\Delta}$ such that $d = f(b) = \alpha^*(c)$.

$$\begin{array}{ccc} \Delta^m & \xrightarrow{\alpha_*} & \Delta^k \\ {\scriptstyle b}\downarrow & \overset{a}{\swarrow} & \downarrow{\scriptstyle c} \\ X & \xrightarrow{f} & Y \end{array} \qquad \begin{array}{ccc} X_k & \xrightarrow{\alpha^*} & X_m \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ Y_k & \xrightarrow{\alpha^*} & Y_m \end{array}$$

We can assume $\alpha\beta = 1_{[k]}$. Let $a := \beta^* b \in X_k$, then $f(a) = \beta^* f(b) = \beta^* d = (\alpha\beta)^* c = c$, so $d = \alpha^* f(a) = f(\alpha^* a)$, and since $\alpha^* a \in (\mathrm{sk}_n X)_m$, we see that $d \in (f(\mathrm{sk}_n X))_m$.[22]

The second statement follows from the first if we take f to be the inclusion map. □

Proposition 1.5.5. *For $0 \leqslant n \leqslant m-2$, we have $\mathrm{sk}_n \Lambda^m_k = \mathrm{sk}_n \partial\Delta^m = \mathrm{sk}_n \Delta^m$. Moreover,*

$$\mathrm{sk}_{m-2} \Delta^m = \bigcup_{j \in [m], i \in [m-1]} d^j d^i \Delta^{m-2}.$$

Proof. Since $\mathrm{sk}_{m-1} \Delta^m = \partial\Delta^m$, we have $\mathrm{sk}_n \partial\Delta^m = \mathrm{sk}_n \Delta^m$. We only need to treat the case when $n = m-2$. Let $f : \coprod_{i \in [m]} \Delta^{m-1} \to \Delta^m$ be the map which is d^i on the i-th summand, then by the result above,

$$f(\coprod_{i \in [m]} \mathrm{sk}_{m-2} \Delta^{m-1}) = \partial\Delta^m \cap \mathrm{sk}_{m-2} \Delta^m = \mathrm{sk}_{m-2} \Delta^m.$$

On the other hand,

$$f(\coprod_{i \in [m]} \mathrm{sk}_{m-2} \Delta^{m-1}) = f(\coprod_{i \in [m]} \partial\Delta^{m-1}) = \bigcup_{j \in [m]} d^j(\partial\Delta^{m-1}) = \bigcup_{j \in [m], i \in [m-1]} d^j d^i \Delta^{m-2}.$$

Similarly, we have

$$\mathrm{sk}_{m-2} \Lambda^m_k = \bigcup_{j \in [m] \setminus \{k\}, i \in [m-1]} d^j d^i \Delta^{m-2}.$$

But by the cosimplicial identities, for $k \in [m], i \in [m-1]$,

$$d^k d^i = \begin{cases} d^i d^{k-1}, & i < k; \\ d^{i+1} d^k, & k \leqslant i, \end{cases}$$

thus $\mathrm{sk}_n \Lambda^m_k = \mathrm{sk}_n \partial\Delta^m = \mathrm{sk}_n \Delta^m$. □

Proposition 1.5.6. *For $X \in \mathsf{s\mathcal{S}et}$, let $D_X : \mathbf{\Delta}^{\mathrm{op}} \to \mathsf{s\mathcal{S}et}, [n] \to X_n$ (the set X_n viewed as a discrete simplicial set), then there is a coequalizer diagram*

$$\coprod_{([m] \to [n]) \in \mathbf{\Delta}} X_n \times \Delta^m \underset{\beta}{\overset{\alpha}{\rightrightarrows}} \coprod_{[n] \in \mathbf{\Delta}} X_n \times \Delta^n \xrightarrow{\gamma} X,$$

where at each simplicial degree k, $\alpha_k(\Delta^n \to X, [k] \to [m]) = (\Delta^n \to X, [k] \to [m] \to [n]), \beta_k(\Delta^n \to X, [k] \to [m]) = (\Delta^m \to \Delta^n \to X, [k] \to [m]), \gamma_k(\Delta^n \to X, [k] \to [n]) = (\Delta^k \to \Delta^n \to X)$.

[22] We don't claim that $\alpha^* a = b$, we only have $f(\alpha^* a) = d = f(b)$.

Proof. It's enough to verify that at each simplicial degree k, there is a coequalizer diagram

$$\coprod_{([m]\to[n])\in\mathbf{\Delta}} X_n\times\Delta^m_k \underset{\beta_k}{\overset{\alpha_k}{\rightrightarrows}} \coprod_{[n]\in\mathbf{\Delta}} X_n\times\Delta^n_k \xrightarrow{\gamma_k} X_k.$$

So let $v:\coprod_{[n]\in\mathbf{\Delta}} X_n\times\Delta^n_k\to L$ be a map of sets with $v\alpha_k=v\beta_k$, i.e. $v(\Delta^n\to X,[k]\to[m]\to[n])=v(\Delta^m\to\Delta^n\to X,[k]\to[m])$; this guarantees that the unique map $u:X_k\to L,(\Delta^k\to X)\mapsto v(\Delta^k\to X,[k]\xrightarrow{1}[k])$ indeed satisfies $u\gamma=v$. □

For $X\in\mathsf{s\mathcal{S}et}$, let $\mathbf{\Delta}X:=(\Delta\downarrow X)$ be its *simplex category*, then (see Proposition 2.1.6)

$$X\cong\operatorname*{colim}_{(\Delta^n\to X)\in\mathbf{\Delta}X}\Delta^n.$$

Let $|\Delta^n|:=\{(t_0,\cdots,t_n)\in\mathbf{R}^{n+1}:t_0,\cdots,t_n\geqslant 0,t_0+\cdots+t_n=1\}\subset\mathbf{R}^{n+1}$ endowed with the subspace topology. We define the *geometric realization* to be the topological space

$$|X|:=\operatorname*{colim}_{(\Delta^n\to X)\in\mathbf{\Delta}X}|\Delta^n|,$$

it is a CW complex. There is an adjunction

$$|\ |:\mathsf{s\mathcal{S}et}\rightleftarrows\mathcal{T}\mathsf{op}:\mathrm{Sing}.$$

Denote $J=\{\Lambda^n_k\hookrightarrow\Delta^n:n\geqslant 1,0\leqslant k\leqslant n\}$ and $I=\{\partial\Delta^n\hookrightarrow\Delta^n:n\geqslant 0\}$.

Theorem 1.5.7 (Kan-Quillen, see [**40**, Theorem 7.10.12 and Example 9.1.13])**.** *The category* $\mathsf{s\mathcal{S}et}$ *of simplicial sets is a simplicial model category with*

- *weak equivalences* $\mathbf{W}$*: those maps* $f:X\to Y$ *in* $\mathsf{s\mathcal{S}et}$ *whose geometric realizations* $|f|:|X|\to|Y|$ *are weak equivalences in* $\mathcal{T}\mathsf{op}$ *(i.e. induce bijections on path components and isomorphisms on all homotopy groups for all choices of base points).*
- *cofibrations* $\mathbf{C}$*: monomorphisms.*
- *fibrations* $\mathbf{F}=\mathrm{RLP}(J)$*, called* Kan fibrations.

The simplicial structure is given by letting $X\otimes K=X\times K, Y^K=\underline{\mathrm{Hom}}(K,Y)=\mathrm{Map}(K,Y)$*. This model structure on* $\mathsf{s\mathcal{S}et}$ *is proper and cofibrantly generated with generating cofibrations* I *and generating trivial cofibrations* J*. Moreover, the adjoint pair*

$$|\ |:\mathsf{s\mathcal{S}et}\rightleftarrows\mathcal{T}\mathsf{op}:\mathrm{Sing}.$$

is a Quillen equivalence (we give the standard model structure on $\mathcal{T}\mathsf{op}$*, as in* [**40**, Theorem 7.10.10]*). Both the unit and counit are natural weak equivalences.*

It's easy to see that any trivial fibration admits a section (and is thus surjective). The geometric realization functor is exact ([**50**, Lemma 2.12]), and the unit and counit of the above adjunction are natural weak equivalences ([**50**, Theorem 2.21]).

A simplicial set K is called *(Kan) fibrant* or a *Kan complex* if the unique map $X\to *=\Delta^0$ is a Kan fibration. We denote by $\mathcal{K}\mathsf{an}$ the full subcategory of $\mathsf{s\mathcal{S}et}$ consisting of Kan complexes. Examples of Kan complexes include discrete simplicial sets (constant presheaves on $\mathbf{\Delta}$, using Proposition 1.5.3, since all the d_i's are identities. Same argument shows that every map between discrete simplicial sets is a fibration), any simplicial group regarded as a simplicial set, and the classifying space of a groupoid.

The category of pointed simplicial sets $\mathsf{s\mathcal{S}et}_*=(*\downarrow\mathsf{s\mathcal{S}et})$ inherits a model category structure from $\mathsf{s\mathcal{S}et}$ (see [**40**, Theorem 7.6.5]). It's in fact a simplicial model category (see [**40**, Example 9.1.14]).

Remark 1.5.8. The functor cosk_n on $\mathsf{s\mathcal{S}et}$ doesn't preserve fibrations: for otherwise, since sk_n preserves cofibrations, $(\mathrm{sk}_n,\mathrm{cosk}_n)$ would be a Quillen pair, so sk_n would preserve trivial cofibrations. But for the trivial cofibration $\Lambda^{n+1}_k\hookrightarrow\Delta^{n+1}$, its image under sk_n is the map $\Lambda^{n+1}_k\hookrightarrow\partial\Delta^{n+1}$, which is not a trivial cofibration.

Proposition 1.5.9. *For* $n>0$*, the standard simplices* Δ^n *are not Kan complexes.*

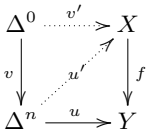
Proof. We define a map $a=(a_0,a_1):\Lambda^2_2\to\Delta^n$ as follows: let $a_0=\begin{pmatrix}0&1\\0&1\end{pmatrix},a_1=\begin{pmatrix}0&1\\1&1\end{pmatrix}:\Delta^1\to\Delta^n$ (it's easy to see that $d_0a_1=d_0a_0=\begin{pmatrix}0\\1\end{pmatrix}:\Delta^0\to\Delta^n$). If there is a map $b:\Delta^2\to\Delta^n$ such that $d_0b=a_0,d_1b=a_1$, then we must have $b=\begin{pmatrix}0&1&2\\1&0&1\end{pmatrix}$, but there is no such a map b in $\mathbf{\Delta}$. □

Proposition 1.5.10 (Quillen)**.** *If* $f:X\to Y$ *is a Kan fibration, then its geometric realization* $|f|:|X|\to|Y|$ *is a Serre fibration in* $\mathcal{T}\mathsf{op}$*.*

Recall that a map is called a *Serre fibration* (or a *weak fibration*) if it has the homotopy lifting property with respect to all CW complexes (or equivalently, just cubes $[0,1]^n$).

Proposition 1.5.11. *If* $f:X\to Y$ *is a Kan fibration and* $f_0:X_0\to Y_0$ *is surjective, then* $f_n:X_n\to Y_n$ *is surjective for every* $n\in\mathbb{N}$*.*

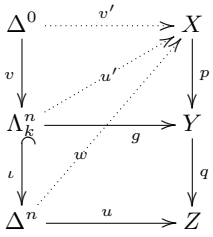
Proof. Given $u \in Y_n$, choose any $v : \Delta^0 \to \Delta^n$, then there exists $v' \in X_0$ such that $f_0(v') = uv$. Since $v : \Delta^0 \to \Delta^n$ is a trivial cofibration, there exists $u' \in X_n$ such that $f_n(u') = u$.



□

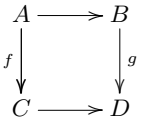
Proposition 1.5.12. *If $p : X \to Y$ is a surjective (in each degree) Kan fibration and $q : Y \to Z$ is a map in* sSet *such that $q \circ p$ is also a Kan fibration, then q is a Kan fibration as well.*

Proof. Given $u \in Z_n$ and maps such that the lower square in the diagram



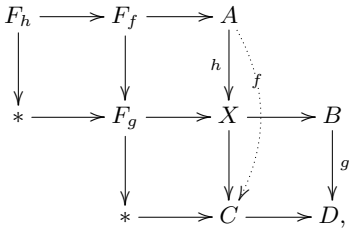
commutes, choose any map $v : \Delta^0 \to \Lambda^n_k$, since $p_0 : X_0 \to Y_0$ is surjective, there exists $v' \in X_0$ such that $pv' = gv$. Since $v : \Delta^0 \to \Lambda^n_k$ is a trivial cofibration, there exists $u' : \Lambda^n_k \to X$ such that the two triangles in the upper square above commute; since the inclusion $\Lambda^n_k \to \Delta^n$ is a trivial cofibration and $q \circ p$ is a fibration, there exists $w : \Delta^n \to X$ such that the two triangles having w as a common edge commute. Then the map $p \circ w : \Delta^n \to Y$ is a correct lifting map for the lower square. □

Proposition 1.5.13 (Fibrewise characterization of homotopy cartesian)**.** *Let*



be a commutative diagram in sSet*. Then it is homotopy cartesian iff for every choice of base point of C, the induced map on homotopy fibres $F_f \to F_g$ is a weak equivalence. If C, D are connected, then it's sufficient for any one choice of base point of C.*

Proof. One direction is clear. For the other, we form the homotopy pullback squares, sequentially constructed from the lower right to upper left (to identify the objects, one uses [**40**, Proposition 13.3.15]):



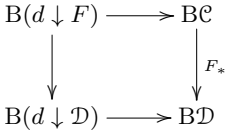
where h is the induced map from A to the homotopy pullback X of the lower-right corner of the original square. Since the map $F_f \to F_g$ is a weak equivalence, F_h is contractible. Then the homotopy exact sequence of the homotopy fibration $F_h \to A \xrightarrow{h} X$ shows that h is a weak equivalence (on each connected component). □

See [**70**, Proposition 3.3.18] for the above result in the setting of topological spaces.

Theorem 1.5.14 ([**40**, Theorem 7.10.13 and Example 9.1.14])**.** *The category* sSet_* *of pointed simplicial sets is a simplicial model category with*

- *weak equivalences* $\mathbf{W}$*: those maps in* sSet_* *which are weak equivalences in* sSet *when forgetting the base points.*
- *fibrations* $\mathbf{F}$*: those maps in* sSet_* *which are fibrations in* sSet *when forgetting the base points.*
- *cofibrations* $\mathbf{C} = \mathrm{LLP}(\mathbf{F})$.

Theorem 1.5.15 (Quillen's Theorem B)**.** *Let $\mathcal{C}, \mathcal{D}$ be small categories and $F : \mathcal{C} \to \mathcal{D}$ a functor. Assume that for every morphism $g : d \to d'$ of $\mathcal{D}$, the induced map $g^* : \mathrm{B}(d' \downarrow F) \to \mathrm{B}(d \downarrow F)$ is a weak equivalence in* sSet*, then for every object d of $\mathcal{D}$, the commutative diagram*



is homotopy cartesian in $\mathsf{s}\mathcal{S}\mathsf{et}$*, and we get a homotopy fibre sequence*

$$\mathrm{B}(d \downarrow F) \to \mathrm{B}\mathcal{C} \xrightarrow{F_*} \mathrm{B}\mathcal{D}$$

in $\mathsf{s}\mathcal{S}\mathsf{et}_*$ *(after suitably choosing base points).*

For a proof, see [**33**, Chapter IV, §5.2].

Theorem 1.5.16 (Quillen's Theorem A)**.** *Let* $\mathcal{C}, \mathcal{D}$ *be small categories and* $F : \mathcal{C} \to \mathcal{D}$ *be a functor. If the simplicial set* $\mathrm{B}(d \downarrow F)$ *is contractible for every object* d *of* $\mathcal{D}$*, then the map* $F_* : \mathrm{B}\mathcal{C} \to \mathrm{B}\mathcal{D}$ *is a weak equivalence in* $\mathsf{s}\mathcal{S}\mathsf{et}$*.*

Finally we recall the discussion at the end of Section 1.3. We have the *smash product* $\wedge$ in $\mathsf{s}\mathcal{S}\mathsf{et}_*$, the category of pointed simplicial sets. For $(X, x), (Y, y) \in \mathsf{s}\mathcal{S}\mathsf{et}_*$, the *simplicial mapping space* $\mathrm{Map}_*((X, x), (Y, y)) \in \mathsf{s}\mathcal{S}\mathsf{et}$ is defined by the cartesian square

$$\begin{array}{ccc} \mathrm{Map}_*((X,x),(Y,y)) & \longrightarrow & \mathrm{Map}(X,Y) \\ \downarrow & \lrcorner & \downarrow{\scriptstyle x^*} \\ \Delta^0 & \xrightarrow{y} & \mathrm{Map}(*,Y), \end{array}$$

in $\mathsf{s}\mathcal{S}\mathsf{et}$, where the right vertical arrow in degree n is given by mapping $g : X \times \Delta^n \to Y$ to the composite $\Delta^n = \Delta^0 \times \Delta^n \xrightarrow{x \times 1} X \times \Delta^n \xrightarrow{g} Y$. So $\mathrm{Map}_*((X, x), (Y, y))_n = \mathsf{s}\mathcal{S}\mathsf{et}_*((X, x) \wedge (\Delta^n)_+, (Y, y))$. Then for $K \in \mathsf{s}\mathcal{S}\mathsf{et}$, $\mathrm{Map}_*(K_+, (Y, y)) = \mathrm{Map}(K, Y)$ and we have the following adjunction

$$(X, x) \wedge (-) : \mathsf{s}\mathcal{S}\mathsf{et}_* \rightleftarrows \mathsf{s}\mathcal{S}\mathsf{et}_* : \mathrm{Map}_*((X, x), -).$$

Define $(X, x) \wedge K := (X, x) \times K_+, (X, x)^K = \mathrm{Map}_*(K_+, (X, x)) = \mathrm{Map}(K, X)$ (naturally pointed), then these give $\mathsf{s}\mathcal{S}\mathsf{et}_*$ the structure of a simplicial model category, see [**40**, Example 9.1.14].

1.6. Simplicial homotopy

Let $f, g : K \to X$ be maps in $\mathsf{s}\mathcal{S}\mathsf{et}$, then a *simplicial homotopy* $h : f \Rightarrow g$ is a commutative diagram

$$\begin{array}{ccccc} K \times \Delta^0 = K & & & & \\ \downarrow{\scriptstyle 1\times d^1} & \searrow^{f} & & & \\ K \times \Delta^1 & & \xrightarrow{h} & & X. \\ \uparrow{\scriptstyle 1\times d^0} & \nearrow_{g} & & & \\ K \times \Delta^0 = K & & & & \end{array}$$

This means that the two composites $\Delta^0 \overset{d^1}{\underset{d^0}{\rightrightarrows}} \Delta^1 \xrightarrow{h^\flat} X^K$ are f and g. If $i : L \hookrightarrow K$ is an inclusion, $\alpha : L \to X$ is a fixed map and $fi = gi = \alpha$, then we say $h : f \Rightarrow g$ (rel L) if $h : f \Rightarrow g$ and moreover the diagram

$$\begin{array}{ccc} L \times \Delta^1 & \xrightarrow{\mathrm{pr}_L} & L \\ \downarrow{\scriptstyle i\times 1} & & \downarrow{\scriptstyle \alpha} \\ K \times \Delta^1 & \xrightarrow{h} & X \end{array}$$

commutes, i.e. $h \circ (i \times 1) = \alpha \circ \mathrm{pr}_L$ is the *constant homotopy (at* α*).* This can also be written as $i^* h^\flat = \alpha \in (X^L)_0$.

$$\begin{array}{ccccc} \Delta^0 & \overset{d^1}{\underset{d^0}{\rightrightarrows}} & \Delta^1 & \xrightarrow{h^\flat} & X^K \\ & & \downarrow & & \downarrow{\scriptstyle i^*} \\ & & \Delta^0 & \xrightarrow{\alpha} & X^L \end{array}$$

Remark 1.6.1. If we define F_α via the pullback diagram

$$\begin{array}{ccc} F_\alpha & \longrightarrow & X^K \\ \downarrow & \lrcorner & \downarrow{\scriptstyle i^*} \\ \Delta^0 & \xrightarrow{\alpha} & X^L, \end{array}$$

then by exponential law, homotopy of maps $K \to X$ (rel L) is equivalent to homotopy of maps $\Delta^0 \to F_\alpha$, these correspond to elements in $\pi_0(F_\alpha, *)$, this reduces general homotopies to homotopies of vertices. Note that, by general theory on simplicial model categories, if X is fibrant, then so is X^K (see for example [**40**, Proposition 9.3.9]).

Proposition 1.6.2. *If* X *is fibrant, then simplicial homotopy and simplicial homotopy* rel L *are both equivalence relations.*

For $X \in \mathsf{s}\mathcal{S}\mathsf{et}, \alpha : \Delta^n \to X$ an n-simplex, we denote $\partial\alpha := (d_0\alpha, \cdots, d_n\alpha) \in (X_{n-1})^{n+1}$. For any $k \in [n]$, let $\iota_k = \iota_k^n : [0] \to [n], 0 \mapsto k$. We call $\alpha_k := X(\iota_k)(\alpha) = \iota_k^*\alpha$ the *k-th vertex* of $\alpha \in X_n$. For a map $\theta : [n] \to [m]$ in $\mathbf{\Delta}$, we have $\theta \circ \iota_k^n = \iota_{\theta(k)}^m$, thus for $\beta \in X_m$, we have $(\theta^*\beta)_k = \beta_{\theta(k)}$.

If $X \in \mathsf{s}\mathcal{S}\mathsf{et}$ is fibrant, we define $\pi_0(X)$ to be the set of homotopy classes of vertices $* = \Delta^0 \xrightarrow{v} X$, called the set of *path components of X*. If $\pi_0(X) = \{*\}$ consisting of a single element, we say that X is *connected*. When there a base point $v \in X_0$, we define $\pi_0(X, v) := \pi_0(X)$ to be the pointed set with base point the component of the vertex v. From the above relations, it's clear that for $\alpha \in X_n$, if $[\alpha_k] = [v] \in \pi_0(X)$, then $[\alpha_l] = [v] \in \pi_0(X), \forall l \in [n]$. For any $v \in X_0$, we define $C(v) = C([v])$ to be the simplicial set which consists of those simplices all of whose vertices are simplicially homotopic to v, we call it the *simplicial path component of X* of the vertex v, or of the element $[v] \in \pi_0(X)$. Clearly $C(v) = C([v])$ only depends on $[v] \in \pi_0(X)$ and it is also fibrant. Moreover, we have $X = \coprod_{[v]\in\pi_0(X)} C([v])$. It's then clear that for $A \in \mathcal{S}\mathsf{et} \subset \mathsf{s}\mathcal{S}\mathsf{et}$, we have $\mathsf{s}\mathcal{S}\mathsf{et}(X, A) \cong \mathcal{S}\mathsf{et}(\pi_0 X, A)$, where a map $X \to A$ corresponds to the induced map $\pi_0 X \to \pi_0 A = A$.

Any map of the form $\varnothing \to Y$ is trivially a Kan fibration, so Kan fibration is not necessarily surjective but the following result says that it's not far from being surjective: once we discard those components which are disjoint with the image (which are irrelvant), the map becomes surjective (still a fibration).

Proposition 1.6.3. *If $f : X \to Y$ is a Kan fibration, $X \neq \varnothing$, Y is fibrant and connected, then f is surjective in each degree.*

In general, for any Kan fibration $f : X \to Y$, the image $f(X)$ is a union of simplicial path components of Y, and $Y = f(X) \coprod Y'$, where Y' is also a union of simplicial path components of Y. The pullback of f along the incusions $f(X) \hookrightarrow Y$ and $Y' \hookrightarrow Y$ are the maps $X \to f(X)$ and $\varnothing \to Y'$, hence both are fibrations, and $f : X \to Y$ is the disjoint union of these two maps.

Thus if X, Y are Kan complexes, then a Kan fibration $f : X \to Y$ is surjective in each degree iff the induced map $f_* : \pi_0(X) \to \pi_0(Y)$ is surjective.

Proof. It's enough to show that $f_0 : X_0 \to Y_0$ is surjective. Fix $e_0 \in X_0$ and let $b_0 = f(e_0) \in Y_0$. For any $b_1 \in Y_0$, since $\pi_0(Y) = \{*\}$, there exists a generalized interval (see [**40**, Definition 9.5.5 and Proposition 9.5.24]) and a map $\alpha : J \to Y$ such that $\alpha i_0 = b_0, \alpha i_1 = b_1$, where i_0, i_1 are the two endpoints of J.[23]

$$\begin{array}{ccc} \Delta^0 & \overset{e_0}{\dashrightarrow} & X \\ {\scriptstyle i_0}\downarrow & \overset{\theta}{\nearrow} & \downarrow{\scriptstyle f} \\ J & \xrightarrow{\alpha} & Y \end{array}$$

As i_0 is a trivial cofibration, there is a lifting map θ as in the above diagram. Thus $f(\theta i_1) = \alpha i_1 = b_1$.

The general statement is now clear. □

The following is a special case of a *Milnor exact sequence* ([**33**, Chapter VI, Proposition 2.15]).

Proposition 1.6.4. *Let $\cdots \to E_n \to E_{n-1} \to \cdots \to E_2 \to E_1 \to E_0$ be a sequence of Kan fibrations in which all E_n are Kan complexes. Assume that all the maps $\pi_1 E_n \to \pi_1 E_{n-1}$ are surjective for any choice of base point in E_n, then we have a canonical isomorphism $\pi_0(\lim_{n\in\mathbb{N}^{\mathrm{op}}} E_n) \xrightarrow{\cong} \lim_{n\in\mathbb{N}^{\mathrm{op}}} \pi_0(E_n)$.*

For a map $f : X \to Y$ in $\mathsf{s}\mathcal{S}\mathsf{et}$, we say that a map $g : Y \to X$ is a *homotopy section* of f if $f \circ g$ is homotopic to 1_Y.

Proposition 1.6.5. *If $f : X \to Y$ is a Kan fibration, and $g : Y \to X$ is a homotopy section of f, then there is a section $g' : Y \to X$ of f such that g, g' are homotopic.*

Proof. Choose a map $h : Y \times \Delta^1 \to Y$ with $h \circ (1_Y \times d^1) = fg, h \circ (1_Y \times d^0) = 1_Y$. Since $1_Y \times d^1$ is a trivial cofibration, we can find a lifting in the following diagram

$$\begin{array}{ccc} Y = Y \times \Delta^0 & \xrightarrow{g} & X \\ {\scriptstyle 1_Y\times d^1}\downarrow & \overset{\theta}{\nearrow} & \downarrow{\scriptstyle f} \\ Y \times \Delta^1 & \xrightarrow{h} & Y, \end{array}$$

take $g' = \theta d^0$, then $fg' = 1_Y$ and θ is a homotopy from g to g'. □

There is an adjunction

$$\pi_0 : \mathsf{s}\mathcal{S}\mathsf{et} \rightleftarrows \mathcal{S}\mathsf{et} : \iota,$$

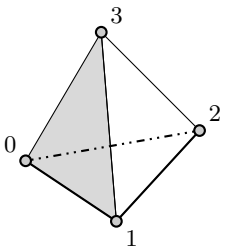
where $\iota : \mathcal{S}\mathsf{et} \to \mathsf{s}\mathcal{S}\mathsf{et}$ is the inclusion.

For fixed vertex $\Delta^0 \xrightarrow{v} X$ and $n > 0$, define $\pi_n(X, v)$ to be the set of homotopy classes of maps $\Delta^n \xrightarrow{\alpha} X$ (rel $\partial\Delta^n$) such that $\alpha|_{\partial\Delta^n} = v$, i.e. such that the diagram

$$\begin{array}{ccc} \partial\Delta^n & \longrightarrow & \Delta^0 \\ \downarrow & & \downarrow{\scriptstyle v} \\ \Delta^n & \xrightarrow{\alpha} & X \end{array}$$

[23] In fact, by horn extension property, we can take $J = \Delta^1$.

commutes. So we have $\pi_n(X,v) = [S^n,(X,v)]_{\mathsf{s\mathfrak{S}et}_*}$. Note that by Remark 1.6.1, $\pi_n(X,v) = \pi_0(F,v)$, where F is the fibre of the map $X^{\Delta^n} \to X^{\partial\Delta^n}$ over v.



For $[\alpha],[\beta] \in \pi_n(X,v)$, let $v_i = v, 0 \leqslant i \leqslant n-2, v_{n-1} = \alpha, v_{n+1} = \beta$. These data give a morphism $\alpha \vee_v \beta = (v_0,\cdots,v_{n-1},v_{n+1}) : \Lambda^{n+1}_n \to X$, which extends to a map $\omega : \Delta^{n+1} \to X$. We define $[\alpha]+[\beta] := [d_n\omega] \in \pi_n(X,v)$ (so $\partial\omega = (v,\cdots,v,\alpha,d_n\omega,\beta)$). Here is a picture for $n=2$: The tetrahedron represents ω, the face $(0,1,2)$ represents β, $(0,2,3)$ represents α, $(1,2,3)$ represents the degenerated face at v, and the face $(0,1,3)$ represents $d_2\omega$.

Proposition 1.6.6. *If X is fibrant, the above operation $+$ is well-defined on $\pi_n(X,v)$, making $(\pi_n(X,v),+)$ a group for all $n \geqslant 1$, which is abelian if $n \geqslant 2$. The identity e of the groups are represented by the simplices $\Delta^n \to \Delta^0 \xrightarrow{v} X$.*

In fact, one can show that $\pi_n(X,v) = \pi_n(|X|,v)$, where on the right, the π_n is the homotopy group defined in algebraic topology for topological spaces. For this and many other interesting comparison results like this, see [**62**, §16].

Let $X \in \mathsf{s\mathfrak{S}et}$ be fibrant. We define its *fundamental groupoid* $\pi_f(X)$ to be the groupoid with objects X_0, and arrows $x \to y$ to be the homotopy classes of maps $\omega : \Delta^1 \to X$ (rel $\partial\Delta^1$) with $\omega d^1 = x, \omega d^0 = y$, denoted $[\omega] : x \to y$.

For $[v_2] : x \to y, [v_0] : y \to z$, define $[v_0] \circ [v_2] := [d_1\xi]$, where $\xi \in X_2$ and $d_i\xi = v_i, i = 0,2$, this makes $\pi_f(X)$ a groupoid. And for any vertex $x \in X_0$, we have $\pi_f(X)(x,x) = \pi_1(X,x)$ with identity $s_0x = (\Delta^1 \xrightarrow{s^0} \Delta^0 \xrightarrow{x} X)$ (take $\xi = (\Delta^2 \xrightarrow{s^0} \Delta^1 \xrightarrow{v_0} X)$, then $\partial\xi = (v_0,v_0,s_0x)$).

Given $[\alpha] \in \pi_n(X,x) (n \geqslant 1)$ and $[\omega] \in (\pi_f X)(x,y)$, let $\{0\} = d^1(\Delta^0) \subset \Delta^1$. We have the following commutative diagram

$$\begin{array}{ccc} (\partial\Delta^n \times \Delta^1) \cup (\Delta^n \times \{0\}) & & \\ \downarrow & \searrow^{(\omega\circ \mathrm{pr}_2,\alpha)} & \\ \Delta^n \times \Delta^1 & \xrightarrow{\exists h_{(\omega,\alpha)}} & X. \\ \uparrow{\scriptstyle 1\times d^0} & \nearrow_{\omega_*\alpha} & \\ \Delta^n \times \Delta^0 & & \end{array}$$

Proposition 1.6.7 ([**33**, Chapter I, Proposition 8.1])**.** *The assignment $[\alpha] \mapsto [\omega_*\alpha] = [h_{(\omega,\alpha)} \circ (1 \times d^0)]$ defines a group homomorphism $[\omega]_* : \pi_n(X,x) \to \pi_n(X,y)$, functorial in $[\omega]$. Thus we get a functor $\pi_f X \to \mathcal{G}\mathsf{r}, x \mapsto \pi_n(X,x)$. Restricting to $x = y$ we get a (left) action of $\pi_1(X,x)$ on $\pi_n(X,x)$ for all $n \geqslant 1$, and for $n = 1$, this is just the conjugation action of $\pi_1(X,v)$ on itself.*

For a Kan complex X and a vertex $* \in X_0$, define the *path space* $\mathrm{P}X$ via the pullback diagram

$$\begin{array}{ccc} \mathrm{P}X & \xrightarrow{i_X} & X^{\Delta^1} \\ \downarrow & \lrcorner & \downarrow{\scriptstyle (d^0)^*} \\ \Delta^0 & \xrightarrow{*} & X^{\Delta^0} = X. \end{array}$$

Define the map $\pi : \mathrm{P}X \to X$ to be the composition $(d^1)^* \circ i_X$, it is a fibration and $\pi_i(\mathrm{P}X,v) = 0$ for all $i \geqslant 0$ and vertex $v \in (\mathrm{P}X)_0$. Define the *loop space* ΩX of X at a vertex $* \in X_0$ to be the fiber of π over $* \in X_0$. We have

$$(\Omega X)_n = \{\Delta^1 \xrightarrow{\gamma} X^{\Delta^n} : \gamma(\partial\Delta^1) = *\}.$$

The map $\pi : \mathrm{P}X \to X$ fits into a pullback diagram

$$\begin{array}{ccc} \mathrm{P}X & \xrightarrow{i_X} & X^{\Delta^1} \\ {\scriptstyle\pi}\downarrow & \lrcorner & \downarrow{\scriptstyle i^*} \\ X & \xrightarrow{*} & X^{\partial\Delta^1} = X \times X, \end{array}$$

where i^* is a fibration of fibrant objects, so $\pi : \mathrm{P}X \to X$ is a fibration of fibrant objects as well. Moreover, we have canonical isomorphisms $\pi_n(X) \cong \pi_{n-1}(\Omega X) \cong \cdots \cong \pi_1(\Omega^{n-1}X) \cong \pi_0(\Omega^n X)$.

There are natural (right) actions of $\pi_1(X,v)$ on all $\pi_n(X,v), n \geqslant 1$ which we now describe. Let $[\alpha] \in \pi_n(X,v) = \pi_{n-1}(\Omega X), [\gamma] \in \pi_1(X,v) = \pi_0(\Omega X)$, let $[\bar\gamma] = [\gamma]^{-1} \in \pi_1(X,v) = \pi_0(\Omega X)$, so we have maps $\alpha : S^{n-1} \to \Omega X, \gamma, \bar\gamma : S^0 \to \Omega X$ in $\mathsf{s\mathfrak{S}et}_*$. Define $\bar\gamma * \alpha * \gamma : S^{n-1} \xrightarrow{\cong} S^0 \wedge S^{n-1} \wedge S^0 \xrightarrow{(\bar\gamma,\alpha,\gamma)} \Omega X$, and letting $[\alpha] \cdot [\gamma] := [\bar\gamma * \alpha * \gamma]$. For $n = 1$, this is just the conjugation action of $\pi_1(X,v)$ on itself. The (right) action described here is just the (left) action given by the functor $\pi_f X \to \mathcal{G}\mathsf{r}, x \mapsto \pi_n(X,x)$ above, by converting left actions to right actions in the usual way.

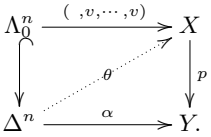
For a Kan fibration $p : X \to Y$ of Kan complexes, define *its fiber F over $* \in Y_0$* via the pullback diagram

$$\begin{array}{ccc} F & \xrightarrow{i} & X \\ \downarrow & \lrcorner & \downarrow{\scriptstyle p} \\ \Delta^0 & \xrightarrow{*} & Y. \end{array}$$

For $v \in F_0$ and $[\alpha] \in \pi_n(Y,*)(n \geqslant 1)$, there is a well-defined element

$$\partial[\alpha] := [d_0\theta] \in \pi_{n-1}(F,v)$$

determined by the lifting diagram



We obtain a homotopy fiber sequence in sSet_*:

$$(F,v) \xrightarrow{i} (X,v) \xrightarrow{p} (Y,*).$$

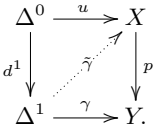
In the following, exactness of a sequence of pointed sets (we view a group as a pointed set with base point the identity e of the group) means that the image of a map equals the preimage of the base point of the next map.

Theorem 1.6.8 ([**33**, Chapter I, Lemma 7.3])**.** *The map $\partial : \pi_n(Y,*) \to \pi_{n-1}(F,v)$, called the* boundary map*, is a group homomorphism for each $n \geqslant 2$, which fits into a long exact sequence*

$$\cdots \to \pi_n(F,v) \xrightarrow{i_*} \pi_n(X,v) \xrightarrow{p_*} \pi_n(Y,*) \xrightarrow{\partial} \pi_{n-1}(F,v) \to \cdots \to \pi_1(Y,*) \xrightarrow{\partial} \pi_0(F,v) \xrightarrow{i_*} \pi_0(X,v) \xrightarrow{p_*} \pi_0(Y,*).$$

This sequence is natural in the fiber sequence $F \to X \xrightarrow{p} Y$.

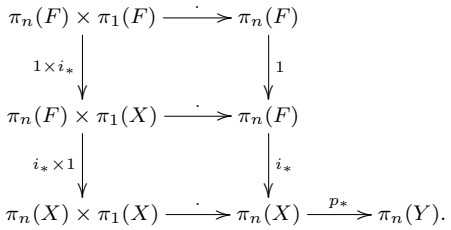
There is a right action of $\pi_1(Y,*)$ on $\pi_0(F,v)$ given by $[u]\cdot[\gamma] = [\tilde{\gamma}d^0]$, where $\tilde{\gamma}$ is a lifting map in the diagram



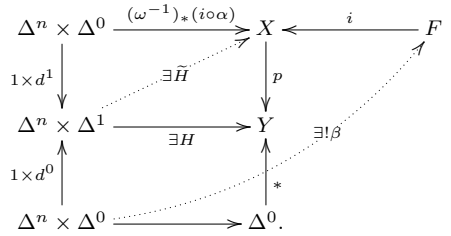
Moreover, the exactness at $\pi_0(F,v)$ can be strengthened as follows: for any $[u],[u'] \in \pi_0(F,v)$, $i_*([u]) = i_*([u']) \in \pi_0(X,v)$ iff $[u]$ and $[u']$ are in the same orbit of this $\pi_1(Y,*)$-action, i.e., $i_*^{-1}(i_*([u])) = [u]\cdot\pi_1(Y,*) \subset \pi_0(F,v)$. In words, two points in the fibre are in the same path component of the total space iff they are in the same orbit of the action by the fundamental group of the base.

For any $[u] \in \pi_0(F,v)$, its stabilizer is $\mathrm{Stab}([u]) = p_*(\pi_1(X,u)) \subset \pi_1(Y,*)$. The map $i_* : \pi_0(F,v) \to \pi_0(X,v)$ is injective on the fiber (orbit) $i_*^{-1}(i_*([u])) = [u]\cdot\pi_1(Y,*)$, i.e. $i_*^{-1}(i_*([u])) = \{[u]\}$, iff $\mathrm{Stab}([u]) = p_*(\pi_1(X,u)) = \pi_1(Y,*)$; $i_* : \pi_0(F,v) \to \pi_0(X,v)$ is injective iff $p_*(\pi_1(X,u)) = \pi_1(Y,*)$ for every $[u] \in \mathrm{im}(\pi_0(F,v) \xrightarrow{i_*} \pi_0(X,v))$.

Remark 1.6.9. In the above, there are also natural (right) actions of $\pi_1(X,v)$ on all $\pi_n(Y,*), \pi_n(F,v), n \geqslant 1$, making the above long exact sequence a long exact sequence of (right) $\pi_1(X,v)$-modules/sets (give the last 3 terms the trivial actions). The action on $\pi_n(Y,*)$ is the composition $\pi_n(Y)\times\pi_1(X) \xrightarrow{1\times p_*} \pi_n(Y)\times\pi_1(Y) \xrightarrow{\cdot} \pi_n(Y)$, while the action on $\pi_n(F,v)$ fits into the following commutative diagram (in the lower square, the image of the lower-left composition lies in $\ker p_* = \mathrm{im}\, i_*$)

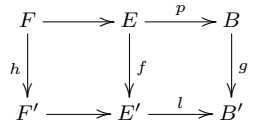


Indeed, given $[\alpha] \in \pi_n(F), [\omega] \in \pi_n(X)$, we have $[(\omega^{-1})_*(i\circ\alpha)] = [h_{(\omega^{-1}, i\circ\alpha)} \circ (1\times d^0)] \in \pi_n(X)$, whose image under p_* is the identity element, so there exists a map H fitting into the following commutative diagram



Taking a lifting map $\widetilde{H}$, we have $p\circ(\widetilde{H}\circ(1\times d^0)) = H\circ(1\times d^0) = *$ is the constant map to the base point, so there exists a (unique if H and $\widetilde{H}$ are given) map $\beta : \Delta^n \to F$ such that $\widetilde{H}\circ(1\times d^0) = i\circ\beta$, then $[\alpha]\cdot[\omega] = [\beta]$. These actions are distributive with the additions in π_n for $n \geqslant 2$.

The long exact sequence is natural: if



is a map of fibrations, then we have a ladder of long exact sequences. In fact, all the terms f_*, g_*, h_* in the resulting ladder are equivariant with respect to the induced map $f_* : \pi_1 E \to \pi_1 E'$ (for f_* and g_*, this is easy; for h_*, this is essentially given by [**63**, Theorem in Chapter 9, §5 (p. 67)]).

Of course, these right actions can be converted to left actions in the usual way: $a \cdot x = x \cdot a^{-1}$.

One can also define the action by the fundamental groupoid of the total space on the "local systems" by considering homotopy groups based at all vertices, as in [**63**, Theorem in Chapter 9, §5], and similar results as above hold.

The following easily checked result is sometimes useful.

Theorem 1.6.10. *Let $q : E \to B$ be a Kan fibration of Kan complexes, assume the two vertices $b_0, b_1 \in B_0$ are in the same path component, write $F_j := q^{-1}(b_j), j = 0, 1$ for the fibers. Choose a path $\omega_0 : \Delta^1 \to B$ with $\omega_0 d^1 = b_0, \omega_0 d^0 = b_1$. This gives a group isomorphism*

$$(\omega_0)_* : \pi_1(B, b_0) \to \pi_1(B, b_1)$$

given by $[\omega] \mapsto [\bar{\omega}_0] \cdot [\omega] \cdot [\omega_0]$, where $\bar{\omega}_0$ is the "path" reversing the direction of ω_0.

We also define a map of sets

$$(\omega_0)_* : \pi_0 F_0 \to \pi_0 F_1$$

by mapping the class $[a]$ of a vertex a in F_0 to the class $[\widetilde{\omega_0} d^0]$ in F_1, where $\widetilde{\omega_0}$ is any lifting map in the following diagram

$$\begin{array}{ccc} \Delta^0 & \xrightarrow{a} & E \\ {\scriptstyle d^1}\downarrow & \overset{\widetilde{\omega_0}}{\nearrow} & \downarrow{\scriptstyle q} \\ \Delta^1 & \xrightarrow{\omega_0} & B. \end{array}$$

Then with the action given by Theorem 1.6.8, the map $(\omega_0)_* : \pi_0 F_0 \to \pi_0 F_1$ is $\pi_1(B, b_0) \xrightarrow{(\omega_0)_*} \pi_1(B, b_1)$-equivariant. Thus there is a bijection of orbit sets (depending on the choice of $[\omega_0]$)

$$(\omega_0)_* : \pi_0 F_0 / \pi_1(B, b_0) \to \pi_0 F_1 / \pi_1(B, b_1).$$

1.7. Homotopy theory of towers

Let $\mathcal{M}$ be a model category, denote by $\mathrm{Tow}(\mathcal{M})$ the category of diagrams

$$\cdots \xrightarrow{q_{n+1}} X_n \xrightarrow{q_n} X_{n-1} \xrightarrow{q_{n-1}} \cdots \xrightarrow{q_2} X_1 \xrightarrow{q_1} X_0$$

in $\mathcal{M}$, called the *category of towers in* $\mathcal{M}$. If we identify the poset $\mathbb{N}$ with the category

$$0 \to 1 \to 2 \to 3 \to \cdots$$

(with only non-identity maps displayed), then we have $\mathrm{Tow}(\mathcal{M}) = \mathcal{M}^{\mathbb{N}^{\mathrm{op}}}$.

If moreover $\mathcal{M}$ is a simplicial model category, we can set for $X, Y \in \mathrm{Tow}(\mathcal{M}), K \in \mathsf{sSet}$,

$$(X \otimes K)_n := X_n \otimes K, (X^K)_n := (X_n)^K, \mathrm{Map}(X, Y)_n := \mathrm{Tow}(\mathcal{M})(X \otimes \Delta^n, Y),$$

these operations give $\mathrm{Tow}(\mathcal{M})$ the structure of a simplicial category.

Theorem 1.7.1 ([**33**, Chapter VI, Proposition 1.3])**.** *The category $\mathrm{Tow}(\mathcal{M})$ of towers in $\mathcal{M}$ has a model structure, with*

- *weak equivalences $\mathbf{W}$: those maps $f : X \to Y$ that are degree-wise weak equivalences.*
- *cofibrations $\mathbf{C}$: those maps $f : X \to Y$ that are degree-wise cofibrations.*
- *fibrations $\mathbf{F}$: those maps $f : X \to Y$ such that $f_0 : X_0 \to Y_0$ and all the induced maps $X_n \to Y_n \times_{Y_{n-1}} X_{n-1}, n \geqslant 1$ are fibrations.*

If moreover $\mathcal{M}$ is a simplicial model category, then $\mathrm{Tow}(\mathcal{M})$ has a simplicial model structure given by the operations above.

In fact, we can get the above result in a very cheap way, as follows: View the poset $\mathbb{N}^{\mathrm{op}} = \{\cdots \to n \to \cdots \to 1 \to 0\}$ as a Reedy category with $\deg(n) = n, \overrightarrow{\mathbb{N}^{\mathrm{op}}} = \{1_n : n \in \mathbb{N}\}, \overleftarrow{\mathbb{N}^{\mathrm{op}}} = \mathbb{N}^{\mathrm{op}}$. In this case, with notations as in [**40**, Chapter 15], the latching category is $\partial(\overrightarrow{\mathbb{N}^{\mathrm{op}}} \downarrow n) = \varnothing$, while the matching category is

$$\partial(n \downarrow \overleftarrow{\mathbb{N}^{\mathrm{op}}}) = \begin{cases} \varnothing, & n = 0; \\ (0 \to 1 \to 2 \to \cdots \to n-1)^{\mathrm{op}}, & n \geqslant 1. \end{cases}$$

Thus the latching objects and matching objects are given by $\mathrm{L}_n X = \operatorname*{colim}_{\varnothing} X = \varnothing$,

$$\mathrm{M}_n X = \begin{cases} \lim_{\varnothing} X = *, & n = 0; \\ X_{n-1}, & n \geqslant 1, \end{cases}$$

since in the latter case, $n-1$ is an initial object of the matching category. Now the result follows from [**40**, Theorem 15.3.4]. For the theory on Reedy category and Reedy model structure, see [**40**, Chapter 15]. An easy computation shows that $\mathrm{sk}_n X$ is the diagram

$$\cdots \to \varnothing \to \cdots \to \varnothing \to X_n \to X_{n-1} \to \cdots \to X_1 \to X_0$$

and $\mathrm{cosk}_n X$ is the diagram

$$\cdots \to X_n \to \cdots \to X_n \to X_n \to X_{n-1} \to \cdots \to X_1 \to X_0,$$

the natural maps $\mathrm{sk}_n X \to X \to \mathrm{cosk}_n X$ are the obvious ones.

Remark 1.7.2. An object X of $\mathrm{Tow}(\mathcal{M})$ is fibrant iff X_0 is fibrant and each of the canonical maps $q_n : X_n \to X_{n-1}$ $(n \geqslant 1)$ is a fibration in $\mathcal{M}$.

If every object of $\mathcal{M}$ is cofibrant, then every object of $\mathrm{Tow}(\mathcal{M})$ is cofibrant.

Note that the constant diagram functor $c : \mathcal{M} \to \mathrm{Tow}(\mathcal{M})$ preserves weak equivalences and cofibrations, so the adjunction

$$c : \mathcal{M} \rightleftarrows \mathrm{Tow}(\mathcal{M}) : \lim$$

is a Quillen pair, we can take the total right derived functor $\mathbf{R}\lim_{\mathbb{N}^{\mathrm{op}}} = \operatorname{holim}_{\mathbb{N}^{\mathrm{op}}} : \mathrm{Ho}(\mathrm{Tow}(\mathcal{M})) \to \mathrm{Ho}(\mathcal{M})$. For $X \in \mathrm{Tow}(\mathcal{M})$, the value $\operatorname{holim}_{\mathbb{N}^{\mathrm{op}}} X$ is called the *homotopy limit of* X, it's determined up to homotopy equivalence by the following procedure: take a fibrant Y together with a weak equivalence $X \to Y$, then $\operatorname{holim}_{\mathbb{N}^{\mathrm{op}}} X \simeq \lim_{\mathbb{N}^{\mathrm{op}}} Y$. See also Proposition 1.2.16.

Theorem 1.7.3 (Homotopy limits preserve weak equivalences)**.** *If $f : X \to Y$ is a weak equivalence in* $\mathrm{Tow}(\mathcal{M})$*, then the induced map $\operatorname{holim}_{\mathbb{N}^{\mathrm{op}}} f : \operatorname{holim}_{\mathbb{N}^{\mathrm{op}}} X \to \operatorname{holim}_{\mathbb{N}^{\mathrm{op}}} Y$ is a weak equivalence in $\mathcal{M}$.*

Proof. Take a fibrant replacement of Y, then it is also a fibrant replacement of X. □

Definition 1.7.4. Let $X \in \mathsf{sSet}$ be connected, then a *Postnikov tower* for X is a tower in sSet

$$\cdots \xrightarrow{q_{n+1}} \mathrm{P}_n X \xrightarrow{q_n} \mathrm{P}_{n-1} X \xrightarrow{q_{n-1}} \cdots \xrightarrow{q_2} \mathrm{P}_1 X \xrightarrow{q_1} \mathrm{P}_0 X$$

equipped with maps $i_n : X \to \mathrm{P}_n X$ such that $i_{n-1} = q_n i_n, \forall n \geqslant 1$ and for all $v \in X_0, \pi_j(\mathrm{P}_n X, i_n(v)) = 0$ for $j > n$,

$$(i_n)_* : \pi_j(X, v) \xrightarrow{\cong} \pi_j(\mathrm{P}_n X, i_n(v))$$

for $1 \leqslant j \leqslant n$.

Remark 1.7.5. One can also consider Postnikov towers for general simplicial sets by working component-wise, we will restrict to the connected simplicial sets. We may also assume X is fibrant. Indeed, if $X \to Y$ is a fibrant replacement, then a Postnikov tower for Y is also a Postnikov tower for X.

There is a specific model for the Postnikov tower: for X fibrant, take $\mathrm{P}_n X = X[n] = \mathrm{im}(X \to \mathrm{cosk}_n X)$ with $i_n : X \to X[n] \subset \mathrm{cosk}_n X$ the canonical map. Since $(\mathrm{cosk}_n X)_k = \mathsf{sSet}(\mathrm{sk}_n \Delta^k, X)$, this image is canonically identified with the quotient $X/\sim_n$, where for two k-simplices $a, b : \Delta^k \to X$, we require $a \sim_n b$ iff the restriction of a, b to $\mathrm{sk}_n \Delta^k$ are the same. It's clear that $(X[n])_k = X_k, 0 \leqslant k \leqslant n$, thus the canonical map $X \to \lim_{n \in \mathbb{N}^{\mathrm{op}}} X[n]$ is an isomorphism. In categorical terms, we form the *kernel pair* g_0, g_1 of the map $i_n : X \to \mathrm{cosk}_n X$, which fit into a pullback diagram

$$\begin{array}{ccc} Q & \xrightarrow{g_1} & X \\ {\scriptstyle g_0}\downarrow & \lrcorner & \downarrow{\scriptstyle i_n} \\ X & \xrightarrow{i_n} & \mathrm{cosk}_n X, \end{array}$$

then $\mathrm{P}_n X = X[n]$ is the coequalizer of g_0, g_1.

Theorem 1.7.6 ([**33**, Chapter VI, Theorem 3.5])**.** *If X is a non-empty Kan complex, then so is each $X[n] = X/\sim_n$; the maps $i_n : X \to X[n]$ and $q_n : X[n] \to X[n-1]$ are surjective Kan fibrations, and the map $X \to \lim_{n \in \mathbb{N}^{\mathrm{op}}} X[n]$ is an isomorphism. The components of $X[0]$ are contractible.*

The tower

$$\cdots \xrightarrow{q_{n+1}} X[n] \xrightarrow{q_n} X[n-1] \xrightarrow{q_{n-1}} \cdots \xrightarrow{q_2} X[1] \xrightarrow{q_1} X[0]$$

in sSet *is a Postnikov tower for X.*

For any $v \in X_0$, define $E_n(X, v) := i_n^{-1}([v]) \subset X$, called (by Moore) the n-th Eilenberg subcomplex of X over v*, it fits into the following pullback diagram*

$$\begin{array}{ccc} E_n(X, v) & \longrightarrow & X \\ \downarrow & \lrcorner & \downarrow{\scriptstyle i_n} \\ \Delta^0 & \xrightarrow{[v]} & X[n], \end{array}$$

or equivalently, there is a fibre sequence $E_n(X, v) \to X \xrightarrow{i_n} X[n]$*. The Eilenberg subcomplex $E_n(X, v)$ is fibrant and n-connected: $\pi_j(E_n(X, v)) = 0, 0 \leqslant j \leqslant n$ (in other words, the map $i_n : X \to X[n]$ is n-connected).*

Proof. From the previous discussion, we already know that the map $X \to \lim_{n \in \mathbb{N}^{\mathrm{op}}} X[n]$ is an isomorphism. The map $i_n : X \to X[n]$ is surjective by definition. Now we show that it is a fibration. Given a solid-arrow diagram

$$\begin{array}{ccc} \Lambda^m_k & \xrightarrow{u} & X \\ {\scriptstyle \iota}\downarrow & \overset{v'}{\nearrow} & \downarrow{\scriptstyle i_n} \\ \Delta^m & \xrightarrow{[v]} & X[n], \end{array}$$

where ι is the inclusion map and $v \in X_m$, we may assume $u = (y_0, \cdots, \hat{y_k}, \cdots, y_m)$ with $y_j \in X_{m-1}, d_i y_j = d_{j-1} y_i, \forall i < j, i, j \neq k$. The commutativity of the square means that $[d_j v] = [y_j] \in X[n]_{m-1}, \forall j \neq k$.

If $m \leqslant n$, then $(i_n)_m = 1_{X_m}$, thus the lifting map v' exists.

If $m = n+1$, then $[d_j v] = [y_j] \in X[n]_n = X_n, \forall j \neq k$, but $d_j v, y_j \in X_n$, thus there is no need to take the classes $[\ ]$ and we indeed have $d_j v = y_j, \forall j \neq k$. Thus $v' := v$ is a lifting map of the square above.

If $m > n+1$, then since X is a Kan complex, there is a map v' shown as the dotted arrow above making the upper triangle commute, i.e. $d_j v' = y_j$, so $[d_j v] = [y_j] = [d_j v'] \in X[n]_{m-1}, j \neq k$. It then suffices to show that $[v'] = i_n(v') = [v]$, i.e. $v'|_{\mathrm{sk}_n \Delta^m} = v|_{\mathrm{sk}_n \Delta^m}$. This is because $\mathrm{sk}_n \Lambda^m_k = \mathrm{sk}_n \Delta^m$ (see Proposition 1.5.5), and $[d_j v] = [d_j v'] \in X[n]_{m-1}, j \neq k$ implies that $v'|_{\mathrm{sk}_n \Lambda^m_k} = v|_{\mathrm{sk}_n \Lambda^m_k}$. Thus v' is a lifting map we want.

By definition it's clear that $i_{n-1} = q_n i_n, \forall n \geqslant 1$, and since all the maps i_n are surjective fibrations, the maps q_n will also be fibrations by Proposition 1.5.12. The same argument shows that each $X[n]$ is fibrant.

The fact that $\pi_j(X[n], i_n(v)) = 0$ for $j > n$ and the n-connectedness of $E_n(X, v)$ follow easily from the definition of homotopy groups, see **[33**, Chapter VI, Lemma 3.6**]**. In particular, $\pi_j(X[0]) = 0, \forall j \geqslant 1$, hence the components of $X[0]$ are contractible. The remaining statements follow from the homotopy long exact sequence of the fibre sequence $E_n(X, v) \to X \xrightarrow{i_n} X[n]$. □

Below, we will need the notion of *Eilenberg-Mac Lane spaces*, which we refer to **[33**, p. 163**]**.

Theorem 1.7.7 (**[33**, Chapter VI, Corollary 3.7**]**)**.** *Let $X \in \mathsf{sSet}$ be a connected Kan complex, $v \in X_0$ and let $K(n)$ be the fiber of the projection $q_n : \mathrm{P}_n X \to \mathrm{P}_{n-1} X$ at $i_n(v)$. Then there is a weak equivalence $K(n) \to K(\pi_n X, n)$. In particular, there is a weak equivalence $\mathrm{P}_1 X \simeq K(1) \simeq K(\pi_1 X, 1) = \mathrm{B}(\pi_1 X)$. Thus there is a homotopy fibre sequence*

$$\mathrm{K}(\pi_n X, n) \to X[n] \xrightarrow{q_n} X[n-1].$$

Definition 1.7.8. Let $f : E \to B$ be a map in sSet, then a *Moore-Postnikov tower* for f is a tower in sSet

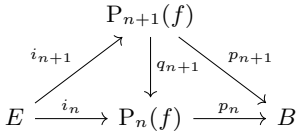
$$\cdots \xrightarrow{q_{n+1}} \mathrm{P}_n(f) \xrightarrow{q_n} \mathrm{P}_{n-1}(f) \xrightarrow{q_{n-1}} \cdots \xrightarrow{q_2} \mathrm{P}_1(f) \xrightarrow{q_1} \mathrm{P}_0(f)$$

equipped with maps $i_n : E \to \mathrm{P}_n(f),\ p_n : \mathrm{P}_n(f) \to B$ such that

- $i_{n-1} = q_n i_n, p_{n-1} q_n = p_n, \forall n \geqslant 1$.
- $p_n i_n = f, \forall n \in \mathbb{N}$.
- For any $v \in E_0$ and $j \leqslant n$, $(i_n)_* : \pi_j(E, v) \xrightarrow{\cong} \pi_j(\mathrm{P}_n(f), i_n(v))$.
- For any $v \in E_0$ and $j \geqslant n+2$, $(p_n)_* : \pi_j(\mathrm{P}_n(f), i_n(v)) \xrightarrow{\cong} \pi_j(B, f(v))$.
- For any $v \in E_0$, there is an exact sequence

  $$0 \to \pi_{n+1}(\mathrm{P}_n(f), i_n(v)) \xrightarrow{(p_n)_*} \pi_{n+1}(B, f(v)) \xrightarrow{\partial} \pi_n(F_v, v),$$

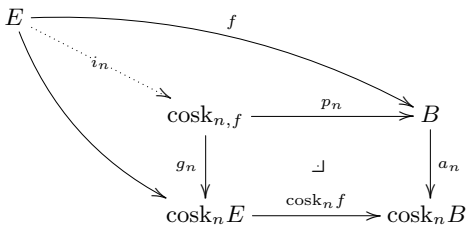
  where F_v is the homotopy fibre of f at $f(v)$, and ∂ is the connecting homomorphism in the homotopy long exact sequence of the homotopy fibre sequence $F_v \to E \xrightarrow{f} B$.



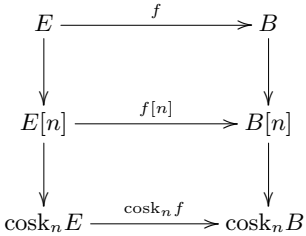
Remark 1.7.9. By the exactness of the sequence $\pi_{n+1}(E, v) \xrightarrow{f_*} \pi_{n+1}(B, f(v)) \xrightarrow{\partial} \pi_n(F_v, v)$, the last point above implies that $\mathrm{im}\ f_* = \ker\ \partial = \pi_{n+1}(\mathrm{P}_n(f), i_n(v))$, thus $\pi_{n+1}(E, v) \xrightarrow{(i_n)_*} \pi_{n+1}(\mathrm{P}_n(f), i_n(v)) \subset \pi_{n+1}(B, f(v))$ is surjective, i.e. the map $i_n : E \to \mathrm{P}_n(f)$ is an $(n+1)$-equivalence (and i_n is n-connected).

A Moore-Postnikov tower for the map $E \to *$ is a Postnikov tower for E.

We define $\mathrm{cosk}_{n,f}$, called the *relative n-coskeleton of f*, via the cartesian square



Suppose now that B is fibrant and $f : E \to B$ is a fibration, we have the following commutative diagram

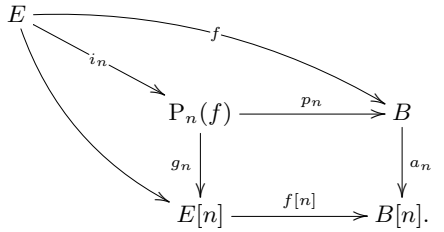


Define $\mathrm{P}_n(f) = \mathrm{im}(E \xrightarrow{i_n} E[n] \times_{B[n]} B \subset \mathrm{cosk}_{n,f}) = \mathrm{im}(E \xrightarrow{i_n} \mathrm{cosk}_{n,f}) = i_n(E) \subset \mathrm{cosk}_{n,f}$, we get a canonical map $i_n : E \to \mathrm{P}_n(f)$. The image of the map $E \to E[n] \times_{B[n]} B$ is canonically identified with the quotient $E/\approx_n$, where for

two k-simplices $a, b : \Delta^k \to X$, we require $a \approx_n b$ iff $f(a) = f(b)$ and the restriction of a, b to $\mathrm{sk}_n\Delta^k$ are the same; in fact, that map is factored as $E \twoheadrightarrow E/\approx_n \rightarrowtail E[n] \times_{B[n]} B$, the first map is given by $u \mapsto [u]$ and the second is given by $[u] \mapsto ([u], f(u))$. We thus have

$$\mathrm{P}_n(f) = (E/\approx_n) \subset E[n] \times_{B[n]} B \subset \mathrm{cosk}_{n,f}.$$

It's clear that $(\mathrm{P}_n(f))_k = E_k, 0 \leqslant k \leqslant n$, and thus the canonical map $E \to \lim\limits_{n\in\mathbb{N}^{\mathrm{op}}} \mathrm{P}_n(f)$ is an isomorphism. We have the following commutative diagram



In concrete terms, the set of m-simplices of $\mathrm{P}_n(f) = E/\approx_n$ consists of ordered pairs $([u], f(u)) \in E[n] \times_{B[n]} B$ (denoted $[u]$ when considered as in $E/\approx_n$), where $u \in E_m$. The maps are given by

$$i_n(u) = ([u], f(u)), p_n([u], f(u)) = f(u), g_n([u], f(u)) = [u], (f[n])([u]) = [f(u)], a_n(b) = [b].$$

Since $a_n f$ is a fibration and the map $E \to E[n]$ is a surjective fibration, we find that $f[n]$ is also a fibration.

For a fixed vertex $v \in E_0$, we have $p_n^{-1}(f(v)) = F_v[n]$. There are obvious maps $q_n : \mathrm{P}_n(f) \to \mathrm{P}_{n-1}(f), n \geqslant 1$, satisfying $i_{n-1} = q_n i_n$. It's also clear that $p_n = p_0 q_1 \cdots q_n = p_{n-1} q_n, \forall n \geqslant 1$.

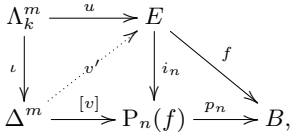
Theorem 1.7.10. *In the above, the tower*

$$\cdots \xrightarrow{q_{n+1}} \mathrm{P}_n(f) \xrightarrow{q_n} \mathrm{P}_{n-1}(f) \xrightarrow{q_{n-1}} \cdots \xrightarrow{q_2} \mathrm{P}_1(f) \xrightarrow{q_1} \mathrm{P}_0(f)$$

is a Moore-Postnikov tower for f. All the maps i_n, p_n, q_n are surjective fibrations, and the map $E \to \lim\limits_{n\in\mathbb{N}^{\mathrm{op}}} \mathrm{P}_n(f)$ is an isomorphism.

Moreover, for any fixed $v \in E_0$, the fibre $F(q_n) = q_n^{-1}(*)$ of q_n $(n \geqslant 1)$ over the vertex $* = ([v], f(v)) \in \mathrm{P}_{n-1}(f)_0$ is weakly equivalent to $\mathrm{K}(\pi_n F_v, n)$.

Proof. From the previous discussion, we already know that the map $E \to \lim\limits_{n\in\mathbb{N}^{\mathrm{op}}} \mathrm{P}_n(f)$ is an isomorphism. We now show that the maps i_n are fibrations, in parallel with the proof of Theorem 1.7.6. Given a solid-arrow diagram



where ι is the inclusion map, $v \in E_m$ and $u = (y_0, \cdots, \hat{y_k}, \cdots, y_m)$ with $y_j \in E_{m-1}, d_i y_j = d_{j-1} y_i, \forall i < j, i, j \neq k$. The commutativity of the square means that $[d_j v] = [y_j] \in \mathrm{P}_n(f)_{m-1}, \forall j \neq k$.

If $m \leqslant n$, then $(i_n)_m = 1_{E_m}$, thus the lifting map v' exists.

If $m = n+1$, then $[d_j v] = [y_j] \in \mathrm{P}_n(f)_n = E_n, \forall j \neq k$, but $d_j v, y_j \in E_n$, thus there is no need to take the classes $[\ ]$ and we indeed have $d_j v = y_j, \forall j \neq k$. Thus $v' := v$ is a lifting map of the square above.

If $m > n+1$, then since f is a Kan fibration, there is a lifting map v' of the outer contour, i.e. $f(v') = f(v), d_j v' = y_j$, so $[d_j v] = [y_j] = [d_j v'] \in \mathrm{P}_n(f)_{m-1}, j \neq k$; in other words:

$$f(d_j v') = f(d_j v), d_j v'|_{\mathrm{sk}_n\Delta^{m-1}} = d_j v|_{\mathrm{sk}_n\Delta^{m-1}} : \mathrm{sk}_n\Delta^{m-1} \to E.$$

Let α be the composition

$$\coprod_{j\in[m]\setminus\{k\}} \mathrm{sk}_n\Delta^{m-1} \hookrightarrow \coprod_{j\in[m]\setminus\{k\}} \Delta^{m-1} \to \Delta^m,$$

where the second map is d^j on the j-th summand. Then we have $v' \circ \alpha = v \circ \alpha$. Since $\mathrm{im}\,\alpha = \bigcup_{j\in[m]\setminus\{k\}} \mathrm{sk}_n d^j \Delta^{m-1} = \mathrm{sk}_n\Lambda_k^m = \mathrm{sk}_n\Delta^m$ for $m > n+1$ (see Proposition 1.5.5), we conclude that $v'|_{\mathrm{sk}_n\Delta^m} = v|_{\mathrm{sk}_n\Delta^m}$, thus v' is in fact a lifting map of the square.

Thus the maps i_n are indeed surjective fibrations, and so are the maps i_n, p_n, q_n, by Proposition 1.5.12.

For a fixed vertex $v \in E_0$, we have $p_n^{-1}(f(v)) = F_v[n]$. We thus get a *fibre sequence*

$$F_v[n] \to \mathrm{P}_n(f) \xrightarrow{p_n} B.$$

The last two points in the definition of Moore-Postnikov tower now follow easily from the homotopy long exact sequence for fibrations (Theorem 1.6.8), together with the results on Postnikov towers for Kan complexes (Theorem 1.7.6).

Since $p_n i_n = f$, by Definition 1.3.1, there is a fibre sequence

$$F(i_n) \to F_v \to F_v[n],$$

where $F(i_n) = i_n^{-1}([v], f(v))$ is the fibre of i_n at the vertex $([v], f(v))$. By the results on Postnikov towers for Kan complexes and the homotopy long exact sequence for this fibration, we conclude that $\pi_j(F(i_n), v) = 0, 0 \leqslant j \leqslant n$ and hence we get the third point: $(i_n)_* : \pi_j(E, v) \xrightarrow{\cong} \pi_j(\mathrm{P}_n(f), i_n(v)), 0 \leqslant j \leqslant n$, from the fibre sequence

$$F(i_n) \to E \to \mathrm{P}_n(f).$$

Similarly, by $p_{n-1}q_n = p_n, \forall n \geqslant 1$, we get the fibre sequence

$$F(q_n) \to F_v[n] \xrightarrow{q_n} F_v[n-1] \ (n \geqslant 1),$$

which gives the last statement in the theorem. □

We state some conclusions drawn from the above proof:

Proposition 1.7.11 ([**33**, Chapter VI, Lemma 3.12 and Corollary 3.13])**.** *In the above, let F be the fibre of f at some vertex b of B, then the fibre of $p_n : \mathrm{P}_n(f) \to B$ at b is the n-th stage $F[n]$ in the Postnikov tower for F. We have fibre sequences*

$$F[n] \to \mathrm{P}_n(f) \xrightarrow{p_n} B,$$
$$F(q_n) \to F[n] \xrightarrow{q_n} F[n-1] \ (n \geqslant 1),$$

and homotopy fibre sequences

$$\mathrm{K}(\pi_n F, n) \to \mathrm{P}_n(f) \xrightarrow{q_n} \mathrm{P}_{n-1}(f) \ (n \geqslant 1).$$

If F is connected, then $p_0 : \mathrm{P}_0(f) \to B$ is a weak equivalence.

Remark 1.7.12. The above results on Postnikov towers and Moore-Postnikov towers already appear in the early literature [**62**, §8], especially Proposition 8.2, 8.8 and Theorem 8.4, 8.9 of that book.

The map q_n is a principal fibration iff $\pi_1(E)$ acts trivially on $\pi_n(F)$. See [**37**, Theorem 4.71] in the topological setting and see also Theorem 1.8.13 for more general and precise results.

1.8. Cohomology with local coefficients and K(M,n)-fibrations

We fix a groupoid G (e.g. a group), then its classifying space BG is a Kan complex. We will consider the diagram category $\mathsf{s}\mathcal{S}\mathsf{et}^G$ and the comma category $(\mathsf{s}\mathcal{S}\mathsf{et} \downarrow \mathrm{B}G)$. If we give $\mathsf{s}\mathcal{S}\mathsf{et}^G$ the projective model structure, then it becomes a simplicial model category. If G is connected, fix $v \in G$ and let $\pi = G_v := G(v,v)$ be the automorphic group at v, then we have equivalences of groupoids $i : \pi \rightleftarrows G : r$ (i is the inclusion, $ri = 1_\pi$), which induces equivalences of model categories $i^* : \mathsf{s}\mathcal{S}\mathsf{et}^G \rightleftarrows \mathsf{s}\mathcal{S}\mathsf{et}^\pi : r^*$.

We mention that equivariant homotopy and cohomology theory in the topological setting, somewhat parallel to some results below, are developed in [**64**].

Proposition 1.8.1 ([**33**, Chapter VI, Lemma 4.2, Corollary 4.4])**.** *Let $p : Z \to Y$ be a map in the diagram category $\mathsf{s}\mathcal{S}\mathsf{et}^G$. If for each $v \in G$, the map $p_v : Z_v \to Y_v$ is a Kan fibration, then so is the induced map $p_* : \mathrm{hocolim}_G Z \to \mathrm{hocolim}_G Y$.*

If for each $v \in G$, Y_v is a Kan complex, then the canonical map $\mathrm{hocolim}_G Y \to \mathrm{B}G$ is a Kan fibration.

A *G-local coefficient system* is a functor $A : G \to \mathcal{A}\mathsf{b}$; see also [**59**, 005E] for some other description. We can form the diagram $\mathrm{K}(A,n) : G \to \mathsf{s}\mathcal{A}\mathsf{b}$ by object-wise performing the usual functorial Eilenberg-Mac Lane space construction. $\mathrm{K}(A,n)$ is fibrant in $\mathsf{s}\mathcal{S}\mathsf{et}^G$.

Then for an object $\phi : X \to \mathrm{B}G$ of $(\mathsf{s}\mathcal{S}\mathsf{et} \downarrow \mathrm{B}G)$, we have the group $C^n_G(X,A)$, whose elements are called the *n-cochain in X with coefficients in A*, which are (set) maps $\alpha : X_n \to \coprod_{v \in G} A_v$ with $\alpha(v) \in A_{\phi(v)(0)}$ ([**33**, p.332]), with pointwise addition. They assemble (for all n) to a cosimplicial abelian group $C^*_G(X,A)$, the corresponding cohomology groups (of the associated Moore complex) is denoted $\mathrm{H}^*_G(X;A)$, called the *cohomology of X with local coefficients in A*.

The assignment $v \mapsto \mathrm{B}(G \downarrow v)$ yields a diagram $G \to \mathsf{s}\mathcal{S}\mathsf{et}$, and for each $v \in G$, the map $\mathrm{B}(G \downarrow v) \to \mathrm{B}G$ is a Kan fibration. Setting

$$\widetilde{X}_v := X \times_{\mathrm{B}G} \mathrm{B}(G \downarrow v),$$

we get a diagram $\widetilde{X} \in \mathsf{s}\mathcal{S}\mathsf{et}^G$, called the *covering system for $\phi : X \to \mathrm{B}G$*.

Remark 1.8.2. If G is the fundamental groupoid of X and $\phi : X \to \mathrm{B}G$ is the canonical map, then $\widetilde{X}_v$ is a copy of the universal cover of (the component of v in) X.

If G is a group, then v is the unique object of G and $\mathrm{B}(G \downarrow v)$ is contractible with free G-action, hence the map $\mathrm{B}(G \downarrow v) \to \mathrm{B}G$ is in fact the universal projection $\mathrm{E}G \to \mathrm{B}G$, thus the map $\widetilde{X} \to X$ is a covering map with deck transformation group G (or a G-torsor over X).

Proposition 1.8.3. *We have the Quillen adjunction*

$$\widetilde{\ }: (\mathsf{s}\mathcal{S}\mathsf{et} \downarrow \mathrm{B}G) \rightleftarrows \mathsf{s}\mathcal{S}\mathsf{et}^G : \mathrm{hocolim}_G.$$

If G is a group, then this is a Quillen equivalence.

The first part is [**33**, Chapter VI, Lemma 4.6, 4.2]. The second part follows from [**45**, Corollary 1.3.6] and the fact that the counit $\mathrm{E}G \times_{\mathrm{B}G} (\mathrm{E}G \times_G Y) = \mathrm{E}G \times Y \to Y$ is a weak equivalence for every (fibrant) $Y \in \mathsf{s}\mathcal{S}\mathsf{et}^G$.

Remark 1.8.4. If G is a connected groupoid, fix $v \in G$ and let $\pi = G_v := G(v,v)$, let $i : \pi \rightleftarrows G : r$ be the equivalence of groupoids as before (i is the inclusion, $ri = 1_\pi$). For a diagram $F \in \mathsf{s}\mathcal{S}\mathsf{et}^G$, let $F_v := F \circ i \in \mathsf{s}\mathcal{S}\mathsf{et}^\pi$. Then the map

$$\mathrm{hocolim}_\pi F_v \to \mathrm{hocolim}_G F$$

is a weak equivalence. Moreover (by [**33**, Chapter IV, Example 1.10]), $\mathrm{hocolim}_\pi F_v \cong \mathrm{E}\pi \times_\pi F_v$.

If G is a group acting additively on an abelian group A (i.e. A is a G-module), then (by [**33**, Chapter IV, Example 1.10]) $\mathrm{hocolim}_G \mathrm{K}(A,n) \cong \mathrm{E}G \times_G \mathrm{K}(A,n) =: \mathrm{K}^G(A,n)$ is the *twisted Eilenberg-Mac Lane space*. There is a canonical map

$$s = s_G : \mathrm{B}G \to \mathrm{K}^G(M,n)$$

induced by the inculsion of the base point 0 of $\mathrm{K}(M,n)$ into $\mathrm{K}(M,n)$.

For any diagram $Y \in \mathsf{s}\mathcal{S}\mathsf{et}^G$, we have the cosimplicial abelian group $\hom_G(Y,A)$ given by $\hom_G(Y,A)^n := \mathcal{S}\mathsf{et}^G(Y_n, A)$.

Proposition 1.8.5 ([**33**, Chapter VI, Corollary 4.8]). *There is a natural isomorphism in* $\mathsf{c}\mathcal{A}\mathsf{b}$*:*

$$C^*_G(X,A) \cong \hom_G(\widetilde{X}, A).$$

Theorem 1.8.6 ([**33**, Chapter VI, Theorems 4.10, 4.11]). *For an object* $\phi : X \to \mathrm{B}G$ *of* $(\mathsf{s}\mathcal{S}\mathsf{et} \downarrow \mathrm{B}G)$ *and a* G*-local coefficient system* $A : G \to \mathcal{A}\mathsf{b}$*, there are natural isomorphisms*

$$[X, \mathrm{hocolim}_G \mathrm{K}(A,n)]_{(\mathsf{s}\mathcal{S}\mathsf{et}\downarrow \mathrm{B}G)} \cong [\widetilde{X}, \mathrm{K}(A,n)]_G \xrightarrow{\cong} \mathrm{H}^n_G(X;A) \cong [\mathbb{Z}\widetilde{X}, A[n]]_{\mathrm{Ch}^G_+},$$

where we write $[-,-]_G$ *for the homotopy classes in* $\mathsf{s}\mathcal{S}\mathsf{et}^G$ *and* $[-,-]_{\mathrm{Ch}^G_+}$ *for the chain homotopy classes in* Ch^G_+*, the category of chain complexes of* G*-diagrams of abelian groups, and* $\mathbb{Z}\widetilde{X}$ *is the Moore complex associated to the* G*-diagram of simplicial sets* $\widetilde{X}$*. The arrow is defined by sending a class of a map* $f : \tilde{X} \to \mathrm{K}(A,n)$ *to the class of the cocycle* $f_n : X_n \to (\mathrm{hocolim}_G A)_n$*.*

If G *is connected, fix* $v \in G$ *and let* $\pi := G(v,v)$ *be the automorphism group at* v*, then restriction to* π*-spaces gives a commutative diagram of group isomorphisms*

$$\begin{array}{ccc} [\widetilde{X}, \mathrm{K}(A,n)]_G & \longrightarrow & \mathrm{H}^n_G(X;A) \\ \downarrow{\scriptstyle i^*} & & \downarrow{\scriptstyle i^*} \\ [\widetilde{X}_v, \mathrm{K}(A_v,n)]_\pi & \longrightarrow & \mathrm{H}^n_\pi(\widetilde{X}_v; A_v). \end{array}$$

Thus $\mathrm{H}^*_G(\mathrm{B}G;A) \cong \mathrm{H}^*_\pi(\pi; A_v)$ *is the usual group cohomology.*

If G *is a group and* A *is a* G*-module, then for* $X \in (\mathsf{s}\mathcal{S}\mathsf{et} \downarrow \mathrm{B}G)$*,*

$$[X, \mathrm{K}^G(A,n)]_{(\mathsf{s}\mathcal{S}\mathsf{et}\downarrow \mathrm{B}G)} \cong [\widetilde{X}, \mathrm{K}(A,n)]_G \xrightarrow{\cong} \mathrm{H}^n_G(X;A) \cong [\mathbb{Z}\widetilde{X}, A[n]]_{\mathrm{Ch}^G_+}$$

is the G-equivariant cohomology of $\widetilde{X} = X \times_{\mathrm{B}G} \mathrm{E}G \in \mathsf{s}\mathcal{S}\mathsf{et}^G$ (equipped with the G-action inherited from that of $\mathrm{E}G$) with coefficients in A. *(Here* Ch^G_+ *is the category of chain complexes of* G*-modules,* $\mathbb{Z}\widetilde{X}$ *is the Moore complex associated to the* G*-simplicial set* $\widetilde{X}$*.) In the* G*-equivariant cohomology group, we have* $[s \circ \phi] = 0$*.* [24] *Thus a map* $f : X \to \mathrm{K}^G(A,n)$ *represents* 0 *iff it's homotopic (over* $\mathrm{B}G$*) to* $s \circ \phi$*.*

Remark 1.8.7. If G is a group and A is a G-module, then for $Y \in \mathsf{s}\mathcal{S}\mathsf{et}^G$, we may define the *$G$-equivariant cohomology* $\mathrm{H}^*_G(Y;A)$ *of Y with coefficients in the G-module A* by

$$\mathrm{H}^n_G(Y;A) := [Y, \mathrm{K}(A,n)]_G.$$

By the Quillen equivalence in Proposition 1.8.3, we easily find that $\mathrm{H}^n_G(Y;A) \cong \mathrm{H}^n_G(\mathrm{E}G \times_G Y; A)$ (the RHS being cohomology with local coefficients; the notation here is confusing, though). So for computations, we should first take a cofibrant replacement of Y then compute homotopy classes in $\mathsf{s}\mathcal{S}\mathsf{et}^G$ ([**33**, Chapter VI, Definition 4.15]). This generalizes the result of **??** by taking G to be the trivial group.

An object of the pointed model category $(\mathsf{s}\mathcal{S}\mathsf{et} \downarrow \mathrm{B}G)_*$ is a map $\phi : X \to \mathrm{B}G$ together with a section $s : \mathrm{B}G \to X$ of ϕ. For instance, if $X = \mathrm{hocolim}_G \mathrm{K}(A,n)$ with $A \in \mathcal{A}\mathsf{b}^G$ a local coefficient system, take $s(\sigma) = (0,\sigma)$, yielding an object of $(\mathsf{s}\mathcal{S}\mathsf{et} \downarrow \mathrm{B}G)_*$. In the notation of [**82**, p. 85], $(\mathsf{s}\mathcal{S}\mathsf{et} \downarrow \mathrm{B}G)_* = \mathsf{s}\mathcal{S}\mathsf{et}/\!/\mathrm{B}G$.

We say that an n-cochain $\alpha \in C^*_G(X,A)$ in X with coefficients in A is *reduced* if $\alpha \circ s = 0$. Such cochains form a cosimplicial abelian subgroup $\widetilde{C}^*_G(X,A)$ of $C^*_G(X,A)$, the resulting cohomology groups are denoted $\widetilde{\mathrm{H}}^*_G(X,A)$, called the *reduced cohomology of X with local coefficients in A*. There is the split short exact sequence of cosimplicial abelian groups

$$0 \to \widetilde{C}^*_G(X,A) \to C^*_G(X,A) \xrightarrow{s^*} C^*_G(\mathrm{B}G, A) \to 0$$

and the split short exact sequence of graded abelian groups

$$0 \to \widetilde{\mathrm{H}}^*_G(X;A) \to \mathrm{H}^*_G(X;A) \xrightarrow{s^*} \mathrm{H}^*_G(\mathrm{B}G;A) \to 0.$$

Theorem 1.8.8 ([**33**, Chapter VI, Lemma 4.13]). *For an object* $X \in (\mathsf{s}\mathcal{S}\mathsf{et} \downarrow \mathrm{B}G)_*$ *and a* G*-local coefficient system* $A : G \to \mathcal{A}\mathsf{b}$*, there is a natural isomorphism*

$$[X, \mathrm{hocolim}_G \mathrm{K}(A,n)]_{(\mathsf{s}\mathcal{S}\mathsf{et}\downarrow \mathrm{B}G)_*} \xrightarrow{\cong} \widetilde{\mathrm{H}}^n_G(X;A).$$

One can also define the G-equivariant homology and suitable relative cohomology for a pair (with suitable assumption), see [**33**, Chapter VI, §4] for more information.

Now let Y be a connected simplicial set with fundamental groupoid G, let $f : Y \to X$ be a map in $(\mathsf{s}\mathcal{S}\mathsf{et} \downarrow \mathrm{B}G)$. For each vertex v of Y, let F_v be the homotopy fibre of f over $f(v)$. For $n \geqslant 1$, we have the relative homotopy group $\pi_n(f,v) := \pi_{n-1}(F_v, v)$, yielding a G-local system $\pi_n(f)$; we also have the obvious G-local system $\pi_n(Y)$. If moreover G is equivalent to the fundamental groupoid of X, there is a long exact sequence of G-local systems

$$\cdots \to \pi_{n+1}(f) \to \pi_n(Y) \xrightarrow{f_*} \pi_n(X) \to \pi_n(f) \to \cdots.$$

Theorem 1.8.9 ([**33**, Chapter VI, Lemma 5.4]). *In the above situation, assume moreover that X is also connected, $n \geqslant 2$ and that for all vertex v of Y, the map $f_* : \pi_k(Y,v) \to \pi_k(X, f(v))$ is an isomorphism for $k < n$ and a surjection for $k = n$ (i.e. F_v is $(n-1)$-connected); so that G is equivalent to the fundamental groupoid of X. Then for any G-local coefficient system $A : G \to \mathcal{A}\mathsf{b}$, there is an isomorphism*

$$\mathrm{H}^k_G(X;A) \xrightarrow{f^*} \mathrm{H}^k_G(Y;A), k < n$$

[24] Here $s = s_G : \mathrm{B}G \to \mathrm{K}^G(M,n)$ is the canonical map. Since $s \circ \phi : X \to \mathrm{K}^G(A,n) = \mathrm{hocolim}_G \mathrm{K}(A,n)$ is the map induced by the trivial maps $x \mapsto 0 \in \mathrm{K}(A,n)$.

and an exact sequence

$$0 \to \mathrm{H}^n_G(X;A) \xrightarrow{f^*} \mathrm{H}^n_G(Y;A) \to \mathcal{A}\mathsf{b}^G(\pi_{n+1}(f),A) \xrightarrow{d} \mathrm{H}^{n+1}_G(X;A) \xrightarrow{f^*} \mathrm{H}^{n+1}_G(Y;A),$$

natural in the map f and the G-local coefficient system $A : G \to \mathcal{A}\mathsf{b}$.

*In particular, taking $A = \pi_{n+1}(f)$ we have the k-*invariant of *f given by*

$$k(f) := d(1_{\pi_{n+1}(f)}) \in \mathrm{H}^{n+1}_G(X;\pi_{n+1}(f)).$$

If $f' : Y' \to X'$ is another such a map in $(\mathsf{s}\mathcal{S}\mathsf{et} \downarrow \mathrm{B}G)$ and $\varphi : X \to X', \psi : Y \to Y'$ are maps inducing isomorphisms on the fundamental groupoids (so that all these four maps induce isomorphisms on fundamental groupoids which are identified with G) with $[\varphi \circ f] = [f' \circ \psi] \in \mathrm{Ho}(\mathsf{s}\mathcal{S}\mathsf{et} \downarrow \mathrm{B}G)$, then $\psi_* k(f) = \varphi^* k(f') \in \mathrm{H}^{n+1}_G(X;\pi_{n+1}(f'))$.

$$\begin{array}{ccc} Y & \xrightarrow{f} & X \\ {\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle\varphi} \\ Y' & \xrightarrow{f'} & X' \end{array} \quad \Longrightarrow \quad \begin{array}{ccc} X & \xrightarrow{k(f)} & \mathrm{hocolim}_G \mathrm{K}(\pi_{n+1}(f), n+1) \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\psi_*} \\ X' & \xrightarrow{k(f')} & \mathrm{hocolim}_G \mathrm{K}(\pi_{n+1}(f'), n+1) \end{array}$$

The last assertion above results from the naturality in f and A, which implies that

$$\psi_* k(f) = d[\psi] = \varphi^* k(f') \in \mathrm{H}^{n+1}_G(X;\pi_{n+1}(f')),$$

here $[\psi] \in \mathcal{A}\mathsf{b}^G(\pi_{n+1}(f),\pi_{n+1}(f'))$ is the map on G-local coefficient systems induced by ψ.

Fix a fibration $p : \mathrm{L}(A,n) \to \mathrm{K}(A,n)$ with contractible total space, functorial in the abelian group A. Then for a G-local coefficient system $A : G \to \mathcal{A}\mathsf{b}$, we get diagrams $\mathrm{L}(A,n), \mathrm{K}(A,n) \in \mathsf{s}\mathcal{A}\mathsf{b}^G$.

Theorem 1.8.10 ([**33**, Chapter VI, Lemma 5.8])**.** *Let X be a connected simplicial set with fundamental groupoid G, and $A : G \to \mathcal{A}\mathsf{b}$ be a G-local coefficient system. Let $[\theta] \in \mathrm{H}^{n+1}_G(X,A) = [X, \mathrm{hocolim}_G \mathrm{K}(A,n+1)]_{(\mathsf{s}\mathcal{S}\mathsf{et}\downarrow \mathrm{B}G)}$, $n \geqslant 2$ and form the pullback diagram*

$$\begin{array}{ccc} Z & \longrightarrow & \mathrm{hocolim}_G \mathrm{L}(A,n+1) \\ {\scriptstyle p}\downarrow & \lrcorner & \downarrow \\ X & \xrightarrow{\theta} & \mathrm{hocolim}_G \mathrm{K}(A,n+1). \end{array}$$

Then the k-invariant $k(p) = [\theta] \in \mathrm{H}^{n+1}_G(X,A)$.

We can apply this to the situation of Theorem 1.8.9 with $A = \pi_{n+1}(f)$ and $[\theta] = k(f)$, in this case we have the following result ([**33**, Chapter VI, Theorem 5.9]).

Theorem 1.8.11. *In the situation of Theorem 1.8.9, the given map $f : Y \to X$ factors through a map $g : Y \to Z$, as shown in the following diagram in $(\mathsf{s}\mathcal{S}\mathsf{et} \downarrow \mathrm{B}G)$:*

$$\begin{array}{ccccc} & & Z & \longrightarrow & \mathrm{hocolim}_G \mathrm{L}(A,n+1) \\ & {\scriptstyle g}\nearrow & {\scriptstyle p}\downarrow & \lrcorner & \downarrow \\ Y & \xrightarrow{f} & X & \xrightarrow{k(f)} & \mathrm{hocolim}_G \mathrm{K}(A,n+1). \end{array}$$

We have $k(p) = k(f)$, and the (usually non-unique) map $g : Y \to Z$ induces an isomorphism

$$g_* : \pi_{n+1}(f) \to \pi_{n+1}(p).$$

In particular, if $F_v \simeq \mathrm{K}(A,n)$ (for all vertex v), then $g : Y \to Z$ is a weak equivalence and thus we have a homotopy cartesian diagram

$$\begin{array}{ccc} Y & \longrightarrow & \mathrm{hocolim}_G \mathrm{L}(A,n+1) \\ {\scriptstyle f}\downarrow & & \downarrow \\ X & \xrightarrow{\theta} & \mathrm{hocolim}_G \mathrm{K}(A,n+1) \end{array}$$

with $[\theta] = k(f) \in \mathrm{H}^{n+1}_G(X;A)$ a uniquely determined cohomology class—the k-invariant of f.

We now discuss some fundamental results and constructions for simplicial sets, identifying explicitly the homotopy colimits in the above diagrams and hence an explicit model of the universal Eilenberg-Mac Lane fibrations.

Let $q : (E,v) \to (B,b)$ be a Kan fibration of pointed Kan complexes, with fiber F over $b \in B_0$. Assume q admits a section $s : B \to E$ (we don't require $s(b) = v$). We denote this by the diagram

$$F \longrightarrow E \underset{s}{\overset{q}{\rightleftarrows}} B. \tag{1.9}$$

The commutative diagrams

$$\begin{array}{ccc} E^{\Delta^1} & \xrightarrow{(\partial\Delta^1 \hookrightarrow \Delta^1)^*} & E^{\partial\Delta^1} \\ {\scriptstyle q^{\Delta^1}}\downarrow & & \downarrow{\scriptstyle q^{\partial\Delta^1}} \\ B^{\Delta^1} & \xrightarrow[(\partial\Delta^1 \hookrightarrow \Delta^1)^*]{} & B^{\partial\Delta^1} \end{array} \quad \text{and} \quad \begin{array}{ccc} E & \xrightarrow{(sq,1)} & E^{\partial\Delta^1} = E \times E \\ \downarrow & & \downarrow{\scriptstyle q^{\partial\Delta^1} = q \times q} \\ B^{\Delta^1} & \xrightarrow[(\partial\Delta^1 \hookrightarrow \Delta^1)^*]{} & B^{\partial\Delta^1} = B \times B \end{array}$$

induce maps

$$\alpha : E^{\Delta^1} \to B^{\Delta^1} \times_{B^{\partial\Delta^1}} E^{\partial\Delta^1} \text{ and } \beta : E \to B^{\Delta^1} \times_{B^{\partial\Delta^1}} E^{\partial\Delta^1},$$

where the vertical map $E \to B^{\Delta^1}$ is the composite $E \xrightarrow{q} B = B^{\Delta^0} \to B^{\Delta^1}$ (the latter arrow is given by taking "constant paths", namely it is the map $(\Delta^1 \to \Delta^0)^*$).

We make E^{Δ^1} and B^{Δ^1} pointed by taking as base points the composites $\Delta^1 \to \Delta^0 \xrightarrow{v} E$ and $\Delta^1 \to \Delta^0 \xrightarrow{b} B$ respectively; the other spaces also has similar base points. Then the map $\alpha : E^{\Delta^1} \to B^{\Delta^1} \times_{B^{\partial\Delta^1}} E^{\partial\Delta^1}$ is a fibration (see [**40**, Proposition 9.3.8]) with fibre over the above chosen base point the loop space ΩF. Now we define a map u via the pull-back diagram

$$\begin{array}{ccc} \mathrm{P}_B E & \longrightarrow & E^{\Delta^1} \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle \alpha} \\ E & \xrightarrow[\beta]{} & B^{\Delta^1} \times_{B^{\partial\Delta^1}} E^{\partial\Delta^1} \end{array} \qquad \text{and a fibre sequence} \qquad \Omega F \longrightarrow \mathrm{P}_B E \xrightarrow{u} E. \tag{1.10}$$

Decoding the definition, we see that, geometrically, the *relative path space* $\mathrm{P}_B E$ is the space whose points (vertices) are pairs (e, σ) with $e \in E$ and σ a path in E from the point $sq(e)$ to e, totally lying inside the fibre of $q : E \to B$ over $q(e)$; the $q^{\Delta^1} : E^{\Delta^1} \to B^{\Delta^1}$ component of α in the pull-back is taking care of the paths lying totally inside some fibres. The map $u(e, \sigma) = e$ is obtained by taking the end points of such paths. We also have a map

$$\nu : B \to \mathrm{P}_B E$$

by taking "constant paths" at points of B (rigiously, one defines it by a commutative diagram and using the universal property of the pull-back in eq. (1.10)). One then checks that $u\nu = s$ and

$$(qu)\nu = \mathrm{id}_B, \nu(qu) \sim \mathrm{id}_{\mathrm{P}_B E},$$

hence B is a retract of $\mathrm{P}_B E$ and there is a homotopy equivalence pair $qu : \mathrm{P}_B E \rightleftarrows B : \nu$; geometrically, the homotopy equivalence can be given by shrinking every path to its starting point.

We also have the *relative loop space* $\Omega_B E$ via the pull-back diagram

$$\begin{array}{ccc} \Omega_B E & \longrightarrow & \mathrm{P}_B E \\ {\scriptstyle \Omega q}\downarrow & & \downarrow{\scriptstyle u} \\ B & \xrightarrow[s]{} & E, \end{array} \tag{1.11}$$

we get another fibration $\Omega q : \Omega_B E \to B$ with fibre ΩF, the total space $\Omega_B E$ has points (vertices) the loops σ in E based at points in B, totally lying inside the fibre of $q : E \to B$; Ωq has a canonical section $s' : B \to \Omega_B E$ (given by taking "constant paths").

Note that if B is just a point, then all these constructions specialize to the usual path fibration $\Omega E \to \mathrm{P}E \to E$.

Now we assume again that G is a (discrete) group and that M is a G-module via a group homomorphism $\rho : G \to \mathrm{Aut}(M)$. Then G acts on $\mathrm{K}(M, n), n \geqslant 2$. As before, we have the twisted Eilenberg-Mac Lane space

$$\mathrm{K}^G(M, n) := \mathrm{E}G \times_G \mathrm{K}(M, n),$$

fitting into a fibre sequence $\mathrm{K}(M, n) \to \mathrm{K}^G(M, n) \xrightarrow{q} \mathrm{B}G$, which admits a section

$$s = s_G : \mathrm{B}G \to \mathrm{K}^G(M, n)$$

induced by the inculsion of the base point of $\mathrm{K}(M, n)$ into $\mathrm{K}(M, n)$. It's easy to find that

$$\pi_j(\mathrm{K}^G(M, n)) = \begin{cases} \pi_1 \mathrm{B}G = G, & j = 1; \\ \pi_j(\mathrm{K}(M, n)) = M, & j = n; \\ 0, & j \neq 1, n. \end{cases}$$

We are thus in the situation of eq. (1.9):

$$\mathrm{K}(M, n) \longrightarrow \mathrm{K}^G(M, n) \underset{s}{\overset{q}{\rightleftarrows}} \mathrm{B}G. \tag{1.12}$$

The previous constructions thus yield a fibre sequence

$$\Omega \mathrm{K}(M, n) \simeq \mathrm{K}(M, n-1) \longrightarrow \mathrm{P}^G(M, n) \xrightarrow{u} \mathrm{K}^G(M, n), \tag{1.13}$$

where $\mathrm{P}^G(M, n) := \mathrm{P}_{\mathrm{B}G} \mathrm{K}^G(M, n)$ in our previous notation. Recall that there is a pair of homotopy equivalences

$$qu : \mathrm{P}^G(M, n) \rightleftarrows \mathrm{B}G : \nu.$$

Thus we get a homotopy fibre sequence

$$\mathrm{K}(M, n-1) \longrightarrow \mathrm{B}G \xrightarrow{s} \mathrm{K}^G(M, n), \tag{1.14}$$

We already know that $\mathrm{hocolim}_G \mathrm{K}(M, n+1) \cong \mathrm{K}^G(M, n+1)$, for the same reason (and using Proposition 1.2.10), we see that $\mathrm{hocolim}_G \mathrm{L}(M, n+1) \cong \mathrm{hocolim}_G * \cong \mathrm{B}G \xrightarrow{\simeq} \mathrm{P}^G(M, n+1)$.

Theorem 1.8.12. *The fibration $u = u_G : \mathrm{P}^G(M,n+1) \to \mathrm{K}^G(M,n+1), n \geqslant 2$ in eq. (1.13) is a* universal K(M,n)-*fibration with G as fundamental group of the total space (and the base space), i.e. for any* K(M,n)*-fibration $p : E \to B$ with $\pi_1 E \cong G$ inducing isomorphism on fundamental groups (where E, B are assumed to be connected), there is a unique element $[k] = k(p) \in \mathrm{H}^{n+1}_G(B;M)$ such that the fibration $p : E \to B$ is equivalent to the pull-back of $u : \mathrm{P}^G(M,n+1) \to \mathrm{K}^G(M,n+1)$ along k. Thus we have* homotopy cartesian squares

$$\begin{array}{ccc} E & \longrightarrow & \mathrm{P}^G(M,n+1) \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle u} \\ B & \xrightarrow[k]{} & \mathrm{K}^G(M,n+1) \end{array} \qquad \textit{and} \qquad \begin{array}{ccc} E & \longrightarrow & \mathrm{B}G \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle s} \\ B & \xrightarrow[k]{} & \mathrm{K}^G(M,n+1). \end{array} \tag{1.15}$$

Thus $p : E \to B$ is the "homotopy fibre" of $k : B \to \mathrm{K}^G(M,n+1)$ in $(\mathsf{sSet} \downarrow \mathrm{B}G)$. *Moreover, the following results hold:*

(1) *$p : E \to B$ is a principal fibration iff the action of G on M is trivial. (And in this case, $\mathrm{H}^{n+1}_G(B;M) = \mathrm{H}^{n+1}(B;M)$, in which the k-invariant lives.)*
(2) *$p : E \to B$ admits a (homotopy) section iff $k(p) = 0 \in \mathrm{H}^{n+1}_G(B;M)$.*

Proof. The first part follows from our general Theorem 1.8.11.

(1) If the action of G on M is trivial, then the map u in the above diagram is just the map $\mathrm{B}G \times \mathrm{PK}(M,n+1) \to \mathrm{B}G \times \mathrm{K}(M,n+1)$, which is a principal fibration with fibre K(M,n).
Conversely, if $p : E \to B$ is a principal K(M,n)-fibration, consider the map of fibrations

$$\begin{array}{ccccc} F_p & \longrightarrow & E & \xrightarrow{\quad p \quad} & B \\ {\scriptstyle k'}\downarrow & & \downarrow{\scriptstyle r} & & \downarrow{\scriptstyle k} \\ F_l & \longrightarrow & \mathrm{PK}(M,n+1) & \xrightarrow{\ l\ } & \mathrm{K}(M,n+1), \end{array}$$

the map $k'_* : \pi_n F_p \to \pi_n F_l$ is an isomorphism and is r_*-equivariant (Remark 1.6.9), where $r_* : \pi_1 E \to \pi_1 \mathrm{PK}(M,n+1) = *$ is the trivial map, hence k'_* is $\pi_1 E$-invariant. Since k'_* is an isomorphism, the $\pi_1 E$-action on $\pi_n F_p = M$ must be trivial.
(2) This follows easily from the last statement of Theorem 1.8.6 and Proposition 1.6.5.

□

From these results, together with the existence result of Moore-Postnikov systems for maps in sSet, we are finally able to give a rather complete description of the Moore-Postnikov decomposition of a map.

Theorem 1.8.13 (Moore-Postnikov systems)**.** *Let $F \to E \xrightarrow{q} B$ be a fibre sequence in sSet_* (with the Kan-Quillen model structure), in which F, E, B are all connected. Denote $\pi_1 E =: G$. Then there is the* Moore-Postnikov tower

$$E \to \cdots \xrightarrow{q_{n+1}} E^{[n]} \xrightarrow{q_n} E^{[n-1]} \xrightarrow{q_{n-1}} \cdots \xrightarrow{q_2} E^{[1]} \xrightarrow{q_1} E^{[0]} \xrightarrow{q_0} B,$$

in sSet_*, and maps $i_n : E \to E^{[n]}$, $p_n : E^{[n]} \to B$ with the following properties:

(1) *$i_{n-1} = q_n i_n, p_{n-1} q_n = p_n, \forall n \geqslant 1$.*
(2) *$p_n i_n = q, \forall n \in \mathbb{N}$.*

$$\begin{array}{ccccc} & & E^{[n+1]} & & \\ & \overset{i_{n+1}}{\nearrow} & \downarrow{\scriptstyle q_{n+1}} & \overset{p_{n+1}}{\searrow} & \\ E & \xrightarrow{i_n} & E^{[n]} & \xrightarrow{p_n} & B \end{array}$$

(3) *The maps $q_n, n \geqslant 1$, are fibrations; q_0 is a weak equivalence.*
(4) *The homotopy fibre $F(q_n)$ of q_n $(n \geqslant 1)$ is weakly equivalent to* K$(\pi_n F, n)$*, hence for each $n \geqslant 1$ we have a homotopy fibre sequence*
$$\mathrm{K}(\pi_n F, n) \to E^{[n]} \xrightarrow{q_n} E^{[n-1]}.$$
(5) *The map $E \to \operatorname{holim}_{n \in \mathbb{N}^{\mathrm{op}}} E^{[n]} = \lim_{n \in \mathbb{N}^{\mathrm{op}}} E^{[n]}$ is a weak equivalence.*

Specializing to the case $E = *$ resp. $B = *$, one gets the Whitehead tower *resp.* Postnikov tower. *Moreover, the following results hold:*

(6) *There are homotopy fibre sequences*
$$F[n] \to E^{[n]} \xrightarrow{p_n} B,$$
$$F(q_n) \to F[n] \xrightarrow{q'_n} F[n-1] \ (n \geqslant 1),$$
where the map q'_n is induced from $q_n : E^{[n]} \to E^{[n-1]}$ and $F[n]$ is the n-th stage of the Postnikov tower for F.
(7) *For any $j \leqslant n$, $(i_n)_* : \pi_j E \xrightarrow{\cong} \pi_j(E^{[n]})$.*
(8) *For any $j \geqslant n+2$, $(p_n)_* : \pi_j(E^{[n]}) \xrightarrow{\cong} \pi_j B$.*
(9) *There are exact sequences*
$$0 \to \pi_{n+1}(E^{[n]}) \xrightarrow{(p_n)_*} \pi_{n+1} B \xrightarrow{\partial} \pi_n F, n \geqslant 0,$$
where ∂ is the connecting homomorphism in the homotopy long exact sequence of the homotopy fibre sequence $F \to E \xrightarrow{q} B$.

(10) *There are homotopy pullback diagrams in* $(\mathsf{s}\mathcal{S}\mathsf{et} \downarrow \mathrm{B}G)$

$$\begin{array}{ccc} E^{[n]} & \longrightarrow & \mathrm{B}G \\ {\scriptstyle q_n}\downarrow & & \downarrow{\scriptstyle s_n} \\ E^{[n-1]} & \xrightarrow{k_{n+1}} & \mathrm{K}^G(\pi_n F, n+1), \end{array}$$

for a unique k-invariant $[k_{n+1}] \in \mathrm{H}^{n+1}_G(E^{[n-1]}; \pi_n F)$*, for all* $n \geqslant 2$*. Moreover,* $q_n : E^{[n]} \to E^{[n-1]}$ *is a principal fibration iff the action of* G *on* $\pi_n F$ *is trivial.*

Remark 1.8.14. Note that in the existence result of the Moore-Postnikov tower (Theorem 1.7.10), from $E^{[1]} = \mathrm{P}_1(f)$ to higher stages in the tower, each space has the same fundamental group G and each map $q_n, n \geqslant 2$ is a $\mathrm{K}(\pi_n F, n)$-fibration, so Theorem 1.8.12 applies to give the statement (10).

For some other (weaker) form of Moore-Postnikov decomposition, see **[75, 37]**.

Remark 1.8.15. It may be more natural to study the category of simplicial sets over $\mathrm{B}G$ containing it as retract, namely $\mathsf{s}\mathcal{S}\mathsf{et}//\mathrm{B}G$ in the notation of **[82**, p. 85**]**.

Chapter 2

Simplicial (pre)sheaves on a Grothendieck site and their local homotopy theory

The goal of this chapter is to describe the homotopy behaviour of simplicial (pre)sheaves on a Grothendieck site. We also discuss briefly the notions of hypercoverings of simplicial presheaves on a Grothendieck site and various fundamental properties of hypercoverings of simplicial presheaves, with emphasis on their application in the local model category structure on the category of simplicial presheaves on a Grothendieck site (mostly following **[50, 26]**).

2.1. Presheaves on a (small) category

We fix a (small) category $\mathcal{C}$, and we let $\mathcal{E}$ be another category (not necessarily small). A *presheaf* on $\mathcal{C}$ with value in $\mathcal{E}$ is a functor $F : \mathcal{C}^{\mathrm{op}} \to \mathcal{E}$. The presheaves on $\mathcal{C}$ with value in $\mathcal{E}$ form the *presheaf category* $\mathcal{P}\mathsf{re}(\mathcal{C}, \mathcal{E})$, morphisms being the natural transformations between functors. We also denote $\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C}, \mathcal{E}) := \mathcal{P}\mathsf{re}(\mathcal{C}, \mathsf{s}\mathcal{E})$, called the *category of simplicial presheaves on* $\mathcal{C}$ *with value in* $\mathcal{E}$. If $\mathcal{E} = \mathcal{S}\mathsf{et}$, we just denote them by $\mathcal{P}\mathsf{re}(\mathcal{C})$ and $\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})$, called the *category of presheaves* resp. *simplicial sheaves on* $\mathcal{C}$. Of course, viewing every set as a (discrete) simplicial set, we have the natural embedding $\mathcal{P}\mathsf{re}(\mathcal{C}) \hookrightarrow \mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})$. By the result of Example A.2.7, $\mathcal{P}\mathsf{re}(\mathcal{C})$ and $\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})$are both bicomplete, (co)limits are given in a sectionwise manner.

For $F \in \mathcal{P}\mathsf{re}(\mathcal{C}, \mathcal{E})$ and an object $U \in \mathcal{C}$, denote $F|_U \in \mathcal{P}\mathsf{re}(\mathcal{C}/U, \mathcal{E})$ to be the composition $(\mathcal{C}/U)^{\mathrm{op}} \xrightarrow{q^{\mathrm{op}}} \mathcal{C}^{\mathrm{op}} \xrightarrow{F} \mathcal{E}$, where $\mathcal{C}/U = (\mathcal{C} \downarrow U)$ is the comma category and $q : \mathcal{C}/U \to \mathcal{C}, (V \to U) \mapsto V$ is the forgetful functor. For a map $f : F \to G$ in $\mathcal{P}\mathsf{re}(\mathcal{C}, \mathcal{E})$, we then get the *restricting map* $f|_U : F|_U \to G|_U$ in $\mathcal{P}\mathsf{re}(\mathcal{C}/U, \mathcal{E})$.

There is a functor $\underline{\mathrm{Hom}} : \mathcal{P}\mathsf{re}(\mathcal{C}, \mathcal{E}) \times \mathcal{P}\mathsf{re}(\mathcal{C}, \mathcal{E}) \to \mathcal{P}\mathsf{re}(\mathcal{C})$ given by letting

$$\underline{\mathrm{Hom}}(E, F)(U) := \mathcal{P}\mathsf{re}(\mathcal{C}/U, \mathcal{E})(E|_U, F|_U) \in \mathcal{S}\mathsf{et}$$

for $E, F \in \mathcal{P}\mathsf{re}(\mathcal{C}, \mathcal{E})$ and for $U \in \mathcal{C}$; the restriction maps are the obvious ones.

In particular, this gives the category $\mathcal{P}\mathsf{re}(\mathcal{C})$ an *internal hom* $\underline{\mathrm{Hom}} = \underline{\mathrm{Hom}}^{\mathcal{C}} : \mathcal{P}\mathsf{re}(\mathcal{C})^{\mathrm{op}} \times \mathcal{P}\mathsf{re}(\mathcal{C}) \to \mathcal{P}\mathsf{re}(\mathcal{C})$. There is also a functor $\times : \mathcal{P}\mathsf{re}(\mathcal{C}) \times \mathcal{P}\mathsf{re}(\mathcal{C}) \to \mathcal{P}\mathsf{re}(\mathcal{C})$ defined by sectionwise product. They satisfy the adjoint relation

$$\mathcal{P}\mathsf{re}(\mathcal{C})(A \times E, F) \cong \mathcal{P}\mathsf{re}(\mathcal{C})(A, \underline{\mathrm{Hom}}(E, F)).$$

Example 2.1.1 (Yoneda embedding)**.** *For $X \in \mathcal{C}$, we have the functor*

$$\begin{aligned} h_X : \mathcal{C}^{\mathrm{op}} &\to \mathcal{S}\mathsf{et}, \\ U &\mapsto \mathcal{C}(U, X). \end{aligned}$$

Then $h_X \in \mathcal{P}\mathsf{re}(\mathcal{C})$, it's the representable functor with representative X*, or* the presheaf represented by X*. The* Yoneda lemma *asserts that for any $F \in \mathcal{P}\mathsf{re}(\mathcal{C})$, there is a canonical isomorphism*

$$\begin{aligned} \mathcal{P}\mathsf{re}(\mathcal{C})(h_X, F) &\xrightarrow{\cong} F(X), \\ (\varphi : h_X \Rightarrow F) &\mapsto \varphi_X(1_X). \end{aligned}$$

In particular, $\mathcal{P}\mathsf{re}(\mathcal{C})(h_X, h_Y) \cong \mathcal{C}(X, Y)$, and thus the functor

$$\begin{aligned} h : \mathcal{C} &\to \mathcal{P}\mathsf{re}(\mathcal{C}), \\ X &\mapsto h_X \end{aligned}$$

is fully faithful, called the Yoneda embedding of $\mathcal{C}$.

Since colimits in $\mathcal{P}\mathsf{re}(\mathcal{C})$ are sectionwise, we see that for any object V in $\mathcal{C}$ and any functor $F : I \to \mathcal{P}\mathsf{re}(\mathcal{C})$,

$$\mathcal{P}\mathsf{re}(\mathcal{C})\big(h_V, \operatorname*{colim}_{i \in I} F_i\big) \cong \big(\operatorname*{colim}_{i \in I} F_i\big)(V) \cong \operatorname*{colim}_{i \in I} F_i(V) \cong \operatorname*{colim}_{i \in I} \mathcal{P}\mathsf{re}(\mathcal{C})(h_V, F_i),$$

i.e. any morphism $h_V \to \operatorname*{colim}_{i \in I} F_i$ factors through some F_i (I is a small category), this can be rephrased by saying that the representables are small objects in the presheaf category. In particular, for any objects $V, V_i, i \in I$ in $\mathcal{C}$, any morphism $h_V \to \coprod_{i \in I} h_{V_i}$ factors through some (unique) h_{V_i}. This is not true if the representable presheaf h_V is replaced by a general presheaf.

Remark 2.1.2. Using Yoneda lemma, we see that $\underline{\mathrm{Hom}}(E, F)(U) \cong \mathcal{P}\mathsf{re}(\mathcal{C})(E \times h_U, F)$. If $\mathcal{C}$ has products of any pair of two objects, then $\underline{\mathrm{Hom}}(h_V, F) \cong F(V \times -)$.

The presheaf category $\mathcal{P}\mathsf{re}(\mathcal{C})$ has a final object $*$, all of whose sections are single points $\{*\}$. If $\mathcal{C}$ has a final object $*$, then $* = h_*$, thus the next result follows easily from Yoneda lemma.

Proposition 2.1.3. *For any $F \in \mathcal{P}\mathsf{re}(\mathcal{C})$, there is a canonical isomorphism $\mathcal{P}\mathsf{re}(\mathcal{C})(*, F) \cong \lim_{U\in\mathcal{C}} F(U)$, the common value is called the* set of global sections of F. *This can be rephrased as $\operatorname{colim}_{U\in\mathcal{C}} h_U \cong *$.*[1] *If $\mathcal{C}$ has a final object $*$, then the set of global sections of F is canonically isomorphic to $F(*)$.*

Remark 2.1.4. From this, we see that

$$\mathcal{P}\mathsf{re}(\mathcal{C})(*, \underline{\mathrm{Hom}}(E, F)) \cong \lim_{U\in\mathcal{C}} \underline{\mathrm{Hom}}(E, F)(U) \cong \lim_{U\in\mathcal{C}} \mathcal{P}\mathsf{re}(\mathcal{C})(E \times h_U, F) \cong \mathcal{P}\mathsf{re}(\mathcal{C})(E \times \operatorname{colim}_{U\in\mathcal{C}} h_U, F) \cong \mathcal{P}\mathsf{re}(\mathcal{C})(E \times *, F),$$

thus the set of global sections of the internal hom is the same as the set of presheaf morphisms:

$$\mathcal{P}\mathsf{re}(\mathcal{C})(*, \underline{\mathrm{Hom}}(E, F)) \cong \mathcal{P}\mathsf{re}(\mathcal{C})(E, F).$$

Definition 2.1.5. For a presheaf $F : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}\mathsf{et}$, define its *category of elements* to be the category $\mathcal{C}_F = \mathrm{El}_F(\mathcal{C}^{\mathrm{op}}) = (h \downarrow F) = (\mathcal{C} \downarrow F)$. The objects are natural transformations $a_V : h_V \Rightarrow F$ (which can be identified with an element $a_V \in F(V)$). A morphism $a_V \to a_{V'}$ is a morphism $V \to V'$ such that $F(V \to V')(a_{V'}) = a_V$.

Tautologically, the map $\coprod_{V\in\mathcal{C}_F} h_V \to F$ is a sectionwise surjection. The following is a more precise result about this.

Proposition 2.1.6. *There is a canonical isomorphism*

$$F \cong \operatorname*{colim}_{\mathcal{C}_F}(h \circ q)$$

in $\mathcal{P}\mathsf{re}(\mathcal{C})$, where q is the functor

$$q : \mathcal{C}_F \to \mathcal{C}, (a_V : h_V \Rightarrow F) \mapsto V.$$

This means that every presheaf is canonically a colimit of representable presheaves:

$$F \cong \operatorname*{colim}_{(a_V : h_V \Rightarrow F)\in\mathcal{C}_F} h_V.$$

Proof. For any $G \in \mathcal{P}\mathsf{re}(\mathcal{C})$, we have the following natural isomorphisms in $\mathcal{S}\mathsf{et}$:

$$\begin{aligned}
&\mathcal{P}\mathsf{re}(\mathcal{C})(\operatorname*{colim}_{\mathcal{C}_F}(h \circ q), G)\\
\cong &\lim_{\mathcal{C}_F} \mathcal{P}\mathsf{re}(\mathcal{C})(h \circ q, G) \cong \lim_{(a_V : h_V \Rightarrow F)\in\mathcal{C}_F} \mathcal{P}\mathsf{re}(\mathcal{C})(h_V, G) \cong \lim_{(a_V : h_V \Rightarrow F)\in\mathcal{C}_F} G(V) \cong \lim_{\mathcal{C}_F}(G \circ q^{\mathrm{op}}) = \mathcal{P}\mathsf{re}(\mathcal{C}_F)(*, G \circ q^{\mathrm{op}})\\
\cong &\{(a_V : h_V \Rightarrow F) \mapsto (b_V \in G(V)) : G(V \to V')(b_{V'}) = b_V, F(V \to V')(a_{V'}) = a_V\}\\
\cong &\{(F(V) \to G(V))_{V\in\mathcal{C}} : \text{ the diagram } \square_{(V\to V')} \text{ commutes }, \forall (V \to V') \in \mathcal{C}\} \cong \mathcal{P}\mathsf{re}(\mathcal{C})(F, G),
\end{aligned}$$

where $\square_{(V\to V')}$ is the diagram

$$\begin{array}{ccc} F(V') & \longrightarrow & G(V') \\ \downarrow & & \downarrow \\ F(V) & \longrightarrow & G(V). \end{array}$$

Yoneda lemma for $\mathcal{P}\mathsf{re}(\mathcal{C})$ gives the desired result. □

For $F, G \in \mathcal{P}\mathsf{re}(\mathcal{C})$, write $G(F) := \mathcal{P}\mathsf{re}(\mathcal{C})(F, G)$, then there is a canonical isomorphism $G(F) \cong \lim_{(h_V \Rightarrow F)\in\mathcal{C}_F} G(V)$. Thus if G is a presheaf of (abelian) groups, then G is an (abelian) group object of $\mathcal{P}\mathsf{re}(\mathcal{C})$.

Remark 2.1.7. In $\mathcal{P}\mathsf{re}(\mathcal{C})$,

monomorphisms=sectionwise injections, epimorphisms=sectionwise surjections, isomorphisms=sectionwise bijections.

For monomorphisms, this follows easily from the Yoneda lemma for $\mathcal{P}\mathsf{re}(\mathcal{C})$, while the eresult for epimorphisms requires a more refined argument, see The Stacks Project [**96**, *Tag 00V5*] for a short proof. This is particularly true for $\mathsf{s}\mathcal{S}\mathsf{et} = \mathcal{P}\mathsf{re}(\mathbf{\Delta})$ and for $\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C}) = \mathcal{P}\mathsf{re}(\mathcal{C} \times \mathbf{\Delta})$.

2.2. Sieves, Grothendieck topologies and Grothendieck sites

For an object $X \in \mathcal{C}$, a *sieve of X* is a subfunctor $\mathcal{U} \subset h_X$. Thus $\mathcal{U}$ can be identified with a family of morphisms $V \to X$, namely the union of all sets $\mathcal{U}(V)$, that is stable by precompositions (like "right ideal"). We denote by $\mathrm{Sie}(X)$ the class of all sieves of X. The *union* and *intersection* of a family of sieves of X is defined in the obvious (sectionwise) way. A sieve is in particular a presheaf, thus:

Proposition 2.2.1. *For a sieve $\mathcal{U}$ of X, we have*

$$\mathcal{U} \cong \operatorname*{colim}_{(a_V : h_V \Rightarrow \mathcal{U})\in\mathcal{C}_\mathcal{U}} h_V \cong \operatorname*{colim}_{(V\to X)\in\mathcal{U}} h_V,$$

here in the last term we view the family of morphisms $\mathcal{U}$ as a full subcategory of the comma category $(\mathcal{C} \downarrow X)$ in taking colimits.

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[1] See the discussion at the beginning of Section 1.1. Cf. the trivial fact: $\operatorname*{colim}_{(U\to X)\in(\mathcal{C}\downarrow X)} h_U \cong h_X$.

If $F \in \mathcal{P}\mathsf{re}(\mathcal{C})$, we define the *sections* of F on $\mathcal{U}$ to be the set $F(\mathcal{U}) := \mathcal{P}\mathsf{re}(\mathcal{C})(\mathcal{U}, F)$, this is consistent with the usual notation by Yoneda lemma. The above proposition yields

$$F(\mathcal{U}) = \mathcal{P}\mathsf{re}(\mathcal{C})(\mathcal{U}, F) \cong \lim_{(V\to X)\in\mathcal{U}} \mathcal{P}\mathsf{re}(\mathcal{C})(h_V, F) \cong \lim_{(V\to X)\in\mathcal{U}} F(V).$$

The inclusion $\mathcal{U} \subset h_X$ yields a canonical *restriction map* $F(X) \to F(\mathcal{U})$.

Given a morphism $f : Y \to X$ in $\mathcal{C}$ and a sieve $\mathcal{U}$ of X, define *the pull-back of* $\mathcal{U}$ *along* f to be the sieve $f^*\mathcal{U} = \mathcal{U}\times_{h_X} h_Y$ of Y, it consists of arrows to Y whose composite with f is in the sieve $\mathcal{U}$. If $f \in \mathcal{U}$, then $f^*\mathcal{U} = h_Y$. For two families of morphisms R, S, all of whose morphisms have codomain X, we say that S *refines* R if for each $(V \to X) \in S$, there exists some $(U \to X) \in R$ such that the morphism $V \to X$ factors through $U \to X$, i.e. $V \to X$ is a composition $V \to U \to X$.

Given a family of morphisms $R = \{(V_i \xrightarrow{\varphi_i} X) : i \in I\}$ with common codomain X, which can be viewed as a full subcategory of the comma category $(\mathcal{C} \downarrow X)$ (when taking (co)limits over R, we always think of R as a category in this way); *the sieve* (R) *generated by the family* R is the sieve $(R) := \bigcup_{i\in I} \varphi_i^* h_{V_i} = \mathrm{im}(\coprod_{i\in I} h_{V_i} \xrightarrow{\coprod_{i\in I}\varphi_i} h_X) \subset h_X$ of X. So $(R) = \{\varphi_i \circ g : i \in I, g \in h_{V_i}\}$ is the largest refinement of the family R (like the "right ideal generated by R").

Thus S refines R iff $(S) \subset (R)$ iff (S) refines (R), so the sieve (S) is indeed more refined ("has smaller holes") than the sieve (R): less arrows go through (S). We order $\mathrm{Sie}(X)$ as follows: for $\mathcal{U}, \mathcal{V} \in \mathrm{Sie}(X), \mathcal{U} \leqslant \mathcal{V}$ iff $\mathcal{U} \supset \mathcal{V}$. In this way $\mathrm{Sie}(X)$ becomes a poset and thus a category in the usual way.

Tracking carefully the data involved, we find that for each object U of $\mathcal{C}$, there is a coequalizer diagram in $\mathcal{S}\mathsf{et}$:

$$\coprod_{(i,j)\in I^2} (h_{V_i} \times_{h_X} h_{V_j})(U) \rightrightarrows \coprod_{i\in I} h_{V_i}(U) \to (R)(U),$$

hence we get a coequalizer diagram in $\mathcal{P}\mathsf{re}(\mathcal{C})$:

$$\boxed{\coprod_{(i,j)\in I^2} (h_{V_i} \times_{h_X} h_{V_j}) \rightrightarrows \coprod_{i\in I} h_{V_i} \to (R)}$$

and equivalently[2], for every $F \in \mathcal{P}\mathsf{re}(\mathcal{C})$, an equalizer diagram of sections in $\mathcal{S}\mathsf{et}$:

$$\boxed{F((R)) \to \prod_{i\in I} F(V_i) \rightrightarrows \prod_{(i,j)\in I^2} \mathcal{P}\mathsf{re}(\mathcal{C})(h_{V_i} \times_{h_X} h_{V_j}, F).}$$

Definition 2.2.2. A *Grothendieck topology* τ on $\mathcal{C}$ is the assignment of a family $\mathrm{Cov}_\tau(X)$ of sieves of X for every object X of $\mathcal{C}$, called *covering sieves of* X, subject to the following conditions:

- For any X, $h_X \in \mathrm{Cov}_\tau(X)$.
- If $\mathcal{U} \in \mathrm{Cov}_\tau(X)$, then for any morphism $f : Y \to X$ in $\mathcal{C}$, the pull-back $f^*\mathcal{U} \in \mathrm{Cov}_\tau(Y)$.
- For $\mathcal{U} \in \mathrm{Cov}_\tau(X)$ and $\mathcal{V} \subset h_X$, if $f^*\mathcal{V} \in \mathrm{Cov}_\tau(Y)$ for every morphism $f : Y \to X$ in $\mathcal{U}$, then $\mathcal{V} \in \mathrm{Cov}_\tau(X)$.

A *Grothendieck site*, or a *site*, is a small category $\mathcal{C}$, equipped with a Grothendieck topology τ, denoted $(\mathcal{C}, \tau)$ or $\mathcal{C}_\tau$. A family of morphisms $R = \{(V_i \xrightarrow{\varphi_i} X) : i \in I\}$ with common codomain X such that the generated sieve $(R) \in \mathrm{Cov}_\tau(X)$ is called a *(τ-)cover* of X.

We say that an object $X \in \mathcal{C}$ *(τ-)quasi-compact* if any covering sieve $\mathcal{U} \in \mathrm{Cov}_\tau(X)$ of X is finitely generated (i.e. generated by a finite family of arrows). We say the topology τ or the site $(\mathcal{C}, \tau)$ is *noetherian* if every object $X \in \mathcal{C}$ is quasi-compact.

Example 2.2.3. *For a small category $\mathcal{C}$, if we define $\mathrm{Cov}_\tau(X) = \{h_X\}$ for every object X, this gives a Grothendieck topology on $\mathcal{C}$, called the* chaotic topology *or* indiscrete topology *on $\mathcal{C}$. It's the smallest (coarsest) Grothendieck topology (among all the Grothendieck topologies) on $\mathcal{C}$.*

Proposition 2.2.4. *For a Grothendieck site $(\mathcal{C}, \tau)$ and any object $X \in \mathcal{C}$, we have*

- *If $\mathcal{U} \subset \mathcal{V} \subset h_X$ and $\mathcal{U} \in \mathrm{Cov}_\tau(X)$, then $\mathcal{V} \in \mathrm{Cov}_\tau(X)$. In words, a sieve refined by a covering sieve is a covering sieve.*
- *If $\mathcal{U}, \mathcal{V} \in \mathrm{Cov}_\tau(X)$, then $\mathcal{U} \cap \mathcal{V} \in \mathrm{Cov}_\tau(X)$.*
- *If $\mathcal{U} \in \mathrm{Cov}_\tau(X)$, and for each $(U_\varphi \xrightarrow{\varphi} X) \in \mathcal{U}$ we are given a covering sieve $\mathcal{V}_\varphi \in \mathrm{Cov}_\tau(U_\varphi)$, then*
$$\mathcal{V} = \mathcal{U} * \{\mathcal{V}_\varphi\} := \{\varphi \circ \psi : \varphi \in \mathcal{U}, \psi \in \mathcal{V}_\varphi\} \in \mathrm{Cov}_\tau(X).$$[3]
- *For any family of morphisms $R = \{(V_i \xrightarrow{\varphi_i} X) : i \in I\}$ such that for any morphism $U \to X$, there exists $\mathcal{V}_U \in \mathrm{Cov}_\tau(U)$ such that for any $(V \to U) \in \mathcal{V}_U$, the composition $V \to U \to X$ factors through some morphism in R (so $h_X * \{\mathcal{V}_U\} \subset (R)$, i.e. R refines $h_X * \{\mathcal{V}_U\}$), then $(R) \in \mathrm{Cov}_\tau(X)$.*

The second result says that $\mathrm{Cov}_\tau(X)$ is filtered: any two covering sieves of X have a (largest) common refinement, namely their intersection. Note that $\mathrm{Cov}_\tau(X)$ is a subposet as well as a subcategory of $\mathrm{Sie}(X)$.

Definition 2.2.5. If $\mathcal{C}$ has fiber products, *a Grothendieck pre-topology* J, or *a basis for a Grothendieck topology* on $\mathcal{C}$ is given by assigning a collection J_X of *covering families* of maps $\{U_i \to X : i \in I\}$ for index sets I, for every object X of $\mathcal{C}$ (every such family in J_X is called *a covering family of* X), subject to the following conditions:

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[2] By considering $\mathcal{P}\mathsf{re}(\mathcal{C})(-, F)$. See the discussion at the beginning of Section 1.1. This equivalence reflects the two characterizations of a diagram of presheaves being a colimit diagram: one is sectionwise, the other is via Yoneda lemma.

[3] It's clear by definiton that $\mathcal{V}_\varphi \subset \varphi^*\mathcal{V} \subset h_{U_\varphi}, \forall\varphi \in \mathcal{U}$. So $\varphi^*\mathcal{V} \in \mathrm{Cov}_\tau(U_\varphi), \forall\varphi \in \mathcal{U}$.

- For any X and $f : X' \to X$ an isomorphism, then the single family $\{f\}$ is a covering family of X.
- If $\{U_i \to X : i \in I\} \in J_X$, then for any morphism $f : Y \to X$ in $\mathcal{C}$, the pullback family $\{U_i \times_X Y \to Y : i \in I\}$ is a covering family of Y.
- If $\{U_i \to X : i \in I\} \in J_X$ and $\{V_{i,j} \to U_i : j \in I_i\} \in J_{U_i}$, then the composited family $\{V_{i,j} \to X : j \in I_i\}$ is a covering family of X.

Proposition 2.2.6. *For a small category $\mathcal{C}$ having fiber products, a Grothendieck pre-topology J determines a unique Grothendieck topology τ, called the* Grothendieck topology generated by J, *given by: for a sieve $\mathcal{U}$ of X, $\mathcal{U} \in \mathrm{Cov}_\tau(X)$ iff $\mathcal{U}$ is refined by some covering family $\{U_i \to X : i \in I\}$, i.e. the sieve $(\{U_i \to X : i \in I\})$ is contained in $\mathcal{U}$.*

Conversely, given a Grothendieck topology τ on $\mathcal{C}$, there is a unique maximal Grothendieck pre-topology J generating τ, given by: for a family of morphisms $R = \{(V_i \xrightarrow{\varphi_i} X) : i \in I\}$ with common codomain X, $R \in J_X$ iff $(R) \in \mathrm{Cov}_\tau(X)$, i.e. R refines some covering sieve of X.

For a thorough treatment of sieves, Grothendieck topologies and the theory of sheaves (in the next section), see the monograph [**61**] or [**3**].

2.3. Sheaves on a Grothendieck site

In this section, we fix a Grothendieck site $(\mathcal{C}, \tau)$. Let $F \in \mathcal{P}\mathsf{re}(\mathcal{C})$. We say that F is τ-*separated* if for any object $X \in \mathcal{C}$ and any $\mathcal{U} \in \mathrm{Cov}_\tau(X)$, the canonical restriction map $F(X) \to F(\mathcal{U})$ is injective.

We say that F is a τ-*sheaf* if for any object $X \in \mathcal{C}$ and any $\mathcal{U} \in \mathrm{Cov}_\tau(X)$, the canonical restriction map $F(X) \to F(\mathcal{U})$ is bijective.

We denote $\mathcal{S}\mathsf{hv}(\mathcal{C}, \tau) = \mathcal{S}\mathsf{hv}(\mathcal{C}_\tau)$ for the full subcategory of $\mathcal{P}\mathsf{re}(\mathcal{C})$ consisting of τ-sheaves.

If τ is the chaotic topology on $\mathcal{C}$, we have $\mathcal{S}\mathsf{hv}(\mathcal{C}_\tau) = \mathcal{P}\mathsf{re}(\mathcal{C})$. Conversely, if $\mathcal{S}\mathsf{hv}(\mathcal{C}_\tau) = \mathcal{P}\mathsf{re}(\mathcal{C})$, then τ is the chaotic topology (an easy consequence of Yoneda lemma applied to the category $\mathcal{P}\mathsf{re}(\mathcal{C})$).

On the other hand, if $\mathcal{C}$ is equipped with the discrete topology (or trivial topology) τ, then $\mathcal{S}\mathsf{hv}(\mathcal{C}_\tau) = \{*\}$ (by applying the sheaf condition to the empty covering sieve $\varnothing \in \mathrm{Cov}_\tau(X) = \mathrm{Sie}(X)$ for every object X). The converse is also true; in fact, the correspondence between Grothendieck topologies and Grothendieck topoi is bijective (see [**3**] or *the Stacks project, tag/00ZP*), so a strict refinement of a topology has strictly less sheaves.

A topology τ on a category $\mathcal{C}$ is called *subcanonical topology* if every representable functor is a τ-sheaf. The *canonical topology* on $\mathcal{C}$ is the Grothendieck topology on $\mathcal{C}$ which is the largest subcanonical topology; a covering sieve is a sieve generated by *a universal effective epimorphism family* (provided fibre products exists in $\mathcal{C}$).[4]

Example 2.3.1. *For a Grothendieck site $(\mathcal{C}, \tau)$ and any object $U \in \mathcal{C}$, the comma category $\mathcal{C}/U = (\mathcal{C} \downarrow U)$ inherits a topology $\tau|_U$, called the* restricted topology on U, *such that for a family $\mathcal{U} = \{V_i \to X\}$ of maps to $X \xrightarrow{g} U$ in $\mathcal{C}/U$, we have $\mathcal{U} \in \mathrm{Cov}_U(X \xrightarrow{g} U) = \mathrm{Cov}_{\tau|_U}(X \xrightarrow{g} U)$ iff $\mathcal{U} = \{V_i \to X\} \in \mathrm{Cov}_\tau(X)$ (when forgetting the maps to U).*

For $F \in \mathcal{P}\mathsf{re}(\mathcal{C})$ and an object $U \in \mathcal{C}$, we already have $F|_U \in \mathcal{P}\mathsf{re}(\mathcal{C}/U)$. If $F \in \mathcal{S}\mathsf{hv}(\mathcal{C}, \tau)$, then $F|_U \in \mathcal{S}\mathsf{hv}(\mathcal{C}/U, \tau|_U)$.

Proposition-Definition 2.3.2 (Sheafification)**.** Let $F \in \mathcal{P}\mathsf{re}(\mathcal{C})$. For any object $X \in \mathcal{C}$, define

$$F^+(X) := \operatorname*{colim}_{\mathcal{U} \in \mathrm{Cov}_\tau(X)} F(\mathcal{U}).$$

It is a filtered colimit, hence the functor $+ : \mathcal{P}\mathsf{re}(\mathcal{C}) \to \mathcal{P}\mathsf{re}(\mathcal{C})$ commutes with all finite limits.

This defines a presheaf $F^+ \in \mathcal{P}\mathsf{re}(\mathcal{C})$, which is τ-separated. There is a natural morphism $F \to F^+$. Moreover, for $G \in \mathcal{S}\mathsf{hv}(\mathcal{C}_\tau)$, the induced map $\mathcal{P}\mathsf{re}(\mathcal{C})(F^+, G) \to \mathcal{P}\mathsf{re}(\mathcal{C})(F, G)$ is a bijection. Thus if $F \in \mathcal{S}\mathsf{hv}(\mathcal{C}_\tau)$, then $F \cong F^+$.

If $F \in \mathcal{P}\mathsf{re}(\mathcal{C})$ is τ-separated, then $F^+ \in \mathcal{P}\mathsf{re}(\mathcal{C})$ is a τ-sheaf.

We define the τ-*sheafification of F* to be the τ-sheaf $aF = F^\sharp := (F^+)^+$, sometimes called the *associated sheaf of F*, there are natural morphisms $F \to F^+ \to F^\sharp$. We have the following adjunction

$$a : \mathcal{P}\mathsf{re}(\mathcal{C}) \rightleftarrows \mathcal{S}\mathsf{hv}(\mathcal{C}_\tau) : \iota,$$

where ι is the inclusion of sheaves into presheaves. The constant sheaf $a(*) = *$ is a terminal object in both $\mathcal{P}\mathsf{re}(\mathcal{C})$ and $\mathcal{S}\mathsf{hv}(\mathcal{C})$.

The τ-sheafification functor $a = a_\tau : \mathcal{P}\mathsf{re}(\mathcal{C}) \to \mathcal{S}\mathsf{hv}(\mathcal{C}_\tau)$ commutes with all finite limits (i.e. a is left exact hence is exact) and thus preserves monomorphisms.

Remark 2.3.3. A sieve $\mathcal{U} \subset h_X$ is a covering sieve iff the sheafification $\mathcal{U}^\sharp \to h_X^\sharp$ is an isomorphism (see *the Stacks project, tag/00ZO*).

Remark 2.3.4. A presheaf F is τ-separated iff the map $F \to F^+$ is a sectionwise monomorphism.

Remark 2.3.5. The sheafification functor $a : \mathcal{P}\mathsf{re}(\mathcal{C}) \to \mathcal{S}\mathsf{hv}(\mathcal{C}_\tau)$ is *not injective* (on objects) in general, since $a(F) \cong a(a(F))$. As a less trivial example, we take the space $\mathbb{R}$ of real numbers with the usual topology, consider the sheaf F of continuous real-valued functions on $\mathbb{R}$ and let G be the presheaf with $G(U)$ being the set of bounded continuous functions on an open subset U. Let $H = F/G$, it's a non-trivial abelian presheaf, while each stalk is trivial (as every continuous function is locally bounded), hence the associated sheaf aH is trivial. See Proposition 3.1.13 and Proposition 3.1.15 for other counterexamples in geometric situations.

In fact, the sheafification functor $a : \mathcal{P}\mathsf{re}(\mathcal{C}) \to \mathcal{S}\mathsf{hv}(\mathcal{C}_\tau)$ can change a presheaf $F \in \mathcal{P}\mathsf{re}(\mathcal{C})$ to a considerably different functor $a_\tau F = F^\sharp : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}\mathsf{et}$. For example, if τ is the discrete topology on $\mathcal{C}$, then $a_\tau F = *$ for every $F \in \mathcal{P}\mathsf{re}(\mathcal{C})$. For two topologies $\tau' \subset \tau$ on $\mathcal{C}$ and $F \in \mathcal{S}\mathsf{hv}(\mathcal{C}_{\tau'})$, the τ-sheaf $a_\tau F$ viewed as a τ'-sheaf could be drastically different from the given τ'-sheaf F.

[4] The canonical topology whose covering sieves are universal effective epimorphism families is defined to forcing the representables to be sheaves. Covering sieves in any subcanonical topology are universal effective epimorphism families.

Proposition 2.3.6. *Let $(\mathcal{C},\tau)$ be a Grothendieck site, where $\mathcal{C}$ is a category of schemes which is closed under fibre products of schemes and τ is a topology on $\mathcal{C}$ finer than the Zariski topology, all of whose covers are collections of morphisms that are open for the Zariski topology and surjective on points.*

Lat A be a set and $\underline{A}$ the associated constant sheaf on $\mathcal{C}$ with the Zariski topology. Then $\underline{A}$ is already a (constant) sheaf on $(\mathcal{C},\tau)$.

Proof. Let $Y \xrightarrow{f} X$ be a τ-cover on the site $\mathcal{C}$. Note that $\underline{A}(X)$ is the set of locally constant functions with values in A, hence can be identified with all disjoint A-indexed open covers of X (given by mapping $v \in \underline{A}(X)$ to the family $\{v^{-1}(a)\}_{a\in A}$).

$$\begin{array}{ccccc} Y\times_X Y & \overset{p_1}{\underset{p_2}{\rightrightarrows}} & Y & \xrightarrow{f} & X \\ & & {\scriptstyle u}\downarrow & \swarrow_{v} & \\ & & A & & \end{array}$$

Given any $u \in \underline{A}(Y)$ with $up_1 = up_2$, we want to find a (unique) locally constant function $v \in \underline{A}(X)$ with $vf = u$. Of course, we have to define v by letting $v^{-1}(a) = f(u^{-1}(a))$ (note that f is an open map), provided that it is well-defined, that is, $f(u^{-1}(a)) \cap f(u^{-1}(b)) = \varnothing, \forall a \neq b \in A$.

We show this now: Otherwise, there would exist $x \in X$ and $y_1, y_2 \in f^{-1}(x)$ with $u(y_1) = a, u(y_2) = b$. By scheme theory, there exists $z \in Y \times_X Y$ with $p_1(z) = y_1, p_2(z) = y_2$, thus $a = u(y_1) = up_1(z) = up_2(z) = u(y_2) = b$, a contraction. □

Remark 2.3.7. Most topologies used in algebraic geometry satisfy the condition in Proposition 2.3.6, for example, τ can be the Nisnevich, étale, fppf, fpqc, h, and *v-topology*.

Proposition 2.3.8. *Let $F \in \mathcal{P}\mathsf{re}(\mathcal{C})$, then $F \in \mathcal{S}\mathsf{hv}(\mathcal{C}_\tau)$ iff for any τ-cover $R = \{(V_i \xrightarrow{\varphi_i} X) : i \in I\}$ (i.e. the generated sieve $(R) \in \mathrm{Cov}_\tau(X)$), the diagram*

$$F(X) \cong \mathcal{P}\mathsf{re}(\mathcal{C})(h_X, F) \to \prod_{i\in I} \mathcal{P}\mathsf{re}(\mathcal{C})(h_{V_i}, F) \rightrightarrows \prod_{(i,j)\in I^2} \mathcal{P}\mathsf{re}(\mathcal{C})(h_{V_i}\times_{h_X} h_{V_j}, F)$$

is an equalizer diagram. In particular, the map $F(X) \to \prod_{i\in I} F(V_i), x \mapsto (\varphi_i^* x)$ is injective.

In this case, given $v_i \in F(V_i), i \in I$ such that the diagrams

$$\begin{array}{ccc} h_{V_i}\times_{h_X} h_{V_j} & \longrightarrow & h_{V_i} \\ \downarrow & & \downarrow{\scriptstyle v_i} \\ h_{V_j} & \xrightarrow{v_j} & F \end{array}$$

commute for all $i, j \in I$, there is a unique map $u : h_X \to F$ such that $v_i = u \circ \varphi_i \in \mathcal{P}\mathsf{re}(\mathcal{C})(h_{V_i}, F)$.

If the Grothendieck site $(\mathcal{C},\tau)$ is defined by a basis J, then $F \in \mathcal{S}\mathsf{hv}(\mathcal{C}_\tau)$ iff for any object $X \in \mathcal{C}$ and any covering family $R = \{(V_i \xrightarrow{\varphi_i} X) : i \in I\}$ of X, the diagram

$$F(X) \to \prod_{i\in I} F(V_i) \rightrightarrows \prod_{(i,j)\in I^2} F(V_i \times_X V_j)$$

is an equalizer diagram.

Proposition 2.3.9. *Let $\mathcal{C}$ be a Grothendieck site (we suppress the data of the Grothendieck topology τ in the notation).*

- *The sheaf category $\mathcal{S}\mathsf{hv}(\mathcal{C}) = \mathcal{S}\mathsf{hv}(\mathcal{C}_\tau)$ is bicomplete. Limits are formed sectionwise, colimits are formed (sectionwise) in the presheaf category $\mathcal{P}\mathsf{re}(\mathcal{C})$ then sheafifyed.*
- *Every monomorphism in $\mathcal{S}\mathsf{hv}(\mathcal{C})$ is an equalizer.*
- *If $\mathcal{C}$ has an initial object $\varnothing$ which is covered by the empty sieve, and $F \in \mathcal{S}\mathsf{hv}(\mathcal{C}_\tau, \mathcal{E})$ ($\mathcal{E}$ is a complete category, with final object $*$), then $F(\varnothing) \cong *$.*[5]

Definition 2.3.10. For $A \in \mathcal{S}\mathsf{hv}\mathcal{A}\mathsf{b}(\mathcal{C}_\tau)$, we denote its torsion subsheaf $\ker(A \xrightarrow{n} A)$ by ${}_nA$. We say that A is a *torsion sheaf* if it is the τ-sheafification of a presheaf of abelian groups, all of whose sections are torsion abelian groups. Equivalently, the canonical morphism

$$\mathrm{colim}({}_nA) \to A$$

is an isomorphism of sheaves, where the colimit is indexed by positive integers ordered by divisibility ([**88**, p. 146]). In this case, A is the τ-sheafification of the (torsion, τ-separated) presheaf $U \mapsto \mathrm{colim}({}_n(A(U)))$.

If $A \in \mathcal{S}\mathsf{hv}\mathcal{A}\mathsf{b}(\mathcal{C}_\tau)$ is a torsion sheaf and $X \in \mathcal{C}$ is τ-quasi-compact, then the abelian group $A(X)$ is torsion. If the site $(\mathcal{C},\tau)$ is noetherian, then all sections of A are torsion abelian groups.

For a prime number ℓ, we say that A is an *ℓ-torsion sheaf* if it is the τ-sheafification of a presheaf of abelian groups, all of whose sections are ℓ-torsion abelian groups. If the site $(\mathcal{C},\tau)$ is noetherian, this is equivalent to that for each $X \in \mathcal{C}$, the abelian group $A(X)$ is ℓ-torsion (annihilated by a power of ℓ).

We denote

$$A(\ell) := \mathrm{colim}({}_{\ell^n}A),$$

[5] This is forced by the equalizer diagram $F(\varnothing) \to * \rightrightarrows *$, but the equalizer of the diagram $* \rightrightarrows *$ is $*$. This need not be the case for preshsaves, e.g. the constant functor with value a non-final object in $\mathcal{E}$.

where the colimit is ordered by divisibility of positive integers; we call it the *ℓ-primary component* of A. It's an ℓ-torsion sheaf associated to the ℓ-torsion presheaf $U \mapsto \mathrm{colim}({}_{\ell^n}(A(U)))$. We have a canonical homomorphism

$$\bigoplus_{\ell} A(\ell) \to A,$$

where the direct sum is over all prime numbers.

Definition 2.3.11. Let $\mathcal{C}$ be a Grothendieck site and let $f : F \to G$ be a morphism in $\mathcal{P}\mathsf{re}(\mathcal{C})$.

- The morphism $f : F \to G$ is said to be a *(τ-)local epimorphism* if for any object $X \in \mathcal{C}$ and any $x \in G(X)$, there exists a covering sieve $\mathcal{U} \in \mathrm{Cov}_\tau(X)$ such that for any $(U \xrightarrow{\varphi} X) \in \mathcal{U}$, there exists some $y_\varphi \in F(U)$ such that $f_U(y_\varphi) = \varphi^*(x)$.[6]

$$\begin{array}{ccc} h_U & \xrightarrow{\varphi} & h_X \\ {\scriptstyle y_\varphi}\vdots & & \downarrow{\scriptstyle x} \\ F & \xrightarrow{f} & G \end{array} \qquad \begin{array}{ccc} \mathcal{P}\mathsf{re}(\mathcal{C})(\coprod_{i\in I} h_{U_i}, F) = \prod_{i\in I} F(U_i) \ni (y_i) \\ \downarrow{\scriptstyle f=\prod_{i\in I} f_{U_i}} \\ x \in G(X) \longrightarrow \mathcal{P}\mathsf{re}(\mathcal{C})(\coprod_{i\in I} h_{U_i}, G) = \prod_{i\in I} G(U_i) \ni (x|_{U_i}) \end{array}$$

- The morphism $f : F \to G$ is said to be a *(τ-)local monomorphism* if for any object $X \in \mathcal{C}$ and any $y, y' \in F(X)$ with $f_X(y) = f_X(y')$, there exists a covering sieve $\mathcal{U} \in \mathrm{Cov}_\tau(X)$ such that $\varphi^*(y) = \varphi^*(y')$ for any $(U \xrightarrow{\varphi} X) \in \mathcal{U}$.

$$h_U \xrightarrow{\varphi} h_X \underset{y'}{\overset{y}{\rightrightarrows}} F \xrightarrow{f} G$$

- The morphism $f : F \to G$ is said to be a *(τ-)local isomorphism* if it is a (τ-)local epimorphism and a (τ-)local monomorphism. Using Proposition 2.2.4 and the equation $f_U(\varphi^*(y)) = \varphi^* f_X(y)$, we find that a morphism $f : F \to G$ is a local isomorphism iff for any object $X \in \mathcal{C}$ and any $x \in G(X)$, there exists a covering sieve $\mathcal{U} \in \mathrm{Cov}_\tau(X)$ such that for any $(U \xrightarrow{\varphi} X) \in \mathcal{U}$, there exists a *unique* $y_\varphi \in F(U)$ such that $f_U(y_\varphi) = \varphi^*(x)$.

Proposition 2.3.12. *Let $\mathcal{C}$ be a Grothendieck site and let $f : F \to G$ be a morphism in $\mathcal{P}\mathsf{re}(\mathcal{C})$. Denote by $af = f^\sharp : F^\sharp \to G^\sharp$ the τ-sheafification map in $\mathcal{S}\mathsf{hv}(\mathcal{C})$. Then the map $f^\sharp$ in $\mathcal{S}\mathsf{hv}(\mathcal{C})$ is an isomorphism (*resp. *monomorphism,* resp. *epimorphism) iff the map f in $\mathcal{P}\mathsf{re}(\mathcal{C})$ is a local isomorphism (*resp. *local monomorphism,* resp. *local epimorphism).*

Remark 2.3.13. For any $F \in \mathcal{P}\mathsf{re}(\mathcal{C})$, the canonical map $F \to F^\sharp$ is a local isomorphism. So for any section $x \in F^\sharp(X)$, there exists a τ-covering family $R = \{(U_i \xrightarrow{\varphi_i} X) : i \in I\}$ such that the fibre of $\prod_{i\in I} F(U_i) \to \prod_{i\in I} F^\sharp(U_i)$ over $(x|_{U_i})$ is a singleton.

Adapted properly, most of the above discussions hold true with $\mathcal{S}\mathsf{hv}(\mathcal{C}, \mathcal{E})$ in place of $\mathcal{S}\mathsf{hv}(\mathcal{C})$ for $\mathcal{E}$ a complete category.

The following result gives some intuition of *covering* sieves.

Proposition 2.3.14. *Given a Grothendieck site $\mathcal{C}$ and a family of morphisms $R = \{(V_i \xrightarrow{\varphi_i} X) : i \in I\}$. Then the induced morphism $\coprod_{i\in I} h_{V_i} \to h_X$ in $\mathcal{P}\mathsf{re}(\mathcal{C})$ is a local epimorphism iff $(R) \in \mathrm{Cov}_\tau(X)$. Thus a subfunctor $\mathcal{U} \subset h_X$ is a covering sieve iff the inclusion $\mathcal{U} \to h_X$ is a local epimorphism (hence a local isomorphism).*

Proof. If $(R) \in \mathrm{Cov}_\tau(X)$, take $\mathcal{U} = x^*(R)$ in the definition of local epimorphism, then it's easy to verify that the morphism $\coprod_{i\in I} h_{V_i} \to h_X$ indeed satisfies the condition of being a local epimorphism. The converse follows easily from the last point of Proposition 2.2.4, using the fact that any morphism $h_V \to \coprod_{i\in I} h_{V_i}$ factors through some h_{V_i}. □

Thus it's tempting to call a local epimorphism a *generalized cover*, in fact this is the terminology for example in **[26]**. With Proposition 2.1.6, we get easily the following result.

Corollary 2.3.15. *Let $\varphi : F \to h_X$ be a morphism in $\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})$ with $X \in \mathcal{C}$, denote by $(F) = (\varphi)$ the set of arrows $U \to X$ such that the map $h_U \to h_X$ factors through φ, so (F) is a sieve of X. We have $(F) \in \mathrm{Cov}_\tau(X) \Leftrightarrow \varphi : F \to h_X$ is a local epimorphism.*

Proposition 2.3.16. *In $\mathcal{S}\mathsf{hv}(\mathcal{C})$, monomorphisms=local monomorphisms=sectionwise injections, epimorphisms=local epimorphisms, isomorphisms=local isomorphisms=sectionwise bijections. So a sheaf isomorphism is the same as a sheaf morphism that is both a sheaf monomorphism and a sheaf epimorphism.*

Proof. For monomorphisms, this is because for all objects $U \in \mathcal{C}$ there are canonical isomorphisms

$$F(U) \cong \mathcal{P}\mathsf{re}(\mathcal{C})(h_U, F) \cong \mathcal{S}\mathsf{hv}(\mathcal{C})((h_U)^\sharp, F).$$

Let $f : F \to G$ be a local epimorphism, and let $G \underset{v}{\overset{u}{\rightrightarrows}} H$ be morphisms in $\mathcal{S}\mathsf{hv}(\mathcal{C})$ such that $u \circ f = v \circ f$. Then it's easy to see from definition that for any object $X \in \mathcal{C}$ and any $x \in G(X)$, there exists a covering sieve $\mathcal{U} \in \mathrm{Cov}_\tau(X)$ with $\varphi^* u(x) = \varphi^* v(x)$ for any $\varphi \in \mathcal{U}$. Thus $u(x) = v(x)$ and $u = v$. Conversely, let $f : F \to G$ be an epimorphism, let $H := G/\mathrm{im}\ f \in \mathcal{S}\mathsf{hv}(\mathcal{C})$. Define $G \underset{v}{\overset{u}{\rightrightarrows}} H$ respectively to be the sheafifications of the presheaf morphisms $x \mapsto [x]$ and $x \mapsto [*]$ for any object $X \in \mathcal{C}$ and any $x \in G(X)$. Then clearly $u \circ f = v \circ f$. Suppose that f is not a local epimorphism, then there exists an object $X \in \mathcal{C}$ and $x \in G(X)$ which is not locally an image under f, thus $u(x)$ is not trivial in $H(X)$, but $v(x)$ is trivial, so $u \neq v$. Then f would not be an epimorphism. □

[6] This is the case if for any object $X \in \mathcal{C}$, there exist a covering sieve $\mathcal{U} \in \mathrm{Cov}_\tau(X)$ and a map $y : \mathcal{U} \to F$ such that $fy = x(\mathcal{U} \hookrightarrow h_X)$.

A *Grothendieck topos* $\mathcal{E}$ is a category that is equivalent to the sheaf category $\mathcal{S}\mathsf{hv}(\mathcal{C}) = \mathcal{S}\mathsf{hv}(\mathcal{C}_\tau)$ for some (small) Grothendieck site $\mathcal{C}$. We will focus on Grothendieck topoi of the form $\mathcal{S}\mathsf{hv}(\mathcal{C})$. Examples include all the presheaf categories, since with the chaotic topology on $\mathcal{C}$, we have $\mathcal{P}\mathsf{re}(\mathcal{C}) = \mathcal{S}\mathsf{hv}(\mathcal{C})$; other common examples include the category of sheaves on a topological space, the simplicial presheaf category $\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C}) = \mathcal{P}\mathsf{re}(\mathcal{C}\times\mathbf{\Delta})$ and the simplicial sheaf category $\mathsf{s}\mathcal{S}\mathsf{hv}(\mathcal{C}) = \mathcal{S}\mathsf{hv}(\mathcal{C}\times\mathbf{\Delta})$ on a site $(\mathcal{C},\tau)$, where we give $\mathbf{\Delta}$ the chaotic topology and $\mathcal{C}\times\mathbf{\Delta}$ the product topology. Note that on the one point space $*$, we have $\mathcal{S}\mathsf{hv}(*) = \mathcal{S}\mathsf{et}$.

We just mention that there is *Giraud's theorem* characterizing a category being a Grothendieck topos.

Recall that a functor preserving finite limits is said to be *left exact*, a functor preserving finite colimits is said to be *right exact*, and a functor preserving both finite limits and finite colimits is said to be *exact*.

Definition 2.3.17. Let $\mathcal{C},\mathcal{D}$ be Grothendieck sites, a *geometric morphism* $f : \mathcal{S}\mathsf{hv}(\mathcal{C}) \to \mathcal{S}\mathsf{hv}(\mathcal{D})$ of Grothendieck topoi consists of an adjoint pair

$$f^* : \mathcal{S}\mathsf{hv}(\mathcal{D}) \rightleftarrows \mathcal{S}\mathsf{hv}(\mathcal{C}) : f_*,$$

where f^*, called the *inverse image functor*, preserves finite limits (i.e. f^* is left exact hence is exact). The functor f_* is called the *direct image functor* of this geometric morphism.

Let $\mathcal{C},\mathcal{D}$ be small categories, then a functor $f : \mathcal{D} \to \mathcal{C}$ yields a functor $f_* : \mathcal{P}\mathsf{re}(\mathcal{C}) \to \mathcal{P}\mathsf{re}(\mathcal{D}), F \mapsto F \circ f^{\mathrm{op}}$. It has a left adjoint f^p given by

$$f^pG(U) = \operatorname*{colim}_{(f(V)\to U)\in(f^{\mathrm{op}}\downarrow U)} G(V),$$

where $(f^{\mathrm{op}} \downarrow U) = (U \downarrow f)^{\mathrm{op}}$. Formally, f^pG is the left Kan extension $\mathrm{Lan}_{f^{\mathrm{op}}}(G)$.

$$\begin{array}{ccc} \mathcal{D}^{\mathrm{op}} & \xrightarrow{G} & \mathcal{S}\mathsf{et} \\ {\scriptstyle f^{\mathrm{op}}}\downarrow & \nearrow_{f^pG} & \\ \mathcal{C}^{\mathrm{op}} & & \end{array}$$

Definition 2.3.18. Let $\mathcal{C},\mathcal{D}$ be Grothendieck sites, then a functor $f : \mathcal{D} \to \mathcal{C}$ is called a *site morphism* if

- f_* is *continuous* in the sense that it maps sheaves to sheaves. Thus we get a functor
$$f_* : \mathcal{S}\mathsf{hv}(\mathcal{C}) \to \mathcal{S}\mathsf{hv}(\mathcal{D}), F \mapsto F \circ f^{\mathrm{op}}.$$
- In the adjunction
$$f^p : \mathcal{P}\mathsf{re}(\mathcal{D}) \rightleftarrows \mathcal{P}\mathsf{re}(\mathcal{C}) : f_*,$$
f^p is (left) exact (i.e. it preserves finite limits).

In this case, if we define $f^* = a \circ f^p$, then we get an adjunction

$$f^* : \mathcal{S}\mathsf{hv}(\mathcal{D}) \rightleftarrows \mathcal{S}\mathsf{hv}(\mathcal{C}) : f_*$$

which gives a geometric morphism $f : \mathcal{S}\mathsf{hv}(\mathcal{C}) \to \mathcal{S}\mathsf{hv}(\mathcal{D})$.

When $\mathcal{D} = *$, a site morphism $f : * \to \mathcal{C}$ gives an adjunction

$$f^* : \mathcal{S}\mathsf{et} \rightleftarrows \mathcal{S}\mathsf{hv}(\mathcal{C}) : f_* = \Gamma,$$

where f^* sends a set M to the constant sheaf associated with M, and $f_* = \Gamma$ is the global section functor.

Definition 2.3.19. A *point* of a Grothendieck topos $\mathcal{S}\mathsf{hv}(\mathcal{C})$ is a geometric morphism $x : \mathcal{S}\mathsf{et} \to \mathcal{S}\mathsf{hv}(\mathcal{C})$. Thus there is an adjoint pair

$$x^* : \mathcal{S}\mathsf{hv}(\mathcal{C}) \rightleftarrows \mathcal{S}\mathsf{et} : x_*.$$

Then the set $F_x := x^*F$ is called *the stalk of F at the point x*. More generally, for any presheaf $G \in \mathcal{P}\mathsf{re}(\mathcal{C})$, we define *the stalk of F at the point x* to be the set $F_x := x^*(aF) = (F^\sharp)_x$.

We say $\mathcal{S}\mathsf{hv}(\mathcal{C})$ *has enough points* if there exists a *set* of points $x_i : \mathcal{S}\mathsf{et} \to \mathcal{S}\mathsf{hv}(\mathcal{C}), i \in I$ such that the functor $(x_i^*) : \mathcal{S}\mathsf{hv}(\mathcal{C}) \to \mathcal{S}\mathsf{et}^I$ is faithful.

Proposition 2.3.20 ([**50**, Lemma 3.26])**.** *Let $f : \mathcal{S}\mathsf{hv}(\mathcal{C}) \to \mathcal{S}\mathsf{hv}(\mathcal{D})$ be a geometric morphism between Grothendieck topoi, then the following are equivalent:*

- *The functor $f^* : \mathcal{S}\mathsf{hv}(\mathcal{D}) \to \mathcal{S}\mathsf{hv}(\mathcal{C})$ is faithful.*
- *The functor f^* reflects isomorphisms (we then say the functor f^* is* conservative*): for a morphism u in $\mathcal{S}\mathsf{hv}(\mathcal{D})$, if f^*u is an isomorphism, then so is u.*
- *The functor f^* reflects epimorphisms.*
- *The functor f^* reflects monomorphisms.*

2.4. Projective and injective model structures on the simplicial (pre)sheaf category

We fix a (small) category $\mathcal{C}$. In this section, we will describe some model structures on $\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})$, the category of simplicial presheaves on $\mathcal{C}$. The following special case of Example A.2.7 is also a (simplicial) degreewise Yoneda lemma.

Proposition 2.4.1. *There is an adjunction*

$$L_U : \mathsf{s}\mathcal{S}\mathsf{et} \rightleftarrows \mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C}) : \Gamma_U,$$

for every object $U \in \mathcal{C}$, where $L_U(K) = h_U \times K, \Gamma_U(F) = F(U)$.

Remark 2.4.2. Since any $X \in \mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})$ is a functor $X : (\mathcal{C}\times\mathbf{\Delta})^{\mathrm{op}} \to \mathcal{S}\mathsf{et}$, and for every object $U \in \mathcal{C}$ and $n \in \mathbb{N}$, we have $h_{(U,[n])} = h_U \times \Delta^n = L_U(\Delta^n)$, we see that every simplicial presheaf is a colimit of functors of the form $L_U(\Delta^n)$.

Definition 2.4.3. There is a simplicial structure on $\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})$, given as follows: for $K \in \mathsf{s}\mathcal{S}\mathsf{et}$, $X, Y \in \mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})$,

$$X^K(U) := X(U)^K,\ (X \otimes K)(U) := X(U) \times K \in \mathsf{s}\mathcal{S}\mathsf{et}$$

for every object $U \in \mathcal{C}$ and with simplicial mapping spaces $\mathrm{Map}_{\mathcal{C}}(X, Y) = \mathrm{Map}(X, Y)$ given by

$$\mathrm{Map}(X, Y)_n := \mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})(X \otimes \Delta^n, Y),$$

the simplicial operators are induced from the cosimplicial operators on the Δ^n's. We have $\mathrm{Map}(h_U, X) \cong X(U)$ by Yoneda lemma. Again, $\mathrm{Map}(*, X)$ is the *simplicial set of global sections of* X and $\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})(*, X)$ is the set of global sections of X_0, or the *set of (base) points of* X.

The category $\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})$ also has an *internal hom*, $\underline{\mathrm{Hom}}$, also called the *internal function complex*, given by letting

$$\underline{\mathrm{Hom}}(X, Y)(U) := \mathrm{Map}(X|_U, Y|_U) = \mathrm{Map}_{\mathcal{C}/U}(X|_U, Y|_U) \in \mathsf{s}\mathcal{S}\mathsf{et}$$

for $X, Y \in \mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})$ and $U \in \mathcal{C}$. There is also a functor $\times : \mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C}) \times \mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C}) \to \mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})$ defined by sectionwise product. They satisfy the adjoint-like relation

$$\mathrm{Map}(A \times X, Y) \cong \mathrm{Map}(A, \underline{\mathrm{Hom}}(X, Y))$$

and hence

$$\underline{\mathrm{Hom}}(A \times X, Y) \cong \underline{\mathrm{Hom}}(A, \underline{\mathrm{Hom}}(X, Y)).$$

In particular, we have $\underline{\mathrm{Hom}}(X, Y)(U) \cong \mathrm{Map}(X \times h_U, Y)$, $\underline{\mathrm{Hom}}(*, Y) \cong Y$ and as in the $\mathcal{P}\mathsf{re}(\mathcal{C})$ case, we have the result about global sections of the internal hom:

$$\mathrm{Map}(*, \underline{\mathrm{Hom}}(X, Y)) \cong \mathrm{Map}(X, Y), \mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})(*, \underline{\mathrm{Hom}}(X, Y)) \cong \mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})(X, Y).$$

If we view a simplicial set K as a *constant simplicial presheaf*, then $X \times K = X \otimes K, \underline{\mathrm{Hom}}(K, X) = X^K$ for all $X \in \mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})$.

If G is a simplicial presheaf of (abelian) groups, then G is an (abelian) group object of $\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})$ and for any $A \in \mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})$, we have $\mathrm{Map}(A, G) \in \mathsf{s}\mathcal{G}\mathsf{r}$ (*resp.* $\mathrm{Map}(A, G) \in \mathsf{s}\mathcal{A}\mathsf{b}$) and $\underline{\mathrm{Hom}}(A, G) \in \mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C}, \mathcal{G}\mathsf{r})$ (*resp.* $\underline{\mathrm{Hom}}(A, G) \in \mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C}, \mathcal{A}\mathsf{b})$).

There are similar formulas in the simplicial model category $\mathsf{s}\mathcal{S}\mathsf{hv}(\mathcal{C})$.

Note that if X, Y are sheaves, then so is $\underline{\mathrm{Hom}}(X, Y)$.

For any $X \in \mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})$, we have $X_n = (X^{\Delta^n})_0, n \geqslant 0$ and $\mathrm{M}_n X = (\mathrm{cosk}_{n-1} X)_n = (\underline{\mathrm{Hom}}(\partial\Delta^n, X))_0 = (X^{\partial\Delta^n})_0, n \geqslant 1$.

Proposition-Definition 2.4.4. There is a model structure on $\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})$, called the *projective model structure*, with

- weak equivalences $\mathbf{W}$: sectionwise weak equivalences.
- fibrations $\mathbf{F}$: sectionwise fibration.
- cofibrations $\mathbf{C} = \mathrm{LLP}(\mathbf{W} \cap \mathbf{F})$.

This projective model structure $\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})_{\mathrm{proj}}$ is cofibrantly generated, with generating cofibrations the maps $L_U(\partial\Delta^n) \to L_U(\Delta^n)$ for all $n \in \mathbb{N}$ and $U \in \mathcal{C}$, and generating trivial cofibrations the maps $L_U(\Lambda^n_k) \to L_U(\Delta^n)$ for all $n \geqslant 1, 0 \leqslant k \leqslant n$ and $U \in \mathcal{C}$.

Proposition-Definition 2.4.5. There is a model structure on $\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})$, called the *Heller injective model structure*, with

- weak equivalences $\mathbf{W}$: sectionwise weak equivalences.
- cofibrations $\mathbf{C}$: sectionwise cofibrations.
- fibrations $\mathbf{F} = \mathrm{RLP}(\mathbf{W} \cap \mathbf{C})$.

It's a proper and cofibrantly generated simplicial model category. We write $\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})_{\mathrm{inj}}$ to indicate this injective model structure.

For details of the above results, see [**50**, §2.3].

For $(X, x), (Y, y) \in \mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})_*$, we have the *pointed simplicial mapping space* $\mathrm{Map}_*((X, x), (Y, y)) \in \mathsf{s}\mathcal{S}\mathsf{et}_*$, pointed by the composite $X \to * \xrightarrow{y} Y$, defined in eqs. (1.5) and (1.6): it fits into the equalizer diagram

$$\mathrm{Map}_*((X, x), (Y, y)) \longrightarrow \mathrm{Map}(X, Y) \underset{y_*}{\overset{x^*}{\rightrightarrows}} \mathrm{Map}(*, Y). \tag{2.1}$$

We have $\mathrm{Map}_*(A_+, (Y, y)) \cong \mathrm{Map}(A, Y)$ for $A \in \mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C}), (Y, y) \in \mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})_*$. We can also define the *pointed internal function complex* $\underline{\mathrm{Hom}}_*((X, x), (Y, y)) \in \mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})_*$ via the cartesian square of simplicial presheaves

$$\begin{array}{ccc} \underline{\mathrm{Hom}}_*((X, x), (Y, y)) & \longrightarrow & \underline{\mathrm{Hom}}(X, Y) \\ \downarrow & & \downarrow{\scriptstyle x^*} \\ * & \xrightarrow[y]{} & \underline{\mathrm{Hom}}(*, Y) \cong Y, \end{array} \tag{2.2}$$

it's pointed by the unique zero map $X \to * \xrightarrow{y} Y$. Then for any $A \in \mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})$, we have $\underline{\mathrm{Hom}}_*(A_+, (Y, y)) \cong \underline{\mathrm{Hom}}(A, Y)$ as pointed simplicial presheaves (pointed by $A \to * \xrightarrow{y} Y$). And there is the equalizer diagram

$$\underline{\mathrm{Hom}}_*((X, x), (Y, y)) \longrightarrow \underline{\mathrm{Hom}}(X, Y) \underset{y_*}{\overset{x^*}{\rightrightarrows}} \underline{\mathrm{Hom}}(*, Y) = Y. \tag{2.3}$$

So $\mathrm{Map}_*((X, x), -), \underline{\mathrm{Hom}}_*((X, x), -)$ preserve limits and for $U \in \mathcal{C}, \underline{\mathrm{Hom}}_*((X, x), (Y, y))(U) = \mathrm{Map}_*(X|_U, Y|_U) \in \mathsf{s}\mathcal{S}\mathsf{et}_*$. We also have $\underline{\mathrm{Hom}}_*(\Delta^0_+, (Y, y)) \cong (Y, y), \underline{\mathrm{Hom}}_*(K_+, (Y, y)) \cong (Y, y)^K$ as pointed simplicial presheaves. More generally, $\underline{\mathrm{Hom}}_*(A_+, (Y, y)) \cong \underline{\mathrm{Hom}}(A, Y)$ for $A \in \mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C}), (Y, y) \in \mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})_*$.

Recall also the discussion at the end of Section 1.3, in the category $\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})_*$ of pointed simplicial presheaves on $\mathcal{C}$, we have the *smash product* $\wedge$. The smash product $(X, x) \wedge (Y, y)$ is defined as the sectionwise smash product of simplicial sets $U \mapsto (X, x)(U) \wedge (Y, y)(U)$—the smash product $(X, x) \wedge (Y, y)$ corepresents maps from $X \times Y$ that are

base-point-preserving separately in each variable. So for $X, Y \in \mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})$, we have $(X \times Y)_+ = X_+ \wedge Y_+$, and for any $K \in \mathsf{s}\mathcal{S}\mathsf{et}$ (viewed as a constant simplicial presheaf), we have $(X, x) \wedge K_+ \cong (X \otimes K)/(x \otimes K)$ as pointed simplicial presheaves.

There is the following adjunction

$$(X, x) \wedge (-) : \mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})_* \rightleftarrows \mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})_* : \underline{\mathrm{Hom}}_*((X, x), -).$$

It's a Quillen adjunction for Jardine's local model structure (which we discuss in Section 2.6). These operations $\wedge, \underline{\mathrm{Hom}}_*$ turn $\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})_*$ into a *closed symmetric monoidal category* with unit $S^0 = \partial\Delta^1 = \Delta^0_+$. The *suspension* is given by $\Sigma X := S^1 \wedge X$ and the *looping* is given by $\Omega X := \underline{\mathrm{Hom}}_*(S^1, X)$, where $S^1 := \Delta^1/\partial\Delta^1$ (viewed as a constant pointed simplicial presheaf).

The operations $\otimes, \mathrm{Map}_*$ and power (which are the sectionwise operations as given at the end of Section 1.5 and also appear here) form a simplicial structure on $\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})_*$. Applying $\mathsf{s}\mathcal{S}\mathsf{et}(\Delta^n, -)$ to eq. (2.1) we get

$$\mathrm{Map}_*((X, x), (Y, y))_n \cong \mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})_*((X, x) \wedge \Delta^n_+, (Y, y)).$$

From this and the above adjunction, we obtain the "enriched adjunction" relations

$$\mathrm{Map}_*(A \wedge X, Y) \cong \mathrm{Map}_*(A, \underline{\mathrm{Hom}}_*(X, Y)),\ \underline{\mathrm{Hom}}_*(A \wedge X, Y) \cong \underline{\mathrm{Hom}}_*(A, \underline{\mathrm{Hom}}_*(X, Y)) \tag{2.4}$$

for $A, X, Y \in \mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})_*$. (The second isomorphism follows from the first.) Thus

$$\mathrm{Map}_*(X|_U, Y|_U) = \underline{\mathrm{Hom}}_*(X, Y)(U) = \mathrm{Map}(h_U, \underline{\mathrm{Hom}}_*(X, Y)) = \mathrm{Map}_*((h_U)_+, \underline{\mathrm{Hom}}_*(X, Y)) = \mathrm{Map}_*((h_U)_+ \wedge X, Y).$$

2.5. Local weak equivalences and local fibrations

Let $\mathcal{C}$ be a Grothendieck site. For $X \in \mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})_*$, define the presheaves $\pi^{\mathrm{pr}}_n X$ given by $(\pi^{\mathrm{pr}}_n X)(U) = \pi_n(X(U))$, for all $n \geqslant 0$, called the *presheaves of homotopy sets/groups of* X. Their sheafification $\pi_n X := a(\pi^{\mathrm{pr}}_n X)$ are called the *sheaves of homotopy sets/groups of* X.

Definition 2.5.1. The map $f : X \to Y$ in $\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})$ is called a *local weak equivalence* if

- $f_* : \pi_0 X \xrightarrow{\cong} \pi_0 Y$ in $\mathcal{S}\mathsf{hv}(\mathcal{C})$.
- $f_* : \pi_n(X|_U, x) \xrightarrow{\cong} \pi_n(Y|_U, f(x))$ in $\mathcal{S}\mathsf{hv}(\mathcal{C}/U)$ for all $n \geqslant 1, U \in \mathcal{C}$ and $x \in X_0(U)$.

Proposition 2.5.2. *If $f : X \to Y$ is a local weak equivalence in $\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})$, then its restriction $f|_U : X|_U \to Y|_U$ is a local weak equivalence in $\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C}/U)$ for every object $U \in \mathcal{C}$.*

Proposition 2.5.3 ([**50**, Corollary 4.22])**.** *Given a map $f : X \to Y$ in $\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})$, the following are equivalent:*

- *f is a local weak equivalence.*
- *$f^+ : X^+ \to Y^+$ is a local weak equivalence.*
- *$f^\sharp : X^\sharp \to Y^\sharp$ is a local weak equivalence.*
- *$f_* : \mathrm{Ex}^\infty(X) \to \mathrm{Ex}^\infty(Y)$ is a local weak equivalence.*

Let $i : K \to L$ be a map in $\mathsf{s}\mathcal{S}\mathsf{et}$ and $f : X \to Y$ be a map in $\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})$. Then we say that f has the *local (right) lifting property* (abbr. LRLP) with respect to i if for every object $U \in \mathcal{C}$ and every commutative diagram

$$\begin{array}{ccc} K & \longrightarrow & X(U) \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle f_U} \\ L & \longrightarrow & Y(U) \end{array}$$

in $\mathsf{s}\mathcal{S}\mathsf{et}$, there exists a covering sieve $\mathcal{U} \in \mathrm{Cov}_\tau(U)$ such that for any $(V \xrightarrow{\varphi} U) \in \mathcal{U}$, the dotted arrow in the diagram

$$\begin{array}{ccccc} K & \longrightarrow & X(U) & \xrightarrow{\varphi^*} & X(V) \\ {\scriptstyle i}\downarrow & \nearrow & & & \downarrow{\scriptstyle f_V} \\ L & \longrightarrow & Y(U) & \xrightarrow{\varphi^*} & Y(V) \end{array}$$

exists (making the two triangles commute). Note that, by adjunction, these diagrams are equivalent to the following two:

$$\begin{array}{ccc} U \otimes K & \longrightarrow & X \\ {\scriptstyle \mathrm{id}_U \otimes i}\downarrow & & \downarrow{\scriptstyle f} \\ U \otimes L & \longrightarrow & Y, \end{array} \qquad \begin{array}{ccccc} V \otimes K & \longrightarrow & U \otimes K & \longrightarrow & X \\ {\scriptstyle \mathrm{id}_V \otimes i}\downarrow & & & \nearrow & \downarrow{\scriptstyle f} \\ V \otimes L & & \longrightarrow & & U \otimes L \longrightarrow Y. \end{array}$$

For a set E of maps in $\mathsf{s}\mathcal{S}\mathsf{et}$, we write $\mathrm{LRLP}(E)$ for the class of maps in $\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})$ that have the local (right) lifting property with respect to every map in E. The class $\mathrm{LRLP}(E)$ is closed under composition and base change.

Proposition 2.5.4 ([**50**, Lemma 4.8])**.** *Let $i : K \to L$ be a map in $\mathsf{s}\mathcal{S}\mathsf{et}$ and $f : X \to Y$ be a map in $\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})$. Then $f \in \mathrm{LRLP}(i)$ iff the degree 0 level of the map*

$$X^L \xrightarrow{(i^*, f_*)} X^K \times_{Y^K} Y^L$$

induced by the commutative diagram

$$\begin{array}{ccc} X^L & \xrightarrow{i^*} & X^K \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle f_*} \\ Y^L & \xrightarrow{i^*} & Y^K \end{array}$$

is a local epimorphism (in $\mathcal{P}\mathsf{re}(\mathcal{C})$). (Note that $(X^K)_0$ is the presheaf $U \mapsto \mathsf{s}\mathcal{S}\mathsf{et}(K, X(U))$.)

Corollary 2.5.5. *Let $i : K \to L$ be a map in $\mathsf{s}\mathcal{S}\mathsf{et}$ and $f : X \to Y$ be a map in $\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})$, let $f^\sharp : X^\sharp \to Y^\sharp$ be the sheafification map in $\mathsf{s}\mathcal{S}\mathsf{hv}(\mathcal{C})$. Then $f \in \mathrm{LRLP}(i)$ iff $f^\sharp \in \mathrm{LRLP}(i)$.*

Definition 2.5.6. A map $f : X \to Y$ in $\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})$ is called a *local fibration* if $f \in \mathrm{LRLP}(\{\Lambda^n_k \hookrightarrow \Delta^n : n \geqslant 1, 0 \leqslant k \leqslant n\})$. Clearly, sectionwise fibrations are local fibrations.

A map $f : X \to Y$ in $\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})$ is called a *local trivial fibration* (or a *hypercovering*) if $f \in \mathrm{LRLP}(\{\partial\Delta^n \hookrightarrow \Delta^n : n \geqslant 0\})$. Clearly, A map which is sectionwise fibration and sectionwise weak equivalence is a local trivial fibration.

A map $j : A \to B$ in $\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})$ is called a *cofibration* if it is a sectionwise cofibration (i.e., a monomorphism).

Proposition 2.5.7. *The following results hold:*

- *A map $f : X \to Y$ in $\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})$ is a local (trivial) fibration iff its sheafification $f^\sharp$ is.*
- *A sectionwise Kan fibration is a local fibration.*
- *For any $X \in \mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})$, the two natural maps $X \to X^+ \to X^\sharp$ are local trivial fibrations.*
- *The inverse image functor of a geometric morphism between Grothendieck topoi preserves local fibrations as well as local trivial fibrations.*

Proposition 2.5.8 ([**50**, Theorem 4.32])**.** *A map $f : X \to Y$ in $\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})$ is a local trivial fibration iff it is a local fibration and a local weak equivalence.*

Proposition 2.5.9. *If a topos $\mathcal{S}\mathsf{hv}(\mathcal{C})$ has enough points $x_i, i \in I$, then a map $f : X \to Y$ in $\mathsf{s}\mathcal{S}\mathsf{hv}(\mathcal{C})$ (or $\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})$) is a local weak equivalence/local (trivial) fibration/monomorphism in $\mathsf{s}\mathcal{S}\mathsf{hv}(\mathcal{C})$ iff all the induced maps $x_i^* f : x_i^* X \to x_i^* Y$ on stalks (or on all stalks) are weak equivalences/(trivial) Kan fibrations/monomorphisms of simplicial sets.*

Proposition 2.5.10 ([**50**, Lemma 4.37])**.** *The pullback of a local weak equivalence along a local fibration is a local weak equivalence.*

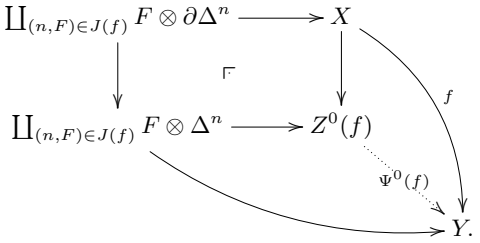
Proposition 2.5.11. *Let $\mathcal{S}$ be a set of objects in $\mathcal{P}\mathsf{re}(\mathcal{C})$ such that for any object U of $\mathcal{C}$, there is a local epimorphism $F \to U$ with F being a coproduct of some objects in $\mathcal{S}$. Then there exist a functor $\Phi = \Phi_{\mathcal{S}} : \mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C}) \to \mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})$ and a natural transformation $\eta = \eta^{\mathcal{S}} : \Phi_{\mathcal{S}} \Rightarrow \mathrm{id}$ with the following properties:*

(1) *For any $n \in \mathbb{Z}_{\geqslant 0}$ and any object Y of $\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})$, the presheaf $\Phi(Y)_n$ is a coproduct of some objects in $\mathcal{S}$.*
(2) *For any object Y of $\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})$, the map $\eta_Y : \Phi(Y) \to Y$ is a local trivial fibration.*

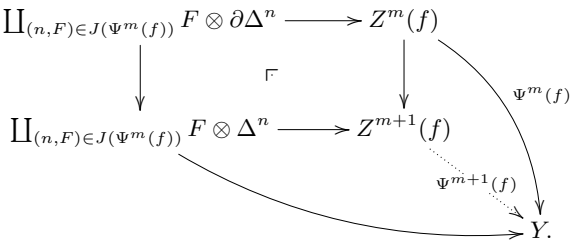
Proof. For any morphism $f : X \to Y$ in $\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})$, we denote $J(f) = J_{\mathcal{S}}(f)$ the set of all commutative diagrams

$$\begin{array}{ccc} F \otimes \partial\Delta^n & \longrightarrow & X \\ {\scriptstyle \mathrm{id}_F \otimes i}\downarrow & & \downarrow{\scriptstyle f} \\ F \otimes \Delta^n & \longrightarrow & Y \end{array}$$

with $n \in \mathbb{Z}_{\geqslant 0}, F \in \mathcal{S}$ and by abuse of notation we denote such a diagram by the pair (n, F) (though one such a pair (n, F) may corresponds to many commutative diagrams as above). We define a presheaf $Z^0(f)$ and a map $\Psi^0(f) : Z^0(f) \to Y$ via the push-out diagram



Assume $m \geqslant 0$ and $\Psi^m(f) : Z^m(f) \to Y$ is defined, we then inductively define a presheaf $Z^{m+1}(f)$ and a map $\Psi^{m+1}(f) : Z^{m+1}(f) \to Y$ via the push-out diagram



It yields a push-out (of presheaves of sets) at each simplicial degree $k \in \mathbb{Z}_{\geqslant 0}$, so we have

$$Z^{m+1}(f)_k = Z^m(f)_k \coprod \Big(\coprod_{(n,F)\in J(\Psi^m(f))} \coprod_{\Delta^n_k \setminus (\partial\Delta^n)_k} F \Big).$$

Moreover, each of the map $Z^m(f) \to Z^{m+1}(f)$ is an inclusion and $\Psi^{m+1}(f)$ extends $\Psi^m(f)$.

Now we start with any object Y of $\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})$ and apply the above procedure to the map $f : \varnothing \to Y$. We observe that in this situation we have an identification $J(f) = \{(F \to Y) : F \in \mathcal{S}\}$, so $Z^0(f) = \coprod_{F\to Y} F$ is a coproduct of objects in $\mathcal{S}$.

We define $\Phi(Y) = \Phi_{\mathcal{S}}(Y) := \operatorname*{colim}_{m\in\mathbb{N}} Z^m(\varnothing \to Y)$, and let $\eta_Y : \Phi(Y) \to Y$ be the induced map, which extends all the maps $\Psi^m(f)$. By universal property of coproducts, we get a functor $\Phi = \Phi_{\mathcal{S}} : \mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C}) \to \mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})$ and a natural transformation $\eta = \eta^{\mathcal{S}} : \Phi_{\mathcal{S}} \Rightarrow \mathrm{id}$. Clearly $\Phi(Y) = \Phi_{\mathcal{S}}(Y) = \bigcup_{m\in\mathbb{N}} Z^m(\varnothing \to Y)$ is a coproduct of objects in $\mathcal{S}$.

To show that $\eta_Y : \Phi(Y) \to Y$ is a local trivial fibration, we need to show that, given any object U of $\mathcal{C}$ and any commutative diagram

$$\begin{array}{ccc} U\otimes\partial\Delta^k & \longrightarrow & \Phi(Y) \\ \downarrow & & \downarrow{\scriptstyle \eta_Y} \\ U\otimes\Delta^k & \longrightarrow & Y, \end{array}$$

there exists a covering sieve $\mathcal{U} \in \mathrm{Cov}_\tau(U)$ such that for any $(V \to U) \in \mathcal{U}$, there exists a dotted arrow in the diagram

$$\begin{array}{ccccc} V\otimes\partial\Delta^k & \longrightarrow & U\otimes\partial\Delta^k & \longrightarrow & \Phi(Y) \\ \downarrow & \nearrow & & & \downarrow{\scriptstyle \eta_Y} \\ V\otimes\Delta^k & \longrightarrow & U\otimes\Delta^k & \longrightarrow & Y \end{array}$$

making it commute. Indeed, by the discussion at the end of Example 2.1.1, the given diagram factors as

$$\begin{array}{ccccc} U\otimes\partial\Delta^k & \longrightarrow & Z^m(f) & \longrightarrow & \Phi(Y) \\ \downarrow & & {\scriptstyle \Psi^m(f)}\downarrow & \swarrow{\scriptstyle \eta_Y} & \\ U\otimes\Delta^k & \longrightarrow & Y & & \end{array}$$

for some $m \in \mathbb{N}$, here and below f is again the map $\varnothing \to Y$. Take a (multi-)set $\mathcal{F}$ of objects in $\mathcal{S}$ such that there is a local epimorphism $\coprod_{F\in\mathcal{F}} F \to U$. We get the following diagram

$$\begin{array}{ccccccc} \coprod_{F\in\mathcal{F}} F\otimes\partial\Delta^k & \longrightarrow & U\otimes\partial\Delta^k & \longrightarrow & Z^m(f) & \longrightarrow & \Phi(Y) \\ \downarrow & & \downarrow & & {\scriptstyle \Psi^m(f)}\downarrow & \swarrow{\scriptstyle \eta_Y} & \\ \coprod_{F\in\mathcal{F}} F\otimes\Delta^k & \longrightarrow & U\otimes\Delta^k & \longrightarrow & Y. & & \end{array}$$

Take $\mathcal{U} = (\coprod_{F\in\mathcal{F}} F)$ (see Corollary 2.3.15), which is a covering sieve of U. We get the following diagram, whose two rows (start from the second term of each row) factor the two rows of the previous diagram (by defining diagram of $\Psi^{m+1}(f)$):

$$\begin{array}{ccccccccc} V\otimes\partial\Delta^k & \longrightarrow & \coprod_{F\in\mathcal{F}} F\otimes\partial\Delta^k & \longrightarrow & \coprod_{(n,F)\in J(\Psi^m(f))} & & F\otimes\partial\Delta^n & \longrightarrow & Z^m(f) \longrightarrow \Phi(Y) \\ \downarrow & & \downarrow & & \downarrow & & & & \downarrow \quad\nearrow \quad\downarrow{\scriptstyle \eta_Y} \\ V\otimes\Delta^k & \longrightarrow & \coprod_{F\in\mathcal{F}} F\otimes\Delta^k & \longrightarrow & \coprod_{(n,F)\in J(\Psi^m(f))} & & F\otimes\Delta^n & \longrightarrow & Z^{m+1}(f) \xrightarrow{\Psi^{m+1}(f)} Y. \end{array}$$

Then for any $(V \to U) \in \mathcal{U}$, the unique map from $V \otimes \Delta^k$ to $\Phi(Y)$—composite of four arrows—in the above diagram gives a desired local lifting map (dotted arrow). □

Remark 2.5.12. The set of representable presheaves trivially satisfies the requirement for $\mathcal{S}$ in the condition above, hence is an example of $\mathcal{S}$. This technical result is a variant of [**69**, Lemma 2.1.16] (where the authors give a sketchy proof), the following result corresponds to [**69**, Lemma 2.1.18].

Proposition 2.5.13. *Let $Y \in \mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C}), X_0 \in \mathcal{P}\mathsf{re}(\mathcal{C})$, and let $p_0 : X_0 \to Y_0$ be a local epimorphism of presheaves, then there exists a local trivial fibration $p : X \to Y$ in $\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})$, whose degree 0 component is p_0. If X_0 is a coproduct of representable presheaves, then one can choose X in such a way that X_n is a coproduct of representable presheaves for every $[n] \in \mathbf{\Delta}$.*

Proof. One proceeds the same as the proof of the previous proposition, with $\mathcal{S}$ being the set of representable presheaves: the map $p_0 : X_0 \to Y_0$ can be viewed as a map $p_0 : \mathrm{cs}_* X_0 \to Y$ in $\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})$ by composing with various monomorphisms $(s_0)^n : Y_0 \to Y_n$ (see the discussion before Remark 1.1.4). Then we inductively define $p_m : X_m \to Y, m \geqslant 1$, by considering only those commutative diagrams with $h_U \otimes \partial\Delta^n \to h_U \otimes \Delta^n$, U objects of $\mathcal{C}$ and $n > 0$. Then

define $X := \operatorname{colim}_{m\in\mathbb{N}} X_m, p : X \to Y$ the induced map. It's not hard to see that the proof of the previous proposition works well in this situation (with p in place of η_Y and p_m in place of $\Psi^{m+1}(f)$). □

2.6. Local model structure on the simplicial (pre)sheaf category

Throughout this section, we fix a Grothendieck site $\mathcal{C}$.

Proposition-Definition 2.6.1 ([**50**, Theorem 5.8, 5.9])**.** There is a model structure on $\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})$, called the *(Jardine) injective model structure*, with

- weak equivalences $\mathbf{W}$: local weak equivalences.
- cofibrations $\mathbf{C}$: sectionwise cofibrations (i.e. monomorphisms).
- fibrations $\mathbf{F} = \mathrm{RLP}(\mathbf{W} \cap \mathbf{C})$, called *(Jardine) injective fibrations* or *global fibrations*.

There is a model structure on $\mathsf{s}\mathcal{S}\mathsf{hv}(\mathcal{C})$, also called the *(Joyal) injective model structure*, with

- weak equivalences $\mathbf{W}$: those maps in $\mathsf{s}\mathcal{S}\mathsf{hv}(\mathcal{C})$ that are local weak equivalences in $\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})$.
- cofibrations $\mathbf{C}$: those maps in $\mathsf{s}\mathcal{S}\mathsf{hv}(\mathcal{C})$ that are cofibrations in $\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})$=monomorphisms.
- fibrations $\mathbf{F} = \mathrm{RLP}(\mathbf{W} \cap \mathbf{C})$, also called *(Joyal) injective fibrations* or *global fibrations*.

The (Jardine-Joyal) injective model structure on $\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})$ and $\mathsf{s}\mathcal{S}\mathsf{hv}(\mathcal{C})$ are both cofibrantly generated and proper. To emphasis these model structures, we denote them by $\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})_J$ and $\mathsf{s}\mathcal{S}\mathsf{hv}(\mathcal{C})_J$ respectively. They are simplicial model categories, where the simplicial structure are given as in Definition 2.4.3. Moreover, the adjoint pair

$$a : \mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})_J \rightleftarrows \mathsf{s}\mathcal{S}\mathsf{hv}(\mathcal{C}_\tau)_J : \iota$$

is a Quillen equivalence, both of which preserve local weak equivalences. Each class of $\mathbf{W}, \mathbf{C}, \mathbf{F}$ for $\mathsf{s}\mathcal{S}\mathsf{hv}(\mathcal{C})_J$ is the intersection of $\mathsf{s}\mathcal{S}\mathsf{hv}(\mathcal{C})$ with that for $\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})_J$.

Proposition 2.6.2 ([**50**, Lemma 5.12])**.** *Let $p : X \to Y$ be a map in $\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})$.*

- *If p is an injective fibration, then it is a sectionwise Kan fibration (hence a local fibration).*
- *If p is a trivial injective fibration, then it is a sectionwise trivial Kan fibration (hence a local trivial fibration).*
- *Any $F \in \mathcal{S}\mathsf{hv}(\mathcal{C}) \subset \mathsf{s}\mathcal{S}\mathsf{hv}(\mathcal{C})$ is (Joyal) injective fibrant.*

Remark 2.6.3. Local weak equivalence+(Jardine) injective fibration≠local weak equivalence+local fibration.

Definition 2.6.4. A *(Jardine) injective fibrant model* of $X \in \mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})$ is a local weak equivalence $j : X \to Z$, where $Z \in \mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})_J$ is (Jardine) injective fibrant. It's a fibrant replacement/approximation of X in the (Jardine) injective model structure of $\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})$. Since the (Jardine) injective model structure on $\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})$ or $\mathsf{s}\mathcal{S}\mathsf{hv}(\mathcal{C})$ is cofibrantly generated, one can make a functorial choice $j : X \to GX$ of injective fibrant models on $\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})$ or $\mathsf{s}\mathcal{S}\mathsf{hv}(\mathcal{C})$ (see [**50**, Remark 5.14]).

We say that X *satisfies descent* or *has the descent property* if it has a (Jardine) injective fibrant model $j : X \to Z$ that is a sectionwise weak equivalence. Clearly, any (Jardine) injective fibrant satisfies descent.

Proposition 2.6.5 ([**50**, Corollary 5.13])**.** *If $X, Y \in \mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})$ are (Jardine) injective fibrant and $f : X \to Y$ is a local weak equivalence, then f is a sectionwise weak equivalence.*

Proposition 2.6.6 ([**50**, Corollary 5.15])**.** *If $X \in \mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})$ satisfies descent and $j : X \to Z$ is a (Jardine) injective fibrant model of X, then j is a sectionwise weak equivalence.*

The following result follows directly from the existence of the (Jardine) injective model structure of $\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})_J$.

Proposition 2.6.7 ([**50**, p.103])**.** *Let $f : X \to Y$ be a local weak equivalence in $\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})$ and Z (Jardine) injective fibrant. Then the map*

$$f^* : \mathrm{Map}(Y, Z) \to \mathrm{Map}(X, Z)$$

is a weak equivalence in $\mathsf{s}\mathcal{S}\mathsf{et}$.

Similar result also holds for the restriction $f|_U : X|_U \to Y|_U$ for every object $U \in \mathcal{C}$, thus the map

$$f^* : \underline{\mathrm{Hom}}(Y, Z) \to \underline{\mathrm{Hom}}(X, Z)$$

is a sectionwise weak equivalence in $\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})$.

Proposition 2.6.8 ([**50**, Corollary 4.41])**.** *Let $i : A \to B, j : C \to D$ be cofibrations in the (Jardine) injective model structure of $\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})$, then the map*

$$(A \times D) \coprod_{A\times C} (B \times C) \xrightarrow{(i_*, j_*)} B \times D$$

induced by the commutative diagram

$$\begin{array}{ccc} A \times C & \xrightarrow{j_*} & A \times D \\ {\scriptstyle i_*}\downarrow & & \downarrow{\scriptstyle i_*} \\ B \times C & \xrightarrow{j_*} & B \times D \end{array}$$

is a cofibration that is a local weak equivalence if either i or j is.

Proposition 2.6.9 ([**50**, Corollary 5.19]). *Let $i : A \to B$ be a cofibration and $p : X \to Y$ a fibration in $\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})_J$, then the map*

$$\underline{\mathrm{Hom}}(B, X) \xrightarrow{i^* \times p_*} \underline{\mathrm{Hom}}(A, X) \times_{\underline{\mathrm{Hom}}(A,Y)} \underline{\mathrm{Hom}}(B, Y)$$

induced by the commutative diagram

$$\begin{array}{ccc} \underline{\mathrm{Hom}}(B, X) & \xrightarrow{i^*} & \underline{\mathrm{Hom}}(A, X) \\ \downarrow{\scriptstyle p_*} & & \downarrow{\scriptstyle p_*} \\ \underline{\mathrm{Hom}}(B, Y) & \xrightarrow{i^*} & \underline{\mathrm{Hom}}(A, Y) \end{array}$$

is a (Jardine) injective fibration which is a local weak equivalence if either i or p is a local weak equivalence.

Proposition 2.6.10 ([**50**, Lemma 5.20]). *Given in $\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})$ a cartesian diagram*

$$\begin{array}{ccc} Z \times_Y X & \longrightarrow & X \\ \downarrow & \lrcorner & \downarrow{\scriptstyle \pi} \\ Z & \longrightarrow & Y \end{array}$$

where the map π is a local fibration, it is homotopy cartesian in the model category $\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})_J$.

Let $\mathcal{C}, \mathcal{D}$ be Grothendieck sites, $\pi : \mathcal{S}\mathsf{hv}(\mathcal{C}) \to \mathcal{S}\mathsf{hv}(\mathcal{D})$ a geometric morphism, then its inverse and direct image functors induce an adjoint pair

$$\pi^* : \mathsf{s}\mathcal{S}\mathsf{hv}(\mathcal{D}) \rightleftarrows \mathsf{s}\mathcal{S}\mathsf{hv}(\mathcal{C}) : \pi_*.$$

Proposition 2.6.11 ([**50**, Corollary 4.12, 4.15]). *The inverse image functor $\pi^* : \mathsf{s}\mathcal{S}\mathsf{hv}(\mathcal{D}) \to \mathsf{s}\mathcal{S}\mathsf{hv}(\mathcal{C})$ of a geometric morphism π preserves local (trivial) fibrations of simplicial sheaves.*

The associated sheaf functor preserves and reflects local (trivial) fibrations of simplicial presheaves.

Proposition 2.6.12 ([**50**, Corollary 5.22]). *The adjoint pair*

$$\pi^* : \mathsf{s}\mathcal{S}\mathsf{hv}(\mathcal{D})_J \rightleftarrows \mathsf{s}\mathcal{S}\mathsf{hv}(\mathcal{C})_J : \pi_*$$

is a Quillen pair for the (Joyal) injective model structures on $\mathsf{s}\mathcal{S}\mathsf{hv}(\mathcal{D})$ and on $\mathsf{s}\mathcal{S}\mathsf{hv}(\mathcal{C})$, and the inverse image functor π^* preserves cofibrations as well as local weak equivalences.

Let $f : \mathcal{D} \to \mathcal{C}$ be a site morphism, then by the discussion of Section 2.3, there is an adjunction

$$f^p : \mathcal{P}\mathsf{re}(\mathcal{D}) \rightleftarrows \mathcal{P}\mathsf{re}(\mathcal{C}) : f_*$$

and it yields a geometric morphism $f : \mathcal{S}\mathsf{hv}(\mathcal{C}) \to \mathcal{S}\mathsf{hv}(\mathcal{D})$.

Proposition 2.6.13 ([**50**, Corollary 5.24]). *The adjoint pair*

$$f^p : \mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{D})_J \rightleftarrows \mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})_J : f_*$$

is a Quillen pair for the (Jardine) injective model structures on $\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{D})$ and on $\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})$, and the functor f^p preserves cofibrations as well as local weak equivalences.

2.7. Hypercoverings and descent

In this section, we still focus on the category $\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})$ of simplicial presheaves on a (small) Grothendieck site $(\mathcal{C}, \tau)$, studying homotopical descent properties of simplicial presheaves on $\mathcal{C}$.

Definition 2.7.1. A map $f : X \to Y$ in $\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})$ is called a *Reedy local epimorphism* if all of its relative matching maps $\widetilde{\mathrm{M}}_n(f) : X_n \to \mathrm{M}_n X \times_{\mathrm{M}_n Y} Y_n$ are local epimorphisms. For $n = 0$, this map is $f_0 : X_0 \to Y_0$; for $n \geqslant 1$, this map is $\widetilde{\mathrm{M}}_n(f) : X_n \to (\mathrm{cosk}_{n-1} X)_n \times_{(\mathrm{cosk}_{n-1} Y)_n} Y_n = \mathrm{M}_{n,f}$.

If $f : X \to Y$ is a map in $\mathcal{P}\mathsf{re}(\mathcal{C})$ viewed as one in $\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})$, then f is a Reedy local epimorphism iff f is an isomorphism of presheaves.

Theorem 2.7.2. *A map $f : X \to Y$ in $\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})$ is a Reedy local epimorphism iff it is a local trivial fibration in $\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})$.*

Proof. This follows easily from Proposition 2.5.4 ([**50**, Lemma 4.8]). □

Definition 2.7.3 ([**26**, Definition 4.2]). Let X be an object of $\mathcal{C}$. A (τ-)*hypercovering (of X)* or a (τ-)*hypercover (of X)* is a local trivial fibration $U \to X$ in $\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})$ for which each U_n is a coproduct of representables. Sectionwise applying the result Proposition 1.5.3, we see that for $U \in \mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})$ with each U_n a coproduct of representables, a map $U \to X$ in $\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})$ is a hypercover of X iff all the presheaf maps $U_0 \to X, U_1 \to U_0 \times_{h_X} U_0$ and $U_n = (U^{\Delta^n})_0 \to (U^{\partial\Delta^n})_0 = (\mathrm{cosk}_{n-1} U)_n, n \geqslant 2$ are local epimorphisms.

We denote the class of all hypercovers by $\mathrm{HC}(\mathcal{C}, \tau)$, it's in general a proper class.

Definition 2.7.4 ([**26**, Definition 6.1]). A collection of hypercovers S is said to be *dense* if every hypercover $U \to X$ in $\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})$ can be refined by a hypercover $V \to X$ in S.

Proposition 2.7.5 ([**26**, Proposition 6.6]). *The class of all hypercovers $\mathrm{HC}(\mathcal{C}, \tau)$ has a subset S which is dense.*

Definition 2.7.6 ([**26**, Definition 4.10])**.** A Reedy local epimorphism $f : X \to Y$ of Y is called *bounded by* $n \in \mathbb{N}$ or is *n-bounded* if its relative matching maps $\widetilde{\mathrm{M}}_k(f) : X_k \to (\mathrm{cosk}_{k-1}X)_k \times_{(\mathrm{cosk}_{k-1}Y)_k} Y_k = \mathrm{M}_{k,f}$ are isomorphisms for all $k > n$. The smallest such $n \in \mathbb{N}$ is then called the *height* of the Reedy local epimorphism $f : X \to Y$, denoted $\mathrm{ht}(f)$. These notions in particular apply to hypercovers, we denote the height of a hypercover $U \to X$ by $\mathrm{ht}(U)$. So for $n \geqslant 1$, a hypercover $U \to X$ is n-bounded iff the maps $U_k \to (\mathrm{cosk}_{k-1}U)_k$ are isomorphisms of sheaves for all $k > n$.

Example 2.7.7 (Čech covers=hypercoverings of height 0)**.** *Let X be an object of $\mathcal{C}$, let $\mathcal{U} = \{(U_i \to X) : i \in I\}$ be a τ-cover of X. We denote $\mathcal{U}_{i_0\cdots i_n} := h_{U_{i_0}} \times_{h_X} \cdots \times_{h_X} h_{U_{i_n}}$ for $i_0, \cdots, i_n \in I, n \in \mathbb{N}$. When fibre products in $\mathcal{C}$ exist, each $\mathcal{U}_{i_0\cdots i_n}$ is representable. We define the* Čech cover $\check{\mathrm{C}}(\mathcal{U}) \in \mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})$ *by letting* $\check{\mathrm{C}}(\mathcal{U})_n := \coprod_{(i_0,\cdots,i_n)\in I^{n+1}} \mathcal{U}_{i_0\cdots i_n}$*, the face maps d_i and degeneracy maps s_j are given by various projections and diagonal maps (together with identities), for example, the map $d_0 : \check{\mathrm{C}}(\mathcal{U})_1 \to \check{\mathrm{C}}(\mathcal{U})_0$ on a summand $h_{U_i} \times_{h_X} h_{U_j}$ is the projection $h_{U_i} \times_{h_X} h_{U_j} \to h_{U_j}$ followed by the inclusion, the map $s_0 : \check{\mathrm{C}}(\mathcal{U})_0 \to \check{\mathrm{C}}(\mathcal{U})_1$ on a summand h_{U_i} is the diagonal map $h_{U_i} \to h_{U_i} \times_{h_X} h_{U_i}$ followed by the inclusion.*

By considering on sections (evaluating at any object $V \in \mathcal{C}$) it's easy to see that $\operatorname{colim}_{[n]\in\mathbf{\Delta}^{\mathrm{op}}} \check{\mathrm{C}}(\mathcal{U})_n \cong \pi_0\check{\mathrm{C}}(\mathcal{U}) \cong (\mathcal{U})$ (the sieve generated by $\mathcal{U}$). The canonical map $\check{\mathrm{C}}(\mathcal{U}) \to (\mathcal{U})$ is a sectionwise weak equivalence (see [**26**, Proposition A.1]*).*

It's clear that the presheaf map $\check{\mathrm{C}}(\mathcal{U})_0 \to X$ is a local epimorphism and $\check{\mathrm{C}}(\mathcal{U})_1 \to \check{\mathrm{C}}(\mathcal{U})_0 \times_X \check{\mathrm{C}}(\mathcal{U})_0$ is an isomorphism. For $n \geqslant 2$, sectionwise applying the result Proposition 1.5.3, we see that there is in $\mathcal{P}\mathsf{re}(\mathcal{C})$ the equalizer diagram

$$(\mathrm{cosk}_{n-1}U)_n \to \prod_{0\leqslant l\leqslant n} U_{n-1} \rightrightarrows \prod_{0\leqslant i<j\leqslant n} U_{n-2}$$

with $U = \check{\mathrm{C}}(\mathcal{U}) \in \mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})$. On the other hand, evaluating at any object $V \in \mathcal{C}$, it's easy to see — read off by hand — that $\check{\mathrm{C}}(\mathcal{U})_n$ fits into the same diagram (as the first term), we see $\check{\mathrm{C}}(\mathcal{U})_n \to (\mathrm{cosk}_{n-1}\check{\mathrm{C}}(\mathcal{U}))_n$ is also an isomorphism.

Thus when fibre products in $\mathcal{C}$ exist, the Čech cover $\check{\mathrm{C}}(\mathcal{U}) \to X$ is a hypercovering of X of height 0. Conversely, if a hypercovering $U \to X$ has height 0, reverse the above reasoning (since the above equalizer diagram exists for any $U \in \mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})$) one sees that U is (isomorphic to) the Čech nerve *associated to the morphism $U_0 \to h_X$.*

All in all, a hypercovering $U \to X$ has height 0 iff it is (isomorphic to) a Čech cover of X.

Proposition 2.7.8 ([**26**, Lemma 6.3])**.** *Let $\mathcal{C} \subset \mathcal{C}'$ be two classes of maps in a model category $\mathcal{M}$ such that the left Bousfield localizations $\mathrm{L}_{\mathcal{C}}\mathcal{M}, \mathrm{L}_{\mathcal{C}'}\mathcal{M}$ exist. The two localizations are the same iff, for any fibration in $\mathrm{L}_{\mathcal{C}}\mathcal{M}$ between $\mathcal{C}$-local objects $X \to Y$, if it is a $\mathcal{C}'$-local equivalence, then it is a $\mathcal{C}$-local equivalence.*

Theorem 2.7.9 ([**26**, Theorem 6.2])**.** *Let S be a class of hypercovers which contains a* set *that is dense (e.g. we can take S to be the class $\mathrm{HC}(\mathcal{C}, \tau)$). Then the Bousfield localizations $\mathrm{L}_S\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})_{\mathrm{proj}}, \mathrm{L}_S\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})_{\mathrm{inj}}$ exist, which are respectively the left Bousfield localizations of $\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})_{\mathrm{proj}}, \mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})_{\mathrm{inj}}$ with respect to the class of local weak equivalences in $\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})$ (so $\mathrm{L}_S\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})_{\mathrm{inj}} = \mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})_J$). We write them respectively as $\mathrm{L}_\tau\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})_{\mathrm{proj}}$ and $\mathrm{L}_\tau\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})_{\mathrm{inj}} = \mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})_J$. Moreover, we have the Quillen equivalence*

$$1 : \mathrm{L}_\tau\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})_{\mathrm{proj}} \rightleftarrows \mathrm{L}_\tau\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})_{\mathrm{inj}} : 1.$$

Since trivial Kan fibrations of simplicial sets are surjective, by Yoneda lemma, we see that *all representable functors are cofibrant in $\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})_{\mathrm{proj}}$ hence in $\mathrm{L}_\tau\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})_{\mathrm{proj}}$.*

Definition 2.7.10 ([**26**, Definition 4.3])**.** We say a simplicial presheaf F *satisfies descent* for a hypercover $U \to X$ if there is a fibrant replacement $\hat{F}$ of F in $\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})_{\mathrm{proj}}$ (see Section 1.2) such that the map $\hat{F}(X) \to \operatorname{holim}_{[n]\in\mathbf{\Delta}} \hat{F}(U_n) =: \operatorname{holim}_{\mathbf{\Delta}} \hat{F}(U_\bullet)$ is a weak equivalence in sSet, where $\hat{F}(U_\bullet) \in \mathsf{csSet}$ is given by $[n] \mapsto \hat{F}(U_n)$. Since the category $\mathrm{FibAp}(F)$ of fibrant replacements of F is contractible ([**40**, Theorem 14.6.2 (2)]) hence non-empty and connected, so by Proposition 1.2.10 we see that if this is the case for one fibrant replacement $\hat{F}$ of F, then it is true for every such $\hat{F}$.[7]

Theorem 2.7.11 ([**26**, Corollary 7.1])**.** *Let S be a class of hypercovers which contains a* set *that is dense. Let $F \in \mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})$.*

- *F is fibrant in $\mathrm{L}_\tau\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})_{\mathrm{proj}}$ iff F is fibrant in $\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})_{\mathrm{proj}}$ (i.e. F is a presheaf of Kan complexes) and it satisfies descent for all hypercovers in S.*
- *F is fibrant in $\mathrm{L}_\tau\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})_{\mathrm{inj}}$ iff F is fibrant in $\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})_{\mathrm{inj}}$ and it satisfies descent for all hypercovers in S.*

Remark 2.7.12. Recall that the Bousfield-Kan map gives a sectionwise weak equivalence $\operatorname{hocolim}_{\mathbf{\Delta}^{\mathrm{op}}} D_U \xrightarrow{\simeq} U$, where $D_U : \mathbf{\Delta}^{\mathrm{op}} \to \mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})_{\mathrm{inj}}, [n] \to U_n$. So for a fibrant replacement $\hat{F}$ of F in $\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})_{\mathrm{inj}}$, we have $\operatorname{holim}_{\mathbf{\Delta}} \hat{F}(U_\bullet) = \operatorname{holim}_{\mathbf{\Delta}} \mathrm{Map}(D_U, \hat{F}) \simeq \mathrm{Map}(\operatorname{hocolim}_{\mathbf{\Delta}^{\mathrm{op}}} D_U, \hat{F}) \simeq \mathrm{Map}(U_\bullet, \hat{F}) \simeq \mathrm{RMap}(U_\bullet, F)$ (here RMap is computed in $\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})_{\mathrm{inj}}$, note that $U_\bullet$ is cofibrant). So the second statement above can be paraphrased as saying: F is fibrant in $\mathrm{L}_\tau\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})_{\mathrm{inj}}$ iff F is fibrant in $\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})_{\mathrm{inj}}$ and $\mathrm{RMap}(X, F) \simeq \mathrm{RMap}(U_\bullet, F)$ for all hypercovers $U \to X$ in S. Once knowing the left Bousfield localizations exist, this theorem follows directly from Proposition 1.4.15.

The following result says that the descent property here coincides with that given in Definition 2.6.4; in fact, part of this result is a special case of Proposition 1.4.6. It is consistent with [**50**, Example 5.42].

Theorem 2.7.13. *Let $F \in \mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})$. The following are equivalent:*

(1) *F satisfies descent for all hypercovers (or for all hypercovers in a dense subset of the class $\mathrm{HC}(\mathcal{C}, \tau)$).*

[7] Let $R : \mathsf{sSet} \to \mathsf{sSet}$ be a functorial fibrant replacement for sSet, then we can take $\hat{F} = R \circ F$, so the property of satisfying descent for $U \to X$ is in fact not related to any model structure on $\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})$ (but of course hypercoverings depend on the topology of $\mathcal{C}$).

(2) *There is a fibrant replacement* $j : F \to \hat{F}$ *of* F *in* $\mathrm{L}_\tau \mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})_{\mathrm{proj}}$ *that is a sectionwise weak equivalence.*
(3) *There is a fibrant replacement* $j' : F \to \hat{F}'$ *of* F *in* $\mathrm{L}_\tau \mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})_{\mathrm{inj}}$ *that is a sectionwise weak equivalence.*
(4) *For every local trivial fibration* $U \to V$*, the induced map* $\mathrm{RMap}(V, F) \to \mathrm{RMap}(U, F)$ *(computed in* $\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})_{\mathrm{inj}}$*) is an isomorphism in* $\mathcal{H} = \mathrm{Ho}(\mathsf{s}\mathcal{S}\mathsf{et})$.
(5) *The associated sheaf* $aF = F^\sharp$ *satisfies descent for all hypercovers.*
(6) *There exists a sectionwise weak equivalence* $\iota : F \to I$ *with* I *fibrant in* $\mathrm{L}_\tau \mathsf{s}\mathcal{S}\mathsf{hv}(\mathcal{C})_{\mathrm{inj}} = \mathsf{s}\mathcal{S}\mathsf{hv}(\mathcal{C})_J$.
(7) *Any local weak equivalence* $F \to I$ *with* I *fibrant in* $\mathrm{L}_\tau \mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})_{\mathrm{inj}} = \mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})_J$ *is a sectionwise weak equivalence.*

Proof. (1) $\Leftrightarrow$ (2): this follows easily from the previous theorem, which says that a presheaf of Kan complexes $\hat{F}$ is fibrant in $\mathrm{L}_\tau \mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})_{\mathrm{proj}}$ iff the map $\hat{F}(X) \to \operatorname{holim}_{\mathbf{\Delta}} \hat{F}(U_\bullet)$ is a weak equivalence in $\mathsf{s}\mathcal{S}\mathsf{et}$.

(2) $\Leftrightarrow$ (3): just note that j' exists and we can take $j = j'$ (since a fibrant replacement in $\mathrm{L}_\tau \mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})_{\mathrm{inj}}$ is one in $\mathrm{L}_\tau \mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})_{\mathrm{proj}}$).

(3) $\Rightarrow$ (4): this is a consequence of Proposition 2.6.7, as $\mathrm{RMap}(U, F) \cong \mathrm{Map}(U, \hat{F}')$ (as j' is a sectionwise weak equivalence, $\hat{F}'$ is also a fibrant replacement of F in $\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})_{\mathrm{inj}}$; we have used that every object in $\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})_{\mathrm{inj}}$ is cofibrant).

(4) $\Rightarrow$ (1): in light of the previous remark, this is obvious.

The equivalence of (5) with (1) $\sim$ (4) follows from the equivalence (1) $\Leftrightarrow$ (4).

Clearly (6) $\Rightarrow$ (3) $\Rightarrow$ (2). Now for (3) $\Rightarrow$ (6). Let $a\hat{F}' = \hat{F}'^\sharp \to I$ be a fibrant replacement in $\mathrm{L}_\tau \mathsf{s}\mathcal{S}\mathsf{hv}(\mathcal{C})_{\mathrm{inj}} = \mathsf{s}\mathcal{S}\mathsf{hv}(\mathcal{C})_J$, then the composite $\iota : F \xrightarrow{j'} \hat{F}' \to a\hat{F}' = \hat{F}'^\sharp \to I$ is also a fibrant replacement in $\mathrm{L}_\tau \mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})_{\mathrm{inj}}$ hence is a sectionwise weak equivalence by (3).

The equivalence of (7) with the others is then clear, in light of Proposition 2.6.5. □

In the following we write $\mathrm{c}_* Z \in \mathsf{cs}\mathcal{S}\mathsf{et}$ for the corresponding constant cosimplicial simplicial set, for $Z \in \mathsf{s}\mathcal{S}\mathsf{et}$.

Definition 2.7.14 ([**26**, Definition 7.2])**.** We say that a map $f : F \to G$ in $\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})$ *satisfies descent* for a hypercover $U \to X$ if there is a fibrant replacement $\hat{f} : \hat{F} \to \hat{G}$ of f in $\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})_{\mathrm{proj}}$ such that the map $F(X) \to \operatorname{holim}_{\mathbf{\Delta}} \mathrm{c}_* G(X) \times_{\hat{G}(U)} \hat{F}(U)$ is a weak equivalence in $\mathsf{s}\mathcal{S}\mathsf{et}$. In other words, this says that the diagram

$$\begin{array}{ccc} F(X) & \longrightarrow & \operatorname{holim}_{\mathbf{\Delta}} \hat{F}(U) \\ {\scriptstyle f_X}\downarrow & & \downarrow \\ G(X) & \longrightarrow & \operatorname{holim}_{\mathbf{\Delta}} \hat{G}(U) \end{array}$$

is homotopy cartesian in $\mathsf{s}\mathcal{S}\mathsf{et}$ ([**26**, Remark 7.3]).

Theorem 2.7.15 ([**26**, Theorem 7.8])**.** *Let* S *be a class of hypercovers which contains a* set *that is dense. Let* $f : F \to G$ *be a map in* $\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})$.

- f *is a fibration in* $\mathrm{L}_\tau \mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})_{\mathrm{proj}}$ *iff* f *is a fibration in* $\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})_{\mathrm{proj}}$ *(i.e.* f *is a sectionwise fibration of simplicial sets) and it satisfies descent for all hypercovers in* S.[8]
- f *is a fibration in* $\mathrm{L}_\tau \mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})_{\mathrm{inj}}$ *iff* f *is a fibration in* $\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})_{\mathrm{inj}}$ *and it satisfies descent for all hypercovers in* S.

Remark 2.7.16. These results equally hold for $\mathsf{s}\mathcal{S}\mathsf{hv}(\mathcal{C}, \tau)$, which we omit.

From these results, we see that for the projective version of the model structure, the fibrants and the fibrations are easier to describe than the injective version, at least one reason is that the fibrations in $\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})_{\mathrm{proj}}$ are just sectionwise fibrations, while those in $\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})_{\mathrm{inj}}$ are not easy to characterize.

2.8. Torsors and their classifications

In this section, we discuss the notion of torsors and some basic results about torsors.

Definition 2.8.1. Let $\mathcal{C} = (\mathcal{C}, \tau)$ be a site. Let G be a sheaf of (possibly non-commutative) groups on $\mathcal{C}$. A *(left) pseudo torsor*, or more precisely a *pseudo G-torsor*, is a sheaf of sets F on $\mathcal{C}$ endowed with an action $a : G \times F \to F$ such that

- whenever $F(U)$ is nonempty the action $G(U) \times F(U) \to F(U)$ is simply transitive (i.e. free and transitive).

This is equivalent to the following condition:

- the morphism of sheaves $G \times F \to F \times F$ given on local sections by $(g, y) \mapsto (y, a(g, y))$ is an isomorphism.

A *morphism of pseudo G-torsors* $F \to F'$ is simply a morphism of sheaves of sets which is G-equivariant.

A *torsor*, more precisely a *G-torsor*, is a pseudo G-torsor such that

- for every $U \in \mathrm{Ob}(\mathcal{C})$ there exists a covering family $\{U_i \to U\}_{i \in I}$ of U such that $F(U_i)$ is nonempty for all $i \in I$.

This is equivalent to:

- there exists a local epimorphism $\coprod_{i \in I} h_{U_i} \to *$ in $\mathcal{P}\mathsf{re}(\mathcal{C})$ such that $F(U_i) \neq \varnothing$ for all $i \in I$.

(Indeed, if so, then for any $U \in \mathrm{Ob}(\mathcal{C})$, let $X = h_U \times \coprod_{i \in I} h_{U_i}$, then the map $X \to U$ is a local epimorphism. Composing with the sectionwise surjection $\coprod_{V \in \mathcal{C}_X} h_V \to X$ we obtain a local epimorphism $\coprod h_V \to U$. By assumption, there is a morphism $\coprod_{i \in I} h_{U_i} \to F$, we thus get a morphism $\coprod_{V \in \mathcal{C}_X} h_V \to X \to \coprod_{i \in I} h_{U_i} \to F$, so $\prod_{V \in \mathcal{C}_X} F(V) = \mathcal{P}\mathsf{re}(\mathcal{C})(\coprod_{V \in \mathcal{C}_X} h_V, F) \neq \varnothing$. The converse is obvious since $\coprod_{U \in \mathrm{ob}(\mathcal{C})} h_{U_i} \to *$ is a local epimorphism.)

The *trivial G-torsor* is the sheaf G endowed with the natural left G-action.

[8] To be compared with Proposition 1.4.17.

A *morphism of G-torsors* is just a morphism of pseudo G-torsors. Any morphism of torsors is automatically an isomorphism. We denote by $\mathcal{T}\mathsf{ors}_{(\mathcal{C},\tau)}(G)$ or $\mathcal{T}\mathsf{ors}_{\mathcal{C}}(G)$ (or $\mathcal{T}\mathsf{ors}_{\tau}(G)$) the category of G-torsors on the site $(\mathcal{C},\tau)$, which is a groupoid.

For any $U \in \mathrm{Ob}(\mathcal{C})$, a (pseudo) G-torsor F gives a (pseudo) $G|_U$-torsor $F|_U$ on $(\mathcal{C}\downarrow U) = (\mathcal{C}/U)$ (or briefly, on U) by restriction. So the restriction map $\mathcal{S}\mathsf{hv}(\mathcal{C}) \to \mathcal{S}\mathsf{hv}(\mathcal{C}/U)$ yields a functor $\mathcal{T}\mathsf{ors}_{\mathcal{C}}(G) \to \mathcal{T}\mathsf{ors}_{\mathcal{C}/U}(G|_U)$.

Using the isomorphisms of sheaves $\tilde{\pi}_0(\mathrm{E}G \times_G F) \cong F/G$ and $\tilde{\pi}_i(\mathrm{E}G \times_G F) \cong *$ for $i > 0$, we get the following homotopical characterization of torsors.

Proposition 2.8.2 ([**50**, Lemma 9.1])**.** *Let $\mathcal{C}$ be a site. Let G be a sheaf of groups on $\mathcal{C}$ and let F be a sheaf of sets on $\mathcal{C}$ endowed with an action $G \times F \to F$. Under these data, the following are equivalent:*

(1) *F is a G-torsor.*
(2) *The action $G \times F \to F$ is free and the canonical map $F/G \to *$ is an isomorphism of sheaves.*
(3) *The simplicial sheaf map $\mathrm{E}G \times_G F \to *$ is a local weak equivalence.*

Lemma 2.8.3. *Let $\mathcal{C}$ be a site. Let G be a sheaf of groups on $\mathcal{C}$. A G-torsor F is trivial if and only if the set of global sections of F, $\Gamma(\mathcal{C}, F) \neq \varnothing$.*

For an object $U \in \mathcal{C}$, F trivializes over U iff $F(U) \neq \varnothing$.

The following is reproduced from *First cohomology and torsors* in the Stacks Project [**96**].

Lemma 2.8.4. *Let $\mathcal{C}$ be a site. Let A be an abelian sheaf on $\mathcal{C}$. There is a canonical bijection between the set of isomorphism classes of A-torsors and the first cohomology group $\mathrm{H}^1(\mathcal{C}; A)$.*

Proof. Let F be an A-torsor. Consider the free abelian sheaf $\mathbb{Z}[F]$ on F. It is the sheafification of the rule which associates to $U \in \mathrm{Ob}(\mathcal{C})$ the collection of finite formal sums $\sum n_i[s_i]$ with $n_i \in \mathbb{Z}$ and $s_i \in F(U)$. There is a natural map

$$\sigma : \mathbb{Z}[F] \longrightarrow \underline{\mathbb{Z}}$$

which to a local section $\sum n_i[s_i]$ associates $\sum n_i$. The kernel of σ is generated by sections of the form $[s] - [s']$. There is a canonical map $a : \ker(\sigma) \to A$ which maps $[s] - [s'] \mapsto h$ where h is the local section of A such that $h \cdot s = s'$. Consider the pushout diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ker(\sigma) & \longrightarrow & \mathbb{Z}[F] & \longrightarrow & \underline{\mathbb{Z}} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle a} & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A & \longrightarrow & \mathscr{E} & \longrightarrow & \underline{\mathbb{Z}} & \longrightarrow & 0 \end{array}$$

Here $\mathscr{E}$ is the extension obtained by pushout. From the long exact cohomology sequence associated to the lower short exact sequence we obtain an element $\xi = \xi_F \in \mathrm{H}^1(\mathcal{C}; A)$ by applying the boundary operator to $1 \in \mathrm{H}^0(\mathcal{C}; \underline{\mathbb{Z}})$.

Conversely, given $\xi \in \mathrm{H}^1(\mathcal{C}; A)$ we can associate to ξ a torsor as follows. Choose an embedding $A \to \mathscr{I}$ of A into an injective abelian sheaf $\mathscr{I}$. We set $\mathscr{Q} = \mathscr{I}/A$ so that we have a short exact sequence

$$0 \longrightarrow A \longrightarrow \mathscr{I} \longrightarrow \mathscr{Q} \longrightarrow 0$$

The element ξ is the image of a global section $q \in \mathrm{H}^0(\mathcal{C}; \mathscr{Q})$ because $\mathrm{H}^1(\mathcal{C}; \mathscr{I}) = 0$. Let $F \subset \mathscr{I}$ be the subsheaf (of sets) of sections that map to q in the sheaf $\mathscr{Q}$. It is easy to verify that F is an A-torsor.

One may check that the above two constructions are inverse to each other. □

By definition, the pointed set $\pi_0(\mathcal{T}\mathsf{ors}_{\mathcal{C}}(G))$ is the set of isomorphism classes of G-torsors, denoted $\mathrm{H}^1(\mathcal{C}; G)$. The following result describes $\mathrm{H}^1(\mathcal{C}; G)$ in terms of homotopy classes in the Jardine injective model structure on $\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})$ (Section 2.6).

Theorem 2.8.5 ([**50**, Theorem 9.8])**.** *Let $(\mathcal{C},\tau)$ be a site, let $G \in \mathcal{S}\mathsf{hv}\mathcal{G}\mathsf{r}(\mathcal{C},\tau)$, then there is a bijection*

$$\mathrm{H}^1(\mathcal{C}; G) \cong [*, \mathrm{B}G].$$

There is a presheaf of groupoids $\mathbf{Tors}^G_{\mathcal{C}} \in \mathcal{P}\mathsf{re}\mathcal{G}\mathsf{rpd}(\mathcal{C})$ given by $\mathbf{Tors}^G_{\mathcal{C}}(U) := \mathcal{T}\mathsf{ors}_{\mathcal{C}/U}(G|_U)$ and a map $j : G \to \mathbf{Tors}^G_{\mathcal{C}}$ such that for any object $x \in G(U)$, $j(U)(x)$ is the *trivial* $G|_U$-torsor $\hom_{G|_U}(-, x)$ on $\mathcal{C}/U$.

We fix a Grothendieck site $(\mathcal{C},\tau)$. We discuss briefly the homotopy theory of $\mathcal{P}\mathsf{re}\mathcal{G}\mathsf{rpd}(\mathcal{C})$ and $\mathcal{S}\mathsf{hv}\mathcal{G}\mathsf{rpd}(\mathcal{C})$, the category of (pre)sheaves of groupoids on $\mathcal{C}$.

We say that a map $f : G \to H$ in $\mathcal{P}\mathsf{re}\mathcal{G}\mathsf{rpd}(\mathcal{C})$ is a *local weak equivalence* (resp. *(Jardine) injective fibration*) if the induced map $f_* : \mathrm{B}G \to \mathrm{B}H$ is a local weak equivalence (resp. (Jardine) injective fibration) in $\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})_J$. A map $f : G \to H$ in $\mathcal{P}\mathsf{re}\mathcal{G}\mathsf{rpd}(\mathcal{C})$ is a *(Jardine) cofibration* if it has the left lifting property with respect to all trivial (Jardine) injective fibrations.

We have the following adjunction

$$\pi : \mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C}) \rightleftarrows \mathcal{P}\mathsf{re}\mathcal{G}\mathsf{rpd}(\mathcal{C}) : \mathrm{B},$$

where the left adjoint is the fundamental groupoid functor.

The category $\mathcal{P}\mathsf{re}\mathcal{G}\mathsf{rpd}(\mathcal{C})$ is a simplicial category, with simplicial mapping spaces given by

$$\mathrm{Map}(G, H)_n := \mathcal{P}\mathsf{re}\mathcal{G}\mathsf{rpd}(\mathcal{C})(G \times \pi(\Delta^n), H).$$

There is a natural isomorphism

$$\mathrm{Map}(G, H) \cong \mathrm{Map}(\mathrm{B}G, \mathrm{B}H)$$

given by sending a simplex $\phi : G \times [n] \to H$ to the composite

$$\mathrm{B}G \times \Delta^n \xrightarrow{1\times\eta} \mathrm{B}G \times \mathrm{B}\pi(\Delta^n) \cong \mathrm{B}(G \times \pi(\Delta^n)) \xrightarrow{\phi_*} \mathrm{B}H.$$

Same definitions and constructions apply to the category $\mathsf{Shv}\mathcal{G}\mathsf{rpd}(\mathcal{C})$.

Proposition 2.8.6 ([**50**, Lemma 9.18])**.** *The functor $\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C}) \to \mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C}), X \mapsto \mathrm{B}\pi X$ preserves local weak equivalences and for any $G \in \mathcal{P}\mathsf{re}\mathcal{G}\mathsf{rpd}(\mathcal{C})$, the counit map $\pi\mathrm{B}G \to G$ is an isomorphism.*

Proposition 2.8.7 ([**50**, Proposition 9.19, 9.20])**.** *With the above definitions and constructions, the categories $\mathcal{P}\mathsf{re}\mathcal{G}\mathsf{rpd}(\mathcal{C})$ and $\mathcal{S}\mathsf{hv}\mathcal{G}\mathsf{rpd}(\mathcal{C})$ become cofibrantly generated proper simplicial model categories, and the adjunction*

$$a : \mathcal{P}\mathsf{re}\mathcal{G}\mathsf{rpd}(\mathcal{C}) \rightleftarrows \mathcal{S}\mathsf{hv}\mathcal{G}\mathsf{rpd}(\mathcal{C}) : \iota$$

is a Quillen equivalence.

The adjunction

$$\pi : \mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})_J \rightleftarrows \mathcal{P}\mathsf{re}\mathcal{G}\mathsf{rpd}(\mathcal{C}) : \mathrm{B}$$

is a Quillen pair.

For the following result, note that any $G \in \mathcal{P}\mathsf{re}\mathcal{G}\mathsf{rpd}(\mathcal{C})$ gives rise to two presheaves $\mathrm{Ob}(G), \mathrm{Mor}(G) \in \mathcal{P}\mathsf{re}(\mathcal{C})$ by the composites $\mathcal{C}^{\mathrm{op}} \xrightarrow{G} \mathcal{G}\mathsf{rpd} \underset{\mathrm{Mor}}{\overset{\mathrm{Ob}}{\rightrightarrows}} \mathcal{S}\mathsf{et}$. They are sheaves if $G \in \mathcal{S}\mathsf{hv}\mathcal{G}\mathsf{rpd}(\mathcal{C})$.

Proposition 2.8.8 ([**50**, Corollary 9.22])**.** *A map $f : G \to H$ in $\mathcal{P}\mathsf{re}\mathcal{G}\mathsf{rpd}(\mathcal{C})$ is a local weak equivalence iff the canonical map*

$$\mathrm{Mor}(G) \to (\mathrm{Ob}(G) \times \mathrm{Ob}(G)) \times_{(\mathrm{Ob}(H)\times\mathrm{Ob}(H))} \mathrm{Mor}(H)$$

is a local isomorphism and

$$\pi_0(G) \to \pi_0(H)$$

is a local epimorphism (or a local isomorphism).

A sheaf of groupoids H is called a *stack* if it satisfies *descent* for the above injective model structure on $\mathcal{S}\mathsf{hv}\mathcal{G}\mathsf{rpd}(\mathcal{C})$, namely if every injective fibrant model $j : H \to H'$ is a sectionwise weak equivalence.

If $j : H \to H'$ is a fibrant model of $H \in \mathcal{S}\mathsf{hv}\mathcal{G}\mathsf{rpd}(\mathcal{C})$, then $j_* : \mathrm{B}H \to \mathrm{B}H'$ is a fibrant model of $\mathrm{B}H \in \mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})_J$. So $H \in \mathcal{S}\mathsf{hv}\mathcal{G}\mathsf{rpd}(\mathcal{C})$ is a stack iff $\mathrm{B}H \in \mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})_J$ satisfies descent.

For any $G \in \mathcal{S}\mathsf{hv}\mathcal{G}\mathsf{rpd}(\mathcal{C})$, we call any of its fibrant model $j : G \to H$ a *stack completion* or *associated stack* of G (since H is a stack). Similar notions also apply for presheaves of groupoids. In this way, stacks are identified with homotopy types of (pre)sheaves of groupoids.

Theorem 2.8.9. *Let $(\mathcal{C}, \tau)$ be a site, let $G \in \mathcal{S}\mathsf{hv}\mathcal{G}\mathsf{r}(\mathcal{C}, \tau)$.*

1. *The induced map $\mathrm{B}G \xrightarrow{j_*} \mathrm{B}(\mathbf{Tors}^G_{\mathcal{C}})$ in $\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})_J$ is a local weak equivalence and $\mathrm{B}(\mathbf{Tors}^G_{\mathcal{C}})$ satisfies descent.*
2. *The map $j : G \to \mathbf{Tors}^G_{\mathcal{C}}$ is a local weak equivalence in $\mathcal{P}\mathsf{re}\mathcal{G}\mathsf{rpd}(\mathcal{C})$ and $\mathbf{Tors}^G_{\mathcal{C}} \in \mathcal{P}\mathsf{re}\mathcal{G}\mathsf{rpd}(\mathcal{C})$ (given by $\mathbf{Tors}^G_{\mathcal{C}}(U) = \mathcal{T}\mathsf{ors}_{\mathcal{C}/U}(G|_U)$) is a stack (i.e. it satisfies descent).*
3. *For any object $U \in \mathcal{C}$ and any object $u \in \mathbf{Tors}^G_{\mathcal{C}}(U) = \mathcal{T}\mathsf{ors}_{\mathcal{C}/U}(G|_U)$ of the groupoid $\mathcal{T}\mathsf{ors}_{\mathcal{C}/U}(G|_U)$, we have*

$$\begin{aligned} &\mathrm{RMap}(U, \mathrm{B}G) \simeq \mathrm{B}(\mathbf{Tors}^G_{\mathcal{C}})(U) = \mathrm{B}\mathcal{T}\mathsf{ors}_{\mathcal{C}/U}(G|_U) \in \mathsf{s}\mathcal{S}\mathsf{et}, \\ &\pi\mathrm{RMap}(U, \mathrm{B}G) \cong \mathbf{Tors}^G_{\mathcal{C}}(U) = \mathcal{T}\mathsf{ors}_{\mathcal{C}/U}(G|_U) \in \mathcal{G}\mathsf{rpd}, \\ &\pi_1(\mathrm{RMap}(U, \mathrm{B}G), u) \cong \mathrm{Aut}_{\mathcal{T}\mathsf{ors}_{\mathcal{C}/U}(G|_U)}(u) \in \mathcal{G}\mathsf{r}. \end{aligned} \tag{2.5}$$

There is a canonical sectionwise weak equivalence

$$\mathbf{R}\Omega_s\mathrm{B}G \simeq G. \tag{2.6}$$

Choose the trivial $G|_U$-torsor $0^G_U := G|_U$ as canonical base point of $\mathrm{RMap}(U, \mathrm{B}G) \simeq \mathrm{B}\mathcal{T}\mathsf{ors}_{\mathcal{C}/U}(G|_U) \in \mathsf{s}\mathcal{S}\mathsf{et}$. Then

$$\pi_n(\mathrm{RMap}(U, \mathrm{B}G), 0^G_U) \cong \begin{cases} \mathrm{H}^1(U; G), & n = 0; \\ G(U), & n = 1; \\ 0, & n > 1. \end{cases} \tag{2.7}$$

Proof. The statements (1) and (2) are [**50**, Proposition 9.26 and Corollary 9.27] (in fact we are in a special case, Jardine treats the more general setting with $G \in \mathcal{S}\mathsf{hv}\mathcal{G}\mathsf{rpd}(\mathcal{C})$); see also [**13**, Lemma 2.3.2]. The first two relations of (3) follow from (1) and the fact that $\pi\mathrm{B}H \cong H$ for any $H \in \mathcal{P}\mathsf{re}\mathcal{G}\mathsf{rpd}(\mathcal{C})$ (we take $H = \mathbf{Tors}^G_{\mathcal{C}}$ here); the third follows from the second.

For the sectionwise weak equivalence $\mathbf{R}\Omega_s\mathrm{B}G \simeq G$, we note that by the third relation above, $\pi_1(\mathrm{RMap}(U, \mathrm{B}G), 0^G_U) \cong \mathrm{Aut}_{\mathcal{T}\mathsf{ors}_{\mathcal{C}/U}(G|_U)}(0^G_U) = \mathrm{Aut}_{\mathcal{T}\mathsf{ors}_{\mathcal{C}/U}(G|_U)}(G|_U) \cong G(U)$ (the last isomorphism is given by mapping $a \in G(U)$ to the automorphism of $0^G_U = G|_U$ given by $x \mapsto xa^{-1}$) and so $\mathbf{R}\Omega_s\mathrm{B}G \simeq \Omega_s\mathrm{B}(\mathbf{Tors}^G_{\mathcal{C}})$ is given by

$$U \mapsto \Omega_s\mathrm{B}(\mathbf{Tors}^G_{\mathcal{C}})(U) \simeq \Omega_s C_{0^G_U}\mathrm{B}(\mathbf{Tors}^G_{\mathcal{C}})(U) \simeq \Omega_s\mathrm{B}\,\mathrm{Aut}_{\mathcal{T}\mathsf{ors}_{\mathcal{C}/U}(G|_U)}(0^G_U) \cong \mathrm{Aut}_{\mathcal{T}\mathsf{ors}_{\mathcal{C}/U}(G|_U)}(0^G_U) \cong G(U),$$

where $C_{0^G_U}\mathrm{B}(\mathbf{Tors}^G_{\mathcal{C}})(U)$ is the component of 0^G_U in $\mathrm{B}(\mathbf{Tors}^G_{\mathcal{C}})(U)$ and we have used the equivalence of groupoids in Remark 1.8.4, giving the weak equivalence $C_{0^G_U}\mathrm{B}(\mathbf{Tors}^G_{\mathcal{C}})(U) \simeq \mathrm{B}\,\mathrm{Aut}_{\mathcal{T}\mathsf{ors}_{\mathcal{C}/U}(G|_U)}(0^G_U)$.

The last formula about $\pi_n(\mathrm{RMap}(U, \mathrm{B}G), 0^G_U)$ is then clear. □

Remark 2.8.10. For a general torsor $u \in \mathbf{Tors}^G_{\mathcal{C}}(U) = \mathcal{T}\mathsf{ors}_{\mathcal{C}/U}(G|_U)$, the automorphism group $\mathrm{Aut}_{\mathcal{T}\mathsf{ors}_{\mathcal{C}/U}(G|_U)}(u)$ is usually different from $G(U)$. The argument above doesn't work, as we can't multiply on the right by an element of a section of that torsor. The reason is essentially that a trivial G-torsor is a (G,G)-bitorsor.

2.9. Homotopical interpretation of sheaf cohomology

Fix a site $\mathcal{C} = (\mathcal{C}, \tau)$, and let $M \in \mathcal{P}\mathsf{re}\mathcal{A}\mathsf{b}(\mathcal{C}), n \in \mathbb{N}$. The associated sheaf $aM = M^\sharp = \mathrm{K}(M^\sharp, 0)$ is fibrant in $\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})$. For $X \in \mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C}), Y = (Y, y) \in \mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})_*$, we define the *$n$-th cohomology of X with coefficients in M* to be

$$\mathrm{H}^n(X; M) = \mathrm{H}^n_\tau(X; M) := [X, \mathrm{K}(M, n)]_{\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})}$$

and the *reduced cohomology of Y with coefficients in M* to be

$$\widetilde{\mathrm{H}}^n(Y; M) = \widetilde{\mathrm{H}}^n_\tau(Y; M) := [Y, \mathrm{K}(M, n)]_{\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})_*},$$

where we are using Jardine's local model structure on $\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})$ and $\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})_*$. So $\widetilde{\mathrm{H}}^n(\Sigma Y; M) = \widetilde{\mathrm{H}}^{n-1}(Y; M)$ and for $X \in \mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})$, we have $\mathrm{H}^n(X; M) = \widetilde{\mathrm{H}}^n(X_+; M)$. Note that since there is a local weak equivalence $\mathrm{K}(M, n) \to \mathrm{K}(M^\sharp, n)$, it doesn't matter if we replace M by $M^\sharp$ everywhere. For a proof of the fact that the cohomology so defined coincides with usual sheaf cohomology, see [**50**, Theorem 8.26]. If $U, X \in \mathcal{C}, (X \to U) \in (\mathcal{C}/U)$, then $\mathrm{H}^n_{\tau|_U}((X \to U); M|_U) \cong \mathrm{H}^n_\tau(X; M)$ (the LHS is computed for $(\mathcal{C}/U)$ with the induced topology $\tau|_U$); cf. [**88**, Chapter I, Corollary (3.8.2)].

When we write $\mathrm{H}^n(Y; M)$, we will mean the (non-reduced) cohomology of Y (forgetting the base point).

If $X \subset Z$, we define $\mathrm{H}^n(Z, X; M) = \mathrm{H}^n_\tau(Z, X; M)$, the *cohomology of the pair (Z, X) with coefficients in M*, to be the reduced cohomology of Z/X (note that it is naturally pointed), the (homotopy) cofiber of the inclusion $X \hookrightarrow Z$:

$$\mathrm{H}^n(Z, X; M) := \widetilde{\mathrm{H}}^n(Z/X; M) = [Z/X, \mathrm{K}(M, n)]_{\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})_*}.$$

So $\mathrm{H}^n(X, \varnothing; M) = \widetilde{\mathrm{H}}^n(X_+; M) = \mathrm{H}^n(X; M)$ and $\widetilde{\mathrm{H}}^n(Y; M) = \mathrm{H}^n(Y, *; M)$. All these are abelian groups.

We say that the inclusion $i : Y \hookrightarrow Z$ *splits* if there exists a map $p : Z \to Y$ such that $pi = \mathrm{id}_Y$.

The next result, familiar from classical topology, follows easily by repeatedly applying the general abstract formalism in Theorem 1.3.3 (or [**45**, Chapter 6]). The long exact sequence for cohomology of a pair follows from that for the triple $\varnothing \subset Y \subset Z$.

Theorem 2.9.1 (Long exact sequence for cohomology of a triple)**.** *Given simplicial presheaves $X \subset Y \subset Z$, we have the long exact sequence of cohomology groups*

$$0 \to \mathrm{H}^0(Z, Y; M) \to \mathrm{H}^0(Z, X; M) \to \mathrm{H}^0(Y, X; M) \to \mathrm{H}^1(Z, Y; M) \to \cdots$$

$$\cdots \to \mathrm{H}^n(Z, Y; M) \to \mathrm{H}^n(Z, X; M) \to \mathrm{H}^n(Y, X; M) \to \mathrm{H}^{n+1}(Z, Y; M) \to \cdots,$$

natural in the triple $X \subset Y \subset Z$.

In particular, there is the long exact sequence for cohomology of a pair *(Z, Y):*

$$0 \to \mathrm{H}^0(Z, Y; M) \to \mathrm{H}^0(Z; M) \to \mathrm{H}^0(Y; M) \to \mathrm{H}^1(Z, Y; M) \to \cdots$$

$$\cdots \to \mathrm{H}^n(Z, Y; M) \to \mathrm{H}^n(Z; M) \to \mathrm{H}^n(Y; M) \to \mathrm{H}^{n+1}(Z, Y; M) \to \cdots.$$

If the inclusion $Y \hookrightarrow Z$ splits, then the long exact sequence reduces to split short exact sequences

$$0 \to \mathrm{H}^n(Z, Y; M) \to \mathrm{H}^n(Z; M) \to \mathrm{H}^n(Y; M) \to 0, n \geqslant 0.$$

Applying the split case to the inclusion $y : * \hookrightarrow Y$ for a pointed simplicial presheaf $Y \in \mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})_*$ and noting that $\mathrm{H}^0(*; M) = \mathrm{H}^0(*; M^\sharp) = M^\sharp(*)$ is the abelian group of global sections of the sheaf $M^\sharp$ and $\mathrm{H}^n(*; M) = \mathrm{H}^n(\mathcal{C}; M)$ for $n \geqslant 1$, we get the following relation, as in classical algebraic topology.

Corollary 2.9.2. *There are split short exact sequences*

$$0 \to \widetilde{\mathrm{H}}^0(Y; M) \to \mathrm{H}^0(Y; M) \to M^\sharp(*) \to 0,$$

and

$$0 \to \widetilde{\mathrm{H}}^n(Y; M) \to \mathrm{H}^n(Y; M) \to \mathrm{H}^n(\mathcal{C}; M) \to 0$$

for $n \geqslant 1$. These sequences are natural in $Y \in \mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})_*$ and $M \in \mathcal{P}\mathsf{re}\mathcal{A}\mathsf{b}(\mathcal{C})$.

The following result appears in [**88**, Chapter I, Propositions (3.4.2) and (3.4.3)].

Theorem 2.9.3. *The inclusion functor $\iota : \mathcal{S}\mathsf{hv}\mathcal{A}\mathsf{b}(\mathcal{C}) \to \mathcal{P}\mathsf{re}\mathcal{A}\mathsf{b}(\mathcal{C})$ is left exact, and its higher derived functor $\mathbf{R}^q\iota = \mathcal{H}^q : \mathcal{S}\mathsf{hv}\mathcal{A}\mathsf{b}(\mathcal{C}) \to \mathcal{P}\mathsf{re}\mathcal{A}\mathsf{b}(\mathcal{C})$ is given by*

$$\mathcal{H}^q(A)(U) = \mathrm{H}^q(U; A), A \in \mathcal{S}\mathsf{hv}\mathcal{A}\mathsf{b}(\mathcal{C}), U \in \mathcal{C}.$$

Moreover, $a(\mathcal{H}^q(A)) = \mathcal{H}^q(A)^+ = 0$ if $q > 0$.

Some authors call a sheaf $A \in \mathcal{S}\mathsf{hv}\mathcal{A}\mathsf{b}(\mathcal{C})$ *flabby* if $\mathcal{H}^q(A) = 0$ for all $q > 0$ ([**88**, Chapter I, Corollary (3.5.3)]). The direct image functor of a site morphism sends a flabby sheaf to a flabby sheaf ([**88**, Chapter I, Proposition (3.7.2)]). See [**88**, pp. 70-73] for some interesting results on some incarnations of the *Leray spectral sequence*.

The following comparison result is taken from [**88**, Chapter I, Theorem (3.9.1) and Corollary (3.9.3)].

Theorem 2.9.4 (Comparison lemma)**.** *Let $i : \mathcal{C}' \hookrightarrow \mathcal{C}$ be a full subcategory with a topology τ' (hence we have a site $(\mathcal{C}', \tau')$). Assume*

- *For any τ-cover $R = \{(V_i \xrightarrow{\varphi_i} U) : i \in I\}$ of U, if $V_i, U \in \mathcal{C}'$, then R is a τ'-cover.*
- *For any $U \in \mathcal{C}$, there exists a τ-cover $R = \{(V_i \xrightarrow{\varphi_i} U) : i \in I\}$ with $V_i \in \mathcal{C}'$.*

Then the adjunctions

$$i^* : \mathcal{Shv}(\mathcal{C}', \tau') \rightleftarrows \mathcal{Shv}(\mathcal{C}, \tau) : i_*, i^* : \mathcal{Shv}\mathcal{Ab}(\mathcal{C}', \tau') \rightleftarrows \mathcal{Shv}\mathcal{Ab}(\mathcal{C}, \tau) : i_*$$

are pairs of quasi-inverse equivalences.

Moreover, for any $U \in \mathcal{C}', A \in \mathcal{Shv}\mathcal{Ab}(\mathcal{C}, \tau), A' \in \mathcal{Shv}\mathcal{Ab}(\mathcal{C}', \tau')$*, we have functorial isomorphisms*

$$\mathrm{H}^n_{\tau'}(U; i_*A) \cong \mathrm{H}^n_{\tau}(U; A), \mathrm{H}^n_{\tau'}(U; A') \cong \mathrm{H}^n_{\tau}(U; i^*A').$$

See also [**88**, Chapter I, Proposition (3.10.2)] for an interesting comparison result on noetherian topologies.

For a noetherian site $(\mathcal{C}, \tau)$ and $U \in \mathcal{C}$, the functor $\mathrm{H}^n_\tau(U; -) : \mathcal{Shv}\mathcal{Ab}(\mathcal{C}, \tau) \to \mathcal{Ab}$ commutes with pseudofiltered colimits and all small direct sums ([**88**, Chapter I, Theorem (3.11.1) and Corollary (3.11.2)]); pseudofiltered colimits of sheaves in the sheaf category are also colimits in the presheaf category.

Theorem 2.9.5 (Leray spectral sequences)**.** *Let* $(\mathcal{E}, \tau_\mathcal{E}) \xrightarrow{g} (\mathcal{D}, \tau_\mathcal{D}) \xrightarrow{f} (\mathcal{C}, \tau_\mathcal{C})$ *be site morphisms.*

(1) *For any* $A \in \mathcal{Shv}\mathcal{Ab}(\mathcal{C}, \tau_\mathcal{C})$*, we have the biregular* Leray spectral sequence *in* $\mathcal{Shv}\mathcal{Ab}(\mathcal{E}, \tau_\mathcal{E})$

$$\mathrm{E}_2^{ij} = \mathbf{R}^i g_*(\mathbf{R}^j f_*(A)) \Longrightarrow \mathbf{R}^{i+j}(gf)_*(A). \tag{2.8}$$

(2) *For any* $A \in \mathcal{Shv}\mathcal{Ab}(\mathcal{C}, \tau_\mathcal{C}), U \in \mathcal{D}$*, we have the biregular* Leray spectral sequence *in* $\mathcal{Ab}$

$$\mathrm{E}_2^{ij} = \mathrm{H}^i_{\tau_\mathcal{D}}(U; \mathbf{R}^j f_*(A)) \Longrightarrow \mathrm{H}^{i+j}_{\tau_\mathcal{C}}(f(U); A). \tag{2.9}$$

If R is a presheaf of commutative rings on the site $(\mathcal{C}, \tau)$, then we denote by $\mathcal{Ch}(R)$ the category of unbounded (homological) chain complexes of presheaves of R-modules; let $\mathcal{Ch}_+(R)$ be its full subcategory consisting of those complexes all of whose negative-degree parts are 0 and denote the inclusion by $\tau^* : \mathcal{Ch}_+(R) \hookrightarrow \mathcal{Ch}(R)$. If $R = \mathbb{Z}$, we write $\mathcal{Ch}(\mathbb{Z}) = \mathcal{Ch}, \mathcal{Ch}_+(\mathbb{Z}) = \mathcal{Ch}_+$. For any $D \in \mathcal{Ch}(R), n \in \mathbb{Z}$, we define the *shift* $D[n]$ by letting $D[n]_i = D_{n+i}$ and the *good truncation* $\tau(D) \in \mathcal{Ch}_+(R)$ by letting $\tau(D)_0 := \ker(D_0 \xrightarrow{\partial_0} D_{-1}), \tau(D)_i := D_i$ for $i > 0$. We obtain an adjunction

$$\tau^* : \mathcal{Ch}_+(R) \rightleftarrows \mathcal{Ch}(R) : \tau$$

with $\tau\tau^* = \mathrm{id}_{\mathcal{Ch}_+(R)}$. See [**50**, §8.2] for more on related constructions.

Given $D \in \mathcal{Ch}, n \in \mathbb{N}$, we define

$$\mathrm{K}(D, n) := \Gamma(\tau(D[-n])).$$

If $A \in \mathcal{Shv}\mathcal{Ab}(\mathcal{C}), n \in \mathbb{N}$ and $A \to J(= J_0 \to J_{-1} \to \cdots)$ is an injective resolution[9] of A in $\mathcal{Shv}\mathcal{Ab}(\mathcal{C})$, then the map $\mathrm{K}(A, n) \to \mathrm{K}(J, n)$ is a local weak equivalence and $\mathrm{K}(J, n)$ satisfies descent ([**50**, Lemma 8.24 and Theorem 8.25]). Moreover, by [**50**, Lemma 8.24], $\mathrm{H}^n_\tau(X; A) = 0$ for $n > 0$ if $A \in \mathcal{Shv}\mathcal{Ab}(\mathcal{C})$ is injective, so

$$\mathrm{H}^n_\tau(X; -) = \mathbf{R}^n \mathrm{H}^0_\tau(X; -) : \mathcal{Shv}\mathcal{Ab}(\mathcal{C}) \to \mathcal{Ab}$$

by uniqueness characterization of derived functors.

Proposition 2.9.6. *For* $A \in \mathcal{PreAb}(\mathcal{C}), X \in \mathsf{s}\mathcal{Pre}(\mathcal{C}), U \in \mathcal{C}, n \in \mathbb{N}$*, there are canonical isomorphisms*

$$\pi_i \mathbf{R}\underline{\mathrm{Hom}}(X, \mathrm{K}(A, n))(U) = [S^i \wedge U_+, \mathbf{R}\underline{\mathrm{Hom}}(X, \mathrm{K}(A, n))]_{\mathsf{s}\mathcal{Pre}(\mathcal{C})_*} \cong \begin{cases} \mathrm{H}^{n-i}_{\tau|_U}(X|_U; A|_U), & 0 \leqslant i \leqslant n; \\ 0, & i > n. \end{cases}$$

There is an isomorphism

$$(\mathbf{R}\Omega_s)^i \mathrm{K}(A, n) \cong \begin{cases} \mathrm{K}(A, n-i), & 0 \leqslant i \leqslant n; \\ *, & i > n. \end{cases}$$

in the homotopy category $\mathrm{Ho}((\mathsf{s}\mathcal{Pre}(\mathcal{C})_J)_*)$ *(i.e. the two sides are locally weak equivalent as objects in the Jardine model category* $(\mathsf{s}\mathcal{Pre}(\mathcal{C})_J)_*$*). Moreover, the homotopy sheaves are given by*

$$\pi_i(\mathrm{K}(A, n)) \cong \begin{cases} A^\sharp, & i = n; \\ 0, & i \neq n. \end{cases}$$

Proof. The proof of [**50**, Proposition 8.32] gives an isomorphism $\mathbf{R}\Omega_s \mathrm{K}(A, n) \cong \mathrm{K}(A, n-1)$ in $\mathrm{Ho}((\mathsf{s}\mathcal{Pre}(\mathcal{C})_J)_*)$. To get the general isomorphism in the second statement, we iterate this and note that $\mathbf{R}\Omega_s A \cong *$.

The first bijection then follows from this (use also the result of [**50**, p.217] that the cohomology is the same for presheaf coefficients and for the associated sheaf coefficients), since by eq. (1.7), we have

$$[S^i \wedge X_+, \mathrm{K}(A, n)]_{\mathsf{s}\mathcal{Pre}(\mathcal{C})_*} \cong [X_+, (\mathbf{R}\Omega_s)^i \mathrm{K}(A, n)]_{(\mathsf{s}\mathcal{Pre}(\mathcal{C})_J)_*} \cong [X_+, \mathrm{K}(A, n-i)]_{(\mathsf{s}\mathcal{Pre}(\mathcal{C})_J)_*} \cong \mathrm{H}^{n-i}_\tau(X; A)$$

if $0 \leqslant i \leqslant n$ and is trivial if $i > n$, and by results in Section 2.4, $\mathrm{RMap}(X \times U, Y) \simeq \mathrm{RMap}(X|_U, Y|_U) \in \mathsf{s}\mathcal{S}\mathsf{et}$, giving that

$$\mathrm{H}^n_\tau(X \times U; A) \cong \mathrm{H}^n_{\tau|_U}(X|_U; A|_U).$$

Taking $X = *$ in the first isomorphism we get the presheaves of homotopy groups (which is a reinterpretation of [**50**, Corollary 8.33]), the n-th term is easy, since the presheaf is already A, and consequently we get $\pi_n(\mathrm{K}(A, n)) \cong A^\sharp$. It's also clear that $\pi_i(\mathrm{K}(A, n)) = 0, i > n$. For $0 \leqslant i \leqslant n$, $\pi_i(\mathrm{K}(A, n)) \cong \pi_0(\mathrm{K}(A, n-i))$ as the two sides are the sheafification of the same presheaf. So we only need to show

$$\pi_0(\mathrm{K}(A, n)) = 0, \forall n > 0.$$

This is the same as saying that for $n > 0$, the sheafification of the presheaf $U \mapsto \mathrm{H}^n_\tau(U; A)$ is trivial. This is because that sheafification is just the n-th higher direct image functor of the identity functor of $\mathcal{Shv}\mathcal{Ab}(\mathcal{C})$, which is exact hence all higher direct image functors are zero. □

[9] The abelian category $\mathcal{Shv}\mathcal{Ab}(\mathcal{C})$ has enough injectives!

Chapter 3

Elements of motivic homotopy theory

After the long walk on Jardine's local homotopy theory in last chapter, we are able to collect some important results concerning unstable motivic homotopy theory and Hermitian K-theory in this chapter. We try to give a short quick introduction of the motivic homotopy theory of Morel-Voevodsky and bird's eye view of this theory but don't mean to be complete. In particular, most results are without proofs.

3.1. The construction of the (unstable) motivic homotopy category

We fix a quasi-compact quasi-separated (*qcqs* for short) scheme S. We write $\mathcal{S}\mathsf{ch}/S$ for the category of S-schemes, and $\mathcal{S}\mathsf{m}_S$ for the category of S-schemes which are finitely presented and smooth over S. Note that $\mathcal{S}\mathsf{m}_S$ is essentially small.

Now we introduce a Grothendieck topology on $\mathcal{S}\mathsf{m}_S$, which will be required to define the motivic homotopy category.

Definition 3.1.1. The *Nisnevich topology* on $\mathcal{S}\mathsf{m}_S$ is the topology with a basis given as follows: for any $X \in \mathcal{S}\mathsf{m}_S$, J_X is the collection of all the finite sets of étale morphisms $\{p_i : U_i \to X\}_{i\in I}$ such that there exists a finite sequence $\varnothing \subset Z_n \subset Z_{n-1} \subset \cdots \subset Z_1 \subset Z_0 = X$ of finitely presented closed subschemes of X such that the induced morphism

$$\coprod_{i\in I} p_i^{-1}(Z_m - Z_{m+1}) \to (Z_m - Z_{m+1})$$

admits a section for all $0 \leqslant m \leqslant n-1$, where each Z_m is endowed with the reduced structure. Note that a finite set of étale morphisms $\{p_i : U_i \to X\}_{i\in I}$ is a Nisnevich cover iff the single morphism $\coprod_{i\in I} p_i : \coprod_{i\in I} U_i \to X$ is a Nisnevich cover. In this way, we get the *Nisnevich site* $(\mathcal{S}\mathsf{m}_S, \mathrm{Nis})$. The Nisnevich topology is coarser than the étale topology (and finer than the Zariski topology), so it's subcanonical and every representable presheaf is in fact a sheaf on $(\mathcal{S}\mathsf{m}_S, \mathrm{Nis})$.

We denote by $\mathrm{L}_{\mathrm{Nis}}\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{S}\mathsf{m}_S)_{\mathrm{proj}}$ and by $\mathrm{L}_{\mathrm{Nis}}\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{S}\mathsf{m}_S)_{\mathrm{inj}}$ the left Bousfield localizations of the category simplicial presheaves with respect to the Nisnevich-hypercovers. The (pointed) homotopy classes will be denoted by $[-,-]_{\mathrm{Nis}(,*)}$. Write $\mathrm{L}_{\mathrm{Nis}}$ for a functorial fibrant replacement in one of them (usually it doesn't matter which model structure we are considering).

Definition 3.1.2. Let $f : U \to X$ be a morphism of schemes, $x \in X$. We say that f is *completely decomposed* (*cd* for short) at $x \in X$ if there exists $u \in U$ such that $f(u) = x$ and the residue field extension $\kappa_x \to \kappa_u$ is an isomorphism, i.e. there is a lifting map as shown:

$$\begin{array}{ccc} & & U \\ & \nearrow & \downarrow f \\ \operatorname{Spec}\kappa_x & \longrightarrow & X. \end{array}$$

Proposition 3.1.3. *An étale morphism $f : U \to X$ is cd at a point $x \in X$ iff the morphism $U \times_X \operatorname{Spec}\mathscr{O}^h_{X,x} \to \operatorname{Spec}\mathscr{O}^h_{X,x}$ has a section, where $\mathscr{O}^h_{X,x}$ is the henselization of the local ring $\mathscr{O}_{X,x}$.*

Proposition 3.1.4. *If S is noetherian and of finite Krull dimension, then a finite set of étale morphisms $\{p_i : U_i \to X\}_{i\in I}$ is a Nisnevich cover iff for each point $x \in X$, there is an $i \in I$ such that the map p_i is cd at x.*

Definition 3.1.5. A *Nisnevich square* or a *elementary distinguished square* is a pullback diagram

$$\begin{array}{ccc} U \times_X V & \longrightarrow & V \\ \downarrow & & \downarrow p \\ U & \xrightarrow{\ i\ } & X \end{array} \tag{Q}$$

in which i is an open immersion, p is étale and the restriction $p^{-1}(X-U) \xrightarrow{p} (X-U)$ is an isomorphism with both sides being equipped with the reduced scheme structures.

Proposition 3.1.6. *Given a Nisnevich square* (Q)*, the set $\{i, p\}$ form a Nisnevich cover of X. Moreover, viewed as a diagram of sheaves,* (Q) *is cocartesian (hence bicartesian) in $\mathcal{S}\mathsf{hv}(\mathcal{S}\mathsf{m}_S, \mathrm{Nis})$. Thus for any sheaf F, the diagram of sets*

$$\begin{array}{ccc} F(X) & \xrightarrow{F(p)} & F(V) \\ {\scriptstyle F(i)}\downarrow & & \downarrow \\ F(U) & \longrightarrow & F(U \times_X V) \end{array} \tag{FQ}$$

is cartesian.

Viewed as a diagram of presheaves, (Q) *is homotopy cocartesian in $\mathrm{L}_{\mathrm{Nis}}\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{S}\mathsf{m}_S)_{\mathrm{proj}}$.*

Remark 3.1.7. A Nisnevich square (Q) viewing as a diagram of presheaves is not necessarily cocartesian and for a presheaf F, the diagram (FQ) is not necessarily cartesian.

We use $\mathcal{Shv}(\mathcal{Sm}_S, \mathrm{Nis}, \mathcal{Ab})$ or $\mathcal{Shv}\mathcal{Ab}(\mathcal{Sm}_S)_{\mathrm{Nis}}$ to denote the category of sheaves of abelian groups on the Nisnevich site $(\mathcal{Sm}_S, \mathrm{Nis})$ (similarly for other notations).

Below we denote by $\mathbb{Z}[X]$ the free abelian sheaf on h_X. The functor $\mathbb{Z}[-]$ is left adjoint to the forgetful functor $\mathcal{Shv}(\mathcal{Sm}_S, \mathrm{Nis}, \mathcal{Ab}) \to \mathcal{Shv}(\mathcal{Sm}_S, \mathrm{Nis})$, so it preserves colimits. For $A \in \mathcal{Shv}(\mathcal{Sm}_S, \mathrm{Nis}, \mathcal{Ab})$, we have the cohomology groups $\mathrm{H}^i_{\mathrm{Nis}}(X, A) = \mathrm{Ext}^i(\mathbb{Z}[X], A)$ with cofficients in the sheaf of abelian groups A.

Corollary 3.1.8. *Given a Nisnevich square* (Q)*, we have a coequalizer diagram*

$$U \times_X V \rightrightarrows U \coprod V \xrightarrow{i \coprod p} X,$$

in $\mathcal{Shv}(\mathcal{Sm}_S, \mathrm{Nis})$ *(the first two arrows are the projections followed by the canonical maps of* U, V *into* $U \coprod V$*) and an exact sequence of sheaves of abelian groups*

$$0 \to \mathbb{Z}[U \times_X V] \to \mathbb{Z}[U] \oplus \mathbb{Z}[V] \to \mathbb{Z}[X] \to 0.$$

This gives a long exact sequence of abelian groups

$$\cdots \to \mathrm{H}^i_{\mathrm{Nis}}(X; A) \xrightarrow{\text{restriction}} \mathrm{H}^i_{\mathrm{Nis}}(U; A) \oplus \mathrm{H}^i_{\mathrm{Nis}}(V; A) \xrightarrow{-} \mathrm{H}^i_{\mathrm{Nis}}(U \times_X V; A) \to \mathrm{H}^{i+1}_{\mathrm{Nis}}(X; A) \to \cdots,$$

natural in $A \in \mathcal{Shv}\mathcal{Ab}(\mathcal{Sm}_S, \mathrm{Nis})$*.*

Theorem 3.1.9. *Let* $\mathcal{E}$ *be a complete category and let* $F \in \mathcal{P}\mathsf{re}(\mathcal{Sm}_S, \mathcal{E})$*. Then* F *is a sheaf (for the Nisnevich topology) iff*

- $F(\varnothing) = *$ *is a final object in* $\mathcal{E}$*,*
- *for any Nisnevich square as* (Q)*, the corresponding square* (FQ) *is cartesian in* $\mathcal{E}$*.*

Definition 3.1.10. Let $F \in \mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{Sm}_S)$. We say that F satisfies the *(Nisnevich) Brown-Gersten property* (*(Nisnevich) BG-property* for short) or the *Nisnevich excision property* if

- $F(\varnothing)$ is a weakly contractible,
- for any Nisnevich square as (Q), the corresponding square (FQ) is homotopy cartesian in $\mathsf{s}\mathcal{S}\mathsf{et}$.

Theorem 3.1.11. *Assume that* S *is a noetherian scheme of finite Krull dimension, then for* $F \in \mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{Sm}_S)$*, the following are equivalent:*

1. *There is a fibrant replacement* $j : F \to \hat{F}$ *of* F *in* $\mathrm{L}_{\mathrm{Nis}}\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})_{\mathrm{proj/inj}}$ *that is a sectionwise weak equivalence.*
2. F *satisfies the BG-property (Nisnevich excision).*
3. F *satisfies descent for all hypercovers (for the Nisnevich topology).*
4. *For any hypercover* $U \to X$*, the map* $\mathrm{RMap}(X, F) \to \mathrm{RMap}(U, F)$ *is a weak equivalence of simplicial sets (here* RMap *is computed in* $\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})_{\mathrm{inj}}$*).*

Proof. This is a corollary of our characterization of simplicial presheaves satisfying descent property in Theorem 2.7.13, together with the results in [**12**, Remark 3.1.4 and Theorem 3.2.5], cf. [**69**, Lemma 3.1.18] or [**50**, Theorem 5.39 and p.181]. For a discussion of this result, see the proof of [**1**, Proposition 3.53] (but note that the statement of that proposition is incorrect). □

Corollary 3.1.12. *Assume that* S *is a noetherian scheme of finite Krull dimension. If* $F \in \mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{Sm}_S)_*$ *satisfies the BG-property, then for any* $U \in \mathcal{Sm}_S$*, there are weak equivalences of simplicial sets*

$$\mathrm{RMap}(U_+, F) \simeq F(U).$$

So for all $n \geqslant 0$*, we have natural isomorphisms*

$$\pi_n(F(U)) \cong [S^n, \mathrm{RMap}(U_+, F)]_{\mathsf{s}\mathcal{S}\mathsf{et}_*} \cong [\Sigma^n U_+, F]_{\mathrm{Nis},*} \cong [U_+, \Omega^n F]_{\mathrm{Nis},*}.^1$$

For $X \in \mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{Sm}_S)_*$ a pointed simplicial presheaf, we define its *(simplicial) homotopy sheaf* $\pi_n(X)$ to be the Nisnevich sheafification of the presheaf

$$U \mapsto [S^n \wedge U_+, X]_{\mathrm{Nis},*}.$$

So $\pi_0(X)$ is the Nisnevich sheafification of the presheaf $U \mapsto [U, X]_{\mathrm{Nis}}$. For $n \in \mathbb{N}$, we say that $X \in \mathcal{Spc}^{\mathbb{A}^1}_*(S)$ is *simplicially* n*-connected* if $\pi_j(X) = *, 0 \leqslant j \leqslant n$.

Let G be a sheaf of groups on the Nisnevich site $(\mathcal{Sm}_S, \mathrm{Nis})$. Applying our functorial construction for all sections, there is the *universal* G*-torsor* $q : \mathrm{E}G \to \mathrm{B}G$.

Similarly, the functorial model of Eilenberg-Mac Lane spaces $\mathrm{K}(A, n) \in \mathsf{s}\mathcal{S}\mathsf{et}$ extends by sectionwise construction to the presheaf setting.

For any $X \in \mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{Sm}_S)$, we denote by $\mathrm{PF}_G(X) =: \mathrm{H}^1_{\mathrm{Nis}}(X, G)$ the set of isomorphism classes of G-torsors over X.

We restate the following two results which appeared in last chapter.

Proposition 3.1.13 ([**69**, Proposition 4.1.15, 4.1.16])**.** *Let* G *be a sheaf of groups on the Nisnevich site* $(\mathcal{Sm}_S, \mathrm{Nis})$*,* $X \in \mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{Sm}_S)$ *and* $U \in \mathcal{Sm}_S$*. There are canonical bijections*

$$[X, \mathrm{B}G]_{\mathrm{Nis}} \cong \mathrm{H}^1_{\mathrm{Nis}}(X; G)$$

[1] Here Ω should mean the derived functor $\mathbf{R}\Omega_s$.

and

$$[S^n \wedge U_+, \mathrm{B}G]_{\mathrm{Nis},*} \cong \begin{cases} \mathrm{H}^1_{\mathrm{Nis}}(U, G), & n = 0; \\ G(U), & n = 1; \\ *, & n > 1. \end{cases}$$

Thus the map $\mathbf{R}\Omega_s \mathrm{B}G \to \pi_0(\mathbf{R}\Omega_s \mathrm{B}G) = G$ *is a local weak equivalence and the homotopy sheaves*[2] *are given by*

$$\pi_n(\mathrm{B}G) \cong \begin{cases} G, & n = 1; \\ 0, & n \neq 1. \end{cases}$$

Remark 3.1.14. For concrete examples, see [**13**, Example 2.3.4].

In fact, the map $\mathbf{R}\Omega_s \mathrm{B}G \to \pi_0(\mathbf{R}\Omega_s \mathrm{B}G) = G$ is a sectionwise weak equivalence by Theorem 2.8.9, see also [**13**, Lemma 2.3.2].

Proposition 3.1.15. *Let* A *be a sheaf of abelian groups on the Nisnevich site* $(\mathcal{S}\mathsf{m}_S, \mathrm{Nis})$, $X \in \mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{S}\mathsf{m}_S)$ *and* $n \geqslant 1$. *Then there is a canonical bijection*

$$[S^i \wedge X_+, \mathrm{K}(A, n)]_{\mathrm{Nis},*} \cong \begin{cases} \mathrm{H}^{n-i}_{\mathrm{Nis}}(X; A), & 0 \leqslant i \leqslant n; \\ 0, & i > n, \end{cases}$$

and an isomorphism

$$(\mathbf{R}\Omega_s)^i \mathrm{K}(A, n) \cong \begin{cases} \mathrm{K}(A, n-i), & 0 \leqslant i \leqslant n; \\ *, & i > n. \end{cases}$$

in the homotopy category $\mathrm{Ho}((\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{S}\mathsf{m}_S)_J)_*)$ *(i.e. the two sides are locally weak equivalent as objects in the Jardine model category* $(\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{S}\mathsf{m}_S)_J)_*$*). Moreover, the homotopy sheaves are given by*

$$\pi_i(\mathrm{K}(A, n)) \cong \begin{cases} A, & i = n; \\ 0, & i \neq n. \end{cases}$$

Definition 3.1.16. The *(unstable) motivic model category* of S is the left Bousfield localization of $\mathrm{L}_{\mathrm{Nis}}\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{S}\mathsf{m}_S)_{\mathrm{proj}}$ with respect to the class of projections $X \times \mathbb{A}^1 \to X$, $X \in \mathcal{S}\mathsf{m}_S$. [3] Write

$$\mathcal{S}\mathsf{pc}^{\mathbb{A}^1}(S) := \mathrm{L}_{\mathbb{A}^1}\mathrm{L}_{\mathrm{Nis}}\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{S}\mathsf{m}_S)_{\mathrm{proj}}$$

for the resulting motivic model category. Its homotopy category is denoted by $\mathcal{H}^{\mathbb{A}^1}(S)$, called the $\mathbb{A}^1$*-homotopy category* of S. The homotopy classes will be denoted by $[-,-]_{\mathbb{A}^1}$. The weak equivalences in $\mathcal{S}\mathsf{pc}^{\mathbb{A}^1}(S)$ are called $\mathbb{A}^1$*-weak equivalences* and the fibrations in $\mathcal{S}\mathsf{pc}^{\mathbb{A}^1}(S)$ are called $\mathbb{A}^1$*-fibrations*.

The pointed version will be denoted $\mathcal{S}\mathsf{pc}^{\mathbb{A}^1}_*(S)$, called the *pointed motivic model category* of S. Its homotopy category is denoted by $\mathcal{H}^{\mathbb{A}^1}_*(S)$, called the *pointed* $\mathbb{A}^1$*-homotopy category* of S. The homotopy classes will be denoted by $[-,-]_{\mathbb{A}^1,*}$.

Note that if we only interested in the $\mathbb{A}^1$-homotopy category, we can work equally well with the injective version, and/or the simplicial sheaves (as in [**69**]), since all these are Quillen equivalent, yielding equivalent homotopy categories; so each of them can be used as a model for the $\mathbb{A}^1$-homotopy theory.

The fibrant objects in $\mathcal{S}\mathsf{pc}^{\mathbb{A}^1}(S)$ will be called $\mathbb{A}^1$*-spaces* or $\mathbb{A}^1$*-local spaces*, which are simplicial presheaves F satisfying the conditions in Theorem 3.1.11 and moreover such that the map $F(U) \to F(U \times \mathbb{A}^1)$ is a weak equivalence for every $U \in \mathcal{S}\mathsf{m}_S$ (we say that F is $\mathbb{A}^1$*-invariant*). The weak equivalences in $\mathcal{S}\mathsf{pc}^{\mathbb{A}^1}(S)$ are called $\mathbb{A}^1$*-(local) weak equivalences*. We have similar notions in the pointed version.

For any $K \in \mathsf{s}\mathcal{S}\mathsf{et}$, we define $\widetilde{K} \in \mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{S}\mathsf{m}_S)$ by $\varnothing \mapsto *, U \mapsto K$ for non-empty $U \in \mathcal{S}\mathsf{m}_S$ (the presheaf $\widetilde{K}$ is sometimes called the constant presheaf associated to K, but it's not so by our convention; anyway, the sheafification of $\widetilde{K}$ and of what we call the constant presheaf associated to K are the same, both are the constant sheaf associated to K). It clearly satisfies the BG-property and is $\mathbb{A}^1$-invariant, hence by Theorem 3.1.11, if K is a Kan complex, then $\widetilde{K}$ is fibrant in $\mathcal{S}\mathsf{pc}^{\mathbb{A}^1}(S) = \mathrm{L}_{\mathbb{A}^1}\mathrm{L}_{\mathrm{Nis}}\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{S}\mathsf{m}_S)_{\mathrm{proj}}$. It's easy to see that two maps between Kan complexes are (simplicially) homotopic in $\mathsf{s}\mathcal{S}\mathsf{et}$ iff the induced maps on the simplicial presheaves so constructed are homotopic in $\mathcal{S}\mathsf{pc}^{\mathbb{A}^1}(S)$. Thus *the homotopy category of* $\mathsf{s}\mathcal{S}\mathsf{et}$ *is embedded in the motivic homotopy category.*

There is an $\mathbb{A}^1$*-localization functor* $\mathrm{L}_{\mathbb{A}^1} : \mathcal{S}\mathsf{pc}^{\mathbb{A}^1}(S) \to \mathcal{S}\mathsf{pc}^{\mathbb{A}^1}(S)$ in the sense that there is a natural transformation $\mathrm{id} \Rightarrow \mathrm{L}_{\mathbb{A}^1}$ such that for any $X \in \mathcal{S}\mathsf{pc}^{\mathbb{A}^1}(S)$, the map $X \to \mathrm{L}_{\mathbb{A}^1}X$ is a trivial cofibration in $\mathcal{S}\mathsf{pc}^{\mathbb{A}^1}(S)$ and $\mathrm{L}_{\mathbb{A}^1}X$ is fibrant in $\mathcal{S}\mathsf{pc}^{\mathbb{A}^1}(S)$ (see [**69**, Definition 2.3.18], where it's called an $\mathbb{A}^1$*-resolution functor*). Of course, this is just a functorial fibrant replacement in this model category. By general results on model categories, we have $[A, X]_{\mathbb{A}^1} = [A, \mathrm{L}_{\mathbb{A}^1}X]_{\mathrm{Nis}}$ for any $A, X \in \mathcal{S}\mathsf{pc}^{\mathbb{A}^1}(S)$ with A cofibrant.

If $S = \mathrm{Spec}(k), k$ a commutative ring, we simply write $\mathcal{S}\mathsf{m}_k$ for $\mathcal{S}\mathsf{m}_S$ and write $\mathcal{H}^{\mathbb{A}^1}_{(*)}(k)$ for $\mathcal{H}^{\mathbb{A}^1}_{(*)}(S)$; similar for other notations.

By Proposition 3.1.6, for a Nisnevich square (Q), the natural map $V/(U \times_X V) \to X/U$ is an $\mathbb{A}^1$-weak equivalence.

---

[2] To get the claimed local weak equivalence, we use the above results to see that this map induces isomorphisms on all homotopy sheaves. For the definition of Ω_s, see Section 3.2. By sheafifying the above result we get the homotopy sheaves. Note that our torsors are Nisnevich locally trivial, so $\pi_0(\mathrm{B}G) = 0$.

[3] We are localizing at a smaller class of maps than that in [**69**, Definition 2.3.1], for a discussion about the equivalence of our definition here with that in [**69**], see [**69**, Proposition 2.3.19] or the discussion in [**49**] before [**49**, Lemma 1.6]. Here one may need the fact that the cofibrations and fibrant objects completely determine a model structure, supposing it exists (see [**74**, Theorem 15.3.1] for an easy argument).

We define a simplicial ring $R_\bullet$ by letting $R_n := \mathbb{Z}[x_0, \cdots, x_n]/\big(\sum_{i=0}^n x_i - 1\big)$ with face maps

$$d_i : R_n \to R_{n-1}, (x_0, \cdots, x_n) \mapsto (x_0, \cdots, x_{i-1}, 0, x_{i+1}, \cdots, x_{n-1})$$

and degeneracy maps

$$s_j : R_n \to R_{n+1}, (x_0, \cdots, x_n) \mapsto (x_0, \cdots, x_{j-1}, x_j + x_{j+1}, x_{j+2}, \cdots, x_{n+1}).$$

(For general $\varphi : [n] \to [m]$ in $\mathbf{\Delta}$, we have $\varphi^*(x_i) = \sum_{j \in \varphi^{-1}(i)} x_j$.) This gives a cosimplicial scheme $\mathbb{A}^\bullet = \mathbb{A}^\bullet_{\mathbb{Z}} := \mathrm{Spec}(R_\bullet)$ and we denote $\mathbb{A}^\bullet_S := \mathbb{A}^\bullet_{\mathbb{Z}} \times S \in \mathsf{c}\mathcal{S}\mathsf{m}_S$. Hence for each $n \in \mathbb{N}$, there is a non-canonical isomorphism of S-schemes $\mathbb{A}^n_S \cong \mathbb{A}^n_S$.

For $U \in \mathcal{S}\mathsf{m}_S$ and $F \in \mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{S}\mathsf{m}_S)$, we have $U \times_S \mathbb{A}^\bullet_S \cong \mathbb{A}^\bullet_U$ and we get a bisimplicial set $F(U \times_S \mathbb{A}^\bullet_S) = F(\mathbb{A}^\bullet_U) \in \mathsf{ss}\mathcal{S}\mathsf{et}$, and we write $(\mathrm{Sing}^{\mathbb{A}^1} F)(U) := \mathrm{diag}\, F(\mathbb{A}^\bullet_U)$ for its diagonal, so $(\mathrm{Sing}^{\mathbb{A}^1} F)_n(U) = F_n(\mathbb{A}^n_U)$; in particular, the vertex set of $\mathrm{Sing}^{\mathbb{A}^1} F$ is identified with $F_0(U) = F(U)_0 \cong \mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{S}\mathsf{m}_S)(U, F)$. The map $\mathbb{A}^n_U = U \times_S \mathbb{A}^n_S \to U \times_S \mathbb{A}^0_S = U$ admits a section, hence there is a canonical injection $F \to \mathrm{Sing}^{\mathbb{A}^1} F$. Then we get the *singular* $\mathbb{A}^1$*-construction* $\mathrm{Sing}^{\mathbb{A}^1} F \in \mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{S}\mathsf{m}_S)$ and the *singular* $\mathbb{A}^1$*-functor* $\mathrm{Sing}^{\mathbb{A}^1} : \mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{S}\mathsf{m}_S) \to \mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{S}\mathsf{m}_S)$ with a natural transformation $\mathrm{id} \Rightarrow \mathrm{Sing}^{\mathbb{A}^1}$. In fact, $\mathrm{Sing}^{\mathbb{A}^1}$ commutes with limits, it preserves $\mathbb{A}^1$-fibrations, and for any $F \in \mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{S}\mathsf{m}_S)$, the map $F \to \mathrm{Sing}^{\mathbb{A}^1} F$ is a trivial $\mathbb{A}^1$-cofibration ([**69**, §2.3.2]).

One can then choose the $\mathbb{A}^1$-localization functor to be

$$\mathrm{L}_{\mathbb{A}^1} = \mathrm{L}_{\mathrm{Nis}} \circ (\mathrm{L}_{\mathrm{Nis}} \circ \mathrm{Sing}^{\mathbb{A}^1})^{\circ \mathbb{N}} \circ \mathrm{L}_{\mathrm{Nis}}$$

which commutes with finite products (see [**69**, Lemma 2.3.20, 3.2.6]).

Definition 3.1.17 ([**69**, Example 3.2.4])**.** An S-scheme X is called $\mathbb{A}^1$*-rigid* if, as a sheaf, it is fibrant in the motivic model category $\mathcal{S}\mathsf{pc}^{\mathbb{A}^1}(S)$. As any sheaf (of sets) is fibrant in $\mathrm{L}_{\mathrm{Nis}}\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{S}\mathsf{m}_S)_{\mathrm{proj}}$, this is equivalent to $(\mathcal{S}\mathsf{ch}/S)(U \times \mathbb{A}^1, X) \to (\mathcal{S}\mathsf{ch}/S)(U, X)$ is a bijection for any $U \in \mathcal{S}\mathsf{m}_S$. Two $\mathbb{A}^1$-rigid S-schemes are isomorphic as schemes iff they are $\mathbb{A}^1$-weakly equivalent. So through Yoneda embedding, the category of $\mathbb{A}^1$-rigid S-schemes is fully-faithfully embedded into the $\mathbb{A}^1$-homotopy category $\mathcal{H}^{\mathbb{A}^1}(S)$.

If the base scheme S is a reduced scheme of finite Krull dimension, then $\mathbb{G}_m$ is $\mathbb{A}^1$-rigid (for this, observe that for any $U \in \mathcal{S}\mathsf{m}_S$, we have $\mathscr{O}_{U \times \mathbb{A}^1}(U \times \mathbb{A}^1)^\times \cong \mathscr{O}_U(U)^\times$, essentially because $(R[t])^\times \cong R^\times$ for any commutative ring R with trivial nilradical: $\sum_{i=0}^n a_i t^i \in R[t]$ is a unit iff $a_0 \in R^\times$ and $a_1, \cdots, a_n$ are nilpotent).

Corollary 3.1.18. *Assume that S is a noetherian scheme of finite Krull dimension. If $F \in \mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{S}\mathsf{m}_S)_*$ satisfies the BG-property and is $\mathbb{A}^1$-invariant (such F is said to satisfy* motivic descent*), then for any $U \in \mathcal{S}\mathsf{m}_S$, there are weak equivalences of simplicial sets*

$$\mathrm{RMap}(U_+, F) \simeq F(U).$$

So for all $n \geqslant 0$, we have natural isomorphisms

$$\pi_n(F(U)) \cong [S^n, \mathrm{RMap}(U_+, F)]_{\mathsf{s}\mathcal{S}\mathsf{et}_*} \cong [\Sigma^n U_+, F]_{\mathbb{A}^1,*} \cong [U_+, (\mathbf{R}\Omega_s)^n F]_{\mathbb{A}^1,*}.$$

Remark 3.1.19. To deduce this result, it's crucial to have that all representable functors are cofibrant in $\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})_{\mathrm{proj/inj}}$ hence in $\mathrm{L}_\tau \mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})_{\mathrm{proj/inj}}$.

Simplicial presheaves satisfy descent or motivic descent are closed under filtered colimits (see [**50**, p. 181]).

This result is quite useful in interpreting some theories in terms of (pointed) $\mathbb{A}^1$-homotopy theory—their representability in the pointed $\mathbb{A}^1$-homotopy category $\mathcal{H}^{\mathbb{A}^1}_*(S)$, examples include algebraic K-theory ([**69**]), (higher) Grothendieck-Witt groups ([**78, 79, 81**]), and affine representability results for vector bundles ([**12**, Theorem 5.2.3]). We will come back to these examples later.

For a sheaf of abelian groups A, we define its *contraction* A_{-1} by $A_{-1}(U) := \ker(A(U \times \mathbb{G}_m) \to A(U))$ induced by the map $U = U \times \{1\} \hookrightarrow U \times \mathbb{G}_m$, or equivalently, $A_{-1}(U) := \mathrm{coker}(A(U) \to A(U \times \mathbb{G}_m))$ induced by the projection $U \times \mathbb{G}_m \to U$.[4] The contraction A_{-1} is also a sheaf of abelian groups. We define inductively $A_{-(n+1)} = (A_{-n})_{-1}, n \in \mathbb{N}$ (by convention, $A_0 = A_{-0} = A$).

Following Morel [**68**, Definition 1.7], we call a presheaf of sets F on $(\mathcal{S}\mathsf{m}_S, \mathrm{Nis})$ $\mathbb{A}^1$*-invariant* if the (injective) map $F(U) \to F(U \times \mathbb{A}^1)$ induced by the projection $U \times \mathbb{A}^1 \to U$ is a bijection for every $U \in \mathcal{S}\mathsf{m}_S$. A sheaf of groups G on $(\mathcal{S}\mathsf{m}_S, \mathrm{Nis})$ is *strongly* $\mathbb{A}^1$*-invariant* if the presheaf $\mathrm{H}^i_{\mathrm{Nis}}(-; G)$ is $\mathbb{A}^1$-invariant for $i = 0, 1$. A sheaf of abelian groups M on $(\mathcal{S}\mathsf{m}_S, \mathrm{Nis})$ is *strictly* $\mathbb{A}^1$*-invariant* if the presheaf $\mathrm{H}^i_{\mathrm{Nis}}(-; M)$ is $\mathbb{A}^1$-invariant for every $i \in \mathbb{N}$. We usually use boldface letters like $\mathbf{G}, \mathbf{A}, \mathbf{M}$ to indicate that the relevant sheaves of (abelian) groups are strongly (strictly) $\mathbb{A}^1$-invariant.

We denote by $\mathcal{G}\mathsf{r}^{\mathbb{A}^1}_S$ the category of strongly $\mathbb{A}^1$-invariant sheaves of groups and by $\mathcal{A}\mathsf{b}^{\mathbb{A}^1}_S$ the category of strictly $\mathbb{A}^1$-invariant sheaves of abelian groups on the Nisnevich site $(\mathcal{S}\mathsf{m}_S, \mathrm{Nis})$, the latter is an abelian category. Moreover, the contraction defines an exact functor $(-)_{-1} : \mathcal{A}\mathsf{b}^{\mathbb{A}^1}_S \to \mathcal{A}\mathsf{b}^{\mathbb{A}^1}_S$.

For a field k, the categories $\mathcal{G}\mathsf{r}^{\mathbb{A}^1}_k$ and $\mathcal{A}\mathsf{b}^{\mathbb{A}^1}_k$ are closed under extensions and taking kernels ([**11**, Lemma 3.1.13]). Note that a non-constant sheaf can have trivial contraction: the n-torsion part ${}_n\mathbf{K}^M_1 \in \mathcal{A}\mathsf{b}^{\mathbb{A}^1}_k$ is a non-constant sheaf but $({}_n\mathbf{K}^M_1)_{-1} = {}_n\mathbb{Z} = 0$.

For a Kan complex K, there is an associated simplicial presheaf $\widetilde{K}$ as above, which is fibrant in $\mathcal{S}\mathsf{pc}^{\mathbb{A}^1}(S) = \mathrm{L}_{\mathbb{A}^1}\mathrm{L}_{\mathrm{Nis}}\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{S}\mathsf{m}_S)_{\mathrm{proj}}$. We thus see (in light of Proposition 3.1.22 below) that any constant sheaf associated to an

[4] Here we use the following fact: In an abelian category, let $\alpha : A \to B, \beta : B \to A$ be morphisms with $\alpha\beta = \mathrm{id}_B$, then the morphism $\ker\alpha \to \mathrm{coker}\beta = A/\mathrm{im}(\beta), a \mapsto [a]$ is an isomorphism.

(abelian) group is strongly (strictly) $\mathbb{A}^1$-invariant, yielding embeddings $\mathcal{G}\mathsf{r} \hookrightarrow \mathcal{G}\mathsf{r}_S^{\mathbb{A}^1}, \mathcal{A}\mathsf{b} \hookrightarrow \mathcal{A}\mathsf{b}_S^{\mathbb{A}^1}$. If $S = \mathrm{Spec}(k)$ for a perfect field k, then these embeddings are fully faithful (since a morphism between constant sheaves on the Nisnevich site $(\mathcal{S}\mathsf{m}_k, \mathrm{Nis})$ is determined by its effect on global sections, which gives back the groups where the constant sheaves take values, in light of Proposition 2.3.6).

Remark 3.1.20. If k is a perfect field and $G \in \mathcal{G}\mathsf{r}_k^{\mathbb{A}^1}$, then G is *unramified* in the sense of [**68**, Definition 2.1] (see [**68**, Remark 2.8]).

Theorem 3.1.21 ([**68**, Theorem 5.46])**.** *Let k be perfect field and let M be a sheaf of abelian groups on the Nisnevich site $(\mathcal{S}\mathsf{m}_k, \mathrm{Nis})$. Then $M \in \mathcal{G}\mathsf{r}_k^{\mathbb{A}^1}$ iff $M \in \mathcal{A}\mathsf{b}_k^{\mathbb{A}^1}$.*

Using Theorem 3.1.9 and Proposition 1.4.6 (3), we easily show the following results.

Proposition 3.1.22. *The following results hold.*

(1) *A sheaf of sets F is $\mathbb{A}^1$-invariant iff F is $\mathbb{A}^1$-fibrant (or $\mathbb{A}^1$-local).*
(2) *A sheaf of groups G is strongly $\mathbb{A}^1$-invariant iff some (hence any) fibrant replacement of $\mathrm{B}G = \mathrm{K}(G,1)$ in $\mathrm{L}_{\mathrm{Nis}}\mathsf{sPre}(\mathcal{S}\mathsf{m}_S)_{\mathrm{proj}}$ (which we can take to be $\mathrm{L}_{\mathrm{Nis}}\mathrm{B}G$) is $\mathbb{A}^1$-fibrant (or $\mathbb{A}^1$-local).*
(3) *A sheaf of abelian groups M is strictly $\mathbb{A}^1$-invariant iff some (hence any) fibrant replacement of $\mathrm{K}(M,n)$ in $\mathrm{L}_{\mathrm{Nis}}\mathsf{sPre}(\mathcal{S}\mathsf{m}_S)_{\mathrm{proj}}$ (which we can take to be $\mathrm{L}_{\mathrm{Nis}}\mathrm{K}(M,n)$) is $\mathbb{A}^1$-fibrant (or $\mathbb{A}^1$-local) for every $n \in \mathbb{N}$.*

For $X \in \mathcal{S}\mathsf{pc}_*^{\mathbb{A}^1}(S)$ a pointed simplicial presheaf, we define its *$\mathbb{A}^1$-homotopy sheaf* $\pi_n^{\mathbb{A}^1}(X)$ to be the Nisnevich sheafification of the presheaf

$$U \mapsto [S^n \wedge U_+, X]_{\mathbb{A}^1,*}.$$

So $\pi_0^{\mathbb{A}^1}(X)$ is the Nisnevich sheafification of the presheaf $U \mapsto [U, X]_{\mathbb{A}^1}$ and we have $\pi_n^{\mathbb{A}^1}(X) = \pi_n(\mathrm{L}_{\mathbb{A}^1}X)$. Thus there are canonical maps of sheaves $\pi_n(X) \to \pi_n^{\mathbb{A}^1}(X)$ from the (simplicial) homotopy sheaves. Moreover, the map

$$\pi_0(X) \twoheadrightarrow \pi_0^{\mathbb{A}^1}(X)$$

is an epimorphism ([**69**, Corollary 2.3.22]), which implies the $n = 0$ case of the unstable $\mathbb{A}^1$-connectivity theorem below (Theorem 3.1.30). We also call $\pi_1^{\mathbb{A}^1}(X)$ the *$\mathbb{A}^1$-fundamental sheaf of X*.

For $n \in \mathbb{N}$, we say that $X \in \mathcal{S}\mathsf{pc}_*^{\mathbb{A}^1}(S)$ is *$\mathbb{A}^1$-n-connected* if $\pi_j^{\mathbb{A}^1}(X) = *, 0 \leqslant j \leqslant n$ (this is equivalent to: some $\mathbb{A}^1$-invariant replacement of X is simplicially n-connected).

The $\mathbb{A}^1$-homotopy sheaves can be used to characterize $\mathbb{A}^1$-weak equivalences between $\mathbb{A}^1$-connected spaces, just as the Whitehead theorem for $\mathcal{T}\mathsf{op}$ (CW-complexes) or sSet; see [**69**, Proposition 3.2.14].

Proposition 3.1.23. *Let X be an $\mathbb{A}^1$-rigid S-scheme. Then we have*

$$\pi_n(X) \cong \pi_n^{\mathbb{A}^1}(X) \cong \begin{cases} X, & n = 0; \\ *, & n > 0. \end{cases}$$

Theorem 3.1.24 ([**68**, Theorem 6.1 and Corollary 6.2])**.** *Let k be perfect field, let $X \in \mathcal{S}\mathsf{pc}_*^{\mathbb{A}^1}(k)$ be a pointed simplicial presheaf. Then $\pi_1^{\mathbb{A}^1}(X) \in \mathcal{G}\mathsf{r}_k^{\mathbb{A}^1}$ and $\pi_n^{\mathbb{A}^1}(X) \in \mathcal{A}\mathsf{b}_k^{\mathbb{A}^1}$ for $n > 1$.*

Conversely, we have the following nice result, which is a consequence of Propositions 3.1.13, 3.1.15 and 3.1.22. It says that, on the one hand, all strongly (strictly) $\mathbb{A}^1$-invariant sheaves of (abelian) groups arise as $\mathbb{A}^1$-homotopy sheaves of some spaces; on the other hand, the (simplicial) Eilenberg-Mac Lane spaces correspond to strongly (strictly) $\mathbb{A}^1$-invariant sheaves of (abelian) groups are indeed Eilenberg-Mac Lane spaces in the $\mathbb{A}^1$-homotopy category. Thus (only) strongly (strictly) $\mathbb{A}^1$-invariant sheaves of (abelian) groups play well in $\mathbb{A}^1$-homotopy theory.

Theorem 3.1.25 (Motivic-homotopical Eilenberg-Mac Lane spaces)**.** *Let $G \in \mathcal{G}\mathsf{r}_S^{\mathbb{A}^1}$ and $M \in \mathcal{A}\mathsf{b}_S^{\mathbb{A}^1}$, then for $i, n \in \mathbb{N}$,*

$$\pi_i^{\mathbb{A}^1}(\mathrm{B}G) \cong \begin{cases} G, & i = 1; \\ 0, & i \neq 1. \end{cases} \qquad \pi_i^{\mathbb{A}^1}(\mathrm{K}(M,n)) \cong \begin{cases} M, & i = n; \\ 0, & i \neq n. \end{cases}$$

Proposition 3.1.26. *Let G be a sheaf of groups on the Nisnevich site $(\mathcal{S}\mathsf{m}_S, \mathrm{Nis})$, then*

$$\pi_0(\mathrm{B}G) = \pi_0^{\mathbb{A}^1}(\mathrm{B}G) = *.$$

If $S = \mathrm{Spec}(k)$ for a perfect field k and $G \in \mathcal{G}\mathsf{r}_k^{\mathbb{A}^1}$, then $\pi_1(\mathrm{B}G) = \pi_1^{\mathbb{A}^1}(\mathrm{B}G) = G$ and for any $\mathbb{A}^1$-0-connected $X \in \mathcal{S}\mathsf{pc}_*^{\mathbb{A}^1}(k)$, the map $[X, \mathrm{B}G]_{\mathbb{A}^1,*} \to \mathcal{G}\mathsf{r}_k^{\mathbb{A}^1}(\pi_1^{\mathbb{A}^1}(X), G)$ given by taking $\pi_1^{\mathbb{A}^1}$ is a bijection.

Proof. Follows easily from Propositions 3.1.13, 3.1.22 and [**68**, Lemma B.7.1)]. □

Corollary 3.1.27. *Let k be a perfect field, then we have the following adjunction*

$$\pi_1^{\mathbb{A}^1} \circ \mathrm{B} : \mathcal{S}\mathsf{hv}\mathcal{G}\mathsf{r}(\mathcal{S}\mathsf{m}_k)_{\mathrm{Nis}} \rightleftarrows \mathcal{G}\mathsf{r}_k^{\mathbb{A}^1} : \iota,$$

where B *is given by taking simplicial classifying spaces and ι is the forgetful functor.*

Proposition 3.1.28. *If k is a perfect field and $M \in \mathcal{A}\mathsf{b}_k^{\mathbb{A}^1}$, then for $n \geqslant 1$, $\pi_n(\mathrm{K}(M,n)) = \pi_n^{\mathbb{A}^1}(\mathrm{K}(M,n)) = M$ and for any $\mathbb{A}^1$-$(n-1)$-connected $X \in \mathcal{S}\mathsf{pc}_*^{\mathbb{A}^1}(k)$, the map $[X, \mathrm{K}(M,n)]_{\mathbb{A}^1,*} \to \mathcal{A}\mathsf{b}_k^{\mathbb{A}^1}(\pi_n^{\mathbb{A}^1}(X), M)$ given by taking $\pi_n^{\mathbb{A}^1}$ is a bijection.*

Proof. The case $n = 1$ is included in Proposition 3.1.26. For $n > 1$, it follows from Propositions 3.1.15, 3.1.22 and [**68**, Lemma B.7.2)] (by checking that the map of taking $\pi_n^{\mathbb{A}^1}$ is surjective and injective). □

Corollary 3.1.29. *Let k be a perfect field, then we have the following adjunction*

$$\pi_n^{\mathbb{A}^1} \circ \mathrm{K}(-,n) : \mathcal{S}\mathsf{hv}\mathcal{A}\mathsf{b}(\mathcal{S}\mathsf{m}_k)_{\mathrm{Nis}} \rightleftarrows \mathcal{A}\mathsf{b}_k^{\mathbb{A}^1} : \iota,$$

where ι is the forgetful functor (and the left adjoint is essentially independent of $n \geqslant 1$, since the right adjoint ι is independent of n).

Theorem 3.1.30 (Unstable $\mathbb{A}^1$-connectivity theorem [**68**, Theorem 6.38])**.** *Let k be a perfect field, let $X \in \mathcal{S}\mathsf{pc}_*^{\mathbb{A}^1}(k)$ be a pointed simplicial presheaf. If X is simplicially n-connected ($n \geqslant 0$), then it is $\mathbb{A}^1$-n-connected.*

The following result is [**68**, Corollary 6.3], it's also an easy consequence of the unstable $\mathbb{A}^1$-connectivity theorem above and [**68**, Theorem 6.60], by letting n goes to infinity there.

Theorem 3.1.31. *Let k be perfect field, let $X \in \mathcal{S}\mathsf{pc}_*^{\mathbb{A}^1}(k)$ be a pointed simplicial presheaf, which is simplicially connected: $\pi_0(X) = *$. Then the following are equivalent:*

- *All the (simplicial) homotopy sheaves of X are strongly $\mathbb{A}^1$-invariant: $\pi_1(X) \in \mathcal{G}\mathsf{r}_k^{\mathbb{A}^1}, \pi_n(X) \in \mathcal{A}\mathsf{b}_k^{\mathbb{A}^1}$ for $n > 1$.*
- *The $\mathbb{A}^1$-localization map $X \to \mathrm{L}_{\mathbb{A}^1} X$ is a local weak equivalence.*

Corollary 3.1.32. *Let k be a perfect field. For $\mathbf{G} \in \mathcal{G}\mathsf{r}_k^{\mathbb{A}^1}, \mathbf{M} \in \mathcal{A}\mathsf{b}_k^{\mathbb{A}^1}, n \geqslant 2$, the $\mathbb{A}^1$-localization map $\mathrm{K}^{\mathbf{G}}(\mathbf{M}, n) \to \mathrm{L}_{\mathbb{A}^1}\mathrm{K}^{\mathbf{G}}(\mathbf{M}, n)$ is a local weak equivalence.*

For a homotopy fibre sequence $F \to E \to B$ in $\mathrm{L}_{\mathrm{Nis}}\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{S}\mathsf{m}_S)_{*,\mathrm{proj}}$, we have short exact sequences

$$0 \to \pi_n(F)/\mathrm{im}\delta_{n+1} \to \pi_n(E) \to \mathrm{im}\delta_n \to 0$$

for $n \geqslant 2$, where $\delta_n : \pi_n B \to \pi_{n-1} F$ is the connecting map. If F and B are $\mathbb{A}^1$-local, this gives that $\pi_n(E) \in \mathcal{A}\mathsf{b}_k^{\mathbb{A}^1}$ for $n \geqslant 2$, since $\mathcal{A}\mathsf{b}_k^{\mathbb{A}^1}$ is an abelian category that is closed under extension. Thus if $\pi_0(E) = *, \pi_1(E) \in \mathcal{G}\mathsf{r}_k^{\mathbb{A}^1}, \pi_1(F) \in \mathcal{A}\mathsf{b}_k^{\mathbb{A}^1}$, we will have that the $\mathbb{A}^1$-localization map $E \to \mathrm{L}_{\mathbb{A}^1} E$ is a local weak equivalence. In fact, the following more general result holds ([**14**, Lemma 2.2.10]).

Theorem 3.1.33. *Let k be perfect field, let $F \to E \to B$ be a homotopy fibre sequence in $\mathrm{L}_{\mathrm{Nis}}\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{S}\mathsf{m}_S)_{*,\mathrm{proj}}$. Assume the $\mathbb{A}^1$-localization maps $F \to \mathrm{L}_{\mathbb{A}^1} F, B \to \mathrm{L}_{\mathbb{A}^1} B$ are local weak equivalences and $\pi_0(E) = *$. Then the $\mathbb{A}^1$-localization map $E \to \mathrm{L}_{\mathbb{A}^1} E$ is a local weak equivalence and $F \to E \to B$ is an $\mathbb{A}^1$-homotopy fibre sequence.*

3.2. Some useful facts about the (unstable) motivic homotopy category

Recall the discussion at the end of Sections 1.3 and 2.4: We have the *smash product* $\wedge$ and internal hom $\underline{\mathrm{Hom}}_*$ in $\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{S}\mathsf{m}_S)_*$, fitting into the following Quillen adjunction

$$(X,x) \wedge (-) : \mathrm{L}_{\mathrm{Nis}}\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{S}\mathsf{m}_S)_{*,\mathrm{proj}} \rightleftarrows \mathrm{L}_{\mathrm{Nis}}\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{S}\mathsf{m}_S)_{*,\mathrm{proj}} : \underline{\mathrm{Hom}}_*((X,x),-),$$

which passes to a Quillen adjunction

$$(X,x) \wedge (-) : \mathcal{S}\mathsf{pc}_*^{\mathbb{A}^1}(S) \rightleftarrows \mathcal{S}\mathsf{pc}_*^{\mathbb{A}^1}(S) : \underline{\mathrm{Hom}}_*((X,x),-).$$

Hence we get the following natural isomorphisms: for $(X,x),(Y,y),(Z,z) \in \mathcal{S}\mathsf{pc}_*^{\mathbb{A}^1}(S)$,

$$[(X,x) \wedge^{\mathrm{L}} (Z,z), (Y,y)]_{\mathbb{A}^1,*} \cong [(X,x), \mathrm{R}\underline{\mathrm{Hom}}_*((Z,z),(Y,y))]_{\mathbb{A}^1,*}.$$

See [**1**, Proposition 4.36] for a discussion.

We have the *simplicial circle* $S^{1,0} = S^1_s = \Delta^1/\partial\Delta^1$ and the *Tate circle* $S^{1,1} = S^1_t = \mathbb{G}_m$ (pointed by the rational point 1). For integers $a \geqslant b \geqslant 0$, we have the *bigraded spheres* $S^{a,b} = (S^{1,1})^{\wedge b} \wedge (S^{1,0})^{\wedge(a-b)} = (S^1_t)^{\wedge b} \wedge (S^1_s)^{\wedge(a-b)}$. Thus $S^{a,b} \wedge S^{c,d} = S^{a+c,b+d}$. The usual simplicial sphere S^n is thus $S^{n,0}$.

For $X \in \mathcal{S}\mathsf{pc}_*^{\mathbb{A}^1}(S)$ a pointed simplicial presheaf and $i,j \in \mathbb{N}$, we define its *bigraded (simplicial) homotopy sheaf* $\pi_{i,j}(X)$ to be the Nisnevich sheafification of the presheaf

$$U \mapsto [S^{i+j,j} \wedge U_+, X]_{\mathrm{Nis},*} = [S^i \wedge \mathbb{G}_m^{\wedge j} \wedge U_+, X]_{\mathrm{Nis},*}.$$

We also define the *bigraded $\mathbb{A}^1$-homotopy sheaf* $\pi_{i,j}^{\mathbb{A}^1}(X)$ to be the Nisnevich sheafification of the presheaf

$$U \mapsto [S^{i+j,j} \wedge U_+, X]_{\mathbb{A}^1,*} = [S^i \wedge \mathbb{G}_m^{\wedge j} \wedge U_+, X]_{\mathbb{A}^1,*}.$$

So $\pi_n(X) = \pi_{n,0}(X)$ and $\pi_{i,j}^{\mathbb{A}^1}(X) = \pi_i^{\mathbb{A}^1}(\mathbf{R}\Omega^j_{\mathbb{G}_m} X), \pi_n^{\mathbb{A}^1}(X) = \pi_{n,0}^{\mathbb{A}^1}(X)$.

Like in $\mathsf{s}\mathcal{S}\mathsf{et}_*$, we set $\Sigma = \Sigma_s := S^1 \wedge (-), \Omega = \Omega_s = \Omega_{S^1} := \underline{\mathrm{Hom}}_*(S^1,-) : \mathcal{S}\mathsf{pc}_*^{\mathbb{A}^1}(S) \to \mathrm{L}_{\mathrm{Nis}}\mathcal{S}\mathsf{pc}_*^{\mathbb{A}^1}(S)$ (where we view S^1 as a constant simplicial presheaf) to be the *simplicial suspension* and the *simplicial looping* functors. We sometimes use the same symbols to denote their derived functors. We also have the *T-suspension functor* $\Sigma_T := (\mathbb{P}^1, \infty) \wedge (-)$. We denote $\Omega_{\mathbb{G}_m} := \underline{\mathrm{Hom}}_*(\mathbb{G}_m,-), \Omega_T := \underline{\mathrm{Hom}}_*(T,-) : \mathcal{S}\mathsf{pc}_*^{\mathbb{A}^1}(S) \to \mathcal{S}\mathsf{pc}_*^{\mathbb{A}^1}(S)$. Then $\Omega_T \simeq \Omega_{S^1}\Omega_{\mathbb{G}_m} \simeq \Omega_{\mathbb{G}_m}\Omega_{S^1}$, and for $\mathbf{M} \in \mathcal{A}\mathsf{b}_S^{\mathbb{A}^1}, n \geqslant 1$, we have $\Omega_{S^1}\mathrm{K}(\mathbf{M},n) \simeq \mathrm{K}(\mathbf{M},n-1), \Omega_{\mathbb{G}_m}\mathrm{K}(\mathbf{M},n) \simeq \mathrm{K}(\mathbf{M}_{-1},n), \Omega_T\mathrm{K}(\mathbf{M},n) \simeq \mathrm{K}(\mathbf{M}_{-1},n-1)$.[5] If $S = \mathrm{Spec}(k)$ for a perfect field k and X is $\mathbb{A}^1$-connected, then $\pi_{i,j}^{\mathbb{A}^1}(X) = \pi_i^{\mathbb{A}^1}(\mathbf{R}\Omega^j_{\mathbb{G}_m} X) = (\pi_i^{\mathbb{A}^1}(X))_{-j}, i \geqslant 1, j \geqslant 0$ (see [**68**, Theorem 6.13]).[6]

The following can be found in [**1**, §4.6], obtained by manipulating homotopy push-out diagrams, taking advantage of the $\mathbb{A}^1$-contractibility of $\mathbb{A}^n$.

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[5] To obtain the result for $\Omega_{\mathbb{G}_m}$, we find by the last formula in Section 2.4 that $\Omega_{\mathbb{G}_m}\mathrm{K}(\mathbf{M},n)(U) = \mathrm{Map}_*((h_U)_+ \wedge \mathbb{G}_m, \mathrm{K}(\mathbf{M},n))$ for any $U \in \mathcal{S}\mathsf{m}_S$. Since there is a split exact sequence $0 \to \mathbb{Z}(U) \to \mathbb{Z}(U \times \mathbb{G}_m) \to \widetilde{\mathbb{Z}}((h_U)_+ \wedge \mathbb{G}_m) \to 0$, taking mapping space to $\mathrm{K}(\mathbf{M},n)$ in $\mathsf{s}\mathcal{P}\mathsf{re}\mathcal{A}\mathsf{b}(\mathcal{C})$ we see that $\mathrm{Map}_*((h_U)_+ \wedge \mathbb{G}_m, \mathrm{K}(\mathbf{M},n)) = \mathrm{Map}(\widetilde{\mathbb{Z}}((h_U)_+ \wedge \mathbb{G}_m), \mathrm{K}(\mathbf{M},n))$ is the kernel of the map of simplicial abelian groups $\mathrm{K}(\mathbf{M}(U \times \mathbb{G}_m), n) \to \mathrm{K}(\mathbf{M}(U), n) = \mathrm{K}(\mathbf{M}_{-1}(U), n)$, hence the result.

[6] It's important to have $\mathbb{A}^1$-connectedness here, but see [**8**, Corollary 2.3] for other special cases.

Proposition 3.2.1. *In the pointed motivic model category $\mathcal{S}\mathsf{pc}^{\mathbb{A}^1}_*(S)$ there are the following $\mathbb{A}^1$-weak equivalences:*

$$\Sigma\mathbb{G}_m \simeq (\mathbb{P}^1,\infty) \simeq \mathbb{A}^1/(\mathbb{A}^1\setminus\{0\}) \simeq \mathbb{P}^1/\mathbb{A}^1,$$
$$S^{2n-1,n} \simeq \mathbb{A}^n\setminus\{0\},$$
$$(\mathbb{P}^1,\infty)^{\wedge n} \simeq \Sigma(\mathbb{A}^n\setminus\{0\}) \simeq S^{2n,n} \simeq S^n\wedge\mathbb{G}_m^{\wedge n} \simeq \mathbb{A}^n/(\mathbb{A}^n\setminus\{0\}), n\geqslant 1.$$

Let $E\to X$ be a vector bundle with sheaf of sections $\mathscr{E}$ which is a finite rank locally free $\mathscr{O}_X$-module (so $E = \mathbb{V}_X(\mathscr{E}) := \mathrm{Spec}_X(\mathrm{Sym}^\bullet_{\mathscr{O}_X}\mathscr{E}^\vee), \mathbb{P}(E) = \mathrm{Proj}_X(\mathrm{Sym}^\bullet_{\mathscr{O}_X}\mathscr{E}^\vee)$). The zero section gives a closed embedding $X\to E$. We define the *Thom space* as the pointed sheaf $\mathrm{Th}(E) = \mathrm{Th}(E/X) := E/(E-X)$.

Proposition 3.2.2. *Let $X\in\mathcal{S}\mathsf{m}_S$ and let $E\to X$ be a vector bundle. Then, there is an isomorphism $\mathrm{Th}(E/X)\cong \mathbb{P}(E\oplus\mathbb{A}^1_X)/(\mathbb{P}(E\oplus\mathbb{A}^1_X)-X)$ in the pointed category of sheaves $\mathcal{S}\mathsf{hv}(\mathcal{S}\mathsf{m}_S,\mathrm{Nis})_*$ (here $E\oplus\mathbb{A}^1_X := E\times_X\mathbb{A}^1_X$).*

Proposition 3.2.3 ([**69**, Theorem 3.2.17])**.** *Let S be a noetherian scheme of finite Krull dimension.*

(1) *Let $X\in\mathcal{S}\mathsf{m}_S$, then there is a canonical isomorphism in $\mathcal{S}\mathsf{hv}(\mathcal{S}\mathsf{m}_S,\mathrm{Nis})_*$ ($\simeq$ means $\mathbb{A}^1$-weak equivalence):*
$$\mathrm{Th}(\mathbb{A}^n_X/X) \simeq \Sigma^n_T X_+ \cong (\mathbb{P}^1,\infty)^{\wedge n}\wedge X_+.$$

(2) *Let $X_1,X_2\in\mathcal{S}\mathsf{m}_S$ and let $E_1\to X_1, E_2\to X_2$ be vector bundles. Then there is a canonical isomorphism in $\mathcal{S}\mathsf{hv}(\mathcal{S}\mathsf{m}_S,\mathrm{Nis})_*$:*
$$\mathrm{Th}(E_1\times E_2/X_1\times X_2)\cong \mathrm{Th}(E_1/X_1)\wedge\mathrm{Th}(E_2/X_2).$$

(3) *Let $X\in\mathcal{S}\mathsf{m}_S$ and let $E\to X$ be a vector bundle, denote $i:\mathbb{P}(E)\to\mathbb{P}(E\oplus\mathbb{A}^1_X)$ the closed embedding at infinity. Then there is an $\mathbb{A}^1$-weak equivalence*
$$\mathbb{P}(E\oplus\mathbb{A}^1_X)/\mathbb{P}(E) \xrightarrow{\simeq} \mathrm{Th}(E/X)$$
and thus we have an $\mathbb{A}^1$-homotopy cofibre sequence
$$\mathbb{P}(E)\to\mathbb{P}(E\oplus\mathbb{A}^1_X)\to\mathrm{Th}(E/X).$$
In particular there is an $\mathbb{A}^1$-weak equivalence
$$\mathbb{P}^n/\mathbb{P}^{n-1}\xrightarrow{\simeq} S^{2n,n}.$$
Moreover, let $j: E\to\mathbb{P}(E\oplus\mathbb{A}^1_X)$ be the open complement of i, then there is a morphism $q:\mathbb{P}(E\oplus\mathbb{A}^1_X)\setminus j(X)\to\mathbb{P}(E)$ such that $qi=\mathrm{id}$ which is a Zariski locally trivial $\mathbb{A}^1$-bundle.

Proposition 3.2.4. *Let $p:E\to X$ be a map in $\mathcal{S}\mathsf{m}_S$ which a Zariski locally trivial affine bundle. Then $p:E\to X$ is an $\mathbb{A}^1$-weak equivalence.*

Corollary 3.2.5. *Let k be a field, then mapping a matrix to its last column gives an $\mathbb{A}^1$-weak equivalence in $\mathcal{S}\mathsf{pc}^{\mathbb{A}^1}(k)$*
$$\mathrm{SL}_{n+1}/\mathrm{SL}_n \simeq \mathbb{A}^{n+1}\setminus\{0\}, n\geqslant 1.$$

Recall that for a closed immersion $i:Z\to X$ with ideal sheaf $\mathscr{I}$, the *normal bundle* is $\mathrm{N}_{Z/X} := \mathbb{V}_Z(\mathscr{N}_{Z/X})$ over Z, where $\mathscr{N}_{Z/X} := \mathscr{C}^\vee_{Z/X} = \mathscr{H}om_{\mathscr{O}_Z}(\mathscr{C}_{Z/X},\mathscr{O}_Z)$ is the *normal sheaf* and $\mathscr{C}_{Z/X} := \mathscr{I}/\mathscr{I}^2$ is the *conormal sheaf*. If i is a regular embedding, then $\mathscr{C}_{Z/X},\mathscr{N}_{Z/X}$ are locally free $\mathscr{O}_Z$-modules of finite rank.

Theorem 3.2.6 (Homotopy purity)**.** *Let S be a noetherian scheme of finite Krull dimension, let $i:Z\to X$ be a (regular) embedding in $\mathcal{S}\mathsf{m}_S$. Then there is an $\mathbb{A}^1$-weak equivalence*
$$X/(X-i(Z))\xrightarrow{\simeq}\mathrm{Th}(\mathrm{N}_{Z/X}).$$

Remark 3.2.7. For a proof of this result, see [**69**, Theorem 3.2.23]. This result can be viewed as a *tubular neighbourhood theorem* in the motivic setting (note however that there is no such result for schemes in the Zariski site).

The following result about the long exact sequence of sheaves of motivic homotopy groups associated to an $\mathbb{A}^1$-homotopy fibre sequence is of fundamental importance in motivic homotopy theory.

Theorem 3.2.8. *Let k be a perfect field, let $F\to E\xrightarrow{q} B$ be an $\mathbb{A}^1$-fibre sequence in the motivic model category $\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{S}\mathsf{m}_k)_*$. There are the* boundary map *maps $\partial:\pi^{\mathbb{A}^1}_n B\to\pi^{\mathbb{A}^1}_{n-1}F$, which are a homomorphisms of sheaves of group for $n\geqslant 2$, fitting into a long exact sequence*
$$\cdots\to\pi^{\mathbb{A}^1}_n F\xrightarrow{i_*}\pi^{\mathbb{A}^1}_n E\xrightarrow{q_*}\pi^{\mathbb{A}^1}_n B\xrightarrow{\partial}\pi^{\mathbb{A}^1}_{n-1}F\to\cdots\to\pi^{\mathbb{A}^1}_1 B\xrightarrow{\partial}\pi^{\mathbb{A}^1}_0 F\xrightarrow{i_*}\pi^{\mathbb{A}^1}_0 E\xrightarrow{q_*}\pi^{\mathbb{A}^1}_0 B.$$
This sequence is natural in the $\mathbb{A}^1$-fibre sequence $F\to E\xrightarrow{q} B$. There is a (left) action of $\pi^{\mathbb{A}^1}_1 B$ on $\pi^{\mathbb{A}^1}_0 F$.

Moreover, the exactness at $\pi^{\mathbb{A}^1}_0 F$ can be strengthened as follows: for any local sections $[u],[u']$ of $\pi^{\mathbb{A}^1}_0 F$, $i_*([u]) = i_*([u'])\in\pi^{\mathbb{A}^1}_0 E$ iff $[u]$ and $[u']$ are in the same orbit of some local sections of the $\pi^{\mathbb{A}^1}_1 B$-action.

Now we are ready to state the following result about Moore-Postnikov towers in the motivic model categories, which, as in the classical topological setting, is a fundamental tool in obstruction theory. We only state the result for $\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{S}\mathsf{m}_k)_*$ but it also works for $\mathsf{s}\mathcal{S}\mathsf{hv}(\mathcal{S}\mathsf{m}_k)_*$.

Theorem 3.2.9 (Moore-Postnikov systems in motivic model category)**.** *Let k be a perfect field, let $F\to E\xrightarrow{q} B$ be an $\mathbb{A}^1$-fibre sequence in the motivic model category $\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{S}\mathsf{m}_k)_*$, with all of F,E,B being $\mathbb{A}^1$-connected and $\mathbb{A}^1$-fibrant. Denote $\pi^{\mathbb{A}^1}_1 E =: \mathbf{G}\in\mathcal{G}\mathsf{r}^{\mathbb{A}^1}_k$. Then there is the* Moore-Postnikov tower
$$E\to\cdots\xrightarrow{q_{n+1}}E_n\xrightarrow{q_n}E_{n-1}\xrightarrow{q_{n-1}}\cdots\xrightarrow{q_2}E_1\xrightarrow{q_1}E_0\xrightarrow{p_0}B,$$
in $\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{S}\mathsf{m}_k)_*$, and maps $i_n:E\to E_n$, $p_n:E_n\to B$ with the following properties:

(1) *All the spaces E_n are $\mathbb{A}^1$-fibrant, the maps $q_n, n \geqslant 1$ are $\mathbb{A}^1$-fibrations, and $p_0 : E_0 \to B$ is an $\mathbb{A}^1$-weak equivalence.*
(2) $i_{n-1} = q_n i_n, p_{n-1} q_n = p_n, \forall n \geqslant 1$.
(3) $p_n i_n = q, \forall n \in \mathbb{N}$.

$$\begin{array}{ccccc} & & E_{n+1} & & \\ & \overset{i_{n+1}}{\nearrow} & \downarrow{\scriptstyle q_{n+1}} & \overset{p_{n+1}}{\searrow} & \\ E & \xrightarrow{i_n} & E_n & \xrightarrow{p_n} & B \end{array}$$

(4) *The $\mathbb{A}^1$-homotopy fibre $F(q_n)$ of q_n $(n \geqslant 1)$ is $\mathbb{A}^1$-weakly equivalent to $\mathrm{K}(\pi_n^{\mathbb{A}^1} F, n)$, hence for each $n \geqslant 1$ we have an $\mathbb{A}^1$-homotopy fibre sequence*

$$\mathrm{K}(\pi_n^{\mathbb{A}^1} F, n) \to E_n \xrightarrow{q_n} E_{n-1}.$$

(5) *There are homotopy pullback diagrams in $(\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{S}\mathsf{m}_k) \downarrow \mathrm{B}\mathbf{G})$ (with model structure induced from the Morel-Voevodsky motivic model structure on $\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{S}\mathsf{m}_k)$)*

$$\begin{array}{ccc} E_n & \longrightarrow & \mathrm{B}\mathbf{G} \\ \downarrow{\scriptstyle q_n} & & \downarrow{\scriptstyle s_n} \\ E_{n-1} & \xrightarrow{k_{n+1}} & \mathrm{K}^{\mathbf{G}}(\pi_n^{\mathbb{A}^1} F, n+1), \end{array}$$

for a unique $[k_{n+1}] \in [E_{n-1}, \mathrm{K}^{\mathbf{G}}(\pi_n^{\mathbb{A}^1} F, n+1)]_{\mathbb{A}^1}$, for all $n \geqslant 2$.
(6) *The map $E \to \operatorname{holim}_{n \in \mathbb{N}^{\mathrm{op}}} E_n$ is a local weak equivalence hence an $\mathbb{A}^1$-weak equivalence.*

Specializing to the case $E = *$ or $B = *$, one gets respectively the Whitehead tower *and* Postnikov tower. *Moreover, by the above existence results, we have*

(7) *For any $j \leqslant n$, $(i_n)_* : \pi_j^{\mathbb{A}^1} E \xrightarrow{\cong} \pi_j^{\mathbb{A}^1} E_n$.*
(8) *For any $j \geqslant n+2$, $(p_n)_* : \pi_j^{\mathbb{A}^1} E_n \xrightarrow{\cong} \pi_j^{\mathbb{A}^1} B$.*
(9) *There are exact sequences*

$$0 \to \pi_{n+1}^{\mathbb{A}^1} E_n \xrightarrow{(p_n)_*} \pi_{n+1}^{\mathbb{A}^1} B \xrightarrow{\partial} \pi_n^{\mathbb{A}^1} F, n \geqslant 0,$$

where ∂ is the connecting homomorphism in the homotopy long exact sequence of the homotopy fibre sequence $F \to E \xrightarrow{q} B$.
(10) *There are $\mathbb{A}^1$-homotopy fibre sequences*

$$F[n] \to E_n \xrightarrow{p_n} B,$$

$$F(q_n) \to F[n] \xrightarrow{q'_n} F[n-1] \ (n \geqslant 1),$$

where the map q'_n is induced from $q_n : E_n \to E_{n-1}$ and $F[n]$ is the n-th stage of the Postnikov tower for F.

Remark 3.2.10. We make a few comments about the above result. First of all, we can assume from the beginning that $E \xrightarrow{q} B$ is a genuine $\mathbb{A}^1$-fibration. The construction before $\mathbb{A}^1$-localization is done in **[68**, Appendix B**]**, the proof relies on some finiteness result, which can be found in **[69**, Theorem 2.1.37**]** and **[68**, Appendix B**]**, and for a modern discussion, see **[58**, Theorem 2.24 and Corollary 2.25**]** or **[57**, Chapter 7**]**, which says that the $(\infty,1)$-topos of $(\infty,1)$-sheaves on the Nisnevich site of S has homotopy dimension (in the sense of **[57]**) bounded above by $\dim(S)$, and in particular the $(\infty,1)$-topos of $(\infty,1)$-sheaves is hypercomplete.

Then by Morel's tower construction for the Jardine model category, or for $\mathrm{L}_{\mathrm{Nis}}\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{S}\mathsf{m}_S)_{*,\mathrm{proj}}$, we have a Moore-Postnikov tower (E'_n) (on the simplicial level), for which the $\mathbb{A}^1$-localization maps $E'_n \to \mathrm{L}_{\mathbb{A}^1} E'_n$ are *local weak equivalences* for all n by analogs of (7)-(9) in that setting, which yield that $\pi_j E'_n \in \mathcal{G}\mathrm{r}_k^{\mathbb{A}^1}$ for all j, and Theorem 3.1.31 (note $\pi_n F = \pi_n^{\mathbb{A}^1} F$ etc.). We then inductively factorize the map $E'_n \to E'_{n-1} \to E_{n-1}$ to $E'_n \to E_n \xrightarrow{q_n} E_{n-1}$, the first map $E'_n \to E_n$ is an $\mathbb{A}^1$-weak equivalence and hence necessarily a *local weak equivalence*, q_n is an $\mathbb{A}^1$-fibration. Hence (E'_n) and (E_n) have the same homotopy limit in $\mathrm{L}_{\mathrm{Nis}}\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{S}\mathsf{m}_S)_{*,\mathrm{proj}}$. While the homotopy limit of (E_n) in $\mathrm{L}_{\mathrm{Nis}}\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{S}\mathsf{m}_S)_{*,\mathrm{proj}}$ is also its homotopy limit in the motivic model category. We obtain the delicate homotopy convergence statement (6).

Note that we don't assume that q induces an isomorphism on $\pi_1^{\mathbb{A}^1}$ or the homotopy fibre F to be $\mathbb{A}^1$-simply connected. This is seen by using Morel's results applied to the first stage $i_1 : E \to E_1$ (suitably reindex the stages), which then satisfies Morel's assumption, hence the homotopy convergence in our more general setting follows from Morel's results as well.

Statement (5) needs a detailed study of fibrations with Eilenberg-Mac Lane spaces as fibers, the base of the universal fibration is a kind of twisted Eilenberg-Mac Lane space; then the given fibration is pulled-back along the k-invariant—the homotopy class of a map from the base to the universal base, see **[68**, Appendix B**]**. The claimed homotopy pullback diagrams are the same as those given in **[68**, Appendix B**]**, since all the vertices in the diagrams are locally weak equivalent to their $\mathbb{A}^1$-localization. In essence, this implies that the towers in the Moore-Postnikov systems are then constructed stage by stage via these k-invariants. (In statement (5), we don't deal with the initial stage $E_1 \xrightarrow{q_1} E_0$ of the tower as we won't use it , but the universal fibration does exist, it's rather complicated. For a description in classical topological setting, see **[38, 39, 16, 75, 37]**. Cf. Theorem 1.8.12.)

Other statements can be formally deduced from the long exact sequence of sheaves of motivic homotopy groups associated to an $\mathbb{A}^1$-homotopy fibre sequence (Theorem 3.2.8).

As in the classical topological setting, the Whitehead tower and Postnikov tower corresponding respectively to the special case $E = *$ and $B = *$, the precise statement of which we omit, are also quite often encountered in the study of motivic homotopy theory, see e.g. **[7]**.

This kind of result is also stated in **[7, 9]** in slightly different forms, where it's used to deduce splitting results for vector bundles over low dimensional smooth affine schemes over a perfect field.

Example 3.2.11 (Examples of $\mathbb{A}^1$-homotopy sheaves)**.** *Here we summarize some explicit results of $\mathbb{A}^1$-homotopy sheaves and/or homotopy sheaves of some spheres, algebraic groups and classifying spaces. Throughout, k is a perfect field, and all $\mathbb{A}^1$-fibre sequences are in the motivic model category $\mathcal{S}\mathsf{pc}_*^{\mathbb{A}^1}(k)$.*

(1) *There is an $\mathbb{A}^1$-fibre sequence*

$$\mathbb{G}_m \to \mathrm{E}\mathbb{G}_m \to \mathrm{B}\mathbb{G}_m.$$

We have, for $i, j \geqslant 0$,

$$\pi_i(\mathbb{G}_m) \cong \pi_i^{\mathbb{A}^1}(\mathbb{G}_m) \cong \begin{cases} \mathbb{G}_m, & i = 0; \\ 0, & i > 0. \end{cases} \qquad \pi_{i,j}^{\mathbb{A}^1}(\mathrm{B}\mathbb{G}_m) \cong \begin{cases} \mathbb{G}_m, & (i,j) = (1,0); \\ \mathbb{Z}, & (i,j) = (1,1); \\ 0, & \textit{else}. \end{cases}$$

For details, see [**8**, Example 2.4].

(2) *For $n \geqslant 2$, we have*

$$\pi_i^{\mathbb{A}^1}(\mathbb{A}^n \setminus \{0\}) \cong \begin{cases} 0, & i < n-1; \\ \mathbf{K}_n^{\mathrm{MW}}, & i = n-1. \end{cases} \qquad \pi_i^{\mathbb{A}^1}(\mathbb{P}^n) \cong \begin{cases} \mathbb{G}_m, & i = 1; \\ 0, & i = 0 \text{ or } 1 < i < n; \\ \mathbf{K}_{n+1}^{\mathrm{MW}}, & i = n. \end{cases}$$

See [**4**, Proposition 2.22 and Lemma 2.23]. *There are universal $\mathbb{A}^1$-coverings (hence $\mathbb{A}^1$-fiber sequences) $\mathbb{G}_m \to \mathbb{A}^n \setminus \{0\} \to \mathbb{P}^{n-1}$ (see* [**68**, Lemma 7.5 and Theorem 7.13]*). The sheaf $\pi_1^{\mathbb{A}^1}(\mathbb{P}^1)$ is highly non-commutative (see* [**68**, §7.3]*) and there is a central extension*

$$0 \to \mathbf{K}_2^{\mathrm{MW}} \to \pi_1^{\mathbb{A}^1}(\mathbb{P}^1) \to \mathbb{G}_m \to 0.$$

(3) *For $n \geqslant 1$, SL_n is $\mathbb{A}^1$-connected and BSL_n is $\mathbb{A}^1$-1-connected. The sequence $\mathrm{SL}_n \hookrightarrow \mathrm{GL}_n \xrightarrow{\det} \mathbb{G}_m$ induces an $\mathbb{A}^1$-fibre sequence*

$$\mathrm{BSL}_n \to \mathrm{BGL}_n \to \mathrm{B}\mathbb{G}_m.$$

We have $\pi_1^{\mathbb{A}^1}(\mathrm{BGL}_n) \cong \mathbb{G}_m$ and the inclusion $\mathrm{SL}_n \hookrightarrow \mathrm{GL}_n$ induces an isomorphism $\pi_i^{\mathbb{A}^1}(\mathrm{BSL}_n) \cong \pi_i^{\mathbb{A}^1}(\mathrm{BGL}_n)$ for $i > 1$.

Moreover, $\pi_1^{\mathbb{A}^1}(\mathrm{SL}_2) \cong \mathbf{K}_2^{\mathrm{MW}}$ and $\pi_1^{\mathbb{A}^1}(\mathrm{SL}_n) \cong \mathbf{K}_2^{\mathrm{M}}$ for $n \geqslant 3$. For details, see [**4**, Lemma 3.9, 3.10, Proposition 3.11 and Theorem 3.12]. *Cf.* [**68**, Remark 1.28].

(4) *There are $\mathbb{A}^1$-fibre sequences*

$$\mathbb{A}^{n+1} \setminus \{0\} \simeq \mathrm{SL}_{n+1}/\mathrm{SL}_n \to \mathrm{BSL}_n \to \mathrm{BSL}_{n+1}$$

and

$$\mathbb{A}^{n+1} \setminus \{0\} \simeq \mathrm{GL}_{n+1}/\mathrm{GL}_n \to \mathrm{BGL}_n \to \mathrm{BGL}_{n+1}.$$

This gives the stability *result: the map*

$$\pi_i^{\mathbb{A}^1}(\mathrm{BSL}_n) \to \pi_i^{\mathbb{A}^1}(\mathrm{BSL}_{n+1})$$

is an epimorphism for $1 \leqslant i \leqslant n$ and an isomorphism for $1 \leqslant i \leqslant n-1$.

(5) *For $n \geqslant 1$, we have*

$$\pi_i(\mathrm{BGL}_n) \cong \begin{cases} \mathrm{GL}_n, & i = 1; \\ *, & i \neq 1. \end{cases} \qquad \pi_i^{\mathbb{A}^1}(\mathrm{BGL}_2) \cong \begin{cases} *, & i = 0; \\ \mathbb{G}_m, & i = 1; \\ \mathbf{K}_2^{\mathrm{MW}}, & i = 2. \end{cases} \qquad \pi_i^{\mathbb{A}^1}(\mathrm{BGL}_n) \cong \begin{cases} *, & i = 0; \\ \mathbb{G}_m, & i = 1; \\ \mathbf{K}_2^{\mathrm{M}}, & i = 2 \end{cases} \quad (n \geqslant 3).$$

Thus BGL_2 is not $\mathbb{A}^1$-fibrant (or $\mathbb{A}^1$-local) and by Proposition 3.1.26, $\mathrm{GL}_2 \notin \mathcal{G}\mathrm{r}_k^{\mathbb{A}^1}$. In general, the $\mathbb{A}^1$-localization map $\mathrm{BGL}_n \to \mathrm{L}_{\mathbb{A}^1}\mathrm{BGL}_n$ is not a local weak equivalence and hence BGL_n doesn't satisfy descent for all Nisnevich hypercovers or the Nisnevich BG-property.

In some sense, $\mathbb{A}^1$-homotopy theory acts as a magical prism, separating a single homotopy sheaf of $\mathrm{K}(G, n)$ for a Nisnevich sheaf of (abelian) groups to a sequence of strongly (strictly) $\mathbb{A}^1$-invariant sheaves of (abelian) groups—its $\mathbb{A}^1$-homotopy sheaves. This reflects the complexity of $\mathbb{A}^1$-homotopy theory. In view of Proposition 3.1.26 and 3.1.28, one should think of this as an advantage rather than a disadvantage, as one can then use our Moore-Postnikov decomposition result to study problems stage by stage, once we know enough about the $\mathbb{A}^1$-homotopy groups involved.

By Proposition 3.1.13, we see that for any $U \in \mathcal{S}\mathsf{m}_S$, there is a canonical bijection

$$[U, \mathrm{BGL}_n]_{\mathrm{Nis}} \cong \mathrm{H}^1_{\mathrm{Nis}}(U; \mathrm{GL}_n),$$

the right hand side is the set of isomorphism classes of Nisnevich locally trivial GL_n-torsors over U (here one may need the fact that GL_n is a *special* group, so étale hence Nisnevich locally trivial GL_n-torsors are also Zariski locally trivial), thus is identified with the set $\mathcal{V}_n(U)$ of isomorphism classes of rank n vector bundles on U. As BGL_n is not $\mathbb{A}^1$-fibrant in general, we don't have $[U, \mathrm{BGL}_n]_{\mathrm{Nis}} \cong [U, \mathrm{BGL}_n]_{\mathbb{A}^1}$, and if this is the case, then we would have $\mathcal{V}_n(U) \cong \mathcal{V}_n(U \times \mathbb{A}^1)$, which says that the Bass-Quillen conjecture holds for U.

This cannot be true for arbitrary U. We give a simple example here (cf. [**34**, §(11.17)]): Let k be a field and $U = \mathbb{P}^1_k$ with standard affine open cover $U^+ = \mathrm{Spec}(k[t]), U^- = \mathrm{Spec}(k[t^{-1}])$ (so $U^+ \cap U^- = \mathrm{Spec}(k[t, t^{-1}])$). Let $\mathbb{A}^1_k = \mathrm{Spec}(k[u])$, then $g(t,u) := \begin{pmatrix} t^{-1}u & u^2 - t \\ t^{-1} & u \end{pmatrix} \in \mathrm{GL}_2(k[t, t^{-1}, u])$ defines a Čech 1-cocycle on $\mathbb{P}^1_k \times_k \mathbb{A}^1_k$ with respect to the open cover $(U^+ \times_k \mathbb{A}^1_k, U^- \times_k \mathbb{A}^1_k)$, which gives a rank 2 vector bundle V on $\mathbb{P}^1_k \times_k \mathbb{A}^1_k$ (by gluing the trivial rank 2 vector bundles

over $U^+ \times_k \mathbb{A}^1_k$ and $U^- \times_k \mathbb{A}^1_k$ via $g(t,u)$). It's easy to find that $g(t,0) = \begin{pmatrix} 0 & -t \\ t^{-1} & 0 \end{pmatrix} \in \mathrm{GL}_2(k[t,t^{-1}])$ is cohomologous to $\begin{pmatrix} t & \\ & t^{-1} \end{pmatrix} \in \mathrm{GL}_2(k[t,t^{-1}])$, hence $V|_{\mathbb{P}^1_k \times \{0\}} \cong \mathscr{O}(1) \oplus \mathscr{O}(-1)$, while $g(t,1) = \begin{pmatrix} t^{-1} & 1-t \\ t^{-1} & 1 \end{pmatrix} \in \mathrm{GL}_2(k[t,t^{-1}])$ is cohomologous to $\begin{pmatrix} t & \\ & 1 \end{pmatrix} \in \mathrm{GL}_2(k[t,t^{-1}])$, hence $V|_{\mathbb{P}^1_k \times \{0\}} \cong \mathscr{O}(1) \oplus \mathscr{O}$. By Grothendieck's classification of vector bundles over $\mathbb{P}^1_k$ we see that $V|_{\mathbb{P}^1_k \times \{0\}} \ncong V|_{\mathbb{P}^1_k \times \{1\}}$ (or computing the dimension of the spaces of global sections on the two sides). So V is not in the image of the pull-back map $\mathcal{V}_2(\mathbb{P}^1_k) \to \mathcal{V}_2(\mathbb{P}^1_k \times_k \mathbb{A}^1_k)$.

On the other hand, if U is a smooth affine scheme over a perfect field k, then work of Morel **[68]** and later Asok-Hoyois-Wendt **[12]** shows that this is indeed the case, and BGL_n represents the functor $\mathcal{V}_n$, after restricting to smooth affine schemes over a perfect field k (this is a very lucky fact). These are deduced by studying the affine Brown-Gersten property. We summarize the related results briefly in the next section.

3.3. Affine BG-property and affine representability results in $\mathbb{A}^1$-homotopy theory

In this section, we collect some results about affine representability results in $\mathbb{A}^1$-homotopy theory, pioneered by Morel **[68]**, Schlichting **[80]** and further developed by Asok-Hoyois-Wendt **[12, 13]**. Most of the results below are from **[12, 13]**, though sometimes we don't make it precise where a result is in those papers, and sometimes we only state a result convenient for what we need (hence less general than the result in those papers).

We use R_{Nis} (resp. R_{Zar}) to denote a fibrant replacement functor of $\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{S}\mathsf{m}_S)$ with Nisnevich (resp. Zariski) local model structure (endowing $\mathcal{S}\mathsf{m}_S$ with the Nisnevich (resp. Zariski) topology).

We denote by $\mathcal{S}\mathsf{m}_S^{\mathrm{aff}}$ the full subcategory of $\mathcal{S}\mathsf{m}_S$ consisting of (absolutely) affine schemes. It inherits a Grothendieck topology from the Nisnevich site $(\mathcal{S}\mathsf{m}_S, \mathrm{Nis})$. The resulting topology is called the *affine Nisnevich topology* on $\mathcal{S}\mathsf{m}_S^{\mathrm{aff}}$. We get the *affine Nisnevich site* $(\mathcal{S}\mathsf{m}_S^{\mathrm{aff}}, \mathrm{AffNis})$. We similarly have the *affine Zariski site* $(\mathcal{S}\mathsf{m}_S^{\mathrm{aff}}, \mathrm{AffZar})$ (which is just the Zariski topology).

Using some technical results about Henselian pairs, it's shown ([**12**, Proposition 2.3.2]) that the affine Nisnevich topology on $\mathcal{S}\mathsf{m}_S^{\mathrm{aff}}$ is given by a *cd-structure* consisting of cartesian squares of the form

$$\begin{array}{ccc} \mathrm{Spec}(B_f) & \longrightarrow & \mathrm{Spec}(B) \\ \downarrow & & \downarrow{\scriptstyle \pi} \\ \mathrm{Spec}(A_f) & \xrightarrow{i} & \mathrm{Spec}(A) \end{array} \tag{Qaff}$$

with $f \in A, \pi$ is étale and induces an isomorphism $A/f \cong B/f$, and where $\mathrm{Spec}(A) \in \mathcal{S}\mathsf{m}_S^{\mathrm{aff}}$. Similar results hold for $(\mathcal{S}\mathsf{m}_S^{\mathrm{aff}}, \mathrm{AffZar})$ with B replaced by A_g ([**12**, Proposition 2.1.3]) with $(f,g) = Af + Ag = A$.

The affine representability results presented in this section can be seen as a study of the relationship among the affine Nisnevich site $(\mathcal{S}\mathsf{m}_S^{\mathrm{aff}}, \mathrm{AffNis})$, the affine Zariski site $(\mathcal{S}\mathsf{m}_S^{\mathrm{aff}}, \mathrm{AffZar})$ and the Nisnevich site $(\mathcal{S}\mathsf{m}_S, \mathrm{Nis})$.

Definition 3.3.1 ([**12**, Definition 3.2.1])**.** Let $F \in \mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{S}\mathsf{m}_S^{\mathrm{aff}})$. We say that F satisfies the *affine (Nisnevich) Brown-Gersten property* (*affine (Nisnevich) BG-property* for short) or the *affine Nisnevich excision property* if

- $F(\varnothing)$ is a weakly contractible,
- for any affine Nisnevich square as (Qaff), the corresponding square F(Qaff) is homotopy cartesian in $\mathsf{s}\mathcal{S}\mathsf{et}$.

We say that $F \in \mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{S}\mathsf{m}_S)$ has this property if its restriction to $\mathcal{S}\mathsf{m}_S^{\mathrm{aff}}$ has this property.

We similarly have the *affine Zariski BG-property* or the *affine Zariski excision property.*

There is a Quillen pair

$$i^* : \mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{S}\mathsf{m}_S)_{\mathrm{inj}} \rightleftarrows \mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{S}\mathsf{m}_S^{\mathrm{aff}})_{\mathrm{inj}} : i_*,$$

where i^* preserves (sectionwise) weak equivalences and cofibrations, with derived adjunction $(i^*, \mathbf{R}i_*)$, where the right derived functor is given by

$$\mathbf{R}i_*F(V) = \operatorname*{hocolim}_{U \in (\mathcal{S}\mathsf{m}_S^{\mathrm{aff}} \downarrow V)} F(U)$$

for $F \in \mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{S}\mathsf{m}_S^{\mathrm{aff}}), V \in \mathcal{S}\mathsf{m}_S$, and is fully faithful ([**12**, Lemma 3.3.1]). The functors $(i^*, \mathbf{R}i_*)$ preserve those simplicial presheaves having (affine) BG-property, and they restrict to an equivalence between the full sub-categories of the homotopy categories consist of these simplicial presheaves ([**12**, Lemma 3.3.2]); we can call $\mathbf{R}i_*i^*F$ for $F \in \mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{S}\mathsf{m}_S)$ the *affine replacement* of F ([**12**, Remark 3.3.3]).

Theorem 3.3.2 ([**12**, Theorem 3.3.4])**.** *Let $F \in \mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{S}\mathsf{m}_S)$.*

1. *If F has the affine Zariski excision property, then the map $F(X) \to (R_{\mathrm{Zar}}F)(X)$ is a weak equivalence for every $X \in \mathcal{S}\mathsf{m}_S^{\mathrm{aff}}$.*
2. *If F has the affine Nisnevich excision property, then $R_{\mathrm{Zar}}F$ has the Nisnevich excision property.*

Recall from Section 3.1 that we have the singular $\mathbb{A}^1$-construction $\mathrm{Sing}^{\mathbb{A}^1} : \mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{S}\mathsf{m}_S) \to \mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{S}\mathsf{m}_S)$ given by $(\mathrm{Sing}^{\mathbb{A}^1} F)(U) := \mathrm{diag}\, F(\not{\Delta}^\bullet_U)$ for $U \in \mathcal{S}\mathsf{m}_S$ and $F \in \mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{S}\mathsf{m}_S)$, so $(\mathrm{Sing}^{\mathbb{A}^1} F)_n(U) = F_n(\not{\Delta}^n_U)$. There is a canonical trivial $\mathbb{A}^1$-cofibration $F \to \mathrm{Sing}^{\mathbb{A}^1} F$.

Theorem 3.3.3 ([**12**, Theorem 4.2.3])**.** *Let $F \in \mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{S}\mathsf{m}_S)$ (resp. $F \in \mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{S}\mathsf{m}_S^{\mathrm{aff}})$) be such that for every $U \in \mathcal{S}\mathsf{m}_S$ (resp. $U \in \mathcal{S}\mathsf{m}_S^{\mathrm{aff}}$), the map $\pi_0(F(U)) \to \pi_0(F(U \times \mathbb{A}^1))$ is a bijection (i.e. the presheaf $\pi_0 \circ F : U \mapsto \pi_0(F(U))$ is $\mathbb{A}^1$-invariant). If F has the (affine) Nisnevich excision property, then $\mathrm{Sing}^{\mathbb{A}^1} F$ has the same property.*

Proposition 3.3.4 ([**12**, Lemma 5.1.2]). *Assume $F \in \mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{S}\mathsf{m}_S)$ has the affine Zariski excision property. If F is $\mathbb{A}^1$-invariant on $\mathcal{S}\mathsf{m}_S^{\mathrm{aff}}$, then it is $\mathbb{A}^1$-invariant (on $\mathcal{S}\mathsf{m}_S$).*

Theorem 3.3.5 ([**12**, Theorem 5.1.3]). *Assume $F \in \mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{S}\mathsf{m}_S)$ has the affine Zariski excision property. If for every $U \in \mathcal{S}\mathsf{m}_S^{\mathrm{aff}}$, $\pi_0(F(U)) \to \pi_0(F(U \times \mathbb{A}^1))$ is a bijection, then $R_{\mathrm{Zar}} \operatorname{Sing}^{\mathbb{A}^1} F$ has Nisnevich excision property and is $\mathbb{A}^1$-invariant (on $\mathcal{S}\mathsf{m}_S$). Moreover, for every $U \in \mathcal{S}\mathsf{m}_S^{\mathrm{aff}}$, we have a canonical isomorphism*

$$\pi_0(F(U)) \xrightarrow{\cong} [U, F]_{\mathbb{A}^1}.$$

For $r \in \mathbb{N}, N \geqslant r$, let $\mathrm{Gr}_{r,N}$ be the Grassmannian of r planes in a dimension N affine space, and denote the colimit over all N by Gr_r, called the *infinite Grassmannian*. The tautological vector bundles on $\mathrm{Gr}_{r,N}$ induces a "vector bundle" (or a GL_r-torsor) on Gr_r. For every $X \in \mathcal{S}\mathsf{m}_S$, we have a canonical map $\mathrm{Hom}(X, \mathrm{Gr}_r) = \mathrm{Gr}_r(X) \to \mathcal{V}_r(X)$ (where $\mathcal{V}_r(X)$ is the set of isomorphism classes of rank r vector bundles on X), which is surjective if $X \in \mathcal{S}\mathsf{m}_k^{\mathrm{aff}}$. The following result is (a special case of) [**12**, Theorem 5.2.3].

Theorem 3.3.6 (Affine representability for vector bundles). *Let k be a perfect field and let $U \in \mathcal{S}\mathsf{m}_k^{\mathrm{aff}}$. Then, there is a bijection*

$$\mathcal{V}_r(U) \xrightarrow{\cong} [U, \mathrm{Gr}_r]_{\mathbb{A}^1}$$

fitting into the following commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}(U, \mathrm{Gr}_r) & & \\ \downarrow & \searrow & \\ \mathcal{V}_r(U) & \xrightarrow{\cong} & [U, \mathrm{Gr}_r]_{\mathbb{A}^1}, \end{array}$$

where the other two arrows are the canonical ones, both of which are surjective.

Thus we have a natural isomorphism of functors

$$\mathcal{V}_r(-) \cong [-, \mathrm{Gr}_r]_{\mathbb{A}^1} : (\mathcal{S}\mathsf{m}_k^{\mathrm{aff}})^{\mathrm{op}} \to \mathcal{S}\mathsf{et}.$$

We say that $F \in \mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{S}\mathsf{m}_S)$ is $\mathbb{A}^1$-*naive* if the canonical map $\operatorname{Sing}^{\mathbb{A}^1} F \to \mathrm{L}_{\mathbb{A}^1} F$ restricts to a sectionwise weak equivalence on $\mathcal{S}\mathsf{m}_S^{\mathrm{aff}}$ (see [**13**, Definition 2.1.1]). If $F \in \mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{S}\mathsf{m}_S)_*$ is $\mathbb{A}^1$-naive, then for every $n \in \mathbb{N}$ and $U \in \mathcal{S}\mathsf{m}_S^{\mathrm{aff}}$, we have an isomorphism

$$\pi_n((\operatorname{Sing}^{\mathbb{A}^1} F)(U)) \cong [S^n \wedge U_+, F]_{\mathbb{A}^1,*}.$$

See [**13**, Lemma 4.2.4] for an interesting result about $\mathbb{A}^1$-naivety.

Proposition 3.3.7 ([**13**, Proposition 2.1.3]). *An object $F \in \mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{S}\mathsf{m}_S)$ is $\mathbb{A}^1$-naive iff $\operatorname{Sing}^{\mathbb{A}^1} F$ has the affine Nisnevich excision property. If this is the case, then $R_{\mathrm{Zar}} \operatorname{Sing}^{\mathbb{A}^1} F$ has the Nisnevich excision property and is $\mathbb{A}^1$-invariant (on $\mathcal{S}\mathsf{m}_S$).*

Proposition 3.3.8 ([**13**, Proposition 2.2.1 and Corollary 2.2.2]). *Let $F \to G \to H$ be a sequence in $\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{S}\mathsf{m}_S)_*$ (or $\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{S}\mathsf{m}_S^{\mathrm{aff}})_*$) that is a sectionwise homotopy fiber sequence. If the presheaf $\pi_0 \circ H : U \mapsto \pi_0(H(U))$ is $\mathbb{A}^1$-invariant, then the induced sequence*

$$\operatorname{Sing}^{\mathbb{A}^1} F \to \operatorname{Sing}^{\mathbb{A}^1} G \to \operatorname{Sing}^{\mathbb{A}^1} H$$

is also a sectionwise homotopy fiber sequence. Moreover, the canonical map

$$\operatorname{Sing}^{\mathbb{A}^1} \mathbf{R}\Omega_s H \to \mathbf{R}\Omega_s \operatorname{Sing}^{\mathbb{A}^1} H$$

is a sectionwise weak equivalence.

Theorem 3.3.9 ([**13**, Theorem 2.2.4, 2.2.5]). *Let $F \to G \to H$ be a sequence in $\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{S}\mathsf{m}_S)_*$ that is a sectionwise homotopy fiber sequence.*

1. *Assume that G, H satisfy affine Nisnevich excision, and that the presheaves $\pi_0 \circ G, \pi_0 \circ H$ are $\mathbb{A}^1$-invariant on $\mathcal{S}\mathsf{m}_S^{\mathrm{aff}}$. Then F is $\mathbb{A}^1$-naive.*
2. *Assume that H satisfies affine Nisnevich excision, and that the presheaf $\pi_0 \circ H$ is $\mathbb{A}^1$-invariant on $\mathcal{S}\mathsf{m}_S^{\mathrm{aff}}$. Then $F \to G \to H$ is an $\mathbb{A}^1$-fiber sequence, i.e. $\mathrm{L}_{\mathbb{A}^1} F \to \mathrm{L}_{\mathbb{A}^1} G \to \mathrm{L}_{\mathbb{A}^1} H$ is a sectionwise homotopy fiber sequence.*

For a commutative ring R, let SO_n be the split special orthogonal group over R.

Proposition 3.3.10 ([**13**, Lemma 3.1.7]). *Let R be a commutative ring with $2 \in R^\times$ and let SO_n be the R-subgroup scheme of GL_n consisting of automorphisms of the standard hyperbolic form q_n. Assume that $n \geqslant 3$.*

1. *The quotient $\mathrm{SO}_n / \mathrm{SO}_{n-1}$ is isomorphic to a quadric hypersurface in $\mathbb{A}^n_R$ defined by $q_n = 1$.*
2. *The projection morphism $\mathrm{SO}_n \to \mathrm{SO}_n / \mathrm{SO}_{n-1}$ gives SO_n the structure of a Zariski locally trivial SO_{n-1}-torsor over the quotient hypersurface.*

Theorem 3.3.11 (Affine homotopy invariance for torsors). *Let k be an infinite field, let G be an isotropic (see* [**13**, Definition 3.3.5]*) reductive k-group. Then $\mathrm{H}^1_{\mathrm{Nis}}(-; G)$ is $\mathbb{A}^1$-invariant on $\mathcal{S}\mathsf{m}_k^{\mathrm{aff}}$. So $\mathrm{H}^1_{\mathrm{Nis}}(-; G)$ is also $\mathbb{A}^1$-invariant on $\mathcal{S}\mathsf{m}_S^{\mathrm{aff}}$ for any $S \in \mathcal{S}\mathsf{m}_k^{\mathrm{aff}}$.*

This corresponds to [**13**, Theorem 3.3.7]; if moreover k is perfect, then it has a converse, see [**13**, Remark 3.3.8]. We also mention that there are more general results for some classical group schemes in [**13**, Theorem 3.3.1 and Theorem 3.3.3]. These kinds of results need a detailed study of certain group schemes.

The following affine representability result for torsors is [**13**, Theorems 2.3.5 and 4.1.3]. There are more general results for classical groups $\mathrm{SL}_n, \mathrm{Sp}_{2n}$ in [**13**, Theorems 4.1.1 and 4.1.2], with weaker (though more technical) assumptions, giving affine representability results in $\mathbb{A}^1$-homotopy theory for oriented vector bundles and symplectic vector bundles.

Theorem 3.3.12 (Affine representability for Nisnevich locally trivial torsors)**.** *Let G be a finitely presented, smooth S-group scheme, such that the presheaf $\mathrm{H}^1_{\mathrm{Nis}}(-;G)$ is $\mathbb{A}^1$-invariant on $\mathcal{S}\mathsf{m}_S^{\mathrm{aff}}$. Then*

1. *$R_{\mathrm{Zar}}\,\mathrm{Sing}^{\mathbb{A}^1}\,\mathrm{B}_{\mathrm{Nis}}G$ has the Nisnevich excision property and is $\mathbb{A}^1$-invariant (on $\mathcal{S}\mathsf{m}_S$), where $\mathrm{B}_{\mathrm{Nis}}G = R_{\mathrm{Nis}}\mathrm{B}G$ is a fibrant replacement of $\mathrm{B}G$ in $\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{S}\mathsf{m}_S)_J$.*
2. *For $U \in \mathcal{S}\mathsf{m}_S^{\mathrm{aff}}$, there is a bijection*

$$\mathrm{H}^1_{\mathrm{Nis}}(U;G) \xrightarrow{\cong} [U, \mathrm{B}G]_{\mathbb{A}^1},$$

giving a natural isomorphism of functors

$$\mathrm{H}^1_{\mathrm{Nis}}(-;G) \cong [-, \mathrm{B}G]_{\mathbb{A}^1} : (\mathcal{S}\mathsf{m}_S^{\mathrm{aff}})^{\mathrm{op}} \to \mathcal{S}\mathsf{et}.$$

The assumption is satisfied if $S = \mathrm{Spec}(k)$ for an infinite field k, and G is an isotropic reductive k-group.

Theorem 3.3.13 (Affine representability for homogeneous spaces)**.** *Let G be a finitely presented, smooth S-group scheme and $H \subset G$ a finitely presented, smooth, closed S-subgroup scheme, such that the quotient G/H exists as an S-scheme. Suppose that the map $G \to G/H$ is Nisnevich locally split and that the presheaves $\mathrm{H}^1_{\mathrm{Nis}}(-,G), \mathrm{H}^1_{\mathrm{Nis}}(-,H)$ are $\mathbb{A}^1$-invariant on $\mathcal{S}\mathsf{m}_S^{\mathrm{aff}}$. Then G/H is $\mathbb{A}^1$-naive. Thus we have an isomorphism*

$$\pi_0((\mathrm{Sing}^{\mathbb{A}^1} G/H)(U)) \cong [U, G/H]_{\mathbb{A}^1}.$$

This result corresponds to [**13**, Theorem 2.4.2]. There are various concrete situations where it applies: [**13**, Theorem 4.2.1, 4.2.2, 4.2.10, 4.2.12, 4.3.1].

Proposition 3.3.14 ([**13**, Corollary 4.2.6])**.** *Let k be a perfect field, R be a smooth k-algebra. Then $\mathbb{A}^n_R \setminus 0$ is $\mathbb{A}^1$-naive. Thus for $U \in \mathcal{S}\mathsf{m}_R^{\mathrm{aff}}$, there is a canonical bijection*

$$\pi_0((\mathrm{Sing}^{\mathbb{A}^1} \mathbb{A}^n_R \setminus 0)(U)) \cong [U, \mathbb{A}^n_R \setminus 0]_{\mathbb{A}^1}.$$

See [**13**, Proposition 4.2.8] for a result about connectivity of $(\mathrm{Sing}^{\mathbb{A}^1} \mathbb{A}^n_R \setminus 0)(U)$.

For a commutative ring R and a smooth linear R-group scheme G, the *Karoubi-Villamayor nonstable K-theory functor* is given by

$$\mathrm{KV}^G_{i+1}(U) := \pi_i((\mathrm{Sing}^{\mathbb{A}^1} G)(U)), U \in \mathcal{S}\mathsf{m}_R, i \geqslant 1.$$

Theorem 3.3.15 ([**13**, Theorem 4.3.1])**.** *Let k be an infinite field, let G be an isotropic reductive k-group. Then G is $\mathbb{A}^1$-naive and for any $U \in \mathcal{S}\mathsf{m}_k^{\mathrm{aff}}$, there are canonical isomorphisms*

$$\mathrm{KV}^G_{i+1}(U) \cong [S^i \wedge U_+, G]_{\mathbb{A}^1,*}.$$

Theorem 3.3.16 ([**13**, Theorem 4.3.3])**.** *Let k be an infinite field, let G be an isotropic reductive k-group. Then for all $n \geqslant 0$, the Zariski sheafification of the presheaf*

$$U \mapsto \pi_n((\mathrm{Sing}^{\mathbb{A}^1} G)(U))$$

on $\mathcal{S}\mathsf{m}_k$ is a Nisnevich sheaf, which is a strongly $\mathbb{A}^1$-invariant sheaf of groups.

3.4. Elements of Hermitian K-theory

In this section, we collect some fundamental results of Schlichting's papers [**77, 78, 79, 81**] about Hermitian K-theory (aka. higher Grothendieck-Witt groups) in the general framework of exact categories with weak equivalences and duality and also specialized to results about schemes.

In the following, we fix an exact category with weak equivalences and duality $(\mathcal{E}, w, *, \eta)$ (see [**78**, §2.3]). To such a quadruple, we associate the *Grothendieck-Witt space* $\mathrm{GW}(\mathcal{E}, w, *, \eta)$ ([**78**, §2.7], see also the "hermitian Q-construction" Q^h in [**77**, §4.1] in the setting of exact categories with duality). Its homotopy groups $\pi_i\mathrm{GW}(\mathcal{E}, w, *, \eta) =: \mathrm{GW}_i(\mathcal{E}, w, *, \eta) (i \geqslant 0)$ are called *higher Grothendieck-Witt groups* of $(\mathcal{E}, w, *, \eta)$. This construction is functorial with respect to $(\mathcal{E}, w, *, \eta)$ ([**78**, §2.8]). The 0-th group $\mathrm{GW}_0(\mathcal{E}, w, *, \eta)$ has a concrete presentation ([**78**, Definition 1] and [**78**, Proposition 3, p.369], or [**77**, Proposition 4.11]).

Also, to an exact category with weak equivalences $(\mathcal{E}, w)$, we can associate a category with weak equivalences and duality $(\mathcal{H}\mathcal{E}, w)$ such that its Grothendieck-Witt space $\mathrm{GW}(\mathcal{H}\mathcal{E}, w)$ is homotopy equivalent to the *K-theory space* $\mathrm{K}(\mathcal{E}, w)$ ([**78**, Proposition 1, p.360]).

We have the exact category with weak equivalences and duality of *admissible short complexes* $(\mathrm{sCx}(\mathcal{E}), w, *, \eta)$ ([**78**, §3.1]).

Theorem 3.4.1 (Additivity for short complexes ([**78**, Theorem 4]))**.** *Let $(\mathcal{E}, w, *, \eta)$ be an exact category with weak equivalences and duality, then there is a homotopy equivalence $\mathrm{GW}(\mathrm{sCx}(\mathcal{E}), w, *, \eta) \simeq \mathrm{GW}(\mathcal{E}, w, *, \eta) \times \mathrm{K}(\mathcal{E}, w)$.*

See [**77**, Theorem 7.1, 7.2] for additivity results on exact categories with duality.

Theorem 3.4.2 (Cofinality ([**78**, Theorem 7, p.383])). *Let $(\mathcal{E}, w, *, \eta)$ be an exact category with weak equivalences and duality which has a symmetric cone ([**78**, Definition 4, p.373]). Let $A \subset \mathrm{K}_0(\mathcal{E}, w)$ be a subgroup closed under the duality action on $\mathrm{K}_0(\mathcal{E}, w)$, and let $\mathcal{E}_A \subset \mathcal{E}$ be the full subcategory of those objects whose class in $\mathrm{K}_0(\mathcal{E}, w)$ belongs to A, it inherits the structure of an exact category with weak equivalences and duality. The induced map on Grothendieck-Witt spaces $\mathrm{GW}(\mathcal{E}_A, w, *, \eta) \to \mathrm{GW}(\mathcal{E}, w, *, \eta)$ induces isomorphisms on π_i for $i \geqslant 1$ and a monomorphism on π_0.*

See [**77**, Corollary 5.2] for cofinality result on exact categories with duality.

Let $(\mathcal{E}, *, \eta)$ be an exact category with duality (see [**77**, Definition 2.1]). We can view it as an exact category with weak equivalences and duality $(\mathcal{E}, i, *, \eta)$, with i being the class of isomorphisms in $\mathcal{E}$. Let $\mathrm{Ch}^{\mathrm{b}}(\mathcal{E})$ be the associated category of bounded (cohomological) chain complexes, whose objects are of the form

$$(E, \mathrm{d}) : \cdots \to E_{n-1} \xrightarrow{\mathrm{d}_{n-1}} E_n \xrightarrow{\mathrm{d}_n} E_{n+1} \to \cdots, \mathrm{d}_n \mathrm{d}_{n-1} = 0.$$

Writing $\mathrm{quis} = \mathrm{quis}(\mathcal{E})$ for the class of quasi-isomorphisms, we get an exact category with weak equivalences $(\mathrm{Ch}^{\mathrm{b}}(\mathcal{E}), \mathrm{quis})$. For each $n \in \mathbb{Z}$, the duality $(*, \eta)$ of $\mathcal{E}$ induces a (naive) duality $(*^n, \eta^n)$ on $\mathrm{Ch}^{\mathrm{b}}(\mathcal{E})$ as follows: for a map of chain complexes $f : (E, \mathrm{d}) \to (E', \mathrm{d})$,

$$(E^{*^n})_i = (E_{-i-n})^*, (\mathrm{d}^{*^n})_i = (\mathrm{d}_{-i-n})^*, (f^{*^n})_i = (f_{-i-n})^*, (\eta_E^n)_i = (-1)^{\frac{n(n-1)}{2}} \eta_{E_i}.$$

We get an exact category with weak equivalences and duality $(\mathrm{Ch}^{\mathrm{b}}(\mathcal{E}), \mathrm{quis}, *^n, \eta^n)$ ([**78**, §6.1]). For $n = 0$, we simply write $(*, \eta)$ for $(*^n, \eta^n)$.

Proposition 3.4.3 ([**78**, Proposition 6]). *For an exact category with duality $(\mathcal{E}, *, \eta)$, the natural duality preserving exact functor $(\mathcal{E}, i, *, \eta) \to (\mathrm{Ch}^{\mathrm{b}}(\mathcal{E}), \mathrm{quis}, *, \eta)$ induces a homotopy equivalence of Grothendieck-Witt spaces*

$$\mathrm{GW}(\mathcal{E}, *, \eta) \simeq \mathrm{GW}(\mathrm{Ch}^{\mathrm{b}}(\mathcal{E}), \mathrm{quis}, *, \eta).$$

Now we turn to the categories obtained from schemes ([**78**, §8.1]). Let X be a scheme, $\mathscr{A} = \mathscr{A}_X$ a quasi-coherent $\mathscr{O}_X$-algebra (not necessarily commutative, and the stalk of $\mathscr{A}$ need not be a local ring) with involution $\bar{()} : \mathscr{A} \to \mathscr{A}^{\mathrm{op}}$, L a line bundle on X, $Z \subset X$ a closed subscheme with open complement $U = X \setminus Z$, and let $n \in \mathbb{Z}$. Let $\mathcal{V}\mathsf{ect}(\mathscr{A})$ be the category of $\mathscr{A}$-vector bundles, i.e. those quasi-coherent left $\mathscr{A}$-modules F for which $F(V)$ is a finitely generated projective $\mathscr{A}(V)$-module for every open affine $V \subset X$ (such an $\mathscr{A}$-vector bundle F need not be locally free). Of course, if $\mathscr{A} = \mathscr{O}_X$, then $\mathcal{V}\mathsf{ect}(\mathscr{A})$ is the usual exact category of vector bundles on X.

A *strictly perfect complex* of $\mathscr{A}$-modules is a bounded left dg $\mathscr{A}$-modules M with each M_n being an $\mathscr{A}$-vector bundle. We get the category $\mathrm{sPerf}(\mathscr{A})$ of strictly perfect complex of $\mathscr{A}$-modules.

We define $\mathscr{A}L[n] := \mathscr{A} \otimes_{\mathscr{O}_X} L[n]$, it's a dg $\mathscr{A}$-bimodule given on local sections by the formula $a(x \otimes l)b = (-1)^{|b|\cdot|l|} axb \otimes l$. There is a dg $\mathscr{A}$-bimodule isomorphism $i : \mathscr{A}L[n] \to (\mathscr{A}L[n])^{\mathrm{op}}$ given by $i(a \otimes l) = \bar{a} \otimes l$, clearly $i^{\mathrm{op}} \circ i = 1$. Then for $\epsilon = \pm 1$, $(\mathscr{A}L[n], \epsilon i)$ is a duality coefficient for $\mathscr{A}$-Mod. For $M \in \mathrm{sPerf}(\mathscr{A})$, define $M^{\sharp^n_{\epsilon L}} := [M^{\mathrm{op}}, \mathscr{A}L[n]]_{\mathscr{A}}$ (see [**78**, §7.2] for the notation), which is also a strictly perfect complex of $\mathscr{A}$-modules. We obtain an exact category with weak equivalences and duality $(\mathrm{sPerf}(\mathscr{A}), \mathrm{quis}, *, \sharp^n_{\epsilon L}, \mathrm{can}_{\mathscr{A}L[n], \epsilon i})$. If $n = 0$, or $\epsilon = 1$, or $L = \mathscr{O}_X$, we will omit the corresponding symbol from the notation. By restriction, we get an exact category with duality $(\mathcal{V}\mathsf{ect}(\mathscr{A}), \sharp_{\epsilon L})$.

We say that $M \in \mathrm{sPerf}(\mathscr{A})$ *has cohomological support in Z* if the restriction $M|_U$ of M to U is acyclic. Write $\mathrm{sPerf}(\mathscr{A}$ on $Z)$ for the category of all strictly perfect complexes of $\mathscr{A}$-modules M having cohomological support in Z. Again by restriction, there is an exact category with duality $(\mathrm{sPerf}(\mathscr{A}$ on $Z), \mathrm{quis}, \sharp^n_{\epsilon L}, \mathrm{can}_{\mathscr{A}L[n], \epsilon i})$. Its Grothendieck-Witt space is denoted ${}_{\epsilon}\mathrm{GW}^n(\mathscr{A}$ on $Z, L)$ (again we can omit some symbol from the notation in special cases, if $Z = X$, we omit "on Z" also).

Proposition 3.4.4 ([**78**, Corollary 1, p.408]). *There are functorial homotopy equivalences of Grothendieck-Witt spaces*

$$\mathrm{GW}^{4n}(\mathscr{A}, L) \simeq \mathrm{GW}(\mathcal{V}\mathsf{ect}(\mathscr{A}), \sharp_L, \mathrm{can}_L), \mathrm{GW}^{4n+2}(\mathscr{A}, L) \simeq \mathrm{GW}(\mathcal{V}\mathsf{ect}(\mathscr{A}), \sharp_L, -\mathrm{can}_L).$$

We say that a scheme X *has an ample family of line bundles* if there is a finite set of line bundles $\{L_1, \cdots, L_n\}$ on X, with sections $s_i \in \Gamma(X, L_i)$ such that the open sets X_{s_i} are affine and cover X (examples include schemes quasi-projective over an affine scheme, or regular separated noetherian schemes). In this case, X has a basis for its topology consisting of open affine sets of this form. Such a scheme X is necessarily qcqs, and any of its subscheme also has an ample family of line bundles. The category $\mathrm{Qcoh}(\mathscr{A})$ of quasi-coherent left $\mathscr{A}$-modules is a Grothendieck abelian category with a generating set of the form $\mathscr{A} \otimes_{\mathscr{O}_X} L_i^k, k \leqslant 0, 1 \leqslant i \leqslant n$ (in the sense that every quasi-coherent left $\mathscr{A}$-module admits an epimorphism from one which is a direct sum of -modules in this generating set). We denote by $\mathsf{D}\mathrm{Qcoh}(\mathscr{A})$ the (unbounded) derived category of $\mathrm{Qcoh}(\mathscr{A})$, and if $Z \subset X$ is a closed subscheme with quasi-compact open complement $U = X \setminus Z$, we write $\mathsf{D}_Z\mathrm{Qcoh}(\mathscr{A})$ for the full triangulated subcategory of those complexes of quasi-coherent left $\mathscr{A}$-modules whose restriction to U are acyclic (see [**78**, §9.4, p.414]).

For our later use in Chapter 4, we focus on affine schemes, over which every line bundle is ample, so the results can be applied there.

Theorem 3.4.5 (Localization ([**78**, Theorem 10, p.409])). *Let X be a scheme with an ample family of line bundles, $Z \subset X$ a closed subscheme with quasi-compact open complement $j : U = X \setminus Z \hookrightarrow X$. Then for every $n \in \mathbb{Z}$, there is a homotopy fibration of Grothendieck-Witt spaces*

$$\mathrm{GW}^n(\mathscr{A}$ on $Z, L) \to \mathrm{GW}^n(\mathscr{A}, L) \to \mathrm{GW}^n(\mathscr{A}|_U, j^*L).$$

Theorem 3.4.6 (Zariski excision ([**78**, Theorem 11])). *Let X be a scheme with an ample family of line bundles, $Z \subset X$ a closed subscheme with quasi-compact open complement. Then for every $n \in \mathbb{Z}$ and every quasi-compact open subscheme $j : V \hookrightarrow X$ with $Z \subset V$, restriction induces a homotopy equivalence of Grothendieck-Witt spaces*

$$\mathrm{GW}^n(\mathscr{A} \text{ on } Z, L) \simeq \mathrm{GW}^n(\mathscr{A}|_V \text{ on } Z, j^*L).$$

Proposition 3.4.7 ([**78**, Proposition 8, p.416]). *Let X be a scheme with an ample family of line bundles $\{L_1, \cdots, L_n\}$, with sections $s_i \in \Gamma(X, L_i)$ such that the open sets X_{s_i} are affine and cover X. Then $\mathsf{DQcoh}(\mathscr{A})$ is compactly generated (*[**78**, §9.5]*) by the set of objects $\{\mathscr{A} \otimes_{\mathscr{O}_X} L_i^k[j] : j \in \mathbb{Z}, k \leqslant 0, 1 \leqslant i \leqslant n\}$.*

Moreover, the inclusion $\mathcal{V}\mathsf{ect}(\mathscr{A}) \subset \mathrm{Qcoh}(\mathscr{A})$ induces a fully faithful triangle functor $\mathsf{D}^{\mathrm{b}}\mathcal{V}\mathsf{ect}(\mathscr{A}) \subset \mathsf{DQcoh}(\mathscr{A})$, with essential image the full triangulated subcategory $\mathsf{D}^{\mathrm{c}}\mathrm{Qcoh}(\mathscr{A})$ of compact objects in $\mathsf{DQcoh}(\mathscr{A})$.

Proposition 3.4.8 ([**78**, Lemma 16, p.418]). *Let X be a scheme with an ample family of line bundles, $Z \subset X$ a closed subscheme with quasi-compact open complement $X \setminus Z$, $j : U \hookrightarrow X$ the inclusion of a quasi-compact open subscheme containing Z. Then we have an equivalence of triangulated categories*

$$j^* : \mathsf{D}_Z\mathrm{Qcoh}(\mathscr{A}) \to \mathsf{D}_Z\mathrm{Qcoh}(\mathscr{A}|_U)$$

with quasi-inverse $\mathbf{R}j_*$.

Proposition 3.4.9 ([**78**, Proposition 9, p.419]). *Let X be a scheme with an ample family of line bundles, $Z \subset X$ a closed subscheme with quasi-compact open complement $X \setminus Z$, $j : U \hookrightarrow X$ the inclusion of a quasi-compact open subscheme.*

(a) *Restriction of vector bundles induces a cofinal and fully faithful triangulated functor*

$$j^* : \mathsf{D}(\mathrm{sPerf}(\mathscr{A} \text{ on } Z), U\text{-quis}) \hookrightarrow \mathsf{D}(\mathrm{sPerf}(\mathscr{A}|_U \text{ on } Z \cap U), U\text{-quis}).$$

(b) *The triangulated category $\mathsf{D}_Z\mathrm{Qcoh}(\mathscr{A})$ is compactly generated and the triangulated functor*

$$\mathsf{D}(\mathrm{sPerf}(\mathscr{A} \text{ on } Z), \mathrm{quis}) \to \mathsf{D}_Z\mathrm{Qcoh}(\mathscr{A})$$

is fully faithful with essential image the full triangulated subcategory $\mathsf{D}^{\mathrm{c}}_Z\mathrm{Qcoh}(\mathscr{A})$ of compact objects in $\mathsf{D}_Z\mathrm{Qcoh}(\mathscr{A})$.

(c) *If $Z \subset U$, then restriction of vector bundles induces an equivalence of triangulated categories*

$$j^* : \mathsf{D}(\mathrm{sPerf}(\mathscr{A} \text{ on } Z), \mathrm{quis}) \hookrightarrow \mathsf{D}(\mathrm{sPerf}(\mathscr{A}|_U \text{ on } Z \cap U), \mathrm{quis}).$$

Proposition 3.4.10 ([**78**, Corollary 2, p.421]). *Let X be a scheme with an ample family of line bundles, $Z \subset X$ a closed subscheme with quasi-compact open complement $X \setminus Z$, $j : U \hookrightarrow X$ the inclusion of a quasi-compact open subscheme. Let M be a chain complex of quasi-coherent left $\mathscr{A}$-modules such that the restriction $j^*M \in \mathrm{sPerf}(\mathscr{A}|_U \text{ on } Z \cap U)$. If the class $[j^*M] \in \mathrm{K}_0(\mathscr{A}|_U \text{ on } Z \cap U)$ is in the image of the map $\mathrm{K}_0(\mathscr{A} \text{ on } Z) \to \mathrm{K}_0(\mathscr{A}|_U \text{ on } Z \cap U)$, then there is a U-quasi-isomorphism $A \to M$ with $A \in \mathrm{sPerf}(\mathscr{A} \text{ on } Z)$.*

Proposition 3.4.11 ([**78**, Proposition 10]). *Let X be a scheme with an ample family of line bundles, $Z \subset X$ a closed subscheme with quasi-compact open complement $X \setminus Z$, $j : U \hookrightarrow X$ the inclusion of a quasi-compact open subscheme. Then for any line bundle L on X and any $n \in \mathbb{Z}$, restriction defines a non-singular exact form functor*

$$(\mathrm{sPerf}(\mathscr{A} \text{ on } Z), U\text{-quis}), \sharp^n_L) \to (\mathrm{sPerf}(\mathscr{A}|_U \text{ on } Z \cap U), U\text{-quis}), \sharp^n_{j^*L})$$

which induce isomorphisms on higher Grothendieck-Witt groups $\mathrm{GW}_i, i \geqslant 1$ and a monomorphism on GW_0.

If $Z \subset U$, then this form functor induces isomorphisms for all higher Grothendieck-Witt groups $\mathrm{GW}_i, i \geqslant 0$.

Proposition 3.4.12 ([**78**, Theorem 13, p.424]). *Let X be a scheme with an ample family of line bundles, $Z \subset X$ a closed subscheme with quasi-compact open complement $X \setminus Z$, L a line bundle on X and $n \in \mathbb{Z}$. Then there is a homotopy fibration of Grothendieck-Witt spaces with contractible total space*

$$\mathrm{GW}^n(\mathscr{A} \text{ on } Z, L) \to \mathrm{GW}^n(C\mathscr{A} \text{ on } Z, L) \to \mathrm{GW}^n(S\mathscr{A} \text{ on } Z, L).$$

(See [**78**, §10.1, p.423] *for the definition of the rings C, S.)*

For a scheme X with an ample family of line bundles, $Z \subset X$ a closed subscheme with quasi-compact open complement $X \setminus Z$, L a line bundle on X and $n \in \mathbb{Z}$, the *Grothendieck-Witt spectrum* $\mathbb{GW}^n(\mathscr{A} \text{ on } Z, L)$ of symmetric spaces over $\mathscr{A}$ with coefficients in the n-th shifted line bundle $L[n]$ and support in Z is the sequence $\mathrm{GW}^n(S^k\mathscr{A} \text{ on } Z, L), k \in \mathbb{N}$, with bonding maps $\mathrm{GW}^n(\mathscr{A} \text{ on } Z, L) \xrightarrow{\simeq} \Omega\mathrm{GW}^n(S\mathscr{A} \text{ on } Z, L)$. It's an Ω-spectrum ([**78**, Definition 8, p.430]). We have

$$\pi_i\mathbb{GW}^n(\mathscr{A} \text{ on } Z, L) = \begin{cases} \mathrm{GW}^n_i(\mathscr{A} \text{ on } Z, L), & i \geqslant 0; \\ \mathrm{GW}^n_0(S^{-i}\mathscr{A} \text{ on } Z, L), & i \leqslant 0. \end{cases}$$

Moreover, there are natural homotopy equivalences of spectra $\mathbb{GW}^n(\mathscr{A} \text{ on } Z, L) \simeq \mathbb{GW}^{n+4}(\mathscr{A} \text{ on } Z, L)$. Below we will need the notion of model structure on the category of spectra, see [**46**] for a rather general treatment about this.

Theorem 3.4.13 (Localization ([**78**, Theorem 14, p.431])). *Let X be a scheme with an ample family of line bundles, $Z \subset X$ a closed subscheme with quasi-compact open complement $j : U = X \setminus Z \hookrightarrow X$. Then for every line bundle L on X and $n \in \mathbb{Z}$, there is a homotopy fibration of Grothendieck-Witt spectra*

$$\mathbb{GW}^n(\mathscr{A} \text{ on } Z, L) \to \mathbb{GW}^n(\mathscr{A}, L) \to \mathbb{GW}^n(\mathscr{A}|_U, j^*L).$$

Theorem 3.4.14 (Zariski excision ([**78**, Theorem 15])). *Let X be a scheme with an ample family of line bundles, $Z \subset X$ a closed subscheme with quasi-compact open complement $X \setminus Z$, L a line bundle on X and $n \in \mathbb{Z}$. Then for every open subscheme $j : V \hookrightarrow X$ containing Z, restriction of vector-bundles induces a homotopy equivalence of Grothendieck-Witt spectra*

$$\mathbb{GW}^n(\mathscr{A} \text{ on } Z, L) \simeq \mathbb{GW}^n(\mathscr{A}|_V, j^*L).$$

Theorem 3.4.15 (Mayer-Vietoris for Zariski open covers ([**78**, Theorem 16])). *Let X be a scheme with an ample family of line bundles, $U, V \subset X$ are quasi-compact open subscheme with $U \cup V = X$, L a line bundle on X and $n \in \mathbb{Z}$. Then restriction of vector bundles induces a homotopy cartesian diagram of Grothendieck-Witt spectra*

$$\begin{array}{ccc} \mathbb{GW}^n(\mathscr{A}, L) & \longrightarrow & \mathbb{GW}^n(\mathscr{A}|_U, L) \\ \downarrow & & \downarrow \\ \mathbb{GW}^n(\mathscr{A}|_V, L) & \longrightarrow & \mathbb{GW}^n(\mathscr{A}|_{U\cap V}, L). \end{array}$$

Corollary 3.4.16 (([**78**, Theorem 1])). *Let X be a scheme with an ample family of line bundles, $U, V \subset X$ are quasi-compact open subscheme with $U \cup V = X$, then there is a long exact sequence*

$$\cdots \to \mathrm{GW}_{i+1}(U \cap V) \to \mathrm{GW}_i(X) \to \mathrm{GW}_i(U) \oplus \mathrm{GW}_i(V) \to \mathrm{GW}_i(U \cap V) \to \mathrm{GW}_{i-1}(X) \to \cdots .$$

We mention the following interesting general results about the "hermitian Q-construction" Q^h of exact categories with duality and their Grothendieck-Witt space in [**77**]: Theorem 6.1 (on Dévissage), Theorem 8.2, Lemma 8.10, Proposition 8.11, Corollary 8.18, Theorem 9.7 and Theorem 9.11.

In [**79**], Schlichting develops (shifted) Grothendieck-Witt groups for dg categories with weak equivalences and duality ([**79**, Definition 1.37]) and for of triangulated categories with duality ([**79**, Definition 3.1]), basically along the same line as above. The 0-th Grothendieck-Witt groups are defined in [**79**, Definitions 1.39, 3.5] via presentations.

To a dg category with weak equivalences and duality it is associated a triangulated category with duality ([**79**, Lemma 3.6]), having isomorphic 0-th Grothendieck-Witt groups ([**79**, Proposition 3.8]). The shifted version is defined in [**79**, Definition 3.11], and again having isomorphic 0-th Grothendieck-Witt groups ([**79**, Corollary 3.13]).

In all degrees the Grothendieck-Witt groups are given in [**79**, Definitions 5.4, 5.7] via (shifted, when 2 is invertible in the category) Grothendieck-Witt spectra, being stable homotopy groups of these spectra. There is the cup-product pairing of these spectra ([**79**, §5.4]). Cup-product with the Bott element η gives an algebraic Bott sequence ([**79**, Theorem 6.1]) and isomorphisms between different Grothendieck-Witt and Witt groups ([**79**, Proposition 6.3]). We record the former below.

Theorem 3.4.17 (Algebraic Bott sequence / Karoubi periodicity sequence). *Let $\mathcal{A}$ be a dg category with weak equivalences and duality for which $\frac{1}{2} \in \mathcal{A}$. Then the sequence of spectra*

$$\mathrm{GW}^{[n]}(\mathcal{A}) \xrightarrow{F} \mathrm{K}(\mathcal{A}) \xrightarrow{H} \mathrm{GW}^{[n+1]}(\mathcal{A}) \xrightarrow{\eta\smile} S^1 \wedge \mathrm{GW}^{[n]}(\mathcal{A})$$

is an exact triangle in the homotopy category of spectra.

There are interesting relations between the spectra associated to certain functors between the dg categories with weak equivalences and duality in [**79**, Lemma 6.4, Theorem 6.5, Theorem 6.6, Proposition 6.8]. One defines the L-theory spectra as the localization of the Grothendieck-Witt spectra with respect to the Bott element η, see [**79**, Definition 7.1, Proposition 7.2].

There are bispectra $\mathbb{GW}$, called *Karoubi-Grothendieck-Witt spectra* associated to dg categories with weak equivalences and duality in which 2 is invertible ([**79**, Definition 8.6]), whose (stable) homotopy groups are given in [**79**, Proposition 8.7]. Again there are interesting relations between the $\mathbb{GW}$ bispectra associated to certain functors between the dg categories with weak equivalences and duality in [**79**, Theorem 8.9, Theorem 8.10, Proposition 8.15] and an algebraic Bott sequence ([**79**, Theorem 8.11]), the (stabilized) L-theory spectra ([**79**, Definition 8.12]).

For a scheme X with an ample family of line bundles as before with 2 invertible in $\Gamma(X, \mathscr{O}_X)$, when specialized to the case $\mathscr{A} = \mathscr{A}_X = \mathscr{O}_X$, then $\mathrm{sPerf}(X \text{ on } Z) := \mathrm{sPerf}(\mathscr{A} \text{ on } Z)$ is just the category of bounded complexes of vector bundles (locally free $\mathscr{O}_X$-modules of finite ranks); below we write X in place of $\mathscr{A}$ in our notations before. Thus there are the n-th shifted *Grothendieck-Witt spectra* and *Karoubi-Grothendieck-Witt spectra* of X with coefficients in a line bundle L: $\mathrm{GW}^{[n]}(X, L)$ and $\mathbb{GW}^{[n]}(X, L)$. Their (stable) homotopy groups are denoted by $\mathrm{GW}_i^{[n]}(X, L)$ and $\mathbb{GW}_i^{[n]}(X, L)$ ([**79**, Definition 9.1]). For $i \geqslant 0$, they are isomorphic ([**79**, Proposition 9.3]).

We record the following important results in [**79**, §9, 10], which are relevant in $\mathbb{A}^1$-homotopy theory.

Theorem 3.4.18 (Localization for vector bundle Grothendieck-Witt groups (for $\mathbb{GW}$) [**79**, Theorem 9.5]). *Let X be a scheme with an ample family of line bundles and assume 2 is invertible in $\Gamma(X, \mathscr{O}_X)$. Let $Z \subset X$ be a closed subscheme with quasi-compact open complement $j : U = X \setminus Z \hookrightarrow X$. Then for every line bundle L on X and $n \in \mathbb{Z}$, there is a homotopy fibration of Karoubi-Grothendieck-Witt spectra*

$$\mathbb{GW}^{[n]}(X \text{ on } Z, L) \to \mathbb{GW}^{[n]}(X, L) \to \mathbb{GW}^{[n]}(U, j^*L).$$

Theorem 3.4.19 (Nisnevich Mayer-Vietoris for $\mathbb{GW}$ [**79**, Theorem 9.6]). *Given a Nisnevich square*

$$\begin{array}{ccc} V & \longrightarrow & Y \\ \downarrow & & \downarrow p \\ U & \xrightarrow{j} & X, \end{array}$$

where j is an open immersion, p is étale and the restriction $Y - V \xrightarrow{p} (X - U)$ is an isomorphism with both sides being equipped with the reduced scheme structures. Assume that X has an ample family of line bundles and that 2 is invertible

in $\Gamma(X,\mathscr{O}_X)$*. Let* $Z\subset X$ *be a closed subscheme with quasi-compact open complement* $j: U=X\setminus Z\hookrightarrow X$*. Then for every line bundle* L *on* X *and* $n\in\mathbb{Z}$*, there is a homotopy cartesian diagram of Karoubi-Grothendieck-Witt spectra*

$$\begin{array}{ccc}\mathbb{GW}^{[n]}(X,L) & \longrightarrow & \mathbb{GW}^{[n]}(U,L|_U)\\ \downarrow & & \downarrow\\ \mathbb{GW}^{[n]}(Y,p^*L) & \longrightarrow & \mathbb{GW}^{[n]}(V,p^*L|_V).\end{array}$$

Theorem 3.4.20 (Nisnevich descent for $\mathbb{GW}$ [**79**, Theorem 9.7])**.** *Let* X *be a scheme of finite Krull dimension with an ample family of line bundles and* 2 *is invertible in* $\Gamma(X,\mathscr{O}_X)$*. Then for every line bundle* L *on* X *and* $n\in\mathbb{Z}$*, the map on global sections of a globally fibrant replacement for the Nisnevich topology of the Karoubi-Grothendieck-Witt spectra functor* $\mathbb{GW}(-,L)$ *on the small Nisnevich site on* X *is a stable equivalence:*

$$\mathbb{GW}^{[n]}(X,L)\xrightarrow{\simeq}\mathbb{H}_{\mathrm{Nis}}\big(X,\mathbb{GW}^{[n]}(-,L)\big).$$

Theorem 3.4.21 (Homotopy Invariance for GW [**79**, Theorem 9.8])**.** *Let* X *be a scheme with an ample family of line bundles. Assume that* 2 *is invertible in* $\Gamma(X,\mathscr{O}_X)$*. Then for every line bundle* L *on* X *and* $n\in\mathbb{Z}$*, the projection* $p: X\times\mathbb{A}^1\to X$ *induces a stable equivalence of Grothendieck-Witt spectra*

$$\mathrm{GW}^{[n]}(X,L)\xrightarrow{\simeq}\mathrm{GW}^{[n]}(X\times\mathbb{A}^1,p^*L).$$

Theorem 3.4.22 (Mayer-Vietoris for regular blow-ups of $\mathbb{GW}$ [**79**, Theorem 9.9])**.** *Consider a cartesian diagram of schemes*

$$\begin{array}{ccc}Y' & \xrightarrow{j} & X'\\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p}\\ Y & \xrightarrow{i} & X,\end{array}$$

where i *is a closed immersion and* p *is the blow-up of* X *along* i*. Assume that* X *has an ample family of line bundles and that* 2 *is invertible in* $\Gamma(X,\mathscr{O}_X)$ *(then the same is true for each of* Y,X',Y'*). If* i *is a regular embedding of codimension* d*, then for every line bundle* L *on* X *and* $n\in\mathbb{Z}$*, the above diagram induces a homotopy cartesian diagram of Karoubi-Grothendieck-Witt spectra*

$$\begin{array}{ccc}\mathbb{GW}^{[n]}(X,L) & \longrightarrow & \mathbb{GW}^{[n]}(Y,i^*L)\\ \downarrow & & \downarrow\\ \mathbb{GW}^{[n]}(X',p^*L) & \longrightarrow & \mathbb{GW}^{[n]}(Y',j^*p^*L).\end{array}$$

3.5. A little stable motivic homotopy theory

In this section, we discuss briefly the theory of T-spectra on the pointed model category $\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})_*$ for a given site $(\mathcal{C},\tau)$, which is the study of the stable model category $\mathcal{S}\mathsf{pt}_T(\mathcal{C}_\tau):=\mathcal{S}\mathsf{pt}_T(\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})_*)$. We are mainly interested in the case when the *parameter object* $T=S^1$ or $T=(\mathbb{P}^1,\infty)$, following [**50**, Chapter 10]. We mention also that there is the general theory of T-spectra on a pointed model category $\mathcal{M}$ (with $T\in\mathcal{M}$ a parameter object)—the study of the stable model category $\mathcal{S}\mathsf{pt}_T(\mathcal{M})$, see e.g. [**46**].

For a fixed parameter object $T\in\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})_*$, a *$T$-spectrum* is a sequence $X=(X^n)_{n\in\mathbb{N}}$ in $\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})_*$, together with *bonding maps* $\sigma_n: T\wedge X^n\to X^{n+1}$; X^n is called the *level* n part of X. A morphism of two T-spectra $f: X\to Y$ is a sequence $f^n: X^n\to Y^n$ of morphisms in $\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})_*$ which respect the respective bonding maps: $f^{n+1}\sigma_n=\sigma_n(T\wedge f^n)$. We obtain the category $\mathcal{S}\mathsf{pt}_T(\mathcal{C}_\tau)$ of *T-spectra*. The category $\mathcal{S}\mathsf{pt}_{S^1}(\mathcal{C}_\tau)$ of *S^1-spectra* is the same as the category of presheaves of spectra as in [**50**, §10.1]. In stable motivic homotopy theory, we are mainly using $T=S^1,\mathbb{G}_m$ or $\mathbb{P}^1$.

We say that a morphism of T-spectra $f: X\to Y$ is a *level weak equivalence* (resp. *level fibration*) if each map $f^n: X^n\to Y^n$ is a weak equivalence (resp. fibration) in the Jardine model structure on $\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})_*$.

For $K\in\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})_*$ and $X=(X^n)\in\mathcal{S}\mathsf{pt}_T(\mathcal{C}_\tau)$, we define $X\wedge K\in\mathcal{S}\mathsf{pt}_T(\mathcal{C}_\tau)$ by letting $(X\wedge K)^n:=X^n\wedge K$ with bonding maps given by $\sigma_n\wedge K$. We also define $\Omega_K X\in\mathcal{S}\mathsf{pt}_T(\mathcal{C}_\tau)$ (called the *"real" T-loop spectrum* in [**50**, p. 374] with different notation) by letting $(\Omega_K X)^n:=\underline{\mathrm{Hom}}_*(K,X^n)=\Omega_K(X^n)$ with bonding maps $T\wedge\underline{\mathrm{Hom}}_*(K,X^n)\to\underline{\mathrm{Hom}}_*(K,X^{n+1})$ given as the composite $T\wedge\underline{\mathrm{Hom}}_*(K,X^n)\xrightarrow{\tau}\underline{\mathrm{Hom}}_*(K,X^n)\wedge T\to\underline{\mathrm{Hom}}_*(K,T\wedge X^n)\xrightarrow{(\sigma_n)_*}\underline{\mathrm{Hom}}_*(K,X^{n+1})$, where τ is given by flipping the smash factors and the middle arrow is adjoint to the map $\underline{\mathrm{Hom}}_*(K,X^n)\xrightarrow{T\wedge}\underline{\mathrm{Hom}}_*(T\wedge K,T\wedge X^n)$.

The T-spectra category $\mathcal{S}\mathsf{pt}_T(\mathcal{C}_\tau)$ has a simplicial structure given as follows: for $K\in\mathsf{s}\mathcal{S}\mathsf{et}$, we define $X\otimes K:=X\wedge K_+, X^K:=\Omega_{K_+}X\in\mathcal{S}\mathsf{pt}_T(\mathcal{C}_\tau)$; for $X,Y\in\mathcal{S}\mathsf{pt}_T(\mathcal{C}_\tau)$, $\mathrm{Map}(X,Y)_n=\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})_*(X\wedge\Delta^n_+,Y)$. With level weak equivalences and level fibrations, they make the T-spectra category $\mathcal{S}\mathsf{pt}_T(\mathcal{C}_\tau)$ a cofibrantly generated proper simplicial model category ([**50**, Proposition 10.15]). We call this model structure the *strict model structure*. In fact, it's more than a simplicial model category: the functor $\wedge:\mathcal{S}\mathsf{pt}_T(\mathcal{C}_\tau)\times\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})_*\to\mathcal{S}\mathsf{pt}_T(\mathcal{C}_\tau)$ is a Quillen bifunctor in the sense of [**45**, Definition 4.2.1] (see [**50**, Lemma 10.16]). Related to this, there is a bifunctor $\mathscr{H}om:\mathcal{S}\mathsf{pt}_T(\mathcal{C}_\tau)^{\mathrm{op}}\times\mathcal{S}\mathsf{pt}_T(\mathcal{C}_\tau)\to\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})_*$ given by $\mathscr{H}om(A,X)(U)=\mathrm{Map}(A\wedge U_+,X)$.

Given $K\in\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{C})_*$, we have the *suspension T-spectrum* $\Sigma^\infty_T K$ with $(\Sigma^\infty_T K)^n=T^{\wedge n}\wedge K$, where $T^{\wedge n}=S^0, T^{\wedge n}=T\wedge\cdots\wedge T$ (n-fold smash power). The *sphere T-spectrum* is $S_T=\Sigma^\infty_T S^0$. We have $\Sigma^\infty_T K=S_T\wedge K$.

The strict model structure on the T-spectra category $\mathcal{S}\mathsf{pt}_T(\mathcal{C}_\tau)$ is not interesting for our purpose, since it depends loosely on the parameter object T for which smashing with T is not invertible in the homotopy category. It's more useful (and is used as an intermediate step) to stabilize and localize with respect to some cofibration $f: A\to B$ in

$\mathsf{sPre}(\mathcal{C})_*$ (typically the map $0 : * \to \mathbb{A}^1$ when working on motivic homotopy theory), for which we refer to [**50**, Theorem 10.20, Examples 10.22 and 10.23, Theorem 10.36], where the author presents very general results about the *stable model structure* and the *f-local stable model structure* on the T-spectra category $\mathcal{S}\mathsf{pt}_T(\mathcal{C}_\tau)$ and their nice homotopical properties. The stabilization procedure is to make the smashing with T and the shifting operations on T-spectra invertible in the resulting stable homotopy category; the f-localization is to make f invertible. There are very general results of Jardine to make these precise, which we don't spell out; rather, see [**50**, §10.3 and §10.4] or [**49**] for some detailed treatment. We only state the following result ([**50**, Corollary 10.26]) for later use.

Proposition 3.5.1. *A T-spectrum $X = (X^n) \in \mathcal{S}\mathsf{pt}_T(\mathcal{C}_\tau)$ is stable f-fibrant iff X is level f-fibrant and the adjoint bonding maps $(\sigma_n)_* : X^n \to \Omega_T X^{n+1}$ are local weak equivalence (or sectionwise weak equivalence) of pointed simplicial presheaves.*

Specializing to the stable motivic homotopy theory, in which case $\mathcal{C} = \mathcal{S}\mathsf{m}_S$ with the Nisnevich topology (where S is a noetherian scheme of finite Krull dimension) and $T = \mathbb{P}^1 = (\mathbb{P}^1, \infty)$, f is $0 : * \to \mathbb{A}^1$, we obtain the stable motivic model category of motivic $\mathbb{P}^1$-spectra, whose homotopy category we denote by $\mathcal{SH}(S)$. The above result then becomes the following.

Proposition 3.5.2. *A motivic $\mathbb{P}^1$-spectrum $X = (X^n) \in \mathcal{S}\mathsf{pt}_{\mathbb{P}^1}(\mathcal{S}\mathsf{m}_S)$ is stable $\mathbb{A}^1$-fibrant iff X is level $\mathbb{A}^1$-fibrant and the adjoint bonding maps $(\sigma_n)_* : X^n \to \Omega_{\mathbb{P}^1} X^{n+1}$ are local weak equivalences (or even sectionwise weak equivalences) of pointed simplicial presheaves.*

For more on the foundations of the stable motivic homotopy category, see [**50**, Examples 10.38, 10.39 and 10.41] and [**49**].

As the ordinary stable homotopy theory, the stable motivic homotopy category is quite useful, it is the suitable category in which many important theories are representable, among which are: the motivic cohomology theory, the algebraic K-theory (and Hermitian K-theory) and the algebraic cobordism. All these can be found in Voevodsky's excellent 1998 ICM address [**92**]. We state here the result for the algebraic K-theory for later use.

Proposition 3.5.3. *There is a motivic $\mathbb{P}^1$-spectrum* $\mathrm{KGL} = (\mathrm{BGL} \times \mathbb{Z}) \in \mathcal{S}\mathsf{pt}_{\mathbb{P}^1}(\mathcal{S}\mathsf{m}_S)$ *all of whose levels are* $\mathrm{BGL} \times \mathbb{Z}$. *This motivic $\mathbb{P}^1$-spectrum* KGL *represents Quillen's algebraic K-theory if the base scheme S is regular: for any $X \in \mathcal{S}\mathsf{m}_S$, we have a canonical bijection*

$$\mathrm{K}_i(X) \xrightarrow{\cong} \pi_i \mathrm{RMap}(\Sigma^\infty_{\mathbb{P}^1} X_+, \mathrm{KGL}) \cong [S^i \wedge X_+, \mathrm{BGL} \times \mathbb{Z}]_{\mathbb{A}^1,*}.$$

Also, Schlichting's work on Hermitian K-theory (or higher Grothendieck-Witt groups) yields the following motivic $\mathbb{P}^1$-spectrum (see [**81**, Theorem 3]).

Proposition 3.5.4. *Assume S is a regular noetherian separated scheme of finite Krull dimension with $2 \in \mathscr{O}_S(S)^\times$. Then there is a motivic $\mathbb{P}^1$-spectrum* $\mathrm{KGW} = (\mathrm{GW}^n) \in \mathcal{S}\mathsf{pt}_{\mathbb{P}^1}(\mathcal{S}\mathsf{m}_S)$, *whose levels are 4-periodic:* $\mathrm{GW}^n \cong \mathrm{GW}^{n+4}$ *and we have*

$$\mathrm{GW}^n \cong \begin{cases} \mathrm{GrO} \times \mathbb{Z}, & n = 0, \\ \mathrm{Sp/GL}, & n = 1, \\ \mathrm{BSp} \times \mathbb{Z}, & n = 2, \\ \mathrm{O/GL}, & n = 3, \end{cases} \quad \textit{and} \quad \Omega_{S^1} \mathrm{GW}^n \cong \begin{cases} \mathrm{O}, & n = 0, \\ (\mathrm{GL/O})_{\text{ét}}, & n = 1, \\ \mathrm{Sp}, & n = 2, \\ \mathrm{GL/Sp}, & n = 3. \end{cases}$$

Here $(\mathrm{GL/O})_{\text{ét}}$ *denotes the étale (or scheme theoretic) quotient and* GrO *is the* infinite orthogonal Grassmannian *(over S).*

Moreover,

$$\mathrm{GW}^n_i(X) = [S^i \wedge X_+, \mathrm{GW}^n]_{\mathbb{A}^1,*}, \forall X \in \mathcal{S}\mathsf{m}_S.$$

Chapter 4

Enumeration results of non-stable vector bundles

We already stated in the introduction the Serre splitting and Bass cancellation theorems in the classical theory of commutative rings and finitely generated projective modules. In this chapter, we apply the super-sophisticated formalism discussed so far to some related problems. Precisely, we re-examine the cancellation property of a projective module by translating the problem into the motivic homotopy language, and study in detail the homotopy type of the related mapping spaces, yielding an algebraic description of the set of isomorphism classes of vector bundles (of a given rank) which represent a given stable vector bundle. Then using cohomological tools, we give cancellation results for projective modules. We also study cancellation properties of symplectic vector bundles in a similar fashion.

4.1. Homotopy type of some motivic spaces and some derived mapping spaces

Recall from Example 3.2.11 that we have the following $\mathbb{A}^1$-homotopy fiber sequence (see [**68**, §8.2]):

$$\mathbb{A}^{n+1} \setminus 0 \to \mathrm{BGL}_n \to \mathrm{BGL}_{n+1}.$$

From this and the computation of the first few $\mathbb{A}^1$-homotopy sheaves of the motivic sphere $\mathbb{A}^{n+1} \setminus 0$ (see [**68**, Corollary 6.43])

$$\pi_i^{\mathbb{A}^1}(\mathbb{A}^{n+1} \setminus 0) = \begin{cases} 0, & i < n, \\ \mathbf{K}^{\mathrm{MW}}_{n+1}, & i = n, \end{cases}$$

we see that the induced map of $\mathbb{A}^1$-homotopy sheaves

$$\pi_j^{\mathbb{A}^1} \mathrm{BGL}_n \to \pi_j^{\mathbb{A}^1} \mathrm{BGL}_{n+1}$$

is an isomorphism if $j < n$ and a surjection if $j = n$.

Thus also, the induced map

$$\pi_j^{\mathbb{A}^1} \mathrm{BGL}_n \to \pi_j^{\mathbb{A}^1} \mathrm{BGL}$$

is an isomorphism if $j < n$ and an epimorphism if $j = n$.

Let F_n be the $\mathbb{A}^1$-homotopy fiber of the canonical map $\varphi_n : \mathrm{BGL}_n \to \mathrm{BGL}$, so that we have an $\mathbb{A}^1$-homotopy fiber sequence

$$F_n \to \mathrm{BGL}_n \to \mathrm{BGL}, .$$

From the above results, we see that

$$\pi_j^{\mathbb{A}^1} F_n = 0, j < n.$$

We now compute the next $\mathbb{A}^1$-homotopy sheaf of F_n. By Section 1.3 (or [**45**, §6.1-§6.5]), there is a commutative diagram

$$\begin{array}{ccccc} \mathbb{A}^{n+1} \setminus 0 & \xrightarrow{i} & F_n & \longrightarrow & F_{n+1} \\ \| & & \downarrow & & \downarrow \\ \mathbb{A}^{n+1} \setminus 0 & \longrightarrow & \mathrm{BGL}_n & \longrightarrow & \mathrm{BGL}_{n+1} \\ & & \downarrow & & \downarrow \\ & & \mathrm{BGL} & = & \mathrm{BGL} \end{array}$$

where the four 3-term rows and columns are $\mathbb{A}^1$-homotopy fiber sequences, with some extra equivariance properties (which we omit to state here). From the $\mathbb{A}^1$-homotopy fiber sequence in the first row we get an exact sequence

$$\pi^{\mathbb{A}^1}_{n+1} F_{n+1} \to \pi^{\mathbb{A}^1}_n (\mathbb{A}^{n+1} \setminus 0) \to \pi^{\mathbb{A}^1}_n F_n \to \pi^{\mathbb{A}^1}_n F_{n+1} = 0,$$

so the map

$$\mathbf{K}^{\mathrm{MW}}_{n+1} = \pi^{\mathbb{A}^1}_n (\mathbb{A}^{n+1} \setminus 0) \to \pi^{\mathbb{A}^1}_n F_n$$

and hence also

$$\pi^{\mathbb{A}^1}_{n+1}(\mathbb{A}^{n+2} \setminus 0) \to \pi^{\mathbb{A}^1}_{n+1} F_{n+1}$$

are epimorphisms. And so

$$\pi^{\mathbb{A}^1}_n F_n \cong \mathrm{coker}(\pi^{\mathbb{A}^1}_{n+1} F_{n+1} \to \pi^{\mathbb{A}^1}_n (\mathbb{A}^{n+1} \setminus 0)) \cong \mathrm{coker}(\pi^{\mathbb{A}^1}_{n+1}(\mathbb{A}^{n+2} \setminus 0) \to \pi^{\mathbb{A}^1}_n (\mathbb{A}^{n+1} \setminus 0)).$$

The composite $\pi^{\mathbb{A}^1}_{n+1}(\mathbb{A}^{n+2} \setminus 0) \to \pi^{\mathbb{A}^1}_{n+1} F_{n+1} \to \pi^{\mathbb{A}^1}_n (\mathbb{A}^{n+1} \setminus 0)$ is the map

$$\mathbf{K}^{\mathrm{MW}}_{n+2} \to \mathbf{K}^{\mathrm{MW}}_{n+1}$$

discussed in [**8**, Lemma 3.5], since, by the naturality of the connecting homomorphism of the upper right square of the previous diagram and its upper left square with n replaced by $n+1$, we obtain a commutative diagram

$$\begin{array}{ccccc}
\pi^{\mathbb{A}^1}_{n+1}(\mathbb{A}^{n+2}\setminus 0) & \xrightarrow{i_*} & \pi^{\mathbb{A}^1}_{n+1}(F_{n+1}) & \longrightarrow & \pi^{\mathbb{A}^1}_{n}(\mathbb{A}^{n+1}\setminus 0) \\
\| & & \downarrow & & \| \\
\pi^{\mathbb{A}^1}_{n+1}(\mathbb{A}^{n+2}\setminus 0) & \xrightarrow{\delta_{n+1}} & \pi^{\mathbb{A}^1}_{n}(\mathrm{SL}_{n+1}) & \xrightarrow{q_n} & \pi^{\mathbb{A}^1}_{n}(\mathbb{A}^{n+1}\setminus 0),
\end{array}$$

where q_n and δ_{n+1} are the maps in [**8**, Lemma 3.5]. Thus that composite map is 0 if n is even and is multiplication by η if n is odd; so

$$\mathrm{im}(\pi^{\mathbb{A}^1}_{n+1}F_{n+1}\to\pi^{\mathbb{A}^1}_{n}(\mathbb{A}^{n+1}\setminus 0))\cong\begin{cases}0, & n \text{ even};\\ \eta\mathbf{K}^{\mathrm{MW}}_{n+2}, & n \text{ odd}.\end{cases}$$

Thus

$$\pi^{\mathbb{A}^1}_{n}F_n\cong\mathrm{coker}(\mathbf{K}^{\mathrm{MW}}_{n+2}\to\mathbf{K}^{\mathrm{MW}}_{n+1})\cong\begin{cases}\mathbf{K}^{\mathrm{MW}}_{n+1}, & n \text{ even};\\ \mathbf{K}^{\mathrm{MW}}_{n+1}/\eta\mathbf{K}^{\mathrm{MW}}_{n+2}=\mathbf{K}^{\mathrm{M}}_{n+1}, & n \text{ odd}.\end{cases}$$

Moreover, by the $\mathbb{A}^1$-homotopy fiber sequence $\mathbb{A}^{n+1}\setminus 0\to F_n\to F_{n+1}$ we get exact sequences

$$\pi^{\mathbb{A}^1}_{n+1}(\mathbb{A}^{n+1}\setminus 0)\to\pi^{\mathbb{A}^1}_{n+1}F_n\to\pi^{\mathbb{A}^1}_{n+1}F_{n+1}=\mathbf{K}^{\mathrm{M}}_{n+2}\xrightarrow{0}\pi^{\mathbb{A}^1}_{n}(\mathbb{A}^{n+1}\setminus 0), n \text{ even},$$

$$\pi^{\mathbb{A}^1}_{n+1}(\mathbb{A}^{n+1}\setminus 0)\to\pi^{\mathbb{A}^1}_{n+1}F_n\to\pi^{\mathbb{A}^1}_{n+1}F_{n+1}=\mathbf{K}^{\mathrm{MW}}_{n+2}\xrightarrow{\eta}\mathbf{K}^{\mathrm{MW}}_{n+1}=\pi^{\mathbb{A}^1}_{n}(\mathbb{A}^{n+1}\setminus 0), n \text{ odd},$$

and since $\ker(\mathbf{K}^{\mathrm{MW}}_{n+2}\xrightarrow{\eta}\mathbf{K}^{\mathrm{MW}}_{n+1})\cong 2\mathbf{K}^{\mathrm{M}}_{n+2}$, we get exact sequences

$$\begin{cases}\pi^{\mathbb{A}^1}_{n+1}(\mathbb{A}^{n+1}\setminus 0)\to\pi^{\mathbb{A}^1}_{n+1}F_n\to\mathbf{K}^{\mathrm{M}}_{n+2}=\pi^{\mathbb{A}^1}_{n+1}F_{n+1}\to 0, & n \text{ even};\\ \pi^{\mathbb{A}^1}_{n+1}(\mathbb{A}^{n+1}\setminus 0)\to\pi^{\mathbb{A}^1}_{n+1}F_n\to 2\mathbf{K}^{\mathrm{M}}_{n+2}\to 0, & n \text{ odd}.\end{cases}\tag{4.1}$$

As $\pi^{\mathbb{A}^1}_j F_n=0$ for $j<n$, by the Moore-Postnikov decomposition in $\mathbb{A}^1$-homotopy theory stated in Theorem 3.2.9, we can factorize the map $\mathrm{BGL}_n\to\mathrm{BGL}$ as $\mathrm{BGL}_n\to E=E_n\to\mathrm{BGL}$ and the map $E=E_n\to\mathrm{BGL}$ fits into a homotopy cartesian diagram

$$\begin{array}{ccc}
E=E_n & \longrightarrow & \mathrm{B}\mathbb{G}_m \\
\downarrow & & \downarrow \\
\mathrm{BGL}=E_{n-1} & \xrightarrow{k_{n+1}} & \mathrm{K}^{\mathbb{G}_m}(\pi^{\mathbb{A}^1}_n F_n, n+1),
\end{array}$$

for a unique $[k_{n+1}]\in[E_{n-1},\mathrm{K}^{\mathbb{G}_m}(\pi^{\mathbb{A}^1}_n F_n,n+1)]_{\mathbb{A}^1}$, if $n\geqslant 2$. For any $j\leqslant n$, we have $\pi^{\mathbb{A}^1}_j\mathrm{BGL}_n\cong\pi^{\mathbb{A}^1}_j E_n$.

If $n\geqslant 3$ is odd, then $\mathbb{G}_m$ acts trivially on $\pi^{\mathbb{A}^1}_n F_n=\mathbf{K}^{\mathrm{M}}_{n+1}$. Indeed, by the paragraph on [**9**, pp. 1056-1057], this action is through the morphism

$$\mathbb{G}_m\to(\mathbf{K}^{\mathrm{MW}}_0)^\times, u\mapsto\langle u\rangle:=1+\eta[u]$$

then the multiplication $\mathbf{K}^{\mathrm{MW}}_0\times\mathbf{K}^{\mathrm{MW}}_{n+1}\to\mathbf{K}^{\mathrm{MW}}_{n+1}$ and the quotient $\mathbf{K}^{\mathrm{MW}}_{n+1}\to\mathbf{K}^{\mathrm{MW}}_{n+1}/\eta\mathbf{K}^{\mathrm{MW}}_{n+2}=\mathbf{K}^{\mathrm{M}}_{n+1}$; but $\eta\mathbf{K}^{\mathrm{MW}}_{n+2}$ is mapped to 0 in $\mathbf{K}^{\mathrm{M}}_{n+1}$.

So the above homotopy cartesian diagram reduces to an $\mathbb{A}^1$-homotopy fiber sequence

$$E=E_n\to\mathrm{BGL}=E_{n-1}\xrightarrow{\theta}\mathrm{K}(\mathbf{K}^{\mathrm{M}}_{n+1},n+1).\tag{4.2}$$

whence a principal $\mathbb{A}^1$-homotopy fiber sequence

$$\mathrm{K}(\mathbf{K}^{\mathrm{M}}_{n+1},n)\to E=E_n\to\mathrm{BGL}=E_{n-1}.$$

The $\mathbb{A}^1$-homotopy class of the map θ is the universal $(n+1)$-st Chern class $c_{n+1}\in\mathrm{H}^{n+1}(\mathrm{BGL};\mathbf{K}^{\mathrm{M}}_{n+1})=\mathrm{CH}^{n+1}(\mathrm{BGL})$ (see [**10**, Example 5.2 and Proposition 5.8]).

Let k be a perfect field, A be a smooth affine k-algebra of Krull dimension $d\geqslant 3$, and $X=\mathrm{Spec}(A)$. Then by the $\mathbb{A}^1$-homotopy long exact sequence and crawling up the Moore-Postnikov tower of the map $\mathrm{BGL}_n\to\mathrm{BGL}$, one easily finds that the map

$$[X,\mathrm{BGL}_n]_{\mathbb{A}^1}\to[X,E]_{\mathbb{A}^1}$$

is surjective if $n\geqslant d-1$, and is bijective if $n\geqslant d$.

We turn to describe some nice properties of certain derived mapping spaces in the Morel-Voevodsky motivic model category, which bridge the problems about algebraic vector bundles with those of H-spaces, and finally transfer our enumeration problem about algebraic vector bundles to a pure topological question.

Theorem 4.1.1. *Let $H=(H_0,H_1,H_2,\cdots)$ be a motivic T-spectrum ($T=\mathbb{P}^1\simeq S^1\wedge\mathbb{G}_m$) for a base scheme S. Define $H'=(H'_0,H'_1,H'_2,\cdots)$ by $H'_n:=\mathbf{R}\underline{\mathrm{Hom}}_*(\mathbb{G}_m^{\wedge n},H_n)$. Then*

1. *$H'=(H'_0,H'_1,H'_2,\cdots)$ is a motivic S^1-spectrum, and $H'_0\simeq\mathrm{L}_{\mathbb{A}^1}H_0$.*
2. *H_0 is an abelian group object in the pointed $\mathbb{A}^1$-homotopy category $\mathcal{H}^{\mathbb{A}^1}_*(S)$.*
3. *For any $X\in\mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{S}\mathsf{m}_S)_*$, the derived mapping space $\mathrm{RMap}_*(X,H_0)\in\mathsf{s}\mathcal{S}\mathsf{et}_*$ is an ∞-loop space and hence all its components are weakly equivalent.*

Proof. We have $H'_0 = \mathbf{R}\underline{\mathrm{Hom}}_*(\mathbb{G}_m^{\wedge 0}, H_0) = \mathbf{R}\underline{\mathrm{Hom}}_*(S^0, H_0) \simeq \underline{\mathrm{Hom}}_*(S^0, \mathrm{L}_{\mathbb{A}^1} H_0) = \mathrm{L}_{\mathbb{A}^1} H_0$.

By performing a fibrant replacement, we may assume that H is a fibrant motivic T-spectrum, so that each $H_n \in \mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{S}\mathsf{m}_S)_*$ is $\mathbb{A}^1$-fibrant and the adjoint bonding maps $H_n \to \underline{\mathrm{Hom}}_*(\mathbb{P}^1, H_{n+1})$ are $\mathbb{A}^1$-weak equivalences (even local weak equivalences or sectionwise weak equivalences, see Proposition 3.5.2). Thus

$$\begin{aligned} H'_n &= \underline{\mathrm{Hom}}_*(\mathbb{G}_m^{\wedge n}, H_n) \simeq \underline{\mathrm{Hom}}_*(\mathbb{G}_m^{\wedge n}, \underline{\mathrm{Hom}}_*(S^1 \wedge \mathbb{G}_m, H_{n+1})) \\ &\cong \underline{\mathrm{Hom}}_*(S^1 \wedge \mathbb{G}_m^{\wedge n+1}, H_{n+1}) \cong \underline{\mathrm{Hom}}_*(S^1, \underline{\mathrm{Hom}}_*(\mathbb{G}_m^{\wedge n+1}, H_{n+1})) = \Omega H'_{n+1}, \end{aligned}$$

showing that $H' = (H'_0, H'_1, H'_2, \cdots)$ is a motivic S^1-spectrum, and therefore (1) is proved.

For (2), just note that for any $X \in \mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{S}\mathsf{m}_S)_*$, the set $[X, H_0]_{\mathbb{A}^1,*} = [X, H'_0]_{\mathbb{A}^1,*} = [X, \Omega^2 H'_2]_{\mathbb{A}^1,*}$ is an abelian group (any term in a motivic S^1-spectrum is an abelian group object in $\mathcal{H}_*^{\mathbb{A}^1}(S)$).

For (3), note that for all $K \in \mathsf{s}\mathcal{S}\mathsf{et}_*$, $X \in \mathsf{s}\mathcal{P}\mathsf{re}(\mathcal{S}\mathsf{m}_S)_*$, there are canonical isomorphisms

$$[K, \mathrm{RMap}_*(X, H_0)]_{\mathsf{s}\mathcal{S}\mathsf{et}_*} \cong [K \wedge X, H_0]_{\mathbb{A}^1,*},$$

the right hand side are abelian groups by (2). □

Corollary 4.1.2. *The space* BGL *is an abelian group object in the pointed $\mathbb{A}^1$-homotopy category $\mathcal{H}_*^{\mathbb{A}^1}(S)$ and for any $X \in \mathcal{S}\mathsf{m}_S$, all the components of the derived mapping space* $\mathrm{RMap}(X, \mathrm{BGL}) \in \mathsf{s}\mathcal{S}\mathsf{et}_*$ *are weakly equivalent.*

If $S = \mathrm{Spec}(k)$ for a perfect field k, then BSL *is an abelian group object in the pointed $\mathbb{A}^1$-homotopy category $\mathcal{H}_*^{\mathbb{A}^1}(k)$ and all the components of the derived mapping space* $\mathrm{RMap}(X, \mathrm{BSL}) \in \mathsf{s}\mathcal{S}\mathsf{et}_*$ *are weakly equivalent.*

Proof. Since the motivic T-spectrum KGL representing algebraic K-theory has term $\mathrm{BGL} \times \mathbb{Z}$ in each level (Proposition 3.5.3), by the previous result we see that $\mathrm{BGL} \times \mathbb{Z}$ is an abelian group object in $\mathcal{H}^{\mathbb{A}^1}(S)$. We conclude by noting that the projection $\mathrm{BGL} \times \mathbb{Z} \to \mathbb{Z}$ is a homomorphism of abelian group objects with kernel BGL.

If $S = \mathrm{Spec}(k)$, note that there is an $\mathbb{A}^1$-homotopy fiber sequence $\mathrm{BSL} \to \mathrm{BGL} \to \mathrm{B}\mathbb{G}_m = \mathrm{K}(\mathbf{K}_1^{\mathrm{M}}, 1)$ (thanks to the fact that the Picard group of a normal scheme is $\mathbb{A}^1$-invariant: $\mathrm{Pic}(X \times \mathbb{A}^1) \cong \mathrm{Pic}(X)$, yielding that $\mathbb{G}_m \in \mathcal{G}\mathsf{r}_k^{\mathbb{A}^1}$), realizing BSL as the kernel of $\mathrm{BGL} \to \mathrm{K}(\mathbf{K}_1^{\mathrm{M}}, 1)$ (note that the second arrow splits), thus BSL is an abelian group object as well. □

This is to be compared with the fact in classical topology that for "non-stable" groups, the path-components of $\mathrm{Map}(X, \mathrm{B}G)$—whose homotopy types are closely related with gauge groups—may represent (infinitely) many distinct homotopy types (see e.g. **[84]**). The following result appears in **[65**, Lemma 9.2.3] in the topological situation.

Proposition 4.1.3. *Let $(B, 0) \in \mathsf{s}\mathcal{S}\mathsf{et}_*$ be an abelian group object in the pointed homotopy category* $\mathrm{Ho}(\mathsf{s}\mathcal{S}\mathsf{et}_*)$ *(i.e. an abelian H-group) with a binary operation $m : (B, 0) \times (B, 0) \to (B, 0), (b, b') \mapsto bb'$. Denote its path components by $B_\xi, \xi \in \pi_0 B$. Then*

$$(B, 0) \simeq (B_0, 0) \times \pi_0(B, 0)$$

as abelian H-groups.

Proof. Fix a base point b_ξ in each component B_ξ with $b_0 = 0$. Then we easily find that the maps

$$B \to B_0 \times \pi_0(B, 0), (b \in B_\xi) \mapsto (bb_{-\xi}, \xi)$$

and

$$B_0 \times \pi_0(B, 0) \to B, (b, \xi) \mapsto bb_\xi$$

are homomorphisms of H-groups and are homotopy inverse to each other. □

4.2. Enumeration results on vector bundles of critical rank

In this section, we take advantage of the fact that $\mathrm{RMap}(X, \mathrm{BGL})$ is an H-space for every scheme X (obtained from last section) to deduce some enumeration results on vector bundles of critical rank. In particular, an enumeration formula of $\mathcal{V}_n(X, \xi)$ (the set of isomorphism classes of rank-n vector bundles which represent a given stable vector bundle ξ) is given. Then we use a delicate result of Suslin to give an upper bound of $\mathcal{V}_n(X, \xi)$, triviality of which would lead to cancellation result. We also discuss in some important cases where that upper bound is trivial.

Recall first that we have the $\mathbb{A}^1$-homotopy fiber sequence

$$F_n \to \mathrm{BGL}_n \xrightarrow{\varphi = \varphi_n} \mathrm{BGL}.$$

We have the induced map on $\mathbb{A}^1$-homotopy classes

$$\varphi_* : [X, \mathrm{BGL}_n]_{\mathbb{A}^1} \to [X, \mathrm{BGL}]_{\mathbb{A}^1}$$

and we want to describe the inverse-image under φ_* of a class in the right hand side.

Given $\xi \in [X, \mathrm{BGL}]_{\mathbb{A}^1}$, let

$$T_\xi := m(-, \xi)_* : \pi_1(\mathrm{RMap}(X, \mathrm{BGL}), 0) \to \pi_1(\mathrm{RMap}(X, \mathrm{BGL}), \xi)$$

be the isomorphism on fundamental groups induced by $m(-, \xi) : \mathrm{RMap}(X, \mathrm{BGL})_0 \to \mathrm{RMap}(X, \mathrm{BGL})_\xi$ as introduced in Proposition 4.1.3. We also denote by $\mathcal{V}_n(X, \xi) := \varphi_*^{-1}(\xi) \subset \mathcal{V}_n(X)$ the set of isomorphism classes of rank-n vector bundles which represent ξ.

The following is one of our first main enumeration results.

Theorem 4.2.1. *Let $n = d$ be odd, let ξ be a stable vector bundle over X, whose classifying map is still denoted by $\xi : X \to \mathrm{BGL}$. Denote by*

$$\Delta(c_{n+1}, \xi) := \Omega c_{n+1} \circ T_\xi : \mathrm{K}_1(X) = [X, \Omega\mathrm{BGL}]_{\mathbb{A}^1} \to [X, \mathrm{K}(\mathbf{K}^{\mathrm{M}}_{n+1}, n)]_{\mathbb{A}^1} = \mathrm{H}^n(X; \mathbf{K}^{\mathrm{M}}_{n+1})$$

the induced homomorphism on fundamental groups, where c_{n+1} is the $(n+1)$-st universal Chern class. Then there is a bijection

$$\mathcal{V}_n(X, \xi) \cong \mathrm{coker}\big(\mathrm{K}_1(X) = [X, \Omega\mathrm{BGL}]_{\mathbb{A}^1} \xrightarrow{\Delta(c_{n+1}, \xi)} [X, \mathrm{K}(\mathbf{K}^{\mathrm{M}}_{n+1}, n)]_{\mathbb{A}^1} = \mathrm{H}^n(X; \mathbf{K}^{\mathrm{M}}_{n+1})\big).$$

Proof. We know that there is a principal $\mathbb{A}^1$-homotopy fiber sequence

$$\mathrm{K}(\mathbf{K}^{\mathrm{M}}_{n+1}, n) \to E \to \mathrm{BGL}$$

classified by the $\mathbb{A}^1$-homotopy class of the universal $(n+1)$-st Chern class $c_{n+1} \in \mathrm{H}^{n+1}(\mathrm{BGL}; \mathbf{K}^{\mathrm{M}}_{n+1}) = \mathrm{CH}^{n+1}(\mathrm{BGL})$. So we have a homotopy fiber sequence

$$\mathrm{RMap}(X, E) \to \mathrm{RMap}(X, \mathrm{BGL}) \xrightarrow{c_{n+1}} \mathrm{RMap}(X, \mathrm{K}(\mathbf{K}^{\mathrm{M}}_{n+1}, n+1))$$

in $\mathsf{s}\mathcal{S}\mathsf{et}_*$.

For $n = d = \dim(X)$, we have $c_{n+1}(\xi) = c_{d+1}(\xi) = 0$ as Chern classes vanish above the rank of the vector bundle. So by the previous discussion and Theorem 1.6.8, the lifting set $\varphi_*^{-1}(\xi)$ is in bijection with the orbit set of the action of $\pi_1(\mathrm{RMap}(X, \mathrm{BGL}), \xi)$ on the abelian group $[X, \mathrm{K}(\mathbf{K}^{\mathrm{M}}_{n+1}, n+1)]_{\mathbb{A}^1} = \mathrm{H}^{n+1}(X; \mathbf{K}^{\mathrm{M}}_{n+1})$, which is the cokernel of the homomorphism

$$\pi_1(\mathrm{RMap}(X, \mathrm{BGL}), \xi) \xrightarrow{(c_{n+1})_*} \pi_1(\mathrm{RMap}(X, \mathrm{K}(\mathbf{K}^{\mathrm{M}}_{n+1}, n+1)), 0) = \mathrm{H}^n(X; \mathbf{K}^{\mathrm{M}}_{n+1}),$$

and since the map T_ξ is an isomorphism by Proposition 4.1.3, the map Ωc_{n+1} above and the composite $\Omega c_{n+1} \circ T_\xi$ have isomorphic cokernel, the claim follows. □

As usual, we have the universal j-th Chern classes $c_j \in \mathrm{H}^j(\mathrm{BGL}; \mathbf{K}^{\mathrm{M}}_j) = \mathrm{CH}^j(\mathrm{BGL})$ for $j \in \mathbb{N}$ (see e.g. [**90**, Chapter 2]).

For the result below, we have the *exterior product* $c_r \times c_{n-r} := \mu(c_r \otimes c_{n-r})$, where μ is the map

$$[\mathrm{BGL}, \mathrm{K}(\mathbf{K}^{\mathrm{M}}_r, r)]_{\mathbb{A}^1} \otimes [\mathrm{BGL}, \mathrm{K}(\mathbf{K}^{\mathrm{M}}_{n-r}, n-r)]_{\mathbb{A}^1} \to [\mathrm{BGL} \times \mathrm{BGL}, \mathrm{K}(\mathbf{K}^{\mathrm{M}}_n, n)]_{\mathbb{A}^1}$$

induced by the multiplication in the graded sheaves of Milnor K-groups $\mathbf{K}^{\mathrm{M}}_*$ (see the end of §3).

Theorem 4.2.2. *The universal Chern classes satisfy*

$$m^* c_n = \sum_{r=0}^{n} (c_r \times c_{n-r}) \in [\mathrm{BGL} \times \mathrm{BGL}, \mathrm{K}(\mathbf{K}^{\mathrm{M}}_n, n)]_{\mathbb{A}^1}. \tag{4.3}$$

Proof. Since the universal r-th Chern class $c_r : \mathrm{BGL} \to \mathrm{K}(\mathbf{K}^{\mathrm{M}}_r, r)$ factor through the inclusion $\mathrm{BGL}_n \to \mathrm{BGL}$ for each $n \geqslant r$ and stabilize, both sides in the formula above are determined by their restriction to $\mathrm{BGL}_n \times \mathrm{BGL}_n$. And we have $\mathrm{CH}^*(\mathrm{BGL}_n) = \mathbb{Z}[c_1, \cdots, c_n]$ ([**90**, Theorem 2.13]).

By the Chow Künneth formula in [**89**, §6] (since BGL_n can be approximated by "linear varieties", the Grassmannians; see also [**90**, Chapter 17] and [**91**] for more discussions on when the Chow Künneth formula holds), we may write

$$\mathrm{CH}^*(\mathrm{BGL}_n \times \mathrm{BGL}_n) = \mathbb{Z}[c_i \times c_j : 1 \leqslant i, j \leqslant n]$$

with $|c_i \times c_j| = i + j$.

On the other hand, for any $U \in \mathcal{S}\mathsf{m}_k$ and $(\xi, \xi') \in [U, \mathrm{BGL}]_{\mathbb{A}^1} \times [U, \mathrm{BGL}]_{\mathbb{A}^1}$, as $m_*(\xi, \xi') = \xi + \xi'$, by Whitney sum formula for Chern classes, we have

$$(m^* c_n)(\xi, \xi') = c_n m_*(\xi, \xi') = c_n(\xi + \xi') = \sum_{r=0}^{n} c_r(\xi) \cdot c_{n-r}(\xi') = \sum_{r=0}^{n} (c_r \times c_{n-r})(\xi, \xi').$$

Now we take $U = X/G$, where X is an open subscheme of a representation $\mathbb{A}^N_k$ of $G = \mathrm{GL}_n \times \mathrm{GL}_n$ for big enough N such that $\mathbb{A}^N_k \setminus X$ is G-invariant and such that G acts freely on X (see [**90**, §2.2]), then $\mathrm{CH}^*(\mathrm{BGL}_n \times \mathrm{BGL}_n) = \mathrm{CH}^*(U)$ ([**90**, Theorem 2.5]) which is induced by a map $u = (\xi, \xi') : U \to \mathrm{BGL}_n \times \mathrm{BGL}_n$ classifying the principal G-bundle $X \to U$ (so that we can consider the induced map u^* as the identity on Chow rings). Then the above formula becomes

$$u^* m^* c_n = u^* \sum_{r=0}^{n} (c_r \times c_{n-r}),$$

proving (4.3). □

We have the *suspension homomorphism*

$$\sigma : \mathrm{CH}^j(\mathrm{BGL}) = [\mathrm{BGL}, \mathrm{K}(\mathbf{K}^{\mathrm{M}}_j, j)]_{\mathbb{A}^1, *} \to [\Omega\mathrm{BGL}, \Omega\mathrm{K}(\mathbf{K}^{\mathrm{M}}_j, j)]_{\mathbb{A}^1, *} = \widetilde{\mathrm{H}}^{j-1}(\mathrm{GL}; \mathbf{K}^{\mathrm{M}}_j)$$

for $j \geqslant 1$, induced by the loop functor Ω which is given by

$$\sigma([\theta]) = [\Omega\theta].$$

Note that in the above, it doesn't matter whether we use pointed or unpointed $\mathbb{A}^1$-homotopy classes provided $j > 2$, see [**8**, Lemma 2.1]. In fact this is also the case for $j = 2$ by Corollary 2.9.2, since k is a field and so $\mathrm{H}^n(*; \mathbf{M}) = 0$ for any $\mathbf{M} \in \mathcal{A}\mathsf{b}^{\mathbb{A}^1}_k$ and $n \geqslant 1$.

Theorem 4.2.3. *Let $\xi \in [X, \mathrm{BGL}]_{\mathbb{A}^1}$ be such that $c_{n+1}(\xi) = 0$. Then the map*

$$\Delta(c_{n+1}, \xi) : \mathrm{K}_1(X) = [X, \Omega\mathrm{BGL}]_{\mathbb{A}^1} \to [X, \mathrm{K}(\mathbf{K}^{\mathrm{M}}_{n+1}, n)]_{\mathbb{A}^1} = \mathrm{H}^n(X; \mathbf{K}^{\mathrm{M}}_{n+1})$$

is given by

$$\Delta(c_{n+1}, \xi)(\beta) = (\Omega c_{n+1})(\beta) + \sum_{r=1}^{n}((\Omega c_r)(\beta)) \cdot c_{n+1-r}(\xi). \tag{4.4}$$

Similar statements hold with BSL *in place of* BGL.

Proof. Since $m^* c_{n+1}(-, \xi) = c_{n+1} m(-, \xi) : \mathrm{RMap}(X, \mathrm{BGL}) \to \mathrm{RMap}(X, \mathrm{K}(\mathbf{K}^{\mathrm{M}}_{n+1}, n+1))$, we may see

$$(m^* c_{n+1}(-, \xi))_* = (c_{n+1})_* T_\xi$$

as maps $\pi_1(\mathrm{RMap}(X, \mathrm{BGL}), 0) \to \pi_1(\mathrm{RMap}(X, \mathrm{K}(\mathbf{K}^{\mathrm{M}}_{n+1}, n+1)), 0) = \mathrm{H}^n(X; \mathbf{K}^{\mathrm{M}}_{n+1})$. Note that these derived mapping spaces RMap and relevant maps are well-defined up to homotopy.

On the other hand, since

$$m^* c_{n+1} = \sum_{r=0}^{n+1} c_r \times c_{n+1-r} \in [\mathrm{BGL} \times \mathrm{BGL}, \mathrm{K}(\mathbf{K}^{\mathrm{M}}_{n+1}, n+1)]_{\mathbb{A}^1},$$

by taking $\mathrm{RMap}(X, -)$ on the two sides we see that the following two maps in $\mathtt{s\mathcal{S}et}_*$ are homotopic:

$$m^* c_{n+1}(-, \xi) \simeq \sum_{r=0}^{n+1} c_r(-) \cdot c_{n+1-r}(\xi) : \mathrm{RMap}(X, \mathrm{BGL}) \to \mathrm{RMap}(X, \mathrm{K}(\mathbf{K}^{\mathrm{M}}_{n+1}, n+1)). \tag{4.5}$$

Here the map $c_r(-) \cdot c_{n+1-r}(\xi)$ is the composite

$$\mathrm{RMap}(X, \mathrm{BGL}) \xrightarrow{(c_r(-), c_{n+1-r}(\xi))} \mathrm{RMap}(X, \mathrm{K}(\mathbf{K}^{\mathrm{M}}_r, r)) \times \mathrm{RMap}(X, \mathrm{K}(\mathbf{K}^{\mathrm{M}}_{n+1-r}, n+1-r))$$
$$\to \mathrm{RMap}(X \times X, \mathrm{K}(\mathbf{K}^{\mathrm{M}}_{n+1}, n+1)) \xrightarrow{\Delta^*} \mathrm{RMap}(X, \mathrm{K}(\mathbf{K}^{\mathrm{M}}_{n+1}, n+1)),$$

where the second coordinate of the first arrow is given by

$$\mathrm{RMap}(X, \mathrm{BGL}) \to \Delta^0 \xrightarrow{c_{n+1-r}(\xi)} \mathrm{RMap}(X, \mathrm{K}(\mathbf{K}^{\mathrm{M}}_{n+1-r}, n+1-r)),$$

the second arrow is the obvious map and the third is induced by the diagonal of X. From this description it's clear that

$$\Omega(c_r(-) \cdot c_{n+1-r}(\xi)) \simeq (\Omega c_r(-)) \cdot c_{n+1-r}(\xi) : \Omega\mathrm{RMap}(X, \mathrm{BGL}) \to \mathrm{RMap}(X, \mathrm{K}(\mathbf{K}^{\mathrm{M}}_{n+1}, n)).$$

Additions are also preserved: for any $[a], [b] \in [\mathrm{RMap}(X, \mathrm{BGL}), \mathrm{RMap}(X, \mathrm{K}(\mathbf{K}^{\mathrm{M}}_{n+1}, n+1))]_{\mathtt{s\mathcal{S}et}_*}$, we have $[a]_* + [b]_* \simeq ([a] + [b])_* : \Omega\mathrm{RMap}(X, \mathrm{BGL}) \to \mathrm{RMap}(X, \mathrm{K}(\mathbf{K}^{\mathrm{M}}_{n+1}, n))$ (the subscript $*$ refers the effects on loop spaces). Indeed, the sum $[a] + [b]$ is represented by the composite

$$\mathrm{RMap}(X, \mathrm{BGL}) \xrightarrow{\Delta} \mathrm{RMap}(X, \mathrm{BGL}) \times \mathrm{RMap}(X, \mathrm{BGL}) \xrightarrow{a \times b} \mathrm{RMap}(X, \mathrm{K}(\mathbf{K}^{\mathrm{M}}_{n+1}, n+1))$$
$$\times \mathrm{RMap}(X, \mathrm{K}(\mathbf{K}^{\mathrm{M}}_{n+1}, n+1)) \xrightarrow{+} \mathrm{RMap}(X, \mathrm{K}(\mathbf{K}^{\mathrm{M}}_{n+1}, n+1)),$$

where Δ refers the diagonal map and "+" is the H-group operation of $\mathrm{RMap}(X, \mathrm{K}(\mathbf{K}^{\mathrm{M}}_{n+1}, n+1))$ (see [**2**, p. 38]); taking Ω everywhere gives a similar composite for $[a]_* + [b]_* \in [\Omega\mathrm{RMap}(X, \mathrm{BGL}), \mathrm{RMap}(X, \mathrm{K}(\mathbf{K}^{\mathrm{M}}_{n+1}, n))]_{\mathtt{s\mathcal{S}et}_*}$, which says exactly that $[a]_* + [b]_* = ([a] + [b])_* \in [\Omega\mathrm{RMap}(X, \mathrm{BGL}), \mathrm{RMap}(X, \mathrm{K}(\mathbf{K}^{\mathrm{M}}_{n+1}, n))]_{\mathtt{s\mathcal{S}et}_*}$. So applying $\pi_0\Omega = \pi_1$ to eq. (4.5) we find

$$\Delta(c_{n+1}, \xi) = c_{n+1} \circ T_\xi = (c_{n+1})_* T_\xi = (m^* c_{n+1}(-, \xi))_* = \sum_{r=0}^{n+1} (\Omega c_r)(-) \cdot c_{n+1-r}(\xi).$$

The BSL case is also valid by Corollary 4.1.2. □

As $c_1(\xi) = c_1(\det \xi)$, we find that

$$\Delta(c_{n+1}, \det \xi)(\beta) = (\Omega c_{n+1})(\beta) + (\Omega c_n)(\beta) \cdot c_1(\xi).$$

If $c_2(\xi) = \cdots = c_n(\xi) = 0$, then $\Delta(c_{n+1}, \xi) = \Delta(c_{n+1}, \det \xi)$; so for $P \in \mathcal{V}_r(X)$ ($X = \mathrm{Spec}(A)$ has odd dimesion $d \geqslant 3$) with $c_2(P) = \cdots = c_d(P) = 0$, we have: P is cancellative iff $\det(P) \oplus A^{d-1}$ is cancellative. (Recall from the Introduction that for a commutative ring R, a finitely generated projective R-module P (of constant rank n) is *cancellative* if for any $m > 0$, $P \oplus R^m \cong Q \oplus R^m$ for some R-module Q implies $P \cong Q$.)

We also have the following obvious corollary.

Corollary 4.2.4. *Assume that* $\dim X = d \geqslant 3$ *is odd. Let $\xi, \xi' \in [X, \mathrm{BGL}]_{\mathbb{A}^1}$. If $c_j(\xi) = c_j(\xi'), 1 \leqslant j \leqslant d$, then ξ and ξ' have the same number of representatives in $\mathcal{V}_d(X)$. In particular, if ξ is cancellative, then so is ξ'.*

We next turn to more refined computations, precisely via Suslin's matrix construction, we provide a lower bound of the image of the homomorphism $\Delta(c_{d+1}, \xi)$ in our enumeration formula above (and hence an upper bound of its cokernel; see Theorem 4.2.9 below). Following [**5**], we denote

$$A_{2n+1} := k[x_0, \cdots, x_n, y_0, \cdots, y_n]/(\sum_{i=0}^{n} x_i y_i - 1)$$

and $Q_{2n+1} := \mathrm{Spec}(A_{2n+1})$. The projection to the first $n+1$ coordinates gives a morphism $Q_{2n+1} \to \mathbb{A}^{n+1} \setminus 0 = \mathbb{A}^{n+1}_k \setminus 0$, which is an $\mathbb{A}^n$-fibration hence an $\mathbb{A}^1$-weak equivalence.

Let A be a k-algebra. To any A-point (a,b) of Q_{2n+1}, Suslin associated (systemically and inductively) a matrix $\alpha_{n+1}(a,b) \in \mathrm{SL}_{2^n}(A) \subset \mathrm{GL}_{2^n}(A)$, yielding a k-morphism

$$\alpha_{n+1} : Q_{2n+1} \to \mathrm{GL}_{2^n} \hookrightarrow \mathrm{GL}.$$

Moreover, Suslin gave a systematic way to reduce the matrix $\alpha_{n+1}(a,b) \in \mathrm{SL}_{2^n}(A) \subset \mathrm{GL}_{2^n}(A)$ via elementary matrix operations (though non-explicitly) to an element $\beta_{n+1}(a,b) \in \mathrm{SL}_{n+1}(A) \subset \mathrm{GL}_{n+1}(A)$, whose first row can be designated to be $(a_0^{n!}, a_1, \cdots, a_n)$ if $a = (a_0, a_1, \cdots, a_n)$—(a special case of) the famous *Suslin's $n!$ theorem* (see **[86, 55]**). This gives a k-morphism

$$\beta_{n+1} : Q_{2n+1} \to \mathrm{SL}_{n+1} \subset \mathrm{GL}_{n+1}.$$

Note that we have $[\alpha_{n+1}(a,b)] = [\beta_{n+1}(a,b)] \in \mathrm{GL}(A)/\mathrm{E}(A) = \mathrm{K}_1(A)$ by construction.

$$\begin{array}{ccccc} & & & & \mathrm{GL}_{n+1} \\ & & & \overset{\beta_{n+1}}{\nearrow} & \big\uparrow \\ X & \xrightarrow{(a,b)} & Q_{2n+1} & \xrightarrow{\alpha_{n+1}} & \mathrm{GL}_{2^n} \end{array}$$

With the $\mathbb{A}^1$-weak equivalences $\mathbb{A}^{n+1} \setminus 0 \simeq Q_{2n+1}$, we obtain a morphism

$$\alpha_{n+1} : \mathbb{A}^{n+1} \setminus 0 \to \mathrm{GL} \simeq \Omega\mathrm{BGL}$$

in $\mathcal{H}_*^{\mathbb{A}^1}(k)$ (base points suitably chosen). Taking adjunction we get a morphism

$$\alpha_{n+1}^\sharp : (\mathbb{P}^1)^{\wedge(n+1)} \simeq \Sigma(\mathbb{A}^{n+1} \setminus 0) \to \mathrm{BGL}$$

in $\mathcal{H}_*^{\mathbb{A}^1}(k)$. Under the canonical isomorphism

$$[\mathbb{A}^{n+1} \setminus 0, \mathrm{K}(\mathbf{K}_r^{\mathrm{M}}, r-1)]_{\mathbb{A}^1,*} \xrightarrow{\cong} [(\mathbb{P}^1)^{\wedge(n+1)}, \mathrm{K}(\mathbf{K}_r^{\mathrm{M}}, r)]_{\mathbb{A}^1,*},$$

the class $(\Omega c_r)(\alpha_{n+1})$ corresponds to $c_r(\alpha_{n+1}^\sharp)$.

$$\mathbb{A}^{n+1} \setminus 0 \xrightarrow{\alpha_{n+1}} \Omega\mathrm{BGL} \xrightarrow{\Omega c_r} \mathrm{K}(\mathbf{K}_r^{\mathrm{M}}, r-1) \leftrightsquigarrow (\mathbb{P}^1)^{\wedge(n+1)} \xrightarrow{\alpha_{n+1}^\sharp} \mathrm{BGL} \xrightarrow{c_r} \mathrm{K}(\mathbf{K}_r^{\mathrm{M}}, r)$$

As $c_r(\alpha_{n+1}^\sharp) \in \widetilde{\mathrm{H}}^r((\mathbb{P}^1)^{\wedge(n+1)}; \mathbf{K}_r^{\mathrm{M}})$ and by **[8**, Lemma 4.5**]**,

$$\widetilde{\mathrm{H}}^r((\mathbb{P}^1)^{\wedge(n+1)}; \mathbf{K}_r^{\mathrm{M}}) = \widetilde{\mathrm{H}}^{r-1}(\mathbb{A}^{n+1} \setminus 0; \mathbf{K}_r^{\mathrm{M}}) \cong \begin{cases} 0, & r \neq n+1; \\ (\mathbf{K}_r^{\mathrm{M}})_{-(n+1)}(\mathrm{Spec}\ k) = \mathbf{K}_0^{\mathrm{M}}(\mathrm{Spec}\ k) = \mathbb{Z}, & r = n+1 \end{cases}$$

(where $\mathbf{M}_{-i}$ denotes the i-th contraction of a sheaf $\mathbf{M}$ and we have used **[8**, Lemma 2.7**]**), we obtain the following.

Proposition 4.2.5. $(\Omega c_r)(\alpha_{n+1}) = (\Omega c_r)(\beta_{n+1}) = 0$ *for* $r \neq n+1$.

Remark 4.2.6. By eq. (4.7) below, we see that $(\Omega c_{n+1})(\alpha_{n+1})$ is $\pm n!$ in $(\mathbf{K}_0^{\mathrm{M}})(\mathrm{Spec}\ k) = \mathbb{Z}$.

An easy exercise of scheme theory shows that for any k-algebra A, we have $\mathrm{Hom}_k(\mathrm{Spec}\ A, \mathbb{A}^{n+1} \setminus 0) \cong \mathrm{Um}_{n+1}(A)$ (the set of unimodular rows in A of length $n+1$). Suppose now that $X := \mathrm{Spec}\ A$ is smooth over k. With some effort, one can show the following (see **[27**, Theorem 2.1**]** or Proposition C.2.12).

Proposition 4.2.7. *Assume* $2 \leqslant \dim A = d \leqslant n$, *then* $\mathrm{H}^n(\mathrm{Spec}\ A; \mathbf{K}_{n+1}^{\mathrm{MW}}) \cong [\mathrm{Spec}\ A, \mathbb{A}^{n+1} \setminus 0]_{\mathbb{A}^1} \cong \mathrm{Um}_{n+1}(A)/\mathrm{E}_{n+1}(A)$. *Here* $\mathrm{E}_{n+1}(A)$ *denotes the group of elementary matrices of size* $n+1$, *with its natural action on rows of length* $n+1$.

Moreover, let $\mathrm{pr}_1 : \mathrm{GL}_{n+1} \to \mathbb{A}^{n+1} \setminus 0$ *be the projection to the first row, then we obtain a morphism*

$$\psi_{n+1} : \mathbb{A}^{n+1} \setminus 0 \to \mathbb{A}^{n+1} \setminus 0, (x_0, x_1, \cdots, x_n) \mapsto (x_0^{n!}, x_1, \cdots, x_n)$$

in $\mathcal{H}_*^{\mathbb{A}^1}(k)$, *being the composite* $\mathbb{A}^{n+1} \setminus 0 \simeq Q_{2n+1} \xrightarrow{\beta_{n+1}} \mathrm{GL}_{n+1} \xrightarrow{\mathrm{pr}_1} \mathbb{A}^{n+1} \setminus 0$. *Then the induced map*

$$(\psi_{n+1})_* : [\mathrm{Spec}\ A, \mathbb{A}^{n+1} \setminus 0]_{\mathbb{A}^1} \to [\mathrm{Spec}\ A, \mathbb{A}^{n+1} \setminus 0]_{\mathbb{A}^1}$$

on the cohomotopy set is given by

$$(\psi_{n+1})_*([a_0, a_1, \cdots, a_n]) = [a_0^{n!}, a_1, \cdots, a_n] = \frac{n!}{2} h \cdot [a_0, a_1, \cdots, a_n]$$

under the above isomorphism, for $(a_0, \cdots, a_n) \in \mathrm{Um}_{n+1}(A)$ *(where* $[a_0, \cdots, a_n]$ *denotes its class in the orbit set* $\mathrm{Um}_{n+1}(A)/\mathrm{E}_{n+1}(A)$ *and* $h = \langle 1, -1 \rangle$*).*

Remark 4.2.8. Let $\omega : \mathbb{A}^{n+1} \setminus 0 \to \mathrm{K}(\mathbf{K}_{n+1}^{\mathrm{MW}}, n)$ be the morphism determined by the projection to the first non-trivial (n-th) Postnikov tower, and let $[\omega] \in \mathrm{H}^n(\mathbb{A}^{n+1} \setminus 0; \mathbf{K}_{n+1}^{\mathrm{MW}})$ be the cohomology class it represents. Then $\mathrm{H}^n(\mathbb{A}^{n+1} \setminus 0; \mathbf{K}_{n+1}^{\mathrm{MW}}) \cong \mathbf{K}_0^{\mathrm{MW}}(k)$ is a rank-1 free $\mathbf{K}_0^{\mathrm{MW}}(k)$-module generated by $[\omega]$. Furthermore,

$$\psi_{n+1}^*[\omega] = \frac{n!}{2} h \cdot [\omega].$$

Then we get the last statement as follows: Let $a = (a_0, \cdots, a_n) \in \mathrm{Um}_{n+1}(A)$, then $\omega_*(\psi_{n+1})_*([a]) = [\omega \circ \psi_{n+1} \circ a] = (\psi_{n+1}^*[\omega])[a] = (\frac{n!}{2} h \cdot [\omega])[a] = \omega_*(\frac{n!}{2} h[a])$, and so $(\psi_{n+1})_*([a]) = \frac{n!}{2} h[a]$ (using Postnikov tower argument to see that $\omega_* : [X, \mathbb{A}^{n+1} \setminus 0]_{\mathbb{A}^1} \to \mathrm{H}^n(X; \mathbf{K}_{n+1}^{\mathrm{MW}})$ is a bijection, since $d \leqslant n$).

$$X = \mathrm{Spec}\ A \xrightarrow{a} \mathbb{A}^{n+1} \setminus 0 \xrightarrow{\psi_{n+1}} \mathbb{A}^{n+1} \setminus 0 \xrightarrow{\omega} \mathrm{K}(\mathbf{K}_{n+1}^{\mathrm{MW}}, n)$$

In fact, let $V \subset \mathbb{A}^{n+1} \setminus 0$ be the closed subvariety defined by $x_1 = \cdots = x_n = 0$, so $V \cong \mathbb{A}^1 \setminus 0, k(V) \cong k(x_0)$. The homology of the portion

$$\bigoplus_{y \in (\mathbb{A}^{n+1}\setminus 0)^{(n-1)}} \mathbf{K}_2^{\mathrm{MW}}(\kappa_y) \to \bigoplus_{y \in (\mathbb{A}^{n+1}\setminus 0)^{(n)}} \mathbf{K}_1^{\mathrm{MW}}(\kappa_y) \to \bigoplus_{y \in (\mathbb{A}^{n+1}\setminus 0)^{(n+1)}} \mathbf{K}_0^{\mathrm{MW}}(\kappa_y)$$

of the Rost-Schmid complex computes $\mathrm{H}^n(\mathbb{A}^{n+1} \setminus 0; \mathbf{K}_{n+1}^{\mathrm{MW}})$, and one finds that $[x_0] \in \mathbf{K}_1^{\mathrm{MW}}(k(V))$ is indeed a cycle. Moreover, in the localization exact sequence

$$0 = \mathrm{H}^n(\mathbb{A}^{n+1}; \mathbf{K}_{n+1}^{\mathrm{MW}}) \to \mathrm{H}^n(\mathbb{A}^{n+1} \setminus 0; \mathbf{K}_{n+1}^{\mathrm{MW}}) \xrightarrow{\partial} \mathrm{H}^{n+1}_{\{0\}}(\mathbb{A}^{n+1}; \mathbf{K}_{n+1}^{\mathrm{MW}}) = \mathbf{K}_0^{\mathrm{MW}}(k) \to 0$$

we have $\partial[x_0] = \langle 1 \rangle \in \mathbf{K}_0^{\mathrm{MW}}(k)$, so $\mathrm{H}^n(\mathbb{A}^{n+1} \setminus 0; \mathbf{K}_{n+1}^{\mathrm{MW}}) = \mathbf{K}_0^{\mathrm{MW}}(k) \cdot [x_0]$. By the relation $[a^n] = n_\epsilon[a]$ in $\mathbf{K}_1^{\mathrm{MW}}$ we see $[x_0^{n!}] = \frac{n!}{2}h[x_0]$ so $\partial[x_0^{n!}] = \frac{n!}{2}h$. Thus

$$\psi_{n+1}^*[x_0] = \frac{n!}{2}h \cdot [x_0].$$

For an abelian group H and an integer m, we write $H/m := H/mH$ for the quotient abelian group. We say that H is *m-divisible* if $H/m = 0$, i.e. if $mH = H$.

Theorem 4.2.9. *Let k be a perfect field with $\mathrm{char}(k) \neq 2$. Assume that A is a smooth k-algebra of odd Krull dimension $d \geqslant 3$. Let $X = \mathrm{Spec}\ A$, $\xi \in [X, \mathrm{BGL}]_{\mathbb{A}^1}$.*

(1) *We have $d! \cdot \mathrm{H}^d(X; \mathbf{K}_{d+1}^{\mathrm{M}}) \subset \mathrm{im}\Delta(c_{d+1}, \xi)$.*
(2) *There is a surjective homomorphism*

$$\mathrm{H}^d(X; \mathbf{K}_{d+1}^{\mathrm{M}})/d! \twoheadrightarrow \mathrm{coker}\Delta(c_{d+1}, \xi) \cong \mathcal{V}_d(X, \xi).$$

(3) *If $\mathrm{H}^d(X; \mathbf{K}_{d+1}^{\mathrm{M}})$ is $d!$-divisible, then $\mathcal{V}_d(X, \xi) = 0$. So in this case, any rank d vector bundle is cancellative. Moreover, the map*

$$(c_{d+1})_* : \pi_1(\mathrm{RMap}(X, \mathrm{BGL}), \xi) \to \pi_1(\mathrm{RMap}(X, \mathrm{K}(\mathbf{K}_{d+1}^{\mathrm{M}}, d+1)), 0) = \mathrm{H}^d(X; \mathbf{K}_{d+1}^{\mathrm{M}})$$

is surjective for every $\xi \in [X, \mathrm{BGL}]_{\mathbb{A}^1}$.

Proof. (1) By [**8**, Proposition 2.6], there is an exact sequence

$$0 \to \mathbf{I}^{d+2} \cong \eta\mathbf{K}_{d+2}^{\mathrm{MW}} \to \mathbf{K}_{d+1}^{\mathrm{MW}} \to \mathbf{K}_{d+1}^{\mathrm{M}} \to 0,$$

where $\mathbf{I} = \ker(\mathbf{GW} = \mathbf{K}_0^{\mathrm{MW}} \to \mathbf{K}_0^{\mathrm{M}}) = \eta\mathbf{K}_1^{\mathrm{MW}}$ is the fundamental ideal and $\mathbf{I}^{d+2}$ its power. Since X has $\mathbb{A}^1$-cohomological dimension at most $\dim X = d$, we see $\mathrm{H}^{d+1}(X; \mathbf{I}^{d+2}) = 0$, hence the map $\tau : \mathrm{H}^d(X; \mathbf{K}_{d+1}^{\mathrm{MW}}) \to \mathrm{H}^d(X; \mathbf{K}_{d+1}^{\mathrm{M}})$ given by reducing coefficients is a surjection. So we can write any element of $\mathrm{H}^d(X; \mathbf{K}_{d+1}^{\mathrm{M}})$ as $\tau([a, b])$ with $(a, b) \in Q_{2d+1}(A), a = (a_0, a_1, \cdots, a_d) \in \mathrm{Um}_{d+1}(A)$.

$$\begin{array}{ccc}
[X, Q_{2d+1}]_{\mathbb{A}^1} = \mathrm{Um}_{d+1}(A)/\mathrm{E}_{d+1}(A) = \mathrm{H}^d(X; \mathbf{K}_{d+1}^{\mathrm{MW}}) & \xrightarrow{\ \tau\ } & \mathrm{H}^d(X; \mathbf{K}_{d+1}^{\mathrm{M}}) \\
{\scriptstyle (\beta_{d+1})_*}\downarrow \uparrow{\scriptstyle (\mathrm{pr}_1)_*} & \nearrow & \uparrow{\scriptstyle \Delta(c_{d+1}, \xi)} \\
[\beta_{d+1}(a, b)] \in [X, \mathrm{GL}_{d+1}]_{\mathbb{A}^1} & \longrightarrow & [X, \mathrm{GL}]_{\mathbb{A}^1} = \mathrm{K}_1(X) \ni [\beta_{d+1}(a, b)]
\end{array}$$

We will show that the following equality holds:

$$d! \cdot \tau([a, b]) = \pm\Delta(c_{d+1}, \xi)([\beta_{d+1}(a, b)]). \tag{4.6}$$

Indeed, by Proposition 4.2.7 (note that $h = \eta \cdot [-1] + 2$ becomes 2 in $\mathbf{K}_*^{\mathrm{M}}$, yielding that $\tau(\frac{d!}{2}h) = d!$),

$$d! \cdot \tau([a, b]) = \tau(d! \cdot [a, b]) = \tau((\psi_{d+1})_*[a, b]) = \tau((\mathrm{pr}_1)_*[\beta_{d+1}(a, b)]) = \tau([a_0^{d!}, a_1, \cdots, a_d]).$$

While Proposition 4.2.5 tells us that

$$\Delta(c_{d+1}, \xi)([\beta_{d+1}(a, b)]) = (\Omega c_{d+1})([\beta_{d+1}(a, b)]).$$

We are thus reduced to showing that

$$\tau([a_0^{d!}, a_1, \cdots, a_d]) = \tau((\mathrm{pr}_1)_*[\beta_{d+1}(a, b)]) = \pm(\Omega c_{d+1})([\beta_{d+1}(a, b)]). \tag{4.7}$$

We will prove more generally that

$$\tau([(1, 0, \cdots, 0) \cdot \beta]) = \pm(\Omega c_{d+1})([\beta]), \text{for } \beta \in \mathrm{SL}_{d+1}(A) \subset \mathrm{SL}(A). \tag{4.8}$$

For this, note that by Theorem 3.2.9 (10), we have an $\mathbb{A}^1$-homotopy fibre sequence

$$(\mathrm{BGL}_d)^{[d]} \xrightarrow{p_d} \mathrm{BGL}_{d+1} \xrightarrow{e_{d+1}} \mathrm{K}(\pi_d^{\mathbb{A}^1}(\mathbb{A}^{d+1} \setminus 0), d+1) = \mathrm{K}(\mathbf{K}_{d+1}^{\mathrm{MW}}, d+1)$$

(where e_{d+1} is the relevant k-invariant) hence also

$$\mathrm{K}(\mathbf{K}_{d+1}^{\mathrm{MW}}, d) \to (\mathrm{BSL}_d)^{[d]} \xrightarrow{p_d} \mathrm{BSL}_{d+1}.$$

As $(\mathbb{A}^{d+1} \setminus 0)[d] = \mathrm{K}(\mathbf{K}_{d+1}^{\mathrm{MW}}, d)$, by Theorem 3.2.9 (10) and naturality statement of (the dual of) [**45**, Proposition 6.5.3], we get a map of $\mathbb{A}^1$-homotopy fibre sequences

$$\begin{array}{ccccccc}
\mathrm{GL}_{d+1} & \xrightarrow{\mathrm{pr}_1} & \mathbb{A}^{d+1} \setminus 0 & \longrightarrow & \mathrm{BGL}_d & \longrightarrow & \mathrm{BGL}_{d+1} \\
\| & & \downarrow{\scriptstyle \tau'} & & \downarrow & & \| \\
\mathrm{GL}_{d+1} & \xrightarrow{\Omega e_{d+1}} & \mathrm{K}(\mathbf{K}_{d+1}^{\mathrm{MW}}, d) & \longrightarrow & (\mathrm{BGL}_d)^{[d]} & \xrightarrow{p_d} & \mathrm{BGL}_{d+1}.
\end{array}$$

In this diagram, the map $\mathrm{GL}_{d+1} \simeq \mathbf{R}\Omega\mathrm{BGL}_{d+1} \xrightarrow{\mathrm{pr}_1} \mathbb{A}^{d+1} \setminus 0$ is indeed "projection to the first row", since after applying $[U, -]_{\mathbb{A}^1}$, the induced map on homotopy is given by the natural action of $\mathrm{SL}_{d+1}(U)/\mathrm{E}_{d+1}(U)$ on the class of the base point $(1, 0, \cdots, 0) \in (\mathbb{A}^{d+1} \setminus 0)(k)$.

Thus $\Omega e_{d+1} = \tau' \circ \mathrm{pr}_1 : \mathrm{GL}_{d+1} \to \mathrm{K}(\mathbf{K}^{\mathrm{MW}}_{d+1}, d)$ (in $\mathcal{H}^{\mathbb{A}^1}_*(k)$), composing with the maps $X \xrightarrow{\beta} \mathrm{SL}_{d+1}$ and $\mathrm{K}(\mathbf{K}^{\mathrm{MW}}_{d+1}, d) \to \mathrm{K}(\mathbf{K}^{\mathrm{M}}_{d+1}, d)$ we find that, after reducing coefficients, $\Omega e_{d+1}([\beta])$ equals to $\tau([(1, 0, \cdots, 0) \cdot \beta])$. While by [**10**, Theorem 1 and Proposition 5.8], our k-invariant e_{d+1} is the (universal) Euler class (up to a unit in $\mathrm{GW}(k)$), whose reduction coincides with the universal Chern class c_{d+1} (up to sign, as a unit in $\mathrm{GW}(k) = \mathbf{K}^{\mathrm{MW}}_0(k)$ is mapped to a unit in $\mathbf{K}^{\mathrm{M}}_0(k) \cong \mathbb{Z}$), establishing our eq. (4.8).

Statements (2) and (3) then follow easily, with the help of the last part of Theorem 1.6.8. □

Remark 4.2.10. If k is algebraically closed ($k = \bar{k}$), then using the Rost-Schmid complex (see [**22, 29**] for some nice expositions on related notions and results), one easily finds that $\mathrm{H}^d(X; \mathbf{K}^{\mathrm{M}}_{d+1})$ is $d!$-divisible, as it is a quotient of the direct sum of groups of the form $\mathbf{K}^{\mathrm{M}}_1(\kappa_x) = \kappa_x^\times = k^\times$ which is $d!$-divisible (x ranges over closed points of X). Of course, for the same reason, we have the more refined result that $\mathrm{H}^d(X; \mathbf{K}^{\mathrm{M}}_{d+1})$ is $d!$-divisible if $\kappa_x^\times$ is $d!$-divisible (i.e. $\kappa_x = (\kappa_x)^{d!}$) for all closed points $x \in X$.

More generally, $\mathrm{H}^d(X; \mathbf{K}^{\mathrm{M}}_{d+1})$ is $d!$-divisible if k has cohomological dimension at most 1 (see for instance [**28**, Theorem 2.2]), as a consequence of Voevodsky's confirmation of the motivic Bloch-Kato conjecture (or *norm residue isomorphism theorem*), a highly non-trivial result.

Then we treat the case when $n = d$ is even. To still get a principal $\mathbb{A}^1$-homotopy fiber sequence, we need to restrict ourselves to the case of oriented vector bundles—namely those classified by homotopy classes of maps to BSL or BSL_d (since $\pi_1^{\mathbb{A}^1}\mathrm{BSL}_d = 0$, as opposed to the fact that $\pi_1^{\mathbb{A}^1}\mathrm{BGL}_d = \mathbb{G}_m$).

Note first that our discussions up to here in this chapter are still valid if we replace GL with SL everywhere, essentially because we also have the $\mathbb{A}^1$-homotopy fiber sequence

$$\mathbb{A}^{n+1} \setminus 0 \to \mathrm{BSL}_n \to \mathrm{BSL}_{n+1}.$$

The only difference is that the obstruction class (namely the k-invariant θ) is different from the case when $n = d$ is odd.

We now identify the k-invariant θ. By the functoriality of the Moore-Postnikov tower, applied to the square

$$\begin{array}{ccc} \mathrm{BSL}_d & \longrightarrow & \mathrm{BSL}_{d+1} \\ \| & & \downarrow \\ \mathrm{BSL}_d & \longrightarrow & \mathrm{BSL} \end{array}$$

(factoring the rows up to the first non-trivial stage E', E) we have the following map of $\mathbb{A}^1$-homotopy fibre sequences (when deleting the last column)

$$\begin{array}{ccccccccc} \mathrm{K}(\mathbf{K}^{\mathrm{MW}}_{d+1}, d) & \longrightarrow & E' & \longrightarrow & \mathrm{BSL}_{d+1} & \xrightarrow{e_{d+1}} & \mathrm{K}(\mathbf{K}^{\mathrm{MW}}_{d+1}, d+1) & \xrightarrow{\tau} & \mathrm{K}(\mathbf{K}^{\mathrm{M}}_{d+1}, d+1) \\ \| & & \downarrow & & \downarrow{\scriptstyle s_{d+1}} & & \| & & \| \\ \mathrm{K}(\mathbf{K}^{\mathrm{MW}}_{d+1}, d) & \longrightarrow & E & \longrightarrow & \mathrm{BSL} & \xrightarrow{\theta} & \mathrm{K}(\mathbf{K}^{\mathrm{MW}}_{d+1}, d+1) & \xrightarrow{\tau} & \mathrm{K}(\mathbf{K}^{\mathrm{M}}_{d+1}, d+1). \end{array}$$

So $\tau e_{d+1} = c_{d+1} = \tau\theta s_{d+1}$ (as Chern classes stabilize, we can write $c_{d+1} = \tau\theta$).

Note that as $\dim X = d$, we have $\mathrm{SK}_1(X) = [X, \Omega\mathrm{BSL}]_{\mathbb{A}^1} = [X, \mathrm{SL}]_{\mathbb{A}^1} \xrightarrow{(s_{d+1})_*} [X, \mathrm{SL}_{d+1}]_{\mathbb{A}^1}$. We thus have a commutative diagram (since τ is induced by the homomorphism $\mathbf{K}^{\mathrm{MW}}_{d+1} \to \mathbf{K}^{\mathrm{M}}_{d+1}$, we can move τ out; cf. [**47**, (1.1)])

$$\begin{array}{ccc} [X, \mathrm{SL}]_{\mathbb{A}^1} & \xrightarrow{\Delta(\theta, \xi)} & \mathrm{H}^d(X; \mathbf{K}^{\mathrm{MW}}_{d+1}) \\ \| & & \downarrow{\scriptstyle \tau} \\ [X, \mathrm{SL}]_{\mathbb{A}^1} & \xrightarrow{\Delta(c_{d+1}, \xi)} & \mathrm{H}^d(X; \mathbf{K}^{\mathrm{M}}_{d+1}). \end{array}$$

Again by the exact sequence

$$0 \to \mathbf{I}^{d+2} \to \mathbf{K}^{\mathrm{MW}}_{d+1} \to \mathbf{K}^{\mathrm{M}}_{d+1} \to 0,$$

we see that the right vertical map $\tau : \mathrm{H}^d(X; \mathbf{K}^{\mathrm{MW}}_{d+1}) \to \mathrm{H}^d(X; \mathbf{K}^{\mathrm{M}}_{d+1})$ is surjective (since X has $\mathbb{A}^1$-cohomological dimension at most d).

If we further assume that the 2-cohomological dimension of our base field k (perfect and $\mathrm{char}(k) \neq 2$) is at most 2: $\mathrm{c.d.}_2(k) \leqslant 2$, then by (the proof of) [**28**, Theorem 2.1] (using the Gersten-Witt complex of $X = \mathrm{Spec}\ A$ and assuming $d \geqslant 3$), we have $\mathrm{H}^d(X; \mathbf{I}^{d+2}) = 0$. So in this case, the right vertical map $\tau : \mathrm{H}^d(X; \mathbf{K}^{\mathrm{MW}}_{d+1}) \to \mathrm{H}^d(X; \mathbf{K}^{\mathrm{M}}_{d+1})$ is an isomorphism.

Finally, we are able to give results similar to those in the odd dimension case (whose proof is almost verbatim the same after suitably changing notations).

Similarly as before, we have the induced map on $\mathbb{A}^1$-homotopy classes

$$\varphi_* : [X, \mathrm{BSL}_n]_{\mathbb{A}^1} \to [X, \mathrm{BSL}]_{\mathbb{A}^1}$$

and want to enumerate its fibers. We also use $\mathcal{V}^\circ_n(X)$ to denote the set of isomorphism classes of rank n *oriented* vector bundle over X.

Theorem 4.2.11. *Assume that the base field k is perfect and* $\mathrm{char}(k) \neq 2$ *with* $\mathrm{c.d.}_2(k) \leqslant 2$. *Let $n = d \geqslant 4$ be even, let ξ be a stable oriented vector bundle over X, whose classifying map is still denoted $\xi : X \to \mathrm{BSL}$. Then there is a bijection*

$$\varphi_*^{-1}(\xi) \cong \mathrm{coker}\big(\mathrm{SK}_1(X) = [X, \Omega\mathrm{BSL}]_{\mathbb{A}^1} \xrightarrow{\Delta(c_{d+1},\xi)} [X, \mathrm{K}(\mathbf{K}^{\mathrm{M}}_{d+1}, d)]_{\mathbb{A}^1} = \mathrm{H}^d(X; \mathbf{K}^{\mathrm{M}}_{d+1})\big).$$

Proposition 4.2.12. *Assume that the condition of Theorem 4.2.11 is satisfied.*

(1) *Let $\xi, \xi' \in [X, \mathrm{BSL}]_{\mathbb{A}^1}$. If $c_j(\xi) = c_j(\xi'), 1 \leqslant j \leqslant d$, then ξ and ξ' has the same number of representatives in $\mathcal{V}^\circ_d(X)$.*
(2) *Let $\xi, \xi' \in \mathcal{V}^\circ_d(X)$, sharing the same total Chern class. If ξ is cancellative, then so is ξ'.*

Theorem 4.2.13. *Assume that the base field k is perfect and* $\mathrm{char}(k) \neq 2$ *with* $\mathrm{c.d.}_2(k) \leqslant 2$. *Assume that A is a smooth k-algebra of even Krull dimension $d \geqslant 4$. Let $X = \mathrm{Spec}\ A$, $\xi \in [X, \mathrm{BSL}]_{\mathbb{A}^1}$.*

(1) *We have $d! \cdot \mathrm{H}^d(X; \mathbf{K}^{\mathrm{M}}_{d+1}) \subset \mathrm{im}\Delta(c_{d+1}, \xi)$.*
(2) *There is a surjective homomorphism*

$$\mathrm{H}^d(X; \mathbf{K}^{\mathrm{M}}_{d+1})/d! \twoheadrightarrow \mathrm{coker}\Delta(c_{d+1}, \xi) \cong \mathcal{V}_d(X, \xi).$$

(3) *If $\mathrm{H}^d(X; \mathbf{K}^{\mathrm{M}}_{d+1})$ is $d!$-divisible, then $\mathrm{coker}\Delta(c_{d+1}, \xi) = 0$. So in this case, any rank d oriented vector bundle is cancellative. Moreover, the map*

$$(c_{d+1})_* : \pi_1(\mathrm{RMap}(X, \mathrm{BSL}), \xi) \to \pi_1(\mathrm{RMap}(X, \mathrm{K}(\mathbf{K}^{\mathrm{MW}}_{d+1}, d+1)), 0) = \mathrm{H}^d(X; \mathbf{K}^{\mathrm{MW}}_{d+1}) \xrightarrow{\tau} \mathrm{H}^d(X; \mathbf{K}^{\mathrm{M}}_{d+1})$$

is surjective for every $\xi \in [X, \mathrm{BSL}]_{\mathbb{A}^1}$.

Remark 4.2.14. For the even rank case, our assumption on the 2-cohomological dimension of the base field k cannot be omitted in order to get $\mathrm{H}^d(X; \mathbf{I}^{d+2}) = 0$ so that $\tau : \mathrm{H}^d(X; \mathbf{K}^{\mathrm{MW}}_{d+1}) \to \mathrm{H}^d(X; \mathbf{K}^{\mathrm{M}}_{d+1})$ is an isomorphism: if we take $A = \mathbb{R}[x, y, z]/(x^2+y^2+z^2-1)$, then $\mathrm{H}^d(\mathrm{Spec}\ A; \mathbf{I}^{d+2}) \neq 0$ (note that $\mathrm{c.d.}_2(\mathbb{R}) = \infty$).

On the other hand, quite a lot of fields satisfy our assumption, e.g. any finite field (with odd characteristic), any algebraically closed field, or any field of the form $L(t)$ or $L(t_1, t_2)$ for an algebraically closed field L with $\mathrm{char}(L) = 0$.

4.3. Enumeration results on vector bundles below critical rank

As in the last section, we take advantage of the fact that $\mathrm{RMap}(X, \mathrm{BSL})$ is an H-space for every scheme X to deduce some enumeration results on vector bundles below critical rank. The case for bundles below critical rank is more complicated than that in the last section, as the relevant spaces are less connected, we need a two-stage Moore-Postnikov factorization to analyze the lifting problem on each stage. In particular, if the base field k is algebraically closed, an enumeration formula of the lifting set for the first stage is given. Then after a careful discussion of the second stage, we prove cancellation of vector bundles of rank $d-1$ (admitting Asok-Fasel conjecture 4.3.6).

Assume that the base field k is perfect and that $\mathrm{char}(k) \neq 2$, A is a smooth affine k-algebra of Krull dimension $d \geqslant 3$, and $X = \mathrm{Spec}(A)$. Let ξ be a stable oriented vector bundle over X, whose classifying map is still denoted $\xi : X \to \mathrm{BSL}$. We will now investigate the isomorphism classes of rank $n = d-1$ (oriented) vector bundles that are stably equivalent to the given ξ (if they exist).

If $d \geqslant 4$ is even, we have the $\mathbb{A}^1$-homotopy fiber sequence

$$F_{d-1} \to \mathrm{BSL}_{d-1} \xrightarrow{\varphi = \varphi_{d-1}} \mathrm{BSL}.$$

We consider the two-stage Moore-Postnikov factorization (Theorem 3.2.9) of the map $\varphi : \mathrm{BSL}_{d-1} \to \mathrm{BSL}$,

$$\begin{array}{ccccc}
F = F_{d-1} & \longrightarrow & \mathrm{BSL}_{d-1} & & \\
 & & \downarrow{\scriptstyle q'} & & \\
\mathrm{K}(\pi^{\mathbb{A}^1}_d F_{d-1}, d) & \longrightarrow & E' & & \\
 & & \downarrow{\scriptstyle q} & & \\
\mathrm{K}(\mathbf{K}^{\mathrm{M}}_d, d-1) & \longrightarrow & E & \xrightarrow{\theta'} & \mathrm{K}(\pi^{\mathbb{A}^1}_d F_{d-1}, d+1) \\
 & \nearrow{\scriptstyle \xi_E} & \downarrow{\scriptstyle p} & & \\
X & \xrightarrow{\xi} & \mathrm{BSL} & \xrightarrow{\theta} & \mathrm{K}(\mathbf{K}^{\mathrm{M}}_d, d).
\end{array} \tag{4.9}$$

By the properties listed in Theorem 3.2.9, it's easy to see that the map $q' : \mathrm{BSL}_{d-1} \to E'$ induces a bijection

$$q'_* : [X, \mathrm{BSL}_{d-1}]_{\mathbb{A}^1} \to [X, E']_{\mathbb{A}^1}.$$

We thus need only to find reasonable conditions, under which the maps

$$p_* : [X, E]_{\mathbb{A}^1} \to [X, \mathrm{BSL}]_{\mathbb{A}^1}$$

and

$$q_* : [X, E']_{\mathbb{A}^1} \to [X, E]_{\mathbb{A}^1}$$

are injective.

As before, we have the following commutative diagram

$$\begin{array}{ccc} \mathrm{BSL}_d & \xrightarrow{e_d} & \mathrm{K}(\mathbf{K}^{\mathrm{MW}}_d, d) \\ {\scriptstyle s_d}\downarrow & & \downarrow{\scriptstyle \tau} \\ \mathrm{BSL} & \xrightarrow{\theta} & \mathrm{K}(\mathbf{K}^{\mathrm{M}}_d, d). \end{array}$$

Thus $\theta s_d = \tau e_d = c_d$ by [**10**, Example 5.2 and Proposition 5.8] as before.

On the other hand, by Serre's splitting theorem, we can write $\xi = (s_d)_*([\xi_d])$ for some $[\xi_d] \in [X, \mathrm{BSL}_d]_{\mathbb{A}^1}$. So $\theta_*([\xi]) = (\theta s_d)_*[\xi_d] = (\tau e_d)_*[\xi_d] = c_d(\xi_d) = c_d(\xi)$ (as Chern classes stabilizes) and we can write $\theta = c_d$.

Note that whenever ξ is represented by a rank $d-1$ vector bundle, we will have $c_d(\xi) = 0$. So reasoning as in Theorem 4.2.1 and using Theorem 4.2.3, we obtain one step to our main enumeration result in the case $n = d-1$—the following description of $p_*^{-1}([\xi])$.

Proposition 4.3.1. *Let k be a perfect field with $\mathrm{char}(k) \neq 2$, A a smooth affine k-algebra of even Krull dimension $d \geqslant 4$, and $X = \mathrm{Spec}(A)$, let ξ be a stable oriented vector bundle over X, whose classifying map is still denoted $\xi : X \to \mathrm{BSL}$. If ξ is represented by a rank $d-1$ vector bundle, there is a bijection*

$$p_*^{-1}(\xi) \cong \mathrm{coker}\big(\mathrm{SK}_1(X) = [X, \Omega\mathrm{BSL}]_{\mathbb{A}^1} \xrightarrow{\Delta(c_d,\xi)} [X, \mathrm{K}(\mathbf{K}^{\mathrm{M}}_d, d-1)]_{\mathbb{A}^1} = \mathrm{H}^{d-1}(X; \mathbf{K}^{\mathrm{M}}_d)\big).$$

The homomorphism $\Delta(c_d, \xi)$ is given as follows: for $\beta \in \mathrm{SK}_1(X) = [X, \mathrm{SL}]_{\mathbb{A}^1}$,

$$\Delta(c_d, \xi)\beta = (\Omega c_d)(\beta) + \sum_{r=1}^{d-1} ((\Omega c_r)(\beta)) \cdot c_{d-r}(\xi).$$

Below, for a sheaf of abelian groups $\mathbf{K}$ and an integer m, we denote by $\mathbf{K}/m := \mathbf{K}/m\mathbf{K}$ for the mod-m quotient sheaf and ${}_m\mathbf{K} := \ker(\mathbf{K} \xrightarrow{m} m\mathbf{K})$ for the subsheaf killed by m. Since the contraction functor $(-)_{-1} : \mathcal{A}\mathsf{b}_k^{\mathbb{A}^1} \to \mathcal{A}\mathsf{b}_k^{\mathbb{A}^1}$ is exact, the constructions of the mod-m quotient and the subsheaves killed by m are preserved by (iterated) contractions. We also write μ_m for the étale sheaf of m-th roots of 1, and $\mu_m^{\otimes n}$ for its n-th tensor power. Denote $\bar{\mathbf{I}}^j := \mathbf{I}^j/\mathbf{I}^{j+1} \cong \mathbf{K}^{\mathrm{M}}_j/2$. The proof of the following result is adapted from that of [**30**, Proposition 6.1].

Proposition 4.3.2. *Assume that the base field k is algebraically closed, $X = \mathrm{Spec}(A)$ is a connected smooth affine k-scheme of dimension d. Then the group $\mathrm{H}^{d-1}(X; \mathbf{K}^{\mathrm{M}}_d)$ is divisible prime to $\mathrm{char}(k)$:*

$$\mathrm{H}^{d-1}(X; \mathbf{K}^{\mathrm{M}}_d)/m = 0$$

for any $m \in \mathbb{Z}_{>0}$ with $\mathrm{char}(k) \nmid m$. In particular, if $\mathrm{char}(k) = 0$ or $\mathrm{char}(k) \geqslant d$, then $\mathrm{H}^{d-1}(X; \mathbf{K}^{\mathrm{M}}_d)$ is $(d-1)!$-divisible: $\mathrm{H}^{d-1}(X; \mathbf{K}^{\mathrm{M}}_d)/(d-1)! = 0$.

Proof. Writing $m = \ell_1^{r_1} \cdots \ell_s^{r_s}$, with each ℓ_i prime, $\ell_i \neq \mathrm{char}(k)$ and $r_i \in \mathbb{N}$, we see that it suffices to consider the case $m = \ell^r$. Let ℓ be a prime number and $\ell \neq \mathrm{char}(k)$, let $r \in \mathbb{N}$. Consider the short exact sequences

$$0 \to {}_{\ell^r}\mathbf{K}^{\mathrm{M}}_d \to \mathbf{K}^{\mathrm{M}}_d \xrightarrow{\ell^r} \ell^r\mathbf{K}^{\mathrm{M}}_d \to 0$$

and

$$0 \to \ell^r\mathbf{K}^{\mathrm{M}}_d \to \mathbf{K}^{\mathrm{M}}_d \to \mathbf{K}^{\mathrm{M}}_d/\ell^r \to 0.$$

Using the Rost-Schmid complex we see that $\mathrm{H}^d(X; {}_{\ell^r}\mathbf{K}^{\mathrm{M}}_d)$ is a quotient of a direct sum of groups of the form ${}_{\ell^r}\mathbf{K}^{\mathrm{M}}_0(\kappa_x) \cong {}_{\ell^r}\mathbb{Z} = 0$ (over all closed points $x \in X^{(d)}$) as $\mathbb{Z}$ is torsion-free, thus $\mathrm{H}^d(X; {}_{\ell^r}\mathbf{K}^{\mathrm{M}}_d) = 0$. We then have exact sequences

$$\mathrm{H}^{d-1}(X; \mathbf{K}^{\mathrm{M}}_d) \to \mathrm{H}^{d-1}(X; \ell^r\mathbf{K}^{\mathrm{M}}_d) \to \mathrm{H}^d(X; {}_{\ell^r}\mathbf{K}^{\mathrm{M}}_d) = 0$$

and

$$\mathrm{H}^{d-1}(X; \ell^r\mathbf{K}^{\mathrm{M}}_d) \to \mathrm{H}^{d-1}(X; \mathbf{K}^{\mathrm{M}}_d) \to \mathrm{H}^{d-1}(X; \mathbf{K}^{\mathrm{M}}_d/\ell^r).$$

Splicing together we get an exact sequence

$$\mathrm{H}^{d-1}(X; \mathbf{K}^{\mathrm{M}}_d) \xrightarrow{\ell^r} \mathrm{H}^{d-1}(X; \mathbf{K}^{\mathrm{M}}_d) \to \mathrm{H}^{d-1}(X; \mathbf{K}^{\mathrm{M}}_d/\ell^r)$$

and hence

$$0 \to \mathrm{H}^{d-1}(X; \mathbf{K}^{\mathrm{M}}_d)/\ell^r \to \mathrm{H}^{d-1}(X; \mathbf{K}^{\mathrm{M}}_d/\ell^r).$$

Therefore to prove $\mathrm{H}^{d-1}(X; \mathbf{K}^{\mathrm{M}}_d)/\ell^r = 0$, it suffices to prove that $\mathrm{H}^{d-1}(X; \mathbf{K}^{\mathrm{M}}_d/\ell^r) = 0$.

For $j, n \in \mathbb{N}$, let $\mathscr{H}^j(n) = (\mathbf{R}^j i_*)\mu_{\ell^r}^{\otimes n}$ be the Zariski sheaf associated to the presheaf $U \mapsto \mathrm{H}^j_{\text{ét}}(U; \mu_{\ell^r}^{\otimes n})$ (where i is the inclusion of the Zariski site into the étale site). We have the biregular *Bloch-Ogus spectral sequence* (see [**21**] or eq. (B.1); it's one incarnation of the *Leray spectral sequence*)

$$\mathrm{E}_2^{ij} = \mathrm{H}^i_{\mathrm{Zar}}(X; \mathscr{H}^j(n)) \Longrightarrow \mathrm{H}^{i+j}_{\text{ét}}(X; \mu_{\ell^r}^{\otimes n}).$$

The term $\mathrm{E}_2^{ij} = \mathrm{H}^i_{\mathrm{Zar}}(X; \mathscr{H}^j(n))$ can be computed as the i-th cohomology of the *Gersten complex*

$$\mathrm{H}^j_{\text{ét}}(\kappa_\eta; \mu_{\ell^r}^{\otimes n}) \to \cdots \to \bigoplus_{x \in X^{(i)}} \mathrm{H}^{j-i}_{\text{ét}}(\kappa_x; \mu_{\ell^r}^{\otimes n-i}) \to \cdots,$$

where $\eta \in X^{(0)}$ is the generic point of X.

Since $k = \bar{k}$, the cohomological dimension $\mathrm{c.d.}(\kappa_x) \leqslant d - \dim(\mathscr{O}_{X,x})$ ([**83**, §4.2, Proposition 11]), and we get $\mathrm{H}^{j-i}_{\text{ét}}(\kappa_x; \mu_{\ell^r}^{\otimes n-i}) = 0$ for $x \in X^{(i)}$ if $j - i > d - i$. Thus $\mathrm{E}_2^{ij} = \mathrm{H}^i_{\mathrm{Zar}}(X; \mathscr{H}^j(n)) = 0$ if $i > d = \dim\ X$ or $j > d$ or $i > j$. Hence in the filtration of the converging term $\mathrm{H}^{2d-1}_{\text{ét}}(X; \mu_{\ell^r}^{\otimes n})$, the only (possibly) non-trivial term is $\mathrm{E}_2^{d-1,d} =$

$\mathrm{H}^{d-1}_{\mathrm{Zar}}(X;\mathscr{H}^d(n))$. While by [**67**, Chapter VI, Theorem 7.2], $\mathrm{H}^{2d-1}_{\text{ét}}(X;\mu_{\ell^r}^{\otimes n})=0$ since X is affine over $k=\bar{k}$. Thus $\mathrm{E}_2^{d-1,d}=\mathrm{H}^{d-1}_{\mathrm{Zar}}(X;\mathscr{H}^d(n))=0$ as well.

There is a commutative diagram ([**20**, Theorem 2.3])

$$\begin{array}{ccccc}
\bigoplus_{x\in X^{(d-2)}}\mathrm{K}^{\mathrm{M}}_2(\kappa_x)/\ell^r & \longrightarrow & \bigoplus_{x\in X^{(d-1)}}\mathrm{K}^{\mathrm{M}}_1(\kappa_x)/\ell^r & \longrightarrow & \bigoplus_{x\in X^{(d)}}\mathrm{K}^{\mathrm{M}}_0(\kappa_x)/\ell^r \\
\downarrow & & \downarrow & & \downarrow \\
\bigoplus_{x\in X^{(d-2)}}\mathrm{H}^2_{\text{ét}}(\kappa_x;\mu_{\ell^r}^{\otimes 2}) & \longrightarrow & \bigoplus_{x\in X^{(d-1)}}\mathrm{H}^1_{\text{ét}}(\kappa_x;\mu_{\ell^r}) & \longrightarrow & \bigoplus_{x\in X^{(d)}}\mathrm{H}^0_{\text{ét}}(\kappa_x;\mathbb{Z}/\ell^r),
\end{array}$$

where the vertical maps are isomorphisms by [**66**] (or Voevodsky's confirmation of the motivic Bloch-Kato conjecture). The homology of the middle terms in the two rows compute $\mathrm{H}^{d-1}(X;\mathbf{K}^{\mathrm{M}}_d/\ell^r)$ and respectively $\mathrm{H}^{d-1}_{\mathrm{Zar}}(X;\mathscr{H}^d(d))(=0)$. Thus

$$\mathrm{H}^{d-1}(X;\mathbf{K}^{\mathrm{M}}_d/\ell^r)=\mathrm{H}^{d-1}_{\mathrm{Zar}}(X;\mathscr{H}^d(d))=0.$$

We are done. □

Let's now treat the case when the dimension of X is odd, still assuming $k=\bar{k}$. By the exact sequence

$$0\to\mathbf{I}^{d+1}\to\mathbf{K}^{\mathrm{MW}}_d\xrightarrow{\tau}\mathbf{K}^{\mathrm{M}}_d\to 0$$

we get an exact sequence

$$\mathrm{H}^{d-1}(X;\mathbf{K}^{\mathrm{MW}}_d)\xrightarrow{\tau}\mathrm{H}^{d-1}(X;\mathbf{K}^{\mathrm{M}}_d)\to\mathrm{H}^d(X;\mathbf{I}^{d+1}).$$

the Rost-Schmid complex for $\mathbf{I}^{d+1}$ says that $\mathrm{H}^d(X;\mathbf{I}^{d+1})$ is a subquotient of $\bigoplus_{x\in X^{(d)}}\mathbf{I}(\kappa_x)=\bigoplus_{x\in X^{(d)}}\mathbf{I}(\bar{k})=0$, thus $\mathrm{H}^d(X;\mathbf{I}^{d+1})=0$ and τ is surjective as well. In fact, more is true: by Voevodsky's confirmation of the Milnor conjecture, we have an isomorphism of sheaves of abelian groups $\bar{\mathbf{I}}^{d+j}\cong\mathscr{H}^{d+j}(d+j)$ $(j\geqslant 1)$, where $\mathscr{H}^{d+j}(d+j)$ is the Zariski sheaf associated to the presheaf $U\mapsto\mathrm{H}^{d+j}_{\text{ét}}(U;\mu_2^{\otimes d+j})$; for reason of cohomological dimension, $\mathscr{H}^{d+j}(d+j)|_X=0$ $(j\geqslant 1)$ (restricting to the Zariski site of X). Thus we have $\bar{\mathbf{I}}^j|_X=0, j>d, \mathbf{I}^{d+1}|_X=\mathbf{I}^{d+2}|_X=\cdots$.

The Arason-Pfister Hauptsatz gives $\bigcap_{j\geqslant 1}\mathbf{I}^{d+j}=0$, thus $\mathbf{I}^{d+1}|_X=0$ and so $\tau:\mathbf{K}^{\mathrm{MW}}_j|_X\to\mathbf{K}^{\mathrm{M}}_j|_X$ is in fact an isomorphism for every $j\geqslant d$. This suffices to conclude that the induced maps on cohomologies $\tau:\mathrm{H}^i(X;\mathbf{K}^{\mathrm{MW}}_j)\to\mathrm{H}^i(X;\mathbf{K}^{\mathrm{M}}_j)$ for $j\geqslant d=\dim X$ are isomorphisms, since these sheaves are strictly $\mathbb{A}^1$-invariant, Nisnevich and Zariski cohomologies of X coincide (and are computed by Rost-Schmid complexes).

The exact sequence

$$0\to 2\mathbf{K}^{\mathrm{M}}_j\to\mathbf{K}^{\mathrm{M}}_j\to\bar{\mathbf{I}}^j\to 0$$

gives isomorphisms

$$\mathrm{H}^i(X;2\mathbf{K}^{\mathrm{M}}_j)\xrightarrow{\cong}\mathrm{H}^i(X;\mathbf{K}^{\mathrm{M}}_j), j>d.$$

We summarize the results as follows:

$$\begin{cases}\mathrm{H}^i(X;\mathbf{I}^j)=0, \mathrm{H}^i(X;2\mathbf{K}^{\mathrm{M}}_j)\cong\mathrm{H}^i(X;\mathbf{K}^{\mathrm{M}}_j), j>d;\\ \tau:\mathrm{H}^i(X;\mathbf{K}^{\mathrm{MW}}_j)\xrightarrow{\cong}\mathrm{H}^i(X;\mathbf{K}^{\mathrm{M}}_j), j\geqslant d.\end{cases}\tag{4.10}$$

Proposition 4.3.3. *Let k be an algebraically closed field with $\mathrm{char}(k)\neq 2$, be a smooth affine k-algebra of Krull dimension $d\geqslant 3$ and $X=\mathrm{Spec}(A)$. Let further ξ be a stable oriented vector bundle over X, whose classifying map is still denoted $\xi:X\to\mathrm{BSL}$. If ξ is represented by a rank $d-1$ vector bundle, then there is a bijection*

$$p_*^{-1}(\xi)\cong\mathrm{coker}\big(\mathrm{SK}_1(X)=[X,\Omega\mathrm{BSL}]_{\mathbb{A}^1}\xrightarrow{\Delta(c_d,\xi)}[X,\mathrm{K}(\mathbf{K}^{\mathrm{M}}_d,d-1)]_{\mathbb{A}^1}=\mathrm{H}^{d-1}(X;\mathbf{K}^{\mathrm{M}}_d)\big).$$

The homomorphism $\Delta(c_d,\xi)$ is given as follows: for $\beta\in\mathrm{SK}_1(X)=[X,\mathrm{SL}]_{\mathbb{A}^1}$,

$$\Delta(c_d,\xi)\beta=(\Omega c_d)(\beta)+\sum_{r=1}^{d-1}((\Omega c_r)(\beta))\cdot c_{d-r}(\xi).$$

So $\Delta(c_d,\xi)([\beta_{d+1}(a,b)])=0$ for all A-points (a,b) of Q_{2d+1}.

Proof. We already treated in Proposition 4.3.1 the case when d is even. For d odd, since $\pi^{\mathbb{A}^1}_{d-1}F_{d-1}\cong\mathbf{K}^{\mathrm{MW}}_d$, we have a similar two-stage Moore-Postnikov factorization as in eq. (4.9), with $\mathbf{K}^{\mathrm{M}}_d$ replaced by $\mathbf{K}^{\mathrm{MW}}_d$ there.

There are the following commutative diagrams:

$$\begin{array}{ccc}\mathrm{BSL}_d & \xrightarrow{e_d} & \mathrm{K}(\mathbf{K}^{\mathrm{MW}}_d,d)\\ {\scriptstyle s_d}\downarrow & \overset{\theta}{\nearrow} & \downarrow{\scriptstyle\tau}\\ \mathrm{BSL} & \xrightarrow{c_d} & \mathrm{K}(\mathbf{K}^{\mathrm{M}}_d,d),\end{array}\qquad\text{and hence}\qquad\begin{array}{ccc}\mathrm{SK}_1(X) & \xrightarrow{\Delta(\theta,\xi)} & \mathrm{H}^{d-1}(X;\mathbf{K}^{\mathrm{MW}}_d)\\ & \searrow^{\Delta(c_d,\xi)} & \downarrow{\scriptstyle\tau}\\ & & \mathrm{H}^{d-1}(X;\mathbf{K}^{\mathrm{M}}_d).\end{array}$$

Since $\tau:\mathbf{K}^{\mathrm{MW}}_d|_X\to\mathbf{K}^{\mathrm{M}}_d|_X$ is an isomorphism, so is the right vertical map. Thus $\Delta(\theta,\xi)$ and $\Delta(c_d,\xi)$ are essentially the same. Therefore the result for the d odd case holds as with the case when d is even in Proposition 4.3.1.

The last statement follows from Proposition 4.2.5. □

Remark 4.3.4. Since $c_d(\xi)=\tau\theta(\xi)$, we see that ξ lifts to a class in $[X,E]_{\mathbb{A}^1}$ iff $c_d(\xi)=0$. While θ'_* maps $[X,E]_{\mathbb{A}^1}$ to 0, hence no further obstruction. We thus get Murthy's splitting result [**71**] for oriented rank d vector bundles: Let X be a smooth affine variety of dimension d over an algebraically closed field k, then an oriented rank d vector bundle ξ over X splits off a trivial line bundle iff $c_d(\xi)=0$.

Theorem 4.3.5. *Assume $k = \bar{k}$ and $\mathrm{char}(k) \neq 2$. Let A be a smooth k-algebra of Krull dimension $d \geqslant 3$. Let $X = \mathrm{Spec}\ A$, $\xi \in [X, \mathrm{BSL}]_{\mathbb{A}^1}$ which is represented by a rank $d-1$ vector bundle.*

(1) *We have $(d-1)! \cdot \mathrm{H}^{d-1}(X; \mathbf{K}^{\mathrm{M}}_d) \subset \mathrm{im}\Delta(c_d, \xi)$.*
(2) *There is a surjective homomorphism*

$$\mathrm{H}^{d-1}(X; \mathbf{K}^{\mathrm{M}}_d)/(d-1)! \twoheadrightarrow \mathrm{coker}\Delta(c_d, \xi).$$

(3) *If $\mathrm{char}(k) = 0$ or $\mathrm{char}(k) \geqslant d$, then the lifting set $p_*^{-1}(\xi) \subset [X, E]_{\mathbb{A}^1}$ is a singleton. So the map*

$$\theta_* : \pi_1(\mathrm{RMap}(X, \mathrm{BSL}), \xi) \to \pi_1(\mathrm{RMap}(X, \mathrm{K}(\mathbf{K}^{\mathrm{M}}_d, d)), 0) = \mathrm{H}^{d-1}(X; \mathbf{K}^{\mathrm{M}}_d)$$

is surjective.

Proof. This is along the same line as the proof of Theorem 4.2.9. We only briefly write down some points. Using the Postnikov tower of $\mathbb{A}^d \setminus 0$, it's easy to see that we have a surjective map $[X, \mathbb{A}^d \setminus 0]_{\mathbb{A}^1} \twoheadrightarrow \mathrm{H}^{d-1}(X; \mathbf{K}^{\mathrm{MW}}_d)$. Thus the composite

$$\mathrm{Um}_d(A)/\mathrm{E}_d(A) = [X, Q_{2d-1}]_{\mathbb{A}^1} = [X, \mathbb{A}^d \setminus 0]_{\mathbb{A}^1} \twoheadrightarrow \mathrm{H}^{d-1}(X; \mathbf{K}^{\mathrm{MW}}_d) \xrightarrow{\tau} \mathrm{H}^{d-1}(X; \mathbf{K}^{\mathrm{M}}_d)$$

is surjective. So every element in $\mathrm{H}^{d-1}(X; \mathbf{K}^{\mathrm{M}}_d)$ is the image of some $[a, b]$ with $(a, b) \in Q_{2d-1}(A), a = (a_1, \cdots, a_d) \in \mathrm{Um}_d(A)$ which we write as $\tau([a, b])$.

$$\begin{array}{ccccc} & & & & \mathrm{SL}_d \\ & & & \overset{\beta_d}{\nearrow} & \big\uparrow \\ X & \xrightarrow{(a,b)} & Q_{2d-1} & \xrightarrow{\alpha_d} & \mathrm{SL}_{2^{d-1}} \end{array}$$

We will show

$$(d-1)! \cdot \tau([a, b]) = \pm\Delta(c_d, \xi)([\beta_d(a, b)]).$$

As in the proof of Theorem 4.2.9 (from eq. (4.7) to the end, where X essentially plays no role), we have

$$\tau((\mathrm{pr}_1)_*[\beta]) = \tau([(1, 0, \cdots, 0) \cdot \beta]) = \pm(\Omega c_d)([\beta]), \text{for } \beta \in \mathrm{SL}_d(A) \subset \mathrm{SL}(A)$$

and

$$(\Omega c_r)(\alpha_d) = \begin{cases} 0, & r \neq d; \\ \pm(d-1)! \in \mathbf{K}^{\mathrm{M}}_0(k) = \mathbb{Z}, & r = d. \end{cases}$$

We get

$$\Delta(c_d, \xi)([\beta_d(a, b)]) = (\Omega c_d)([\beta_d(a, b)]).$$

Thus

$$\begin{aligned} (d-1)! \cdot \tau([a, b]) &= \tau((d-1)! \cdot [a, b]) = \tau((\psi_d)_*[a, b]) = \tau((\mathrm{pr}_1)_*[\beta_d(a, b)]) \\ &= \pm(\Omega c_d)([\beta_d(a, b)]) = \pm\Delta(c_d, \xi)([\beta_d(a, b)]). \end{aligned}$$

This finishes proving (1). Statements (2) and (3) then follow easily from (1), the divisibility result in Proposition 4.3.2 and the last part of Theorem 1.6.8. □

Finally we study the map $q_* : [X, E']_{\mathbb{A}^1} \to [X, E]_{\mathbb{A}^1}$. By the discussion in Section 4.1, we get exact sequences

$$\begin{cases} \pi^{\mathbb{A}^1}_d(\mathbb{A}^d \setminus 0) \to \pi^{\mathbb{A}^1}_d F_{d-1} \to \mathbf{K}^{\mathrm{M}}_{d+1} = \pi^{\mathbb{A}^1}_d F_d \to 0, & d \text{ odd}; \\ \pi^{\mathbb{A}^1}_d(\mathbb{A}^d \setminus 0) \to \pi^{\mathbb{A}^1}_d F_{d-1} \to 2\mathbf{K}^{\mathrm{M}}_{d+1} \to 0, & d \text{ even}, \end{cases} \tag{4.11}$$

where in the d even case, the term $2\mathbf{K}^{\mathrm{M}}_{d+1}$ sits in an exact sequence $0 \to 2\mathbf{K}^{\mathrm{M}}_{d+1} \to \mathbf{K}^{\mathrm{MW}}_{d+1} = \pi^{\mathbb{A}^1}_d F_d \to \mathbf{I}^{d+1} \to 0$ ([**8**, Proposition 2.6]), and $\mathbf{I}^{d+1}|_X = 0$ if $k = \bar{k}$, showing that the canonical homomorphism $2\mathbf{K}^{\mathrm{M}}_{d+1} \to \mathbf{K}^{\mathrm{MW}}_{d+1} = \pi^{\mathbb{A}^1}_d F_d$ induces an isomorphism $\mathrm{H}^d(X; 2\mathbf{K}^{\mathrm{M}}_{d+1}) \cong \mathrm{H}^d(X; \pi^{\mathbb{A}^1}_d F_d)$. Using the fact that the Nisnevich cohomological dimension of X is bounded above by $\dim(X) = d$, we get exact sequences for highest degree cohomology:

$$\begin{cases} \mathrm{H}^d(X; \pi^{\mathbb{A}^1}_d(\mathbb{A}^d \setminus 0)) \to \mathrm{H}^d(X; \pi^{\mathbb{A}^1}_d F_{d-1}) \to \mathrm{H}^d(X; \mathbf{K}^{\mathrm{M}}_{d+1}) \to 0, & d \text{ odd}; \\ \mathrm{H}^d(X; \pi^{\mathbb{A}^1}_d(\mathbb{A}^d \setminus 0)) \to \mathrm{H}^d(X; \pi^{\mathbb{A}^1}_d F_{d-1}) \to \mathrm{H}^d(X; 2\mathbf{K}^{\mathrm{M}}_{d+1}) \to 0, & d \text{ even}. \end{cases}$$

If $k = \bar{k}$, then these two exact sequences become one:

$$\mathrm{H}^d(X; \pi^{\mathbb{A}^1}_d(\mathbb{A}^d \setminus 0)) \to \mathrm{H}^d(X; \pi^{\mathbb{A}^1}_d F_{d-1}) \to \mathrm{H}^d(X; \mathbf{K}^{\mathrm{M}}_{d+1}) \cong \mathrm{H}^d(X; \pi^{\mathbb{A}^1}_d F_d) \to 0, d \geqslant 3. \tag{4.12}$$

We now invoke the following conjecture of Asok-Fasel describing $\pi^{\mathbb{A}^1}_d(\mathbb{A}^d \setminus 0)$ (see [**6**, Conjecture 7]).

Conjecture 4.3.6 (Asok-Fasel)**.** *Let k be a perfect field with $\mathrm{char}(k) \neq 2$, then there is a sequence*

$$\mathbf{K}^{\mathrm{M}}_{d+2}/24 \to \pi^{\mathbb{A}^1}_d(\mathbb{A}^d \setminus 0) \to \mathbf{GW}^d_{d+1} \to 0 \tag{4.13}$$

of homomorphisms in $\mathcal{A}\mathsf{b}^{\mathbb{A}^1}_k$ which is exact at $\pi^{\mathbb{A}^1}_d(\mathbb{A}^d \setminus 0)$ and also becomes exact at $\mathbf{GW}^d_{d+1}$ after $d-3$-fold contractions. This gives an exact sequence

$$\mathrm{H}^d(X; \mathbf{K}^{\mathrm{M}}_{d+2}/24) \to \mathrm{H}^d(X; \pi^{\mathbb{A}^1}_d(\mathbb{A}^d \setminus 0)) \to \mathrm{H}^d(X; \mathbf{GW}^d_{d+1}) \to 0 \tag{4.14}$$

if X is a smooth k-scheme of dimension d.

The exact sequence on cohomologies above follows easily from the statement on contracted sheaves by a diagram chase using Rost-Schmid complexes. This conjecture is stably true after d-fold contractions, as confirmed by the recently published work **[76]** of Röndigs-Spitzweck-Østvær.

If $k = \bar{k}$, then any generator of the group $\mathbf{K}^{\mathrm{M}}_2(k)$ can be written in the form $\{a^{24}, b\} = 24\{a, b\}$ by the group law of Milnor K-theory. Thus $\mathbf{K}^{\mathrm{M}}_2(k)/24 = 0$. Since $\mathrm{H}^d(X; \mathbf{K}^{\mathrm{M}}_{d+2}/24)$ is a subquotient of $\bigoplus_{x \in X^{(d)}} \mathbf{K}^{\mathrm{M}}_2(\kappa_x)/24 \cong \bigoplus_{x \in X^{(d)}} \mathbf{K}^{\mathrm{M}}_2(k)/24 = 0$, we see $\mathrm{H}^d(X; \mathbf{K}^{\mathrm{M}}_{d+2}/24) = 0$ and so if Conjecture 4.3.6 holds, then

$$\mathrm{H}^d(X; \pi^{\mathbb{A}^1}_d(\mathbb{A}^d \setminus 0)) \xrightarrow{\cong} \mathrm{H}^d(X; \mathbf{GW}^d_{d+1}). \tag{4.15}$$

We have the sheafified Karoubi periodicity sequences (Theorem 3.4.17)

$$\mathbf{K}^{\mathrm{Q}}_{d+1} \xrightarrow{H} \mathbf{GW}^d_{d+1} \xrightarrow{\eta} \mathbf{GW}^{d-1}_d \xrightarrow{f} \mathbf{K}^{\mathrm{Q}}_d$$

in $\mathcal{A}\mathsf{b}^{\mathbb{A}^1}_k$, which is exact. Let $\mathbf{A} := \mathrm{im}(H), \mathbf{B} := \mathrm{im}(\eta)$, then we have exact sequences

$$\mathbf{K}^{\mathrm{Q}}_{d+1} \xrightarrow{H} \mathbf{A} \to 0,$$

$$0 \to \mathbf{A} \to \mathbf{GW}^d_{d+1} \xrightarrow{\eta} \mathbf{B} \to 0,$$

yielding exact sequences on cohomologies:

$$\mathrm{H}^d(X; \mathbf{K}^{\mathrm{M}}_{d+1}) \cong \mathrm{H}^d(X; \mathbf{K}^{\mathrm{Q}}_{d+1}) \to \mathrm{H}^d(X; \mathbf{A}) \to 0,$$

$$\mathrm{H}^d(X; \mathbf{A}) \to \mathrm{H}^d(X; \mathbf{GW}^d_{d+1}) \to \mathrm{H}^d(X; \mathbf{B}) \to 0$$

and hence

$$\mathrm{H}^d(X; \mathbf{K}^{\mathrm{M}}_{d+1}) \to \mathrm{H}^d(X; \mathbf{GW}^d_{d+1}) \to \mathrm{H}^d(X; \mathbf{B}) \to 0.$$

Contracting the sheafified Karoubi periodicity sequence d-times we get an exact sequence $\mathbf{K}^{\mathrm{M}}_1 \xrightarrow{H} \mathbf{GW}^0_1 \xrightarrow{\eta} \mathbf{B}_{-d} \to 0$. While the composite $\mathbf{K}^{\mathrm{M}}_1 \xrightarrow{H} \mathbf{GW}^3_1 \xrightarrow{f} \mathbf{K}^{\mathrm{M}}_1$ is multiplication by 2, we see that the composite $2\mathbf{K}^{\mathrm{M}}_1 \hookrightarrow \mathbf{K}^{\mathrm{M}}_1 \xrightarrow{H} \mathbf{GW}^0_1$ is 0. So, we have an exact sequence $\mathbf{K}^{\mathrm{M}}_1/2 \xrightarrow{H} \mathbf{GW}^0_1 \xrightarrow{\eta} \mathbf{B}_{-d} \to 0$, which splits into two: $\mathbf{K}^{\mathrm{M}}_1/2 \xrightarrow{H} \mathbf{A}_{-d} \to 0, 0 \to \mathbf{A}_{-d} \to \mathbf{GW}^0_1 \xrightarrow{\eta} \mathbf{B}_{-d} \to 0$. Using again the Rost-Schmid complexes we find exact sequences $\mathrm{H}^d(X; \mathbf{K}^{\mathrm{M}}_{d+1}/2) \to \mathrm{H}^d(X; \mathbf{A}) \to 0$ and $\mathrm{H}^d(X; \mathbf{A}) \to \mathrm{H}^d(X; \mathbf{GW}^d_{d+1}) \to \mathrm{H}^d(X; \mathbf{B}) \to 0$. Finally, we obtain an exact sequence

$$0 = \mathrm{H}^d(X; \mathbf{K}^{\mathrm{M}}_{d+1}/2) \to \mathrm{H}^d(X; \mathbf{GW}^d_{d+1}) \to \mathrm{H}^d(X; \mathbf{B}) \to 0. \tag{4.16}$$

By **[9**, Lemma 3.6.3**]**, we have $\mathrm{H}^d(X; \mathbf{B}) \cong \mathrm{Ch}^d(X)$, where $\mathrm{Ch}^d(X) = \mathrm{H}^d(X; \mathbf{K}^{\mathrm{M}}_d/2) \cong \mathrm{CH}^d(X)/2$ is the group of mod-2 codimension-d cycle classes on X. Since $k = \bar{k}$, we have $\mathrm{Ch}^d(X) = 0$ (in fact, $\mathrm{CH}^d(X)$ is uniquely divisible, see e.g. **[85]** for a discussion). Thus $\mathrm{H}^d(X; \mathbf{GW}^d_{d+1}) = 0$ and so $\mathrm{H}^d(X; \pi^{\mathbb{A}^1}_d(\mathbb{A}^d \setminus 0)) = 0$ (assuming Conjecture 4.3.6). Using eq. (4.12), we find

$$\mathrm{H}^d(X; \pi^{\mathbb{A}^1}_d F_{d-1}) \xrightarrow{\cong} \mathrm{H}^d(X; \pi^{\mathbb{A}^1}_d F_d) \cong \mathrm{H}^d(X; \mathbf{K}^{\mathrm{M}}_{d+1}) \cong [X, \mathbb{A}^{d+1} \setminus 0]_{\mathbb{A}^1} \cong [X, Q_{2d+1}]_{\mathbb{A}^1}, d \geqslant 3. \tag{4.17}$$

Theorem 4.3.7. *Assume that $k = \bar{k}$ and $\mathrm{char}(k) \neq 2$. Let A be a smooth k-algebra of Krull dimension $d \geqslant 3$, let $X = \mathrm{Spec}\, A$.* Assume Conjecture 4.3.6 holds. *Then the map $q_* : [X, E']_{\mathbb{A}^1} \to [X, E]_{\mathbb{A}^1}$ is a bijection (here E, E' are the relevant stages of Moore-Postnikov factorization for $\varphi : \mathrm{BSL}_{d-1} \to \mathrm{BSL}$ as in diagram* (4.9)*).*

Proof. The $\mathbb{A}^1$-homotopy fibre sequence

$$\mathrm{K}(\pi^{\mathbb{A}^1}_d F_{d-1}, d) \to E' \xrightarrow{q} E \xrightarrow{\theta'} \mathrm{K}(\pi^{\mathbb{A}^1}_d F_{d-1}, d+1),$$

shows that q_* is surjective, and gives a homotopy fibre sequence in sSet_*:

$$\mathrm{RMap}(X, E') \xrightarrow{q_*} \mathrm{RMap}(X, E) \xrightarrow{\theta'_*} \mathrm{RMap}(X, \mathrm{K}(\pi^{\mathbb{A}^1}_d F_{d-1}, d+1)).$$

By the last part of Theorem 1.6.8, to show injectivity of q_*, we need to show: for any $\xi \in [X, \mathrm{BSL}]_{\mathbb{A}^1}$ which is represented by a rank $d-1$ vector bundle (or equivalently, $c_d(\xi) = 0$), let $\xi_E \in [X, E]_{\mathbb{A}^1}$ be the unique lifting of ξ as in eq. (4.9) (so $\theta'_*(\xi_E) = 0 \in \mathrm{H}^{d+1}(X; \pi^{\mathbb{A}^1}_d F_{d-1})$), then the map

$$\theta'_* : \pi_1(\mathrm{RMap}(X, E), \xi_E) \to \pi_1(\mathrm{RMap}(X, \mathrm{K}(\pi^{\mathbb{A}^1}_d F_{d-1}, d+1)), 0) = \mathrm{H}^d(X; \pi^{\mathbb{A}^1}_d F_{d-1})$$

is surjective.

Consider now the comparison diagram of Moore-Postnikov towers

$$\begin{array}{ccccccccc}
F_{d-1} & \longrightarrow & \mathrm{BSL}_{d-1} & \longrightarrow & E' & \xrightarrow{q} & E & \xrightarrow{p} & \mathrm{BSL} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle p} & & \| \\
F_d & \longrightarrow & \mathrm{BSL}_d & \longrightarrow & \tilde{E} & \longrightarrow & \mathrm{BSL} & = & \mathrm{BSL},
\end{array}$$

where $\tilde{E}$ is the first stage in the Moore-Postnikov tower of the map $\mathrm{BSL}_d \to \mathrm{BSL}$. By functoriality, the first stage k-invariants (that of the column of E) gives a commutative square

$$\begin{array}{ccc} E & \xrightarrow{\theta'} & \mathrm{K}(\pi_d^{\mathbb{A}^1} F_{d-1}, d+1) \\ {\scriptstyle p}\downarrow & & \downarrow \\ \mathrm{BSL} & \xrightarrow{\tilde{\theta}} & \mathrm{K}(\pi_d^{\mathbb{A}^1} F_d, d+1), \end{array} \tag{4.18}$$

where $\tilde{\theta}$ is the k-invariant "θ" in the rank d case, we write it as $\tilde{\theta}$ to distinguish it from the k-invariant "θ" in the rank $d-1$ case here) and the right vertical map is induced by the map $F_{d-1} \to F_d$. Applying $\mathrm{RMap}(X, -)$ we obtain a commutative diagram

$$\begin{array}{ccc} \mathrm{RMap}(X, E) & \xrightarrow{\theta'} & \mathrm{RMap}(X, \mathrm{K}(\pi_d^{\mathbb{A}^1} F_{d-1}, d+1)) \\ {\scriptstyle p}\downarrow & & \downarrow \\ \mathrm{RMap}(X, \mathrm{BSL}) & \xrightarrow{\tilde{\theta}} & \mathrm{RMap}(X, \mathrm{K}(\pi_d^{\mathbb{A}^1} F_d, d+1)), \end{array} \tag{4.19}$$

and hence

$$\begin{array}{ccccc} c'' \in \pi_1(\mathrm{RMap}(X,E), \xi_E) & & \xrightarrow{\theta'_*} & & \pi_1(\mathrm{RMap}(X, \mathrm{K}(\pi_d^{\mathbb{A}^1} F_{d-1}, d+1)), 0) \ni c \\ {\scriptstyle p_*}\downarrow & & & & \downarrow{\scriptstyle \cong} \\ T_\xi[\beta_{d+1}(a,b)] \in \pi_1(\mathrm{RMap}(X, \mathrm{BSL}), \xi) & & \xrightarrow{\tilde{\theta}_* = (c_{d+1})_*} & & \pi_1(\mathrm{RMap}(X, \mathrm{K}(\pi_d^{\mathbb{A}^1} F_d, d+1)), 0) \ni c \\ {\scriptstyle T_\xi}\uparrow & & & & \| \\ [\beta_{d+1}(a,b)] \in \pi_1(\mathrm{RMap}(X, \mathrm{BSL}), 0) & & \xrightarrow{\Delta(c_{d+1}, \xi)} & & \mathrm{H}^d(X; \mathbf{K}^{\mathrm{M}}_{d+1}) \ni c \\ {\scriptstyle \beta_{d+1}}\uparrow & & & \nearrow_{\pm d!} & \\ [X, Q_{2d+1}]_{\mathbb{A}^1} \cong \mathrm{H}^d(X; \mathbf{K}^{\mathrm{MW}}_{d+1}) \ni c' = [a,b], & & & & \end{array} \tag{4.20}$$

where the arrow $\tilde{\theta}_*$ is surjective by Theorems 4.2.9 and 4.2.13, and the right vertical maps are isomorphisms by eq. (4.17); the middle square commutes by Theorem 4.2.3, and the lower triangle is given by eq. (4.6), the arrow labeled by $\pm d!$ is surjective (see Remark 4.2.10) . So for any $c \in \pi_1(\mathrm{RMap}(X, \mathrm{K}(\pi_d^{\mathbb{A}^1} F_{d-1}, d+1)), 0) = \mathrm{H}^d(X; \mathbf{K}^{\mathrm{M}}_{d+1})$, we can find $c' = [a,b] \in [X, Q_{2d+1}]_{\mathbb{A}^1}$ with $c = \pm d! \cdot [a,b] = \Delta(c_{d+1}, \xi)([\beta_{d+1}(a,b)])$.

Theorem 4.2.3 (again) and the last statement of Proposition 4.3.3 tell that

$$\theta_*(T_\xi[\beta_{d+1}(a,b)]) = (c_d)_*(T_\xi[\beta_{d+1}(a,b)]) = \Delta(c_d, \xi)([\beta_{d+1}(a,b)]) = 0,$$

where θ is the k-invariant in the rank $d-1$ case, which corresponds to the Chern class c_d.

On the other hand, the fiber sequence

$$\mathrm{RMap}(X, E)_{\xi_E} \xrightarrow{p} \mathrm{RMap}(X, \mathrm{BSL})_\xi \xrightarrow{\theta} \mathrm{RMap}(X, \mathrm{K}(\mathbf{K}_d^{\mathrm{M(W)}}, d))_0$$

in $\mathsf{s\mathfrak{S}et}_*$, where the subscripts refer the corresponding components, induces another such sequence

$$\Omega_{\xi_E}\mathrm{RMap}(X, E) \xrightarrow{p} \Omega_\xi \mathrm{RMap}(X, \mathrm{BSL}) \xrightarrow{\theta} \mathrm{RMap}(X, \mathrm{K}(\mathbf{K}_d^{\mathrm{M(W)}}, d-1)).$$

Thus $T_\xi[\beta_{d+1}(a,b)] \in \ker(\theta_*) = \mathrm{im}(p_*)$ and there exists $c'' \in \pi_1(\mathrm{RMap}(X, E), \xi_E)$ with $p_*(c'') = T_\xi[\beta_{d+1}(a,b)]$, which then satisfies $\theta'_*(c'') = c$, proving that the map

$$\theta'_* : \pi_1(\mathrm{RMap}(X, E), \xi_E) \to \pi_1(\mathrm{RMap}(X, \mathrm{K}(\pi_d^{\mathbb{A}^1} F_{d-1}, d+1)), 0) = \mathrm{H}^d(X; \pi_d^{\mathbb{A}^1} F_{d-1})$$

is surjective. Hence q_* is injective as well. □

We finally arrive at the following cancellation result for (oriented) rank $d-1$ vector bundles over a smooth affine variety of dimension d, admitting Asok-Fasel conjecture. (As before, we let $\varphi = \varphi_{d-1} : \mathrm{BSL}_{d-1} \to \mathrm{BSL}$ be the stabilizing map.)

Theorem 4.3.8. *Assume that the base field k is algebraically closed and that $\mathrm{char}(k) \neq 2$. Let A be a smooth k-algebra of Krull dimension $d \geqslant 3$. Let $X = \mathrm{Spec}\ A$, $\xi \in [X, \mathrm{BSL}]_{\mathbb{A}^1}$ which is represented by a rank $d-1$ vector bundle (or equivalently, $c_d(\xi) = 0$). Let $\varphi_* : [X, \mathrm{BSL}_{d-1}]_{\mathbb{A}^1} \to [X, \mathrm{BSL}]_{\mathbb{A}^1}$ be the stabilizing map. If $\mathrm{char}(k) = 0$ or $\mathrm{char}(k) \geqslant d$, then the lifting set $\mathcal{V}_{d-1}(X, \xi) = \varphi_*^{-1}(\xi) \subset [X, \mathrm{BSL}_{d-1}]_{\mathbb{A}^1}$ is a singleton, provided Conjecture 4.3.6 holds. In other words, every oriented rank $d-1$ vector bundle over X is cancellative.*

Remark 4.3.9. It's important to assume k is algebraically closed ($k = \bar{k}$) here: there is Mohan Kumar's examples **[52]** of rank $d-1$ stably free modules which are not free, when the base field is a C_1-field but not algebraically closed.

4.4. Cancellation properties of symplectic vector bundles

In this section, we give enumeration results and in particular study cancellation properties of symplectic vector bundles, using the ideas and methods of the previous sections.

By Theorem 3.3.12 or [**13**, Theorems 2.3.5, 3.3.3, and 4.1.2] (see also [**72**]), the set of classes of rank $2n$ symplectic vector bundles over a smooth affine scheme over a perfect field is represented by the motivic space BSp_{2n}.

The map $\mathrm{Sp}_{2n} \to \mathrm{Sp}_{2n+2}, A \mapsto \begin{pmatrix} A & \\ & \mathrm{I}_2 \end{pmatrix}$ induces an $\mathbb{A}^1$-homotopy fiber sequence

$$\mathbb{A}^{2n+2} \setminus 0 \simeq \mathrm{Sp}_{2n+2}/\mathrm{Sp}_{2n} \to \mathrm{BSp}_{2n} \to \mathrm{BSp}_{2n+2}.$$

Since

$$\pi_i^{\mathbb{A}^1}(\mathbb{A}^{2n+2} \setminus 0) = \begin{cases} 0, & i < 2n+1, \\ \mathbf{K}^{\mathrm{MW}}_{2n+2}, & i = 2n+1, \end{cases}$$

we find that the map $\pi_i^{\mathbb{A}^1}\mathrm{BSp}_{2n} \to \pi_i^{\mathbb{A}^1}\mathrm{BSp}_{2n+2}$ is an isomorphism if $i < 2n+1$ and a surjection if $i = 2n+1$. Thus also, the induced map $\pi_i^{\mathbb{A}^1}\mathrm{BSp}_{2n} \to \pi_i^{\mathbb{A}^1}\mathrm{BSp}$ is an isomorphism if $i < 2n+1$ and a surjection if $i = 2n+1$ (see also [**7**, Theorems 2.6 and 3.3]).

Let F'_n be the $\mathbb{A}^1$-homotopy fiber of the canonical map $\varphi : \mathrm{BSp}_{2n} \to \mathrm{BSp}$. We then have an $\mathbb{A}^1$-homotopy fiber sequence

$$F'_n \to \mathrm{BSp}_{2n} \xrightarrow{\varphi} \mathrm{BSp},$$

and from the above results we see that

$$\pi_j^{\mathbb{A}^1} F'_n = 0, j < 2n+1.$$

As before, we have an $\mathbb{A}^1$-homotopy fiber sequence

$$\mathbb{A}^{2n+2} \setminus 0 \to F'_n \to F'_{n+1},$$

which then gives $\pi^{\mathbb{A}^1}_{2n+1}(\mathbb{A}^{2n+2} \setminus 0) \xrightarrow{\cong} \pi^{\mathbb{A}^1}_{2n+1}F'_n$ and we obtain

$$\pi_i^{\mathbb{A}^1} F'_n = \begin{cases} 0, & i < 2n+1, \\ \mathbf{K}^{\mathrm{MW}}_{2n+2}, & i = 2n+1. \end{cases}$$

The motivic space $\mathrm{BSp} \times \mathbb{Z}$ represents symplectic K-theory in the motivic homotopy category $\mathcal{H}^{\mathbb{A}^1}(k)$, and so $\pi_i^{\mathbb{A}^1}\mathrm{BSp} \cong \mathbf{K}^{\mathrm{Sp}}_i, i \geqslant 1$ and $\pi_1^{\mathbb{A}^1}\mathrm{BSp} \cong \mathbf{K}^{\mathrm{Sp}}_1 = 0$. Moreover, there is a motivic $T^{\wedge 4}$-spectrum with term $\mathrm{BSp} \times \mathbb{Z}$ at each level (see e.g. [**81**, Theorem 3], or by the same result, we see that $\mathrm{BSp} \times \mathbb{Z}$ is a term of a motivic T-spectrum, as stated in Proposition 3.5.4), yielding as before the following result.

Corollary 4.4.1. *If k is a perfect field, then* BSp *is an abelian group object in the pointed $\mathbb{A}^1$-homotopy category $\mathcal{H}^{\mathbb{A}^1}_*(k)$ and for any $X \in \mathcal{S}\mathsf{m}_k$, all the components of the derived mapping space* $\mathrm{RMap}(X, \mathrm{BSp}) \in \mathsf{s}\mathcal{S}\mathsf{et}_*$ *are weakly equivalent.*

Thus there is a canonical map $m : \mathrm{BSp} \times \mathrm{BSp} \to \mathrm{BSp}$ in the pointed $\mathbb{A}^1$-homotopy category $\mathcal{H}^{\mathbb{A}^1}_*(k)$ giving the abelian group object structure of BSp. Given $\xi \in [X, \mathrm{BSp}]_{\mathbb{A}^1}$, let

$$T_\xi := m(-, \xi)_* : \pi_1(\mathrm{RMap}(X, \mathrm{BSp}), 0) \to \pi_1(\mathrm{RMap}(X, \mathrm{BSp}), \xi)$$

be the isomorphism on fundamental groups induced by $m(-, \xi) : \mathrm{RMap}(X, \mathrm{BSp})_0 \to \mathrm{RMap}(X, \mathrm{BSp})_\xi$ as introduced in Proposition 4.1.3.

Similarly as before, we denote by $\mathcal{V}^{\mathrm{Sp}}_{2n}(X, \xi) := \varphi_*^{-1}(\xi)$ the set of isomorphism classes of rank-$2n$ symplectic vector bundles which represent ξ.

Let $E = E_{2n}$ be the first non-trivial stage of the Moore-Postnikov tower of the canonical map $\mathrm{BSp}_{2n} \to \mathrm{BSp}$. Then we obtain a principal $\mathbb{A}^1$-homotopy fiber sequence

$$\mathrm{K}(\mathbf{K}^{\mathrm{MW}}_{2n+2}, 2n+1) \to E \to \mathrm{BSp}$$

classified by a k-invariant $b_{n+1} \in \widetilde{\mathrm{CH}}^{2n+2}(\mathrm{BSp}) = \mathrm{H}^{2n+2}(\mathrm{BSp}; \mathbf{K}^{\mathrm{MW}}_{2n+2})$ (which is given by a map $b_{n+1} : \mathrm{BSp} \to \mathrm{K}(\mathbf{K}^{\mathrm{MW}}_{2n+2}, 2n+2)$ in $\mathcal{H}^{\mathbb{A}^1}(k)$).

We have a cartesian suqare

$$\begin{array}{ccc} \mathrm{Sp}_{2n} & \longrightarrow & \mathrm{Sp}_{2n+2} \\ \downarrow & & \downarrow \\ \mathrm{SL}_{2n+1} & \longrightarrow & \mathrm{SL}_{2n+2}, \end{array}$$

giving a commutative diagram

$$\begin{array}{ccc} \mathrm{BSp}_{2n} & \longrightarrow & \mathrm{BSp}_{2n+2} \\ \downarrow & & \downarrow \\ \mathrm{BSL}_{2n+1} & \longrightarrow & \mathrm{BSL}_{2n+2}, \end{array}$$

which in turn, by naturality of k-invariants, yields a commutative diagram

$$\begin{array}{ccc} \mathrm{BSp}_{2n+2} & \xrightarrow{b_{n+1}} & \mathrm{K}(\mathbf{K}^{\mathrm{MW}}_{2n+2}, 2n+2) \\ \downarrow & & \| \\ \mathrm{BSL}_{2n+2} & \xrightarrow{e_{2n+2}} & \mathrm{K}(\mathbf{K}^{\mathrm{MW}}_{2n+2}, 2n+2), \end{array} \tag{4.21}$$

where $e_{2n+2} \in \widetilde{\mathrm{CH}}^{2n+2}(\mathrm{BSL}_{2n+2}) = \mathrm{H}^{2n+2}(\mathrm{BSL}_{2n+2}; \mathbf{K}^{\mathrm{MW}}_{2n+2})$ is the universal Euler class. The universal Euler class restricts to the universal *Borel class*, see e.g. [**44**, Proposition 4.3], where it's called the *Pontryagin class* and denoted $\mathrm{p}_{n+1}(\mathrm{Sp}_{2n+2})$; we follow the nowadays common nomenclature—the Borel class, as in the newest version of [**72**]. Thus the k-invariant $b_{n+1} \in \widetilde{\mathrm{CH}}^{2n+2}(\mathrm{BSp}) = \mathrm{H}^{2n+2}(\mathrm{BSp}; \mathbf{K}^{\mathrm{MW}}_{2n+2})$ is exactly the $n+1$-st Borel class.

The Borel classes b_i's satisfy a Whitney formula (or rather, Cartan sum formula, see [**44**, Theorem 4.10] or [**72**]):

$$b_n(\xi + \xi') = \sum_{r=0}^{n} b_r(\xi) \cdot b_{n-r}(\xi')$$

for any $U \in \mathcal{S}\mathsf{m}_k$ and $(\xi, \xi') \in [U, \mathrm{BSp}]_{\mathbb{A}^1} \times [U, \mathrm{BSp}]_{\mathbb{A}^1}$. And so

$$m^* b_n = \sum_{r=0}^{n} (b_r \times b_{n-r}) \in [\mathrm{BSp} \times \mathrm{BSp}, \mathrm{K}(\mathbf{K}^{\mathrm{MW}}_{2n}, 2n)]_{\mathbb{A}^1}, \tag{4.22}$$

where $b_r \times b_{n-r} := \mu(b_r \otimes b_{n-r})$, μ being the obvious map

$$[\mathrm{BSp}, \mathrm{K}(\mathbf{K}^{\mathrm{MW}}_{2r}, 2r)]_{\mathbb{A}^1} \otimes [\mathrm{BSp}, \mathrm{K}(\mathbf{K}^{\mathrm{MW}}_{2n-2r}, 2n-2r)]_{\mathbb{A}^1} \to [\mathrm{BSp} \times \mathrm{BSp}, \mathrm{K}(\mathbf{K}^{\mathrm{MW}}_{2n}, 2n)]_{\mathbb{A}^1}$$

induced by the multiplication in the graded sheaves of Milnor-Witt K-groups $\mathbf{K}^{\mathrm{MW}}_*$.

If k is a perfect field and $X = \mathrm{Spec}(A)$ is a smooth affine k-scheme with $3 \leqslant \dim(X) = d \leqslant 2n+1$, then an easy argument using Moore-Postnikov tower shows that any stable symplectic vector bundle ξ over X is represented by a rank $2n$ symplectic vector bundle $\xi_{2n} \in [X, \mathrm{BSp}_{2n}]_{\mathbb{A}^1}$, i.e. $\xi : X \to \mathrm{BSp}$ factors through the canonical map $\varphi : \mathrm{BSp}_{2n} \to \mathrm{BSp}$ (in the $\mathbb{A}^1$-homotopy category $\mathcal{H}^{\mathbb{A}^1}(k)$). If $d < 2n+1$, then another Moore-Postnikov tower argument shows that the representative symplectic vector bundle $\xi_{2n} \in [X, \mathrm{BSp}_{2n}]_{\mathbb{A}^1}$ is unique. So the main problem is to consider in the case $d = 2n+1$, the uniqueness of the lifting, or the cancellation property of the symplectic vector bundle ξ_{2n}.

As in Theorems 4.2.1 and 4.2.3, we obtain the following enumeration result on symplectic vector bundles near critical rank (of course, only the $d = 2n+1$ case is interesting; the $d < 2n+1$ cases are almost trivial by the Moore-Postnikov tower argument).

Theorem 4.4.2. *Let k be a perfect field with $\mathrm{char}(k) \neq 2$ and let $X = \mathrm{Spec}(A)$ be a smooth affine k-scheme of dimension d with $3 \leqslant d \leqslant 2n+1$. Let ξ be a stable symplectic vector bundle over X, whose classifying map is still denoted by $\xi : X \to \mathrm{BSp}$. Denote by*

$$\Delta(b_{n+1}, \xi) := b_{n+1} \circ T_\xi : \mathrm{KSp}_1(X) = [X, \Omega\mathrm{BSp}]_{\mathbb{A}^1} \to [X, \mathrm{K}(\mathbf{K}^{\mathrm{MW}}_{2n+2}, 2n+1)]_{\mathbb{A}^1} = \mathrm{H}^{2n+1}(X; \mathbf{K}^{\mathrm{MW}}_{2n+2})$$

the induced homomorphism on fundamental groups. Then there is a bijection

$$\mathcal{V}^{\mathrm{Sp}}_{2n}(X, \xi) \cong \mathrm{coker}\big(\mathrm{KSp}_1(X) = [X, \Omega\mathrm{BSp}]_{\mathbb{A}^1} \xrightarrow{\Delta(b_{n+1}, \xi)} [X, \mathrm{K}(\mathbf{K}^{\mathrm{MW}}_{2n+2}, 2n+1)]_{\mathbb{A}^1} = \mathrm{H}^{2n+1}(X; \mathbf{K}^{\mathrm{MW}}_{2n+2})\big).$$

Moreover, the map

$$\Delta(b_{n+1}, \xi) : \mathrm{KSp}_1(X) = [X, \Omega\mathrm{BSp}]_{\mathbb{A}^1} \to [X, \mathrm{K}(\mathbf{K}^{\mathrm{MW}}_{2n+2}, 2n+1)]_{\mathbb{A}^1} = \mathrm{H}^{2n+1}(X; \mathbf{K}^{\mathrm{MW}}_{2n+2})$$

is given by

$$\Delta(b_{n+1}, \xi)(\beta) = (\Omega b_{n+1})(\beta) + \sum_{r=1}^{n} ((\Omega b_r)(\beta)) \cdot b_{n+1-r}(\xi). \tag{4.23}$$

We again turn to more refined computations in order to give cancellation results on symplectic vector bundles near critical rank. As discussed above, we only need to consider the case $d = 2n+1$ and we do assume this in the sequel.

Consider the following commutative diagram in $\mathcal{H}^{\mathbb{A}^1}(k)$:

$$\begin{array}{ccccccc} & & & & \mathrm{SL}_{d+1} & \xrightarrow{H} & \mathrm{Sp}_{2(d+1)} \subset \mathrm{Sp} \xrightarrow{\Omega b_r} \mathrm{K}(\mathbf{K}^{\mathrm{MW}}_{2r}, 2r-1) \\ & & & \overset{\beta_{d+1}}{\nearrow} & \uparrow & & \\ X & \xrightarrow{(a,b)} & Q_{2d+1} & \xrightarrow{\alpha_{d+1}} & \mathrm{SL}_{2^d}, & & \end{array}$$

where $H : \mathrm{SL}_r \to \mathrm{Sp}_{2r}, A \mapsto \begin{pmatrix} A & \\ & (A^T)^{-1} \end{pmatrix}$. So

$$(\Omega b_r)(H\beta_{d+1}) \in \widetilde{\mathrm{H}}^{2r-1}(Q_{2d+1}; \mathbf{K}^{\mathrm{MW}}_{2r}) \cong \begin{cases} 0, & 2r \neq d+1; \\ \mathbf{K}^{\mathrm{MW}}_0(k) = \mathrm{GW}(k), & 2r = d+1 \ (r = n+1). \end{cases}$$

It follows that $(\Omega b_r)(H\beta_{d+1}) = 0, \forall r \neq n+1$ and that

$$\Delta(b_{n+1}, \xi)(H\beta_{d+1}) = (\Omega b_{n+1})(H\beta_{d+1}). \tag{4.24}$$

Let $F : \mathrm{Sp} \hookrightarrow \mathrm{SL}$ be the "forgetful" map induced by the inclusions $\mathrm{Sp}_{2r} \hookrightarrow \mathrm{SL}_{2r}$, then we have the induced maps

$$\mathrm{SK}_1(X) \xrightarrow{H} \mathrm{KSp}_1(X) \xrightarrow{F} \mathrm{SK}_1(X).$$

By **[86**, Lemma 5.3], $[\alpha_{d+1}(a,b)] + [\alpha_{d+1}(a,b)^T] = 0$ in $\mathrm{SK}_1(X)$, for any A-point (a,b) of Q_{2d+1} (since the matrix I_r in **[86**, Lemma 5.3] is in $\mathrm{SL}_{2^r}(\mathbb{Z})$, hence elementary). Thus

$$FH([\alpha_{d+1}(a,b)]) = \left[\begin{pmatrix} \alpha_{d+1}(a,b) & \\ & (\alpha_{d+1}(a,b)^T)^{-1} \end{pmatrix}\right] = [\alpha_{d+1}(a,b)] - [\alpha_{d+1}(a,b)^T] = 2[\alpha_{d+1}(a,b)].$$

As $[\alpha_{d+1}(a,b)] = [\beta_{d+1}(a,b)] \in \mathrm{SK}_1(X)$, we also get

$$FH([\beta_{d+1}(a,b)]) = 2[\beta_{d+1}(a,b)].$$

The diagram eq. (4.21) then yields the following commutative diagrams

$$\begin{array}{ccc} \mathrm{BSp} & \xrightarrow{b_{n+1}} & \mathrm{K}(\mathbf{K}^{\mathrm{MW}}_{d+1}, d+1) \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle \tau} \\ \mathrm{BSL} & \xrightarrow{c_{d+1}} & \mathrm{K}(\mathbf{K}^{\mathrm{M}}_{d+1}, d+1), \end{array} \quad \text{and hence} \quad \begin{array}{ccc} \mathrm{Sp} & \xrightarrow{\Omega b_{n+1}} & \mathrm{K}(\mathbf{K}^{\mathrm{MW}}_{d+1}, d) \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle \tau} \\ \mathrm{BSL} & \xrightarrow{\Omega c_{d+1}} & \mathrm{K}(\mathbf{K}^{\mathrm{M}}_{d+1}, d). \end{array} \tag{4.25}$$

Composing the latter with the map $H : \mathrm{SL} \to \mathrm{Sp}$ and applying $[X, -]_{\mathbb{A}^1}$ we see that

$$\tau(\Omega b_{n+1})H = (\Omega c_{d+1})FH : \mathrm{SK}_1(X) \to \mathrm{H}^d(X; \mathbf{K}^{\mathrm{M}}_{d+1}).$$

Considering the effect on $[\beta_{d+1}(a,b)] \in \mathrm{SK}_1(X)$ for any A-point (a,b) of Q_{2d+1} with $a = (a_0, a_1, \cdots, a_d)$, we find that

$$\tau(\Omega b_{n+1})H([\beta_{d+1}(a,b)]) = 2(\Omega c_{d+1})([\beta_{d+1}(a,b)]) = \pm 2 \cdot \tau([a_0^{d!}, a_1, \cdots, a_d]) = \tau(\pm 2 \cdot \frac{d!}{2} h \cdot [(a,b)]) = \tau(\pm d! \cdot h \cdot [(a,b)])$$

by Proposition 4.2.7 and eq. (4.7), where $[(a,b)]$ denotes the image of $(a,b) \in Q_{2d+1}(A)$ in $[X, Q_{2d+1}]_{\mathbb{A}^1} \cong \mathrm{H}^d(X; \mathbf{K}^{\mathrm{MW}}_{d+1})$.

As noted before, if $\mathrm{char}(k) \neq 2$ and $\mathrm{c.d.}_2(k) \leqslant 2$, then $\mathrm{H}^d(X; \mathbf{I}^{d+2}) = 0$ and the reduction map $\tau : \mathrm{H}^d(X; \mathbf{K}^{\mathrm{MW}}_{d+1}) \to \mathrm{H}^d(X; \mathbf{K}^{\mathrm{M}}_{d+1})$ is an isomorphism (if $\mathrm{c.d.}(k) \leqslant 1$, one can also use the Arason-Pfister Hauptsatz and the motivic Bloch-Kato conjecture to show that $\mathbf{I}^{d+2}|_X = 0$). Then we must have

$$\Delta(b_{n+1}, \xi)H([\beta_{d+1}(a,b)]) = (\Omega b_{n+1})H([\beta_{d+1}(a,b)]) = \pm d! \cdot h \cdot [(a,b)].$$

This says that $d! \cdot h \cdot \mathrm{H}^d(X; \mathbf{K}^{\mathrm{MW}}_{d+1}) \subset \mathrm{im}\ \Delta(b_{n+1}, \xi)$, $(2 \cdot d!) \cdot \mathrm{H}^d(X; \mathbf{K}^{\mathrm{M}}_{d+1}) \subset \mathrm{im}(\tau\Delta(b_{d+1}, \xi))$. Since h goes to 2 under τ and $\tau : \mathrm{H}^d(X; \mathbf{K}^{\mathrm{MW}}_{d+1}) \to \mathrm{H}^d(X; \mathbf{K}^{\mathrm{M}}_{d+1})$ is an isomorphism, in fact we have $(d! \cdot h) \cdot \mathrm{H}^d(X; \mathbf{K}^{\mathrm{MW}}_{d+1}) = (2 \cdot d!) \cdot \mathrm{H}^d(X; \mathbf{K}^{\mathrm{MW}}_{d+1}) \subset \mathrm{im}\ \Delta(b_{d+1}, \xi)$. We summarize the discussion as follows.

Theorem 4.4.3. *Let k be a perfect field with $\mathrm{char}(k) \neq 2$ and $\mathrm{c.d.}_2(k) \leqslant 2$. Let $X = \mathrm{Spec}(A)$ be a smooth affine k-scheme of dimension $d = 2n+1 \geqslant 3$. Let ξ be a stable symplectic vector bundle over X, whose classifying map is still denoted by $\xi : X \to \mathrm{BSp}$.*

1. *We have $(2 \cdot d!) \cdot \mathrm{H}^d(X; \mathbf{K}^{\mathrm{M}}_{d+1}) \subset \mathrm{im}(\tau\Delta(b_{n+1}, \xi))$.*
2. *There is a surjective homomorphism*
$$\mathrm{H}^d(X; \mathbf{K}^{\mathrm{M}}_{d+1})/2 \cdot d! \twoheadrightarrow \mathrm{coker}\Delta(b_{n+1}, \xi) \cong \mathcal{V}^{\mathrm{Sp}}_{2n}(X, \xi).$$
3. *If $\mathrm{H}^d(X; \mathbf{K}^{\mathrm{M}}_{d+1})$ is $d!$-divisible, then $\mathrm{coker}\Delta(b_{n+1}, \xi) = 0$. So in this case, any rank $2n = d-1$ symplectic vector bundle is cancellative. Moreover, the map*
$$(b_{n+1})_* : \pi_1(\mathrm{RMap}(X, \mathrm{BSp}), \xi) \to \pi_1(\mathrm{RMap}(X, \mathrm{K}(\mathbf{K}^{\mathrm{MW}}_{d+1}, d+1)), 0) = \mathrm{H}^d(X; \mathbf{K}^{\mathrm{MW}}_{d+1})$$
is surjective for every $\xi \in [X, \mathrm{BSp}]_{\mathbb{A}^1}$.
4. *If $\mathrm{c.d.}(k) \leqslant 1$, then ξ is represented by a unique symplectic vector bundle over X of rank $2n = d-1$.*

For statement (3), note that if an abelian group is $d!$-divisible, then it is also $2 \cdot d!$-divisible. The last statement above is again guaranteed by Voevodsky's confirmation of the motivic Bloch-Kato conjecture as in Remark 4.2.10.

4.5. Further directions

Let k be a perfect field, let A be a smooth affine k-algebra of dimension d and let $X := \mathrm{Spec}(A)$. The results we obtain so far are apparently not satisfactory (e.g. the restriction on the cohomological dimension and orientability) if we want to have a complete picture about cancellation properties of projective modules.

On the other hand, recent work of Das-Tikader-Zinna **[23]**, relying heavily on results about Euler class groups, gives an enumeration result on stably trivial vector bundles over smooth real affine algebras of top rank. Precisely, let A be a smooth affine $\mathbb{R}$-algebra of dimension d and let $X := \mathrm{Spec}(A)$. Then the real points $X(\mathbb{R})$ form a d-dimensional smooth real manifold. Let $\mathcal{C}$ be the (finite) set of *compact* connected components of $X(\mathbb{R})$. Assuming $X(\mathbb{R})$ is *orientable*, then the set of isomorphism classes of rank d (finitely generated) stably free projective A-modules is isomorphic to $\mathbb{Z}^{\oplus|\mathcal{C}|}$ if d is even, while it has the structure of an $\mathbb{F}_2$-vector space of dimension no bigger than $|\mathcal{C}|$ if d is odd ([**23**, Theorem 1.2]).

In view of this, we raise the following three-fold direction for future work.

1. In **[25]**, for the case when the dimension d is even, we assumed that the base field k is perfect and $\mathrm{char}(k) \neq 2$ with $\mathrm{c.d.}_2(k) \leqslant 2$. We will try to relax that assumption.

   Also, in the above case in **[25]**, we assumed that the projective A-modules we consider have trivial determinant. We will try to weaken this assumption also.

(2) We will try to generalize the enumeration result of Das-Tikader-Zinna [**23**] to all projective A-modules (A a smooth affine $\mathbb{R}$-algebra of dimension d), at least for those with trivial determinant.

Our strategy is as follows. As mentioned above, [**47**] gives nice enumeration results on topological vector bundles over CW-complexes of top rank, and the work [**25**] gives parallel enumeration results on algebraic vector bundles over smooth affine algebras of top rank. We will proceed by comparing algebraic vector bundles and topological vector bundles (via the *real realization* functor from the motivic homotopy category to the classical homotopy category). Diagrammatically, we will study the relationship between the two rows of the commutative square

$$\begin{array}{ccc} \mathcal{V}_d(X) & \longrightarrow & \widetilde{\mathrm{K}}_0(X) \\ \downarrow & & \downarrow \\ \mathcal{V}_d^{\mathrm{top}}(X) & \longrightarrow & \widetilde{\mathrm{K}}_0^{\mathrm{top}}(X) \end{array}$$

where $\mathcal{V}_d(X)$ (resp. $\mathcal{V}_d^{\mathrm{top}}(X)$) denotes the set of isomorphism classes of rank d algebraic (resp. topological) vector bundles over X (resp. $X(\mathbb{R})$), and $\widetilde{\mathrm{K}}_0(X)$ (resp. $\widetilde{\mathrm{K}}_0^{\mathrm{top}}(X)$) denotes the group of stably equivalent classes of algebraic (resp. topological) vector bundles over X (resp. $X(\mathbb{R})$).

In fact, [**29**] already gives nice results on passing from the algebraic realm to the topological one, though further work need to be done for our purpose. In light of results of Das-Tikader-Zinna [**23**] and Fasel [**29**], we believe that, the enumeration of real algebraic vector bundles (of top rank) is essentially a topological problem.

More preferably, we may try to reprove the main enumeration result of [**23**] using motivic methods, without stably free assumption from the beginning.

(3) More ambitiously, we will try to eliminate the condition in the cancellation result of [**25**], in the case of $k = \bar{k}$, for rank $d-1$ algebraic vector bundles. Precisely, we will try to prove in that case that

$$\mathrm{H}^d(X; \pi_d^{\mathbb{A}^1}(\mathbb{A}^d \setminus 0)) = 0,$$

which is (apparently from the argument in [**25**]) enough to confirm cancellation for rank $d-1$ algebraic vector bundles unconditionally; or even more ambitiously, the Asok-Fasel Conjecture 4.3.6: if k be a perfect field with $\mathrm{char}(k) \neq 2$, then the sequence

$$\mathbf{K}^{\mathrm{M}}_{d+2}/24 \to \pi_d^{\mathbb{A}^1}(\mathbb{A}^d \setminus 0) \to \mathbf{GW}^d_{d+1} \to 0 \tag{4.13}$$

becomes exact after d-fold contractions.

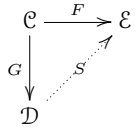
Appendix A

Kan extensions

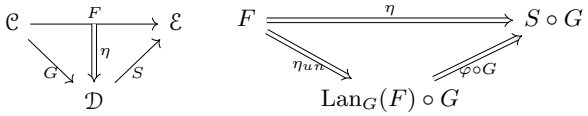
In this appendix, we collect some results on Kan extensions, part of the abstract nonsense. The slogan is that a Kan extension is very effective in creating adjunctions and is very rewarding, which should be apparent from the examples we give; in fact, there are many more, from algebraic geometry and abstract homotopy theory, and presumably, everything related to category theory, after all "All concepts are Kan Extensions".

A.1. Generalities

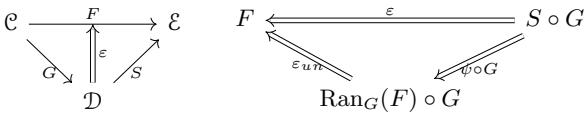
We fix categories $\mathcal{C}, \mathcal{D}, \mathcal{E}$ and functors $F : \mathcal{C} \to \mathcal{E}, G : \mathcal{C} \to \mathcal{D}$. We want to "extend" F to a functor $S : \mathcal{D} \to \mathcal{E}$ (by imagining G as an embedding). In general, this is not possible. And the notion of Kan extensions is the "best possible approximating solution".



Definition A.1.1. A **left Kan extension** of F along G is a functor $\mathrm{Lan}_G(F) : \mathcal{D} \to \mathcal{E}$, together with a universal natural transformation $\eta_{un} : F \Rightarrow \mathrm{Lan}_G(F) \circ G$ such that for any other functor $S : \mathcal{D} \to \mathcal{E}$ and natural transformation $\eta : F \Rightarrow S \circ G$, there exists a unique natural transformation $\varphi : \mathrm{Lan}_G(F) \Rightarrow S$ such that $\eta = (\varphi \circ G) \circ \eta_{un}$.



Dually, a **right Kan extension** of F along G is a functor $\mathrm{Ran}_G(F) : \mathcal{D} \to \mathcal{E}$, together with a universal natural transformation $\varepsilon_{un} : \mathrm{Ran}_G(F) \circ G \Rightarrow F$ such that for any other functor $S : \mathcal{D} \to \mathcal{E}$ and natural transformation $\varepsilon : S \circ G \Rightarrow F$, there exists a unique natural transformation $\psi : S \Rightarrow \mathrm{Ran}_G(F)$ such that $\varepsilon = \varepsilon_{un} \circ (\psi \circ G)$.



If $\mathrm{Lan}_G(F)$ exists for every functor $F : \mathcal{C} \to \mathcal{E}$, then we have an adjoint pair

$$\mathrm{Lan}_G : \mathcal{E}^{\mathcal{C}} \rightleftarrows \mathcal{E}^{\mathcal{D}} : G^*,$$

where $G^*(S) = S \circ G$. Precisely, there is a natural isomorphism

$$\mathcal{E}^{\mathcal{D}}(\mathrm{Lan}_G(F), S) \cong \mathcal{E}^{\mathcal{C}}(F, S \circ G).$$

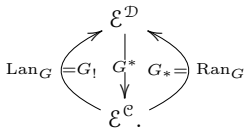
Similarly, if $\mathrm{Ran}_G(F)$ exists for every functor $F : \mathcal{C} \to \mathcal{E}$, then we have an adjoint pair

$$G^* : \mathcal{E}^{\mathcal{D}} \rightleftarrows \mathcal{E}^{\mathcal{C}} : \mathrm{Ran}_G.$$

Precisely, there is a natural isomorphism

$$\mathcal{E}^{\mathcal{C}}(S \circ G, F) \cong \mathcal{E}^{\mathcal{D}}(S, \mathrm{Ran}_G(F)).$$

Schematically, in these cases we have the following adjoint pairs:



This is a fertile source of adjoint pairs, as can be seen from the following examples.

Example A.1.2 (Derived functors)**.** *Consider a model category $\mathcal{M}$ with the canonical (localization) functor to its homotopy category $\gamma_{\mathcal{M}} : \mathcal{M} \to \mathrm{Ho}(\mathcal{M})$. For any functor $F : \mathcal{M} \to \mathcal{N}$, its* left derived functor $\mathbf{L}F : \mathrm{Ho}(\mathcal{M}) \to \mathcal{N}$ *is defined as the* right *Kan extension* $\mathrm{Ran}_{\gamma_{\mathcal{M}}}(F)$*. Thus it's equipped with a universal natural transformation $\varepsilon_{un} : \mathbf{L}F \circ \gamma_{\mathcal{M}} \Rightarrow F$.*

Similarly, its right derived functor $\mathbf{R}F : \mathrm{Ho}(\mathcal{M}) \to \mathcal{N}$ *is defined as the* left *Kan extension* $\mathrm{Lan}_{\gamma_{\mathcal{M}}}(F)$*. It's equipped with a universal natural transformation $\eta_{un} : F \Rightarrow \mathbf{R}F \circ \gamma_{\mathcal{M}}$.*

When $\mathcal{N}$ is also a model category with the localization functor $\gamma_{\mathcal{N}} : \mathcal{N} \to \mathrm{Ho}(\mathcal{N})$, the (total) left derived functor $\mathbf{L}F : \mathrm{Ho}(\mathcal{M}) \to \mathrm{Ho}(\mathcal{N})$ *is defined as the* right *Kan extension* $\mathrm{Ran}_{\gamma_{\mathcal{M}}}(\gamma_{\mathcal{N}} \circ F)$*. Thus it's equipped with a universal natural*

transformation $\varepsilon_{un} : \mathbf{L}F \circ \gamma_{\mathcal{M}} \Rightarrow \gamma_{\mathcal{N}} \circ F$. *And the* (total) right derived functor $\mathbf{R}F : \mathrm{Ho}(\mathcal{M}) \to \mathrm{Ho}(\mathcal{N})$ *is defined as the* left *Kan extension* $\mathrm{Lan}_{\gamma_{\mathcal{M}}}(\gamma_{\mathcal{N}} \circ F)$. *It's equipped with a universal natural transformation* $\eta_{un} : \gamma_{\mathcal{N}} \circ F \Rightarrow \mathbf{R}F \circ \gamma_{\mathcal{M}}$.

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{F} & \mathcal{N} \\ {\scriptstyle \gamma_{\mathcal{M}}}\downarrow & & \downarrow{\scriptstyle \gamma_{\mathcal{N}}} \\ \mathrm{Ho}(\mathcal{M}) & \underset{\mathbf{R}F}{\xrightarrow{\mathbf{L}F}} & \mathrm{Ho}(\mathcal{N}) \end{array}$$

For more about derived functors, see [**74**, Definition 2.1.17] *and* [**40**, §8.4].

A.2. Examples and some useful facts

In this section, we present some interesting examples and some useful facts about Kan extensions.

Example A.2.1 (Representations (co)induced by representations of subgroups)**.** *Let G be a (discrete) group, we can view it as a groupoid with one object $*$, then for a field k, a k-representation of the group G is the same thing as a functor $G \to \mathcal{V}\mathsf{ect}_k$, thus $\mathcal{R}\mathsf{ep}_k(G) = (\mathcal{V}\mathsf{ect}_k)^G$.*

Let H be a subgroup of G, viewed as a subcategory of G. Then the embedding $H \hookrightarrow G$ induces the restriction functor $\mathrm{res}^G_H : \mathcal{R}\mathsf{ep}_k(G) \to \mathcal{R}\mathsf{ep}_k(H)$ sending a G-module V to V itself viewed as an H-module. In this case Kan extensions fit into the following adjoint pairs

$$\begin{array}{ccc} & \mathcal{R}\mathsf{ep}_k(G) & \\ {\scriptstyle \mathrm{Ind}^G_H}\swarrow & \downarrow{\scriptstyle \mathrm{res}^G_H} & \searrow{\scriptstyle \mathrm{Coind}^G_H} \\ & \mathcal{R}\mathsf{ep}_k(H), & \end{array}$$

where for an H-module V, $\mathrm{Ind}^G_H(V) = k[G] \otimes_{k[H]} V, \mathrm{Coind}^G_H(V) = \mathrm{Hom}_H(k[G], V)$.

Example A.2.2 (Geometric realization)**.** *Consider the standard cosimplicial space $|\Delta^\bullet| : \mathbf{\Delta} \to \mathcal{T}\mathsf{op}$, then the geometric realization $|\ \ | : \mathcal{S}\mathsf{set} \to \mathcal{T}\mathsf{op}$ (see Section 1.5) is the left Kan extension of $|\Delta^\bullet|$ along the Yoneda embedding $\mathbf{\Delta} \hookrightarrow \mathcal{S}\mathsf{set}$.*

$$\begin{array}{ccc} \mathbf{\Delta} & \xrightarrow{|\Delta^\bullet|} & \mathcal{T}\mathsf{op} \\ \downarrow & \nearrow{\scriptstyle |\ \ |} & \\ \mathcal{S}\mathsf{set} & & \end{array}$$

Now consider the *left comma category* $(F \downarrow d)$ for an object d in $\mathcal{D}$, whose objects are pairs (c, f) with c an object in $\mathcal{C}$ and $f : F(c) \to d$ a morphism in $\mathcal{D}$, and whose morphisms are the obvious commutative triangles (i.e. a morphism $h : c \to c'$ such that $f = f' \circ F(h)$). We have the functor $j_d : (F \downarrow d) \to \mathcal{C}, (c, f) \mapsto c$, and the natural transformation $\eta : F \circ j_d \Rightarrow \mathrm{const}_d$ with $\eta_{(c,f)} = f : F(c) \to d$, as in the following diagram:

$$\begin{array}{ccc} (F \downarrow d) & \xrightarrow{j_d} & \mathcal{C} \\ \downarrow & \overset{\eta}{\swarrow} & \downarrow{\scriptstyle F} \\ * & \xrightarrow[d]{\mathrm{const}_d} & \mathcal{D}. \end{array}$$

Varying d in $\mathcal{D}$ we get the *global (left) comma category* $(F \downarrow \mathcal{D}) = (F \downarrow 1_{\mathcal{D}})$, with objects triples (c, d, f), c an object in $\mathcal{C}$, d an object in $\mathcal{D}$ and $f : F(c) \to d$ a morphism in $\mathcal{D}$, and whose morphisms are the obvious commutative squares (i.e. a pair of morphisms (h, g) with $h : c \to c'$ in $\mathcal{C}$ and $g : d \to d'$ in $\mathcal{D}$ such that $g \circ f = f' \circ F(h)$). There is the functor $j : (F \downarrow \mathcal{D}) \to \mathcal{C} \times \mathcal{D}, (c, d, f) \mapsto (c, d)$.

The global comma category $(F \downarrow \mathcal{D})$ can be identified with the *category of elements* of the functor $\overline{F} : \mathcal{C}^{\mathrm{op}} \times \mathcal{D} \to \mathcal{S}\mathsf{et}, (c, d) \mapsto \mathcal{D}(F(c), d)$. [1]

Dually, there are the *right comma category* $(d \downarrow F)$ for an object d in $\mathcal{D}$ and the *global (right) comma category* $(\mathcal{D} \downarrow F) = (1_{\mathcal{D}} \downarrow F)$.

For d an object in $\mathcal{D}$ we have the functor $j_d : (G \downarrow d) \to \mathcal{C}, (c, f) \mapsto c$. Thus we get a $(G \downarrow d)$-diagram $F \circ j_d$ in $\mathcal{E}$.

Theorem A.2.3. *If the colimit $\operatorname{colim}_{(G\downarrow d)}(F \circ j_d)$ exists for every object d in $\mathcal{D}$ (in particular if $\mathcal{C}$ is small and $\mathcal{E}$ is cocomplete), then the left Kan extension $\mathrm{Lan}_G(F) : \mathcal{D} \to \mathcal{E}$ exists, and there is a canonical isomorphism*

$$\mathrm{Lan}_G(F)(d) \cong \operatorname*{colim}_{(G\downarrow d)}(F \circ j_d).$$

If moreover the functor $G : \mathcal{C} \to \mathcal{D}$ is fully faithful (i.e. a full embedding), then the universal natural transformation $\eta_{un} : F \Rightarrow \mathrm{Lan}_G(F) \circ G$ is an isomorphism of functors.

Theorem A.2.4. *If the limit $\lim_{(d\downarrow G)}(F \circ j_d)$ exists for every object d in $\mathcal{D}$ (in particular if $\mathcal{C}$ is small and $\mathcal{E}$ is complete), then the right Kan extension $\mathrm{Ran}_G(F) : \mathcal{D} \to \mathcal{E}$ exists, and there is a canonical isomorphism*

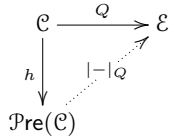
$$\mathrm{Ran}_G(F)(d) \cong \lim_{(d\downarrow G)}(F \circ j_d).$$

[1] For the notion of category of elements, see the nLab article *category of elements*. For more about the related constructions, see the wikipedia article *Comma category*.

If moreover the functor $G : \mathcal{C} \to \mathcal{D}$ is fully faithful (i.e. a full embedding), then the universal natural transformation $\varepsilon_{un} : \mathrm{Ran}_G(F) \circ G \Rightarrow F$ is an isomorphism of functors (it's in fact the counit for the adjunction $G^* \dashv \mathrm{Ran}_G$, see [**60**, Chapter X, §3, Corollary 3]*).*

Remark A.2.5. The above results say that Kan extensions along full embeddings are genuine extensions. Note however that not all Kan extensions can be expressed as above, since the codomain category may be not (co)complete, e.g. the homotopy category $\mathrm{Ho}(\mathcal{M})$ in Example A.1.2, which has notoriously few limits and colimits (see [**74**, Remark 1.3.6]). For more on these results, see [**74**, Chapter 1] or [**60**, Chapter X].

Example A.2.6 (Yoneda extensions)**.** *Let $\mathcal{C}$ be a small category, $\mathcal{E}$ a cocomplete category, $Q : \mathcal{C} \to \mathcal{E}$ a functor. Let $h : \mathcal{C} \to \mathcal{P}\mathsf{re}(\mathcal{C})$ be the Yoneda embedding (see Example 2.1.1). Let $|-|_Q = \mathrm{Lan}_h Q : \mathcal{P}\mathsf{re}(\mathcal{C}) \to \mathcal{E}$ be the left Kan extension (computed via colimits), it preserves colimits.*



Since the Yoneda embedding is fully faithful, we have $|h_V|_Q = Q(V)$ for any object $V \in \mathcal{C}$. Since for any $X \in \mathcal{P}\mathsf{re}(\mathcal{C})$ we have $X \cong \operatorname{colim}_{(a_V : h_V \Rightarrow X) \in \mathcal{C}_X} h_V$ (see Proposition 2.1.6), we get $|X|_Q \cong \operatorname{colim}_{(a_V : h_V \Rightarrow X) \in \mathcal{C}_X} Q(V)$.

We define a functor $\mathrm{Sing}_Q : \mathcal{E} \to \mathcal{P}\mathsf{re}(\mathcal{C})$ by setting $\mathrm{Sing}_Q(e) = h_e \circ Q$, i.e. $(\mathrm{Sing}_Q e)(U) = \mathcal{E}(Q(U), e)$ for any objects $U \in \mathcal{C}, e \in \mathcal{E}$. They form an adjunction

$$|-|_Q : \mathcal{P}\mathsf{re}(\mathcal{C}) \rightleftarrows \mathcal{E} : \mathrm{Sing}_Q.$$

Indeed, for any $X \in \mathcal{P}\mathsf{re}(\mathcal{C})$ and $e \in \mathcal{E}$, we have the following natural isomorphisms in $\mathcal{S}\mathsf{et}$:

$$\begin{aligned}\mathcal{P}\mathsf{re}(\mathcal{C})(X, \mathrm{Sing}_Q(e)) \cong & \mathcal{P}\mathsf{re}(\mathcal{C})(\operatorname{colim} h_V, \mathrm{Sing}_Q(e)) \cong \lim \mathcal{P}\mathsf{re}(\mathcal{C})(h_V, \mathrm{Sing}_Q(e)) \cong \lim \mathrm{Sing}_Q(e)(V) \\ \cong & \lim \mathcal{E}(Q(V), e) \cong \mathcal{E}(\operatorname{colim} Q(V), e) \cong \mathcal{E}(|X|_Q, e).\end{aligned}$$

This generalizes the construction of geometric realization above (with $\mathcal{C} = \mathbf{\Delta}, \mathcal{E} = \mathcal{T}\mathsf{op}$).

Even more is true: the above construction sending a functor $Q \in \mathcal{E}^{\mathcal{C}}$ to the adjoint pair $(|-|_Q, \mathrm{Sing}_Q)$ defines an equivalence of categories[2]

$$\mathcal{E}^{\mathcal{C}} \xrightarrow{\simeq} \mathrm{Adj}(\mathcal{P}\mathsf{re}(\mathcal{C}), \mathcal{E}).$$

This construction is sometimes refered to as Yoneda extensions *(see* [**35**]*, Digression 1.8).*

Replacing $\mathcal{E}$ by $\mathcal{E}^{\mathrm{op}}$ and passing to the opposite categories, we get an equivalence

$$\mathcal{E}^{\mathcal{C}^{\mathrm{op}}} \xrightarrow{\simeq} \mathrm{Adj}(\mathcal{E}, \mathcal{P}\mathsf{re}(\mathcal{C})^{\mathrm{op}}).$$

In particular, for $A^\bullet \in \mathsf{c}\mathcal{E} = \mathcal{E}^{\mathbf{\Delta}}$ and $Y_\bullet \in \mathsf{s}\mathcal{E} = \mathcal{E}^{\mathbf{\Delta}^{\mathrm{op}}}$, we get the adjunctions

$$A^\bullet \otimes - : \mathsf{s}\mathcal{S}\mathsf{et} \rightleftarrows \mathcal{E} : \mathcal{E}(A^\bullet, -)$$

and

$$(Y_\bullet^-)^{\mathrm{op}} = \mathrm{Hom}(-, Y_\bullet)^{\mathrm{op}} : \mathsf{s}\mathcal{S}\mathsf{et} \rightleftarrows \mathcal{E}^{\mathrm{op}} : \mathcal{E}(-, Y_\bullet).$$

If $\mathcal{E}$ is pointed and bicomplete, then we have similar equivalences and we denote the adjunctions by

$$A^\bullet \wedge - : \mathsf{s}\mathcal{S}\mathsf{et}_* \rightleftarrows \mathcal{E} : \mathcal{E}(A^\bullet, -)$$

and

$$(Y_\bullet^-)^{\mathrm{op}} = \mathrm{Hom}_*(-, Y_\bullet)^{\mathrm{op}} : \mathsf{s}\mathcal{S}\mathsf{et}_* \rightleftarrows \mathcal{E}^{\mathrm{op}} : \mathcal{E}(-, Y_\bullet).$$

In this case, we have a natural isomorpism $A^\bullet \wedge K_+ \cong A^\bullet \otimes K$ (on the right hand side, $\otimes$ denote the operation in the unpointed case as above). See [**45**, Corollary 3.1.6].

See also [**57**, Example 2.2.2.4, 2.2.2.5] *for adjunctions obtained in this way, which have important applications in the theory of ∞-categories.*

Example A.2.7 (Evaluation has two-sides adjoints)**.** *Let $\mathcal{I}, \mathcal{C}$ be small categories, and let $\mathcal{E}$ be a bicomplete category, then for any fixed object c in $\mathcal{C}$, we have the functor $G_c : \mathcal{I} \to \mathcal{I} \times \mathcal{C}, i \mapsto (i, c)$. The functor $\Gamma_c = G_c^* : \mathcal{E}^{\mathcal{I} \times \mathcal{C}} \to \mathcal{E}^{\mathcal{I}}$ is the "evaluation" at c. By the above results, it has a left adjoint L_c and a right adjoint R_c, hence the functor Γ_c preserves all limits and all colimits (evaluation preserves all (co)limits). Spelling this out, we have:*

Let $\mathcal{Y} : \mathcal{I} \to \mathcal{E}^{\mathcal{C}}$ be a diagram in $\mathcal{E}^{\mathcal{C}}$ indexed by a small category $\mathcal{I}$, and write $\mathcal{Y}(c) : \mathcal{I} \to \mathcal{E}$ for the induced $\mathcal{I}$-diagram in $\mathcal{E}$ for every object c in $\mathcal{C}$. Then the colimit $\operatorname{colim}_{\mathcal{I}} \mathcal{Y}$ is pointwise, i.e., for $Y \in \mathcal{E}^{\mathcal{C}}$, and given (compatible) maps $\mathcal{Y}_i \to Y$ in $\mathcal{E}^{\mathcal{C}}$ for every $i \in \mathcal{I}$, the induced map $\operatorname{colim}_{\mathcal{I}} \mathcal{Y} \xrightarrow{\cong} Y$ is an isomorphism if and only if the induced map $\operatorname{colim}_{\mathcal{I}} \mathcal{Y}(c) \xrightarrow{\cong} Y(c)$ is an isomorphism for every object c in $\mathcal{C}$. Briefly we have

$$\boxed{\operatorname{colim}_{\mathcal{I}}(\mathcal{Y}(c)) \cong (\operatorname{colim}_{\mathcal{I}} \mathcal{Y})(c), \lim_{\mathcal{I}}(\mathcal{Y}(c)) \cong (\lim_{\mathcal{I}} \mathcal{Y})(c).}$$

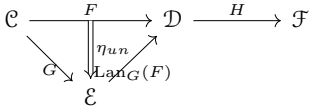
[2] There will be set-theoretic issue (both sides won't be locally small categories in general) which we ignore, since one can make the stated equivalence into a 1-1 correspondence, so if $\mathcal{C}$ is small, then $\mathcal{E}^{\mathcal{C}}$ is locally small and hence so is the right side.

This in particular implies that in the presheaf category, (co)limits are formed sectionwise. The functor Γ_c is exact (on the diagram category, note however that in the category of sheaves, the functor taking sections at some object is not exact). If $\mathcal{E}$ is a (co)complete category, then so is the functor (diagram) category $\mathcal{E}^{\mathcal{C}}$. Moreover, we have

$$L_c X = X \otimes h^c : (i,d) \mapsto \coprod_{\mathcal{C}(c,d)} X_i,$$

$$R_c X = X^{h_c} : (i,d) \mapsto \prod_{\mathcal{C}(d,c)} X_i.$$

Definition A.2.8. A functor $H : \mathcal{D} \to \mathcal{F}$ *preserves* the left Kan extension $(\mathrm{Lan}_G(F), \eta_{un})$ if the pair $(H \circ \mathrm{Lan}_G(F), H \circ \eta_{un})$ is the left Kan extension of the functor $H \circ F$ along G.



Theorem A.2.9 ([**74**, Lemma 1.3.3])**.** *Any left adjoint preserves left Kan extensions.*

Appendix B

Étale sites and étale cohomology

In this appendix, we briefly present some basic facts about étale cohomology. We intend to be minimal rather than to be complete, so only those are useful for our purpose in the text are presented.

B.1. A leisurely excursion to étale sites

Let R be a commutative ring and A be a commutative R-algebra given by a ring homomorphism $\varphi : R \to A$, let $\mathfrak{p} \in \mathrm{Spec}(A), \mathfrak{q} = \varphi^{-1}(\mathfrak{p}) \in \mathrm{Spec}(R)$. We say that $\mathfrak{p}$ *is unramified over* R or A/R is *unramified at* $\mathfrak{p}$ if $\mathfrak{p}A_{\mathfrak{p}} = \mathfrak{q}A_{\mathfrak{p}}$ and the field extension $\kappa(\mathfrak{p})/\kappa(\mathfrak{q})$ is separable algebraic. Otherwise, we say that A/R is *ramified at* $\mathfrak{p}$. If A/R is unramified at $\mathfrak{p}$ and $A_{\mathfrak{p}}/R_{\mathfrak{q}}$ is flat, we say that A/R is *étale at* $\mathfrak{p}$. If A/R is unramified (étale) at $\mathfrak{p}$ for every $\mathfrak{p} \in \mathrm{Spec}(A)$, we say A/R is an *unramified* (*étale*) algebra. Let $V_{A/R} \subset \mathrm{Spec}(A)$ be the set of those $\mathfrak{p} \in \mathrm{Spec}(A)$ at which A/R is ramified, call it the *ramification locus* of A/R. The set of those $\mathfrak{p} \in \mathrm{Spec}(A)$ at which A/R is étale is called the *étale locus* of A/R.

If A/R is *essentially of finite type* (i.e., A is a localization of a finitely-generated R algebra), then $V_{A/R} = \mathrm{supp}(\Omega^1_{A/R})$ (is closed in $\mathrm{Spec}(A)$); A/R is unramified iff $\Omega^1_{A/R} = 0$. If R is noetherian, then the étale locus of A/R is open in $\mathrm{Spec}(A)$. See [**54**, Corollary 6.10].

More generally, let $f : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ be a morphism of schemes and let $x \in X, y = f(x) \in Y$. We say that f *is unramified at* x if there exists an affine open neighbourhood $U = \mathrm{Spec}(A)$ of x in X and affine open $V = \mathrm{Spec}(R)$ of y in Y with $f(U) \subset V$ such that the induced ring map $R \to A$ is unramified. See *the Stacks project, tag/02G3* for a discussion on unramified morphisms of schemes.

We say that a morphism of schemes $f : X \to Y$ is *étale* at a point $x \in X$ if f is unramified at $x \in X$, locally of finite presentation at $x \in X$ and is flat (smooth) at $x \in X$. We say that $f : X \to Y$ is an *étale morphism* if it is étale at every point $x \in X$. A morphism $f : X \to Y$ is étale iff it is smooth of relative dimension 0, iff it is flat, locally of finite presentation and $\Omega^1_{X/Y} = 0$. Étale morphisms are closed under composites and base change but are not necessarily separated (the morphism from the affine line with a double origin to the affine line (sending both origins to the same point—the origin of the target) is a non-separated étale morphism); if $f : X \to Y$ is an étale morphism of schemes and Y is reduced (resp. regular) then so is X (*the Stacks project, tag/03PA*). See *the Stacks project, tag/02GH* for more discussions on étale morphisms of schemes.

The *small étale site* $X_{\text{ét}}$ is the site of étale X-schemes, with covers given by surjective families of X-morphisms (any X-morphism in $X_{\text{ét}}$ is necessarily étale). The small étale site $X_{\text{ét}}$ is subcanonical; in fact, for any X-scheme U, we have $U_X := (\mathcal{Sch}/X)(-, U) \in \mathcal{Shv}(X_{\text{ét}})$ ([**88**, Chapter II, Theorem (3.1.2)]) but different X-schemes may represent the same sheaf on $X_{\text{ét}}$: as an easy counter-example, let k be a field and consider $X := \mathbb{A}^1_k = \mathrm{Spec}(k[t]), Z := V(t) = \{0\} \hookrightarrow X$, then both Z and the empty scheme $\varnothing$ represent the empty sheaf $\varnothing$ on the small étale site $X_{\text{ét}}$ (since any étale morphism $U \to X$ has an open image hence cannot factor through $Z = \{0\} \hookrightarrow X$). If one wants to have a non-empty sheaf, just observe by the above that both $Z \coprod X$ and the scheme X represent the final sheaf $*$ on $X_{\text{ét}}$.

If $F \in \mathcal{Shv}(X_{\text{ét}})$ is represented by $U \in X_{\text{ét}}$, then for any morphism of schemes $f : Y \to X$, $f^*F \in \mathcal{Shv}(Y_{\text{ét}})$ is represented by $U \times_X Y \in Y_{\text{ét}}$ ([**88**, Chapter II, Proposition (3.1.3)]). If $\mathscr{M} \in \mathcal{Qcoh}(X)$ is a quasi-coherent $\mathscr{O}_X$-module, we can associate an étale sheaf $\mathscr{M}_{\text{ét}} = W(\mathscr{M}) \in \mathcal{Shv}\mathcal{Ab}(X_{\text{ét}})$ by $\mathscr{M}_{\text{ét}}(V \xrightarrow{\varphi} X) = \Gamma(V, \varphi^{-1}\mathscr{M} \otimes_{\varphi^{-1}\mathscr{O}_X} \mathscr{O}_V)$. The functor

$$\mathcal{Qcoh}(X) \to \mathcal{Shv}\mathcal{Ab}(X_{\text{ét}}), \mathscr{M} \mapsto \mathscr{M}_{\text{ét}}$$

is additive and left exact. We have $(\mathscr{O}_X)_{\text{ét}} = \mathbb{G}_{a,X}$. If X is an S-scheme, then $(\Omega^1_{X/S})_{\text{ét}}(U) = \Gamma(U, \Omega^1_{U/S})$ ([**67**, Chapter II, Proposition 1.3]). For any abelian group A, we associate the constant sheaf $\underline{A}_X$ with $\underline{A}_X(V) = A^{\pi_0(V)}$ ($\pi_0(V)$ is the set of connected components of $V \in X_{\text{ét}}$); it's represented by the étale group scheme $\coprod_A X$ (using that V is the coproduct of the schemes in $\pi_0(V)$). We have the adjunction

$$(\underline{\ })_X : \mathcal{Ab} \rightleftarrows \mathcal{Shv}\mathcal{Ab}(X_{\text{ét}}) : \Gamma(X, -).$$

Clearly, for $M \in \mathcal{Shv}\mathcal{Ab}(X_{\text{ét}})$, we have $\mathcal{Shv}\mathcal{Ab}(X_{\text{ét}})(\underline{(\mathbb{Z}/n)}_X, M) \cong {}_nM(X) = \ker(M(X) \xrightarrow{n} M(X))$.

There are also the multiplicative group $\mathbb{G}_{m,X} \in \mathcal{Shv}\mathcal{Ab}(X_{\text{ét}})$ and its subsheaf of n-th roots of unity $\mu_{n,X} \in \mathcal{Shv}\mathcal{Ab}(X_{\text{ét}})$ given by $\mathbb{G}_{m,X}(U) = \mathscr{O}_U(U)^\times, \mu_{n,X}(U) = \{s \in \mathscr{O}_U(U) : s^n = 1\}$. They fit into an exact sequence

$$0 \to \mu_{n,X} \to \mathbb{G}_{m,X} \xrightarrow{n} \mathbb{G}_{m,X}.$$

The sheaves $\underline{(\mathbb{Z}/n)}_X$ and $\mu_{n,X} \in \mathcal{Shv}\mathcal{Ab}(X_{\text{ét}})$ are (étale) locally isomorphic if n is different from the characteristics of all residue fields of all points in X; they are isomorphic iff $\mathscr{O}_X(X)^\times$ contains a *primitive* n-th roots of unity ([**88**, Chapter II, (3.1.4)]).

B.2. Étale cohomology

Using the formalism of Section 2.9, for any $U \in X_{\text{ét}}, A \in \mathcal{Shv}\mathcal{Ab}(X_{\text{ét}}), n \in \mathbb{N}$, we have the *étale cohomology groups* $\mathrm{H}^n_{\text{ét}}(U; A)$.

If the scheme X has characteristic p (a prime number): $p \cdot \mathscr{O}_{X,x} = 0, \forall x \in X$, we have an injection $\underline{(\mathbb{Z}/p)}_X \hookrightarrow \mathbb{G}_{a,X}$. We also have the *Frobenius morphism* $\mathrm{Frob} : \mathbb{G}_{a,X} \to \mathbb{G}_{a,X}, s \mapsto s^p$. We have the exact *Artin-Schreier sequence*

$$0 \to \underline{(\mathbb{Z}/p)}_X \to \mathbb{G}_{a,X} \xrightarrow{\mathrm{Id}-\mathrm{Frob}} \mathbb{G}_{a,X} \to 0.$$

in $\mathcal{Shv}\mathcal{Ab}(X_{\text{ét}})$ ([**88**, Chapter II, Theorem (4.2.1)]). The associated long exact sequence gives a short exact sequence

$$0 \to \mathscr{O}_X(X)/(\mathrm{Id} - \mathrm{Frob})\mathscr{O}_X(X) \to \mathrm{H}^1_{\text{ét}}(X; \mathbb{Z}/p) \to \mathrm{H}^1_{\text{ét}}(X; \mathbb{G}_{a,X})^{\mathrm{Frob}} \to 0.$$

If X is a reduced proper scheme over a separably closed field k of characteristic p, then

$$\mathrm{H}^1_{\text{ét}}(X; \mathbb{Z}/p) \cong \mathrm{H}^1_{\text{ét}}(X; \mathbb{G}_{a,X})^{\mathrm{Frob}}.$$

If $X = \mathrm{Spec}(R)$ is affine of characteristic p, then

$$\mathrm{H}^i_{\text{ét}}(X; \mathbb{Z}/p) = \begin{cases} R/(\mathrm{Id} - \mathrm{Frob})R, & i = 1, \\ 0, & i > 1. \end{cases}$$

We also denote by X_{Zar} the *small Zariski site* of X. There is an (obvious) site morphism

$$\iota : X_{\mathrm{Zar}} \to X_{\text{ét}},$$

giving adjunctions

$$\iota^* : \mathcal{Shv}(X_{\mathrm{Zar}}) \rightleftarrows \mathcal{Shv}(X_{\text{ét}}) : \iota_*, \iota^* : \mathcal{Shv}\mathcal{Ab}(X_{\mathrm{Zar}}) \rightleftarrows \mathcal{Shv}\mathcal{Ab}(X_{\text{ét}}) : \iota_*.$$

We have the biregular *Leray spectral sequence* (or *Bloch-Ogus spectral sequence*) in $\mathcal{Ab}$

$$\mathrm{E}_2^{ij} = \mathrm{H}^i_{\mathrm{Zar}}(X; \mathbf{R}^j\iota_*(A)) \Longrightarrow \mathrm{H}^{i+j}_{\text{ét}}(X; A), \tag{B.1}$$

functorial in $A \in \mathcal{Shv}\mathcal{Ab}(X_{\text{ét}})$.

For $\mathscr{M} \in \mathcal{Qcoh}(X)$, we have $\iota_*(\mathscr{M}_{\text{ét}}) = \mathscr{M}$. The edge homomorphisms $\mathrm{H}^i_{\mathrm{Zar}}(X; \mathscr{M}) \to \mathrm{H}^i_{\text{ét}}(X; \mathscr{M}_{\text{ét}})$ for the spectral sequence in eq. (B.1) are isomorphisms. So if X is affine, then $\mathrm{H}^i_{\text{ét}}(X; \mathscr{M}_{\text{ét}}) = 0$ for $i > 0$ ([**88**, Chapter II, Theorem (4.1.2)]).

We have $\mathbf{R}^1(\iota_*\mathbb{G}_{m,X}) = 0$ (see [**88**, Chapter II, Theorem (4.3.1) and Lemma (4.3.2)]), which implies *Hilbert's theorem 90*:

$$\mathrm{H}^1_{\text{ét}}(X; \mathbb{G}_{m,X}) \cong \mathrm{Pic}(X).$$

If $n \in \mathbb{Z}$ is invertible on X: $n \in (\mathscr{O}_{X,x})^\times, \forall x \in X$, then we have the exact *Kummer sequence*

$$0 \to \mu_{n,X} \to \mathbb{G}_{m,X} \xrightarrow{n} \mathbb{G}_{m,X} \to 0$$

in $\mathcal{Shv}\mathcal{Ab}(X_{\text{ét}})$ ([**88**, Chapter II, Theorem (4.4.1)]). The associated long exact sequence gives a short exact sequence

$$0 \to \mathscr{O}_X(X)^\times/n \to \mathrm{H}^1_{\text{ét}}(X; \mu_{n,X}) \to {}_n\mathrm{Pic(X)} \to 0.$$

If X is a reduced proper scheme over a separably closed field k and $n \in \mathbb{Z}$ is invertible on X, then

$$\mathrm{H}^1_{\text{ét}}(X; \mu_{n,X}) \cong {}_n\mathrm{Pic(X)}.$$

If $X = \mathrm{Spec}(R)$ with R a local ring, then $\mathrm{H}^1_{\text{ét}}(X; \mu_{n,X}) \cong R^\times/n$.

The *big étale site* $\acute{\mathrm{E}}\mathrm{t}(X)$ of a scheme X is the site of all X-schemes $\mathcal{Sch}/X$, with covers $\mathrm{Cov}_\tau(U)$ for $U \in \mathcal{Sch}/X$ given by surjective families $\{U_i \to U : i \in I\}$ of étale X-morphisms. We have a site morphism

$$i : X_{\text{ét}} \to \acute{\mathrm{E}}\mathrm{t}(X),$$

giving adjunctions

$$\mathrm{EXT} = i^* : \mathcal{Shv}(X_{\text{ét}}) \rightleftarrows \mathcal{Shv}(\acute{\mathrm{E}}\mathrm{t}(X)) : i_* = \mathrm{Res}, \mathrm{EXT} = i^* : \mathcal{Shv}\mathcal{Ab}(X_{\text{ét}}) \rightleftarrows \mathcal{Shv}\mathcal{Ab}(\acute{\mathrm{E}}\mathrm{t}(X)) : i_* = \mathrm{Res},$$

where for $F \in \mathcal{Shv}(X_{\text{ét}})$, the sheaf $\mathrm{EXT}(F)$ is the sheafification of the presheaf

$$\acute{\mathrm{E}}\mathrm{t}(X) \ni U \mapsto \operatorname*{colim}_{V \in (U \downarrow i)} F(V),$$

where $(U \downarrow i)$ is the *comma category* given by the pair of functors $[0] \xrightarrow{U} \acute{\mathrm{E}}\mathrm{t}(X) \xleftarrow{i} X_{\text{ét}}$. We have the following comparison result for the sheaf theory on the big and small étale sites ([**88**, Chapter II, Theorem (3.3.1)], with the fact that $(X_{\text{ét}} \downarrow U) = U_{\text{ét}}, (\acute{\mathrm{E}}\mathrm{t}(X) \downarrow U) = \acute{\mathrm{E}}\mathrm{t}(U)$ for $U \in X_{\text{ét}}$), which explains the focus on the small étale site.

Proposition B.2.1. *For any $F \in \mathcal{Shv}(X_{\text{ét}})$, the unit $F \to \mathrm{Res}(\mathrm{EXT}(F))$ of the adjunction $(\mathrm{EXT}, \mathrm{Res})$ is an isomorphism, the functor* EXT *is fully faithful, and the functor* Res *is exact.*

Moreover, for any $A \in \mathcal{Shv}\mathcal{Ab}(X_{\text{ét}}), B \in \mathcal{Shv}\mathcal{Ab}(\acute{\mathrm{E}}\mathrm{t}(X)), U \in X_{\text{ét}}$, there are canonical isomorphisms

$$\mathrm{H}^n_{\text{ét}}(X; A) \cong \mathrm{H}^n_{\acute{\mathrm{E}}\mathrm{t}(X)}(X; \mathrm{EXT}(A)), \mathrm{H}^n_{\text{ét}}(X; \mathrm{Res}(B)) \cong \mathrm{H}^n_{\acute{\mathrm{E}}\mathrm{t}(X)}(X; B); \mathrm{H}^n_{\text{ét}}(U; A) \cong \mathrm{H}^n_{\acute{\mathrm{E}}\mathrm{t}(X)}(U; \mathrm{EXT}(A)).$$

Given a morphism of schemes $f : X \to Y$, it induces a site morphism

$$f_{\text{ét}} : Y_{\text{ét}} \to X_{\text{ét}}, (V \to Y) \mapsto (V \times_Y X \to X),$$

giving adjunctions

$$f^* : \mathcal{Shv}(Y_{\text{ét}}) \rightleftarrows \mathcal{Shv}(X_{\text{ét}}) : f_*, f^* : \mathcal{Shv}\mathcal{Ab}(Y_{\text{ét}}) \rightleftarrows \mathcal{Shv}\mathcal{Ab}(X_{\text{ét}}) : f_*,$$

where $f_*(A)(V) = A(f_{\text{ét}}(V)) = A(V \times_Y X)$; the first adjunction is a geometric morphism in topos-theoretic sense. If f is an immersion, then the counit $f^* f_*(A) \to A$ is a natural isomorphism; the functor $f_* : \mathcal{Shv}\mathcal{Ab}(X_{\text{ét}}) \to \mathcal{Shv}\mathcal{Ab}(Y_{\text{ét}})$ is fully faithful ([**88**, Chapter II, Proposition (8.1.1)]).The *higher direct image* is given as follows: for $A \in \mathcal{Shv}\mathcal{Ab}(X_{\text{ét}}), \mathbf{R}^q f_*(A) \in \mathcal{Shv}\mathcal{Ab}(Y_{\text{ét}})$ is the sheaf associated to the presheaf $V \mapsto \mathrm{H}^q_{\text{ét}}(V \times_Y X; A)$. For $V \in Y_{\text{ét}}$, we have the biregular *Leray spectral sequence* in $\mathcal{Ab}$

$$\mathrm{E}_2^{ij} = \mathrm{H}^i_{\text{ét}}(V; \mathbf{R}^j f_*(A)) \Longrightarrow \mathrm{H}^{i+j}_{\text{ét}}(V \times_Y X; A), \tag{B.2}$$

functorial in $A \in \mathcal{Shv}\mathcal{Ab}(X_{\text{ét}})$. Using its edge morphisms and adjunction, we obtain, for $A \in \mathcal{Shv}\mathcal{Ab}(X_{\text{ét}}), B \in \mathcal{Shv}\mathcal{Ab}(Y_{\text{ét}})$, canonical homomorphisms

$$\begin{cases} \mathrm{H}^i_{\text{ét}}(V; f_*(A)) \to \mathrm{H}^i_{\text{ét}}(V \times_Y X; A), \\ \mathrm{H}^i_{\text{ét}}(V; B) \to \mathrm{H}^i_{\text{ét}}(V \times_Y X; f^*(B)). \end{cases} \tag{B.3}$$

If $g : Y \to Z$ is another morphism of schemes, we have the biregular *Leray spectral sequence* in $\mathcal{Shv}\mathcal{Ab}(Z_{\text{ét}})$

$$\mathrm{E}_2^{ij} = \mathbf{R}^i g_*(\mathbf{R}^j f_*(A)) \Longrightarrow \mathbf{R}^{i+j}(gf)_*(A), \tag{B.4}$$

functorial in $A \in \mathcal{Shv}\mathcal{Ab}(X_{\text{ét}})$. Using its edge morphisms, we obtain the *base change morphisms*

$$\begin{cases} \mathbf{R}^i g_*(f_*(A)) \to \mathbf{R}^i(gf)_*(A), \\ \mathbf{R}^i(gf)_*(A) \to g_*(\mathbf{R}^i f_*(A)). \end{cases} \tag{B.5}$$

Given a cartesian diagram of schemes

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ {\scriptstyle v'}\downarrow & & \downarrow{\scriptstyle v} \\ X & \xrightarrow{f} & Y, \end{array} \tag{B.6}$$

we get the composite $\mathbf{R}^i f_* \to (\mathbf{R}^i f_*)(v'_* v'^*) \to (\mathbf{R}^i(fv')_*)v'^* = (\mathbf{R}^i(vf')_*)v'^* \to v_*(\mathbf{R}^i f'_*)v'^*$, its adjoint is the *base change morphism* $v^* \mathbf{R}^i f_* \to (\mathbf{R}^i f'_*)v'^*$. If it is a natural isomorphism for all such cartesian diagrams, we say that $\mathbf{R}^i f_*$ *commutes with arbitrary base change*.

We also have the *restricted étale site* $X_{\text{fpét}}$, which is the full subcategory of finitely presented étale X-schemes (a morphism is *finitely presented* or *of finitely presentation* is it is qcqs and locally of finitely presentation). If X is quasi-compact, the restricted étale site $X_{\text{fpét}}$ is noetherian ([**88**, Chapter II, Lemma (1.5.1)]). There is a site morphism

$$i : X_{\text{fpét}} \to X_{\text{ét}},$$

giving adjunctions

$$\text{ext} = i^* : \mathcal{Shv}(X_{\text{fpét}}) \rightleftarrows \mathcal{Shv}(X_{\text{ét}}) : i_* = \text{res}, \text{ext} = i^* : \mathcal{Shv}\mathcal{Ab}(X_{\text{fpét}}) \rightleftarrows \mathcal{Shv}\mathcal{Ab}(X_{\text{ét}}) : i_* = \text{res}.$$

If the scheme X is quasi-separated, then (ext, res) are mutually quasi-inverse equivalences of categories ([**88**, Chapter II, Proposition (1.5.2)]). We then easily obtain the following result ([**88**, Chapter II, Corollary (1.5.3)]).

Proposition B.2.2. *If X is a qcqs scheme (e.g. noetherian or affine), then the functor $\mathrm{H}^n_{\text{ét}}(X; -) : \mathcal{Shv}\mathcal{Ab}(X_{\text{ét}}) \to \mathcal{Ab}$ commutes with pseudofiltered colimits and all small direct sums.*

A *geometric point of* X is a morphism $u : P = \mathrm{Spec}(\Omega) \to X$ with Ω a separably closed field. It is the same as the image point $x \in X$ of u, with an embedding $\kappa_x \to \Omega$; we then say that the geometric point P lies over x. The morphism $u : \mathrm{Spec}(\Omega) \to X$ induces a site morphism

$$u_{\text{ét}} : X_{\text{ét}} \to \mathrm{Spec}(\Omega)_{\text{ét}}, (V \to X) \mapsto (V \times_X \mathrm{Spec}(\Omega) \to \mathrm{Spec}(\Omega)),$$

giving adjunctions

$$u^* : \mathcal{Shv}(X_{\text{ét}}) \rightleftarrows \mathcal{Shv}(\mathrm{Spec}(\Omega)_{\text{ét}}) \cong \mathcal{Set} : u_*, u^* : \mathcal{Shv}\mathcal{Ab}(X_{\text{ét}}) \rightleftarrows \mathcal{Shv}\mathcal{Ab}(\mathrm{Spec}(\Omega)_{\text{ét}}) \cong \mathcal{Ab} : u_*.$$

So a geometric point of X is a point of $\mathcal{Shv}(X_{\text{ét}})$ in topos-theoretic sense, we thus have the *stalk* $A_P = A_u = u^* A$ for $A \in \mathcal{Shv}(X_{\text{ét}})$ or $\mathcal{Shv}\mathcal{Ab}(X_{\text{ét}})$.

For $U \in X_{\text{ét}}$ we have $u^* h_U = (\mathcal{Sch}/X)(\mathrm{Spec}(\Omega), U)$; we have $u^* \mu_{n,X} = \mu_n(\Omega)$; for any abelian group A, we have $u^* \underline{A}_X = A$.

If $v : P' \to P$ is an X-morphism of geometric points of X, then it induces an isomorphism $A_{P'} \xrightarrow{\cong} A_P$ on stalks; if $f : X \to Y$ is a morphism of schemes, then the composite $f(u) : f(P) = \mathrm{Spec}(\Omega) \xrightarrow{u} X \xrightarrow{f} Y$ is a geometric point of Y and we have $(f^* B)_P \cong B_{f(P)}$ ([**88**, Chapter II, Proposition (5.2)]). Thus regarding geometric points and stalks for étale sheaves, we only need to choose a separable closure κ_x^s of κ_x for each $x \in X$; we denote the resulting geometric point by $\bar{x} : \mathrm{Spec}(\kappa_x^s) \to X$. The stalks A_P of all geometric points P with image point $x \in X$ are then (non-canonically) isomorphic to $A_{\bar{x}}$. The topos $\mathcal{Shv}(X_{\text{ét}})$ has enough points, given by the set of geometric points $\{\bar{x} : x \in X\}$ ([**88**, Chapter II, Theorem (5.6)]).

We have the *category* $\mathcal{N}\mathrm{b}_X(P)$ *of étale neighborhoods of* P *in* X, the objects are pairs (V, v), with $V \in X_{\text{ét}}$ and $v \in (\mathcal{S}\mathsf{ch}/X)(P, V)$; morphisms are objects in the category $(P \downarrow \mathcal{S}\mathsf{ch} \downarrow X)$. The category $\mathcal{N}\mathrm{b}_X(P)$ is cofiltered. We have a canonical isomorphism ([**88**, Chapter II, Proposition (5.4)])

$$\operatorname*{colim}_{(V,v)\in\mathcal{N}\mathrm{b}_X(P)^{\mathrm{op}}} A(V) \xrightarrow{\cong} A_P.$$

For any $s \in A(V)$, we denote s_P the image in A_P of s. If $A \in \mathcal{S}\mathsf{hv}\mathcal{A}\mathsf{b}(X_{\text{ét}})$ and $s \in A(X)$, we call

$$\mathrm{supp}(s) := \{x \in X : s_{\bar{x}} \neq 0\}$$

the *support of* s, it's Zariski-closed in X. We also denote $\mathrm{supp}(A)$ the closure in X of the set of $x \in X$ such that $A_{\bar{x}} \neq 0$, call it the *support of* A; it's the complement of the largest open subset of X on which A restricts to the zero sheaf.

We similarly have the notion of *étale neighborhoods of an* X*-scheme in* X.

In analog, for $x \in X$, we have the *category* $\mathcal{N}\mathrm{b}_X(x)$ *of étale neighborhoods of* x *in* X, the objects are pairs (V, v), with $V \in X_{\text{ét}}$ and $v \in V$ maps to x with $\kappa_x \xrightarrow{\cong} \kappa_v$; morphisms are objects in the category $(\mathrm{Spec}(\kappa_x) \downarrow \mathcal{S}\mathsf{ch} \downarrow X)$. The category $\mathcal{N}\mathrm{b}_X(x)$ is cofiltered.

If $f : X \to Y$ is a morphism of schemes, then for $A \in \mathcal{S}\mathsf{hv}\mathcal{A}\mathsf{b}(X_{\text{ét}})$ and Q a geometric point of Y, we have

$$\mathbf{R}^i f_*(A)_Q \cong \operatorname*{colim}_{(V,v)\in\mathcal{N}\mathrm{b}_Y(Q)^{\mathrm{op}}} \mathrm{H}^i_{\text{ét}}(V \times_Y X; A).$$

The *geometric fiber of* X *over* $y \in Y$ *(or over* $\bar{y}$*)* is defined to be

$$X_{\bar{y}} := X \times_Y \mathrm{Spec}(\kappa_y^s) = X_y \otimes_{\kappa_y} \kappa_y^s.$$

If $f : X \to Y$ is a finite morphism, then $X_y = \coprod_{x\in X_y} \mathrm{Spec}(\mathscr{O}_{X_y,x}) = \coprod_{x\in X_y} \mathrm{Spec}(\mathscr{O}_{X,x} \otimes_{\mathscr{O}_{Y,y}} \kappa_y)$ (has discrete topology), so we have

$$X_{\bar{y}} \cong \coprod_{x\in X_y} \mathrm{Spec}(\mathscr{O}_{X,x} \otimes_{\mathscr{O}_{Y,y}} \kappa_y^s).$$

Also, $(X_y)_{\mathrm{red}} = \coprod_{x\in X_y} \mathrm{Spec}(\kappa_x)$, $(X_{\bar{y}})_{\mathrm{red}} = \coprod_{x\in X_y} \mathrm{Spec}((\kappa_x^s)^{[\kappa_x:\kappa_y]_s}) = \coprod_{x\in X_y} \coprod_{[\kappa_x:\kappa_y]_s} \mathrm{Spec}(\kappa_x^s)$, both have discrete topology. For each $x \in X_y$, there are exact $[\kappa_x : \kappa_y]_s$ points of $X_{\bar{y}}$ with image x,[1] all of them are geometric points of X with image x, i.e. $\sharp(X_{\bar{y}} \to X_y)^{-1}(x) = [\kappa_x : \kappa_y]_s$.

If moreover $f : X \to Y$ is finite unramified (so that $\mathscr{O}_{X_y,x} = \kappa_x$), then as κ_y-schemes we have

$$X_{\bar{y}} \cong \coprod_{x\in X_y} \mathrm{Spec}((\kappa_x^s)^{[\kappa_x:\kappa_y]}) = \coprod_{x\in X_y} \coprod_{[\kappa_x:\kappa_y]} \mathrm{Spec}(\kappa_x^s).$$

A (commutative) local ring is called a *henselian local ring* if it satisfies Hensel's lemma. A local ring that is integral over a henselian local ring is henselian; every quotient of a henselian local ring is henselian. The inclusion functor of henselian local rings into all local rings has a left adjoint $A \mapsto A^h$, called *henselization*.

For any local ring $(A, \mathfrak{m})$, its henselization A^h is a faithfully flat A-algebra with maximal ideal $\mathfrak{m}A^h$, and the map $A/\mathfrak{m} \to A^h/\mathfrak{m}A^h$ on residue fields is an isomorphism; A^h is noetherian iff A is.

A local ring $(A, \mathfrak{m})$ is called *a strict henselian local ring*, or *strict local*, if it is a henselian local ring and the residue field $A/\mathfrak{m}$ is separably closed. Given an embedding $A/\mathfrak{m} = k \to \Omega$ with Ω a separably closed field, there is a strict henselian local ring A^{sh}, called the *strict henselization of* A *with respect to* $k \to \Omega$, which can be given by a certain universal property.

For any local ring $(A, \mathfrak{m})$, its strict henselization A^{sh} is a faithfully flat A-algebra with maximal ideal $\mathfrak{m}A^{sh}$, and the map $A/\mathfrak{m} \to A^{sh}/\mathfrak{m}A^{sh}$ exibits $A^{sh}/\mathfrak{m}A^{sh}$ as a separable closure of $A/\mathfrak{m}$; A^{sh} is noetherian iff A is.

If R is a (strict) henselian local ring and A is a finite R-algebra (so that A is a semi-local ring), then the canonical map $A \to \prod_{\mathfrak{m}\in\mathrm{Max}(A)} A_{\mathfrak{m}}$ is an isomorphism and each $A_{\mathfrak{m}}$ is (strict) henselian.

For more description, see [**88**, Chapter II, (6.1)] or the *Stacks project, tag/03QD* and *tag/0BSK*.

Let X be a scheme and $u : P = \mathrm{Spec}(\Omega) \to X$ (with Ω a separably closed field) a geometric point of X at x. We have

$$\mathscr{O}_{X,P} := (\mathbb{G}_{a,X})_P \cong \operatorname*{colim}_{(V,v)\in\mathcal{N}\mathrm{b}_X(P)^{\mathrm{op}}} \mathscr{O}_V(V) \cong \operatorname*{colim}_{(V,v)\in\mathcal{N}\mathrm{b}_X(P)^{\mathrm{op}}} \mathscr{O}_{V,v(P)} \cong \mathscr{O}^{sh}_{X,x}$$

is the strict henselization of $\mathscr{O}_{X,x}$ with respect to $\kappa_x \to \Omega$. We call $\mathscr{O}_{X,P}$ the *strict local ring of* X *at the geometric point* P, and the scheme $X(P) := \mathrm{Spec}(\mathscr{O}_{X,P})$ is called the *strict localization of* X *at the geometric point* P. There are canonical morphisms $j_P : X(P) = \mathrm{Spec}(\mathscr{O}_{X,P}) \to \mathrm{Spec}(\mathscr{O}_{X,x}) \xrightarrow{j_x} X$.

Similarly, $\mathscr{O}^h_{X,x} \cong \operatorname*{colim}_{(V,v)\in\mathcal{N}\mathrm{b}_X(x)^{\mathrm{op}}} \mathscr{O}_V(V) \cong \operatorname*{colim}_{(V,v)\in\mathcal{N}\mathrm{b}_X(x)^{\mathrm{op}}} \mathscr{O}_{V,v(x)}$ (see the *Stacks project, tag/02LD*).

For $A \in \mathcal{S}\mathsf{hv}(X_{\text{ét}})$ or $\mathcal{S}\mathsf{hv}\mathcal{A}\mathsf{b}(X_{\text{ét}})$, we have ([**88**, Chapter II, Proposition (6.2.2)])

$$A_P \cong \Gamma(X(P), j_P^*(A)).$$

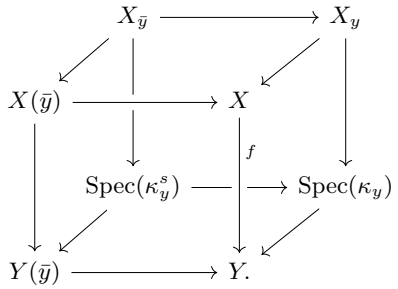
Let $f : X \to Y$, $A \in \mathcal{S}\mathsf{hv}\mathcal{A}\mathsf{b}(X_{\text{ét}})$ be a morphism of schemes and let $y \in Y$. Let $Y(\bar{y}) = \mathrm{Spec}(\mathscr{O}_{Y,\bar{y}}) \cong \lim_{(V,v)\in\mathcal{N}\mathrm{b}^{\mathrm{aff}}_Y(\bar{y})} V$ be the strict localization of Y at $\bar{y}$ (cf. the Zariski analog: the Zariski localization $Y(y) := \mathrm{Spec}(\mathscr{O}_{Y,y}) \cong \lim_{V\in\mathcal{N}\mathrm{b}_{Y_{\mathrm{Zar}}}(y)} V =$

[1] For a finite extension L/k, we have $L \otimes_k k^s \cong L \otimes_{L_s} (L_s \otimes_k k^s) \cong (L \otimes_{L_s} k^s)^{[L:k]_s} \cong (L^s)^{[L:k]_s}$ as k-algebras, by Proposition 2.5 in Brian Conrad's *handout on Inseparable extensions*, where $L_s := L \cap k^s$ is the subfield of elements in L that are separable over k and $[L : k]_s = [L_s : k]$ is the separable degree of the field extension L/k; note that k^s is also a separable closure of L_s.

$\bigcap_{V\in \mathcal{N}\mathrm{b}_{Y_{\mathrm{Zar}}}(y)} V$ is the intersection of all open neighborhoods of y in Y, consisting of the points in Y generalizing y) and let

$$X(\bar{y}) := X\times_Y Y(\bar{y}) \cong \lim_{(V,v)\in \mathcal{N}\mathrm{b}^{\mathrm{aff}}_Y(\bar{y})} X\times_Y V, A(\bar{y}) := (X(\bar{y})\to X)^*A \in \mathcal{S}\mathsf{hv}_{\mathcal{A}\mathsf{b}}(X(\bar{y})_{\text{ét}}).$$

We have $X(\bar{y})\otimes_{\mathscr{O}_{Y,\bar{y}}}\kappa^s_y \cong X_{\bar{y}}$, and $X_{\bar{y}}$ is the closed fiber $X(\bar{y})_{\bar{y}}$ of the morphism $X(\bar{y})\to Y(\bar{y})$, we just write $X_{\bar{y}} = X(\bar{y})_{\bar{y}}$. We heve the following commutative cube, all four vertical faces are cartesian squares:



If $f: X\to Y$ is a finite morphism, then the projection $X(\bar{y})\to Y(\bar{y})$ is also finite (hence affine), the closed fiber $X_{\bar{y}}$ consists of all the finitely many closed points of $X(\bar{y})$ (and is a discrete topological space). By [**67**, Chapter I, Theorem 4.2 (b)] or [**88**, Chapter II, Theorem (6.1.1) ii)] (with the fact that any finite field extension of a separably closed field is separably closed), we have

$$X(\bar{y}) \cong \coprod_{a\in X_{\bar{y}}} \mathrm{Spec}(\mathscr{O}_{X(\bar{y}),a}) = \coprod_{x\in X_y}\ \coprod_{a\in (X_{\bar{y}}\to X_y)^{-1}(x)} \mathrm{Spec}(\mathscr{O}_{X(\bar{y}),a}),$$

the second identification is seen by decomposing the set $X_{\bar{y}}$ as $X_{\bar{y}} = \coprod_{\bar{x}\in X_{\bar{y}}}(X_{\bar{y}}\to X_y)^{-1}(x)$; each ring $\mathscr{O}_{X(\bar{y}),a}$—the Zariski local ring of the scheme $X(\bar{y})$ at its point a—is a strict henselian local ring (as a finite $\mathscr{O}_{Y,\bar{y}}$-algebra). By the *Stacks project, tag/08HV* (for our purpose here, we can assume Y and hence X are affine, since $f: X\to Y$ is finite), we have

$$\mathscr{O}_{X(\bar{y}),a} \cong \mathscr{O}_{X,\bar{x}}.$$

Thus

$$X(\bar{y}) \cong \coprod_{\bar{x}\in X_{\bar{y}}} \mathrm{Spec}(\mathscr{O}_{X,\bar{x}}) = \coprod_{x\in X_y}\ \coprod_{\bar{x}\in (X_{\bar{y}}\to X_y)^{-1}(x)} \mathrm{Spec}(\mathscr{O}_{X,\bar{x}}),$$

As $\sharp(X_{\bar{y}}\to X_y)^{-1}(x) = [\kappa_x:\kappa_y]_s$ for each $x\in X_y$, we obtain

$$X(\bar{y}) \cong \coprod_{x\in X_y} \mathrm{Spec}((\mathscr{O}_{X,\bar{x}})^{[\kappa_x:\kappa_y]_s}) = \coprod_{x\in X_y}\ \coprod_{[\kappa_x:\kappa_y]_s} \mathrm{Spec}(\mathscr{O}_{X,\bar{x}}).$$

Recall the notion of torsion sheaves and ℓ-torsion sheaves in Definition 2.3.10. For a scheme X and $A\in \mathcal{S}\mathsf{hv}_{\mathcal{A}\mathsf{b}}(X_{\text{ét}})$, A is torsion iff all stalks $A_{\bar{x}}$ are torsion abelian groups. For every $n\in\mathbb{N}$, the sheaf $\mu_{n,X}$ is torsion; if A is a torsion abelian group, the constant sheaf $\underline{A}_X$ is torsion.

If X is a qcqs scheme, $A\in \mathcal{S}\mathsf{hv}_{\mathcal{A}\mathsf{b}}(X_{\text{ét}})$ is an (ℓ-)torsion sheaf, then all the étale cohomology groups $\mathrm{H}^n_{\text{ét}}(X;A), n\in\mathbb{N}$, are ($\ell$-)torsion abelian groups ([**88**, Chapter II, Proposition (9.1.2)]). For a morphism of schemes $f: X\to Y$ and $A\in \mathcal{S}\mathsf{hv}_{\mathcal{A}\mathsf{b}}(X_{\text{ét}}), B\in \mathcal{S}\mathsf{hv}_{\mathcal{A}\mathsf{b}}(Y_{\text{ét}})$, if B is torsion, then so is f^*B; if A is (ℓ-)torsion and f is qcqs, then all the sheaves $\mathbf{R}^n f_*A, n\in\mathbb{N}$, are ($\ell$-)torsion ([**88**, Chapter II, Proposition (9.1.3)]). Let $j: U\to X$ be an open immersion and $i: Y\to X$ a closed immersion, then $j_!, i^!$ map (ℓ-)torsion sheaves to (ℓ-)torsion sheaves; if $X\setminus Y$ is retro-compact, then $\mathbf{R}^n i^!, n\in\mathbb{N}$, map ($\ell$-)torsion sheaves to ($\ell$-)torsion sheaves ([**88**, Chapter II, (9.1.4)]).

Now let X be a noetherian scheme, let $x\in X$ which gives a canonical morphism $i: \mathrm{Spec}(\kappa_x)\to X$. Then for any $A\in \mathcal{S}\mathsf{hv}_{\mathcal{A}\mathsf{b}}(\mathrm{Spec}(\kappa_x)_{\text{ét}})$ and $q>0$, the sheaf $\mathbf{R}^q i_*(A)\in \mathcal{S}\mathsf{hv}_{\mathcal{A}\mathsf{b}}(X_{\text{ét}})$ is torsion. (Indeed, as $i: \mathrm{Spec}(\kappa_x)\to X$ is qcqs, for any $y\in X$ we have $\mathbf{R}^q i_*(A)_{\bar{y}} \cong \mathrm{H}^q_{\text{ét}}(X(\bar{y})\times_X \mathrm{Spec}(\kappa_x); A(\bar{y}))$. The scheme $X(\bar{y})\times_X \mathrm{Spec}(\kappa_x)$ is the fiber of the morphism $X(\bar{y})\to X$ at x, which is empty if $x\notin \overline{\{y\}}$ while is a finite coproduct of the spectra of separable algebraic extensions of κ_x by [**36**, Proposition (18.8.12) (ii)]). Results on Galois cohomology tells that the resulting groups are torsion.) Moreover, the cohomology group $\mathrm{H}^q_{\text{ét}}(X; i_*(A))$ is torsion (see [**88**, Chapter II, p. 149]).

If X is a regular noetherian scheme, the cohomology groups $\mathrm{H}^q_{\text{ét}}(X;\mathbb{G}_{m,X})$ are torsion for $q\geqslant 2$.

The canonical homomorphism

$$\oplus A(\ell) \xrightarrow{\cong} A$$

(the direct sum is over all prime numbers) is an isomorphism ([**88**, Chapter II, Remark (9.1.6)]).

Let X be a qcqs scheme, assume $A\in \mathcal{S}\mathsf{hv}_{\mathcal{A}\mathsf{b}}(X_{\text{ét}})$ is torsion, we have

$$\mathrm{H}^q_{\text{ét}}(X;A) \cong \bigoplus_{\ell} \mathrm{H}^q_{\text{ét}}(X;A(\ell))$$

and this identifies $\mathrm{H}^q_{\text{ét}}(X;A(\ell))$ as the ℓ-primary component of $\mathrm{H}^q_{\text{ét}}(X;A)$:

$$\mathrm{H}^q_{\text{ét}}(X;A(\ell)) \cong \mathrm{H}^q_{\text{ét}}(X;A)(\ell).$$

The *ℓ-cohomological dimension* $\mathrm{c.d.}_\ell(X)$ is the smallest natural number n (or ∞ if such an n doesn't exist) such that $\mathrm{H}^q_{\text{ét}}(X;A)=0$ for all $q>n$ and all ℓ-torsion sheaves A, or equivalently, $\mathrm{H}^q_{\text{ét}}(X;A)(\ell)=0$ for all $q>n$ and all torsion sheaves A. The *cohomological dimension* $\mathrm{c.d.}(X)$ is defined as

$$\mathrm{c.d.}(X):=\sup_\ell \mathrm{c.d.}_\ell(X).$$

It's the smallest natural number n (or ∞ if such an n doesn't exist) such that $\mathrm{H}^q_{\text{ét}}(X;A)=0$ for all $q>n$ and all torsion sheaves A.

Since the immersion $X_{\mathrm{red}}\to X$ induces an equivalence of categories $\mathcal{Shv}\mathcal{Ab}((X_{\mathrm{red}})_{\text{ét}})\to\mathcal{Shv}\mathcal{Ab}(X_{\text{ét}})$, we see

$$\mathrm{c.d.}_\ell(X)=\mathrm{c.d.}_\ell(X_{\mathrm{red}}),\mathrm{c.d.}(X)=\mathrm{c.d.}(X_{\mathrm{red}}).$$

Let X be a scheme. We say that $A\in\mathcal{Shv}\mathcal{Ab}(X_{\text{ét}})$ is *locally constant* if there is a covering family $\{V_i\to X\}$ of X in $X_{\text{ét}}$ such that the restrictions $A|_{V_i}\in\mathcal{Shv}\mathcal{Ab}((V_i)_{\text{ét}})$ are constant. For any morphism of schemes $f:X\to Y$, the functor f^* takes locally constant sheaves on $Y_{\text{ét}}$ to locally constant sheaves on $X_{\text{ét}}$ (however f_* usually doesn't).

If A is a finite étale commutative group scheme over X, the sheaf h_A represented by A is locally constant, its stalk $(\underline{A}_X)_{\bar{x}}$ is finite and equals the group $A(\kappa_x^s)$; conversely, any locally constant sheaf on $X_{\text{ét}}$ is isomorphic to a étale commutative group scheme over X, which is finite over X if the given sheaf has finite stalks ([**88**, Chapter II, Propositions (9.2.2) and (9.2.3)]).

If k is a field, we fix a separable closure k^s of k and let $G:=\mathrm{Gal}(k^s/k)$. Then for an abelian group A, the constant sheaf on $\mathrm{Spec}(k)_{\text{ét}}$ with stalk A corresponds to the trivial G-module structure on A; a locally constant sheaf with stalk A corresponds to a continuous G-module structure on A which factors through a finite quotient of G.

We say that $A\in\mathcal{Shv}\mathcal{Ab}(X_{\text{ét}})$ is *finite* if all stalks $A_{\bar{x}}$ are finite abelian groups and for $A\in\mathcal{Shv}\mathcal{Ab}(X_{\text{ét}})$ finite, we say that A is *of order prime to n* for some $n\in\mathbb{N}$ if all stalks $A_{\bar{x}}$ are finite abelian groups with order prime to n. In the following, we write $A|_Z:=i^*(A)$ for a subscheme Z with canonical immersion $i:Z\to X$. For the notion of *constructible subsets* in a scheme or a topological space, see [**34**, (10.5) and (10.10)].

Let X be a noetherian scheme with ring of rational functions $K=\prod_{y\in X^{(0)}}\mathscr{O}_{X,y}$ which is an artinian ring, where the product is over all maximal points of X (i.e. the generic points of irreducible components of X). We have

$$\mathrm{c.d.}_\ell(K)=\mathrm{c.d.}_\ell(K_{\mathrm{red}})=\max\{\mathrm{c.d.}_\ell(\kappa_y):y\in X^{(0)}\}.$$

Let $j:\mathrm{Spec}(K)=\coprod_{y\in X^{(0)}}\mathrm{Spec}(\mathscr{O}_{X,y})\to X$ be the canonical morphism, then $\mathrm{c.d.}_\ell(\mathrm{Spec}(K)\times_X X(\bar{x}))\leqslant\mathrm{c.d.}_\ell(K),\forall x\in X$ ([**88**, Chapter II, Lemma (10.2.2)]). For $A\in\mathcal{Shv}\mathcal{Ab}(\mathrm{Spec}(K)_{\text{ét}})$ we have

$$(\mathbf{R}^qj_*A)_{\bar{y}}\cong\begin{cases}A_{\bar{y}}, & q=0;\\ 0, & q>0,\end{cases}$$

for any maximal point y of X ([**88**, Chapter II, Lemma (10.1.4)]).

Let $(R,\mathfrak{m},k:=R/\mathfrak{m},K:=\mathrm{Frac}(R))$ be a henselian DVR with $k=\bar{k},\mathrm{char}(k)=\mathrm{char}(K)=p>0$, let $G:=\mathrm{Gal}(K^s/K)$ and let A be a finite continuous G-module of order prime to p. Then the Galois cohomology groups $\mathrm{H}^q(G;A)$ are finite (and $\mathrm{H}^q(G;A)=0,q>1$).

If X is an algebraic scheme over a field k with $\mathrm{char}(k)=p>0$, then

$$\begin{cases}\mathrm{c.d.}_\ell(X)\leqslant\mathrm{c.d.}_\ell(k)+2\dim(X), & \ell\neq p;\\ \mathrm{c.d.}_p(X)\leqslant\dim(X)+1.\end{cases}$$

If moreover $k=k^s$, then $\mathrm{c.d.}(X)\leqslant 2\dim(X)$; if X is also affine, then $\mathrm{c.d.}(X)\leqslant\dim(X)$.

The base change morphism $v^*\mathbf{R}^if_*(A)\to(\mathbf{R}^if'_*)v'^*(A)$ associated to the square eq. (B.6) is an isomorphism if $f:X\to Y$ is a proper morphism of schemes and $A\in\mathcal{Shv}\mathcal{Ab}(X_{\text{ét}})$ is torsion ([**88**, Chapter II, Theorem (11.3.1)]); in this case, for any $y\in Y$ we have $\mathbf{R}^if_*(A)_{\bar{y}}\cong\mathrm{H}^i_{\text{ét}}(X_{\bar{y}};(X_{\bar{y}}\to X)^*A),\forall i\geqslant 0$. If $f:X\to Y$ is proper of relative dimension $\leqslant n$ and $A\in\mathcal{Shv}\mathcal{Ab}(X_{\text{ét}})$ is torsion, then $\mathbf{R}^if_*(A)=0,i>2n$.

If $k\subset k'$ are separably closed fields and X is a proper k-scheme, let $X':=X\otimes_k k'$, let $A\in\mathcal{Shv}\mathcal{Ab}(X_{\text{ét}})$ be a torsion sheaf, let $A':=(X'\to X)^*A\in\mathcal{Shv}\mathcal{Ab}(X'_{\text{ét}})$. Then $\mathrm{H}^i_{\text{ét}}(X;A)\cong\mathrm{H}^i_{\text{ét}}(X';A'),\forall i\geqslant 0$ ([**88**, Chapter II, Corollary (11.3.4)]).

The base change morphism $v^*\mathbf{R}^if_*(A)\to(\mathbf{R}^if'_*)v'^*(A)$ associated to the square eq. (B.6) is an isomorphism also in the following case: $f:X\to Y$ is a qcqs morphism of schemes, $v:Y'\to Y$ is smooth and $A\in\mathcal{Shv}\mathcal{Ab}(X_{\text{ét}})$ is ℓ-torsion for some prime ℓ invertible on Y ([**88**, Chapter II, Theorem (11.3.5)]).

If X is a proper k-scheme, $k=k^s$ and $A\in\mathcal{Shv}\mathcal{Ab}(X_{\text{ét}})$ is a constructible sheaf, then all the étale cohomology groups $\mathrm{H}^q_{\text{ét}}(X;A),q\geqslant 0$ are finite.

If X is a smooth algebraic scheme over a field k with $k=k^s,\mathrm{char}(k)=p>0$ and $A\in\mathcal{Shv}\mathcal{Ab}(X_{\text{ét}})$ is a locally constant finite sheaf of order prime to p, then all the étale cohomology groups $\mathrm{H}^q_{\text{ét}}(X;A),q\geqslant 0$ are finite ([**88**, Chapter II, Theorem (11.4.3)]).

Appendix C

Unimodular elements in projective modules

In this appendix, we present briefly some results on unimodular elements in projective modules, with relation to the "naive" $\mathbb{A}^1$-homotopy classes. These are quite useful since in important cases, the "naive" $\mathbb{A}^1$-homotopy classes coincide with the "genuine" $\mathbb{A}^1$-homotopy classes. We present more than we need: some results are not used in the text but are included because the author considers them to be interesting.

C.1. Some useful facts

For a commutative ring R and $r \in \mathbb{N}$, we denote by $\mathcal{V}_r(R)$ the set of isomorphism classes of finitely generated projective R-modules of constant rank r; we identify it with the set of isomorphism classes of rank r vector bundles over $\mathrm{Spec}(R)$. We also write $\mathcal{V}(R) := \bigcup_{r\in\mathbb{N}} \mathcal{V}_r(R)$. For $n \geqslant 1$, we have canonical maps

$$\mathcal{V}_r(R) \to \mathcal{V}_r(R[t_1, \cdots, t_n]), P \mapsto P[t_1, \cdots, t_n] := P \otimes_R R[t_1, \cdots, t_n]$$

and

$$\mathcal{V}_r(R[t_1, \cdots, t_n]) \to \mathcal{V}_r(R), \tilde{P} \mapsto \tilde{P}/(t_1, \cdots, t_n)\tilde{P} = \tilde{P} \otimes_{R[t_1,\cdots,t_n]} R,$$

induced by the ring maps $R \hookrightarrow R[t_1, \cdots, t_n]$ and $R[t_1, \cdots, t_n] \to R, t_j \mapsto 0$, respectively. Geometrically, they respectively correspond to pullback of vector bundles along the canonical projection $\mathbb{A}^n_R \to \mathrm{Spec}(R)$ and its zero-section $\mathrm{Spec}(R) \to \mathbb{A}^n_R$. So the first map is injective and the second is surjective, with composite being the identity map of $\mathcal{V}_r(R)$.

Proposition C.1.1. *Let $P, Q \in \mathcal{V}_r(R)$ and $\varphi : P \to Q$ a homomorphism of R-module. Then φ is an isomorphism iff $\det\varphi : \det P \to \det Q = \bigwedge^r_R Q$ is an isomorphism.*

Proof. As finitely generated projective R-modules are locally free, we can consider locally over $\mathrm{Spec}(R)$ and hence reduce to the case when both P, Q are free, then the result is just the usual linear algebra. □

The following is an answer to the Bass-Quillen conjecture by H. Lindel **[56]** in great generality, see also **[73]** for some other form.

Theorem C.1.2 (Bass-Quillen conjecture)**.** *Let R be a regular ring essentially of finite type over a field k and let $n \geqslant 1$ be an integer. Then the two maps $\mathcal{V}_r(R) \to \mathcal{V}_r(R[t_1, \cdots, t_n])$ and $\mathcal{V}_r(R[t_1, \cdots, t_n]) \to \mathcal{V}_r(R)$ defined above are bijections and inverse to each other.*

Theorem C.1.3 ([**94**, Theorem 3.3])**.** *Let R be a regular ring essentially of finite type over a field k and let $r \geqslant 3$ be an integer. If $\varphi(t_1, \cdots, t_n) \in \mathrm{Aut}_{R[t_1,\cdots,t_n]}(R[t_1, \cdots, t_n]^{\oplus r}), n \geqslant 1$ and $\varphi(0, \cdots, 0) = \mathrm{id}$, then $\varphi(t_1, \cdots, t_n) \in \mathrm{E}_r(R[t_1, \cdots, t_n])$.*

For $P \in \mathcal{V}(R)$, we say $\varphi \in \mathrm{Aut}_R(P \oplus R)$ is an *elementary automorphism* of $P \oplus R$ if it is in the subgroup of $\mathrm{Aut}_R(P \oplus R)$ generated by automorphisms of the form $\begin{pmatrix} \mathrm{id}_P & 0 \\ \xi & 1 \end{pmatrix}$ and $\begin{pmatrix} \mathrm{id}_P & v \\ 0 & 1 \end{pmatrix}$, for $v \in P, \xi \in P^\vee$. These two kinds of maps are called *elementary transvections* of $P \oplus R$ (they are indeed transvections). Note that

$$\begin{pmatrix} \mathrm{id}_P & 0 \\ \xi & 1 \end{pmatrix}\begin{pmatrix} p \\ a \end{pmatrix} = \begin{pmatrix} p \\ \langle\xi, p\rangle + a \end{pmatrix}, \begin{pmatrix} \mathrm{id}_P & v \\ 0 & 1 \end{pmatrix}\begin{pmatrix} p \\ a \end{pmatrix} = \begin{pmatrix} p + av \\ a \end{pmatrix}.$$

The group generated by the set of all elementary automorphisms of $P \oplus R$ is denoted $\mathrm{E}(P \oplus R)$.

For any $\sigma \in \mathrm{E}(P \oplus R)$, there exists $\tilde{\sigma}(t) \in \mathrm{E}(P[t] \oplus R[t])$ such that $\tilde{\sigma}(0) = \mathrm{id}_{P\oplus R}, \tilde{\sigma}(1) = \sigma$. (Write σ as a finite product of elementary transvections as above, then replace ξ or v there by ξt or vt, we get the desired $\tilde{\sigma}(t)$.)

On the other hand, it's clear from definition that if $\psi(t) \in \mathrm{E}(P[t] \oplus R[t])$, then for any $t_0 \in R$, the evaluation $\psi(t_0) \in \mathrm{E}(P \oplus R)$.

For $u, v \in P \oplus R$, we write $u \sim v$ or $u \sim_{\mathrm{E}} v$ if there is $\varphi \in \mathrm{E}(P \oplus R)$ such that $u = \varphi(v)$. This is clearly an equivalence relation. If $u \sim v$, then $u \in \mathrm{Um}(P \oplus R)$ iff $v \in \mathrm{Um}(P \oplus R)$. (Here for a module M, we set $\mathrm{Um}(M) := \{x \in M : \exists \xi \in \mathrm{Hom}_R(M, R) with \xi(x) = 1\}$, as in the next section.) We have $(p, a) \sim (p, \langle\xi, p\rangle + a) \sim (p + av, a), \forall v \in P, \xi \in P^\vee$ and $(p, a) \sim (p, b)$ if $a - b \in P^\vee p$.

The following surprising fact is one of the main results in **[15]**, which in particular says that the group $\mathrm{E}(P \oplus R)$ of elementary automorphisms doesn't depend on the direct sum decomposition of $P \oplus R$ as a module.

Theorem C.1.4 (Elementary=Transvections)**.** *Assume* rank $P = n \geqslant 2$. *The group $\mathrm{E}(P \oplus R)$ coincides with the subgroup of $\mathrm{Aut}_R(P \oplus R)$ generated by the set of all transvections of $P \oplus R$. Moreover, $\mathrm{E}(P \oplus R) \lhd \mathrm{Aut}_R(P \oplus R)$.*

We will need the following local-global principle from **[15**, Theorem 3.6].

Theorem C.1.5 (Local-Global Principle)**.** *Let $\varphi(t) \in \mathrm{Aut}_{R[t]}(P[t] \oplus R[t])$. If $\varphi(0) = \mathrm{id}_{P\oplus R}$ and for every $\mathfrak{m} \in \mathrm{Max}(R)$ (the spectrum of maximal ideals in R), the localization $\varphi_{\mathfrak{m}}(t) \in \mathrm{E}(P_{\mathfrak{m}}[t] \oplus R_{\mathfrak{m}}[t])$, then $\varphi(t) \in \mathrm{E}(P[t] \oplus R[t])$.*

Proposition C.1.6. *Let R be a regular ring essentially of finite type over a field k, P be a finitely generated projective R-module of constant rank $n \geqslant 2$. For $\varphi_0, \varphi_1 \in \mathrm{Aut}_R(P \oplus R)$, the following are equivalent.*

(1) *There exists $\sigma \in \mathrm{E}(P \oplus R)$ such that $\varphi_1 = \varphi_0\sigma$.*
(2) *There exists $\varphi(t) \in \mathrm{Aut}_{R[t]}(P[t] \oplus R[t])$ such that $\varphi(0) = \varphi_0, \varphi(1) = \varphi_1$.*
(3) *There exists $\psi(t) \in \mathrm{E}(P[t] \oplus R[t])$ such that $\varphi_1 = \varphi_0\psi(1)$ (and $\psi(0) = \mathrm{id}_{P\oplus R}$).*

Proof. (1) $\Rightarrow$ (2): Take $\tilde{\sigma}(t) \in \mathrm{E}(P[t] \oplus R[t])$ such that $\tilde{\sigma}(0) = \mathrm{id}_{P\oplus R}, \tilde{\sigma}(1) = \sigma$. Define $\varphi(t) := \varphi_0\tilde{\sigma}(t)$, then $\varphi(0) = \varphi_0, \varphi(1) = \varphi_1$.

(2) $\Rightarrow$ (3): Consider the element $\psi(t) := \varphi_0^{-1}\varphi(t) \in \mathrm{Aut}_{R[t]}(P[t] \oplus R[t])$, we have $\psi(0) = \mathrm{id}$ and localizing at all $\mathfrak{p} \in \mathrm{Spec}R$ and using the fact that $P_\mathfrak{p} \oplus R_\mathfrak{p}$ is free of rank $n + 1 \geqslant 3$, we find by Theorem C.1.3 that the localizations of $\psi(t)$ at all $\mathfrak{p} \in \mathrm{Spec}R$ are in $\mathrm{E}_{n+1}(R_\mathfrak{p}[t]) = \mathrm{E}(P_\mathfrak{p}[t] \oplus R_\mathfrak{p}[t])$. By the local-global principle in Theorem C.1.5, we see $\psi(t) \in \mathrm{E}(P[t] \oplus R[t])$ and $\varphi_0^{-1}\varphi_1 = \psi(1)$.

(3) $\Rightarrow$ (1): Assume $\psi(t) \in \mathrm{E}(P[t] \oplus R[t])$ and $\varphi_1 = \varphi_0\psi(1)$, so we have $\sigma := \psi(1) \in \mathrm{E}(P \oplus R)$ and $\varphi_1 = \varphi_0\sigma$. □

C.2. Unimodular elements and elementary automorphisms

Let R be a noetherian commutative ring of dimension $d \geqslant 1$, P be a finitely generated projective R-module of constant rank n. Let $P^\vee := \mathrm{Hom}_R(P, R)$ be the dual. We say that an element $x \in P$ is a *unimodular element* if there exists $\xi \in P^\vee$ such that $\langle \xi, x\rangle = 1$, here $\langle -, -\rangle : P^\vee \times P \to R$ is the natural pairing (evaluation map). The set of all unimodular elements of P is denoted by $\mathrm{Um}(P) = \mathrm{Um}_R(P)$.

Any element $x \in P$ can be viewed as a map $x : R \to P, r \mapsto rx$. If $x \in \mathrm{Um}(P)$ and $\langle \xi, x\rangle = 1$, then $x : R \to P$ is a *split monomorphism* as $\xi \circ x = \mathrm{id}_R$; in this case, $\mathrm{coker}(R \xrightarrow{x} P) = P/Rx$ is also projective and $P \cong R \oplus \mathrm{coker}(x)$.

For $\varphi \in \mathrm{Aut}_R(P)$, we define $\varphi^*, \varphi^\vee \in \mathrm{Aut}_R(P^\vee)$ by letting

$$\langle \varphi^*(\xi), x\rangle := \langle \xi, \varphi(x)\rangle, x \in P, \xi \in P^\vee$$

and $\varphi^\vee := (\varphi^*)^{-1}$. One easily checks that

$$\langle \varphi^\vee(\xi), \varphi(x)\rangle = \langle \xi, x\rangle, x \in P, \xi \in P^\vee.$$

As finitely generated projective modules are *reflexive*, the above two formulas in fact say that any one of $\varphi, \varphi^*, \varphi^\vee$ determines the other two.

It's clear that if $x \in \mathrm{Um}(P)$, then $\varphi(x) \in \mathrm{Um}(P)$ for every $\varphi \in \mathrm{Aut}_R(P)$. If $(v, a) \in \mathrm{Um}(P \oplus R)$, then $(v, a^r) \in \mathrm{Um}(P \oplus R)$ for every $r \geqslant 1$.

In the following, we write elements in any direct sum as "row vectors" to save spaces, while we write them as "column vectors" when we perform "matrix products". So we can write

$$\mathrm{End}_R(P \oplus R) = \begin{pmatrix} \mathrm{End}_R(P) & P \\ P^\vee & R \end{pmatrix}.$$

For $x \in P$ we denote $P^\vee x := \{\langle \xi, x\rangle : \xi \in P^\vee\}$which is an ideal of R. Then $x \in \mathrm{Um}(P) \Leftrightarrow P^\vee x = R$. So for $x = (x', a) \in P \oplus R$ with $x' \in P, a \in R$, we have $x \in \mathrm{Um}(P \oplus R) \Leftrightarrow P^\vee x' + Ra = R$.

A map $\varphi \in \mathrm{Aut}_R(P)$ is called a *transvection* if $\varphi = \mathrm{id}_P + \langle \xi, -\rangle v$ for some $v \in P, \xi \in P^\vee$ with $\langle \xi, v\rangle = 0$ and either $v \in \mathrm{Um}(P)$ or $\xi \in \mathrm{Um}(P^\vee)$. It's easy to see that for a transvection $\varphi = \mathrm{id}_P + \langle \xi, -\rangle v$, we have $\xi \circ \varphi = \xi$ and its inverse is also a transvection $\varphi^{-1} = \mathrm{id}_P - \langle \xi, -\rangle v = \mathrm{id}_P + \langle -\xi, -\rangle v$. We denote the set of transvection of P by $\mathrm{Trans}(P)$. The conjugation of a transvection is a transvection: for $\sigma \in \mathrm{Aut}_R(P)$, we have $\sigma(\mathrm{id}_P + \langle \xi, -\rangle v)\sigma^{-1} = \mathrm{id}_P + \langle \sigma^\vee(\xi), -\rangle\sigma(v)$.

Proposition C.2.1. *For a finitely generated projective R-module P, denote by $\mathrm{LDS}(P)$ the set of rank 1 free direct summands of P. Then the map*

$$\mathrm{Um}(P) \to \mathrm{LDS}(P), v \mapsto Rv$$

is surjective, whose fibre over Rv is $R^\times v \cong R^\times$ (the orbit of $v \in \mathrm{Um}(P)$ under the natural action of $R^\times$). Thus P splits out a trivial line bundle iff P admits a unimodular element.

Proof. First of all, the definition of the map makes sense: take $\xi \in P^\vee$ with $\langle \xi, v\rangle = 1$, then $\xi : P \to R$ is a left splitting morphism of the short exact sequence

$$0 \to R \to P \to Q \to 0,$$

where the map $R \to P$ is $r \mapsto rv$, and $P \to Q := P/Rv$ is the quotient map. Thus Rv is a rank 1 free submodule of P and by the *splitting lemma*, the above exact sequence yields a direct sum decomposition $P = Rv \oplus Q'$, where $Q' \cong Q$ is the image of a right splitting morphism $Q \to P$.

It's also clear that for any direct sum decomposition $P = L \oplus Q$ with $L \in \mathrm{LDS}(P)$ a rank 1 free submodule, we have $L = Rv$ for a generator $v \in L$, and $v \in \mathrm{Um}(P)$. So the map is surjective.

If $Ru = Rv$, then clearly $u \in R^\times v$. As $\langle \xi, v\rangle = 1$, we have $R^\times v \cong R^\times$ (bijection). □

Proposition C.2.2. *Let $x \in P, \xi \in P^\vee$ and $\langle \xi, x\rangle = 1$. Then the map $\varphi : \ker(\xi) \to \mathrm{coker}(x), y \mapsto [y]$ is an isomorphism.*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ker(\xi) & \longrightarrow & P & \longrightarrow & R & \longrightarrow & 0 \\ & & & & \| & \overset{\varphi}{\searrow} & & & \\ 0 & \longrightarrow & R & \xrightarrow{x} & P & \longrightarrow & \mathrm{coker}(x) & \longrightarrow & 0 \end{array}$$

Proof. Let $y \in \ker(\xi) \subset P$. If $\varphi(y) = 0 \in \operatorname{coker}(x) = P/Rx$, then $y = rx$ for some $r \in R$. So $r = \langle \xi, rx\rangle = \langle \xi, y\rangle = 0$ as $y \in \ker(\xi)$. So φ is injective.

For any $y \in P$, let $r = \langle \xi, y\rangle, y' = y - rx$, then $y' \in \ker(\xi)$ and $y = y' + rx \equiv y' (\mathrm{mod} Rx)$, so $[y] = [y'] = \varphi(y')$, φ is surjective as well. □

Proposition C.2.3. *Assume* $\dim R = d, \operatorname{rank} P = n$. *Let* $p, q \in P, a, b \in R$.

(1) *If* $(p, a), (p, 1-a) \in \mathrm{Um}(P \oplus R)$, *then* $(p, a(1-a)) \in \mathrm{Um}(P \oplus R)$.
(2) *For any* $(p, a), (q, b) \in \mathrm{Um}(P \oplus R)$, *we have* $(p, q, ab) \in \mathrm{Um}(P \oplus P \oplus R)$.
(3) *If* $1 \leqslant d \leqslant n$ *and* $(p, a, b) \in \mathrm{Um}(P \oplus R \oplus R)$, *then there exists* $\varphi \in \mathrm{Aut}_R(P \oplus R \oplus R)$ *such that* $\varphi(p, a, b) = (0, 0, 1)$.
(4) *If* $1 \leqslant d \leqslant n-1$ *and* $(p, a) \in \mathrm{Um}(P \oplus R)$, *then there exists* $v \in P$ *such that* $p + av \in \mathrm{Um}(P)$. *If we take* $\xi \in P^\vee$ *such that* $\langle \xi, p + av\rangle = 1$, *then*

$$\begin{pmatrix} \mathrm{id}_P & -(p+av) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathrm{id}_P & 0 \\ (1-a)\xi & 1 \end{pmatrix} \begin{pmatrix} \mathrm{id}_P & v \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p \\ a \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Proof. In the following, we use repeatedly that $Ia + I(1-a) = I$, for any ideal I of R and $a \in R$.

(1) We take $\xi, \eta \in P^\vee, r, s \in R$ with $\langle \xi, p\rangle + ra = 1 = \langle \eta, p\rangle + s(1-a)$, then it's easy to see that $\langle (1-a)\xi + a\eta, p\rangle + (r+s)a(1-a) = 1$. This also follows from (2) with $p = q, b = 1-a$.

(2) By assumption, $P^\vee p + Ra = P^\vee q + Rb = R$. So $Rb = b(P^\vee p) + Rab \subset P^\vee p + Rab, Rb(1-a) \subset P^\vee p + P^\vee q + Rab$. Similarly, $Ra(1-b) \subset P^\vee p + P^\vee q + Rab$. Thus

$$\begin{aligned} & P^\vee p + P^\vee q + Rab \\ =& P^\vee p + P^\vee q + Rab + Ra(1-b) + Rb(1-a) + Rab \\ =& P^\vee p + P^\vee q + (Rab + Ra(1-b)) + (Rab + Rb(1-a)) \\ =& P^\vee p + P^\vee q + Ra + Rb = R + R = R. \end{aligned}$$

(3) Consider the following diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R & \xrightarrow{i} & P \oplus R \oplus R & \longrightarrow & Q & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \psi} & & \\ 0 & \longrightarrow & R & \xrightarrow{j} & P \oplus R \oplus R & \longrightarrow & P \oplus R & \longrightarrow & 0, \end{array}$$

where $i(r) = r(p, a, b), j(r) = r(0, 0, 1)$ the maps $P \oplus R \oplus R \to Q$ and $P \oplus R \oplus R \to P \oplus R$ are the quotient maps. By assumption, i is split, hence both rows are split exact sequences and Q is projective. Since $n + 1 > d, P \oplus R$ is cancellative, thus there exists an isomorphism $\psi : Q \to P \oplus R$.

Now choosing a splitting map λ of the quotient map $P \oplus R \oplus R \to Q$, we can define a map $\varphi : P \oplus R \oplus R = R(p, a, b) \oplus \mathrm{im}\lambda \to P \oplus R \oplus R$ by $\varphi(r(p, a, b)) = (0, 0, r)$ and $\varphi(\lambda(q)) = (\psi(q), 0)$, making the displayed diagram commutative and satisfying $\varphi(p, a, b) = (0, 0, 1)$. By the 5-lemma, $\varphi \in \mathrm{Aut}_R(P \oplus R \oplus R)$.

(4) This result follows from [**18**, Theorem 9.1] by applying it in the case $A = \Lambda = R, Q = R, \alpha = (p, a), \mathfrak{a} = (0)$ (note that by our assumption, $f\text{-}\operatorname{rank} P = n$ in Bass's notation in [**18**]).

□

We say that $(u, a) \in \mathrm{Um}(P \oplus R)$ is *completable* if there exists $\sigma \in \mathrm{Aut}_R(P \oplus R)$ such that $\sigma(0, 1) = (u, a)$, i.e. σ is of the form $\begin{pmatrix} ? & u \\ ? & a \end{pmatrix}$.

Proposition C.2.4. *For* $(u, a), (v, b) \in \mathrm{Um}(P \oplus R)$, *the following are equivalent.*

(1) $\operatorname{coker}(R \xrightarrow{(u,a)} P \oplus R) \cong \operatorname{coker}(R \xrightarrow{(v,b)} P \oplus R)$.
(2) *There exists* $\sigma \in \mathrm{Aut}_R(P \oplus R)$ *such that* $\sigma(u, a) = (v, b)$.

In particular, taking $(v, b) = (0, 1)$ *we see that* (u, a) *is completable iff* $\operatorname{coker}(u, a) \cong P$.

Proof. Assume $\rho : \operatorname{coker}(u, a) \xrightarrow{\cong} \operatorname{coker}(v, b)$, consider the following diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R & \underset{(\xi\ a')}{\overset{\binom{u}{a}}{\rightleftarrows}} & P \oplus R & \underset{\varphi'}{\overset{\varphi}{\rightleftarrows}} & \operatorname{coker}(u, a) & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle \rho} & & \\ 0 & \longrightarrow & R & \xrightarrow{\binom{v}{b}} & P \oplus R & \underset{\psi'}{\overset{\psi}{\rightleftarrows}} & \operatorname{coker}(v, b) & \longrightarrow & 0. \end{array}$$

As $(u, a) : R \to P \oplus R$ is a split monomorphism, we can also choose right splitting morphisms $\varphi' : \operatorname{coker}(u, a) \to P \oplus R$ and $\psi' : \operatorname{coker}(v, b) \to P \oplus R$. Then $P \oplus R = R(u, a) \oplus \mathrm{im}\varphi'$ and we define $\sigma : P \oplus R = R(u, a) \oplus \mathrm{im}\varphi' \to P \oplus R$ by letting $\sigma(ru, ra) = (rv, rb)$ and $\sigma\varphi' = \psi\rho$. It's easy to see the map σ so defined makes the whole diagram above commutative. Then 5-lemma (or snake lemma) tells that σ is an isomorphism, so $\sigma \in \mathrm{Aut}_R(P \oplus R)$ and $\sigma(u, a) = (v, b)$. The converse is clear. □

Corollary C.2.5. *For a fixed commutative ring* R *and a finitely generated projective* R*-module* P, *the following are equivalent.*

(1) *For any (projective) R-module Q,*

$$P \oplus R \cong Q \oplus R \Rightarrow P \cong Q.$$

(2) *The natural action of* $\mathrm{Aut}_R(P \oplus R)$ *on* $\mathrm{Um}(P \oplus R)$ *is transitive.*
(3) *Every element in* $\mathrm{Um}(P \oplus R)$ *is completable.*

Proof. From the previous result, we see that (2) $\Leftrightarrow$ (3). To see (1) $\Leftrightarrow$ (3), let $\varphi : P \oplus R \xrightarrow{\cong} Q \oplus R$ with $\varphi(u, a) = (0, 1)$, then $Q = \mathrm{coker}(0, 1) \cong \mathrm{coker}(u, a)$. □

Proposition C.2.6. *Let $(v, a) \in P \oplus R, (\xi, b) \in P^\vee \oplus R$ and $\langle (\xi, b), (v, a) \rangle = \langle \xi, v \rangle + ab = 1$. Then the following are equivalent.*

(1) *(v, a) is completable.*
(2) $\mathrm{coker}(R \xrightarrow{(v,a)} P \oplus R) \cong P$.
(3) $\ker(P \oplus R \xrightarrow{(\xi,b)} R) \cong P$.
(4) *(ξ, b) is completable: there exists $\tau \in \mathrm{Aut}_R(P \oplus R)$ such that $\tau^*(0, 1) = (\xi, b)$. (So τ is of the form $\begin{pmatrix} ? & ? \\ \xi & b \end{pmatrix}$.)*

If $\begin{pmatrix} \rho & v \\ \eta & a \end{pmatrix}^{-1} = \begin{pmatrix} \rho' & v' \\ \eta' & a' \end{pmatrix} \in \mathrm{Aut}_R(P \oplus R)$, *then we have short exact sequences*

$$0 \longrightarrow R \xrightarrow{\begin{pmatrix} v \\ a \end{pmatrix}} P \oplus R \xrightarrow{(\rho', v')} P \longrightarrow 0$$

and

$$0 \longleftarrow R \xleftarrow[(\eta', a')]{} P \oplus R \xleftarrow[\begin{pmatrix} \rho \\ \eta \end{pmatrix}]{} P \longleftarrow 0,$$

where the lower one splits the upper one.

Proof. (1) $\Leftrightarrow$ (2) follows from Proposition C.2.4, (3) $\Leftrightarrow$ (4) is by similar argument, using the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ker(\xi, b) & \longrightarrow & P \oplus R & \longrightarrow & R & \longrightarrow & 0 \\ & & \downarrow & & \vdots & & \| & & \\ 0 & \longrightarrow & P & \longrightarrow & P \oplus R & \longrightarrow & R & \longrightarrow & 0. \end{array}$$

(2) $\Leftrightarrow$ (3) follows from Proposition C.2.2. □

Remark C.2.7. Once we get some *specific condition* for completability, like $(\sigma, u) : \mathrm{coker}(R \xrightarrow{(v,a)} P \oplus R) \xrightarrow{\cong} P$ kind of universally depends on $(\xi, b), (v, a)$, then wen can get Suslin's $n!$ theorem in this setting, from Suslin's result.

Proposition C.2.8. *Assume* $\dim R = d, \mathrm{rank}\ P = n$. *Let $u, u' \in P, a, b, a', b' \in R$.*

(1) *If $(u, a) \in \mathrm{Um}(P \oplus R)$ and $ab - 1, ab' - 1 \in P^\vee u$, then $b - b' \in P^\vee u$.*
(2) *Assume $(u, a), (u', a') \in \mathrm{Um}(P \oplus R)$ and $ab - 1 \in P^\vee u, a'b' - 1 \in P^\vee u'$. If $(u, a) \sim (u', a')$, then $(u, b) \sim (u', b')$.*

Proof. (1) This follows from $b - b' = (ab - 1)b' - (ab' - 1)b$.

(2) Case 1: $\begin{pmatrix} u' \\ a' \end{pmatrix} = \begin{pmatrix} u \\ \langle \xi, u \rangle + a \end{pmatrix}$. Then $ab - 1, ab' - 1 = a'b' - 1 - b'\langle \xi, u \rangle \in P^\vee u$, so $b - b' \in P^\vee u$ and thus

$$\begin{pmatrix} u' \\ b' \end{pmatrix} = \begin{pmatrix} u \\ b' \end{pmatrix} \sim \begin{pmatrix} u \\ b' + (b - b') \end{pmatrix} = \begin{pmatrix} u \\ b \end{pmatrix}.$$

Case 2: $\begin{pmatrix} u' \\ a' \end{pmatrix} = \begin{pmatrix} u + av \\ a \end{pmatrix}$. We have $ab - 1 \in P^\vee u, a'b' - 1 = ab' - 1 \in P^\vee(u + av)$ and we can write $ab' - 1 = \langle -\xi, u + av \rangle$ so that $a(b' + \langle \xi, v \rangle) - 1 \in P^\vee u$ and by (1), $b' + \langle \xi, v \rangle - b \in P^\vee u$.

Let

$$\varphi := \mathrm{id}_{P \oplus R} + \begin{pmatrix} -av \\ \langle \xi, v \rangle \end{pmatrix} (\xi, a) = \begin{pmatrix} \mathrm{id}_P - a\langle \xi, - \rangle v & -a^2 v \\ \langle \xi, v \rangle \xi & 1 + a\langle \xi, v \rangle \end{pmatrix}.$$

Since $(\xi, a) \begin{pmatrix} -av \\ \langle \xi, v \rangle \end{pmatrix} = 0, (\xi, a) \in \mathrm{Um}(P^\vee \oplus R)$, we see that φ is a transvection and then $\varphi \in \mathrm{E}(P \oplus R)$. A direct computation now gives

$$\varphi \begin{pmatrix} u' \\ b' \end{pmatrix} = \begin{pmatrix} u \\ b' + \langle \xi, v \rangle \end{pmatrix} \sim \begin{pmatrix} u \\ b' + \langle \xi, v \rangle - (b' + \langle \xi, v \rangle - b) \end{pmatrix} = \begin{pmatrix} u \\ b \end{pmatrix},$$

thus again $(u, b) \sim (u', b')$.

In general, we can refine the relation $(u, a) \sim (u', a')$ by finitely many $\sim$ with each adjacent pair being one of the two cases above. The result follows.

□

Proposition C.2.9. *Assume* $\dim R = d, \mathrm{rank}\ P = n, 1 \leqslant d \leqslant 2n - 1$. *Then for any $u, v \in \mathrm{Um}(P \oplus R)$, there exist $w \in P, x \in R$ such that $u \sim (w, x), v \sim (w, 1 - x)$.*

In particular, for any $u \in \mathrm{Um}(P \oplus R)$, there exist $w \in P, x \in R$ such that $u \sim (w, x) \sim (w, 1 - x)$.

Proof. Let $u=(p,a), v=(q,b)$, then $(p,q,ab)\in \mathrm{Um}(P\oplus P\oplus R)$. Since $1\leqslant d\leqslant 2n-1$, we have $(p+abp', q+abq')\in \mathrm{Um}(P\oplus P)$ for some $p', q'\in P$.

As $u=(p,a)\sim (p+abp',a), v=(q,b)\sim (q+abq',b)$, we may and we do assume from the beginning that $(p,q)\in \mathrm{Um}(P\oplus P)$. So we have $\langle \xi, p\rangle + \langle \eta, q\rangle = 1-(a+b)$ for some $\xi,\eta\in P^\vee$, so $(a+\langle \xi,p\rangle)+(b+\langle \eta,q\rangle)=1$.

Since $u=(p,a)\sim (p, a+\langle \xi,p\rangle), v=(q,b)\sim (q, b+\langle \eta, q\rangle)$, we will assume $a+b=1$. Now define $w:=p-a(p-q)=q+(1-a)(p-q)\in P$, then we have $u=(p,a)\sim (w,a), v=(q,1-a)\sim (w,1-a)$. □

Remark C.2.10. If we trace back the above proof carefully, we find that we can take

$$x=a+\langle \xi, p+abp'\rangle, w=(1-x)(p+abp')+x(q+abq'),$$

where one takes $p', q'\in P, \xi,\eta\in P^\vee$ satisfying $(p+abp', q+abq')\in \mathrm{Um}(P\oplus P), \langle \xi, p+abp'\rangle + \langle \eta, q+abq'\rangle = 1-(a+b)$, so $1-x=b+\langle \eta, q+abq'\rangle$.

Proposition C.2.11. *Assume that R is an integral domain. Let $p,q\in P, a,b\in R$ be such that $(p,a),(p,1-a),(q,b),(q,1-b)\in \mathrm{Um}(P\oplus R)$. If $(p,a)\sim (q,b),(p,1-a)\sim (q,1-b)$, then $(p,a(1-a))\sim (q,b(1-b))$.*

Proof. We may assume that for some $r,s,r',s'\in R$ we have $ra-1, s(1-a)-1\in P^\vee p, r'b-1, s'(1-b)-1\in P^\vee q$. The proof of Proposition C.2.3 (1) shows that $(r+s)a(1-a)-1\in P^\vee p, (r'+s')b(1-b)-1\in P^\vee q$.

Since $(p,a)\sim (q,b),(p,1-a)\sim (q,1-b)$, Proposition C.2.8 tells us $(p,r)\sim (q,r'),(p,s)\sim (q,s')$, and so $(p,r+s)\sim (q,r'+s')$. As $(r+s)a(1-a)-1\in P^\vee p, (r'+s')b(1-b)-1\in P^\vee q$, we have $(p,a(1-a))\sim (q,b(1-b))$ by Proposition C.2.8 (2) again. □

Proposition C.2.12. *Let R be a regular ring essentially of finite type over a field k and P a finitely generated projective R-module of constant rank $n\geqslant 2$. For $(u,a),(v,b)\in \mathrm{Um}(P\oplus R)$, the following are equivalent.*

1. *There exists $\sigma\in \mathrm{E}(P\oplus R)$ such that $\sigma(u,a)=(v,b)$.*
2. *There exists $v(t)\in \mathrm{Um}(P[t]\oplus R[t])$ such that $v(0)=(u,a), v(1)=(v,b)$.*

Proof. (1) $\Rightarrow$ (2): Take $\tilde{\sigma}(t)\in \mathrm{E}(P[t]\oplus R[t])$ such that $\tilde{\sigma}(0)=\mathrm{id}_{P\oplus R}, \tilde{\sigma}(1)=\sigma$. Define $v(t):=\tilde{\sigma}(t)(u,a)$, since $(u,a)\in \mathrm{Um}(P\oplus R)\subset \mathrm{Um}(P[t]\oplus R[t])$, we have $v(t)\in \mathrm{Um}(P[t]\oplus R[t])$ and $v(0)=(u,a), v(1)=(v,b)$.

(2) $\Rightarrow$ (1): Assume that $v(t)\in \mathrm{Um}(P[t]\oplus R[t])$ satisfies $v(0)=(u,a), v(1)=(v,b)$. By Lindel's form of Bass-Quillen conjecture (Theorem C.1.2), there exists $Q\in \mathcal{V}_n(R)$ such that $\mathrm{coker}(v(t))\cong Q[t]$. Then necessarily $Q=Q(0)=\mathrm{coker}(v(0))$ (evaluating at $t=0$ is right exact) and so $Q[t]=\mathrm{coker}(v(0)\otimes \mathrm{id}_{R[t]})$ (since $R[t]$ is a flat R-algebra). Consider the following diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R[t] & \xrightarrow{v(t)} & P[t]\oplus R[t] & \longrightarrow & Q[t] & \longrightarrow & 0 \\ & & \| & & \vdots{\scriptstyle \psi(t)} & & \| & & \\ 0 & \longrightarrow & R[t] & \xrightarrow{v(0)\otimes \mathrm{id}_{R[t]}} & P[t]\oplus R[t] & \longrightarrow & Q[t] & \longrightarrow & 0. \end{array}$$

Choose a splitting map $\lambda(t),\mu(t)$ of the two quotient maps $P[t]\oplus R[t]\to Q[t]$ in the two rows; note that it may happen that $\lambda(t)\neq \mu(t)$ as $v(t)\neq v(0)\otimes \mathrm{id}_{R[t]}$ in general (in fact, one can take $\mu(t)=\lambda(0)\otimes \mathrm{id}_{R[t]}$). We can define $\psi(t): P[t]\oplus R[t]=R[t]v(t)\oplus \lambda(Q[t])\to P[t]\oplus R[t]$ by letting $\psi(t)(v(t))=v(0), \psi(t)\circ\lambda(t)=\mu(t)$. It's easy to see that the dotted arrow $\psi(t)$ makes the above diagram commutative and $\psi(0)=\mathrm{id}_{P\oplus R}$. By 5-lemma, we have $\psi(t)\in \mathrm{Aut}_{R[t]}(P[t]\oplus R[t])$. So, as in the proof of Proposition C.1.6, we have $\psi(t)\in \mathrm{E}(P[t]\oplus R[t])$ and finally $\sigma:=\psi(1)^{-1}\in \mathrm{E}(P\oplus R), \sigma(v(0))=v(1)$. □

The following proposition is a slight variation of [**53**, Lemma 2].

Proposition C.2.13. *Let R be a commutative ring and P a finitely generated projective R-module. For any $(u,a)\in \mathrm{Um}(P\oplus R), v\in P$, we have $(u+tav, a^r)\in \mathrm{Um}(P[t]\oplus R[t])$.*

If moreover R is a regular ring essentially of finite type over a field k and P is a finitely generated projective R-module of constant rank $n\geqslant 2$, then there exists $\sigma\in \mathrm{E}(P\oplus R)$ such that $\sigma(u,a^r)=(u+av,a^r)$.

Proof. For $0\leqslant i\leqslant r$, as $(u,a^{r-i})\in \mathrm{Um}(P\oplus R)$, there exists $b_i\in R, \xi_i\in P^\vee$ such that

$$\langle \xi_0, u\rangle + a^r b_0 = 1$$
$$\langle \xi_i, u\rangle + a^{r-i} b_i = -\langle \xi_{i-1}, v\rangle, 1\leqslant i\leqslant r,$$

with $\xi_r=0, b_r=-\langle \xi_{r-1}, v\rangle$. So $\langle \xi_i, u\rangle + \langle \xi_{i-1}, v\rangle + a^{r-i}b_i=0$, for $1\leqslant i\leqslant r$.

Let

$$b(t):=\sum_{i=0}^r b_i t^i\in R[t], \xi(t):=\sum_{i=0}^r a^i\xi_i t^i\in (P^\vee)[t]=(P[t])^\vee,$$

then

$$\langle \xi(t), u+tav\rangle + a^r b(t)=\sum_{i=0}^r a^i(\langle \xi_i, u\rangle + \langle \xi_{i-1}, v\rangle + a^{r-i}b_i)t^i = 1+0t+\cdots+0t^r=1,$$

thus $(u+tav, a^r)\in \mathrm{Um}(P[t]\oplus R[t])$.

The second result then follows from the previous proposition. □

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