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# On the Bertini theorem in Arakelov geometry

Xiaozong Wang

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Xiaozong Wang. On the Bertini theorem in Arakelov geometry. Algebraic Geometry [math.AG].  
Université Paris-Saclay, 2020. English. NNT : 2020UPASM015 . tel-03010687

**HAL Id: tel-03010687**

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Submitted on 17 Nov 2020

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# On the Bertini theorem in Arakelov geometry

**Thèse de doctorat de l'Université Paris-Saclay**

École doctorale n° 574, École doctorale de Mathématiques  
Hadamard (EDMH)

Spécialité de doctorat: Mathématiques fondamentales  
Unité de recherche: Université Paris-Saclay, CNRS, Laboratoire de  
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Thèse présentée et soutenue à Orsay, le 23 septembre 2020, par

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# Remerciements

Tout d'abord, je tiens à exprimer ma gratitude à mon directeur de thèse, François Charles. Je ne saurais que sous-estimer son influence sur mon apprentissage des mathématiques depuis mon projet de M1. L'achèvement de cette thèse est dû largement à sa grande disponibilité, son ample générosité et son enthousiasme mathématique durant ces années de préparation. Les discussions qu'on a eu m'ont beaucoup nourri d'intuitions mathématiques. D'avoir été son étudiant, c'est pour moi un grand honneur et en même temps une expérience très fructifiant et très agréable.

Je suis sincèrement reconnaissant à Pascal Autissier et Shou-Wu Zhang d'avoir accepté de lire cette thèse. Leurs remarques et commentaires m'ont été très chers. Je remercie également Jean-Benoît Bost, Antoine Chambert-Loir et Mathilde Herblot de m'avoir fait l'honneur d'accepter d'être membre de ce jury.

Ces années témoignent de nombreux discussions mathématiques pleines d'idées. Si je ne suis pas capable de citer tous les mathématiciens desquels j'ai appris, j'aimerais du moins remercier Emiliano Ambrosi, Amadou Bah, Olivier Benoît, Yang Cao, Jean-Louis Colliot-Thélène, Étienne Fouvry, Wille Liu, Emanuele Macrì, Salim Tayou, Yeping Zhang, Xiaolei Zhao, pour tout ce qu'ils m'ont partagé généreusement.

La préparation de cette thèse est accompagnée d'une expérience précieuse en tant que l'organisateur du Séminaire Mathjeunes. J'aimerais exprimer mes gratitude à Jie Lin et Cong Xue de m'avoir passé le relais de l'organisation, et aussi à Zicheng Qian et Yu Min d'avoir partagé cette responsabilité avec moi. Je veux remercier tous mes amis qui ont accepté de donner des exposés à ce séminaires durant ces années. Ces exposés qui m'ont permis d'élargir ma vision des mathématiques m'ont été très précieux. J'ai aussi beaucoup profité des séminaires et des groupes de travail, je suis reconnaissant à leur organisateurs. Je veux remercier Yongqi Liang qui m'a invité à organiser une conférence à Hefei ensemble. C'était une expérience inoubliable.

Je voudrais exprimer mes remerciements à mes professeurs qui m'ont aidé avant le début de ma recherche mathématique. C'est grâce aux cours et au projets que j'ai suivi d'Anna Cadoret, Gaëtan Chenevier, Laurent Clozel, Charles Favre, Bruno Klingler, Alena Pirutka et Claire Voisin que j'ai pu approfondir ma connaissance sur les mathématiques. Ma venue en France doit beaucoup à mes professeurs à l'Université de Shandong, notamment Xijun Hu, Hualin Huang, Shige Peng, Penghui Wang, etc.

Mon séjour à Orsay durant la thèse est entouré des bons moments mathématiques et quotidiens. C'est un grand plaisir pour moi d'avoir partagé un bureau avec Anne-Edgar, Romain, Sasha, Vincent et Yanis, et d'avoir eu des discussions intéressantes avec eux. Les moments très fréquents où on accueille Guillaume et Hugo dans le bureau me sont inoubliables. Je remercie Guillaume, Hugo et Jeanne d'avoir fait des efforts particuliers pour me tirer de mon bureau à midi et me mettre à table avec les autres doctorants pendant le déjeuner. Je remercie chaleureusement Bingxiao, Céline, Changzhen, Cyril, Dorian, Elyes, Elio, Jean, Louise, Lucien, Ning, Pierre, Pierre-Louis, Ruoci, Yisheng, Yoël, Zhangchi, Zhixiang, qui ont rendu ces années à Orsay pleines de joie. Je remercie aussi Daxin, Hongjie, Huajie, Jiaming, Jiandi, Jie, Juanyong, Ming-

chen, Peiyi, Quentin, Tiago, Yanbo, Yi, Zhixin, d'avoir rendu mes années en France encore plus belles. J'aimerais aussi remercier Jingxian qui m'a accompagné durant la préparation de cette thèse.

Enfin, je remercie mes parents pour leur soutien constant.

*C'est pourquoi l'analyse de la finitude ne cesse de revendiquer contre l'historicisme la part que celui-ci avait négligée : elle a pour objet de faire surgir, au fondement de toutes les positivités et avant elles, la finitude qui les rend possibles (...)*

---

M. Foucault, *Les mots et les choses*



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*à mes parents*



# Chapitre 1

## Introduction (en français)

Dans cette thèse on étudie les théorèmes de type Bertini, c'est-à-dire les théorèmes qui confirment, sur un schéma ayant une certaine propriété géométrique, l'existence de sous-schémas fermés qui possèdent la même propriété géométrique.

Le résultat principal de cette thèse est le Théorème 1.3.4. Prenons une variété arithmétique projective régulière  $\mathcal{X}$  munie d'un faisceau hermitien ample  $\overline{\mathcal{L}} = (\mathcal{L}, \|\cdot\|)$  (voir la Section 1.2 pour les définitions précises). On considère les sections globales  $\sigma$  de  $\overline{\mathcal{L}}^{\otimes d}$  telles que  $\|\sigma\| < 1$  et dont le diviseur n'a pas de point singulier sur la fibre  $\mathcal{X}_p$  au-dessus d'aucun nombre premier  $p \leq e^{\varepsilon d}$ . Ici  $\varepsilon$  est une constante positive. Le Théorème 1.3.4 dit que la proportion de telles sections dans l'ensemble des sections de norme strictement plus petite que 1 tend vers  $\zeta_{\mathcal{X}}(1 + \dim \mathcal{X})^{-1}$  quand  $d$  tend vers l'infini.

### 1.1 Théorèmes de Bertini sur un corps

Le théorème de Bertini classique est le suivant :

**Théorème 1.1.1.** *Soit  $X$  une variété lisse quasi-projective de dimension  $m$  sur un corps infini  $k$ . Soit  $f : X \rightarrow \mathbb{P}_k^n$  une immersion. Alors  $f^{-1}(H)$  est lisse de dimension  $m - 1$  sur  $k$  pour un hyperplan général  $H$  de  $\mathbb{P}_k^n$ . En particulier, il existe un hyperplan  $H$  de  $\mathbb{P}_k^n$  tel que  $f^{-1}(H)$  est lisse de dimension  $m - 1$  sur  $k$ .*

**Remarque.** Les hyperplans de  $\mathbb{P}_k^n$  sont paramétrés par l'ensemble des  $k$ -points de la grassmannienne  $G(1, n + 1)$ . Cette grassmannienne est isomorphe à  $\mathbb{P}_k^n$ . Dire qu'un hyperplan général vérifie une certaine propriété, c'est dire qu'il existe un ouvert de Zariski non-vide  $U \subset G(1, n + 1)$  tel que tout hyperplan  $H_y$  correspondant à un  $k$ -point  $y \in U(k)$  vérifie cette propriété.

Comme  $k$  est un corps infini, un ouvert non-vide  $U \subset G(1, n + 1)$  contient un nombre infini de  $k$ -points. Ce théorème fournit ainsi une infinité d'hyperplans  $H$  tels que  $f^{-1}(H)$  est lisse de dimension  $m - 1$  sur  $k$ . Appliquant ce théorème de manière répétée, on obtient des sous-variétés lisses de  $X$  de toute dimension plus petite que  $m$ .

Il existe aussi des théorèmes de même type concernant d'autres propriétés géométriques. Le livre de Jouanolou a présenté quelques résultats. Nous les citons ci-dessous :

**Théorème 1.1.2** (Corollaire 6.11 de [Jo83]). *Soient  $k$  un corps infini,  $X$  un  $k$ -schéma de type fini, et  $f : X \rightarrow \mathbb{P}_k^n$  un  $k$ -morphisme.*

- i) Supposons  $k$  de caractéristique 0 ou  $f$  non-ramifié, et  $X$  lisse (resp. géométriquement réduit). Alors pour un  $H \in G(1, n+1)$  général, le  $k$ -schéma  $f^{-1}(H)$  est lisse (resp. géométriquement réduit).
- ii) Supposons  $\dim f(X) \geq 2$ , et  $X$  géométriquement irréductible. Alors pour un  $H \in G(1, n+1)$  général, le  $k$ -schéma  $f^{-1}(H)$  est géométriquement irréductible.

**Remarque.** Pour tout corps  $k$  (pas forcément infini), on peut trouver cet ouvert non-vide  $U$  de  $G(1, n+1)$  paramétrant les hyperplans  $H$  de  $\mathbb{P}_k^n$  tels que  $f^{-1}(H)$  a la même propriété géométrique que  $X$ , mais c'est l'infinitude du corps  $k$  qui garantit que  $U$  contient des  $k$ -points. Si  $k$  est un corps fini, il est possible que cet ouvert n'ait pas de point rationnel.

Dans [Po04], B. Poonen a montré que si on considère les hypersurfaces de degré suffisamment grand, on peut trouver, pour  $X$  lisse de dimension  $m$  et  $f : X \rightarrow \mathbb{P}_k^n$  une immersion, des hypersurfaces  $H$  telles que  $f^{-1}(H) = X \cap H$  est lisse de dimension  $m-1$ . En effet, ce qu'il a montré est que l'ensemble des hypersurfaces dont l'intersection schématique avec  $X$  est lisse de dimension  $m-1$  a une densité au sens suivant :

**Théorème 1.1.3** (Poonen, Theorem 1.1 de [Po04]). *Soit  $\mathbb{F}_q$  un corps fini de caractéristique  $p \neq q$  éléments. Soit  $X$  un sous-schéma lisse de  $\mathbb{P}_{\mathbb{F}_q}^n$  de dimension  $m$ . On a*

$$\lim_{d \rightarrow \infty} \frac{\#\{\sigma \in H^0(\mathbb{P}_{\mathbb{F}_q}^n, \mathcal{O}(d)) ; \operatorname{div} \sigma \cap X \text{ est lisse de dimension } m-1\}}{\#H^0(\mathbb{P}_{\mathbb{F}_q}^n, \mathcal{O}(d))} = \zeta_X(m+1)^{-1} > 0,$$

où  $\zeta_X$  est la fonction zêta de  $X$  définie par

$$\zeta_X(s) = \prod_{x \in |X|} (1 - q^{-s \deg x})^{-1}.$$

**Remarque.** Dans cette thèse, l'intersection des schémas se comprend comme au sens schématique.

Charles et Poonen a montré dans [CP16] un théorème de même type pour l'irréductibilité géométrique :

**Théorème 1.1.4** (Theorem 1.1 de [CP16]). *Soit  $X$  un sous-schéma géométriquement irréductible de  $\mathbb{P}_{\mathbb{F}_q}^n$ . Si  $\dim X \geq 2$ , on a*

$$\lim_{d \rightarrow \infty} \frac{\#\{\sigma \in H^0(\mathbb{P}_{\mathbb{F}_q}^n, \mathcal{O}(d)) ; \operatorname{div} \sigma \cap X \text{ est géométriquement irréductible}\}}{\#H^0(\mathbb{P}_{\mathbb{F}_q}^n, \mathcal{O}(d))} = 1.$$

**Remarque.** Si  $X$  est projectif, le théorème de Poonen est équivalent à l'assertion que pour un faisceau très ample  $\mathcal{L}$  sur  $X$ , on a

$$\lim_{d \rightarrow \infty} \frac{\#\{\sigma \in H^0(X, \mathcal{L}^{\otimes d}) ; \operatorname{div} \sigma \text{ est lisse de dimension } m-1\}}{\#H^0(X, \mathcal{L}^{\otimes d})} = \zeta_X(m+1)^{-1}.$$

En effet, soit  $X \hookrightarrow \mathbb{P}^n$  le plongement induit par le système linéaire complet de  $\mathcal{L}$ . Alors pour tout entier positif  $d$ , une section  $\sigma \in H^0(\mathbb{P}_{\mathbb{F}_q}^n, \mathcal{O}(d))$  est telle que  $\operatorname{div} \sigma \cap X$  est lisse de dimension  $m-1$  si et seulement si la restriction  $\sigma|_X \in H^0(X, \mathcal{L}^{\otimes d})$  est telle que  $\operatorname{div}(\sigma|_X)$  est lisse de dimension  $m-1$ . Autrement dit, l'ensemble  $\{\sigma \in H^0(\mathbb{P}_{\mathbb{F}_q}^n, \mathcal{O}(d)) ; \operatorname{div} \sigma \cap X \text{ est lisse de dimension } m-1\}$

est l'image réciproque de l'ensemble  $\{\sigma \in H^0(X, \mathcal{L}^{\otimes d}) ; \text{div}\sigma \text{ est lisse de dimension } m - 1\}$  par la restriction naturelle

$$H^0(\mathbb{P}_{\mathbb{F}_q}^n, \mathcal{O}(d)) \longrightarrow H^0(X, \mathcal{L}^{\otimes d}).$$

La surjectivité de cette restriction pour  $d$  grand induit l'égalité des proportions

$$\begin{aligned} & \frac{\#\{\sigma \in H^0(\mathbb{P}_{\mathbb{F}_q}^n, \mathcal{O}(d)) ; \text{div}\sigma \cap X \text{ est lisse de dimension } m - 1\}}{\#H^0(\mathbb{P}_{\mathbb{F}_q}^n, \mathcal{O}(d))} \\ = & \frac{\#\{\sigma \in H^0(X, \mathcal{L}^{\otimes d}) ; \text{div}\sigma \text{ est lisse de dimension } m - 1\}}{\#H^0(X, \mathcal{L}^{\otimes d})}, \end{aligned}$$

ce qui entraîne l'équivalence.

Notre premier résultat est une généralisation de ce théorème. On montre que la condition d'être très ample peut se remplacer par être ample, voir Theorem 3.0.1.

**Théorème 1.1.5.** *Soit  $\mathbb{F}_q$  un corps fini. Soient  $Y$  un schéma projectif de dimension  $n$  sur  $\mathbb{F}_q$ , et  $X$  un sous-schéma lisse de  $Y$  de dimension  $m$ . Soit  $\mathcal{L}$  un faisceau ample sur  $Y$ . Supposons qu'il existe un sous-schéma ouvert lisse quasi-compact  $U$  de  $Y$  contenant  $X$ . Alors on a*

$$\lim_{d \rightarrow \infty} \frac{\#\{\sigma \in H^0(Y, \mathcal{L}^{\otimes d}) ; \text{div}\sigma \cap X \text{ est lisse de dimension } m - 1\}}{\#H^0(Y, \mathcal{L}^{\otimes d})} = \zeta_X(m + 1)^{-1}.$$

## 1.2 Positivité en géométrie d'Arakelov

Dans la géométrie arithmétique, les objets qui nous intéressent sont les variétés arithmétiques. Une *variété arithmétique* est un schéma intègre séparé de type fini plat sur  $\text{Spec } \mathbb{Z}$ . L'analogie entre les variétés arithmétiques et les variétés algébriques définies sur un corps remonte à Weil, qui a comparé dans [We39] le théorème de Minkowski pour les courbes arithmétiques et le théorème de Riemann-Roch pour les courbes projectives définies sur  $\mathbb{C}$ . Cette idée est ensuite suivie par Arakelov, qui a établi dans [Ar74] et [Ar78] une théorie d'intersection pour les surfaces arithmétiques régulières.

Rappelons que  $\mathcal{X}$  est *régulière en un point fermé*  $x$  si on a

$$\dim_{\kappa(x)} \frac{\mathfrak{m}_{\mathcal{X},x}}{\mathfrak{m}_{\mathcal{X},x}^2} = \dim \mathcal{X},$$

où  $\mathfrak{m}_{\mathcal{X},x}$  est l'idéal maximal de la fibre  $\mathcal{O}_{\mathcal{X},x}$  du faisceau structural du schéma  $\mathcal{X}$  sur  $x$  et  $\dim \mathcal{X}$  est la dimension de  $\mathcal{X}$  en tant qu'un schéma ;  $\mathcal{X}$  est *singulière en*  $x$  si elle n'est pas régulière en  $x$ .

Si  $\mathcal{X}$  est régulière en tout point fermé de  $\mathcal{X}$ , on dit qu'elle est *régulière*. Elle est *singulière* si elle n'est pas régulière. Si  $\mathcal{X}$  est singulière en  $x$ , on dit que  $x$  est un *point singulier* de  $\mathcal{X}$ .

L'idée d'Arakelov est de "compactifier" une surface arithmétique régulière  $\mathcal{X}$  en enrichissant la structure complexe naturelle sur sa fibre complexe  $\mathcal{X}_{\mathbb{C}} = \mathcal{X} \times_{\text{Spec } \mathbb{Z}} \text{Spec } \mathbb{C}$ . À un faisceau inversible  $\mathcal{L}$  sur  $\mathcal{X}$ , il ajoute une structure hermitienne sur  $\mathcal{L}_{\mathbb{C}}$ , qui est la restriction de  $\mathcal{L}$  sur  $\mathcal{X}_{\mathbb{C}}$ . On dénote par  $\overline{\mathcal{L}} = (\mathcal{L}, \|\cdot\|)$  le faisceau inversible  $\mathcal{L}$  muni d'une métrique hermitienne  $\|\cdot\|$  sur  $\mathcal{L}_{\mathbb{C}}$ . Le groupe de Picard arithmétique, noté  $\widehat{\text{Pic}}(\mathcal{X})$  est le groupe des classes de faisceaux inversibles hermitiens sur  $\mathcal{X}$  pour des isomorphismes préservant les métriques.

Soit  $\overline{\mathcal{L}}$  et  $\overline{\mathcal{M}}$  deux faisceaux inversibles hermitiens. On suppose qu'il existe des sections globales non-nulles  $\sigma \in H^0(\mathcal{X}, \overline{\mathcal{L}})$  et  $\tau \in H^0(\mathcal{X}, \overline{\mathcal{M}})$  telles que  $\text{div}\sigma$  et  $\text{div}\tau$  n'ont pas de composante en

commun. Une conséquence de cette hypothèse est que quand on restreint sur  $\mathcal{X}_{\mathbb{C}}$ ,  $\operatorname{div}\sigma|_{\mathcal{X}_{\mathbb{C}}}$  et  $\operatorname{div}\tau|_{\mathcal{X}_{\mathbb{C}}}$  sont tous les deux un nombre fini de points (comptés avec multiplicité) et que

$$(\operatorname{div}\sigma|_{\mathcal{X}_{\mathbb{C}}}) \cap (\operatorname{div}\tau|_{\mathcal{X}_{\mathbb{C}}}) = \emptyset.$$

On note par  $c_1(\overline{\mathcal{M}}_{\mathbb{C}})$  la première classe de Chern de  $\overline{\mathcal{M}}_{\mathbb{C}}$ . C'est une  $(1, 1)$ -forme différentielle sur  $\mathcal{X}_{\mathbb{C}}$  qui est égale à  $-\operatorname{dd}^c \log\|\tau\|^2$  en dehors de  $\operatorname{div}\tau|_{\mathcal{X}_{\mathbb{C}}}$ . On définit

$$(\sigma.\tau)_{\text{fin}} = \sum_{x \in |\mathcal{X}|} \log \#(\mathcal{O}_x / (\sigma, \tau)_x),$$

où  $(\sigma, \tau)_x$  est l'idéal de  $\mathcal{O}_x$  engendré par l'image de  $\sigma$  et  $\tau$  par des isomorphismes quelconques  $\mathcal{L}_x \xrightarrow{\sim} \mathcal{O}_x$  et  $\mathcal{M}_x \xrightarrow{\sim} \mathcal{O}_x$ . L'idéal engendré de  $\mathcal{O}_x$  ne dépend pas des isomorphismes choisis. En écrivant  $\operatorname{div}\sigma|_{\mathcal{X}_{\mathbb{C}}} = \sum_i n_i P_i$  avec  $P_i \in |\mathcal{X}_{\mathbb{C}}|$ , on définit

$$(\sigma.\tau)_{\infty} = - \sum_i n_i \log\|\tau(P_i)\| - \int_{\mathcal{X}(\mathbb{C})} (\log\|\sigma|_{\mathcal{X}_{\mathbb{C}}}\|) c_1(\overline{\mathcal{M}}_{\mathbb{C}}).$$

Le nombre d'intersection de  $\overline{\mathcal{L}}$  et  $\overline{\mathcal{M}}$  est défini par

$$\overline{\mathcal{L}}.\overline{\mathcal{M}} = (\sigma.\tau)_{\text{fin}} + (\sigma.\tau)_{\infty}.$$

Cette définition peut s'étendre à tous les faisceaux inversibles hermitiens. Ensuite Arakelov a déduit le théorème suivant :

**Théorème 1.2.1.** [Ar74] *La construction plus haut induit un accouplement*

$$\begin{aligned} \widehat{\operatorname{Pic}}(\mathcal{X}) \times \widehat{\operatorname{Pic}}(\mathcal{X}) &\longrightarrow \mathbb{R} \\ (\overline{\mathcal{L}}, \overline{\mathcal{M}}) &\longmapsto \overline{\mathcal{L}}.\overline{\mathcal{M}}. \end{aligned}$$

De plus, cet accouplement est bilinéaire et symétrique.

Cet accouplement est défini comme l'accouplement d'intersection arithmétique sur une surface arithmétique régulière.

Avec cette théorie d'intersection et la technique de calculer le déterminant de la cohomologie d'un faisceau inversible, Deligne montre dans [De87] un théorème de Riemann-Roch arithmétique pour les surfaces arithmétiques.

En particulier, ce théorème de Riemann-Roch arithmétique a une conséquence sur l'existence des petites sections montré par Faltings dans [Fa84]. Nous citons cette conséquence sous la forme dans [So89] :

**Théorème 1.2.2.** [So89, Théorème 6] *Soit  $\overline{\mathcal{L}}$  un faisceau inversible hermitien sur une surface arithmétique régulière  $\mathcal{X}$ , tel que la forme de Chern  $c_1(\overline{\mathcal{L}})$  soit strictement positive. Pour tout fibré vectoriel hermitien  $\overline{\mathcal{E}}$  sur  $\mathcal{X}$ , et tout nombre réel  $\varepsilon > 0$ , il existe un entier  $d_0$  tel que si  $d \geq d_0$ ,  $\overline{\mathcal{E}} \otimes \overline{\mathcal{L}}^{\otimes d}$  possède une section globale non-nulle  $\sigma$  telle que*

$$\|\sigma\| < \exp\left(\left(\varepsilon - \frac{\overline{\mathcal{L}}.\overline{\mathcal{L}}}{2 \operatorname{deg} \overline{\mathcal{L}}}\right)d\right),$$

où

$$\|\sigma\| = \|\sigma\|_{\infty} = \sup_{x \in \mathcal{X}(\mathbb{C})} \|\sigma(x)\|.$$

Le théorème de Riemann-Roch arithmétique est ensuite généralisé par Gillet et Soulé dans [GS92] pour les variétés arithmétiques régulières de dimension arbitraire, en faisant intervenir les groupes de Chow arithmétiques construits dans [GS90]. Les courants de Green et les classes caractéristiques pour les fibrés vectoriels hermitiens établies dans [GS90b] et [GS90c] jouent des rôles importants dans la construction des groupes de Chow arithmétiques. Voir [Bo91] pour une introduction à ce théorème.

On considère l'amplitude arithmétique définie comme suit :

**Définition.** Soit  $M$  un espace analytique complexe. Soit  $\bar{L} = (L, \|\cdot\|)$  un faisceau inversible hermitien sur  $M$ , où  $\|\cdot\|$  est une métrique hermitienne continue sur  $L$ . Alors  $\bar{L}$  est dit *semipositif* si pour toute section  $\sigma$  de  $\bar{L}$  sur un ouvert  $U$  de  $M$ , la fonction  $-\log \|\sigma\|$  est plurisousharmonique sur  $U$ .

**Définition.** Soit  $\mathcal{X}$  une variété arithmétique projective, et soit  $\bar{\mathcal{L}}$  un faisceau inversible hermitien sur  $\mathcal{X}$ . On dit que  $\bar{\mathcal{L}}$  est *ample* sur  $\mathcal{X}$  s'il vérifie les trois conditions suivantes :

- i)  $\mathcal{L}$  est ample sur  $\mathcal{X}$  ;
- ii)  $\bar{\mathcal{L}}$  est semipositif sur l'espace analytique complexe  $\mathcal{X}(\mathbb{C})$  ;
- iii) pour tout  $d \gg 1$ ,  $H^0(\mathcal{X}, \bar{\mathcal{L}}^{\otimes d})$  est engendré par les sections de norme strictement plus petite que 1.

Cette définition est d'abord formulée pour un faisceau inversible hermitien sur une surface arithmétique dont la fibre générique est régulière par Shouwu Zhang dans [Zh92]. Zhang montre dans le même article que dans le cadre des faisceaux inversibles hermitiens sur une surface arithmétique dont la fibre générique est régulière, l'amplitude est équivalente à la positivité ( $\bar{\mathcal{L}}$  est positif si  $\bar{\mathcal{L}}$  est semipositif sur la variété complexe  $\mathcal{X}(\mathbb{C})$ ,  $\bar{\mathcal{L}} \cdot \bar{\mathcal{L}} > 0$  et  $\widehat{\deg} \bar{\mathcal{L}}|_D > 0$  pour tout diviseur  $D$ ). Cette équivalence est le théorème de Nakai-Moishezon arithmétique. La version générale pour toute variété arithmétique est montrée ensuite dans [Zh95].

Dans la géométrie d'Arakelov, l'ensemble

$$H_{\text{Ar}}^0(\mathcal{X}, \bar{\mathcal{L}}^{\otimes d}) := \{\sigma \in H^0(\mathcal{X}, \bar{\mathcal{L}}^{\otimes d}) ; \|\sigma\| < 1\}$$

est l'analogue arithmétique de l'espace vectoriel des sections globales dans le cadre des faisceaux sur une variété sur un corps, et l'analogue arithmétique de sa dimension est le nombre

$$h_{\text{Ar}}^0(\mathcal{X}, \bar{\mathcal{L}}^{\otimes d}) := \log \#H_{\text{Ar}}^0(\mathcal{X}, \bar{\mathcal{L}}^{\otimes d}).$$

Si  $\bar{\mathcal{L}}$  est un faisceau hermitien ample sur une variété arithmétique  $\mathcal{X}$ , la croissance de  $h_{\text{Ar}}^0(\mathcal{X}, \bar{\mathcal{L}}^{\otimes d})$  en fonction de  $d$  vérifie aussi une formule de Hilbert-Samuel arithmétique comme pour les faisceaux amples sur une variété.

**Théorème 1.2.3.** Soient  $\mathcal{X}$  une variété arithmétique projective de dimension  $n$ ,  $\bar{\mathcal{L}}$  un faisceau hermitien ample sur  $\mathcal{X}$  et  $\bar{\mathcal{M}}$  un faisceau hermitien localement libre de rang  $r$  sur  $\mathcal{X}$ . Alors quand  $d$  tend vers  $\infty$ ,

$$h_{\text{Ar}}^0(\mathcal{X}, \bar{\mathcal{L}}^{\otimes d} \otimes \bar{\mathcal{M}}) = \frac{r}{n!} \bar{\mathcal{L}}^n d^n + o(d^n).$$

Cette formule est détaillée dans le Chapitre 4 de cette thèse.

Il existe aussi un théorème d'indice de Hodge arithmétique, montré par Moriwaki dans [Mo09], dont l'hypothèse est automatiquement satisfaite par un faisceau hermitien ample.



### 1.3 Théorème de Bertini arithmétique

Soit  $\mathcal{X}$  une variété arithmétique, c'est-à-dire un schéma intègre séparé qui est plat de type fini sur  $\text{Spec } \mathbb{Z}$ . Dans le cas arithmétique, on s'intéresse aussi à une version analogue du théorème de Bertini sur la lissité comme dans le cas sur des corps finis. Pour un faisceau inversible ample  $\mathcal{L}$  sur une variété arithmétique projective  $\mathcal{X}$  qui est lisse sur  $\text{Spec } \mathbb{Z}$ , on voudrait définir une bonne densité pour un sous-ensemble  $\mathcal{P} \subset \bigcup_{d \geq 0} \mathbb{H}^0(\mathcal{X}, \mathcal{L}^{\otimes d})$  telle que la densité des sections  $\sigma \in \bigcup_{d \geq 0} \mathbb{H}^0(\mathcal{X}, \mathcal{L}^{\otimes d})$  dont le diviseur est lisse est positive. Cela impliquera l'existence des sections globales  $\sigma$  ayant le diviseur lisse  $\text{div } \sigma$  pour  $d$  suffisamment grand. Mais comme Poonen a expliqué dans [Po04, Section 5.7], la condition de lissité est trop forte. On a besoin de considérer la régularité au lieu de la lissité pour obtenir une densité potentiellement strictement positive.

On peut trouver au moins deux approches pour une bonne densité dans notre situation : l'une est la densité définie par Poonen dans [Po04] pour un sous-ensemble de  $\bigcup_{d \geq 0} \mathbb{H}^0(\mathbb{P}_{\mathbb{Z}}^n, \mathcal{O}(d))$ , et l'autre est la densité définie par Bhargava, Shankar and Wang dans [BSW16] pour un sous-ensemble de l'ensemble des polynômes à une variable unitaire à valeurs entières.

Dans la Section 5 de [Po04], Poonen a établi une densité pour  $\mathcal{O}(1)$  sur l'espace projectif  $\mathbb{P}_{\mathbb{Z}}^n$ . Soit  $\mathcal{P} = \bigcup_{d \geq 0} \mathcal{P}_d$  un sous-ensemble de  $\bigcup_{d \geq 0} \mathbb{H}^0(\mathbb{P}_{\mathbb{Z}}^n, \mathcal{O}(d))$ , où  $\mathcal{P}_d \subset \mathbb{H}^0(\mathbb{P}_{\mathbb{Z}}^n, \mathcal{O}(d))$ . Pour tout  $d$ , on a une  $\mathbb{Z}$ -base naturelle de  $\mathbb{H}^0(\mathbb{P}_{\mathbb{Z}}^n, \mathcal{O}(d))$  qui est composée des monômes de degré  $d$ . Pour simplifier les notations, on les note par  $f_{d,1}, \dots, f_{d,h_d}$ , où  $h_d = h^0(\mathbb{P}_{\mathbb{Q}}^n, \mathcal{O}(d))$ . Toute section  $f \in \mathbb{H}^0(\mathbb{P}_{\mathbb{Z}}^n, \mathcal{O}(d))$  peut s'écrire  $f = \sum_{i=1}^{h_d} a_i f_{d,i}$  avec  $a_i \in \mathbb{Z}$ . Poonen définit la densité supérieure  $\bar{\mu}_{\mathbb{P},d}(\mathcal{P})$  de  $\mathcal{P}_d$  comme

$$\bar{\mu}_{\mathbb{P},d}(\mathcal{P}_d) = \max_{\tau \in \mathfrak{S}_{h_d}} \limsup_{B_{\tau(1)} \rightarrow \infty} \cdots \limsup_{B_{\tau(h_d)} \rightarrow \infty} \frac{\#\left(\mathcal{P}_d \cap \left\{ \sum_{i=1}^{h_d} a_i f_{d,i} \in \mathbb{H}^0(\mathbb{P}_{\mathbb{Z}}^n, \mathcal{O}(d)) ; |a_i| \leq B_i, \forall i \right\}\right)}{\#\left\{ \sum_{i=1}^{h_d} a_i f_{d,i} \in \mathbb{H}^0(\mathbb{P}_{\mathbb{Z}}^n, \mathcal{O}(d)) ; |a_i| \leq B_i, \forall i \right\}}.$$

Ici  $\mathfrak{S}_{h_d}$  est le groupe symétrique de  $h_d$  symboles. La densité supérieure de  $\mathcal{P}$  est alors définie comme

$$\bar{\mu}_{\mathbb{P}}(\mathcal{P}) = \limsup_{d \rightarrow \infty} \bar{\mu}_{\mathbb{P},d}(\mathcal{P}_d).$$

La densité inférieure de  $\mathcal{P}$  est définie de façon similaire, et la densité de  $\mathcal{P}$  existe si ses densités supérieure et inférieure coïncident. Avec cette définition de densité, Poonen a montré le théorème suivant :

**Théorème 1.3.1** (Poonen, Theorem 5.1 de [Po04]). *Quand  $\mathcal{X}$  est un sous-schéma régulier de dimension  $m$  de  $\mathbb{P}_{\mathbb{Z}}^n$ , admettant la conjecture abc et une conjecture supplémentaire qui vaut au moins quand  $\mathcal{X}$  est projectif, la densité du sous-ensemble des  $f \in \bigcup_{d \geq 0} \mathbb{H}^0(\mathbb{P}_{\mathbb{Z}}^n, \mathcal{O}(d))$  telle que  $\text{div}(f) \cap \mathcal{X}$  est régulier de dimension  $m - 1$  est  $\zeta_{\mathcal{X}}(m + 1)^{-1}$ .*

**Remarque.** Dans la démonstration de Poonen, la conjecture abc sert à montrer que pour tout  $d$  fixé, la densité supérieure des sections globales dont le diviseur a un point singulier sur une fibre au-dessus d'un nombre premier  $p \geq M$  avec  $M > 0$  tend vers 0 quand  $M$  tend vers l'infini. La démonstration de cela suit l'idée de A. Granville dans [Gr98] que pour un polynôme  $f(x) \in \mathbb{Z}[x]$  on peut obtenir un contrôle asymptotique des facteurs carrés premiers de  $f(n)$  par la norme de  $n \in \mathbb{Z}$ . Poonen généralise cette idée au cas des polynômes à plusieurs variables dans [Po03].

Essentiellement, dans chaque degré  $d$ , on obtient la densité  $\bar{\mu}_d$  en calculant la limite coefficient par coefficient. L'action du groupe symétrique ajoute la condition que l'ordre des coefficients peut être arbitraire. Ce théorème nous permet de trouver les sections globales  $f \in \mathbb{H}^0(\mathbb{P}_{\mathbb{Z}}^n, \mathcal{O}(d))$  telles

que  $\text{div}(f) \cap \mathcal{X}$  est régulier de dimension  $m - 1$  pour  $d$  suffisamment grand.

Dans [BSW16], Bhargava, Shankar et Wang considèrent les polynômes à une variable unitaires et à coefficients entiers  $f(x) = x^d + a_1x^{d-1} + \dots + a_d \in V_d^{\text{mon}}(\mathbb{Z})$  tels que  $\mathbb{Z}[x]/(f(x))$  est l'anneau des entiers du corps  $\mathbb{Q}[x]/(f(x))$ . Cette condition signifie exactement que quand on homogénéise  $f$  en une section globale

$$F(X, Y) = X^d + a_1X^{d-1}Y + \dots + a_dY^d \in H^0(\mathbb{P}_{\mathbb{Z}}^1, \mathcal{O}(d)),$$

$\text{div}(F)$  est un diviseur régulier de  $\mathbb{P}_{\mathbb{Z}}^1$ .

Le coefficient de  $X^d$  dans  $F(X, Y)$  étant 1, pour tout premier  $p$ ,  $F(X, Y) \bmod p$  est un polynôme à coefficients dans  $\mathbb{F}_p$  qui est homogène de degré  $d$  et qui n'est pas divisible par  $Y$ . Alors pour tout  $p$ ,  $\text{div}(F)$  ne contient pas le point à l'infini  $\infty_{\mathbb{F}_p} = \text{div}(Y) \cap \mathbb{P}_{\mathbb{F}_p}^1$  sur la fibre  $\mathbb{P}_{\mathbb{F}_p}^1$ . Cela signifie que l'intersection de  $\text{div}(F)$  avec le diviseur à l'infini  $\infty_{\mathbb{Z}} = \text{div}(Y)$  est vide, donc  $\text{div}(F)$  est inclus dans l'espace affine  $\mathbb{A}_{\mathbb{Z}}^1 = \mathbb{P}_{\mathbb{Z}}^1 - \text{div}(Y)$ . Alors on peut identifier  $\text{div}(F)$  avec  $\text{div}(f)$ , où  $f$  est comme plus haut, et  $\text{div}(F) \simeq \text{Spec } \mathbb{Z}[x]/(f(x))$ . Notons que le schéma  $\text{Spec } \mathbb{Z}[x]/(f(x))$  est régulier si et seulement si  $\mathbb{Z}[x]/(f(x))$  est normal. La régularité de  $\text{Spec } \mathbb{Z}[x]/(f(x))$  est donc équivalente au fait que  $\mathbb{Z}[x]/(f(x))$  est l'anneau des entiers de  $\mathbb{Q}[x]/(f(x))$ . Dans leur article, fixant le degré  $d > 1$ , ils ordonnent les polynômes unitaires à coefficients entiers  $f(x) = x^d + a_1x^{d-1} + \dots + a_d$  par une fonction de hauteur

$$H(f) := \max\{|a_i|^{\frac{1}{i}}\},$$

et calculent la densité d'un sous-ensemble  $\mathcal{P}_d \subset V_d^{\text{mon}}(\mathbb{Z})$  par

$$\mu_{H,d}(\mathcal{P}_d) = \lim_{R \rightarrow \infty} \frac{\#\{\mathcal{P}_d \cap \{f \in V_d^{\text{mon}}(\mathbb{Z}) ; H(f) \leq R\}\}}{\#\{f \in V_d^{\text{mon}}(\mathbb{Z}) ; H(f) \leq R\}}.$$

Identifiant  $V_d^{\text{mon}}(\mathbb{Z})$  avec l'ensemble  $\{F \in H^0(\mathbb{P}_{\mathbb{Z}}^1, \mathcal{O}(d)) ; \text{div}(F) \cap \infty_{\mathbb{Z}} = \emptyset\}$  par homogénéisation, la densité de  $\mathcal{P}_d$  peut se comprendre comme

$$\mu_{H,d}(\mathcal{P}_d) = \lim_{R \rightarrow \infty} \frac{\#\{\mathcal{P}_d \cap \{F \in H^0(\mathbb{P}_{\mathbb{Z}}^1, \mathcal{O}(d)) ; \text{div}(F) \cap \infty_{\mathbb{Z}} = \emptyset, H(F) \leq R\}}{\#\{F \in H^0(\mathbb{P}_{\mathbb{Z}}^1, \mathcal{O}(d)) ; \text{div}(F) \cap \infty_{\mathbb{Z}} = \emptyset, H(F) \leq R\}}.$$

On peut alors reformuler [BSW16, Theorem 1.2] comme suivant :

**Théorème 1.3.2** (Theorem 1.2 de [BSW16]). *Pour un  $d > 1$  fixé, soit*

$$\mathcal{P}_d := \{F \in H^0(\mathbb{P}_{\mathbb{Z}}^1, \mathcal{O}(d)) ; \text{div}(F) \cap \infty_{\mathbb{Z}} = \emptyset, \text{div}(F) \text{ est régulier de dimension } 1\}.$$

Alors, on a

$$\mu_{H,d}(\mathcal{P}_d) = \zeta(2)^{-1}.$$

**Remarque.** Ce résultat est similaire au théorème de Poonen. En effet, si on note que  $\zeta(2)^{-1}$  peut s'exprimer par des valeurs de la fonction zêta de la droite affine sur  $\text{Spec } \mathbb{Z}$

$$\zeta_{\mathbb{A}_{\mathbb{Z}}^1}(s) = \prod_p \zeta_{\mathbb{A}_{\mathbb{F}_p}^1}(s) = \prod_p \frac{1}{1 - p^{1-s}},$$

c'est-à-dire,

$$\zeta(2)^{-1} = \prod_p (1 - p^{-2}) = \zeta_{\mathbb{A}_{\mathbb{Z}}^1}(3)^{-1},$$

le Théorème 1.3.2 nous dit que pour tout  $d > 1$ , la densité du sous-ensemble  $\mathcal{P}_d$  de

$$\{F \in H^0(\mathbb{P}_{\mathbb{Z}}^1, \mathcal{O}(d)) ; \text{div } F \cap \infty_{\mathbb{Z}} = \emptyset\}$$

composé des sections dont le diviseur est régulier est égale à

$$\zeta_{\mathbb{P}_{\mathbb{Z}}^1 - \infty_{\mathbb{Z}}} (1 + \dim(\mathbb{P}_{\mathbb{Z}}^1 - \infty_{\mathbb{Z}}))^{-1} = \zeta_{\mathbb{A}_{\mathbb{Z}}^1}(3)^{-1}.$$

On retrouve un énoncé similaire à celui du Théorème 1.3.1. Mais comme on ne considère que des sections globales dont le diviseur est disjoint de  $\infty_{\mathbb{Z}}$ , on ne peut pas retrouver le théorème de Poonen pour  $\mathcal{X} = \mathbb{A}_{\mathbb{Z}}^1$  dans  $\mathbb{P}_{\mathbb{Z}}^1$ .

Ces deux types de densités sont définies en faisant intervenir une base de  $H^0(\mathbb{P}_{\mathbb{Z}}^n, \mathcal{O}(d))$  (une base de  $H^0(\mathbb{P}_{\mathbb{Z}}^1, \mathcal{O}(d))$  dans le cas de Bhargava, Shankar et Wang). Dans le cas des espaces projectifs, on a un choix naturel pour une telle base, qui est la base composée des monômes de degré correspondant. Mais une telle base canonique n'existe pas quand on considère  $H^0(\mathcal{X}, \mathcal{L}^{\otimes d})$  pour une variété arithmétique projective  $\mathcal{X}$  générale munie d'un faisceau inversible ample  $\mathcal{L}$ . De plus, il n'est pas suffisant de considérer les faisceaux inversibles amples pour comprendre la nature arithmétique des schémas sur  $\text{Spec } \mathbb{Z}$ . On a besoin de l'amplitude arithmétique présentée dans la Section 1.2 qui induit des bonnes propriétés de finitude.

On aimerait donc penser à une version arithmétique du théorème de Bertini dans le contexte de la géométrie d'Arakelov, notamment avec la notion des faisceaux hermitiens amples sur des variétés arithmétiques.

Pour un faisceau hermitien ample  $\overline{\mathcal{L}}$  fixé, on dit qu'un sous-ensemble

$$\mathcal{P} \subset \bigcup_{d \geq 0} H_{\text{Ar}}^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes d})$$

est de densité  $\rho$  pour un certain  $0 \leq \rho \leq 1$  si

$$\lim_{d \rightarrow \infty} \frac{\#(\mathcal{P} \cap H_{\text{Ar}}^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes d}))}{\#H_{\text{Ar}}^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes d})} = \rho.$$

On définit la densité supérieure et la densité inférieure de la même façon. Quand elles existent, on écrit la densité, la densité supérieure et la densité inférieure de  $\mathcal{P}$  par  $\mu(\mathcal{P})$ ,  $\overline{\mu}(\mathcal{P})$  and  $\underline{\mu}(\mathcal{P})$ , respectivement.

Dans [Ch17], Charles montre le théorème de Bertini arithmétique sur l'irréductibilité dans ce contexte, dont l'énoncé est le suivant :

**Théorème 1.3.3.** *Soit  $\mathcal{X}$  une variété arithmétique projective, et soit  $\overline{\mathcal{L}}$  un faisceau inversible hermitien ample sur  $\mathcal{X}$ . Soient  $\mathcal{Y}$  un schéma intègre de type fini sur  $\text{Spec } \mathbb{Z}$  et  $f : \mathcal{Y} \rightarrow \mathcal{X}$  un morphisme génériquement lisse sur son image. On suppose de plus que la dimension de l'image de  $\mathcal{Y}$  est plus grande ou égale à 2. Soit  $\varepsilon > 0$  un nombre réel. Alors l'ensemble*

$$\left\{ \sigma \in \bigcup_{d > 0} H^0(\mathcal{X}, \overline{\mathcal{L}}(\varepsilon)^{\otimes d}) ; \|\sigma|_{f(\mathcal{Y}(\mathbb{C}))}\|_{\infty} < 1 \text{ et } \text{div}(f^*\sigma)_{\text{horiz}} \text{ est irréductible} \right\}$$

a densité 1 dans  $\{\sigma \in \bigcup_{d > 0} H^0(\mathcal{X}, \overline{\mathcal{L}}(\varepsilon)^{\otimes d}) ; \|\sigma|_{f(\mathcal{Y}(\mathbb{C}))}\|_{\infty} < 1\}$ .

Ce résultat peut se comparer à celui de Breuillard et Varjú dans [BV19] pour les polynômes à coefficients dans  $\{0, 1\}$ . Breuillard et Varjú montre que si on admet l'hypothèse de Riemann pour la fonction zeta de Dedekind  $\zeta_K$  pour tout corps de nombres de la forme  $K = \mathbb{Q}(a)$  pour une racine  $a$  d'un polynôme à coefficients dans  $\{0, 1\}$ , la proportion des polynômes irréductibles dans l'ensemble des polynômes de degré  $d$  à coefficients dans  $\{0, 1\}$  tels que  $P(0) \neq 0$  tend vers 1 quand  $d$  tend vers l'infini. (Pour chaque  $d$ , cet ensemble est fini.)

Dans cette thèse, on montre le théorème suivant :

**Théorème 1.3.4.** *Soit  $\mathcal{X}$  une variété arithmétique régulière projective de dimension  $n$ , et soit  $\overline{\mathcal{L}}$  un faisceau hermitien ample sur  $\mathcal{X}$ . Il existe une constante  $\varepsilon > 0$  telle qu'en écrivant*

$$\mathcal{P}_{d,p \leq e^{\varepsilon d}} := \left\{ \sigma \in H_{\text{Ar}}^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes d}) ; \begin{array}{l} \text{div} \sigma \text{ n'a pas de point singulier de} \\ \text{caractéristique résiduelle plus petite que } e^{\varepsilon d} \end{array} \right\}$$

et  $\mathcal{P}_A = \bigcup_{d \geq 0} \mathcal{P}_{d,p \leq e^{\varepsilon d}}$ , on a

$$\mu(\mathcal{P}_A) = \zeta_{\mathcal{X}}(1+n)^{-1},$$

où  $\zeta_{\mathcal{X}}(s)$  est la fonction zêta

$$\zeta_{\mathcal{X}}(s) = \prod_{x \in |\mathcal{X}|} (1 - \#\kappa(x)^{-s})^{-1}.$$

Ici  $\kappa(x)$  est le corps résiduel de  $x$ , et la caractéristique résiduelle d'un point fermé  $x$  dans  $\mathcal{X}$  est la caractéristique de son corps résiduel.

**Théorème 1.3.5.** *Soit  $\mathcal{X}$  une variété arithmétique régulière projective de dimension  $n$ , et soit  $\overline{\mathcal{L}}$  un faisceau hermitien ample sur  $\mathcal{X}$ . Notons*

$$\mathcal{P}_d := \left\{ \sigma \in H_{\text{Ar}}^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes d}) ; \text{div} \sigma \text{ est régulier} \right\}$$

et  $\mathcal{P} = \bigcup_{d \geq 0} \mathcal{P}_d$ . On a

$$\overline{\mu}(\mathcal{P}) \leq \zeta_{\mathcal{X}}(1+n)^{-1},$$

où  $\overline{\mu}(\mathcal{P})$  est la densité supérieure de  $\mathcal{P}$ .

*Démonstration.* Si une section  $\sigma \in H_{\text{Ar}}^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes d})$  est telle que  $\text{div} \sigma$  est régulier, alors en particulier son diviseur n'a pas de point singulier de caractéristique résiduelle plus petite que  $e^{\varepsilon d}$  avec la constante  $\varepsilon$  comme dans le théorème plus haut. Alors naturellement  $\mathcal{P}_d \subset \mathcal{P}_{d,p \leq e^{\varepsilon d}}$  et  $\mathcal{P} \subset \mathcal{P}_A$ . Donc on a

$$\overline{\mu}(\mathcal{P}) \leq \mu(\mathcal{P}_A) = \zeta_{\mathcal{X}}(1+n)^{-1}.$$

□

Le Théorème 1.3.4 a aussi un corollaire que l'on va montrer à la fin de cette thèse :

**Corollaire 1.3.1.** *Soit  $\mathcal{X}$  une variété arithmétique régulière projective de dimension  $n$ , et soit  $\overline{\mathcal{L}}$  un faisceau hermitien ample sur  $\mathcal{X}$ . Il existe une constante  $c > 1$  telle que pour tout  $R > 1$  on a*

$$\lim_{d \rightarrow \infty} \frac{\# \left\{ \sigma \in H^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes d}) ; \begin{array}{l} \|\sigma\|_{\infty} < R^d, \text{ Sing}(\text{div} \sigma) \text{ n'a pas de point} \\ \text{de caractéristique résiduelle plus petite que } (cR)^{\frac{d}{2}} \end{array} \right\}}{\# \left\{ \sigma \in H^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes d}) ; \|\sigma\|_{\infty} < R^d \right\}} = \zeta_{\mathcal{X}}(1+n)^{-1}$$

Dans [Au01], [Au02], Autissier a donné un autre analogue arithmétique des théorèmes de Bertini. Il montre comme cas particulier que si  $\mathcal{X}$  est une variété arithmétique de dimension  $n$  sur un anneau des entiers  $\mathcal{O}_K$  (où ici  $K$  est un corps de nombres), et  $\overline{\mathcal{L}}$  un faisceau hermitien très ample sur  $\mathcal{X}$ , alors il existe une extension finie  $L$  de  $K$  et une section  $\sigma \in H^0(\mathcal{X}_{\mathcal{O}_L}, \overline{\mathcal{L}})$  telle que en écrivant  $g : \text{Spec } \mathcal{O}_L \rightarrow \text{Spec } \mathcal{O}_K$  le morphisme induit de  $\mathcal{O}_K \hookrightarrow \mathcal{O}_L$ , pour tout point fermé  $b \in \text{Spec } \mathcal{O}_L$ , la fibre du diviseur  $(\text{div } \sigma)_b$  est lisse si  $\mathcal{X}_{g(b)}$  l'est. De plus, on peut borner la hauteur de  $\text{div } \sigma$  définie par  $\overline{\mathcal{L}}$  en fonction de celle de  $\mathcal{X}$ ,  $\deg_{\mathcal{L}_{\mathbb{Q}}} \mathcal{X}_{\mathbb{Q}}$ ,  $n$  et une constante effective ne dépendant que de  $\overline{\mathcal{L}}$  et  $n$ .

Ce résultat est plus fort que le notre au sens où le diviseur qu'il donne vient d'une section globale du faisceau  $\overline{\mathcal{L}}$ , et de plus le diviseur vérifie la condition sur la lissité au lieu de la régularité. Le désavantage de ce résultat est qu'il est indispensable de passer aux changements de base finis pour trouver une telle section globale. En particulier, si  $\mathcal{X}$  est définie sur  $\text{Spec } \mathbb{Z}$ , il y a très peu de chance qu'on puisse trouver un diviseur défini sur  $\text{Spec } \mathbb{Z}$  par la méthode d'Autissier. Notre résultat, en revanche, permet de trouver des diviseurs qui sont définis sur  $\text{Spec } \mathbb{Z}$  si la variété arithmétique que l'on considère l'est.

## 1.4 Méthode de la preuve

Soient  $\mathcal{X}$  une variété arithmétique projective régulière de dimension  $n$ , et  $\overline{\mathcal{L}}$  un faisceau hermitien ample sur  $\mathcal{X}$ .

On dénote

$$\mathcal{P}_{d,p \leq d^{\frac{1}{n+1}}} := \left\{ \sigma \in H_{\text{Ar}}^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes d}) ; \begin{array}{l} \text{div } \sigma \text{ n'a pas de point singulier de caractéristique} \\ \text{résiduelle plus petite que } d^{\frac{1}{n+1}} \end{array} \right\},$$

et

$$\mathcal{Q}_d^m := \left\{ \sigma \in H_{\text{Ar}}^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes d}) ; \begin{array}{l} \text{div } \sigma \text{ a un point singulier de caractéristique} \\ \text{résiduelle entre } d^{\frac{1}{n+1}} \text{ et } e^{\varepsilon d} \end{array} \right\}.$$

On montre le Théorème 1.3.4 en donnant les deux estimées suivantes pour  $d$  suffisamment grand :

$$\left| \frac{\#\mathcal{P}_{d,p \leq d^{\frac{1}{n+1}}}}{\#H_{\text{Ar}}^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes d})} - \zeta_{\mathcal{X}}(n+1)^{-1} \right| = O(d^{-\frac{1}{n+1}}), \quad (1.1)$$

et

$$\frac{\#\mathcal{Q}_d^m}{\#H_{\text{Ar}}^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes d})} = O(d^{-\frac{1}{n+1}}). \quad (1.2)$$

Ici la constante intervenant dans le grand  $O$  ne dépend que de  $\mathcal{X}$  et  $\overline{\mathcal{L}}$ .

En effet, si on note que

$$\mathcal{P}_{d,p \leq e^{\varepsilon d}} \subset \mathcal{P}_{d,p \leq d^{\frac{1}{n+1}}} \subset \mathcal{P}_{d,p \leq e^{\varepsilon d}} \cup \mathcal{Q}_d^m,$$

on a

$$\begin{aligned} & \left| \frac{\#\mathcal{P}_{d,p \leq e^{\varepsilon d}}}{\#H_{\text{Ar}}^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes d})} - \zeta_{\mathcal{X}}(n+1)^{-1} \right| \\ & \leq \left| \frac{\#\mathcal{P}_{d,p \leq d^{\frac{1}{n+1}}}}{\#H_{\text{Ar}}^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes d})} - \zeta_{\mathcal{X}}(n+1)^{-1} \right| + \left| \frac{\#\mathcal{P}_{d,p \leq e^{\varepsilon d}}}{\#H_{\text{Ar}}^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes d})} - \frac{\#\mathcal{P}_{d,p \leq d^{\frac{1}{n+1}}}}{\#H_{\text{Ar}}^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes d})} \right| \\ & \leq \left| \frac{\#\mathcal{P}_{d,p \leq d^{\frac{1}{n+1}}}}{\#H_{\text{Ar}}^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes d})} - \zeta_{\mathcal{X}}(n+1)^{-1} \right| + \frac{\#\mathcal{Q}_d^m}{\#H_{\text{Ar}}^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes d})} = O(d^{-\frac{1}{n+1}}), \end{aligned}$$

qui tend vers 0 quand  $d$  tend vers l'infini.

On donne d'abord l'idée de la preuve de l'estimée (1.1). Notons que le diviseur  $\text{div}\sigma$  d'une section  $\sigma \in H^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes d})$  est singulier sur un point fermé  $x \in |\mathcal{X}_p|$  si et seulement si

$$\dim_{\kappa(x)} \frac{\mathfrak{m}_{\text{div}\sigma, x}}{\mathfrak{m}_{\text{div}\sigma, x}^2} = \dim \mathcal{X} = n.$$

Cette condition ne dépend que de l'image de  $\sigma$  dans  $H^0(\mathcal{X}_{p^2}, \overline{\mathcal{L}}^{\otimes d})$  par la restriction, où  $\mathcal{X}_{p^2} = \mathcal{X} \times_{\text{Spec } \mathbb{Z}} \text{Spec } \mathbb{Z}/p^2$ . Autrement dit, si on fixe  $d$ , le lieu singulier du diviseur  $\text{div}\sigma$  d'une section  $\sigma \in H^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes d})$  sur les fibres  $\mathcal{X}_p$  avec  $p \leq d^{\frac{1}{n+1}}$  ne dépend que de sa restriction  $\sigma \bmod M_d$  dans  $H^0(\mathcal{X}_{M_d}, \overline{\mathcal{L}}^{\otimes d})$ , où  $M_d = \prod_{p \leq d^{\frac{1}{n+1}}} p^2$ . On veut transformer la proportion de  $\mathcal{P}_{d, p \leq d^{\frac{1}{n+1}}}$  dans  $H_{\text{Ar}}^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes d})$  en celle de l'ensemble

$$\mathcal{P}'_{d, M_d} := \left\{ \sigma' \in H^0(\mathcal{X}_{M_d}, \overline{\mathcal{L}}^{\otimes d}) ; \forall p \leq d^{\frac{1}{n+1}}, \forall x \in |\mathcal{X}_p|, \dim_{\kappa(x)} \frac{\mathfrak{m}_{\text{div}\sigma', x}}{\mathfrak{m}_{\text{div}\sigma', x}^2} = n - 1 \right\}.$$

L'étude de la restriction modulo  $N$  des sections globales dans le chapitre 4 nous fournit

$$\left| \frac{\#\mathcal{P}_{d, p \leq d^{\frac{1}{n+1}}}}{\#H_{\text{Ar}}^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes d})} - \frac{\#\mathcal{P}'_{d, M_d}}{\#H^0(\mathcal{X}_{M_d}, \overline{\mathcal{L}}^{\otimes d})} \right| \leq e^{-\eta d}$$

où  $\eta$  est une constante positive. Alors l'estimée (1.1) vaut si on prouve

$$\left| \frac{\#\mathcal{P}'_{d, M_d}}{\#H^0(\mathcal{X}_{M_d}, \overline{\mathcal{L}}^{\otimes d})} - \zeta_{\mathcal{X}}(n+1)^{-1} \right| = O(d^{-\frac{1}{n+1}}).$$

L'étude de la convergence de  $\prod_{p \leq d^{\frac{1}{n+1}}} \zeta_{\mathcal{X}_p}(n+1)^{-1}$  vers  $\zeta_{\mathcal{X}}(n+1)^{-1}$  dans le chapitre 5 qui nous donne

$$\left| \prod_{p \leq d^{\frac{1}{n+1}}} \zeta_{\mathcal{X}_p}(n+1)^{-1} - \zeta_{\mathcal{X}}(n+1)^{-1} \right| = O(d^{-\frac{1}{n+1}}).$$

Cela nous conduit à prouver

$$\left| \frac{\#\mathcal{P}'_{d, M_d}}{\#H^0(\mathcal{X}_{M_d}, \overline{\mathcal{L}}^{\otimes d})} - \prod_{p \leq d^{\frac{1}{n+1}}} \zeta_{\mathcal{X}_p}(n+1)^{-1} \right| = O(d^{-\frac{1}{n+1}}).$$

Selon le lemme chinois, on a un isomorphisme

$$H^0(\mathcal{X}_{M_d}, \overline{\mathcal{L}}^{\otimes d}) \simeq \prod_{p \leq d^{\frac{1}{n+1}}} H^0(\mathcal{X}_{p^2}, \overline{\mathcal{L}}^{\otimes d}).$$

Si on écrit

$$\mathcal{P}'_{d, p^2} := \left\{ \sigma \in H^0(\mathcal{X}_{p^2}, \overline{\mathcal{L}}^{\otimes d}) ; \forall x \in |\mathcal{X}_{p^2}|, \dim_{\kappa(x)} \frac{\mathfrak{m}_{\text{div}\sigma, x}}{\mathfrak{m}_{\text{div}\sigma, x}^2} = n - 1 \right\},$$

le sous-ensemble  $\mathcal{P}'_{d,M_d}$  de  $H^0(\mathcal{X}_{M_d}, \overline{\mathcal{L}}^{\otimes d})$  correspond exactement au sous-ensemble  $\prod_{p \leq d^{\frac{1}{n+1}}} \mathcal{P}'_{d,p^2}$  de  $\prod_{p \leq d^{\frac{1}{n+1}}} H^0(\mathcal{X}_{p^2}, \overline{\mathcal{L}}^{\otimes d})$  par l'isomorphisme plus haut.

On montre d'abord l'estimée

$$\left| \frac{\#\mathcal{P}'_{d,p^2}}{\#H^0(\mathcal{X}_{p^2}, \overline{\mathcal{L}}^{\otimes d})} - \zeta_{\mathcal{X}_p}(n+1)^{-1} \right| = O\left(\left(\frac{d}{p}\right)^{-\frac{2}{n}}\right) \quad (1.3)$$

pour tout  $p \leq d^{\frac{1}{n+1}}$ , suivant la démonstration du Théorème 1.1.5. Les étapes de la preuve sont les suivantes :

1. D'abord, on montre que la proportion du sous-ensemble des sections  $\sigma \in H^0(\mathcal{X}_{p^2}, \overline{\mathcal{L}}^{\otimes d})$  qui vérifient  $\dim_{\kappa(x)} \frac{\mathfrak{m}_{\text{div}\sigma, x}^{\text{div}\sigma, x}}{\mathfrak{m}_{\text{div}\sigma, x}^2} = n-1$  pour tout point fermé  $x$  de degré plus petit ou égal à un entier  $r$  est égale à

$$\prod_{x \in |\mathcal{X}|, \deg x \leq r} (1 - \#\kappa(x)^{-(1+n)})$$

quand  $r$  n'est pas très grand. On donne une borne supérieure  $r_d$  pour  $r$  qui dépend de  $d$  où cette proportion vaut pour tout  $0 < r \leq r_d$ .

2. On montre ensuite que pour un entier  $N$  fixé, la proportion des  $\sigma \in H^0(\mathcal{X}_{p^2}, \overline{\mathcal{L}}^{\otimes d})$  telles qu'il existe un point fermé  $x$  de degré entre  $r_d$  et  $\frac{d}{nN}$  où  $\dim_{\kappa(x)} \frac{\mathfrak{m}_{\text{div}\sigma, x}^{\text{div}\sigma, x}}{\mathfrak{m}_{\text{div}\sigma, x}^2} \neq n-1$  tend vers 0 quand  $d$  tend vers l'infini.
3. On trouve une constante  $N(p)$  qui dépend de  $p$  telle que la proportion des  $\sigma \in H^0(\mathcal{X}_{p^2}, \overline{\mathcal{L}}^{\otimes d})$  vérifiant la condition qu'il existe un point fermé  $x$  de degré strictement plus grand que  $\frac{d}{nN(p)}$  où  $\dim_{\kappa(x)} \frac{\mathfrak{m}_{\text{div}\sigma, x}^{\text{div}\sigma, x}}{\mathfrak{m}_{\text{div}\sigma, x}^2} \neq n-1$  tend vers 0 quand  $d$  tend vers l'infini.
4. Finalement, l'estimée (1.3) suit si on rassemble ces trois estimées.

On compare ensuite  $\prod_{p \leq d^{\frac{1}{n+1}}} \frac{\#\mathcal{P}'_{d,p^2}}{\#H^0(\mathcal{X}_{p^2}, \overline{\mathcal{L}}^{\otimes d})}$  avec  $\prod_{p \leq d^{\frac{1}{n+1}}} \zeta_{\mathcal{X}_p}(n+1)^{-1}$  et puis on montre que

$$\left| \prod_{p \leq d^{\frac{1}{n+1}}} \frac{\#\mathcal{P}'_{d,p^2}}{\#H^0(\mathcal{X}_{p^2}, \overline{\mathcal{L}}^{\otimes d})} - \prod_{p \leq d^{\frac{1}{n+1}}} \zeta_{\mathcal{X}_p}(n+1)^{-1} \right| = O\left(\sum_{p \leq d^{\frac{1}{n+1}}} \left(\frac{d}{p}\right)^{-\frac{2}{n}}\right),$$

ce qui nous permet de conclure la preuve de l'estimée (1.1).

L'estimée (1.2)

$$\frac{\#\mathcal{Q}_d^m}{\#H_{\text{Ar}}^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes d})} = O(d^{-\frac{1}{n+1}})$$

se déduit du résultat suivant : il existe une constante  $c > 0$  telle que pour tout  $d \gg 1$  et tout nombre premier  $p$  tel que  $\mathcal{X}_p$  est lisse et irréductible, écrivant

$$\mathcal{Q}_{d,p^2} := \left\{ \sigma \in H^0(\mathcal{X}_{p^2}, \overline{\mathcal{L}}^{\otimes d}) ; \exists x \in |\mathcal{X}_{p^2}|, \dim_{\kappa(x)} \frac{\mathfrak{m}_{\text{div}\sigma, x}^{\text{div}\sigma, x}}{\mathfrak{m}_{\text{div}\sigma, x}^2} = n \right\},$$

on a

$$\frac{\#\mathcal{Q}_{d,p^2}}{\#H^0(\mathcal{X}_{p^2}, \overline{\mathcal{L}}^{\otimes d})} \leq c \cdot p^{-2}. \quad (1.4)$$

En effet, pour  $d$  suffisamment grand, si  $p \leq e^{\varepsilon d}$ , on montre dans la Section 4.2 que

$$\frac{\#\left\{\sigma \in H_{\text{Ar}}^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes d}), ; \sigma \bmod p^2 \in \mathcal{Q}_{d,p^2}\right\}}{\#H_{\text{Ar}}^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes d})} \leq 4 \frac{\#\mathcal{Q}_{d,p^2}}{\#H^0(\mathcal{X}_{p^2}, \overline{\mathcal{L}}^{\otimes d})}.$$

Comme

$$\mathcal{Q}_d^m \subset \bigcup_{d^{\frac{1}{n+1}} \leq p \leq e^{\varepsilon d}} \left\{\sigma \in H_{\text{Ar}}^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes d}), ; \sigma \bmod p^2 \in \mathcal{Q}_{d,p^2}\right\},$$

on obtient l'estimée (1.2) en faisant la somme des  $\frac{\#\mathcal{Q}_{d,p^2}}{\#H^0(\mathcal{X}_{p^2}, \overline{\mathcal{L}}^{\otimes d})}$ .

On montre l'inégalité (1.4) en suivant la preuve du Lemme 5.9 de [Po04]. On montre d'abord que la proportion des sections dans  $H^0(\mathcal{X}_p, \overline{\mathcal{L}}^{\otimes d})$  dont le diviseur a le lieu singulier de dimension positive est majorée par un multiple de  $p^{-2}$  quand  $d$  est suffisamment grand (cette condition sur  $d$  ne dépend pas de  $p$ ). Ceci est un corollaire de l'assertion que si on fixe un diviseur horizontal régulier, la proportion des sections dans  $H^0(\mathcal{X}_p, \overline{\mathcal{L}}^{\otimes d})$  dont le diviseur a le lieu singulier dont l'intersection avec le diviseur choisi est non-vide est majorée par un multiple de  $p^{-2}$  quand  $d$  est suffisamment grand. Ensuite, on montre que la proportion des sections  $\sigma$  dans  $H^0(\mathcal{X}_{p^2}, \overline{\mathcal{L}}^{\otimes d})$  où on peut trouver un point fermé  $x$  de  $\mathcal{X}_p$  de degré plus petit que ou égal à  $\frac{d}{nN(p)}$  avec  $\dim_{\kappa(x)} \frac{\mathfrak{m}_{\text{div}\sigma, x}^{\text{div}\sigma, x}}{\mathfrak{m}_{\text{div}\sigma, x}^2} \neq n-1$  est majorée par un multiple de  $p^{-2}$  et la même conclusion vaut pour la proportion des sections  $\sigma$  dans  $H^0(\mathcal{X}_{p^2}, \overline{\mathcal{L}}^{\otimes d})$  où le lieu singulier de  $\text{div}\sigma|_{\mathcal{X}_p}$  est de dimension 0 et où on peut trouver un point fermé  $x$  de  $\mathcal{X}_p$  de degré plus grand que ou égal à  $\frac{d}{nN(p)}$  avec  $\dim_{\kappa(x)} \frac{\mathfrak{m}_{\text{div}\sigma, x}^{\text{div}\sigma, x}}{\mathfrak{m}_{\text{div}\sigma, x}^2} \neq n-1$ . Cela implique l'inégalité (1.4) et donc l'estimée (1.2) suit.

## 1.5 Organisation du texte

Dans le Chapitre 3 on montre le Théorème 1.1.5, qui est une version généralisée du théorème de lissité de Bertini sur un corps fini démontré par Poonen (cité comme le Théorème 1.1.3).

Dans le Chapitre 4, on parle de l'amplitude arithmétique en général. On rappelle dans la Section 4.1 les définitions précises concernant l'amplitude. Dans la Section 4.2 on présente deux résultats concernant la réduction modulo  $N$  ( $N \in \mathbb{Z}_{>0}$ ) des sections globales d'une puissance d'un faisceau hermitien ample. Le premier des deux résultats est montré par Charles dans [Ch17].

Les chapitres suivants se vouent à la démonstration du Théorème 1.3.4.

Dans le Chapitre 5, on présente les calculs de la vitesse de convergence de

$$\prod_{x \in |\mathcal{X}_p|, \deg x \leq r} (1 - \#\kappa(x))^{-(n+1)}$$

vers  $\zeta_{\mathcal{X}_p}(n+1)^{-1}$  quand  $r \rightarrow \infty$ , et celle de  $\prod_{p \leq R} \zeta_{\mathcal{X}_p}(n+1)^{-1}$  vers  $\zeta_{\mathcal{X}}(n+1)^{-1}$  quand  $R \rightarrow \infty$  pour une variété arithmétique  $\mathcal{X}$ .

La démonstration du Théorème 1.3.4 s'appuie sur une estimée effective de la proportion des sections globales dont le diviseur n'a pas de point singulier sur une seule fibre. On calcule cette estimée dans le Chapitre 6. Cette estimée peut se réduire à calculer, pour une variété arithmétique projective  $\mathcal{X}$  de dimension  $n$  munie d'un faisceau hermitien ample  $\overline{\mathcal{L}}$ , la proportion des  $\sigma \in H^0(\mathcal{X}_{p^2}, \overline{\mathcal{L}}^{\otimes d})$  telles que pour tout point fermé,  $x \in \text{div}\sigma$ ,  $\dim_{\kappa(x)} \frac{\mathfrak{m}_{\text{div}\sigma, x}^{\text{div}\sigma, x}}{\mathfrak{m}_{\text{div}\sigma, x}^2} = n-1$ , où

$$\mathcal{X}_{p^2} = \mathcal{X} \times_{\text{Spec } \mathbb{Z}} \text{Spec } (\mathbb{Z}/p^2).$$



Ce calcul suit la démonstration de Poonen du théorème de lissité de Bertini sur un corps fini dans [Po04]. Avec un choix des entiers positifs  $r_{p,d}, N(p)$ , où  $r_{p,d}$  dépend de  $p, d$  et  $N(p)$  ne dépend que de  $p$ , on donne des estimées de la proportion des sections  $\sigma \in H^0(\mathcal{X}_{p^2}, \overline{\mathcal{L}}^{\otimes d})$  dont le diviseur a un point singulier de degré plus petit que ou égale à  $r_{p,d}$ , entre  $r_{p,d}$  et  $\frac{d}{nN(p)}$ , et plus grand que ou égal à  $\frac{d}{nN(p)}$ , respectivement. On conclut en mettant ensemble ces trois estimées.

L'estimée sur une seule fibre s'étend facilement à plusieurs fibres. Ce résultat nous permet de montrer que l'on peut assembler toutes les fibres sur  $p$  avec  $p < d^{\frac{1}{n+1}}$  sans nuire à la convergence de la proportion des  $\sigma \in H_{\text{Ar}}^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes d})$  telles que  $\text{div} \sigma$  n'a pas de point singulier sur ces fibres. Dans le Chapitre 7, on montre que la proportion des sections globales strictement effectives (c'est-à-dire de norme strictement plus petite que 1) de  $\overline{\mathcal{L}}^{\otimes d}$  dont le diviseur n'a pas de point singulier de caractéristique résiduelle plus petite que  $d^{\frac{1}{n+1}}$  tend vers  $\zeta_{\mathcal{X}}(1+n)^{-1}$  quand  $d$  tend vers l'infini.

Finalement, dans le Chapitre 8, on montre qu'il existe des constantes  $\varepsilon > 0$  et  $c > 0$  telle que pour tout  $p \leq e^{\varepsilon d}$  dont la fibre au-dessus est lisse et irréductible (ces deux conditions sont satisfaites pour tout  $p$  sauf un nombre fini), la proportion des sections globales strictement effectives de  $\overline{\mathcal{L}}^{\otimes d}$  dont le diviseur a des points singuliers sur cette fibre est plus petite que  $cp^{-2}$ . Ici la constante  $\varepsilon$  est telle que  $2\varepsilon$  vérifie la condition dans la Proposition 4.2.3. Ainsi la proportion des  $\sigma \in H_{\text{Ar}}^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes d})$  telles que  $\text{div} \sigma$  a des points singuliers sur la fibre  $\mathcal{X}_p$  pour un  $d^{\frac{1}{n+1}} \leq p \leq e^{\varepsilon d}$  est majorée par

$$\sum_{d^{\frac{1}{n+1}} \leq p \leq e^{\varepsilon d}} cp^{-2}.$$

Cette proportion tend vers 0 quand  $d$  tend vers infini. On déduit le Théorème 1.3.4 de cette estimée et de l'estimée (1.1). Pour finir, on montre le Corollaire 1.3.1 en appliquant le Théorème 1.3.4.

## 1.6 Notations

1. Pour un ensemble fini  $S$ , on note  $\#S$  son cardinal.
2. Pour un nombre réel positif  $x$ , on note

$$\lfloor x \rfloor = \max\{n \in \mathbb{Z} ; n \leq x\}, \quad \lceil x \rceil = \min\{n \in \mathbb{Z} ; n \geq x\}.$$

3. Soient  $f, g : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$  deux fonctions continues telles que  $f(0) = g(0) = 0$ . On dit  $f = O(g)$  s'il existe  $c > 0$  et  $\varepsilon > 0$  tel que pour tout  $0 < x < \varepsilon$  on a

$$|f(x)| \leq c \cdot g(x).$$

On dit  $f \sim g$  si  $f = O(g)$  et  $g = O(f)$ .

4. Si  $f : X \rightarrow Y$  est un plongement fermé, on écrit

$$f : X \hookrightarrow Y.$$

# Chapitre 2

## Introduction

In this thesis, we study theorems of Bertini type, i.e. theorems which confirm, on a scheme satisfying a certain geometric property, the existence of closed subschemes satisfying the same geometric property.

The main result of this thesis is Theorem 2.1.1. Take a regular projective arithmetic variety  $\mathcal{X}$  equipped with an ample Hermitian line bundle  $\overline{\mathcal{L}} = (\mathcal{L}, \|\cdot\|)$  (see Chapter 4 for precise definitions). We consider the global sections  $\sigma$  of  $\overline{\mathcal{L}}^{\otimes d}$  such that  $\|\sigma\| < 1$  and that the divisor of  $\sigma$  does not have singular point on the fiber  $\mathcal{X}_p$  over a prime number  $p \leq e^{\varepsilon d}$ . Here  $\varepsilon$  is a positive constant. Theorem 2.1.1 tells us that the proportion of such sections in the set of global sections of norm strictly smaller than 1 tends to  $\zeta_{\mathcal{X}}(1 + \dim \mathcal{X})^{-1}$  when  $d$  tends to infinity.

The classical Bertini theorem states that if  $X$  is a smooth quasi-projective variety of dimension  $m$  over an infinite field  $k$  embedded into some projective space  $\mathbb{P}_k^n$ , the intersection of  $X$  with a general hyperplane of  $\mathbb{P}_k^n$  is smooth of dimension  $m - 1$ . Here general means that the set of hyperplanes satisfying this property is the set of  $k$ -points of a non-empty open subscheme  $U$  of the dual projective space  $(\mathbb{P}_k^n)^\vee$  of  $\mathbb{P}_k^n$ . This open subscheme  $U$  of the dual projective space exists regardless of the conditions on the base field, but it's the infiniteness of the field  $k$  that guarantees the existence of infinitely many  $k$ -points in  $U$ . We have similar theorems on reducedness, irreducibility, connectedness, etc. A good reference for these results is [Jo83].

When  $k$  is a finite field, this theorem still gives us a non-empty open subscheme of  $(\mathbb{P}_k^n)^\vee$  parametrizing hyperplanes whose intersection with  $X$  is smooth, but may fail to give such a hyperplane as the open subscheme may have no  $k$ -point. In [Po04], B. Poonen proved that if we consider the proportion of hypersurfaces of degree  $d$  whose intersection with  $X$  is smooth of dimension  $m - 1$  among all the degree  $d$  hypersurfaces, this proportion tends to

$$\zeta_X(m + 1) = \prod_{x \in |X|} (1 - q^{-(m+1) \deg x})^{-1}$$

when  $d$  tends to infinity. In [CP16], Charles and Poonen also considered hypersurfaces of degree  $d$  of  $\mathbb{P}_k^n$  whose intersection with an irreducible subscheme  $X$  of dimension at least 2 is still irreducible, and proved that the proportion of such hypersurfaces tends to 1 when  $d$  tends to infinity.

It is also of interest to have a good analogue of Bertini smoothness theorem for quasi-projective schemes over  $\text{Spec } \mathbb{Z}$ . But as Poonen explained in [Po04, Section 5.7], smoothness condition is too strong in the arithmetic situation. We need to consider regularity rather than smoothness. In the same article, Poonen established a density for subsets of  $\bigcup_{d \geq 0} H^0(\mathbb{P}_{\mathbb{Z}}^n, \mathcal{O}(d))$  and showed

that for a regular subscheme  $\mathcal{X}$  of  $\mathbb{P}_{\mathbb{Z}}^n$ , assuming the *abc* conjecture and an auxiliary conjecture, the density of sections  $f \in \bigcup_{d \geq 0} H^0(\mathbb{P}_{\mathbb{Z}}^n, \mathcal{O}(d))$  such that  $\text{div } f \cap \mathcal{X}$  is regular is  $\zeta_{\mathcal{X}}(\dim \mathcal{X} + 1)^{-1}$ . Poonen's result depends on the embedding of  $\mathcal{X}$  into  $\mathbb{P}_{\mathbb{Z}}^n$  and the choice of a coordinate system in  $\mathbb{P}_{\mathbb{Z}}^n$ . It would be better to have a more general result without an explicit choice of an embedding into some projective space. This leads us to consider a similar result in the setting of Arakelov geometry.

## 2.1 Main theorems

Let  $\mathcal{X}$  be a regular projective arithmetic variety, i.e. an integral separated scheme which is regular and flat, projective of finite type over  $\text{Spec } \mathbb{Z}$ . For an ample line bundle  $\mathcal{L}$  on  $\mathcal{X}$ , we want to define a good density for a subset  $\mathcal{P} \subset \bigcup_{d \geq 0} H^0(\mathcal{X}, \mathcal{L}^{\otimes d})$  so that the density of the subset of sections  $\sigma \in \bigcup_{d \geq 0} H^0(\mathcal{X}, \mathcal{L}^{\otimes d})$  whose divisor is regular is positive. This will imply the existence of global sections  $\sigma$  with regular divisor  $\text{div } \sigma$  for sufficiently large  $d$ .

Recall that  $\mathcal{X}$  is *regular at a closed point*  $x$  if we have

$$\dim_{\kappa(x)} \frac{\mathfrak{m}_{\mathcal{X},x}}{\mathfrak{m}_{\mathcal{X},x}^2} = \dim \mathcal{X},$$

where  $\mathfrak{m}_{\mathcal{X},x}$  is the maximal ideal of the stalk  $\mathcal{O}_{\mathcal{X},x}$  of the structure sheaf of scheme  $\mathcal{X}$  on  $x$  and  $\dim \mathcal{X}$  is the dimension of  $\mathcal{X}$  as a scheme;  $\mathcal{X}$  is *singular at*  $x$  if it is not regular at  $x$ . If  $\mathcal{X}$  is regular at all closed points of  $\mathcal{X}$  it is called *regular*. It is *singular* if not regular. If  $\mathcal{X}$  is singular at  $x$ , we say that  $x$  is a *singular point* of  $\mathcal{X}$ .

In order to get good positivity properties of the ample line bundles on arithmetic varieties, we add on them a Hermitian structure and consider the notion of arithmetic ampleness for Hermitian line bundles on projective arithmetic varieties. The arithmetic ampleness is developed by Henri Gillet and Christophe Soulé in [GS92], and by Shouwu Zhang in [Zh92] (for arithmetic surfaces) and [Zh95]. Assume that  $\mathcal{X}$  is a projective arithmetic variety. An ample Hermitian line bundle  $\overline{\mathcal{L}} = (\mathcal{L}, \|\cdot\|)$  on  $\mathcal{X}$  is an ample line bundle  $\mathcal{L}$  equipped with a Hermitian metric  $\|\cdot\|$  on the restriction  $\mathcal{L}_{\mathbb{C}}$  to the fiber  $\mathcal{X}(\mathbb{C})$  with additional positivity conditions. For such  $\overline{\mathcal{L}}$  on  $\mathcal{X}$ , we consider the set of *strictly effective sections*

$$H_{\text{Ar}}^0(\mathcal{X}, \overline{\mathcal{L}}) := \{\sigma \in H^0(\mathcal{X}, \overline{\mathcal{L}}) ; \|\sigma\|_{\infty} < 1\}$$

as an analogue of  $H^0(X, \mathcal{L})$  for an ample line bundle  $\mathcal{L}$  on a projective variety  $X$  defined over a field, and

$$h_{\text{Ar}}^0(\mathcal{X}, \overline{\mathcal{L}}) := \log(\#H_{\text{Ar}}^0(\mathcal{X}, \overline{\mathcal{L}}))$$

as an analogue of  $h^0(X, \mathcal{L})$ . Here

$$\|\sigma\|_{\infty} = \sup_{z \in \mathcal{X}(\mathbb{C})} \|\sigma(z)\|.$$

We will give a precise definition of an ample Hermitian line bundle and discuss some of its properties in Chapter 4.

For a fixed ample Hermitian line bundle  $\overline{\mathcal{L}}$ , we say that a subset  $\mathcal{P}$  of  $\bigcup_{d \geq 0} H_{\text{Ar}}^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes d})$  has density  $\rho$  for some  $0 \leq \rho \leq 1$  if

$$\lim_{d \rightarrow \infty} \frac{\#(\mathcal{P} \cap H_{\text{Ar}}^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes d}))}{\#H_{\text{Ar}}^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes d})} = \rho.$$

We define the upper density and lower density in the same way. We denote the density, the upper density and the lower density of  $\mathcal{P}$ , when exist, by  $\mu(\mathcal{P})$ ,  $\bar{\mu}(\mathcal{P})$  and  $\underline{\mu}(\mathcal{P})$ , respectively.

Our main results are the following :

**Theorem 2.1.1.** *Let  $\mathcal{X}$  be a regular projective arithmetic variety of dimension  $n$ , and let  $\bar{\mathcal{L}}$  be an ample Hermitian line bundle on  $\mathcal{X}$ . There exists a constant  $\varepsilon > 0$  such that by denoting*

$$\mathcal{P}_{d,p \leq e^{\varepsilon d}} := \left\{ \sigma \in H_{\text{Ar}}^0(\mathcal{X}, \bar{\mathcal{L}}^{\otimes d}) ; \begin{array}{l} \text{div} \sigma \text{ has no singular point of residual} \\ \text{characteristic smaller than } e^{\varepsilon d} \end{array} \right\}$$

and  $\mathcal{P}_A = \bigcup_{d \geq 0} \mathcal{P}_{d,p \leq e^{\varepsilon d}}$ , we have

$$\mu(\mathcal{P}_A) = \zeta_{\mathcal{X}}(1+n)^{-1},$$

where  $\zeta_{\mathcal{X}}(s)$  is the zeta function

$$\zeta_{\mathcal{X}}(s) = \prod_{x \in |\mathcal{X}|} (1 - \#\kappa(x)^{-s})^{-1}.$$

Here  $\kappa(x)$  is the residual field of  $x$ , and the residual characteristic of a closed point  $x$  in  $\mathcal{X}$  is the characteristic of its residue field.

**Theorem 2.1.2.** *Let  $\mathcal{X}$  be a regular projective arithmetic variety of dimension  $n$ , and let  $\bar{\mathcal{L}}$  be an ample Hermitian line bundle on  $\mathcal{X}$ . Set*

$$\mathcal{P}_d := \left\{ \sigma \in H_{\text{Ar}}^0(\mathcal{X}, \bar{\mathcal{L}}^{\otimes d}) ; \text{div} \sigma \text{ is regular} \right\}$$

and  $\mathcal{P} = \bigcup_{d \geq 0} \mathcal{P}_d$ . We have

$$\bar{\mu}(\mathcal{P}) \leq \zeta_{\mathcal{X}}(1+n)^{-1},$$

where  $\bar{\mu}(\mathcal{P})$  is the upper density of  $\mathcal{P}$ .

*Proof.* If a section  $\sigma \in H_{\text{Ar}}^0(\mathcal{X}, \bar{\mathcal{L}}^{\otimes d})$  is such that  $\text{div} \sigma$  is regular, then in particular it has no singular point of residual characteristic smaller than  $e^{\varepsilon d}$  with constant  $\varepsilon$  as in Theorem 2.1.1. So naturally  $\mathcal{P}_d \subset \mathcal{P}_{d,p \leq e^{\varepsilon d}}$  and  $\mathcal{P} \subset \mathcal{P}_A$ . Therefore we have

$$\bar{\mu}(\mathcal{P}) \leq \mu(\mathcal{P}_A) = \zeta_{\mathcal{X}}(1+n)^{-1}.$$

□

**Corollary 2.1.3.** *Let  $\mathcal{X}$  be a regular projective arithmetic variety of dimension  $n$ , and let  $\bar{\mathcal{L}}$  be an ample Hermitian line bundle on  $\mathcal{X}$ . There exists a constant  $c > 1$  such that for any  $R > 1$  we have*

$$\lim_{d \rightarrow \infty} \frac{\# \left\{ \sigma \in H^0(\mathcal{X}, \bar{\mathcal{L}}^{\otimes d}) ; \begin{array}{l} \|\sigma\|_{\infty} < R^d, \text{Sing}(\text{div} \sigma) \text{ has no point} \\ \text{of residual characteristic smaller than } (cR)^{\frac{d}{2}} \end{array} \right\}}{\# \left\{ \sigma \in H^0(\mathcal{X}, \bar{\mathcal{L}}^{\otimes d}) ; \|\sigma\|_{\infty} < R^d \right\}} = \zeta_{\mathcal{X}}(1+n)^{-1}.$$

It is also using this notion of density that Charles proved in [Ch17] the analogous Bertini irreducibility theorem for arithmetic varieties, which says that if  $\mathcal{X}$  is an irreducible arithmetic variety of dimension at least 2 and  $\bar{\mathcal{L}}$  an ample Hermitian line bundle on  $\mathcal{X}$ , then the set of global

sections in  $\bigcup_{d \geq 0} H^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes d})$  whose divisor is irreducible has density 1. The result of Charles can also be compared to the result of Breuillard and Varjú in [BV19] for polynomials only with coefficients in 0 and 1. Breuillard and Varjú showed that if we admit the Riemann hypothesis for the Dedekind zeta function  $\zeta_K$  for all number fields of the form  $K = \mathbb{Q}(a)$  for some root  $a$  of a polynomial with 0, 1 coefficients, the density of the subset of irreducible polynomials in the set of polynomials  $P(X)$  of 1 variable with 0, 1 coefficients such that  $P(0) \neq 0$  is 1. (In each degree  $d$ , such polynomials are finite in number, so the density can be defined by the limit of proportion when  $d$  tends to infinity.)

## 2.2 Comparison with earlier results

We compare our result with some existing results.

We first recall the result of Poonen that we already mentioned. In [Po04] Poonen established a density for  $\bigcup_{d \geq 0} H^0(\mathbb{P}_{\mathbb{Z}}^n, \mathcal{O}(d))$  on the projective space  $\mathbb{P}_{\mathbb{Z}}^n$ . Let  $\mathcal{P} = \bigcup_{d \geq 0} \mathcal{P}_d$  be a subset of  $\bigcup_{d \geq 0} H^0(\mathbb{P}_{\mathbb{Z}}^n, \mathcal{O}(d))$ , where  $\mathcal{P}_d \subset H^0(\mathbb{P}_{\mathbb{Z}}^n, \mathcal{O}(d))$ . For any  $d$ , we have a natural  $\mathbb{Z}$ -basis of  $H^0(\mathbb{P}_{\mathbb{Z}}^n, \mathcal{O}(d))$  which is composed of all monomials of degree  $d$ . For simplicity of notations we denote them by  $f_{d,1}, \dots, f_{d,h_d}$  where  $h_d = h^0(\mathbb{P}_{\mathbb{Q}}^n, \mathcal{O}(d))$ . Any section  $f \in H^0(\mathbb{P}_{\mathbb{Z}}^n, \mathcal{O}(d))$  can be written as  $f = \sum_{i=1}^{h_d} a_i f_{d,i}$  with some  $a_i \in \mathbb{Z}$  for each  $i$ . Poonen defines the upper density of  $\mathcal{P}_d$  as

$$\bar{\mu}_{\mathcal{P},d}(\mathcal{P}) = \max_{\tau \in \mathfrak{S}_{h_d}} \limsup_{B_{\tau(1)} \rightarrow \infty} \cdots \limsup_{B_{\tau(h_d)} \rightarrow \infty} \frac{\#\left(\mathcal{P}_d \cap \left\{ \sum_{i=1}^{h_d} a_i f_{d,i} \in H^0(\mathbb{P}_{\mathbb{Z}}^n, \mathcal{O}(d)) ; |a_i| \leq B_i, \forall i \right\}\right)}{\#\left\{ \sum_{i=1}^{h_d} a_i f_{d,i} \in H^0(\mathbb{P}_{\mathbb{Z}}^n, \mathcal{O}(d)) ; |a_i| \leq B_i, \forall i \right\}}.$$

Here  $\mathfrak{S}_{h_d}$  is the symmetric group of  $h_d$  symbols. The upper density of  $\mathcal{P}$  is then defined by

$$\bar{\mu}_{\mathcal{P}}(\mathcal{P}) = \limsup_{d \rightarrow \infty} \bar{\mu}_{\mathcal{P},d}(\mathcal{P}_d).$$

The lower density of  $\mathcal{P}$  is defined similarly and the density of  $\mathcal{P}$  exists if its upper and lower density coincide. Using this density, Poonen proved the following theorem :

**Theorem 2.2.1** (Poonen, Theorem 5.1 of [Po04]). *When  $\mathcal{X}$  is a regular subscheme of dimension  $m$  of  $\mathbb{P}_{\mathbb{Z}}^n$ , assuming the abc conjecture and a supplementary conjecture which holds at least when  $\mathcal{X}$  is projective, the density of the set of sections  $f \in \bigcup_{d \geq 0} H^0(\mathbb{P}_{\mathbb{Z}}^n, \mathcal{O}(d))$  such that  $\text{div}(f) \cap \mathcal{X}$  is regular of dimension  $m - 1$  is  $\zeta_{\mathcal{X}}(m + 1)^{-1}$ .*

**Remark.** In Poonen's proof, the *abc* conjecture is used to show that for any fixed  $d$ , the upper density of global sections whose divisor has a singular point on a fiber over a prime number  $p \geq M$  with  $M > 0$  tends to 0 when  $M$  tends to infinity. The proof of this follows the idea of Granville in [Gr98] that for a polynomial  $f(x) \in \mathbb{Z}[x]$  we can get an asymptotic control of the prime squarefactors of  $f(n)$  by the norm of  $n \in \mathbb{Z}$ . Poonen generalized this idea to the case of multivariable polynomials in [Po03].

Essentially, in each degree  $d$ , we get the density  $\bar{\mu}_d$  by taking the limit of coefficients one by one. The action of symmetric group adds the condition that the order of coefficients can be arbitrary. This theorem permits us to find global sections  $f \in H^0(\mathbb{P}_{\mathbb{Z}}^n, \mathcal{O}(d))$  such that  $\text{div}(f) \cap \mathcal{X}$  is regular of dimension  $m - 1$  for sufficiently high  $d$ .

The density defined by Poonen depends on a choice of coordinates of  $\mathbb{P}_{\mathbb{Z}}^n$ . These coordinates determines which global sections are monomials in each  $H^0(\mathbb{P}_{\mathbb{Z}}^n, \mathcal{O}(d))$ . His method is hard to be

applied to a more general case, for example when we consider an arithmetic variety other than  $\mathbb{P}_{\mathbb{Z}}^n$  equipped with an ample line bundle which may not be very ample. Moreover, the size of the global sections with regular divisor cannot be controled using this method. The global sections having regular divisor may have very large coefficients as polynomials. Finally, the *abc* conjecture is powerfully used in his proof. Without it, the proof can give control of sections whose divisor does not have singular points of finitely many fixed residual characteristics, but cannot give the limit of the proportion of global sections of  $\mathcal{O}(d)$  whose divisor has no singular point of residual characteristic smaller than  $e^{\varepsilon d}$  for a constant  $\varepsilon$  as we do.

In [BSW16], Bhargava, Shankar and Wang proved that monic integer polynomials of one variable  $f(x) = x^d + a_1x^{d-1} + \dots + a_d \in V_d^{\text{mon}}(\mathbb{Z})$  such that  $\mathbb{Z}[x]/(f(x))$  is the ring of integers of the field  $\mathbb{Q}[x]/(f(x))$  has density  $\zeta(2)^{-1}$ . Here the density is constructed using the size of the coefficients of polynomials.

This result can be viewed as a version of Bertini regularity theorem for  $\mathbb{P}_{\mathbb{Z}}^1$ . In fact, the condition that  $\mathbb{Z}[x]/(f(x))$  is the ring of integers of the field  $\mathbb{Q}[x]/(f(x))$  means exactly that when we homogenize  $f$  to the global section

$$F(X, Y) = X^d + a_1X^{d-1}Y + \dots + a_dY^d \in H^0(\mathbb{P}_{\mathbb{Z}}^1, \mathcal{O}(d)),$$

$\text{div}(F)$  is a regular divisor of  $\mathbb{P}_{\mathbb{Z}}^1$ . Indeed, the coefficient of  $X^d$  in  $F(X, Y)$  guaranties that the intersection of  $\text{div}(F)$  with the infinity divisor  $\infty_{\mathbb{Z}} = \text{div}(Y)$  is empty, so  $\text{div}(F)$  is contained in the affine space  $\mathbb{A}_{\mathbb{Z}}^1 = \mathbb{P}_{\mathbb{Z}}^1 - \text{div}(Y)$ . Then we can identify  $\text{div}(F)$  with  $\text{div}(f)$  where  $f$  is the above monic polynomial, and then get an isomorphism of schemes  $\text{div}(F) \simeq \text{Spec } \mathbb{Z}[x]/(f(x))$ . Note that the scheme  $\text{Spec } \mathbb{Z}[x]/(f(x))$  is regular if and only if  $\mathbb{Z}[x]/(f(x))$  is normal. The regularity of  $\text{Spec } \mathbb{Z}[x]/(f(x))$  is hence equivalent to the fact that  $\mathbb{Z}[x]/(f(x))$  is the ring of integers of  $\mathbb{Q}[x]/(f(x))$ .

In their paper, fixing the degree  $d > 1$ , they order the monic integer polynomials

$$f(x) = x^d + a_1x^{d-1} + \dots + a_d$$

by a height function

$$H(f) := \max\{|a_i|^{\frac{1}{i}}\},$$

and calculate the density of a subset  $\mathcal{P}_d \subset V_d^{\text{mon}}(\mathbb{Z})$  by

$$\mu_{H,d}(\mathcal{P}_d) = \lim_{R \rightarrow \infty} \frac{\#\{\mathcal{P}_d \cap \{f \in V_d^{\text{mon}}(\mathbb{Z}) ; H(f) \leq R\}\}}{\#\{f \in V_d^{\text{mon}}(\mathbb{Z}) ; H(f) \leq R\}}.$$

Identifying  $V_d^{\text{mon}}(\mathbb{Z})$  with the set  $\{F \in H^0(\mathbb{P}_{\mathbb{Z}}^1, \mathcal{O}(d)) ; \text{div } F \cap \infty_{\mathbb{Z}} = \emptyset\}$  by homogenization, the density of  $\mathcal{P}_d$  can be understood as

$$\mu_{H,d}(\mathcal{P}_d) = \lim_{R \rightarrow \infty} \frac{\#\{\mathcal{P}_d \cap \{F \in H^0(\mathbb{P}_{\mathbb{Z}}^1, \mathcal{O}(d)) ; \text{div } F \cap \infty_{\mathbb{Z}} = \emptyset, H(F) \leq R\}}{\#\{F \in H^0(\mathbb{P}_{\mathbb{Z}}^1, \mathcal{O}(d)) ; \text{div } F \cap \infty_{\mathbb{Z}} = \emptyset, H(F) \leq R\}}.$$

We can then reformulate the result in [BSW16] as follows :

**Theorem 2.2.2** (Theorem 1.2 of [BSW16]). *For a fixed  $d > 1$ , set*

$$\mathcal{P}_d := \{F \in H^0(\mathbb{P}_{\mathbb{Z}}^1, \mathcal{O}(d)) ; \text{div } F \cap \infty_{\mathbb{Z}} = \emptyset, \text{div } F \text{ is regular of dimension } 1\}.$$

*Then we have*

$$\mu_{H,d}(\mathcal{P}_d) = \zeta(2)^{-1}.$$

**Remark.** This result is similar to Poonen's theorem. In fact, if we note that  $\zeta(2)^{-1}$  can be expressed by values of the zeta function of the affine line over  $\text{Spec } \mathbb{Z}$

$$\zeta_{\mathbb{A}_{\mathbb{Z}}^1}(s) = \prod_p \zeta_{\mathbb{A}_{\mathbb{Z}_p}^1} = \prod_p \frac{1}{1 - p^{1-s}},$$

which is,

$$\zeta(2)^{-1} = \prod_p (1 - p^{-2}) = \zeta_{\mathbb{A}_{\mathbb{Z}}^1}(3)^{-1},$$

then the theorem tells us that, for any  $d > 1$ , the density of the subset  $\mathcal{P}_d$  of sections with regular divisor of the set

$$\{F \in H^0(\mathbb{P}_{\mathbb{Z}}^1, \mathcal{O}(d)) ; \text{div } F \cap \infty_{\mathbb{Z}} = \emptyset\}$$

is equal to

$$\zeta_{\mathbb{P}_{\mathbb{Z}}^1 - \infty_{\mathbb{Z}}} (1 + \dim(\mathbb{P}_{\mathbb{Z}}^1 - \infty_{\mathbb{Z}}))^{-1} = \zeta_{\mathbb{A}_{\mathbb{Z}}^1}(3)^{-1}.$$

This is a statement similar to Theorem 2.2.1. But as we only consider global sections whose divisor is disjoint of  $\infty_{\mathbb{Z}}$ , we can not recover Poonen's theorem for  $\mathcal{X} = \mathbb{A}_{\mathbb{Z}}^1$  in  $\mathbb{P}_{\mathbb{Z}}^1$ .

The result of Bhargava, Shankar and Wang surpasses our result for  $\mathbb{P}_{\mathbb{Z}}^1$  in the sense that for any  $d > 1$ , they can actually find global sections of  $H^0(\mathbb{P}_{\mathbb{Z}}^1, \mathcal{O}(d))$  whose divisor is regular without auxiliary assumptions. Neither can we get such a strong statement using Poonen's method. But the method of Bhargava, Shankar and Wang is hard to be generalized to other situations. Their proof depends on the monogenicity of the finite  $\mathbb{Z}$ -algebra  $\mathbb{Z}[x]/(f(x))$ . They constructed a map from the moduli space of monogenic finite  $\mathbb{Z}$ -algebras of length  $d$  to the space of symmetric  $n \times n$  matrices quotient by the action of group  $SO(A_0)$ , where  $A_0$  is the  $n \times n$  anti-diagonal matrix. This map is then used in the article to turn the counting of monic polynomials to the counting of special orbits in this quotient space. Due to the construction of this map, it is difficult to release the monic condition in their theorem so as to get a result for all polynomials with coefficients in  $\mathbb{Z}$ . It is even more difficult to generalize this method to regular arithmetic varieties other than  $\mathbb{P}_{\mathbb{Z}}^1$ .

In [Au01], [Au02], Autissier showed another arithmetic analogue of the Bertini theorems. He proved as a particular case that if  $\mathcal{X}$  is an arithmetic variety of dimension  $n$  over an integer ring  $\mathcal{O}_K$  (where  $K$  here is a number field), and  $\overline{\mathcal{L}}$  a very ample Hermitian line bundle on  $\mathcal{X}$ , then there exists a finite extension  $L$  of  $K$  and a section  $\sigma \in H^0(\mathcal{X}_{\mathcal{O}_L}, \overline{\mathcal{L}})$  such that by writing  $g : \text{Spec } \mathcal{O}_L \rightarrow \text{Spec } \mathcal{O}_K$  the morphism induced by  $\mathcal{O}_K \hookrightarrow \mathcal{O}_L$ , for any closed point  $b \in \text{Spec } \mathcal{O}_L$ , the fiber  $(\text{div } \sigma)_b$  of the divisor  $\text{div } \sigma$  is smooth if  $\mathcal{X}_{g(b)}$  is. Moreover, we can bound the height of  $\text{div } \sigma$  (defined by  $\overline{\mathcal{L}}$ ) in terms of the height of  $\mathcal{X}$ ,  $\text{deg}_{\mathcal{L}_{\mathbb{Q}}} \mathcal{X}_{\mathbb{Q}}$ ,  $n$  and an effective constant which is only dependent of  $\overline{\mathcal{L}}$  and  $n$ .

This result is stronger than ours in the sense where the divisor that he gives comes from a global section of the sheaf  $\overline{\mathcal{L}}$  but not  $\overline{\mathcal{L}}^{\otimes d}$  for a large  $d$ , and moreover the divisor satisfies the smoothness condition rather than the regularity condition. The disadvantage of this result is that it need to pass to finite base changes to find such a global section. In particular, if  $\mathcal{X}$  is defined over  $\text{Spec } \mathbb{Z}$ , there's little chance that we can find a divisor satisfying the smoothness condition in the statement which is defined over  $\text{Spec } \mathbb{Z}$  by Autissier's method. Our result, on the other hand, provides divisors which are defined over  $\text{Spec } \mathbb{Z}$  if so is the arithmetic variety that we consider.

## 2.3 Organisation of the text

In Chapter 3 we prove a generalized version of Poonen's Bertini smoothness theorem over a finite field, replacing the very ample condition by ample condition.

In Chapter 4, we talk about the arithmetic ampleness in general. We recall in Section 4.1 the precise definitions concerning ampleness as well as some properties of arithmetic ample line bundles such as the arithmetic Hilbert-Samuel formula. In Section 4.2 we present two results concerning the modulo  $N$  ( $N \in \mathbb{Z}_{>0}$ ) reduction of the global sections of a power of an ample Hermitian line bundle. The first of the two results is proved by Charles in [Ch17].

The following chapters are aimed at proving Theorem 2.1.1.

In Chapter 5, we present the computations of the speed of convergence of

$$\prod_{x \in |\mathcal{X}_p|, \deg x \leq r} (1 - \#\kappa(x)^{-(n+1)})$$

to  $\zeta_{\mathcal{X}_p}(n+1)^{-1}$  when  $r \rightarrow \infty$ , and that of  $\prod_{p \leq R} \zeta_{\mathcal{X}_p}(n+1)^{-1}$  to  $\zeta_{\mathcal{X}}(n+1)^{-1}$  when  $R \rightarrow \infty$  for an arithmetic variety  $\mathcal{X}$ .

The proof of Theorem 2.1.1 relies on an effective estimate of proportion of global sections whose divisor has no singular point on one single fiber. We compute this estimate in Chapter 6. This estimate can be reduced to computing, for a projective arithmetic variety  $\mathcal{X}$  of dimension  $n$  with an ample Hermitian line bundle  $\overline{\mathcal{L}}$ , the proportion of  $\sigma \in H^0(\mathcal{X}_{p^2}, \overline{\mathcal{L}}^{\otimes d})$  such that for any closed point  $x \in \text{div} \sigma$ ,  $\dim_{\kappa(x)} \frac{\mathfrak{m}_{\text{div} \sigma, x}^2}{\mathfrak{m}_{\text{div} \sigma, x}} = n - 1$ , where

$$\mathcal{X}_{p^2} = \mathcal{X} \times_{\text{Spec } \mathbb{Z}} \text{Spec } \mathbb{Z}/p^2.$$

We follow Poonen's proof of the Bertini smoothness theorem over finite fields, in [Po04]. With a choice of positive integers  $r_{p,d}, N(p)$ , where  $r_{p,d}$  depends on  $p, d$  and  $N(p)$  depends only on  $p$ , we give estimates of the proportion of sections  $\sigma \in H^0(\mathcal{X}_{p^2}, \overline{\mathcal{L}}^{\otimes d})$  whose divisor has a singular point of degree smaller than or equal to  $r_{p,d}$ , between  $r_{p,d}$  and  $\frac{d}{nN(p)}$  and larger than or equal to  $\frac{d}{nN(p)}$ , respectively. Then we conclude by putting together these three estimates.

The estimate on one single fiber can be easily extended to finitely many fibers. The effective estimate permit us to show that we can gather all fibers over  $p$  such that  $p \leq d^{\frac{1}{n+1}}$  without ruining the convergence of the proportion of  $\sigma \in H_{\text{Ar}}^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes d})$  such that  $\text{div} \sigma$  has no singular point on all these fibers. In Chapter 7, we show that the proportion of the strictly effective global sections (which means sections of norm strictly smaller than 1) of  $\overline{\mathcal{L}}^{\otimes d}$  whose divisor does not have singular point of residual characteristic smaller than  $d^{\frac{1}{n+1}}$  tends to  $\zeta_{\mathcal{X}}(1+n)^{-1}$  when  $d$  tends to infinity.

Finally, in Chapter 8, we show that there exists constants  $c > 0$  and  $\varepsilon > 0$  with  $2\varepsilon$  satisfying Proposition 4.2.3 such that for any prime  $p \leq e^{\varepsilon d}$  such that the fiber over  $p$  is smooth and irreducible (these two conditions are satisfied by all but finitely many  $p$ ), the proportion of strictly effective global sections of  $\overline{\mathcal{L}}^{\otimes d}$  whose divisor has singular points on this fiber is smaller than  $cp^{-2}$ . Consequently the proportion of  $\sigma \in H_{\text{Ar}}^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes d})$  such that  $\text{div} \sigma$  has singular points on the fiber  $\mathcal{X}_p$  for some  $d^{\frac{1}{n+1}} \leq p \leq e^{\varepsilon d}$  is bounded above by

$$\sum_{d^{\frac{1}{n+1}} \leq p \leq e^{\varepsilon d}} cp^{-2}.$$



This proportion tends to 0 when  $d$  tends to infinity. We deduce Theorem 2.1.1 from this estimate and the estimate of Chapter 7 for  $p < d^{\frac{1}{n+1}}$ . To conclude, we show Corollary 2.1.3 by applying Theorem 2.1.1.

## 2.4 Notations

1. For a finite set  $S$ , we note  $\#S$  its cardinality.
2. For a positive real number  $x$ , we note

$$\lfloor x \rfloor = \max\{n \in \mathbb{Z} ; n \leq x\}, \quad \lceil x \rceil = \min\{n \in \mathbb{Z} ; n \geq x\}.$$

3. Let  $f, g : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$  be two real continuous functions such that  $f(0) = g(0) = 0$ . We say  $f = O(g)$  if there exist  $c > 0$  and  $\varepsilon > 0$  such that for any  $0 < x < \varepsilon$  we have

$$|f(x)| \leq c \cdot g(x).$$

We say  $f \sim g$  if  $f = O(g)$  and  $g = O(f)$ .

4. If  $f : X \rightarrow Y$  is a closed embedding, we denote it by

$$f : X \hookrightarrow Y.$$

## Chapitre 3

# Bertini smoothness theorem over finite fields

In this chapter, we prove a slightly generalized version of B. Poonen's Bertini theorem over finite fields. The precise statement is the following :

**Theorem 3.0.1.** *Let  $\mathbb{F}_q$  be a finite field of characteristic  $p$ . Let  $Y$  be a projective scheme of dimension  $n$  over  $\mathbb{F}_q$ , and  $X$  a smooth subscheme of  $Y$  of dimension  $m$ . Let  $\mathcal{L}$  be an ample line bundle on  $Y$ . Assume that there exists a smooth quasi-compact open subscheme  $U$  in  $Y$  containing  $X$ . Set*

$$\mathcal{P}_d := \{\sigma \in \mathbf{H}^0(Y, \mathcal{L}^{\otimes d}) ; \operatorname{div} \sigma \cap X \text{ is smooth of dimension } m-1\}$$

and  $\mathcal{P} = \bigcup_{d \geq 0} \mathcal{P}_d$ . We have

$$\mu(\mathcal{P}) = \lim_{d \rightarrow \infty} \frac{\#\mathcal{P}_d}{\#\mathbf{H}^0(Y, \mathcal{L}^{\otimes d})} = \zeta_X(m+1)^{-1} > 0.$$

Here  $\zeta_X$  is the zeta function

$$\zeta_X(s) = \prod_{x \in |X|} (1 - q^{-s \deg x})^{-1}$$

**Remark.** If we take  $Y = \mathbb{P}_{\mathbb{F}_q}^n$ ,  $\mathcal{L} = \mathcal{O}(1)$ , we get Poonen's theorem.

Note that for a  $\sigma \in \mathbf{H}^0(Y, \mathcal{L}^{\otimes d})$ ,  $\operatorname{div} \sigma \cap X$  is smooth if and only if it is non-singular at every closed point of  $\operatorname{div} \sigma \cap X$ . To prove this theorem, we class the closed points of  $X$  by their degree, so that for each degree there exist only finitely many closed points. In Poonen's proof, he classifies the closed points into three parts for each  $\mathcal{L}^{\otimes d}$ , which are the following : closed points of degree smaller than or equal to a chosen positive integer  $r$ , closed points of degree between  $r$  and  $\frac{d}{m+1}$ , and closed points of degree bigger than  $\frac{d}{m+1}$ . Then he estimates the number of sections in  $\mathbf{H}^0(Y, \mathcal{L}^{\otimes d})$  whose divisor has singular points in these parts, respectively.

Our proof follows his method in a faithful way. But we need more explicite bounds for bad sections, so as to get the speed of convergence for the final limit. For technical reasons, we need the following result :

**Lemma 3.0.2.** *Let  $\mathcal{L}$  be an ample line bundle on a projective scheme  $Y$  over a field  $k$ . Then there exists a positive integer  $N$  such that*

- i)  $\mathcal{L}^{\otimes d}$  is very ample for all  $d \geq N$  ;*
- ii) for any  $a, b \geq N$ , the natural morphism*

$$H^0(Y, \mathcal{L}^{\otimes a}) \otimes H^0(Y, \mathcal{L}^{\otimes b}) \longrightarrow H^0(Y, \mathcal{L}^{\otimes(a+b)})$$

*is surjective.*

*Proof.* This is a classical result. The first statement is part of [La04, Theorem 1.2.6], and the second statement can be deduced directly from [La04, Theorem 1.8.3].  $\square$

We choose a positive integer  $r$ , an integer  $N$  satisfying this lemma and depending possibly on  $q$ , and set

$$\mathcal{P}_{d, \leq r} = \{ \sigma \in H^0(Y, \mathcal{L}^{\otimes d}) ; \forall x \in X, \deg x \leq r, \text{div} \sigma \cap X \text{ is smooth of dimension } m-1 \text{ at } x \}$$

$$\mathcal{Q}_{d, > r}^{\text{med}} = \{ \sigma \in H^0(Y, \mathcal{L}^{\otimes d}) ; \exists x \in X, r < \deg x \leq \frac{d}{(m+1)N}, \text{div} \sigma \cap X \text{ is singular at } x \}$$

$$\mathcal{Q}_d^{\text{high}} = \{ \sigma \in H^0(Y, \mathcal{L}^{\otimes d}) ; \exists x \in X, \deg x \geq \frac{d}{(m+1)N}, \text{div} \sigma \cap X \text{ is singular at } x \}.$$

Then clearly

$$\mathcal{P}_d \subset \mathcal{P}_{d, \leq r} \subset \mathcal{P}_d \cup \mathcal{Q}_{d, > r}^{\text{med}} \cup \mathcal{Q}_d^{\text{high}}.$$

We give bound for the proportion of these three sets.

### 3.1 Singular points of small degree

**Lemma 3.1.1.** *Let  $Y$  be a projective scheme over  $\mathbb{F}_q$ ,  $\mathcal{L}$  an ample line bundle over  $Y$ . Let  $Z$  be a finite subscheme of  $Y$ . Let  $N$  be a positive integer satisfying Lemma 3.0.2. Then the restriction morphism*

$$\phi_{d, Z} : H^0(Y, \mathcal{L}^{\otimes d}) \longrightarrow H^0(Z, \mathcal{L}^{\otimes d})$$

*is surjective for all  $d \geq Nh_Z$ , where  $h_Z = \dim_{\mathbb{F}_q} H^0(Z, \mathcal{O}_Z)$ .*

*Proof.* If  $\mathcal{L}$  is very ample, by [Po04, Lemma 2.1],  $\phi_{d, Z}$  is surjective when  $d \geq h_Z - 1$ , and this lemma is also true. When  $\mathcal{L}$  is only ample, for any  $\delta_0 \geq N$ ,  $\mathcal{L}^{\otimes \delta_0}$  is very ample and  $\phi_{d\delta_0, Z}$  is surjective for any  $d \geq h_Z - 1$ . Now for any  $d \geq Nh_Z$ , we can find  $s_d \geq h_Z - 1$  and  $N \leq r_d < 2N$  such that  $d = s_d N + r_d$ . By Lemma 3.0.2, we have a surjection

$$H^0(Y, \mathcal{L}^{\otimes s_d N}) \otimes H^0(Y, \mathcal{L}^{\otimes r_d}) \longrightarrow H^0(Y, \mathcal{L}^{\otimes d}).$$

Moreover, since  $Z$  is finite, for all  $d \geq 0$ , we have  $H^0(Z, \mathcal{L}^{\otimes d}) \simeq H^0(Z, \mathcal{O}_Z)$ . These isomorphisms are not canonical, but can give us an isomorphism

$$H^0(Z, \mathcal{L}^{\otimes s_d N}) \otimes_{H^0(Z, \mathcal{O}_Z)} H^0(Z, \mathcal{L}^{\otimes r_d}) \longrightarrow H^0(Z, \mathcal{L}^{\otimes d})$$

which makes the following diagram commutative :

$$\begin{array}{ccc} H^0(Y, \mathcal{L}^{\otimes s_d N}) \otimes H^0(Y, \mathcal{L}^{\otimes r_d}) & \longrightarrow & H^0(Y, \mathcal{L}^{\otimes d}) \\ \downarrow & & \downarrow \\ H^0(Z, \mathcal{L}^{\otimes s_d N}) \otimes_{H^0(Z, \mathcal{O}_Z)} H^0(Z, \mathcal{L}^{\otimes r_d}) & \xrightarrow{\sim} & H^0(Z, \mathcal{L}^{\otimes d}) \end{array}$$

Thus it suffices to show that the left vertical morphism is surjective. Since  $Y$  is projective,  $H^0(Y, \mathcal{L}^{\otimes ra})$  is of finite  $\mathbb{F}_q$ -dimension. Let  $H^0(Y, \mathcal{L}^{\otimes ra}) = \bigoplus_i \mathbb{F}_q t_i$ . As  $\mathcal{L}^{\otimes ra}$  is very ample, it is globally generated. So for each  $z \in |Z|$ , we can find one  $t_i$  in the set of generators such that  $t_i(z) \neq 0$ . Since surjectivity is stable under field base change, replacing  $\mathbb{F}_q$  by a finite field extension, we can assume that there exists a linear combination  $t = \sum_i a_i t_i$  such that  $t(z) \neq 0$  for any  $z \in |Z|$ . Then we have

$$\begin{array}{ccc} H^0(Y, \mathcal{L}^{\otimes s_d N}) \otimes \mathbb{F}_q t^{\mathbb{C}} & \longrightarrow & H^0(Y, \mathcal{L}^{\otimes s_d N}) \otimes H^0(Y, \mathcal{L}^{\otimes ra}) \\ \downarrow & & \downarrow \\ H^0(Z, \mathcal{L}^{\otimes s_d N}) \otimes \mathbb{F}_q t|_Z & \longrightarrow & H^0(Z, \mathcal{L}^{\otimes s_d N}) \otimes_{H^0(Z, \mathcal{O}_Z)} H^0(Z, \mathcal{L}^{\otimes ra}) \end{array}$$

By our construction, the section  $t$  trivializes  $H^0(Z, \mathcal{L}^{\otimes ra})$ . So the bottom morphism is an isomorphism. Hence the right vertical morphism is surjective. By the commutativity of the first diagram, the morphism

$$H^0(Y, \mathcal{L}^{\otimes d}) \longrightarrow H^0(Z, \mathcal{L}^{\otimes d})$$

is also surjective, which is what we need to show.  $\square$

With this lemma, we can control the proportion of  $\mathcal{P}_{d, \leq r}$ .

**Proposition 3.1.2.** *Let  $Y$  be a projective scheme of dimension  $n$  over  $\mathbb{F}_q$  equipped with an ample line bundle  $\mathcal{L}$ . Let  $X$  be a subscheme smooth over  $\mathbb{F}_q$  of dimension  $m$  of  $Y$ . Set*

$\mathcal{P}_{d, \leq r} = \{\sigma \in H^0(Y, \mathcal{L}^{\otimes d}) ; \forall x \in X, \deg x \leq r, \operatorname{div} \sigma \cap X \text{ is smooth of dimension } m-1 \text{ at } x\}$ ,  
and  $\mathcal{P}_{\leq r} = \bigcup_{d \geq 0} \mathcal{P}_{d, \leq r}$ . We have

$$\mu(\mathcal{P}_{\leq r}) = \lim_{d \rightarrow \infty} \frac{\#\mathcal{P}_{d, \leq r}}{\#H^0(Y, \mathcal{L}^{\otimes d})} = \prod_{\deg x \leq r} \left(1 - q^{-(m+1) \deg x}\right)$$

In fact, with a positive integer  $N$  satisfying Lemma 3.0.2, for any

$$d \geq N \left( \sum_{\deg x \leq r} (1+m) \deg x \right),$$

we have

$$\frac{\#\mathcal{P}_{d, \leq r}}{\#H^0(Y, \mathcal{L}^{\otimes d})} = \prod_{\deg x \leq r} \left(1 - q^{-(m+1) \deg x}\right).$$

*Proof.* For any closed point  $x$  of  $X$ , let  $x'$  be the closed subscheme in  $X$  defined by  $\mathfrak{m}_x^2$ , where  $\mathfrak{m}_x$  is the ideal sheaf of  $x$  in  $X$ . Then  $x'$  is the first order infinitesimal neighbourhood of  $x$  in  $X$ . Let  $X'_{\leq r}$  be the union of the closed subschemes  $x'$  for all  $x \in X$  with  $\deg x \leq r$ . Note that the number of closed points of  $X$  with degree smaller than or equal to  $r$  is finite. This union is a disjoint finite union. So  $X'_{\leq r}$  is a finite subscheme of  $X$  defined by the ideal sheaf  $\prod_{\deg x \leq r} \mathfrak{m}_x^2$ . Hence

$$H^0(X'_{\leq r}, \mathcal{O}_{X'_{\leq r}}) = H^0\left(X'_{\leq r}, \prod_{\deg x \leq r} \mathcal{O}_{x'}\right) = \prod_{\deg x \leq r} H^0(X, \mathcal{O}_X / \mathfrak{m}_x^2).$$

Since  $X$  is smooth over  $\mathbb{F}_q$ , for any closed point  $x$ , we have

$$\dim_{\kappa(x)} H^0(X, \mathfrak{m}_x / \mathfrak{m}_x^2) = m.$$

Hence

$$\dim_{\mathbb{F}_q} \mathrm{H}^0(X, \mathcal{O}_X/\mathfrak{m}_x^2) = (1+m) \deg x,$$

and we have

$$\dim_{\mathbb{F}_q} \mathrm{H}^0(X'_{\leq r}, \mathcal{O}_{X'_{\leq r}}) = \sum_{\deg x \leq r} (1+m) \deg x$$

Apply Lemma 3.1.1 to the case  $Z = X'_{\leq r}$ . We get that the morphism

$$\mathrm{H}^0(Y, \mathcal{L}^{\otimes d}) \longrightarrow \mathrm{H}^0(X'_{\leq r}, \mathcal{L}^{\otimes d})$$

is surjective if  $d \geq N \left( \sum_{\deg x \leq r} (1+m) \deg x \right)$ .

Note that for a section  $\sigma \in \mathrm{H}^0(Y, \mathcal{L}^{\otimes d})$ , the intersection  $\mathrm{div} \sigma \cap X$  is singular at a closed point  $x \in X$  if and only if the image of  $\sigma$  in  $\mathrm{H}^0(x', \mathcal{L}^{\otimes d})$  by the restriction map

$$\mathrm{H}^0(Y, \mathcal{L}^{\otimes d}) \longrightarrow \mathrm{H}^0(x', \mathcal{L}^{\otimes d})$$

is zero. So  $\mathrm{div} \sigma \cap X$  has no singular point of degree smaller than or equal to  $r$  if and only if its restriction to  $\mathrm{H}^0(X'_{\leq r}, \mathcal{O}_{X'_{\leq r}}) \simeq \prod_{\deg x \leq r} \mathrm{H}^0(x', \mathcal{L}^{\otimes d})$  lies in the subset  $\prod_{\deg x \leq r} (\mathrm{H}^0(x', \mathcal{L}^{\otimes d}) - \{0\})$ .

So for any  $d \geq N \left( \sum_{\deg x \leq r} (1+m) \deg x \right)$ ,

$$\begin{aligned} \frac{\#\mathcal{P}_{d, \leq r}}{\#\mathrm{H}^0(Y, \mathcal{L}^{\otimes d})} &= \frac{\#\prod_{\deg x \leq r} (\mathrm{H}^0(X, \mathcal{O}_X/\mathfrak{m}_x^2) - \{0\})}{\#\prod_{\deg x \leq r} \mathrm{H}^0(X, \mathcal{O}_X/\mathfrak{m}_x^2)} \\ &= \frac{\prod_{\deg x \leq r} (q^{(m+1) \deg x} - 1)}{\prod_{\deg x \leq r} q^{(m+1) \deg x}} \\ &= \prod_{\deg x \leq r} \left( 1 - q^{-(m+1) \deg x} \right). \end{aligned}$$

This shows the result. □

## 3.2 Singular points of medium degree

**Lemma 3.2.1.** *Let  $Y$  be a projective scheme over  $\mathbb{F}_q$  of dimension  $n$  and  $X$  a smooth subscheme of  $Y$  of dimension  $m$ . Let  $\mathcal{L}$  be an ample line bundle of  $Y$ . Let  $N$  be a positive integer verifying Lemma 3.0.2. For a fixed  $d$ , let  $x \in X$  be a closed point of  $X$  of degree  $e$  such that*

$$e \leq \lfloor \frac{d}{N(m+1)} \rfloor.$$

*Then the proportion of  $\sigma \in \mathrm{H}^0(Y, \mathcal{L}^{\otimes d})$  such that  $\mathrm{div} \sigma \cap X$  is not smooth of dimension  $m-1$  at  $x$  is  $q^{-(m+1)e}$ .*

*Proof.* Let  $x'$  be the first order infinitesimal neighbourhood of  $x$  in  $X$ . Apply Lemma 3.1.1 to the case  $Z = x'$ . We obtain that the restriction map

$$\mathrm{H}^0(Y, \mathcal{L}^{\otimes d}) \longrightarrow \mathrm{H}^0(x', \mathcal{L}^{\otimes d})$$

is surjective when

$$Nh^0(x', \mathcal{L}^{\otimes d}) = N(m+1) \deg x \leq d,$$

that is, when

$$\deg x \leq \lfloor \frac{d}{N(m+1)} \rfloor.$$

Since a section  $\sigma \in H^0(Y, \mathcal{L}^{\otimes d})$  is such that  $\text{div} \sigma \cap X$  is singular at  $x$  if and only if the image of  $\sigma$  in  $H^0(x', \mathcal{L}^{\otimes d})$  is 0. Hence when the degree condition for  $x$  is satisfied, the proportion of such sections is equal to

$$\frac{1}{\#H^0(x', \mathcal{L}^{\otimes d})} = q^{-(m+1)e},$$

Thus we get the result.  $\square$

**Proposition 3.2.2.** *Let  $Y$  be a projective scheme over  $\mathbb{F}_q$  of dimension  $n$  and  $X$  a smooth subscheme of  $Y$  of dimension  $m$ . Let  $\mathcal{L}$  be an ample line bundle of  $Y$ . Let  $N$  be a positive integer verifying Lemma 3.0.2. Set*

$$\mathcal{Q}_{d,>r}^{\text{med}} = \{\sigma \in H^0(Y, \mathcal{L}^{\otimes d}) ; \exists x \in X, r < \deg x \leq \frac{d}{(m+1)N}, \text{div} \sigma \cap X \text{ is singular at } x\}.$$

Then there exists a constant  $c_0$  such that

$$\frac{\#\mathcal{Q}_{d,>r}^{\text{med}}}{\#H^0(Y, \mathcal{L}^{\otimes d})} \leq 2c_0q^{-r}.$$

In particular,

$$\bar{\mu}(\mathcal{Q}_{d,>r}^{\text{med}}) \leq 2c_0q^{-r}.$$

*Proof.* Identifying  $Y$  to a closed subscheme of a projective space, we can see  $X$  as a subscheme of the same projective space. By [LW54], we can find a constant  $c_0 > 0$  such that for any  $e \geq 1$ ,

$$\#X(\mathbb{F}_{q^e}) \leq c_0q^{me}.$$

Let  $N$  be the positive integer as in the previous lemma. Then the lemma tells us that if  $x \in X$  is a closed point of degree  $e \leq \lfloor \frac{d}{(m+1)N} \rfloor$ , the proportion of sections  $\sigma \in H^0(Y, \mathcal{L}^{\otimes d})$  such that  $\text{div} \sigma \cap X$  is singular at  $x$  is  $q^{-(m+1)e}$ . Therefore we have

$$\begin{aligned} & \frac{\#\mathcal{Q}_{d,>r}^{\text{med}}}{\#H^0(Y, \mathcal{L}^{\otimes d})} \\ & \leq \sum_{e=r+1}^{\lfloor \frac{d}{(m+1)N} \rfloor} \#X(\mathbb{F}_{q^e}) \cdot q^{-(m+1)e} \\ & < \sum_{e=r}^{\infty} \#X(\mathbb{F}_{q^e}) q^{-(m+1)e} \\ & \leq \sum_{e=r}^{\infty} c_0q^{em} \cdot q^{-(m+1)e} = \sum_{e=r}^{\infty} c_0q^{-e} = \frac{c_0q^{-r}}{1-q^{-1}}. \end{aligned}$$

Since  $q \geq 2$ , we have  $1 - q^{-1} \geq \frac{1}{2}$  and  $\frac{c_0q^{-r}}{1-q^{-1}} \leq 2c_0q^{-r}$ . Hence we get  $\frac{\#\mathcal{Q}_{d,>r}^{\text{med}}}{\#H^0(Y, \mathcal{L}^{\otimes d})} \leq 2c_0q^{-r}$ . This implies our result.  $\square$

### 3.3 Singular points of high degree

**Lemma 3.3.1.** *Let  $Y$  be a projective scheme over  $\mathbb{F}_q$  of dimension  $n$ . Let  $Z$  be a finite closed subscheme of  $Y$  whose support is included in the smooth locus of  $Y$ . Let  $\mathcal{L}$  be an ample line bundle over  $Y$  and let  $N$  be a positive integer satisfying Lemma 3.0.2. After replacing  $\mathbb{F}_q$  by a finite extension of  $\mathbb{F}_q$  if needed, we can find a linear subspace  $V \subset H^0(Y, \mathcal{L}^N)$  of dimension  $n+1$  such that the rational map*

$$\varphi : Y \dashrightarrow \mathbb{P}(V^\vee),$$

with  $V^\vee$  the dual space of  $V$ , is dominant, and that  $\varphi$  induces a closed embedding of  $Z$  in  $\mathbb{P}(V^\vee)$ .

*Proof.* We may assume that  $|Z| = \{z_1, \dots, z_l\}$ . It suffices to find a linear subspace  $V \subset H^0(Y, \mathcal{L}^N)$  of dimension  $n+1$  such that the induced rational map

$$\varphi : Y \dashrightarrow \mathbb{P}(V^\vee)$$

is defined and étale on a neighbourhood of  $Z$  in  $Y$ , and satisfies the condition that  $\varphi(z_i) \neq \varphi(z_j)$  for any  $z_i \neq z_j \in |Z|$ .

Since  $\mathcal{L}^{\otimes N}$  is very ample on  $Y$ , we can first embed  $Y$  in  $\mathbb{P}^K = \mathbb{P}(H^0(Y, \mathcal{L}^{\otimes N})^\vee)$ . Replacing  $\mathbb{F}_q$  by a larger finite field if needed, we can find a section  $s \in H^0(Y, \mathcal{L}^{\otimes N})$  which is non-zero at any point of  $|Z|$ . Write  $U_1 := Y - \text{div}s$ . The embedding of  $Y$  in  $\mathbb{P}^K$  induces an embedding of  $U_1$  in  $\mathbb{A}^K$ . The hyperplane  $\mathbb{P}^K - \mathbb{A}^K$  is defined by the section  $s \in H^0(\mathbb{P}^K, \mathcal{O}(1)) = H^0(Y, \mathcal{L}^{\otimes N})$ . Moreover, the scheme  $Z$ , being a closed subscheme of  $U_1$ , is also embedded in  $\mathbb{A}^K$ . To finish the proof, we only need to find a projection  $\mathbb{A}^K \rightarrow \mathbb{A}^n$  which is étale when restricted to a neighbourhood of  $Z$  in  $U_1$  and injective when restricted to  $Z$ . In fact, we show that a general projection satisfies these two conditions. Here general means all projections  $\mathbb{A}^K \rightarrow \mathbb{A}^n$  contained in a non-empty open subscheme of  $\text{Gr}(n, K)$ , which is the moduli space of such projections.

For a general projection  $\mathbb{A}^K \rightarrow \mathbb{A}^n$ , the composition  $\varphi : U_1 \rightarrow \mathbb{A}^K \rightarrow \mathbb{A}^n$  is étale on a neighbourhood of  $Z$  in  $U_1$ . To see this, we show that for a general projection  $\mathbb{A}^K \rightarrow \mathbb{A}^n$ ,  $\varphi : U_1 \rightarrow \mathbb{A}^K \rightarrow \mathbb{A}^n$  is étale at any point of  $Z$ . For  $z_i \in Z$ , the exact sequence

$$0 \longrightarrow C_{U_1/\mathbb{A}^K, z_i} \longrightarrow \Omega_{\mathbb{A}^K, z_i} \longrightarrow \Omega_{U_1, z_i} \longrightarrow 0$$

splits, and  $\Omega_{U_1, z_i}$  is free of rank  $n$  by hypothesis. Therefore for any projection  $\mathbb{A}^K \rightarrow \mathbb{A}^n$ , the composition  $\varphi : U_1 \rightarrow \mathbb{A}^K \rightarrow \mathbb{A}^n$  induces a morphism on differential sheaves

$$\Omega_{\mathbb{A}^n, \varphi(z_i)} \longrightarrow \Omega_{\mathbb{A}^K, z_i} \longrightarrow \Omega_{U_1, z_i}.$$

By the Jacobian criterion, a general projection  $\mathbb{A}^K \rightarrow \mathbb{A}^n$  induces an isomorphism

$$\Omega_{\mathbb{A}^n, \varphi(z_i)} \xrightarrow{\sim} \Omega_{U_1, z_i}.$$

As  $Z$  is a finite scheme, a general projection  $\mathbb{A}^K \rightarrow \mathbb{A}^n$  is étale at any point of  $Z$ . Moreover, if a  $\mathbb{A}^K \rightarrow \mathbb{A}^n$  sends two different points  $z_i, z_j$  of  $Z$  to the same point, then it contracts the  $\mathbb{A}^1$  containing  $z_i, z_j$ . Projections contracting a certain fixed line is contained in a strictly closed subscheme of the moduli space of projections  $\mathbb{A}^K \rightarrow \mathbb{A}^n$ . Thus a general projection do not contract this line. Since  $Z$  is finite, there are only finitely many lines in  $\mathbb{A}^K$  joining two of points in  $Z$ . Therefore a general projection  $\mathbb{A}^K \rightarrow \mathbb{A}^n$  sends the set of points  $|Z|$  injectively to  $\mathbb{A}^n$ , hence injective when restricted to  $Z$ . Such a projection induces a rational map  $\mathbb{P}^K \dashrightarrow \mathbb{P}^n$ , which shows our result. □

**Lemma 3.3.2.** *Let  $Y, Z, \mathcal{L}, N$  be as in the above lemma. Set  $h_Z = \dim_{\mathbb{F}_q} H^0(Z, \mathcal{O}_Z)$ . Then for any  $d > 2N$  the proportion of global sections  $\sigma \in H^0(Y, \mathcal{L}^{\otimes d})$  which are sent to 0 by the restriction morphism*

$$H^0(Y, \mathcal{L}^{\otimes d}) \longrightarrow H^0(Z, \mathcal{L}^{\otimes d})$$

*is at most  $q^{-\min(\lfloor \frac{d}{N} \rfloor, h_Z)}$ .*

*Proof.* By the previous lemma, we can find a finite extension  $\mathbb{F}$  of  $\mathbb{F}_q$  and a subspace  $V \subset H^0(Y_{\mathbb{F}}, \mathcal{L}^{\otimes N})$  of dimension  $n + 1$  which induces a dominant rational map

$$\varphi : Y_{\mathbb{F}} \dashrightarrow \mathbb{P}(V^{\vee})$$

such that  $\varphi|_{Z_{\mathbb{F}}}$  is injective. Now we use this rational map to show that

$$\dim_{\mathbb{F}} \text{Im} (H^0(Y_{\mathbb{F}}, \mathcal{L}^{\otimes d}) \longrightarrow H^0(Z_{\mathbb{F}}, \mathcal{L}^{\otimes d})) \geq \min(\lfloor \frac{d}{N} \rfloor, h_Z).$$

As the dimension of the image is invariant under field base change, we then get

$$\dim_{\mathbb{F}_q} \text{Im} (H^0(Y, \mathcal{L}^{\otimes d}) \longrightarrow H^0(Z, \mathcal{L}^{\otimes d})) \geq \min(\lfloor \frac{d}{N} \rfloor, h_Z).$$

Let  $(\sigma_0, \dots, \sigma_n)$  be a base of  $V$  and  $H_0 = \text{div} \sigma_0$  in  $Y_{\mathbb{F}}$ . The sections  $\sigma_i$  can also be regarded as global sections of  $\mathcal{O}(1)$  on  $\mathbb{P}(V^{\vee})$ . This way we can identify  $\mathbb{P}(V^{\vee}) - \text{div} \sigma_0$  with  $\mathbb{A}^n$  with coordinates  $x_1 = \frac{\sigma_1}{\sigma_0}, \dots, x_n = \frac{\sigma_n}{\sigma_0}$ . Then the rational map  $\varphi$  can be represented by a morphism

$$\varphi : Y_{\mathbb{F}} - H_0 \longrightarrow \mathbb{A}^n.$$

Moreover, we can assume that  $Z_{\mathbb{F}}$  is a closed subscheme of  $Y_{\mathbb{F}} - H_0$ .

For all  $r > 0$  with  $(r + 1)N \leq d$ , the sheaf  $\mathcal{L}^{\otimes d} \otimes \mathcal{O}(-rH_0) \simeq \mathcal{L}^{\otimes (d-rN)}$  is very ample on  $Y_{\mathbb{F}}$ . So we can find a section  $\sigma_{H_0} \in H^0(Y_{\mathbb{F}}, \mathcal{L}^{\otimes d})$  which vanishes of order  $r$  along  $H_0$  but does not vanish identically on  $Y_{\mathbb{F}}$ . Let  $P \in \mathbb{F}[x_1, \dots, x_n]$  be a polynomial of total degree smaller than or equal to  $r$ . Then the section  $\varphi^*(P) \cdot \sigma_{H_0} \in H^0(Y_{\mathbb{F}} - H_0, \mathcal{L}^{\otimes d})$  extends to a global section on  $Y_{\mathbb{F}}$ . As  $\varphi$  is dominant, linearly independent polynomials of degree smaller than or equal to  $r$  induce linearly independent sections in  $H^0(Y_{\mathbb{F}} - H_0, \mathcal{L}^{\otimes d})$ , hence in  $H^0(Y_{\mathbb{F}}, \mathcal{L}^{\otimes d})$ . Thus we get a injective homomorphism

$$\mathbb{F}[x_1, \dots, x_n]^{\leq r} \hookrightarrow H^0(Y_{\mathbb{F}}, \mathcal{L}^{\otimes d}).$$

Moreover, we can choose an isomorphism  $H^0(Z_{\mathbb{F}}, \mathcal{L}^{\otimes d}) \xrightarrow{\sim} H^0(Z_{\mathbb{F}}, \mathcal{O}_{Z_{\mathbb{F}}})$  so that the following diagram commutes :

$$\begin{array}{ccc} \mathbb{F}[x_1, \dots, x_n]^{\leq r} & \hookrightarrow & H^0(Y_{\mathbb{F}}, \mathcal{L}^{\otimes d}) \\ \downarrow & & \downarrow \\ H^0(\varphi(Z_{\mathbb{F}}), \mathcal{O}_{\varphi(Z_{\mathbb{F}})}) & & H^0(Z_{\mathbb{F}}, \mathcal{L}^{\otimes d}) \\ & \searrow \sim & \swarrow \sim \\ & H^0(Z_{\mathbb{F}}, \mathcal{O}_{Z_{\mathbb{F}}}) & \end{array}$$

Then we have

$$\begin{aligned} & \dim_{\mathbb{F}} \text{Im} (H^0(Y_{\mathbb{F}}, \mathcal{L}^{\otimes d}) \longrightarrow H^0(Z_{\mathbb{F}}, \mathcal{L}^{\otimes d})) \\ & \geq \dim_{\mathbb{F}} \text{Im} (\mathbb{F}[x_1, \dots, x_n]^{\leq r} \longrightarrow H^0(\varphi(Z_{\mathbb{F}}), \mathcal{O}_{\varphi(Z_{\mathbb{F}})})) \\ & \geq \min(h_Z, r + 1), \end{aligned}$$



where the last inequality follows from Lemma 2.5 of [Po04]. Now choose  $r = \lfloor \frac{d}{N} \rfloor - 1$ , which is possible as  $(r+1)N = \lfloor \frac{d}{N} \rfloor \cdot N \leq d$ . Then the above inequality becomes

$$\dim_{\mathbb{F}} \operatorname{Im} (\mathrm{H}^0(Y_{\mathbb{F}}, \mathcal{L}^{\otimes d}) \longrightarrow \mathrm{H}^0(Z_{\mathbb{F}}, \mathcal{L}^{\otimes d})) \geq \min(h_Z, \lfloor \frac{d}{N} \rfloor).$$

This induces, as said above, that

$$\dim_{\mathbb{F}_q} \operatorname{Im} (\mathrm{H}^0(Y, \mathcal{L}^{\otimes d}) \longrightarrow \mathrm{H}^0(Z, \mathcal{L}^{\otimes d})) \geq \min(\lfloor \frac{d}{N} \rfloor, h_Z).$$

Therefore the proportion of global sections  $\sigma \in \mathrm{H}^0(Y, \mathcal{L}^{\otimes d})$  which are sent to 0 by the restriction morphism  $\mathrm{H}^0(Y, \mathcal{L}^{\otimes d}) \longrightarrow \mathrm{H}^0(Z, \mathcal{L}^{\otimes d})$  is at most  $q^{-\min(\lfloor \frac{d}{N} \rfloor, h_Z)}$ .  $\square$

**Proposition 3.3.3.** *Let  $\mathbb{F}_q$  be a finite field of characteristic  $p$ . Let  $Y$  be a projective scheme of dimension  $n$  over  $\mathbb{F}_q$ , and  $X$  a smooth subscheme of  $Y$  of dimension  $m$ . Assume that there exists a smooth quasi-compact open subscheme  $U$  in  $Y$  containing  $X$ . Let  $\mathcal{L}$  be an ample line bundle on  $Y$ , and let  $N$  be a sufficiently large integer. Set*

$$\mathcal{Q}_d^{\text{high}} = \{\sigma \in \mathrm{H}^0(Y, \mathcal{L}^{\otimes d}) ; \exists x \in X, \deg x \geq \frac{d}{(m+1)N}, \operatorname{div} \sigma \cap X \text{ is singular at } x\}$$

and  $\mathcal{Q}^{\text{high}} = \bigcup_{d \geq 0} \mathcal{Q}_d^{\text{high}}$ . There exists a constant  $c > 0$  only depending on  $X, U$  and the choice of  $N$ , independent of  $d$ , such that

$$\frac{\#\mathcal{Q}_d^{\text{high}}}{\#\mathrm{H}^0(Y, \mathcal{L}^{\otimes d})} = O(d^m \cdot q^{-c\frac{d}{p}}).$$

Here the constant involved in the big  $O$  only depends on the sheaf  $\mathcal{L}$ , and the schemes  $\overline{X} = X \times_{\operatorname{Spec} \mathbb{F}_q} \operatorname{Spec} \overline{\mathbb{F}_q}$ ,  $\overline{Y} = Y \times_{\operatorname{Spec} \mathbb{F}_q} \operatorname{Spec} \overline{\mathbb{F}_q}$ , where  $\overline{\mathbb{F}_q}$  is any algebraic closure of  $\mathbb{F}_q$ , hence is independent of  $d, q$ . In particular,

$$\overline{\mu}(\mathcal{Q}^{\text{high}}) = \limsup_{d \rightarrow \infty} \frac{\#\mathcal{Q}_d^{\text{high}}}{\#\mathrm{H}^0(Y, \mathcal{L}^{\otimes d})} = 0.$$

We need some reduction before proving this proposition.

**Lemma 3.3.4.** *Let  $\{U_\alpha\}_{\alpha \in I}$  be a finite open covering of  $U$ . If the proposition is true for all  $X \cap U_\alpha$ , then it is also true for  $X$ .*

*Proof.* If  $\{U_\alpha\}_{\alpha \in I}$  is a finite open covering of  $U$ , then  $X = \bigcup_{\alpha \in I} (X \cap U_\alpha)$ . If we write

$$\mathcal{Q}_{d,\alpha}^{\text{high}} = \{\sigma \in \mathrm{H}^0(Y, \mathcal{L}^{\otimes d}) ; \exists x \in X \cap U_\alpha, \deg x \geq \frac{d}{(m+1)N}, \operatorname{div} \sigma \cap X \cap U_\alpha \text{ is singular at } x\},$$

then we have

$$\mathcal{Q}_d^{\text{high}} \subset \bigcup_{\alpha \in I} \mathcal{Q}_{d,\alpha}^{\text{high}}.$$

Hence if the proposition is true for all  $X \cap U_\alpha$ , we can find constants  $c_{U_\alpha}$  for each  $U_\alpha$  such that

$$\frac{\#\mathcal{Q}_{d,\alpha}^{\text{high}}}{\#\mathrm{H}^0(Y, \mathcal{L}^{\otimes d})} = O(d^m \cdot q^{-c_{U_\alpha} \frac{d}{p}}),$$

Then setting  $c = \min_{\alpha} \{c_{U_{\alpha}}\}$ , we get

$$\frac{\#\mathcal{Q}_d^{\text{high}}}{\#\mathbf{H}^0(Y, \mathcal{L}^{\otimes d})} \leq \sum_{\alpha \in I} \frac{\#\mathcal{Q}_{d,\alpha}^{\text{high}}}{\#\mathbf{H}^0(Y, \mathcal{L}^{\otimes d})} = O(d^m \cdot q^{-c\frac{d}{p}}).$$

□

Therefore we may replace  $X$  by one of the  $X \cap U_{\alpha}$ . In particular, we have the following :

**Corollary 3.3.5.** *We may assume that the smooth quasi-compact open subscheme  $U$  containing  $X$  in the statement satisfies the following condition : there exist  $t_1, \dots, t_n \in \mathbf{H}^0(U, \mathcal{O}_Y)$  such that  $X$  is defined by  $t_{m+1} = \dots = t_n = 0$ , and that*

$$\Omega_{U/\mathbb{F}_q}^1 \simeq \bigoplus_{i=1}^n \mathcal{O}_U dt_i, \quad \Omega_{X/\mathbb{F}_q}^1 \simeq \bigoplus_{i=1}^m \mathcal{O}_X dt_i.$$

*Proof.* As the two condition in the corollary is satisfied locally at any point of  $X$ , the corollary follows from Lemma 3.3.4 as  $U$  is quasi-compact. □

**Lemma 3.3.6.** *For any positive integer  $M$ , with a choice of an integer  $N_0 \geq M$ , we can cover  $U$  by finitely many open subschemes  $U'$  satisfying the condition that we can find a section  $\tau \in \mathbf{H}^0(Y, \mathcal{L}^{\otimes(N_0+1)})$  and sections  $\tau_1, \dots, \tau_s \in \mathbf{H}^0(Y, \mathcal{L}^{\otimes N_0})$  such that*

$$U' = Y - \text{div}(\tau) = \bigcup_{j=1}^s (Y - \text{div}(\tau_j)).$$

*Proof.* First, take an integer  $N'_0 > 0$  satisfying Lemma 3.0.2 and that  $\mathcal{I}_{Y-U} \otimes \mathcal{L}^{\otimes d}$  is globally generated for all  $d \geq N'_0$ . Here  $\mathcal{I}_{Y-U}$  is the ideal sheaf of the closed subscheme  $Y - U$  with the reduced induced structure. We may then choose non-zero sections

$$\tau'_1, \dots, \tau'_t \in \mathbf{H}^0(Y, \mathcal{I}_{Y-U} \otimes \mathcal{L}^{\otimes N'_0}) \subset \mathbf{H}^0(Y, \mathcal{L}^{\otimes N'_0}),$$

generating  $\mathcal{I}_{Y-U} \otimes \mathcal{L}^{\otimes N'_0}$ . This means that set theoretically, we have  $Y - U = \bigcap_i \text{div}(\tau'_i)$ . In other words,

$$U = \bigcup_i (Y - \text{div}(\tau'_i)),$$

where  $Y - \text{div}(\tau'_i)$  are open subschemes of  $Y$ . Without loss of generality, we may assume that  $U$  itself is one of such open subschemes, i.e. there exists a section  $\tau' \in \mathbf{H}^0(Y, \mathcal{L}^{\otimes N'_0})$  such that  $U = Y - \text{div}(\tau')$ . We denote  $\text{div}(\tau')$  by  $D$ .

Now set  $N_0 = rN'_0 - 1$  for some positive integer  $r$  such that  $N_0 \geq M$  and that the sheaf  $\mathcal{I}_D \otimes \mathcal{L}^{\otimes N_0}$  is globally generated. Then in particular we can find sections  $\tau_1, \dots, \tau_s \in \mathbf{H}^0(Y, \mathcal{I}_D \otimes \mathcal{L}^{\otimes N_0}) \subset \mathbf{H}^0(Y, \mathcal{L}^{\otimes N_0})$  such that  $D = \bigcap_{j=1}^s \text{div}(\tau_j)$  set theoretically. This suggests that

$$U = \bigcup_{j=1}^s (Y - \text{div}(\tau_j)).$$

We also set  $\tau = (\tau')^r \in \mathbf{H}^0(Y, \mathcal{L}^{\otimes(N_0+1)})$ . In this situation we still have  $D = \text{div}(\tau)$  set theoretically. The section  $\tau$  and sections  $\tau_1, \dots, \tau_s$  are then what we need in the lemma. □

**Corollary 3.3.7.** *We may assume that the smooth quasi-compact open subscheme  $U$  containing  $X$  in the statement satisfies the condition in Corollary 3.3.5 and the condition that we can find a section  $\tau \in H^0(Y, \mathcal{L}^{\otimes(N_0+1)})$  and sections  $\tau_1, \dots, \tau_s \in H^0(Y, \mathcal{L}^{\otimes N_0})$  such that*

$$U = Y - \operatorname{div}(\tau) = \bigcup_{j=1}^s (Y - \operatorname{div}(\tau_j)).$$

*Proof.* This is a direct consequence of Lemma 3.3.4 and Lemma 3.3.6.  $\square$

For any  $d > 0$  and any  $1 \leq j \leq s$ , consider the morphism

$$\begin{aligned} \Phi_j : H^0(Y, \mathcal{L}^{\otimes d}) &\longrightarrow H^0(U, \mathcal{O}_Y) \\ \sigma &\longmapsto \frac{\sigma \cdot \tau_j^d}{\tau^d}. \end{aligned}$$

For simplicity of notations, we don't distinguish morphisms  $\Phi_j$  for different  $d$ . This will not cause any confusion as the source of  $\Phi_j$  will be clear by the context. Then for any  $\sigma \in H^0(Y, \mathcal{L}^{\otimes d})$ , we have

$$\operatorname{div} \sigma \cap U = \bigcap_{1 \leq j \leq s} \operatorname{div} \Phi_j(\sigma).$$

Let  $\partial_i \in \operatorname{Der}_{\mathbb{F}_q}(\mathcal{O}_U, \mathcal{O}_U) \simeq \operatorname{Hom}_{\mathcal{O}_U}(\Omega_{U/\mathbb{F}_q}^1, \mathcal{O}_U)$  be the dual of  $dt_i \in H^0(U, \Omega_{U/\mathbb{F}_q}^1)$ . Then a global section  $\sigma \in H^0(Y, \mathcal{L}^{\otimes d})$  is such that  $\operatorname{div} \sigma \cap X$  is singular at a closed point  $x \in X$  if and only if for a  $U_j$  containing  $x$ , we have

$$\Phi_j(\sigma)(x) = (\partial_1 \Phi_j(\sigma))(x) = \dots = (\partial_m \Phi_j(\sigma))(x) = 0.$$

We want to show that there exists a positive integer  $N_1$  such that for each  $i$ ,  $(\partial_i \Phi_j(\sigma)) \cdot \tau^{d+N_1}$  can be extended to a global section of  $\mathcal{L}^{\otimes(N_0+1)(d+N_1)}$ . To show this, we need to study the derivation map

$$d : H^0(U, \mathcal{O}_Y) \longrightarrow H^0(U, \Omega_{Y/\mathbb{F}_q}^1).$$

For any section  $f \in H^0(U, \mathcal{O}_Y)$ , we denote its image under the derivation by  $df \in H^0(U, \Omega_{Y/\mathbb{F}_q}^1)$ .

**Lemma 3.3.8.** *In the setting of the above corollary, there exists a positive integer  $d_0$  such that when  $d \geq d_0$ , for any  $f \in H^0(U, \mathcal{O}_Y)$ , if  $f \cdot \tau^d$  can be extended to a global section in  $H^0(Y, \mathcal{L}^{\otimes(N_0+1)d})$ , then  $df \cdot \tau^{d+1}$  can be extended to a global section in  $H^0(Y, \Omega_{Y/\mathbb{F}_q}^1 \otimes \mathcal{L}^{\otimes(N_0+1)(d+1)})$ .*

*Proof.* By assumption, the sheaf  $\mathcal{L}^{\otimes(N_0+1)}$  is very ample. So it induces a closed embedding

$$Y \hookrightarrow \mathbb{P} \left( H^0(Y, \mathcal{L}^{\otimes(N_0+1)}) \right).$$

To simplify the notation, we denote  $\mathbb{P} \left( H^0(Y, \mathcal{L}^{\otimes(N_0+1)}) \right)$  by  $\mathbb{P}^{K_0}$  with homogeneous coordinates  $T_0, T_1, \dots, T_{K_0}$ . In particular, let  $T_0$  be the section corresponding to  $\tau$  in  $H^0(\mathbb{P}^{K_0}, \mathcal{O}(1))$ . Therefore the closed embedding  $Y \hookrightarrow \mathbb{P}^{K_0}$  identifies  $U$  with a closed subscheme of  $\mathbb{A}^{K_0} = \mathbb{P}^{K_0} - \operatorname{div} T_0$ . Write  $x_i = \frac{T_i}{T_0}$ . Then  $x_1, \dots, x_{K_0}$  form a system of coordinates of  $\mathbb{A}^{K_0}$ . Let  $d_0 > 0$  be an integer such that for any  $d \geq d_0$ , the restriction morphism

$$H^0(\mathbb{P}^{K_0}, \mathcal{O}(d)) \longrightarrow H^0(Y, \mathcal{L}^{\otimes(N_0+1)d})$$

is surjective. Take  $f \in H^0(U, \mathcal{O}_Y)$  and  $d \geq d_0$  such that  $f \cdot \tau^d$  can be extended to a global section in  $H^0(Y, \mathcal{L}^{\otimes(N_0+1)d})$ . The surjectivity of the above restriction morphism suggests that we can

find  $\tilde{F} \in H^0(\mathbb{P}^{K_0}, \mathcal{O}(d))$  whose restriction to  $H^0(Y, \mathcal{L}^{\otimes(N_0+1)d})$  is the chosen extension of  $f \cdot \tau^d$ .  
Set

$$\tilde{f} = \frac{\tilde{F}}{T_0^d} \in H^0(\mathbb{A}^{K_0}, \mathcal{O}_{\mathbb{P}^{K_0}}).$$

Then  $\tilde{f}$  has image  $f$  when restricted to  $U$ . So  $\tilde{f} = H^0(\mathbb{A}^{K_0}, \mathcal{O}_{\mathbb{P}^{K_0}})$  is a section such that  $\tilde{f} \cdot T_0^d$  can be extended to a section in  $H^0(\mathbb{P}^{K_0}, \mathcal{O}(d))$  and that  $\tilde{f}|_U = f$ .

Now we consider the derivation

$$H^0(\mathbb{A}^{K_0}, \mathcal{O}_{\mathbb{P}^{K_0}}) \longrightarrow H^0(\mathbb{A}^{K_0}, \Omega_{\mathbb{P}^{K_0}/\mathbb{F}_q}^1)$$

sending  $\tilde{f}$  to

$$d\tilde{f} = \sum_{i=1}^{K_0} \left( \frac{\partial}{\partial x_i} \tilde{f} \right) dx_i.$$

The fact that  $\tilde{f} \cdot T_0^d$  can be extended to a section in  $H^0(\mathbb{P}^{K_0}, \mathcal{O}(d))$  means that  $\tilde{f}$  is a polynomial of total degree smaller than or equal to  $d$  in  $x_1, \dots, x_{K_0}$ . As for each  $i$ ,  $\frac{\partial}{\partial x_i} \tilde{f}$  is of degree strictly smaller than  $\tilde{f}$  if non-zero, all  $\left( \frac{\partial}{\partial x_i} \tilde{f} \right) \cdot T_0^{d-1}$  can be extended to a global section in  $H^0(\mathbb{P}^{K_0}, \mathcal{O}(d-1))$ . Note that we have a short exact sequence

$$0 \longrightarrow \Omega_{\mathbb{P}^{K_0}/\mathbb{F}_q}^1 \longrightarrow \mathcal{O}_{\mathbb{P}^{K_0}/\mathbb{F}_q}(-1)^{\oplus(K_0+1)} \longrightarrow \mathcal{O}_{\mathbb{P}^{K_0}/\mathbb{F}_q} \longrightarrow 0$$

identifying  $\Omega_{\mathbb{P}^{K_0}/\mathbb{F}_q}^1$  as a locally free subsheaf of  $\mathcal{O}_{\mathbb{P}^{K_0}/\mathbb{F}_q}(-1)^{\oplus(K_0+1)}$ . Under this identification, for each  $1 \leq i \leq K_0$  we can write

$$dx_i = d\left(\frac{T_i}{T_0}\right) = \frac{1}{T_0} e_i - \frac{T_i}{T_0^2} e_0,$$

where  $e_0, \dots, e_{K_0}$  is a chosen basis of  $\mathcal{O}_{\mathbb{P}^{K_0}/\mathbb{F}_q}(-1)^{\oplus(K_0+1)}$ . Therefore for each  $1 \leq i \leq K_0$ ,  $dx_i \cdot T_0^2$  can be extended to a global section in  $H^0(\mathbb{P}^{K_0}, \Omega_{\mathbb{P}^{K_0}/\mathbb{F}_q}^1 \otimes \mathcal{O}(2))$ . As a consequence,  $d\tilde{f} \cdot T_0^{d+1}$  can be extended to a global section in  $H^0(\mathbb{P}^{K_0}, \Omega_{\mathbb{P}^{K_0}/\mathbb{F}_q}^1 \otimes \mathcal{O}(d+1))$ .

When restricted to  $Y$ , we have a natural morphism

$$\left( \Omega_{\mathbb{P}^{K_0}/\mathbb{F}_q}^1 \otimes \mathcal{O}(d+1) \right) \Big|_Y \simeq \left( \Omega_{\mathbb{P}^{K_0}/\mathbb{F}_q}^1 \right) \Big|_Y \otimes \mathcal{L}^{\otimes(d+1)} \xrightarrow{r} \Omega_{Y/\mathbb{F}_q}^1 \otimes \mathcal{L}^{\otimes(d+1)},$$

which gives us a section  $r(d\tilde{f} \cdot T_0^{d+1}) \in H^0(Y, \Omega_{Y/\mathbb{F}_q}^1 \otimes \mathcal{L}^{\otimes(N_0+1)(d+1)})$ . To finish the proof, it suffices to show that

$$r\left(d\tilde{f} \cdot T_0^{d+1}\right) \Big|_U = df \cdot \tau^{d+1}.$$

But as  $r$  is induced by  $\left( \Omega_{\mathbb{P}^{K_0}/\mathbb{F}_q}^1 \right) \Big|_Y \longrightarrow \Omega_{Y/\mathbb{F}_q}^1$ , it commutes with the multiplication by  $T_0, \tau$ .

So the equality is clear. Therefore  $r\left(d\tilde{f} \cdot T_0^{d+1}\right)$  is an extension of  $df \cdot \tau^{d+1}$  in  $H^0(Y, \Omega_{Y/\mathbb{F}_q}^1 \otimes \mathcal{L}^{\otimes(N_0+1)(d+1)})$ .  $\square$

**Lemma 3.3.9.** *There exists a positive integer  $N'_1$  which only depends on the sections  $t_1, \dots, t_n \in H^0(U, \mathcal{O}_Y)$  and  $\tau \in H^0(Y, \mathcal{L}^{\otimes(N_0+1)})$  satisfying the following condition : if a section  $f \in H^0(U, \mathcal{O}_Y)$  is such that  $df \cdot \tau^d$  can be extended to a global section in  $H^0(Y, \Omega_{Y/\mathbb{F}_q}^1 \otimes \mathcal{L}^{\otimes(N_0+1)d})$ , then for any  $1 \leq i \leq m$  and any  $\delta \geq N'_1$ , the section  $\partial_i f \cdot \tau^{d+\delta}$  can be extended to a global section of  $\mathcal{L}^{\otimes(N_0+1)(d+\delta)}$ . If this condition is satisfied by  $N'_1$ , it is also satisfied by any integer bigger than  $N'_1$ .*

*Proof.* Note that for each  $1 \leq i \leq n$ , the dual  $\partial_i \in \text{Der}_{\mathbb{F}_q}(\mathcal{O}_U, \mathcal{O}_U) \simeq \text{Hom}_{\mathcal{O}_U}(\Omega_{U/\mathbb{F}_q}^1, \mathcal{O}_U)$  of  $dt_i \in \text{H}^0(U, \Omega_{U/\mathbb{F}_q}^1)$  can be regarded as a section in  $\text{H}^0\left(U, \mathcal{H}om_{\mathcal{O}_Y}(\Omega_{Y/\mathbb{F}_q}^1, \mathcal{O}_Y)\right)$ . Since the sheaf  $\mathcal{H}om_{\mathcal{O}_Y}(\Omega_{Y/\mathbb{F}_q}^1, \mathcal{O}_Y)$  is coherent and that  $U$  is the complement of  $\text{div}\tau$  in  $Y$ , there exists an integer  $N'_{1,i} > 0$  such that for any  $\delta \geq N'_{1,i}$ ,  $\partial_i \cdot \tau^\delta \in \text{H}^0\left(U, \mathcal{H}om_{\mathcal{O}_Y}(\Omega_{Y/\mathbb{F}_q}^1, \mathcal{O}_Y) \otimes \mathcal{L}^{(N_0+1)\delta}\right)$  can be extended to a global section in  $\text{H}^0\left(Y, \mathcal{H}om_{\mathcal{O}_Y}(\Omega_{Y/\mathbb{F}_q}^1, \mathcal{O}_Y) \otimes \mathcal{L}^{(N_0+1)\delta}\right)$ . Set

$$N'_1 = \max\{N'_{1,i} ; 1 \leq i \leq n\}.$$

Then when  $\delta \geq N'_1$ , for any  $1 \leq i \leq n$  the section  $\partial_i \cdot \tau^\delta \in \text{H}^0\left(U, \mathcal{H}om_{\mathcal{O}_Y}(\Omega_{Y/\mathbb{F}_q}^1, \mathcal{O}_Y) \otimes \mathcal{L}^{(N_0+1)\delta}\right)$  can be extended to a global section in

$$\text{H}^0\left(Y, \mathcal{H}om_{\mathcal{O}_Y}(\Omega_{Y/\mathbb{F}_q}^1, \mathcal{O}_Y) \otimes \mathcal{L}^{(N_0+1)\delta}\right) \simeq \text{Hom}(\Omega_{Y/\mathbb{F}_q}^1, \mathcal{L}^{(N_0+1)\delta}).$$

Note that

$$\text{Hom}(\Omega_{Y/\mathbb{F}_q}^1, \mathcal{L}^{(N_0+1)\delta}) \simeq \text{Hom}(\Omega_{Y/\mathbb{F}_q}^1 \otimes \mathcal{L}^d, \mathcal{L}^{(N_0+1)(d+\delta)}).$$

The global section

$$(\partial_i \cdot \tau^\delta)(df \cdot \tau^d) \in \text{H}^0(Y, \mathcal{L}^{(N_0+1)(d+\delta)})$$

satisfies

$$((\partial_i \cdot \tau^\delta)(df \cdot \tau^d)) \Big|_U = (\partial_i f) \cdot \tau^{d+\delta},$$

which means that  $(\partial_i f) \cdot \tau^{d+\delta}$  can be extended to a global section of  $\mathcal{L}^{(N_0+1)(d+\delta)}$ . Hence we conclude.  $\square$

**Lemma 3.3.10.** *There exists a positive integer  $N_1$  which only depends on the sheaf  $\mathcal{L}$ , the sections  $t_1, \dots, t_n \in \text{H}^0(U, \mathcal{O}_Y)$  and  $\tau \in \text{H}^0(Y, \mathcal{L}^{\otimes(N_0+1)})$  such that for any  $\sigma \in \text{H}^0(Y, \mathcal{L}^{\otimes d})$ , any  $1 \leq i \leq m$ ,  $1 \leq j \leq s$ , the section  $(\partial_i \Phi_j(\sigma)) \cdot \tau^{d+\delta}$  can be extended to a global section of  $\mathcal{L}^{\otimes(N_0+1)(d+\delta)}$  for any  $\delta \geq N_1$ . If this condition is satisfied by  $N_1$ , it is also satisfied by any integer bigger than  $N_1$ .*

*Proof.* For any  $\sigma \in \text{H}^0(Y, \mathcal{L}^{\otimes d})$ , as  $\Phi_j(\sigma) = \frac{\sigma \cdot \tau^d}{\tau^d}$ , evidently  $\Phi_j(\sigma) \cdot \tau^d$  can be extended to a global section in  $\text{H}^0(Y, \mathcal{L}^{\otimes(N_0+1)d})$ . Let  $d_0$  be the constant defined in Lemma 3.3.8. In particular,  $\Phi_j(\sigma) \cdot \tau^{d+d_0}$  can also be extended to a global section in  $\text{H}^0(Y, \mathcal{L}^{\otimes(N_0+1)(d+d_0)})$ . Then by Lemma 3.3.8  $d\Phi_j(\sigma) \cdot \tau^{d+d_0+1}$  can be extended to a global section in  $\text{H}^0(Y, \Omega_{Y/\mathbb{F}_q}^1 \otimes \mathcal{L}^{\otimes(N_0+1)(d+d_0+1)})$ . Set  $N_1 = N'_1 + d_0 + 1$ . So when  $\delta \geq N_1$ ,  $\delta - d_0 - 1 \geq N'_1$ . Applying Lemma 3.3.9, we obtain that for any  $\delta \geq N_1$ , the section

$$(\partial_i \Phi_j(\sigma)) \cdot \tau^{d+\delta} = (\partial_i \Phi_j(\sigma)) \cdot \tau^{(d+d_0+1)+(\delta-d_0-1)}$$

can be extended to a global section in  $\text{H}^0(Y, \mathcal{L}^{\otimes(N_0+1)(d+\delta)})$ .  $\square$

Now we assume that  $p$  is the characteristic of  $\mathbb{F}_q$ , and that  $(N_0 + 1, p) = 1$ . We also assume that  $N_1 \geq N_0$ . Take  $N = (N_0 + 1)(N_1 + p - 1) + p$ . For any  $d \geq N$ , there exists a  $N_1 \leq l_d < N_1 + p$  such that

$$d \bmod p \equiv (N_0 + 1)l_d \bmod p.$$

Note  $k_d = \frac{1}{p}[d - (N_0 + 1)l_d]$ . Then for any  $d \geq N$ , fixing extensions of  $\tau^{l_d}$  and of  $t_i \tau^{l_d}$  to global sections for all  $1 \leq i \leq m$ , we can construct morphisms of groups

$$\begin{aligned} \text{H}^0(Y, \mathcal{L}^{\otimes k_d}) &\longrightarrow \text{H}^0(Y, \mathcal{L}^{\otimes d}) \\ \beta &\longmapsto \beta^p \cdot t_i \tau^{l_d} \end{aligned}$$

for any  $1 \leq i \leq m$ , and

$$\begin{aligned} \mathrm{H}^0(Y, \mathcal{L}^{\otimes k_d}) &\longrightarrow \mathrm{H}^0(Y, \mathcal{L}^{\otimes d}) \\ \gamma &\longmapsto \gamma^p \cdot \tau^{l_d}. \end{aligned}$$

We construct a surjective morphism

$$\mathrm{H}^0(Y, \mathcal{L}^{\otimes d}) \times \left( \prod_{i=1}^m \mathrm{H}^0(Y, \mathcal{L}^{\otimes k_d}) \right) \times \mathrm{H}^0(Y, \mathcal{L}^{\otimes k_d}) \longrightarrow \mathrm{H}^0(Y, \mathcal{L}^{\otimes d})$$

which sends  $(\sigma_0, (\beta_1, \dots, \beta_m), \gamma)$  to

$$\sigma = \sigma_0 + \sum_{i=1}^m \beta_i^p t_i \tau^{l_d} + \gamma^p \tau^{l_d}.$$

Naturally, this morphism commutes with all  $\Phi_j$  :

$$\Phi_j(\sigma) = \Phi_j(\sigma_0) + \sum_{i=1}^m \Phi_j(\beta_i)^p t_i \Phi_j(\tau)^{l_d} + \Phi_j(\gamma)^p \Phi_j(\tau)^{l_d}.$$

Since

$$\partial_i[\Phi_j(\beta_i)^p t_i \Phi_j(\tau)^{l_d}] = \Phi_j(\beta_i)^p \Phi_j(\tau)^{l_d} + l_d \Phi_j(\beta_i)^p t_i \Phi_j(\tau)^{l_d-1} \cdot \partial_i \Phi_j(\tau),$$

and for  $i' \neq i$ ,

$$\partial_i[\Phi_j(\beta_{i'})^p t_{i'} \Phi_j(\tau)^{l_d}] = l_d \Phi_j(\beta_{i'})^p t_{i'} \Phi_j(\tau)^{l_d-1} \cdot \partial_i \Phi_j(\tau),$$

the differential of  $\Phi_j(\sigma)$  can be written as

$$\begin{aligned} \partial_i \Phi_j(\sigma) &= \left[ \sum_{i'=1}^m l_d \Phi_j(\beta_{i'})^p t_{i'} \Phi_j(\tau)^{l_d-1} + l_d \Phi_j(\gamma)^p \Phi_j(\tau)^{l_d-1} \right] \cdot \partial_i \Phi_j(\tau) \\ &\quad + \partial_i \Phi_j(\sigma_0) + \Phi_j(\beta_i)^p \Phi_j(\tau)^{l_d} \\ &= \partial_i \Phi_j(\sigma_0) + \frac{l_d(\Phi_j(\sigma) - \Phi_j(\sigma_0))}{\Phi_j(\tau)} \partial_i \Phi_j(\tau) + \Phi_j(\beta_i)^p \Phi_j(\tau)^{l_d} \end{aligned}$$

To prove Proposition 3.3.3, for a section  $\sigma \in \mathrm{H}^0(Y, \mathcal{L}^{\otimes d})$ , we need to study the singular locus of  $\mathrm{div} \sigma \cap X$ . Since

$$\mathrm{div} \sigma \cap U = \bigcup_{1 \leq j \leq s} (U_j \cap \mathrm{div} \Phi_j(\sigma)),$$

and  $X \subset U$ , we have

$$\mathrm{Sing}(\mathrm{div} \sigma \cap X) \subset \bigcup_{1 \leq j \leq s} (U_j \cap \mathrm{Sing}(\mathrm{div}(\Phi_j(\sigma)) \cap X)).$$

Note that for a  $\sigma \in \mathrm{H}^0(Y, \mathcal{O}_Y)$ ,  $\mathrm{div}(\Phi_j(\sigma)) \cap X$  is singular at a point  $x \in X$  if and only if

$$\Phi_j(\sigma)(x) = \partial_1 \Phi_j(\sigma)(x) = \dots = \partial_m \Phi_j(\sigma)(x) = 0$$

by conditions on  $\partial_i$ ,  $1 \leq i \leq n$ . Therefore we have

$$\mathrm{Sing}(\mathrm{div}(\Phi_j(\sigma)) \cap X) = \mathrm{div}(\Phi_j(\sigma)) \cap \mathrm{div}(\partial_1 \Phi_j(\sigma)) \cap \dots \cap \mathrm{div}(\partial_m \Phi_j(\sigma))$$

in  $U$ .

Now for any  $(\sigma_0, (\beta_1, \dots, \beta_m), \gamma) \in \mathbb{H}^0(Y, \mathcal{L}^{\otimes d}) \times \left( \prod_{i=1}^m \mathbb{H}^0(Y, \mathcal{L}^{\otimes k_d}) \right) \times \mathbb{H}^0(Y, \mathcal{L}^{k_d})$ , set

$$g_{j,i}(\sigma_0, \beta_i) = \partial_i \Phi_j(\sigma_0) - \frac{l_d \Phi_j(\sigma_0)}{\Phi_j(\tau)} \partial_i \Phi_j(\tau) + \Phi_j(\beta_i)^p \Phi_j(\tau)^{l_d},$$

and

$$W_{j,i} := X \cap U_j \cap \{g_{j,1} = \dots = g_{j,i} = 0\}.$$

Then for any  $\sigma = \sigma_0 + \sum_{i=1}^m \beta_i^p t_i \tau^{l_d} + \gamma^p \tau^{l_d}$ , comparing the expressions of  $g_{j,i}$  and  $\partial_i \Phi_j$ , which gives

$$g_{j,i}(\sigma_0, \beta_i) = \partial_i \Phi_j(\sigma) - \frac{l_d \Phi_j(\sigma)}{\Phi_j(\tau)} \partial_i \Phi_j(\tau),$$

we have

$$g_{j,i}(\sigma_0, \beta_i)|_{\text{div } \Phi_j(\sigma)} = \partial_i \Phi_j(\sigma)|_{\text{div } \Phi_j(\sigma)}.$$

Applying Lemma 3.3.10 to the section  $\sigma \in \mathbb{H}^0(Y, \mathcal{L}^{\otimes d})$ , we know that  $(\partial_i \Phi_j(\sigma)) \cdot \tau^{d+\delta}$  can be extended to a global section of  $\mathcal{L}^{\otimes (N_0+1)(d+\delta)}$  for any  $\delta \geq N_1$ . By the same lemma, the section  $\partial_i \Phi_j(\tau) \cdot \tau^{(N_0+1)+\delta}$  extends to a global section of  $\mathcal{L}^{\otimes (N_0+1)((N_0+1)+\delta)}$  for any  $\delta \geq N_1$ . Note that on  $U$  we have

$$\begin{aligned} \frac{l_d \Phi_j(\sigma)}{\Phi_j(\tau)} &= l_d \frac{\sigma \cdot \tau_j^d}{\tau^d} \cdot \left( \frac{\tau \cdot \tau_j^{N_0+1}}{\tau^{N_0+1}} \right)^{-1} \\ &= l_d \frac{\sigma \cdot \tau_j^{d-N_0-1}}{\tau^{d-N_0}}. \end{aligned}$$

So the section

$$\frac{l_d \Phi_j(\sigma)}{\Phi_j(\tau)} \partial_i \Phi_j(\tau) \cdot \tau^{d+\delta} = \left( \frac{l_d \Phi_j(\sigma)}{\Phi_j(\tau)} \cdot \tau^{d-N_0} \right) \cdot \left( \partial_i \Phi_j(\tau) \cdot \tau^{(N_0+1)+(\delta-1)} \right)$$

extends to a global section of  $\mathcal{L}^{\otimes (N_0+1)(d+\delta)}$  for any  $\delta \geq N_1 + 1$ . Therefore the section

$$g_{j,i}(\sigma_0, \beta_i) \cdot \tau^{d+\delta} = \left( \partial_i \Phi_j(\sigma) - \frac{l_d \Phi_j(\sigma)}{\Phi_j(\tau)} \partial_i \Phi_j(\tau) \right) \cdot \tau^{d+\delta} \in \mathbb{H}^0(U, \mathcal{L}^{\otimes (N_0+1)(d+\delta)})$$

can be extended to a global section in  $\mathbb{H}^0(Y, \mathcal{L}^{\otimes (N_0+1)(d+\delta)})$  for any  $\delta \geq N_1 + 1$ .

**Lemma 3.3.11.** *For  $0 \leq i \leq m-1$ , with a fixed choice of  $\sigma_0, \beta_1, \dots, \beta_i$  such that  $\dim W_{j,i} \leq m-i$ , the proportion of  $\beta_{i+1}$  in  $\mathbb{H}^0(Y, \mathcal{L}^{\otimes k_d})$  such that  $\dim W_{j,i+1} \leq m-i-1$  is  $1 - O(d^i \cdot q^{2 - \frac{d}{(N_0+1)N_1 p}})$ , where  $p$  is the characteristic of  $\mathbb{F}_q$  and the constant involved is independent of  $d, q$ .*

*Proof.* Let  $V_1, \dots, V_s$  be the  $(m-i)$ -dimensional  $\mathbb{F}_q$ -irreducible components of the reduced scheme  $(W_{j,i})_{\text{red}}$ . The closure of the  $V_i$ 's in  $Y$  are contained in the set of  $(m-i)$ -dimensional  $\mathbb{F}_q$ -irreducible components of  $\overline{X} \cap \text{div } g_{j,1} \tau^{d+N_1+1} \cap \dots \cap \text{div } g_{j,i} \tau^{d+N_1+1}$ . Since the sections  $g_{j,l} \tau^{d+N_1+1}$  are global sections of  $\mathbb{H}^0(Y, \mathcal{L}^{\otimes (N_0+1)(d+N_1+1)})$ , and that  $\mathcal{L}^{\otimes (N_0+1)}$  induces a closed embedding of  $Y$  into  $\mathbb{P}(\mathbb{H}^0(Y, \mathcal{L}^{\otimes (N_0+1)})^\vee)$ , the sections  $g_{j,l} \tau^{d+N_1+1}$  can be extended uniquely to sections of

$$\mathbb{H}^0(\mathbb{P}(\mathbb{H}^0(Y, \mathcal{L}^{\otimes (N_0+1)})^\vee), \mathcal{O}(d + N_1 + 1)).$$

Applying refined Bézout's theorem (see [Fu84, Theorem 12.3] for a precise statement), we get

$$s \leq (\deg \overline{X})(\deg g_{j,1}\tau^{d+N_1+1}) \cdots (\deg g_{j,i}\tau^{d+N_1+1}) = (\deg \overline{X})(d+N_1+1)^i = O(d^i),$$

where coefficients involved in  $O(d^i)$  only depends on  $\deg \overline{X}$ . Since for  $1 \leq e \leq s$  we have  $\dim V_e \geq 1$ , so for each  $V_e$  there exists a  $t_i$  such that  $t_i|_{V_e}$  is not constant. We want to bound

$$G_{e,j}^{\text{bad}} := \{\beta_{i+1} \in H^0(Y, \mathcal{L}^{\otimes k_d}) ; g_{j,i+1}(\sigma_0, \beta_{i+1}) \text{ is identically 0 on } V_e\}.$$

Note that if  $\beta_{i+1}, \beta'_{i+1} \in G_{e,j}^{\text{bad}}$ , then  $\beta_{i+1} - \beta'_{i+1}$  is identically 0 on  $V_e$ . In fact, as on  $V_e$

$$\begin{aligned} & g_{j,i+1}(\sigma_0, \beta_{i+1}) - g_{j,i+1}(\sigma_0, \beta'_{i+1}) \\ &= \Phi_j(\beta_{i+1})^p \Phi_j(\tau)^{l_d} - \Phi_j(\beta'_{i+1})^p \Phi_j(\tau)^{l_d} \\ &= \Phi_j(\beta_{i+1} - \beta'_{i+1})^p \Phi_j(\tau)^{l_d}, \end{aligned}$$

and  $\Phi_j(\tau)$  is everywhere non-zero, we have that if  $G_{e,j}^{\text{bad}} \neq \emptyset$ , then it is a coset of the subspace of sections of  $H^0(Y, \mathcal{L}^{\otimes k_d})$  which vanishes on  $V_e$ . When  $d$  is large, we can decompose  $k_d$  by  $k_d = k_{d,1}(N_0+1) + k_{d,2}N_0$  with  $k_{d,1}, k_{d,2} \geq 0$  and  $k_{d,2}$  minimal among all the decompositions. Note that if  $k_{d,2} \geq N_0+1$ , we can replace  $k_{d,2}$  by  $k_{d,2} - (N_0+1)$  and  $k_{d,1}$  by  $k_{d,1} + N_0$ , which gives another decomposition of  $k_d$ . So when  $k_{d,2}$  is minimal, we have  $k_{d,2} \leq N_0$  and therefore

$$k_{d,1} \geq \frac{k_d - N_0^2}{N_0 + 1}.$$

Then the sections

$$\tau^{k_{d,1}} \tau_j^{k_{d,2}}, t_i \tau^{k_{d,1}} \tau_j^{k_{d,2}}, \dots, t_i^{\lfloor \frac{k_{d,1}}{N_1} \rfloor} \tau^{k_{d,1}} \tau_j^{k_{d,2}} \in H^0(Y, \mathcal{L}^{\otimes k_d})$$

restricting to  $V_e$  are linearly independent. So the codimension of the subspace of sections in  $H^0(Y, \mathcal{L}^{\otimes k_d})$  vanishing on  $V_e$  is bigger than or equal to  $\lfloor \frac{k_{d,1}}{N_1} \rfloor + 1$ . As  $\lfloor \frac{k_{d,1}}{N_1} \rfloor + 1 \geq \frac{k_{d,1}}{N_1}$ , this implies that the probability that  $g_{j,i+1}$  vanishes on one of the  $V_e$ 's is at most

$$s \cdot q^{-\lfloor \frac{k_{d,1}}{N_1} \rfloor - 1} \leq s \cdot q^{-\frac{k_{d,1}}{N_1}}.$$

As  $k_{d,1} \geq \frac{k_d - N_0^2}{N_0 + 1}$ ,  $N_1 \geq N_0$  and

$$k_d = \frac{1}{p} [d - (N_0 + 1)l_d] > \frac{1}{p} [d - (N_0 + 1)(N_1 + p)],$$

we get

$$\begin{aligned} s \cdot q^{-\frac{k_{d,1}}{N_1}} &\leq s \cdot q^{-\frac{k_d - N_0^2}{(N_0 + 1)N_1}} \\ &\leq s \cdot q^{\frac{N_0^2}{(N_0 + 1)N_1} - \frac{d - (N_0 + 1)(N_1 + p)}{(N_0 + 1)N_1 p}} \\ &\leq s \cdot q^{\frac{N_0}{N_1} + \frac{N_1 + p}{N_1 p} - \frac{d}{(N_0 + 1)N_1 p}} \\ &\leq s \cdot q^{2 - \frac{d}{(N_0 + 1)N_1 p}} \\ &= O(d^i q^{2 - \frac{d}{(N_0 + 1)N_1 p}}), \end{aligned}$$

where the constant involved in only depends on the degree of  $\overline{X}$  as a closed subscheme of  $\mathbb{P}(H^0(Y, \mathcal{L}^{\otimes (N_0 + 1)})^\vee)$ , hence is independent of  $d$  and  $q$ . Since  $\dim W_{j,i+1} \leq m - i - 1$  if and only if  $g_{j,i+1}$  does not vanishing on any  $V_e$ , we get that the proportion of  $\beta_{i+1}$  in  $H^0(Y, \mathcal{L}^{\otimes k_d})$  such that  $\dim W_{j,i+1} \leq m - i - 1$  is  $1 - O(d^i \cdot q^{2 - \frac{d}{(N_0 + 1)N_1 p}})$ .  $\square$



**Lemma 3.3.12.** *With a fixed choice of  $\sigma_0, \beta_1, \dots, \beta_m$  such that  $W_{j,m}$  is finite, we have for  $d$  sufficiently large, the proportion of  $\gamma$  in  $H^0(Y, \mathcal{L}^{\otimes kd})$  such that*

$$\text{div}\sigma \cap W_{j,m} \cap \left\{ x \in |X| ; \deg x \geq \frac{d}{(m+1)N} \right\} = \emptyset$$

is

$$1 - O(d^m q^{-\frac{d}{(m+1)N}}),$$

where  $\sigma = \sigma_0 + \sum_{i=1}^m \beta_i^p t_i \tau^{l_d} + \gamma^p \tau^{l_d}$  and the constant involved only depends on the degree of  $\bar{X}$  as a closed subscheme of  $\mathbb{P}(H^0(Y, \mathcal{L}^{\otimes(N_0+1)})^\vee)$ , hence is independent of  $d, q$ .

*Proof.* Applying once more Bézout's theorem, we obtain that

$$\#W_{j,m} = O(d^m)$$

with constant involved independent of  $q$ . For any  $x \in W_{j,m}$ , the set  $H^{\text{bad}}$  of sections  $\gamma \in H^0(Y, \mathcal{L}^{\otimes kd})$  such that  $x$  is contained in  $\text{div}\sigma$  with  $\sigma = \sigma_0 + \sum_{i=1}^m \beta_i^p t_i \tau^{l_d} + \gamma^p \tau^{l_d}$  is a coset of

$$\text{Ker}(\text{ev}_x \circ \Phi_j : H^0(Y, \mathcal{L}^{\otimes kd}) \rightarrow \kappa(x)),$$

where  $\kappa(x)$  is the residual field of  $x$  and  $\text{ev}_x$  is the evaluation at  $x$ . If moreover  $\deg x \geq \frac{d}{(m+1)N}$ , Lemma 3.3.2 tells us that

$$\frac{\#H^{\text{bad}}}{\#H^0(Y, \mathcal{L}^{\otimes kd})} \leq q^{-\min(\lfloor \frac{k_d}{N_0} \rfloor, \frac{d}{(m+1)N})}.$$

Thus when  $d$  tends to infinity, the proportion of sections  $\gamma \in H^0(Y, \mathcal{L}^{\otimes kd})$  such that for  $\sigma = \sigma_0 + \sum_{i=1}^m \beta_i^p t_i \tau^{l_d} + \gamma^p \tau^{l_d}$ ,

$$\text{div}\sigma \cap W_{j,m} \cap \left\{ x \in |X| ; \deg x \geq \frac{d}{(m+1)N} \right\} \neq \emptyset,$$

is bounded above by

$$\#W_{j,m} \cdot q^{-\min(\lfloor \frac{k_d}{N_0} \rfloor, \frac{d}{(m+1)N})} = O(d^m q^{-\min(\lfloor \frac{k_d}{N_0} \rfloor, \frac{d}{(m+1)N})}),$$

where the constant involved is independent of  $d, q$ . Since  $k_d = \frac{1}{p}[d - (N_0 + 1)l_d]$  with  $N_1 \leq l_d < N_1 + p$ , we have  $k_d \geq \frac{1}{p}[d - (N_0 + 1)(N_1 + p)]$ . Then

$$\lfloor \frac{k_d}{N_0} \rfloor \geq \frac{d - (N_0 + 1)(N_1 + p)}{N_0 p} - 1 \geq \frac{d - (N_0 + 1)(N_1 + 2p)}{N_0 p} \geq \frac{d}{2N_0 p}$$

for large  $d$ . However, since  $N = (N_0 + 1)(N_1 + p - 1) + p$ , we have

$$\frac{d}{(m+1)N} = \frac{d}{(m+1)[(N_0 + 1)(N_1 + p - 1) + p]} \leq \frac{d}{2(N_0 + 1)(N_1 + p - 1)}.$$

When  $d$  is large, clearly

$$\frac{d}{2(N_0 + 1)(N_1 + p - 1)} \leq \frac{d}{2N_0 p}.$$

Therefore  $\lfloor \frac{k_d}{N_0} \rfloor \geq \frac{d}{(m+1)N}$ , and the proportion of sections  $\gamma \in H^0(Y, \mathcal{L}^{\otimes k_d})$  such that for  $\sigma = \sigma_0 + \sum_{i=1}^m \beta_i^p t_i \tau^{l_d} + \gamma^p \tau^{l_d}$ ,

$$\operatorname{div} \sigma \cap W_{j,m} \cap \{x \in |X| ; \deg x \geq \frac{d}{(m+1)N}\} = \emptyset,$$

is

$$1 - O(d^m q^{-\frac{d}{(m+1)N}}),$$

where the constant involved is independent of  $d, q$ . □

*Proof of Proposition 3.3.3.* Choose

$$(\sigma_0, (\beta_1, \dots, \beta_m), \gamma) \in H^0(Y, \mathcal{L}^{\otimes d}) \times \left( \prod_{i=1}^m H^0(Y, \mathcal{L}^{\otimes k_d}) \right) \times H^0(Y, \mathcal{L}^{k_d})$$

uniformly at random. Lemma 3.3.11 and 3.3.12 show that as  $d \rightarrow \infty$ , writing

$$\sigma = \sigma_0 + \sum_{i=1}^m \beta_i^p t_i \tau^{l_d} + \gamma^p \tau^{l_d},$$

the proportion of  $(\sigma_0, (\beta_1, \dots, \beta_m), \gamma)$  such that

$$\dim W_{j,i} = m - i, \quad 0 \leq i \leq m$$

and

$$\operatorname{div} \sigma \cap W_{j,m} \cap \left\{ x \in |X| ; \deg x \geq \frac{d}{(m+1)N} \right\} = \emptyset,$$

is

$$\begin{aligned} & \left[ \prod_{i=0}^{m-1} \left( 1 - O(d^i \cdot q^{2 - \frac{d}{(N_0+1)N_1 p}}) \right) \right] \cdot \left( 1 - O(d^m q^{-\frac{d}{(m+1)N}}) \right) \\ &= \left( 1 - O(d^{m-1} \cdot q^{2 - \frac{d}{(N_0+1)N_1 p}}) \right) \cdot \left( 1 - O(d^m q^{-\frac{d}{(m+1)N}}) \right). \end{aligned}$$

Since for  $d$  sufficiently large,

$$\begin{aligned} \frac{d}{(m+1)N} &= \frac{d}{(m+1)[(N_0+1)(N_1+p-1)+p]} \\ &\geq \frac{d}{(m+1)[(N_0+2)(N_1+p-1)]} \\ &\geq \frac{d}{(m+1)(N_0+2)N_1 p} \end{aligned}$$

and

$$\frac{d}{(N_0+1)N_1 p} - 2 \geq \frac{d}{2(N_0+2)N_1 p} \geq \frac{d}{(m+1)(N_0+2)N_1 p}$$

the probability above can be written as

$$\begin{aligned} & \left( 1 - O\left( d^{m-1} \cdot q^{-\frac{d}{(m+1)(N_0+2)N_1 p}} \right) \right) \cdot \left( 1 - O\left( d^m \cdot q^{-\frac{d}{(m+1)(N_0+2)N_1 p}} \right) \right) \\ &= 1 - O\left( d^m \cdot q^{-\frac{d}{(m+1)(N_0+2)N_1 p}} \right). \end{aligned}$$

On the other hand, for  $\sigma = \sigma_0 + \sum_{i=1}^m \beta_i^p t_i \tau^{l_d} + \gamma^p \tau^{l_d}$ , as

$$g_{j,i}(\sigma_0, \beta_i)|_{\text{div } \Phi_j(\sigma)} = \partial_i \Phi_j(\sigma)|_{\text{div } \Phi_j(\sigma)},$$

we have

$$\begin{aligned} & \text{Sing}(\text{div}(\Phi_j(\sigma)) \cap X) \cap U_j \\ &= \text{div}(\Phi_j(\sigma)) \cap U_j \cap \{\partial_1 \Phi_j(\sigma) = \cdots = \partial_m \Phi_j(\sigma) = 0\} \\ &= \text{div}(\Phi_j(\sigma)) \cap U_j \cap \{g_{j,1}(\sigma_0, \beta_1) = \cdots = g_{j,m}(\sigma_0, \beta_m) = 0\} \\ &= \text{div}(\Phi_j(\sigma)) \cap W_{j,m}. \end{aligned}$$

Since  $\sigma = \sigma_0 + \sum_{i=1}^m \beta_i^p t_i \tau^{l_d} + \gamma^p \tau^{l_d}$  defines a surjective homomorphism of groups

$$\mathbb{H}^0(Y, \mathcal{L}^{\otimes d}) \times \left( \prod_{i=1}^m \mathbb{H}^0(Y, \mathcal{L}^{\otimes k_d}) \right) \times \mathbb{H}^0(Y, \mathcal{L}^{k_d}) \longrightarrow \mathbb{H}^0(Y, \mathcal{L}^{\otimes d}),$$

we obtain that when  $d \rightarrow \infty$ , the proportion of  $\sigma \in \mathbb{H}^0(Y, \mathcal{L}^{\otimes d})$  such that

$$\text{Sing}(\text{div}(\Phi_j(\sigma)) \cap X) \cap U_j \cap \left\{ x \in |X| ; \deg x \geq \frac{d}{(m+1)N} \right\} = \emptyset$$

is

$$1 - O\left(d^m \cdot q^{-\frac{d}{(m+1)(N_0+2)N_1 p}}\right).$$

Since

$$\text{Sing}(\text{div} \sigma \cap X) \subset \bigcup_j (U_j \cap \text{Sing}(\text{div}(\Phi_j(\sigma)) \cap X)),$$

setting  $c = \frac{1}{(m+1)(N_0+2)N_1}$ , we have when  $d \rightarrow \infty$ , the proportion of  $\sigma \in \mathbb{H}^0(Y, \mathcal{L}^{\otimes d})$  such that

$$\text{Sing}(\text{div}(\sigma) \cap X) \cap \left\{ x \in |X| ; \deg x \geq \frac{d}{(m+1)N} \right\} = \emptyset$$

is

$$1 - O\left(d^m \cdot q^{-c \frac{d}{p}}\right),$$

which finishes the proof of Proposition 3.3.3.  $\square$

**Corollary 3.3.13.** *In the same setting as in Proposition 3.3.3, there exists a constant  $c > 0$  independent of  $d, q$  such that*

$$\frac{\#\left\{ \sigma \in \mathbb{H}^0(Y, \mathcal{L}^{\otimes d}) ; \dim\left(\text{Sing}(\text{div} \sigma \cap X)\right) > 0 \right\}}{\#\mathbb{H}^0(Y, \mathcal{L}^{\otimes d})} = O\left(d^m \cdot q^{-c \frac{d}{p}}\right),$$

where the constant involved is independent of  $d, q$ .

*Proof.* This follows directly from Proposition 3.3.3 once we notice that

$$\left\{ \sigma \in \mathbb{H}^0(Y, \mathcal{L}^{\otimes d}) ; \dim\left(\text{Sing}(\text{div} \sigma \cap X)\right) > 0 \right\} \subset \mathcal{Q}_d^{\text{high}}$$

$\square$

### 3.4 Proof of Bertini smoothness theorem over finite fields

*Proof of Theorem 3.0.1.* The zeta function  $\zeta_X(s)$  is convergent for  $s > \dim X$ . So in particular  $\zeta_X(m+1)^{-1} = \prod_{x \in |X|} (1 - q^{-(m+1)\deg x})$  is convergent. By Proposition 3.1.2,

$$\lim_{r \rightarrow \infty} \mu(\mathcal{P}_{\leq r}) = \zeta_X(m+1)^{-1}.$$

On the other hand, by construction of  $\mathcal{P}_{d, \leq r}$ ,  $\mathcal{Q}_{d, > r}^{\text{med}}$ ,  $\mathcal{Q}_d^{\text{high}}$ , we have

$$\mathcal{P}_d \subset \mathcal{P}_{d, \leq r} \subset \mathcal{P}_d \cup \mathcal{Q}_{d, > r}^{\text{med}} \cup \mathcal{Q}_d^{\text{high}}.$$

Hence

$$\left| \frac{\#\mathcal{P}_d}{\#\mathbf{H}^0(Y, \mathcal{L}^{\otimes d})} - \frac{\#\mathcal{P}_{d, \leq r}}{\#\mathbf{H}^0(Y, \mathcal{L}^{\otimes d})} \right| \leq \frac{\#\mathcal{Q}_{d, > r}^{\text{med}}}{\#\mathbf{H}^0(Y, \mathcal{L}^{\otimes d})} + \frac{\#\mathcal{Q}_d^{\text{high}}}{\#\mathbf{H}^0(Y, \mathcal{L}^{\otimes d})}.$$

When  $d \rightarrow \infty$ , by Proposition 3.2.2 and Proposition 3.3.3 we have

$$\frac{\#\mathcal{Q}_{d, > r}^{\text{med}}}{\#\mathbf{H}^0(Y, \mathcal{L}^{\otimes d})} + \frac{\#\mathcal{Q}_d^{\text{high}}}{\#\mathbf{H}^0(Y, \mathcal{L}^{\otimes d})} = O(q^{-r}) + O(d^m \cdot q^{-c \frac{d}{p}}).$$

Hence  $\bar{\mu}(\mathcal{P})$  and  $\underline{\mu}(\mathcal{P})$  differ from  $\mu(\mathcal{P}_{\leq r})$  by at most  $\bar{\mu}(\mathcal{Q}_{> r}^{\text{med}}) + \bar{\mu}(\mathcal{Q}_d^{\text{high}}) = O(q^{-r})$ . So letting  $r$  tend to  $\infty$ , we get

$$\mu(\mathcal{P}) = \lim_{r \rightarrow \infty} \mu(\mathcal{P}_{\leq r}) = \zeta_X(m+1)^{-1}.$$

□



# Chapitre 4

## Arithmetic ampleness

In this chapter, we discuss arithmetic ampleness for arithmetic varieties, i.e. integral separated schemes which are flat and of finite type over  $\text{Spec } \mathbb{Z}$ . This notion is established in [Zh92] and [Zh95].

### 4.1 Recall of basic properties

**Definition.** Let  $M$  be a complex analytic space. Let  $\bar{L} = (L, \|\cdot\|)$  be a Hermitian line bundle on  $M$ , where  $\|\cdot\|$  is a continuous Hermitian metric on  $L$ . Then  $\bar{L}$  is said to be *semipositive* if for any section  $\sigma$  of  $\bar{L}$  on any open subset  $U$  of  $M$  such that  $\sigma$  does not vanish on any point of  $U$ , the function  $-\log \|\sigma\|$  is plurisubharmonic on  $U$ .

**Remark.** If  $M$  is a complex manifold and the Hermitian metric  $\|\cdot\|$  on  $\bar{L}$  is of differentiability class  $C^2$ , then saying that  $\bar{L}$  is semipositive is equivalent to saying that for any section  $s$  of  $\bar{L}$  on any open subset  $U$  of  $M$  such that  $s$  does not vanish on any point of  $U$ ,

$$\sqrt{-1}\partial\bar{\partial}(-\log \|s\|)$$

is a non-negative  $(1,1)$ -form.

**Definition.** Let  $\mathcal{X}$  be a projective arithmetic variety, which means an arithmetic variety projective over  $\text{Spec } \mathbb{Z}$ , and let  $\bar{\mathcal{L}}$  be a Hermitian line bundle on  $\mathcal{X}$ . We say that  $\bar{\mathcal{L}}$  is *ample* on  $\mathcal{X}$  if it verifies the following three conditions :

- i)  $\mathcal{L}$  is ample over  $\text{Spec } \mathbb{Z}$ ;
- ii)  $\bar{\mathcal{L}}$  is semipositive on the complex analytic space  $\mathcal{X}(\mathbb{C})$ ;
- iii) for any  $d \gg 1$ ,  $H^0(\mathcal{X}, \bar{\mathcal{L}}^{\otimes d})$  is generated by sections of norm strictly smaller than 1.

For any Hermitian line bundle  $\bar{\mathcal{L}} = (\mathcal{L}, \|\cdot\|)$  and any real number  $\delta$ , we note  $\bar{\mathcal{L}}(\delta)$  the Hermitian line bundle  $(\mathcal{L}, \|\cdot\| e^{-\delta})$ . If  $\bar{\mathcal{L}}$  is ample, it is easy to see that  $\bar{\mathcal{L}}(\delta)$  is also ample for any  $\delta > 0$ .

A useful result concerning ample Hermitian line bundles is the following proposition, which is a simple version of [Ch17, Prop. 2.3] :

**Proposition 4.1.1.** *Let  $\mathcal{X}$  be a projective arithmetic variety, and let  $\bar{\mathcal{L}}$  be an ample Hermitian line bundle on  $\mathcal{X}$ . There exists a positive constant  $\varepsilon_0$  such that for any large enough integer  $d$ ,  $H^0(\mathcal{X}, \bar{\mathcal{L}}^{\otimes d})$  has a basis consisting of sections with norm smaller than  $e^{-\varepsilon_0 d}$ .*

Now we recall the arithmetic Hilbert-Samuel theorem for arithmetic ample line bundles, which is proved by Gillet and Soulé in [GS92] using arithmetic Riemann-Roch theorem for arithmetic varieties with smooth generic fiber. There is also a proof given by Abbes and Bouche in [AB95] without the application of the arithmetic Riemann-Roch theorem. The generalization without the smoothness condition on the generic fiber is proved by Zhang in [Zh95].

**Theorem 4.1.2.** *Let  $\mathcal{X}$  be a projective arithmetic variety of absolute dimension  $n$ ,  $\overline{\mathcal{L}}$  an ample Hermitian line bundle on  $\mathcal{X}$  and  $\overline{\mathcal{M}}$  a Hermitian vector bundle of rank  $r$  on  $\mathcal{X}$ . As  $d$  tends to  $\infty$ , we have*

$$h_{\text{Ar}}^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes d} \otimes \overline{\mathcal{M}}) = \frac{r}{n!} \overline{\mathcal{L}}^n d^n + o(d^n).$$

*Proof.* The statement can be found in [Yu08, Corollary 2.7 (1)]. We get this by combining two results. The first one is the estimate of  $\chi_{\text{sup}}(\overline{\mathcal{L}}^{\otimes d} \otimes \overline{\mathcal{M}})$ , which is the logarithm of the covolume of the lattice  $H^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes d} \otimes \overline{\mathcal{M}})$  for the sup norm, given by Shouwu Zhang in [Zh95, Theorem (1.4)] as

$$\chi_{\text{sup}}(\overline{\mathcal{L}}^{\otimes d} \otimes \overline{\mathcal{M}}) = \frac{r}{n!} \overline{\mathcal{L}}^n d^n + o(d^n).$$

The second result is a consequence of [GS91, Theorem 2], which says that

$$|h_{\text{Ar}}^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes d} \otimes \overline{\mathcal{M}}) - \chi_{\text{sup}}(\overline{\mathcal{L}}^{\otimes d} \otimes \overline{\mathcal{M}})| = O(d^{n-1} \log d).$$

□

## 4.2 Restriction modulo $N$ of sections

**Lemma 4.2.1.** *Let  $\mathcal{X}$  be a projective arithmetic variety, and let  $\overline{\mathcal{L}}$  be an ample Hermitian line bundle on  $\mathcal{X}$ . There is a positive integer  $d_0$  such that when  $d \geq d_0$ , we have*

$$H^0(\mathcal{X}_N, \overline{\mathcal{L}}^{\otimes d}) \simeq H^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes d}) / (N \cdot H^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes d}))$$

for any positive integer  $N$ , where  $\mathcal{X}_N = \mathcal{X} \times_{\text{Spec } \mathbb{Z}} \text{Spec } \mathbb{Z}/N\mathbb{Z}$ .

*Proof.* We first prove the lemma for the case when  $N$  is a prime number. By the flatness of the structure morphism  $\mathcal{X} \rightarrow \text{Spec } \mathbb{Z}$ , the Euler characteristic  $\chi(\overline{\mathcal{L}}_p^{\otimes d})$  of the fiber  $\mathcal{X}_p$  satisfies

$$\chi(\overline{\mathcal{L}}_p^{\otimes d}) = \chi(\overline{\mathcal{L}}_{\mathbb{Q}}^{\otimes d})$$

for any prime  $p$ . We choose a  $d_1$  such that  $h^i(\mathcal{X}_{\mathbb{Q}}, \overline{\mathcal{L}}^{\otimes d}) = 0$  for any  $d \geq d_1$  and any  $i > 0$ , which implies  $h^i(\mathcal{X}_p, \overline{\mathcal{L}}^{\otimes d}) = 0$  for all but finitely many  $p$ . Let  $d_2 \in \mathbb{Z}_{>0}$  be such that  $h^i(\mathcal{X}_p, \overline{\mathcal{L}}^{\otimes d}) = 0$  for each of the remaining primes  $p$  and any  $i > 0$ . Such a  $d_2$  exists as the restriction sheaf  $\overline{\mathcal{L}}|_{\mathcal{X}_p}$  is ample for any prime  $p$ . Then we can take  $d_0 = \max(d_1, d_2)$ . For any  $d \geq d_0$ , we have

$$h^0(\mathcal{X}_p, \overline{\mathcal{L}}^{\otimes d}) = \chi(\overline{\mathcal{L}}_p^{\otimes d}) = \chi(\overline{\mathcal{L}}_{\mathbb{Q}}^{\otimes d}) = h^0(\mathcal{X}_{\mathbb{Q}}, \overline{\mathcal{L}}^{\otimes d}).$$

As the function  $p \mapsto h^0(\mathcal{X}_p, \overline{\mathcal{L}}^{\otimes d})$  is a semicontinuous function, the constancy of  $h^0(\mathcal{X}_p, \overline{\mathcal{L}}^{\otimes d})$  implies that

$$H^0(\mathcal{X}_p, \overline{\mathcal{L}}^{\otimes d}) \simeq H^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes d}) / (p \cdot H^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes d}))$$

for any prime number  $p$ .

Now any positive integer  $N$  has a decomposition  $N = \prod_{i=1}^r p_i^{s_i}$  for some prime numbers  $p_i$  and positive integers  $s_i$ . Then we have

$$\mathcal{X}_N = \mathcal{X} \times_{\text{Spec } \mathbb{Z}} \text{Spec } (\mathbb{Z}/N\mathbb{Z}) = \prod_{i=1}^r \mathcal{X} \times_{\text{Spec } \mathbb{Z}} \text{Spec } (\mathbb{Z}/(p_i^{s_i}\mathbb{Z})) = \prod_{i=1}^r \mathcal{X}_{p_i^{s_i}}.$$

Hence for any  $d$ ,

$$\mathrm{H}^0(\mathcal{X}_N, \bar{\mathcal{L}}^{\otimes d}) = \mathrm{H}^0\left(\prod_{i=1}^r \mathcal{X}_{p_i^{s_i}}, \bar{\mathcal{L}}^{\otimes d}\right) = \prod_{i=1}^r \mathrm{H}^0(\mathcal{X}_{p_i^{s_i}}, \bar{\mathcal{L}}^{\otimes d}).$$

On the other hand,

$$\mathrm{H}^0(\mathcal{X}, \bar{\mathcal{L}}^{\otimes d}) / (N \cdot \mathrm{H}^0(\mathcal{X}, \bar{\mathcal{L}}^{\otimes d})) \simeq \prod_{i=1}^r \mathrm{H}^0(\mathcal{X}, \bar{\mathcal{L}}^{\otimes d}) / (p_i^{s_i} \cdot \mathrm{H}^0(\mathcal{X}, \bar{\mathcal{L}}^{\otimes d})).$$

Hence it suffices to prove the case when  $N = p^s$  for some prime number  $p$  and some positive integer  $s$ . The case when  $s = 1$  is proved above, so we may assume that  $s > 1$ . Let

$$\varphi_{d,p^s} : \mathrm{H}^0(\mathcal{X}, \bar{\mathcal{L}}^{\otimes d}) / (p^s \cdot \mathrm{H}^0(\mathcal{X}, \bar{\mathcal{L}}^{\otimes d})) \longrightarrow \mathrm{H}^0(\mathcal{X}_{p^s}, \bar{\mathcal{L}}^{\otimes d})$$

be the map induced by  $\mathrm{H}^0(\mathcal{X}, \bar{\mathcal{L}}^{\otimes d}) \longrightarrow \mathrm{H}^0(\mathcal{X}_{p^s}, \bar{\mathcal{L}}^{\otimes d})$ . Note that  $\mathrm{H}^0(\mathcal{X}_{p^s}, \bar{\mathcal{L}}^{\otimes d})$  is a free  $\mathbb{Z}/p^s\mathbb{Z}$ -module of rank equal to  $\dim_{\mathbb{F}_p} \mathrm{H}^0(\mathcal{X}_p, \bar{\mathcal{L}}^{\otimes d})$ . So by cardinality reason  $\varphi_{d,p^s}$  is an isomorphism if and only if it is injective. By the proven isomorphism  $\mathrm{H}^0(\mathcal{X}, \bar{\mathcal{L}}^{\otimes d}) / (p \cdot \mathrm{H}^0(\mathcal{X}, \bar{\mathcal{L}}^{\otimes d})) \simeq \mathrm{H}^0(\mathcal{X}_p, \bar{\mathcal{L}}^{\otimes d})$ , we have  $\mathrm{Ker}(\varphi)/p \cdot \mathrm{Ker}(\varphi) = 0$ , which suggests  $\mathrm{Ker}(\varphi) = 0$  by Nakayama's lemma. Hence we conclude.  $\square$

We have two results concerning the restriction modulo  $N$  map.

**Proposition 4.2.2.** *Let  $\mathcal{X}$  be a projective arithmetic variety, and let  $\bar{\mathcal{L}}$  be an ample Hermitian line bundle on  $\mathcal{X}$ . Let  $0 < \alpha_0 < 1$  be a real number. There exists a positive constant  $\eta$  such that for any  $d \in \mathbb{Z}_{>0}$  large enough, if  $N \in \mathbb{Z}_{>0}$  is bounded above by  $\exp(d^{\alpha_0})$ , then the following holds :*

i) *the restriction morphism*

$$\psi_{d,N} : \mathrm{H}_{\mathrm{Ar}}^0(\mathcal{X}, \bar{\mathcal{L}}^{\otimes d}) \longrightarrow \mathrm{H}^0(\mathcal{X}_N, \bar{\mathcal{L}}^{\otimes d})$$

*is surjective, where  $\mathcal{X}_N = \mathcal{X} \times_{\text{Spec } \mathbb{Z}} \text{Spec } \mathbb{Z}/N\mathbb{Z}$ ;*

ii) *for any two sections  $s, s'$  in  $\mathrm{H}^0(\mathcal{X}_N, \bar{\mathcal{L}}^{\otimes d})$ , we have*

$$\frac{|\#\psi_{d,N}^{-1}(s) - \#\psi_{d,N}^{-1}(s')|}{\#\psi_{d,N}^{-1}(s)} \leq e^{-\eta d}.$$

This proposition is a reformulation of [Ch17, Proposition 2.14]. Here the constant  $\alpha_0$  can be any real number between 0 and 1.

**Proposition 4.2.3.** *Let  $\mathcal{X}$  be a projective arithmetic variety, and let  $\bar{\mathcal{L}}$  be an ample Hermitian line bundle on  $\mathcal{X}$ . Let  $\varepsilon_0$  be a constant as in Proposition 4.1.1, and choose a constant  $\varepsilon < \varepsilon_0$ . For a positive integer  $N$ , let*

$$\psi_{d,N} : \mathrm{H}_{\mathrm{Ar}}^0(\mathcal{X}, \bar{\mathcal{L}}^{\otimes d}) \longrightarrow \mathrm{H}^0(\mathcal{X}_N, \bar{\mathcal{L}}^{\otimes d})$$



be the restriction map, where  $\mathcal{X}_N = \mathcal{X} \times_{\text{Spec } \mathbb{Z}} \text{Spec}(\mathbb{Z}/N)$ . When  $d$  is large enough, for any odd integer  $N \leq e^{\varepsilon d}$  and for any subset  $E \subset H^0(\mathcal{X}_N, \overline{\mathcal{L}}^{\otimes d})$ , we have

$$\frac{\#\psi_{d,N}^{-1}(E)}{\#\mathbb{H}_{\text{Ar}}^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes d})} \leq 4 \frac{\#E}{\#H^0(\mathcal{X}_N, \overline{\mathcal{L}}^{\otimes d})}.$$

*Proof.* Note that  $H^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes d})$  is a free  $\mathbb{Z}$ -module for any  $d$ . For simplicity of notation, we write  $h = \text{rk}(H^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes d}))$ . We may assume that  $d$  is large enough so that for any positive integer  $N$  we have  $H^0(\mathcal{X}_N, \overline{\mathcal{L}}^{\otimes d}) \simeq H^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes d}) / (N \cdot H^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes d}))$  by Lemma 4.2.1. Let  $(\sigma_1, \dots, \sigma_h)$  be a  $\mathbb{Z}$ -basis of  $H^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes d})$  such that

$$\|\sigma_j\| < e^{-\varepsilon_0 d}, \quad \forall j \in \{1, \dots, h\}.$$

For an odd integer  $N$  such that  $0 < N \leq e^{\varepsilon d}$  with a fixed  $\varepsilon < \varepsilon_0$ , we set

$$D_{d,N} = \left\{ \sigma = \sum_{j=1}^h \lambda_j \sigma_j ; |\lambda_j| \leq \frac{N}{2}, \lambda_j \in \mathbb{R} \right\} \subset H^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes d}) \otimes_{\mathbb{Z}} \mathbb{R}.$$

Then we have

$$D_{d,N} \cap H^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes d}) = \left\{ \sigma = \sum_{j=1}^h \lambda_j \sigma_j ; -\frac{N-1}{2} \leq \lambda_j \leq \frac{N-1}{2}, \lambda_j \in \mathbb{Z} \right\},$$

and

$$H^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes d}) \otimes_{\mathbb{Z}} \mathbb{R} = N \cdot H^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes d}) + D_{d,N}.$$

Moreover, for any  $\sigma \in D_{d,N}$ , we have a bound for the norm of  $\sigma$

$$\|\sigma\| \leq h \frac{N}{2} e^{-\varepsilon_0 d}.$$

The existence of such a basis is guaranteed by Proposition 4.1.1. In particular, when  $d$  is large enough, as  $N \leq e^{\varepsilon d}$  with  $\varepsilon < \varepsilon_0$ , any  $\sigma \in D_{d,N}$  satisfies

$$\|\sigma\| \leq \frac{h}{2} e^{(\varepsilon - \varepsilon_0)d} < 1.$$

Note that by the expression of  $D_{d,N} \cap H^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes d})$ , we have a 1-1 correspondence between elements in  $D_{d,N} \cap H^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes d})$  and elements in  $H^0(\mathcal{X}_N, \overline{\mathcal{L}}^{\otimes d})$  induced by the restriction modulo  $N$  map. So the map

$$\psi_{d,N} : \mathbb{H}_{\text{Ar}}^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes d}) \longrightarrow H^0(\mathcal{X}_N, \overline{\mathcal{L}}^{\otimes d})$$

is surjective for such  $N$ .

For any  $R \in \mathbb{R}_+$ , we set

$$B_d(R) = \left\{ \sigma \in H^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes d}) \otimes_{\mathbb{Z}} \mathbb{R} ; \|\sigma\| < R \right\}.$$

Then in particular, we have

$$B_d(1) \cap H^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes d}) = \mathbb{H}_{\text{Ar}}^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes d}).$$

For any element  $\sigma \in \mathbf{H}^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes d}) \otimes_{\mathbb{Z}} \mathbb{R}$ , we can find a  $\sigma' \in N \cdot \mathbf{H}^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes d})$  such that  $\sigma - \sigma' \in D_{d,N}$ . If moreover  $\sigma \in B_d(1)$ , we have

$$\|\sigma'\| \leq \|\sigma\| + \|\sigma - \sigma'\| < 1 + h \frac{N}{2} e^{-\varepsilon_0 d}.$$

Thus we have two inclusions

$$\begin{aligned} B_d(1) &\subset \left( B_d\left(1 + \frac{Nh}{2} e^{-\varepsilon_0 d}\right) \cap N \cdot \mathbf{H}^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes d}) \right) + D_{d,N}; \\ \mathbf{H}_{\text{Ar}}^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes d}) &\subset \left( B_d\left(1 + \frac{Nh}{2} e^{-\varepsilon_0 d}\right) \cap N \cdot \mathbf{H}^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes d}) \right) + D_{d,N} \cap \mathbf{H}^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes d}). \end{aligned}$$

Note that any element in  $\mathbf{H}^0(\mathcal{X}_N, \overline{\mathcal{L}}^{\otimes d})$  has exactly one preimage in  $D_{d,N} \cap \mathbf{H}^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes d})$ . The number of sections  $\sigma \in D_{d,N} \cap \mathbf{H}^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes d})$  such that  $\psi_{d,N}(\sigma) \in E$  is equal to  $\#E$ . Then by the above inclusion, we have

$$\begin{aligned} \#\psi_{d,N}^{-1}(E) &= \#\{\sigma \in \mathbf{H}_{\text{Ar}}^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes d}) ; \psi_{d,N}(\sigma) \in E\} \\ &\leq \# \left( B_d\left(1 + \frac{Nh}{2} e^{-\varepsilon_0 d}\right) \cap N \cdot \mathbf{H}^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes d}) \right) \cdot \#E. \end{aligned}$$

Now we bound  $\# \left( B_d\left(1 + \frac{Nh}{2} e^{-\varepsilon_0 d}\right) \cap N \cdot \mathbf{H}^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes d}) \right)$ . If  $\sigma \in B_d\left(1 + \frac{Nh}{2} e^{-\varepsilon_0 d}\right) \cap N \cdot \mathbf{H}^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes d})$ , any  $\sigma' \in \sigma + D_{d,N}$  satisfies

$$\|\sigma'\| \leq \|\sigma\| + \|\sigma - \sigma'\| < 1 + \frac{Nh}{2} e^{-\varepsilon_0 d} + \frac{Nh}{2} e^{-\varepsilon_0 d} = 1 + Nhe^{-\varepsilon_0 d}.$$

Hence

$$\left( B_d\left(1 + \frac{Nh}{2} e^{-\varepsilon_0 d}\right) \cap N \cdot \mathbf{H}^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes d}) \right) + D_{d,N} \subset B_d(1 + Nhe^{-\varepsilon_0 d}),$$

and in particular, we have

$$\text{Vol} \left( \left( B_d\left(1 + \frac{Nh}{2} e^{-\varepsilon_0 d}\right) \cap N \cdot \mathbf{H}^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes d}) \right) + D_{d,N} \right) \leq \text{Vol} (B_d(1 + Nhe^{-\varepsilon_0 d})).$$

If  $\sigma_1, \sigma_2$  are two distinct elements in  $B_d\left(1 + \frac{Nh}{2} e^{-\varepsilon_0 d}\right) \cap N \cdot \mathbf{H}^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes d})$ , the intersection  $(\sigma_1 + D_{d,N}) \cap (\sigma_2 + D_{d,N})$  is either empty or a subset in  $\mathbf{H}^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes d}) \otimes_{\mathbb{Z}} \mathbb{R}$  of dimension smaller than  $h$ . In particular, the intersection always has volume 0. Therefore we have

$$\begin{aligned} &\text{Vol} \left( \left( B_d\left(1 + \frac{Nh}{2} e^{-\varepsilon_0 d}\right) \cap N \cdot \mathbf{H}^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes d}) \right) + D_{d,N} \right) \\ &= \# \left( B_d\left(1 + \frac{Nh}{2} e^{-\varepsilon_0 d}\right) \cap N \cdot \mathbf{H}^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes d}) \right) \cdot \text{Vol}(D_{d,N}). \end{aligned}$$

From this equality, we can bound  $\# \left( B_d\left(1 + \frac{Nh}{2} e^{-\varepsilon_0 d}\right) \cap N \cdot \mathbf{H}^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes d}) \right)$  as

$$\begin{aligned} &\# \left( B_d\left(1 + \frac{Nh}{2} e^{-\varepsilon_0 d}\right) \cap N \cdot \mathbf{H}^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes d}) \right) \\ &= \frac{\text{Vol} \left( \left( B_d\left(1 + \frac{Nh}{2} e^{-\varepsilon_0 d}\right) \cap N \cdot \mathbf{H}^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes d}) \right) + D_{d,N} \right)}{\text{Vol}(D_{d,N})} \\ &\leq \frac{\text{Vol} (B_d(1 + Nhe^{-\varepsilon_0 d}))}{\text{Vol}(D_{d,N})} \end{aligned}$$

Now set

$$D_{d,1} = \left\{ \sigma = \sum_{j=1}^h \lambda_j \sigma_j ; |\lambda_j| \leq \frac{1}{2}, \lambda_j \in \mathbb{R} \right\} \subset H^0(\mathcal{X}, \bar{\mathcal{L}}^{\otimes d}) \otimes_{\mathbb{Z}} \mathbb{R}.$$

We get similarly that for any  $\sigma \in D_{d,1}$ ,  $\|\sigma\| \leq \frac{h}{2}e^{-\varepsilon_0 d}$ . If  $\sigma \in H^0(\mathcal{X}, \bar{\mathcal{L}}^{\otimes d}) \otimes_{\mathbb{Z}} \mathbb{R}$  is such that  $\|\sigma\| < 1 - \frac{h}{2}e^{-\varepsilon_0 d}$ , then we can find a section  $\sigma' \in H^0(\mathcal{X}, \bar{\mathcal{L}}^{\otimes d})$  such that  $\sigma \in \sigma' + D_{d,1}$ ; moreover, we get

$$\|\sigma'\| \leq \|\sigma\| + \|\sigma - \sigma'\| < 1 - \frac{h}{2}e^{-\varepsilon_0 d} + \frac{h}{2}e^{-\varepsilon_0 d} = 1.$$

Thus we have

$$B_d(1 - \frac{h}{2}e^{-\varepsilon_0 d}) \subset H_{\text{Ar}}^0(\mathcal{X}, \bar{\mathcal{L}}^{\otimes d}) + D_{d,1}.$$

So similarly we have

$$\text{Vol}\left(B_d(1 - \frac{h}{2}e^{-\varepsilon_0 d})\right) \leq \#H_{\text{Ar}}^0(\mathcal{X}, \bar{\mathcal{L}}^{\otimes d}) \cdot \text{Vol}(D_{d,1})$$

Note that for any  $R > 0$ ,

$$\text{Vol}(B_d(R)) = R^h \text{Vol}(B_d(1)),$$

and

$$\text{Vol}(D_{d,N}) = N^h \text{Vol}(D_{d,1}).$$

Hence

$$\begin{aligned} \frac{\text{Vol}(B_d(1 + Nhe^{-\varepsilon_0 d}))}{\text{Vol}(D_{d,N})} &= \left( \frac{1 + Nhe^{-\varepsilon_0 d}}{1 - \frac{h}{2}e^{-\varepsilon_0 d}} \right)^h \frac{\text{Vol}(B_d(1 - \frac{h}{2}e^{-\varepsilon_0 d}))}{N^h \cdot \text{Vol}(D_{d,1})} \\ &\leq \left( \frac{1 + Nhe^{-\varepsilon_0 d}}{1 - \frac{h}{2}e^{-\varepsilon_0 d}} \right)^h \frac{\#H_{\text{Ar}}^0(\mathcal{X}, \bar{\mathcal{L}}^{\otimes d}) \cdot \text{Vol}(D_{d,1})}{N^h \cdot \text{Vol}(D_{d,1})} \\ &= N^{-h} \left( \frac{1 + Nhe^{-\varepsilon_0 d}}{1 - \frac{h}{2}e^{-\varepsilon_0 d}} \right)^h \#H_{\text{Ar}}^0(\mathcal{X}, \bar{\mathcal{L}}^{\otimes d}). \end{aligned}$$

Since  $N \leq e^{\varepsilon d}$ , we have

$$\begin{aligned} (1 + Nhe^{-\varepsilon_0 d})^h &= \exp(Nh^2e^{-\varepsilon_0 d} + O(N^2h^3e^{-2\varepsilon_0 d})) \\ &= 1 + Nh^2e^{-\varepsilon_0 d} + O(N^2h^4e^{-2\varepsilon_0 d}) \\ &\leq 1 + h^2e^{(\varepsilon - \varepsilon_0)d} + O(h^4e^{2(\varepsilon - \varepsilon_0)d}). \end{aligned}$$

As  $\varepsilon < \varepsilon_0$  and that the rank  $h = \text{rk}(H^0(\mathcal{X}, \bar{\mathcal{L}}^{\otimes d}))$  grows polynomially with  $d$ , when  $d$  is sufficiently large,

$$(1 + Nhe^{-\varepsilon_0 d})^h \leq 1 + 2h^2e^{(\varepsilon - \varepsilon_0)d} \leq 2.$$

Similarly, we have when  $d$  is sufficiently large,

$$(1 - \frac{h}{2}e^{-\varepsilon_0 d})^h = 1 - \frac{h^2}{2}e^{-\varepsilon_0 d} + O(h^4e^{-2\varepsilon_0 d}) \geq 1 - h^2e^{-\varepsilon_0 d} \geq \frac{1}{2}.$$

Hence we have

$$\left( \frac{1 + Nhe^{-\varepsilon_0 d}}{1 - \frac{h}{2}e^{-\varepsilon_0 d}} \right)^h \leq \frac{1 + 2h^2e^{(\varepsilon - \varepsilon_0)d}}{1 - h^2e^{-\varepsilon_0 d}} \leq 4$$

for any  $N < e^{\varepsilon d}$ . Therefore, we can bound  $\# \left( B_d(1 + \frac{Nh}{2}e^{-\varepsilon_0 d}) \cap N \cdot \mathbf{H}^0(\mathcal{X}, \bar{\mathcal{L}}^{\otimes d}) \right)$  by

$$\begin{aligned} & \# \left( B_d(1 + \frac{Nh}{2}e^{-\varepsilon_0 d}) \cap N \cdot \mathbf{H}^0(\mathcal{X}, \bar{\mathcal{L}}^{\otimes d}) \right) \\ & \leq \frac{\text{Vol}(B_d(1 + Nhe^{-\varepsilon_0 d}))}{\text{Vol}(D_{d,N})} \\ & \leq N^{-h} \left( \frac{1 + Nhe^{-\varepsilon_0 d}}{1 - \frac{h}{2}e^{-\varepsilon_0 d}} \right)^h \# \mathbf{H}_{\text{Ar}}^0(\mathcal{X}, \bar{\mathcal{L}}^{\otimes d}) \\ & \leq 4N^{-h} \cdot \# \mathbf{H}_{\text{Ar}}^0(\mathcal{X}, \bar{\mathcal{L}}^{\otimes d}), \end{aligned}$$

So finally we have

$$\begin{aligned} \#\psi_{d,N}^{-1}(E) & \leq \# \left( B_d(1 + \frac{Nh}{2}e^{-\varepsilon_0 d}) \cap N \cdot \mathbf{H}^0(\mathcal{X}, \bar{\mathcal{L}}^{\otimes d}) \right) \cdot \#E \\ & \leq 4N^{-h} \cdot \left( \# \mathbf{H}_{\text{Ar}}^0(\mathcal{X}, \bar{\mathcal{L}}^{\otimes d}) \right) \cdot \#E. \end{aligned}$$

Note that by Lemma 4.2.1, when  $d$  is large enough,  $\mathbf{H}^0(\mathcal{X}_N, \bar{\mathcal{L}}^{\otimes d}) \simeq \mathbf{H}^0(\mathcal{X}, \bar{\mathcal{L}}^{\otimes d}) / (N \cdot \mathbf{H}^0(\mathcal{X}, \bar{\mathcal{L}}^{\otimes d}))$ . So for such  $d$  we have  $\#\mathbf{H}^0(\mathcal{X}_N, \bar{\mathcal{L}}^{\otimes d}) = N^h$ . Hence

$$\frac{\#\psi_{d,N}^{-1}(E)}{\#\mathbf{H}_{\text{Ar}}^0(\mathcal{X}, \bar{\mathcal{L}}^{\otimes d})} \leq 4 \frac{\#E}{\#\mathbf{H}^0(\mathcal{X}_N, \bar{\mathcal{L}}^{\otimes d})}$$

and we conclude. □

**Remark.** When  $d$  is large enough,  $\mathbf{H}^0(\mathcal{X}, \bar{\mathcal{L}}^{\otimes d})$  is a free  $\mathbb{Z}$ -module such that

$$h = \text{rk}(\mathbf{H}^0(\mathcal{X}, \bar{\mathcal{L}}^{\otimes d})) = \dim_{\mathbb{Q}} \mathbf{H}^0(\mathcal{X}_{\mathbb{Q}}, \bar{\mathcal{L}}^{\otimes d}) = \chi(\mathcal{X}_{\mathbb{Q}}, \mathcal{L}^{\otimes d})$$

as  $\bar{\mathcal{L}}$  is ample.

The asymptotic Riemann-Roch Theorem tells us then that

$$h = \frac{((\mathcal{L}|_{\mathcal{X}_{\mathbb{Q}}})^{n-1})}{(n-1)!} d^{n-1} + O(d^{n-2}),$$

with  $((\mathcal{L}|_{\mathcal{X}_{\mathbb{Q}}})^{n-1})$  the intersection number of  $n-1$  copies of  $\mathcal{L}|_{\mathcal{X}_{\mathbb{Q}}}$ , whereas

$$h_{\text{Ar}}^0(\mathcal{X}, \bar{\mathcal{L}}^{\otimes d} \otimes \bar{\mathcal{M}}) = \frac{r}{n!} \bar{\mathcal{L}}^n d^n + o(d^n)$$

by Theorem 4.1.2.



## Chapitre 5

# Convergence of special values of zeta functions

Let  $\mathcal{X}$  be an arithmetic scheme of absolute dimension  $n$ . We fix from now on a constant  $c_0 > 0$  such that for any prime integer  $p$  and any  $e \in \mathbb{Z}_{>0}$ ,

$$\#\mathcal{X}(\mathbb{F}_{p^e}) = \#\mathcal{X}_p(\mathbb{F}_{p^e}) \leq c_0 p^{(n-1)e},$$

where  $n$  is the absolute dimension of  $\mathcal{X}$  (so  $\mathcal{X}_p$  is of dimension  $n - 1$ ). Such a constant exists by the Lang-Weil estimates in [LW54]. A good introduction to the function  $\#\mathcal{X}(\mathbb{F}_{p^e})$  and its properties is Serre's book [Se12].

We know that the zeta function

$$\zeta_{\mathcal{X}}(s) = \prod_{x \in |\mathcal{X}|} (1 - \#\kappa(x)^{-s})^{-1}$$

is absolutely convergent for any  $s \in \mathbb{C}$  satisfying  $\operatorname{Re}(s) > n$ . Moreover, the zeta function of  $\mathcal{X}$  is the product of the zeta function of all its fibers, i.e. we have

$$\zeta_{\mathcal{X}}(s) = \prod_{p \text{ prime}} \zeta_{\mathcal{X}_p}(s).$$

For later use, we calculate in this section the speed of convergence of

$$\prod_{x \in |\mathcal{X}_p|, \deg x \leq r} (1 - \#\kappa(x)^{-(n+1)})$$

to  $\zeta_{\mathcal{X}_p}(n+1)^{-1}$  when  $r \rightarrow \infty$ , and that of  $\prod_{p \leq R} \zeta_{\mathcal{X}_p}(n+1)^{-1}$  to  $\zeta_{\mathcal{X}}(n+1)^{-1}$  when  $R \rightarrow \infty$ .

**Lemma 5.0.1.** *For any prime number  $p$  and any positive integer  $e \geq 1$ , we have*

$$-\log(1 - p^{-e}) < 2p^{-e}.$$

*In particular, for any closed point  $x$  on an arithmetic variety  $\mathcal{X}$  and any integer  $e \geq 1$ , we have*

$$-\log(1 - \#\kappa(x)^{-e}) < 2 \cdot \#\kappa(x)^{-e}.$$

*Proof.* For any real number  $0 < t \leq \frac{\sqrt{5}-1}{2}$ , we have  $-\log(1-t) \leq 2\log(1+t)$ . Indeed, when  $0 < t \leq \frac{\sqrt{5}-1}{2}$ , we have

$$-1 < t^2 + t - 1 = \left(t + \frac{1}{2}\right)^2 - \frac{5}{4} \leq 0.$$

Then

$$(1+t)^2(1-t) = 1 - (t^3 + t^2 - t) = 1 - t(t^2 + t - 1) \geq 1,$$

which implies

$$\frac{1}{1-t} \leq (1+t)^2,$$

i.e.

$$-\log(1-t) \leq 2\log(1+t).$$

Since for any  $t > 0$ ,  $\log(1+t) < t$ , we have for  $0 < t \leq \frac{\sqrt{5}-1}{2}$ ,

$$-\log(1-t) \leq 2t.$$

As  $\frac{\sqrt{5}-1}{2} > \frac{1}{2}$ , any prime  $p$  and positive integer  $r \geq 1$  satisfy  $p^{-r} \leq \frac{\sqrt{5}-1}{2}$ . Hence we conclude.  $\square$

**Lemma 5.0.2.** *Let  $\mathcal{X}$  be an arithmetic scheme of absolute dimension  $n$ . For any prime number  $p$  and any positive integer  $r \geq 1$ , we have*

$$\left| \prod_{x \in |\mathcal{X}_p|, \deg x \leq r} \left(1 - p^{-(n+1)\deg x}\right) - \zeta_{\mathcal{X}_p}(n+1)^{-1} \right| \leq 4c_0 p^{-2(r+1)}.$$

*Proof.* By Lemma 5.0.1, for any closed point  $x$  of  $\mathcal{X}$ ,

$$-\log(1 - \#\kappa(x)^{-(n+1)}) < 2 \cdot \#\kappa(x)^{-(n+1)}$$

We have

$$\begin{aligned} \sum_{x \in |\mathcal{X}_p|, \deg x > r} \left(-\log(1 - p^{-(n+1)\deg x})\right) &< 2 \sum_{x \in |\mathcal{X}_p|, \deg x > r} p^{-(n+1)\deg x} \\ &\leq 2 \sum_{e=r+1}^{\infty} \#\mathcal{X}(\mathbb{F}_{p^e}) p^{-(n+1)e} \\ &\leq 2 \sum_{e=r+1}^{\infty} c_0 p^{(n-1)e} \cdot p^{-(n+1)e} \\ &= 2c_0 \sum_{e=r+1}^{\infty} p^{-2e} \\ &= 2c_0 \frac{p^{-2(r+1)}}{1-p^{-2}} \\ &< 4c_0 p^{-2(r+1)}. \end{aligned}$$

On the other hand, for any  $x \in |\mathcal{X}_p|$ ,

$$\left(1 - p^{-(n+1)\deg x}\right) < 1.$$

Hence

$$\begin{aligned}
& \left| \prod_{x \in |\mathcal{X}_p|, \deg x \leq r} \left(1 - p^{-(n+1) \deg x}\right) - \zeta_{\mathcal{X}_p}(n+1)^{-1} \right| \\
&= \prod_{x \in |\mathcal{X}_p|, \deg x \leq r} \left(1 - p^{-(n+1) \deg x}\right) \cdot \left(1 - \prod_{x \in |\mathcal{X}_p|, \deg x > r} \left(1 - p^{-(n+1) \deg x}\right)\right) \\
&< 1 - \prod_{x \in |\mathcal{X}_p|, \deg x > r} \left(1 - p^{-(n+1) \deg x}\right) \\
&= 1 - \exp\left(\sum_{x \in |\mathcal{X}_p|, \deg x > r} \log\left(1 - p^{-(n+1) \deg x}\right)\right).
\end{aligned}$$

By the above computation, we have

$$\begin{aligned}
& \left| \prod_{x \in |\mathcal{X}_p|, \deg x \leq r} \left(1 - p^{-(n+1) \deg x}\right) - \zeta_{\mathcal{X}_p}(n+1)^{-1} \right| \\
&< 1 - \exp\left(-4c_0 p^{-2(r+1)}\right) \\
&< 4c_0 p^{-2(r+1)}
\end{aligned}$$

as for any  $t > 0$ ,  $e^{-t} > 1 - t$ . Therefore we conclude.  $\square$

**Lemma 5.0.3.** *Let  $\mathcal{X}$  be an arithmetic scheme of absolute dimension  $n$ . For any prime number  $p$ , we have*

$$0 < \log \zeta_{\mathcal{X}_p}(n+1) \leq 4c_0 p^{-2}.$$

*Proof.* In fact, we have

$$\begin{aligned}
0 < \log \zeta_{\mathcal{X}_p}(n+1) &= \log\left(\prod_{x \in |\mathcal{X}_p|} (1 - \#\kappa(x)^{-(n+1)})^{-1}\right) \\
&= (-1) \sum_{x \in |\mathcal{X}_p|} \log\left(1 - \#\kappa(x)^{-(n+1)}\right) \\
&< \left(\sum_{x \in |\mathcal{X}_p|} 2\#\kappa(x)^{-(n+1)}\right) \\
&< 2 \sum_{e=1}^{\infty} \#\mathcal{X}(\mathbb{F}_{p^e}) \cdot p^{-e(n+1)},
\end{aligned}$$

where the third line uses Lemma 5.0.1. By the choice of  $c_0$  at the beginning of this section, for any  $e$ ,

$$\#\mathcal{X}(\mathbb{F}_{p^e}) = \#\mathcal{X}_p(\mathbb{F}_{p^e}) \leq c_0 p^{(n-1)e}.$$

Then we have

$$\begin{aligned}
\sum_{e=1}^{\infty} \#\mathcal{X}(\mathbb{F}_{p^e}) \cdot p^{-e(n+1)} &\leq c_0 \sum_{e=1}^{\infty} p^{-2e} \\
&\leq \frac{c_0 p^{-2}}{1 - p^{-2}} < 2c_0 p^{-2}.
\end{aligned}$$



Hence we conclude. □

**Lemma 5.0.4.** *When  $R \in \mathbb{Z}_{>0}$  is large enough, we have*

$$\left| \prod_{p \leq R} \zeta_{\mathcal{X}_p}(n+1)^{-1} - \zeta_{\mathcal{X}}(n+1)^{-1} \right| < 8c_0 \zeta_{\mathcal{X}}(n+1)^{-1} \cdot R^{-1}.$$

*Proof.* Since  $\zeta_{\mathcal{X}}(s) = \prod_p \zeta_{\mathcal{X}_p}(s)$ , for a positive integer  $R$  we have

$$\begin{aligned} \left| \prod_{p \leq R} \zeta_{\mathcal{X}_p}(n+1)^{-1} - \zeta_{\mathcal{X}}(n+1)^{-1} \right| &= \zeta_{\mathcal{X}}(n+1)^{-1} \cdot \left| \prod_{p > R} \zeta_{\mathcal{X}_p}(n+1) - 1 \right| \\ &= \zeta_{\mathcal{X}}(n+1)^{-1} \cdot \left| \exp \left( \sum_{p > R} \log \zeta_{\mathcal{X}_p}(n+1) \right) - 1 \right|. \end{aligned}$$

Since by Lemma 5.0.3  $0 < \log \zeta_{\mathcal{X}_p}(n+1) \leq 4c_0 p^{-2}$ , we have

$$\begin{aligned} 0 < \sum_{p > R} \log \zeta_{\mathcal{X}_p}(n+1) &< 4c_0 \sum_{p > R} p^{-2} \\ &< 4c_0 \sum_{k > R} k^{-2} \\ &< 4c_0 \int_R^{\infty} x^{-2} dx \\ &= 4c_0 R^{-1}. \end{aligned}$$

When  $t \in \mathbb{R}$  is sufficiently small, we have  $e^t - 1 < 2t$ . Therefore when  $R$  is sufficiently large,

$$\begin{aligned} \left| \prod_{p \leq R} \zeta_{\mathcal{X}_p}(n+1)^{-1} - \zeta_{\mathcal{X}}(n+1)^{-1} \right| &= \zeta_{\mathcal{X}}(n+1)^{-1} \cdot \left| \exp \left( \sum_{p > R} \log \zeta_{\mathcal{X}_p}(n+1) \right) - 1 \right| \\ &< \zeta_{\mathcal{X}}(n+1)^{-1} \cdot \left| \exp(4c_0 R^{-1}) - 1 \right| \\ &< 8c_0 R^{-1} \zeta_{\mathcal{X}}(n+1)^{-1}. \end{aligned}$$

□

## Chapitre 6

# Effective computations on a single fiber

In this chapter, for a regular projective arithmetic variety  $\mathcal{X}$  of dimension  $n$  equipped with an ample Hermitian line bundle  $\bar{\mathcal{L}}$ , we calculate the density of the set of global sections in  $H_{\text{Ar}}^0(\mathcal{X}, \bar{\mathcal{L}}^{\otimes d})$  whose divisor has no singular point lying on a fiber  $\mathcal{X}_p$  for a fixed prime integer  $p$  when  $d \rightarrow \infty$ . Note that this density differs from the density of sections in  $H^0(\mathcal{X}_p, \mathcal{L}^{\otimes d})$  whose divisor is smooth over  $\mathbb{F}_p$ . This is because when a global section  $\sigma \in H^0(\mathcal{X}, \bar{\mathcal{L}}^{\otimes d})$  is such that  $\text{div}\sigma$  has no singular point on  $\mathcal{X}_p$ , it is still possible that its image by the restriction map  $\phi_{d,p} : H^0(\mathcal{X}, \bar{\mathcal{L}}^{\otimes d}) \rightarrow H^0(\mathcal{X}_p, \mathcal{L}^{\otimes d})$  is such that  $\text{div}\phi_{d,p}(\sigma)$  is singular.

Indeed, let  $x$  be a closed point on the fiber  $\mathcal{X}_p$  with maximal ideal  $\mathfrak{m}_x$  as a closed subscheme of  $\mathcal{X}$ . We may assume that  $\mathcal{X}_p$  is smooth over  $\mathbb{F}_p$ . The maximal ideal of  $x$  as a closed point of  $\mathcal{X}_p$  is  $\mathfrak{m}_{\mathcal{X}_p, x} = \mathfrak{m}_x / (p \cdot \mathcal{O}_{\mathcal{X}, x})$ . For any  $\sigma \in H^0(\mathcal{X}, \bar{\mathcal{L}}^{\otimes d})$ , its divisor  $\text{div}\sigma$  is singular at  $x$  if and only if  $\sigma$  is contained in  $H^0(\mathcal{X}, \bar{\mathcal{L}}^{\otimes d} \otimes \mathfrak{m}_x^2)$ , where we identify  $H^0(\mathcal{X}, \bar{\mathcal{L}}^{\otimes d} \otimes \mathfrak{m}_x^2)$  with a sub- $\mathbb{Z}$ -module of  $H^0(\mathcal{X}, \bar{\mathcal{L}}^{\otimes d})$  by regarding  $\bar{\mathcal{L}}^{\otimes d} \otimes \mathfrak{m}_x^2$  as a subsheaf of  $\bar{\mathcal{L}}^{\otimes d}$ . This is equivalent to the condition that, denoting  $x'$  the closed subscheme of  $\mathcal{X}$  defined by the ideal sheaf  $\mathfrak{m}_x^2$ , the image of  $\sigma$  by the restriction map  $H^0(\mathcal{X}, \bar{\mathcal{L}}^{\otimes d}) \rightarrow H^0(x', \mathcal{L}^{\otimes d})$  is 0. Similarly, denoting  $x''$  the closed subscheme of  $\mathcal{X}_p$  defined by  $\mathfrak{m}_{\mathcal{X}_p, x}^2$ ,  $\text{div}\phi_{d,p}(\sigma)$  is singular at  $x$  if and only if the image of  $\sigma$  by the restriction map  $H^0(\mathcal{X}, \bar{\mathcal{L}}^{\otimes d}) \rightarrow H^0(x'', \mathcal{L}^{\otimes d})$  is 0. Note that as  $x$  is a regular point of  $\mathcal{X}$ ,

$$\#H^0(x', \mathcal{L}^{\otimes d}) = \#H^0(x', \mathcal{O}_{x'}) = (\#\kappa(x))^{1+n}.$$

Similarly, since  $\mathcal{X}_p$  is smooth,

$$\#H^0(x'', \mathcal{L}^{\otimes d}) = \#H^0(x'', \mathcal{O}_{x''}) = (\#\kappa(x))^{1+(n-1)} = (\#\kappa(x))^n.$$

Moreover, by the definition of  $x'$  and  $x''$ , the restriction map  $H^0(\mathcal{X}, \bar{\mathcal{L}}^{\otimes d}) \rightarrow H^0(x'', \mathcal{L}^{\otimes d})$  factors through

$$H^0(x', \mathcal{L}^{\otimes d}) \rightarrow H^0(x'', \mathcal{L}^{\otimes d}).$$

Therefore we have a strict inclusion

$$\text{Ker} \left( H^0(\mathcal{X}, \bar{\mathcal{L}}^{\otimes d}) \rightarrow H^0(x', \mathcal{L}^{\otimes d}) \right) \subsetneq \text{Ker} \left( H^0(\mathcal{X}, \bar{\mathcal{L}}^{\otimes d}) \rightarrow H^0(x'', \mathcal{L}^{\otimes d}) \right)$$

which implies that it is possible that  $\text{div}\phi_{d,p}(\sigma)$  is singular at  $x$  while  $\text{div}\sigma$  is regular at  $x$ .

**Example.** Consider  $\mathbb{P}_{\mathbb{Z}}^2$  together with the ample line bundle  $\mathcal{O}(1)$  on it. Then  $X^2 + 5Y^2 - Z^2$  is a global section in  $H^0(\mathbb{P}_{\mathbb{Z}}^2, \mathcal{O}(2))$ . The restriction  $\phi_{2,5}(X^2 + 5Y^2 - Z^2)$  in  $H^0(\mathbb{P}_{\mathbb{F}_5}^2, \mathcal{O}(2))$  is equal to  $X^2 - Z^2$ . So  $\text{div}(\phi_{2,5}(X^2 + 5Y^2 - Z^2))$  has a singular point  $P = [0, 1, 0] \in \mathbb{P}_{\mathbb{F}_5}^2$ . But  $P$  is not a singular point of  $\text{div}(X^2 + 5Y^2 - Z^2)$ . Indeed, consider the open affine neighbourhood  $\mathbb{A}_{\mathbb{Z}}^2 = \mathbb{P}_{\mathbb{Z}}^2 - \text{div}(Y)$  of  $P$ . The ideal sheaf  $\mathfrak{m}_{\mathbb{A}_{\mathbb{Z}}^2, P}$  of  $P$  in  $\mathbb{A}_{\mathbb{Z}}^2$  is generated by  $\frac{X}{Y}, \frac{Z}{Y}, 5 \in H^0(\mathbb{A}_{\mathbb{Z}}^2, \mathcal{O}_{\mathbb{A}_{\mathbb{Z}}^2})$ . Then  $\mathfrak{m}_{\mathbb{A}_{\mathbb{Z}}^2, P}^2$  is generated by  $\frac{X^2}{Y^2}, \frac{Z^2}{Y^2}, 25, \frac{XZ}{Y^2}, \frac{5X}{Y}, \frac{5Z}{Y}$ . Let  $P'$  be the first order infinitesimal neighbourhood of  $P$  in  $\mathbb{P}_{\mathbb{Z}}^2$ . Then  $P'$  can be regarded as a closed subscheme of  $\mathbb{A}_{\mathbb{Z}}^2$  defined by  $\mathfrak{m}_{\mathbb{A}_{\mathbb{Z}}^2, P}^2$ . Note that

$$X^2 + 5Y^2 - Z^2 = \left(\frac{X^2}{Y^2} - \frac{Z^2}{Y^2} + 5\right)Y^2.$$

The image of  $X^2 + 5Y^2 - Z^2$  in  $H^0(P', \mathcal{O}(2))$  is  $5 \cdot Y^2$ , which is non-zero. So  $P$  is a singular point of  $\text{div}(\phi_{2,5}(X^2 + 5Y^2 - Z^2))$ , but it is not a singular point of  $\text{div}(X^2 + 5Y^2 - Z^2)$ .

## 6.1 Main result

We write

$$\mathcal{P}_{d,p} := \left\{ \sigma \in H_{\text{Ar}}^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes d}) ; \text{div } \sigma \text{ has no singular point on } \mathcal{X}_p \right\}.$$

**Theorem 6.1.1.** *Let  $\mathcal{X}$  be a regular projective arithmetic variety of absolute dimension  $n$ , and let  $\overline{\mathcal{L}}$  be an ample Hermitian line bundle on  $\mathcal{X}$ . There exists a constant  $C > 1$  such that for any large enough integer  $d$  and any prime number  $p$  verifying  $Cnp^n < d$ , we have*

$$\left| \frac{\#\mathcal{P}_{d,p}}{\#H_{\text{Ar}}^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes d})} - \zeta_{\mathcal{X}_p}(n+1)^{-1} \right| = O\left(\left(\frac{d}{p}\right)^{-\frac{2}{n}}\right),$$

where the constant involved in big  $O$  is independent of  $d, p$ .

To prove the result, it suffices to prove the following proposition :

**Proposition 6.1.2.** *Define*

$$\mathcal{P}'_{d,p^2} := \left\{ \sigma' \in H^0(\mathcal{X}_{p^2}, \overline{\mathcal{L}}^{\otimes d}) ; \forall x \in |\text{div } \sigma'|, \dim_{\kappa(x)} \frac{\mathfrak{m}_{\text{div } \sigma', x}}{\mathfrak{m}_{\text{div } \sigma', x}^2} = n - 1 \right\},$$

where  $\mathcal{X}_{p^2} = \mathcal{X} \times_{\text{Spec } \mathbb{Z}} (\text{Spec } \mathbb{Z}/p^2\mathbb{Z})$ . Then there exists a constant  $C > 1$  such that for any prime number  $p$  verifying  $Cnp^n < d$ , we have

$$\left| \frac{\#\mathcal{P}'_{d,p^2}}{\#H^0(\mathcal{X}_{p^2}, \overline{\mathcal{L}}^{\otimes d})} - \zeta_{\mathcal{X}_p}(n+1)^{-1} \right| = O\left(\left(\frac{d}{p}\right)^{-\frac{2}{n}}\right),$$

where the constant involved in big  $O$  is independent of  $d, p$ .

*Proof of Theorem 6.1.1.* Assuming this proposition, by Proposition 4.2.2, for any  $0 < \alpha_0 < 1$  we can find a constant  $\eta > 0$  such that when  $d$  is large enough, for any prime number  $p$  such that  $p^2 \leq \exp(d^{\alpha_0})$  and any  $\sigma'_1, \sigma'_2 \in H^0(\mathcal{X}_{p^2}, \overline{\mathcal{L}}^{\otimes d})$ ,

$$\frac{\left| \# \left( \phi_{d,p^2}^{-1}(\sigma'_1) \cap H_{\text{Ar}}^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes d}) \right) - \# \left( \phi_{d,p^2}^{-1}(\sigma'_2) \cap H_{\text{Ar}}^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes d}) \right) \right|}{\# \left( \phi_{d,p^2}^{-1}(\sigma'_1) \cap H_{\text{Ar}}^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes d}) \right)} \leq e^{-\eta d},$$

where  $\phi_{d,p^2} : \mathbb{H}^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes d}) \rightarrow \mathbb{H}^0(\mathcal{X}_{p^2}, \overline{\mathcal{L}}^{\otimes d})$  is the restriction map. When we take the sum over all  $\sigma'_2 \in \mathbb{H}^0(\mathcal{X}_{p^2}, \overline{\mathcal{L}}^{\otimes d})$ , as

$$\bigcup_{\sigma'_2 \in \mathbb{H}^0(\mathcal{X}_{p^2}, \overline{\mathcal{L}}^{\otimes d})} \left( \phi_{d,p^2}^{-1}(\sigma'_2) \cap \mathbb{H}_{\text{Ar}}^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes d}) \right) = \mathbb{H}_{\text{Ar}}^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes d})$$

we get

$$\begin{aligned} & \left| \# \left( \phi_{d,p^2}^{-1}(\sigma'_1) \cap \mathbb{H}_{\text{Ar}}^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes d}) \right) \cdot \left( \# \mathbb{H}^0(\mathcal{X}_{p^2}, \overline{\mathcal{L}}^{\otimes d}) \right) - \# \mathbb{H}_{\text{Ar}}^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes d}) \right| \\ \leq & \sum_{\sigma'_2 \in \mathbb{H}^0(\mathcal{X}_{p^2}, \overline{\mathcal{L}}^{\otimes d})} \left| \# \left( \phi_{d,p^2}^{-1}(\sigma'_1) \cap \mathbb{H}_{\text{Ar}}^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes d}) \right) - \# \left( \phi_{d,p^2}^{-1}(\sigma'_2) \cap \mathbb{H}_{\text{Ar}}^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes d}) \right) \right| \\ \leq & \sum_{\sigma'_2 \in \mathbb{H}^0(\mathcal{X}_{p^2}, \overline{\mathcal{L}}^{\otimes d})} e^{-\eta d} \cdot \# \left( \phi_{d,p^2}^{-1}(\sigma'_1) \cap \mathbb{H}_{\text{Ar}}^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes d}) \right) \\ = & e^{-\eta d} \cdot \# \left( \phi_{d,p^2}^{-1}(\sigma'_1) \cap \mathbb{H}_{\text{Ar}}^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes d}) \right) \cdot \# \mathbb{H}^0(\mathcal{X}_{p^2}, \overline{\mathcal{L}}^{\otimes d}) \end{aligned}$$

Dividing both side of the inequality by  $\# \mathbb{H}^0(\mathcal{X}_{p^2}, \overline{\mathcal{L}}^{\otimes d})$ , we get

$$\left| \# \left( \phi_{d,p^2}^{-1}(\sigma') \cap \mathbb{H}_{\text{Ar}}^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes d}) \right) - \frac{\# \mathbb{H}_{\text{Ar}}^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes d})}{\# \mathbb{H}^0(\mathcal{X}_{p^2}, \overline{\mathcal{L}}^{\otimes d})} \right| \leq e^{-\eta d} \cdot \# \left( \phi_{d,p^2}^{-1}(\sigma') \cap \mathbb{H}_{\text{Ar}}^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes d}) \right).$$

Since  $\mathcal{P}_{d,p}$  is exactly the preimage of  $\mathcal{P}'_{d,p^2}$  in  $\mathbb{H}_{\text{Ar}}^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes d})$ , summing up over all  $\sigma' \in \mathcal{P}'_{d,p^2}$ , we get

$$\begin{aligned} & \left| \# \mathcal{P}_{d,p} - \frac{\# \mathbb{H}_{\text{Ar}}^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes d})}{\# \mathbb{H}^0(\mathcal{X}_{p^2}, \overline{\mathcal{L}}^{\otimes d})} \cdot \# \mathcal{P}'_{d,p^2} \right| \\ = & \left| \# \bigcup_{\sigma' \in \mathcal{P}'_{d,p^2}} \left( \phi_{d,p^2}^{-1}(\sigma') \cap \mathbb{H}_{\text{Ar}}^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes d}) \right) - \frac{\# \mathbb{H}_{\text{Ar}}^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes d})}{\# \mathbb{H}^0(\mathcal{X}_{p^2}, \overline{\mathcal{L}}^{\otimes d})} \cdot \# \mathcal{P}'_{d,p^2} \right| \\ \leq & \sum_{\sigma' \in \mathcal{P}'_{d,p^2}} \left| \# \left( \phi_{d,p^2}^{-1}(\sigma') \cap \mathbb{H}_{\text{Ar}}^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes d}) \right) - \frac{\# \mathbb{H}_{\text{Ar}}^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes d})}{\# \mathbb{H}^0(\mathcal{X}_{p^2}, \overline{\mathcal{L}}^{\otimes d})} \right| \\ \leq & \sum_{\sigma' \in \mathcal{P}'_{d,p^2}} e^{-\eta d} \cdot \# \left( \phi_{d,p^2}^{-1}(\sigma') \cap \mathbb{H}_{\text{Ar}}^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes d}) \right) \\ = & e^{-\eta d} \cdot \# \mathcal{P}_{d,p} \leq e^{-\eta d} \cdot \# \mathbb{H}_{\text{Ar}}^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes d}). \end{aligned}$$

Dividing this inequality by  $\# \mathbb{H}_{\text{Ar}}^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes d})$ , we get

$$\left| \frac{\# \mathcal{P}_{d,p}}{\# \mathbb{H}_{\text{Ar}}^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes d})} - \frac{\# \mathcal{P}'_{d,p^2}}{\# \mathbb{H}^0(\mathcal{X}_{p^2}, \overline{\mathcal{L}}^{\otimes d})} \right| \leq e^{-\eta d}.$$

Therefore assuming Proposition 6.1.2, we have

$$\begin{aligned}
& \left| \frac{\#\mathcal{P}_{d,p}}{\mathbb{H}_{\text{Ar}}^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes d})} - \zeta_{\mathcal{X}_p}(n+1)^{-1} \right| \\
& \leq \left| \frac{\#\mathcal{P}_{d,p}}{\#\mathbb{H}_{\text{Ar}}^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes d})} - \frac{\#\mathcal{P}'_{d,p^2}}{\#\mathbb{H}^0(\mathcal{X}_{p^2}, \overline{\mathcal{L}}^{\otimes d})} \right| + \left| \frac{\#\mathcal{P}'_{d,p^2}}{\mathbb{H}^0(\mathcal{X}_{p^2}, \overline{\mathcal{L}}^{\otimes d})} - \zeta_{\mathcal{X}_p}(n+1)^{-1} \right| \\
& = O(e^{-\eta d}) + O\left(\left(\frac{d}{p}\right)^{-\frac{2}{n}}\right) = O\left(\left(\frac{d}{p}\right)^{-\frac{2}{n}}\right).
\end{aligned}$$

□

The proof of Proposition 6.1.2 follows the method of Poonen for his proof of the Bertini theorem over finite fields in [Po04].

We will prove Proposition 6.1.2 through the following steps.

1. In Section 6.2 we will calculate the proportion of  $\sigma \in \mathbb{H}^0(\mathcal{X}_{p^2}, \overline{\mathcal{L}}^{\otimes d})$  such that  $\dim_{\kappa(x)} \frac{\mathfrak{m}_{\text{div}\sigma, x}}{\mathfrak{m}_{\text{div}\sigma, x}^2} = n-1$  for any closed point  $x$  of degree  $\leq r$  for an integer  $r$ . This proportion equals to

$$\prod_{x \in |\mathcal{X}|, \deg x \leq r} (1 - \#\kappa(x)^{-(1+n)})$$

for  $r$  not too big. We will give a bound  $r_d$  for  $r$  depending on  $d$  where this proportion is valide for any  $0 < r \leq r_d$ .

2. Then in Section 6.3, we will show that for some integer constant  $N$ , the proportion of  $\sigma \in \mathbb{H}^0(\mathcal{X}_{p^2}, \overline{\mathcal{L}}^{\otimes d})$  such that there exists a closed point  $x$  of degree between  $r_d$  and  $\frac{d}{nN}$  where the condition  $\dim_{\kappa(x)} \frac{\mathfrak{m}_{\text{div}\sigma, x}}{\mathfrak{m}_{\text{div}\sigma, x}^2} = n-1$  is not satisfied tends to 0 when  $d$  tends to infinity.
3. In Section 6.4, we will show the following : there exists a constant  $N(p)$  depending on  $p$  such that the proportion of  $\sigma \in \mathbb{H}^0(\mathcal{X}_{p^2}, \overline{\mathcal{L}}^{\otimes d})$  which satisfy the condition that there exists a closed point  $x$  of degree larger than  $\frac{d}{nN(p)}$  where we have  $\dim_{\kappa(x)} \frac{\mathfrak{m}_{\text{div}\sigma, x}}{\mathfrak{m}_{\text{div}\sigma, x}^2} \neq n-1$  tends to 0 when  $d$  tends to infinity.
4. In Section 6.5, we will put these three estimates together to get an effective estimate of proportion of global sections whose divisor has no singular point on one single fiber.

In the following, we need a relative version of Lemma 3.0.2 :

**Lemma 6.1.3.** *Let  $\mathcal{L}$  be an ample line bundle on a projective scheme  $\mathcal{Y}$  flat over an open subscheme  $\mathcal{S} = \text{Spec } R$  of  $\text{Spec } \mathbb{Z}$ . Then there exists a positive integer  $N$  such that*

- i)  $\mathcal{L}^{\otimes d}$  is very ample for all  $d \geq N$  ;
- ii) for any  $a, b \geq N$ , the natural morphism of  $R$ -modules

$$\mathbb{H}^0(\mathcal{Y}, \mathcal{L}^{\otimes a}) \otimes \mathbb{H}^0(\mathcal{Y}, \mathcal{L}^{\otimes b}) \longrightarrow \mathbb{H}^0(\mathcal{Y}, \mathcal{L}^{\otimes(a+b)})$$

is surjective.

*Proof.* It suffices to take the integer  $N$  such that Lemma 3.0.2 holds for the generic fiber  $\mathcal{Y}_{\mathbb{Q}}$  and that  $\mathbb{H}^0(\mathcal{Y}, \mathcal{L}^{\otimes d})$  is torsion free for any  $d \geq N$ . □

## 6.2 Singular points of small degree

We need a lemma :

**Lemma 6.2.1.** *Let  $Z$  be a closed subscheme of  $\mathcal{X}_{p^2}$  of dimension 0, and let  $N$  be a positive integer such that  $\mathcal{L}^{\otimes N}$  is very ample. The restriction morphism*

$$H^0(\mathcal{X}_{p^2}, \mathcal{L}^{\otimes d}) \longrightarrow H^0(Z, \mathcal{L}^{\otimes d})$$

*is surjective when  $d \geq Nh_Z$ , where  $h_Z = \dim_{\mathbb{F}_p}(H^0(Z, \mathcal{O}_Z) \otimes_{\mathbb{Z}/p^2\mathbb{Z}} \mathbb{F}_p)$ .*

*Proof.* Let  $C_d$  be the cokernel of the restriction map. Then  $C_d \otimes_{\mathbb{Z}/p^2\mathbb{Z}} \mathbb{F}_p$  is the cokernel of

$$H^0(\mathcal{X}_p, \mathcal{L}^{\otimes d}) \longrightarrow H^0(Z_p, \mathcal{L}^{\otimes d}) = H^0(Z, \mathcal{L}^{\otimes d}) \otimes_{\mathbb{Z}/p^2\mathbb{Z}} \mathbb{F}_p.$$

When  $d \geq Nh_Z$ , by Lemma 3.1.1, we have  $C_d \otimes_{\mathbb{Z}/p^2\mathbb{Z}} \mathbb{F}_p = 0$ . Then by the short exact sequence

$$0 \longrightarrow pC_d \longrightarrow C_d \longrightarrow C_d \otimes_{\mathbb{Z}/p^2\mathbb{Z}} \mathbb{F}_p \longrightarrow 0,$$

we get  $pC_d = C_d$ . Applying Nakayama's lemma to  $C_d$ , considered as a  $\mathbb{Z}/p^2\mathbb{Z}$ -module, we get  $C_d = 0$ . Thus the surjectivity of the restriction map in the lemma holds when  $d \geq Nh_Z$ .  $\square$

**Lemma 6.2.2.** *Set*

$$\mathcal{P}'_{d,p^2,\leq r} := \left\{ \sigma' \in H^0(\mathcal{X}_{p^2}, \overline{\mathcal{L}}^{\otimes d}) ; \forall x \in |\operatorname{div} \sigma'| \text{ of degree } \leq r, \dim_{\kappa(x)} \frac{\mathfrak{m}_{\operatorname{div} \sigma', x}}{\mathfrak{m}_{\operatorname{div} \sigma', x}^2} = n - 1 \right\}.$$

*When  $d$  is sufficiently large, for any positive integer  $r$  satisfying  $2c_0Nnrp^{(n-1)r} \leq d$ , we have*

$$\frac{\#\mathcal{P}'_{d,p^2,\leq r}}{\#H^0(\mathcal{X}_{p^2}, \overline{\mathcal{L}}^{\otimes d})} = \prod_{x \in |\mathcal{X}_p|, \deg x \leq r} \left( 1 - p^{-(n+1)\deg x} \right).$$

*Proof.* For any closed point  $x \in \mathcal{X}_{p^2}$ , let  $x'$  be the closed subscheme of  $\mathcal{X}_{p^2}$  defined by the square of the maximal ideal of  $x$ . Then  $x'$  is the first order infinitesimal neighborhood of  $x$  in  $\mathcal{X}_{p^2}$ . We have  $x' \simeq \operatorname{Spec}(\mathcal{O}_{\mathcal{X}_{p^2}, x} / \mathfrak{m}_{\mathcal{X}_{p^2}, x}^2)$ . A section  $\sigma' \in H^0(\mathcal{X}_{p^2}, \overline{\mathcal{L}}^{\otimes d})$  is such that  $\operatorname{div} \sigma'$  contains  $x$  and that  $\dim_{\kappa(x)} \frac{\mathfrak{m}_{\operatorname{div} \sigma', x}}{\mathfrak{m}_{\operatorname{div} \sigma', x}^2} = n$  if and only if the restriction map

$$H^0(\mathcal{X}_{p^2}, \overline{\mathcal{L}}^{\otimes d}) \longrightarrow H^0(x', \overline{\mathcal{L}}^{\otimes d})$$

sends  $\sigma'$  to 0. For a positive integer  $r$ , let  $\mathcal{X}'_{p^2, \leq r}$  be the disjoint union

$$\mathcal{X}'_{p^2, \leq r} = \coprod_{x \in |\mathcal{X}_{p^2}|, \deg x \leq r} x'.$$

Then we have a natural isomorphism

$$H^0(\mathcal{X}'_{p^2, \leq r}, \overline{\mathcal{L}}^{\otimes d}) \simeq \prod_{x \in |\mathcal{X}_{p^2}|, \deg x \leq r} H^0(x', \overline{\mathcal{L}}^{\otimes d}).$$

A section  $\sigma' \in H^0(\mathcal{X}_{p^2}, \overline{\mathcal{L}}^{\otimes d})$  is such that  $\text{div} \sigma'$  is regular at all closed points  $x$  of degree  $\leq r$  if and only if its image in  $H^0(\mathcal{X}'_{p^2, \leq r}, \overline{\mathcal{L}}^{\otimes d})$  lies in the subset which by the above natural isomorphism corresponds to

$$\prod_{x \in |\mathcal{X}_{p^2}|, \deg x \leq r} \left( H^0(x', \overline{\mathcal{L}}^{\otimes d}) - \{0\} \right).$$

To get the result, we need to study the surjectivity of the restriction map

$$H^0(\mathcal{X}_{p^2}, \mathcal{L}^{\otimes d}) \longrightarrow H^0(\mathcal{X}'_{p^2, \leq r}, \overline{\mathcal{L}}^{\otimes d}) \simeq H^0(\mathcal{X}_{p^2}, \mathcal{O}_{\mathcal{X}'_{p^2, \leq r}} \otimes \mathcal{L}^{\otimes d}).$$

When  $n = 1$ , the number of closed points of  $|\mathcal{X}_{p^2}|$  is bounded above by  $\#\mathcal{X}(\mathbb{C})$ . We have

$$\begin{aligned} \dim_{\mathbb{F}_p} H^0(\mathcal{X}'_{p^2, \leq r}, \mathcal{O}_{\mathcal{X}'_{p^2, \leq r}}) \otimes_{\mathbb{Z}/p^2\mathbb{Z}} \mathbb{F}_p &= \dim_{\mathbb{F}_p} H^0\left(\mathcal{X}_{p^2}, \prod_{\deg x \leq r} \mathcal{O}_{x'}\right) \otimes_{\mathbb{Z}/p^2\mathbb{Z}} \mathbb{F}_p \\ &= \sum_{\deg x \leq r} \dim_{\mathbb{F}_p} H^0(x', \mathcal{O}_{x'}) \otimes_{\mathbb{Z}/p^2\mathbb{Z}} \mathbb{F}_p \\ &= \sum_{\deg x \leq r} ((1-1) + 1) \deg x \\ &\leq \sum_{x \in |\mathcal{X}_{p^2}|} \deg x < \infty \end{aligned}$$

for any  $r > 0$ . So the restriction map is always surjective when  $d$  is sufficiently large by Lemma 6.2.1. When  $n > 1$ , we have

$$\begin{aligned} \dim_{\mathbb{F}_p} H^0(\mathcal{X}'_{p^2, \leq r}, \mathcal{O}_{\mathcal{X}'_{p^2, \leq r}}) \otimes_{\mathbb{Z}/p^2\mathbb{Z}} \mathbb{F}_p &= \dim_{\mathbb{F}_p} H^0\left(\mathcal{X}_{p^2}, \prod_{\deg x \leq r} \mathcal{O}_{x'}\right) \otimes_{\mathbb{Z}/p^2\mathbb{Z}} \mathbb{F}_p \\ &= \sum_{\deg x \leq r} \dim_{\mathbb{F}_p} H^0(x', \mathcal{O}_{x'}) \otimes_{\mathbb{Z}/p^2\mathbb{Z}} \mathbb{F}_p \\ &= \sum_{\deg x \leq r} ((n-1) + 1) \deg x \\ &\leq n \sum_{e=1}^r \#\mathcal{X}_{p^2}(\mathbb{F}_{p^e}) e \\ &\leq n \sum_{e=1}^r c_0 p^{(n-1)e} \\ &\leq nc_0 r \cdot \sum_{e=1}^r p^{(n-1)e} \\ &\leq nc_0 r \frac{p^{(n-1)(r+1)} - 1}{p^{n-1} - 1}. \end{aligned}$$

By Lemma 6.2.1, when  $d \geq N \cdot \left( nc_0 r \frac{p^{(n-1)(r+1)} - 1}{p^{n-1} - 1} \right)$  with  $N$  as in the lemma, the restriction map

$$H^0(\mathcal{X}_{p^2}, \mathcal{L}^{\otimes d}) \longrightarrow H^0\left(\mathcal{X}_{p^2}, \mathcal{O}_{\mathcal{X}'_{p^2, \leq r}} \otimes \mathcal{L}^{\otimes d}\right)$$

is surjective. In particular, since we have

$$nc_0r \frac{p^{(n-1)(r+1)} - 1}{p^{n-1} - 1} \leq nc_0r \frac{p^{(n-1)(r+1)}}{\frac{1}{2}p^{n-1}} = 2nc_0rp^{(n-1)r},$$

the surjectivity of the restriction holds for  $r, d$  verifying

$$2c_0Nnrp^{(n-1)r} \leq d.$$

For such  $r, d$ , we have

$$\frac{\#\mathcal{P}'_{d,p^2,\leq r}}{\#\mathrm{H}^0(\mathcal{X}_{p^2}, \mathcal{L}^{\otimes d})} = \prod_{x \in |\mathcal{X}_p|, \deg x \leq r} \left(1 - p^{-(n+1)\deg x}\right).$$

□

### 6.3 Singular points of medium degree

Let  $N$  be an integer satisfying Lemma 6.1.3. Set

$$\mathcal{Q}_{d,p^2,r}^{\mathrm{med}} := \left\{ \sigma' \in \mathrm{H}^0(\mathcal{X}_{p^2}, \bar{\mathcal{L}}^{\otimes d}) ; \exists x \in |\mathrm{div} \sigma'| \text{ of degree } r < \deg x \leq \frac{d}{nN}, \dim_{\kappa(x)} \frac{\mathfrak{m}_{\mathrm{div} \sigma', x}}{\mathfrak{m}_{\mathrm{div} \sigma', x}^2} = n \right\}$$

**Lemma 6.3.1.** *We have for  $r \geq 1$ ,*

$$\frac{\#\mathcal{Q}_{d,p^2,r}^{\mathrm{med}}}{\#\mathrm{H}^0(\mathcal{X}_{p^2}, \mathcal{L}^{\otimes d})} < 2c_0p^{-2(r+1)},$$

where the constant  $c_0$  is as defined in Chapter 5.

*Proof.* For any closed point  $x$  in  $\mathcal{X}_{p^2}$ , applying Lemma 6.2.1 to the first order infinitesimal neighborhood  $x'$  of  $x$  in  $\mathcal{X}_{p^2}$ , we get that the restriction morphism

$$\mathrm{H}^0(\mathcal{X}_{p^2}, \mathcal{L}^{\otimes d}) \longrightarrow \mathrm{H}^0(\mathcal{X}_{p^2}, \mathcal{O}_{x'} \otimes \mathcal{L}^{\otimes d})$$

is surjective when

$$N(n \deg x) \leq d,$$

which is when  $\deg x \leq \frac{d}{nN}$ . We can then estimate the proportion of elements in  $\mathcal{Q}_{d,p^2,r}^{\mathrm{med}}$  by

$$\begin{aligned} \frac{\#\mathcal{Q}_{d,p^2,r}^{\mathrm{med}}}{\#\mathrm{H}^0(\mathcal{X}_{p^2}, \mathcal{L}^{\otimes d})} &\leq \sum_{r < \deg x \leq \lfloor \frac{d}{nN} \rfloor} \frac{\#\mathrm{Ker}(\mathrm{H}^0(\mathcal{X}_{p^2}, \mathcal{L}^{\otimes d}) \longrightarrow \mathrm{H}^0(\mathcal{X}_{p^2}, \mathcal{O}_{x'} \otimes \mathcal{L}^{\otimes d}))}{\#\mathrm{H}^0(\mathcal{X}_{p^2}, \mathcal{L}^{\otimes d})} \\ &\leq \sum_{e=r+1}^{\lfloor \frac{d}{nN} \rfloor} \#\mathcal{X}(\mathbb{F}_{p^e}) p^{-(n+1)e} \\ &\leq \sum_{e=r+1}^{\infty} c_0 p^{(n-1)e} p^{-(n+1)e} \\ &\leq c_0 \sum_{e=r+1}^{\infty} p^{-2e} \\ &= \frac{c_0 p^{-2(r+1)}}{1 - p^{-2}} < 2c_0 p^{-2(r+1)}. \end{aligned}$$

□



## 6.4 Singular points of large degree

**Proposition 6.4.1.** *Let  $0 < \alpha < 1$  be a constant. There exist positive integers  $N_0, N_1$  only depending on  $\mathcal{X}$  and  $\bar{\mathcal{L}}$  such that for any  $p \leq d^\alpha$ , denoting*

$$N(p) = (N_0 + 1)(N_1 + p - 1) + p,$$

and

$$\mathcal{Q}_{d,p^2}^{\text{high}} := \left\{ \sigma' \in \mathbb{H}^0(\mathcal{X}_{p^2}, \bar{\mathcal{L}}^{\otimes d}) ; \exists x \in |\text{div} \sigma'| \text{ of degree } \geq \frac{d}{nN(p)}, \dim_{\kappa(x)} \frac{\mathfrak{m}_{\text{div} \sigma', x}}{\mathfrak{m}_{\text{div} \sigma', x}^2} = n \right\},$$

we have

$$\frac{\#\mathcal{Q}_{d,p^2}^{\text{high}}}{\#\mathbb{H}^0(\mathcal{X}_{p^2}, \mathcal{L}^{\otimes d})} = O\left(d^n p^{-c_1 \frac{d}{p}}\right),$$

where  $c_1$  and the constant involved in big  $O$  are independent of  $d, p$  and  $\alpha$ .

In particular, we have

$$\lim_{d \rightarrow \infty} \frac{\#\mathcal{Q}_{d,p^2}^{\text{high}}}{\#\mathbb{H}^0(\mathcal{X}_{p^2}, \mathcal{L}^{\otimes d})} = 0.$$

**Remark.** When  $d$  is large enough, for any  $p \leq d^\alpha$ ,

$$d^n p^{-c_1 \frac{d}{p}} \leq d^n d^{-c_1 \alpha d^{1-\alpha}} = d^{n-c_1 \alpha d^{1-\alpha}}.$$

As  $\alpha < 1$ ,  $d^{n-c_1 \alpha d^{1-\alpha}}$  tends to 0 when  $d$  tends to infinity. So the above proportion  $\frac{\#\mathcal{Q}_{d,p^2}^{\text{high}}}{\#\mathbb{H}^0(\mathcal{X}_{p^2}, \mathcal{L}^{\otimes d})}$  is always near 0 for any  $p \leq d^\alpha$  when  $d$  is large enough.

When  $\mathcal{X}$  is regular, so is its generic fiber  $\mathcal{X}_{\mathbb{Q}}$ , which is equivalent to say that  $\mathcal{X}_{\mathbb{Q}}$  is smooth over  $\mathbb{Q}$ . This implies that we can find an open subset  $\mathcal{S}$  of  $\text{Spec } \mathbb{Z}$  such that  $\mathcal{X}_{\mathcal{S}}$  is smooth over  $\mathcal{S}$ . We will give a uniform control of the proportion of  $\mathcal{Q}_{d,p^2}^{\text{high}}$  for primes  $p \in \mathcal{S}$  such that  $2c_0 N(p) n p^{n-1} \leq d$ . As  $\text{Spec } \mathbb{Z} - \mathcal{S}$  is a finite scheme, the set of primes  $p$  where the proportion of  $\mathcal{Q}_{d,p^2}^{\text{high}}$  is not controled is finite in number. We then give independent control of the proportion of  $\mathcal{Q}_{d,p^2}^{\text{high}}$  for each fiber with constants possibly depending on  $p$ . The finiteness of such  $p$  permits us to get a uniform control for all primes  $p$  verifying  $2c_0 N(p) n p^{n-1} \leq d$ .

Thus Proposition 6.4.1 is implied by the following two propositions :

**Proposition 6.4.2.** *Fix a constant  $0 < \alpha < 1$ . For any prime  $p \leq d^\alpha$  such that  $\mathcal{X}_p$  is smooth over  $\mathbb{F}_p$  we have*

$$\frac{\#\mathcal{Q}_{d,p^2}^{\text{high}}}{\#\mathbb{H}^0(\mathcal{X}_{p^2}, \mathcal{L}^{\otimes d})} = O\left(d^{n-1} p^{-c_1 \frac{d}{p}}\right),$$

where  $c_1$  and the constant involved in big  $O$  are independent of  $d, p$  and  $\alpha$ .

**Proposition 6.4.3.** *Fix a constant  $0 < \alpha < 1$ . For any prime number  $p \leq d^\alpha$  with possibly singular  $\mathcal{X}_p$ , we have*

$$\frac{\#\mathcal{Q}_{d,p^2}^{\text{high}}}{\#\mathbb{H}^0(\mathcal{X}_{p^2}, \mathcal{L}^{\otimes d})} = O\left(d^n p^{-c'_1 \frac{d}{p}}\right),$$

where  $c'_1$  and the constant involved in big  $O$  are independent of  $d, \alpha$ , but may depend on  $p$ .

We prove first the smooth case. Before proving Proposition 6.4.2, we need some preparation.

**Lemma 6.4.4.** *Let  $\mathcal{S}$  be an open subscheme of  $\text{Spec } \mathbb{Z}$  such that  $\mathcal{X}_{\mathcal{S}}$  is smooth over  $\mathcal{S}$ . We can find a finite cover of  $\mathcal{X}_{\mathcal{S}}$  by open subschemes  $U$  verifying the following conditions :*

1) *we can find  $t_1, \dots, t_{n-1} \in H^0(U, \mathcal{O}_U)$  such that*

$$\Omega_{U/\mathcal{S}}^1 \simeq \bigoplus_{i=1}^{n-1} \mathcal{O}_U dt_i;$$

2) *for any positive integer  $M$ , we can choose a constant  $N_0 \geq M$  satisfying Lemma 6.1.3 such that there exists a  $\tau_0 \in H^0(\mathcal{X}_{\mathcal{S}}, \mathcal{L}^{\otimes(N_0+1)})$  such that  $\mathcal{X}_{\mathcal{S}} - U = \text{div} \tau_0$ ;*

3) *with the same  $N_0$  as above, there exist  $\tau_1, \dots, \tau_k \in H^0(\mathcal{X}_{\mathcal{S}}, \mathcal{L}^{\otimes N_0})$  such that  $U = \bigcup_{1 \leq j \leq k} (\mathcal{X}_{\mathcal{S}} - \text{div} \tau_j)$ .*

*Proof.* As  $\Omega_{\mathcal{X}_{\mathcal{S}}/\mathcal{S}}^1$  is a locally free sheaf of rank  $n-1$ , we can find a finite open covering  $\{U_{\beta}\}$  of  $\mathcal{X}_{\mathcal{S}}$  and sections  $t_{\beta,1}, \dots, t_{\beta,n-1} \in H^0(U_{\beta}, \mathcal{O}_{U_{\beta}})$  such that

$$\Omega_{U_{\beta}/\mathcal{S}}^1 \simeq \bigoplus_{i=1}^{n-1} \mathcal{O}_{U_{\beta}} dt_{\beta,i}.$$

Hence we may assume the condition (1) for  $U$ . Moreover, if the condition (1) is satisfied by  $U$ , then it is also satisfied by any open subscheme of  $U$ .

If  $U$  is an open subscheme of  $\mathcal{X}_{\mathcal{S}}$  satisfying the condition (1), we will show that it can be covered by finitely many open subschemes  $\{U_{\alpha}\}$  of  $U$  which satisfy the conditions (2) and (3). This will finish the proof of the lemma.

Since  $\bar{\mathcal{L}}$  is ample, we may take a positive integer  $N'_0 > 0$  satisfying Lemma 6.1.3 such that for any  $d \geq N'_0$ , the sheaf  $\mathcal{I}_{\mathcal{X}_{\mathcal{S}}-U} \otimes \bar{\mathcal{L}}^{\otimes d}$  is globally generated, where  $\mathcal{I}_{\mathcal{X}_{\mathcal{S}}-U}$  is the ideal sheaf of  $\mathcal{X}_{\mathcal{S}} - U$  with the induced reduced structure. We may then choose non-zero sections

$$\tau'_1, \dots, \tau'_s \in H^0(\mathcal{X}_{\mathcal{S}}, \mathcal{I}_{\mathcal{X}_{\mathcal{S}}-U} \otimes \bar{\mathcal{L}}^{\otimes N'_0}) \subset H^0(\mathcal{X}_{\mathcal{S}}, \bar{\mathcal{L}}^{\otimes N'_0}),$$

generating  $\mathcal{I}_{\mathcal{X}_{\mathcal{S}}-U} \otimes \bar{\mathcal{L}}^{\otimes N'_0}$ . This means that set theoretically, we have  $\mathcal{X}_{\mathcal{S}} - U = \bigcap_i \text{div}(\tau'_i)$ . In other words, we get a finite cover of  $U$  :

$$U = \bigcup_i (\mathcal{X}_{\mathcal{S}} - \text{div}(\tau'_i)),$$

where  $\mathcal{X}_{\mathcal{S}} - \text{div}(\tau'_i)$  are open subschemes of  $\mathcal{X}_{\mathcal{S}}$ . Note that for each  $i$ ,  $\mathcal{X}_{\mathcal{S}} - \text{div}(\tau'_i)$  satisfies the condition (1) of the lemma. Without loss of generality, we may replace  $U$  by one of the subschemes  $\mathcal{X}_{\mathcal{S}} - \text{div}(\tau'_i)$ , i.e. we assume that there is a section  $\tau'_0 \in H^0(\mathcal{X}_{\mathcal{S}}, \bar{\mathcal{L}}^{\otimes N'_0})$  such that  $U = \mathcal{X}_{\mathcal{S}} - \text{div}(\tau'_0)$ . We denote  $\text{div}(\tau'_0)$  by  $D$ .

Now set  $N_0 = rN'_0 - 1$  for some positive integer  $r$  such that  $N_0 \geq M$  and that the sheaf  $\mathcal{I}_D \otimes \bar{\mathcal{L}}^{\otimes N_0}$  is globally generated. Then in particular we can find sections  $\tau_1, \dots, \tau_k \in H^0(\mathcal{X}_{\mathcal{S}}, \mathcal{I}_D \otimes \bar{\mathcal{L}}^{\otimes N_0}) \subset H^0(\mathcal{X}_{\mathcal{S}}, \bar{\mathcal{L}}^{\otimes N_0})$  such that  $D = \bigcap_{j=1}^k \text{div}(\tau_j)$  set theoretically. We also set  $\tau_0 = (\tau'_0)^r \in H^0(\mathcal{X}_{\mathcal{S}}, \bar{\mathcal{L}}^{\otimes(N_0+1)})$ . In this situation we still have  $D = \text{div}(\tau_0)$  set theoretically. This means exactly  $U = \bigcup_{1 \leq j \leq k} (\mathcal{X}_{\mathcal{S}} - \text{div} \tau_j)$ . The section  $\tau_0$  and sections  $\tau_1, \dots, \tau_k$  are then what we need for conditions (2) and (3) in the lemma.  $\square$

**Lemma 6.4.5.** For an open subscheme  $U$  of  $\mathcal{X}_S$ , set

$$\mathcal{Q}_{d,p,U}^{\text{high}} = \left\{ \sigma \in \mathbb{H}^0(\mathcal{X}_p, \overline{\mathcal{L}}^{\otimes d}) ; \text{div } \sigma \text{ has a singular point in } U \cap \text{div } \sigma \text{ of degree } \geq \frac{d}{nN(p)} \right\}.$$

Then Proposition 6.4.1 holds if we have

$$\frac{\#\mathcal{Q}_{d,p,U}^{\text{high}}}{\#\mathbb{H}^0(\mathcal{X}_p, \overline{\mathcal{L}}^{\otimes d})} = O\left(d^{n-1}p^{-c_{U,1}\frac{d}{p}}\right),$$

for all  $U$  verifying the conditions in Lemma 6.4.4, where the constant  $c_{U,1}$  only depends on  $\mathcal{X}, \overline{\mathcal{L}}$  and  $U$ .

*Proof.* Let  $\{U_\alpha\}$  be a finite cover of  $\mathcal{X}_S$  where all the  $U_\alpha$  are open subschemes of  $\mathcal{X}_S$  satisfying the conditions in Lemma 6.4.4. Then for any  $p \in \mathcal{S}$ , we get a finite open cover  $\{U_{\alpha,p^2}\}$  of  $\mathcal{X}_{p^2}$ . For an open set  $U$  of  $\mathcal{X}_S$  flat over  $\mathcal{S}$ , set

$$\mathcal{Q}_{d,p^2,U}^{\text{high}} = \left\{ \sigma' \in \mathbb{H}^0(\mathcal{X}_{p^2}, \overline{\mathcal{L}}^{\otimes d}) ; \exists x \in |\text{div } \sigma' \cap U| \text{ of degree } \geq \frac{d}{nN(p)}, \dim_{\kappa(x)} \frac{\mathfrak{m}_{\text{div } \sigma', x}}{\mathfrak{m}_{\text{div } \sigma', x}^2} = n \right\}.$$

To bound  $\mathcal{Q}_{d,p^2,U}^{\text{high}}$ , it suffices to bound  $\mathcal{Q}_{d,p^2,U_\alpha}^{\text{high}}$  for all  $U_\alpha$  in the covering.

Note that for any  $\sigma \in \mathbb{H}^0(\mathcal{X}_{p^2}, \overline{\mathcal{L}}^{\otimes d})$  and any  $x \in |\text{div } \sigma|$ , we have an exact sequence

$$p\mathcal{O}_{\text{div } \sigma} \longrightarrow \frac{\mathfrak{m}_{\text{div } \sigma, x}}{\mathfrak{m}_{\text{div } \sigma, x}^2} \longrightarrow \frac{\mathfrak{m}_{\text{div } \bar{\sigma}, x}}{\mathfrak{m}_{\text{div } \bar{\sigma}, x}^2} \longrightarrow 0$$

where  $\bar{\sigma} = \sigma \bmod p$  is the restriction in  $\mathbb{H}^0(\mathcal{X}_p, \overline{\mathcal{L}}^{\otimes d})$  and  $\text{div } \bar{\sigma}$  is the divisor in  $\mathcal{X}_p$ . Therefore

$$\dim_{\kappa(x)} \frac{\mathfrak{m}_{\text{div } \bar{\sigma}, x}}{\mathfrak{m}_{\text{div } \bar{\sigma}, x}^2} \geq \dim_{\kappa(x)} \frac{\mathfrak{m}_{\text{div } \sigma, x}}{\mathfrak{m}_{\text{div } \sigma, x}^2} - 1$$

In particular, if  $\dim_{\kappa(x)} \frac{\mathfrak{m}_{\text{div } \sigma, x}}{\mathfrak{m}_{\text{div } \sigma, x}^2} = n = \dim \mathcal{X}_S$ , then as

$$\dim_{\kappa(x)} \frac{\mathfrak{m}_{\text{div } \sigma, x}}{\mathfrak{m}_{\text{div } \sigma, x}^2} - 1 \leq \dim_{\kappa(x)} \frac{\mathfrak{m}_{\text{div } \bar{\sigma}, x}}{\mathfrak{m}_{\text{div } \bar{\sigma}, x}^2} \leq \dim \mathcal{X}_p = n - 1,$$

we have  $\dim_{\kappa(x)} \frac{\mathfrak{m}_{\text{div } \bar{\sigma}, x}}{\mathfrak{m}_{\text{div } \bar{\sigma}, x}^2} = n - 1$ , which means that  $x$  is a singular point of  $\text{div } \bar{\sigma}$ .

Then for a section  $\sigma \in \mathbb{H}^0(\mathcal{X}_{p^2}, \overline{\mathcal{L}}^{\otimes d})$ ,  $\sigma \in \mathcal{Q}_{d,p^2,U}^{\text{high}}$  implies  $\bar{\sigma} \in \mathcal{Q}_{d,p,U}^{\text{high}}$ . Thus

$$\frac{\#\mathcal{Q}_{d,p^2,U}^{\text{high}}}{\#\mathbb{H}^0(\mathcal{X}_{p^2}, \overline{\mathcal{L}}^{\otimes d})} \leq \frac{\#\{\sigma \in \mathbb{H}^0(\mathcal{X}_{p^2}, \overline{\mathcal{L}}^{\otimes d}) ; \bar{\sigma} \in \mathcal{Q}_{d,p,U}^{\text{high}}\}}{\#\mathbb{H}^0(\mathcal{X}_{p^2}, \overline{\mathcal{L}}^{\otimes d})} = \frac{\#\mathcal{Q}_{d,p,U}^{\text{high}}}{\#\mathbb{H}^0(\mathcal{X}_p, \overline{\mathcal{L}}^{\otimes d})}.$$

It suffices then to bound  $\#\mathcal{Q}_{d,p,U_\alpha}^{\text{high}}$  for  $U_\alpha$  in the covering. Here all  $U_\alpha$  satisfies the conditions in Lemma 6.4.4. Such a finite covering exists by Lemma 6.4.4. If for any  $\alpha$ , the estimate

$$\frac{\#\mathcal{Q}_{d,p,U_\alpha}^{\text{high}}}{\#\mathbb{H}^0(\mathcal{X}_p, \overline{\mathcal{L}}^{\otimes d})} = O\left(d^{n-1}p^{-c_{U_\alpha,1}\frac{d}{p}}\right)$$

holds, then setting  $c_1 = \min_\alpha \{c_{U_\alpha,1}\} > 0$ , we have

$$\frac{\#\mathcal{Q}_{d,p^2,U}^{\text{high}}}{\#\mathbb{H}^0(\mathcal{X}_{p^2}, \overline{\mathcal{L}}^{\otimes d})} \leq \sum_\alpha \frac{\#\mathcal{Q}_{d,p,U_\alpha}^{\text{high}}}{\#\mathbb{H}^0(\mathcal{X}_p, \overline{\mathcal{L}}^{\otimes d})} = O\left(d^{n-1}p^{-c_1\frac{d}{p}}\right).$$

□

Now for an open subscheme  $U$  as in Lemma 6.4.4, we get morphisms

$$\begin{aligned} \Phi_j : \mathbb{H}^0(\mathcal{X}_S, \mathcal{L}^{\otimes d}) &\longrightarrow \mathbb{H}^0(U, \mathcal{O}_U) \\ \sigma &\longmapsto \frac{\sigma \cdot \tau_j^d}{\tau_0^d} \end{aligned}$$

for any  $d \in \mathbb{Z}_{>0}$  and  $1 \leq j \leq k$ .

On the other hand, Lemma 3.3.10 tells us that there exists a positive integer  $N_1$  such that for any  $\sigma \in \mathbb{H}^0(\mathcal{X}_p, \mathcal{L}^{\otimes d})$ , any  $1 \leq i \leq n-1$ , the section  $(\partial_i \Phi_j(\sigma)) \cdot \tau_{0,p}^{d+\delta}$  extends to a global section in  $\mathbb{H}^0(\mathcal{X}_p, \mathcal{L}^{\otimes (N_0+1)(d+\delta)})$  for any  $\delta \geq N_1$ . Here  $\tau_{0,p}$  is the restriction of  $\tau_0$  modulo  $p$  in  $\mathbb{H}^0(\mathcal{X}_p, \mathcal{L}^{\otimes (N_0+1)})$ .

In fact, we may choose  $N_1$  to be  $N_1 = N'_1 + d_0 + 1$  where  $N'_1$  is such that when  $\delta \geq N'_1$ , for any  $1 \leq i \leq n$  the section  $\partial_i \cdot \tau_0^\delta \in \mathbb{H}^0\left(U, \mathcal{H}om_{\mathcal{O}_{\mathcal{X}_S}}(\Omega_{\mathcal{X}_S/S}^1, \mathcal{O}_{\mathcal{X}_S}) \otimes \mathcal{L}^{\otimes (N_0+1)\delta}\right)$  can be extended to a global section in

$$\mathbb{H}^0\left(\mathcal{X}_S, \mathcal{H}om_{\mathcal{O}_{\mathcal{X}_S}}(\Omega_{\mathcal{X}_S/S}^1, \mathcal{O}_{\mathcal{X}_S}) \otimes \bar{\mathcal{L}}^{\otimes (N_0+1)\delta}\right) \simeq \mathbb{H}om(\Omega_{\mathcal{X}_S/S}^1, \bar{\mathcal{L}}^{\otimes (N_0+1)\delta}),$$

and where  $d_0$  is such that for any  $d \geq d_0$ , the restriction morphism

$$\mathbb{H}^0\left(\mathbb{P}\left(\mathbb{H}^0(\mathcal{X}_S, \bar{\mathcal{L}}^{\otimes (N_0+1)})\right), \mathcal{O}(d)\right) \longrightarrow \mathbb{H}^0(\mathcal{X}_S, \bar{\mathcal{L}}^{\otimes (N_0+1)d})$$

is surjective. So  $N_1$  is again independent of  $d$  and  $p$ .

We enlarge  $N_0$  if necessary so that it verifies the following conditions :

- 1)  $N_0 + 1$  is a power of a prime number ;
- 2) for any  $d \geq N_0$ ,  $(\mathcal{L}|_{\mathcal{X}_S})^{\otimes d}$  is very ample ;
- 3) for any  $a, b \geq N_0$ , we have a surjective morphism

$$\mathbb{H}^0(\mathcal{X}_S, \mathcal{L}^{\otimes a}) \otimes_{\mathbb{H}^0(S, \mathcal{O}_S)} \mathbb{H}^0(\mathcal{X}_S, \mathcal{L}^{\otimes b}) \longrightarrow \mathbb{H}^0(\mathcal{X}_S, \mathcal{L}^{\otimes (a+b)}).$$

We prove the following result :

**Lemma 6.4.6.** *For any prime  $p \in \mathcal{S}$ , take*

$$N(p) = (N_0 + 1)(N_1 + p - 1) + p = p(N_0 + 2) + (N_0 + 1)(N_1 - 1).$$

*With notation as in Proposition 6.4.1, if  $p$  and  $N(p)$  satisfy  $2c_0N(p)np^{n-1} \leq d$ , then we have*

$$\frac{\#\mathcal{Q}_{d,p,U}^{\text{high}}}{\#\mathbb{H}^0(\mathcal{X}_p, \mathcal{L}^{\otimes d})} = O\left(d^{n-1}p^{-c_1 \frac{d}{p}}\right),$$

*where  $c_1$  and the constant involved in big  $O$  are independent of  $d, p$ .*

By Lemma 6.4.5, this implies Proposition 6.4.2.

*Proof of Lemma 6.4.6.* For each  $p \in \mathcal{S}$ , if  $d \geq N(p)$ ,  $d$  has a unique decomposition

$$d = pk_{p,d} + (N_0 + 1)l_{p,d}$$

with  $N_1 \leq l_{p,d} < N_1 + p$ . We have a surjective map

$$\mathbb{H}^0(\mathcal{X}_p, \mathcal{L}^{\otimes d}) \times \left( \prod_{i=1}^{n-1} \mathbb{H}^0(\mathcal{X}_p, \mathcal{L}^{\otimes k_{p,d}}) \right) \times \mathbb{H}^0(\mathcal{X}_p, \mathcal{L}^{k_{p,d}}) \longrightarrow \mathbb{H}^0(\mathcal{X}_p, \mathcal{L}^{\otimes d})$$

which sends  $(\sigma_0, (\beta_1, \dots, \beta_{n-1}), \gamma)$  to

$$\sigma = \sigma_0 + \sum_{i=1}^{n-1} \beta_i^p t_i \tau_{0,p}^{l_d} + \gamma^p \tau_{0,p}^{l_d},$$

where  $\tau_{0,p}$  is the restriction of  $\tau_0$  modulo  $p$  in  $H^0(\mathcal{X}_p, \mathcal{L}^{\otimes(N_0+1)})$ . Thus

$$\Phi_{j,p}(\sigma) = \Phi_{j,p}(\sigma_0) + \sum_{i=1}^{n-1} \Phi_{j,p}(\beta_i)^p t_i \Phi_{j,p}(\tau_{0,p})^{l_d} + \Phi_{j,p}(\gamma)^p \Phi_{j,p}(\tau_{0,p})^{l_d}$$

in  $H^0(U \cap \mathcal{X}_p, \mathcal{O}_{U \cap \mathcal{X}_p})$ , where  $\Phi_{j,p} = \Phi_j|_{U \cap \mathcal{X}_p}$ . As  $\tau_{0,p}$  is nowhere zero on  $U$ , we have

$$\text{Sing}(\text{div}\sigma) \cap (U - \text{div}\tau_{j,p}) = \text{Sing}(\text{div}\Phi_{j,p}(\sigma)) \cap (U - \text{div}\tau_{j,p})$$

and hence

$$\text{Sing}(\text{div}\sigma) \cap U \subset \bigcup_{j=1}^k \text{Sing}(\text{div}\Phi_{j,p}(\sigma)).$$

Since

$$\partial_i[\Phi_{j,p}(\beta_i)^p t_i \Phi_{j,p}(\tau_{0,p})^{l_d}] = \Phi_{j,p}(\beta_i)^p \Phi_{j,p}(\tau_{0,p})^{l_d} + l_d \Phi_{j,p}(\beta_i)^p t_i \Phi_{j,p}(\tau_{0,p})^{l_d-1} \cdot \partial_i \Phi_{j,p}(\tau_{0,p}),$$

and for any  $i' \neq i$ ,

$$\partial_i[\Phi_{j,p}(\beta_{i'})^p t_{i'} \Phi_{j,p}(\tau_{0,p})^{l_d}] = l_d \Phi_{j,p}(\beta_{i'})^p t_{i'} \Phi_{j,p}(\tau_{0,p})^{l_d-1} \cdot \partial_i \Phi_{j,p}(\tau_{0,p}),$$

we have

$$\begin{aligned} \partial_i \Phi_{j,p}(\sigma) &= \partial_i \Phi_{j,p}(\sigma_0) + \sum_{i'=1}^{n-1} \partial_i [\Phi_{j,p}(\beta_{i'})^p t_{i'} \Phi_{j,p}(\tau_{0,p})^{l_d}] + \partial_i [\Phi_{j,p}(\gamma)^p \Phi_{j,p}(\tau_{0,p})^{l_d}] \\ &= \left[ \sum_{i'=1}^{n-1} l_d \Phi_{j,p}(\beta_{i'})^p t_{i'} \Phi_{j,p}(\tau_{0,p})^{l_d-1} + l_d \Phi_{j,p}(\gamma)^p \Phi_{j,p}(\tau_{0,p})^{l_d-1} \right] \cdot \partial_i \Phi_{j,p}(\tau_{0,p}) \\ &\quad + \partial_i \Phi_{j,p}(\sigma_0) + \Phi_{j,p}(\beta_i)^p \Phi_{j,p}(\tau_{0,p})^{l_d} \\ &= \partial_i \Phi_{j,p}(\sigma_0) + \frac{l_d(\Phi_{j,p}(\sigma) - \Phi_{j,p}(\sigma_0))}{\Phi_{j,p}(\tau_{0,p})} \partial_i \Phi_{j,p}(\tau_{0,p}) + \Phi_{j,p}(\beta_i)^p \Phi_{j,p}(\tau_{0,p})^{l_d}. \end{aligned}$$

Now set

$$g_{p,j,i}(\sigma_0, \beta_i) = \partial_i \Phi_{j,p}(\sigma_0) - \frac{l_d \Phi_{j,p}(\sigma_0)}{\Phi_{j,p}(\tau_{0,p})} \partial_i \Phi_{j,p}(\tau_{0,p}) + \Phi_{j,p}(\beta_i)^p \Phi_{j,p}(\tau_{0,p})^{l_d},$$

and

$$W_{p,j,i} := \mathcal{X}_p \cap U \cap \{g_{p,j,1} = \dots = g_{p,j,i} = 0\}.$$

Then for any  $\sigma = \sigma_0 + \sum_{i=1}^{n-1} \beta_i^p t_i \tau_{0,p}^{l_d} + \gamma^p \tau_{0,p}^{l_d}$ , comparing the expressions of  $g_{p,j,i}$  and  $\partial_i \Phi_{j,p}$  we have

$$g_{p,j,i}(\sigma, \beta_i) = \partial_i \Phi_{j,p}(\sigma) - \frac{l_d \Phi_{j,p}(\sigma)}{\Phi_{j,p}(\tau_{0,p})} \partial_i \Phi_{j,p}(\tau_{0,p}),$$

and hence

$$g_{p,j,i}(\sigma_0, \beta_i)|_{\text{div } \Phi_{j,p}(\sigma)} = \partial_i \Phi_{j,p}(\sigma)|_{\text{div } \Phi_{j,p}(\sigma)}.$$

Moreover, any section  $g_{p,j,i}(\sigma_0, \beta_i) \cdot \tau_{0,p}^{d+\delta} \in \mathbb{H}^0(\mathcal{X}_p \cap U, \mathcal{L}^{\otimes(N_0+1)(d+\delta)})$  can be extended to a global section in  $\mathbb{H}^0(\mathcal{X}_p, \mathcal{L}^{\otimes(N_0+1)(d+\delta)})$  for any  $\delta \geq N_1 + 1$ . In fact, we know already that the section  $\partial_i \Phi_{j,p}(\sigma) \cdot \tau_{0,p}^{d+\delta} \in \mathbb{H}^0(\mathcal{X}_p \cap U, \mathcal{L}^{\otimes(N_0+1)(d+\delta)})$  can be extended to a global section in  $\mathbb{H}^0(\mathcal{X}_p, \mathcal{L}^{\otimes(N_0+1)(d+\delta)})$  for any  $\delta \geq N_1$ . On  $\mathcal{X}_p \cap U$  we have

$$\begin{aligned} \frac{l_d \Phi_{j,p}(\sigma)}{\Phi_{j,p}(\tau_{0,p})} &= l_d \frac{\sigma \cdot \tau_{j,p}^d}{\tau_{0,p}^d} \cdot \left( \frac{\tau \cdot \tau_{j,p}^{N_0+1}}{\tau_{0,p}^{N_0+1}} \right)^{-1} \\ &= l_d \frac{\sigma \cdot \tau_{j,p}^{d-N_0-1}}{\tau_{0,p}^{d-N_0}}. \end{aligned}$$

By Lemma 3.3.10, the section  $\partial_i \Phi_{j,p}(\tau_{0,p}) \cdot \tau_{0,p}^{(N_0+1)+\delta}$  can be extended to a global section in  $\mathbb{H}^0(\mathcal{X}_p, \mathcal{L}^{\otimes(N_0+1)((N_0+1)+\delta)})$  for any  $\delta \geq N_1$ , so the section

$$\frac{l_d \Phi_{j,p}(\sigma)}{\Phi_{j,p}(\tau_{0,p})} \partial_i \Phi_{j,p}(\tau_{0,p}) \cdot \tau_{0,p}^{d+\delta} = \left( \frac{l_d \Phi_{j,p}(\sigma)}{\Phi_{j,p}(\tau_{0,p})} \cdot \tau_{0,p}^{d-N_0} \right) \cdot \left( \partial_i \Phi_{j,p}(\tau_{0,p}) \cdot \tau_{0,p}^{(N_0+1)+(\delta-1)} \right)$$

extends to a global section of  $\mathcal{L}^{\otimes(N_0+1)(d+\delta)}$  for any  $\delta \geq N_1 + 1$ . Therefore the section

$$g_{j,i}(\sigma_0, \beta_i) \cdot \tau^{d+\delta} = \left( \partial_i \Phi_j(\sigma) - \frac{l_d \Phi_j(\sigma)}{\Phi_j(\tau)} \partial_i \Phi_j(\tau) \right) \cdot \tau^{d+\delta} \in \mathbb{H}^0(U, \mathcal{L}^{\otimes(N_0+1)(d+\delta)})$$

can be extended to a global section in  $\mathbb{H}^0(Y, \mathcal{L}^{\otimes(N_0+1)(d+\delta)})$  for any  $\delta \geq N_1 + 1$ .

**Lemma 6.4.7.** *When  $d$  is sufficiently large, the proportion of*

$$(\sigma_0, (\beta_1, \dots, \beta_{n-1}), \gamma) \in \mathbb{H}^0(\mathcal{X}_p, \mathcal{L}^{\otimes d}) \times \left( \prod_{i=1}^{n-1} \mathbb{H}^0(\mathcal{X}_p, \mathcal{L}^{\otimes k_{p,d}}) \right) \times \mathbb{H}^0(\mathcal{X}_p, \mathcal{L}^{k_{p,d}})$$

such that for  $\sigma = \sigma_0 + \sum_{i=1}^{n-1} \beta_i^p t_i \tau_{0,p}^{l_d} + \gamma^p \tau_{0,p}^{l_d}$ ,

$$\text{div } \sigma \cap W_{p,n-1,j} \cap \left\{ x \in |\mathcal{X}_p| ; \deg x \geq \frac{d}{nN(p)} \right\} = \emptyset,$$

is

$$1 - O\left(d^{n-1} p^{-c_1 \frac{d}{p}}\right),$$

with a constant  $c_1$  depending only on  $N_0, N_1$  and the dimension  $n$ .

*Proof.* Apply Lemma 3.3.11 to the case  $Y = \mathcal{X}_p$  and  $X = U \cap \mathcal{X}_p$ . We obtain that for  $0 \leq i \leq n-2$ , with a fixed choice of  $\sigma_0, \beta_1, \dots, \beta_i$  such that  $\dim W_{p,i,j} \leq n-1-i$ , the proportion of  $\beta_{i+1}$  in  $\mathbb{H}^0(\mathcal{X}_p, \mathcal{L}^{\otimes k_{p,d}})$  such that  $\dim W_{p,i+1,j} \leq n-2-i$  is  $1 - O\left(d^i \cdot p^{2 - \frac{d}{(N_0+1)N_1 p}}\right)$ , where the constant involved depends only on the degree of  $\mathcal{X}_p$  when embedded in  $\mathbb{P}(\mathbb{H}^0(\mathcal{L}_p^{\otimes(N_0+1)})^\vee)$  (this degree is independent of  $p$ ), hence is independent of  $d, p$ . In particular, the proportion of  $(\sigma_0, (\beta_1, \dots, \beta_{n-1}))$  such that  $W_{p,n-1,j}$  is finite is

$$\prod_{i=0}^{n-2} \left( 1 - O\left(d^i \cdot p^{2 - \frac{d}{(N_0+1)N_1 p}}\right) \right) = 1 - O\left(d^{n-2} \cdot p^{2 - \frac{d}{(N_0+1)N_1 p}}\right).$$

And then Lemma 3.3.12 tells us that for fixed  $(\sigma_0, (\beta_1, \dots, \beta_{n-1}))$  making  $W_{p,n-1,j}$  finite, the proportion of  $\gamma \in H^0(\mathcal{X}_p, \mathcal{L}^{\otimes k_{p,d}})$  such that for  $\sigma = \sigma_0 + \sum_{i=1}^{n-1} \beta_i^p t_i \tau_{0,p}^{l_d} + \gamma^p \tau_{0,p}^{l_d}$ ,

$$\operatorname{div} \sigma \cap W_{p,n-1,j} \cap \left\{ x \in |\mathcal{X}_p| ; \deg x \geq \frac{d}{nN(p)} \right\} = \emptyset,$$

is

$$1 - O(d^{n-1} p^{-\frac{d}{nN(p)}}),$$

where the constant involved is independent of  $d, p$ .

Therefore for large enough  $d$ , the proportion of

$$(\sigma_0, (\beta_1, \dots, \beta_{n-1}), \gamma) \in H^0(\mathcal{X}_p, \mathcal{L}^{\otimes d}) \times \left( \prod_{i=1}^{n-1} H^0(\mathcal{X}_p, \mathcal{L}^{\otimes k_{p,d}}) \right) \times H^0(\mathcal{X}_p, \mathcal{L}^{k_{p,d}})$$

such that

$$\operatorname{div} \sigma \cap W_{p,n-1,j} \cap \left\{ x \in |\mathcal{X}_p| ; \deg x \geq \frac{d}{nN(p)} \right\} = \emptyset,$$

where  $\sigma = \sigma_0 + \sum_{i=1}^{n-1} \beta_i^p t_i \tau_{0,p}^{l_d} + \gamma^p \tau_{0,p}^{l_d}$ , is

$$\begin{aligned} & \left( \prod_{i=0}^{n-2} \left( 1 - O(d^i \cdot p^{2 - \frac{d}{(N_0+1)N_1 p}}) \right) \right) \cdot \left( 1 - O(d^{n-1} p^{-\frac{d}{nN(p)}}) \right) \\ &= \left( 1 - O(d^{n-2} \cdot p^{2 - \frac{d}{(N_0+1)N_1 p}}) \right) \cdot \left( 1 - O(d^{n-1} p^{-\frac{d}{nN(p)}}) \right) \\ &= \left( 1 - O(d^{n-2} \cdot p^{2 - \frac{d}{(N_0+1)N_1 p}}) \right) \cdot \left( 1 - O(d^{n-1} p^{-\frac{d}{nN(p)}}) \right) \\ &= 1 - O \left( \max \left( d^{n-2} p^{2 - \frac{d}{(N_0+1)N_1 p}}, d^{n-1} p^{-\frac{d}{nN(p)}} \right) \right), \end{aligned}$$

As  $N(p) = p(N_0 + 2) + (N_0 + 1)(N_1 - 1)$ , when  $d$  is sufficiently large, we have

$$\begin{aligned} \frac{d}{nN(p)} &= \frac{d}{n[p(N_0 + 2) + (N_0 + 1)(N_1 - 1)]} \\ &\geq \frac{d}{n[(N_0 + 2)(N_1 + p - 1)]} \\ &\geq \frac{d}{n(N_0 + 2)N_1 p}, \end{aligned}$$

and

$$\frac{d}{(N_0 + 1)N_1 p} - 2 \geq \frac{d}{2n(N_0 + 2)N_1 p}.$$

Therefore when  $d$  tends to infinity,

$$\begin{aligned} & \left( \prod_{i=0}^{n-2} \left( 1 - O(d^i \cdot p^{2 - \frac{d}{(N_0+1)N_1 p}}) \right) \right) \cdot \left( 1 - O(d^{n-1} p^{-\frac{d}{nN(p)}}) \right) \\ &= 1 - O \left( \max \left( d^{n-2} p^{2 - \frac{d}{(N_0+1)N_1 p}}, d^{n-1} p^{-\frac{d}{nN(p)}} \right) \right) \\ &= 1 - O \left( d^{n-1} p^{-\frac{d}{2n(N_0+2)N_1 p}} \right) \\ &= 1 - O \left( d^{n-1} p^{-c_1 \frac{d}{p}} \right) \end{aligned}$$

with constant  $c_1 = \frac{1}{2n(N_0+2)N_1}$  which then depending only on  $N_0, N_1$  and the dimension  $n$ .  $\square$

On the other hand, for such  $\sigma = \sigma_0 + \sum_{i=1}^{n-1} \beta_i^p t_i \tau_{0,p}^{l_d} + \gamma^p \tau_{0,p}^{l_d} \in \mathbf{H}^0(\mathcal{X}_p, \mathcal{L}^{\otimes d})$ , we have

$$\text{Sing}(\text{div}\sigma) \cap U \subset \bigcup_{j=1}^k \text{Sing}(\text{div}\Phi_{j,p}(\sigma))$$

and

$$\begin{aligned} \text{Sing}(\text{div}\Phi_{j,p}(\sigma)) &= \text{div}\Phi_{j,p}(\sigma) \cap \{\partial_1 \Phi_{j,p}(\sigma) = \cdots = \partial_{n-1} \Phi_{j,p}(\sigma) = 0\} \\ &= \text{div}\Phi_{j,p}(\sigma) \cap \{g_{p,j,1}(\sigma_0, \beta_1) = \cdots = g_{p,j,n-1}(\sigma_0, \beta_{n-1}) = 0\} \\ &= \text{div}\Phi_{j,p}(\sigma) \cap W_{p,n-1,j} \\ &= \text{div}\sigma \cap W_{p,n-1,j} \end{aligned}$$

as

$$g_{p,j,i}(\sigma_0, \beta_i)|_{\text{div}\Phi_{j,p}(\sigma)} = \partial_i \Phi_{j,p}(\sigma)|_{\text{div}\Phi_{j,p}(\sigma)}.$$

Since the homomorphism of groups

$$\mathbf{H}^0(\mathcal{X}_p, \mathcal{L}^{\otimes d}) \times \left( \prod_{i=1}^{n-1} \mathbf{H}^0(\mathcal{X}_p, \mathcal{L}^{\otimes k_{p,d}}) \right) \times \mathbf{H}^0(\mathcal{X}_p, \mathcal{L}^{k_{p,d}}) \longrightarrow \mathbf{H}^0(\mathcal{X}_p, \mathcal{L}^{\otimes d})$$

sending  $(\sigma_0, (\beta_1, \dots, \beta_{n-1}), \gamma)$  to  $\sigma = \sigma_0 + \sum_{i=1}^{n-1} \beta_i^p t_i \tau_{0,p}^{l_d} + \gamma^p \tau_{0,p}^{l_d}$  is surjective, Lemma 6.4.7 implies that

$$\frac{\#\left\{ \sigma \in \mathbf{H}^0(\mathcal{X}_p, \mathcal{L}^{\otimes d}) ; \text{Sing}(\text{div}\sigma) \cap U \cap \left\{ x \in |\mathcal{X}_p| ; \deg x \geq \frac{d}{nN(p)} \right\} = \emptyset \right\}}{\#\mathbf{H}^0(\mathcal{X}_p, \mathcal{L}^{\otimes d})} = 1 - O\left(d^{n-1} p^{-c_1 \frac{d}{p}}\right).$$

which means that the proportion of  $\sigma \in \mathbf{H}^0(\mathcal{X}_p, \mathcal{L}^{\otimes d})$  such that  $\text{div}\sigma$  has no singular point of degree strictly larger than  $\frac{d}{nN(p)}$ , that is, elements not contained in  $\mathcal{Q}_{d,p,U}^{\text{high}}$ , is  $1 - O\left(d^{n-1} p^{-c_1 \frac{d}{p}}\right)$  with a constant  $c_1$  depending only on  $N_0, N_1$  and  $n$ . We therefore conclude that

$$\frac{\#\mathcal{Q}_{d,p,U}^{\text{high}}}{\#\mathbf{H}^0(\mathcal{X}_p, \mathcal{L}^{\otimes d})} = O\left(d^{n-1} p^{-c_1 \frac{d}{p}}\right).$$

with a possibly smaller  $c_1$ .

Now that we have proved Lemma 6.4.6 except for the only prime number  $p_0$  dividing  $N_0 + 1$ , we can run the same process with another constant  $N'_0 > N_0$  such that  $(N'_0 + 1, N_0 + 1) = 1$ . We get a control for the proportion of  $\mathcal{Q}_{d,p_0,U}^{\text{high}}$  with different constants as we have

$$\mathcal{Q}_{d,p_0,U}^{\text{high}} \subset \left\{ \sigma' \in \mathbf{H}^0(\mathcal{X}_{p_0^2}, \overline{\mathcal{L}}^{\otimes d}) ; \exists x \in |\text{div}\sigma'| \text{ of degree } \geq \frac{d}{nN'(p_0)}, \dim_{\kappa(x)} \frac{\mathfrak{m}_{\text{div}\sigma',x}}{\mathfrak{m}_{\text{div}\sigma',x}^2} = n \right\},$$

where

$$N'(p_0) = (N'_0 + 1)(N_1 + p_0 - 1) + p_0 > (N_0 + 1)(N_1 + p_0 - 1) + p_0 = N(p_0).$$

So by modifying the constant  $c_1$  and the constant involved in the big  $O$ , this case can be included in the uniform control. Therefore we proved Lemma 6.4.6.  $\square$



Now we proceed to prove the Proposition 6.4.3.

*Proof of Proposition 6.4.3.* The main problem for controlling the proportion of  $\mathcal{Q}_{d,p^2}^{\text{high}}$  for  $p \notin \mathcal{S}$  is that  $\mathcal{X}_p$  over  $p$  might be singular. We decompose  $\mathcal{X}_p$  into regular and singular part :

$$\mathcal{X}_p = U_p \cup Z_p$$

where  $Z_p = \text{Sing}(\mathcal{X}_p)$  is the singular locus of  $\mathcal{X}_p$  and  $U_p = \mathcal{X}_p - Z_p$ . As  $\mathcal{X}$  is regular, for a closed point  $x$  in  $\mathcal{X}_p$ , if  $x \in U_p$ ,  $\dim_{\kappa(x)} \mathfrak{m}_{\mathcal{X}_p,x} / \mathfrak{m}_{\mathcal{X}_p,x}^2 = n - 1$ ; if  $x \in Z_p$ ,  $\dim_{\kappa(x)} \mathfrak{m}_{\mathcal{X}_p,x} / \mathfrak{m}_{\mathcal{X}_p,x}^2 = \dim_{\kappa(x)} \mathfrak{m}_{\mathcal{X},x} / \mathfrak{m}_{\mathcal{X},x}^2 = n$ . Set

$$\begin{aligned} \mathcal{Q}_{d,U_p}^{\text{high}} &= \left\{ \sigma \in \text{H}^0(\mathcal{X}_p, \mathcal{L}^{\otimes d}) ; \exists x \in |\text{div} \sigma \cap U_p| \text{ of degree } \geq \frac{d}{nN(p)}, \dim_{\kappa(x)} \frac{\mathfrak{m}_{\text{div} \sigma, x}}{\mathfrak{m}_{\text{div} \sigma, x}^2} = n - 1 \right\}, \\ \mathcal{Q}_{d,Z_p}^{\text{high}} &= \left\{ \sigma \in \text{H}^0(\mathcal{X}_p, \mathcal{L}^{\otimes d}) ; \exists x \in |\text{div} \sigma \cap Z_p| \text{ of degree } \geq \frac{d}{nN(p)}, \dim_{\kappa(x)} \frac{\mathfrak{m}_{\text{div} \sigma, x}}{\mathfrak{m}_{\text{div} \sigma, x}^2} = n \right\}. \end{aligned}$$

For a section  $\sigma \in \text{H}^0(\mathcal{X}_{p^2}, \mathcal{L}^{\otimes d})$ , assume that  $\text{div} \sigma$  contains a closed point  $x$  with  $\dim_{\kappa(x)} \frac{\mathfrak{m}_{\text{div} \sigma, x}}{\mathfrak{m}_{\text{div} \sigma, x}^2} = n$ . Let  $\bar{\sigma} = \sigma \bmod p$  be its image in  $\text{H}^0(\mathcal{X}_p, \mathcal{L}^{\otimes d})$ . Then if  $x \in U_p$ ,  $x$  is also a singular point of  $\text{div} \bar{\sigma} \cap U_p$ , i.e.  $\dim_{\kappa(x)} \frac{\mathfrak{m}_{\text{div} \bar{\sigma}, x}}{\mathfrak{m}_{\text{div} \bar{\sigma}, x}^2} = n - 1$ ; if  $x \in Z_p$ , we have then  $\dim_{\kappa(x)} \frac{\mathfrak{m}_{\text{div} \bar{\sigma}, x}}{\mathfrak{m}_{\text{div} \bar{\sigma}, x}^2} = n$ . So we have

$$\{\bar{\sigma} ; \sigma \in \mathcal{Q}_{d,p^2}^{\text{high}}\} \subset \mathcal{Q}_{d,U_p}^{\text{high}} \cup \mathcal{Q}_{d,Z_p}^{\text{high}},$$

hence

$$\frac{\#\mathcal{Q}_{d,p^2}^{\text{high}}}{\#\text{H}^0(\mathcal{X}_{p^2}, \mathcal{L}^{\otimes d})} \leq \frac{\#\mathcal{Q}_{d,U_p}^{\text{high}}}{\#\text{H}^0(\mathcal{X}_p, \mathcal{L}^{\otimes d})} + \frac{\#\mathcal{Q}_{d,Z_p}^{\text{high}}}{\#\text{H}^0(\mathcal{X}_p, \mathcal{L}^{\otimes d})}$$

We can bound the first term  $\frac{\#\mathcal{Q}_{d,U_p}^{\text{high}}}{\#\text{H}^0(\mathcal{X}_p, \mathcal{L}^{\otimes d})}$  by exactly the same method as in the proof of Lemma 6.4.6. The second term can be bounded by a slightly different way.

As now  $(\Omega_{\mathcal{X}_p/\mathbb{F}_p}^1)|_{Z_p}$  is locally free of rank  $n$ , we cover an open neighbourhood of  $Z_p$  by open subschemes  $V_{Z_p, \alpha}$  where we can find  $t_{\alpha,1}, \dots, t_{\alpha,n} \in \text{H}^0(V_{Z_p, \alpha}, \mathcal{O}_{V_{Z_p, \alpha}})$  such that the image  $\overline{dt}_i$  of  $dt_i$  in  $(\Omega_{V_{Z_p, \alpha}/\mathbb{F}_p}^1)|_{V_{Z_p, \alpha} \cap Z_p}$  satisfies

$$(\Omega_{V_{Z_p, \alpha}/\mathbb{F}_p}^1)|_{V_{Z_p, \alpha} \cap Z_p} \simeq \bigoplus_{i=1}^n \mathcal{O}_{V_{Z_p, \alpha} \cap Z_p} \overline{dt}_i.$$

Then choosing convenient constants  $N'_0 \geq N_0$ ,  $N'_1 \geq N_1$  and setting

$$N'(p) = (N'_0 + 1)(N'_1 + p - 1) + p,$$

the same process as in the proof of Lemma 6.4.6 gives us that the proportion of  $\mathcal{Q}_{d,Z_p}^{\text{high}}$ , being a subset of

$$\left\{ \sigma \in \text{H}^0(\mathcal{X}_p, \overline{\mathcal{L}}^{\otimes d}) ; \exists x \in |\text{div} \sigma \cap Z_p| \text{ of degree } \geq \frac{d}{nN'(p)}, \dim_{\kappa(x)} \frac{\mathfrak{m}_{\text{div} \sigma, x}}{\mathfrak{m}_{\text{div} \sigma, x}^2} = n \right\}$$

is bounded by

$$\begin{aligned}
& 1 - \left( \prod_{i=0}^n (1 - O(d^i \cdot p^{2 - \frac{d}{(N_0+1)N_1^i p}})) \right) \cdot \left( 1 - O(d^n p^{-\frac{d}{nN_1^i(p)}}) \right) \\
&= 1 - \left( 1 - O(d^n \cdot p^{2 - \frac{d}{(N_0+1)N_1^i p}}) \right) \cdot \left( 1 - O(d^n p^{-\frac{d}{nN_1^i(p)}}) \right) \\
&= O\left(d^n p^{-c_1' \frac{d}{p}}\right).
\end{aligned}$$

Thus Proposition 6.4.3 is proved.  $\square$

## 6.5 Proof of Proposition 6.1.2

*Proof.* As in the previous section, let  $N_0$  be an integer verifying the following conditions :

- 1)  $N_0 + 1$  is a power of a prime number ;
- 2) for any  $d \geq N_0$ ,  $(\mathcal{L}|_{\mathcal{X}_S})^{\otimes d}$  is relatively very ample ;
- 3) for any  $a, b \geq N_0$ , we have a surjective morphism

$$\mathrm{H}^0(\mathcal{X}_S, \mathcal{L}^{\otimes a}) \otimes_{\mathrm{H}^0(S, \mathcal{O}_S)} \mathrm{H}^0(\mathcal{X}_S, \mathcal{L}^{\otimes b}) \longrightarrow \mathrm{H}^0(\mathcal{X}_S, \mathcal{L}^{\otimes(a+b)}).$$

Let  $N_1$  be as in Proposition 6.4.1. For each  $p \in \mathcal{S}$ , take

$$N(p) = (N_0 + 1)(N_1 + p - 1) + p = p(N_0 + 2) + (N_0 + 1)(N_1 - 1).$$

In particular,  $N(p)$  also satisfies the conditions (2) and (3) above. By Lemma 6.2.2, for any positive integer  $r$  which satisfies  $2c_0 N(p) n r p^{(n-1)r} \leq d$ , we have

$$\frac{\#\mathcal{P}'_{d, p^2, \leq r}}{\#\mathrm{H}^0(\mathcal{X}_p, \mathcal{L}^{\otimes d})} = \prod_{x \in |\mathcal{X}_p|, \deg x \leq r} \left( 1 - p^{-(n+1) \deg x} \right).$$

Let  $r_{p,d}$  be the largest  $r$  verifying this condition. In order to have  $r_{p,d} \geq 1$ , we need

$$2c_0 N(p) n p^{n-1} \leq d.$$

Let  $C = 2c_0(N_0 + 3)$ . When  $d$  is larger than  $2c_0 n(N_0 + 3)(N_0 + 1)^n(N_1 - 1)^n$ , if  $p$  satisfies  $Cnp^n < d$ , then either  $p < (N_0 + 1)(N_1 - 1)$ , in which case we have

$$\begin{aligned}
2c_0 N(p) n p^{n-1} &= 2c_0 \left[ p(N_0 + 2) + (N_0 + 1)(N_1 - 1) \right] n p^{n-1} \\
&< 2c_0 \left[ (N_0 + 2)(N_0 + 1)(N_1 - 1) + (N_0 + 1)(N_1 - 1) \right] n p^{n-1} \\
&= 2c_0 (N_0 + 3)(N_0 + 1)(N_1 - 1) n p^{n-1} \\
&< 2c_0 n (N_0 + 3)(N_0 + 1)^n (N_1 - 1)^n \leq d;
\end{aligned}$$

or  $(N_0 + 1)(N_1 - 1) \leq p < Cnp^n < d$ , so that

$$\begin{aligned}
2c_0 N(p) n p^{n-1} &= 2c_0 \left[ p(N_0 + 2) + (N_0 + 1)(N_1 - 1) \right] n p^{n-1} \\
&\leq 2c_0 (N_0 + 3) p n p^{n-1} \\
&= Cnp^n < d.
\end{aligned}$$

So the above condition is satisfied, hence  $r_{p,d} \geq 1$ .

Since

$$\mathcal{P}'_{d,p^2} \subset \mathcal{P}'_{d,p^2, \leq r_{p,d}} \subset \mathcal{P}'_{d,p^2} \cup \mathcal{Q}_{d,p^2, r_{p,d}}^{\text{med}} \cup \mathcal{Q}_{d,p^2}^{\text{high}},$$

we have

$$\begin{aligned} & \left| \frac{\#\mathcal{P}'_{d,p^2}}{\#\mathbb{H}^0(\mathcal{X}_{p^2}, \mathcal{L}^{\otimes d})} - \zeta_{\mathcal{X}_p}(n+1)^{-1} \right| \\ & \leq \left| \frac{\#\mathcal{P}'_{d,p^2}}{\#\mathbb{H}^0(\mathcal{X}_{p^2}, \mathcal{L}^{\otimes d})} - \frac{\#\mathcal{P}'_{d,p^2, \leq r_{p,d}}}{\#\mathbb{H}^0(\mathcal{X}_{p^2}, \mathcal{L}^{\otimes d})} \right| + \left| \frac{\#\mathcal{P}'_{d,p^2, \leq r_{p,d}}}{\#\mathbb{H}^0(\mathcal{X}_{p^2}, \mathcal{L}^{\otimes d})} - \zeta_{\mathcal{X}_p}(n+1)^{-1} \right| \\ & \leq \frac{\#\mathcal{Q}_{d,p^2, r_{p,d}}^{\text{med}}}{\#\mathbb{H}^0(\mathcal{X}_{p^2}, \mathcal{L}^{\otimes d})} + \frac{\#\mathcal{Q}_{d,p^2}^{\text{high}}}{\#\mathbb{H}^0(\mathcal{X}_{p^2}, \mathcal{L}^{\otimes d})} + \left| \zeta_{\mathcal{X}_p}(n+1)^{-1} - \prod_{x \in |\mathcal{X}_{p^2}|, \deg x \leq r_{p,d}} (1 - p^{-(n+1) \deg x}) \right|. \end{aligned}$$

By Lemma 6.3.1,

$$\frac{\#\mathcal{Q}_{d,p^2, r_{p,d}}^{\text{med}}}{\#\mathbb{H}^0(\mathcal{X}_{p^2}, \mathcal{L}^{\otimes d})} < 2c_0 p^{-2(r_{p,d}+1)}$$

By the choice of  $r_{p,d}$ , we have

$$2c_0 N(p) n(r_{p,d}+1) \cdot p^{(n-1)(r_{p,d}+1)} > d.$$

So

$$\begin{aligned} p^{-(r_{p,d}+1)} &= \left( p^{n(r_{p,d}+1)} \right)^{-\frac{1}{n}} \\ &< \left( (r_{p,d}+1) p^{(n-1)(r_{p,d}+1)} \right)^{-\frac{1}{n}} \\ &< \left( \frac{d}{2c_0 N(p)} \right)^{-\frac{1}{n}} \\ &= O\left( \left( \frac{d}{p} \right)^{-\frac{1}{n}} \right). \end{aligned}$$

Therefore we have

$$\frac{\#\mathcal{Q}_{d,p^2, r_{p,d}}^{\text{med}}}{\#\mathbb{H}^0(\mathcal{X}_{p^2}, \mathcal{L}^{\otimes d})} = O\left( \left( \frac{d}{p} \right)^{-\frac{2}{n}} \right),$$

where the coefficient involved in is independent of  $d, p$ .

By Proposition 6.4.1, we have

$$\frac{\#\mathcal{Q}_{d,p^2}^{\text{high}}}{\#\mathbb{H}^0(\mathcal{X}_{p^2}, \mathcal{L}^{\otimes d})} = O\left( d^n p^{-c_1 \frac{d}{p}} \right),$$

where again the coefficient involved in is independent of  $d, p$ .

Note that Lemma 5.0.2 shows

$$\left| \prod_{x \in |\mathcal{X}_p|, \deg x \leq r_{p,d}} (1 - p^{-(n+1) \deg x}) - \zeta_{\mathcal{X}_p}(n+1)^{-1} \right| \leq 4c_0 p^{-2(r_{p,d}+1)} = O\left( \left( \frac{d}{p} \right)^{-\frac{2}{n}} \right).$$

Finally, by putting together all these three inequalities, we get

$$\begin{aligned}
& \left| \frac{\#\mathcal{P}'_{d,p^2}}{\#\mathbf{H}^0(\mathcal{X}_{p^2}, \mathcal{L}^{\otimes d})} - \zeta_{\mathcal{X}_p}(n+1)^{-1} \right| \\
& \leq \frac{\#\mathcal{Q}_{d,p^2,r_p,d}^{\text{med}}}{\#\mathbf{H}^0(\mathcal{X}_{p^2}, \mathcal{L}^{\otimes d})} + \frac{\#\mathcal{Q}_{d,p^2}^{\text{high}}}{\#\mathbf{H}^0(\mathcal{X}_{p^2}, \mathcal{L}^{\otimes d})} + \left| \zeta_{\mathcal{X}_p}(n+1)^{-1} - \prod_{x \in |\mathcal{X}_{p^2}|, \deg x \leq r_p, d} (1 - p^{-(n+1) \deg x}) \right| \\
& = O\left(\left(\frac{d}{p}\right)^{-\frac{2}{n}}\right) + O\left(d^n p^{-c_1 \frac{d}{p}}\right) + O\left(\left(\frac{d}{p}\right)^{-\frac{2}{n}}\right) \\
& = O\left(\left(\frac{d}{p}\right)^{-\frac{2}{n}}\right)
\end{aligned}$$

where the coefficient involved in is independent of  $d, p$ , which is what we need to show.  $\square$



# Chapitre 7

## Singular points of small residual characteristic

In this chapter, we will show the following result :

**Proposition 7.0.1.** *Let  $\mathcal{X}$  be a regular projective arithmetic variety of absolute dimension  $n$ , and let  $\bar{\mathcal{L}}$  be an ample Hermitian line bundle on  $\mathcal{X}$ . Set*

$$\mathcal{P}_{d,p \leq d^{\frac{1}{n+1}}} := \left\{ \sigma \in H_{\text{Ar}}^0(\mathcal{X}, \bar{\mathcal{L}}^{\otimes d}) ; \begin{array}{l} \text{div } \sigma \text{ has no singular point of residual} \\ \text{characteristic smaller than } d^{\frac{1}{n+1}} \end{array} \right\}.$$

When  $d$  is sufficiently large, we have

$$\left| \frac{\#\mathcal{P}_{d,p \leq d^{\frac{1}{n+1}}}}{\#H_{\text{Ar}}^0(\mathcal{X}, \bar{\mathcal{L}}^{\otimes d})} - \zeta_{\mathcal{X}}(n+1)^{-1} \right| = O(d^{-\frac{1}{n+1}}).$$

Here the constant involved in the big  $O$  depends only on  $\mathcal{X}$ .

In particular, denoting  $\mathcal{P}_B = \bigcup_{d>0} \mathcal{P}_{d,p \leq d^{\frac{1}{n+1}}}$ , we have

$$\mu(\mathcal{P}_B) = \zeta_{\mathcal{X}}(n+1)^{-1}.$$

### 7.1 Union of a finite number of fibers

Let  $p, q$  be two different prime numbers. Then we have

$$\mathcal{X}_{p^2q^2} = \mathcal{X} \times_{\text{Spec } \mathbb{Z}} \text{Spec}(\mathbb{Z}/p^2q^2\mathbb{Z}) \simeq \mathcal{X}_{p^2} \amalg \mathcal{X}_{q^2}.$$

For any  $d \geq 0$ , we have an isomorphism

$$\lambda_{p^2q^2} : H^0(\mathcal{X}_{p^2q^2}, \bar{\mathcal{L}}^{\otimes d}) \xrightarrow{\sim} H^0(\mathcal{X}_{p^2}, \bar{\mathcal{L}}^{\otimes d}) \times H^0(\mathcal{X}_{q^2}, \bar{\mathcal{L}}^{\otimes d}).$$

For any  $\sigma \in H_{\text{Ar}}^0(\mathcal{X}, \bar{\mathcal{L}}^{\otimes d})$ ,  $\sigma \in \mathcal{P}_{d,p} \cap \mathcal{P}_{d,q}$  if and only if the restriction map

$$\psi_{d,p^2q^2} : H_{\text{Ar}}^0(\mathcal{X}, \bar{\mathcal{L}}^{\otimes d}) \longrightarrow H^0(\mathcal{X}_{p^2q^2}, \bar{\mathcal{L}}^{\otimes d})$$

sends  $\sigma$  to an element in the set  $\lambda_{p^2q^2}^{-1}(\mathcal{P}'_{d,p^2} \times \mathcal{P}'_{d,q^2})$ . Therefore we have

$$\lim_{d \rightarrow \infty} \frac{\#(\mathcal{P}_{d,p} \cap \mathcal{P}_{d,q})}{\#\mathbb{H}_{\text{Ar}}^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes d})} = \zeta_{\mathcal{X}_p}(n+1)^{-1} \zeta_{\mathcal{X}_q}(n+1)^{-1}.$$

More generally, for any finite set  $I$  of prime numbers  $p$ , we have

$$\lim_{d \rightarrow \infty} \frac{\#(\bigcap_{p \in I} \mathcal{P}_{d,p})}{\#\mathbb{H}_{\text{Ar}}^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes d})} = \prod_{p \in I} \zeta_{\mathcal{X}_p}(n+1)^{-1}.$$

By Lemma 4.2.1, we may only consider  $d > 0$  such that for any positive integer  $N$ , we have

$$\mathbb{H}^0(\mathcal{X}_N, \overline{\mathcal{L}}^{\otimes d}) \simeq \mathbb{H}^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes d}) / (N \cdot \mathbb{H}^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes d})).$$

Fix a positive constant  $\frac{3}{4} < \alpha_0 < 1$ . By Proposition 4.2.2, when  $d$  is large enough, for any  $N < e^{d^{\alpha_0}}$ , the map

$$\psi_{d,N} : \mathbb{H}_{\text{Ar}}^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes d}) \longrightarrow \mathbb{H}^0(\mathcal{X}_N, \overline{\mathcal{L}}^{\otimes d})$$

is surjective and there exists a positive constant  $\eta$  with

$$\frac{|\#\psi_{d,N}^{-1}(\sigma) - \#\psi_{d,N}^{-1}(\sigma')|}{\#\psi_{d,N}^{-1}(\sigma)} \leq e^{-\eta d}.$$

for any two sections  $\sigma, \sigma'$  in  $\mathbb{H}^0(\mathcal{X}_N, \overline{\mathcal{L}}^{\otimes d})$ .

For a positive integer  $r$ , take  $N_r = \prod_{p \leq r} p^2$ .

**Lemma 7.1.1.** *Let  $C$  be the constant in Theorem 6.1.1. For any large enough integer  $d$ , and for any integer  $r$  verifying  $Cnr^n < d$  with  $n = \dim \mathcal{X}$  and  $N_r < e^{d^{\alpha_0}}$ , we have*

$$\left| \frac{\# \left( \bigcap_{p \leq r} \mathcal{P}_{d,p} \right)}{\#\mathbb{H}_{\text{Ar}}^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes d})} - \prod_{p \leq r} \zeta_{\mathcal{X}_p}(n+1)^{-1} \right| = O \left( \frac{\sum_{p \leq r} p^{\frac{2}{n}}}{d^{\frac{2}{n}}} \right).$$

*Proof.* The Chinese remainder theorem implies that

$$\begin{aligned} \mathbb{H}^0(\mathcal{X}_{N_r}, \overline{\mathcal{L}}^{\otimes d}) &\simeq \mathbb{H}^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes d}) \otimes_{\mathbb{Z}} \mathbb{Z}/N_r\mathbb{Z} \\ &\simeq \mathbb{H}^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes d}) \otimes_{\mathbb{Z}} \left( \prod_{p \leq r} \mathbb{Z}/p^2\mathbb{Z} \right) \\ &\simeq \prod_{p \leq r} \mathbb{H}^0(\mathcal{X}_{p^2}, \overline{\mathcal{L}}^{\otimes d}). \end{aligned}$$

Moreover we have  $\mathcal{X}_{N_r} = \coprod_{p \leq r} \mathcal{X}_{p^2}$ . Set

$$E_{d,r} := \left\{ \sigma \in \mathbb{H}^0(\mathcal{X}_{N_r}, \overline{\mathcal{L}}^{\otimes d}) ; \forall x \in |\text{div} \sigma|, \dim_{\kappa(x)} \frac{\mathfrak{m}_{\text{div} \sigma, x}}{\mathfrak{m}_{\text{div} \sigma, x}^2} = n-1 \right\}.$$

Then a section  $\sigma \in \mathbb{H}^0(\mathcal{X}_{N_r}, \overline{\mathcal{L}}^{\otimes d})$  is contained in  $E_{d,r}$  if and only if for any  $p \leq r$ , its restriction  $\sigma|_{\mathcal{X}_{p^2}}$  is contained in  $\mathcal{P}'_{d,p^2}$ . In particular, a section  $\sigma \in \mathbb{H}^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes d})$  satisfies  $\psi_{d,N_r}(\sigma) \in E_{d,r}$

if and only if  $\psi_{d,p^2}(\sigma) \in \mathcal{P}'_{d,p^2}$ , for all  $p \leq r$ , which means exactly that this  $\sigma$  is contained in  $\bigcap_{p \leq r} \mathcal{P}_{d,p}$ . On the other hand, still by the Chinese remainder theorem,

$$\frac{\#E_{d,r}}{\#\mathbb{H}^0(\mathcal{X}_{N_r}, \overline{\mathcal{L}}^{\otimes d})} = \prod_{p \leq r} \frac{\#\mathcal{P}'_{d,p^2}}{\#\mathbb{H}^0(\mathcal{X}_{p^2}, \mathcal{L}^{\otimes d})}.$$

As said above, with some positive constant  $\eta$ , we have

$$\frac{|\#\psi_{d,N_r}^{-1}(\sigma) - \#\psi_{d,N_r}^{-1}(\sigma')|}{\#\psi_{d,N_r}^{-1}(\sigma)} \leq e^{-\eta d},$$

with  $\sigma, \sigma'$  in  $\mathbb{H}^0(\mathcal{X}_{N_r}, \overline{\mathcal{L}}^{\otimes d})$ . Fixing one  $\sigma$ , we can sum up for all  $\sigma' \in E_{d,r}$  and get

$$\begin{aligned} & \left| \#E_{d,r} - \frac{\#\left(\bigcap_{p \leq r} \mathcal{P}_{d,p}\right)}{\#\psi_{d,N_r}^{-1}(\sigma)} \right| \\ &= \left| \frac{\left(\#E_{d,r} \cdot \#\psi_{d,N_r}^{-1}(\sigma)\right) - \#\psi_{d,N_r}^{-1}(E_{d,r})}{\#\psi_{d,N_r}^{-1}(\sigma)} \right| \\ &\leq \sum_{\sigma' \in E_{d,r}} \frac{|\#\psi_{d,N_r}^{-1}(\sigma) - \#\psi_{d,N_r}^{-1}(\sigma')|}{\#\psi_{d,N_r}^{-1}(\sigma)} \\ &\leq \#E_{d,r} \cdot e^{-\eta d}, \end{aligned}$$

where the last inequality follows from Proposition 4.2.2. We can also write this inequality as

$$\left| \#\bigcap_{p \leq r} \mathcal{P}_{d,p} - \left(\#E_{d,r} \cdot \#\psi_{d,N_r}^{-1}(\sigma)\right) \right| \leq \left(\#E_{d,r} \cdot \#\psi_{d,N_r}^{-1}(\sigma)\right) e^{-\eta d}.$$

Now we take the sum for all  $\sigma \in \mathbb{H}^0(\mathcal{X}_{N_r}, \overline{\mathcal{L}}^{\otimes d})$  and get

$$\begin{aligned} & \left| \left(\#\bigcap_{p \leq r} \mathcal{P}_{d,p}\right) \cdot \#\mathbb{H}^0(\mathcal{X}_{N_r}, \overline{\mathcal{L}}^{\otimes d}) - \left(\#E_{d,r} \cdot \#\mathbb{H}_{\text{Ar}}^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes d})\right) \right| \\ &= \left| \sum_{\sigma \in \mathbb{H}^0(\mathcal{X}_{N_r}, \overline{\mathcal{L}}^{\otimes d})} \left(\#\bigcap_{p \leq r} \mathcal{P}_{d,p} - \left(\#E_{d,r} \cdot \#\psi_{d,N_r}^{-1}(\sigma)\right)\right) \right| \\ &\leq \sum_{\sigma \in \mathbb{H}^0(\mathcal{X}_{N_r}, \overline{\mathcal{L}}^{\otimes d})} \left| \left(\#\bigcap_{p \leq r} \mathcal{P}_{d,p} - \left(\#E_{d,r} \cdot \#\psi_{d,N_r}^{-1}(\sigma)\right)\right) \right| \\ &\leq \sum_{\sigma \in \mathbb{H}^0(\mathcal{X}_{N_r}, \overline{\mathcal{L}}^{\otimes d})} \left(\#E_{d,r} \cdot \#\psi_{d,N_r}^{-1}(\sigma)\right) e^{-\eta d} \\ &= \left(\#E_{d,r} \cdot \#\mathbb{H}_{\text{Ar}}^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes d})\right) e^{-\eta d} \\ &\leq \left(\#\mathbb{H}^0(\mathcal{X}_{N_r}, \overline{\mathcal{L}}^{\otimes d}) \cdot \#\mathbb{H}_{\text{Ar}}^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes d})\right) e^{-\eta d}. \end{aligned}$$



Dividing both side by  $\#H^0(\mathcal{X}_{N_r}, \overline{\mathcal{L}}^{\otimes d}) \cdot \#H_{\text{Ar}}^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes d})$ , we get

$$\left| \frac{\#\bigcap_{p \leq r} \mathcal{P}_{d,p}}{\#H_{\text{Ar}}^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes d})} - \frac{\#E_{d,r}}{\#H^0(\mathcal{X}_{N_r}, \overline{\mathcal{L}}^{\otimes d})} \right| \leq e^{-\eta d}.$$

Since we already know that

$$\frac{\#E_{d,r}}{\#H^0(\mathcal{X}_{N_r}, \overline{\mathcal{L}}^{\otimes d})} = \prod_{p \leq r} \frac{\#\mathcal{P}'_{d,p^2}}{\#H^0(\mathcal{X}_{p^2}, \mathcal{L}^{\otimes d})},$$

the inequality can be written as

$$\left| \frac{\#\bigcap_{p \leq r} \mathcal{P}_{d,p}}{\#H_{\text{Ar}}^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes d})} - \prod_{p \leq r} \frac{\#\mathcal{P}'_{d,p^2}}{\#H^0(\mathcal{X}_{p^2}, \mathcal{L}^{\otimes d})} \right| \leq e^{-\eta d}.$$

Thus to finish the proof, it suffices to show the following lemma :

**Lemma 7.1.2.** *Under the condition of Lemma 7.1.1, we have*

$$\left| \prod_{p \leq r} \frac{\#\mathcal{P}'_{p^2,d}}{\#H^0(\mathcal{X}_{p^2}, \mathcal{L}^{\otimes d})} - \prod_{p \leq r} \zeta_{\mathcal{X}_p}(1+n)^{-1} \right| = O\left(\frac{\sum_{p \leq r} p^{\frac{2}{n}}}{d^{\frac{2}{n}}}\right).$$

*Proof.* By Proposition 6.1.2, for any prime number  $p$  verifying  $Cnp^n < d$ , we have

$$\frac{\#\mathcal{P}'_{d,p^2}}{\#H^0(\mathcal{X}_{p^2}, \mathcal{L}^{\otimes d})} = \zeta_{\mathcal{X}_p}(1+n)^{-1} + O\left(\left(\frac{d}{p}\right)^{-\frac{2}{n}}\right)$$

with the constant involved in big  $O$  independent of  $p$  and  $d$ . Therefore we can calculate the product as

$$\begin{aligned} \prod_{p \leq r} \frac{\#\mathcal{P}'_{d,p^2}}{\#H^0(\mathcal{X}_{p^2}, \mathcal{L}^{\otimes d})} &= \prod_{p \leq r} \left( \zeta_{\mathcal{X}_p}(1+n)^{-1} + O\left(\left(\frac{d}{p}\right)^{-\frac{2}{n}}\right) \right) \\ &= \prod_{p \leq r} \zeta_{\mathcal{X}_p}(1+n)^{-1} + \sum_{p \leq r} O\left(\left(\frac{p}{d}\right)^{\frac{2}{n}}\right) \\ &= \prod_{p \leq r} \zeta_{\mathcal{X}_p}(1+n)^{-1} + O\left(\frac{\sum_{p \leq r} p^{\frac{2}{n}}}{d^{\frac{2}{n}}}\right). \end{aligned}$$

□

The above lemma shows the result, as

$$\begin{aligned}
& \left| \frac{\#\bigcap_{p \leq r} \mathcal{P}_{d,p}}{\#\mathbb{H}_{\text{Ar}}^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes d})} - \prod_{p \leq r} \zeta_{\mathcal{X}_p}(n+1)^{-1} \right| \\
& \leq \left| \frac{\#\bigcap_{p \leq r} \mathcal{P}_{d,p}}{\#\mathbb{H}_{\text{Ar}}^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes d})} - \prod_{p \leq r} \frac{\#\mathcal{P}'_{p^2,d}}{\#\mathbb{H}^0(\mathcal{X}_{p^2}, \overline{\mathcal{L}}^{\otimes d})} \right| + \left| \prod_{p \leq r} \frac{\#\mathcal{P}'_{p^2,d}}{\#\mathbb{H}^0(\mathcal{X}_{p^2}, \overline{\mathcal{L}}^{\otimes d})} - \prod_{p \leq r} \zeta_{\mathcal{X}_p}(1+n)^{-1} \right| \\
& = O(e^{-nd}) + O\left(\frac{\sum_{p \leq r} p^{\frac{2}{n}}}{d^{\frac{2}{n}}}\right) \\
& = O\left(\frac{\sum_{p \leq r} p^{\frac{2}{n}}}{d^{\frac{2}{n}}}\right).
\end{aligned}$$

□

## 7.2 Bound on number of fibers

Now we prove that we can choose  $r = d^{\frac{1}{n+1}}$ . As

$$\mathcal{P}_{d, p \leq d^{\frac{1}{n+1}}} = \bigcap_{p \leq d^{\frac{1}{n+1}}} \mathcal{P}_{d,p},$$

we will in fact prove the following :

**Lemma 7.2.1.** *For large enough integer  $d$ , we have*

$$\left| \frac{\#\mathcal{P}_{d, p \leq d^{\frac{1}{n+1}}}}{\#\mathbb{H}_{\text{Ar}}^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes d})} - \prod_{p \leq d^{\frac{1}{n+1}}} \zeta_{\mathcal{X}_p}(n+1)^{-1} \right| = O(d^{-\frac{1}{n+1}}).$$

*Proof.* Note that

$$N_r = \prod_{p \leq r} p^2 \leq \prod_{k=1}^r k^2 = (r!)^2$$

and that  $r! < r^r = \exp(r \log r)$ . If  $2r \log r < d^{\alpha_0}$ , we get  $N_r < \exp(d^{\alpha_0})$ . Then by Proposition 4.2.2 the restriction map

$$\psi_{d, N_r} : \mathbb{H}_{\text{Ar}}^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes d}) \longrightarrow \mathbb{H}^0(\mathcal{X}_{N_r}, \overline{\mathcal{L}}^{\otimes d})$$

is surjective. If moreover  $r$  satisfies  $Cnr^n < d$  as in Theorem 6.1.1, then by Lemma 7.1.1, we have

$$\left| \frac{\#\left(\bigcap_{p \leq r} \mathcal{P}_{d,p}\right)}{\#\mathbb{H}_{\text{Ar}}^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes d})} - \prod_{p \leq r} \zeta_{\mathcal{X}_p}(n+1)^{-1} \right| = O\left(\frac{\sum_{p \leq r} p^{\frac{2}{n}}}{d^{\frac{2}{n}}}\right).$$

Now as above,

$$\sum_{p \leq r} p^{\frac{2}{n}} \leq \sum_{k=1}^r k^{\frac{2}{n}} < r \cdot r^{\frac{2}{n}} = r^{\frac{n+2}{n}}.$$

Thus for  $r = d^{\frac{1}{n+1}}$  we have

$$\sum_{p \leq d^{\frac{1}{n+1}}} p^{\frac{2}{n}} < d^{\frac{n+2}{n(n+1)}} = O(d^{\frac{n+2}{n(n+1)}}).$$

It's easy to see that  $r = d^{\frac{1}{n+1}}$  also satisfies conditions  $2r \log r < d^{\alpha_0}$ ,  $Cnr^n < d$  for large  $d$ . For this  $r$ , we have

$$\left| \frac{\#\mathcal{P}_{d,p \leq d^{\frac{1}{n+1}}}}{\#\mathbb{H}_{\text{Ar}}^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes d})} - \prod_{p \leq d^{\frac{1}{n+1}}} \zeta_{\mathcal{X}_p}(n+1)^{-1} \right| = O\left(\frac{\sum_{p \leq d^{\frac{1}{n+1}}} p^{\frac{2}{n}}}{d^{\frac{2}{n}}}\right) = O(d^{\frac{n+2}{n(n+1)} - \frac{2}{n}}) = O(d^{-\frac{1}{n+1}}).$$

□

### 7.3 Proof of Proposition 7.0.1

*Proof.* Apply Lemma 5.0.4 and take  $R = d^{\frac{1}{n+1}}$ . We get

$$\left| \prod_{p \leq d^{\frac{1}{n+1}}} \zeta_{\mathcal{X}_p}(n+1)^{-1} - \zeta_{\mathcal{X}}(n+1)^{-1} \right| = O(d^{-\frac{1}{n+1}}).$$

Combining this with Lemma 7.2.1, we get

$$\begin{aligned} & \left| \frac{\#\mathcal{P}_{d,p \leq d^{\frac{1}{n+1}}}}{\#\mathbb{H}_{\text{Ar}}^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes d})} - \zeta_{\mathcal{X}}(n+1)^{-1} \right| \\ & \leq \left| \frac{\#\mathcal{P}_{d,p \leq d^{\frac{1}{n+1}}}}{\#\mathbb{H}_{\text{Ar}}^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes d})} - \prod_{p \leq d^{\frac{1}{n+1}}} \zeta_{\mathcal{X}_p}(n+1)^{-1} \right| + \left| \prod_{p \leq d^{\frac{1}{n+1}}} \zeta_{\mathcal{X}_p}(n+1)^{-1} - \zeta_{\mathcal{X}}(n+1)^{-1} \right| \\ & = O(d^{-\frac{1}{n+1}}) + O(d^{-\frac{1}{n+1}}) = O(d^{-\frac{1}{n+1}}), \end{aligned}$$

which proves Proposition 7.0.1. □

# Chapitre 8

## Final step

In this section, we prove Theorem 2.1.1. The main step is to show the following proposition :

**Proposition 8.0.1.** *Let  $\mathcal{X}$  be a regular projective arithmetic variety of dimension  $n$ , and let  $\mathcal{L}$  be an ample line bundle on  $\mathcal{X}$ . Then there exists a constant  $c > 0$  such that for any  $d \gg 1$  and any prime number  $p$  such that  $\mathcal{X}_p$  is smooth and irreducible, denoting*

$$\mathcal{Q}_{d,p^2} := \left\{ \sigma \in H^0(\mathcal{X}_{p^2}, \mathcal{L}^{\otimes d}) ; \exists x \in |\mathcal{X}_{p^2}|, \dim_{\kappa(x)} \frac{\mathfrak{m}_{\text{div}\sigma, x}}{\mathfrak{m}_{\text{div}\sigma, x}^2} = n \right\},$$

we have

$$\frac{\#\mathcal{Q}_{d,p^2}}{\#H^0(\mathcal{X}_{p^2}, \mathcal{L}^{\otimes d})} \leq c \cdot p^{-2}.$$

### 8.1 Divisors with higher dimensional singular locus

**Lemma 8.1.1.** *Let  $\mathcal{X}$  be an irreducible projective scheme of dimension  $n$  over  $\text{Spec } \mathbb{Z}$ . Let  $\mathcal{L}$  be an ample line bundle on  $\mathcal{X}$ . For any large enough  $d$  and any prime number  $p$  such that  $\mathcal{X}_p$  is smooth of dimension  $n - 1$  over  $\mathbb{F}_p$ , if  $\sigma \in H^0(\mathcal{X}_p, \mathcal{L}^{\otimes d})$  is such that  $\text{Sing}(\text{div}\sigma)$  is finite, then*

$$\#\text{Sing}(\text{div}\sigma) = O(d^{n-1}),$$

where the constant involved does not depend on  $d$  or  $p$ .

*Proof.* We take the construction in Section 6.4. Let  $\mathcal{S}$  be the maximal open subscheme of  $\text{Spec } \mathbb{Z}$  such that  $\mathcal{X}_{\mathcal{S}} = \mathcal{X} \times_{\text{Spec } \mathbb{Z}} \mathcal{S}$  is smooth over  $\mathcal{S}$ . So  $\mathcal{S}$  contains all prime numbers  $p$  such that  $\mathcal{X}_p$  is smooth of dimension  $n - 1$  over  $\mathbb{F}_p$ . Applying Lemma 6.4.4, we may assume that there exists a positive integer  $N$  and an open cover of  $\mathcal{X}_{\mathcal{S}}$  by  $\mathcal{X}_{\mathcal{S}} = \bigcup_{\alpha \in A} U_{\alpha}$  making the following conditions valide :

1. the sheaf  $\mathcal{L}^{\otimes d}$  is very ample for any  $d \geq N$  ;
2. there exists  $\tau_{\alpha} \in H^0(\mathcal{X}_{\mathcal{S}}, \mathcal{L}^{\otimes(N+1)})$  such that

$$\mathcal{X}_{\mathcal{S}} - U_{\alpha} = \text{div}\tau_{\alpha} ;$$

3. there exist  $\tau_{\alpha,1}, \dots, \tau_{\alpha,k_{\alpha}} \in H^0(\mathcal{X}_{\mathcal{S}}, \mathcal{L}^{\otimes N})$  such that

$$U_{\alpha} = \bigcup_{1 \leq j \leq k_{\alpha}} (\mathcal{X}_{\mathcal{S}} - \text{div}\tau_{\alpha,j}) ;$$

4. for any  $\alpha \in A$ , there exist  $t_{\alpha,1}, \dots, t_{\alpha,n-1} \in H^0(U_\alpha, \mathcal{O}_{U_\alpha})$  such that

$$\Omega_{U_\alpha/S}^1 \simeq \bigoplus_{i=1}^{n-1} \mathcal{O}_{U_\alpha} dt_{\alpha,i},$$

We note  $\partial_{\alpha,i} \in \text{Der}_{\mathcal{O}_S}(\mathcal{O}_{U_\alpha}, \mathcal{O}_{U_\alpha}) \simeq \text{Hom}(\Omega_{U_\alpha/S}^1, \mathcal{O}_{U_\alpha})$  the dual of  $dt_{\alpha,i}$ .

Now we take one arbitrary  $U$  among the  $U_\alpha$ 's in the open cover, and we drop the subscript  $\alpha$  for simplicity of notation. For any  $1 \leq j \leq k$ , we have the restriction morphism

$$\Phi_j : H^0(\mathcal{X}_S, \mathcal{L}^{\otimes d}) \longrightarrow H^0(U, \mathcal{O}_U)$$

sending  $\sigma$  to  $\frac{\sigma \tau_j^d}{\tau^d}$ . Then for any  $\sigma \in H^0(\mathcal{X}_S, \mathcal{L}^{\otimes d})$ ,  $\partial_i \Phi_j(\sigma)$  is a section in  $H^0(U, \mathcal{O}_U)$ . For any  $p \in \mathcal{S}$ ,  $\Phi_j$  induces

$$\Phi_{p,j} : H^0(\mathcal{X}_p, \mathcal{L}^{\otimes d}) \longrightarrow H^0(U_p, \mathcal{O}_{U_p})$$

such that for any  $\sigma' \in H^0(\mathcal{X}_p, \mathcal{L}^{\otimes d})$ ,  $\partial_i \Phi_{p,j}(\sigma')$  is a section in  $H^0(U_p, \mathcal{O}_{U_p})$ . Here  $U_p = U \cap \mathcal{X}_p$ . Then Lemma 3.3.10 tells that we can find a positive integer  $N_1$  such that for any  $\delta \geq N_1$ , any  $p \in \mathcal{S}$  and any  $\sigma \in H^0(\mathcal{X}_p, \mathcal{L}^{\otimes d})$ , the section  $(\partial_i \Phi_{p,j}(\sigma)) \cdot \tau^{d+\delta}$  extends to a global section in  $H^0(\mathcal{X}_p, \mathcal{L}^{\otimes(N+1)(d+\delta)})$ . Since  $\mathcal{L}^{\otimes(N+1)}$  is very ample,  $(\partial_i \Phi_{p,j}(\sigma)) \cdot \tau^{d+\delta}$  can also be regarded as a global section of

$$H^0\left(\mathbb{P}(H^0(\mathcal{X}_p, \mathcal{L}^{\otimes(N+1)})^\vee), \mathcal{O}(d+\delta)\right).$$

Then since  $\tau$  is nowhere 0 on  $U \supset (\mathcal{X}_S - \text{div}\tau_{\alpha,j})$ , for any  $\sigma \in H^0(\mathcal{X}_p, \mathcal{L}^{\otimes d})$ , we have

$$\begin{aligned} & (\mathcal{X}_S - \text{div}\tau_{\alpha,j}) \cap \text{Sing}(\text{div}\sigma) \\ \subset & \text{Sing}(\text{div}\Phi_j(\sigma)) \\ = & \text{div}\Phi_j(\sigma) \cap \left( \bigcap_{i=1}^{n-1} \text{div}(\partial_i \Phi_j(\sigma)) \right) \\ = & (\mathcal{X}_S - \text{div}\tau_{\alpha,j}) \cap \text{div}(\sigma^{N+1}) \cap \left( \bigcap_{i=1}^{n-1} \text{div}(\partial_i \Phi_j(\sigma) \cdot \tau^{d+N_1}) \right). \end{aligned}$$

On the other hand, we have

$$\sigma^{N+1} \in H^0(\mathbb{P}(H^0(\mathcal{X}_p, \mathcal{L}^{\otimes(N+1)})^\vee), \mathcal{O}(d)).$$

Denote the degree of  $\mathcal{X}_p$  as a closed subscheme of  $\mathbb{P}(H^0(\mathcal{X}_p, \mathcal{L}^{\otimes(N+1)})^\vee)$  by  $\text{deg}_{\mathcal{L}^{\otimes(N+1)}}(\mathcal{X}_p)$ . If  $\sigma \in H^0(\mathcal{X}_p, \mathcal{L}^{\otimes d})$  is such that  $\text{Sing}(\text{div}\sigma)$  is finite, then  $(\mathcal{X}_S - \text{div}\tau_{\alpha,j}) \cap \text{Sing}(\text{div}\sigma)$  is finite and we can find  $n-1$  divisors among the  $n$  ones appeared in the above intersection such that the intersection of these  $n-1$  divisors and  $\mathcal{X}_S - \text{div}\tau_{\alpha,j}$  is finite. Obviously this intersection contains  $(\mathcal{X}_S - \text{div}\tau_{\alpha,j}) \cap \text{Sing}(\text{div}\sigma)$ . Applying refined Bézout's theorem [Fu84, Theorem 12.3], we get

$$\#(\text{Sing}(\text{div}\sigma) \cap (\mathcal{X}_S - \text{div}\tau_{\alpha,j})) \leq (\text{deg}_{\mathcal{L}^{\otimes(N+1)}}(\mathcal{X}_p))(d+N_1)^{n-1} = O(d^{n-1}),$$

where coefficients involved in  $O(d^{n-1})$  is independent of  $p$  when  $d$  is large enough ( $\text{deg}_{\mathcal{L}^{\otimes(N+1)}}(\mathcal{X}_p)$  is independent of  $p$ ). Therefore we have

$$\#\text{Sing}(\text{div}\sigma) \leq \sum_{\alpha \in A} \sum_{j=1}^{k_\alpha} \#(\text{Sing}(\text{div}\sigma) \cap (\mathcal{X}_S - \text{div}\tau_{\alpha,j})) = O(d^{n-1})$$

with coefficients involved in  $O(d^{n-1})$  independent of  $p$  when  $d$  is large enough.  $\square$

The following lemma is a generalization of Lemma 5.9 in [Po04], where Poonen shows that for an integral quasi-projective scheme  $X$  generically smooth over  $\mathbb{Z}$  equipped with a very ample line bundle inducing an immersion  $X \hookrightarrow \mathbb{P}_{\mathbb{Z}}^n$  for some  $n > 0$ , if the generic fiber  $\overline{X}_{\mathbb{Q}}$  of the Zariski closure  $\overline{X}$  of  $X$  in  $\mathbb{P}_{\mathbb{Z}}^n$  has at most isolated singular points, then there exists  $c > 0$  such that if  $d, p$  are sufficiently large, then

$$\frac{\#\{\sigma \in H^0(\mathbb{P}_p^n, \mathcal{O}(d)) ; \dim(\text{Sing}(\text{div}\sigma|_{X_p})) > 0\}}{\#H^0(\mathbb{P}_p^n, \mathcal{O}(d))} < \frac{c}{p^2}.$$

We prove the same conclusion for the case when  $X$  is projective and equipped with an ample line bundle, in place of a very ample line bundle.

**Lemma 8.1.2.** *Let  $\mathcal{X}$  be a scheme of dimension  $n$  which is projective over  $\text{Spec } \mathbb{Z}$  and which satisfies that there exists an open subscheme  $\mathcal{S}$  of  $\text{Spec } \mathbb{Z}$  such that  $\mathcal{X}_{\mathcal{S}}$  is irreducible and smooth over  $\mathcal{S}$ . Let  $\mathcal{L}$  be an ample line bundle on  $\mathcal{X}$ . Then there exists a constant  $c_H > 0$  such that for any  $d \gg 1$  and any prime number  $p$  where  $\mathcal{X}_p$  is smooth and irreducible of dimension  $n - 1$ , we have*

$$\frac{\#\{\sigma \in H^0(\mathcal{X}_p, \mathcal{L}^{\otimes d}) ; \dim(\text{Sing}(\text{div}\sigma)) > 0\}}{\#H^0(\mathcal{X}_p, \mathcal{L}^{\otimes d})} \leq c_H \cdot p^{-2}.$$

**Remark.** If the prime  $p$  is fixed, by Corollary 3.3.13 there exists a constant  $c > 0$  such that

$$\frac{\#\{\sigma \in H^0(\mathcal{X}_p, \mathcal{L}^{\otimes d}) ; \dim(\text{Sing}(\text{div}\sigma)) > 0\}}{\#H^0(\mathcal{X}_p, \mathcal{L}^{\otimes d})} = O(d^{n-1} \cdot p^{-c\frac{d}{p}}).$$

When  $d$  is sufficiently large, we deduce from it that

$$\frac{\#\{\sigma \in H^0(\mathcal{X}_p, \mathcal{L}^{\otimes d}) ; \dim(\text{Sing}(\text{div}\sigma)) > 0\}}{\#H^0(\mathcal{X}_p, \mathcal{L}^{\otimes d})} \leq p^{-2}.$$

So Corollary 3.3.13 gives a better bound on the proportion of sections whose divisor has positive dimensional singular locus. But this bound is well behaved only when  $d$  is much larger than the prime  $p$ . In this lemma, the bound we give is independent of the choice of  $p$ . In particular, it is valide even when  $p$  is much bigger than  $d$ .

*Proof.* We choose a constant  $N \in \mathbb{Z}_{>0}$  verifying Lemma 6.1.3. If  $\mathcal{X}$  is of dimension 1, then  $\dim(\mathcal{X}_p) = 0$  for any prime  $p$  and the conclusion holds automatically. When  $\mathcal{X}$  is of dimension 2, for  $p$  such that  $\mathcal{X}_p$  is smooth and irreducible of dimension 1 (which is satisfied for all but finitely many  $p$ ), if  $\sigma \in H^0(\mathcal{X}_p, \mathcal{L}^{\otimes d})$  is such that  $\dim(\text{Sing}(\text{div}\sigma)) > 0$ , then  $\text{Sing}(\text{div}\sigma) = \mathcal{X}_p$ , which is impossible unless  $\sigma = 0$ . This means in the case of dimension 2, when  $d$  is large enough we always have

$$\frac{\#\{\sigma \in H^0(\mathcal{X}_p, \mathcal{L}^{\otimes d}) ; \dim(\text{Sing}(\text{div}\sigma)) > 0\}}{\#H^0(\mathcal{X}_p, \mathcal{L}^{\otimes d})} = \frac{1}{\#H^0(\mathcal{X}_p, \mathcal{L}^{\otimes d})} = p^{-h^0(\mathcal{X}_p, \mathcal{L}^{\otimes d})} < p^{-2}.$$

So the lemma is true when  $\dim \mathcal{X} \leq 2$ . We prove the higher dimensional case by induction. Assume that for any scheme  $\mathcal{Y}$  of dimension smaller than  $n$  which is projective over  $\text{Spec } \mathbb{Z}$  with  $\mathcal{Y}_{\mathcal{S}_y}$  irreducible and smooth over some open subscheme  $\mathcal{S}_y$  of  $\text{Spec } \mathbb{Z}$ , and which is equipped with an ample line bundle  $\mathcal{M}$ , there exists a constant  $c_{\mathcal{Y}, \mathcal{M}} > 0$  such that for any  $d \gg 1$  and any prime  $p$  such that  $\mathcal{Y}_p$  is smooth and irreducible of dimension  $\dim \mathcal{Y} - 1$ , we have

$$\frac{\#\{\sigma \in H^0(\mathcal{Y}_p, \mathcal{M}^{\otimes d}) ; \dim(\text{Sing}(\text{div}\sigma)) > 0\}}{\#H^0(\mathcal{Y}_p, \mathcal{M}^{\otimes d})} \leq c_{\mathcal{Y}, \mathcal{M}} \cdot p^{-2}.$$

By the classical Bertini theorem over  $\mathbb{Q}$ , we can find a section  $\sigma_{\mathcal{D}} \in H^0(\mathcal{X}_{\mathbb{Q}}, \mathcal{L}^{\otimes N})$  whose divisor  $\mathcal{D}_{\mathbb{Q}}$  is a smooth and irreducible divisor of  $\mathcal{X}_{\mathbb{Q}}$ . By possibly replacing  $\sigma_{\mathcal{D}}$  by a multiple of it, we may assume that  $\sigma_{\mathcal{D}}$  is in fact a section of  $H^0(\mathcal{X}, \mathcal{L}^{\otimes N})$ . Then the divisor  $\mathcal{D} = \text{div}\sigma_{\mathcal{D}}$  on  $\mathcal{X}$  has no singular point on the generic fiber  $\mathcal{X}_{\mathbb{Q}}$ . Let  $\mathcal{S}$  be an open subscheme of  $\text{Spec } \mathbb{Z}$  such that  $\mathcal{X}_{\mathcal{S}}$  is smooth and irreducible of dimension  $n - 1$ . By restricting to a smaller  $\mathcal{S}$  we may assume that  $\mathcal{X}_{\mathcal{S}}$  does not contain any vertical component of  $\mathcal{D}$ , and that  $\mathcal{D}_{\mathcal{S}} = \mathcal{D} \cap \mathcal{X}_{\mathcal{S}}$  is smooth over  $\mathcal{S}$ . We may assume moreover that for any  $p \in \mathcal{S}$ ,  $\mathcal{X}_p$  and  $\mathcal{D}_p$  are both smooth and irreducible such that  $\dim \mathcal{X}_p = n - 1$  and  $\dim \mathcal{D}_p = n - 2$ . For the rest of the proof, we fix the divisor  $\mathcal{D}$ . Note that  $\mathcal{D}$  together with the open subscheme  $\mathcal{S}$  of  $\text{Spec } \mathbb{Z}$  and the restriction sheaf  $\mathcal{L}|_{\mathcal{D}}$  also verifies the assumption of the lemma.

For any prime  $p \in \text{Spec } \mathbb{Z} - \mathcal{S}$  such that  $\mathcal{X}_p$  is smooth and irreducible of dimension  $n - 1$ , we can find a constant  $c$  by Corollary 3.3.13, such that

$$\frac{\#\{\sigma \in H^0(\mathcal{X}_p, \mathcal{L}^{\otimes d}) ; \dim(\text{Sing}(\text{div}\sigma)) > 0\}}{\#H^0(\mathcal{X}_p, \mathcal{L}^{\otimes d})} = O(d^{n-1} \cdot p^{-c\frac{d}{p}}).$$

So when  $d$  is sufficiently large, the right side can be bounded above by  $p^{-2}$ . Since  $\text{Spec } \mathbb{Z} - \mathcal{S}$  is a finite scheme, when  $d$  is sufficiently large, for any  $p \in \text{Spec } \mathbb{Z} - \mathcal{S}$  such that  $\mathcal{X}_p$  is smooth and irreducible of dimension  $n - 1$ , we have

$$\frac{\#\{\sigma \in H^0(\mathcal{X}_p, \mathcal{L}^{\otimes d}) ; \dim(\text{Sing}(\text{div}\sigma)) > 0\}}{\#H^0(\mathcal{X}_p, \mathcal{L}^{\otimes d})} \leq p^{-2}.$$

Hence it suffices to prove the lemma for primes  $p \in \mathcal{S}$ .

Now let  $p \in \mathcal{S}$ . If a section  $\sigma \in H^0(\mathcal{X}_p, \mathcal{L}^{\otimes d})$  is such that

$$\dim(\text{Sing}(\text{div}\sigma)) > 0,$$

then as  $\mathcal{X}_p$  is irreducible and projective by the assumption on  $\mathcal{S}$ , we have

$$\text{Sing}(\text{div}\sigma) \cap \mathcal{D}_p \neq \emptyset.$$

By induction hypothesis, we know that there exists a constant  $c_{\mathcal{D}} > 0$  such that if  $d$  is sufficiently large, then for any  $p \in \mathcal{S}$ , we have

$$\frac{\#\{\sigma \in H^0(\mathcal{D}_p, \mathcal{L}^{\otimes d}) ; \dim(\text{Sing}(\text{div}\sigma)) > 0\}}{\#H^0(\mathcal{D}_p, \mathcal{L}^{\otimes d})} \leq c_{\mathcal{D}} \cdot p^{-2}.$$

As  $\mathcal{L}$  is ample on  $\mathcal{X}$ , when  $d$  is large enough, the restriction map

$$H^0(\mathcal{X}, \mathcal{L}^{\otimes d}) \longrightarrow H^0(\mathcal{D}, \mathcal{L}^{\otimes d})$$

is surjective. So for such  $d$ , the morphism

$$H^0(\mathcal{X}_p, \mathcal{L}^{\otimes d}) \longrightarrow H^0(\mathcal{D}_p, \mathcal{L}^{\otimes d})$$

is surjective for any  $p \in \mathcal{S}$ , and hence

$$\frac{\#\{\sigma \in H^0(\mathcal{X}_p, \mathcal{L}^{\otimes d}) ; \dim(\text{Sing}(\text{div}\sigma \cap \mathcal{D}_p)) > 0\}}{\#H^0(\mathcal{X}_p, \mathcal{L}^{\otimes d})} \leq c_{\mathcal{D}} \cdot p^{-2}.$$

We need to bound sections  $\sigma \in H^0(\mathcal{X}_p, \mathcal{L}^{\otimes d})$  such that  $\text{Sing}(\text{div}\sigma) \cap \mathcal{D}_p$  is finite and non-empty. Since  $\mathcal{D}$  is of dimension  $n-1$ , let  $c'_0 > 0$  be a constant such that

$$\#\mathcal{D}(\mathbb{F}_{p^e}) \leq c'_0 p^{(n-2)e}$$

for any prime number  $p$  and any integer  $e \geq 1$ . For any closed point  $x \in |\mathcal{D}_p|$  of degree  $e \leq \frac{d}{Nn}$ , we have by Lemma 3.2.1, the proportion of  $\sigma \in H^0(\mathcal{X}_p, \mathcal{L}^{\otimes d})$  such that  $\text{div}\sigma$  is singular at  $x$  is  $p^{-ne}$ . Then we have

$$\begin{aligned} & \frac{\#\{\sigma \in H^0(\mathcal{X}_p, \mathcal{L}^{\otimes d}) ; \exists x \in |\text{Sing}(\text{div}\sigma) \cap \mathcal{D}_p|, \deg x \leq \lfloor \frac{d}{Nn} \rfloor\}}{\#H^0(\mathcal{X}_p, \mathcal{L}^{\otimes d})} \\ & \leq \sum_{x \in |\mathcal{D}_p|, \deg x \leq \lfloor \frac{d}{Nn} \rfloor} p^{-n \deg x} \\ & < \sum_{e=1}^{\lfloor \frac{d}{Nn} \rfloor} \#\mathcal{D}(\mathbb{F}_{q^e}) p^{-ne} \\ & \leq \sum_{e=1}^{\lfloor \frac{d}{Nn} \rfloor} c'_0 p^{(n-2)e} \cdot p^{-ne} \\ & \leq \frac{c'_0 p^{-2}}{1-p^{-2}} < 2c'_0 p^{-2}. \end{aligned}$$

If a section  $\sigma \in H^0(\mathcal{X}_p, \mathcal{L}^{\otimes d})$  whose divisor has positive dimensional singular locus is not included in the above two cases, then it satisfies the two following conditions at the same time :

- $\text{Sing}(\text{div}\sigma \cap \mathcal{D}_p)$  is a finite set ;
- if  $x$  is a closed point of  $\text{Sing}(\text{div}\sigma) \cap \mathcal{D}_p \subset \text{Sing}(\text{div}\sigma \cap \mathcal{D}_p)$ , then  $\deg x \geq \frac{d}{Nn}$ .

Then to finish the proof, it suffices to show that we can find a constant  $c_3 > 0$  such that when  $d$  is large enough, for any  $p \in \mathcal{S}$ ,

$$\frac{\#\{\sigma \in H^0(\mathcal{X}_p, \mathcal{L}^{\otimes d}) ; \text{Sing}(\text{div}\sigma \cap \mathcal{D}_p) \text{ finite}, \exists x \in \text{Sing}(\text{div}\sigma) \cap \mathcal{D}_p, \deg x \geq \frac{d}{Nn}\}}{\#H^0(\mathcal{X}_p, \mathcal{L}^{\otimes d})} < c_3 p^{-2}.$$

For large enough  $d$ , consider the surjective morphism

$$H^0(\mathcal{X}_p, \mathcal{L}^{\otimes d}) \times H^0(\mathcal{X}_p, \mathcal{L}^{\otimes (d-N)}) \longrightarrow H^0(\mathcal{X}_p, \mathcal{L}^{\otimes d})$$

which sends  $(\sigma_0, \sigma_1) \in H^0(\mathcal{X}_p, \mathcal{L}^{\otimes d}) \times H^0(\mathcal{X}_p, \mathcal{L}^{\otimes (d-N)})$  to the section  $\sigma = \sigma_0 + \sigma_1 \cdot \sigma_{\mathcal{D}}$ . For any  $\sigma = \sigma_0 + \sigma_1 \cdot \sigma_{\mathcal{D}}$  in  $H^0(\mathcal{X}_p, \mathcal{L}^{\otimes d})$ , the singular locus of  $\text{div}\sigma \cap \mathcal{D}_p$  is independent of  $\sigma_1$ , i.e.

$$\text{Sing}(\text{div}\sigma \cap \mathcal{D}_p) = \text{Sing}(\text{div}\sigma_0 \cap \mathcal{D}_p).$$

If  $(\sigma_0, \sigma_1) \in H^0(\mathcal{X}_p, \mathcal{L}^{\otimes d}) \times H^0(\mathcal{X}_p, \mathcal{L}^{\otimes (d-N)})$  is such that  $\text{Sing}(\text{div}\sigma_0 \cap \mathcal{D}_p)$  is finite, we assume that  $\text{Sing}(\text{div}\sigma_0 \cap \mathcal{D}_p) = \{x_1, \dots, x_l\}$ . Applying Lemma 8.1.1 to  $\mathcal{D}$ , we have  $l = O(d^{n-2})$  with coefficients depending on  $\mathcal{L}$  and  $\mathcal{D}$  but not  $p$ . For a fixed  $\sigma_0$  and any  $x_i \in \text{Sing}(\text{div}\sigma_0 \cap \mathcal{D}_p)$ , let  $x'_i$  be the first order infinitesimal neighbourhood of  $x_i$  in  $\mathcal{X}_p$ . If  $\text{div}(\sigma_0 + \sigma_1 \sigma_{\mathcal{D}_p})$  is singular at  $x_i$ , then the image of  $\sigma_0 + \sigma_1 \sigma_{\mathcal{D}_p}$  in  $H^0(x'_i, \mathcal{L}^{\otimes d})$  by the natural restriction morphism is 0. Let  $\mathfrak{m}_{\mathcal{X}_p, x_i}$  be the ideal sheaf of  $x_i$  in  $\mathcal{X}_p$ . We have a natural exact sequence of sheaves on  $\mathcal{X}_p$

$$0 \longrightarrow \mathcal{L}^{\otimes (d-N)} \longrightarrow \mathcal{L}^{\otimes d} \longrightarrow \mathcal{L}^{\otimes d} \otimes \mathcal{O}_{\mathcal{D}} \longrightarrow 0,$$



where the morphism  $\mathcal{L}^{\otimes(d-N)} \rightarrow \mathcal{L}^{\otimes d}$  is the multiplication by  $\sigma_{\mathcal{D}}$ . Restricting this exact sequence of sheaves to the closed subscheme  $x'_i$ , we get a right exact sequence

$$\frac{\mathcal{O}_{\mathcal{X}_p}}{\mathfrak{m}_{\mathcal{X}_p, x_i}^2} \otimes \mathcal{L}^{\otimes(d-N)} \rightarrow \frac{\mathcal{O}_{\mathcal{X}_p}}{\mathfrak{m}_{\mathcal{X}_p, x_i}^2} \otimes \mathcal{L}^{\otimes d} \rightarrow \frac{\mathcal{O}_{\mathcal{X}_p}}{\mathfrak{m}_{\mathcal{X}_p, x_i}^2 + \mathcal{I}_{\mathcal{D}}} \otimes \mathcal{L}^{\otimes d} \rightarrow 0,$$

where  $\mathcal{I}_{\mathcal{D}}$  is the ideal sheaf of  $\mathcal{D}$ . Note that the sheaf  $\frac{\mathfrak{m}_{\mathcal{X}_p, x_i}}{\mathfrak{m}_{\mathcal{X}_p, x_i}^2} \otimes \mathcal{L}^{\otimes(d-N)}$  is contained in the kernel of the first morphism as  $x_i \in \mathcal{D}$ . So the above right exact sequence induces the following right exact sequence

$$\frac{\mathcal{O}_{\mathcal{X}_p}}{\mathfrak{m}_{\mathcal{X}_p, x_i}} \otimes \mathcal{L}^{\otimes(d-N)} \rightarrow \frac{\mathcal{O}_{\mathcal{X}_p}}{\mathfrak{m}_{\mathcal{X}_p, x_i}^2} \otimes \mathcal{L}^{\otimes d} \rightarrow \frac{\mathcal{O}_{\mathcal{X}_p}}{\mathfrak{m}_{\mathcal{X}_p, x_i}^2 + \mathcal{I}_{\mathcal{D}}} \otimes \mathcal{L}^{\otimes d} \rightarrow 0.$$

Since  $\mathcal{D}$  is nonsingular at  $x_i$ , the multiplication by  $\sigma_{\mathcal{D}}$  morphism

$$\frac{\mathcal{O}_{\mathcal{X}_p}}{\mathfrak{m}_{\mathcal{X}_p, x_i}^2} \otimes \mathcal{L}^{\otimes(d-N)} \rightarrow \frac{\mathcal{O}_{\mathcal{X}_p}}{\mathfrak{m}_{\mathcal{X}_p, x_i}^2} \otimes \mathcal{L}^{\otimes d}$$

is not a zero map. This implies that the induced morphism

$$\frac{\mathcal{O}_{\mathcal{X}_p}}{\mathfrak{m}_{\mathcal{X}_p, x_i}} \otimes \mathcal{L}^{\otimes(d-N)} \rightarrow \frac{\mathcal{O}_{\mathcal{X}_p}}{\mathfrak{m}_{\mathcal{X}_p, x_i}^2} \otimes \mathcal{L}^{\otimes d}$$

is not zero neither. As the left term is a  $\kappa(x)$  linear space of dimension 1, this morphism is in fact injective and hence we have the following short exact sequence

$$0 \rightarrow \frac{\mathcal{O}_{\mathcal{X}_p}}{\mathfrak{m}_{\mathcal{X}_p, x_i}} \otimes \mathcal{L}^{\otimes(d-N)} \rightarrow \frac{\mathcal{O}_{\mathcal{X}_p}}{\mathfrak{m}_{\mathcal{X}_p, x_i}^2} \otimes \mathcal{L}^{\otimes d} \rightarrow \frac{\mathcal{O}_{\mathcal{X}_p}}{\mathfrak{m}_{\mathcal{X}_p, x_i}^2 + \mathcal{I}_{\mathcal{D}}} \otimes \mathcal{L}^{\otimes d} \rightarrow 0.$$

Now, with the same notation as above,  $x_i \in \text{Sing}(\text{div}\sigma_0 \cap \mathcal{D}_p)$  means that the image of the restriction of  $\sigma_0$  in  $\frac{\mathcal{O}_{\mathcal{X}_p}}{\mathfrak{m}_{\mathcal{X}_p, x_i}^2 + \mathcal{I}_{\mathcal{D}}} \otimes \mathcal{L}^{\otimes d}$  is 0. If  $x_i \in \text{Sing}(\text{div}\sigma_0) \cap \mathcal{D}_p$ , then  $\sigma_0 + \sigma_1\sigma_{\mathcal{D}}$  has image 0 by the restriction to  $\frac{\mathcal{O}_{\mathcal{X}_p}}{\mathfrak{m}_{\mathcal{X}_p, x_i}^2} \otimes \mathcal{L}^{\otimes d}$ . By the above exact sequence, this is a condition on  $\sigma_1(x_i) \in \text{H}^0(x_i, \mathcal{L}^{\otimes(d-N)})$ . By the exactness of the sequence, there is only one value of  $\sigma_1(x_i) \in \text{H}^0(x_i, \mathcal{L}^{\otimes(d-N)})$  which makes  $\text{div}(\sigma_0 + \sigma_1\sigma_{\mathcal{D}})$  singular at  $x_i$ . If moreover we have  $\deg x_i \geq \frac{d}{Nn}$ , then for any  $\beta \in \text{H}^0(x_i, \mathcal{L}^{\otimes(d-N)})$ , by Lemma 3.3.2 the number of sections in  $\text{H}^0(\mathcal{X}_p, \mathcal{L}^{\otimes(d-N)})$  whose image in  $\text{H}^0(x_i, \mathcal{L}^{\otimes(d-N)})$  by the restriction map is  $\beta$  is bounded above by

$$p^{-\min(\lfloor \frac{d}{N} \rfloor, \deg x_i)} \cdot \#\text{H}^0(\mathcal{X}_p, \mathcal{L}^{\otimes(d-N)}) \leq p^{-\frac{d}{Nn}} \cdot \#\text{H}^0(\mathcal{X}_p, \mathcal{L}^{\otimes(d-N)}).$$

Therefore, we have

$$\begin{aligned} & \frac{\#\left\{ \sigma \in \text{H}^0(\mathcal{X}_p, \mathcal{L}^{\otimes d}) ; \begin{array}{l} \text{Sing}(\text{div}\sigma \cap \mathcal{D}_p) \text{ finite,} \\ \exists x \in \text{Sing}(\text{div}\sigma) \cap \mathcal{D}_p, \deg x \geq \frac{d}{Nn} \end{array} \right\}}{\#\text{H}^0(\mathcal{X}_p, \mathcal{L}^{\otimes d})} \\ &= \frac{\#\left\{ (\sigma_0, \sigma_1) ; \begin{array}{l} \text{Sing}(\text{div}(\sigma_0 + \sigma_1\sigma_{\mathcal{D}}) \cap \mathcal{D}_p) \text{ finite,} \\ \exists x \in \text{Sing}(\text{div}(\sigma_0 + \sigma_1\sigma_{\mathcal{D}})) \cap \mathcal{D}_p, \deg x \geq \frac{d}{Nn} \end{array} \right\}}{\#\text{H}^0(\mathcal{X}_p, \mathcal{L}^{\otimes d}) \cdot \#\text{H}^0(\mathcal{X}_p, \mathcal{L}^{\otimes(d-N)})} \\ &\leq \frac{\#\text{H}^0(\mathcal{X}_p, \mathcal{L}^{\otimes d}) \cdot \left[ O(d^{n-2})p^{-\frac{d}{Nn}} \#\text{H}^0(\mathcal{X}_p, \mathcal{L}^{\otimes(d-N)}) \right]}{\#\text{H}^0(\mathcal{X}_p, \mathcal{L}^{\otimes d}) \cdot \#\text{H}^0(\mathcal{X}_p, \mathcal{L}^{\otimes(d-N)})} \\ &= O(d^{n-2})p^{-\frac{d}{Nn}}. \end{aligned}$$

Obviously, the last term is bounded above by  $p^{-2}$  when  $d$  is large enough. Hence we finish the proof.  $\square$

## 8.2 Proof of Proposition 8.0.1

*Proof.* We fix a positive integer  $N$  that verifies Lemma 6.1.3. Let  $p$  be a prime such that  $\mathcal{X}_p$  is smooth and irreducible. By Lemma 6.2.1, for any closed point  $x \in |\mathcal{X}_p|$  verifying the condition  $d \geq N(n \deg x)$ , i.e.  $\deg x \leq \frac{d}{Nn}$ , the restriction morphism

$$\varphi_{p^2,x} : H^0(\mathcal{X}_{p^2}, \mathcal{L}^{\otimes d}) \longrightarrow H^0(x', \mathcal{L}^{\otimes d}),$$

is surjective, where  $x'$  is the first order infinitesimal neighbourhood of  $x$  in  $\mathcal{X}$ . Therefore the proportion of global sections in  $H^0(\mathcal{X}_{p^2}, \mathcal{L}^{\otimes d})$  whose divisor satisfies  $\dim_{\kappa(x)} \frac{\mathfrak{m}_{\text{div}\sigma,x}}{\mathfrak{m}_{\text{div}\sigma,x}^2} = n$  is equal to

$$\frac{\#\text{Ker } \varphi_{p^2,x}}{\#H^0(\mathcal{X}_{p^2}, \mathcal{L}^{\otimes d})} = p^{-(n+1) \deg x}.$$

Then with the constant  $c_0$  defined in Chapter 5, we have

$$\begin{aligned} & \frac{\#\left\{ \sigma \in H^0(\mathcal{X}_{p^2}, \mathcal{L}^{\otimes d}) ; \exists x \in |\mathcal{X}_{p^2}|, \deg x \leq \frac{d}{Nn}, \dim_{\kappa(x)} \frac{\mathfrak{m}_{\text{div}\sigma,x}}{\mathfrak{m}_{\text{div}\sigma,x}^2} = n \right\}}{\#H^0(\mathcal{X}_{p^2}, \mathcal{L}^{\otimes d})} \\ & \leq \sum_{x \in |\mathcal{X}_{p^2}|, \deg x \leq \frac{d}{Nn}} p^{-(n+1) \deg x} \\ & \leq \sum_{e=1}^{\lfloor \frac{d}{Nn} \rfloor} \#\mathcal{X}_{p^2}(\mathbb{F}_{p^e}) p^{-(n+1)e} \\ & \leq \sum_{e=1}^{\lfloor \frac{d}{Nn} \rfloor} c_0 p^{(n-1)e} \cdot p^{-(n+1)e} \\ & \leq \sum_{e=1}^{\infty} c_0 p^{-2e} = c_0 p^{-2} + \sum_{e=2}^{\infty} c_0 p^{-2e} \leq 2c_0 p^{-2}. \end{aligned}$$

Note that Lemma 8.1.2 tells us that in  $H^0(\mathcal{X}_{p^2}, \mathcal{L}^{\otimes d})$ , the proportion of sections whose divisor has positive dimensional singular locus is bounded above by  $c_H p^{-2}$ . Consequently, as the restriction

$$\varphi_{d,p} : H^0(\mathcal{X}_{p^2}, \mathcal{L}^{\otimes d}) \longrightarrow H^0(\mathcal{X}_p, \mathcal{L}^{\otimes d})$$

is surjective, the proportion of sections  $\sigma \in H^0(\mathcal{X}_{p^2}, \mathcal{L}^{\otimes d})$  such that

$$\dim \text{Sing}(\text{div}(\sigma|_{\mathcal{X}_p})) > 0$$

is also bounded above by  $c_H p^{-2}$ . To finish the proof, it suffices to bound the proportion of sections in the set

$$\left\{ \sigma \in H^0(\mathcal{X}_{p^2}, \mathcal{L}^{\otimes d}) ; \text{Sing}(\text{div}\sigma|_{\mathcal{X}_p}) \text{ finite}, \exists x \in |\mathcal{X}_{p^2}|, \deg x \geq \frac{d}{Nn}, \dim_{\kappa(x)} \frac{\mathfrak{m}_{\text{div}\sigma,x}}{\mathfrak{m}_{\text{div}\sigma,x}^2} = n \right\}$$

Now we take a subset  $E_{d,p} \subset H^0(\mathcal{X}_{p^2}, \mathcal{L}^{\otimes d})$  such that the restriction map  $\varphi_{d,p}$  induces a bijection from  $E_{d,p}$  to  $H^0(\mathcal{X}_p, \mathcal{L}^{\otimes d})$ . For example, if we choose a  $\mathbb{Z}/p^2\mathbb{Z}$ -basis of  $H^0(\mathcal{X}_{p^2}, \mathcal{L}^{\otimes d})$ ,

we can take  $E_{d,p}$  to be the set of sections having coefficients in  $\{0, 1, \dots, p-1\} \subset \mathbb{Z}/p^2\mathbb{Z}$  when written as linear combination of sections in this basis. Then any section  $\sigma \in H^0(\mathcal{X}_{p^2}, \mathcal{L}^{\otimes d})$  can be written uniquely as

$$\sigma = \sigma_1 + p\sigma_2$$

for some  $\sigma_1, \sigma_2 \in E_{d,p}$ . Then  $\text{Sing}(\text{div}(\sigma|_{\mathcal{X}_p})) = \text{Sing}(\text{div}(\sigma_1|_{\mathcal{X}_p}))$ . Now let  $\sigma$  be a section in  $H^0(\mathcal{X}_{p^2}, \mathcal{L}^{\otimes d})$  such that  $\text{Sing}(\text{div}(\sigma|_{\mathcal{X}_p}))$  is finite. We may assume that as a set,

$$|\text{Sing}(\text{div}(\sigma|_{\mathcal{X}_p}))| = \{x_1, \dots, x_l\}.$$

Then by Lemma 8.1.1, we have

$$l = O(d^{n-1}).$$

Moreover, for  $i \in \{1, \dots, l\}$ ,  $\text{div}(\sigma_1 + p\sigma_2)$  is singular at  $x_i$  if and only if the image of  $\sigma_1 + p\sigma_2$  in  $H^0(x'_i, \mathcal{L}^{\otimes d})$  is 0, where  $x'_i$  is the first order infinitesimal neighbourhood of  $x_i$  in  $\mathcal{X}_{p^2}$ . Let  $\mathfrak{m}_{x_i}$  be the ideal sheaf of  $x_i$  in  $\mathcal{X}_{p^2}$ . Then  $x'_i$  is defined by the ideal sheaf  $\mathfrak{m}_{x_i}^2$ . Now we have a right exact sequence of sheaves

$$\frac{\mathcal{O}_{\mathcal{X}_{p^2}}}{\mathfrak{m}_{x_i}^2} \otimes \mathcal{L}^{\otimes d} \longrightarrow \frac{\mathcal{O}_{\mathcal{X}_{p^2}}}{\mathfrak{m}_{x_i}^2} \otimes \mathcal{L}^{\otimes d} \longrightarrow \frac{\mathcal{O}_{\mathcal{X}_{p^2}}}{\mathfrak{m}_{x_i}^2 + p\mathcal{O}_{\mathcal{X}_{p^2}}} \otimes \mathcal{L}^{\otimes d} \longrightarrow 0,$$

where the first morphism is the multiplication by  $p$ . Note that  $\frac{\mathfrak{m}_{x_i}}{\mathfrak{m}_{x_i}^2} \otimes \mathcal{L}^{\otimes d} \subset \frac{\mathcal{O}_{\mathcal{X}_{p^2}}}{\mathfrak{m}_{x_i}^2} \otimes \mathcal{L}^{\otimes d}$  is contained in the kernel of the first morphism, we obtain an exact sequence

$$\frac{\mathcal{O}_{\mathcal{X}_{p^2}}}{\mathfrak{m}_{x_i}} \otimes \mathcal{L}^{\otimes d} \longrightarrow \frac{\mathcal{O}_{\mathcal{X}_{p^2}}}{\mathfrak{m}_{x_i}^2} \otimes \mathcal{L}^{\otimes d} \longrightarrow \frac{\mathcal{O}_{\mathcal{X}_{p^2}}}{\mathfrak{m}_{x_i}^2 + p\mathcal{O}_{\mathcal{X}_{p^2}}} \otimes \mathcal{L}^{\otimes d} \longrightarrow 0.$$

Note that  $\mathcal{X}$  is regular. The multiplication by  $p$  map

$$\frac{\mathcal{O}_{\mathcal{X}_{p^2}}}{\mathfrak{m}_{x_i}^2} \otimes \mathcal{L}^{\otimes d} \longrightarrow \frac{\mathcal{O}_{\mathcal{X}_{p^2}}}{\mathfrak{m}_{x_i}^2} \otimes \mathcal{L}^{\otimes d}$$

cannot be zero. So is  $\frac{\mathcal{O}_{\mathcal{X}_{p^2}}}{\mathfrak{m}_{x_i}} \otimes \mathcal{L}^{\otimes d} \longrightarrow \frac{\mathcal{O}_{\mathcal{X}_{p^2}}}{\mathfrak{m}_{x_i}^2} \otimes \mathcal{L}^{\otimes d}$ . But if this morphism is not zero, it must be injective as  $\frac{\mathcal{O}_{\mathcal{X}_{p^2}}}{\mathfrak{m}_{x_i}} \otimes \mathcal{L}^{\otimes d}$  is in fact a sheaf supported on  $x_i$  where its stalk is a  $\kappa(x_i)$ -vector space of dimension 1. Hence we get a short exact sequence

$$0 \longrightarrow \frac{\mathcal{O}_{\mathcal{X}_{p^2}}}{\mathfrak{m}_{x_i}} \otimes \mathcal{L}^{\otimes d} \longrightarrow \frac{\mathcal{O}_{\mathcal{X}_{p^2}}}{\mathfrak{m}_{x_i}^2} \otimes \mathcal{L}^{\otimes d} \longrightarrow \frac{\mathcal{O}_{\mathcal{X}_{p^2}}}{\mathfrak{m}_{x_i}^2 + p\mathcal{O}_{\mathcal{X}_{p^2}}} \otimes \mathcal{L}^{\otimes d} \longrightarrow 0.$$

Since these sheaves are all supported on  $x_i$ , this sequence induces the following exact sequence of groups

$$0 \longrightarrow H^0(x_i, \mathcal{L}^{\otimes d}) \longrightarrow H^0(x'_i, \mathcal{L}^{\otimes d}) \longrightarrow H^0(x'_i \cap \mathcal{X}_p, \mathcal{L}^{\otimes d}) \longrightarrow 0.$$

It tells us that there is only one value of  $H^0(x_i, \mathcal{L}^{\otimes d})$  for  $\sigma_2(x_i)$  which makes  $\dim_{\kappa(x)} \frac{\mathfrak{m}_{\text{div}\sigma, x}^{\text{div}\sigma, x}}{\mathfrak{m}_{\text{div}\sigma, x}^2} = n$  for  $\sigma = \sigma_1 + p\sigma_2$ . Note that Lemma 3.3.2 tells us that when  $d$  is large enough, for any  $x \in |\mathcal{X}_{p^2}|$  verifying  $\deg x \geq \frac{d}{N_n}$  and any value  $s \in H^0(x, \mathcal{L}^{\otimes d})$ , we have

$$\frac{\#\{\sigma \in H^0(\mathcal{X}_p, \mathcal{L}^{\otimes d}) ; \sigma(x) = s\}}{\#H^0(\mathcal{X}_p, \mathcal{L}^{\otimes d})} \leq p^{-\lfloor \frac{d}{N} \rfloor} \leq p^{1 - \frac{d}{N}}.$$

Since  $\varphi_{d,p}$  induces a bijection between  $E_{d,p}$  and  $H^0(\mathcal{X}_p, \mathcal{L}^{\otimes d})$ , we have

$$\#\{\sigma \in E_{d,p} ; \sigma(x) = s\} \leq p^{1-\frac{d}{N}} \cdot \#H^0(\mathcal{X}_p, \mathcal{L}^{\otimes d}).$$

Therefore, when  $d$  is large enough, we have

$$\begin{aligned} & \frac{\#\left\{ \sigma \in H^0(\mathcal{X}_{p^2}, \mathcal{L}^{\otimes d}) ; \begin{array}{l} \text{Sing}(\text{div}(\sigma|_{\mathcal{X}_p})) \text{ finite, } \exists x \in |\mathcal{X}_{p^2}|, \\ \deg x \geq \frac{d}{Nn}, \dim_{\kappa(x)} \frac{m_{\text{div}\sigma, x}}{m_{\text{div}\sigma, x}^2} = n \end{array} \right\}}{\#H^0(\mathcal{X}_{p^2}, \mathcal{L}^{\otimes d})} \\ &= \frac{\#\left\{ (\sigma_1, \sigma_2) \in E_{d,p} \times E_{d,p} ; \begin{array}{l} \text{Sing}(\text{div}(\sigma_1|_{\mathcal{X}_p})) \text{ finite, } \exists x \in |\mathcal{X}_{p^2}|, \deg x \geq \frac{d}{Nn}, \\ \dim_{\kappa(x)} \frac{m_{\text{div}\sigma, x}}{m_{\text{div}\sigma, x}^2} = n, \text{ with } \sigma = \sigma_1 + p\sigma_2 \end{array} \right\}}{\#H^0(\mathcal{X}_{p^2}, \mathcal{L}^{\otimes d})} \\ &\leq \frac{\#E_{d,p} \cdot \left[ O(d^{n-1})p^{1-\frac{d}{N}} \#H^0(\mathcal{X}_p, \mathcal{L}^{\otimes d}) \right]}{\#H^0(\mathcal{X}_{p^2}, \mathcal{L}^{\otimes d})} \\ &= O(d^{n-1})p^{1-\frac{d}{N}} \leq p^{-2}. \end{aligned}$$

This finishes the proof.  $\square$

### 8.3 Proof of Theorem 2.1.1

Note that in Chapter 7 we have shown that the proportion of sections in  $H_{\text{Ar}}^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes d})$  whose divisor has no singular point of residual characteristic smaller than  $d^{\frac{1}{n+1}}$  tends to  $\zeta_{\mathcal{X}}(n+1)^{-1}$  already. Theorem 2.1.1 can be reduced to the following :

**Proposition 8.3.1.** *Let  $\mathcal{X}$  be a regular projective arithmetic variety of absolute dimension  $n$ , and let  $\overline{\mathcal{L}}$  be an ample Hermitian line bundle on  $\mathcal{X}$ . Let  $\varepsilon > 0$  be a constant such that  $2\varepsilon < \varepsilon_0$ , where  $\varepsilon_0$  is defined in Proposition 4.1.1. Set*

$$\mathcal{Q}_d^m := \left\{ \sigma \in H_{\text{Ar}}^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes d}) ; \begin{array}{l} \text{div} \sigma \text{ has a singular point of residual} \\ \text{characteristic between } d^{\frac{1}{n+1}} \text{ and } e^{\varepsilon d} \end{array} \right\}.$$

When  $d$  is sufficiently large, we have

$$\frac{\#\mathcal{Q}_d^m}{\#H_{\text{Ar}}^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes d})} = O(d^{-\frac{1}{n+1}}).$$

Here the constant involved in the big  $O$  depends only on  $\mathcal{X}$  and  $\overline{\mathcal{L}}$ .

In particular, denoting  $\mathcal{Q}^m = \bigcup_{d>0} \mathcal{Q}_d^m$ , we have

$$\mu(\mathcal{Q}^m) = 0.$$

*Proof.* Since for any  $\sigma \in H_{\text{Ar}}^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes d})$ ,  $\text{div} \sigma$  has a singular point on the fiber  $\mathcal{X}_p$  implies  $\sigma \bmod p^2 \in \mathcal{Q}_{d,p^2}$ , we have

$$\frac{\#\mathcal{Q}_d^m}{\#H_{\text{Ar}}^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes d})} \leq \sum_{d^{\frac{1}{n+1}} \leq p \leq e^{\varepsilon d}} \frac{\#\{\sigma \in H_{\text{Ar}}^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes d}) ; \sigma \bmod p^2 \in \mathcal{Q}_{d,p^2}\}}{\#H_{\text{Ar}}^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes d})}.$$

As  $p \geq d^{\frac{1}{n+1}}$  implies that  $p^2$  is odd, we can apply Proposition 4.2.3 to the case  $N = p^2$  and  $E = \mathcal{Q}_{d,p^2}$ , and obtain that for any constant  $0 < \varepsilon' < \varepsilon_0$ , when  $d$  is large enough and  $p^2 \leq e^{\varepsilon'd}$ , we have

$$\begin{aligned} & \frac{\#\{\sigma \in \mathbf{H}_{\text{Ar}}^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes d}); \sigma \bmod p^2 \in \mathcal{Q}_{d,p^2}\}}{\#\mathbf{H}_{\text{Ar}}^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes d})} \\ & \leq 4p^{-2\text{rk}(\mathbf{H}^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes d}))} \cdot \#\mathcal{Q}_{d,p^2} \\ & = 4 \frac{\#\mathcal{Q}_{d,p^2}}{\#\mathbf{H}^0(\mathcal{X}_{p^2}, \mathcal{L}^{\otimes d})}. \end{aligned}$$

Since  $\mathcal{X}$  is a regular arithmetic variety, it is irreducible and generically smooth. So if  $d$  is large enough, for any prime number  $p \geq d^{\frac{1}{n+1}}$ ,  $\mathcal{X}_p$  is irreducible and smooth over  $\mathbb{F}_p$ . Then Proposition 8.0.1 tells us that there exists a constant  $c > 0$  such that

$$\frac{\#\mathcal{Q}_{d,p^2}}{\#\mathbf{H}^0(\mathcal{X}_{p^2}, \mathcal{L}^{\otimes d})} \leq cp^{-2}.$$

Therefore by setting  $\varepsilon = \frac{1}{2}\varepsilon'$ , we conclude with

$$\begin{aligned} \frac{\#\mathcal{Q}_d^m}{\#\mathbf{H}_{\text{Ar}}^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes d})} & \leq \sum_{d^{\frac{1}{n+1}} \leq p \leq e^{\varepsilon d}} \frac{\#\{\sigma \in \mathbf{H}_{\text{Ar}}^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes d}); \sigma \bmod p^2 \in \mathcal{Q}_{d,p^2}\}}{\#\mathbf{H}_{\text{Ar}}^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes d})} \\ & \leq \sum_{d^{\frac{1}{n+1}} \leq p \leq e^{\varepsilon d}} 4 \frac{\#\mathcal{Q}_{d,p^2}}{\#\mathbf{H}^0(\mathcal{X}_{p^2}, \mathcal{L}^{\otimes d})} \\ & \leq \sum_{d^{\frac{1}{n+1}} \leq p \leq e^{\varepsilon d}} 4cp^{-2} \\ & = 4c \left( \sum_{d^{\frac{1}{n+1}} \leq p \leq \infty} p^{-2} \right) \\ & < 4cd^{-\frac{1}{n+1}}, \end{aligned}$$

which is the statement of the proposition.  $\square$

*Proof of Theorem 2.1.1.* Since  $\mathcal{P}_A \subset \mathcal{P}_B \subset \mathcal{P}_A \cup \mathcal{Q}^m$ , we get that

$$|\mu(\mathcal{P}_A) - \mu(\mathcal{P}_B)| \leq \mu(\mathcal{Q}^m) = 0.$$

Therefore we have

$$\mu(\mathcal{P}_A) = \mu(\mathcal{P}_B) = \zeta_{\mathcal{X}}(1+n)^{-1}.$$

This finishes the proof.  $\square$

## 8.4 Proof of Corollary 2.1.3

*Proof.* Let  $\mathcal{X}$  be a regular projective arithmetic variety of dimension  $n$ , and let  $\overline{\mathcal{L}}$  be an ample line bundle on  $\mathcal{X}$ . By Proposition 4.1.1, there exists a positive constant  $\varepsilon_0$  such that for any

large enough integer  $d$ ,  $H^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes d})$  has a basis consisting of sections with norm smaller than  $e^{-\varepsilon_0 d}$ . Choose the constant  $c$  to be a real number satisfying  $1 < c < e^{\varepsilon_0}$ . For any  $R > 1$ , set  $\overline{\mathcal{L}}' = (\mathcal{L}, \|\cdot\|')$  where  $\|\cdot\|' = \|\cdot\| R^{-1}$ . Since  $(\mathcal{L}, \|\cdot\| e^{-\delta})$  is ample for any  $\delta > 0$ , the Hermitian line bundle  $\overline{\mathcal{L}}'$  is also ample. Then by construction, for any large enough integer  $d$ ,  $H^0(\mathcal{X}, \overline{\mathcal{L}}'^{\otimes d})$  has a basis consisting of sections with norm smaller than  $R^{-d} e^{-\varepsilon_0 d} = e^{-(\varepsilon_0 + \log R)d}$ . Set  $\varepsilon = \frac{1}{2} \log(cR)$ . Then we have

$$\varepsilon = \frac{1}{2}(\log c + \log R) < \frac{1}{2}(\varepsilon_0 + \log R),$$

and we can apply Theorem 2.1.1 to  $\overline{\mathcal{L}}'$  with constant  $\varepsilon$  chosen as above. Then the density result is exactly what we need to prove. □



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**Titre:** Autour du théorème de Bertini en géométrie d'Arakelov

**Mots clés:** Théorème de Bertini, Géométrie d'Arakelov, Théorie d'intersection, Amplitude arithmétique

**Résumé:** Cette thèse a pour objet l'étude des propriétés géométriques des variétés arithmétiques. Plus précisément, nous nous intéressons à l'existence des sous-schémas projectifs réguliers sur une variété arithmétique projective régulière. Nous étendons d'abord un résultat de Poonen. Nous prouvons notamment qu'étant donné une variété projective lisse  $X$  sur un corps fini et un faisceau ample  $\mathcal{L}$  au-dessus de  $X$ , la proportion des sections glob-

ales de  $\mathcal{L}^{\otimes d}$  ayant un diviseur lisse tend vers  $\zeta_X(1 + \dim X)^{-1}$  quand  $d$  tend vers l'infini. Nous montrons ensuite que pour une variété arithmétique projective régulière  $\mathcal{X}$  muni d'un faisceau hermitien ample  $\bar{\mathcal{L}}$ , la proportion des sections globales de norme infinie strictement plus petite que 1 de  $\bar{\mathcal{L}}^{\otimes d}$  dont le diviseur n'a pas de point singulier sur la fibre  $\mathcal{X}_p$  au-dessus d'aucun nombre premier  $p \leq e^{\varepsilon d}$  tend vers  $\zeta_{\mathcal{X}}(1 + \dim \mathcal{X})^{-1}$  quand  $d$  tend vers l'infini.

**Title:** On the Bertini theorem in Arakelov geometry

**Keywords:** Bertini theorem, Arakelov geometry, Intersection theory, Arithmetic ampleness

**Abstract:** The purpose of this thesis is to study the geometric properties of the arithmetic varieties. More precisely, we are interested in the existence of regular projective subschemes of a regular projective arithmetic variety. First we extend a result of Poonen. In particular, we prove that given a smooth projective variety  $X$  over a finite field and an ample line bundle  $\mathcal{L}$  on  $X$ , the proportion of global sections

of  $\mathcal{L}^{\otimes d}$  which has a smooth divisor tends to  $\zeta_X(1 + \dim X)^{-1}$  when  $d$  tends to infinity. Then we show that for a regular projective arithmetic variety  $\mathcal{X}$  equipped with an ample Hermitian line bundle  $\bar{\mathcal{L}}$ , the proportion of global sections of supremum norm strictly smaller than 1 of  $\bar{\mathcal{L}}^{\otimes d}$  whose divisor does not have a singular point on the fiber  $\mathcal{X}_p$  over any prime  $p \leq e^{\varepsilon d}$  tends to  $\zeta_{\mathcal{X}}(1 + \dim \mathcal{X})^{-1}$  as  $d$  tends to infinity.

