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# Some contributions to backward stochastic differential equations and applications

Arij Manai

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ECOLE DOCTORALE N° 601

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de l'Information et de la Communication*

Spécialité : *Mathématiques*

Par

**Arij MANAI**

**« Some contributions to backward stochastic differential equations  
and applications »**

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*À mon père et mon grand-père que j'aime tant.*



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# INTRODUCTION

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## 1.1 Expected utility maximisation problem

### 1.1.1 An overview of the theory of backward SDEs

#### Backward SDEs in a continuous framework

**Classical results** Motivated by the study of stochastic control problems, backward stochastic differential equations (BSDEs in short) attracted a lot of attention in the last two decades. The interest given to this field is due to its important connections with the pricing of contingent claims and stochastic optimisation problems in mathematical finance. Backward stochastic differential equations were firstly introduced by Bismut [15] in 1973 who used a linear driver that appeared naturally when studying the adjoint equations of stochastic optimal control problems. Later, this notion has been generalized by Pardoux and Peng [103] to the Lipschitz case.

To define this special class of stochastic differential equations, take:

- A filtered probability space  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})$  whose filtration satisfies the usual conditions of right continuity and completeness and generated by a  $d$ -dimensional Brownian motion  $B$ .
- $\mathcal{H}^2 = \{ \text{progressively measurable process } Z \text{ s.t. } \mathbb{E}[\int_0^T |Z_s|^2 ds] < \infty \}$ .
- $\mathcal{S}^2 = \{ \text{progressively measurable process } Y \text{ s.t. } \mathbb{E}[\sup_{t \in [0, T]} |Y_t|^2] < \infty \}$ .
- $\mathcal{S}^\infty$  is the set of all progressively measurable a.s. bounded processes  $Y$ .
- A function  $f : (s, \omega, y, z) \rightarrow f(s, \omega, y, z)$  which is measurable w.r.t  $\mathcal{P} \times \mathcal{B}(\mathbb{R}^k) \times \mathcal{B}(\mathbb{R}^{k \times d})$ . Here,  $\mathcal{P}$ ,  $\mathcal{B}(\mathbb{R}^k)$  and  $\mathcal{B}(\mathbb{R}^{k \times d})$  are respectively the  $\sigma$ -field of  $\mathcal{F}$ -progressively measurable sets on  $\Omega \times [0, T]$ , the Borel  $\sigma$ -field on  $\mathbb{R}^k$  and the Borel  $\sigma$ -field of  $\mathbb{R}^{k \times d}$ .
- A positive and finite terminal time  $T$ .
- An  $\mathcal{F}_T$ -measurable random variable  $\xi$ .

Then

$$Y_t = \xi + \int_t^T f_s(Y_s, Z_s) ds - \int_t^T Z_s dB_s, \quad t \in [0, T], \mathbb{P}\text{-a.s.}, \quad (1.1)$$

is called a *Backward Stochastic Differential Equation* associated to  $(f, \xi)$ . The function  $f$  is called *the driver* or *the generator* and the random variable  $\xi$  is called *the terminal value*. Later, we shall write  $\text{BSDE}(f, \xi)$  to refer to the BSDE whose generator is  $f$  and terminal condition is  $\xi$ .

**Definition 1.1.1.** *A solution to the BSDE (1.1) is a pair of  $\mathcal{F}_t$ -adapted processes  $(Y_t, Z_t)_{0 \leq t \leq T} \in \mathcal{S}^2(\mathbb{R}^k) \times \mathcal{H}^2(\mathbb{R}^{k \times d})$  such that (1.1) holds true a.s..*

The theory of BSDEs has received considerable attention and a large number of authors has been interested to this topic. The first important results was given by Pardoux and Peng in their seminal paper [103]. The authors generalized such equations to the non-linear Lipschitz case and proved, in a Brownian framework, the following existence and uniqueness results.

**Theorem 1.1.2.** *(Pardoux, Peng [103]) Assume that  $\{f(t, 0, 0), t \in [0, T]\} \in \mathcal{H}^2$  and that for some constant  $C > 0$ ,  $f$  satisfies the following Lipschitz condition: for all  $t \in [0, T]$  and  $(y, z), (y', z') \in \mathbb{R}^k \times \mathbb{R}^{k \times d}$ .*

$$(L) \quad |f_t(y, z) - f_t(y', z')| \leq C(|y - y'| + |z - z'|).$$

*Then for every  $\xi \in \mathbb{L}^2(\mathcal{F}_T)$ , there exists a unique solution  $(Y, Z) \in \mathcal{S}^2 \times \mathcal{H}^2$  to the BSDE  $(f, \xi)$ .*

Now, we state a comparison result which is very interesting and useful for the construction of solutions with more general drivers. Roughly speaking, a comparison theorem for BSDEs allows one to compare solutions to BSDEs as soon as one can compare the terminal conditions. The main technique to prove this result is a linearization procedure which is part of the standard machinery. This theorem is a powerful tool in the study of 1-dimensional BSDEs and it plays the role of “maximum principle” in the PDE theory. This crucial property of BSDEs was introduced by Peng [105] and later generalized by El Karoui et al. [43].

**Theorem 1.1.3.** *(El Karoui, Peng, Quenez [43], El Karoui, Hamadene, Matoussi [40]) Let  $(Y, Z)$  and  $(Y', Z')$  be solutions of two BSDEs associated to  $(f, \xi)$  and  $(f', \xi')$  where only  $f$  is assumed to satisfy Assumption (L).*

*Assume further that  $\xi \leq \xi'$   $\mathbb{P}$ -a.s. and  $f(t, Y'_t, Z'_t) \leq f'(t, Y'_t, Z'_t)$   $d\mathbb{P} \otimes dt$ -a.s., then*

$$Y_t \leq Y'_t, \quad \forall t \in [0, T], \quad \mathbb{P}\text{-a.s.} \quad (1.2)$$

Later, the interest on BSDEs has been increased steadily and efforts have been made to relax the assumptions on the coefficients under which the existence and eventually the unique-

ness of the solution are guaranteed.

In [34], Darling and Pardoux relaxed the Lipschitz continuity assumption and replace it by a monotonicity condition. More precisely they studied the case of BSDEs with random terminal time. They provide an existence and uniqueness result for a generator which is Lipschitz with respect to  $z$  and monotone with respect to  $y$ .

Besides, Lepeltier and San Martin [76] proved that, in the one dimensional case, it is sufficient to assume that the generator  $f$  is continuous and has a linear growth. However, they only provide the existence of a minimal solution via inf-convolution approximation techniques. Note that in this case, uniqueness does not hold in general. In fact, one needs stronger assumptions on the generator to ensure the uniqueness of the solution. For more details about these conditions, we refer the reader to [85] and [79].

**Continuous Quadratic Backward SDEs** The main innovation on BSDEs since their introduction is probably the study of quadratic BSDEs which turns out to play an important role in applications especially in analyzing the problem of portfolio optimisation under constraints and dynamic risk measures. The quadratic non-linearity in the driver is described by a second order polynomial growth in the control variable  $z$  of the driver i.e.

$$|f(t, y, z)| \leq |l_t| + c_t |y| + \frac{\delta}{2} |z|^2, \quad dt \otimes d\mathbb{P} \text{ a.s.}, \quad (1.3)$$

where  $\delta$  is a positive constant and  $c$  and  $l$  are integrable adapted processes.

In this part, we shall define and recall some important results about quadratic BSDEs. The fundamental problem about studying this type of equations was proving the existence and uniqueness of a solution. Results about this topic can be divided into two parts.

**The case of bounded terminal condition** The first general theoretical results about quadratic BSDEs with bounded terminal condition were obtained by Kobylanski in her PhD thesis and her seminal paper [69]. The author proved existence and uniqueness results for such equations in a Brownian filtration. Her approach relies on an exponential transformation as to come back to a better known framework of BSDEs with a coefficient with a linear growth and then achieves a regularization procedure to pass to the limit. The main difficulty of this approach relies on proving the strong convergence of the martingale parts without imposing very strong assumptions. This method relies heavily on a procedure called the *monotone stability* that we give below.

**Lemma 1.1.4.** (Kobylanski, [69]) *Let  $(f, \xi)$  be a set of parameters and let  $(f^n, \xi^n)$  a sequence of parameters such that*

- The sequence  $(f^n)_n$  converges to  $f$  locally uniformly on  $[0, T] \times \mathbb{R} \times \mathbb{R}^d$  for each  $n \in \mathbb{N}$ ,  $(\xi^n)_n \in L^\infty(\Omega)$  and  $(\xi^n)_n$  converges to  $\xi$  in  $L^\infty(\Omega)$ .

- There exists  $k : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  such that  $k \in L^1([0, T])$  and there exists  $C > 0$  such that

$$\forall n \in \mathbb{N}, \forall (t, y, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d, |f^n(t, y, z)| \leq k_t + C|z|^2.$$

- For each  $n$  the BSDE with parameter  $(f^n, \xi^n)$  has a solution  $(Y^n, Z^n) \in \mathcal{S}^\infty(\mathbb{R}) \times \mathcal{H}^2(\mathbb{R}^d)$  such that the sequence  $(Y^n)_n$  is monotonic and there exists  $M > 0$  such that for all  $n \in \mathbb{N}$ ,  $\|Y^n\|_\infty \leq M$ .

Then, there exists a pair of processes  $(Y, Z) \in \mathcal{S}^\infty(\mathbb{R}) \times \mathcal{H}^2(\mathbb{R}^d)$  such that

- $Y^n$  converges uniformly to  $Y$  on  $[0, T]$  i.e.  $\lim_{n \rightarrow \infty} \sup_{t \in [0, T]} |Y_t^n - Y_t| = 0$ ,  $\mathbb{P}$ -a.s.

- $Z^n$  converges to  $Z$  in  $\mathcal{H}^2$  i.e.  $\lim_{n \rightarrow \infty} \mathbb{E} \left[ \int_0^T (Z_s^n - Z_s)^2 ds \right] = 0$ .

In the Lipschitz-quadratic case, existence and uniqueness of the solution is obtained by Tevzadze in [116] using a totally different approach. More precisely, the author used a fixed point argument to obtain the existence for small terminal values and then pastes solutions together in the general bounded case. The main advantage of this method is that it avoids us to prove the complicated convergence result of the martingale part as in Kobylansky's method.

In [37], Delbaen, Hu and Bao investigated the case when the driver  $f$  has super-quadratic growth with respect to the variable  $z$  which is mathematically traduced as follows

$$\lim_{|z| \rightarrow \infty} \frac{f(z)}{|z|^2} = \infty. \quad (1.4)$$

Their work is divided into two parts. In the first one, they prove the problem is generally ill-posed in the sense that, given a generator satisfying (1.4), there exists a bounded terminal condition for which the associated BSDE does not admit any bounded solution and contrarily, when the BSDE admits a bounded solution, there exist infinitely many bounded solutions for that BSDE. They also notice that the monotone stability result which plays a crucial role in the study of quadratic BSDEs does not hold any more. In the second part, they show that the problem become well-posed in a Markovian setting. To prove the existence of a solution in this setting, they use the well-known result that makes a link between the BSDE( $f, \xi$ ) when  $f$  is quadratic or Lipschitz and the following PDE

$$\begin{cases} \frac{\partial u}{\partial t}(t, x) + \frac{1}{2} \text{Tr}(\sigma \sigma^T(t, x) \frac{\partial^2 u}{\partial x^2}(t, x)) + \frac{\partial u}{\partial x}(t, x) b(t, x) - g(-u_x(t, x) \sigma(t, x)) = 0, \\ u(T, x) = \Phi(x), \end{cases} \quad (1.5)$$

and show that, when  $\Phi$  is bounded and continuous, the BSDE admits a solution.

Briand and Elie provide in [23] a simple approach to solve quadratic BSDEs with bounded terminal condition. Using only direct probabilistic arguments, the authors recover the existence result earlier obtained by Kobylanski. In their approach, they rely heavily on the BMO martingales theory which was used in [55] for the first time in the context of BSDEs. They consider a bounded terminal condition and a Lipschitz-quadratic generator as in [116]. However, instead of a fixed point method, they used an approximation procedure based on Malliavin calculus which leads to a very simple and direct proof.

**The case of unbounded terminal condition** It is well-known that the boundedness of the terminal value requirement, present in all the previous works, is a very strong condition. However, from the point of view of applications, the boundedness of terminal value seems to be very restrictive and not realistic. Moreover, from a theoretical point of view, it is not a necessary condition to obtain a solution. To better illustrate that, let us consider the following example.

$$Y_t = \xi + \int_t^T \frac{|Z_s|^2}{2} - \int_t^T Z_s dB_s, \quad t \in [0, T], \quad \mathbb{P}\text{-a.s.} \quad (1.6)$$

The explicit solution of the quadratic BSDE above is  $Y_t = \ln \mathbb{E}[\exp(\xi) | \mathcal{F}_t]$ . On this simple example, it is obvious to see that the existence of exponential moments of the terminal condition is sufficient to construct a solution to our BSDE (1.6).

Based on this observation, Briand and Hu [24] get rid of the boundedness condition and prove that the existence of an exponential moment of  $\xi$  is a sufficient condition to construct a solution. Under the following assumptions:

**Assumption 1.1.5.** *There exists  $\alpha \geq 0, \beta \geq 0, \gamma > 0$  such that  $\mathbb{P}$ -a.s.*

- (i)  $\forall t \in [0, T], (y, z) \mapsto f(t, y, z)$  is continuous,
- (ii)  $\forall (t, y, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d, |f(t, y, z)| \leq \alpha + \beta|y| + \frac{\gamma}{2}|z|^2,$
- (iii)  $\mathbb{E}[e^{\gamma e^{\beta T} |\xi|}] < \infty.$

Briand and Hu establish the following existence [24] result.

**Theorem 1.1.6** (Briand, Hu [24]). *Let Assumption 1.1.5 hold. Then, the BSDE(1.1) admits at least a solution  $(Y, Z)$ .*

Later, the authors focused on the question of uniqueness for such BSDEs. Trying to fill the gap of uniqueness, they figure out that a stronger assumption is needed on  $z$ . Indeed, the generator has to be convex or concave in its  $z$  component. With this further assumption, it is possible to obtain a comparison theorem to solutions of unbounded quadratic BSDEs.

The existence result in [24] has been revisited later by El Karoui and Barrieu [7] who proposed a completely different approach and consider a forward point of view to treat directly the question of convergence and stability results for a special class of quadratic semi-martingales. The advantage of this method is that the authors obtained a powerful existence result using the stability of certain families of semimartingales which is, compared to the methods used in the literature, simpler and easier to achieve.

**Remark 1.1.7.** *Note that when the terminal condition is bounded, a crucial advantage is that  $\int_0^\cdot Z_s dB_s$  is a BMO-martingale. This property combined with a local lipschitz condition can be used to prove the uniqueness, see for instance [23, 55, 84, 96]. However, this property does not remain true when the terminal value is not bounded. In fact, in this case,  $\int_0^\cdot Z_s dB_s$  is in general not anymore a BMO-martingale.*

### Backward SDEs in a discontinuous framework

**Classical results** We consider in this part a filtered probability space  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})$  whose filtration satisfies the usual hypothesis of completeness and right continuity. We suppose that its filtration is generated by an  $\mathbb{R}^d$ -valued Brownian motion  $B$  and an independent integer valued random measure  $\mu(dt, de)$  defined on  $\mathbb{R}^+ \times E$  with a compensator  $\lambda(dt, de)$ .  $\tilde{\Omega} := \Omega \times [0, T] \times E$  is equipped with the  $\sigma$ -field  $\tilde{\mathcal{P}} := \mathcal{P} \times \mathcal{E}$  where  $\mathcal{P}$  denotes the predictable  $\sigma$ -field on  $\Omega \times [0, T]$  and  $\mathcal{E}$  is the Borel  $\sigma$ -field on  $E$ .

We will also suppose that  $\lambda(dt, de)$  is absolutely continuous with respect to the Lebesgue measure  $dt$  i.e.

$$\lambda(dt, de) = \nu(de)dt,$$

where  $\nu(de)$  is a positive measure satisfying

$$\int_E (1 \wedge |e|^2) \lambda(de) < \infty.$$

Further, we denote by  $\tilde{\mu}$  the compensated jump measure (a martingale)

$$\tilde{\mu}(dt, de) = \mu(dt, de) - \lambda(dt, de) = \mu(dt, de) - \nu(de)dt. \quad (1.7)$$

We also assume that both of the Brownian motion  $B$  and the compensated random measure  $\tilde{\mu}$  of the integer-valued random measure  $\mu$  have the weak predictable representation property with respect to the filtration  $(\mathcal{F}_t)_{0 \leq t \leq T}$ . More precisely, every square integrable local martingale  $M$  has the following representation

$$M_t = M_0 + \int_0^t Z_s dB_s + \int_0^t \int_E U_s(e) \tilde{\mu}(ds, de), \quad 0 \leq t \leq T, \mathbb{P}\text{-a.s.},$$

where  $Z$  is a predictable process such that

$$\int_0^T |Z_s|^2 ds < +\infty, \mathbb{P}\text{-a.s.},$$

and  $U_s(e) : \tilde{\Omega} \rightarrow \mathbb{R}$  is a predictable function such that

$$\int_0^t \int_E |U_s(e)|^2 \nu(de) ds < +\infty.$$

This leads to generalize in a natural way the BSDE (1.1) to the jump case in the following way: We will say that the triplet of  $\mathbb{F}$ -adapted processes  $(Y, Z, U)$  is a solution to the BSDE with a generator  $f$  and a terminal value  $\xi$  if we have

$$Y_t = \xi + \int_t^T f_s(Y_s, Z_s, U_s) ds - \int_t^T Z_s dB_s + \int_t^T \int_E U_s(e) \tilde{\mu}(ds, de), \quad t \in [0, T], \mathbb{P}\text{-a.s.}, \quad (1.8)$$

where  $f$  and  $\xi$  are defined as follows

- $f : [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \times \mathcal{L}^2(\nu) \rightarrow \mathbb{R}$  is a measurable function with respect to the  $\sigma$ -field  $\mathcal{P} \times \mathcal{B}(\mathbb{R}) \times \mathcal{B}(\mathbb{R}^d) \times \mathcal{B}(\mathcal{L}^2(\nu))$ .
- $\xi$  is an  $\mathbb{R}$ -valued  $\mathcal{F}_T$ -measurable random variable.

This generalization of BSDEs to a setting with jumps enlarges the scope of their applications, for instance to insurance modeling, in which jumps are inherent (see for instance Liu and Ma [80]).

The main difference between these equations (BSDEs with jumps) and BSDEs in continuous filtration is that these one involves a second stochastic integral with respect to the compensated random measure  $\tilde{\mu}$  whose intergrand  $U$ , differently from  $Z$ , takes values in an infinite dimensional function space instead of an Euclidean space.

Let us define the following spaces

- $S^\infty = \{\text{adapted processes } Y \text{ with RCLL paths s.t. } \text{ess sup}_t |Y_t| < \infty \text{ a.s.}\}.$
- $\mathbb{H}_\nu^2 = \{\text{measurable processes } U \text{ s.t. } \mathbb{E}\left(\int_0^T \int_E |U_s(e)|^2 \nu(de) ds\right) < \infty\}.$

**Definition 1.1.8.** *A solution to the BSDE with jumps (1.8) is a triple of processes  $(Y, Z, U)$  satisfying (1.8) and it belongs to  $S^\infty \times \mathcal{H}^2 \times \mathbb{H}_\nu^2$ .*

A large number of authors investigated the question of existence and uniqueness of solutions for such equations under different assumptions.

Tang and Li [115] were the first to provide an existence and uniqueness result to the BSDE with jumps (1.8) which generalizes the earlier work of Pardoux and Peng [103] in the continuous setting.

**Theorem 1.1.9** (Tang, Li [115]). *Assume that  $f$  satisfies the following condition*

$$\mathbb{E}\left[\int_t^T |f_s(0, 0, 0)|^2 dt\right] < +\infty,$$

*and that for some constant  $C > 0$ , it satisfies the following Lipschitz condition:*

$$|f_t(y, z, u) - f_t(y', z', u')| \leq K(|y - y'| + |z - z'| + \|u - u'\|)$$

*for all  $t \in [0, T]$  and  $(y, z, u), (y', z', u') \in \mathbb{R} \times \mathbb{R}^d \times L^2(\nu)$ . Then for every  $\xi \in \mathbb{L}^2(\mathcal{F}_T)$ , there exists a unique solution  $(Y, Z, U) \in \mathcal{S}^2 \times \mathcal{H}^2 \times \mathbb{H}_\nu^2$  to the BSDE (1.8).*

This result has been later improved by Pardoux in [101] where the author proved the existence of a unique solution to the BSDE( $f, \xi$ ) in the  $k$ -dimensional case ( $k \in \mathbb{N}^*$ ) under the following assumption.

**Assumption 1.1.10.**

- (i)  $f$  is Lipschitz with respect to  $z$  and  $u$  and monotonic with respect to  $y$ .
- (ii)  $f$  is continuous with respect to  $y$  and there exist an  $\mathbb{R}^+$ -valued adapted process  $(\phi_t)_{0 \leq t \leq T}$  and a constant  $K > 0$  such that

$$\mathbb{E}\left[\int_0^T \phi_s^2 ds\right] < \infty \text{ and } |f(t, y, z, u)| \leq \phi_t + K(|y| + \|z\| + (\int_E |u(e)|^2 \nu(de))^{\frac{1}{2}}).$$

In 1995, Barles, Buckdahn and Pardoux [6] provided a comparison theorem as well as some links between BSDEs and non-linear parabolic integral-partial differential equations, generalizing some results of [103] to the case of jumps. We highlight the fact that their comparison result was obtained under a strong assumption on the generator which has the following form

$$f_t(y, z, u) := h_t(y, z, \int_E u(e) \bar{\gamma}_s(e) \nu(de)), \tag{1.9}$$

where  $\bar{\gamma} : \Omega \times [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$  is measurable and satisfies  $0 \leq \bar{\gamma}(e) \leq C(1 \wedge |e|), \forall e \in \mathbb{R}$  and  $h : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$  is measurable and satisfies

$$(\omega, t, y, z, q) \mapsto h(\omega, t, y, z, q)$$

**Assumption 1.1.11.**

- (i)  $\mathbb{E}\left(\int_0^T |h(s, 0, 0, 0)|^2 ds\right) < +\infty.$
- (ii)  $h$  is Lipschitz with respect to  $y, z$  and  $q.$
- (iii)  $h$  is non-decreasing with respect to  $q.$

In order to weaken the previous conditions on the generator and obtain the comparison result, Royer introduced the following  $(A_\gamma)$ -condition in [112].

**Assumption 1.1.12.  $(A_\gamma)$ -condition**

There exists  $-1 < C_1 \leq 0$  and  $C_2 \geq 0$  such that  $\forall y \in \mathbb{R}, \forall z \in \mathbb{R}^d, \forall u, u' \in L^2(\nu)$

$$f_t(y, z, u) - f_t(y, z, u') \leq \int_E \gamma_t^{y, z, u, u'}(e)(u - u')(e)\nu(de), \quad (1.10)$$

where  $\gamma^{y, z, u, u'} : \Omega \times [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$  is measurable with respect to all the variables and satisfies  $C_1(1 \wedge |e|) \leq \gamma_t(e) \leq C_2(1 \wedge |e|).$

The main difference between the  $\gamma$  and the  $\bar{\gamma}$  in (1.9) is that  $\gamma$  is allowed to depend on  $y, z, u$  and  $u'$  and it can be negative as soon as it remains larger than  $C_1(1 \wedge |e|).$  Thanks to the  $(A_\gamma)$ -condition, Royer obtained the following result.

**Theorem 1.1.13.** (Royer [112]) Consider two BSDEs with jumps with parameters  $(f^1, \xi^1)$  and  $(f^2, \xi^2)$  such that  $f^1$  satisfies 1.1.10,  $f^2$  satisfies 1.1.12 and the terminal conditions  $\xi^1$  and  $\xi^2 \in L^2(\mathcal{F}_T).$  Denote by  $(Y^1, Z^1, U^1)$  and  $(Y^2, Z^2, U^2)$  the respective solutions. If  $\xi^1 \leq \xi^2$  and  $f^1(Y^1, Z^1, U^1) \leq f^2(Y^1, Z^1, U^1),$   $\mathbb{P}$ -a.s., then  $\forall t \in [0, T], Y_t^1 \leq Y_t^2.$

Latar, Becherer emphasizes in [9] that the Lipschitz assumption on the generator used in the previous works is very restrictive and that a square integrability property of the solutions is not enough in applications. Hence, he extends results of Tang and Li [115] and Barles, Buckdahn and Pardoux [6] on square integrable solutions to solutions with more integrability and with a random measure which is possibly inhomogeneous in time but of finite jump activity which turns out to be, from the point of view of applications, more convenient. His result also covers a family of generators that satisfy a certain monotonicity property but need not to be Lipschitz in the jump component. The author uses these results in order to solve an exponential optimisation problem in a model with non-predictable jump risk.

**Quadratic Backward SDEs with jumps** In contrast to the diffusion setup, when it comes to quadratic BSDEs in a discontinuous setting, the literature has been rather small. The only existing results until recently concern particular cases of quadratic BSDEs that appear in utility

maximisation or indifference pricing problems in a jump framework.

Non lipschitz-quadratic generators in  $Z$  have been studied by Morlais in [97, 98] where the author proved the existence of the solution of a special quadratic BSDE with jumps who's generator is given by

$$f_s(z, u) = \inf_{\pi} \left( \frac{\alpha}{2} |\pi \sigma_s - (z + \frac{\theta}{\alpha})|^2 + |u - \pi \beta_s|_{\alpha} \right) - \theta_s z - \frac{|\theta|^2}{2\alpha}, \quad (1.11)$$

for a bounded terminal condition. The appearance of this kind of quadratic BSDE with jumps comes from a practical financial issues.

The general case of quadratic BSDEs with jumps when the terminal value is unbounded is studied by Ngoupeyou in his PhD thesis [100] and the subsequent papers by El Karoui, Matoussi and Ngoupeyou [41] and by Jeanblanc, Matoussi and Ngoupeyou [59] where the authors solve a utility maximisation problem from terminal wealth and intermediate control under model uncertainty. Thus, to solve such a problem, they extend the techniques developed in [7] to a jump setting and managed to obtain existence of the solutions for quadratic BSDEs in a discontinuous framework with an unbounded terminal value.

Recently, Kazi-Tani, Possamai and Zhou [109], extend the fixed point methodology of Tevzadze [116] to a discontinuous setting. In fact, the authors prove an existence result for a terminal condition  $\xi$  having a small  $\|\cdot\|_{\infty}$ -norm then for any  $\xi$  in  $\mathbb{L}^{\infty}$  by splitting  $\xi$  in pieces having a small enough norm and thus they just past the obtained solutions to obtain a solution of the equation.

### 1.1.2 BSDEs and connection with semi-linear PDEs

In this part, we recall briefly the relationship between BSDEs and partial differential equations (PDEs for short). Most of these results appear to be useful for problems in financial mathematics, stochastic control and differential games (see [43] and [48] ) and provide a probabilistic interpretation for semi-linear partial differential equations etc and mainly in the problem of pricing American options that we will consider later. We also refer the reader to El Karoui, Hamadene and Matoussi [40] for some applications.

The link between BSDEs and semi-linear PDEs can be established by considering Markovian BSDEs where the randomness of the generator and the terminal condition comes from a diffusion process  $(X_s, 0 \leq s \leq T)$  which is the strong solution of a standard Ito's stochastic

differential equations

$$X_t = x + \int_0^t b(X_s)ds + \int_0^t \sigma(s, X_s)dW_s, t \in [0, T], \mathbb{P}\text{-a.s.}$$

Consider the following semi-linear PDE

$$\begin{cases} \frac{\partial u}{\partial t}(t, x) + \mathcal{L}u(t, x) + f(t, x, u(t, x), \nabla u(t, x)\sigma(t, x)) = 0, (t, x) \in [0, T] \times \mathbb{R}^d \\ u(T, x) = g(x), \end{cases} \quad (1.12)$$

where  $\mathcal{L}$  is the second order differential operator associated to  $X$  given by

$$\mathcal{L}u(t, x) := \frac{1}{2} \text{Tr}[\sigma(t, x)\sigma'(t, x)\nabla^2 u(t, x)] + b(t, x) \cdot \nabla u(t, x),$$

Under suitable assumptions on the  $f, g, b$  and  $\sigma$ , the PDE (1.12) has a classical smooth solution and then the processes  $(Y, Z) = (Y_t, Z_t)_{t \in [0, T]} := (u(t, X_t), \nabla u(t, X_t)\sigma(t, X_t))_{t \in [0, T]}$  solves the following BSDE

$$Y_t = g(X_T) + \int_t^T f_s(X_s, Y_s, Z_s)ds - \int_t^T Z_s dW_s, t \in [0, T], \mathbb{P}\text{-a.s.}$$

This link gives a probabilistic interpretation for solutions of the semi-linear PDE (1.12) using the solution of the BSDE and generalizes the Feynman-Kac formula to a semi-linear case. From that interpretation, one can use probabilistic methods for numerical simulations of solutions of semi-linear PDEs.

### 1.1.3 Expected utility maximisation problem and stability analysis

#### Expected utility maximisation theory

Financial theory in general and mathematical finance in particular aim to describe and understand the behavior of a rational agent faced with uncertain evolution of asset prices. In this case, the mathematical tool used to solve this problem is *the expected utility maximisation theory* which is also called *the portfolio choice theory*.

This problem has been firstly introduced by Von Neumann and Morgenstern in [95] where the authors suppose that the preferences of an investor can be represented by a utility function  $U : \mathbb{R} \rightarrow \mathbb{R}$  which describes the risk preferences of the agent when endowed with an amount of wealth  $x \in \mathbb{R}$ . This function is always increasing and concave so that the agent always prefers more wealth to less and the higher is his wealth, the agent is less sensitive to variations in it.

Think of an agent who starts at time  $t$  with an initial capital  $x$ , trades in the market in a

self-financing way and has the obligation of paying the amount  $\xi$  at time  $T$  represented by an  $\mathcal{F}_T$ -random variable. Naturally, the agent's objective is to maximize her/his expected utility  $\mathbb{E}[U(X_T)]$  over all possible random variables  $X_T$  she/he can generate by implementing different trading strategies. Mathematically, he/she aims to solve the following optimisation problem

$$u(t, x) = \sup_{\pi \in \mathcal{A}} \mathbb{E}[U(X_T^{x, \pi} - \xi)], \quad (1.13)$$

where  $\pi$  is the trading strategy,  $\mathcal{A}$  is the set of all admissible trading strategies on  $[t, T]$  and  $\xi$  denotes the random liability that he must deliver at time  $T$ .

The problem (1.13) has been firstly addressed by Merton in [90, 91] when the risky assets follow a Black-Scholes model in a complete market, the utility function is of power type and no constraints are made on the strategy's set  $\mathcal{A}$ .

In the literature, the main questions that have been treated concerning this problem are essentially related to:

- The characterization of the value process.
- The existence and uniqueness of optimal strategies.
- The characterization of the optimal strategies.

A huge number of authors tried to solve the problem (1.13). To do so, two main approaches have been used. In the following, we describe briefly these two methods.

**The convex duality approach** In this approach, one looks at a dual problem associated to the primal one (1.13). It consists of a minimization problem with an objective function given by a functional of the convex conjugate of  $U$  which we denote by  $V$  and it is defined by

$$V(y) = \sup_{x > 0} (U(x) - xy), \forall y > 0. \quad (1.14)$$

The dual domain consists of the set of super martingale defaltors for  $S$  which is an enlargement of the set of densities of equivalent local martingales measures for  $S$ . Under the so called *reasonable asymptotic elasticity assumption* which ensures that  $U$  has power growth for large values, the dual problem admits a solution called the *dual optimizer*  $\hat{Y}$ . In fact, it is the solution of a minimization problem given by

$$v(y) = \inf_{\mathbb{Q} \in \mathcal{M}} \mathbb{E}[V(y) \frac{d\mathbb{Q}}{d\mathbb{P}}],$$

where  $\mathcal{M}$  is the set of probability measures  $\mathbb{Q}$  equivalent to  $\mathbb{P}$  under which the price  $S$  is a  $\mathbb{Q}$ -local martingale with finite V-entropy (i.e.  $\mathbb{E}[\frac{d\mathbb{Q}}{d\mathbb{P}}] < \infty$ ). The existence of a solution to (1.13) is

then retrieved via standard arguments from convex analysis.

The convex duality method is originally initiated by Bismut [15] and it has been firstly adopted by Karatzas et al. [62] and [49] in a complete market model i.e. under the existence of a unique equivalent martingale measure. Later, Karatzas et al. [63] get rid of this restrictive and unrealistic assumption on the market model (completeness) and solved the problem in an incomplete market.

In the general setting of semimartingales, we quote Kramkov and Schachermayer [70] and Schachermayer [114]. Delbaen et al. [36] give a duality result between the optimal strategy for the exponential utility maximisation and the martingale measure which minimize the relative entropy with respect to the real world measure  $\mathbb{P}$ .

Unfortunately, the convex duality approach provides only the existence of optimal strategies results but it doesn't give any characterization of the value process or the optimal strategies.

Another drawback of this approach comes from the perspective of numerical approximation. More precisely, it does not allow so far computations and simulations of value functions and optimal strategies.

**The stochastic approach** This approach is provided by an interpretation of the martingale optimality principle using backward stochastic differential equations via stochastic dynamic control techniques. Recall that, for a long time, these techniques had been used in finance only in a Markovian setting. In [111], Rouge and El Karoui proved that the solution of a utility maximisation problem can be characterized as a solution of a BSDE when the agent preferences are of exponential type and the market is complete. In this case, the authors compute the value function by means of BSDEs assuming restrictively that the strategies are constrained in a convex cone.

Since then, the problem has been considered by numerous authors and a standard approach has been to try converting the utility maximisation problem into a BSDE type stochastic control problem. This work has been extended later by Imkeller, Hu and Muller in [55] to the case of power and logarithmic utility functions when admissible strategies take their values in a closed subset which might be non-convex. In this work, the authors rely on *the martingale optimality principle* given below.

**Martingale optimality principle** Construct a family of processes  $(R_t^\pi)_{0 \leq t \leq T}$  where  $\pi \in \mathcal{A}$  satisfying

(i)  $R_T^\pi = U(X_T^\pi - \xi)$  for all  $\pi \in \mathcal{A}$ .

(ii)  $R_0^\pi = R_0$  for all  $\pi \in \mathcal{A}$ .

(iii)  $R^\pi$  is a supermartingale for all  $\pi \in \mathcal{A}$  and there exists an optimal strategy  $\pi^* \in \mathcal{A}$  such that  $R^{\pi^*}$  is a martingale.

Suppose that the family  $(R_t^\pi)_{0 \leq t \leq T}$  satisfying (i), (ii) and (iii) above exists, then

$$\mathbb{E}[U(X_T^\pi - \xi)] = \mathbb{E}[R_T^\pi] \leq R_0^\pi = u(x) = \mathbb{E}[R_0^{\pi^*}] = \mathbb{E}[U(X_T^{\pi^*} - \xi)]. \quad (1.15)$$

Therefore,  $\pi^*$  is the optimal strategy of (1.13). Relying on the relation (1.15) and on results in [55] and [111], we obtain the equivalence between solving the optimisation problem (1.13) and a BSDE

$$Y_t = \xi + \int_t^T f_s(Y_s, Z_s) ds - \int_t^T Z_s dB_s, \quad t \in [0, T], \quad \mathbb{P}\text{-a.s.}$$

The problem of utility maximisation with a general utility function was solved by Horst et al. [54]. For the case of continuous and discontinuous filtration with an exponential utility function see Mania and Schweizer [84], Morlais [96] and Becherer [9].

The main advantage of this approach in the portfolio choice problems is that, contrary to convex duality methods, BSDEs can also deal with non-convex trading constraints. Another advantage of using BSDEs is that their solutions can be computed numerically efficiently by Monte Carlo simulation.

Nonetheless, when it comes to portfolio choice problems in a setting with jumps, the literature is far less abounding.

### Stability analysis

As in any optimisation problem, once the expected utility problem is solved, the second problem might concern stability issues. A large number of authors investigate the behavior of their problem's solution under perturbations of the different input variables. Perturbations here may represent for example uncertain evolution of asset prices or changes in the investor's preferences. To deal with this kind of situations, stability analysis has been recently developed to understand and describe the behavior of the investor faced to such problems. The main question here is then the following: How is the agent's behavior affected by changes in his problem's input parameters?

In the literature, two main types of stability results have been considered:

- **Stability with respect to the agent's preferences:** More precisely, when the one considers a sequence of utility functions  $U^n$  which converges to  $U$  in some sense and then investigates the convergence of the optimal objects. The stability analysis of solutions with respect to risk preferences and initial capital was initiated by Jouini and Napp in [60]. The authors consider an Itô-process model and study the stability of the optimal investment-consumption

strategy  $(\pi, c)$  with respect to the choice of the utility function defined as in [91] as follows

$$U(c, X) = \mathbb{E} \left[ \int_0^t u(t, c_t) dt + V(X) \right].$$

Their utility maximisation problem is the following

$$\sup_{(\pi, c) \in \mathcal{A}(x)} U(c, X_T^{x, \pi, c}),$$

where  $\mathcal{A}(x)$  is the set of admissible investment-consumption strategies. The authors consider a sequence of utility functions that converges pointwise and satisfies some growth property and prove the convergence of the optimal wealth and consumption a.s. and in  $L^p$  for  $p \geq 1$  as well. They also show that, if the marginal utilities are convex, the  $L^1$ -convergence of the optimal investment process in the general case as well as its convergence in a Markov setting can be obtained.

In [71], Larsen extends results of [60] to the case of incomplete markets with continuous semimartingale dynamics and give a weaker convergence result. However, he only proves the convergence in probability of the optimal wealth.

Karadzas and Zitkovic in [66] proved a stability result for the utility maximisation problem in a general semimartingale setting in the presence of liquid assets and random endowments. In this work, the authors keep the financial market's input and the random endowment fixed and make perturbations on the investor's preferences ( i.e. his utility function and the subjective probability). More precisely, they vary  $U$  and  $\mathbb{P}$  by means of sequences  $U^n$  with a limiting function  $U$  and  $\mathbb{P}^n$  with the limiting probability measure  $\mathbb{P}$ . They make assumptions on the mode of convergence of these sequences in the following way

$$\left\{ \begin{array}{l} \forall n \in \mathbb{N}, \mathbb{P}^n \sim \mathbb{P}, \lim_{n \rightarrow +\infty} \mathbb{P}^n = \mathbb{P} \text{ in total variation ,} \\ \lim_{n \rightarrow +\infty} U^n = U \text{ pointwise.} \end{array} \right.$$

Stability of the exponential utility maximisation problem with respect to small perturbations on the agent's preferences is studied in [119] where Xing considers two different setting. In the first part, the author considers a general semimartingale model where random endowments are present and proved the stability under the following conditions

$$\left\{ \begin{array}{l} \text{The sequence of utility function defined on } \mathbb{R} \text{ converges pointwise to the exponential utility:} \\ \lim_{\delta \rightarrow 0} U_\delta(x) = -\exp(-x)^2, \\ \text{The ratio of marginal utilities is bounded from above and away from zero, uniformly in } \delta \end{array} \right.$$

• **Stability with respect to perturbations in the model:** More precisely, it is the case when the utility function is fixed and the underlying market inputs vary.

In [72], Larsen and Zitkovic studied the stability with respect to perturbations in the market price of risk for a fixed volatility and for a utility function defined on the positive real line. Their problem is set in a general filtration (i.e. right continuous and complete) generating one-dimensional continuous local martingale  $M$ . Variations on the model are made by the sequence  $(\lambda_n)_{n \in \mathbb{N}}$  of the market price of risk processes which amounts the following sequence of stock price process

$$dS_t^{\lambda_n} = \lambda_n(t)d\langle M \rangle_t + dM(t),$$

where  $\langle M \rangle = (\langle M \rangle_t)_{t \in [0, T]}$  is its quadratic variation. Their utility maximisation problem is then

$$u^{\lambda^n}(x) = \sup_X \mathbb{E}[U(X_T^{\lambda^n})],$$

where  $U$  is a utility function defined on  $\mathbb{R}_+$ . When  $\lambda^n(t)$  converges in an appropriate topology and under a  $V$ -compactness assumption, the authors proved the stability of  $u^{\lambda^n}$  and the terminal wealth process  $X_T^{x, \lambda^n}$  when the probability measure  $\mathbb{P}$  is fixed. The  $V$ -compactness condition is the following

**Assumption 1.1.14.** *The set  $\{V(Z_T^n), n \in \mathbb{N}\}$  is uniformly integrable where, for  $n \geq 1$ ,  $Z^n$  is defined by  $Z_t^n = \mathcal{E}(-\lambda^n \cdot M)_t = \exp(-\int_0^t \lambda_u dM_u - \frac{1}{2} \int_0^t \lambda_u^2 d\langle M \rangle_u)$  (the  $n$ -th minimal martingale measure) and  $V$  denotes the convex dual of  $U$  which is defined as in (1.14).*

in [8], Bayraktar and Kravitz solve the same problem of [72] when the utility function is defined on the whole real line working on the typical example of exponential utility. In their work, they provide a stability result of the exponential utility maximisation problem and proved that in addition to the  $V$ -compactness assumption, made in [72] and used to establish a lower semi-continuity, they need another condition related to a local BMO hypothesis in order to establish an upper semi-continuity. More precisely they made the following regularity assumption

**Assumption 1.1.15.** *There exists a sequence of stopping times  $\tau_j \uparrow T$  such that, for each  $j$ ,*

$$\sup_n \|(\lambda^n \cdot M)^{\tau_j}\|_{BMO_2} < \infty,$$

where  $BMO_2$  is the set of martingales  $R$  (not necessarily continuous) where there exists a constant  $r$  such that  $\mathbb{E}[|R_T - R_\tau|^2 | \mathcal{F}_\tau]^{\frac{1}{2}} \leq r$  for all stopping times  $\tau \in [0, T]$ .

Under this assumption, Bayraktar and Kravitz obtained the following stability result.

**Theorem 1.1.16.** *(Bayraktar, Kravitz [8]) Suppose that  $Z_T^n \rightarrow Z_T^\infty$  in probability and  $Z^\infty$  is a martingale. Suppose also that the  $V$ -compactness Assumption 1.1.14 and the regularity Assumption 1.1.15 are satisfied. Then  $u^n(x) \rightarrow u^\infty(x)$  pointwise and hence uniformly.*

Note that both [8] and [72] consider risky assets with continuous price process and no random endowment.

In [45], Frei relies on a BSDE stability result, in a general semimartingale setting, to study two different utility maximisation problems. Indeed, in the first part he proved the stability of the utility maximisation problem when perturbations are made in the payoff and the constraints set. More precisely, the author considers an investor, with exponential preferences, trading in the following markets

$$dS_s^j = S_s^j(dM_s^j + \sum_{i=1}^d \lambda_s^i d\langle M^j, M^i \rangle_s), \quad 0 \leq s \leq T, \quad S_0^j > 0, \quad \text{for } j = 1, \dots, d,$$

where  $M$  is a local martingale and  $\lambda$  is the market price of risk. His utility maximisation problem is given by

$$V_t^{H,C} = \text{ess sup}_{\pi \in \mathcal{A}_t^C} \mathbb{E}[U(X_T^{x,\pi} + H) | \mathcal{F}_t], \quad (1.16)$$

where  $\pi$  represents a trading strategy,  $H$  a bounded random variable representing the payoff,  $\mathcal{A}_t^C$  is the set of all admissible strategies and  $C$  is a constraints set which is a closed subset of  $\mathbb{R}^d$  containing zero. He varies the payoff  $H$  and the constraint's set  $C$  by means of sequences  $(H^n)_{n \in \mathbb{N}}$  and  $(C^n)_{n \in \mathbb{N}}$ . Hence he proves that, under the following assumptions,

$$\left\{ \begin{array}{l} \text{Convergence of the payoff sequence: } \lim_{n \rightarrow \infty} H^n = H^\infty \text{ a.s.,} \\ \text{The sets } C^n \subset \mathbb{R}^d \text{ are Wijsman-convergent to a limiting set } C^\infty \subset \mathbb{R}^d, \end{array} \right.$$

the value function  $V^{H^n, C^n}$  converges a.s. to  $V^{H^\infty, C^\infty}$ . He also proves the convergence of the indifference price function.

In the second part of [45], the author extends the result to the case of a fixed market price of risk and a varying underlying correlation factor between the traded and the non-traded securities.

In [65], Kardaras considers a financial market with continuous-path asset prices and proves the stability of the numeraire portfolio with respect to the filtration representing informations available to the agent, the probability measure representing the state of nature and the constraints set representing possible restrictions on the investment strategies, faced by the agent. More precisely, he makes infinitesimal changes on these parameter and proves the convergence of the log-wealth of the numeraire portfolios.

Mocha and Westray proved, in [94], the stability of the value function of the utility maximisation problem associated to a sequence of markets via BSDEs. Indeed, the authors established a relationship between the optimisation problem and the solution of a quadratic semimartingale BSDE and then established the continuity of the optimizers with respect to the input parameters.

### 1.1.4 Main results and contributions

Our starting point in this part is an optimisation problem: An investor trading in a sequence of particular financial markets, that we precise later, aims to maximize his expected utility. Naturally, his objective is to find the optimal strategy in order to reach his goal. Once this step is achieved, the investor might wonder: *What happens if the input parameters of the markets change suddenly?*

In the second chapter of this thesis, our aim is to prove the stability of the utility maximisation problem in nonequivalent markets with respect to perturbations in both drifts and volatilities, in a continuous and discontinuous setting using dynamic methods.

The basic purpose of this chapter is to find assumptions which guarantee that over all markets, the value function converges. More precisely, we seek sufficient conditions on the market parameters that ensure the following

$$u^n(x) = \mathbb{E}[U(X_T^{n,\pi^n} - \xi)] \rightarrow u^\infty(x) = \mathbb{E}[U(X_T^{\infty,\pi^\infty} - \xi)], \text{ when } n \rightarrow +\infty \quad (1.17)$$

#### Motivation, formulation of the problem in a continuous setting and contribution

We adopt the idea that the market model's inputs are important in the utility maximisation problem. The motivation behind this work comes from the recent paper [117] where the author studied the stability of the utility maximisation problem with random endowment and indifference prices for a sequence of financial markets in an incomplete Brownian setting. The novelty lies in the nonequivalence of markets, in which the volatility of asset prices as well as the drift varies. Degeneracies arise from the presence of nonequivalence. When the utility function is defined on the positive real line, Weston shows via a counterexample that the expected utility maximisation problem can be unstable. However, she proved a positive stability result for utility functions defined on the entire real line (e.g. exponential utility).

Let us start by describing briefly the method in [117]. The sequence of nonequivalent financial markets is taken as follows

$$dS_t^n = \lambda_t^n d\langle M^n \rangle_t + dM_t^n, \quad (1.18)$$

where, for  $n \geq 1$ ,  $M^n = \sum_{i=1}^d (\sigma^{n,i} \cdot B^i)$ ,  $B = (B^1, \dots, B^d)$  is a  $d$ -dimensional Brownian motion defined on  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ ,  $\sigma^n = (\sigma^{n,1}, \dots, \sigma^{n,d})$  where  $\sigma^{n,i} \in \mathbb{L}^2([0, T])$ ,  $\lambda^n |\sigma^n|^2 \in \mathbb{L}^1([0, T])$  and finally a contingent claim  $f$  given by a bounded random variable. The utility function  $U : \mathbb{R} \rightarrow \mathbb{R}$  is supposed to be finite on the whole real line. The primal optimisation problem associated to an investor starting with an initial capital  $x$ , trading in the sequence of financial markets  $S^n$  via

a strategy  $H$ , is the following

$$u^n(x) = \sum_{H \in \mathcal{H}^n} \mathbb{E}[U(x + (H.S^n)_T + f)], x \in \mathbb{R}, \quad (1.19)$$

where  $\mathcal{H}^n$  is the set of all  $S^n$ -integrable trading strategies  $H$  such that, for all  $t$ , there exists  $K$  satisfying  $(H.S^n)_t \geq -K$ .

The dual optimisation problem associated to (1.19) for the  $S^n$ -markets is defined, for  $n \geq 1$  by

$$v_n(y) = \inf_{\mathbb{Q} \in \mathcal{M}_y^n} \mathbb{E}\left[V\left(y \frac{d\mathbb{Q}}{d\mathbb{P}}\right) + y \frac{d\mathbb{Q}}{d\mathbb{P}} f\right], y > 0, \quad (1.20)$$

where  $\mathcal{M}_y^n$  is the set of measures  $\mathbb{Q} \sim \mathbb{P}$  having finite V-entropy i.e.  $\mathbb{E}[V(\frac{d\mathbb{Q}}{d\mathbb{P}})] < \infty$ .

In order to prove the stability of the value function (1.19), various assumptions have been made in [117]. More precisely, she made assumptions on the sequence of markets and the limiting market.

In fact, Weston first makes convergence assumptions on the sequence of markets in the following way.

**Assumption 1.1.17.** *The sequences  $(M^n)_{n \geq 1}$  and  $(\lambda^n.M^n)_{n \geq 1}$  converge respectively to  $M^\infty$  and  $(\lambda^\infty.M^\infty)$  in the semimartingale topology as  $n \rightarrow \infty$ .*

The second assumption was made on the minimal martingale density process  $Z_T^n = \frac{d\mathbb{Q}^n}{d\mathbb{P}}$ , where  $S^n$  is a  $\mathbb{Q}^n$ -local martingale, as follows.

**Assumption 1.1.18.** *For  $n \geq 1$ , each minimal martingale density process  $Z^n$  is a  $\mathbb{P}$ -martingale.*

The author also requires the following non-degeneracy assumption.

**Assumption 1.1.19.**  $\sum_{i=1}^d (\sigma_t^{\infty,i})^2 \neq 0$ , for all  $t \in [0, T]$ ,  $\mathbb{P}$ -a.s.

An assumption on the limiting dual problem has been made in the following way.

**Assumption 1.1.20.** *The limiting dual problem can be expressed by*

$$v_\infty(y) = \inf_{L \in \mathcal{B}} \mathbb{E}\left[yV(Z_T^\infty \mathcal{E}(L)_T) + yZ_T^\infty \mathcal{E}(L)_T f\right], y > 0.$$

Finally, a V-compactness Assumption 1.1.14 is also made as in [72], [8]. Under the conditions above, the author obtained the following stability result.

**Theorem 1.1.21.** (Weston [117]) *Suppose that the sequence of markets satisfy 1.1.17, 1.1.18 and 1.1.14. Suppose that the limiting market satisfies Assumptions 1.1.19 and 1.1.20. Then the value function (1.19) converges i.e.*

$$\lim_{n \rightarrow \infty} u_n(x) = u_\infty(x). \quad (1.21)$$

To prove that, the author establishes a lower and upper semi-continuity result for the sequence of primal and dual value functions. More precisely, she proves in a first step that

$$u_\infty(x) \leq \liminf_{n \rightarrow \infty} u_n(x),$$

and in a second step that

$$v_\infty(y) \geq \limsup_{n \rightarrow \infty} v_n(y).$$

In the first part of this thesis, we consider, for  $n \geq 1$ , a sequence of financial markets consisting of one risk-free asset with zero interest rate and  $d$  risky assets  $S^{n,d} = (S^{n,1}, S^{n,2}, \dots, S^{n,d})$  with dynamics

$$\begin{cases} dS_t^n &= S_t^n \sigma_t^n (\theta_t^n dt + dB_t), \\ S_0^n &= 1, \end{cases} \quad (1.22)$$

where  $\theta_t^n = (\theta_t^{n,1}, \dots, \theta_t^{n,d})$  and  $\sigma_t^n = (\sigma_t^{n,1}, \dots, \sigma_t^{n,d})$  are bounded and progressively measurable processes. We assume that  $\sigma^n$  is invertible with bounded inverse process for all  $n \geq 1$ . Financially,  $\theta^n$  is the sequence of the market price of risk.

In the following, we define the notion of trading strategies in our context.

**Definition 1.1.22.** For each  $n \geq 1$ , we call a trading strategy every  $d$ -dimensional  $\mathcal{F}$ -predictable process  $\pi_t^n = (\pi_t^{n,1}, \dots, \pi_t^{n,d})$  where  $\pi^{n,i}$  describes the amount of money invested in each stock  $i$  at time  $t$  in each  $S^n$ -market and such that the stochastic integral  $\int_0^\cdot \pi_s \frac{dS_s}{S_s}$  is well defined i.e. when

$$\mathbb{E} \left( \sum_{i=1}^d \int_0^T \left| \frac{\pi_s^i}{S_s^i} \right|^2 d\langle S^i \rangle_s \right) < \infty.$$

The situation is the following: consider a financial agent who begins with an initial endowment  $x > 0$  and who trades in the sequence of markets (1.22) according to a trading strategy  $\pi^n$ . His associated wealth process is then given at any time  $t$  by

$$X_t^{n,\pi} = x + \sum_{i=1}^d \int_0^t \pi_u^{n,i} \frac{dS_u^{n,i}}{S_u^{n,i}} = x + \int_0^t \pi_u^n \sigma_u^n (\theta_u^n du + dB_u), \quad \forall t \in [0, T], \quad \mathbb{P}\text{-a.s.} \quad (1.23)$$

We also assume that the investor pays a liability  $\xi$  at maturity  $T$ , that is to say an  $\mathcal{F}_T$ -measurable random variable which could represent the value of any option or contract maturing at time  $T$ . Therefore, at time  $T$ , the wealth process becomes

$$X_T^{n,\pi,x} - \xi = x + \int_0^T \pi_u^n \frac{dS_u^n}{S_u^n} - \xi = x + \int_0^T \pi_u^n \sigma_u^n (\theta_u^n du + dB_u) - \xi, \quad \mathbb{P}\text{-a.s.}$$

In this part, we assume that the agent's preferences are modeled by an increasing and concave function  $U$  which is, in this part, supposed to be defined on  $\mathbb{R}^+$ . We take the typical example of

power utility function defined as follows

$$U(x) = \frac{x^\gamma}{\gamma}, \gamma \in ]-\infty, 0[ \cup (0, 1), \forall x > 0. \quad (1.24)$$

Due to the domain restriction of the power utility function, we shall define the wealth process in a multiplicative way taking as control  $\rho_t^n = \frac{X_t^n}{\pi_t^n}$  which represents the part of wealth invested in the stock.

The objective of the investor is to maximize the expected utility of her terminal wealth  $\mathbb{E}[U(X_T^{n,\pi,x} - \xi)]$  over all her possible admissible strategies. We now state the definition of admissible trading strategies.

**Definition 1.1.23.** For  $n \geq 1$ , a  $d$ -dimensional predictable process  $\pi^n = (\pi_t^n)_{0 \leq t \leq T}$  is called an admissible trading strategy and we write  $\pi_t^n \in \tilde{\mathcal{A}}_t$  if it satisfies

1.  $\pi_t^n$  is  $M^n$ -integrable i.e.

$$\mathbb{E}\left[\int_0^t |\pi_s^n|^2 d\langle M, M \rangle_s\right] < \infty \text{ i.e. } \mathbb{E}\left[\int_0^t |\pi_s^n \sigma_s^n|^2 ds\right] < \infty.$$

2. For all  $n \geq 1$ ,  $\pi^n \in C$ ,  $\mathbb{P}$ -a.s. where  $C$  is an  $\mathbb{R}^d$ -closed set containing zero and it is called the constraint set.

First formulation of the value process The agent is then facing the following optimisation problem

$$\begin{aligned} u^n(x) &:= \sup_{\rho \in \tilde{\mathcal{A}}_t} \mathbb{E}[U(X_T^{\rho,x,n} - \xi)] \\ &= \sup_{\rho \in \tilde{\mathcal{A}}_t} \mathbb{E}\left[U\left(x + \int_0^T X_s^{\rho,x,n} \rho_s^n \sigma_s^n (\theta_s^n ds + dB_s) - \xi\right)\right]. \end{aligned}$$

For simplicity, we make the following change of variables:  $p_t^n = \pi_t^n \sigma_t^n$ ,  $C_t^n = C \sigma_t^n$  and  $\mathcal{A}_t^n = \tilde{\mathcal{A}}_t \sigma_t^n$  and get the following second formulation of the problem.

Second formulation of the value process The agent aims to solve the following problem

$$u^n(x) = \sup_{p \in \tilde{\mathcal{A}}_t^n} \mathbb{E}\left[\frac{x^\gamma}{\gamma} \mathcal{E}\left(\int_0^t \gamma p_s^n (\theta_s^n ds + dB_s)\right)\right], \quad (1.25)$$

where  $p_t^n \in \tilde{\mathcal{A}}_t^n$  iff  $\mathbb{E}\left[\int_0^T |p_s^n|^2 ds\right] < \infty$  and  $p_t^n \in C_t^n$  ( $C_t^n$  is also closed).

The approach adopted here for determining the value function  $u^n$  and the optimal control  $p^{n*}$  is classical and based on the dynamic programming principle as in [55]. More precisely, it

consists on the martingale optimality principle in order to obtain a characterization of the value process in terms of BSDE in the following way.

**Theorem 1.1.24.** (Imkeller, Müller, Hu [55]) For  $n \geq 1$ , the value function of the optimisation problem (1.25) is given by

$$u^n(x) = \frac{x^\gamma}{\gamma} \exp(Y_0^n), \quad \text{for } x > 0,$$

where  $Y_0^n$  is defined by the unique solution  $(Y^n, Z^n)$  of the BSDE

$$Y_t^n = \xi - \int_t^T f^n(s, Z_s^n) ds - \int_t^T Z_s^n dB_s, \quad 0 \leq t \leq T, \mathbb{P}\text{-a.s.}$$

with

$$f_t^n(z) = \frac{\gamma(1-\gamma)}{2} \text{dist}^2\left(\frac{1}{1-\gamma}(z + \theta_t^n), C_t^n\right) - \frac{\gamma|z + \theta_t^n|^2}{2(1-\gamma)} - \frac{1}{2}|z|^2. \quad (1.26)$$

There exists an optimal trading strategy  $p^{n*} \in \tilde{\mathcal{A}}^n$  with

$$p^{n*} \in \Pi_{C_t^n(\omega)}\left(\frac{1}{1-\gamma}(Z_t^n + \theta_t^n)\right).$$

This theorem makes the link between the optimisation problem that we consider and the solution of a quadratic BSDE which encodes the dynamic value process.

In order to prove our stability result, we make different assumptions on the convergence of the constraints sets and the sequences of drifts and volatility and we obtain the following result. For more details about this conditions, we refer the reader to Subsection 2.2.3.

**Theorem 1.1.25.** Under suitable assumptions, the solution  $(Y^n, Z^n)$  of the BSDE  $(f^n, \xi)$  converges to the solution  $(Y^\infty, Z^\infty)$  of the BSDE  $(f^\infty, \xi)$  defined by

$$Y_t^\infty = \xi - \int_t^T f_s^\infty(Y_s^\infty, Z_s^\infty) ds - \int_t^T Z_s^\infty dB_s, \quad 0 \leq t \leq T, \mathbb{P}\text{-a.s.}$$

with

$$f_t^\infty(z) = \frac{\gamma(1-\gamma)}{2} \text{dist}^2\left(\frac{1}{1-\gamma}(z + \theta_t^\infty), C_t^\infty\right) - \frac{\gamma|z + \theta_t^\infty|^2}{2(1-\gamma)} - \frac{1}{2}|z|^2, \quad (1.27)$$

in the following sense

$$\mathbb{E}\left[\exp\left(\sup_{0 \leq t \leq T} |Y_t^n - Y_t^\infty|\right) + \left(\int_0^T |Z_s^n - Z_s^\infty|^2 ds\right)\right] \xrightarrow{n \rightarrow \infty} 1.$$

Then,  $Y_0^n$  converges to a deterministic  $Y_0^\infty$  and  $\lim_{n \rightarrow +\infty} u^n(x) = \frac{x^\gamma}{\gamma} \exp(Y_0^\infty)$ .

The proof of the theorem above is based on results in [25]. In fact, we rely on the fact that the generator is quadratic and convex in its  $z$  component and use Proposition 7 in [25] to prove

the convergence of the BSDE's solution which amounts directly the stability of our optimisation problem.

### Motivation, formulation of the problem in discontinuous setting and contribution

So far, trajectories of the underlying assets have been assumed to be continuous. Diffusion processes like (1.22) can't generate discontinuous paths since the Brownian motion is everywhere continuous. This is clearly unrealistic since stock prices expose sudden price movements when reacting to good or bad news. To reproduce a more realistic behavior of the price process that also accounts for such movements in the price, dynamics (1.22) will be modified by adding a jump component.

In the second part of chapter 2, we extend the result obtained above to a discontinuous framework. We start by adding jumps in the sequence of the asset's price (1.22) and study the behavior of the value process by making a link with quadratic backward SDEs with jumps. Our model is analogous to the previous one, the only difference stems in assuming here that the price process has jumps. In fact, we consider the same setting introduced in 1.1.1. The sequence of markets becomes

$$dS_t^n = S_{t-}^n \left( \sigma_t^n (\theta_t^n dt + dB_t) + \int_E \beta_s^n(e) \tilde{\mu}(ds, de) \right), t \in [0, T], \mathbb{P}\text{-a.s.} \quad (1.28)$$

Here, the processes  $\sigma^n$  and  $\theta^n$  are defined as in the previous part and  $\beta^n$  is assumed to be a bounded predictable process greater than -1 to ensure the positivity of the exponential martingale  $\mathcal{E}(\beta \cdot \tilde{\mu})$ ,  $\mathbb{P}$ -a.s. and consequently the positivity of the price process.

The utility maximisation problem faced by the investor in this case is the following

$$u^n(x) = \sup_{\pi \in \mathcal{H}} \mathbb{E} \left[ U \left( x + \int_t^T \pi_s^n \frac{dS_s^n}{S_{s-}^n} + \xi \right) \right], \quad (1.29)$$

where  $U$  is an exponential utility function defined as follows

$$U(x) = -\exp(-\alpha x), \text{ for } \alpha > 0. \quad (1.30)$$

The payoff  $\xi$  is an  $\mathcal{F}_T$  random variable and  $\mathcal{H}^n$  is the set of all predictable  $\mathbb{R}^n$ -valued trading strategies which take their values in a constraint set  $\mathcal{C}^n$ . Applying again the dynamic programming principle described above, we make the link between the utility maximisation problem (1.29) and a quadratic BSDE with jumps as in [97, 98].

**Theorem 1.1.26.** *For all  $n \geq 1$ , the expression of the value process (1.29) is given by*

$$u_t^n(x) = -\exp(-\alpha(x - Y_0^n)),$$

where  $(Y_t^n, Z_t^n, U_t^n)$  is the solution of the JBSDE( $f^n, \xi$ )

$$Y_t^n = \xi + \int_t^T f_s^n(Y_s^n, Z_s^n, U_s^n) ds - \int_t^T Z_s^n dB_s - \int_t^T \int_E U_s^n(e) \tilde{\mu}(ds, de), 0 \leq t \leq T, \mathbb{P}\text{-a.s.}$$

whose generator is defined as follows

$$f_s^n(z, u) = \inf_{\pi^n \in C^n} \left[ \frac{\alpha}{2} |\pi_s^n \sigma_s^n - (z + \frac{\theta_s^n}{\alpha})|^2 + \int_E g_\alpha(u_s(e) - \pi_s^n \beta_s^n(e)) \nu(de) \right] - \theta_s^n z - \frac{|\theta_s^n|^2}{2\alpha}. \quad (1.31)$$

Now, we make again suitable assumptions on the sequence of the input parameters which leads us to the second main result of this part. For more details about the convergence modes of the inputs, we refer the reader to 2.3.3. Our first step consists on proving the convergence of the strategies  $\pi^n$ . To do so, we rely on results in [67].

**Lemma 1.1.27.** *Under suitable assumptions, the sequence of strategies  $(\pi_t^n)_{n \geq 1}$  converges to  $\pi_t$  in  $\mathcal{H}^2$ .*

The next step is devoted to prove the stability of the solution of the quadratic BSDE with jumps related to the optimisation problem. In fact, we make the needed conditions on the input parameters in order to ensure the convergence of the value process of the problem. For more details about these assumptions, see Subsection 2.3.3.

**Theorem 1.1.28.** *Under suitable assumptions, the sequence of processes  $(f_t^n)_{n \geq 1}$  defined in (1.31) converges pointwise to  $f_t$  and  $(Y^n, Z^n, U^n)$  converges to  $(Y, Z, U)$  in  $S^\infty \times \mathcal{H}^2 \times \mathbb{H}^2$ . In particular,  $Y_0^n$  is deterministic and it converges  $\mathbb{P}$ -a.s. to  $Y_0$ .*

This result entails directly the convergence of the sequence of value processes associated to the exponential utility maximisation problem.

**Theorem 1.1.29.** *Let  $(u^n)_{n \in \mathbb{N}}$  defined by*

$$u_t^n(x) = -\exp(-\alpha(x - Y_t^n)),$$

be the sequence of dynamic value functions of the constrained utility maximisation problem (1.29). Then, under suitable assumptions, we have

$$\lim_{n \rightarrow \infty} \mathbb{E} \left[ \sup_{t \in [0, T]} |u_t^n(x) - u_t(x)| \right] = 0. \quad (1.32)$$

In particular, the sequence of the static value functions  $u_0^n(x) = -\exp(-\alpha(x - Y_0^n))$  remains stable.

This work is concertized in the preprint [83].

## 1.2 Pricing American options

The third chapter of this thesis is devoted to a work that has been done at CEMRACS 2017 (Centre d'été Mathématiques de Recherche Avancée en Calcul Scientifique) which is a scientific event of the SMAI (the french Society of Applied and Industrial Mathematics). The CEMRACS 2017 consisted of six weeks from 17 July to 25 August 2017 and took place at CIRM, Luminy, Marseille. In the first week, a summer school on numerical methods for the analysis of stochastic models associated with stochastic control problems, uncertainty propagation analysis or mean-field type interaction phenomena, was proposed. The remaining five weeks were intensive long research sessions on different research projects. More precisely, numerical projects have been proposed by research departments or industrials that want to explore a new numerical method about a specific problem. The project described in this chapter deals with numerical methods for the pricing of American options. This work is concertized by a published paper [18] in ESAIM Proceedings.

### 1.2.1 Problem formulation

One of the most important problems in the option pricing theory is the valuation and optimal exercise of derivatives of American type. This type of derivative exists in all major financial markets where buyers and sellers meet to participate in the trade of assets at prices determined by the forces of supply and demand. It includes equity, foreign exchange, energy, insurance etc. However, the problem of pricing and optimal exercise of American options remain one of the most challenging in derivatives finance.

An American call or put option gives its owner the right but not the obligation to purchase (respectively, sell) the underlying asset for the strike price at any time until expiry which makes its pricing and hedging mathematically challenging and few closed form solutions have been found.

To put the problem in its mathematical context, we may consider the case of a single stock (non-dividend paying) market under the famous Black and Scholes setting. Namely, let  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})$  be a filtered probability space carrying a standard one dimensional Brownian motion  $W$ . The stock price  $X_t$  varies as follows

$$dX(s) = rX(s)ds + \sigma X(s)dW_s, \quad s \in (t, T], \quad (1.33)$$

for which the solution  $X_t$  is defined by

$$X_s = x \exp \left( \left( r - \frac{\sigma^2}{2} \right) (s - t) + \sigma (W_s - W_t) \right), \quad s \geq t,$$

under the risk natural probability. Here,  $x > 0$  is the stock price at time  $t$ ,  $r > 0$  is the risk-free interest rate and  $\sigma > 0$  is the volatility. Then, the arbitrage free value at time  $t$  of an American option maturing at  $T \geq t$  is given by

$$V(t, x) = \sup_{\tau \in \mathcal{T}_{[t, T]}} \mathbb{E}[e^{-r(\tau-t)} g(X_\tau)], \quad (1.34)$$

where  $\mathcal{T}_{[t, T]}$  is the collection of  $[t, T]$ -valued stopping times, and  $g$  is the payoff function, say continuous, see e.g. [17] and the references therein. Typical examples are

$$g(x') = \begin{cases} (x' - K)^+, & \text{for a call option} \\ (K - x')^+, & \text{for a put option,} \end{cases} \quad (1.35)$$

where  $K > 0$  denotes the strike price. Define, for  $\epsilon > 0$ , the following stopping time

$$\tau_\epsilon = \tau_\epsilon^{t, x} := \inf \{ s \in [t, T] / V(s, X(s)) \leq g(X(s)) + \epsilon \}. \quad (1.36)$$

In the following, we give the dynamic programming principle which will be very useful later.

**Proposition 1.2.1.** *(Dynamic programming principle)*

- Any stopping times  $t \leq \theta \leq \tau_\epsilon$  satisfies

$$V(t, x) = \mathbb{E}[e^{-r(\theta-t)} V(\theta, X(\theta))]. \quad (1.37)$$

- For all stopping times  $\theta \in [t, T]$ , we have

$$V(t, x) \geq \mathbb{E}[e^{-r(\theta-t)} V(\theta, X(\theta))]. \quad (1.38)$$

- For  $\epsilon = 0$ ,  $\tau_0$  is an optimal stopping time and for all  $t \leq s \leq \tau_0$ , the process  $e^{-r(s-t)} V(s, X(s))$  is a martingale.

In addition to that, one can derive from (1.37) the following dynamic programming principle for the optimal stopping problem: For any stopping time  $\theta \in [t, T]$ , we have

$$V(t, x) = \sup_{t \leq \tau \leq T} \mathbb{E} \left[ 1_{\tau < \theta} e^{-r(\tau-t)} g(X(\tau)) + 1_{\tau \geq \theta} e^{-r(\theta-t)} V(\theta, X(\theta)) \right]. \quad (1.39)$$

To find the value  $V$  of the American option, there exist different methods in the literature:

- Duality approach formulation : Rogers [110], Anderson and Broadie [2] and Kogan and Hough [50].
- Integral equation formulation: Kim [68], Jacka [57].
- Free boundary formulation: McKean [88].
- Variational inequalities: Bensoussan and Lions [12], Jaillet, Lamberton and Lapeyre [58].
- Early exercise premium formulation: Carr, Jarrow and Myneni [32].

Among the formulations cited above, we give a brief description of the two major approaches to solve the American option pricing problem.

### The free-boundary formulation:

The connexion between pricing American options and free boundary problems was given by Samuelson in [113] and it was mathematically studied by McKean in [88]. This approach consists in looking for the option's value and a boundary that splits the domain into a continuation region where the option's value satisfies a differential equation and a stopping region where the value is equal to a known function.

To solve this problem, the author writes the American option price explicitly up to knowing a certain function which is the optimal stopping boundary.

By construction,  $V(\cdot, X) \geq g(X)$ , and the option should be exercised only when we have  $V(\cdot, X) \leq g(X)$ . This leads to define the following two regions:

- the continuation region:

$$C = \{(t, x) \in [0, T) \times (0, \infty) : V(t, x) > g(x)\}.$$

- the stopping (or the exercise) region:

$$S = \{(t, x) \in [0, T) \times (0, \infty) : V(t, x) = g(x)\}.$$

These are the basics of the common formulation of the American option price as a free boundary problem, which already appears in McKean [88]:  $V$  solves a heat-equation type linear parabolic problem on  $C$  and equals  $g$  on  $S$ , with the constraint of being always greater than  $g$ .

American options and related problems such as optimal stopping and free boundary have been studied in a jump diffusion model see for example, [107, 108].

### The quasi-variational formulation

Another formulation is based on the quasi-variational approach of Bensoussan and Lions [12] in which the replication price of an American option with an exercise payoff  $g(x)$  solves (at least in the viscosity solution sense) the quasi-variational partial differential equation

$$\begin{cases} \min (ru(t, x) - \mathcal{L}_{BS}u(t, x), u(t, x) - g(x)) = 0, & \forall x \in [0, T) \times [0, \infty), \\ u(t, x) = g(x), & \forall x \in [0, \infty), \end{cases} \quad (1.40)$$

where  $\mathcal{L}_{BS}$  is the Dynkin operator associated to  $X$ :

$$\mathcal{L}_{BS} = \partial_t + rx\partial_x + \frac{1}{2}\sigma^2x^2\partial_x^2.$$

This problem can be equivalently written, for  $(t, x) \in [0, T) \times [0, +\infty)$  as follows

$$\begin{cases} (u(t, x) - g(x))(\mathcal{L}_{BS}u(t, x) - ru(t, x)) = 0, \\ \mathcal{L}_{BS}u(t, x) - ru(t, x) \leq 0, \\ u(t, x) \geq g(x). \end{cases} \quad (1.41)$$

In this formulation, the option price  $V$  is the unique solution of the variational inequality (1.40).

### Another formulation: A viscosity solution approach

In the third part of the thesis, we focus on another formulation that can be found in [13], see also [14] and the references therein. We focus on American options of type call and put for which the payoff function is given by

$$g(x) = \begin{cases} (x - K)^+, & \text{for a call option} \\ (K - x)^+, & \text{for a put option.} \end{cases} \quad (1.42)$$

The main advantage of this formulation is that, contrary to the free boundary (resp. the quasi-variational) formulation described above, there is no free boundary to determine (resp. no "side constraints" that need to be verified) and it gives an easy to implement numerical scheme for computing the value of an American option.

This approach relies essentially on the dynamic programming principle given in 1.2.1 and allows the American option valuation problem to be stated in terms of a semilinear Black and Scholes partial differential equation set on a fixed domain, that is to say: one seeks to find a

function  $u = u(t, x)$  that satisfies

$$\begin{cases} ru(t, x) - \mathcal{L}_{BS}u(t, x) = q(t, x), & (t, x) \in [0, T) \times (0, +\infty), \\ u(T, x) = g(x), & x \in (0, +\infty), \end{cases} \quad (1.43)$$

where  $q$  is a nonlinear reaction term defined as

$$q(x, u(t, x)) = c(x)H(g(x) - u(t, x)) = \begin{cases} 0 & \text{if } g(x) < u(t, x), \\ c(x) & \text{if } g(x) \geq u(t, x). \end{cases}$$

Here,  $c$  is a certain cash flow function, e.g.  $c = rK$  for a put option and  $H$  is the Heaviside function. In this case, Benth et al. [13] proved the following result.

**Theorem 1.2.2.** (Benth et al., [13]) *The American option (call and put) valuation problem is equivalent to finding a function  $u(t, x) : [0, T) \times [0, +\infty) \rightarrow \mathbb{R}$  satisfying (1.43) in the sense of viscosity solutions.*

Note that the semilinear Black and Scholes equation (1.43) does not make sense if we consider classical solutions because of the discontinuity of  $y \rightarrow q(x, y)$ . It has to be considered in the discontinuous viscosity solution sense, see e.g. Crandall, Ishii and Lions [33]. Namely, even if  $V$  is continuous, the supersolution property should be stated in terms of the lower-semicontinuous envelope of  $q$ , the other way round for the subsolution property. This means in particular that the supersolution and subsolution properties are not defined with respect to the same operator.

Still, thanks to the very specific monotonicity of  $y \rightarrow q(x, y)$ , it is proved in [13] that, within the Black and Scholes model, the American option price is the unique solution of (1.43) in the appropriate sense. In fact, the authors proved the following result.

**Theorem 1.2.3.** (Benth et al., [13]) *The value function  $V(t, x)$  defined in (1.34) is a viscosity solution of the terminal value problem (1.43).*

**Remark 1.2.4.** (i) *Note that, even with all these methods and others in the literature, it is not possible to find closed-form solutions to American option valuation problems. A possible way to solve the problem is to resort to numerical schemes to find  $V$ .*

(ii) *Let us point out that, the different methods cited above and others, will lead to different numerical schemes. For more details about mathematical and numerical aspects of the two methods above, we refer the reader to [39], [64] and [99] and the references therein.*

## 1.2.2 Valuation of American-style options and BSDEs

### Reflected BSDEs and connection with valuing American options

It is well-known that the price of an American option corresponds to the solution of reflected backward stochastic differential equation (RBSDE in short). These equations were introduced by El Karoui, Kapoudjian, Pardoux, Peng and Quenez in [42]. The formulation of this problem is the following: Given a filtered probability space  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})$  generated by  $d$ -dimensional Brownian motion  $W$ , given also an obstacle process  $L$ , a progressively measurable generator  $f$  and an  $\mathcal{F}_t$ -measurable terminal condition  $\xi$ , a solution to a one dimensional RBSDE consists in a triple  $(Y, Z, K)$  of processes taking value in  $\mathbb{R} \times \mathbb{R}^d \times \mathbb{R}$  such that

$$\begin{cases} Y_t = \xi + \int_t^T f(Y_s, Z_s) ds - \int_t^T Z_s dW_s + K_T - K_t, t \in [0, T], \mathbb{P}\text{-a.s.} \\ Y_t \geq L_t, t \in [0, T], \mathbb{P}\text{-a.s.} \\ \int_0^T (Y_s - L_s) dK_s = 0, \mathbb{P}\text{-a.s.} \end{cases}$$

The process  $K$  is non-decreasing and null at 0 and this process is added in order to push the solution  $Y$  upwards, so that it may remain above the stochastic process  $L$  called *the obstacle*. The last condition means that the process  $K$  acts only when the solution  $Y$  reaches the obstacle  $L$ , and provides the uniqueness of the solution. The uniqueness and existence of the solution are proved in [42] using a fixed point argument and by an approximation via penalization techniques.

The connection between reflected BSDEs and American options was studied by El Karoui, Pardoux and Quenez [42]. Since the solution of a reflected BSDE is forced to be above the obstacle, it illustrates the fact that the price of an American option is always greater than the payoff of the option. Their idea is to consider the strategy wealth portfolio  $(Y_t, \pi_t)$  as a pair of adapted processes which satisfies the following BSDE:

$$-dY_t = b(t, Y_t, \pi_t) dt - \pi_t' \sigma_t dW_t, t \in [0, T], \mathbb{P}\text{-a.s.},$$

where  $b$  is an  $\mathbb{R}$ -valued convex and Lipschitz function with respect to  $(y, \pi)$  and the volatility matrix  $\sigma$  is invertible and its inverse  $\sigma_t^{-1}$  is bounded.

In complete market, the problem of pricing an American option at time  $t$  consists in determining a stopping time  $\tau \geq t$  and a payoff exercise  $\tilde{L}_t$  where

$$\tilde{L}_\tau = L_\tau 1_{\{\tau < T\}} + \xi 1_{\{\tau = T\}}.$$

The price of an American option  $(\tilde{L}_s, 0 \leq s \leq T)$  at time  $t$  is given by

$$Y_t = \text{ess sup}_{\tau \in \Gamma_t} Y_t(\tau, \tilde{L}_\tau).$$

Then the connection between American options and reflected BSDEs is given by the following theorem.

**Theorem 1.2.5** (El Karoui, Pardoux, Quenez, [42]). *There exists  $\pi$  and a non-decreasing continuous process  $K$  such that for all  $t \in [0, T]$*

$$\left\{ \begin{array}{l} Y_t = \xi + \int_t^T b(s, Y_s, \pi_s) ds - \int_t^T \pi_s \sigma_s dW_s + K_T - K_t, 0 \leq t \leq T, \mathbb{P}\text{-a.s.} \\ Y_t \geq L_t, \mathbb{P}\text{-a.s.} \\ \int_0^T (Y_s - L_s) dK_s = 0, \mathbb{P}\text{-a.s.} \end{array} \right.$$

Furthermore, the stopping time  $D_t = \inf\{s \geq t, Y_s = L_s\} \wedge T$  is optimal after  $t$ .

### 1.2.3 Main results and contributions

In this part of the thesis, we study the problem of pricing American options from both theoretical and numerical point of view. More precisely we address two numerical methods for the pricing of American options. Both methods are based upon an alternative representation of the option's value in terms of viscosity solution of a parabolic equation with a nonlinear reaction term.

As explained above, there exist several characterizations for the price  $V$  (1.34) of an American option within a diffusion model. Notably, when the payoff function is given by  $g$  as in (1.42),  $V$  can be shown to be a viscosity solution of the following semi-linear PDE

$$\left\{ \begin{array}{l} rV - \partial_t V - \mathcal{L}V - c(x)1_{\{V(t,x) \leq g(x)\}} = 0, \forall (t, x) \in [0, T) \times (0, +\infty), \\ V(T, x) = g(x), \forall x \in (0, +\infty), \end{array} \right. \quad (1.44)$$

where  $\mathcal{L}$  is the infinitesimal generator of the underlying process and  $c$  is a function depending on  $g$  and on the model coefficients. When the underlying model is Black-Scholes and  $g$  is a call or a put payoff, this result was proved by Benth, Karlsen and Reikvam as pointed out in Theorem 1.2.2.

### Theoretical contributions and new results

Our first contribution in this part is to extend results of [13] to a general multidimensional complete diffusion model (for  $d$  assets) namely, a multidimensional local volatility model with corre-

lation and for more general payoff functions.

We consider a financial market with  $d$  stocks, on  $(\Omega, \mathcal{F}, \mathbb{P})$ , whose prices process  $X^{t,x}$  evolves according to

$$X^{t,x} = x + \int_t^\cdot r X_s^{t,x} ds + \int_t^\cdot \sigma(s, X_s^{t,x}) dW_s, \quad (1.45)$$

in which  $r \in \mathbb{R}$  is a constant denoting the risk free interest rate and  $\sigma : [0, T] \times (0, \infty)^d \rightarrow \mathbb{R}^{d \times d}$  is a matrix valued function that is assumed to be continuous and uniformly Lipschitz in its second component. We also assume that  $\bar{\sigma} : (t', x') \in [0, T] \times (0, \infty)^d \rightarrow \text{diag}[x']^{-1} \sigma(t', x')$  is bounded and uniformly Lipschitz in its second component, where  $\text{diag}[x']$  stands for the diagonal matrix with  $i$ -th diagonal entry equal to the  $i$ -th component of  $x'$ . This implies that  $X^{t,x}$  takes values in  $(0, \infty)^d$  whenever  $x \in (0, \infty)^d$ .

In this part, we will focus on the formulation of the problem as in [13] that we have already explained in Subsection 1.2.1. In this formulation, the problem is stated in terms of semilinear Black and Scholes partial differential equation set in a fixed domain, namely:

$$\begin{cases} r\varphi - \mathcal{L}\varphi = q(\cdot, \varphi), & \text{on } [0, T) \times (0, +\infty)^d, \\ \varphi(T, \cdot) = g, & \text{on } (0, +\infty)^d, \end{cases} \quad (1.46)$$

for a suitable reaction term  $q$  defined on  $(0, \infty)^d \times \mathbb{R}$  by:

$$q(x, y) = \begin{cases} 0 & \text{if } g(x) < y, \\ c(x) & \text{if } g(x) \geq y. \end{cases}, \quad (x, y) \in (0, \infty)^d \times \mathbb{R}.$$

The Dynkin operator  $\mathcal{L}$  above is here defined, for a smooth function  $\varphi$ , by

$$\mathcal{L}\varphi(t', x') = \partial_t \varphi(t', x') + \langle r x', D\varphi(t', x') \rangle + \frac{1}{2} \text{Tr}[\sigma \sigma^\top D^2 \varphi](t', x'). \quad (1.47)$$

This semilinear Black and Scholes equation does not make sense if one considers classical solutions because of the discontinuity of the term  $y \rightarrow q(x, y)$ . It has to be considered in the discontinuous viscosity solution sense, see e.g. Crandall, Ishii and Lions [33].

From a theoretical point of view, the notion of viscosity solution requires some care since it involves the notions of lower and upper semi-continuous envelopes. More precisely, for a general payoff function  $g$  and due to the discontinuity of the reaction term  $q(x, y)$ , the notion of viscosity solution of (1.44) has to be adapted. Importantly, this nonlinear reaction term  $q$ , is singular as it involves a Heaviside function.

As done in [13], the notions of super- resp. sub-solutions are considered for two different equations, in which  $q$  is replaced by its lower-semicontinuous envelope  $q_*$ , resp. its upper-

semicontinuous envelope  $q^*$ . When the payoff function  $g$  is  $C^2$  on a set containing the time sections of the exercise region  $\{x : V(t, x) = g(x)\}$ , as in the case of put and call options, then  $c = rg - \mathcal{L}g$ . In the more general setting that we are considering in this part, our main result relies on the following assumption on the cash flow function  $c$ :

**Assumption 1.2.6.** *The map  $c : (0, \infty)^d \rightarrow \mathbb{R}_+$  is continuous with polynomial growth. Moreover,  $g$  is a viscosity subsolution of  $r\varphi - \mathcal{L}\varphi - c = 0$  of on  $\{(t, x) \in [0, T) \times (0, \infty)^d : V(t, x) = g(x)\}$ .*

Under 1.2.6, we extend results in [13] to a more general model and provide theoretical justification for more general payoff functions, possibly on several underlying.

**Theorem 1.2.7.** *Under Assumption 1.2.6 on  $c$ ,  $V$  is a viscosity solution of (1.46). Moreover,  $V$  has a polynomial growth.*

The proof of the theorem above follows arguments of [13]. Indeed, it relies on proving that  $V$  is a supersolution (rep. a subsolution) of (1.46). The super solution property is standard and direct while for the subsolution property, we use dynamic programming principle (1.2.1) and Assumption 1.2.6.

Then, we complement the viscosity solution property with a comparison principle as in [13] in order to prove that  $V$  is the unique viscosity solution of (1.46) with polynomial growth.

**Proposition 1.2.8.** *Assume that conditions of Theorem 1.2.7 hold. Let  $v$  and  $w$  be respectively a supersolution and a subsolution of (1.46), with polynomial growth. Then,  $v \geq w$  on  $[0, T) \times (0, \infty)^d$ .*

**Numerical results** Different approaches have been proposed, in the literature, in order to solve BSDEs numerically. Classically, the numerical method used to solve BSDEs is based on a backward iteration where every step requires the computation of conditional expectations, for instance, see [20] and [122]. To compute these conditional expectations, a regression method is generally used. However, the main drawback of this method, from a practical point of view, is that it is very costly and suffers from the so called "curse of dimensionality". Moreover, finding good regressors is quite difficult and time-consuming especially in the case of multi-assets portfolio. This leads to the introduction of new numerical methods based on branching diffusion describing a marked Galton-Waston random tree.

This numerical algorithm is based on a pure forward simulation of branching processes which were introduced by Henry-Labordere [51] and Henry-Labordere, Tan and Touzi [53]. The particularity of this algorithm is that it avoids the estimation of conditional expectations which turns out to be very advantageous in practice. In the next section, we give a brief description of this method that we will use in our numerical approaches.

**Branching diffusions** Branching diffusion theory was firstly introduced by McKean in [87] where the author gives a probabilistic representation of Kolmogorov-Petrovskii-Piskunov PDE (called KPP equation) and more generally for semi-linear PDEs of the following type

$$\begin{cases} \partial_t u(t, x) + \mathcal{L}u(t, x) + \sum_{k=0}^{\infty} p_k u^k(t, x) = 0, & (t, x) \in \mathbb{R}^+ \times \mathbb{R}^d, \\ u(T, x) = g(x), & x \in \mathbb{R}^d, \end{cases} \quad (1.48)$$

where  $\mathcal{L}$  is an Itô operator and  $(p_k)_k$  is a probability mass sequence i.e it must satisfy the following restrictive conditions

$$\sum_{k=0}^{\infty} p_k = 1, 0 \leq p_k \leq 1. \quad (1.49)$$

The probabilistic interpretation is the following: The branching process starts with a particle at the origin (time 0), performs an Itô diffusion on  $\mathbb{R}^d$  with generator  $\mathcal{L}$ , dies after a mean  $\beta$  exponential time called "the branching rate" and produces  $k$  i.i.d. descendants with a probability  $p_k$ . Then the descendants perform independent Itô diffusions with the same generator  $\mathcal{L}$  driven by independent Brownian motions. Every descendant dies and reproduces i.i.d. descendants independently after independent exponential time, etc. Note that the case when  $k = 0$  means that the particle dies without generating descendants. We continue to simulate particles until it hit the space boundary or is alive at time  $t$ . The mathematical construction of this process is given in [53] where the authors explain in details the three steps of this construction.

### Local polynomial approximation

Another important tool in our numerical methods is the local polynomial approximation proposed by Bouchard et al. in [19]. This method relies on the link between the PDE (1.48) and the following BSDE with a polynomial driver and a terminal condition  $g(W_T)$ :

$$Y_t = g(W_T) + \int_t^T \sum_{k \geq 0} p_k (Y_t)^k dt - \int_t^T Z_t W_t, \mathbb{P}\text{-a.s.}, \quad (1.50)$$

in which  $W$  is a Brownian motion. Indeed, the  $Y$ 's component of this BSDE satisfies  $Y_t = u(\cdot, W_t)$  and it can be estimated using the branching process based Feynman-Kac representation of (1.48) via a pure forward Monte-Carlo scheme.

The idea consists of using this representation to solve BSDEs with Lipschitz drivers and approximating this driver by local polynomials to avoid the explosion of the sequence of the approximating BSDEs and this is due to the fact that these polynomials are defined on small arbitrary time intervals.

In practice, this method needs to be combined with a Picard iteration scheme. In fact, the choice of the polynomial will depend on the position of the solution  $Y$  but contrary to the classical Picard scheme for BSDEs, a precise estimation of the whole path of the solution in each Picard iteration is not needed. The reason is that, once the local polynomial is fixed on a partition  $(A_i)_i$  of  $\mathbb{R}$ , we only need to know in which partition  $A_i$  the solution lives at certain branching times of the underlying branching processes.

Concretely, the authors in [19] proved that for a BSDE of the form (1.50), the function  $u(t, x)$  has the following form

$$u(t, x) = \mathbb{E} \left[ \frac{g(W_T)}{\bar{F}(T-t)} 1_{\tau \geq T-t} + 1_{\tau < T-t} \sum_k \frac{p_k}{\rho(\tau)} u(t + \tau, W_{t+\tau})^k \right], \quad (1.51)$$

where  $\tau$  is an independent exponentially distributed random variable with density  $\rho$ .

### Numerical contribution and new results

The main contribution in this part of the thesis concerns the numerical side. More precisely, we exploit the connection between non-linear PDEs and BSDEs in order to give two numerical methods for the pricing of American options based on the representation of the option's value in the form of a viscosity solution of a parabolic equation with a nonlinear reaction term. To give a probabilistic interpretation, we use the fact that the solution of (1.46) is formally related to the solution  $(Y, Z) \in \mathcal{S}^2 \times \mathcal{H}^2$  of the following BSDE

$$Y = \exp(-rT)g(X_T) + \int_0^T e^{-rs} q(X_s, e^{rs} Y_s) ds - \int_0^T Z_s dB_s, \mathbb{P}\text{-a.s.}, \quad (1.52)$$

by the following relation:  $e^{-r\cdot} V(\cdot, X) = Y$ .

As it can be seen, the associated BSDE with nonlinear reaction term given above is not well posed due to the discontinuity of  $q$ . However, for the purpose of numerical approximation, it can be smoothed out in order to solve numerically the mollified BSDE.

We show that any sequence of BSDEs with properly mollified drivers  $q_n$  defines a sequence of value functions  $V_n$  that converge pointwise to the desired solution  $V$  (the option's value). More precisely, mollifying the reaction term is a very needed to restore well-posedness of the BSDE and we show that the solution of the mollified BSDEs converge to the value function of the American option when the initial point of the forward process is fixed.

**Proposition 1.2.9.** *Let the conditions of Theorem 1.2.7 hold. Let  $(q_n)_{n \geq 1}$  be a sequence of continuous functions on  $(0, \infty)^d \times \mathbb{R}$  that are Lipschitz in their last component. Assume also that  $(q_n)_{n \geq 1}$  is uniformly bounded by a function with polynomial growth in its first component*

and linear growth in its last component. Assume further that

$$\limsup_{\substack{n \rightarrow \infty \\ (x', y') \rightarrow (x, y)}} q_n(x', y') \leq q^*(x, y) \quad \text{and} \quad \liminf_{\substack{n \rightarrow \infty \\ (x', y') \rightarrow (x, y)}} q_n(x', y') \geq q_*(x, y), \quad (1.53)$$

for all  $(x, y) \in (0, \infty)^d \times \mathbb{R}$ . For  $(t, x) \in [0, T] \times (0, \infty)^d$ , let  $(Y^{t,x,n})_{n \geq 1}$  be such that

$$Y_s^{t,x,n} = \mathbb{E}_s[e^{-rT}g(X_T^{t,x}) + \int_s^T e^{-ru}q_n(X_u^{t,x}, e^{ru}Y_u^{t,x,n})du],$$

for  $s \in [t, T]$ , and set  $V_n(t, x) := e^{rt}Y_t^{t,x,n}$ . Then,  $(V_n)_{n \geq 1}$  converges pointwise to  $V$  as  $n \rightarrow \infty$ .

Once a smoothing procedure is done, we are back to essentially solving a BSDE. So, we address two approaches to solve the problem numerically:

**The method (A): Local polynomial approximation and branching processes** The first approach relies on fixing a mollified driver and then approximate it by local polynomials so that branching particle methods for BSDEs with polynomial drivers can be applied. More precisely, we mollify the Heaviside function that appears in the nonlinear reaction term  $q$  and approximate the mollified coefficient by polynomials in order to apply a pure forward Monte Carlo algorithm, based upon branching and Picard iteration, for the estimation of  $V_n$ . This method is inspired from [19]. The choice of the polynomial form in this method depends on the space position of the solution  $Y$ . The main steps of this approach are the following:

- (1) Approximate the Heaviside function  $H : z \mapsto 1_{\{z \geq 0\}}$  by a sequence of Lipschitz functions  $(H^n)_{n \geq 1}$  and define  $q_n$  by

$$q_n : (x, y) \mapsto c(x)H_n(g(x) - y).$$

- (2) Approximate  $q_n$  by a map  $(x, y) \mapsto \bar{q}_n(x, y, y)$  of a local polynomial form:

$$\bar{q}_n : (x, y, y') \mapsto \sum_{j=1}^{j_0} \sum_{l=0}^{l_0} a_{j,l}(x)y^l\phi(y'), \quad (1.54)$$

where  $(a_{j,l}, \phi_j)_{l \leq l_0, j \leq j_0}$  is a family of continuous and bounded maps satisfying

$$|a_{j,l}| \leq C_{l_0}, \quad |\phi_j(y'_1) - \phi_j(y'_2)| \leq L_\phi|y'_1 - y'_2|, \quad \text{and} \quad |\phi_j| \leq 1, \quad (1.55)$$

for all  $y'_1, y'_2 \in \mathbb{R}$ ,  $j \leq j_0$  and  $l \leq l_0$  and some constants  $C_{l_0}, L_\phi \geq 0$ . In (1.54), the  $(a_{j,l}(x))_{l \leq l_0}$  can be interpreted as the coefficient of a polynomial approximation of  $q_n$  on

a subset  $A_j$ , where  $(A_j)_{j \leq j_0}$  form a partition of  $\mathbb{R}$  and the  $\phi_j$ 's as smoothing kernels that allow one to pass in a Lipschitz way from one part of the partition to another one.

(3) Consider the sequence of BSDE

$$\begin{aligned} \bar{Y}_s^{t,x,n,k+1} = & \mathbb{E}_s[e^{-rT} g(X_T^{t,x})] \\ & + \mathbb{E}\left[\int_s^T e^{-ru} \bar{q}_n(X_u^{t,x}, e^{ru} \bar{Y}_u^{t,x,n,k+1}, e^{ru} \bar{Y}_u^{t,x,n,k}) du\right], \quad k \geq 1, \end{aligned}$$

with  $\bar{Y}^{t,x,n,1}$  given as an initial prior (e.g.  $e^r \cdot g(X^{t,x})$ ). Given  $\bar{Y}^{t,x,n,k}$ ,  $\bar{Y}^{t,x,n,k+1}$  solves a BSDE with polynomial driver that can be estimated by using branching processes as in the Feynman-Kac representation of the Kolmogorov-Petrovskii-Piskunov equation, see [51, 52]. For more details, we refer the interested reader to [19].

**The method (B): A randomization procedure** The second approach relies on constructing a sequence of mollified drivers by randomization of the indicator function. Actually, we only add an independent noise in the definition of  $q$  which also have the effect of smoothing it out. More precisely,

$$q_n(x, y) = \mathbb{E}[c(x) 1_{y \leq g(x) + \frac{\epsilon}{n}}],$$

where  $\epsilon$  is a properly chosen positive independent random variable. Then, evaluate the non-linear Feynman-Kac representation for the BSDE with driver  $q_n(x, y)$  via a backward algorithm. It is done by using a very simple version of the algorithm in [19].

Unfortunately, in practice, the first method is very unstable. This is due to the fact that this method was essentially dedicated, as explained in [19], to situations where the driver is smooth which makes the local polynomial's coefficients small and the support of the smoothing kernels large and do not intersect to much. As it can be seen, none of these requirements are met in the case of the indicator function which explains the failure of this algorithm in our case. This work is concretized in the published paper [18].

### 1.3 Mean field forward backward SDE with jumps and application for storage in smart grids

The Last chapter of this thesis is dedicated to the study of fully coupled mean field forward-backward stochastic differential equations (MF-FBSDE in short) with jumps and applications in the energy storage field.

### 1.3.1 Forward backward stochastic differential equations

**Forward-backward SDE** A forward-backward stochastic differential equation is the following system:

$$\begin{cases} X_t = X_0 + \int_0^t b_s(X_s, Y_s, Z_s)ds + \int_0^t \sigma_s(X_s, Y_s, Z_s)dW_s, t \in [0, T], \mathbb{P}\text{-a.s.}, \\ Y_t = g(X_T) + \int_t^T h_s(X_s, Y_s, Z_s)ds - \int_t^T Z_s dW_s, t \in [0, T], \mathbb{P}\text{-a.s.}, \end{cases} \quad (1.56)$$

where  $W$  is a standard Brownian motion and the coefficients  $b$ ,  $\sigma$ ,  $g$  and  $h$  are progressively measurable functions defined on appropriate spaces and that can be random.

To solve the FBSDE (1.56), various methods have been proposed in the literature. Let us mention the three main approaches that have been developed:

- *The method of contraction mapping:* This method has been introduced by Antonelli [3] and has been detailed later by Pardoux and Tang in [104]. In his PhD thesis, Antonelli studied the well-posedness of these equations over a sufficiently small time duration and provides a counterexample to show that, when the time duration is large, the solvability of such equations may fail.
- *The 4-step scheme method:* This method has been initiated by Ma et al. in [82] and Ma and Yong [81] who developed it under the very strong assumption requiring that the coefficients of the forward equation are non-degenerate and deterministic. This way to solve FBSDEs can be considered as a sort of combination between PDE methods and probability methods. Using these tools, the authors established an existence and uniqueness result for fully coupled FBSDEs on an arbitrary given time interval.

The main advantage of this approach is that it removes the restriction on the time duration by allowing arbitrary time duration. However, it requires a Markovian structure, a high regularity on the coefficients and a non-degeneracy of the forward diffusion.

This result was later improved by Delarue in [35] where the author relaxed the regularity conditions that the 4-step scheme required.

- *The method of continuation:* This method was introduced in [56] and [106], and it allows the non-Markovian structure. However, it requires monotonicity conditions on the coefficients. It was developed later in [120] and [121].

**Forward-backward SDEs with jumps** The notion of classical fully coupled forward backward stochastic differential equations has been naturally extended to the jump setting. The literature

about this topic is rather small.

In [118, 123], Wu and Zhen extended the results of [120] and [106] to the case of fully coupled FBSDE with jumps. The authors studied a fully coupled FBSDEs driven by both a Brownian motion and a Poisson random measure under the monotonicity condition. More precisely, in [123], Zhen obtained the existence and uniqueness of the solution for such fully coupled FBSDEs with jumps and in [118] and Wu proved the existence and uniqueness of the solution as well as a comparison theorem over a stochastic interval.

### 1.3.2 Mean field game and extended mean field game theory

Mean field game theory (MFG in short) is devoted to the analysis of differential games with infinitely many players. Historically, it goes back to the early works of Kac [61] and McKean [89] in the 1950's. It was initially suggested in order to study the behavior of a large number of mutually-interacting particles in different fields of physical science. e.g. the derivation of Boltzmann or Vlasov equations in the kinetic gas theory. Roughly speaking, in large population dynamic games, where it is not possible for a player to collect detailed state informations about all other players, this theory tells that one just needs to implement strategies based on the distribution of the other players.

Now, we explain the mean field game system in more details. The typical model of mean field games is the following system

$$\begin{cases} -\partial_t u - \nu \Delta u + H(x, m, Du) = 0, & \text{in } \mathbb{R}^d \times (0, T), \\ \partial_t m - \nu \Delta m - \operatorname{div}(D_p H(x, m, Du)m) = 0, & \text{in } \mathbb{R}^d \times (0, T), \\ m(0) = m_0, u(x, T) = G(x, m(T)), \end{cases} \quad (1.57)$$

where  $\nu$  is a nonnegative parameter. The first equation has to be understood backward in time and the second one is forward in time. There are two crucial structure conditions for this system:

- The convexity of  $H = H(x, m, p)$  with respect to  $p$  which implies that the first equation (a Hamilton-Jacobi equation) is associated with an optimal control problem. This first equation shall be the value function associated with a typical small player.
- $m_0$  and  $m_t$  are probability or the density of probability measures.

The interpretation of this system is the following. An average agent controls the stochastic differential equation

$$dX_t = \alpha_t dt + \sqrt{2\nu} B_t, \quad (1.58)$$

where  $B_t$  is a standard Brownian motion and he aims at minimizing the following quantity

$$\mathbb{E} \left[ \int_0^T \frac{1}{2} L(X_s, m(s), \alpha_s) ds + G(X_T, m(T)) \right], \quad (1.59)$$

where  $L$  is the Fenchel conjugate of the convex function  $H$  with respect to  $p$  and as it can be seen, the evolution of the measure  $m(s)$  enters as a parameter in the cost function.

In the literature, there are many approaches to solve the problem of differential games with an infinite number of agents.

- A first approach consists of looking at the limit of Nash equilibrium in differential games with a large number of players and try to pass to the limit as this number goes to infinity.
- A second approach relies on finding the equations that should be satisfied by Nash equilibria of differential games with infinitely many players and to prove that the resulting solutions of these equations allow to solve differential games with finitely many players.

The starting point was the pioneering recent papers of Lasry and Lions [73–75, 78] who enlarged considerably the horizon for applications of mean field problems. They extended this approach to problems in economics, finance and also the theory of stochastic differential games where they introduced a general mathematical modeling approach of situations where a large number of particles is involved.

Since then, the literature on the MFG has grown considerably: Many authors work on this subject, for instance, in the survey by Gomes and Saud [46] and in the monograph by Bensoussan, Frehse and Yam [11].

In 2009, nonlinear mean-field backward stochastic differential equations have been investigated in the work of Buckdahn, Djehiche Li and Peng [26]. Since then, the theory of mean-field forward-backward stochastic differential equations as well as the theory of the associated partial differential equations of mean-field type has been intensively studied in the literature.

Buckdahn, Li and Peng [27] studied a mean field problem in a Markovian setting. In one hand, the authors investigated the existence and uniqueness of the mean field BSDEs in more general setting. In fact, unlike [26], they consider that the coefficients are not necessarily deterministic. In the other hand, they give a comparison principle of this new type of BSDEs and study a decoupled mean-field FBSDEs and its relation with PDEs.

In [92], Min, Peng and Qin studied a new type of equations whose coefficients depend on the state of the solution processes as well as their expected values. They called this type of equations *fully coupled FBSDES* and they proved, under some monotonicity conditions, the existence and uniqueness of a square integrable adapted solution. For a complete overview about this topic, we refer the interested reader to [11, 28–31].

### 1.3.3 Main results and contributions

In the last part of thesis, our main purpose is to obtain existence and uniqueness results for a general class of fully coupled forward-backward stochastic differential equations of mean-field type (MF-FBSDE in short) with jumps under weak monotonicity conditions and without the non-degeneracy assumption on the forward equation. This is accomplished by suggesting an implicit approximation scheme that is shown to converge to the solution of the system of MF-FBSDE with jumps. Then, we provide an application in the field of storage in smart grids.

Let us first introduce the Wasserstein distance between two probability measures.

For any random variable  $X$  on  $(\Omega, \mathcal{F}, \mathbb{P})$ , we denote by  $\mathbb{P}_X$  its probability law under  $\mathbb{P}$ . We denote by  $\mathcal{M}_2(\mathbb{R}^d)$  the set of probability measures on  $\mathbb{R}^d$  with finite moments of order 2 equipped with the 2-Wasserstein distance

$$\begin{aligned} \mathcal{W}_2(\mu, \mu') &:= \inf\left\{\left(\int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^2 F(dx, dy)\right)^{\frac{1}{2}}, F \in \mathcal{M}_2(\mathbb{R}^d \times \mathbb{R}^d) \text{ with marginals } \mu, \mu'\right\} \\ &:= \inf\left\{(\mathbb{E}|\xi - \xi'|^2)^{\frac{1}{2}} : \mu = \mathcal{L}(\xi), \mu' = \mathcal{L}(\xi')\right\}, \end{aligned}$$

where  $\mathcal{L}(\xi)$  and  $\mathcal{L}(\xi')$  are respectively the law of  $\xi$  and  $\xi'$  and the infimum is taken over  $F \in \mathcal{M}_2(\mathbb{R}^d \times \mathbb{R}^d)$  with marginals  $\mu$  and  $\mu'$ .

Notice that if  $X^1$  and  $X^2$  are random variables of order 2 with values in  $\mathbb{R}^d$ , then we have the following inequality involving the Wasserstein metric between the laws of the square integrable random variables  $X^1$  and  $X^2$  and their  $L^2$ -distance:

$$\mathcal{W}_2(\mathbb{P}_{X^1}, \mathbb{P}_{X^2}) \leq \left[\mathbb{E}|X^1 - X^2|^2\right]^{\frac{1}{2}}. \quad (1.60)$$

After introducing the needed elements, our purpose is to prove existence and uniqueness results of the following system of fully-coupled forward backward SDEs with jumps

$$(S) \begin{cases} X_t = X_0 + \int_0^t b_s(X_s, Y_s, Z_s, K_s, \mathbb{P}_{(X_s, Y_s)}) ds + \int_0^t \sigma_s(X_s, Y_s, Z_s, K_s, \mathbb{P}_{(X_s, Y_s)}) dW_s \\ \quad + \int_0^t \int_E \beta(s, X_{s-}, Y_{s-}, Z_s, K_s, \mathbb{P}_{(X_s, Y_s)}) \tilde{\pi}(ds, de), \quad 0 \leq t \leq T, \mathbb{P}\text{-a.s.} \\ Y_t = g(X_T, \mathbb{P}_{X_T}) + \int_t^T h_s(X_s, Y_s, Z_s, K_s, \mathbb{P}_{(X_s, Y_s)}) ds - \int_t^T Z_s dW_s - \int_t^T \int_E K_s(e) \tilde{\pi}(ds, de). \end{cases}$$

This system has been studied in [38] without the jump component. In this part, we extend this result to a more general setting. To do so, we start by making the following Lipschitz continuity assumptions on the coefficients.

1. The functions  $b, h, \sigma$  and  $\beta$  are Lipschitz in  $(x, y, z, k)$ . That is, there exists a constant  $C > 0$  such that for all  $t \in [0, T]$ ,  $u = (x, y, z, k)$ ,  $u' = (x', y', z', k') \in \mathbb{R}^{d+d+d \times d} \times \mathbb{L}^0(\mathcal{B}(E), \eta)$  and  $\nu, \nu' \in \mathbb{M}_2(\mathbb{R}^d \times \mathbb{R}^d)$ ,

$$|b(t, u, \nu) - b(t, u', \nu')| + |h(t, u, \nu) - h(t, u', \nu')| + |\sigma(t, u, \nu) - \sigma(t, u', \nu')| + |\beta(t, u, \nu) - \beta(t, u', \nu')| \leq C \left[ |x - x'| + |y - y'| + \|z - z'\| + |k - k'|_{\mathcal{L}^2(\eta)} + \mathcal{W}_2(\nu, \nu') \right].$$

2. For  $\phi \in \{b, h, g, \sigma, \beta\}$ ,  $\phi$  is Lipschitz with respect to  $x, y, z, k$  and  $\nu$  with  $C_\phi^x, C_\phi^y, C_\phi^z, C_\phi^k$  and  $C_\phi^\nu$  as the Lipschitz constants.

3. The function  $g : \Omega \times \mathbb{R}^d \times \mathbb{M}_2(\mathbb{R}^d) \rightarrow \mathbb{R}^d$  is Lipschitz in  $(x, \mu)$  i.e. there exists  $C > 0$  such that for all  $x, x' \in \mathbb{R}^d$  and for all  $\mu, \mu' \in \mathbb{M}_1(\mathbb{R}^d)$ ,

$$|g(x, \mu) - g(x', \mu')| \leq C(|x - x'| + \mathcal{W}_2(\mu, \mu')), \quad \mathbb{P}\text{-a.s.} \quad (1.61)$$

The first main result of this part is Theorem 1.3.1 given below, where we prove an existence and uniqueness result of the solution of our system **(S)** under Assumption **(H1)**.

$$(H1) \left\{ \begin{array}{l} (i) \text{ There exists } k > 0, \text{ s.t } \forall t \in [0, T], \nu \in \mathbb{M}_1(\mathbb{R}^d \times \mathbb{R}^d), u, u' \in \mathbb{R}^{d+d+d \times d} \times \mathbb{L}^0(\mathcal{B}(E), \eta), \\ \quad \mathcal{A}(t, u, u', \nu) \leq -k|x - x'|^2, \quad \mathbb{P}\text{-a.s.} \\ (ii) \text{ There exists } k' > 0, \text{ s.t } \forall \nu \in \mathbb{M}_2(\mathbb{R}^d \times \mathbb{R}^d), x, x' \in \mathbb{R}^d \\ \quad (g(x, \nu) - g(x', \nu)) \cdot (x - x') \geq k'|x - x'|^2, \quad \mathbb{P}\text{-a.s.} \end{array} \right.$$

**Theorem 1.3.1.** *Under Assumption **(H1)**, there exists a unique solution  $U = (X, Y, Z, K)$  of the mean field FBSDE with jumps **(S)**.*

To prove the existence part, we use an approximation scheme based on perturbations of the forward SDE of the system **(S)**. More precisely, we introduce  $\delta \in ]0, 1]$  and consider a sequence  $(X^n, Y^n, Z^n, K^n)$  of processes defined recursively in the following way:

$(X^0, Y^0, Z^0, K^0) = (0, 0, 0, 0)$  and for  $n \geq 1$ ,  $U^n = (X^n, Y^n, Z^n, K^n)$  satisfies, for every  $t \in [0, T]$

$$\left\{ \begin{array}{l} X_t^{n+1} = X_0 + \int_0^t [b_s(U_s^{n+1}, \nu_s^n) - \delta(Y_s^{n+1} - Y_s^n)] ds + \int_0^t [\sigma_s(U_s^{n+1}, \nu_s^n) - \delta(Z_s^{n+1} - Z_s^n)] dW_s \\ \quad + \int_0^t \int_E (\beta_s(U_s^{n+1}, \nu_s^n) - \delta(K_s^{n+1} - K_s^n)) \tilde{\pi}(ds, de), \\ Y_t^{n+1} = g(X_T^{n+1}, \mu_T^n) - \int_t^T h_s(U_s^{n+1}, \nu_s^n) ds - \int_t^T Z_s^{n+1} dW_s - \int_t^T \int_E K_s^{n+1}(e) \tilde{\pi}(ds, de), \end{array} \right.$$

where  $\nu_t^n = \mathbb{P}_{(X_t^n, Y_t^n)}$  and  $\mu_T^n = \mathbb{P}_{X_T^n}$ . Once we show that the following inequality

$$\mathbb{E}[|X_t^{n+1} - X_t^n|^2] + \mathbb{E}\left[\int_0^T |U_s^{n+1} - U_s^n|^2 ds\right] \leq \frac{\theta}{\gamma} \mathbb{E}[|X_t^n - X_t^{n-1}|^2] + \mathbb{E}\left[\int_0^T |U_s^n - U_s^{n-1}|^2 ds\right], \quad (1.62)$$

where

$$\begin{cases} \gamma := \min(k' - \frac{C_g^\nu \epsilon}{2}, k - \frac{\bar{\epsilon} C_h^\nu}{2}, (\delta - \frac{\kappa \delta}{2} - \frac{\bar{\epsilon} C_b^\nu}{2}), (\delta - \frac{\kappa \delta}{2} - \frac{\bar{\epsilon} C_\sigma^\nu}{2}), (\delta - \frac{\kappa \delta}{2} - \frac{\bar{\epsilon} C_\beta^\nu}{2})) \\ \theta = \max(\frac{C_g^\nu}{2\epsilon}, -\frac{C_h^\nu + C_b^\nu + C_\sigma^\nu + C_\beta^\nu}{2\epsilon} + \frac{\delta}{2\kappa}), \end{cases} \quad (1.63)$$

is a contraction, we obtain the desired result.

In the uniqueness part, we suppose that **(S)** has two solutions  $(X, Y, Z, K)$  and  $(X', Y', Z', K')$ . It suffices to apply Itô's formula to  $(X'_t - X_t)(Y'_t - Y_t)$  and then find an upper (resp. a lower) bound to  $\mathbb{E}[(X'_t - X_t)(Y'_t - Y_t)]$  to conclude that the solution is unique.

In the second part, we aim to prove the existence and uniqueness of the solution of **(S)** under the following weaker assumptions.

$$(H2) \begin{cases} (i) \text{ There exists } k > 0, \text{ s.t } \forall t \in [0, T], \nu \in \mathbb{M}_2(\mathbb{R}^d \times \mathbb{R}^d), u, u' \in \mathbb{R}^{d+d+d \times d}, \\ \quad \mathcal{A}(t, u, u', \nu) \leq -k(|y - y'|^2 + \|z - z'\|^2 + |k - k'|_{\mathcal{L}^2(\eta)}), \text{ } \mathbb{P}\text{-a.s.} \\ (ii) \text{ There exists } k' > 0, \text{ s.t } \forall \nu \in \mathbb{M}_2(\mathbb{R}^d \times \mathbb{R}^d), x, x' \in \mathbb{R}^d \\ \quad (g(x, \nu) - g(x', \nu)) \cdot (x - x') \geq k'|x - x'|^2, \text{ } \mathbb{P}\text{-a.s.} \end{cases}$$

To do so, we start by proving the following a priori estimate.

**Lemma 1.3.2.** *Let  $(Y', Z', K')$  be another solution of the system **(S)**. Then, under **(H2)**, we have the following estimates*

$$\mathbb{E}\left[\int_0^T |\Delta X_s|^2 ds\right] \leq \frac{\exp(t \Upsilon^1) - \Upsilon^1}{\Upsilon^1} [\Upsilon^2 E\left[\int_0^t |\Delta Y_s|^2 ds\right] + \Upsilon^3 E\left[\int_0^t |\Delta Z_s|^2 ds\right] + \Upsilon^4 E\left[\int_0^t |\Delta K_s|_{\mathcal{L}^2(\eta)}^2 ds\right]],$$

where

$$\begin{cases} \Upsilon^1 := 3 + 2C_b^x + 5(C_\sigma^x)^2 + 5(C_\beta^x)^2 + 2C_b^\nu + 5(C_\sigma^\nu)^2 + 5(C_\beta^\nu)^2 \\ \Upsilon^2 := (C_b^y)^2 + 5(C_\sigma^y)^2 + 5(C_\beta^y)^2 + C_b^\nu + 5(C_\sigma^\nu)^2 + 5(C_\beta^\nu)^2 \\ \Upsilon^3 := (C_b^z)^2 + 5(C_\sigma^z)^2 + 5(C_\beta^z)^2 \\ \Upsilon^4 := (C_b^k)^2 + 5(C_\sigma^k)^2 + 5(C_\beta^k)^2. \end{cases}$$

Our second main result in this part is the following.

**Theorem 1.3.3.** *Under Assumption (H2), there exists a unique solution  $U = (X, Y, Z, K)$  of the mean field FBSDE with jumps (S).*

We emphasize that the proof of this result is very similar to the one of Theorem 1.3.3. However, the only difference is that perturbations here are made on the backward SDE of the system (S) in the following way

$$\left\{ \begin{array}{l} X_t^{n+1} = X_0 + \int_0^t b_s(U_s^{n+1}, \nu_s^n) ds + \int_0^t \sigma_s(U_s^{n+1}, \nu_s^n) dW_s + \int_0^t \int_E \beta_s(U_s^{n+1}, \nu_s^n) \tilde{\pi}(ds, de), \\ Y_t^{n+1} = g(X_T^{n+1}, \mu_T^n) + \delta(X_T^{n+1} - X_T^n) - \int_t^T [h_s(U_s^{n+1}, \nu_s^n) + \delta(X_s^{n+1} - X_s^n)] ds \\ \quad - \int_t^T Z_s^{n+1} dW_s - \int_t^T \int_E K_s^{n+1}(e) \tilde{\pi}(ds, de), \end{array} \right.$$

with  $\mu_T^n = \mathbb{P}_{X_T^n}$  and  $\nu_t^n = \mathbb{P}_{(X_t^n, Y_t^n)}$ .

**Application: Storage in smart grids** After establishing the existence and uniqueness results, we give an application in storage in smart grids. More precisely, we provide a stylized quantitative model for a power system with distributed local energy generation and storage.

This system is modeled as a network connecting a large number of nodes. Each node has a local electricity consumption, a local electricity production and manages a local storage device as in [1]. However, unlike [1], we assume that the production of energy is unpredictable due to its dependence on environmental conditions such as the sun, the speed of the wind etc. which are intermittent and irregular. This leads to include a jump component in the net power production of each node.

We consider that the aim of each node is to minimize its own cost of electricity consumption by controlling the storage device and we assume that the spot price level reflects the instantaneous global consumption. In a non-cooperative game setting, we are led to the analysis of a non-zero sum stochastic game with N players and to the search of Nash-equilibria.

To illustrate that, we formulate and solve an Extended Mean Field Game type control (EMFG) with common noise. Note that mean field type control (MFC in short) is different from the mean field game (MFG). For a comparison between the two concepts, we refer the reader to the book of Bensoussan, Frehse and Yam [11] as well as the article of Carmona, Delarue, and Lachapelle [31]. We prove that the EMFG associated to this power network game admits a unique solution which can be characterized by solving an associated FBSDE. In the particular case where the cost structure is quadratic and the pricing rule is linear, the FBSDE characterizing the solution of the EMFG can be solved explicitly. This work is concretized in the preprint [86].

# UTILITY MAXIMIZATION PROBLEM IN NONEQUIVALENT MARKETS: A BSDE APPROACH

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## 2.1 Introduction

Our aim in this chapter is to provide a second point of view about the stability of the utility maximization in nonequivalent markets in continuous framework and especially in the case of discontinuous filtration (information including jumps). The main motivation behind this work comes from the result stated in [117] where the author investigated the same problem but uses a completely different method to solve it. In fact, the author uses a duality approach and her idea consists on providing conditions on the utility function and the sequence of markets in order to obtain the convergence of the value functions and the indifference prices. In this chapter, we consider a similar setup to [117] and we characterize dynamically the value function using the martingale optimality principle. Then, thanks to the properties of the drivers in both cases (continuous and discontinuous), we succeed to prove the stability of the value process in the two frameworks.

The outline is as follows. We start by providing a complete description of the market and recalling briefly some definitions in Subsection 2.2.1. Later, we establish the link between our optimization problem which is of power type and BSDEs. More precisely, we characterize dynamically the value function using stochastic control techniques namely the dynamic programming principle and consequently we link the problem, in continuous setting, to a special quadratic BSDE. Using this connection and making the needed assumptions on the input parameters and their convergence mode, we provide a first stability result in Subsection 2.2.3 when the filtration is continuous. Section 2.3 is devoted to the investigation of the same problem when the market include jumps. To do so, we start by introducing the financial market model which includes jumps generated by a random measure, giving some notations and then describing the optimization problem associated to an exponential utility function and making the link with

quadratic BSDEs with jumps using the same stochastic control techniques. Besides, after giving the properties of the BSDE's driver, we make hypotheses (see Assumption 2.3.4) on the input parameters of our market model and their mode of convergence to insure the stability of the value function which is established in Subsection 2.3.3.

## 2.2 Stability of the utility maximisation problem in nonequivalent markets : A continuous framework

In this section, we study the stability of the utility maximisation problem associated to a sequence of non-equivalent financial markets. More precisely, we choose a utility function defined on the positive real line and we prove the convergence of the associated value process.

### 2.2.1 The model formulation and the optimization problem

Throughout this part, we work in a filtered probability space  $(\Omega, \mathbb{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})$  where the filtration is generated by an  $\mathbb{R}^d$ -valued Brownian motion. We will explain the financial context by providing here all the definitions and common assumptions. We consider a sequence of financial markets consisting in  $d + 1$  assets: one risk free asset with zero interest rate and  $d$  risky assets. The price process  $S^n$  of the  $d$  risky assets evolves according to the following equation

$$\frac{dS_t^{n,i}}{S_t^{n,i}} = dM_t^{n,i} + \sum_{j=1}^d \lambda_t^{n,j} d\langle M^{n,i}, M^{n,j} \rangle_t, \quad 0 \leq t \leq T, \quad S_0^{n,i} = 1, \quad \text{for } i = 1, \dots, d,$$

where  $\lambda = (\lambda_t^n)_{0 \leq t \leq T}$  is an  $\mathbb{R}^d$ -valued predictable process, called the market price of risk, satisfying

$$\int_0^T (\lambda_s^n)^\top d\langle M^n \rangle_s \lambda_s^n < \infty, \quad a.s.,$$

and  $\langle M^n \rangle$  denotes the quadratic variation of the local martingale  $M^n$  where  $M^n$  is a  $d$ -dimensional continuous martingale,  $\langle M^{n,i}, M^{n,j} \rangle_t$  stands for the  $j$ -th column of the  $\mathbb{R}^{d \times d}$  matrix valued process  $\langle M^n \rangle$ . In this sequel, we take the following particular dynamics of the price process  $S^n$

$$dS_t^n = \lambda_t^n |\sigma_t^n|^2 dt + \sigma_t^n dB_t, \quad 0 \leq t \leq T \quad \text{and } S_0^n = 1.$$

Define  $\mu_t^n = \lambda_t^n |\sigma_t^n|^2$  and the market price of risk  $\theta_t^n = (\sigma_t^n)^\top (\sigma_t^n \sigma_t^n)^\top)^{-1} \mu_t^n$ . Consequently, we are dealing with a sequence of markets with price dynamics  $S^n$  of the following form

$$\begin{cases} dS_t^n = S_t^n \sigma_t^n (\theta_t^n dt + dB_t), \quad t \in [0, T], \mathbb{P}\text{-a.s.} \\ S_0^n = 1. \end{cases} \quad (2.1)$$

Note that  $\mu^{n,i}$  and  $\sigma^{n,i}$  are respectively  $\mathbb{R}$  and  $\mathbb{R}^{1 \times d}$  valued predictable uniformly bounded processes,  $\sigma^n$  is of full rank (i.e.  $\det(\sigma_t^n (\sigma_t^n)^{\text{tr}})$ ) and  $\theta^{n,i}$  is then an  $\mathbb{R}^d$ -valued predictable uniformly bounded process as well.

**Definition 2.2.1.** A predictable  $\mathbb{R}^d$ -valued process  $\pi^n = (\pi_t^n)_{t \in [0, T]}$  is called a self-financing trading strategy if it satisfies

- $\pi_s^n \in C$ ,  $\mathbb{P}$ -a.s. where  $C$  is the  $\mathbb{R}^d$ -valued constraints set.
- The wealth process  $X^{\pi, n}$  associated to an agent with an initial capital  $x$  at time  $t$  and running a strategy  $\pi^n$  is defined as follows

$$X_t^{\pi, n} = x + \int_0^t \sum_{i=1}^d \frac{\pi_u^{n,i}}{S_u^{n,i}} dS_u^{n,i} = x + \int_0^t \pi_s^n \sigma_s^n (dB_s + \theta_s^n ds), \quad t \in [0, T].$$

We emphasize that in the definition above, each component  $\pi^{n,i}$  of the trading strategy describes the amount of money invested in the  $i$ -th asset  $S^{n,i}$  for  $i = 1, \dots, d$ . We also precise that the presence of constraints on the strategies entails the incompleteness of the market. More precisely, a strategy allowing the replicability of a contingent claim does not necessarily exists.

We will also suppose that the investor has to pay ( or receive) a liability  $\xi$  at time  $T$ , that is to say an  $\mathcal{F}_T$ -measurable random variable which could represent the value of an option or a contract maturing at time  $T$ .

In this context, we define the utility maximisation problem which aims at giving the expression of the value process defined at any time  $t$  by

$$u^n(x) = \sup_{\pi^n \in \mathcal{A}_t} \mathbb{E} \left[ U(X_T^{\pi, n} - \xi) \right], \quad (2.2)$$

where  $U$  is a non-decreasing concave function (utility function) and  $\mathcal{A}_t$  is the set of admissible strategies. Let us define the admissibility of the strategies in our context.

**Definition 2.2.2.** Let  $C$  be a closed set in  $\mathbb{R}^{1 \times d}$  which contains zero. The set of admissible strategies consists of all  $d$ -dimensional predictable processes  $\pi_t^n = (\pi_t^n)_{t \in [0, T]}$  which satisfy  $\mathbb{E} \left[ \int_0^t |\pi_s^n \sigma_s^n|^2 ds \right] < \infty$  and  $\pi_t^n \in C$ ,  $\mathbb{P}$ -a.s..

**Power utility** In this part, we study the stability of the value process when the agent preferences are given by a power utility function. More precisely

$$U(x) = \frac{x^\gamma}{\gamma}, \quad x \in ]0, +\infty[, \quad \gamma \in ]0, 1[. \quad (2.3)$$

Note that the case of power utility function is different from the others. In fact, the definition of trading strategies is a bit different. In this context, a constrained trading strategy is an  $\mathbb{R}^d$ -valued process  $\rho^n \in C$  where  $\rho^{n,i}$  stands for the part of the wealth invested in stock  $i$ . The wealth process in this case is then defined in the following way

$$X_t^{\rho,n} = x + \int_0^t \sum_{i=1}^d X_s^{\rho,n} \frac{\rho_s^n}{S_s^{n,i}} dS_s^{\rho,i} = x + \int_0^t X_s^{\rho,n} \rho_s^n \sigma_s^n (dB_s + \theta_s^n ds). \quad (2.4)$$

One particularity of the power utility function is that it allows us to write the wealth process in a multiplicative way

$$X_t^{\rho,n} = x \mathcal{E} \left( \int_0^t \rho_s^n \sigma_s^n (\theta_s^n ds + dB_s) \right), \quad t \in [0, T].$$

**Remark 2.2.3.** We emphasize that, as mentioned in [55], it is more convenient to take

$$p_t^n = \rho_t^n \sigma_t^n \text{ and } C_t^n(\omega) = C \sigma_t^n(\omega), \quad t \in [0, T].$$

This entails the following second formulation of the utility maximisation problem

$$u_n(x) := \sup_{p \in \mathcal{A}_t^n} \mathbb{E}[U(x + \int_0^T X_s^{p,n} p_s^n (\theta_s^n ds + dB_s) - \xi)]. \quad (2.5)$$

The new set of admissible strategies  $\tilde{\mathcal{A}}_t^n$  is the set of all  $\mathbb{R}^{1 \times d}$ -valued predictable processes  $(p_t^n)_{0 \leq t \leq T}$  where  $p_t^n \in \tilde{\mathcal{A}}_t^n$  iff:  $\mathbb{E} \int_0^T |p_s^n|^2 ds < \infty$   $\mathbb{P}$ -a.s. and  $p_t^n(\omega) \in C_t^n(\omega)$  and such that the set  $C_t^n(\omega)$  is also closed.

## 2.2.2 A BSDE description of the value process

Our aim in this part is to make a connexion between the optimisation problem (2.5) and a backward stochastic differential equation. To do so, we rely on dynamic programming principle as in [55].

**Notations and preliminaries about quadratic BSDEs** Let us first introduce the following norms and spaces that will be often used in the present sequel

- For  $a \in \mathbb{R}^d$  and  $C$  is a subset of  $\mathbb{R}^d$ , we define the distance between  $a$  and  $C$  as follows

$$\text{dist}_C(a) = \min_{b \in C} |a - b|.$$

- The set  $\Pi_C(a)$  consists of all the elements of  $C$  at with the minimum is obtained i.e.

$$\Pi_C(a) = \{b \in C : |a - b| = \text{dist}_C(a)\}.$$

- $\mathcal{S}^2$  is the space of  $\mathbb{R}$ -valued continuous and  $\mathcal{F}_t$ -progressively measurable processes  $Y := (Y_s)_{s \leq T}$  such that

$$\mathbb{E} \left[ \sup_{0 \leq t \leq T} |Y_t|^2 \right] < \infty.$$

- $\mathcal{H}^2$  is the space of  $\mathbb{R}^d$ -valued and  $\mathcal{F}_t$ -progressively measurable processes  $Z := (Z_s)_{s \leq T}$  such that

$$\mathbb{E} \left[ \int_0^T |Z_s|^2 ds \right] < \infty.$$

Consider the following BSDE

$$Y_t = \xi + \int_t^T f_s(Y_s, Z_s) ds - \int_t^T Z_s dB_s, 0 \leq t \leq T, \mathbb{P}\text{-a.s.} \quad (2.6)$$

where  $\xi$  is a real-valued  $\mathcal{F}_T$ -measurable random variable called the terminal value and  $f : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^{1 \times d}$  is a  $\mathcal{P} \times \mathcal{B}(\mathbb{R}) \times \mathcal{B}(\mathbb{R}^{1 \times d})$ -measurable random function called the generator. Here  $\mathcal{P}$ ,  $\mathcal{B}(\mathbb{R})$  and  $\mathcal{B}(\mathbb{R}^{1 \times d})$  denote respectively the  $\sigma$ -field of all predictable sets of  $\Omega \times [0, T]$ , the Borel field of  $\mathbb{R}$  and  $\mathbb{R}^d$ . We write  $\text{BSDE}(f, \xi)$  to refer to the BSDE with generator  $f$  and terminal value  $\xi$ .

A solution to the BSDE (2.6) is a pair of  $\mathcal{F}_t$ -adapted processes  $(Y, Z) \in \mathcal{S}^2 \times \mathcal{H}^2$  such that (2.6) holds true  $\mathbb{P}$ -a.s.. Consider now the following assumptions

**Assumption 2.2.4.** *The terminal condition  $\xi$  is a bounded random variable.*

**Assumption 2.2.5.** *There exists a non-negative predictable process  $\alpha$  satisfying  $\int_0^T \alpha_s ds < \infty$  and constants  $\beta, \gamma > 0$ , such that*

$$|f_t(y, z)| \leq \alpha_t + \beta|y| + \gamma|z|^2, \forall (y, z) \in \mathbb{R} \times \mathbb{R}^{1 \times d}.$$

Under the Assumptions 2.2.4 and 2.2.5, the  $\text{BSDE}(f, \xi)$  is said to be quadratic. One important motivation to study this type of BSDE is that it appears naturally when using the dynamic method to solve the utility maximization problem. To prove the existence and uniqueness of a solution of a scalar quadratic BSDE with a bounded terminal value, Rouge and El Karoui [111] and Hu, Imkeller and Muller [55] used results of Kobylansky [69] on the existence and they used comparison arguments for quadratic BSDEs driven by a Brownian motion to prove the uniqueness of the solution. In the Lipschitz-quadratic case, existence and uniqueness of the solution is obtained by Tevzadze in [116] via a Picard iteration argument. This result was extended to multidimensional  $Y$  components for small terminal conditions. Notice that a strong requirement present in all the previous articles is that the terminal value is bounded. Later,

Briand and Hu [24, 25] get rid of this condition and replaced it by the assumption that they only need exponential moments and established the existence and the uniqueness of a solution but unlike the previous works, the driver is convex in  $z$ .

**The dynamic method** In order to characterize the value process (2.2) via quadratic BSDEs, we use the following martingale optimality principle as in [55].

Martingale optimality principle Construct a family of processes  $(R_t^{\pi,n})_{0 \leq t \leq T}$  where  $\pi^n \in \mathcal{A}_t$  satisfying

- (i)  $R_T^{\pi,n} = U(X_T^{\pi,n} - \xi)$  for all  $\pi^n \in \mathcal{A}_t$ .
- (ii)  $R_0^{\pi,n} = R_0^n$  for all  $\pi^n \in \mathcal{A}_t$ .
- (iii)  $R^{\pi,n}$  is a supermartingale for all  $\pi^{n*} \in \mathcal{A}$  and there exists an optimal strategy  $\pi^{n*} \in \mathcal{A}_t$  such that  $R^{\pi^{n*}}$  is a martingale.

Relying on this technique, a link between the power optimisation problem and quadratic BSDEs has been made in the following way.

**Theorem 2.2.6.** ([55] Imkeller, Muller, Hu) *The value function of the optimization problem (2.5) is given, for  $n \geq 1$ , by*

$$u^n(x) = \frac{x^\gamma}{\gamma} \exp(Y_0^n), \quad \text{for } x > 0,$$

where  $Y_0^n$  is defined by the unique solution  $(Y^n, Z^n)$  of the BSDE

$$Y_t^n = \xi - \int_t^T f_s^n(Z_s^n) ds - \int_t^T Z_s^n dB_s, \quad \mathbb{P}\text{-a.s.} \quad (2.7)$$

with

$$f_t^n(z) = \frac{\gamma(1-\gamma)}{2} \text{dist}^2\left(\frac{1}{1-\gamma}(z + \theta_t^n), C_t^n\right) - \frac{\gamma|z + \theta_t^n|^2}{2(1-\gamma)} - \frac{1}{2}|z|^2. \quad (2.8)$$

There exists an optimal trading strategy  $p^{n*} \in \tilde{\mathcal{A}}^n$  with

$$p_t^{n*} \in \Pi_{C_t^n(\omega)}\left(\frac{1}{1-\gamma}(Z_t^n + \theta_t^n)\right).$$

### 2.2.3 Stability of the value process

In this part, we investigate the stability of a utility maximization problem for a utility function of power type where misspecifications on the model are denoted by variations of the coefficients of the risky assets. In fact, given an initial capital  $x > 0$ , our question is the following: *Under which conditions on the input parameters of the sequence of markets  $S^n$  and their convergence*

mode will the solution  $(Y^n, Z^n)$  of the BSDE  $(f^n, \xi)$  and consequently the corresponding value process  $u^n$  converge? We first make the following assumptions.

**Assumption 2.2.7.** For  $n \geq 1$ ,  $C^n$  is a sequence of nonempty closed subsets of  $\mathbb{R}^d$  such that

$$\lim_{n \rightarrow \infty} \text{dist}(\cdot, C^n) = \text{dist}(\cdot, C^\infty), \quad dt \otimes d\mathbb{P}\text{-a.s.}$$

**Assumption 2.2.8.** The sequences of the drift and volatility are uniformly bounded and converge, for  $n \geq 1$ , in the following sense

$$\lim_{n \rightarrow \infty} \mu_t^n = \mu_t^\infty, \quad dt \otimes d\mathbb{P}\text{-a.s.},$$

and

$$\lim_{n \rightarrow \infty} \sigma_t^n = \sigma_t^\infty, \quad dt \otimes d\mathbb{P}\text{-a.s.}$$

• Let us precise that the type of convergence in Assumption 2.2.7 is called the Wijsman convergence. More precisely, we say that  $C^n \rightarrow C$  as  $n \rightarrow +\infty$  in the Wijsman topology with respect to the metric  $d$  if

$$\text{dist}_{C^n}(x) \rightarrow \text{dist}_{C^\infty}(x), \quad \forall x \in \mathbb{R}^d.$$

• We call  $C^\infty$  the closed set limit of the set sequence  $(C^n)_{n \in \mathbb{N}}$  i.e.  $\lim_{n \rightarrow \infty} C^n = C^\infty$  if

$$C^\infty = \bigcap_n \left( \bigcup_{k \geq n} C^k \right) = \bigcup_n \left( \bigcap_{k \geq n} C^k \right).$$

• Let us also recall the following result about Wijsman convergence.

**Proposition 2.2.9.** [10]. The following assertions are equivalent

• The sequence  $(C^n)_{n \in \mathbb{N}}$  of closed nonempty sets converges to  $C^\infty$  :

$$\lim_{n \rightarrow \infty} C^n = C^\infty$$

• The sequence  $(\text{dist}(\cdot, C^n))_{n \in \mathbb{N}}$  of functions converges pointwise to  $\text{dist}(\cdot, C^\infty)$

For more details about Wijsman convergence we refer the reader to [10].

• Under Assumption 2.2.8,  $\theta_t^n$  is uniformly bounded and  $\lim_{n \rightarrow \infty} \theta_t^n = \theta_t^\infty$ ,  $dt \otimes d\mathbb{P}\text{-a.s.}$

Now, we give stability result associated to our optimization problem stated in a continuous framework.

**Theorem 2.2.10.** Under Assumptions 2.2.7 and 2.2.8, the solution  $(Y^n, Z^n)$  of the BSDE  $(f^n, \xi)$

(2.7) converges to the solution  $(Y^\infty, Z^\infty)$  of the BSDE  $(f^\infty, \xi)$  defined by

$$Y_t^\infty = \xi - \int_t^T f^\infty(s, Y_s^\infty, Z_s^\infty) ds - \int_t^T Z_s^\infty dB_s,$$

with

$$f_t^\infty(z) = \frac{\gamma(1-\gamma)}{2} \text{dist}^2\left(\frac{1}{1-\gamma}(z + \theta_t^\infty), C_t^\infty\right) - \frac{\gamma|z + \theta_t^\infty|^2}{2(1-\gamma)} - \frac{1}{2}|z|^2, \quad (2.9)$$

in the following sense

$$\mathbb{E}\left[\exp\left(\sup_{0 \leq t \leq T} |Y_t^n - Y_t^\infty|\right) + \left(\int_0^T |Z_s^n - Z_s^\infty|^2 ds\right)\right] \xrightarrow{n \rightarrow \infty} 1.$$

Then,  $Y_0^n$  converges to a deterministic  $Y_0^\infty$  and  $\lim_{n \rightarrow +\infty} u^n(x) = \frac{x^\gamma}{\gamma} \exp(Y_0^\infty)$ .

**Proof.** The proof is straightforward and follows from Proposition 7 in [25]. In fact, the sequence of drivers  $(f^n)_n$  defined in (2.8) is convex in  $z$  and has a quadratic growth in  $z$  then it suffices to prove that it converges  $dt \otimes \mathbb{P}$ -a.s, for each  $z \in \mathbb{R}^d$  to  $f^\infty$  defined in (2.9). To proceed, it is evident that  $(z + \theta_t^n)^2 \xrightarrow{n \rightarrow \infty} (z + \theta_t)^\infty$ ,  $dt \otimes d\mathbb{P}$ -a.s., then it suffices to prove that

$$\text{dist}\left(\frac{1}{1-\gamma}(z + \theta_t^n), C_t^n\right) \xrightarrow{n \rightarrow \infty} \text{dist}\left(\frac{1}{1-\gamma}(z + \theta_t^\infty), C_t^\infty\right). \quad (2.10)$$

To do so, we use the fact that

$$\begin{aligned} \text{dist}\left(\frac{1}{1-\gamma}(z + \theta_t^n), C_t^n\right) &\leq \left\| \frac{1}{1-\gamma}(z + \theta_t^n) - \frac{1}{1-\gamma}(z + \theta_t^\infty) \right\| + \text{dist}\left(\frac{1}{1-\gamma}(z + \theta_t^\infty), C_t^n\right) \\ &\leq \frac{1}{1-\gamma} \|\theta_t^n - \theta_t^\infty\| + \text{dist}\left(\frac{1}{1-\gamma}(z + \theta_t^\infty), C_t^n\right) \end{aligned}$$

To get (2.10), we use Assumption 2.2.8 and again the fact that

$$\begin{aligned} \text{dist}\left(\frac{1}{1-\gamma}(z + \theta_t^\infty), C_t^n\right) &\leq \left\| \frac{1}{1-\gamma}(z + \theta_t^\infty) - \frac{1}{1-\gamma}(z + \theta_t^n) \right\| + \text{dist}\left(\frac{1}{1-\gamma}(z + \theta_t^n), C_t^n\right) \\ &\xrightarrow{n \rightarrow \infty} \text{dist}\left(\frac{1}{1-\gamma}(z + \theta_t^\infty), C_t^\infty\right) \\ &\leq \lim_{n \rightarrow \infty} \text{dist}\left(\frac{1}{1-\gamma}(z + \theta_t^n), C_t^n\right) \end{aligned}$$

which implies the desired result and concludes the proof.

**Remark 2.2.11.** In continuous setting, stability results were proved in the literature using different methods and under different assumptions. In fact, in [45], Frei's result relies on a pointwise convergence of the generator to study a specific stability problem for an exponential investor and this result has been used later to study markets with a fixed price of risk and a varying

correlation factor between the traded and non traded assets. In [93], the authors impose a convergence in probability of  $\int |f_s^n(Z) - f_s^\infty(Z)| ds$  and convergence in exponential moments of  $\sup |Y_s^n - Y_s^\infty|$  with an unbounded terminal condition. Barrieu and El Karoui imposed monotonicity in  $n$  and local uniform convergence of the drivers.

## 2.3 Stability of the utility maximisation problem in non equivalent markets: A discontinuous framework

So far, trajectories of the underlying assets have been assumed to be continuous. However, diffusion like this cannot generate discontinuous paths since the noise component  $B_t$  is continuous. In reality, stock prices are exposed to sudden movements due to several reasons which, clearly, make this representation unrealistic. So, in order to reproduce a more realistic behavior of the price process which take into consideration such movements, we will add a jump component to the price dynamic.

In this section, we will extend the results obtained above to a framework including jumps. To do so, we start by adding a jump component in the sequence of the asset prices (2.1) and then prove the stability of the associated value process by making a link with a specific sequence of backward SDEs with jumps.

### 2.3.1 The model formulation and the optimization problem

**Notations and preliminaries about the jump setting** Our model here is analogous to the previous one, the only difference stems in assuming here that the price process has jumps. We highlight that, here, unlike most works in the literature, where the jump measure is assumed to be a Lévy measure, we allow the compensator of the jump to be a random measure.

Hence, we consider a probability space  $(\Omega, \mathbb{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})$  which is now generated by the following two independent processes:

- A standard one dimensional Brownian motion  $(B_t)_{t \in [0, T]}$ ,
- An integer valued random measure  $\mu$  independent of  $B$  defined on  $[0, T] \times E$  and we denote the associated counting measure by  $\mu(ds, de)$  defined as follows

$$\begin{aligned} \mu(dt, de) : \Omega \times [0, T] \times E &\rightarrow \mathcal{B}([0, T]) \times \mathcal{E} \\ (\omega, ds, de) \rightarrow \mu(\omega, ds, de) &= \sum_{t \in [0, s]}^{\Delta X_t \neq 0} \delta_{(t, \Delta X_t)}(dt, de). \end{aligned}$$

The corresponding compensator is given by  $\nu(\omega, ds, dx)$ . We will suppose that  $\nu(\omega, ds, dx)$  is absolutely continuous with respect to the Lebesgue measure i.e.

$$\nu(ds, de) = \nu(de)ds,$$

where  $\nu(de)$  is a positive  $\sigma$ -finite measure satisfying

$$\int_E (1 \wedge |e|)^2 \nu(de) < \infty. \quad (2.11)$$

Hereafter, we suppose that the filtration is generated by  $B$  and  $\mu$  and satisfies the usual conditions of completeness and right continuity. Furthermore, we denote by  $\tilde{\mu}$  the compensated jump measure, which is a martingale, as follows

$$\tilde{\mu}(ds, de) = \mu(ds, de) - \nu(ds, de).$$

Let  $\mathcal{P}$  denotes the predictable  $\sigma$ -field on  $\Omega \times [0, T]$  and  $\tilde{\mathcal{P}} := \mathcal{P} \otimes \mathcal{B}(E)$  denotes the  $\sigma$ -field on  $\Omega \times [0, T] \times E$  and introduce the following norms and spaces that will be often used in the present work

- $\mathcal{P}$  (resp.  $\mathbf{P}$ ) is the  $\sigma$ -algebra of  $\mathbb{F}$ -progressively measurable (resp.  $\mathbb{F}$ -predictable) sets on  $\Omega \times [0, T]$ .
- $\mathcal{S}^2$  is the space of RCLL processes  $Y := (Y_s)_{s \leq T}$  such that

$$\mathbb{E} \left[ \sup_{0 \leq t \leq T} |Y_t|^2 \right] < +\infty,$$

- $\mathcal{S}^\infty$  is the set of RCLL processes  $Y$  such that

$$\mathbb{E} \left[ \sup_{0 \leq t \leq T} |Y_t| \right] < +\infty,$$

- $\mathcal{L}^2(\nu)$  is the space of Borel measurable functions  $(\varphi(e))_{e \in E} : E \rightarrow \mathbb{R}$  such that

$$\int_E |\varphi(e)|^2 \nu(de) < +\infty.$$

- $\mathbb{H}_\nu^2$  is the space of  $\tilde{\mathcal{P}}$ -measurable processes  $U := (U_s)_{s \leq T}$  such that

$$\mathbb{E} \left[ \int_0^T \int_E |U_s(e)|^2 \nu(de) ds \right] < +\infty.$$

We also assume that, with respect to the filtration  $(\mathcal{F}_t)_{0 \leq t \leq T}$ , both the martingale  $B$  and the

compensated random measure  $\tilde{\mu}$  have a weak predictable representation. More precisely, every local martingale  $M$  has the following representation:

$$M_t = M_0 + \int_0^t Z_s dB_s + \int_0^t \int_E U_s(e) \tilde{\mu}(ds, de), \quad 0 \leq t \leq T,$$

where  $Z$  and  $U$  are predictable processes such that

$$\mathbb{E}\left[\int_0^T |Z_s|^2 ds\right] < \infty, \quad \text{and} \quad \mathbb{E}\left[\int_0^T \int_E |U_s(e)|^2 \nu(de) ds\right] < +\infty, \quad \mathbb{P}\text{-a.s.}$$

In contrast to Section 2.2, we consider a market consisting in one non-risky asset with zero interest rate and a single risky asset whose price evolves in a non-continuous way

$$\begin{cases} dS_t^n = S_{t-}^n \left( \sigma_t^n (\theta_t^n dt + dB_t) + \int_E \beta_s^n(e) \tilde{\mu}(ds, de) \right), & t \in [0, T], \quad \mathbb{P}\text{-a.s.} \\ S_0^n = 1, \end{cases} \quad (2.12)$$

where  $\beta^n$  is a predictable bounded process satisfying  $\beta^n > -1$  in order to ensure the  $\mathbb{P}$ -a.s. positivity of the price process.  $\sigma^n$  and  $\theta^n$  satisfy the same properties as in Section 2.2.

Consider an investor who starts with an initial capital  $x$  and runs a trading strategy  $\pi^n$  in order to maximize his final expected utility. Concretely, he/she aims to solve the following problem

$$v_t^n(x) = \sup_{\pi^n \in \mathcal{A}} \mathbb{E}[U(X_T^{\pi^n} - \xi)]. \quad (2.13)$$

The admissibility criterion in this case is defined in the following way.

**Definition 2.3.1.** *The set of admissible trading strategies  $\mathcal{A}$  consists of all  $\mathbb{F}$ -predictable processes  $\pi^n = (\pi_t^n)_{0 \leq t \leq T}$  which satisfy*

$$\mathbb{E}\left[\int_0^T |\pi_t^n \sigma_t^n|^2 dt + \int_0^T \int_E |\pi_t^n \beta_t^n(e)|^2 \nu(de) dt\right] < +\infty, \quad \mathbb{P}\text{-a.s.} \quad (2.14)$$

and  $\pi^n \in C$ ,  $dt \otimes d\mathbb{P}$ -a.s. ( $C$  is a closed set containing zero).

The wealth process of the investor, in this case, is given by the following equation

$$X_t^{\pi^n} = x + \int_0^t \pi_s^n \sigma_s^n (\theta_s^n ds + dB_s) + \int_0^t \int_E \pi_s^n \beta_s^n(e) \tilde{\mu}(ds, de), \quad t \in [0, T], \quad \mathbb{P}\text{-a.s.} \quad (2.15)$$

### 2.3.2 Exponential utility case and link with BSDEs with jumps

For a market with stock prices as in (2.12), consider the expected utility maximization problem (2.13) when the utility function is of exponential type

$$U(x) = -\exp(-\alpha x), \quad x \in \mathbb{R}, \quad \alpha > 0. \quad (2.16)$$

Hence, the expected utility maximisation problem is the following

$$v_t^n(x) = \sup_{\pi \in \mathcal{A}} \mathbb{E}[-\exp(-\alpha(x + \int_0^T \pi_s^n \frac{dS_s^n}{S_s^n} - \xi))]. \quad (2.17)$$

Note that, as in the first section, the random liability  $\xi$  satisfies Assumption 2.2.4.

It is well known that the value process  $v^n$  and the optimal trading strategy  $\pi^n$  can be fully described by a backward stochastic differential with jumps (BSDEJ in short). In fact, relying on the same martingale optimality principle used in Section 2.2, we obtain the following characterization of the optimisation problem in terms of a BSDEJ.

**Theorem 2.3.2** (Morlais [97], [98]). *For all  $n \geq 1$ , the expression of the value process (2.17) is given by*

$$u^n(x) = -\exp(-\alpha(x - Y_0^n)), \quad (2.18)$$

where  $(Y_t^n, Z_t^n, U_t^n)$  is the solution of the BSDE( $f^n, \xi^n$ ) given by

$$Y_t^n = \xi^n + \int_t^T f_s^n(Z_s^n, U_s^n) ds - \int_t^T Z_s^n dB_s - \int_t^T \int_{\mathbb{R}/\{0\}} U_s^n(x) \tilde{\mu}(ds, dx), \quad (2.19)$$

whose generator is defined as follows

$$f_s^n(z, u) = \inf_{\pi \in C} \left[ \frac{\alpha}{2} |\pi \sigma_s^n - (z + \frac{\theta_s^n}{\alpha})|^2 + |(u(s, e) - \pi \beta_s^n(e))|_\alpha \right] - \theta_s^n z - \frac{|\theta_s^n|^2}{2\alpha}, \quad (2.20)$$

where  $|\cdot|_\alpha$  refers to the convex functional defined by:  $|u|_\alpha = \int_E \frac{e^{\alpha u(e)} - \alpha u(e) - 1}{\alpha} \nu(de)$ .

We emphasize that the generator (2.20) of the BSDEJ associated to the optimisation problem (2.17) has the following properties

1. **Continuity property:** For all  $t \in [0, T]$ ,  $(z, u) \mapsto f_t^n(z, u)$  is continuous  $\mathbb{P}$ -a.s.
2. **Convexity property:**  $(z, u) \mapsto f_t^n(z, u)$  is a convex function, that is  $\forall \lambda \in (0, 1)$ :

$$f_t^n(\lambda z + (1 - \lambda)z', \lambda u + (1 - \lambda)u') \leq \lambda f_t^n(z, u) + (1 - \lambda) f_t^n(z', u').$$

3. **Growth property:** There exists a constant  $\alpha > 0$  such that  $f$  has the following growth

$$|f_t^n(z, u)| \leq \frac{\alpha}{2}|z|^2 + |u|_\alpha, \mathbb{P}\text{-a.s.} \quad (2.21)$$

4.  $(A_\gamma)$ -**condition:** There exists  $-1 < C_1 \leq 0$  and  $C_2 \geq 0$  s.t.  $\forall(y, z) \in \mathbb{R} \times \mathbb{R}^d, \forall u, u' \in L^2(\nu)$

$$f_t^n(y, z, u) - f_t^n(y, z, u') \leq \int_E \gamma_t^{y, z, u, u'}(e)(u - u')(e)\nu(de), \mathbb{P}\text{-a.s.} \quad (2.22)$$

where  $\gamma^{y, z, u, u'} : \Omega \times [0, T] \times E \rightarrow \mathbb{R}$  is measurable w.r.t. all the variables and satisfies  $C_1(1 \wedge |e|) \leq \gamma_t(e) \leq C_2(1 \wedge |e|)$ .

For more details about how  $f^n$  satisfies the properties above, we refer the reader to [97].

Let us also mention that, in contrast to the diffusion setup, when it comes to quadratic BSDEs in a discontinuous setting, the literature has been rather small. The only existing results until recently concern particular cases of quadratic BSDEs that appears in utility maximization or indifference pricing problems in a jump framework. Non Lipschitz-quadratic generators in  $Z$  have been studied by Morlais in [97], [98] where the author proved the existence of the solution of a special quadratic BSDE with jumps who's generator is given by Theorem 2.3.2.

**A comparison result** In this part, we give a comparison result for BSDEs with jumps satisfying the properties given above.

**Theorem 2.3.3.** *Let  $(Y_t, Z_t, U_t)$  be a solution to the BSDE  $(f, \xi)$  and  $(Y'_t, Z'_t, U'_t)$  a solution of the BSDE  $(f', \xi')$  such that  $\xi$  and  $\xi'$  are bounded random variables. Moreover, assume that  $\mathbb{P}$ -a.s., we have*

$$(i) \ \xi \leq \xi',$$

$$(ii) \ \forall(t, y, z, u) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d \times L^2(\nu), \ f_t(y, z, u) \leq f'_t(y, z, u).$$

Then,  $\mathbb{P}$ -a.s., for each  $t \in [0, T]$ , we have  $Y_t \leq Y'_t$ .

**Proof.** In the proof, we adapt the idea used in [25] in a continuous setting to get the comparison result above. In fact, we use the convexity property of the generator in  $(z, u)$  to estimate  $Y_t - \theta Y'_t$  for all  $\theta \in (0, 1)$ . We make the following change of variables

$$P_t = \exp(c\delta Y_t), \ Q_t = c \exp(c\delta Y_t)\delta Z_t \text{ and } J_t = e^{c\delta Y_t} (e^{c\delta U_t} - 1),$$

where

$$\delta Y_t = Y_t - \theta Y'_t, \ \delta Z_t = Z_t - \theta Z'_t \text{ and } \delta U_t = U_t - \theta U'_t.$$

We obtain, by applying the Itô's formula to  $P_t$ , the following

$$P_t = P_T + \underbrace{\int_t^T cP_s \left( \delta F_s - \frac{1}{2}c(\delta Z_s)^2 - \int_E \frac{1}{c}(\exp(c\delta U_s(e)) - c\delta U_s(e) - 1)\nu(de) \right) ds}_{G_s} - \int_t^T Q_s dB_s - \int_t^T \int_E J_s \tilde{\mu}(ds, de), 0 \leq t \leq T, \mathbb{P}\text{-a.s.} \quad (2.23)$$

where  $\delta F_t = f(t, z, u) - \theta f'(t, z', u')$ .

As  $f$  is convex in  $(z, u)$ , let us introduce

$$Z_t := \theta Z'_t + (1 - \theta) \frac{Z_t - \theta Z'_t}{(1 - \theta)} \quad \text{and} \quad U_t := \theta U'_t + (1 - \theta) \frac{U_t - \theta U'_t}{(1 - \theta)}.$$

Then, thanks to the convexity property,  $f$  can be easily bounded from above in the following way

$$\begin{aligned} f_t(Z_t, U_t) &= f_t\left(\theta Z'_t + (1 - \theta) \frac{Z_t - \theta Z'_t}{(1 - \theta)}, \theta U'_t + (1 - \theta) \frac{U_t - \theta U'_t}{(1 - \theta)}\right) \\ &\leq \theta f_t(Z'_t, U'_t) + (1 - \theta) f\left(\frac{Z_t - \theta Z'_t}{(1 - \theta)}, \frac{U_t - \theta U'_t}{(1 - \theta)}\right) \\ &\leq \theta f_t(Z'_t, U'_t) + \frac{\alpha}{2(1 - \theta)} |\delta Z_t|^2 + (1 - \theta) \left| \frac{\delta U_t}{1 - \theta} \right|_\alpha, \end{aligned}$$

which entails that

$$\delta F_t \leq \theta \delta f_t + \frac{\alpha}{2(1 - \theta)} |\delta Z_t|^2 + (1 - \theta) \left| \frac{\delta U_t}{1 - \theta} \right|_\alpha, \quad \text{where } \delta f_t = (f - f')(t, Z'_t, U'_t).$$

Using the inequalities above, we have

$$G_t \leq \theta \delta f_t + \frac{\alpha}{2(1 - \theta)} |\delta Z_t|^2 + (1 - \theta) \left| \frac{\delta U_t}{1 - \theta} \right|_\alpha - \frac{1}{2}c(\delta Z_t)^2 - |\delta U_t|_c.$$

Thus, taking  $c = \frac{\alpha}{1 - \theta}$ , we get rid of the dependence on the component  $U$ . Indeed, the generator of the new BSDE (2.23) can be bounded as follows

$$G_t \leq \frac{\alpha}{1 - \theta} \theta P_t \delta f_t,$$

and we obtain

$$P_t \leq P_T + \int_t^T \frac{\alpha}{1 - \theta} \theta P_s \delta f_s - \int_t^T Q_s dB_s - \int_t^T \int_E J_s(e) \tilde{\mu}(ds, de), 0 \leq t \leq T, \mathbb{P}\text{-a.s.}$$

Introduce, for all  $n \geq 1$ , the following stopping time

$$\tau_n = \inf\{u \geq t, \int_t^u |Q_s|^2 ds \geq n\} \wedge \inf\{u \geq t, \int_t^u |J_s(e)|^2 \nu(de) \geq n\} \wedge T,$$

and the process  $D_t = \exp\left(\int_0^t \frac{\alpha}{1-\theta} \theta \delta f_s ds\right)$  to define the processes

$$\tilde{P}_t = D_t P_t, \quad \tilde{Q}_t = D_t Q_t \quad \text{and} \quad \tilde{J}_t = D_t J_t.$$

Applying Itô's formula to  $\tilde{P}_t$  for any stopping time  $0 \leq t \leq \tau_n \leq T$  gives

$$\tilde{P}_t \leq \tilde{P}_{\tau_n} - \int_t^{\tau_n} \tilde{Q}_s dB_s - \int_t^{\tau_n} \int_E \tilde{J}_s \tilde{\mu}(ds, de), \quad 0 \leq t \leq T, \mathbb{P}\text{-a.s.}$$

Taking the conditional expectation with respect to  $\mathcal{F}_t$  in the last inequality gives

$$\tilde{P}_t \leq \mathbb{E}\left[\tilde{P}_{\tau_n} \middle| \mathcal{F}_t\right].$$

Therefore,

$$P_t \leq \mathbb{E}\left[\exp\left(\int_t^{\tau_n} \frac{\alpha}{1-\theta} \theta \delta f_s ds\right) P_{\tau_n} \middle| \mathcal{F}_t\right].$$

Letting  $n$  to  $\infty$  goes to infinity

$$P_t \leq \mathbb{E}\left[\exp\left(\int_t^T \frac{\alpha}{1-\theta} \theta \delta f_s ds\right) \exp\left(\frac{\alpha}{1-\theta} (\xi - \theta \xi')\right) \middle| \mathcal{F}_t\right].$$

Using the fact that  $(\xi - \theta \xi') = (1-\theta)\xi + \theta(\xi - \theta \xi') \leq (1-\theta)|\xi| + \theta(\xi - \xi')$  and the assumptions of the theorem, one can deduce that

$$P_t \leq \mathbb{E}\left[\exp(\alpha|\xi|) \middle| \mathcal{F}_t\right],$$

which gives

$$Y_t - \theta Y'_t \leq \frac{1-\theta}{\alpha} \log \mathbb{E}\left[\exp(\alpha|\xi|) \middle| \mathcal{F}_t\right].$$

Finally, it suffices to send  $\theta$  to 1 to get the comparison result.

### 2.3.3 Stability result

In this part, we shall prove the stability of the sequence of dynamic value functions of the constrained utility maximization problem (2.17) associated to the sequence of nonequivalent markets (2.12). The term "stability" here is used in the sense that: if for a given convergent sequence of inputs, we get a convergent sequence of outputs, then we will say that the stability holds. In the present work, we make perturbations on the sequences of the input parameters of

the market to describe an uncertain evolution of the asset prices and investigate the behavior of the sequence of value functions. So, we aim to answer the following question: *What are the assumptions to make on the sequence of the inputs parameters and their mode of convergence to guarantee the stability of the value process?"*

To prove the stability of the value process, we use its characterization via BSDEs with jumps given in 2.3.2. In fact, to prove the convergence of  $v^n$ , we shall prove the convergence of the first component  $Y^n$  of the BSDE's solution. So our main focus is on the backward stochastic differential equation with jumps that encodes the dynamic value process and on transferring new results on quadratic semimartingale BSDEs to the portfolio choice problem, in particular to its stability properties.

**Assumption 2.3.4.**

**(H1)** For  $n \geq 1$ , the terminal conditions  $\xi^n$  are bounded uniformly in  $\mathbb{L}^\infty$  and it converges in the following sense

$$\lim_{n \rightarrow \infty} \xi^n = \xi \text{ a.s.} \quad (2.24)$$

**(H2)** The  $\mathbb{R}$ -valued constraint set  $C$  is compact.

**(H3)** For  $n \geq 1$ , the sequences of the market price of risk, volatility and the predictable processes  $\beta^n$  satisfy the following

- $(\theta_t^n)_{n \geq 1}$  are bounded in  $\mathbb{L}^2$  and  $\lim_{n \rightarrow \infty} \int_0^t |\theta_s^n - \theta_s|^2 ds = 0$ ,  $dt \otimes d\mathbb{P}$ -a.s.
  - $(\sigma_t^n)_{n \geq 1}$  are non-singular valued and converges to  $\sigma_t$  in  $\mathbb{L}^2$  for all  $t \in [0, T]$ ,  $dt \otimes d\mathbb{P}$ -a.s.
  - $\lim_{n \rightarrow \infty} \beta_t^n = \beta_t$  in  $L^2(\nu)$ .
- (2.25)

**Lemma 2.3.5.** Under Assumption **(H1)** and **(H3)**, the sequence of strategies  $(\pi^n)_{n \geq 1}$  converges to  $\pi$  in  $\mathcal{H}^2$ .

**Proof.** • The dynamics of the wealth process (2.15) can be considered as a BSDE with jumps. More precisely it can be written as follows

$$X_t^{\pi,n} = X_T^{\pi,n} + \int_t^T (-\pi_s^n \sigma_s^n \theta_s^n) ds - \int_t^T \pi_s^n \sigma_s^n dB_s - \int_t^T \int_E \pi_s^n \beta_s^n(e) \tilde{\mu}(ds, de), \quad 0 \leq t \leq T, \mathbb{P}\text{-a.s.} \quad (2.26)$$

where  $X_t^{\pi,n}$  is the wealth at time  $t$  and  $X_T^{\pi,n}$  is the value of the portfolio at time  $T$  which might be enough to guarantee a contingent claim  $\xi$ . Now, we set

$$Z_s^n := \pi_s^n \sigma_s^n, \quad U_s^n(e) := Z_s^n \beta_s^n(e) (\sigma_s^n)^{-1} \quad \text{and} \quad g_s^n(Z_s^n) := -\theta_s^n Z_s^n.$$

So (2.26) becomes

$$X_t^{\pi,n} = X_T^{\pi,n} - \int_t^T \theta_s^n Z_s^n ds - \int_t^T Z_s^n dB_s - \int_t^T \int_E U_s^n(e) \tilde{\mu}(ds, de), \quad 0 \leq t \leq T, \mathbb{P}\text{-a.s.} \quad (2.27)$$

This BSDE is linear in  $z^n$  which permits us to obtain an explicit solution. In fact, its solution is given by the following expression

$$X_t^n = \mathbb{E}[\Gamma_{t,T}^n \xi^n / \mathcal{F}_t] = \frac{1}{\Gamma_{0,t}^n} \mathbb{E}[\Gamma_{0,T}^n \xi^n / \mathcal{F}_t], \quad t \in [0, T], \mathbb{P}\text{-a.s.}$$

where, for each  $t \in [0, T]$ ,  $(\Gamma_{t,s}^n)_{0 \leq s \leq T}$  is the adjoint process and it is the unique solution of the following forward SDE

$$\begin{cases} d\Gamma_{t,s}^n = \Gamma_{t,s}^n \theta_s^n dB_s, \\ \Gamma_{t,t}^n = 1, \end{cases}$$

and  $\Gamma_{t,s}^n = \exp(-\frac{1}{2} \int_t^s (\theta_u^n)^2 du + \int_t^s \theta_u^n dB_u)$ .

In order to prove the convergence of the strategies  $(\pi_t^n)_{n \geq 1}$ , the idea consists in proving the stability of the solution  $(X_t^n, \pi_t^n \sigma_t^n, U_t^n)_{0 \leq t \leq T}$  of the linear BSDE (2.27). To do so, notice that

$$X_t^n - X_t = \mathbb{E}[\xi^n \Gamma_{t,T}^n - \xi \Gamma_{t,T} | \mathcal{F}_t] = \mathbb{E}[(\xi^n - \xi) \Gamma_{t,T}^n + \xi (\Gamma_{t,T}^n - \Gamma_{t,T}) | \mathcal{F}_t].$$

We have  $(\theta_t^n)_{n \geq 1}$  is a sequence of bounded processes in  $\mathbb{L}^2$  which satisfy **(H3)**, then  $\Gamma_{t,T}^n \rightarrow \Gamma_{t,T}$  as  $n \rightarrow +\infty$ . Then it remains, under **(H1)**, to apply Hölder inequality together with a dominated convergence theorem to obtain the desired convergence.

• Another method to prove the result consists in making the two following change of measures

$$d\mathbb{Q}^n = \Gamma_{0,T}^n d\mathbb{P} \quad \text{and} \quad d\mathbb{Q} = \Gamma_{0,T} d\mathbb{P}.$$

To get the desired result, we have to suppose that probability measure  $\mathbb{Q}^n$  converges as follows:

$\lim_{n \rightarrow +\infty} \mathbb{Q}^n = \mathbb{Q}$  in total variation. We emphasize that, we have

$$\frac{d\mathbb{Q}^n}{d\mathbb{Q}} = \exp\left(-\frac{1}{2} \int_0^T [(\theta_s^n)^2 - \theta_s^2] ds + \int_0^T (\theta_s^n - \theta_s) dB_s\right). \quad (2.28)$$

In this case, as long as  $\lim_{n \rightarrow \infty} \int_0^T |\theta_s^n - \theta_s|^2 ds$  in probability then  $\lim_{n \rightarrow \infty} \mathbb{Q}^n = \mathbb{Q}$  in total variation.

The convergence of the other components of the BSDE's solution are obtained directly via an application of Itô's formula.

**Proposition 2.3.6.** *Let Assumptions **(H1)**, **(H2)** and **(H3)** hold true. Then, the sequence of processes  $(f_t^n)_{n \geq 1}$  defined in (2.20) converges pointwise to the  $f_t$  a.s. and  $(Y^n, Z^n, U^n)$  converges*

to  $(Y, Z, U)$  in  $S^\infty \times \mathcal{H}^2 \times \mathbb{H}_\nu^2$ . In particular,  $Y_0^n$  is deterministic and it converges  $\mathbb{P}$ -a.s. to  $Y_0$ .

**Proof.** We divide the proof in two steps: the first one concerns the convergence of the sequence of drivers and the second one concerns the convergence of the BSDE's solution.

Step 1 The sequence of processes  $(f_t^n)_{n \geq 1}$  given by (2.20) converges a.s. to  $f_t$ . In fact, since it is the infimum of a convex and continuous function in  $(z, u)$  then there exists a unique minimum of  $f^n$  on  $C$  that is reached by an optimal strategy  $\pi^{n,*}$ . Then, (2.20) can be written as follows

$$f_s^n(z, u) = \frac{\alpha}{2} |\pi_s^{n,*} \sigma_s^n - (z + \frac{\theta_s^n}{\alpha})|^2 + |u - \pi_s^{n,*} \beta_s^n|_\alpha - \theta_s^n z - \frac{|\theta_s^n|^2}{2\alpha}. \quad (2.29)$$

Assumption **(H3)** in (2.3.4) together with the convergence of the strategies proved in (2.3.5) lead to the convergence of sequence of processes  $(f_t^n)_{n \geq 1}$ .

Step 2 Our aim now is to prove the convergence of  $(Y^n, Z^n, U^n)$ , solution of the BSDE (2.19), in the following sense

$$\sup_{0 \leq t \leq T} (|Y_t^n - Y_t|) + \left( \int_0^T |Z_s^n - Z_s|^2 ds \right) + \left( \int_0^T |U_s^n(e) - U_s(e)|^2 \nu(de) ds \right) \rightarrow 0.$$

In order to prove the desired result, our aim is to estimate the quantity  $Y_t^n - \theta Y_t$  for  $\theta \in (0, 1)$ . So our starting point consists on exploiting, for  $n \geq 1$ , the convexity of the generator  $f^n$  in  $(z^n, u^n)$ , by making, for some positive constant  $c$ , the following change of variables

$$P_t^n = \exp(c\delta_n Y_t), \quad Q_t^n = cP_t \delta_n Z_t \quad \text{and} \quad J_t^n = P_t (\exp(c\delta_n U_t) - 1),$$

where

$$\delta_n F_t = f_t^n(Z_t^n, U_t^n) - \theta f_t(Z_t, U_t), \quad \delta_n Y_t = Y_t^n - \theta Y_t, \quad \delta_n Z_t = Z_t^n - \theta Z_t \quad \text{and} \quad \delta_n U_t = U_t^n - \theta U_t.$$

Now, we apply Itô's formula to the process  $P_t^n$  which gives

$$\begin{aligned} P_t^n &= P_T^n + \int_t^T cP_s^n [(\delta_n F_s - \frac{1}{2}c(\delta_n Z_s)^2 - \int_E \frac{1}{c}(\exp(c\delta_n U_s(e)) - c\delta_n U_s(e) - 1)\nu(de)] ds \\ &\quad - \int_t^T cP_s^n \delta_n Z_s dB_s - \int_t^T \int_E P_s^n (\exp(c\delta_n U_s(e)) - 1) \tilde{\mu}(ds, de) \\ &= P_T^n + \int_t^T G_s^n ds - \int_t^T Q_s^n dB_s - \int_t^T \int_E J_s^n(e) \tilde{\mu}(ds, de), \end{aligned} \quad (2.30)$$

where we denote by  $G_s^n$  the generator of the BSDE above i.e.

$$G_t^n = cP_t^n \left[ (\delta_n F_t - \frac{1}{2}c(\delta_n Z_t)^2 - \int_E \frac{1}{c}(\exp(c\delta_n U_t(e)) - c\delta_n U_t(e) - 1)\nu(de) \right]. \quad (2.31)$$

Since  $f^n$  is convex in  $(z^n, u^n)$ , we have

$$f_t^n(Z_t^n, U_t^n) = f_t^n\left(\theta Z_t + (1-\theta)\frac{Z_t^n - \theta Z_t}{1-\theta}, \theta U_t + (1-\theta)\frac{U_t^n - \theta U_t}{1-\theta}\right) \quad (2.32)$$

$$\leq \theta f_t^n(Z_t, U_t) + (1-\theta)f_t^n\left(\frac{Z_t^n - \theta Z_t}{1-\theta}, \frac{U_t^n - \theta U_t}{1-\theta}\right). \quad (2.33)$$

Then

$$\delta_n F_t \leq \theta \delta_n f_t + (1-\theta)f_t^n\left(\frac{\delta_n Z_t}{1-\theta}, \frac{\delta_n U_t}{1-\theta}\right) \quad \text{where } \delta_n f_t := (f_t^n - f_t)(Z_t, U_t).$$

As  $f^n$  satisfies the growth property (2.21), then

$$\delta_n F_s \leq \theta \delta_n f_s + \frac{\alpha}{2(1-\theta)}|\delta_n Z_s|^2 + (1-\theta)\left|\frac{\delta_n U_s}{1-\theta}\right|^\alpha.$$

Now, by taking the non-negative constant  $c = \frac{\alpha}{1-\theta}$  in equation (2.30), we have the following inequality

$$G_t^n \leq \frac{\alpha}{1-\theta}\theta P_t^n \delta_n f_t.$$

Using the inequality above, we get that

$$P_t^n \leq \mathbb{E}\left[P_T^n + \frac{\alpha\theta}{1-\theta}\int_t^T P_s^n |\delta_n f_s| ds \middle| \mathcal{F}_t\right].$$

Now, notice that the quantities  $P_t^n$  and  $P_T^n$  can be upper bounded as follows

$$P_s^n = \exp\left(\frac{\alpha}{1-\theta}(Y_s^n - \theta Y_t)\right) \leq \sup_{0 \leq t \leq T} \exp\left(\frac{\alpha}{1-\theta}(|Y_t^n| + |Y_t|)\right),$$

$$P_T^n = \exp\left(\frac{\alpha}{1-\theta}(\xi^n - \theta \xi)\right) \leq \exp\left(\frac{\alpha}{1-\theta}(|\xi^n - \theta \xi| \vee |\xi - \theta \xi^n|)\right),$$

which allows the derivation the following inequality

$$Y_t^n - \theta Y_t \leq \frac{1-\theta}{\alpha} \log \mathbb{E}\left[\exp\left(\frac{\alpha}{1-\theta}(|\xi^n - \theta \xi| \vee |\xi - \theta \xi^n|)\right) + \int_t^T \frac{\alpha\theta}{1-\theta} \sup_{0 \leq t \leq T} \exp\left(\frac{\alpha}{1-\theta}(|Y_t^n| + |Y_t|)\right) |\delta_n f_s| ds \middle| \mathcal{F}_t\right].$$

Now, using the inequality  $\log(x) \leq x$  and subtracting  $Y_t$  from both sides, we obtain

$$Y_t^n - Y_t \leq (\theta - 1)|Y_t| + \frac{1-\theta}{\alpha} \mathbb{E}\left[\exp\left(\frac{\alpha}{1-\theta}(|\xi^n - \theta \xi| \vee |\xi - \theta \xi^n|)\right) \middle| \mathcal{F}_t\right] \quad (2.34)$$

$$+ \theta \mathbb{E}\left[\int_t^T \sup_{0 \leq t \leq T} \left[\exp\left(\frac{\alpha}{1-\theta}(|Y_t^n| + |Y_t|)\right)\right] |\delta_n f_s| ds \middle| \mathcal{F}_t\right].$$

By symmetry, we obtain the same for  $Y_t - \theta Y_t^n$  and we finally obtain that

$$\begin{aligned} |Y_t - Y_t^n| &\leq (\theta - 1)(|Y_t^n| + |Y_t|) + \frac{1 - \theta}{\alpha} \mathbb{E} \left[ \exp \left( \frac{\alpha}{1 - \theta} (|\xi^n - \theta \xi| \vee |\xi - \theta \xi^n|) \right) \middle| \mathcal{F}_t \right] \\ &\quad + \theta \mathbb{E} \left[ \int_0^T \sup_{0 \leq t \leq T} \left[ \exp \left( \frac{\alpha}{1 - \theta} (|Y_t^n| + |Y_t|) \right) \right] |\delta_n f_s| ds \middle| \mathcal{F}_t \right]. \end{aligned}$$

Taking into account that the process  $Y_t^n$  is bounded for all  $n \geq 1$ ,  $|\delta_n f_s|$  converges to 0 as proved in the first step and letting  $\theta$  goes to 1 and  $n$  goes to infinity, we obtain that

$$\lim_{n \rightarrow \infty} \sup_{0 \leq t \leq T} |Y_t^n - Y_t| = 0, \mathbb{P}\text{-a.s.}$$

Now, in order to prove the convergence of the martingale parts, we apply Itô's formula to the process  $(Y_t^n - Y_t)^2$  which gives

$$\begin{aligned} (Y_t^n - Y_t)^2 &= (\xi^n - \xi^2) - 2 \int_t^T (Y_s^n - Y_s)(f_s^n(Z_s, U_s) - f_s(Z_s, U_s)) ds \\ &\quad + 2 \int_t^T (Y_s^n - Y_s)(Z_s^n - Z_s) dB_s - 2 \int_t^T \int_E (Y_s^n - Y_s)(U_s^n(e) - U_s(e)) \tilde{\mu}(ds, de) \\ &\quad - \int_t^T (Z_s^n - Z_s)^2 ds - \int_t^T \int_E (U_s^n(e) - U_s(e))^2 \nu(de) ds. \end{aligned}$$

Taking the expectation of the expression above gives

$$\begin{aligned} \mathbb{E}[(Y_t^n - Y_t)^2] &= \mathbb{E}[(\xi^n - \xi)^2] + 2\mathbb{E} \left[ \int_t^T (Y_s^n - Y_s)(f_s^n(Z_s^n, U_s^n) - f_s(Z_s, U_s)) ds \right] \\ &\quad - \mathbb{E} \left[ \int_t^T (Z_s^n - Z_s)^2 ds \right] - \mathbb{E} \left[ \int_t^T \int_E (U_s^n(e) - U_s(e))^2 \nu(de) ds \right]. \end{aligned}$$

Hence

$$\begin{aligned} &\mathbb{E} \left[ \int_0^T |Z_s^n - Z_s|^2 ds \right] + \mathbb{E} \left[ \int_0^T \int_E |U_s^n(e) - U_s(e)|^2 \nu(de) ds \right] \\ &\leq \mathbb{E}[(\xi^n - \xi)^2] + 2\mathbb{E} \left[ \sup_{s \in [0, T]} |Y_s^n - Y_s| \int_0^T |f_s^n(Z_s^n, U_s^n) - f_s(Z_s, U_s)| ds \right] \\ &\leq \mathbb{E}[(\xi^n - \xi)^2] + 2\|Y_s^n - Y_s\|_{\mathcal{S}^\infty} \mathbb{E} \left[ \int_0^T |f_s^n(Z_s^n, U_s^n) - f_s(Z_s, U_s)| ds \right]. \end{aligned}$$

Under Assumption **(H1)** and by using dominated convergence theorem, the sequene of terminal conditions  $(\xi^n)_{n \geq 1}$  converges as follows  $\lim_{n \rightarrow \infty} \mathbb{E}[(\xi^n - \xi)^2] = 0$ . Further, since  $f^n$  and  $f$  satisfy the growth-property (2.21), then

$$\mathbb{E} \left[ \int_0^T |f_s^n(Z_s^n, U_s^n) - f_s(Z_s, U_s)| ds \right] \leq \mathbb{E} \left[ \int_0^T \left( \frac{\alpha}{2} (|Z_s^n|^2 + |Z_s|^2) + |U_s^n|_\alpha + |U_s|_\alpha \right) ds \right],$$

where the right hand side is obviously finite. Finally, we obtain

$$\lim_{n \rightarrow \infty} \mathbb{E} \left[ \int_0^T |Z_s^n - Z_s|^2 ds \right] + \mathbb{E} \left[ \int_0^T \int_E |U_s^n(e) - U_s(e)|^2 \nu(de) ds \right] \leq 0, \quad (2.35)$$

which concludes the proof.

**Theorem 2.3.7.** *Let Assumption 2.3.4 hold true and let  $(u^n)_{n \in \mathbb{N}}$  defined by*

$$u_t^n(x) = -\exp(-\alpha(x - Y_t^n)), \quad (2.36)$$

*be the sequence of dynamic value functions of the constrained utility maximization problem (2.17). Then*

$$\lim_{n \rightarrow \infty} \mathbb{E} \left[ \sup_{t \in [0, T]} |u_t^n(x) - u_t(x)| \right] = 0. \quad (2.37)$$

*In particular, the sequence of the static value functions  $u_0^n(x) = -\exp(-\alpha(x - Y_0^n))$  remains stable.*

**Proof.** The proof is straightforward and it is a consequence of Proposition 2.3.6. In fact, it suffices to see that

$$\begin{aligned} \mathbb{E} \left[ \sup_{0 \leq t \leq T} |u_t^n(x) - u_t(x)| \right] &= \mathbb{E} \left[ \sup_{0 \leq t \leq T} |e^{(-\alpha(x - Y_t^n))} - e^{(-\alpha(x - Y_t))}| \right] \\ &= e^{-\alpha x} \mathbb{E} \left[ \sup_{0 \leq t \leq T} e^{\alpha Y_t} (e^{\alpha(Y_t^n - Y_t)} - 1) \right] \\ &\leq e^{-\alpha x} \mathbb{E} \left[ e^{\alpha \sup_t |Y_t|} (e^{\alpha \sup_t |Y_t^n - Y_t|} - 1) \right]. \end{aligned}$$

When  $n$  goes to infinity, the right hand side goes to 0 and the value process remains stable.



# MONTE-CARLO METHODS FOR THE PRICING OF AMERICAN OPTIONS: A SEMI-LINEAR BSDE POINT OF VIEW

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## 3.1 Introduction

In this chapter, we are interested in the problem of pricing American options from a theoretical and numerical side. To put the problem in a mathematical context, let us first consider the case of a single stock (non-dividend paying) market under the famous Black and Scholes setting, [16]. Namely, let  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$  be a filtered probability space carrying a standard one dimensional Brownian motion  $W$  and let us model the stock price process  $X$  as

$$X_s = x \exp\left(\left(r - \frac{\sigma^2}{2}\right)(s - t) + \sigma(W_s - W_t)\right), \quad s \geq t,$$

under the risk natural probability. Here,  $x > 0$  is the stock price at time  $t$ ,  $r > 0$  is the risk-free interest rate and  $\sigma > 0$  is the volatility. Then, the arbitrage free value at time  $t$  of an American option maturing at  $T \geq t$  is given by

$$V(t, x) = \sup_{\tau \in \mathcal{T}_{[t, T]}} \mathbb{E}[e^{-r(\tau-t)} g(X_\tau)], \quad (3.1)$$

where  $\mathcal{T}_{[t, T]}$  is the collection of  $[t, T]$ -valued stopping times, and  $g$  is the payoff function, say continuous, see e.g. [17] and the references therein. Typical examples are

$$g(x') = \begin{cases} (x' - K)^+, & \text{for a call option} \\ (K - x')^+, & \text{for a put option,} \end{cases}$$

where  $K > 0$  denotes the strike price. By construction,  $V(\cdot, X) \geq g(X)$ , and the option should be exercised only when  $V(\cdot, X) \leq g(X)$ . This leads to define the following two regions:

- The continuation region:

$$C = \{(t, x) \in [0, T) \times (0, \infty) : V(t, x) > g(x)\}.$$

- The stopping (or the exercise) region:

$$S = \{(t, x) \in [0, T) \times (0, \infty) : V(t, x) = g(x)\}.$$

These are the basics of the common formulation of the American option price as a free boundary problem, which already appears in McKean [88]:  $V$  solves a heat-equation type linear parabolic problem on  $C$  and equals  $g$  on  $S$ , with the constraint of being always greater than  $g$ . Another formulation is based on the quasi-variational approach of Bensoussan and Lions [12]: the price solves (at least in the viscosity solution sense) the quasi-variational partial differential equation

$$\begin{cases} \min(r\varphi - \mathcal{L}_{BS}\varphi, \varphi - g) = 0, & \text{on } [0, T) \times (0, \infty) \\ \varphi(T, \cdot) = g, & \text{on } (0, \infty), \end{cases}$$

in which  $\mathcal{L}_{BS}$  is the Dynkin operator associated to  $X$ :

$$\mathcal{L}_{BS} = \partial_t + rxD + \frac{1}{2}\sigma^2 x^2 D^2,$$

where  $D$  and  $D^2$  are the Jacobian and Hessian operators.

In this Chapter, we focus the formulation that can be found in [13], see also [14] and the references therein. The American option valuation problem can be stated in terms of a semilinear Black and Scholes partial differential equation set on a fixed domain, namely:

$$\begin{cases} r\varphi - \mathcal{L}_{BS}\varphi = q(\cdot, \varphi), & \text{on } [0, T) \times (0, +\infty) \\ \varphi(T, \cdot) = g, & \text{on } (0, \infty) \end{cases} \quad (3.2)$$

where  $q$  is a nonlinear reaction term defined as

$$q(x, \varphi(t, x)) = c(x)H(g(x) - \varphi(t, x)) = \begin{cases} 0 & \text{if } g(x) < \varphi(t, x) \\ c(x) & \text{if } g(x) \geq \varphi(t, x), \end{cases}$$

in which  $c$  is a certain cash flow function, e.g.  $c = rK$  for a put option, and  $H$  is the Heaviside function.

Note that this semilinear Black and Scholes equation does not make sense if we consider classical solutions because of the discontinuity of  $y \rightarrow q(x, y)$ . It has to be considered in the discontinuous viscosity solution sense, see e.g. Crandall, Ishii and Lions [33]. More precisely,

the difficulty stems in the fact that the nonlinear reaction term  $q(x, V(t, x)) = c(x)\mathbf{1}_{V(t,x) \leq g(x)}$  is singular as it involves an indicator function. This discontinuity makes it hard to handle from the theoretical and numerical side and the notion of viscosity solution of (3.2) requires some care and it has to be adapted. As done in [13], the notions of super (resp. subsolution) are in fact considered for two different equations, in which  $q$  is replaced by its lower-semicontinuous envelope  $q_*$  (resp. its upper-semicontinuous envelope  $q^*$ ).

Although, we succeed to prove that the value function  $V$  of the optimal stopping problem for the American option is the unique viscosity solution of (3.2) with polynomial growth. Note that this semilinear Black and Scholes equation does not make sense if we consider classical solutions because of the discontinuity of  $y \mapsto q(x, y)$ . It has to be considered in the discontinuous viscosity solution sense, see e.g. Crandall, Ishii and Lions [33].

Namely, even if  $V$  is continuous, the supersolution property should be stated in terms of the lower-semicontinuous envelope of  $q$ , the other way round for the subsolution property. This means in particular that the super- and subsolution properties are not defined with respect to the same operator. Still, thanks to the very specific monotonicity of  $y \rightarrow q(x, y)$ , it is proved in [13] that, within the Black and Scholes model, the American option price is the unique solution of (3.2) in the appropriate sense.

In this chapter, we first extend the characterization of [13] in terms of (3.2) to a general payoff function and to a general market model. Then, we suggest two numerical schemes based on this formulation. The general idea consists in (formally) identifying the solution  $V$  of (3.2) to the solution  $(Y, Z)$  of the backward stochastic differential equation

$$Y = e^{-rT}g(X_T) + \int_0^T e^{-rs}q(X_s, e^{rs}Y_s)ds - \int_0^T Z_s dW_s, \mathbb{P}\text{-a.s.},$$

by  $e^{-r\cdot}V(\cdot, X) = Y$ .

In the first algorithm, we follow the approach of Bouchard et al. [19] and approximate the nonlinear driver  $q$  by local polynomials so as to be able to apply an extended version of the pure forward branching processes based Feynman-Kac representation of the Kolmogorov-Petrovskii-Piskunov equation, see [51, 52].

Unfortunately, our numerical experiments show that this algorithm is quite unstable, see Section 3.3.1.

In the second algorithm, we do not try to approximate  $q$  by local polynomials but in place regularize it with a noise by replacing  $q(X, e^r Y)$  by  $c(X)\mathbf{1}_{\{g(X)+\epsilon \geq e^r Y\}}$ , in which  $\epsilon$  is an independent random variable. When the variance of  $\epsilon$  vanishes, this provides a converging estimator. For  $\epsilon$  given, the corresponding  $Y$  is estimated by using the approach of Bouchard et al. [19] with (random) polynomial  $(t, x, y, y') \mapsto c(x)\mathbf{1}_{\{g(x)+\epsilon \geq e^{rt}y'\}}$  and particles that can only die (with-

out creating any children). This algorithm turns out to be very precise, see Section 3.3.2.

### 3.2 Non-linear parabolic equation representation

From now on, we take  $\Omega$  as the space of  $\mathbb{R}^d$ -valued continuous maps on  $[0, T]$  starting at 0, endowed with the Wiener measure  $\mathbb{P}$ . We let  $W$  denote the canonical process and let  $(\mathcal{F}_t)_{t \leq T}$  be its completed filtration. Given  $t \in [0, T]$  and  $x \in (0, \infty)^d$ , we consider a financial market with  $d$  stocks whose prices process  $X^{t,x}$  evolves according to

$$X^{t,x} = x + \int_t^\cdot r X_s^{t,x} ds + \int_t^\cdot \sigma(s, X_s^{t,x}) dW_s, \mathbb{P}\text{-a.s.} \quad (3.3)$$

in which  $r \in \mathbb{R}$  is a constant<sup>1</sup>, the risk free interest rate, and  $\sigma : [0, T] \times (0, \infty)^d \mapsto \mathbb{R}^{d \times d}$  is a matrix valued-function that is assumed to be continuous and uniformly Lipschitz in its second component. We also assume that  $\bar{\sigma} : (t', x') \in [0, T] \times (0, \infty)^d \mapsto \text{diag}[x']^{-1} \sigma(t', x')$  is uniformly Lipschitz in its second component and bounded, where  $\text{diag}[x']$  stands for the diagonal matrix with  $i$ -th diagonal entry equal to the  $i$ -th component of  $x'$ . This implies that  $X^{t,x}$  takes values in  $(0, \infty)^d$  whenever  $x \in (0, \infty)^d$ .

We also assume that  $\mathbb{P}$  is the only (equivalent) probability measure under which  $e^{-r(\cdot-t)} X^{t,x}$  is a (local) martingale, for  $(t, x) \in [0, T] \times (0, \infty)^d$ . Then, given a continuous payoff function  $g : (0, \infty)^d \rightarrow \mathbb{R}$ , with polynomial growth, the price of the American option with payoff  $g$  is given by

$$V(t, x) = \sup_{\tau \in \mathcal{T}_{[t, T]}} \mathbb{E}[e^{-r(\tau-t)} g(X_\tau^{t,x})],$$

in which  $\mathcal{T}_{[t, T]}$  is the collection of  $[t, T]$ -valued stopping times. See [17].

**Remark 3.2.1.** *The fact that  $(t, x) \in [0, T] \times \mathbb{R}_+^d \mapsto V(t, x)$  is continuous with polynomial growth follows from standard estimates under the above assumptions. In particular, the set  $\{(t, x) \in [0, T] \times \mathbb{R}_+^d : V(t, x) = g(x)\}$  is closed.*

The aim of this section is to prove that  $V$  is a viscosity solution of the non-linear parabolic equation

$$\begin{aligned} r\varphi - \mathcal{L}\varphi - q(\cdot, \varphi) &= 0 && \text{on } [0, T) \times (0, \infty)^d \\ \varphi(T, \cdot) &= g && \text{on } (0, +\infty)^d, \end{aligned} \quad (3.4)$$

for a suitable reaction function  $q$  on  $(0, \infty)^d \times \mathbb{R}$ . In the above,  $\mathcal{L}$  denotes the Dynkin operator

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1. It should be clear that this assumption is only made for simplicity. Also note that a dividend rate could be added at no cost.

associated to (3.3):

$$\mathcal{L}\varphi(t', x') = \partial_t \varphi(t', x') + \langle rx', D\varphi(t', x') \rangle + \frac{1}{2} \text{Tr}[\sigma \sigma^\top D^2 \varphi](t', x'),$$

for a smooth function  $\varphi$ . To be more precise, we define the function  $q$  by

$$q(x, y) = \begin{cases} 0 & \text{if } g(x) < y \\ c(x) & \text{if } g(x) \geq y \end{cases}, \quad (x, y) \in (0, \infty)^d \times \mathbb{R},$$

where  $c$  is a measurable map satisfying the following Assumption 3.2.2.

**Assumption 3.2.2.** *The map  $c : (0, \infty)^d \mapsto \mathbb{R}_+$  is continuous with polynomial growth. Moreover,  $g$  is a viscosity subsolution of  $r\varphi - \mathcal{L}\varphi - c = 0$  on  $\{(t, x) \in [0, T] \times (0, \infty)^d : V(t, x) = g(x)\}$ .*

Before providing examples of such a function  $c$ , let us make some important observations.

**Remark 3.2.3.** *First,  $\{(t, x) \in [0, T] \times (0, \infty)^d : V(t, x) = g(x)\} \subset \{x \in (0, \infty)^d : g(x) > 0\}$  if  $V > 0$  on  $[0, T] \times (0, \infty)^d$ , which is typically the case in practice (e.g. because  $g$  is non-negative and the probability that  $g(X) > 0$  on  $[0, T]$  is positive). In particular, if  $g$  is  $C^2$  on  $\{g > 0\}$  then one can choose  $c = [rg - \mathcal{L}g]^+$  on  $\{g > 0\}$ . Second, if  $g$  is convex, then it can not be touched from above by a  $C^2$  function at a point at which it is not  $C^1$ , which implies that one can forget some singularity points in the verification of Assumption 3.2.2 above.*

In Section 3.3, we shall suggest Monte-Carlo based numerical methods for the computation of  $V$ . One can then try to minimize the variance of the estimator over the choice of  $c$ . However, it seems natural to choose the function  $c$  so that  $g$  is actually a viscosity solution of  $r\varphi - \mathcal{L}\varphi - c = 0$  on  $\{(t, x) \in [0, T] \times (0, \infty)^d : V(t, x) = g(x)\}$ . In the numerical study of Section 3.3, this choice coincides with the  $c$  with the minimal absolute value, which intuitively should correspond to the one minimizing the variance of the Monte-Carlo estimator. We leave the theoretical study of this variance minimization problem to future researches.

**Example 3.2.1.** *Let us consider the following examples in which  $\bar{\sigma}$  is a constant matrix with  $i$ -th lines  $\bar{\sigma}^i$ . Fix  $K, K_1, K_2 > 0$  with  $K_1 < K_2$ .*

- *For  $d = 1$  and a put  $g : x \in (0, \infty) \mapsto [K - x]^+$ , the function  $c$  is given by the constant  $rK$ . This is one of the cases treated in [13].*
- *For  $d = 1$  and a strangle  $g : x \in (0, \infty) \mapsto [K_1 - x]^+ + [x - K_2]^+$ , the function  $c$  can be any continuous function equal to  $rK_1$  on  $(0, K_1)$  and equal to  $-rK_2$  on  $(K_2, \infty)$ , whenever  $V > 0$ .*
- *For  $d = 2$  and a put on arithmetic mean  $g : x \in (0, \infty)^2 \mapsto [K - \frac{1}{2} \sum_{i=1}^2 x^i]^+$ , we can take  $c = rK$ .*

- For  $d = 2$  and a put on geometric mean  $g : x \in (0, \infty)^2 \mapsto [K - \sqrt{x^1 x^2}]^+$ ,  $c$  can be taken as

$$x \in (0, \infty)^2 \mapsto [rK - \frac{1}{8}(\|\bar{\sigma}^1\|^2 + \|\bar{\sigma}^2\|^2 - 2\langle \bar{\sigma}^1, \bar{\sigma}^2 \rangle)\sqrt{x^1 x^2}]^+.$$

Since  $q$  is discontinuous, we need to consider (3.4) in the sense of viscosity solutions for discontinuous operators. More precisely, let  $q_*$  and  $q^*$  denote the lower- and upper-semicontinuous envelopes of  $q$ . We say that a lower-semicontinuous function  $v$  is a viscosity supersolution of (3.4) if it is a viscosity supersolution of

$$\begin{cases} r\varphi - \mathcal{L}\varphi - q_*(\cdot, \varphi) = 0 & \text{on } [0, T) \times (0, \infty)^d \\ \varphi(T, \cdot) = g & \text{on } (0, +\infty)^d. \end{cases}$$

Similarly, we say that an upper-semicontinuous function  $v$  is a viscosity subsolution of (3.4) if it is a viscosity subsolution of

$$\begin{cases} r\varphi - \mathcal{L}\varphi - q^*(\cdot, \varphi) = 0 & \text{on } [0, T) \times (0, \infty)^d \\ \varphi(T, \cdot) = g & \text{on } (0, +\infty)^d. \end{cases}$$

We say that a continuous function is a viscosity solution of (3.4) if it is both a viscosity super- and subsolution.

Then, we have the following characterization of the American option price, which extends the result of [13] to our context. Recall Remark 3.2.1.

**Theorem 3.2.4.** *Let  $c$  be as in Assumption 3.2.2. Then,  $V$  is a viscosity solution of (3.4). It has a polynomial growth.*

**Proof.** We just follow the arguments of [13].

a. First note that  $V \geq g$ , so that<sup>2</sup>  $q_*(\cdot, V) = 0$ . Hence, the supersolution property is equivalent to being a supersolution of

$$r\varphi - \mathcal{L}\varphi = 0 \text{ on } [0, T) \times (0, \infty)^d \text{ and } \varphi(T, \cdot) = g \text{ on } (0, +\infty)^d,$$

which is standard.

b. Fix  $(t, x) \in [0, T) \times (0, \infty)^d$  and a smooth function  $\varphi$  such that  $(t, x)$  achieves a maximum on  $[0, T) \times (0, \infty)^d$  of  $V - \varphi$  and  $(V - \varphi)(t, x) = 0$ . If  $t = T$ , then the required result holds by definition. We now assume that  $t < T$ . If  $(t, x)$  belongs to the open set  $C := \{V > g\}$ , recall

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2. Note that this is an important consequence of using  $q_*$  instead of  $q$ .

Remark 3.2.1, then one can find a  $[t, T]$ -valued stopping time  $\tau$  such that  $(\cdot \wedge \tau, X_{\cdot \wedge \tau}^{t,x}) \in C$ , and it follows from the dynamic programming principle, see e.g. [21], that

$$\varphi(t, x) \leq \mathbb{E} \left[ e^{-r(\tau_\varepsilon - t)} \varphi(\tau_\varepsilon, X_{\tau_\varepsilon}) \right],$$

in which  $\tau_\varepsilon := \tau \wedge (t + \varepsilon)$  for  $\varepsilon > 0$ . Then, standard arguments lead to

$$0 \geq r\varphi(t, x) - \mathcal{L}\varphi(t, x) = r\varphi(t, x) - \mathcal{L}\varphi(t, x) - q^*(x, \varphi(t, x)).$$

Let us now assume that  $(t, x) \in S := \{V = g\}$ . In particular,  $\varphi(t, x) = V(t, x) = g(x)$  and therefore  $q^*(x, \varphi(t, x)) = q^*(x, V(t, x)) = c(x)$ . Since  $V \geq g$ ,  $(t, x)$  is also a maximum of  $g - \varphi$  and  $\varphi$  satisfies

$$0 \geq r\varphi(t, x) - \mathcal{L}\varphi(t, x) - c(x) = r\varphi(t, x) - \mathcal{L}\varphi(t, x) - q^*(x, \varphi(t, x)),$$

by Assumption 3.2.2.

This viscosity solution property can be complemented with a comparison principle as in [13]. Combined with Theorem 3.2.4, it shows that  $V$  is the unique viscosity solution of (3.4) with polynomial growth.

**Proposition 3.2.5.** *Let the conditions of Theorem 3.2.4 hold. Let  $v$  and  $w$  be respectively a super- and a subsolution of (3.4), with polynomial growth. Then,  $v \geq w$  on  $[0, T] \times (0, \infty)^d$ .*

**Proof.** We adapt the arguments of [13]. As usual, one can assume without loss of generality that  $r > 0$ , upon replacing  $v$  by  $(t, x) \mapsto e^{-\rho t}v(t, x)$  and  $w$  by  $(t, x) \mapsto e^{-\rho t}w(t, x)$  for some  $\rho > |r|$ . Fix  $p \geq 1$  and  $C > 0$  such that  $|v(t, x)| + |w(t, x)| \leq C(1 + \|x\|^p)$  for all  $(t, x) \in [0, T] \times (0, \infty)^d$ . Set  $\psi(t, x) := e^{-\kappa t}(1 + \|x\|^{2p})$  for  $(t, x) \in [0, T] \times (0, \infty)^d$ , for some  $\kappa$  large enough so that  $\psi$  is a supersolution of  $-\mathcal{L}\psi = 0$  on  $[0, T] \times (0, \infty)^d$ , which is possible since  $\bar{\sigma}$  is bounded. Set

$$\phi_n^\varepsilon(t, x, y) := w(t, y) - v(t, x) - n\|x - y\|^{2p} - \lambda\psi(t, y) - \frac{\varepsilon}{\prod_{i=1}^d x^i} - \frac{\varepsilon}{\prod_{i=1}^d y^i},$$

for  $n \geq 1$ ,  $\varepsilon > 0$ ,  $(t, x, y) \in [0, T] \times (0, \infty)^{2d}$ , and a given  $\lambda > 0$ . Assume that  $\sup_{[0, T] \times (0, \infty)^{2d}} (w - v) > 0$ . Then one can find  $\varepsilon_0, \lambda > 0$  and  $\delta > 0$  such that

$$\sup_{[0, T] \times (0, \infty)^{2d}} \phi_n^\varepsilon \geq \delta, \quad \text{for } \varepsilon \in (0, \varepsilon_0) \text{ and } n \geq 1. \quad (3.5)$$

Clearly,  $\phi_n^\varepsilon$  admits a maximum point  $(t_n^\varepsilon, x_n^\varepsilon, y_n^\varepsilon)$  on  $[0, T] \times (0, \infty)^{2d}$ .

Moreover, it follows from standard arguments that  $(t_n^\varepsilon, x_n^\varepsilon, y_n^\varepsilon)$  converges to some  $(t_n, x_n, y_n) \in$

$[0, T] \times \mathbb{R}_+^d$  as  $\varepsilon \rightarrow 0$ , possibly along a subsequence, and that

$$\lim_{\varepsilon \rightarrow 0} \left( \frac{\varepsilon}{\prod_{i=1}^d (x_n^\varepsilon)^i} + \frac{\varepsilon}{\prod_{i=1}^d (y_n^\varepsilon)^i} \right) = 0, \quad \lim_{n \rightarrow \infty} n \|x_n - y_n\|^{2p} = 0, \quad (3.6)$$

$$\lim_{\varepsilon \rightarrow 0} (w(t_n^\varepsilon, y_n^\varepsilon), v(t_n^\varepsilon, x_n^\varepsilon)) = (w(t_n, y_n), v(t_n, x_n)), \quad (3.7)$$

$$\lim_{n \rightarrow \infty} y_n = \hat{y}, \quad \text{for some } \hat{y} \in \mathbb{R}_+^d, \quad (3.8)$$

possibly along subsequences, see e.g. [17, Proof of Theorem 4.5] and [33]. Combining Ishii's Lemma, see e.g. [33], with the super- and subsolution properties of  $v$ ,  $\psi$  and  $w$ , we obtain

$$\begin{aligned} 0 \geq & r(w(t_n^\varepsilon, y_n^\varepsilon) - v(t_n^\varepsilon, x_n^\varepsilon)) - q^*(y_n^\varepsilon, w(t_n^\varepsilon, y_n^\varepsilon)) + q_*(x_n^\varepsilon, v(t_n^\varepsilon, x_n^\varepsilon)) \\ & - O(n \|x_n^\varepsilon - y_n^\varepsilon\|^{2p}) - \eta_\varepsilon^n, \end{aligned}$$

in which, thanks to the left-hand side of (3.6),  $\eta_\varepsilon^n \rightarrow 0$  as  $\varepsilon \rightarrow 0$ , for all  $n \geq 1$ . By the right-hand side of (3.6), the discussion just above it, and (3.7), sending  $\varepsilon \rightarrow 0$  and then  $n \rightarrow \infty$  leads to

$$0 \geq \limsup_{n \rightarrow \infty} \{r(w(t_n, y_n) - v(t_n, x_n)) - q^*(y_n, w(t_n, y_n)) + q_*(x_n, v(t_n, x_n))\},$$

and therefore

$$\liminf_{n \rightarrow \infty} \{q^*(y_n, w(t_n, y_n)) - q_*(x_n, v(t_n, x_n))\} \geq r\delta,$$

by (3.5). Recall that  $c$  is non-negative and that  $w(t_n, y_n) - v(t_n, x_n) \geq \delta$  by (3.5). If, along a subsequence,  $g(x_n) > v(t_n, x_n)$  for all  $n$ , then  $q^*(y_n, w(t_n, y_n)) - q_*(x_n, v(t_n, x_n)) \leq c(y_n) - c(x_n)$  for all  $n$ , leading to a contradiction since  $c(x_n) - c(y_n) \rightarrow 0$  as  $n \rightarrow \infty$  (recall (3.6) and (3.8)) and  $r > 0$ . If, along a subsequence,  $g(x_n) \leq v(t_n, x_n)$  for all  $n$ , then  $g(y_n) \leq v(t_n, x_n) + \delta/2 \leq w(t_n, y_n) - \delta/2$  for all  $n$  large enough and the above liminf is also non-positive. A contradiction too.

### 3.3 Monte-Carlo estimation

The solution of (3.4) is formally related to the solution  $(Y, Z) \in \mathbf{S}_2 \times \mathbf{L}_2$  of the backward stochastic differential equation

$$Y = e^{-rT} g(X_T) + \int_0^T e^{-rs} q(X_s, e^{rs} Y_s) ds - \int_0^T Z_s dW_s, \quad \mathbb{P}\text{-a.s.},$$

by  $e^{-r \cdot} V(\cdot, X) = Y$ . In the above,  $\mathbf{S}_2$  denotes the space of adapted processes  $\xi$  such that  $\mathbb{E}[\sup_{[0, T]} \|\xi\|^2] < \infty$  and  $\mathbf{L}_2$  denotes the space of predictable processes  $\xi$  such that  $\mathbb{E}[\int_0^T \|\xi_t\|^2 dt] < \infty$ .

$\infty$ .

**Remark 3.3.1.** Note that, if  $(Y, Z)$  satisfies the above BSDE, then

$$Y_0 = \mathbb{E}[e^{-rT}g(X_T) + \int_0^T e^{-rs}q(X_s, e^{rs}Y_s)ds].$$

In the case where  $c = rg - \mathcal{L}g$ , on  $\{(t, x) \in [0, T] \times (0, \infty)^d : V(t, x) = g(x)\}$ , this corresponds to the early exercise premium formula. Recall Assumption 3.2.2 and see [13, Section 6].

In practice the above BSDE is not well-posed because  $q$  is not continuous. However, it can be smoothed out for the purpose of numerical approximations. In the following, we write  $\mathbb{E}_s[\cdot]$  to denote the expectation given  $\mathcal{F}_s$ ,  $s \leq T$ .

**Proposition 3.3.2.** Let the condition of Theorem 3.2.4 hold. Let  $(q_n)_{n \geq 1}$  be a sequence of continuous functions on  $(0, \infty)^d \times \mathbb{R}$  that are Lipschitz in their last component<sup>3</sup>. Assume that  $(q_n)_{n \geq 1}$  is uniformly bounded by a function with polynomial growth in its first component and linear growth in its last component. Assume further that

$$\limsup_{\substack{n \rightarrow \infty \\ (x', y') \rightarrow (x, y)}} q_n(x', y') \leq q^*(x, y) \quad \text{and} \quad \liminf_{\substack{n \rightarrow \infty \\ (x', y') \rightarrow (x, y)}} q_n(x', y') \geq q_*(x, y), \quad (3.9)$$

for all  $(x, y) \in (0, \infty)^d \times \mathbb{R}$ . For  $(t, x) \in [0, T] \times (0, \infty)^d$ , let  $(Y^{t,x,n})_{n \geq 1}$  be such that

$$Y_s^{t,x,n} = \mathbb{E}_s[e^{-rT}g(X_T^{t,x}) + \int_s^T e^{-ru}q_n(X_u^{t,x}, e^{ru}Y_u^{t,x,n})du],$$

for  $s \in [t, T]$ , and set  $V_n(t, x) := e^{rt}Y_t^{t,x,n}$ . Then,  $(V_n)_{n \geq 1}$  converges pointwise to  $V$  as  $n \rightarrow \infty$ .

**Proof.** Each BSDE associated to  $q_n$  admits a unique solution  $(Y^{t,x,n}, Z^{t,x,n}) \in \mathbf{S}_2 \times \mathbf{L}_2$ , and it is standard to show that  $V_n$  is a continuous viscosity solution of

$$r\varphi - \mathcal{L}\varphi - q_n(\cdot, \varphi) = 0 \text{ on } [0, T] \times (0, \infty)^d \text{ and } \varphi(T, \cdot) = g \text{ on } (0, \infty)^d.$$

Moreover,  $(V_n)_{n \geq 1}$  has (uniformly) polynomial growth, thanks to the uniform polynomial growth assumption on  $(q_n)_{n \geq 1}$ . See e.g. [102]. By stability and (3.9), see e.g. [5], it follows that the relaxed limsup  $V^*$  and liminf  $V_*$  of  $(V_n)_{n \geq 1}$  are respectively sub- and super-solutions of (3.4). By Proposition 3.2.5,  $V^* \leq V \leq V_*$  and therefore equality holds among the three functions.

Therefore, up to a smoothing procedure, we are back to essentially solving a BSDE. In the next two sections, we propose two approaches. The first one consists in smoothing  $q$  into a smooth function  $q_n$  to which we apply the local polynomial approximation procedure of

3. See below for examples.

[19]. This allows us to use a pure forward Monte-Carlo method for the estimation of  $V_n$ , based on branching processes. In the second approach, we only add an independent noise in the definition of  $q$ , which also has the effect of smoothing it out, and then use a very simple version of the algorithm in [19]. As our numerical experiments show, the first approach is quite unstable while the second one is very efficient.

### 3.3.1 Local polynomial approximation and branching processes

Given Proposition 3.3.2, it is tempting to estimate the American option price by using the recently developed Monte-Carlo method for BSDEs, see [22] and the references therein. Here, we propose to use the forward approach suggested by [19], which is based on the use of branching processes coupled (in theory) with Picard iterations.

The first step consists in approximating the Heaviside function  $H : z \mapsto 1_{\{z \geq 0\}}$  by a sequence of Lipschitz functions  $(H_n)_{n \geq 1}$  and to define  $q_n$  by

$$q_n : (x, y) \mapsto c(x)H_n(g(x) - y).$$

Then,  $q_n$  is approximated by a map  $(x, y) \mapsto \bar{q}_n(x, y, y)$  of local polynomial form:

$$\bar{q}_n : (x, y, y') \mapsto \sum_{j=1}^{j_0} \sum_{l=0}^{l_0} a_{j,l}(x) y^l \phi_j(y'), \quad (3.10)$$

where  $(a_{j,l}, \phi_j)_{l \leq l_0, j \leq j_0}$  is a family of continuous and bounded maps satisfying

$$|a_{j,l}| \leq C_{l_0}, \quad |\phi_j(y'_1) - \phi_j(y'_2)| \leq L_\phi |y'_1 - y'_2| \quad \text{and} \quad |\phi_j| \leq 1,$$

for all  $y'_1, y'_2 \in \mathbb{R}$ ,  $j \leq j_0$  and  $l \leq l_0$ , for some constants  $C_{l_0}, L_\phi \geq 0$ . The elements of  $(a_{j,l}(x))_{l \leq l_0}$  should be interpreted as the coefficients of a polynomial approximation of  $q_n$  on a subset  $A_j$ , in which  $(A_j)_{j \leq j_0}$  forms a partition of  $\mathbb{R}$  and the  $\phi_j$ 's as smoothing kernels that allow one to pass in a Lipschitz way from one part of the partition to another one, see [19].

Then, one can consider the sequence of BSDEs

$$\begin{aligned} \bar{Y}_s^{t,x,n,k+1} &= \mathbb{E}_s[e^{-rT} g(X_T^{t,x})] \\ &+ \mathbb{E}\left[\int_s^T e^{-ru} \bar{q}_n(X_u^{t,x}, e^{ru} \bar{Y}_u^{t,x,n,k+1}, e^{ru} \bar{Y}_u^{t,x,n,k}) du\right], \quad k \geq 1, \end{aligned}$$

with  $\bar{Y}^{t,x,n,1}$  given as an initial prior (e.g.  $e^r \cdot g(X^{t,x})$ ). Given  $\bar{Y}^{t,x,n,k}$ ,  $\bar{Y}^{t,x,n,k+1}$  solves a BSDE with polynomial driver that can be estimated by using branching processes as in the Feynman-Kac representation of the Kolmogorov-Petrovskii-Piskunov equation, see [51, 52]. We refer to [19] for more details.

In practice, we use the Method A of [19, Section 3]. We perform a numerical experiment in dimension 1, with a time horizon of one year, and a risk-free interest rate set at 6%. We consider the Black and Scholes model with one single stock whose volatility is 40%. We price a put option which strike is  $K := 40$ . At the money, the American option price is around 5.30, while the European option is worth 5.05. In view of Example 3.2.1, we take  $c = rK^4$ . We first smooth the driver with a centered Gaussian density with variance  $\kappa^{-2}$ , so as to replace it by  $0.5rKe^{-rt}\text{erfc}(\kappa*(y - e^{-rt}g(x)))$  with  $\kappa = 10$ . See Figure 3.1. Then, we apply a quadratic spline approximation. In actual computation, as it is impossible to apply spline approximation on the whole half real line, we limited the domain of  $y$  for the driver function to  $[0, 40(1 - e^{-0.06})]$ . We partition this bounded domain into 20 intervals with equal-distant points and define a piecewise polynomial on this domain by assigning a quadric polynomial to each intervals. Finally, we match the values and derivatives of our piecewise polynomial at each point of the grid to the original function (except at the right-end where the derivative is assumed to be zero). The truncation of domain will not alter the computational result as our limited domain includes the maximum payoff for the put option. The resulting approximation is indistinguishable from the original function displayed on Figure 3.1.

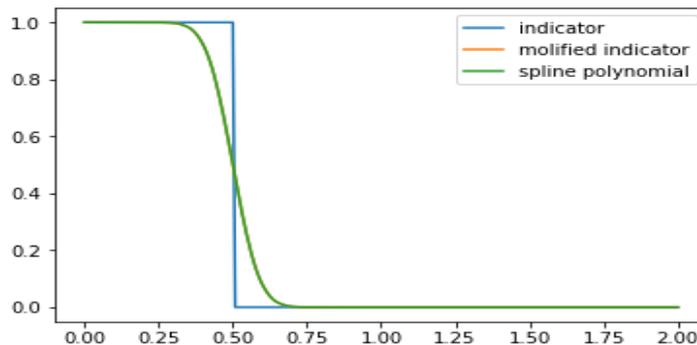


Figure 3.1 – Approximation of the Heaviside function.

We also partition  $[0, T]$  in 10 periods. As for the grid in the  $x$ -component, we use a 25-point uniform space-grid on the interval  $[e^{-20}, 80]$ . We estimate the early exercise value by first using 1.000 Monte-Carlo paths. As can be seen on Figure 3.2, the results are not good and this does not improve much with a higher number of simulations. The algorithm turns out to be quite unstable and not accurate. It remains pretty unstable even for a large number of simulated paths. This is not so surprising. Indeed, as explained in [19], their approach is dedicated to situations where the driver functions is rather smooth, so that the local polynomial's coefficients  $(a_{j,l})_{j,l}$  are small, and the supports of the  $\phi_j$ 's are large and do not intersect too much. Since

4. Note that, for this payoff, the constant  $rK$  is the function with the smallest absolute value among the functions  $c$  satisfying the requirements of Assumption 3.2.2.

we are approximating the Heaviside function, none of these requirements are met.

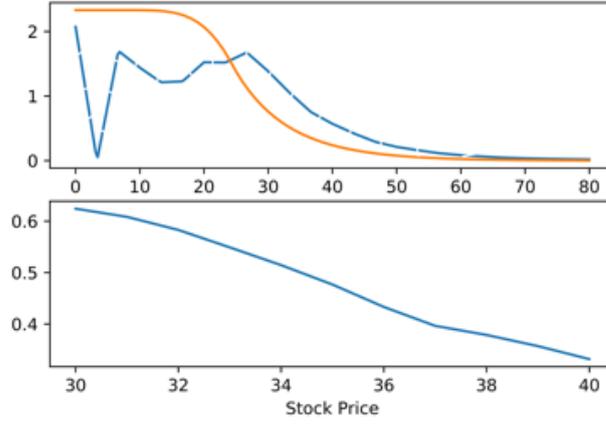


Figure 3.2 – Branching with local polynomial approximation. Upper graph: Early exercise premium (plain line obtained by a pde solver, dashed line estimated). Lower graph: Error on the early exercise premium estimation.

### 3.3.2 Driver randomization

In this second approach, we enlarge the state space so as to introduce an independent integrable random variable  $\epsilon$  with density  $f$  such that  $z \mapsto (1 + |z|)f'(z)$  is integrable. We assume that the interior of the support of  $f$  is of the form  $(m_\epsilon, M_\epsilon)$  with  $-\infty \leq m_\epsilon < M_\epsilon \leq \infty$ . Then, we define the sequence of random maps

$$\tilde{q}_n(x, y) := c(x)1_{\{g(x) + \frac{\epsilon}{n} \geq y\}}$$

as well as

$$q_n(x, y) := c(x)n \left\{ [g(x) + M_\epsilon/n - y]^+ f(M_\epsilon) - [g(x) + m_\epsilon/n - y]^+ f(m_\epsilon) \right\} - c(x)n \int [g(x) + z/n - y]^+ f'(z) dz$$

so that

$$q_n(x, y) = \mathbb{E}[\tilde{q}_n(x, y)]$$

for  $n \geq 1$ . If  $c$  is non-negative, continuous and has polynomial growth, then the sequence  $(q_n)_{n \geq 1}$  matches the requirements of Proposition 3.3.2.

We now let  $\tau$  be an independent exponentially distributed random variable with density  $\rho$

and cumulative distribution  $1 - \bar{F}$ . Then,  $Y^{t,x,n}$  defined as in Proposition 3.3.2 satisfies

$$Y_s^{t,x,n} = \mathbb{E}_s \left[ e^{-rT} \frac{g(X_T^{t,x})}{\bar{F}(T-t)} \mathbf{1}_{\{T-t \leq \tau\}} + \mathbf{1}_{\{T-t > \tau\}} \frac{e^{-r\tau} \tilde{q}_n(X_{t+\tau}^{t,x}, e^{r\tau} Y_{t+\tau}^{t,x,n})}{\rho(\tau)} \right].$$

This can be viewed as a branching based representation in which particles die at an exponential time. When a particle die before  $T$ , we give it the (random) mark  $\tilde{q}_n(X_{t+\tau}^{t,x}, e^{r\tau} Y_{t+\tau}^{t,x,n})$ . In terms of the representation of Section 3.3.1, this corresponds to  $j_0 = 1$ ,  $l_0 = 0$ , to replacing  $a_{1,0}(x)\phi_1(y')$  by  $\tilde{q}_n(x, y')$ , and to not using a Picard iteration scheme.

On a finite time grid  $\pi \subset [0, T]$  containing  $\{0, T\}$ , it can be approximated by the sequence  $v_n^\pi$  defined by  $v_n^\pi(T, \cdot) = g$  and

$$\begin{aligned} v_n^\pi(t, x) = & \mathbb{E} \left[ e^{-rT} \frac{g(X_T^{t,x})}{\bar{F}(T-t)} \mathbf{1}_{\{T-t \leq \tau\}} \right] \\ & + \mathbb{E} \left[ \mathbf{1}_{\{T-t > \tau\}} \frac{e^{-r\tau} \tilde{q}_n(X_{\phi_{t+\tau}^\pi}^{t,x}, e^{r\tau} v_n^\pi(\phi_{t+\tau}^\pi, X_{\phi_{t+\tau}^\pi}^{t,x}))}{\rho(\tau)} \right], \end{aligned} \quad (3.11)$$

where  $\phi_s^\pi := \inf\{s' \geq s : s' \in \pi\}$  for  $s \leq T$ . Showing that  $v_n^\pi(\phi_t^\pi, x)$  converges point-wise to  $Y_t^{t,x,n}$  as the modulus of  $\pi$  vanishes can be done by working along the lines of [4, Section 4.3] or [44]. In view of Proposition 3.3.2,  $v_n^\pi$  converges point-wise to  $V$  as  $|\pi| \rightarrow 0$  and  $n \rightarrow \infty$ . A similar analysis could be performed when considering a grid in space, which will be necessary in practice.

Then, (3.11) provides a natural backward algorithm: given a space-time grid  $\Pi := (t_i, x_j)_{i,j}$ , (3.11) can be used to compute  $v_n^\pi(t_i, x_j)$  given the already computed values of  $v_n^\pi$  at the later times in the grid, by replacing the expectation by a Monte-Carlo counterpart.

Let us now consider a put option pricing problem within the Black-Scholes model as in the previous section. The interest rate is 6%, the volatility is 20% and the strike is 25. The partition  $\pi$  of  $[0, T]$  is uniform with 100 time steps. However, we update  $v_n^\pi$  only every 10 time steps (and consider that it is constant in time in between).

The fine grid  $\pi$  is therefore only used to approximate  $X_\tau^{t,x}$  by  $X_{\phi_\tau^\pi}^{t,x}$  accurately. We use a 40-points equidistant space-grid on the interval  $[5, 50]$ . The random variable  $\epsilon/n$  is exponentially distributed, with mean equal to  $10^{-100}$ , while  $\tau$  has mean 0.6. In Figures 3.3, 3.4 and 3.5, we provide the estimated prices, the estimated early exercise premium as well as the corresponding relative errors. The statistics are based on 50 independent trials.

The reference values are computed with an implicit scheme for the associated pde, with regular grids of 500 points in space and 1.000 points in time (we also provide the European

option price in the top-left graph, for comparison).

The relative errors are capped to 10% or 40% for ease of readability. These graphs show that the numerical method is very efficient. The relative error for a stock price higher than 30/35 are not significant since it corresponds to option prices very close to 0. For 10.000 simulated paths, it takes 12 seconds for one estimation of the whole price curve with a R code running on a Macbook 2014, 2.5 GHz Intel Core i7, with 4 physical cores.

We next consider a strangle with strikes 25 and 27, see Example 3.2.1. Again, there is only one possible choice for  $c$  on  $g > 0$ , see Remark 3.2.3. The results obtained with 50.000 sample paths are displayed in Figures 3.6. Note that we do not use any variance reduction technique in these experiments.

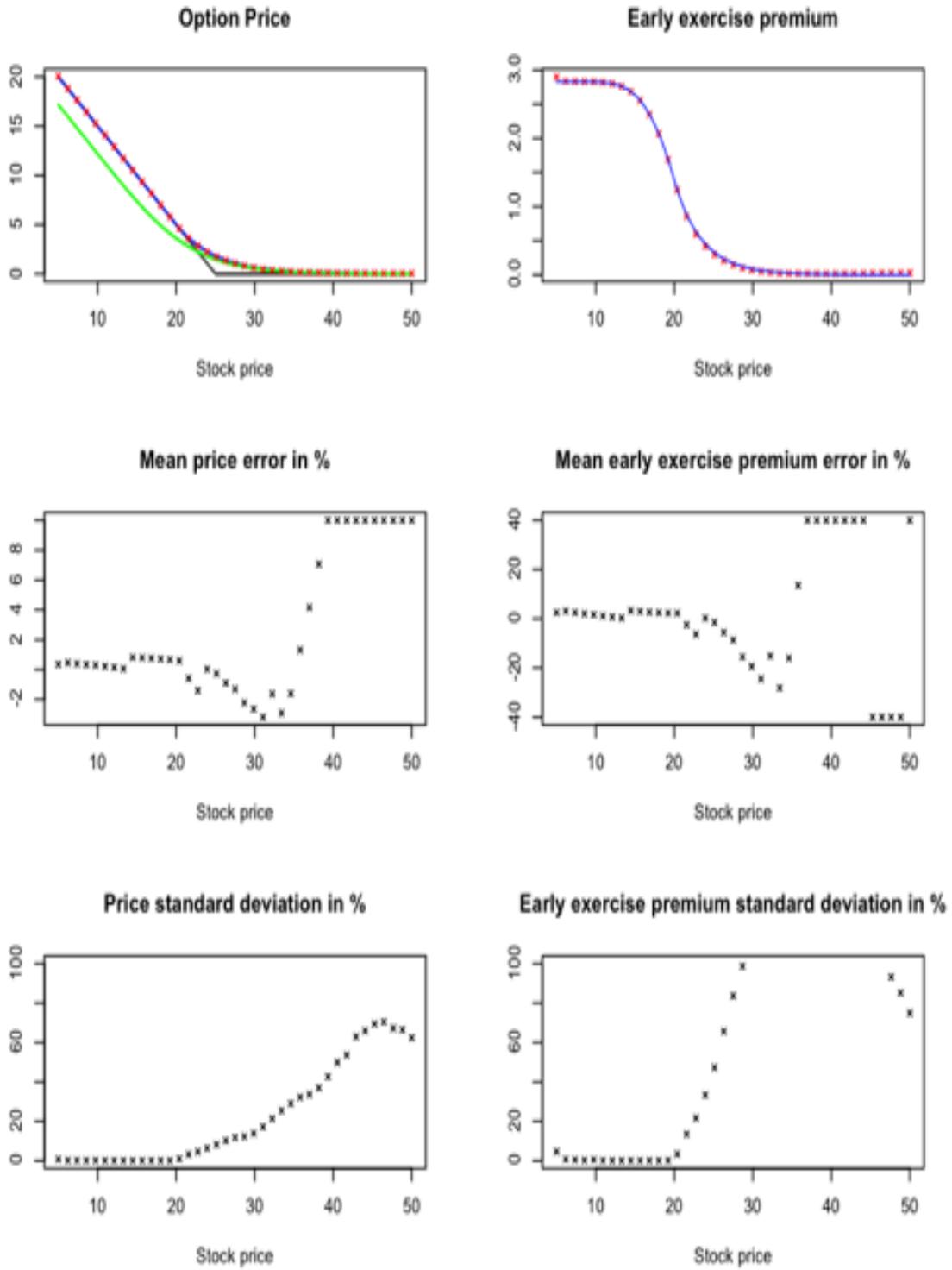


Figure 3.3 – Branching with indicator driver. Put option, 1.000 sample paths. Plain lines=true values, crosses=estimations.

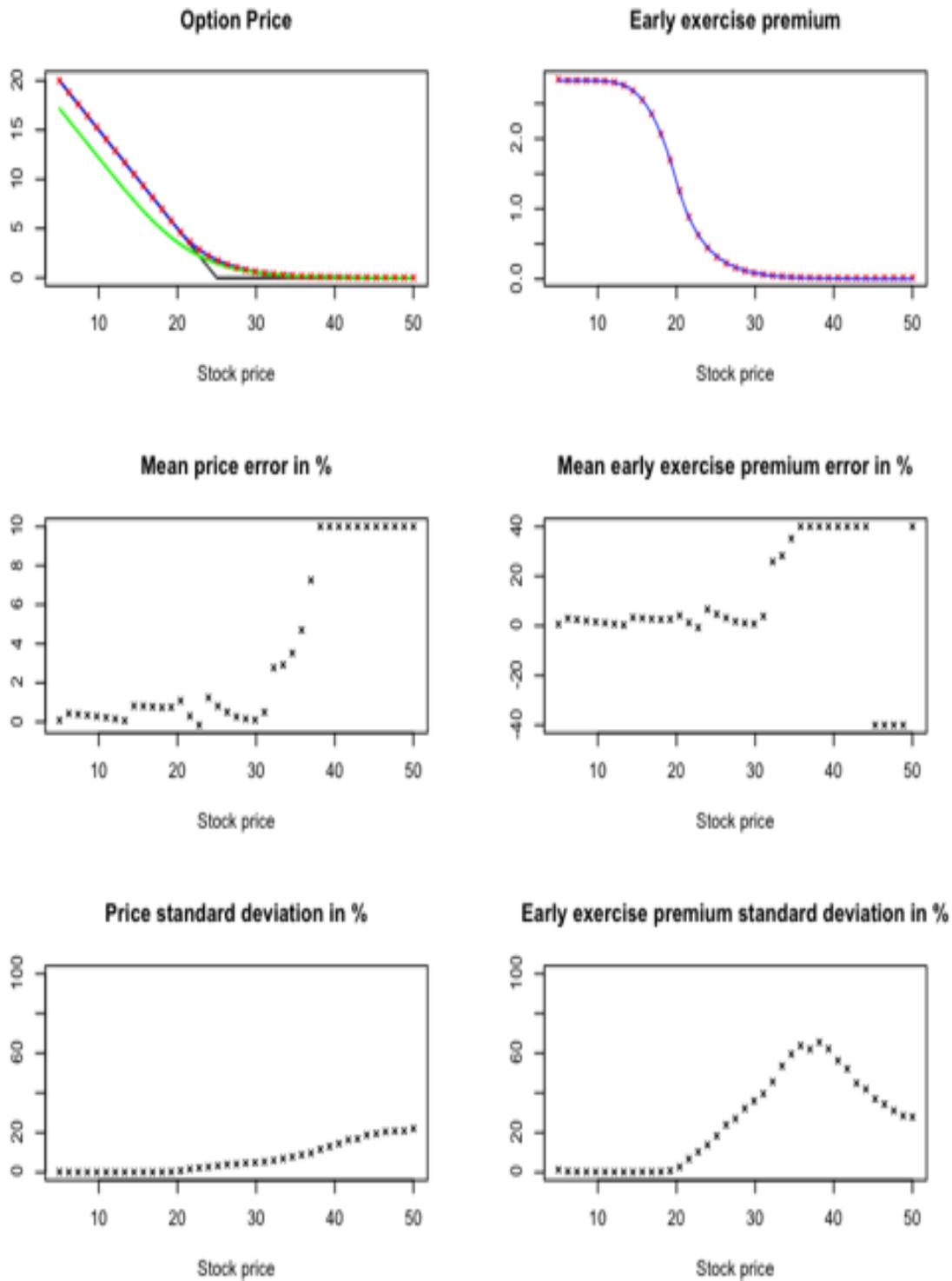


Figure 3.4 – Branching with indicator driver. Put option, 10.000 sample paths. Plain lines=true values, crosses=estimations.

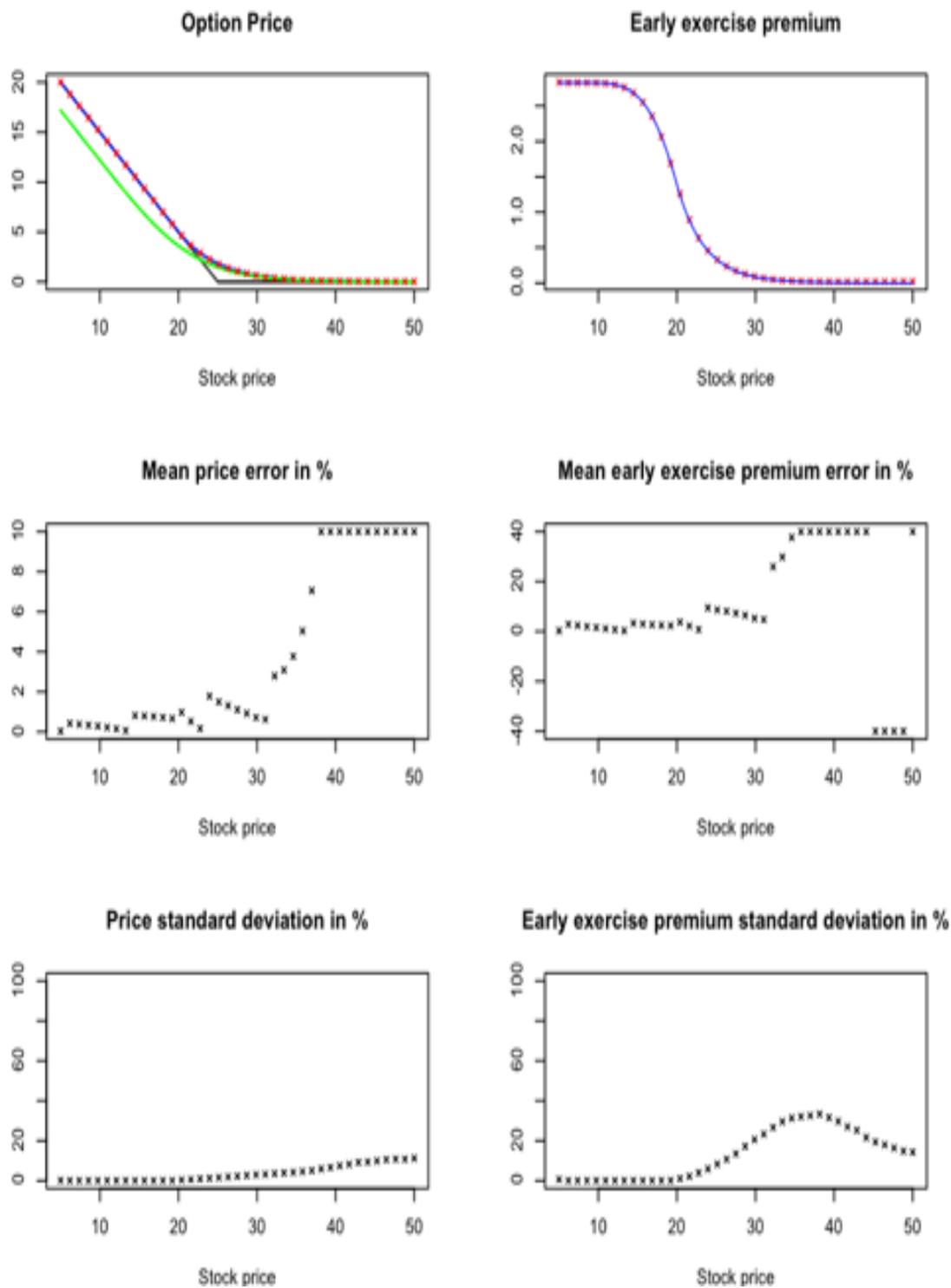


Figure 3.5 – Branching with indicator driver. Put option, 50.000 sample paths. Plain lines=true values, crosses=estimations.

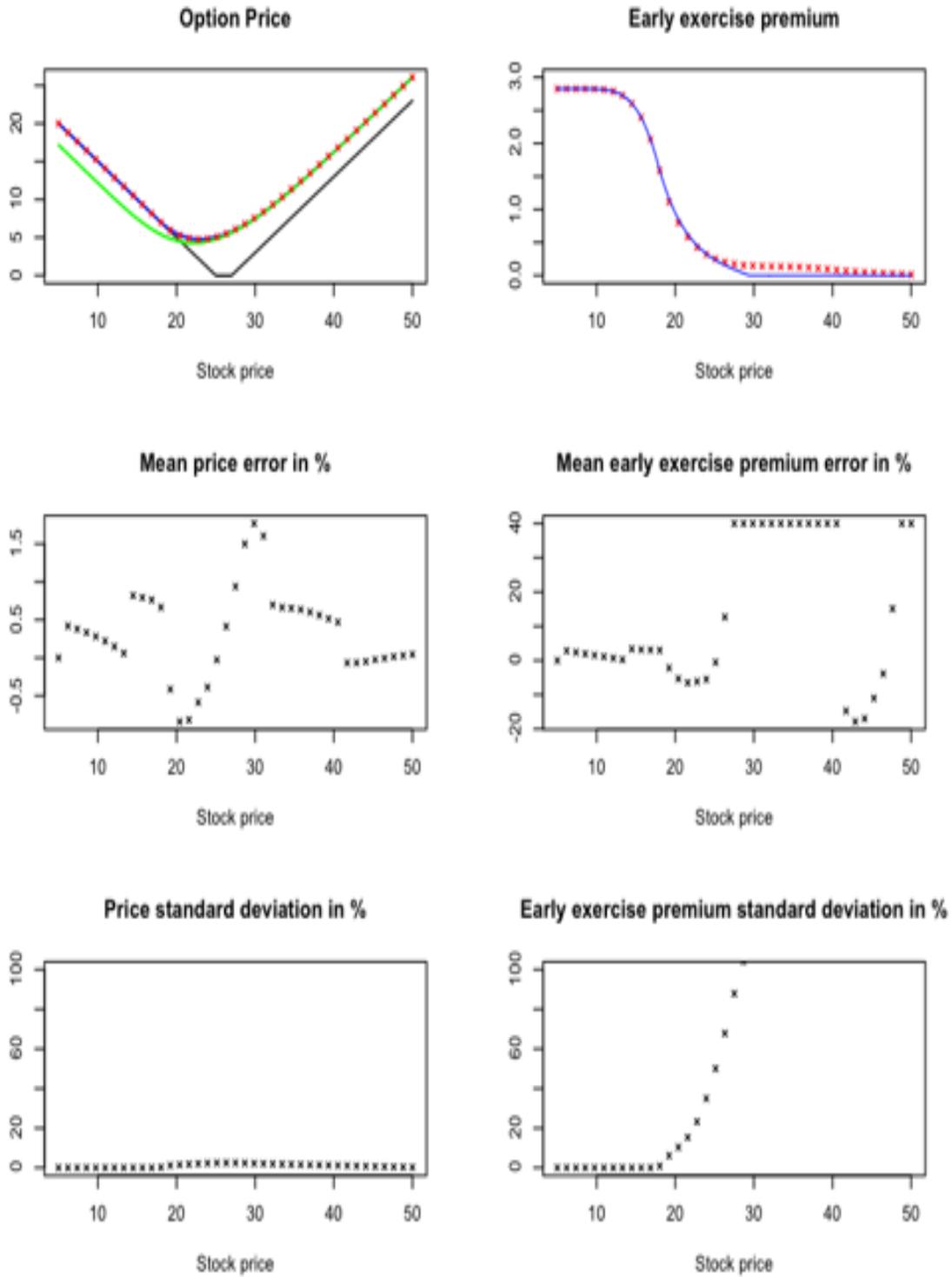


Figure 3.6 – Branching with indicator driver. Strangle option, 50.000 sample paths. Plain lines=true values, crosses=estimations.

# MEAN FIELD FORWARD-BACKWARD SDE WITH JUMPS AND STORAGE PROBLEM IN SMART GRIDS

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## 4.1 Introduction

In this chapter, we are interested in a general class of fully-coupled mean field forward-backward stochastic differential equations with jumps. More precisely, we would like to provide existence and uniqueness results to the following system

$$\left\{ \begin{array}{l} X_t = X_0 + \int_0^t b_s(X_s, Y_s, Z_s, K_s(x), \mathbb{P}_{(X_s, Y_s)}) ds + \int_0^t \sigma_s(X_s, Y_s, Z_s, K_s(x), \mathbb{P}_{(X_s, Y_s)}) dW_s \\ \quad + \int_0^t \int_E \beta_s(X_s, Y_s, Z_s, K_s(x), \mathbb{P}_{(X_s, Y_s)}) \tilde{\pi}(ds, de), \quad t \in [0, T], \mathbb{P}\text{-a.s.} \\ Y_t = g(X_T, \mathbb{P}_{X_T}) - \int_t^T h_s(X_s, Y_s, Z_s, K_s(x), \mathbb{P}_{(X_s, Y_s)}) ds - \int_t^T Z_s dW_s - \int_t^T \int_E K_s(e) \tilde{\pi}(ds, de), \end{array} \right. \quad (4.1)$$

where  $W$  and  $\pi$  are respectively a  $d$ -dimensional standard Brownian motion and an integer-valued random jump measure with a compensator  $\nu$  defined on a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ . Under Lipschitz continuity and monotonicity assumptions on the coefficients, we derive two existence and uniqueness results for the system (4.1) under two different assumptions. We emphasize that we do not require non-degeneracy of the diffusion coefficients of the forward process and we allow it to depend on  $Z$  and  $K$ .

Our approach to solve this problem is by suggesting an implicit approximation scheme which is shown to converge to the solutions of the McKean-Vlasov FBSDE with jumps.

Our second contribution is a study of an extended mean field game type control associated to the problem of energy storage in smart grids.

The outline of this chapter is as follows. After recalling briefly some notations, we define, in Section 4.3, the system of fully coupled forward backward SDE with jumps and we suggest

existence and uniqueness results under different Assumptions **(H1)** and **(H2)**.

Then, in Section 4.4, we consider a stylized model for a power network with distributed local power generation and storage. This model has been considered in [1] where the system is modeled as network connecting a large number of nodes, where each node is characterized by a local electricity consumption, has a local electricity production and manages a local storage device. In this part, in contrast to [1], we take account of the unpredictability of the energy production. In fact, it depends on intermittent and irregular environmental conditions and meteorological forecasts (sun, wind..).

To illustrate this phenomena, we include a jump component in our analysis. When the number of nodes is infinite, we link the problem to an extended mean field game type control which unique solution is characterized through solving an associated Forward Backward SDEs.

In Section 4.4.3, we consider an example of one node where the cost structure is quadratic and the pricing rule is linear and we give an explicit solution of the EMFG via its connection to a FBSDE with jumps. Finally, in the Appendix 4.5, we make some extensions of the results in [47].

## 4.2 Framework: Notations and setting

We consider a filtered probability space  $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$  on which the filtration  $\mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$  satisfies the usual conditions of completeness and right continuity. On this stochastic basis, let  $W$  a  $d$ -dimensional Brownian motion and  $\pi(\omega, dt, de)$  an independent integer valued random measure defined on  $([0, T] \times E, \mathcal{B}([0, T]) \otimes \mathcal{B}(E))$ , with compensator  $\eta(\omega, dt, de)$ .

The predictable  $\sigma$ -field on  $\Omega \times [0, T]$  is denoted by  $\mathcal{P}$  and  $\tilde{\mathcal{P}} = \mathcal{P} \otimes \mathcal{B}(E)$  is the respective  $\sigma$ -field on  $\tilde{\Omega} = \Omega \times [0, T] \times E$ . For a  $\sigma$ -finite measure  $\lambda$  on  $(E, \mathcal{B}(E))$  satisfying  $\int_E 1 \wedge |e|^2 \lambda(de) < \infty$  and a bounded  $\tilde{\mathcal{P}}$ -measurable non negative density function  $\zeta$ , we will assume that the compensator  $\eta$  is absolutely continuous with respect to  $\lambda \otimes dt$  such that

$$\eta(\omega, dt, de) = \zeta(\omega, t, e) \lambda(de) dt, \quad 0 \leq \zeta \leq C_\eta, \quad \text{for some constant } C_\eta.$$

Finally, we will denote by  $\tilde{\pi}$  the compensated measure of  $\pi$  as

$$\tilde{\pi}(\omega, dt, de) = \pi(\omega, dt, de) - \eta(\omega, dt, de).$$

For any random variable  $X$  on  $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ , we denote by  $\mathbb{P}_X$  its probability law under  $\mathbb{P}$ . We denote by  $\mathcal{M}_2(\mathbb{R}^d)$  the set of probability measures on  $\mathbb{R}^d$  with finite moments of order 2 equipped

with the 2-Wassertein distance

$$\begin{aligned} \mathcal{W}_2(\mu, \mu') &:= \inf\left\{\left(\int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^2 F(dx, dy)\right)^{\frac{1}{2}}, F \in \mathcal{M}_2(\mathbb{R}^d \times \mathbb{R}^d) \text{ with marginals } \mu, \mu'\right\} \\ &:= \inf\left\{(\mathbb{E}|\xi - \xi'|^2)^{\frac{1}{2}} : \mu = \mathcal{L}(\xi), \mu' = \mathcal{L}(\xi')\right\}, \end{aligned}$$

where  $\mathcal{L}(\xi)$  and  $\mathcal{L}(\xi')$  are respectively the law of  $\xi$  and  $\xi'$  and the infimum is taken over  $F \in \mathcal{M}_2(\mathbb{R}^d \times \mathbb{R}^d)$  with marginals  $\mu$  and  $\mu'$ .

Notice that if  $X^1$  and  $X^2$  are random variables of order 2 with values in  $\mathbb{R}^d$ , then we have the following inequality involving the Wasserstein metric between the laws of the square integrable random variables  $X^1$  and  $X^2$  and their  $L^2$ - distance:

$$\mathcal{W}_2(\mathbb{P}_{X^1}, \mathbb{P}_{X^2}) \leq \left[\mathbb{E}|X^1 - X^2|^2\right]^{\frac{1}{2}}. \quad (4.2)$$

Now, we define the spaces of processes which will be used in the present work.

- $\mathbb{H}^2$  is the space of all  $\mathbb{R}^d$ -valued and  $\mathcal{F}_t$ -progressively measurable process such that

$$\|Z\|_{\mathbb{H}^2}^2 := \mathbb{E} \int_0^T |Z_s|^2 ds < +\infty, \mathbb{P}\text{-a.s.}$$

- $\mathbb{H}_\eta^2$  is the space of all predictable processes such that

$$\|K\|_\eta^2 := \mathbb{E} \int_0^T \int_{\mathbb{E}} |K_s(e)|^2 \eta(de, ds) < +\infty, \mathbb{P}\text{-a.s.}$$

- For  $k, \bar{k}$  in the space  $\mathbb{L}^0(\mathcal{B}(E), \eta)$  of all  $\mathcal{B}(E)$ -measurable functions with the topology of convergence in measure, we define

$$|k - \bar{k}|_t^2 = \int_E |k(e) - \bar{k}(e)|^2 \zeta(t, e) \lambda(de).$$

- For  $u = (x, y, z, k) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{d \times d} \times \mathbb{L}^0(\mathcal{B}(E), \eta)$ , we set  $\|u\|^2 := |x|^2 + |y|^2 + \|z\|^2 + |k|_t^2$

Finally, we will assume  $W$  and  $\eta$  jointly have the weak predictable representation property with respect to the filtration  $(\mathcal{F}_t)_{0 \leq t \leq T}$ . This means that every square integrable martingale  $\mathcal{M}$  has a the following representation,

$$\mathcal{M} = \mathcal{M}_0 + \int Z dB + K \star \tilde{\pi},$$

where  $Z \in \mathbb{H}^2$  and  $K \in \mathbb{H}_\eta^2$ .

Finally, we set  $\mathbb{R}^{m+m+m \times m} = \mathbb{R}^m \times \mathbb{R}^m \times L(\mathbb{R}^m, \mathbb{R}^m)$ . For  $x, y \in \mathbb{R}^d$ ,  $x \cdot y$  denotes the scalar product and for  $x, y \in L(\mathbb{R}^m, \mathbb{R}^m)$ ,  $[x, y] = \sum_1^m x^j y^j$  where  $x^j$  (resp.  $y^j$ ) refers to the  $j$ -th columns of  $x$  (resp.  $y$ ). However, we suppressed the bracket for notational simplicity.

### 4.3 The system of forward-backward SDE with jumps of McKean-Vlasov type

In this section, we study the solvability of the following fully coupled mean-field forward backward SDE with jumps driven by a Brownian motion  $B$  and an integer valued independent random jump measure  $\tilde{\pi}$

$$(\mathbf{S}) \left\{ \begin{array}{l} X_t = X_0 + \int_0^t b_s(X_s, Y_s, Z_s, K_s, \mathbb{P}_{(X_s, Y_s)}) ds + \int_0^t \sigma_s(X_s, Y_s, Z_s, K_s, \mathbb{P}_{(X_s, Y_s)}) dW_s \\ \quad + \int_0^t \int_E \beta(s, X_{s-}, Y_{s-}, Z_s, K_s, \mathbb{P}_{(X_s, Y_s)}) \tilde{\pi}(ds, de), \quad t \in [0, T], \mathbb{P}\text{-a.s.} \\ Y_t = g(X_T, \mathbb{P}_{X_T}) + \int_t^T h_s(X_s, Y_s, Z_s, K_s, \mathbb{P}_{(X_s, Y_s)}) ds - \int_t^T Z_s dW_s - \int_t^T \int_E K_s(e) \tilde{\pi}(ds, de). \end{array} \right.$$

where  $(X, Y, Z, K)$  is an  $\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^{d \times d} \times \mathbb{L}^0(\mathcal{B}(E), \eta)$ -valued adapted processes and  $\mathbb{P}_{(X_t, Y_t)}$  is the marginal distribution of  $(X_t, Y_t)$ .

We require that the coefficients of the system **(S)** satisfy the following assumptions.

#### Assumption 4.3.1. Lipschitz Assumptions

- 1- The functions  $b, h, \sigma$  and  $\beta$  are Lipschitz in  $(x, y, z, k, \nu)$  i.e. there exists a constant  $C > 0$  such that for all  $t \in [0, T]$ ,  $u = (x, y, z, k)$ ,  $u' = (x', y', z', k') \in \mathbb{R}^{d+d+d \times d} \times \mathbb{L}^0(\mathcal{B}(E), \eta)$  and  $\nu, \nu' \in \mathcal{M}_2(\mathbb{R}^d \times \mathbb{R}^d)$ ,

$$\begin{aligned} & |b(t, u, \nu) - b(t, u', \nu')| + |h(t, u, \nu) - h(t, u', \nu')| + |\sigma(t, u, \nu) - \sigma(t, u', \nu')| \\ & + |\beta(t, u, \nu) - \beta(t, u', \nu')| \leq C[|x - x'| + |y - y'| + \|z - z'\| + |k - k'|_t + \mathcal{W}_2(\nu, \nu')]. \end{aligned}$$

- 2- The function  $g : \Omega \times \mathbb{R}^d \times \mathcal{M}_2(\mathbb{R}^d) \rightarrow \mathbb{R}^d$  is Lipschitz in  $(x, \mu)$  i.e. there exists  $C > 0$  such that for all  $x, x' \in \mathbb{R}^d$  and for all  $\mu, \mu' \in \mathcal{M}_2(\mathbb{R}^d)$ ,

$$|g(x, \mu) - g(x', \mu')| \leq C(|x - x'| + \mathcal{W}_2(\mu, \mu')), \quad \mathbb{P}\text{-a.s.} \quad (4.3)$$

- 3- For  $\phi \in \{b, h, g, \sigma, \beta\}$ ,  $\phi$  is Lipschitz with respect to  $x, y, z, k$  and  $\nu$  with  $C_\phi^x, C_\phi^y, C_\phi^z, C_\phi^k$  and  $C_\phi^\nu$  as the Lipschitz constants.

For  $u = (x, y, z, k)$  and  $u' = (x', y', z', k') \in \mathbb{R}^{d+d+d \times d} \times \mathbb{L}^0(\mathcal{B}(E), \eta)$ ,  $\nu \in \mathcal{M}_2(\mathbb{R}^d \times \mathbb{R}^d)$  we define

the operator  $\mathcal{A}$  in the following way

$$\begin{aligned} \mathcal{A}(t, u, u', \nu) &= (b(s, u, \nu) - b(s, u', \nu)) \cdot (y - y') + (h(s, u, \nu) - h(s, u', \nu)) \cdot (x - x') \\ &\quad + [(\sigma(s, u, \nu) - \sigma(s, u', \nu)), (z - z')] \\ &\quad + \int_E (\beta(s, u, \nu) - \beta(s, u', \nu))(k - k')(e) \eta(ds, de). \end{aligned}$$

### 4.3.1 Existence and uniqueness under (H1)

In this part, we shall prove the existence and uniqueness of the solution of the system **(S)** under the following assumption:

$$\text{(H1)} \quad \left\{ \begin{array}{l} (i) \text{ There exists } k > 0, \text{ s.t } \forall t \in [0, T], \nu \in \mathcal{M}_2(\mathbb{R}^d \times \mathbb{R}^d), u, u' \in \mathbb{R}^{d+d+d \times d} \times \mathbb{L}^0(\mathcal{B}(E), \eta), \\ \quad \mathcal{A}(t, u, u', \nu) \leq -k|x - x'|^2, \mathbb{P}\text{-a.s.} \\ (ii) \text{ There exists } k' > 0, \text{ s.t } \forall \nu \in \mathcal{M}_2(\mathbb{R}^d \times \mathbb{R}^d), x, x' \in \mathbb{R}^d \\ \quad (g(x, \nu) - g(x', \nu)) \cdot (x - x') \geq k'|x - x'|^2, \mathbb{P}\text{-a.s.} \end{array} \right.$$

We start by giving a key estimate for the difference of two solutions of the mean-field fully coupled FBSDEs with jumps **(S)** satisfying **(H1)**.

**Lemma 4.3.2.** *Let  $(Y', Z', K')$  another solution of the system **(S)**. Then, under **(H1)**, we have the following estimates*

$$\mathbb{E}[|Y_s - Y'_s|^2] \leq \Theta^1 \mathbb{E}[|X_T - X'_T|^2] + \Theta^2 \int_0^T \mathbb{E}|X_s - X'_s|^2 ds, \quad (4.4)$$

$$\mathbb{E}\left[\int_0^T (|Z_s - Z'_s|^2 + |K_s - K'_s|^2) ds\right] \leq \bar{\Theta}^1 \mathbb{E}[|X_T - X'_T|^2] + \bar{\Theta}^2 \int_0^T \mathbb{E}|X_s - X'_s|^2 ds, \quad (4.5)$$

where

$$\left\{ \begin{array}{l} \bar{\Theta}^1 = 2[(C_h^x + (C_h^z)^2 + (C_h^k)^2 + 2C_h^\nu + 2C_h^y)]\Theta^1 + 2(C_g^x + C_g^\nu)^2 \\ \bar{\Theta}^2 = 2[(C_h^x + (C_h^z)^2 + (C_h^k)^2 + 2C_h^\nu + 2C_h^y)]\Theta^2 + 2(C_h^x + C_h^\nu) \\ \Theta^1 = e^{(C_h^x + (C_h^z)^2 + (C_h^k)^2 + 2C_h^\nu + 2C_h^y)T} (C_g^x + C_g^\nu)^2 \\ \Theta^2 = e^{(C_h^x + (C_h^z)^2 + (C_h^k)^2 + 2C_h^\nu + 2C_h^y)T} (C_h^x + C_h^\nu). \end{array} \right.$$

**Proof.** For simplicity, we shall make the following notations that will be used all along this chapter:  $\Delta X = X' - X$ ,  $\Delta Y = Y' - Y$ ,  $\Delta Z = Z' - Z$ ,  $\Delta K = K' - K$ .

- We start by proving the first estimate. Let us consider the following processes

$$\begin{aligned}\zeta_s^1 &= \frac{h(X'_s, Y'_s, Z'_s, K'_s, \mathbb{P}_{(X'_s, Y'_s)}) - h(X_s, Y'_s, Z'_s, K'_s, \mathbb{P}_{(X'_s, Y'_s)})}{X'_s - X_s} 1_{\{X_s \neq X'_s\}} \\ \zeta_s^2 &= \frac{h(X_s, Y'_s, Z'_s, K'_s, \mathbb{P}_{(X'_s, Y'_s)}) - h(X_s, Y_s, Z'_s, K'_s, \mathbb{P}_{(X'_s, Y'_s)})}{Y'_s - Y_s} 1_{\{Y_s \neq Y'_s\}} \\ \zeta_s^3 &= \frac{h(X_s, Y_s, Z'_s, K'_s, \mathbb{P}_{(X'_s, Y'_s)}) - h(X_s, Y_s, Z_s, K'_s, \mathbb{P}_{(X'_s, Y'_s)})}{\|Z'_s - Z_s\|^2} (Z'_s - Z_s) 1_{\{Z_s \neq Z'_s\}},\end{aligned}$$

which are respectively bounded by  $C_h^x, C_h^y, C_h^z$  due to the Lipschitz assumption on  $h$ . We apply Itô's formula to the process  $|\Delta Y|^2$  and we obtain

$$\begin{aligned}\mathbb{E}[|\Delta Y_t|^2] &= \mathbb{E}[g(X'_T, \mathbb{P}_{X'_T}) - g(X_T, \mathbb{P}_{X_T})]^2 + 2\mathbb{E} \int_t^T \Delta Y_s [\zeta_s^1 \Delta X_s + \zeta_s^2 \Delta Y_s + \zeta_s^3 \Delta Z_s] ds \\ &- \mathbb{E} \left[ \int_t^T \int_E (|\Delta Y_{s-} + \Delta K_s(e)|^2 - |\Delta Y_{s-}|^2) \tilde{\pi}(de, ds) - \int_t^T \|\Delta Z_s\|^2 ds \right] \\ &- \mathbb{E} \left[ \int_t^T \int_E (|\Delta Y_{s-} + \Delta K_s(e)|^2 - |\Delta Y_{s-}|^2 - 2|\Delta Y_{s-} \Delta K_s(e)|) \eta(de, ds) - \int_t^T 2\Delta Y_s \Delta Z_s dW_s \right] \\ &+ 2\mathbb{E} \left[ \int_t^T \Delta Y_s [h(U'_s, \mathbb{P}_{(X'_s, Y'_s)}) - h(U'_s, \mathbb{P}_{(X_s, Y_s)})] ds \right] \\ &+ 2\mathbb{E} \left[ \int_t^T \Delta Y_s [h(X_s, Y_s, Z_s, K'_s, \mathbb{P}_{(X'_s, Y'_s)}) - h(X_s, Y_s, Z_s, K_s, \mathbb{P}_{(X'_s, Y'_s)})] ds \right].\end{aligned}$$

Since the stochastic integrals are true martingales, we conclude that

$$\begin{aligned}\mathbb{E}[|\Delta Y_t|^2] &+ \mathbb{E} \left[ \int_t^T \|\Delta Z_s\|^2 ds \right] + \mathbb{E} \left[ \int_t^T \int_E |\Delta U_s(e)|^2 \eta(de, ds) \right] \\ &= \mathbb{E}[|g(X'_T, \mathbb{P}_{X'_T}) - g(X_T, \mathbb{P}_{X_T})|^2] + 2\mathbb{E} \int_t^T \Delta Y_s [\zeta_s^1 \Delta X_s + \zeta_s^2 \Delta Y_s + \zeta_s^3 \Delta Z_s] ds \\ &+ \mathbb{E} \left[ \int_t^T \Delta Y_s [h(X_s, Y_s, Z_s, K'_s, \mathbb{P}_{(X'_s, Y'_s)}) - h(X_s, Y_s, Z_s, K_s, \mathbb{P}_{(X'_s, Y'_s)})] ds \right] \\ &+ 2\mathbb{E} \left[ \int_t^T \Delta Y_s [(U_s, \mathbb{P}_{(X'_s, Y'_s)}) - h(U'_s, \mathbb{P}_{(X_s, Y_s)})] ds \right].\end{aligned}\tag{4.6}$$

Using the Lipschitz property of  $h$ , we obtain that

$$\begin{aligned}\mathbb{E}[|\Delta Y_t|^2] &+ \int_t^T \|\Delta Z_s\|^2 ds + \int_t^T \int_E |\Delta K_s(e)|^2 \eta(de, ds) \leq \mathbb{E}[|g(X'_T, \mathbb{P}_{X'_T}) - g(X_T, \mathbb{P}_{X_T})|^2] \\ &+ 2\mathbb{E} \int_t^T \Delta Y_s [C_h^x |\Delta X_s| + C_h^y |\Delta Y_s| + C_h^z |\Delta Z_s| + C_h^k |\Delta K_s|] ds \\ &+ 2\mathbb{E} \left[ \int_t^T \Delta Y_s [h(U'_s, \mathbb{P}_{(X'_s, Y'_s)}) - h(U'_s, \mathbb{P}_{(X_s, Y_s)})] ds \right].\end{aligned}\tag{4.7}$$

Notice that, in one hand, we have

$$2\Delta Y_s[h(U'_s, \mathbb{P}_{(X'_s, Y'_s)}) - h(U'_s, \mathbb{P}_{(X_s, Y_s)})] \leq 2C_h^\nu |\Delta Y_s| (\sqrt{\mathbb{E}[|\Delta X_s|^2]} + \sqrt{\mathbb{E}[|\Delta Y_s|^2]}), \quad (4.8)$$

and in the other hand, we have

$$\begin{aligned} & 2\Delta Y_s[C_h^x |\Delta X_s| + C_h^y |\Delta Y_s| + C_h^z |\Delta Z_s| + C_h^k |\Delta K_s|] ds \\ & \leq C_h^x |\Delta Y_s|^2 + C_h^x |\Delta X_s|^2 + (C_h^z)^2 |\Delta Y_s|^2 + |\Delta Z_s|^2 + (C_h^k)^2 |\Delta Y_s|^2 + |\Delta K_s|^2 + 2C_h^y |\Delta Y_s|^2. \end{aligned} \quad (4.9)$$

Moreover, by Young inequality and the Lipschitz property on  $g$  we obtain

$$\begin{aligned} |g(X'_T, \mathbb{P}_{X'_T}) - g(X_T, \mathbb{P}_{X_T})|^2 &= |g(X'_T, \mathbb{P}_{X'_T}) - g(X_T, \mathbb{P}_{X'_T}) + g(X_T, \mathbb{P}_{X'_T}) - g(X_T, \mathbb{P}_{X_T})|^2 \\ &\leq |C_g^x |X'_T - X_T| + C_g^\nu \mathcal{W}_2(\mu', \mu)|^2 \\ &\leq (C_g^x)^2 |\Delta X_T|^2 + (C_g^\nu)^2 |\Delta X_T|^2 + 2C_g^x C_g^\nu |\Delta X_T| \mathcal{W}_2(\mu', \mu) \\ &\leq (C_g^x)^2 |\Delta X_T|^2 + (C_g^\nu)^2 |\Delta X_T|^2 + C_g^x C_g^\nu |\Delta X_T|^2 + C_g^x C_g^\nu \mathcal{W}_2^2(\mu', \mu) \\ &\leq (C_g^x)^2 |\Delta X_T|^2 + (C_g^\nu)^2 |\Delta X_T|^2 + 2C_g^x C_g^\nu |\Delta X_T|^2 \\ &\leq (C_g^x + C_g^\nu)^2 |\Delta X_T|^2. \end{aligned} \quad (4.10)$$

Now, plugging (4.10), (4.9) and (4.8) in (4.7) yields

$$\begin{aligned} \mathbb{E}[|\Delta Y_t|^2] &\leq (C_g^x + C_g^\nu)^2 \mathbb{E}[|\Delta X_T|^2] + \int_t^T (C_h^x + C_h^\nu) \mathbb{E}[|\Delta X_s|^2] ds \\ &\quad + \mathbb{E}\left[\int_t^T (C_h^x + (C_h^z)^2 + (C_h^k)^2 + 2C_h^\nu + 2C_h^y) |\Delta Y_s|^2 ds\right]. \end{aligned}$$

Finally, Gronwall's lemma implies

$$\mathbb{E}[|\Delta Y_s|^2] \leq e^{[(C_h^x + (C_h^z)^2 + (C_h^k)^2 + 2C_h^\nu + 2C_h^y)T]} \left[ (C_g^x + C_g^\nu)^2 \mathbb{E}[|\Delta X_T|^2] + (C_h^x + C_h^\nu) \int_0^T \mathbb{E}[|\Delta X_s|^2] ds \right],$$

and we obtain the following inequality

$$\mathbb{E}[|\Delta Y_s|^2] \leq \Theta^1 \mathbb{E}[|\Delta X_T|^2] + \Theta^2 \int_0^T \mathbb{E}[|\Delta X_s|^2] ds. \quad (4.11)$$

• Let us now prove the second estimate. Recalling (4.7) and noting that

$$\begin{cases} 2C_h^z |\Delta Y_s| |\Delta Z_s| \leq 2(C_h^z)^2 |\Delta Y_s|^2 + \frac{1}{2} |\Delta Z_s|^2 \\ 2C_h^k |\Delta Y_s| |\Delta K_s| \leq 2(C_h^k)^2 |\Delta Y_s|^2 + \frac{1}{2} |\Delta K_s|^2, \end{cases}$$

we obtain

$$\begin{aligned} \frac{1}{2}\mathbb{E}\left[\int_t^T (|\Delta Z_s|^2 + |\Delta K_s|^2)ds\right] &\leq (C_g^x + C_g^\nu)^2\mathbb{E}[|\Delta X_T|^2] + \int_t^T (C_h^x + C_h^\nu)\mathbb{E}|\Delta X_s|^2 ds \\ &+ \mathbb{E}\left[\int_t^T (C_h^x + 2(C_h^z)^2 + 2(C_h^k + C_h^\nu)^2 + 2C_h^y)|\Delta Y_s|^2 ds\right]. \end{aligned}$$

Henceforth, making the following notations

$$\begin{aligned} \bar{\Theta}^1 &= 2[(C_h^x + (C_h^z)^2 + (C_h^k)^2 + 2C_h^\nu + 2C_h^y)]\Theta^1 + 2(C_g^x + C_g^\nu)^2 \\ \bar{\Theta}^2 &= 2[(C_h^x + (C_h^z)^2 + (C_h^k)^2 + 2C_h^\nu + 2C_h^y)]\Theta^2 + 2(C_h^x + C_h^\nu), \end{aligned}$$

we obtain the desired result

$$\mathbb{E}\left[\int_t^T (|\Delta Z_s|^2 + |\Delta K_s|^2)ds\right] \leq \bar{\Theta}^1\mathbb{E}[|\Delta X_T|^2] + \bar{\Theta}^2 \int_0^T \mathbb{E}|\Delta X_s|^2 ds. \quad (4.12)$$

These previous estimates allow us to prove the following uniqueness result of the solution of the mean-field FBSDE with jumps **(S)**.

**Proposition 4.3.3.** *Under **(H1)**, there exists a unique solution  $U = (X, Y, Z, K)$  of the mean field FBSDE with jumps **(S)**.*

**Proof.** Suppose that **(S)** has another solution  $U' = (X', Y', Z', K')$ . Applying Itô's formula to the product  $\Delta X_t \Delta Y_t$  gives

$$d(\Delta X_t \Delta Y_t) = \Delta X_t d(\Delta Y_t) + \Delta Y_t d(\Delta X_t) + d\langle \Delta X_t, \Delta Y_t \rangle_t$$

Taking the conditional expectation, we obtain

$$\begin{aligned} \Gamma_T &= \mathbb{E}[\Delta X_T \Delta Y_T] = \mathbb{E}\left[\int_0^T \{(b(s, U_s, \nu_s) - b(s, U'_s, \nu'_s))\Delta Y_s \right. \\ &+ (h(s, U_s, \nu_s) - h(s, U'_s, \nu'_s))\Delta X_s + (\sigma(s, U_s, \nu_s) - \sigma(s, U'_s, \nu'_s))\Delta Z_s\} ds \\ &+ \int_0^T \int_E (\beta(s, U_s, \nu_s) - \beta(s, U'_s, \nu'_s))\Delta K_s \eta(ds, de)\bigg] + \mathbb{E}\left[\int_0^T \Delta X_s \Delta Z_s dW_s\right] \\ &+ \int_0^T \int_E \Delta X_s \Delta K_s \tilde{\pi}(ds, de)\bigg] + \mathbb{E}\left[\int_0^T \Delta Y_s (\sigma(s, U_s, \nu) - \sigma(s, U'_s, \nu'_s)) dW_s\right] \\ &+ \mathbb{E}\left[\int_0^T \int_E \Delta Y_s (\beta(s, U_s, \nu_s) - \beta(s, U'_s, \nu'_s)) \tilde{\pi}(de, ds)\right]. \end{aligned}$$

Let us observe that the local martingale  $\int_0^t \Delta X_s \Delta Z_s dW_s + \int_0^t \Delta X_s \Delta K_s \tilde{\pi}(ds, de)$  is a true  $(\mathbb{P}, \mathcal{F})$  martingale.

Indeed, using the BDG inequality with the help of the square integrability of  $\Delta Y$ ,  $\Delta Z$  and  $\Delta K$ , we get

$$\begin{aligned} \mathbb{E}\left[\sup_{0 \leq t \leq T} \left| \int_0^t \Delta X_s \Delta Z_s dW_s \right|\right] &\leq C \mathbb{E}\left[\sup_{0 \leq t \leq T} |\Delta X_t|^2 \int_0^T |\Delta Z_s|^2 ds\right]^{\frac{1}{2}} \\ &\leq C(\mathbb{E}[\sup_{0 \leq t \leq T} |\Delta X_t|^2] + \mathbb{E}[\int_0^T |\Delta Z_s|^2 ds]) < +\infty, \end{aligned}$$

and

$$\begin{aligned} \mathbb{E}\left[\sup_{0 \leq t \leq T} \left| \int_0^t \int_E \Delta X_s \Delta K_s \tilde{\pi}(de, ds) \right|\right] &\leq C \mathbb{E}\left[\int_0^T \int_E |\Delta X_s \Delta K_s|^2 \eta(de, ds)\right]^{\frac{1}{2}} \\ &\leq C(\mathbb{E}[\sup_{0 \leq t \leq T} |\Delta X_t|^2] + \mathbb{E}[\int_0^T \int_E |\Delta K_s|^2 \eta(de, ds)]) < +\infty. \end{aligned}$$

In the same way, we can prove that

$$\int_0^T \Delta Y_s (\sigma(s, U_s, \nu) - \sigma(s, U'_s, \nu'_s)) dW_s + \int_0^T \int_E \Delta Y_s (\beta(s, U_s, \nu_s) - \beta(s, U'_s, \nu'_s)) \tilde{\pi}(de, ds),$$

is a  $(\mathbb{P}, \mathcal{F})$ -martingale.

Afterwards, we study each term separately. Let us start by the term  $\Delta X_T \Delta Y_T$ : In one hand, using **(H1)**, we make the following computation

$$\begin{aligned} \Gamma_T &= \mathbb{E}[(\Delta X_T)(g(X'_T, \mathbb{P}_{X'_T}) - g(X_T, \mathbb{P}_{X_T}))] \\ &\geq \mathbb{E}[k' |\Delta X_T|^2 - C_g^\nu (|\Delta X_T| \cdot \mathcal{W}_2(\mathbb{P}_{X'_T}, \mathbb{P}_{X_T}))] \\ &\geq k' \mathbb{E}[|\Delta X_T|^2] - C_g^\nu \mathbb{E}[|\Delta X_T|] \mathbb{E}[|\Delta X_T|^2]^{\frac{1}{2}} \\ &\geq (k' - C_g^\nu) \mathbb{E}[|\Delta X_T|^2]. \end{aligned} \tag{4.13}$$

On the other hand, we have

$$\begin{aligned} \Gamma_T &\leq \mathbb{E}\left[\int_0^T \{ \mathcal{A}(s, U_s, U'_s, \nu_s) + ((b(s, U'_s, \nu_s) - b(s, U_s, \nu'_s)) \cdot \Delta Y_s \right. \\ &\quad \left. + (h(s, U'_s, \nu_s) - h(s, U_s, \nu'_s)) \cdot \Delta X_s + (\sigma(s, U'_s, \nu_s) - \sigma(s, U_s, \nu'_s)) \Delta Z_s \} ds \right. \\ &\quad \left. + \int_0^T \int_E (\beta(s, U'_s, \nu_s) - \beta(s, U_s, \nu'_s)) \Delta K_s \eta(ds, de) \right]. \end{aligned}$$

The Lipschitz assumption together with Young inequality ( $ab \leq \frac{1}{2}(a^2 + b^2)$ ) imply that

$$\begin{aligned} \Gamma_T &\leq \mathbb{E} \left[ \int_0^T [\mathcal{A}(s, U_s, U'_s, \nu) + (C_h^\nu |\Delta X_s| + C_f^\nu |\Delta Y_s| + C_\sigma^\nu |\Delta Z_s| + C_\beta^\nu |\Delta K_s|) \mathcal{W}_2(\nu_s, \nu'_s)] ds \right] \\ &\leq \mathbb{E} \left[ \int_0^T -k |\Delta X_s|^2 ds \right] + \frac{1}{2} \mathbb{E} \left[ \int_0^T (C_h^\nu |\Delta X_s|^2 + C_h^\nu \mathcal{W}_2^2(\nu'_s, \nu_s) + C_b^\nu |\Delta Y_s|^2 + C_b^\nu \mathcal{W}_2^2(\nu'_s, \nu_s)) ds \right] \\ &\quad + \frac{1}{2} \mathbb{E} \left[ \int_0^T (C_\sigma^\nu \|\Delta Z_s\|^2 + C_\sigma^\nu \mathcal{W}_2^2(\nu'_s, \nu_s)) ds \right] + \frac{1}{2} \mathbb{E} \left[ \int_0^T (C_\beta^\nu |\Delta K_s|^2 + C_\beta^\nu \mathcal{W}_2^2(\nu'_s, \nu_s)) ds \right]. \end{aligned}$$

Using the following inequality

$$\mathcal{W}_2^2(\nu'_s, \nu_s) \leq \mathbb{E}[|\Delta X_s|^2] + E[|\Delta Y_s|^2], \quad (4.14)$$

we obtain that

$$\begin{aligned} \Gamma_T &\leq \mathbb{E} \left[ \int_0^T -k |\Delta X_s|^2 ds \right] + \mathbb{E} \left[ \int_0^T [C_h^\nu + \frac{1}{2}(C_b^\nu + C_\sigma^\nu + C_\beta^\nu)] |\Delta X_s|^2 ds \right] \\ &\quad + \mathbb{E} \left[ \int_0^T [C_b^\nu + \frac{1}{2}(C_h^\nu + C_\sigma^\nu + C_\beta^\nu)] |\Delta Y_s|^2 ds \right] \\ &\quad + \frac{1}{2} C_\sigma^\nu \mathbb{E} \left[ \int_0^T \|\Delta Z_s\|^2 ds \right] + \frac{1}{2} C_\beta^\nu \mathbb{E} \left[ \int_0^T |\Delta K_s|^2 ds \right]. \end{aligned}$$

Now, using the estimates in Lemma 4.3.5, we get

$$\begin{aligned} \Gamma_T &\leq \mathbb{E} \left[ \int_0^T (-k + [C_h^\nu + \frac{1}{2}(C_b^\nu + C_\sigma^\nu + C_\beta^\nu)] |\Delta X_s|^2 ds \right. \\ &\quad + [C_b^\nu + \frac{1}{2}(C_h^\nu + C_\sigma^\nu + C_\beta^\nu)] (\Theta^1 \mathbb{E}[|\Delta X_T|^2] + \Theta^2 \int_0^T \mathbb{E}[|\Delta X_s|^2] ds) \\ &\quad \left. + \frac{1}{2}(C_\sigma^\nu \vee C_\beta^\nu) (\bar{\Theta}^1 \mathbb{E}[|\Delta X_T|^2] + \bar{\Theta}^2 \int_0^T \mathbb{E}[|\Delta X_s|^2] ds) \right]. \end{aligned}$$

Hence, (4.13) gives that

$$\begin{aligned} (k' - C_g^\nu) \mathbb{E}[|\Delta X_T|^2] &\leq \left( \Theta^1 [C_b^\nu + \frac{1}{2}(C_h^\nu + C_\sigma^\nu + C_\beta^\nu)] + \frac{1}{2}(C_\sigma^\nu \vee C_\beta^\nu) \bar{\Theta}^1 \right) \mathbb{E}[|\Delta X_T|^2] \\ &\quad + \left( -k + [C_h^\nu + \frac{1}{2}(C_b^\nu + C_\sigma^\nu + C_\beta^\nu)] + [C_b^\nu + \frac{1}{2}(C_h^\nu + C_\sigma^\nu + C_\beta^\nu)] \Theta^2 + \frac{1}{2}(C_\sigma^\nu \vee C_\beta^\nu) \bar{\Theta}^2 \right) \mathbb{E} \left[ \int_0^T |\Delta X_s|^2 ds \right]. \end{aligned} \quad (4.15)$$

Henceforth, taking

$$\begin{aligned} (k' - C_g^\nu) &\geq \Theta^1 [C_b^\nu + \frac{1}{2}(C_h^\nu + C_\sigma^\nu + C_\beta^\nu)] + \frac{1}{2}(C_\sigma^\nu \vee C_\beta^\nu) \bar{\Theta}^1, \\ k &\geq [C_h^\nu + \frac{1}{2}(C_b^\nu + C_\sigma^\nu + C_\beta^\nu)] + [C_b^\nu + \frac{1}{2}(C_h^\nu + C_\sigma^\nu + C_\beta^\nu)] \Theta^2 + \frac{1}{2}(C_\sigma^\nu \vee C_\beta^\nu) \bar{\Theta}^2, \end{aligned}$$

we obtain that  $X_T = X'_T$ , and  $\forall t \in [0, T]$ ,  $X'_t = X_t$   $\mathbb{P}$ -a.s. Hence,  $(Y, Z, K)$  and  $(Y, Z, K)$  are two

solutions of

$$Y_t = g(X_T, \mathbb{P}_{X_T}) + \int_t^T h_s(X_s, Y_s, Z_s, K_s, \mathbb{P}_{(X_s, Y_s)}) ds - \int_t^T Z_s dW_s - \int_t^T \int_E K_s(e) \tilde{\pi}(ds, de).$$

However, this mean field BSDEs with jumps admits a unique solution (see [77]). Therefore, the system **(S1)** admits a unique solution.

**Theorem 4.3.4.** *Under Assumption (H1), there exists a solution  $U = (X, Y, Z, K)$  of the mean field FBSDE with jumps system **(S)**.*

**Proof.** In order to prove the existence of the solution, we use an approximation scheme based on perturbations of the forward equation. Let  $\delta \in ]0, 1]$  and consider a sequence  $(X^n, Y^n, Z^n, K^n)$  of processes defined recursively by  $(X^0, Y^0, Z^0, K^0) = (0, 0, 0, 0)$  and for  $n \geq 1$ ,  $U^n = (X^n, Y^n, Z^n, K^n)$  satisfies

$$\begin{cases} X_t^{n+1} = X_0 + \int_0^t [b_s(U_s^{n+1}, \nu_s^n) - \delta(Y_s^{n+1} - Y_s^n)] ds + \int_0^t [\sigma_s(U_s^{n+1}, \nu_s^n) - \delta(Z_s^{n+1} - Z_s^n)] dW_s \\ \quad + \int_0^t \int_E (\beta_s(U_s^{n+1}, \nu_s^n) - \delta(K_s^{n+1} - K_s^n)) \tilde{\pi}(ds, de), \\ Y_t^{n+1} = g(X_T^{n+1}, \mu_T^n) - \int_t^T h_s(U_s^{n+1}, \nu_s^n) ds - \int_t^T Z_s^{n+1} dW_s - \int_t^T \int_E K_s^{n+1}(e) \tilde{\pi}(ds, de). \end{cases} \quad (4.16)$$

Hereafter, we shall use the following simplified notations: For  $n \geq 1$ ,  $t \in [0, T]$ , we set

$$\hat{X}_t^{n+1} := X_t^{n+1} - X_t^n, \quad \hat{Y}_t^{n+1} := Y_t^{n+1} - Y_t^n, \quad \hat{Z}_t^{n+1} := Z_t^{n+1} - Z_t^n, \quad \hat{K}_t^{n+1} := K_t^{n+1} - K_t^n$$

and for a function  $\phi = \{b, h, \sigma, \beta\}$ , we set

$$\hat{\phi}_t^{n+1} := \phi(t, U_t^{n+1}, \nu_t^n) - \phi(t, U_t^n, \nu_t^{n-1}), \quad \tilde{\phi}_t^n := \phi(t, U_t^n, \nu_t^n) - \phi(t, U_t^n, \nu_t^{n-1}).$$

We first apply Itô's formula to the product  $\hat{X}^{n+1} \hat{Y}^{n+1}$

$$\begin{aligned} \mathbb{E}[\hat{X}_T^{n+1} \hat{Y}_T^{n+1}] &= \mathbb{E}\left[\int_0^T \hat{Y}_s^{n+1} [\hat{b}_s^{n+1} - \delta(\hat{Y}_s^{n+1} - \hat{Y}_s^n)] ds\right] + \mathbb{E}\left[\int_0^T \hat{Y}_s^{n+1} [\hat{\sigma}_s^{n+1} - \delta(\hat{Z}_s^{n+1} - \hat{Z}_s^n)] dW_s\right] \\ &\quad + \mathbb{E}\left[\int_0^T \int_E \hat{Y}_s^{n+1} [\hat{\beta}_s^{n+1} - \delta(\hat{K}_s^{n+1}(e) - \hat{K}_s^n(e))] \tilde{\pi}(de, ds)\right] + \mathbb{E}\left[\int_0^T \hat{X}_s^{n+1} \hat{h}_s^{n+1} ds\right] \\ &\quad - \mathbb{E}\left[\int_0^T \hat{X}_s^{n+1} \hat{Z}_s^{n+1} dW_s\right] - \mathbb{E}\left[\int_0^T \int_E \hat{X}_s^{n+1} \hat{K}_s^{n+1}(e) \tilde{\pi}(de, ds)\right] \\ &\quad + \mathbb{E}\left[\int_0^T (\hat{\sigma}_s^{n+1} - \delta(\hat{Z}_s^{n+1} - \hat{Z}_s^n), \hat{Z}_s^{n+1}) ds\right] + \mathbb{E}\left[\int_0^T \int_E \hat{K}_s^{n+1} (\hat{\beta}_s^{n+1} - \delta(\hat{K}_s^{n+1} - \hat{K}_s^n)) \eta(de, ds)\right]. \end{aligned}$$

Using the BDG inequality, we can easily see that the stochastic integrals in the above expres-

sion are a true martingale. Hence we obtain

$$\begin{aligned} \mathbb{E}[\hat{X}_T^{n+1}\hat{Y}_T^{n+1}] &= \mathbb{E}\left[\int_0^T \hat{Y}_s^{n+1}[\hat{b}_s^{n+1} - \delta(\hat{Y}_s^{n+1} - \hat{Y}_s^n)]ds\right] + \mathbb{E}\left[\int_0^T \hat{X}_s^{n+1}\hat{h}_s^{n+1}ds\right] \\ &+ \mathbb{E}\left[\int_0^T (\hat{\sigma}_s^{n+1} - \delta(\hat{Z}_s^{n+1} - \hat{Z}_s^n), \hat{Z}_s^{n+1})ds\right] + \mathbb{E}\left[\int_0^T \int_E \hat{K}_s^{n+1}(\hat{\beta}_s^{n+1} - \delta(\hat{K}_s^{n+1} - \hat{K}_s^n))\eta(de, ds)\right]. \end{aligned}$$

Rearranging terms, we get

$$\begin{aligned} \delta\mathbb{E}\left[\int_0^T \hat{Y}_s^{n+1}\hat{Y}_s^n ds + \int_0^T \hat{Z}_s^{n+1}\hat{Z}_s^n ds + \int_0^T \int_E \hat{K}_s^{n+1}\hat{K}_s^n\eta(de, ds)\right] &= \mathbb{E}[\hat{X}_T^{n+1}\hat{Y}_T^{n+1}] \\ &- \mathbb{E}\left[\int_0^T \hat{X}_s^{n+1}\hat{h}_s^{n+1} + \hat{Y}_s^{n+1}\hat{b}_s^{n+1} + \hat{Z}_s^{n+1}\hat{\sigma}_s^{n+1}\right]ds + \int_0^T \int_E \hat{K}_s^{n+1}\hat{\beta}_s^{n+1}\eta(de, ds). \\ &+ \delta\mathbb{E}\left[\int_0^T |\hat{Y}_s^{n+1}|^2 + \|\hat{Z}_s^{n+1}\|^2 + |\hat{K}_s^{n+1}|_s^2 ds\right]. \end{aligned} \quad (4.17)$$

Since  $\hat{X}_T^{n+1}\hat{Y}_T^{n+1} = \hat{X}_T^{n+1}[g(X_T^{n+1}, \mu_T^n) - g(X_T^n, \mu_T^{n-1})]$ , we have from **(H1)**

$$\begin{aligned} \hat{X}_T^{n+1}\hat{Y}_T^{n+1} &= \hat{X}_T^{n+1}[g(X_T^{n+1}, \mu_T^n) - g(X_T^n, \mu_T^{n-1})] \\ &= \hat{X}_T^{n+1}[g(X_T^{n+1}, \mu_T^n) - g(X_T^n, \mu_T^n)] + \hat{X}_T^{n+1}[g(X_T^n, \mu_T^n) - g(X_T^n, \mu_T^{n-1})] \\ &\geq k'|\hat{X}_T^n|^2 - C_g^\nu|\hat{X}_T^n|\mathcal{W}_2(\mu_T^n, \mu_T^{n-1}). \end{aligned}$$

Using (4.2) and the elementary inequality :  $\forall \epsilon > 0, 2ab \leq \epsilon^{-1}a^2 + \epsilon b^2$ , we obtain

$$\mathbb{E}[\hat{X}_T^{n+1}\hat{Y}_T^{n+1}] \geq (k' - \frac{C_g^\nu \epsilon}{2})\mathbb{E}[|\hat{X}_T^{n+1}|^2] - \frac{C_g^\nu}{2\epsilon}\mathbb{E}[|\hat{X}_T^n|^2]. \quad (4.18)$$

In the other hand, using once again **(H1)**

$$\begin{aligned} &\int_0^T [\hat{X}_s^{n+1}\hat{h}_s^{n+1} + \hat{Y}_s^{n+1}\hat{b}_s^{n+1} + \hat{Z}_s^{n+1}\hat{\sigma}_s^{n+1}]ds + \int_0^T \int_E \hat{K}_s^{n+1}\hat{\beta}_s^{n+1}\eta(de, ds) \\ &= \int_0^T [\mathcal{A}(s, U_s^{n+1}, U_s^n, \nu^n) + \hat{Y}_s^{n+1}\bar{b}_s^n + \hat{X}_s^{n+1}\bar{h}_s^n + \hat{Z}_s^{n+1}\bar{\sigma}_s^n]ds + \int_0^T \int_E \hat{K}_s^{n+1}\bar{\beta}_s^n\eta(de, ds). \\ &\leq -k \int_0^T |\hat{X}_s^{n+1}|^2 ds + \int_0^T [C_h^\nu|\hat{X}_s^{n+1}| + C_b^\nu|\hat{Y}_s^{n+1}| + C_\sigma^\nu\|\hat{Z}_s^{n+1}\| + C_\beta^\nu|\hat{K}_s^{n+1}|_s]\mathcal{W}_2(\nu^n, \nu^{n-1})ds. \end{aligned}$$

Using Young inequality :  $\forall \tilde{\epsilon} > 0, 2ab \leq \tilde{\epsilon}^{-1}a^2 + \tilde{\epsilon}b^2$ , we obtain

$$\begin{aligned} &\int_0^T [\hat{X}_s^{n+1}\hat{h}_s^{n+1} + \hat{Y}_s^{n+1}\hat{b}_s^{n+1} + \hat{Z}_s^{n+1}\hat{\sigma}_s^{n+1}]ds + \int_0^T \int_E \hat{K}_s^{n+1}\hat{\beta}_s^{n+1}\eta(de, ds) \\ &\leq (\frac{\tilde{\epsilon}C_h^\nu}{2} - k) \int_0^T |\hat{X}_s^{n+1}|^2 ds + \frac{\tilde{\epsilon}}{2} \int_0^T (C_b^\nu|\hat{Y}_s^{n+1}|^2 + C_\sigma^\nu\|\hat{Z}_s^{n+1}\|^2 + C_\beta^\nu|\hat{K}_s^{n+1}|_s^2)ds \\ &+ \frac{C_h^\nu + C_b^\nu + C_\sigma^\nu + C_\beta^\nu}{2\tilde{\epsilon}}\mathcal{W}_2^2(\nu_s^n, \nu_s^{n-1}). \end{aligned}$$

Notice that  $W_2^2(\nu_s^n, \nu_s^{n-1}) \leq \mathbb{E}[|\hat{X}_s^n|^2 + |\hat{Y}_s^n|^2]$ . Hence, taking the conditional expectation in the expression above, we obtain

$$\begin{aligned} & \mathbb{E}\left[\int_0^T [\hat{X}_s^{n+1}\hat{h}^{n+1}(s) + \hat{Y}_s^{n+1}\hat{b}^{n+1}(s) + \hat{Z}_s^{n+1}\hat{\sigma}^{n+1}(s)]ds + \int_0^T \int_E \hat{K}_s^{n+1}\hat{\beta}_s^{n+1}\eta(de, ds)\right] \\ & \leq \left(\frac{\tilde{\epsilon}C_h^\nu}{2} - k\right)\mathbb{E}\left[\int_0^T |\hat{X}_s^{n+1}|^2 ds\right] + \frac{\tilde{\epsilon}}{2}\mathbb{E}\left[\int_0^T (C_b^\nu|\hat{Y}_s^{n+1}|^2 + C_\sigma^\nu\|\hat{Z}_s^{n+1}\|^2 + C_\beta^\nu|\hat{K}_s^{n+1}|_s^2)ds\right] \\ & \quad + \frac{C_h^\nu + C_b^\nu + C_\sigma^\nu + C_\beta^\nu}{2\tilde{\epsilon}}\mathbb{E}\left[\int_0^T (|\hat{X}_s^n|^2 + |\hat{Y}_s^n|^2)ds\right]. \end{aligned} \quad (4.19)$$

In addition,

$$\begin{aligned} & \mathbb{E}\left[\int_0^T (\hat{Y}_s^{n+1}\hat{Y}_s^n + \hat{Z}_s^{n+1}\hat{Z}_s^n)ds + \int_0^T \int_E \hat{K}_s^{n+1}\hat{K}_s^n\eta(de, ds)\right] \\ & \leq \frac{\kappa}{2}\mathbb{E}\left[\int_0^T |\hat{Y}_s^{n+1}|^2 + \|\hat{Z}_s^{n+1}\|^2 + |\hat{K}_s^{n+1}|_s^2 ds\right] + \frac{1}{2\kappa}\mathbb{E}\left[\int_0^T |\hat{Y}_s^n|^2 + \|\hat{Z}_s^n\|^2 + |\hat{K}_s^n|_s^2 ds\right]. \end{aligned} \quad (4.20)$$

Plugging (4.18), (4.19) and (4.20) in (4.17) we obtain

$$\begin{aligned} & \left(k' - \frac{C_g^\nu\epsilon}{2}\right)\mathbb{E}[|\hat{X}_T^{n+1}|^2] - \frac{C_g^\nu}{2\epsilon}\mathbb{E}[|\hat{X}_T^n|^2] + \delta\mathbb{E}\left[\int_0^T |\hat{Y}_s^{n+1}|^2 + \|\hat{Z}_s^{n+1}\|^2 + |\hat{K}_s^{n+1}|_s^2 ds\right] \\ & \quad + \left(-\frac{\tilde{\epsilon}C_h^\nu}{2} + k\right)\mathbb{E}\left[\int_0^T |\hat{X}_s^{n+1}|^2 ds\right] - \frac{\tilde{\epsilon}}{2}\mathbb{E}\left[\int_0^T C_b^\nu|\hat{Y}_s^{n+1}|^2 + C_\sigma^\nu\|\hat{Z}_s^{n+1}\|^2 + C_\beta^\nu|\hat{K}_s^{n+1}|_s^2 ds\right] \\ & \quad - \frac{C_h^\nu + C_b^\nu + C_\sigma^\nu + C_\beta^\nu}{2\epsilon}\mathbb{E}\left[\int_0^T |\hat{X}_s^n|^2 + |\hat{Y}_s^n|^2 ds\right] \\ & \leq \frac{\delta\kappa}{2}\mathbb{E}\left[\int_0^T |\hat{Y}_s^{n+1}|^2 + \|\hat{Z}_s^{n+1}\|^2 + |\hat{K}_s^{n+1}|_s^2 ds\right] + \frac{\delta}{2\kappa}\mathbb{E}\left[\int_0^T |\hat{Y}_s^n|^2 + \|\hat{Z}_s^n\|^2 + |\hat{K}_s^n|_s^2 ds\right]. \end{aligned}$$

Rearranging terms we get

$$\begin{aligned} & \left(k' - \frac{C_g^\nu\epsilon}{2}\right)\mathbb{E}[|\hat{X}_T^{n+1}|^2] + \left(k - \frac{\tilde{\epsilon}C_h^\nu}{2}\right)\mathbb{E}\left[\int_0^T |\hat{X}_s^{n+1}|^2 ds\right] + \left(\delta - \frac{\kappa\delta}{2} - \frac{\tilde{\epsilon}C_f^\nu}{2}\right)\mathbb{E}\left[\int_0^T |\hat{Y}_s^{n+1}|^2 ds\right] \\ & \quad + \left(\delta - \frac{\kappa\delta}{2} - \frac{\tilde{\epsilon}C_\sigma^\nu}{2}\right)\mathbb{E}\left[\int_0^T \|\hat{Z}_s^{n+1}\|^2 ds\right] + \left(\delta - \frac{\kappa\delta}{2} - \frac{\tilde{\epsilon}C_\beta^\nu}{2}\right)\mathbb{E}\left[\int_0^T |\hat{K}_s^{n+1}|_s^2 ds\right] \\ & \leq \frac{C_g^\nu}{2\epsilon}\mathbb{E}[|\hat{X}_T^n|^2] + \frac{C_h^\nu + C_b^\nu + C_\sigma^\nu + C_\beta^\nu}{2\epsilon}\mathbb{E}\left[\int_0^T |\hat{X}_s^n|^2 ds\right] \\ & \quad + \frac{\delta}{2\kappa}\mathbb{E}\left[\int_0^T \|\hat{Z}_s^n\|^2 + |\hat{K}_s^n|_s^2 ds\right] + \left(\frac{C_h^\nu + C_b^\nu + C_\sigma^\nu + C_\beta^\nu}{2\epsilon} + \frac{\delta}{2\kappa}\right)\mathbb{E}\left[\int_0^T |\hat{Y}_s^n|^2 ds\right]. \end{aligned}$$

Setting

$$\begin{cases} \gamma := \min\left(k' - \frac{C_g^\nu\epsilon}{2}, k - \frac{\tilde{\epsilon}C_h^\nu}{2}, \left(\delta - \frac{\kappa\delta}{2} - \frac{\tilde{\epsilon}C_b^\nu}{2}\right), \left(\delta - \frac{\kappa\delta}{2} - \frac{\tilde{\epsilon}C_\sigma^\nu}{2}\right), \left(\delta - \frac{\kappa\delta}{2} - \frac{\tilde{\epsilon}C_\beta^\nu}{2}\right)\right) \\ \theta = \max\left(\frac{C_g^\nu}{2\epsilon}, -\frac{C_h^\nu + C_b^\nu + C_\sigma^\nu + C_\beta^\nu}{2\epsilon} + \frac{\delta}{2\kappa}\right), \end{cases} \quad (4.21)$$

we obtain that

$$\mathbb{E}[|\hat{X}_T^{n+1}|^2] + \mathbb{E}\left[\int_0^T \|\hat{U}_s^{n+1}\|^2 ds\right] \leq \frac{\theta}{\gamma} (\mathbb{E}[|\hat{X}_T^n|^2] + \mathbb{E}\left[\int_0^T \|\hat{U}_s^n\|^2 ds\right]). \quad (4.22)$$

Choosing  $\tilde{\epsilon}$  and  $\epsilon$  so that  $\theta < \gamma$ , the inequality becomes a contraction. Thus,  $(\hat{X}_T^n)_{n \geq 0}$  is a Cauchy sequence in  $\mathbb{H}^2(\Omega, \mathbb{P})$  and  $(\hat{X}^n)_{n \geq 0}, (\hat{Y}^n)_{n \geq 0}, (\hat{Z}^n)_{n \geq 0}$  and  $(\hat{K}^n)_{n \geq 0}$  are Cauchy sequences respectively in  $\mathbb{H}^2([0, T], \Omega, dt \otimes d\mathbb{P})$  and  $\mathbb{H}_\eta^2([0, T], \Omega, dt \otimes d\eta)$ . Hence, if  $X, Y, Z$  and  $K$  are the respective limits of these sequences, passing to the limit in (4.16), we see that  $(X, Y, Z, K)$  is a solution of (4.3).

### 4.3.2 Existence and uniqueness under (H2)

Our second main result is an extension to the case where the datas satisfy a weaker monotonicity assumptions. We adopt here a common strategy which is the Picard approach: we construct a schema based on small perturbation. This helps us to construct the contracting maps and therefore deduce the existence of a unique solution of the system **(S)**. Consider the following assumption

$$(H2) \left\{ \begin{array}{l} (i) \text{ There exists } k > 0, \text{ s.t } \forall t \in [0, T], \nu \in \mathcal{M}_2(\mathbb{R}^d \times \mathbb{R}^d), u, u' \in \mathbb{R}^{d+d+d \times d} \times \mathbb{L}^0(\mathcal{B}(E), \eta), \\ \quad A(t, u, u', \nu) \leq -k(|y - y'|^2 + \|z - z'\|^2 + |k - k'|_s), \mathbb{P}\text{-a.s.} \\ (ii) \text{ There exists } k' > 0, \text{ s.t } \forall \nu \in \mathcal{M}_2(\mathbb{R}^d \times \mathbb{R}^d), x, x' \in \mathbb{R}^d \\ \quad (g(x, \nu) - g(x', \nu)) \cdot (x - x') \geq k'|x - x'|^2, \mathbb{P}\text{-a.s.} \end{array} \right.$$

As in the previous section we will give a useful a priori estimate.

**Lemma 4.3.5.** *Let  $(Y', Z', K')$  another solution of the the system **(S)**. Then, under **(H2)** we have the following estimates*

$$\mathbb{E}\left[\int_0^T |\Delta X_s|^2 ds\right] \leq \frac{\exp(T \cdot \Upsilon^1) - \Upsilon^1}{\Upsilon^1} [\Upsilon^2 \mathbb{E}\left[\int_0^T |\Delta Y_s|^2 ds\right] + \Upsilon^3 \mathbb{E}\left[\int_0^T |\Delta Z_s|^2 ds\right] + \Upsilon^4 \mathbb{E}\left[\int_0^T |\Delta K_s|_s^2 ds\right],$$

where

$$\left\{ \begin{array}{l} \Upsilon^1 := 3 + 2C_b^x + 5(C_\sigma^x)^2 + 5(C_\beta^x)^2 + 2C_b^y + 5(C_\sigma^y)^2 + 5(C_\beta^y)^2 \\ \Upsilon^2 := (C_b^y)^2 + 5(C_\sigma^y)^2 + 5(C_\beta^y)^2 + C_b^z + 5(C_\sigma^z)^2 + 5(C_\beta^z)^2 \\ \Upsilon^3 := (C_b^z)^2 + 5(C_\sigma^z)^2 + 5(C_\beta^z)^2 \\ \Upsilon^4 := (C_b^k)^2 + 5(C_\sigma^k)^2 + 5(C_\beta^k)^2, \end{array} \right.$$

**Proof.** Applying Itô formula to  $|\Delta X|^2$ , we compute using the Lipschitz assumption

$$\begin{aligned} \mathbb{E}[|\Delta X_t|^2] &\leq 2\mathbb{E}\left[\int_0^t |\Delta X_s|(C_b^x|\Delta X_s| + C_b^y|\Delta Y_s| + C_b^z\|\Delta Z_s\| + C_b^k|\Delta K_s|_{\mathcal{L}^2(\eta)} + C_b^\nu\mathcal{W}_2(\nu'_s, \nu_s))ds\right] \\ &+ 5\mathbb{E}\left[\int_0^t [(C_\sigma^x)^2|\Delta X_s|^2 + (C_\sigma^y)^2|\Delta Y_s|^2 + (C_\sigma^z)^2\|\Delta Z_s\|^2 + (C_\sigma^k)^2|\Delta K_s|_{\mathcal{L}^2(\eta)}^2 + (C_\sigma^\nu)^2\mathcal{W}_2^2(\nu'_s, \nu_s)]ds\right] \\ &+ 5\mathbb{E}\left[\int_0^t [(C_\beta^x)^2|\Delta X_s|^2 + (C_\beta^y)^2|\Delta Y_s|^2 + (C_\beta^z)^2\|\Delta Z_s\|^2 + (C_\beta^k)^2|\Delta K_s|_{\mathcal{L}^2(\eta)}^2 + (C_\beta^\nu)^2\mathcal{W}_2^2(\nu'_s, \nu_s)^2]\right] \end{aligned}$$

Then, we apply Young inequality and we obtain

$$\begin{aligned} \mathbb{E}[|\Delta X_t|^2] &\leq \left(3 + 2C_b^x + 5(C_\sigma^x)^2 + 5(C_\beta^x)^2 + 2C_b^\nu + 5(C_\sigma^\nu)^2 + 5(C_\beta^\nu)^2\right)\mathbb{E}\left[\int_0^t |\Delta X_s|^2 ds\right] \\ &+ \left((C_b^y)^2 + 5(C_\sigma^y)^2 + 5(C_\beta^y)^2 + C_b^\nu + 5(C_\sigma^\nu)^2 + 5(C_\beta^\nu)^2\right)\mathbb{E}\left[\int_0^t |\Delta Y_s|^2 ds\right] \\ &+ \left((C_b^z)^2 + 5(C_\sigma^z)^2 + 5(C_\beta^z)^2\right)\mathbb{E}\left[\int_0^t \|\Delta Z_s\|^2 ds\right] + \left((C_b^k)^2 + 5(C_\sigma^k)^2 + 5(C_\beta^k)^2\right)\mathbb{E}\left[\int_0^t |\Delta K_s|_s^2 ds\right]. \end{aligned}$$

Thus, taking  $\Upsilon^1$ ,  $\Upsilon^2$ ,  $\Upsilon^3$ , and  $\Upsilon^4$  as in (4.23) and applying Gronwall lemma imply

$$\mathbb{E}[|\Delta X_t|^2] \leq \frac{\exp(t\Upsilon^1) - \Upsilon^1}{\Upsilon^1} \left[ \Upsilon^2 \mathbb{E}\left[\int_0^t |\Delta Y_s|^2 ds\right] + \Upsilon^3 \mathbb{E}\left[\int_0^t \|\Delta Z_s\|^2 ds\right] + \Upsilon^4 \mathbb{E}\left[\int_0^t |\Delta K_s|_s^2 ds\right] \right],$$

which gives the desired result.

**Proposition 4.3.6.** *Under Assumption (H2), there exists a unique solution  $(X, Y, Z, K)$  of the FBSDE with jumps system (4.3).*

**Proof.** Let  $U = (X, Y, Z, K)$  and  $U' = (X', Y', Z', K')$  be two solutions of the mean-field FBSDE with jumps system (S). Using the same notation as in Proposition 4.3.3, We have as proved earlier in (4.13)

$$\Gamma_T \geq (k' - C_g^\nu)\mathbb{E}[|\Delta X_T|^2]. \quad (4.23)$$

On the other hand, using (H2) and the Lipschitz assumption, we compute

$$\begin{aligned} \Gamma_T &\leq \mathbb{E}\left[-k \int_0^T (|\Delta Y_s|^2 + \|\Delta Z_s\|^2 + |\Delta K_s|^2) ds + \int_0^T [C_h^\nu + \frac{1}{2}(C_b^\nu + C_\sigma^\nu + C_\beta^\nu)]|\Delta X_s|^2 ds\right] \\ &+ \mathbb{E}\left[\int_0^T [C_b^\nu + \frac{1}{2}(C_h^\nu + C_\sigma^\nu + C_\beta^\nu)]|\Delta Y_s|^2 ds + C_\sigma^\nu \int_0^T \|\Delta Z_s\|^2 ds + C_\beta^\nu \int_0^T |\Delta K_s|_s^2 ds\right]. \end{aligned}$$

Combining (4.3.5) and (4.23) we obtain

$$\begin{aligned}
 & (k' - C_g^\nu) \mathbb{E}[|\Delta X_T|^2] + k \mathbb{E} \left[ \int_0^T (|\Delta Y_s|^2 + |\Delta Z_s|^2 + |\Delta K_s|_s^2) ds \right] \\
 & \leq \left[ \Upsilon^\nu \frac{\exp(t \cdot \Upsilon^1) - \Upsilon^1}{\Upsilon^1} \Upsilon^2 + (C_b^\nu + \frac{1}{2}(C_h^\nu + C_\sigma^\nu + C_\beta^\nu)) \right] \mathbb{E} \left[ \int_0^t |\Delta Y_s|^2 ds \right] \\
 & + \left[ \Upsilon^\nu \frac{\exp(t \cdot \Upsilon^1) - \Upsilon^1}{\Upsilon^1} \Upsilon^3 + \frac{C_\sigma^\nu}{2} \right] \mathbb{E} \left[ \int_0^t |\Delta Z_s|^2 ds \right] \\
 & + \left[ \Upsilon^\nu \frac{\exp(t \cdot \Upsilon^1) - \Upsilon^1}{\Upsilon^1} \Upsilon^4 + \frac{C_\beta^\nu}{2} \right] \mathbb{E} \left[ \int_0^t |\Delta K_s|_s^2 ds \right],
 \end{aligned}$$

where  $\Upsilon^\nu := [C_h^\nu + \frac{1}{2}(C_b^\nu + C_\sigma^\nu + C_\beta^\nu)]$ . Choosing the Lipschitz constants small enough to obtain

$$\begin{aligned}
 k & > \left[ \Upsilon^\nu \frac{\exp(t \cdot \Upsilon^1) - \Upsilon^1}{\Upsilon^1} \Upsilon^2 + (C_b^\nu + \frac{1}{2}(C_h^\nu + C_\sigma^\nu + C_\beta^\nu)) \right] \\
 k & > \left[ \Upsilon^\nu \frac{\exp(t \cdot \Upsilon^1) - \Upsilon^1}{\Upsilon^1} \Upsilon^3 + \frac{C_\sigma^\nu}{2} \right], \quad k > \left[ \Upsilon^\nu \frac{\exp(t \cdot \Upsilon^1) - \Upsilon^1}{\Upsilon^1} \Upsilon^4 + \frac{C_\beta^\nu}{2} \right],
 \end{aligned}$$

and  $k' - C_g^\nu > 0$ . Thus, we have

$$(k' - C_g^\nu) \mathbb{E}[|\Delta X_T|^2] + k \int_0^T (|Y'_s - Y_s|^2 + \|Z'_s - Z_s\|^2 + |K'_s - K_s|_s^2) ds \leq 0.$$

This implies that  $X'_T = X_T$  and for all  $t \in [0, T]$ ,  $X'_t = X_t$ ,  $Y'_t = Y_t$ ,  $Z'_t = Z_t$  and  $K'_t = K_t$ ,  $\mathbb{P}$ -a.s which gives the desired result.

**Theorem 4.3.7.** *Under Assumption (H2), there exists a solution  $(X, Y, Z, K)$  of the FBSDE with jumps (4.3).*

**Proof.** Following the same approach as in Proposition 4.3.4, we use an approximation scheme based on perturbation. However, perturbations here are made in the backward SDE with jumps. Let  $\delta > 0$  and consider a sequence  $(X^n, Y^n, Z^n, K^n)$  of processes defined recursively by :  $(X^0, Y^0, Z^0, K^0) = (0, 0, 0, 0)$  and for  $n \geq 1$ ,  $U^n = (X^n, Y^n, Z^n, K^n)$  satisfies

$$\left\{ \begin{aligned}
 X_t^{n+1} &= X_0 + \int_0^t b_s(U_s^{n+1}, \nu_s^n) ds + \int_0^t \sigma_s(U_s^{n+1}, \nu_s^n) dW_s + \int_0^t \int_E \beta_s(U_s^{n+1}, \nu_s^n) \tilde{\pi}(ds, de), \\
 Y_t^{n+1} &= g(X_T^{n+1}, \mu_T^n) + \delta(X_T^{n+1} - X_T^n) - \int_t^T [h_s(U_s^{n+1}, \nu_s^n) + \delta(X_s^{n+1} - X_s^n)] ds \\
 &\quad - \int_t^T Z_s^{n+1} dW_s - \int_t^T \int_E K_s^{n+1}(e) \tilde{\pi}(ds, de).
 \end{aligned} \right. \tag{4.24}$$

with  $\mu_T^n = \mathbb{P}_{X_T^n}$ ,  $\nu_t^n = \mathbb{P}_{(X_t^n, Y_t^n)}$ .

We keep the same notation as in Theorem 4.3.4 and we apply Itô formula to the product

$$\hat{X}_s^{n+1}\hat{Y}_s^{n+1}.$$

$$\begin{aligned} \hat{X}_T^{n+1}\hat{Y}_T^{n+1} - \hat{X}_0^{n+1}\hat{Y}_0^{n+1} &= \int_0^T \hat{Y}_s^{n+1}\hat{b}_s^{n+1}ds + \int_0^T \hat{Y}_s^{n+1}\hat{\sigma}_s^{n+1}dW_s \\ &+ \int_0^T \hat{X}_s^{n+1}[\hat{h}_s^{n+1} - \delta(\hat{X}_s^{n+1} - \hat{X}_s^n)]ds - \int_0^T \hat{X}_s^{n+1}\hat{Z}_s^{n+1}dW_s - \int_0^T \int_E \hat{X}_s^{n+1}\hat{K}_s^{n+1}(e)\tilde{\pi}(de, ds) \\ &+ \int_0^T (\hat{\sigma}_s^{n+1}, \hat{Z}_s^{n+1})ds + \int_0^T \int_E \hat{K}_s^{n+1}\hat{\beta}_s^{n+1}\eta(de, ds) + \int_0^T \int_E \hat{Y}_s^{n+1}\hat{\beta}_s^{n+1}\tilde{\pi}(de, ds). \end{aligned} \quad (4.25)$$

Notice that, since the terminal condition is given by

$$\hat{Y}_T^{n+1} = [g(X_T^{n+1}, \mu_T^n) - g(X_T^n, \mu_T^{n-1})] + \delta(X_T^{n+1} - X_T^n),$$

we rewrite the above equation as follows

$$\begin{aligned} &\mathbb{E}[\hat{X}_s^{n+1}(g(X_T^{n+1}, \mu_T^n) - g(X_T^n, \mu_T^{n-1})) + \delta(|\hat{X}_T^{n+1}|^2 - \hat{X}_T^{n+1}\hat{X}_T^n)] \\ &= \mathbb{E}[\int_0^T \hat{Y}_s^{n+1}\hat{b}_s^{n+1}ds + \int_0^T \hat{X}_s^{n+1}\hat{h}_s^{n+1}ds + \int_0^T (\hat{\sigma}_s^{n+1}, \hat{Z}_s^{n+1})ds] \\ &+ \mathbb{E}[\int_0^T \int_E \hat{K}_s^{n+1}\hat{\beta}_s^{n+1}\eta(de, ds) - \delta(\int_0^T |\hat{X}_s^{n+1}|^2ds - \int_0^T \hat{X}_s^{n+1}\hat{X}_s^n ds)]. \end{aligned}$$

Using Assumption **(H2)** and Young's inequality, we obtain

$$\begin{aligned} \mathbb{E}[\hat{X}_T^{n+1}(g(X_T^{n+1}, \mu_T^n) - g(X_T^n, \mu_T^{n-1}))] &= \mathbb{E}[\hat{X}_T^{n+1}(g(X_T^{n+1}, \mu_T^n) - g(X_T^n, \mu_T^n))] \\ &+ \mathbb{E}[\hat{X}_T^{n+1}(g(X_T^n, \mu_T^n) - g(X_T^n, \mu_T^{n-1}))] \\ &\geq -C_g^\nu \mathbb{E}[|\hat{X}_T^{n+1}|] \mathcal{W}_2(\mu_T^n, \mu_T^{n-1}) + k' \mathbb{E}[|\hat{X}_T^{n+1}|^2] \\ &\geq (k' - \frac{C_g^\nu \epsilon}{2}) \mathbb{E}[|\hat{X}_T^{n+1}|^2] - \frac{2C_g^\nu}{\epsilon} \mathcal{W}_2^2(\mu_T^n, \mu_T^{n-1}) \\ &\geq (k' - \frac{C_g^\nu \epsilon}{2}) \mathbb{E}[|\hat{X}_T^{n+1}|^2] - \frac{2C_g^\nu}{\epsilon} \mathbb{E}[|\hat{X}_T^n|^2]. \end{aligned} \quad (4.26)$$

Besides, classical linearization technics imply that

$$\begin{aligned} &\mathbb{E}[\int_0^T [\hat{Y}_s^{n+1}\hat{b}_s^{n+1} + \hat{X}_s^{n+1}\hat{h}_s^{n+1}(\hat{\sigma}_s^{n+1}, \hat{Z}_s^{n+1})]ds + \int_0^T \int_E \hat{K}_s^{n+1}\hat{\beta}_s^{n+1}\eta(de, ds)] \\ &= \mathbb{E}[\int_0^T [\mathcal{A}(s, U_s^{n+1}, U_s^n, \nu_s^n) + \hat{Y}_s^{n+1}\bar{b}_s^n + \hat{X}_s^{n+1}\bar{h}_s^n + \hat{Z}_s^{n+1}\bar{\sigma}_s^n]ds \\ &+ \mathbb{E}[\int_0^T \int_E \hat{K}_s^{n+1}\bar{\beta}_s^n\eta(de, ds)]. \end{aligned}$$

Once again, Assumption **(H2)** and Young inequality give

$$\begin{aligned}
 & \mathbb{E} \left[ \int_0^T [\hat{Y}_s^{n+1} \hat{b}_s^{n+1} + \hat{X}_s^{n+1} \hat{h}_s^{n+1} + (\hat{\sigma}_s^{n+1}, \hat{Z}_s^{n+1})] ds + \int_0^T \int_E \hat{K}_s^{n+1} \hat{\beta}_s^{n+1} \eta(de, ds) \right] \\
 & \leq -k \mathbb{E} \left[ \int_0^T |\hat{Y}_s^{n+1}|^2 + \|\hat{Z}_s^{n+1}\|^2 + |\hat{K}_s^{n+1}|_s^2 ds \right] + \left( \frac{C_b^\nu \alpha}{2} - k \right) \mathbb{E} \left[ \int_0^T |\hat{Y}_s^{n+1}|^2 ds \right] \\
 & + \mathbb{E} \left[ \int_0^T [C_h^\nu |\hat{X}_s^{n+1}| + C_b^\nu |\hat{Y}_s^{n+1}| + C_\sigma^\nu \|\hat{Z}_s^{n+1}\| + C_\beta^\nu |\hat{K}_s^{n+1}|_{\mathcal{L}^2(\eta)}] \mathcal{W}_2(\nu_s^n, \nu_s^{n-1}) ds \right] \\
 & \leq \frac{C_h^\nu \alpha}{2} \mathbb{E} \left[ \int_0^T |\hat{X}_s^{n+1}|^2 ds \right] + \frac{C_h^\nu + C_b^\nu + C_\sigma^\nu + C_\beta^\nu}{2\alpha} \mathbb{E} [ (|\hat{X}_s^n|^2 + |\hat{Y}_s^n|^2) ds ] \\
 & + \left( \frac{C_\sigma^\nu \alpha}{2} - k \right) \mathbb{E} \left[ \int_0^T \|\hat{Z}_s^{n+1}\|^2 ds \right] + \left( \frac{C_\beta^\nu \alpha}{2} - k \right) \mathbb{E} \left[ \int_0^T |\hat{K}_s^{n+1}|_s^2 ds \right]. \tag{4.27}
 \end{aligned}$$

Therefore, we obtain from (4.26) and (4.27)

$$\begin{aligned}
 & \mathbb{E} \left[ \left( k' - \frac{c_g^\nu \epsilon}{2} + \frac{\delta}{2} \right) |\hat{X}_T^{n+1}|^2 + \int_0^T \left( -\frac{C_h^\nu \alpha}{2} + \delta - \frac{\delta \rho}{2} \right) |\hat{X}_s^{n+1}|^2 ds + \int_0^T \left( \frac{C_b^\nu \alpha}{2} - k \right) |\hat{Y}_s^{n+1}|^2 ds \right] \\
 & + \mathbb{E} \left[ \int_0^T \left( \frac{C_\sigma^\nu \alpha}{2} - k \right) |\hat{Z}_s^{n+1}|^2 ds \right] + \mathbb{E} \left[ \int_0^T \left( \frac{C_\beta^\nu \alpha}{2} - k \right) |\hat{K}_s^{n+1}|_s^2 ds \right] \\
 & \leq \mathbb{E} \left[ \left( \frac{2C_g^\nu}{\epsilon} + \frac{\delta}{2} \right) |\hat{X}_T^n|^2 \right] + \int_0^T \left( \frac{C_h^\nu + C_b^\nu + C_\sigma^\nu + C_\beta^\nu}{2\alpha} + \frac{\delta}{2\rho} \right) |\hat{X}_s^n|^2 ds \\
 & + \mathbb{E} \left[ \int_0^T \frac{C_h^\nu + C_b^\nu + C_\sigma^\nu + C_\beta^\nu}{2\alpha} |\hat{Y}_s^n|^2 ds \right].
 \end{aligned}$$

Henceforth,

$$\tilde{\gamma} \mathbb{E} [ |\hat{X}_T^{n+1}|^2 + \int_0^T \|\hat{U}_s^{n+1}\|^2 ds ] \leq \tilde{\theta} \mathbb{E} [ |\hat{X}_T^n|^2 + \int_0^T \|\hat{U}_s^n\|^2 ds ]. \tag{4.28}$$

Choosing  $\tilde{\epsilon}$ ,  $\alpha$  and  $\epsilon$  so that  $\frac{\tilde{\theta}}{\tilde{\gamma}} < 1$ , the inequality (4.28) becomes a contraction. Thus,  $(\hat{X}_T^n)_{n \geq 0}$  is a Cauchy sequence in  $\mathbb{H}^2(\Omega, \mathbb{P})$  and  $(\hat{X}^n)_{n \geq 0}$ ,  $(\hat{Y}^n)_{n \geq 0}$ ,  $(\hat{Z}^n)_{n \geq 0}$  and  $(\hat{K}^n)_{n \geq 0}$  are Cauchy sequences respectively in  $\mathbb{H}^2([0, T], \Omega, dt \otimes d\mathbb{P})$  and  $\mathbb{H}_\eta^2([0, T], \Omega, dt \otimes d\eta)$ . Hence, if  $X, Y, Z$  and  $K$  are the respective limits of these sequences, passing to the limit in (4.16), we see that  $(X, Y, Z, K)$  is a solution of (4.24).

## 4.4 Application: Storage problem

### 4.4.1 Description of the model

We consider a stylized model for a power grid with distributed local energy generation and storage. The grid connects  $N$  nodes indexed by  $i = 1, \dots, N$ . Each node is characterized by two state variables:

- The storage level  $S_t$  representing the total energy available in the storage device.
- The net power production of the energy (photovoltaic panels, diesel energy,..) that each nodes produces after all costs subtracted  $Q_t$ .

We assume that the nodes forming this grid can be partitioned in  $\Gamma$  different groups: the nodes in the same group  $\gamma$  share the same characteristics of local net power production and storage, yet these characteristics vary from one group to the other.

We denote by  $N_\gamma$  the number of nodes in group  $\gamma$  so that  $N = \sum_{\gamma=1}^{\Gamma} N_\gamma$  and we define  $\pi^\gamma = N_\gamma/N$  as the ratio of the population size of region  $\gamma$  to the whole population. We shall abusively write  $i \in \gamma$  to signify that the node  $i$  is in region  $\gamma$ .

In order to model the dynamics of the state variables, we consider a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  carrying  $N + 1$  Brownian motions  $B^0, B^1, \dots, B^N$  and a Poisson process defined on  $[0, T] \times \Omega \times \mathbb{R}^*$  to which is associated a counting measure  $\hat{N}(de, dt) = n(de)dt$ . We suppose that the predictable measure  $n(de)$  is positive, finite and satisfies the following integrability condition

$$\int_{\mathbb{R}^*} (1 \wedge |e|)^2 n(de) < \infty. \quad (4.29)$$

We also consider  $N + 1$  independant Poisson measures  $(N^0, N^1, \dots, N^N)$  a  $N$  independant identically distributed random variables  $x_0^i = (s_0^i, q_0^i)$  which are independant from  $B^0, N^0$  the  $B^i$  and the  $N^i$ . We denote by  $\mathcal{F} = \{\mathcal{F}_t\}$  the filtration defined by

$$\mathcal{F}_t = \sigma\{(s_0^i, q_0^i, q_0^0), B_s^0, B_s^i, N^0, N^i \text{ where } i = 1, \dots, N, s \leq t\},$$

and the filtration  $\mathcal{F}^0 = \{\mathcal{F}_t^0\}$  generated by  $B^0$  and  $N^0$ . We also denote by  $\mathcal{A}$  the set of  $\mathcal{F}$ -adapted real-valued processes  $a = \{a_t\}$  such that  $\mathbb{E}[\int_0^T |a_u|^2] < \infty$ . Let us now define the dynamics of the state variables.

- The power production of the energy  $Q_t^i$  of each node  $i \in \{1, \dots, N\}$  in the region  $\gamma$  at time  $t$  is modeled in the following way:

$$\begin{cases} dQ_t^i = \mu^\gamma(t, Q_t^i)dt + dM_t^i + dM_t^0 \\ Q_0^i = q_0^i, \end{cases} \quad (4.30)$$

where

$$\begin{cases} dM_t^i = \sigma^\gamma(t, Q_t^i)dB_t^i + \int_E \beta^\gamma(t, e, Q_{t-}^i) \tilde{N}^i(dt, de), \\ dM_t^0 = \sigma^0(t, Q_t^0)dB_t^0 + \int_E \beta^0(t, e, Q_{t-}^0) \tilde{N}^0(dt, de). \end{cases}$$

- The battery level  $S_t^i$  of the node  $i$  in the region  $\gamma$  is controlled through a storage action  $\alpha^{\gamma,i} \in \mathcal{A}$  according to

$$\begin{cases} S_t^i = S_0^i + \int_0^t \alpha_s^i ds, \\ 0 \leq S_t^i \leq S_{\max}. \end{cases} \quad (4.31)$$

The quantity  $Q_t^i - \alpha_t^i$  is the net injection of the node. It can be either positive or negative:

- If  $Q_t^i - \alpha_t^i$  is positive: It corresponds to electricity being sold from the node  $i$  to the grid.
- If  $Q_t^i - \alpha_t^i$  is negative: It corresponds to electricity being bought by the node  $i$  from the grid.

**Remark 4.4.1.** *In our framework, in contrast with the paper of Alasseur et al [1], we assume that the production of energy is unpredictable. This is due to its dependence on environmental conditions such as the sun, the speed of the wind which are intermittent and irregular which is traduced by including the jump component in our analyses. We will also assume that the storage level will be enforced by a constraint. In other words, we assume that there is a maximal level for which the battery can support.  $S_{\max}$  is the battery's maximum instantaneous power output.*

As in [1], we include a micro grid system indexed by 0 called the "rest of the world", which is characterized by one state variable, its *local net power production*  $Q_t^0$ , and which does not possess any storage. The net production of the rest of the world is given by

$$\begin{cases} dQ_t^0 = \mu^0(t, Q_t^0)dt + \sigma^0(t, Q_t^0)dB_t^0 + \int_E \beta^0(t, e, Q_{t-}^0) \tilde{N}^0(dt, de), \\ Q_0^0 = q_0^0. \end{cases} \quad (4.32)$$

In our model  $B_0$  and  $\tilde{N}^0$  represent a common signal which affects the energy demand of the whole grid. Then for each  $i$ ,  $\sigma^\gamma : \mathbb{R} \rightarrow \mathbb{R}$  and  $\beta^\gamma$  are given functions which allow to model how the node  $i$  of region  $\gamma$  is affected by the common signal  $B_t^0$  and  $N_t^0$ . We assume that the rest of the world is only affected by this common signal  $B_t^0$  and  $N_t^0$ .

### **Electricity spot price**

We make the assumption that the electricity price per Watt-hour depends on the instantaneous demand. When the strategy  $\alpha = (\alpha^1, \dots, \alpha^N) \in \mathcal{A}^N$  is implemented, the spot price is given by

$$P_t^{N,\alpha} = p \left( -Q_t^0 - \sum_{i=1}^N \eta(Q_t^i - \alpha_t^i) \right), \quad (4.33)$$

where  $p(\cdot)$  is the exogenous inverse demand function for electricity and  $\eta$  is a scaling parameter which weights the contribution of each individual node  $i$  to the whole system.

We assume that  $p$  is a strictly increasing function. Since the energy model here is given through a macro grid system connecting a large number of small nodes  $i$ , we shall consider the limit when  $N \rightarrow \infty$  and  $\eta \rightarrow 0$ . Here we assume that  $\eta = \frac{1}{N}$ . So, the spot price can be written as follows

$$P_t^{N,\alpha} = p \left( -Q_t^0 - \sum_{i=1}^N \frac{1}{N} (Q_t^i - \alpha_t^i) \right), \quad (4.34)$$

where  $\frac{1}{N} \sum_{i=1}^N (Q_t^i - \alpha_t^i)$  is the averaged net injections.

### The control problem

We consider a finite time horizon  $T > 0$ . When the control action  $\alpha = (\alpha^1, \dots, \alpha^N)$  is implemented, the cost incurred at the node  $i$  in the region  $\gamma$  is given by

$$J^{i,\gamma,N}(\alpha) = \mathbb{E} \left[ \int_0^T P_t^{N,\alpha} \cdot (\alpha_t^i - Q_t^i) + L_T^\gamma(Q_t^i, \alpha_t^i) + L_S(S_t^{i,\alpha^i}, \alpha_t^i) dt + g(S_T^{i,\alpha^i}) \right], \quad (4.35)$$

where  $L_T^\gamma, L_S : \mathbb{R} \rightarrow \mathbb{R}$  and  $g : \mathbb{R} \rightarrow \mathbb{R}$  are continuous functions.

- $P_t^{N,\alpha}(\alpha_t^i - Q_t^i)$  represents the current volumetric cost (resp. profit) of electricity consumed (resp. produced) at the spot price  $P_t^{N,\alpha}$ .
- $L_S(S_t^{i,\alpha^i}, \alpha_t^i)$  represent the current and it is assumed to be identical in all the regions  $\gamma$ .
- $L_T^\gamma(Q_t^i, \alpha_t^i)$  is the volumetric charge. This electricity cost is closely related to the power that the system requires in peak hours and hence produce enough power to satisfy the highest level of peak demand.

The rest of the world incurs only energy and transmission costs

$$J^{0,N}(\alpha) = \mathbb{E} \left[ \int_0^T -P_t^{N,\alpha} \cdot Q_t^0 + L_T^0(Q_t^0, 0) dt \right]. \quad (4.36)$$

### Central Planner control problem

The central planner aims to dictate a storage rule:  $\alpha = (\alpha^1, \dots, \alpha^N)$  in order to minimize the *egalitarian* cost function between the nodes and the rest of the world

$$J^{C,N}(\alpha) = J^{0,N}(\alpha) + \sum_{i=1}^N \frac{1}{N} J^{i,\gamma,N}(\alpha).$$

where  $1/N$  is the scaling parameter which weights the contribution of each individual node to

the system. The cost function  $J^{C,N}(\alpha)$  can also be written as

$$J^{C,N}(\alpha) = J^{0,N}(\alpha) + \sum_{\gamma=1}^{\Gamma} \pi^{\gamma} \sum_{i=1}^{N_{\gamma}} \frac{1}{N_{\gamma}} J^{i,\gamma,N}(\alpha).$$

**Definition 4.4.2** (Optimal coordinated plan). *We say that  $\hat{\alpha} = (\hat{\alpha}^1, \dots, \hat{\alpha}^N) \in \mathcal{A}^N$  is an optimal coordinated plan if:  $\hat{\alpha} = \operatorname{argmin}_{\alpha \in \mathcal{A}^N} J^{C,N,\eta}(\alpha)$ .*

**Assumption 4.4.3.**

- The current cost  $(s, q, \alpha) \mapsto L_T^{\gamma}(q, \alpha) + L_S(s, \alpha)$  is strictly convex with respect to  $(s, \alpha)$ . The terminal cost  $s \mapsto g(s)$  is strictly convex with respect to  $s$ .
- There exists some constant  $C > 0$  such that

$$|L_T^{\gamma}(q, a)| + |L_S(s, a)| + |g(s)| \leq C (|q|^2 + |s|^2 + |a|^2)$$

- The functions  $L_T^{\gamma}$ ,  $L_S$  and  $g$  are continuously differentiable and their derivatives are a Lipschitz continuous functions.
- The coefficients  $\mu^0$  and  $\sigma^0$  (respectively  $\mu^{\gamma}$  and  $\sigma^{\gamma}$ ) are Lipschitz continuous functions and with linear growth in the state variable.

**4.4.2 Reformulation: Mean field type control problem**

In this section we consider a filtered probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  carrying  $\Gamma$  standard Brownian motions  $\{B^{\gamma}, \gamma = 1, \dots, \Gamma\}$  and an independent Poisson random measure  $\{N^{\gamma}, \gamma = 1, \dots, \Gamma\}$  which are mutually independent and independent from the filtration  $\mathbb{F}^0$ . We shall use the following notation. If  $\xi = \{\xi_t\}$  is an  $\mathcal{F}$ -adapted process, then  $\bar{\xi} = \{\bar{\xi}_t\}$  denotes the process defined by  $\bar{\xi}_t := \mathbb{E}[\xi_t | \mathcal{F}_t^0]$ .

Let  $x_0 = (s_0, q_0) = (x_0^{\gamma} = (s_0^{\gamma}, q_0^{\gamma}))_{1 \leq \gamma \leq \Gamma}$  be a random vector which is independent from  $\mathcal{F}^0$ . Let  $Q^0$  and  $Q^{\gamma}$  be the production processes defined by

$$\begin{aligned} Q^{\gamma} &= q_0^{\gamma} + \int_0^t \mu^{\gamma}(u, Q_u^{\gamma}) du + \int_0^t \sigma^{\gamma}(u, Q_u^{\gamma}) dB_u^{\gamma} + \int_0^t \sigma^{\gamma,0}(u, Q_u^{\gamma}) dB_u^0 \\ &\quad + \int_0^t \int_E \beta^{\gamma}(u, e, Q_{u-}^{\gamma}) \tilde{N}^{\gamma}(du, de) + \int_0^t \int_E \beta^{\gamma,0}(u, e, Q_{u-}^0) \tilde{N}^0(du, de). \\ Q_t^0 &= q_0^0 + \int_0^t \mu^0(u, Q_u^0) du + \int_0^t \sigma^0(u, Q_u^0) dB_u^0 + \int_0^t \int_E \beta^{\gamma}(u, e, Q_{u-}^0) \tilde{N}^0(du, de). \end{aligned} \quad (4.37)$$

If  $\bar{\nu} = (\bar{\nu}^1, \dots, \bar{\nu}^\Gamma)$  is an  $\mathcal{F}^0$ -adapted  $\mathbb{R}^\Gamma$ -valued process, we denote

$$P_t^{\bar{\nu}} = p \left( -Q_t^0 - \sum_{\gamma \in \Gamma} \pi^\gamma \left( \mathbb{E}[Q_t^\gamma | \mathcal{F}_t^0] - \bar{\nu}_t^\gamma \right) \right). \quad (4.38)$$

Now, for  $\bar{\nu}^0 = (\bar{\nu}^{1,0}, \dots, \bar{\nu}^{\Gamma,0})$ ,  $\alpha = (\alpha^1, \dots, \alpha^\Gamma)$  and for each  $\gamma = 1, \dots, \Gamma$ , we consider the two following cost functions

$$J_{x_0}^\gamma(\alpha^\gamma, \bar{\nu}) = \mathbb{E} \int_0^T \left[ P_t^{\bar{\nu}}(\alpha_t^\gamma - Q_t^\gamma) + L_T^\gamma(Q_t^\gamma, \alpha_t^\gamma) + L_S(S_t^\gamma, \alpha_t^\gamma) \right] dt + \mathbb{E}[g(S_T^\gamma)]. \quad (4.39)$$

$$J_{x_0}^C(\alpha) = \mathbb{E} \int_0^T [-P_t^{\bar{\alpha}} Q_t^0 + L_T^0(Q_t^0, 0)] dt + \sum_{\gamma=1}^{\Gamma} \pi^\gamma J_{x_0}^\gamma(\alpha^\gamma, \bar{\alpha}_t). \quad (4.40)$$

where  $S_t^\gamma = s_0^\gamma + \int_0^t \alpha_u^\gamma du$ .

**Definition 4.4.4** (Mean field Nash equilibrium). *Let  $x_0 = (s_0, q_0)$  be a random vector independent from  $\mathcal{F}^0$ . We say that  $\alpha^* = \{\alpha^{\gamma,*}, 1 \leq \gamma \leq \Gamma\}$  is a mean field Nash equilibrium if, for each  $\gamma$ ,  $\alpha^{\gamma,*}$  minimizes the function  $\alpha^\gamma \mapsto J_{x_0}^\gamma(\alpha^\gamma, \{\mathbb{E}[\alpha_t^* | \mathcal{F}_t^0]\})$ .*

**Definition 4.4.5** (Mean field optimal control). *Let  $x_0 = (s_0, q_0)$  be a random vector independent from  $\mathcal{F}^0$ . We say that  $\hat{\alpha} = \{\hat{\alpha}^\gamma, 1 \leq \gamma \leq \Gamma\}$  is a mean field optimal control if,  $\hat{\alpha}$  minimizes the function  $\alpha \mapsto J_{x_0}^C(\alpha)$ .*

### Characterization of mean field Nash equilibrium

**Proposition 4.4.6.** *Let  $\bar{\nu}$  be a given  $\mathcal{F}^0$ -adapted  $\mathbb{R}^\Gamma$ -valued process. Then there exists a unique control  $(\alpha^{1,*}, \dots, \alpha^{\Gamma,*}) = \alpha^*(\bar{\nu}, x_0)$  such that*

- *For each  $\gamma \in 1, \dots, \Gamma$ ,  $\alpha^{\gamma,*}$  minimizes the function  $\alpha^\gamma \mapsto J_{x_0}^\gamma(\alpha^\gamma, \bar{\nu})$ .*
- *If  $(S^{\gamma,*}, Q^\gamma)$  is the state process corresponding to the initial data condition  $x_0^\gamma$ , to the control  $\alpha^{\gamma,*}$ , and to the dynamic above, then there exists a unique adapted solution  $(Y^{\gamma,*}, Z^{0,\gamma,*}, Z^{\gamma,*}, V^{\gamma,*}, V^{0,\gamma,*})$  of the BDSE with jumps*

$$\begin{aligned} Y_t^{\gamma,*} &= \partial_s g(S_T^{\gamma,*}) + \int_0^T \partial_s L_S(S_t^{\gamma,*}, \alpha_t^{\gamma,*}) dt + \int_0^T Z_t^{0,\gamma,*} dB_t^0 + Z_t^{\gamma,*} dB_t^\gamma \\ &+ \int_0^T \int_E V_t^{\gamma,*}(e) \tilde{N}^\gamma(dt, de) + \int_0^T \int_E V_t^{0,\gamma,*}(e) \tilde{N}^0(dt, de), \end{aligned} \quad (4.41)$$

satisfying the coupling condition

$$Y_t^{\gamma,*} + P_t^{\bar{\nu}} + \partial_\alpha L_T^\gamma(Q_t^\gamma, \alpha_t^{\gamma,*}) + \partial_\alpha L_S(S_t^{\gamma,*}, \alpha_t^{\gamma,*}) = 0. \quad (4.42)$$

Conversely, assume that there exists  $(\alpha^{\gamma,*}, S^{\gamma,*}, Y^{\gamma,*}, Z^{0,\gamma,*}, Z^{\gamma,*}, V^{\gamma,*}, V^{0,\gamma,*})$  which satisfy the

coupling condition (4.42) as well as the FBSDEJ, then  $\alpha^{\gamma,*}$  is the optimal control minimizing  $J_{x_0}^\gamma(\alpha^\gamma, \bar{\nu})$  and  $S^{\gamma,*}$  is the optimal trajectory.

• If in addition:  $\forall \gamma = 1, \dots, \Gamma \mathbb{E}[\alpha_t^{\gamma,*} | \mathcal{F}_t^0] = \bar{\nu}_t^{\gamma,0}$ , then  $\alpha^*$  is a mean field Nash equilibrium.

**Proof.** • Since the dynamic programming principal does not work in this context, our proof consists on the classical Pontryagin's maximum principle where the characterization of the Mean field Nash equilibrium is given by the associated McKean-Vlasov FBSDEs. Using the fact that  $J_{x_0}^\gamma$  is a strictly convex coercive function and Gateaux-differentiable (see Assumption (4.4.3)), we have

$$d_\beta J_{x_0}^\gamma(\cdot, \bar{\nu}) := 0. \quad (4.43)$$

We start by computing the functional directional derivative of  $J_{x_0}^\gamma(\cdot, \bar{\nu})$

$$d_\beta J_{x_0}^\gamma(\cdot, \bar{\nu}) = \mathbb{E} \left[ \int_0^T [P_u^\beta + \partial_\alpha L_T^\gamma(Q_u^\gamma, \alpha_u^\gamma) + \partial_\alpha L_S(S_u^\gamma, \alpha_u^\gamma) + \partial_u L_S(S_u^\gamma, \alpha_u^\gamma)] \beta_u du \right] + \mathbb{E} \left[ \tilde{S}_T^\beta \partial_s g(S_T^\gamma) \right]. \quad (4.44)$$

Hence, there exists a unique optimal control  $\alpha^{\gamma,*} = \alpha^{\gamma,*}(\bar{\nu}, x_0)$  satisfying the following Euler optimality condition

$$\mathbb{E} \left[ \int_0^T [P_u^\beta + \partial_\alpha L_T^\gamma(Q_u^\gamma, \alpha_u^\gamma) + \partial_\alpha L_S(S_u^\gamma, \alpha_u^\gamma) + \partial_s L_S(S_u^\gamma, \alpha_u^\gamma)] \beta_u du + \tilde{S}_T^\beta \partial_s g(S_T^\gamma) \right] = 0. \quad (4.45)$$

We denote by  $S^{\gamma,*}$  the optimal trajectory associated to  $\alpha^{\gamma,*}$ . Applying Itô Tanaka formula to  $S_t^\beta Y_t^{\gamma,*}$  we get

$$\begin{aligned} \tilde{S}_t^\beta Y_t^{\gamma,*} &= \tilde{S}_T^\beta Y_T^{\gamma,*} + \int_t^T Y_s^{\gamma,*} \beta_s ds - \int_t^T \tilde{S}_s^\beta \partial_s L_S(S_s^{\gamma,*}, \alpha_s^{\gamma,*}) ds + \int_0^T \tilde{S}_s^\beta Z_s^{0,\gamma,*} dB_s^0 \\ &+ \int_t^T \tilde{S}_s^\beta Z_s^{\gamma,*} dB_s^\gamma + \int_0^T \int_E \tilde{S}_s^\beta V_s^{\gamma,*}(e) \tilde{N}^\gamma(ds, de) + \int_0^T \int_E \tilde{S}_s^\beta V_s^{0,\gamma,*}(e) \tilde{N}^{0,\gamma}(ds, de) \end{aligned}$$

Taking the conditional expectation in the equation above, we obtain

$$\mathbb{E} \left[ \tilde{S}_T^\beta Y_T^{\gamma,*} \right] = \mathbb{E} \left[ \int_t^T Y_s^{\gamma,*} \beta_s ds - \int_t^T \tilde{S}_s^\beta \partial_s L_S(S_s^{\gamma,*}, \alpha_s^{\gamma,*}) ds \right]. \quad (4.46)$$

Taking into account the terminal the terminal condition  $Y_T^* = \partial_s g(S_T^{\gamma,*})$  and the Euler optimality condition (4.45), the previous equation leads to

$$\mathbb{E} \left[ \int_0^T \left( Y_s^{\gamma,*} + P_s^\beta + \partial_\alpha L_T^\gamma(Q_s^\gamma, \alpha_s^{\gamma,*}) + \partial_\alpha L_S(S_s^{\gamma,*}, \alpha_s^{\gamma,*}) \right) \beta_s ds \right] = 0. \quad (4.47)$$

• Suppose that  $(\alpha^{\gamma,*}, S^{\gamma,*}, Y^{\gamma,*}, Z^{0,\gamma,*}, Z^{\gamma,*}, V^{\gamma,*}, V^{0,\gamma,*})$  is a solution of the following coupled

Forward-Backward SDE with jumps.

$$\left\{ \begin{array}{l} Y_T^{\gamma,*} = \partial_s g(S_T^{\gamma,*}) + \int_0^T \partial_s L_S(S_s^{\gamma,*}, \alpha_s^{\gamma,*}) ds + \int_0^T Z_s^{0,\gamma,*} dB_s^0 + Z_s^{\gamma,*} dB_s^\gamma \\ \quad + \int_0^T \int_E V_s^{\gamma,*}(e) \tilde{N}(ds, de) + \int_0^T \int_E V_s^{0,\gamma,*}(e) \tilde{N}^0(ds, de). \\ Q_T^\gamma = q_0^\gamma + \int_0^T \mu^\gamma(s, Q_s^\gamma) ds + \int_0^T \sigma^\gamma(s, Q_s^\gamma) dB_s^\gamma + \int_0^T \sigma^{\gamma,0}(s, Q_s^\gamma) dB_s^0 \\ \quad + \int_0^T \int_E \beta^\gamma(s, e, Q_{s-}^\gamma) \tilde{N}^\gamma(ds, de) + \int_0^T \int_E \beta^{\gamma,0}(s, e, Q_{s-}^{\gamma,0}) \tilde{N}^0(ds, de) \\ S_t^\gamma = s_0^\gamma + \int_0^t \alpha_s^\gamma ds. \end{array} \right. \quad (4.48)$$

with

$$Y_s^{\gamma,*} + P_s^{\bar{\nu}} + \partial_\alpha L_T^\gamma(Q_s^\gamma, \alpha_s^{\gamma,*}) + \partial_s L_S(S_s^{\gamma,*}, \alpha_s^{\gamma,*}) = 0,$$

The idea is then to compute the the gâteaux derivative of  $J_{x_0}^{\gamma, \text{MFG}}(\cdot, \bar{\nu})$  at  $\alpha^{\gamma,*}$  to obtain zero and then from the strict convexity of  $J_{x_0}^{\gamma, \text{MFG}}(\cdot, \bar{\nu})$  we obtain the desire result.

### Characterization of mean field optimal controls

**Proposition 4.4.7.** *Assume that  $\hat{\alpha} = (\hat{\alpha}^1, \dots, \hat{\alpha}^\Gamma)$  minimizes the functional  $J_{x_0}^C(\alpha)$ , and denote by  $\hat{S} = (\hat{S}^1, \dots, \hat{S}^\Gamma)$  is the corresponding controlled trajectory. Then there exists a unique adapted solution  $(\hat{Y} = (\hat{Y}^1, \dots, \hat{Y}^\Gamma), \hat{Z} = (\hat{Z}^1, \dots, \hat{Z}^\Gamma), \hat{Z}^0 = (\hat{Z}^{0,1}, \dots, \hat{Z}^{0,\Gamma}), \hat{V} = (\hat{V}^1, \dots, \hat{V}^\Gamma), \hat{V}^0 = (\hat{V}^{0,1}, \dots, \hat{V}^{0,\Gamma}))$  of the BDSE*

$$\left\{ \begin{array}{l} d\hat{Y}_t^\gamma = -\partial_s L_S(\hat{S}_t^\gamma, \hat{\alpha}_t^\gamma) dt + \hat{Z}_t^{0,\gamma} dB_t^0 + \hat{Z}_t^\gamma dB_t^\gamma + \int_E V_t^\gamma(e) \tilde{N}(dt, de) + \int_E V_t^{0,\gamma}(e) \tilde{N}^0(dt, de) \\ \hat{Y}_T^\gamma = \partial_s g(\hat{S}_T^\gamma). \end{array} \right. \quad (4.49)$$

satisfying the coupling condition: for all  $\gamma = 1, \dots, \Gamma$

$$\begin{aligned} 0 = & \hat{Y}_t^\gamma + \partial_\alpha L_T^\gamma(Q_t^\gamma, \hat{\alpha}_t^\gamma) + \partial_\alpha L_S(\hat{S}_t, \hat{\alpha}_t^\gamma) + P_t^{\bar{\alpha}} \\ & - p' \left( -Q_t^0 - \Pi_\Gamma \cdot (\bar{Q}_t - \bar{\alpha}_t) \right) \left( -Q_t^0 - \Pi_\Gamma \cdot (\bar{Q}_t - \bar{\alpha}_t) \right), \end{aligned} \quad (4.50)$$

with  $\bar{\alpha}_t = \mathbb{E}[\hat{\alpha}_t | \mathcal{F}_t^0]$  and  $\Pi_\Gamma = (\pi_1, \dots, \pi_\Gamma)^T$ . Conversely, suppose that  $(\hat{S}, \hat{\alpha}, \hat{Y}, \hat{Z}^0, \hat{Z})$  is an adapted solution to the forward-backward system (4.31)-(4.49) with the coupling condition (4.50), then  $\hat{\alpha}$  is the optimal control minimizing  $J_{x_0}^{\text{MFC}}(\alpha)$  and  $\hat{S}$  is the optimal trajectory.

**Proof.** Exactly as [1], we only prove the necessary condition of Pontryagin's maximum principle for optimality. The sufficient condition could be proven exactly as it is done in Proposition 4.4.6. Thanks to Assumption 4.4.3 insures that the cost function  $\alpha \in \mathcal{A} \mapsto J_{x_0}^C(\alpha)$  is Gâteaux

differentiable with Gâteaux derivative given by

$$\begin{aligned} d_{\beta} J_{x_0}^C(\alpha) &= \gamma \pi^{\gamma} \mathbb{E} \left[ \partial_s g(S_T^{\gamma}) \tilde{S}_T^{\beta\gamma} + \int_0^T \partial_s L_S(S_u^{\gamma}, \alpha_u^{\gamma}) S_u^{\beta\gamma} du \right] \\ &\quad + \gamma \pi^{\gamma} \mathbb{E} \left[ \int_0^T \left\{ P_u^{\bar{\alpha}} + \partial_{\alpha} L_T^{\gamma}(Q^{\gamma}, \alpha_t^{\gamma}) + \partial_{\alpha} L_S(S^{\gamma}, \alpha^{\gamma}) \right\} \beta_u^{\gamma} du \right] \\ &\quad - \sum_{\gamma} \pi^{\gamma} \mathbb{E} \left[ \int_0^T p' \left( -Q_u^0 - \Pi_{\Gamma} \cdot (\bar{Q}_u - \bar{\alpha}_u) \right) \left( -Q_u^0 - \Pi_{\Gamma} \cdot \{ \bar{Q}_u - \bar{\alpha}_u \} \right) \beta_u^{\gamma} du \right], \end{aligned}$$

where  $\tilde{S}_u^{\beta\gamma}$  is the process defined by

$$d\tilde{S}_u^{\beta\gamma} = \beta_u^{\gamma} du, \quad \tilde{S}_0^{\beta\gamma} = 0.$$

Hence the optimal control  $\hat{\alpha}$  satisfies the Euler optimality condition: for all  $\beta = (\beta^1, \dots, \beta^{\Gamma})$

$$\begin{aligned} 0 &= \gamma \pi^{\gamma} \mathbb{E} \left[ \partial_s g(S_T^{\gamma}) \tilde{S}_T^{\beta\gamma} + \int_0^T \partial_s L_S(S_u^{\gamma}, \alpha_u^{\gamma}) S_u^{\beta\gamma} du \right] \\ &\quad + \gamma \pi^{\gamma} \mathbb{E} \left[ \int_0^T \left\{ P_u^{\bar{\alpha}} + \partial_{\alpha} L_T^{\gamma}(Q_u^{\gamma}, \alpha_u^{\gamma}) + \partial_{\alpha} L_S(S_u^{\gamma}, \alpha_u^{\gamma}) \right\} \beta_u^{\gamma} du \right] \\ &\quad - \sum_{\gamma} \pi^{\gamma} \mathbb{E} \left[ \int_0^T \left\{ p' \left( -Q_u^0 - \Pi_{\Gamma} \cdot (\bar{Q}_u^0 - \bar{\alpha}_u^0) \right) \left( -Q_u^0 - \Pi_{\Gamma} \cdot (\bar{Q}_u - \bar{\alpha}_u) \right) \right\} \beta_u^{\gamma} du \right], \end{aligned}$$

• Now, let  $(\hat{Y}, \hat{Z}, \hat{Z}^0, \hat{V}^{\gamma}, \hat{V}^{0,\gamma})$  be the unique solution to the BSDE with jump (4.49), and let  $\hat{S}$  be the state process associated to the optimal control  $\hat{\alpha}$ , applying Itô formula, we obtain

$$\sum_{\gamma} \pi^{\gamma} \mathbb{E} \left[ \hat{Y}_T^{\gamma} \tilde{S}_T^{\beta\gamma} \right] = \sum_{\gamma} \pi^{\gamma} \mathbb{E} \left[ \int_0^T \left\{ -\partial_s L_S(\hat{S}_u, \hat{\alpha}_u) + \beta_u^{\gamma} \hat{Y}_u^{\gamma} \right\} du \right].$$

Taking into account the terminal condition  $\hat{Y}_T^{\gamma} = \partial_s g(\hat{S}_T^{\gamma})$  and the Euler Optimality condition for  $\hat{\alpha}$  we get: for all  $\beta = (\beta^1, \dots, \beta^{\Gamma}) \in \mathcal{A}^{\Gamma}$ :

$$\begin{aligned} 0 &= \gamma \pi^{\gamma} \mathbb{E} \left[ \int_0^T \left\{ \hat{Y}_u^{\gamma} + P_u^{\bar{\alpha}} + \partial_{\alpha} L_T^{\gamma}(Q_u^{\gamma}, \hat{\alpha}_u) + \partial_{\alpha} L_S(\hat{S}_u, \hat{\alpha}_u) \right. \right. \\ &\quad \left. \left. - p' \left( -Q_u^0 - \Pi_{\Gamma} \cdot (\bar{Q}_u - \bar{\alpha}_u) \right) \left( -Q_u^0 - \Pi_{\Gamma} \cdot (\bar{Q}_u - \bar{\alpha}_u) \right) \right\} \beta_u^{\gamma} du \right]. \end{aligned}$$

We deduce the coupling condition (4.50).

**Proposition 4.4.8.** Assume that  $\hat{\alpha}$  is a mean field optimal control for the problem with a pricing

rule  $p$ . Then  $\hat{\alpha}$  is a mean field Nash equilibrium for the MFG problem with pricing rule

$$p^{\text{MFG}}(x) = p(x) + xp'(x). \quad (4.51)$$

**Proof.** This result follows straightforward from comparing the two coupling condition (4.50) and (4.42) since the two McKean-Vlasov BSDEs (4.41) and (4.49) are of the same form.

#### 4.4.3 Explicit solution of the MFC with 1 region

In this section, we provide an example where an explicit solution of the MFC problem is obtained. We consider a linear pricing rule of the following form

$$p(x) = p_0 + p_1x. \quad (4.52)$$

The storage cost  $L_S$  is defined by: For  $A_1 < 0, A_2 > 0, C < 0$ ,

$$L_S(s, \alpha) = A_1s + \frac{A_2}{2}s^2 + \frac{C}{2}\alpha^2.$$

For some given positive constant  $\{K^\gamma\}_{\gamma=1}^\Gamma$ , the transmission cost  $L_T^\gamma$  is defined by

$$L_T^\gamma(q, \alpha) = \frac{K^\gamma}{2}(q - \alpha)^2.$$

For some constants  $B_1$  and  $B_2 > 0$ , the terminal cost

$$g(s) = \frac{B_2}{2} \left( s - \frac{B_1}{B_2} \right)^2.$$

Now, we will consider the simple case of one region, i.e. when  $\pi = 1$ . we aim to find an explicit solution to the MFC problem associated to the linear quadratic case.

**Step 1** In this first step, we use the forward-backward system (4.49)-(4.31) and the coupling condition (4.50) in order to get the optimal control  $\bar{\alpha}$  and the optimal trajectory  $\bar{S}$  associated to one node in this region. We have

$$\begin{cases} d\bar{S}_t = \bar{\alpha}_t dt, & \bar{S}_0 = 0, \\ d\bar{Y}_t = -(A_2\bar{S}_t + A_1)dt + \bar{Z}_t^0 dB_t^0 + \int_E \bar{V}_t^{\gamma,*}(e) \tilde{N}(dt, de), & \bar{Y}_T = B_2\bar{S}_T - B_1. \end{cases} \quad (4.53)$$

To find the optimal control  $\bar{\alpha}$ , we use firstly the coupling condition (4.50) to obtain

$$\bar{Y}_t - K(\bar{Q}_t - \bar{\alpha}_t) + C\bar{\alpha}_t + P_t^{\bar{\alpha}} - p'(-Q_t^0 - \bar{Q}_t + \bar{\alpha}_t)(-Q_t^0 - \bar{Q}_t + \bar{\alpha}_t) = 0, \quad (4.54)$$

where  $Q^0$  and  $Q$  are defined by (4.37). Now, Proposition 3.3 in [1] and the linear form of  $p$  in (4.52) give

$$P_t^{\bar{\alpha}} = p_0 + 2p_1(-Q_t^0 - \bar{Q}_t + \bar{\alpha}_t), \quad (4.55)$$

and we obtain the following expression of the optimal control  $\bar{\alpha}$ :

$$\begin{aligned} \bar{\alpha}_t &= -\frac{1}{K+C+p_1} \left( \bar{Y}_t + p_0 - p_1 Q_t^0 - (p_1 + K) \bar{Q}_t \right) \\ &= -\Delta(\bar{Y}_t + b_t), \end{aligned}$$

where  $\Delta = \frac{1}{K+C+p_1}$  and  $b_t = p_0 - p_1 Q_t^0 - (p_1 + K) \bar{Q}_t$ .

We expect the solution of the FBSDE (4.53) to be affine. It has the following form:

$$\bar{Y}_t = \bar{\phi}_t \bar{S}_t + \bar{\psi}_t, \quad (4.56)$$

where  $\phi$  and  $\psi$  are deterministic functions. Computing  $d\bar{Y}_t$  from this expression, we obtain

$$d\bar{Y}_t = \bar{S}_t(-\Delta \bar{\phi}_t^2 + \dot{\bar{\phi}}_t)dt - \Delta \bar{\phi}_t(\bar{\psi}_t dt + b_t)dt + \dot{\bar{\psi}}_t. \quad (4.57)$$

Identifying the two expressions of  $d\bar{Y}_t$  we get, in one hand, that

$$\dot{\bar{\phi}}_t - \Delta \bar{\phi}_t^2 + A_2 = 0, \quad \bar{\phi}(T) = B_2 \quad (4.58)$$

which is a Riccati equation. In the other hand, we obtain that  $\psi$  is the unique solution of the BSDE

$$d\bar{\psi}_t = \Delta \bar{\phi}_t(\bar{\psi}_t + b_t)dt - A_1 dt + \bar{Z}_t^0 dB_t^0 + \int_E \bar{V}_t^{\gamma, \star}(e) \tilde{N}(dt, de). \quad (4.59)$$

Consequently, substituting  $b_t$ , we get the following BSDE

$$d\bar{\psi}_t = \Delta \bar{\phi}_t(\bar{\psi}_t + \bar{P}_t)dt + \bar{Z}_t^0 dB_t^0 + \int_E \bar{V}_t^{\gamma, \star}(e) \tilde{N}(dt, de), \quad \psi_T = -B_1. \quad (4.60)$$

This allows us to find the expression of the approximated electricity price. In fact, in one hand we have

$$\begin{aligned} \bar{\psi}_t &= \Delta \bar{\phi}_t(\bar{\psi}_t + \bar{P}_t)dt + \bar{Z}_t^0 dB_t^0 + \int_E \bar{V}_t^{\gamma, \star}(e) \tilde{N}(dt, de) \\ &= \Delta \bar{\phi}_t(\bar{Y}_t - \bar{\phi}_t \bar{S}_t + \bar{P}_t)dt + \bar{Z}_t^0 dB_t^0 + \int_E \bar{V}_t^{\gamma, \star}(e) \tilde{N}(dt, de) \\ &= \Delta \bar{\phi}_t \bar{Y}_t dt - \Delta(\bar{\phi}_t)^2 \bar{S}_t dt + \int_E \bar{V}_t^{\gamma, \star}(e) \tilde{N}(dt, de). \end{aligned}$$

In the other hand, we have that  $d\bar{\psi}_t = d\bar{Y}_t - \dot{\bar{\phi}}_t \bar{S}_t dt - \bar{\phi}_t d\bar{S}_t$ . So, we obtain

$$-(A_2 \bar{S}_t dt + A_1) dt - \Delta \bar{\phi}_t \bar{P}_t dt = \dot{\bar{\phi}}_t \bar{S}_t dt + \bar{\phi}_t d\bar{S}_t + \Delta \bar{\phi}_t \bar{Y}_t - \Delta (\bar{\phi}_t)^2 \bar{S}_t dt.$$

Finally, using the Riccati equation (4.58) in the equation above, we obtain directly the following price expression

$$\bar{P}_t = -\frac{A_1}{\Delta \bar{\phi}_t} + b_t.$$

As it can be seen,  $\bar{\Psi}$  is the solution of linear BSDE with jumps. So it has the following expression

$$\bar{\Psi}_t = \mathbb{E}\left[-\Gamma_{t,T} B_1 + \int_t^T \Gamma_{t,u} \Delta \bar{\phi}_u \bar{P}_u du \mid \mathcal{F}_t^0\right],$$

where  $\Gamma_{t,T}$  is the adjoint process and in this case, it is the solution of

$$d\Gamma_{t,s} = \Gamma_{t,s} \Delta \bar{\phi}_s ds,$$

which is  $\Gamma_{t,s} = \exp(\int_t^s \Delta \bar{\phi}_s ds)$ . Consequently,  $\bar{\Psi}_t$  is given by

$$\bar{\Psi}_t = -B_1 \exp\left\{-\int_t^T \Delta \bar{\phi}(u) du\right\} - \mathbb{E}\left[\int_t^T \Delta \bar{\phi}(u) \exp\left\{-\int_t^u \Delta \bar{\phi}(s) ds\right\} \bar{P}_u du \mid \mathcal{F}_t^0\right].$$

The function  $\bar{\phi}$  is given by

$$\bar{\phi}(t) = -\frac{\rho}{\Delta} \frac{e^{-\rho(T-t)}(-B_2 \Delta + \rho) - e^{\rho(T-t)}(B_2 \Delta + \rho)}{e^{-\rho(T-t)}(-B_2 \Delta + \rho) + e^{\rho(T-t)}(B_2 \Delta + \rho)} \quad \text{with } \rho := \sqrt{A_2 \Delta},$$

Now, to find  $\bar{S}_t$ , it suffices to solve the following simple EDO

$$d\bar{S}_t = \left[-\Delta \bar{\phi}_t \bar{S}_t - \Delta \left(\bar{\Psi}_t + \bar{P}_t + \frac{A_1}{\Delta \bar{\phi}_t}\right)\right] dt, \quad (4.61)$$

for which the solution is given BY

$$\bar{S}_t = \bar{S}_0 \exp\left(\int_0^t -\Delta \bar{\phi}_s ds\right) - \Delta \int_0^t \exp\left(\int_u^t -\Delta \bar{\phi}_s ds\right) \left(\bar{\Psi}_u + \bar{P}_u + \frac{A_1}{\Delta \bar{\phi}_u}\right) du. \quad (4.62)$$

As  $\bar{S}_0 = 0$ , the solution is then

$$\bar{S}_t = -\Delta \int_0^t \exp\left\{-\int_u^t \Delta \bar{\phi}(s) ds\right\} \left(\bar{P}_u + \bar{\Psi}_u + \frac{A_1}{\Delta \bar{\phi}(u)}\right) du.$$

**Step 2** Once we obtain all the optimal elements of one node in the first step, we use the FBSDE (4.41) and the coupling condition (4.42) to find the optimal objects associated to one region

containing a number of identical nodes. Using the FBSDE (4.41), we have

$$\begin{aligned} dS_t &= -\delta \left( Y_t + P_t + \frac{A_1}{\delta \phi(t)} \right) dt, \quad S_0 = s_0, \\ dY_t &= -(A_2 S_t + A_1) dt + Z_t^0 dB_t^0 + Z_t dB_t + \int_E V_t(e) \tilde{N}(dt, de), \quad Y_T = B_2 S_T - B_1. \end{aligned}$$

Again, we look at a solution of the form

$$Y_t = \varphi(t) S_t + \psi_t,$$

and using the coupling condition (4.42), we obtain the expression the optimal control  $\alpha$ . In fact,

$$Y_t - K(Q_t - \alpha_t) + C\alpha_t + P_t^{\bar{\alpha}} - p_1(-Q_t^0 - Q_t - \alpha_t) = 0$$

where  $P_t^{\bar{\alpha}} = p_0 + 2p_1(-Q_t^0 - \bar{Q}_t + \bar{\alpha}_t)$ . Then

$$\begin{aligned} \alpha_t &= -\delta(Y_t + p_0 - KQ_t - 2p_1(Q_t^0 + \bar{Q}_t - \bar{\alpha}_t) + \frac{A_1}{\delta \phi_t}) \\ &= -\delta(Y_t + P_t + \frac{A_1}{\delta \phi_t}), \end{aligned}$$

where  $\delta = \frac{-1}{K+C}$ ,  $P_t = p_0 - KQ_t - 2p_1(Q_t^0 + \bar{Q}_t - \bar{\alpha}_t) - \frac{A_1}{\delta \phi_t}$ . Once again, we aim to find a solution to the FBSDE above which has the following form

$$Y_t = \varphi(t) S_t + \psi_t,$$

where  $\varphi$  and  $\psi$  can be explicitly calculated in the same way as before. In fact,  $\psi$  is the solution of the following equation

$$d\psi_t = -\delta \varphi_t (\psi_t dt + P_t) + Z_t^0 dB_t^0 + Z_t dB_t + \int_E V_t(e) \tilde{N}(de, dt) \quad (4.63)$$

and  $\varphi_t$  satisfies the following Riccati equation

$$\delta \varphi_t^2 - \dot{\varphi}_t - A_2 = 0 \quad (4.64)$$

whose solution is given by

$$\varphi(t) = -\frac{\rho}{\delta} \frac{e^{-\rho(T-t)}(-B_2\delta + \rho) - e^{\rho(T-t)}(B_2\delta + \rho)}{e^{-\rho(T-t)}(-B_2\delta + \rho) + e^{\rho(T-t)}(B_2\delta + \rho)} \quad \text{with } \rho := \sqrt{A_2\delta},$$

As it can be seen,  $\psi_t$  is the solution of the following BSDE driven by a 2-dimensional Brownian

motion

$$d\psi_t = -\delta\varphi_t(\psi_t + P_t)dt + \tilde{Z}_t d\tilde{B}_t + \int_E V_t(e) \tilde{N}(de, ds), \quad (4.65)$$

where  $\tilde{Z}_t = (Z_t^0, Z_t)$  and  $\tilde{B}_t = (B_t^0, B_t)$ .

Finally

$$S_t = s_0 \exp \left\{ - \int_0^t \delta\varphi(u) du \right\} - \delta \int_0^t \exp \left\{ - \int_u^t \delta\varphi(s) ds \right\} \left( P_u + \psi_u + \frac{A_1}{\delta\phi(u)} \right) du.$$

**Remark 4.4.9.** Notice that in the example that we treated above, we only assume the presence of a common noise  $B^0$  (no common jump  $\tilde{N}^0$ ) in order to simplify computations. However, we emphasize that the presence of the common jump  $\tilde{N}^0$  make just little changes in the proof.

## 4.5 Appendix

In this section, we extend some of the results of Hamadène [47] concerning FBSDEs in the Brownian setting to the case of jumps. Let us note that arguments of proof are close to the one used by Hamadène in [47] with some minor modifications due to jumps setting. However, we still provide the proof of existence. We look for the solution of the following fully coupled forward-backward SDE with jumps

$$(\mathbf{S}) \begin{cases} X_t = X_0 + \int_0^t b_s(X_s, Y_s, Z_s, K_s) ds + \int_0^t \sigma_s(X_s, Y_s, Z_s, K_s(e)) dW_s \\ \quad + \int_0^t \int_E \beta_s(X_{s-}, Y_{s-}, Z_s, K_s(e)) \tilde{\pi}(ds, de), \quad 0 \leq t \leq T, \mathbb{P}\text{-a.s.} \\ Y_t = g(X_T, \mathbb{P}_{X_T}) - \int_t^T h_s(X_s, Y_s, Z_s, K_s) ds - \int_t^T Z_s dW_s - \int_t^T \int_E K_s(e) \tilde{\pi}(ds, de), \end{cases}$$

We assume the following assumptions:

- The functions  $f, h, \sigma, \beta$  defined on  $\mathbb{R}^{d+d+d \times d} \times \mathbb{L}^0(\mathcal{B}(E), \nu)$  are Lipschitz in  $(x, y, z, k)$  and uniformly in  $\omega \in \Omega$ .

- The function  $g$  is defined on  $\Omega \times \mathbb{R}^d$  and valued in  $\mathbb{R}^d$  such that for any  $x \in \mathbb{R}^d$ ,  $g$  is  $\mathcal{F}_T$ -measurable and square integrable. Moreover,  $g$  is Lipschitz in  $x$  and uniformly in  $\omega \in \Omega$ .

Finally, for  $u = (x, y, z, k)$  and  $u' = (x', y', z', k') \in \mathbb{R}^{d+d+d \times d}$ , we define the function  $\mathcal{A}$  as follows

$$\begin{aligned} \bar{A}(t, u, u') &= [b_t(x, y, z, k) - b_t(x', y', z', k')](y - y') + [h_t(x, y, z, k) - h_t(x', y', z', k')](x - x') \\ &\quad + [\sigma_t(x, y, z, k) - \sigma_t(x', y', z', k')](z - z') + \int_E (\beta_t(x, y, z, k) - \beta_t(x', y', z', k'))(k - k')(e) \eta(dt, de). \end{aligned}$$

We make the following assumption

**Assumption 4.5.1.**

$$(\bar{H}1) \left\{ \begin{array}{l} (i) \text{ There exists } k > 0, \text{ s.t. } \forall t \in [0, T], \nu \in \mathcal{M}_2(\mathbb{R}^d \times \mathbb{R}^d), u, u' \in \mathbb{R}^{d+d+d \times d} \times \mathbb{L}^0(\mathcal{B}(E), \eta), \\ \quad \bar{A}(t, u, u') \leq -k|x - x'|^2, \mathbb{P}\text{-a.s.} \\ (ii) \text{ There exists } k' > 0, \text{ s.t. } \forall \nu \in \mathcal{M}_2(\mathbb{R}^d \times \mathbb{R}^d), x, x' \in \mathbb{R}^d \\ \quad (g(x, \nu) - g(x', \nu)) \cdot (x - x') \geq k'|x - x'|^2, \mathbb{P}\text{-a.s.} \\ (iii) \min\{k, k'\} \geq C(\bar{C} + 1). \end{array} \right.$$

**Proposition 4.5.2.** *Under Assumption ( $\bar{H}1$ ), there exists a unique solution  $U = (X, Y, Z, K)$  of the FBSDE with jumps (4.5).*

**Proof.** In the following proof, we will use the notation  $\bar{C}$  to denote a generic constant that may change from line to line and that depends in an implicit way on  $T$  and the Lipschitz constants. The key point of the proof is to consider a sequence  $U^n = (X^n, Y^n, Z^n, K^n)$  of processes defined recursively by :  $(X^0, Y^0, Z^0, K^0) = (0, 0, 0, 0)$  and for  $n \geq 1$ ,  $U^n = (X^n, Y^n, Z^n, K^n)$  satisfies, for all  $t \in [0, T]$  and  $\delta \in ]0, 1]$ , the following system

$$\left\{ \begin{array}{l} X_t^{n+1} = x + \int_0^t \left( f_s(X_s^{n+1}, Y_s^{n+1}, Z_s^{n+1}, K_s^{n+1}) - \delta(Y_s^{n+1} - Y_s^n) \right) ds \\ \quad + \int_0^t \left( \sigma_s(X_s^{n+1}, Y_s^{n+1}, Z_s^{n+1}, K_s^{n+1}) - \delta(Y_s^{n+1} - Y_s^n) \right) dW_s \\ \quad + \int_0^t \int_E \left( \beta_s(X_s^{n+1}, Y_s^{n+1}, Z_s^{n+1}, K_s^{n+1}) - \delta(K_s^{n+1} - K_s^n) \right) \tilde{\pi}(ds, de), \\ Y_t^{n+1} = g(X_T^{n+1}) - \int_t^T h_s(X_s^{n+1}, Y_s^{n+1}, Z_s^{n+1}, K_s^{n+1}) ds - \int_t^T Z_s^{n+1} dW_s - \int_t^T \int_E K_s^{n+1}(e) \tilde{\pi}(ds, de). \end{array} \right.$$

For  $n \geq 1, t \in [0, T]$ , we consider the following processes

$$\hat{X}_t^{n+1} := X_t^{n+1} - X_t^n, \quad \hat{Y}_t^{n+1} := Y_t^{n+1} - Y_t^n, \quad \hat{Z}_t^{n+1} := Z_t^{n+1} - Z_t^n, \quad \hat{K}_t^{n+1} := K_t^{n+1} - K_t^n.$$

and for a function  $\phi = \{f, h, \sigma, \beta\}$ , we set

$$\hat{\phi}_t^{n+1} := \phi(t, U_t^{n+1}, \nu_t^n) - \phi(t, U_t^n, \nu_t^{n-1}), \quad \tilde{\phi}_t^n := \phi(t, U_t^n, \nu_t^n) - \phi(t, U_t^n, \nu_t^{n-1}).$$

In order to prove the existence of the solution, we will show that  $(X^n, Y^n, Z^n, K^n)_{n \geq 0}$  is a

Cauchy sequence. First, we apply Itô's formula to  $\hat{X}^{n+1}\hat{Y}^{n+1}$  and take the expectation

$$\begin{aligned} \mathbb{E}[\hat{X}_T^{n+1}\hat{Y}_T^{n+1}] &= \mathbb{E}\left[\int_0^T \hat{Y}_s^{n+1}[\hat{f}_s^{n+1} - \delta(\hat{Y}_s^{n+1} - \hat{Y}_s^n)]ds + \mathbb{E}\left[\int_0^T \hat{X}_s^{n+1}\hat{h}_s^{n+1}ds\right] \right. \\ &\quad \left. + \mathbb{E}\left[\int_0^T (\hat{\sigma}_s^{n+1} - \delta(\hat{Z}_s^{n+1} - \hat{Z}_s^n), \hat{Z}_s^{n+1})ds + \int_0^T \int_E \hat{K}_s^{n+1}(\hat{\beta}_s^{n+1} - \delta(\hat{K}_s^{n+1} - \hat{K}_s^n))\eta(de, ds)\right]\right]. \end{aligned}$$

Rearranging terms, we get

$$\begin{aligned} &\mathbb{E}[\hat{X}_T^{n+1}(g(X_T^{n+1}) - g(X_T^n))] + \delta\mathbb{E}\left[\int_0^T |\hat{Y}_s^{n+1}|^2 + \|\hat{Z}_s^{n+1}\|^2 + |\hat{K}_s^{n+1}|_s^2 ds\right] \\ &- \mathbb{E}\left[\int_0^T \hat{X}_s^{n+1}\hat{h}_s^{n+1} + \hat{Y}_s^{n+1}\hat{f}_s^{n+1} + \hat{\sigma}^{n+1}(s)\hat{Z}_s^{n+1} ds + \int_0^T \int_E \hat{K}_s^{n+1}\hat{\beta}_s^{n+1}\eta(de, ds)\right] \\ &= \delta\mathbb{E}\left[\int_0^T \hat{Y}_s^{n+1}\hat{Y}_s^n + \hat{Z}_s^{n+1}\hat{Z}_s^n ds + \int_0^T \int_E \hat{K}_s^{n+1}\hat{K}_s^n\eta(de, ds)\right]. \end{aligned} \quad (4.66)$$

Using Assumption 4.5.1, we get

$$\begin{aligned} &\mathbb{E}\left[k'|\hat{X}_T^{n+1}|^2 + \delta\int_0^T |\hat{Y}_s^{n+1}|^2 + \|\hat{Z}_s^{n+1}\|^2 + |\hat{K}_s^{n+1}|_s^2 ds + k\int_0^T |\hat{X}_s^{n+1}|^2 ds\right] \\ &\leq \delta\mathbb{E}\left[\int_0^T \hat{Y}_s^{n+1}\hat{Y}_s^n + \hat{Z}_s^{n+1}\hat{Z}_s^n ds + \int_0^T \int_E \hat{K}_s^{n+1}\hat{K}_s^n\eta(de, ds)\right]. \end{aligned} \quad (4.67)$$

In addition, the elementary inequality  $ab \leq a^2/2 + b^2/2$  yields to

$$\begin{aligned} &\mathbb{E}\left[\int_0^T \hat{Y}_s^{n+1}\hat{Y}_s^n + \hat{Z}_s^{n+1}\hat{Z}_s^n ds\right] + \int_0^T \int_E \hat{K}_s^{n+1}\hat{K}_s^n \eta(de, ds) \\ &\leq \frac{1}{2}\mathbb{E}\left[\int_0^T |\hat{Y}_s^{n+1}|^2 + \|\hat{Z}_s^{n+1}\|^2 + |\hat{K}_s^{n+1}|_s^2 ds\right] \\ &\quad + \frac{1}{2}\mathbb{E}\left[\int_0^T |\hat{Y}_s^n|^2 + \|\hat{Z}_s^n\|^2 + |\hat{K}_s^n|_s^2 ds\right]. \end{aligned} \quad (4.68)$$

Pugging (4.68) in (4.67), we obtain

$$k'\mathbb{E}[|\hat{X}_T^{n+1}|^2] + k\mathbb{E}\left[\int_0^T |\hat{X}_s^{n+1}|^2 ds\right] + \frac{\delta}{2}\mathbb{E}\left[\int_0^T |\hat{Y}_s^{n+1}|^2 + \|\hat{Z}_s^{n+1}\|^2 + |\hat{K}_s^{n+1}|_s^2 ds\right] \quad (4.69)$$

$$\leq \frac{\delta}{2}\left[\mathbb{E}\left[\int_0^T |\hat{Y}_s^n|^2 + \|\hat{Z}_s^n\|^2 + |\hat{K}_s^n|_s^2 ds\right]\right]. \quad (4.70)$$

Now, using Itô's formula to  $|\hat{Y}^n|^2$ , we obtain classically that

$$\forall n \geq 1, \exists C^1 > 0, \mathbb{E}\left[\int_0^T |\hat{Y}_s^n|^2 + \|\hat{Z}_s^n\|^2 + |\hat{K}_s^n|_s^2 ds\right] \leq C^1\left(\mathbb{E}[|\hat{X}_T^n|^2] + \int_0^T |\hat{X}_s^n|^2 ds\right).$$

Plugging this estimates in (4.69), we obtain by choosing  $\delta := \min(k, k')/\tilde{C}$ ,

$$\mathbb{E}[|\hat{X}_T^{n+1}|^2] + \mathbb{E}\left[\int_0^T |\hat{X}_s^{n+1}|^2 ds\right] \leq \frac{1}{2^n} [\mathbb{E}|\hat{X}_T^1|^2 + \mathbb{E}\int_0^T |\hat{X}_s^1|^2 ds].$$

Besides, we also have for all  $n \geq 1$

$$\mathbb{E}\left[\int_0^T |\hat{Y}_s^n|^2 + \|\hat{Z}_s^n\|^2 + |\hat{K}_s^n|^2 ds\right] \leq \frac{C^1}{2^{n-1}} \left(\mathbb{E}\left[|\hat{X}_T^1|^2 + \int_0^T |\hat{X}_s^1|^2 ds\right]\right).$$

Thus,  $(\hat{X}_T^n)_{n \geq 0}$  is a Cauchy sequence in  $\mathbb{H}^2(\Omega, \mathbb{P})$  and  $(\hat{X}^n)_{n \geq 0}, (\hat{Y}^n)_{n \geq 0}, (\hat{Z}^n)_{n \geq 0}$  and  $(\hat{K}^n)_{n \geq 0}$  are Cauchy sequences respectively in  $\mathbb{H}^2([0, T], \Omega, dt \otimes d\mathbb{P})$  and  $\mathbb{H}_\eta^2([0, T], \Omega, dt \otimes d\eta)$ . Hence, if  $X, Y, Z$  and  $K$  are the respective limits of these sequences, passing to the limit in (4.5), gives us the desired result.

In order to prove that the system (4.5) has a unique solution, we suppose that  $(\bar{X}, \bar{Y}, \bar{Z}, \bar{K})$  is also solution of (4.5). Let  $\bar{X} = X^1 - X^2, \bar{Y} = Y^1 - Y^2, \bar{Z} = Z^1 - Z^2, \bar{K} = K^1 - K^2$ .

Apply Itô's formula to the product  $\bar{X}\bar{Y}$ , it follows straightforwardly from Assumption ( $\bar{H}1$ ) that

$$k' \mathbb{E}[|\bar{X}_T|^2] + k \mathbb{E}\left[\int_0^T |\bar{X}_s|^2 ds\right] \leq 0,$$

and conclude that  $\bar{X}_T = 0, \bar{X}_t = 0, \forall t \in [0, T], \mathbb{P}$ -a.s. Thus, in order to finish the proof we need to show that  $(\bar{Y}, \bar{Z}, \bar{K}) = (0, 0, 0)$ . This is actually a simple consequence of [123].

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**Titre :** Quelques contributions aux équations différentielles stochastiques rétrogrades et leurs applications

**Mots clés :** Equations différentielles stochastiques rétrogrades, maximisation d'utilité, solutions de viscosité, options Américaines, jeux à champs moyen, EDS progressive-rétrograde.

**Résumé :** Cette thèse est consacrée à l'étude des équations différentielles stochastiques rétrogrades (EDSR) et leurs applications. Dans le chapitre 1, on étudie le problème de maximisation de l'utilité de la richesse terminale où le prix de l'actif peut être discontinu sous des contraintes sur les stratégies de l'agent. Nous nous concentrons sur l'EDSR dont la solution représente l'utilité maximale, ce qui permet de transférer des résultats sur les EDSR quadratiques, en particulier les résultats de stabilité, au problème de maximisation d'utilité. Dans le chapitre 2, on considère le problème de valorisation d'options Américaines des points de vue théorique et numérique en s'appuyant sur la représentation du prix de l'option comme solution de viscosité d'une équation parabolique non linéaire. Nous étendons le résultat prouvé

pour un put ou call Américain à un cas plus général dans un cadre multidimensionnel. Nous proposons deux schémas numériques inspirés par les processus de branchement. Nos expériences numériques montrent que l'approximation du générateur discontinu, associé à l'EDP, par des polynômes locaux n'est pas efficace tandis qu'une simple procédure de randomisation donne de très bon résultats. Dans le chapitre 3, nous prouvons des résultats d'existence et d'unicité pour une classe générale d'équations forward-backward à champs moyen sous une condition de monotonicité faible et une hypothèse non-dégénérescence sur l'équation forward et on donne une application dans le domaine de stockage d'énergie dans le cas où la production d'électricité est imprévisible.

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**Title :** Some contributions to backward stochastic differential equations and applications

**Keywords :** Backward stochastic differential equations, utility maximisation, viscosity solutions, American options, mean-field games, forward-backward SDEs

**Abstract :** This thesis is dedicated to the study of backward stochastic differential equations (BSDEs) and their applications. In chapter 1, we study the problem of maximizing the utility from terminal wealth where the stock price may jump and there are investment constraints on the agent's strategies. We focus on the BSDE whose solution represents the maximal utility, which allows transferring results on quadratic BSDEs, in particular the stability results, to the problem of utility maximisation. In chapter 2, we consider the problem of pricing American options from theoretical and numerical sides based upon an alternative representation of the value of the option in the form of a viscosity solution of a parabolic equation with a nonlinear reaction term. We extend the viscosity solution characterization proved for

call/put American option prices to the case of a general payoff function in a multi-dimensional setting. We address two new numerical schemes inspired by the branching processes. Our numerical experiments show that approximating the discontinuous driver of the associated reaction/diffusion PDE by local polynomials is not efficient, while a simple randomization procedure provides very good results. In chapter 3, we provide existence and uniqueness results for a general class of coupled mean-field forward-backward SDEs with jumps under weak monotonicity conditions and without the non-degeneracy assumption on the forward equation and we give an application in the field of storage in smart grids in the case where the production of electricity is unpredictable.