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Étude mathématique de la dérivation de l'équation de Boltzmann dans un domaine à bord

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Thèse de doctorat de Mathématiques
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et
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Titre : Étude mathématique de la dérivation de l'équation de Boltzmann dans un domaine à bord.

Résumé : Ce travail s'inscrit dans le cadre de l'analyse des équations cinétiques, qui sont un cas particulier d'équations aux dérivées partielles.
L'équation de Boltzmann se propose de modéliser des gaz fortement dilués. Ses solutions présentent un comportement intéressant, aussi bien du point de vue physique que mathématique : elles évoluent de façon irréversible vers un état d'équilibre bien déterminé. Cependant, l'équation de Boltzmann est obtenue formellement à partir d'une modélisation microscopique de la matière, où les interactions entre les atomes constituant le gaz étudié sont temporellement réversibles. De ce paradoxe apparent est née la motivation d'une obtention (on parle de dérivation) mathématiquement rigoureuse de l'équation de Boltzmann.
Ce manuscrit s'attache à obtenir cette dérivation dans le cadre d'un domaine à bord (précédemment obtenue par Lanford dans le cas d'un domaine sans bord, qui est historiquement le premier résultat d'une dérivation rigoureuse de l'équation de Boltzmann) avec lequel les particules, modélisées par des sphères dures, interagissent par réflexion spéculaire. Le domaine étudié ici sera le demi-espace. Cette dérivation s'appuie sur l'étude de suites d'équations, les hiérarchies BBGKY et de Boltzmann. Après une présentation de l'obtention formelle de ces hiérarchies dans le cas d'un domaine à bord, un soin tout particulier a été apporté la définition rigoureuse de ces hiérarchies dans un cadre fonctionnel approprié. Dans une seconde partie, la preuve de la convergence des solutions de la hiérarchie BBGKY vers celles de la hiérarchie de Boltzmann est présentée en détail. En particulier, le contrôle géométrique des recollisions dans le cas du demi-espace est traité de la façon la plus exhaustive possible.

Mots clés : Analyse, équations aux dérivées partielles, équations cinétiques, équation de Boltzmann, dérivation de l'équation de Boltzmann, théorème de Lanford, hiérarchie BBGKY, sphères dures.

Title : Mathematical derivation of the Boltzmann equation with boundary condition.

Abstract : This work belongs to the field of kinetic equations, a particular case of Partial Differential Equations.
The Boltzmann equation is a model introduced in order to describe diluted gases. The solutions of this equation present an interesting behaviour: they converge irreversibly towards a determined equilibrium. However, the Boltzmann equation is formally obtained using a microscopic description of the matter, in which the interactions between the atoms constituting the gas are time-reversible. From this paradox has emerged the question of a mathematically rigorous derivation of the Boltzmann equation.
In this work one has obtained such a derivation in the case of a domain with a boundary (Lanford obtained this derivation in the case when the domain is the whole Euclidean space, which was historically speaking the first rigorous derivation of the Boltzmann equation), by prescribing the specular reflexion as the way the particles and the boundary of the domain interact. The domain here will be the half-space.
This derivation is using sequences of equations, namely the BBGKY and the Boltzmann hierarchies. After a presentation of the formal derivation of those hierarchies in the case of a domain with boundary, one has carefully addressed the rigorous definition of those hierarchies, detailing the appropriate functional setting. In a second part, the proof of the convergence of the solutions of the BBGKY hierarchy towards the solutions of the Boltzmann hierarchy is presented exhaustively. In particular, the geometrical control of the recollisions in the case of the half-space has been written in the most comprehensive way.

Keywords : Analysis, PDEs, kinetic equations, Boltzmann equation, derivation of the Boltzmann equation, Lanford's theorem, BBGKY hierarchy, hard spheres.

À ma famille.

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Introduction

Ce travail a pour objet l'obtention rigoureuse de l'équation de Boltzmann à partir d'une modélisation particulaire de la matière constituant un gaz, qui évolue à l'intérieur d'un volume délimité par une paroi.

Dans cette introduction, on présentera brièvement au cours d'une première partie l'équation de Boltzmann, son histoire, sa forme et sa signification. Dans cette même partie, on abordera quelques propriétés élémentaires vérifiées par les solutions de cette équation, et on décrira leurs interprétations physiques.
Dans une deuxième partie, on présentera un bref état de l'art sur l'équation de Boltzmann, et les grands problèmes ouverts la concernant. En particulier, on introduira la question centrale autour de laquelle s'articule entièrement ce travail : le problème de l'obtention rigoureuse (on parle alors de *dérivation*) de l'équation de Boltzmann.
Enfin, dans une troisième partie, les résultats réunis dans ce travail seront présentés.

Chapitre 1

L'équation de Boltzmann

1.1 Le contexte

L'équation de Boltzmann s'inscrit dans le cadre de la théorie des équations aux dérivées partielles qui ont trait à la physique, et en particulier la théorie des équations cinétiques.
Les équations aux dérivées partielles de la physique mésoscopique et macroscopique forment d'une famille particulière d'équations aux dérivées partielles. Elles apparaissent lorsque l'on tente de comprendre le comportement d'un fluide, bien souvent un gaz ou un plasma, en effectuant une limite à partir d'une description particulaire, c'est-à-dire une description à l'échelle microscopique par les constituants élémentaires de ce fluide. Autrement dit, en considérant un très grand nombre d'atomes, molécules ou ions, constituant le fluide étudié, ou de façon plus générale, de particules, qui évoluent en suivant des lois qui décrivent leurs interactions, avec le milieu ambiant d'une part, et entre particules d'autre part, on cherche à étudier le comportement global de cet ensemble de particules élémentaires. L'idée remonte au moins au XVIIIe siècle, au cours duquel le physicien et mathématicien suisse Daniel Bernoulli, décédé à Bâle, a publié en 1738 le traité *Hydrodynamica, sive de Viribus et Motibus Fluidorum commentarii* [5]. On parle ici de comprendre le comportement d'un fluide, en terme de mouvements et de phénomènes que l'on observe à échelle humaine. Ces comportements peuvent par exemple être décrits par des températures, des pressions, ou des vitesses d'écoulement. Bernoulli établit par exemple dans son traité un lien entre la pression qu'exerce un gaz sur la paroi du récipient qui le contient, et les chocs contre cette paroi des particules qui constituent ce gaz.
Une telle équation est en fait un pari : on espère comprendre le monde par le comportement d'éléments, invisibles individuellement à l'échelle des sensations humaines, mais qui agissant ensemble produisent des effets mesurables. C'est en ce sens que l'on parle d'un passage de modèles microscopiques vers des modèles mésoscopiques ou macroscopiques.
Par ailleurs, cette recherche de conséquences sur le comportement d'un tout (à

une échelle dépassant de loin celle à laquelle évoluent les constituants élémentaires et très nombreux de ce tout) à partir du comportement individuel et bien compris de ses constituants semble être une approche particulièrement féconde. De fait, ces équations constituent un outil de choix pour étudier des phénomènes aussi éloignés a priori que sont la propagation d'épidémies dans une population ou le mouvement des galaxies dans l'univers.

Enfin, une équation cinétique est une équation dont l'inconnue, qui sera notée f, est une fonction qui dépend de trois paramètres : le temps, noté t, une position de l'espace (souvent à trois dimensions, en vue des applications concrètes), notée x, et enfin une vitesse, caractérisée par une orientation, un sens, et une magnitude, ce qui peut aussi se représenter par un point de l'espace (une fois encore, en général à trois dimensions), notée v.
La quantité

$$f(t, x, v)$$

représente le nombre de particules constituant le gaz étudié qui se trouvent, à l'instant t, à la position x et se déplaçant à la vitesse v. Cette fonction f porte donc naturellement le nom de *densité de présence dans l'espace des phases*. Tout l'objet des équations cinétiques est de décrire l'évolution de cette densité de présence.
Dans tout ce travail, on se contentera d'étudier des particules décrites seulement par leur vitesse et leur position. En particulier, toutes les particules seront supposées identiques (avec une même masse), c'est-à-dire que le fluide étudié ne sera composé que d'une espèce. Par ailleurs, les particules représenteront des atomes, et non des molécules. Ainsi, l'énergie interne de chaque particule (qui sert à décrire, par exemple, les vibrations internes à certaines molécules) ne sera pas prise en compte. Enfin, on ne considérera ici que des particules neutres, qui n'interagissent pas avec un champ électrique ou magnétique.

1.2 La quantité décrite par l'équation de Boltzmann et l'obtention formelle de cette équation

L'équation de Boltzmann est une équation cinétique particulière. Avant de la décrire, on commence par présenter le modèle microscopique qui permet de l'obtenir.

1.2.1 Principe d'inertie et transport libre

Si dans un premier temps, on affecte à chaque particule du gaz une vitesse à l'instant initial $t = 0$, et si l'on suppose que les particules ne sont soumises qu'au principe d'inertie, qui signifie exactement que ces particules conservent leur vitesse pour toujours, et avancent donc en ligne droite à vitesse constante, on en déduit alors directement la position de chaque particule, à n'importe quel instant. En effet, si on note x la position d'une particule à l'instant initial $t = 0$

et v sa vitesse au même instant, sa position à n'importe quel instant t sera donnée par la quantité

$$x + tv.$$

Ainsi, si la répartition de l'ensemble des particules à l'instant initial est connue (c'est-à-dire, si on suppose connues la position et la vitesse de chacune des particules constituant le gaz étudié au temps $t = 0$), on peut calculer la densité de présence du gaz à l'instant initial, que l'on notera f_0 (et qui est donc une fonction qui ne dépend que de la position x et de la vitesse v). Et d'autre part, puisque l'on connaît la position et la vitesse de chaque particule, à chaque instant, on est donc en mesure de connaître la densité de présence à tout instant. Elle est donnée par l'expression

$$(t, x, v) \mapsto f(t, x, v) = f_0(x - tv, v).$$

Si on suppose la donnée initiale suffisamment régulière, on vérifie facilement qu'elle est solution de l'équation aux dérivées partielles suivante :

$$\partial_t f + v \cdot \nabla_x f = 0, \tag{1.1}$$

appelée équation de transport libre. Cette équation décrit donc le comportement d'une densité de présence d'un nuage de particules qui évoluent en suivant le principe d'inertie.

1.2.2 Le modèle des sphères dures

Il convient d'étudier des modèles moins simplistes que ce dernier. En suivant le même cheminement que Ludwig Boltzmann, on va donc supposer que les particules composant le gaz interagissent entre elles.

Mais comment les particules peuvent-elles interagir entre elles ? L'heuristique peut être d'un certain secours pour répondre à cette question, délicate au demeurant, puisque s'il n'est pas aisé d'observer les atomes ou les molécules, il est sans doute encore plus ardu d'observer directement des particules interagir entre elles. Un exemple relativement simple d'interactions entre constituants élémentaires d'un système, chacun des constituants suivant le principe d'inertie lorsqu'il est suffisamment éloigné de tous les autres constituants, est donné par des boules roulant sur un billard, et entrant de temps à autre en collision avec une autre boule. On va donc supposer, sans vergogne, que les particules, constituant le gaz que l'on étudie, évoluent selon les mêmes lois. Autrement dit, on va supposer que les particules sont des sphères ayant toutes le même rayon, qui sera noté $\varepsilon > 0$, et qui lorsqu'elles entrent en collision, voient leurs vitesses modifiées de telle sorte que le choc soit parfaitement élastique.
Bien entendu, il ne s'agit que d'un modèle, obtenu par analogie, et retenu pour sa simplicité : il est donc critiquable. Ainsi, d'autres modèles, plus réalistes, mais aussi plus complexes, s'attachent à décrire avec plus de soin les interactions entre les molécules qui constituent un fluide. On renvoie par exemple à

[23][1] où des modèles d'interaction basés sur des potentiels, comme le modèle de Lennard-Jones, sont présentés.

Pour définir des interactions qui représentent des collisions élastiques, on peut poser la loi suivante de modification des vitesses au cours d'une collision entre deux particules, qui se produit lorsque les centres x_1 et x_2 d'une paire de particules sont séparés par une distance d'exactement ε. Si l'on note v et v_* les vitesses respectives des deux particules avant la collision, aussi appelées vitesses *pré-collisionnelles*, on définit

$$v' = v - (v - v_*) \cdot \omega\omega \tag{1.2}$$

et

$$v'_* = v_* + (v - v_*) \cdot \omega\omega, \tag{1.3}$$

où v' et v'_* représentent les vitesses respectives des deux particules après la collision, et ω le vecteur normalisé de la position relative de la seconde particule par rapport à la première, c'est-à-dire :

$$\omega = \frac{x_2 - x_1}{\left|x_2 - x_1\right|}.$$

On parlera de vitesses *post-collisionnelles* pour v' et v'_*, et de *paramètre angulaire de collision* pour ω.
Dire que les particules entrent en collision au temps τ_0 signifie d'une part que

$$\left|x_1(\tau_0) - x_2(\tau_0)\right| = \varepsilon \qquad \text{(condition de contact).}$$

D'autre part, puisque les trajectoires respectives des particules s'expriment sous la forme

$$x_1(\tau) = x_1(\tau_0) + (\tau - \tau_0)v$$

et

$$x_2(\tau) = x_2(\tau_0) + (\tau - \tau_0)v_*,$$

au moins sur un court intervalle de temps avant la collision, dire que la collision a lieu signifie que la distance entre les particules va en diminuant avant l'impact, c'est-à-dire que :

$$\left[\frac{\mathrm{d}}{\mathrm{d}\tau}\left|x_1(\tau) - x_2(\tau)\right|\right]_{\tau=\tau_0^-} < 0,$$

[1] Voir la section 4 "Generalizations" du chapitre II "THE BOLTZMANN EQUATION", et en particulier la discussion qui débute page 59.

mais puisque l'expression explicite des trajectoires juste avant l'impact permet d'obtenir :

$$\begin{aligned}\left[\frac{\mathrm{d}}{\mathrm{d}\tau}\big|x_1(\tau)-x_2(\tau)\big|^2\right]_{\tau=\tau_0^-} &= \left[\frac{\mathrm{d}}{\mathrm{d}\tau}\Big|\big(x_1(\tau_0)-x_2(\tau_0)\big)+(\tau-\tau_0)\big(v-v_*\big)\Big|^2\right]_{\tau=\tau_0^-}\\ &= \left[\frac{\mathrm{d}}{\mathrm{d}\tau}\Big(\big|x_1(\tau_0)-x_2(\tau_0)\big|^2\right.\\ &\left.+2(\tau-\tau_0)\big(x_1(\tau_0)-x_2(\tau_0)\big)\cdot\big(v-v_*\big)+(\tau-\tau_0)^2\big|v-v_*\big|^2\Big)\right]_{\tau=\tau_0^-}\\ &= 2\big(x_1(\tau_0)-x_2(\tau_0)\big)\cdot\big(v-v_*\big),\end{aligned}$$

et que la distance entre les particules est bien entendu une quantité strictement positive, la condition se réécrit :

$$\left[\frac{\mathrm{d}}{\mathrm{d}\tau}\big|x_1(\tau)-x_2(\tau)\big|\right]_{\tau=\tau_0^-}<0 \quad\Leftrightarrow\quad \big(x_1(\tau_0)-x_2(\tau_0)\big)\cdot\big(v-v_*\big)<0$$

(condition de configuration pré-collisionnelle).

On observe en particulier, grâce aux expressions explicites (1.2) et (1.3) des vitesses post-collisionnelles, que l'on a :

$$\begin{aligned}\frac{\big(x_1(\tau_0)-x_2(\tau_0)\big)}{\big|x_1(\tau_0)-x_2(\tau_0)\big|}\cdot\big(v'-v'_*\big) &= -\omega\cdot\big(v-v_*-2(v-v_*)\cdot\omega\omega\big)\\ &= \omega\cdot\big(v-v_*\big),\end{aligned}$$

si bien que

$$\big(x_1(\tau_0)-x_2(\tau_0)\big)\cdot\big(v'-v'_*\big)>0,$$

ou encore

$$\left[\frac{\mathrm{d}}{\mathrm{d}\tau}\big|x_1(\tau)-x_2(\tau)\big|\right]_{\tau=\tau_0^+}>0,$$

ce qui montre que les particules vont avoir tendance à s'écarter après la collision. Par ailleurs, si l'on calcule la quantité de mouvement des deux particules après la collision (en divisant par les masses des particules, que l'on aura supposées égales) :

$$\begin{aligned}v'+v'_* &= \big(v-(v-v_*)\cdot\omega\omega\big)+\big(v_*+(v-v_*)\cdot\omega\omega\big)\\ &= v+v_*,\end{aligned}\tag{1.4}$$

ainsi que l'énergie cinétique post-collisionnelle du même système :

$$\begin{aligned}\frac{|v'|^2}{2}+\frac{|v'_*|^2}{2} &= \frac{1}{2}\Big(|v|^2-2\big((v-v_*)\cdot\omega\big)\big(v\cdot\omega\big)+\big|(v-v_*)\cdot\omega\big|^2\\ &\qquad +|v_*|^2+2\big((v-v_*)\cdot\omega\big)\big(v_*\cdot\omega\big)+\big|(v-v_*)\cdot\omega\big|^2\Big)\\ &= \frac{1}{2}\Big(|v|^2-2\big((v-v_*)\cdot\omega\big)^2+|v_*|^2+2\big|(v-v_*)\cdot\omega\big|^2\Big)\\ &= \frac{|v|^2}{2}+\frac{|v_*|^2}{2}, \end{aligned} \tag{1.5}$$

on peut observer que l'on a construit un modèle qui transforme les paires de vitesses pré-collisionnelles en vitesses post-collisionnelles, et qui préserve la quantité de mouvement et l'énergie cinétique au cours du temps (dans le cadre de collisions entre molécules monoatomiques, où l'énergie interne microscopique n'apparaît pas), ce qui est parfaitement satisfaisant.

On peut remarquer que des modèles de collision inélastiques existent aussi (voir par exemple [6], ainsi que [64][2] pour un tour d'horizon de références sur le sujet). Mais il semble que le problème de la dérivation de l'équation de Boltzmann à partir d'un tel modèle soit encore un probème ouvert et qu'il s'avère des plus délicats.

1.2.3 Le cas d'un domaine à bord

La spécificité de ce travail réside dans la présence d'un obstacle. De nombreuses références ([25], [26],[64]) incluent dans leur présentation introductive la possibilité de considérer un fluide évoluant autour d'un obstacle, mais il n'existe pas, à la connaissance de l'auteur de ce travail, de preuve rigoureuse de la dérivation de l'équation de Boltzmann pour des domaines à bord. Par exemple le théorème de Lanford, résultat principal de [34][3] est énoncé dans le cas où le domaine est l'espace euclidien $\mathbb{R}^d$ tout entier, ou bien le tore $\mathbb{T}^d$.
Si cette fois on souhaite obtenir une dérivation de l'équation de Boltzmann pour un domaine à bord, il convient de choisir un modèle pour les interactions entre les particules et le bord du domaine.
Il existe bien des façons de procéder, mais la plus simple consiste en le choix de la réflexion spéculaire sur le bord de l'obstacle. Cette réflexion, aussi appelée loi de Snell-Descartes (en référence au phénomène optique de réflexion sur un miroir), est la plus facile à concevoir et à manipuler à bien des égards, ne serait-ce que parce que le modèle heuristique des boules de billard est la transcription de l'observation de phénomènes se produisant à échelle humaine.
À l'échelle microscopique, on va donc supposer que lorsqu'une particule du système, à la position $x(\tau)$ au temps τ, atteint l'obstacle (que l'on notera Ω) au

[2] Voir en particulier le paragraphe "Enskog equation", section 2.1 "Derivation issues : problems of separation of scales" du dernier chapitre.
[3] Voir la section 3.1 "Lanford and King's theorems".

temps τ_0, c'est-à-dire que

$$x(\tau_0) \in \partial\Omega,$$

sa vitesse $v(\tau_0^-)$ juste avant l'impact, qui vérifie la condition :

$$v(\tau_0^-) \cdot n(x(\tau_0)) < 0,$$

(où $n(x(\tau_0))$ est le vecteur normal sortant à l'obstacle au point $x(\tau_0)$), est modifiée par la symétrie orthogonale suivante :

$$v(\tau_0^+) = v(\tau_0^-) - 2\big(v(\tau_0^-) \cdot n(x(\tau_0))\big)n(x(\tau_0)). \tag{1.6}$$

La particule effectue donc un rebond contre l'obstacle Ω, et bien entendu, la vitesse de la particule immédiatement après le rebond vérifie

$$v(\tau_0^+) \cdot n(x(\tau_0)) > 0.$$

À l'échelle macroscopique, cette interaction avec l'obstacle se traduit par la condition de bord suivante, vérifiée par la densité de présence en tout point x du bord de l'obstacle :

$$\forall x \in \partial\Omega,\ \forall v \in \mathbb{R}^d \text{ tel que } v \cdot n(x) > 0,\ f(x,v) = f\big(x, v - 2\big(v \cdot n(x)\big)v\big). \tag{1.7}$$

L'attrait de ce modèle pour sa simplicité est contrebalancé par quelques défauts importants. On renvoie à [64][4] pour une discussion sur les limitations de ce modèle, ainsi que pour un panorama d'autres modèles.
Notamment, une limitation significative réside en le fait qu'il est établi depuis longtemps (au moins depuis Maxwell) que le modèle de la réflexion spéculaire ne traduit pas de façon satisfaisante les interactions gaz-paroi réelles. Dans [51][5], James Clerk Maxwell introduisit et justifia l'utilisation d'un modèle hybride, où une certaine proportion des particules subit une simple réflexion spéculaire contre l'obstacle, tandis que le reste est absorbé par la paroi puis réémis de façon diffuse, en suivant une loi gaussienne donnée (on parle de diffusion de Maxwell), pour tout point x appartenant au bord du domaine, par la formule suivante :

$$f(x,v) = \Big(\int_{v' \cdot n(x) < 0} f(x,v')\big|v' \cdot n(x)\big|\,\mathrm{d}v'\Big)M_w(v),$$

où $n(x)$ est le vecteur normal à l'obstacle en x, et M_w est la maxwellienne :

$$M_w(x) = \frac{\exp\Big(-\frac{|v|^2}{2T_w}\Big)}{\big(2\pi\big)^{\frac{d-1}{2}} T_w^{\frac{d+1}{2}}},$$

où T_w est la température correspondant à l'équilibre thermodynamique de la paroi, et d est la dimension de l'espace euclidien ambiant $\mathbb{R}^d$ qui contient le domaine dans lequel évolue le fluide considéré. Le lecteur pourra consulter [23][6] pour une présentation moderne de ce modèle.

[4] Voir la section 1.5 "Boundary conditions" du chapitre I.

[5] En particulier, une discussion sur le modèle à choisir pour l'interaction entre le gaz et la paroi se trouve dans l'appendice de cette référence.

[6] Voir la section 5 "Maxwell's boundary conditions. Accomodation coefficients" du chapitre III.

1.2.4 L'équation de Boltzmann

On peut maintenant aborder le problème de la dérivation formelle de l'équation de Boltzmann, à partir du modèle basé sur les interactions particulaires décrites plus haut.
En ce qui concerne le contexte historique, il convient de noter que les travaux fondateurs de August Karl Krönig ([47]) datant de 1856, puis ceux obtenus de façon indépendante par Rudolf Clausius ([28]) en 1857, ont conduit Maxwell en 1860, dans [52], à établir la loi de distribution des vitesses des particules d'un gaz à l'équilibre thermodynamique. Il montra que les vitesses sont réparties selon une loi gaussienne particulière, qui s'écrit sous la forme :

$$M(v) = \exp\big(- Av \cdot v + b \cdot v + c\big),$$

où $A = \lambda I_d$ est la matrice d'une homothétie, b est un vecteur de $\mathbb{R}^d$, et c est un nombre réel. De tels profils portent aujourd'hui son nom : dans le contexte de la théorie cinétique, on parle de *maxwelliennes*.
Dans la continuité des travaux de Maxwell, Ludwig Boltzmann généralise ses résultats dans le cadre d'un gaz hors de l'équilibre thermodynamique, dans l'article [13] publié en 1872, alors qu'il était professeur à Graz. Ainsi écrit-il pour la première fois l'équation qui porte son nom et qui est le principal objet d'étude de ce présent travail.
Bien des références traitent de la dérivation formelle de cette équation. Le lecteur pourra consulter, outre les écrits historiques [13] et [14] de Boltzmann lui-même, les références plus récentes [26][7], [23][8], ou encore [37][9].
Bien que la dérivation formelle de l'équation de Boltzmann ne sera pas présentée ici, on s'arrêtera malgré tout sur les hypothèses qui permettent de l'obtenir. En particulier, des hypothèses supplémentaires à celles déjà vérifiées par le modèle des sphères dures seront requises. Par ailleurs, les propriétés vérifiées par le modèle choisi ne lui seront pas propres, de telle sorte que d'autres modèles microscopiques permettent d'obtenir aussi l'équation de Boltzmann.

On rappelle que l'on a complexifié le modèle simpliste des particules qui n'évoluent que par inertie, et qui avait permis d'obtenir l'équation de transport libre (1.1) vérifiée par la densité de présence f, qui s'écrivait

$$\partial_t f = -v \cdot \nabla_x f,$$

et qui signifie donc que, pour un instant t, pour une position x et une vitesse v données, la seule façon pour la densité de particule à l'instant t et en (x, v) d'évoluer au cours du temps provient d'un apport de particules se déplaçant à la vitesse v, et atteignant le voisinage de la position x au temps t.
Dans le cas de l'équation de Boltzmann, les collisions entre particules décrites

[7] Voir les trois premières sections du chapitre II : "Informal Derivation of the Boltzmann Equation".
[8] Voir les cinq premières sections du chapitre II "THE BOLTZMANN EQUATION".
[9] Voir les sections 1.1.2 "A formal "derivation" of the Boltzmann Equation", 1.2 "The Form of the Collision Operator", et 1.3 "The Hard Sphere Case".

par le modèle des sphères dures représentent une autre façon de modifier le nombre de particules se déplaçant à la vitesse v, et donc de modifier la valeur de la densité de présence. On s'attend donc à ce que la densité vérifie une équation de la forme :

$$\partial_t f(t,x,v) = -v \cdot \nabla_x f(t,x,v) + Q(t,x,v), \tag{1.8}$$

où Q est à déterminer.
Les hypothèses nécessaires pour obtenir l'expression du terme Q, et donc en fin de compte l'équation de Boltzmann, sont les suivantes. Le lecteur pourra consulter par exemple les références [26][10] pour une discussion sur la signification physique des hypothèses requises pour l'obtention de l'équation de Boltzmann, et en particulier en ce qui concerne la condition de chaos moléculaire, ou [64][11] pour une présentation des modèles autres que celui des sphères dures qui vérifient aussi les hypothèses qui suivent.

1. Les particules interagissent essentiellement via des collisions binaires, c'est-à-dire que la probabilité qu'une collision impliquant trois particules se produise est négligeable devant celles impliquant seulement deux particules.
   Cette hypothèse implique que le terme Q écrit dans l'équation précédente ne fait intervenir que la densité de présence de deux particules, que l'on va noter $f^{(2)}$. Les densités de présence d'un plus grand nombre de particules n'ont pas d'effet sur ce terme. Autrement dit, on doit avoir une expression de la forme
   $$Q(t,x,v) = Q(f^{(2)})(t,x,v).$$

2. Les effets des collisions sur la densité de présence sont localisées en temps et en espace. Ceci signifie qu'au temps t, à la position x et pour la vitesse v, seules les particules présentes au voisinage de la position x au temps t auront un impact significatif sur la valeur de la densité de présence en (t,x,v).
   En particulier, ce sera le cas pour un modèle tel que le temps d'interaction entre deux particules est bref, et que la distance maximale à laquelle les particules peuvent interagir est négligeable, ou autrement dit, petite devant l'échelle spatiale typique du problème. Il est clair d'une part que le modèle des sphères dures vérifie cette condition, et d'autre part que d'autres modèles satisfont aussi cette hypothèse. Le modèle des sphères dures est d'ailleurs un des plus délicats à traiter au vu des autres modèles candidats vérifiant cette hypothèse, puisque le potentiel d'interaction est singulier, en ce sens que les interactions entre particules se limitent aux parties $\left\{\left|x_i - x_j\right| = \varepsilon\right\}$ de l'espace des phases. Le potentiel est concentré sur ces sous-variétés, et est en ce sens singulier.

[10] En particulier, les sections 2.2 "Boltzmann's Argument in a Modern Perspective" et 2.3 "Molecular Chaos. Critique and Justification".
[11] En particulier : la section 1.3 "Boltzmann's collision operator".

On peut citer par exemple [34][12], référence dans laquelle la dérivation rigoureuse de l'équation de Boltzmann à partir d'un modèle de particules interagissant par le biais d'un potentiel régulier, défini sur un support petit, est traitée à part égale avec le modèle des sphères dures.
Une tentative intéressante de relaxation de l'hypothèse de localisation du support du potentiel régissant les interactions entre particules, dans le cadre de la dérivation rigoureuse de l'équation de Boltzmann *linéaire*, est proposée dans [3]. Nathalie Ayi a besoin d'un potentiel avec une très forte décroissance[13] en la variable d'espace x (en $\exp\big(\exp\big(\exp\big(-|x|^k\big)\big)\big)$).
Dans [64][14], Cédric Villani fait remarquer que, bien qu'il semble paradoxal d'étudier des modèles microscopiques basés sur des interactions à longue portée entre les particules, et devant conduire à des modèles macroscopiques traduisant des interactions localisées, il n'y a en fait pas de contradiction, puisque les effets macroscopiques des interactions à longue distance peuvent être négligeables.
On peut aussi citer [30], qui présente une dérivation rigoureuse de l'équation de Boltzmann linéaire à partir d'un modèle basé sur une particule évoluant entre des obstacles fixes agissant sur elle à longue distance. Dans cet article et contrairement à celui de Nathalie Ayi, les interactions ont malgré tout un support borné, dont la borne est envoyée à l'infini avec le nombre de particules du système. Cependant, le résultat de convergence obtenu vers les solutions de l'équation de Boltzmann linéaire est global en temps, ce qui n'est pas le cas dans [3].
Avec cette hypothèse, le terme Q peut en fait se réecrire

$$\begin{aligned} Q(t,x,v) = \int_{v_*}\int_{v'}\int_{v'_*} \Big[& P\big((v',v'_*)\to(v,v_*)\big) f^{(2)}(t,x,v',x,v'_*) \\ & - P\big((v,v_*)\to(v',v'_*)\big) f^{(2)}(t,x,v,x,v_*)\Big]\,\mathrm{d}v'_*\,\mathrm{d}v'\,\mathrm{d}v_*, \end{aligned}$$

(le fait d'intégrer seulement sur les variables de vitesse traduit la localisation en temps et en espace) où $P\big((v',v'_*)\to(v,v_*)\big)$ représente la probabilité que deux particules, qui se trouvent au même instant au voisinage du même point, avec des vitesses respectives v' et v'_*, entrent en collision et voient leurs vitesses respectives modifiées pour devenir v et v_*. Ce genre de collision fait bien entendu augmenter le nombre de particules ayant la vitesse v, et ces collisions seront d'autant plus nombreuses qu'il y aura un nombre important de couples de particules avec des vitesses respectives v' et v'_*, ce qui explique le terme

$$P\big((v',v'_*)\to(v,v_*)\big) f^{(2)}(t,x,v',x,v'_*),$$

qui est un terme positif.
En revanche, une collision impliquant une particule avec une vitesse pré-

[12] En particulier, la partie III "The case of short range potentials".
[13] Voir la section 4 "Terms associated to the long-range part of the potential" pour plus de détails.
[14] Voir la section 1.4 "Collision kernels".

collisionnelle v va nécessairement faire décroître le nombre de particules avec cette vitesse, ce qui explique le terme de perte

$$P\big((v,v_*)\to(v',v'_*)\big)f^{(2)}(t,x,v,x,v_*),$$

retranché dans l'intégrale ci-dessus.

3. La quantité de mouvement et l'énergie cinétique sont conservées lors des collisions. Dans le cas du modèle des sphères dures, les calculs présentés plus haut ((1.4) et (1.5)) montrent que cette hypothèse est vérifiée.
Avec cette hypothèse, le terme de Q se réécrit encore :

$$\begin{aligned}Q(t,x,v)=\int_{v_*}\int_{v'}\int_{v'_*}\Big[&P\big((v',v'_*)\to(v,v_*)\big)f^{(2)}(t,x,v',x,v'_*)\\&-P\big((v,v_*)\to(v',v'_*)\big)f^{(2)}(t,x,v,x,v_*)\Big]\\&\times\mathbb{1}_{v+v_*=v'+v'_*,\frac{|v|^2}{2}+\frac{|v_*|^2}{2}=\frac{|v'|^2}{2}+\frac{|v'_*|^2}{2}}\,\mathrm{d}v'_*\,\mathrm{d}v'\,\mathrm{d}v_*.\end{aligned}$$

4. Les collisions sont *micro-réversibles*,
c'est-à-dire que la dynamique définie à l'échelle particulaire est réversible en temps. C'est bien sûr le cas pour le modèle des sphères dures. Cette hypothèse implique que

$$P\big((v',v'_*)\to(v,v_*)\big)=P\big((v,v_*)\to(v',v'_*)\big),$$

et donc le terme Q se réécrit

$$\begin{aligned}Q(t,x,v)=\int_{v_*}\int_{v'}\int_{v'_*}&P\big((v,v_*)\to(v',v'_*)\big)\\&\times\Big[f^{(2)}(t,x,v',x,v'_*)-f^{(2)}(t,x,v,x,v_*)\Big]\\&\times\mathbb{1}_{v+v_*=v'+v'_*,\frac{|v|^2}{2}+\frac{|v_*|^2}{2}=\frac{|v'|^2}{2}+\frac{|v'_*|^2}{2}}\,\mathrm{d}v'_*\,\mathrm{d}v'\,\mathrm{d}v_*.\end{aligned}$$

5. L'hypothèse du chaos moléculaire de Boltzmann est vérifiée.
Cette hypothèse, aussi connue sous le nom de *Stosszahlansatz* et au sens très subtil (voir à ce sujet la discussion présentée dans [64][15]), signifie que les vitesses de deux particules sur le point d'entrer en collision sont décorrélées, et donc que la densité de présence d'un couple de particules se tensorise, autrement dit :

$$f^{(2)}(t,x,v,x,v_*)=f(t,x,v)f(t,x,v_*).$$

Avec ces cinq hypothèses, et en utilisant une paramétrisation appropriée, on obtient l'équation de Boltzmann, qui s'écrit :

$$\partial_t f+v\cdot\nabla_x f=Q(f,f), \tag{1.9}$$

[15] Elle se trouve à la fin de la section 2.1 "Mathematical validity of the Boltzmann equation" du premier chapitre.

où $Q(f,f)$ représente ce que l'on appelle souvent le *terme de collision*, et qui admet pour écriture explicite l'expression suivante :

$$\int_{\mathbb{R}^d_{v_*}} \int_{\mathbb{S}^{d-1}_\omega} B(v-v_*,\omega)\Big[f(t,x,v')f(t,x,v'_*) - f(t,x,v)f(t,x,v_*)\Big]\,\mathrm{d}\omega\,\mathrm{d}v_*, \tag{1.10}$$

avec B le *noyau de collision*, dont l'expression dépend du modèle d'interactions entre particules choisi, et qui ne dépend que de $v-v_*$ et de ω, le paramètre angulaire introduit dans (1.2) et (1.3) pour définir les vitesses post-collisionnelles. En réalité, vu l'invariance galiléenne du problème, il ne dépend en fait que de la norme $|v-v_*|$, et du produit scalaire $(v-v_*)\cdot\omega$.
Dans le cas des sphères dures (voir [23][16] ou [37][17]), le noyau de collision est donné par l'expression

$$|\omega\cdot(v-v_*)|.$$

On renvoie le lecteur à [64][18] pour une discussion sur les noyaux de collisions provenant d'autres modèles.
On a donc obtenu une équation aux dérivées partielles, composée d'un terme de transport, et d'un terme intégral, quadratique en l'inconnue f. Bien souvent, on trouvera dans les références la notation $Q(f,f)$ pour le terme de collision, dans le but d'insister sur son caractère quadratique en f.

1.3 Résultats élémentaires sur l'équation de Boltzmann

1.3.1 Lois de conservation

On l'a vu, l'équation de Boltzmann a pour principal souci la description de la matière à l'échelle mésoscopique ou macroscopique, en se basant sur une modélisation raisonnable de la matière à l'échelle microscopique, qui prend en compte des contraintes physiques telles que la conservation de la quantité de mouvement, ou de l'énergie cinétique à cette échelle au cours du temps. L'étude du terme de collision va apporter quelques résultats intéressants quant au comportement qualitatif des solutions de l'équation de Boltzmann.

Puisque l'inconnue f de l'équation de Boltzmann est une densité de présence de particules, c'est-à-dire que $f(t,x,v)$ représente à l'instant t, la densité de particules au voisinage du point x, avec une vitesse qui se trouve au voisinage de v, on va s'intéresser aux quantités

$$\int_x \int_v f(t,x,v)\,\mathrm{d}v\,\mathrm{d}x,$$

[16] Voir la section 3 "The Boltzmann equation for rigid spheres" du chapitre II.
[17] Voir la section 1.3 "The Hard Sphere Case".
[18] En particulier, voir la section 1.4 "Collision kernels" du premier chapitre.

$$\int_x \int_v v f(t,x,v)\,\mathrm{d}v\,\mathrm{d}x$$

et

$$\int_x \int_v \frac{|v|}{2}^2 f(t,x,v)\,\mathrm{d}v\,\mathrm{d}x,$$

qui représentent respectivement la masse, la quantité de mouvement et l'énergie cinétique totales du gaz. Pour ce faire, on va poser une quantité plus générale

$$\int_x \int_v f(t,x,v)\varphi(v)\,\mathrm{d}v\,\mathrm{d}x,$$

qui conduit naturellement à étudier une formulation faible de l'équation de Boltzmann (voir [64] pour une discussion historique sur cette manipulation).

Soit donc f une solution de l'équation de Boltzmann, supposée suffisamment régulière pour que les manipulations suivantes soient valides, et φ une fonction régulière qui ne dépend que de la vitesse v. On écrit

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_x \int_v f(t,x,v)\varphi(v)\,\mathrm{d}v\,\mathrm{d}x &= \int_x \int_v \big(\partial_t f\big)(t,x,v)\varphi(v)\,\mathrm{d}v\,\mathrm{d}x \\ &= \int_x \int_v \Big(-v\cdot\nabla_x f + Q(f,f)\Big)(t,x,v)\varphi(v)\,\mathrm{d}v\,\mathrm{d}x.\end{aligned}$$

Le premier terme du membre de droite peut se réécrire

$$\int_v -v\cdot\Big(\int_x \nabla_x f(t,x,v)\,\mathrm{d}x\Big)\varphi(v)\,\mathrm{d}v,$$

et donc si la solution f vérifie une condition d'intégrabilité par rapport à la variable d'espace x, ou si le domaine est compact (par exemple si $x \in \mathbb{T}^d$, où $\mathbb{T}^d$ est le tore $\mathbb{R}^d/\mathbb{Z}^d$), et si les conditions éventuelles au bord du domaine d'intégration en la variable de position x le permettent, ce terme s'annule, et on trouve

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_x \int_v f(t,x,v)\varphi(v)\,\mathrm{d}v = \int_x \int_v Q(f,f)(t,x,v)\varphi(v)\,\mathrm{d}v\,\mathrm{d}x.$$

On sera donc en mesure de déterminer l'ensemble des quantités conservées par l'équation au cours du temps si l'on est en mesure de déterminer l'ensemble des fonctions φ qui annulent la quantité

$$\int_x \int_v Q(f,f)(t,x,v)\varphi(v)\,\mathrm{d}v\,\mathrm{d}x.$$

De telles fonctions φ s'appellent des *invariants de collision*.
En utilisant les propriétés de symétrie vérifiées par le terme de collision (voir

par exemple [26][19]), on trouve

$$\int_x \int_v Q(f,f)(t,x,v)\varphi(v)\,\mathrm{d}v\,\mathrm{d}x$$
$$= \frac{1}{4}\int_v \int_{v_*} \int_\omega B(v-v_*,\omega)\big(f(v')f(v'_*) - f(v)f(v_*)\big)$$
$$\times \big(\varphi(v)+\varphi(v_*)-\varphi(v')-\varphi(v'_*)\big)\,\mathrm{d}\omega\,\mathrm{d}v_*\,\mathrm{d}v. \tag{1.11}$$

On est donc conduit à étudier, comme Boltzmann l'avait déjà remarqué ([11] et [12]), l'équation suivante :

$$\varphi(v)+\varphi(v_*) = \varphi(v')+\varphi(v'_*). \tag{1.12}$$

On peut montrer que les seules solutions qui vérifient l'égalité (1.12) sont les combinaisons linéaires des fonctions

$$1, v_i \text{ et } \frac{|v|}{2}^2,$$

où $1 \leq i \leq d$.

On a donc montré que l'équation de Boltzmann conserve la masse totale, la quantité de mouvement et l'énergie cinétique totale, c'est-à-dire

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_x \int_v f(t,x,v)\,\mathrm{d}v\,\mathrm{d}x = 0, \tag{1.13}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_x \int_v v f(t,x,v)\,\mathrm{d}v\,\mathrm{d}x = 0, \tag{1.14}$$

et

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_x \int_v \frac{|v|}{2}^2 f(t,x,v)\,\mathrm{d}v\,\mathrm{d}x = 0. \tag{1.15}$$

Du reste, la détermination des invariants de collision a été obtenue pour des hypothèses de plus en plus faibles au cours des années. Le lecteur pourra consulter les références suivantes, classées par ordre chronologique : [39], [40], [19], [24], [2], [26][20]. On peut aussi consulter l'article récent [16], qui adapte une preuve déjà présente dans [29] et utilisée pour étudier l'équation de Landau.

1.3.2 Entropie et flèche du temps

Un des principaux attraits de l'équation de Boltzmann réside dans l'information intrinsèque qu'elle contient : la flèche du temps. On peut s'apercevoir en effet

[19] Voir la section 3.1 "Collision Invariants" du chapitre III.
[20] Voir la section 3.1 "Collision Invariants" du chapitre III, à partir de la page 36.

que l'on peut associer à chaque solution de l'équation une quantité qui dépend du temps, et qui est systématiquement décroissante, et que cette quantité n'est stationnaire que lorsque la solution en question a atteint un profil bien particulier.

En effet, soit f une solution de l'équation de Boltzmann. On lui associe, en suivant par exemple les références [26][21] ou [55][22], la quantité suivante :

$$H(f)(t) = \int_x \int_v f(t,x,v) \ln f(t,x,v) \, \mathrm{d}v \, \mathrm{d}x, \tag{1.16}$$

que l'on appelle l'*entropie*. Il convient du reste de noter que la tradition dans l'étude mathématique de l'équation de Boltzmann veut que l'on prenne pour définition de l'entropie l'opposé de l'entropie utilisée dans les travaux de physique.
La dérivée par rapport au temps de cette quantité vérifie (au niveau formel, et sous réserve que les conditions aux limites au bord du domaine en la variable d'espace x le permettent)

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} H(f)(t) &= \int_x \int_v \partial_t \big(f \ln f\big)(t,x,v) \, \mathrm{d}v \, \mathrm{d}x \\
&= \int_x \int_v \partial_t f(t,x,v) \big(\ln f(t,x,v) + 1\big) \, \mathrm{d}v \, \mathrm{d}x \\
&= - \int_x \int_v v \cdot \big(\nabla_x f\big)(t,x,v) \ln f(t,x,v) \, \mathrm{d}v \, \mathrm{d}x \\
&\qquad + \int_x \int_v Q(f,f)(t,x,v) \ln f(t,x,v) \, \mathrm{d}v \, \mathrm{d}x,
\end{aligned}$$

et puisque

$$\nabla_x \big(f \ln f\big) = \big(\nabla_x f\big) \ln f + f \nabla_x \big(\ln f\big) = \nabla_x f \big(\ln f + 1\big),$$

on trouve

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} H(f)(t) = \int_x \int_v v \cdot \nabla_x f \, \mathrm{d}v \, \mathrm{d}x - \int_x \int_v v \cdot \nabla_x \big(f \ln f\big)(t,x,v) \, \mathrm{d}v \, \mathrm{d}x \\
+ \int_x \int_v Q(f,f)(t,x,v) \ln f(t,x,v) \, \mathrm{d}v \, \mathrm{d}x,
\end{aligned}$$

et donc

$$\frac{\mathrm{d}}{\mathrm{d}t} H(f)(t) = \int_x \int_v Q(f,f)(t,x,v) \ln f(t,x,v) \, \mathrm{d}v \, \mathrm{d}x. \tag{1.17}$$

[21] Voir la section 3.4 "The H-Theorem" du chapitre III.
[22] Voir la section 1.1.2 "H Functional and H Theorem" du premier chapitre.

De la même façon que les symétries permettent d'obtenir l'équation (1.11), le membre de droite de la précédente équation (1.17) se réécrit

$$\begin{aligned}\int_x\int_v Q(f,f)(t,x,v)\ln f(t,x,v)\,\mathrm{d}v\,\mathrm{d}x \\ = -\frac{1}{4}\int_x\int_v\int_{v_*}\int_\omega B(v-v_*,\omega) \\ \times\big(f(v')f(v'_*)-f(v)f(v_*)\big)\ln\Big(\frac{f(v')f(v'_*)}{f(v)f(v_*)}\Big)\,\mathrm{d}\omega\,\mathrm{d}v_*\,\mathrm{d}v\,\mathrm{d}x.\end{aligned} \tag{1.18}$$

L'inégalité élémentaire $\forall\ x>0,\ (x-1)\ln(x)\geq 0$ appliquée à la quantité

$$x=\frac{f(v')f(v'_*)}{f(v)f(v_*)}$$

(sous réserve que la solution f de l'équation de Boltzmann est strictement positive, ce qui n'est pas un problème trivial : voir [64][23]), on en déduit que la dérivée temporelle de l'entropie $H(f)$ est négative.

La décroissance de l'entropie constitue ce que l'on appelle le théorème H, publié par Ludwig Boltzmann en 1872, que l'on peut trouver à la fin de [13]. De fait, puisque l'entropie associée à une solution ne peut que décroître, il devient alors essentiel de comprendre, d'une part, à quoi ressemblent les solutions pour lesquelles l'entropie associée est stationnaire (on parle d'*équilibres thermodynamiques* pour de telles solutions), et d'autre part, vers quoi convergent les solutions lorsque le temps avance, c'est-à-dire lorsque leur entropie décroît. En particulier, est-ce que les solutions convergent vers les équilibres thermodynamiques ?

On pourrait déterminer les équilibres thermodynamiques en procédant comme dans la section précédente, puisqu'une solution f dotée d'une entropie constante vérifie :

$$\begin{aligned}0=\frac{\mathrm{d}}{\mathrm{d}t}H(f)(t)=-\frac{1}{4}\int_x\int_v\int_{v_*}\int_\omega B(v-v_*,\omega) \\ \times\big(f(v')f(v'_*)-f(v)f(v_*)\big)\ln\Big(\frac{f(v')f(v'_*)}{f(v)f(v_*)}\Big)\,\mathrm{d}\omega\,\mathrm{d}v_*\,\mathrm{d}v\,\mathrm{d}x,\end{aligned}$$

et donc le logarithme $\ln f$ de cette solution satisfait à l'équation (1.12), ce qui permet d'en déduire une expression de f. Mais le lecteur pourra aussi consulter [55], où est donnée dès la deuxième page[24] une détermination des équilibres thermodynamiques par une méthode de minimisation sous contraintes. On minimise

[23] En particulier, la section 6 "Lower bounds" du chapitre II. Quant à l'affirmation que la positivité des solutions constitue encore un problème non trivial, Cédric Villani écrit à propos de cette question (dans l'introduction du second chapitre de la même référence) : "As for the strict positivity, the matter is not very clear yet".

[24] Dans la section introductive "Foreword".

tout naturellement l'entropie, sous les contraintes (1.13), (1.14) et (1.15) de la section précédente, c'est-à-dire que l'on tient compte des quantités conservées par les solutions au cours du temps, et le théorème des multiplicateurs de Lagrange permet de conclure.
Chacune de ces méthodes conduit à la détermination des équilibres thermodynamiques (et on retrouve d'ailleurs le résultat, publié dans [52], de James C. Maxwell), qui sont par ailleurs des solutions *stationnaires* à l'équation de Boltzmann (dans le sens où elles ne dépendent pas du temps), données par l'expression :

$$f(x,v) = \lambda \exp\left(b \cdot v + c|v|^2\right),$$

où λ et c sont des nombres réels et b un vecteur de $\mathbb{R}^d$.

1.3.3 Le paradoxe de l'apparition de l'irréversibilité

On achève cette section par quelques commentaires. L'équation de Boltzmann décrit des phénomènes irréversibles (c'est le sens du théorème H), et en ce sens, elle constitue un outil de choix pour les physiciens, puisque les phénomènes irréversibles sont légions dans la nature.

FIGURE 1.1 : Trajectoire et flèche du temps. On voit que les particules qui constituent le jet d'eau suivent une trajectoire parabolique, symétrique par rapport à la verticale. Ce qui permet de déduire que les gouttes d'eau vont de la gauche vers la droite n'est donc pas la forme de la trajectoire qu'elles suivent, mais la forme générale du jet d'eau, qui va en se délitant : un tel processus est irréversible. Photographie d'Adèle Godefroy.

Mais bien que ce constat suscite l'intérêt pour cette équation, il est en même temps source d'embarras. En effet, l'équation de Boltzmann est obtenue à partir d'une modélisation particulaire de la matière basée sur des phénomènes réversibles. On discerne alors un paradoxe important : comment un système intégralement régi par des phénomènes réversibles à l'échelle microscopique peut-il présenter un comportement global irréversible ?
Cette question, apparemment naïve, est en fait extrêmement profonde, et du reste de nombreux contemporains de Boltzmann se sont interrogés sur ce sujet. On peut citer notamment les écrits de Johann Josef Loschmidt [50] et Ernst Zermelo [68]. On pourra aussi consulter la référence plus moderne [26][25] pour une discussion sur les paradoxes soulevés par ces deux auteurs.

[25] Voir les sections 3.5 "Loschmidt's Paradox" et 3.6 "Poincaré's Recurrence and Zermelo's Paradox".

Chapitre 2

État de l'art sur l'équation de Boltzmann

2.1 Le problème de Cauchy

Dans le chapitre précédent, on s'est évertué à énumérer des propriétés qualitatives des solutions de l'équation de Boltzmann, avec la détermination des quantités conservées, comme la masse totale du fluide étudié, ou au contraire avec l'étude de quantités qui ne font que décroître, comme l'entropie.
Cependant, la résolution du problème de Cauchy associé à l'équation de Boltzmann (et qui peut conduire par la suite à s'interroger sur le caractère de problème bien posé au sens de Hadamard ([42]), question fondamentale lorsque l'on étudie une équation issue de la physique) n'a pas encore été présenté : étant donnée une densité de présence initiale $(x, v) \mapsto f_0(x, v)$ définie sur $U \times \mathbb{R}^d$ (où U est un ouvert de $\mathbb{R}^d$), existe-t-il, dans un premier temps, une solution au système

$$\begin{cases} \partial_t f + v \cdot \nabla_x f &= Q(f, f), \\ f(0, x, v) = f_0, \end{cases} \tag{2.1}$$

muni d'une condition aux limites, et si oui, dans quel espace fonctionnel ? Dans un second temps, on pourra s'interroger sur la question de l'unicité de la solution d'un tel problème.
Pour un tour d'horizon des différentes méthodes et des principaux résultats, on pourra consulter le livre [37] pour une étude approfondie du problème de Cauchy associé à l'équation de Boltzmann. Il convient de préciser qu'il existe deux grandes familles de résultats dans le cadre de la résolution du problème de Cauchy (2.1), qui vont être présentées brièvement. Ces résultats ne sont malgré tout qu'une réponse partielle à la résolution du problème de Cauchy associé à l'équation de Boltzmann, encore largement ouvert.

2.1.1 Théorie perturbative

On a remarqué précédemment que les maxwelliennes sont des solutions stationnaires de l'équation de Boltzmann (1.9), en plus d'être des équilibres thermodynamiques. L'idée de la théorie perturbative consiste à considérer des données initiales proches de ces équilibres. En imaginant que les solutions issues de ces données initiales restent proches des équilibres, on peut espérer que la non-linéarité du terme de collision ait des effets négligeables devant le terme de transport. En d'autres termes, on essaye de linéariser l'équation. Seiji Ukai obtint le premier dans [61] et [62] le résultat suivant, qui concerne l'équation de Boltzmann dans le tore, dans le cas inhomogène en espace, en utilisant cette méthode. Il considère les espaces de Banach suivants :

$$H_{l,\beta} = \big\{(x,v) \mapsto u(x,v) \ / \ \big(1+|v|\big)^{\beta} u \in L^{\infty}\big(\mathbb{R}^3_v, H^l\big(\mathbb{T}^3_x\big)\big)\big\},$$

munis des normes

$$\|u\|_{l,\beta} = \sup_{v\in\mathbb{R}^3} (1+|v|)^{\beta} \|u(\cdot,v)\|_{H^l(\mathbb{T}^3)},$$

où l et β sont des réels strictement positifs, et $H^l\big(\mathbb{T}^3_x\big)$ est l'espace de Sobolev sur le tore à trois dimensions $\mathbb{T}^3$.

Théorème 1 (Ukai [61], 1974)**.** *On considère un noyau de collision $B(v-v_*,\omega)$ qui vérifie*

$$0 \le B(u,\omega) \le b_0 |\cos\omega| \big(|u| + |u|^{-\delta}\big)$$

avec $b_0 > 0$, $0 \le \delta < 1$, et

$$\int_0^{\pi} B(u,\omega) \sin\omega \, \mathrm{d}\omega \ge b_1 \frac{|u|}{1+|u|}$$

avec $b_1 > 0$.
Pour tous nombres réels $l \ge 1/2$, $\beta \ge 3/2$, $\varepsilon > 0$ et $\gamma > 0$ (avec γ assez petit), il existe deux nombres strictement positifs α_1 et α_2 tel que si la donnée initiale f_0 s'écrit

$$f_0 = e^{-|v|^2} + e^{-\frac{|v|}{2}^2} u_0$$

avec u_0 qui appartient à l'espace $H_{l+1+\varepsilon,\beta+1+\varepsilon}$ et tel que

$$\|u_0\|_{l+1+\varepsilon,\beta+1+\varepsilon} \le \alpha_1$$

(ainsi qu'une condition supplémentaire, détaillée dans [61]), alors l'équation de Boltzmann admet une unique solution f en temps global *qui s'écrit*

$$f = e^{-|v|^2} + e^{-\frac{|v|}{2}^2} u$$

avec u qui appartient à l'espace fonctionnel :

$$L^{\infty}\big([0,+\infty[_t, H_{l+1+\varepsilon,\beta+1+\varepsilon}\big) \cap \mathcal{C}^0\big([0,+\infty[_t, H_{l+1,\beta+1}\big) \cap \mathcal{C}^1\big([0,+\infty[_t, H_{l,\beta}\big),$$

et qui vérifie :

$$\|u(t)\|_{l+1+\varepsilon,\beta+1+\varepsilon} \le \alpha_2 e^{-\gamma t}. \tag{2.2}$$

Des progrès continuels sont accomplis dans cette direction depuis ce travail, et on peut citer par exemple la référence récente [43] qui contient un résultat similaire au théorème de Ukai, complété par une estimation de la vitesse de convergence de la solution perturbée vers l'équilibre thermodynamique, de même nature que le contrôle (2.2).
Par ailleurs, on peut aussi citer les deux articles [18] et [17], qui traitent de la théorie perturbative de l'équation de Boltzmann dans des domaines à bord.
Enfin, on mentionne l'article [41]. Bien qu'il ne s'inscrive pas à proprement parler dans le cadre de la théorie perturbative, cet article présente un résultat de régularité pour les solutions de l'équation de Boltzmann dans un domaine à bord, ici en l'occurrence, un domaine convexe.

Il existe d'autre part une théorie perturbative "autour du vide", c'est-à-dire que dans ce cadre, les données initiales sont supposées proches de la solution nulle, qui est en particulier un équilibre trivial de l'équation de Boltzmann (c'est en ce sens que l'on parle encore ici de théorie perturbative). On peut citer le résultat [4] d'existence et d'unicité de solutions globales à l'équation de Boltzmann, dû à Nicola Bellomo et Giuseppe Toscani. Contrairement à de nombreux résultats de théorie perturbative dans tout l'espace, qui demandent une forte décroissance en la variable de position de la distribution initiale du gaz, dans [4], on peut se contenter d'imposer une décroissance en position en l'inverse d'un polynôme. Il est par contre impératif de conserver une décroissance gaussienne en vitesse de la donnée initiale.

2.1.2 Les solutions renormalisées de DiPerna et Lions

À la fin des années 1980, Ronald DiPerna et Pierre-Louis Lions ont considéré une version très affaiblie de l'équation de Boltzmann. Dans [32], ils procèdent à un changement d'inconnue, non linéaire, et considèrent l'équation

$$\partial_t g + v \cdot \nabla_x g = \frac{1}{1+f} Q(f,f), \tag{2.3}$$

où $g = \ln(1+f)$, qui est bien entendu équivalente formellement à l'équation de Boltzmann (1.9). Si f est solution de cette nouvelle équation au sens des distributions, on parle de *solution renormalisée*. Dans [33], DiPerna et Lions parviennent à montrer que chaque terme de l'équation précédente (2.3) fait sens dans l'espace fonctionnel L^1_{loc}.
Cette méthode permet d'obtenir un résultat d'existence globale, pour des données initiales quelconques (et en particulier, qui peuvent être loin de l'équilibre thermodynamique), et le lecteur trouvera dans [33][1] le théorème suivant.

Théorème 2 (DiPerna, Lions [33], 1989)**.** *On suppose que le noyau de collision B vérifie, pour tout $R < +\infty$:*

$$\frac{1}{1+|v|^2} \int_{|z-v|\leq R} \int_\omega B(z,\omega)\,\mathrm{d}\omega\,\mathrm{d}z \underset{|v|\to+\infty}{\longrightarrow} 0.$$

[1] Voir la section VIII "Global existence".

Pour toute donnée initiale f_0 positive presque partout sur $\mathbb{R}^d \times \mathbb{R}^d$ et telle que

$$\int_v \int_x f_0\big(1 + |x|^2 + |v|^2 + |\ln f_0|\big)\,\mathrm{d}x\,\mathrm{d}v < +\infty,$$

il existe une fonction $f \in \mathcal{C}^0\big([0,+\infty[, L^1\big(\mathbb{R}^d \times \mathbb{R}^d\big)\big)$ qui vérifie $f(0,\cdot,\cdot) = f_0$ et qui est une solution renormalisée de l'équation de Boltzmann (1.9).

Ce théorème assure l'existence d'une solution globale en temps pour un choix très important de données initiales. Malgré tout, le problème de l'unicité, mais aussi celui de la régularité des solutions renormalisées sont aujourd'hui toujours ouverts.

2.2 Le comportement des solutions en temps long

On l'a vu, l'entropie d'une solution de l'équation de Boltzmann ne peut que décroître, et seules les maxwelliennes sont des solutions dont l'entropie n'évolue pas au cours du temps : autrement dit, on a caractérisé les équilibres thermodynamiques.
Une question naturelle surgit alors : est-ce que chaque solution converge vers un équilibre thermodynamique ? Et si oui, est-il possible de quantifier la vitesse de convergence ?
Cette question, bien que fondamentale dans la compréhension qualitative des solutions de l'équation de Boltzmann, est assez éloignée du cadre de ce travail. On donne ici simplement quelques références, la plus complète sur le sujet étant sans doute [55], et une brève discussion sur une conjecture importante, qui peut désormais être considérée comme pratiquement résolue.
La convergence vers l'équilibre des solutions, si convergence vers l'équilibre il y a effectivement, doit être quantifiée à l'aide d'une certaine mesure. On renvoie à [55][2], qui permet de comprendre pourquoi on va étudier la différence d'une solution f et de son équilibre thermodynamique associé M^f avec la quantité, notée $H\big(f|M^f\big)$, et définie par :

$$\int_x \int_v \big(f - M^f\big) \ln M^f \,\mathrm{d}v\,\mathrm{d}x.$$

On peut d'ailleurs parler d'équilibre thermodynamique *associé* à une solution, puisque la masse, la quantité de mouvement et l'énergie cinétique étant conservées au cours du temps, si une solution converge vers un équilibre, cet équilibre doit naturellement avoir mêmes masse, quantité de mouvement et énergie cinétique. Or ces quantités scalaires (au nombre de 5 en dimension 3, et $d+2$ dans le cas général) caractèrisent entièrement les maxwelliennes (voir encore [55][3] pour cette caractérisation).
Dans [22], Carlo Cercignani propose une conjecture qui porte aujourd'hui son

[2] Voir la section 1.1.3 "What this Course is About : Convergence to Equilibrium".
[3] Dans Foreword.

nom, et qui énonce que l'écart entre une solution et sa maxwellienne doit être régi par une relation de la forme :

$$\frac{\mathrm{d}}{\mathrm{d}t}H(f) \geq KH\big(f|M^f\big), \tag{2.4}$$

où K est une constante qui peut dépendre de f. Dans la version initiale de la conjecture, elle ne dépendait que de la masse, la quantité de mouvement et l'énergie cinétique de f. Une telle inégalité, de type Gronwall, impliquerait immédiatement une vitesse de convergence exponentielle de la solution vers l'équilibre thermodynamique qui lui est associé, dans le contexte spatialement homogène. Ce type d'inégalité est communément appelé 'Entropie - Production d'Entropie", ou "EEP" en anglais.

Cette conjecture dut être perpétuellement revue, puisque des contre-exemples contredisant des versions de plus en plus faibles furent trouvés. Le théorème le plus marquant est sans doute celui que le lecteur trouvera dans [7]. Il énonce que, peu importe la constante K choisie, il est possible de trouver une donnée initiale f qui ne va pas converger vers son équilibre M^f à la vitesse prescrite par (2.4). On peut même imposer que les données initiales soient minorées par une maxwellienne, avoir une très forte décroissance à l'infini, ou bien une très grande régularité en norme d'espaces de Sobolev, rien n'y fait : on pourra toujours trouver un contre-exemple violant (2.4).
Malgré tout, en 1992 Eric Carlen et Maria Carvalho obtinrent la première inégalité de production d'entropie dans [20]. Ils prouvèrent qu'il existe une fonction réelle θ, très plate au voisinage de 0, telle que

$$\frac{\mathrm{d}}{\mathrm{d}t}H(f) \geq \theta\big(H\big(f|M^f\big)\big).$$

Puis Cédric Villani et Giuseppe Toscani (voir [59]) obtinrent en 1999, pour un noyau de collision qui vérifie $B(v-v_*,\omega) \geq K_B\big(1+|v-v_*|\big)^{-\beta}$, et pour tout réel ε strictement positif, une inégalité de production d'entropie de la forme

$$\frac{\mathrm{d}}{\mathrm{d}t}H(f) \geq K(\varepsilon)H\big(f|M^f\big)^{1+\varepsilon},$$

qui implique une vitesse de convergence sous-exponentielle de la solution vers son équilibre, et enfin en 2003 Cédric Villani parvint à relaxer la condition sur le noyau de collision, et donc à couvrir un grand nombre de situations physiques pertinentes, dans un article [65] titré, de manière évocatrice : "Cercignani's conjecture is sometimes true and always almost true."

Dans le cas inhomogène, la convergence vers l'équilibre est plus délicate à traiter. En effet, on sait que les solutions doivent converger vers des maxwelliennes, mais dans ce cadre, les paramètres λ, b et c définissant les maxwelliennes par la formule

$$M(v) = \exp\big(-(\lambda I_d v)\cdot v + bv + c\big)$$

dépendent a priori de la variable de position x. Si l'on considère un domaine borné et des conditions aux limites bien choisies, on peut alors montrer que les équilibres sont en fait des maxwelliennes globales (c'est-à-dire dont les précédents paramètres ne dépendent pas de la position). La question se pose alors de déterminer la vitesse de convergence des solutions vers de tels équilibres.
En particulier, l'étude quantitative de ce comportement en temps long d'une solution de l'équation de Boltzmann, pour des solutions régulières du problème avec conditions aux limites de réflexion spéculaire dans un ouvert borné a été traitée dans l'article [31] de Laurent Desvillettes et Cédric Villani. Les outils et les résultats présentés dans ce travail sont à l'origine du concept d'hypocoercivité. On pourra à ce sujet consulter le mémoire [66].

2.3 Les problèmes de dérivation

Au cours du Congrès international des mathématiciens de 1900, qui eut lieu à Paris, David Hilbert proposa sa célèbre liste de vingt-trois problèmes non résolus [44], qui devait guider la recherche mathématique pour les décennies à venir.
Le sixième problème de Hilbert est intitulé "Traitement mathématique des axiomes en physique". Il ajoute en particulier :

"*Les travaux de Boltzmann sur les principes de la mécanique suggèrent le problème de développer mathématiquement les processus limitatifs, juste esquissés, qui mènent de la vision atomiste aux lois du mouvement du continu*".

Cet énoncé ouvre les hostilités, et invite les mathématiciens à se saisir de la validité rigoureuse de l'équation de Boltzmann, à la fois comme objet d'étude en lui-même, mais aussi en tant qu'étape intermédiaire vers des objectifs plus ambitieux, comme l'axiomatisation de la mécanique des fluides.
Deux axes principaux ont été suivis. On présentera brièvement le programme de Kac, qui n'est pas l'objet de ce travail. Ensuite, on introduira le programme de Grad, le résultat de Lanford et comment le présent travail s'intègre dans ce programme de recherche.

2.3.1 Le programme de Kac : une approche probabiliste

Contrairement au programme de Grad, le programme de Kac repose sur la dérivation de l'équation *spatialement homogène* de Boltzmann, c'est-à-dire que les solutions ne dépendent pas de la variable d'espace x, si bien que l'équation étudiée ici s'écrit :

$$\partial_t f = Q(f, f). \tag{2.5}$$

Puisque la variable d'espace n'intervient pas, Mark Kac introduisit en 1956 dans [45] un modèle de trajectoires, appelé le modèle de Boltzmann-Kac, qui permet malgré tout de faire évoluer les vitesses des particules au cours du temps. Ces vitesses ne sont pas modifiées lors de collisions particulaires décrites

précedemment, c'est-à-dire lorsqu'une certaine condition sur la position relative de deux particules est vérifiée, ce qui ne fait plus sens ici, mais elles sont modifiées par un processus stochastique. On choisit aléatoirement un couple de particules i et j et, selon un processus de Poisson, les temps auxquels les vitesses des particules vont être modifiées. On choisit enfin le paramètre angulaire $\sigma \in \mathbb{S}^2$ au hasard, en suivant une loi à densité qui dépend du noyau de collision B (voir (1.10)), et on modifie les vitesses du couple de particules choisies comme suit :

$$v_i' = \frac{v_i + v_j}{2} + \frac{|v_j - v_i|}{2}\sigma$$

et

$$v_j' = \frac{v_i + v_j}{2} - \frac{|v_j - v_i|}{2}\sigma.$$

Ce processus conserve lui aussi l'énergie cinétique et la quantité de mouvement du système (il s'agit simplement d'une autre paramétrisation des collisions conservant ces quantités).

Finalement, l'objet d'étude est la fonction de répartition F_N de N particules évoluant suivant le modèle de Boltzmann-Kac. On suppose au temps initial que cette fonction de répartition est f_0-chaotique (en suivant la définition de Kac), où f_0 est la donnée initiale du problème de Cauchy (2.1). Il convient alors de montrer que la première marginale de la fonction de répartition F_N converge vers une solution f de (2.5).

On renvoie par exemple à [54] pour une présentation moderne du programme de Kac.

2.3.2 Le théorème de Lanford : une approche déterministe

Le programme de Grad trouve son origine dans l'obtention d'une suite d'équations vérifiées par les marginales $f_N^{(s)}$ (avec $1 \leq s \leq N$) d'une fonction de répartition d'un système de N ($N \in \mathbb{N}^*$) sphères dures de diamètre ε. Cette suite d'équations est obtenue à partir de l'équation de Liouville, et s'appelle la *hiérarchie BBGKY*, qui doit son nom aux travaux des physiciens et chimistes Nikolaï Bogolioubov [10], Max Born et Herbert Green [15], John Kirkwood [46] et Jacques Yvon [67]. Grad obtient alors dans [38] une relation vérifiée par la première marginale de la fonction de répartition du système de sphères dures, et observe que dans la limite de Boltzmann-Grad $N \to +\infty$, $N\varepsilon^{d-1} = 1$, et avec quelques autres manipulations, cette équation n'est autre que l'équation de Boltzmann.

Une de ces manipulations consiste à utiliser la propagation du chaos, ce qui signifie que si la suite des données initiales $\left(f_{N,0}^{(s)}\right)_{1\leq s\leq N}$ est approximativement tensorisée, c'est-à-dire que cette suite est donnée par les marginales de la distribution initiale :

$$f_{N,0}^{(N)}(Z_N) = \left(\mathcal{Z}_N^{-1}\mathbb{1}_{Z_N\in\mathcal{D}_N^\varepsilon}\right)\prod_{i=1}^{N} f_{N,0}^{(1)}(z_i) \tag{2.6}$$

(où $Z_N = (z_1, \ldots, z_N)$ est la configuration du système dans l'espace des phases $\mathcal{D}_N^\varepsilon$ d'un système à N particules de rayon $\varepsilon/2$, $\mathbb{1}_{Z_N\in\mathcal{D}_N^\varepsilon}$ est la fonction indicatrice

de cet espace des phases $\mathcal{D}_N^\varepsilon$, encodant en particulier la condition d'exclusion $|x_i - x_j| > \varepsilon$ pour tous $1 \leq i < j \leq N$, ce qui signifie que les particules de rayon $\varepsilon/2$ ne se chevauchent pas, et enfin où $\mathcal{Z}_N^{-1}$ est simplement un facteur de normalisation), alors cette tensorisation se propage aussi à la suite de solutions $\big(t \mapsto f_N^{(s)}(t, \dots)\big)_{1 \leq s \leq N}$ pour tout temps dans la limite $N \to +\infty$.

Il convient de noter que la limite de Boltzmann-Grad $N \to +\infty$, $N\varepsilon^{d-1} = 1$ a une signification physique importante. Le fait que la quantité $N\varepsilon^{d-1}$ reste constante lorsque N varie signifie que, indépendamment du nombre de particules considérées, le temps moyen de parcours libre d'une particule reste constant. Autrement dit, une particule parcourra en moyenne une distance fixée avant d'entrer en collision avec une autre particule. Avant cette collision, la particule s'est donc déplacée librement, en ligne droite. Une discussion un peu plus poussée sur la signification de cette limite est proposée dans la section 1.1.7.
La limite de Boltzmann-Grad implique alors que le volume occupé par l'ensemble des particules du système, qui est de l'ordre de $N\varepsilon^d$, est en fait de l'ordre de ε ou, ce qui est équivalent, de l'ordre de $N^{-\frac{1}{d-1}}$, et tend donc vers 0 lorsque N tend vers l'infini : seule une infime partie du volume occupé par le fluide est en fait occupée par des particules. C'est une des raisons pour laquelle l'équation de Boltzmann est utilisée dans le cadre de la modélisation de gaz très dilués.
Par ailleurs, dans le cadre de la résolution du sixième problème de Hilbert mentionné plus haut, on utilise cette limite de Boltzmann-Grad de façon cruciale. D'abord en se fixant un nombre α tel que $N\varepsilon^{d-1} = \alpha$, on obtient l'équation de Boltzmann à partir d'une modélisation particulaire de la matière : tout l'objet de ce travail est de montrer en quel sens cette étape peut être rigoureusement franchie. Dans la littérature anglophone, cette étape est souvent appelée "low density limit".
Bien qu'il ne s'agisse pas du sujet du présent travail, il ne semblait pas raisonnable de ne pas mentionner la seconde étape de la résolution du sixième problème de Hilbert : le passage du modèle cinétique au modèle continu. L'objectif final de Hilbert était en effet de parvenir à justifier les équations de la mécanique des fluides. Il est en effet possible d'achever ce programme en prenant appui sur le résultat de "low density limit" obtenu, et de retrouver, d'abord formellement, les équations de Navier-Stokes à partir de l'équation de Boltzmann, en effectuant ce que l'on appelle une limite hydrodynamique. On va faire tendre à son tour la quantité α vers l'infini, ce qui signifie que le temps moyen de parcours libre des particules du système va tendre vers zéro. La signification physique de cette limite est que les particules vont s'entrechoquer à une cadence très élevée. De nombreux résultats ont été obtenus dans le sens du passage rigoureux à la limite hydrodynamique. On renvoie le lecteur à la référence [56] sur ce sujet.

Quant à la justification rigoureuse de l'obtention de l'équation de Boltzmann à partir d'un modèle particulaire déterministe, Carlo Cercignani dans [21] (en 1972), puis Oscar Lanford dans [49] (en 1975) obtiennent les premiers résultats mathématiques dans le sens de la résolution du programme de Grad. En par-

ticulier, Lanford remarque que la propagation du chaos peut être obtenue par une étude précise des trajectoires des sphères dures, et apporte une preuve rigoureuse de l'apparition de l'irréversibilité. Pour la première fois, une preuve mathématique démontre la validité rigoureuse de l'équation de Boltzmann localement en temps, apportant une avancée décisive à la résolution du sixième problème de Hilbert. Les paradoxes tels que ceux mentionnés à la fin de la section 1.3.3 page 27 ne peuvent plus être vus comme des arguments invalidant l'équation de Boltzmann, et les propriétés de ses solutions a priori surprenantes changent de statut, passant d'apparentes contradictions qui réduisent l'intérêt du modèle à celui de phénomènes complexes au sens profond.
Le théorème de Lanford s'énonce comme suit.

Théorème 3 (Lanford [49], 1975)**.** *Soit β un nombre réel strictement positif, et soit $f_0 : \mathbb{R}^{2d} \to \mathbb{R}_+$ une densité de probabilité continue qui vérifie*

$$\left\| (x,v) \mapsto f_0(x,v) \exp\big(\frac{\beta}{2}|v|^2\big) \right\|_{L^\infty(\mathbb{R}^{2d})} < +\infty.$$

On considère le système de N sphères dures de diamètre ε (régi par la dynamique introduite dans la section 1.2.2) décrit par la fonction de répartition $f_N : \mathbb{R}^{2dN} \to \mathbb{R}_+$, distribuées initialement selon la densité f_0 de façon indépendante (au sens où la suite des distributions initiales de s particules, avec $1 \leq s \leq N$, est donnée par les marginales de l'expression (2.6)*).*
Alors, il existe un temps $T > 0$ qui ne dépend que de β et de μ où

$$\exp(-\mu) = \left\| (x,v) \mapsto f_0(x,v) \exp\big(\frac{\beta}{2}|v|^2\big) \right\|_{L^\infty(\mathbb{R}^{2d})},$$

tel que, dans la limite de Boltzmann-Grad :

$$N \to +\infty, \quad N\varepsilon^{d-1} = 1,$$

la première marginale $f_N^{(1)}$ de la fonction de répartition f_N converge vers la solution f du problème de Cauchy (2.1) *associé à l'equation de Boltzmann, avec $\big|(v - v_*) \cdot \omega\big|_+$ pour noyau de collision $B(v - v_*, \omega)$, au sens des observables, c'est-à-dire que pour tout compact K de l'espace des positions $\mathbb{R}^d$ et pour toute fonction test $v \in \mathbb{R}^d \mapsto \varphi(v) \in \mathbb{R}$:*

$$\left\| \mathbb{1}_K(x) \Big| \int_{\mathbb{R}^d} \big(f - f_N^{(1)}\big)(t,x,v)\varphi(v)\,\mathrm{d}v \Big| \right\|_{L^\infty([0,T]_t \times \mathbb{R}^d_x)} \underset{N \to +\infty}{\longrightarrow} 0.$$

La preuve de Lanford, qui représente une avancée conceptuelle majeure, a par la suite été complétée à de nombreuses reprises et beaucoup d'auteurs ont apporté une contribution significative à l'achèvement de cet édifice particulièrement complexe. On peut citer l'article [60] de Kôhei Uchiyama, le livre [26] de Carlo Cercignani, Reinhard Illner et Mario Pulvirenti[4], ou encore la contribution de

[4]Voir le chapitre 4, et en particulier les sections 4.4 "Rigorous Validity of the Boltzmann Equation" et 4.5 "Validity of the Boltzmann Equation for a Rare Cloud of Gas in the Vacuum".

Herbert Spohn avec [58]. Une étude des arbres de même type, que l'on doit à Carlo Cercignani, Viktor Ivanovitch Gerasimenko et Dmitri Ya. Petrina, basée sur un contrôle de la taille des "trajectoires pathologiques", c'est-à-dire des trajectoires de sphères dures rendant difficile la comparaison avec les trajectoires associées obtenues dans la limite de Boltzmann-Grad, a ouvert la voie vers des preuves de plus en plus quantitatives. Cette étude est présentée dans le livre [25].

Enfin, les travaux de Thierry Bodineau, Isabelle Gallagher, Laure Saint-Raymond, Sergio Simonella et Benjamin Texier ont apporté une contribution décisive au programme de Grad. En particulier, alors que le théorème de Lanford a été présenté pour la première fois en 1975, l'article [34] de 2013 peut être considéré comme la forme la plus aboutie de la preuve de ce théorème obtenue jusqu'à présent. Une remarque fondamentale concernant le théorème de Lanford est l'intervalle de temps sur lequel sa conclusion est valide : contrairement aux résultats présentés dans la section 2.1, la convergence de la première marginale n'a lieu que sur un intervalle de temps petit, donné par la régularité de la donnée initiale. En 2016, le plan de preuve de Lanford permet à Bodineau, Gallagher et Saint-Raymond dans [8] d'obtenir une convergence sur un intervalle de temps arbitrairement grand de la fonction de distribution d'une particule marquée qui évolue dans un gaz à l'équilibre thermodynamique, vers la solution de l'équation de Boltzmann linéaire. Ce résultat est à comparer avec celui de Giovanni Gallavotti [35], qui obtenait la convergence de la fonction de distribution d'une particule marquée qui évoluait au milieu d'obstacles fixés distribués de façon aléatoire, vers la solution de l'équation de Boltzmann linéaire.
Ce même groupe d'auteurs a par ailleurs publié un article ([9]) qui revient sur l'apparition de l'irréversibilité lors du passage à la limite de Boltzmann-Grad.

Chapitre 3

Apports et structure de ce travail

Ce travail s'inscrit dans le programme de Grad, et vise à obtenir une preuve du théorème de Lanford dans le cas particuler d'un domaine à bord. Au fur et à mesure du temps consacré à la résolution de ce problème, l'accent a été mis progressivement sur une rédaction aussi exhaustive que possible des arguments déployés pour démontrer le théorème de Lanford. Et puisque ce travail s'appuie en particulier sur l'article phare [34], le travail d'adaptation de la preuve a permis de mettre à jour quelques éléments à détailler dans la très longue et très technique démonstration du théorème de Lanford. Les nouveautés concernent d'une part l'obtention d'un théorème dérivation rigoureux de l'équation de Boltzmann dans le cas d'un domaine à bord, et d'autre part les commentaires faits sur la preuve présentée dans [34].

3.1 Le théorème de Lanford dans le demi-espace

S'agissant de la motivation à l'origine du présent manuscrit, on va démontrer un résultat analogue au théorème 4, dans le cas où les particules évoluent dans un domaine à bord. Ce travail est exécuté dans le cadre de sphères dures évoluant dans le demi-espace, et qui interagissent par réflexion spéculaire avec le bord de l'obstacle, qui est donc ici un demi-plan.
Comme il a été précisé plus haut, on suivra la preuve de Lanford, qui s'appuie sur une étude minutieuse des trajectoires suivies par les sphères dures. Ce plan de preuve incite à considérer dans un premier temps un problème simplifié par rapport au cas d'un obstacle général : on restreint d'abord l'étude au cas où l'obstacle est convexe. En effet, dans ce cas si une particule rebondit une première fois contre l'obstacle, elle ne pourra pas rebondir à nouveau contre ce dernier sans que sa vitesse ne soit modifiée, c'est-à-dire, sans que cette particule ne soit entrée en collision avec une autre particule du système. Ce simple constat simplifie grandement l'étude des trajectoires, alors qu'en présence d'un obstacle

quelconque on pourrait parfaitement imaginer des particules rebondissant en cascade contre l'obstacle, demandant un traitement analytique supplémentaire. Finalement, on se contentera dans ce travail d'étudier le cas où l'obstacle est le demi-espace. A priori, on remarque que l'obstacle dont le bord est un hyperplan est, parmi les obstacles convexes, le pire d'une certaine façon. Le pire, en ce sens que la convexité a un effet diffusif : si deux particules suivent des trajectoires parallèles avant d'entrer toutes les deux en collision avec l'obstacle, après rebond les trajectoires seront divergentes, de sorte que les particules en question auront tendance à s'écarter l'une de l'autre, prévenant ainsi toute recollison. Lorsque le bord de l'obstacle est plat, on est dans la situation critique où l'effet diffusif est nul, et l'on pourrait alors redouter des complications relatives aux trajectoires que suivent les particules.
Cependant, le demi-espace simplifie encore davantage l'étude du problème. En effet, dans ce cas, non seulement la convexité permet d'obtenir des trajectoires simples, mais en plus elles deviennent explicites, ce qui facilite d'autant plus les calculs de recollisions.

Les nouveautés présentées ici et relatives à la présence de l'obstacle sont les suivantes.

- Le caractère bien posé, presque partout, du problème de Cauchy pour un système de sphères dures évoluant dans un domaine à bord a été l'occasion de revisiter la preuve d'Alexander [1], et conduit à l'obtention de l'énoncé de la Proposition 2, section 1.2 page 55.

- L'obtention de la hiérarchie BBGKY dans le cas de la présence d'un obstacle suit les travaux fondamentaux des pionniers de la hiérarchie BBGKY. La présentation de cette dérivation est très proche de celle de [34]. Il convient de remarquer que le terme B (voir l'équation (2.14) page 71) est spécifique au cas du domaine à bord, et que si un autre modèle d'interaction avec le bord du domaine avait été choisi, ce terme aurait été radicalement différent.

- Le coeur du travail présenté ici est l'adaptation au cas du domaine à bord des lemmes géométriques ayant pour but de contrôler les trajectoires pathologiques au sens de [25]. La présence d'un obstacle complique sérieusement le lemme de tir présenté dans [34][1] ainsi que la proposition fondamentale qui empêche l'apparition de recollision lorsque l'on ajoute une autre particule à un système de sphères dures qui n'aurait pas subi de recollision sans cet ajout[2]. Ces modifications dans le cas de la présence d'un obstacle sont soigneusement détaillées dans la section 12.2, débutant page 333. Il est à noter que pour que l'analogue de la Proposition 12.1.1 de [34] fonctionne, on a dû introduire un cut-off supplémentaire en la proximité des particules au moment d'une collision. Ce cut-off est traité dans la même

[1] Voir la section 12.2 "Geometrical lemmas".
[2] Voir la section 12.1 "Stability of good configurations by adjunction of collisional particles".

section 12.2, et il ne change pas fondamentalement la vitesse de convergence obtenue dans le résultat final. En revanche, il nécessite l'écriture d'un lemme supplémentaire de scattering, très similaire au lemme 12.2.2 de [34][3]. Ce nouveau lemme est présenté dans la section 12.2.5 page 368.

- Enfin, un soin tout particulier a été apporté à la définition rigoureuse du cadre fonctionnel dans lequel on définit puis on résout la hiérarchie BBGKY et la hiérarchie de Boltzmann. C'est l'objet des sections 5, 6 and 7. En particulier, la plus longue partie[4] concerne la définition rigoureuse de l'opérateur de transport-collision de la hiérarchie BBGKY.
  Il est à noter que tous les résultats intermédiaires présentés et les arguments développés dans ce travail au sujet de la définition rigoureuse de l'opérateur de transport-collision de la hiérarchie BBGKY peuvent déjà être trouvés dans [34][5], sous une forme plutôt succinte. Ici, on s'est attaché à détailler chacun de ces arguments, puisque l'étape de la définition rigoureuse de la hiérarchie BBGKY est à la fois cruciale et délicate.
  À ce stade du texte, il est sans doute utile de préciser une dernière chose à ce sujet : on montre dans la section 5.1 que la hiérarchie BBGKY fait sens pour des fonctions continues en temps, à valeurs dans les fonctions L^∞ sur l'espace des phases, et qui vérifient une condition de décroissance en vitesse : c'est la conclusion du Théorème 1 page 137. En particulier, on ne sera pas contraint de se contenter de solutions au sens faible de la hiérarchie BBGKY.

3.2 La preuve du théorème de Lanford détaillée et commentée

On l'a vu, le théorème de Lanford (théorème 4), obtenu dans [49], et démontré de plus en plus en détail, d'abord dans [26], puis dans [25] et enfin dans [34], donne la convergence de la fonction de distribution du système de sphères dures vers la solution de l'équation de Boltzmann, valide sur un temps court.
On s'intéresse à présent à la description de la topologie utilisée pour décrire cette convergence. Les auteurs de [34] ont obtenu une convergence localement uniforme en les variables de temps et d'espace, mais une convergence faible en vitesse. Cette limitation n'altère nullement l'intérêt pratique de ce théorème, puisque les quantités décrites par les équations cinétiques ne sont observables par l'expérience qu'au travers de leurs moments, autrement dit, on ne peut mesurer que des intégrales contre des fonctions de la vitesse v des solutions des équations cinétiques. Cependant, l'étude des contrôles obtenus sur les intégrales de $f - f_N^{(1)}$ contre des observables (c'est-à-dire des fonctions φ de la vitesse v)

[3] Voir la section 12.2.2 "Modification of bad trajectories by hard sphere reflection."
[4] Voir la section 5.1.
[5] Voir la section 5.1 "Rigorous formulation of the BBGKY hierarchy".

montre que, systématiquement, les bornes comprennent le terme

$$\big|\varphi\big|_{L^\infty(\mathbb{R}^d)}.$$

Ce terme semble suggérer qu'une amélioration de la convergence est possible, et d'un point de vue mathématique, il n'y a a priori aucune raison de ne pas énoncer le résultat le plus général possible. Dans ce travail, on présente une convergence plus forte, à savoir une convergence localement uniforme en toutes les variables : de temps, d'espace et de vitesse. Cependant, cette convergence n'a pas lieu sur l'espace des phases tout entier. Les compacts choisis ne doivent pas contenir de vitesses rasantes (c'est-à-dire que les configurations contenues dans les compacts considérés sont toutes telles que leur produit scalaire avec le vecteur e_1, qui est le vecteur normal unitaire sortant de l'obstacle, ne soit pas trop proche de zéro). Cette condition d'exclusion est due à la préparation des données initiales, première étape du travail qui vise à éliminer les trajectoires qui présentent des recollisions. Une autre limitation est que les compacts ne peuvent pas intersecter le bord de l'obstacle. On perd donc une information quant au comportement des solutions au voisinage de l'obstacle.

Par ailleurs, on peut apporter quelques commentaires quant aux hypothèses requises dans [34].

- Les théorèmes fondamentaux 6 et 7 de [34][6], qui assurent l'existence et l'unicité de solutions respectives aux hiérarchies BBGKY et de Boltzmann, reposent de façon cruciale sur un argument de point fixe, dû à Ukai ([63]). Dans le cas de la hiérarchie de Boltzmann, pour chaque équation de cette hiérarchie le cadre fonctionnel choisi dans [34] (comme dans toutes les références précédentes) est celui des fonctions continues par rapport au temps, à valeurs dans les fonctions continues sur l'espace des phases. Or la hiérarchie de Boltzmann, dans sa forme intégrale (voir la section 4.2 page 83), est composée notamment du premier terme

  $$\mathcal{T}_t^{s,0}\big(f^{(s)}(0,\cdot)\big),$$

  qui fait intervenir le transport libre (avec conditions de bord). La continuité seule de la donnée initiale se révèle insuffisante pour obtenir une comparaison uniforme en espace de la quantité

  $$\Big|\mathcal{T}_t^{s,0}\big(f^{(s)}(0,\cdot)\big) - f^{(s)}(0,\cdot)\Big|.$$

  La section 5.2, et en particulier la sous-section 5.2.4 page 179 propose donc un cadre fonctionnel un peu plus restrictif, mais qui permet en contrepartie de présenter une version nouvelle de l'argument du point fixe.

- D'autre part, l'argument du point fixe de Seiji Ukai ([63]) repose sur une inégalité de contraction dans un espace fonctionnel bien choisi.

[6] Voir la section 5.2 "Functional spaces and statement of the results".

On rappelle que les hiérarchies sont des suites d'équations, où chacune de ces équations donne une relation entre deux marginales. Chaque équation des hiérarchies fait donc sens dans un espace fonctionnel qui lui est propre (c'est l'objet des sections 6.1 et 6.2). Il convient donc, une fois que l'espace fonctionnel de chaque équation est donné (voir la section 7.1.1), de lier entre eux ces espaces en considérant une topologie sur la famille formée par ces espaces (voir la section 7.1.2). On va alors définir chacune des hiérarchies sur une famille de tels espaces fonctionnels. De plus, on travaille avec des fonctions qui dépendent du temps. À chaque instant, on a donc une famille d'espaces fonctionnels, famille dans laquelle, par exemple pour le cas de la hiérarchie BBGKY, vit la suite de marginales $\left(f_N^{(s)}(t,\cdot)\right)_{1\leq s\leq N}$. À chaque instant on peut associer à la famille $\left(f_N^{(s)}(t,\cdot)\right)_{1\leq s\leq N}$ une norme d'espace de Banach

$$\left\|\left(f_N^{(s)}(t,\cdot)\right)_{1\leq s\leq N}\right\|_{\cdot,\widetilde{\beta}(t),\widetilde{\mu}(t)}.$$

Dans la littérature, ces espaces sont introduits dans la section 5.2 "Functional spaces and statement of the results" dans [34]. L'inégalité de contraction due à Ukai, et qui doit être vérifiée par les hiérarchies vues comme opérateurs agissant sur ces suites d'espaces fonctionnels, repose sur une utilisation subtile de la décroissance par rapport au temps des poids qui permettent de définir les espaces fonctionnels sur lesquels les hiérarchies BBGKY et de Boltzmann sont bien définies (un poids $\widetilde{\beta}$ pour un contrôle gaussien sur le profil en vitesse de chacune des marginales, introduit dans les définitions 23 page 206 et 24 page 206, et un poids $\widetilde{\mu}$ pour un contrôle de décroissance des normes de la s-ème marginale quand s devient grand, introduit dans les définitions 25 page 207 et 26 page 207).
Par ailleurs, dans [63], aucune condition de continuité par rapport au temps n'est choisie, tandis que dans [34], on demande à avoir une continuité par rapport au temps à valeurs dans les espaces de Banach définis précédemment, c'est-à-dire, si l'on considère par exemple le cas de la hiérarchie BBGKY, que l'on impose la condition :

$$\forall t\in\,]0,T],\ \lim_{u\to t^-}\left\|\left(f_N^{(s)}(t)\right)_{1\leq s\leq N}-\left(f_N^{(s)}(u)\right)_{1\leq s\leq N}\right\|_{N,\varepsilon,\widetilde{\beta}(t),\widetilde{\mu}(t)}=0. \tag{3.1}$$

Cette condition de continuité fait sens physiquement, d'une part car on a peine à imaginer les fonctions étranges qui seraient dans un tel espace, sans condition de continuité, et d'autre part car exiger une continuité temporelle pour des phénomènes physiques ne semble pas, dans ce cadre, déraisonnable.
On montre dans ce travail, à la section 8.2 page 251, que cette condition de continuité engendre des complications lorsque l'on tente d'obtenir l'inégalité de contraction de Ukai[7]. On propose dans ce travail deux pistes :

[7]Dans la référence [34], il s'agit de l'inégalité (5.4.4) de la section 5.4 "Continuity estimates".

la première, d'utiliser un poids $\widetilde{\mu}$ (qui a vocation à lier les espaces dans lesquels se trouvent les marginales) plus fort, et la seconde, d'affaiblir la continuité en temps. Ces deux topologies sont introduites dans la section 7.1.3 et étudiées par la suite. On verra que, vu les conditions que devront vérifier les données initiales, la deuxième option sera la plus pertinente. Deux versions de l'inégalité de contraction de Ukai (une pour chacune des topologies introduites) sont proposées dans ce travail : c'est l'objet de la section 8.3.

- Enfin, pour obtenir une quantification de la vitesse de convergence de la première marginale vers la solution de l'équation de Boltzmann, le lemme 14.2.3 de [34][8] joue un rôle crucial puisqu'il a pour objet de contrôler l'erreur due à la divergence des trajectoires dans le cas d'une donnée initiale tensorisée. On revient sur ce résultat, et on propose dans ce travail une version plus quantitative de ce lemme, avec des hypothèses un peu plus fortes, présentée dans la section 16.1.3 page 507. Le lecteur pourra aussi consulter la remarque page 515, qui revient sur les hypothèses plus fortes de ce nouveau lemme.

3.3 Le résultat principal

On présente maintenant le résultat principal de ce texte.

Théorème 4 (Théorème de Lanford dans le demi-espace)**.** *Soit β un nombre réel strictement positif, et soit $f_0 : \big\{x \in \mathbb{R}^d \ / \ x \cdot e_1 \geq 0\big\} \times \mathbb{R}^d \to \mathbb{R}_+$ une densité de probabilité continue, qui vérifie :*

$$f(x,v) \underset{|(x,v)| \to +\infty}{\longrightarrow} 0,$$

$$\left\| (x,v) \mapsto f_0(x,v) \exp\big(\frac{\beta}{2}|v|^2\big) \right\|_{L^\infty(\{x \in \mathbb{R}^d / x \cdot e_1 \geq 0\} \times \mathbb{R}^d)} < +\infty,$$

et

$$\forall x \in \big\{x \in \mathbb{R}^d \ / \ x \cdot e_1 = 0\big\},\ \forall v \in \mathbb{R}^d \text{ tel que } v \cdot e_1 > 0, f(x,v) = f\big(x, v - 2\big(v \cdot e_1\big)v\big)$$

(où e_1 est le premier vecteur de la base canonique).
On considère dans $\big\{x \in \mathbb{R}^d \ / \ x \cdot e_1 > \varepsilon/2\big\} \times \mathbb{R}^d$ le système de N sphères dures de diamètre ε, avec réfléxion spéculaire (voir la condition (1.6)*), décrit par la fonction de répartition $f_N : \mathbb{R}^{2dN} \to \mathbb{R}_+$, distribuées initialement selon la densité f_0 de façon indépendante (au sens où la suite des distributions initiales de s particules, avec $1 \leq s \leq N$, est donnée par les marginales de l'expression* (2.6)*).*
Alors, il existe un temps $T > 0$ qui ne dépend que de β et de μ où

$$\exp(-\mu) = \left\| (x,v) \mapsto f_0(x,v) \exp\big(\frac{\beta}{2}|v|^2\big) \right\|_{L^\infty(\{x \in \mathbb{R}^d / x \cdot e_1 \geq 0\} \times \mathbb{R}^d)},$$

[8] Il s'agit du dernier lemme de contrôle des termes d'erreurs. Voir la section 14.2.3 "Error coming from the divergence of trajectories".

tel que, dans la limite de Boltzmann-Grad :

$$N \to +\infty, \quad N\varepsilon^{d-1} = 1,$$

la première marginale $f_N^{(1)}$ *de la fonction de répartition* f_N *converge localement uniformément sur*

$$[0,T] \times \big\{x \in \mathbb{R}^d \ / \ x \cdot e_1 > 0\big\} \times \big\{v \in \mathbb{R}^d \ / \ v \cdot e_1 \neq 0\big\}$$

vers la solution f *du problème de Cauchy associé à l'équation de Bolzmann avec condition de bord*

$$\begin{cases} \partial_t f + v \cdot \nabla_x f &= Q(f,f), \\ f(0,\cdot,\cdot) &= f_0(\cdot,\cdot), \\ f(x,v) &= f\big(x, v - 2\big(v \cdot e_1\big)v\big) \\ & \forall x \in \big\{x \in \mathbb{R}^d \ / \ x \cdot e_1 = 0\big\}, \ \forall v \in \mathbb{R}^d \ \textit{tel que} \ v \cdot e_1 > 0, \end{cases}$$

avec $\big|(v - v_*) \cdot \omega\big|_+$ *pour noyau de collision* $B(v - v_*, \omega)$*, c'est-à-dire que pour tout compact* K *de la partie* $\big\{x \in \mathbb{R}^d \ / \ x \cdot e_1 > 0\big\} \times \big\{v \in \mathbb{R}^d \ / \ v \cdot e_1 \neq 0\big\}$ *de l'espace des phases :*

$$\Big\| \mathbb{1}_K(x,v)\big(f - f_N^{(1)}\big)(t,x,v) \Big\|_{L^\infty([0,T]_t \times \{x \cdot e_1 > 0\} \times \{v \cdot e_1 \neq 0\})} \underset{N \to +\infty}{\longrightarrow} 0.$$

Si de plus la racine carrée de la donnée initiale $\sqrt{f_0}$ *est lipschitzienne par rapport à la variable de position* x*, alors il existe un réel strictement positif* a *tel que, dans la limite de Boltzmann-Grad :*

$$\Big\| \mathbb{1}_K(x,v)\big(f - f_N^{(1)}\big)(t,x,v) \Big\|_{L^\infty([0,T]_t \times \{x \cdot e_1 > 0\} \times \{v \cdot e_1 \neq 0\})} = O(\varepsilon^a).$$

3.4 Perspectives

On conclut cette introduction par quelques pistes de recherches futures suggérées par ce travail.

On s'est rapidement contenté d'un obstacle très simple pour l'étude de la validié du théorème de Lanford dans un domaine à bord. Dans un premier temps, il semble très naturel d'essayer d'obtenir la preuve de ce théorème lorsque les particules évoluent autour d'un obstacle convexe, d'abord régulier, puis plus général. Le cas de l'obstacle régulier ne devrait pas poser de difficulté insurmontable. Par ailleurs, un travail généralisant le lemme de tir 27 page 335 dans ce cadre est en préparation. En revanche, comment par exemple traiter les singularités du domaine ? Ou encore les obstacles qui ne sont pas convexes ? Assistera-t-on à une accumulation de particules dans les éventuelles aspérités de l'obstacle ?
On peut aussi se pencher sur la topologie pour laquelle le résultat de convergence a été obtenu. En particulier, on peut s'interroger sur la possibilité de démontrer une convergence sur tout compact de l'espace des phases, et non plus seulement

pour des compacts qui ne contiennent pas de vitesses rasantes ou des positions se trouvant au bord de l'obstacle. En particulier, si les compacts que l'on s'autorise à considérer peuvent contenir des positions au bord de l'obstacle, cela impliquerait une bien meilleure compréhension du comportement des solutions de l'équation de Boltzmann au bord du domaine.

D'autres généralisations sont aussi des sujets d'étude naturels : comment procéder si les particules ne suivent pas la loi de Snell-Descartes lorsqu'elles atteignent le bord du domaine, mais évoluent selon un modèle hybride, partiellement diffusif ? Un changement radical dans le plan de preuve semble s'imposer dans ce cas.
Enfin, on peut aussi étudier le cas d'un obstacle qui n'est plus fixe, mais qui évolue, par exemple, à la fois sous l'effet sa propre inertie et de la gravité, et des interactions du nuage de particules qui l'entourent.

Un autre sujet d'étude sans doute passionnant est celui de l'étude du problème de Cauchy d'un système de particules qui subit des collisions inélastiques. À la connaissance de l'auteur, il n'y a pour l'instant aucun résultat satisfaisant obtenu dans cette direction.

Part I

From the dynamics of particles to the relevant hierarchies

Chapter 1

Dynamics of a finite number of hard spheres

1.1 Intuitive description of the dynamics

One considers a system of N ($N \in \mathbb{N}^*$) spherical particles of mass 1 and of radius $\varepsilon/2$ (with $\varepsilon > 0$) evolving outside of a convex, closed obstacle Ω of the Euclidean space $\mathbb{R}^d$ ($d \geq 2$), such that $\partial\Omega$ is of class $\mathcal{C}^1$.
All along this work, the dimension of the ambiant Euclidean space will be fixed, and denoted by d, which is an integer larger or equal to 2.
The dynamics of those N particles is the following. At $t = 0$, each particle i of the system starts from its initial position $x_i \in \mathbb{R}^d$, such that $|x_i - x_j| > \varepsilon$ for all $j \neq i$ and $d(x_i, \Omega) > \varepsilon/2$ for all $1 \leq i \leq N$. Moreover, each particle has an initial velocity $v_i \in \mathbb{R}^d$. Let $x_i(t)$ and $v_i(t)$ respectively denote the position and velocity at time t of the particle i, with $x_i(0) = x_i$ and $v_i(0) = v_i$.

1.1.1 Definition of the free flow

Following Newton's first law, the particles are moving along a straight line with a constant velocity, as long as no interaction (with another particle or with the obstacle Ω) changes the trajectories, that is :

$$x_i(t) = x_i + tv_i \tag{1.1}$$

for all $1 \leq i \leq N$, and $t > 0$ such that $|x_i(t) - x_j(t)| = |(x_i - x_j) + t(v_i - v_j)| > \varepsilon$ for all $i \neq j$, and $d(x_i + tv_i, \Omega) > \varepsilon/2$ for all i.

Of course, the dynamics of the particles will not be the free transport defined in (1.1) all the time. Depending on the initial configuration (that is, the value of $(x_1, v_1, ..., x_N, v_N)$), there can exist times such that the trajectory of some particles can be changed, that is $x_i(t)$ will not be equal anymore to $x_i + tv_i$.

Such times will be called *events*. There are two possible kinds of events, defined below.

1.1.2 Interaction with the boundary of the domain : the flow with boundary condition

The first kind of event that one wants to describe happens when a particle bounces against the boundary of the domain, that is when

$$d(x_i + t_0 v_i, \Omega) = \varepsilon/2 \tag{1.2}$$

for some $1 \leq i \leq N$ and $t_0 > 0$.
In that case, the velocity of any particle satisfying the distance condition (1.2) in this configuration is changed according to the reflexion law of Snell-Descartes, that is the velocity immediately before the contact with Ω $v_i(t_0^-)$ is instantaneously replaced by

$$v_i' = v_i(t_0^+) = v_i(t_0^-) - 2v_i(t_0^-) \cdot n(\tilde{x}_i(t_0))n(\tilde{x}_i(t_0)),$$

for all $1 \leq i \leq N$ such that $d(x_i + t_0 v_i, \Omega) = \varepsilon/2$, where $\tilde{x}_i(t_0)$ is the only point of Ω such that $|x_i(t_0) - \tilde{x}_i(t_0)| = \varepsilon/2$ (this point exists by definition of t_0 and if the obstacle Ω is a closed set of the Euclidean space, and it is unique if the obstacle Ω is a convex set), and $n(\tilde{x}_i(t_0))$ is the outgoing unitary normal vector to Ω at $\tilde{x}_i(t_0)$, defined as soon as $\partial\Omega$ is of class $\mathcal{C}^1$. Then, for $t \geq t_0$ with $t - t_0$ small enough, one has

$$x_i(t) = x_i(t_0) + (t - t_0)v_i',$$

while the velocities of all the particles not in contact at time t_0 with the obstacle remain of course unchanged.

1.1.3 Interaction between the particles : the hard sphere flow

The second kind of event that one considers is the collision between two particles. It occurs when there exists $t_1 > 0$ and $1 \leq i \neq j \leq N$ such that (denoting by $v_i(t_1^-)$ and $v_j(t_1^-)$ the velocities of particles i and j immediately before collision)

$$\big|\big(x_i(t_1) - x_j(t_1)\big) + t_1\big(v_i(t_1^-) - v_j(t_1^-)\big)\big| = \varepsilon.$$

The velocities $v_i(t_1^-)$ and $v_j(t_1^-)$ of those two particles, called *pre-collisional*, are respectively replaced by the following velocities v_i' and v_j', called *post-collisional* (one uses the notation v_i and v_j instead of $v_i(t_1^-)$ and $v_j(t_1^-)$ here) :

$$\begin{cases} v_i' &= v_i - (v_i - v_j) \cdot \omega_{i,j}\omega_{i,j}, \\ v_j' &= v_j + (v_i - v_j) \cdot \omega_{i,j}\omega_{i,j}, \end{cases}$$

with

$$\omega_{i,j} = \frac{x_i(t_1) - x_j(t_1)}{|x_i(t_1) - x_j(t_1)|}.$$

Then, for $t \geq t_1$ with $t - t_1$ small enough, one has

$$x_i(t) = x_i(t_1) + (t - t_1)v_i',\ v_i(t) = v_i',$$

and

$$x_j(t) = x_j(t_1) + (t - t_1)v_j',\ v_j(t) = v_j',$$

and this modification is done for all couples (i, j) of particles verifying the condition $\big|(x_i - x_j) + t(v_i - v_j)\big| = \varepsilon$ (assuming that for all times, the couples of particles satisfying this assumption constitute a family of disjointed couples : one will show later that the initial configurations leading, for some time t, to a situation such that this is not satisfied, are of measure 0), while the velocities of the particles which are not in contact with another particle are not modified.

1.1.4 The scattering mapping

One is led to define a mapping acting on pairs of velocities, based on the transition from pre to post-collisional configurations.

Definition 1 (Scattering mapping)**.** *For any vector ω of the sphere $\mathbb{S}^{d-1}$ (called* the angular parameter*), one defines the* scattering mapping $\mathfrak{S}_\omega$ *(or simply $\mathfrak{S}$) as the following function :*

$$\mathfrak{S}_\omega : (v, v_*) \mapsto \big(v - \big((v - v_*) \cdot \omega\big)\omega, v_* + \big((v - v_*) \cdot \omega\big)\omega\big). \tag{1.3}$$

The pair of velocities obtained from the original pair (v, v_*) after the application of the scattering mapping will usually be denoted (v', v_*'), rather than $\mathfrak{S}_\omega(v, v_*)$ (this notation omitting the angular parameter).

One sees that the change of velocities of a pair of colliding particles i and j at time t defined in the previous section is given by the scattering mapping, applied for the particular angular parameter

$$\omega = \frac{x_i(t) - x_j(t)}{|x_i(t) - x_j(t)|}.$$

One gives briefly some obvious elementary properties verified by the scattering mapping.

Proposition 1 (Elementary properties of the scattering mapping)**.** *For any angular parameter ω :*

- *the scattering mapping is an involution, that is*

$$\mathfrak{S}_\omega \circ \mathfrak{S}_\omega = id,$$

- *it sends the pre-collisional pairs of velocities to the post-collisional ones and conversely, that is*

$$(v - v_*) \cdot \omega < 0 \Leftrightarrow (v' - v'_*) \cdot \omega > 0,$$

- *it verifies the two following conservation laws :*

$$v' + v'_* = v + v_*,$$

and

$$\frac{|v'|^2}{2} + \frac{|v'_*|^2}{2} = \frac{|v|^2}{2} + \frac{|v_*|^2}{2}.$$

Remark 1. *The two conservation laws have in fact an important physical meaning : the first one, which is a vector relation, means that the* momentum is preserved *by scattering mapping, while the second one, which is a scalar relation, means that the* kinetic energy is preserved *by this mapping.*

1.1.5 The notations for the configurations

The motion of each particle of the system is entirely determined, at any time t, by its position and its velocity at this time t. Therefore, the configuration of the whole system is entirely determined by a single vector, which is the collection of all those positions and velocities. Such a vector is called a *configuration.*

Definition 2 (Configurations)**.** *Let N be a positive integer. For any collection of N vectors $x_1, \ldots, x_N$ lying in $\mathbb{R}^d$ (representing the respective positions of the N particles of the system), and $v_1, \ldots, v_N$ lying in $\mathbb{R}^d$ (representing the respective velocities of the N particles of the system), one will call* configuration of the system *the following vector, denoted*

$$Z_N = \big(x_1, v_1, \ldots, x_N, v_N\big). \tag{1.4}$$

One will denote the collection of positions of the configuration Z_N, that is $\big(x_1, \ldots, x_N\big)$, as

$$X_N, \tag{1.5}$$

and the collection of velocities of this configuration, that is $\big(v_1, \ldots, v_N\big)$, as

$$V_N. \tag{1.6}$$

Moreover, the specific position (respectively velocity) of the particle i, that is x_i (respectively v_i), will be denoted as

$$Z_N^{X,i} \text{ or } X_N^i \quad (\text{respectively} \quad Z_N^{V,i} \text{ or } V_N^i). \tag{1.7}$$

Finally, when the scattering operator (see Definition 1 above) is applied to the pair of velocities of the particles i and j of the configuration Z_N, one will denote the new configuration

$$\big(x_1, v_1, \ldots, x_i, v_i', \ldots, x_j, v_j', \ldots, x_N, v_N\big),$$

where v_i' and v_j' are the image by the scattering mapping $\mathfrak{S}_\omega$ (for some angular parameter ω which is not written in the notations, but usually easily identifiable in the context) as

$$\big(Z_N\big)'_{i,j}, \quad \text{or } \big(x_1, v_1, \ldots, x_N, v_N\big)'_{i,j}. \tag{1.8}$$

1.1.6 Introducing the free and the hard sphere transports (with boundary condition)

In this section, one will introduce some notation for the transport operators used in the sequel.

Definition 3 (Free transport with boundary condition)**.** *For any positive integer s (denoting the number of particles), any configuration*

$$Z_s = \big(x_1, v_1, \ldots, x_s, v_s\big) \in \big(\mathbb{R}^d\big)^{2s},$$

such that, for all $1 \leq i \leq s$:

$$d(x_i, \Omega) \geq 0,$$

(the particles have to lie outside the obstacle Ω) and for any real number t (denoting time), one will denote by

$$T_t^{s,0}(Z_s) \tag{1.9}$$

the image of the configuration Z_s by the free transport dynamics with boundary condition described above, after a time t (that is, the particles have a radius of size 0 and move in straight lines with a constant velocity except when they bounce against the boundary of the obstacle Ω).

Definition 4 (Hard sphere transport with boundary condition)**.** *For any positive integer s (denoting the number of particles), any strictly positive number ε (denoting the diameter of the particles) and any configuration*

$$Z_s = \big(x_1, v_1, \ldots, x_s, v_s\big) \in \big(\mathbb{R}^d\big)^{2s},$$

such that, for all $1 \leq i \leq s$:

$$d(x_i, \Omega) \geq \varepsilon/2,$$

and for all $1 \leq i < j \leq s$:

$$\big|x_i - x_j\big| \geq \varepsilon,$$

and for any real number t (denoting time), one will denote by

$$T_t^{s,\varepsilon}(Z_s) \tag{1.10}$$

the image of the configuration Z_s by the hard sphere dynamics described above, after a time t (that is, the particles move in straight lines with a constant velocity except when they bounce against the boundary of the obstacle Ω, or when they collide with each other).

In order to define properly the dynamics described in Section 1.1, one needs some preliminary work, that one now presents.

1.1.7 The link between the number and the radius of the particles : the low density limit

So far, no condition was set linking the number N of particles of the system of hard spheres, and the radius $\varepsilon/2$ of those hard spheres. Following Grad in [38][1], one will assume in a formal way that, in average, the particles will travel in straight lines, between two collisions, for a fixed distance, called the *mean free path*. In dimension d, this condition can be translated into the following scaling :

$$N\varepsilon^{d-1} = 1, \tag{1.11}$$

and, together with the limit $N \to +\infty$, the condition (1.11) is called the *Boltzmann-Grad scaling*, or the *Boltzmann-Grad limit*.

The formal link between the mean free path and the scaling condition (1.11) is the following. One assumes that all the particles of the system are travelling with a normalized velocity inside a subset A of the Euclidean space $\mathbb{R}^d$. Considering only one particle i of the system, which is considered to be, one recalls, a sphere of radius $\varepsilon/2$, since its velocity is of norm 1, during a time interval of length 1, this particle i will cover a cylinder of volume $C(d)\varepsilon^{d-1}$, where $C(d)$ is a constant depending only on the dimension. If this particle i collides with another particle j, this means that the particle j has to cross the cylinder covered by the particle i during the time interval $[0, 1]$. Since there are $N - 1$ other particles, the possibility for a collision between the particle i and another particle, that is the possibility for the particle i to move in a straight line only during a time interval contained in $[0, 1]$ is proportional to $(N - 1)\varepsilon^{d-1}$, that is, it is of order $N\varepsilon^{d-1}$ when N is large, so that the Boltzmann-Grad scaling is recovered.
One emphasizes here on the fact that this discussion is only formal. However, this discussion enables to recover the scaling in some natural way.

A consequence of this scaling is that one describes a diluted gas. Indeed, if the quantity $N\varepsilon^{d-1}$ is equal to 1, the quantity $N\varepsilon^d$, which is proportional to

[1] See the Section 2 "Elementary Properties of the Boltzmann Equation", and especially the paragraph starting page 342 concerning spherical molecules. Grad is himself quoting Chapman and Cowling ([27]) and Maxwell ([53]).

the volume occupied by the N spherical particles of the system, is therefore equal to

$$C(d)N\varepsilon^d = C(d)\varepsilon = C(d)\big(1/N\big)^{\frac{1}{d-1}},$$

with $C(d)$ denoting a constant depending only on the dimension, and goes to zero as N goes to infinity. In other words, only a tiny part of the space in which lies the gas is filled with the molecules. This is the reason why the Boltzmann-Grad scaling is sometimes called the *low-density limit*.

1.2 Rigorous definition of the dynamics of the hard spheres

1.2.1 Condition on the obstacle Ω : general discussion

One considers particles which follow the dynamics of hard spheres, that is they are balls with a given diameter $\varepsilon > 0$. Although this parameter ε is meant to go to zero, one wants to be able to define properly the dynamics of balls of diameter ε, for all $\varepsilon < \varepsilon_0$ (with $\varepsilon_0 > 0$ given, depending on the obstacle).
The dynamics will be well-defined only if, for a ball centered on $x(t)$ colliding with $\partial\Omega$, there is only one point belonging to $\partial\Omega \cap B_\varepsilon(x(t))$.
In the case when $d = 2$, because $\partial\Omega$ is a $\mathcal{C}^1$ curve, it can be seen at each point as the graph of some function. The previous condition is then equivalent to the fact that the function which represents the boundary has a second derivative smaller than $1/\varepsilon$ for all ε small enough. Then one sees that the dynamics of one hard sphere of radius smaller than ε_0 is well-defined outside Ω if and only if the curvature of the boundary $\partial\Omega$ is smaller than $1/\varepsilon_0$.
Then it seems convenient to study only subsets of $\mathbb{R}^d$ such that their boundaries are $\mathcal{C}^2$ curves, with a bounded curvature.
In this work, one will in fact focus on one of the easiest possible setting : the obstacle will be the half-space, and therefore the boundary of the domain will be an hyperplane of the Euclidean space $\mathbb{R}^d$.
A natural extension of this work would consist in considering obstacles which are the closure of convex open sets of the Euclidean space.

1.2.2 Elimination of a zero-measure set of configurations

One will check here that for almost all initial configurations, the dynamics introduced in Sections 1.1.2 and 1.1.3 is well defined. One first introduces the phase space.
One needs to choose initial configurations such that none of the particles crosses any other particle of the system, that is particles have to start far enough one from the other, while none of the particles is crossing the obstacle either. One will introduce a notation for this subset, called the *phase space*, since it will be in this subset that the configuration describing the state of the system of particles will evolve.

Definition 5 (The phase space for N hard spheres of radius $\varepsilon/2$)**.** *For any positive integer N (denoting the number of particles of the system) and any strictly positive number ε (denoting the diameter of the particles), one will call the* phase space for N hard spheres of radius $\varepsilon/2$ *evolving around the obstacle $\Omega \subset \mathbb{R}^d$ as the subset of $\mathbb{R}^{2dN}$ defined as :*

$$\mathcal{D}_N^\varepsilon = \Big\{ Z_N \in \Big(\Big(\overline{\Omega + B(0, \varepsilon/2)} \Big)^c \Big)^N \times \mathbb{R}^{dN} \ / \\ \forall\ 1 \leq i < j \leq N,\ |x_i - x_j| > \varepsilon \Big\}, \tag{1.12}$$

that is, in the setting of this work, when $\Omega = \big\{ x \in \mathbb{R}^d \ x \cdot e_1 > \varepsilon/2 \big\}$:

$$\mathcal{D}_N^\varepsilon = \Big\{ Z_N \in \big(\big\{ x \cdot e_1 > \varepsilon/2 \big\} \times \mathbb{R}^d \big)^N \ / \\ \forall\ 1 \leq i < j \leq N,\ \big|x_i - x_j\big| > \varepsilon \Big\},$$

where e_1 denotes the first vector of the canonical basis of the Euclidean space $\mathbb{R}^d$.

Unfortunately, some configurations could lead to an ill-defined dynamics. One first observes that in the case of a bouncing against the obstacle, the velocity is modified, following the Snell-Descartes law introduced in Section 1.1.2 page 50. In particular, the velocity after the bouncing depends on the point of contact between the particle and the obstacle. Thanks to the convexity of the obstacle (in fact any obstacle with a bounded curvature provides the same property up to assuming that the size of the particles is small enough), none of the initial configurations could lead to a situation in which a particle touches at the same time the obstacle at two or more different points.
However, some initial configurations could lead to a situation in which a particle collides with two other particles (or more) at the same time. One should also consider an initial configuration leading to a situation in which a particle bounces against the obstacle, and collides with at least another particle at the same time. With those two kinds of initial configuration, the dynamics is ill-defined, indeed : two velocities or more are assigned to the same particle at the time of the considered event. There is no other possible cause of an ill-defined dynamics when the obstacle is the half-space. Indeed, for the case of the grazing velocities, that is, if a particle has a velocity which is parallel to the wall, its trajectory will be well defined since by hypothesis the particle starts from a position far enough from the obstacle, and then will remain far enough from it, as long as the velocity of the particle is not modified. And in this situation, the only way to modify this velocity is with a collision with another particle.
One calls *pathological* a trajectory issued from an initial configuration leading to an ill-defined dynamics.
Fortunately, those initial configurations represent only a very small part of the set of all initial configurations, the size of which can be controlled (one will show below that it is a zero-measure set).

One of the first results stated in [34][2], and based on a proof written in [1], shows that the pathological trajectories are very rare in the case of particles moving freely in the whole space or in the torus. One introduces a variable δ, which is meant to be small, and which will relax a bit the condition "*at the same time*". Indeed, instead of evaluating the size of the set of initial configurations leading to two collisions at the same time involving the same particle, one evaluates the size of the initial configurations leading to two collisions implying the same particle in a small time interval, of size δ. Some trajectories issued from those initial configurations, forming a set of size controlled by δ, may be ill-defined, but one can assert that all the trajectories issued from the other initial configurations are well defined.

Here, of course, one has to consider other pathological trajectories due to the presence of the obstacle. A similar result to the one of [34] is nevertheless obtained in this section following the original reference [1], in the case of particles evolving outside of an obstacle, showing that the initial configurations which lead to at least two *events* during an interval of size δ represent a small part of all the initial configurations. One recalls that an event is either a collision between two particles or a bouncing of a particle with the boundary of the obstacle Ω.

Proposition 2 (Measure of the initial configurations leading to pathological trajectories, and accumulation of events)**.** *Let Ω be the half-space, that is :*

$$\Omega = \big\{x \in \mathbb{R}^d \ / \ x \cdot e_1 \leq 0\big\},$$

where e_1 denotes the first vector of the canonical basis of the Euclidean space $\mathbb{R}^d$, let N be an integer larger than 3 and $0 < \varepsilon \leq 1$ be a strictly positive number. Then the two following assertions hold.

- *The set of initial configurations contained in the phase space for N hard spheres $\mathcal{D}_N^\varepsilon$ leading to a pathological trajectory during the time interval $\mathbb{R}_+$ is of measure zero.*
- *For every initial configuration $Z_N \in \mathcal{D}_N^\varepsilon$, one considers the subset $\mathcal{E}(Z_N)$ of $\mathbb{R}_+$ composed of all the times of the events of this dynamics during the largest time interval where it is well-defined. Then for every initial configuration Z_N outside a subset of the phase space for N hard spheres $\mathcal{D}_N^\varepsilon$, which is of measure zero, $\mathcal{E}(Z_N)$ is a discrete set.*

Remark 2. *The last point of Proposition 2 means that there is no infinite accumulation of events.*

Proof. One starts by proving the first point. The proof of this point, as in the article [34], relies first on estimates which evaluate the size of the set of initial configurations leading to several events in small time intervals. A crucial tool here is the cut-off in very long distances between the particles, and very high

[2]See Proposition 4.1.1, in Section "The N-particle flow".

energy of the system of particles.

Let $\delta \in]0,1]$, $\rho \geq 1$ and $R \geq 1$ be three strictly positive real numbers. One starts by removing configurations featuring large distances between the particles. One removes also configurations with positions far from $x = 0$, or with high energies (that is, norms of the velocities). This means that one considers initial configurations Z_N (see Definition 2 page 52) taken in :

$$\mathcal{D}^{\varepsilon}_{N,\rho,R} = \mathcal{D}^{\varepsilon}_{N} \cap \Big(\big(B_{\mathbb{R}^d}(0,\rho)\big)^N \times B_{\mathbb{R}^{dN}}(0,R) \Big)$$

(see Definition 5 page 56). For every integer $1 \leq i \leq N$, and any pair $1 \leq j \neq k \leq N$ with $j \neq i$ and $k \neq i$, one considers :

$$\mathcal{A}^i_{j,k}(\delta) = \left\{ Z_N \in \mathcal{D}^{\varepsilon}_{N,\rho,R} \ / \ |x_i - x_j| \leq 2\delta R + \varepsilon, \ |x_i - x_k| \leq 2\delta R + \varepsilon \right\}.$$

If Z_N does not belong to this subset of initial configurations, it is not possible yet to claim that the dynamics will be well defined for the whole time interval $[0,\delta]$, but it is clear that the particle i will not be able to collide with the particles j and k during this time interval. Indeed, if one assumes that there exist two times t_1 and t_2 in $[0,\delta]$ such that the dynamics of the system of N particles is well defined on the time interval $[0, \max(t_1,t_2)]$ and such that :

$$\begin{cases} |x_i(t_1) - x_j(t_1)| = \varepsilon, \\ |x_i(t_2) - x_k(t_2)| = \varepsilon, \end{cases}$$

then one would get :

$$|x_i - x_j| \leq |x_i - x_i(t_1)| + |x_i(t_1) - x_j(t_1)| + |x_j(t_1) - x_j| \leq 2\delta R + \varepsilon,$$

since the velocity of each particle has to be smaller in norm than R. Similarly one would get $|x_i - x_k| \leq 2\delta R + \varepsilon$.
On the other hand, one notices that $|x_i - x_j| > \varepsilon$ and $|x_i - x_k| > \varepsilon$ because the initial configuration lies in $\mathcal{D}^{\varepsilon}_N$. Then the size of the set $\mathcal{A}^i_{j,k}$ is bounded by :

$$|B_{\mathbb{R}^d}(0,\rho)|^{N-2} \big(|B_{\mathbb{R}^d}(0, 2\delta R + \varepsilon)| - |B_{\mathbb{R}^d}(0,\varepsilon)| \big)^2 |B_{\mathbb{R}^{dN}}(0,R)|,$$

and using the fact that :

$$\big(|B_{\mathbb{R}^d}(0, 2\delta R + \varepsilon)| - |B_{\mathbb{R}^d}(0,\varepsilon)| \big)^2 = C(d) \big((2\delta R + \varepsilon)^d - \varepsilon^d \big)^2,$$

with $C(d)$ denoting a constant depending only on the dimension d, one can state that the size of the set $\mathcal{A}^i_{j,k}$ is bounded by :

$$\left| \mathcal{A}^i_{j,k} \right| \leq C_1(d,N) \rho^{d(N-2)} R^{dN} \Big(\sum_{k=1}^{d} \binom{d}{k} \big(2\delta R\big)^k \varepsilon^{d-k} \Big)^2, \tag{1.13}$$

with $C_1(d,N)$ denoting a constant depending only on the the dimension d and the number of particles N.
Similarly, for all integers $1 \leq i \neq j \leq N$, one considers :

$$\mathcal{B}^i_j(\delta) = \left\{ Z_N \in \mathcal{D}^{\varepsilon}_{N,\rho,R} \ / \ |x_i - x_j| \leq 2\delta R + \varepsilon, d(\Omega, x_i) \leq \delta R + \varepsilon/2 \right\}.$$

As for the subset $\mathcal{A}^i_{j,k}(\delta)$, the set $\mathcal{B}^i_j(\delta)$ is such that, if Z_N does not belong to $\mathcal{B}^i_j(\delta)$, then the particle i will not collide with the particle j nor bounce against the obstacle during the time interval $[0,\delta]$.
Indeed, if one assumes that there exist two times $t_1 \in [0,\delta]$ and $t_2 \in [0,\delta]$ such that the dynamics of the system of N particles is well defined on the time interval $[0, \max(t_1,t_2)]$ and such that :

$$\begin{cases} |x_i(t_1) - x_j(t_1)| = \varepsilon, \\ d(\Omega, x_i(t_2)) = \varepsilon/2, \end{cases}$$

then one gets on the one hand, as above :

$$|x_i - x_j| \leq 2\delta R + \varepsilon.$$

On the other hand, one gets :

$$d(\Omega, x_i) \leq \delta R + \varepsilon/2,$$

since for every $y \in \Omega$:

$$d(\Omega, x_i) \leq d(y, x_i) \leq d(y, x_i(t_2)) + d(x_i(t_2), x_i) \leq d(y, x_i(t_2)) + \delta R$$

because the velocity of all the particles is bounded by R. Taking the infimum of the quantity $d(y, x_i(t_2)) + \delta R$ for all the y belonging to Ω, one obtains the result.
The size of the set $\mathcal{B}^i_j$ is also bounded by the following expression :

$$\begin{aligned} |B_{\mathbb{R}^d}(0,\rho)|^{N-2} &\left(|B_{\mathbb{R}^d}(0, 2\delta R + \varepsilon)| - |B_{\mathbb{R}^d}(0,\varepsilon)|\right) |B_{\mathbb{R}^{d-1}}(0,\rho)| \\ &\times |[\varepsilon/2, \delta R + \varepsilon/2]|\, |B_{\mathbb{R}^{dN}}(0,R)| \\ &= C_2(d,N)\rho^{d(N-1)-1} R^{dN+1} \delta \sum_{k=1}^{d} \binom{d}{k} (2\delta R)^k \varepsilon^{d-k}, \end{aligned} \tag{1.14}$$

with $C_2(d,N)$ denoting a constant depending only on the dimension d and the number of particles N.
Here one notices that the condition $d(\Omega, x_i) \leq \delta R + \varepsilon/2$ defining the set $\mathcal{B}^i_j(\delta)$ is very easy to transform in a simple condition when the obstacle is the half-space.

Now one is able to define a subset of $\mathcal{D}^\varepsilon_{N,\rho,R}$ composed of initial configurations which are all leading to a dynamics well defined on the whole time interval $[0,\delta]$. Indeed, if one considers :

$$\mathcal{D}^\varepsilon_{N,\rho,R} \setminus \mathcal{C}^\varepsilon_{N,\rho,R}(\delta, 1)$$

with

$$\mathcal{C}^\varepsilon_{N,\rho,R}(\delta,1) = \Big(\Big(\bigcup_{\substack{1\leq i\leq N \\ 1\leq j\neq k\leq N,\ j\neq i,\ k\neq i}} \mathcal{A}^i_{j,k}(\delta) \Big) \cup \Big(\bigcup_{\substack{1\leq i\leq N \\ 1\leq j\leq N,\ j\neq i}} \mathcal{B}^i_j(\delta) \Big) \Big),$$

then for every initial configuration Z_N taken in this set, no particle can collide with two different particles during the time interval $[0, \delta]$, and there is no particle which can collide with another particle and bounce against the obstacle Ω during the same time interval.
Moreover, the size of the excluded set is small, that is one has :

$$\begin{aligned}
\left|\mathcal{C}^{\varepsilon}_{N,\rho,R}(\delta,1)\right| &= \left|\left(\left(\bigcup_{\substack{1\le i\le N\\ 1\le j\ne k\le N,\ j\ne i,\ k\ne i}} \mathcal{A}^i_{j,k}(\delta)\right)\cup\left(\bigcup_{\substack{1\le i\le N\\ 1\le j\le N,\ j\ne i}} \mathcal{B}^i_j(\delta)\right)\right)\right| \\
&\le C_1(d,N)N\binom{N-1}{2}\rho^{d(N-2)}R^{dN}\left(\sum_{k=1}^{d}\binom{d}{k}(2\delta R)^k\varepsilon^{d-k}\right)^2 \\
&\quad + C_2(d,N)N(N-1)\rho^{d(N-1)-1}R^{dN+1}\delta\sum_{k=1}^{d}\binom{d}{k}(2\delta R)^k\varepsilon^{d-k} \\
&\le C_3(d,N)R^{dN+2}\left(\rho^{d(N-2)}\varepsilon^{2(d-1)}+\rho^{d(N-1)-1}\varepsilon^{d-1}\right)\delta^2 \qquad (1.15)
\end{aligned}$$

for δ small enough with ε, R and ρ fixed, with C_3 denoting a constant depending only on the dimension d and the number of particles N, because when $\varepsilon, \delta \le 1$ and $R \ge 1$, if one assumes that :

$$\delta \le \delta_0 = \min_{1\le k\le d}\frac{\varepsilon}{2R}\left(\frac{d}{\binom{d}{k}}\right)^{1/(k-1)},$$

one sees immediately that :

$$\binom{d}{k}(2\delta R)^k\varepsilon^{d-k} \le \binom{d}{1}2\delta R\varepsilon^{d-1},$$

so that the last inequality becomes obvious.

One will now use the set constructed just above to prove that the dynamics is well defined on the whole half real line $\mathbb{R}_+$ for almost every initial configurations.
First, let t be a given strictly positive constant. One will prove that the size of the initial configurations leading to an ill-defined dynamics before the time t is of measure zero among all the initial configurations.
One chooses $\delta > 0$ such that t/δ is a positive integer, say m. On the one hand, up to excluding a "small" subset of $\mathcal{D}^{\varepsilon}_{N,\rho,R}$, one knows that for every initial configuration, the dynamics is well-defined until the time δ. On the other hand, by the definition of the dynamics, one knows that the energy of the system is constant and is smaller than R. But since the initial positions of the vector Z_N are all taken in $B_{\mathbb{R}^d}(0,\rho)$, at time δ, the initial configuration Z_N becomes a new configuration which belongs to

$$\mathcal{D}^{\varepsilon}_N \cap \left(\left(B_{\mathbb{R}^d}(0,\rho+\delta R)\right)^N \times B_{\mathbb{R}^{dN}}(0,R)\right),$$

that is $\mathcal{D}^{\varepsilon}_{N,\rho+\delta R,R}$, according to the previous notation. Remembering that $\varepsilon \leq 1$, the bound (1.15) can be rewritten since

$$\rho^{d(N-2)}\varepsilon^{2(d-1)} \leq \frac{\rho^{d(N-1)-1}}{\rho^{d-1}}\varepsilon^{d-1} \leq \rho^{d(N-1)-1}\varepsilon^{d-1},$$

so that

$$\left(\rho^{d(N-2)}\varepsilon^{2(d-1)} + \rho^{d(N-1)-1}\varepsilon^{d-1}\right) \leq 2\rho^{d(N-1)-1}\varepsilon^{d-1}. \tag{1.16}$$

Then, following the same steps as above and thanks to the bound (1.15) rewritten with (1.16), and up to excluding a small subset $\mathcal{C}^{\varepsilon}_{N,\rho+\delta R,R}(\delta,1)$ of $\mathcal{D}^{\varepsilon}_{N,\rho+\delta R,R}$ of size bounded by

$$C_4(d,N)R^{dN+2}\left(\rho+\delta R\right)^{d(N-1)-1}\varepsilon^{d-1}\delta^2,$$

(with $C_4(d,N) = 2C_3(d,N)$), the dynamics is then well-defined until 2δ. Since the hard sphere flow preserves the measure (see for example [26][3]), it is possible to exclude from the set of the initial configuration a subset denoted by

$$\mathcal{C}^{\varepsilon}_{N,\rho,R}(\delta,2), \tag{1.17}$$

which is the union of two subsets. The first subset, that is $\mathcal{C}^{\varepsilon}_{N,\rho,R}(\delta,1)$, ensures that the dynamics is well-defined until δ. The second subset, which is :

$$\left(T^{N,\varepsilon}_{-\delta}\left(\mathcal{C}^{\varepsilon}_{N,\rho+\delta R,R}(\delta,1)\right)\right) \cap \left(B_{\mathbb{R}^d}(0,R)^N \times B_{\mathbb{R}^{dN}}(0,R)\right),$$

ensures that it is well-defined until 2δ ($T^{N,\varepsilon}_{*}$ denoting the hard sphere flow of N particles of radius ε).
By induction, up to excluding a subset $\mathcal{C}^{\varepsilon}_{N,\rho,R}(\delta,m)$ which has a size smaller than

$$C_4(d,N)R^{dN+2}\varepsilon^{d-1}\delta^2\sum_{k=0}^{m-1}\left((\rho+k\delta R)^{d(N-1)-1}\right),$$

the dynamics is well defined on the whole time interval $[0,t]$. Taking $\rho = R$, the size of the excluded set is then bounded by (remembering that $t = m\delta$) :

$$C_4(d,N)R^{d(2N-1)+1}\varepsilon^{d-1}t(1+(m-1)\delta)^{d(N-1)-1}\delta \leq C(d,N,t,\varepsilon)R^{d(2N-1)+1}\delta. \tag{1.18}$$

Considering the subset of the initial configurations

$$\mathcal{F}^{\varepsilon}_{N,R}(t) = \bigcap_{\delta>0\ /\ (t/\delta)\ \in\ \mathbb{N}^*} \mathcal{C}^{\varepsilon}_{N,R,R}(\delta,t/\delta),$$

that is

$$\bigcap_{m\ \in\ \mathbb{N}^*} \mathcal{C}^{\varepsilon}_{N,R,R}(t/m,m),$$

[3]In particular, see Appendix 4.A "More About Hard-Sphere Dynamics".

which is of course of measure zero thanks to the bound (1.18), one knows that every initial configuration taken in

$$\mathcal{D}^{\varepsilon}_{N,R,R} = \mathcal{D}^{\varepsilon}_{N} \cap \Big(\left(B_{\mathbb{R}^d}(0,R)\right)^N \times B_{\mathbb{R}^{dN}}(0,R) \Big)$$

and outside the set

$$\mathcal{F}^{\varepsilon}_{N,R}(t)$$

is well-defined for the whole time interval $[0,t]$.
It is now sufficient to consider a sequence $(R_n)_{n\in\mathbb{N}}$ going to infinity with $R_0 \geq 1$, and :

$$\mathcal{F}^{\varepsilon}_{N}(t) = \bigcup_{n\in\mathbb{N}} \mathcal{F}^{\varepsilon}_{N,R_n}(t),$$

which is of measure zero as a countable union of subsets of measure zero, and which contains all the initial configurations (without any condition on the sizes of the positions and velocities) leading to an ill-defined dynamics before the time t.
Finally, since the time t has been chosen arbitrarily, one considers the set :

$$\mathcal{F}^{\varepsilon}_{N} = \bigcup_{p\in\mathbb{N}^*} \mathcal{F}^{\varepsilon}_{N}(pt),$$

which is once again of measure zero as a countable union of subsets of measure zero. If taken outside this subset $\mathcal{F}^{\varepsilon}_{N}$, any initial configuration $Z_N \in \mathcal{D}^{\varepsilon}_{N}$ leads to a well-defined dynamics, and the first point of the proposition is proved.

For the second point, one will consider an initial configuration of $\mathcal{D}^{\varepsilon}_{N}$, which is not an element of $\mathcal{F}^{\varepsilon}_{N}$. This subset is, as shown in the previous point, of measure zero, and the dynamics from this initial configuration is well-defined on the whole time interval $\mathbb{R}_+$.
On the one hand, if one assumes that $\mathcal{E}(Z_N)$ admits an accumulation point, say $t_0 > 0$ (that is there exists a sequence of events $(t_k)_{k\in\mathbb{N}}$ such that $t_k \underset{k\to+\infty}{\longrightarrow} t_0$), since there is only a finite number of particles (here : N), there exists $1 \leq N_0 \leq N$ such that it is possible to extract a subsequence $(t_{\varphi(k)})_{k\in\mathbb{N}}$ of $(t_k)_{k\in\mathbb{N}}$, in such a way that all the events of this subsequence correspond to the particle N_0, that is each event $t_{\varphi(k)}$ is either a collision between N_0 and another particle, or it is a bouncing of the particle N_0 against the obstacle.
On the other hand, going back to the definition of the subset $\mathcal{F}^{\varepsilon}_{N}$, the fact that $Z_N \notin \mathcal{F}^{\varepsilon}_{N}$ exactly means that :

$$\forall t > 0,\ \forall\, p \in \mathbb{N}^*,\ \forall\, n \in \mathbb{N},\ Z_N \notin \mathcal{F}^{\varepsilon}_{N,R_n}(pt),$$

where $\mathcal{F}^{\varepsilon}_{N,\rho_n,R_n}(pt)$ is defined in the proof of the first point, that is :

$$\forall t > 0,\ \forall\, p \in \mathbb{N}^*,\ \forall\, n \in \mathbb{N},\ \exists\, m \in \mathbb{N}^*\ /\ Z_N \in \mathcal{C}^{\varepsilon}_{N,R_n}(pt/m, m).$$

In particular, one chooses $t = 2t_0$, where t_0 is the accumulation point of the set of events $\mathcal{E}(Z_N)$ mentionned above, so that $t_0 \in]0,t[$, and then there exists

$k_0 \in \mathbb{N}$ so that for all $k \geq k_0$, one has :

$$t_{\varphi(k)} \in]0, t[.$$

Since the sequence $\big(R_n\big)_{n\in\mathbb{N}}$ is increasing and tends to infinity, there exists $n_0 \in \mathbb{N}$ such that :

$$Z_N \in \big(B_{\mathbb{R}^d}(0, R_{n_0})\big)^N \times B_{\mathbb{R}^{dN}}(0, R_{n_0}).$$

Finally, for $t = 2t_0$, $n = n_0$ and $p = 1$, there exists $m_0 \in \mathbb{N}^*$ such that :

$$Z_N \in \mathcal{C}^{\varepsilon}_{N,R_{n_0}}(t/m_0, m_0).$$

So all the events concerning the particle N_0 are separated by a time interval of length larger than t/m_0, which is obviously a contradiction with the fact that t_0 is an accumulation point of events concerning this particle. The second point of the proposition is then proved. □

Remark 3. *Even if the propostion is stated in the particular case when the obstacle is the half-space, it certainly holds for a more general convex obstacle, namely if one considers the closure of a convex open set of the Euclidean space $\mathbb{R}^d$ with a $\mathcal{C}^2$ boundary such that the curvature of this boundary is bounded. The proof presented here should be essentially the same, except some small additionnal geometric arguments. In particular, the size of the set $\mathcal{B}^i_j$ introduced in the proof depends on the obstacle.*
At this level, it is interesting to notice that it is possible to show Proposition 2 for a flow of hard spheres outside of a set a little bit more general than a convex set, at least in dimension 2. Indeed, it is sufficient to assume that there exists $\alpha > 0$ such that, for every $x \in \partial\Omega \subset \mathbb{R}^2$, Ω is outside of the convex subset bounded by the parabola of vertex x and of focal length α. It is easy to show under such an assumption that if a particle collides twice with the boundary $\partial\Omega$ during a time interval of size δ, then the initial speed of this particle has to lie in a set of measure of order δ.

Chapter 2

The BBGKY hierarchy for the dynamics of N hard spheres

Now that the motion of the particles has been described, one will study the behaviour of density functions of a system of such particles. In fact one will see that it is possible to link the marginals of those density functions by a sequence of equations, called the *BBGKY hierarchy*.

2.1 Formal derivation of the BBGKY hierarchy

2.1.1 Introducing the marginals of the distribution function, and the Liouville equation

One introduces here the notation for the marginals of a distribution function of a system of N particles. In the following, one will need a slightly more general definition of the marginals of a nonnegative function, namely : one will define the s-th marginal, for any s positive integer.

Definition 6 (Marginals of a distribution function)**.** *Let N be a positive integer, and ε be a strictly positive real number. One considers a nonnegative, integrable function f_N defined on the phase space for N hard spheres of radius $\varepsilon/2$ (see Definition 5 page 56).*
For every integer $1 \leq s \leq N$, one will call the s-th marginal of the function f_N, *the function denoted by $f_N^{(s)}$, and defined as :*

$$f_N^{(s)}(Z_s) = \int_{\mathbb{R}^{2d(N-s)}} f_N(Z_s, z_{s+1}, \ldots, z_N) \mathbb{1}_{Z_N \in \mathcal{D}_N^\varepsilon} \, \mathrm{d}z_{s+1} \ldots \mathrm{d}z_N.$$

For every integer $s > N$, one will call the s-th marginal of the nonnegative function f_N, *the function denoted by $f_N^{(s)}$, and defined as the zero function defined*

on $\mathcal{D}_s^\varepsilon$.

One immediately sees that the s-th marginal is defined on $\mathcal{D}_s^\varepsilon \subset \mathbb{R}^{2ds}$. Moreover, for any positive integer N, any strictly positive real number ε, and for every integers $1 \leq s \leq N-1$, the marginals satisfy the following property :

$$f_N^{(s)}(Z_s) = \int_{\mathbb{R}^{2d}} f_N^{(s+1)}(Z_s, z_{s+1})\,\mathrm{d}z_{s+1}.$$

One recalls that one is considering the simple case of the half-space, that is

$$\Omega = \big\{x \in \mathbb{R}^d \ / \ x \cdot e_1 \leq 0\big\}.$$

In this configuration, it is clear that the condition of regularity on $\partial\Omega$ (discussed in Section 1.2.1 page 55) is fulfilled so that the dynamics of N hard spheres, for any $N \in \mathbb{N}^*$, is well defined, and thanks to Proposition 2 page 57, one knows that the set of the configurations leading to a pathological trajectory is of measure zero.

Let $N \in \mathbb{N}^*$ and $\varepsilon > 0$ be fixed. One recalls that $\mathcal{D}_N^\varepsilon$ denotes the phase space for N hard spheres, which is the set

$$\begin{aligned}\mathcal{D}_N^\varepsilon =& \Big\{Z_N \in \mathbb{R}^{2dN} \ / \ \forall\ 1 \leq i \neq j \leq N, |x_i - x_j| > \varepsilon\Big\}\\ &\cap \Big\{Z_N \in \mathbb{R}^{2dN} \ / \ \forall\ 1 \leq i \leq N, x_i \cdot e_1 > \varepsilon/2\Big\}.\end{aligned}$$

One considers the nonnegative solution f_N of the *Liouville equation* associated to the transport of N hard spheres of radius $\varepsilon/2$ outside the obstacle Ω, defined in the phase space $\mathcal{D}_N^\varepsilon$. This equation writes :

$$\begin{aligned}&\forall t \geq 0, \forall Z_N \in \mathcal{D}_N^\varepsilon,\\ &\qquad \partial_t f_N(t, Z_N) + \sum_{i=1}^N v_i \cdot \nabla_{x_i} f_N(t, Z_N) = 0. \qquad (2.1)\end{aligned}$$

The reader can see that the boundary conditions are not prescribed yet. This is the purpose of the following paragraph.

2.1.2 The boundary conditions of the hard sphere dynamics

Boundary conditions are added to equation (2.1) in order to define f_N when its argument is about to enter the phase space, fitting with the hard sphere dynamics. In other words, those boundary conditions provide a rigorous way of defining the collisions between particles as well as the bouncings of the particles against the boundary of the obstacle Ω.

In order to write down those boundary conditions, one defines an involution,

denoted by χ_N^ε, on the boundary of the phase space $\partial\mathcal{D}_N^\varepsilon$. One first observes that for $Z_N = \big(x_1, v_1, \dots, x_N, v_N\big) \in \partial\mathcal{D}_N^\varepsilon$, either there exist $1 \le i \ne j \le N$ such that $|X_N^i - X_N^j| = \varepsilon$, or there exists $1 \le i \le N$ such that $X_N^i \cdot e_1 = \varepsilon/2$. The map χ_N^ε is defined almost everywhere on $\partial\mathcal{D}_N^\varepsilon$. It acts only on the velocity variables of the configurations, in the way described in the following Definition.

Definition 7 (Boundary conditions for the hard sphere dynamics)**.** *Let s be a positive integer and ε be a strictly positive number. One defines the* boundary conditions for the hard sphere dynamics of s particles of radius $\varepsilon/2$ around the obstacle Ω *as the following function.*

$$\chi_s^\varepsilon : \left\{ \begin{array}{ll} \partial\mathcal{D}_s^\varepsilon & \to \partial\mathcal{D}_s^\varepsilon, \\ Z_s & \mapsto \chi_s^\varepsilon(Z_s), \end{array} \right. \tag{2.2}$$

with

$$\chi_N^\varepsilon(Z_s)^X = (Z_s)^X$$

and is such that if there exist $1 \le i \ne j \le s$ such that $|x_i - x_j| = \varepsilon$, $|x_k - x_l| > \varepsilon$ for all couples $(k,l) \ne (i,j)$, and $x_k \cdot e_1 > \varepsilon/2$ for all $1 \le k \le s$:

$$\left\{ \begin{array}{ll} \chi_s^\varepsilon(Z_s)^{V,i} & = v_i - (1/\varepsilon^2)(v_i - v_j)\cdot(x_i - x_j)(x_i - x_j), \\ \chi_s^\varepsilon(Z_s)^{V,j} & = v_j + (1/\varepsilon^2)(v_i - v_j)\cdot(x_i - x_j)(x_i - x_j), \\ \chi_s^\varepsilon(Z_s)^{V,k} & = v_k \text{ for all } 1 \le k \le s, k \ne i, k \ne j, \end{array} \right.$$

using the notations (1.7) *in Definition 2 page 52 to describe the positions and the velocities of a given configuration. In other words, the velocities of the pair of particles i and j are modified using the scattering mapping (see Definition 1 page 51) with the angular parameter*

$$\frac{x_i - x_j}{\big|x_i - x_j\big|} = \frac{x_i - x_j}{\varepsilon},$$

that is

$$\big(\chi_s^\varepsilon(Z_s)^{V,i}, \chi_s^\varepsilon(Z_s)^{V,j}\big) = \mathfrak{S}_{(x_i - x_j)/\varepsilon}\big(Z_s^{V,i}, Z_s^{V,j}\big) = \mathfrak{S}_{(x_i - x_j)/\varepsilon}\big(v_i, v_j\big).$$

If there exists $1 \le i \le s$ such that $x_i \cdot e_1 = \varepsilon/2$, $x_j \cdot e_1 > \varepsilon/2$ for all $j \ne i$, and $|x_j - x_k| > \varepsilon$ for all $1 \le j < k \le s$:

$$\left\{ \begin{array}{l} \chi_s^\varepsilon(Z_s)^{V,i} = v_i - 2v_i \cdot e_1 e_1, \\ \chi_s^\varepsilon(Z_s)^{V,j} = v_j \text{ for all } 1 \le j \le s, j \ne i. \end{array} \right.$$

The map χ_N^ε describes in a rigorous way how the particles collide with each other or are reflected on the wall.

The map χ_N^ε is defined properly on the set $\widehat{\partial\mathcal{D}_N^\varepsilon}$, defined as :

$$\widehat{\partial\mathcal{D}_N^\varepsilon} = \Big(\bigcup_{1 \le i \le N} B_i\Big) \cup \Big(\bigcup_{1 \le i < j \le N} C_{i,j}\Big) \tag{2.3}$$

with, for all $1 \leq i \leq N$:

$$B_i = \Big\{ Z_N \in \mathbb{R}^{2dN} \ / \ x_i \cdot e_1 = \varepsilon/2, \\ \forall\, j \neq i,\ x_j \cdot e_1 > \varepsilon/2,\ \forall\, 1 \leq k < l \leq N,\ |x_j - x_k| > \varepsilon \Big\}, \tag{2.4}$$

(corresponding to the configurations in which the only event is a bouncing of the particle i against the wall), and for all $1 \leq i < j \leq N$:

$$C_{i,j} = \Big\{ Z_N \in \mathbb{R}^{2dN} \ / \ |x_i - x_j| = \varepsilon, \\ \forall\, 1 \leq k < l \leq N \ / \ (k,l) \neq (i,j),\ |x_k - x_l| > \varepsilon, \\ \forall\, 1 \leq m \leq N,\ x_m \cdot e_1 > \varepsilon/2 \Big\}, \tag{2.5}$$

(corresponding to the configurations in which the only event is a collision between the two particles i and j), and one observes that the map χ_N^ε is an involution on this set.

Then, once the map χ_N^ε is properly defined on the boundary of the phase space except for a subset of measure zero, namely, on the set $\widehat{\partial \mathcal{D}_N^\varepsilon}$ of full measure, one defines the boundary condition associated to Equation (2.1) page 66. To do so, one splits into two parts $\widehat{\partial \mathcal{D}_N^\varepsilon}$, separating the "incoming" and the "outgoing" configurations.
One writes

$$B_i^{\ in} = \Big\{ Z_N \in \mathbb{R}^{2dN} \ / \ x_i \cdot e_1 = \varepsilon/2, \\ \forall\, j \neq i,\ x_j \cdot e_1 > \varepsilon/2,\ \forall\, 1 \leq k < l \leq N,\ |x_j - x_k| > \varepsilon, \\ \text{and } v_i \cdot e_1 > 0 \Big\},$$

(corresponding to a configuration in which only the particle i is in contact with the obstacle, and such that, for any strictly positive time, this particle will be at a distance strictly larger than $\varepsilon/2$ from the obstacle, so that if the configuration is transpored by the hard sphere flow for a small enough, strictly positive time, the image of this configuration will lie in the phase space $\mathcal{D}_N^\varepsilon$. This is an incoming configuration),
and

$$C_{i,j}^{\ in} = \Big\{ Z_N \in \mathbb{R}^{2dN} \ / \ |x_i - x_j| = \varepsilon, \\ \forall\, 1 \leq k < l \leq N \ / \ (k,l) \neq (i,j),\ |x_k - x_l| > \varepsilon, \\ \forall\, 1 \leq m \leq N,\ x_m \cdot e_1 > \varepsilon/2, \\ \text{and } (x_i - x_j) \cdot (v_i - v_j) > 0 \Big\},$$

(corresponding to a configuration in which only the pair of particles i and j are colliding one with another, and such that, for any small enough, strictly positive

time, the distance between those two particles will be strictly larger than ε, this is also an incoming configuration).
Defining finally

$$\big(\widehat{\partial\mathcal{D}_N^\varepsilon}\big)^{in} = \Big(\bigcup_{1\le i\le N} B_i^{\,in}\Big) \cup \Big(\bigcup_{1\le i<j\le N} C_{i,j}^{\,in}\Big), \tag{2.6}$$

one sets the boundary condition associated to Equation (2.1) page 66 as :

$$\begin{aligned} &\forall t \ge 0,\, \forall Z_N \in \big(\widehat{\partial\mathcal{D}_N^\varepsilon}\big)^{in}, \\ &\qquad f_N(t, Z_N) = f_N(t, \chi_N^\varepsilon(Z_N)). \end{aligned} \tag{2.7}$$

The so-called BBGKY hierarchy is a system consisting of a finite number of equations holding in a weak sense, that will be detailed below, built at the formal level from the Liouville equation (2.1) page 66 with the boundary conditions (2.7), assuming in addition the *indistinguishibility of the particles*, that is :

$$\forall \sigma \in \mathfrak{S}_N,\ f_N(t, Z_{\sigma(N)}) = f_N(t, Z_N), \tag{2.8}$$

with $Z_{\sigma(N)}$ denoting naturally $\big(z_{\sigma(1)}, \dots, z_{\sigma(N)}\big)$, and $\mathfrak{S}_N$ denoting the symmetric group on the finite set $\big\{1, \dots, N\big\}$.
One observes that indistinguishibility is propagated in time (at the formal level) by the Liouville equation (2.1) with the boundary conditions (2.7), so that it is sufficient to assume this indistinguishibility at the level of the initial data only.

Let s be a positive integer smaller than N. Let $\phi_s \in \mathcal{C}_c^\infty\big(\mathbb{R}_+ \times \mathbb{R}^{2ds}, \mathbb{R}\big)$, such that ϕ_s satisfies the following equality, which is a boundary condition in the phase space of s particles :

$$\forall t \ge 0,\ \forall Z_s \in \big(\widehat{\partial\mathcal{D}_s^\varepsilon}\big)^{in},\ \phi_s(t, Z_s) = \phi_s(t, \chi_s^\varepsilon(Z_s)). \tag{2.9}$$

Starting from the Liouville equation (2.1) page 66, one sees that

$$\int_{\mathbb{R}_+\times\mathbb{R}^{2dN}} \Big[\partial_t f_N + \sum_{i=1}^N v_i \cdot \nabla_{x_i} f_N\Big](t, Z_N)\phi_s(t, Z_s)\mathbb{1}_{Z_N\in\mathcal{D}_N^\varepsilon}\,\mathrm{d}Z_N\,\mathrm{d}t = 0. \tag{2.10}$$

One first observes that, without considering the integration with respect to the time variable, the second term in (2.10), denoted by A, can be rewritten as follows :

$$
\begin{aligned}
A = & \int_{\mathbb{R}^{2dN}} \sum_{i=1}^{N} v_i \cdot \nabla_{x_i} f_N(t, Z_N) \phi_s(t, Z_s) \mathbb{1}_{Z_N \in \mathcal{D}_N^\varepsilon} \, \mathrm{d}Z_N \\
= & \int_{\mathbb{R}^{2dN}} \sum_{i=1}^{N} v_i \cdot \big[\nabla_{x_i}\big(f_N \phi_s\big) - f_N \nabla_{x_i} \phi_s\big](t, Z_N) \mathbb{1}_{Z_N \in \mathcal{D}_N^\varepsilon} \, \mathrm{d}Z_N \\
= & \sum_{i=1}^{N} \int_{\partial \mathcal{D}_N^\varepsilon} f_N(t, Z_N) \phi_s(t, Z_s) v_i \cdot n_{\partial \mathcal{D}_N^\varepsilon} \, \mathrm{d}S \\
& - \sum_{i=1}^{s} \int_{\mathbb{R}^{2dN}} v_i \cdot \nabla_{x_i} \phi_s(t, Z_s) f_N(t, Z_N) \mathbb{1}_{Z_N \in \mathcal{D}_N^\varepsilon} \, \mathrm{d}Z_N,
\end{aligned}
$$

where $n_{\partial \mathcal{D}_N^\varepsilon}$ denotes the normal unitary outgoing vector at any point belonging to the boundary $\partial \mathcal{D}_N^\varepsilon$, and $\mathrm{d}S$ the natural measure on the hypersurface $\partial \mathcal{D}_N^\varepsilon$. Using the decomposition of the boundary of the phase space according to (2.3) (and throwing away the set $\partial \mathcal{D}_N^\varepsilon \backslash \widehat{\partial \mathcal{D}_N^\varepsilon}$ of measure zero), into the subsets B_j, or $C_{j,k}$, respectively defined in (2.4) and (2.5) just above, with notations for the normal unitary outgoing vectors to B_j and $C_{j,k}$ explained in formulas (2.12) and (2.13) just below, one gets

$$
\begin{aligned}
A = & \sum_{i=1}^{N} \int_{\mathbb{R}^{dN}_{V_N}} v_i \cdot \sum_{j=1}^{N} \int_{x_j \cdot e_1 = \varepsilon/2} f_N(t, Z_N) \phi_s(t, Z_s) n^i_{x_j \cdot e_1 = \varepsilon/2} \, \mathrm{d}S \\
& + \sum_{i=1}^{N} \int_{\mathbb{R}^{dN}_{V_N}} v_i \cdot \sum_{1 \le j < k \le N} \int_{|x_j - x_k| = \varepsilon} f_N(t, Z_N) \phi_s(t, Z_s) n^i_{|x_j - x_k| = \varepsilon} \, \mathrm{d}S \\
& - \sum_{i=1}^{s} \int_{\mathbb{R}^{2dN}_{Z_N}} v_i \cdot \nabla_{x_i} \phi_s(t, Z_s) f_N(t, Z_N) \mathbb{1}_{Z_N \in \mathcal{D}_N^\varepsilon} \, \mathrm{d}Z_N
\end{aligned} \tag{2.11}
$$

with the explicit expressions of the normal vectors :

$$
n_{x_j \cdot e_1 = \varepsilon/2} = \big(\underbrace{0, \dots, 0}_{\mathbb{R}^d_{x_1}}, \dots, \underbrace{-1, 0, \dots, 0}_{\mathbb{R}^d_{x_j}}, \dots, \underbrace{0, \dots, 0}_{\mathbb{R}^d_{x_N}}\big) \in \mathbb{R}^{dN},
$$

so that

$$
n^i_{x_j \cdot e_1 = \varepsilon/2} = -\delta^i_j e_1 \in \mathbb{R}^d, \tag{2.12}
$$

where δ^i_j denotes the Kronecker symbol, and :

$$
n_{|x_j - x_k| = \varepsilon} = \frac{\sqrt{2}}{2\varepsilon} \big(\underbrace{0, \dots, 0}_{\mathbb{R}^d_{x_1}}, \dots, \underbrace{x_k - x_j}_{\mathbb{R}^d_{x_j}}, \dots, \underbrace{x_j - x_k}_{\mathbb{R}^d_{x_k}}, \dots, \underbrace{0, \dots, 0}_{\mathbb{R}^d_{x_N}}\big) \in \mathbb{R}^{dN},
$$

so that

$$
n^i_{|x_j - x_k| = \varepsilon} = \frac{\sqrt{2}}{2\varepsilon} \big(\delta^i_k - \delta^i_j\big)(x_j - x_k) \in \mathbb{R}^d. \tag{2.13}
$$

The first term of the right-hand side of (2.11), denoted as B, cancels. Indeed using the explicit form (2.12) of the normal vector to the boundary, one splits the domain of integration between

$$\big\{Z_N \in \mathbb{R}^{2dN} \ / \ x_j \cdot e_1 = \varepsilon/2, \ v_j \cdot e_1 < 0\big\},$$

(which corresponds to a pre-bouncing situation against the obstacle for the particle j), and

$$\big\{Z_N \in \mathbb{R}^{2dN} \ / \ x_j \cdot e_1 = \varepsilon/2, \ v_j \cdot e_1 > 0\big\},$$

(which corresponds to a post-bouncing situation for the same particle). Note that the remaining subset $\big\{Z_N \in \mathbb{R}^{2dN} \ / \ x_j \cdot e_1 = \varepsilon/2, \ v_j \cdot e_1 = 0\big\}$ is of measure zero as a manifold of codimension 2. Then

$$\begin{aligned}
B &= \sum_{i=1}^{N} \int_{\mathbb{R}^{dN}_{V_N}} v_i \cdot \sum_{j=1}^{N} \int_{x_j \cdot e_1 = \varepsilon/2} f_N(t, Z_N) \phi_s(t, Z_s) n^i_{x_j \cdot e_1 = \varepsilon/2} \, \mathrm{d}S \\
&= - \sum_{i=1}^{N} \int_{\mathbb{R}^{dN}_{V_N}} v_i \cdot e_1 \int_{x_i \cdot e_1 = \varepsilon/2} f_N(t, Z_N) \phi_s(t, Z_s) \, \mathrm{d}S \\
&= - \sum_{i=1}^{N} \int\limits_{\substack{x_i \cdot e_1 = \varepsilon/2 \\ v_i \cdot e_1 < 0}} f_N(t, Z_N) \phi_s(t, Z_s) v_i \cdot e_1 \, \mathrm{d}S \\
&\qquad\qquad - \sum_{i=1}^{N} \int\limits_{\substack{x_i \cdot e_1 = \varepsilon/2 \\ v_i \cdot e_1 > 0}} f_N(t, Z_N) \phi_s(t, Z_s) v_i \cdot e_1 \, \mathrm{d}S, \qquad (2.14)
\end{aligned}$$

so that, performing the change of variable $Z_N \mapsto \chi^\varepsilon_N(Z_N)$ in the last term, one finds :

$$\begin{aligned}
B &= - \sum_{i=1}^{N} \int\limits_{\substack{x_i \cdot e_1 = \varepsilon/2 \\ v_i \cdot e_1 < 0}} f_N(t, Z_N) \phi_s(t, Z_s) v_i \cdot e_1 \, \mathrm{d}S \\
&- \sum_{i=1}^{N} \int\limits_{\substack{x_i \cdot e_1 = \varepsilon/2 \\ v_i \cdot e_1 < 0}} f_N\big(t, \chi^\varepsilon_N(Z_N)\big) \\
&\qquad \times \phi_s\big(t, \chi^\varepsilon_N(Z_N)^{X,1}, \chi^\varepsilon_N(Z_N)^{V,1}, \dots, \chi^\varepsilon_N(Z_N)^{X,s}, \chi^\varepsilon_N(Z_N)^{V,s}\big) \\
&\qquad\qquad \times \big(\chi^\varepsilon_N(Z_N)^{V,i}\big) \cdot e_1 \, \mathrm{d}S \\
&= 0,
\end{aligned}$$

since

$$\big(\chi^\varepsilon_N(Z_N)^{V,i}\big) \cdot e_1 = \big(v_i - 2 v_i \cdot e_1\big) \cdot e_1 = -v_i \cdot e_1$$

by definition of χ_N^ε on the subset B_i of the boundary.

The second term obtained in (2.11), denoted as C, is then rewritten as follows, using again the explicit form (2.13) of the normal vector to the boundary :

$$\begin{aligned} C &= \sum_{i=1}^{N} \int_{\mathbb{R}^{dN}_{V_N}} v_i \cdot \sum_{1\le j<k\le N} \int_{|x_j-x_k|=\varepsilon} f_N(t,Z_N)\phi_s(t,Z_s) n^i_{|x_j-x_k|=\varepsilon}\,\mathrm{d}S \\ &= -\sum_{1\le i<j\le N} \int_{\mathbb{R}^{dN}_{V_N}} \frac{\sqrt{2}}{2\varepsilon}(v_i-v_j)\cdot \int_{|x_i-x_j|=\varepsilon} (x_i-x_j) f_N(t,Z_N)\phi_s(t,Z_s)\,\mathrm{d}S. \end{aligned}$$

Splitting the sum in three terms, one obtains :

$$\begin{aligned} C = -\Big(\sum_{1\le i<j\le s} + \sum_{\substack{1\le i\le s\\ s+1\le j\le N}} + \sum_{\substack{s+1\le i\\ <j\le N}}\Big) \int_{\{|x_i-x_j|=\varepsilon\}_{X_N}\times\mathbb{R}^{dN}_{V_N}} \frac{\sqrt{2}}{2\varepsilon}(v_i-v_j)\cdot(x_i-x_j) \\ \times f_N(t,Z_N)\phi_s(t,Z_s)\,\mathrm{d}S. \end{aligned} \tag{2.15}$$

The first and the third sums in equation (2.15) cancel, again due to the definition of χ. To be more explicit, for the first sum for example, one has to separate the domain of integration between pre and post-collisional configurations, and after changing the variables in the post-collisional term using the definition of χ, the cancellation happens.
For the second sum of the right-hand side of (2.15), one uses that the particles are indistinguishable, that is one uses the condition (2.8) page 69. One obtains :

$$-(N-s)\sum_{1\le i\le s} \int_{\{|x_i-x_{s+1}|=\varepsilon\}_{X_N}\times\mathbb{R}^{dN}_{V_N}} \frac{\sqrt{2}}{2\varepsilon}(v_{s+1}-v_i)\cdot(x_{s+1}-x_i) f_N(t,Z_N)\phi_s(t,Z_s)\,\mathrm{d}S.$$

The last step consists in rewriting x_{s+1} as $x_i+\omega_\varepsilon$, with ω_ε belonging to the hypersphere $\mathbb{S}^{d-1}(0,\varepsilon)$ or radius ε, and expressing the measure $\mathrm{d}S$ explicitly, namely :

$$\mathrm{d}S = \sqrt{2}\,\mathrm{d}\widehat{X_N}^{s+1}\,\mathrm{d}\mathbb{S}^{d-1}_{x_{s+1}}(0,\varepsilon)\,\mathrm{d}V_N,$$

where $\mathrm{d}\widehat{X_N}^{s+1}$ is the Lebesgue measure on $\mathbb{R}^{d(N-1)}$ (space of X_N in which the coordinate x_{s+1} is eliminated).

Finally, gathering all the previous results, one finds that, for $1 \leq s \leq N-1$:

$$
\begin{aligned}
&\int_{\mathbb{R}^{2dN}} f_N(0,Z_N)\phi_s(0,Z_s)\mathbb{1}_{Z_N\in\mathcal{D}_N^\varepsilon}\,\mathrm{d}Z_N \\
&+\int_0^{+\infty}\int_{\mathbb{R}^{2dN}} f_N(t,Z_N)\partial_t\phi_s(t,Z_s)\mathbb{1}_{Z_N\in\mathcal{D}_N^\varepsilon}\,\mathrm{d}Z_N\,\mathrm{d}t \\
&+\sum_{i=1}^{s}\int_0^{+\infty}\int_{\mathbb{R}^{2dN}} f_N(t,Z_N)v_i\cdot\nabla_{x_i}\phi_s(t,Z_s)\mathbb{1}_{Z_N\in\mathcal{D}_N^\varepsilon}\,\mathrm{d}Z_N\,\mathrm{d}t \\
&+(N-s)\varepsilon^{d-1}\sum_{i=1}^{s}\int_0^{+\infty}\int_{\mathbb{R}^{2ds}}\int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}}\int_{\mathbb{R}^{d(N-s-1)}}\omega\cdot\big(v_{s+1}-v_i\big) \\
&\qquad\times f_N\big(t,Z_s,x_i+\varepsilon\omega,v_{s+1},x_{s+2},v_{s+2},\dots,x_N,v_N\big)\phi_s(t,Z_s) \\
&\qquad\qquad\times\mathbb{1}_{Z_N\in\mathcal{D}_N^\varepsilon}\big(Z_s,x_i+\varepsilon\omega,v_{s+1},x_{s+2},v_{s+2},\dots,x_N,v_N\big) \\
&\qquad\qquad\qquad\mathrm{d}x_{s+2}\,\mathrm{d}v_{s+2}\dots\mathrm{d}x_N\,\mathrm{d}v_N\,\mathrm{d}\omega\,\mathrm{d}v_{s+1}\,\mathrm{d}Z_s\,\mathrm{d}t \\
&=0,
\end{aligned}
$$

that is

$$
\begin{aligned}
&\int_{\mathbb{R}^{2ds}}\phi_s(0,Z_s)\Big(\int_{\mathbb{R}^{2d(N-s)}} f_N(0,Z_N)\mathbb{1}_{Z_N\in\mathcal{D}_N^\varepsilon}\,\mathrm{d}z_{s+1}\dots\mathrm{d}z_N\Big)\,\mathrm{d}Z_s \\
&+\int_0^{+\infty}\int_{\mathbb{R}^{2ds}}\partial_t\phi_s(t,Z_s)\Big(\int_{\mathbb{R}^{2d(N-s)}} f_N(t,Z_N)\mathbb{1}_{Z_N\in\mathcal{D}_N^\varepsilon}\,\mathrm{d}z_{s+1}\dots\mathrm{d}z_N\Big)\,\mathrm{d}Z_s\,\mathrm{d}t \\
&+\sum_{i=1}^{s}\int_0^{+\infty}\int_{\mathbb{R}^{2ds}} v_i\cdot\nabla_{x_i}\phi_s(t,Z_s) \\
&\qquad\qquad\times\Big(\int_{\mathbb{R}^{2d(N-s)}} f_N(t,Z_N)\mathbb{1}_{Z_N\in\mathcal{D}_N^\varepsilon}\,\mathrm{d}z_{s+1}\dots\mathrm{d}z_N\Big)\,\mathrm{d}Z_s\,\mathrm{d}t \\
&+(N-s)\varepsilon^{d-1}\sum_{i=1}^{s}\int_0^{+\infty}\int_{\mathbb{R}^{2ds}}\int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}}\omega\cdot\big(v_{s+1}-v_i\big)\phi_s(t,Z_s) \\
&\quad\times\Big(\int_{\mathbb{R}^{d(N-s-1)}} f_N\big(t,Z_s,x_i+\varepsilon\omega,v_{s+1},x_{s+2},v_{s+2},\dots,x_N,v_N\big) \\
&\qquad\qquad\times\mathbb{1}_{Z_N\in\mathcal{D}_N^\varepsilon}\big(Z_s,x_i+\varepsilon\omega,v_{s+1},x_{s+2},v_{s+2},\dots,x_N,v_N\big) \\
&\qquad\qquad\qquad\mathrm{d}x_{s+2}\,\mathrm{d}v_{s+2}\dots\mathrm{d}x_N\,\mathrm{d}v_N\Big) \\
&\qquad\qquad\qquad\qquad\mathrm{d}\omega\,\mathrm{d}v_{s+1}\,\mathrm{d}Z_s\,\mathrm{d}t \\
&=0.
\end{aligned}
$$

According to Definition 6 page 65, one can rewrite the last equation using the marginals of the solution f_N of the Liouville equation :

$$
\begin{aligned}
&\int_{\mathbb{R}^{2ds}} \phi_s(0,Z_s) f_N^{(s)}(0,Z_s)\,\mathrm{d}Z_s + \int_0^{+\infty}\int_{\mathbb{R}^{2ds}} \partial_t\phi_s(t,Z_s) f_N^{(s)}(t,Z_s)\,\mathrm{d}Z_s\,\mathrm{d}t \\
&\quad + \sum_{i=1}^{s}\int_0^{+\infty}\int_{\mathbb{R}^{2ds}} v_i\cdot\nabla_{x_i}\phi_s(t,Z_s) f_N^{(s)}(t,Z_s)\,\mathrm{d}Z_s\,\mathrm{d}t \\
&\quad + (N-s)\varepsilon^{d-1}\sum_{i=1}^{s}\int_0^{+\infty}\int_{\mathbb{R}^{2ds}}\int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}} \omega\cdot\big(v_{s+1}-v_i\big)\phi_s(t,Z_s) \\
&\qquad\qquad\qquad\qquad \times f_N^{(s+1)}(Z_s, x_i+\varepsilon\omega, v_{s+1})\,\mathrm{d}\omega\,\mathrm{d}v_{s+1}\,\mathrm{d}Z_s\,\mathrm{d}t \\
&= 0.
\end{aligned}
$$

This means that the following equation holds on $\mathbb{R}_+\times\mathcal{D}_s^\varepsilon$:

$$
\partial_t f_N^{(s)} + \sum_{i=1}^{s} v_i\cdot\nabla_{x_i} f_N^{(s)} = \mathcal{C}_{s,s+1}^{N,\varepsilon} f_N^{(s+1)},
$$

in the weak sense, that is, testing against smooth, compactly supported functions verifying the boundary condition (2.9), with additional boundary conditions properly written in the formal Definition 9 below, and where $\mathcal{C}_{s,s+1}^{N,\varepsilon}$ is the s-th collision operator, denoting

$$
\begin{aligned}
\mathcal{C}_{s,s+1}^{N,\varepsilon} f_N^{(s+1)}(t,Z_s) = \sum_{i=1}^{s}(N-s)\varepsilon^{d-1}\int_{\mathbb{S}^{d-1}\times\mathbb{R}^d} &\omega\cdot(v_{s+1}-v_i) \\
&\times f_N^{(s+1)}(t,Z_s,x_i+\varepsilon\omega,v_{s+1})\mathrm{d}\omega\mathrm{d}v_{s+1}.
\end{aligned}
$$

Nevertheless, the quantity written here is not entirely satisfactory, since the marginal $f_N^{(s+1)}$ is evaluated on incoming configurations of the phase space for some part of the domain of integration. Namely, for any integer $1\le i\le s$, for the i-th term of the sum, when $\omega\cdot(v_{s+1}-v_i)>0$, the configuration $\big(Z_s, x_i+\varepsilon\omega, v_{s+1}\big)$ is an incoming configuration. Therefore, one will use the boundary condition satisfied by the marginals so that by definition

$$
\begin{aligned}
&\int_{\mathbb{S}^{d-1}\times\mathbb{R}^d} \omega\cdot(v_{s+1}-v_i) f_N^{(s+1)}(t,Z_s,x_i+\varepsilon\omega,v_{s+1})\mathrm{d}\omega\mathrm{d}v_{s+1} \\
&\quad = \int_{\mathbb{S}^{d-1}\times\mathbb{R}^d} \big[\omega\cdot(v_{s+1}-v_i)\big]_+ f_N^{(s+1)}(t,Z_s,x_i+\varepsilon\omega,v_{s+1})\mathrm{d}\omega\mathrm{d}v_{s+1} \\
&\qquad - \int_{\mathbb{S}^{d-1}\times\mathbb{R}^d} \big[\omega\cdot(v_{s+1}-v_i)\big]_- f_N^{(s+1)}(t,Z_s,x_i+\varepsilon\omega,v_{s+1})\mathrm{d}\omega\mathrm{d}v_{s+1} \\
&\quad = \int_{\mathbb{S}^{d-1}\times\mathbb{R}^d} \big[\omega\cdot(v_{s+1}-v_i)\big]_+ \\
&\qquad\qquad\qquad \times f_N^{(s+1)}(t,x_1,v_1,\dots,x_i,v_i',\dots,x_i+\varepsilon\omega,v_{s+1}')\mathrm{d}\omega\mathrm{d}v_{s+1} \\
&\qquad - \int_{\mathbb{S}^{d-1}\times\mathbb{R}^d} \big[\omega\cdot(v_{s+1}-v_i)\big]_- f_N^{(s+1)}(t,Z_s,x_i+\varepsilon\omega,v_{s+1})\mathrm{d}\omega\mathrm{d}v_{s+1}.
\end{aligned}
$$

Definition 8 (s-th collision operator of the BBGKY hierarchy)**.** *Let N and s be two positive integers, and ε be a strictly positive real number. Then one defines the* s-th collision operator of the BBGKY hierarchy *as the operator associating to any function $f_N^{(s+1)}$ depending on $t \in \mathbb{R}$ and $Z_{s+1} \in \mathcal{D}_{s+1}^{\varepsilon}$, a function $\mathcal{C}_{s,s+1}^{N,\varepsilon} f_N^{(s+1)}$ depending on $t \in \mathbb{R}$ and on $Z_s \in \mathcal{D}_s^{\varepsilon}$ (one stresses the fact that the image of $f_N^{(s+1)}$ by this collision operator is defined on a time interval times the phase space of s particles instead of $s+1$ particles) such that for $s \leq N-1$*

$$\begin{aligned}
&\mathcal{C}_{s,s+1}^{N,\varepsilon} f_N^{(s+1)}(t, Z_s) \\
&\quad = \sum_{i=1}^{s} (N-s)\varepsilon^{d-1} \Bigg[\int_{\mathbb{S}^{d-1}\times\mathbb{R}^d} \big[\omega \cdot (v_{s+1} - v_i)\big]_+ \\
&\qquad\qquad \times f_N^{(s+1)}(t, x_1, v_1, \ldots, x_i, v_i', \ldots, x_i + \varepsilon\omega, v_{s+1}') \, d\omega \, dv_{s+1} \\
&\qquad\qquad - \int_{\mathbb{S}^{d-1}\times\mathbb{R}^d} \big[\omega \cdot (v_{s+1} - v_i)\big]_- \\
&\qquad\qquad\qquad \times f_N^{(s+1)}(t, Z_s, x_i + \varepsilon\omega, v_{s+1}) \, d\omega \, dv_{s+1} \Bigg],
\end{aligned} \tag{2.16}$$

where v_i' and v_{s+1}' are the images by the scattering mapping introduced in Definition 1 page 51 of the pair of velocities v_i and v_{s+1}, with the angular parameter $(x_i + \varepsilon\omega) - x_i = \omega$, and such that for $s \geq N$

$$\mathcal{C}_{s,s+1}^{N,\varepsilon} = 0.$$

Remark 4.

One notices that the equation verified by $f_N^{(s)}$ for $s = N$ is exactly the Liouville equation (2.1)*.*

The previous definition is only formal : here one has not detailed the functional space in which the marginals $f_N^{(s)}$ have to be chosen in order to obtain a meaningful expression for the collision operator of the BBGKY hierarchy. This is the purpose of Section 5 starting page 87 below.

2.2 Definition of the BBGKY hierarchy

One defines the BBGKY hierarchy, obtained at the formal level in the previous section, from the Liouville equation (2.1) with the boundary condition (2.7), and the additional assumption of indistinguishibility (2.8).

Definition 9 (BBGKY hierarchy)**.** *Let T be a positive number, $N \in \mathbb{N}^*$ and $\varepsilon > 0$. One defines the* BBGKY hierarchy *as the finite collection of the following*

equations, for $1 \leq s \leq N-1$ *:*

$$\partial_t f_N^{(s)} + \sum_{i=1}^{s} v_i \cdot \nabla_{x_i} f_N^{(s)} = \mathcal{C}_{s,s+1}^{N,\varepsilon} f_N^{(s+1)}, \tag{2.17}$$

holding in $[0,T] \times \mathcal{D}_s^\varepsilon$*, and for* $s = N$ *:*

$$\partial_t f_N^{(N)} + \sum_{i=1}^{N} v_i \cdot \nabla_{x_i} f_N^{(N)} = 0,$$

holding in $[0,T] \times \mathcal{D}_N^\varepsilon$*, with the additional boundary conditions :*

$$f_N^{(s)}(t, Z_s) = f_N^{(s)}(t, \chi_s^\varepsilon(Z_s)) \tag{2.18}$$

(using the notation χ_s^ε *introduced in Definition 7 page 67) for all* $1 \leq s \leq N$*,* $t \in [0,T]$ *and* $Z_s \in \left(\widehat{\partial \mathcal{D}_s^\varepsilon}\right)^{in}$ *(defined in* (2.6) *page 69).*

Chapter 3

The Boltzmann hierarchy

3.1 Formal limit of the BBGKY hierarchy

Noticing that the collision operator $\mathcal{C}_{s,s+1}^{N,\varepsilon}$ depends on N and ε, and that the quantity $(N-s)\varepsilon^{d-1}$ is naturally involved in the definition of this operator, one is led to letting ε tend to zero, keeping $N\varepsilon^{d-1}$ constant (say $N\varepsilon^{d-1}=1$). This is the Boltzmann-Grad asymptotics, introduced by Grad in [38], and discussed in the previous Section 1.1.7. The condition $N\varepsilon^{d-1}=1$, in addition to its physical meaning, is convenient in the limit $\varepsilon \to 0$ in order to obtain a formal limit of the BBGKY hierarchy which is not the free transport equation.
Considering the collision operator of the BBGKY hierarchy introduced in (2.16), one performs in its second term the change of variables $\omega \to -\omega$, so that, using the identity

$$\big[\omega \cdot (v_{s+1}-v_i)\big]_- = \big[(-\omega)\cdot(v_{s+1}-v_i)\big]_+,$$

one finds

$$\begin{aligned}&\int_{\mathbb{S}^{d-1}\times\mathbb{R}^d} \big[\omega\cdot(v_{s+1}-v_i)\big]_- f_N^{(s+1)}(t,Z_s,x_i+\varepsilon\omega,v_{s+1})\mathrm{d}\omega\mathrm{d}v_{s+1}\\ &= \int_{\mathbb{S}^{d-1}\times\mathbb{R}^d} \big[\omega\cdot(v_{s+1}-v_i)\big]_+ f_N^{(s+1)}(t,Z_s,x_i-\varepsilon\omega,v_{s+1})\mathrm{d}\omega\mathrm{d}v_{s+1}.\end{aligned}$$

Letting ε go to zero, the term $(N-s)\varepsilon^{d-1}$ tends to 1 according to the Boltzmann-Grad scaling, and the positions $x_i+\varepsilon\omega$ and $x_i-\varepsilon\omega$ are replaced by x_i. One obtains therefore a formal limit operator $\mathcal{C}_{s,s+1}^0$ from the expression of the collision operator of the BBGKY hierarchy, defined as

$$\begin{aligned}\mathcal{C}_{s,s+1}^0 f^{(s+1)}(t,Z_s) &= \sum_{i=1}^{s}\int_{\mathbb{S}^{d-1}\times\mathbb{R}^d}[\omega\cdot(v_{s+1}-v_i)]_+\\ &\times\Big(f^{(s+1)}(t,x_1,...,x_i,v_i',...,x_s,v_s,x_i,v_{s+1}')-f^{(s+1)}(t,Z_s,x_i,v_{s+1})\Big)\,\mathrm{d}\omega\mathrm{d}v_{s+1},\end{aligned}$$

where v_i' and v_{s+1}' are the images by the scattering mapping introduced in Definition 1 page 51 of the pair of velocities v_i and v_{s+1}, with the angular

parameter ω.
One emphasizes here that the use of the boundary condition (2.18) satisfied by the marginals, and the change of variables $\omega \to -\omega$ in the domain of the integral corresponding to the incoming configurations has to be performed *before* the formal limit $\varepsilon \to 0$. Indeed, if one starts by the limit $\varepsilon \to 0$, the formal limit of the collision operator of the BBGKY hierarchy is :

$$\int_{\mathbb{S}^{d-1}\times\mathbb{R}^d} \omega \cdot (v_{s+1} - v_i) f_N^{(s+1)}(t, Z_s, x_i, v_{s+1}) \mathrm{d}\omega \mathrm{d}v_{s+1},$$

so that it is not possible to use the boundary condition satisfied by $f_N^{(s+1)}$, and the change of variables $\omega \to -\omega$ in the domain $\omega \cdot (v_{s+1} - v_i) < 0$ provides that the formal limit of the collision operator is zero (and then the formal limit of the equations of the BBGKY hierarchy are the free transport equations) if the limit $\varepsilon \to 0$ is performed without caution.

3.2 Definition of the Boltzmann hierarchy

From the previous discussion, one defines the limiting collision operator obtained above. One emphasizes here on the fact that this definition is only formal.

Definition 10 (s-th collision operator of the Boltzmann hierarchy)**.** *Let s be a positive integer. Then one defines the* s-th collision operator of the Boltzmann hierarchy *as the operator associating to any function $f^{(s+1)}$ depending on $t \in \mathbb{R}$ and on $Z_{s+1} \in \left(\Omega^c \times \mathbb{R}^d\right)^{s+1}$, a function $\mathcal{C}^0_{s,s+1} f^{(s+1)}$ depending on $t \in \mathbb{R}$ and on $Z_s \in \left(\Omega^c \times \mathbb{R}^d\right)^s$ such that*

$$\begin{aligned}\mathcal{C}^0_{s,s+1} f^{(s+1)}(t, Z_s) = \sum_{i=1}^{s} \int_{\mathbb{S}^{d-1}\times\mathbb{R}^d} \big[\omega \cdot (v_{s+1} - v_i)\big]_+ \\ \times \Big(f^{(s+1)}(t, x_1, v_1, ..., x_i, v_i', ..., x_s, v_s, x_i, v_{s+1}') \\ - f^{(s+1)}(t, Z_s, x_i, v_{s+1}) \Big) d\omega\, dv_{s+1}\end{aligned} \tag{3.1}$$

where v_i' and v_{s+1}' are the images by the scattering mapping introduced in Definition 1 page 51 of the pair of velocities v_i and v_{s+1}, with the angular parameter ω.

One introduces also boundary conditions in the Boltzmann-Grad limit, very similar to the boundary conditions of the hard sphere dynamics (see Definition 7 page 67). In particular here, the radius of the particles of the system has been sent to zero, therefore only a set of measure zero can contain initial configurations leading to a collision between two particles. As a consequence, the boundary condition previously defined when $|x_i - x_j| = \varepsilon$ vanishes here. Besides, the particles can still bounce against the obstacle : this is the only remaining boundary condition in the limit. In other words, one has recovered here the free flow with boundary condition.

Definition 11 (Boundary conditions for the free flow dynamics)**.** *Let s be a positive integer. One defines the* boundary conditions for the free flow dynamics of s particles around the obstacle Ω *as the following function.*

$$\chi_s^0 : \left\{ \begin{array}{ll} \partial\big(\Omega^c \times \mathbb{R}^d\big)^s & \to \partial\big(\Omega^c \times \mathbb{R}^d\big)^s, \\ Z_s & \mapsto \chi_s^0(Z_s), \end{array} \right. \tag{3.2}$$

with

$$\chi_s^0(Z_s)^X = (Z_s)^X$$

and such that if there exists $1 \leq i \leq s$ such that $x_i \cdot e_1 = 0$, and $x_j \cdot e_1 > 0$ for all $j \neq i$:

$$\left\{ \begin{array}{l} \chi_s^0(Z_s)^{V,i} = v_i - 2v_i \cdot e_1 e_1, \\ \chi_s^0(Z_s)^{V,j} = v_j \text{ for all } 1 \leq j \leq s, j \neq i. \end{array} \right.$$

The hierarchy obtained as a formal limit when $\varepsilon \to 0$, $N\varepsilon^{d-1} = 1$ of the BBGKY hierarchy will be called the Boltzmann hierarchy.
One notes that a significant difference with the BBGKY hierarchy lies in the infinite number of equations involved in the Boltzmann hierarchy. Once again the following definition is only formal at this step of the work.

Definition 12 (Boltzmann hierarchy)**.** *Let T be a strictly positive number. One defines the* Boltzmann hierarchy *as the infinite collection of the following equations, for $s \geq 1$:*

$$\partial_t f^{(s)} + \sum_{i=1}^{s} v_i \cdot \nabla_{x_i} f^{(s)} = \mathcal{C}_{s,s+1}^0 f^{(s+1)} \tag{3.3}$$

holding in $[0,T] \times \big(\Omega^c \times \mathbb{R}^d\big)^s$, with the additional boundary conditions :

$$f^{(s)}(t, Z_s) = f^{(s)}(t, \chi_s^0(Z_s)) \tag{3.4}$$

(using the notation χ_s^0 introduced in Definition 11) for all $s \geq 1$, $t \in [0,T]$ and $Z_s \in \partial\big(\Omega^c \times \mathbb{R}^d\big)^s$).

If one assumes that $f^{(2)}$ is tensorized, that is if there exists $f^{(1)}$ such that

$$f^{(2)}(t, x_1, v_1, x_2, v_2) = f^{(1)}(t, x_1, v_1) f^{(1)}(t, x_2, v_2)$$

for all $t \in \mathbb{R}_+$, all $x_1, x_2 \in \{x \in \mathbb{R}^d \ / \ x \cdot e_1 > 0\}$ and all $v_1, v_2 \in \mathbb{R}^d$, and if one assumes moreover that $f^{(1)}$ and $f^{(2)}$ satisfy the first equation of the Boltzmann hierarchy, that is :

$$\partial_t f^{(1)}(t, x_1, v_1) + v_1 \cdot \nabla_{x_1} f^{(1)}(t, x_1, v_1) = \mathcal{C}_{1,2}^0 f^{(2)}(t, x_1, v_1),$$

one obtains in fact that

$$\begin{aligned} \mathcal{C}_{1,2}^0 f^{(2)}(t, x_1, v_1) = & \int_{\mathbb{S}^{d-1} \times \mathbb{R}^d} [\omega \cdot (v_2 - v_1)]_+ \\ & \times \Big(f^{(2)}(t, x_1, v_1', x_1, v_2') - f^{(2)}(t, x_1, v_1, x_1, v_2) \Big) \, \mathrm{d}\omega \mathrm{d}v_2 \\ = & \int_{\mathbb{S}^{d-1} \times \mathbb{R}^d} [\omega \cdot (v_2 - v_1)]_+ \Big(f_1'^{(1)} f_2'^{(1)} - f_1^{(1)} f_2^{(1)} \Big) \, \mathrm{d}\omega \mathrm{d}v_2 \end{aligned}$$

with the obvious notations :

$$\begin{cases} f_1'^{(1)} = f^{(1)}(t, x_1, v_1'), \\ f_2'^{(1)} = f^{(1)}(t, x_1, v_2'), \\ f_1^{(1)} = f^{(1)}(t, x_1, v_1), \\ f_2^{(1)} = f^{(1)}(t, x_1, v_2). \end{cases}$$

In other words, with all those assumptions, the function $f^{(1)}$ turns out to be a solution of the Boltzmann equation. So, formally, the first marginal of a distribution function of N hard spheres of radius $\varepsilon/2$ tends, in a certain sense, to a solution of the Boltzmann equation.
In fact, for any positive integer s, if one considers a solution f of the Boltzmann equation, and the tensorized function

$$f^{(s)}\big(z_1, \dots, z_s\big) = f(z_1) \dots f(z_s),$$

one knows ([25][1]) that the infinite sequence of functions $\big(f^{(s)}\big)_{s \geq 1}$ is a solution of the Boltzmann hierarchy (3.3).
The core of this work consists then in defining properly this interesting hierarchy, and showing its proximity with the BBGKY hierarchy in a sense that will be made more precise in the sequel.

[1] In particular, one can refer to the first section of Chapter IV : "Introduction : On the Boltzmann-Grad limit".

Chapter 4

The integrated forms of the hierarchies

One now presents a mild (or integrated) version of the BBGKY and Boltzmann hierarchies, which will be useful when one wants to show, first, that those hierarchies admit solutions, and the rigorous convergence of the solutions of the BBGKY hierarchy towards the solutions of the Boltzmann hierarchy.

4.1 Integrated form of the BBGKY hierarchy

One starts from a sequence $F_N = \left(f_N^{(s)}\right)_{s\geq 1}$ which is (formally) a solution of the BBGKY hierarchy, that is

$$\begin{aligned}
&\partial_t f_N^{(s)}(t, Z_s) + \sum_{i=1}^{s} v_i \cdot \nabla_{x_i} f_N^{(s)}(t, Z_s) \\
&\quad = (N-s)\varepsilon^{d-1} \sum_{i=1}^{s} \int_{\mathbb{S}^{d-1}\times\mathbb{R}^d} \omega \cdot (v_{s+1} - v_i) f_N^{(s+1)}(t, Z_s, x_i + \varepsilon\omega, v_{s+1}) \mathrm{d}\omega \mathrm{d}v_{s+1}
\end{aligned}$$

for all $Z_s \in \mathcal{D}_s^\varepsilon$ and $t \in [0, T]$ for some T strictly positive, and such that $f_N^{(s)}$ satisfies also the boundary condition (2.18).

As in [34][1], one then performs formal computations in order to get the mild formulation that will be needed in the sequel, and using the notation (1.10) introduced in Definition 4 page 53 for the hard sphere transport with boundary

[1]See in particular the end of Section 4.3 "Weak formulation of Liouville's equation", and the end of Section 4.4 "The Boltzmann hierarchy and the Boltzmann equation".

condition, one finds :

$$\begin{aligned} f_N^{(s)}(t,Z_s) &= f_N^{(s)}(0,T_{-t}^{s,\varepsilon}(Z_s)) \\ &\quad + (N-s)\varepsilon^{d-1}\sum_{i=1}^{s}\int_0^t\int_{\mathbb{S}^{d-1}\times\mathbb{R}^d}\omega\cdot(v_{s+1}-v_i) \\ &\qquad \times f_N^{(s+1)}(u,T_{u-t}^{s,\varepsilon}(Z_s),T_{u-t}^{s,\varepsilon}(Z_s)_i^X+\varepsilon\omega,v_{s+1})\mathrm{d}\omega\mathrm{d}v_{s+1}\mathrm{d}u. \end{aligned}$$

It is possible to rewrite again the last equation in a more synthetic form, which will be the integrated form of the BBGKY hierarchy. First, one needs the following definition.

Definition 13 (Hard sphere flow)**.** *Let s be a positive integer. One defines the* hard sphere flow *as the following mapping, associating to a function depending on $u\in\mathbb{R}$ (seen as a time variable) and $Z_s\in\mathcal{D}_s^\varepsilon$ (seen as a variable lying in the phase space of s particles of radius $\varepsilon/2$) another function depending on t,u ($t\in\mathbb{R}$) and Z_s, such that :*

$$\begin{cases} \mathbb{R}\times\mathbb{R}\times\mathcal{D}_s^\varepsilon & \to \mathcal{D}_s^\varepsilon, \\ \quad\big(t,u,Z_s\big) & \mapsto \mathcal{T}_t^{s,\varepsilon}f^{(s)}(u,Z_s), \end{cases} \tag{4.1}$$

with

$$\big(\mathcal{T}_t^{s,\varepsilon}f^{(s)}\big)(u,Z_s) = f^{(s)}(u,T_{-t}^{s,\varepsilon}(Z_s))$$

for all $t,u\in\mathbb{R}$ and $Z_s\in\mathcal{D}_s^\varepsilon$.

Remark 5. *From the hard sphere transport, which is defined as a mapping acting on configurations of a system of particles, one just defined a mapping acting on functions, possibly depending on time. If the function on which the hard sphere flow is acting does not depend on time, one sees however that this hard sphere flow defines a new function, with a time dependency.*

Then, the mild formulation of the BBGKY hierarchy can be properly defined in the following way.

Definition 14 (Integrated form of the BBGKY hierarchy)**.** *Let N be a positive integer and $\varepsilon>0$. For a sequence $F_N=(f_N^{(s)})_{s\geq 1}$ of nonnegative functions such that*

$$\forall s\geq 1,\ \ f_N^{(s)}\in\mathcal{C}([0,T],L^\infty(\mathcal{D}_s^\varepsilon)),$$

F_N is said to be satisfying the integrated form of the BBGKY hierarchy *if for all $s\geq 1$:*

$$f_N^{(s)}(t,Z_s) = \big(\mathcal{T}_t^{s,\varepsilon}f_N^{(s)}(0,\cdot)\big)(Z_s) + \int_0^t \mathcal{T}_{t-u}^{s,\varepsilon}\mathcal{C}_{s,s+1}^{N,\varepsilon}f_N^{(s+1)}(u,Z_s)\,du \tag{4.2}$$

in some sense that has to be clarified, with $\mathcal{C}_{s,s+1}^{N,\varepsilon}$ denoting the BBGKY collision operator defined in 8 page 75.

Remark 6. *One notices here the important role of the "initial data", that is the values of the function $f_N^{(s)}$ at $t=0$.*

4.2 Integrated form of the Boltzmann hierarchy

The same computation performed in the previous section 4.1 for the BBGKY hierarchy, starting from the differential form of the general equation (3.3) page 79 defining the Boltzmann hierarchy, provides also a mild (or integrated) form of the Boltzmann hierarchy.

Definition 15 (Free flow with boundary condition)**.** *Let s be a positive integer. One defines the* free flow with boundary condition *as the following mapping, associating to a function depending on $u \in \mathbb{R}$ (seen as a time variable) and $Z_s \in \left(\overline{\Omega^c} \times \mathbb{R}^d\right)^s$ (seen as a variable lying in the phase space of s particles of radius zero) another function depending on t, u ($t \in \mathbb{R}$) and Z_s, such that :*

$$\left\{ \begin{array}{rl} \mathbb{R} \times \mathbb{R} \times \left(\Omega^c \times \mathbb{R}^d\right)^s & \to \left(\Omega^c \times \mathbb{R}^d\right)^s, \\ \left(t, u, Z_s\right) & \mapsto \mathcal{T}_t^{s,0} f^{(s)}(u, Z_s), \end{array} \right. \tag{4.3}$$

with

$$\left(\mathcal{T}_t^{s,0} f^{(s)}\right)(u, Z_s) = f^{(s)}(u, T_{-t}^{s,0}(Z_s))$$

for all $t, u \in \mathbb{R}$ and $Z_s \in \left(\Omega^c \times \mathbb{R}^d\right)^s$.

Definition 16 (Integrated form of the Boltzmann hierarchy)**.** *Let T be a strictly positive number. For a sequence $F = (f^{(s)})_{s \geq 1}$ of nonnegative functions such that*

$$\forall s \geq 1, \ f^{(s)} \in \mathcal{F}\left([0, T] \times \left(\Omega^c \times \mathbb{R}^d\right)^s\right),$$

F is said to be satisfying the integrated form of the Boltzmann hierarchy *if for all $s \geq 1$:*

$$f_N^{(s)}(t, Z_s) = \mathcal{T}_t^{s,0}\left(f_N^{(s)}(0, \cdot)\right)(Z_s) + \int_0^t \mathcal{T}_{t-u}^{s,0} \mathcal{C}_{s,s+1}^0 f^{(s+1)}(u, Z_s) du, \tag{4.4}$$

in some sense that has to be clarified, with $\mathcal{C}_{s,s+1}^0$ denoting the Boltzmann collision operator defined in (3.1)*, Definition 10 page 78.*

Part II

Functional setting for the existence and comparison of the solutions of the hierarchies

Chapter 5

About the rigorous definition and basic properties of the collision operator

Previously, one has introduced the collision operators, respectively for particles of radius $\varepsilon/2$, and in the Boltzmann-Grad limit. Since those operators are defined using an integral over a submanifold of dimension $(d-1)+d$ (see Definitions 8 page 75 and 10 page 78 : one integrates over $\mathbb{S}^{d-1}\times\mathbb{R}^d$), if the function on which the collision operators is regular enough, no problem occurs.
One will see in the sequel that the functions on which the collision operator of the BBGKY hierarchy will act will not be continuous in general. One will consider instead L^p functions, and defining this operator on such a class of functions will be the challenging purpose tackled in Section 5.1, starting just below.

One notices that in the limit, that is considering the Boltzmann hierarchy (4.4), $x_{s+1} = x_i + \varepsilon\omega$ is replaced by x_i. Here arises then another singularity, since the sphere $x_i + \varepsilon\mathbb{S}^{d-1}$ is replaced by the point x_i : the submanifold $\left\{x_i + \varepsilon\omega \ / \ \omega \in \mathbb{S}^{d-1}\right\}$, of codimension 1 in $\mathbb{R}^d$, is replaced by $\{x_i\}$, which is of codimension d in $\mathbb{R}^d$. It will prevent to define the collision operator of the Boltzmann operator on L^p spaces with the method presented used in the case of the BBGKY hierarchy. Therefore, one will restrict the domain of the collision operator of the Boltzmann hierarchy to the continuous functions.
Besides, in the case of the Boltzmann hierarchy, one sees that even if the functions $f^{(s)}$ are continuous, the collision operator is in fact acting on functions composed with the free flow with boundary condition. This can be another source of discontinuity, and this will be discussed below, in Section 5.2.

5.1 Definition of the transport-collision operator for the BBGKY hierarchy

Let N be a given positive integer, ε be a strictly positive real number, and s be a positive integer which verifies $s \leq N-1$.
First, one will split the operator $\mathcal{C}^{N,\varepsilon}_{s,s+1}$ into simple elements. One writes :

$$\mathcal{C}^{N,\varepsilon}_{s,s+1} = (N-s)\sum_{i=1}^{s}\left(\mathcal{C}^{\varepsilon}_{s,s+1,+,i} - \mathcal{C}^{\varepsilon}_{s,s+1,-,i}\right)$$

with :

$$\mathcal{C}^{\varepsilon}_{s,s+1,\pm,i}\varphi^{(s+1)} = \varepsilon^{d-1}\int\limits_{\mathbb{S}^{d-1}_{\omega}\times\mathbb{R}^{d}_{v_{s+1}}} \left(\omega\cdot(v_{s+1}-v_i)\right)_{\pm}\varphi^{(s+1)}(Z_s, x_i+\varepsilon\omega, v_{s+1})\,\mathrm{d}\omega\,\mathrm{d}v_{s+1}.$$

In order to try to prove that the elementary operators $\mathcal{C}^{\varepsilon}_{s,s+1,\pm,i}$ are well defined on L^∞ (which is somehow an usual setting, see for example [26][1] for the homogeneous Boltzmann equation, and especially [25][2] for the BBGKY hierarchy), one will cut off large positions and large velocities in order to consider instead L^1 functions, and the Fubini-Tonelli theorem will enable to conclude. The main idea is to use the integration with respect to time to "add the missing dimension", so that the integral of $\varphi^{(s+1)}$, which will be a L^∞ function, will be well-defined.

More explicitly, let R_1 and R_2 be two strictly positive numbers, which will be chosen as very large.
If one integrates crudely $\mathcal{C}^{\varepsilon}_{s,s+1,\pm,i}$ on all the remaining coordinates of the phase space (that is if one integrates with respect to the Lebesgue measure $\mathrm{d}Z_s$), one will only obtain an integral over a submanifold of the phase space $\mathcal{D}^{\varepsilon}_{s+1}$ of codimension 1, so that a dimension is still missing, yet the trace of a L^∞ function is not well-defined on such a submanifold.
But the integrated form of the BBGKY hierarchy (4.2) (see page 82) obtained in the previous part involves, beside the collision integral, the hard sphere transport, so it is natural to consider those two objects together. Then, one will study the following quantity :

$$\int_0^T \int\limits_{\mathcal{D}^{\varepsilon}_{s}} \mathcal{C}^{\varepsilon}_{s,s+1,\pm,i}\mathcal{T}^{s+1,\varepsilon}_{t}\varphi^{(s+1)}\,\mathrm{d}Z_s\,\mathrm{d}t,$$

for $T>0$ given, which is the integrated with respect to time hard sphere flow (see Definition 13 page 82) composed with the collision operator for particles of diameter ε.

[1] See in particular Section 6.3.
[2] See in particular Chapter III "The Initial Value Problem for L^∞ Data: Thermodynamic Limit.

As already noticed, a cut-off in the domain of integration will play a crucial role in the sequel : first in order to enable to integrate L^∞ functions on compact subsets of the phase space, and second to simplify the expression of the hard sphere transport. Considering therefore initial configurations inside a bounded subset of the phase space, the time interval $[0,T]$ can be divided into smaller intervals on which the flow of $s+1$ hard spheres of diameter ε coincides with the free flow, that is on those intervals, the particles move in straight lines, with a constant velocity.
So let δ belong to $]0,1[$. If the kinetic energy of a system of $s+1$ particles is bounded, or, equivalently, the norm of the vector composed of all the velocities of the particles of the system, is bounded, say, by R_2, it is easy to exclude the parts of the domain of integration which contain initial configurations which can lead to a change in the velocities before δ. Following [34][3], one is therefore naturally led to consider *truncated operators* as follows. When there is no obstacle, which is the setting considered in [34], the authors introduced[4] the truncated phase space

$$\begin{aligned}\partial\mathcal{D}_\delta^{i,s+1,\pm} = \Big\{ Z_{s+1} \in B_{\mathbb{R}^{2d(s+1)}}(0,R) \ / \ |x_i - x_{+1}| = \varepsilon, \\ \pm(v_i - v_{s+1})\cdot(x_i - x_{s+1}) > 0, \\ \text{and } \forall\ (k,l) \neq (i,s+1),\ |x_k - x_l| > \varepsilon + \delta R \Big\}.\end{aligned}$$

In the case of the particles moving outside of an obstacle in the Euclidean space, one denotes by :

$$\mathfrak{D} = \mathfrak{D}(\varepsilon, R_1, R_2, \delta) \subset [0,T] \times \mathbb{S}^{d-1} \times \mathbb{R}^d \times \mathcal{D}_s^\varepsilon$$

the domain of the truncated collision operator, defined as

$$\begin{aligned}\mathfrak{D}(\varepsilon, R_1, R_2, \delta) = \Big(\prod_{\substack{1\le j<k\le s+1\\(j,k)\neq(i,s+1)}} \mathbb{1}_{|x_j - x_k| > \varepsilon + \sqrt{2}\delta R_2} \Big) \Big(\prod_{1\le l\le s+1} \mathbb{1}_{d(x_l,\Omega) > \varepsilon/2 + \delta R_2} \Big) \\ \times \mathbb{1}_{X_{s+1}\in B_{\mathbb{R}^{d(s+1)}}(0,R_1)} \mathbb{1}_{V_{s+1}\in B_{\mathbb{R}^{d(s+1)}}(0,R_2)},\end{aligned} \tag{5.1}$$

and one sets :

$$\begin{aligned}&\mathcal{C}^\varepsilon_{s,s+1,\pm,i}(R_1,R_2,\delta)\varphi^{(s+1)}(Z_s) \\ &\quad= \int_{\mathbb{S}^{d-1}_\omega \times \mathbb{R}^d_{v_{s+1}}} \varepsilon^{d-1} \left(\omega\cdot(v_{s+1} - v_i)\right)_\pm \varphi^{(s+1)}(Z_s, x_i + \varepsilon\omega, v_{s+1}) \\ &\qquad\times \Big(\prod_{\substack{1\le j<k\le s+1\\(j,k)\neq(i,s+1)}} \mathbb{1}_{|x_j - x_k| > \varepsilon + \sqrt{2}\delta R_2} \Big) \Big(\prod_{1\le l\le s+1} \mathbb{1}_{d(x_l,\Omega) > \varepsilon/2 + \delta R_2} \Big) \\ &\qquad\times\ \mathbb{1}_{X_{s+1}\in B_{\mathbb{R}^{d(s+1)}}(0,R_1)} \mathbb{1}_{V_{s+1}\in B_{\mathbb{R}^{d(s+1)}}(0,R_2)}\, \mathrm{d}\omega\, \mathrm{d}v_{s+1}.\end{aligned}$$

[3]See Section 5.1.1 : A local system of coordinates near the boundary.
[4]At the very end of Section 5.1.1.

One will discuss quickly this last expression.
If the particles are separated by a distance larger than $\varepsilon + \sqrt{2}\delta R_2$, it is clear that they will not be able to collide with each other during the time interval $[0, \delta]$. Indeed one has, for any pair of particles $1 \leq i < j \leq s+1$:

$$\begin{aligned}
&\big|\big(T_{-t}^{s+1,\varepsilon}(Z_{s+1})\big)^{X,i} - \big(T_{-t}^{s+1,\varepsilon}(Z_{s+1})\big)^{X,j}\big| \\
&\qquad \geq \big|\big(T_{0}^{s+1,\varepsilon}(Z_{s+1})\big)^{X,i} - \big(T_{0}^{s+1,\varepsilon}(Z_{s+1})\big)^{X,j}\big| \\
&\qquad\quad - \big|\big(T_{-t}^{s+1,\varepsilon}(Z_{s+1})\big)^{X,i} - \big(T_{0}^{s+1,\varepsilon}(Z_{s+1})\big)^{X,i}\big| \\
&\qquad\quad - \big|\big(T_{-t}^{s+1,\varepsilon}(Z_{s+1})\big)^{X,j} - \big(T_{0}^{s+1,\varepsilon}(Z_{s+1})\big)^{X,j}\big|, \qquad (5.2)
\end{aligned}$$

following the notations introduced in Definition 4 page 53, the first term of the right-hand side being of course :

$$\big(T_{0}^{s+1,\varepsilon}(Z_{s+1})\big)^{X,i} - \big(T_{0}^{s+1,\varepsilon}(Z_{s+1})\big)^{X,j} = x_i - x_j,$$

the two other ones being bounded from above by the largest distance crossed by a particle during a time smaller than δ, that is :

$$\big|\big(T_{-t}^{s+1,\varepsilon}(Z_{s+1})\big)^{X,i} - \big(T_{0}^{s+1,\varepsilon}(Z_{s+1})\big)^{X,i}\big| \leq \delta \max_{u\in[0,\delta]} \big|\big(T_{-u}^{s+1,\varepsilon}(Z_{s+1})\big)^{V,i}\big|,$$

and similarly :

$$\big|\big(T_{-t}^{s+1,\varepsilon}(Z_{s+1})\big)^{X,j} - \big(T_{0}^{s+1,\varepsilon}(Z_{s+1})\big)^{X,j}\big| \leq \delta \max_{u\in[0,\delta]} \big|\big(T_{-u}^{s+1,\varepsilon}(Z_{s+1})\big)^{V,j}\big|.$$

Remembering that one has, for all $u \in \mathbb{R}$:

$$\sum_{k=1}^{s+1} \big|\big(T_{-u}^{s+1,\varepsilon}(Z_{s+1})\big)^{V,k}\big|^2 = \sum_{k=1}^{s+1} \big|V_{s+1}\big|^2 \leq R_2^2,$$

by conservation of the kinetic energy along the trajectories of the hard sphere flow, one has :

$$\begin{aligned}
&\Big(\big|\big(T_{-t}^{s+1,\varepsilon}(Z_{s+1})\big)^{X,i} - \big(T_{0}^{s+1,\varepsilon}(Z_{s+1})\big)^{X,i}\big| \\
&\qquad + \big|\big(T_{-t}^{s+1,\varepsilon}(Z_{s+1})\big)^{X,j} - \big(T_{0}^{s+1,\varepsilon}(Z_{s+1})\big)^{X,j}\big|\Big)^2 \\
&\qquad = \big|\big(T_{-t}^{s+1,\varepsilon}(Z_{s+1})\big)^{X,i} - \big(T_{0}^{s+1,\varepsilon}(Z_{s+1})\big)^{X,i}\big|^2 \\
&\qquad\quad + 2\big|\big(T_{-t}^{s+1,\varepsilon}(Z_{s+1})\big)^{X,i} - \big(T_{0}^{s+1,\varepsilon}(Z_{s+1})\big)^{X,i}\big| \\
&\qquad\qquad \cdot \big|\big(T_{-t}^{s+1,\varepsilon}(Z_{s+1})\big)^{X,j} - \big(T_{0}^{s+1,\varepsilon}(Z_{s+1})\big)^{X,j}\big| \\
&\qquad\quad + \big|\big(T_{-t}^{s+1,\varepsilon}(Z_{s+1})\big)^{X,j} - \big(T_{0}^{s+1,\varepsilon}(Z_{s+1})\big)^{X,j}\big|^2 \\
&\qquad \leq 2\big|\big(T_{-t}^{s+1,\varepsilon}(Z_{s+1})\big)^{X,i} - \big(T_{0}^{s+1,\varepsilon}(Z_{s+1})\big)^{X,i}\big|^2 \\
&\qquad\quad + 2\big|\big(T_{-t}^{s+1,\varepsilon}(Z_{s+1})\big)^{X,j} - \big(T_{0}^{s+1,\varepsilon}(Z_{s+1})\big)^{X,j}\big|^2 \\
&\qquad \leq 2\delta^2 R_2^2,
\end{aligned}$$

so that the following quantity is bounded from below for all $0 \leq \delta \leq t$:

$$\left|\left(T_{-t}^{s+1,\varepsilon}(Z_{s+1})\right)^{X,i} - \left(T_{-t}^{s+1,\varepsilon}(Z_{s+1})\right)^{X,j}\right| > \varepsilon,$$

which implies of course that no collision may occur between the particles i and j.

Remark 7. *The cut-off in distance may look like complicated since one can immediately bound from above the distance crossed by the two particles as follows :*

$$\begin{aligned}\left|\left(T_{-t}^{s+1,\varepsilon}(Z_{s+1})\right)^{X,i} - \left(T_{0}^{s+1,\varepsilon}(Z_{s+1})\right)^{X,i}\right| \\ + \left|\left(T_{-t}^{s+1,\varepsilon}(Z_{s+1})\right)^{X,j} - \left(T_{0}^{s+1,\varepsilon}(Z_{s+1})\right)^{X,j}\right| \leq 2R_2.\end{aligned}$$

However, replacing the constant 2 *by* $\sqrt{2}$ *is optimal.*

One has also to take into account the possibility of a bouncing against the obstacle Ω. To do so, by imposing that $d(x_l, \Omega) > \varepsilon/2 + \delta R_2$, it is clear that no particle can bounce against the obstacle during the same time interval $[0, \delta]$.

Remark 8. *There is here a slight abuse in the notations used to define the cut-off in the expression of the truncated collision operator defined just above. Indeed, one used the notation :*

$$|x_j - x_k| > \varepsilon + \delta R_2$$

for all couples (j, k) *except* $(j, k) = (i, s+1)$*. However,* x_{s+1} *does not belong to the integration variables, and* x_{s+1} *has to be understood in the sense of :*

$$x_{s+1} = x_i + \varepsilon\omega.$$

Finally, one will study the quantity :

$$\int_0^\delta \int_{Z_s \in \mathcal{D}_s^\varepsilon} \mathcal{C}_{s,s+1,\pm,i}^\varepsilon(R_1, R_2, \delta) \mathcal{T}_t^{s+1,\varepsilon} \varphi^{(s+1)} \, \mathrm{d}Z_s \, \mathrm{d}t. \tag{5.3}$$

5.1.1 Definition of the truncated collision operator on L^∞ functions

On the restricted domain of the collision operator $\mathcal{C}_{s,s+1,\pm,i}^\varepsilon$, and for a small time interval, say $[0, \delta]$, the hard sphere transport coincides with the free flow composed with, depending on whether the configuration $\left(Z_s, x_i + \varepsilon\omega, v_{s+1}\right)$ (which belongs of course to the boundary $\partial\mathcal{D}_{s+1}^\varepsilon$ of the phase space, since the distance between the particles i and $s+1$ is by definition exactly ε) is pre-collisional (that is if $\omega \cdot (v_{s+1} - v_i) < 0$) or post-collisional (that is if $\omega \cdot (v_{s+1} - v_i) > 0$), the scattering operator. In both cases, this free transport can be seen as a change of variables on the product of the phase space of s particles $\mathcal{D}_s^\varepsilon$ with $\mathbb{S}^{d-1} \times \mathbb{R}^d$ and with the time interval $[0, \delta]$, which is in fact the domain on which the transport-collision operator will be integrated.

The interesting fact is, on the one hand, that it is possible to send, by this transport, which is explicitly (in the simplest case without scattering) :

$$(Z_s, t, \omega, v_{s+1}) \mapsto (X_s - tV_s, V_s, x_i + \varepsilon\omega - tv_{s+1}, v_{s+1}),$$

the complicated domain :

$$[0, \delta]_t \times \left(\mathcal{D}_s^\varepsilon\right)_{Z_s} \times \mathbb{S}_\omega^{d-1} \times \mathbb{R}_{v_{s+1}}^d$$

into the simpler one :

$$\mathcal{D}_{s+1}^\varepsilon.$$

On the other hand, and following the previous idea, one sees that it will be crucial to know the Jacobian determinant of this transport. One will see that it will be some natural quantity, which appears in the Boltzmann equation, and the BBGKY and Boltzmann hierarchies.
In the two following propositions one will study this Jacobian determinant, in the pre and post-collisional cases.

Proposition 3. *Consider the following function :*

$$S_s^- : \begin{cases} \mathcal{D}_s^\varepsilon \times [0, \delta] \times \mathbb{S}^{d-1} \times \mathbb{R}^d & \to \mathbb{R}^{2d(s+1)}, \\ \left(Z_s, t, \omega, v_{s+1}\right) & \mapsto \widetilde{Z}_{s+1} \\ & \quad = \left(X_s - tV_s, V_s, x_i + \varepsilon\omega - tv_{s+1}, v_{s+1}\right). \end{cases}$$

Then the function S_s^- is one-to-one on the subset of $\mathcal{D}_s^\varepsilon \times [0, \delta] \times \mathbb{S}^{d-1} \times \mathbb{R}^d$ composed of the pre-collisional configurations (that is such that $\omega \cdot (v_{s+1} - v_i) < 0$), and transforms the measure $\varepsilon^{d-1}\left|\omega \cdot (v_{s+1} - v_i)\right| \mathrm{d}Z_s \, \mathrm{d}t \, \mathrm{d}\omega \, \mathrm{d}v_{s+1}$ in the Lebesgue measure on $\mathbb{R}^{2d(s+1)}$. In other words, the absolute value of the Jacobian determinant of S_s^- is equal to :

$$\varepsilon^{d-1}\left|\omega \cdot (v_{s+1} - v_i)\right|.$$

As an illustration of the general result stated in the previous proposition, one will compute directly the Jacobian determinant of the function S_s^- in the case when the dimension is equal to 2, then 3, which are of course the most physically meaningful cases. The proof of Proposition 3 will be provided afterwards.

In the case when $d = 2$, $\mathbb{S}^{d-1}$ is just the circle of center 0 and of radius 1. Then it is of course parametrized by :

$$\omega = (\cos\theta, \sin\theta),$$

with $\theta \in [0, 2\pi]$. The Jacobian matrix of S_s^- writes therefore :

$$Jac(S_s^-) = \left(\begin{array}{c|c|c|c|c}
I_{2s} & -tI_{2s} & -V_s & 0 & 0 \\
\hline
0 & I_{2s} & 0 & 0 & 0 \\
\hline
0 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} 0 & 0 & -v_{s+1} & \begin{matrix} -\varepsilon \sin\theta \\ \varepsilon \cos\theta \end{matrix} & \begin{matrix} -t & 0 \\ 0 & -t \end{matrix} \\
\hline
0 & 0 & 0 & 0 & \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}
\end{array}\right). \tag{5.4}$$

Denoting $v_i = (v_{i,1}, v_{i,2})$ and $v_{s+1} = (v_{s+1,1}, v_{s+1,2})$, and simplifying the determinant of the matrix (5.4) (see the proof below for this step, detailed in the general case), one finds

$$J(S_s^-) = \begin{vmatrix} 1 & 0 & -v_{i,1} & 0 \\ 0 & 1 & -v_{i,2} & 0 \\ 1 & 0 & -v_{s+1,1} & -\varepsilon \sin\theta \\ 0 & 1 & -v_{s+1,2} & \varepsilon \cos\theta \end{vmatrix}$$

that is, substracting the first line to the third one, and the second line to the fourth one :

$$J(S_s^-) = \begin{vmatrix} 1 & 0 & -v_{i,1} & 0 \\ 0 & 1 & -v_{i,2} & 0 \\ 0 & 0 & v_{i,1} - v_{s+1,1} & -\varepsilon \sin\theta \\ 0 & 0 & v_{i,2} - v_{s+1,2} & \varepsilon \cos\theta \end{vmatrix}$$

and then developping twice the determinant with respect to the first column :

$$\begin{aligned} J(S_s^-) &= \begin{vmatrix} v_{i,1} - v_{s+1,1} & -\varepsilon \sin\theta \\ v_{i,2} - v_{s+1,2} & \varepsilon \cos\theta \end{vmatrix} \\ &= \varepsilon\big(\cos\theta(v_{i,1} - v_{s+1,1}) + \sin\theta(v_{i,2} - v_{s+1,2})\big) = \varepsilon\omega \cdot (v_i - v_{s+1}), \end{aligned}$$

which is the result stated in the proposition when $d = 2$, since one recalls that the natural measure on the circle $\mathbb{S}^1$ is just $\mathrm{d}\theta$ in the case of the chosen system of coordinates.

Now in the case $d = 3$, $\mathbb{S}^{d-1}$ is the usual three-dimensional sphere, and one can for example parametrize it by :

$$\omega = (\cos\theta\cos\phi, \cos\theta\sin\phi, \sin\theta),$$

with $\theta \in [-\pi/2, \pi/2]$ and $\phi \in [0, 2\pi]$. Denoting $v_i = (v_{i,1}, v_{i,2}, v_{i,3})$ and $v_{s+1} = (v_{s+1,1}, v_{s+1,2}, v_{s+1,3})$, and writing directly the Jacobian determinant of S_s^- after the early simplifications performed in the general proof, one finds :

$$J(S_s^-) = \begin{vmatrix} 1 & 0 & 0 & -v_{i,1} & 0 & 0 \\ 0 & 1 & 0 & -v_{i,2} & 0 & 0 \\ 0 & 0 & 1 & -v_{i,3} & 0 & 0 \\ 0 & 0 & 0 & v_{i,1} - v_{s+1,1} & -\varepsilon\sin\theta\cos\phi & -\varepsilon\cos\theta\sin\phi \\ 0 & 0 & 0 & v_{i,2} - v_{s+1,2} & -\varepsilon\sin\theta\sin\phi & \varepsilon\cos\theta\cos\phi \\ 0 & 0 & 0 & v_{i,3} - v_{s+1,3} & \cos\theta & 0, \end{vmatrix}$$

so that three developments of the determinant with respect to the first column, and a rough use of the well-known explicit formula for a 3×3 determinant provide

$$\begin{aligned} J(S_s^-) &= \begin{vmatrix} v_{i,1} - v_{s+1,1} & -\varepsilon\sin\theta\cos\phi & -\varepsilon\cos\theta\sin\phi \\ v_{i,2} - v_{s+1,2} & -\varepsilon\sin\theta\sin\phi & \varepsilon\cos\theta\cos\phi \\ v_{i,3} - v_{s+1,3} & \cos\theta & 0 \end{vmatrix} \\ &= \varepsilon^2\big(-(v_{i,3} - v_{s+1,3})\cos\theta\sin\theta\cos^2\phi - (v_{i,2} - v_{s+1,2})\cos^2\theta\sin\phi \\ &\qquad - (v_{i,3} - v_{s+1,3})\cos\theta\sin\theta\sin^2\phi - (v_{i,1} - v_{s+1,1})\cos^2\theta\cos\phi\big) \\ &= -\varepsilon^2\cos\theta\big((v_{i,1} - v_{s+1,1})\cos\theta\cos\phi + (v_{i,2} - v_{s+1,2})\cos\theta\sin\phi \\ &\qquad + (v_{i,3} - v_{s+1,3})\sin\theta(\cos^2\phi + \sin^2\phi)\big) \\ &= -\varepsilon^2\big(\omega\cdot(v_i - v_{s+1})\big)\cos\theta. \end{aligned}$$

Once again, this is the result stated in the proposition, since the measure on the sphere $\mathbb{S}^2$ is, in the case of the chosen parametrization, $\cos\theta\,\mathrm{d}\theta\,\mathrm{d}\phi$.

One presents now the proof of Proposition 3, in the general case (that is, for any dimension $d \geq 2$).

Proof of Proposition 3. One starts to show that the function S_s^- is one-to-one. For a configuration $\widetilde{Z}_{s+1} = \big(\widetilde{x}_1, \widetilde{v}_1, \ldots, \widetilde{x}_i, \widetilde{v}_i, \ldots, \widetilde{x}_{s+1}, \widetilde{v}_{s+1}\big)$ belonging to the image by S_s^- of the subset composed of the pre-collisional configurations, there exists at least one time t_1 such that

$$\big|(\widetilde{x}_i + t_1\widetilde{v}_i) - (\widetilde{x}_{s+1} + t_1\widetilde{v}_{s+1})\big| = \varepsilon \tag{5.5}$$

and such that

$$\big((\widetilde{x}_{s+1} + t_1\widetilde{v}_{s+1}) - (\widetilde{x}_i + t_1\widetilde{v}_i)\big)\cdot(\widetilde{v}_{s+1} - \widetilde{v}_i) < 0 \tag{5.6}$$

by definition of the image by S_s^- of the pre-collisional configurations. Such a time, fulfilling the two conditions (5.5) and (5.6), is actually unique, since the function $t \mapsto \left|(\widetilde{x}_i + t_k\widetilde{v}_i) - (\widetilde{x}_{s+1} + t_k\widetilde{v}_{s+1})\right|^2$ is a quadratic, positive and non constant (since one knows that its derivative is non zero at t_1 thanks to (5.6)) function of t, so that there exist at most two times t_k (with $k = 1$ or 2) such that

$$\left|(\widetilde{x}_i + t_k\widetilde{v}_i) - (\widetilde{x}_{s+1} + t_k\widetilde{v}_{s+1})\right| = \varepsilon,$$

and one has for the other time t_2 :

$$\big((\widetilde{x}_{s+1} + t_2\widetilde{v}_{s+1}) - (\widetilde{x}_i + t_2\widetilde{v}_i)\big) \cdot (\widetilde{v}_{s+1} - \widetilde{v}_i) > 0$$

(since the minimum of the function $t \mapsto \left|(\widetilde{x}_i + t_k\widetilde{v}_i) - (\widetilde{x}_{s+1} + t_k\widetilde{v}_{s+1})\right|^2$ lies between t_1 and t_2).
One sees therefore that S_s^- is one-to-one from the pre-collisional configurations, taking $\big(Z_s, t, \omega, v_{s+1}\big)$ as the inverse of the configuration $\widetilde{Z}_{s+1}$ with

$$\forall\ 1 \le j \le s+1,\ v_j = \widetilde{v}_j,$$

$$t = t_1,$$

$$\omega = (\widetilde{x}_{s+1} + t_1\widetilde{v}_{s+1}) - (\widetilde{x}_i + t_1\widetilde{v}_i),$$

$$\forall\ 1 \le j \le s, s \ne i,\ x_j = \widetilde{x}_j + t_1\widetilde{v}_j = \widetilde{x}_j + tv_j$$

and finally

$$x_i = \widetilde{x}_i + t_1\widetilde{v}_i - \varepsilon\big((\widetilde{x}_{s+1} + t_1\widetilde{v}_{s+1}) - (\widetilde{x}_i + t_1\widetilde{v}_i)\big) = \widetilde{x}_i + tv_i - \varepsilon\omega.$$

Now, one studies the Jacobian determinant of S_s^-. Using spherical coordinates, that is using the following parametrization of the sphere $\mathbb{S}^{d-1}$:

$$\begin{cases} \omega_1 = \sin\theta_1 \sin\theta_2 \ldots \sin\theta_{d-1}, \\ \omega_2 = \sin\theta_1 \sin\theta_2 \ldots \cos\theta_{d-1}, \\ \ldots \\ \omega_{d-2} = \sin\theta_1 \sin\theta_2 \cos\theta_3, \\ \omega_{d-1} = \sin\theta_1 \cos\theta_2, \\ \omega_d = \cos\theta_1, \end{cases} \tag{5.7}$$

where

$$\omega = (\omega_1, \ldots, \omega_d)$$

denotes a general element of $\mathbb{S}^{d-1}$, one computes the Jacobian matrix $Jac(S_s^-)$ of the mapping S_s^- using the coordinates $(Z_s, t, \theta_1, \ldots, \theta_{d-1}, v_{s+1}) \in \mathcal{D}_s^\varepsilon \times [0, \delta] \times$

$[0, 2\pi] \times [0, \pi]^{d-2} \times \mathbb{R}^d$. One finds :

$$Jac(S_s^-) = \left(\begin{array}{ccccc} I_{ds} & -tI_{ds} & -V_s & 0 & 0 \\ 0 & I_{ds} & 0 & 0 & 0 \\ 0 \quad I_d \quad 0 & 0 & -v_{s+1} & \varepsilon Jac(\omega) & -tI_d \\ 0 & 0 & 0 & 0 & I_d \end{array}\right), \quad (5.8)$$

where $Jac(\omega)$ denotes the Jacobian matrix of the previous parametrization (5.7) of the sphere, for any $\omega = (\omega_1, \omega_2, \ldots, \omega_d) \in \mathbb{S}^{d-1}$, that is :

$$\begin{pmatrix} \cos\theta_1 \sin\theta_2 \ldots \sin\theta_{d-1} & \sin\theta_1 \cos\theta_2 \ldots \sin\theta_{d-1} & \ldots & \sin\theta_1 \sin\theta_2 \ldots \cos\theta_{d-1} \\ \cos\theta_1 \sin\theta_2 \ldots \cos\theta_{d-1} & \sin\theta_1 \cos\theta_2 \ldots \cos\theta_{d-1} & \ldots & -\sin\theta_1 \sin\theta_2 \ldots \sin\theta_{d-1} \\ \cos\theta_1 \sin\theta_2 \ldots \cos\theta_{d-2} & \sin\theta_1 \cos\theta_2 \ldots \cos\theta_{d-2} & \ldots & 0 \\ \vdots & \vdots & \iddots & 0 \\ \cos\theta_1 \cos\theta_2 & -\sin\theta_1 \sin\theta_2 & \ldots & 0 \\ -\sin\theta_1 & 0 & \ldots & 0 \end{pmatrix}.$$

Computing the Jacobian determinant $J(S_s^-)$ of the Jacobian matrix $Jac(S_s^-)$, one finds, after developing with respect to the lines $ds + 1$ until $2ds$ the determinant of the matrix (5.8), and then with respect to the last d lines :

$$J(S_s^-) = \left|\begin{array}{ccccc} I_{ds} & -V_s & 0 \\ 0 \quad I_d \quad 0 & -v_{s+1} & \varepsilon Jac(\omega) \end{array}\right|, \quad (5.9)$$

then one develops the determinant (5.9) with respect to the ds first columns,

except the ones which contain more than one entry different from 0, and finds :

$$J(S_s^-) = \left\| \begin{array}{c|c|c} I_d & -v_i & 0 \\ \hline I_d & -v_{s+1} & \varepsilon Jac(\omega) \end{array} \right\|, \tag{5.10}$$

and since the two matrices :

$$I_d \text{ and } \left(\begin{array}{c|c} -v_{s+1} & \varepsilon Jac(\omega) \end{array} \right)$$

commute, one has :

$$J(S_s^-) = \left| I_d \cdot \left(\begin{array}{c|c} -v_{s+1} & \varepsilon Jac(\omega) \end{array} \right) - \left(\begin{array}{c|c} -v_i & 0 \end{array} \right) \cdot I_d \right|$$

(or even simpler : after substracting the first d lines of (5.10), respectively, to the last d ones, and developing the determinant), and then

$$J(S_s^-) = \left\| \begin{array}{c|c} v_i - v_{s+1} & \varepsilon Jac(\omega) \end{array} \right\|.$$

Then, if one denotes by $\bigwedge J_\omega$ the generalized cross product of the columns of the Jacobian matrix $Jac(\omega)$, which is defined by duality as :

$$\forall v \in \mathbb{R}^d,\ \det\big(Jac(\omega), v\big) = \bigwedge J_\omega \cdot v,$$

one sees that :

$$\begin{aligned} J(S_s^-) = \varepsilon^{d-1} \det(v_i - v_{s+1}, Jac(\omega)) &= (-\varepsilon)^{d-1} \det\big(Jac(\omega), v_i - v_{s+1}\big) \\ &= (-\varepsilon)^{d-1} \bigwedge J_\omega \cdot (v_i - v_{s+1}). \end{aligned}$$

Finally, on the one hand the vector $\bigwedge J_\omega$ is built as a vector which is orthogonal to any tangent vector to the sphere at the point ω, so that in this case :

$$\bigwedge J_\omega \ // \ \omega,$$

and then immediately :

$$\bigwedge J_\omega = \pm \Big| \bigwedge J_\omega \Big| \omega.$$

On the other hand, the magnitude of the vector $\bigwedge J_\omega$ is the volume of the parallelotope defined by the columns of the Jacobian matrix $Jac(\omega)$ of the parametrization of the sphere. That is why it is used in order to define the measure of the surface of the sphere. In other words, the measure $\mathrm{d}\omega$ means exactly that :

$$\Big| \bigwedge J_\omega \Big| \, \mathrm{d}\theta_1 \dots \theta_{d-1}$$

in coordinates.
In the end, one obtains : $|J(S_s^-)| = \varepsilon^{d-1} \big| \omega \cdot (v_{s+1} - v_i) \big|$. □

One can guess that this proposition will be used only in half of the phase space, more precisely the subset of the phase space that corresponds to pre-collisional configurations, that is those for which $\omega \cdot (v_{s+1} - v_i) < 0$. In this case, at least for a small time, the backwards hard sphere flow coincides with the free flow.
In the other half of the phase space, corresponding to post-collisional configurations, that is those for which $\omega \cdot (v_{s+1} - v_i) > 0$, if $|x_i - x_{s+1}| = \varepsilon$, the scattering (see Definition 1 page 51) is used to define pre-collisional velocities, so that the backwards hard sphere flow coincides in that case with the free flow with the pre-collisional velocities instead of the post-collisional ones. As Proposition 3 gives the Jacobian determinant of the free transport in the pre-collisional case, the following Proposition studies the post-collisional situation, and provides the expression of the Jacobian determinant of the free transport composed with the scattering operator.

Proposition 4. *Consider the following function :*

$$S_s^+ : \begin{cases} \mathcal{D}_s^\varepsilon \times [0,\delta] \times \mathbb{S}^{d-1} \times \mathbb{R}^d & \to \mathbb{R}^{2d(s+1)} \\ \big(Z_s, t, \omega, v_{s+1}\big) & \mapsto \widetilde{Z}_{s+1} \\ & \quad = \big(X_s - tV_s', V_s', x_i + \varepsilon\omega - tv_{s+1}', v_{s+1}'\big), \end{cases}$$

where V_s' denotes $(v_1, \dots, v_i', \dots, v_s)$, and v_i' and v_{s+1}' are the post-collisional velocities of v_i and v_{s+1}, that is v_i' and v_{s+1}' denotes the image by the scattering mapping $\mathfrak{S}$ (see Definition 1 page 51) of the pair of velocities v_i and v_{s+1} with the angular parameter ω.
Then the function S_s^+ is one-to-one on the subset of $\mathcal{D}_s^\varepsilon \times [0,\delta] \times \mathbb{S}^{d-1} \times \mathbb{R}^d$ composed of the post-collisional configurations (that is such that $\omega \cdot (v_{s+1} - v_i) > 0$), and transforms the measure

$$\varepsilon^{d-1} \big| \omega \cdot (v_{s+1} - v_i) \big| \, \mathrm{d}Z_s \, \mathrm{d}t \, \mathrm{d}\omega \, \mathrm{d}v_{s+1}$$

in the Lebesgue measure on $\mathbb{R}^{2d(s+1)}$. In other words, the absolute value of the Jacobian determinant of S_s^+ is equal to :

$$\varepsilon^{d-1} \big| \omega \cdot (v_{s+1} - v_i) \big|.$$

Proof. The proof of the fact that S_s^+ is one-to-one is obtained exactly as in the proof of Proposition 3.
Concerning the absolute value of the Jacobian determinant of this function, the proof relies strongly on the previous Proposition 3 on the one hand, and on the other hand on a property of the scattering operator.
Indeed, if $\omega \cdot (v_{s+1} - v_i) > 0$ the function S_s^+ can of course be seen as the backwards free-flow applied to the pre-collisional situation associated to the post-collisional situation $(Z_s, x_i + \varepsilon\omega, v_{s+1})$, in other words

$$S_s^+ = S_s^- \circ \mathfrak{C},$$

where $\mathfrak{C}$ denotes the function

$$\mathfrak{C} : \left\{ \begin{array}{ll} \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}^{d-1} & \to \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}^{d-1} \\ (v_1, v_2, \omega) & \mapsto (v_1' = v_1 - (v_1 - v_2) \cdot \omega\omega, v_2' = v_2 + (v_1 - v_2) \cdot \omega\omega, \omega) \end{array} \right.$$

(one can notice the similarity of the scattering function defined in Definition 1 page 51).
Then, once the Jacobian determinant of the function $\mathfrak{C}$ is computed, the result will be obvious using the result of the previous proposition.
One can immediately see that the definition of the function $\mathfrak{C}$ makes sense on a larger subset of $\mathbb{R}^{3d}$, indeed the expression :

$$\mathfrak{C}(v_1, v_2, \omega)$$

makes sense on $\mathbb{R}^d \times \mathbb{R}^d \times \big(\mathbb{R}^d \backslash \{0\}\big)$ (it makes even sense on the whole space $\mathbb{R}^{3d}$, but it will not be useful here). One will use the fact that $\mathbb{R}^d \backslash \{0\}$ is transformed in $\mathbb{S}^{d-1} \times \mathbb{R}_+^*$ by using the polar coordinates.
The function

$$\mathfrak{C} : \left\{ \begin{array}{ll} \mathbb{R}^d \times \mathbb{R}^d \times \big(\mathbb{R}^d \backslash \{0\}\big) & \to \mathbb{R}^d \times \mathbb{R}^d \times \big(\mathbb{R}^d \backslash \{0\}\big) \\ (v_1, v_2, u) & \mapsto (v_1 - (v_1 - v_2) \cdot uu, v_2 + (v_1 - v_2) \cdot uu, u) \end{array} \right.$$

has a differential that can be easily computed since for every $h_1, h_2, h_3 \in \mathbb{R}^d$, one has :

$$\left\{ \begin{array}{ll} (v_1 + h_1) - \big((v_1 + h_1) - v_2\big) \cdot uu & = v_1 - (v_1 - v_2) \cdot uu + h_1 - h_1 \cdot uu, \\ v_1 - \big(v_1 - (v_2 + h_2)\big) \cdot uu & = v_1 - (v_1 - v_2) \cdot uu + h_2 \cdot uu, \\ v_1 - (v_1 - v_2) \cdot \big(u + h_3\big)\big(u + h_3\big) & = v_1 - (v_1 - v_2) \cdot uu - (v_1 - v_2) \cdot uh_3 \\ & \quad -(v_1 - v_2) \cdot h_3 u - (v_1 - v_2) \cdot h_3 h_3, \end{array} \right.$$

and

$$\left\{ \begin{array}{ll} v_2 + \big((v_1 + h_1) - v_2\big) \cdot uu & = v_2 + (v_1 - v_2) \cdot uu + h_1 \cdot uu, \\ (v_2 + h_2) + \big(v_1 - (v_2 + h_2)\big) \cdot uu & = v_2 + (v_1 - v_2) \cdot uu + h_2 - h_2 \cdot uu, \\ v_2 + (v_1 - v_2) \cdot \big(u + h_3\big)\big(u + h_3\big) & = v_2 + (v_1 - v_2) \cdot uu + (v_1 - v_2) \cdot uh_3 \\ & \quad +(v_1 - v_2) \cdot h_3 u + (v_1 - v_2) \cdot h_3 h_3, \end{array} \right.$$

so that the differential of $\mathfrak{C}$ can be expressed with the following Jacobian matrix at every point $(v_1, v_2, u) \in \mathbb{R}^d \times \mathbb{R}^d \times \left(\mathbb{R}^d \backslash \{0\}\right)$:

$$\left(\begin{array}{|c|c|c|}
\hline
I_d - u \otimes u & u \otimes u & (v_2 - v_1) \cdot u I_d + (v_2 - v_1) \otimes u \\
\hline
u \otimes u & I_d - u \otimes u & -(v_2 - v_1) \cdot u I_d - (v_2 - v_1) \otimes u \\
\hline
0 & 0 & I_d \\
\hline
\end{array}\right). \tag{5.11}$$

The Jacobian determinant $J(\mathfrak{C})$ of the matrix (5.11) can be easily computed, since one can develop it with respect to the last d lines, it remains :

$$J(\mathfrak{C}) = \left\|\begin{array}{|c|c|}
\hline
I_d - u \otimes u & u \otimes u \\
\hline
u \otimes u & I_d - u \otimes u \\
\hline
\end{array}\right\| = \left\|\begin{array}{|c|c|}
\hline
I_d & I_d \\
\hline
u \otimes u & I_d - u \otimes u \\
\hline
\end{array}\right\|,$$

or again :

$$J(\mathfrak{C}) = \left\|\begin{array}{|c|c|}
\hline
I_d & 0 \\
\hline
u \otimes u & I_d - 2u \otimes u \\
\hline
\end{array}\right\| = \left\|\begin{array}{|c|}
\hline
I_d - 2u \otimes u \\
\hline
\end{array}\right\| = 1 + (-2u) \cdot u,$$

the last identity being obtained thanks to the property of the tensor product stating that for all vectors a and b, one has

$$\det\left(I_d + a \otimes b\right) = 1 + a \cdot b.$$

The result of the proposition is then obtained by computing the product of the two determinants $J(S_s^-)$ and $J(\mathfrak{C})$, and by noticing that in the case studied here, the vector u lies in fact in the sphere $\mathbb{S}^{d-1}$, so that one has of course

$$1 + (-2u) \cdot u = 1 - 2 = -1.$$

□

Those two propositions will be useful to define properly the truncated transport-collision operators, which is the object of the following lemma.

Lemma 1 (Definition of the truncated in time, position and velocity transport-collision operator, and L^1 regularity for L^∞ functions)**.** *Let s be a positive integer, $\varepsilon \leq 1$ and T be two strictly positive numbers. Finally, let R_1, R_2 be two strictly positive numbers, and δ be a strictly positive number such that :*

$$\delta \leq \delta_0(d, \varepsilon, R_2) = \min_{1 \leq j \leq d} \left[\frac{\varepsilon}{2R_2} \Big(\frac{d}{\binom{d}{j}} \Big)^{\frac{1}{j-1}} \right]. \tag{5.12}$$

Then for every integer $1 \leq i \leq s$, and for every function $\varphi^{(s+1)} \in L^\infty\big(\mathcal{D}^\varepsilon_{s+1}\big)$, the function

$$(t, Z_s) \mapsto \mathcal{C}^\varepsilon_{s,s+1,\pm,i}(R_1, R_2, \delta) \mathcal{T}^{s+1,\varepsilon}_t \varphi^{(s+1)}$$

defined as :

$$\begin{aligned}
\mathcal{C}^\varepsilon_{s,s+1,\pm,i}(R_1,R_2,\delta)\mathcal{T}^{s+1,\varepsilon}_t \varphi^{(s+1)}(Z_s) = \int_{\mathbb{S}^{d-1}_\omega \times \mathbb{R}^d_{v_{s+1}}} & \varepsilon^{d-1} \left(\omega \cdot (v_{s+1} - v_i)\right)_\pm \\
& \times \mathcal{T}^{s+1,\varepsilon}_t \varphi^{(s+1)}(Z_s, x_i + \varepsilon\omega, v_{s+1}) \\
& \times \Bigg(\prod_{\substack{1 \leq j < k \leq s+1 \\ (j,k) \neq (i,s+1)}} \mathbb{1}_{|x_j - x_k| > \varepsilon + \sqrt{2}\delta R_2} \Bigg) \Bigg(\prod_{1 \leq l \leq s+1} \mathbb{1}_{d(x_l, \Omega) > \varepsilon/2 + \delta R_2} \Bigg) \\
& \times \ \mathbb{1}_{X_{s+1} \in B_{\mathbb{R}^{d(s+1)}}(0,R_1)} \mathbb{1}_{V_{s+1} \in B_{\mathbb{R}^{d(s+1)}}(0,R_2)} \, \mathrm{d}\omega \, \mathrm{d}v_{s+1}
\end{aligned}$$

is well defined as a function of $L^1\big([0,T] \times \mathcal{D}^\varepsilon_s\big)$, and the following control on its L^1 norm holds :

$$\begin{aligned}
\Big| \mathcal{C}^\varepsilon_{s,s+1,\pm,i}(R_1,R_2,\delta)\mathcal{T}^{s+1,\varepsilon}_t \varphi^{(s+1)} \Big|_{L^1([0,T] \times \mathcal{D}^\varepsilon_s)} & \\
\leq C(d,s) \varepsilon^{d-1} T (R_1 + \delta R_2)^{ds} R_2^{d(s+1)+1} \big| \varphi^{(s+1)} \big|_{L^\infty(\mathcal{D}^\varepsilon_{s+1})}. & \tag{5.13}
\end{aligned}$$

Proof. Here one starts by the pre-collisional case, that is one studies the operator $\mathcal{C}^\varepsilon_{s,s+1,-,i}(R_1, R_2, \delta)$. If the configuration $\big(Z_s, x_i + \varepsilon\omega, v_{s+1}\big)$, belonging to

$$\begin{aligned}
\Big(\bigcap_{\substack{1 \leq j < k \leq s+1 \\ (j,k) \neq (i,s+1)}} \big\{ |x_j - x_k| > \varepsilon + \sqrt{2}\delta R_2 \big\} \Big) \cap \Big(\bigcap_{1 \leq l \leq s+1} \big\{ d(x_l, \Omega) > \varepsilon/2 + \delta R_2 \big\} \Big) & \\
\cap \ \Big(\big(B_{\mathbb{R}^{d(s+1)}}(0, R_1)\big)_{X_{s+1}} \times \big(B_{\mathbb{R}^{d(s+1)}}(0, R_2)\big)_{V_{s+1}} \Big), &
\end{aligned}$$

is assumed to be a pre-collisional configuration, that is if $\omega \cdot (v_{s+1} - v_i) < 0$, according to the definition of the restricted domain of the truncated collision operator, one knows that the backwards flow of $s+1$ hard spheres of diameter ε coincides with the backwards free flow on this domain, at least for any time t belonging to $[0, \delta]$, that is :

$$T_{-t}^{s+1,\varepsilon}\big(Z_s, x_i + \varepsilon\omega, v_{s+1}\big) = S_s^-\big(Z_s, x_i + \varepsilon\omega, v_{s+1}\big)$$

for all $t \in [0, \delta]$. Then, using Proposition 3 page 92 and the theorem of change of variables, one has :

$$\begin{aligned}
&\int_{\mathcal{I}_{s,i}^{\varepsilon,-}(R_1,R_2,\delta)} \Big|\varphi^{(s+1)}\big(\widetilde{Z}_{s+1}\big)\Big| \,\mathrm{d}\widetilde{Z}_{s+1} \\
&\quad = \int_0^\delta \int_{\mathcal{D}_s^\varepsilon} \Big|\mathcal{C}_{s,s+1,-,i}^{\varepsilon}(R_1, R_2, \delta)\varphi^{(s+1)}\big(S_s^-(Z_s, x_i + \varepsilon\omega, v_{s+1})\big)\Big| \,\mathrm{d}Z_s \,\mathrm{d}t,
\end{aligned}$$

with $\mathcal{I}_{s,i}^{\varepsilon,-}(R_1, R_2, \delta)$ denoting the image by S_s^- of the domain of integration :

$$\begin{aligned}
&\times \Big(\Big(\bigcap_{\substack{1\le j<k\le s+1\\(j,k)\neq(i,s+1)}} \{|x_j - x_k| > \varepsilon + \sqrt{2}\delta R_2\}\Big) \\
&\cap \Big(\bigcap_{1\le l\le s+1} \{d(x_l, \Omega) > \varepsilon/2 + \delta R_2\}\Big) \cap \{|X_{s+1}| < R_1\} \cap \{|V_{s+1}| < R_2\}\Big) \\
&\qquad\qquad \subset \mathcal{D}_s^\varepsilon \times [0, \delta] \times \mathbb{S}^{d-1} \times \mathbb{R}^d.
\end{aligned}$$

On the one hand, in particular, if $|X_{s+1}| < R_1$ (where, as it is precised in the remark 8 page 91, X_{s+1} denotes $(x_1, \ldots, x_s, x_i + \varepsilon\omega)$), $|V_{s+1}| < R_2$ and $t \in [0, \delta]$, then one sees that :

$$|X_{s+1} - tV_{s+1}| < R_1 + \delta R_2. \tag{5.14}$$

On the other hand, one has :

$$\varepsilon \le |(x_i + \varepsilon\omega - tv_{s+1}) - (x_i - tv_i)|, \tag{5.15}$$

since the configuration is assumed to be pre-collisional, that is $\omega \cdot (v_{s+1} - v_i) < 0$. Finally, this distance can be bounded from above writing :

$$\begin{aligned}
|\varepsilon\omega - t(v_{s+1} - v_i)|^2 &= \varepsilon^2 - 2t\varepsilon\omega \cdot (v_{s+1} - v_i) + t^2|v_{s+1} - v_i|^2 \\
&\le \varepsilon^2 + 2\varepsilon t|v_{s+1} - v_i| + t^2|v_{s+1} - v_i|^2 = (\varepsilon + t|v_{s+1} - v_i|)^2 \\
&\le (\varepsilon + 2\delta R_2)^2.
\end{aligned} \tag{5.16}$$

Using the three last inequalities, it is possible to bound the size of the set $\mathcal{I}_{s,i}^{\varepsilon,-}(R_1, R_2, \delta)$. More precisely, if one considers an element

$$\widetilde{Z}_{s+1} = \big(\widetilde{x}_1, \widetilde{v}_1, \ldots, \widetilde{x}_{s+1}, \widetilde{v}_{s+1}\big)$$

belonging to the image $\mathcal{I}_{s,i}^{\varepsilon,-}(R_1,R_2,\delta)$, the inequality (5.14) enables to bound the ds first components $\widetilde{x}_1,\dots,\widetilde{x}_s$ (that is, the components of the s first positions in the image of S_s^-, contained in the phase space of $s+1$ particles) of any element of $\mathcal{I}_{s,i}^{\varepsilon,-}(R_1,R_2,\delta)$. The bound on the ds following components $\widetilde{v}_1,\dots,\widetilde{v}_s$ and the d last components $\widetilde{v}_{s+1}$ (that is, the components of the velocities in the phase space of $s+1$ particles) is a direct consequence of the cut-off $|V_{s+1}|\leq R_2$. Finally, the two inequalities (5.15) and (5.16) enable to bound the norm of the components of the $s+1$ positions in the phase space, that is one has :

$$\mathcal{I}_{s,i}^{\varepsilon,-}(R_1,R_2,\delta)\subset B_{\mathbb{R}^{ds}}(0,R_1+\delta R_2)\times B_{\mathbb{R}^{ds}}(0,R_2)\times B_{\mathbb{R}^d}(0,R_1+\delta R_2)\times B_{\mathbb{R}^d}(0,R_2)$$

with, the following extra condition, consequence of the two inequalities (5.15) and (5.16) :

$$\widetilde{x}_{s+1}\in\big(B(\widetilde{x}_i,\varepsilon+2\delta R_2)\backslash B(\widetilde{x}_i,\varepsilon)\big).$$

It is then easy to obtain a bound on the size of $\mathcal{I}_{s,i}^{\varepsilon,-}(R_1,R_2,\delta)$:

$$\Big|\mathcal{I}_{s,i}^{\varepsilon,-}(R_1,R_2,\delta)\Big|\leq C(d,s)(R_1+\delta R_2)^{ds}R_2^{d(s+1)}\big|B_{\mathbb{R}^d}(0,\varepsilon+2\delta R_2)\backslash B_{\mathbb{R}^d}(0,\varepsilon)\big|, \tag{5.17}$$

with $C(d,s)$ a constant depending only on the dimension and the number of particles s, and then immediately :

$$\begin{aligned}\int_{\mathcal{I}_{s,i}^{\varepsilon,-}(R_1,R_2,\delta)}\Big|\varphi^{(s+1)}\big(\widetilde{Z}_{s+1}\big)\Big|\,\mathrm{d}\widetilde{Z}_{s+1}&\leq C(d,s)(R_1+\delta R_2)^{ds}R_2^{d(s+1)}\\&\quad\times\big|B_{\mathbb{R}^d}(0,\varepsilon+2\delta R_2)\backslash B_{\mathbb{R}^d}(0,\varepsilon)\big|\cdot\big|\varphi^{(s+1)}\big|_{L^\infty(\mathcal{D}_{s+1}^\varepsilon)}.\end{aligned}$$

Using the same argument as in the proof of Proposition 2 page 57, one has :

$$\begin{aligned}\big|B_{\mathbb{R}^d}(0,\varepsilon+2\delta R_2)\backslash B_{\mathbb{R}^d}(0,\varepsilon)\big|&=C(d)\big((\varepsilon+2\delta R_2)^d-\varepsilon^d\big)\\&=C(d)\sum_{j=1}^d\binom{d}{j}\varepsilon^{d-j}(2\delta R_2)^j,\end{aligned}$$

with $C(d)$ denoting a constant depending only on the dimension d, this measure being smaller than

$$C(d)\delta\varepsilon^{d-1}R_2$$

if δ is small enough. More precisely, this bound will hold under condition 5.12, which is

$$\delta\leq\delta_0(d,\varepsilon,R_2)=\min_{1\leq j\leq d}\left[\frac{\varepsilon}{2R_2}\Big(\frac{d}{\binom{d}{j}}\Big)^{\frac{1}{j-1}}\right],$$

since in this case the sum can be controlled as follows :

$$\begin{aligned}\sum_{j=1}^{d}\binom{d}{j}\varepsilon^{d-j}(2\delta R_2)^j &\leq 2\delta R_2\sum_{j=1}^{d}\binom{d}{j}\varepsilon^{d-j}(2\delta R_2)^{j-1}\\ &\leq 2\delta R_2\sum_{j=1}^{d}\binom{d}{j}\varepsilon^{d-j}\left(\varepsilon\Big(\frac{d}{\binom{d}{j}}\Big)^{\frac{1}{j-1}}\right)^{j-1}\\ &\leq C(d)\delta R_2\varepsilon^{d-1},\end{aligned}$$

using in the last step that $\varepsilon \leq 1$. So if $\delta \leq \delta_0$, one finds that :

$$\begin{aligned}&\int_{\mathcal{I}^{\varepsilon,-}_{s,i}(R_1,R_2,\delta)}\Big|\varphi^{(s+1)}\big(\widetilde{Z}_{s+1}\big)\Big|\,\mathrm{d}\widetilde{Z}_{s+1}\\ &\quad\leq C(d,s)\delta\varepsilon^{d-1}(R_1+\delta R_2)^{ds}R_2^{d(s+1)+1}\big|\varphi^{(s+1)}\big|_{L^\infty(\mathcal{D}^\varepsilon_{s+1})},\end{aligned}\tag{5.18}$$

assuming, of course, that $\varphi^{(s+1)}$ is a L^∞ function.

For the post-collisional case, that is for $\mathcal{C}^\varepsilon_{s,s+1,+,i}(R_1,R_2,\delta)$, the configuration $\big(Z_s, x_i+\varepsilon\omega, v_{s+1}\big)$ is now a post-collisional configuration, which corresponds to the case when $\omega\cdot(v_{s+1}-v_i)>0$, and then the backwards flow of $s+1$ hard spheres of diameter ε coincides with the backwards free flow composed this time with the scattering, on the restricted domain of the truncated transport-collision operator on the whole time interval $[0,\delta]$, that is :

$$T^{s+1,\varepsilon}_{-t}(Z_s, x_i+\varepsilon\omega, v_{s+1}) = S^+_s(Z_s, x_i+\varepsilon\omega, v_{s+1})$$

for all $t\in[0,\delta]$ on $\mathfrak{D}(\varepsilon,R_1,R_2,\delta)$ (introduced in (5.1) page 89). Instead of using Proposition 3, one uses Proposition 4 and the theorem of change of variables. From that point, the proof becomes identical, since the image $\mathcal{I}^{\varepsilon,+}_{s,i}$ of the restricted domain of the truncated collision operator by the application S^+_s will verify the same estimates as $\mathcal{I}^{\varepsilon,-}_{s,i}$, and provide the same L^1 control.

From now on, one will consider the pre and the post-collisional configurations at once, and decompose the time interval $[0,T]$ in $\lceil T/\delta\rceil$ smaller intervals, of size δ (except possibly the last, which can be even smaller). Here, the notation $\lceil x\rceil$ means that one takes the smallest integer larger than x, and $\lfloor x\rfloor$ means that one takes the largest integer smaller than x. Then, on each interval, which writes $[n\delta,(n+1)\delta]$ (except possibly $[\big(\lfloor T/\delta\rfloor\big)\delta, T]$ for the last one), one writes :

$$\begin{aligned}&\int_{n\delta}^{(n+1)\delta}\int_{\mathcal{D}^\varepsilon_s}\Big|\mathcal{C}^\varepsilon_{s,s+1,\pm,i}(R_1,R_2,\delta)\mathcal{T}^{s+1,\varepsilon}_t\varphi^{(s+1)}\big(Z_{s+1}\big)\Big|\,\mathrm{d}Z_s\,\mathrm{d}t\\ &\quad=\int_{n\delta}^{(n+1)\delta}\int_{\mathcal{D}^\varepsilon_s}\Big|\mathcal{C}^\varepsilon_{s,s+1,\pm,i}(R_1,R_2,\delta)\mathcal{T}^{s+1,\varepsilon}_{t-n\delta}\mathcal{T}^{s+1,\varepsilon}_{n\delta}\varphi^{(s+1)}\big(Z_{s+1}\big)\Big|\,\mathrm{d}Z_s\,\mathrm{d}t,\end{aligned}$$

and by change of variables $u = t - n\delta$:

$$= \int_0^\delta \int_{\mathcal{D}_s^\varepsilon} \left| \mathcal{C}^\varepsilon_{s,s+1,\pm,i}(R_1, R_2, \delta) \mathcal{T}_u^{s+1,\varepsilon} \mathcal{T}_{n\delta}^{s+1,\varepsilon} \varphi^{(s+1)}\big(Z_{s+1}\big) \right| \mathrm{d}Z_s \, \mathrm{d}u. \tag{5.19}$$

Thanks to the inequality :

$$\big|\mathcal{T}_{n\delta}^{s+1,\varepsilon} \varphi^{(s+1)}\big|_{L^\infty(\mathcal{D}^\varepsilon_{s+1})} \leq \big|\varphi^{(s+1)}\big|_{L^\infty(\mathcal{D}^\varepsilon_{s+1})},$$

which is nothing else than the preservation of the L^∞ norm by the hard sphere transport (see [26][5]), one sees that, using together the inequalities (5.18) (applied to the function $\mathcal{T}_{n\delta}^{s+1,\varepsilon} \varphi^{(s+1)}$) and (5.19), one has :

$$\begin{aligned} &\int_{n\delta}^{(n+1)\delta} \int_{\mathcal{D}_s^\varepsilon} \left| \mathcal{C}^\varepsilon_{s,s+1,\pm,i}(R_1, R_2, \delta) \mathcal{T}_t^{s+1,\varepsilon} \varphi^{(s+1)}\big(Z_{s+1}\big) \right| \mathrm{d}Z_s \, \mathrm{d}t \\ &\qquad \leq C(d,s)\delta\varepsilon^{d-1}(R_1 + \delta R_2)^{ds} R_2^{d(s+1)+1} \left|\varphi^{(s+1)}\right|_{L^\infty\left(\mathcal{D}^\varepsilon_{s+1}\right)}. \end{aligned} \tag{5.20}$$

Then, it is possible to sum the inequalities (5.20) for all n between 0 (which is, of course, in fact just the inequality (5.18)) and $\lceil T/\delta \rceil - 1$. Choosing δ small enough (that is fulfilling the condition (5.12) stated in the lemma) and such that T/δ is an integer, one has $[\big(\lfloor T/\delta \rfloor, \big), T] = [T - \delta, T]$ and one obtains :

$$\begin{aligned} &\int_0^T \int_{\mathcal{D}_s^\varepsilon} \left| \mathcal{C}^\varepsilon_{s,s+1,\pm,i}(R_1, R_2, \delta) \mathcal{T}_t^{s+1,\varepsilon} \varphi^{(s+1)}\big(Z_{s+1}\big) \right| \mathrm{d}Z_s \, \mathrm{d}t \\ &= \sum_{j=0}^{T/\delta - 1} \int_{j\delta}^{(j+1)\delta} \int_{\mathcal{D}_s^\varepsilon} \left| \mathcal{C}^\varepsilon_{s,s+1,\pm,i}(R_1, R_2, \delta) \mathcal{T}_t^{s+1,\varepsilon} \varphi^{(s+1)}\big(Z_{s+1}\big) \right| \mathrm{d}Z_s \, \mathrm{d}t \\ &\leq C(d,s)\varepsilon^{d-1} T (R_1 + \delta R_2)^{ds} R_2^{d(s+1)+1} \left|\varphi^{(s+1)}\right|_{L^\infty\left(\mathcal{D}^\varepsilon_{s+1}\right)}. \end{aligned} \tag{5.21}$$

Finally, one has shown that the expression (5.3) page 91 makes sense thanks to the change of variables induced by the free transport, providing that the quantity

$$\int_{\mathcal{I}^{\varepsilon,-}_{s,i}(R_1,R_2,\delta)} \left|\varphi^{(s+1)}\big(\widetilde{Z}_{s+1}\big)\right| \mathrm{d}\widetilde{Z}_{s+1}$$

makes sense itself. Since one has shown that $\mathcal{I}^{\varepsilon,-}_{s,i}$ is bounded in the phase space, it is enough to assume that $\varphi^{(s+1)}$ belongs to $L^\infty\big(\mathcal{D}^\varepsilon_{s+1}\big)$ to be integrable on

[5] In particular, see Appendix 4.A "More About Hard-Sphere Dynamics".

this domain.
Moreover, one has shown that the function :

$$(t, Z_s) \mapsto \mathcal{C}^{\varepsilon}_{s,s+1,\pm,i}(R_1, R_2, \delta)\mathcal{T}^{s+1,\varepsilon}_t \varphi^{(s+1)}$$

is, first thanks to the Fubini-Tonelli theorem, a measurable function, and second thanks to the Fubini-Lebesgue theorem, an integrable function, such that its L^1 norm is controlled by the L^∞ norm of the function $\varphi^{(s+1)} \in L^\infty\big(\mathcal{D}^{\varepsilon}_{s+1}\big)$, exactly as it is shown in the last inequality.

In summary, the truncated transport-collision operator is well defined on the L^∞ functions, and sends such functions into L^1 functions. Lemma 1 is proved. □

5.1.2 Control of the L^∞ norm of the truncated transport-collision operator

Now that the truncated transport-collision operator is well-defined on $L^\infty\big(\mathcal{D}^{\varepsilon}_{s+1}\big)$, one wants to obtain an operator which sends functions from some Lebesgue space into the same space (with less variables, of course, since the collision operator has to be seen as an integration over the last velocity variable and over a parameter angle). More precisely, since this operator makes sense on the almost everywhere bounded functions, one will study its L^∞ norm.

From now on, one will consider a measurable subset A of $[0, \delta] \times \mathcal{D}^{\varepsilon}_s$, of finite measure.
Indeed, for an integrable function f on a mesured space (X, μ), one knows that if there exists a constant C such that, for any measurable subset A of finite measure, one has :

$$\int_X \mathbb{1}_A \big|f\big| \,\mathrm{d}\mu \leq C\big|A\big|,$$

then one can assert that f is almost everywhere bounded by the constant C.

Using the notation (5.1) page 89 for the domain of the truncated transport-collision operator, one has written previously :

$$\begin{aligned}\int_0^\delta \int_{\mathcal{D}^{\varepsilon}_s} \int_{\mathbb{S}^{d-1}_\omega} \int_{\mathbb{R}^d_{v_{s+1}}} & \varepsilon^{d-1}\big(\omega \cdot (v_{s+1} - v_i)\big)_\pm \mathcal{T}^{s+1,\varepsilon}_t \varphi^{(s+1)}(Z_s, x_i + \varepsilon\omega, v_{s+1}) \\ & \times \mathbb{1}_{\mathfrak{D}(\varepsilon, R_1, R_2, \delta)} \,\mathrm{d}v_{s+1} \,\mathrm{d}\omega \,\mathrm{d}Z_s \,\mathrm{d}t \\ = \int_{S^{\pm}_s(\mathfrak{D})} & \varphi^{(s+1)}\big(\widetilde{Z}_{s+1}\big) \,\mathrm{d}\widetilde{Z}_{s+1}.\end{aligned}$$

One will now replace the term $\mathbb{1}_{\mathfrak{D}}$ by $\mathbb{1}_{\mathfrak{D}\cap B}$, with :

$$B = A \times \mathbb{S}^{d-1} \times B_{\mathbb{R}^d}(0, R_2),$$

in order to obtain a control on the L^∞ norm of the truncated collision operator. Here, there is an abuse of notation, since the variables defining the subset B are

not given in a consistent order according to the definition of $\mathfrak{D}$. Nevertheless, this abuse of notation is convenient in order to regroup the variables ω and v_{s+1} of the domain of integration of the collision operator on the one hand, and the remaining variables t and Z_s on which the indicator function $\mathbb{1}_A$ acts on the other hand.
One wants to study :

$$\int\limits_{A\subset[0,\delta]\times\mathcal{D}_s^\varepsilon} \Big|\mathcal{C}^\varepsilon_{s,s+1,\pm,i}(R_1,R_2,\delta)\mathcal{T}_t^{s+1,\varepsilon}\varphi^{(s+1)}\Big|\,\mathrm{d}Z_s\,\mathrm{d}t,$$

this quantity being well-defined thanks to Lemma 1 page 101. One now shows the following result.

Lemma 2 (L^∞ regularity of the truncated in time, position and velocity transport-collision operator for L^∞ functions)**.** *Let s be a positive integer, $\varepsilon\le 1$ and T be two strictly positive numbers. Let R_1,R_2 be two strictly positive numbers, and δ be a strictly positive number such that :*

$$\delta\le\delta_0(d,\varepsilon,R_2)=\min_{1\le j\le d}\left[\frac{\varepsilon}{2R_2}\Big(\frac{d}{\binom{d}{j}}\Big)^{\frac{1}{j-1}}\right].$$

Then for every integer $1\le i\le s$, and for every function $\varphi^{(s+1)}\in L^\infty\big(\mathcal{D}^\varepsilon_{s+1}\big)$, the function :

$$(t,Z_s)\mapsto\mathcal{C}^\varepsilon_{s,s+1,\pm,i}(R_1,R_2,\delta)\mathcal{T}_t^{s+1,\varepsilon}\varphi^{(s+1)},$$

defined as in Lemma 1 page 101 verifies the two following estimates :

- *for every measurable subset A of $[0,T]\times\mathcal{D}_s^\varepsilon$ of finite measure, one has :*

$$\int\limits_{A\subset[0,T]\times\mathcal{D}_s^\varepsilon}\Big|\mathcal{C}^\varepsilon_{s,s+1,\pm,i}(R_1,R_2,\delta)\mathcal{T}_t^{s+1,\varepsilon}\varphi^{(s+1)}\Big|\,\mathrm{d}Z_s\,\mathrm{d}t \le C(d,s)\varepsilon^{d-1}R_2^{d+1}|A|\big|\varphi^{(s+1)}\big|_{L^\infty(\mathcal{D}^\varepsilon_{s+1})}, \tag{5.22}$$

- *in addition :*

$$\big|\mathcal{C}^\varepsilon_{s,s+1,\pm,i}(R_1,R_2,\delta)\mathcal{T}_t^{s+1,\varepsilon}\varphi^{(s+1)}\big|_{L^\infty([0,T]\times\mathcal{D}_s^\varepsilon)} \le C(d,s)\varepsilon^{d-1}R_2^{d+1}\big|\varphi^{(s+1)}\big|_{L^\infty(\mathcal{D}^\varepsilon_{s+1})}. \tag{5.23}$$

Proof. Let A be a measurable subset of $[0,\delta]\times\mathcal{D}_s^\varepsilon$, of finite measure, and let B be defined by :

$$B=A\times\mathbb{S}^{d-1}\times B_{\mathbb{R}^d}(0,R_2).$$

One will assume in addition that $B\subset\mathfrak{D}$ (so that it will not be necessary to consider $\mathfrak{D}\cap B$, but just B as domain of the integral). Since, for the whole

time interval $[0,\delta]$, the backwards hard sphere flow coincides on $\mathfrak{D}$ with the backwards free flow in the case of a pre-collisional configuration, and coincides with the backwards free flow composed with the scattering in the case of a post-collisional configuration, this is still true on B, and one has obviously :

$$\begin{aligned}\int_0^\delta \int_{\mathcal{D}_s^\varepsilon} \int_{\mathbb{S}^{d-1}_\omega} \int_{\mathbb{R}^d_{v_{s+1}}} \varepsilon^{d-1}\big(\omega\cdot(v_{s+1}-v_i)\big)_\pm \mathcal{T}_t^{s+1,\varepsilon}\varphi^{(s+1)}\mathbb{1}_B \,\mathrm{d}v_{s+1}\,\mathrm{d}\omega\,\mathrm{d}Z_s\,\mathrm{d}t \\ = \int_{S_s^\pm(B)} \varphi^{(s+1)}\big(\widetilde{Z}_{s+1}\big)\,\mathrm{d}\widetilde{Z}_{s+1}.\end{aligned}$$

Then, since the L^∞ norm of $\varphi^{(s+1)}$ is controlled by hypothesis, it will be enough to control the size of $S_s^\pm(B)$.
Then, writing :

$$\big|S_s^\pm(B)\big| = \int \mathbb{1}_{S_s^\pm(B)},$$

one gets, using the fact that $S_s^\pm$ is one-to-one onto its image, and denoting $\big(S_s^\pm\big)^{-1}$ its inverse on this image :

$$\begin{aligned}\big|S_s^\pm(B)\big| &= \int_{\mathcal{D}_{s+1}^\varepsilon} \mathbb{1}_B \circ \big(S_s^\pm\big)^{-1}\,\mathrm{d}\widetilde{Z}_{s+1} \\ &= \int_{\mathcal{D}_s^\varepsilon\times[0,\delta]\times\mathbb{S}^{d-1}_\omega\times\mathbb{R}^d_{v_{s+1}}} \mathbb{1}_B\cdot\big|J\big(S_s^\pm\big)\big|\,\mathrm{d}v_{s+1}\,\mathrm{d}\omega\,\mathrm{d}Z_s\,\mathrm{d}t \\ &= \int_{A\subset\mathcal{D}_s^\varepsilon\times[0,\delta]} \left[\int_{\mathbb{S}^{d-1}_\omega\times B_{\mathbb{R}^d_{v_{s+1}}}(0,R_2)} \varepsilon^{d-1}\big|\omega\cdot(v_{s+1}-v_i)\big|\,\mathrm{d}v_{s+1}\,\mathrm{d}\omega\right]\mathrm{d}Z_s\,\mathrm{d}t \\ &\le \varepsilon^{d-1}\big|\mathbb{S}^{d-1}\big|\int_A\int_{B_{\mathbb{R}^d_{v_{s+1}}}(0,R_2)} |v_{s+1}-v_i|\,\mathrm{d}v_{s+1}\,\mathrm{d}Z_s\,\mathrm{d}t \\ &\le 2R_2\varepsilon^{d-1}\big|\mathbb{S}^{d-1}\big|\int_A\int_{B_{\mathbb{R}^d_{v_{s+1}}}(0,R_2)} \mathrm{d}v_{s+1}\,\mathrm{d}Z_s\,\mathrm{d}t\end{aligned}$$

since if $Z_s=(\dots,v_i,\dots)$ belongs to A, then $|v_i|\le R_2$. One has :

$$\begin{aligned}\int_{A\subset[0,\delta]\times\mathcal{D}_s^\varepsilon} \Big|\mathcal{C}^\varepsilon_{s,s+1,\pm,i}(R_1,R_2,\delta)\mathcal{T}_t^{s+1,\varepsilon}\varphi^{(s+1)}\Big|\,\mathrm{d}Z_s\,\mathrm{d}t \\ \le C(d,s)\varepsilon^{d-1}R_2^{d+1}|A|\big|\varphi^{(s+1)}\big|_{L^\infty(\mathcal{D}_{s+1}^\varepsilon)}.\end{aligned} \tag{5.24}$$

On the one hand, if one considers, for all $m>0$ the subset A_m defined as

follows :

$$A_m = \Big\{(t, Z_s) \in [0,\delta] \times \mathcal{D}_s^\varepsilon \ / \ \Big|\big(\mathcal{C}^\varepsilon_{s,s+1,\pm,i}(R_1,R_2,\delta)\mathcal{T}_t^{s+1,\varepsilon}\varphi^{(s+1)}\big)(Z_s)\Big| \geq C(d,s)\varepsilon^{d-1}R_2^{d+1}\big|\varphi^{(s+1)}\big|_{L^\infty(\mathcal{D}^\varepsilon_{s+1})} + m\Big\},$$

which is well defined since the truncated transport-collision operator is a measurable function, and which is of finite measure, since it is integrable, thanks to inequality (5.13), one gets thanks to the inequality (5.24) :

$$\int_{A_m} \Big|\mathcal{C}^\varepsilon_{s,s+1,\pm,i}(R_1,R_2,\delta)\mathcal{T}_t^{s+1,\varepsilon}\varphi^{(s+1)}\Big| \,\mathrm{d}Z_s\,\mathrm{d}t \leq C(d,s)\varepsilon^{d-1}R_2^{d+1}|A_m|\big|\varphi^{(s+1)}\big|_{L^\infty(\mathcal{D}^\varepsilon_{s+1})}. \tag{5.25}$$

On the other hand, since the truncated collision operator is integrable on $[0,\delta]\times\mathcal{D}_s^\varepsilon$, one obtains by definition of A_m :

$$\Big(C(d,s)\varepsilon^{d-1}R_2^{d+1}\big|\varphi^{(s+1)}\big|_{L^\infty(\mathcal{D}^\varepsilon_{s+1})} + m\Big)|A_m| \leq \int_0^\delta \int_{Z_s\in\mathcal{D}_s^\varepsilon} \mathbb{1}_{A_m}\big|\mathcal{C}^\varepsilon_{s,s+1,\pm,i}(R_1,R_2,\delta)\mathcal{T}_t^{s+1,\varepsilon}\varphi^{(s+1)}\big| \,\mathrm{d}Z_s\,\mathrm{d}t, \tag{5.26}$$

so that inequalities (5.25) and (5.26) imply together that for any $m > 0$, one has necessarily :

$$|A_m| = 0.$$

In other words, one has :

$$\Big|\mathcal{C}^\varepsilon_{s,s+1,\pm,i}(R_1,R_2,\delta)\mathcal{T}_t^{s+1,\varepsilon}\varphi^{(s+1)}\Big|(Z_s) \leq C(d,s)\varepsilon^{d-1}R_2^{d+1}\big|\varphi^{(s+1)}\big|_{L^\infty(\mathcal{D}^\varepsilon_{s+1})}$$

almost everywhere on $[0,\delta]\times\mathcal{D}_s^\varepsilon$.
In the same way, for all measurable subsets A_n of $[n\delta,(n+1)\delta]\times\mathcal{D}_s^\varepsilon$ of finite measure, one denotes :

$$A_n^\delta = \big\{(u, Z_s) \in [0,\delta]\times\mathcal{D}_s^\varepsilon \ / \ (u+n\delta, Z_s) \in A_n\big\}.$$

One has of course :

$$|A_n^\delta| = |A_n|, \tag{5.27}$$

and by the change of variables $u = t - n\delta$, one gets :

$$\begin{aligned} I_n &= \int_{A_n} \big|\mathcal{C}^\varepsilon_{s,s+1,\pm,i}(R_1,R_2,\delta)\mathcal{T}_t^{s+1,\varepsilon}\varphi^{(s+1)}\big| \,\mathrm{d}Z_s\,\mathrm{d}t \\ &= \int_{A_n} \big|\mathcal{C}^\varepsilon_{s,s+1,\pm,i}(R_1,R_2,\delta)\mathcal{T}_{t-n\delta}^{s+1,\varepsilon}\mathcal{T}_{n\delta}^{s+1,\varepsilon}\varphi^{(s+1)}\big| \,\mathrm{d}Z_s\,\mathrm{d}t \\ &= \int_{A_n^\delta} \big|\mathcal{C}^\varepsilon_{s,s+1,\pm,i}(R_1,R_2,\delta)\mathcal{T}_u^{s+1,\varepsilon}\mathcal{T}_{n\delta}^{s+1,\varepsilon}\varphi^{(s+1)}\big| \,\mathrm{d}Z_s\,\mathrm{d}u. \end{aligned}$$

Using the inequality (5.24), one gets :

$$I_n \leq C(d,s)\varepsilon^{d-1}R_2^{d+1}|A_n^\delta|\big|\mathcal{T}_{n\delta}^{s+1,\varepsilon}\varphi^{(s+1)}\big|_{L^\infty(\mathcal{D}_{s+1}^\varepsilon)},$$

and then using the equality (5.27) and the control on the L^∞ norm for the hard sphere flow :

$$\begin{aligned} I_n &\leq C(d,s)\varepsilon^{d-1}R_2^{d+1}|A_n|\big|\mathcal{T}_{n\delta}^{s+1,\varepsilon}\varphi^{(s+1)}\big|_{L^\infty(\mathcal{D}_{s+1}^\varepsilon)} \\ &\leq C(d,s)\varepsilon^{d-1}R_2^{d+1}|A_n|\big|\varphi^{(s+1)}\big|_{L^\infty(\mathcal{D}_{s+1}^\varepsilon)}. \end{aligned}$$

This enables to show, as above, that :

$$\Big|\mathcal{C}_{s,s+1,\pm,i}^\varepsilon(R_1,R_2,\delta)\mathcal{T}_t^{s+1,\varepsilon}\varphi^{(s+1)}\Big|(Z_s) \leq C(d,s)\varepsilon^{d-1}R_2^{d+1}\big|\varphi^{(s+1)}\big|_{L^\infty(\mathcal{D}_{s+1}^\varepsilon)}$$

almost everywhere on $[n\delta,(n+1)\delta]\times\mathcal{D}_s^\varepsilon$, for all $0\leq n\leq T/\delta-1$, where δ has been chosen small enough so that it fulfills the condition (5.12) stated in Lemma 1 page 101, and such that T/δ is an integer.

Finally, one has shown that the truncated transport-collision operator associates to any function $\varphi^{(s+1)}$ of $L^\infty\big(\mathcal{D}_{s+1}^\varepsilon\big)$ a function of $L^\infty\big([0,T]\times\mathcal{D}_s^\varepsilon\big)$. Moreover, this operator is continuous on L^∞, and satisfies the explicit estimate :

$$\begin{aligned} &\big|\mathcal{C}_{s,s+1,\pm,i}^\varepsilon(R_1,R_2,\delta)\mathcal{T}_t^{s+1,\varepsilon}\varphi^{(s+1)}\big|_{L^\infty([0,T]\times\mathcal{D}_s^\varepsilon)} \\ &\qquad\qquad \leq C(d,s)\varepsilon^{d-1}R_2^{d+1}\big|\varphi^{(s+1)}\big|_{L^\infty(\mathcal{D}_{s+1}^\varepsilon)}, \end{aligned}$$

which is exactly the estimate (5.23) of the lemma. Lemma 2 is proved. □

5.1.3 Removing the truncation parameter related to the time variable

Limit of the sequence of operators for R_1, R_2 given, when δ goes to 0

One now considers two parameters δ,δ' such that :

$$0<\delta'<\delta\leq 1,$$

and the new truncated collision operator :

$$\mathcal{C}_{s,s+1,\pm,i}^\varepsilon(R_1,R_2,\delta',\delta)\mathcal{T}_t^{s+1,\varepsilon},$$

defined as :

$$\mathcal{C}_{s,s+1,\pm,i}^\varepsilon(R_1,R_2,\delta')\mathcal{T}_t^{s+1,\varepsilon}-\mathcal{C}_{s,s+1,\pm,i}^\varepsilon(R_1,R_2,\delta)\mathcal{T}_t^{s+1,\varepsilon}.$$

One then observes that :

$$\begin{aligned}
&\int_0^{\delta'} \int_{Z_s \in \mathcal{D}_s^\varepsilon} \left| \mathcal{C}^\varepsilon_{s,s+1,\pm,i}(R_1,R_2,\delta',\delta)\mathcal{T}_t^{s+1,\varepsilon}\varphi^{(s+1)} \right| \mathrm{d}Z_s \,\mathrm{d}t \\
&\quad = \int_0^{\delta'} \int_{Z_s \in \mathcal{D}_s^\varepsilon} \varepsilon^{d-1} \Bigg| \int_{\mathbb{S}_\omega^{d-1}\times\mathbb{R}^d_{v_{s+1}}} \big(\omega\cdot(v_{s+1}-v_i)\big)_\pm \mathcal{T}_t^{s+1,\varepsilon}\varphi^{(s+1)}(Z_s, x_i+\varepsilon\omega, v_{s+1}) \\
&\qquad \times \Bigg(\Bigg(\prod_{\substack{1\le j<k\le s+1\\(j,k)\neq(i,s+1)}} \mathbb{1}_{|x_j-x_k|>\varepsilon+\sqrt{2}\delta' R_2} \Bigg) \Bigg(\prod_{1\le l\le s+1} \mathbb{1}_{d(x_l,\Omega)>\varepsilon/2+\delta' R_2} \Bigg) \\
&\qquad - \Bigg(\prod_{\substack{1\le j<k\le s+1\\(j,k)\neq(i,s+1)}} \mathbb{1}_{|x_j-x_k|>\varepsilon+\sqrt{2}\delta R_2} \Bigg) \Bigg(\prod_{1\le l\le s+1} \mathbb{1}_{d(x_l,\Omega)>\varepsilon/2+\delta R_2} \Bigg) \Bigg) \\
&\qquad \times \mathbb{1}_{X_{s+1}\in B_{\mathbb{R}^{d(s+1)}}(0,R_1)} \mathbb{1}_{V_{s+1}\in B_{\mathbb{R}^{d(s+1)}}(0,R_2)} \,\mathrm{d}\omega\,\mathrm{d}v_{s+1} \Bigg| \,\mathrm{d}Z_s\,\mathrm{d}t.
\end{aligned}$$

One will now work to obtain a control on the L^1 norm of the difference

$$\mathcal{C}^\varepsilon_{s,s+1,\pm,i}(R_1,R_2,\delta',\delta)\mathcal{T}_t^{s+1,\varepsilon},$$

in order to prove that the original truncated transport-collision operator

$$\mathcal{C}^\varepsilon_{s,s+1,\pm,i}\mathcal{T}_t^{s+1,\varepsilon}$$

constitutes a Cauchy sequence in L^1 as δ varies and converges to zero. Notice that one will not be able to obtain such a result in the L^∞ topology. The result is the following.

Lemma 3 (L^1 norm of the difference of two truncated in time, position and velocity operators for L^∞ functions, with the same position and velocity cut-off parameters)**.** *Let s be a positive integer, $\varepsilon \le 1$ and T be two strictly positive numbers. Let R_1, R_2 be two strictly positive numbers, and $\delta' < \delta$ be two strictly positive numbers such that :*

$$\delta \le \delta_0(d,R_1,R_2) = \min_{1\le j\le d} \left[\frac{\varepsilon}{2R_2} \Big(\frac{d}{\binom{d}{j}} \Big)^{\frac{1}{j-1}} \right].$$

Then for every integer $1 \le i \le s$, and for every function $\varphi^{(s+1)} \in L^\infty\big(\mathcal{D}^\varepsilon_{s+1}\big)$, the function :

$$\mathcal{C}^\varepsilon_{s,s+1,\pm,i}(R_1,R_2,\delta',\delta)\mathcal{T}_t^{s+1,\varepsilon}\varphi^{(s+1)}$$

defined as

$$\begin{aligned}
&\mathcal{C}^\varepsilon{}_{s,s+1,\pm,i}(R_1,R_2,\delta',\delta)\mathcal{T}_t^{s+1,\varepsilon}\varphi^{(s+1)} \\
&\quad = \mathcal{C}^\varepsilon_{s,s+1,\pm,i}(R_1,R_2,\delta)\mathcal{T}_t^{s+1,\varepsilon}\varphi^{(s+1)} - \mathcal{C}^\varepsilon_{s,s+1,\pm,i}(R_1,R_2,\delta')\mathcal{T}_t^{s+1,\varepsilon}\varphi^{(s+1)},
\end{aligned}$$

verifies the following L^1 estimate :

$$\int_0^T \int_{Z_s \in \mathcal{D}_s^\varepsilon} \left| \mathcal{C}^\varepsilon_{s,s+1,\pm,i}(R_1, R_2, \delta', \delta) \mathcal{T}_t^{s+1,\varepsilon} \varphi^{(s+1)} \right| \mathrm{d}Z_s \, \mathrm{d}t$$
$$\leq C(d, s, \varepsilon, R_1, R_2) \delta T \left| \varphi^{(s+1)} \right|_{L^\infty}. \qquad (5.28)$$

Proof. First, thanks to Lemma 1 page 101, the difference of the transport-collision operators $\mathcal{C}^\varepsilon_{s,s+1,\pm,i}(R_1, R_2, \delta', \delta) \mathcal{T}_t^{s+1,\varepsilon} \varphi^{(s+1)}$ is well-defined and is integrable.
One will now try to simplify the notations. For every $\delta > 0$, one denotes by E_δ the following subset of $[0, T] \times \mathcal{D}_s^\varepsilon \times \mathbb{S}^{d-1} \times \mathbb{R}^d$:

$$E_\delta = \Big(\bigcap_{\substack{1 \leq j < k \leq s+1 \\ (j,k) \neq (i, s+1)}} \big\{ |x_j - x_k| > \varepsilon + \sqrt{2} \delta R_2 \big\} \Big)$$
$$\cap \Big(\bigcap_{1 \leq l \leq s+1} \big\{ d(x_l, \Omega) > \varepsilon/2 + \delta R_2 \big\} \Big),$$

(that is : E_δ is the subset of the domain of the truncated transport-collision operator in time before δ, encoding a condition on the particles, which prevents collisions or bouncings against the obstacle during the time interval $[0, \delta]$). One notices first, if $\delta' < \delta$, if one has $|x_j - x_k| > \varepsilon + \sqrt{2}\delta R_2$, then obviously $|x_j - x_k| > \varepsilon + \sqrt{2}\delta' R_2$, and similarly $d(x_l, \Omega) > \varepsilon/2 + \delta R_2$ implies $d(x_l, \Omega) > \varepsilon/2 + \delta' R_2$, so that the following inclusion holds :

$$E_\delta \subset E_{\delta'}.$$

In other words, the family of subsets $\big(E_\delta\big)_{\delta > 0}$ is decreasing with δ.
Secondly, the new truncated transport-collision operator

$$\mathcal{C}^\varepsilon_{s,s+1,\pm,i}(R_1, R_2, \delta', \delta) \mathcal{T}_t^{s+1,\varepsilon}$$

integrated over $[0, \delta'] \times \mathcal{D}_s^\varepsilon$ can be rewritten more simply as :

$$\int_0^{\delta'} \int_{Z_s \in \mathcal{D}_s^\varepsilon} \varepsilon^{d-1} \Bigg| \int_{\mathbb{S}^{d-1}_\omega \times \mathbb{R}^d_{v_{s+1}}} \big(\omega \cdot (v_{s+1} - v_i)\big)_\pm \mathcal{T}_t^{s+1,\varepsilon} \varphi^{(s+1)}(Z_s, x_i + \varepsilon\omega, v_{s+1}) \mathbb{1}_{E_{\delta'} \setminus E_\delta}$$
$$\times \mathbb{1}_{X_{s+1} \in B_{\mathbb{R}^{d(s+1)}}(0, R_1)} \mathbb{1}_{V_{s+1} \in B_{\mathbb{R}^{d(s+1)}}(0, R_2)} \, \mathrm{d}\omega \, \mathrm{d}v_{s+1} \Bigg| \mathrm{d}Z_s \, \mathrm{d}t.$$

An element of $E_{\delta'} \setminus E_\delta$ is such that all the particles are initially separated by a sufficiently large distance (here : $\varepsilon + \sqrt{2}\delta' R_2$) and are initially sufficiently far from the obstacle (here at a distance larger than $\varepsilon/2 + \delta' R_2$). On the other hand, there exists at least one particle close to the obstacle (here : at a distance smaller than $\varepsilon/2 + \delta R_2$), or a pair of particles which are close (here : at a distance smaller than $\varepsilon + \sqrt{2}\delta R_2$).

If a particle is close to the obstacle, say, the particle l_0 with $1 \leq l_0 \leq s+1$, one has that :

$$\varepsilon/2 + \delta' R_2 < d(x_{l_0}, \Omega) \leq \varepsilon/2 + \delta R_2.$$

Since in the setting of this work the obstacle Ω is the half space $\{x \in \mathbb{R}^d \ / \ x \cdot e_1 = 0\}$, this condition exactly means that :

$$\varepsilon/2 + \delta' R_2 < x_{l_0} \cdot e_1 \leq \varepsilon/2 + \delta R_2.$$

One wants here a control on the image by $S_s^{\pm}$ of the position of a particle l, with $l < s+1$, since one has already a control on the position of the particle $s+1$: this particle will remain close to the particle i.
Assuming therefore that l_0 verifies $l_0 < s+1$, it is clear that when $t \leq \delta'$, the backwards hard sphere flow coincides with the backwards free flow because they coincide on $[0, \delta]$, so that for $0 \leq t \leq \delta'$, one has :

$$\varepsilon/2 < (x_{l_0} - t v_{l_0}) \cdot e_1 \leq \varepsilon/2 + \delta R_2 + \delta' R_2.$$

Remembering that $|x_{l_0}| < R_1$, one gets that :

$$\widetilde{x}_{l_0} \in [\varepsilon/2, \varepsilon/2 + (\delta + \delta') R_2] \times B_{\mathbb{R}^{d-1}}(0, R_1 + \delta' R_2),$$

which is of course a subset of size bounded by :

$$C(d)\big(R_1 + \delta' R_2\big)^{d-1} (\delta + \delta') R_2 \leq C(d)\big(R_1 + \delta' R_2\big)^{d-1} R_2 \delta,$$

using the fact that $\delta' < \delta$, which bounds from above the size of the set in which x_{l_0} lies. On the contrary, if $l_0 = s+1$, that is if :

$$\varepsilon/2 + \delta' R_2 < x_{s+1} \cdot e_1 \leq \varepsilon/2 + \delta R_2,$$

one is able to deduce also a condition on the position of the particle i after being transported by $S_s^{\pm}$. Indeed, since one has by definition

$$x_{s+1} = x_i + \varepsilon \omega,$$

one obtains for every time $0 \leq t \leq \delta'$:

$$\varepsilon/2 \leq (x_i + \varepsilon\omega - t v_i) \cdot e_1 < \varepsilon/2 + \delta R_2 + \delta' R_2,$$

that is :

$$\varepsilon/2 - \varepsilon\omega \cdot e_1 \leq (x_i - t v_i) \cdot e_1 < \varepsilon/2 - \varepsilon\omega \cdot e_1 + (\delta + \delta') R_2.$$

One has to be careful here since the first inequality may be insufficient to conclude. Remembering that the initial configuration was taken in E'_δ, one also knows that :

$$d(x_i, \Omega) > \varepsilon/2 + \delta' R_2,$$

that is, for every $0 \leq t \leq \delta'$:

$$(x_i - tv_i) \cdot e_1 > \varepsilon/2.$$

Finally, one has shown that :

$$(x_i - tv_i) \cdot e_1 \in [\varepsilon/2 - \varepsilon\omega \cdot e_1, \varepsilon/2 - \varepsilon\omega \cdot e_1 + (\delta + \delta')R_2] \cap [\varepsilon/2, +\infty[.$$

The second part of the intersection of the right-hand side of the previous condition shows that the particle i is not too close to the obstacle before a time δ' (so that the transport of hard spheres coincides with $S_s^{\pm}$).
In any case, the quantity $(x_i - tv_i) \cdot e_1$ lies in an interval of size smaller than $(\delta + \delta')R_2$. The same argument as above, that is $x_i \in B_{\mathbb{R}^d}(0, R_1)$, enables to assert that $\widetilde{x}_i$ lies in a subset of the phase space of $s+1$ particles with a size controlled by

$$C(d)(R_1 + \delta' R_2)^{d-1}(\delta + \delta')R_2 \leq C(d)(R_1 + \delta' R_2)^{d-1} R_2 \delta.$$

One now assumes that a pair of particles are close to each other, say, the two particles j_0 and k_0 with $j_0 < k_0$, and $(j_0, k_0) \neq (i, s+1)$. There are here two possibilities. Either the particle k_0 is the particle $s+1$, or not. If $k_0 \neq s+1$, one has that :

$$\varepsilon + \sqrt{2}\delta' R_2 < |x_{j_0} - x_{k_0}| \leq \varepsilon + \sqrt{2}\delta R_2.$$

As above, one has to be careful. If both j_0 and k_0 are different from $s+1$, then again, after a time $t \leq \delta'$, the backwards hard sphere flow coincides with the backwards free flow, and moreover, one has on the one hand

$$\varepsilon < |(x_{j_0} - tv_{j_0}) - (x_{k_0} - tv_{k_0})|,$$

and on the other hand

$$|(x_{j_0} - tv_{j_0}) - (x_{k_0} - tv_{k_0})| \leq \varepsilon + \sqrt{2}\delta R_2 + \sqrt{2}\delta' R_2.$$

In other words, for all $0 \leq t \leq \delta'$, the position $x_{j_0} - tv_{j_0}$ belongs to the difference of the balls :

$$B_{\mathbb{R}^d}(x_{k_0} - tv_{k_0}, \varepsilon + \sqrt{2}(\delta + \delta')R_2) \backslash B_{\mathbb{R}^d}(x_{k_0} - tv_{k_0}, \varepsilon),$$

which has size

$$C(d)\big((\varepsilon + \sqrt{2}(\delta + \delta')R_2)^d - \varepsilon^d\big) = C(d)\sum_{p=1}^{d}\binom{d}{p}\varepsilon^{d-p}\big(\sqrt{2}R_2\big)^p(\delta + \delta')^p,$$

which can be bounded from above when $\varepsilon \leq 1$ by :

$$C(d)\varepsilon\sum_{p=1}^{d}\binom{d}{p}\big(\sqrt{2}R_2\big)^p(\delta + \delta')^p.$$

Then when δ is small enough, that is smaller than :

$$\delta_0(d, R_2) = \min_{1\le p\le d} \Big[\frac{d}{\binom{d}{p}} \Big(\frac{1}{\sqrt{2}R_2} \Big)^{p-1} \Big], \tag{5.29}$$

it can be again be bounded from above by :

$$C(d)\varepsilon R_2(\delta + \delta') \le C(d)\varepsilon R_2 \delta.$$

If now $k_0 = s+1$, then one deduces that $j_0 \neq i$ since by hypothesis $(j_0, k_0) \neq (i, s+1)$. One can get a control on the position of the particle j_0, depending on the position of the particle $s+1 = k_0$, depending in turn on the position of the particle i. More explicitly, one shows easily, as above, that

$$(x_{j_0} - tv_{j_0}) \in B_{\mathbb{R}^d}\big(x_{s+1} - tv_{s+1}, \varepsilon + \sqrt{2}(\delta+\delta')R_2\big) \backslash B_{\mathbb{R}^d}\big(x_{s+1} - tv_{s+1}, \varepsilon\big),$$

and the control of the position $\widetilde{x}_{s+1}$ of the particle $s+1$ follows.

As before, and in any case (that is if this is the case of a particle close to the obstacle, or if this is the case of two particles close to each other), one knows moreover that for all $t \le \delta'$:

$$\varepsilon \le \big|(x_{s+1} - tv_{s+1}) - (x_i - tv_i)\big| \le \varepsilon + \sqrt{2}\delta' R_2$$

in the pre-collisional case, and

$$\varepsilon \le \big|(x_{s+1} - tv'_{s+1}) - (x_i - tv'_i)\big| \le \varepsilon + \sqrt{2}\delta' R_2$$

in the post-collisional case. One recalls that the factor $\sqrt{2}$ comes from the bound on the velocities, namely $\big|V_{s+1}\big| \le R_2$, which implies :

$$\begin{aligned} |v_{s+1} - v_i|^2 = |v_{s+1}|^2 - 2v_i \cdot v_{s+1} + |v_i|^2 &\le |v_{s+1}|^2 + 2|v_i||v_{s+1}| + |v_i|^2 \\ &\le 2|v_{s+1}|^2 + |v_i|^2 \le 2R_2^2. \end{aligned}$$

In summary, as above, if one denotes by $\mathcal{I}^{\varepsilon,\pm}_{s,i}(R_1, R_2, \delta', \delta)$ the image by $S^{\pm}_s$ of the domain of integration :

$$\begin{aligned} [0,\delta'] \times \Big(E_{\delta'} \backslash E_\delta \cap \big\{|X_{s+1}| < R_1\big\} \cap \big\{|V_{s+1}| < R_2\big\} \Big) \\ \subset [0,\delta'] \times \mathcal{D}^\varepsilon_s \times \mathbb{S}^{d-1} \times \mathbb{R}^d, \end{aligned}$$

in any case (pre or post-collisional, that is either for $\mathcal{C}^\varepsilon_{s,s+1,-,i}$ or for $\mathcal{C}^\varepsilon_{s,s+1,+,i}$), one has then

$$\begin{aligned} \mathcal{I}^{\varepsilon,\pm}_{s,i}(R_1, R_2, \delta', \delta) \in B_{\mathbb{R}^{ds}}\big(0, R_1 + \delta' R_2\big) \times B_{\mathbb{R}^{ds}}\big(0, R_2\big) \\ \times B_{\mathbb{R}^d}\big(0, R_1 + \delta' R_2\big) \times B_{\mathbb{R}^d}\big(0, R_2\big), \end{aligned}$$

with in addition, for every $\widetilde{Z}_{s+1}$ denoted as :

$$\widetilde{Z}_{s+1} = (\widetilde{x}_1, \dots, \widetilde{x}_{k_0}, \dots, \widetilde{x}_{l_0}, \dots, \widetilde{V}_s, \widetilde{x}_{s+1}, \widetilde{v}_{s+1}) \in \mathcal{I}^{\varepsilon,\pm}_{s,i}(R_1, R_2, \delta', \delta),$$

the two following extra conditions :

- either there exist an integer $1 \leq l_0 \leq s$ and a real number α such that :

$$\widetilde{x}_{l_0} \cdot e_1 \in [\varepsilon/2 + \alpha, \varepsilon/2 + \alpha + (\delta + \delta')R_2] \cap [\varepsilon/2, +\infty[,$$

  or there exist $1 \leq j_0 < k_0 \leq s+1$ with $j_0 \neq i$ if $k_0 = s+1$ such that :

$$\widetilde{x}_{j_0} \in \big(B_{\mathbb{R}^d}(\widetilde{x}_{k_0}, \varepsilon + \sqrt{2}(\delta + \delta')R_2) \backslash B_{\mathbb{R}^d}(\widetilde{x}_{k_0}, \varepsilon)\big),$$

- and, in any case :

$$\widetilde{x}_{s+1} \in \big(B(\widetilde{x}_i, \varepsilon + \sqrt{2}\delta' R_2) \backslash B(\widetilde{x}_i, \varepsilon)\big).$$

One obtains then that :

$$\begin{aligned}
\big|\mathcal{I}^{\varepsilon,\pm}_{s,i}(R_1, R_2, \delta', \delta)\big| \leq & \overbrace{\big(C_1(d(s-1))(R_1 + \delta' R_2)^{d(s-1)}\big)}^{\text{conditions on } \widetilde{x}_1, \ldots, \widetilde{x}_{l_0-1}, \widetilde{x}_{l_0+1}, \ldots, \widetilde{x}_s} \\
& \times \overbrace{\big(C_2(d-1)(R_1 + \delta' R_2)^{d-1}(\delta' + \delta)R_2\big)}^{\text{condition on } \widetilde{x}_{l_0}} \\
& \times \overbrace{\big(C_3(d)\big((\varepsilon + \sqrt{2}\delta' R_2)^d - \varepsilon^d\big)\big)}^{\text{condition on } \widetilde{x}_{s+1}} \overbrace{C_4(d(s+1))R_2^{d(s+1)}}^{\text{condition on } \widetilde{V}_{s+1}} \\
& + \overbrace{\big(C_1(d(s-1))(R_1 + \delta' R_2)^{d(s-1)}\big)}^{\text{conditions on } \widetilde{x}_1, \ldots, \widetilde{x}_{j_0-1}, \widetilde{x}_{j_0+1}, \ldots, \widetilde{x}_s} \\
& \times \overbrace{\big(C_3(d)\big((\varepsilon + \sqrt{2}(\delta' + \delta)R_2)^d - \varepsilon^d\big)\big)}^{\text{condition on } \widetilde{x}_{j_0}} \\
& \times \overbrace{\big(C_3(d)\big((\varepsilon + \sqrt{2}\delta' R_2)^d - \varepsilon^d\big)\big)}^{\text{condition on } \widetilde{x}_{s+1}} \overbrace{C_4(d(s+1))R_2^{d(s+1)}}^{\text{condition on } \widetilde{V}_{s+1}},
\end{aligned}$$

or again

$$\begin{aligned}
\big|\mathcal{I}^{\varepsilon,\pm}_{s,i}(R_1, R_2, \delta', \delta)\big| \leq & \, C(d, s, \varepsilon)\delta'(\delta' + \delta)R_2^{d(s+1)+2}(R_1 + \delta' R_2)^{d(s-1)+d-1} \\
& + C(d, s, \varepsilon)\delta'(\delta' + \delta)R_2^{d(s+1)+2}(R_1 + \delta' R_2)^{d(s-1)}
\end{aligned}$$

for δ small enough (with an explicit bound on the largest δ verifying the inequality above, depending on d, s and ε, given by the condition (5.29)), that is :

$$\big|\mathcal{I}^{\varepsilon,\pm}_{s,i}(R_1, R_2, \delta', \delta)\big| \leq C(d, s, \varepsilon)\delta'\delta(R_1 + R_2)^{ds-1}R_2^{d(s+1)+2},$$

so that :

$$\begin{aligned}
\int_0^{\delta'} \int_{Z_s \in \mathcal{D}^\varepsilon_s} & \Big|\mathcal{C}^\varepsilon_{s,s+1,\pm,i}(R_1, R_2, \delta', \delta)\mathcal{T}^{s+1,\varepsilon}_t \varphi^{(s+1)}\Big| \, \mathrm{d}Z_s \, \mathrm{d}t \\
& \leq C(d, s, \varepsilon, R_1, R_2)\delta'\delta\big|\varphi^{(s+1)}\big|_{L^\infty\left(\mathcal{D}^\varepsilon_{s+1}\right)}.
\end{aligned}$$

Proceeding as above, that is cutting the time interval $[0,T]$ into pieces of size δ', controlling the L^∞ norm of the hard sphere flow applied to $\varphi^{(s+1)}$ and summing the L^1 norm of the truncated transport-collision operator on all subintervals of size δ', one immediately gets :

$$\int_0^T \int_{Z_s \in \mathcal{D}_s^\varepsilon} \left| \mathcal{C}^\varepsilon_{s,s+1,\pm,i}(R_1,R_2,\delta',\delta)\mathcal{T}_t^{s+1,\varepsilon}\varphi^{(s+1)} \right| \mathrm{d}Z_s \, \mathrm{d}t \\ \leq C(d,s,\varepsilon,R_1,R_2)\delta T \left|\varphi^{(s+1)}\right|_{L^\infty}. \tag{5.30}$$

Lemma 3 is proved. □

One is now able to define the limit of the truncated transport-collision operator, when the condition on the cut-off related to time is relaxed, that is when δ goes to zero.

Lemma 4 (Definition of the truncated in position and velocity transport-collision operator, and L^1, L^∞ regularity for L^∞ functions)**.** *Let s be a positive integer, $\varepsilon \leq 1$ and T be two strictly positive numbers. Let also R_1, R_2 be two strictly positive numbers.*
Then for every integer $1 \leq i \leq s$, and for every function $\varphi^{(s+1)} \in L^\infty\big(\mathcal{D}^\varepsilon_{s+1}\big)$, the sequence of functions :

$$\Big(\mathcal{C}^\varepsilon_{s,s+1,\pm,i}(R_1,R_2,\delta)\mathcal{T}_t^{s+1,\varepsilon}\varphi^{(s+1)}\Big)_{\delta>0}$$

is a converging sequence in $L^1\big([0,T]\times\mathcal{D}^\varepsilon_s\big)$. Moreover, its limit, denoted by :

$$\mathcal{C}^\varepsilon_{s,s+1,\pm,i}(R_1,R_2)\mathcal{T}_t^{s+1,\varepsilon}\varphi^{(s+1)},$$

lies in $L^\infty\big([0,T]\times\mathcal{D}^\varepsilon_s\big)$, and verifies :

- *the following L^∞ estimate :*

$$\left|\mathcal{C}^\varepsilon_{s,s+1,\pm,i}(R_1,R_2)\mathcal{T}_t^{s+1,\varepsilon}\varphi^{(s+1)}\right|_{L^\infty\left([0,T]\times\mathcal{D}^\varepsilon_s\right)} \\ \leq C(d,s)\varepsilon^{d-1}R_2^{d+1}\left|\varphi^{(s+1)}\right|_{L^\infty\left(\mathcal{D}^\varepsilon_{s+1}\right)}, \tag{5.31}$$

- *and the following convergence result :*

$$\mathcal{C}^\varepsilon_{s,s+1,\pm,i}(R_1,R_2,\delta)\mathcal{T}_t^{s+1,\varepsilon}\varphi^{(s+1)} \underset{\delta\to 0}{\overset{*}{\rightharpoonup}} \mathcal{C}^\varepsilon_{s,s+1,\pm,i}(R_1,R_2)\mathcal{T}_t^{s+1,\varepsilon}\varphi^{(s+1)} \tag{5.32}$$

  in $L^\infty\big([0,T]\times\mathcal{D}^\varepsilon_s\big)$.

Proof. First of all, for $\delta > 0$ small enough the sequence :

$$\Big(\mathcal{C}^\varepsilon_{s,s+1,\pm,i}(R_1,R_2,\delta)\mathcal{T}_t^{s+1,\varepsilon}\varphi^{(s+1)}\Big)_{\delta>0}$$

is well-defined in $L^1\big([0,T]\times\mathcal{D}_s^\varepsilon\big)$ thanks to Lemma 1 page 101. Then, the inequality (5.28) stated in Lemma 3 means that this sequence is a Cauchy sequence in $L^1\big([0,T]\times\mathcal{D}_s^\varepsilon\big)$, so thanks to the Riesz-Fischer theorem it converges as δ goes to zero towards a limit belonging also to $L^1\big([0,T]\times\mathcal{D}_s^\varepsilon\big)$, which will be denoted :

$$\mathcal{C}^\varepsilon_{s,s+1,\pm,i}(R_1,R_2)\mathcal{T}_t^{s+1,\varepsilon}\varphi^{(s+1)}.$$

One will now show that this limit belongs in fact also to the space $L^\infty\big([0,T]\times\mathcal{D}_s^\varepsilon\big)$. It is a direct consequence of the inequality (5.22) stated in Lemma 2 page 107, which holds in the limit $\delta\to 0$. Indeed, for all measurable subsets $A\subset[0,T]\times\mathcal{D}_s^\varepsilon$ of finite measure, one can write :

$$\begin{aligned}
&\int\limits_{A\subset[0,T]\times\mathcal{D}_s^\varepsilon}\Big|\mathcal{C}^\varepsilon_{s,s+1,\pm,i}(R_1,R_2)\mathcal{T}_t^{s+1,\varepsilon}\varphi^{(s+1)}\Big|\,\mathrm{d}Z_s\,\mathrm{d}t\\
&\qquad\le \int\limits_{A\subset[0,T]\times\mathcal{D}_s^\varepsilon}\Big|\mathcal{C}^\varepsilon_{s,s+1,\pm,i}(R_1,R_2)\mathcal{T}_t^{s+1,\varepsilon}\varphi^{(s+1)}\\
&\qquad\qquad\qquad - \mathcal{C}^\varepsilon_{s,s+1,\pm,i}(R_1,R_2,\delta)\mathcal{T}_t^{s+1,\varepsilon}\varphi^{(s+1)}\Big|\,\mathrm{d}Z_s\,\mathrm{d}t\\
&\qquad\quad + \int\limits_{A\subset[0,T]\times\mathcal{D}_s^\varepsilon}\Big|\mathcal{C}^\varepsilon_{s,s+1,\pm,i}(R_1,R_2,\delta)\mathcal{T}_t^{s+1,\varepsilon}\varphi^{(s+1)}\Big|\,\mathrm{d}Z_s\,\mathrm{d}t\\
&\qquad\le \Big|\mathcal{C}^\varepsilon_{s,s+1,\pm,i}(R_1,R_2)\mathcal{T}_t^{s+1,\varepsilon}\varphi^{(s+1)}\\
&\qquad\qquad\qquad - \mathcal{C}^\varepsilon_{s,s+1,\pm,i}(R_1,R_2,\delta)\mathcal{T}_t^{s+1,\varepsilon}\varphi^{(s+1)}\Big|_{L^1([0,T]\times\mathcal{D}_s^\varepsilon)}\\
&\qquad\quad + C(d,s)\varepsilon^{d-1}R_2^{d+1}|A|\big|\varphi^{(s+1)}\big|_{L^\infty(\mathcal{D}^\varepsilon_{s+1})},
\end{aligned}$$

and the first term in this inequality can be chosen as small as one wants choosing δ small enough, so that :

$$\begin{aligned}
&\int\limits_{A\subset[0,T]\times\mathcal{D}_s^\varepsilon}\Big|\mathcal{C}^\varepsilon_{s,s+1,\pm,i}(R_1,R_2)\mathcal{T}_t^{s+1,\varepsilon}\varphi^{(s+1)}\Big|\,\mathrm{d}Z_s\,\mathrm{d}t\\
&\qquad\qquad\le C(d,s)\varepsilon^{d-1}R_2^{d+1}|A|\big|\varphi^{(s+1)}\big|_{L^\infty(\mathcal{D}^\varepsilon_{s+1})}.
\end{aligned}\tag{5.33}$$

Now, following exactly the same path as above to obtain the L^∞ control of the first truncated operator (in time, position and velocity), one can now easily show that $\mathcal{C}^\varepsilon_{s,s+1,\pm,i}(R_1,R_2)\mathcal{T}_t^{s+1,\varepsilon}\varphi^{(s+1)}$ lies in $L^\infty\big([0,T]\times\mathcal{D}_s^\varepsilon\big)$ and :

$$\begin{aligned}
&\big|\mathcal{C}^\varepsilon_{s,s+1,\pm,i}(R_1,R_2)\mathcal{T}_t^{s+1,\varepsilon}\varphi^{(s+1)}\big|_{L^\infty([0,T]\times\mathcal{D}_s^\varepsilon)}\\
&\qquad\le C(d,s)\varepsilon^{d-1}R_2^{d+1}\big|\varphi^{(s+1)}\big|_{L^\infty(\mathcal{D}^\varepsilon_{s+1})}.
\end{aligned}\tag{5.34}$$

Finally it remains to show that this L^1 limit, which is also L^∞, is indeed the limit of the sequence $\big(\mathcal{C}^\varepsilon_{s,s+1,\pm,i}(R_1,R_2,\delta)\mathcal{T}_t^{s+1,\varepsilon}\varphi^{(s+1)}\big)_{\delta>0}$ in some topology on

$L^\infty\big([0,T]\times\mathcal{D}_s^\varepsilon\big)$. It is based on the following general argument.
One has obtained that the sequence

$$\big(\mathcal{C}^\varepsilon_{s,s+1,\pm,i}(R_1,R_2,\delta)\mathcal{T}_t^{s+1,\varepsilon}\varphi^{(s+1)}\big)_{\delta>0}$$

is :

- bounded uniformly in δ in L^∞ (this is the control (5.23) page 107),
- a converging sequence in L^1 (this is the first part of Lemma 4),
- its limit belongs also to L^∞ (this is the control (5.31) of Lemma 4).

Those three hypotheses are enough to obtain in the general case a weak-$*$ convergence in L^∞. Indeed, let (X,μ) be a mesured space. For any $g\in L^1(X)$, and any converging sequence $\big(f_n\big)_{n\geq 0}\in\big(L^1(X)\big)^{\mathbb{N}}$ towards a certain function $f\in L^1(X)$, with

$$\big|f\big|_{L^\infty}\leq F,\ \forall n\in\mathbb{N},\ \big|f_n\big|_{L^\infty}\leq F$$

for some strictly positive number F, one knows that

$$\int_X \mathbb{1}_{|g|>M}\,g\,\mathrm{d}\mu\underset{M\to+\infty}{\longrightarrow}0$$

(in fact, one knows that the rate of convergence towards zero of this quantity with respect to M is at least of order $1/M$, which is nothing more but the Markov inequality, but here in fact one does not need a quantitative rate of convergence to obtain the weak-$*$ convergence). Therefore, one writes :

$$\begin{aligned}\Big|\int_X g(f_n-f)\,\mathrm{d}\mu\Big|&\leq\int_X\big|g(f_n-f)\big|\,\mathrm{d}\mu\\&\leq\int_X\mathbb{1}_{|g|\leq M}\big|g(f_n-f)\big|\,\mathrm{d}\mu+\int_X\mathbb{1}_{|g|>M}\big|g(f_n-f)\big|\,\mathrm{d}\mu\\&\leq M\int_X\big|f_n-f\big|\,\mathrm{d}\mu+2F\int_X\mathbb{1}_{|g|>M}\big|g\big|\,\mathrm{d}\mu.\end{aligned}$$

Choosing M large enough, the second term of the upper bound found just above can be chosen as small as one wants. Therefore, choosing the index n of the sequence large enough, the first term can be chosen also as small as one wants, hence the weak-$*$ convergence in L^∞. Lemma 4 is proved.
One notes that the third hypothesis providing the general result (about the limit belonging to L^∞) is in fact a consequence of the two first hypotheses. Indeed, if a sequence of functions converges in L^1 towards some limit, it is possible to extract a subsequence which converges almost everywhere towards this limit. Therefore, each term of the subsequence is almost everywhere bounded by hypothesis, and so is the limit obtained previously in L^1, which turns out to be therefore also in L^∞. $\square$

Remark 9. *In the last point of the proof of Lemma 4 concerning the weak-$*$ convergence in L^∞, one followed a fast and general argument.*
The reader may find in [34][6] another argument to obtain this weak-$*$ convergence, based on the Markov inequality. In [34], the argument is only sketched. Although all the ideas were presented in this reference, one proposes here for the sake of completeness a detailed version of this proof.

For $\delta > 0$ small enough, one introduces the set :

$$I_\delta = \Big\{ (t, Z_s) \in [0,T] \times \mathcal{D}_s^\varepsilon \ / \\ \Big| \mathcal{C}^\varepsilon_{s,s+1,\pm,i}(R_1,R_2,\delta)\mathcal{T}_t^{s+1,\varepsilon}\varphi^{(s+1)} - \mathcal{C}^\varepsilon_{s,s+1,\pm,i}(R_1,R_2)\mathcal{T}_t^{s+1,\varepsilon}\varphi^{(s+1)} \Big| > \sqrt{\delta} \ \Big\}, \tag{5.35}$$

which is of course measurable since the truncated transport-collision operators $\mathcal{C}^\varepsilon_{s,s+1,\pm,i}(R_1,R_2,\delta)\mathcal{T}_t^{s+1,\varepsilon}\varphi^{(s+1)}$ applied to $\varphi^{(s+1)}$ and the limit of this sequence of truncated transport-collision operators $\mathcal{C}^\varepsilon_{s,s+1,\pm,i}(R_1,R_2)\mathcal{T}_t^{s+1,\varepsilon}\varphi^{(s+1)}$ applied to $\varphi^{(s+1)}$ are measurable functions, and in addition the set I_δ is of finite measure since those functions are integrable.
Using the Markov inequality, one knows that the size of I_δ can be controlled as follows :

$$\big|I_\delta\big| \leq \frac{1}{\sqrt{\delta}} \Big| \mathcal{C}^\varepsilon_{s,s+1,\pm,i}(R_1,R_2,\delta)\mathcal{T}_t^{s+1,\varepsilon}\varphi^{(s+1)} \\ - \mathcal{C}^\varepsilon_{s,s+1,\pm,i}(R_1,R_2)\mathcal{T}_t^{s+1,\varepsilon}\varphi^{(s+1)} \Big|_{L^1([0,T]\times\mathcal{D}_s^\varepsilon)},$$

and using the inequality (5.28) *page 112, which holds in the limit $\delta' \to 0$, one gets*

$$\big|I_\delta\big| \leq C(d,s,\varepsilon,R_1,R_2,T)\sqrt{\delta}\big|\varphi^{(s+1)}\big|_{L^\infty(\mathcal{D}^\varepsilon_{s+1})}. \tag{5.36}$$

Now, let f be an element of $L^1\big([0,T]\times\mathcal{D}_s^\varepsilon\big)$. Since the integrand of the following integral is a product of a L^1 function with a L^∞ one, it is well defined, and one

[6] See Subsection 5.1.3 "Removing the truncation", in the version of the article with Section "Rigorous formulation of the BBGKY hierarchy".

has :

$$\Big| \int_{[0,T]\times\mathcal{D}_s^\varepsilon} \Big(\mathcal{C}^\varepsilon_{s,s+1,\pm,i}(R_1,R_2,\delta)\mathcal{T}_t^{s+1,\varepsilon}\varphi^{(s+1)} - \mathcal{C}^\varepsilon_{s,s+1,\pm,i}(R_1,R_2)\mathcal{T}_t^{s+1,\varepsilon}\varphi^{(s+1)}\Big) f \,\mathrm{d}Z_s\,\mathrm{d}t\Big|$$

$$\leq \Big|\mathcal{C}^\varepsilon_{s,s+1,\pm,i}(R_1,R_2,\delta)\mathcal{T}_t^{s+1,\varepsilon}\varphi^{(s+1)} - \mathcal{C}^\varepsilon_{s,s+1,\pm,i}(R_1,R_2)\mathcal{T}_t^{s+1,\varepsilon}\varphi^{(s+1)}\Big|_{L^\infty([0,T]\times\mathcal{D}_s^\varepsilon)} \int_{I_\delta} \big|f\big|\,\mathrm{d}Z_s\,\mathrm{d}t$$

$$+ \sqrt{\delta}\int_{(I_\delta)^c} \big|f\big|\,\mathrm{d}Z_s\,\mathrm{d}t$$

$$\leq \Big(C(d,s)\varepsilon^{d-1}R_2^{d+1}\big|\varphi^{(s+1)}\big|_{L^\infty(\mathcal{D}^\varepsilon_{s+1})}\int_{I_\delta}\big|f\big|\Big) + \Big(\sqrt{\delta}\big|f\big|_{L^1([0,T]\times\mathcal{D}_s^\varepsilon)}\Big),$$

thanks to the two similar inequalities (5.23) *and* (5.34) *which control the* L^∞ *norm of* $\mathcal{C}^\varepsilon_{s,s+1,\pm,i}(R_1,R_2)\mathcal{T}_t^{s+1,\varepsilon}$*, and since* f *is integrable and the size of* I_δ *goes to zero as shown by* (5.36)*, one has :*

$$\int_{I_\delta}\big|f\big| \underset{\delta\to 0}{\longrightarrow} 0,$$

which finishes to show that :

$$\mathcal{C}^\varepsilon_{s,s+1,\pm,i}(R_1,R_2,\delta)\mathcal{T}_t^{s+1,\varepsilon}\varphi^{(s+1)} \underset{\delta\to 0}{\overset{*}{\rightharpoonup}} \mathcal{C}^\varepsilon_{s,s+1,\pm,i}(R_1,R_2)\mathcal{T}_t^{s+1,\varepsilon}\varphi^{(s+1)}$$

in $L^\infty\big([0,T]\times\mathcal{D}_s^\varepsilon\big)$.

Defining the truncated transport-collision operators for time-dependent functions, when R_1 and R_2 given

So far, one has defined $\mathcal{C}^\varepsilon_{s,s+1,\pm,i}(R_1,R_2)\mathcal{T}_t^{s+1,\varepsilon}$ on L^∞ functions which do not depend on time. In the following, it will be useful to define this operator on time-dependent functions. More precisely, one will show that it can be defined on $\mathcal{C}\big([0,T],L^\infty\big(\mathcal{D}^\varepsilon_{s+1}\big)\big)$, taking its values in $L^\infty\big([0,T]\times\mathcal{D}_s^\varepsilon\big)$, with additional controls. Those controls will be necessary in the sequel, in order to obtain, after an integration in time, an operator from $\mathcal{C}\big([0,T],L^\infty\big(\mathcal{D}^\varepsilon_{s+1}\big)\big)$ to $\mathcal{C}\big([0,T],L^\infty\big(\mathcal{D}^\varepsilon_{s}\big)\big)$, in which it is possible to use a fixed point argument.

The first step consists in defining the truncated transport-collision operator for piecewise constant in time functions, taking their values in $L^\infty\big(\mathcal{D}^\varepsilon_{s+1}\big)$, in a way which obviously extends the definition for constant in time functions, as it was done in Lemma 1 page 101. The piecewise constant in time functions are dense into $\mathcal{C}\big([0,T],L^\infty\big(\mathcal{D}^\varepsilon_{s+1}\big)\big)$, so that in the sequel the definition of the truncated transport-collision operator will be extended by continuity to this space.

One can wonder why is it mandatory to use piecewise constant in time functions. The reason is the following. The definition of the transport-collision operator relies on the change of variable induced by the map S_s^- (or S_s^+, depending on the sign $\pm$ of $\mathcal{C}^\varepsilon_{s,s+1,\pm,i}$), see Propositions 3 page 92 and 4 page 98 (for the definition of those maps and the study of the change of variable). The domain of this map is

$$\mathcal{D}_s^\varepsilon \times [0,\delta] \times \mathbb{S}^{d-1} \times \mathbb{R}^d,$$

and in particular any element of this domain is a product of an element Z_s of the phase space of s particles with a time t, and with two other elements which do not matter here. This time coordinate does not exist anymore in the image of those maps S_s^- and S_s^+, since the image is a subset of the phase space of $s+1$ particles. Therefore, one cannot perform directly this change of variable for a time-dependent function.

Lemma 5 (Definition of the truncated in time, position and velocity transport–collision operator, and L^1, weak L^∞ and L^∞ regularity for piecewise constant in time functions)**.** *Let s be a positive integer, $\varepsilon \leq 1$ and T be two strictly positive numbers. Let R_1, R_2 be two strictly positive numbers, and δ be a strictly positive number such that :*

$$\delta \leq \delta_0(d,\varepsilon,R_2) = \min_{1\leq j\leq d} \left[\frac{\varepsilon}{2R_2} \Big(\frac{d}{\binom{d}{j}} \Big)^{\frac{1}{j-1}} \right]. \tag{5.37}$$

Then for every integer $1 \leq i \leq s$, and for every function

$$\varphi_{PC}^{(s+1)} \in L^\infty\big([0,T], \big(\mathcal{D}_{s+1}^\varepsilon\big)\big)$$

such that

$$\varphi_{PC}^{(s+1)} = \sum_{p=1}^{P} \mathbb{1}_{[t_{p-1},t_p[} \alpha_p$$

with $\alpha_p \in L^\infty\big(\mathcal{D}_{s+1}^\varepsilon\big)$, the function :

$$(t, Z_s) \mapsto \mathcal{C}^\varepsilon_{s,s+1,\pm,i}(R_1,R_2,\delta)\mathcal{T}_t^{s+1,\varepsilon}\varphi_{PC}^{(s+1)},$$

defined as :

$$\begin{aligned}\mathcal{C}^\varepsilon_{s,s+1,\pm,i}(R_1,R_2,\delta)&\mathcal{T}_t^{s+1,\varepsilon}\varphi_{PC}^{(s+1)}(t,Z_s)\\ &= \sum_{p=1}^{P} \mathbb{1}_{[t_{p-1},t_p[} \mathcal{C}^\varepsilon_{s,s+1,\pm,i}(R_1,R_2,\delta)\mathcal{T}_t^{s+1,\varepsilon}\alpha_p\end{aligned}$$

can be defined in $L^1 \cap L^\infty\big([0,T]\times\mathcal{D}_s^\varepsilon\big)$. Moreover, it satisfies the two following estimates :

- *control of the L^1 norm :*

$$\Big|\mathcal{C}^{\varepsilon}_{s,s+1,\pm,i}(R_1,R_2,\delta)\mathcal{T}^{s+1,\varepsilon}_t\varphi^{(s+1)}_{PC}\Big|_{L^1([0,T]\times\mathcal{D}^{\varepsilon}_s)}$$
$$\leq C(d,s)\varepsilon^{d-1}T(R_1+\delta R_2)^{ds}R_2^{d(s+1)+1}\big|\varphi^{(s+1)}_{PC}\big|_{L^\infty([0,T],L^\infty(\mathcal{D}^{\varepsilon}_{s+1}))}, \tag{5.38}$$

- *control of the L^∞ norm :*

$$\Big|\mathcal{C}^{\varepsilon}_{s,s+1,\pm,i}(R_1,R_2,\delta)\mathcal{T}^{s+1,\varepsilon}_t\varphi^{(s+1)}_{PC}\Big|_{L^\infty([0,T]\times\mathcal{D}^{\varepsilon}_s)}$$
$$\leq C(d,s)\varepsilon^{d-1}R_2^{d+1}\big|\varphi^{(s+1)}_{PC}\big|_{L^\infty([0,T],L^\infty(\mathcal{D}^{\varepsilon}_{s+1}))}. \tag{5.39}$$

Proof. Let $\varphi^{(s+1)}_{PC}\in L^\infty\big([0,T],\big(\mathcal{D}^{\varepsilon}_{s+1}\big)\big)$ be a piecewise constant in time function, that is :

$$\varphi^{(s+1)}_{PC}=\sum_{p=1}^{P}\mathbb{1}_{[t_{p-1},t_p[}\alpha_p,$$

with $\alpha_p\in L^\infty\big(\mathcal{D}^{\varepsilon}_{s+1}\big)$ for all $1\leq p\leq P$, and $t_0=0, t_P=T$, $\big(t_p\big)_{1\leq p\leq P}$ being an increasing sequence of strictly positive numbers. For a time-dependent, piecewise constant function $\varphi^{(s+1)}_j$, the following function is obviously still defined and integrable :

$$\sum_{p=1}^{P}\mathbb{1}_{[t_{p-1},t_p[}\mathcal{C}^{\varepsilon}_{s,s+1,\pm,i}(R_1,R_2,\delta)\mathcal{T}^{s+1,\varepsilon}_t\alpha_p.$$

Indeed, Lemma 1 page 101 asserts that for all $1\leq p\leq P$, each function

$$\mathcal{C}^{\varepsilon}_{s,s+1,\pm,i}(R_1,R_2,\delta)\mathcal{T}^{s+1,\varepsilon}_t\alpha_p$$

is well-defined and integrable, and $\mathbb{1}_{[t_{p-1},t_p[}$ (which is an abuse of notation for $\mathbb{1}_{[t_{p-1},t_p[\times\mathcal{D}^{\varepsilon}_s}$) is of course a L^∞ function.
One now obtains the announced L^1 control. One has :

$$\int_0^\delta\int_{\mathcal{D}^{\varepsilon}_s}\int_{\mathbb{S}^{d-1}_\omega}\int_{\mathbb{R}^d_{v_{s+1}}}\varepsilon^{d-1}\big(\omega\cdot(v_{s+1}-v_i)\big)_\pm\Big|\mathcal{T}^{s+1,\varepsilon}_t\Big(\sum_{p=1}^{P}\alpha_p\mathbb{1}_{[t_{p-1},t_p[}\Big)\Big|$$
$$\times\,\mathbb{1}_{\mathfrak{D}}\,\mathrm{d}v_{s+1}\,\mathrm{d}\omega\,\mathrm{d}Z_s\,\mathrm{d}t$$
$$=\sum_{p=1}^{P}\int_{[0,\delta]\cap[t_{p-1},t_p[}\int_{\mathcal{D}^{\varepsilon}_s}\int_{\mathbb{S}^{d-1}_\omega}\int_{\mathbb{R}^d_{v_{s+1}}}\varepsilon\big(\omega\cdot(v_{s+1}-v_i)\big)_\pm\big|\alpha_p\circ S^\pm_s\big|$$
$$\times\,\mathbb{1}_{\mathfrak{D}}\,\mathrm{d}v_{s+1}\,\mathrm{d}\omega\,\mathrm{d}Z_s\,\mathrm{d}t,$$

with $S^\pm_s$ defined and studied in Propositions 3 page 92 and 4 page 98, and $\mathfrak{D}$ denoting the domain of the truncated in time, position and velocity collision

operator, as defined in (5.1), page 89. The fact that $S_s^{\pm}$ is one-to-one on its domain $[0,\delta]\times\mathcal{D}_s^{\varepsilon}\times\mathbb{S}_{\omega}^{d-1}\times\mathbb{R}_{v_{s+1}}^{d}$ enables to write that

$$\mathcal{I}_{s,i}^{\varepsilon,\pm}(R_1,R_2,\delta)=\bigsqcup_{p=1}^{P}\mathcal{I}_{s,i}^{\varepsilon,\pm}(R_1,R_2,\delta,p)$$

where $\mathcal{I}_{s,i}^{\varepsilon,\pm}(R_1,R_2,\delta,p)$ denotes the image by $S_s^{\pm}$ of

$$\mathfrak{D}\cap\Big(\big([0,\delta]\cap[t_{p-1},t_p[\big)\times\mathcal{D}_s^{\varepsilon}\times\mathbb{S}_{\omega}^{d-1}\times\mathbb{R}_{v_{s+1}}^{d}\Big),$$

and, as above, $\mathcal{I}_{s,i}^{\varepsilon,\pm}(R_1,R_2,\delta)$ denotes the image by $S_s^{\pm}$ of

$$\mathfrak{D}\cap\Big([0,\delta]\times\mathcal{D}_s^{\varepsilon}\times\mathbb{S}_{\omega}^{d-1}\times\mathbb{R}_{v_{s+1}}^{d}\Big).$$

On this set, one knows that the hard sphere transport coincides with the map $S_s^{\pm}$, by change of variables (see Propositions 3 page 92 and 4 page 98), so that

$$\begin{aligned}
&\int_0^{\delta}\int_{\mathcal{D}_s^{\varepsilon}}\mathcal{C}_{s,s+1,\pm,i}^{\varepsilon}\mathcal{T}_t^{s+1,\varepsilon}\varphi_{PC}^{(s+1)}\,\mathrm{d}Z_s\,\mathrm{d}t\\
&=\sum_{p=1}^{P}\int_{[0,\delta]\cap[t_{p-1},t_p[}\int_{\mathcal{D}_s^{\varepsilon}}\int_{\mathbb{S}_{\omega}^{d-1}}\int_{\mathbb{R}_{v_{s+1}}^{d}}\varepsilon\big(\omega\cdot(v_{s+1}-v_i)\big)_{\pm}\big|\alpha_p\circ S_s^{\pm}\big|\\
&\qquad\qquad\times\mathbb{1}_{\mathfrak{D}}\,\mathrm{d}v_{s+1}\,\mathrm{d}\omega\,\mathrm{d}Z_s\,\mathrm{d}t\\
&=\sum_{p=1}^{P}\int_{Z_{s+1}\in\mathcal{I}_{s,i}^{\varepsilon,\pm}(R_1,R_2,\delta,p)}\big|\alpha_p\big|\,\mathrm{d}Z_{s+1}\qquad\le\max_{1\le p\le P}\big|\alpha_p\big|_{L^{\infty}(\mathcal{D}_{s+1}^{\varepsilon})}\sum_{p=1}^{P}\Big|\mathcal{I}_{s,i}^{\varepsilon,\pm}(R_1,R_2,\delta,p)\Big|\\
&\le\big|\varphi_{PC}^{(s+1)}\big|_{L^{\infty}([0,T],L^{\infty}(\mathcal{D}_{s+1}^{\varepsilon}))}\Big|\mathcal{I}_{s,i}^{\varepsilon,\pm}(R_1,R_2,\delta)\Big|.
\end{aligned}\tag{5.40}$$

Again, the Fubini-Lebesgue theorem shows that

$$\sum_{p=1}^{P}\mathbb{1}_{[t_{p-1},t_p[}\int_{\mathbb{S}_{\omega}^{d-1}}\int_{\mathbb{R}_{v_{s+1}}^{d}}\varepsilon^{d-1}\big(\omega\cdot(v_{s+1}-v_i)\big)_{\pm}\big(\alpha_p\circ S_s^{\pm}\big)\mathbb{1}_{\mathfrak{D}}\,\mathrm{d}v_{s+1}\,\mathrm{d}\omega$$

is a measurable function defined almost everywhere and is moreover a L^1 function on $[0,\delta]\times\mathcal{D}_s^{\varepsilon}$, with an additional control on its L^1 norm on $[0,\delta]\times\mathcal{D}_s^{\varepsilon}$, thanks to the right-hand side of (5.40). Besides the control of the L^{∞} norm of the hard sphere transport operator

$$\big|\mathcal{T}_t^{s+1,\varepsilon}\alpha_p\big|_{L^{\infty}(\mathcal{D}_{s+1}^{\varepsilon})}=\big|\alpha_p\big|_{L^{\infty}(\mathcal{D}_{s+1}^{\varepsilon})}$$

for all $t\in[0,T]$ and all $1\le p\le P$ enables, for the particular choices of $t=n\delta$, for any integer $1\le n\le T/\delta$, to remove the restriction on the time interval $[0,\delta]$, and to extend the previous control of the L^1 norm on the whole

domain $[0,T]\times\mathcal{D}_s^\varepsilon$. One obtains therefore, thanks to the estimate (5.17) page 103 controlling the size of $\left|\mathcal{I}_{s,i}^{\varepsilon,\pm}(R_1,R_2,\delta)\right|$, the following control of the L^1 norm on $[0,T]\times\mathcal{D}_s^\varepsilon$ of the previous function, which can be rewritten as :

$$\int_0^T\int_{\mathcal{D}_s^\varepsilon}\left|\mathcal{C}_{s,s+1,\pm,i}^\varepsilon(R_1,R_2,\delta)\mathcal{T}_t^{s+1,\varepsilon}\varphi_{PC}^{(s+1)}(t,Z_s)\right|\mathrm{d}Z_s\,\mathrm{d}t \\ \leq C(d,s)\varepsilon^{d-1}T(R_1+\delta R_2)^{ds}R_2^{d(s+1)+1}\left|\varphi_{PC}^{(s+1)}\right|_{L^\infty([0,T],L^\infty(\mathcal{D}_{s+1}^\varepsilon))}. \tag{5.41}$$

Then, as above, the control of the L^∞ norm of the truncated transport-collision operator for time-dependent functions is obtained with a weak control of the L^∞ norm. More precisely, for any measurable subset A of finite measure of $[0,T]\times\mathcal{D}_s^\varepsilon$, one divides A as follows :

$$A=\bigsqcup_{\substack{1\leq n\leq T/\delta\\1\leq p\leq P}}\Big(A\cap\Big([(n-1)\delta,n\delta]\times\mathcal{D}_s^\varepsilon\Big)\cap\Big([t_{p-1},t_p[\times\mathcal{D}_s^\varepsilon\Big)\Big).$$

The set $A\cap\Big([(n-1)\delta,n\delta]\times\mathcal{D}_s^\varepsilon\Big)\cap\Big([t_{p-1},t_p[\times\mathcal{D}_s^\varepsilon\Big)$ will be denoted by $A_{n,p}^\delta$. Similarly to (5.22), one obtains :

$$\int_{A_{n,p}^\delta}\left|\mathcal{C}_{s,s+1,\pm,i}^\varepsilon(R_1,R_2,\delta)\mathcal{T}_t^{s+1,\varepsilon}\varphi_{PC}^{(s+1)}\right|\mathrm{d}Z_s\,\mathrm{d}t \\ \leq C(d,s)\varepsilon^{d-1}R_2^{d+1}|A_{n,p}^\delta|\left|\alpha_p\right|_{L^\infty}.$$

Summing over all n and p, one gets :

$$\int_A\left|\mathcal{C}_{s,s+1,\pm,i}^\varepsilon(R_1,R_2,\delta)\mathcal{T}_t^{s+1,\varepsilon}\varphi_{PC}^{(s+1)}\right|\mathrm{d}Z_s\,\mathrm{d}t \\ \leq C(d,s)\varepsilon^{d-1}R_2^{d+1}|A|\left|\varphi_{PC}^{(s+1)}\right|_{L^\infty([0,T],L^\infty(\mathcal{D}_{s+1}^\varepsilon))}. \tag{5.42}$$

One deduces from (5.42) that the same inequality as (5.23) page 107 holds, for time-dependent and piecewise constant functions :

$$\left|\mathcal{C}_{s,s+1,\pm,i}^\varepsilon(R_1,R_2,\delta)\mathcal{T}_t^{s+1,\varepsilon}\varphi_{PC}^{(s+1)}\right|_{L^\infty([0,T]\times\mathcal{D}_s^\varepsilon)} \\ \leq C(d,s)\varepsilon^{d-1}R_2^{d+1}\left|\varphi_{PC}^{(s+1)}\right|_{L^\infty([0,T],L^\infty(\mathcal{D}_{s+1}^\varepsilon))}. \tag{5.43}$$

Lemma 5 is proved. □

Lemma 6 (Definition of the truncated in time, position and velocity transport-collision operator in L^∞ for functions of $\mathcal{C}\big([0,T],L^\infty\big(\mathcal{D}_{s+1}^\varepsilon\big)\big)$). *Let s be a*

positive integer, $\varepsilon \leq 1$ and T be two strictly positive numbers. Let R_1, R_2 be two strictly positive numbers, and δ be a strictly positive number such that :

$$\delta \leq \delta_0(d, \varepsilon, R_2) = \min_{1 \leq j \leq d} \left[\frac{\varepsilon}{2R_2} \Big(\frac{d}{\binom{d}{j}} \Big)^{\frac{1}{j-1}} \right].$$

Then for every integer $1 \leq i \leq s$, and for every converging sequence $\big(\varphi_j^{(s+1)}\big)_{j \geq 0}$ of $L^\infty\big([0,T], \big(\mathcal{D}_{s+1}^\varepsilon\big)\big)$ of piecewise constant in time functions, the sequence

$$\Big(\mathcal{C}_{s,s+1,\pm,i}^\varepsilon(R_1, R_2, \delta) \mathcal{T}_t^{s+1,\varepsilon} \varphi_j^{(s+1)} \Big)_{j \geq 0}$$

is also a converging sequence in $L^\infty\big([0,T] \times \mathcal{D}_{s+1}^\varepsilon\big)$.
In particular, for every function $\varphi^{(s+1)} \in \mathcal{C}\big([0,T], L^\infty\big(\mathcal{D}_{s+1}^\varepsilon\big)\big)$, one defines

$$\mathcal{C}_{s,s+1,\pm,i}^\varepsilon(R_1, R_2, \delta) \mathcal{T}_t^{s+1,\varepsilon} \varphi^{(s+1)}$$

as the limit in $L^\infty\big([0,T] \times \mathcal{D}_s^\varepsilon\big)$ of the sequence

$$\Big(\mathcal{C}_{s,s+1,\pm,i}^\varepsilon(R_1, R_2, \delta) \mathcal{T}_t^{s+1,\varepsilon} \varphi_j^{(s+1)} \Big)_{j \geq 0},$$

for any sequence $\big(\varphi^{(s+1)}\big)_{j \geq 0}$ in $L^\infty\big([0,T], \big(\mathcal{D}_{s+1}^\varepsilon\big)\big)$ of piecewise constant in time functions, converging towards $\varphi^{(s+1)}$ in $L^\infty\big([0,T], \big(\mathcal{D}_{s+1}^\varepsilon\big)\big)$.

The result of this lemma follows from the theorem of extension of a continuous, linear map defined on a dense subset (here : the subset composed of the piecewise constant in time functions) of a complete metric space (here : $\mathcal{C}\big([0,T], L^\infty\big(\mathcal{D}_{s+1}^\varepsilon\big)\big)$).

Now one wants to recover the L^1 and the L^∞ controls on the limit

$$\mathcal{C}_{s,s+1,\pm,i}^\varepsilon(R_1, R_2, \delta) \mathcal{T}_t^{s+1,\varepsilon} \varphi^{(s+1)}$$

for any function $\varphi^{(s+1)} \in \mathcal{C}\big([0,T], L^\infty\big(\mathcal{D}_{s+1}^\varepsilon\big)\big)$.

Lemma 7 (L^1 and L^∞ regularity of the truncated in time, position and velocity transport-collision operator for time-dependent functions)**.** *Let s be a positive integer, $\varepsilon \leq 1$ and T be two strictly positive numbers. Let R_1, R_2 be two strictly positive numbers, and δ be a strictly positive number such that :*

$$\delta \leq \delta_0(d, \varepsilon, R_2) = \min_{1 \leq j \leq d} \left[\frac{\varepsilon}{2R_2} \Big(\frac{d}{\binom{d}{j}} \Big)^{\frac{1}{j-1}} \right].$$

Then for any integer $1 \leq i \leq s$, and for any function

$$\varphi^{(s+1)} \in \mathcal{C}\big([0,T], L^\infty\big(\mathcal{D}_{s+1}^\varepsilon\big)\big),$$

the function

$$\mathcal{C}_{s,s+1,\pm,i}^\varepsilon(R_1, R_2, \delta) \mathcal{T}_t^{s+1,\varepsilon} \varphi^{(s+1)}$$

defined in Lemma 6 satisfies the two following estimates :

- *control of the L^1 norm :*

$$\Big|\mathcal{C}^{\varepsilon}_{s,s+1,\pm,i}(R_1,R_2,\delta)\mathcal{T}^{s+1,\varepsilon}_t\varphi^{(s+1)}\Big|_{L^1([0,T]\times\mathcal{D}^{\varepsilon}_s)}$$
$$\leq C(d,s)\varepsilon^{d-1}T(R_1+\delta R_2)^{ds}R_2^{d(s+1)+1}\big|\varphi^{(s+1)}\big|_{L^\infty([0,T],L^\infty(\mathcal{D}^{\varepsilon}_{s+1}))}, \tag{5.44}$$

- *control of the L^∞ norm :*

$$\Big|\mathcal{C}^{\varepsilon}_{s,s+1,\pm,i}(R_1,R_2,\delta)\mathcal{T}^{s+1,\varepsilon}_t\varphi^{(s+1)}\Big|_{L^\infty([0,T]\times\mathcal{D}^{\varepsilon}_s)}$$
$$\leq C(d,s)\varepsilon^{d-1}R_2^{d+1}\big|\varphi^{(s+1)}\big|_{L^\infty([0,T],L^\infty(\mathcal{D}^{\varepsilon}_{s+1}))}. \tag{5.45}$$

Proof. Of course, the L^∞ control (5.39) page 123, true for the piecewise constant in time functions, stated in Lemma 5, holds in the limit $j\to+\infty$, thanks to the L^∞ convergence of the sequence

$$\Big(\mathcal{C}^{\varepsilon}_{s,s+1,\pm,i}(R_1,R_2,\delta)\mathcal{T}^{s+1,\varepsilon}_t\varphi^{(s+1)}_j\Big)_{j\geq 0},$$

for any sequence $\big(\varphi^{(s+1)}_j\big)_{j\geq 0}$ of piecewise constant in time functions converging towards $\varphi^{(s+1)}$.
Moreover, one can get a L^1 control on this limit. Indeed, the inequality (5.38) (establishing the L^1 control on the truncated collision operator defined for piecewise constant functions) stated in Lemma 5 page 122 shows that the sequence

$$\Big(\mathcal{C}^{\varepsilon}_{s,s+1,\pm,i}(R_1,R_2,\delta)\mathcal{T}^{s+1,\varepsilon}_t\varphi^{(s+1)}_j\Big)_j$$

is also a Cauchy sequence of the functional space $L^1\big([0,T],L^\infty(\mathcal{D}^{\varepsilon}_{s+1})\big)$, so that, by the Riesz-Fischer theorem, this sequence converges towards a limit in this space denoted by

$$\widetilde{\mathcal{C}}^{\varepsilon}_{s,s+1,\pm,i}(R_1,R_2,\delta)\mathcal{T}^{s+1,\varepsilon}_t\varphi^{(s+1)}.$$

For this limit, one sees immediately that the L^1 control (5.38) page 123 obtained for each term of the sequence holds also in the limit $j\to+\infty$, that is one has :

$$\int_0^T\int_{\mathcal{D}^{\varepsilon}_s}\Big|\widetilde{\mathcal{C}}^{\varepsilon}_{s,s+1,\pm,i}(R_1,R_2,\delta)\mathcal{T}^{s+1,\varepsilon}_t\big(\varphi^{(s+1)}(t,Z_s)\big)\Big|\,\mathrm{d}Z_s\,\mathrm{d}t$$
$$\leq C(d,s)\varepsilon^{d-1}T(R_1+\delta R_2)^{ds}R_2^{d(s+1)+1}\big|\varphi^{(s+1)}\big|_{L^\infty([0,T],L^\infty(\mathcal{D}^{\varepsilon}_{s+1}))}. \tag{5.46}$$

One notices that those two limits are equal, since the L^∞ convergence implies the almost everywhere convergence, while the L^1 convergence implies the almost everywhere convergence up to extracting a subsequence. In other words, one has

$$\widetilde{\mathcal{C}}^{\varepsilon}_{s,s+1,\pm,i}(R_1,R_2,\delta)\mathcal{T}^{s+1,\varepsilon}_t\varphi^{(s+1)}=\mathcal{C}^{\varepsilon}_{s,s+1,\pm,i}(R_1,R_2,\delta)\mathcal{T}^{s+1,\varepsilon}_t\varphi^{(s+1)}, \tag{5.47}$$

and then the limit satisfies the L^1 and the L^∞ controls (5.46) and (5.39). Lemma 7 is proved. □

Going back to the removal of the truncation parameters, it is possible to remove the truncation parameter δ for the truncated collision operator defined on time-dependent functions following exactly the same steps as in the section starting page 110 and in the proof of Lemma 3. One checks in the proof of the following lemma that those steps can still be followed in this more general setting.

Lemma 8 (Definition of the truncated in position and velocity transport-collision operator, and L^1, L^∞ regularity for functions of $\mathcal{C}\big([0,T], L^\infty\big(\mathcal{D}^\varepsilon_{s+1}\big)\big)$). *Let s be a positive integer, $\varepsilon \leq 1$ and T be two strictly positive numbers. Let R_1, R_2 be two strictly positive numbers.*
Then for any integer $1 \leq i \leq s$, and for any function :

$$\varphi^{(s+1)} \in \mathcal{C}\big([0,T], L^\infty\big(\mathcal{D}^\varepsilon_{s+1}\big)\big),$$

the sequence of functions :

$$\Big(\mathcal{C}^\varepsilon_{s,s+1,\pm,i}(R_1,R_2,\delta)\mathcal{T}^{s+1,\varepsilon}_t\varphi^{(s+1)}\Big)_{\delta>0}$$

is a converging sequence in $L^1\big([0,T]\times\mathcal{D}^\varepsilon_s\big)$. Moreover, its limit, denoted by :

$$\mathcal{C}^\varepsilon_{s,s+1,\pm,i}(R_1,R_2)\mathcal{T}^{s+1,\varepsilon}_t\varphi^{(s+1)},$$

is bounded almost everywhere, and verifies :

- *the following L^∞ estimate :*

$$\begin{aligned}\big|\mathcal{C}^\varepsilon_{s,s+1,\pm,i}(R_1,R_2)\mathcal{T}^{s+1,\varepsilon}_t\varphi^{(s+1)}\big|_{L^\infty([0,T]\times\mathcal{D}^\varepsilon_s)}&\\ \leq C(d,s)\varepsilon^{d-1}R_2^{d+1}\big|\varphi^{(s+1)}\big|_{L^\infty([0,T],L^\infty(\mathcal{D}^\varepsilon_{s+1}))},&\end{aligned}\tag{5.48}$$

- *and the following convergence result :*

$$\mathcal{C}^\varepsilon_{s,s+1,\pm,i}(R_1,R_2,\delta)\mathcal{T}^{s+1,\varepsilon}_t\varphi^{(s+1)} \underset{\delta\to 0}{\overset{*}{\rightharpoonup}} \mathcal{C}^\varepsilon_{s,s+1,\pm,i}(R_1,R_2)\mathcal{T}^{s+1,\varepsilon}_t\varphi^{(s+1)}\tag{5.49}$$

  in $L^\infty\big([0,T]\times\mathcal{D}^\varepsilon_s\big)$.

Proof. The analog of the inequality (5.28) page 112, which controls the L^1 norm of the difference of the truncated collision operators for δ and δ', holds for piecewise constant in time functions, that is :

$$\begin{aligned}\int_0^T\int_{Z_s\in\mathcal{D}^\varepsilon_s}\Big|\mathcal{C}^\varepsilon_{s,s+1,\pm,i}(R_1,R_2,\delta',\delta)\mathcal{T}^{s+1,\varepsilon}_t\varphi^{(s+1)}_j(t,Z_s)\Big|\,\mathrm{d}Z_s\,\mathrm{d}t&\\ \leq C(d,s,\varepsilon,R_1,R_2)\delta T\big|\varphi^{(s+1)}_j\big|_{L^\infty([0,T],L^\infty(\mathcal{D}^\varepsilon_{s+1}))},&\end{aligned}\tag{5.50}$$

for all $\varphi_j^{(s+1)} = \sum_{p=1}^{P_j} \alpha_{j,p} \mathbb{1}_{[t_{j,(p-1)},t_{j,p}[}$, piecewise constant in time functions taking its values in $L^\infty\big(\mathcal{D}_{s+1}^\varepsilon\big)$. One has shown the L^1 convergence of the sequence $\Big(\mathcal{C}_{s,s+1,\pm,i}^\varepsilon(R_1,R_2,\delta)\mathcal{T}_t^{s+1,\varepsilon}\varphi_j^{(s+1)}\Big)_{j\geq 0}$ towards its limit, so that the last inequality holds for every function continuous in time, taking its values in $L^\infty\big(\mathcal{D}_{s+1}^\varepsilon\big)$:

$$\int_0^T \int_{Z_s\in\mathcal{D}_s^\varepsilon} \Big|\mathcal{C}_{s,s+1,\pm,i}^\varepsilon(R_1,R_2,\delta',\delta)\mathcal{T}_t^{s+1,\varepsilon}\varphi^{(s+1)}(t,Z_s)\Big|\,\mathrm{d}Z_s\,\mathrm{d}t \leq C(d,s,\varepsilon,R_1,R_2)\delta T\big|\varphi^{(s+1)}\big|_{L^\infty([0,T],L^\infty(\mathcal{D}_{s+1}^\varepsilon))} \tag{5.51}$$

for all $\varphi^{(s+1)} \in \mathcal{C}\big([0,T],L^\infty(\mathcal{D}_{s+1}^\varepsilon)\big)$. Similarly, the L^1 convergence enables to state immediately an L^∞ control on the operator obtained in the limit $\delta \to 0$, which holds in the limit. In other words, one has :

$$\Big|\mathcal{C}_{s,s+1,\pm,i}^\varepsilon(R_1,R_2)\mathcal{T}_t^{s+1,\varepsilon}\varphi^{(s+1)}\Big|_{L^\infty([0,T]\times\mathcal{D}_s^\varepsilon)} \leq C(d,s)\varepsilon^{d-1}R_2^{d+1}\big|\varphi^{(s+1)}\big|_{L^\infty([0,T],L^\infty(\mathcal{D}_{s+1}^\varepsilon))}. \tag{5.52}$$

The proof of the weakly-$*$ convergence in L^∞ of the sequence

$$\Big(\mathcal{C}_{s,s+1,\pm,i}^\varepsilon(R_1,R_2,\delta)\mathcal{T}_t^{s+1,\varepsilon}\varphi^{(s+1)}(t,\cdot)\Big)_{\delta>0}$$

towards $\mathcal{C}_{s,s+1,\pm,i}^\varepsilon(R_1,R_2)\mathcal{T}_t^{s+1,\varepsilon}\varphi^{(s+1)}(t,\cdot)$ is a consequence of the convergence in L^1, and the bound (5.45) of Lemma 7. Lemma 8 is proved. □

5.1.4 Removing the truncation parameters in position and velocity

Finally, one will remove the two remaining truncation parameters in large distances R_1, and large velocities R_2. On the one hand, this will complete the study of the definition of the transport-collision operator, and on the other hand a satisfactory functional setting will naturally appear.

Limit of the sequence of operators $\mathcal{C}_{s,s+1,\pm,i}^\varepsilon(R_1,R_2)\mathcal{T}_t^{s+1,\varepsilon}$ when R_1 goes to infinity, with R_2 given : an almost everywhere convergence

The first step here consists in removing the truncation parameter in high distance R_1. The idea is the following. In fact, it is the same to consider a truncated transport-collision operator in R_1 and R_2 acting on L^∞ functions on the whole phase space, and a transport-collision operator, truncated only in high velocities, that is in R_2, according to the previous notations, acting on L^∞ functions, compactly supported with respect to the position variable.
Therefore, after checking that, for any ball in which lies the position variable,

there exists $R_{1,0}$ large enough such that for all $R_{1,0} \leq R_1, R_1'$, on the one hand $\mathcal{C}^{\varepsilon}_{s,s+1,\pm,i}(R_1,R_2)\mathcal{T}_t^{s+1,\varepsilon}$, and $\mathcal{C}^{\varepsilon}_{s,s+1,\pm,i}(R_1',R_2)\mathcal{T}_t^{s+1,\varepsilon}$ on the other hand coincide on this ball, one will define the truncated transport-collision operator without truncation in position $\mathcal{C}^{\varepsilon}_{s,s+1,\pm,i}(R_2)\mathcal{T}_t^{s+1,\varepsilon}\varphi^{(s+1)}$ as the almost everywhere limit of the sequence of functions $\big(\mathcal{C}^{\varepsilon}_{s,s+1,\pm,i}(R_1,R_2)\mathcal{T}_t^{s+1,\varepsilon}\varphi^{(s+1)}\big)_{R_1>0}$. Finally, the operator obtained in the limit will take its values in L^∞, since the L^∞ control of the truncated collision operator does not depend on the truncation parameter in high position R_1.
First, one studies the support of $\mathcal{C}^{\varepsilon}_{s,s+1,\pm,i}(R_1,R_2,\delta)\mathcal{T}_t^{s+1,\varepsilon}\varphi^{(s+1)}$ with respect to the position variable. This is the object of the following lemma.

Lemma 9 (Support of the difference of two truncated in time, position and velocity transport-collision operators with R_2 and δ given)**.** *Let s be a positive integer, ε and T be two strictly positive numbers. Let R_1, R_1', R_2 and δ be four strictly positive numbers such that :*

$$R_1 < R_1',$$

and :

$$\delta \leq \delta_0(d,\varepsilon,R_2) = \min_{1\leq j\leq d}\left[\frac{d}{\binom{d}{j}}\Big(\frac{\varepsilon}{2R_2}\Big)^{j-1}\right].$$

If in addition :

$$R_1 \geq \sqrt{2s}\varepsilon,$$

then for every integer $1 \leq i \leq s$, and for every function

$$\varphi^{(s+1)} \in \mathcal{C}\big([0,T], L^\infty\big(\mathcal{D}^{\varepsilon}_{s+1}\big)\big),$$

the support of the difference of the truncated transport-collision operators :

$$\mathcal{C}^{\varepsilon}_{s,s+1,\pm,i}(R_1',R_2,\delta)\mathcal{T}_t^{s+1,\varepsilon}\varphi^{(s+1)} - \mathcal{C}^{\varepsilon}_{s,s+1,\pm,i}(R_1,R_2,\delta)\mathcal{T}_t^{s+1,\varepsilon}\varphi^{(s+1)}$$

is contained in $[0,T]\times\big\{(X_s,V_s)\in\mathcal{D}^{\varepsilon}_s \;/\; \big(\frac{R_1}{\sqrt{2}}-\varepsilon\sqrt{s}\big)\leq |X_s| < R_1'\big\} \subset [0,T]\times\mathcal{D}^{\varepsilon}_s$.

Proof. If one assumes that the vector X_{s+1} encoding the positions of the $s+1$ particles has a norm larger than R_1, and using the fact that the position of the $s+1$ particle is by definition $x_i+\varepsilon\omega$, one has :

$$\begin{aligned} R_1^2 \leq |X_{s+1}|^2 = |X_s|^2 + |x_i+\varepsilon\omega|^2 &= \sum_{j=1}^{s}|x_j|^2 + |x_i|^2 + 2\varepsilon\omega\cdot x_i + \varepsilon^2 \\ &\leq 2\sum_{j=1}^{s}|x_j|^2 + 4\varepsilon|x_i| + 2\varepsilon^2 s, \end{aligned}$$

thanks to the Cauchy-Schwarz inequality, and then :

$$\frac{R_1^2}{2} \leq \sum_{j=1}^{s}|x_j|^2 + 2\varepsilon\sum_{j=1}^{s}|x_j| + \varepsilon^2 s \leq \sum_{j=1}^{s}|x_j|^2 + 2\varepsilon\sqrt{s}\sqrt{\sum_{j=1}^{s}|x_j|^2} + \varepsilon^2 s,$$

thanks again to the Cauchy-Schwarz inequality applied on the vectors $(1,\dots,1)$ and $(|x_1|,\dots,|x_s|)$, so that one has :

$$\frac{R_1^2}{2} \leq \big(|X_s| + \varepsilon\sqrt{s}\big)^2,$$

that is, in other words :

$$\frac{R_1}{\sqrt{2}} - \varepsilon\sqrt{s} \leq |X_s|.$$

For the end of the proof, one will denote

$$R_1^* = R_1^*(s,\varepsilon,R_1) = \frac{R_1}{\sqrt{2}} - \varepsilon\sqrt{s}.$$

Now, one considers a function $\varphi^{(s+1)} \in \mathcal{C}\big([0,T], L^\infty\big(\mathcal{D}_{s+1}^\varepsilon\big)\big)$. There exists then a sequence $\big(\varphi_j^{s+1}\big)_{j\geq 0}$ of piecewise constant in time functions belonging to $L^\infty\big([0,T], L^\infty\big(\mathcal{D}_{s+1}^\varepsilon\big)\big)$ which converges towards $\varphi^{(s+1)}$ uniformly with, as usual, $\varphi_j^{(s+1)}$ denoting

$$\varphi_j^{(s+1)} = \sum_{p=1}^{P_j} \mathbb{1}_{[t_{j,(p-1)},t_{j,p}[}\alpha_{j,p},$$

with $\alpha_{j,p} \in L^\infty\big(\mathcal{D}_{s+1}^\varepsilon\big)$ for every $1 \leq p \leq P_j$ and $j \in \mathbb{N}$.
Up to assuming that δ is small enough (that is verifying the condition in Lemma 5 page 122), one knows (thanks to Lemma 5) that $\mathcal{C}_{s,s+1,\pm,i}^\varepsilon(R_1',R_2,\delta)\mathcal{T}_t^{s+1,\varepsilon}\varphi_j^{(s+1)}$ and $\mathcal{C}_{s,s+1,\pm,i}^\varepsilon(R_1,R_2,\delta)\mathcal{T}_t^{s+1,\varepsilon}\varphi_j^{(s+1)}$ are integrable over $[0,T]\times\mathcal{D}_s^\varepsilon$. Therefore, one considers the following quantity :

$$\begin{aligned}\int_0^T \int_{\mathcal{D}_s^\varepsilon} \Big|&\mathcal{C}_{s,s+1,\pm,i}^\varepsilon(R_1',R_2,\delta)\mathcal{T}_t^{s+1,\varepsilon}\varphi_j^{(s+1)} \\ &- \mathcal{C}_{s,s+1,\pm,i}^\varepsilon(R_1,R_2,\delta)\mathcal{T}_t^{s+1,\varepsilon}\varphi_j^{(s+1)}\Big|(t,Z_s) \\ &\times \Big[\mathbb{1}_{\{|X_s|<R_1^*\}} + \mathbb{1}_{\{R_1'\leq|X_s|\}}\Big]\,\mathrm{d}Z_s\,\mathrm{d}t. \qquad (5.53)\end{aligned}$$

It is of course 0, which finishes the study of the support of the difference of two truncated collision operators.
Now one will consider the previous quantity (5.53) and show that it is still zero in the limit $j\to+\infty$, that is it is still true for any function $\varphi^{(s+1)} \in \mathcal{C}\big([0,T], L^\infty\big(\mathcal{D}_s^\varepsilon\big)\big)$. Thanks to Lemma 7 page 126, one knows that the functions $\mathcal{C}_{s,s+1,\pm,i}^\varepsilon(R_1,R_2,\delta)\mathcal{T}_t^{s+1,\varepsilon}\varphi^{(s+1)}$ and $\mathcal{C}_{s,s+1,\pm,i}^\varepsilon(R_1',R_2,\delta)\mathcal{T}_t^{s+1,\varepsilon}\varphi^{(s+1)}$ are integrable. The quantity

$$\int_0^T \int_{\mathcal{D}_s^\varepsilon} \Big| \mathcal{C}^\varepsilon_{s,s+1,\pm,i}(R_1', R_2, \delta) \mathcal{T}_t^{s+1,\varepsilon} \varphi^{(s+1)} - \mathcal{C}^\varepsilon_{s,s+1,\pm,i}(R_1, R_2, \delta) \mathcal{T}_t^{s+1,\varepsilon} \varphi^{(s+1)} \Big| (t, Z_s) \times \Big[\mathbb{1}_{\{|X_s| < R_1^*\}} + \mathbb{1}_{\{R_1' \le |X_s|\}} \Big] \, \mathrm{d}Z_s \, \mathrm{d}t.$$

is then well defined. Besides, since one considers a difference of integrable functions, for any $\alpha > 0$, there exists $R(\alpha) > 0$ such that

$$\int_0^T \int_{\mathcal{D}_s^\varepsilon} \Big| \mathcal{C}^\varepsilon_{s,s+1,\pm,i}(R_1', R_2, \delta) \mathcal{T}_t^{s+1,\varepsilon} \varphi^{(s+1)} - \mathcal{C}^\varepsilon_{s,s+1,\pm,i}(R_1, R_2, \delta) \mathcal{T}_t^{s+1,\varepsilon} \varphi^{(s+1)} \Big| (t, Z_s) \times \Big[\mathbb{1}_{\{|X_s| < R_1^*\}} + \mathbb{1}_{\{R_1' \le |X_s|\}} \Big] \mathbb{1}_{\{|Z_s| > R\}} \, \mathrm{d}Z_s \, \mathrm{d}t \le \alpha/2.$$

On the other hand, one can easily control the following quantity :

$$\int_0^T \int_{\mathcal{D}_s^\varepsilon} \Big| \mathcal{C}^\varepsilon_{s,s+1,\pm,i}(R_1', R_2, \delta) \mathcal{T}_t^{s+1,\varepsilon} \varphi^{(s+1)} - \mathcal{C}^\varepsilon_{s,s+1,\pm,i}(R_1, R_2, \delta) \mathcal{T}_t^{s+1,\varepsilon} \varphi^{(s+1)} \Big| (t, Z_s) \times \Big[\mathbb{1}_{\{|X_s| < R_1^*\}} + \mathbb{1}_{\{R_1' \le |X_s|\}} \Big] \mathbb{1}_{\{|Z_s| \le R\}} \, \mathrm{d}Z_s \, \mathrm{d}t,$$

thanks to the convergence of the sequences

$$\big(\mathcal{C}^\varepsilon_{s,s+1,\pm,i}(R_1, R_2, \delta) \mathcal{T}_t^{s+1,\varepsilon} \varphi_j^{(s+1)} \big)_{j \ge 0}$$

and

$$\big(\mathcal{C}^\varepsilon_{s,s+1,\pm,i}(R_1', R_2, \delta) \mathcal{T}_t^{s+1,\varepsilon} \varphi_j^{(s+1)} \big)_{j \ge 0},$$

respectively towards

$$\mathcal{C}^\varepsilon_{s,s+1,\pm,i}(R_1, R_2, \delta) \mathcal{T}_t^{s+1,\varepsilon} \varphi^{(s+1)}$$

and

$$\mathcal{C}^\varepsilon_{s,s+1,\pm,i}(R_1', R_2, \delta) \mathcal{T}_t^{s+1,\varepsilon} \varphi^{(s+1)}$$

in the $L^\infty\big([0,T], L^\infty\big(\mathcal{D}_{s+1}^\varepsilon\big)\big)$ norm obtained in Lemma 6 page 125. Indeed one has that

$$\int_0^T \int_{\mathcal{D}_s^\varepsilon} \Big| \mathcal{C}^\varepsilon_{s,s+1,\pm,i}(R_1', R_2, \delta) \mathcal{T}_t^{s+1,\varepsilon} \varphi^{(s+1)} - \mathcal{C}^\varepsilon_{s,s+1,\pm,i}(R_1, R_2, \delta) \mathcal{T}_t^{s+1,\varepsilon} \varphi^{(s+1)} \Big| (t, Z_s) \times \Big[\mathbb{1}_{\{|X_s| < R_1^*\}} + \mathbb{1}_{\{R_1' \le |X_s|\}} \Big] \mathbb{1}_{\{|Z_s| \le R\}} \, \mathrm{d}Z_s \, \mathrm{d}t$$

is smaller than

$$\begin{aligned}
&\int_0^T \int_{\mathcal{D}_s^\varepsilon} \Big| \mathcal{C}^\varepsilon_{s,s+1,\pm,i}(R_1', R_2, \delta) \mathcal{T}_t^{s+1,\varepsilon} \Big[\varphi^{(s+1)} - \varphi_j^{(s+1)} \Big] \Big| (t, Z_s) \\
&\qquad \times \Big[\mathbb{1}_{\{|X_s| < R_1^*\}} + \mathbb{1}_{\{R_1' \le |X_s|\}} \Big] \mathbb{1}_{\{|Z_s| \le R\}} \, \mathrm{d}Z_s \, \mathrm{d}t \\
&+ \int_0^T \int_{\mathcal{D}_s^\varepsilon} \Big| \mathcal{C}^\varepsilon_{s,s+1,\pm,i}(R_1, R_2, \delta) \mathcal{T}_t^{s+1,\varepsilon} \Big[\varphi_j^{(s+1)} - \varphi^{(s+1)} \Big] (t, Z_s) \Big| \\
&\qquad \times \Big[\mathbb{1}_{\{|X_s| < R_1^*\}} + \mathbb{1}_{\{R_1' \le |X_s|\}} \Big] \mathbb{1}_{\{|Z_s| \le R\}} \, \mathrm{d}Z_s \, \mathrm{d}t \\
&+ \int_0^T \int_{\mathcal{D}_s^\varepsilon} \Big| \mathcal{C}^\varepsilon_{s,s+1,\pm,i}(R_1', R_2, \delta) \mathcal{T}_t^{s+1,\varepsilon} \varphi_j^{(s+1)} \\
&\qquad - \mathcal{C}^\varepsilon_{s,s+1,\pm,i}(R_1, R_2, \delta) \mathcal{T}_t^{s+1,\varepsilon} \varphi_j^{(s+1)} \Big| (t, Z_s) \\
&\qquad\qquad \times \Big[\mathbb{1}_{\{|X_s| < R_1^*\}} + \mathbb{1}_{\{R_1' \le |X_s|\}} \Big] \mathbb{1}_{\{|Z_s| \le R\}} \, \mathrm{d}Z_s \, \mathrm{d}t.
\end{aligned}$$

The previous quantity (5.53) is zero, therefore the third term in the expression above is zero, for any element $\varphi_j^{(s+1)}$ of the approximating sequence of $\varphi^{(s+1)}$. The two other ones are controlled in the same following way. For the first one, denoted T_1, one has of course :

$$\begin{aligned}
T_1 = &\int_0^T \int_{\mathcal{D}_s^\varepsilon} \Big| \mathcal{C}^\varepsilon_{s,s+1,\pm,i}(R_1', R_2, \delta) \mathcal{T}_t^{s+1,\varepsilon} \Big[\varphi^{(s+1)} - \varphi_j^{(s+1)} \Big] \Big| (t, Z_s) \\
&\qquad \times \Big[\mathbb{1}_{\{|X_s| < R_1^*\}} + \mathbb{1}_{\{R_1' \le |X_s|\}} \Big] \mathbb{1}_{\{|Z_s| \le R\}} \, \mathrm{d}Z_s \, \mathrm{d}t \\
&\le \Big| \mathcal{C}^\varepsilon_{s,s+1,\pm,i}(R_1', R_2, \delta) \mathcal{T}_t^{s+1,\varepsilon} \Big[\varphi^{(s+1)} - \varphi_j^{(s+1)} \Big] \Big|_{L^\infty([0,T], L^\infty(\mathcal{D}_{s+1}^\varepsilon))} \\
&\qquad \times \int_0^T \int_{\mathcal{D}_s^\varepsilon} \mathbb{1}_{\{|Z_s| \le R\}} \, \mathrm{d}Z_s \, \mathrm{d}t \\
&\le C(d,s) R^{2ds} T \Big| \mathcal{C}^\varepsilon_{s,s+1,\pm,i}(R_1', R_2, \delta) \mathcal{T}_t^{s+1,\varepsilon} \Big[\varphi^{(s+1)} - \varphi_j^{(s+1)} \Big] \Big|_{L^\infty([0,T], L^\infty(\mathcal{D}_{s+1}^\varepsilon))}.
\end{aligned}$$

One can then choose $j_0 = j_0(d, s, T, \alpha, R)$ such that, for all $j \ge j_0$, one will have

$$\begin{aligned}
C(d,s) R^{2ds} T \Big| \mathcal{C}^\varepsilon_{s,s+1,\pm,i}(R_1', R_2, \delta) \mathcal{T}_t^{s+1,\varepsilon} \Big[\varphi^{(s+1)} - \varphi_j^{(s+1)} \Big] \Big|_{L^\infty([0,T], L^\infty(\mathcal{D}_{s+1}^\varepsilon))} \\
\le \alpha/4.
\end{aligned}$$

For such j, one will have then :

$$\begin{aligned}
&\int_0^T \int_{\mathcal{D}_s^\varepsilon} \Big| \mathcal{C}^\varepsilon_{s,s+1,\pm,i}(R_1', R_2, \delta) \mathcal{T}_t^{s+1,\varepsilon} \varphi^{(s+1)} \\
&\qquad - \mathcal{C}^\varepsilon_{s,s+1,\pm,i}(R_1, R_2, \delta) \mathcal{T}_t^{s+1,\varepsilon} \varphi^{(s+1)} \Big| (t, Z_s) \\
&\qquad\qquad \times \Big[\mathbb{1}_{\{|X_s| < R_1^*\}} + \mathbb{1}_{\{R_1' \le |X_s|\}} \Big] \, \mathrm{d}Z_s \, \mathrm{d}t \le \alpha.
\end{aligned}$$

Noticing that the left-hand side does not depend on j, one has in fact that :

$$\int_0^T \int_{\mathcal{D}_s^\varepsilon} \Big| \mathcal{C}^\varepsilon_{s,s+1,\pm,i}(R_1', R_2, \delta) \mathcal{T}_t^{s+1,\varepsilon} \varphi^{(s+1)} - \mathcal{C}^\varepsilon_{s,s+1,\pm,i}(R_1, R_2, \delta) \mathcal{T}_t^{s+1,\varepsilon} \varphi^{(s+1)} \Big| (t, Z_s) \times \Big[\mathbb{1}_{\{|X_s| < R_1^*\}} + \mathbb{1}_{\{R_1' \leq |X_s|\}} \Big] \, \mathrm{d}Z_s \, \mathrm{d}t = 0,$$

which concludes the proof of Lemma 9. □

Thanks to Lemma 9, one is now able to recover the convergence almost everywhere of the sequence $\big(\mathcal{C}^\varepsilon_{s,s+1,\pm,i}(R_1, R_2) \mathcal{T}_t^{s+1,\varepsilon} \varphi^{(s+1)}\big)_{R_1 > 0}$, using the study of the support of the difference

$$\big(\mathcal{C}^\varepsilon_{s,s+1,\pm,i}(R_1', R_2) - \mathcal{C}^\varepsilon_{s,s+1,\pm,i}(R_1, R_2)\big) \mathcal{T}_t^{s+1,\varepsilon} \varphi^{(s+1)},$$

which is nothing more than the result of the previous lemma, in the limit $\delta \to 0$.

Lemma 10 (Support of the difference of two truncated in position and velocity transport-collision operators with R_2 given)**.** *Let s be a positive integer, ε and T be two strictly positive numbers. Let R_1, R_1' and R_2 be three strictly positive numbers such that :*

$$R_1 < R_1'.$$

If in addition :

$$R_1 \geq \sqrt{2s}\varepsilon,$$

then for every integer $1 \leq i \leq s$, and for every function

$$\varphi^{(s+1)} \in \mathcal{C}\big([0,T], L^\infty\big(\mathcal{D}_{s+1}^\varepsilon\big)\big),$$

the support of the difference of the truncated transport-collision operators :

$$\mathcal{C}^\varepsilon_{s,s+1,\pm,i}(R_1', R_2) \mathcal{T}_t^{s+1,\varepsilon} \varphi^{(s+1)} - \mathcal{C}^\varepsilon_{s,s+1,\pm,i}(R_1, R_2) \mathcal{T}_t^{s+1,\varepsilon} \varphi^{(s+1)}$$

is contained in $[0,T] \times \big\{ (X_s, V_s) \in \mathcal{D}_s^\varepsilon \ / \ \big(\frac{R_1}{\sqrt{2}} - \varepsilon\sqrt{s}\big) \leq |X_s| < R_1' \big\} \subset [0,T] \times \mathcal{D}_s^\varepsilon$.

Proof. Thanks to Lemma 8 page 128, one knows that the sequences

$$\big(\mathcal{C}^\varepsilon_{s,s+1,\pm,i}(R_1', R_2, \delta) \mathcal{T}_t^{s+1,\varepsilon} \varphi^{(s+1)}\big)_{\delta > 0}$$

and

$$\big(\mathcal{C}^\varepsilon_{s,s+1,\pm,i}(R_1, R_2, \delta) \mathcal{T}_t^{s+1,\varepsilon} \varphi^{(s+1)}\big)_{\delta > 0}$$

are converging sequences in $L^1\big([0,T] \times \mathcal{D}_s^\varepsilon\big)$, towards their respective limits $\mathcal{C}^\varepsilon_{s,s+1,\pm,i}(R_1', R_2) \mathcal{T}_t^{s+1,\varepsilon} \varphi^{(s+1)}$ and $\mathcal{C}^\varepsilon_{s,s+1,\pm,i}(R_1', R_2) \mathcal{T}_t^{s+1,\varepsilon} \varphi^{(s+1)}$.
Let then α be a strictly positive number. There exists a strictly positive number $\delta_0(\alpha)$ such that, for every $0 < \delta \leq \delta_0$, one has :

$$\Big| \mathcal{C}^\varepsilon_{s,s+1,\pm,i}(R_1', R_2, \delta) \mathcal{T}_t^{s+1,\varepsilon} \varphi^{(s+1)} - \mathcal{C}^\varepsilon_{s,s+1,\pm,i}(R_1', R_2) \mathcal{T}_t^{s+1,\varepsilon} \varphi^{(s+1)} \Big|_{L^1\left([0,T] \times \mathcal{D}_s^\varepsilon\right)} \leq \alpha/2,$$

and

$$\left|\mathcal{C}^{\varepsilon}_{s,s+1,\pm,i}(R_1,R_2,\delta)\mathcal{T}^{s+1,\varepsilon}_t\varphi^{(s+1)}-\mathcal{C}^{\varepsilon}_{s,s+1,\pm,i}(R_1,R_2)\mathcal{T}^{s+1,\varepsilon}_t\varphi^{(s+1)}\right|_{L^1\left([0,T]\times\mathcal{D}^{\varepsilon}_s\right)} \leq \alpha/2.$$

One has then immediately :

$$\begin{aligned}
&\int_0^T\int_{\mathcal{D}^{\varepsilon}_s}\Big|\mathcal{C}^{\varepsilon}_{s,s+1,\pm,i}(R_1',R_2)\mathcal{T}^{s+1,\varepsilon}_t\varphi^{(s+1)} \\
&\qquad - \mathcal{C}^{\varepsilon}_{s,s+1,\pm,i}(R_1,R_2)\mathcal{T}^{s+1,\varepsilon}_t\varphi^{(s+1)}\Big|(t,Z_s) \\
&\qquad\qquad \times\left[\mathbb{1}_{\{|X_s|<R_1^*\}}+\mathbb{1}_{\{R_1'\leq|X_s|\}}\right]\mathrm{d}Z_s\,\mathrm{d}t \\
&\leq \int_0^T\int_{\mathcal{D}^{\varepsilon}_s}\Big|\mathcal{C}^{\varepsilon}_{s,s+1,\pm,i}(R_1',R_2)\mathcal{T}^{s+1,\varepsilon}_t\varphi^{(s+1)} \\
&\qquad - \mathcal{C}^{\varepsilon}_{s,s+1,\pm,i}(R_1',R_2,\delta)\mathcal{T}^{s+1,\varepsilon}_t\varphi^{(s+1)}\Big|(t,Z_s)\,\mathrm{d}Z_s\,\mathrm{d}t \\
&+\int_0^T\int_{\mathcal{D}^{\varepsilon}_s}\Big|\mathcal{C}^{\varepsilon}_{s,s+1,\pm,i}(R_1',R_2,\delta)\mathcal{T}^{s+1,\varepsilon}_t\varphi^{(s+1)} \\
&\qquad - \mathcal{C}^{\varepsilon}_{s,s+1,\pm,i}(R_1,R_2,\delta)\mathcal{T}^{s+1,\varepsilon}_t\varphi^{(s+1)}\Big|(t,Z_s) \\
&\qquad\qquad \times\left[\mathbb{1}_{\{|X_s|<R_1^*\}}+\mathbb{1}_{\{R_1'\leq|X_s|\}}\right]\mathrm{d}Z_s\,\mathrm{d}t \\
&+\int_0^T\int_{\mathcal{D}^{\varepsilon}_s}\Big|\mathcal{C}^{\varepsilon}_{s,s+1,\pm,i}(R_1,R_2,\delta)\mathcal{T}^{s+1,\varepsilon}_t\varphi^{(s+1)} \\
&\qquad - \mathcal{C}^{\varepsilon}_{s,s+1,\pm,i}(R_1,R_2)\mathcal{T}^{s+1,\varepsilon}_t\varphi^{(s+1)}\Big|(t,Z_s)\,\mathrm{d}Z_s\,\mathrm{d}t.
\end{aligned}$$

Thanks to Lemma 9, the second term above is zero, while the first and the third are smaller than $\alpha/2$ for $\delta\leq\delta_0$ thanks to the L^1 convergence of the sequences in δ. In other words :

$$\begin{aligned}
\int_0^T\int_{\mathcal{D}^{\varepsilon}_s}&\left|\mathcal{C}^{\varepsilon}_{s,s+1,\pm,i}(R_1',R_2)\mathcal{T}^{s+1,\varepsilon}_t\varphi^{(s+1)}-\mathcal{C}^{\varepsilon}_{s,s+1,\pm,i}(R_1,R_2)\mathcal{T}^{s+1,\varepsilon}_t\varphi^{(s+1)}\right| \\
&\times\left[\mathbb{1}_{\{|X_s|<R_1^*\}}+\mathbb{1}_{\{R_1'\leq|X_s|\}}\right]\mathrm{d}Z_s\,\mathrm{d}t \\
&\leq\alpha.
\end{aligned}$$

The left-hand side does not depend on δ, so that it is in fact zero, which ends the proof of Lemma 10. □

The following lemma, which is an easy consequence of Lemma 10, provides then the almost everywhere convergence of the sequence of truncated collision operators in position and velocity, while the truncated parameter in position is relaxed.

Lemma 11 (Definition of the truncated in velocity transport-collision operator, and L^∞ regularity for functions of $\mathcal{C}\big([0,T], L^\infty\big(\mathcal{D}^\varepsilon_{s+1}\big)\big)$). *Let s be a positive integer, ε and T be two strictly positive numbers. Let also R_2 be a strictly positive number.*
Then for every integer $1 \leq i \leq s$, and for every function

$$\varphi^{(s+1)} \in \mathcal{C}\big([0,T], L^\infty\big(\mathcal{D}^\varepsilon_{s+1}\big)\big),$$

the sequence of functions

$$\big(\mathcal{C}^\varepsilon_{s,s+1,\pm,i}(R_1,R_2)\mathcal{T}^{s+1,\varepsilon}_t\varphi^{(s+1)}\big)_{R_1>0}$$

converges almost everywhere on $[0,T]\times\mathcal{D}^\varepsilon_s$ towards a measurable function. Moreover, its limit, denoted

$$\mathcal{C}^\varepsilon_{s,s+1,\pm,i}(R_2)\mathcal{T}^{s+1,\varepsilon}_t\varphi^{(s+1)},$$

is bounded almost everywhere, satisfies the following equality :

$$\mathcal{C}^\varepsilon_{s,s+1,\pm,i}(R_2)\mathcal{T}^{s+1,\varepsilon}_t\varphi^{(s+1)} = \mathcal{C}^\varepsilon_{s,s+1,\pm,i}(R_1,R_2)\mathcal{T}^{s+1,\varepsilon}_t\varphi^{(s+1)}$$

almost everywhere on $\big\{(t,Z_s) \in [0,T]\times\mathcal{D}^\varepsilon_s \ / \ |X_s| \leq \frac{R_1}{\sqrt{2}} - \varepsilon\sqrt{s}\big\}$ for every $R_1 \geq \sqrt{2s}\varepsilon$, and verifies the following L^∞ estimate :

$$\begin{aligned}\Big|\mathcal{C}^\varepsilon_{s,s+1,\pm,i}(R_2)\mathcal{T}^{s+1,\varepsilon}_t\varphi^{(s+1)}\Big|_{L^\infty([0,T]\times\mathcal{D}^\varepsilon_s)} \\ \leq C(d,s)\varepsilon^{d-1}R_2^{d+1}\big|\varphi^{(s+1)}\big|_{L^\infty([0,T],L^\infty(\mathcal{D}^\varepsilon_{s+1}))}.\end{aligned} \tag{5.54}$$

Limit of the sequence of operators $\mathcal{C}^\varepsilon_{s,s+1,\pm,i}(R_2)\mathcal{T}^{s+1,\varepsilon}_t$ when R_2 goes to infinity : a Cauchy sequence assuming an additional decrease in velocity

The last remaining truncation parameter that has to be relaxed is then R_2, the truncation parameter with respect to the velocity variable. One will have to set additional hypotheses on the function on which the transport-collision acts, in order to obtain a well-defined operator without any cut-off.
To stress the fact that one is considering functions of $L^\infty\big([0,T]\times\mathcal{D}^\varepsilon_{s+1}\big)$ satisfying additional assumptions, one changes the notations, and the function on which the transport-collision is acting will be denoted $h^{(s+1)}$ instead of $\varphi^{(s+1)}$. The notation $f^{(s+1)}$ will be devoted to the solutions of the Boltzmann hierarchy. One will assume that the function $h^{(s+1)}$ is uniformly bounded on the phase space by a function which depends essentially only on velocity. In the following, this bound will have to depend on time too.

Remark 10. *If one considers the subset of $\mathcal{C}\big([0,T], L^\infty\big(\mathcal{D}^\varepsilon_{s+1}\big)\big)$ composed of functions $h^{(s+1)}$ which fulfill the following hypotheses, that is for all $t \in [0,T]$ and for almost every $Z_{s+1} \in \mathcal{D}^\varepsilon_{s+1}$,*

$$|h^{(s+1)}\big(t,Z_{s+1}\big)| \leq g_{s+1}\big(t,|V_{s+1}|\big),$$

then this set of functions is not a vector space, since for $\lambda \in \mathbb{R}$ large enough, the condition :

$$|\lambda h^{(s+1)}| \leq g_{s+1}\big(t, |V_{s+1}|\big)$$

will be of course wrong for general $h^{(s+1)}$ and g_{s+1}.
This is an issue, since it will be convenient to work with a functional space which is a vector space. Then, one will instead consider functions $h^{(s+1)}$ of $\mathcal{C}\big([0,T], L^\infty\big(\mathcal{D}^\varepsilon_{s+1}\big)\big)$ which verify :

$$\Big|\frac{h^{(s+1)}(t, Z_{s+1})}{g_{s+1}(t, |V_{s+1}|)}\Big|_{L^\infty([0,T], L^\infty(\mathcal{D}^\varepsilon_{s+1}))} < +\infty.$$

Of course, this condition, in order to make sense, imply to assume that g_{s+1} is strictly positive .
With this condition, one has defined a sub-Banach space of $\mathcal{C}\big([0,T], L^\infty\big(\mathcal{D}^\varepsilon_{s+1}\big)\big)$.

Finally, one can state (and prove) the crucial theorem which will conclude this section.

Theorem 1 (Definition of the collision operator of the BBGKY hierarchy for functions of $\mathcal{C}\big([0,T], L^\infty\big(\mathcal{D}^\varepsilon_{s+1}\big)\big)$ decaying sufficiently fast at infinity in the velocity variables)**.** *Let s be a positive integer, ε and T be two strictly positive numbers.*
Let in addition $g_{s+1} : [0,T] \times \mathbb{R}_+ \to \mathbb{R}_+$ be a function verifying :

- *$(t,x) \mapsto g_{s+1}(t,x)$ is measurable and almost everywhere strictly positive,*
- *for all $x \in \mathbb{R}_+$, the function*

$$t \mapsto g_{s+1}(t,x) \tag{5.55}$$

  is increasing,

- *for all $t \in [0,T]$ and almost every $(v_1, \ldots, v_s) \in \mathbb{R}^{ds}$, the function*

$$v_{s+1} \mapsto |V_{s+1}| g_{s+1}\big(t, |V_{s+1}|\big) \tag{5.56}$$

  is integrable on $\mathbb{R}^d$,

- *for all $t \in [0,T]$, the function*

$$(v_1, \ldots, v_s) \mapsto \int_{\mathbb{R}^d} |V_{s+1}| g_{s+1}\big(t, |V_{s+1}|\big)\, \mathrm{d}v_{s+1} \tag{5.57}$$

  is bounded almost everywhere, and

$$\Big\| \int_{\mathbb{R}^d} \mathbb{1}_{|V_{s+1}| \geq R} |V_{s+1}| g_{s+1}\big(t, |V_{s+1}|\big)\, \mathrm{d}v_{s+1} \Big\|_{L^\infty_t L^\infty_{V_s}} \tag{5.58}$$

  converges to zero as R goes to infinity.

Then for every integer $1 \leq i \leq s$*, and for every function*

$$h^{(s+1)} \in \mathcal{C}\big([0,T], L^\infty\big(\mathcal{D}^\varepsilon_{s+1}\big)\big) \tag{5.59}$$

such that

$$\Big|\frac{h^{(s+1)}(t,Z_{s+1})}{g_{s+1}\big(t,|V_{s+1}|\big)}\Big|_{L^\infty([0,T],L^\infty(\mathcal{D}^\varepsilon_{s+1}))} < +\infty,$$

the sequence

$$\Big(\mathcal{C}^\varepsilon_{s,s+1,\pm,i}(R_2)\mathcal{T}^{s+1,\varepsilon}_t h^{(s+1)}\Big)_{R_2>0}$$

converges in $L^\infty\big([0,T]\times\mathcal{D}^\varepsilon_s\big)$ *as* R_2 *goes to infinity towards a limit which will be denoted* $\mathcal{C}^\varepsilon_{s,s+1,\pm,i}\mathcal{T}^{s+1,\varepsilon}_t h^{(s+1)}$*, so that in other words :*

$$\mathcal{C}^\varepsilon_{s,s+1,\pm,i}(R_2)\mathcal{T}^{s+1,\varepsilon}_t h^{(s+1)} \underset{R_2\to+\infty}{\overset{L^\infty_{t,Z_s}}{\longrightarrow}} \mathcal{C}^\varepsilon_{s,s+1,\pm,i}\mathcal{T}^{s+1,\varepsilon}_t h^{(s+1)}.$$

Moreover, the following inequality holds almost everywhere on $[0,T]\times\mathcal{D}^\varepsilon_s$ *:*

$$\begin{aligned}\Big|\mathcal{C}^\varepsilon_{s,s+1,\pm,i}\mathcal{T}^{s+1,\varepsilon}_t h^{(s+1)}(t,Z_s)\Big| \leq \varepsilon^{d-1}\frac{\big|\mathbb{S}^{d-1}\big|}{2}\Big|\frac{h^{(s+1)}(t,Z_{s+1})}{g_{s+1}\big(t,|V_{s+1}|\big)}\Big|_{L^\infty(\mathcal{D}^\varepsilon_{s+1})} \\ \times\int_{\mathbb{R}^d}\big(|v_i|+|v_{s+1}|\big)g_{s+1}\big(t,|V_{s+1}|\big)\,\mathrm{d}v_{s+1}.\end{aligned} \tag{5.60}$$

Proof. The proof will be divided into two main parts :

- one starts (page 140) by the convergence of the sequence of operators truncated in velocity, and hence the rigorous definition of the operator without any truncation,
- then (page 157), one will study the stability of the control of the decrease in velocity, that is one will prove the inequality (5.60).

The main idea of the first part, that is the proof of the convergence, is to show that the sequence of operators :

$$\big(\mathcal{C}^\varepsilon_{s,s+1,\pm,i}(R_2)\mathcal{T}^{s+1,\varepsilon}_t h^{(s+1)}\big)_{R_2>0}$$

is a Cauchy sequence of the complete space $L^\infty\big([0,T]\times\mathcal{D}^\varepsilon_s\big)$. On the one hand, as it was done above, a control of the difference :

$$\mathcal{C}^\varepsilon_{s,s+1,\pm,i}(R_2')\mathcal{T}^{s+1,\varepsilon}_t h^{(s+1)} - \mathcal{C}^\varepsilon_{s,s+1,\pm,i}(R_2)\mathcal{T}^{s+1,\varepsilon}_t h^{(s+1)}$$

in the L^∞ norm will be obtained thanks to the same control obtained for the truncated in position, velocity and time operator acting on piecewise constant in time functions, on which the operator is defined as a usual integral. In other words, one will have, in order to obtain meaningful quantities, to start from the

beginning of the definition of the collision operator in order to obtain the control that one wants to show, and extend this definition step by step, removing the truncation parameters one by one using the lemmas obtained before.
On the other hand, this control will be deduced, as usual too, from a weak L^∞ control. For this first part of the proof, one provides here a summary for the convenience of the reader, since the proof is quite long, technical, but similar to what was done before in many aspects.

- Page 140 : careful description of the domain corresponding to the difference of the operators

$$\mathcal{C}^{\varepsilon}_{s,s+1,\pm,i}(R_1, R_2', \delta)\mathcal{T}_t^{s+1,\varepsilon} \text{ and } \mathcal{C}^{\varepsilon}_{s,s+1,\pm,i}(R_1, R_2, \delta)\mathcal{T}_t^{s+1,\varepsilon}$$

  with δ, R_1, R_2, R_2' four strictly positive numbers such that $R_2 < R_2'$. In particular, if the transport-collision is applied to piecewise constant in time functions, it is defined as an usual integral, and this part is devoted to the description of the three subsets in which this domain will be divided into, namely A_1, A_2 and A_3, defined respectively by the conditions (5.63) page 141, (5.64) page 142 and (5.62) page 141.

- Page 142 : weak L^∞ control of the transport-collision operator restricted on the subset A_1 for $h^{(s+1)}_{PC}$ piecewise constant in time (inequality (5.65) page 143).

- Page 143 : weak L^∞ control of the transport-collision operator restricted on the subset A_2 for $h^{(s+1)}$ piecewise constant in time (inequality (5.66) page 144).

- Page 144 : weak L^∞ control of the transport-collision operator restricted on the subset A_3 for $h^{(s+1)}$ piecewise constant in time (inequality (5.72) page 150). One insists on the fact that this part relies on new arguments, namely on the one hand, the control of operator when there exists a bound decreasing in high velocities, and on the other hand, the modification of the proof of this control when the bound also depends on time.

The controls for piecewise constant in time functions and for each of the three parts are then gathered together in the inequality (5.73) page 150.

- Page 150 : weak L^∞ control for the three parts, for $h^{(s+1)}$ belonging to $\mathcal{C}\big([0,T], \big(\mathcal{D}^{\varepsilon}_{s+1}\big)\big)$, with truncation in time, position and velocity (inequality (5.78) page 154).

- Page 154 : weak L^∞ control for the three parts when the truncation parameters in time and position are relaxed (inequality (5.81) page 157), and conclusion of the first part of the proof.

The second part of the proof, starting page 157, follows the same plan, starting by the sought inequality obtained for the truncated transport-collision operator applied to piecewise constant in time functions, and then by relaxing all the

truncation parameters. As above, one provides the summary of the different steps of the second part.

- Page 157 : presentation of the inequality (5.82) page 158, that will be proved in the following section of the proof, and why it is equivalent to the inequality (5.60) stated in the lemma.
- Page 158 : proof of the inequality (5.85) page 164. As for the first part of the proof, this section is clearly the most crucial, since it relies on a careful study of the bound and the use of the hypothesis about the increasing in time property of the profile g_{s+1}.
- Page 164 : finally, exactly as for the first part of the proof, one will prove the inequality (5.82) for continuous in time functions, without any truncation parameter involved in the transport-collision operator.

First part of the proof

Description of the domain of the difference of operators for two different truncation parameters in velocity

Back to the definition of the first truncated collision operators for time-dependent, piecewise constant functions (see Section 5.1.3, especially page 121, and Lemma 5 page 122), one will study, for $\delta > 0$, $0 < R_1$ and $0 < R_2 < R_2'$, and for such piecewise constant in time functions :

$$h_{PC}^{(s+1)} = \sum_{p=1}^{P} \alpha_p \mathbb{1}_{[t_{j,(p-1)}, t_{j,p}[} \tag{5.61}$$

with $\alpha_p \in L^\infty\big(\mathcal{D}_{s+1}^\varepsilon\big)$, the difference :

$$\mathcal{C}_{s,s+1,\pm,i}^\varepsilon(R_1, R_2', \delta)\mathcal{T}_t^{s+1,\varepsilon} h_{PC}^{(s+1)} - \mathcal{C}_{s,s+1,\pm,i}^\varepsilon(R_1, R_2, \delta)\mathcal{T}_t^{s+1,\varepsilon} h_{PC}^{(s+1)}.$$

In particular, one will study the domain of the difference of those operators. Denoting by A the domain of the collision operator $\mathcal{C}_{s,s+1,\pm,i}^\varepsilon(R_1, R_2, \delta)$:

$$A = \Big(\prod_{\substack{1 \le j < k \le s+1 \\ (j,k) \neq (i,s+1)}} \mathbb{1}_{|x_j - x_k| > \varepsilon + \delta R_2} \Big) \Big(\prod_{1 \le l \le s+1} \mathbb{1}_{d(x_l, \Omega) > \varepsilon/2 + \delta R_2} \Big) \times \mathbb{1}_{X_{s+1} \in B_{\mathbb{R}^{d(s+1)}}(0, R_1)} \mathbb{1}_{V_{s+1} \in B_{\mathbb{R}^{d(s+1)}}(0, R_2)},$$

and similarly denoting by A' the domain of $\mathcal{C}_{s,s+1,\pm,i}^\varepsilon(R_1, R_2', \delta)$ (which is exactly the same except that R_2 is just replaced by R_2') :

$$A' = \Big(\prod_{\substack{1 \le j < k \le s+1 \\ (j,k) \neq (i,s+1)}} \mathbb{1}_{|x_j - x_k| > \varepsilon + \delta R_2'} \Big) \Big(\prod_{1 \le l \le s+1} \mathbb{1}_{d(x_l, \Omega) > \varepsilon/2 + \delta R_2'} \Big) \times \mathbb{1}_{X_{s+1} \in B_{\mathbb{R}^{d(s+1)}}(0, R_1)} \mathbb{1}_{V_{s+1} \in B_{\mathbb{R}^{d(s+1)}}(0, R_2')},$$

one has to notice the two following points.
First, one has

$$\left|\mathbb{1}_{A'} - \mathbb{1}_{A}\right| \leq \mathbb{1}_{A'\backslash(A\cap A')} + \mathbb{1}_{A\backslash(A\cap A')}.$$

Second, for R_1, R_2 and R_2' given, when δ goes to zero, one has in the limit :

$$A \subset A'.$$

More precisely, since by definition,

$$\begin{aligned}
A \cap A' = & \bigcap_{\substack{1\leq j<k\leq s+1\\(j,k)\neq(i,s+1)}} \left\{|x_j - x_k| > \varepsilon + \delta R_2\right\} \cap \bigcap_{1\leq l\leq s+1} \left\{d(x_l,\Omega) > \varepsilon/2 + \delta R_2\right\} \\
& \cap B_{X_{s+1}}(0,R_1) \cap B_{V_{s+1}}(0,R_2) \\
& \cap \bigcap_{\substack{1\leq j<k\leq s+1\\(j,k)\neq(i,s+1)}} \left\{|x_j - x_k| > \varepsilon + \delta R_2'\right\} \cap \bigcap_{1\leq l\leq s+1} \left\{d(x_l,\Omega) > \varepsilon/2 + \delta R_2'\right\} \\
& \cap B_{X_{s+1}}(0,R_1) \cap B_{V_{s+1}}(0,R_2') \\
= & \bigcap_{\substack{1\leq j<k\leq s+1\\(j,k)\neq(i,s+1)}} \left\{|x_j - x_k| > \varepsilon + \delta R_2'\right\} \cap \bigcap_{1\leq l\leq s+1} \left\{d(x_l,\Omega) > \varepsilon/2 + \delta R_2'\right\} \\
& \cap B_{X_{s+1}}(0,R_1) \cap B_{V_{s+1}}(0,R_2),
\end{aligned}$$

one sees immediately that $Z_{s+1} \in A\backslash(A\cap A')$ if and only if

$$\left\{\begin{array}{l}
\forall j<k \ / \ (j,k)\neq(i,s+1),\ |x_j - x_k| > \varepsilon + \delta R_2, \\
\forall l,\ d(x_l,\Omega) > \varepsilon/2 + \delta R_2, \\
|X_{s+1}| < R_1, \\
|V_{s+1}| < R_2, \\
\exists\ (j_0,k_0)\neq(i,s+1) \ / \ |x_{j_0} - x_{k_0}| \leq \varepsilon + \delta R_2' \\
\qquad\qquad \text{or } \exists\ l_0 \ / \ d(x_{l_0},\Omega) \leq \varepsilon/2 + \delta R_2'.
\end{array}\right. \tag{5.62}$$

Similarly, one has that $Z_{s+1} \in A'\backslash(A\cap B)$ if and only if

$$\left\{\begin{array}{l}
\forall j<k \ / \ (j,k)\neq(i,s+1),\ |x_j - x_k| > \varepsilon + \delta R_2', \\
\forall l,\ d(x_l,\Omega) > \varepsilon/2 + \delta R_2', \\
|X_{s+1}| < R_1, \\
|V_{s+1}| < R_2', \\
|V_{s+1}| \geq R_2.
\end{array}\right.$$

Then, the subset $A\backslash(A\cap A')$ can be divided into two parts. Indeed, one has

$$A\backslash(A\cap A') = A_1 \cup A_2,$$

with $Z_{s+1} \in A_1$ if and only if

$$\left\{\begin{array}{l}
\forall j<k \ / \ (j,k)\neq(i,s+1),\ |x_j - x_k| > \varepsilon + \delta R_2, \\
\forall l,\ d(x_l,\Omega) > \varepsilon/2 + \delta R_2, \\
|X_{s+1}| < R_1, \\
|V_{s+1}| < R_2, \\
\exists\ (j_0,k_0)\neq(i,s+1) \ / \ |x_{j_0} - x_{k_0}| \leq \varepsilon + \delta R_2',
\end{array}\right. \tag{5.63}$$

and $Z_{s+1} \in A_2$ if and only if

$$\begin{cases} \forall j < k \ / \ (j,k) \neq (i, s+1), \ |x_j - x_k| > \varepsilon + \delta R_2, \\ \forall l, \ d(x_l, \Omega) > \varepsilon/2 + \delta R_2, \\ |X_{s+1}| < R_1, \\ |V_{s+1}| < R_2, \\ \exists \ l_0 \ / \ d(x_{l_0}, \Omega) \leq \varepsilon/2 + \delta R_2'. \end{cases} \tag{5.64}$$

For the sake of simplicity, the subset $A' \backslash (A \cap A')$ will be denoted A_3, defined therefore in (5.62). From this point, three parts have to be controlled, which are the integrals of respective domains A_1, A_2 and $A_3 = A' \backslash (A \cap A')$. Only the last one will remain in the limit $\delta \to 0$, while the others will vanish.
More precisely, one considers a subset D of finite measure of $[0,T] \times \mathcal{D}_s^\varepsilon$, in order to obtain, as it was done several time above, a weak L^∞ control of the difference of transport-collision operators with two different truncation parameters in velocity.

Control for the part A_1

For the first part A_1, that is the one such that there exists a pair of particles (j_0, k_0) satisfying :

$$\varepsilon + \delta R_2 < |x_{j_0} - x_{k_0}| \leq \varepsilon + \delta R_2',$$

one has, with D denoting any measurable subset of $[0,T] \times \mathcal{D}_s^\varepsilon$ of finite measure :

$$\begin{aligned}
&\int_0^\delta \int_{\mathcal{D}_s^\varepsilon} \mathbb{1}_D \left| \int_{\mathbb{S}_\omega^{d-1} \times \mathbb{R}_{v_{s+1}}^d} \varepsilon^{d-1} \big(\omega \cdot (v_{s+1} - v_i)\big)_\pm \mathcal{T}_t^{s+1,\varepsilon} h_{PC}^{(s+1)} \mathbb{1}_{A_1} \, \mathrm{d}\omega \, \mathrm{d}v_{s+1} \right| \mathrm{d}Z_s \, \mathrm{d}t \\
&\leq \sum_{p=1}^P \int_{\substack{[0,\delta] \\ \cap [t_{(p-1)}, t_p[}} \int_{\mathcal{D}_s^\varepsilon} \mathbb{1}_D \int_{\mathbb{S}_\omega^{d-1} \times \mathbb{R}_{v_{s+1}}^d} \left| \varepsilon^{d-1} \big(\omega \cdot (v_{s+1} - v_i)\big)_\pm \mathcal{T}_t^{s+1,\varepsilon} \alpha_p \mathbb{1}_{A_1} \right| \mathrm{d}\omega \, \mathrm{d}v_{s+1} \, \mathrm{d}Z_s \, \mathrm{d}t \\
&\leq \sum_{p=1}^P \int_{\mathcal{D}_{s+1}^\varepsilon} \big|\alpha_p(\widetilde{Z}_{s+1})\big| \mathbb{1}_{S_s^\pm \left(A_1 \cap \left[\left(\left(([0,\delta] \cap [t_{(p-1)}, t_p[) \times \mathcal{D}_s^\varepsilon\right) \cap D\right) \times \mathbb{S}^{d-1} \times \mathbb{R}^d\right]\right)} \, \mathrm{d}\widetilde{Z}_{s+1} \\
&\leq \big|h_{PC}^{(s+1)}\big|_{L^\infty([0,T], L^\infty(\mathcal{D}_{s+1}^\varepsilon))} \int_{\mathcal{D}_{s+1}^\varepsilon} \mathbb{1}_{S_s^\pm(A_1 \cap (D \times \mathbb{S}^{d-1} \times \mathbb{R}^d))} \, \mathrm{d}\widetilde{Z}_{s+1} \\
&\leq \big|h_{PC}^{(s+1)}\big|_{L^\infty([0,T], L^\infty(\mathcal{D}_{s+1}^\varepsilon))} \int_{\mathcal{D}_{s+1}^\varepsilon} \mathbb{1}_{S_s^\pm(A_1)} \, \mathrm{d}\widetilde{Z}_{s+1}.
\end{aligned}$$

Of course, the image $S_s^\pm(A_1)$ is contained in $B_{X_{s+1}}(0, R_1 + \delta R_2) \times B_{V_{s+1}}(0, R_2)$, and since there exists $(j_0, k_0) \neq (i, s+1)$ such that :

$$\varepsilon + \delta R_2 < |x_{j_0} - x_{k_0}| \leq \varepsilon + \delta R_2',$$

then for all $0 \leq t \leq \delta$, one has

$$\varepsilon < |(x_{j_0} - tv_{j_0}) - (x_{k_0} - tv_{k_0})| \leq \varepsilon + \delta(R_2 + R_2').$$

As previously, one has to be careful about the possible equality $j_0 = s+1$ or $k_0 = s+1$.
Keeping in mind that for all $0 \leq t \leq \delta$ one has also

$$\varepsilon \leq (x_{s+1} - tv_{s+1}) - (x_i - tv_i)| < \varepsilon + \delta R_2,$$

so that $S_s^{\pm}(A_1)$ is contained in a subset of finite measure of $\mathcal{D}_{s+1}^{\varepsilon}$, of which the measure is smaller than

$$\begin{aligned} C(d,s)(R_1 + \delta R_2)^{d(s-1)} \varepsilon^2 \delta^2 R_2^d (R_2 + R_2')^d R_2^{d(s+1)} \\ \leq C(d,s,\varepsilon)\delta^2 (R_1 + \delta R_2)^{d(s-1)} R_2^{d(s+2)} (R_2')^d. \end{aligned}$$

As above, summing on all subintervals of $[0,T]$ of size δ, that is summing right-hand sides which individually are of order δ^2, provides a control of order δ, so that one gets :

$$\begin{aligned} &\left| \mathbb{1}_D \int_{\mathbb{S}_\omega^{d-1} \times \mathbb{R}_{v_{s+1}}^d} \varepsilon^{d-1} \big(\omega \cdot (v_{s+1} - v_i)\big)_{\pm} \mathcal{T}_t^{s+1,\varepsilon} h_{PC}^{(s+1)} \mathbb{1}_{A_1} \, \mathrm{d}\omega \, \mathrm{d}v_{s+1} \right|_{L^1([0,T] \times \mathcal{D}_s^{\varepsilon})} \\ &\leq C(d,s,\varepsilon)\delta (R_1 + \delta R_2)^{d(s-1)} R_2^{d(s+2)} (R_2')^d \big| h_{PC}^{(s+1)} \big|_{L^\infty([0,T], L^\infty(\mathcal{D}_{s+1}^{\varepsilon}))}. \end{aligned} \tag{5.65}$$

(the only difference with the previous inequality is the power of δ).

Control for the part A_2

For the second part A_2, one splits again as usual the time interval $[0,T]$ in several smaller intervals of length δ. On each of them, one studies carefully the consequences of the definition of A_2 : there exists a particle, say l_0 such that $\varepsilon/2 + \delta R_2 < d(x_{l_0}, \Omega) \leq \varepsilon/2 + \delta R_2'$.
Since one assumed that the obstacle Ω is the half-plane $\{x \in \mathbb{R}^d \ / \ x \cdot e_1 = 0\}$, the condition :

$$\varepsilon/2 + \delta R_2 < d(x_{l_0}, \Omega) \leq \varepsilon/2 + \delta R_2'$$

means :

$$\varepsilon/2 + \delta R_2 < x_{l_0} \cdot e_1 \leq \varepsilon/2 + \delta R_2'.$$

Then for all $0 \leq t \leq \delta$, one has :

$$\varepsilon/2 < (x_{l_0} - tv_{l_0}) \cdot e_1 \leq \varepsilon/2 + \delta R_2' + \delta R_2.$$

If $l_0 \neq s+1$, the condition above will be useful. But as above, if $l_0 = s+1$, that is if the condition on the closeness of the particle with respect to the obstacle holds on the particle $s+1$, one has to deduce a bound on the position of another

particle, here on the particle i. As above again, one can show that if $l_0 = s+1$, then :

$$(x_i - tv_i)\cdot e_1 \in [\varepsilon/2 - \varepsilon\omega\cdot e_1, \varepsilon/2 - \varepsilon\omega\cdot e_1 + \delta(R_2 + R_2')] \cap [\varepsilon/2, +\infty[.$$

The extra condition on the position of $x_{s+1} - tv_{s+1}$, now usual, enables to state that $S_s^{\pm}(A_2)$ is contained in a subset of finite measure of $\mathcal{D}_{s+1}^{\varepsilon}$, of which the measure of which is smaller than

$$\begin{aligned} C(d,s)(R_1+\delta R_2)^{d(s-1)}\delta(R_2+R_2')(R_1+\delta R_2)^{d-1}\varepsilon\delta R_2 R_2^{d(s+1)} \\ \leq C(d,s,\varepsilon)\delta^2(R_1+\delta R_2)^{ds-1}R_2^{d(s+1)+1}R_2'. \end{aligned}$$

Again, summing on all subintervals of $[0,T]$ of size δ, one will still lose the power 2 over δ in the last expression, but the term in δ will be conserved, so that

$$\begin{aligned} &\Big|\mathbb{1}_D \int_{\mathbb{S}_\omega^{d-1}\times\mathbb{R}_{v_{s+1}}^d} \varepsilon^{d-1}\big(\omega\cdot(v_{s+1}-v_i)\big)_{\pm}\mathcal{T}_t^{s+1,\varepsilon}h_{PC}^{(s+1)}\mathbb{1}_{A_2}\,\mathrm{d}\omega\,\mathrm{d}v_{s+1}\Big|_{L^1\left([0,T]\times\mathcal{D}_s^{\varepsilon}\right)} \\ &\qquad \leq C(d,s,\varepsilon)\delta(R_1+\delta R_2)^{ds-1}R_2^{d(s+1)+1}R_2'\big|h_{PC}^{(s+1)}\big|_{L^\infty([0,T],L^\infty(\mathcal{D}_{s+1}^{\varepsilon}))}. \end{aligned} \tag{5.66}$$

Control for the part A_3

For the last part $A_3 = A'\backslash(A\cap A')$, that is the one such that, in particular, the velocity variable of all the particles is controlled by

$$R_2 \leq |V_{s+1}| < R_2',$$

one will have to use crucially an additional hypothesis on the decay at infinity of $h_{PC}^{(s+1)}$ in the velocity variables. Since $S_s^{\pm}\big(\{R_2 \geq |V_{s+1}|\}\big) = \{R_2 \geq |\widetilde{V}_{s+1}|\}$, and of course :

$$A'\backslash(A\cap A') \subset \{R_2 \leq |V_{s+1}|\},$$

if one denotes :

$$Q_3^{\delta} = \int_0^{\delta}\int_{\mathcal{D}_s^{\varepsilon}} \mathbb{1}_D\Big|\int_{\mathbb{S}_\omega^{d-1}\times\mathbb{R}_{v_{s+1}}^d} \varepsilon^{d-1}\big(\omega\cdot(v_{s+1}-v_i)\big)_{\pm}h_{PC}^{(s+1)}\big(S_s^{\pm}\big)\mathbb{1}_{A'\backslash(A\cap A')}\,\mathrm{d}v_{s+1}\,\mathrm{d}\omega\Big|\,\mathrm{d}Z_s\,\mathrm{d}t$$

one simply writes, thanks to the change of variable introduced in Propositions 3 page 92 or 4 page 98 (depending on the sign $\pm$) :

$$Q_3^{\delta} \leq \sum_{p=1}^{P}\int_{\mathcal{D}_{s+1}^{\varepsilon}} \mathbb{1}_{S_s^{\pm}(D_{\delta,p}^{A'})}\big|\alpha_p(\widetilde{Z}_{s+1})\big|\,\mathrm{d}\widetilde{Z}_{s+1},$$

where $D_{\delta,p}^{A'}$ denotes the image by the function $S_s^{\pm}$ of the subset $A'\backslash(A\cap A')$ intersected with the arbitrary subset $D\times\mathbb{S}^{d-1}\times\mathbb{R}^d$ and the product of time interval $[0,\delta]\cap[t_{p-1},t_p[$ with the others variables of integration, that is :

$$D_{\delta,p}^{A'} = \Big(A'\backslash(A\cap A')\Big)\cap\Big(\Big(\big(([0,\delta]\cap[t_{p-1},t_p[)\times\mathcal{D}_s^{\varepsilon}\big)\cap D\Big)\times\mathbb{S}^{d-1}\times\mathbb{R}^d\Big).$$

One now uses the additional hypothesis (5.59) on the decrease in velocity variables of $h_{PC}^{(s+1)}$, that is one assumes that :

$$\big|h_{PC}^{(s+1)}\big(t,\widetilde{Z}_{s+1}\big)\big| \leq \left|\frac{h_{PC}^{(s+1)}(t,\widetilde{Z}_{s+1})}{g_{s+1}\big(t,|\widetilde{V}_{s+1}|\big)}\right|_{L^\infty(\mathcal{D}_{s+1}^\varepsilon)} g_{s+1}\big(t,|\widetilde{V}_{s+1}|\big)$$

almost everywhere on the phase space of $s+1$ particles and for every time $t \in [0,T]$. Using the explicit expression (5.61) of the function $h_{PC}^{(s+1)}$, this means that for all $1 \leq p \leq P$, for all $t \in [t_{p-1}, t_p[$:

$$\big|\alpha_p\big(\widetilde{Z}_{s+1}\big)\big| \leq \left|\frac{h_{PC}^{(s+1)}(t,\widetilde{Z}_{s+1})}{g_{s+1}\big(t,|\widetilde{V}_{s+1}|\big)}\right|_{L^\infty(\mathcal{D}_{s+1}^\varepsilon)} g_{s+1}\big(t,|\widetilde{V}_{s+1}|\big) \tag{5.67}$$

almost everywhere on the phase space of $s+1$ particles, with

$$v_{s+1} \mapsto \big|V_{s+1}\big| g_{s+1}\big(t,|V_{s+1}|\big)$$

integrable for all $t \in [0,T]$ and almost every $(v_1,\ldots,v_s)$ (thanks to the hypothesis controlling the quantity (5.56)) and :

$$(t,v_1,\ldots,v_s) \mapsto \int_{\mathbb{R}^d} |V_{s+1}| g_{s+1}\big(t,|V_{s+1}|\big)\,\mathrm{d}v_{s+1}$$

bounded almost everywhere (thanks to the hypothesis controlling the hypothesis (5.57)).
Of course, one wants to use the bound (5.67) in order to compare the quantities

$$\begin{aligned}\sum_{p=1}^P \int_0^\delta \int_{\mathcal{D}_s^\varepsilon} \int_{\mathbb{S}_\omega^{d-1}\times\mathbb{R}_{v_{s+1}}^d} & \varepsilon^{d-1}\big(\omega\cdot(v_{s+1}-v_i)\big)_\pm \\ & \times \mathbb{1}_{D_{\delta,p}^{A'}} \Big|h_{PC}^{(s+1)}\big(S_s^\pm\big)\Big|\,\mathrm{d}v_{s+1}\,\mathrm{d}\omega\,\mathrm{d}Z_s\,\mathrm{d}t \\ = \sum_{p=1}^P \int_{\mathcal{D}_{s+1}^\varepsilon} & \mathbb{1}_{S_s^\pm(D_{\delta,p}^{A'})} \big|\alpha_p(\widetilde{Z}_{s+1})\big|\,\mathrm{d}\widetilde{Z}_{s+1}\end{aligned} \tag{5.68}$$

and

$$\begin{aligned}&\int_0^\delta \int_{\mathcal{D}_s^\varepsilon} \mathbb{1}_D \int_{\mathbb{S}_\omega^{d-1}\times\mathbb{R}_{v_{s+1}}^d} \mathbb{1}_{R_2\leq|V_{s+1}|}\varepsilon^{d-1}\big|V_{s+1}\big| g_{s+1}\big(t,\big|\big(S_s^\pm\big)^V\big|\big)\,\mathrm{d}v_{s+1}\,\mathrm{d}\omega\,\mathrm{d}Z_s\,\mathrm{d}t \\ &= \varepsilon^{d-1}\big|\mathbb{S}^{d-1}\big| \int_0^\delta \int_{\mathcal{D}_s^\varepsilon} \mathbb{1}_D \int_{\mathbb{R}_{v_{s+1}}^d} \mathbb{1}_{R_2\leq|V_{s+1}|}\big|V_{s+1}\big| g_{s+1}\big(t,|V_{s+1}|\big)\,\mathrm{d}v_{s+1}\,\mathrm{d}Z_s\,\mathrm{d}t.\end{aligned} \tag{5.69}$$

One recalls that the quantity (5.68) is defined by the expression on the right-hand side of the equation (since α_p is only a L^∞ function, one recalls that an integral with respect to only some of its variables is not clearly well-defined without an additional work), while the quantity (5.69) is a usual Lebesgue integral). So to compare the two quantities and use the bound (5.67), one has to perform the same change of variables (the one described in Propositions 3 page 92 or 4 page 98). This is not possible here, for the same reason which leads to work with piecewise constant in time functions if one wants to define the transport-collision operator for time-dependent functions : this change of variable does not work for general functions depending on time.
However, one assumed that the bound g_{s+1} is increasing with respect to time. Besides, the quantity :

$$\left|\frac{h_{PC}^{(s+1)}(t,\widetilde{Z}_{s+1})}{g_{s+1}\big(t,|\widetilde{V}_{s+1}|\big)}\right|_{L^\infty(\mathcal{D}^\varepsilon_{s+1})}$$

is assumed to be uniformly bounded on $[0,T]$. As a consequence, it is possible to obtain an interesting bound on $\big|\alpha_p\big(\widetilde{Z}_{s+1}\big)\big|$, which is sharp enough on the one hand, and which does not depend on time on the other hand. More precisely, since the previous inequality (5.67) holds on the whole time interval $[t_{p-1},t_p[$, one chooses $t=t_{p-1}$ so that, for all $1\le p\le P$:

$$\begin{aligned}\big|\alpha_p\big(\widetilde{Z}_{s+1}\big)\big| &\le \left|\frac{h_{PC}^{(s+1)}(t_{p-1},\widetilde{Z}_{s+1})}{g_{s+1}\big(t_{p-1},|\widetilde{V}_{s+1}|\big)}\right|_{L^\infty(\mathcal{D}^\varepsilon_{s+1})} g_{s+1}\big(t_{p-1},|\widetilde{V}_{s+1}|\big)\\ &\le \left|\frac{h_{PC}^{(s+1)}(t,\widetilde{Z}_{s+1})}{g_{s+1}\big(t,|\widetilde{V}_{s+1}|\big)}\right|_{L^\infty([0,T],L^\infty(\mathcal{D}^\varepsilon_{s+1}))} g_{s+1}\big(t_{p-1},|\widetilde{V}_{s+1}|\big).\end{aligned}$$

One has then immediately :

$$\begin{aligned}Q_3^\delta \le \sum_{p=1}^P \int_{\mathcal{D}^\varepsilon_{s+1}} \mathbb{1}_{S_s^\pm(D^{A'}_{\delta,q})} &\left|\frac{h_{PC}^{(s+1)}(t,\widetilde{Z}_{s+1})}{g_{s+1}\big(t,|\widetilde{V}_{s+1}|\big)}\right|_{L^\infty([0,T],L^\infty(\mathcal{D}^\varepsilon_{s+1}))}\\ &\times g_{s+1}\big(t_{p-1},|\widetilde{V}_{s+1}|\big)\,\mathrm{d}\widetilde{Z}_{s+1}. \qquad (5.70)\end{aligned}$$

Now, in the integrand of the bound the dependency of g_{s+1} on the time variable has been replaced by an evaluation at time t_{p-1}, so that, denoting as usual $V_{s+1}=\big(V_s,v_{s+1}\big)$, one can perform the change of variable backwards :

$$\begin{aligned}Q_3^\delta \le \sum_{p=1}^P \int_0^\delta \mathbb{1}_{[t_{p-1},t_p[} \int_{\mathcal{D}^\varepsilon_s} \mathbb{1}_D \int_{\mathbb{S}^{d-1}_\omega\times\mathbb{R}^d_{v_{s+1}}} &\varepsilon^{d-1}\big(\omega\cdot(v_{s+1}-v_i)\big)_\pm \mathbb{1}_{A'\setminus(A\cap A')}\\ &\times g_{s+1}\big(t_{p-1},|V_{s+1}|\big)\,\mathrm{d}v_{s+1}\,\mathrm{d}\omega\,\mathrm{d}Z_s\,\mathrm{d}t.\end{aligned}$$

Thanks to the Cauchy-Schwarz and then the triangular inequalities on the scalar product $\omega \cdot (v_{s+1} - v_i)$ which provides :

$$\big|\omega \cdot (v_{s+1} - v_i)\big| \leq |v_{s+1} - v_i| \leq |v_i| + |v_{s+1}|,$$

and then again the Cauchy-Schwarz inequality applied to the scalar product of the two vectors $(|v_i|, |v_{s+1}|)$ and $(1,1)$ of $\mathbb{R}^2$, which gives of course :

$$|v_i| + |v_{s+1}| = (|v_i|, |v_{s+1}|) \cdot (1,1) \leq \sqrt{1^2 + 1^2}\sqrt{|v_i|^2 + |v_{s+1}|^2} \leq \sqrt{2}|V_{s+1}|,$$

one obtains

$$Q_3^\delta \leq \sum_{p=1}^{P} \sqrt{2}\varepsilon^{d-1} \int_0^\delta \mathbb{1}_{[t_{p-1},t_p[} \int_{\mathcal{D}_s^\varepsilon} \mathbb{1}_D \int_{\mathbb{S}_\omega^{d-1} \times \mathbb{R}^d_{v_{s+1}}} \mathbb{1}_{A' \setminus (A \cap A')} \big|V_{s+1}\big| \times g_{s+1}\big(t_{p-1}, |V_{s+1}|\big)\, \mathrm{d}v_{s+1}\, \mathrm{d}\omega\, \mathrm{d}Z_s\, \mathrm{d}t.$$

Besides one sees that the last upper bound for Q_3^δ is well defined as the integral of the product of an indicator function of a subset of finite measure, that is a L^1 function, and a L^∞ function, since for all V_s and all integer $1 \leq p \leq P$, the function :

$$v_{s+1} \mapsto |V_{s+1}| g_{s+1}\big(t_{p-1}, |V_{s+1}|\big)$$

is assumed to be integrable (hypothesis (5.56)), and for all $1 \leq p \leq P$, the function :

$$V_{s+1} \mapsto \int_{\mathbb{R}^d} \big|V_{s+1}\big| g_{s+1}\big(t_{p-1}, |V_{s+1}|\big)$$

is assumed to be essentially bounded (hypothesis (5.57)).
Then using that $A' \setminus (A \cap A') \subset \{|V_{s+1}| \geq R_2\}$, and the fact that the integrand of the upper bound does not depend anymore of the variable ω, one finds :

$$Q_3^\delta \leq \sqrt{2}\varepsilon^{d-1}\big|\mathbb{S}^{d-1}\big| \left| \frac{h_{PC}^{(s+1)}(t, \widetilde{Z}_{s+1})}{g_{s+1}\big(t, |\widetilde{V}_{s+1}|\big)} \right|_{L^\infty([0,T], L^\infty(\mathcal{D}_{s+1}^\varepsilon))}$$
$$\times \sum_{p=1}^{P} \int_0^\delta \mathbb{1}_{[t_{p-1},t_p[} \int_{\mathcal{D}_s^\varepsilon} \mathbb{1}_D \int_{\mathbb{R}^d_{v_{s+1}}} \mathbb{1}_{|V_{s+1}| \geq R_2} \big|V_{s+1}\big| g_{s+1}\big(t_{p-1}, |V_{s+1}|\big)\, \mathrm{d}v_{s+1}\, \mathrm{d}Z_s\, \mathrm{d}t,$$

and finally, using the fact that g_{s+1} has been assumed to be increasing in time (hypothesis (5.55)), that is for all $V_{s+1} \in \mathbb{R}^{d(s+1)}$ and all $t \in [t_{p-1}, t_p[$:

$$g_{s+1}\big(t_{p-1}, |V_{s+1}|\big) \leq g_{s+1}\big(t, |V_{s+1}|\big),$$

one obtains :

$$Q_3^\delta \leq \sqrt{2}\varepsilon^{d-1}\big|\mathbb{S}^{d-1}\big| \left| \frac{h_{PC}^{(s+1)}(t, \widetilde{Z}_{s+1})}{g_{s+1}\big(t, |\widetilde{V}_{s+1}|\big)} \right|_{L^\infty([0,T], L^\infty(\mathcal{D}_{s+1}^\varepsilon))}$$
$$\times \sum_{p=1}^{P} \int_0^\delta \mathbb{1}_{[t_{p-1},t_p[} \int_{\mathcal{D}_s^\varepsilon} \mathbb{1}_D \int_{\mathbb{R}^d_{v_{s+1}}} \mathbb{1}_{|V_{s+1}| \geq R_2} \big|V_{s+1}\big| g_{s+1}\big(t, |V_{s+1}|\big)\, \mathrm{d}v_{s+1}\, \mathrm{d}Z_s\, \mathrm{d}t,$$

so that in the end it is possible to gather the different terms of the sum as :

$$\sum_{p=1}^{P} \mathbb{1}_{[t_{p-1},t_p[\cap[0,\delta]} = \mathbb{1}_{[0,\delta]},$$

that is :

$$Q_3^\delta \leq \sqrt{2}\varepsilon^{d-1}\left|\mathbb{S}^{d-1}\right| \left| \frac{h_{PC}^{(s+1)}(t,\widetilde{Z}_{s+1})}{g_{s+1}\big(t,|\widetilde{V}_{s+1}|\big)} \right|_{L^\infty([0,T],L^\infty(\mathcal{D}_{s+1}^\varepsilon))}$$
$$\times \int_0^\delta \int_{\mathcal{D}_s^\varepsilon} \mathbb{1}_D \int_{\mathbb{R}^d_{v_{s+1}}} \mathbb{1}_{|V_{s+1}|\geq R_2} \big|V_{s+1}\big| g_{s+1}\big(t,|V_{s+1}|\big)\,\mathrm{d}v_{s+1}\,\mathrm{d}Z_s\,\mathrm{d}t. \tag{5.71}$$

For the sake of simplicity, one will denote the constant $\sqrt{2}\varepsilon^{d-1}\big|\mathbb{S}^{d-1}\big|$ as $C_2(d,\varepsilon)$.

One has considered only the first part $[0,\delta]$ of the time interval $[0,T]$. For the whole time interval, one decomposes the subset of finite mesure $D \subset [0,T]\times\mathcal{D}_s^\varepsilon$ and one denotes

$$Q_3^T = \int_0^T \int_{\mathcal{D}_s^\varepsilon} \mathbb{1}_D$$
$$\times \left| \int_{\mathbb{S}^{d-1}_\omega \times \mathbb{R}^d_{v_{s+1}}} \varepsilon^{d-1}\big(\omega\cdot(v_{s+1}-v_i)\big)_\pm \mathcal{T}_t^{s+1,\varepsilon} h_{PC}^{(s+1)} \mathbb{1}_{A'\setminus(A\cap A')}\,\mathrm{d}v_{s+1}\,\mathrm{d}\omega \right| \mathrm{d}Z_s\,\mathrm{d}t$$

and then one simply writes

$$Q_3^T = \sum_{n=0}^{T/\delta-1} \int_{n\delta}^{(n+1)\delta} \int_{\mathcal{D}_s^\varepsilon} \mathbb{1}_D$$
$$\times \left| \int_{\mathbb{S}^{d-1}_\omega \times \mathbb{R}^d_{v_{s+1}}} \varepsilon^{d-1}\big(\omega\cdot(v_{s+1}-v_i)\big)_\pm \mathcal{T}_t^{s+1,\varepsilon} h_{PC}^{(s+1)} \mathbb{1}_{A'\setminus(A\cap A')}\,\mathrm{d}v_{s+1}\,\mathrm{d}\omega \right| \mathrm{d}Z_s\,\mathrm{d}t$$

so that, by the change of variable $u = t - n\delta$ in each term of the sum, one obtains :

$$Q_3^T = \sum_{n=0}^{T/\delta-1} \int_0^\delta \int_{\mathcal{D}_s^\varepsilon} \mathbb{1}_{D_{\delta,n}}$$
$$\times \left| \int_{\mathbb{S}^{d-1}_\omega \times \mathbb{R}^d_{v_{s+1}}} \varepsilon^{d-1}\big(\omega\cdot(v_{s+1}-v_i)\big)_\pm \mathcal{T}_{u-n\delta}^{s+1,\varepsilon} h_{PC}^{(s+1)} \mathbb{1}_{A'\setminus(A\cap A')}\,\mathrm{d}v_{s+1}\,\mathrm{d}\omega \right| \mathrm{d}Z_s\,\mathrm{d}t$$

thanks to the conservation of the L^1 norm by the hard sphere transport, with $D_{\delta,n}$ denoting the subset of $[0,\delta]\times\mathcal{D}_s^\varepsilon$ defined as the translation in the time variable of the subset $D\cap[n\delta,(n+1)\delta]$, that is $(u,Z_s)\in D_{\delta,n}$ if and only if

$(u+n\delta, Z_s) \in D \cap [n\delta,(n+1)\delta]$. Each term is then bounded from above in the same way as (5.71) so that one obtains :

$$Q_3^T \leq C_2(d,\varepsilon) \sum_{n=0}^{T/\delta-1} \left| \frac{\mathcal{T}_{t-n\delta}^{s+1,\varepsilon} h_{PC}^{(s+1)}(t,\widetilde{Z}_{s+1})}{g_{s+1}\big(t,|\widetilde{V}_{s+1}|\big)} \right|_{L^\infty([0,T],L^\infty(\mathcal{D}_{s+1}^\varepsilon))} \int_0^\delta \int_{\mathcal{D}_s^\varepsilon} \mathbb{1}_{D_{n,\delta}}$$
$$\times \int_{\mathbb{R}^d_{v_{s+1}}} \mathbb{1}_{R_2 \leq |V_{s+1}|} \big|V_{s+1}\big| g_{s+1}\big(u+n\delta,|V_{s+1}|\big)\, \mathrm{d}v_{s+1}\, \mathrm{d}Z_s\, \mathrm{d}u.$$

The hard sphere transport preserves also the L^∞ norm so that for all integer n, and the bound g_{s+1} is invariant under its action, so that :

$$\begin{aligned}
&\left| \frac{\mathcal{T}_{t-n\delta}^{s+1,\varepsilon} h_{PC}^{(s+1)}(t,\widetilde{Z}_{s+1})}{g_{s+1}\big(t,|\widetilde{V}_{s+1}|\big)} \right|_{L^\infty([0,T],L^\infty(\mathcal{D}_{s+1}^\varepsilon))} \\
&\qquad = \left| \frac{h_{PC}^{(s+1)}\big(t, T_{n\delta-t}^{s+1,\varepsilon}\big(\widetilde{Z}_{s+1}\big)\big)}{g_{s+1}\big(t, T_{n\delta-t}^{s+1,\varepsilon}\big(\widetilde{Z}_{s+1}\big)^V\big)} \right|_{L^\infty([0,T],L^\infty(\mathcal{D}_{s+1}^\varepsilon))} \\
&\qquad = \left| \frac{h_{PC}^{(s+1)}\big(t,\widetilde{Z}_{s+1}\big)}{g_{s+1}\big(t,|\widetilde{V}_{s+1}|\big)} \right|_{L^\infty([0,T],L^\infty(\mathcal{D}_{s+1}^\varepsilon))},
\end{aligned}$$

and finally, using the fact that the quantity

$$\int_{\mathbb{R}^d_{v_{s+1}}} \big|V_{s+1}\big| g_{s+1}\big(t,|V_{s+1}|\big) \mathbb{1}_{R_2 \leq |V_{s+1}|}\, \mathrm{d}v_{s+1}$$

is uniformly controlled in the velocity *and* in time variables by hypotheses, one gets :

$$\begin{aligned}
Q_3^T &\leq C_2(d,\varepsilon)\Big(\sum_{n=0}^{T/\delta-1} \int_0^\delta \int_{\mathcal{D}_s^\varepsilon} \mathbb{1}_{D_{n,\delta}}\, \mathrm{d}Z_s\, \mathrm{d}t \Big) \left| \frac{h_{PC}^{(s+1)}\big(t,\widetilde{Z}_{s+1}\big)}{g_{s+1}\big(t,|\widetilde{V}_{s+1}|\big)} \right|_{L^\infty([0,T],L^\infty(\mathcal{D}_{s+1}^\varepsilon))} \\
&\qquad \times \left\| \int_{\mathbb{R}^d} \mathbb{1}_{|V_{s+1}| \geq R_2} |V_{s+1}| g_{s+1}\big(t,|V_{s+1}|\big)\, \mathrm{d}v_{s+1} \right\|_{L^\infty([0,T],L^\infty_{V_{s+1}}(\mathbb{R}^{d(s+1)}))} \\
&\leq C_2(d,\varepsilon)|D| \left| \frac{h_{PC}^{(s+1)}\big(t,\widetilde{Z}_{s+1}\big)}{g_{s+1}\big(t,|\widetilde{V}_{s+1}|\big)} \right|_{L^\infty([0,T],L^\infty(\mathcal{D}_{s+1}^\varepsilon))} \\
&\qquad \times \left\| \int_{\mathbb{R}^d} \mathbb{1}_{|V_{s+1}| \geq R_2} |V_{s+1}| g_{s+1}\big(t,|V_{s+1}|\big)\, \mathrm{d}v_{s+1} \right\|_{L^\infty([0,T],L^\infty_{V_{s+1}}(\mathbb{R}^{d(s+1)}))}.
\end{aligned}$$

In summary, the fact that the quantity

$$\left\| \int_{\mathbb{R}^d} \mathbb{1}_{|V_{s+1}| \geq R} |V_{s+1}| g_{s+1}\big(t,|V_{s+1}|\big)\, \mathrm{d}v_{s+1} \right\|_{L^\infty([0,T],L^\infty_{V_{s+1}}(\mathbb{R}^{d(s+1)}))}$$

converges to zero as R goes to infinity, implies that one has obtained :

$$\begin{aligned}
&\Big|\mathbb{1}_D \int_{\mathbb{S}^{d-1}_\omega \times \mathbb{R}^d_{v_{s+1}}} \varepsilon^{d-1}\big(\omega\cdot(v_{s+1}-v_i)\big)_\pm \mathcal{T}_t^{s+1,\varepsilon} h_{PC}^{(s+1)} \mathbb{1}_{A'\setminus(A\cap A')}\,\mathrm{d}\omega\,\mathrm{d}v_{s+1}\Big|_{L^1([0,T]\times\mathcal{D}^\varepsilon_s)} \\
&\quad \le C(d,\varepsilon)|D| \left|\frac{h_{PC}^{(s+1)}\big(t,\widetilde{Z}_{s+1}\big)}{g_{s+1}\big(t,|\widetilde{V}_{s+1}|\big)}\right|_{L^\infty([0,T],L^\infty(\mathcal{D}^\varepsilon_{s+1}))} \\
&\qquad \times \left\|\int_{\mathbb{R}^d} \mathbb{1}_{|V_{s+1}|\ge R_2}|V_{s+1}| g_{s+1}\big(t,|V_{s+1}|\big)\,\mathrm{d}v_{s+1}\right\|_{L^\infty([0,T],L^\infty_{V_{s+1}}(\mathbb{R}^{d(s+1)}))}.
\end{aligned} \tag{5.72}$$

Therefore one has obtained a weak L^∞ control of the differences of the operators

$$\mathcal{C}^\varepsilon_{s,s+1,\pm,i}(R_1,R_2',\delta)\mathcal{T}_t^{s+1,\varepsilon} - \mathcal{C}^\varepsilon_{s,s+1,\pm,i}(R_1,R_2,\delta)\mathcal{T}_t^{s+1,\varepsilon}$$

defined on piecewise constant in time functions, which can be summarized, gathering the results of the inequalities (5.65), (5.66) and especially (5.72) as :

$$\begin{aligned}
&\Big|\mathbb{1}_D\Big[\mathcal{C}^\varepsilon_{s,s+1,\pm,i}(R_1,R_2',\delta)\mathcal{T}_t^{s+1,\varepsilon} h_{PC}^{(s+1)} \\
&\qquad - \mathcal{C}^\varepsilon_{s,s+1,\pm,i}(R_1,R_2,\delta)\mathcal{T}_t^{s+1,\varepsilon} h_{PC}^{(s+1)}\Big]\Big|_{L^1([0,T]\times\mathcal{D}^\varepsilon_s)} \\
&\qquad \le C_1(d,s,\varepsilon)\delta(R_1+\delta R_2)^{d(s-1)} R_2^{d(s+1)+1} R_2' \\
&\qquad\qquad \times \Big(R_2^{d-1}(R_2')^{d-1} + (R_1+\delta R_2)^{d-1}\Big)|h_{PC}^{(s+1)}|_{L^\infty([0,T],L^\infty(\mathcal{D}^\varepsilon_{s+1}))} \\
&\qquad + C_2(d,\varepsilon)|D| \left|\frac{h_{PC}^{(s+1)}\big(t,\widetilde{Z}_{s+1}\big)}{g_{s+1}\big(t,|\widetilde{V}_{s+1}|\big)}\right|_{L^\infty([0,T],L^\infty(\mathcal{D}^\varepsilon_{s+1}))} \\
&\qquad \times \left\|\int_{\mathbb{R}^d} \mathbb{1}_{|V_{s+1}|\ge R_2}|V_{s+1}| g_{s+1}\big(t,|V_{s+1}|\big)\,\mathrm{d}v_{s+1}\right\|_{L^\infty([0,T],L^\infty_{V_{s+1}}(\mathbb{R}^{d(s+1)}))}.
\end{aligned} \tag{5.73}$$

Now, one has to check that this control holds in the different limits.

From the control for piecewise constant in time functions to the control for functions of $\mathcal{C}\big([0,T],L^\infty\big(\mathcal{D}^\varepsilon_{s+1}\big)\big)$

First, one will obtain the inequality (5.73) above for all continuous functions in time, that is one will establish the control on the following quantity :

$$\mathcal{C}^\varepsilon_{s,s+1,\pm,i}(R_1,R_2',\delta)\mathcal{T}_t^{s+1,\varepsilon} h^{(s+1)} - \mathcal{C}^\varepsilon_{s,s+1,\pm,i}(R_1,R_2,\delta)\mathcal{T}_t^{s+1,\varepsilon} h^{(s+1)}$$

(that is after taking the limit $j\to+\infty$ for an approximating sequence of piecewise constant in time functions $\big(h_j^{(s+1)}\big)_{j\ge0}$ which converges towards $h^{(s+1)}$

belonging to $\mathcal{C}\big([0,T], L^\infty\big(\mathcal{D}^\varepsilon_{s+1}\big)\big)$.

So let $h^{(s+1)}$ be a function belonging to $\mathcal{C}\big([0,T], L^\infty\big(\mathcal{D}^\varepsilon_{s+1}\big)\big)$, and let $\big(h^{(s+1)}_j\big)_{j\geq 0}$ be a sequence of piecewise constant in time functions belonging to the functional space $L^\infty\big([0,T], L^\infty\big(\mathcal{D}^\varepsilon_{s+1}\big)\big)$, and converging towards $h^{(s+1)}$. For the terms with domains A_1 and A_2, without any additional hypothesis than the uniform convergence, the following inequality holds in the limit thanks to the density of the piecewise constant in time functions in $\mathcal{C}\big([0,T], L^\infty\big(\mathcal{D}^\varepsilon_{s+1}\big)\big)$, and thanks to the extension theorem of a continuous linear application defined on a dense subspace into a complete space (here $L^1\big([0,T]\times\mathcal{D}^\varepsilon_s\big)$. One obtains, on the one hand the rigorous definition of the following quantities, and on the other hand, the bounds :

$$\begin{aligned}&\Big|\mathbb{1}_D\int_{\mathbb{S}^{d-1}_\omega\times\mathbb{R}^d_{v_{s+1}}}\varepsilon^{d-1}\big(\omega\cdot(v_{s+1}-v_i)\big)_\pm\mathcal{T}^{s+1,\varepsilon}_t h^{(s+1)}\mathbb{1}_{A_1}\,\mathrm{d}\omega\,\mathrm{d}v_{s+1}\Big|_{L^1([0,T]\times\mathcal{D}^\varepsilon_s)}\\&\qquad\leq C(d,s,\varepsilon)\delta(R_1+\delta R_2)^{d(s-1)}R_2^{d(s+2)}(R_2')^d\big|h^{(s+1)}\big|_{L^\infty([0,T],L^\infty(\mathcal{D}^\varepsilon_{s+1}))},\end{aligned}\tag{5.74}$$

and

$$\begin{aligned}&\Big|\mathbb{1}_D\int_{\mathbb{S}^{d-1}_\omega\times\mathbb{R}^d_{v_{s+1}}}\varepsilon^{d-1}\big(\omega\cdot(v_{s+1}-v_i)\big)_\pm\mathcal{T}^{s+1,\varepsilon}_t h^{(s+1)}\mathbb{1}_{A_2}\,\mathrm{d}\omega\,\mathrm{d}v_{s+1}\Big|_{L^1([0,T]\times\mathcal{D}^\varepsilon_s)}\\&\qquad\leq C(d,s,\varepsilon)\delta(R_1+\delta R_2)^{ds-1}R_2^{d(s+1)+1}R_2'\big|h^{(s+1)}\big|_{L^\infty([0,T],L^\infty(\mathcal{D}^\varepsilon_{s+1}))},\end{aligned}\tag{5.75}$$

so that the two inequalities (5.65) and (5.66) hold in the limit $j\to+\infty$, that is they hold also for depending on time, continuous function $h^{(s+1)}$ belonging to the functional space $\mathcal{C}\big([0,T], L^\infty\big(\mathcal{D}^\varepsilon_{s+1}\big)\big)$.

For the third term, that is the integral with $A_3 = A'\backslash(A\cap A')$ as domain, if one assumes in addition that $h^{(s+1)}$ verifies also that for all $t\in[0,T]$ and almost everywhere on $\mathcal{D}^\varepsilon_{s+1}$:

$$|h^{(s+1)}(t,Z_{s+1})|\leq g_{s+1}\big(t,|V_{s+1}|\big),$$

with g_{s+1} verifying the assumptions (5.55), (5.56) and (5.58) of the theorem, then for any sequence $\big(h^{(s+1)}_j\big)_{j\geq 0}$ of piecewise constant in time functions taking their values in $L^\infty\big(\mathcal{D}^\varepsilon_{s+1}\big)$ and converging towards $h^{(s+1)}$ in $L^\infty_t L^\infty_{Z_s}$ norm, it is possible to modify it into a sequence of piecewise constant in time functions still converging towards $h^{(s+1)}$ and such that, for all $j\geq 0$:

$$|h^{(s+1)}_j(t,Z_{s+1})|\leq g_{s+1}\big(t,|V_{s+1}|\big)\tag{5.76}$$

for all $t \in [t_{j,(p-1)}, t_{j,p}[$ and almost everywhere on $\mathcal{D}^{\varepsilon}_{s+1}$.
Then the quantity

$$\left| \mathbb{1}_D \int_{\mathbb{S}^{d-1}_{\omega} \times \mathbb{R}^d_{v_{s+1}}} \varepsilon^{d-1} \big(\omega \cdot (v_{s+1} - v_i)\big)_{\pm} \mathcal{T}^{s+1,\varepsilon}_t h^{(s+1)} \mathbb{1}_{A' \setminus (A \cap A')} \, \mathrm{d}\omega \, \mathrm{d}v_{s+1} \right|_{L^1([0,T] \times \mathcal{D}^{\varepsilon}_s)}$$

makes sense since one has

$$\begin{aligned} &\left| \mathbb{1}_D \int_{\mathbb{S}^{d-1}_{\omega} \times \mathbb{R}^d_{v_{s+1}}} \varepsilon^{d-1} \big(\omega \cdot (v_{s+1} - v_i)\big)_{\pm} \mathcal{T}^{s+1,\varepsilon}_t h^{(s+1)} \mathbb{1}_{A' \setminus (A \cap A')} \, \mathrm{d}\omega \, \mathrm{d}v_{s+1} \right|_{L^1([0,T] \times \mathcal{D}^{\varepsilon}_s)} \\ &\quad = \left| \mathbb{1}_D \cdot \mathcal{C}^{\varepsilon}_{s,s+1,\pm,i}(R_1, R_2', \delta) \mathcal{T}^{s+1,\varepsilon}_t \big(\mathbb{1}_{A' \setminus (A \cap A')} \cdot h^{(s+1)}\big) \right|_{L^1([0,T] \times \mathcal{D}^{\varepsilon}_s)}, \end{aligned}$$

the right-hand side being well defined since if $h^{(s+1)}$ belongs to the space $\mathcal{C}\big([0,T], L^{\infty}\big(\mathcal{D}^{\varepsilon}_{s+1}\big)\big)$, then $\mathbb{1}_{A' \setminus (A \cap A')} \cdot h^{(s+1)}$ too. Another way to see the rigorous definition of the quantity is the following. The two sides of the equations make sense and are equal in the particular case when $h^{(s+1)}$ is a piecewise constant in time function, so the equality holds in the limit when $h^{(s+1)}$ is approximated by a sequence of such functions.
By linearity, one gets then :

$$\begin{aligned} &\left| \mathbb{1}_D \int_{\mathbb{S}^{d-1}_{\omega} \times \mathbb{R}^d_{v_{s+1}}} \varepsilon^{d-1} \big(\omega \cdot (v_{s+1} - v_i)\big)_{\pm} \mathcal{T}^{s+1,\varepsilon}_t h^{(s+1)} \mathbb{1}_{A' \setminus (A \cap A')} \, \mathrm{d}\omega \, \mathrm{d}v_{s+1} \right|_{L^1([0,T] \times \mathcal{D}^{\varepsilon}_s)} \\ &\leq \left| \mathbb{1}_D \cdot \mathcal{C}^{\varepsilon}_{s,s+1,\pm,i}(R_1, R_2', \delta) \mathcal{T}^{s+1,\varepsilon}_t \Big(\mathbb{1}_{A' \setminus (A \cap A')} \cdot \big(h^{(s+1)} - h^{(s+1)}_{PC}\big)\Big) \right|_{L^1([0,T] \times \mathcal{D}^{\varepsilon}_s)} \\ &\qquad + \left| \mathbb{1}_D \cdot \mathcal{C}^{\varepsilon}_{s,s+1,\pm,i}(R_1, R_2', \delta) \mathcal{T}^{s+1,\varepsilon}_t \big(\mathbb{1}_{A' \setminus (A \cap A')} \cdot h^{(s+1)}_{PC}\big) \right|_{L^1([0,T] \times \mathcal{D}^{\varepsilon}_s)} \end{aligned}$$

for any piecewise constant in time function $h^{(s+1)}_{PC}$. In particular, for any of the approximating sequence $\big(h^{(s+1)}_j\big)_{j \geq 0}$ converging towards $h^{(s+1)}$ and verifying the dominating property (5.76), one has, after controlling the first term of the right-hand side with the help of Lemma 7 page 126, which controls the L^{∞} norm of the truncated in position, velocity and time collision operator of any function belonging to $\mathcal{C}\big([0,T], L^{\infty}\big(\mathcal{D}^{\varepsilon}_{s+1}\big)\big)$, and after controlling the second one

with (5.72) :

$$\begin{aligned}
&\Big| \mathbb{1}_D \int_{\mathbb{S}^{d-1}_\omega \times \mathbb{R}^d_{v_{s+1}}} \varepsilon^{d-1} \big(\omega \cdot (v_{s+1} - v_i)\big)_\pm \mathcal{T}_t^{s+1,\varepsilon} h^{(s+1)} \mathbb{1}_{A' \setminus (A \cap A')} \, \mathrm{d}\omega \, \mathrm{d}v_{s+1} \Big|_{L^1([0,T] \times \mathcal{D}^\varepsilon_s)} \\
&\quad \leq C_1(d,s) \varepsilon^{d-1} R_2^{d+1} |D| \big| h^{(s+1)} - h_j^{(s+1)} \big|_{L^\infty([0,T], L^\infty(\mathcal{D}^\varepsilon_{s+1}))} \\
&\qquad + C_2(d,\varepsilon) |D| \left| \frac{h_{PC}^{(s+1)}\big(t, \widetilde{Z}_{s+1}\big)}{g_{s+1}\big(t, |\widetilde{V}_{s+1}|\big)} \right|_{L^\infty([0,T], L^\infty(\mathcal{D}^\varepsilon_{s+1}))} \\
&\qquad\quad \times \left\| \int_{\mathbb{R}^d} \mathbb{1}_{|V_{s+1}| \geq R_2} |V_{s+1}| g_{s+1}\big(t, |V_{s+1}|\big) \, \mathrm{d}v_{s+1} \right\|_{L^\infty([0,T], L^\infty_{V_{s+1}}(\mathbb{R}^{d(s+1)}))} .
\end{aligned}$$

Of course, the first term of the right-hand side of the previous inequality can be chosen as small as one wants, so that the inequality (5.72) holds in the limit, that is for any function $h^{(s+1)}$ belonging to $\mathcal{C}\big([0,T], L^\infty\big(\mathcal{D}^\varepsilon_{s+1}\big)\big)$, one has :

$$\begin{aligned}
&\Big| \mathbb{1}_D \int_{\mathbb{S}^{d-1}_\omega \times \mathbb{R}^d_{v_{s+1}}} \varepsilon^{d-1} \big(\omega \cdot (v_{s+1} - v_i)\big)_\pm \mathcal{T}_t^{s+1,\varepsilon} h^{(s+1)} \mathbb{1}_{A' \setminus (A \cap A')} \, \mathrm{d}\omega \, \mathrm{d}v_{s+1} \Big|_{L^1([0,T] \times \mathcal{D}^\varepsilon_s)} \\
&\quad \leq C_2(d,\varepsilon) |D| \left| \frac{h_{PC}^{(s+1)}\big(t, \widetilde{Z}_{s+1}\big)}{g_{s+1}\big(t, |\widetilde{V}_{s+1}|\big)} \right|_{L^\infty([0,T], L^\infty(\mathcal{D}^\varepsilon_{s+1}))} \\
&\qquad \times \left\| \int_{\mathbb{R}^d} \mathbb{1}_{|V_{s+1}| \geq R_2} |V_{s+1}| g_{s+1}\big(t, |V_{s+1}|\big) \, \mathrm{d}v_{s+1} \right\|_{L^\infty([0,T], L^\infty_{V_{s+1}}(\mathbb{R}^{d(s+1)}))} .
\end{aligned} \tag{5.77}$$

After splitting the domain of the difference of truncated collision operators $\mathcal{C}^\varepsilon_{s,s+1,\pm,i}(R_1, R_2', \delta)\mathcal{T}_t^{s+1,\varepsilon}$ and $\mathcal{C}^\varepsilon_{s,s+1,\pm,i}(R_1, R_2, \delta)\mathcal{T}_t^{s+1,\varepsilon}$ in the three terms studied above and gathering the inequalities (5.74), (5.75) and (5.77), one finally

obtains :

$$
\begin{aligned}
&\Bigg|\mathbb{1}_D\Big[\mathcal{C}^{\varepsilon}_{s,s+1,\pm,i}(R_1,R_2',\delta)\mathcal{T}^{s+1,\varepsilon}_t h^{(s+1)} \\
&\qquad\qquad - \mathcal{C}^{\varepsilon}_{s,s+1,\pm,i}(R_1,R_2,\delta)\mathcal{T}^{s+1,\varepsilon}_t h^{(s+1)}\Big]\Bigg|_{L^1([0,T]\times\mathcal{D}^{\varepsilon}_s)} \\
&\quad\leq C_1(d,s,\varepsilon)\delta(R_1+\delta R_2)^{d(s-1)}R_2^{d(s+1)+1}R_2' \\
&\qquad\qquad \times\Big(R_2^{d-1}(R_2')^{d-1}+(R_1+\delta R_2)^{d-1}\Big)|h^{(s+1)}|_{L^\infty([0,T],L^\infty(\mathcal{D}^{\varepsilon}_{s+1}))} \\
&\qquad + C_2(d,\varepsilon)|D|\left|\frac{h^{(s+1)}_{PC}\big(t,\widetilde{Z}_{s+1}\big)}{g_{s+1}\big(t,|\widetilde{V}_{s+1}|\big)}\right|_{L^\infty([0,T],L^\infty(\mathcal{D}^{\varepsilon}_{s+1}))} \\
&\qquad\quad \times\left\|\int_{\mathbb{R}^d}\mathbb{1}_{|V_{s+1}|\geq R_2}|V_{s+1}|g_{s+1}\big(t,|V_{s+1}|\big)\,\mathrm{d}v_{s+1}\right\|_{L^\infty([0,T],L^\infty_{V_{s+1}}(\mathbb{R}^{d(s+1)}))},
\end{aligned}
\tag{5.78}
$$

that is, the inequality (5.73) holds for any continuous in time function.

Removing the truncation parameters in time and position

Now one removes the truncation parameters in time δ, and position R_1. For the truncation parameter δ, one will simply use Lemma 8 page 128. Indeed, for all strictly positive numbers R_1, R, the sequence :

$$
\big(\mathcal{C}^{\varepsilon}_{s,s+1,\pm,i}(R_1,R,\delta)\mathcal{T}^{s+1,\varepsilon}_t h^{(s+1)}\big)_{\delta>0}
$$

is a converging sequence in $L^1\big([0,T]\times\mathcal{D}^{\varepsilon}_s\big)$. If one denotes

$$
\begin{aligned}
Q_{R_2,R_2'} = \Bigg|&\mathbb{1}_D\Big[\mathcal{C}^{\varepsilon}_{s,s+1,\pm,i}(R_1,R_2')\mathcal{T}^{s+1,\varepsilon}_t h^{(s+1)} \\
&\qquad - \mathcal{C}^{\varepsilon}_{s,s+1,\pm,i}(R_1,R_2)\mathcal{T}^{s+1,\varepsilon}_t h^{(s+1)}\Big]\Bigg|_{L^1([0,T]\times\mathcal{D}^{\varepsilon}_s)},
\end{aligned}
$$

one has

$$
\begin{aligned}
Q_{R_2,R_2'} \leq & \Big| \mathbb{1}_D \Big[\mathcal{C}^{\varepsilon}_{s,s+1,\pm,i}(R_1,R_2')\mathcal{T}_t^{s+1,\varepsilon} h^{(s+1)} \\
& \qquad - \mathcal{C}^{\varepsilon}_{s,s+1,\pm,i}(R_1,R_2',\delta)\mathcal{T}_t^{s+1,\varepsilon} h^{(s+1)} \Big] \Big|_{L^1([0,T]\times\mathcal{D}^{\varepsilon}_s)} \\
& + \Big| \mathbb{1}_D \Big[\mathcal{C}^{\varepsilon}_{s,s+1,\pm,i}(R_1,R_2',\delta)\mathcal{T}_t^{s+1,\varepsilon} h^{(s+1)} \\
& \qquad - \mathcal{C}^{\varepsilon}_{s,s+1,\pm,i}(R_1,R_2,\delta)\mathcal{T}_t^{s+1,\varepsilon} h^{(s+1)} \Big] \Big|_{L^1([0,T]\times\mathcal{D}^{\varepsilon}_s)} \\
& + \Big| \mathbb{1}_D \Big[\mathcal{C}^{\varepsilon}_{s,s+1,\pm,i}(R_1,R_2,\delta)\mathcal{T}_t^{s+1,\varepsilon} h^{(s+1)} \\
& \qquad - \mathcal{C}^{\varepsilon}_{s,s+1,\pm,i}(R_1,R_2)\mathcal{T}_t^{s+1,\varepsilon} h^{(s+1)} \Big] \Big|_{L^1([0,T]\times\mathcal{D}^{\varepsilon}_s)}
\end{aligned}
$$

which implies immediately thanks to the inequality (5.78) :

$$
\begin{aligned}
Q_{R_2,R_2'} \leq & \Big| \mathcal{C}^{\varepsilon}_{s,s+1,\pm,i}(R_1,R_2')\mathcal{T}_t^{s+1,\varepsilon} h^{(s+1)} \\
& \qquad - \mathcal{C}^{\varepsilon}_{s,s+1,\pm,i}(R_1,R_2',\delta)\mathcal{T}_t^{s+1,\varepsilon} h^{(s+1)} \Big|_{L^1([0,T]\times\mathcal{D}^{\varepsilon}_s)} \\
& + C(d,s,\varepsilon)\delta(R_1+\delta R_2)^{d(s-1)} R_2^{d(s+1)+1} R_2' \\
& \qquad \times \Big(R_2^{d-1}(R_2')^{d-1} + (R_1+\delta R_2)^{d-1} \Big) |h^{(s+1)}|_{L^\infty([0,T],L^\infty(\mathcal{D}^{\varepsilon}_{s+1}))} \\
& + C_2(d,\varepsilon)|D| \left| \frac{h^{(s+1)}_{PC}\big(t,\widetilde{Z}_{s+1}\big)}{g_{s+1}\big(t,|\widetilde{V}_{s+1}|\big)} \right|_{L^\infty([0,T],L^\infty(\mathcal{D}^{\varepsilon}_{s+1}))} \\
& \times \left\| \int_{\mathbb{R}^d} \mathbb{1}_{|V_{s+1}|\geq R_2} |V_{s+1}| g_{s+1}\big(t,|V_{s+1}|\big)\, \mathrm{d}v_{s+1} \right\|_{L^\infty([0,T],L^\infty_{V_{s+1}}(\mathbb{R}^{d(s+1)}))} \\
& + \Big| \mathcal{C}^{\varepsilon}_{s,s+1,\pm,i}(R_1,R_2,\delta)\mathcal{T}_t^{s+1,\varepsilon} h^{(s+1)} \\
& \qquad - \mathcal{C}^{\varepsilon}_{s,s+1,\pm,i}(R_1,R_2)\mathcal{T}_t^{s+1,\varepsilon} h^{(s+1)} \Big|_{L^1([0,T]\times\mathcal{D}^{\varepsilon}_s)} .
\end{aligned}
$$

Taking $\delta > 0$ small enough, one can choose the first, the second and the fourth terms as small as one desires, so one has shown that :

$$\begin{aligned}
&\Bigg| \mathbb{1}_D \Big[\mathcal{C}^{\varepsilon}_{s,s+1,\pm,i}(R_1, R_2') \mathcal{T}_t^{s+1,\varepsilon} h^{(s+1)} \\
&\qquad\qquad - \mathcal{C}^{\varepsilon}_{s,s+1,\pm,i}(R_1, R_2) \mathcal{T}_t^{s+1,\varepsilon} h^{(s+1)} \Big] \Bigg|_{L^1([0,T]\times\mathcal{D}^{\varepsilon}_s)} \\
&= Q_{R_2,R_2'} \\
&\leq C_2(d,\varepsilon)|D| \left| \frac{h^{(s+1)}_{PC}\big(t, \widetilde{Z}_{s+1}\big)}{g_{s+1}\big(t, |\widetilde{V}_{s+1}|\big)} \right|_{L^\infty([0,T], L^\infty(\mathcal{D}^{\varepsilon}_{s+1}))} \\
&\qquad \times \left\| \int_{\mathbb{R}^d} \mathbb{1}_{|V_{s+1}|\geq R_2} |V_{s+1}| g_{s+1}\big(t, |V_{s+1}|\big) \, \mathrm{d}v_{s+1} \right\|_{L^\infty([0,T], L^\infty_{V_{s+1}}(\mathbb{R}^{d(s+1)}))} .
\end{aligned} \tag{5.79}$$

One sees then that the previous inequality holds in the limit $R_1 \to +\infty$. Indeed, one has :

- thanks to Lemma 11 page 136, for every $R > 0$, the sequence :

  $$\big(\mathcal{C}^{\varepsilon}_{s,s+1,\pm,i}(R_1, R) \mathcal{T}_t^{s+1,\varepsilon} h^{(s+1)} \big)_{R_1>0}$$

  converges almost everywhere towards the almost everywhere bounded function $\mathcal{C}^{\varepsilon}_{s,s+1,\pm,i}(R) \mathcal{T}_t^{s+1,\varepsilon} h^{(s+1)}$,

- thanks to Lemma 10 page 134, for every $R_1, R > 0$, the following inequality holds almost everywhere :

  $$\big| \mathcal{C}^{\varepsilon}_{s,s+1,\pm,i}(R_1, R) \mathcal{T}_t^{s+1,\varepsilon} h^{(s+1)} \big| \leq \big| \mathcal{C}^{\varepsilon}_{s,s+1,\pm,i}(R) \mathcal{T}_t^{s+1,\varepsilon} h^{(s+1)} \big|,$$

- and the right-hand side of the last inequality (5.79) does not depend on the truncation parameter R_1.

Finally, for every measurable subset D of $[0,T] \times \mathcal{D}^{\varepsilon}_s$ of finite measure, the function :

$$\mathbb{1}_D \big| \mathcal{C}^{\varepsilon}_{s,s+1,\pm,i}(R) \mathcal{T}_t^{s+1,\varepsilon} h^{(s+1)} \big|$$

is integrable. Therefore, thanks to the dominated convergence theorem :

$$\begin{aligned}
&\Big|\mathbb{1}_D\Big[\mathcal{C}^{\varepsilon}_{s,s+1,\pm,i}(R_2')\mathcal{T}_t^{s+1,\varepsilon}h^{(s+1)} \\
&\qquad\qquad - \mathcal{C}^{\varepsilon}_{s,s+1,\pm,i}(R_2)\mathcal{T}_t^{s+1,\varepsilon}h^{(s+1)}\Big]\Big|_{L^1([0,T]\times\mathcal{D}^{\varepsilon}_s)} \\
&\quad\le C_2(d,\varepsilon)|D|\left|\frac{h^{(s+1)}_{PC}\big(t,\widetilde{Z}_{s+1}\big)}{g_{s+1}\big(t,|\widetilde{V}_{s+1}|\big)}\right|_{L^\infty([0,T],L^\infty(\mathcal{D}^{\varepsilon}_{s+1}))} \\
&\qquad\times\left\|\int_{\mathbb{R}^d}\mathbb{1}_{|V_{s+1}|\ge R_2}|V_{s+1}|g_{s+1}\big(t,|V_{s+1}|\big)\,\mathrm{d}v_{s+1}\right\|_{L^\infty([0,T],L^\infty_{V_{s+1}}(\mathbb{R}^{d(s+1)}))}.
\end{aligned} \tag{5.80}$$

This last result implies, following an argument now familiar to the reader, that the difference of the truncated in velocity collision operators verifies :

$$\begin{aligned}
&\Big|\mathcal{C}^{\varepsilon}_{s,s+1,\pm,i}(R_2')\mathcal{T}_t^{s+1,\varepsilon}h^{(s+1)} - \mathcal{C}^{\varepsilon}_{s,s+1,\pm,i}(R_2)\mathcal{T}_t^{s+1,\varepsilon}h^{(s+1)}\Big|_{L^\infty([0,T]\times\mathcal{D}^{\varepsilon}_s)} \\
&\quad\le C_2(d,\varepsilon)\left|\frac{h^{(s+1)}_{PC}\big(t,\widetilde{Z}_{s+1}\big)}{g_{s+1}\big(t,|\widetilde{V}_{s+1}|\big)}\right|_{L^\infty([0,T],L^\infty(\mathcal{D}^{\varepsilon}_{s+1}))} \\
&\qquad\times\left\|\int_{\mathbb{R}^d}\mathbb{1}_{|V_{s+1}|\ge R_2}|V_{s+1}|g_{s+1}\big(t,|V_{s+1}|\big)\,\mathrm{d}v_{s+1}\right\|_{L^\infty([0,T],L^\infty_{V_{s+1}}(\mathbb{R}^{d(s+1)}))}.
\end{aligned} \tag{5.81}$$

This last inequality shows that the sequence $\big(\mathcal{C}^{\varepsilon}_{s,s+1,\pm,i}(R_2)\mathcal{T}_t^{s+1,\varepsilon}h^{(s+1)}\big)_{R_2>0}$ is a Cauchy sequence in the complete space $L^\infty\big([0,T]\times\mathcal{D}^{\varepsilon}_s\big)$ providing the additional hypotheses on the decay with respect to the velocity variables, so that it converges in this functional space, which finishes the proof of the rigorous definition of the transport-collision operator $\mathcal{C}^{\varepsilon}_{s,s+,\pm,i}\mathcal{T}_t^{s+1,\varepsilon}$.

Second part of the proof

Now one will obtain the stability, by the transport-collision operator of the BBGKY hierarchy, of the control of the decrease in velocity, that is one will prove the inequality (5.60) stated in the theorem, page 138.

Description of the inequality of stability (5.60) **of the theorem**

The path is very similar to the one leading to the inequality (5.73) page 150. One takes a measurable subset D of $[0,T]\times\mathcal{D}^{\varepsilon}_s$, and a strictly positive number

R. One will show that

$$\int_0^T \int_{\mathcal{D}_s^\varepsilon} \mathbb{1}_D \Big| \mathbb{1}_{|X_s|\leq R}\, \mathcal{C}^\varepsilon_{s,s+1,\pm,i} \mathcal{T}_t^{s+1,\varepsilon} h^{(s+1)}(t,Z_s)\Big| \,\mathrm{d}Z_s\,\mathrm{d}t$$
$$\leq \frac{\varepsilon^{d-1}}{2}\big|\mathbb{S}^{d-1}\big| \int_0^T \int_{\mathcal{D}_s^\varepsilon} \mathbb{1}_D \int_{\mathbb{R}^d_{v_{s+1}}} \big(|v_i|+|v_{s+1}|\big) g_{s+1}\big(t,|V_{s+1}|\big)\,\mathrm{d}v_{s+1}\,\mathrm{d}Z_s\,\mathrm{d}t, \tag{5.82}$$

which enables again to obtain an L^∞ inequality because D is arbitrary, in other words, (5.82) is equivalent to the following assertion, for almost every $(t,Z_s)\in [0,T]\times\mathcal{D}_s^\varepsilon$:

$$\Big|\mathbb{1}_{|X_s|\leq R}\, \mathcal{C}^\varepsilon_{s,s+1,\pm,i} \mathcal{T}_t^{s+1,\varepsilon} h^{(s+1)}(t,Z_s)\Big|$$
$$\leq \frac{\varepsilon^{d-1}}{2}\big|\mathbb{S}^{d-1}\big| \int_{\mathbb{R}^d_{v_{s+1}}} \big(|v_i|+|v_{s+1}|\big) g_{s+1}\big(t,|V_{s+1}|\big)\,\mathrm{d}v_{s+1}.$$

And since the cut-off parameter in position R is arbitrary too, the last inequality holding for all R is of course equivalent to the inequality (5.60) of the theorem.

Proof of the inequality for sequences of piecewise constant in time functions approximating a continuous one

For any $h^{(s+1)}\in\mathcal{C}\big([0,T],L^\infty\big(\mathcal{D}^\varepsilon_{s+1}\big)\big)$ such that for all $t\in[0,T]$ and almost every $Z_{s+1}\in\mathcal{D}^\varepsilon_{s+1}$:

$$\big|h^{(s+1)}(t,Z_{s+1})\big|\leq g_{s+1}\big(t,|V_{s+1}|\big),$$

there exists a sequence of piecewise constant in time functions $\big(h_j^{(s+1)}\big)_{j\geq 0}$ with, for every $j\in\mathbb{N}$:

$$h_j^{(s+1)} = \sum_{p=1}^{P_j} \alpha_{j,p}\mathbb{1}_{[t_{j,(p-1)},t_{j,p}[},$$

and $\alpha_{j,p}\in L^\infty\big(\mathcal{D}^\varepsilon_{s+1}\big)$ for all $1\leq p\leq P_j$, such that :

$$h_j^{(s+1)} \underset{j\to+\infty}{\overset{L^\infty_t L^\infty_{Z_{s+1}}}{\longrightarrow}} h^{(s+1)}.$$

From this sequence, it is then possible to modify it in order to keep the properties of convergence in $\mathcal{C}\big([0,T],L^\infty\big(\mathcal{D}^\varepsilon_{s+1}\big)\big)$ and the fact that each term of this sequence is still piecewise constant in time, with in addition that, for all $j\in\mathbb{N}$, for all $t\in[0,T]$ and for almost every $Z_{s+1}\in\mathcal{D}^\varepsilon_{s+1}$:

$$\big|h_j^{(s+1)}(t,Z_{s+1})\big|\leq \big|h^{(s+1)}(t,Z_{s+1})\big|\leq g_{s+1}\big(t,|V_{s+1}|\big).$$

One has then, for every $j\geq 0$, for every R_1 and R_2 strictly positive numbers, and for every $\delta>0$ verifying the condition (5.37) of Lemma 5 page 122, that

the truncated transport-collision operator is well-defined and if one denotes

$$Q_T = \int_0^T \int_{\mathcal{D}_s^\varepsilon} \mathbb{1}_D \Big| \mathcal{C}^\varepsilon_{s,s+1,\pm,i}(R_1,R_2,\delta)\mathcal{T}_t^{s+1,\varepsilon} h_j^{(s+1)} \Big| (t,Z_s)\,\mathrm{d}Z_s\,\mathrm{d}t,$$

one has, after the usual decomposition of the time interval such that the hard sphere and the free transports coincide :

$$\begin{aligned} Q_T &= \sum_{n=0}^{T/\delta-1} \int_{n\delta}^{(n+1)\delta} \int_{\mathcal{D}_s^\varepsilon} \mathbb{1}_D \Big| \mathcal{C}^\varepsilon_{s,s+1,\pm,i}(R_1,R_2,\delta)\mathcal{T}_t^{s+1,\varepsilon} h_j^{(s+1)} \Big| (t,Z_s)\,\mathrm{d}Z_s\,\mathrm{d}t \\ &= \sum_{n=0}^{T/\delta-1} \sum_{p=1}^{P_j} \int_{n\delta}^{(n+1)\delta} \int_{\mathcal{D}_s^\varepsilon} \mathbb{1}_D \mathbb{1}_{[t_{j,p-1},t_{j,p}[\times\mathcal{D}_s^\varepsilon} \\ &\qquad \times \Bigg| \int_{\mathbb{S}_\omega^{d-1}\times\mathbb{R}^d_{v_{s+1}}} \varepsilon^{d-1}\big(\omega\cdot(v_{s+1}-v_i)\big)_\pm \mathcal{T}_t^{s+1,\varepsilon}\alpha_{j,p}\mathbb{1}_{\mathfrak{D}(R_1,R_2,\delta)}\,\mathrm{d}v_{s+1}\,\mathrm{d}\omega \Bigg| \mathrm{d}Z_s\,\mathrm{d}t, \end{aligned}$$

where $\mathfrak{D}(R_1,R_2,\delta)$ denotes the domain of the truncated in time, position and velocity transport-collision operator, defined by the equation (5.1) page 89. Taking the crude bound :

$$\begin{aligned} Q_T \le \sum_{n=0}^{T/\delta-1} \sum_{p=1}^{P_j} \int_{n\delta}^{(n+1)\delta} \int_{\mathcal{D}_s^\varepsilon} \mathbb{1}_D \mathbb{1}_{[t_{j,p-1},t_{j,p}[\times\mathcal{D}_s^\varepsilon} \\ \times \int_{\mathbb{S}_\omega^{d-1}\times\mathbb{R}^d_{v_{s+1}}} \varepsilon^{d-1}\big(\omega\cdot(v_{s+1}-v_i)\big)_\pm \Big|\mathcal{T}_t^{s+1,\varepsilon}\alpha_{j,p}\Big| \mathbb{1}_{\mathfrak{D}(R_1,R_2,\delta)}\,\mathrm{d}v_{s+1}\,\mathrm{d}\omega\,\mathrm{d}Z_s\,\mathrm{d}t \end{aligned}$$

and then applying the change of variable $u = t - n\delta$ in each term of the first sum as it was done several times before, one gets

$$\begin{aligned} Q_T \le \sum_{n=0}^{T/\delta-1} \sum_{p=1}^{P_j} \int_0^\delta \int_{\mathcal{D}_s^\varepsilon} \mathbb{1}_{D_{n,\delta}} \mathbb{1}_{[t_{j,p-1}-n\delta,t_{j,p}-n\delta[\times\mathcal{D}_s^\varepsilon} \\ \times \int_{\mathbb{S}_\omega^{d-1}\times\mathbb{R}^d_{v_{s+1}}} \varepsilon^{d-1}\big(\omega\cdot(v_{s+1}-v_i)\big)_\pm \Big|\mathcal{T}_{u+n\delta}^{s+1,\varepsilon}\alpha_{j,p}\Big| \mathbb{1}_{\mathfrak{D}(R_1,R_2,\delta)}\,\mathrm{d}v_{s+1}\,\mathrm{d}\omega\,\mathrm{d}Z_s\,\mathrm{d}u, \end{aligned}$$

where $D_{n,\delta}$ denotes, as above, the translation in time of the measurable part D, that is :

$$(u,Z_s) \in D_{n,\delta} \text{ if and only if } (u+n\delta, Z_s) \in D \cap [n\delta,(n+1)\delta].$$

Recalling that the domain $\mathfrak{D}(R_1,R_2,\delta)$ (also denoted $\mathfrak{D}$ at some points in the following for the sake of simplicity) has been chosen in order to have the following equality almost everywhere for all $0 \le u \le \delta$:

$$T_{-u}^{s+1,\varepsilon}\alpha_{j,p}\big(Z_s, x_i+\varepsilon\omega, v_{s+1}\big) = S_s^\pm\big(Z_s, x_i+\varepsilon\omega, v_{s+1}\big),$$

(where the free transport $S_s^{\pm}$ is defined and studied in details in Section 5.1.1 page 91), the last inequality can then be rewritten as :

$$Q_T \leq \sum_{n=0}^{T/\delta-1} \sum_{p=1}^{P_j} \int_0^{\delta} \int_{\mathcal{D}_s^{\varepsilon}} \mathbb{1}_{D_{n,\delta}} \mathbb{1}_{[t_{j,p-1}-n\delta, t_{j,p}-n\delta[\times \mathcal{D}_s^{\varepsilon}} \times \int_{\mathbb{S}_{\omega}^{d-1}\times\mathbb{R}_{v_{s+1}}^{d}} \varepsilon^{d-1}\big(\omega\cdot(v_{s+1}-v_i)\big)_{\pm} \Big|\mathcal{T}_{n\delta}^{s+1,\varepsilon}\alpha_{j,p}\big(S_s^{\pm}\big(Z_s, x_i+\varepsilon\omega, v_{s+1}\big)\big)\Big| \times \mathbb{1}_{\mathfrak{D}(R_1,R_2,\delta)}\,\mathrm{d}v_{s+1}\,\mathrm{d}\omega\,\mathrm{d}Z_s\,\mathrm{d}u.$$

Now the change of variables studied in Propositions 3 and 4, respectively pages 92 and 98, respectively for the two cases $\omega\cdot(v_{s+1}-v_i)>0$ and $\omega\cdot(v_{s+1}-v_i)<0$, provides, if one denotes :

$$D_{\delta,n,p} = \Big(D_{n,\delta} \cap \big([t_{j,p-1}-n\delta, t_{j,p}-n\delta[\,\cap\,[0,\delta]\times\mathcal{D}_s^{\varepsilon}\big)\Big) \times \mathbb{S}^{d-1}\times\mathbb{R}^d \cap \mathfrak{D}(R_1,R_2,\delta),$$

the following inequality :

$$Q_T \leq \sum_{n=0}^{T/\delta-1} \sum_{p=1}^{P_j} \int_{\mathcal{D}_{s+1}^{\varepsilon}} \mathbb{1}_{S_s^{\pm}(D_{\delta,n,p})} \big|\mathcal{T}_{n\delta}^{s+1,\varepsilon}\alpha_{j,p}\big(\widetilde{Z}_{s+1}\big)\big|\,\mathrm{d}\widetilde{Z}_{s+1}.$$

The hypothesis on the uniform bound in time and position verified by $h^{(s+1)}$, and then by construction by each term of the sequence $h_j^{(s+1)}$ means that, for all $u\in[t_{j,p-1},t_{j,p}[$ and for almost every $\widetilde{Z}_{s+1}\in\mathcal{D}_{s+1}^{\varepsilon}$:

$$\begin{aligned}\big|\mathcal{T}_{n\delta}^{s+1,\varepsilon}\alpha_{j,p}(\widetilde{Z}_{s+1})\big| &\leq \left|\frac{\alpha_{j,p}\big(T_{-n\delta}^{s+1,\varepsilon}\big(Z_{s+1}\big)\big)}{g_{s+1}\big(u,\big|\big(T_{-n\delta}^{s+1,\varepsilon}\big(Z_{s+1}\big)\big)^V\big|\big)}\right|_{L^{\infty}(\mathcal{D}_{s+1}^{\varepsilon})} \times g_{s+1}\big(u,\big|\big(T_{-n\delta}^{s+1,\varepsilon}\big(\widetilde{Z}_{s+1}\big)\big)^V\big|\big)\\ &\leq \left|\frac{\alpha_{j,p}(Z_{s+1})}{g_{s+1}\big(u,|V_{s+1}|\big)}\right|_{L^{\infty}(\mathcal{D}_{s+1}^{\varepsilon})} g_{s+1}\big(u,|\widetilde{V}_{s+1}|\big),\end{aligned}$$

thanks to the conservation of the kinetic energy and the L^1 norm by the hard sphere transport.
However, the following bound on Q_T :

$$Q_T \leq \sum_{n=0}^{T/\delta-1} \sum_{p=1}^{P_j} \left|\frac{\alpha_{j,p}(Z_{s+1})}{g_{s+1}\big(u,|V_{s+1}|\big)}\right|_{L^{\infty}(\mathcal{D}_{s+1}^{\varepsilon})} \times \int_{\mathcal{D}_{s+1}^{\varepsilon}} \mathbb{1}_{S_s^{\pm}(D_{\delta,n,p})} g_{s+1}\big(u,|\widetilde{V}_{s+1}|\big)\,\mathrm{d}\widetilde{Z}_{s+1},$$

even if it is of course true, is not satisfactory under this form. The reason is the same as in the discussion in the section about the control of the part A_3, in the first part of the proof, starting page 144. Indeed, one wants to apply the change of variable backwards, since the bound g_{s+1} has a good behaviour with respect to the hard sphere transport (since the velocity variable of the bound g_{s+1} is in fact only the kinetic energy variable, which is preserved by the transport). And as in the section for A_3, one has to tackle the dependency on time of g_{s+1}, otherwise the last control will just be useless.
Nevertheless, the trick used to write previously the control (5.70) page 146 will not be sharp enough here, since one wants to obtain a bound on the transport-collision operator which depends on time (see the inequality (5.60) stated in the theorem), and in particular, on the quantity :

$$\left|\frac{h_j^{(s+1)}(t, Z_{s+1})}{g_{s+1}\big(t, |V_{s+1}|\big)}\right|_{L^\infty(\mathcal{D}^\varepsilon_{s+1})},$$

which cannot be then controlled as brutally as what was done previously by :

$$\left|\frac{h_j^{(s+1)}(t, Z_{s+1})}{g_{s+1}\big(t, |V_{s+1}|\big)}\right|_{L^\infty([0,T], L^\infty(\mathcal{D}^\varepsilon_{s+1}))}.$$

The idea to obtain the control is however somehow similar as the idea used in (5.70). The difference lies in the introduction of another parameter. For all $1 \leq p \leq P_1$, one will divide the time interval :

$$[t_{j,p-1}, t_{j,p}[$$

in Q parts of same size (with Q a positive integer), that is one writes :

$$[t_{j,p-1}, t_{j,p}[= \bigcup_{q=0}^{Q-1} \Big[t_{j,p-1} + q\frac{(t_{j,p} - t_{j,p-1})}{Q}, t_{j,p-1} + (q+1)\frac{(t_{j,p} - t_{j,p-1})}{Q}\Big[.$$

For the convenience of the reader, one will denote

$$t^q_{j,p-1} = t_{j,p-1} + q\frac{(t_{j,p} - t_{j,p-1})}{Q}. \tag{5.83}$$

The fact that the bound g_{s+1} is increasing with respect to the time variable enables to write that for all $t \in [t^q_{j,p-1}, t^{q+1}_{j,p-1}[$:

$$\begin{aligned}\big|\mathcal{T}^{s+1,\varepsilon}_{n\delta}\alpha_{j,p}(\widetilde{Z}_{s+1})\big| &\leq \left|\frac{\alpha_{j,p}(Z_{s+1})}{g_{s+1}\big(u, |V_{s+1}|\big)}\right|_{L^\infty(\mathcal{D}^\varepsilon_{s+1})} g_{s+1}\big(u, |\widetilde{V}_{s+1}|\big)\\ &\leq \left|\frac{\alpha_{j,p}(Z_{s+1})}{g_{s+1}\big(t^q_{j,p-1}, |V_{s+1}|\big)}\right|_{L^\infty(\mathcal{D}^\varepsilon_{s+1})} g_{s+1}\big(t^{q+1}_{j,p-1}, |\widetilde{V}_{s+1}|\big),\end{aligned}$$

so that one can control Q_T writing :

$$Q_T \leq \sum_{n=0}^{T/\delta-1} \sum_{p=1}^{P_j} \sum_{q=1}^{Q-1} \left| \frac{\alpha_{j,p}(Z_{s+1})}{g_{s+1}\big(t^q_{j,p-1}, |V_{s+1}|\big)} \right|_{L^\infty(\mathcal{D}^\varepsilon_{s+1})} \times \int_{\mathcal{D}^\varepsilon_{s+1}} \mathbb{1}_{S^\pm_s(D_{\delta,n,p,q})} g_{s+1}\big(t^{q+1}_{j,p-1}, |\widetilde{V}_{s+1}|\big)\, \mathrm{d}\widetilde{Z}_{s+1},$$

with $D_{\delta,n,p,q}$ denoting :

$$D_{\delta,n,p} = \Big(D_{n,\delta} \cap \big([t^q_{j,p-1} - n\delta, t^{q+1}_{j,p-1} - n\delta[\cap [0,\delta] \times \mathcal{D}^\varepsilon_s\big)\Big) \times \mathbb{S}^{d-1} \times \mathbb{R}^d \cap \mathfrak{D}(R_1, R_2, \delta). \tag{5.84}$$

One understands the great advantage of this new bound : on the one hand, one has removed the dependency with respect to time, so that the change of variables backwards can be used here, and on the other hand, one can hope to recover a very sharp control choosing the parameter Q larger and larger.
So applying then the two changes of variable backwards, one obtains, first with the change of variable $S^\pm_s$:

$$Q_T \leq \sum_{n=0}^{T/\delta-1} \sum_{p=1}^{P_j} \sum_{q=1}^{Q-1} \int_0^\delta \mathbb{1}_{[t^q_{j,p-1}, t^{q+1}_{j,p-1}[} \int_{\mathcal{D}^\varepsilon_s} \mathbb{1}_{D_{n,\delta}} \left| \frac{\alpha_{j,p}(Z_{s+1})}{g_{s+1}\big(t^q_{j,p-1}, |V_{s+1}|\big)} \right|_{L^\infty(\mathcal{D}^\varepsilon_{s+1})} \times \int_{\mathbb{S}^{d-1}_\omega \times \mathbb{R}^d_{v_{s+1}}} \varepsilon^{d-1}\big(\omega \cdot (v_{s+1} - v_i)\big)_\pm g_{s+1}\big(t^{q+1}_{j,p-1}, |V_{s+1}|\big)\, \mathrm{d}v_{s+1}\, \mathrm{d}\omega\, \mathrm{d}Z_s\, \mathrm{d}u$$

(one has deleted also the domain $\mathfrak{D}(R_1, R_2, \delta)$ of the truncated transport-collision operator, which is of no use anymore, and which shows that this control will behave well when the truncated parameters will be removed), and then with the change of time variable $u = t - n\delta$:

$$Q_T \leq \sum_{n=0}^{T/\delta-1} \sum_{p=1}^{P_j} \sum_{q=1}^{Q-1} \int_{n\delta}^{(n+1)\delta} \mathbb{1}_{[t^q_{j,p-1}, t^{q+1}_{j,p-1}[} \int_{\mathcal{D}^\varepsilon_s} \mathbb{1}_D \left| \frac{\alpha_{j,p}(Z_{s+1})}{g_{s+1}\big(t^q_{j,p-1}, |V_{s+1}|\big)} \right|_{L^\infty(\mathcal{D}^\varepsilon_{s+1})} \times \int_{\mathbb{S}^{d-1}_\omega \times \mathbb{R}^d_{v_{s+1}}} \varepsilon^{d-1}\big(\omega \cdot (v_{s+1} - v_i)\big)_\pm g_{s+1}\big(t^{q+1}_{j,p-1}, |V_{s+1}|\big)\, \mathrm{d}v_{s+1}\, \mathrm{d}\omega\, \mathrm{d}Z_s\, \mathrm{d}t$$

which can obviously be rewritten after grouping the terms of the first sum in n :

$$Q_T \leq \sum_{p=1}^{P_j} \sum_{q=1}^{Q-1} \int_0^T \mathbb{1}_{[t^q_{j,p-1}, t^{q+1}_{j,p-1}[} \int_{\mathcal{D}^\varepsilon_s} \mathbb{1}_D \left| \frac{\alpha_{j,p}(Z_{s+1})}{g_{s+1}\big(t^q_{j,p-1}, |V_{s+1}|\big)} \right|_{L^\infty(\mathcal{D}^\varepsilon_{s+1})} \times \int_{\mathbb{S}^{d-1}_\omega \times \mathbb{R}^d_{v_{s+1}}} \varepsilon^{d-1}\big(\omega \cdot (v_{s+1} - v_i)\big)_\pm g_{s+1}\big(t^{q+1}_{j,p-1}, |V_{s+1}|\big)\, \mathrm{d}v_{s+1}\, \mathrm{d}\omega\, \mathrm{d}Z_s\, \mathrm{d}t.$$

Now, after a Cauchy-Schwarz inequality on the scalar product $\omega \cdot (v_{s+1} - v_i)$ in the integrand, and noticing that the condition $\pm\big(\omega \cdot (v_{s+1} - v_i)\big) \geq 0$ in the integrand implies that the domain for the angle parameter variable ω is in fact, for $v_{s+1} - v_i$ fixed :

$$\mathbb{S}^{d-1} \cap \big\{ \pm \big(\omega \cdot (v_{s+1} - v_i)\big) \geq 0 \big\},$$

one gets :

$$Q_T \leq \varepsilon^{d-1} \sum_{p=1}^{P_j} \int_0^T \mathbb{1}_{[t_{j,p-1}, t_{j,p}[} \int_{\mathcal{D}_s^\varepsilon} \mathbb{1}_D \left| \frac{\alpha_{j,p}(Z_{s+1})}{g_{s+1}\big(t_{j,p-1}^q, |V_{s+1}|\big)} \right|_{L^\infty(\mathcal{D}_{s+1}^\varepsilon)}$$
$$\times \int_{\mathbb{R}^d_{v_{s+1}}} \int_{\mathbb{S}^{d-1} \cap \left\{ \pm \left(\omega \cdot (v_{s+1} - v_i)\right) \geq 0 \right\}} \big(|v_i| + |v_{s+1}|\big) g_{s+1}\big(t_{j,p-1}^{q+1}, |V_{s+1}|\big) \, \mathrm{d}\omega \, \mathrm{d}v_{s+1} \, \mathrm{d}Z_s \, \mathrm{d}t.$$

the integrand being integrable thanks to the inequality :

$$|v_i| + |v_{s+1}| \leq |V_{s+1}| + |v_i| \leq \sqrt{2} \big| \big(V_s, v_{s+1}\big) \big|,$$

and since now the integrand does not depend on the integration variable ω, and one has :

$$\big| \mathbb{S}^{d-1} \cap \big\{ \pm \big(\omega \cdot (v_{s+1} - v_i)\big) \geq 0 \big\} \big| = \frac{\big|\mathbb{S}^{d-1}\big|}{2},$$

one gets in the end :

$$Q_T \leq \varepsilon^{d-1} \frac{\big|\mathbb{S}^{d-1}\big|}{2} \sum_{p=1}^{P_j} \sum_{q=1}^{Q-1} \int_{t_{j,p-1}^q}^{t_{j,p-1}^{q+1}} \int_{\mathcal{D}_s^\varepsilon} \mathbb{1}_D \left| \frac{\alpha_{j,p}(Z_{s+1})}{g_{s+1}\big(t_{j,p-1}^q, |V_{s+1}|\big)} \right|_{L^\infty(\mathcal{D}_{s+1}^\varepsilon)}$$
$$\times \int_{\mathbb{R}^d_{v_{s+1}}} \big(|v_i| + |v_{s+1}|\big) g_{s+1}\big(t_{j,p-1}^{q+1}, |V_{s+1}|\big) \, \mathrm{d}v_{s+1} \, \mathrm{d}Z_s \, \mathrm{d}t.$$

Finally, Q is arbitrary, and one has, for every $1 \leq p \leq P_j$ that the sequence of functions

$$\sum_{q-1}^{Q-1} \mathbb{1}_{[t_{j,p-1}^q, t_{j,p-1}^{q+1}[} \left| \frac{\alpha_{j,p}(Z_{s+1})}{g_{s+1}\big(t_{j,p-1}^q, |V_{s+1}|\big)} \right|_{L^\infty(\mathcal{D}_{s+1}^\varepsilon)} g_{s+1}\big(t_{j,p-1}^{q+1}, |V_{s+1}|\big)$$

converges almost everywhere towards

$$\left| \frac{\alpha_{j,p}(Z_{s+1})}{g_{s+1}\big(t, |V_{s+1}|\big)} \right|_{L^\infty(\mathcal{D}_{s+1}^\varepsilon)} g_{s+1}\big(t, |V_{s+1}|\big)$$

since the function g_{s+1} was assumed to be increasing with respect to the time variable. Again, thanks to the increasing property of g_{s+1}, the integrand of the last bound of Q_T is bounded from above by the function :

$$\varepsilon^{d-1}\frac{\left|\mathbb{S}^{d-1}\right|}{2}\mathbb{1}_D\left|\frac{h^{(s+1)_j}\big(t,Z_{s+1}\big)}{g_{s+1}\big(t,|V_{s+1}|\big)}\right|_{L^\infty([0,T],L^\infty(\mathcal{D}^\varepsilon_{s+1}))}\big(|v_i|+|v_{s+1}|\big)g_{s+1}\big(T,|V_{s+1}|\big),$$

which is integrable thanks to the hypotheses of regularity on the bound g_{s+1} set in the statement of the theorem. So the dominated convergence theorem enables to assert, when Q goes to infinity, that :

$$\begin{aligned}Q_T \leq \varepsilon^{d-1}\frac{\left|\mathbb{S}^{d-1}\right|}{2}\sum_{p=1}^{P_j}\sum_{q=1}^{Q-1}\int_0^T\int_{\mathcal{D}^\varepsilon_s}\mathbb{1}_D\left|\frac{h_j^{(s+1)}\big(t,Z_{s+1}\big)}{g_{s+1}\big(t^q_{j,p-1},|V_{s+1}|\big)}\right|_{L^\infty(\mathcal{D}^\varepsilon_{s+1})}\\ \times\int_{\mathbb{R}^d_{v_{s+1}}}\big(|v_i|+|v_{s+1}|\big)g_{s+1}\big(t,|V_{s+1}|\big)\,\mathrm{d}v_{s+1}\,\mathrm{d}Z_s\,\mathrm{d}t.\end{aligned}$$

In the previous control, D is arbitrary and can be replaced by :

$$D\cap|X^s|\leq R.$$

This change provides that, for all $R>0$ and $j\geq 0$:

$$\begin{aligned}&\int_0^T\int_{\mathcal{D}^\varepsilon_s}\mathbb{1}_D\mathbb{1}_{|X_s|\leq R}\left|\mathcal{C}^\varepsilon_{s,s+1,\pm,i}(R_1,R_2,\delta)\mathcal{T}^{s+1,\varepsilon}_t h_j^{(s+1)}(t,Z_s)\right|\mathrm{d}Z_s\,\mathrm{d}t\\ &\quad\leq\varepsilon^{d-1}\frac{\left|\mathbb{S}^{d-1}\right|}{2}\int_0^T\int_{\mathcal{D}^\varepsilon_s}\mathbb{1}_D\int_{\mathbb{R}^d_{v_{s+1}}}\big(|v_i|+|v_{s+1}|\big)g_{s+1}\big(t,|V_{s+1}|\big)\,\mathrm{d}v_{s+1}\,\mathrm{d}Z_s\,\mathrm{d}t.\end{aligned}\tag{5.85}$$

Proof of the inequality for continuous in time functions, and removal of the truncation parameters

To finish the proof of Inequality (5.60) of the theorem, one will now just use the different results of convergence obtained along the work of the rigorous definition of the BBGKY transport-collision operator.

For any measurable subset D of $[0,T]\times\mathcal{D}^\varepsilon_s$ of finite measure and for any strictly positive number R, one decomposes the quantity

$$\int_0^T\int_{\mathcal{D}^\varepsilon_s}\mathbb{1}_D\mathbb{1}_{|X_s|\leq R}\left|\mathcal{C}^\varepsilon_{s,s+1,\pm,i}\mathcal{T}^{s+1,\varepsilon}_t\ h^{(s+1)}(t,Z_s)\right|\mathrm{d}Z_s\,\mathrm{d}t$$

(this quantity is well defined thanks to the first point of the theorem, proved in the first part of the proof) as follows :

$$
\begin{aligned}
&\int_0^T \int_{\mathcal{D}_s^\varepsilon} \mathbb{1}_D \mathbb{1}_{|X_s|\leq R} \left|\mathcal{C}^\varepsilon_{s,s+1,\pm,i}\mathcal{T}_t^{s+1,\varepsilon} h^{(s+1)}(t,Z_s)\right| \mathrm{d}Z_s\,\mathrm{d}t \\
\leq \quad & \int_0^T \int_{\mathcal{D}_s^\varepsilon} \mathbb{1}_D \mathbb{1}_{|X_s|\leq R} \Big|\mathcal{C}^\varepsilon_{s,s+1,\pm,i}\mathcal{T}_t^{s+1,\varepsilon} h^{(s+1)}(t,Z_s) \\
& \qquad\qquad - \mathcal{C}^\varepsilon_{s,s+1,\pm,i}(R_2)\mathcal{T}_t^{s+1,\varepsilon} h^{(s+1)}(t,Z_s)\Big| \mathrm{d}Z_s\,\mathrm{d}t \\
& + \int_0^T \int_{\mathcal{D}_s^\varepsilon} \mathbb{1}_D \mathbb{1}_{|X_s|\leq R} \Big|\mathcal{C}^\varepsilon_{s,s+1,\pm,i}(R_2)\mathcal{T}_t^{s+1,\varepsilon} h^{(s+1)}(t,Z_s) \\
& \qquad\qquad - \mathcal{C}^\varepsilon_{s,s+1,\pm,i}(R_1,R_2)\mathcal{T}_t^{s+1,\varepsilon} h^{(s+1)}(t,Z_s)\Big| \mathrm{d}Z_s\,\mathrm{d}t \\
& + \int_0^T \int_{\mathcal{D}_s^\varepsilon} \mathbb{1}_D \mathbb{1}_{|X_s|\leq R} \Big|\mathcal{C}^\varepsilon_{s,s+1,\pm,i}(R_1,R_2)\mathcal{T}_t^{s+1,\varepsilon} h^{(s+1)}(t,Z_s) \\
& \qquad\qquad - \mathcal{C}^\varepsilon_{s,s+1,\pm,i}(R_1,R_2,\delta)\mathcal{T}_t^{s+1,\varepsilon} h^{(s+1)}(t,Z_s)\Big| \mathrm{d}Z_s\,\mathrm{d}t \\
& + \int_0^T \int_{\mathcal{D}_s^\varepsilon} \mathbb{1}_D \mathbb{1}_{|X_s|\leq R} \Big|\mathcal{C}^\varepsilon_{s,s+1,\pm,i}(R_1,R_2,\delta)\mathcal{T}_t^{s+1,\varepsilon} h^{(s+1)}(t,Z_s) \\
& \qquad\qquad - \mathcal{C}^\varepsilon_{s,s+1,\pm,i}(R_1,R_2,\delta)\mathcal{T}_t^{s+1,\varepsilon} h_j^{(s+1)}(t,Z_s)\Big| \mathrm{d}Z_s\,\mathrm{d}t \\
& + \int_0^T \int_{\mathcal{D}_s^\varepsilon} \mathbb{1}_D \mathbb{1}_{|X_s|\leq R} \Big|\mathcal{C}^\varepsilon_{s,s+1,\pm,i}(R_1,R_2,\delta)\mathcal{T}_t^{s+1,\varepsilon} h_j^{(s+1)}(t,Z_s)\Big| \mathrm{d}Z_s\,\mathrm{d}t.
\end{aligned}
$$

On the one hand, the last term is of course controlled thanks to the inequality (5.85), which concluded the previous section of this part of the proof. On the other hand, the four other ones can be chosen as small as one wants, choosing carefully the truncation parameters R_1, R_2 and δ. Indeed, for any strictly positive number α, one has the following results.

- the first term is bounded by :

  $$|D|\Big|\mathcal{C}^\varepsilon_{s,s+1,\pm,i}\mathcal{T}_t^{s+1,\varepsilon} h^{(s+1)} - \mathcal{C}^\varepsilon_{s,s+1,\pm,i}(R_2)\mathcal{T}_t^{s+1,\varepsilon} h^{(s+1)}\Big|_{L^\infty([0,T]\times\mathcal{D}_s^\varepsilon)},$$

  which can be chosen smaller than $\alpha/3$ for R_2 big enough thanks to the convergence of the sequence of truncated transport-collision operators :

  $$\Big(\mathcal{C}^\varepsilon_{s,s+1,\pm,i}(R_2)\mathcal{T}_t^{s+1,\varepsilon} h^{(s+1)}\Big)_{R_2>0},$$

  which is the first point of Theorem 1 page 137, just shown in the first part of this very proof.
  Let R_2^0 be such R_2, fixed until the end of the proof.

- Thanks to Lemma 11 page 136 about the rigorous definition of the truncated in velocity transport-collision operator, one knows that for any

strictly positive numbers R_2, R, the following equality holds almost everywhere on $|X_s| \leq R$ for R_1 big enough :

$$\mathcal{C}^{\varepsilon}_{s,s+1,\pm,i}(R_2)\mathcal{T}^{s+1,\varepsilon}_t h^{(s+1)} = \mathcal{C}^{\varepsilon}_{s,s+1,\pm,i}(R_1,R_2)\mathcal{T}^{s+1,\varepsilon}_t h^{(s+1)},$$

that is one has :

$$\mathbb{1}_{|X_s|\leq R}\Big|\mathcal{C}^{\varepsilon}_{s,s+1,\pm,i}(R_2)\mathcal{T}^{s+1,\varepsilon}_t h^{(s+1)}(t,Z_s) \\ - \mathcal{C}^{\varepsilon}_{s,s+1,\pm,i}(R_1,R_2)\mathcal{T}^{s+1,\varepsilon}_t h^{(s+1)}(t,Z_s)\Big| = 0$$

almost everywhere on $[0,T]\times\mathcal{D}^{\varepsilon}_s$, in other words the integrand of the second term of the previous decomposition is zero almost everywhere for R_1 big enough, and so the second term is zero itself for such R_1. In particular, for $R_2 = R_2^0$ chosen in the last point, there exists some R_1 verifying the previous equality almost everywhere.
Let R_1^0 be such R_1, fixed until the end of the proof.

- Thanks to Lemma 8 page 128, one knows that for any strictly positive numbers R_1 and R_2, the sequence of operators

$$\Big(\mathcal{C}^{\varepsilon}_{s,s+1,\pm,i}(R_1,R_2,\delta)\mathcal{T}^{s+1,\varepsilon}_t h^{(s+1)}\Big)_{\delta>0}$$

is a converging sequence as δ goes to zero in $L^1\big([0,T]\times\mathcal{D}^{\varepsilon}_s\big)$. In particular for $R_1 = R_1^0$ and $R_2 = R_2^0$, it is then possible to choose δ small enough such that :

$$\begin{aligned}
&\int_0^T\int_{\mathcal{D}^{\varepsilon}_s}\mathbb{1}_D\mathbb{1}_{|X_s|\leq R}\Big|\mathcal{C}^{\varepsilon}_{s,s+1,\pm,i}(R_1,R_2)\mathcal{T}^{s+1,\varepsilon}_t h^{(s+1)}(t,Z_s) \\
&\qquad\qquad - \mathcal{C}^{\varepsilon}_{s,s+1,\pm,i}(R_1,R_2,\delta)\mathcal{T}^{s+1,\varepsilon}_t h^{(s+1)}(t,Z_s)\Big|\,\mathrm{d}Z_s\,\mathrm{d}t \\
&\leq \int_0^T\int_{\mathcal{D}^{\varepsilon}_s}\Big|\mathcal{C}^{\varepsilon}_{s,s+1,\pm,i}(R_1,R_2)\mathcal{T}^{s+1,\varepsilon}_t h^{(s+1)}(t,Z_s) \\
&\qquad\qquad - \mathcal{C}^{\varepsilon}_{s,s+1,\pm,i}(R_1,R_2,\delta)\mathcal{T}^{s+1,\varepsilon}_t h^{(s+1)}(t,Z_s)\Big|\,\mathrm{d}Z_s\,\mathrm{d}t \\
&\leq \alpha/3.
\end{aligned}$$

In particular, for $R_1 = R_1^0$ and $R_2 = R_2^0$ fixed in the two previous points, let δ^0 be such δ, fixed for the end of the proof, which verifies the last inequality.

- Finally the fourth term of the previous decomposition is controlled thanks to the L^∞ estimate (5.48) of the truncated in time, position and velocity transport-collision operator, stated in Lemma 8 page 128. By linearity of

the transport-collision operator, one has :

$$\begin{aligned}
&\int_0^T \int_{\mathcal{D}_s^\varepsilon} \mathbb{1}_D \mathbb{1}_{|X_s|\leq R} \Big| \mathcal{C}^\varepsilon_{s,s+1,\pm,i}(R_1^0, R_2^0, \delta^0) \mathcal{T}_t^{s+1,\varepsilon} h^{(s+1)}(t, Z_s) \\
&\qquad\qquad - \mathcal{C}^\varepsilon_{s,s+1,\pm,i}(R_1^0, R_2^0, \delta^0) \mathcal{T}_t^{s+1,\varepsilon} h_j^{(s+1)}(t, Z_s) \Big| \, \mathrm{d}Z_s \, \mathrm{d}t \\
&\leq |D| \Big| \mathcal{C}^\varepsilon_{s,s+1,\pm,i}(R_1^0, R_2^0, \delta^0) \mathcal{T}_t^{s+1,\varepsilon} \Big(h^{(s+1)} - h_j^{(s+1)} \Big) \Big|_{L^\infty([0,T]\times\mathcal{D}_s^\varepsilon)} \\
&\leq C(d,s) \varepsilon^{d-1} (R_2^0)^{d+1} |D| \big| h^{(s+1)} - h_j^{(s+1)} \big|_{L^\infty([0,T], L^\infty(\mathcal{D}_{s+1}^\varepsilon))}.
\end{aligned}$$

Now, one has just to choose j_0 large enough so that the last upper bound is itself smaller than $\alpha/3$.

Gathering all those controls together, one sees that :

$$\begin{aligned}
&\int_0^T \int_{\mathcal{D}_s^\varepsilon} \mathbb{1}_D \mathbb{1}_{|X_s|\leq R} \, \Big| \mathcal{C}^\varepsilon_{s,s+1,\pm,i} \mathcal{T}_t^{s+1,\varepsilon} h^{(s+1)}(t, Z_s) \Big| \, \mathrm{d}Z_s \, \mathrm{d}t \\
&\leq \alpha/3 + 0 + \alpha/3 + \alpha/3 \\
&\quad + \varepsilon^{d-1} \frac{|\mathbb{S}^{d-1}|}{2} \int_0^T \int_{\mathcal{D}_s^\varepsilon} \mathbb{1}_D \int_{\mathbb{R}^d_{v_{s+1}}} \big(|v_i| + |v_{s+1}| \big) g_{s+1}\big(t, |V_{s+1}|\big) \, \mathrm{d}v_{s+1} \, \mathrm{d}Z_s \, \mathrm{d}t,
\end{aligned}$$

with α being arbitrary, so that the inequality (5.82) page 158 is proven, which concludes the proof of the inequality (5.60) of the theorem.
Theorem 1 is proved. □

Remark 11. *If one denotes $X_{\varepsilon,s,g_{s+1}}$ the Banach space defined as the subset of $\mathcal{C}\big([0,T], L^\infty\big(\mathcal{D}_{s+1}^\varepsilon\big)\big)$ composed of the functions $h^{(s+1)}$ such that :*

$$\Big| \frac{h^{(s+1)}(t, Z_{s+1})}{g_{s+1}\big(t, |V_{s+1}|\big)} \Big|_{L^\infty([0,T], L^\infty(\mathcal{D}_{s+1}^\varepsilon))} < +\infty,$$

one has shown that the collision operator is well-defined on $X_{\varepsilon,s,g_{s+1}}$, and takes its values in $L^\infty\big([0,T] \times \mathcal{D}_s^\varepsilon\big)$, with in addition a boundedness property on the velocity variable, uniform on position but depending on time.

5.2 Definition of the collision-transport operator for the Boltzmann hierarchy, and stability of the relevant functional spaces

Even if there is an additional difficuly to define the collision-transport operator for the Boltzmann hierarchy when the particles evolve around an obstacle regarding the case of the whole Euclidean space, that is without any obstacle, its definition is *much* simpler than the one of the collision operator for the BBGKY

hierarchy. The difficulty comes from the fact that the free flow takes into account the boundary conditions when there is an obstacle, and therefore the particles are not travelling in straight lines due to the bouncing under the action of the transport introduced in Definition 3 page 53.

In the case without obstacle (the reader may refer to [34][7], for example), the Boltzmann hierarchy is defined with a collision operator composed with the free transport, without any boundary condition. Without any obstacle, the free transport preserves continuity, so that the collision-transport operator is easy to define if one decides to work in spaces of continuous functions, up to showing that the collision operator applied to a continuous function makes sense. When there is an obstacle, the free transport is defined with boundary conditions (here : when a particle bounces against the obstacle, its velocity is changed according to the Snell-Descartes law). Such conditions imply a loss of continuity in general. However, the relevant quantity used to define the Boltzmann hierarchy is an *integral* of the collision-transport operator.
So one will show that the collision operator makes sense on some functional space, and preserves continuity. This is the purpose of Subsection 5.2.1, and of the following Lemma 12.
Then, in Subsection 5.2.2 one will discuss the loss of continuity generated by the free flow with boundary condition. In the sequel, this loss will be addressed thanks to boundary conditions satisfied by the solutions of Boltzmann hierarchy.
Afterwards in Subsection 5.2.3, one will show that, even if the functions on which the integrated in time collision-transport operator of the Boltzmann hierarchy acts do not satisfy any boundary condition, the integral over the time variable of the free transport with boundary condition enables to recover the lost continuity. This is the purpose of Lemma 13 page 173.

One recalls the general equation composing the Boltzmann hierarchy (see Section 3 page 77 for more details) :

$$f^{(s)}(t, Z_s) = \mathcal{T}_t^{s,0} f^{(s)}(0, Z_s) + \int_0^t \mathcal{T}_{t-u}^{s,0} \mathcal{C}_{s,s+1}^0 f^{(s+1)}(u, Z_s)\, \mathrm{d}u,$$

with the last integral being explicitly :

$$\begin{aligned}
&\int_0^t \mathcal{T}_{t-u}^{s,0} \mathcal{C}_{s,s+1}^0 f^{(s+1)}(u, Z_s) = \int_0^t \mathcal{C}_{s,s+1}^0 f^{(s+1)}\big(u, T_{t-u}^{s,0}(Z_s)\big)\, \mathrm{d}u \\
&= \sum_{i=1}^{s} \int_0^t \int_{\mathbb{S}_\omega^{d-1} \times \mathbb{R}_{v_{s+1}}^d} \big[\omega \cdot (v_{s+1} - v_i)\big]_+ \Bigg[f^{(s+1)}\Big(u, \Big(T_{u-t}^{s,0}(Z_s), \big(T_{u-t}^{s,0}(Z_s)\big)^{X,i}, v_{s+1}\Big)'_{i,s+1}\Big) \\
&\qquad - f^{(s+1)}(u, T_{u-t}^{s,0}(Z_s), \big(T_{u-t}^{s,0}(Z_s)\big)^{X,i}, v_{s+1}) \Bigg]\, \mathrm{d}\omega\, \mathrm{d}v_{s+1}\, \mathrm{d}u, \qquad (5.86)
\end{aligned}$$

[7] See in particular Section 4.4 "The Boltzmann hierarchy and the Boltzmann equation".

with the notations $\big(T^{s,0}_{u-t}(Z_s)\big)^{X,i}$ and $\Big(T^{s,0}_{u-t}(Z_s), \big(T^{s,0}_{u-t}(Z_s)\big)^{X,i}, v_{s+1}\Big)'_{i,s+1}$ introduced in the points (1.7) and (1.8) of Definition 2 page 52, and in Definition 3 page 53.

5.2.1 Stability of continuous functions by the collision operator of the Boltzmann hierarchy

Lemma 12 (Stability of the subset of continuous functions decaying sufficiently fast at infinity in the velocity variables)**.** *Let T be a strictly positive number. One considers a measurable function :*

$$g_{s+1} : [0,T] \times \mathbb{R}_+ \to \mathbb{R}_+$$

strictly positive almost everywhere such that for all $t \in [0,T]$

$$(v_1, \ldots, v_{s+1}) \mapsto |V_{s+1}| g_{s+1}\big(t, |V_{s+1}|\big) \in L^1\big(\mathbb{R}^{d(s+1)}\big).$$

Then for every continuous function $f^{(s+1)} \in \mathcal{C}\big([0,T] \times \big(\Omega^c \times \mathbb{R}^d\big)^{s+1}\big)$ such that

$$\left| \frac{f^{(s+1)}\big(t, Z_{s+1}\big)}{g_{s+1}\big(t, |V_{s+1}|\big)} \right|_{L^\infty([0,T], L^\infty((\Omega^c \times \mathbb{R}^d)^{s+1}))} < +\infty,$$

one has that for every $(u, Z_s) \in [0,T] \times \big(\Omega^c \times \mathbb{R}^d\big)^s$, the collision operator of the Boltzmann hierarchy (introduced in Definition 10 page 78) defined as

$$\begin{aligned} \mathcal{C}^0_{s,s+1} f^{(s+1)}(u, Z_s) = \sum_{i=1}^{s} \int_{\mathbb{S}^{d-1}_\omega \times \mathbb{R}^d_{v_{s+1}}} \big[\omega \cdot (v_{s+1} - v_i)\big]_+ \\ \times \Big[f^{(s+1)}\big(u, \big(Z_s, x_i, v_{s+1}\big)'_{i,s+1}\big) - f^{(s+1)}(u, Z_s, x_i, v_{s+1}) \Big] \, \mathrm{d}\omega \, \mathrm{d}v_{s+1} \end{aligned} \tag{5.87}$$

is well-defined as a converging integral, and the function

$$\left\{ \begin{array}{rcl} [0,T] \times \big(\Omega^c \times \mathbb{R}^d\big)^s & \to & \mathbb{R}, \\ (u, Z_s) & \mapsto & \mathcal{C}^0_{s,s+1} f^{(s+1)}(u, Z_s), \end{array} \right.$$

is a continuous function such that

$$\left| \frac{\mathcal{C}^0_{s,s+1} f^{(s+1)}(t, Z_s)}{\displaystyle\int_{\mathbb{R}^d} g_{s+1}\big(t, |V_{s+1}|\big) \, \mathrm{d}v_{s+1}} \right|_{L^\infty([0,T], L^\infty((\Omega^c \times \mathbb{R}^d)^s))} < +\infty.$$

Moreover, if $f^{(s+1)}$ belongs to $\mathcal{C}\big([0,T] \times \big(\overline{\Omega^c} \times \mathbb{R}^d\big)^{s+1}\big)$ (Ω^c, which is open, has been replaced by its closure $\overline{\Omega^c}$), then the function :

$$\left\{ \begin{array}{rcl} [0,T] \times \big(\overline{\Omega^c} \times \mathbb{R}^d\big)^s & \to & \mathbb{R}, \\ (u, Z_s) & \mapsto & \mathcal{C}^0_{s,s+1} f^{(s+1)}(u, Z_s), \end{array} \right.$$

is well defined and continuous, such that :

$$\left|\frac{\mathcal{C}^0_{s,s+1} f^{(s+1)}(t, Z_s)}{\displaystyle\int_{\mathbb{R}^d} g_{s+1}\big(t, |(V_s, v_{s+1})|\big)\,\mathrm{d}v_{s+1}}\right|_{L^\infty([0,T], L^\infty((\overline{\Omega^c}\times\mathbb{R}^d)^s))} < +\infty. \tag{5.88}$$

Proof. The proof is the same for the case where $f^{(s+1)}$ is defined and continuous on $\big(\Omega^c\times\mathbb{R}^d\big)^{s+1}$, or $\big(\overline{\Omega^c}\times\mathbb{R}^d\big)^{s+1}$, and this is a simple consequence of the Leibniz integral rule theorem.
First, for every $(u, Z_s) \in [0,T]\times\big(\Omega^c\times\mathbb{R}^d\big)^s$, the quantity $\mathcal{C}^0_{s,s+1} f^{(s+1)}$ is indeed well defined, since one has, for every $(\omega, v_{s+1}) \in \mathbb{S}^{d-1}\times\mathbb{R}^d$:

$$\begin{aligned}
&\left|\big[\omega\cdot(v_{s+1}-v_i)\big]_+\Big[f^{(s+1)}\big(u, (Z_s, x_i, v_{s+1})'_{i,s+1}\big) - f^{(s+1)}(u, Z_s, x_i, v_{s+1})\Big]\right| \\
&\quad\le \big(|v_i|+|v_{s+1}|\big)\Big(\big|f^{(s+1)}\big(u, (Z_s, x, v_{s+1})'_{i,s+1}\big)\big| + \big|f^{(s+1)}(u, Z_s, x_i, v_{s+1})\big|\Big) \\
&\quad\le \big(|v_i|+|v_{s+1}|\big)\left|\frac{f^{(s+1)}(u, Z_{s+1})}{g_{s+1}\big(u, |V_{s+1}|\big)}\right|_{L^\infty(Z_{s+1})} \\
&\qquad\qquad\times\Big(g_{s+1}\big(u, |(v_1,\dots,v'_i,\dots,v'_{s+1})|\big) + g_{s+1}\big(u, |V_{s+1}|\big)\Big) \\
&\quad\le 2\big(|v_i|+|v_{s+1}|\big)\left|\frac{f^{(s+1)}(u, Z_{s+1})}{g_{s+1}\big(u, |V_{s+1}|\big)}\right|_{L^\infty(Z_{s+1})} g_{s+1}\big(u, |V_{s+1}|\big).
\end{aligned} \tag{5.89}$$

So one notices here that each term of the sum defining $\mathcal{C}^0_{s,s+1} f^{(s+1)}(u, Z_s)$ is an integral of a function bounded in absolute value by an integrable function since by hypothesis, for all $t\in[0,T]$ the function $V_{s+1}\mapsto |V_{s+1}| g_{s+1}\big(t, |V_{s+1}|\big)$ is measurable and integrable over $\mathbb{R}^{d(s+1)}$, so by the Fubini-Tonelli theorem, for all $t\in[0,T]$ and almost every $V_s\in\mathbb{R}^{ds}$, the function :

$$v_{s+1}\mapsto g_{s+1}\big(t, |(V_s, v_{s+1})|\big)$$

is a measurable and integrable function.
One has obtained moreover that :

$$\begin{aligned}
\big|\mathcal{C}^0_{s,s+1} f^{(s+1)}(u, Z_s)\big| &\le \sum_{i=1}^{s}\int_{\mathbb{S}^{d-1}_\omega\times\mathbb{R}^d_{v_{s+1}}} \big(|v_i|+|v_{s+1}|\big)\left|\frac{f^{(s+1)}(u, Z_{s+1})}{g_{s+1}\big(u, |V_{s+1}|\big)}\right|_{L^\infty_{Z_{s+1}}} \\
&\qquad\times\Big(g_{s+1}\big(u, |(v_1,\dots,v'_i,\dots,v'_{s+1})|\big) + g_{s+1}\big(u, |V_{s+1}|\big)\Big)\,\mathrm{d}\omega\,\mathrm{d}v_{s+1} \\
&\le 2s|\mathbb{S}^{d-1}|\left|\frac{f^{(s+1)}(u, Z_{s+1})}{g_{s+1}\big(u, |V_{s+1}|\big)}\right|_{L^\infty_{Z_{s+1}}} \\
&\qquad\times\int_{\mathbb{R}^d_{v_{s+1}}} \big(|v_i|+|v_{s+1}|\big) g_{s+1}\big(u, |V_{s+1}|\big)\,\mathrm{d}v_{s+1},
\end{aligned}$$

which proves the control (5.88).
Finally, for the continuity of the collision operator (5.87), this is also a consequence of the bound (5.89) on the integrand, which is uniform in the parameters (u, Z_s), joint with the fact that the function

$$(u, Z_s, \omega, v_{s+1}) \mapsto \big|\omega \cdot (v_{s+1} - v_i)\big|_+ \\ \times \Big[f^{(s+1)}\big(u, \big(Z_s, x_i, v_{s+1}\big)'_{i,s+1}\big) - f^{(s+1)}(u, Z_s, x_i, v_{s+1}) \Big]$$

is of course a continuous function, so the Leibniz integral rule provides the conclusion. □

Remark 12. *The extra condition about the continuity of $f^{(s+1)}$ on the whole closure of the phase space may look strange, but it will be necessary in the following, when the transport operator will be applied.*

5.2.2 Loss of continuity induced by the free transport with boundary conditions

Now, one is about to address the problem of the loss of continuity induced by the free transport with boundary collision. First, one will give an explicit example of such loss of continuity, in a very simple case.

If one considers the initial distribution function f_0 defined on $\mathbb{R}_+^* \times \mathbb{R}$ (here the position and velocity variables will lie in spaces of dimension 1, that is $x \in \mathbb{R}_+^*$ and $v \in \mathbb{R}$, so that in this very simple case, the obstacle is the half line $\{x \leq 0\}$) defined as :

$$\begin{cases} \mathbb{R}_+^* \times \mathbb{R} & \to & \mathbb{R}, \\ \quad (x, v) & \mapsto & f_0(x, v) = v, \end{cases}$$

one sees immediately that f_0 is continuous. Of course such function does not fulfill the conditions of the previous lemma providing the rigorous definition of the collision operator, and moreover the phase space of this function has a dimension which is so small that this object has no chance to be relevant for the study of statistical mechanics in general, but this toy model will be interesting enough to show the process of loss of continuity.
If one applies to f_0 the free transport with boundary condition for a time, say 1, one observes the following thing. Since $\big(\mathcal{T}_1^{1,0} f_0\big)(x, v) = f_0\big(T_{-1}^{1,0}(x, v)\big)$, for every $x > 0$ and $v \leq 0$ (see Definition 15 page 83 for the notation $\mathcal{T}^{1,0}$), one has :

$$x - v > 0,$$

so that $T_{-1}^{1,0}(x, v) = (x - v, v)$ (that is such an initial configuration never leads to a bouncing against the obstable $x = 0$ at time $t = 1$).
But if $v > 0$, depending on the sign of $x - v$, three situations may happen.

- as above, if $x - v > 0$, then $T_{-1}^{1,0}(x, v) = (x - v, v)$

- if $x - v = 0$, it means that starting from the initial configuration (x, v), the particle will be at position 0 at time 1, that is a bouncing against the obstacle occurs at this time, and therefore the free transport with boundary condition $t \mapsto T^{1,0}_{-t}(x, v)$ is discontinuous at $t = 1$, inducing a discontinuity for the function $(t, x, v) \mapsto \mathcal{T}^{1,0}_{-t} f_0(x, v)$ (since the position of the particle has reached the boundary of the domain, and then its velocity will change at this time : v will be changed into $-v$).

- if $x - v < 0$, then a collision has already happened, so that $T^{1,0}_{-1}(x, v) \neq (x - v, v)$. The collision happened at the time t when $x - t_{x,v} v = 0$, that is $t_{x,v} = x/v$, and then the position of the particle at time 1 is then :

$$x - t_{x,v} v - (1 - t_{x,v})(-v) = x - 2t_{x,v} v + v = -x + v,$$

so that in fact $T^{1,0}_{-1}(x, v) = (-x + v, -v)$.

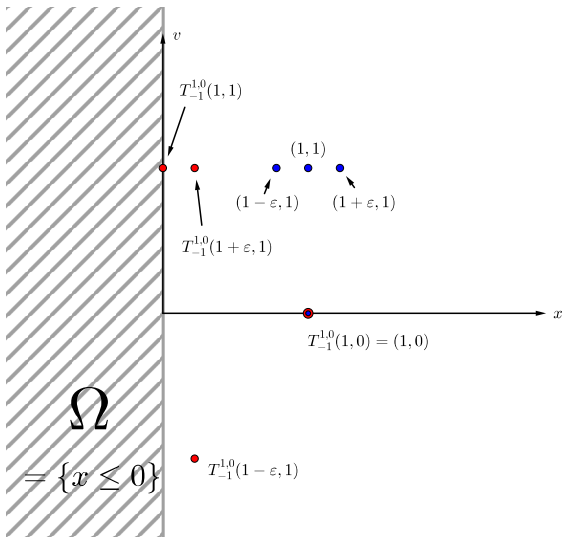

Figure 5.1: The boundary condition induces a jump in the phase space during a bouncing.

If one considers now the special configuration $(1, 1)$ in the phase space, one sees that for $\varepsilon > 0$, as small as one desires, the two positions $(1 + \varepsilon, 1)$ and $(1 - \varepsilon, 1)$ are respectively in the first case and in the last case in the discussion just above. In particular, one obtains :

$$\big(\mathcal{T}^{1,0}_1 f_0\big)(1 + \varepsilon, 1) = f_0(T^{1,0}_{-1}(1 + \varepsilon, 1)) = f_0(\varepsilon, 1) = 1,$$

while

$$\big(\mathcal{T}^{1,0}_1 f_0\big)(1 - \varepsilon, 1) = f_0(T^{1,0}_{-1}(1 - \varepsilon, 1)) = f_0(\varepsilon, -1) = -1.$$

In particular $\mathcal{T}^{1,0}_1 f_0$ is not continuous at the configuration $(1, 1)$ of the phase space. For the specific case of the free transport applied for a time 1, one sees

that the set of discontinuity points of $\mathcal{T}_1^{1,0} f_0$ is exactly composed by the points (x, v) such that $x = v$, that is :

$$\left\{(x, x) \ / \ x \in \mathbb{R}_+^*\right\}.$$

This half line could be seen as the image in the phase space of the boundary of the obstacle by the free transport. In general, that is for an arbitrary time $t > 0$, the set of discontinuity points of $\mathcal{T}_t^{1,0} f_0$ is :

$$\left\{(x, x/t) \ / \ x \in \mathbb{R}_+^*\right\}.$$

5.2.3 Stability of continuous functions by the integrated in time collision-transport operator of the Boltzmann hierarchy

Thanks to the very simple example discussed above, one sees that there is no hope to recover continuity after applying the free transport with boundary collision on general continuous functions such as $\mathcal{C}_{s,s+1}^0 f^{(s+1)}$.

However, even if $\mathcal{T}_{t-u}^{s,0} \mathcal{C}_{s,s+1}^0 f^{(s+1)}$ cannot be regular for general continuous functions $f^{(s+1)}$, one will see thanks to the following lemma, up to assuming a continuity hypothesis on the function $f^{(s+1)}$ on the boundary of the obstacle Ω (which is *not* a boundary condition, linking two different values of $f^{(s+1)}$, evaluated respectively at two points of the boundary of the phase space, such as the condition (3.4) page 79) that the quantity

$$\int_0^t \mathcal{T}_{t-u}^{s,0} \mathcal{C}_{s,s+1}^0 f^{(s+1)}(u, Z_s) \, \mathrm{d}u$$

is smooth, proving therefore a regularizing effect of the collision-transport operator integrated in time of the Boltzmann hierarchy.

One will see along the proof of the following lemma why the assumption about the continuity of $f^{(s+1)}$ on the closure of the phase space is necessary, that is, on the boundary of the obstacle Ω.

Lemma 13 (Stability by the integrated in time collision-transport operator of the Boltzmann hierarchy of the subset of continuous functions decaying sufficiently fast at infinity in the velocity variables)**.** *Let T be a strictly positive number. For every continuous function*

$$f^{(s)} \in \mathcal{C}\big([0, T] \times \big(\overline{\Omega^c} \times \mathbb{R}^d\big)^s\big),$$

and for every $(u, Z_s) \in [0, T] \times \big(\overline{\Omega^c} \times \mathbb{R}^d\big)^s$, the quantity

$$\int_0^t \mathcal{T}_{t-u}^{s,0} f^{(s)}(u, Z_s) \, \mathrm{d}u$$

is well-defined, and the function

$$\left\{\begin{array}{rcl} [0,T]\times\left(\overline{\Omega^c}\times\mathbb{R}^d\right)^s & \to & \mathbb{R}, \\ (t,Z_s) & \mapsto & \displaystyle\int_0^t \mathcal{T}^{s,0}_{t-u} f^{(s)}(u,Z_s)\,\mathrm{d}u, \end{array}\right.$$

is a continuous function.
Moreover, if there exists a measurable function

$$g_s : [0,T]\times\mathbb{R}_+ \to \mathbb{R}_+$$

strictly positive almost everywhere such that for all $t\in[0,T]$ *:*

$$V_s \mapsto |V_s| g_s\big(t,|V_s|\big) \in L^1\big(\mathbb{R}^{ds}\big),$$

such that if for all $t\in[0,T]$, $f^{(s)}$ *satisfies*

$$\left|\frac{f^{(s)}(t,Z_s)}{g_s\big(t,|V_s|\big)}\right|_{L^\infty((\overline{\Omega^s}\times\mathbb{R}^d)^s)} < +\infty,$$

then $\displaystyle\int_0^t \mathcal{T}^{s,0}_{t-u} f^{(s)}\,\mathrm{d}u$ *verifies also for all* $t\in[0,T]$ *and almost every* $Z_s\in\left(\overline{\Omega^c}\times\mathbb{R}^d\right)^s$ *:*

$$\int_0^t \mathcal{T}^{s,0}_{t-u} f^{(s)}(u,Z_s)\,\mathrm{d}u \le \int_0^t \left|\frac{f^{(s)}(u,Z_s)}{g_s\big(u,|V_s|\big)}\right|_{L^\infty_{Z_s}} g_s\big(u,\big|V_s\big|\big)\,\mathrm{d}u. \tag{5.90}$$

Proof. One starts by showing that the quantity $\int_0^t \mathcal{T}^{s,0}_{t-u} f^{(s)}(u,Z_s)\,\mathrm{d}u$ is well-defined for every $(t,Z_s)\in[0,T]\times\left(\overline{\Omega^c}\times\mathbb{R}^d\right)^s$.
First, if $t=0$, of course the quantity is 0. Now one focuses on the more interesting case $t>0$. There is only a finite number $J(Z_s)$ of times t_j ($\in\mathbb{R}$) such that there exists $1\le i_j\le s$ satisfying the condition $\left(T^{s,0}_{t_i}(Z_s)\right)^{X,i_j}\in\partial\Omega$, since one considers only a finite number s of particles, and since the obstacle is convex, each particle can bounce against the obstacle at most once.
It is then clear that the interval $[0,T]$ can be divided, up to excluding a finite subset of points contained in $\{t_1,\dots,t_{J(Z_s)}\}$, into sub-intervals $]t_j,t_{j+1}[$ such that

$$\left\{\begin{array}{rcl} [0,T] & \to & \left(\Omega^c\times\mathbb{R}^d\right)^s, \\ u & \mapsto & T^{s,0}_{-u}(Z_s) \end{array}\right.$$

is continuous on each sub-interval $]t_j,t_{j+1}[$, with a limit when $v\to t_j^+$ or $v\to t_j^-$.
In fact, it is important to keep in mind that the discontinuity only appears in the velocity variable, at the time of a bouncing against the obstacle.
One assumes in addition that the function $f^{(s)}$ is *continuous at the boundary* of the phase space, which means that one assumes that $f^{(s)}$ is continuous on

$$[0,T]\times\left(\overline{\Omega^c}\times\mathbb{R}^d\right)^s,$$

(and not only on the open subset of $\mathbb{R}^{2ds}$ $[0,T]\times\left(\Omega^c\times\mathbb{R}^d\right)^s$). Here the points of the boundary of the phase space are just the configurations such that one of the particle lies on the boundary of the obstacle, that is there exists at least one positive integer $1\leq i\leq s$ such that $x_i=\left(Z_s\right)^{X,i}\in\partial\Omega$, or in other words, in the present context of the half space : $x_i\cdot e_1=0$. The condition of continuity at the boundary means that for any point Z_s belonging to the boundary of the phase space and for any time $t\in[0,T]$, the quantity

$$f^{(s)}(t,\widetilde{Z}_s)$$

has a finite limit when $\widetilde{Z}_s\to Z_s$, then this additional hypothesis is enough to deduce that the function $u\mapsto\mathcal{T}^{s,0}_{t-u}f^{(s)}(u,Z_s)=f^{(s)}(u,T^{s,0}_{u-t}(Z_s))$ is piecewise continuous on $[0,T]$, with finite limits to the left and to the right of any point of discontinuity, so that for every $t\in[0,T]$, its integral on the interval $[0,t]$

$$\int_0^t\mathcal{T}^{s,0}_{t-u}f^{(s)}(u,Z_s)\,\mathrm{d}u=\int_0^t f^{(s)}\left(u,T^{s,0}_{u-t}(Z_s)\right)\mathrm{d}u$$

is well defined, thanks to the Riemann integral theory.
Now, the continuity is obtained by a dominated convergence argument. Let (t,Z_s) be any point of $[0,T]\times\left(\overline{\Omega^c}\times\mathbb{R}^d\right)^s$, one will show that the function

$$(t,Z_s)\mapsto\int_0^t\mathcal{T}^{s,0}_{t-u}f^{(s)}(u,Z_s)\,\mathrm{d}u$$

is continuous at this point.
Let α be a strictly positive real number such that $B_{\mathbb{R}^d}(x_i,\alpha)\subset\Omega^c$ for every integer $1\leq i\leq s$. One defines a neighbourhood of the point Z_s in the phase space, and one considers all the configurations reached from this neighbourhood during the time interval $[0,T]$. In other words, one considers the "tube" based on the neighbourhood around Z_s. One considers the image $T^{s,0}_{[-T,0]}(\mathcal{V}(Z_s))$ of

$$\mathcal{V}(Z_s)=\left(\overline{B_{\mathbb{R}^d}(x_i,\alpha)\times B_{\mathbb{R}^d}(v_i,\alpha)}\right)^s\subset\left(\Omega^c\times\mathbb{R}^d\right)^s$$

by the free transport with boundary conditions for the whole time interval $[0,T]$, that is one defines :

$$\begin{aligned}T^{s,0}_{[-T,0]}(\mathcal{V}(Z_s))=\Big\{Z'_s\in&\left(\Omega^c\times\mathbb{R}^d\right)^s\ /\\&\exists\ t\in[0,T],\widetilde{Z}_s\in\mathcal{V}(Z_s)\ /\ T^{s,0}_{-t}(\widetilde{Z}_s)=Z'_s\Big\}.\end{aligned}$$

This subset of configurations is a compact set of the phase space, and thanks to the continuity on $f^{(s)}$ on the whole phase space and its boundary, $f^{(s)}$ is bounded on $T^{s,0}_{[-T,0]}(\mathcal{V}(Z_s))$.
Now, for any other $(\tilde{t},\widetilde{Z}_s)\in[0,T]\times\left(\overline{\Omega^c}\times\mathbb{R}^d\right)^s$ (one assumes without loss of

generality that $t \leq \tilde{t}$), one has

$$\left| \int_0^{\tilde{t}} \mathcal{T}^{s,0}_{\tilde{t}-u} f^{(s)}(u, \widetilde{Z_s})\, \mathrm{d}u - \int_0^t \mathcal{T}^{s,0}_{t-u} f^{(s)}(u, Z_s)\, \mathrm{d}u \right|$$
$$\leq \left| \int_t^{\tilde{t}} \mathcal{T}^{s,0}_{\tilde{t}-u} f^{(s)}(u, \widetilde{Z_s})\, \mathrm{d}u \right| + \left| \int_0^t \left[\mathcal{T}^{s,0}_{\tilde{t}-u} f^{(s)}(u, \widetilde{Z_s}) - \mathcal{T}^{s,0}_{t-u} f^{(s)}(u, Z_s) \right] \mathrm{d}u \right|.$$

For every $\varepsilon > 0$, there exists $\beta(\varepsilon) > 0$ such that if $\tilde{t} - t < \beta$ and $|\widetilde{Z}_s - Z_s| < \beta$, one has

$$\left| \int_t^{\tilde{t}} \mathcal{T}^{s,0}_{\tilde{t}-u} f^{(s)}(u, \widetilde{Z}_s)\, \mathrm{d}u \right| < \varepsilon/2.$$

Indeed, choosing $\widetilde{Z}_s \in \mathcal{V}(Z_s)$, that is taking $|\widetilde{Z}_s - Z_s| < \alpha$, one has

$$T^{s,0}_{-t}(\widetilde{Z}_s) \in T^{s,0}_{[-T,0]}(\mathcal{V}(Z_s))$$

for all $t \in [0, T]$, so that one has :

$$\left| \int_t^{\tilde{t}} \mathcal{T}^{s,0}_{\tilde{t}-u} f^{(s)}(u, \widetilde{Z}_s)\, \mathrm{d}u \right| \leq \int_t^{\tilde{t}} \left| f^{(s)}\big(u, T^{s,0}_{\tilde{t}-u}(\widetilde{Z}_s)\big) \right| \mathrm{d}u$$
$$\leq (\tilde{t} - t) \Big(\max_{[0,T] \times T^{s,0}_{[-T,0]}(\mathcal{V}(Z_s))} f^{(s)} \Big),$$

which is smaller than $\varepsilon/2$ when

$$\tilde{t} - t < \frac{\varepsilon}{2\Big(\max_{[0,T] \times T^{s,0}_{[-T,0]}(\mathcal{V}(Z_s))} f^{(s)} \Big)},$$

so that it is enough to choose :

$$\beta \leq \min \left(\alpha, \frac{\varepsilon}{2\Big(\max_{[0,T] \times T^{s,0}_{[-T,0]}(\mathcal{V}(Z_s))} f^{(s)} \Big)} \right).$$

For the second term, one uses the fact that the free transport with boundary conditions is continuous in the sense that for a configuration of the phase space Z_s, with a finite number of times of discontinuity $t_1, \ldots, t_{J(Z_s)}$ in the velocity variable, one has that if $\widetilde{Z_s} \to Z_s$, then the sequence of functions

$$T^{s,0}_{[-T,0]}(\widetilde{Z}_s) : \left\{ \begin{array}{rcl} [0,T] & \to & \big(\overline{\Omega^c} \times \mathbb{R}^d\big)^s, \\ t & \mapsto & T^{s,0}_{-t}(\widetilde{Z}_s), \end{array} \right.$$

converges almost everywhere towards the function

$$T^{s,0}_{[-T,0]}(Z_s) : \left\{ \begin{array}{rcl} [0,T] & \to & \big(\overline{\Omega^c} \times \mathbb{R}^d\big)^s, \\ t & \mapsto & T^{s,0}_{-t}(Z_s). \end{array} \right.$$

In fact, one can be more accurate, and say that the convergence holds for any time except for the times of discontinuity of the trajectory from the initial configuration Z_s.
Besides, for $|\widetilde{Z}_s - Z_s| < \alpha$, one has

$$\begin{aligned}
&\left| \int_0^t \Big[\mathcal{T}^{s,0}_{\tilde{t}-u} f^{(s)}(u, \widetilde{Z}_s) - \mathcal{T}^{s,0}_{t-u} f^{(s)}(u, Z_s) \Big] \, \mathrm{d}u \right| \\
&\qquad \leq \int_0^t \left| \mathcal{T}^{s,0}_{\tilde{t}-u} f^{(s)}(u, \widetilde{Z}_s) - \mathcal{T}^{s,0}_{t-u} f^{(s)}(u, Z_s) \right| \mathrm{d}u \\
&\qquad \leq \int_0^t \left| \mathcal{T}^{s,0}_{\tilde{t}-u} f^{(s)}(u, \widetilde{Z}_s) - \mathcal{T}^{s,0}_{t-u} f^{(s)}(u, \widetilde{Z}_s) \right| \mathrm{d}u \\
&\qquad\qquad + \int_0^t \left| \mathcal{T}^{s,0}_{t-u} f^{(s)}(u, \widetilde{Z}_s) - \mathcal{T}^{s,0}_{t-u} f^{(s)}(u, Z_s) \right| \mathrm{d}u. \qquad (5.91)
\end{aligned}$$

The integrand of the first term of the right-hand side of the last inequality (5.91) is

$$f^{(s)}\big(u, T^{s,0}_{u-\tilde{t}}(\widetilde{Z}_s)\big) - f^{(s)}\big(u, T^{s,0}_{u-t}(\widetilde{Z}_s)\big).$$

Using the fact that the norm of the vector composed of all the velocities of the system of particles is not modified along the trajectories, one can control the length of the trajectory of the particles. In other words, it is easy to bound the following difference :

$$\left| T^{s,0}_{u-\tilde{t}}(\widetilde{Z}_s) - T^{s,0}_{u-t}(\widetilde{Z}_s) \right| \leq (\tilde{t} - t) \big| \widetilde{V}_s \big|,$$

where $\widetilde{V}_s$ denotes the vector composed with all the velocities of the configuration $\widetilde{Z}_s$.
But of course, for every $u \in [0, T]$ and every $\widetilde{Z}_s \in \mathcal{V}(Z_s)$, one has

$$T^{s,0}_{u-\tilde{t}}(\widetilde{Z}_s), \ T^{s,0}_{u-\tilde{t}}(Z_s) \in T^{s,0}_{[-T,0]}\big(\mathcal{V}(Z_s)\big),$$

and since $f^{(s)}$ is continuous, it is uniformly continuous on the compact set $[0, T] \times T^{s,0}_{[-T,0]}\big(\mathcal{V}(Z_s)\big)$, and there exists $\gamma > 0$ such that if $|\tilde{t} - t| < \gamma$, then for every $u \in [0, T]$ and every $\widetilde{Z}_s \in \mathcal{V}(Z_s)$:

$$\left| f^{(s)}\big(u, T^{s,0}_{u-\tilde{t}}(\widetilde{Z}_s)\big) - f^{(s)}\big(u, T^{s,0}_{u-t}(\widetilde{Z}_s)\big) \right| < \frac{\varepsilon}{4T},$$

and then of course :

$$\int_0^t \left| \mathcal{T}^{s,0}_{\tilde{t}-u} f^{(s)}(u, \widetilde{Z}_s) - \mathcal{T}^{s,0}_{t-u} f^{(s)}(u, \widetilde{Z}_s) \right| \mathrm{d}u \leq \int_0^T \frac{\varepsilon}{4T} \, \mathrm{d}u = \frac{\varepsilon}{4}.$$

Finally, the second term of the right-hand side of the inequality (5.91) is controlled thanks to the dominated convergence theorem. Indeed, the integrand

$$\left| \mathcal{T}^{s,0}_{t-u} f^{(s)}(u, \widetilde{Z}_s) - \mathcal{T}^{s,0}_{t-u} f^{(s)}(u, Z_s) \right|$$

can be obviously uniformly bounded on $[0,T] \times T^{s,0}_{[-T,0]}\big(\mathcal{V}(Z_s)\big)$ by

$$\max_{[0,T]\times T^{s,0}_{[-T,0]}(\mathcal{V}(Z_s))} f^{(s)},$$

which is of course integrable on the compact interval $[0,t] \subset [0,T]$. The control then holds thanks to the convergence almost everywhere of $\mathcal{T}^{s,0}_{t-u} f^{(s)}(u, \widetilde{Z}_s)$ towards $\mathcal{T}^{s,0}_{t-u} f^{(s)}(u, Z_s)$ when $\widetilde{Z}_s \to Z_s$, that is there exists $\delta > 0$ such that if $\big|\widetilde{Z}_s - Z_s\big| < \delta$, then :

$$\int_0^t \left| \mathcal{T}^{s,0}_{t-u} f^{(s)}(u, \widetilde{Z}_s) - \mathcal{T}^{s,0}_{t-u} f^{(s)}(u, Z_s) \right| \mathrm{d}u < \frac{\varepsilon}{4},$$

so that the continuity of the integrated collision-transport of the Boltzmann hierarchy is proved.
Assuming now that there exists g_s fulfilling the hypothesis of boundedness on

$$\frac{f^{(s)}(t, Z_s)}{g_s\big(t, \big|V_s\big|\big)},$$

one has of course almost everywhere that

$$\begin{aligned}\int_0^t \mathcal{T}^{s,0}_{t-u} f^{(s)}(u, Z_s)\, \mathrm{d}u &\leq \int_0^t \left| \frac{f^{(s)}(u, Z_s)}{g_s\big(u, |V_s|\big)} \right|_{L^\infty((\overline{\Omega^c}\times\mathbb{R}^d)^s)} g_s\big(u, \big|\big(T^{s,0}_{t-u}(Z_s)\big)^V\big|\big)\, \mathrm{d}u \\ &\leq \int_0^t \left| \frac{f^{(s)}(u, Z_s)}{g_s\big(u, |V_s|\big)} \right|_{L^\infty((\overline{\Omega^c}\times\mathbb{R}^d)^s)} g_s\big(u, \big|V_s\big|\big)\, \mathrm{d}u,\end{aligned}$$

so Lemma 13 is entirely proved. □

In fact, in the sequel, the hypothesis of Lemma 13 page 173 of continuity at the boundary will be always fulfilled, since one will consider functions satisfying the stronger boundary condition

$$f^{(s)}(t, Z_s) = f^{(s)}(t, \widehat{Z}_s),$$

where Z_s is a configuration such that at least one of its particles is on the boundary of the obstacle, that is in the case of the half space, when there exists at least one integer $1 \leq i \leq s$ such that

$$x_i \cdot e_1 = 0,$$

with a post-bouncing velocity, that is with

$$v_i \cdot e_1 > 0$$

and where $\widehat{Z}_s$ is the same configuration except that the velocity of the particles on the boundary have been replaced by

$$\widehat{v}_i = \mathcal{S}_0(v_i),$$

the orthogonal symmetry with respect to the boundary of the obstacle, that is the wall $x \cdot e_1 = 0$. Actually, one has in fact naturally recovered the boundary conditions already given in Definition 11 page 79.

5.2.4 The need for a decrease in the entire phase space variable

The loss of uniform continuity in time in the phase space variable

In Section 6.1 one defined properly the transport-collision operator for the BBGKY hierarchy, proceeding in two steps. First, one considered L^∞ functions over the phase space of $s+1$ particles, without any dependence on time. Afterwards, the natural way to extend the result to time-dependent functions was to use piecewise constant in time functions, and then consider the closure of this subset inside the larger functional space $L^\infty\big([0,T], L^\infty\big(\mathcal{D}^\varepsilon_{s+1}\big)\big)$. Finally, the functional space chosen for the rigorous definition of the transport-collision operator of the BBGKY hierarchy was

$$\mathcal{C}\big([0,T], L^\infty\big(\mathcal{D}^\varepsilon_{s+1}\big)\big),$$

with an additional hypothesis of decay in the velocity variable.

For the integrated in time collision-transport operator of the Boltzmann hierarchy, one has seen that the same hypothesis concerning a decrease in the velocity variable is sufficient, and this hypothesis was important in order to be able to assert that the integral (12) page 169 defining the collision operator is converging. But more important here : one has seen that the continuity is preserved, that is if one considers a function $f^{(s+1)}$ belonging to the functional space

$$\mathcal{C}\big([0,T] \times \big(\overline{\Omega^c} \times \mathbb{R}^d\big)^{s+1}\big),$$

(with a decrease in the velocity variable), then its image by the integrated in time collision-transport operator of the Boltzmann hierarchy has the same regularity, that is

$$(t, Z_s) \mapsto \int_0^t \mathcal{T}^{s,0}_{t-u}\mathcal{C}^0_{s,s+1} f^{(s+1)}(u, Z_s)\,\mathrm{d}u \in \mathcal{C}\big([0,T] \times \big(\overline{\Omega^c} \times \mathbb{R}^d\big)^s\big)$$

(this is a simple consequence of Lemmas 12 page 169 and 13 page 173 together).

One recalls that one of the main objectives of this work is to prove a convergence of the solutions of the BBGKY hierarchy towards the solutions of the Boltzmann hierarchy. As a consequence, it is mandatory to find a functional setting for each hierarchy, such that the solutions of the two hierarchies share the same properties. Therefore, one is led to choosing the strongest regularity for the two hierarchies.
However an important problem arises from this additional assumption of continuity (it is important to note here that the elements of $\mathcal{C}\big([0,T], \mathcal{C}\big(\big(\overline{\Omega^c} \times \mathbb{R}^d\big)^s\big)\big)$ have much better regularity than the elements of $\mathcal{C}\big([0,T] \times \big(\overline{\Omega^c} \times \mathbb{R}^d\big)^s\big)$), since

for two times t_1 and t_2 close, one has :

$$\begin{aligned}
&\left| \int_0^{t_2} \mathcal{T}^{s,0}_{t_2-u} \mathcal{C}^0_{s,s+1} f^{(s+1)}(u, Z_s)\,\mathrm{d}u - \int_0^{t_1} \mathcal{T}^{s,0}_{t_1-u} \mathcal{C}^0_{s,s+1} f^{(s+1)}(u, Z_s)\,\mathrm{d}u \right| \\
&\qquad \leq \left| \int_{t_1}^{t_2} \mathcal{T}^{s,0}_{t_2-u} \mathcal{C}^0_{s,s+1} f^{(s+1)}(u, Z_s)\,\mathrm{d}u \right| \\
&\qquad\qquad + \int_0^{t_1} \left| \mathcal{T}^{s,0}_{t_2-u} \mathcal{C}^0_{s,s+1} f^{(s+1)}(u, Z_s) - \mathcal{T}^{s,0}_{t_1-u} \mathcal{C}^0_{s,s+1} f^{(s+1)}(u, Z_s) \right| \mathrm{d}u.
\end{aligned}$$

The first term is controlled without any problem, since the hypothesis on the uniform in time and position decrease in the velocity variable implies in particular that the integrand is bounded, and when $t_2 \to t_1$, it is then clear that the first term goes to zero.
Nevertheless, the second term causes trouble. Indeed, without any additional assumption, it seems impossible to assert that it converges to zero, uniformly in the phase space variable, as t_2 converges to t_1.
In other words, the continuity in time seems to be lost.

Besides, considering the Boltzmann hierarchy in its totality :

$$f^{(s)}(t, Z_s) = \big(\mathcal{T}^{s,0}_t f^{(s)}(0, \cdot)\big)(Z_s) + \int_0^t \mathcal{T}^{s,0}_{t-u} \mathcal{C}^0_{s,s+1} f^{(s+1)}(u, Z_s)\,\mathrm{d}u,$$

in order to obtain something consistent, one wants to have the same regularity for each term. In particular, one desires to have :

$$(t, Z_s) \mapsto \big(\mathcal{T}^{s,0}_t f^{(s)}(0, \cdot)\big)(Z_s) \in \mathcal{C}\big([0, T], \mathcal{C}\big(\big(\overline{\Omega^c} \times \mathbb{R}^d\big)^s\big)\big).$$

Unfortunately, this cannot be true in general for any continuous initial datum $f^{(s)}(0, Z_s)$.
Indeed, if one chooses the case without obstacle (even if it is possible to construct such counter-examples when there is a convex obstacle on which the particles bounce) and if one considers the simple example of a continuous distribution function for one particle $f_0^{(1)} \in \mathcal{C}\big(\overline{\Omega^c} \times \mathbb{R}^d\big)$ such that :

- $\big|f_0^{(1)}(x, v)\big| \leq 1$ for all $(x, v) \in \overline{\Omega^c} \times \mathbb{R}^d$,
- $f^{(1)}(x_k, e_1) = 1$, where x_k denotes the vector $(k, 0, \ldots, 0) \in \mathbb{R}^d$ and e_1 denotes the first vector of the canonical basis of the Euclidean space $\mathbb{R}^d$,
- $f^{(1)}(x, e_1) = 0$ for all $x \in B(x_k, 1/2) \backslash B(x_k, 1/2^k)$,

then it is clear that, for any strictly positive time t, even very small, one can choose a position x_k such that :

$$\big|f^{(1)}(x_k - te_1, e_1) - f^{(1)}(x_k, e_1)\big| = 1,$$

that is, in other words :

$$\left|\mathcal{T}_t^{1,0} f^{(1)}(x_k, e_1) - f^{(1)}(x_k, e_1)\right| = 1,$$

which implies of course that :

$$\left|\mathcal{T}_t^{1,0} f^{(1)} - f^{(1)}\right|_\infty \underset{t\to 0}{\not\to} 0.$$

As a consequence, there is no hope for general continuous initial datum to recover the continuity in time after the application of the free transport (with boundary conditions or not), and then no hope to obtain a well-posed Boltzmann hierarchy in the functional space $\mathcal{C}\big([0,T], \mathcal{C}\big(\big(\overline{\Omega^c} \times \mathbb{R}^d\big)^s\big)\big)$ (even with an additional decrease in the velocity variable, as it was assumed in order to define the BBGKY hierarchy).

The strategy to recover uniform in the phase space variable continuity in time

The phenomenon behind this loss of continuity in time is quite easy to identify : it is a problem of concentration of the derivative. In other words, the problem is that, even though a continuous function is bounded, its derivative can reach very high values.
Of course, this problem is specific to the unbounded setting. In the torus for example, no such example of functions can be found.

To prevent this problem, there are mainly two different solutions.

- One can either ask in addition that the space derivative of the functions involved to solve the Boltzmann hierarchy is bounded. To be more precise, since one works with continuous functions only, so that they cannot be differentiated in general, one can work with Lipschitz functions. However, from the use of this additional assumption emerge two problems. First, the uniform norm used to compare functions involved to define the collision-transport operators does not see the Lipschitz nature of functions, so one should instead consider another norm, an then work again to check if the collision operator makes sense for both of the hierarchies. Second, and it is more or less the same kind of problem, the subset of the Lispchitz functions is not closed in the set of the continuous functions for the uniform norm. Since the argument of existence and uniqueness of solutions of the hierarchies is based on a fixed point argument, as it will be detailed in the sequel, the setting of a Banach space is essential, but this structure is lost with this additional assumption.

- Or one can ask in addition that the functions go to zero when the phase space variable becomes large. Behind, this is the idea of using the theory of continuous functions defined on compact sets : if the compactness is not fulfilled, one will just ask in addition that, for any $\varepsilon > 0$, there exists

a compact set in the phase space such that, outside this set, the function is smaller than ε, that is negligible. This explains why one did not need any other assumptions than the continuity in the case of particles evolving inside a torus, which is of course a compact manifold (see [34]).
The great advantage of this choice is that the set of continuous functions that go to zero at infinity is closed in the set of continuous functions, and then it is a Banach space. Moreover, the fact that the initial data, which are in the context of the statistical physics density functions, go to zero at infinity in the position variable can be understood as if there is no mass at infinity, which is a quite satisfying assumption.

- One can in fact state a more general, and then less demanding, assumption on the initial data. In fact it is enough to assume that the initial data are uniformly continuous with respect to the position variable. This setting will not be used in this work, however since the subset of uniformly continuous functions is closed for the uniform norm in the set of continuous functions, it is also an interesting setting.

It is now clear that the two problems identified in the previous paragraph, at the beginning of this Section 5.2.4 are now tackled with this additional assumption. Indeed, for any function $\widetilde{f}^{(s)}$ given by the expression

$$\widetilde{f}^{(s)}(t, Z_s) = \big(\mathcal{T}_t^{s,0} f_0^{(s)}\big)(Z_s) + \int_0^t \mathcal{T}_{t-u}^{s,0} \mathcal{C}_{s,s+1}^0 f^{(s+1)}(u, Z_s)\,\mathrm{d}u,$$

with $f_0^{(s)}$ and $f^{(s+1)}$ continuous and vanishing at infinity, if one wants to control uniformly in the phase space variable Z_s the difference

$$\widetilde{f}^{(s)}(t_1, Z_s) - \widetilde{f}^{(s)}(t_2, Z_s)$$

as $t_2 - t_1$ goes to zero, first one uses the decrease at infinity of the initial datum $f_0^{(s)}$ and of the function $f^{(s+1)}$. Outside a ball of large radius in the phase space,$f^{(s+1)}$ and $f_0^{(s)}$ are uniformly small, and so are $\mathcal{T}_t^{s,0} f_0^{(s)}(Z_s)$, and $\displaystyle\int_0^t \mathcal{T}_{t-u}^{s,0} \mathcal{C}_{s,s+1}^0 f^{(s+1)}(u, Z_s)\,\mathrm{d}u$.

Now for the remaining part of the phase space, one notices that in any compact set the uniform continuity of continuous functions can be used, wich enables to evaluate uniformly in the phase space variable :

$$\mathcal{T}_{t_1-t_2}^{s,0} f^{(s)}(Z_s) - f^{(s)}(Z_s),$$

and

$$\int_0^{t_2} \mathcal{T}_{t_2-u}^{s,0} \mathcal{C}_{s,s+1}^0 f^{(s+1)}(u, Z_s)\,\mathrm{d}u - \int_0^{t_1} \mathcal{T}_{t_1-u}^{s,0} \mathcal{C}_{s,s+1}^0 f^{(s+1)}(u, Z_s)\,\mathrm{d}u,$$

so that the continuity in time, uniformly in the phase space variable, is recovered. The following lemma presents the result rigorously.

Lemma 14 (Uniform in the phase space variable continuity in time for continuous functions verifying decreasing hypothesis with respect to the velocity variable and vanishing at infinity)**.** *Let T be a strictly positive number, and s be a positive integer. One considers two measurable functions $g_s : \mathbb{R}_+ \to \mathbb{R}_+$ and $g_{s+1} : [0,T] \times \mathbb{R}_+ \to \mathbb{R}_+$ strictly positive almost everywhere such that for all $t \in [0,T]$:*

$$V_s \mapsto |V_s| g_s\big(t, |V_s|\big) \in L^1\big(\mathbb{R}^{ds}\big),$$

and

$$V_{s+1} \mapsto |V_{s+1}| g_{s+1}\big(t, |V_{s+1}|\big) \in L^1\big(\mathbb{R}^{d(s+1)}\big).$$

Then for every continuous functions :

$$f_0^{(s)} \in \mathcal{C}_0\big(\big(\overline{\Omega^c} \times \mathbb{R}^d\big)^s\big),\ f^{(s+1)} \in \mathcal{C}_0\big([0,T] \times \big(\overline{\Omega^c} \times \mathbb{R}^d\big)^{s+1}\big)$$

(where $\mathcal{C}_0(X)$ denotes of course the continuous functions on X vanishing at infinity) such that

$$\left| \frac{f^{(s)}(Z_s)}{g_s\big(t, |V_{s+1}|\big)} \right|_{L^\infty((\overline{\Omega^c} \times \mathbb{R}^d)^s)} < +\infty$$

and

$$\left| \frac{f^{(s+1)}(t, Z_{s+1})}{g_{s+1}\big(t, |V_{s+1}|\big)} \right|_{L^\infty([0,T], L^\infty((\overline{\Omega^c} \times \mathbb{R}^d)^s))} < +\infty,$$

the continuous functions

$$\begin{cases} [0,T] \times \big(\overline{\Omega^c} \times \mathbb{R}^d\big)^s & \to \mathbb{R}, \\ \qquad\qquad (t, Z_s) & \mapsto \mathcal{T}_t^{s,0} f_0^{(s)}(t, Z_s) \end{cases}$$

and

$$\begin{cases} [0,T] \times \big(\overline{\Omega^c} \times \mathbb{R}^d\big)^s & \to \mathbb{R}, \\ \qquad\qquad (t, Z_s) & \mapsto \int_0^t \mathcal{T}_{t-u}^{s,0} \mathcal{C}_{s,s+1}^0 f^{(s+1)}(u, Z_s)\, \mathrm{d}u \end{cases}$$

belong to the space $\mathcal{C}_0\big([0,T] \times \big(\overline{\Omega^c} \times \mathbb{R}^d\big)^s\big)$ of uniform in the phase space variable continuous in time functions.

Remark 13. *One finishes this section with a short discussion about some general considerations related to functions vanishing at infinity in some variables, and depending in addition on another parameter.*

- *The functional space $\mathcal{C}_0\big([0,T] \times \big(\overline{\Omega^c} \times \mathbb{R}^d\big)^s\big)$ of continuous functions on $[0,T] \times \big(\overline{\Omega^c} \times \mathbb{R}^d\big)^s$ vanishing at infinity is defined as the set of continuous functions $f^{(s)}$ on $[0,T] \times \big(\overline{\Omega^c} \times \mathbb{R}^d\big)^s$ such that :*

  $$|f^{(s)}(t, Z_s)| \underset{|(t,Z_s)| \to +\infty}{\longrightarrow} 0.$$

  The time variable t belongs to the bounded interval $[0,T]$, so of course t cannot go to infinity. The notation $|(t, Z_s)| \to +\infty$ emphasizes on the fact

that the convergence to zero has to hold not only in the case when t is fixed.
This space must not be confused with the space of continuous functions on $[0,T] \times \left(\overline{\Omega^c} \times \mathbb{R}^d\right)^s$ such that, for all $t \in [0,T]$, one has :

$$|f^{(s)}(t, Z_s)| \underset{|Z_s| \to +\infty}{\longrightarrow} 0,$$

since the last one is strictly larger than $\mathcal{C}_0\big([0,T] \times \left(\overline{\Omega^c} \times \mathbb{R}^d\right)^s\big)$.
Indeed, it is easy to construct an element belonging to the last space while it is not in the first, replacing $\left(\overline{\Omega^c} \times \mathbb{R}^d\right)^s$ by $\mathbb{R}$ in order to simplify the setting. To do so, one can consider the function :

$$f = \begin{cases} [0,T] \times \mathbb{R} & \to \mathbb{R}, \\ \quad (t, Z_s) & \mapsto f(t,x) = \mathbb{1}_{[0, \frac{1}{2(|x|+1)}]} 2(|x|+1)t \\ & \qquad\qquad + \mathbb{1}_{]\frac{1}{2(|x|+1)}, \frac{1}{|x|+1}]} 2(1 - (|x|+1)t). \end{cases}$$

- *The elements $f^{(s)}$ of the functional space $\mathcal{C}_0\big([0,T] \times \left(\overline{\Omega^c} \times \mathbb{R}^d\right)^s\big)$ verify the stronger vanishing property :*

$$\forall \alpha > 0, \exists R > 0 \ / \ \forall t \geq 0, \ |Z_s| \geq R \Rightarrow |f^{(s)}(t, Z_s)| \leq \alpha,$$

  which asserts that the convergence to zero is uniform in the time variable t. Besides, for all functions f of $\mathcal{C}_0\big([0,T] \times \left(\overline{\Omega^c} \times \mathbb{R}^d\right)^s\big)$ and for all $t \in [0,T]$, $Z_s \mapsto f^{(s)}(t, Z_s)$ is uniformly continuous.

- *This uniform continuity enables to state the following equality between functional spaces :*

$$\mathcal{C}_0\big([0,T] \times \left(\overline{\Omega^c} \times \mathbb{R}^d\right)^s\big) = \mathcal{C}\big([0,T], \mathcal{C}_0\big(\left(\overline{\Omega^c} \times \mathbb{R}^d\right)^s\big)\big).$$

Chapter 6

About the rigorous definition of the hierarchies

One has seen in Section 4.1 page 81 that the BBGKY hierarchy can be rewritten in an integrated form, that is one has :

$$f_N^{(s)}(t, Z_s) = \mathcal{T}_t^{s,\varepsilon} f_N^{(s)}(0, Z_s) + \int_0^t \mathcal{T}_{t-u}^{s,\varepsilon} \mathcal{C}_{s,s+1}^{N,\varepsilon} f_N^{(s+1)}(u, Z_s)\, \mathrm{d}u, \tag{6.1}$$

for N given, and any $1 \leq s \leq N-1$, exactly as the Boltzmann hierarchy, which is composed of the very similar sequence of equations :

$$f^{(s)}(t, Z_s) = \mathcal{T}_t^{s,\varepsilon} f^{(s)}(0, Z_s) + \int_0^t \mathcal{T}_{t-u}^{s,0} \mathcal{C}_{s,s+1}^{0} f^{(s+1)}(u, Z_s)\, \mathrm{d}u, \tag{6.2}$$

for every $s \geq 1$.

On the one hand, the results obtained in Section 5.2 provide a functional setting in which the Boltzmann hierarchy is well defined. In other words, one found that if $f^{(s)}$ and $f^{(s+1)}$ belong respectively to

$$\mathcal{C}_0\big([0,T] \times \big(\overline{\Omega^c} \times \mathbb{R}^d\big)^s\big)$$

and

$$\mathcal{C}_0\big([0,T] \times \big(\overline{\Omega^c} \times \mathbb{R}^d\big)^{s+1}\big),$$

with in addition for $t \in [0,T]$

$$\left|\frac{f^{(s)}(t, Z_s)}{g_s\big(t, \big|V_s\big|\big)}\right|_{L^\infty((\overline{\Omega^c} \times \mathbb{R}^d)^s)} < +\infty \text{ and } \left|\frac{f^{(s+1)}(t, Z_{s+1})}{g_{s+1}\big(t, \big|V_{s+1}\big|\big)}\right|_{L^\infty((\overline{\Omega^c} \times \mathbb{R}^d)^{s+1})} < +\infty,$$

with

$$V_s \mapsto \big|V_s\big| g_s\big(t, |V_s|\big) \text{ and } V_{s+1} \mapsto \big|V_{s+1}\big| g_{s+1}\big(t, |V_{s+1}|\big)$$

both integrable, then each term of the equation (6.2) makes sense, and one obtained therefore a consistent equation.
On the other hand, before Section 5.1 dealing with the definition of the collision operator of the BBGKY hierarchy, the equation (6.1) was only formal. Besides, this operator alone was not defined, one only provided a definition for the collision operator composed with the hard sphere flow :

$$\mathcal{C}_{s,s+1}^{N,\varepsilon}\mathcal{T}_u^{s+1,\varepsilon}h_N^{(s+1)}$$

(see Theorem 1 page 137).
In the following Section 6.1, one defines the integrated in time transport-collision-transport of the BBGKY hierarchy

$$\int_0^t \mathcal{T}_{t-u}^{s,\varepsilon}\mathcal{C}_{s,s+1}^{N,\varepsilon}f_N^{(s+1)}(u,Z_s)\,\mathrm{d}u,$$

this definition finishing therefore to provide a rigorous definition of all the terms involved in the writing of the BBGKY hierarchy (6.1) (since the hard sphere transport preserves the measure, the term $\big(\mathcal{T}_t^{s,\varepsilon}f_N^{(s)}(0,\cdot)\big)(Z_s)$ is not a source of trouble). Using the functional spaces on which each equation of the BBGKY hierarchy is defined, this section will be concluded with Theorem 2 (page 194), providing a functional setting on which the BBGKY hierarchy will be defined.
In Section 6.2, one will write for the sake of completeness the analog of Theorem 2 for the Boltzmann hierarchy (see Theorem 3 page 197), even if this result could have been written previously, since all the tools were already obtained in the previous Section 5.2.

6.1 The case of the BBGKY hierarchy

6.1.1 Definition of the conjugate BBGKY hierarchy and stability on the functional spaces involved

In Section 5, one has shown that the collision operator is well-defined on the functional space $\mathcal{C}\big([0,T],L^\infty\big(\mathcal{D}_{s+1}^\varepsilon\big)\big)$, assuming in addition a decrease in high velocities, if one composes it with the hard sphere transport (this is the purpose of Section 5.1, see its concluding Theorem 1 page 137).
From the finite sequence of equations :

$$\forall\ 1\le s\le N-1,\ f_N^{(s)}(t,\cdot)=\mathcal{T}_t^{s,\varepsilon}f_N^{(s)}(0,\cdot)+\int_0^t\mathcal{T}_{t-u}^{s,\varepsilon}\mathcal{C}_{s,s+1}^{N,\varepsilon}f_N^{(s+1)}(u,\cdot)\,\mathrm{d}u,$$

one defines formally a new hierarchy, such that the unknown is now $h_N^{(s)}$, defined as

$$h_N^{(s)}=\mathcal{T}_{-t}^{s,\varepsilon}f_N^{(s)},$$

in order to use Theorem 1. In the following, the theorems of existence and uniqueness will be stated for this new hierarchy, holding for the conjugate sequence $\big(h_N^{(s)}\big)_{1\le s\le N}$, since the work done in Section 5 cannot be performed on

the original integrated equations of the BBGKY hierarchy (6.1).
Rewriting the generic equation of the BBGKY hierarchy with $h_N^{(s)}$ and $h_N^{(s+1)}$ instead of $f_N^{(s)}$ and $f_N^{(s+1)}$, one obtains :

$$\mathcal{T}_t^{s,\varepsilon} h_N^{(s)}(t,\cdot) = \mathcal{T}_t^{s,\varepsilon} f_N^{(s)}(0,\cdot) + \int_0^t \mathcal{T}_{t-u}^{s,\varepsilon} \mathcal{C}_{s,s+1}^{N,\varepsilon} \mathcal{T}_u^{s+1,\varepsilon} h_N^{(s+1)}(u,\cdot)\,\mathrm{d}u,$$

and finally applying the hard sphere transport for a time t to the last equation, one gets :

$$h_N^{(s)}(t,\cdot) = f_N^{(s)}(0,\cdot) + \int_0^t \mathcal{T}_{-u}^{s,\varepsilon} \mathcal{C}_{s,s+1}^{N,\varepsilon} \mathcal{T}_u^{s+1,\varepsilon} h_N^{(s+1)}(u,\cdot)\,\mathrm{d}u,$$

which will be the relevant object in the following.

Definition 17 (Integrated form of the conjugate BBGKY hierarchy)**.** *Let N be a positive integer and T and ε be two strictly positive real numbers. For two sequences of functions :*

$$F_{N,0} = \big(f_{N,0}^{(s)}\big)_{1\le s\le N} \in \Big(L^\infty(\mathcal{D}_s^\varepsilon)\Big)_{1\le s\le N}$$

and

$$H_N = \big(h_N^{(s)}\big)_{1\le s\le N} \in \Big(\mathcal{C}\big([0,T], L^\infty(\mathcal{D}_s^\varepsilon)\big)\Big)_{1\le s\le N},$$

of functions, H_N is said to be verifying the the integrated form of the conjugate BBGKY hierarchy *with initial datum $F_{N,0}$ if for all $1 \le s \le N-1$:*

$$h_N^{(s)}(t,Z_s) = f_{N,0}^{(s)}(Z_s) + \int_0^t \mathcal{T}_{-u}^{s,\varepsilon} \mathcal{C}_{s,s+1}^{N,\varepsilon} \mathcal{T}_u^{s+1,\varepsilon} h_N^{(s+1)}(u,Z_s)\,\mathrm{d}u, \tag{6.3}$$

and for $s = N$:

$$h_N^{(N)}(t,Z_N) = f_{N,0}^{(N)}(Z_N).$$

6.1.2 Stability of the continuous in time functions by the integrated in time transport-collision-transport of the BBGKY hierarchy

On the one hand, one has only shown so far that the transport-collision operator of the BBGKY hierarchy takes its values in the functional space

$$L^\infty\big([0,T]\times\mathcal{D}_s^\varepsilon\big)$$

(see Theorem 1 page 137), and on the other hand, the relevant functional space for the image of the transport-collision-transport operator turns out to be $L^1\big([0,T],X\big)$, with X some Banach space, as it is required by the equation (6.3) written in Definition 17.
Finally, remembering that the BBGKY hierarchy consists in N equations, one

shall work, not only on functions satisfying any equation separately, but on sequences of functions such that each term verifies an equation. Besides, the BBGKY hierarchy is a system of equations which are not closed, except the last one. Indeed, for any positive integer $s \leq N-1$, each equation involves $f_N^{(s)}$ and $f_N^{(s+1)}$. So if one assumes that $f_N^{(s+1)}$ belongs to $\mathcal{C}\big([0,T], L^\infty\big(\mathcal{D}_{s+1}^\varepsilon\big)\big)$ with additional hypotheses on the decay in high velocities, one wants to be able to define the transport-collision-transport operator such that the equation (6.3) is consistent, that is such that each side of the equation belongs to the same functional space. In other words, one wants to prove that :

$$\int_0^t \mathcal{T}_{-u}^{s,\varepsilon} \mathcal{C}_{s,s+1}^{N,\varepsilon} \mathcal{T}_u^{s+1,\varepsilon} h_N^{(s+1)} \, \mathrm{d}u \in \mathcal{C}\big([0,T], L^\infty\big(\mathcal{D}_s^\varepsilon\big)\big),$$

with the same decrease in high velocity. This is the purpose of the following lemma.

Lemma 15 (Stability by the integrated in time transport-collision-transport operator of the BBGKY hierarchy of the subset of continuous in time L^∞ functions on the phase space decaying sufficiently fast at infinity in the velocity variables)**.** *Let s be a positive integer, ε and T be two strictly positive numbers. Let in addition $g_{s+1} : [0,T] \times \mathbb{R}_+ \to \mathbb{R}_+$ a function satisfying :*

- *$(t,x) \mapsto g_{s+1}(t,x)$ is a measurable, almost everywhere strictly positive function,*
- *for all $t \in [0,T]$ and almost every $V_s \in \mathbb{R}^{ds}$, the function :*

$$v_{s+1} \mapsto |V_{s+1}| g_{s+1}\big(t, |V_{s+1}|\big)$$

  is integrable,
- *the function :*

$$(t, V_s) \mapsto \int_{\mathbb{R}^d} |V_{s+1}| g_{s+1}\big(t, |V_{s+1}|\big) \, \mathrm{d}v_{s+1}$$

  is bounded almost everywhere, and

$$\left| \int_{\mathbb{R}^d} \mathbb{1}_{|V_{s+1}| \geq R} |V_{s+1}| g_{s+1}\big(t, |V_{s+1}|\big) \, \mathrm{d}v_{s+1} \right|_{L^\infty(\mathbb{R}^{ds})}$$

  converges to zero as R goes to infinity.

Then for every integer $1 \leq i \leq s$, any sign $+$ or $-$, and for every function $h^{(s+1)} \in \mathcal{C}\big([0,T], L^\infty\big(\mathcal{D}_{s+1}^\varepsilon\big)\big)$ such that

$$\left| \frac{h^{(s+1)}(t, Z_{s+1})}{g_{s+1}\big(t, |V_{s+1}|\big)} \right|_{L^\infty([0,T], L^\infty(\mathcal{D}_{s+1}^\varepsilon))} < +\infty,$$

the function :

$$(t, Z_s) \mapsto \int_0^t \mathcal{T}_{-u}^{s,\varepsilon} \mathcal{C}_{s,s+1,\pm,i}^{\varepsilon} \mathcal{T}_u^{s+1,\varepsilon} h^{(s+1)}(u, Z_s)\, \mathrm{d}u.$$

is well-defined and belongs to the functional space $\mathcal{C}\big([0,T], L^\infty\big(\mathcal{D}_s^\varepsilon\big)\big)$.
In addition, one has, for every $u \leq t \in [0,T]$ *and for almost every* $Z_s \in \mathcal{D}_s^\varepsilon$ *:*

$$\begin{aligned}\left| \int_0^t \mathcal{T}_{-u}^{s,\varepsilon} \mathcal{C}_{s,s+1,\pm,i}^{\varepsilon} \mathcal{T}_u^{s+1,\varepsilon} h^{(s+1)}(u, Z_s)\, \mathrm{d}u \right| & \\ \leq \varepsilon^{d-1} \frac{|\mathbb{S}^{d-1}|}{2} & \\ \times \int_u^t \left| \frac{h^{(s+1)}(\tau, Z_{s+1})}{g_{s+1}\big(\tau, |V_{s+1}|\big)} \right|_{L^\infty(\mathcal{D}_{s+1}^\varepsilon)} \int_{\mathbb{R}^d} \big(|v_i| + |v_{s+1}|\big) g_{s+1}\big(\tau, |V_{s+1}|\big)\, \mathrm{d}v_{s+1}\, \mathrm{d}\tau. & \end{aligned} \tag{6.4}$$

Proof. The proof will start by the definition of the function, for any $t \in [0,T]$:

$$Z_s \mapsto \int_0^t \mathcal{T}_{-u}^{s,\varepsilon} \mathcal{C}_{s,s+1,\pm,i}^{\varepsilon} \mathcal{T}_u^{s+1,\varepsilon} h^{(s+1)}(u, Z_s)\, \mathrm{d}u.$$

First, thanks to Theorem 1 page 137, the function :

$$(u, Z_s) \mapsto \mathcal{C}_{s,s+1,\pm,i}^{\varepsilon} \mathcal{T}_u^{s+1,\varepsilon} h^{(s+1)}(u, Z_s)$$

is almost everywhere well defined, measurable and is essentially bounded.
Besides, since the hard sphere transport, well defined on the whole phase space except for a subset of measure zero and for any time $t \in [0,T]$ (in fact, for every $t \in \mathbb{R}$), is continuous on this interval and preserves the measure, the map

$$\left\{ \begin{array}{rcl} [0,T] \times \mathcal{D}_s^\varepsilon & \to & [0,T] \times \mathcal{D}_s^\varepsilon, \\ (u, Z_s) & \mapsto & \big(u, T_u^{s,\varepsilon}(Z_s)\big), \end{array} \right.$$

is also defined and continuous almost everywhere on $[0,T] \times \mathcal{D}_s^\varepsilon$ and preserves the measure. In particular, the function

$$\left\{ \begin{array}{rcl} [0,T] \times \mathcal{D}_s^\varepsilon & \to & \mathbb{R}, \\ (u, Z_s) & \mapsto & \mathcal{C}_{s,s+1,\pm,i}^{\varepsilon} \mathcal{T}_u^{s+1,\varepsilon} h^{(s+1)}\big(u, T_{-u}^{s,\varepsilon}(Z_s)\big) \\ & & \quad = \mathcal{T}_{-u}^{s,\varepsilon} \mathcal{C}_{s,s+1,\pm,i}^{\varepsilon} \mathcal{T}_u^{s+1,\varepsilon} h^{(s+1)}(u, Z_s), \end{array} \right.$$

is almost everywhere well defined, measurable, essentially bounded and one has, for almost every $u \in [0,T]$:

$$\left| \mathcal{C}_{s,s+1,\pm,i}^{\varepsilon} \mathcal{T}_u^{s+1,\varepsilon} h^{(s+1)} \right|_{L^\infty([0,T] \times \mathcal{D}_s^\varepsilon)} = \left| \mathcal{T}_{-u}^{s,\varepsilon} \mathcal{C}_{s,s+1,\pm,i}^{\varepsilon} \mathcal{T}_u^{s+1,\varepsilon} h^{(s+1)} \right|_{L^\infty([0,T] \times \mathcal{D}_s^\varepsilon)}.$$

So for any measurable subset A of $[0,T] \times \mathcal{D}_s^\varepsilon$ of finite measure, the function :

$$\mathbb{1}_A \mathcal{T}_{-u}^{s,\varepsilon} \mathcal{C}_{s,s+1,\pm,i}^{\varepsilon} \mathcal{T}_u^{s+1,\varepsilon} h^{(s+1)}(u, \cdot)$$

is measurable, and integrable as a product of an integrable function with an essentially bounded function, and one has of course :

$$\begin{aligned}\int_0^T \int_{\mathcal{D}_s^\varepsilon} & \mathbb{1}_A \mathcal{T}_{-u}^{s,\varepsilon} \mathcal{C}_{s,s+1,\pm,i}^\varepsilon \mathcal{T}_u^{s+1,\varepsilon} h^{(s+1)} \,\mathrm{d}Z_s \,\mathrm{d}u \\ & \leq |A| \Big| \mathcal{T}_{-u}^{s,\varepsilon} \mathcal{C}_{s,s+1,\pm,i}^\varepsilon \mathcal{T}_u^{s+1,\varepsilon} h^{(s+1)} \Big|_{L^\infty([0,T]\times\mathcal{D}_s^\varepsilon)} \\ & \leq |A| \Big| \mathcal{C}_{s,s+1,\pm,i}^\varepsilon \mathcal{T}_u^{s+1,\varepsilon} h^{(s+1)} \Big|_{L^\infty([0,T]\times\mathcal{D}_s^\varepsilon)}. \end{aligned} \tag{6.5}$$

In particular, for every measurable subset B of $\mathcal{D}_s^\varepsilon$ of finite measure and for every $t \in [0,T]$, the function

$$\mathbb{1}_{[0,t]\times B} \mathcal{T}_{-u}^{s,\varepsilon} \mathcal{C}_{s,s+1,\pm,i}^\varepsilon \mathcal{T}_u^{s+1,\varepsilon} h^{(s+1)}(u,\cdot)$$

is integrable on $[0,T]\times\mathcal{D}_s^\varepsilon$, so by the Fubini theorem, the function

$$\begin{aligned}\mathbb{1}_B(Z_s) \int_0^T & \mathbb{1}_{[0,t]} \mathcal{T}_{-u}^{s,\varepsilon} \mathcal{C}_{s,s+1,\pm,i}^\varepsilon \mathcal{T}_u^{s+1,\varepsilon} h^{(s+1)}(u,\cdot) \,\mathrm{d}u \\ & = \mathbb{1}_B(Z_s) \int_0^t \mathcal{T}_{-u}^{s,\varepsilon} \mathcal{C}_{s,s+1,\pm,i}^\varepsilon \mathcal{T}_u^{s+1,\varepsilon} h^{(s+1)}(u,\cdot) \,\mathrm{d}u \end{aligned}$$

is almost everywhere well-defined and is integrable. Moreover, the inequality (6.5) can be rewritten here as

$$\begin{aligned}\int_{\mathcal{D}_s^\varepsilon} & \Big| \mathbb{1}_B(Z_s) \int_0^t \mathcal{T}_{-u}^{s,\varepsilon} \mathcal{C}_{s,s+1,\pm,i}^\varepsilon \mathcal{T}_u^{s+1,\varepsilon} h^{(s+1)}(u,Z_s) \,\mathrm{d}u \Big| \,\mathrm{d}Z_s \\ & \leq \int_0^T \int_{\mathcal{D}_s^\varepsilon} \mathbb{1}_{[0,t]\times B} \Big| \mathcal{T}_{-u}^{s,\varepsilon} \mathcal{C}_{s,s+1,\pm,i}^\varepsilon \mathcal{T}_u^{s+1,\varepsilon} h^{(s+1)}(u,Z_s) \Big| \,\mathrm{d}Z_s \,\mathrm{d}u \\ & \leq t|B| \Big| \mathcal{C}_{s,s+1,\pm,i}^\varepsilon \mathcal{T}_u^{s+1,\varepsilon} h^{(s+1)} \Big|_{L^\infty([0,T]\times\mathcal{D}_s^\varepsilon)} \\ & \leq T|B| \Big| \mathcal{C}_{s,s+1,\pm,i}^\varepsilon \mathcal{T}_u^{s+1,\varepsilon} h^{(s+1)} \Big|_{L^\infty([0,T]\times\mathcal{D}_s^\varepsilon)}. \end{aligned}$$

On the other hand, for every $t \in [0,T]$, one can define the measurable function

$$Z_s \mapsto \int_0^t \mathcal{T}_{-u}^{s,\varepsilon} \mathcal{C}_{s,s+1,\pm,i}^\varepsilon \mathcal{T}_u^{s+1,\varepsilon} h^{(s+1)}(u,Z_s) \,\mathrm{d}u$$

as the almost everywhere limit of the sequence of functions

$$Z_s \mapsto \mathbb{1}_{B_k} \int_0^t \mathcal{T}_{-u}^{s,\varepsilon} \mathcal{C}_{s,s+1,\pm,i}^\varepsilon \mathcal{T}_u^{s+1,\varepsilon} h^{(s+1)}(u,Z_s) \,\mathrm{d}u,$$

where $\big(B_k\big)_{k\geq 0}$ denotes any covering sequence of compact subsets of the phase space $\mathcal{D}_s^\varepsilon$. Of course this defines a function which does not depend on the covering

sequence of compact subsets. This function verifies then, for any measurable subset B of the phase space $\mathcal{D}_s^\varepsilon$ of finite measure, the same inequality as above, that is

$$\int_{\mathcal{D}_s^\varepsilon} \Big| \mathbb{1}_B(Z_s) \int_0^t \mathcal{T}_{-u}^{s,\varepsilon} \mathcal{C}_{s,s+1,\pm,i}^\varepsilon \mathcal{T}_u^{s+1,\varepsilon} h^{(s+1)}(u, Z_s)\, \mathrm{d}u \Big| \, \mathrm{d}Z_s \\ \leq T|B| \Big| \mathcal{C}_{s,s+1,\pm,i}^\varepsilon \mathcal{T}_u^{s+1,\varepsilon} h^{(s+1)} \Big|_{L^\infty([0,T]\times\mathcal{D}_s^\varepsilon)},$$

which proves that, for every $t \in [0,T]$, the function

$$Z_s \mapsto \int_0^t \mathcal{T}_{-u}^{s,\varepsilon} \mathcal{C}_{s,s+1,\pm,i}^\varepsilon \mathcal{T}_u^{s+1,\varepsilon} h^{(s+1)}(u, Z_s)\, \mathrm{d}u$$

is in fact essentially bounded, and one has :

$$\Big| \int_0^t \mathcal{T}_{-u}^{s,\varepsilon} \mathcal{C}_{s,s+1,\pm,i}^\varepsilon \mathcal{T}_u^{s+1,\varepsilon} h^{(s+1)}(u, \cdot)\, \mathrm{d}u \Big|_{L^\infty(\mathcal{D}_s^\varepsilon)} \\ \leq T \Big| \mathcal{C}_{s,s+1,\pm,i}^\varepsilon \mathcal{T}_u^{s+1,\varepsilon} h^{(s+1)} \Big|_{L^\infty([0,T]\times\mathcal{D}_s^\varepsilon)},$$

which concludes the first part of the proof.
Now, one will show that the map

$$\left\{ \begin{array}{rcl} [0,T] & \to & L^\infty\big(\mathcal{D}_s^\varepsilon\big) \\ t & \mapsto & \Big(Z_s \mapsto \displaystyle\int_0^t \mathcal{T}_{-u}^{s,\varepsilon} \mathcal{C}_{s,s+1,\pm,i}^\varepsilon \mathcal{T}_u^{s+1,\varepsilon} h^{(s+1)}(u, Z_s)\, \mathrm{d}u \Big) \end{array} \right.$$

is in fact continuous. This is again an easy consequence of the inequality (6.5). For any $t_1, t_2 \in [0,T]$ (one will assume that $t_1 < t_2$ without loss of generality), and for any measurable subset B of finite measure of the phase space $\mathcal{D}_s^\varepsilon$, one has :

$$\begin{aligned} &\int_{\mathcal{D}_s^\varepsilon} \mathbb{1}_B(Z_s) \Big| \int_0^{t_2} \mathcal{T}_{-u}^{s,\varepsilon} \mathcal{C}_{s,s+1,\pm,i}^\varepsilon \mathcal{T}_u^{s+1,\varepsilon} h^{(s+1)}(u, Z_s)\, \mathrm{d}u \\ &\qquad\qquad - \int_0^{t_1} \mathcal{T}_{-u}^{s,\varepsilon} \mathcal{C}_{s,s+1,\pm,i}^\varepsilon \mathcal{T}_u^{s+1,\varepsilon} h^{(s+1)}(u, Z_s)\, \mathrm{d}u \Big| \, \mathrm{d}Z_s \\ &= \int_{\mathcal{D}_s^\varepsilon} \mathbb{1}_B(Z_s) \Big| \int_0^T \mathbb{1}_{[t_1,t_2]} \mathcal{T}_{-u}^{s,\varepsilon} \mathcal{C}_{s,s+1,\pm,i}^\varepsilon \mathcal{T}_u^{s+1,\varepsilon} h^{(s+1)}(u, Z_s)\, \mathrm{d}u \Big| \, \mathrm{d}Z_s \\ &\leq \int_0^T \int_{\mathcal{D}_s^\varepsilon} \mathbb{1}_{[t_1,t_2]\times B} \Big| \mathcal{T}_{-u}^{s,\varepsilon} \mathcal{C}_{s,s+1,\pm,i}^\varepsilon \mathcal{T}_u^{s+1,\varepsilon} h^{(s+1)}(u, Z_s) \Big| \, \mathrm{d}Z_s\, \mathrm{d}u \\ &\leq |t_2 - t_1| \cdot |B| \Big| \mathcal{C}_{s,s+1,\pm,i}^\varepsilon \mathcal{T}_u^{s+1,\varepsilon} h^{(s+1)} \Big|_{L^\infty([0,T]\times\mathcal{D}_s^\varepsilon)}. \end{aligned}$$

This shows that the difference of the two essentially bounded functions :

$$\int_0^{t_2} \mathcal{T}_{-u}^{s,\varepsilon}\mathcal{C}_{s,s+1,\pm,i}^{\varepsilon}\mathcal{T}_u^{s+1,\varepsilon} hi^{(s+1)}(u,Z_s)\,\mathrm{d}u \\ - \int_0^{t_1} \mathcal{T}_{-u}^{s,\varepsilon}\mathcal{C}_{s,s+1,\pm,i}^{\varepsilon}\mathcal{T}_u^{s+1,\varepsilon} h^{(s+1)}(u,Z_s)\,\mathrm{d}u$$

has its L^∞ norm controlled as follows :

$$\Big| \int_0^{t_2} \mathcal{T}_{-u}^{s,\varepsilon}\mathcal{C}_{s,s+1,\pm,i}^{\varepsilon}\mathcal{T}_u^{s+1,\varepsilon} h^{(s+1)}(u,\cdot)\,\mathrm{d}u \\ - \int_0^{t_1} \mathcal{T}_{-u}^{s,\varepsilon}\mathcal{C}_{s,s+1,\pm,i}^{\varepsilon}\mathcal{T}_u^{s+1,\varepsilon} h^{(s+1)}(u,\cdot)\,\mathrm{d}u\Big|_{L^\infty(\mathcal{D}_s^\varepsilon)} \\ \leq |t_2-t_1| \Big|\mathcal{C}_{s,s+1,\pm,i}^{\varepsilon}\mathcal{T}_u^{s+1,\varepsilon} h^{(s+1)}\Big|_{L^\infty([0,T]\times\mathcal{D}_s^\varepsilon)},$$

which implies in particular the continuity in time in the L^∞ norm.
Finally, one will show that the decay property at infinity in the velocitie variable verified by $h^{(s+1)}$ is propagated to the integrated in time transport-collision-transport operator applied to $h^{(s+1)}$.
Thanks to the inequality (5.60) stated in Theorem 1 page 138, if one denotes :

$$Q(t,Z_s) = \left| \int_u^t \mathcal{T}_{-\tau}^{s,\varepsilon}\mathcal{C}_{s,s+1,\pm,i}^{\varepsilon}\mathcal{T}_\tau^{s+1,\varepsilon} h^{(s+1)}(\tau,Z_s)\,\mathrm{d}\tau \right|,$$

one has immediately that, for all $u \leq t \in [0,T]$ and almost every $Z_s \in \mathcal{D}_s^\varepsilon$:

$$Q = \left| \int_u^t \mathcal{C}_{s,s+1,\pm,i}^{\varepsilon}\mathcal{T}_v^{s+1,\varepsilon} h^{(s+1)}\big(v, T_v^{s,\varepsilon}(Z_s)\big)\,\mathrm{d}v \right| \\ \leq \varepsilon^{d-1}\frac{\big|\mathbb{S}^{d-1}\big|}{2} \\ \times \int_u^t \left| \frac{h^{(s+1)}(\tau,Z_{s+1})}{f_{s+1}\big(\tau,|V_{s+1}|\big)} \right|_{L^\infty_{Z_{s+1}}} \int_{\mathbb{R}^d} \big(|v_i|+|v_{s+1}|\big) g_{s+1}\big(v,|V_{s+1}|\big)\,\mathrm{d}v_{s+1}\,\mathrm{d}\tau,$$

where V_{s+1} denotes as usual $\big(V_s, v_{s+1}\big)$, this equality holding true since :

$$\Big|\big(T_u^{s,\varepsilon}(Z_s)\big)^V\Big| = |V_s|,$$

by the conservation of the kinetic energy by the hard sphere transport. The proof of the theorem is complete. □

6.1.3 Definition of the appropriate functional spaces for the BBGKY hierarchy

It is time to introduce again some notations to simplify the statement of the following Theorem 2 about the definition of the BBGKY hierarchy.

Definition 18 (X^0_{ε,s,g_s}, the functional space of the $L^\infty\big(\mathcal{D}^\varepsilon_s\big)$ functions bounded in velocity by the function g_s)**.** *Let s be a positive integer, ε and T be two strictly positive numbers.*
Let in addition $g_s : \mathbb{R}_+ \to \mathbb{R}_+$ be a function such that :

$$r \mapsto g_s(r)$$

is a measurable, almost everywhere strictly positive function.
One denotes by X^0_{ε,s,g_s} the subset of $L^\infty\big(\mathcal{D}^\varepsilon_s\big)$ composed of the functions $f_0^{(s)}$ satisfying :

$$\left| Z_s \mapsto \frac{f_0^{(s)}(Z_s)}{g_s\big(|V_s|\big)} \right|_{L^\infty(\mathcal{D}^\varepsilon_s)} < +\infty.$$

Definition 19 ($\widetilde{X}_{\varepsilon,s,g_s}$, the functional space of the continuous in time functions taking their values in $L^\infty\big(\mathcal{D}^\varepsilon_s\big)$ uniformly bounded in velocity by the function g_s)**.** *Let s be a positive integer, ε and T be two strictly positive numbers.*
Let in addition $g_s : \mathbb{R}_+ \to \mathbb{R}_+$ be a function such that :

$$r \mapsto g_s(r)$$

is a measurable, almost everywhere strictly positive function.
One denotes by $\widetilde{X}_{\varepsilon,s,g_s}$ the subset of the functional space $\mathcal{C}\big([0,T], L^\infty\big(\mathcal{D}^\varepsilon_s\big)\big)$ composed of the functions $f^{(s)}$ satisfying, for all $t \in [0,T]$:

$$\left| Z_s \mapsto \frac{f^{(s)}(t, Z_s)}{g_s\big(|V_s|\big)} \right|_{L^\infty(\mathcal{D}^\varepsilon_s)} < +\infty.$$

Remark 14. *Those notations may look heavy, therefore one provides a brief description of them, since those notations will be used extensively in the sequel. For the functional space X^0_{ε,s,g_s} of the initial data introduced in Definition 18, the exponent 0 means of course that this space is for the initial data of the hierarchy.*
On the contrary, the tilde in the notation $\widetilde{X}_{\varepsilon,s,g_s}$ of the functional space introduced in Definition 19 means that one considers functions depending on time, and this notation for such functional spaces involving time will be used until the end of this work.

6.1.4 Rigorous definition of the BBGKY hierarchy

One has now all the required tools to define properly the BBGKY hierarchy, as an operator acting on a finite product of functional spaces, into itself. The following theorem is an easy consequence of Lemma 15 page 188. The only important idea here is to find a function which can be integrated several times, each iterate of integration satisfying for a given positive integer s the conditions on the function g_{s+1} of Lemma 15, so that those iterates will play the role of the uniform bound in velocity for the functions composed with the integrated in time transport-collision-transport operator of the BBGKY hierarchy.

Theorem 2 (Rigorous definition of the BBGKY hierarchy for a general family of functional spaces)**.** *Let N be a positive integer, ε and T two strictly positive numbers.*
Let in addition $g : \mathbb{R}_+ \to \mathbb{R}_+$ be a function verifying :

- *$r \mapsto g(r)$ is a measurable, almost everywhere strictly positive function,*
- *the function :*
$$V_N \mapsto \big(1 + |V_N|^2\big)^{N/2} g\big(|V_N|\big)$$
is integrable,
- *for every integer $1 \leq s \leq N$:*
$$\left| \int_{\mathbb{R}^{d(N-s+1)}} \mathbb{1}_{|V_N| \geq R} |V_N|^{N-s+1} g\big(|V_N|\big) \, \mathrm{d}v_s \ldots \mathrm{d}v_N \right|_{L^\infty(\mathbb{R}^{d(s-1)})}$$
converges to zero as R goes to infinity.

Then (using Definition 18 page 193), for every sequence of initial data :
$$\big(f_{N,0}^{(s)}\big)_{1 \leq s \leq N} \in \big(X^0_{\varepsilon,s,g_s}\big)_{1 \leq s \leq N},$$
and (using Definition 19 page 193) for every sequence of functions :
$$\big(f_N^{(s)}\big)_{1 \leq s \leq N} \in \big(\widetilde{X}_{\varepsilon,s,g_s}\big)_{1 \leq s \leq N}$$
with $g_N = g$ and g_s denoting, for $1 \leq s \leq N-1$:
$$g_s : Z_s \mapsto \int_{\mathbb{R}^{d(N-s)}} |V_N|^{N-s} g\big(|V_N|\big) \, \mathrm{d}v_{s+1} \ldots \mathrm{d}v_N,$$
that is such that, for every $1 \leq s \leq N$:
$$\big(f_N^{(s)}\big)_{1 \leq s \leq N} \in \Big(\mathcal{C}\big([0,T], L^\infty\big(\mathcal{D}_s^\varepsilon\big)\big)\Big)_{1 \leq s \leq N}$$
with in addition, for all $t \in [0,T]$ and for all $1 \leq s \leq N-1$:
$$\left| \frac{f_N^{(s)}(t, Z_s)}{\displaystyle\int_{\mathbb{R}^{d(N-s)}} |V_N|^{N-s} g\big(|V_N|\big) \, \mathrm{d}v_{s+1} \ldots \mathrm{d}v_N} \right|_{L^\infty(\mathcal{D}_s^\varepsilon)} < +\infty,$$
and
$$\left| \frac{f_N^{(N)}(t, Z_N)}{g\big(|V_N|\big)} \right|_{L^\infty(\mathcal{D}_N^\varepsilon)} < +\infty,$$
one has that the BBGKY operator
$$\big(f_N^{(s)}\big)_{1 \leq s \leq N} \mapsto \Big(f_{N,0}^{(s)} + \mathbb{1}_{s \leq N-1} \int_0^t \mathcal{T}_{-u}^{s,\varepsilon} \mathcal{C}_{s,s+1}^{N,\varepsilon} \mathcal{T}_u^{s+1,\varepsilon} f_N^{(s+1)}(u, \cdot) \, \mathrm{d}u\Big)_{1 \leq s \leq N}$$
is well defined on $\big(\widetilde{X}_{\varepsilon,s,g_s}\big)_{1 \leq s \leq N}$, and takes its values in the same space.

6.2 The case of the Boltzmann hierarchy

6.2.1 Notations for the appropriate functional spaces for the Boltzmann hierarchy

One does the same work as for the BBGKY hierarchy, in order to simplify the statement of Theorem 3 below. First, one starts, as in the previous section, by introducing some convenient notations.

An important difference with the BBGKY hierarchy lies in the boundary condition verified by the solutions of the Boltzmann hierarchy. Indeed, in Section 5.1, no boundary condition was required to define properly the integrated in time transport-collision operator of the BBGKY hierarchy. On the contrary, such a condition was mandatory to define properly the Boltzmann hierarchy (see Section 5.2, and in particular Section 5.2.3).

Definition 20 (X^0_{0,s,g_s}, the functional space of continuous functions on $\left(\overline{\Omega^c} \times \mathbb{R}^d\right)^s$ vanishing at infinity and bounded in velocity by the function g_s)**.** *Let s be a positive integer, T be a strictly positive number.*
Let in addition $g_s : \mathbb{R}_+ \to \mathbb{R}_+$ be a function such that :

$$r \mapsto g_s(r)$$

is a measurable, almost everywhere strictly positive function.
One denotes by X^0_{0,s,g_s} the subset of the functional space $\mathcal{C}_0\left(\left(\overline{\Omega^c} \times \mathbb{R}^d\right)^s\right)$ composed of the functions $f_0^{(s)}$ satisfying the boundary condition :

$$f_0^{(s)}(Z_s) = f_0^{(s)}(\chi_s^0(Z_s)) \tag{6.6}$$

for all Z_s belonging to the boundary of $\left(\overline{\Omega^c} \times \mathbb{R}^d\right)^s$ (that is such that there exists at least an integer $1 \leq i \leq s$ such that $x_i \cdot e_1 = 0$ and $v_i \cdot e_1 > 0$, with χ_s^0 defined in Definition 11 page 79), and the following decay in the velocity variable :

$$\left| Z_s \mapsto \frac{f_0^{(s)}(Z_s)}{g_s\left(|V_s|\right)} \right|_{L^\infty(\mathbb{R}^{ds})} < +\infty.$$

Definition 21 ($\widetilde{X}_{0,s,g_s}$, the functional space of continuous functions on $[0,T] \times \left(\overline{\Omega^c} \times \mathbb{R}^d\right)^s$ vanishing at infinity uniformly bounded in velocity by the function g_s)**.** *Let s be a positive integer, T be a strictly positive number.*
Let in addition $g_s : \mathbb{R}_+ \to \mathbb{R}_+$ be a function such that :

$$r \mapsto g_s(r)$$

is a measurable, almost everywhere strictly positive function.
One denotes by $\widetilde{X}_{0,s,g_s}$ the subset of the functional space $\mathcal{C}_0\left([0,T] \times \left(\overline{\Omega^c} \times \mathbb{R}^d\right)^s\right)$ composed of the functions $f^{(s)}$ satisfying for all $t \in [0,T]$ the boundary condition :

$$f^{(s)}(t, Z_s) = f^{(s)}(t, \chi_s^0(Z_s)) \tag{6.7}$$

for all Z_s belonging to the boundary of $\left(\overline{\Omega^c} \times \mathbb{R}^d\right)^s$ (that is such that there exists at least an integer $1 \leq i \leq s$ such that $x_i \cdot e_1 = 0$ and $v_i \cdot e_1 > 0$, with χ_s^0 defined in Definition 11 page 79), and the following decay in the velocity variable :

$$\left| Z_s \mapsto \frac{f^{(s)}(t, Z_s)}{g_s\big(|V_s|\big)} \right|_{L^\infty(\mathbb{R}^{ds})} < +\infty.$$

Remark 15. *The only difference with the notations of the spaces introduced for the BBGKY hierarchy lies in the first index : ε is replaced by zero, which means that the particle have a radius zero.*

6.2.2 A slightly different way of defining the sequence of functional spaces for the Boltzmann hierarchy

The setting chosen in order to define the finite sequence of functional spaces on which the BBGKY hierarchy is defined relies on a single function, denoted g in the statement of Theorem 2 page 194. This function, depending only on the velocity variable, is meant to bound uniformly the distribution functions of the N particles. Moreover, integrated with respect to the $N - s$ last velocity variables, the function obtained is meant to bound the s-th marginal, for $s \leq N$ particles. One defined therefore a finite family of functional spaces, each marginal lying in an element of this family, and this function g had to be chosen so that the BBGKY hierarchy sends actually this finite sequence of functional spaces into itself.
Of course, this cannot be done for the Boltzmann hierarchy, since this hierarchy involves an infinite number of equations, and then an infinite sequence of functional spaces. It would imply that, if one tries to follow the idea used for the BBGKY hierarchy, to find a function with an infinite number of variables, such that one can obtain, by integrating this function, all of the weights in velocity used to define each of the functional spaces forming the infinite sequence on which the Boltzmann hierarchy should be defined.

However, the idea of choosing a single function in velocity (in order to be integrated several times to bound the profile of the marginals) is convenient, but not mandatory to define this sequence of functional spaces on which the hierarchies will act. Indeed, going back to the inequalities (5.88) of Lemma 12 page 169, and (5.90) of Lemma 13 page 173, one sees that if the Boltzmann hierarchy is well defined on the sequence of functional spaces $\left(\widetilde{X}_{0,s,g_s}\right)_{s\geq 1}$ (for example, it is sufficient to assume that $V_s \mapsto |V_s| g_s\big(|V_s|\big)$ is integrable for every positive integer s), the k-th term of the image of the sequence $\left(f^{(s)}\right)_{s\geq 1}$ by the Boltzmann operator, which is :

$$\mathcal{T}_t^{k,0} f_0^{(k)} + \int_0^t \mathcal{T}_{t-u}^{k,0} \mathcal{C}_{k,k+1}^0 f^{(k+1)} \, \mathrm{d}u$$

will belong to the space $\widetilde{X}_{0,k,g_k}$ if one has :

- of course that :
$$f_0^{(k)} \in X_{0,k,g_k},$$
- and that :
$$\int_{\mathbb{R}^d_{v_{k+1}}} |V_{k+1}| g_{k+1}\big(|V_{k+1}|\big)\, \mathrm{d}v_{k+1} \leq g_k\big(|V_k|\big). \tag{6.8}$$

Indeed, the first inequality ensures that $t \mapsto \mathcal{T}_t^{k,0} f_0^{(k)}$ belongs to $\widetilde{X}_{0,k,g_k}$, while the second one, with the two inequalities (5.88) page 169, and (5.90) page 173 proves that the integrated in time collision-transport term belongs also to $\widetilde{X}_{0,k,g_k}$.
This inequality (6.8) will be then the condition used to define the infinite sequence of functional spaces on which the Boltzmann operator will act.

6.2.3 The definition of the Boltzmann hierarchy

As for the BBGKY hierarchy, the following theorem is a consequence of the previous lemmas. However, one will point out an important difference in a very short proof below the statement.

Theorem 3 (Definition of the Boltzmann hierarchy on the infinite product of the functional spaces $\Big(X_{0,s,g_s}\Big)_{s\geq 1}$)**.** *Let T be a strictly positive number.*
Let in addition $\big(g_s\big)_{s\geq 1}$ be a sequence of functions $g_s : \mathbb{R}_+ \to \mathbb{R}_+$ verifying :

- *for every $s \geq 1$, $r \mapsto g_s(r)$ is a measurable, almost everywhere strictly positive function,*
- *for every integer $s \geq 1$, the function :*
$$V_s \mapsto |V_s| g_s\big(|V_s|\big)$$
is integrable,
- *for every integer $s \geq 1$:*
$$\int_{\mathbb{R}^d_{v_{s+1}}} |V_{s+1}| g_{s+1}\big(|V_{s+1}|\big)\, \mathrm{d}v_{s+1} \leq g_s\big(|V_s|\big).$$

Then for every sequence of initial data :
$$\big(f_0^{(s)}\big)_{s\geq 1} \in \big(X_{0,s,g_s}\big)_{s\geq 1},$$
(where $\big(X_{0,s,g_s}\big)_{s\geq 1}$ is introduced in Definition 20 page 195) and for every sequence of functions :
$$\big(f^{(s)}\big)_{s\geq 1} \in \big(\widetilde{X}_{0,s,g_s}\big)_{s\geq 1}$$

(where $\left(\widetilde{X}_{0,s,g_s}\right)_{s\geq 1}$ *is introduced in Definition 21 page 195), the Boltzmann operator :*

$$\left(f^{(s)}\right)_{s\geq 1} \mapsto \left(\mathcal{T}_t^{s,0} f_0^{(s)} + \int_0^t \mathcal{T}_{t-u}^{s,0} \mathcal{C}_{s,s+1}^0 f^{(s+1)}(u,\cdot)\,\mathrm{d}u\right)_{s\geq 1}$$

is well-defined on $\left(\widetilde{X}_{0,s,g_s}\right)_{s\geq 1}$*, and takes its values in the same space.*

Proof. The assumption on the boundary condition (6.7) satisfied by each term of the sequence $f^{(s)}$ (according to Definition 21 page 195) is stronger than the assumption of continuity on $\left(\overline{\Omega^c}\times\mathbb{R}^d\right)^s$, required to apply Lemma 13 page 173. Therefore, applying together Lemmas 12 page 169 and 13, one sees that the second term of the Boltzmann hierarchy, namely the integrated in time collision-transport operator

$$f^{(s+1)} \mapsto \left((t,Z_s) \mapsto \int_0^t \mathcal{T}_{t-u}^{s,0} \mathcal{C}_{s,s+1}^0 f^{(s+1)}(u,\cdot)\,\mathrm{d}u\right)$$

sends the space $\widetilde{X}_{0,s+1,g_{s+1}}$ into $\widetilde{X}_{0,s,g_s}$, and the decrease in velocity is also recovered.

For the other term, and this is specific to the case of the Boltzmann hierarchy, the boundary condition (6.6) stated in Definition 21 is important to obtain that

$$(t,Z_s) \mapsto \left(\mathcal{T}_t^{s,0} f_0^{(s)}\right)(Z_s)$$

is continuous, uniformly in Z_s, that is, is an element of

$$\mathcal{C}\left([0,T], L^\infty\left(\left(\overline{\Omega^c}\times\mathbb{R}^d\right)^s\right)\right).$$

Besides, the decay with respect to the velocity variable and the vanishing properties at infinity are immediately recovered. □

6.3 About the functions used to define the spaces $X^0_{\cdot,s,g_s}$ and $\widetilde{X}_{\cdot,s,g_s}$

6.3.1 Compatibility between the functions defining the spaces for the BBGKY and the Boltzmann hierarchies

So far, one has proven in this section the rigorous definition of the BBGKY (Theorem 2 page 194) and the Boltzmann operators (Theorem 3 page 197), and one has shown in addition that those operators send their respective domains into themselves.
However to do so, one required that each equation of the two hierarchies is defined on a functional space composed of functions verifying a uniform decrease

in velocity, namely, for each functional space, its functions had to be bounded by a profile depending only on the velocity variables, and those profiles had to satisfy a compatibility condition : the bound defining the decrease for the $s+1$-th equation should be integrable, and after an integration one should recover the bound defining the decrease for the s-th equation. As a summary, one assumed for the BBGKY hierarchy that there exists a function :

$$g : \mathbb{R}_+ \to \mathbb{R}_+$$

such that :

- for all $1 \leq s \leq N$, the function :

$$\begin{cases} \mathbb{R}^{dN} & \to \mathbb{R}_+, \\ V_N & \mapsto |V_N|^s g\big(|V_N|\big) \end{cases}$$

  is integrable and for all $2 \leq s \leq N$ the function :

$$(v_1, \dots, v_{s-1}) \mapsto \int_{\mathbb{R}^{d(N-s+1)}} |V_N|^{N-s+1} g\big(|V_N|\big) \, \mathrm{d}v_s \dots \mathrm{d}v_N$$

  is bounded almost everywhere,

- for all $1 \leq s \leq N$:

$$\Big| \int_{\mathbb{R}^{d(N-s+1)}} \mathbb{1}_{|V_N| \geq R} |V_N|^{N-s+1} g\big(|V_N|\big) \, \mathrm{d}v_s \dots \mathrm{d}v_N \Big|_{L^\infty(\mathbb{R}^{d(s-1)})}$$

  goes to zero as R goes to infinity.

For the Boltzmann hierarchy, one has assumed that there exists a sequence $\big(g_s\big)_{s \geq 1}$ of functions :

$$g_s : \mathbb{R}_+ \to \mathbb{R}_+$$

such that :

- for every $s \geq 1$, $x \mapsto g_s(x)$ is a measurable, almost everywhere strictly positive function,

- for every integer $s \geq 1$, the function :

$$V_s \mapsto |V_s| g_s\big(|V_s|\big)$$

  is integrable,

- for every integer $s \geq 1$:

$$\int_{\mathbb{R}^d_{v_{s+1}}} |V_{s+1}| g_{s+1}\big(|V_{s+1}|\big) \, \mathrm{d}v_{s+1} \leq g_s\big(|V_s|\big).$$

The final goal of this work is to compare solutions of the BBGKY hierarchy to solutions of the Boltzmann hierarchy. To do so, those solutions have to belong to similar spaces, in the sense that one wants to compare elements of the sequence of functional spaces $\left(\widetilde{X}_{\varepsilon,s,g_s}\right)_{1\leq s\leq N}$ with elements of $\left(\widetilde{X}_{0,s,g_s}\right)_{s\geq 1}$, the two sequences of functional spaces being defined with the same functions g_s bounding uniformly the profile with respect to the velocity variable. To be more explicit, the idea is to consider $\left(\widetilde{X}_{0,s,g_s}\right)_{s\geq 1}$ as fixed, while the space $\left(\widetilde{X}_{\varepsilon,s,g_s}\right)_{1\leq s\leq N}$ is the one containing approximating elements of the solution of the Boltzmann hierachy, this approximation being more and more accurate as the number of particles N grows, that is as the length of the sequence of functional spaces composing $\left(\widetilde{X}_{\varepsilon,s,g_s}\right)_{1\leq s\leq N}$ grows.
This consideration leads to considering the sequence $\left(g_s\right)_{s\geq 1}$ defining the space on which the Boltzmann operator acts as given, and one wants to obtain for any positive integer N, from the finite sequence $g_1,\ldots,g_N$, the required properties that this finite sequence has to verify in order to define properly the BBGKY operator on this sequence of functional spaces.
In other words, if one assumes that the sequence $\left(g_s\right)_{s\geq 1}$ verifies the assumptions, recalled above, used to define the Boltzmann hierarchy, is it enough to obtain that the assumptions used to define the BBGKY hierarchy hold true ? Clearly, it is not. Finally, one will work with sequence of functions $\left(g_s\right)_{s\geq 1}$ verifying the following hypotheses.

Definition 22 (Hypotheses for the joint definition of the BBGKY and Boltzmann operators)**.** *Let $\left(g_s\right)_{s\geq 1}$ be a sequence of functions $g_s:\mathbb{R}_+\to\mathbb{R}_+$. This sequence $\left(g_s\right)_{s\geq 1}$ will be called* hierarchies-admissible *if it verifies the following hypotheses :*

1. *for every $N\geq 1$, $r\mapsto g_N(r)$ is a measurable, almost everywhere strictly positive function,*
2. *for every integer $N\geq 1$, the function*
$$V_N\mapsto\left(1+|V_N|^2\right)^{N/2}g_N\left(|V_N|\right)$$
is integrable,
3. *for every integer $N\geq 1$, and for every integer $1\leq s\leq N$:*
$$\frac{1}{g_N\left(|V_N|\right)}\int_{\mathbb{R}^{d_s}}|V_{N+s}|^s g_{N+s}\left(|V_{N+s}|\right)\mathrm{d}v_{N+1}\ldots\mathrm{d}v_{N+s}$$
is bounded almost everywhere.
4. *for every integer $N\geq 1$, the function :*
$$V_{s-1}\mapsto\int_{\mathbb{R}^{d(N-s+1)}}|V_N|^{N-s+1}g\left(|V_N|\right)\mathrm{d}v_s\ldots\mathrm{d}v_N$$
is bounded almost everywhere,

5. *for every integer $N \geq 1$, and for every integer $1 \leq s \leq N$:*

$$\Big| \int_{\mathbb{R}^{d(N-s+1)}} \mathbb{1}_{|V_N| \geq R} |V_N|^{N-s+1} g_N\big(|V_N|\big) \, \mathrm{d}v_s \dots \mathrm{d}v_N \Big|_{L^\infty(\mathbb{R}^{d(s-1)})}$$

goes to zero as R goes to infinity.

With those hypotheses together, Theorems 2 page 194 and 3 page 197 ensure that the BBGKY and the Boltzmann operators and respectively well defined on $\big(\widetilde{X}_{\varepsilon,s,g_s}\big)_{1 \leq s \leq N}$, and $\big(\widetilde{X}_{0,s,g_s}\big)_{s \geq 1}$.

6.3.2 Existence of functions verifying the hypotheses of the definition of the spaces $X^0_{\cdot,s,g_s}$ and $\widetilde{X}_{\cdot,s,g_s}$

An important question remains : is it possible to find a sequence of functions $\big(g_s\big)_{s \geq 1}$ which verifies the hypotheses of Definition 22 ?
Luckily, the answer is affirmative. This is the object of the following lemma.

Lemma 16 (Gaussians are hierarchies-admissible)**.** *Let $\big(\beta_s\big)_{s \geq 1}$ be an increasing sequence of positive numbers, and $\big(g_s\big)_{s \geq 1}$ be the sequence of functions defined as, for every $s \geq 1$:*

$$g_s = \begin{cases} \mathbb{R}_+ & \to \mathbb{R}_+ \\ x & \mapsto \exp\big(-\beta_s \frac{x^2}{2}\big). \end{cases}$$

Then this sequence is hierarchies-admissible.

Proof. The two first points of Definition 22 are clearly verified.
From the easy inequality :

$$\begin{aligned}
&\frac{1}{g_N\big(|V_N|\big)} \int_{\mathbb{R}^{ds}} |V_{N+s}|^s g_{N+s}\big(|V_{N+s}|\big) \, \mathrm{d}v_{N+1} \dots \mathrm{d}v_{N+s} \\
&\quad = \exp\Big(\beta_N \frac{|V_N|^2}{2}\Big) \int_{\mathbb{R}^{ds}} |V_{N+s}|^s \exp\Big(-\beta_{N+s} \frac{|V_{N+s}|^2}{2}\Big) \, \mathrm{d}v_{N+1} \dots \mathrm{d}v_{N+s} \\
&\quad = \exp\Big(\beta_N \frac{|V_N|^2}{2}\Big) \int_{\mathbb{R}^{ds}} |V_{N+s}|^s \exp\Big(-\beta_{N+s} \frac{|V_N|^2}{2}\Big) \\
&\qquad\qquad \times \exp\Big(-\beta_{N+s} \frac{\sum_{i=N+1}^{N+s} |v_i|^2}{2}\Big) \, \mathrm{d}v_{N+1} \dots \mathrm{d}v_{N+s} \\
&\quad = \exp\Big(-(\beta_{N+s} - \beta_N) \frac{|V_N|^2}{2}\Big) \\
&\qquad\qquad \times \int_{\mathbb{R}^{ds}} |V_{N+s}|^s \exp\Big(-\beta_{N+s} \frac{\sum_{i=N+1}^{N+s} |v_i|^2}{2}\Big) \, \mathrm{d}v_{N+1} \dots \mathrm{d}v_{N+s} \\
&\quad \leq \exp\Big(-(\beta_{N+s} - \beta_N) \frac{|V_N|^2}{2}\Big) \\
&\qquad\qquad \times \int_{\mathbb{R}^{ds}} |V_{N+s}|^s \exp\Big(-\beta_{N+s} \frac{\sum_{i=N+1}^{N+s} |v_i|^2}{2}\Big) \, \mathrm{d}v_{N+1} \dots \mathrm{d}v_{N+s}
\end{aligned}$$

and using the fact that :

$$|V_{N+s}|^s = \Big(|V_N|^2 + \sum_{i=N+1}^{N+s} |v_i|^2\Big)^{s/2} \leq \Big(|V_N| + \sum_{i=N+1}^{N+s} |v_i|\Big)^s$$
$$\leq 2^s\Big(|V_N|^s + \Big(\sum_{i=N+1}^{N+s} |v_i|\Big)^s\Big),$$

one obtains the third point, and the fourth is proven the same way.
The last one is obtained, on the one hand after noticing that :

$$|V_N|^{N-s+1} g_N\big(|V_N|\big) = \Big(|V_N|^{N-s+1} \exp\Big(-\frac{\beta_N}{2}|V_N|^2\Big)\Big) \exp\Big(-\frac{\beta_N}{2}|V_N|^2\Big),$$

with $V_N \mapsto |V_N|^{N-s+1} \exp\Big(-\frac{\beta_N}{2}|V_N|^2\Big)$ a radial continuous function, vanishing at infinity, so that there exists $C(d, \beta_N, s) > 0$ such that for all $V_N \in \mathbb{R}^{dN}$:

$$|V_N|^{N-s+1} \exp\Big(-\frac{\beta_N}{2}|V_N|^2\Big) \leq C(d, \beta_N, s),$$

so that the term implying the power of the norm of V_N is under control.
On the other hand, noticing the following inclusion of sets holds :

$$\Big\{V_N \in \mathbb{R}^{dN} \ / \ |V_N|^2 \geq R^2\Big\}$$
$$\subset \Big\{V_N \in \mathbb{R}^{dN} \ / \ |V_{s-1}|^2 \geq R^2/2 \text{ or } \sum_{i=s}^{N} |v_i|^2 \geq R^2/2\Big\},$$

that is :

$$\Big\{V_N \in \mathbb{R}^{dN} \ / \ |V_N|^2 \geq R^2\Big\}$$
$$\subset \Big\{V_N \in \mathbb{R}^{dN} \ / \ |V_{s-1}|^2 \geq R^2/2\Big\} \cup \Big\{V_N \in \mathbb{R}^{dN} \ / \ \sum_{i=s}^{N} |v_i|^2 \geq R^2/2\Big\},$$

one has :

$$\int_{\mathbb{R}^{d(N-s+1)}} \mathbb{1}_{|V_N| \geq R} |V_N|^{N-s+1} g_N\big(|V_N|\big)\, \mathrm{d}v_s \ldots \mathrm{d}v_N$$
$$\leq \mathbb{1}_{|V_{s-1}|^2 \geq R^2/2} \int_{\mathbb{R}^{d(N-s+1)}} C(d, \beta_N, s) \exp\Big(-\frac{\beta_N}{2}|V_N|^2\Big)\, \mathrm{d}v_s \ldots \mathrm{d}v_N$$
$$+ \int_{\mathbb{R}^{d(N-s+1)}} \mathbb{1}_{\sum_{i=s}^{N} |v_i|^2 \geq R^2/2} C(d, \beta_N, s) \exp\Big(-\frac{\beta_N}{2}|V_N|^2\Big)\, \mathrm{d}v_s \ldots \mathrm{d}v_N.$$

The first term can again be bounded by :

$$\exp\Big(-\frac{\beta_N}{4}R^2\Big) \int_{\mathbb{R}^{d(N-s+1)}} C(d, \beta_N, s) \exp\Big(-\frac{\beta_N}{2}\sum_{i=s}^{N} |v_i|^2\Big)\, \mathrm{d}v_s \ldots \mathrm{d}v_N,$$

which does not depend on the variable V_{s-1} and goes then uniformly to zero as R goes to infinity since the integrand is integrable and does not depend on any parameter.
The second term can be bounded by :

$$C(d,\beta_N,s)\int_{\mathbb{R}^{d(N-s+1)}} \mathbb{1}_{\sum_{i=s}^N |v_i|^2 \geq R^2/2} \exp\Big(-\frac{\beta_N}{2}\sum_{i=s}^N |v_i|^2\Big)\,\mathrm{d}v_s\ldots\mathrm{d}v_N,$$

which does not depend on the variable V_{s-1}, and goes similarly to zero as R goes to infinity as the tail of the integral of an integrable function. $\square$

Chapter 7

Final functional spaces

7.1 Definition of the functional spaces

Thanks to the work done in Sections 5 and 6, one has seen that the relevant functional spaces which give a rigorous sense to the conjugate BBGKY hierarchy :

$$\forall\ 1 \leq s \leq N-1 :$$
$$h_N^{(s)}(t, Z_s) = f_{N,0}^{(s)}(Z_s) + \int_0^t \mathcal{T}_{-u}^{s,\varepsilon} \mathcal{C}_{s,s+1}^{N,\varepsilon} \mathcal{T}_u^{s+1,\varepsilon} h_N^{(s+1)}(u, Z_s)\, \mathrm{d}u,$$

and the Boltzmann hierarchy :

$$\forall\ s \geq 1 :$$
$$f^{(s)}(t, Z_s) = \mathcal{T}_t^{s,0} f_0^{(s)}(Z_s) + \int_0^t \mathcal{T}_{t-u}^{s,0} \mathcal{C}_{s,s+1}^{0} f^{(s+1)}(u, Z_s)\, \mathrm{d}u$$

are respectively $\left(X_{\varepsilon,s,g_s}\right)_{1\leq s\leq N}$, and $\left(X_{0,s,g_s}\right)_{s\geq 1}$, with $\left(g_s\right)_{s\geq 1}$ a sequence which is hierarchy-admissible in the sense of Definition 22 page 200.
However, and since now the main goal of this section is to obtain the existence and uniqueness of a solution to each hierarchy, this regularity, which may look somehow strong, is not enough to obtain easily a fixed point result. So one is led to introduce new functional spaces, more regular, on which the fixed point argument will hold.
The main difference with the example of the gaussian given at the end of the previous section, where the regularity in velocity changed with the number of particles involved, lies in the fact that this regularity will change according to time.

7.1.1 Definition of the spaces $X_{\varepsilon,s,\beta}$ and $X_{0,s,\beta}$, the functions of the phase space of s particles bounded by a gaussian in the velocity variables

One starts by the definition of the first kind of functional space, in which each marginal will lie. First, one introduces the relevant space for the BBGKY hierarchy, that is for functions defined on the phase space $\mathcal{D}_s^\varepsilon$ of s particles and of diameter ε.

Definition 23 (Norm $|\cdot|_{\varepsilon,s,\beta}$, space $X_{\varepsilon,s,\beta}$)**.** *Let ε and $\beta > 0$ be two strictly positive numbers and s be a positive integer. For any function $h^{(s)}$ belonging to $L^\infty\big(\mathcal{D}_s^\varepsilon\big)$, one defines :*

$$|h^{(s)}|_{\varepsilon,s,\beta} = \sup_{Z_s \in \mathcal{D}_s^\varepsilon} \operatorname{ess} \left[\big|h^{(s)}(Z_s)\big| \exp\left(\frac{\beta}{2} \sum_{i=1}^{s} |v_i|^2 \right) \right],$$

and the space $X_{\varepsilon,s,\beta}$ as the space of the functions of $L^\infty\big(\mathcal{D}_s^\varepsilon\big)$ with a finite $|\cdot|_{\varepsilon,s,\beta}$ norm, that is :

$$X_{\varepsilon,s,\beta} = \Big\{ h^{(s)} \in L^\infty\left(\mathcal{D}_s^\varepsilon\right) \ / \ |h^{(s)}|_{\varepsilon,s,\beta} < +\infty \Big\}.$$

Similarly, one defines the same norm for the Boltzmann hierarchy, in the case where the particles have radius zero, and then are seen as points.

Definition 24 (Norm $|\cdot|_{0,s,\beta}$, space $X_{0,s,\beta}$)**.** *Let β be a strictly positive number and s be a positive integer. For any function $f^{(s)}$ belonging to $\mathcal{C}_0\big(\big(\overline{\Omega^c} \times \mathbb{R}^d\big)^s\big)$, one defines :*

$$|f^{(s)}|_{0,s,\beta} = \sup_{Z_s \in \left(\overline{\Omega^c} \times \mathbb{R}^d\right)^s} \left[\big|f^{(s)}(Z_s)\big| \exp\left(\frac{\beta}{2} \sum_{i=1}^{s} |v_i|^2 \right) \right],$$

and the space $X_{0,s,\beta}$ as the space of the continuous functions vanishing at infinity defined on $\big(\overline{\Omega^c} \times \mathbb{R}^d\big)^s$ with a finite $|\cdot|_{0,s,\beta}$ norm, that is :

$$X_{0,s,\beta} = \big\{ f^{(s)} \in \mathcal{C}_0\big(\big(\overline{\Omega^c} \times \mathbb{R}^d\big)^s\big) \ / \ |f^{(s)}|_{0,s,\beta} < +\infty \big\},$$

and satisfying the following boundary condition

$$f^{(s)}(Z_s) = f^{(s)}(\chi_s^0(Z_s))$$

for all Z_s belonging to the boundary of $\big(\overline{\Omega^c} \times \mathbb{R}^d\big)^s$, that is such that there exists at least an integer $1 \leq i \leq s$ such that $x_i \cdot e_1 = 0$ and $v_i \cdot e_1 > 0$, with χ_s^0 introduced in Definition 11 page 79.

One has immediately the two following propositions.

Proposition 5. *For any $\beta > 0$, $\varepsilon > 0$ and any positive integer s, $X_{\varepsilon,s,\beta}$ is a Banach space with respect to the norm $|\cdot|_{\varepsilon,s,\beta}$.*

Proposition 6. *For any $\beta > 0$ and any positive integer s, $X_{0,s,\beta}$ is a Banach space with respect to the norm $|\cdot|_{0,s,\beta}$.*

7.1.2 Definition of the spaces $\mathbf{X}_{\varepsilon,\beta,\mu^\alpha}$ and $\mathbf{X}_{0,\beta,\mu^\alpha}$, the sequence of functions of $X_{\cdot,s,\beta}$ with an exponential weight with respect to the number of particles

From those functional spaces, which are nothing more than functions defined on the phase space of s particles, bounded from above by a gaussian in the velocity variable, one will define spaces of sequences of elements of such functional spaces, with a varying number s of particles.

One stresses the fact that one defines in fact *a family* of such spaces, with, in addition to a real parameter μ, which is the activity of the solution from a physical point of view, another parameter α, strictly positive, and which will be taken equal to 1 or 2 in the sequel. The choice of this parameter α and the consequences of this choice are discussed in Section 8.2 page 251, and it leads to define two kinds of continuity with respect to time, as done in the following in Definitions 27 and 29 for the BBGKY hierarchy, and Definitions 28 and 30 for the Boltzmann hierarchy. About the question of notations, the parameter α appears in the notation of the spaces $\mathbf{X}$ and $\widetilde{\mathbf{X}}$ as the exponent of the index μ in the sequel.

Definition 25 (Norm $\|\cdot\|_{N,\varepsilon,\beta,\mu^\alpha}$, space $\mathbf{X}_{N,\varepsilon,\beta,\mu^\alpha}$)**.** *Let N be a positive integer. Let ε an β be two strictly positive numbers, μ be a real number and $\alpha > 0$ be a strictly positive number. For any finite sequence $H_N = \left(h_N^{(s)}\right)_{1\leq s\leq N}$ of functions $h_N^{(s)}$ of $X_{\varepsilon,s,\beta}$, one defines :*

$$\|H_N\|_{N,\varepsilon,\beta,\mu^\alpha} = \max_{1\leq s\leq N}\Big(|h_N^{(s)}|_{\varepsilon,s,\beta}\exp(s^\alpha\mu)\Big),$$

and the space $\boldsymbol{X}_{N,\varepsilon,\beta,\mu^\alpha}$ as the space of the finite sequences $H_N = \left(h_N^{(s)}\right)_{1\leq s\leq N}$ such that for every $1 \leq s \leq N$, $h_N^{(s)}$ belongs to $X_{\varepsilon,s,\beta}$, and such that the sequence $\left(h_N^{(s)}\right)_{1\leq s\leq N}$ has a finite $\|\cdot\|_{N,\varepsilon,\beta,\mu^\alpha}$ norm, that is :

$$\boldsymbol{X}_{N,\varepsilon,\beta,\mu^\alpha} = \Big\{H_N = \left(h_N^{(s)}\right)_{1\leq s\leq N} \in \left(X_{\varepsilon,s,\beta}\right)_{1\leq s\leq N} \ / \ \|H_N\|_{N,\varepsilon,\beta,\mu^\alpha} < +\infty\Big\}.$$

Definition 26 (Norm $\|\cdot\|_{0,\beta,\mu^\alpha}$, space $\mathbf{X}_{0,\beta,\mu^\alpha}$)**.** *Let β be a strictly positive number, μ be a real number and α be a strictly positive number. For any infinite sequence $F = \left(f^{(s)}\right)_{s\geq 1}$ of functions $f^{(s)}$ of $X_{0,s,\beta}$, one defines :*

$$\|F\|_{0,\beta,\mu^\alpha} = \sup_{s\geq 1}\Big(|f^{(s)}|_{0,s,\beta}\exp(s^\alpha\mu)\Big),$$

and the space $\boldsymbol{X}_{0,\beta,\mu^\alpha}$ as the space of the infinite sequences $\left(f^{(s)}\right)_{s\geq 1}$ such that for every $s \geq 1$, $f^{(s)}$ belongs to $X_{0,s,\beta}$, and such that the sequence $\left(f^{(s)}\right)_{s\geq 1}$ has a finite $\|\cdot\|_{0,\beta,\mu^\alpha}$ norm, that is :

$$\boldsymbol{X}_{0,\beta,\mu^\alpha} = \Big\{F = \left(f^{(s)}\right)_{s\geq 1} \in \left(X_{0,s,\beta}\right)_{s\geq 1} \ / \ \|F\|_{0,\beta,\mu^\alpha} < +\infty\Big\}.$$

The following result describes the embeddings that exist between the spaces $X_{\varepsilon,s,\beta}$ for different parameters β on the one hand, and on the other hand between the spaces $\mathbf{X}_{N,\varepsilon,\beta,\mu^\alpha}$ for different parameters β and μ.

Proposition 7 (Embeddings of the spaces $X_{\varepsilon,s,\beta}$, and $\mathbf{X}_{N,\varepsilon,\beta,\mu^\alpha}$)**.** *Let s be a positive integer and ε be a strictly positive number.*

- *For any $\beta \leq \beta'$, one has :*

$$X_{\varepsilon,s,\beta'} \subset X_{\varepsilon,s,\beta},$$

and if $h^{(s)}$ belongs to $X_{\varepsilon,s,\beta'}$, one has :

$$|h^{(s)}|_{\varepsilon,s,\beta} \leq |h^{(s)}|_{\varepsilon,s,\beta'}. \tag{7.1}$$

- *For any $\beta \leq \beta'$, any $\mu \leq \mu'$ and any $\alpha > 0$, one has :*

$$\boldsymbol{X}_{N,\varepsilon,\beta',\mu'^\alpha} \subset \boldsymbol{X}_{N,\varepsilon,\beta,\mu^\alpha},$$

and if $\big(h_N^{(s)}\big)_{1\leq s\leq N}$ belongs to $\boldsymbol{X}_{N,\varepsilon,\beta',\mu'^\alpha}$, one has :

$$\Big\|\big(h_N^{(s)}\big)_{1\leq s\leq N}\Big\|_{N,\varepsilon,\beta,\mu^\alpha} \leq \Big\|\big(h_N^{(s)}\big)_{1\leq s\leq N}\Big\|_{N,\varepsilon,\beta',\mu'^\alpha}. \tag{7.2}$$

Proof. The first inequality (7.1) and the first inclusion are both a direct consequence of the easy following computation. If the function $h^{(s)}$ belongs to the space $X_{\varepsilon,s,\beta'}$, then the following quantity is bounded almost everywhere on the phase space $\mathcal{D}_s^\varepsilon$, and one has :

$$\begin{aligned}
&\big|h^{(s)}(Z_s)\big| \exp\Big(\frac{\beta'}{2}\sum_{i=1}^s |v_i|^2\Big) \\
&\qquad = \big|h^{(s)}(Z_s)\big| \exp\Big(\frac{\beta'-\beta}{2}\sum_{i=1}^s |v_i|^2\Big) \exp\Big(\frac{\beta}{2}\sum_{i=1}^s |v_i|^2\Big),
\end{aligned}$$

and the quantity $\frac{\beta'-\beta}{2}\sum_{i=1}^s |v_i|^2$ is strictly positive, so that of course :

$$\begin{aligned}
\big|h^{(s)}(Z_s)\big| \exp\Big(\frac{\beta}{2}\sum_{i=1}^s |v_i|^2\Big) &\leq \big|h^{(s)}(Z_s)\big| \exp\Big(\frac{\beta'}{2}\sum_{i=1}^s |v_i|^2\Big) \\
&\leq \big|h^{(s)}\big|_{\varepsilon,s,\beta'}.
\end{aligned}$$

The second inequality (7.2) and the second inclusion are obtained in the same way for a sequence $\big(h_N^{(s)}\big)_{1\leq s\leq N}$ belonging to $\mathbf{X}_{N,\varepsilon,\beta',\mu'^\alpha}$. Indeed, for every integer $1 \leq s \leq N$, one writes :

$$\begin{aligned}
&\big|h_N^{(s)}\big| \exp\Big(\frac{\beta'}{2}\sum_{i=1}^s |v_i|^2\Big) \exp(s^\alpha \mu') \\
&\qquad = \big|h_N^{(s)}\big| \exp\Big(\frac{\beta}{2}\sum_{i=1}^s |v_i|^2\Big) \exp(s^\alpha \mu) \exp\Big(\frac{\beta'-\beta}{2}\sum_{i=1}^s |v_i|^2\Big) \exp\big(s^\alpha(\mu'-\mu)\big),
\end{aligned}$$

and one notices that the quantity :

$$\frac{\beta' - \beta}{2} \sum_{i=1}^{s} |v_i|^2 + (\mu' - \mu) s^\alpha$$

is strictly positive by hypothesis. □

Similarly, the analog of this proposition holds for the spaces $X_{0,s,\beta}$, and $\mathbf{X}_{0,\beta,\mu^\alpha}$.

Proposition 8 (Embeddings of the spaces $X_{0,s,\beta}$, and $\mathbf{X}_{0,\beta,\mu^\alpha}$)**.** *Let s be a positive integer.*

- *For any $\beta \leq \beta'$, one has :*

$$X_{0,s,\beta'} \subset X_{0,s,\beta},$$

and if $f^{(s)}$ belongs to $X_{0,s,\beta'}$, one has :

$$|f^{(s)}|_{0,s,\beta} \leq |f^{(s)}|_{0,s,\beta'}. \tag{7.3}$$

- *For any $\beta \leq \beta'$, any $\mu \leq \mu'$ and any $\alpha > 0$, one has :*

$$\boldsymbol{X}_{0,\beta',\mu'^\alpha} \subset \boldsymbol{X}_{0,\beta,\mu^\alpha},$$

and if $\left(f^{(s)}\right)_{s\geq 1}$ belongs to $\boldsymbol{X}_{0,\beta',\mu'^\alpha}$, one has :

$$\left\| \left(f^{(s)}\right)_{s\geq N} \right\|_{0,\beta,\mu^\alpha} \leq \left\| \left(f^{(s)}\right)_{s\geq 1} \right\|_{0,\beta',\mu'^\alpha}. \tag{7.4}$$

7.1.3 Definition of the spaces $\widetilde{\mathbf{X}}_{\varepsilon,\widetilde{\beta},\widetilde{\mu}^\alpha}$ and $\widetilde{\mathbf{X}}_{0,\widetilde{\beta},\widetilde{\mu}^\alpha}$, the functions of sequences belonging to $\mathbf{X}_{\cdot,\widetilde{\beta}(t),\widetilde{\mu}(t)^\alpha}$ at time t

One will define now spaces of time-dependent functions. It will be important in the sequel to enable a loss of regularity when time grows, this loss being described as sequences $\left(f^{(s)}(t)\right)_s$ belonging to larger and larger spaces as the time parameter t grows. It means in the context that the parameters β and μ, respectively in the gaussian and exponential weights, as introduced in Definitions 23 and 25 for the BBGKY hierarchy, 24 and 26 for the Boltzmann hierarchy, are non increasing with respect to time. With such weights, the growth of the functional spaces as the time variable is increasing is described in Proposition 7 and 8.

From this point, there are two possibilities to define the relevant spaces of time-dependent functions taking their values in $\mathbf{X}_{\cdot,\widetilde{\beta}(t),\widetilde{\mu}(t)}$, depending on the value of α. One will see that the value of the parameter α has to be balanced with the regularity with respect to time, so that the spaces remain stable under the action of the collision operators of the two hierarchies.

The case of $\alpha = 2$, and uniform continuity in time in the parameter s

One follows here the definition[1] given in the erratum version of the article [34], in the sense that one requires the continuity in time, uniformly in the parameter s. If one wants to preserve this continuity property under the action of the collision operators of the hierarchies, it is important to choose $\alpha > 1$. In this work, one will choose $\alpha = 2$. Further explanations on this statement are given in Section 8.2 page 251

Definition 27 (Norm $|||\cdot|||_{N,\varepsilon,\tilde{\beta},\tilde{\mu}^2}$, space $\widetilde{\mathbf{X}}_{N,\varepsilon,\tilde{\beta},\tilde{\mu}^2}$)**.** *Let N be a positive integer. Let ε be a strictly positive number. For any $T > 0$, any strictly positive, non increasing function $\tilde{\beta}$, any non increasing function $\tilde{\mu}$, both defined on $[0,T]$, and any function*

$$\widetilde{H}_N : \left\{ \begin{array}{rcl} [0,T] & \to & \bigcup_{t\in[0,T]} \boldsymbol{X}_{N,\varepsilon,\widetilde{\beta}(t),\widetilde{\mu}^2(t)}, \\ t & \mapsto & \widetilde{H}_N(t) = \left(h_N^{(s)}(t)\right)_{1\le s\le N}, \end{array} \right.$$

such that $\widetilde{H}_N(t) \in \boldsymbol{X}_{N,\varepsilon,\widetilde{\beta}(t),\widetilde{\mu}^2(t)}$ for all $t \in [0,T]$, one defines

$$\left|\left|\left|\widetilde{H}_N\right|\right|\right|_{N,\varepsilon,\tilde{\beta},\tilde{\mu}^2} = \sup_{0\le t\le T} \left\|\widetilde{H}_N(t)\right\|_{N,\varepsilon,\tilde{\beta}(t),\tilde{\mu}^2(t)},$$

and the space $\widetilde{\boldsymbol{X}}_{N,\varepsilon,\tilde{\beta},\tilde{\mu}^2}$ as the space of functions

$$\widetilde{H}_N : \left\{ \begin{array}{rcl} [0,T] & \to & \bigcup_{t\in[0,T]} \boldsymbol{X}_{N,\varepsilon,\widetilde{\beta}(t),\widetilde{\mu}^2(t)} \\ t & \mapsto & \widetilde{H}_N(t) = \left(h_N^{(s)}(t)\right)_{1\le s\le N}, \end{array} \right.$$

such that $\widetilde{H}_N(t) \in \boldsymbol{X}_{N,\varepsilon,\widetilde{\beta}(t),\widetilde{\mu}^2(t)}$ for all $t \in [0,T]$, with a finite $|||\cdot|||_{N,\varepsilon,\widetilde{\beta},\widetilde{\mu}^2}$, and verifying a left continuity in time hypothesis :

$$\forall t \in]0,T],\ \lim_{u\to t^-} \left\|\widetilde{H}_N(t) - \widetilde{H}_N(u)\right\|_{N,\varepsilon,\widetilde{\beta}(t),\widetilde{\mu}^2(t)} = 0, \tag{7.5}$$

that is :

$$\widetilde{\boldsymbol{X}}_{N,\varepsilon,\tilde{\beta},\tilde{\mu}^2} = \left\{ \widetilde{H}_N : \left\{ \begin{array}{rcl} [0,T] & \to & \bigcup_{t\in[0,T]} \boldsymbol{X}_{N,\varepsilon,\widetilde{\beta}(t),\widetilde{\mu}^2(t)} \\ t & \mapsto & \widetilde{H}_N(t) = \left(h_N^{(s)}(t)\right)_{1\le s\le N}, \end{array} \right. \right. \ /$$
$$\forall t \in [0,T], \widetilde{H}_N(t) \in \boldsymbol{X}_{N,\varepsilon,\widetilde{\beta}(t),\widetilde{\mu}^2(t)},\ \left|\left|\left|\widetilde{H}_N\right|\right|\right|_{N,\varepsilon,\widetilde{\beta},\widetilde{\mu}^2} < +\infty,$$
$$\left. \text{and}\ \ \forall t \in]0,T],\ \lim_{u\to t^-} \left\|\widetilde{H}_N(t) - \widetilde{H}_N(u)\right\|_{N,\varepsilon,\widetilde{\beta}(t),\widetilde{\mu}^2(t)} = 0 \right\}.$$

Remark 16. *One has to notice that, to be meaningful, the continuity property* (7.5) *in the last definition uses Proposition 7 page 208 concerning the embeddings of the space of sequences, together with the crucial fact that the functions $\widetilde{\beta}$ and $\widetilde{\mu}$ are assumed to be non increasing.*

[1] See the last Definition 5.2.4 of Section 5.2 "Functional spaces and statement of the results".

One defines finally the space of time-dependent sequences of functions (that is, a space of functions of sequences of functions) for the Boltzmann hierarchy, exactly in the same way as above.

Definition 28 (Norm $|||\cdot|||_{0,\tilde{\beta},\tilde{\mu}^2}$, space $\widetilde{\mathbf{X}}_{0,\tilde{\beta},\tilde{\mu}^2}$)**.** *For any $T > 0$, any strictly positive, non increasing function $\tilde{\beta}$ and any non increasing function $\tilde{\mu}$, both defined on $[0,T]$, and any function :*

$$\widetilde{F} : \left\{ \begin{array}{rcl} [0,T] & \to & \bigcup_{t\in[0,T]} \boldsymbol{X}_{N,\varepsilon,\widetilde{\beta}(t),\widetilde{\mu}^2(t)} \\ t & \mapsto & \widetilde{F}(t) = \left(f^{(s)}(t)\right)_{s\geq 1}, \end{array} \right.$$

such that $\widetilde{F}(t) \in \boldsymbol{X}_{0,\widetilde{\beta}(t),\widetilde{\mu}^2(t)}$ for all $t \in [0,T]$, one defines

$$\left|\left|\left|\widetilde{F}\right|\right|\right|_{0,\tilde{\beta},\tilde{\mu}^2} = \sup_{0\leq t\leq T} \left\|\widetilde{F}(t)\right\|_{0,\tilde{\beta}(t),\tilde{\mu}^2(t)},$$

and the space $\widetilde{\boldsymbol{X}}_{0,\tilde{\beta},\tilde{\mu}^2}$ as the space of functions

$$\widetilde{F} : \left\{ \begin{array}{rcl} [0,T] & \to & \bigcup_{t\in[0,T]} \boldsymbol{X}_{0,\widetilde{\beta}(t),\widetilde{\mu}^2(t)} \\ t & \mapsto & \widetilde{F}(t) = \left(f^{(s)}(t)\right)_{s\geq 1}, \end{array} \right.$$

such that $\widetilde{F}(t) \in \boldsymbol{X}_{0,\widetilde{\beta}(t),\widetilde{\mu}^2(t)}$ for all $t \in [0,T]$, with a finite $|||\cdot|||_{0,\widetilde{\beta},\widetilde{\mu}^2}$, and verifying a left continuity in time hypothesis :

$$\forall t \in \,]0,T], \ \lim_{u\to t^-} \left\|\widetilde{F}(t) - \widetilde{F}(u)\right\|_{0,\widetilde{\beta}(t),\widetilde{\mu}^2(t)} = 0, \tag{7.6}$$

that is :

$$\widetilde{\boldsymbol{X}}_{0,\tilde{\beta},\tilde{\mu}^2} = \left\{ \widetilde{F} : \left\{ \begin{array}{rcl} [0,T] & \to & \bigcup_{t\in[0,T]} \boldsymbol{X}_{0,\widetilde{\beta}(t),\widetilde{\mu}^2(t)}, \\ t & \mapsto & \widetilde{F}(t) = \left(f^{(s)}(t)\right)_{s\geq 1}, \end{array} \right. \ / \right.$$
$$\forall t \in [0,T], \widetilde{F}(t) \in \boldsymbol{X}_{0,\widetilde{\beta}(t),\widetilde{\mu}^2(t)}, \ \left|\left|\left|\widetilde{F}\right|\right|\right|_{0,\widetilde{\beta},\widetilde{\mu}^2} < +\infty,$$
$$\left. \text{and} \ \ \forall t \in \,]0,T], \ \lim_{u\to t^-} \left\|\widetilde{F}(t) - \widetilde{F}(u)\right\|_{0,\widetilde{\beta}(t),\widetilde{\mu}^2(t)} = 0 \right\}.$$

The case of $\alpha = 1$, and continuity in time for every integer s

It is also possible to consider a weaker weight, that is choosing $\alpha = 1$, and asking a less restrictive condition of continuity in time : instead of having a continuity in the $\|\cdot\|_{\cdot,\widetilde{\beta}(t),\widetilde{\mu}(t)^1}$ norm at time t, one will ask instead, for any positive integer s, a continuity in the $|\cdot|_{\cdot,s,\widetilde{\beta}(t)}$-norm at time t.

Definition 29 (Norm $|||\cdot|||_{N,\varepsilon,\tilde{\beta},\tilde{\mu}^1}$, space $\widetilde{\mathbf{X}}_{N,\varepsilon,\tilde{\beta},\tilde{\mu}^1}$)**.** *Let N be a positive integer, ε be a strictly positive number. For any $T > 0$, any strictly positive, non*

increasing function $\tilde{\beta}$, any non increasing function $\tilde{\mu}$, both defined on $[0,T]$, and any function

$$\widetilde{H}_N : \left\{ \begin{array}{rcl} [0,T] & \to & \bigcup_{t\in[0,T]} \boldsymbol{X}_{N,\varepsilon,\widetilde{\beta}(t),\widetilde{\mu}^1(t)}, \\ t & \mapsto & \widetilde{H}_N(t) = \left(h_N^{(s)}(t)\right)_{1\le s\le N}, \end{array} \right.$$

such that $\widetilde{H}_N(t) \in \boldsymbol{X}_{N,\varepsilon,\widetilde{\beta}(t),\widetilde{\mu}^1(t)}$ for all $t \in [0,T]$, one defines

$$\left|\left|\left|\widetilde{H}_N\right|\right|\right|_{N,\varepsilon,\tilde{\beta},\tilde{\mu}^1} = \sup_{0\le t\le T} \left|\left|\widetilde{H}_N(t)\right|\right|_{N,\varepsilon,\tilde{\beta}(t),\tilde{\mu}^1(t)},$$

and the space $\widetilde{\boldsymbol{X}}_{N,\varepsilon,\tilde{\beta},\tilde{\mu}^1}$ as the space of functions

$$\widetilde{H}_N : \left\{ \begin{array}{rcl} [0,T] & \to & \bigcup_{t\in[0,T]} \boldsymbol{X}_{N,\varepsilon,\widetilde{\beta}(t),\widetilde{\mu}^1(t)} \\ t & \mapsto & \widetilde{H}_N(t) = \left(h_N^{(s)}(t)\right)_{1\le s\le N}, \end{array} \right.$$

such that $\widetilde{H}_N(t) \in \boldsymbol{X}_{N,\varepsilon,\widetilde{\beta}(t),\widetilde{\mu}^1(t)}$ for all $t \in [0,T]$, with a finite $|||\cdot|||_{N,\varepsilon,\widetilde{\beta},\widetilde{\mu}^1}$ norm, and verifying a left continuity in time hypothesis :

$$\forall t \in]0,T],\ \forall\ 1 \le s \le N,\quad \lim_{u\to t^-} \left|h_N^{(s)}(t) - h_N^{(s)}(u)\right|_{\varepsilon,s,\widetilde{\beta}(t)} = 0, \tag{7.7}$$

that is :

$$\widetilde{\boldsymbol{X}}_{N,\varepsilon,\tilde{\beta},\tilde{\mu}^1} = \left\{ \widetilde{H}_N : \left\{ \begin{array}{rcl} [0,T] & \to & \bigcup_{t\in[0,T]} \boldsymbol{X}_{N,\varepsilon,\widetilde{\beta}(t),\widetilde{\mu}^1(t)} \\ t & \mapsto & \widetilde{H}_N(t) = \left(h_N^{(s)}(t)\right)_{1\le s\le N}, \end{array} \right. \right. \quad /$$
$$\forall t \in [0,T], \widetilde{H}_N(t) \in \boldsymbol{X}_{N,\varepsilon,\widetilde{\beta}(t),\widetilde{\mu}^1(t)},\ \left|\left|\left|\widetilde{H}_N\right|\right|\right|_{N,\varepsilon,\widetilde{\beta},\widetilde{\mu}^1} < +\infty,$$
$$\left. \text{and}\ \ \forall t \in]0,T],\ \forall\ 1 \le s \le N,\ \lim_{u\to t^-} \left|h_N^{(s)}(t) - h_N^{(s)}(u)\right|_{\varepsilon,s,\widetilde{\beta}(t)} = 0 \right\}.$$

Similarly, one introduces the same kind of spaces for the Boltzmann hierarchy.

Definition 30 (Norm $|||\cdot|||_{0,\tilde{\beta},\tilde{\mu}^1}$, space $\widetilde{\mathbf{X}}_{0,\tilde{\beta},\tilde{\mu}^1}$)**.** *For any $T > 0$, any strictly positive, non increasing function $\tilde{\beta}$ and any non increasing function $\tilde{\mu}$ both defined on $[0,T]$, and any function*

$$\widetilde{F} : \left\{ \begin{array}{rcl} [0,T] & \to & \bigcup_{t\in[0,T]} \boldsymbol{X}_{N,\varepsilon,\widetilde{\beta}(t),\widetilde{\mu}^1(t)} \\ t & \mapsto & \widetilde{F}(t) = \left(f^{(s)}(t)\right)_{s\ge 1}, \end{array} \right.$$

such that $\widetilde{F}(t) \in \boldsymbol{X}_{0,\widetilde{\beta}(t),\widetilde{\mu}^1(t)}$ for all $t \in [0,T]$, one defines

$$\left|\left|\left|\widetilde{F}\right|\right|\right|_{0,\tilde{\beta},\tilde{\mu}^1} = \sup_{0\le t\le T} \left|\left|\widetilde{F}(t)\right|\right|_{0,\tilde{\beta}(t),\tilde{\mu}^1(t)},$$

and the space $\widetilde{\boldsymbol{X}}_{0,\tilde{\beta},\tilde{\mu}^1}$ *as the space of functions*

$$\widetilde{F}:\left\{\begin{array}{rcl}[0,T] & \to & \bigcup_{t\in[0,T]}\boldsymbol{X}_{0,\widetilde{\beta}(t),\widetilde{\mu}^1(t)}\\ t & \mapsto & \widetilde{F}(t)=\left(f^{(s)}(t)\right)_{s\geq 1},\end{array}\right.$$

such that $\widetilde{F}(t)\in\boldsymbol{X}_{0,\widetilde{\beta}(t),\widetilde{\mu}^1(t)}$ *for all* $t\in[0,T]$*, with a finite* $|||\cdot|||_{0,\widetilde{\beta},\widetilde{\mu}^1}$*, and verifying a left continuity in time hypothesis :*

$$\forall t\in\,]0,T],\ \forall\ 1\leq s\leq N,\ \lim_{u\to t^-}\left|h_N^{(s)}(t)-h_N^{(s)}(u)\right|_{0,s,\widetilde{\beta}(t)}=0, \tag{7.8}$$

that is :

$$\widetilde{\boldsymbol{X}}_{0,\tilde{\beta},\tilde{\mu}^1}=\Bigg\{\widetilde{F}:\left\{\begin{array}{rcl}[0,T] & \to & \bigcup_{t\in[0,T]}\boldsymbol{X}_{0,\widetilde{\beta}(t),\widetilde{\mu}^1(t)},\\ t & \mapsto & \widetilde{F}(t)=\left(f^{(s)}(t)\right)_{s\geq 1},\end{array}\right.\ /$$
$$\forall t\in[0,T],\widetilde{F}(t)\in\boldsymbol{X}_{0,\widetilde{\beta}(t),\widetilde{\mu}^2(t)},\ \left|\left|\left|\widetilde{F}\right|\right|\right|_{0,\widetilde{\beta},\widetilde{\mu}^2}<+\infty,$$
$$\text{and}\ \ \forall\ t\in\,]0,T],\ \forall 1\leq s\leq N,\ \lim_{u\to t^-}\left|f^{(s)}(t)-f^{(s)}(u)\right|_{0,s,\widetilde{\beta}(t)}=0\Bigg\}.$$

7.2 Banach space structure of the spaces $\widetilde{\mathbf{X}}_{N,\varepsilon,\widetilde{\beta},\widetilde{\mu}^\alpha}$ and $\widetilde{\mathbf{X}}_{0,\widetilde{\beta},\widetilde{\mu}^\alpha}$

In order to use the Banach-Caccioppoli fixed-point theorem, one needs to work on complete spaces. Fortunately, the spaces $\widetilde{\mathbf{X}}_{N,\varepsilon,\widetilde{\beta},\widetilde{\mu}^\alpha}$ and $\widetilde{\mathbf{X}}_{0,\widetilde{\beta},\widetilde{\mu}^\alpha}$ defined in the previous section satisfy this regularity property.

Theorem 4 (Completeness of the space $\widetilde{\mathbf{X}}_{N,\varepsilon,\widetilde{\beta},\widetilde{\mu}^\alpha}$)**.** *Let N be a positive integer. Let ε be a strictly positive number. For any $T>0$, any strictly positive, non increasing function $\tilde{\beta}$ and any non increasing function $\tilde{\mu}$, both defined on $[0,T]$, and for $\alpha=1$ or 2, the space $\widetilde{\boldsymbol{X}}_{N,\varepsilon,\widetilde{\beta},\widetilde{\mu}^\alpha}$ is a Banach space.*

Proof. It is easy to show that the space $\widetilde{\mathbf{X}}_{N,\varepsilon,\widetilde{\beta},\widetilde{\mu}^\alpha}$ is indeed a vector space. To obtain completeness, one will take, without surprise, an arbitrary Cauchy sequence of $\widetilde{\mathbf{X}}_{N,\varepsilon,\widetilde{\beta},\widetilde{\mu}^\alpha}$ in order to prove that it is converging in this same space.

So let $\left(t\mapsto\left(h_N^{(s),n}(t,\cdot)\right)_{1\leq s\leq N}\right)_{n\geq 0}$ be a Cauchy sequence of the space $\widetilde{\mathbf{X}}_{N,\varepsilon,\widetilde{\beta},\widetilde{\mu}^\alpha}$. One will recall the complicated definition of the space of functions of sequences. It implies in particular that, for all $\delta>0$, there exists an integer $n_0(\delta)$ such that for all integers $p,q\geq n_0$:

$$\left|\left|\left|t\mapsto\left(h_N^{(s),p}(t,\cdot)-h_N^{(s),q}(t,\cdot)\right)_{1\leq s\leq N}\right|\right|\right|_{N,\varepsilon,\widetilde{\beta},\widetilde{\mu}^\alpha}<\delta,$$

that is, according to Definitions 27 page 210 and 29 page 211 depending on the value of α, for all $t \in [0,T]$ and for all $p,q \geq n_0$:

$$\Big\| \big(h_N^{(s),p}(t,\cdot) - h_N^{(s),q}(t,\cdot)\big)_{1\leq s\leq N} \Big\|_{N,\varepsilon,\widetilde{\beta}(t),\widetilde{\mu}^\alpha(t)} < \delta.$$

According to Definition 25 page 207, this is equivalent to asserting that, for all $t \in [0,T]$ and all integer $1 \leq s \leq N$, and all $p,q \geq n_0$:

$$\Big| h_N^{(s),p}(t,\cdot) - h_N^{(s),q}(t,\cdot) \Big|_{\varepsilon,s,\widetilde{\beta}(t)} < \delta \exp(-s^\alpha \widetilde{\mu}(t)).$$

Finally, according to Definition 23 page 206, one has in fact assumed that, for all $t \in [0,T]$, any integer $1 \leq s \leq N$, almost all $Z_s \in \mathcal{D}_s^\varepsilon$, and all $p,q \geq n_0$:

$$\Big| h_N^{(s),p}(t,Z_s) - h_N^{(s),q}(t,Z_s) \Big| \exp\Big(\frac{\widetilde{\beta}(t)}{2}\sum_{i=1}^s |v_i|^2\Big) < \delta \exp(-s^\alpha \widetilde{\mu}(t)).$$

In particular, for all $\eta > 0$, $t \in [0,T]$ and any integer $1 \leq s \leq N$, there exists $\delta(\eta,t,s) > 0$ such that :

$$\delta(\varepsilon,t,s)\exp(-s^\alpha \widetilde{\mu}(t)) < \eta.$$

For this particular δ, there exists $n_0(\delta)$ such that for every integers $p,q \geq n_0$, one has almost everywhere on $\mathcal{D}_s^\varepsilon$:

$$\Big| h_N^{(s),p}(t,Z_s) - h_N^{(s),q}(t,Z_s) \Big| \exp\Big(\frac{\widetilde{\beta}(t)}{2}\sum_{i=1}^s |v_i|^2\Big) < \eta.$$

This shows that, for every $t \in [0,T]$ and $1 \leq s \leq N$ fixed, the sequence of functions of $L^\infty\big(\mathcal{D}_s^\varepsilon\big)$:

$$\Big(Z_s \mapsto h_N^{(s),n}(t,Z_s)\exp\Big(\frac{\widetilde{\beta}(t)}{2}\sum_{i=1}^s |v_i|^2\Big)\Big)_{n\geq 0}$$

is a Cauchy sequence in $L^\infty\big(\mathcal{D}_s^\varepsilon\big)$. Thanks to the Riesz-Fischer theorem, one knows that the space $L^\infty\big(\mathcal{D}_s^\varepsilon\big)$ is complete, so that for every $t \in [0,T]$ and every $1 \leq s \leq N$, this sequence converges in $L^\infty\big(\mathcal{D}_s^\varepsilon\big)$ towards a limit, which will be denoted

$$Z_s \mapsto k_N^{(s)}(t,Z_s).$$

If one considers the function :

$$Z_s \mapsto h_N^{(s)}(t,Z_s) = k_N^{(s)}(t,Z_s)\exp\Big(-\frac{\widetilde{\beta}(t)}{2}\sum_{i=1}^s |v_i|^2\Big),$$

then of course the function :

$$Z_s \mapsto h_N^{(s)}(t,Z_s)\exp\Big(\frac{\widetilde{\beta}(t)}{2}\sum_{i=1}^s |v_i|^2\Big).$$

is bounded almost everywhere, which means by definition, that for every $t \in [0,T]$ and every $1 \leq s \leq N$, the function :

$$Z_s \mapsto h_N^{(s)}(t, Z_s)$$

belongs to the space $X_{\varepsilon,s,\widetilde{\beta}(t)}$ defined in Definition 23.
As usual, it remains to show, on the one hand, that the limit function of the sequences :

$$t \mapsto \left(h_N^{(s)}(t,\cdot)\right)_{1\leq s\leq N}$$

belongs indeed to the space $\widetilde{\mathbf{X}}_{N,\varepsilon,\widetilde{\beta},\widetilde{\mu}^\alpha}$, and on the other hand, that the sequence converges towards this limit function.
So for any $t \in [0,T]$ and $1 \leq s \leq N$ fixed, one studies first the quantity :

$$\left|h_N^{(s)}(t,\cdot)\right|_{\varepsilon,s,\widetilde{\beta}(t)},$$

which can be bounded from above by :

$$\left|h_N^{(s)}(t,\cdot) - h_N^{(s),p}(t,\cdot)\right|_{\varepsilon,s,\widetilde{\beta}(t)} + \left|h_N^{(s),p}(t,\cdot)\right|_{\varepsilon,s,\widetilde{\beta}(t)},$$

for any $p \in \mathbb{N}$. But by the convergence in $X_{\varepsilon,s,\widetilde{\beta}(t)}$ of the sequence :

$$\left(h_N^{(s),p}(t,\cdot)\right)_{p\geq 0},$$

towards $h_N^{(s)}(t,\cdot)$, the first term of the last upper bound can be chosen, choosing p big enough, as small as one wants, while the sequence :

$$\left(t \mapsto \left(h_N^{(s),n}(t,\cdot)\right)_{1\leq s\leq N}\right)_{n\geq 0}$$

is bounded in the vector space $\widetilde{\mathbf{X}}_{N,\varepsilon,\widetilde{\beta},\widetilde{\mu}^\alpha}$ as a Cauchy sequence, that is one has, for every $t \in [0,T]$, $1 \leq s \leq N$ and $p \in \mathbb{N}$:

$$\left|h_N^{(s),p}(t,\cdot)\right|_{\varepsilon,s,\widetilde{\beta}(t)} \leq \sup_{n\in\mathbb{N}} \left|\!\left|\!\left|\left(t \mapsto \left(h_N^{(s),n}(t,\cdot)\right)_{1\leq s\leq N}\right)_{n\geq 0}\right|\!\right|\!\right|_{N,\varepsilon,\widetilde{\beta},\widetilde{\mu}^\alpha} \exp(-s^\alpha \widetilde{\mu}(t)),$$

In summary, the control of the norm $|\!|\!|\cdot|\!|\!|_{N,\varepsilon,\widetilde{\beta},\widetilde{\mu}^\alpha}$ holds in the limit $p \to +\infty$, that is, for all $t \in [0,T]$ and $1 \leq s \leq N$:

$$\left|h_N^{(s)}(t,\cdot)\right|_{\varepsilon,s,\widetilde{\beta}(t)} \leq \sup_{n\in\mathbb{N}} \left|\!\left|\!\left|\left(t \mapsto \left(h_N^{(s),n}(t,\cdot)\right)_{1\leq s\leq N}\right)_{n\geq 0}\right|\!\right|\!\right|_{N,\varepsilon,\widetilde{\beta},\widetilde{\mu}^\alpha} \exp(-s^\alpha \widetilde{\mu}(t)),$$

which means exactly that the limit $t \mapsto \left(h_N^{(s)}(t,\cdot)\right)_{1\leq s\leq N}$ has a finite $|\!|\!|\cdot|\!|\!|_{N,\varepsilon,\widetilde{\beta},\widetilde{\mu}^\alpha}$ norm.
One notices here that the value of the parameter α has no impact here on the

control on the $|||\cdot|||_{N,\varepsilon,\widetilde{\beta},\widetilde{\mu}^\alpha}$.

Nevertheless, the proof that the limit belongs to the space $\widetilde{\mathbf{X}}_{N,\varepsilon,\widetilde{\beta},\widetilde{\mu}^\alpha}$ is not utterly complete. Indeed, Definition 27 in the case when $\alpha = 2$ or Definition 29 in the case when $\alpha = 1$ require also a continuity with respect to time. This continuity is again obtained with the continuity of each term of the Cauchy sequence. Indeed, for any $\delta > 0$, there exists $n_0 \in \mathbb{N}$ such that :

$$\left|\left|\left| t \mapsto \left(h_N^{(s)}(t,\cdot) - h_N^{(s),n_0}(t,\cdot)\right)_{1\le s\le N} \right|\right|\right|_{N,\varepsilon,\widetilde{\beta},\widetilde{\mu}} < \delta/4. \tag{7.9}$$

In the case when $\alpha = 2$, for any $t \in]0,T]$ and this same $\delta > 0$, there exists also $0 \le u_0(\delta) < t$ such that for all $u_0 \le u \le t$, one has (by definition of the fact that each term of the Cauchy sequence $\left(t \mapsto \left(h_N^{(s),n}(t,\cdot)\right)_{1\le s\le N}\right)_{n\ge 0}$ belongs to the space $\widetilde{\mathbf{X}}_{N,\varepsilon,\widetilde{\beta},\widetilde{\mu}^2}$, so that each term (and in particular the term n_0) is left-continuous in time for all $t \in]0,T]$ with respect to the norm $\|\cdot\|_{N,\varepsilon,\widetilde{\beta}(t),\widetilde{\mu}(t)^2}$), that is according to the property (7.5) of Definition 27 page 210 :

$$\left\|\left(h_N^{(s),n_0}(t,\cdot) - h_N^{(s),n_0}(u,\cdot)\right)_{1\le s\le N}\right\|_{N,\varepsilon,\widetilde{\beta}(t),\widetilde{\mu}(t)^2} < \delta/2.$$

Writing then :

$$\begin{aligned}
&\left\|\left(h_N^{(s)}(t,\cdot) - h_N^{(s)}(u,\cdot)\right)_{1\le s\le N}\right\|_{N,\varepsilon,\widetilde{\beta}(t),\widetilde{\mu}(t)^2} \\
&\quad\le \left\|\left(h_N^{(s)}(t,\cdot) - h_N^{(s),n_0}(t,\cdot)\right)_{1\le s\le N}\right\|_{N,\varepsilon,\widetilde{\beta}(t),\widetilde{\mu}(t)^2} \\
&\qquad + \left\|\left(h_N^{(s),n_0}(t,\cdot) - h_N^{(s),n_0}(u,\cdot)\right)_{1\le s\le N}\right\|_{N,\varepsilon,\widetilde{\beta}(t),\widetilde{\mu}(t)^2} \\
&\qquad + \left\|\left(h_N^{(s),n_0}(u,\cdot) - h_N^{(s)}(u,\cdot)\right)_{1\le s\le N}\right\|_{N,\varepsilon,\widetilde{\beta}(t),\widetilde{\mu}(t)^2},
\end{aligned}$$

and using, on the one hand for the third term the inequality (7.2) of Proposition 7 page 208, which provides since the weights $\widetilde{\beta}$ and $\widetilde{\mu}$ are non increasing :

$$\begin{aligned}
&\left\|\left(h_N^{(s),n_0}(u,\cdot) - h_N^{(s),n_0}(u,\cdot)\right)_{1\le s\le N}\right\|_{N,\varepsilon,\widetilde{\beta}(t),\widetilde{\mu}(t)^2} \\
&\qquad\qquad \le \left\|\left(h_N^{(s),n_0}(u,\cdot) - h_N^{(s),n_0}(u,\cdot)\right)_{1\le s\le N}\right\|_{N,\varepsilon,\widetilde{\beta}(u),\widetilde{\mu}(u)^2},
\end{aligned}$$

and on the other hand, the uniform control in time (7.9) for the first and the third term, one has then :

$$\left\|\left(h_N^{(s)}(t,\cdot) - h_N^{(s)}(u,\cdot)\right)_{1\le s\le N}\right\|_{N,\varepsilon,\widetilde{\beta}(t),\widetilde{\mu}(t)^\alpha} < \delta/4 + \delta/2 + \delta/4 = \delta,$$

so that the left continuity in time for all $t \in]0,T]$ with respect to the norm $\|\cdot\|_{N,\varepsilon,\widetilde{\beta}(t),\widetilde{\mu}(t)^\alpha}$ holds, and then one has proved that the limit :

$$t \mapsto \left(h_N^{(s)}(t,\cdot)\right)_{1\le s\le N}$$

belongs to $\widetilde{\mathbf{X}}_{N,\varepsilon,\widetilde{\beta},\widetilde{\mu^\alpha}}$, with in addition :

$$\left|\left|\left|\Big(t \mapsto \big(h_N^{(s)}(t,\cdot)\big)_{1\le s\le N}\Big)_{n\ge 0}\right|\right|\right|_{N,\varepsilon,\widetilde{\beta},\widetilde{\mu^\alpha}} \le \sup_{n\in\mathbb{N}} \left|\left|\left|\Big(t \mapsto \big(h_N^{(s),n}(t,\cdot)\big)_{1\le s\le N}\Big)_{n\ge 0}\right|\right|\right|_{N,\varepsilon,\widetilde{\beta},\widetilde{\mu^\alpha}}.$$

In the case when $\alpha = 1$, for the continuity property, one has to check that, for every integer $1 \le s \le N$, the quantity :

$$\big|h_N^{(s)}(t,\cdot) - h_N^{(s)}(u,\cdot)\big|_{\varepsilon,s,\widetilde{\beta}(t)}$$

goes to zero as u tends to t from the left, which is obtained thanks to the continuity of each term $h_N^{(s),n_0}$ of the sequence $\big(h_N^{(s),n_0}\big)_{1\le s\le N}$ with respect to the $|\cdot|_{\varepsilon,s,\widetilde{\beta}(t)}$ norm, for a well chosen term of the approximating sequence $\big(\big(h_N^{(s),n}\big)_{1\le s\le N}\big)_{n\ge 0}$ (as in the case when $\alpha = 2$).

As for the last part of the proof, regardless the value of the parameter α since in this very part, it will play no role, one has to show that the Cauchy sequence converges towards the obtained limit. The inequality :

$$\left|\Big(h_N^{(s),p}(t,Z_s) - h_N^{(s),q}(t,Z_s)\Big)\exp\Big(\frac{\widetilde{\beta}(t)}{2}\sum_{i=1}^{s}|v_i|^2\Big)\right| < \delta\exp(-s^\alpha\widetilde{\mu}(t)),$$

taken in the limit $q \to +\infty$ provides, for every $t \in [0,T]$ and every $1 \le s \le N$:

$$\big|h_N^{(s),p}(t,\cdot) - h_N^{(s)}(t,\cdot)\big|_{\varepsilon,s,\widetilde{\beta}(t)} < \delta\exp(-s^\alpha\widetilde{\mu}(t)).$$

In other words, one has that, for all $p \ge n_0(\delta)$:

$$\left|\left|\left|t \mapsto \Big(h_N^{(s),p}(t,\cdot) - h_N^{(s)}(t,\cdot)\Big)_{1\le s\le N}\right|\right|\right|_{N,\varepsilon,\widetilde{\beta},\widetilde{\mu^\alpha}} < \delta.$$

But since δ was chosen arbitrarily, it means that the Cauchy sequence is converging towards an element of $\widetilde{\mathbf{X}}_{N,\varepsilon,\widetilde{\beta},\widetilde{\mu^\alpha}}$. □

The analog of Theorem 4 holds of course for the space $\widetilde{\mathbf{X}}_{0,\widetilde{\beta},\widetilde{\mu}}$.

Theorem 5 (Completeness of the space $\widetilde{\mathbf{X}}_{0,\widetilde{\beta},\widetilde{\mu}}$)**.** *For any $T > 0$, any strictly positive, non increasing function $\tilde{\beta}$ and any non increasing function $\tilde{\mu}$, both defined on $[0,T]$, the space $\widetilde{\mathbf{X}}_{0,\widetilde{\beta},\widetilde{\mu}}$ is a Banach space.*

The proof of this theorem is essentially the same as for the previous theorem. The only difference lies in the fact that the space $\widetilde{\mathbf{X}}_{0,\widetilde{\beta},\widetilde{\mu^\alpha}}$ is built with continuous functions vanishing at infinity, but those two properties hold in the uniform limit.

Chapter 8

Fixed point theorem for the BBGKY and the Boltzmann operators

Along the previous Chapter 7, one has introduced the useful functional spaces $\widetilde{\mathbf{X}}_{N,\varepsilon,\widetilde{\beta},\widetilde{\mu}^{\alpha}}$ and $\widetilde{\mathbf{X}}_{0,\widetilde{\beta},\widetilde{\mu}^{\alpha}}$, and then proved that they are complete. The purpose of this chapter is now to show that there exists, in those spaces, for any well chosen initial datum, a unique solution to the BBGKY and Boltzmann hierarchies on a non trivial time interval.

To do so, this chapter will be divided into three parts.
The first one will be devoted to showing that the BBGKY and the Boltzmann operators, again up to choosing correctly the initial data, take their values respectively in the spaces $\widetilde{\mathbf{X}}_{N,\varepsilon,\widetilde{\beta},\widetilde{\mu}^{\alpha}}$ and $\widetilde{\mathbf{X}}_{0,\widetilde{\beta},\widetilde{\mu}^{\alpha}}$ if one assumes that the functions of sequences

$$t \mapsto \big(h_N^{(s)}(t,\cdot)\big)_{1\leq s\leq N}$$

and

$$t \mapsto \big(f^{(s)}(t,\cdot)\big)_{s\geq 1}$$

are chosen in the same spaces, which means, in other words, that the BBGKY and the Boltzmann operators send the spaces $\widetilde{\mathbf{X}}_{N,\varepsilon,\widetilde{\beta},\widetilde{\mu}^{\alpha}}$ and $\widetilde{\mathbf{X}}_{0,\widetilde{\beta},\widetilde{\mu}^{\alpha}}$ into themselves.
The second part will be devoted to showing that the BBGKY and the Boltzmann operators, that is the functions (if one denotes $t \mapsto \big(h_N^{(s)}(t,\cdot)\big)_{1\leq s\leq N}$ by H_N, and $t \mapsto \big(f^{(s)}(t,\cdot)\big)_{s\geq 1}$ by F) :

$$H_N \mapsto \Big(t \mapsto \big(f_{N,0}^{(s)}(\cdot) + \mathbb{1}_{s\leq N-1}\int_0^t \mathcal{T}_{-u}^{s,\varepsilon}\mathcal{C}_{s,s+1}^{N,\varepsilon}\mathcal{T}_u^{s+1,\varepsilon}h_N^{(s+1)}(u,\cdot)\,\mathrm{d}u\big)_{1\leq s\leq N}\Big)$$

and

$$F \mapsto \Big(t \mapsto \big(\mathcal{T}_t^{s,0} f_0^{(s)}(\cdot) + \int_0^t \mathcal{T}_{t-u}^{s,0} \mathcal{C}_{s,s+1}^0 f^{(s+1)}(u,\cdot)\,\mathrm{d}u\big)_{s\geq 1}\Big)$$

are in fact respectively contracting mappings on $\widetilde{\mathbf{X}}_{N,\varepsilon,\widetilde{\beta},\widetilde{\mu}^\alpha}$ and $\widetilde{\mathbf{X}}_{0,\widetilde{\beta},\widetilde{\mu}^\alpha}$. The condition to obtain this phenomenom of contraction will rely on a careful choice of time T and of the functions $\widetilde{\beta}$ and $\widetilde{\mu}$.
Finally, the third part will be devoted to the statement of the theorems of existence and uniqueness of a solution to the BBGKY and the Boltzmann hierarchies in the spaces $\widetilde{\mathbf{X}}_{N,\varepsilon,\widetilde{\beta},\widetilde{\mu}^\alpha}$ and $\widetilde{\mathbf{X}}_{0,\widetilde{\beta},\widetilde{\mu}^\alpha}$.

8.1 Image of the spaces $\widetilde{\mathbf{X}}_{N,\varepsilon,\widetilde{\beta},\widetilde{\mu}^\alpha}$ and $\widetilde{\mathbf{X}}_{0,\widetilde{\beta},\widetilde{\mu}^\alpha}$ by the BBGKY and the Boltzmann operators

8.1.1 The BBGKY operator

One starts by setting a definition for the BBGKY operator, already introduced informally along this work.

Definition 31 (BBGKY operator)**.** *For any sequence of initial data $\big(f_{N,0}^{(s)}\big)_{1\leq s\leq N}$ belonging to the space $\boldsymbol{X}_{N,\varepsilon,\widetilde{\beta}(0),\widetilde{\mu}(0)^\alpha}$ and any function of sequences $H_N = t \mapsto \big(h_N^{(s)}(t,\cdot)\big)_{1\leq s\leq N}$ belonging to the space $\widetilde{\boldsymbol{X}}_{N,\varepsilon,\widetilde{\beta},\widetilde{\mu}^\alpha}$, one defines the function of sequences :*

$$t \mapsto \big(f_{N,0}^{(s)}(\cdot) + \mathbb{1}_{s\leq N-1}\int_0^t \mathcal{T}_{-u}^{s,\varepsilon}\mathcal{C}_{s,s+1}^{N,\varepsilon}\mathcal{T}_u^{s+1,\varepsilon} h_N^{(s+1)}(u,\cdot)\,\mathrm{d}u\big)_{1\leq s\leq N}, \tag{8.1}$$

denoted by :

$$\mathfrak{E}_{N,\varepsilon}\Big(\big(f_{N,0}^{(s)}\big)_{1\leq s\leq N}, H_N\Big).$$

The functions which associates $\mathfrak{E}_{N,\varepsilon}\Big(\big(f_{N,0}^{(s)}\big)_{1\leq s\leq N}, H_N\Big)$ to the function of sequences H_N is called the BBGKY operator with sequence of initial data $\big(f_{N,0}^{(s)}\big)_s$.

Remark 17. *One can already notice that this BBGKY operator is well defined thanks to the work done about the definition of the integrated in time transport-collision-transport operator. The question addressed in this section is to determine its image, when restricted to the space $\widetilde{\boldsymbol{X}}_{N,\varepsilon,\widetilde{\beta},\widetilde{\mu}^\alpha}$.*

Since the goal of this section is to show that, if $H_N \in \widetilde{\mathbf{X}}_{N,\varepsilon,\widetilde{\beta},\widetilde{\mu}^\alpha}$, one has :

$$t \mapsto \Big(f_{N,0}^{(s)}(\cdot) + \mathbb{1}_{s\leq N-1}\int_0^t \mathcal{T}_{-u}^{s,\varepsilon}\mathcal{C}_{s,s+1}^{N,\varepsilon}\mathcal{T}_u^{s+1,\varepsilon} h_N^{(s+1)}(u,\cdot)\,\mathrm{d}u\Big)_{1\leq s\leq N} \in \widetilde{\mathbf{X}}_{N,\varepsilon,\widetilde{\beta},\widetilde{\mu}^\alpha},$$

that is, using the notation just introduced above :

$$\mathfrak{E}_{N,\varepsilon}\Big(\big(f_{N,0}^{(s)}\big)_{1\leq s\leq N}, H_N\Big) \in \widetilde{\mathbf{X}}_{N,\varepsilon,\widetilde{\beta},\widetilde{\mu}^\alpha},$$

one will prove the two following lemmas, each one addressing the regularity of a term of the BBGKY operator written here.

Lemma 17. *Let N be a positive integer. Let ε be a strictly positive number. For $\alpha = 1$ or 2, for any $T > 0$, any strictly positive, non increasing function $\tilde{\beta}$ and any non increasing function $\tilde{\mu}$, both defined on $[0,T]$, if the initial datum :*

$$\left(f_{N,0}^{(s)}\right)_{1\leq s\leq N}$$

belongs to the space $\boldsymbol{X}_{N,\varepsilon,\widetilde{\beta}(0),\widetilde{\mu}(0)^\alpha}$, then the constant function of sequences :

$$t \mapsto \left(f_{N,0}^{(s)}\right)_{1\leq s\leq N}$$

belongs to the space $\widetilde{\boldsymbol{X}}_{N,\varepsilon,\widetilde{\beta},\widetilde{\mu}^\alpha}$, and one has moreover that :

$$\left|\left|\left|t \mapsto \left(f_{N,0}^{(s)}\right)_{1\leq s\leq N}\right|\right|\right|_{N,\varepsilon,\widetilde{\beta},\widetilde{\mu}^\alpha} \leq \left|\left|\left(f_{N,0}^{(s)}\right)_{1\leq s\leq N}\right|\right|_{N,\varepsilon,\widetilde{\beta}(0),\widetilde{\mu}(0)^\alpha}. \tag{8.2}$$

Proof. The hypothesis $\left(f_{N,0}^{(s)}\right)_{1\leq s\leq N} \in \mathbf{X}_{N,\varepsilon,\widetilde{\beta}(0),\widetilde{\mu}(0)^\alpha}$ implies in particular that the $\|\cdot\|_{N,\varepsilon,\widetilde{\beta}(0),\widetilde{\mu}(0)^\alpha}$ norm of the element $\left(f_{N,0}^{(s)}\right)_{1\leq s\leq N}$ is finite. If one considers now the constant function :

$$F_{N,0} : \left\{ \begin{array}{rcl} [0,T] & \to & \mathbf{X}_{N,\varepsilon,\widetilde{\beta}(0),\widetilde{\mu}(0)^\alpha}, \\ t & \mapsto & \left(f_{N,0}^{(s)}\right)_{1\leq s\leq N}, \end{array} \right.$$

the inequality (7.2) of Proposition 7 page 208 provides, thanks to the fact that the two functions $\widetilde{\beta}$ and $\widetilde{\mu}$ are non increasing, that for all $t \in [0,T]$, the $\|\cdot\|_{N,\varepsilon,\widetilde{\beta}(t),\widetilde{\mu}(t)^\alpha}$ norm of $\left(f_{N,0}^{(s)}\right)_{1\leq s\leq N} \in \mathbf{X}_{N,\varepsilon,\widetilde{\beta}(0),\widetilde{\mu}(0)^\alpha}$ is finite, and one has :

$$\left|\left|\left(f_{N,0}^{(s)}\right)_{1\leq s\leq N}\right|\right|_{N,\varepsilon,\widetilde{\beta}(t),\widetilde{\mu}(t)^\alpha} \leq \left|\left|\left(f_{N,0}^{(s)}\right)_{1\leq s\leq N}\right|\right|_{N,\varepsilon,\widetilde{\beta}(0),\widetilde{\mu}(0)^\alpha},$$

so that the constant function $t \mapsto \left(f_{N,0}^{(s)}\right)_{1\leq s\leq N}$ has, by definition (see Definitions 29 or 27 depending on the value of α, for the definition of the $|||\cdot|||_{N,\varepsilon,\widetilde{\beta},\widetilde{\mu}^\alpha}$ norm) a finite $|||\cdot|||_{N,\varepsilon,\widetilde{\beta},\widetilde{\mu}^\alpha}$ norm, and one has :

$$\left|\left|\left|t \mapsto \left(f_{N,0}^{(s)}\right)_{1\leq s\leq N}\right|\right|\right|_{N,\varepsilon,\widetilde{\beta},\widetilde{\mu}^\alpha} \leq \left|\left|\left(f_{N,0}^{(s)}\right)_{1\leq s\leq N}\right|\right|_{N,\varepsilon,\widetilde{\beta}(0),\widetilde{\mu}(0)^\alpha},$$

so that one has obtained the inequality (8.2) of the lemma.

The only missing point to conclude the proof of the lemma is the left continuity in time for any $t \in]0,T]$. Depending on the value of α, one has two different conditions of continuity (see Definition 27 page 210 for the case $\alpha = 2$, Definition 29 page 211 for the case $\alpha = 1$).

In the case when $\alpha = 2$, the left continuity in time is with respect to the $\|\cdot\|_{N,\varepsilon,\widetilde{\beta}(t),\widetilde{\mu}(t)}$ norm, that is, one has to show that for any $t \in]0,T]$:

$$\lim_{u \to t^-} \|F_{N,0}(t) - F_{N,0}(u)\|_{N,\varepsilon,\widetilde{\beta}(t),\widetilde{\mu}(t)^2} = 0.$$

But for any $t \in]0,T]$ and any $0 \leq u \leq t$, one has of course :

$$F_{N,0}(t) = F_{N,0}(u)$$

since the function $F_{N,0}$ is constant in time, and then one has :

$$\|F_{N,0}(t) - F_{N,0}(u)\|_{N,\varepsilon,\widetilde{\beta}(t),\widetilde{\mu}(t)^2} = 0,$$

and then the continuity is proved in the case when $\alpha = 2$.
In the case when $\alpha = 1$, the left continuity in time is, for every integer $1 \leq s \leq N$, with respect to the $|\cdot|_{\varepsilon,s,\widetilde{\beta}(t)}$, that is, one has to show that for any integer $1 \leq s \leq N$ and for any $t \in]0,T]$:

$$\lim_{u \to t^-} \left|f_{N,0}^{(s)}(t) - f_{N,0}^{(s)}(u)\right|_{\varepsilon,s,\widetilde{\beta}(0)} = 0.$$

But, exactly as in the case when $\alpha = 2$, one has $F_{N,0}^{(s)}(t) = F_{N,0}^{(s)}(u) = f_{N,0}^{(s)}$, which provides immediately the result on the limit, and the continuity is proved also in the case $\alpha = 1$, which ends the proof of the lemma. □

Stability by the integral term of the BBGKY hierarchy of the space $\widetilde{\mathbf{X}}_{N,\varepsilon,\widetilde{\beta},\widetilde{\mu}^2}$ (case $\alpha = 2$)

Here one focuses on the study of the image of the space $\widetilde{\mathbf{X}}_{N,\varepsilon,\widetilde{\beta},\widetilde{\mu}^2}$ (see Definition 27 page 210, and in particular the continuity condition (7.5)) by the integral term of the BBGKY hierarchy. For the preservation of the finiteness of the $|||\cdot|||_{N,\varepsilon,\widetilde{\beta},\widetilde{\mu}^2}$ norm, one follows the proof of Ukai[1] ([63]), which is now classical and integrated in the proof of [34][2]. Besides, one focuses on the preservation of the continuity in time (7.5), which is a crucial verification in order to state that the BBGKY and Boltzmann operators preserve the functional spaces which one desires to apply a fixed-point argument on. That is why it is important here to consider a strong weight, that is to consider the case $\alpha = 2$.

Lemma 18. *Let N be a positive integer and ε be a strictly positive number. For any $T > 0$, any $\beta_0 > 0$, $\mu_0 > 0$ and $\lambda > 0$ such that :*

$$\beta_0 - \lambda T > 0 \text{ and } \mu_0 - \lambda T > 0,$$

if the function of sequences :

$$t \mapsto \left(h_N^{(s)}(t,\cdot)\right)_{1 \leq s \leq N}$$

[1]See Section 4 "Uniform Estimates", and in particular Lemma 4.2 and 4.3.
[2]See Section 5.4 "Continuity estimates".

belongs to the space $\widetilde{\boldsymbol{X}}_{N,\varepsilon,\widetilde{\beta}_\lambda,\widetilde{\mu}^2_\lambda}$ *with* $\widetilde{\beta}_\lambda$ *and* $\widetilde{\mu}_\lambda$ *denoting :*

$$\widetilde{\beta}_\lambda : \left\{ \begin{array}{rcl} [0,T] & \to & \mathbb{R}^*_+, \\ t & \mapsto & \widetilde{\beta}_\lambda(t) = \beta_0 - \lambda t, \end{array} \right.$$

and

$$\widetilde{\mu}_\lambda : \left\{ \begin{array}{rcl} [0,T] & \to & \mathbb{R}^*_+, \\ t & \mapsto & \widetilde{\mu}_\lambda(t) = \mu_0 - \lambda t, \end{array} \right.$$

then the function of sequences :

$$t \mapsto \Big(\mathbb{1}_{s \le N-1} \int_0^t \mathcal{T}^{s,\varepsilon}_{-u} \mathcal{C}^{N,\varepsilon}_{s,s+1} \mathcal{T}^{s+1,\varepsilon}_u h^{(s+1)}_N(u,\cdot)\,\mathrm{d}u \Big)_{1 \le s \le N}$$

belongs to the same space $\widetilde{\boldsymbol{X}}_{N,\varepsilon,\widetilde{\beta}_\lambda,\widetilde{\mu}^2_\lambda}$*. In addition, one has the following inequality*

$$\begin{aligned} \Bigg\| t \mapsto \Big(\mathbb{1}_{s \le N-1} \int_0^t \mathcal{T}^{s,\varepsilon}_{-u} \mathcal{C}^{N,\varepsilon}_{s,s+1} \mathcal{T}^{s+1,\varepsilon}_u h^{(s+1)}_N(u,\cdot)\,\mathrm{d}u \Big)_{1 \le s \le N} \Bigg\|_{N,\varepsilon,\widetilde{\beta}_\lambda,\widetilde{\mu}^2_\lambda} \\ \le C(d,N,\varepsilon) \widetilde{\beta}_\lambda(T)^{-d/2} \exp\big(-3\widetilde{\mu}_\lambda(T)\big) \frac{\big(1+\widetilde{\beta}_\lambda(T)^{-1/2}\big)}{\lambda} \\ \times \Big\| \big(h^{(s)}_N\big)_{1 \le s \le N} \Big\|_{N,\varepsilon,\widetilde{\beta}_\lambda,\widetilde{\mu}^2_\lambda}, \end{aligned} \tag{8.3}$$

with $C(d,N,\varepsilon)$ *denoting :*

$$C(d,N,\varepsilon) = C'(d) N \varepsilon^{d-1},$$

and $C'(d)$ *being a constant which depends only on the dimension* d*.*

Proof. First, one will show that the function of sequences :

$$t \mapsto \Big(\mathbb{1}_{s \le N-1} \int_0^t \mathcal{T}^{s,\varepsilon}_{-u} \mathcal{C}^{N,\varepsilon}_{s,s+1} \mathcal{T}^{s+1,\varepsilon}_u h^{(s+1)}_N(u,\cdot)\,\mathrm{d}u \Big)_{1 \le s \le N}$$

has a finite $\|\!|\cdot|\!\|_{N,\varepsilon,\widetilde{\beta}_\lambda,\widetilde{\mu}^2_\lambda}$ norm.
One starts by recalling that why the functional spaces involved in the statement of the lemma and this quantity are well defined. For any $T > 0$, $\beta_0 > 0$, $\mu_0 \in \mathbb{R}$ and $\lambda > 0$ such that $\beta_0 - \lambda T > 0$, one has of course

$$\forall\ 0 \le t \le T,\ \widetilde{\beta}_\lambda(t) = \beta_0 - \lambda t > 0,$$

and the functions $\widetilde{\beta}_\lambda$ and $\widetilde{\mu}_\lambda$ are non increasing, so that the space $\widetilde{\mathbf{X}}_{N,\varepsilon,\widetilde{\beta}_\lambda,\widetilde{\mu}^2_\lambda}$ is well defined (thanks to the work done in Section 7 page 205).
Now, by hypothesis since the function :

$$t \mapsto \big(h^{(s)}_N(t,\cdot)\big)_{1 \le s \le N}$$

belongs to the space $\widetilde{\mathbf{X}}_{N,\varepsilon,\widetilde{\beta}_\lambda,\widetilde{\mu}_\lambda^2}$, it is in particular an element of the sequence of the functional spaces $\Big(\mathcal{C}\big([0,T],L^\infty\big(\mathcal{D}_s^\varepsilon\big)\big)\Big)_{1\le s\le N}$, and each term $h_N^{(s)}$ is uniformly bounded with respect to the position variable by the increasing family of gaussians :

$$(t,Z_s)\mapsto C\exp\big(-\frac{\widetilde{\beta}(t)}{2}\sum_{i=1}^s|v_i|^2\big)$$

for some constant C large enough.
In particular each term $h_N^{(s)}$ satisfies all the hypotheses of Lemma 15 page 188, so that the quantity :

$$\int_0^t \mathcal{T}_{-u}^{s,\varepsilon}\mathcal{C}_{s,s+1}^{N,\varepsilon}\mathcal{T}_u^{s+1,\varepsilon}h_N^{(s+1)}(u,\cdot)\,\mathrm{d}u$$

is well defined, which shows that the function of sequences :

$$t\mapsto\Big(\mathbb{1}_{s\le N-1}\int_0^t \mathcal{T}_{-u}^{s,\varepsilon}\mathcal{C}_{s,s+1}^{N,\varepsilon}\mathcal{T}_u^{s+1,\varepsilon}h_N^{(s+1)}(u,\cdot)\,\mathrm{d}u\Big)_{1\le s\le N}$$

is well defined, without any additional assumptions on the regularity of the functions $\widetilde{\beta}_\lambda$ and $\widetilde{\mu}_\lambda$ except the fact that they have to be non increasing, and strictly positive for the first one.

To address the problem of its $|||\cdot|||_{N,\varepsilon,\widetilde{\beta}_\lambda,\widetilde{\mu}_\lambda^2}$ norm, one uses the bound (6.4) obtained in the same Lemma 15 page 188, which states that for any $t\in[0,T]$ and for any $1\le s\le N-1$, and Q denoting :

$$Q=\Big|\int_0^t \mathcal{T}_{-u}^{s,\varepsilon}\mathcal{C}_{s,s+1}^{N,\varepsilon}\mathcal{T}_u^{s+1,\varepsilon}h_N^{(s+1)}(u,Z_s)\,\mathrm{d}u\Big|,$$

and remembering that $\mathcal{C}_{s,s+1}^{N,\varepsilon}$ is defined as :

$$\mathcal{C}_{s,s+1}^{N,\varepsilon}=(N-s)\sum_{i=1}^s\Big(\mathcal{C}_{s,s+1,+,i}^\varepsilon-\mathcal{C}_{s,s+1,-,i}^\varepsilon\Big)$$

one has almost everywhere on the phase space $\mathcal{D}_s^\varepsilon$:

$$\begin{aligned}Q&\le(N-s)\sum_{i=1}^s\Bigg[\Big|\int_0^t \mathcal{T}_{-u}^{s,\varepsilon}\mathcal{C}_{s,s+1,+,i}^\varepsilon\mathcal{T}_u^{s+1,\varepsilon}h_N^{(s+1)}(u,Z_s)\,\mathrm{d}u\Big|\\&\qquad+\Big|\int_0^t \mathcal{T}_{-u}^{s,\varepsilon}\mathcal{C}_{s,s+1,-,i}^\varepsilon\mathcal{T}_u^{s+1,\varepsilon}h_N^{(s+1)}(u,Z_s)\,\mathrm{d}u\Big|\Bigg]\\&\le 2(N-s)\varepsilon^{d-1}\frac{|\mathbb{S}^{d-1}|}{2}\sum_{i=1}^s\int_0^t\Big|h_N^{(s+1)}(u,Z_{s+1})\exp\Big(\frac{\widetilde{\beta}_\lambda(u)}{2}\sum_{j=1}^{s+1}|v_j|^2\Big)\Big|_{L^\infty(\mathcal{D}_{s+1}^\varepsilon)}\\&\qquad\times\int_{\mathbb{R}^d}\big(|v_i|+|v_{s+1}|\big)\exp\Big(-\frac{\widetilde{\beta}_\lambda(u)}{2}\sum_{j=1}^{s+1}|v_j|^2\Big)\,\mathrm{d}v_{s+1}\,\mathrm{d}u.\end{aligned}$$

Using the definition of the norm $|\cdot|_{\varepsilon,s,\widetilde{\beta}_\lambda(t)}$ (Definition 23 page 206), and denoting $C(d,N,\varepsilon)=N\varepsilon^{d-1}|\mathbb{S}^{d-1}|$, one finds :

$$\begin{aligned}Q\leq\ & C(d,N,\varepsilon)\int_0^t \big|h_N^{(s+1)}(u,\cdot)\big|_{\varepsilon,s+1,\widetilde{\beta}_\lambda(u)}\\ &\times\int_{\mathbb{R}^d}\Big(\sum_{i=1}^s|v_i|+s|v_{s+1}|\Big)\exp\Big(-\frac{\widetilde{\beta}_\lambda(u)}{2}\sum_{j=1}^{s+1}|v_j|^2\Big)\,\mathrm{d}v_{s+1}\,\mathrm{d}u.\end{aligned}$$

Splitting the integral in the two following parts :

$$\begin{aligned}Q_1=\sum_{i=1}^s|v_i|\int_0^t\big|h_N^{(s+1)}(u,\cdot)\big|_{\varepsilon,s+1,\widetilde{\beta}_\lambda(u)}\exp\Big(-\frac{\widetilde{\beta}_\lambda(u)}{2}\sum_{j=1}^{s+1}|v_j|^2\Big)\\ \times\int_{\mathbb{R}^d}\exp\Big(-\frac{\widetilde{\beta}_\lambda(u)}{2}|v_{s+1}|^2\Big)\,\mathrm{d}v_{s+1}\,\mathrm{d}u,\end{aligned}$$

and

$$\begin{aligned}Q_2=s\int_0^t\big|h_N^{(s+1)}(u,\cdot)\big|_{\varepsilon,s+1,\widetilde{\beta}_\lambda(u)}\exp\Big(-\frac{\widetilde{\beta}_\lambda(u)}{2}\sum_{j=1}^{s+1}|v_j|^2\Big)\\ \times\int_{\mathbb{R}^d}|v_{s+1}|\exp\Big(-\frac{\widetilde{\beta}_\lambda(u)}{2}|v_{s+1}|^2\Big)\,\mathrm{d}v_{s+1}\,\mathrm{d}u,\end{aligned}$$

and performing the change of variables $w_{s+1}=\sqrt{\frac{\widetilde{\beta}_\lambda(u)}{2}}v_{s+1}$ in the integral over $\mathbb{R}^d$, one obtains :

$$\begin{aligned}Q_1&=\sum_{i=1}^s|v_i|\int_0^t\big|h_N^{(s+1)}(u,\cdot)\big|_{\varepsilon,s+1,\widetilde{\beta}_\lambda(u)}\exp\Big(-\frac{\widetilde{\beta}_\lambda(u)}{2}\sum_{j=1}^{s+1}|v_j|^2\Big)\\ &\qquad\times\bigg(\int_{\mathbb{R}^d}\Big(\frac{2}{\widetilde{\beta}_\lambda(u)}\Big)^{d/2}\exp\big(-|w_{s+1}|^2\big)\,\mathrm{d}w_{s+1}\bigg)\,\mathrm{d}u\\ &=C_1(d)\sum_{i=1}^s|v_i|\int_0^t\Big(\frac{2}{\widetilde{\beta}_\lambda(u)}\Big)^{d/2}\big|h_N^{(s+1)}(u,\cdot)\big|_{\varepsilon,s+1,\widetilde{\beta}_\lambda(u)}\\ &\qquad\times\exp\Big(-\frac{\widetilde{\beta}_\lambda(u)}{2}\sum_{j=1}^{s+1}|v_j|^2\Big)\,\mathrm{d}u,\end{aligned}$$

with

$$C_1(d)=\Big(\int_{-\infty}^{+\infty}\exp(-x^2)\,\mathrm{d}x\Big)^d$$

and :

$$Q_2 = s\int_0^t \big|h_N^{(s+1)}(u,\cdot)\big|_{\varepsilon,s+1,\widetilde{\beta}_\lambda(u)} \exp\Big(-\frac{\widetilde{\beta}_\lambda(u)}{2}\sum_{j=1}^{s+1}|v_j|^2\Big)$$
$$\times\Big(\sqrt{\frac{2}{\widetilde{\beta}_\lambda(u)}}^{d+1}\int_{\mathbb{R}^d}\sqrt{\widetilde{\beta}_\lambda(u)}2|v_{s+1}|\exp\Big(-\frac{\widetilde{\beta}_\lambda(u)}{2}|v_{s+1}|^2\Big)\sqrt{\frac{\widetilde{\beta}_\lambda(u)}{2}}^{d}\,\mathrm{d}v_{s+1}\Big)\,\mathrm{d}u$$
$$= sC_2(d)\int_0^t\Big(\sqrt{\frac{2}{\widetilde{\beta}_\lambda(u)}}\Big)^{d+1}\big|h_N^{(s+1)}(u,\cdot)\big|_{\varepsilon,s+1,\widetilde{\beta}_\lambda(u)}\exp\Big(-\frac{\widetilde{\beta}_\lambda(u)}{2}\sum_{j=1}^{s+1}|v_j|^2\Big)\,\mathrm{d}u$$

with

$$C_2(d) = \int_{\mathbb{R}^d}|w_{s+1}|\exp\big(-|w_{s+1}|^2\big)\,\mathrm{d}w_{s+1}.$$

Remembering that one wants to obtain a bound on the $|||\cdot|||_{N,\varepsilon,\widetilde{\beta}_\lambda,\widetilde{\mu}_\lambda^2}$ norm of the function of sequences :

$$t\mapsto\Big(\mathbb{1}_{s\le N-1}\int_0^t\mathcal{T}_{-u}^{s,\varepsilon}\mathcal{C}_{s,s+1}^{N,\varepsilon}\mathcal{T}_u^{s+1,\varepsilon}h_N^{(s+1)}(u,\cdot)\,\mathrm{d}u\Big)_{1\le s\le N},$$

one will consider the product of Q with

$$\exp\Big(\frac{\widetilde{\beta}(t)}{2}\sum_{i=1}^{s}|v_i|^2\Big)\times\exp\big(s^2\widetilde{\mu}(t)\big),$$

in order to bound that product uniformly in $t\in[0,T]$ and $1\le s\le N-1$. One gets, denoting

$$Q(t,s,Z_s) = \exp\Big(\frac{\widetilde{\beta}_\lambda(t)}{2}\sum_{i=1}^{s}|v_i|^2\Big)\exp\big(s^2\widetilde{\mu}_\lambda(t)\big)$$
$$\times\Big|\int_0^t\mathcal{T}_{-u}^{s,\varepsilon}\mathcal{C}_{s,s+1}^{N,\varepsilon}\mathcal{T}_u^{s+1,\varepsilon}h_N^{(s+1)}(u,Z_s)\,\mathrm{d}u\Big|,$$

$$Q(t,s,Z_s)\ \le\ C_3(d,N,\varepsilon)\int_0^t\Big(\sum_{i=1}^{s}|v_i|+s\widetilde{\beta}_\lambda(u)^{-1/2}\Big)\widetilde{\beta}_\lambda(u)^{-d/2}$$
$$\times\big|h_N^{(s+1)}(t,\cdot)\big|_{\varepsilon,s+1,\widetilde{\beta}_\lambda(t)}\exp\Big(\frac{\big(\widetilde{\beta}_\lambda(t)-\widetilde{\beta}_\lambda(u)\big)}{2}\sum_{i=1}^{s}|v_i|^2+s^2\widetilde{\mu}_\lambda(t)\Big)\,\mathrm{d}u,$$

with $C_3(d,N,\varepsilon)$ denoting $C(d,N,\varepsilon)\max(C_1(d),C_2(d))$.
Using now the definition of the $\|\cdot\|_{N,\varepsilon,\widetilde{\beta}_\lambda(t),\widetilde{\mu}_\lambda(t)^2}$ norm (Definition 25 page 207), one has, for every time $u\in[0,T]$ and every integer $1\le s\le N-1$:

$$\exp\big((s+1)^2\widetilde{\mu}_\lambda(u)\big)\big|h_N^{(s+1)}(u,\cdot)\big|_{\varepsilon,s+1,\widetilde{\beta}_\lambda(u)}\le\Big\|\big(h_N^{(s)}(u,\cdot)\big)_{1\le s\le N}\Big\|_{N,\varepsilon,\widetilde{\beta}_\lambda(u),\widetilde{\mu}_\lambda(u)},$$

and the definition of the $|||\cdot|||_{N,\varepsilon,\widetilde{\beta}_\lambda,\widetilde{\mu}^2_\lambda}$ norm (Definition 27 page 210), one has for every time $u \in [0,T]$:

$$\Big\|\big(h_N^{(s)}(u,\cdot)\big)_{1\le s\le N}\Big\|_{N,\varepsilon,\widetilde{\beta}_\lambda(u),\widetilde{\mu}_\lambda(u)^2} \le \Big|\Big|\Big|\big(h_N^{(s)}\big)_{1\le s\le N}\Big|\Big|\Big|_{N,\varepsilon,\widetilde{\beta}_\lambda,\widetilde{\mu}^2_\lambda},$$

so that it provides the bound on $Q(t,s,Z_s)$:

$$\begin{aligned} Q(t,s,Z_s) \le C_3(d,N,\varepsilon)&\Big|\Big|\Big|\big(h_N^{(s+1)}\big)_{1\le s\le N}\Big|\Big|\Big|_{N,\varepsilon,\widetilde{\beta}_\lambda,\widetilde{\mu}^2_\lambda}\int_0^t\Big(\sum_{i=1}^s|v_i|+s\widetilde{\beta}_\lambda(u)^{-1/2}\Big)\\ &\times\widetilde{\beta}_\lambda(u)^{-d/2}\exp\Big(\frac{\big(\widetilde{\beta}_\lambda(t)-\widetilde{\beta}_\lambda(u)\big)}{2}\sum_{i=1}^s|v_i|^2+s^2\widetilde{\mu}_\lambda(t)-(s+1)^2\widetilde{\mu}_\lambda(u)\Big)\,\mathrm{d}u. \end{aligned} \tag{8.4}$$

Except for the loss of the possible cancellations between gain and loss terms in the collision operator, that is one has written that

$$\Big|\mathcal{C}^{N,\varepsilon}_{s,s+1,i}\Big| = \Big|\mathcal{C}^{N,\varepsilon}_{s,s+1,i,+}-\mathcal{C}^{N,\varepsilon}_{s,s+1,i,-}\Big| \le \Big|\mathcal{C}^{N,\varepsilon}_{s,s+1,i,+}\Big|+\Big|\mathcal{C}^{N,\varepsilon}_{s,s+1,i,-}\Big|,$$

which is of course not sharp at all, one was in the rest of the computation as careful as possible. And here, one sees the need to add assumptions on the non increasing functions $\widetilde{\beta}$ and $\widetilde{\mu}$. Indeed, the last bound (8.4) on $Q(t,s,Z_s)$ is not uniform in s, nor in the phase space variable Z_s. If one considers very simple, non increasing functions for $\widetilde{\beta}$ and $\widetilde{\mu}$, for example if one assumes that they are constant, that is for all $t\in[0,T]$:

$$\widetilde{\beta}(t)=\beta_0$$

and

$$\widetilde{\mu}(t)=\mu_0,$$

the bound (8.4) becomes :

$$\begin{aligned} Q(t,s,Z_s) \le C_3(d,N,\varepsilon)&\Big|\Big|\Big|\big(h_N^{(s+1)}\big)_{1\le s\le N}\Big|\Big|\Big|_{N,\varepsilon,\widetilde{\beta},\widetilde{\mu}^2}\Big(\sum_{i=1}^s|v_i|+s\beta_0^{-1/2}\Big)\\ &\times\beta_0^{-d/2}\int_0^t\exp\big(-\mu_0\big)\,\mathrm{d}u, \end{aligned}$$

which will never provide the fact that the BBGKY operator sends the space $\widetilde{\mathbf{X}}_{N,\varepsilon,\widetilde{\beta},\widetilde{\mu}^2}$ into itself in those conditions, since the bound depends strongly on s and Z_s.

However, looking carefully at the integral in the bound (8.4) :

$$\begin{aligned} &\int_0^t\Big(\sum_{i=1}^s|v_i|+s\widetilde{\beta}(u)^{-1/2}\Big)\widetilde{\beta}(u)^{-d/2}\\ &\qquad\times\exp\Big(\frac{\widetilde{\beta}(t)-\widetilde{\beta}(u)}{2}\sum_{i=1}^s|v_i|^2+s^2\big(\widetilde{\mu}(t)-\widetilde{\mu}(u)\big)-(2s+1)\widetilde{\mu}(u)\Big)\,\mathrm{d}u, \end{aligned}$$

one has noticed just before that if $\widetilde{\beta}$ and $\widetilde{\mu}$ are not chosen wisely, then the term

$$\sum_{i=1}^{s} |v_i| + s\widetilde{\beta}(u)^{-1/2}$$

ruins immediately any chance to get a bound independent on s and Z_s. In other words, one has to compensate this term. But the integral is close to the expression $\int y' \exp(y)$, so using elementary differential calculus, it seems possible to control the other term, next to the exponential term under the integral.
Choosing then simple functions $\widetilde{\beta} = \widetilde{\beta}_\lambda$ and $\widetilde{\mu} = \widetilde{\mu}_\lambda$ that can lead to an explicit computation, that is taking :

$$\widetilde{\beta}_\lambda(t) = \beta_0 - \lambda t$$

and

$$\widetilde{\mu}_\lambda(t) = \mu_0 - \lambda t,$$

which are of course non increasing for $\lambda > 0$, the bound (8.4) can be rewritten as follows :

$$Q(t,s,Z_s) \leq C_3(d,N,\varepsilon) \Big|\Big|\Big| \big(h_N^{(s+1)}\big)_{1\leq s\leq N} \Big|\Big|\Big|_{N,\varepsilon,\widetilde{\beta}_\lambda,\widetilde{\mu}_\lambda^2} \int_0^t \Big(\sum_{i=1}^{s} |v_i| + s\widetilde{\beta}_\lambda(u)^{-1/2} \Big)$$
$$\times\, \widetilde{\beta}_\lambda(u)^{-d/2} \exp\Big(\frac{\lambda(u-t)}{2} \sum_{i=1}^{s} |v_i|^2 + s^2\lambda(u-t) - (2s+1)\widetilde{\mu}_\lambda(u) \Big)\, \mathrm{d}u.$$

About the properties verified by the weight functions $\widetilde{\beta}_\lambda$ and $\widetilde{\mu}_\lambda$, one recalls that those two functions have to be chosen non increasing in order to use the embedding property of the $X_{\varepsilon,s,\widetilde{\beta}_\lambda(t)}$ and $\mathbf{X}_{N,\varepsilon,\widetilde{\beta}_\lambda(t),\widetilde{\mu}_\lambda(t)^2}$ spaces, so that the condition of left continuity in time (7.5), given in the definition 27 of the $\widetilde{\mathbf{X}}_{N,\varepsilon,\widetilde{\beta}_\lambda,\widetilde{\mu}_\lambda}$ page 210 makes sense.
On the other hand, one recalls that the fact that the function $\widetilde{\beta}_\lambda$ is strictly positive on the whole time interval $[0,T]$ is mandatory in order to get a control in the high velocity on $h_N^{(s)}$, so that the integrated in time transport-collision-transport of this function is well defined (see section 6.1.2 and Lemma 15 page 188).
Finally, the condition on the positivity of the function $\widetilde{\mu}_\lambda$ is a consequence of the fact that, for the next step of the proof, that is the one about the left continuity in time of the integral term of the BBGKY operator, one needs a strong weight on the number of particles. This weight will impose then to have positivity for $\widetilde{\mu}_\lambda$ in order to recover the fact that the integral term of the BBGKY operator has a finite $|||\cdot|||_{\varepsilon,\widetilde{\beta}_\lambda,\widetilde{\mu}_\lambda^2}$ norm. About the need of this weight, and what kind of continuity one can obtain if the weight is not assumed to be that strong, the reader can refer to the first paragraph of Section 8.2 page 251 below.

The functions $u \mapsto \widetilde{\beta}_\lambda(u)^{-1/2}$, $u \mapsto \widetilde{\beta}_\lambda(u)^{-d/2}$ and $u \mapsto \exp\big(-(2s+1)\widetilde{\mu}_\lambda(u)\big)$ being all increasing, the following bound holds :

$$Q(t,s,Z_s) \le C_3(d,N,\varepsilon)\Big|\!\Big|\!\Big|\big(h_N^{(s+1)}\big)_{1\le s\le N}\Big|\!\Big|\!\Big|_{N,\varepsilon,\widetilde{\beta}_\lambda,\widetilde{\mu}_\lambda^2}\int_0^t\Big(\sum_{i=1}^s|v_i|+s\widetilde{\beta}_\lambda(T)^{-1/2}\Big)$$
$$\times\,\widetilde{\beta}_\lambda(T)^{-d/2}\exp\Big(\frac{\lambda(u-t)}{2}\sum_{i=1}^s|v_i|^2+s^2\lambda(u-t)-(2s+1)\widetilde{\mu}_\lambda(T)\Big)\,\mathrm{d}u,$$

that is :

$$Q(t,s,Z_s) \le C_3(d,N,\varepsilon)\Big|\!\Big|\!\Big|\big(h_N^{(s+1)}\big)_{1\le s\le N}\Big|\!\Big|\!\Big|_{N,\varepsilon,\widetilde{\beta}_\lambda,\widetilde{\mu}_\lambda^2}\widetilde{\beta}_\lambda(T)^{-d/2}$$
$$\times\exp\big(-(2s+1)\widetilde{\mu}_\lambda(T)\big)$$
$$\times\int_0^t\Big(\sum_{i=1}^s|v_i|+s\widetilde{\beta}_\lambda(T)^{-1/2}\Big)\exp\Big(\lambda(u-t)\Big(\sum_{i=1}^s\frac{|v_i|^2}{2}+s^2\Big)\Big)\,\mathrm{d}u.$$

The integral of the last inequality can be then computed explicitly as follows :

$$\int_0^t\Big(\sum_{i=1}^s|v_i|+s\widetilde{\beta}_\lambda(T)^{-1/2}\Big)\exp\Big(\lambda(u-t)\Big(\sum_{i=1}^s\frac{|v_i|^2}{2}+s^2\Big)\Big)\,\mathrm{d}u$$
$$=\Big(\sum_{i=1}^s|v_i|+s\widetilde{\beta}_\lambda(T)^{-1/2}\Big)\left[\frac{\exp\Big(\lambda(u-t)\Big(\sum_{i=1}^s\frac{|v_i|^2}{2}+s^2\Big)\Big)}{\lambda\Big(\sum_{i=1}^s\frac{|v_i|^2}{2}+s^2\Big)}\right]_0^t$$
$$\le\frac{\sum_{i=1}^s|v_i|+s\widetilde{\beta}_\lambda(T)^{-1/2}}{\lambda\Big(\sum_{i=1}^s\frac{|v_i|^2}{2}+s^2\Big)}.$$

Splitting again in two terms the bound obtained for the integral, if one denotes

$$I_1=\frac{\sum_{i=1}^s|v_i|}{\lambda\Big(\sum_{i=1}^s\frac{|v_i|^2}{2}+s^2\Big)}$$

and

$$I_2=\frac{s\widetilde{\beta}_\lambda(T)^{-1/2}}{\lambda\Big(\sum_{i=1}^s\frac{|v_i|^2}{2}+s^2\Big)},$$

one obtains, using the Cauchy-Schwarz inequality for the two vectors

$$(|v_1|,\dots,|v_s|)$$

and

$$(1,\dots,1)$$

a control of the term I_1 and an obvious bound from below for the denominator of the term I_2 :

$$\begin{aligned} I_1 + I_2 &\leq \frac{\sqrt{\big(\sum_{i=1}^s |v_i|^2\big)\big(\sum_{i=1}^s 1^2\big)}}{\lambda\Big(\sum_{i=1}^s \frac{|v_i|^2}{2} + s^2\Big)} + \frac{s\widetilde{\beta}_\lambda(T)^{-1/2}}{s^2\lambda} \\ &\leq \frac{1}{\lambda}\Big(\frac{\frac{1}{2}\big(\sum_{i=1}^s |v_i|^2 + s\big)}{\sum_{i=1}^s \frac{|v_i|^2}{2} + s^2} + \widetilde{\beta}_\lambda(T)^{-1/2}\Big) \\ &\leq \frac{1}{\lambda}\big(1 + \widetilde{\beta}_\lambda(T)^{-1/2}\big). \end{aligned}$$

In summary, one has obtained :

$$\begin{aligned} Q(t,s,Z_s) \leq C_3(d,N,\varepsilon)\widetilde{\beta}_\lambda(T)^{-d/2} \exp\big(-(2s+1)\widetilde{\mu}_\lambda(T)\big)\frac{\big(1+\widetilde{\beta}_\lambda(T)^{-1/2}\big)}{\lambda} \\ \times \Big|\!\Big|\!\Big|\big(h_N^{(s+1)}\big)_{1\leq s\leq N}\Big|\!\Big|\!\Big|_{N,\varepsilon,\widetilde{\beta}_\lambda,\widetilde{\mu}_\lambda^2}. \end{aligned} \tag{8.5}$$

Finally, the fact that the function $\widetilde{\mu}_\lambda$ has been chosen strictly positive is used crucially here to write that, for any positive integer s :

$$\exp\big(-(2s+1)\widetilde{\mu}_\lambda(T)\big) \leq \exp\big(-3\widetilde{\mu}_\lambda(T)\big).$$

Recalling that $Q(t,s,Z_s)$ is defined as :

$$\begin{aligned} Q(t,s,Z_s) = \exp\Big(\frac{\widetilde{\beta}_\lambda(t)}{2}\sum_{i=1}^s |v_i|^2\Big)\exp\big(s^2\widetilde{\mu}_\lambda(t)\big) \\ \times \Big|\int_0^t \mathcal{T}_{-u}^{s,\varepsilon}\mathcal{C}_{s,s+1}^{N,\varepsilon}\mathcal{T}_u^{s+1,\varepsilon}h_N^{(s+1)}(u,Z_s)\,\mathrm{d}u\Big|, \end{aligned}$$

and noticing that the right-hand side of the last inequality (8.5), combined with the last remark about the positivity of $\widetilde{\mu}_\lambda$, does not depend on s nor Z_s, one has in fact shown that :

$$\begin{aligned} \Big|\!\Big|\!\Big|t \mapsto \Big(\int_0^t \mathcal{T}_{-u}^{s,\varepsilon}\mathcal{C}_{s,s+1}^{N,\varepsilon}\mathcal{T}_u^{s+1,\varepsilon}h_N^{(s+1)}(u,\cdot)\,\mathrm{d}u\Big)_{1\leq s\leq N}\Big|\!\Big|\!\Big|_{N,\varepsilon,\widetilde{\beta}_\lambda,\widetilde{\mu}_\lambda^2} \\ \leq C_3(d,N,\varepsilon)\widetilde{\beta}_\lambda(T)^{-d/2}\exp\big(-3\widetilde{\mu}_\lambda(T)\big)\frac{\big(1+\widetilde{\beta}_\lambda(T)^{-1/2}\big)}{\lambda} \\ \times \Big|\!\Big|\!\Big|\big(h_N^{(s+1)}\big)_{1\leq s\leq N}\Big|\!\Big|\!\Big|_{N,\varepsilon,\widetilde{\beta}_\lambda,\widetilde{\mu}_\lambda^2}, \end{aligned}$$

up to assuming that $\widetilde{\beta}_\lambda$ and $\widetilde{\mu}_\lambda$ are given by the expressions :

$$\widetilde{\beta}_\lambda(t) = \beta_0 - \lambda t$$

and

$$\widetilde{\mu}_\lambda(t) = \mu_0 - \lambda t,$$

and with the only extra conditions $\beta_0 - \lambda T > 0$ and $\mu_0 - \lambda T > 0$, and the inequality (8.3) of the Lemma on the $|||\cdot|||_{\varepsilon,\widetilde{\beta}_\lambda,\widetilde{\mu}^2_\lambda}$ norm is proven.

In other words, the integral term of the image by the BBGKY operator of any element of the functional space $\widetilde{\mathbf{X}}_{N,\varepsilon,\widetilde{\beta}_\lambda,\widetilde{\mu}^2_\lambda}$ has a finite $|||\cdot|||_{N,\varepsilon,\widetilde{\beta}_\lambda,\widetilde{\mu}^2_\lambda}$ norm.

To finish showing that the integrated in time transport-collision-transport operator sends the space $\widetilde{\mathbf{X}}_{N,\varepsilon,\widetilde{\beta}_\lambda,\widetilde{\mu}^2_\lambda}$ into itself, it remains only to check the left continuity for all $t \in]0,T]$ with respect to the $\|\cdot\|_{N,\varepsilon,\widetilde{\beta}_\lambda(t),\widetilde{\mu}_\lambda(t)^2}$ norm.
One considers then, for any $t \in]0,T]$, for any $0 \leq u \leq t$, for any integer $1 \leq s \leq N-1$, and for any $Z_s \in \mathcal{D}^\varepsilon_s$ the difference :

$$Q = \Bigg| \int_0^t \mathcal{T}^{s,\varepsilon}_{-\tau} \mathcal{C}^{N,\varepsilon}_{s,s+1} \mathcal{T}^{s+1,\varepsilon}_{\tau} h^{(s+1)}_N(\tau, Z_s)\,\mathrm{d}\tau \\ - \int_0^u \mathcal{T}^{s,\varepsilon}_{-\tau} \mathcal{C}^{N,\varepsilon}_{s,s+1} \mathcal{T}^{s+1,\varepsilon}_{\tau} h^{(s+1)}_N(\tau, Z_s)\,\mathrm{d}\tau \Bigg|.$$

The quantity Q can be controlled, as above, thanks to the inequality (6.4) of Lemma 15 page 188, in the particular case of :

$$g_{s+1}(\tau, |V_{s+1}|) = \exp\Big(- \frac{\widetilde{\beta}_\lambda(\tau)}{2} |V_{s+1}|^2 \Big),$$

taking carefully into account the bounds of the integral with respect to time. One obtains then :

$$\Bigg| \int_u^t \mathcal{T}^{s,\varepsilon}_{-\tau} \mathcal{C}^{N,\varepsilon}_{s,s+1} \mathcal{T}^{s+1,\varepsilon}_{\tau} h^{(s+1)}_N(\tau, Z_s)\,\mathrm{d}\tau \Bigg| \leq 2(N-s) \sum_{i=1}^{s} \varepsilon^{d-1} \frac{|\mathbb{S}^{d-1}|}{2} \\ \times \int_u^t \left| \frac{h^{(s+1)}(\tau, Z_{s+1})}{g_{s+1}\big(\tau, |V_{s+1}|\big)} \right|_{L^\infty(\mathcal{D}^\varepsilon_{s+1})} \int_{\mathbb{R}^d} \big(|v_i| + |v_{s+1}|\big) g_{s+1}\big(\tau, |V_{s+1}|\big)\,\mathrm{d}v_{s+1}\,\mathrm{d}\tau \\ \leq (N-s)\varepsilon^{d-1} |\mathbb{S}^{d-1}| \\ \times \int_u^t \big|h^{(s+1)}(\tau,\cdot)\big|_{\varepsilon,s,\widetilde{\beta}(\tau)} \int_{\mathbb{R}^d} \Big(\sum_{i=1}^{s} |v_i| + s|v_{s+1}| \Big) \exp\Big(- \frac{\widetilde{\beta}_\lambda(\tau)}{2} |V_{s+1}| \Big)\,\mathrm{d}v_{s+1}\,\mathrm{d}\tau.$$

As above, using the definition of the $\|\cdot\|_{N,\varepsilon,\widetilde{\beta}_\lambda(\tau),\widetilde{\mu}_\lambda(\tau)^2}$ norm, that is :

$$\big|h^{(s+1)}(\tau,\cdot)\big|_{\varepsilon,s,\widetilde{\beta}_\lambda(\tau)} \leq \Big\| \big(h^{(s)}_N(\tau,\cdot)\big)_{1\leq s\leq N} \Big\|_{N,\varepsilon,\widetilde{\beta}_\lambda(\tau),\widetilde{\mu}_\lambda(\tau)^2} \exp(-(s+1)^2 \widetilde{\mu}_\lambda(\tau)) \\ \leq \Big|\Big| \big(h^{(s)}_N\big)_{1\leq s\leq N} \Big|\Big|_{N,\varepsilon,\widetilde{\beta}_\lambda,\widetilde{\mu}^2_\lambda} \exp(-(s+1)^2 \widetilde{\mu}_\lambda(\tau))$$

and multiplying the quantity Q by the product of weights

$$\exp\Big(\frac{\widetilde{\beta}_\lambda(t)}{2} |V_s|^2 \Big) \exp(s^2 \widetilde{\mu}_\lambda(t)),$$

one has :

$$
\begin{aligned}
Q\exp\Big(\frac{\widetilde{\beta}_\lambda(t)}{2}|V_s|^2\Big)\exp(s^2\widetilde{\mu}_\lambda(t)) &\le C(d,N,\varepsilon)\Big|\!\Big|\!\Big|\big(h_N^{(s)}\big)_{1\le s\le N}\Big|\!\Big|\!\Big|_{N,\varepsilon,\widetilde{\beta}_\lambda,\widetilde{\mu}_\lambda^2}\\
&\quad\times\int_u^t\Big(\sum_{i=1}^s|v_i|+s\widetilde{\beta}_\lambda(T)^{-1/2}\Big)\widetilde{\beta}_\lambda(T)^{-d/2}\\
&\qquad\times\exp\Big(\frac{\lambda}{2}(\tau-t)\sum_{i=1}^s|v_i|^2+s^2\widetilde{\mu}_\lambda(t)-(s+1)^2\widetilde{\mu}_\lambda(\tau)\Big)\,\mathrm{d}\tau.
\end{aligned}
$$

Using the fact that the function $\widetilde{\mu}_\lambda$ is non increasing, one has :

$$
\begin{aligned}
&\exp\Big(\frac{\lambda}{2}(\tau-t)\sum_{i=1}^s|v_i|^2+s^2\widetilde{\mu}_\lambda(t)-(s+1)^2\widetilde{\mu}_\lambda(\tau)\Big)\\
&=\exp\Big(\frac{\lambda}{2}(\tau-t)\sum_{i=1}^s|v_i|^2+s^2(\widetilde{\mu}_\lambda(t)-\widetilde{\mu}_\lambda(\tau))-(2s+1)\widetilde{\mu}_\lambda(\tau)\Big)\\
&\le\exp(-\widetilde{\mu}_\lambda(T))\exp(-2s\widetilde{\mu}_\lambda(T))\exp\Big(\frac{\lambda}{2}(\tau-t)\sum_{i=1}^s|v_i|^2+s^2(\widetilde{\mu}_\lambda(t)-\widetilde{\mu}_\lambda(\tau))\Big).
\end{aligned}
$$

If one denotes then Q' the quantity :

$$
Q'=Q\exp\Big(\frac{\widetilde{\beta}_\lambda(t)}{2}|V_s|^2\Big)\exp(s^2\widetilde{\mu}_\lambda(t)),
$$

one has in fact :

$$
\begin{aligned}
Q'\le C(d,N,\varepsilon)\widetilde{\beta}_\lambda(T)^{-d/2}\exp(-\widetilde{\mu}_\lambda(T))&\Big|\!\Big|\!\Big|\big(h_N^{(s)}\big)_{1\le s\le N}\Big|\!\Big|\!\Big|_{N,\varepsilon,\widetilde{\beta}_\lambda,\widetilde{\mu}_\lambda^2}\\
&\times\Big(\sum_{i=1}^s|v_i|+s\widetilde{\beta}_\lambda(T)^{-1/2}\Big)\exp(-2s\widetilde{\mu}_\lambda(T))\\
&\times\int_u^t\exp\Big(\frac{\lambda}{2}(\tau-t)\sum_{i=1}^s|v_i|^2+\lambda s^2(\tau-t)\Big)\,\mathrm{d}\tau.
\end{aligned}
$$

Here, the key argument is the use of the Cauchy-Schwarz inequality in the integral. One writes :

$$
\begin{aligned}
&\int_u^t\exp\Big(\frac{\lambda}{2}(\tau-t)\sum_{i=1}^s|v_i|^2+\lambda s^2(\tau-t)\Big)\,\mathrm{d}\tau\\
&\le\sqrt{\int_u^t 1^2\,\mathrm{d}\tau}\sqrt{\int_u^t\exp\Big(\frac{\lambda}{2}(\tau-t)\sum_{i=1}^s|v_i|^2+\lambda s^2(\tau-t)\Big)^2\,\mathrm{d}\tau}\\
&\le\sqrt{t-u}\frac{1}{\sqrt{\lambda\big(\sum_{i=1}^s|v_i|^2+2s^2\big)}}.
\end{aligned}
$$

The quantity Q' is then controlled by :

$$Q' \leq C(d,N,\varepsilon)\widetilde{\beta}_\lambda(T)^{-d/2}\exp(-\widetilde{\mu}_\lambda(T))\Big|\Big|\Big|\big(h_N^{(s)}\big)_{1\leq s\leq N}\Big|\Big|\Big|_{N,\varepsilon,\widetilde{\beta}_\lambda,\widetilde{\mu}_\lambda^2}$$
$$\times\Big(\sum_{i=1}^{s}|v_i|+s\widetilde{\beta}_\lambda(T)^{-1/2}\Big)\frac{\exp(-2s\widetilde{\mu}_\lambda(T))}{\sqrt{\lambda\big(\sum_{i=1}^{s}|v_i|^2+2s^2\big)}}\sqrt{t-u}.$$

The left continuity will be proved if one is able to show that the quantity :

$$\Big(\sum_{i=1}^{s}|v_i|+s\widetilde{\beta}_\lambda(T)^{-1/2}\Big)\frac{\exp(-2s\widetilde{\mu}_\lambda(T))}{\sqrt{\lambda\big(\sum_{i=1}^{s}|v_i|^2+2s^2\big)}}$$

is uniformly bounded in s and Z_s. The first term, denoted as Q_1, that is :

$$Q_1=\sum_{i=1}^{s}|v_i|\frac{\exp(-2s\widetilde{\mu}_\lambda(T))}{\sqrt{\lambda\big(\sum_{i=1}^{s}|v_i|^2+2s^2\big)}},$$

can be controlled as, thanks to the Cauchy-Schwarz inequality applied to the vectors $\big(|v_1|,\ldots,|v_s|\big)$ and $(1,\ldots,1)$ of $\mathbb{R}^s$:

$$Q_1\leq\exp(-2s\widetilde{\mu}_\lambda(T))\frac{\sqrt{s}\sqrt{\sum_{i=1}^{s}|v_i|^2}}{\sqrt{\lambda\big(\sum_{i=1}^{s}|v_i|^2+2s^2\big)}}.$$

Then, with an obvious bound from below in the denominator, one gets :

$$Q_1\leq\frac{1}{\sqrt{\lambda}}\sqrt{s}\exp(-2s\widetilde{\mu}_\lambda(T)).$$

Thanks now to the crucial hypothesis $\widetilde{\mu}_\lambda(T)>0$, one can state that the sequence of strictly positive numbers :

$$\big(\sqrt{s}\exp(-2s\widetilde{\mu}_\lambda(T))\big)_{s\geq1}$$

is bounded by some constant $C(\widetilde{\mu}_\lambda(T))$ since

$$\sqrt{s}\exp(-2s\widetilde{\mu}_\lambda(T))\underset{s\to+\infty}{\longrightarrow}0,$$

so that in fact

$$Q_1\leq\frac{C(\widetilde{\mu}_\lambda(T))}{\sqrt{\lambda}}.$$

For the second term Q_2 defined as :

$$Q_2=s\widetilde{\beta}_\lambda(T)^{-1/2}\frac{\exp(-2s\widetilde{\mu}_\lambda(T))}{\sqrt{\lambda\big(\sum_{i=1}^{s}|v_i|^2+2s^2\big)}},$$

one has easily, since $\exp(-2s\widetilde{\mu}_\lambda(T))<1$ and $\sum_{i=1}^{s}|v_i|^2\geq0$, that :

$$Q_2\leq\widetilde{\beta}_\lambda(T)^{-1/2}\frac{s}{\sqrt{2\lambda s^2}}=\frac{\widetilde{\beta}_\lambda(T)^{-1/2}}{\sqrt{2\lambda}}.$$

Here one has noticed the crucial role played by the power 2 over s in the denominator (in fact, it just has to be strictly larger than 1) : without it, it was not possible to bound uniformly the quantity Q_2.
Gathering the controls on Q_1 and Q_2 provides then for Q' :

$$Q' \leq C(d,N,\varepsilon)\frac{\widetilde{\beta}_\lambda(T)^{-d/2}\big(C(\widetilde{\mu}_\lambda(T))+\widetilde{\beta}_\lambda(T)^{-1/2}\big)\exp(-\widetilde{\mu}_\lambda(T))}{\sqrt{\lambda}} \times \left|\!\left|\!\left|\big(h_N^{(s)}\big)_{1\leq s\leq N}\right|\!\right|\!\right|_{N,\varepsilon,\widetilde{\beta}_\lambda,\widetilde{\mu}_\lambda^2}\sqrt{t-u},$$

and the left continuity for all time $t \in]0,T]$ is proved. □

The two previous Lemmas 17 page 221 and 18 page 222 together show that the BBGKY operator sends indeed the space $\widetilde{\mathbf{X}}_{N,\varepsilon,\widetilde{\beta}_\lambda,\widetilde{\mu}_\lambda^2}$ into itself, up to assuming some additional hypotheses on the weights $\widetilde{\beta}_\lambda$ and $\widetilde{\mu}_\lambda$.

Remark 18. *One can notice here, along the part of the proof devoted to the left continuity in time of the integral term of the BBGKY operator, that the hypothesis on the uniform in s left continuity of the argument $\big(h_N^{(s)}\big)_{1\leq s\leq N}$ was not used. Therefore, the integral term of the BBGKY operator has a regularizing effect, in the sense that, in the case when $\alpha = 2$, the hypothesis on the left continuity in time has not to be assumed to be recovered for the image of the integral term of the operator.*
One notices also that if the parameter α is strictly larger than 1 (and not exactly 2), the proof above still works. One has only to replace the Cauchy-Schwarz inequality by the Hölder inequality, providing therefore the continuity in time, with a rate of convergence a bit different (it is still a power of $t-u$).

Stability by the integral term of the BBGKY hierarchy of the space $\widetilde{\mathbf{X}}_{N,\varepsilon,\widetilde{\beta},\widetilde{\mu}^1}$ (case $\alpha = 1$)

In this paragraph, one does the same work on the image by the integral term of the BBGKY hierarchy, but this time of the space $\widetilde{\mathbf{X}}_{N,\varepsilon,\widetilde{\beta},\widetilde{\mu}^1}$ (see Definition 29 page 211, and in particular the continuity condition (7.7)). After the complete statement of the Lemma, only the parts of the proof that are different from the proof of Lemma 18 will be written. One stresses here the fact that there is no positivity condition of μ_0 nor of the function $\widetilde{\mu}$ in this particular case $\alpha = 1$.

Lemma 19. *Let N be a positive integer and ε be a strictly positive number. For any $T > 0$, any $\beta_0 > 0$, $\mu_0 \in \mathbb{R}$ and $\lambda > 0$ such that :*

$$\beta_0 - \lambda T > 0,$$

if the function of sequences :

$$t \mapsto \big(h_N^{(s)}(t,\cdot)\big)_{1\leq s\leq N}$$

belongs to the space $\widetilde{\boldsymbol{X}}_{N,\varepsilon,\widetilde{\beta}_\lambda,\widetilde{\mu}^1_\lambda}$ *with* $\widetilde{\beta}_\lambda$ *and* $\widetilde{\mu}_\lambda$ *denoting :*

$$\widetilde{\beta}_\lambda : \left\{ \begin{array}{rcl} [0,T] & \to & \mathbb{R}_+^*, \\ t & \mapsto & \widetilde{\beta}_\lambda(t) = \beta_0 - \lambda t, \end{array} \right.$$

and

$$\widetilde{\mu}_\lambda : \left\{ \begin{array}{rcl} [0,T] & \to & \mathbb{R}, \\ t & \mapsto & \widetilde{\mu}_\lambda(t) = \mu_0 - \lambda t, \end{array} \right.$$

then the function of sequences :

$$t \mapsto \Big(\mathbb{1}_{s \le N-1} \int_0^t \mathcal{T}^{s,\varepsilon}_{-u} \mathcal{C}^{N,\varepsilon}_{s,s+1} \mathcal{T}^{s+1,\varepsilon}_u h_N^{(s+1)}(u,\cdot)\,\mathrm{d}u \Big)_{1 \le s \le N}$$

belongs to the same space $\widetilde{\boldsymbol{X}}_{N,\varepsilon,\widetilde{\beta}_\lambda,\widetilde{\mu}^1_\lambda}$*. In addition, one has the following inequality :*

$$\begin{aligned} \Bigg\| t \mapsto \Big(\mathbb{1}_{s \le N-1} \int_0^t \mathcal{T}^{s,\varepsilon}_{-u} \mathcal{C}^{N,\varepsilon}_{s,s+1} \mathcal{T}^{s+1,\varepsilon}_u h_N^{(s+1)}(u,\cdot)\,\mathrm{d}u \Big)_{1 \le s \le N} \Bigg\|_{N,\varepsilon,\widetilde{\beta}_\lambda,\widetilde{\mu}^1_\lambda} \\ \le C(d,N,\varepsilon) \widetilde{\beta}_\lambda(T)^{-d/2} \exp(-\widetilde{\mu}_\lambda(T)) \frac{(1+\widetilde{\beta}_\lambda(T)^{-1/2})}{\lambda} \\ \times \Big\| \big(h_N^{(s)}\big)_{1 \le s \le N} \Big\|_{N,\varepsilon,\widetilde{\beta}_\lambda,\widetilde{\mu}^1_\lambda} \end{aligned} \tag{8.6}$$

with $C(d,N,\varepsilon)$ *denoting :*

$$C(d,N,\varepsilon) = C'(d) N \varepsilon^{d-1}$$

and $C'(d)$ *being a constant which depends only on the dimension* d*.*

Proof. The rigorous definition of the quantity studied in this lemma is obtained in the same way as for the previous one.
For the finiteness of the $\|\!|\cdot|\!\|_{N,\varepsilon,\widetilde{\beta}_\lambda,\widetilde{\mu}^1_\lambda}$ norm, the relevant analog of the quantity previously denoted $Q(t,s,Z_s)$, will be here :

$$\begin{aligned} Q(t,s,Z_s) = \exp\Big(\frac{\widetilde{\beta}_\lambda(t)}{2} \sum_{i=1}^s |v_i|^2 \Big) \exp(s\widetilde{\mu}_\lambda(t)) \\ \times \Big| \int_0^t \mathcal{T}^{s,\varepsilon}_{-u} \mathcal{C}^{N,\varepsilon}_{s,s+1} \mathcal{T}^{s+1,\varepsilon}_u h_N^{(s+1)}(u,Z_s)\,\mathrm{d}u \Big| \end{aligned}$$

(the power 2 over s in the last exponential of the first line has been of course replaced by 1, in order to control the $\|\!|\cdot|\!\|_{N,\varepsilon,\widetilde{\beta}_\lambda,\widetilde{\mu}^1_\lambda}$ norm introduced in Definition 29 page 211), and this quantity will verify the analog of the inequality (8.4) page 227, with the same arguments as in the proof of the previous Lemma, so that

one obtains in this case :

$$Q(t,s,Z_s) \leq C_3(d,N,\varepsilon)\Big|\!\Big|\!\Big|\big(h_N^{(s+1)}\big)_{1\leq s\leq N}\Big|\!\Big|\!\Big|_{N,\varepsilon,\widetilde{\beta}_\lambda,\widetilde{\mu}_\lambda^1}\int_0^t\Big(\sum_{i=1}^s|v_i|+s\widetilde{\beta}(u)^{-1/2}\Big)$$
$$\times\,\widetilde{\beta}_\lambda(u)^{-d/2}\exp\Big(\frac{(\widetilde{\beta}_\lambda(t)-\widetilde{\beta}_\lambda(u))}{2}\sum_{i=1}^s|v_i|^2+s\widetilde{\mu}_\lambda(t)-(s+1)\widetilde{\mu}_\lambda(u)\Big)\,\mathrm{d}u \tag{8.7}$$

(here again, the crucial change lies in the replacement of s^2 by s). According to the affine expressions chosen for the functions $\widetilde{\beta}_\lambda$ and $\widetilde{\mu}_\lambda$, the last inequality becomes :

$$Q(t,s,Z_s) \leq C_3(d,N,\varepsilon)\Big|\!\Big|\!\Big|\big(h_N^{(s+1)}\big)_{1\leq s\leq N}\Big|\!\Big|\!\Big|_{N,\varepsilon,\widetilde{\beta}_\lambda,\widetilde{\mu}_\lambda^1}\int_0^t\Big(\sum_{i=1}^s|v_i|+s\widetilde{\beta}_\lambda(u)^{-1/2}\Big)$$
$$\times\,\widetilde{\beta}_\lambda(u)^{-d/2}\exp\Big(\frac{\lambda(u-t)}{2}\sum_{i=1}^s|v_i|^2+s\lambda(u-t)-\widetilde{\mu}_\lambda(u)\Big)\,\mathrm{d}u.$$

The functions $\widetilde{\beta}_\lambda$ and $\widetilde{\mu}_\lambda$ being non increasing, one obtains :

$$Q(t,s,Z_s) \leq C(d,N,\varepsilon)\widetilde{\beta}_\lambda(T)^{-d/2}\exp(-\widetilde{\mu}_\lambda(T))\frac{(1+\widetilde{\beta}_\lambda(T)^{-1/2})}{\lambda}$$
$$\times\,\Big|\!\Big|\!\Big|\big(h_N^{(s)}\big)_{1\leq s\leq N}\Big|\!\Big|\!\Big|_{N,\varepsilon,\widetilde{\beta}_\lambda,\widetilde{\mu}_\lambda^1},$$

which provides the control on the $|\!|\!|\cdot|\!|\!|_{N,\varepsilon,\widetilde{\beta}_\lambda,\widetilde{\mu}_\lambda^1}$ norm of the integral term of the BBGKY operator, and the inequality (8.6) of the lemma.

As for the left continuity in time, one considers, for each $1\leq s\leq N$ fixed

$$Q\exp\Big(\frac{\widetilde{\beta}_\lambda(t)}{2}|V_s|^2\Big)\exp(s\widetilde{\mu}_\lambda(t)),$$

instead of :

$$Q\exp\Big(\frac{\widetilde{\beta}_\lambda(t)}{2}|V_s|^2\Big)\exp(s^2\widetilde{\mu}_\lambda(t))$$

(one is considering here the case when $\alpha=1$), with :

$$Q=\Big|\int_0^t\mathcal{T}_{-\tau}^{s,\varepsilon}\mathcal{C}_{s,s+1}^{N,\varepsilon}\mathcal{T}_\tau^{s+1,\varepsilon}h_N^{(s+1)}(\tau,Z_s)\,\mathrm{d}\tau$$
$$-\int_0^u\mathcal{T}_{-\tau}^{s,\varepsilon}\mathcal{C}_{s,s+1}^{N,\varepsilon}\mathcal{T}_\tau^{s+1,\varepsilon}h_N^{(s+1)}(\tau,Z_s)\,\mathrm{d}\tau\Big|$$
$$=\Big|\int_u^t\mathcal{T}_{-\tau}^{s,\varepsilon}\mathcal{C}_{s,s+1}^{N,\varepsilon}\mathcal{T}_\tau^{s+1,\varepsilon}h_N^{(s+1)}(\tau,Z_s)\,\mathrm{d}\tau\Big|.$$

Once again, using the definition of the norms $|\cdot|_{\varepsilon,s,\widetilde{\beta}_\lambda(t)}$, $\|\cdot\|_{N,\varepsilon,\widetilde{\beta}_\lambda,\widetilde{\mu}_\lambda(t)^1}$ and $|||\cdot|||_{N,\varepsilon,\widetilde{\beta}_\lambda,\widetilde{\mu}_\lambda}$, and the fact that the functions $\widetilde{\beta}_\lambda$ and $\widetilde{\mu}_\lambda$ are decreasing, one finds, similarly as above :

$$\begin{aligned}
Q\exp\Big(\frac{\widetilde{\beta}_\lambda(t)}{2}|V_s|^2\Big)\exp(s\widetilde{\mu}_\lambda(t)) &\le C(d,N,\varepsilon)\Big|\Big|\Big|\big(h_N^{(s)}\big)_{1\le s\le N}\Big|\Big|\Big|_{N,\varepsilon,\widetilde{\beta}_\lambda,\widetilde{\mu}_\lambda^1}\\
&\quad\times\int_u^t\Big(\sum_{i=1}^s|v_i|+s\widetilde{\beta}_\lambda(T)^{-1/2}\Big)\widetilde{\beta}_\lambda(T)^{-d/2}\\
&\qquad\times\exp\Big(\frac{\lambda}{2}(\tau-t)\sum_{i=1}^s|v_i|^2+s\widetilde{\mu}_\lambda(t)-(s+1)\widetilde{\mu}_\lambda(\tau)\Big)\,\mathrm{d}\tau
\end{aligned}$$

and then

$$\begin{aligned}
Q\exp\Big(\frac{\widetilde{\beta}_\lambda(t)}{2}|V_s|^2\Big)\exp(s\widetilde{\mu}_\lambda(t)) &\le C(d,N,\varepsilon)\widetilde{\beta}_\lambda(T)^{-d/2}\Big|\Big|\Big|\big(h_N^{(s)}\big)_{1\le s\le N}\Big|\Big|\Big|_{N,\varepsilon,\widetilde{\beta}_\lambda,\widetilde{\mu}_\lambda^1}\\
&\quad\times\exp(-\widetilde{\mu}_\lambda(T))\Big(\sum_{i=1}^s|v_i|+s\widetilde{\beta}_\lambda(T)^{-1/2}\Big)\\
&\qquad\times\int_u^t\exp\Big(\frac{\lambda}{2}(\tau-t)\sum_{i=1}^s|v_i|^2+\lambda s(\tau-t)\Big)\,\mathrm{d}\tau.
\end{aligned}$$

Once again, the Cauchy-Schwarz inequality provides

$$\begin{aligned}
&\int_u^t\exp\Big(\frac{\lambda}{2}(\tau-t)\sum_{i=1}^s|v_i|^2+\lambda s(\tau-t)\Big)\,\mathrm{d}\tau\\
&\le\sqrt{\int_u^t 1^2\,\mathrm{d}\tau}\sqrt{\int_u^t\exp\Big(\frac{\lambda}{2}(\tau-t)\sum_{i=1}^s|v_i|^2+\lambda s(\tau-t)\Big)^2\,\mathrm{d}\tau}\\
&\le\sqrt{t-u}\,\frac{1}{\sqrt{\lambda\big(\sum_{i=1}^s|v_i|^2+2s\big)}}.
\end{aligned}$$

In this case, one obtains :

$$\begin{aligned}
Q\exp\Big(\frac{\widetilde{\beta}_\lambda(t)}{2}|V_s|^2\Big)\exp(s\widetilde{\mu}_\lambda(t)) &\le C(d,N,\varepsilon)\Big|\Big|\Big|\big(h_N^{(s)}\big)_{1\le s\le N}\Big|\Big|\Big|_{N,\varepsilon,\widetilde{\beta}_\lambda,\widetilde{\mu}_\lambda^1}\\
&\quad\times\int_u^t\Big(\sum_{i=1}^s|v_i|+s\widetilde{\beta}_\lambda(T)^{-1/2}\Big)\widetilde{\beta}_\lambda(T)^{-d/2}\\
&\qquad\times\exp\Big(\frac{\lambda}{2}(\tau-t)\sum_{i=1}^s|v_i|^2+s\widetilde{\mu}_\lambda(t)-(s+1)\widetilde{\mu}_\lambda(\tau)\Big)\,\mathrm{d}\tau\\
&\le C(d,N,\varepsilon)\widetilde{\beta}_\lambda(T)^{-d/2}\exp(-\widetilde{\mu}_\lambda(T))\Big|\Big|\Big|\big(h_N^{(s)}\big)_{1\le s\le N}\Big|\Big|\Big|_{N,\varepsilon,\widetilde{\beta}_\lambda,\widetilde{\mu}_\lambda^1}\\
&\quad\times\frac{\sum_{i=1}^s|v_i|+s\widetilde{\beta}_\lambda(T)^{-1/2}}{\sqrt{\lambda\big(\sum_{i=1}^s|v_i|^2+2s\big)}}\sqrt{t-u}.
\end{aligned}$$

The last fraction above will not have its numerator compensated by its denominator uniformly in s, however it is not important in the setting $\alpha = 1$, since one wants, for every s fixed, a bound which has to be uniform only in the variable Z_s of the phase space. One has :

$$\frac{\sum_{i=1}^s |v_i|}{\sqrt{\lambda\Big(\sum_{i=1}^s |v_i|^2 + 2s\Big)}} \leq \frac{\sqrt{s}\sqrt{\sum_{i=1}^s |v_i|^2}}{\sqrt{\lambda}\sqrt{\sum_{i=1}^s |v_i|^2}} = \sqrt{\frac{s}{\lambda}},$$

and :

$$\frac{s\widetilde{\beta}_\lambda(T)^{-1/2}}{\sqrt{\lambda\Big(\sum_{i=1}^s |v_i|^2 + 2s\Big)}} \leq \frac{s\widetilde{\beta}_\lambda(T)^{-1/2}}{\sqrt{2\lambda s}} = \widetilde{\beta}_\lambda(T)^{-1/2}\sqrt{\frac{s}{2\lambda}},$$

so that the left continuity in time, for every $1 \leq s \leq N$, is recovered in the $|\cdot|_{\varepsilon,s,\widetilde{\beta}_\lambda(t)}$ norm, since one has :

$$\begin{aligned}
&\left|\int_u^t \mathcal{T}_{-\tau}^{s,\varepsilon}\mathcal{C}_{s,s+1}^{N,\varepsilon}\mathcal{T}_\tau^{s+1,\varepsilon}h_N^{(s+1)}(\tau, Z_s)\,\mathrm{d}\tau\right| \exp\Big(\frac{\widetilde{\beta}_\lambda(t)}{2}|V_s|^2\Big) \\
&\leq C(d,N,\varepsilon)\widetilde{\beta}_\lambda(T)^{-1/2}\exp(-2\widetilde{\mu}_\lambda(T))\frac{\big(1+\widetilde{\beta}_\lambda(T)^{-1/2}\big)}{\sqrt{\lambda}}\Big|\!\Big|\!\Big|\big(h_N^{(s)}\big)_{1\leq s\leq N}\Big|\!\Big|\!\Big|_{N,\varepsilon,\widetilde{\beta}_\lambda,\widetilde{\mu}_\lambda^1} \\
&\qquad\qquad \times\sqrt{s}\sqrt{t-u},
\end{aligned}$$

that is, for every $1 \leq s \leq N$:

$$\big|h_N^{(s)}(t) - h_N^{(s)}(u)\big|_{\varepsilon,s,\widetilde{\beta}_\lambda(t)} \underset{u\to t^-}{\longrightarrow} 0,$$

and then the proof of the lemma is complete. □

8.1.2 The Boltzmann operator

In this section, one gives a definition of the Boltzmann operator, in an analogous way as for the BBGKY one.

Definition 32 (Boltzmann operator)**.** *For any sequence of initial data* $\big(f_0^{(s)}\big)_{s\geq 1}$ *belonging to the space* $\boldsymbol{X}_{0,\widetilde{\beta}(0),\widetilde{\mu}(0)^\alpha}$ *and any function of sequences* $F = t \mapsto \big(f^{(s)}(t,\cdot)\big)_{s\geq 1}$ *belonging to the space* $\widetilde{\boldsymbol{X}}_{0,\widetilde{\beta},\widetilde{\mu}^\alpha}$*, one defines the function of sequences :*

$$t \mapsto \Big(\mathcal{T}_t^{s,0}f_0^{(s)}(\cdot) + \int_0^t \mathcal{T}_{t-u}^{s,0}\mathcal{C}_{s,s+1}^0 f^{(s+1)}(u,\cdot)\,\mathrm{d}u\Big)_{s\geq 1}, \tag{8.8}$$

denoted by :

$$\mathfrak{E}_0\Big(\big(f_0^{(s)}\big)_{s\leq 1}, F\Big).$$

The functions which associates $\mathfrak{E}_0\Big(\big(f_0^{(s)}\big)_{s\geq 1}, F\Big)$ *to the function of sequences* F *is called the* Boltzmann operator with sequence of initial data $\big(f_0^{(s)}\big)_s$.

Exactly as in the previous section, one will show that the Boltzmann operator :

$$F \mapsto \Big(t \mapsto \big(\mathcal{T}_t^{s,0} f_0^{(s)}(\cdot) + \int_0^t \mathcal{T}_{t-u}^{s,0} \mathcal{C}_{s,s+1}^0 f^{(s+1)}(u,\cdot)\,\mathrm{d}u\big)_{s\geq 1}\Big), \tag{8.9}$$

with $F = t \mapsto \big(f^{(s)}(t,\cdot)\big)_{s\geq 1}$, sends the space $\widetilde{\mathbf{X}}_{0,\widetilde{\beta},\widetilde{\mu}^\alpha}$ into itself, and as in the previous section, one will study the two terms of this operator, that is :

$$F \mapsto \Big(t \mapsto \big(\mathcal{T}_t^{s,0} f_0^{(s)}(\cdot)\big)_{s\geq 1}\Big)$$

and

$$F \mapsto \Big(t \mapsto \big(\int_0^t \mathcal{T}_{t-u}^{s,0} \mathcal{C}_{s,s+1}^0 f^{(s+1)}(u,\cdot)\,\mathrm{d}u\big)_{s\geq 1}\Big),$$

which is the purpose of the two following lemmas.

The first lemma will require a meaningful and important additional hypothesis to obtain the control of the image of the function of sequences $t \mapsto \big(\mathcal{T}^{s,0} f_0^{(s)}\big)_{s\geq 1}$, namely the boundary condition for the sequence of initial data.

Lemma 20. *Let $\alpha = 1$ or 2. For any $T > 0$, any $\widetilde{\beta}$ and $\widetilde{\mu}$ two decreasing functions both defined on $[0,T]$, if the initial datum :*

$$\big(f_0^{(s)}\big)_{s\geq 1}$$

belongs to the space $\boldsymbol{X}_{0,\widetilde{\beta}(0),\widetilde{\mu}(0)^\alpha}$, and verifying in addition the boundary condition, for all $s \geq 1$:

$$f_0^{(s)}\big(\chi_s^0(Z_s)\big) = f_0^{(s)}(Z_s)$$

for every Z_s such that there exists $1 \leq i \leq s$ such that $x_i \cdot e_1 = 0$ (where χ_s^0 is introduced in Definition 11 page 79), then the function of sequences :

$$t \mapsto \big(\mathcal{T}_t^{s,0} f_0^{(s)}(\cdot)\big)_{s\geq 1}$$

belongs to the space $\widetilde{\boldsymbol{X}}_{0,\widetilde{\beta},\widetilde{\mu}^\alpha}$, and one has moreover that :

$$\Big\|\Big\|t \mapsto \big(\mathcal{T}_t^{s,0} f_0^{(s)}\big)_{s\geq N}\Big\|\Big\|_{0,\widetilde{\beta},\widetilde{\mu}^\alpha} \leq \Big\|\big(f_0^{(s)}\big)_{s\geq 1}\Big\|_{0,\widetilde{\beta}(0),\widetilde{\mu}(0)^\alpha}. \tag{8.10}$$

Proof. First one recalls that, for the case of the Boltzmann hierarchy, the space $\widetilde{\mathbf{X}}_{0,\widetilde{\beta},\widetilde{\mu}^\alpha}$ is defined in particular as a subspace of the continuous functions on $\big(\overline{\Omega^c} \times \mathbb{R}^d\big)^s$, vanishing at infinity.
However, one cannot just consider functions of $\mathcal{C}\big(\big(\overline{\Omega^c} \times \mathbb{R}^d\big)\big)$ for initial data. Indeed, the discussion of the section 5.2.2 page 171 has shown that for general continuous functions $f_0^{(s)}$, if t is non zero, the function :

$$\mathcal{T}_t^{s,\varepsilon} f_0^{(s)}$$

can be discontinuous. Nevertheless, if one restricts the set in which the initial datum $f_0^{(s)}$ is chosen, that is if one considers in addition continuous functions on :

$$\left(\overline{\Omega^c} \times \mathbb{R}^d\right)^s$$

verifying the boundary condition

$$f_0^{(s)}\big(\chi_s^0(Z_s)\big) = f_0^{(s)}(Z_s),$$

for any configuration of the phase space Z_s belonging to the boundary of the domain, that is such there exists at least one integer $1 \leq i \leq s$ such that $x_i \in \partial\Omega = \{x \cdot e_1 = 0\}$ (where one has used the notations of Definition 11 page 79), one will be able to show that for any number t, the function :

$$Z_s \mapsto \mathcal{T}_t^{s,0} f_0^{(s)}(Z_s)$$

is continuous on the whole phase space

$$\left(\overline{\Omega^c} \times \mathbb{R}^d\right)^s.$$

There are only two cases. Either the configuration $T_{-t}^{s,0}(Z_s)$ does not belong to the boundary of the domain, or it does.
In the first case, there exists a neighbourhood $\mathcal{V}(Z_s)$ of Z_s such that the function

$$\left\{\begin{array}{ll} \mathcal{V}(Z_s) & \to \left(\overline{\Omega^c} \times \mathbb{R}^d\right)^s, \\ \quad Z_s^* & \mapsto T_{-t}^{s,0}(Z_s^*) \end{array}\right.$$

is continuous, and so by composition, the function :

$$Z_s^* \mapsto \mathcal{T}_t^{s,0} f_0^{(s)}(Z_s^*) = f_0^{(s)}\big(T_{-t}^{s,0}(Z_s^*)\big)$$

is continuous on the neighbourhood $\mathcal{V}(Z_s)$, so that the function is continuous at Z_s.
In the second case, that is if Z_s belongs to the boundary of the domain, there exists at least one integer $1 \leq i_0 \leq s$ such that one has :

$$\big(T_{-t}^{s,0}(Z_s)\big)^{X,i_0} \cdot e_1 > 0.$$

One denotes $1 \leq i_1 < \cdots < i_p \leq s$ the numbers of the particles bouncing against the obstacle at time t, that is such that :

$$\big(T_{-t}^{s,0}(Z_s)\big)^{X,i_k} \cdot e_1 > 0.$$

For each particle i_k lying at the configuration $(x', v') \in \overline{\Omega^c} \times \mathbb{R}^d$ in the phase space of one particle, its configuration belongs to exactly one of those three subsets :

- $\big\{(x',v') \in \overline{\Omega^c} \times \mathbb{R}^d \ / \ (x'-tv')\cdot e_1 > 0\big\}$, that is the particle has not bounced against the obstacle during the time interval $[0,t]$. In this case, for such a particle, one has :

$$\big(T^{s,0}_{-t}(x',v')\big)^{i_k} = (x'-tv',v').$$

- $\big\{(x',v') \in \overline{\Omega^c} \times \mathbb{R}^d \ / \ (x'-tv')\cdot e_1 = 0\big\}$, that is the particle bounces against the obstacle exactly at time t. In this case, for such a particle, one has, if $v'\cdot e_1 < 0$ (using the convention for the definition of the free transport with boundary condition, such that any bouncing configuration is pre-bouncing) :

$$T^{s,0}_{-t}(x',v') = (x'-tv',v'),$$

and if $v'\cdot e_1 > 0$:

$$T^{s,0}_{-t}(x',v') = (x'-tv',-v').$$

- and finally $\big\{(x',v') \in \overline{\Omega^c} \times \mathbb{R}^d \ / \ (x'-tv')\cdot e_1 < 0\big\}$, that is the particle has already bounced against the obstacle before time t. In this case, for such a particle, one has :

$$T^{s,0}_{-t}(x',v') = \big(x'-2x'\cdot e_1e_1 + t(v'-2v'\cdot e_1e_1), v'-2v'\cdot e_1e_1\big).$$

So one considers then a neighbourhood of the point Z_s, which belongs to the boundary of the domain, and one divides this neighbourhood in the three subparts described above. The goal is to show that for a point Z_s^* of the neighbourhood, such that each of the particles in this configuration belongs to any of the three subparts, close enough to Z_s, the quantity :

$$\big|f^{(s)}\big(T^{s,0}_{-t}(Z_s)\big) - f^{(s)}\big(T^{s,0}_{-t}(Z_s^*)\big)\big|$$

can be chosen as small as one wants.
One denotes I_1 the subset of particles of the configuration Z_s^* which belong to the first subpart, I_2 the subset of particles which belong to the second part, and of course I_3 the subset of particles which belong to the third part.
One has then :

$$\begin{aligned}\big|f^{(s)}\big(T^{s,0}_{-t}(Z_s)\big) - f^{(s)}\big(T^{s,0}_{-t}(Z_s^*)\big)\big| &\leq \big|f^{(s)}\big(T^{s,0}_{-t}(Z_s)\big) - f^{(s)}\big(\widehat{T^{s,0}_{-t}(Z_s^*)}\big)\big| \\ &\quad + \big|f^{(s)}\big(\widehat{T^{s,0}_{-t}(Z_s^*)}\big) - f^{(s)}\big(T^{s,0}_{-t}(Z_s^*)\big)\big|,\end{aligned}$$

where the configuration $\widehat{T^{s,0}_{-t}(Z_s^*)}$ denotes the vector of the phase space of s particles defined as follows :

- if the particle i_k belongs to I_1 or I_2, then the configuration of the particle i_k in the configuration $\widehat{T^{s,0}_{-t}(Z_s^*)}$ is the one of the configuration Z_s^*, that is

$$\big(\widehat{T^{s,0}_{-t}(Z_s')}\big)^{i_k} = \big(T^{s,0}_{-t}(Z_s')\big)^{i_k},$$

- if the particle i_k belongs to I_3, then the *position* of the particle i_k in the configuration $\widehat{T_{-t}^{s,0}(Z_s^*)}$ is the one of the configuration Z_s, while its *velocity* is the one of the configuration Z_s^*, that is :

$$\big(\widehat{T_{-t}^{s,0}(Z_s^*)}\big)^{X,i_k} = \big(T_{-t}^{s,0}(Z_s)\big)^{X,i_k} \text{ and } \big(\widehat{T_{-t}^{s,0}(Z_s^*)}\big)^{V,i_k} = \big(T_{-t}^{s,0}(Z_s^*)\big)^{V,i_k}.$$

This way of writing the difference $f^{(s)}\big(T_{-t}^{s,0}(Z_s)\big) - f^{(s)}\big(T_{-t}^{s,0}(Z_s^*)\big)$ shows the continuity of the function $\mathcal{T}_t^{s,0} f_0^{(s)}$, since one has obtained, for particles i_k belonging to I_1 and I_2 :

$$\begin{aligned}
&\Big|f^{(s)}\big(T_{-t}^{s,0}(Z_s)\big) - f^{(s)}\big(\widehat{T_{-t}^{s,0}(Z_s^*)}\big)\Big| + \Big|f^{(s)}\big(\widehat{T_{-t}^{s,0}(Z_s^*)}\big) - f^{(s)}\big(T_{-t}^{s,0}(Z_s^*)\big)\Big| \\
&= \Big|f^{(s)}\big(\dots, \big(T_{-t}^{s,0}(Z_s)\big)^{i_k}, \dots\big) - f^{(s)}\big(\dots, \big(\widehat{T_{-t}^{s,0}(Z_s^*)}\big)^{i_k}, \dots\big)\Big| \\
&\qquad + \Big|f^{(s)}\big(\dots, \big(\widehat{T_{-t}^{s,0}(Z_s^*)}\big)^{i_k}, \dots\big) - f^{(s)}\big(\dots, \big(T_{-t}^{s,0}(Z_s^*)\big)^{i_k}, \dots\big)\Big| \\
&= \Big|f^{(s)}\big(\dots, \big(T_{-t}^{s,0}(Z_s)\big)^{i_k}, \dots\big) - f^{(s)}\big(\dots, \big(T_{-t}^{s,0}(Z_s^*)\big)^{i_k}, \dots\big)\Big| \\
&\qquad + \Big|f^{(s)}\big(\dots, \big(T_{-t}^{s,0}(Z_s^*)\big)^{i_k}, \dots\big) - f^{(s)}\big(\dots, \big(T_{-t}^{s,0}(Z_s)\big)^{i_k}, \dots\big)\Big|,
\end{aligned}$$

so that for the second term, the components of the particles belonging to I_1 or I_2 are the same. For the first term, one has :

$$\begin{aligned}
&\Big|f^{(s)}\big(\dots, \big(T_{-t}^{s,0}(Z_s)\big)^{i_k}, \dots\big) - f^{(s)}\big(\dots, \big(T_{-t}^{s,0}(Z_s^*)\big)^{i_k}, \dots\big)\Big| \\
&= \Big|f^{(s)}\big(\dots, x_{i_k} - tv_{i_k}, \big(T_{-t}^{s,0}(Z_s)\big)^{V,i_k}\big), \dots\big) \\
&\qquad - f^{(s)}\big(\dots, x_{i_k}^* - tv_{i_k}^*, \big(T_{-t}^{s,0}(Z_s^*)\big)^{V,i_k}\big), \dots\big)\Big|.
\end{aligned}$$

Here, if i_k belongs to I_1, then it is clear that $\big(T_{-t}^{s,0}(Z_s^*)\big)^{V,i_k}\big) = v_{i_k}^*$. But if i_k belongs to I_2, one can have :

$$\big(T_{-t}^{s,0}(Z_s^*)\big)^{V,i_k}\big) = v_{i_k}^* \text{ or } v_{i_k}^* - 2v_{i_k}^* \cdot e_1 e_1,$$

and similarly, the same problem occurs for Z_s. But for a neighbourhood of Z_s chosen small enough, one can state that $\big(T_{-t}^{s,0}(Z_s^*)\big)^{V,i_k}\big)$ and $\big(T_{-t}^{s,0}(Z_s^*)\big)^{V,i_k}\big)$ are of the same form (that is, there are either both of the form v, or both of the form $v - 2v \cdot e_1 e_1$).
The comparison of the components of particles belonging to I_3 is very similar, and then thanks to the boundary condition and the continuity of the initial datum $f_0^{(s)}$, one has is fact obtained the continiuity of the function $\mathcal{T}_t^{s,0} f_0^{(s)}$ at the point Z_s if Z_s belongs to the boundary of the domain, so in summary, one has shown the continuity everywhere.
The fact that the $|||\cdot|||_{0,\widetilde{\beta},\widetilde{\mu}^\alpha}$ norm of the function of sequences $t \mapsto \big(\mathcal{T}_t^{s,0} f_0^{(s)}(\cdot)\big)_{s\geq 1}$ is finite is easy to prove.

Indeed, for every nonnegative time t, every positive integer $s \geq 1$ and every $Z_s \in \left(\overline{\Omega^c} \times \mathbb{R}^d\right)^s$, one has that, since the weights $\widetilde{\beta}$ and $\widetilde{\mu}$ are non increasing :

$$\begin{aligned}
&\left|\mathcal{T}_t^{s,0} f_0^{(s)}(Z_s)\right| \exp\left(\frac{\widetilde{\beta}(t)}{2}|V_i|^2\right) \exp(s^\alpha \widetilde{\mu}(t)) \\
&\leq \left|\mathcal{T}_t^{s,0} f_0^{(s)}(Z_s)\right| \exp\left(\frac{\widetilde{\beta}(0)}{2}|V_i|^2\right) \exp(s^\alpha \widetilde{\mu}(0)) \\
&\quad = \left|f_0^{(s)}\left(T_{-t}^{s,0}(Z_s)\right)\right| \exp\left(\frac{\widetilde{\beta}(0)}{2}\left|\left(T_{-t}^{s,0}(Z_s)\right)^V\right|^2\right) \exp(s^\alpha \widetilde{\mu}(0)) \\
&\leq \left\|\left(f_0^{(s)}\right)_{s\geq 1}\right\|_{0,\widetilde{\beta}(0),\widetilde{\mu}(0)^\alpha},
\end{aligned}$$

this last bound being finite by hypothesis on the regularity of the sequence of initial data $\left(f_0^{(s)}\right)_{s\geq 1}$, so that the inequality (8.10) of the lemma is proved.

To conclude the proof of the lemma, the left continuity for every time $t \in]0,T]$ has to be verified. For the case when $\alpha = 2$, it is with respect to the norm $\|\cdot\|_{0,\widetilde{\beta}(t),\widetilde{\mu}(t)^\alpha}$, if $\alpha = 1$, it is, for every $s \geq 1$, with respect to the $|\cdot|_{0,s,\widetilde{\beta}(0)}$ norm.

One writes the proof only in the case when $\alpha = 2$ (see Definitions 26 page 207 and 28 page 211), since for $\alpha = 1$ the arguments are exactly the same. For $t \in]0,T]$, one will show that :

$$\left\|\left(\mathcal{T}_t^{s,0} f_0^{(s)} - \mathcal{T}_u^{s,0} f_0^{(s)}\right)_{s\geq 1}\right\|_{0,\widetilde{\beta}(t),\widetilde{\mu}(t)^2} \underset{u\to t^-}{\longrightarrow} 0,$$

that is, in other words, for every $\varepsilon > 0$, there exists $u_0(\varepsilon) \in [0,t[$ such that for every $u \in [u_0,t]$, any positive integer s and any configuration Z_s of the phase space of s particles, one has :

$$\left|\mathcal{T}_t^{s,0} f_0^{(s)}(Z_s) - \mathcal{T}_u^{s,0} f_0^{(s)}(Z_s)\right| \exp\left(\frac{\widetilde{\beta}(t)}{2}|V_s|^2\right) \exp(s^2 \widetilde{\mu}(t)) < \varepsilon.$$

First, one will reduce the problem and consider only a finite number of particles. To do so, if one denotes :

$$Q = \left|\mathcal{T}_t^{s,0} f_0^{(s)}(Z_s) - \mathcal{T}_u^{s,0} f_0^{(s)}(Z_s)\right| \exp\left(\frac{\widetilde{\beta}(t)}{2}|V_s|^2\right) \exp(s^2 \widetilde{\mu}(t)),$$

then one has just to notice that, for any positive integer s and any configuration Z_s :

$$Q \leq \left[\left|f_0^{(s)}\left(T_{-t}^{s,0}(Z_s)\right)\right| + \left|f_0^{(s)}\left(T_{-u}^{s,0}(Z_s)\right)\right|\right] \exp\left(\frac{\widetilde{\beta}(t)}{2}|V_s|^2\right) \exp(s^2 \widetilde{\mu}(t)),$$

and then by conservation of the kinetic energy along the trajectories of the free transport with boundary condition, and the fact the function $\widetilde{\beta}$ is non increasing,

one has :

$$\begin{aligned}
Q &\leq \left|f_0^{(s)}\big(T_{-t}^{s,0}(Z_s)\big)\right| \exp\Big(\frac{\widetilde{\beta}(t)}{2}\Big|\big(T_{-t}^{s,0}(Z_s)\big)^V\Big|^2\Big) \exp(s^2\widetilde{\mu}(t)) \\
&\qquad + \left|f_0^{(s)}\big(T_{-u}^{s,0}(Z_s)\big)\right| \exp\Big(\frac{\widetilde{\beta}(t)}{2}\Big|\big(T_{-u}^{s,0}(Z_s)\big)^V\Big|^2\Big) \exp(s^2\widetilde{\mu}(t)) \\
&\leq \left|f_0^{(s)}\big(T_{-t}^{s,0}(Z_s)\big)\right| \exp\Big(\frac{\widetilde{\beta}(0)}{2}\Big|\big(T_{-t}^{s,0}(Z_s)\big)^V\Big|^2\Big) \exp(s^2\widetilde{\mu}(t)) \\
&\qquad + \left|f_0^{(s)}\big(T_{-u}^{s,0}(Z_s)\big)\right| \exp\Big(\frac{\widetilde{\beta}(0)}{2}\Big|\big(T_{-u}^{s,0}(Z_s)\big)^V\Big|^2\Big) \exp(s^2\widetilde{\mu}(t)),
\end{aligned}$$

and then using the fact that, by hypothesis, for every positive integer s the quantity :

$$\big|f_0^{(s)}(Z_s)\big| \exp\Big(\frac{\widetilde{\beta}(0)}{2}|V_s|^2\Big)$$

is uniformly bounded in Z_s by $\big|f_0^{(s)}\big|_{0,s,\widetilde{\beta}(0)}$, one obtains :

$$Q \leq 2\big|f_0^{(s)}\big|_{0,s,\widetilde{\beta}(0)} \exp(s^2\widetilde{\mu}(t)).$$

Since the function $\widetilde{\mu}$ is decreasing and t is strictly positive, one has that :

$$\widetilde{\mu}(t) - \widetilde{\mu}(0) < 0,$$

so that on the one hand :

$$\exp\big(s^2(\widetilde{\mu}(t) - \widetilde{\mu}(0)\big) \underset{s\to+\infty}{\longrightarrow} 0,$$

and on the other hand :

$$\begin{aligned}
Q &\leq 2\big|f_0^{(s)}\big|_{0,s,\widetilde{\beta}(0)} \exp(s^2\widetilde{\mu}(0)) \exp\big(s^2(\widetilde{\mu}(t) - \widetilde{\mu}(0)\big) \exp(s^2\widetilde{\mu}(0)) \\
&\leq 2\Big\|\big(f_0^{(s)}\big)_{s\geq 1}\Big\|_{0,\widetilde{\beta}(0),\widetilde{\mu}(0)^2} \exp\big(s^2(\widetilde{\mu}(t) - \widetilde{\mu}(0)\big),
\end{aligned}$$

the last inequality holding thanks to the fact that the sequence of initial data belongs to the space $\mathbf{X}_{0,\widetilde{\beta}(0),\widetilde{\mu}(0)^2}$.
One can therefore find a positive integer $s_0(\varepsilon) = s_0$ such that, for all $s \geq s_0$, one has

$$\exp\big(s^2(\widetilde{\mu}(t) - \widetilde{\mu}(0)\big) \leq \frac{\varepsilon}{2}\Big\|\big(f_0^{(s)}\big)_{s\geq 1}\Big\|^{-1}_{0,\widetilde{\beta}(0),\widetilde{\mu}(0)^2}. \tag{8.11}$$

For such s, the quantity Q is then uniformly bounded in Z_s by ε, and so one has only to consider the remaining cases, that is the finite number of cases when $1 \leq s < s_0$.

In the same way, for the remaining cases $1 \leq s < s_0$, one will reduce the problem to the case of a bounded energy. One writes again :

$$\begin{aligned} Q &= \Big|\mathcal{T}_t^{s,0} f_0^{(s)}(Z_s) - \mathcal{T}_u^{s,0} f_0^{(s)}(Z_s)\Big| \exp\Big(\frac{\widetilde{\beta}(t)}{2}|V_s|^2\Big) \exp(s^2\widetilde{\mu}(t)) \\ &\leq \Big|f_0^{(s)}\big(T_{-t}^{s,0}(Z_s)\big)\Big| \exp\Big(\frac{\widetilde{\beta}(t)}{2}|V_s|^2\Big) \exp(s^2\widetilde{\mu}(t)) \\ &\qquad + \Big|f_0^{(s)}\big(T_{-u}^{s,0}(Z_s)\big)\Big| \exp\Big(\frac{\widetilde{\beta}(t)}{2}|V_s|^2\Big) \exp(s^2\widetilde{\mu}(t)), \end{aligned}$$

and using once more that the function $\widetilde{\mu}$ is decreasing and the conservation of the kinetic energy along the trajectories of the free transport, one gets :

$$\begin{aligned} Q \leq \Big(&\Big|f_0^{(s)}\big(T_{-t}^{s,0}(Z_s)\big)\Big| \exp\Big(\frac{\widetilde{\beta}(0)}{2}\Big|\big(T_{-t}^{s,0}(Z_s)\big)^V\Big|^2\Big) \\ &+ \Big|f_0^{(s)}\big(T_{-u}^{s,0}(Z_s)\big)\Big| \exp\Big(\frac{\widetilde{\beta}(0)}{2}\Big|\big(T_{-t}^{s,0}(Z_s)\big)^V\Big|^2\Big)\Big) \\ &\times \exp\Big(\frac{\big(\widetilde{\beta}(t) - \widetilde{\beta}(0)\big)}{2}|V_s|^2\Big) \exp(s^2\widetilde{\mu}(0)), \end{aligned}$$

so that thanks to the fact that the sequence of initial data belongs to the space $\mathbf{X}_{0,\widetilde{\beta}(0),\widetilde{\mu}(0)^2}$, one obtains :

$$\begin{aligned} Q &\leq 2\big|f_0^{(s)}\big|_{0,s,\widetilde{\beta}(0)} \exp(s^2\widetilde{\mu}(0)) \exp\Big(\frac{\big(\widetilde{\beta}(t) - \widetilde{\beta}(0)\big)}{2}|V_s|^2\Big) \\ &\leq 2\Big\|\big(f_0^{(s)}\big)_{s\geq 1}\Big\|_{0,\widetilde{\beta}(0),\widetilde{\mu}(0)^2} \exp\Big(\frac{\big(\widetilde{\beta}(t) - \widetilde{\beta}(0)\big)}{2}|V_s|^2\Big). \end{aligned}$$

Thanks to the fact that one considers now only a finite number of integers s, there exists a strictly positive number $R(\varepsilon, s_0) = R(\varepsilon)$ such that for every $1 \leq s < s_0$, and for every $V_s \in \mathbb{R}^{ds}$ such that

$$|V_s| \geq R,$$

one has

$$\exp\Big(\frac{\big(\widetilde{\beta}(t) - \widetilde{\beta}(0)\big)}{2}|V_s|^2\Big) \leq \frac{\varepsilon}{2}\Big\|\big(f_0^{(s)}\big)_{s\geq 1}\Big\|^{-1}_{0,\widetilde{\beta}(0),\widetilde{\mu}(0)^2}. \tag{8.12}$$

This means that for such $1 \leq s < s_0$ and configurations with an energy larger than R^2, the quantity Q is again bounded by ε.
Finally, for s smaller than s_0 and for an energy lower than R, one will use the fact that the functions $f_0^{(s)}$ are uniformly continuous as functions vanishing at infinity. Recalling that the quantity Q is given by :

$$Q = \Big|\mathcal{T}_t^{s,0} f_0^{(s)}(Z_s) - \mathcal{T}_u^{s,0} f_0^{(s)}(Z_s)\Big| \exp\Big(\frac{\widetilde{\beta}(t)}{2}|V_s|^2\Big) \exp(s^2\widetilde{\mu}(t)),$$

it can be rewritten as :

$$Q = \Big|\mathcal{T}_t^{s,0} f_0^{(s)}(Z_s) - \mathcal{T}_{u-t}^{s,0}\mathcal{T}_t^{s,0} f_0^{(s)}(Z_s)\Big| \exp\Big(\frac{\widetilde{\beta}(t)}{2}|V_s|^2\Big) \exp(s^2\widetilde{\mu}(t)),$$

or again :

$$Q = \Big|\mathcal{T}_t^{s,0} f_0^{(s)}(Z_s) - \mathcal{T}_t^{s,0} f_0^{(s)}\big(T_{t-u}^{s,0}(Z_s)\big)\Big| \exp\Big(\frac{\widetilde{\beta}(t)}{2}|V_s|^2\Big) \exp(s^2\widetilde{\mu}(t)).$$

Using the fact that one is now considering the case of the bounded energy, one has :

$$Q \leq \Big|\mathcal{T}_t^{s,0} f_0^{(s)}(Z_s) - \mathcal{T}_t^{s,0} f_0^{(s)}\big(T_{t-u}^{s,0}(Z_s)\big)\Big| \exp\Big(\frac{\widetilde{\beta}(t)}{2}R^2\Big) \exp(s^2\widetilde{\mu}(t)).$$

From the fact that

$$\big|Z_s - T_{t-u}^{s,0}(Z_s)\big| \leq (t-u)|V_s| \leq (t-u)R,$$

and the fact that the initial data are verifying the boundary condition (this point is important, since the free transport is not continuous in its velocity variables), one knows that the function

$$Z_s \mapsto \mathcal{T}_t^{s,0} f_0^{(s)}(Z_s)$$

is continuous, and therefore uniformly continuous since it vanishes at infinity. Thanks to the uniform continuity of each function $\mathcal{T}_t^{s,0} f_0^{(s)}$ for $1 \leq s < s_0$, there exists then $u_0(\varepsilon, s_0, R) = u_0 \in [0, t[$ such that for all $u_0 \leq u \leq t$, all positive integers $1 \leq s \leq s_0$ and all configurations Z_s of the phase space of s particles, one has :

$$\Big|\mathcal{T}_t^{s,0} f_0^{(s)}(Z_s) - \mathcal{T}_t^{s,0} f_0^{(s)}\big(T_{t-u}^{s,0}(Z_s)\big)\Big| \leq \exp\Big(-\frac{\widetilde{\beta}(t)}{2}R^2\Big) \exp(-s^2\widetilde{\mu}(t))\varepsilon, \tag{8.13}$$

and then for such conditions that the quantity Q is again bounded by ε.
So, in summary, one has shown that, for any $\varepsilon > 0$, there exist a positive integer $s_0(\varepsilon)$ and a strictly positive number $R(\varepsilon, s_0)$ such that, uniformly in the positive integer s (if it is larger or equal to s_0, this is the control (8.11)) and uniformly in the configuration Z_s of the phase space of s particles (if its energy is larger than R^2, this is the control (8.12), and for all the remaining cases, this is the last control (8.13)), the quantity Q is smaller than ε, up to choosing u close enough to t, that is one has obtained the left continuity in time with respect to the $\|\cdot\|_{0,\widetilde{\beta}(0),\widetilde{\mu}(0)^2}$ norm.
Hence the lemma is entirely proved. □

Remark 19. *Here one has to notice that it was mandatory to assume that the weights $\widetilde{\beta}$ and $\widetilde{\mu}$ are strictly decreasing, and not only non increasing, in order in*

particular to recover the left continuity in time with respect to the $\|\cdot\|_{0,\widetilde{\beta}(t),\widetilde{\mu}(t)^\alpha}$ *norm.*
Besides, the last part of the proof, about continuity in time, was written for a weight in s^2*. However, if the power above* s *were only* 1*, the proof would have been the same.*

Finally, the two following lemmas will address the question of the image of the second term of the Boltzmann operator, namely the integrated in time collision-transport operator, in the two particular cases of the spaces $\widetilde{\mathbf{X}}_{0,\widetilde{\beta},\widetilde{\mu}^\alpha}$, for $\alpha = 2$, then $\alpha = 1$. One recalls that the first one deals with the case of a strong weight with respect to the number of particles s and a uniform in s continuity in time, while the second one focuses on the case of a continuity in time which is not uniform in s. The statements of thoses two lemmas are very similar to lemmas 18 and 19, and so are their proofs, since whatever the hierarchy considered is, the integrated in time transport-collision (or transport-collision-transport) operators satisfy the same continuity properties on the spaces $\widetilde{\mathbf{X}}_{\cdot,\widetilde{\beta}_\lambda,\widetilde{\mu}_\lambda}$. One will then just write the statements of those lemmas in a first time, and then point out briefly the differences in the proofs.

Stability by the integral term of the Boltzmann hierarchy of the space $\widetilde{\mathbf{X}}_{0,\widetilde{\beta},\widetilde{\mu}^2}$ (case $\alpha = 2$)

Lemma 21. *For any* $T > 0$*, any* $\beta_0 > 0$*,* $\mu_0 > 0$ *and* $\lambda > 0$ *such that :*

$$\beta_0 - \lambda T > 0 \text{ and } \mu_0 - \lambda T > 0,$$

if the function of sequences :

$$t \mapsto \left(f^{(s)}(t,\cdot)\right)_{s\geq 1}$$

belongs to the space $\widetilde{\boldsymbol{X}}_{0,\widetilde{\beta}_\lambda,\widetilde{\mu}_\lambda^2}$ *with* $\widetilde{\beta}_\lambda$ *and* $\widetilde{\mu}_\lambda$ *denoting :*

$$\widetilde{\beta}_\lambda : \left\{ \begin{array}{rcl} [0,T] & \to & \mathbb{R}_+^*, \\ t & \mapsto & \widetilde{\beta}_\lambda(t) = \beta_0 - \lambda t, \end{array} \right.$$

and

$$\widetilde{\mu}_\lambda : \left\{ \begin{array}{rcl} [0,T] & \to & \mathbb{R}_+^*, \\ t & \mapsto & \widetilde{\mu}_\lambda(t) = \mu_0 - \lambda t, \end{array} \right.$$

then the function of sequences :

$$t \mapsto \left(\int_0^t \mathcal{T}_{t-u}^{s,\varepsilon} \mathcal{C}_{s,s+1}^0 f^{(s+1)}(u,\cdot)\,\mathrm{d}u \right)_{s\geq 1}$$

belongs to the same space $\widetilde{\boldsymbol{X}}_{0,\widetilde{\beta}_\lambda,\widetilde{\mu}^2_\lambda}$. *In addition, one has the following inequality :*

$$\begin{aligned}\Big|\!\Big|\!\Big| t \mapsto \Big(\int_0^t \mathcal{T}^{s,\varepsilon}_{t-u}\mathcal{C}^0_{s,s+1} f^{(s+1)}(u,\cdot)\,\mathrm{d}u\Big)_{s\geq 1}\Big|\!\Big|\!\Big|_{0,\widetilde{\beta}_\lambda,\widetilde{\mu}^2_\lambda} \\ \leq C(d)\widetilde{\beta}_\lambda(T)^{-d/2}\exp\big(-3\widetilde{\mu}_\lambda(T)\big)\frac{\big(1+\widetilde{\beta}_\lambda(T)^{-1/2}\big)}{\lambda} \\ \times \Big|\!\Big|\!\Big|\big(f^{(s)}\big)_{s\geq 1}\Big|\!\Big|\!\Big|_{0,\widetilde{\beta}_\lambda,\widetilde{\mu}^2_\lambda}, \quad (8.14)\end{aligned}$$

with $C(d)$ *denoting a constant which depends only on the dimension* d.

Stability by the integral term of the Boltzmann hierarchy of the space $\widetilde{\mathbf{X}}_{0,\widetilde{\beta},\widetilde{\mu}^1}$ (case $\alpha = 1$)

Lemma 22. *For any* $T > 0$, *any* $\beta_0 > 0$, $\mu_0 \in \mathbb{R}$ *and* $\lambda > 0$ *such that :*

$$\beta_0 - \lambda T > 0,$$

if the function of sequences :

$$t \mapsto \big(f^{(s)}(t,\cdot)\big)_{s\geq 1}$$

belongs to the space $\widetilde{\boldsymbol{X}}_{0,\widetilde{\beta}_\lambda,\widetilde{\mu}^1_\lambda}$ *with* $\widetilde{\beta}_\lambda$ *and* $\widetilde{\mu}_\lambda$ *denoting :*

$$\widetilde{\beta}_\lambda : \left\{ \begin{array}{rcl} [0,T] & \to & \mathbb{R}^*_+, \\ t & \mapsto & \widetilde{\beta}_\lambda(t) = \beta_0 - \lambda t, \end{array}\right.$$

and

$$\widetilde{\mu}_\lambda : \left\{ \begin{array}{rcl} [0,T] & \to & \mathbb{R}, \\ t & \mapsto & \widetilde{\mu}_\lambda(t) = \mu_0 - \lambda t, \end{array}\right.$$

then the function of sequences :

$$t \mapsto \Big(\int_0^t \mathcal{T}^{s,\varepsilon}_{t-u}\mathcal{C}^0_{s,s+1} f^{(s+1)}(u,\cdot)\,\mathrm{d}u\Big)_{s\geq 1}$$

belongs to the same space $\widetilde{\boldsymbol{X}}_{0,\widetilde{\beta}_\lambda,\widetilde{\mu}^1_\lambda}$. *In addition, one has the following inequality :*

$$\begin{aligned}\Big|\!\Big|\!\Big| t \mapsto \Big(\int_0^t \mathcal{T}^{s,\varepsilon}_{t-u}\mathcal{C}^0_{s,s+1} f^{(s+1)}(u,\cdot)\,\mathrm{d}u\Big)_{s\geq 1}\Big|\!\Big|\!\Big|_{0,\widetilde{\beta}_\lambda,\widetilde{\mu}^1_\lambda} \\ \leq C(d)\widetilde{\beta}_\lambda(T)^{-d/2}\exp(-\widetilde{\mu}_\lambda(T))\frac{\big(1+\widetilde{\beta}_\lambda(T)^{-1/2}\big)}{\lambda} \\ \times \Big|\!\Big|\!\Big|\big(f^{(s)}\big)_{s\geq 1}\Big|\!\Big|\!\Big|_{0,\widetilde{\beta}_\lambda,\widetilde{\mu}^1_\lambda} \quad (8.15)\end{aligned}$$

with $C(d)$ *denoting a constant which depends only on the dimension* d.

The changes in the proof in the case of the integral part of the Boltzmann operator

For the sake of completeness, one provides in what follows the points that have to be changed in the proof of Lemmas 18 and 19 in order to obtain Lemmas 21 and 22.

Proof. One starts the proof by the fact that, for every $t \in [0,T]$ and for every $s \geq 1$, the function :

$$Z_s \mapsto \int_0^t \mathcal{T}^{s,0}_{t-u}\mathcal{C}^0_{s,s+1} f^{(s+1)}(u, Z_s)\,\mathrm{d}u$$

is a continuous function on $\left(\overline{\Omega^c} \times \mathbb{R}^d\right)^s$ vanishing at infinity.
This is a consequence of the results of the section 5.2 page 167, namely, Lemmas 12 page 169 and 13 page 173.
Now, it remains to check that the function of sequences :

$$t \mapsto \Big(\int_0^t \mathcal{T}^{s,0}_{t-u}\mathcal{C}^0_{s,s+1} f^{(s+1)}(u,\cdot)\,\mathrm{d}u\Big)_{s\geq 1}$$

has a finite $|||\cdot|||_{0,\widetilde{\beta},\widetilde{\mu}^\alpha}$ norm, and the quantity, for all $0 \leq u \leq t$ with $t \in]0,T]$:

$$t \mapsto \Big(\int_0^t \mathcal{T}^{s,0}_{t-u}\mathcal{C}^0_{s,s+1} f^{(s+1)}(u,\cdot)\,\mathrm{d}u\Big)_{s\geq 1}$$

has a $\|\cdot\|_{0,\widetilde{\beta}(t),\widetilde{\mu}(t)^\alpha}$ norm which goes to zero when u goes to t.
For those two controls, one denotes :

$$Q = \Big|\int_u^t \mathcal{T}^{s,0}_{t-\tau}\mathcal{C}^0_{s,s+1} f^{(s+1)}(\tau, Z_s)\,\mathrm{d}\tau\Big| \exp\Big(\frac{\widetilde{\beta}(t)}{2}\sum_{j=1}^s |v_j|^2\Big)\exp(s^\alpha\widetilde{\mu}(t))$$

and one will make explicit the expression of Q. For all $0 \leq u \leq t \leq T$, for all positive integer s and for all configuration $Z_s \in \left(\overline{\Omega^c} \times \mathbb{R}^d\right)^s$:

$$\begin{aligned} Q = \Big|\int_u^t \sum_{i=1}^s \int_{\mathbb{S}^{d-1}_\omega \times \mathbb{R}^d_{v_{s+1}}} &\big(\omega\cdot(v_{s+1}-v_i)\big)_+ \\ &\times \Big(f^{(s+1)}\big(\tau, \Big(T^{s,0}_{u-t}(Z_s), \big(T^{s,0}_{u-t}(Z_s)\big)^{X,i}, v_{s+1}\Big)'_{i,s+1}\big) \\ &\qquad - f^{(s+1)}\big(\tau, T^{s,0}_{u-t}(Z_s), \big(T^{s,0}_{u-t}(Z_s)\big)^{X,i}, v_{s+1}\big)\Big)\Big)\,\mathrm{d}\tau\Big| \\ &\times \exp\Big(\frac{\widetilde{\beta}(t)}{2}\sum_{j=1}^s |v_j|^2\Big)\exp(s^\alpha\widetilde{\mu}(t)), \end{aligned} \tag{8.16}$$

where one has used the notations detailed in the point (1.8) of Definition 2 page 52. This quantity can be roughly bounded from above writing :

$$
\begin{aligned}
Q \leq \sum_{i=1}^{s} \int_u^t \int_{\mathbb{S}_\omega^{d-1}\times\mathbb{R}^d_{v_{s+1}} \cap\, \omega\cdot(v_{s+1}-v_i)>0} & \big(\omega\cdot(v_{s+1}-v_i)\big)_+ \big|f^{(s+1)}(\tau,\cdot)\big|_{0,s+1,\widetilde{\beta}(\tau)} \\
& \times \Big(\exp\Big(-\frac{\widetilde{\beta}(\tau)}{2}\Big|\Big(T^{s,0}_{u-t}(Z_s),\big(T^{s,0}_{u-t}(Z_s)\big)^{X,i},v_{s+1}\Big)'_{i,s+1}\Big|^2\Big) \\
& \qquad + \exp\Big(-\frac{\widetilde{\beta}(\tau)}{2}\Big|\Big(T^{s,0}_{u-t}(Z_s),\big(T^{s,0}_{u-t}(Z_s)\big)^{X,i},v_{s+1}\Big)\Big|^2\Big)\Big)\,\mathrm{d}\tau \\
& \times \exp\Big(\frac{\widetilde{\beta}(t)}{2}\sum_{j=1}^{s}|v_j|^2\Big)\exp(s^\alpha\widetilde{\mu}(t)),
\end{aligned}
$$

the right-hand side of the last inequality being equal, thanks to the conservation of the kinetic energy along the trajectories of the free transport and during the collisions, to :

$$
\begin{aligned}
\sum_{i=1}^{s} \int_u^t \int_{\mathbb{S}_\omega^{d-1}\times\mathbb{R}^d_{v_{s+1}} \cap\, \omega\cdot(v_{s+1}-v_i)>0} & 2\big(\omega\cdot(v_{s+1}-v_i)\big)_+ \big|f^{(s+1)}(\tau,\cdot)\big|_{0,s+1,\widetilde{\beta}(\tau)} \exp\Big(-\frac{\widetilde{\beta}(\tau)}{2}\big|V_{s+1}\big|^2\Big)\,\mathrm{d}\tau \\
& \times \exp\Big(\frac{\widetilde{\beta}(t)}{2}\sum_{j=1}^{s}|v_j|^2\Big)\exp(s^\alpha\widetilde{\mu}(t)),
\end{aligned}
$$

so that, after applying the Cauchy-Schwarz inequality to the scalar product $\omega\cdot(v_{s+1}-v_i)$, one obtains :

$$
\begin{aligned}
Q \leq \sum_{i=1}^{s} \int_u^t \int_{\mathbb{S}_\omega^{d-1}\times\mathbb{R}^d_{v_{s+1}} \cap\, \omega\cdot(v_{s+1}-v_i)>0} & 2\big(|v_i|+|v_{s+1}|\big)_+ \big|f^{(s+1)}(\tau,\cdot)\big|_{0,s+1,\widetilde{\beta}(\tau)} \exp\Big(-\frac{\widetilde{\beta}(\tau)}{2}\big|V_{s+1}\big|^2\Big)\,\mathrm{d}\tau \\
& \times \exp\Big(\frac{\widetilde{\beta}(t)}{2}\sum_{j=1}^{s}|v_j|^2\Big)\exp(s^\alpha\widetilde{\mu}(t)),
\end{aligned}
$$

and then using the control of the $|||\cdot|||_{0,\widetilde{\beta},\widetilde{\mu}^\alpha}$ of the function of sequences $t\mapsto \big(f^{(s)}(t,\cdot)\big)_{s\geq 1}$, one has :

$$
\big|f^{(s+1)}(\tau,\cdot)\big|_{0,s+1,\widetilde{\beta}(\tau)} \leq \Big|\Big|\Big|\big(f^{(s)}\big)_{s\geq 1}\Big|\Big|\Big|_{0,\widetilde{\beta},\widetilde{\mu}^\alpha} \exp(-(s+1)^\alpha\widetilde{\mu}(\tau)),
$$

which provides in the end :

$$
\begin{aligned}
Q \leq \sum_{i=1}^{s} \int_u^t \int_{\mathbb{S}_\omega^{d-1}\times\mathbb{R}^d_{v_{s+1}} \cap\, \omega\cdot(v_{s+1}-v_i)>0} & 2\big(|v_i|+|v_{s+1}|\big)_+ \exp(-(s+1)^\alpha\widetilde{\mu}(\tau))\exp\Big(-\frac{\widetilde{\beta}(\tau)}{2}\big|V_{s+1}\big|^2\Big)\,\mathrm{d}\tau \\
& \times \exp\Big(\frac{\widetilde{\beta}(t)}{2}\sum_{j=1}^{s}|v_j|^2\Big)\exp(s^\alpha\widetilde{\mu}(t))\Big|\Big|\Big|\big(f^{(s)}\big)_{s\geq 1}\Big|\Big|\Big|_{0,\widetilde{\beta},\widetilde{\mu}^\alpha}.
\end{aligned}
$$

The last upper bound obtained for Q is then exactly the same as the one obtained in the inequality (8.4) of the proof of Lemma 18 page 227 in the case when $\alpha = 2$, and in the case when $\alpha = 1$, it is the same as the upper bound of the inequality (8.7) of Lemma 19 page 236. So, the same assumptions on the weights $\widetilde{\beta}$ and $\widetilde{\mu}$ provide the same conclusions, and then the two lemmas about the integral term of the Boltzmann operator are proved. □

So, as in the previous section, Lemma 20 and Lemma 21 together show that the Boltzmann operator sends the space $\widetilde{\mathbf{X}}_{0,\widetilde{\beta},\widetilde{\mu}^2}$ into itself for the case $\alpha = 2$, and in the case $\alpha = 1$, Lemma 20 and Lemma 22 together show that this operator sends the space $\widetilde{\mathbf{X}}_{0,\widetilde{\beta},\widetilde{\mu}^1}$ into itself, when the weights $\widetilde{\beta}$ and $\widetilde{\mu}$ are wisely chosen (note that the choice on those weights is more restrictive in the case $\alpha = 2$).

8.2 About the strength of the weight in s and its consequences on the continuity in time for the BBGKY and the Boltzmann operators

Here one can wonder why such a stronger weight was used to define the $\|\cdot\|_{\cdot,\widetilde{\beta}(t),\widetilde{\mu}(t)^2}$ norm (one stresses that one is focusing on the case $\alpha = 2$). Recall that, for the BBGKY hierarchy of the Boltzmann hierarchy indistinctly, one has defined, along Definition 27 for the BBGKY hierarchy, and Definition 28 for the Boltzmann hierarchy, for a sequence $\left(f^{(s)}\right)_s$ of functions $f^{(s)}$ of the phase space of s particles :

$$\left\|\left(f^{(s)}\right)_s\right\|_{\cdot,\beta,\mu^\alpha} = \sup_s \left(\left|f^{(s)}\right|_{\cdot,\beta,\mu} \exp(s^2\mu)\right).$$

The square over s, as the fact that the function $\widetilde{\mu}$ has to be strictly positive, were crucially used to recover the left continuity in time for the $\|\cdot\|_{\cdot,\widetilde{\beta}(t),\widetilde{\mu}(t)^2}$ norm of the image of the integral term of the BBGKY or the Boltzmann operators.
Since the continuity of the solutions of the Boltzmann and the BBGKY hierarchies are important, a question naturally arises from that proof : is this hypothesis really necessary ? Then, one will discuss the pros and the cons of those two different continuity hypotheses preserved by the integrated in time transport-collision-transport operator.

8.2.1 About the necessity of a strong weight for the uniform in s continuity setting

A condition set to belong to the spaces $\widetilde{\mathbf{X}}_{N,\varepsilon,\widetilde{\beta},\widetilde{\mu}^2}$ and $\widetilde{\mathbf{X}}_{0,\widetilde{\beta},\widetilde{\mu}^2}$ was the left continuity in time for the $\|\cdot\|_{\cdot,\widetilde{\beta}(t),\widetilde{\mu}(t)^2}$ norm, that is one has to show that, for all $t \in]0,T]$ and $0 \leq u \leq t$:

$$\lim_{u \to t^-} \left\|\widetilde{G}(t) - \widetilde{G}(u)\right\|_{\cdot,\widetilde{\beta}(t),\widetilde{\mu}(t)^2} = 0.$$

One will take the example of the BBGKY operator, the same problems arising exactly in the same way for the Boltzmann operator. To obtain the left continuity in time for the integral term of the BBGKY operator, if one considers the proof of Lemma 18 page 222, but with weaker weights, that is if one works with a function of sequences $t \mapsto \big(h_N^{(s)}\big)_{1\leq s\leq N}$ verifying only that, uniformly in $t \in [0,T]$ and $1 \leq s \leq N$:

$$\big|h_N^{(s)}(t,\cdot)\big|_{\varepsilon,s,\widetilde{\beta}(t)} \exp(s\widetilde{\mu}(t)) \leq C$$

with C a strictly positive constant (which is the norm introduced in Definition 25 page 207, in the case $\alpha = 1$), one wants to show the left continuity in time, uniformly in s (which is the continuity hypothesis set to define the spaces $\widetilde{\mathbf{X}}_{\varepsilon,\widetilde{\beta},\widetilde{\mu}^2}$, see Definition 27 page 210) :

$$\lim_{u\to t^-} \left\| \Big(\mathbb{1}_{s\leq N-1} \int_u^t \mathcal{T}_{-\tau}^{s,\varepsilon} \mathcal{C}_{s,s+1}^{N,\varepsilon} \mathcal{T}_{-\tau}^{s+1,\varepsilon} h_N^{(s+1)}(\tau,\cdot)\,\mathrm{d}\tau \Big)_{1\leq s\leq N} \right\|_{N,\varepsilon,\widetilde{\beta}(t),\widetilde{\mu}(t)^1} = 0,$$

that is, here in the context of the weaker weight :

$$\lim_{u\to t^-} \left[\max_{1\leq s\leq N-1} \Big| \int_u^t \mathcal{T}_{-\tau}^{s,\varepsilon} \mathcal{C}_{s,s+1}^{N,\varepsilon} \mathcal{T}_{-\tau}^{s+1,\varepsilon} h_N^{(s+1)}(\tau,\cdot)\,\mathrm{d}\tau \Big|_{\varepsilon,s,\widetilde{\beta}(t)} \exp(s\widetilde{\mu}(t)) \right] = 0.$$

In other words, one wonders if the left continuity in time, uniformly in the number of particles s, can be conserved if the weight $\exp(s^2\mu)$ is replaced by $\exp(s\mu)$, this continuity property being of course more restrictive than the one chosen to define the spaces $\widetilde{\mathbf{X}}_{N,\varepsilon,\widetilde{\beta},\widetilde{\mu}^1}$ (see Definition 29 page 211).
The problem, following the proof of Lemma 18, is then solved if one is able to show that the following quantity Q goes to zero, uniformly in s and Z_s :

$$Q = \int_u^t \Big(\sum_{i=1}^s |v_i| + s \Big) \exp\Big(\frac{\lambda}{2}(\tau - t) \sum_{i=1}^s |v_i|^2 + s(\widetilde{\mu}(t) - \widetilde{\mu}(\tau)) \Big)\,\mathrm{d}\tau,$$

that is, using the definition of the function $\widetilde{\mu}$:

$$Q = \int_u^t \Big(\sum_{i=1}^s |v_i| + s \Big) \exp\Big(\frac{\lambda}{2}(\tau - t) \sum_{i=1}^s |v_i|^2 + s(\tau - t) \Big)\,\mathrm{d}\tau.$$

However, this last quantity can be computed explicitly, and for s and Z_s fixed, one obtains :

$$Q = \frac{\sum_{i=1}^s |v_i| + s}{\lambda\Big(\frac{1}{2}\sum_{i=1}^s |v_i|^2 + s\Big)} \Big(1 - \exp\Big(\frac{\lambda}{2}(u - t) \sum_{i=1}^s |v_i|^2 + s(u - t) \Big) \Big),$$

so that :

$$Q \underset{u\to t}{\sim} \frac{1}{\lambda} \Big(\sum_{i=1}^s |v_i| + s \Big)(t - u).$$

In other words, there is no hope to obtain simultaneously a uniform control in s and Z_s on the quantity Q, and a control on a convergence towards zero as $u \to t^-$, hence the necessity of a stronger weight to obtain the left continuity in time for the BBGKY operator, and for the same reason, for the Boltzmann hierarchy.
In other words, the uniform in s left continuity in time can be recovered after an application of the integrated in time transport-collision-transport operator only if one is in the case $\alpha = 2$ (or, more generally, in the case $\alpha > 1$), and in the case $\alpha = 1$, one can consider only a left continuity in time, for each s.

8.2.2 About the advantages and the disadvantages of the two different continuity in time hypotheses

One will see in the following that the left continuity in time, uniform in s, that is the one obtained in the case when $\alpha = 2$, will provide a strong result in the preparation phase of the comparison of the solutions of the two hierarchies. Namely : the cut-offs in high number of collisions, large energy and small time difference between collisions will be controlled in the $|||\cdot|||_{\cdot,\widetilde{\beta},\widetilde{\mu}^\alpha}$ norm.
To be explicit, one will see that the two first cut-offs (the first, concerning the cut-off in high number of adjunctions, addressed in Section 11.1, and stated in 11 page 278, and the second, concerning the cut-off in high energies, addressed in Section 11.2, and stated in 12 page 281) hold whatever the value of α is. But one can compare the difference of the strength of the results 14 page 301 and 15 page 308, stated in Section 11.3 dealing with the cut-off in small time difference between the adjunctions, those two results depending crucially on the value of α, and providing two very different kinds of convergence.

However, an important step in the comparison of the two solutions lies in the choice of initial data for the Boltzmann hierarchy. On the one hand, one will see that it will be very convenient, for several crucial reasons, to work with tensorized initial data (besides, they are also the most easy to consider). On the other hand, there are only few initial data which are, at the same time, tensorized and belonging to the space $\mathbf{X}_{0,\beta_0,\mu_0^2}$, due in particular to the fact that the function $\widetilde{\mu}$ has to remain strictly positive in this setting, which is a very demanding condition.
Indeed, if one considers the sequence of initial data $\big(f_0^{(s)}\big)_{s\geq 1}$, obtained by the tensorization of the gaussian

$$g(z) = \exp\Big(-\frac{\beta}{2}|v|^2\Big)$$

(with $z = (x, v) \in \overline{\Omega^c} \times \mathbb{R}^d$, and $v \in \mathbb{R}^d$), that is, for all $s \geq 1$,

$$f_0^{(s)}(Z_s) = g^{\otimes s}(Z_s) = \prod_{i=1}^{s} \exp\Big(-\frac{\beta}{2}|v_i|^2\Big),$$

one sees immediately, if $\mu > 0$, that :

$$\left|f^{(s)}\right|_{0,s,\beta} \exp\left(s\mu\right) = \exp\left(s\mu\right) \underset{s\to+\infty}{\longrightarrow} +\infty$$

so that

$$\sup_{s\geq 1} \left|f^{(s)}\right|_{0,s,\beta} \exp\left(s\mu\right) = \left|\left(f^{(s)}\right)_{s\geq 1}\right|_{0,\beta,\mu} = +\infty.$$

So one can consider instead some normalization of this tensorized datum, such as

$$\widetilde{f}^{(s)} = C(s) g^{\otimes s},$$

with

$$C(s) \leq \exp\left(-s\mu\right),$$

but in this case, for s large enough, $\widetilde{f}^{(s)}$ is *not* a density, which is a too restrictive setting for physical applications.
For the less strong continuity, that is the one obtained in the case $\alpha = 1$, the advantages and the disadvantages are exactly exchanged : the control of the effects of the cut-offs in the preparation phase will be only obtained, for each number of particles s fixed, in the $|\cdot|_{\cdot,s,\widetilde{\beta}(t)}$ norm.
Conversely, in this setting, there is no condition on the sign of the function $\widetilde{\mu}$, and then of course a much larger class of initial data can be chosen in order to apply Theorem 6.

8.3 Contraction estimates on the collision operators

The previous section 8.1 was devoted to showing that the BBGKY and the Boltzmann hierarchies preserve respectively the spaces $\widetilde{\mathbf{X}}_{N,\varepsilon,\widetilde{\beta},\widetilde{\mu}^\alpha}$ and $\widetilde{\mathbf{X}}_{0,\widetilde{\beta},\widetilde{\mu}^\alpha}$, for $\alpha = 1$ or 2. One has seen that a sufficient condition to obtain this stability is to assume that the functions $\widetilde{\beta}$ and $\widetilde{\mu}$ are affine, strictly positive (in fact, if $\alpha = 1$, it is sufficient to require only that $\widetilde{\beta}$ strictly positive) on $[0, T]$ and decreasing.
The following section will be devoted to investigating a sufficient condition on those functions $\widetilde{\beta}$ and $\widetilde{\mu}$ to obtain, not only operators from the spaces $\widetilde{\mathbf{X}}_{N,\varepsilon,\widetilde{\beta},\widetilde{\mu}^\alpha}$ and $\widetilde{\mathbf{X}}_{0,\widetilde{\beta},\widetilde{\mu}^\alpha}$ into themselves, but in addition such that those operators are contracting mappings, of course in order to apply a fixed point theorem.

Remark 20. *Here one will use crucially the setting of the Boltzmann-Grad limit, which is, one recalls, an assumption on the limit of the quantity :*

$$N\varepsilon^{d-1}.$$

The goal of this assumption is, first, to obtain the same bound on the norm of the BBGKY and Boltzmann operators, for $\beta_0 > 0$ and $\mu_0 > 0$ given, and second, to obtain especially a control on those norm which is uniform in the number of particles N, which, one recalls, is to be sent to infinity.

One starts by two statements of a contraction inequality holding for the BBGKY operator, one for each case depending on the values of α.

Lemma 23 (Contraction lemma for the BBGKY operator)**.** *Let α be 1 or 2. Let N be a positive integer and ε be a strictly. In the Boltzmann-Grad limit :*

$$N\varepsilon^{d-1} = 1,$$

for any strictly positive real number β_0 and any (strictly positive in the case $\alpha = 2$) real number μ_0, there exist two strictly positive real numbers T and λ such that $\beta_0 - \lambda T$ is strictly positive, and in the case $\alpha = 2$, $\mu_0 - \lambda T$ is also strictly positive, and such that, if one defines the two functions :

$$\widetilde{\beta}_\lambda = \begin{cases} [0,T] & \to \mathbb{R}_+^*, \\ t & \mapsto \widetilde{\beta}_\lambda(t) = \beta_0 - \lambda t, \end{cases}$$

and, in the case $\alpha = 2$

$$\widetilde{\mu}_\lambda = \begin{cases} [0,T] & \to \mathbb{R}_+^*, \\ t & \mapsto \widetilde{\mu}_\lambda(t) = \mu_0 - \lambda t, \end{cases}$$

(here $\widetilde{\mu}_\lambda$ has to be strictly positive), or in the case $\alpha = 1$

$$\widetilde{\mu}_\lambda = \begin{cases} [0,T] & \to \mathbb{R}, \\ t & \mapsto \widetilde{\mu}_\lambda(t) = \mu_0 - \lambda t, \end{cases}$$

then for any sequence $\big(f_{N,0}^{(s)}\big)_{1\le s\le N}$ of initial data belonging to $\boldsymbol{X}_{N,\varepsilon,\beta_0,\mu_0^\alpha}$, the integral term of the BBGKY operator associated to this sequence of initial data has a $|||\cdot|||_{N,\varepsilon,\widetilde{\beta}_\lambda,\widetilde{\mu}_\lambda^\alpha}$ norm smaller than 1/2.

In other words, for every $H_N = t \mapsto \big(h_N^{(s)}(t,\cdot)\big)_{1\le s\le N} \in \widetilde{\boldsymbol{X}}_{N,\varepsilon,\widetilde{\beta}_\lambda,\widetilde{\mu}_\lambda^\alpha}$, the following control holds :

$$\left|\!\left|\!\left| t \mapsto \Big(\mathbb{1}_{s\le N-1}\int_0^t \mathcal{T}_{-u}^{s,\varepsilon}\mathcal{C}_{s,s+1}^{N,\varepsilon}\mathcal{T}_u^{s+1,\varepsilon}h_N^{(s+1)}(u,\cdot)\,\mathrm{d}u\Big)_{1\le s\le N}\right|\!\right|\!\right|_{N,\varepsilon,\widetilde{\beta}_\lambda,\widetilde{\mu}_\lambda^\alpha} \le \frac{1}{2}|||H_N|||_{N,\varepsilon,\widetilde{\beta}_\lambda,\widetilde{\mu}_\lambda^\alpha}. \tag{8.17}$$

Proof. One writes the proof in detail for the case $\alpha = 2$ only, since in the case $\alpha = 1$, the arguments are essentially the same. In the case $\alpha = 1$, one has to work with the quantity :

$$T \mapsto Q(T) = C(d)\big(\beta_0 - \lambda T\big)^{-d/2}\exp\big(-\big(\mu_0 - \lambda T\big)\big)\frac{\big(1 + \big(\beta_0 - \lambda T\big)^{-1/2}\big)}{\lambda},$$

given by the control (8.6) of Lemma 19 page 234 instead of what follows. Besides, the reader can find a detailed proof for the case $\alpha = 1$ in [34][3].

[3]See Section 5.4 "Continuity estimates", and in particular the very end of the proof of Lemma 5.4.3.

For any $\beta_0 > 0$ and $\mu_0 > 0$, and any $\lambda > 0$, one can choose T such that $\beta_0 - \lambda T$ and $\mu_0 - \lambda T$ are both strictly positive. For example, one can choose :

$$T(\beta_0, \mu_0, \lambda) = \min\Big(\frac{9\beta_0}{10\lambda}, \frac{9\mu_0}{10\lambda}\Big). \tag{8.18}$$

This condition means exactly that $\widetilde{\beta}(t) \geq \beta_0/10 > 0$ and $\widetilde{\mu}(t) \geq \mu_0/10 > 0$ for all time $t \in [0, T]$.

For such λ and T, Lemma 18 page 222 can be applied, and one has, thanks to the inequality (8.3) of that very Lemma, that the $|||\cdot|||_{N,\varepsilon,\widetilde{\beta},\widetilde{\mu}^2}$ norm of the integral term of the BBGKY operator is smaller than Q, with Q defined as :

$$Q = C(d)\widetilde{\beta}_\lambda(T)^{-d/2} \exp\big(-3\widetilde{\mu}_\lambda(T)\big)\frac{\big(1+\widetilde{\beta}_\lambda(T)^{-1/2}\big)}{\lambda}$$

for some constant $C(d)$ depending only on the dimension.
Of course, the goal of the proof is to find λ and T such that the quantity Q is smaller than $1/2$.
Recalling that

$$\begin{aligned} Q &= C(d)\widetilde{\beta}_\lambda(T)^{-d/2} \exp\big(-3\widetilde{\mu}_\lambda(T)\big)\frac{\big(1+\widetilde{\beta}_\lambda(T)^{-1/2}\big)}{\lambda} \\ &= C(d)\big(\beta_0 - \lambda T\big)^{-d/2} \exp\big(-3\big(\mu_0 - \lambda T\big)\big)\frac{\big(1+\big(\beta_0 - \lambda T\big)^{-1/2}\big)}{\lambda} \\ &= Q(T), \end{aligned}$$

one notices first, if one sees here Q as a function of T, that :

$$Q(0) = C(d)\beta_0^{-d/2}\exp(-3\mu_0)\frac{\big(1+\beta_0^{-1/2}\big)}{\lambda},$$

so, for β_0 and μ_0 given, one sees immediately that $Q(0)$ can be chosen as small as one wants up to choosing λ large enough.
So let $\lambda = \lambda(\beta_0, \mu_0)$ such that :

$$Q(0) \leq 1/4.$$

Since the function :

$$T \mapsto Q(T) = C(d)\big(\beta_0 - \lambda T\big)^{-d/2} \exp\big(-3\big(\mu_0 - \lambda T\big)\big)\frac{\big(1+\big(\beta_0 - \lambda T\big)^{-1/2}\big)}{\lambda}$$

is well defined and continuous around 0, one can then find a strictly positive T (here : $T = T(\beta_0, \mu_0, \lambda) = T(\beta_0, \mu_0)$), smaller than the bound $\min\Big(\frac{9\beta_0}{10\lambda}, \frac{9\mu_0}{10\lambda}\Big)$ (which granted the positivity of $\widetilde{\beta}$ and $\widetilde{\mu}$) such that, for every $t \in [0, T]$:

$$Q(t) \leq 1/2,$$

so that the lemma is proven. $\square$

Similarly, for the Boltzmann operator, the proof relies on Lemma 21 for the case when $\alpha = 2$, and on Lemma 22 for the case when $\alpha = 1$, which are the analogs of Lemmas 18 and 19 for the integral term of the BBGKY operator, and the result is essentially the same.

Lemma 24 (Contraction lemma for the Boltzmann operator)**.** *Let α be 1 or 2. For any strictly positive real number β_0 and any (strictly positive in the case $\alpha = 2$) real number μ_0, there exist two strictly positive real numbers T and λ such that $\beta_0 - \lambda T$ is positive, and in the case $\alpha = 2$, $\mu_0 - \lambda T$ is also strictly positive, and such that if one defines the two functions :*

$$\widetilde{\beta}_\lambda = \begin{cases} [0,T] & \to \mathbb{R}_+^*, \\ t & \mapsto \widetilde{\beta}_\lambda(t) = \beta_0 - \lambda t, \end{cases}$$

and, in the case $\alpha = 2$

$$\widetilde{\mu}_\lambda = \begin{cases} [0,T] & \to \mathbb{R}_+^*, \\ t & \mapsto \widetilde{\mu}_\lambda(t) = \mu_0 - \lambda t, \end{cases}$$

(here $\widetilde{\mu}_\lambda$ has to be strictly positive), or in the case $\alpha = 1$

$$\widetilde{\mu}_\lambda = \begin{cases} [0,T] & \to \mathbb{R}, \\ t & \mapsto \widetilde{\mu}_\lambda(t) = \mu_0 - \lambda t, \end{cases}$$

then for any sequence $\left(f_0^{(s)}\right)_{s\geq 1}$ of initial data belonging to $\boldsymbol{X}_{0,\beta_0,\mu_0^\alpha}$, the integral term of the Boltzmann operator associated to this sequence of initial data has a $|||\cdot|||_{N,\varepsilon,\widetilde{\beta}_\lambda,\widetilde{\mu}_\lambda^\alpha}$ norm smaller than $1/2$.
In other words, for every $F = t \mapsto \left(f^{(s)}(t,\cdot)\right)_{s\geq 1} \in \widetilde{\boldsymbol{X}}_{0,\widetilde{\beta}_\lambda,\widetilde{\mu}_\lambda^\alpha}$, the following control holds :

$$\left|\left|\left| t \mapsto \Big(\int_0^t \mathcal{T}_{t-u}^{s,\varepsilon} \mathcal{C}_{s,s+1}^0 f^{(s+1)}(u,\cdot)\,\mathrm{d}u \Big)_{s\geq 1} \right|\right|\right|_{0,\widetilde{\beta}_\lambda,\widetilde{\mu}_\lambda^\alpha} \leq \frac{1}{2} |||F|||_{0,\widetilde{\beta}_\lambda,\widetilde{\mu}_\lambda^\alpha}. \tag{8.19}$$

Remark 21. *Although Lemmas 23 and 24 are stated independently, it is important to stress the fact that, thanks to the additional hypothesis of the Boltzmann-Grad limit, the time interval $[0,T]$ on which the respective integral term of the BBGKY and the Boltzmann operators has a $|||\cdot|||_{\cdot,\widetilde{\beta}_\lambda,\widetilde{\mu}_\lambda^\alpha}$ norm smaller than $1/2$ is the same (since it depends in fact only on β_0 and μ_0). In particular, this time interval* does not *depend on the number of particles N, and then one does not run the risk anymore of seeing the size of the time interval on which the operators are contracting mappings vanishing while this number of particles N is sent to infinity.*

Chapter 9

Existence and uniqueness of solutions to the hierarchies

One is now able to state one of the two key results of this work.

Theorem 6 (Joint local in time existence and uniqueness of solutions to the BBGKY and Boltzmann hierarchies)**.** *Let α be 1 or 2. Let β_0 be a strictly positive real number and μ_0 a (strictly positive in the case $\alpha = 2$) real number. There exist a time $T > 0$, a strictly positive decreasing function $\tilde{\beta}$ and a (strictly positive in the case $\alpha = 2$) decreasing function $\tilde{\mu}$ defined on $[0, T]$ such that :*

$$\tilde{\beta}(0) = \beta_0, \ \tilde{\mu}(0) = \mu_0,$$

and such that for any positive integer N and any strictly positive number $\varepsilon > 0$ verifying the Boltzmann-Grad limit :

$$N\varepsilon^{d-1} = 1,$$

any pair of sequences of initial data $F_{N,0} = \big(f_{N,0}^{(s)}\big)_{1\leq s\leq N}$ and $F_0 = \big(f_0^{(s)}\big)_{s\geq 1}$ belonging respectively to $\boldsymbol{X}_{N,\varepsilon,\beta_0,\mu_0^\alpha}$ and $\boldsymbol{X}_{0,\beta_0,\mu_0^\alpha}$ give rise respectively to a unique solution $H_N = t \mapsto \big(h_N^{(s)}(t,\cdot)\big)_{1\leq s\leq N}$ in $\widetilde{\boldsymbol{X}}_{N,\varepsilon,\tilde{\beta},\tilde{\mu}^\alpha}$ to the BBGKY hierarchy with initial datum $F_{N,0}$ and $F = t \mapsto \big(f^{(s)}(t,\cdot)\big)_{s\geq 1}$ in $\widetilde{\boldsymbol{X}}_{0,\tilde{\beta},\tilde{\mu}^\alpha}$ to the Boltzmann hierarchy with initial datum F_0, that is, using the notation introduced in Definitions 31 page 220 and 32 page 238, there exists a unique pair of elements H_N and F belonging respectively to the spaces $\widetilde{\boldsymbol{X}}_{N,\varepsilon,\widetilde{\beta},\widetilde{\mu}^\alpha}$ and $\widetilde{\boldsymbol{X}}_{0,\widetilde{\beta},\widetilde{\mu}^\alpha}$ such that, for every $t \in [0, T]$:

$$H_N(t) = \mathfrak{E}_{N,\varepsilon}\big(F_{N,0}, H_N\big)(t),$$

and

$$F(t) = \mathfrak{E}_0\big(F_0, F\big)(t).$$

Moreover, the decreasing functions $\widetilde{\beta} = \widetilde{\beta}_\lambda$ and $\widetilde{\mu} = \widetilde{\mu}_\lambda$ are affine, given by the expressions

$$\widetilde{\beta}_\lambda = \begin{cases} [0,T] & \to \mathbb{R}_+^*, \\ t & \mapsto \widetilde{\beta}_\lambda(t) = \beta_0 - \lambda t, \end{cases}$$

and

$$\widetilde{\mu}_\lambda = \begin{cases} [0,T] & \to \mathbb{R} \ (\mathbb{R}_+^* \text{ in the case } \alpha = 2), \\ t & \mapsto \widetilde{\mu}_\lambda(t) = \mu_0 - \lambda t, \end{cases}$$

with λ and T given in Lemmas 23 page 255 and 24 page 257.

Proof. The idea of the proof is very simple : one will show that the BBGKY and the Boltzmann operators are contracting mappings on the respective Banach spaces $\widetilde{\mathbf{X}}_{N,\varepsilon,\widetilde{\beta},\widetilde{\mu}^\alpha}$ and $\widetilde{\mathbf{X}}_{0,\widetilde{\beta},\widetilde{\mu}^\alpha}$, so the Banach-Caccioppoli will provide the result of the theorem. The work will be done only for the BBGKY operator and the case $\alpha = 2$, since, whatever α is, any result used for this operator has its equivalent for the Boltzmann hierarchy, and all the arguments can be directly used for the study of the Boltzmann operator in the same way.

First, Lemmas 17 page 221 and 18 page 222 together show, up to defining two particular weights $\widetilde{\beta}$ and $\widetilde{\mu}$ on some non empty time interval $[0,T]$ (those weights and time interval being independent of the positive integer N and the strictly positive number ε assuming that one is working in the Boltzmann-Grad limit), that the BBGKY operator (defined in Definition 31 page 220) associated to any sequence of initial data $\big(f_{N,0}^{(s)}\big)_{1\le s\le N}$ of the space $\mathbf{X}_{N,\varepsilon,\beta_0,\mu_0^2}$ is well defined and continuous on the space $\widetilde{\mathbf{X}}_{N,\varepsilon,\widetilde{\beta},\widetilde{\mu}^2}$, and sends this space into itself.

One fixes then the sequence of initial data $F_{N,0} = \big(f_{N,0}^{(s)}\big)_{1\le s\le N} \in \mathbf{X}_{N,\varepsilon,\beta_0,\mu_0^2}$, and takes two functions of sequences $H_{N,1} = t \mapsto \big(h_{N,1}^{(s)}(t,\cdot)\big)_{1\le s\le N}$ and $H_{N,2} = t \mapsto \big(h_{N,2}^{(s)}(t,\cdot)\big)_{1\le s\le N}$. One will consider the difference :

$$\mathfrak{E}_{N,0}\big(F_{N,0}, H_{N,1}\big) - \mathfrak{E}_{N,0}\big(F_{N,0}, H_{N,2}\big)$$

which is by definition equal to :

$$\begin{aligned} t \mapsto & \Big(f_{N,0}^{(s)}(\cdot) + \mathbb{1}_{s\le N-1}\int_0^t \mathcal{T}_{-u}^{s,\varepsilon}\mathcal{C}_{s,s+1}^{N,\varepsilon}\mathcal{T}_u^{s+1,\varepsilon}h_{N,1}^{(s+1)}(u,\cdot)\,\mathrm{d}u\Big)_{1\le s\le N} \\ & - \Big(f_{N,0}^{(s)}(\cdot) + \mathbb{1}_{s\le N-1}\int_0^t \mathcal{T}_{-u}^{s,\varepsilon}\mathcal{C}_{s,s+1}^{N,\varepsilon}\mathcal{T}_u^{s+1,\varepsilon}h_{N,2}^{(s+1)}(u,\cdot)\,\mathrm{d}u\Big)_{1\le s\le N}, \end{aligned}$$

that is, since the first terms with the initial data obviously cancel :

$$\begin{aligned} & \mathfrak{E}_{N,0}\big(F_{N,0}, H_{N,1}\big) - \mathfrak{E}_{N,0}\big(F_{N,0}, H_{N,2}\big) \\ & \quad = t \mapsto \Big(\mathbb{1}_{s\le N-1}\int_0^t \mathcal{T}_{-u}^{s,\varepsilon}\mathcal{C}_{s,s+1}^{N,\varepsilon}\mathcal{T}_u^{s+1,\varepsilon}h_{N,1}^{(s+1)}(u,\cdot)\,\mathrm{d}u \\ & \qquad\qquad - \mathbb{1}_{s\le N-1}\int_0^t \mathcal{T}_{-u}^{s,\varepsilon}\mathcal{C}_{s,s+1}^{N,\varepsilon}\mathcal{T}_u^{s+1,\varepsilon}h_{N,2}^{(s+1)}(u,\cdot)\,\mathrm{d}u\Big)_{1\le s\le N}, \end{aligned}$$

and then, by linearity of the transport-collision-transport integrated in time :

$$\begin{aligned}&\mathfrak{E}_{N,0}\big(F_{N,0},H_{N,1}\big)-\mathfrak{E}_{N,0}\big(F_{N,0},H_{N,2}\big)\\&\qquad=t\mapsto\Big(\mathbb{1}_{s\leq N-1}\int_0^t\mathcal{T}_{-u}^{s,\varepsilon}\mathcal{C}_{s,s+1}^{N,\varepsilon}\mathcal{T}_u^{s+1,\varepsilon}\big(h_{N,1}^{(s+1)}(u,\cdot)-h_{N,2}^{(s+1)}(u,\cdot)\big)\,\mathrm{d}u\Big)_{1\leq s\leq N}.\end{aligned}$$

Finally, only the norm of the integral term of the BBGKY operator will play a role for the contracting property of the operator. Moreover, for any β_0 and μ_0 strictly positive, Lemma 23 page 255 shows that it is possible to find a non empty time interval $[0,T]$ and to define two weights $\widetilde{\beta}$ and $\widetilde{\mu}$ with additional hypotheses satisfied by those weights on this time interval such that, on the space $\widetilde{\mathbf{X}}_{N,\varepsilon,\widetilde{\beta},\widetilde{\mu}^2}$, the integral term of the BBGKY operator $\mathfrak{E}_{N,\varepsilon}$ has a $|||\cdot|||_{N,\varepsilon,\widetilde{\beta},\widetilde{\mu}^2}$ norm smaller than $1/2$. The following quantity is then well defined and verifies obviously :

$$\begin{aligned}&\big|\big|\big|\mathfrak{E}_{N,0}\big(F_{N,0},H_{N,1}\big)-\mathfrak{E}_{N,0}\big(F_{N,0},H_{N,2}\big)\big|\big|\big|_{N,\varepsilon,\widetilde{\beta},\widetilde{\mu}^2}\\=&\Bigg|\Bigg|\Bigg|\Big(\mathbb{1}_{s\leq N-1}\int_0^t\mathcal{T}_{-u}^{s,\varepsilon}\mathcal{C}_{s,s+1}^{N,\varepsilon}\mathcal{T}_u^{s+1,\varepsilon}\big(h_{N,1}^{(s+1)}(u,\cdot)-h_{N,2}^{(s+1)}(u,\cdot)\big)\,\mathrm{d}u\Big)_{1\leq s\leq N}\Bigg|\Bigg|\Bigg|_{N,\varepsilon,\widetilde{\beta},\widetilde{\mu}^2}\\&\qquad\leq\frac{1}{2}\Bigg|\Bigg|\Bigg|\big(h_{N,1}^{(s+1)}-h_{N,2}^{(s+1)}\big)_{1\leq s\leq N}\Bigg|\Bigg|\Bigg|_{N,\varepsilon,\widetilde{\beta},\widetilde{\mu}^2}=\frac{1}{2}|||H_{N,1}-H_{N,2}|||_{N,\varepsilon,\widetilde{\beta},\widetilde{\mu}^2}.\end{aligned}$$

In other words, the BBGKY operator associated to the sequence of initial data $F_{N,0}$ is Lipschitz with a Lipschitz constant smaller than 1, that is it is a contracting mapping. □

Part III

Convergence of the solutions of the BBGKY hierarchy towards the solutions of the Boltzmann hierarchy

Now that the problem of the existence (and uniqueness) of solutions of the Boltzmann and the BBGKY hierarchies has been addressed, one will work on the comparison of those solutions. The goal of this part is to show that, in the appropriate topology, the solutions of the BBGKY hierarchy converge towards the solutions of the Boltzmann hierarchy.

The two next chapters (Chapters 10 and 11) will be devoted to preliminary simplifications. One gives here a brief description of those preliminary simplifications.

- In Chapter 10, starting page 267, the solutions of the hierarchies found previously will be rewritten in order to have an explicit expression, using only the operators defined in Subsections 8.1.1 page 220 and 8.1.2 page 238 (see Definitions 31 page 220 and 32 page 238) and the initial data. One will see that the solutions will have the aspect of a sum of iterates of the transport-collision-transport operators for the BBGKY hierarchy, and of collision-transport operators for the Boltzmann hierarchy, applied only on the elements of the sequences of initial data.

- In Chapter 11, starting page 277, one will consider truncated versions of the sums described above, and study the effects of the corresponding cut-offs. The idea is, on the one hand, to consider dynamics on the time interval $[0, T]$ implying only a bounded energy, a bounded number of collisions, and such that those collisions are clearly separated in time. This section will be a first preparation for the effective comparison of the solutions of the two hierarchies, performed carefully at the end of this part. On the other hand, one will show that the difference between the original sums and the truncated ones can be chosen as small as one wants up to choosing wisely the truncation parameters.

The main arguments of the effective comparison are then presented in Chapters 12 and 13, and are based on a geometrical point of view of the convergence of the trajectories described by the BBGKY hierarchy, towards the trajectories described by the Boltzmann hierarchy.

- Chapter 12, starting page 309, is devoted to the last cut-offs required to compare easily the trajectories described by the two hierarchies.

- In Chapter 13, starting page 449, one provides a quantitative comparison of the trajectories described by the two hierarchies.

The results of those two last chapters enable finally to state the convergence of the main terms of the decomposition of the solutions of the BBGKY hierarchy towards the main terms of the decomposition of the solutions of the Boltzmann hierarchy : this is the purpose of Section 13.3, starting page 475. The first result obtained is only qualitative at this step, but it will be improved below.
The last error terms, which constitute the difference between the solutions and

the main terms of the decomposition of those solutions, are controlled in Chapter 14.

Finally, in Chapter 15 , all the controls obtained are gathered, and one finally presents the first convergence result of this work : Theorem 7, page 493. This result is also only qualitative, but asserts the convergence of the solutions of the BBGKY hierarchy towards the solutions of the Boltzmann hierarchy.
Considering more restrictive hypotheses (in particular, concerning the initial data), one presents in Chapter 16 the final result of this work, Lanford's theorem for a domain with boundary conditions (Theorem 8 page 517), which asserts in the Boltzmann-Grad limit $N \to +\infty$, $N\varepsilon^{d-1} = 1$ the convergence of the first marginal of a distribution function of a system of N hard spheres evolving around an obstacle, towards the solution of the Boltzmann equation (this is from this solution that the initial distribution functions of the system of N hard spheres are built). One notices that this theorem, unlike Theorem 7, provides a quantitative convergence towards the solutions of the Boltzmann equation.

Chapter 10

Rewriting the solutions of the hierarchies in terms of the initial data

Starting from the integrated forms of the conjugate BBGKY hierarchy (see the sequence of equations (6.3) page 187) and the Boltzmann hierarchy (see the sequence of equations (4.4) page 83), it is possible to rewrite those equations so that for all positive integers N and s, $h_N^{(s)}$ and $f^{(s)}$ are rewritten in terms of initial data. This new expression is called the *iterated Duhamel formula*.
One starts by the easier case of the solutions of the BBGKY hierarchy, since it implies only a finite number of equations and initial data. The case of the Boltzmann hierarchy will require a small additional work.

10.1 The iterated Duhamel formula for the BBGKY hierarchy

The following proposition expresses the solutions of the BBGKY hierarchy only in terms of initial data, after introducing new notations for the sake of simplicity.

Definition 33 (Notations for the integrated in time transport-collision-transport operator of the BBGKY hierarchy)**.** *Let α be 1 or 2. For all positive integers N and s and any sequence of functions $\left(f_N^{(s)}\right)_{1\leq s\leq N}$ belonging to the space $\widetilde{X}_{\varepsilon,\widetilde{\beta},\widetilde{\mu}^\alpha}$, one will denote its image by the integrated in time transport-collision-transport operator of the BBGKY hierarchy :*

$$t \mapsto \int_0^t \mathcal{T}_{-u}^{s,\varepsilon}\mathcal{C}_{s,s+1}^{N,\varepsilon}\mathcal{T}_u^{s+1,\varepsilon} f_{N,0}^{(s+1)}\,\mathrm{d}u$$

as :

$$t \mapsto \big(\mathcal{I}_s^{N,\varepsilon} f_{N,0}^{(s)}\big)(t,\cdot). \tag{10.1}$$

The k-th iterate of this operator applied on any function $f_{N,0}^{(s+k)} \in X_{\varepsilon,s+k,\beta_0}$, that is :

$$t \mapsto \int_0^t \mathcal{T}_{-t_1}^{s,\varepsilon} \mathcal{C}_{s,s+1}^{N,\varepsilon} \mathcal{T}_{t_1}^{s+1,\varepsilon} \int_0^{t_1} \mathcal{T}_{-t_2}^{s+1,\varepsilon} \mathcal{C}_{s+1,s+2}^{N,\varepsilon} \mathcal{T}_{t_2}^{s+2,\varepsilon} \dots$$
$$\int_0^{t_{k-1}} \mathcal{T}_{-t_k}^{s+k-1,\varepsilon} \mathcal{C}_{s+k-1,s+k}^{N,\varepsilon} \mathcal{T}_{t_k}^{s+k,\varepsilon} f_{N,0}^{(s+k)} \, \mathrm{d}t_k \dots \mathrm{d}t_2 \, \mathrm{d}t_1,$$

which can be denoted as :

$$t \mapsto \Big(\mathcal{I}_s^{N,\varepsilon} \circ \mathcal{I}_{s+1}^{N,\varepsilon} \circ \dots \circ \mathcal{I}_{s+k-1}^{N,\varepsilon} f_{N,0}^{(s+k)}\Big)(t,\cdot),$$

thanks to the new notations (10.1)*, will be in fact often denoted as :*

$$t \mapsto \big(\mathcal{I}_{s,s+k-1}^{N,\varepsilon} f_{N,0}^{(s+k)}\big)(t,\cdot). \tag{10.2}$$

(The second subscript index describes the number of iterations).

Remark 22. *In the notation* (10.2)*, the function $f_{N,0}^{(s)}$ does not depend on time. Indeed $\mathcal{I}_{s,s+k-1}^{N,\varepsilon}$ will be applied in practice to the initial data.*

Proposition 9 (Iterated Duhamel formula for the solution of the BBGKY hierarchy)**.** *Let $\alpha = 1$ or 2. Let N be a positive integer and ε be a strictly positive number. In the Boltzmann-Grad limit :*

$$N\varepsilon^{d-1} = 1,$$

for any strictly positive number β_0, any (strictly positive in the case $\alpha = 2$) number μ_0, and for any sequence of initial data :

$$F_{N,0} = \big(f_{N,0}^{(s)}\big)_{1 \le s \le N}$$

belonging to the space $\boldsymbol{X}_{N,\varepsilon,\beta_0,\mu_0^\alpha}$, the unique solution of the integrated form of the conjugate BBGKY hierarchy with initial datum $F_{N,0}$ on the time interval $[0,T]$ (given by Theorem 6 page 259) is the sequence of functions (using the notations (10.1) *and* (10.2) *introduced in Definition 33) :*

$$H_N = t \mapsto \Big(f_{N,0}^{(s)} + \sum_{k=1}^{N-s} \mathbb{1}_{s \le N-k} \big(\mathcal{I}_{s,s+k-1}^{N,\varepsilon} f_{N,0}^{(s+k)}\big)(t,\cdot)\Big)_{1 \le s \le N}. \tag{10.3}$$

Proof. One considers a positive integer N and a strictly positive number ε. In the Boltzmann-Grad limit :

$$N\varepsilon^{d-1} = 1,$$

thanks to Theorem 6 page 259, for any strictly positive number β_0, any (strictly positive if $\alpha = 2$) number μ_0, and for any sequence of initial data $F_{N,0} = \big(f_{N,0}^{(s)}\big)_{1\leq s\leq N}$, there exists a unique solution $H_N = \big(h_{N,0}^{(s)}\big)_{1\leq s\leq N}$ on a non empty time interval $[0,T]$ of the integrated form of the conjugate BBGKY hierarchy with initial datum $F_{N,0}$ (see Definition 17 page 187 for the rigorous definition of the BBGKY hierarchy).

One has then, according to the theorem quoted :

$$H_N = \mathfrak{E}_{N,\varepsilon}\big(F_{N,0}, H_N\big),$$

which is equivalent to say that, for every time $t \in [0,T]$ and every integer $1 \leq s \leq N$:

$$h_N^{(s)}(t,\cdot) = f_{N,0}^{(s)} + \mathbb{1}_{s\leq N-1}\int_0^t \mathcal{T}_{-u}^{s,\varepsilon}\mathcal{C}_{s,s+1}^{N,\varepsilon}\mathcal{T}_u^{s+1,\varepsilon}h_N^{(s+1)}(u,\cdot)\,\mathrm{d}u. \tag{10.4}$$

In particular, the last equation of the N ones described in (10.4) writes exactly, for all $t \in [0,T]$:

$$h_N^{(N)}(t,\cdot) = f_{N,0}^{(N)}, \tag{10.5}$$

so that the last term of the solution of the BBGKY is entirely determined, and expressed only with the initial data. Besides, one can combine the last and the penultimate terms described in (10.4). One obtains, for every time $t \in [0,T]$:

$$\begin{aligned} h_N^{(N-1)}(t,\cdot) &= f_{N,0}^{(N-1)} + \int_0^t \mathcal{T}_{-u}^{N-1,\varepsilon}\mathcal{C}_{N-1,N}^{N,\varepsilon}\mathcal{T}_u^{N,\varepsilon}h_N^{(N)}(u,\cdot)\,\mathrm{d}u \\ &= f_{N,0}^{(N-1)} + \int_0^t \mathcal{T}_{-u}^{N-1,\varepsilon}\mathcal{C}_{N-1,N}^{N,\varepsilon}\mathcal{T}_u^{N,\varepsilon}f_{N,0}^{(N)}\,\mathrm{d}u, \end{aligned} \tag{10.6}$$

so the penultimate term of the solution of the BBGKY hierarchy is also determined, using only initial data and the integrated in time transport-collision-transport operator.
One sees then immediately that it is possible to iterate the process, thanks to the linearity of the integrated in time transport-collision-transport operator, in order to obtain the same description for any term of the solution of the BBGKY

hierarchy. For $h_N^{(N-2)}$, one obtains :

$$\begin{aligned}
h_N^{(N-2)}(t,\cdot) &= f_{N,0}^{(N-2)} + \int_0^t \mathcal{T}_{-t_1}^{N-2,\varepsilon}\mathcal{C}_{N-2,N-1}^{N,\varepsilon}\mathcal{T}_{t_1}^{N-1,\varepsilon}h_N^{(N-1)}(t_1,\cdot)\,\mathrm{d}t_1 \\
&= f_{N,0}^{(N-2)} + \int_0^t \mathcal{T}_{-t_1}^{N-2,\varepsilon}\mathcal{C}_{N-2,N-1}^{N,\varepsilon}\mathcal{T}_{t_1}^{N-1,\varepsilon} \\
&\qquad \Big(f_{N,0}^{(N-1)} + \int_0^{t_1} \mathcal{T}_{-t_2}^{N-1,\varepsilon}\mathcal{C}_{N-1,N}^{N,\varepsilon}\mathcal{T}_{t_2}^{N,\varepsilon}f_{N,0}^{(N)}\,\mathrm{d}t_2\Big)\,\mathrm{d}t_1 \\
&= f_{N,0}^{(N-2)} + \int_0^t \mathcal{T}_{-t_1}^{N-2,\varepsilon}\mathcal{C}_{N-2,N-1}^{N,\varepsilon}\mathcal{T}_{t_1}^{N-1,\varepsilon}f_{N,0}^{(N-1)}\,\mathrm{d}t_1 \\
&+ \int_0^t \mathcal{T}_{-t_1}^{N-2,\varepsilon}\mathcal{C}_{N-2,N-1}^{N,\varepsilon}\mathcal{T}_{t_1}^{N-1,\varepsilon}\int_0^{t_1} \mathcal{T}_{-t_2}^{N-1,\varepsilon}\mathcal{C}_{N-1,N}^{N,\varepsilon}\mathcal{T}_{t_2}^{N,\varepsilon}f_{N,0}^{(N)}\,\mathrm{d}t_2\,\mathrm{d}t_1.
\end{aligned}$$

One can therefore iterate the process without any difficuly, and using the notations just introduced in Definition 33 to simplify the presentation, one obtains in the end the result stated in Proposition 9. □

10.2 The iterated Duhamel formula for the Boltzmann hierarchy

Of course, the same work can be done for the Boltzmann hierarchy. This is the purpose of the following proposition. As above, one introduces new notations for the sake of simplicity, after the presentation of a short computation which will give an idea of how to process in this case.
For any positive integer s, for a solution $\big(f^{(s)}\big)_{s\geq 1}$ of the Boltzmann hierarchy, that is satisfying the equations (32) page 238, one has :

$$f^{(s)}(t,\cdot) = \mathcal{T}_t^{s,0}f_0^{(s)}(\cdot) + \int_0^t \mathcal{T}_{t-u}^{s,0}\mathcal{C}_{s,s+1}^0 f^{(s+1)}(t_1,\cdot)\,\mathrm{d}u,$$

and :

$$f^{(s+1)}(t,\cdot) = \mathcal{T}_t^{s+1,0}f_0^{(s+1)}(\cdot) + \int_0^t \mathcal{T}_{t-u}^{s+1,0}\mathcal{C}_{s+1,s+2}^0 f^{(s+2)}(u,\cdot)\,\mathrm{d}u,$$

so that, combining those two equations, one obtains :

$$\begin{aligned}
f^{(s+1)} &= \mathcal{T}_t^{s,0}f_0^{(s)} + \int_0^t \mathcal{T}_{t-t_1}^{s,0}\mathcal{C}_{s,s+1}^0\big(\mathcal{T}_{t_1}^{s+1,0}f_0^{(s+1)}(\cdot) \\
&\qquad + \int_0^{t_1} \mathcal{T}_{t_1-t_2}^{s+1,0}\mathcal{C}_{s+1,s+2}^0 f^{(s+2)}(t_2,\cdot)\,\mathrm{d}t_2\big)\,\mathrm{d}t_1 \\
&= \mathcal{T}_t^{s,0}f_0^{(s)} + \int_0^t \mathcal{T}_{t-t_1}^{s,0}\mathcal{C}_{s,s+1}^0\mathcal{T}_{t_1}^{s+1,0}f_0^{(s+1)}(\cdot)\,\mathrm{d}t_1 \\
&\qquad + \int_0^t \mathcal{T}_{t-t_1}^{s,0}\mathcal{C}_{s,s+1}^0\int_0^{t_1} \mathcal{T}_{t_1-t_2}^{s+1,0}\mathcal{C}_{s+1,s+2}^0 f^{(s+2)}(t_2,\cdot)\,\mathrm{d}t_2\,\mathrm{d}t_1.
\end{aligned}$$

Definition 34 (Notations for the integrated in time collision-transport operator of the Boltzmann hierarchy)**.** *For any positive integer s and any function of sequences $t \mapsto \left(f^{(s+1)}\right)_{s\geq 1}$ belonging to the space $X_{0,\widetilde{\beta},\widetilde{\mu^\alpha}}$, one will denote the image of its s-th term by the integrated in time collision-transport operator of the Boltzmann hierarchy :*

$$t \mapsto \int_0^t \mathcal{T}^{s,0}_{t-u}\mathcal{C}^0_{s,s+1} f^{(s+1)}(u,\cdot)\,\mathrm{d}u$$

as :

$$t \mapsto \left(\mathcal{I}^0_s f^{(s)}\right)(t,\cdot). \tag{10.7}$$

The k-th iterate of the integrated in time transport-collision operator :

$$t \mapsto \int_0^t \mathcal{T}^{s,0}_{t-t_1}\mathcal{C}^{N,\varepsilon}_{s,s+1}\int_0^{t_1}\mathcal{T}^{s,0}_{t_1-t_2}\mathcal{C}^0_{s+1,s+2}\cdots \\ \int_0^{t_{k-1}}\mathcal{T}^{s+k-1,0}_{t_{k-1}-t_k}\mathcal{C}^0_{s+k-1,s+k} f^{(s+k)}(t_k,\cdot)\,\mathrm{d}t_k\ldots\mathrm{d}t_2\,\mathrm{d}t_1,$$

will be denoted as :

$$t \mapsto \left(\mathcal{I}^0_{s,s+k-1} f^{(s+k)}\right)(t,\cdot). \tag{10.8}$$

The fact that, unlike the BBGKY hierarchy, the Boltzmann hierarchy is not studied after a conjugation in time, has absolutely no impact here. However, the fact that the Boltzmann hierarchy implies an infinite number of functions makes the result a bit harder to obtain. Indeed, for the conjugate BBGKY hierarchy, since one considers only a finite number N of particles, at some point, iterating the integrated in time transport-collision-transport forces to consider the last term of the hierarchy, which is simply given by the initial datum $h_N^{(N)} = f_{N,0}^{(N)}$. In the case of the Boltzmann hierarchy, the process will never stop.

Nevertheless, the analog of equation (10.3) of Proposition 9 will hold for the Boltzmann hierarchy. Heuristically, remembering that the Boltzmann hierarchy is obtained from the BBGKY hierarchy after sending the number of particles N to infinity (in the Boltzmann-Grad limit), one expects that the following formula for the Boltzmann hierarchy holds :

$$\begin{aligned} f^{(s)}(t,\cdot) &= \mathcal{T}^{s,0}_t f^{(s)}_0(\cdot) + \sum_{k=1}^{+\infty}\left(\mathcal{I}^0_{s,s+k-1} g_0^{(s+k)}\right)(t,\cdot) \\ &= g_0^{(s)}(t,\cdot) + \sum_{k=1}^{+\infty}\left(\mathcal{I}^0_{s,s+k-1} g_0^{(s+k)}\right)(t,\cdot), \end{aligned}$$

where $g_0^{(s)}(t,\cdot)$ denotes $\mathcal{T}^{s,0}_t f^{(s)}_0(\cdot)$.

Proposition 10 (Duhamel formula for the Boltzmann hierarchy)**.** *Let $\alpha = 1$ or 2. For any strictly positive number β_0, any (strictly positive in the case $\alpha = 2$) number μ_0, and for any sequence of initial data :*

$$F_0 = \left(f_0^{(s)}\right)_{s\geq 1}$$

belonging to the space $\boldsymbol{X}_{0,\beta_0,\mu_0^\alpha}$, the unique solution of the integrated form of the Boltzmann hierarchy with initial datum F_0 on the time interval $[0,T]$ (given by Theorem 6 page 259) is the sequence of functions (using the notations (10.7) *and* (10.8) *introduced in Definition 34 page 271) :*

$$F = t \mapsto \left(\mathcal{T}_t^{s,0} f_0^{(s)}(\cdot) + \sum_{k=1}^{+\infty} \left(\mathcal{I}^0_{s,s+k-1}\left(u \mapsto \mathcal{T}_u^{s+k,0} f_0^{(s+k)}\right)\right)(t,\cdot)\right)_{s\geq 1}. \quad (10.9)$$

Proof. First, one will prove that the function (of sequences) :

$$\widehat{F} = t \mapsto \left(\mathcal{T}_t^{s,0} f_0^{(s)}(\cdot) + \sum_{k=1}^{+\infty} \left(\mathcal{I}^0_{s,s+k-1}\left(u \mapsto \mathcal{T}_u^{s+k,0} f_0^{(s+k)}\right)\right)(t,\cdot)\right)_{s\geq 1}$$

is indeed well defined. To do so, one considers, for any positive integer n, the finite sum :

$$F^n = t \mapsto \left(\mathcal{T}_t^{s,0} f_0^{(s)}(\cdot) + \sum_{k=1}^{n} \left(\mathcal{I}^0_{s,s+k-1}\left(u \mapsto \mathcal{T}_u^{s+k,0} f_0^{(s+k)}\right)\right)(t,\cdot)\right)_{s\geq 1}.$$

This function F^n is itself well defined and is an element of $\widetilde{\mathbf{X}}_{0,\widetilde{\beta},\widetilde{\mu}^\alpha}$. Indeed, its first term

$$t \mapsto \left(\mathcal{T}_t^{s,0} f_0^{(s)}\right)_{s\geq 1}$$

is an element of $\widetilde{\mathbf{X}}_{0,\widetilde{\beta},\widetilde{\mu}^\alpha}$ because F_0 is an element of $\mathbf{X}_{0,\beta_0,\mu_0}$, and thanks to Lemma 20 page 239.
For the other terms, starting again from the fact that

$$t \mapsto \left(\mathcal{T}_t^{s,0} f_0^{(s)}\right)_{s\geq 1}$$

is an element of $\widetilde{\mathbf{X}}_{0,\widetilde{\beta},\widetilde{\mu}^\alpha}$, and using the fact that several iterations of the integrated in time collision-transport operator of the Boltzmann hierarchy, which is the integral term of the Boltzmann operator sends the space $\widetilde{\mathbf{X}}_{0,\widetilde{\beta},\widetilde{\mu}^\alpha}$ into itself (thanks to Lemma 21 in the case $\alpha = 2$, and thanks to Lemma 22 in the case $\alpha = 1$), one obtains that

$$\left(\mathcal{I}^0_{s,s+k-1}\left(u \mapsto \mathcal{T}_u^{s+k,0} f_0^{(s+k)}\right)\right)(t,\cdot)$$

is itself an element of $\widetilde{\mathbf{X}}_{0,\widetilde{\beta},\widetilde{\mu}^\alpha}$.
Moreover, the sequence (of functions of sequences...) :

$$F^n = \left(t \mapsto \left(\mathcal{T}_t^{s,0} f_0^{(s)}(\cdot) + \sum_{k=1}^{n} \left(\mathcal{I}^0_{s,s+k-1}\left(u \mapsto \mathcal{T}_u^{s+k,0} f_0^{(s+k)}\right)\right)(t,\cdot)\right)_{s\geq 1}\right)_{n\geq 1},$$

is a Cauchy sequence of the Banach space $\widetilde{\mathbf{X}}_{0,\widetilde{\beta},\widetilde{\mu}^{\alpha}}$ up to choose carefully the weights $\widetilde{\beta}$ and $\widetilde{\mu}$. Indeed, for any positive integers $p < q$, one has :

$$\begin{aligned}
F^q - F^p &= \Big(t \mapsto \Big(\mathcal{T}_t^{s,0} f_0^{(s)}(\cdot) + \sum_{k=1}^{q} \Big(\mathcal{I}^0_{s,s+k-1}\big(u \mapsto \mathcal{T}_u^{s+k,0} f_0^{(s+k)}\big)\Big)(t,\cdot)\Big)_{s\geq 1}\Big) \\
&\quad - \Big(t \mapsto \Big(\mathcal{T}_t^{s,0} f_0^{(s)}(\cdot) + \sum_{k=1}^{p} \Big(\mathcal{I}^0_{s,s+k-1}\big(u \mapsto \mathcal{T}_u^{s+k,0} f_0^{(s+k)}\big)\Big)(t,\cdot)\Big)_{s\geq 1}\Big) \\
&= t \mapsto \Big(\sum_{k=p+1}^{q} \Big(\mathcal{I}^0_{s,s+k-1}\big(u \mapsto \mathcal{T}_u^{s+k,0} f_0^{(s+k)}\big)\Big)(t,\cdot)\Big)_{s\geq 1},
\end{aligned}$$

so that, evaluating the distance between two terms of the sequence $\big(F^n\big)_{n\geq 1}$:

$$\begin{aligned}
&|||F^q - F^p|||_{0,\widetilde{\beta},\widetilde{\mu}^{\alpha}} \\
&\quad \leq \sum_{k=p+1}^{q} \Big|\Big|\Big| t \mapsto \Big(\Big(\mathcal{I}^0_{s,s+k-1}\big(u \mapsto \mathcal{T}_u^{s+k,0} f_0^{(s+k)}\big)\Big)(t,\cdot)\Big)_{s\geq 1} \Big|\Big|\Big|_{0,\widetilde{\beta},\widetilde{\mu}^{\alpha}},
\end{aligned}$$

and since one is considering the unique solution of the Boltzmann hierarchy given by Theorem 6 page 259, the weights $\widetilde{\beta}$ and $\widetilde{\mu}$ are given by

$$\widetilde{\beta} = \widetilde{\beta}_{\lambda}$$

and

$$\widetilde{\mu} = \widetilde{\mu}_{\lambda}$$

as in Lemma 24 page 257, so that the right-hand side of the last inequality being controlled (thanks to the inequality (8.19) of the same Lemma 24) with :

$$\begin{aligned}
|||F^q - F^p|||_{0,\widetilde{\beta}_{\lambda},\widetilde{\mu}^{\alpha}_{\lambda}} &\leq \sum_{k=p+1}^{q} \Big(\frac{1}{2}\Big)^k \Big|\Big|\Big| t \mapsto \big(\mathcal{T}_t^{s,0} f_0^{(s)}\big)_{s\geq 1} \Big|\Big|\Big|_{0,\widetilde{\beta}_{\lambda},\widetilde{\mu}^{\alpha}_{\lambda}} \\
&\leq \Big(\sum_{k=p+1}^{+\infty} \Big(\frac{1}{2}\Big)^k\Big) \|F_0\|_{0,\beta_0,\mu_0^{\alpha}} = \frac{1}{2^p} \|F_0\|_{0,\beta_0,\mu_0^{\alpha}}.
\end{aligned}$$

The sequence $\big(F^n\big)_{n\geq 1}$ converges then in the space $\widetilde{\mathbf{X}}_{0,\widetilde{\beta}_{\lambda},\widetilde{\mu}^{\alpha}_{\lambda}}$, which finishes to show that the quantity :

$$\widehat{F} = t \mapsto \Big(\mathcal{T}_t^{s,0} f_0^{(s)}(\cdot) + \sum_{k=1}^{+\infty} \Big(\mathcal{I}^0_{s,s+k-1}\big(u \mapsto \mathcal{T}_u^{s+k,0} f_0^{(s+k)}\big)\Big)(t,\cdot)\Big)_{s\geq 1}$$

is meaningful, as the limit of the sequence $\big(F_n\big)_{n\geq 1}$.
Now, one will show that the function $\widehat{F}$ verifies the fixed point equation

$$\mathfrak{E}_0\big(F_0, \widehat{F}\big) = \widehat{F},$$

where, one recalls, the Boltzmann operator is introduced in Definition 31 page 220. Therefore, using the uniqueness of the solution of this fixed point equation (as stated in Theorem 6 page 259), one will deduce that $F = \widehat{F}$ and the Lemma will be entirely proved.

One considers then the quantity $\mathfrak{E}_0\big(F_0, F^n\big)$, where the terms of the sequence F^n will be denoted $f^{n,(s)}$ for all $s \geq 1$, that is :

$$F^n = \big(f^{n,(s)}\big)_{s\geq 1}.$$

One has by definition of the Boltzmann operator $\mathfrak{E}_0$:

$$\mathfrak{E}_0\big(F_0, F^n\big) = t \mapsto \Big(\mathcal{T}_t^{s,0} f_0^{(s)} + \int_0^t \mathcal{T}_{t-u}^{s,0}\big(\mathcal{C}_{s,s+1}^0 f^{n,(s+1)}\big)(u,\cdot)\,\mathrm{d}u\Big)_{s\geq 1}$$

where $f^{n,(s+1)}$ is given by the expression

$$f^{n,(s+1)}(t,\cdot) = \mathcal{T}_t^{s,0} f_0^{(s)}(\cdot) + \sum_{k=1}^n \Big(\mathcal{I}_{s,s+k-1}^0\big(u \mapsto \mathcal{T}_u^{s+k,0} f_0^{(s+k)}\big)\Big)(t,\cdot),$$

so that

$$\begin{aligned}\mathfrak{E}_0\big(F_0, F^n\big) = t \mapsto \Big(\mathcal{T}_t^{s,0} f_0^{(s)}(\cdot) + \int_0^t \mathcal{T}_{t-u}^{s,0}\Big[\mathcal{C}_{s,s+1}^0 \\ \Big(v \mapsto \mathcal{T}_v^{s+1,0} f_0^{(s+1)}(\cdot) \\ + \sum_{k=1}^n \big(\mathcal{I}_{s+1,s+k}^0\big(\tau \mapsto \mathcal{T}_\tau^{s+k+1,0} f_0^{(s+k+1)}\big)\big)(v,\cdot)\Big)\Big](u,\cdot)\,\mathrm{d}u\Big)_{s\geq 1}\end{aligned}$$

then using the linearity of the transport and the collision operators :

$$\begin{aligned}\mathfrak{E}_0\big(F_0, F^n\big) = t \mapsto \Big(\mathcal{T}_t^{s,0} f_0^{(s)}(\cdot) \\ + \int_0^t \mathcal{T}_{t-u}^{s,0}\Big[\mathcal{C}_{s,s+1}^0\Big(v \mapsto \mathcal{T}_v^{s+1,0} f_0^{(s+1)}(\cdot)\Big)\Big](u,\cdot)\,\mathrm{d}u \\ + \sum_{k=1}^n \int_0^t \mathcal{T}_{t-u}^{s,0} \\ \Big[\mathcal{C}_{s,s+1}^0\Big(v \mapsto \Big(\mathcal{I}_{s+1,s+k}^0\big(\tau \mapsto \mathcal{T}_\tau^{s+k+1,0} f_0^{(s+k+1)}\big)\Big)(v,\cdot)\Big)\Big](u,\cdot)\,\mathrm{d}u\Big)_{s\geq 1},\end{aligned}$$

and now, recalling the definition of the free flow acting on functions (see Definition 15 page 83), that is :

$$\big(\mathcal{T}_t^{s,0} g^{(s)}\big)(u, Z_s) = g^{(s)}\big(u, T_{-t}^{s,0}(Z_s)\big),$$

one can rewrite the equation using the notation introduced in Definition 34 :

$$\begin{aligned}
\mathfrak{E}_0\big(F_0, F^n\big) &= t \mapsto \Big(\mathcal{T}_t^{s,0} f_0^{(s)}(\cdot) + \mathcal{I}_s^0\big(u \mapsto \mathcal{T}_u^{s+1,0} f_0^{(s+1)}\big)(t,\cdot) \\
&\qquad + \sum_{k=1}^{n} \mathcal{I}_s^0\Big(u \mapsto \big(\mathcal{I}_{s+1,s+k}^0\big(\tau \mapsto \mathcal{T}_\tau^{s+k+1,0} f_0^{(s+k+1)}\big)\big)(u,\cdot)\Big)(t,\cdot)\Big)_{s\geq 1} \\
&= t \mapsto \Big(\mathcal{T}_t^{s,0} f_0^{(s)}(\cdot) + \Big(\mathcal{I}_s^0\big(u \mapsto \mathcal{T}_u^{s+1,0} f_0^{(s+1)}\big)\Big)(t,\cdot) \\
&\qquad\qquad + \sum_{k=1}^{n} \Big(\mathcal{I}_{s,s+k}^0\big(u \mapsto \mathcal{T}_u^{s+k+1,0} f_0^{(s+k+1)}\big)\Big)(t,\cdot)\Big)_{s\geq 1} \\
&= t \mapsto \Big(\mathcal{T}_t^{s,0} f_0^{(s)}(\cdot) + \sum_{k=1}^{n+1} \Big(\mathcal{I}_{s,s+k-1}^0\big(u \mapsto \mathcal{T}_u^{s+k,0} f_0^{(s+k)}\big)\Big)(t,\cdot)\Big)_{s\geq 1}.
\end{aligned}$$

Finally, one has :

$$\mathfrak{E}_0\big(F_0, F^n\big) = F^{n+1},$$

so by letting n going to infinity, and thanks to the continuity of the Boltzmann operator associated to the initial datum F_0 $\mathfrak{E}_0(F_0,\cdot)$ (this continuity is a consequence of the contracting property (8.19) of the integral term of the Boltzmann operator, stated in Lemma 24 page 257), and thanks to the fact that :

$$F^{n+1} \underset{n\to+\infty}{\longrightarrow} \widehat{F}$$

one obtains that :

$$\mathfrak{E}_0\big(F_0, \widehat{F}\big) = \widehat{F},$$

concluding therefore the proof of the proposition. □

Chapter 11

Cut-off in parameters describing the dynamics of the particles

In the case of the Boltzmann hierarchy, thanks to the equation (10.9) of Proposition 10 page 272, one knows that the sequence of functions $F = t \mapsto \left(f^{(s)}\right)_{s\geq 1}$ belonging to $\widetilde{\mathbf{X}}_{0,\widetilde{\beta},\widetilde{\mu}^{\alpha}}$ is the solution of the Boltzmann hierarchy with initial datum F_0 if and only if F is defined by the formula

$$F = t \mapsto \Big(\mathcal{T}_t^{s,0} f_0^{(s)}(\cdot) + \sum_{k=1}^{+\infty} \Big(\mathcal{I}^0_{s,s+k-1}\big(u \mapsto \mathcal{T}_u^{s+k,0} f_0^{(s+k)}\big)\Big)(t,\cdot)\Big)_{s\geq 1}.$$

One will see that this infinite sum of iterations of transport-collision operators applied on functions decreasing quickly with respect to the velocity variable (since for every positive integer s and for every time $t \in [0,T]$, the function $Z_s \mapsto \mathcal{T}_t^{s,0} f_0^{(s)}(Z_s)$ belongs to the space $X_{0,s,\widetilde{\beta}(t)}$) can be divided in a principal part, and some remainders, taking into account the cases of respectively, a high number of iterations (see Section 11.1 page 277), configurations with a large energy (see Section 11.2 page 280), and iterations of the transport-collision operators at two successive times which are close (see Section 11.3 page 290), all of them being controlled independently and carefully in the following sections. The main term will be composed then by a finite number of iterations, clearly separated in time, of the transport-collision-transport operator applied on the elements of the initial data, after removing the configurations of the phase space with a large energy.

11.1 Cut-off in high number of collisions

This section is devoted to the cut-off in high number of collisions.

Proposition 11 (Cut-off in high number of collisions)**.** *Let $\alpha = 1$ or 2. Let β_0 be a strictly positive number and μ_0 be a real number (which has to be strictly positive too in the case when $\alpha = 2$). For any positive integer n, any positive integer N and any strictly positive number $\varepsilon > 0$ verifying the Boltzmann-Grad limit :*

$$N\varepsilon^{d-1} = 1,$$

and any couple of sequences of initial data $F_{N,0} = \big(f_{N,0}^{(s)}\big)_{1\leq s\leq N}$ and $F_0 = \big(f_0^{(s)}\big)_{s\geq 1}$ belonging respectively to $\boldsymbol{X}_{N,\varepsilon,\beta_0,\mu_0^\alpha}$ and $\boldsymbol{X}_{0,\beta_0,\mu_0^\alpha}$, the respective unique solutions $H_N \in \widetilde{\boldsymbol{X}}_{N,\varepsilon,\widetilde{\beta}_\lambda,\widetilde{\mu}_\lambda^\alpha}$ to the BBGKY hierarchy with initial datum $F_{N,0}$ and $F \in \widetilde{\boldsymbol{X}}_{0,\widetilde{\beta}_\lambda,\widetilde{\mu}_\lambda^\alpha}$ to the Boltzmann hierarchy with inital datum F_0 (where $\widetilde{\beta}_\lambda$, $\widetilde{\mu}_\lambda$, H_N and F are given by Theorem 6 page 259) satisfy :

$$\left\|\!\left| H_N - \Big(f_{N,0}^{(s)} + \sum_{k=1}^{n} \mathbb{1}_{s\leq N-k}\big(\mathcal{I}_{s,s+k-1}^{N,\varepsilon} f_{N,0}^{(s+k)}\big)\Big)_{1\leq s\leq N} \right\|\!\right|_{N,\varepsilon,\widetilde{\beta}_\lambda,\widetilde{\mu}_\lambda^\alpha} \leq \left(\frac{1}{2}\right)^n \|F_{N,0}\|_{N,\varepsilon,\beta_0,\mu_0^\alpha}, \quad (11.1)$$

and

$$\left\|\!\left| F - \Big(t \mapsto \mathcal{T}_t^{s,0} f_0^{(s)} + \sum_{k=1}^{n} \big(\mathcal{I}_{s,s+k-1}^{0}\big(u \mapsto \mathcal{T}_u^{s+k,0} f_0^{(s+k)}\big)\big)(t,\cdot)\Big)_{s\geq 1} \right\|\!\right|_{0,\widetilde{\beta}_\lambda,\widetilde{\mu}_\lambda^\alpha} \leq \left(\frac{1}{2}\right)^n \|F_0\|_{0,\beta_0,\mu_0^\alpha}. \quad (11.2)$$

Proof. The proof is exactly the same, whether $\alpha = 1$ or 2, or whether one considers the solution of the BBGKY hierarchy and one wants to prove the inequality (11.1), or the solution of the Boltzmann hierarchy and the inequality (11.2). The proof will be therefore written only in the case when $\alpha = 2$, for the solution of the Boltzmann hierarchy.
Using the Duhamel formula (10.9), one knows that the unique solution of the Boltzmann hierarchy in the space $\widetilde{\mathbf{X}}_{0,\widetilde{\beta}_\lambda,\widetilde{\mu}_\lambda^2}$ can be expressed as :

$$F = \lim_{m\to+\infty} \Big(t \mapsto \mathcal{T}_t^{s,0} f_0^{(s)} + \sum_{k=1}^{m} \big(\mathcal{I}_{s,s+k-1}^{0}\big(u \mapsto \mathcal{T}_u^{s+k,0} f_0^{(s+k)}\big)\big)(t,\cdot)\Big)_{s\geq 1}.$$

On the one hand, for any positive integer n given, the quantity :

$$\begin{aligned} Q_{n,m} = &\Big(t \mapsto \mathcal{T}_t^{s,0} f_0^{(s)} + \sum_{k=1}^{m} \big(\mathcal{I}_{s,s+k-1}^{0}\big(u \mapsto \mathcal{T}_u^{s+k,0} f_0^{(s+k)}\big)\big)(t,\cdot)\Big)_{s\geq 1} \\ &- \Big(t \mapsto \mathcal{T}_t^{s,0} f_0^{(s)} + \sum_{k=1}^{n} \big(\mathcal{I}_{s,s+k-1}^{0}\big(u \mapsto \mathcal{T}_u^{s+k,0} f_0^{(s+k)}\big)\big)(t,\cdot)\Big)_{s\geq 1} \end{aligned}$$

converges when m goes to infinity towards :

$$F - \Big(t \mapsto \mathcal{T}_t^{s,0} f_0^{(s)} + \sum_{k=1}^{n} \big(\mathcal{I}^0_{s,s+k-1}\big(u \mapsto \mathcal{T}_u^{s+k,0} f_0^{(s+k)}\big)\big)(t,\cdot)\Big)_{s\geq 1}$$

in the space $\widetilde{\mathbf{X}}_{0,\widetilde{\beta}_\lambda,\widetilde{\mu}^2_\lambda}$. On the other hand, this quantity $Q_{n,m}$ can be of course rewritten as :

$$Q_{n,m} = \Big(t \mapsto \sum_{k=n+1}^{m} \big(\mathcal{I}^0_{s,s+k-1}\big(u \mapsto \mathcal{T}_u^{s+k,0} f_0^{(s+k)}\big)\big)(t,\cdot)\Big)_{s\geq 1}.$$

The inequality (8.19) of the Contraction Lemma 24 page 257 (in general, depending on the situation, one has to use the appropriate lemma among Lemmas 23 or 24) applied to the sequence of conjugate initial data

$$\big(h^{(s)}\big)_{s\geq 1} = \big(t \mapsto \mathcal{T}_t^{s,0} f_0^{(s)}\big)_{s\geq 1},$$

which is an element of $\widetilde{\mathbf{X}}_{0,\tilde{\beta}_\lambda,\tilde{\mu}^2_\lambda}$ as Lemma 20 page 239 shows it, provides :

$$\begin{aligned}
|||Q_{n,m}|||_{0,\widetilde{\beta}_\lambda,\widetilde{\mu}^2_\lambda} &\leq \sum_{k=n+1}^{m} \Big|\Big|\Big| t \mapsto \Big(\mathcal{I}^0_{s,s+k-1}\big(u \mapsto \mathcal{T}_u^{s+k,0} f_0^{(s+k)}\big)(t,\cdot)\Big)_{s\geq 1} \Big|\Big|\Big|_{0,\widetilde{\beta}_\lambda,\widetilde{\mu}^2_\lambda} \\
&\leq \sum_{k=n+1}^{m} \frac{1}{2^k} \Big|\Big|\Big| t \mapsto \Big(\mathcal{T}_t^{s,0} f_0^{(s)}(t,\cdot)\Big)_{s\geq 1} \Big|\Big|\Big|_{0,\widetilde{\beta}_\lambda,\widetilde{\mu}^2_\lambda} \\
&\leq \Big(\sum_{k=n+1}^{+\infty} \frac{1}{2^k}\Big) \Big|\Big|\Big| t \mapsto \Big(\mathcal{T}_t^{s,0} f_0^{(s)}(t,\cdot)\Big)_{s\geq 1} \Big|\Big|\Big|_{0,\widetilde{\beta}_\lambda,\widetilde{\mu}^2_\lambda} \\
&\leq \frac{1}{2^n} \Big|\Big|\Big| t \mapsto \Big(\mathcal{T}_t^{s,0} f_0^{(s)}(t,\cdot)\Big)_{s\geq 1} \Big|\Big|\Big|_{0,\widetilde{\beta}_\lambda,\widetilde{\mu}^2_\lambda}.
\end{aligned}$$

Finally, the inequality (8.10) of Lemma 20 provides a control of the $|||\cdot|||_{0,\widetilde{\beta}_\lambda,\widetilde{\mu}^2_\lambda}$ norm of the transported sequence of initial data, that is :

$$\Big|\Big|\Big| t \mapsto \Big(\mathcal{T}_t^{s,0} f_0^{(s)}\Big)_{s\geq 1} \Big|\Big|\Big|_{0,\widetilde{\beta}_\lambda,\widetilde{\mu}^2_\lambda} \leq \Big|\Big|\big(f_0^{(s)}\big)_{s\geq 1}\Big|\Big|_{0,\beta_0,\mu_0^2}.$$

One obtains that :

$$|||Q_{n,m}|||_{0,\widetilde{\beta}_\lambda,\widetilde{\mu}^2_\lambda} \leq \frac{1}{2^n} \Big|\Big|\big(f_0^{(s)}\big)_{s\geq 1}\Big|\Big|_{0,\beta_0,\mu_0^2},$$

hence the lemma is proved, by letting m go to infinity. □

If one denotes the solution of the BBGKY hierarchy after the cut-off for more than n collisions as :

$$H_N^n = t \mapsto \Big(f_{N,0}^{(s)}(\cdot) + \sum_{k=1}^{n} \mathbb{1}_{s\leq N-k} \big(\mathcal{I}^{N,\varepsilon}_{s,s+k-1} f_{N,0}^{(s+k)}\big)(t,\cdot)\Big)_{1\leq s\leq N},$$

one has shown that, uniformly in the number of particles N and their diameter ε, in the Boltzmann-Grad limit :

$$|||H_N - H_N^n|||_{N,\varepsilon,\widetilde{\beta}_\lambda,\widetilde{\mu}_\lambda^\alpha} \leq \frac{1}{2^n} \|F_{N,0}\|_{N,\varepsilon,\beta_0,\mu_0^\alpha},$$

that is the solution of the BBGKY hierarchy can be approximated in the $|||\cdot|||_{N,\varepsilon,\widetilde{\beta}_\lambda,\widetilde{\mu}_\lambda^\alpha}$ norm as much as one wants, by only a finite sum of iterates of the integrated in time transport-collision-transport operator applied to the initial data.
One introduces therefore the following notations for the sake of simplicity.

Definition 35 (Cut-off in high number of collisions of the solutions of the hierarchies)**.** *For any sequence of initial data $F_{N,0} = \big(f_{N,0}^{(s)}\big)_{1\leq s\leq N}$ belonging to $\boldsymbol{X}_{N,\varepsilon,\beta_0,\mu_0^\alpha}$, one defines the* truncated in high number of collisions solution of the conjugate BBGKY hierarchy *associated to the initial datum $F_{N,0}$ as the function :*

$$H_N^n = t \mapsto \Big(f_{N,0}^{(s)}(\cdot) + \sum_{k=1}^{n} \mathbb{1}_{s\leq N-k}\big(\mathcal{I}_{s,s+k-1}^{N,\varepsilon} f_{N,0}^{(s+k)}\big)(t,\cdot)\Big)_{1\leq s\leq N}.$$

For any sequence of initial data $F_0 = \big(f_0^{(s)}\big)_{s\geq 1}$ belonging to $\boldsymbol{X}_{0,\beta_0,\mu_0^\alpha}$, one defines the truncated in high number of collisions solution of the Boltzmann hierarchy *associated to the initial datum F_0 as the function :*

$$F^n = t \mapsto \Big(\mathcal{T}_t^{s,0} f_0^{(s)} + \sum_{k=1}^{n} \big(\mathcal{I}_{s,s+k-1}^{0}\big(u \mapsto \mathcal{T}_u^{s+k,0} f_0^{(s+k)}\big)\big)(t,\cdot)\Big)_{s\geq 1}.$$

11.2 Cut-off in large energy

One now splits the sequences of initial data into two terms, with a condition on the energy of the initial configuration. Namely, one will write :

$$\big(f_{N,0}^{(s)}\big)_{1\leq s\leq N} = \big(f_{N,0}^{(s)}\mathbb{1}_{|V_s|\leq R}\big)_{1\leq s\leq N} + \big(f_{N,0}^{(s)}\mathbb{1}_{|V_s|>R}\big)_{1\leq s\leq N},$$

and one will control any finite sum of iterations of the integrated in time transport-collision-transport operator applied on the second term of this decomposition, that is the part of the sequence of initial data with large energy configurations.
As in the previous section, one introduces new notations in order to simplify the statement of the following lemma.

Definition 36 (Cut-off in large energy of the solutions of the hierarchies)**.** *For any sequence of initial data $F_{N,0} = \big(f_{N,0}^{(s)}\big)_{1\leq s\leq N}$ belonging to $\boldsymbol{X}_{N,\varepsilon,\beta_0,\mu_0^\alpha}$, one defines the* truncated in high number of collisions and in large energy solution

of the conjugate BBGKY hierarchy *associated to the initial datum* $F_{N,0}$ *as the function :*

$$H_N^{n,R} = t \mapsto \Big(f_{N,0}^{(s)}(\cdot) \mathbb{1}_{|V_s| \leq R} + \sum_{k=1}^{n} \mathbb{1}_{s \leq N-k} \big(\mathcal{I}_{s,s+k-1}^{N,\varepsilon} \big(f_{N,0}^{(s+k)} \mathbb{1}_{|V_{s+k}| \leq R} \big) \big)(t,\cdot) \Big)_{1 \leq s \leq N}.$$

For any sequence of initial data $F_0 = \big(f_0^{(s)} \big)_{s \geq 1}$ *belonging to* $\boldsymbol{X}_{0,\beta_0,\mu_0^\alpha}$*, one defines the* truncated in high number of collisions and in large energy solution of the Boltzmann hierarchy *associated to the initial datum* F_0 *as the function :*

$$F^{n,R} = t \mapsto \Big(\mathcal{T}_t^{s,0} f_0^{(s)}(\cdot) \mathbb{1}_{|V_s| \leq R} + \sum_{k=1}^{n} \big(\mathcal{I}_{s,s+k-1}^{N,\varepsilon} \big(u \mapsto \mathcal{T}_u^{s+k,0} f_{N,0}^{(s+k)} \mathbb{1}_{|V_{s+k}| \leq R} \big) \big)(t,\cdot) \Big)_{s \geq 1}.$$

Proposition 12 (Cut-off in large energy configurations)**.** *Let* $\alpha = 1$ *or* 2*. Let* β_0 *be a strictly positive number and* μ_0 *be a real number (which has to be strictly positive in the case* $\alpha = 2$*). There exists an affine, strictly positive, decreasing function* $\widetilde{\beta}' < \widetilde{\beta}$ *defined on* $[0,T]$ *(where* $\widetilde{\beta} = \widetilde{\beta}_\lambda$ *is given by Theorem 6 page 259) and two constants :*

$$C_1(d,\beta_0,\mu_0) \text{ and } C_2(d,\beta_0,\mu_0),$$

depending only on the dimension d *and on the numbers* β_0 *and* μ_0*, such that for any positive integer* n*, any strictly positive number* R*, any positive integer* N *and any strictly positive number* $\varepsilon > 0$ *verifying the Boltzmann-Grad limit :*

$$N\varepsilon^{d-1} = 1,$$

and any pair of sequences of initial data $F_{N,0} = \big(f_{N,0}^{(s)} \big)_{1 \leq s \leq N}$ *and* $F_0 = \big(f_0^{(s)} \big)_{s \geq 1}$ *belonging respectively to* $\boldsymbol{X}_{N,\varepsilon,\beta_0,\mu_0^\alpha}$ *and* $\boldsymbol{X}_{0,\beta_0,\mu_0^\alpha}$*, the truncated in high number of collisions solution of the BBGKY hierarchy* $H_N^n \in \widetilde{\boldsymbol{X}}_{N,\varepsilon,\widetilde{\beta}_\lambda,\widetilde{\mu}_\lambda^\alpha}$ *associated to the initial datum* $F_{N,0}$ *and the truncated in high number of collisions solution of the Boltzmann hierarchy* $F^n \in \widetilde{\boldsymbol{X}}_{0,\widetilde{\beta}_\lambda,\widetilde{\mu}_\lambda^\alpha}$ *associated to the inital datum* F_0 *(where* $\widetilde{\beta}_\lambda$ *and* $\widetilde{\mu}_\lambda$ *are given by Theorem 6 page 259) satisfy :*

$$\Big|\Big| \Big| H_N^n - H_N^{n,R} \Big| \Big|\Big|_{N,\varepsilon,\widetilde{\beta}',\widetilde{\mu}_\lambda^\alpha} \leq C_1 \exp\big(- C_2 R^2 \big) \| F_{N,0} \|_{N,\varepsilon,\widetilde{\beta}'(0),\mu_0^\alpha}, \tag{11.3}$$

and

$$\big|\big| \big| F^n - F^{n,R} \big| \big|\big|_{0,\widetilde{\beta}',\widetilde{\mu}_\lambda^\alpha} \leq C_1 \exp\big(- C_2 R^2 \big) \| F_0 \|_{0,\widetilde{\beta}'(0),\mu_0^\alpha}. \tag{11.4}$$

Proof. One will write the proof only in the case $\alpha = 2$ and for the BBGKY hierarchy, since for the other cases $\alpha = 1$ or for the Boltzmann hierarchy, the

arguments are the same.
Back to the control of the norm of the integral part of the BBGKY operator, that is the inequality (8.3) of Lemma 18 page 222 (in general, depending on the situation, one has to use the appropriate lemma among Lemmas 19, 18, 22 or 21), one knows that for every strictly positive numbers β_0, μ_0, T and λ such that $\beta_0 - \lambda T$ and $\mu_0 - \lambda T$ are strictly positive, the $|||\cdot|||_{N,\varepsilon,\widetilde{\beta}_\lambda,\widetilde{\mu}_\lambda^2}$ norm of the integral term of the BBGKY hierarchy, that is of :

$$t \mapsto \Big(\mathbb{1}_{s\leq N-1}\big(\mathcal{I}_s^{N,\varepsilon} f_{N,0}^{(s+1)}\big)(t,\cdot)\Big)_{1\leq s\leq N}$$

is smaller than :

$$C(d,N,\varepsilon)(\beta_0-\lambda T)^{-d/2}\exp\big(-3(\mu_0-\lambda T)\big) \\ \times \frac{\big(1+(\beta_0-\lambda T)^{-1/2}\big)}{\lambda}\Big|\Big|\Big|\big(f_N^{(s)}\big)_{1\leq s\leq N}\Big|\Big|\Big|_{N,\varepsilon,\widetilde{\beta}_\lambda,\widetilde{\mu}_\lambda^2}.$$

However, the setting is more restrictive, since in order to apply the fixed point theorem which provides the existence and the uniqueness of the solution H_N of the BBGKY hierarchy, one has chosen, according to Lemma 23 page 255 (again, in general, depending on the situation, one has to use the appropriate lemma among Lemmas 23 or 24), λ and T such that in addition one has :

$$C(d,N,\varepsilon)(\beta_0-\lambda T)^{-d/2}\exp\big(-3(\mu_0-\lambda T)\big)\frac{\big(1+(\beta_0-\lambda T)^{-1/2}\big)}{\lambda}\leq\frac{1}{2}.$$

Noticing that the condition :

$$\beta_0-\lambda T>0$$

holds true and that the function :

$$\begin{aligned}\beta\mapsto & C(d,N,\varepsilon)(\beta-\lambda T)^{-d/2}\exp\big(-3(\mu_0-\lambda T)\big)\frac{\big(1+(\beta-\lambda T)^{-1/2}\big)}{\lambda}\\ &= C(d,N,\varepsilon)\exp\big(-3(\mu_0-\lambda T)\big)(\beta-\lambda T)^{-d/2}\frac{\big(1+(\beta-\lambda T)^{-1/2}\big)}{\lambda}\end{aligned}$$

is continuous, there exists $\beta_0'<\beta_0$ such that :

$$\beta_0'-\lambda T>0,$$

and

$$C(d,N,\varepsilon)(\beta_0'-\lambda T)^{-d/2}\exp\big(-3(\mu_0-\lambda T)\big)\frac{\big(1+(\beta_0'-\lambda T)^{-1/2}\big)}{\lambda}<1.$$

One recalls here that since one works in the setting of the Boltzmann-Grad limit $N\varepsilon^{d-1}=1$, the choice of β_0' does not depend on the parameters N or ε, because one has here, as it is stated in Lemma 18 (in general, depending on the

situation, one has to use the appropriate lemma among Lemmas 19, 18, 22 or 21) :

$$C(d,N,\varepsilon) \leq C'(d).$$

One denotes then :

$$C(d,N,\varepsilon)(\beta_0' - \lambda T)^{-d/2} \exp\big(-3(\mu_0 - \lambda T)\big) \frac{\big(1 + (\beta_0' - \lambda T)^{-1/2}\big)}{\lambda} = \theta,$$

with $\theta \in]0,1[$, and one defines :

$$\widetilde{\beta}' = \begin{cases} [0,T] & \to \mathbb{R}_+^*, \\ \quad t & \mapsto \widetilde{\beta}'(t) = \beta_0' - \lambda t. \end{cases}$$

Using those notations, one has of course that for all $t \in [0,T]$:

$$0 < \widetilde{\beta}'(t) < \widetilde{\beta}_\lambda(t),$$

and one has in fact shown that the integral term of the BBGKY hierarchy, and in fact more generally its iterates :

$$\Big(t \mapsto \big(h_N^{(s)}(t,\cdot)\big)_{1\leq s\leq N}\Big) \\ \mapsto \Big(t \mapsto \Big(\mathbb{1}_{s\leq N-k}\big(\mathcal{I}_{s,s+k-1}^{N,\varepsilon}\big(u \mapsto h_N^{(s+k)}(u,\cdot)\big)\big)(t,\cdot)\Big)_{1\leq s\leq N}\Big)$$

are all well defined on the space $\widetilde{\mathbf{X}}_{N,\varepsilon,\widetilde{\beta}',\widetilde{\mu}_\lambda}$ (where the first weight $\widetilde{\beta}_\lambda$ has been replaced by $\widetilde{\beta}'$), take their values into this same space, and verify the contracting property :

$$\Big|\!\Big|\!\Big| t \mapsto \Big(\mathbb{1}_{s\leq N-k}\big(\mathcal{I}_{s,s+k-1}^{N,\varepsilon}\big(u \mapsto h_N^{(s+k)}(u,\cdot)\big)\big)(t,\cdot)\Big)_{1\leq s\leq N} \Big|\!\Big|\!\Big|_{N,\varepsilon,\widetilde{\beta}',\widetilde{\mu}_\lambda^2} \\ \leq \theta^k \Big|\!\Big|\!\Big| t \mapsto \big(h_N^{(s)}(t,\cdot)\big)_{1\leq s\leq N} \Big|\!\Big|\!\Big|_{N,\varepsilon,\widetilde{\beta}',\widetilde{\mu}_\lambda^2}. \quad (11.5)$$

Considering finally the difference of functions of sequences $H_N^n - H_N^{n,R}$, one has

$$H_N^n - H_N^{n,R} = \Big(t \mapsto \Big(f_{N,0}^{(s)}(\cdot) + \sum_{k=1}^n \mathbb{1}_{s\leq N-k}\big(\mathcal{I}_{s,s+k-1}^{N,\varepsilon}\big(f_{N,0}^{(s+k)}\big)\big)(t,\cdot)\Big)_{1\leq s\leq N}\Big) \\ - \Big(t \mapsto \Big(f_{N,0}^{(s)}(\cdot)\mathbb{1}_{|V_s|\leq R} + \sum_{k=1}^n \mathbb{1}_{s\leq N-k}\big(\mathcal{I}_{s,s+k-1}^{N,\varepsilon}\big(f_{N,0}^{(s+k)}\mathbb{1}_{|V_{s+k}|\leq R}\big)\big)(t,\cdot)\Big)_{1\leq s\leq N}\Big) \\ = \Big(t \mapsto \Big(f_{N,0}^{(s)}(\cdot)\mathbb{1}_{|V_s|> R} + \sum_{k=1}^n \mathbb{1}_{s\leq N-k}\big(\mathcal{I}_{s,s+k-1}^{N,\varepsilon}\big(f_{N,0}^{(s+k)}\mathbb{1}_{|V_{s+k}|> R}\big)\big)(t,\cdot)\Big)_{1\leq s\leq N}\Big)$$

by linearity of the iterates of the integrated in time transport-collision-transport operator, so that one has :

$$\begin{aligned}
\Big|\!\Big|\!\Big| H_N^n - H_N^{n,R} \Big|\!\Big|\!\Big|_{N,\varepsilon,\widetilde{\beta}',\widetilde{\mu}_\lambda^2} &\leq \Big|\!\Big|\!\Big| t \mapsto \big(f_{N,0}^{(s)} \mathbb{1}_{|V_s|>R}\big)_{1\leq s\leq N} \Big|\!\Big|\!\Big|_{N,\varepsilon,\widetilde{\beta}',\widetilde{\mu}_\lambda^2} \\
&+ \Big|\!\Big|\!\Big| t \mapsto \Big(\sum_{k=1}^n \mathbb{1}_{s\leq N-k} \big(\mathcal{I}_{s,s+k-1}^{N,\varepsilon}\big(f_{N,0}^{(s+k)} \mathbb{1}_{|V_{s+k}|>R}\big)\big)(t,\cdot)\Big)_{1\leq s\leq N} \Big|\!\Big|\!\Big|_{N,\varepsilon,\widetilde{\beta}',\widetilde{\mu}_\lambda^2} \\
&\leq \Big|\!\Big|\!\Big| t \mapsto \big(f_{N,0}^{(s)} \mathbb{1}_{|V_s|>R}\big)_{1\leq s\leq N} \Big|\!\Big|\!\Big|_{N,\varepsilon,\widetilde{\beta}',\widetilde{\mu}_\lambda^2} \\
&+ \sum_{k=1}^n \Big|\!\Big|\!\Big| t \mapsto \Big(\mathbb{1}_{s\leq N-k} \big(\mathcal{I}_{s,s+k-1}^{N,\varepsilon}\big(f_{N,0}^{(s+k)} \mathbb{1}_{|V_{s+k}|>R}\big)\big)(t,\cdot)\Big)_{1\leq s\leq N} \Big|\!\Big|\!\Big|_{N,\varepsilon,\widetilde{\beta}',\widetilde{\mu}_\lambda^2} \\
&\leq \Big|\!\Big|\!\Big| t \mapsto \big(f_{N,0}^{(s)} \mathbb{1}_{|V_s|>R}\big)_{1\leq s\leq N} \Big|\!\Big|\!\Big|_{N,\varepsilon,\widetilde{\beta}',\widetilde{\mu}_\lambda^2} \\
&+ \sum_{k=1}^n \theta^k \Big|\!\Big|\!\Big| t \mapsto \big(f_{N,0}^{(s)} \mathbb{1}_{|V_s|>R}\big)_{1\leq s\leq N} \Big|\!\Big|\!\Big|_{N,\varepsilon,\widetilde{\beta}',\widetilde{\mu}_\lambda^2}
\end{aligned}$$

thanks to the previous inequality (11.5), that is :

$$\Big|\!\Big|\!\Big| H_N^n - H_N^{n,R} \Big|\!\Big|\!\Big|_{N,\varepsilon,\widetilde{\beta}',\widetilde{\mu}_\lambda^2} \leq \frac{1}{1-\theta} \Big|\!\Big|\!\Big| t \mapsto \big(f_{N,0}^{(s)} \mathbb{1}_{|V_s|>R}\big)_{1\leq s\leq N} \Big|\!\Big|\!\Big|_{N,\varepsilon,\widetilde{\beta}',\widetilde{\mu}_\lambda^2}.$$

Applying then the inequality (8.2) of Lemma 17 page 221 to the constant function of sequences :

$$t \mapsto \big(f_{N,0}^{(s)} \mathbb{1}_{|V_s|>R}\big)_{1\leq s\leq N},$$

one immediately obtains that :

$$\Big|\!\Big|\!\Big| H_N^n - H_N^{n,R} \Big|\!\Big|\!\Big|_{N,\varepsilon,\widetilde{\beta}',\widetilde{\mu}_\lambda} \leq \frac{1}{1-\theta} \Big|\!\Big| \big(f_{N,0}^{(s)} \mathbb{1}_{|V_s|>R}\big)_{1\leq s\leq N} \Big|\!\Big|_{N,\varepsilon,\beta_0',\mu_0^2}. \tag{11.6}$$

Finally, since for every integer $1 \leq s \leq N$ and almost everywhere on the phase space of s particles $\mathcal{D}_s^\varepsilon$:

$$\begin{aligned}
&\big|\mathbb{1}_{|V_s|>R} f_{N,0}^{(s)}(Z_s)\big| \exp\Big(\frac{\beta_0'}{2}|V_s|^2\Big) \\
&\qquad = \mathbb{1}_{|V_s|>R} \exp\Big(\frac{\beta_0'-\beta_0}{2}|V_s|^2\Big) \big|f_{N,0}^{(s)}(Z_s)\big| \exp\Big(\frac{\beta_0}{2}|V_s|^2\Big),
\end{aligned}$$

and since $\beta_0' < \beta_0$, one gets

$$\begin{aligned}
&\big|\mathbb{1}_{|V_s|>R} f_{N,0}^{(s)}(Z_s)\big| \exp\Big(\frac{\beta_0'}{2}|V_s|^2\Big) \\
&\qquad \leq \mathbb{1}_{|V_s|>R} \exp\Big(\frac{\beta_0'-\beta_0}{2}R^2\Big) \big|f_{N,0}^{(s)}(Z_s)\big| \exp\Big(\frac{\beta_0}{2}|V_s|^2\Big),
\end{aligned}$$

so that

$$\big|\mathbb{1}_{|V_s|>R} f_{N,0}^{(s)}\big|_{\varepsilon,s,\beta_0'} \leq \exp\Big(\frac{\beta_0'-\beta_0}{2}R^2\Big) \big|f_{N,0}^{(s)}\big|_{\varepsilon,s,\beta_0}.$$

Multiplying the last inequality by $\exp(s^2\mu_0)$ and taking the supremum on the number of particles s, one obtains that, since by hypothesis the sequence of initial data belongs to the space $\mathbf{X}_{N,\varepsilon,\beta_0,\mu_0}$:

$$\left\| \left(f_{N,0}^{(s)} \mathbb{1}_{|V_s|>R}\right)_{1\leq s\leq N} \right\|_{N,\varepsilon,\beta_0',\mu_0^2} \leq \exp\left(\frac{\beta_0'-\beta_0}{2}R^2\right) \left\| \left(f_{N,0}^{(s)}\right)_{1\leq s\leq N} \right\|_{N,\varepsilon,\beta_0,\mu_0^2}. \tag{11.7}$$

Collecting the inequalities (11.6) and (11.7) together, and denoting :

$$C_1 = \frac{1}{1-\theta},$$

and

$$C_2 = \frac{\beta_0'-\beta_0}{2},$$

one obtains in the end the inequality stated in the lemma, since the constant θ depends only on β_0', μ_0, λ, T and the dimension, the quantities λ, T and β_0' depending themselves only on β_0, μ_0, and the dimension d. □

Remark 23. *It is important here to notice that in order to be able to compare solutions before and after a truncation in large energy and to obtain a small remainder as the parameter of cut-off becomes larger and larger, one has to pay a price in the strength of the* $|||\cdot|||_{\cdot,\widetilde{\beta},\widetilde{\mu}}$ *norm, in the sense that one is only able to compare solutions taken in the space with some weight* $\widetilde{\beta}$ *in a norm defined with a less strong weight* $\widetilde{\beta}' < \widetilde{\beta}$.

One proves here a simple result of propagation of finiteness of the energy by the integrated in time (transport-) collision-transport operators for the two hierarchies, which will be useful in the sequel, and in particular in the estimates related to the next cut-off.

Proposition 13 (Stability of the cut-off in large energy by the integral terms of the BBGKY and Boltzmann operators)**.** *Let* $\alpha = 1$ *or* 2. *For any* $T > 0$, *any* $\beta_0 > 0$, $\mu_0 > 0$ *and* $\lambda > 0$ *such that :*

$$\beta_0 - \lambda T > 0 \ (\textit{and } \mu_0 - \lambda T > 0 \textit{ if } \alpha = 2),$$

for any strictly positive number $R > 0$, *if one defines* $\widetilde{\beta}_\lambda$ *and* $\widetilde{\mu}_\lambda$ *as the functions*

$$\widetilde{\beta}_\lambda : \left\{ \begin{array}{rcl} [0,T] & \to & \mathbb{R}_+^*, \\ t & \mapsto & \widetilde{\beta}_\lambda(t) = \beta_0 - \lambda t, \end{array} \right.$$

and

$$\widetilde{\mu}_\lambda : \left\{ \begin{array}{rcl} [0,T] & \to & \mathbb{R}, \\ t & \mapsto & \widetilde{\mu}_\lambda(t) = \mu_0 - \lambda t, \end{array} \right.$$

one has :

- *for the integral term of the BBGKY operator, for any positive integer N and any strictly positive number ε, and for any function of sequences :*

$$t \mapsto \big(h_N^{(s)}(t,\cdot)\big)_{1\leq s\leq N}$$

belonging to the space $\widetilde{\boldsymbol{X}}_{N,\varepsilon,\widetilde{\beta}_\lambda,\widetilde{\mu}_\lambda^\alpha}$, one has the following equality :

$$\begin{aligned} t \mapsto & \Big(\mathbb{1}_{s\leq N-1}\big(\mathcal{I}_s^{N,\varepsilon}\big(h_N^{(s+1)}\mathbb{1}_{|V_{s+1}|\leq R}\big)\big)(t,\cdot)\Big)_{1\leq s\leq N} \\ & = t \mapsto \Big(\mathbb{1}_{|V_s|\leq R}\mathbb{1}_{s\leq N-1}\big(\mathcal{I}_s^{N,\varepsilon}\big(h_N^{(s+1)}\mathbb{1}_{|V_{s+1}|\leq R}\big)(t,\cdot)\big)\Big)_{1\leq s\leq N}. \end{aligned} \tag{11.8}$$

- *for the integral term of the Boltzmann operator, for any function of sequences :*

$$t \mapsto \big(f^{(s)}(t,\cdot)\big)_{s\geq 1}$$

belonging to the space $\widetilde{\boldsymbol{X}}_{0,\widetilde{\beta}_\lambda,\widetilde{\mu}_\lambda^\alpha}$, one has the following equality :

$$\begin{aligned} t \mapsto & \Big(\big(\mathcal{I}_s^0\big(f^{(s+1)}\mathbb{1}_{|V_{s+1}|\leq R}\big)\big)(t,\cdot)\Big)_{s\geq 1} \\ & = t \mapsto \Big(\mathbb{1}_{|V_s|\leq R}\big(\mathcal{I}_s^0\big(f^{(s+1)}\mathbb{1}_{|V_{s+1}|\leq R}\big)\big)(t,\cdot)\Big)_{s\geq 1}. \end{aligned} \tag{11.9}$$

Proof. First, one will provide a proof for the last inequality, concerning the Boltzmann operator. Secondly, a proof for the BBGKY operator will be proposed, separately, since it requires different arguments and tools.
The proof is quite straightforward in the case of the Boltzmann operator. Indeed, for every positive integer s, the following quantity is a classical integral of a continuous function. For any strictly positive number R and any positive integer s, one goes back to the definition of the integrated in time collision-transport operator of the Boltzmann hierarchy. It provides :

$$\begin{aligned} & \int_0^t \mathcal{T}_{t-u}^{s,0}\mathcal{C}_{s,s+1}^0\big(f^{(s+1)}(u,Z_s)\mathbb{1}_{|V_{s+1}|\leq R}\big)\,\mathrm{d}u \\ & \qquad = \sum_{i=1}^s \int_0^t \int_{\mathbb{S}_\omega^{d-1}\times\mathbb{R}^d_{v_{s+1}}} \big(\omega\cdot\big(v_{s+1}-\big(T_{u-t}^{s,0}(Z_s)\big)^{V,i}\big)\big)_+ \\ & \times\Big(\Big(f^{(s+1)}\mathbb{1}_{|V_{s+1}|\leq R}\Big)\Big(u,\Big(T_{u-t}^{s,0}(Z_s),\big(T_{u-t}^{s,0}(Z_s)\big)^{X,i},v_{s+1}\Big)'_{i,s+1}\Big) \\ & \qquad -\Big(f^{(s+1)}\mathbb{1}_{|V_{s+1}|\leq R}\Big)\Big(u,T_{u-t}^{s,0}(Z_s),\big(T_{u-t}^{s,0}(Z_s)\big)^{X,i},v_{s+1}\Big)\Big)\,\mathrm{d}\omega\,\mathrm{d}v_{s+1}\,\mathrm{d}u \end{aligned}$$

using first the definition of the free flow (see Definition 15 page 83) and then the definition of the collision operator of the Boltzmann hierarchy (see Definition

10 page 78), where the following quantity

$$\Big(T^{s,0}_{u-t}(Z_s), \big(T^{s,0}_{u-t}(Z_s)\big)^{X,i}, v_{s+1}\Big)'_{i,s+1}$$

denotes the configuration obtained from the configuration

$$\big(T^{s,0}_{u-t}(Z_s), \big(T^{s,0}_{u-t}(Z_s)\big)^{X,i}, v_{s+1}\big),$$

after applying the scattering operator (see Definition 1 page 51) for the pair of particles i and $s+1$, that is replacing the velocities

$$\big(T^{s,0}_{u-t}(Z_s)\big)^{V,i} \text{ and } v_{s+1},$$

by

$$\Big(\big(\big(T^{s,0}_{u-t}(Z_s)\big)^{V,i}\big)', v'_{s+1}\Big),$$

with ω as angular parameter.
Now since in the integrand appears the function $f^{(s+1)}$ multiplied with the indicator function

$$\mathbb{1}_{|V_{s+1}|\leq R},$$

if the vectors composed of the velocities of the configurations

$$\Big(T^{s,0}_{u-t}(Z_s), \big(T^{s,0}_{u-t}(Z_s)\big)^{X,i}, v_{s+1}\Big)'_{i,s+1} \text{ and } \big(T^{s,0}_{u-t}(Z_s), \big(T^{s,0}_{u-t}(Z_s)\big)^{X,i}, v_{s+1}\big)$$

have a norm larger than R, that is if the vectors

$$\big(\big(T^{s,0}_{u-t}(Z_s)\big)^{V,1}, \ldots, \big(\big(T^{s,0}_{u-t}(Z_s)\big)^{V,i}\big)', \ldots, v'_{s+1}\big) \text{ and } \big(T^{s,0}_{u-t}(Z_s)^V, v_{s+1}\big)$$

have a norm larger than R, then the integrand is zero.
But for any configuration $Z_s \in \big(\overline{\Omega^c} \times \mathbb{R}^d\big)^s$ such that $\big|V_s\big| > R$, and for any $v_{s+1} \in \mathbb{R}^d$, one has that, on the one hand :

$$\begin{aligned}\Big|\Big(\Big(T^{s,0}_{u-t}(Z_s),&\big(T^{s,0}_{u-t}(Z_s)\big)^{X,i}, v_{s+1}\Big)'_{i,s+1}\Big)^V\Big| \\ &= \Big|\Big(T^{s,0}_{u-t}(Z_s), \big(T^{s,0}_{u-t}(Z_s)\big)^{X,i}, v_{s+1}\Big)^V\Big|\end{aligned}$$

thanks to the conservation of the kinetic energy during collisions between the particles, and on the other hand :

$$\begin{aligned}\Big|\Big(T^{s,0}_{u-t}(Z_s), \big(T^{s,0}_{u-t}(Z_s)\big)^{X,i}, v_{s+1}\Big)^V\Big|^2 &= \Big|\Big(T^{s,0}_{u-t}(Z_s)\Big)^V\Big|^2 + |v_{s+1}|^2 \\ &= \big|V_s\big|^2 + |v_{s+1}|^2\end{aligned}$$

thanks to the conservation of the kinetic energy along the trajectory of the free transport with boundary conditions, so that :

$$\Big|\Big(T^{s,0}_{u-t}(Z_s), \big(T^{s,0}_{u-t}(Z_s)\big)^{X,i}, v_{s+1}\Big)^V\Big|^2 \geq \big|V_s\big|^2 > R^2.$$

So if $\left|V_s\right| > R$, both configurations :

$$\Big(T^{s,0}_{u-t}(Z_s), \big(T^{s,0}_{u-t}(Z_s)\big)^{X,i}, v_{s+1}\Big)'_{i,s+1}$$

and

$$\Big(T^{s,0}_{u-t}(Z_s), \big(T^{s,0}_{u-t}(Z_s)\big)^{X,i}, v_{s+1}\Big)$$

belong to the domain :

$$\big\{Z_{s+1} \ / \ \left|V_{s+1}\right| > R\big\},$$

so that the integrand :

$$\begin{aligned}
&\big(\omega \cdot \big(v_{s+1} - \big(T^{s,0}_{u-t}(Z_s)\big)^{V,i}\big)\big)_+ \\
&\qquad \times \Big(\Big(f^{(s+1)}\mathbb{1}_{|V_{s+1}|\leq R}\Big)\big(u, \Big(T^{s,0}_{u-t}(Z_s), \big(T^{s,0}_{u-t}(Z_s)\big)^{X,i}, v_{s+1}\Big)'_{i,s+1}\big) \\
&\qquad\qquad - \Big(f^{(s+1)}\mathbb{1}_{|V_{s+1}|\leq R}\Big)\big(u, T^{s,0}_{u-t}(Z_s), \big(T^{s,0}_{u-t}(Z_s)\big)^{X,i}, v_{s+1}\big)\Big)
\end{aligned}$$

is zero, for any Z_s such that $\left|V_s\right| > R$, for all $(\omega, v_{s+1}) \in \mathbb{S}^{d-1} \times \mathbb{R}^d$.
In other words, one has indeed shown the following equality, for any positive integer s, any time $t \in [0,T]$ and any configuration Z_s :

$$\begin{aligned}
&\int_0^t \mathcal{T}^{s,0}_{t-u}\mathcal{C}^0_{s,s+1}\big(f^{(s+1)}(u,Z_s)\mathbb{1}_{|V_{s+1}|\leq R}\big)\,\mathrm{d}u \\
&\qquad = \mathbb{1}_{|V_s|\leq R}\Big(\int_0^t \mathcal{T}^{s,0}_{t-u}\mathcal{C}^0_{s,s+1}\big(f^{(s+1)}(u,Z_s)\mathbb{1}_{|V_{s+1}|\leq R}\big)\,\mathrm{d}u\Big),
\end{aligned}$$

which is exactly the equality (11.9) stated in the lemma, in the case of the Boltzmann operator.

For the integral part of the BBGKY operator, one recalls that the transport-collision operator is *not* an integral, except for the special cases of regular functions $h^{(s+1)}$ (see Section 5 page 87). However, if the function $h^{(s+1)}$ is bounded by a weight which depends on time (as the elements of the sequences belonging to the spaces $\widetilde{\mathbf{X}}_{N,\varepsilon,\widetilde{\beta}_\lambda,\widetilde{\mu}_\lambda}$), one knows that the image by the integrated in time transport-collision-transport collision of such function can be controlled by a classical Lebesgue integral of the weight, as Lemma 15 page 188 shows it.
If the function of sequences :

$$t \mapsto \big(h^{(s)}_N(t,\cdot)\big)_{1\leq s\leq N}$$

belongs to the space $\widetilde{\mathbf{X}}_{N,\varepsilon,\widetilde{\beta}_\lambda,\widetilde{\mu}^\alpha_\lambda}$, it means in particular that for all integer $1 \leq s \leq N$ and for all $t \in [0,T]$, the quantity :

$$\left|h^{(s)}_N(t,Z_s)\right| \exp\Big(\frac{\widetilde{\beta}(t)}{2}|V_s|^2\Big)$$

is almost everywhere finite on the phase space $\mathcal{D}_s^\varepsilon$. It implies immediately that, for any strictly positive number R, the quantity

$$\mathbb{1}_{|V_s|\leq R}\big|h_N^{(s)}(t,Z_s)\big|\exp\Big(\frac{\widetilde{\beta}(t)}{2}|V_s|^2\Big)$$

is also almost everywhere finite. Then one can apply Lemma 15 to the function :

$$h^{(s+1)}=(t,Z_{s+1})\mapsto \mathbb{1}_{|V_{s+1}|\leq R}\big|h_N^{(s+1)}(t,Z_{s+1})\big|,$$

with the bound controlling the decrease in velocity :

$$g_{s+1}=(t,V_{s+1})\mapsto \mathbb{1}_{|V_{s+1}|\leq R}\exp\Big(-\frac{\widetilde{\beta}(t)}{2}|V_{s+1}|^2\Big),$$

which verifies all the assumptions of the lemma, since

$$(t,V_{s+1})\mapsto \exp\Big(-\frac{\widetilde{\beta}(t)}{2}|V_{s+1}|^2\Big)$$

also verifies all of them and bounds from above g_{s+1}. The inequality (6.4) of Lemma 15 provides then :

$$\begin{aligned}\Big|\big(\mathcal{I}_s^{N,\varepsilon}\big(h_N^{(s+1)}\mathbb{1}_{|V_{s+1}|>R}\big)\big)(t,\cdot)\Big| &\leq (N-s)\varepsilon^{d-1}\big|\mathbb{S}^{d-1}\big|\int_0^t\big|h_N^{(s+1)}(u,\cdot)\big|_{\varepsilon,s+1,\widetilde{\beta}(u)}\\ &\times\int_{\mathbb{R}^d}\mathbb{1}_{|V_{s+1}|\leq R}\big(\sum_{i=1}^s|v_i|+s|v_{s+1}|\big)\exp\Big(-\frac{\widetilde{\beta}(u)}{2}|V_{s+1}|^2\Big)\,\mathrm{d}v_{s+1}\,\mathrm{d}u.\end{aligned}$$

For

$$V_{s+1}=\big(V_s,v_{s+1}\big),$$

writing simply :

$$|V_{s+1}|^2=|V_s|^2+|v_{s+1}|^2\geq|V_s|^2,$$

one immediately sees that :

$$|V_s|>R\Rightarrow|V_{s+1}|>R,$$

so that in other words :

$$\{Z_{s+1}\in\mathcal{D}_{s+1}^\varepsilon\ /\ |V_s|>R\}\subset\{Z_{s+1}\in\mathcal{D}_{s+1}^\varepsilon\ /\ |V_{s+1}|>R\},$$

or again :

$$\mathbb{1}_{|V_s|>R}\leq\mathbb{1}_{|V_{s+1}|>R}.$$

One obtains then immediately :

$$
\begin{aligned}
&\mathbb{1}_{|V_s|>R}\Big|\big(\mathcal{I}_s^{N,\varepsilon}\big(h_N^{(s+1)}\mathbb{1}_{|V_{s+1}|>R}\big)\big)(t,\cdot)\Big| \\
&\quad\le (N-s)\varepsilon^{d-1}\big|\mathbb{S}^{d-1}\big|\mathbb{1}_{|V_s|>R}\int_0^t \big|h_N^{(s+1)}(u,\cdot)\big|_{\varepsilon,s+1,\widetilde{\beta}(u)} \\
&\qquad\times\int_{\mathbb{R}^d}\mathbb{1}_{|V_{s+1}|\le R}\big(\sum_{i=1}^s|v_i|+s|v_{s+1}|\big)\exp\Big(-\frac{\widetilde{\beta}(u)}{2}|V_{s+1}|^2\Big)\,\mathrm{d}v_{s+1}\,\mathrm{d}u \\
&\quad\le (N-s)\varepsilon^{d-1}\big|\mathbb{S}^{d-1}\big|\int_0^t \big|h_N^{(s+1)}(u,\cdot)\big|_{\varepsilon,s+1,\widetilde{\beta}(u)} \\
&\times\int_{\mathbb{R}^d}\big(\mathbb{1}_{|V_{s+1}|>R}\mathbb{1}_{|V_{s+1}|\le R}\big)\big(\sum_{i=1}^s|v_i|+s|v_{s+1}|\big)\exp\Big(-\frac{\widetilde{\beta}(u)}{2}|V_{s+1}|^2\Big)\,\mathrm{d}v_{s+1}\,\mathrm{d}u,
\end{aligned}
$$

which implies in particular that

$$
t\mapsto \mathbb{1}_{|V_s|>R}\big(\mathcal{I}_s^{N,\varepsilon}\big(h_N^{(s+1)}\mathbb{1}_{|V_{s+1}|\le R}\big)\big)(t,\cdot)
$$

is the zero function, and then

$$
\begin{aligned}
\big(\mathcal{I}_s^{N,\varepsilon}\big(h_N^{(s+1)}\mathbb{1}_{|V_{s+1}|\le R}\big)\big)(t,\cdot) &= \mathbb{1}_{|V_s|\le R}\big(\mathcal{I}_s^{N,\varepsilon}\big(h_N^{(s+1)}\mathbb{1}_{|V_{s+1}|\le R}\big)\big)(t,\cdot) \\
&\quad+\mathbb{1}_{|V_s|>R}\big(\mathcal{I}_s^{N,\varepsilon}\big(h_N^{(s+1)}\mathbb{1}_{|V_{s+1}|\le R}\big)\big)(t,\cdot) \\
&= \mathbb{1}_{|V_s|\le R}\big(\mathcal{I}_s^{N,\varepsilon}\big(h_N^{(s+1)}\mathbb{1}_{|V_{s+1}|\le R}\big)\big)(t,\cdot),
\end{aligned}
$$

which concludes the proof of the inequality (11.8), hence the lemma is entirely proved. □

11.3 Cut-off in small time difference between two collisions

One finishes by the cut-off in small time difference between two collisions, which will be the last of the preliminary cut-offs.
After two iterations of the integrated in time transport-collision operator, one gets the formula :

$$
\int_0^t \mathcal{T}_{t-t_1}^{s,0}\mathcal{C}_{s,s+1}^0\int_0^{t_1}\mathcal{T}_{t_1-t_2}^{s+1,0}\mathcal{C}_{s+1,s+2}^0 f^{(s+2)}(t_2,\cdot)\,\mathrm{d}t_2\,\mathrm{d}t_1
$$

It means that, after an application of the transport of s particles during a time $t-t_1$, one adds a particle. After that, one transports again the $s+1$ particles during a time t_1-t_2, and finally adds a last particle, so that one finishes with $s+2$ particles at time $t-(t-t_1)-(t_1-t_2)=t_1-t_2$. One sees that the length of the time interval between the two collisions is t_1-t_2. It can be arbitrarily small, which may cause troubles in the sequel. The cut-off in small difference between two collisions means that one will bound from below the difference t_1-t_2.

Let $\delta > 0$ be a small parameter. One recalls that, thanks to the Duhamel formula (see Propositions 9 page 268 and 10 page 272), the solutions of the hierarchies are written as sums of iterations of the integrated in time (transport-) collision-transport operator. For the case of the BBGKY hierarchy, each term is written, using the notations (10.2) introduced in Definition 33 page 268, in the following way :

$$\big(\mathcal{I}^{N,\varepsilon}_{s,s+k-1} f^{(s+k)}_{N,0}\big)(t,\cdot),$$

which is defined as :

$$\int_0^t \mathcal{T}^{s,\varepsilon}_{-t_1} \mathcal{C}^{N,\varepsilon}_{s,s+1} \mathcal{T}^{s+1,\varepsilon}_{t_1} \int_0^{t_1} \mathcal{T}^{s,\varepsilon}_{-t_2} \mathcal{C}^{N,\varepsilon}_{s+1,s+2} \mathcal{T}^{s+2,\varepsilon}_{t_2} \dots$$
$$\int_0^{t_{k-1}} \mathcal{T}^{s+k-1,\varepsilon}_{-t_k} \mathcal{C}^{N,\varepsilon}_{s+k-1,s+k} \mathcal{T}^{s+k,\varepsilon}_{t_k} f^{(s+k)}_{N,0} \, \mathrm{d}t_k \dots \mathrm{d}t_2 \, \mathrm{d}t_1.$$

If one wants to separate in time, by a lower bound of size δ, the applications of the collision operators $\mathcal{C}^{N,\varepsilon}_{s+j,s+j+1}$, it means that one asks that :

- $t_1 - t_2 \geq \delta$ in order to separate in time $\mathcal{C}^{N,\varepsilon}_{s,s+1}$ and $\mathcal{C}^{N,\varepsilon}_{s+1,s+2}$,
- $t_2 - t_3 \geq \delta$ in order to separate $\mathcal{C}^{N,\varepsilon}_{s+1,s+2}$ and $\mathcal{C}^{N,\varepsilon}_{s+2,s+3}$,
- ...
- and finally, $t_{k-1} - t_k \geq \delta$, in order to separate $\mathcal{C}^{N,\varepsilon}_{s+k-2,s+k-1}$ and $\mathcal{C}^{N,\varepsilon}_{s+k-1,s+k}$.

Taking all those conditions and starting by the last one, one sees that in particular, they imply that

- $t_{k-1} \geq \delta$,
- $t_{k-2} \geq \delta + t_{k-1} \geq 2\delta$
- ...
- $t_1 \geq (k-1)\delta$.

It will be useful in the following to require in addition a time separation between the time t of evaluation of the solution of the hierarchy (this time t appearing in the upper bound of the first integral defining the solution of the hierarchy in the Duhamel formula (10.3) page 268), and the first application of the collision operator $\mathcal{C}^{N,\varepsilon}_{s,s+1}$. In other words, one will set the additional condition

$$t - t_1 \geq \delta.$$

Regrouping all the conditions, one sees that in particular

$$t \geq k\delta.$$

This separation in time requires then to cut the time domain on which one integrates the transport-collision-transport operators, and to control the size of the removed small time intervals.
One introduces therefore new notations. They are rather natural in the sense that the additional index and exponent added in this new notation correspond to the new bounds of the time interval on which the (transport-) collision-transport operator is integrated.

Definition 37 (Notations for the integrated in time, truncated in time, transport-collision-transport operator of the BBGKY hierarchy)**.** *For any positive integer N, any strictly positive number ε, any function $t \mapsto \big(f_N^{(s)}(t,\cdot)\big)_{1\leq s\leq N}$ belonging to the space $\widetilde{\boldsymbol{X}}_{N,\varepsilon,\widetilde{\beta},\widetilde{\mu}^\alpha}$, any integer $1 \leq s \leq N$, and any mesurable functions*

$$a : \left\{ \begin{array}{ll} \mathbb{R}_+ & \to \mathbb{R}_+, \\ t & \mapsto a(t), \end{array} \right. \quad \text{and } b : \left\{ \begin{array}{ll} \mathbb{R}_+ & \to \mathbb{R}_+, \\ t & \mapsto b(t), \end{array} \right.$$

one will denote its image by the integrated in time transport-collision-transport operator, truncated in time, of the BBGKY hierarchy (with $[a,b] \subset [0,t]$) :

$$t \mapsto \int_0^t \mathbb{1}_{u\geq a(t)} \mathbb{1}_{u\leq b(t)} \mathcal{T}_{-u}^{s,\varepsilon} \mathcal{C}_{s,s+1}^{N,\varepsilon} \mathcal{T}_u^{s+1,\varepsilon} f_N^{(s+1)}(u,\cdot)\,\mathrm{d}u$$

as :

$$t \mapsto \big(\big(\mathcal{I}_s^{N,\varepsilon}\big)_{a(t)}^{b(t)} f_N^{(s+1)}\big)(t,\cdot). \tag{11.10}$$

One will also consider additional conditions for the cut-off in the time variable, namely :

$$t \mapsto \mathbb{1}_{t<c} \int_a^b \mathcal{T}_{-u}^{s,\varepsilon} \mathcal{C}_{s,s+1}^{N,\varepsilon} \mathcal{T}_u^{s+1,\varepsilon} f_N^{(s+1)}(u,\cdot)\,\mathrm{d}u$$

and

$$t \mapsto \mathbb{1}_{t\geq c} \int_a^b \mathcal{T}_{-u}^{s,\varepsilon} \mathcal{C}_{s,s+1}^{N,\varepsilon} \mathcal{T}_u^{s+1,\varepsilon} f_N^{(s+1)}(u,\cdot)\,\mathrm{d}u$$

which will be respectively denoted as :

$$t \mapsto \big({}_c^- \big(\mathcal{I}_s^{N,\varepsilon}\big)_a^b f_N^{(s+1)}\big)(t,\cdot) \quad \text{and} \quad t \mapsto \big({}_c^+ \big(\mathcal{I}_s^{N,\varepsilon}\big)_a^b f_N^{(s+1)}\big)(t,\cdot). \tag{11.11}$$

Remark 24. *When there is an additional condition $t \leq c$, the dependency on t of a and b is dropped in the notations. The reason is that in practice, in the case of this additional condition, the two first conditions $u \geq a$ and $u \leq b$ will be given by functions a and b which do not depend on t.*

One introduces the same notations for the Boltzmann hierarchy.

Definition 38 (Notations for the integrated in time, truncated in time, collision-transport operator of the Boltzmann hierarchy)**.** *For any function of sequences $t \mapsto \big(f^{(s)}(t,\cdot)\big)_{s\geq 1}$ belonging to the space $\widetilde{\boldsymbol{X}}_{0,\widetilde{\beta},\widetilde{\mu}^\alpha}$, any positive integer*

s, and any mesurable functions

$$a : \left\{ \begin{array}{ll} \mathbb{R}_+ & \to \mathbb{R}_+, \\ t & \mapsto a(t), \end{array} \right. \quad \textit{and } b : \left\{ \begin{array}{ll} \mathbb{R}_+ & \to \mathbb{R}_+, \\ t & \mapsto b(t), \end{array} \right.$$

one will denote its image by the integrated in time collision-transport operator, truncated in time, of the Boltzmann hierarchy :

$$t \mapsto \int_0^t \mathbb{1}_{u \geq a(t)} \mathbb{1}_{u \leq b(t)} \mathcal{T}_{t-u}^{s,0} \mathcal{C}_{s,s+1}^0 f^{(s+1)}(u, \cdot) \, \mathrm{d}u$$

as :

$$t \mapsto \big(\big(\mathcal{I}_s^0\big)_{a(t)}^{b(t)} f^{(s+1)} \big)(t, \cdot). \tag{11.12}$$

Similarly as for the BBGKY hierarchy, one will denote :

$$t \mapsto \mathbb{1}_{t<c} \int_a^b \mathcal{T}_{-u}^{s,0} \mathcal{C}_{s,s+1}^0 \mathcal{T}_u^{s+1,0} f^{(s+1)}(u, \cdot) \, \mathrm{d}u$$

and

$$t \mapsto \mathbb{1}_{t \geq c} \int_a^b \mathcal{T}_{-u}^{s,0} \mathcal{C}_{s,s+1}^0 \mathcal{T}_u^{s+1,0} f^{(s+1)}(u, \cdot) \, \mathrm{d}u$$

respectively as :

$$t \mapsto \big({}_c^- \big(\mathcal{I}_s^0\big)_a^b f^{(s+1)} \big)(t, \cdot) \quad \textit{and} \quad t \mapsto \big({}_c^+ \big(\mathcal{I}_s^0\big)_a^b f^{(s+1)} \big)(t, \cdot) \tag{11.13}$$

The decomposition that one has in mind at this point of the work can be pictured thanks to the following Figure 11.1.

One decomposes then (in the example here, in the case of the BBGKY hierarchy) each iterations of the integrated in time transport-collision-transport operator as follows :

$$\begin{aligned} \big(\mathcal{I}_{s,s+k-1}^{N,\varepsilon} \big(f^{(s+k)}\big)\big)(t, \cdot) = \int_0^t \mathcal{T}_{-t_1}^{s,\varepsilon} \mathcal{C}_{s,s+1}^{N,\varepsilon} \mathcal{T}_{t_1}^{s+1,\varepsilon} \int_0^{t_1} \mathcal{T}_{-t_2}^{s+1,\varepsilon} \mathcal{C}_{s+1,s+2}^{N,\varepsilon} \mathcal{T}_{t_2}^{s+2,\varepsilon} \\ \dots \int_0^{t_{k-1}} \mathcal{T}_{-t_k}^{s+k-1,\varepsilon} \mathcal{C}_{s+k-1,s+k}^{N,\varepsilon} \mathcal{T}_{t_k}^{s+k,\varepsilon} f_{N,0}^{(s+k)} \, \mathrm{d}t_k \dots \mathrm{d}t_2 \, \mathrm{d}t_1 \end{aligned}$$

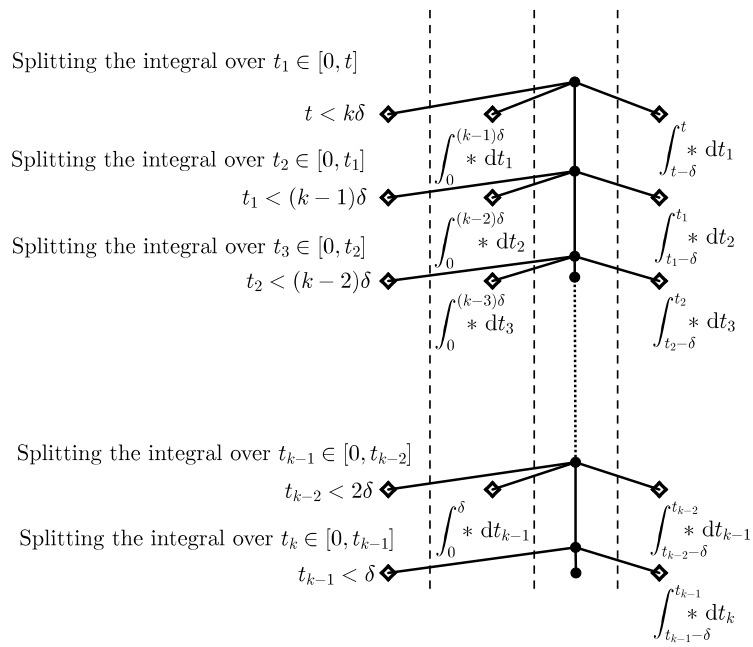

Figure 11.1: The different decompositions of the initial integral $\mathcal{I}_{s,s+k-1}^{\cdot} f_{\cdot}^{(s+k)}$ into the separated in time collisions part and the remainders, represented by squares on the figure

$$
\begin{aligned}
= \Big(& \mathbb{1}_{t<k\delta} \int_0^t \mathcal{T}_{-t_1}^{s,\varepsilon} \mathcal{C}_{s,s+1}^{N,\varepsilon} \mathcal{T}_{t_1}^{s+1,\varepsilon} \\
& \quad + \mathbb{1}_{t\geq k\delta} \Big(\Big[\int_0^{(k-1)\delta} + \int_{(k-1)\delta}^{t-\delta} + \int_{t-\delta}^{t} \Big] \mathcal{T}_{-t_1}^{s,\varepsilon} \mathcal{C}_{s,s+1}^{N,\varepsilon} \mathcal{T}_{t_1}^{s+1,\varepsilon} \Big) \Big) \\
& \Big(\mathbb{1}_{t_1<(k-1)\delta} \int_0^{t_1} \mathcal{T}_{-t_2}^{s+1,\varepsilon} \mathcal{C}_{s+1,s+2}^{N,\varepsilon} \mathcal{T}_{t_2}^{s+2,\varepsilon} \\
& \quad + \mathbb{1}_{t_1\geq (k-1)\delta} \Big(\Big[\int_0^{(k-2)\delta} + \int_{(k-2)\delta}^{t_1-\delta} + \int_{t_1-\delta}^{t_1} \Big] \mathcal{T}_{-t_2}^{s+1,\varepsilon} \mathcal{C}_{s+1,s+2}^{N,\varepsilon} \mathcal{T}_{t_2}^{s+2,\varepsilon} \Big) \Big) \\
& \qquad \cdots \\
& \Big(\mathbb{1}_{t_{k-1}<\delta} \int_0^{t_{k-1}} \mathcal{T}_{-t_k}^{s+k-1,\varepsilon} \mathcal{C}_{s+k-1,s+k}^{N,\varepsilon} \mathcal{T}_{t_k}^{s+k,\varepsilon} \\
& \quad + \mathbb{1}_{t_{k-1}\geq \delta} \Big(\Big[\int_0^{t_{k-1}-\delta} + \int_{t_{k-1}-\delta}^{t_{k-1}} \Big] \mathcal{T}_{-t_k}^{s+k-1,\varepsilon} \mathcal{C}_{s+k-1,s+k}^{N,\varepsilon} \mathcal{T}_{t_k}^{s+k,\varepsilon} \Big) \Big) \\
& \qquad\qquad\qquad\qquad f_{N,0}^{(s+k)}(t_k,\cdot) \, \mathrm{d}t_k \, \mathrm{d}t_{k-1} \ldots \mathrm{d}t_1,
\end{aligned}
$$

so that in fact, considering each of the j iterations (with $1 \leq j \leq k$), using the new notations introduced in Definition 33 page 267, and 37 page 292, one decomposes

$$\begin{aligned}\big(\mathcal{I}^{N,\varepsilon}_{s+j-1} f^{(s+j)}\big)(t_{j-1},\cdot) =& \mathbb{1}_{t_{j-1}<a_{j,2}} \big(\mathcal{I}^{N,\varepsilon}_{s+j-1} f^{(s+j)}\big)(t_{j-1},\cdot) \\ &+ \mathbb{1}_{t_{j-1}\geq a_{j,2}} \sum_{l=2}^{4} \big(\big(\mathcal{I}^{N,\varepsilon}_{s+j-1}\big)^{a_{j,l+1}}_{a_{j,l}} f^{(s+j)}\big)(t_{j-1},\cdot),\end{aligned}$$

with, for the last iteration, that is for application of the integrated in time transport-collision-transport $\displaystyle\int_0^t \mathcal{T}^{s,\varepsilon}_{-t_1} \mathcal{C}^{N,\varepsilon}_{s,s+1} \mathcal{T}^{s+1,\varepsilon}_{t_1}$ to distribution functions of $s+1$ particles (the indices are inversed with the order of the composition of the operators) :

$$c_1 = k\delta, \quad a_{1,1} = 0, \quad a_{1,2} = (k-1)\delta, \quad a_{1,3} = t-\delta, \quad a_{1,4} = t,$$

and, in the general case $1 \leq j \leq k$:

$$c_j = (k-j+1)\delta, \quad a_{j,1} = 0, \quad a_{j,2} = (k-j)\delta, \quad a_{j,3} = t_{j-1}-\delta, \quad a_{j,4} = t_{j-1}$$

(where t_0 denotes t).
In order to obtain a more synthetic expression of the decomposed integrated in time collision-transport operator, one writes :

$$\begin{aligned}\big(\mathcal{I}^{N,\varepsilon}_{s+j-1} f^{(s+j)}\big)(t_{j-1},\cdot) =& \mathbb{1}_{t_{j-1}<c_j} \sum_{l=1}^{3} \big(\big(\mathcal{I}^{N,\varepsilon}_{s+j-1}\big)^{a_{j,l+1}}_{a_{j,l}} f^{(s+j)}\big)(t_{j-1},\cdot) \\ &+ \mathbb{1}_{t_{j-1}\geq c_j} \sum_{l=1}^{3} \big(\big(\mathcal{I}^{N,\varepsilon}_{s+j-1}\big)^{a_{j,l+1}}_{a_{j,l}} f^{(s+j)}\big)(t_{j-1},\cdot),\end{aligned}$$

that is, using the notation (11.11) of Definition 37 :

$$\big(\mathcal{I}^{N,\varepsilon}_{s+j-1} f^{(s+j)}\big)(t_{j-1},\cdot) = \sum_{\pm} \sum_{l=1}^{3} \Big(\big(\tbinom{\pm}{c_j} \big(\mathcal{I}^{N,\varepsilon}_{s}\big)^{a_{j,l+1}}_{a_{j,l}}\big) f^{(s+j)}\Big)(t_{j-1},\cdot).$$

Decomposing the terms describing the solution of the BBGKY hierarchy given by the Duhamel formula, one obtains

$$\begin{aligned}&\big(\mathcal{I}^{N,\varepsilon}_{s,s+k-1} f^{(s+k)}_{N,0}\big)(t,\cdot) \\ &\quad= \Big(\big(\mathcal{I}^{N,\varepsilon}_{s} \circ \mathcal{I}^{N,\varepsilon}_{s+1} \circ \cdots \circ \mathcal{I}^{N,\varepsilon}_{s+k-1}\big) f^{(s+k)}_{N,0}\Big)(t,\cdot) \\ &\quad= \Big(\Big(\sum_{\pm_1} \sum_{l_1=1}^{3} \Big(\tbinom{\pm_1}{c_1} \big(\mathcal{I}^{N,\varepsilon}_{s}\big)^{a_{1,l_1+1}}_{a_{1,l_1}}\Big)\Big) \circ \Big(\sum_{\pm_2} \sum_{l_2=1}^{3} \Big(\tbinom{\pm_2}{c_2} \big(\mathcal{I}^{N,\varepsilon}_{s+1}\big)^{a_{2,l_2+1}}_{a_{2,l_2}}\Big)\Big) \\ &\qquad\circ \cdots \circ \Big(\sum_{\pm_k} \sum_{l_k=1}^{3} \Big(\tbinom{\pm_k}{c_k} \big(\mathcal{I}^{N,\varepsilon}_{s+k-1}\big)^{a_{k,l_k+1}}_{a_{k,l_k}}\Big)\Big) f^{(s+k)}_{N,0}\Big)(t,\cdot),\end{aligned}$$

and then using the linearity of the integrated in time collision-transport operators :

$$\begin{aligned}
&\big(\mathcal{I}^{N,\varepsilon}_{s,s+k-1} f^{(s+k)}_{N,0}\big)(t,\cdot)\\
&\quad = \Big(\sum_{\pm_1}\sum_{l_1=1}^{3}\sum_{\pm_2}\sum_{l_2=1}^{3}\cdots\sum_{\pm_k}\sum_{l_k=1}^{3}\Big(\Big({}^{\pm_1}_{c_1}\big(\mathcal{I}^{N,\varepsilon}_{s}\big)^{a_{1,l_1+1}}_{a_{1,l_1}}\Big)\circ\Big({}^{\pm_2}_{c_2}\big(\mathcal{I}^{N,\varepsilon}_{s+1}\big)^{a_{2,l_2+1}}_{a_{2,l_2}}\Big)\\
&\qquad\qquad \circ\cdots\circ\Big({}^{\pm_k}_{c_k}\big(\mathcal{I}^{N,\varepsilon}_{s+k-1}\big)^{a_{k,l_k+1}}_{a_{k,l_k}}\Big)\Big) f^{(s+k)}_{N,0}\Big)(t,\cdot).
\end{aligned}$$

Of course, the idea is to show that, for small parameters of separation in time δ, the contribution of the elements of the decomposition implying an integral with a time domain of size of order δ provide only a small contribution. In other words, one expects that only the parts of the form :

$$\mathbb{1}_{t_{k-j}\geq j\delta}\int_{j\delta}^{t_{k-j}-\delta}\mathrm{d}t_{k-j+1}$$

provide a significant contribution to the integrated in time collision-transport operator. Hence, for the iteration $\mathcal{I}^{0}_{s,s+k-1}$, one expects that only the term :

$$\begin{aligned}
&\mathbb{1}_{t\geq(k-1)\delta}\int_{(k-1)\delta}^{t}\mathcal{T}^{s,\varepsilon}_{-t_1}\mathcal{C}^{N,\varepsilon}_{s,s+1}\mathcal{T}^{s+1,\varepsilon}_{t_1}\Big(\mathbb{1}_{t_1\geq(k-1)\delta}\int_{(k-2)\delta}^{t_1-\delta}\mathcal{T}^{s+1,\varepsilon}_{-t_2}\mathcal{C}^{N,\varepsilon}_{s+1,s+2}\mathcal{T}^{s+2,\varepsilon}_{t_2}\\
&\quad\cdots\Big(\mathbb{1}_{t_{k-1}\geq\delta}\int_{0}^{t_{k-1}-\delta}\mathcal{T}^{s+k-1,\varepsilon}_{-t_k}\mathcal{C}^{N,\varepsilon}_{s+k-1,s+k}\mathcal{T}^{s+k,\varepsilon}_{t_k}f^{(s+k)}_{N,0}(t_k,\cdot)\,\mathrm{d}t_k\Big)\cdots\Big)\,\mathrm{d}t_1,
\end{aligned}$$

that is, using the notations (11.10) and (11.11) of Definition 37,

$$\Big(\Big({}^{+}_{c_1}\big(\mathcal{I}^{0}_{s}\big)^{a_{1,3}}_{a_{1,2}}\Big)\circ\Big({}^{+}_{c_2}\big(\mathcal{I}^{0}_{s+1}\big)^{a_{2,3}}_{a_{2,2}}\Big)\circ\cdots\circ\Big({}^{+}_{c_k}\big(\mathcal{I}^{0}_{s+k-1}\big)^{a_{k,3}}_{a_{k,2}}\Big)\Big)f^{(s+k)}_{0}$$

will play a significant role.
This will be proved along the next lemmas, and one is led to introduce the following notations.

Definition 39 (Iteration of the integrated in time (transport-) collision-transport operator with collisions separated in time)**.** *For any function $f^{(s+k)}_{N,0}$ belonging to $X_{\varepsilon,s+k,\beta_0}$, one defines the* k-th truncated in high number of collisions, large energy and small separation in time between collisions of the BBGKY hierarchy *as the function :*

$$\begin{aligned}
\mathcal{I}^{N,\varepsilon,\delta}_{s,s+k-1}(f^{(s+k)}_{N,0}) = &\Big(\Big({}^{+}_{c_1}\big(\mathcal{I}^{N,\varepsilon}_{s}\big)^{a_{1,3}}_{a_{1,2}}\Big)\circ\Big({}^{+}_{c_2}\big(\mathcal{I}^{N,\varepsilon}_{s+1}\big)^{a_{2,3}}_{a_{2,2}}\Big)\\
&\circ\cdots\circ\Big({}^{+}_{c_k}\big(\mathcal{I}^{N,\varepsilon}_{s+k-1}\big)^{a_{k,3}}_{a_{k,2}}\Big)\Big)f^{(s+k)}_{N,0},
\end{aligned}\tag{11.14}$$

and for any function $f^{(s+k)}_{0}$ belonging to $X_{0,s+k,\beta_0}$, one defines the k-th truncated in high number of collisions, large energy and small separation in time

between collisions of the Boltzmann hierarchy *as the function :*

$$\mathcal{I}^{0,\delta}_{s,s+k-1}\big(u\mapsto \mathcal{T}^{s+k,0}_u f^{(s+k)}_{N,0}\big) = \Big(\Big(\overset{+}{c_1}\big(\mathcal{I}^0_s\big)^{a_{1,3}}_{a_{1,2}}\Big)\circ\Big(\overset{+}{c_2}\big(\mathcal{I}^0_{s+1}\big)^{a_{2,3}}_{a_{2,2}}\Big) \circ\cdots\circ\Big(\overset{+}{c_k}\big(\mathcal{I}^0_{s+k-1}\big)^{a_{k,3}}_{a_{k,2}}\Big)\Big)\big(u\mapsto \mathcal{T}^{s+k,0}_u f^{(s+k)}_0\big), \quad (11.15)$$

with, for all $1\leq j\leq k$ *:*

$$c_j = (k-j+1)\delta$$

and

$$a_{j,1}=0,\quad a_{j,2}=(k-j)\delta,\quad a_{j,3}=t_{j-1}-\delta,\quad a_{j,4}=t_{j-1}.$$

Definition 40 (Cut-off in small time difference between the collisions of the solutions of the hierarchies)**.** *For any sequence of initial data* $F_{N,0}=\big(f^{(s)}_{N,0}\big)_{1\leq s\leq N}$ *belonging to* $\boldsymbol{X}_{N,\varepsilon,\beta_0,\mu_0^\alpha}$*, one defines (using the notation* (11.14) *of Definition 39) the* truncated in high number of collisions, large energy and small time difference between collisions solution of the conjugate BBGKY hierarchy *associated to the initial datum* $F_{N,0}$ *as the function :*

$$H^{n,R,\delta}_N = t\mapsto \Big(f^{(s)}_{N,0}(\cdot)\mathbb{1}_{|V_s|\leq R} + \sum_{k=1}^n \mathbb{1}_{s\leq N-k}\Big(\mathcal{I}^{N,\varepsilon,\delta}_{s,s+k-1}\big(f^{(s+k)}_{N,0}\mathbb{1}_{|V_{s+k}|\leq R}\big)\Big)(t,\cdot)\Big)_{1\leq s\leq N}.$$

For any sequence of initial data $F_0=\big(f^{(s)}_0\big)_{s\geq 1}$ *belonging to* $\boldsymbol{X}_{0,\beta_0,\mu_0^\alpha}$*, one defines (using the notation* (11.15) *of Definition 39) the* truncated in high number of collisions, large energy and small time difference between collisions solution of the Boltzmann hierarchy *associated to the initial datum* F_0 *as the function :*

$$F^{n,R,\delta} = t\mapsto \Big(\mathcal{T}^{s,0}_t f^{(s)}_0(\cdot)\mathbb{1}_{|V_s|\leq R} + \sum_{k=1}^n \Big(\mathcal{I}^{N,\varepsilon,\delta}_{s,s+k-1}\big(u\mapsto \mathcal{T}^{s+k,0}_u f^{(s+k)}_{N,0}\mathbb{1}_{|V_{s+k}|\leq R}\big)\Big)(t,\cdot)\Big)_{s\geq 1}.$$

Remark 25. *The decomposition described above used strongly the linearity of the integrated in time (transport-) collision-transport operator. However, this decomposition is only formal until one is able to show that the collision-transport operator can be indeed applied to each term of the decomposition. Here, it seems to be a problem, since, for the case of the BBGKY hierarchy (but the problem is the same for the Boltzmann hierarchy), the transport-collision operator of a time-dependent function has been proven to be well defined if this function is an element of* $\mathcal{C}\big([0,T],L^\infty\big(\mathcal{D}^\varepsilon_s\big)\big)$ *(as it was done in Section 5). In particular, the condition of continuity in time was crucial. Besides, multiplying by the indicator function of a time interval breaks immediately this continuity, so in particular, the following terms of the decomposition :*

$$\Big(\overset{+}{a_{k,2}}\big(\mathcal{I}^0_{s+k-1}\big)^{a_{k,3}}_{a_{k,2}}\Big)\big(u\mapsto \mathcal{T}^{s+k,0}_u f^{(s+k)}_0\big)$$

are not continuous in time.
However, this loss of continuity is not a problem, since if one considers carefully the paragraph 5.1.3 page 121 devoted to the definition of the transport-collision operator for time depending-function, one sees that in fact assuming the continuity is somehow too strong, because one considers only functions that belong to the closure of step functions, that is, piecewise constant in time functions, taking their values in $L^\infty\big(\mathcal{D}_s^\varepsilon\big)$. In other words, the transport-collision operator is in fact perfectly defined for piecewise continuous in time functions. The last step which enables to obtain the final operator, that is an integration with respect to time, is therefore not a source of trouble.
In particular, the multiplication of the integrand by an indicator function of a time interval causes no problem, so that one can still consider iterations of the integrated in time transport-collision-transport operator in the case if the cut-off in time described in this section is applied. Of course, the function obtained does not belong to the same space in which the fixed point theorem and the results of existence and uniqueness of the solutions of the hierarchies have been obtained, however it is still meaningful to consider its $|||\cdot|||_{N,\varepsilon,\widetilde{\beta},\widetilde{\mu}}$ norm.

One now states two lemmas, which are in fact nothing more than an intermediate statement of a result obtained to show the continuity in time of the integrated in time transport-collision-transport operator, in the second part of the proof of Lemma 18 page 222 for the case when $\alpha = 2$, and of Lemma 19 page 234 for the case when $\alpha = 1$. They will be useful to obtain the sought control on the difference :

$$\mathcal{I}_{s,s+k-1}^{\cdot}\, f_{\cdot}^{(s+k)} - \mathcal{I}_{s,s+k-1}^{\cdot,\delta}\, f_{\cdot}^{(s+k)}.$$

Lemma 25 (Control of the cut-off in the time domain of the integrated in time (transport-) collision-transport operator, case $\alpha = 2$)**.** *Let T, β_0, μ_0 and λ be four strictly positive numbers such that :*

$$\beta_0 - \lambda T > 0 \quad \textit{and} \quad \mu_0 - \lambda T > 0.$$

One denotes $\widetilde{\beta}$ and $\widetilde{\mu}$ as :

$$\widetilde{\beta} : \left\{ \begin{array}{rcl} [0,T] & \to & \mathbb{R}_+^*, \\ t & \mapsto & \widetilde{\beta}(t) = \beta_0 - \lambda t, \end{array} \right.$$

and

$$\widetilde{\mu} : \left\{ \begin{array}{rcl} [0,T] & \to & \mathbb{R}_+^*, \\ t & \mapsto & \widetilde{\mu}(t) = \mu_0 - \lambda t. \end{array} \right.$$

In the case of the BBGKY hierarchy, for any positive integer N and any strictly positive number $\varepsilon > 0$ in the Boltzmann-Grad limit :

$$N\varepsilon^{d-1} = 1,$$

for any piecewise continuous in time function of sequences $t \mapsto \big(h_N^{(s)}(t,\cdot)\big)_{1\le s\le N}$ defined on $[0,T]$, taking its values in $\big(L^\infty\big(\mathcal{D}_s^\varepsilon\big)\big)_{1\le s\le N}$, and with a finite $|||\cdot|||_{N,\varepsilon,\widetilde{\beta},\widetilde{\mu}^2}$

norm, for any continuous functions

$$a = t \mapsto a(t) < b = t \mapsto b(t)$$

and any number $c \in [0,T]$*, for any integer* $1 \leq s \leq N-1$*, the function :*

$$t \mapsto \big({}^{\pm}_{c}(\mathcal{I}_s^{N,\varepsilon})^{b(t)}_{a(t)}\big) h_N^{(s+1)}$$

is well defined and is piecewise continuous with respect to time. In addition, one has, for any integer $1 \leq s \leq N-1$*, the following inequality :*

$$\begin{aligned}\Big|\!\Big|\!\Big| t \mapsto \Big(\mathbb{1}_{s\leq N-1}\big({}^{-}_{c}(\mathcal{I}_s^{N,\varepsilon})^{t}_{0}\big) h_N^{(s+1)}\Big)_{1\leq s\leq N} \Big|\!\Big|\!\Big|_{\varepsilon,\widetilde{\beta},\widetilde{\mu}^2} &\leq C(d)\widetilde{\beta}(T)^{-d/2}\lambda^{-1/2} \\ &\quad \times \Big|\!\Big|\!\Big| \big(h_N^{(s)}\big)_{1\leq s\leq N} \Big|\!\Big|\!\Big|_{N,\varepsilon,\widetilde{\beta},\widetilde{\mu}^2} \sqrt{c},\end{aligned} \tag{11.16}$$

and for all $t \in [0,T]$ *:*

$$\begin{aligned}\Big\| t \mapsto \Big(\mathbb{1}_{s\leq N-1}\big({}^{+}_{c}(\mathcal{I}_s^{N,\varepsilon})^{b(t)}_{a(t)}\big) h_N^{(s+1)}\Big)_{1\leq s\leq N} \Big\|_{\varepsilon,\widetilde{\beta}(t),\widetilde{\mu}(t)^2} &\leq C(d)\widetilde{\beta}(T)^{-d/2}\lambda^{-1/2} \\ &\quad \times \Big|\!\Big|\!\Big| \big(h_N^{(s)}\big)_{1\leq s\leq N} \Big|\!\Big|\!\Big|_{N,\varepsilon,\widetilde{\beta},\widetilde{\mu}^2} \sqrt{b(t)-a(t)},\end{aligned} \tag{11.17}$$

with $C(d)$ *denoting a constant which depends only on the dimension* d*.*
In the case of the Boltzmann hierarchy, for any piecewise continuous in time function of sequences $t \mapsto \big(f^{(s)}(t,\cdot)\big)_{s\geq N}$ *defined on* $[0,T]$*, taking its values in* $\big(\mathcal{C}_0\big(\big(\overline{\Omega^c} \times \mathbb{R}^d\big)^s\big)\big)_{s\geq 1}$*, and with a finite* $|\!|\!|\cdot|\!|\!|_{0,\widetilde{\beta},\widetilde{\mu}^2}$ *norm, and for any continuous functions*

$$a = t \mapsto a(t) < b = t \mapsto b(t)$$

and any number $c \in [0,T]$*, for any integer* $s \geq 1$*, the function :*

$$t \mapsto \big({}^{\pm}_{c}(\mathcal{I}_s^{0})^{b(t)}_{a(t)}\big) h_N^{(s+1)}$$

is well defined and is piecewise continuous with respect to time. In addition, one has the following inequality :

$$\Big|\!\Big|\!\Big| t \mapsto \Big(\big({}^{-}_{c}(\mathcal{I}_s^{0})^{t}_{0}\big) f^{(s+1)}\Big)_{s\geq 1} \Big|\!\Big|\!\Big|_{0,\widetilde{\beta},\widetilde{\mu}^2} \leq C(d)\widetilde{\beta}(T)^{-d/2}\lambda^{-1/2} \Big|\!\Big|\!\Big| \big(f^{(s)}\big)_{s\geq 1} \Big|\!\Big|\!\Big|_{0,\widetilde{\beta},\widetilde{\mu}^2} \sqrt{c}, \tag{11.18}$$

and for all $t \in [0,T]$ *:*

$$\begin{aligned}\Big\| t \mapsto \Big(\big({}^{+}_{c}(\mathcal{I}_s^{0})^{b(t)}_{a(t)}\big) f^{(s+1)}\Big)_{s\geq 1} \Big\|_{0,\widetilde{\beta}(t),\widetilde{\mu}(t)^2} &\leq C(d)\widetilde{\beta}(T)^{-d/2}\lambda^{-1/2} \\ &\quad \times \Big|\!\Big|\!\Big| \big(f^{(s)}\big)_{s\geq 1} \Big|\!\Big|\!\Big|_{0,\widetilde{\beta},\widetilde{\mu}^2} \sqrt{b(t)-a(t)},\end{aligned} \tag{11.19}$$

with $C(d)$ *denoting a constant which depends only on the dimension* d*.*

Lemma 26 (Control of the cut-off in the time domain of the integrated in time (transport-) collision-transport operator, case $\alpha = 1$)**.** *Let T, β_0 and λ be three strictly positive numbers and μ_0 be a real number such that :*

$$\beta_0 - \lambda T > 0.$$

One denotes $\widetilde{\beta}$ and $\widetilde{\mu}$ as :

$$\widetilde{\beta} : \left\{ \begin{array}{rcl} [0,T] & \to & \mathbb{R}_+^*, \\ t & \mapsto & \widetilde{\beta}(t) = \beta_0 - \lambda t, \end{array} \right.$$

and

$$\widetilde{\mu} : \left\{ \begin{array}{rcl} [0,T] & \to & \mathbb{R}, \\ t & \mapsto & \widetilde{\mu}(t) = \mu_0 - \lambda t. \end{array} \right.$$

In the case of the BBGKY hierarchy, for any positive integer N and any strictly positive number $\varepsilon > 0$ in the Boltzmann-Grad limit :

$$N\varepsilon^{d-1} = 1,$$

for any piecewise continuous with respect to time function of sequences $t \mapsto \big(h_N^{(s)}(t,\cdot)\big)_{1\leq s\leq N}$ defined on $[0,T]$, taking its values in $\big(L^\infty(\mathcal{D}_s^\varepsilon)\big)_{1\leq s\leq N}$, and with a finite $|||\cdot|||_{N,\varepsilon,\widetilde{\beta},\widetilde{\mu}^1}$ norm, and for any continuous functions $a = t \mapsto a(t) < b = t \mapsto b(t)$ and any number $c \in [0,T]$, for any integer $1 \leq s \leq N-1$, the function :

$$t \mapsto \big({}^{\pm}_{c}(\mathcal{I}_s^{N,\varepsilon})^{b(t)}_{a(t)}\big) h_N^{(s+1)}$$

is well-defined and is a piecewise continuous with respect to time function. In addition, one has, for every $1 \leq s \leq N-1$, the following inequality :

$$\begin{aligned} \Big| \big({}^{-}_{c}(\mathcal{I}_s^{N,\varepsilon})^{t}_{0}\big) h_N^{(s+1)} \Big|_{\varepsilon,s,\widetilde{\beta}(t)} &\leq C(d)\widetilde{\beta}(T)^{-d/2} \exp\big(-2\widetilde{\mu}(T)\big) \frac{\big(1+\widetilde{\beta}(T)^{-1/2}\big)}{\sqrt{\lambda}} \\ &\quad \times \Big|\!\Big|\!\Big| \big(h_N^{(s)}\big)_{1\leq s\leq N} \Big|\!\Big|\!\Big|_{N,\varepsilon,\widetilde{\beta},\widetilde{\mu}^1} \sqrt{s}\sqrt{c}, \end{aligned} \tag{11.20}$$

and

$$\begin{aligned} \Big| \big({}^{+}_{c}(\mathcal{I}_s^{N,\varepsilon})^{b(t)}_{a(t)}\big) h_N^{(s+1)} \Big|_{\varepsilon,s,\widetilde{\beta}(t)} &\leq C(d)\widetilde{\beta}(T)^{-d/2} \exp\big(-2\widetilde{\mu}(T)\big) \frac{\big(1+\widetilde{\beta}(T)^{-1/2}\big)}{\sqrt{\lambda}} \\ &\quad \times \Big|\!\Big|\!\Big| \big(h_N^{(s)}\big)_{1\leq s\leq N} \Big|\!\Big|\!\Big|_{N,\varepsilon,\widetilde{\beta},\widetilde{\mu}^1} \sqrt{s}\sqrt{b(t)-a(t)}, \end{aligned} \tag{11.21}$$

with $C(d)$ denoting a constant which depends only on the dimension d.
In the case of the Boltzmann hierarchy, for any piecewise continuous with respect to time function of sequences (of functions) $t \mapsto \big(f^{(s)}(t,\cdot)\big)_{s\geq N}$ defined on $[0,T]$, taking its values in $\big(\mathcal{C}_0\big(\big(\overline{\Omega^c} \times \mathbb{R}^d\big)^s\big)\big)_{s\geq 1}$, and with a finite $|||\cdot|||_{0,\widetilde{\beta},\widetilde{\mu}^2}$ norm, and

for any continuous functions $a = t \mapsto a(t) < b = t \mapsto b(t)$ *and any number* $c \in [0,T]$*, for any integer* $s \geq 1$*, the function :*

$$t \mapsto \big({}^{\pm}_{c}(\mathcal{I}^0_s)^{b(t)}_{a(t)}\big) h_N^{(s+1)}$$

is well defined and is a piecewise continuous with respect to time function. In addition, one has the following inequality :

$$\Big|\big({}^{-}_{c}(\mathcal{I}^0_s)^{t}_{0}\big) f^{(s+1)}\Big|_{0,s,\widetilde{\beta}(t)} \leq C(d)\widetilde{\beta}(T)^{-d/2} \exp\big(-2\widetilde{\mu}(T)\big) \frac{\big(1+\widetilde{\beta}(T)^{-1/2}\big)}{\sqrt{\lambda}} \times \Big|\!\Big|\!\Big|\big(f^{(s)}\big)_{s\geq 1}\Big|\!\Big|\!\Big|_{0,\widetilde{\beta},\widetilde{\mu}^1} \sqrt{s}\sqrt{c}, \tag{11.22}$$

and

$$\Big|\big({}^{+}_{c}(\mathcal{I}^0_s)^{b(t)}_{a(t)}\big) f^{(s+1)}\Big|_{0,s,\widetilde{\beta}(t)} \leq C(d)\widetilde{\beta}(T)^{-d/2} \exp\big(-2\widetilde{\mu}(T)\big) \frac{\big(1+\widetilde{\beta}(T)^{-1/2}\big)}{\sqrt{\lambda}} \times \Big|\!\Big|\!\Big|\big(f^{(s)}\big)_{s\geq 1}\Big|\!\Big|\!\Big|_{0,\widetilde{\beta},\widetilde{\mu}^1} \sqrt{s}\sqrt{b(t)-a(t)}, \tag{11.23}$$

with $C(d)$ *denoting a constant which depends only on the dimension* d*.*

Finally, one can state the two most important propositions of this section, which will make explicit the control of the cut-off in small time difference between the collisions. The first proposition will be devoted to the case when $\alpha = 2$ (that is, for the solutions to the hierarchies in the space $\widetilde{\mathbf{X}}_{\cdot,\widetilde{\beta},\widetilde{\mu}^2}$, see Definitions 27 and 28 page 211), while the second proposition will address the case when $\alpha = 1$ (see Definitions 29 and 30 page 212).

Proposition 14 (Cut-off in small time difference between the collisions, case $\alpha = 2$)**.** *Let* β_0 *and* μ_0 *be two strictly positive number. There exists a constant :*

$$C_3(d, \beta_0, \mu_0)$$

depending only on the dimension and on the numbers β_0 *and* μ_0*, such that for any positive integer* n*, any strictly positive numbers* R *and* δ*, any positive integer* N *and any strictly positive number* $\varepsilon > 0$ *verifying the Boltzmann-Grad limit :*

$$N\varepsilon^{d-1} = 1,$$

and any couple of sequences of initial data $F_{N,0} = \big(f^{(s)}_{N,0}\big)_{1\leq s\leq N}$ *and* $F_0 = \big(f^{(s)}_0\big)_{s\geq 1}$ *belonging respectively to* $\boldsymbol{X}_{N,\varepsilon,\beta_0,\mu_0^2}$ *and* $\boldsymbol{X}_{0,\beta_0,\mu_0^2}$*, the respective truncated in high number of collisions and in large energy solutions* $H_N^{n,R} \in \widetilde{\boldsymbol{X}}_{N,\varepsilon,\widetilde{\beta},\widetilde{\mu}^2}$ *to the BBGKY hierarchy with initial datum* $F_{N,0}$ *and* $F \in \widetilde{\boldsymbol{X}}_{0,\widetilde{\beta},\widetilde{\mu}^2}$ *to the Boltzmann hierarchy with inital datum* F_0*, verifies :*

$$\Big|\!\Big|\!\Big|H_N^{n,R} - H_N^{n,R,\delta}\Big|\!\Big|\!\Big|_{N,\varepsilon,\widetilde{\beta},\widetilde{\mu}^2} \leq C_3 n^{3/2}\sqrt{\delta}\|F_{N,0}\|_{N,\varepsilon,\beta_0,\mu_0^2}, \tag{11.24}$$

and

$$\left|\!\left|\!\left| F_N^{n,R} - F_N^{n,R,\delta} \right|\!\right|\!\right|_{0,\widetilde{\beta},\widetilde{\mu}^2} \leq C_3 n^{3/2}\sqrt{\delta}\|F_0\|_{0,\beta_0,\mu_0^2}. \tag{11.25}$$

Remark 26. *Since the time T and the quantity λ are depending on β_0 and μ_0 (see the proof of Lemma 23 page 255), any expression depending on β_0, μ_0, λ and T can of course be expressed only with β_0 and μ_0, which is in particular the case for the constant C_3 involved in the bounds* (11.24) *and* (11.25) *of the proposition.*

Proof. The proof will be written only for the BBGKY hierarchy, since the case of the Boltzmann hierarchy does not present any difference.
One has :

$$\begin{aligned}
\mathcal{I}_{s,s+k-1}^{N,\varepsilon} f_{N,0}^{(s+k)} &= \left(\mathcal{I}_s^{N,\varepsilon} \circ \mathcal{I}_{s+1}^{N,\varepsilon} \circ \cdots \circ \mathcal{I}_{s+k-2}^{N,\varepsilon} \circ \mathcal{I}_{s+k-1}^{N,\varepsilon}\right) f_{N,0}^{(s+k)} \\
&= \left(\mathcal{I}_s^{N,\varepsilon} \circ \mathcal{I}_{s+1}^{N,\varepsilon} \circ \cdots \circ \mathcal{I}_{s+k-2}^{N,\varepsilon} \circ \left(\sum_{\pm_k} \sum_{l_k=1}^{3} {}^{\pm_k}_{c_k}\left(\mathcal{I}_{s+k-1}^{N,\varepsilon}\right)_{a_{k,l_k}}^{a_{k,l_k+1}}\right)\right) f_{N,0}^{(s+k)} \\
&= \sum_{\pm_k} \sum_{l_k=1}^{3} \left(\mathcal{I}_s^{N,\varepsilon} \circ \mathcal{I}_{s+1}^{N,\varepsilon} \circ \cdots \circ \mathcal{I}_{s+k-2}^{N,\varepsilon} \circ \left({}^{\pm_k}_{c_k}\left(\mathcal{I}_{s+k-1}^{N,\varepsilon}\right)_{a_{k,l_k}}^{a_{k,l_k+1}}\right)\right) f_{N,0}^{(s+k)}
\end{aligned}$$

using the notation (11.11) page 292.
One can then control, on the one hand, the terms obtained in the case when $\pm_k = -$. One has, for every $1 \leq l_k \leq 3$, and almost everywhere on $[0,T] \times \mathcal{D}_{s+k-1}^{\varepsilon}$, thanks to the inequality (11.16) of the previous Lemma 25 :

$$\begin{aligned}
&\left|\!\left|\!\left| t_{k-1} \mapsto \left(\mathbb{1}_{s\leq N-1}\left({}^{-}_{c_k}(\mathcal{I}_s^{N,\varepsilon})_{a_{k,l_k}(t_{k-1})}^{a_{k,l_{k+1}}(t_{k-1})}\right) f_{N,0}^{(s+1)}\right)_{1\leq s\leq N} \right|\!\right|\!\right|_{N,\varepsilon,\widetilde{\beta},\widetilde{\mu}^2} \\
&\qquad \leq C(d)\widetilde{\beta}(T)^{-d/2}\lambda^{-1/2} \left|\!\left|\!\left| \left(f_{N,0}^{(s)}\right)_{1\leq s\leq N} \right|\!\right|\!\right|_{N,\varepsilon,\widetilde{\beta},\widetilde{\mu}^2} \sqrt{\delta},
\end{aligned}$$

using in the end the fact that $a_{k,2} = \delta$ (one has to be careful here : this is specific to the case of the first iteration, since in general $a_{j,2} = (k-j)\delta$).
Choosing then λ and T given by Lemma 23 page 255, the integrated in time transport-collision-transport operator is in this case a contracting mapping as the inequality (8.17) of this very Lemma asserts it, and one obtains therefore for every $1 \leq l_k \leq 3$:

$$\begin{aligned}
&\left|\!\left|\!\left| t_{k-1} \mapsto \left(\mathbb{1}_{s\leq N-k}\left(\mathcal{I}_{s,s+k-2}^{N,\varepsilon} \circ \left({}^{-}_{c_k}\left(\mathcal{I}_{s+k-1}^{N,\varepsilon}\right)_{a_{k,l_k}(t_{k-1}}^{a_{k,l_k+1}(t_{k-1})}\right)\right) f_{N,0}^{(s+k)}\right)_{1\leq s\leq N} \right|\!\right|\!\right|_{N,\varepsilon,\widetilde{\beta},\widetilde{\mu}^2} \\
&\qquad \leq \frac{1}{2^{k-1}} C(d)\widetilde{\beta}(T)^{-d/2}\lambda^{-1/2} \left|\!\left|\!\left| \left(f_{N,0}^{(s)}\right)_{1\leq s\leq N} \right|\!\right|\!\right|_{N,\varepsilon,\widetilde{\beta},\widetilde{\mu}^2} \sqrt{\delta}.
\end{aligned} \tag{11.26}$$

In the same way, for the case when $\pm_k = +$ and $l_k = 1$ or 3, one has for every

$t_{k-1} \in [0,T]$, thanks to the inequality (11.17) of Lemma 25 :

$$\begin{aligned}&\left\| \left(\mathbb{1}_{s\le N-1} \left({}^{+}_{c_k} (\mathcal{I}_s^{N,\varepsilon})^{a_{k,l_{k+1}}(t_{k-1})}_{a_{k,l_k}(t_{k-1})} \right) f_{N,0}^{(s+1)} \right)_{1\le s\le N} \right\|_{N,\varepsilon,\widetilde{\beta}(t_{k-1}),\widetilde{\mu}(t_{k-1})^2} \\ &\quad \le C(d)\widetilde{\beta}(T)^{-d/2}\lambda^{-1/2} \left\| \left(f_{N,0}^{(s)} \right)_{1\le s\le N} \right\|_{N,\varepsilon,\widetilde{\beta},\widetilde{\mu}^2} \sqrt{a_{k,l_{k+1}}(t_{k-1}) - a_{k,l_k}(t_{k-1})},\end{aligned}$$

and recalling that one has chosen $a_{k,1} = 0$, $a_{k,2} = \delta$, $a_{k,3} = t_{k-1} - \delta$, and $a_{k,4} = t_{k-1}$, one finds in fact that, for $l_k = 1$:

$$\begin{aligned}&\left\| \left(\mathbb{1}_{s\le N-1} \left({}^{+}_{c_k} (\mathcal{I}_s^{N,\varepsilon})^{a_{k,2}(t_{k-1})}_{a_{k,1}(t_{k-1})} \right) f_{N,0}^{(s+1)} \right)_{1\le s\le N} \right\|_{N,\varepsilon,\widetilde{\beta}(t_{k-1}),\widetilde{\mu}(t_{k-1})^2} \\ &\quad \le C(d)\widetilde{\beta}(T)^{-d/2}\lambda^{-1/2} \left\| \left(f_{N,0}^{(s)} \right)_{1\le s\le N} \right\|_{N,\varepsilon,\widetilde{\beta},\widetilde{\mu}^2} \sqrt{\delta - 0},\end{aligned}$$

and for $l_k = 3$:

$$\begin{aligned}&\left\| \left(\mathbb{1}_{s\le N-1} \left({}^{+}_{c_k} (\mathcal{I}_s^{N,\varepsilon})^{a_{k,2}(t_{k-1})}_{a_{k,1}(t_{k-1})} \right) f_{N,0}^{(s+1)} \right)_{1\le s\le N} \right\|_{N,\varepsilon,\widetilde{\beta}(t_{k-1}),\widetilde{\mu}(t_{k-1})^2} \\ &\quad \le C(d)\widetilde{\beta}(T)^{-d/2}\lambda^{-1/2} \left\| \left(f_{N,0}^{(s)} \right)_{1\le s\le N} \right\|_{N,\varepsilon,\widetilde{\beta},\widetilde{\mu}^2} \sqrt{t_{k-1} - (t_{k-1} - \delta)},\end{aligned}$$

so that the right-hand side does not depend on time. It is then possible to consider the supremum with respect to the time variable t_{k-1} for the left-hand side to obtain :

$$\begin{aligned}&\left\| t_{k-1} \mapsto \left(\mathbb{1}_{s\le N-1} \left({}^{+}_{c_k} (\mathcal{I}_s^{N,\varepsilon})^{a_{k,l_{k+1}}(t_{k-1})}_{a_{k,l_k}(t_{k-1})} \right) f_{N,0}^{(s+1)} \right)_{1\le s\le N} \right\|_{N,\varepsilon,\widetilde{\beta},\widetilde{\mu}^2} \\ &\qquad \le C(d)\widetilde{\beta}(T)^{-d/2}\lambda^{-1/2} \left\| \left(f_{N,0}^{(s)} \right)_{1\le s\le N} \right\|_{N,\varepsilon,\widetilde{\beta},\widetilde{\mu}^2} \sqrt{\delta}.\end{aligned}$$

As in the case when $\pm_k = -$, several iterations of the contracting integrated in time transport-collision-transport operator provides for $l_k = 1$ or 3 :

$$\begin{aligned}&\left\| \left(\mathbb{1}_{s\le N-k} \mathcal{I}_{s,s+k-2}^{N,\varepsilon} \circ \left({}^{+}_{c_k} \left(\mathcal{I}_{s+k-1}^{N,\varepsilon} \right)^{a_{k,l_k+1}}_{a_{k,l_k}} \right) f_{N,0}^{(s+k)} \right)_{1\le s\le N} \right\|_{N,\varepsilon,\widetilde{\beta},\widetilde{\mu}^2} \\ &\qquad \le \frac{1}{2^{k-1}} C(d)\widetilde{\beta}(T)^{-d/2}\lambda^{-1/2} \left\| \left(f_{N,0}^{(s)} \right)_{1\le s\le N} \right\|_{N,\varepsilon,\widetilde{\beta},\widetilde{\mu}^2} \sqrt{\delta}.\end{aligned} \tag{11.27}$$

The controls (11.26) and (11.27) enable to write that :

$$\begin{aligned}\mathcal{I}_{s,s+k-1}^{N,\varepsilon} f_{N,0}^{(s+k)} &= \left(\mathcal{I}_{s,s+k-2}^{N,\varepsilon} \circ \left({}^{+}_{c_k} \left(\mathcal{I}_{s+k-1}^{N,\varepsilon} \right)^{a_{k,3}}_{a_{k,2}} \right) \right) f_{N,0}^{(s+k)} \\ &\quad + \sum_{\substack{\pm_k \\ 1\le l_k\le 3 \\ (\pm_k, l_k)\ne(+,2)}} \left(\mathcal{I}_{s,s+k-2}^{N,\varepsilon} \circ \left({}^{\pm_k}_{c_k} \left(\mathcal{I}_{s+k-1}^{N,\varepsilon} \right)^{a_{k,l_k+1}}_{a_{k,l_k}} \right) \right) f_{N,0}^{(s+k)},\end{aligned}$$

with each term in the sum of the right-hand side being small in the $|||\cdot|||_{N,\varepsilon,\widetilde{\beta},\widetilde{\mu}}$ norm and controlled by $\sqrt{\delta}$. One gets

$$\begin{aligned}
&\Big|\Big|\Big|\mathcal{I}^{N,\varepsilon}_{s,s+k-1} f^{(s+k)}_{N,0} - \Big(\mathcal{I}^{N,\varepsilon}_{s,s+k-2} \circ \big({}^{+}_{c_k}\big(\mathcal{I}^{N,\varepsilon}_{s+k-1}\big)^{a_{k,3}}_{a_{k,2}}\big)\Big) f^{(s+k)}_{N,0}\Big|\Big|\Big|_{N,\varepsilon,\widetilde{\beta},\widetilde{\mu}^2} \\
&\quad \leq \sum_{\substack{\pm_k \\ 1\leq l_k\leq 3 \\ (\pm_k,l_k)\neq(+,2)}} \Big|\Big|\Big|\big(\mathcal{I}^{N,\varepsilon}_{s,s+k-2} \circ \big({}^{\pm_k}_{c_k}\big(\mathcal{I}^{N,\varepsilon}_{s+k-1}\big)^{a_{k,l_k+1}}_{a_{k,l_k}}\big)\big) f^{(s+k)}_{N,0}\Big|\Big|\Big|_{N,\varepsilon,\widetilde{\beta},\widetilde{\mu}^2} \\
&\quad \leq \sum_{\substack{\pm_k \\ 1\leq l_k\leq 3 \\ (\pm_k,l_k)\neq(+,2)}} \frac{1}{2^{k-1}} C(d)\widetilde{\beta}(T)^{-d/2}\lambda^{-1/2} \Big|\Big|\Big|\big(f^{(s)}_{N,0}\big)_{1\leq s\leq N}\Big|\Big|\Big|_{N,\varepsilon,\widetilde{\beta},\widetilde{\mu}^2} \sqrt{\delta} \\
&\quad \leq \frac{5}{2^{k-1}} C(d)\widetilde{\beta}(T)^{-d/2}\lambda^{-1/2} \Big|\Big|\Big|\big(f^{(s)}_{N,0}\big)_{1\leq s\leq N}\Big|\Big|\Big|_{N,\varepsilon,\widetilde{\beta},\widetilde{\mu}^2} \sqrt{\delta}. \qquad (11.28)
\end{aligned}$$

Then, one only has to iterate the process, by decomposing :

$$\Big(\mathcal{I}^{N,\varepsilon}_{s+k-2} \circ \big({}^{+}_{c_k}\big(\mathcal{I}^{N,\varepsilon}_{s+k-1}\big)^{a_{k,3}}_{a_{k,2}}\big)\Big) f^{(s+k)}_{N,0}$$

in the following parts :

$$\begin{aligned}
&\Big(\big({}^{+}_{c_{k-1}}\big(\mathcal{I}^{N,\varepsilon}_{s+k-2}\big)^{a_{k-1,3}}_{a_{k-1,2}}\big) \circ \big({}^{+}_{c_k}\big(\mathcal{I}^{N,\varepsilon}_{s+k-1}\big)^{a_{k,3}}_{a_{k,2}}\big)\Big) f^{(s+k)}_{N,0} \\
&\quad + \sum_{\substack{\pm_{k-1} \\ 1\leq l_{k-1}\leq 3 \\ (\pm_{k-1},l_{k-1})\neq(+,2)}} \Big(\big({}^{\pm_{k-1}}_{c_{k-1}}\big(\mathcal{I}^{N,\varepsilon}_{s+k-2}\big)^{a_{k-1,l_{k-1}+1}}_{a_{k-1,l_{k-1}}}\big) \circ \big({}^{+}_{c_k}\big(\mathcal{I}^{N,\varepsilon}_{s+k-1}\big)^{a_{k,3}}_{a_{k,2}}\big)\Big) f^{(s+k)}_{N,0}.
\end{aligned}$$

One has to use here the fact that :

$$\begin{aligned}
&\Big|\Big|\Big|t \mapsto \Big(\big({}^{+}_{c_j}\big(\mathcal{I}^{N,\varepsilon}_{s+j-1}\big)^{a_{j,3}(t)}_{a_{j,2}}\big) g^{(s+j)}(t,\cdot)\Big)_{1\leq s\leq N}\Big|\Big|\Big|_{N,\varepsilon,\widetilde{\beta},\widetilde{\mu}^2} \\
&\qquad\qquad \leq \Big|\Big|\Big|t \mapsto \Big(\big(\mathcal{I}^{N,\varepsilon}_{s+j-1} g^{(s+j)}\big)(t,\cdot)\Big)_{1\leq s\leq N}\Big|\Big|\Big|_{N,\varepsilon,\widetilde{\beta},\widetilde{\mu}^2}, \qquad (11.29)
\end{aligned}$$

which holds true since the domain of the integral of the left-hand side is strictly contained in the domain of the integral of the right-hand side.
For the first iteration of the argument, that is for the case $j = k-1$, one will consider

$$g^{(s+k-1)} = \big({}^{+}_{c_k}\big(\mathcal{I}^{N,\varepsilon}_{s+k-1}\big)^{a_{k,3}}_{a_{k,2}}\big)\Big) f^{(s+k)}_{N,0}.$$

It is then possible to apply the inequalities (11.26) or (11.27) depending on the term of the sum considered, replacing $f^{(s+k)}_{N,0}$ by $\Big({}^{\pm_k}_{c_k}\big(\mathcal{I}^{N,\varepsilon}_{s+k-1}\big)^{a_{k,l_k+1}}_{a_{k,l_k}}\Big) f^{(s+k)}_{N,0}$, so that the sum :

$$\sum_{\substack{\pm_{k-1} \\ 1\leq l_{k-1}\leq 3 \\ (\pm_{k-1},l_{k-1})\neq(+,2)}} \Big(\big({}^{\pm_{k-1}}_{c_{k-1}}\big(\mathcal{I}^{N,\varepsilon}_{s+k-2}\big)^{a_{k-1,l_{k-1}+1}}_{a_{k-1,l_{k-1}}}\big) \circ \big({}^{+}_{c_k}\big(\mathcal{I}^{N,\varepsilon}_{s+k-1}\big)^{a_{k,3}}_{a_{k,2}}\big)\Big) f^{(s+k)}_{N,0}$$

is also controlled in the $|||\cdot|||_{N,\varepsilon,\widetilde{\beta},\widetilde{\mu}^2}$ norm, and is of order $\sqrt{2\delta}$ (previously, it was $\sqrt{\delta}$, but the term with $\pm = -$ is in the general case of order $\sqrt{(k-j)\delta}$). The iterations of the contracting integrated in time transport-collision-transport operator provides that the function of sequences :

$$\Bigg(t \mapsto \Bigg(\mathbb{1}_{s\leq N-k} \sum_{\substack{\pm_{k-1} \\ 1\leq l_{k-1}\leq 3 \\ (\pm_{k-1},l_{k-1})\neq(+,2)}} \mathcal{I}^{N,\varepsilon}_{s,s+k-3} \circ \left(\begin{smallmatrix}\pm_{k-1}\\ c_{k-1}\end{smallmatrix}\left(\mathcal{I}^{N,\varepsilon}_{s+k-2}\right)^{a_{k-1,l_{k-1}+1}}_{a_{k-1,l_{k-1}}}\right) \\ \circ \left(\begin{smallmatrix}+\\ c_k\end{smallmatrix}\left(\mathcal{I}^{N,\varepsilon}_{s+k-1}\right)^{a_{k,3}}_{a_{k,2}}\right) f^{(s+k)}_{N,0}\Bigg)(t,\cdot)\Bigg)_{1\leq s\leq N}$$

is bounded in the $|||\cdot|||_{N,\varepsilon,\widetilde{\beta},\widetilde{\mu}^2}$ norm by :

$$\frac{1}{2^{k-1}} C(d)\widetilde{\beta}(T)^{-d/2}\lambda^{-1/2} \left|\left|\left|\left(f^{(s)}_{N,0}\right)_{s\geq 1}\right|\right|\right|_{N,\varepsilon,\widetilde{\beta},\widetilde{\mu}^2} \sqrt{2\delta}, \tag{11.30}$$

and one gets

$$\begin{aligned}
&\Bigg|\Bigg|\Bigg|\mathcal{I}^{N,\varepsilon}_{s,s+k-1} f^{(s+k)}_{N,0} \\
&\quad - \Bigg(\mathcal{I}^{N,\varepsilon}_{s,s+k-3} \circ \left(\begin{smallmatrix}\pm_{k-1}\\ c_{k-1}\end{smallmatrix}\left(\mathcal{I}^{N,\varepsilon}_{s+k-2}\right)^{a_{k-1,l_{k-1}+1}}_{a_{k-1,l_{k-1}}}\right) \circ \left(\begin{smallmatrix}+\\ c_k\end{smallmatrix}\left(\mathcal{I}^{N,\varepsilon}_{s+k-1}\right)^{a_{k,3}}_{a_{k,2}}\right)\Bigg) f^{(s+k)}_{N,0}\Bigg|\Bigg|\Bigg|_{N,\varepsilon,\widetilde{\beta},\widetilde{\mu}^2} \\
&\leq \Bigg|\Bigg|\Bigg|\mathcal{I}^{N,\varepsilon}_{s,s+k-1} f^{(s+k)}_{N,0} - \Bigg(\mathcal{I}^{N,\varepsilon}_{s,s+k-2} \circ \left(\begin{smallmatrix}+\\ c_k\end{smallmatrix}\left(\mathcal{I}^{N,\varepsilon}_{s+k-1}\right)^{a_{k,3}}_{a_{k,2}}\right)\Bigg) f^{(s+k)}_{N,0}\Bigg|\Bigg|\Bigg|_{N,\varepsilon,\widetilde{\beta},\widetilde{\mu}^2} \\
&+ \Bigg|\Bigg|\Bigg|\Bigg(\mathcal{I}^{N,\varepsilon}_{s,s+k-2} \circ \left(\begin{smallmatrix}+\\ c_k\end{smallmatrix}\left(\mathcal{I}^{N,\varepsilon}_{s+k-1}\right)^{a_{k,3}}_{a_{k,2}}\right)\Bigg) f^{(s+k)}_{N,0} \\
&\quad - \Bigg(\mathcal{I}^{N,\varepsilon}_{s,s+k-3} \circ \left(\begin{smallmatrix}\pm_{k-1}\\ c_{k-1}\end{smallmatrix}\left(\mathcal{I}^{N,\varepsilon}_{s+k-2}\right)^{a_{k-1,l_{k-1}+1}}_{a_{k-1,l_{k-1}}}\right) \circ \left(\begin{smallmatrix}+\\ c_k\end{smallmatrix}\left(\mathcal{I}^{N,\varepsilon}_{s+k-1}\right)^{a_{k,3}}_{a_{k,2}}\right)\Bigg) f^{(s+k)}_{N,0}\Bigg|\Bigg|\Bigg|_{N,\varepsilon,\widetilde{\beta},\widetilde{\mu}^2} \\
&\leq \Bigg|\Bigg|\Bigg|\mathcal{I}^{N,\varepsilon}_{s,s+k-1} f^{(s+k)}_{N,0} - \Bigg(\mathcal{I}^{N,\varepsilon}_{s,s+k-2} \circ \left(\begin{smallmatrix}+\\ c_k\end{smallmatrix}\left(\mathcal{I}^{N,\varepsilon}_{s+k-1}\right)^{a_{k,3}}_{a_{k,2}}\right)\Bigg) f^{(s+k)}_{N,0}\Bigg|\Bigg|\Bigg|_{N,\varepsilon,\widetilde{\beta},\widetilde{\mu}^2} \\
&\quad + \sum_{\substack{\pm_{k-1} \\ 1\leq l_{k-1}\leq 3 \\ (\pm_{k-1},l_{k-1})\neq(+,2)}} \Bigg|\Bigg|\Bigg|\mathcal{I}^{N,\varepsilon}_{s,s+k-3} \circ \left(\begin{smallmatrix}\pm_{k-1}\\ c_{k-1}\end{smallmatrix}\left(\mathcal{I}^{N,\varepsilon}_{s+k-2}\right)^{a_{k-1,l_{k-1}+1}}_{a_{k-1,l_{k-1}}}\right) \\
&\qquad \circ \left(\begin{smallmatrix}+\\ c_k\end{smallmatrix}\left(\mathcal{I}^{N,\varepsilon}_{s+k-1}\right)^{a_{k,3}}_{a_{k,2}}\right) f^{(s+k)}_{N,0}\Bigg|\Bigg|\Bigg|_{N,\varepsilon,\widetilde{\beta},\widetilde{\mu}^2},
\end{aligned}$$

that is, using the control (11.28) for the first term of the last upper bound, and

the control (11.30) for the second term :

$$
\begin{aligned}
&\Bigg\|\mathcal{I}^{N,\varepsilon}_{s,s+k-1} f^{(s+k)}_{N,0} \\
&\quad - \Big(\mathcal{I}^{N,\varepsilon}_{s,s+k-3} \circ \big({}^{\pm_{k-1}}_{c_{k-1}} \big(\mathcal{I}^{N,\varepsilon}_{s+k-2}\big)^{a_{k-1,l_{k-1}+1}}_{a_{k-1,l_{k-1}}}\big) \circ \big({}^{+}_{c_k} \big(\mathcal{I}^{N,\varepsilon}_{s+k-1}\big)^{a_{k,3}}_{a_{k,2}}\big)\Big) f^{(s+k)}_{N,0}\Bigg\|_{N,\varepsilon,\widetilde{\beta},\widetilde{\mu}^2} \\
&\leq \frac{5}{2^{k-1}} C(d)\widetilde{\beta}(T)^{-d/2}\lambda^{-1/2} \Big\|\big(f^{(s)}_{N,0}\big)_{1\leq s\leq N}\Big\|_{N,\varepsilon,\widetilde{\beta},\widetilde{\mu}^2} \sqrt{\delta} \\
&\quad + \sum_{\substack{\pm_{k-1} \\ 1\leq l_{k-1}\leq 3 \\ (\pm_{k-1},l_{k-1})\neq(+,2)}} \frac{1}{2^{k-1}} C(d)\widetilde{\beta}(T)^{-d/2}\lambda^{-1/2} \\
&\qquad\qquad \times \Big\|t \mapsto \Big(\Big({}^{+}_{c_k}\big(\mathcal{I}^{N,\varepsilon}_{s+k-1}\big)^{a_{k,3}}_{a_{k,2}}\Big) f^{(s+k)}_{N,0}\Big)_{1\leq s\leq N}\Big\|_{N,\varepsilon,\widetilde{\beta},\widetilde{\mu}^2} \sqrt{2\delta}.
\end{aligned}
$$

Recursively, one is now able to isolate the main term, denoted by

$$
\Big(\big({}^{+}_{c_1}\big(\mathcal{I}^{N,\varepsilon}_{s}\big)^{a_{1,3}}_{a_{1,2}}\big) \circ \big({}^{+}_{c_2}\big(\mathcal{I}^{N,\varepsilon}_{s+1}\big)^{a_{2,3}}_{a_{2,2}}\big) \circ \cdots \circ \big({}^{+}_{c_k}\big(\mathcal{I}^{N,\varepsilon}_{s+k-1}\big)^{a_{k,3}}_{a_{k,2}}\big)\Big) f^{(s+k)}_{N,0}.
$$

One obtains :

$$
\begin{aligned}
\mathcal{I}^{N,\varepsilon}_{s,s+k-1} f^{(s+k)}_{N,0} = &\sum_{\substack{\pm_k \\ 1\leq l_k\leq 3 \\ (\pm_k,l_k)\neq(+,2)}} \Big(\mathcal{I}^{N,\varepsilon}_{s} \circ \cdots \circ \mathcal{I}^{N,\varepsilon}_{s+k-2} \circ \big({}^{\pm_k}_{c_k}\big(\mathcal{I}^{N,\varepsilon}_{s+k-1}\big)^{a_{k,l_k+1}}_{a_{k,l_k}}\big)\Big) f^{(s+k)}_{N,0} \\
&+ \sum_{\substack{\pm_{k-1} \\ 1\leq l_{k-1}\leq 3 \\ (\pm_{k-1},l_{k-1})\neq(+,2)}} \Big(\mathcal{I}^{N,\varepsilon}_{s} \circ \cdots \circ \mathcal{I}^{N,\varepsilon}_{s+k-3} \circ \big({}^{\pm_{k-1}}_{c_{k-1}}\big(\mathcal{I}^{N,\varepsilon}_{s+k-2}\big)^{a_{k-1,l_{k-1}+1}}_{a_{k-1,l_{k-1}}}\big) \\
&\qquad\qquad \circ \big({}^{+}_{c_k}\big(\mathcal{I}^{N,\varepsilon}_{s+k-1}\big)^{a_{k,3}}_{a_{k,2}}\big)\Big) f^{(s+k)}_{N,0} \\
&+ \cdots \\
&+ \sum_{\substack{\pm_1 \\ 1\leq l_1\leq 3 \\ (\pm_1,l_1)\neq(+,2)}} \Big(\big({}^{\pm_1}_{c_1}\big(\mathcal{I}^{N,\varepsilon}_{s}\big)^{a_{1,l_1+1}}_{a_{1,l_1}}\big) \circ \big({}^{+}_{c_2}\big(\mathcal{I}^{N,\varepsilon}_{s+1}\big)^{a_{2,3}}_{a_{2,2}}\big) \\
&\qquad\qquad \circ \cdots \circ \big({}^{+}_{c_k}\big(\mathcal{I}^{N,\varepsilon}_{s+k-1}\big)^{a_{k,3}}_{a_{k,2}}\big)\Big) f^{(s+k)}_{N,0} \\
&+ \Big(\big({}^{+}_{c_1}\big(\mathcal{I}^{N,\varepsilon}_{s}\big)^{a_{1,3}}_{a_{1,2}}\big) \circ \big({}^{+}_{c_2}\big(\mathcal{I}^{N,\varepsilon}_{s+1}\big)^{a_{2,3}}_{a_{2,2}}\big) \\
&\qquad\qquad \circ \cdots \circ \big({}^{+}_{c_k}\big(\mathcal{I}^{N,\varepsilon}_{s+k-1}\big)^{a_{k,3}}_{a_{k,2}}\big)\Big) f^{(s+k)}_{N,0}
\end{aligned}
$$

that is

$$\mathcal{I}^{N,\varepsilon}_{s,s+k-1} f^{(s+k)}_{N,0} = \sum_{\substack{\pm_1,\pm_2,\dots,\pm_k \\ 1\le l_1,\dots,l_k\le 3 \\ (\pm_1,\dots,\pm_k,l_1,\dots,l_k)\neq(+,\dots,+,2,\dots,2)}} \Big(\big({}^{\pm_1}_{c_1}\big(\mathcal{I}^{N,\varepsilon}_{s}\big)^{a_{1,l_1+1}}_{a_{1,l_1}}\big) \circ \cdots \circ \big({}^{\pm_k}_{c_k}\big(\mathcal{I}^{N,\varepsilon}_{s+k-1}\big)^{a_{k,l_k+1}}_{a_{k,l_k}}\big)\Big) f^{(s+k)}_{N,0}$$
$$+ \Big(\big({}^{+}_{c_1}\big(\mathcal{I}^{N,\varepsilon}_{s}\big)^{a_{1,3}}_{a_{1,2}}\big) \circ \big({}^{+}_{c_2}\big(\mathcal{I}^{N,\varepsilon}_{s+1}\big)^{a_{2,3}}_{a_{2,2}}\big) \circ \cdots \circ \big({}^{+}_{c_k}\big(\mathcal{I}^{N,\varepsilon}_{s+k-1}\big)^{a_{k,3}}_{a_{k,2}}\big)\Big) f^{(s+k)}_{N,0},$$

or again, in a more synthetic way and using the notations (10.2) page 268 and (11.14) page 296 :

$$\mathcal{I}^{N,\varepsilon}_{s,s+k-1} f^{(s+k)}_{N,0} = \mathcal{I}^{N,\varepsilon,\delta}_{s,s+k-1} f^{(s+k)}_{N,0}$$
$$+ \sum_{j=0}^{k-1} \sum_{\substack{\pm_{k-j} \\ 1\le l_{k-j}\le 3 \\ (\pm_{k-j},l_{k-j})\neq(+,2)}} \Big(\mathcal{I}^{N,\varepsilon}_{s,s+k-j-2} \circ \big({}^{\pm_{k-j}}_{c_{k-j}}\big(\mathcal{I}^{N,\varepsilon}_{s+k-j-1}\big)^{a_{k-j,l_{k-j}+1}}_{a_{k-j,l_{k-j}}}\big) \circ \mathcal{I}^{N,\varepsilon,\delta}_{s+k-j,s+k-1}\Big) f^{(s+k)}_{N,0}.$$

One has then obtained :

$$\Big|\!\Big|\!\Big| t \mapsto \Big(\big(\mathcal{I}^{N,\varepsilon}_{s,s+k-1} f^{(s+k)}_{N,0}\big)(t,\cdot) - \big(\mathcal{I}^{N,\varepsilon,\delta}_{s,s+k-1} f^{(s+k)}_{N,0}\big)(t,\cdot)\Big)_{1\le s\le N} \Big|\!\Big|\!\Big|_{N,\varepsilon,\widetilde{\beta},\widetilde{\mu}}$$
$$\le \sum_{j=0}^{k-1} \frac{1}{2^{k-1}} C(d)\widetilde{\beta}(T)^{-d/2}\lambda^{-1/2} \Big|\!\Big|\!\Big| \big(f^{(s)}_{N,0}\big)_{s\ge 1} \Big|\!\Big|\!\Big|_{N,\varepsilon,\widetilde{\beta},\widetilde{\mu}^2} \sqrt{k\delta},$$

so that

$$\Big|\!\Big|\!\Big| H^{n,R}_N - H^{n,R,\delta}_N \Big|\!\Big|\!\Big|_{N,\varepsilon,\widetilde{\beta},\widetilde{\mu}}$$
$$\le \sum_{k=1}^{n} \Big|\!\Big|\!\Big| t \mapsto \Big(\big(\mathcal{I}^{N,\varepsilon}_{s,s+k-1} f^{(s+k)}_{N,0}\big)(t,\cdot) - \big(\mathcal{I}^{N,\varepsilon,\delta}_{s,s+k-1} f^{(s+k)}_{N,0}\big)(t,\cdot)\Big)_{1\le s\le N} \Big|\!\Big|\!\Big|_{N,\varepsilon,\widetilde{\beta},\widetilde{\mu}}$$
$$\le \sum_{k=0}^{n} \sum_{j=0}^{k-1} \frac{1}{2^{k-1}} C(d)\widetilde{\beta}(T)^{-d/2}\lambda^{-1/2} \Big|\!\Big|\!\Big| \big(f^{(s)}_{N,0}\big)_{s\ge 1} \Big|\!\Big|\!\Big|_{N,\varepsilon,\widetilde{\beta},\widetilde{\mu}^2} \sqrt{k\delta}$$
$$\le \sum_{k=0}^{n} \frac{1}{2^{k-1}} C(d)\widetilde{\beta}(T)^{-d/2}\lambda^{-1/2} \Big\| \big(f^{(s)}_{N,0}\big)_{s\ge 1} \Big\|_{N,\varepsilon,\widetilde{\beta}_0,\widetilde{\mu}_0^2} n\sqrt{n\delta},$$

hence the lemma is proven. □

For the case $\alpha = 1$, the result on the control on the cut-off in small difference in time between the collisions is of a different nature, since in this situation one is not able to recover a convergence in the $|\!|\!|\cdot|\!|\!|_{\cdot,\widetilde{\beta},\widetilde{\mu}}$ norm of the truncated solutions when the truncation parameter in small time δ goes to zero. However, one will recover an almost everywhere convergence, in the sense that for every number of particles s fixed, the convergence will be obtained in the uniform norm on the phase space of s particles.

Proposition 15 (Cut-off in small time difference between the collisions, case $\alpha = 1$)**.** *Let β_0 be a strictly positive number and μ_0 be a real number. There exists a constant :*

$$C_3(d, \beta_0, \mu_0)$$

depending only on the dimension d and on the numbers β_0 and μ_0, such that for any positive integer n, any strictly positive numbers R and δ, any positive integer N and any strictly positive number $\varepsilon > 0$ verifying the Boltzmann-Grad limit :

$$N\varepsilon^{d-1} = 1,$$

and any couple of sequences of initial data $F_{N,0} = \left(f_{N,0}^{(s)}\right)_{1\leq s\leq N}$ and $F_0 = \left(f_0^{(s)}\right)_{s\geq 1}$ belonging respectively to $\boldsymbol{X}_{N,\varepsilon,\beta_0,\mu_0^1}$ and $\boldsymbol{X}_{0,\beta_0,\mu_0^1}$, the respective truncated in high number of collisions and in large energy solutions $H_N^{n,R} \in \widetilde{\boldsymbol{X}}_{N,\varepsilon,\widetilde{\beta},\widetilde{\mu}^1}$ to the BBGKY hierarchy with initial datum $F_{N,0}$ and $F \in \widetilde{\boldsymbol{X}}_{0,\widetilde{\beta},\widetilde{\mu}^1}$ to the Boltzmann hierarchy with inital datum F_0 verifies, for all integer $1 \leq s \leq N$ and time $t \in [0, T]$:

$$\left|\left(H_N^{n,R}\right)^{(s)}(t,\cdot) - \left(H_N^{n,R,\delta}\right)^{(s)}(t,\cdot)\right|_{\varepsilon,s,\widetilde{\beta}(t)} \leq C_3\|F_{N,0}\|_{N,\varepsilon,\beta_0,\mu_0^1}\sqrt{s}n^{3/2}\sqrt{\delta}, \tag{11.31}$$

and

$$\left|\left(F^{n,R}\right)^{(s)}(t,\cdot) - \left(F^{n,R,\delta}\right)^{(s)}(t,\cdot)\right|_{0,s,\widetilde{\beta}(t)} \leq C_3\|F_0\|_{0,\beta_0,\mu_0^1}\sqrt{s}n^{3/2}\sqrt{\delta}. \tag{11.32}$$

The proof is very similar to the proof of Proposition 14, and it is based on the same decomposition. However, the change of the regularity in the initial data enables only to use Lemma 26 (instead of Lemma 25), so that once the decomposition is done, one can only estimate the size of the remainders when the number s of particles is fixed.

Chapter 12

Final preparation for the comparison

12.1 Decomposition of the solutions into terms of the same type

12.1.1 A geometric view of the iterations of the transport-collision operators and the dominated convergence argument

Since one wants to compare the solutions of the two hierarchies, and since the corresponding equations are given by the iterated Duhamel formulae (10.3) page 268, and (10.9) page 272, one will compare each term of the sum defining the solutions. Recalling that the transport-collision operator $\mathcal{C}^{N,\varepsilon}_{s,s+1}\mathcal{T}^{s+1,\varepsilon}$ of the BBGKY hierarchy (see Section 5.1 page 88) is defined as :

$$\mathcal{C}^{N,\varepsilon}_{s,s+1}\mathcal{T}^{s+1,\varepsilon} = (N-s)\sum_{i=1}^{s}\big(\mathcal{C}^{\varepsilon}_{s,s+1,+,i}\mathcal{T}^{s+1,\varepsilon} - \mathcal{C}^{\varepsilon}_{s,s+1,-,i}\mathcal{T}^{s+1,\varepsilon}\big),$$

each term :

$$\mathcal{I}^{N,\varepsilon}_{s,s+k-1}f^{(s+k)}_{N,0} = \mathcal{I}^{N,\varepsilon}_{s}\circ\cdots\circ\mathcal{I}^{N,\varepsilon}_{s+k-1}f^{(s+k)}_{N,0}$$

obtained in the decomposition provided by the iterated Duhamel formula (10.3) can again be splitted, following the notations introduced in Definition 33 page 267, as follows :

$$\begin{aligned}&\Big(\big(\sum_{\substack{1\le j_1\le s\\ \pm_1}}(\pm_1 1)\mathcal{I}^{N,\varepsilon}_{\substack{s\\ \pm_1,j_1}}\big)\circ\cdots\circ\big(\sum_{\substack{1\le j_k\le s+k-1\\ \pm_k}}(\pm_k 1)\mathcal{I}^{N,\varepsilon}_{\substack{s+k-1\\ \pm_k,j_k}}\big)\Big)f^{(s+k)}_{N,0}\\ &\quad=\sum_J\sum_M(\pm_1 1)\dots(\pm_k 1)\Big(\mathcal{I}^{N,\varepsilon}_{\substack{s\\ \pm_1,j_1}}\circ\cdots\circ\mathcal{I}^{N,\varepsilon}_{\substack{s+k-1\\ \pm_k,j_k}}\Big)f^{(s+k)}_{N,0},\end{aligned}\tag{12.1}$$

where $\mathcal{I}^{N,\varepsilon}_{\substack{s+l-1\\ \pm_l, j_l}}$ denotes

$$(N-s-l+1)\int_0^t \mathcal{T}^{s+l-1,\varepsilon}_{-t_1}\mathcal{C}^{\varepsilon}_{s+l-1,s+l,\pm_l,j_l}\mathcal{T}^{s+l,\varepsilon}_{t_1}\,\mathrm{d}t_1.$$

Of course, the same decomposition can be done for the solutions of the Boltzmann hierarchy. Recalling that the generic equation of the Boltzmann hierarchy was written as :

$$\partial_t f^{(s)} + \sum_{i=1}^{s} v_i\cdot\nabla_{x_i} f^{(s)} = \mathcal{C}^0_{s,s+1} f^{(s+1)},$$

with the collision operator written in the form :

$$\mathcal{C}^0_{s,s+1} f^{(s+1)} = \sum_{i=1}^{s}\int_{\mathbb{S}^{d-1}}\int_{\mathbb{R}^d}\big[\omega\cdot(v_{s+1}-v_i)\big]_+\Big(f^{(s+1)}(t,\big(Z_s,x_i,v_{s+1}\big)'_{i,s+1})$$
$$- f^{(s+1)}(t,Z_s,x_i,v_{s+1})\Big)\,\mathrm{d}v_{s+1}\,\mathrm{d}\omega$$

(see Definition 2 page 52 for the notation $\big(Z_s,x_i,v_{s+1}\big)'_{i,s+1}$, and one recalls that $[x]_+$ and $[x]_-$ denote respectively the positive and the negative part of the real number x).
This collision operator can be rewritten, with the simple change of variable $\omega\to-\omega$, as :

$$\mathcal{C}^0_{s,s+1} f^{(s+1)} = \sum_{i=1}^{s}\Bigg[\int_{\mathbb{S}^{d-1}}\int_{\mathbb{R}^d}\big[\omega\cdot(v_{s+1}-v_i)\big]_+ f^{(s+1)}(t,\big(Z_s,x_i,v_{s+1}\big)'_{i,s+1})$$
$$-\big[\omega\cdot(v_{s+1}-v_i)\big]_- f^{(s+1)}(t,Z_s,x_i,v_{s+1})\,\mathrm{d}v_{s+1}\,\mathrm{d}\omega\Bigg],$$

this new form enabling a termwise comparison between the solutions of the two hierarchies.
One will indeed, in the following, compare each term of this new decomposition to obtain the result of convergence. This is in this sense that the titles of the following Subsection 12.1.3 page 329 and Section 13.3 page 475 have to be understood : this termwise comparison will enable to compare the solutions of the two hierarchies, which are the sums of all those terms.
The idea is the following. Still considering the case of the Boltzmann hierarchy, each term $\mathcal{I}^0_{s,s+k-1} f_0^{(s+k)}$ of the decomposition provided by the Duhamel formula (10.9) (except the first one, which is simply an element of the sequence of the initial data composed with a transport), writes :

$$t\mapsto \int_0^t \mathcal{T}^{s,0}_{t-t_1}\mathcal{C}^0_{s,s+1}\int_0^{t_1}\mathcal{T}^{s+1,0}_{t_1-t_2}\mathcal{C}^0_{s+1,s+2}\cdots$$
$$\int_0^{t_{k-1}}\mathcal{T}^{s+k-1,0}_{t_{k-1}-t_k}\mathcal{C}^0_{s+k-1,s+k}\mathcal{T}^{s+k,0}_{t_k} f_0^{(s+k)}(t_k,\cdot)\,\mathrm{d}t_k\ldots\mathrm{d}t_2\,\mathrm{d}t_1.$$

Explicitly, for the second term (that is, the first one involving an integrated in time collision-transport operator), one has :

$$\int_0^t \mathcal{T}^{s,0}_{t-t_1}\mathcal{C}^0_{s,s+1}\mathcal{T}^{s+1,0}_{t_1} f_0^{(s+1)}(t_1,\cdot)\,\mathrm{d}t_1 = \sum_{j_1=1}^{s}\Big(\Big(\mathcal{I}^0_{s\atop +,j_1} - \mathcal{I}^0_{s\atop -,j_1}\Big)\big(t_1 \mapsto \mathcal{T}^{s+1,0}_{t_1} f_0^{(s+1)}\big)\Big)(t,\cdot)$$

where $\mathcal{I}^0_{s\atop +,j_1} f^{(s+1)}$ and $\mathcal{I}^0_{s\atop -,j_1} f^{(s+1)}$ denote respectively

$$\int_0^t \mathcal{T}^{s,0}_{t-t_1}\Big(\int_{\mathbb{S}^{d-1}_\omega\times\mathbb{R}^d_{v_{s+1}}} \big[\omega\cdot(v_{s+1}-v_{j_1})\big]_+ f^{(s+1)}\big(t_1,(Z_s,x_{j_1},v_{s+1})'_{j_1,s+1}\big)\,\mathrm{d}\omega\,\mathrm{d}v_{s+1}\Big)\,\mathrm{d}t_1$$

and

$$\int_0^t \mathcal{T}^{s,0}_{t-t_1}\Big(\int_{\mathbb{S}^{d-1}_\omega\times\mathbb{R}^d_{v_{s+1}}} \big[\omega\cdot(v_{s+1}-v_{j_1})\big]_- f^{(s+1)}\big(t_1,Z_s,x_{j_1},v_{s+1}\big)\,\mathrm{d}\omega\,\mathrm{d}v_{s+1}\Big)\,\mathrm{d}t_1,$$

that is :

$$\begin{aligned}
&\int_0^t \mathcal{T}^{s,0}_{t-t_1}\mathcal{C}^0_{s,s+1}\mathcal{T}^{s+1,0}_{t_1} f_0^{(s+1)}(t_1,\cdot)\,\mathrm{d}t_1\\
&= \sum_{j_1=1}^{s}\int_0^t\int_{\mathbb{S}^{d-1}_\omega\times\mathbb{R}^d_{v_{s+1}}} \Big[\omega\cdot\Big(v_{s+1}-\big(T^{s,0}_{t_1-t}(Z_s)\big)^{V,j_1}\Big)\Big]_+\\
&\quad\times\Bigg[f_0^{(s+1)}\Big(T^{s+1,0}_{-t_1}\Big(\Big(T^{s,0}_{t_1-t}(Z_s),\big(T^{s,0}_{t_1-t}(Z_s)\big)^{X,j_1},v_{s+1}\Big)'_{j_1,s+1}\Big)\Big)\\
&\qquad - f_0^{(s+1)}\Big(T^{s+1,0}_{-t_1}\Big(T^{s,0}_{t_1-t}(Z_s),\big(T^{s,0}_{t_1-t}(Z_s)\big)^{X,j_1},v_{s+1}\Big)\Big)\Bigg]\,\mathrm{d}\omega\,\mathrm{d}v_{s+1}\,\mathrm{d}t_1,
\end{aligned} \tag{12.2}$$

or again, after the change of variable $\omega \to -\omega$:

$$
\begin{aligned}
&\int_0^t \mathcal{T}^{s,0}_{t-t_1} \mathcal{C}^0_{s,s+1} \mathcal{T}^{s+1,0}_{t_1} f_0^{(s+1)}(t_1,\cdot)\,\mathrm{d}t_1 \\
&= \sum_{j_1=1}^{s} \int_0^t \int_{\mathbb{S}^{d-1}_\omega \times \mathbb{R}^d_{v_{s+1}}} \Big[\omega \cdot \Big(v_{s+1} - \big(T^{s,0}_{t_1-t}(Z_s)\big)^{V,j_1}\Big)\Big]_+ \\
&\qquad \times f_0^{(s+1)}\Big(T^{s+1,0}_{-t_1}\Big(\Big(T^{s,0}_{t_1-t}(Z_s), \big(T^{s,0}_{t_1-t}(Z_s)\big)^{X,j_1}, v_{s+1}\Big)'_{j_1,s+1}\Big)\Big)\,\mathrm{d}\omega\,\mathrm{d}v_{s+1}\,\mathrm{d}t_1 \\
&- \sum_{j_1=1}^{s} \int_0^t \int_{\mathbb{S}^{d-1}_\omega \times \mathbb{R}^d_{v_{s+1}}} \Big[\omega \cdot \Big(v_{s+1} - \big(T^{s,0}_{t_1-t}(Z_s)\big)^{V,j_1}\Big)\Big]_- \\
&\qquad \times f_0^{(s+1)}\Big(T^{s+1,0}_{-t_1}\Big(T^{s,0}_{t_1-t}(Z_s), \big(T^{s,0}_{t_1-t}(Z_s)\big)^{X,j_1}, v_{s+1}\Big)\Big)\,\mathrm{d}\omega\,\mathrm{d}v_{s+1}\,\mathrm{d}t_1. \qquad (12.3)
\end{aligned}
$$

One gives also the third one, for the reader interested in the explicit forms of such terms :

$$
\begin{aligned}
&\int_0^t \mathcal{T}^{s,0}_{t-t_1} \mathcal{C}^0_{s,s+1} \int_0^{t_1} \mathcal{T}^{s+1,0}_{t_1-t_2} \mathcal{C}^0_{s+1,s+2} \mathcal{T}^{s+2,0}_{t_2} f_0^{(s+2)}(t_2,\cdot)\,\mathrm{d}t_2\,\mathrm{d}t_1 \\
&\qquad = \sum_{j_1=1}^{s} \sum_{j_2=1}^{s+1} \Big(\Big(\big(\mathcal{I}^0_{s\,+,j_1} - \mathcal{I}^0_{s\,-,j_1}\big) \circ \big(\mathcal{I}^0_{s+1\,+,j_2} - \mathcal{I}^0_{s+1\,-,j_2}\big)\Big) \\
&\qquad\qquad \Big(t_2 \mapsto \mathcal{T}^{s+2,0}_{t_2} f_0^{(s+2)}\Big)\Big)(t,\cdot),
\end{aligned}
$$

that is, after the change of variables $\omega \to -\omega$:

$$
\begin{aligned}
&\int_0^t \mathcal{T}^{s,0}_{t-t_1} \mathcal{C}^0_{s,s+1} \int_0^{t_1} \mathcal{T}^{s+1,0}_{t_1-t_2} \mathcal{C}^0_{s+1,s+2} \mathcal{T}^{s+2,0}_{t_2} f_0^{(s+2)}(t_2,\cdot)\,\mathrm{d}t_2\,\mathrm{d}t_1 \\
&= \sum_{j_1=1}^{s} \sum_{\pm_1} \sum_{j_2=1}^{s+1} \sum_{\pm_2} \int_0^t \int_{\mathbb{S}^{d-1}_{\omega_1}\times\mathbb{R}^d_{v_{s+1}}} (\pm_1) \Big[\omega_1 \cdot \big(v_{s+1} - \big(T^{s,0}_{t_1-t}(Z_s)\big)^{V,j_1}\big)\Big]_{\pm_1} \\
&\qquad\qquad \times \int_0^{t_1} \int_{\mathbb{S}^{d-1}_{\omega_2}\times\mathbb{R}^d_{v_{s+2}}} (\pm_2) \\
&\Big[\omega_2 \cdot \Big(v_{s+2} - \Big(T^{s+1,0}_{t_2-t_1}\big(\big(T^{s,0}_{t_1-t}(Z_s), \big(T^{s,0}_{t_1-t}(Z_s)\big)^{X,j_1}, v_{s+1}\big)^{\prime(\text{if } \pm_1=+)}_{j_1,s+1}\big)\Big)^{V,j_2}\Big)\Big]_{\pm_2} \\
&\quad \times f_0^{(s+2)}\Big(T^{s+2,0}_{-t_2}\Big(\Big(T^{s+1,0}_{t_2-t_1}\Big(\Big(T^{s,0}_{t_1-t}(Z_s), \big(T^{s,0}_{t_1-t}(Z_s)\big)^{X,j_1}, v_{s+1}\Big)^{\prime(\text{if } \pm_1=+)}_{j_1,s+1}\Big), \\
&\Big(T^{s+1,0}_{t_2-t_1}\Big(\Big(T^{s,0}_{t_1-t}(Z_s), \big(T^{s,0}_{t_1-t}(Z_s)\big)^{X,j_1}, v_{s+1}\Big)^{\prime(\text{if } \pm_1=+)}_{j_1,s+1}\Big)\Big)^{X,j_2}, v_{s+2}\Big)^{\prime(\text{if } \pm_2=+)}_{j_2,s+2}\Big)\Big) \\
&\qquad\qquad\qquad\qquad\qquad\qquad \mathrm{d}\omega_2\,\mathrm{d}v_{s+2}\,\mathrm{d}t_2\,\mathrm{d}\omega_1\,\mathrm{d}v_{s+1}\,\mathrm{d}t_1,
\end{aligned}
\tag{12.4}
$$

or again, focusing on the decomposition described in this section :

$$
\begin{aligned}
&= \sum_{j_1=1}^{s} \sum_{j_2=1}^{s+1} \int_0^t \int_{\mathbb{S}^{d-1}_{\omega_1}\times\mathbb{R}^d_{v_{s+1}}} \Big[\omega_1 \cdot \big(v_{s+1} - \big(T^{s,0}_{t_1-t}(Z_s)\big)^{V,j_1}\big)\Big]_+ \\
&\times \int_0^{t_1} \int_{\mathbb{S}^{d-1}_{\omega_2}\times\mathbb{R}^d_{v_{s+2}}} \Big[\omega_2 \cdot \Big(v_{s+2} - \Big(T^{s+1,0}_{t_2-t_1}\big(\big(T^{s,0}_{t_1-t}(Z_s), \big(T^{s,0}_{t_1-t}(Z_s)\big)^{X,j_1}, v_{s+1}\big)^{\prime}_{j_1,s+1}\big)\Big)^{V,j_2}\Big)\Big]_+ \\
&\times f_0^{(s+2)}\Big(T^{s+2,0}_{-t_2}\Big(\Big(T^{s+1,0}_{t_2-t_1}\Big(\Big(T^{s,0}_{t_1-t}(Z_s), \big(T^{s,0}_{t_1-t}(Z_s)\big)^{X,j_1}, v_{s+1}\Big)^{\prime}_{j_1,s+1}\Big), \\
&\qquad \Big(T^{s+1,0}_{t_2-t_1}\Big(\Big(T^{s,0}_{t_1-t}(Z_s), \big(T^{s,0}_{t_1-t}(Z_s)\big)^{X,j_1}, v_{s+1}\Big)^{\prime}_{j_1,s+1}\Big)\Big)^{X,j_2}, v_{s+2}\Big)^{\prime}_{j_2,s+2}\Big)\Big) \\
&\qquad\qquad\qquad\qquad\qquad\qquad \mathrm{d}\omega_2\,\mathrm{d}v_{s+2}\,\mathrm{d}t_2\,\mathrm{d}\omega_1\,\mathrm{d}v_{s+1}\,\mathrm{d}t_1
\end{aligned}
$$

(terms corresponding to $\pm_1, \pm_2 = +$) (12.5)

$$
\begin{aligned}
&- \sum_{j_1=1}^{s} \sum_{j_2=1}^{s+1} \int_0^t \int_{\mathbb{S}^{d-1}_{\omega_1} \times \mathbb{R}^d_{v_{s+1}}} \Big[\omega_1 \cdot \big(v_{s+1} - \big(T^{s,0}_{t_1-t}(Z_s) \big)^{V,j_1} \big) \Big]_- \\
&\times \int_0^{t_1} \int_{\mathbb{S}^{d-1}_{\omega_2} \times \mathbb{R}^d_{v_{s+2}}} \Big[\omega_2 \cdot \Big(v_{s+2} - \Big(T^{s+1,0}_{t_2-t_1} \big(T^{s,0}_{t_1-t}(Z_s), \big(T^{s,0}_{t_1-t}(Z_s) \big)^{X,j_1}, v_{s+1} \big) \Big)^{V,j_2} \Big) \Big]_+ \\
&\times f_0^{(s+2)} \Big(T^{s+2,0}_{-t_2} \Bigg(\Bigg(T^{s+1,0}_{t_2-t_1} \Big(T^{s,0}_{t_1-t}(Z_s), \big(T^{s,0}_{t_1-t}(Z_s) \big)^{X,j_1}, v_{s+1} \Big), \\
&\qquad \Big(T^{s+1,0}_{t_2-t_1} \Big(T^{s,0}_{t_1-t}(Z_s), \big(T^{s,0}_{t_1-t}(Z_s) \big)^{X,j_1}, v_{s+1} \Big) \Big)^{X,j_2}, v_{s+2} \Bigg)'_{j_2,s+2} \Bigg) \Big) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \mathrm{d}\omega_2 \, \mathrm{d}v_{s+2} \, \mathrm{d}t_2 \, \mathrm{d}\omega_1 \, \mathrm{d}v_{s+1} \, \mathrm{d}t_1
\end{aligned}
$$

(terms corresponding to $\pm_1 = -, \pm_2 = +$)

$$
\begin{aligned}
&- \sum_{j_1=1}^{s} \sum_{j_2=1}^{s+1} \int_0^t \int_{\mathbb{S}^{d-1}_{\omega_1} \times \mathbb{R}^d_{v_{s+1}}} \Big[\omega_1 \cdot \big(v_{s+1} - \big(T^{s,0}_{t_1-t}(Z_s) \big)^{V,j_1} \big) \Big]_+ \\
&\times \int_0^{t_1} \int_{\mathbb{S}^{d-1}_{\omega_2} \times \mathbb{R}^d_{v_{s+2}}} \Big[\omega_2 \cdot \Big(v_{s+2} - \Big(T^{s+1,0}_{t_2-t_1} \big(\big(T^{s,0}_{t_1-t}(Z_s), \big(T^{s,0}_{t_1-t}(Z_s) \big)^{X,j_1}, v_{s+1} \big)'_{j_1,s+1} \big) \Big)^{V,j_2} \Big) \Big]_- \\
&\times f_0^{(s+2)} \Big(T^{s+2,0}_{-t_2} \Bigg(T^{s+1,0}_{t_2-t_1} \Big(\Big(T^{s,0}_{t_1-t}(Z_s), \big(T^{s,0}_{t_1-t}(Z_s) \big)^{X,j_1}, v_{s+1} \Big)'_{j_1,s+1} \Big), \\
&\qquad \Big(T^{s+1,0}_{t_2-t_1} \Big(\Big(T^{s,0}_{t_1-t}(Z_s), \big(T^{s,0}_{t_1-t}(Z_s) \big)^{X,j_1}, v_{s+1} \Big)'_{j_1,s+1} \Big) \Big)^{X,j_2}, v_{s+2} \Bigg) \Big) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \mathrm{d}\omega_2 \, \mathrm{d}v_{s+2} \, \mathrm{d}t_2 \, \mathrm{d}\omega_1 \, \mathrm{d}v_{s+1} \, \mathrm{d}t_1
\end{aligned}
$$

(terms corresponding to $\pm_1 = +, \pm_2 = -$)

$$
\begin{aligned}
&+ \sum_{j_1=1}^{s} \sum_{j_2=1}^{s+1} \int_0^t \int_{\mathbb{S}^{d-1}_{\omega_1} \times \mathbb{R}^d_{v_{s+1}}} \Big[\omega_1 \cdot \big(v_{s+1} - \big(T^{s,0}_{t_1-t}(Z_s) \big)^{V,j_1} \big) \Big]_- \\
&\times \int_0^{t_1} \int_{\mathbb{S}^{d-1}_{\omega_2} \times \mathbb{R}^d_{v_{s+2}}} \Big[\omega_2 \cdot \Big(v_{s+2} - \Big(T^{s+1,0}_{t_2-t_1} \big(T^{s,0}_{t_1-t}(Z_s), \big(T^{s,0}_{t_1-t}(Z_s) \big)^{X,j_1}, v_{s+1} \big) \Big)^{V,j_2} \Big) \Big]_- \\
&\times f_0^{(s+2)} \Big(T^{s+2,0}_{-t_2} \Big(T^{s+1,0}_{t_2-t_1} \big(T^{s,0}_{t_1-t}(Z_s), \big(T^{s,0}_{t_1-t}(Z_s) \big)^{X,j_1}, v_{s+1} \big), \\
&\qquad \Big(T^{s+1,0}_{t_2-t_1} \Big(T^{s,0}_{t_1-t}(Z_s), \big(T^{s,0}_{t_1-t}(Z_s) \big)^{X,j_1}, v_{s+1} \Big) \Big)^{X,j_2}, v_{s+2} \Big) \Big) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \mathrm{d}\omega_2 \, \mathrm{d}v_{s+2} \, \mathrm{d}t_2 \, \mathrm{d}\omega_1 \, \mathrm{d}v_{s+1} \, \mathrm{d}t_1
\end{aligned}
$$

(terms corresponding to $\pm_1, \pm_2 = -$).

One will focus later on a way to simplify the computation and the manipulation of such complicated expressions.

Remark 27. *One notes that a difference between the iterates of the integrated in time (transport-) collision-transport operators of the two hierarchies lies in the presence or not of the prefactors. Namely, in the case of the BBGKY hierarchy, the transport-collision operators defined in Section 5.1 page 88 are composed of the elementary operators $\mathcal{C}^{\varepsilon}_{s,s+1,\pm,i}$, multiplied by the factor $(N-s)$. Of course, when those operators are iterated, a factor*

$$(N-s)(N-s-1)\dots(N-s-k+1)$$

appears. Keeping in mind that the measure used to integrate the transported function $\mathcal{T}^{s+1,\varepsilon}_t f^{(s+1)}$ on which acts the collision operator is

$$\varepsilon^{d-1} \big(\omega \cdot (v_{s+1} - v_i) \big)_{\pm} \, \mathrm{d}\omega \, \mathrm{d}v_{s+1},$$

the prefactor in front of the sum of all the iterated transport-collision-transport operators of the BBGKY hierarchy (for k iterates) is

$$(N-s)(N-s-1)\dots(N-s-k+1)\varepsilon^{k(d-1)}.$$

Back to the sketch of proof for the comparison, taking the example of the second term (written in (12.3) for the Boltzmann hierarchy), one expects that it will be possible to compare the quantities :

$$T^{s+1,0}_{-t_1} \Big(\Big(T^{s,0}_{t_1-t}(Z_s), \big(T^{s,0}_{t_1-t}(Z_s) \big)^{X,j_1}, v_{s+1} \Big)'_{j_1,s+1} \Big)$$

for the Boltzmann hierarchy, and :

$$T^{s+1,\varepsilon}_{-t_1} \Big(\Big(T^{s,\varepsilon}_{t_1-t}(Z_s), \big(T^{s,\varepsilon}_{t_1-t}(Z_s) \big)^{X,j_1}, v_{s+1} \Big)'_{j_1,s+1} \Big)$$

for the BBGKY hierarchy, when ε is small.
Then one wants to compare the integral terms, that is, the elements of the two respective hierarchies implying the same dynamics. In other words, one will compare, pairwise, each term of the decomposition described just above (in the general case, but in an abstract way for the BBGKY hierarchy in (12.1), and in an explicit way for the Boltzmann hierarchy in (12.2) for the case $k = 1$, and in (12.4) for the case $k = 2$) of the solution of the BBGKY hierarchy with the associated term of the solution of the Boltzmann hierarchy.
One can say, somehow, that those elements are *of same type*, since they are obtained, starting from the same initial configuration Z_s, after modifications related to adjunctions of new particles, and choices of pre- or post-collisional configurations.

For the sake of simplicity, the description of this new decomposition was done starting from the Duhamel formula obtained in Proposition 9, that is, before the preliminary cut-offs in high number of collisions (see section 11.1 page 277), large energy (see section 11.2 page 280) and in small time difference between the collisions (see section 11.3 page 290). Of course those cut-offs will play a decisive role in the comparison of the new elementary terms just described, so one will start from the quantities $H_N^{n,R,\delta}$ and $F^{n,R,\delta}$ (see Definition 40 page 297 for the introduction of those two quantities) to define properly the elementary terms. One will then introduce once again new notations, which will take into account the corresponding decomposition.

One starts by a simple convention on the indices, which will simplify the notations for the sum over all the possible decompositions into elements of same type.

Definition 41 (Convention for the indices of the decomposition into elements of same type)**.** *For any positive integers s and k, one denotes the* set of the sequences of selected particles for the adjunctions *by :*

$$\mathfrak{J}_k^s$$

the set of vectors, composed with the generic elements denoted :

$$J_k = (j_1, j_2, \ldots, j_k)$$

such that $j_1, \ldots, j_k$ are all positive integers verifying, for all $1 \leq i \leq k$:

$$1 \leq j_i \leq s + i - 1.$$

For any positive integer k, one denotes by :

$$\mathfrak{M}_k$$

the set of "vectors", composed with the generic elements denoted :

$$M_k = (\pm_1, \ldots, \pm_k).$$

Remark 28. *For an initial configuration $Z_s \in \mathcal{D}_s^\varepsilon$, the set $\mathfrak{J}_k^s$ describes all the possible ways J_k to add recursively particles to the set of s particles starting from the position Z_s : after an evolution following for a while the hard sphere or the free flow, one will choose the particle j_1 to add a particle next to it, let the system evolves again under the hard sphere or the free flow for a while, and then add again a particle next to the particle j_2, and so on.*
Similarly, the set $\mathfrak{M}_k$ describes if one chooses to add, at each step, the particle in a pre or in a post-collisional configuration.

One starts here from the notations (10.1) and (10.7) for the iterates of the (transport)-collision-transport operators, introduced respectively in Definitions 33 page 267 and 34 page 271. One will now refine a bit those notations, and one will also use the notations about the cut-off in small time difference between the collisions, introduced in Definitions 37 page 292, 38 page 292 and 39 page 296.

Definition 42 (Decomposition of the solutions of the BBGKY hierarchy into elements of same type)**.** *For any positive integer s, any integer $1 \leq j \leq s$, any sign $\pm = +$ or $-$, and any function $f_{N,0}^{(s+1)}$ belonging to the space $X_{\varepsilon,s,\beta}$, one will denote its image by the element of the integrated in time transport-collision-transport operator of the BBGKY hierarchy*

$$t \mapsto (N-s) \int_0^t \mathcal{T}_{-t_1}^{s,\varepsilon} \mathcal{C}_{s,s+1,\pm,j}^{\varepsilon} \mathcal{T}_{t_1}^{s+1,\varepsilon} f_{N,0}^{(s+1)} \, \mathrm{d}t_1$$

as :

$$t \mapsto \big(\mathcal{I}_{\substack{s \\ \pm,j}}^{N,\varepsilon} f_{N,0}^{(s+1)}\big)(t,\cdot). \tag{12.6}$$

For any positive integer k, any elements $M_k = (\pm_1, \ldots, \pm_k) \in \mathfrak{M}_k$ and $J_k = (j_1, \ldots, j_k) \in \mathfrak{J}_k^s$, the iterates of such operators, that is :

$$t \mapsto \frac{(N-s)!}{(N-s-k)!} \int_0^t \mathcal{T}_{-t_1}^{s,\varepsilon} \mathcal{C}_{s,s+1,\pm_1,j_1}^{\varepsilon} \mathcal{T}_{t_1}^{s+1,\varepsilon} \int_0^{t_1} \mathcal{T}_{-t_2}^{s+1,\varepsilon} \mathcal{C}_{s,s+1,\pm_2,j_2}^{\varepsilon} \mathcal{T}_{t_2}^{s+2,\varepsilon} \ldots$$
$$\int_0^{t_{k-1}} \mathcal{T}_{-t_k}^{s+k-1,\varepsilon} \mathcal{C}_{s+k-1,s+k,\pm_k,j_k}^{\varepsilon} \mathcal{T}_{t_k}^{s+k,\varepsilon} f_{N,0}^{(s+k)} \, \mathrm{d}t_k \ldots \mathrm{d}t_2 \, \mathrm{d}t_1$$

will be denoted as :

$$t \mapsto \big(\mathcal{I}_{\substack{s,s+k-1 \\ M_k,J_k}}^{N,\varepsilon} f_{N,0}^{(s+k)}\big)(t,\cdot). \tag{12.7}$$

Considering the cut-off in small time difference between collisions as in Definition 37 page 292, one denotes the element of type M_k, J_k of the BBGKY hierarchy *as*

$$t \mapsto \big(\mathcal{I}_{\substack{s,s+k-1 \\ M_k,J_k}}^{N,\varepsilon,\delta} f_{N,0}^{(s+k)}\big)(t,\cdot). \tag{12.8}$$

One introduces the same notations for the Boltzmann hierarchy.

Definition 43 (Decomposition of the solutions of the Boltzmann hierarchy into elements of same type)**.** *For any positive integer s, any integer $1 \leq j \leq s$, any sign $\pm = +$ or $-$, and any function of sequences $t \mapsto \left(f^{(s+1)}\right)_{s\geq 1}$ belonging to the space $X_{0,\widetilde{\beta},\widetilde{\mu}^1}$, one will denote the image of its s-th term by the element of the integrated in time collision-transport operator of the Boltzmann hierarchy :*

$$t \mapsto \int_0^t \mathcal{T}^{s,0}_{t-t_1} \mathcal{C}^0_{s,s+1,\pm,j} f^{(s+1)}(t_1,\cdot)\,\mathrm{d}t_1$$

as :

$$t \mapsto \left(\mathcal{I}^0_{\substack{s\\ \pm,j}} f^{(s+1)}\right)(t,\cdot). \tag{12.9}$$

For any positive integer k, any elements $M_k = (\pm_1,\ldots,\pm_k) \in \mathfrak{M}_k$ and $J_k = (j_1,\ldots,j_k) \in \mathfrak{J}^s_k$, the iterates of such operators, that is :

$$t \mapsto \int_0^t \mathcal{T}^{s,0}_{t-t_1} \mathcal{C}^0_{s,s+1,\pm_1,j_1} \int_0^{t_1} \mathcal{T}^{s+1,0}_{t_1-t_2} \mathcal{C}^0_{s,s+1,\pm_2,j_2} \cdots \\ \int_0^{t_{k-1}} \mathcal{T}^{s+k-1,0}_{t_{k-1}-t_k} \mathcal{C}^0_{s+k-1,s+k,\pm_k,j_k} f^{(s+k)}(t_k,\cdot)\,\mathrm{d}t_k \ldots \mathrm{d}t_2\,\mathrm{d}t_1$$

will be denoted as :

$$t \mapsto \left(\mathcal{I}^0_{\substack{s,s+k-1\\ M_k,J_k}} f^{(s+k)}\right)(t,\cdot). \tag{12.10}$$

Considering the cut-off in small time difference between collisions as in Definition 37, one denotes the element of type M_k, J_k of the Boltzmann hierarchy *as :*

$$t \mapsto \left(\mathcal{I}^{0,\delta}_{\substack{s,s+k-1\\ M_k,J_k}} f^{(s+k)}\right)(t,\cdot). \tag{12.11}$$

Using the notations just introduced, one can decompose the truncated solutions $H^{n,R,\delta}_N$ and $F^{n,R,\delta}$ of the two hierarchies. Starting from the Definition 40 page 297 of the truncated solutions, and decomposing, for all integer $1 \leq l \leq k$ each operator as

$$\mathcal{I}^{\cdot,\delta}_{s+l-1} = \sum_{\pm_l} \sum_{1\leq j_l \leq s+l-1} \mathcal{I}^{\cdot,\delta}_{\substack{s+l-1\\ \pm_l,j_l}},$$

and then by iterations and linearity

$$\begin{aligned}\mathcal{I}^{\cdot,\delta}_{s,s+k-1} &= \mathcal{I}^{\cdot,\delta}_{s} \circ \cdots \circ \mathcal{I}^{\cdot,\delta}_{s+k-1} \\ &= \sum_{\pm_1} \sum_{1\leq j_1\leq s} \cdots \sum_{\pm_k} \sum_{1\leq j_k \leq s+k-1} \mathcal{I}^{\cdot,\delta}_{\substack{s\\ \pm_1,j_1}} \circ \cdots \circ \mathcal{I}^{\cdot,\delta}_{\substack{s+k-1\\ \pm_k,j_k}},\end{aligned}$$

(see Definitions 37 page 292 and 38 page 292 for the notation $\mathcal{I}_{s+l-1}^{\cdot,\delta}$, and 39 page 296 for the notation $\mathcal{I}_{s,s+k-1}^{\cdot,\delta}$) and finally using the convention for the indices introduced in Definition 41 page 316, one obtains :

$$\mathcal{I}_{s,s+k-1}^{\cdot,\delta} = \sum_{M_k \in \mathfrak{M}_k} \sum_{J_k \in \mathfrak{J}_k^s} \mathcal{I}_{s,s+k-1,\,M_k,J_k}^{\cdot,\delta}.$$

It provides

$$\begin{aligned} H_N^{n,R,\delta} &= t \mapsto \Big(f_{N,0}^{(s)}(\cdot) \mathbb{1}_{|V_s| \le R} \\ &\qquad + \sum_{k=1}^{n} \mathbb{1}_{s \le N-k} \big(\mathcal{I}_{s,s+k-1}^{N,\varepsilon,\delta} \big(f_{N,0}^{(s+k)} \mathbb{1}_{|V_{s+k}| \le R} \big) \big)(t,\cdot) \Big)_{1 \le s \le N} \\ &= t \mapsto \Big(f_{N,0}^{(s)}(\cdot) \mathbb{1}_{|V_s| \le R} \\ &+ \sum_{k=1}^{n} \mathbb{1}_{s \le N-k} \sum_{M_k \in \mathfrak{M}_k} \sum_{J_k \in \mathfrak{J}_k^s} \prod_{l=1}^{k} (\pm_l) \big(\mathcal{I}_{s,s+k-1,\,M_k,J_k}^{N,\varepsilon,\delta} \big(f_{N,0}^{(s+k)} \mathbb{1}_{|V_{s+k}| \le R} \big) \big)(t,\cdot) \Big)_{1 \le s \le N} \end{aligned}$$

and

$$\begin{aligned} F^{n,R,\delta} &= t \mapsto \Big(\mathcal{T}_t^{s,0} f_0^{(s)}(\cdot) \mathbb{1}_{|V_s| \le R} \\ &\qquad + \sum_{k=1}^{n} \big(\mathcal{I}_{s,s+k-1,\,M_k,J_k}^{0,\delta} \big(u \mapsto \mathcal{T}_u^{s+k,0} f_0^{(s+k)} \mathbb{1}_{|V_{s+k}| \le R} \big) \big)(t,\cdot) \Big)_{s \ge 1} \\ &= t \mapsto \Big(\mathcal{T}_t^{s,0} f_0^{(s)}(\cdot) \mathbb{1}_{|V_s| \le R} \\ &+ \sum_{k=1}^{n} \sum_{M_k \in \mathfrak{M}_k} \sum_{J_k \in \mathfrak{J}_k^s} \prod_{l=1}^{k} (\pm_l) \big(\mathcal{I}_{s,s+k-1,\,M_k,J_k}^{0,\delta} \big(u \mapsto \mathcal{T}_u^{s+k,0} f_0^{(s+k)} \mathbb{1}_{|V_{s+k}| \le R} \big) \big)(t,\cdot) \Big)_{s \ge 1}. \end{aligned}$$

Introducing a notation for the pseudo-trajectories

For the sake of simplicity, one will introduce once again new notations. The purpose of the following notations is to make readable the decompositions written explicitly in (12.3) page 312 and (12.4) page 313.
For example, the equation (12.4) gives the third term of the sum of the iterated Duhamel formula (10.9) page 272, which is a way to express explicitly the solution of the Boltzmann hierarchy. This quantity is itself decomposed into a sum, with four terms, the first one being given by the equation 12.5 page 313,

namely :

$$\sum_{j_1=1}^{s}\sum_{j_2=1}^{s+1}\int_0^t\int_{\mathbb{S}^{d-1}_{\omega_1}\times\mathbb{R}^d_{v_{s+1}}}\Big[\omega_1\cdot\big(v_{s+1}-\big(T^{s,0}_{t_1-t}(Z_s)\big)^{V,j_1}\big)\Big]_+$$
$$\times\int_0^{t_1}\int_{\mathbb{S}^{d-1}_{\omega_2}\times\mathbb{R}^d_{v_{s+2}}}\Big[\omega_2\cdot\Big(v_{s+2}-\Big(T^{s+1,0}_{t_2-t_1}\big(\big(T^{s,0}_{t_1-t}(Z_s),\big(T^{s,0}_{t_1-t}(Z_s)\big)^{X,j_1},v_{s+1}\big)'_{j_1,s+1}\big)\Big)^{V,j_2}\Big)\Big]_+$$
$$\times f_0^{(s+2)}\Big(T^{s+2,0}_{-t_2}\Big(\Big(T^{s+1,0}_{t_2-t_1}\Big(\Big(T^{s,0}_{t_1-t}(Z_s),\big(T^{s,0}_{t_1-t}(Z_s)\big)^{X,j_1},v_{s+1}\Big)'_{j_1,s+1}\Big),$$
$$\Big(T^{s+1,0}_{t_2-t_1}\Big(\Big(T^{s,0}_{t_1-t}(Z_s),\big(T^{s,0}_{t_1-t}(Z_s)\big)^{X,j_1},v_{s+1}\Big)'_{j_1,s+1}\Big)\Big)^{X,j_2},v_{s+2}\Big)'_{j_2,s+2}\Big)\Big)$$
$$\mathrm{d}\omega_2\,\mathrm{d}v_{s+2}\,\mathrm{d}t_2\,\mathrm{d}\omega_1\,\mathrm{d}v_{s+1}\,\mathrm{d}t_1$$

which corresponds to :

$$\underset{(+,+),(j_1,j_2)}{\mathcal{I}^0_{s,s+1}}\big(t\mapsto\mathcal{T}^{s+2,0}_t f_0^{(s+2)}\big),$$

according to the notations of Definition 43 page 318.

This example should be quite convincing that it is mandatory to introduce a notation to describe the position of a system of particles after several applications of the transport and the adjunction of a new particle.

First, one will however discuss a bit the concept of "pseudo-trajectories". In the case of the Boltzmann hierarchy, the argument taken by the initial datum $f_0^{(s+1)}$ in the first term $\underset{(+),(j_1)}{\mathcal{I}^0_s}\big(t\mapsto\mathcal{T}^{s+1,0}_t f_0^{(s+1)}\big)$ (see (12.3) page 312) is :

$$T^{s+1,0}_{-t_1}\Big(\Big(T^{s,0}_{t_1-t}(Z_s),\big(T^{s,0}_{t_1-t}(Z_s)\big)^{X,j_1},v_{s+1}\Big)'_{j_1,s+1}\Big).$$

If one looks carefully at this quantity, one sees that it is obtained following the steps below, presented for the two hierarchies.

1. First, one chooses an initial datum Z_s, taken in the phase space of s particles.

2. Then this initial datum is transported by $T^{s,0}$ in the case of the Boltzmann hierarchy, and by $T^{s,\varepsilon}$ in the case of the BBGKY hierarchy, during a time t_1-t, that is the configuration Z_s evolves naturally during t_1-t, following the laws of the free transport with boundary condition in the case of the Boltzmann hierarchy, or following the hard sphere transport during the same time in the case of the BBGKY hierarchy.

3. Then, and this is a crucial step, another particle is added to the system of s particles. For the Boltzmann hierarchy, at time $t_1 - t$, the particle $s+1$ will start from the position of the particle j_1, that is $T^{s,0}_{t_1-t}(Z_s)$, and one will choose v_{s+1} as the velocity of this new particle, and ω_1 as the angular parameter. Afterwards, if the following quantity is strictly positive :

$$\omega \cdot \Big(v_{s+1} \big(T^{s,0}_{t_1-t}(Z_s) \big)^{V,j_1} \Big)$$

in the case of the Boltzmann hierarchy, and

$$\omega \cdot \Big(v_{s+1} \big(T^{s,\varepsilon}_{t_1-t}(Z_s) \big)^{V,j_1} \Big)$$

in the case of the BBGKY hierarchy, the scattering (see Definition 1 page 51) is applied to the pair of velocities of the two particles j_1 and $s+1$ of this new configuration of $s+1$ particles, that is one considers the quantity :

$$\Big(T^{s,0}_{t_1-t}(Z_s), \big(T^{s,0}_{t_1-t}(Z_s) \big)^{X,j_1}, v_{s+1} \Big)'_{j_1,s+1}.$$

The use of the scattering implies that the angular parameter ω plays an important role at this step. However, in the case of a pair of particles in a pre-collisional configuration, the angular parameter does not play any role for the Boltzmann hierarchy.
But it does in the case of the BBGKY hierarchy, since the new particle is added at the position :

$$\big(T^{s,0}_{t_1-t}(Z_s) \big)^{X,j_1} + \varepsilon\omega,$$

so that in the case of the hard sphere dynamics, one considers the new system of $s+1$ particles in the configuration :

$$\Big(T^{s,\varepsilon}_{t_1-t}(Z_s), \big(T^{s,\varepsilon}_{t_1-t}(Z_s) \big)^{X,j_1} + \varepsilon\omega, v_{s+1} \Big)'_{j_1,s+1}.$$

One can see here that there is a difference between the different variables used to describe entirely the dynamics. The initial configuration Z_s has to be considered as fixed, while the adjunction parameters, namely the velocity v_{s+1} of the new particle and the angular parameter ω are integration variables : one will consider all the possible trajectories obtained when (ω, v_{s+1}) varies in $\mathbb{S}^{d-1} \times \mathbb{R}^d$. The quantity $t - t_1$ has an intermediate status, since the time t is fixed as Z_s, but t_1 varies in the interval $[0, t]$, and one integrates over this time interval.

4. And finally, one transports during a time t_1 the new system of $s+1$ particles.

The previous steps are illustrated on Figure 12.1 below.

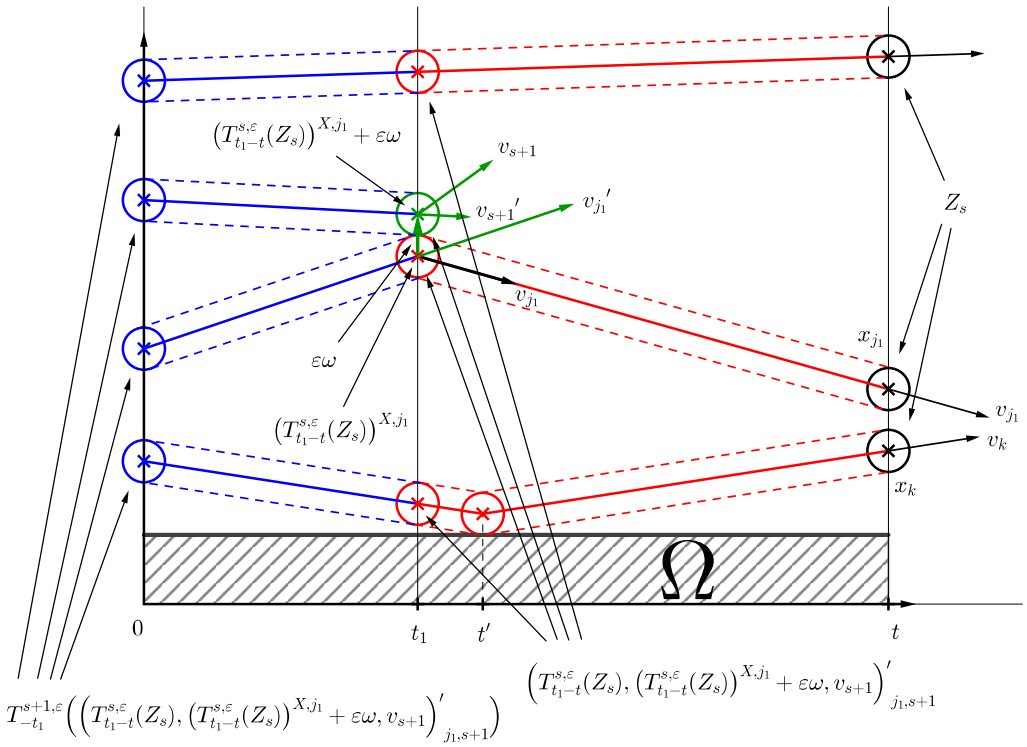

Figure 12.1: Construction, step by step, of a pseudo-trajectory starting from the configuration Z_s, for the BBGKY hierarchy

Remark 29. *The respective steps of the construction of the pseudo-trajectory are in the following colours. In black : the choice of the initial configuration Z_s in the phase space at time t (step 1), in red : the transport of this configuration during $t_1 - t$ (step 2), in green : the adjunction of a new particle and the scattering (step 3) and finally in blue : the last transport of the new configuration of $s+1$ particles, during $-t_1$ (step 4).*
Note that on the figure, at time t', a bouncing of the particle k against the obstacle Ω, which follows the hard sphere flow with boundary condition, has been represented.

Now, one provides a few comments about the terminology of "pseudo-trajectories". This name is used in [34][1], and emphasizes on the fact that the trajectories studied are not directly linked to the physical dynamics of the particles. Indeed, the number of particles of a pseudo-trajectory changes along time. On the other hand, if one studies the s-th marginal of the distribution function of a large number of particles, that is if one considers only the evolution of s of such particles, this object focuses only on the particles involved in the dynamics of those s particles : at the beginning, they evolve as if they are alone, but sometimes,

[1]See Section 7.4 "Reformulation in terms of pseudo-trajectories"

collisions happen, which are represented here by the adjunction of a particle to the system. After the first and before the second ajunction, the system which is considered has $s+1$ particles, evolving again as if they are alone, until the next collision, implying once again that a new particle is added to the system, and so on.

Now that the pseudo-trajectories are properly introduced, it is time to introduce notations for those pseudo-trajectories.

Definition 44 (Notations for the adjunction parameters of the pseudo-trajectories)**.** *Let t be a strictly positive number. For any positive integers s and k, one denotes the* set of sequences of times of adjunction *by :*

$$\mathfrak{T}_k$$

the set composed of the strictly decreasing sequences of k elements belonging to $[0,t]$, that is

$$\mathfrak{T}_k = \Big\{(t_1,\dots,t_k) \in [0,t]^k \ / \ t_1 < \dots < t_k\Big\}.$$

One denotes the sequence of adjunction parameters *by :*

$$\mathfrak{A}_k$$

the set composed of the sequences of k elements belonging to $\mathbb{S}^{d-1} \times \mathbb{R}^d$, that is

$$\mathfrak{A}_k = \Big\{\big((\omega_1, v_{s+1}),\dots,(\omega_k, v_{s+k})\big)\Big\}.$$

Definition 45 (Backwards pseudo-trajectory of the Boltzmann hierarchy)**.** *Let s and k be two positive integers, and t be a strictly positive number.*
For any initial configuration $Z_s \in \big(\{x \in \mathbb{R}^d \ / \ x \cdot e_1 > 0\} \times \mathbb{R}^d\big)^s$, any sequence of times of adjunction $(t_i)_{1\le i\le k} \in \mathfrak{T}_k$, any sequence of selected particles for the adjunctions $J_k \in \mathfrak{J}_k^s$ and any sequence of adjunction parameters $\big(\omega_i, v_{s+i}\big)_{1\le i\le k} \in \mathfrak{A}_k$, one defines the backwards pseudo-trajectory of the Boltzmann hierarchy with adjunction parameters T_k, J_k, A_k, *which will be denoted $Z^0(T_k,J_k,A_k)(\cdot)$ or sometimes to simplify $Z^0(\cdot)$, as follows (the configurations $Z^0_{s,k}$ are defined recursively):*

- *for all $\tau \in \ [t_1,t]$:*

$$Z^0(T_k,J_k,A_k)(\tau) = Z^0_{s,0}(T_k,J_k,A_k)(\tau) = T^{s,0}_{\tau-t}(Z_s),$$

- *for all $\tau \in \ [t_2,t_1[$:*

$$\begin{aligned} Z^0(T_k,J_k,A_k)(\tau) &= Z^0_{s,1}(T_k,J_k,A_k)(\tau) \\ &= T^{s+1,0}_{\tau-t_1}\big(Z^0_{s,0}(T_k,J_k,A_k)(t_1), \big(Z^0_{s,0}(T_k,J_k,A_k)(t_1)\big)^{X,j_1}, v_{s+1}\big) \end{aligned}$$

if $j_1 = -$, *and*

$$Z^0(T_k,J_k,A_k)(\tau) = Z^0_{s,1}(T_k,J_k,A_k)(\tau)$$
$$= T^{s+1,0}_{\tau-t_1}\Big(\big(Z^0_{s,0}(T_k,J_k,A_k)(t_1),\big(Z^0_{s,0}(T_k,J_k,A_k)(t_1)\big)^{X,j_1},v_{s+1}\big)'_{j_1,s+1}\Big)$$

if $j_1 = +$ *(following the notations of Definition 2 page 52),*

- *in general, for all* $\tau \in [t_{k+1},t_k[$ *:*

$$Z^0(T_k,J_k,A_k)(\tau) = Z^0_{s,k}(T_k,J_k,A_k)(\tau)$$
$$= T^{s+k,0}_{\tau-t_k}\big(Z^0_{s,k-1}(T_k,J_k,A_k)(t_k),\big(Z^0_{s,k-1}(T_k,J_k,A_k)(t_k)\big)^{X,j_k},v_{s+k}\big)$$

if $j_k = -$, *and*

$$Z^0(T_k,J_k,A_k)(\tau) = Z^0_{s,k}(T_k,J_k,A_k)(\tau)$$
$$= T^{s+k,0}_{\tau-t_k}\Big(\big(Z^0_{s,k-1}(T_k,J_k,A_k)(t_k),\big(Z^0_{s,k-1}(T_k,J_k,A_k)(t_k)\big)^{X,j_k},v_{s+k}\big)'_{j_k,s+k}\Big)$$

if $j_k = +$.

Similarly for the dynamics of the hard spheres of radius $\varepsilon/2$ (for some $\varepsilon > 0$), one introduces the corresponding notations.

Definition 46 (Backwards pseudo-trajectory of the BBGKY hierarchy)**.** *Let* s *and* k *be two positive integers, and* ε *and* t *be two strictly positive numbers. For any initial configuration* $Z_s \in \mathcal{D}^\varepsilon_s$*, any sequence of times of adjunction* $(t_i)_{1\le i\le k} \in \mathfrak{T}_k$*, any sequence of selected particles for the adjunctions* $J_k \in \mathfrak{J}^s_k$ *and any sequence of adjunction parameters* $\big(\omega_i, v_{s+i}\big)_{1\le i\le k} \in \mathfrak{A}_k$*, one defines the* backwards pseudo-trajectory of the BBGKY hierarchy with adjunction parameters T_k, J_k, A_k*, which will be denoted* $Z^\varepsilon(T_k,J_k,A_k)(\cdot)$ *or sometimes to simplify* $Z^\varepsilon(\cdot)$*, as follows (the* $Z^\varepsilon_{s,k}$ *are defined recursively) :*

- *for all* $\tau \in [t_1,t]$ *:*

$$Z^\varepsilon(T_k,J_k,A_k)(\tau) = Z^\varepsilon_{s,0}(T_k,J_k,A_k)(\tau) = T^{s,\varepsilon}_{\tau-t}(Z_s),$$

- *for all* $\tau \in [t_2,t_1[$ *:*

$$Z^\varepsilon(T_k,J_k,A_k)(\tau) = Z^\varepsilon_{s,1}(T_k,J_k,A_k)(\tau)$$
$$= T^{s+1,\varepsilon}_{\tau-t_1}\big(Z^\varepsilon_{s,0}(T_k,J_k,A_k)(t_1),\big(Z^\varepsilon_{s,0}(T_k,J_k,A_k)(t_1)\big)^{X,j_1}+\varepsilon\omega_1,v_{s+1}\big)$$

if $j_1 = -$, *and*

$$Z^\varepsilon(T_k,J_k,A_k)(\tau) = Z^\varepsilon_{s,1}(T_k,J_k,A_k)(\tau)$$
$$= T^{s+1,\varepsilon}_{\tau-t_1}\Big(\big(Z^\varepsilon_{s,0}(T_k,J_k,A_k)(t_1),$$
$$\big(Z^\varepsilon_{s,0}(T_k,J_k,A_k)(t_1)\big)^{X,j_1}+\varepsilon\omega_1,v_{s+1}\big)'_{j_1,s+1}\Big)$$

if $j_1 = +$,

- *in general, for all* $\tau \in [t_{k+1}, t_k[$ *:*

$$\begin{aligned} Z^\varepsilon(T_k, J_k, A_k)(\tau) &= Z^\varepsilon_{s,k}(T_k, J_k, A_k)(\tau) \\ &= T^{s+k,\varepsilon}_{\tau-t_k}\big(Z^\varepsilon_{s,k-1}(T_k, J_k, A_k)(t_k), \big(Z^\varepsilon_{s,k-1}(T_k, J_k, A_k)(t_k)\big)^{X,j_k} + \varepsilon\omega_k, v_{s+k}\big) \end{aligned}$$

if $j_k = -$*, and*

$$\begin{aligned} Z^\varepsilon(T_k, J_k, A_k)(\tau) &= Z^\varepsilon_{s,k}(T_k, J_k, A_k)(\tau) \\ &= T^{s+k,\varepsilon}_{\tau-t_k}\Big(\big(Z^\varepsilon_{s,k-1}(T_k, J_k, A_k)(t_k), \\ &\qquad \big(Z^\varepsilon_{s,k-1}(T_k, J_k, A_k)(t_k)\big)^{X,j_k} + \varepsilon\omega_k, v_{s+k}\big)'_{j_k,s+k}\Big) \end{aligned}$$

if $j_k = +$*.*

Remark 30. *From the definition of a general pseudo-trajectory*

$$\tau \mapsto Z^{\cdot}_{s,k}(T_k, J_k, A_k)(\tau),$$

it is easy to recover the sequence M_k *encoding the type of this pseudo-trajectory. Indeed, its general term* m_l *is the sign of the quantity*

$$\omega_l \cdot \big(v_{s+l} - v^{\cdot,j_l}_{s,l-1}(t_l)\big).$$

In addition, instead of using the previous notations $\big(Z^{\cdot}(T_k, J_k, A_k)(\tau)\big)^{X,j}$ or $\big(Z^{\cdot}(T_k, J_k, A_k)(\tau)\big)^{V,j}$ to denote respectively the position and the velocity of the particle j of following a pseudo-trajectory, one will introduce another notation to simplify the writing of the elementary terms composing the solutions of the hierarchies.

Definition 47 (Position and velocity of particles of the pseudo-trajectories)**.** *Let* s *and* k *be two positive integers, and* ε *and* t *be two strictly positive numbers. For any initial configuration* $Z_s \in \mathcal{D}^\varepsilon_s$*, any sequence of times of adjunction* $(t_i)_{1\le i\le k} \in \mathfrak{T}_k$*, any sequence of selected particles for the adjunctions* $J_k \in \mathfrak{J}^s_k$ *and any sequence of adjunction parameters* $\big(\omega_i, v_{s+i}\big)_{1\le i\le k} \in \mathfrak{A}_k$*, one will denote respectively by :*

$$x^{0,j}_{s,k}(T_k, J_k, A_k)(\tau) \text{ and } x^{\varepsilon,j}_{s,k}(T_k, J_k, A_k)(\tau) \tag{12.12}$$

the positions of the particle j *of the pseudo-trajectories* $\big(Z^0_{s,k}(T_k, J_k, A_k)(\tau)\big)^{X,j}$ *of the Boltzmann hierarchy and* $\big(Z^\varepsilon_{s,k}(T_k, J_k, A_k)(\tau)\big)^{X,j}$ *of the BBGKY hierarchy.*
Similarly, one will denote respectively by :

$$v^{0,j}_{s,k}(T_k, J_k, A_k)(\tau) \text{ and } v^{\varepsilon,j}_{s,k}(T_k, J_k, A_k)(\tau) \tag{12.13}$$

the velocities of the particle j of the pseudo-trajectories $\big(Z^0_{s,k}(T_k,J_k,A_k)(\tau)\big)^{V,j}$ of the Boltzmann hierarchy and $\big(Z^\varepsilon_{s,k}(T_k,J_k,A_k)(\tau)\big)^{V,j}$ of the BBGKY hierarchy.
Finally, one will denote respectively

$$X^0_{s,k}(T_k,J_k,A_k)(\tau) \text{ and } X^\varepsilon_{s,k}(T_k,J_k,A_k)(\tau) \tag{12.14}$$

the collection

$$\big(x^{0,1}_{s,k}(T_k,J_k,A_k)(\tau),\dots,x^{0,s+k}_{s,k}(T_k,J_k,A_k)(\tau)\big)$$

of the positions of all the particles of the configuration $Z^0_{s,k}(T_k,J_k,A_k)(\tau)$, and the collection

$$\big(x^{\varepsilon,1}_{s,k}(T_k,J_k,A_k)(\tau),\dots,x^{\varepsilon,s+k}_{s,k}(T_k,J_k,A_k)(\tau)\big)$$

of the positions of all the particles of the configuration $Z^\varepsilon_{s,k}(T_k,J_k,A_k)(\tau)$.
Similarly, one denotes

$$V^0_{s,k}(T_k,J_k,A_k)(\tau) \text{ and } V^\varepsilon_{s,k}(T_k,J_k,A_k)(\tau) \tag{12.15}$$

the collection of the velocities of all the particles of the configuration $Z^0_{s,k}(\tau)$, and of the configuration $Z^\varepsilon_{s,k}(\tau)$.

Using those new notations, the first term

$$\underset{(+),(j_1)}{\mathcal{I}^0_s}\big(t\mapsto \mathcal{T}^{s+1,0}_t f_0^{(s+1)}\big)$$

detailed in (12.3) page 312 can now be rewritten as

$$\int_0^t \int_{\mathbb{S}^{d-1}_{\omega_1}\times\mathbb{R}^d_{v_{s+1}}} \Big[\omega_1\cdot\big(v_{s+1}-v^{0,j_1}_{s,0}\big(t_1,j_1,(\omega_1,v_{s+1})\big)(t_1)\big)\Big]_+ \\ \times f_0^{(s+1)}\big(Z^0_{s,1}\big(t_1,j_1,(\omega_1,v_{s+1})\big)(t_1)\big)\,\mathrm{d}\omega_1\,\mathrm{d}v_{s+1}\,\mathrm{d}t_1,$$

and the term

$$\underset{(+,+),(j_1,j_2)}{\mathcal{I}^0_{s,s+1}}\big(t\mapsto \mathcal{T}^{s+2,0}_t f_0^{(s+2)}\big),$$

can be rewritten as

$$\int_0^t \int_{\mathbb{S}^{d-1}_{\omega_{s+1}}\times\mathbb{R}^d_{v_{s+1}}} \Big[\omega_{s+1}\cdot\Big(v_{s+1}-v^{0,j_1}_{s,0}\big((t_1,t_2),(j_1,j_2),\big((\omega_1,v_{s+1}),(\omega_2,v_{s+2})\big)\big)(t_1)\Big)\Big]_+ \\ \times\int_0^{t_1}\int_{\mathbb{S}^{d-1}_{\omega_{s+2}}\times\mathbb{R}^d_{v_{s+2}}} \Big[\omega_{s+2}\cdot\Big(v_{s+2}-v^{0,j_2}_{s,1}\big((t_1,t_2),(j_1,j_2),\big((\omega_1,v_{s+1}),(\omega_2,v_{s+2})\big)\big)(t_2)\Big)\Big]_+ \\ \times f_0^{(s+2)}\big(Z^0_{s,2}\big((t_1,t_2),(j_1,j_2),\big((\omega_1,v_{s+1}),(\omega_2,v_{s+2})\big)\big)(0)\big) \\ \mathrm{d}\omega_{s+2}\,\mathrm{d}v_{s+2}\,\mathrm{d}t_2\,\mathrm{d}\omega_{s+1}\,\mathrm{d}v_{s+1}\,\mathrm{d}t_1.$$

12.1.2 The concept of trees, and the first limitations of the naive approach

As Figure 12.1 page 322 suggests it, a geometrical interpretation of the pseudo-trajectories is natural. The reader may refer to the book [26][2], where a sharp study of the geometrical properties of the trajectories of the two hierarchies play a significant role, or to [34][3], in which the trajectories are rigorously controlled. The reader is also invited to refer to [57], in which a systematic computation on pseudo-trajectories is introduced.
One will call, following for example the reference [34], the geometrical representation (as performed in Figure 12.1) a *tree*.
The terminology should be commented : the representations of the pseudo-trajectories are called trees because following along time τ the positions of all the particles of the system (the number of them increases as τ grows), the picture obtained will look like a tree (possibly a strange one, with broken branches due to the bouncings against the obstacle, or tangled, if two particles of the pseudo-trajectory collide).

The possibility of using the dominated convergence arises from the careful observation of Figure 12.2 page 331 below. One sees that in general, the trees of same type of the respective BBGKY and Boltzmann hierarchies are very close. However, in some situations, the trees are not comparable at all, even if they are of same type.

One of the most serious problems comes out of the recollisions : in the Boltzmann hierarchy, the particles follow the free flow, so that they cannot collide with each other (they have a zero radius). This is not the case for the BBGKY hierarchy, and this difference may generate singularities, in the sense that the trees become radically different, and the pseudo-trajectories cannot in this case be easily compared. This is pictured in the following Figure 12.3 page 332 : at time t_1, a particle is added to particle 2 (the particle adjoined is called therefore particle 3), but a recollision happens at time t_2 between this new particle 3 and the particle 1. This recollision has the immediate effect that the velocities of those particles are changed by scattering, so that one is not able to compare anymore pairwise the particles of the two hierarchies. And as time still decreases, this change in velocity implies of course a strong divergence between the positions of the particles of the two hierarchies : for example, at time 0 the positions $x_{2,1}^{0,3}(0)$ and $x_{2,1}^{\varepsilon,3}(0)$ are now separated by an important distance, and it seems not obvious at all to control this distance as ε goes to zero.

The problem of the recollision is unavoidable, and has nothing to do with the presence of an obstacle. Indeed, the setting chosen in [34] is, either the whole Euclidean space $\mathbb{R}^d$, or the torus $\mathbb{T}^d$, and the core of the proof of the main the-

[2]See Section 4.4 "Rigorous Validity of the Boltzmann Equation", and especially Theorem 4.4.1 and its proof.

[3]See in particular Sections 12 "Elimination of recollisions" and 14 "Convergence proof".

orem of this reference is the control of the recollisions[4].
In the case of a phase space with a boundary, additional difficulties have to be taken into account, and this is the very interest of this work. Indeed, before a bouncing against the obstacle, even if the pair of particles j of the two hierarchies occupy the same position and have the same velocity, that is if $x_{s,k}^{0,j}(\tau) = x_{s,k}^{\varepsilon,j}(\tau)$ and $v_{s,k}^{0,j}(\tau) = v_{s,k}^{\varepsilon,j}(\tau)$, two singularities, of different nature, appears during the bouncing (in this situation, if the particle of one of the two hierarchies bounces, then the other will bounce too). Those phenomena can be observed on Figure 12.4 page 332 below.
The particle of the Boltzmann hierarchy will not in general bounce against the boundary at the same time as the particle of the BBGKY hierarchy. On Figure 12.4, the times of bouncing are denoted respectively by t_0' and t_ε'. As a consequence, during the whole time interval $[t_0', t_\varepsilon']$, the velocities $v_{s,k}^{0,j}(\tau)$ and $v_{s,k}^{\varepsilon,j}(\tau)$ are very different, and therefore preventing the uniform comparison of the pseudo-trajectories. Since it is not reasonable to consider only pseudo-trajectories such that $t_0' = t_\varepsilon'$ for any bouncing in the dynamics, the only hope lies then in the fact that such time interval $[t_0', t_\varepsilon']$ must be small. This subset of pathological times is discussed in Section 12.2.5 page 368 below. Nevertheless, this pathological time interval is not always small. For example, if a particle is grazing the obstacle, it is easy to imagine a very long time separating the respective times of bouncing of this particle, when it follows the pseudo-trajectory of the BBGKY hierarchy, and the pseudo-trajectory of the Boltzmann hierarchy. But if one is able to exclude the grazing collisions, then this time interval should be smaller and smaller as ε goes to zero. One sees then here what kind of new cut-off has to be done.
Another difference appears between the pseudo-trajectories of the Boltzmann and the BBGKY hierarchies : after the bouncing of the same particle against the obstacle (that is when $\tau \leq \min(t_0', t_\varepsilon')$), one sees on Figure 12.4 that the positions $x_{s,k}^{0,j}(0)$ and $x_{s,k}^{\varepsilon,j}(0)$ are separated by a small distance. In the case of the half-plane, this distance does not change along time if no other bouncing or particle addition happens, since one has $v_{s,k}^{0,j}(\tau) = v_{s,k}^{\varepsilon,j}(\tau)$ for $\tau \leq \min(t_0', t_\varepsilon')$. If the obstacle is more general and presents some curvature, then after the two bouncings of the particle for the two dynamics, the velocities $v_{s,k}^{0,j}(\tau)$ and $v_{s,k}^{\varepsilon,j}(\tau)$ can slightly differ, causing therefore an increasing distance between the positions along time. However, this difference should be easily controlled as ε goes to zero.

[4]See the very end of Section 7.4 "Reformulation in terms of pseudo-trajectories" for a sketch of proof, and Section 12 "Elimination of recollisions" for the practical application of this sketch.

12.1.3 Details on the dominated convergence argument : the choice of the trees and sketch of the geometrical part of the proof

This last subsection will summarize the ideas developed in the two previous Subsections 12.1.1 and 12.1.2.

After the decomposition in elements of same type, one has seen that the relevant quantities that have to be compared in order to obtain the convergence of the solutions of the BBGKY hierarchy towards the solutions of the Boltzmann hierarchy are, *formally* (for example, in the case of two iterations)

$$\begin{aligned}\int_0^t \int_{\mathbb{S}^{d-1}_{\omega_{s+1}}\times\mathbb{R}^d_{v_{s+1}}} &\Big[\omega_{s+1}\cdot\big(v_{s+1}-v^{0,j_1}_{s,0}((\pm_1,\pm_2),(j_1,j_2),t_1)\big)\Big]_{\pm_1}\\ &\times\int_0^{t_1}\int_{\mathbb{S}^{d-1}_{\omega_{s+2}}\times\mathbb{R}^d_{v_{s+2}}}\Big[\omega_{s+2}\cdot\big(v_{s+2}-v^{0,j_2}_{s,1}((\pm_1,\pm_2),(j_1,j_2),t_2)\big)\Big]_{\pm_2}\\ &\qquad\times f_0^{(s+2)}\big(Z^0_{s,2}((\pm_1,\pm_2),(j_1,j_2),0)\big)\,\mathrm{d}\omega_{s+2}\,\mathrm{d}v_{s+2}\,\mathrm{d}t_2\,\mathrm{d}\omega_{s+1}\,\mathrm{d}v_{s+1}\,\mathrm{d}t_1.\end{aligned}\tag{12.16}$$

in the case of the Boltzmann hierarchy, and :

$$\begin{aligned}\int_0^t \int_{\mathbb{S}^{d-1}_{\omega_{s+1}}\times\mathbb{R}^d_{v_{s+1}}} &\Big[\omega_{s+1}\cdot\big(v_{s+1}-v^{\varepsilon,j_1}_{s,0}((\pm_1,\pm_2),(j_1,j_2),t_1)\big)\Big]_{\pm_1}\\ &\times\int_0^{t_1}\int_{\mathbb{S}^{d-1}_{\omega_{s+2}}\times\mathbb{R}^d_{v_{s+2}}}\Big[\omega_{s+2}\cdot\big(v_{s+2}-v^{\varepsilon,j_2}_{s,1}((\pm_1,\pm_2),(j_1,j_2),t_2)\big)\Big]_{\pm_2}\\ &\qquad\times f_{N,0}^{(s+2)}\big(Z^\varepsilon_{s,2}((\pm_1,\pm_2),(j_1,j_2),0)\big)\,\mathrm{d}\omega_{s+2}\,\mathrm{d}v_{s+2}\,\mathrm{d}t_2\,\mathrm{d}\omega_{s+1}\,\mathrm{d}v_{s+1}\,\mathrm{d}t_1.\end{aligned}\tag{12.17}$$

in the case of the BBGKY hierarchy.
One used the word "*formally*" in order to emphasize that, in the case of the BBGKY hierarchy, the work done in Chapter II, and especially in Section 5.1 page 88, shows that the iterated operator used to define the solutions described in the iterated Duhamel formula (10.3) page 268 is not defined as a usual integral. Indeed, for the case of the BBGKY hierarchy one recalls that the transport-collision operator is defined as the limit of a sequence of integrals, but it is not clear that this limit can be itself written as an integral.
One noticed that the trees obtained using the dynamics of the free flow and those obtained using the dynamics of the hard sphere flow can dramatically diverge, preventing any geometric comparison in general. However, for the two quantities written above, only specific parts of those two trees have to be close.

Namely, one wants to compare the following (components of the) nodes of the two trees :

- $v_{s,0}^{0,j_1}((\pm_1,\pm_2),(j_1,j_2),t_1)$ with $v_{s,0}^{\varepsilon,j_1}((\pm_1,\pm_2),(j_1,j_2),t_1)$,
- $v_{s,1}^{0,j_2}((\pm_1,\pm_2),(j_1,j_2),t_2)$ with $v_{s,1}^{\varepsilon,j_2}((\pm_1,\pm_2),(j_1,j_2),t_2)$,
- and $Z_{s,2}^{0}((\pm_1,\pm_2),(j_1,j_2),0)$ with $Z_{s,2}^{\varepsilon}((\pm_1,\pm_2),(j_1,j_2),0)$.

Two principles will enable a safe comparison.
First one will remove trees undergoing recollisions in the case of the hard sphere dynamics (since such trees will be absolutely impossible to compare with their analog corresponding to the free flow dynamics).
Second, one will take into account the presence of the obstacle. This is the very core of this work, and also the main difference with the setting studied in [34]. One will see that the obstacle has an impact on how to exclude trees with recollisions, but also, even without recollisions. Indeed, the quantities :

$$v_{s,0}^{0,j_1}((\pm_1,\pm_2),(j_1,j_2),t_1) \text{ and } v_{s,0}^{\varepsilon,j_1}((\pm_1,\pm_2),(j_1,j_2),t_1)$$

cannot be compared, if only one of the particles of the two trees has bounced against the obstacle while the other has not.

One will remove in the time domain the times corresponding to an addition of a particle too close to the obstacle. Of course, something has to be done to show that this removal will generate only a small error term.
For the whole configurations at time 0

$$Z_{s,2}^{0}((\pm_1,\pm_2),(j_1,j_2),0)$$

and

$$Z_{s,2}^{\varepsilon}((\pm_1,\pm_2),(j_1,j_2),0),$$

such a work is not possible anymore, since at this level, the two trees have been entirely determined until the time of the final evaluation of the configuration of the whole system described by the pseudo-trajectory : the time 0.
One recalls that, even if the recollisions have been excluded so that the positions of all the particles are close between the two pseudo-trajectories, the two velocities for the same particle following the BBGKY on the one hand, and the Boltzmann pseudo-trajectory on the other hand, may be very different due to a bouncing, if this bouncing is close in time to the final time evaluation 0. However, this object of concern is not that serious. Even if it seems a very tough problem, implying very sharp geometric lemmas if one wants to control this proximity with the obstacle of all the particles at the final time $t = 0$, one has to keep in mind that the two relevant quantities that have to be compared, according to the expressions (12.16) and (12.17) page 329 above, are not directly the final configurations, but instead the whole expressions :

$$f_0^{(s+2)}\big(Z_{s,2}^{0}((\pm_1,\pm_2),(j_1,j_2),0)\big)$$

and

$$f_{N,0}^{(s+2)}\big(Z_{s,2}^{\varepsilon}((\pm_1,\pm_2),(j_1,j_2),0)\big).$$

Here appears the question of the convergence of the initial data of the BBGKY hierarchy towards the initial data of the Boltzmann hierarchy, but if one can obtain that :

$$f_{N,0}^{(s+2)}\big(Z_{s,2}^{\varepsilon}((\pm_1,\pm_2),(j_1,j_2),0)\big)$$

and

$$f_{0}^{(s+2)}\big(Z_{s,2}^{\varepsilon}((\pm_1,\pm_2),(j_1,j_2),0)\big)$$

are close, then since the distance between the positions of the particles between the two pseudo-trajectories is small, an important difference between the velocities of the same particle following respectively the Boltzmann pseudo-trajectory on the one hand, and the BBGKY hierarchy on the other hand, implies in fact that this particle has bounced against the obstacle at a time very close to the time 0. Therefore, using the boundary condition verified by the initial datum $f_0^{(s+2)}$, the convergence of the element described in (12.17) towards the element described in (12.16) is obtained.

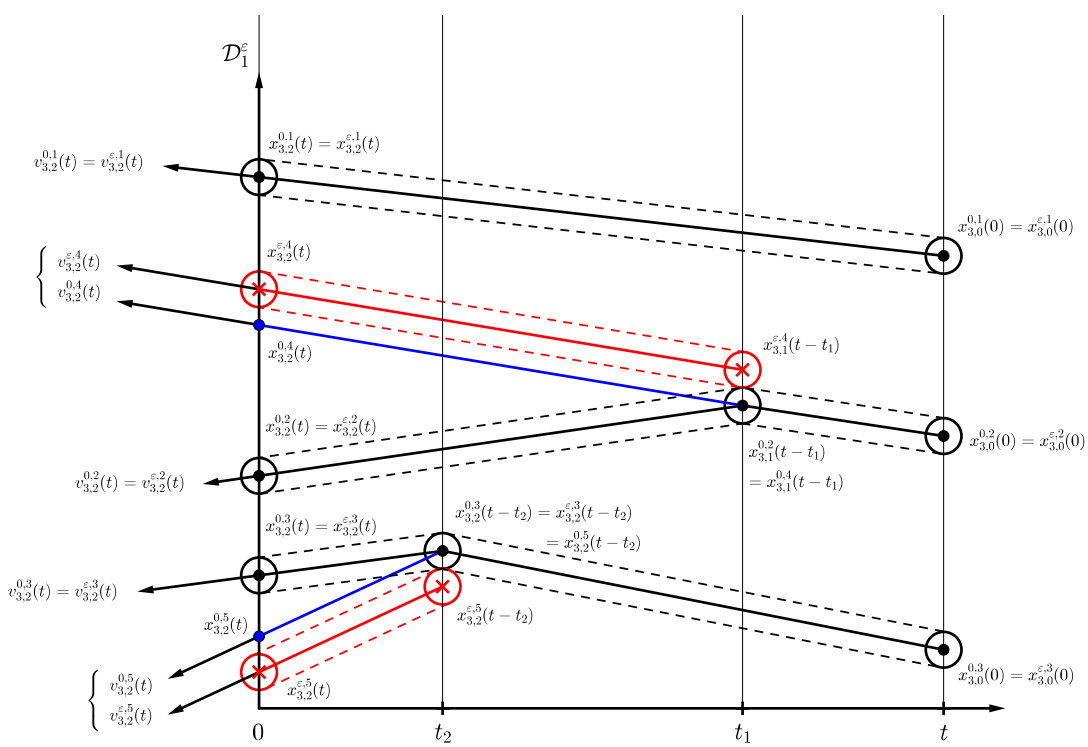

Figure 12.2: Comparison of the trees of the two hierarchies : when the radius of the particles is small, the distance between the particles of the two trees is uniformly controlled.

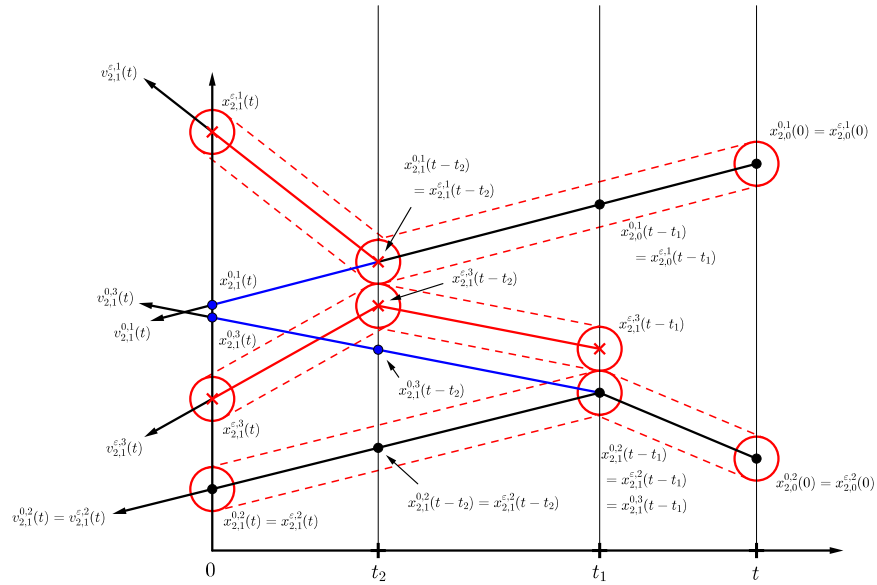

Figure 12.3: Case of a recollision in the dynamics of the hard spheres, and divergence from the dynamics of particles of radius zero, following the free flow.

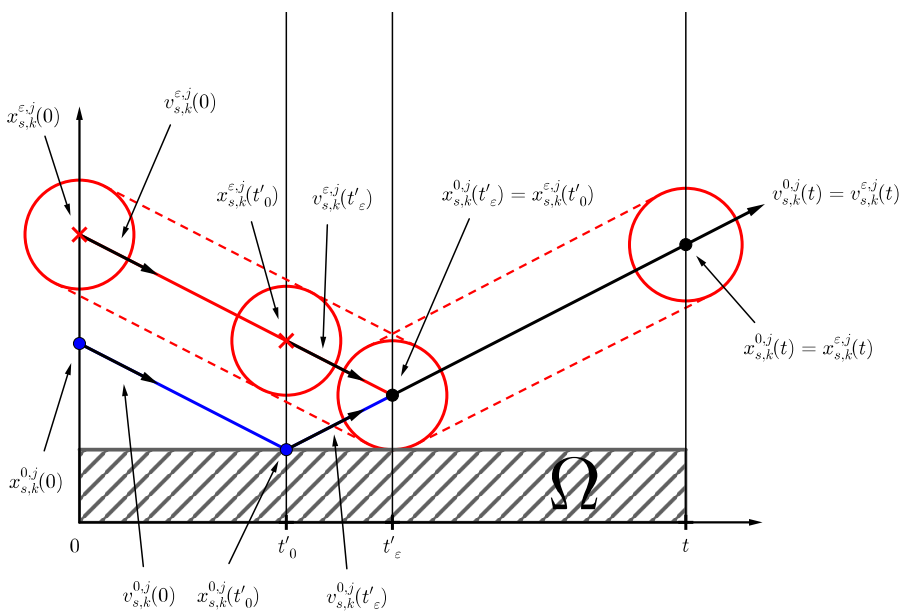

Figure 12.4: The two phenomena of divergence of the pseudo-trajectories appearing during a bouncing against the obstacle.

12.2 Geometric lemmas

This section is then entirely devoted to the elimination of the recollisions and the control of the size of the cut-off of those recollisions.

12.2.1 The concept of good configurations

The following key concept here is the notion of "good configurations", which is the set of initial data such that all particles start from a certain distance to each other, and transported by the hard sphere or the free flow, all particles remain at this distance from each other.

Definition 48 (Good configuration)**.** *Let ε and c be two strictly positive constants and k be a positive integer.*
One defines the set of good configurations for the hard sphere dynamics of k particles, separated by at least c *(here the radius of the particles is $\varepsilon/2$) as the subset of $\mathcal{D}_k^\varepsilon$ composed of the configurations Z_k such that, for all $\tau > 0$ and $1 \leq i \neq j \leq k$:*

$$\left| \left(T_{-\tau}^{k,\varepsilon}(Z_k)\right)^{X,i} - \left(T_{-\tau}^{k,\varepsilon}(Z_k)\right)^{X,j} \right| > c.$$

The set of the good configurations for the hard sphere dynamics of k particles separated by at least c is denoted by $\mathcal{G}_k^\varepsilon(c)$.
Similarly, one defines the set of good configurations for the free flow with boundary condition dynamics of k particles, separated by at least c *(here the radius of the particles is zero) as the subset of $\left(\Omega^c \times \mathbb{R}^d\right)^s$ composed of the configurations Z_k such that, for all $\tau > 0$ and $1 \leq i \neq j \leq k$:*

$$\left| \left(T_{-\tau}^{k,0}(Z_k)\right)^{X,i} - \left(T_{-\tau}^{k,0}(Z_k)\right)^{X,j} \right| > c.$$

The set of the good configurations for the free flow with boundary condition dynamics of k particles separated by at least c is denoted by $\mathcal{G}_k^0(c)$.

Of course, if $c \geq \varepsilon$, then, in particular, none of the configurations of $\mathcal{G}_k^\varepsilon(c)$ leads to a collision between two particles following the hard sphere dynamics, for any strictly positive time. In the case of the half-plane, the hard sphere transport of a particle is very simple. Indeed, if the particle i, of the system starting from the initial configuration Z_k, starts from $x_i \in \Omega^c$ (that is such that $x_i \cdot e_1 > \varepsilon/2$), with initial velocity v_i, there are only two cases :

- either $v_i \cdot e_1 \leq 0$, so that $\left(T_{-t}^{k,\varepsilon}(Z_k)\right)^{V,i} = v_i$ and $\left(T_{-t}^{k,\varepsilon}(Z_k)\right)^{X,i} \cdot e_1 > \varepsilon/2$ (following the notations introduced in Definition 4 page 53) and then

  $$\left(T_{-t}^{k,\varepsilon}(Z_k)\right)^{X,i} = x_i - tv_i$$

  for all $t \geq 0$,

- or $v_i \cdot e_1 > 0$, so that there exists a unique $t_1^i = (x_i \cdot e_1 - \varepsilon/2)/(v_i \cdot e_1) > 0$ such that $x_i(-t_1^i) \cdot e_1 = \varepsilon/2$ and then

$$\left(T_{-t}^{k,\varepsilon}(Z_k)\right)^{V,i} = v_i - \mathbb{1}_{t>t_1^i} \cdot (2v_i \cdot e_1 e_1),$$

$$\left(T_{-t}^{k,\varepsilon}(Z_k)\right)^{X,i} = x_i - tv_i + \left(\int_0^t \mathbb{1}_{s>t_1^i} \mathrm{d}s\right) \cdot (2v_i \cdot e_1 e_1).$$

In the last case, for $t > t_1^i$, one has

$$\begin{aligned}\left(T_{-t}^{k,\varepsilon}(Z_k)\right)^{X,i} &= x_i - tv_i + (t - t_1^i)(2v_i \cdot e_1 e_1) \\ &= x_i - (x_i \cdot e_1 - \varepsilon/2)/(v_i \cdot e_1))(2v_i \cdot e_1 e_1) - t(v_i - 2v_i \cdot e_1 e_1) \\ &= x_i - 2x_i \cdot e_1 e_1 + \varepsilon e_1 - t(v_i - 2v_i \cdot e_1 e_1).\end{aligned}$$

12.2.2 The shooting lemma

One introduces some notations for the orthogonal symmetries with respect to some hyperplanes, such as the wall $x \cdot e_1 = 0$, the boundary of the obstacle Ω.

Definition 49 (Notations for the orthogonal symmetries)**.** *For any vector $x \in \mathbb{R}^d$, one denotes by :*

$$\mathcal{S}_0(x) = x - 2x \cdot e_1 e_1 \tag{12.18}$$

its image by the orthogonal symmetry with respect to the hyperplane $x \cdot e_1 = 0$. Similarly, for any strictly positive number ε, one denotes by :

$$\mathcal{S}_\varepsilon(x) = x - 2x \cdot e_1 e_1 + \varepsilon e_1 \tag{12.19}$$

its image by the orthogonal symmetry with respect to the hyperplane $x \cdot e_1 = \varepsilon/2$.

One starts by introducing a convenient notation for the cylinders of the Euclidean space $\mathbb{R}^d$.

Definition 50 (Notations for the cylinders)**.** *Let v be a vector of $\mathbb{R}^d$, w be a non zero vector of $\mathbb{R}^d$ and ρ be a nonnegative number. One denotes by :*

$$K(v, w, \rho)$$

the cylinder of origin v, of axis w, and of radius ρ, *that is :*

$$K(v, w, \rho) = \Big\{x \in \mathbb{R}^d \ / \ \forall u \in \mathbb{S}^{d-1} \text{ such that } u \cdot w = 0, \ (x - v) \cdot u \leq \rho\Big\}. \tag{12.20}$$

The shooting lemma in the case of the half-plane, first version

Lemma 27 (Shooting lemma in the case of the half-plane, with varying axes)**.** *Let R, δ, ε, $\widetilde{a}$ and ε_0 be five strictly positive numbers, such that :*

$$\varepsilon \leq \widetilde{a}, \quad 2\sqrt{3}\widetilde{a} \leq \varepsilon_0. \tag{12.21}$$

One considers two points $\overline{x}_1, \overline{x}_2 \in \left\{x \in \mathbb{R}^d \ / \ x \cdot e_1 > 0\right\}$ such that $|\overline{x}_1 - \overline{x}_2| \geq \varepsilon_0$, and $v_1 \in B(0,R)$.
Then for all $x_1 \in B(\overline{x}_1, \widetilde{a}) \cap \left\{y \in \mathbb{R}^d \ / \ y \cdot e_1 > \varepsilon/2\right\}$, $x_2 \in B(\overline{x}_2, \widetilde{a}) \cap \left\{y \in \mathbb{R}^d \ / \ y \cdot e_1 > \varepsilon/2\right\}$, and $v_2 \in B(0,R)$, if one denotes :

$$\overline{Z_2} = \left(\overline{x}_1, v_1, \overline{x}_2, v_2\right) \text{ and } Z_2 = \left(x_1, v_1, x_2, v_2\right),$$

1. *if for some $\delta > 0$*

$$v_2 \notin K(v_1, \overline{x}_1 - \overline{x}_2, \varepsilon_0/\delta) \cup K(\mathcal{S}_0(v_1), \mathcal{S}_0(\overline{x}_1) - \overline{x}_2, \varepsilon_0/\delta), \tag{12.22}$$

(using the notation (12.18) *page 334 for $\mathcal{S}_0$) one has for all $\tau \geq \delta$:*

$$\left|\left(T^{2,0}_{-\tau}\left(\overline{Z_2}\right)\right)^{X,1} - \left(T^{2,0}_{-\tau}\left(\overline{Z_2}\right)\right)^{X,2}\right| > \varepsilon_0,$$

or in other words :

$$T^{2,0}_{-\delta}\left(\overline{Z_2}\right) \in \mathcal{G}^0_2(\varepsilon_0),$$

2. *if in addition :*

$$\begin{aligned} v_2 \notin K(v_1, \overline{x}_1 - \overline{x}_2, 12R\widetilde{a}/\varepsilon_0) \cup K(\mathcal{S}_0(v_1), \mathcal{S}_0\big(\overline{x}_1 - \mathcal{S}_\varepsilon(\overline{x}_2)\big), 12R\widetilde{a}/\varepsilon_0) \\ \cup K(\mathcal{S}_0(v_1), \mathcal{S}_\varepsilon(\overline{x}_1) - \overline{x}_2, 12R\widetilde{a}/\varepsilon_0), \end{aligned} \tag{12.23}$$

(using the notation (12.19) *page 334 for $\mathcal{S}_\varepsilon$), one has for all $\tau > 0$:*

$$\left|\left(T^{2,\varepsilon}_{-\tau}(Z_2)\right)^{X,1} - \left(T^{2,\varepsilon}_{-\tau}(Z_2)\right)^{X,2}\right| > \varepsilon,$$

or in other words :

$$Z_2 \in \mathcal{G}^\varepsilon_2(\varepsilon).$$

Proof. One starts by the proof of the condition (12.23), concerning the hard sphere dynamics.

Proof of the condition (12.23)

If one assumes that the set of times τ of collisions is non empty, that is

$$\tilde{\tau}_{coll} = \left\{\tau \geq 0 \ / \ \left|\left(T^{2,\varepsilon}_{-\tau}(Z_2)\right)^{X,1} - \left(T^{2,\varepsilon}_{-\tau}(Z_2)\right)^{X,2}\right| = \varepsilon\right\} \neq \emptyset,$$

one will consider $\tau_0 = \inf \tilde{\tau}_{coll}$, which is an element of $\tilde{\tau}_{coll}$, because all the points of $\tilde{\tau}_{coll}$ are isolated points. Then, by definition of the smallest element of $\tilde{\tau}_{coll}$, the particles have followed the free flow of particles of radius $\varepsilon/2$ before τ_0, with boundary conditions (that is : the particles interact with the obstacle, but not with each other), and then there are only four possible cases, depending on the fact that each particle can bounce against the obstacle before τ_0 or not.

- <u>First subcase</u> :

$$\left(T^{2,\varepsilon}_{-\tau_0}(Z_2)\right)^{X,1} = x_1 - \tau_0 v_1 \text{ and } \left(T^{2,\varepsilon}_{-\tau_0}(Z_2)\right)^{X,2} = x_2 - \tau_0 v_2,$$

this is the case of a collision before a bouncing against the obstacle of any of the two particles. One has

$$|(x_1 - x_2) - \tau_0(v_1 - v_2)| = \varepsilon,$$

and then

$$|(\overline{x}_1 - \overline{x}_2) - \tau_0(v_1 - v_2)| \leq \varepsilon + 2\widetilde{a} \leq 3\widetilde{a}.$$

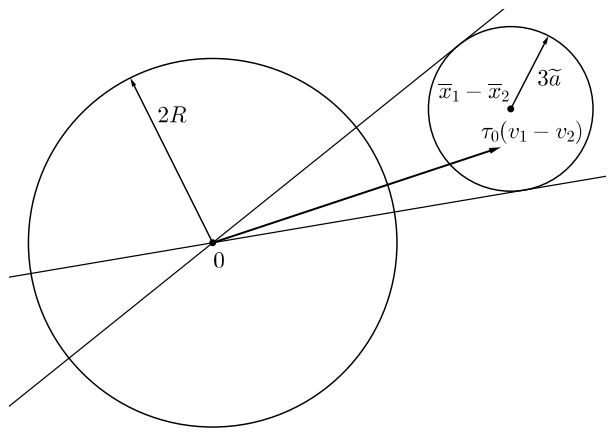

Figure 12.5: Construction of the cone containing the relative velocity $v_1 - v_2$

This means exactly that $v_1 - v_2$ belongs to the ball centered on $(\overline{x}_1 - \overline{x}_2)/\tau_0$, of radius $3\widetilde{a}/\tau_0$, so that $v_1 - v_2$ belongs to the cone of vertex $0 \in \mathbb{R}^d$, based on the ball centered on $\overline{x}_1 - \overline{x}_2$ and of radius $3\widetilde{a}$.

Of course, $v_1 - v_2$ has a norm smaller than $2R$. The intersection of this cone with the ball centered on 0, of radius $2R$ is contained in the cylinder of origin 0, of axis $\overline{x}_1 - \overline{x}_2$ and of radius $\rho > 0$ with

$$\rho = \frac{2R \cdot 3\widetilde{a}}{\sqrt{|\overline{x}_1 - \overline{x}_2|^2 - (3\widetilde{a})^2}}.$$

One sees that

$$\rho \leq \frac{12R\widetilde{a}}{\varepsilon_0}$$

as soon as $\varepsilon_0^2 - 9\widetilde{a}^2 \geq \varepsilon_0^2/4$, that is $\varepsilon_0 \geq 2\sqrt{3}\widetilde{a}$. Since the condition holds on v_2 for v_1 given, one obtains the first term of the subset of the excluded velocities v_2 described in the condition (12.23).

- <u>Second subcase</u> :

$$\Big(T^{2,\varepsilon}_{-\tau_0}(Z_2)\Big)^{X,1} = x_1 - \tau_0 v_1 \text{ and } \Big(T^{2,\varepsilon}_{-\tau_0}(Z_2)\Big)^{X,1} = \mathcal{S}_\varepsilon(x_2 - \tau_0 v_2),$$

this is the case of a collision after a single bouncing against the obstacle, involving the second particle. One has, using the notation (12.19) for symmetries introduced in Definition 49 page 334 :

$$\begin{aligned}\Big|\Big(T^{2,\varepsilon}_{-\tau_0}(Z_2)\Big)^{X,1} - \Big(T^{2,\varepsilon}_{-\tau_0}(Z_2)\Big)^{X,1}\Big| &= \Big|(x_1 - \tau_0 v_1) - \mathcal{S}_\varepsilon(x_2 - \tau_0 v_2)\Big| \\ &= \big|(x_1 - \tau_0 v_1) - (x_2 - 2x_2 \cdot e_1 e_1 + \varepsilon e_1 - \tau_0(v_2 - 2v_2 \cdot e_1 e_1))\big| \\ &= \big|(x_1 - (x_2 - 2x_2 \cdot e_1 e_1) - \varepsilon e_1) - \tau_0(v_1 - (v_2 - 2v_2 \cdot e_1 e_1))\big|.\end{aligned}$$

The orthogonal symmetry $\mathcal{S}_\varepsilon$ with respect to the hyperplane $x \cdot e_1 = \varepsilon/2$ writes :

$$\mathcal{S}_\varepsilon(x) = x - 2(x \cdot e_1)e_1 + \varepsilon e_1.$$

Similarly, the orthogonal symmetry $\mathcal{S}_0$ with respect to the hyperplane $x \cdot e_1 = 0$ writes :

$$\mathcal{S}_0(x) = x - 2(x \cdot e_1)e_1.$$

One can then rewrite the difference between the position of the two particles :

$$\Big|\Big(T^{2,\varepsilon}_{-\tau_0}(Z_2)\Big)^{X,1} - \Big(T^{2,\varepsilon}_{-\tau_0}(Z_2)\Big)^{X,1}\Big| = \Big|\big(x_1 - \mathcal{S}_\varepsilon(x_2)\big) - \tau_0\big(v_1 - \mathcal{S}_0(v_2)\big)\Big|,$$

which is equal to ε by hypothesis, and then using the fact that $\mathcal{S}_0$ is a linear isometry, one obtains :

$$\begin{aligned}\Big|\Big(T^{2,\varepsilon}_{-\tau_0}(Z_2)\Big)^{X,1} - \Big(T^{2,\varepsilon}_{-\tau_0}(Z_2)\Big)^{X,1}\Big| &= \Big|\mathcal{S}_0\big(\big(x_1 - \mathcal{S}_\varepsilon(x_2)\big) - \tau_0\big(v_1 - \mathcal{S}_0(v_2)\big)\big)\Big| \\ &= \Big|\mathcal{S}_0\big(x_1 - \mathcal{S}_\varepsilon(x_2)\big) - \tau_0\big(\mathcal{S}_0(v_1) - v_2\big)\Big| \\ &= \varepsilon.\end{aligned}$$

Then, using the linearity of the symmetry $\mathcal{S}_0$:

$$\begin{aligned}\mathcal{S}_0\big(x_1 - \mathcal{S}_\varepsilon(x_2)\big) &= \mathcal{S}_0\big((x_1 - \overline{x}_1) + \overline{x}_1 - \mathcal{S}_\varepsilon(\overline{x}_2) - \mathcal{S}_\varepsilon(x_2) + \mathcal{S}_\varepsilon(\overline{x}_2)\big) \\ &= \mathcal{S}_0(x_1 - \overline{x}_1) + \mathcal{S}_0\big(\overline{x}_1 - \mathcal{S}_\varepsilon(\overline{x}_2)\big) - \mathcal{S}_0\big(\mathcal{S}_\varepsilon(x_2) - \mathcal{S}_\varepsilon(\overline{x}_2)\big).\end{aligned}$$

Using now the explicit expression of the symmetry $\mathcal{S}_\varepsilon$, one finds :

$$\mathcal{S}_\varepsilon(x_2) - \mathcal{S}_\varepsilon(\overline{x}_2) = \mathcal{S}_0(x_2) - \mathcal{S}_0(\overline{x}_2) = \mathcal{S}_0(x_2 - \overline{x}_2)$$

so that, proceeding as in the previous case, that is using the triangular inequality, one obtains :

$$\begin{aligned}\Big|\mathcal{S}_0\big(\overline{x}_1 - \mathcal{S}_\varepsilon(\overline{x}_2)\big) - \tau_0\big(\mathcal{S}_0(v_1) - v_2\big)\Big| &\leq \varepsilon + \Big|\mathcal{S}_0(x_1 - \overline{x}_1)\Big| + \big|x_2 - \overline{x}_2\big| \\ &\leq 3\widetilde{a}. \qquad (12.24)\end{aligned}$$

This implies that $\mathcal{S}_0(v_1) - v_2$ belongs to the cone of vertex $0 \in \mathbb{R}^d$, based on the ball centered on $\mathcal{S}_0\big(\overline{x}_1 - \mathcal{S}_\varepsilon(\overline{x}_2)\big)$ and of radius $3\widetilde{a}$.

Moreover, for all $\overline{x}_1$ and $\overline{x}_2$ in $\big\{x \in \mathbb{R}^d / x \cdot e_1 > \varepsilon/2\big\}$, one has that :

$$\Big|\overline{x}_1 - \mathcal{S}_\varepsilon(\overline{x}_2)\Big| \geq \big|\overline{x}_1 - \overline{x}_2\big|.$$

Indeed, by continuity there exists a unique $\theta \in]0,1[$ such that

$$(\theta\overline{x}_1 + (1-\theta)\mathcal{S}_\varepsilon(\overline{x}_2)) \cdot e_1 = \varepsilon/2.$$

Denoting by $\overline{x}_\theta$ the quantity $\theta\overline{x}_1 + (1-\theta)\mathcal{S}_\varepsilon(\overline{x}_2)$, one has :

$$\Big|\mathcal{S}_\varepsilon(\overline{x}_2) - \overline{x}_\theta\Big| = \Big|\mathcal{S}_\varepsilon(\overline{x}_2) - \mathcal{S}_\varepsilon(\overline{x}_\theta)\Big| = \Big|\mathcal{S}_0(\overline{x}_2 - \overline{x}_\theta)\Big| = \big|\overline{x}_2 - \overline{x}_\theta\big|,$$

thanks to the explicit expressions (12.19) and (12.18) of the symmetries, and because $\mathcal{S}_\varepsilon$ preserves all points of the hyperplane $\big\{x \in \mathbb{R}^d \ / \ x \cdot e_1 = \varepsilon/2\big\}$, so it preserves in particular $\overline{x}_\theta$. Then using the fact that, by definition, $\overline{x}_\theta$ belongs to the segment $[\overline{x}_1, \mathcal{S}_\varepsilon(\overline{x}_2)]$:

$$\begin{aligned}\Big|\mathcal{S}_\varepsilon(\overline{x}_2) - \overline{x}_1\Big| &= \Big|\mathcal{S}_\varepsilon(\overline{x}_2) - \overline{x}_\theta| + |\overline{x}_\theta - \overline{x}_1\Big| \\ &= \big|\overline{x}_2 - \overline{x}_\theta\big| + \big|\overline{x}_\theta - \overline{x}_1\big| \geq \big|\overline{x}_2 - \overline{x}_1\big|.\end{aligned}$$

Finally, using the same arguments as in the first case, the intersection of the cone defined just above with the ball centered on 0 of radius $2R$ is contained in the cylinder of origin 0, of axis $\mathcal{S}_0\big(\overline{x}_1 - \mathcal{S}_\varepsilon(\overline{x}_2)\big)$ and of the same radius $\rho > 0$ as in the previous case thanks to the fact that

$$|\mathcal{S}_\varepsilon(\overline{x}_2) - \overline{x}_1| \geq |\overline{x}_2 - \overline{x}_1| \geq \varepsilon_0,$$

and then one obtains the second term of the subset of the excluded velocities v_2 described in the condition (12.23).

- <u>Third subcase</u> :

$$\Big(T^{2,\varepsilon}_{-\tau_0}(Z_2)\Big)^{X,1} = \mathcal{S}_\varepsilon(x_1 - \tau_0 v_1) \text{ and } \Big(T^{2,\varepsilon}_{-\tau_0}(Z_2)\Big)^{X,2} = x_2 - \tau_0 v_2,$$

this is the case, very similar to the previous one, of a collision after a single bouncing against the obstacle, involving the first particle. One writes in the same way :

$$\begin{aligned}\Big|\Big(T^{2,\varepsilon}_{-\tau_0}(Z_2)\Big)^{X,1} - \Big(T^{2,\varepsilon}_{-\tau_0}(Z_2)\Big)^{X,2}\Big| &= \Big|\mathcal{S}_\varepsilon(x_1 - \tau_0 v_1) - (x_2 - \tau_0 v_2)\Big| \\ &= \big|(x_1 - 2x_1 \cdot e_1 e_1 + \varepsilon e_1 - \tau_0(v_1 - 2v_1 \cdot e_1 e_1)) - (x_2 - \tau_0 v_2)\big| \\ &= \big|(x_1 - 2x_1 \cdot e_1 e_1 + \varepsilon e_1 - x_2) - \tau_0(v_1 - 2v_1 \cdot e_1 e_1 - v_2)\big| \\ &= \Big|\big(\mathcal{S}_\varepsilon(x_1) - x_2\big) - \tau_0\big(\mathcal{S}_0(v_1) - v_2\big)\Big|,\end{aligned}$$

which is equal, by hypothesis, to ε. Writing, as above (without composing by the symmetry $\mathcal{S}_0$) :

$$\begin{aligned}\mathcal{S}_\varepsilon(x_1) - x_2 &= \big(\mathcal{S}_\varepsilon(x_1) - \mathcal{S}_\varepsilon(\overline{x}_1)\big) + \mathcal{S}_\varepsilon(\overline{x}_1) - \overline{x}_2 + (\overline{x}_2 - x_2)\\ &= \mathcal{S}_0(x_1 - \overline{x}_1) + \mathcal{S}_\varepsilon(\overline{x}_1) - \overline{x}_2 + (\overline{x}_2 - x_2),\end{aligned}$$

so that one obtains that :

$$\Big|\mathcal{S}_\varepsilon(\overline{x}_1) - \overline{x}_2 - \tau_0\big(\mathcal{S}_0(v_1) - v_2\big)\Big| \leq 3\widetilde{a}. \tag{12.25}$$

Then the vector $\mathcal{S}_0(v_1) - v_2$ belongs to the cone of vertex $0 \in \mathbb{R}^d$, based on the ball centered on $\mathcal{S}_\varepsilon(\overline{x}_1) - \overline{x}_2$ and of radius $3\widetilde{a}$. As above, one can show that :

$$\Big|\mathcal{S}_\varepsilon(\overline{x}_1) - \overline{x}_2\Big| \geq \big|\overline{x}_1 - \overline{x}_2\big| \geq \varepsilon_0,$$

so that if ε_0 is large enough beside $\widetilde{a}$, namely, if $\varepsilon_0 \geq 2\sqrt{3}\widetilde{a}$, then the vector $v_2 - \mathcal{S}_0(v_1)$ belongs to the cylinder of origin 0, of axis $\mathcal{S}_\varepsilon(\overline{x}_1) - \overline{x}_2$, and of radius $12R\widetilde{a}/\varepsilon_0$. Then by a simple translation, one obtains the third term of the subset of the excluded velocities v_2 of the condition (12.23).

- <u>Fourth subcase</u> :

$$\Big(T^{2,\varepsilon}_{-\tau_0}(Z_2)\Big)^{X,1} = \mathcal{S}_\varepsilon(x_1 - \tau_0 v_1) \text{ and } \Big(T^{2,\varepsilon}_{-\tau_0}(Z_2)\Big)^{X,2} = \mathcal{S}_\varepsilon(x_2 - \tau_0 v_2),$$

in this case, the two particles have already bounced against the obstacle before being close at time τ_0. One gets very easily, thanks to the explicit expressions (12.19) and (12.18) of the symmetries, that :

$$\begin{aligned}\Big|\Big(T^{2,\varepsilon}_{-\tau_0}(Z_2)\Big)^{X,1} - \Big(T^{2,\varepsilon}_{-\tau_0}(Z_2)\Big)^{X,2}\Big| &= \Big|\mathcal{S}_\varepsilon(x_1 - \tau_0 v_1) - \mathcal{S}_\varepsilon(x_2 - \tau_0 v_2)\Big|\\ &= \Big|\mathcal{S}_0\big((x_1 - \tau_0 v_1) - (x_2 - \tau_0 v_2)\big)\Big| = \varepsilon,\end{aligned}$$

but since the symmetry $\mathcal{S}_0$ is an isometry, the condition of proximity of the two particles is equivalent to :

$$\big|(x_1 - x_2) - \tau_0(v_1 - v_2)\big| = \varepsilon,$$

which is of course the condition studied in detail in the first case, and this condition provides the first term of the condition (12.23), so all the cases have been investigated, and the first part of the proof, dedicated to the study of the condition (12.23), is complete.

Proof of the condition (12.22)

One addresses now the proof of the condition (12.22), concerning the free flow dynamics with boundary condition.

One is now studying a problem for particles of radius zero. Therefore, those particles will not interact with each other, and there will be no trouble with recollisions.
If one assumes that there exists some time τ_0 such that :

$$\Big| \Big(T^{2,0}_{-\tau_0}\big(\overline{Z_2}\big)\Big)^{X,1} - \Big(T^{2,0}_{-\tau_0}\big(\overline{Z_2}\big)\Big)^{X,2} \Big| \leq \varepsilon_0,$$

then one will have to study the same possible subcases as in the first part of the proof, starting page 335.

- <u>First subcase</u> :

$$\Big(T^{2,0}_{-\tau_0}\big(\overline{Z_2}\big)\Big)^{X,1} = \overline{x}_1 - \tau_0 v_1 \text{ and } \Big(T^{2,0}_{-\tau_0}\big(\overline{Z_2}\big)\Big)^{X,2} = \overline{x}_2 - \tau_0 v_2.$$

One has :

$$\big|(\overline{x}_1 - \overline{x}_2) - \tau_0(v_1 - v_2)\big| \leq \varepsilon_0.$$

In particular, for all unit vector n which is orthogonal to $\overline{x}_1 - \overline{x}_2$, one has :

$$\big|n \cdot ((\overline{x}_1 - \overline{x}_2) - \tau_0(v_1 - v_2))\,\big| = \tau_0\big|n \cdot (v_1 - v_2)\big| \leq \varepsilon_0.$$

This means exactly that $v_1 - v_2$ belongs to the cylinder of origin 0, of axis $\overline{x}_1 - \overline{x}_2$ and of radius ε_0/τ_0, and since of course one has by hypothesis :

$$\tau_0 \geq \delta,$$

this cylinder is obviously contained in $K(0, \overline{x}_1 - \overline{x}_2, \varepsilon_0/\delta)$.

- <u>Second subcase</u> :

$$\Big(T^{2,0}_{-\tau_0}\big(\overline{Z_2}\big)\Big)^{X,1} = \overline{x}_1 - \tau_0 v_1 \text{ and } \Big(T^{2,0}_{-\tau_0}\big(\overline{Z_2}\big)\Big)^{X,2} = \mathcal{S}_0(\overline{x}_2 - \tau_0 v_2).$$

Here one assumes that :

$$\Big|\big(\overline{x}_1 - \mathcal{S}_0(\overline{x}_2)\big) - \tau_0\big(v_1 - \mathcal{S}_0(v_2)\big)\Big| \leq \varepsilon_0.$$

Applying the symmetry $\mathcal{S}_0$, which is an isometry, one obtains :

$$\Big|\mathcal{S}_0\Big(\big(\overline{x}_1 - \mathcal{S}_0(\overline{x}_2)\big) - \tau_0\big(v_1 - \mathcal{S}_0(v_2)\big)\Big)\Big| = \Big|\mathcal{S}_0(\overline{x}_1) - \overline{x}_2 - \tau_0\big(\mathcal{S}_0(v_1) - v_2\big)\Big| \leq \varepsilon_0. \tag{12.26}$$

Similarly, for all unit vector n which is orthogonal to $\mathcal{S}_0(\overline{x}_1) - \overline{x}_2$:

$$\tau_0\big|n \cdot (\mathcal{S}_0(v_1) - v_2)\big| \leq \varepsilon_0,$$

so that $\mathcal{S}_0(v_1) - v_2$ belongs to the cylinder of origin 0, axis $\mathcal{S}_0(\overline{x}_1) - \overline{x}_2$ and of radius ε_0/δ, and then the second term of the condition (12.22) is recovered.

- Third subcase :

$$\left(T^{2,0}_{-\tau_0}\left(\overline{Z}_2\right)\right)^{X,1} = \mathcal{S}_0\big(\overline{x}_1 - \tau_0 v_1\big) \text{ and } \left(T^{2,0}_{-\tau_0}\left(\overline{Z}_2\right)\right)^{X,2} = \overline{x}_2 - \tau_0 v_2.$$

This is the symmetrical situation of the previous case. With the same argument, one deduces that $v_2 - \mathcal{S}_0(v_1)$ belongs to the cylinder of origin 0, axis $\mathcal{S}_0\big(\overline{x}_1\big) - \overline{x}_2$ and of radius ε_0/δ, which is the same cylinder as for the previous case, and it is the second term of the condition (12.22).

- Fourth subcase :

$$\left(T^{2,0}_{-\tau_0}\left(\overline{Z}_2\right)\right)^{X,1} = \mathcal{S}_0\big(\overline{x}_1 - \tau_0 v_1\big) \text{ and } \left(T^{2,0}_{-\tau_0}\left(\overline{Z}_2\right)\right)^{X,2} = \mathcal{S}_0\big(\overline{x}_2 - \tau_0 v_2\big).$$

For this last case, the condition :

$$\Big|\mathcal{S}_0\big(\overline{x}_1 - \tau_0 v_1\big) - \mathcal{S}_0\big(\overline{x}_2 - \tau_0 v_2\big)\Big| = \Big|\mathcal{S}_0\left((\overline{x}_1 - \overline{x}_2) - \tau_0(v_1 - v_2)\right)\Big| \leq \varepsilon_0/\delta,$$

is equivalent, because $\mathcal{S}_0$ is an isometry, to :

$$\big|(\overline{x}_1 - \overline{x}_2) - \tau_0(v_1 - v_2)\big| \leq \varepsilon_0/\delta.$$

But this condition is, as for the first part of the proof, already studied, and has already been given by the first term of the condition (12.22), so that the proof of this second part, hence of the whole lemma, is complete. □

Remark 31. *The size of the cylinders which contain the excluded velocities in the previous lemma is small, providing of course that the conditions which link the parameters $\widetilde{a}$, ε and ε_0 are fulfilled. To be more precise, the size of the cylinders is of order :*

$$R^d\left(\frac{\widetilde{a}}{\varepsilon_0}\right)^{d-1} + R\left(\frac{\varepsilon_0}{\delta}\right)^{d-1}.$$

The shooting lemma in the case of the half-plane, second version

Some additional comments can be made about the conditions (12.23) and (12.22) obtained in the previous lemma. Considering the three cylinders obtained for the condition (12.23), that is :

$$K(v_1, \overline{x}_1 - \overline{x}_2, 12Ra/\varepsilon_0),\ K(\mathcal{S}_0(v_1), \mathcal{S}_0\big(\overline{x}_1 - \mathcal{S}_\varepsilon(\overline{x}_2)\big), 12Ra/\varepsilon_0),$$
$$\text{and } K(\mathcal{S}_0(v_1), \mathcal{S}_\varepsilon(\overline{x}_1) - \overline{x}_2, 12Ra/\varepsilon_0),$$

one sees in fact that the two last ones are very close, in the sense that in the limit $\varepsilon \to 0$, they become identical, since their respective axis become :

$$\mathcal{S}_0\big(\overline{x}_1 - \mathcal{S}_\varepsilon(\overline{x}_2)\big) \to \mathcal{S}_0\big(\overline{x}_1 - \mathcal{S}_0(\overline{x}_2)\big) = \mathcal{S}_0(\overline{x}_1) - \overline{x}_2$$

for the first axis, and :

$$\mathcal{S}_\varepsilon(\overline{x}_1) - \overline{x}_2 \to \mathcal{S}_0(\overline{x}_1) - \overline{x}_2$$

for the second axis. One may be tempted then to replace this condition, concerning three cylinders by a condition concerning only two, which is already the case for the condition (12.22).

One notices also in the condition (12.23) that the axes of the cylinders depend on ε. One will see in the following that it may cause trouble when one tries to prove the convergence verified by the solutions of the BBGKY hierarchy towards the Boltzmann hierarchy. The goal of the following lemma is to address the two previous remarks : first, one will relax a bit the condition (12.23) (that is, one will work with cylinders with a larger radius) but such that the conditions, in the limit of the small parameters $\varepsilon, a, \varepsilon_0, \delta \to 0$, describe a subset of the phase space composed of cylinders with fixed axes, that is, which do not depend on the varying parameters $\varepsilon, a, \varepsilon_0$ and δ. Second, one will notice immediately, by doing so, that the relaxation on the radius of the cylinders will indeed enable to consider conditions concerning only two cylinders.

Lemma 28 (Shooting lemma in the case of the half-plane, with fixed axes)**.** *Let R, δ, ε, a and ε_0 be five strictly positive numbers, such that :*

$$\varepsilon \leq a, \quad 2\sqrt{3}a \leq \varepsilon_0. \tag{12.27}$$

One considers two points $\overline{x}_1, \overline{x}_2 \in \left\{x \in \mathbb{R}^d \ / \ x \cdot e_1 > 0\right\}$ such that $|\overline{x}_1 - \overline{x}_2| \geq \varepsilon_0$, and $v_1 \in B(0, R)$.
Then for all $x_1 \in B(\overline{x}_1, a)$, $x_2 \in B(\overline{x}_2, a)$, and $v_2 \in B(0, R)$, if :

$$v_2 \notin K(v_1, \overline{x}_1 - \overline{x}_2, 12Ra/\varepsilon_0) \cup K(\mathcal{S}_0(v_1), \mathcal{S}_0(\overline{x}_1) - \overline{x}_2, 16Ra/\varepsilon_0), \tag{12.28}$$

(using the notations (12.18) *and* (12.19) *for the symmetries), one has for all $\tau > 0$:*

$$\left| \left(T^{2,\varepsilon}_{-\tau}(Z_2)\right)^{X,1} - \left(T^{2,\varepsilon}_{-\tau}(Z_2)\right)^{X,2} \right| > \varepsilon.$$

Proof. The proof of this new version of the shooting lemma is in fact simply obtained from a slight modification of the proof of the first version.
Concerning the proof of the first point, going back to the "conic estimates", that is the part of the proof dedicated to the first point of Lemma 27, first one found that if $\left(T^{2,\varepsilon}_{-\tau_0}(Z_2)\right)^{X,1} = \mathcal{S}_\varepsilon(x_1 - \tau_0 v_1)$ and $\left(T^{2,\varepsilon}_{-\tau_0}(Z_2)\right)^{X,2} = x_2 - \tau_0 v_2$, then (see the inequality (12.25) page 339) :

$$\left| \mathcal{S}_0\big(\overline{x}_1 - \mathcal{S}_\varepsilon(\overline{x}_2)\big) - \tau_0\big(\mathcal{S}_0(v_1) - v_2\big) \right| \leq 3a.$$

Since the goal is to remove the dependency on ε in the axis $\mathcal{S}_0\big(\overline{x}_1 - \mathcal{S}_\varepsilon(\overline{x}_2)\big)$ previously found, one writes (according to the explicit expression of the symmetries (12.18) and (12.19)) :

$$\mathcal{S}_\varepsilon(\overline{x}_2) = \mathcal{S}_0(\overline{x}_2) + \varepsilon e_1.$$

One obtains then easily that :

$$\Big|\mathcal{S}_0\big(\overline{x}_1 - \mathcal{S}_0(\overline{x}_2)\big) - \tau_0\big(\mathcal{S}_0(v_1) - v_2\big)\Big| - \Big|\mathcal{S}_0(\varepsilon e_1)\Big| \\ \leq \Big|\mathcal{S}_0\big(\overline{x}_1 - \mathcal{S}_\varepsilon(\overline{x}_2)\big) - \tau_0\big(\mathcal{S}_0(v_1) - v_2\big)\Big|,$$

so that :

$$\Big|\mathcal{S}_0\big(\overline{x}_1 - \mathcal{S}_0(\overline{x}_2)\big) - \tau_0\big(\mathcal{S}_0(v_1) - v_2\big)\Big| \leq 3a + \varepsilon \leq 4a. \tag{12.29}$$

From this new inequality, following the same path as the one used for the proof of the previous Lemma 27, if one assumes

$$\Big(T^{2,\varepsilon}_{-\tau_0}(Z_2)\Big)^{X,1} = x_1 - \tau_0 v_1,$$

$$\Big(T^{2,\varepsilon}_{-\tau_0}(Z_2)\Big)^{X,2} = \mathcal{S}_\varepsilon(x_2 - \tau_0 v_2),$$

and

$$\Big|\Big(T^{2,\varepsilon}_{-\tau_0}(Z_2)\Big)^{X,1} - \Big(T^{2,\varepsilon}_{-\tau_0}(Z_2)\Big)^{X,2}\Big| = \varepsilon,$$

then the inequality (12.29) implies that the relative velocity $v_2 - \mathcal{S}_0(v_1)$ belongs to the cone of vertex $0 \in \mathbb{R}^d$ and based on the ball $B\big(\mathcal{S}_0\big(\overline{x}_1 - \mathcal{S}_0(\overline{x}_2)\big), 4a\big)$. Since in addition the relative velocity lies in the ball $B(0, 2R)$, one deduces that v_2 belongs to the cylinder $K(\mathcal{S}_0(v_1), \mathcal{S}_0\big(\overline{x}_1 - \mathcal{S}_0(\overline{x}_2)\big), 16Ra/\varepsilon_0)$. This new cylinder plays the role of the second term of the condition (12.28) of Lemma 27.
Secondly, one showed that if $\Big(T^{2,\varepsilon}_{-\tau_0}(Z_2)\Big)^{X,1} = x_1 - \tau_0 v_1$ and $\Big(T^{2,\varepsilon}_{-\tau_0}(Z_2)\Big)^{X,2} = \mathcal{S}_\varepsilon(x_2 - \tau_0 v_2)$, then (see the inequality (12.24) page 337) :

$$\Big|\mathcal{S}_\varepsilon(\overline{x}_1) - \overline{x}_2 - \tau_0\big(\mathcal{S}_0(v_1) - v_2\big)\Big| \leq 3a.$$

Again, writing :

$$\mathcal{S}_\varepsilon(\overline{x}_1) = \mathcal{S}_0(\overline{x}_1) + \varepsilon e_1,$$

one obtains :

$$\Big|\mathcal{S}_0(\overline{x}_1) - \overline{x}_2 - \tau_0\big(\mathcal{S}_0(v_1) - v_2\big)\Big| \leq 3a + \varepsilon \leq 4a.$$

One knows then that this condition implies that the velocity v_2 belongs to the cylinder $K(\mathcal{S}_0(v_1), \mathcal{S}_0(\overline{x}_1) - \overline{x}_2, 16Ra/\varepsilon_0)$. This new cylinder plays the role of the last term of the control (12.23) of Lemma 27, but since of course :

$$\mathcal{S}_0\big(\overline{x}_1 - \mathcal{S}_\varepsilon(\overline{x}_2)\big) = \mathcal{S}_0(\overline{x}_1) - \overline{x}_2,$$

one sees that the two last terms of the control (12.23) of Lemma 27 have been replaced by a single one, which is a cone defined with an axis which *does not* depend on ε, hence the point of the new version of the shooting lemma. □

12.2.3 Effects of the scattering on the excluded cylinders : removing the recollisions after scattering

When the k-th adjunction occurs, that is when a particle is added to a system of $s+k-1$ particles, depending on the sign $\pm$ of the collision operator (introduced in Definitions 42 page 317 and 42 page 317)

$$\mathcal{I}^{\cdot s}_{\pm,j},$$

the configuration is either pre or post collisional. For the BBGKY hierarchy, and in the first case, corresponding to the sign $-$, a particle is added to the particle j_k, at the position $x^{\varepsilon,j_k}_{s,k-1}(t_k)+\varepsilon\omega_k$, one chooses a velocity v_{s+k} (such that $\omega_k\cdot(v^{\varepsilon,j_k}_{s,k-1}(t_k)-v_{s+k})>0$) and one applies only the backwards hard sphere transport until the next adjunction. The trajectories after this adjunction and before the next one (that is, for times $t\leq t_k$ and $t>t_{k+1}$) are represented in red on the following Figure 12.6.

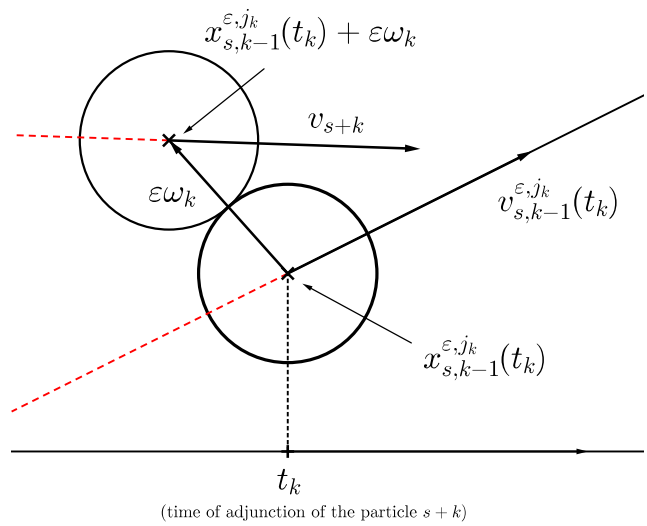

Figure 12.6: Adjunction of a particle to the system, in a pre-collisional configuration.

In the second case, corresponding to the sign $+$, after the adjunction at the position $x^{\varepsilon,j_k}_{s,k-1}+\varepsilon\omega_k$, one will apply the scattering operator (see Definition 1 page 51) to the configuration :

$$\big(x^{\varepsilon,j_k}_{s,k-1}(t_k),v^{\varepsilon,j_k}_{s,k-1}(t_k),x^{\varepsilon,j_k}_{s,k-1}(t_k)+\varepsilon\omega_k,v_{s+k}\big).$$

On Figure 12.7, one sees the trajectories of the particles modified by the scattering operator. In green are pictured the trajectories that the particles would have followed without the effect of this scattering operator, which leads to an

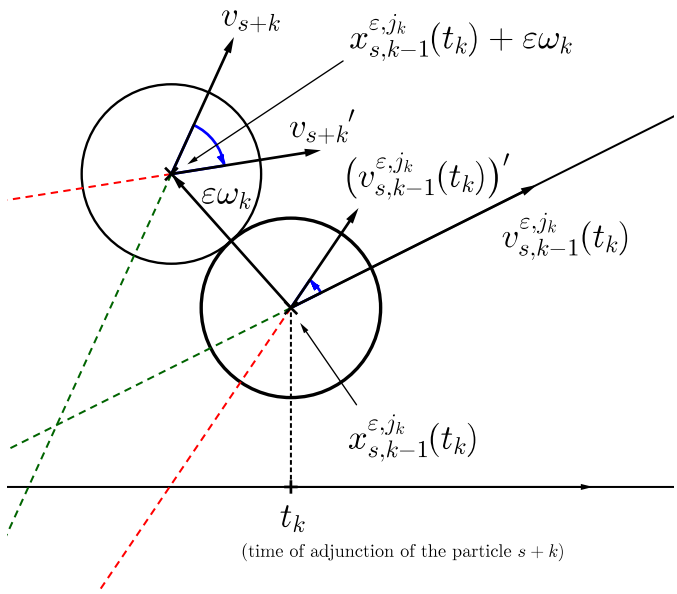

Figure 12.7: Adjunction of a particle to the system, in a post-collisional configuration.

overlapping of the particles for times $t < t_k$. The trajectory obtained after modifying the velocities with the scattering are represented in red. The effect on the pair of velocities $\left(v^{\varepsilon,j_k}_{s,k-1}, v_{s+k}\right)$ of this operator is pictured in blue.
Therefore, in order to keep a system in a good configuration, it is mandatory to obtain a condition on the velocity v_{s+k} (and, if acting only on the velocity variable v_{s+k} is not enough, on the angular parameter ω_k) such that, after the scattering, the two pre-collisional velocities lie outside the pathological cylinders defined with respect to all the other particles of the configuration.
The following lemma focuses on the way the pathological sets are modified by the collisions. One defines :

$$\mathcal{N}^*(w,y,\rho)(v_1) = \Big\{(\omega, v_2) \in \mathbb{S}^{d-1} \times B(0,R) \ / \ (v_2 - v_1)\cdot\omega > 0,$$
$$v_1' \in K(w,y,\rho) \text{ or } v_2' \in K(w,y,\rho)\Big\},$$

where v_1' and v_2' denote, as usual, the post-collisional velocities associated to the pre-collisional velocities v_1 and v_2 and the angular parameter ω. The following lemma is also written in [34].

Lemma 29 (Scattering lemma for cylinders)**.** *There exists a strictly positive constant $C_5(d)$ depending only on the dimension d such that for all strictly positive numbers ρ and R, all vectors $(y,w) \in \mathbb{R}^d \times B(0,R)$, and $v_1 \in B(0,R)$, one has :*

$$|\mathcal{N}^*(w,y,\rho)(v_1)| \leq C_5 R^{d+1/2}\rho^{d-3/2}.$$

One notices that this lemma does not involve the geometry of the domain in which the particles are moving. In particular, the fact that particles are interacting with an obstacle has no consequence here. Therefore, the proof that the reader may find in [34] provides the result.

12.2.4 Adjunction of particles and recollisions

Definition 51 (Stability by adjunction of the good configurations)**.** *Let k be a positive integer, and R, δ, ε, a, ε_0 be five strictly positive numbers. For $\overline{Z}_k \in \mathcal{G}_k^0(\varepsilon_0)$, one defines*

$$\mathcal{B}_k(R, \delta, \varepsilon, a, \varepsilon_0)(\overline{Z}_k) \subset \mathbb{S}^{d-1} \times B(0, R)$$

as the complement *of the set of the elements (ω, v) of $\mathbb{S}^{d-1} \times B(0, R)$ such that, for all*

$$Z_k \in \mathcal{G}_k^\varepsilon(\varepsilon)$$

with, for all $1 \leq i \leq k$

$$\left|\overline{x}_i - x_i\right| = \left|\left(\overline{Z}_k\right)^{X,i} - \left(Z_k\right)^{X,i}\right| \leq a,$$

$$\overline{v}_k = v_k, \text{ that is } \left(\overline{Z}_k\right)^{V,k} = \left(Z_k\right)^{V,k},$$

and for all $1 \leq i \leq k-1$:

$$v_i = \overline{v}_i \text{ or } v_i = \mathcal{S}_0(\overline{v}_i), \text{ that is } (Z_k)^{V,i} = \left(\overline{Z}_k\right)^{V,i} \text{ or } (Z_k)^{V,i} = \mathcal{S}_0\big((\overline{Z}_k)^{V,i}\big)$$

then :

- *if $\omega \cdot (v - \overline{v}_k) < 0$:*
  - *the configuration $(Z_k, x_k + \varepsilon\omega, v)$ does not lead to a further recollision :*

$$\begin{aligned} &\forall \tau > 0, \forall\ 1 \leq i < j \leq k+1, \\ &\quad \left|\left(T_{-\tau}^{k+1,\varepsilon}(Z_k, x_k + \varepsilon\omega, v)\right)^{X,i} - \left(T_{-\tau}^{k+1,\varepsilon}(Z_k, x_k + \varepsilon\omega, v)\right)^{X,j}\right| > \varepsilon, \end{aligned} \tag{12.30}$$

    that is

$$\big(Z_k, x_k + \varepsilon\omega, v\big) \in \mathcal{G}_{k+1}^\varepsilon(\varepsilon),$$

  - *the configuration $\big(\overline{Z}_k, \overline{x}_k, v\big)$ is a good configuration for the free flow dynamics of $k+1$ particles with boundary condition, separated by at least ε_0 (see Definition 48 page 333) after a time δ :*

$$\begin{aligned} &\forall \tau > \delta, \forall\ 1 \leq i < j \leq k+1, \\ &\quad \left|\Big(T_{-\tau}^{k+1,0}\big((\overline{Z}_k, \overline{x}_k, v)\big)\Big)^{X,i} - \Big(T_{-\tau}^{k+1,0}\big((\overline{Z}_k, \overline{x}_k, v)\big)\Big)^{X,j}\right| > \varepsilon_0, \end{aligned} \tag{12.31}$$

    that is

$$T_{-\delta}^{k+1,0}\big(\overline{Z}_k, \overline{x}_k, v\big) \in \mathcal{G}_{k+1}^0(\varepsilon_0),$$

- *if* $\omega \cdot (v - \overline{v}_k) > 0$ *:*
  - *the configuration* $\big(Z_s, x_k + \varepsilon\omega, v\big)'_{k,k+1}$ *(see Definition 1 page 51 and the point* (1.8) *of Definition 2 page 52 for the notations) does not lead to a further recollision :*

$$\begin{aligned}&\forall \tau > 0, \forall\ 1 \le i \ne j \le k+1,\\ &\quad \Big|\Big(T^{k+1,\varepsilon}_{-\tau}\big(\big(Z_k, x_k + \varepsilon\omega, v\big)'_{k,k+1}\big)\Big)^{X,i}\\ &\qquad - \Big(T^{k+1,\varepsilon}_{-\tau}\big(\big(Z_k, x_k + \varepsilon\omega, v\big)'_{k,k+1}\big)\Big)^{X,j}\Big| > \varepsilon, \end{aligned} \tag{12.32}$$

  that is

$$\big(Z_s, x_k + \varepsilon\omega, v\big)'_{k,k+1} \in \mathcal{G}^{\varepsilon}_{k+1}(\varepsilon),$$

  - *the configuration* $\big(\overline{Z}_k, \overline{x}_k, v\big)$ *is a good configuration for the free flow dynamics of* $k+1$ *particles with boundary condition, separated by at least* ε_0 *after a time* δ *:*

$$\begin{aligned}&\forall \tau > \delta, \forall\ 1 \le i < j \le k+1,\\ &\quad \Big|\Big(T^{k+1,0}_{-\tau}\big(\big(\overline{Z}_k, \overline{x}_k, v\big)'_{k,k+1}\big)\Big)^{X,i} - \Big(T^{k+1,0}_{-\tau}\big(\big(\overline{Z}_k, \overline{x}_k, v\big)'_{k,k+1}\big)\Big)^{X,j}\Big| > \varepsilon_0, \end{aligned} \tag{12.33}$$

  that is

$$T^{k+1,0}_{-\delta}\Big(\big(\overline{Z}_k, \overline{x}_k, v\big)'_{k,k+1}\Big) \in \mathcal{G}^{0}_{k+1}(\varepsilon_0).$$

Remark 32. *Here, one wants to obtain an upper bound on the size of the set* $\Big|\mathcal{B}_k\big(R, \delta, \varepsilon, a, \varepsilon_0\big)\big(\overline{Z}_k\big)\Big|$*. Finding a sharp bound will mean that there are a lot of ways to add a particle to a system of* k *particles in a good configuration, which lead to a new system of* $k+1$ *particles in a good configuration.*

One notices that there is an important degree of freedom introduced in Definition 51 : one authorizes a possible small difference between the positions of the configurations $\overline{Z}_k \in \big(\Omega^c \times \mathbb{R}^d\big)^k$ *and the positions of the vector* $Z_k \in \mathcal{D}^{\varepsilon}_k$*, while the velocities of those two configurations are the same, or at most differ by a symmetry (that is* $v_k = \overline{v}_k$ *or* $\mathcal{S}_0(\overline{v}_k)$*).*
The necessity of this additional degree of freedom is actually clear. Indeed, even if one compares trees without recollision, a small difference in the positions of the configurations will appear between the trees of the BBGKY and the Boltzmann hierarchies. It is due, on the one hand, to the radius of the particles, and on the other hand, to the interaction with the obstacle, as it was already noticed in Section 12.1.2 page 327, and represented on Figures 12.2 page 331 and 12.4 page 332.

Proposition 16 (Control of the size of the good configurations by adjunction of particles)**.** *There exists a strictly positive constant* $C(d)$ *depending only on*

the dimension d such that for any positive integer k, and for all $R, \delta, \varepsilon, a, \varepsilon_0, \rho, \eta$ strictly positive real numbers such that

$$2\varepsilon \leq a, \quad 4\sqrt{3}a \leq \varepsilon_0, \quad \varepsilon_0 \leq \eta\delta, \quad 3a \leq \rho, \tag{12.34}$$

$$R \geq 1 \text{ and } \varepsilon_0/\delta \leq 1, \tag{12.35}$$

and for all $\overline{Z}_k \in \mathcal{G}_k^0(\varepsilon_0)$ such that

$$\overline{x}_k \cdot e_1 = \big(\overline{Z}_k\big)^{X,k} \cdot e_1 \geq \rho, \tag{12.36}$$

there exists a measurable subset $\widetilde{\mathcal{B}_k}(R, \delta, \varepsilon, a, \varepsilon_0, \rho, \eta)(\overline{Z}_k) \subset S^{d-1} \times B(0, R)$ such that :

$$\mathcal{B}_k(R, \delta, \varepsilon, a, \varepsilon_0)(\overline{Z}_k) \subset \widetilde{\mathcal{B}_k}(R, \delta, \varepsilon, a, \varepsilon_0, \rho, \eta)(\overline{Z}_k)$$

and

$$\bigcup_{\substack{\overline{Z}_k \in \ \mathcal{G}_k(\varepsilon_0), \\ \overline{x}_k \cdot e_1 \geq \rho}} \{\overline{Z}_k\} \times \widetilde{\mathcal{B}_k}(R, \delta, \varepsilon, a, \varepsilon_0, \rho, \eta)(\overline{Z}_k)$$

is measurable. Moreover, one has :

$$\begin{aligned} &\Big|\widetilde{\mathcal{B}_k}(R, \delta, \varepsilon, a, \varepsilon_0, \rho, \eta)(\overline{Z}_k)\Big| \\ &\quad \leq \ C(d)\Big(\eta^d + R^d\Big(\frac{a}{\rho}\Big)^{d-1} + kR^{2d-1}\Big(\frac{a}{\varepsilon_0}\Big)^{d-3/2} + kR^{d+1/2}\Big(\frac{\varepsilon_0}{\delta}\Big)^{d-3/2}\Big). \end{aligned} \tag{12.37}$$

Proof. For a configuration $\overline{Z}_k \in \mathcal{G}_k^0(\varepsilon_0)$, one wants to study the subset

$$\mathcal{B}_k(R, \delta, \varepsilon, a, \varepsilon_0)(\overline{Z}_k)$$

of $\mathbb{S}^{d-1} \times B(0, R)$.
The hypothesis on $\overline{Z}_k$ means exactly, according to Definition 48 page 333, that for all $\tau > 0$ and all $1 \leq i < j \leq k$:

$$\Big|\Big(T_{-\tau}^{k,0}\big(\overline{Z}_k\big)\Big)^{X,i} - \Big(T_{-\tau}^{k,0}\big(\overline{Z}_k\big)\Big)^{X,j}\Big| > \varepsilon_0.$$

The subset $\mathcal{B}_k(R, \delta, \varepsilon, a, \varepsilon_0)(\overline{Z}_k)$, associated to the configuration $\overline{Z}_k$, according to Definition 51 page 346, is defined as the complement of the subset of the angular and velocity adjunction parameters such that, for all configurations $Z_k \in \mathcal{G}_k^\varepsilon(\varepsilon)$ "close enough" to $\overline{Z}_k$ (that is such that, first the positions of the particles of the configuration Z_k are not too far from the positions of $\overline{Z}_k$, that is :

$$\big|\overline{X}_k - X_k\big| = \big|\big(\overline{Z}_k\big)^X - \big(Z_k\big)^X\big| \leq a,$$

and second, the velocties of the particles of the configuration Z_k, up to the application of the symmetry $\mathcal{S}_0$, are the same (except for the last particle k : in that case, the velocities have to be the same), that is :

$$\overline{v}_k = v_k, \text{ that is } \big(\overline{Z}_k\big)^{V,k} = \big(Z_k\big)^{V,k},$$

and for all $1 \leq i \leq k-1$:

$$\overline{v}_i = v_i \text{ or } \overline{v}_i = \mathcal{S}_0(v_i)),$$

when one "adds" the $(k+1)$-th particle to the system of k particles, the new system of $k+1$ particles, has to satisfy the two following conditions :

- for the hard sphere dynamics, the new particle is added at position $x_k + \varepsilon\omega$, and at velocity v, and the new system of $k+1$ particles $(Z_k, x_k + \varepsilon\omega, v)$ has to verify the condition (12.30) of Definition 51 if $(x_k, v_k, x_k + \varepsilon\omega, v)$ is a pair of pre-collisional particles, and the system $\big(Z_k, x_k + \varepsilon\omega, v\big)'_{k,k+1}$ has to verify the condition (12.32) of Definition 51 if $(x_k, v_k, x_k + \varepsilon\omega, v)$ is a pair of post-collisional particles,
- for the free flow with boundary condition, the new particle is added at position x_k, and at velocity v, and the new system of $k+1$ particles $(Z_k, x_k + \varepsilon\omega, v)$ has to verify the condition (12.31) of Definition 51 if (x_k, v_k, x_k, v) is a pair of pre-collisional particles, and the system $\big(Z_k, x_k, v\big)'_{k,k+1}$ has to verify the condition (12.33) of Definition 51 if (x_k, v_k, x_k, v) is a pair of post-collisional particles.

The conditions on the hard sphere dynamics

One starts by a problem of rigorous definition. Since the hard sphere flow for $k+1$ particles is not defined on the whole space $\big(\{y \cdot e_1 > \varepsilon/2\} \times \mathbb{R}^d\big)^{k+1}$, but only on $\mathcal{D}^{\varepsilon}_{k+1}$ (in fact, after removing a measurable subset of measure zero), one needs to be sure that the configuration :

$$\big(Z_k, x_k + \varepsilon\omega, v\big)$$

belongs to $\mathcal{D}^{\varepsilon}_{k+1}$. One will start by verifying that the configuration does not present an overlapping between two particles, that is one needs to be sure that, for all $1 \leq i \leq k-1$:

$$\big|x_i - (x_k + \varepsilon\omega)\big| > \varepsilon.$$

But on the one hand, one knows that $\overline{Z}_k \in \mathcal{G}^0_k(\varepsilon_0)$, so in particular for all $1 \leq i \leq k-1$:

$$\big|\overline{x}_i - \overline{x}_k\big| \geq \varepsilon_0.$$

On the other hand, since :

$$\big|\overline{X}_k - X_k\big| \leq a,$$

one can see in particular that for all $1 \leq j \leq k$:

$$\big|\overline{x}_j - x_j\big| \leq a.$$

All together, it implies that :

$$\begin{aligned}\big|x_i - (x_k + \varepsilon\omega)\big| &\geq \big|\overline{x}_i - \overline{x}_k\big| - \big|\overline{x}_i - x_i\big| - \big|\overline{x}_k - x_k\big| - \big|\varepsilon\omega\big| \\ &\geq \varepsilon_0 - 2a - \varepsilon \geq \varepsilon_0 - 3a.\end{aligned}$$

The condition

$$5a \leq \varepsilon_0, \tag{12.38}$$

together with $\varepsilon < 2a$ implies that :

$$\big|x_i - (x_k + \varepsilon\omega)\big| > \varepsilon,$$

and as a consequence, the configuration

$$\big(Z_k, x_k + \varepsilon\omega, v\big)$$

does not include any overlapping.
It is also mandatory to check that the position $x_k + \varepsilon\omega$ is far enough from the obstacle, so that the added particle does not overlap the obstacle.
The hypothesis (12.36) of the proposition implies that :

$$(x_k + \varepsilon\omega) \cdot e_1 = \overline{x}_k \cdot e_1 + \big(x_k - \overline{x}_k\big) \cdot e_1 + \varepsilon\omega \cdot e_1 \geq \rho - a - \varepsilon \geq \rho - 2a.$$

Recalling that

$$3a \leq \rho, \tag{12.39}$$

one has therefore, since $\varepsilon/2 < a$:

$$(x_k + \varepsilon\omega) \cdot e_1 > \varepsilon/2,$$

so that the configuration

$$\big(Z_k, x_k + \varepsilon\omega, v\big)$$

belongs indeed to $\mathcal{D}^{\varepsilon}_{k+1}$.

Having checked that the hard sphere transport applied to the new configuration of $k+1$ particles is well defined, one now checks the conditions (12.30) and (12.32).

The pre-collisional case for the hard sphere dynamics : the condition (12.30)

One now focuses on the pre-collisional case (that is $\omega \cdot (v - \overline{v}_k) < 0$), and prove the inequality (12.30).

One assumes therefore that the pair of particles (x_k, v_k) and $(x_k + \varepsilon\omega, v)$ is a pre-collisional pair, so that there is no scattering used at time 0 in the definition of the configuration of $k+1$ particles on which the backwards hard sphere flow will act.
One will remove the adjunction parameters leading to recollision. Here a recollision means that there exists a pair of particles of the new system of $k+1$ particles, such that the two particles of the pair, after following the hard sphere

flow, are at some point at distance ε from each other.
Therefore, if one assumes that the set of times τ of collisions is non empty, that is

$$\begin{aligned}\tilde{\tau}_{\text{coll}}^{k+1} = \bigcup_{1\le i\neq j\le k+1} \Big\{\tau > 0 \ / \ \Big|\Big(T_{-\tau}^{k+1,\varepsilon}\big(Z_k, x_k+\varepsilon\omega, v\big)\Big)^{X,i} \\ - \Big(T_{-\tau}^{k+1,\varepsilon}\big(Z_k, x_k+\varepsilon\omega, v\big)\Big)^{X,j}\Big| = \varepsilon\Big\} \\ \neq \emptyset,\end{aligned}$$

one defines $\tau_0 = \inf \tilde{\tau}_{\text{coll}}^{k+1} = \min \tilde{\tau}_{\text{coll}}^{k+1}$. One has to determine which couple of particles (i,j) can collide, that is

$$\Big|\Big(T_{-\tau_0}^{k+1,\varepsilon}\big(Z_k, x_k+\varepsilon\omega, v\big)\Big)^{X,i} - \Big(T_{-\tau_0}^{k+1,\varepsilon}\big(Z_k, x_k+\varepsilon\omega, v\big)\Big)^{X,j}\Big| = \varepsilon, \qquad (12.40)$$

which would contradict the validity of the condition (12.30) of Definition 51, and what are the consequences on the initial velocities.
For all $0 \le \tau \le \tau_0$, each particle of the system follows the hard sphere flow for a single particle (which takes into account the interaction with the obstacle for a particle of radius $\varepsilon/2$, but not the interactions with other particles), because by definition, before τ_0 the particles are far enough from one another, not to collide. It means in particular that the particles can still have their respective velocity modified by a bouncing against the obstacle (which is assumed to be here the half-space). For any particle l, one will denote as above :

$$\tau_l = \inf\Big\{\tau > 0 \ / \ \Big(T_{-\tau}^{k+1,\varepsilon}(Z_k, x_k+\varepsilon\omega, v)\Big)^{X,l} \cdot e_1 = \varepsilon/2\Big\},$$

the time of first bouncing of the particle l against the obstacle (which may be non finite).

- $\underline{1 \le i < j \le k}$:
  No such couple can verify (12.40), because for all $\tau \le \tau_0$ and $1 \le j \le k$, one has
  $$\Big(T_{-\tau}^{k+1,\varepsilon}(Z_k, x_k+\varepsilon\omega, v)\Big)^{X,l} = \Big(T_{-\tau}^{k,\varepsilon}(Z_k)\Big)^{X,l},$$
  and one assumed that $Z_k \in \mathcal{G}_k^{\varepsilon}(\varepsilon)$.

- $\underline{i = k \text{ and } j = k+1}$:
  Here, several cases have to be distinguished, depending on the possibility of bouncing against the obstacle before the time τ_0, that is before identity (12.40) holds.

  - $\tau_0 \le \min\{\tau_k, \tau_{k+1}\}$:
    If the particles k and $k+1$ do not bounce against the obstacle before τ_0 that is if $\tau_0 \le \min\{\tau_k, \tau_{k+1}\}$, the particles k and $k+1$ follow then the backwards free flow and one has for all $0 \le \tau \le \tau_0$:
    $$T_{-\tau}^{k+1,\varepsilon}((Z_k, x_k+\varepsilon\omega, v))^{X,k} = x_k - \tau v_k,$$

(one used here $v_k = \overline{v}_k$) and

$$T_{-\tau}^{k+1,\varepsilon}((Z_k, x_k + \varepsilon\omega, v))^{X,k+1} = x_k + \varepsilon\omega - \tau v.$$

Then

$$\begin{aligned}&\Big|T_{-\tau}^{k+1,\varepsilon}((Z_k, x_k + \varepsilon\omega, v))^{X,k} - T_{-\tau}^{k+1,\varepsilon}((Z_k, x_k + \varepsilon\omega, v))^{X,k+1}\Big|\\ &= |\varepsilon\omega - \tau(v - v_k)| \geq \varepsilon,\end{aligned}$$

since

$$\begin{aligned}|\varepsilon\omega - \tau(v - \overline{v}_k)|^2 &= \varepsilon^2 - 2\tau\varepsilon\omega \cdot (v - \overline{v}_k) + \tau^2|v - \overline{v}_k|^2\\ &\geq \varepsilon^2,\end{aligned}$$

and even

$$|\varepsilon\omega - \tau(v - \overline{v}_k)| > \varepsilon$$

for all $0 < \tau \leq \tau_0$, up to assuming that $v \neq \overline{v}_k$, that is up to the exclusion of the subset $\mathbb{S}^d \times \{\overline{v}_k\}$, of course of measure zero in $\mathbb{S}^d \times B(0, R)$.

– $\min\{\tau_k, \tau_{k+1}\} < \tau_0 \leq \max\{\tau_k, \tau_{k+1}\}$:
If one has :

$$\min\{\tau_k, \tau_{k+1}\} < \tau_0 \leq \max\{\tau_k, \tau_{k+1}\},$$

that is if only one of the two particles k and $k+1$ has bounced against the obstacle before the time τ_0 in the equation (12.40), and

$$\begin{aligned}\Big|&\Big(T_{-\tau_0}^{k+1,\varepsilon}(Z_k, x_k + \varepsilon\omega, v)\Big)^{X,k}\\ &\qquad - \Big(T_{-\tau_0}^{k+1,\varepsilon}(Z_k, x_k + \varepsilon\omega, v)\Big)^{X,k+1}\Big| = \varepsilon,\end{aligned}$$

then the norm of the difference of the two positions writes

$$\begin{aligned}&\big|\mathcal{S}_\varepsilon\big(x_k - \tau_0\overline{v}_k\big) - \big(x_k + \varepsilon\omega - \tau_0 v\big)\big|\\ &= \big|\big(x_k - \tau_0\overline{v}_k - 2(x_k - \tau_0\overline{v}_k) \cdot e_1 e_1 + \varepsilon e_1\big) - \big(x_k + \varepsilon\omega - \tau_0 v\big)\big|\\ &= \big| - 2x_k \cdot e_1 e_1 - \varepsilon\omega + \varepsilon e_1 - \tau_0\big(\mathcal{S}_0(\overline{v}_k) - v\big)\big| = \varepsilon\end{aligned}$$

or

$$\begin{aligned}&\Big|\big(x_k - \tau_0\overline{v}_k\big) - \mathcal{S}_\varepsilon\big(x_k + \varepsilon\omega - \tau_0 v\big)\Big|\\ &= \Big|\big(x_k - \tau_0\overline{v}_k\big) - \big(x_k + \varepsilon\omega - \tau_0 v\big) + 2\big(x_k + \varepsilon\omega - \tau_0 v\big) \cdot e_1 e_1 - \varepsilon e_1\Big|\\ &= \Big|2x_k \cdot e_1 e_1 + 2\varepsilon\omega \cdot e_1 e_1 - \varepsilon\omega - \varepsilon e_1 - \tau_0\big(\overline{v}_k - \mathcal{S}_0(v)\big)\Big|\\ &= \Big|\mathcal{S}_0\Big(2x_k \cdot e_1 e_1 + 2\varepsilon\omega \cdot e_1 e_1 - \varepsilon\omega - \varepsilon e_1 - \tau_0\big(\overline{v}_k - \mathcal{S}_0(v)\big)\Big)\Big|\\ &= \Big|\mathcal{S}_0\big(2x_k \cdot e_1 e_1\big) + \mathcal{S}_0\big(2\varepsilon\omega \cdot e_1 e_1 - \varepsilon\omega\big) - \mathcal{S}_0\big(\varepsilon e_1\big) - \tau_0\Big(\mathcal{S}_0\big(\overline{v}_k\big) - v)\Big)\Big|\\ &= \Big| - 2x_k \cdot e_1 e_1 - \varepsilon\omega + \varepsilon e_1 - \tau_0\big(\mathcal{S}_0(\overline{v}_k) - v)\big)\Big| = \varepsilon,\end{aligned}$$

so the two possible cases lead to the same equation. One wants a control on the difference $v - \mathcal{S}_0(\overline{v}_k)$ which does not depend on the position of the particle k of the configuration Z_k, nor on the angular parameter ω, but only on the configuration $\overline{Z}_k$. One writes then :

$$\begin{aligned}
\varepsilon &= \Big|2x_k \cdot e_1 e_1 + \varepsilon\omega - \varepsilon e_1 - \tau_0\big(v - \mathcal{S}_0(\overline{v}_k)\big)\Big| \\
&\geq \Big|2\overline{x}_k \cdot e_1 e_1 - \tau_0\big(v - \mathcal{S}_0(\overline{v}_k)\big)\Big| - \big|2x_k \cdot e_1 e_1 - 2\overline{x}_k \cdot e_1 e_1\big| - \big|\varepsilon\omega\big| - \big|\varepsilon e_1\big| \\
&\geq \Big|2\overline{x}_k \cdot e_1 e_1 - \tau_0\big(v - \mathcal{S}_0(\overline{v}_k)\big)\Big| - 2a - 2\varepsilon,
\end{aligned}$$

so that, thanks to the inequality $\varepsilon \leq a$:

$$\Big|2\overline{x}_k \cdot e_1 e_1 - \tau_0\big(v - \mathcal{S}_0(\overline{v}_k)\big)\Big| \leq 5a.$$

Following the same arguments as in the proof of Lemma 27, one deduces that the relative velocity $v - \mathcal{S}_0(\overline{v}_k)$ belongs to the cone of vertex $0 \in \mathbb{R}^d$ and based on the ball $B(2\overline{x}_k \cdot e_1 e_1, 5a)$. Remembering that this relative velocity lies also in the ball $B(0, 2R)$, one deduces that it lies in particular in the cylinder of origin $0 \in \mathbb{R}^d$, of axis $2\overline{x}_k \cdot e_1 e_1$ and, up to assuming that $\overline{x}_k$ is large enough compared to a (in order to have $\big|2\overline{x}_k \cdot e_1 e_1\big| \geq (5a)$, so that the denominator of the following fraction is strictly positive), of radius

$$\frac{2R \cdot 5a}{\sqrt{\big|2\overline{x}_k \cdot e_1 e_1\big|^2 - (5a)^2}},$$

with, as above, $\mathcal{S}_0(x)$ denoting the orthogonal symmetry of the vector $x \in \mathbb{R}^d$ with respect to the hyperplane $\{x \in \mathbb{R}^d \ / \ x \cdot e_1 = 0\}$.
Thanks to the hypothesis on the distance with the obstacle (12.36), that is :

$$\rho \leq \overline{x}_k \cdot e_1,$$

and remembering that :

$$\frac{5}{\sqrt{3}} a \leq \rho, \tag{12.41}$$

that is :

$$\sqrt{4\rho^2 - 25a^2} \geq \rho,$$

one has :

$$\frac{10Ra}{\sqrt{4(\overline{x}_k \cdot e_1)^2 - 25a^2}} \leq \frac{10Ra}{\rho},$$

which bounds from above the radius of the cylinder. So up to the exclusion of the cylinder

$$K\big(\mathcal{S}_0(\overline{v}_k), 2\overline{x}_k \cdot e_1 e_1, 10Ra/\rho\big), \tag{12.42}$$

for every τ such that $0 < \min\{\tau_k, \tau_{k+1}\} < \tau \leq \max\{\tau_k, \tau_{k+1}\}$, one has :

$$\Big|T_{-\tau}^{k+1,\varepsilon}((Z_k, x_k + \varepsilon\omega, v))^{X,k} - T_{-\tau}^{k+1,\varepsilon}((Z_k, x_k + \varepsilon\omega, v))^{X,k+1}\Big| > \varepsilon.$$

The measure of the intersection of the cylinder (12.42) with the ball $B(0, R)$ (in which v lies) is then controlled by

$$C(d)(4R)\Big(\frac{10Ra}{\rho}\Big)^{d-1}. \tag{12.43}$$

– $\tau_0 > \max\{\tau_k, \tau_{k+1}\}$:
Now, if the two particles k and $k+1$ have already bounced against the obstacle, that is if :

$$\tau_0 > \max\{\tau_k, \tau_{k+1}\}$$

and

$$\Big|\Big(T_{-\tau_0}^{k+1,\varepsilon}(Z_k, x_k + \varepsilon\omega, v)\Big)^{X,k} - \Big(T_{-\tau_0}^{k+1,\varepsilon}(Z_k, x_k + \varepsilon\omega, v)\Big)^{X,k+1}\Big| = \varepsilon,$$

then

$$\begin{aligned}
T_{-\tau_0}^{k+1,\varepsilon}((Z_k, x_k + \varepsilon\omega, v))^{X,k} &= \mathcal{S}_\varepsilon\big(x_k - \tau_0\overline{v}_k\big) \\
&= (x_k - \tau_0\overline{v}_k) - 2(x_k - \tau_0\overline{v}_k)\cdot e_1e_1 + \varepsilon e_1 \\
&= \big(x_k - 2x_k\cdot e_1e_1 + \varepsilon e_1\big) - \tau_0\big(\overline{v}_k - 2\overline{v}_k\cdot e_1e_1\big) \\
&= \mathcal{S}_\varepsilon(x_k) - \tau_0\mathcal{S}_0(\overline{v}_k)
\end{aligned}$$

and

$$T_{-\tau_0}^{k+1,\varepsilon}((Z_k, x_k + \varepsilon\omega, v))^{X,k+1} = \mathcal{S}_\varepsilon(x_k + \varepsilon\omega) - \tau_0\mathcal{S}_0(v),$$

so that

$$\begin{aligned}
&\Big|T_{-\tau_0}^{k+1,\varepsilon}((Z_k, x_k + \varepsilon\omega, v))^{X,k} - T_{-\tau_0}^{k+1,\varepsilon}((Z_k, x_k + \varepsilon\omega, v))^{X,k+1}\Big| \\
&= \Big|\big(\mathcal{S}_\varepsilon(x_k) - \tau_0\mathcal{S}_0(\overline{v}_k)\big) - \big(\mathcal{S}_\varepsilon(x_k + \varepsilon\omega) - \tau_0\mathcal{S}_0(v)\big)\Big| \\
&= \Big|\big(x_k - 2x_k\cdot e_1e_1 + \varepsilon e_1 - \tau_0\mathcal{S}_0(\overline{v}_k)\big) \\
&\qquad\qquad - \big(x_k + \varepsilon\omega - 2(x_k + \varepsilon\omega)\cdot e_1e_1 + \varepsilon e_1 - \tau_0\mathcal{S}_0(v)\big)\Big| \\
&= \Big| - \varepsilon\mathcal{S}_0(\omega) - \tau_0\big(\mathcal{S}_0(\overline{v}_k) - \mathcal{S}_0(v)\big)\Big| \\
&= \big|\varepsilon\omega - \tau_0(v - \overline{v}_k)\big|,
\end{aligned}$$

and even

$$\Big| T_{-\tau}^{k+1,\varepsilon}((Z_k, x_k + \varepsilon\omega, v))^{X,k} \\ - T_{-\tau}^{k+1,\varepsilon}((Z_k, x_k + \varepsilon\omega, v))^{X,k+1} \Big| > \varepsilon.$$

This quantity was assumed to be equal to ε, which is not possible except if $v = \overline{v}_k$.

At this point, the condition (12.30) is verified (up to the sets which were eliminated) for the case $i = k$, $j = k + 1$, and the only remaining case that one has to check is when $i < k$ and $j = k + 1$.

- $\underline{i \leq k - 1 \text{ and } j = k + 1}$:
For this last possibility in the choice of the particles which could contradict the condition (12.30), one will simply use the Shooting Lemma 27 page 335 (in particular, one notices that no hypothesis on the distance between the particles and the obstacle is required to apply this lemma). Indeed, for all times τ before the time τ_0 of the first violation of the separation inequality (12.30), the particles do not interact with each other, and then one has of course :

$$\Big(T_{-\tau}^{k+1,\varepsilon}\big(Z_k, x_k + \varepsilon\omega, v\big)\Big)^{X,i} = \Big(T_{-\tau}^{2,\varepsilon}\big(x_i, v_i, x_k + \varepsilon\omega, v\big)\Big)^{X,1}$$

and

$$\Big(T_{-\tau}^{k+1,\varepsilon}\big(Z_k, x_k + \varepsilon\omega, v\big)\Big)^{X,k+1} = \Big(T_{-\tau}^{2,\varepsilon}\big(x_i, v_i, x_k + \varepsilon\omega, v\big)\Big)^{X,2}.$$

Therefore, by hypothesis, one has

$$\big|\overline{x}_i - \overline{x}_k\big| \geq \varepsilon_0,$$

$$\big|\overline{x}_i - x_i\big| \leq a,$$

and

$$\big|\overline{x}_k - (x_k + \varepsilon\omega)\big| \leq a + \varepsilon \leq 2a.$$

One has to be careful here, since there are two possibilities : $v_i = \overline{v}_i$, or $v_i = \mathcal{S}_0(\overline{v}_i)$. Remembering that :

$$4\sqrt{3}a \leq \varepsilon_0, \tag{12.44}$$

and taking $\widetilde{a} = 2a$, one recovers the hypothesis (12.21) of Lemma 27 page 335, so it can be applied there twice (one for each possible value of the velocity

v_i), and therefore, up to assuming that :

$$\begin{aligned} v \notin \Big(& K(\overline{v}_i, \overline{x}_i - \overline{x}_k, 24Ra/\varepsilon_0) \cup K\big(\mathcal{S}_0(\overline{v}_i), \mathcal{S}_0\big(\overline{x}_i - \mathcal{S}_\varepsilon(\overline{x}_k)\big), 24Ra/\varepsilon_0\big) \\ & \cup K\big(\mathcal{S}_0(\overline{v}_i), \mathcal{S}_\varepsilon(\overline{x}_i) - \overline{x}_k, 24Ra/\varepsilon_0\big)\Big) \\ \cup \Big(& K\big(\mathcal{S}_0(\overline{v}_i), \overline{x}_i - \overline{x}_k, 24Ra/\varepsilon_0\big) \cup K\big(\overline{v}_i, \mathcal{S}_0\big(\overline{x}_i - \mathcal{S}_\varepsilon(\overline{x}_k)\big), 24Ra/\varepsilon_0\big) \\ & \cup K\big(\overline{v}_i, \mathcal{S}_\varepsilon(\overline{x}_i) - \overline{x}_k, 24Ra/\varepsilon_0\big)\Big), \end{aligned} \tag{12.45}$$

then $\big(x_i, v_i, x_k + \varepsilon\omega, v\big) \in \mathcal{G}_2^\varepsilon(\varepsilon)$. Therefore, up to the exclusion of those cylinders, the first recollision cannot be between the particles i and $k+1$. The size of the six cylinders (12.45) is controlled by

$$6C(d)(4R)\Big(\frac{24Ra}{\varepsilon_0}\Big)^{d-1} \tag{12.46}$$

Finally, all the possible cases of a recollision from the configuration $\big(Z_k, x_k + \varepsilon\omega, v\big)$ have been investigated so far in the pre-collisional case.

The post-collisional case for the hard sphere dynamics : the condition (12.32)

One finishes the study of the conditions concerning the hard sphere dynamics with the post-collisional case, that is when $\omega \cdot (v - \overline{v}_k) > 0$.

In this case, the particles k and $k+1$ form a post-collisional pair, and according to the definition of the hard sphere dynamics, their respective velocities are changed into $\overline{v}'_k$ and v' (see Definition 1 page 51 in which the scattering operator is introduced) before being transported. Then, one has to do the same work as for the previous part of the proof, since $\big(x_k, v'_k, x_k + \varepsilon\omega, v'\big)$ is a pair of particles in a pre-collisional configuration. One wants to be sure that there is no strictly positive time τ and no couple of integers $1 \le i < j \le k+1$ violating the condition (12.32).
One considers then, as for the pre-collisional configuration, the smallest time τ_0 such that :

$$\begin{aligned} \Big| \Big(T_{-\tau_0}^{k+1,\varepsilon}\big(x_k, v'_k, x_k + \varepsilon\omega, v'\big)\Big)^{X,i} \\ - \Big(T_{-\tau_0}^{k+1,\varepsilon}\big(x_k, v'_k, x_k + \varepsilon\omega, v'\big)\Big)^{X,j} \Big| = \varepsilon. \end{aligned}$$

A significant difference with the pre-collisional case is the following. One will have to exclude not only some velocities v, but also some angular parameters ω.

- $\underline{1 \le i < j \le k-1}$:
  A simple remark, as in the case $1 \le i < j \le k$ for a pre-collisional configuration, enables to solve this case : the change of the velocities produced by the

scattering due to the post-collisional configuration does not affect the velocities of the particle i for $1 \leq i \leq k-1$, and since by hypothesis $Z_k \in \mathcal{G}_k^\varepsilon(\varepsilon)$, the first recollision cannot be between two such particles.
Of course, the scattering has modified the velocity of the particle k, so this argument cannot be used for $j = k$.

- $\underline{1 \leq i \leq k-1 \text{ and } j = k}$:
After the collision, the velocity $\overline{v}_k$ of the particle k is modified into $\overline{v}'_k$ by the scattering operator (see Definition 1 page 51), so one can imagine that the condition (12.32) can be violated, for some $i \leq k-1$ and $j = k$. But since

$$\left|\overline{x}_i - \overline{x}_k\right| \geq \varepsilon_0,$$

and

$$\left|\overline{x}_i - x_i\right| \leq a \text{ and } \left|\overline{x}_k - x_k\right| \leq a,$$

remembering that

$$2\sqrt{3}a \leq \varepsilon_0, \tag{12.47}$$

one can apply twice (for $v_i = \overline{v}_i$, and for $v_i = \mathcal{S}_0\big(\overline{v}_i\big)$) Lemma 27 with $\widetilde{a} = a$, so that up to the exclusion of the following union of cylinders :

$$\begin{aligned}\Big(K\big(\overline{v}_i, \overline{x}_i - \overline{x}_k, 12Ra/\varepsilon_0\big) \cup K\big(\mathcal{S}_0(\overline{v}_i), \mathcal{S}_0\big(\overline{x}_i - \mathcal{S}_\varepsilon(\overline{x}_k)\big), 12Ra/\varepsilon_0\big)\\ \cup K(\mathcal{S}_0(\overline{v}_i), \mathcal{S}_\varepsilon(\overline{x}_i) - \overline{x}_k, 12Ra/\varepsilon_0)\Big)\\ \cup \Big(K\big(\mathcal{S}_0(\overline{v}_i), \overline{x}_i - \overline{x}_k, 12Ra/\varepsilon_0\big) \cup K\big(\overline{v}_i, \mathcal{S}_0\big(\overline{x}_i - \mathcal{S}_\varepsilon(\overline{x}_k)\big), 12Ra/\varepsilon_0\big)\\ \cup K\big(\overline{v}_i, \mathcal{S}_\varepsilon(\overline{x}_i) - \overline{x}_k, 12Ra/\varepsilon_0\big)\Big),\end{aligned} \tag{12.48}$$

in which $\overline{v}'_k$ is chosen, one can assert that if the configuration

$$\big(Z_k, x_k + \varepsilon\omega, v\big)'_{k,k+1}$$

leads to a first recollision, then it will not be between the particles i and k.

- $\underline{1 \leq i \leq k-1 \text{ and } j = k+1}$:
Similarly, one wants to prevent a first recollision between the particles $1 \leq i \leq k-1$ and $j = k+1$.
Thanks to the work done above for the pre-collisional case, if one knows that v' does not belong to the family of cylinders (12.45) page 356, then the first recollision cannot imply the particles $i \leq k-1$ and $k+1$.

- $\underline{i = k \text{ and } j = k+1}$:
Finally, one wants to prevent a first recollision between the particles k and $k+1$. Such a recollision would lead to the existence of a time $\tau_0 > 0$ such

that :

$$\Big|\Big(T^{k+1,\varepsilon}_{-\tau_0}\big(\big(Z_k, x_k+\varepsilon\omega, v_{k+1}\big)'_{k,k+1}\big)\Big)^{X,k} - \Big(T^{k+1,\varepsilon}_{-\tau_0}\big(\big(Z_k, x_k+\varepsilon\omega, v_{k+1}\big)'_{k,k+1}\big)\Big)^{X,k+1}\Big| = \varepsilon.$$

One cannot copy here the work done above in the pre-collisional case : indeed, the trajectories, after the scattering, are defined with the two velocities $\overline{v}'_k$ and v', which both depends on v, and, even more seriously, on ω. Previously, at most one of the two velocities was depending on v.
By hypothesis, since $(x_k, \overline{v}'_k)$ and $(x_k+\varepsilon\omega, \overline{v}'_{k+1})$ form a pre-collisional pair, no recollision can happen if none of the particles, or both particles already have bounced against the obstacle, that is the recollision is possible only if $\min\{\tau_k, \tau_{k+1}\} < \tau_0 \leq \max\{\tau_k, \tau_{k+1}\}$.
In the case when $\min\{\tau_k, \tau_{k+1}\} < \tau_0 < \max\{\tau_k, \tau_{k+1}\}$, one has then, as for the pre-collisional case, and for the two cases $\tau_k = \min\{\tau_k, \tau_{k+1}\}$ or $\tau_{k+1} = \min\{\tau_k, \tau_{k+1}\}$:

$$\big|2x_k\cdot e_1e_1 + \varepsilon\omega - \varepsilon e_1 - \tau_0\big(v' - \mathcal{S}_0\big(\overline{v}'_k\big)\big)\big| = \varepsilon.$$

As for the pre-collisional case, one obtains that the vector $v' - \mathcal{S}_0\big(\overline{v}'_k\big)$ belongs to the cylinder of origin $0 \in \mathbb{R}^d$, of axis $2\overline{x}_k\cdot e_1e_1$, and of radius

$$\frac{10Ra}{\rho}.$$

However this is not a satisfactory condition, since it seems hard to deduce from it something on v itself. It is not even possible to consider ω fixed and to apply the Scattering Lemma for cylinders, since the question of the dependency on ω of the subset excluded by the Scattering Lemma seems not obvious to master at all.
The idea for getting rid of the complicated dependency on v and ω of the difference

$$v' - \mathcal{S}_0\big(\overline{v}'_k\big)$$

appearing in the last recollision equation, and for recovering a condition on v alone, is to write :

$$v' - \mathcal{S}_0\big(\overline{v}'_k\big) = v' - \overline{v}'_k - \big(\mathcal{S}_0(\overline{v}'_k) - \overline{v}'_k\big).$$

On the one hand, using the very definition of the scattering of the velocities $(\overline{v}_k, v)$ with angular parameter ω (see Definition 1 page 51), one has :

$$v' - \overline{v}'_k = \big(v - \overline{v}_k\big) - 2\big(v - \overline{v}_k\big)\cdot\omega\omega.$$

On the other hand, one notices that

$$\mathcal{S}_0(\overline{v}'_k) - \overline{v}'_k$$

is colinear to e_1. But of course, for any vector x, and for any scalar λ, x belongs to the cylinder $K(0, y, \rho)$ if and only if

$$x + \lambda y$$

belongs to the same cylinder (see Definition 50 page 334 for the notation and the definition of the cylinders). Therefore, $v' - \mathcal{S}_0\big(\overline{v}'_k\big)$ belongs to the cylinder

$$K\big(0, e_1, 10Ra/\rho\big) \tag{12.49}$$

if and only if

$$\big(v - \overline{v}_k\big) - 2\big(v - \overline{v}_k\big) \cdot \omega\omega$$

belongs to the same cylinder. At this step, one has recovered a condition on v instead of a condition on v'. However, the condition on the velocity v still depends on ω.

Now one notices that

$$\kappa : x \in \mathbb{R}^d \mapsto x - 2x \cdot \omega\omega \in \mathbb{R}^d$$

is the orthogonal symmetry with respect to ω, which is of course an involution. Therefore,

$$\big(v - \overline{v}_k\big) - 2\big(v - \overline{v}_k\big) \cdot \omega\omega$$

belongs to

$$K(0, e_1, 10Ra/\rho)$$

if and only if

$$\kappa\Big(\big(v - \overline{v}_k\big) - 2\big(v - \overline{v}_k\big) \cdot \omega\omega\Big) \ \Big[= \kappa^2\big(v - \overline{v}_k\big) = v - \overline{v}_k\Big]$$

belongs to

$$\kappa\Big(K(0, e_1, 10Ra/\rho)\Big) = K\big(0, \kappa(e_1), 10Ra/\rho\big).$$

Therefore, if one excludes among the angular and velocity adjunction parameters $\big(\omega, v\big) \in \mathbb{S}^{d-1} \times \mathbb{R}^d$ the subset

$$\Big(\bigcup_{\omega \in \mathbb{S}^{d-1}} \{\omega\} \times K\big(\overline{v}_k, e_1 - 2e_1 \cdot \omega\omega, 10Ra/\rho\big)\Big), \tag{12.50}$$

then the first recollision will not be between the particles k and $k+1$.

The size of the subset (12.50) is then controlled by

$$C(d)(4R)\big(10Ra/\rho\big)^{d-1}, \tag{12.51}$$

with $C(d)$ a constant depending only on the dimension, using the fact that $\big|v - \overline{v}_k\big| \leq 2R$.
One notices here that the Scattering Lemma was not useful for the specific case of the recollision between k and $k+1$.

Back to the cases not totally solved (namely ($1 \leq i \leq k-1$ and $j = k$) and ($1 \leq i \leq k-1$ and $j = k+1$)), one obtained conditions on v' and $\overline{v}'_k$, which have not to belong to cylinders depending on the configuration $\overline{Z}_k$. To be more explicit, one obtained the two unions of cylinders (12.48) page 357 and (12.45) page 356. One sees in fact that the subset (12.48) is contained in the subset (12.45), since they are defined with cylinders with the same origins and the same axis, while the radii are larger in the case of (12.45).
Those exclusion conditions hold on the velocity v', and one wishes to translate this condition into conditions on v and ω. Thanks to the Scattering Lemma for cylinders 29 page 345, it is possible to exclude six subsets of $\mathbb{S}^{d-1} \times B(0,R)$ such that if (ω, v) does not belong to those subsets, then neither v' nor $\overline{v}'_k$ belong to any of the six cylinders described just above. Moreover, the size of each of those subset is also controlled thanks to the Scattering Lemma for cylinders. Namely, the size of each of those subset is smaller than

$$C_5(d) R^{d+1/2} \Big(\frac{24Ra}{\varepsilon_0} \Big)^{d-3/2}$$

where C_5 is the constant, depending only on the dimension d, given by the Scattering Lemma 29, that is :

$$C(d) R^{2d-1} \Big(\frac{a}{\varepsilon_0} \Big)^{d-3/2}, \tag{12.52}$$

with

$$C(d) = 24^{d-3/2} C_5(d).$$

The conditions on the free flow dynamics with boundary condition

One now focuses on the conditions for the free flow dynamics. One wants to show that if $\overline{Z}_k$ belongs to $\mathcal{G}^0_k(\varepsilon_0)$, then except for a small subset of angular and velocity adjunction parameters (ω, v), the new configuration of $k+1$ particles (possibly after the application of the scattering, depending on the pre or post-collisional character of the configuration $(\overline{v}_k, v)$ with angular parameter ω), the new configuration

$$\big(\overline{Z}_k, \overline{x}_k, v\big)$$

is also an element of $\mathcal{G}^0_{k+1}(\varepsilon_0)$ (possibly after a small time).

The pre-collisional case for the free flow dynamics with boundary condition : the condition (12.31)

As for the hard sphere dynamics, one will consider different cases, by studying all the possible ways to choose a pair of particles which contradict the condition (12.31) page 346, and see what it implies on the adjunction parameters.

- $\underline{1 \leq i < j \leq k}$:
  This case is by far the easiest to solve. By hypothesis, the configuration $\overline{Z}_k$

belongs to $\mathcal{G}_k^0(\varepsilon_0)$, which means that for every pair of particles $1 \leq i < j \leq k$ and for any strictly positive time τ :

$$\Big|\Big(T_{-\tau}^{k,0}\big(\overline{Z}_k\big)\Big)^{X,i} - \Big(T_{-\tau}^{k,0}\big(\overline{Z}_k\big)\Big)^{X,k}\Big| > \varepsilon_0.$$

But since the particles do not interact with each other in the free flow with boundary condition dynamics, one has for all $1 \leq i \leq k$ and all $\tau \geq 0$:

$$\Big(T_{-\tau}^{k,0}\big(\overline{Z}_k\big)\Big)^{X,i} = \Big(T_{-\tau}^{k+1,0}\big(\overline{Z}_k, \overline{x}_k, v\big)\Big)^{X,i},$$

so in particular one deduces that for all $\tau \geq \delta$:

$$\Big|\Big(T_{-\tau}^{k+1,0}\big(\overline{Z}_k, \overline{x}_k, v\big)\Big)^{X,i} - \Big(T_{-\tau}^{k+1,0}\big(\overline{Z}_k, \overline{x}_k, v\big)\Big)^{X,j}\Big| \geq \varepsilon_0.$$

- $\underline{i = k \text{ and } j = k+1}$:
  One writes explicitly the trajectory of the free flow with boundary condition here.
  One denotes τ_0 any time such that

$$\Big|\Big(T_{-\tau_0}^{k+1,0}\big(\overline{Z}_k, \overline{x}_k, v\big)\Big)^{X,k} - \Big(T_{-\tau_0}^{k+1,0}\big(\overline{Z}_k, \overline{x}_k, v\big)\Big)^{X,k+1}\Big| \leq \varepsilon_0.$$

  Clearly, if $\Big(\Big(T_{-\tau_0}^{k+1,0}\big(\overline{Z}_k, \overline{x}_k, v\big)\Big)^{X,k}\Big) \cdot e_1 \geq 0$, one has :

$$\Big(T_{-\tau_0}^{k+1,0}\big(\overline{Z}_k, \overline{x}_k, v\big)\Big)^{X,k} = \overline{x}_k - \tau_0 \overline{v}_k,$$

  and if not, one has :

$$\Big(T_{-\tau_0}^{k+1,0}\big(\overline{Z}_k, \overline{x}_k, v\big)\Big)^{X,k} = \mathcal{S}_0\big(\overline{x}_k - \tau_0 \overline{v}_k\big).$$

  Similarly, for the particle $k+1$, if $\Big(\Big(T_{-\tau_0}^{k+1,0}\big(\overline{Z}_k, \overline{x}_k, v\big)\Big)^{X,k+1}\Big) \cdot e_1 \geq 0$, one has :

$$\Big(T_{-\tau_0}^{k+1,0}\big(\overline{Z}_k, \overline{x}_k, v\big)\Big)^{X,k+1} = \overline{x}_k - \tau_0 v,$$

  and if not, one has :

$$\Big(T_{-\tau_0}^{k+1,0}\big(\overline{Z}_k, \overline{x}_k, v\big)\Big)^{X,k+1} = \mathcal{S}_0\big(\overline{x}_k - \tau_0 v\big).$$

  Therefore, the quantity

$$\Big|\Big(T_{-\tau_0}^{k+1,0}\big(\overline{Z}_k, \overline{x}_k, v\big)\Big)^{X,k} - \Big(T_{-\tau_0}^{k+1,0}\big(\overline{Z}_k, \overline{x}_k, v\big)\Big)^{X,k+1}\Big|$$

  can be only equal to

$$\big|\big(\overline{x}_k - \tau_0 \overline{v}_k\big) - \big(\overline{x}_k - \tau_0 v\big)\big| = \tau_0 \big|v - \overline{v}_k\big|$$

or to

$$\big|\mathcal{S}_0\big(\overline{x}_k - \tau_0\overline{v}_k\big) - \big(\overline{x}_k - \tau_0 v\big)\big| = \big|\tau_0\big(v - \mathcal{S}_0(\overline{v}_k)\big) - \big(\mathcal{S}_0(\overline{x}_k) - \overline{x}_k\big)\big|.$$

In the first case, for any strictly positive number η, up to the exclusion of the ball

$$B(\overline{v}_k, \eta)$$

in the set of admissible v contained in $B(0, R)$, one can assume that

$$\big|v - \overline{v}_k\big| > \eta. \tag{12.53}$$

Under this assumption, one has :

$$\Big|\Big(T^{k+1,0}_{-\tau_0}\big(\overline{Z}_k, \overline{x}_k, v\big)\Big)^{X,k} - \Big(T^{k+1,0}_{-\tau_0}\big(\overline{Z}_k, \overline{x}_k, v\big)\Big)^{X,k+1}\Big| = \tau_0\big|v - \overline{v}_k\big| > \tau_0\eta,$$

and if $\tau_0 \geq \delta$, then of course :

$$\Big|\Big(T^{k+1,0}_{-\tau_0}\big(\overline{Z}_k, \overline{x}_k, v\big)\Big)^{X,k} - \Big(T^{k+1,0}_{-\tau_0}\big(\overline{Z}_k, \overline{x}_k, v\big)\Big)^{X,k+1}\Big| > \delta\eta,$$

so a soon as

$$\varepsilon_0 \leq \delta\eta \tag{12.54}$$

(as assumed in the proposition), the separation condition is recovered.
In the second case

$$\begin{aligned}\Big|\Big(T^{k+1,0}_{-\tau_0}\big(\overline{Z}_k, \overline{x}_k, v\big)\Big)^{X,k} &- \Big(T^{k+1,0}_{-\tau_0}\big(\overline{Z}_k, \overline{x}_k, v\big)\Big)^{X,k+1}\Big| \\ &= \big|\tau_0\big(v - \mathcal{S}_0(\overline{v}_k)\big) - \big(\mathcal{S}_0(\overline{x}_k) - \overline{x}_k\big)\big|,\end{aligned}$$

the condition of closeness exactly means that :

$$\big|\tau_0\big(v - \mathcal{S}_0(\overline{v}_k)\big) - \big(\mathcal{S}_0(\overline{x}_k) - \overline{x}_k\big)\big| \leq \varepsilon_0.$$

Noticing that :

$$\mathcal{S}_0(\overline{x}_k) - \overline{x}_k = \overline{x}_k - 2\overline{x}_k \cdot e_1 e_1 - \overline{x}_k = -2\overline{x}_k \cdot e_1 e_1,$$

and proceeding as in the Shooting Lemma, one sees that for any unitary vector u orthogonal to e_1 :

$$\begin{aligned}\tau_0\big|u \cdot \big(v - \mathcal{S}_0(\overline{v}_k)\big)\big| &= \big|u \cdot \big(\tau_0\big(v - \mathcal{S}_0(\overline{v}_k)\big) - \big(\mathcal{S}_0(\overline{x}_k) - \overline{x}_k\big)\big)\big| \\ &\leq \big|\tau_0\big(v - \mathcal{S}_0(\overline{v}_k)\big) - \big(\mathcal{S}_0(\overline{x}_k) - \overline{x}_k\big)\big| \leq \varepsilon_0.\end{aligned}$$

Assuming that $\tau_0 \geq \delta$, this condition is implied by the (less stringent) condition

$$\big|u \cdot \big(v - \mathcal{S}_0(\overline{v}_k)\big)\big| \leq \varepsilon_0/\delta,$$

and since u is any unitary and orthogonal to e_1 vector, this condition exactly means that v belongs to the cylinder

$$K\big(\mathcal{S}_0(\overline{v}_k), e_1, \varepsilon_0/\delta\big). \tag{12.55}$$

In summary, if v does not belong to the subsets described in (12.53) or (12.55), then the particles k and $k+1$ will be at a distance larger than ε_0 after a time δ, and the measure of the union of those two subsets is controlled by :

$$C(d)\Big(\eta^d + R\Big(\frac{\varepsilon_0}{\delta}\Big)^{d-1}\Big). \tag{12.56}$$

- $\underline{1 \leq i \leq k-1 \text{ and } j = k+1}$:
  In this case, the problem of choosing the velocity v in such a way that the particles i ($\leq k-1$) and k stay at a distance large enough from each other will be solved thanks to the first point (12.22) of the Shooting Lemma 27 page 335, which can be applied directly (the first point of the lemma does not require a relation of comparison between a and ε_0, for example). Then, up to the exclusion of the union of the two cylinders :

$$K\big(\overline{v}_i, \overline{x}_i - \overline{x}_k, \varepsilon_0/\delta\big) \cup K\big(\mathcal{S}_0(\overline{v}_i), \mathcal{S}_0(\overline{x}_i) - \overline{x}_k, \varepsilon_0/\delta\big), \tag{12.57}$$

  the particles i and $k+1$ will be at a distance larger than ε_0 after a time δ, which ends the study of the pre-collisional case. The size of the union of those two cylinders is then controlled by

$$2C(d)R\Big(\frac{\varepsilon_0}{\delta}\Big)^{d-1}. \tag{12.58}$$

The post-collisional case for the free flow dynamics with boundary condition : the condition (12.33)

One finally studies the pathological adjunction parameters for the free flow dynamics with boundary condition, in the case of a post-collisional configuration.

- $\underline{1 \leq i < j \leq k-1}$:
  Only the velocities of the particles k and $k+1$ are modified by the scattering, so this case does not present more pathological velocities leading to a small difference in position between the particles $1 \leq i < j \leq k-1$ than the pre-collisional case. In this situation, one will always have

$$\Big|\Big(T_{-\tau}^{k+1,0}\big(\overline{Z}_k, \overline{x}_k, v\big)\Big)^{X,i} - \Big(T_{-\tau}^{k+1,0}\big(\overline{Z}_k, \overline{x}_k, v\big)\Big)^{X,j}\Big| > \varepsilon_0.$$

- $\underline{1 \leq i \leq k-1 \text{ and } j = k}$:
  In this situation, it is sufficient to apply the first point of the Shooting Lemma 27 to the configuration

$$\big(\overline{x}_i, \overline{v}_i, \overline{x}_k, \overline{v}'_k\big).$$

Up to choosing $\overline{v}'_k$ outside of the union of the two cylinders

$$K\big(\overline{v}_i, \overline{x}_i - \overline{x}_k, \varepsilon_0/\delta\big) \cup K\big(\mathcal{S}_0(\overline{v}_i), \mathcal{S}_0(\overline{x}_i) - \overline{x}_k, \varepsilon_0/\delta\big) \tag{12.59}$$

the separation condition (12.33) is obtained for any pair of particles $1 \leq i \leq k-1$ and k.

- $\underline{1 \leq i \leq k-1 \text{ and } k+1}$:
  Exactly as for the previous point, up to choosing v' outside of the union of the two cylinders

$$K\big(\overline{v}_i, \overline{x}_i - \overline{x}_k, \varepsilon_0/\delta\big) \cup K\big(\mathcal{S}_0(\overline{v}_i), \mathcal{S}_0(\overline{x}_i) - \overline{x}_k, \varepsilon_0/\delta\big) \tag{12.60}$$

  the separation condition (12.33) is obtained for any pair of particles $1 \leq i \leq k-1$ and $k+1$.

- $\underline{i = k \text{ and } j = k+1}$:
  As for the pre-collisional case, the violation of the condition (12.33) writes in this case :

$$\tau_0\big|v' - \overline{v}'_k\big| \leq \varepsilon_0$$

  or

$$\big|\tau_0\big(v' - \mathcal{S}_0(\overline{v}'_k)\big) - \big(\mathcal{S}_0(\overline{x}_k) - \overline{x}_k\big)\big| \leq \varepsilon_0.$$

  Assuming that the post-collisional velocity v' does not belong to the ball $B(\overline{v}'_k, \eta)$, one gets :

$$\tau_0\big|v' - \overline{v}'_k\big| > \tau_0\eta.$$

  For $\tau_0 \geq \delta$, one uses the condition (12.54) assumed in the proposition :

$$\varepsilon_0 \leq \delta\eta, \tag{12.61}$$

  so that the condition (12.33) is fulfilled. However, the assumption $v' \notin B(\overline{v}'_k, \eta)$ is not satisfactory since it is not a condition on v and ω. But a simple consequence of the definition of the scattering operator (see Definition 1 page 51) that is the conservation of the kinetic energy in the center-of-mass frame, implies that the following equation holds :

$$\big|v' - \overline{v}'_k\big| = \big|v - \overline{v}_k\big|.$$

  This means of course that v' belongs to the ball $B\big(\overline{v}'_k, \eta\big)$ if and only if v belongs to the ball $B\big(\overline{v}_k, \eta\big)$. Therefore, the condition

$$\big|v - \overline{v}_k\big| > \eta \tag{12.62}$$

  enables, exactly as for the pre-collisional case, to recover the separation condition (12.33).
  Finally studying carefully the condition

$$\big|\tau_0\big(v' - \mathcal{S}_0(\overline{v}'_k)\big) - \big(\mathcal{S}_0(\overline{x}_k) - \overline{x}_k\big)\big| \leq \varepsilon_0,$$

one deduces using the same arguments as above, first (as for the pre-collisional situation with the free flow dynamics) that the vector

$$v' - \mathcal{S}_0(\overline{v}'_k)$$

belongs to the cylinder of origin 0, axis e_1 and of radius ε_0/δ when $\tau_0 \geq \delta$. Second, using again the decomposition (as for the post-collisional situation with the hard sphere dynamics) :

$$v' - \mathcal{S}_0\big(\overline{v}'_k\big) = v' - \overline{v}'_k - \big(\mathcal{S}_0(\overline{v}'_k) - \overline{v}'_k\big),$$

one deduces that the vector

$$\big(v - \overline{v}_k\big) - 2\big(v - \overline{v}_k\big) \cdot \omega\omega$$

belongs to the cylinder

$$K\big(0, e_1, \varepsilon_0/\delta\big),$$

which implies that the vector $v - \overline{v}_k$ belongs to the cylinder

$$K\big(0, e_1 - 2e_1 \cdot \omega\omega, \varepsilon_0/\delta\big).$$

Therefore, one excludes in this case

$$\Big(\bigcup_{\omega \in \mathbb{S}^{d-1}} \{\omega\} \times K\big(0, e_1 - 2e_1 \cdot \omega\omega, \varepsilon_0/\delta\big)\Big). \tag{12.63}$$

The size of the excluded subset (12.63) is then controlled by

$$C(d)(4R)\big(\varepsilon_0/\delta\big)^{d-1}, \tag{12.64}$$

with $C(d)$ a constant depending only on the dimension.

As for the hard sphere dynamics, one will finally apply the Scattering Lemma 29 page 345 to conclude the study of the remaining cases ($1 \leq i \leq k-1$ and $j = k$), and ($1 \leq i \leq k-1$ and $j = k+1$). This lemma states that there exists a subset of $\mathbb{S}^{d-1} \times \mathbb{R}^d$ such that if (ω, v) does not belong to this subset, then v' nor $\overline{v}'_k$ do not belong to the two cylinders, described in (12.59) or in (12.60). Moreover, the size of this subset is controlled by

$$C(d)R^{d+1/2}\Big(\frac{\varepsilon_0}{\delta}\Big)^{d-3/2}. \tag{12.65}$$

Collect of the hypotheses and cut-offs coming from the different cases

In summary, for the hard sphere dynamics, regardless the fact that the configuration is pre or post-collisional, one has assumed :

- the conditions (12.38) page 350 and (12.39) page 350, that is

  $$5a \leq \varepsilon_0 \text{ and } 3a \leq \rho$$

  so that the new configuration $\big(Z_s, x_k + \varepsilon\omega, v\big)$ belongs to $\mathcal{D}^{\varepsilon}_{k+1}$,

- the conditions (12.41) page 353, (12.44) page 355 and (12.47) page 357, that is
$$\frac{5}{\sqrt{3}}a \leq \rho,\ 4\sqrt{3}a \leq \varepsilon_0 \text{ and } 2\sqrt{3}a \leq \varepsilon_0$$
so that outside of the following excluded subsets, the dynamics of the new system of hard spheres will not undergo a recollision.

Thanks to those assumptions, one has excluded the following subsets among the angular and velocity adjunction parameters $(\omega, v) \in \mathbb{S}^{d-1} \times B(0, R)$:

- the cylinder (12.42) page 353 (with a size smaller than (12.51) page 359),
- the six cylinders described in (12.45) page 356 (the size of their union being controlled by (12.52) page 360) for each integer $1 \leq i \leq k-1$, so that one has to remove in fact $6(k-1)$ cylinders from $B(0, R)$,
- the images by the scattering operator of the $12(k-1)$ cylinders given by (12.48) page 357 and (12.45) page 356, corresponding in fact, due to an obvious inclusion, to the images by the scattering of the $6(k-1)$ cylinders (12.45) page 356 (the size of each of those images is controlled by (12.52) page 360),
- the family of cylinders, parametrized with the angular parameter ω lying in the sphere $\mathbb{S}^{d-1}$, and given by (12.50) page 359 this family (which is a subset of $\mathbb{S}^{d-1} \times B(0, R)$, having a measure controlled by (12.51) page 359).

The union of those $12(k-1)+3$ subsets of $\mathbb{S}^{d-1} \times B(0, R)$ will be denoted :
$$\widetilde{\mathcal{B}_k}^{(\varepsilon)}(R, \varepsilon, a, \varepsilon_0, \rho)(\overline{Z}_k),$$
it corresponds therefore to the set of adjunction parameters (ω, v) that one has to exclude to keep the new system of $k+1$ particles in a good configuration for the hard sphere dynamics, separated by at least ε, and it has a size smaller than :
$$C(d)\Big(R^d\Big(\frac{a}{\rho}\Big)^{d-1} + 6(k-1)R^d\Big(\frac{a}{\varepsilon_0}\Big)^{d-1} + 6(k-1)R^{2d-1}\Big(\frac{a}{\varepsilon_0}\Big)^{d-3/2} + R^d\Big(\frac{a}{\rho}\Big)^{d-1}\Big). \tag{12.66}$$

For the free-flow dynamics with boundary condition, one has only assumed (12.54) page 362 and (12.61) page 364, which is in fact only one condition, namely
$$\varepsilon_0 \leq \delta\eta,$$
and one has excluded the following subsets from $\mathbb{S}^{d-1} \times B(0, R)$ for the adjunction parameters (ω, v) :

- the ball of velocities too close to $\overline{v}_k$ (condition (12.53) page 362) and the cylinder (12.55) page 363 (the measure of the union of those subsets being controlled by (12.56) page 363),

- for every integer $1 \leq i \leq k-1$, the pair of cylinders (12.57) page 363 (of which the size is controlled by (12.58) page 363), so in the end, one excludes here $2(k-1)$ cylinders,

- the images by the scattering operator of the $2(k-1)$ cylinders given by the two conditions (giving, for every integer $1 \leq i \leq k-1$, the same pair of cylinders) (12.59) page 364 and (12.60) page 364 (and the size of each of those image is controlled by (12.65) page 365),

- since the exclusion of the ball (12.62) page 364 has already been done previously (condition (12.53) page 362) one finishes by excluding the family of cylinders, parametrized with the angular parameter ω lying in the sphere $\mathbb{S}^{d-1}$, and given by (12.63) page 365 (this family, which is a subset of $\mathbb{S}^{d-1} \times B(0,R)$, having a measure controlled by (12.64) page 365).

This union of those $4k-1$ subsets of $\mathbb{S}^{d-1} \times B(0,R)$ will be denoted :

$$\widetilde{\mathcal{B}_k}^{(\varepsilon_0)}(R,\delta,\varepsilon,\varepsilon_0,\eta)(\overline{Z}_k),$$

and here it corresponds to the set of ajdunction parameters (ω, v) that one has excluded to keep again the new system of $k+1$ particles in a good configuration for the free-flow dynamics with boundary condition, separated by at least ε_0, for any time larger than δ, and it has a size smaller than :

$$C(d)\Big(\eta^d + R\Big(\frac{\varepsilon_0}{\delta}\Big)^{d-1} + 2(k-1)R\Big(\frac{\varepsilon_0}{\delta}\Big)^{d-1} \\ + 2(k-1)R^{d+1/2}\Big(\frac{\varepsilon_0}{\delta}\Big)^{d-3/2} + R\Big(\frac{\varepsilon_0}{\delta}\Big)^{d-1}\Big). \quad (12.67)$$

It is clear at this step that one will consider the set :

$$\widetilde{\mathcal{B}_k}^{(\varepsilon)}(R,\varepsilon,a,\varepsilon_0,\rho)(\overline{Z}_k) \cup \widetilde{\mathcal{B}_k}^{(\varepsilon_0)}(R,\delta,\varepsilon,\varepsilon_0,\eta)(\overline{Z}_k),$$

denoted as :

$$\widetilde{\mathcal{B}_k}(R,\delta,\varepsilon,a,\varepsilon_0,\rho,\eta)(\overline{Z}_k)$$

(the dependency on ε is through the axis of the excluded cylinders), in order to obtain a subset of $\mathbb{S}^{d-1} \times \mathbb{R}^d$ that contains $\mathcal{B}_k(R,\delta,\varepsilon,a,\varepsilon_0)(\overline{Z}_k)$. Besides, the size of $\widetilde{\mathcal{B}_k}(R,\delta,\varepsilon,a,\varepsilon_0,\rho,\eta)(\overline{Z}_k)$ is given by the size of its two parts, the first being controlled by (12.66), and the second by (12.67), hence the control (12.37) of the lemma about the size, up to assuming in the end that :

$$R \geq 1, \ a/\varepsilon_0 \leq 1 \text{ and } \varepsilon_0/\delta \leq 1.$$

Finally, the measurability of

$$\bigcup_{\substack{\overline{Z}_k \in \mathcal{G}_k(\varepsilon_0),\\ \overline{x}_k \cdot e_1 > \rho}} \{\overline{Z}_k\} \times \widetilde{\mathcal{B}_k}(R, \delta, \varepsilon, a, \varepsilon_0, \rho, \eta)(\overline{Z}_k)$$

is a simple consequence of the fact that all the subsets composing $\widetilde{\mathcal{B}_k}$ are given by regular parametrizations, which depend in addition smoothly on the configuration $\overline{Z}_k$ for most of them, and for the others, that is the ones defined with the scattering operator, one recalls that this operator is a continuous involution. □

Remark 33. *It is absolutely crucial here to notice that one needs to introduce a cut-off on the distance between the particle k (on which one adds formally another particle), and the obstacle. In the following, it will be important to evaluate the possible cases in which particles are indeed close to the obstacle, and if possible, control the size of those events.*

12.2.5 Taking into account the cut-off in proximity with the obstacle

In Proposition 16, an important hypothesis is the bound from below concerning the distance between the particle colliding (named k in the proposition) and the obstacle in order to exclude only a set of velocities of small size. Since one cannot have a restrictive condition on the position, one will try to prevent the infringement of this hypothesis by cutting again some small subsets off in the domains of integration of the elements of the decomposition of the solutions.

Translating the condition of proximity with the obstacle in terms of non-grazing trajectories and small pathological times intervals

It is not possible to cut off all the initial configurations leading to a bouncing against the obstacle such that the measure of this cut-off remains small. Indeed, in particular for a single particle it means that one would have to cut all the initial speeds off leading to a collision, that is, half of the sphere in which the initial velocities are chosen.
So the solution is to make small the set of times during which the particle chosen for the adjunction is close to the obstacle. It implies two things : removing the grazing collisions (otherwise one can imagine very large time intervals in which a particle remains close to the obstacle), and once it is done, removing the times which are still problematic, but which are only few now thanks to the cut-off in grazing trajectories.

Considering a small parameter $\alpha \ll 1$, and assuming that a velocity v verifies

$$|v \cdot e_1| > \alpha,$$

then if a particle starts with such a velocity from a position $x \in \{x \cdot e_1 > \varepsilon/2\}$, the size of the time interval such that

$$|T^{1,\varepsilon}_{-\tau}(x,v) \cdot e_1| \leq \rho$$

is then smaller than

$$2\rho/\alpha.$$

If now k particles starting from Z_k are in good configuration and such that $|\left(T^{k,\varepsilon}_{-\tau}(Z_k)\right)^{V,i} \cdot e_1| > \alpha$ for all $i \in \{1, ..., k\}$ and $\tau \geq 0$, and if one adds to the particle k another one (that is one adds a particle starting from $(x_k + \varepsilon\omega, v_{k+1})$) such that (x_k, v_k) and $(x_k + \varepsilon\omega, v_{k+1})$ are in a pre-collisional configuration, it is clear that to fulfill the condition

$$\left|\left(T^{k+1,\varepsilon}_{-\tau}\right)^{X,k+1} \cdot e_1\right| > \alpha,$$

one has just to exclude the set

$$\{v \ / \ |v \cdot e_1| \leq \alpha\}$$

of the ball in which v_{s+1} is chosen. Then, if the radius of the ball in which the velocity v is chosen is, say, R, then the excluded set as a size smaller than

$$C(d)R^{d-1}\alpha, \tag{12.68}$$

with $C(d)$ depending only on the dimension.

Scattering lemma for excluded spaces between two hyperplanes : removing grazing trajectories after the scattering

Now if (x_k, v_k) and $(x_k + \varepsilon\omega, v_{k+1})$ are in a post-collisional configuration, one needs to control the size of the subset of adjunction parameters (ω, v_{k+1}) such that this configuration, after the scattering, contains particles following a grazing trajectory.
In other words, one needs a similar result as the control obtained in Lemma 29 page 345.

Definition 52 (Adjunction parameters inducing grazing collisions after scattering)**.** *For any $v_1 \in B(0,R)$, one will call the subset of the* adjunction parameters inducing grazing collisions after adding a particle to another one with velocity v_1 and after scattering*, and one will denote*

$$\mathcal{N}^*(R,\alpha)(v_1) \subset \mathbb{S}^{d-1} \times B(0,R)$$

for the set defined by :

$$\mathcal{N}^*(R,\alpha)(v_1) = \Big\{(\omega, v_2) \in \mathbb{S}^{d-1} \times B(0,R) \ / \ (v_2 - v_1) \cdot \omega > 0,$$
$$v_1' \in \{|v \cdot e_1| \leq \alpha\} \ \textit{or} \ v_2' \in \{|v \cdot e_1| \leq \alpha\}\Big\}.$$

Since one wants to remove the grazing trajectories, one needs to control the size of this subset $\mathcal{N}^*(R,\alpha)(v_1)$. To do so, one will use the following lemma.

Lemma 30 (Control of the measure of the part of a sphere contained between two parallel hyperplanes)**.** *Let α, r and R be three strictly positive numbers. One recalls that e_1 denotes the first vector of the canonical basis of the Euclidean space $\mathbb{R}^d$.*
In the case $d=2$, if

$$R \geq 1, \quad \alpha \leq \min\left\{1, \left(\frac{R}{6}\right)^2, \left(\frac{1-(\sqrt{3}-\sqrt{2})}{\frac{\sqrt{3}}{2}+\sqrt{2}}\right)^2\right\}, \quad 2\sqrt{\alpha} \leq r \leq 2R,$$

and if

$$\left|x \cdot e_1\right| \leq r - \sqrt{\alpha},$$

one has

$$\left|\left\{y \in \partial B(x,r) \ / \ \left|y \cdot e_1\right| \leq \alpha\right\}\right| \leq \sqrt{R}\alpha^{1/4}. \tag{12.69}$$

In the case $d \geq 3$, there exists a constant $C(d)$ depending only on the dimension such that if $r \leq 2R$, one has

$$\left|\left\{y \in \partial B(x,r) \ / \ \left|y \cdot e_1\right| \leq \alpha\right\}\right| \leq C(d) r^{d-2}\alpha. \tag{12.70}$$

Proof. For α, r and R three strictly positive numbers, $x \in B(0,R)$ and a given ball of $\mathbb{R}^d$, centered on x and of radius r such that :

$$r \leq 2R,$$

one will study the size of the subset consisting of the points of the boundary of the ball $B(x,r)$ contained between the two hyperplanes $\{y \cdot e_1 = \alpha\}$ and $\{y \cdot e_1 = -\alpha\}$. In other words, one studies the size of the set :

$$\{y \in \partial B(x,r) \ / \ |y \cdot e_1| \leq \alpha\}. \tag{12.71}$$

One denotes the quantity $|x \cdot e_1|$ by p. Without loss of generality, one will assume that

$$x \cdot e_1 \geq 0,$$

that is $x \cdot e_1 = p$. There are, depending on the radius r, the distance 2α between the two planes and the distance p between the center of the ball $B(x,r)$ and the plane $\left\{x \in \mathbb{R}^d \ / \ x \cdot e_1 = 0\right\}$, three possible cases.

- The case $r < p - \alpha$: the trival case of the empty intersection.
  If $p - \alpha > r$, one has the configuration represented on the following Figure 12.8, for every $y \in \partial B(x,r) \subset B(x,r)$:

$$\begin{aligned} |y \cdot e_1| &= |x \cdot e_1 + (y-x) \cdot e_1| \\ &\geq |x \cdot e_1| - |(y-x) \cdot e_1| \\ &\geq p - r > \alpha, \end{aligned}$$

so that

$$\big\{y \in \partial B(x,r) \ / \ |y \cdot e_1| \le \alpha\big\} = \emptyset,$$

and one immediately gets

$$\Big|\big\{y \in \partial B(x,r) \ / \ |y \cdot e_1| \le \alpha\big\}\Big| = 0. \tag{12.72}$$

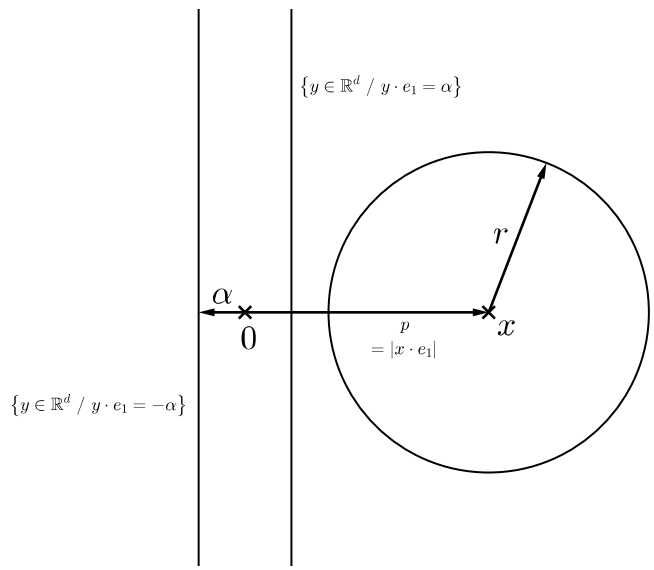

Figure 12.8: The first case $r < p - \alpha$.

- The case $r \ge p - \alpha$, and $\big[r \le p + \alpha$ or $r \le \alpha\big]$: the case of the ball which does not cross the two planes delimiting the space $-\alpha \le x \cdot e_1 \le \alpha$.
  Here there are two subcases, the first one ($r \le p + \alpha$) is represented on the following Figure 12.9. Those two cases are somehow intermediate cases.

  For the subcase $r \le \alpha$, one has obviously

  $$\big\{y \in \partial B(x,r) \ / \ |y \cdot e_1| \le \alpha\big\} \subset \big\{y \in \partial B(x,r)\big\},$$

  so that of course for the subcase $r \ge p - \alpha$, $r < \alpha$, one has

  $$\Big|\big\{y \in \partial B(x,r) \ / \ |y \cdot e_1| \le \alpha\big\}\Big| \le \big|\partial B(x,r)\big| = C(d) r^{d-1}, \tag{12.73}$$

  with $C(d)$ denoting a constant depending only on the dimension d.
  This equation provides, on the one hand, the control (12.69) of the lemma (concerning the case when $d = 2$) in the particular case $r \le \alpha$, since one has

  $$r^{d-1} = r \le \alpha \le \sqrt{R}\alpha^{1/4},$$

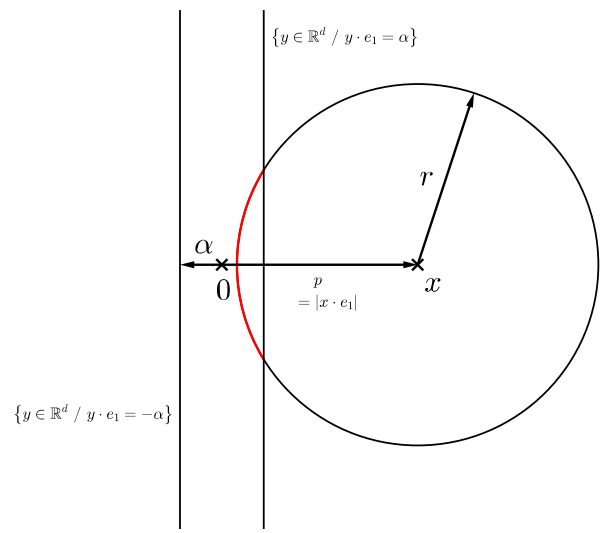

Figure 12.9: The case $p - \alpha \leq r \leq p + \alpha$.

since $\alpha \leq 1$ and $R \geq 1$.
On the other hand, it provides also the control (12.70) of the lemma (concerning the case $d \geq 3$) in the particular case $r \leq \alpha$, since one has immediately, using the hypothesis $r \leq \alpha$:

$$r^{d-1} \leq r^{d-2}\alpha.$$

For the subcase $r \leq p + \alpha$ (which does not exclude the previous subcase $r \leq \alpha$), one will in fact reduce the study of this case to the case in which the ball is crossing entirely the space $-\alpha \leq x \cdot e_1 \leq \alpha$ between the planes if the radius of the ball is large enough, or to the case in which the ball is entirely contained in the space $-\alpha \leq x \cdot e_1 \leq \alpha$ between the two planes, if the radius of the ball is small compared to the distance 2α between the planes.
The idea for this reduction is pictured in Figure 12.10 below.
Dividing the interval $[-\alpha, \alpha]$ (which has to be seen as a subset of the vectorial line generated by the vector e_1) into two parts $[-\alpha, p - r[$ and $[p - r, \alpha]$, the first corresponding to none of the scalar products $y \cdot e_1$, for any y belonging to the boundary of the ball $B(x, r)$. Indeed :

$$\begin{aligned} y \cdot e_1 &= x \cdot e_1 + (y - x) \cdot e_1 \\ &\geq p - |(x - y) \cdot e_1| \\ &\geq p - r. \end{aligned}$$

Then one sees, using $\alpha \leq p - r + 2\alpha$ (which is equivalent to $r \leq p + \alpha$), that the following inclusion holds :

$$\big\{y \in \partial B(x, r) \ / \ |y \cdot e_1| \leq \alpha\big\} \subset \big\{y \in \partial B(x, r) \ / \ p - r \leq y \cdot e_1 \leq p - r + 2\alpha\big\}$$

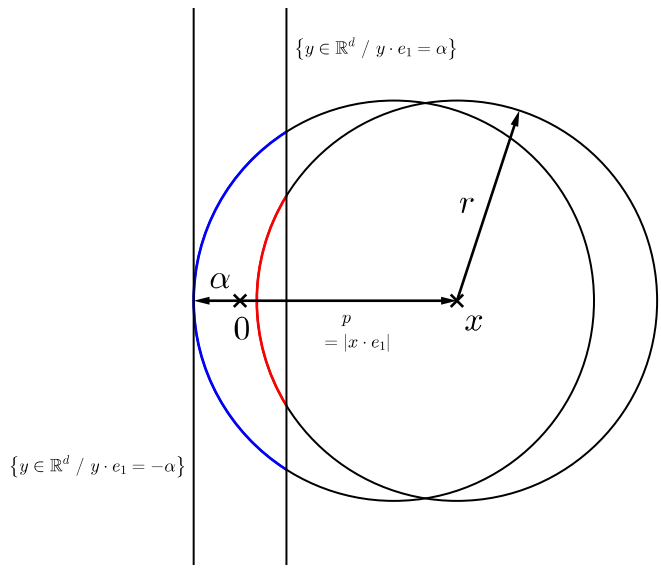

Figure 12.10: An illustration of the idea which enables to state the inequality (12.74).

so that :

$$\Big|\big\{y \in \partial B(x,r) \ / \ |y \cdot e_1| \leq \alpha\big\}\Big|$$

in the case when $p > r - \alpha$ is smaller than the measure of the same set in the case when $p = r - \alpha$. In other words, if $r - \alpha \leq p \leq r + \alpha$, then

$$\Big|\big\{y \in \partial B(x,r) \ / \ |y \cdot e_1| \leq \alpha\big\}\Big| \leq \Big|\big\{y \in \partial B((r-\alpha)\, e_1, r) \ / \ |y \cdot e_1| \leq \alpha\big\}\Big|. \tag{12.74}$$

- The case $r > p + \alpha$ and $r > \alpha$: the case of the ball which crosses the two planes delimiting the space $-\alpha \leq x \cdot e_1 \leq \alpha$.
  In this case, one will compute explicitly the measure of the subset (12.71), with two subcases depending on the dimension d.

  First, one assumes that $d \geq 3$.
  In this case, the explicit computation provides

$$\begin{aligned}\big\{y \in \partial B(x,r) \ / \ |y \cdot e_1| \leq \alpha\big\} &= \int\limits_{(x+r\mathbb{S}^{d-1}) \cap \{|y \cdot e_1| \leq \alpha\}} \mathrm{d}S \\ &= \int\limits_{[p-r,p+r] \cap [-\alpha,\alpha]} \Big|(\sqrt{r^2 - (p-z)^2})\mathbb{S}^{d-2}\Big| \, \mathrm{d}z,\end{aligned}$$

where $x + r\mathbb{S}^{d-1}$ denotes :

$$\left\{y \in \mathbb{R}^d \ / \ \exists\ \omega_d \in \mathbb{S}^{d-1} \ / \ y = x + r\omega_d\right\} = \partial B(x,r),$$

and $(\sqrt{r^2 - (p-y)^2})\mathbb{S}^{d-2}$ denotes the submanifold of codimension 1 of $\mathbb{R}^{d-1}$ defined as :

$$\left\{z \in \mathbb{R}^{d-1} \ / \ \exists\ \omega_{d-1} \in \mathbb{S}^{d-2} \ / \ z = \sqrt{r^2 - (p-y)^2}\omega_{d-1}\right\}.$$

From the hypothesis $p < r - \alpha$, one sees immediately that $p - r < -\alpha$. Remembering that $r > \alpha$, one has :

$$p + r > p + \alpha \geq \alpha,$$

so that of course :

$$[-\alpha, \alpha] \subset [p - r, p + r],$$

and then one can easily complete the computation :

$$\left|\left\{y \in \partial B(x,r) \ / \ |y \cdot e_1| \leq \alpha\right\}\right| = \int_{-\alpha}^{\alpha} (r^2 - (p-y)^2)^{\frac{d-2}{2}} \left|\mathbb{S}^{d-2}\right| \mathrm{d}y,$$

so that one obtains :

$$\left|\left\{y \in \partial B(x,r) \ / \ |y \cdot e_1| \leq \alpha\right\}\right| = C(d) r^{d-1} \int_{-\frac{(\alpha+p)}{r}}^{\frac{\alpha-p}{r}} \left(1 - u^2\right)^{\frac{d-2}{2}} \mathrm{d}u. \quad (12.75)$$

The function :

$$p \mapsto \int_{-\frac{(\alpha+p)}{r}}^{\frac{\alpha-p}{r}} (1 - u^2)^{\frac{d-2}{2}} \mathrm{d}u$$

is defined and differentiable on the whole interval $[0, r - \alpha]$ since $u \mapsto (1 - u^2)^{\frac{d-2}{2}}$ is defined and continuous on $[-1, 1]$, and since $p < r - \alpha$ and $p + r > \alpha$, one obtains $-1 \leq \frac{-\alpha-p}{r}$ and $\frac{\alpha-p}{r} \leq 1$. Thanks to the following inequality :

$$\begin{aligned} \frac{\partial}{\partial p} \int_{-\frac{(\alpha+p)}{r}}^{\frac{\alpha-p}{r}} \left(1 - u^2\right)^{\frac{d-2}{2}} \mathrm{d}u &= \frac{1}{r}\left[\left(1 - \left(\frac{p+\alpha}{r}\right)^2\right)^{\frac{d-2}{2}} - \left(1 - \left(\frac{p-\alpha}{r}\right)^2\right)^{\frac{d-2}{2}}\right] \\ &\leq 0 \end{aligned} \quad (12.76)$$

for $p \geq 0$ (since the function $(1-u^2)^{\frac{d-2}{2}}$ is decreasing), and since $|p - \alpha| \leq p + \alpha$ for all $p \geq 0$, one can write :

$$\int_{-\frac{(\alpha+p)}{r}}^{\frac{\alpha-p}{r}} \left(1 - u^2\right)^{\frac{d-2}{2}} \mathrm{d}u \leq 2 \int_0^{\alpha/r} (1 - u^2)^{\frac{d-2}{2}} \mathrm{d}u \leq 2\alpha/r.$$

This last control concludes the study in the case when $d \geq 3$, $r > \alpha$ and $p \leq r - \alpha$:

$$\left|\left\{y \in \partial B(x,r) \ / \ |y \cdot e_1| \leq \alpha\right\}\right| \leq C(d) r^{d-1} \frac{\alpha}{r} = C(d) r^{d-2} \alpha. \tag{12.77}$$

Moreover, when $r > \alpha$ and $r - \alpha \leq p \leq r + \alpha$ (which was the second case studied), one can complete the result obtained above :

$$\left|\left\{y \in \partial B(x,r) \ / \ |y \cdot e_1| \leq \alpha\right\}\right| \leq C(d) r^{d-2} \alpha,$$

since the right-hand side of the inequality (12.74) page 373, which writes

$$\left|\left\{y \in \partial B((r-\alpha)\, e_1, r) \ / \ |y \cdot e_1| \leq \alpha\right\}\right|$$

is equal, according to the computation (12.75), to the function :

$$p \mapsto C(d) r^{d-1} \int_{-\frac{\alpha+p}{r}}^{\frac{\alpha-p}{r}} (1-u^2)^{\frac{d-2}{2}} \, \mathrm{d}u,$$

evaluated at point $p = r - \alpha$ (since if $x = (r-\alpha)e_1$, then $p = r - \alpha$). One used already the inequality (12.76) to show that this function is decreasing on the interval $[0, r-\alpha]$. So the right-hand side of the inequality (12.74) is bounded by the value of the same function evaluated at $p = 0$, this value being

$$\int_{-\frac{\alpha}{r}}^{\frac{\alpha}{r}} \left(1-u^2\right)^{\frac{d-2}{2}} \, \mathrm{d}u,$$

which is exactly the same quantity used to obtain (12.77).

Second, one assumes that $d = 2$.
One performs the same direct computation as in the case $d \geq 3$. One finds in this case

$$\left|\left\{y \in \partial B(x,r) \ / \ |y \cdot e_1| \leq \alpha\right\}\right| = r \left(\arccos\left(-\frac{(\alpha+p)}{r}\right) - \arccos\left(\frac{\alpha-p}{r}\right)\right).$$

The control on this quantity is obtained in three steps.

- First, one removes the dependency on p (inequality (12.79)).
- Second, one removes the dependency on r (inequality (12.85) page 378).
- Finally, since the right-hand side obtained in (12.85) is a complicated expression given with the inverse function of the cosine, one will simplify this control.

One begins by removing the dependency on p and computes therefore for all $p \geq 0$:

$$\begin{aligned}\frac{\partial}{\partial p}\Big[r\Big(\arccos\Big(-\frac{(\alpha+p)}{r}\Big)-\arccos\Big(\frac{\alpha-p}{r}\Big)\Big)\Big] \\ &= \frac{1}{\sqrt{1-\frac{(p+\alpha)^2}{r^2}}}-\frac{1}{\sqrt{1-\frac{(p-\alpha)^2}{r^2}}} \\ &\geq 0,\end{aligned}$$

so that, if one assumes in addition that

$$p \leq r-\alpha^a \tag{12.78}$$

with $a \in]0,1[$ (in other words, one excludes some values of p, representing a set of size of order α^a), one has :

$$\begin{aligned}r\Big(\arccos\Big(-\frac{(\alpha+p)}{r}\Big)-\arccos\Big(\frac{\alpha-p}{r}\Big)\Big) \\ \leq r\Big(\arccos\Big(\frac{\alpha^a-\alpha}{r}-1\Big)-\arccos\Big(\frac{\alpha^a+\alpha}{r}-1\Big)\Big).\end{aligned} \tag{12.79}$$

Now one removes the dependency on r on the control obtained in (12.79). One considers the right-hand side of this inequality as a function of r. After a computation of its derivative, one can see that the function

$$r \mapsto r\left(\arccos\left(\frac{\alpha^a-\alpha}{r}-1\right)-\arccos\left(\frac{\alpha^a+\alpha}{r}-1\right)\right) \tag{12.80}$$

is increasing. Indeed, the following expression for the derivative of the function (12.80) holds :

$$\begin{aligned}r \mapsto &\Big(\arccos\Big(\frac{\alpha^a-\alpha}{r}-1\Big)-\arccos\Big(\frac{\alpha^a+\alpha}{r}-1\Big)\Big) \\ &+r\left(\left(\frac{\alpha^a-\alpha}{r^2}\right)\frac{1}{\sqrt{1-\left(\frac{\alpha^a-\alpha}{r}-1\right)^2}}-\left(\frac{\alpha^a+\alpha}{r^2}\right)\frac{1}{\sqrt{1-\left(\frac{\alpha^a+\alpha}{r}-1\right)^2}}\right),\end{aligned}$$

and the second term can be rewritten as :

$$\begin{aligned}&r\left(\left(\frac{\alpha^a-\alpha}{r^2}\right)\frac{1}{\sqrt{2\left(\frac{\alpha^a-\alpha}{r}\right)-\left(\frac{\alpha^a-\alpha}{r}\right)^2}}-\left(\frac{\alpha^a+\alpha}{r^2}\right)\frac{1}{\sqrt{2\left(\frac{\alpha^a+\alpha}{r}\right)-\left(\frac{\alpha^a+\alpha}{r}\right)^2}}\right) \\ &=(\alpha^a-\alpha)\frac{1}{\sqrt{2r(\alpha^a-\alpha)-(\alpha^a-\alpha)^2}}-(\alpha^a+\alpha)\frac{1}{\sqrt{2r(\alpha^a+\alpha)-(\alpha^a+\alpha)^2}},\end{aligned}$$

so that the derivative of the function (12.80) is strictly positive for a given r if and only if

$$\Big(\arccos\Big(\frac{\alpha^a-\alpha}{r}-1\Big)-\arccos\Big(\frac{\alpha^a+\alpha}{r}-1\Big)\Big)$$
$$\geq \frac{\alpha^a+\alpha}{\sqrt{2r(\alpha^a+\alpha)-(\alpha^a+\alpha)^2}}-\frac{\alpha^a-\alpha}{\sqrt{2r(\alpha^a-\alpha)-(\alpha^a-\alpha)^2}}. \quad (12.81)$$

On the other hand, with an additional hypothesis on r, namely :

$$2\alpha^a \leq r, \quad (12.82)$$

with the same $a \in]0,1[$ as the one taken for the condition (12.78), if $\alpha \leq 1$, one finds that

$$\frac{\alpha^a-\alpha}{r}-1 \leq \frac{\alpha^a+\alpha}{r}-1 \leq \frac{2\alpha^a}{r}-1 \leq 0.$$

So using the convexity of the function arccos on $[-1,0]$, one has that :

$$\arccos\Big(\frac{\alpha^a-\alpha}{r}-1\Big)-\arccos\Big(\frac{\alpha^a+\alpha}{r}-1\Big)$$
$$\geq \Big(\Big(\frac{\alpha^a+\alpha}{r}-1\Big)-\Big(\frac{\alpha^a-\alpha}{r}-1\Big)\Big)\frac{1}{\sqrt{1-\Big(\frac{\alpha^a+\alpha}{r}-1\Big)^2}},$$

that is :

$$\arccos\Big(\frac{\alpha^a-\alpha}{r}-1\Big)-\arccos\Big(\frac{\alpha^a+\alpha}{r}-1\Big)$$
$$\geq \frac{2\alpha}{r\sqrt{2\Big(\frac{\alpha^a+\alpha}{r}\Big)-\Big(\frac{\alpha^a+\alpha}{r}\Big)^2}}=\frac{2\alpha}{\sqrt{2r(\alpha^a+\alpha)-(\alpha^a+\alpha)^2}}, \quad (12.83)$$

so that it is sufficient to prove that :

$$\frac{2\alpha}{\sqrt{2r(\alpha^a+\alpha)-(\alpha^a+\alpha)^2}}$$
$$\geq \frac{\alpha^a+\alpha}{\sqrt{2r(\alpha^a+\alpha)-(\alpha^a+\alpha)^2}}-\frac{\alpha^a-\alpha}{\sqrt{2r(\alpha^a-\alpha)-(\alpha^a-\alpha)^2}} \quad (12.84)$$

in order to obtain the inequality (12.81).
But the last inequality (12.84) is of course equivalent to :

$$\frac{\alpha^a-\alpha}{\sqrt{2r(\alpha^a-\alpha)-(\alpha^a-\alpha)^2}} \geq \frac{\alpha^a-\alpha}{\sqrt{2r(\alpha^a+\alpha)-(\alpha^a+\alpha)^2}},$$

and assuming that $0<\alpha, a<1$, one sees that $\alpha^a-\alpha>0$, so that the inequality (12.84) is equivalent to :

$$2r(\alpha^a-\alpha)-(\alpha^a-\alpha)^2 \leq 2r(\alpha^a+\alpha)-(\alpha^a+\alpha)^2,$$

that is :

$$\alpha^a \leq r,$$

which is implied by the hypothesis (12.82). Therefore :

$$r\left(\arccos\left(\frac{\alpha^a-\alpha}{r}-1\right)-\arccos\left(\frac{\alpha^a+\alpha}{r}-1\right)\right) \\ \leq R\left(\arccos\left(\frac{\alpha^a-\alpha}{R}-1\right)-\arccos\left(\frac{\alpha^a+\alpha}{R}-1\right)\right) \tag{12.85}$$

if $2\alpha^a \leq r \leq R$.

In order to obtain a result similar to (12.77) in the case when $d = 2$, $2\alpha^a \leq r \leq R$, one needs to remove the inverse function of the cosine in the right-hand side. To do so, one will use the asymptotic equivalence relation

$$\arccos(x) - \pi \underset{-1}{\sim} -\sqrt{2(x+1)}.$$

More precisely, back to the definition of the cosine of a real number using a series, one has :

$$\forall\ |x| \leq 2,\ 1-\frac{x^2}{2} \leq \cos(x) \leq 1-\frac{x^2}{2}+\frac{x^4}{24},$$

since the serie which defines the cosine of x is an alternating series (in fact, it is easy to show by simple arguments of analysis that those two inequalities hold in fact on $\mathbb{R}$). Moreover, using the fact that :

$$\forall x \in [-2,2],\ \frac{x^4}{24} \leq \frac{x^2}{6},$$

one gets :

$$\forall x \in [-2,2],\ 1-\frac{x^2}{2} \leq \cos(x) \leq 1-\frac{x^2}{3}.$$

Those two inequalities are equivalent to :

$$\forall x \in [\pi-2,\pi+2],\ \frac{(x-\pi)^2}{3}-1 \leq \cos(x) \leq \frac{(x-\pi)^2}{2}-1. \tag{12.86}$$

Since one wants to use a theorem of comparison of limits on the inverse function of the cosine around -1, the last inequality is relevant only on $[\pi-2,\pi]$. The function which is the lower bound in the two inequalities above evaluated at $\pi-2$ is $1/3$, which will give the largest interval on which one can deduce an inequality for the inverses of the functions implied in the inequalities (12.86). So taking the inverse of those inequalities, one obtains :

$$\forall x \in [-1,-2/3],\ \pi-\sqrt{3(x+1)} \leq \arccos(x) \leq \pi-\sqrt{2(x+1)}. \tag{12.87}$$

Remark 34. *Notice that one can be more precise for the lower bound of the inverse function of the cosine around* -1*. Indeed, one can use :*

$$\pi - \sqrt{6 - 2\sqrt{3 - 6x}} \leq \arccos(x) \tag{12.88}$$

instead of the left-hand side of the inequality (12.86)*, obtained by the same method but replacing the arbitrarily chosen lower bound of the cosine by the sharper one :*

$$x \mapsto -1 + \frac{(x-\pi)^2}{2} - \frac{(x-\pi)^4}{24},$$

this bound holding true for all $x \in \mathbb{R}$*. However, this lower bound is one to one only on* $[\pi - \sqrt{6}, \pi]$*, onto* $[-1, 1/2]$*, so that the sharper lower bound* (12.88) *is true for all* $x \in [-1, 1/2]$*.*
This bound gives a better rate for the end of the proof, but it will not be useful in the sequel.

Finally, if

$$0 \leq \frac{\alpha^a + \alpha}{R} \leq 1/3, \tag{12.89}$$

the two inequalities (12.87) applied on

$$\arccos\Big(\frac{\alpha^a - \alpha}{R} - 1\Big) \text{ and } \arccos\Big(\frac{\alpha^a + \alpha}{R} - 1\Big)$$

enable to write :

$$\arccos\Big(\frac{\alpha^a - \alpha}{R} - 1\Big) - \arccos\Big(\frac{\alpha^a + \alpha}{R} - 1\Big) \leq \sqrt{3\frac{\alpha^a + \alpha}{R}} - \sqrt{2\frac{\alpha^a - \alpha}{R}}. \tag{12.90}$$

In particular, since $\alpha^a \geq \alpha$ (due to the fact that $\alpha \leq 1$ and $a < 1$), if one has :

$$\alpha \leq \Big(\frac{R}{6}\Big)^{1/a}, \tag{12.91}$$

then the condition (12.89) is fulfilled.
Now, the right-hand side of the inequality (12.90) can be first rewritten as :

$$\frac{\alpha^{a/2}}{\sqrt{R}}\Big(\sqrt{3}\sqrt{1 + \alpha^{1-a}} - \sqrt{2}\sqrt{1 - \alpha^{1-a}}\Big),$$

and then bounded by (since $\alpha < 1$) :

$$\frac{\alpha^{a/2}}{\sqrt{R}}\Big(\sqrt{3}(1 + \frac{1}{2}\alpha^{1-a}) - \sqrt{2}(1 - \alpha^{1-a})\Big)$$

thanks to the concavity of the functions $x \mapsto \sqrt{1+x}$ and $x \mapsto \sqrt{1-x}$, which enables in particular, if $0 \leq x \leq 1$, to write that :

$$\sqrt{1-x} \geq 1 - x.$$

Last, but not least, if one has :

$$\Big(\frac{\sqrt{3}}{2}+\sqrt{2}\Big)\alpha^{1-a}\leq 1-(\sqrt{3}-\sqrt{2}),$$

that is :

$$\alpha\leq\left(\frac{1-(\sqrt{3}-\sqrt{2})}{\frac{\sqrt{3}}{2}+\sqrt{2}}\right)^{\frac{1}{1-a}}, \tag{12.92}$$

the following inequality holds true :

$$\frac{\alpha^{a/2}}{\sqrt{R}}\Big(\sqrt{3}\sqrt{1+\alpha^{1-a}}-\sqrt{2}\sqrt{1-\alpha^{1-a}}\Big)\leq\frac{\alpha^{a/2}}{\sqrt{R}}. \tag{12.93}$$

Gathering the intermediate inequalities (12.79), (12.85), (12.90) and (12.93), if the two conditions (12.78) and (12.82), namely :

$$r\geq 2\alpha^a \text{ and } p\leq r-\alpha^a,$$

are fulfilled ((12.78) used to obtain (12.79), and (12.82) used to obtain (12.85)), if in addition :

$$\alpha\leq\min\Big\{1,\Big(\frac{R}{6}\Big)^{1/a},\left(\frac{1-(\sqrt{3}-\sqrt{2})}{\frac{\sqrt{3}}{2}+\sqrt{2}}\right)^{\frac{1}{1-a}}\Big\}, \tag{12.94}$$

(conditions $\alpha<1$ used from the beginning, (12.91) in order to use the inequality (12.90), (12.92) in order to use the inequality (12.93)) then one has :

$$\begin{aligned}&\Big|\big\{y\in\partial B(x,r)\ /\ |y\cdot e_1|\leq\alpha\big\}\Big|\\&\qquad=r\Big(\arccos\Big(-\frac{(\alpha+p)}{r}\Big)-\arccos\Big(\frac{\alpha-p}{r}\Big)\Big)\leq\sqrt{R}\alpha^{a/2},\end{aligned} \tag{12.95}$$

which concludes the study of the last subcase, that is for $r>p+\alpha$, $r>\alpha$, and for $d=2$. The result of the lemma is obtained taking $a=1/2$.

□

Lemma 31. *There exist two strictly positive constants $C(d)$ and $c(d)$ depending only on the dimension d such that for all strictly positive numbers $\alpha\leq c(d)$ and $R\geq 1$, and all $v\in B(0,R)$, one has :*

$$\Big|\mathcal{N}^*(R,\alpha)(v)\Big|\leq C(d)R^2\alpha^{1/8} \tag{12.96}$$

in the case $d=2$, and

$$\Big|\mathcal{N}^*(R,\alpha)(v)\Big|\leq C(d)R^{d-1}\alpha \tag{12.97}$$

in the case $d\geq 3$.

Proof. The idea of the proof is the following.

- One starts (see the paragraph "Description of the image of the scattering" starting page 381) by a general study of the image of the scattering operator (see Definition 1 page 51). In particular, one obtains geometric relations, on the one hand, between the post-collisional velocities v_1' and v_2' and the pre-collisional velocities v_1 and v_2, and on the other hand, between the difference of the pre and post-collisional velocities ($v_2' - v_2$ and $v_1' - v_1$) and the angular parameter ω.

- Then one discusses (see the paragraph "Sketch of the proof" starting page 381), using a figure, the link that one can observe between the post-collisional velocities when they are contained in the space between the two planes $\{x \in \mathbb{R}^d \ / \ x \cdot e_1 = -\alpha\}$ and $\{x \in \mathbb{R}^d \ / \ x \cdot e_1 = \alpha\}$, and the angular parameter ω.

- One develops then rigorously the general idea described in the previous paragraph, by decomposing the subset $\mathcal{N}^*(R,\alpha)(v_1)$ (see the paragraph "Decomposition of the set $\mathcal{N}^*(R,\alpha)(v_1)$" starting page 383).

- Then one proceeds by controlling the measure of the subsets of

$$\mathcal{N}^*(R,\alpha)(v_1),$$

  each one being defined with v_2 fixed (see the paragraph "Control of the quantities $\left|\mathcal{N}^*(R,\alpha)(v_1,v_2)\right|$" starting page 384).

Description of the image of the scattering

Using the scattering operator introduced in Definition 1 page 51, one starts by writing the post-collisional velocities v_1' and v_2' using the pre-collisional velocities v_1 and v_2.
For the two pre-collisional velocities $v_1, v_2 \in B(0,R)$ and for any angular parameter $\omega \in \mathbb{S}^{d-1}$, the post-colisional velocities v_1' and v_2' verify

$$\frac{v_1' + v_2'}{2} = \frac{v_1 + v_2}{2} \text{ and } |v_1' - v_2'| = |v_1 - v_2|, \tag{12.98}$$

that is v_1' and v_2' are the two vertices of a diameter of the circle which has also $[v_1, v_2]$ as diameter. In other words, for v_1 and v_2 given (and if ω varies in $\mathbb{S}^{d-1}$), the vectors v_1' and v_2' lies in the boundary of the ball centered on $(v_1 + v_2)/2$ and of radius $|v_1 - v_2|/2$.
Moreover, using again the explicit expressions of the post-collisional velocities given in Definition 1, the three vectors $v_1' - v_1$, $v_2' - v_2$ and ω have the same direction.

Sketch of the proof

It is now possible to describe the geometric argument that will be used to control the size of $\mathcal{N}^*(R,\alpha)(v_1)$. One will use for that the following Figure

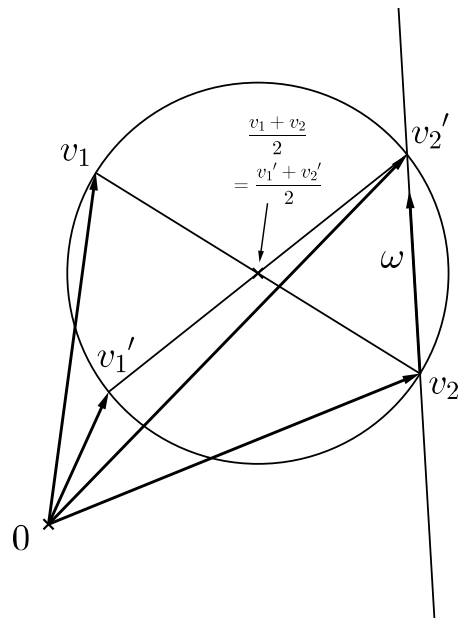

Figure 12.11: Consequence of the two equalities (12.98).

12.12. One recalls that the goal of this lemma is to determine, for a velocity $v_1 \in B(0,R)$ given, a condition on the velocity $v_2 \in B(0,R)$ and the angular parameter $\omega \in \mathbb{S}^{d-1}$ such that, either v_1' or v_2' satisfy

$$\big|v' \cdot e_1\big| \leq \alpha.$$

One chooses to find a condition on ω when v_2 is also fixed. Of course, afterwards one has to study the dependency of this condition on v_2.
Then, if v_2 is also fixed, one knows, thanks to the equations (12.98), if ω varies, that v_1' and v_2' lie in the intersection between the boundary of the ball centered on $(v_1+v_2)/2$ and of radius $\big|v_2-v_1\big|/2$, and the space $-\alpha \leq v \cdot e_1 \leq \alpha$. In the two-dimensional case, this intersection is represented in blue on Figure 12.12.
For such v_1', one sees that the intersection between the segment between v_1 and v_1', and the ball centered on v_1 and of radius $\big|v_2-v_1\big|/2$ lies in the part represented in green on the Figure. In blue, one sees a solid angle defining a cone of vertex $(v_1+v_2)/2$, in green this is a solid angle defining a cone of vertex v_1. Since v_1 lies by definition on the boundary of the ball

$$B\Big(\frac{v_1+v_2}{2}, \frac{\big|v_2-v_1\big|}{2}\Big),$$

the inscribed angle theorem states that the blue angle measures twice the green angle (at least, in the two-dimensional case). And since $v_1'-v_1$ and ω are colinear (and have the same orientation, since $\mathcal{N}^*$ is a subset of $\big\{(v_2-v_1)\cdot\omega > 0\big\} \subset \mathbb{S}^{d-1}_\omega \times B_{v_2}(0,R)$), if v_1' is between the two planes $x \cdot e_1 = -\alpha$ and $x \cdot e_1 = \alpha$, one

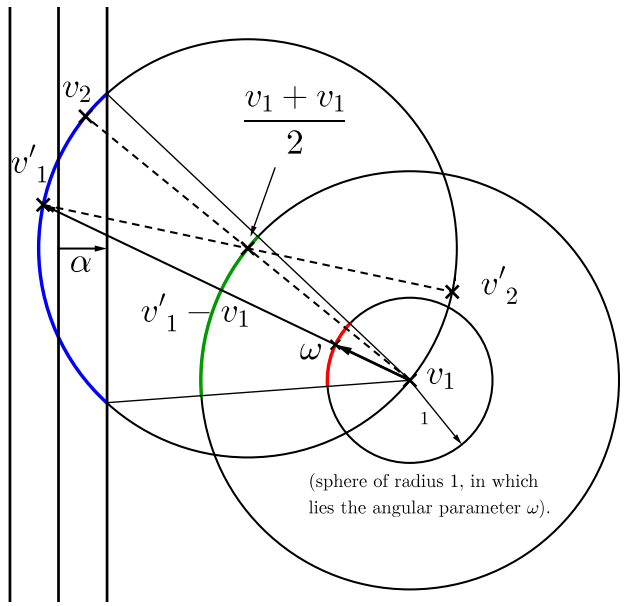

Figure 12.12: The link between the difference of the velocities $v_1' - v_1$ and the angular parameter ω.

deduces that the angular parameter ω has to lie in the part of the ball centered on v_1 represented in red on Figure 12.12.

Decomposition of the set $\mathcal{N}^*(R,\alpha)(v_1)$

Following the sketch of the proof introduced in the previous paragraph, one defines, for v_1 *and* v_2 fixed, the subset of angular parameters lying in the sphere $\mathbb{S}^{d-1}$ as

$$\mathcal{N}^*(R,\alpha)(v_1,v_2) = \Big\{\omega \in \mathbb{S}^{d-1} \ / \ v_1' \in \{|v\cdot e_1| \le \alpha\} \text{ or } v_2' \in \{|v\cdot e_1| \le \alpha\}\Big\}.$$

One has therefore

$$\big|\mathcal{N}^*(R,\alpha)(v_1)\big| = \int_{v_2\in B(0,R)} \big|\mathcal{N}^*(R,\alpha)(v_1,v_2)\big|\,\mathrm{d}v_2.$$

It is now possible to simplify the problem by using symmetries. Indeed, any two-dimensional plane which contains the straight line through the velocity v_1 and orientated by e_1 (the first vector of the canonical basis) enables to consider only a two-dimensional problem, which is always the same whatever the two-dimensional plane chosen, and this problem is in fact exactly the one described on Figure 12.12.

Therefore, for any velocity v_2, one considers the plane which contains the straight line through v_1 and orientated by e_1, and the velocity v_2.

In other words, one considers the orthogonal projection of the velocity v_2 on the wall $v \cdot e_1 = 0$, and one denotes $u(v_2)$ its normalization. Using the polar coordinates in the plane containing e_1 and $u(v_2)$, it is possible to write :

$$\big|\mathcal{N}^*(R,\alpha)(v_1)\big| \leq C(d) \int_{\mathbb{S}^{d-2}} \int_0^{2R} \int_0^{\pi} \rho^{d-1} \big|\mathcal{N}^*(R,\alpha)\big(v_1, v_2(u,\rho,\theta)\big)\big| \, \mathrm{d}\theta \, \mathrm{d}\rho \, \mathrm{d}u, \tag{12.99}$$

with

$$v_2(u,\rho,\theta) = v_1 + \rho(\cos\theta e_1 + \sin\theta u),$$

and $C(d)$ denoting a constant depending only on the dimension.
It remains to study the quantity $\big|\mathcal{N}^*(R,\alpha)(v_1,v_2)\big|$, which is, according to the previous discussion, a subset of the boundary of a ball in the Euclidean space $\mathbb{R}^d$.

Control of the quantities $\big|\mathcal{N}^*(R,\alpha)(v_1,v_2)\big|$

Back to the study of the sets $\mathcal{N}^*(R,\alpha)(v_1,v_2)$, as it was noticed previously, for v_1 and v_2 fixed, the post-collisional velocities v_1' and v_2' lie in the ball centered on $(v_1+v_2)/2$ of radius $\big|v_2 - v_1\big|/2$.
Therefore, considering

$$x = (v_1+v_2)/2,$$

and

$$r = \big|v_2 - v_1\big|/2$$

that is, using the coordinates introduced in the previous section :

$$x = v_1/2 + \big(v_1 + \rho(\cos\theta e_1 + \sin\theta u\big)/2$$

with u orthogonal to e_1, and

$$r = \rho/2,$$

one sees that ω belongs to $\mathcal{N}^*(R,\alpha)(v_1,v_2)$ if and only if v_1' or v_2' belong to

$$\partial B(x,r) \cap \big\{ \ |y \cdot e_1| \leq \alpha \big\}. \tag{12.100}$$

Lemma 30 page 370 provides then a control on the size of the subset (12.100). One needs now to use the link between v_1' (or v_2') and ω. Remembering the argument of the inscribed angle theorem described in the paragraph "Sketch of the proof" page 381 above, one will consider the following map.
For a given point x_0 of the boundary of the ball $B(x,r)$ (that is $\big|x - x_0\big| = r$), one defines the function :

$$\widetilde{\omega}_{x_0} : \begin{cases} \partial B(x,r) \backslash \{x_0\} & \to B(0,1), \\ \qquad\qquad\qquad y & \mapsto \frac{y-x_0}{|y-x_0|}. \end{cases}$$

Thanks to the inscribed angle theorem applied in any plane which contains the straight line through x_0 orientated by the vector e_1, one deduces that the

measure of the image of the subset $\big\{y \in \partial B(x,r) \ / \ \big|y \cdot e_1\big| \leq \alpha\big\}$ is smaller than half the measure of the subset itself $\big\{y \in \partial B(x,r) \ / \ \big|y \cdot e_1\big| \leq \alpha\big\}$, divided by the surface $C(d)r^{d-1}$ of the sphere $\partial B(x,r)$.
But since $v_1' - v_1$ and ω are colinear, with the same orientation, one sees that choosing $x_0 = v_1$ in the definition of the previous map, which sends elements of the boundary of the ball $B(x,r)$ into the ball centered on 0 and of radius 1, one has in fact :

$$\widetilde{\omega}_{v_1}(v_1') = \omega.$$

Similarly :

$$\widetilde{\omega}_{v_2}(v_2') = \omega,$$

so that

$$\begin{aligned}\big|\mathcal{N}^*(R,\alpha)(v_1,v_2)\big| &\leq \Big|\widetilde{\omega}_{v_1}\Big(\big\{y \in \partial B(x,r) \ / \ \big|y \cdot e_1\big| \leq \alpha\big\}\Big)\Big| \\ &\quad + \Big|\widetilde{\omega}_{v_2}\Big(\big\{y \in \partial B(x,r) \ / \ \big|y \cdot e_1\big| \leq \alpha\big\}\Big)\Big| \\ &\leq 2\frac{1}{2C(d)r^{d-1}}\Big|\big\{y \in \partial B(x,r) \ / \ \big|y \cdot e_1\big| \leq \alpha\big\}\Big|. \qquad (12.101)\end{aligned}$$

Now returning to the control of the measure of the set $\mathcal{N}^*(R,\alpha)(v_1)$, in the case $d \geq 3$, using the decomposition (12.99) :

$$\big|\mathcal{N}^*(R,\alpha)(v_1)\big| \leq C(d)\int_{\mathbb{S}^{d-2}}\int_0^{2R}\int_0^{\pi}\rho^{d-1}\big|\mathcal{N}^*(R,\alpha)\big(v_1,v_2(u,\rho,\theta)\big)\big|\,\mathrm{d}\theta\,\mathrm{d}\rho\,\mathrm{d}u$$

and then the controls (12.101), and (12.70) of Lemma 30 page 370 provide

$$\big|\mathcal{N}^*(R,\alpha)(v_1)\big| \leq C(d)\int_{\mathbb{S}^{d-2}}\int_0^{2R}\int_0^{\pi}\rho^{d-1}\Big(\frac{1}{\rho^{d-1}}\rho^{d-2}\alpha\Big)\,\mathrm{d}\theta\,\mathrm{d}\rho\,\mathrm{d}u$$

since, one recalls, $r = \rho/2$. In the end one finds

$$\begin{aligned}\big|\mathcal{N}^*(R,\alpha)(v_1)\big| &\leq C(d)\int_0^{2R}\rho^{d-2}\alpha\,\mathrm{d}\rho \\ &\leq C(d)R^{d-1}\alpha.\end{aligned}$$

In the case $d = 2$, it will be relevant in the sequel to cut-off the small difference between the velocities v_1 and v_2 (as it was done also in Proposition 16 page 347). One considers therefore a strictly positive number b smaller than 1, and one writes :

$$\begin{aligned}\big|\mathcal{N}^*(R,\alpha)(v_1)\big| \leq &\int_{v_2 \in B(v_1,\alpha^b)}\big|\mathcal{N}^*(R,\alpha)(v_1,v_2)\big|\,\mathrm{d}v_2 \\ &+ \int_{v_2 \in B(v_1,2R)\setminus B(v_1,\alpha^b)}\big|\mathcal{N}^*(R,\alpha)(v_1,v_2)\big|\,\mathrm{d}v_2.\end{aligned}$$

Using again the decomposition (12.99), one writes

$$\begin{aligned}\big|\mathcal{N}^*(R,\alpha)(v_1)\big| \leq & \int_{v_2\in B(v_1,\alpha^b)} \big|\mathcal{N}^*(R,\alpha)(v_1,v_2)\big|\,\mathrm{d}v_2 \\ & + C(d)\int_{\alpha^b}^{2R}\int_0^{\pi} \rho\big|\mathcal{N}^*(R,\alpha)\big(v_1,v_2(\rho,\theta)\big)\big|\,\mathrm{d}\theta\,\mathrm{d}\rho.\end{aligned}$$

In order to apply the control (12.69) of Lemma 30, one needs to remove from the integral the centers x of the balls $B(x,r)$ such that

$$\big|x\cdot e_1\big| > r-\sqrt{\alpha}.$$

This condition is equivalent to

$$\big|v_1\cdot e_1 + (\rho\cos\theta)/2\big| > r-\sqrt{\alpha}.$$

One obtained also, along the proof of Lemma 30, thaf if $r < \big|x\cdot e_1\big| - \alpha$ (or, with the notations of Lemma 30 : $r < p-\alpha$), then

$$\Big|\big\{y\in\partial B(x,r)\ /\ \big|y\cdot e_1\big|\leq\alpha\big\}\Big| = 0.$$

Therefore, one decomposes the interval $[0,\pi]$, in which θ lies, in three parts, by the following conditions :

- the first one is given by

$$\big|v_1\cdot e_1 + (\rho\cos\theta)/2\big| > r+\alpha. \tag{12.102}$$

  This case corresponds to the first case $r < p-\alpha$ of the proof of Lemma 30, when the center of the ball $B(x,r)$ is too far from the space

$$\big\{x\in\mathbb{R}^d\ /\ -\alpha\leq x\cdot e_1\leq\alpha\big\},$$

  so that the ball does not cross this space, and then in this case

$$\Big|\mathcal{N}^*(R,\alpha)\big(v_1,v_2(\rho,\theta)\big)\Big| = 0,$$

- the second one is given by

$$r-\sqrt{\alpha} < \big|v_1\cdot e_1 + (\rho\cos\theta)/2\big| \leq r+\alpha. \tag{12.103}$$

  On this interval one has no sharp estimate on the size of $\mathcal{N}^*(R,\alpha)(v_1,v_2)$, but one will control the term thanks to the size of the interval,

- and finally the third one is given by

$$\big|v_1\cdot e_1 + (\rho\cos\theta)/2\big| \leq r-\sqrt{\alpha}. \tag{12.104}$$

  On this interval, one will use (12.69).

Remembering that $r = \rho/2$, the condition (12.103) is equivalent to

$$1 - \frac{2}{\rho}\big(\sqrt{\alpha} + v_1 \cdot e_1\big) < \cos\theta \leq 1 + \frac{2}{\rho}\big(\alpha - v_1 \cdot e_1\big)$$

or

$$-1 - \frac{2}{\rho}\big(\alpha + v_1 \cdot e_1\big) < \cos\theta \leq -1 + \frac{2}{\rho}\big(\sqrt{\alpha} - v_1 \cdot e_1\big),$$

or again, since the cosine function is decreasing on $[0, \pi]$:

$$\begin{aligned}\arccos\Big(\min\big(1, 1 + \frac{2}{\rho}\big(\alpha - v_1 \cdot e_1\big)\big)\Big)& \\ \leq \theta < \arccos\Big(\min\big(1, 1 - \frac{2}{\rho}\big(\sqrt{\alpha} + v_1 \cdot e_1\big)\big)\Big)&\end{aligned}$$

or

$$\begin{aligned}\arccos\Big(\max\big(-1, -1 + \frac{2}{\rho}\big(\sqrt{\alpha} - v_1 \cdot e_1\big)\big)\Big)& \\ \leq \theta < \arccos\Big(\max\big(-1, -1 - \frac{2}{\rho}\big(\alpha + v_1 \cdot e_1\big)\big)\Big).&\end{aligned}$$

As in the end of the proof of Lemma 30, one has to control those differences of inverse function of the cosine. Using the explicit expression of the derivative of the function $x \mapsto \arccos(x)$, one has :

$$\forall -1 < x < 1,\ \frac{\mathrm{d}}{\mathrm{d}x}\arccos(x) = -\frac{1}{\sqrt{1 - x^2}},$$

and then

$$\forall -1 < x < 1,\ \frac{\mathrm{d}^2}{\mathrm{d}x^2}\arccos(x) = -\Big(-\frac{1}{2}\Big)\frac{-2x}{(1 - x^2)^{3/2}} = -\frac{x}{(1 - x^2)^{3/2}}.$$

This quantity being nonnegative for all $-1 < x \leq 0$, the derivative of the inverse function of the cosine is increasing on $]-1, 0]$. It provides, for any

$$-1 \leq x < y < 0,$$

and any

$$0 < \varepsilon \leq y - x,$$

$$\begin{aligned}\arccos(x + \varepsilon) - \arccos(y) &= -\int_{x+\varepsilon}^{y} \frac{\mathrm{d}}{\mathrm{d}t}\arccos(t)\,\mathrm{d}t \\ &= -\int_{-1+\varepsilon}^{-1+(y-x)} \frac{\mathrm{d}}{\mathrm{d}t}\arccos(t)\,\mathrm{d}t \\ &\quad - \int_{x+\varepsilon}^{-1+\varepsilon} \frac{\mathrm{d}}{\mathrm{d}t}\arccos(t)\,\mathrm{d}t \\ &\quad + \int_{y}^{-1+(y-x)} \frac{\mathrm{d}}{\mathrm{d}t}\arccos(t)\,\mathrm{d}t.\end{aligned}$$

Using the hypothesis on ε, one can control the two last term of the right-hand side of the last equation as follows, thanks to the change of variables $t \to t-(y-x)+\varepsilon$ in the last term :

$$\begin{aligned}
-\int_{x+\varepsilon}^{-1+\varepsilon} \frac{\mathrm{d}}{\mathrm{d}t}\arccos(t)\,\mathrm{d}t + \int_{y}^{-1+(y-x)} \frac{\mathrm{d}}{\mathrm{d}t}\arccos(t)\,\mathrm{d}t \\
&= \int_{-1+\varepsilon}^{x+\varepsilon} \frac{\mathrm{d}}{\mathrm{d}t}\arccos(t)\,\mathrm{d}t - \int_{-1+(y-x)}^{y} \frac{\mathrm{d}}{\mathrm{d}t}\arccos(t)\,\mathrm{d}t \\
&= \int_{-1+\varepsilon}^{x+\varepsilon} \Big[\frac{\mathrm{d}}{\mathrm{d}t}\arccos(t) - \frac{\mathrm{d}}{\mathrm{d}t}\arccos(t+(y-x)-\varepsilon)\Big]\,\mathrm{d}t.
\end{aligned}$$

This last quantity is negative, since the integrand is negative, thanks to the fact that the derivative of the inverse function of the cosine is increasing on the interval $]-1,0]$, and that $(y-x)-\varepsilon>0$. Therefore :

$$\begin{aligned}
\arccos(x+\varepsilon)-\arccos(y) &\le -\int_{-1+\varepsilon}^{-1+(y-x)} \frac{\mathrm{d}}{\mathrm{d}t}\arccos(t)\,\mathrm{d}t \\
&\le \arccos(-1+\varepsilon)-\arccos(-1+(y-x)).
\end{aligned}$$

Taking the limit $\varepsilon \to 0$, one gets :

$$\forall -1 \le x \le y < 0,\ \arccos(x)-\arccos(y) \le \arccos(-1)-\arccos(-1+y-x).$$

It provides here that (of course, for the first difference below, one uses the fact that the function $x \mapsto \arccos(x)-\pi/2$ is odd) the two quantities

$$\arccos\Big(\min\big(1, 1-\frac{2}{\rho}(\sqrt{\alpha}+v_1\cdot e_1)\big)\Big) - \arccos\Big(\min\big(1, 1+\frac{2}{\rho}(\alpha-v_1\cdot e_1)\big)\Big)$$

and

$$\arccos\Big(\max\big(-1, -1-\frac{2}{\rho}(\alpha+v_1\cdot e_1)\big)\Big) - \arccos\Big(\max\big(-1, -1+\frac{2}{\rho}(\sqrt{\alpha}-v_1\cdot e_1)\big)\Big)$$

are smaller than

$$\begin{aligned}
\arccos(-1) - \arccos\Big(-1+\frac{2}{\rho}(\alpha+\sqrt{\alpha})\Big) \\
&\le \arccos(-1) - \arccos\Big(-1+\frac{4}{\rho}\sqrt{\alpha}\Big).
\end{aligned}$$

Finally one uses the inequality

$$\forall x \in [-1,-2/3],\ \pi - \sqrt{3(x+1)} \le \arccos(x). \tag{12.105}$$

One finds

$$\max(A_1, A_2) \le \sqrt{3\frac{4}{\rho}\sqrt{\alpha}}, \tag{12.106}$$

where

$$A_1 = \arccos\Big(\min\big(1, 1 - \frac{2}{\rho}(\sqrt{\alpha} + v_1 \cdot e_1)\big)\Big) \\ - \arccos\Big(\min\big(1, 1 + \frac{2}{\rho}(\alpha - v_1 \cdot e_1)\big)\Big)$$

and

$$A_2 = \arccos\Big(\max\big(-1, -1 - \frac{2}{\rho}(\alpha + v_1 \cdot e_1)\big)\Big) \\ - \arccos\Big(\max\big(-1, -1 + \frac{2}{\rho}(\sqrt{\alpha} - v_1 \cdot e_1)\big)\Big),$$

this control holding true if

$$-1 + \frac{4}{\rho}\sqrt{\alpha} \leq -2/3$$

(in order to apply the inequality 12.105), that is

$$\frac{\sqrt{\alpha}}{\rho} \leq 1/12.$$

Recalling that one has split the first integral into two parts using the condition

$$\rho \geq \alpha^b,$$

one finds therefore

$$\begin{aligned} \Big|\big\{\theta \in [0,\pi] \ / \ r - \sqrt{\alpha} < \big|v_1 \cdot e_1 + (\rho\cos\theta)/2\big| \leq r + \alpha\big\}\Big| \\ &\leq 2\big(\arccos(-1) - \arccos\big(-1 + \frac{2}{\rho}\big)\big) \\ &\leq 4\sqrt{3}\alpha^{1/4-b/2}. \end{aligned} \tag{12.107}$$

The decomposition of $\big|\mathcal{N}^*(R,\alpha)(v_1)\big|$ can be rewritten as

$$\begin{aligned} \big|\mathcal{N}^*(R,\alpha)(v_1)\big| \leq & \int_{v_2 \in B(v_1,\alpha^b)} \big|\mathcal{N}^*(R,\alpha)(v_1,v_2)\big| \, \mathrm{d}v_2 \\ & + C(d)\int_{\alpha^b}^{2R}\int_0^{\pi} \rho\big|\mathcal{N}^*(R,\alpha)\big(v_1, v_2(u,\rho,\theta)\big)\big| \, \mathrm{d}\theta \, \mathrm{d}\rho \\ \leq & \int_{v_2 \in B(v_1,\alpha^b)} \big|\mathcal{N}^*(R,\alpha)(v_1,v_2)\big| \, \mathrm{d}v_2 \\ & + C(d)\int_{\alpha^b}^{2R}\int_{\rho/2-\sqrt{\alpha}<|v_1\cdot e_1+(\rho\cos\theta)/2|\leq\rho/2+\alpha} \rho\big|\mathcal{N}^*(R,\alpha)\big(v_1, v_2(u,\rho,\theta)\big)\big| \, \mathrm{d}\theta \, \mathrm{d}\rho \\ & + C(d)\int_{\alpha^b}^{2R}\int_{|v_1\cdot e_1+(\rho\cos\theta)/2|\leq\rho/2-\sqrt{\alpha}} \rho\big|\mathcal{N}^*(R,\alpha)\big(v_1, v_2(u,\rho,\theta)\big)\big| \, \mathrm{d}\theta \, \mathrm{d}\rho, \end{aligned}$$

(the two last terms of the right-hand side are respectively corresponding to the conditions (12.103) and (12.104), while the integral corresponding to the condition (12.102) vanishes), then, since by definition $\mathcal{N}^*(R,\alpha)(v_1,v_2)$ is a subset of the sphere $\mathbb{S}^1$, one bounds directly the integrands of the two first terms respectively by

$$\big|\mathbb{S}^1\big|$$

and

$$2R\big|\mathbb{S}^1\big|,$$

while one uses the control (12.69) of Lemma 30 for the last term, which holds only if

$$R \geq 1,\ \alpha \leq \min\Big\{1, \Big(\frac{R}{6}\Big)^2, \Big(\frac{1-\big(\sqrt{3}-\sqrt{2}\big)}{\frac{\sqrt{3}}{2}+\sqrt{2}}\Big)^2\Big\}$$

and

$$\sqrt{\alpha} \leq r/2 = \rho/4.$$

Since R is larger than 1, the condition on α is in fact a condition which does not depend on R.
Besides, if one takes $b < 1/2$, and if $\rho \geq \alpha^b$, one has

$$\frac{\rho}{4\sqrt{\alpha}} \geq \frac{\alpha^b}{4\sqrt{\alpha}} = \frac{1}{4}\alpha^{b-1/2} \underset{\alpha\to 0}{\longrightarrow} +\infty.$$

This implies immediately that there exists a positive constant $c(b)$ depending only on the parameter b such that for all $0 < \alpha \leq c(b)$, one has

$$\rho/4 \geq \sqrt{\alpha} \text{ and } \frac{\sqrt{\alpha}}{\rho} \leq 1/12$$

for any ρ belonging to the interval $[\alpha^b, 2R]$. For such α smaller than $c(b)$, one can then appy the control (12.69) of Lemma 30 on the one hand, and the control (12.106) on the other hand.
It provides

$$\begin{aligned}\big|\mathcal{N}^*(R,\alpha)(v_1)\big| \leq & \int_{v_2\in B(v_1,\alpha^b)} \big|\mathbb{S}^1\big|\,\mathrm{d}v_2 \\ & + C\int_{\alpha^b}^{2R}\int_{\rho/2-\sqrt{\alpha}<|v_1\cdot e_1+(\rho\cos\theta)/2|\leq\rho/2+\alpha} 2R\big|\mathbb{S}^1\big|\,\mathrm{d}\theta\,\mathrm{d}\rho \\ & + C\int_{\alpha^b}^{2R}\int_{|v_1\cdot e_1+(\rho\cos\theta)/2|\leq\rho/2-\sqrt{\alpha}} \rho\Big(\frac{1}{\rho}\sqrt{R}\alpha^{1/4}\Big)\,\mathrm{d}\theta\,\mathrm{d}\rho,\end{aligned}$$

and finally using the control (12.107) for the size of the domain of the integral in θ for the second term, one obtains

$$\big|\mathcal{N}^*(R,\alpha)(v_1)\big| \leq C\Big(\alpha^{2b} + R^2\alpha^{1/4-b/2} + R^{3/2}\alpha^{1/4}\Big),$$

smaller than

$$CR^2\big(\alpha^{2b} + \alpha^{1/4-b/2}\big)$$

for $R \geq 1$ and $\alpha \leq 1$.
To obtain a quantity which goes to zero as α goes to zero (for R fixed), one is again led to choose $b < 1/2$, for example $b = 1/4$, hence the result stated in the lemma for $d = 2$. □

Remark 35. *The parameter ρ used to describe the radius in the polar coordinates in the previous proof must not be confused with the cut-off parameter in small distance between the obstacle and the particles used, for example, in Proposition 16 page 347, and more generally in the end of this work. Indeed, in the previous proof, the quantity $\sqrt{\alpha}/\rho$ goes to zero as α goes to zero, while in Chapter 15, and especially in the proof of Theorem 7 page 493, the quantity ρ/α can be chosen as small as one wants.*

12.3 Excluding the pathological pseudo-trajecto -ries

Now, all the useful tools such as the geometric lemmas of the two previous sections have been shown, and one is finally able to perform the plan described in Section 12.1.1 page 309 above, and to study the trees introduced in Section 12.1.2 page 327.

12.3.1 Preparation of the initial configurations

One recalls that one has in mind to show that the trajectories with boundary conditions of the hard sphere flow and the free flow are close, up to excluding a small subset among all the variables involved in the definition of the dynamics. One has seen that the significant divergences between those two trajectories come from the recollisions.
The purpose of this section is to remove the first possible source of recollisions, which are the pathological initial configurations (the second source is the pathological way of adding another particle to the system). One has then to prepare the initial configurations Z_s so that the geometric lemmas can be applied.

First, one discusses what was done before, in the article [34]. In their setting, the idea to remove the pathological initial configurations is the same that the one to remove the pathological adjunction parameters, since they consider observables, that is integrals of solutions of the hierarchies integrated against some test functions in the velocity variable, which enable to cut off in this velocity domain the pathological initial configurations. It is indeed possible, since the condition defining pathological initial configurations holds only on the velocity variable. As a consequence, it is possible to control the error done in that step of the proof since the size of the removed subset of the domain is controlled thanks to the geometric lemmas, and besides this method imposes to consider

local uniform convergence in the position variable, but only a weak convergence in the velocity variable.
In this work, one will be able to state a local uniform convergence in all the variables of the phase space, up to removing a small subset of this phase space, which is therefore a stronger result of convergence than the one stated in [34].

Introducing the domain of local uniform convergence

It is important here to notice that the version of the shooting lemma used here is the one with fixed axis (see Lemma 28 page 342), that is, the axes of the cylinders of pathological velocities do not depend on ε. This will be important in the sequel in order to obtain the uniform convergence on every compact set outside some submanifold of the phase space. If the dependency with respect to the parameter ε is not removed, it is not clear that one can take compact sets outside a submanifold, so that the uniform convergence will not be satisfied anymore.
One introduces therefore the set, for which the uniform convergence on every compact set contained in its complementary can be stated.

Definition 53 (Domain of local uniform convergence Ω_s)**.** *Let s be a positive integer. One defines the* domain of local uniform convergence*, denoted as Ω_s, as the subset of the phase space of s particles $\mathcal{D}_s^0 = \big(\{x \in \mathbb{R}^d \ / \ x \cdot e_1 > 0\} \times \mathbb{R}^d\big)^s$ defined as :*

$$Z_s \in \Omega_s \Leftrightarrow Z_s \in \Omega_s^1 \cap \Omega_s^2 \cap \Omega_s^3 \cap \Omega_s^4,$$

that is

$$\Omega_s = \bigcap_{j=1}^{4} \Omega_s^j,$$

with :

$$\begin{cases} \Omega_s^1 &= \Big\{Z_s \in \mathcal{D}_s^0 \ / \ \forall \ 1 \leq i \neq j \leq s, x_i \neq x_j\Big\}, \\ \Omega_s^2 &= \Big\{Z_s \in \mathcal{D}_s^0 \ / \ \forall \ 1 \leq i \leq s, v_i \cdot e_1 \neq 0\Big\}, \\ \Omega_s^3 &= \Big\{Z_s \in \mathcal{D}_s^0 \ / \ \forall \ 1 \leq i < j \leq s, v_j \notin v_i + \mathit{Vect}(x_i - x_j)\Big\}, \\ \Omega_s^4 &= \Big\{Z_s \in \mathcal{D}_s^0 \ / \ \forall 1 \leq i < j \leq s, v_j \notin \mathcal{S}_0(v_i) + \mathit{Vect}\big(\mathcal{S}_0(x_i) - x_j\big)\Big\}, \end{cases}$$

where $\mathcal{S}_0$ denotes the orthogonal symmetry with respect to the first vector of the canonical basis e_1, that is $\mathcal{S}_0(v) = v - 2\big(v \cdot e_1\big)e_1$.

One then defines a family of special subsets of Ω_s, so that any compact subset of Ω_s will be contained in some element of this family, and on which the geometrical lemmas can easily be applied.

Definition 54 (Subsets of preparation of the initial configurations)**.** *One defines, for any positive integer s and any strictly positive numbers ε, ε_0, R, α*

and γ *the* subset of preparation of the initial configurations $\Delta_s(\varepsilon, R, \varepsilon_0, \alpha, \gamma)$ *as the following subset of* Ω_s *:*

$$\Delta_s = \Delta_s(\varepsilon, R, \varepsilon_0, \alpha, \gamma) = \bigcap_{j=1}^{6} \Delta_s^j,$$

with

$$\Delta_s^1 = \Delta_s^1(\varepsilon) = \big\{Z_s \in \mathbb{R}^{2ds} \ / \ \forall 1 \leq i \leq s, \ x_i \cdot e_1 > \varepsilon/2\big\}, \tag{12.108}$$

$$\Delta_s^2 = \Delta_s^2(R) = \big\{Z_s \in \mathbb{R}^{2ds} \ / \ \big|V_s\big| \leq R\big\}, \tag{12.109}$$

$$\Delta_s^3 = \Delta_s^3(\varepsilon_0) = \big\{Z_s \in \mathbb{R}^{2ds} \ / \ \min_{1 \leq i < j \leq s} \big|x_i - x_j\big| \geq \varepsilon_0\big\}, \tag{12.110}$$

$$\Delta_s^4 = \Delta_s^4(\alpha) = \big\{Z_s \in \mathbb{R}^{2ds} \ / \ \min_{1 \leq i \leq s} \big|v_i \cdot e_1\big| \geq \alpha\big\}, \tag{12.111}$$

$$\Delta_s^5 = \Delta_s^5(\gamma) = \big\{Z_s \in \mathbb{R}^{2ds} \ / \ \min_{1 \leq i < j \leq s} d\big(v_j, v_i + \mathit{Vect}(x_i - x_j)\big) \geq \gamma\big\}, \tag{12.112}$$

$$\begin{aligned} \Delta_s^6 &= \Delta_s^6(\gamma) \\ &= \big\{Z_s \in \mathbb{R}^{2ds} \ / \ \min_{1 \leq i < j \leq s} d\big(v_j, \mathcal{S}_0(v_i) + \mathit{Vect}\big(\mathcal{S}_0(x_i) - x_j\big)\big) \geq \gamma\big\}. \end{aligned} \tag{12.113}$$

One can note that each of the six conditions defining Δ_s are of different nature. The first condition (12.108) is mandatory to have particles starting inside the phase space (otherwise, the particles may overlap the obstacle, and the transport would be not defined for such initial configurations).
The third condition (12.110) means that in particular the particles start with a distance large enough between each other. One will be then able, thanks to the second condition (12.109) to use the results stated in Lemma 28 page 342, which provides the small set of pathological velocities to exclude such that all the other configurations starting with velocities outside this pathological set will remain far enough from each other : those configurations are in a good configuration (see Definition 48 page 333). The two last conditions (12.112) and (12.113) are then used to make sure that the velocities are outside the cylinders given by Lemma 28, which contain pathological initial configurations. With those conditions, the system of s particles will face no recollision in the future.
Finally, the fourth condition (12.111) ensures that no particle will follow a grazing trajectory, nor it will be close to the obstacle during a long time. This condition is not necessary to make sure that the initial system, of s particles, will face no recollision. However, this condition will be used when one will add another particle to the system. Indeed, one knows, thanks to Lemma 16 page 347, that if all the particles of the system are far enough from the obstacle and in a good configuration, then it will possible to add another particle so that the new system of $s+1$ particles will be also in a good configuration.

Compact subsets of the domain Ω_s and good configurations : preparing the initial configurations

Proposition 17. *Let s be a positive integer, and let ε, R, δ, ε_0 and α be five strictly positive numbers such that :*

$$2\sqrt{3}\varepsilon \leq \varepsilon_0.$$

Then for any initial configuration $Z_s \in \mathcal{D}_s^0$ such that :

$$Z_s \in \Delta_s\big(\varepsilon, R, \varepsilon_0, \alpha, \max(16R\varepsilon/\varepsilon_0, \varepsilon_0/\delta)\big)$$

one has that :

- *no recollision will happen from the initial configuration Z_s following the hard sphere flow (introduced in Definition 4 page 53), that is*

$$Z_k \in \mathcal{G}_s^\varepsilon(\varepsilon), \tag{12.114}$$

- *after a time δ, the particles of the configuration Z_s, following the free flow with boundary condition (introduced in Definition 3 page 53) will be in a good configuration, separated by at least ε_0, that is*

$$T_{-\delta}^{s,0}(Z_k) \in \mathcal{G}_s^0(\varepsilon_0), \tag{12.115}$$

- *for all $1 \leq i \leq s$ and any strictly positive number $\rho > 0$:*

$$\left|\left\{\tau \geq 0 \ / \ \left|\left(T_{-\tau}^{s,0}(Z_s)\right)^{X,i} \cdot e_1\right| \leq \rho\right\}\right| \leq \frac{2\rho}{\alpha}. \tag{12.116}$$

Remark 36. *Note that in Lemma 28 page 342, Definition 48 page 333 and in Proposition 16 page 347, one authorises a small difference of position between the vectors X_k and $\overline{X}_k$ (this distance is controlled by the quantity denoted previously by a). At the present point of the work, this quantity does not appear : indeed the role of this quantity is to take into account that one is considering configurations such that their respective images by the hard sphere and the free flows are close, but with a possible small difference. Here, one is only at the preparation step : the same initial configuration will be used to start the two dynamics (the dynamics of the hard spheres, and the dynamics of the free flow).*

Proof. The first point (12.114) of the result stated in the proposition is exactly the result (12.28) of Lemma 28 page 342, which can be applied since the condition (12.27) of this lemma is fulfilled. One applies it recursively (that is, for all the particles). Any velocity v_1 can be chosen for the first particle, and once this velocity is chosen, one applies the shooting lemma for fixed v_1, obtaining therefore a condition excluding some velocity v_2, the velocity of the second particle. Then one iterates the process, and excludes some velocities v_3 for the third particles due to the conditions imposed by the shooting lemma, depending on the velocities v_1 and v_2 chosen, and so on. Of course, the shooting lemma can be applied thanks to the following hypotheses :

- $\left|V_s\right| \leq R$,
- $d\big(Z_s, (\Omega_s^1)^c\big) \geq \varepsilon_0$, that is for every $i \neq j$, one has : $\left|x_i - x_j\right| \geq \varepsilon_0$,
- and $\min\big(d\big(Z_s, (\Omega_s^3)^c\big), d\big(Z_s, (\Omega_s^4)^c\big)\big) \geq \max\big(16Ra/\varepsilon_0, 4\varepsilon_0/\delta\big)$, that is for every $j > i$, the velocity v_j chosen for the particle j does not belong to any of the two cylinders of radius $16Ra/\varepsilon_0$ and of respective origins and axis v_i and $x_i - x_j$, and $\mathcal{S}_0(v_i)$ and $\mathcal{S}_0(x_i) - x_j$,

which are the first, the second and (first part of) the last hypotheses of the lemma.
Similarly, the second point (12.115) stated in the proposition is a consequence of the second result of Lemma 27 page 335, using exactly the same hypotheses except the last, replacing the radius $16Ra/\varepsilon_0$ by ε_0/δ.
Finally, the last point corresponds just to excluding the set

$$\{v \in B_{\mathbb{R}^d}(0, R) \ / \ |v \cdot e_1| \leq \alpha\}$$

for every velocity component v_s of Z_s. □

12.3.2 Surgery in time, velocity and angular parameter for the integrated in time (transport-) collision-transport operators

If one considers only pseudo-trajectories starting from initial configurations which are in a good configuration (see Definition 48 page 333), which was prepared by the restriction performed in the previous Section 12.3.1, the goal now is to determine pseudo-trajectories that are still not pathological after the adjunction steps, that is such that there will be no recollision after adding new particles to the system.
Thanks to the geometric lemmas of Section 12.2 page 333, one will aim to remove the sets of all possible velocities (of the new particle) and angular parameters (determining the position of the new particle at the time of adjunction) which will generate a pathological pseudo-trajectory at this step of construction of the tree, so that when a particle will be added to a finite number of other particles in a good configuration, the new set of particles remains in a good configuration and so that no particle stays close too long to the boundary of the obstacle.

The difference with the previous paragraph lies in the fact that this exclusion will take place in the domain of the integrals defining the (transport-) collision-transport operators, and not among the set of the initial configurations. One is then led to study the effects of a cut-off in the domains of the operators of the hierarchies.

Surgery on the domain of the integrated in time collision-transport operator for the Boltzmann hierarchy

This exclusion will have then a different effect from the restriction of the domain of convergence obtained in the previous Section 12.3.1 : instead of giving

conditions on the type of convergence (previously, by excluding some initial configurations, since one saw that the local uniform convergence will not be possible on the whole phase space, but only on Ω_s, that is only when some pathological submanifolds are removed), this time this exclusion will create a small error term due to the cut-off made in the domains of the integrals defining the collision operators.
This cut-off has to be controlled, and this is the purpose of the present paragraph. One starts by a general surgery lemma, detailing the effects of such a cut-off in the domain of the integral of the collision operators, here in the easiest case of the Boltzmann hierarchy, since it is defined as a usual Lebesgue integral and therefore will not require an additional important work to solve the question of the rigorous definition.

On the one hand, in the spirit of the geometric lemmas, one wants to authorize adjunctions of particles to a system in a good configuration only when the particle which will undergo the adjunction is not too close to the obstacle. The way to obtain this property was discussed above, and will be obtained by removing pathological times during which the particle is close to the obstacle (those times are meant to be few due to the fact that the trajectory of the particle is not grazing the obstacle). Therefore, one will have to study the effects of surgery on the time domain of the integrated in time collision-transport operator. Of course, the geometric lemmas describe also which kind of velocity and angular parameter one has to avoid to keep the system with the additional particle in a good configuration, which will imply on the other hand to study the effects of a surgery on the domain of the collision operator itself, that is on the variables v_{s+1} and ω. Once those two cut-offs are made, a simple iteration of the geometric lemmas provides that the hard sphere flow outside the removed subsets is very similar to the free flow, or in other words, one will obtain that the pseudo-trajectories are not pathological.

A last word about the cut-off in time : here the removal of some subsets of the domains of the time integrals will not only have the effect of providing only non pathological pseudo-trajectories. Even if a particle of the system is not meant to undergo an adjunction, if this particle bounces against the obstacle, there will be a small time during which the hard sphere and the free flows will not be easily comparable. Indeed, one of those two dynamics will generate a bouncing first. From this time and until the bouncing against the obstacle prescribed by the other dynamics, the two pseudo-trajectories will not be similar, in the sense that the velocities of the particle will be quite different following one or the other dynamics (see Figure 12.4 page 332). The solution to avoid this problem lies again in cutting-off those pathological times, which will be, again thanks to the cut-off in grazing trajectories, hopefully quite few.

Introducing the decomposition induced by the surgery in adjunction parameters for the Boltzmann hierarchy

One recalls also that one starts from solutions which have already been simplified, thanks to the results of Section 11 page 277. In particular, one considers only (finite numbers of) iterations of the Boltzmann operator with a separation in time between those iterations, applied to functions truncated in high energy.

For the generic post-collisional first integral term of the Duhamel formula ((10.9) page 272) after the cut-offs of Section 11, that is :

$$\mathbb{1}_{t\geq\delta}\int_0^{t-\delta}\int_{\mathbb{S}^{d-1}_\omega\times\mathbb{R}^d_{v_{s+1}}}\Big[\omega\cdot\Big(v_{s+1}-\big(T^{s,0}_{t_1-t}(Z_s)\big)^{V,j_1}\Big)\Big]_+\mathbb{1}_{|V_{s+1}|\leq R}$$
$$\times f_0^{(s+1)}\Big(T^{s+1,0}_{-t_1}\Big(\Big(T^{s,0}_{t_1-t}(Z_s),\big(T^{s,0}_{t_1-t}(Z_s)\big)^{X,j_1},v_{s+1}\Big)'_{j_1,s+1}\Big)\Big)\,\mathrm{d}\omega\,\mathrm{d}v_{s+1}\,\mathrm{d}t_1,$$

one will decompose, finally, as follows :

$$\begin{aligned}
&\mathbb{1}_{t\geq\delta}\int_0^{t-\delta}\mathbb{1}_{U_{j_1}(Z_s)}\int_{\mathbb{S}^{d-1}_\omega\times\mathbb{R}^d_{v_{s+1}}}\Big[\omega\cdot\Big(v_{s+1}-\big(T^{s,0}_{t_1-t}(Z_s)\big)^{V,j_1}\Big)\Big]_+\mathbb{1}_{|V_{s+1}|\leq R}\\
&\qquad\times f_0^{(s+1)}\Big(T^{s+1,0}_{-t_1}\Big(\Big(T^{s,0}_{t_1-t}(Z_s),\big(T^{s,0}_{t_1-t}(Z_s)\big)^{X,j_1},v_{s+1}\Big)'_{j_1,s+1}\Big)\Big)\,\mathrm{d}\omega\,\mathrm{d}v_{s+1}\,\mathrm{d}t_1\\
&+\mathbb{1}_{t\geq\delta}\int_0^{t-\delta}\mathbb{1}_{U_{j_1}(Z_s)^c}\int_{\mathbb{S}^{d-1}_\omega\times\mathbb{R}^d_{v_{s+1}}}\mathbb{1}_{E_{j_1}(T^{s,0}_{t_1-t}(Z_s))}\\
&\qquad\times\Big[\omega\cdot\Big(v_{s+1}-\big(T^{s,0}_{t_1-t}(Z_s)\big)^{V,j_1}\Big)\Big]_+\mathbb{1}_{|V_{s+1}|\leq R}\\
&\qquad\times f_0^{(s+1)}\Big(T^{s+1,0}_{-t_1}\Big(\Big(T^{s,0}_{t_1-t}(Z_s),\big(T^{s,0}_{t_1-t}(Z_s)\big)^{X,j_1},v_{s+1}\Big)'_{j_1,s+1}\Big)\Big)\,\mathrm{d}\omega\,\mathrm{d}v_{s+1}\,\mathrm{d}t_1\\
&+\mathbb{1}_{t\geq\delta}\int_0^{t-\delta}\mathbb{1}_{U_{j_1}(Z_s)^c}\int_{\mathbb{S}^{d-1}_\omega\times\mathbb{R}^d_{v_{s+1}}}\mathbb{1}_{E_{j_1}(T^{s,0}_{t_1-t}(Z_s))^c}\\
&\qquad\times\Big[\omega\cdot\Big(v_{s+1}-\big(T^{s,0}_{t_1-t}(Z_s)\big)^{V,j_1}\Big)\Big]_+\mathbb{1}_{|V_{s+1}|\leq R}\\
&\qquad\times f_0^{(s+1)}\Big(T^{s+1,0}_{-t_1}\Big(\Big(T^{s,0}_{t_1-t}(Z_s),\big(T^{s,0}_{t_1-t}(Z_s)\big)^{X,j_1},v_{s+1}\Big)'_{j_1,s+1}\Big)\Big)\,\mathrm{d}\omega\,\mathrm{d}v_{s+1}\,\mathrm{d}t_1,
\end{aligned}\tag{12.117}$$

where $U_{j_1}(Z_s)$ and $E_{j_1}\big(T^{s,0}_{t_1-t}(Z_s)\big)$ denote respectively the subset of $[0,T]$ composed of the pathological times on the one hand, and the subset of $\mathbb{S}^{d-1}\times\mathbb{R}^d$ composed of the pathological angular parameters and velocities on the other hand. Those subsets will be properly defined below, in the paragraph starting page 432.

The first term, defined with a double condition $[0, t-\delta] \cap U_{j_1}(Z_s)$ on the time domain (with $U_{j_1}(Z_s)$ representing the time interval during which the particle j_1 is close to the obstacle), and the second term, defined with a time domain being the complement of $U_{j_1}(Z_s)$ and with a condition $E_{j_1}(Z_s)$ on the adjunction parameters (ω, v_{s+1}) domain (with $E_{j_1}(Z_s)$ representing the pathological adjunction parameters determined in the geometric lemmas, leading to a system which is not in a good configuration after adding another particle). Both terms are meant to be small, as the size of the subsets $U_s(Z_s)$ and $E_{j_1}(Z_s)$, while the third term will be, on the one hand, the most significant part, and on the other hand, it will correspond to pseudo-trajectories without recollision.

Remark 37. *One emphasizes that the subsets $U_{j_1}(Z_s)$ and $E_{j_1}(T^{s,0}_{t_1-t}(Z_s))$ depend of course on the initial configuration Z_s. The second subset also depends on the time t_1 chosen for adding a particle, indeed the adjunction parameters are of course excluded with respect to the global position of the system of particles at the very time chosen for the adjunction. This is the object of the following discussion, just below the remark.*

The dependency of the subsets excluded by the surgery on the initial configuration forces to be quite careful when one defines the truncated in adjunction parameters, integrated in time collision-transport operator of the Boltzmann hierarchy. The reason is the following. At Z_s fixed, it is enough to ask only that $U_{j_1}(Z_s)$ is measurable. But the regularity for the other subset E_{j_1} is more demanding, since one composes the indicator function of this subset with the transport. One will then require, in addition with the fact that $E_{j_1}(Z_s)$ is measurable for any configuration Z_s, that

$$\bigcup_{Z_s \in (\overline{\Omega^c} \times \mathbb{R}^d)^s} \big(\{Z_s\} \times E_{j_1}(Z_s)\big)$$

is measurable, which will provide, for t and Z_s fixed, that the function

$$t_1 \mapsto \mathbb{1}_{E_{j_1}(T^{s,0}_{t_1-t}(Z_s))}$$

is measurable, since $t_1 \mapsto T^{s,0}_{t_1-t}(Z_s)$ is piecewise continuous, and then the following quantity, for t and Z_s fixed, will be well defined.

Definition 55 (Truncated in adjunction parameters (time, velocity and angular parameter), integrated in time collision-transport operator of the Boltzmann hierarchy)**.** *For any integer $1 \leq j \leq s$, any function $f^{(s+1)} \in \mathcal{C}\big([0,T] \times \big(\overline{\Omega^c} \times \mathbb{R}^d\big)^{s+1}\big)$ with $f^{(s+1)}(t,\cdot) \in X_{0,s+1,\beta}$ for any $t \in [0,T]$, for any function :*

$$U : Z_s \mapsto U(Z_s) \in \mathcal{P}([0, t-\delta])$$

and

$$E : Z_s \mapsto E(Z_s) \in \mathcal{P}\big(\mathbb{S}^{d-1} \times \mathbb{R}^d\big)$$

of measurable subsets $U(Z_s)$ *and* $E(Z_s)$*, one denotes the* truncated in adjunction parameters, integrated in time collision-transport operator of the Boltzmann hierarchy *of type* $(\pm, j)$ *the function :*

$$\mathbb{1}_{t\geq\delta}\int_0^{t-\delta}\mathbb{1}_{U(Z_s)}\int_{\mathbb{S}_\omega^{d-1}\times\mathbb{R}^d_{v_{s+1}}}\mathbb{1}_{E(T^{s,0}_{u-t}(Z_s))}\Big[\omega\cdot\Big(v_{s+1}-\big(T^{s,0}_{u-t}(Z_s)\big)^{V,j}\Big)\Big]_+$$
$$\times f^{(s+1)}(u,\big(T^{s,0}_{u-t}(Z_s),\big(T^{s,0}_{u-t}(Z_s)\big)^{X,j},v_{s+1}\big)'_{j,s+1})\,\mathrm{d}\omega\,\mathrm{d}v_{s+1}\,\mathrm{d}u$$

if $\pm=+$*, and :*

$$\mathbb{1}_{t\geq\delta}\int_0^{t-\delta}\mathbb{1}_{U(Z_s)}\int_{\mathbb{S}_\omega^{d-1}\times\mathbb{R}^d_{v_{s+1}}}\mathbb{1}_{E(T^{s,0}_{u-t}(Z_s))}\Big[\omega\cdot\Big(v_{s+1}-\big(T^{s,0}_{u-t}(Z_s)\big)^{V,j}\Big)\Big]_-$$
$$\times f^{(s+1)}\big(u,T^{s,0}_{u-t}(Z_s),\big(T^{s,0}_{t_1-t}(Z_s)\big)^{X,j},v_{s+1}\big)\,\mathrm{d}\omega\,\mathrm{d}v_{s+1}\,\mathrm{d}u$$

if $\pm=-$ *as :*

$$\mathcal{I}^{0,\delta}_{\substack{s\\\pm,j}}(U,E)f^{(s+1)}(t,Z_s). \tag{12.118}$$

In the case when $E(Z_s)=\mathbb{S}^{d-1}\times\mathbb{R}^d$ *for every* Z_s *(surgery only in the time variable), one will simply denote :*

$$\mathcal{I}^{0,\delta}_{\substack{s\\\pm,j}}\big(U,\mathbb{S}^{d-1}\times\mathbb{R}^d\big)f^{(s+1)}(t,Z_s)=\mathcal{I}^{0,\delta}_{\substack{s\\\pm,j}}(U)f^{(s+1)}(t,Z_s). \tag{12.119}$$

The decomposition (12.117) presented page 397 writes then, following the notations just introduced :

$$\mathcal{I}^{0,\delta}_{\substack{s\\\pm,j}}f^{(s+1)}(t,Z_s)=\mathcal{I}^{0,\delta}_{\substack{s\\\pm,j}}(U_{j_1})f^{(s+1)}(t,Z_s)+\mathcal{I}^{0,\delta}_{\substack{s\\\pm,j}}(U^c_{j_1},E_j)f^{(s+1)}(t,Z_s)$$
$$+\mathcal{I}^{0,\delta}_{\substack{s\\\pm,j}}(U^c_{j_1},E^c_{j_1})f^{(s+1)}(t,Z_s),$$

with only the last term of the right-hand side representing an important contribution in this decomposition.
For any positive integer k, and any generic elements $M_k=(\pm_1,\ldots,\pm_k)\in\mathfrak{M}_k$ and $J_k=(j_1,\ldots,j_k)\in\mathfrak{J}^s_k$, the same decomposition will be then performed for the iterated integrated in time collision-transport operator of the Boltzmann hierarchy of type (M_k,J_k), that is, only formally at this step :

$$\mathcal{I}^{0,\delta}_{\substack{s,s+k-1\\(M_k,J_k)}}f_0^{(s+k)}=\mathcal{I}^{0,\delta}_{\substack{s\\\pm_1,j_1}}\circ\mathcal{I}^{0,\delta}_{\substack{s+1,s+k-1\\(\pm_2,\ldots,\pm_j),(j_2,\ldots,j_k)}}f_0^{(s+k)}$$
$$=\Big[\mathcal{I}^{0,\delta}_{\substack{s\\\pm_1,j_1}}(U_{j_1})+\mathcal{I}^{0,\delta}_{\substack{s\\\pm_1,j_1}}(U^c_{j_1},E_{j_1})+\mathcal{I}^{0,\delta}_{\substack{s\\\pm_1,j_1}}(U^c_{j_1},E^c_{j_1})\Big]\circ\mathcal{I}^{0,\delta}_{\substack{s+1,s+k-1\\(\pm_2,\ldots,\pm_j),(j_2,\ldots,j_k)}}f_0^{(s+k)}$$

(at this step, one has just performed the decomposition described above on the operator $\mathcal{I}^{0,\delta}_s$, which is applied to the function $\mathcal{I}^{0,\delta}_{s+1,s+k-1}f_0^{(s+k)}$). Now one will

iterate this decomposition on the last of the three terms obtained, while the two first ones will be let unchanged :

$$\begin{aligned}\mathcal{I}^{0,\delta}_{\substack{s\\M_k,J_k}} f_0^{(s+k)} =& \Big[\mathcal{I}^{0,\delta}_{\substack{s\\\pm_1,j_1}}(U_{j_1}) + \mathcal{I}^{0,\delta}_{\substack{s\\\pm_1,j_1}}(U^c_{j_1},E_{j_1})\Big] \circ \mathcal{I}^{0,\delta}_{\substack{s+1,s+k-1\\(\pm_2,\dots,\pm_k,j_2,\dots,j_k)}} f_0^{(s+k)}\\ &+ \mathcal{I}^{0,\delta}_{\substack{s\\\pm_1,j_1}}(U^c_{j_1},E^c_{j_1})\Big[\mathcal{I}^{0,\delta}_{\substack{s+1\\\pm_2,j_2}}(U_{j_2}) + \mathcal{I}^{0,\delta}_{\substack{s+1\\\pm_2,j_2}}(U^c_{j_2},E_{j_2})\\ &\qquad + \mathcal{I}^{0,\delta}_{\substack{s+1\\\pm_2,j_2}}(U^c_{j_2},E^c_{j_2})\Big] \circ \mathcal{I}^{0,\delta}_{\substack{s+2,s+k-1\\(\pm_3,\dots,\pm_k),(j_3,\dots,j_k)}} f_0^{(s+k)},\end{aligned}$$

and so on, applying again the decomposition on the last of the three terms just obtained. One finds then :

$$\begin{aligned}\mathcal{I}^{0,\delta}_{\substack{s,s+k-1\\M_k,J_k}} f_0^{(s+k)} =& \Big[\mathcal{I}^{0,\delta}_{\substack{s\\\pm_1,j_1}}(U_{j_1}) + \mathcal{I}^{0,\delta}_{\substack{s\\\pm_1,j_1}}(U^c_{j_1},E_{j_1})\Big] \circ \mathcal{I}^{0,\delta}_{\substack{s+1,s+k-1\\(\pm_2,\dots,\pm_k),(j_2,\dots,j_k)}} f_0^{(s+k)}\\ &+\mathcal{I}^{0,\delta}_{\substack{s\\\pm_1,j_1}}(U^c_{j_1},E^c_{j_1})\Big[\mathcal{I}^{0,\delta}_{\substack{s+1\\\pm_2,j_2}}(U_{j_2}) + \mathcal{I}^{0,\delta}_{\substack{s+1\\\pm_2,j_2}}(U^c_{j_2},E_{j_2})\Big] \circ \mathcal{I}^{0,\delta}_{\substack{s+2,s+k-1\\(\pm_3,\dots,\pm_k),(j_3,\dots,j_k)}} f_0^{(s+k)}\\ &+\dots\\ &+\mathcal{I}^{0,\delta}_{\substack{s\\\pm_1,j_1}}(U^c_{j_1},E^c_{j_1}) \circ \dots \circ \mathcal{I}^{0,\delta}_{\substack{s+k-2\\\pm_{k-1},j_{k-1}}}(U^c_{j_{k-1}},E^c_{j_{k-1}})\\ &\qquad \circ \Big[\mathcal{I}^{0,\delta}_{\substack{s+k-1\\\pm_k,j_k}}(U_{j_k}) + \mathcal{I}^{0,\delta}_{\substack{s+k-1\\\pm_k,j_k}}(U^c_{j_k},E_{j_k})\Big] f_0^{(s+k)}\\ &+\mathcal{I}^{0,\delta}_{\substack{s\\\pm_1,j_1}}(U^c_{j_1},E^c_{j_1}) \circ \dots \circ \mathcal{I}^{0,\delta}_{\substack{s+k-1\\\pm_k,j_k}}(U^c_{j_k},E^c_{j_k}) f_0^{(s+k)}.\end{aligned}$$

Of course, the idea is that, on the right-hand side, each term of the decomposition will be negligible except the last one, which represents the pseudo-trajectories which are not pathological.

Here, however the problem of rigorous definition after surgery occurs again in the case of the iterated operators. Indeed, the main term of the decomposition

of the element $\mathcal{I}^{0,\delta}_{\substack{s,s+1\\(+,+),(j_1,j_2)}}\big(t\mapsto \mathcal{T}^{s+2,0}_t f^{(s+2)}_0\big)$ writes :

$$\begin{aligned}
&\mathbb{1}_{t\geq 2\delta}\int_\delta^{t-\delta}\mathbb{1}_{U_{j_1}(Z_s)^c}\int_{\mathbb{S}^{d-1}_{\omega_1}\times\mathbb{R}^d_{v_{s+1}}}\mathbb{1}_{E_{j_1}(T^{s,0}_{t_1-t}(Z_s))^c}\Big[\omega_1\cdot\big(v_{s+1}-\big(T^{s,0}_{t_1-t}(Z_s)\big)^{V,j_1}\big)\Big]_+\\
&\times\mathbb{1}_{t_1\geq\delta}\int_0^{t_1-\delta}\mathbb{1}_{U_{j_2}((T^{s,0}_{t_1-t}(Z_s),(T^{s,0}_{t_1-t}(Z_s))^{X,j_1},v_{s+1})'_{j_1,s+1})^c}\\
&\times\int_{\mathbb{S}^{d-1}_{\omega_2}\times\mathbb{R}^d_{v_{s+2}}}\mathbb{1}_{E_{j_2}(T^{s+1,0}_{t_2-t_1}((T^{s,0}_{t_1-t}(Z_s),(T^{s,0}_{t_1-t}(Z_s))^{X,j_1},v_{s+1})'_{j_1,s+1}))^c}\\
&\times\Big[\omega_2\cdot\big(v_{s+2}-\Big(T^{s+1,0}_{t_2-t_1}\big(\big(T^{s,0}_{t_1-t}(Z_s),\big(T^{s,0}_{t_1-t}(Z_s)\big)^{X,j_1},v_{s+1}\big)'_{j_1,s+1}\big)\Big)^{V,j_2}\Big]_+\\
&\times\Bigg[f^{(s+2)}_0\Big(T^{s+2,0}_{-t_2}\Bigg(\Bigg(T^{s+1,0}_{t_2-t_1}\Big(\Big(T^{s,0}_{t_1-t}(Z_s),\big(T^{s,0}_{t_1-t}(Z_s)\big)^{X,j_1},v_{s+1}\Big)'_{j_1,s+1},\\
&\qquad\Big(T^{s+1,0}_{t_2-t_1}\Big(\Big(T^{s,0}_{t_1-t}(Z_s),\big(T^{s,0}_{t_1-t}(Z_s)\big)^{X,j_1},v_{s+1}\Big)'_{j_1,s+1}\Big)\Big)^{X,j_2},v_{s+2}\Bigg)'_{j_2,s+2}\Bigg)\Bigg)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\mathrm{d}\omega_2\,\mathrm{d}v_{s+2}\,\mathrm{d}t_2\,\mathrm{d}\omega_1\,\mathrm{d}v_{s+1}\,\mathrm{d}t_1.
\end{aligned}$$

One sees that the dependency of U_{j_2} on its variable has to be regular enough so that the integrand is at least measurable. Nevertheless, this regularity is mandatory, even if one does not consider iterations of the truncated operator, because one does not consider only the number :

$$\mathcal{I}^{0,\delta}_{\substack{s\\\pm,j}}\big(U,\mathbb{S}^{d-1}\times\mathbb{R}^d\big)f^{(s+1)}(t,Z_s)=\mathcal{I}^{0,\delta}_{\substack{s\\\pm,j}}(U)f^{(s+1)}(t,Z_s),$$

for t and Z_s, but of course a function of those variables. So if one desires to consider at least measurable functions, one is led to assume that

$$\bigcup_{Z_s\in(\overline{\Omega^c}\times\mathbb{R}^d)^s}\big(\{Z_s\}\times U(Z_s)\big)$$

is also measurable.

Besides the question of the functional setting has to be discussed a bit. For the Boltzmann hierarchy, one worked with continuous functions. But as soon as a cut-off in high energy and small time difference between adjunctions has been performed, the continuity with respect to the phase space variable on the one hand, and to time on the other hand, have been lost. However, this is not an object of serious concern here : it is not mandatory anymore to work in some Banach space for the cut-offs (it was important before for the fixed point argument).

The only sufficient hypotheses here, in order to iterate the integrated in time collision-transport operator, will be that the subsets removed for the cut-off

will be given by measurable functions, as the integrand, that is f composed with pseudo-trajectories (the measurability of f composed with the pseudo-trajectories is obtained in the case of the Boltzmann hierarchy thanks to the continuity of f). In those conditions, the iterations of the operators after surgery are meaningful, defined as an usual Lebesgue integral.

Definition 56 (Iterated truncated in adjunction parameters (time, velocity and angular parameter), integrated in time collision-transport operator of the Boltzmann hierarchy)**.** *Let k be a positive integer, $J_k = (j_1, J_{k-1})$ be an element of $\mathfrak{J}_k^s$ and $M_k = (\pm_1, M_{k-1})$ be an element of $\mathfrak{M}_k$.*
For any measurable function $f^{(s+k)} : [0,T] \times \left(\overline{\Omega^c} \times \mathbb{R}^d\right)^{s+k} \to \mathbb{R}$ such that $f^{(s+k)}(t,\cdot) \in X_{0,s+k,\beta}$ for any $t \in [0,T]$, for any family of measurable subsets

$$U_{J_k} = (U_{j_1}, \ldots, U_{j_k}) = (U_{j_1}, U_{J_{k-1}})$$

of $[0, t-\delta]^k$ and any family of measurable subsets

$$E_{J_k} = (E_{j_1}, \ldots, E_{j_k}) = (E_{j_1}, E_{J_{k-1}})$$

of $\left(\mathbb{S}^{d-1} \times \mathbb{R}^d\right)^k$, one denotes the iterated, truncated in adjunction parameters, integrated in time collision-transport operator of the Boltzmann hierarchy *of type (M_k, J_k) the function, defined thanks to the previous Definition 55 and by recursion :*

$$\mathbb{1}_{t \geq k\delta} \int_{(k-1)\delta}^{t-\delta} \mathbb{1}_{U_{j_1}(Z_s)} \int_{\mathbb{S}_\omega^{d-1} \times \mathbb{R}^d_{v_{s+1}}} \mathbb{1}_{E_{j_1}(T^{s,0}_{t_1-t}(Z_s))} \left[\omega_1 \cdot \left(v_{s+1} - \left(T^{s,0}_{u-t}(Z_s)\right)^{V,j}\right)\right]_+$$
$$\times \left(\mathcal{I}^{0,\delta}_{\substack{s+1,s+k-1\\ M_{k-1},J_{k-1}}}(U_{J_{k-1}}, E_{J_{k-1}}) f^{(s+k)}\right)(t_1, \left(T^{s,0}_{t_1-t}(Z_s), x_j, v_{s+1}\right)'_{j,s+1}) \, \mathrm{d}\omega \, \mathrm{d}v_{s+1} \, \mathrm{d}u$$

if $\pm = +$, and :

$$\mathbb{1}_{t \geq k\delta} \int_{(k-1)\delta}^{t-\delta} \mathbb{1}_{U_{j_1}(Z_s)} \int_{\mathbb{S}_\omega^{d-1} \times \mathbb{R}^d_{v_{s+1}}} \mathbb{1}_{E_{j_1}(T^{s,0}_{u-t}(Z_s))} \left[\omega \cdot \left(v_{s+1} - \left(T^{s,0}_{u-t}(Z_s)\right)^{V,j}\right)\right]_-$$
$$\times \left(\mathcal{I}^{0,\delta}_{\substack{s+1s+k-1\\ M_{k-1},J_{k-1}}}(U_{J_{k-1}}, E_{J_{k-1}}) f^{(s+1)}(u, Z_s, x_j, v_{s+1})\right) \mathrm{d}\omega \, \mathrm{d}v_{s+1} \, \mathrm{d}u$$

if $\pm = -$, as :

$$\mathcal{I}^{0,\delta}_{\substack{s,s+k-1\\ (M_k,J_k)}}(U_{J_k}, E_{J_k}) f^{(s+k)}. \tag{12.120}$$

One recalls that, formally, one wants to perform the following decomposition :

$$\mathcal{I}^{0,\delta}_{\substack{s,s+k-1\\ (M_k,J_k)}} f^{(s+k)} = \mathcal{I}^{0,\delta}_{\substack{s,s+k-1\\ (M_k,J_k)}}\left(U^c_{J_k}, E^c_{J_k}\right) f^{(s+k)} + \text{remainder terms},$$

where one has denoted :

$$U^c_{J_k} = \big(U^c_{j_1}, \dots, U^c_{j_k}\big)$$

and

$$E^c_{J_k} = \big(E^c_{j_1}, \dots, E^c_{j_k}\big),$$

with the families of measurable subsets :

$$U_{J_k} = \big(U_{j_1}, \dots, U_{j_k}\big),$$

and

$$E_{J_k} = \big(E_{j_1}, \dots, E_{j_k}\big)$$

composed with only small measure respective subsets of the time interval for the U, and of $\mathbb{S}^{d-1} \times \mathbb{R}^d$ for the E.
One can therefore state the two following lemmas, which will respectively control the two types of remainder terms, namely :

$$\mathcal{I}^{0,\delta}_{\substack{s,s+k-1\\ M_k,J_k}}\big(U^c_{J_k}, E^c_{J_k}\big) \circ \mathcal{I}^{0,\delta}_{\substack{s+k\\ \pm_{k+1},j_{k+1}}}(U_{j_{k+1}})$$

and

$$\mathcal{I}^{0,\delta}_{\substack{s,s+k-1\\ M_k,J_k}}\big(U^c_{J_k}, E^c_{J_k}\big) \circ \mathcal{I}^{0,\delta}_{\substack{s+k\\ \pm_{k+1},j_{k+1}}}(U^c_{j_{k+1}}, E_{j_{k+1}}).$$

Surgery in time domain of the integrated in time collision-transport operator for the Boltzmann hierarchy

Lemma 32 (Surgery lemma in the time domain of the iterated, integrated in time collision-transport operator of the Boltzmann hierarchy)**.** *Let s be a positive integer, k be a nonnegative integer, β_0 be a strictly positive number, and μ_0 be a real number.*
Let U_{j_l} be a function of measurable subsets

$$U_{j_l} : Z_{s+l-1} \mapsto U_{j_l}(Z_{s+l-1})$$

of $[(k+p-l)\delta, t-\delta]$, with $1 \le l \le k+1$ and $1 \le j_l \le s+l-1$, such that

$$\bigcup_{Z_{s+l-1} \in (\overline{\Omega^c} \times \mathbb{R}^d)^{s+l-1}} \big(\{Z_{s+l-1}\} \times U_{j_l}(Z_{s+l-1})\big)$$

is measurable and $U_{j_{s+1}}$ has a measure uniformly bounded in Z_{s+k} and in j_{s+1} (by a constant denoted $\big|U\big|$) and let E_{j_l} be a function of measurable subsets

$$E_{j_l} : Z_{s+l-1} \mapsto E_{j_l}(Z_{s+l-1})$$

of $\mathbb{S}^{d-1} \times \mathbb{R}^d$, with $1 \le l \le k$ and $1 \le j_l \le s+l-1$, such that

$$\bigcup_{Z_{s+l-1} \in (\overline{\Omega^c} \times \mathbb{R}^d)^{s+l-1}} \big(\{Z_{s+l-1}\} \times E_{j_l}(Z_{s+l-1})\big)$$

is measurable.
Then there exist two strictly positive real numbers T and λ such that :

$$\beta_0 - \lambda T > 0,$$

and such that for any sequence of regulated functions $\big(f^{(s)}\big)_{s\geq 1}$, with for every $s \geq 1$,

$$f^{(s)} : [0,T] \times \big(\overline{\Omega^c} \times \mathbb{R}^d\big)^s \to \mathbb{R},$$

and with $\big(f^{(s)}\big)_{s\geq 1}$ having a finite $|||\cdot|||_{0,\widetilde{\beta}_\lambda,\widetilde{\mu}^1_\lambda}$ norm, one has that for all $t \in [0,T]$ and any strictly positive number $R > 0$:

$$\Big| \sum_{M_{k+1}} \sum_{J_{k+1}} \prod_{l=1}^{k+1} (\pm_l) \mathcal{I}^{0,\delta}_{\substack{s,s+k-1\\ M_k,J_k}}\big((U_{J_k})^c, E^c_{J_k}\big) \circ \mathcal{I}^{0,\delta}_{\substack{s+k\\ \pm_{k+1},j_{k+1}}} (U_{j_{k+1}})\Big(\mathbb{1}_{|V_{s+k+1}|\leq R} f^{(s+k+1)}\Big)(t,\cdot)\Big|_{0,s,\widetilde{\beta}_\lambda(t)} \leq C(d,\beta_0,\mu_0)\frac{(s+k)}{2^k} R|U| \Big|\Big|\Big|\big(f^{(s)}\big)_{s\geq 1}\Big|\Big|\Big|_{0,\widetilde{\beta}_\lambda,\widetilde{\mu}^1_\lambda} \tag{12.121}$$

with :

$$C(d,\beta_0,\mu_0) = \sqrt{2}\big|\mathbb{S}^{d-1}\big| \exp\big(-\widetilde{\mu}_\lambda(T)\big)\Big(\int_{\mathbb{R}^d} \exp\Big(-\frac{\widetilde{\beta}_\lambda(T)}{2}|v|^2\Big)\,\mathrm{d}v\Big).$$

Remark 38. *One recalls, according to the proof of Lemma 24 page 257, that the two numbers T and λ are functions of the parameters β_0 and μ_0 (they are chosen so that the integrated in time collision-transport operator is a contracting mapping on the space $\widetilde{\boldsymbol{X}}_{0,\widetilde{\beta},\widetilde{\mu}}$), so the notation for the constant involved in the last inequality of the lemma, and in particular, the dependency stated with respect to β_0 and μ_0 only, is clear now.*
One notes moreover that the control of the remainder is only in the $|\cdot|_{0,s,\widetilde{\beta}_\lambda(t)}$ norm, uniformly in t, and not in the $|||\cdot|||_{0,\widetilde{\beta}_\lambda,\widetilde{\mu}_\lambda}$ norm. The limitation is the same that when one tried to cut-off the solutions in small time difference between the adjunctions.

Proof. Writing the definition of the iterated, truncated in adjunction parameters, integrated in time, transport-collision operator

$$\mathcal{I}^{0,\delta}_{\substack{s,s+k-1\\ M_k,J_k}}\big(U^c_{J_k}, E^c_{J_k}\big) \circ \mathcal{I}^{0,\delta}_{\substack{s+k\\ \pm_{k+1},j_{k+1}}} (U_{j_{k+1}})\Big(\mathbb{1}_{|V_{s+k+1}|\leq R} f^{(s+k+1)}\Big),$$

one has, using the notations for the pseudo-trajectories introduced in Definitions

45 page 323 and 47 page 325, that the quantity

$$\Bigg| \sum_{M_{k+1}} \sum_{J_{k+1}} \prod_{l=1}^{k+1} (\pm_l) \mathcal{I}^{0,\delta}_{s,s+k-1\,M_k,J_k}\big(U^c_{J_k}, E^c_{J_k}\big) \circ \mathcal{I}^{0,\delta}_{s+k\,\pm_{k+1},j_{k+1}}(U_{j_{k+1}})\Big(\mathbb{1}_{|V_{s+k+1}|\le R} f^{(s+k+1)}\Big)\Bigg|(t, Z_s)$$

can be rewritten as :

$$\begin{aligned}
&\Bigg| \sum_{M_{k+1}} \sum_{J_{k+1}} \prod_{l=1}^{k+1} (\pm_l) \mathcal{I}^{0,\delta}_{s,s+k-1\,M_k,J_k}\big(U^c_{J_k}, E^c_{J_k}\big) \circ \mathcal{I}^{0,\delta}_{s+k\,\pm_{k+1},j_{k+1}}(U_{j_{k+1}})\Big(\mathbb{1}_{|V_{s+k+1}|\le R} f^{(s+k+1)}\Big)\Bigg|(t, Z_s)\\
&= \mathbb{1}_{t\ge (k+p)\delta}\Bigg| \int_{(k+p-1)\delta}^{t-\delta} \mathbb{1}_{U_{j_1}(Z_s)} \int_{\mathbb{S}^{d-1}_{\omega_1}\times\mathbb{R}^d_{v_{s+1}}} \sum_{\pm_1}\sum_{j_1=1}^{s} (\pm_1)\mathbb{1}_{E^c_{j_1}(Z^0_{s,0}(M_{k+1},J_{k+1},t_1))}\\
&\qquad\qquad \times \Big[\omega_1\cdot\big(v_{s+1} - v^{0,j_1}_{s,0}(M_{k+1},J_{k+1},t_1)\big)\Big]_{\pm_1}\\
&\quad \times \mathbb{1}_{t_1\ge (k+p-1)\delta} \int_{(k+p-2)\delta}^{t_1-\delta} \mathbb{1}_{U_{j_2}(Z^0_{s,1}(M_{k+1},J_{k+1},t_1))} \int_{\mathbb{S}^{d-1}_{\omega_2}\times\mathbb{R}^d_{v_{s+2}}} \sum_{\pm_2}\sum_{j_2=1}^{s+1} (\pm_2)\\
&\qquad\qquad \times \mathbb{1}_{E^c_{j_2}(Z^0_{s,1}(M_{k+1},J_{k+1},t_2))}\Big[\omega_2\cdot\big(v_{s+2} - v^{0,j_2}_{s,1}(M_{k+1},J_{k+1},t_2)\big)\Big]_{\pm_2}\\
&\qquad \dots\\
&\times \mathbb{1}_{t_{k-1}\ge (p+1)\delta} \int_{p\delta}^{t_{k-1}-\delta} \mathbb{1}_{U_{j_k}(Z^0_{s,k-1}(M_{k+1},J_{k+1},t_{k-1}))} \int_{\mathbb{S}^{d-1}_{\omega_k}\times\mathbb{R}^d_{v_{s+k}}} \sum_{\pm_k}\sum_{j_k=1}^{s+k-1} (\pm_k)\\
&\qquad\qquad \times \mathbb{1}_{E^c_{j_k}(Z^0_{s,k-1}(M_{k+1},J_{k+1},t_k))}\Big[\omega_k\cdot\big(v_{s+k} - v^{0,j_k}_{s,k-1}(M_{k+1},J_{k+1},t_k)\big)\Big]_{\pm_k}\\
&\quad \times \mathbb{1}_{t_k\ge p\delta} \int_{(p-1)\delta}^{t_k-\delta} \mathbb{1}_{U_{j_{k+1}}(Z^0_{s,k}(M_{k+1},J_{k+1},t_k))} \int_{\mathbb{S}^{d-1}_{\omega_{k+1}}\times\mathbb{R}^d_{v_{s+k+1}}} \sum_{\pm_{k+1}}\sum_{j_{k+1}=1}^{s+k} (\pm_{k+1})\\
&\qquad\qquad \times \Big[\omega_{k+1}\cdot\big(v_{s+k+1} - v^{0,j_{k+1}}_{s,k}(M_{k+1},J_{k+1},t_{k+1})\big)\Big]_{\pm_{k+1}}\\
&\qquad\qquad\qquad \times \mathbb{1}_{|V_{s+k+1}|\le R} f^{(s+k+1)}\big(t_{k+1}, Z^0_{s,k+1}(M_{k+1},J_{k+1},0)\big)\\
&\qquad\qquad \mathrm{d}\omega_{k+1}\,\mathrm{d}v_{s+k+1}\,\mathrm{d}t_{k+1}\,\mathrm{d}\omega_k\,\mathrm{d}v_{s+k}\,\mathrm{d}t_k\dots\mathrm{d}\omega_2\,\mathrm{d}v_{s+2}\,\mathrm{d}t_2\,\mathrm{d}\omega_1\,\mathrm{d}v_{s+1}\,\mathrm{d}t_1\Bigg|.
\end{aligned}$$

One bounds crudely this quantity thanks to (a massive use of) the triangular

inequality and the removal of most of the indicator functions, so that one obtains

$$\Big|\sum_{M_{k+1}}\sum_{J_{k+1}}\prod_{l=1}^{k+1}(\pm_l)\mathcal{I}^{0,\delta}_{s,s+k-1}\big(U^c_{J_k},E^c_{J_k}\big)$$
$$\circ\,\mathcal{I}^{0,\delta}_{\substack{s+k\\ \pm_{k+1},j_{k+1}}}(U_{j_{k+1}})\Big(\mathbb{1}_{|V_{s+k+1}|\le R}f^{(s+k+1)}\Big)\Big|(t,Z_s)$$
$$\le\int_0^t\int_{\mathbb{S}^{d-1}_{\omega_1}\times\mathbb{R}^d_{v_{s+1}}}\sum_{\pm_1}\sum_{j_1=1}^{s}\Big[\omega_1\cdot\big(v_{s+1}-v^{0,j_1}_{s,0}(M_{k+1},J_{k+1},t_1)\big)\Big]_{\pm_1}$$
$$\times\int_0^{t_1}\int_{\mathbb{S}^{d-1}_{\omega_2}\times\mathbb{R}^d_{v_{s+2}}}\sum_{\pm_2}\sum_{j_2=1}^{s+1}\Big[\omega_2\cdot\big(v_{s+2}-v^{0,j_2}_{s,1}(M_{k+1},J_{k+1},t_2)\big)\Big]_{\pm_2}$$
$$\dots$$
$$\times\int_0^{t_{k-1}}\int_{\mathbb{S}^{d-1}_{\omega_k}\times\mathbb{R}^d_{v_{s+k}}}\sum_{\pm_k}\sum_{j_k=1}^{s+k-1}\Big[\omega_k\cdot\big(v_{s+k}-v^{0,j_k}_{s,k-1}(M_{k+1},J_{k+1},t_k)\big)\Big]_{\pm_k}$$
$$\times\int_0^{t_k}\mathbb{1}_{U_{j_{k+1}}(Z^0_{s,k}(M_{k+1},J_{k+1},t_k))}\int_{\mathbb{S}^{d-1}_{\omega_{k+1}}\times\mathbb{R}^d_{v_{s+k+1}}}\sum_{\pm_{k+1}}\sum_{j_{k+1}=1}^{s+k}$$
$$\times\Big[\omega_{k+1}\cdot\big(v_{s+k+1}-v^{0,j_{k+1}}_{s,k}(M_{k+1},J_{k+1},t_{k+1})\big)\Big]_{\pm_{k+1}}$$
$$\times\,\mathbb{1}_{|V_{s+k+1}|\le R}\Big|f^{(s+k+1)}\big(t_{k+1},Z^0_{s,k+1}(M_{k+1},J_{k+1},t_{k+1})\big)\Big|$$
$$\mathrm{d}\omega_{k+1}\,\mathrm{d}v_{s+k+1}\,\mathrm{d}t_{k+1}\,\mathrm{d}\omega_k\,\mathrm{d}v_{s+k}\,\mathrm{d}t_k\dots\mathrm{d}\omega_2\,\mathrm{d}v_{s+2}\,\mathrm{d}t_2\,\mathrm{d}\omega_1\,\mathrm{d}v_{s+1}\,\mathrm{d}t_1,$$

and then, using the boundedness of $f^{(s+k+1)}(u,\cdot)$ in the $|\cdot|_{0,s+k+1,\widetilde{\beta}_\lambda(u)}$ norm, and the boundedness of $\big(f^{(s)}\big)_{s\ge1}$ in the $|||\cdot|||_{0,\widetilde{\beta}_\lambda,\widetilde{\mu}^1_\lambda}$ norm, that is, one recalls :

$$\big|f^{(s+k+1)}(t_{k+1},V_{s+k+1})\big|\le\Big|\Big|\Big|\big(f^{(s)}\big)_{s\ge1}\Big|\Big|\Big|_{0,\widetilde{\beta}_\lambda,\widetilde{\mu}^1_\lambda}\exp\big(-(s+k+1)\widetilde{\mu}_\lambda(t_{k+1})\big)$$
$$\times\exp\Big(-\frac{\widetilde{\beta}_\lambda(t_{k+1})}{2}\big|V_{s+k+1}\big|^2\Big),$$

so that the quantity

$$\Big|\sum_{M_{k+1}}\sum_{J_{k+1}}\prod_{l=1}^{k+1}(\pm_l)\mathcal{I}^{0,\delta}_{s,s+k-1}\big(U^c_{J_k},E^c_{J_k}\big)$$
$$\circ\,\mathcal{I}^{0,\delta}_{\substack{s+k\\ \pm_{k+1},j_{k+1}}}(U_{j_{k+1}})\Big(\mathbb{1}_{|V_{s+k+1}|\le R}f^{(s+k+1)}\Big)\Big|(t,Z_s)\exp\Big(\frac{\beta_\lambda(t)}{2}|Z_s|^2\Big)$$

can be controlled in the following way.

One has

$$
\begin{aligned}
&\Bigg| \sum_{M_{k+1}} \sum_{J_{k+1}} \prod_{l=1}^{k+1} (\pm_l) \mathcal{I}^{0,\delta}_{s,s+k-1,\,M_k,J_k}\big(U^c_{J_k}, E^c_{J_k}\big) \\
&\qquad \circ \mathcal{I}^{0,\delta}_{s+k,\,\pm_{k+1},j_{k+1}}(U_{j_{k+1}})\Big(\mathbb{1}_{|V_{s+k+1}|\le R} f^{(s+k+1)}\Big)\Bigg|(t, Z_s)\exp\Big(\frac{\beta_\lambda(t)}{2}|Z_s|^2\Big) \\
\le\ & \int_0^t \int_{\mathbb{S}^{d-1}_{\omega_1}\times\mathbb{R}^d_{v_{s+1}}} \sum_{\pm_1}\sum_{j_1=1}^{s} \Big[\omega_1\cdot\big(v_{s+1} - v^{0,j_1}_{s,0}(M_{k+1},J_{k+1},t_1)\big)\Big]_{\pm_1} \\
&\times \int_0^{t_1} \int_{\mathbb{S}^{d-1}_{\omega_2}\times\mathbb{R}^d_{v_{s+2}}} \sum_{\pm_2}\sum_{j_2=1}^{s+1} \Big[\omega_2\cdot\big(v_{s+2} - v^{0,j_2}_{s,1}(M_{k+1},J_{k+1},t_2)\big)\Big]_{\pm_2} \\
&\quad \dots \\
&\times \int_0^{t_{k-1}} \int_{\mathbb{S}^{d-1}_{\omega_k}\times\mathbb{R}^d_{v_{s+k}}} \sum_{\pm_k}\sum_{j_k=1}^{s+k-1} \Big[\omega_k\cdot\big(v_{s+k} - v^{0,j_k}_{s,k-1}(M_{k+1},J_{k+1},t_k)\big)\Big]_{\pm_k} \\
&\times \int_0^{t_k} \mathbb{1}_{U_{j_{k+1}}(Z^0_{s,k}(M_{k+1},J_{k+1},t_k))} \int_{\mathbb{S}^{d-1}_{\omega_{k+1}}\times\mathbb{R}^d_{v_{s+k+1}}} \sum_{\pm_{k+1}}\sum_{j_{k+1}=1}^{s+k} \\
&\qquad\qquad \times \Big[\omega_{k+1}\cdot\big(v_{s+k+1} - v^{0,j_{k+1}}_{s,k}(M_{k+1},J_{k+1},t_{k+1})\big)\Big]_{\pm_{k+1}} \\
&\quad \times \mathbb{1}_{|V_{s+k+1}|\le R} \Big|\Big|\Big|\big(f^{(s)}\big)_{s\ge1}\Big|\Big|\Big|_{0,\widetilde{\beta}_\lambda,\widetilde{\mu}^1_\lambda} \exp\big(-(s+k+1)\widetilde{\mu}_\lambda(t_{k+1})\big) \\
&\qquad\qquad \times \exp\Big(-\frac{\widetilde{\beta}_\lambda(t_{k+1})}{2}\big|V^0_{s,k+1}\big|^2\Big)\,\mathrm{d}\omega_{k+1}\,\mathrm{d}v_{s+k+1}\,\mathrm{d}t_{k+1} \\
&\qquad\qquad\qquad \mathrm{d}\omega_k\,\mathrm{d}v_{s+k}\,\mathrm{d}t_k\dots\mathrm{d}\omega_2\,\mathrm{d}v_{s+2}\,\mathrm{d}t_2\,\mathrm{d}\omega_1\,\mathrm{d}v_{s+1}\,\mathrm{d}t_1.
\end{aligned}
$$

Now one bounds the two last integrals

$$
\begin{aligned}
&\int_0^{t_k} \mathbb{1}_{U_{j_{k+1}}(Z^0_{s,k}(M_{k+1},J_{k+1},t_k))} \int_{\mathbb{S}^{d-1}_{\omega_{k+1}}\times\mathbb{R}^d_{v_{s+k+1}}} \sum_{\pm_{k+1}}\sum_{j_{k+1}=1}^{s+k} \\
&\qquad \times \Big[\omega_{k+1}\cdot\big(v_{s+k+1} - v^{0,j_{k+1}}_{s,k}(M_{k+1},J_{k+1},t_{k+1})\big)\Big]_{\pm_{k+1}} \mathbb{1}_{|V_{s+k+1}|\le R} \\
&\times \exp\big(-(s+k+1)\widetilde{\mu}_\lambda(t_{k+1})\big)\exp\Big(-\frac{\widetilde{\beta}_\lambda(t_{k+1})}{2}\big|V^0_{s,k+1}(M_{k+1},J_{k+1},t_{k+1})\big|^2\Big) \\
&\qquad\qquad \mathrm{d}\omega_{k+1}\,\mathrm{d}v_{s+k+1}\,\mathrm{d}t_{k+1}
\end{aligned}
$$

as follows :

$$\begin{aligned}
&\int_0^{t_k} \mathbb{1}_{U_{j_{k+1}}(Z^0_{s,k}(M_{k+1},J_{k+1},t_k))} \int_{\mathbb{S}^{d-1}_{\omega_{k+1}}\times\mathbb{R}^d_{v_{s+k+1}}} \sum_{\pm_{k+1}} \sum_{j_{k+1}=1}^{s+k} \\
&\qquad\qquad \times \Big[\omega_{k+1}\cdot\big(v_{s+k+1} - v^{0,j_{k+1}}_{s,k}(M_{k+1},J_{k+1},t_{k+1})\big)\Big]_{\pm_{k+1}} \mathbb{1}_{|V_{s+k+1}|\le R} \\
&\times \exp\big(-(s+k+1)\widetilde{\mu}_\lambda(t_{k+1})\big)\exp\Big(-\frac{\widetilde{\beta}_\lambda(t_{k+1})}{2}\big|V^0_{s,k+1}(M_{k+1},J_{k+1},t_{k+1})\big|^2\Big) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \mathrm{d}\omega_{k+1}\,\mathrm{d}v_{s+k+1}\,\mathrm{d}t_{k+1} \\
&\le \exp\big(-(s+k)\widetilde{\mu}_\lambda(t_k)\big)\exp\big(-\widetilde{\mu}_\lambda(T)\big)\int_0^{t_k} \mathbb{1}_{U_{s+k}(Z^0_{s,k}(M_{k+1},J_{k+1},t_k))} \\
&\quad\times \int_{\mathbb{S}^{d-1}_{\omega_{k+1}}\times\mathbb{R}^d_{v_{s+k+1}}} \mathbb{1}_{(\pm_{k+1})\omega_{k+1}\cdot\left(v_{s+k+1}-v^{0,j_{k+1}}_{s,k}(M_{k+1},J_{k+1},t_{k+1})\right)\ge 0}\mathbb{1}_{|V_{s+k+1}|\le R} \\
&\quad\times \sum_{\pm_{k+1}}\sum_{j_{k+1}=1}^{s+k}\Big(\big|v_{s+k+1}\big| + \big|v^{0,j_{k+1}}_{s,k}(M_{k+1},J_{k+1},t_{k+1})\big|\Big) \\
&\qquad\times \exp\Big(-\frac{\widetilde{\beta}_\lambda(t_{k+1})}{2}\big|V^0_{s,k+1}(M_{k+1},J_{k+1},t_{k+1})\big|^2\Big)\,\mathrm{d}\omega_{k+1}\,\mathrm{d}v_{s+k+1}\,\mathrm{d}t_{k+1}
\end{aligned}$$

since the function $\widetilde{\mu}_\lambda$ is decreasing. On the other hand, since one has, first thanks to the very definition of the pseudo-trajectories (see Definition 45 page 323), and second due to the conservation of the kinetic energy by the free transport and the scattering :

$$\begin{aligned}
\big|V^0_{s,k+1}(M_{k+1},J_{k+1},t_{k+1})\big|^2 &= \big|V^0_{s,k}(M_{k+1},J_{k+1},t_{k+1})\big|^2 + \big|v_{s+k+1}\big|^2 \\
&= \big|V^0_{s,k}(M_{k+1},J_{k+1},t_k)\big|^2 + \big|v_{s+k+1}\big|^2 \\
&= \overbrace{\big|V^0_{s,k-1}(M_{k+1},J_{k+1},t_{k-1})\big|^2 + \big|v_{s+k}\big|^2}+\big|v_{s+k+1}\big|^2,
\end{aligned}$$

and so on, so that one has in the end :

$$\big|V^0_{s,k+1}(M_{k+1},J_{k+1},t_{k+1})\big|^2 = \big|V_s\big|^2 + \sum_{l=1}^{k+1}\big|v_{s+l}\big|^2.$$

One deduces then in particular that

$$\Big(\big|v_{s+k+1}\big| + \big|v^{0,j_{k+1}}_{s,k}(M_{k+1},J_{k+1},t_{k+1})\big|\Big)\mathbb{1}_{|V_{s+k+1}|\le R} \le \sqrt{2}R.$$

One finds therefore, since the function $\widetilde{\beta}_\lambda$ is also decreasing :

$$
\begin{aligned}
&\exp\big(-(s+k)\widetilde{\mu}_\lambda(t_k)\big)\exp\big(-\widetilde{\mu}_\lambda(T)\big)\int_0^{t_k}\mathbb{1}_{U_{s+k}(Z^0_{s,k}(M_{k+1},J_{k+1},t_k))}\\
&\quad\times\int_{\mathbb{S}^{d-1}_{\omega_{k+1}}\times\mathbb{R}^d_{v_{s+k+1}}}\mathbb{1}_{(\pm_{k+1})\omega_{k+1}\cdot\left(v_{s+k+1}-v^{0,j_{k+1}}_{s,k}(M_{k+1},J_{k+1},t_{k+1})\right)\geq 0}\mathbb{1}_{|V_{s+k+1}|\leq R}\\
&\quad\times\sum_{\pm_{k+1}}\sum_{j_{k+1}=1}^{s+k}\Big(\big|v_{s+k+1}\big|+\big|v^{0,j_{k+1}}_{s,k}(M_{k+1},J_{k+1},t_{k+1})\big|\Big)\\
&\qquad\times\exp\Big(-\frac{\widetilde{\beta}_\lambda(t_{k+1})}{2}\big|V^0_{s,k+1}(M_{k+1},J_{k+1},t_{k+1})\big|^2\Big)\,\mathrm{d}\omega_{k+1}\,\mathrm{d}v_{s+k+1}\,\mathrm{d}t_{k+1}\\
&\leq\exp\big(-(s+k)\widetilde{\mu}_\lambda(t_k)\big)\exp\big(-\widetilde{\mu}_\lambda(T)\big)\int_0^{t_k}\mathbb{1}_{U_{s+k}(Z^0_{s,k}(M_{k+1},J_{k+1},t_k))}\\
&\qquad\qquad\times\exp\Big(-\frac{\widetilde{\beta}_\lambda(t_k)}{2}\big(\big|V_s\big|^2+\sum_{l=1}^k\big|v_{s+l}\big|^2\big)\Big)\\
&\qquad\times\int_{\mathbb{S}^{d-1}_{\omega_{k+1}}\times\mathbb{R}^d_{v_{s+k+1}}}\sqrt{2}R\;\mathbb{1}_{(\pm_{k+1})\omega_{k+1}\cdot\left(v_{s+k+1}-v^{0,j_{k+1}}_{s,k}(M_{k+1},J_{k+1},t_{k+1})\right)\geq 0}\\
&\qquad\quad\times\sum_{\pm_{k+1}}\sum_{j_{k+1}=1}^{s+k}\exp\Big(-\frac{\widetilde{\beta}_\lambda(T)}{2}\big|v_{s+k+1}\big|^2\Big)\,\mathrm{d}\omega_{k+1}\,\mathrm{d}v_{s+k+1}\,\mathrm{d}t_{k+1},
\end{aligned}
$$

the right-hand side being in fact equal to :

$$
\begin{aligned}
&\sqrt{2}\big|\mathbb{S}^{d-1}\big|\exp\big(-\widetilde{\mu}_\lambda(T)\big)\Big(\int_{\mathbb{R}^d}\exp\Big(-\frac{\widetilde{\beta}_\lambda(T)}{2}\big|v_{s+k+1}\big|^2\Big)\,\mathrm{d}v_{s+k+1}\Big)\\
&\qquad\times\exp\big(-(s+k)\widetilde{\mu}_\lambda(t_k)\big)\exp\Big(-\frac{\widetilde{\beta}_\lambda(t_k)}{2}\big(\big|V_s\big|^2+\sum_{l=1}^k\big|v_{s+l}\big|^2\big)\Big)\\
&\qquad\qquad\times(s+k)R\big|U_{s+k}\big|.
\end{aligned}
$$

One sees finally that the quantity :

$$\int_0^t \int_{\mathbb{S}^{d-1}_{\omega_1}\times\mathbb{R}^d_{v_{s+1}}} \sum_{\pm_1}\sum_{j_1=1}^{s} \Big[\omega_1\cdot\big(v_{s+1}-v^{0,j_1}_{s,0}(M_{k+1},J_{k+1},t_1)\big)\Big]_{\pm_1}$$
$$\times\int_0^{t_1} \int_{\mathbb{S}^{d-1}_{\omega_2}\times\mathbb{R}^d_{v_{s+2}}} \sum_{\pm_2}\sum_{j_2=1}^{s+1} \Big[\omega_2\cdot\big(v_{s+2}-v^{0,j_2}_{s,1}(M_{k+1},J_{k+1},t_2)\big)\Big]_{\pm_2}$$
$$\dots$$
$$\times\int_0^{t_{k-1}} \int_{\mathbb{S}^{d-1}_{\omega_k}\times\mathbb{R}^d_{v_{s+k}}} \sum_{\pm_k}\sum_{j_k=1}^{s+k-1} \Big[\omega_k\cdot\big(v_{s+k}-v^{0,j_k}_{s,k-1}(M_{k+1},J_{k+1},t_k)\big)\Big]_{\pm_k}$$
$$\times\exp\big(-(s+k)\widetilde{\mu}_\lambda(t_k)\big)\exp\Big(-\frac{\widetilde{\beta}_\lambda(t_k)}{2}\big(|V_s|^2+\sum_{l=1}^{k}|v_{s+l}|^2\big)\Big)\,\mathrm{d}\omega_k\,\mathrm{d}v_{s+k}\,\mathrm{d}t_k$$
$$\dots\mathrm{d}\omega_2\,\mathrm{d}v_{s+2}\,\mathrm{d}t_2\,\mathrm{d}\omega_1\,\mathrm{d}v_{s+1}\,\mathrm{d}t_1$$

is nothing more than the usual iterated, integrated in time, collision-transport of the Boltzmann hierarchy, crudely bounded regardless of the possible cancellation effects between pre and post-collisional terms, applied to the function :

$$g_{s+k}:(t_k,V_{s+k})\mapsto\exp\big(-(s+k)\widetilde{\mu}_\lambda(t_k)\big)\exp\Big(-\frac{\widetilde{\beta}_\lambda(t_k)}{2}\big(|V_s|^2+\sum_{l=1}^{s+k}|v_{s+l}|^2\big)\Big),$$

with of course

$$\big(g_s\big)_{s\geq 1}\in\widetilde{\mathbf{X}}_{0,\widetilde{\beta}_\lambda,\widetilde{\mu}_\lambda}.$$

This quantity is then controlled as in Lemma 24 page 257. For β_0 and μ_0 fixed, if T and λ are carefully chosen, one recalls that it is possible to obtain an operator with a $|||\cdot|||_{0,\widetilde{\beta}_\lambda,\widetilde{\mu}_\lambda}$ norm smaller than $1/2$. One obtains therefore in the end, after multiplying by the gaussian weight in velocity :

$$\Big|\sum_{M_{k+1}}\sum_{J_{k+1}}\prod_{l=1}^{k+1}(\pm_l)\mathcal{I}^{0,\delta}_{s,s+k-1}\big(U^c_{J_k},E^c_{J_k}\big)$$
$$\circ\,\mathcal{I}^{0,\delta}_{s+k}{}_{\pm_{k+1},j_{k+1}}(U_{j_{k+1}})\Big(\mathbb{1}_{|V_{s+k+1}|\leq R}f^{(s+k+1)}\Big)\Big|(t,Z_s)\exp\Big(\frac{\beta_\lambda(t)}{2}|Z_s|^2\Big)$$
$$\leq C(d,\beta_0,\mu_0)\frac{(s+k)}{2^k}R|U_{j_{k+1}}|\Big|\Big|\Big|\big(f^{(s)}\big)_{s\geq 1}\Big|\Big|\Big|_{0,\widetilde{\beta}_\lambda,\widetilde{\mu}_\lambda}$$

with :

$$C(d,\beta_0,\mu_0)=\sqrt{2}\big|\mathbb{S}^{d-1}\big|\exp\big(-\widetilde{\mu}_\lambda(T)\big)\Big(\int_{\mathbb{R}^d}\exp\Big(-\frac{\widetilde{\beta}_\lambda(T)}{2}|v_{s+k+1}|^2\Big)\,\mathrm{d}v_{s+k+1}\Big),$$

which is the inequality stated in the lemma. □

Surgery in the angular parameter and velocity domain of the integrated in time collision-transport operator for the Boltzmann hierarchy

Lemma 33 (Surgery lemma in the angular parameter and velocity domain of the iterated, integrated in time, collision-transport operator of the Boltzmann hierarchy)**.** *Let s be a positive integer, k be a nonnegative integer, β_0 be a strictly positive number, and μ_0 be a real number.*
Let U_{j_l} be a function of measurable subsets

$$U_{j_l} : Z_{s+l-1} \mapsto U_{j_l}(Z_{s+l-1})$$

of $[(k+p-l)\delta, t-\delta]$, with $1 \leq l \leq k+1$, such that

$$\bigcup_{Z_{s+l-1} \in (\overline{\Omega^c} \times \mathbb{R}^d)^{s+l-1}} \big(\{Z_{s+l-1}\} \times U_{s+l-1}(Z_{s+l-1})\big)$$

is measurable, and let E_{j_l} be a function of measurable subsets

$$E_{j_l} : Z_{s+l-1} \mapsto E_{j_l}(Z_{s+l-1})$$

of $\mathbb{S}^{d-1} \times \mathbb{R}^d$, with $1 \leq l \leq k+1$ and $1 \leq j_l \leq s+l-1$, such that :

$$\bigcup_{Z_{s+l-1} \in (\overline{\Omega^c} \times \mathbb{R}^d)^{s+l-1}} \big(\{Z_{s+l-1}\} \times E_{j_l}(Z_{s+l-1})\big)$$

is measurable, and $E_{j_{k+1}}$ has a measure uniformly bounded in Z_{s+k} and in j_{k+1} by a constant denoted $\big|E\big|$.
Then there exist two strictly positive real numbers T and λ such that :

$$\beta_0 - \lambda T > 0,$$

and such that for any sequence of regulated functions $\big(f^{(s)}\big)_{s\geq 1}$, with for every $s \geq 1$,

$$f^{(s)} : [0,T] \times \big(\overline{\Omega^c} \times \mathbb{R}^d\big)^{s+k} \to \mathbb{R}$$

and with $\big(f^{(s)}\big)_{s\geq 1}$ having a finite $|||\cdot|||_{0,\widetilde{\beta}_\lambda,\widetilde{\mu}^1_\lambda}$ norm, one has that for all $t \in [0,T]$ and any strictly positive number $R > 0$:

$$\Bigg|\!\Bigg|\!\Bigg|\Bigg(\sum_{M_{k+1}} \sum_{J_{k+1}} \prod_{l=1}^{k+1} (\pm_l) \mathcal{I}^{0,\delta}_{\substack{s,s+k-1\\ M_k,J_k}} \big(U^c_{J_k}, E^c_{J_k}\big) \circ \mathcal{I}^{0,\delta}_{\substack{s+k\\ \pm_{k+1},j_{k+1}}} \big(U^c_{j_{k+1}}, E_{j_{k+1}}\big) \Big(\mathbb{1}_{|V_{s+k+1}|\leq R} f^{(s+k+1)}\Big)\Bigg)_{s\geq 1}\Bigg|\!\Bigg|\!\Bigg|_{0,\widetilde{\beta}_\lambda,\widetilde{\mu}^1_\lambda}$$
$$\leq C(\beta_0,\mu_0) \frac{1}{2^k} R \big|E\big| \Big|\!\Big|\!\Big|\big(f^{(s)}\big)_{s\geq 1}\Big|\!\Big|\!\Big|_{0,\widetilde{\beta}_\lambda,\widetilde{\mu}^1_\lambda} \tag{12.122}$$

with :

$$C(\beta_0,\mu_0) = \sqrt{2} \frac{\exp\big(-\widetilde{\mu}_\lambda(T)\big)}{\lambda}.$$

Proof. As for the proof of the previous lemma, one is in the end led to control the following quantity :

$$\int_0^{t_k} \mathbb{1}_{U^c_{j_{k+1}}(Z^0_{s,k}(M_{k+1},J_{k+1},t_k))} \int_{\mathbb{S}^{d-1}_{\omega_{k+1}}\times\mathbb{R}^d_{v_{s+k+1}}} \sum_{\pm_{k+1}} \sum_{j_{k+1}=1}^{s+k} \mathbb{1}_{E_{j_{k+1}}(Z^0_{s,k}(M_{k+1},J_{k+1},t_{k+1}))}$$
$$\times \Big[\omega_{k+1}\cdot\big(v_{s+k+1} - v^{0,j_{k+1}}_{s,k}(M_{k+1},J_{k+1},t_{k+1})\big)\Big]_{\pm_{k+1}} \mathbb{1}_{|V_{s+k+1}|\le R}$$
$$\times \exp\big(-(s+k+1)\widetilde{\mu}_\lambda(t_{k+1})\big)\exp\Big(-\frac{\widetilde{\beta}_\lambda(t_{k+1})}{2}\big|V^0_{s,k+1}(M_{k+1},J_{k+1},t_{k+1})\big|^2\Big)$$
$$\mathrm{d}\omega_{k+1}\,\mathrm{d}v_{s+k+1}\,\mathrm{d}t_{k+1},$$

denoted here by Q.
This time one will be able to recover a control in the $|||\cdot|||_{0,\widetilde{\beta}_\lambda,\widetilde{\mu}^1_\lambda}$ norm. One computes therefore the product of Q with the relevant weights, namely :

$$\exp\big((s+k)\widetilde{\mu}_\lambda(t_k)\big)\exp\Big(\frac{\widetilde{\beta}_\lambda(t_k)}{2}\big|V_{s+k}\big|^2\Big).$$

One finds :

$$Q\exp\big((s+k)\widetilde{\mu}_\lambda(t_k)\big)\exp\Big(\frac{\widetilde{\beta}_\lambda(t_k)}{2}\big|V_{s+k}\big|^2\Big)$$
$$= \int_0^{t_k} \mathbb{1}_{U^c_{j_{k+1}}(Z^0_{s,k}(M_{k+1},J_{k+1},t_k))} \int_{\mathbb{S}^{d-1}_{\omega_{k+1}}\times\mathbb{R}^d_{v_{s+k+1}}} \sum_{\pm_{k+1}} \sum_{j_{k+1}=1}^{s+k} \mathbb{1}_{E_{j_{k+1}}(Z^0_{s,k}(M_{k+1},J_{k+1},t_{k+1}))}$$
$$\times \Big[\omega_{k+1}\cdot\big(v_{s+k+1} - v^{0,j_{k+1}}_{s,k}(M_{k+1},J_{k+1},t_{k+1})\big)\Big]_{\pm_{k+1}} \mathbb{1}_{|V_{s+k+1}|\le R}$$
$$\times \exp\big(-(s+k+1)\widetilde{\mu}_\lambda(t_{k+1})\big)\exp\big((s+k)\widetilde{\mu}_\lambda(t_k)\big)$$
$$\times \exp\Big(-\frac{\widetilde{\beta}_\lambda(t_{k+1})}{2}\big|V^0_{s,k+1}(M_{k+1},J_{k+1},t_{k+1})\big|^2\Big)\exp\Big(\frac{\widetilde{\beta}_\lambda(t_k)}{2}\big|V_{s+k}\big|^2\Big)$$
$$\mathrm{d}\omega_{k+1}\,\mathrm{d}v_{s+k+1}\,\mathrm{d}t_{k+1},$$

and one drops immediately the indicator function in the time variable, since in the spirit of the whole work, $U_{j_{k+1}}$ is meant to be small and has already provided a control (see the previous Lemma 32), so its complement is very close to the whole time interval $[0,t_k]$, and then no control can be provided from this term. Using, as in the proof of the previous lemma, that :

$$\big(\big|v_{s+k+1}\big| + \big|v^{0,j_{k+1}}_{s,k}(M_{k+1},J_{k+1},t_{k+1})\big|\big)\mathbb{1}_{|V_{s+k+1}|\le R} \le \sqrt{2}R,$$

and :

$$\big|V^0_{s,k+1}(M_{k+1},J_{k+1},t_{k+1})\big|^2 = \big|V_s\big|^2 + \sum_{l=1}^{k+1}\big|v_{s+l}\big|^2 = \big|V_{s+k}\big|^2 + \big|v_{s+k+1}\big|^2,$$

one obtains :

$$
\begin{aligned}
&Q \exp\big((s+k)\widetilde{\mu}_\lambda(t_k)\big) \exp\Big(\frac{\widetilde{\beta}_\lambda(t_k)}{2}\big|V_{s+k}\big|^2\Big) \\
&\leq \exp\big(-\widetilde{\mu}_\lambda(T)\big) \int_0^{t_k} \exp\big((s+k)(\widetilde{\mu}_\lambda(t_k) - \widetilde{\mu}_\lambda(t_{k+1}))\big) \\
&\quad \times \int_{\mathbb{S}^{d-1}_{\omega_{k+1}} \times \mathbb{R}^d_{v_{s+k+1}}} \sqrt{2}R \sum_{j_{k+1}=1}^{s+k} \mathbb{1}_{E_{j_{k+1}}(Z^0_{s,k}(M_{k+1},J_{k+1},t_{k+1}))} \\
&\quad \times \exp\Big(\frac{\widetilde{\beta}_\lambda(t_k) - \widetilde{\beta}_\lambda(t_{k+1})}{2}\big|V_{s+k}\big|^2\Big) \exp\Big(-\frac{\widetilde{\beta}_\lambda(t_{k+1})}{2}\big|v_{s+k+1}\big|^2\Big) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \mathrm{d}\omega_{k+1}\,\mathrm{d}v_{s+k+1}\,\mathrm{d}t_{k+1}
\end{aligned}
$$

(the first sum over the two possible signs of $\pm_{k+1}$ has disappeared since one has in fact :

$$
\begin{aligned}
&\sum_{\pm_{k+1}} \mathbb{1}_{E_{j_{k+1}}(Z^0_{s,k}(M_{k+1},J_{k+1},t_{k+1}))} \\
&\qquad\qquad \times \Big[\omega_{k+1} \cdot \big(v_{s+k+1} - v^{0,j_{k+1}}_{s,k}(M_{k+1},J_{k+1},t_{k+1})\big)\Big]_{\pm_{k+1}} \\
&= \mathbb{1}_{E_{j_{k+1}}(Z^0_{s,k}(M_{k+1},J_{k+1},t_{k+1}))} \Big|\omega_{k+1} \cdot \big(v_{s+k+1} - v^{0,j_{k+1}}_{s,k}(M_{k+1},J_{k+1},t_{k+1})\big)\Big|).
\end{aligned}
$$

One has to notice here that the argument inside the two gaussians in velocity

$$
\begin{aligned}
\frac{\widetilde{\beta}_\lambda(t_k) - \widetilde{\beta}_\lambda(t_{k+1})}{2}\big|V_{s+k}\big|^2 &= \frac{(\beta_0 - \lambda t_k) - (\beta_0 - \lambda t_{k+1})}{2}\big|V_{s+k}\big|^2 \\
&= \frac{\lambda(t_{k+1} - t_k)}{2}\big|V_{s+k}\big|^2
\end{aligned}
$$

and

$$
-\frac{\widetilde{\beta}_\lambda(t_{k+1})}{2}\big|v_{s+k+1}\big|^2
$$

are both negative, so that the term under the integral

$$
\exp\Big(\frac{\widetilde{\beta}_\lambda(t_k) - \widetilde{\beta}_\lambda(t_{k+1})}{2}\big|V_{s+k}\big|^2\Big) \exp\Big(-\frac{\widetilde{\beta}_\lambda(t_{k+1})}{2}\big|v_{s+k+1}\big|^2\Big)
$$

can be bounded from above by 1. Once again, this control can look like brutal, but since one integrates only over the subset $E_{j_{k+1}}$, compact and meant to be small, the decrease at infinity of the gaussian will play no role here.

One has therefore :

$$
\begin{aligned}
&Q\exp\big((s+k)\widetilde{\mu}_\lambda(t_k)\big)\exp\Big(\frac{\widetilde{\beta}_\lambda(t_k)}{2}\big|V_{s+k}\big|^2\Big)\\
&\le \exp\big(-\widetilde{\mu}_\lambda(T)\big)\int_0^{t_k}\exp\big((s+k)(\widetilde{\mu}_\lambda(t_k)-\widetilde{\mu}_\lambda(t_{k+1}))\big)\\
&\times\int_{\mathbb{S}^{d-1}_{\omega_{k+1}}\times\mathbb{R}^d_{v_{s+k+1}}}\sqrt{2}R\sum_{j_{k+1}=1}^{s+k}\mathbb{1}_{E_{j_{k+1}}(Z^0_{s,k}(M_{k+1},J_{k+1},t_{k+1}))}\,\mathrm{d}\omega_{k+1}\,\mathrm{d}v_{s+k+1}\,\mathrm{d}t_{k+1}\\
&\le\sqrt{2}R\exp\big(-\widetilde{\mu}_\lambda(T)\big)\int_0^{t_k}\exp\big((s+k)(\widetilde{\mu}_\lambda(t_k)-\widetilde{\mu}_\lambda(t_{k+1}))\big)\\
&\qquad\qquad\times\sum_{j_{k+1}=1}^{s+k}\big|E_{j_{k+1}}(Z^0_{s,k}(M_{k+1},J_{k+1},t_{k+1}))\big|\,\mathrm{d}t_{k+1},
\end{aligned}
$$

and since the measure of the subset $E_{j_{k+1}}(Z^0_{s,k}(M_{k+1},J_{k+1},t_{k+1}))$ is uniformly bounded in Z_{s+k} and j_{k+1} by $\big|E\big|$, one has :

$$
\begin{aligned}
&Q\exp\big((s+k)\widetilde{\mu}_\lambda(t_k)\big)\exp\Big(\frac{\widetilde{\beta}_\lambda(t_k)}{2}\big|V_{s+k}\big|^2\Big)\\
&\le \exp\big(-\widetilde{\mu}_\lambda(T)\big)\int_0^{t_k}\exp\big((s+k)(\widetilde{\mu}_\lambda(t_k)-\widetilde{\mu}_\lambda(t_{k+1}))\big)\\
&\times\int_{\mathbb{S}^{d-1}_{\omega_{k+1}}\times\mathbb{R}^d_{v_{s+k+1}}}\sqrt{2}R\sum_{j_{k+1}=1}^{s+k}\mathbb{1}_{E_{j_{k+1}}(Z^0_{s,k}(M_{k+1},J_{k+1},t_{k+1}))}\,\mathrm{d}\omega_{k+1}\,\mathrm{d}v_{s+k+1}\,\mathrm{d}t_{k+1}\\
&\le\sqrt{2}R\exp\big(-\widetilde{\mu}_\lambda(T)\big)\int_0^{t_k}\exp\big((s+k)(\widetilde{\mu}_\lambda(t_k)-\widetilde{\mu}_\lambda(t_{k+1}))\big)\\
&\qquad\qquad\times\sum_{j_{k+1}=1}^{s+k}\big|E_{j_{k+1}}(Z^0_{s,k}(M_{k+1},J_{k+1},t_{k+1}))\big|\,\mathrm{d}t_{k+1}\\
&\le\sqrt{2}R\exp\big(-\widetilde{\mu}_\lambda(T)\big)(s+k)\big|E\big|\int_0^{t_k}\exp\big((s+k)(\widetilde{\mu}_\lambda(t_k)-\widetilde{\mu}_\lambda(t_{k+1}))\big)\,\mathrm{d}t_{k+1}.
\end{aligned}
$$

Finally, noticing that the integral term in time can be computed explicitly, one finds :

$$
\begin{aligned}
\int_0^{t_k}\exp\big((s+k)(\widetilde{\mu}_\lambda(t_k)-\widetilde{\mu}_\lambda(t_{k+1}))\big)\,\mathrm{d}t_{k+1}&=\int_0^{t_k}\exp\big(\lambda(s+k)(t_{k+1}-t_k)\big)\,\mathrm{d}t_k\\
&=\left[\frac{1}{\lambda(s+k)}\exp\big(\lambda(s+k)(t_{k+1}-t_k)\right]_0^{t_k}\le\frac{1}{\lambda(s+k)},
\end{aligned}
$$

so that the term depending on s and k in the bound found above is compensated by this integral.

In summary, one has obtained

$$Q\exp\big((s+k)\widetilde{\mu}_\lambda(t_k)\big)\exp\Big(\frac{\widetilde{\beta}_\lambda(t_k)}{2}\big|V_{s+k}\big|^2\Big)\leq\sqrt{2}\frac{\exp\big(-\widetilde{\mu}_\lambda(T)\big)}{\lambda}R\big|E\big|,$$

this bound being uniform in s and Z_{s+k}, which means that

$$|||Q|||_{0,\widetilde{\beta}_\lambda,\widetilde{\mu}_\lambda}\leq\sqrt{2}\frac{\exp\big(-\widetilde{\mu}^1_\lambda(T)\big)}{\lambda}R\big|E\big|.$$

As in the proof of the previous lemma, the control of the whole remainder

$$\sum_{M_{k+1}}\sum_{J_{k+1}}\prod_{l=1}^{k+1}(\pm_l)\mathcal{I}^{0,\delta}_{\substack{s,s+k-1\\M_k,J_k}}\big(U^c_{J_k},E^c_{J_k}\big)$$
$$\circ\,\mathcal{I}^{0,\delta}_{\substack{s+k\\\pm_{k+1},j_{k+1}}}\big(U^c_{j_{k+1}},E_{j_{k+1}}\big)\Big(\mathbb{1}_{|V_{s+k+1}|\leq R}f^{(s+k+1)}\Big),$$

this time in the $|||\cdot|||_{0,\widetilde{\beta}_\lambda,\widetilde{\mu}^1_\lambda}$ norm, is obtained using the fact that the term $\mathcal{I}^{0,\delta}_{\substack{s,s+k-1\\M_k,J_k}}\big(U^c_{J_k},E^c_{J_k}\big)$ is a contracting mapping with respect to this norm. This concludes the proof of the lemma. □

Remark 39. *One notices that the controls* (32) *and* (33) *of the two previous lemmas hold only in the context of functions already truncated in high energy. Without this assumption, one would not have been able to recover the result.*

Surgery on the domains of the integrated in time collision-transport operator of the BBGKY hierarchy

The same result holds for the truncated in adjunction parameters operator of the BBGKY hierarchy. However, the proof is much longer, since one needs to give a sense to this truncated operator. One faced the same problem when one wanted to define properly the transport-collison operator of the BBGKY hierarchy : the following integral

$$\varepsilon^{d-1}\int_{\mathbb{S}^{d-1}_\omega\times\mathbb{R}^d_{v_{s+1}}}(\omega\cdot(v_{s+1}-v_i))_\pm\,\varphi^{(s+1)}(Z_s,x_i+\varepsilon\omega,v_{s+1})\,\mathrm{d}\omega\,\mathrm{d}v_{s+1}$$

did not make sense a priori due to a lack of regularity of the integrand. The problem was then solved along Section 5.1 starting page 88, first by dividing the integral and cutting off the pathological initial configurations leading to a too complicated expression of the hard sphere transport, and second by showing that the parameters of the cut-off, defining sequences of transport-collision operators with respect to thoses parameters, can be relaxed, that is the sequences converge as the parameters of the cut-offs are removed.
In the present case this work cannot be reused directly, but it can be adapted in order to take into account the additional cut-offs in pathological adjunction parameters.
One starts therefore by this problem of rigorous definition.

Rigorous definition of the truncated in adjunction parameters, integrated in time transport-collision-transport of the BBGKY hierarchy

One can wonder why one has to use piecewise constant in time functions, and then a density argument to obtain the result. Instead, one could try to find a subspace of $\mathcal{C}\big([0,T], L^\infty\big(\mathcal{D}_s^\varepsilon\big)\big)$, of regular functions, such that the straightforward computation done for the case of the Boltzmann hierarchy could also be performed, and then conclude by a density argument. The problem is that there is no hope to find, at least continuous functions (with respect to the phase space variable) which form a dense subspace into $\mathcal{C}\big([0,T], L^\infty\big(\mathcal{D}_s^\varepsilon\big)\big)$.

One starts by a first lemma, which states that the truncated in adjunction parameters transport-collision operator of the BBGKY hierarchy is a well-defined function, and which verifies the following stability condition : if the argument $h^{(s+1)}$ of the transport-collision operator is uniformly bounded in velocity by some function decreasing fastly enough at infinity, then the transport-collision operator applied to $h^{(s+1)}$ is bounded in the same way by some other function decreasing at infinity.
Those two properties constitute the analog of Theorem 1 page 137. And, without any surprise, those results are obtained in the same way, that is one will follow exactly the same path as in Section 5.1, starting page 88. To be accurate, one will in fact obtain the same bounds used in several of the lemmas of this section, which will then of course provide the same conclusion. On the one hand, the difference lies only in the checking that the cut-off in the pathological adjunction parameters (that is, the difference in the domains of the integral defining the collision operator) does not prevent to apply the same arguments as in Section 5.1. On the other hand, if the new domain of the integral defining the analog of the collision operator is small, one has to show that it is possible to obtain a control on this new transport-collision operator, truncated in adjunction parameters, which takes into account the size of this small domain. To be more explicit about this control, and it is the main difference with Section 5.1, one will obtain an upper bound on the $|||\cdot|||_{N,\varepsilon,s,\widetilde{\beta}_\lambda,\widetilde{\mu}_\lambda^1}$ norm of the truncated in adjunction parameters, iterated, integrated in time transport-collision-transport operator, which is a function of the size of the excluded subset of adjunction parameters.
One notes that this result will use the crucial hypothesis that one considers functions defined only on a domain of bounded energy.

Lemma 34 (Surgery lemma in the domain of the transport-collision operator of the BBGKY hierarchy)**.** *Let s be a positive integer, ε and T be two strictly positive numbers. Let $g_{s+1} : [0,T] \times \mathbb{R}_+ \to \mathbb{R}_+$ be a function verifying :*

- *$(t,x) \mapsto g_{s+1}(t,x)$ is measurable and almost everywhere non-negative,*
- *for all $x \in \mathbb{R}_+$, the function :*

$$t \mapsto g_{s+1}(t,x)$$

is increasing,

- *for all* $t \in [0,T]$ *and almost every* $(v_1,\dots,v_s) \in \mathbb{R}^{ds}$*, the function :*

$$v_{s+1} \mapsto \big|V_{s+1}\big| g_{s+1}\big(t, \big|V_{s+1}\big|\big)$$

is integrable on $\mathbb{R}^d$*,*

- *for all* $t \in [0,T]$*, the function :*

$$(v_1,\dots,v_s) \mapsto \int_{\mathbb{R}^d} \big|V_{s+1}\big| g_{s+1}\big(t, \big|V_{s+1}\big|\big)\, \mathrm{d}v_{s+1}$$

is bounded almost everywhere, and :

$$\Big| \int_{\mathbb{R}^d} \mathbb{1}_{|V_{s+1}| \geq R} \big|V_{s+1}\big| g_{s+1}\big(t, \big|V_{s+1}\big|\big)\, \mathrm{d}v_{s+1} \Big|_{L^\infty_t L^\infty_{V_s}}$$

converges to zero as R *goes to infinity.*

Let in addition E_j *be a function :*

$$E_j : Z_s \in \mathcal{D}^\varepsilon_s \mapsto E_j(Z_s) \in \mathcal{P}\big(\mathbb{S}^{d-1} \times \mathbb{R}^d\big)$$

such that :

$$\bigcup_{Z_s \in (\overline{\Omega^c} \times \mathbb{R}^d)^s} \big(\{Z_s\} \times E_j(Z_s)\big)$$

is measurable, and such that $E_j(Z_s)$ *is also measurable.*
Then for any integer $1 \leq j \leq s$*, any sign* $\pm = +$ *or* $-$*, and for any regulated function :*

$$h^{(s+1)} \in L^\infty\big([0,T], L^\infty\big(\mathcal{D}^\varepsilon_{s+1}\big)\big)$$

such that for all $t \in [0,T]$*, there exists a nonnegative number* $C(t)$ *such that for almost every* $Z_{s+1} \in \mathcal{D}^\varepsilon_{s+1}$ *:*

$$\big|h^{(s+1)}(t, Z_{s+1})\big| \leq C(t) g_{s+1}\big(t, \big|V_{s+1}\big|\big)$$

(for all t*, the smallest constant* $C(t)$ *verifying this condition will then be denoted as :*

$$\Big| \frac{h^{(s+1)}(t, Z_{s+1})}{g_{s+1}\big(t, \big|V_{s+1}\big|\big)} \Big|_{L^\infty(\mathcal{D}^\varepsilon_{s+1})}),$$

and for any sequence of piecewise constant in time function :

$$\big(h^{(s+1)}_k\big)_{k \geq 0} \in \Big(L^\infty\big([0,T], L^\infty\big(\mathcal{D}^\varepsilon_{s+1}\big)\big)\Big)^{\mathbb{N}}$$

converging in $L^\infty\big([0,T], L^\infty\big(\mathcal{D}^\varepsilon_{s+1}\big)$ *towards* $h^{(s+1)}$*, one has that the sequence (depending on the strictly positive numbers* δ*,* R_1*,* R_2 *and on the integer* k*) of*

integrals :

$$\mathcal{C}^{\varepsilon}_{s,s+1,\pm,j}(\delta,R_1,R_2)(E)\mathcal{T}^{s+1,\varepsilon}_t h^{(s+1)}_k = \varepsilon^{d-1}\int\limits_{\mathbb{S}^{d-1}_\omega\times\mathbb{R}^d_{v_{s+1}}} \big(\omega\cdot(v_{s+1}-v_j)\big)_\pm \mathcal{T}^{s+1,\varepsilon}_t h^{(s+1)}_k \mathbb{1}_{\mathfrak{D}}\mathbb{1}_{E_j}\,\mathrm{d}v_{s+1}\,\mathrm{d}\omega \quad (12.123)$$

(where $\mathfrak{D}=\mathfrak{D}(\varepsilon,R_1,R_2,\delta)$ denotes, as in (5.1) *page 89 the domain of the truncated in time, position and velocity transport-collision operator of the BBGKY hierarchy) is a well-defined sequence of integrable and bounded almost everywhere functions on $[0,T]\times\mathcal{D}^\varepsilon_s$, and :*

1. *it converges in $L^\infty\big([0,T]\times\mathcal{D}^\varepsilon_s\big)$ as the integer j goes to infinity towards a limit denoted as*

$$\mathcal{C}^{\varepsilon}_{s,s+1,\pm,j}(\delta,R_1,R_2)(E_j)\mathcal{T}^{s+1,\varepsilon}_t h^{(s+1)},$$

   which does not depend on the sequence $\big(h^{(s+1)}_k\big)_{k\geq 0}$ and which is also in $L^1\big([0,T]\times\mathcal{D}^\varepsilon_s\big)$,

2. *the sequence of functions $\Big(\mathcal{C}^{\varepsilon}_{s,s+1,\pm,j}(\delta,R_1,R_2)(E_j)\mathcal{T}^{s+1,\varepsilon}_t h^{(s+1)}\Big)_{\delta>0}$ converges in $L^1\big([0,T]\times\mathcal{D}^\varepsilon_s\big)$ as the parameter δ goes to zero towards a limit denoted as*

$$\mathcal{C}^{\varepsilon}_{s,s+1,\pm,j}(R_1,R_2)(E_j)\mathcal{T}^{s+1,\varepsilon}_t h^{(s+1)},$$

   the sequence converging also weak-$*$ in $L^\infty\big([0,T]\times\mathcal{D}^\varepsilon_s\big)$ towards the same limit,

3. *the sequence of functions $\Big(\mathcal{C}^{\varepsilon}_{s,s+1,\pm,j}(R_1,R_2)(E_j)\mathcal{T}^{s+1,\varepsilon}_t h^{(s+1)}\Big)_{R_1>0}$ converges almost everywhere on $[0,T]\times\mathcal{D}^\varepsilon_s$ as the parameter R_1 goes to infinity towards a limit denoted as*

$$\mathcal{C}^{\varepsilon}_{s,s+1,\pm,j}(R_2)(E_j)\mathcal{T}^{s+1,\varepsilon}_t h^{(s+1)}$$

   belonging to $L^\infty\big([0,T]\times\mathcal{D}^\varepsilon_s\big)$,

4. *the sequence of functions $\Big(\mathcal{C}^{\varepsilon}_{s,s+1,\pm,j}(R_2)(E_j)\mathcal{T}^{s+1,\varepsilon}_t h^{(s+1)}\Big)_{R_2>0}$ converges in $L^\infty\big([0,T]\times\mathcal{D}^\varepsilon_s\big)$ as the parameter R_2 goes to infinity towards a limit denoted as*

$$\mathcal{C}^{\varepsilon}_{s,s+1,\pm,j}(E_j)\mathcal{T}^{s+1,\varepsilon}_t h^{(s+1)}.$$

This limit

$$\mathcal{C}^{\varepsilon}_{s,s+1,\pm,j}(E_j)\mathcal{T}^{s+1,\varepsilon}_t h^{(s+1)} \quad (12.124)$$

is called the truncated in adjunction parameters transport-collision operator of the BBGKY hierarchy.

Moreover, this truncated in adjunction parameters operator verifies the additivity property of the domains, that is for every functions of measurable subsets $E_{j,1}$ and $E_{j,2}$ such that for almost every Z_s :

$$\big|E_{j,1}(Z_s)\cap E_{j,2}(Z_s)\big| = 0,$$

one has :

$$\begin{aligned}\mathcal{C}^{\varepsilon}_{s,s+1,\pm,j}&(E_{j,1}\cup E_{j,2})\mathcal{T}^{s+1,\varepsilon}_t h^{(s+1)}\\ &= \mathcal{C}^{\varepsilon}_{s,s+1,\pm,j}(E_{j,1})\mathcal{T}^{s+1,\varepsilon}_t h^{(s+1)} + \mathcal{C}^{\varepsilon}_{s,s+1,\pm,j}(E_{j,2})\mathcal{T}^{s+1,\varepsilon}_t h^{(s+1)},\end{aligned} \tag{12.125}$$

and the two following controls : one has almost everywhere on $[0,T]\times\mathcal{D}^{\varepsilon}_s$:

$$\begin{aligned}\Big|\mathcal{C}^{\varepsilon}_{s,s+1,\pm,j}(E_j)\mathcal{T}^{s+1,\varepsilon}_t h^{(s+1)}(t,Z_s)\Big| &\le \varepsilon^{d-1}\frac{\big|\mathbb{S}^{d-1}\big|}{2}\left|\frac{h^{(s+1)}(t,Z_{s+1})}{g_{s+1}\big(t,|V_{s+1}|\big)}\right|_{L^\infty(\mathcal{D}^{\varepsilon}_{s+1})}\\ &\qquad\times\int_{\mathbb{R}^d}\big(|v_j|+|v_{s+1}|\big)g_{s+1}\big(t,|V_{s+1}|\big)\,\mathrm{d}v_{s+1},\end{aligned} \tag{12.126}$$

and for any strictly positive number R :

$$\begin{aligned}&\Big|\mathcal{C}^{\varepsilon}_{s,s+1,\pm,j}(E_j)\mathcal{T}^{s+1,\varepsilon}_t\Big(\mathbb{1}_{|V_{s+1}|\le R}h^{(s+1)}(t,Z_s)\Big)\Big| \le \sqrt{2}\varepsilon^{d-1}R\\ &\times\left|\frac{h^{(s+1)}(t,Z_{s+1})}{g_{s+1}\big(t,|V_{s+1}|\big)}\right|_{L^\infty(\mathcal{D}^{\varepsilon}_{s+1})}\int_{\mathbb{S}^{d-1}_\omega\times\mathbb{R}^d_{v_{s+1}}}\mathbb{1}_{\pm\omega\cdot(v_{s+1}-v_j)\ge 0}\mathbb{1}_{E_j(Z_s)}g_{s+1}\big(t,|V_{s+1}|\big)\,\mathrm{d}\omega\,\mathrm{d}v_{s+1}.\end{aligned} \tag{12.127}$$

Remark 40. *The first equality* (12.125) *will be of course used in order to decompose the original integrated in time transport-collision-transport operator into two terms. On the one hand, the first term is composed of the pathological adjunction parameters, which has to be seen as a remainder, so that this term will be controlled by the last control* (12.127) *(using therefore the control of the size of the pathological adjunction parameters, stated in Proposition 16 page 347). On the other hand, the second term, which is composed of all the other adjunction parameters (so that if one adds a particle to the system which is in a good configuration, the new system with this new particle stays in a good configuration), will play the role of the main term.*
Finally, the purpose of the second control (12.126) *enables to state that this main term preserves the $|||\cdot|||_{\varepsilon,\widetilde{\beta},\widetilde{\mu^1}}$ norm.*

Proof. One starts the proof by the rigourous definition of the quantity :

$$\begin{aligned}\mathcal{C}^{\varepsilon}_{s,s+1,\pm,i}&(\delta,R_1,R_2)(E)h^{(s+1)}_j\\ &= \varepsilon^{d-1}\int_{\mathbb{S}^{d-1}_\omega\times\mathbb{R}^d_{v_{s+1}}}\big(\omega\cdot(v_{s+1}-v_i)\big)_\pm\mathcal{T}^{s+1,\varepsilon}_t h^{(s+1)}_j\mathbb{1}_{\mathfrak{D}}\mathbb{1}_E\,\mathrm{d}v_{s+1}\,\mathrm{d}\omega,\end{aligned} \tag{12.128}$$

which turns out to be an integrable function on $[0,T]\times\mathcal{D}_s^\varepsilon$.

This point is very similar to the control (5.38) of Lemma 5 page 122 of the section 5.1.3 page 121, and it is obtained following exactly the same steps as the proof of this lemma. The redaction will be quite fast, the reader can go back to the detailed proof of Lemma 5 for the missing arguments here.
The function $h_j^{(s+1)}$ is assumed to be piecewise constant in time, in other words it can be written :

$$h_j^{(s+1)}=\sum_{i=1}^{P_j}\mathbb{1}_{[t_{p-1},t_p[}\alpha_{j,p},$$

where $\alpha_{j,p}$ is a function belonging to $L^\infty\big(\mathcal{D}_{s+1}^\varepsilon\big)$. For the three strictly positive numbers δ, R_1 and R_2 defining the domain $\mathfrak{D}$ of the truncated in time, position and velocity transport-collision operator of the BBGKY hierarchy (defined in (5.1) page 89), one defines the quantity :

$$I_1=\sum_{n=0}^{T/\delta-1}\sum_{p=1}^{P_j}\int_{\mathcal{D}_{s+1}^\varepsilon}\mathbb{1}_{S_s^\pm(E_{\delta,n,p})}\big|\mathcal{T}_{n\delta}^{s+1,\varepsilon}\alpha_{j,p}\big(\widetilde{Z}_{s+1}\big)\big|\,\mathrm{d}\widetilde{Z}_{s+1},$$

with $E_{\delta,n,p}$ denoting :

$$\begin{aligned}&\big([t_{j,p-1}-n\delta,t_{j,p}-n\delta[\,\cap\,[0,\delta]\times\mathcal{D}_s^\varepsilon\big)\times\mathbb{S}^{d-1}\times\mathbb{R}^d\\&\qquad\cap\,\mathfrak{D}(R_1,R_2,\delta)\\&\quad\cap\;[0,\delta]\times\bigcup_{Z_s\in\mathcal{D}_s^\varepsilon}\big(\{Z_s\}\times E(Z_s)\big),\end{aligned}$$

(which is measurable, in particular thanks to the hypothesis on the measurability of $\bigcup_{Z_s\in\mathcal{D}_s^\varepsilon}(\{Z_s\}\times E(Z_s))$) and $S_s^\pm$ introduced and studied in Propositions 3 page 92 and 4 page 98, one sees that this integral I_1 is finite thanks to the conservation of the L^∞ norm by the hard sphere flow :

$$\begin{aligned}I_1&\le\sum_{p=1}^{P_j}\big|\alpha_{j,p}\big|_{L^\infty(\mathcal{D}_{s+1}^\varepsilon)}\sum_{n=0}^{T/\delta-1}\int_{\mathcal{D}_{s+1}^\varepsilon}\mathbb{1}_{S_s^\pm(E_{\delta,n,p})}\,\mathrm{d}\widetilde{Z}_{s+1}\\&\le\big|h_j^{(s+1)}\big|_{L_t^\infty L_{Z_{s+1}}^\infty}\sum_{p=1}^{P_j}\sum_{n=0}^{T/\delta-1}\int_{\mathcal{D}_{s+1}^\varepsilon}\mathbb{1}_{S_s^\pm(T_{\delta,n,p})}\mathbb{1}_{S_s^\pm(\mathfrak{D}(R_1,R_2,\delta))}\,\mathrm{d}\widetilde{Z}_{s+1}\end{aligned}$$

(one just forgets the condition restricting the domain to $E(Z_s)$ in the integral over $\mathbb{S}^{d-1}\times\mathbb{R}^d$) with $T_{\delta,n,p}$ denoting :

$$[t_{j,p-1},t_{j,p}[\,\cap\,[0,\delta]\times\mathcal{D}_s^\varepsilon\times\mathbb{S}^{d-1}\times\mathbb{R}^d,$$

so that :

$$I_1\le\big|h_j^{(s+1)}\big|_{L_t^\infty L_{Z_{s+1}}^\infty}\sum_{n=0}^{T/\delta-1}\int_{\mathcal{D}_{s+1}^\varepsilon}\mathbb{1}_{S_s^\pm(\mathfrak{D}(R_1,R_2,\delta))}\,\mathrm{d}\widetilde{Z}_{s+1}.$$

Then, a careful study of the size of the image of $\mathfrak{D}(R_1, R_2, \delta)$ by the application $S_s^{\pm}$, already done in the proof of Lemma 1, provides the same result (also rewritten in (5.38) in Lemma 5 page 122 for the case of a time-dependent function) in this case, that is :

$$I_1 \leq C(d,s)\varepsilon^{d-1}T(R_1+\delta R_2)^{ds}R_2^{d(s+1)+1}\big|h_j^{(s+1)}\big|_{L_t^\infty L_{Z_{s+1}}^\infty},$$

up to assuming that the condition (5.12) page 101 is fulfilled. This condition can be replaced by the less accurate statement "up to assuming δ small enough for R_1 and R_2 fixed", which is not a problem, since one of the next steps of the program is to consider the limit $\delta \to 0$ (see the proof of Lemma 1 page 101 for more details about the origin of this condition).
I_1 being finite implies therefore that the function :

$$\sum_{n=0}^{T/\delta-1}\sum_{p=1}^{P_j}\mathbb{1}_{S_s^{\pm}(E_{\delta,n,p})}\mathcal{T}_{n\delta}^{s+1,\varepsilon}\alpha_{j,p}\big(\widetilde{Z}_{s+1}\big)$$

is integrable over the phase space of $s+1$ particles, so that by the changes of variable described in Propositions 3 and 4 (and since the domain $\mathfrak{D}$ is designed in order to have the condition $S_s^{\pm} = \mathcal{T}_t^{s+1,\varepsilon}$ when $t \in [0,\delta]$), the function :

$$\sum_{p=1}^{P_j}\mathbb{1}_{[t_{p-1},t_p[}\ \varepsilon^{d-1}\big(\omega\cdot(v_{s+1}-v_i)\big)_{\pm}\mathbb{1}_{E(Z_s)}\mathbb{1}_{\mathfrak{D}(R_1,R_2,\delta)}\mathcal{T}_t^{s+1,\varepsilon}\alpha_{j,p}$$

is integrable over the space product $[0,T]\times\mathcal{D}_s^{\varepsilon}\times\mathbb{S}^{d-1}\times\mathbb{R}^d$. Then by the Fubini theorem, the function :

$$\sum_{p=1}^{P_j}\mathbb{1}_{[t_{p-1},t_p[}\ \varepsilon^{d-1}\int_{\mathbb{S}_\omega^{d-1}\times\mathbb{R}_{v_{s+1}}^d}\big(\omega\cdot(v_{s+1}-v_i)\big)_{\pm}\mathbb{1}_{E(Z_s)}\mathbb{1}_{\mathfrak{D}(R_1,R_2,\delta)}\mathcal{T}_t^{s+1,\varepsilon}\alpha_{j,p}$$

is well-defined, measurable and integrable. In other words, the quantity :

$$\mathcal{C}_{s,s+1,\pm,i}^{\varepsilon}(\delta,R_1,R_2)(E)h_j^{(s+1)}$$

is well-defined and is an integrable function on $[0,T]\times\mathcal{D}_s^{\varepsilon}$.

Now, as for the almost everywhere bound on $[0,T]\times\mathcal{D}_s^{\varepsilon}$, choosing any mesurable subset D of finite measure of $[0,T]\times\mathcal{D}_s^{\varepsilon}$, if one considers the integral :

$$I_2 = \sum_{n=0}^{T/\delta-1}\sum_{p=1}^{P_j}\int_{\mathcal{D}_{s+1}^{\varepsilon}}\mathbb{1}_{S_s^{\pm}(DE_{\delta,n,p})}\big|\mathcal{T}_{n\delta}^{s+1,\varepsilon}\alpha_{j,p}\big(\widetilde{Z}_{s+1}\big)\big|\,\mathrm{d}\widetilde{Z}_{s+1},$$

with :

$$(u,Z_s)\in D_{n,\delta} \text{ if and only if } (u+n\delta,Z_s)\in D\cap[n\delta,(n+1)\delta[,$$

$DE_{\delta,n,p}$ denoting :

$$\Big(D_{n,\delta} \cap \big([t_{j,p-1} - n\delta, t_{j,p} - n\delta[\cap [0,\delta] \times \mathcal{D}_s^\varepsilon\big)\Big) \times \mathbb{S}^{d-1} \times \mathbb{R}^d \\ \cap \mathfrak{D}(R_1, R_2, \delta) \\ \cap \ [0,\delta] \times \bigcup_{Z_s \in \mathcal{D}_s^\varepsilon} \big(\{Z_s\} \times E(Z_s)\big), \tag{12.129}$$

the conservation of the L^∞ norm by the hard sphere flow provides again :

$$I_2 \leq \sum_{p=1}^{P_j} \big|\alpha_{j,p}\big|_{L^\infty(\mathcal{D}_{s+1}^\varepsilon)} \sum_{n=0}^{T/\delta-1} \int_{\mathcal{D}_{s+1}^\varepsilon} \mathbb{1}_{S_s^\pm(DE_{\delta,n,p})} \, \mathrm{d}\widetilde{Z}_{s+1},$$

while the use of the change of variables $S_s^\pm$ described in Propositions 3 and 4 used backwards enables to write :

$$\begin{aligned} I_2 &\leq \big|h_j^{(s+1)}\big|_{L_t^\infty L_{Z_{s+1}}^\infty} \sum_{p=1}^{P_j} \sum_{n=0}^{T/\delta-1} \int_0^\delta \int_{\mathcal{D}_s^\varepsilon} \int_{\mathbb{S}_\omega^{d-1} \times \mathbb{R}_{v_{s+1}}^d} \varepsilon^{d-1} \big(\omega \cdot (v_{s+1} - v_i)\big)_\pm \\ &\qquad\qquad \times \mathbb{1}_{D, E_{\delta,n,p}} \, \mathrm{d}v_{s+1} \, \mathrm{d}\omega \, \mathrm{d}Z_s \, \mathrm{d}u \\ &\leq \big|h_j^{(s+1)}\big|_{L_t^\infty L_{Z_{s+1}}^\infty} \sum_{n=0}^{T/\delta-1} \int_0^\delta \int_{\mathcal{D}_s^\varepsilon} \mathbb{1}_{D_{n,\delta}}(u, Z_s) \int_{\mathbb{S}_\omega^{d-1} \times \mathbb{R}_{v_{s+1}}^d} \varepsilon^{d-1} \big(\omega \cdot (v_{s+1} - v_i)\big)_\pm \\ &\qquad\qquad \times \mathbb{1}_{E(Z_s)} \mathbb{1}_{\mathfrak{D}(R_1,R_2,\delta)} \, \mathrm{d}v_{s+1} \, \mathrm{d}\omega \, \mathrm{d}Z_s \, \mathrm{d}u \\ &\leq \big|h_j^{(s+1)}\big|_{L_t^\infty L_{Z_{s+1}}^\infty} \sum_{n=0}^{T/\delta-1} \int_{n\delta}^{(n+1)\delta} \int_{\mathcal{D}_s^\varepsilon} \mathbb{1}_{D \cap [n\delta,(n+1)\delta]}(t, Z_s) \\ &\qquad \times \int_{\mathbb{S}_\omega^{d-1} \times \mathbb{R}_{v_{s+1}}^d} \varepsilon^{d-1} \big(\omega \cdot (v_{s+1} - v_i)\big)_\pm \mathbb{1}_{\mathfrak{D}(R_1,R_2,\delta)} \, \mathrm{d}v_{s+1} \, \mathrm{d}\omega \, \mathrm{d}Z_s \, \mathrm{d}t \\ &\leq \big|h_j^{(s+1)}\big|_{L_t^\infty L_{Z_{s+1}}^\infty} \int_0^T \int_{\mathcal{D}_s^\varepsilon} \mathbb{1}_D(t, Z_s) \int_{\mathbb{S}_\omega^{d-1} \times \mathbb{R}_{v_{s+1}}^d} \varepsilon^{d-1} \big(\omega \cdot (v_{s+1} - v_i)\big)_\pm \\ &\qquad\qquad \times \mathbb{1}_{\mathfrak{D}(R_1,R_2,\delta)} \, \mathrm{d}v_{s+1} \, \mathrm{d}\omega \, \mathrm{d}Z_s \, \mathrm{d}t. \end{aligned}$$

Finally, using a rough Cauchy-Schwarz and then triangular inequalities on the scalar product $\omega \cdot (v_{s+1} - v_i)$, and remembering that if :

$$(Z_s, t, \omega, v_{s+1}) \in \mathfrak{D}(R_1, R_2, \delta),$$

then in particular v_{s+1} belongs to the ball $B_{\mathbb{R}^d}(0, R_2)$, one finds that :

$$I_2 \leq C(d,s) \varepsilon^{d-1} R_2^{d+1} |D| \big|h_j^{(s+1)}\big|_{L_t^\infty L_{Z_{s+1}}^\infty}.$$

As it was done a lot of times along the section 5.1, and in particular in the paragraph 5.1.2 starting page 106, the previous inequality, implies in particular that, for any measurable subset D of $[0,T]\times\mathcal{D}_s^\varepsilon$ of finite measure :

$$\Big|\mathbb{1}_D\mathcal{C}^\varepsilon_{s,s+1,\pm,i}(\delta,R_1,R_2)(E)h_j^{(s+1)}\Big|_{L^1([0,T]\times\mathcal{D}_s^\varepsilon)}\leq C(d,s,\varepsilon,R_2)|D|\big|h_j^{(s+1)}\big|_{L^\infty_tL^\infty_{Z_{s+1}}},$$

the last control is equivalent to the following control in the L^∞ norm :

$$\Big|\mathcal{C}^\varepsilon_{s,s+1,\pm,i}(\delta,R_1,R_2)(E)h_j^{(s+1)}\Big|_{L^\infty([0,T]\times\mathcal{D}_s^\varepsilon)}\leq C(d,s,\varepsilon,R_2)\big|h_j^{(s+1)}\big|_{L^\infty_tL^\infty_{Z_{s+1}}}.$$

Before proving the four points of the lemma preceeding the equality (12.125) and the controls (12.126) and (12.127) (those points establishing the rigourous definition of the truncated in adjunction parameters operator $\mathcal{C}^\varepsilon_{s,s+1,\pm,i}(E)$), one will in fact prove those equality and controls on the elements $\mathcal{C}^\varepsilon_{s,s+1,\pm,i}(\delta,R_1,R_2)(E)$ that one has just defined, which is the only interesting part of the proof.
The rest, that is, the four points stating the convergence of the operators in the respective limits in j, δ, R_1 and R_2, has been essentially done previously for the case of the operator without truncation in adjunction parameters, and besides one will simply emphasize on the controls which will enable to plug into the proofs already written above.
Similarly, the fact that the equality (12.125) and the controls (12.126) and (12.127) are propagated through those respective limits will be also a consequence of the proofs written previously.

One starts then by the equality (12.125) for $\mathcal{C}^\varepsilon_{s,s+1,\pm,i}(\delta,R_1,R_2)(E)$. If one considers two functions of measurable subsets E_1 and E_2 such that for almost every $Z_s\in\mathcal{D}_s^\varepsilon$, one has :

$$\big|E_1(Z_s)\cap E_2(Z_s)\big|=0,$$

then the following equality holds almost everywhere on $[0,T]\times\mathcal{D}_s^\varepsilon\times\mathbb{S}^{d-1}\times\mathbb{R}^d$:

$$\begin{aligned}&\sum_{p=1}^{P_j}\mathbb{1}_{[t_{p-1},t_p[}\ \varepsilon^{d-1}\big(\omega\cdot(v_{s+1}-v_i)\big)_\pm\mathbb{1}_{E_1(Z_s)\cup E_2(Z_s)}\mathbb{1}_{\mathfrak{D}(R_1,R_2,\delta)}\mathcal{T}_t^{s+1,\varepsilon}\alpha_{j,p}\\&=\sum_{p=1}^{P_j}\mathbb{1}_{[t_{p-1},t_p[}\ \varepsilon^{d-1}\big(\omega\cdot(v_{s+1}-v_i)\big)_\pm\mathbb{1}_{E_1(Z_s)}\mathbb{1}_{\mathfrak{D}(R_1,R_2,\delta)}\mathcal{T}_t^{s+1,\varepsilon}\alpha_{j,p}\\&\qquad+\sum_{p=1}^{P_j}\mathbb{1}_{[t_{p-1},t_p[}\ \varepsilon^{d-1}\big(\omega\cdot(v_{s+1}-v_i)\big)_\pm\mathbb{1}_{E_2(Z_s)})\mathbb{1}_{\mathfrak{D}(R_1,R_2,\delta)}\mathcal{T}_t^{s+1,\varepsilon}\alpha_{j,p}.\end{aligned}$$

According to the definition of the function $\mathcal{C}^\varepsilon_{s,s+1,\pm,i}(\delta,R_1,R_2)(E)h_j^{(s+1)}$, the Fubini theorem applied to each term of this equality provides therefore that one has in fact :

$$\begin{aligned}&\mathcal{C}^\varepsilon_{s,s+1,\pm,i}(\delta,R_1,R_2)(E_1\cup E_2)h_j^{(s+1)}\\&\qquad=\mathcal{C}^\varepsilon_{s,s+1,\pm,i}(\delta,R_1,R_2)(E_1)h_j^{(s+1)}+\mathcal{C}^\varepsilon_{s,s+1,\pm,i}(\delta,R_1,R_2)(E_2)h_j^{(s+1)}.\end{aligned}\tag{12.130}$$

The control (12.126) for $\mathcal{C}^{\varepsilon}_{s,s+1,\pm,i}(\delta,R_1,R_2)(E)$ is very similar to the control (5.60) of Theorem 1 page 137.
Following the second part of the proof of Theorem 1, devoted to the control (5.60) and starting page 157, one sees that the control is obtained for piecewise constant in time functions (see page 158) once the quantity, denoted Q_T, is controlled by :

$$Q_T \leq \sum_{n=0}^{T/\delta-1}\sum_{p=1}^{P_j}\sum_{q=1}^{Q-1}\left|\frac{\alpha_{j,p}(Z_{s+1})}{g_{s+1}\big(t^q_{j,p-1},|V_{s+1}|\big)}\right|_{L^\infty(\mathcal{D}^{\varepsilon}_{s+1})}$$
$$\times\int_{\mathcal{D}^{\varepsilon}_{s+1}}\mathbb{1}_{S^{\pm}_s(D_{\delta,n,p,q})}g_{s+1}\big(t^{q+1}_{j,p-1},|\widetilde{V}_{s+1}|\big)\,\mathrm{d}\widetilde{Z}_{s+1}.$$

But of course the right-hand side of the last inequality bounds also from above the quantity :

$$\int_0^T\int_{\mathcal{D}^{\varepsilon}_s}\mathbb{1}_D\Big|\mathcal{C}^{\varepsilon}_{s,s+1,\pm,i}(R_1,R_2,\delta)(E)\mathcal{T}^{s+1,\varepsilon}_t h^{(s+1)}_j\Big|\,\mathrm{d}Z_s\,\mathrm{d}t,$$

it is a simple consequence of the fact that :

$$DE_{\delta,n,p,q}\subset D_{\delta,n,p,q}$$

following the notations (12.129) page 422 for the set in the left-hand side. The times $t_{j,p-1}$ and $t_{j,p}$ are also replaced by $t^q_{j,p-1}$ and $t^{q+1}_{j,p-1}$, those times being defined in (5.83) page 161, and (5.84) page 162 for the right-hand side, since the set described in the left-hand side is simply defined with an additional restrictive criterion.
From this observation, one can therefore use the rest of the second part of the proof of Theorem 1.

For the last (and the most important of this lemma !) control (12.127), one considers again a measurable subset D of $[0,T]\times\mathcal{D}^{\varepsilon}_s$, and one starts from the quantity :

$$I=\int_0^T\int_{\mathcal{D}^{\varepsilon}_s}\mathbb{1}_D\mathcal{C}^{\varepsilon}_{s,s+1,\pm,i}(R_1,R_2,\delta)(E)\mathcal{T}^{s+1,\varepsilon}_t\Big(\mathbb{1}_{|V_{s+1}|\leq R}h^{(s+1)}_j\Big)\,\mathrm{d}Z_s\,\mathrm{d}t,$$

which is, by definition :

$$I=\sum_{n=0}^{T/\delta-1}\sum_{p=1}^{P_j}\int_{\mathcal{D}^{\varepsilon}_{s+1}}\mathbb{1}_{S^{\pm}_s(DE_{\delta,n,p})}\mathcal{T}^{s+1,\varepsilon}_{n\delta}\big(\mathbb{1}_{|\widetilde{V}_{s+1}|\leq R}\alpha_{j,p}(\widetilde{Z}_{s+1})\big)\,\mathrm{d}\widetilde{Z}_{s+1},$$

with, one recalls, $DE_{\delta,n,p}$ defined in (12.129) above. One introduces, as for the previous control, an additional division of the time interval, that is, as in (5.83) page 161, writing :

$$[t_{j,p-1},t_{j,p}[=\bigcup_{q=0}^{Q-1}\Big[t_{j,p-1}+q\frac{(t_{j,p}-t_{j,p-1})}{Q},t_{j,p-1}+(q+1)\frac{(t_{j,p}-t_{j,p-1})}{Q}\Big[,$$

for Q an arbitrary integer, denoting in addition for the sake of simplicity :

$$t^q_{j,p-1} = t_{j,p-1} + q\frac{(t_{j,p} - t_{j,p-1})}{Q},$$

and one finally replaces $DE_{\delta,n,p}$ by $DE_{\delta,n,p,q}(Q)$, defined as :

$$\begin{aligned}\Big(D_{n,\delta} \ \cap \ \big([t^q_{j,p-1} - n\delta, t^{q+1}_{j,p-1} - n\delta[\cap [0,\delta] \times \mathcal{D}^\varepsilon_s\big)\Big) \times \mathbb{S}^{d-1} \times \mathbb{R}^d \\ \cap \mathfrak{D}(R_1, R_2, \delta) \\ \cap \ [0,\delta] \times \bigcup_{Z_s \in \mathcal{D}^\varepsilon_s} \big(\{Z_s\} \times E(Z_s)\big).\end{aligned} \tag{12.131}$$

This additional division provides then :

$$I = \sum_{n=0}^{T/\delta-1} \sum_{p=1}^{P_j} \sum_{q=0}^{Q-1} \int_{\mathcal{D}^\varepsilon_{s+1}} \mathbb{1}_{S^\pm_s(DE_{\delta,n,p,q}(Q))} \mathcal{T}^{s+1,\varepsilon}_{n\delta}\big(\mathbb{1}_{|\widetilde{V}_{s+1}| \le R} \alpha_{j,p}(\widetilde{Z}_{s+1})\big) \, \mathrm{d}\widetilde{Z}_{s+1}.$$

The bound g_{s+1} being increasing with respect to time variable, one has :

$$\begin{aligned}\Big|\mathcal{T}^{s+1,\varepsilon}_{n\delta}\big(\mathbb{1}_{|\widetilde{V}_{s+1}| \le R} \alpha_{j,p}(\widetilde{Z}_{s+1})\big)\Big| \le \left|\frac{\alpha_{j,p}(Z_{s+1})}{g_{s+1}\big(t^q_{j,p-1}, |V_{s+1}|\big)}\right|_{L^\infty(\mathcal{D}^\varepsilon_{s+1})} \\ \times \mathbb{1}_{|\widetilde{V}_{s+1}| \le R} g_{s+1}\big(t^{q+1}_{j,p-1}, |\widetilde{V}_{s+1}|\big),\end{aligned}$$

so that :

$$\begin{aligned}I \le \sum_{n=0}^{T/\delta-1} \sum_{p=1}^{P_j} \sum_{q=0}^{Q-1} \left|\frac{\alpha_{j,p}(Z_{s+1})}{g_{s+1}\big(t^q_{j,p-1}, |V_{s+1}|\big)}\right|_{L^\infty(\mathcal{D}^\varepsilon_{s+1})} \\ \times \int_{\mathcal{D}^\varepsilon_{s+1}} \mathbb{1}_{S^\pm_s(DE_{\delta,n,p,q}(Q))} \mathbb{1}_{|\widetilde{V}_{s+1}| \le R} \ g_{s+1}\big(t^{q+1}_{j,p-1}, |\widetilde{V}_{s+1}|\big) \, \mathrm{d}\widetilde{Z}_{s+1}.\end{aligned}$$

Using the change of variable $S^\pm_s$ backwards, one obtains then :

$$\begin{aligned}I \le \sum_{p=1}^{P_j} \sum_{q=0}^{Q-1} \int_0^T \mathbb{1}_{[t^q_{j,p-1}, t^{q+1}_{j,-1}[} \int_{\mathcal{D}^\varepsilon_s} \mathbb{1}_D \left|\frac{\alpha_{j,p}(Z_{s+1})}{g_{s+1}\big(t^q_{j,p-1}, |V_{s+1}|\big)}\right|_{L^\infty(\mathcal{D}^\varepsilon_{s+1})} \\ \times \int_{\mathbb{S}^{d-1}_\omega \times \mathbb{R}^d_{v_{s+1}}} \varepsilon^{d-1}\big(\omega \cdot (v_{s+1} - v_i)\big)_\pm \mathbb{1}_{E(Z_s)} \mathbb{1}_{|V_{s+1}| \le R} \ g_{s+1}\big(t^{q+1}_{j,p-1}, |\widetilde{V}_{s+1}|\big) \, \mathrm{d}\omega \, \mathrm{d}v_{s+1} \, \mathrm{d}Z_s \, \mathrm{d}t.\end{aligned}$$

The following equality holds on the domain $\big\{|V_{s+1}| \le R\big\}$:

$$\big|\omega \cdot (v_{s+1} - v_i)\big| \le \sqrt{2}R,$$

and the sequence of functions :

$$\left|\frac{\alpha_{j,p}(Z_{s+1})}{g_{s+1}\big(t^q_{j,p-1},\big|V_{s+1}\big|\big)}\right|_{L^\infty(\mathcal{D}^\varepsilon_{s+1})} g_{s+1}\big(t^{q+1}_{j,p-1},\big|\widetilde{V}_{s+1}\big|\big)$$

converges almost everywhere towards :

$$\left|\frac{h^{(s+1)}(t,Z_{s+1})}{g_{s+1}\big(t,\big|V_{s+1}\big|\big)}\right|_{L^\infty(\mathcal{D}^\varepsilon_{s+1})} g_{s+1}\big(t,\big|\widetilde{V}_{s+1}\big|\big)$$

as Q goes to infinity and is uniformly bounded by the integrable function :

$$\left|\frac{h^{(s+1)}(t,Z_{s+1})}{g_{s+1}\big(t,\big|V_{s+1}\big|\big)}\right|_{L^\infty(\mathcal{D}^\varepsilon_{s+1})} g_{s+1}\big(T,\big|\widetilde{V}_{s+1}\big|\big),$$

so the dominated convergence theorem enables to obtain the control (12.127) for $\mathcal{C}^\varepsilon_{s,s+1,\pm,i}(\delta,R_1,R_2)(E)$. □

It remains now to finish the rigourous definition of the truncated in adjunction parameters operator of the BBGKY hierarchy, that is one will define the integrated in time transport-collision-transport operator, with in addition a truncation for some angular and velocity parameters of another particle adjunction. The proof of the following lemma is just the same, without any change as the one of Lemma 15 page 188. This similarity is not a surprise, the work is the same, and the only important change in this section lies in the integral over $\mathbb{S}^{d-1}\times\mathbb{R}^d$, this change having an effect only in the previous steps implied for the definition of the transport-collision operator of the BBGKY hierarchy, while the other change, namely the surgery in the time domain, will not cause any additional difficulty).

Lemma 35 (Surgery lemma in the time domain for the integrated in time transport-collision-transport operator of the BBGKY hierarchy)**.** *Let s be a positive integer, ε and T be two strictly positive numbers.*
Let $g_{s+1}:[0,T]\times\mathbb{R}_+\to\mathbb{R}_+$ be a function verifying :

- *$(t,x)\mapsto g_{s+1}(t,x)$ is measurable and almost everywhere non-negative,*
- *for all $x\in\mathbb{R}_+$, the function :*

$$t\mapsto g_{s+1}(t,x)$$

  is increasing,
- *for all $t\in[0,T]$ and almost every $(v_1,\dots,v_s)\in\mathbb{R}^{ds}$, the function :*

$$v_{s+1}\mapsto\big|V_{s+1}\big|g_{s+1}\big(t,\big|V_{s+1}\big|\big)$$

  is integrable on $\mathbb{R}^d$,

- *for all $t \in [0,T]$, the function :*

$$(v_1,\dots,v_s) \mapsto \int_{\mathbb{R}^d} \big|V_{s+1}\big| g_{s+1}\big(t,\big|V_{s+1}\big|\big)\,\mathrm{d}v_{s+1}$$

is bounded almost everywhere, and :

$$\Big|\int_{\mathbb{R}^d} \mathbb{1}_{|V_{s+1}|\geq R}\big|V_{s+1}\big| g_{s+1}\big(t,\big|V_{s+1}\big|\big)\,\mathrm{d}v_{s+1}\Big|_{L^\infty_t L^\infty_{V_s}}$$

converges to zero as R goes to infinity.

Let in addition U_j be a function :

$$U_j : Z_s \in \mathcal{D}^\varepsilon_s \mapsto U_j(Z_s) \in \mathcal{P}([0,t])$$

such that

$$\bigcup_{Z_s \in (\overline{\Omega^c}\times\mathbb{R}^d)^s} \big(\{Z_s\}\times U_j(Z_s)\big)$$

is measurable, and such that $U_j(Z_s)$ is also measurable, and let E_j be a function :

$$E_j : Z_s \in \mathcal{D}^\varepsilon_s \mapsto E_j(Z_s) \in \mathcal{P}\big(\mathbb{S}^{d-1}\times\mathbb{R}^d\big)$$

such that

$$\bigcup_{Z_s \in (\overline{\Omega^c}\times\mathbb{R}^d)^s} \big(\{Z_s\}\times E_j(Z_s)\big)$$

is measurable, and such that $E_j(Z_s)$ is also measurable.
Then for any integer $1 \leq j \leq s$, any sign $\pm = +$ or $-$, and for any regulated function :

$$h^{(s+1)} \in L^\infty\big([0,T], L^\infty\big(\mathcal{D}^\varepsilon_{s+1}\big)\big)$$

such that for all $t \in [0,T]$, there exists a nonnegative number $C(t)$ such that for almost every $Z_{s+1} \in \mathcal{D}^\varepsilon_{s+1}$:

$$\big|h^{(s+1)}(t,Z_{s+1})\big| \leq C(t) g_{s+1}\big(t,\big|V_{s+1}\big|\big)$$

(for all t, the smallest constant $C(t)$ verifying this condition will then be denoted as :

$$\Big|\frac{h^{(s+1)}(t,Z_{s+1})}{g_{s+1}\big(t,\big|V_{s+1}\big|\big)}\Big|_{L^\infty(\mathcal{D}^\varepsilon_{s+1})}),$$

one has that the function :

$$(t,Z_s) \mapsto \mathbb{1}_{t\geq(p+1)\delta}\int_{p\delta}^{t-\delta} \mathbb{1}_{U_j(Z_s)}\mathcal{T}^{s,\varepsilon}_{-u}\mathcal{C}^\varepsilon_{s,s+1,\pm,i}(E_j)\mathcal{T}^{s+1,\varepsilon}_u h^{(s+1)}(u,Z_s)\,\mathrm{d}u.$$

is well defined and is a regulated in time function, belonging to the functional space $L^\infty\big([0,T], L^\infty\big(\mathcal{D}^\varepsilon_s\big)\big)$.
This function, denoted :

$$\mathcal{I}^{N,\varepsilon,\delta}_{\substack{s\\ \pm,j}}(U_j,E_j)h^{(s+1)},$$

is called the truncated in adjunction parameters, integrated in time, transport-collision-transport operator of the BBGKY hierarchy*. In the case when* $E_j = \mathbb{S}^{d-1} \times \mathbb{R}^d$ *for every* Z_s *(surgery only in time variable), one will simply denote :*

$$\mathcal{I}^{N,\varepsilon,\delta}_{\substack{s\\ \pm,j}}\big(U_j, \mathbb{S}^{d-1} \times \mathbb{R}^d\big) h^{(s+1)} = \mathcal{I}^{N,\varepsilon,\delta}_{\substack{s\\ \pm,j}}(U_j) h^{(s+1)}.$$

Moreover this truncated in adjunction parameters operator verifies the two following additivity properties of the domains, that is for every functions $E_{j,1}$ *and* $E_{j,2}$ *such that for almost every* Z_s *:*

$$\big|E_{j,1}(Z_s) \cap E_{j,2}(Z_s)\big| = 0,$$

one has :

$$\mathcal{I}^{N,\varepsilon,\delta}_{\substack{s\\ \pm,j}}\big(U_j, E_{j,1} \cup E_{j,2}\big) h^{(s+1)} = \mathcal{I}^{N,\varepsilon,\delta}_{\substack{s\\ \pm,j}}\big(U_j, E_{j,1}\big) h^{(s+1)} + \mathcal{I}^{N,\varepsilon,\delta}_{\substack{s\\ \pm,j}}\big(U_j, E_{j,2}\big) h^{(s+1)}, \tag{12.132}$$

and for every functions $U_{j,1}$ *and* $U_{j,2}$ *such that for almost every* Z_s *:*

$$\big|U_{j,1}(Z_s) \cap U_{j,2}(Z_s)\big| = 0,$$

one has :

$$\mathcal{I}^{N,\varepsilon,\delta}_{\substack{s\\ \pm,j}}\big(U_{j,1} \cup U_{j,2}, E_j\big) h^{(s+1)} = \mathcal{I}^{N,\varepsilon,\delta}_{\substack{s\\ \pm,j}}\big(U_{j,1}, E_j\big) h^{(s+1)} + \mathcal{I}^{N,\varepsilon,\delta}_{\substack{s\\ \pm,j}}\big(U_{j_2}, E_j\big) h^{(s+1)}. \tag{12.133}$$

and the three following controls : one has, for every $u \leq t \in [0,T]$ *and for almost every* $Z_s \in \mathcal{D}^{\varepsilon}_s$ *:*

$$\begin{aligned}\Big|\mathcal{I}^{N,\varepsilon,\delta}_{\substack{s\\ \pm,j}}(U_j, E_j) h^{(s+1)}(t, Z_s)\Big| \leq & \varepsilon^{d-1} \frac{|\mathbb{S}^{d-1}|}{2} \int_0^t \left|\frac{h^{(s+1)}(\tau, Z_{s+1})}{g_{s+1}\big(\tau, |V_{s+1}|\big)}\right|_{L^\infty(\mathcal{D}^{\varepsilon}_{s+1})} \\ & \times \int_{\mathbb{R}^d} \big(|v_i| + |v_{s+1}|\big) g_{s+1}\big(\tau, |V_{s+1}|\big)\, \mathrm{d}v_{s+1}\, \mathrm{d}\tau,\end{aligned} \tag{12.134}$$

$$\begin{aligned}\Big|\mathcal{I}^{N,\varepsilon,\delta}_{\substack{s\\ \pm,j}}(U_j)\Big(\mathbb{1}_{|V_{s+1}| \leq R} h^{(s+1)}\Big)(t, Z_s)\Big| \leq & \ \varepsilon^{d-1} \frac{|\mathbb{S}^{d-1}|}{2} R \\ & \times \int_0^t \mathbb{1}_{U_j(Z_s)} \left|\frac{h^{(s+1)}(\tau, Z_{s+1})}{g_{s+1}\big(\tau, |V_{s+1}|\big)}\right|_{L^\infty(\mathcal{D}^{\varepsilon}_{s+1})} \int_{\mathbb{R}^d} g_{s+1}\big(\tau, |V_{s+1}|\big)\, \mathrm{d}v_{s+1}\, \mathrm{d}\tau,\end{aligned} \tag{12.135}$$

and :

$$\left|\mathcal{I}^{N,\varepsilon,\delta}_{\substack{s\\ \pm,j}}(U_j,E_j)\Big(\mathbb{1}_{|V_{s+1}|\leq R}h^{(s+1)}\Big)(t,Z_s)\right|$$
$$\leq \varepsilon^{d-1}R\int_0^t \mathbb{1}_{U_j(Z_s)}\left|\frac{h^{(s+1)}(\tau,Z_{s+1})}{g_{s+1}\big(\tau,|V_{s+1}|\big)}\right|_{L^\infty(\mathcal{D}^\varepsilon_{s+1})}$$
$$\times\int_{\mathbb{S}^{d-1}_\omega\times\mathbb{R}^d_{v_{s+1}}}\mathbb{1}_{\pm\omega\cdot(v_{s+1}-v_j)\geq 0}\mathbb{1}_{E_j(Z_s)}g_{s+1}\big(\tau,|V_{s+1}|\big)\,\mathrm{d}v_{s+1}\,\mathrm{d}\tau. \tag{12.136}$$

Remark 41. *One recalls that the equation* (12.132) *is non trivial, since the integrated in time transport-collision-transport operator of the BBGKY hierarchy is not defined as an usual integral on the angular parameter and velocity variables, on the contrary on the operator of the Boltzmann hierarchy.*
Besides the equation (12.133) *is easier to obtain, since the integration with respect to time variable of the transport-collision-transport operator is, this time, an usual integral.*

Decomposition induced by the surgery in adjunction parameters for the BBGKY hierarchy
One is now able to decompose, as for the Boltzmann hierarchy, any element of type M_k, J_k of the BBGKY hierarchy. And as for the Boltzmann hierarchy, one will introduce a notation for the iteration of the truncated in adjunction parameters, integrated in time, transport-collision-transport operator.

Definition 57 (Iterated, truncated in adjunction parameters (time, velocity and angular parameter), integrated in time transport-collision-transport operator of the BBGKY hierarchy)**.** *For any positive integer k, any generic elements $J_k = (j_1, J_{k-1}) \in \mathfrak{J}^s_k$ and $M_k = (\pm_1, M_{k-1}) \in \mathfrak{M}_k$ and any regulated in time function $f^{(s+k)} \in L^\infty\big([0,T], L^\infty\big(\mathcal{D}^\varepsilon_{s+1}\big)\big)$ with $f^{(s+k)}$ verifying the bound conditions of previous Lemmas 34 and 35, for any families of measurable subsets*

$$U_{J_k} = (U_{j_1},\ldots,U_{j_k}) = (U_{j_1}, U_{J_{k-1}})$$

of $[0,t-\delta]^k$ and

$$E_{J_k} = (E_{j_1},\ldots,E_{j_k}) = (E_{j_1}, E_{J_{k-1}})$$

of $\big(\mathbb{S}^{d-1}\times\mathbb{R}^d\big)^k$, one denotes the iterated, truncated in adjunction parameters, integrated in time transport-collision-transport operator of the BBGKY hierarchy *of type (M_k, J_k) the function, defined thanks to the result of Lemma 35 and by recursion :*

$$\mathcal{I}^{N,\varepsilon,\delta}_{\substack{s\\(\pm_1,j_1)}}(U_{j_1},E_{j_1})h^{(s+k)}\circ\mathcal{I}^{N,\varepsilon,\delta}_{\substack{s+1,s+k-1\\(M_{k-1},J_{k-1})}}(U_{J_{k-1}},E_{J_{k-1}})h^{(s+k)}$$

as :

$$\mathcal{I}^{N,\varepsilon,\delta}_{\substack{s,s+k-1\\(M_k,J_k)}}(U_{J_k},E_{J_k})h^{(s+k)}. \tag{12.137}$$

Therefore, the two same results as for the Boltzmann hierarchy can be stated in this context. The proof is exactly the same as for the case of the other hierarchy.

Lemma 36 (Surgery lemma in the time domain of the iterated, integrated in time collision-transport operator of the BBGKY hierarchy)**.** *Let ε be a strictly positive number, N and s be two positive integers, k be a nonnegative integer, β_0 be a strictly positive number and μ_0 be a real number.*
Let in addition U_{j_l} be a function of measurable subsets

$$U_{j_l} : Z_{s+l-1} \mapsto U_{j_l}(Z_{s+l-1})$$

of $[(k+p-l)\delta, t-\delta]$, with $1 \le l \le k+1$ and $1 \le j_l \le s+l-1$, such that :

$$\bigcup_{Z_{s+l-1}\in(\overline{\Omega^c}\times\mathbb{R}^d)^{s+l-1}} \big(\{Z_{s+l-1}\}\times U_{j_l}(Z_{s+l-1})\big),$$

is measurable and such that $U_{j_{k+1}}$ has a measure uniformly bounded in Z_{s+k} and in j_{k+1} by a constant denoted $|U|$, and let E_{j_l} be a function of measurable subsets

$$E_{j_l} : Z_{s+l-1} \mapsto E_{j_l}(Z_{s+l-1})$$

of $\mathbb{S}^{d-1}\times\mathbb{R}^d$, with $1 \le l \le k$ and $1 \le j_l \le s+l-1$, such that :

$$\bigcup_{Z_{s+l-1}\in(\overline{\Omega^c}\times\mathbb{R}^d)^{s+l-1}} \big(\{Z_{s+l-1}\}\times E_{j_l}(Z_{s+l-1})\big)$$

is measurable.
Then, in the Boltzmann-Grad limit :

$$N\varepsilon^{d-1} = 1$$

there exist two strictly positive real numbers T and λ such that

$$\beta_0 - \lambda T > 0,$$

and such that for any sequence of regulated in time functions $\big(h_N^{(s)}\big)_{1\le s\le N}$ such that for every s, $h_N^{(s)} \in L^\infty\big([0,T], L^\infty\big(\mathcal{D}_s^\varepsilon\big)\big)$ has a finite $|||\cdot|||_{N,\varepsilon,\widetilde{\beta}_\lambda,\widetilde{\mu}^1_\lambda}$ norm, one has that for all $t\in[0,T]$ and all strictly positive number $R>0$:

$$\Bigg|\sum_{M_{k+1}}\sum_{J_{k+1}}\prod_{l=1}^{k+1}(\pm_l)\mathcal{I}^{N,\varepsilon,\delta}_{\substack{s,s+k-1\\M_k,J_k}}\big((U_{J_k})^c,E^c_{J_k}\big)$$
$$\circ\,\mathcal{I}^{N,\varepsilon,\delta}_{\substack{s+k\\\pm_{k+1},j_{k+1}}}(U_{j_{k+1}})\Big(\mathbb{1}_{|V_{s+k+1}|\le R}h_N^{(s+k+1)}\Big)(t,\cdot)\Bigg|_{\varepsilon,s,\widetilde{\beta}_\lambda(t)}$$
$$\le C(d,\beta_0,\mu_0)\frac{(s+k)}{2^k}R|U|\Big|\Big|\Big|\big(h_N^{(s)}\big)_{1\le s\le N}\Big|\Big|\Big|_{N,\varepsilon,\widetilde{\beta}_\lambda,\widetilde{\mu}^1_\lambda} \tag{12.138}$$

with :

$$C(d,\beta_0,\mu_0)=\sqrt{2}\big|\mathbb{S}^{d-1}\big|\exp\big(-\widetilde{\mu}_\lambda(T)\big)\Big(\int_{\mathbb{R}^d}\exp\Big(-\frac{\widetilde{\beta}_\lambda(T)}{2}\big|v\big|^2\Big)\,\mathrm{d}v\Big).$$

Lemma 37 (Surgery lemma in the angular parameter and velocity domain of the iterated, integrated in time, collision-transport operator of the BBGKY hierarchy)**.** *Let ε be a strictly positive number, N and s be two positive integers, k be a nonnegative integer, β_0 be a strictly positive number, μ_0 be a real number μ_0.*
Let in addition U_{j_l} be a function of measurable subsets :

$$U_{j_l}:Z_{s+l-1}\mapsto U_{j_l}(Z_{s+l-1})$$

of $[(k+p-l)\delta,t-\delta]$, with $1\le l\le k+1$ and $1\le j_l\le s+l-1$, such that :

$$\bigcup_{Z_{s+l-1}\in(\overline{\Omega^c}\times\mathbb{R}^d)^{s+l-1}}\big(\{Z_{s+l-1}\}\times U_{j_l}(Z_{s+l-1})\big),$$

is measurable, and let E_{j_l} be a function of measurable subsets :

$$E_{j_l}:Z_{s+l-1}\mapsto E_{j_l}(Z_{s+l-1})$$

of $\mathbb{S}^{d-1}\times\mathbb{R}^d$, with $1\le l\le k+1$ and $1\le j_l\le s+l-1$, such that :

$$\bigcup_{Z_{s+l-1}\in(\overline{\Omega^c}\times\mathbb{R}^d)^{s+l-1}}\big(\{Z_{s+l-1}\}\times E_{j_l}(Z_{s+l-1})\big)$$

is measurable, and such that $E_{j_{k+1}}$ has a measure uniformly bounded in Z_{s+k} and in j_{k+1} by a constant denoted $\big|E\big|$.
Then, in the Boltzmann-Grad limit :

$$N\varepsilon^{d-1}=1$$

there exist two strictly positive real numbers T and λ such that

$$\beta_0-\lambda T>0,$$

and such that for any sequence of regulated in time functions $\big(h_N^{(s)}\big)_{1\le s\le N}$ such that for every s, $h_N^{(s)}\in L^\infty\big([0,T],L^\infty\big(\mathcal{D}_s^\varepsilon\big)\big)$ has a finite $|||\cdot|||_{N,\varepsilon,\widetilde{\beta}_\lambda,\widetilde{\mu}_\lambda^1}$ norm, one has that for all $t\in[0,T]$ and all strictly positive number $R>0$:

$$\begin{aligned}\Bigg|\!\Bigg|\!\Bigg|\Bigg(\mathbb{1}_{s+k+1\le N}\sum_{M_{k+1}}\sum_{J_{k+1}}\prod_{l=1}^{k+1}(\pm_l)\mathcal{I}^{N,\varepsilon,\delta}_{\substack{s,s+k-1\\M_k,J_k}}\big((U_{J_k})^c,E^c_{J_k}\big)\\ \circ\,\mathcal{I}^{N,\varepsilon,\delta}_{\substack{s+k\\\pm_{k+1},j_{k+1}}}\big(U^c_{j_{k+1}},E_{j_{k+1}}\big)\Big(\mathbb{1}_{|V_{s+k+1}|\le R}h_N^{(s+k+1)}\Big)\Bigg)_{1\le s\le N}\Bigg|\!\Bigg|\!\Bigg|_{N,\varepsilon,\widetilde{\beta}_\lambda,\widetilde{\mu}_\lambda^1}\\ \le C(\beta_0,\mu_0)\frac{1}{2^k}R\big|E\big|\Big|\!\Big|\!\Big|\big(f^{(s)}\big)_{s\ge1}\Big|\!\Big|\!\Big|_{N,\varepsilon,\widetilde{\beta}_\lambda,\widetilde{\mu}_\lambda^1}\end{aligned}\tag{12.139}$$

with :

$$C(\beta_0,\mu_0)=\sqrt{2}\frac{\exp\big(-\widetilde{\mu}_\lambda(T)\big)}{\lambda}.$$

12.3.3 Removing the pathological adjunction parameters in the elements constituting the solutions of the hierarchies

Now that the pathological ajdunction parameters, leading to recollisions, have been determined when a particle is added to a system of particles (see Section 12.2 and especially Proposition 16 page 347), and that modifications of the domains of integration of the operators defining the two hierarchies are controlled (see Section 12.3.2 and especially Propositions 33 page 411 and 37 page 431), one will naturally mix those two results, in order to keep only subsets of the domains of integration producing trees without recollision for the hard sphere dynamics.

Introducing the excluded pathological subsets of adjunction parameters and the elementary terms

One starts by naming the subsets of the domains that will be excluded at each iteration of the operators, first for the velocity and angular adjunction parameters, and then for the time of adjunctions.

Remark 42. *One has to notice here that* the same subsets *are removed from the domains of integration for the two hierarchies.*

Definition 58 (Excluded velocities and angular parameters)**.** *Let k and j_k be two positive integers such that $1 \leq j_k \leq k$ and $\varepsilon, R, \delta, a, \varepsilon_0, \rho, \eta$ and α be eight strictly positive numbers. For every $Z_k \in \Delta_k$, one defines the* set of the excluded velocities and angular parameters of a system of k particles in configuration Z_k, for an adjunction to the particle j_k *as the subset $E_{j_k}(Z_k)$ of the elements $(\omega, v) \in \mathbb{S}^{d-1} \times B(0,R)$ with*

$$\begin{aligned} Ej_k(Z_k) &= Ej_k(R,\delta,\varepsilon,a,\varepsilon_0,\rho,\eta,\alpha)(Z_k) \\ &= \widetilde{\mathcal{B}_k}(R,\delta,\varepsilon,a,\varepsilon_0,\rho,\eta)(Z_k) \\ &\qquad \cup \left(\mathbb{S}^{d-1} \times \left\{v \cdot e_1 \geq \alpha\right\}\right) \cup \mathcal{N}^*(R,\alpha)((Z_k)^{V,j_k}), \end{aligned} \tag{12.140}$$

with $\widetilde{\mathcal{B}_k}(R,\delta,\varepsilon,a,\varepsilon_0,\rho,\eta)(Z_k) \subset \mathbb{S}^{d-1} \times B(0,R)$ given by Proposition 16 page 347, and $\mathcal{N}^*(R,\alpha)((Z_k)^{V,j_k})$ introduced in Definition 52 page 369.

Remark 43. *One notices that the subset $\widetilde{\mathcal{B}_k}(Z_k)$ given by Proposition 16 is defined for a particle added to the system in contact with the particle k. Of course when another particle of the system is chosen for the adjunction, one does not define the same subset of pathological velocities and angular parameters. Nevertheless, there is no loss of generality, since the proof of Proposition 16 is identical, and the control on the size of the analog of $\widetilde{\mathcal{B}_k}(Z_k)$ for another particle chosen for the ajunction is exactly the same.*

It remains now to define the set of the times that one wants to exclude, that is the times such that, for the particle j_k involved in the k-th adjunction, the distance

between the obstacle and this particle is large enough to apply the geometrical lemmas. Those geometrical lemmas will then provide a set of velocities of small size that one will exclude in order to only obtain trees without recollisions, that is such that the particles of the BBGKY hierarchy follow the characteristics of the free transport with boundary condition.

Definition 59 (Excluded times)**.** *Let k and j_k be two positive integers such that $1 \leq j_k \leq k$, and let ρ be a strictly positive number. For every $Z_k \in \left(\overline{\Omega^c} \times \mathbb{R}^d\right)^k$, one defines the* set of the excluded times of a system of k particles in configuration Z_k, for an adjunction to the particle j_k *as the subset $U_{j_k}(t, Z_k)$ of $\mathbb{R}_+$ such that :*

$$\begin{aligned} U_{j_k}(Z_k) &= U_{j_k}(\rho)(Z_k) \\ &= \Big\{\tau \geq 0 \ / \ \left(T^{k,0}_{-\tau}(Z_k)\right)^{X,j_k} \cdot e_1 \leq \rho\Big\}. \end{aligned} \tag{12.141}$$

It is therefore possible to define the terms composing the Boltzmann and the BBGKY solutions thanks to the Duhamel formula (10.9) page 272 and (10.3) page 268, after performing the surgery on the domains, to keep only pseudo-trajectories without any recollision. One will see thanks to the geometrical lemmas that, indeed, those terms after surgery bring the most important contribution to the decomposition of the solutions, while the pathological adjunction parameters in the domains are removed and regrouped in a remainder term, which can be chosen as small as one wants.

Definition 60 (Elementary Boltzmann terms)**.** *For any positive integers s and k, any elements $M_k \in \mathfrak{M}_k$ and $J_k \in \mathfrak{J}^s_k$, any strictly positive numbers $R, \delta, \varepsilon, a, \varepsilon_0, \rho, \eta$ and α and any sequence of initial data $F_0 = \left(f_0^{(s)}\right)_{s\geq 1}$ belonging to $\boldsymbol{X}_{0,\beta_0,\widetilde{\mu}^1}$, one defines the* elementary Boltzmann term of type (M_k, J_k) of the s-th function of the hierarchy *as the function :*

$$\begin{aligned} (t, Z_s) \mapsto \mathcal{I}^{0,\delta}_{\substack{s,s+k-1\\ M_k,J_k}}\big(U^c_{J_k}(\rho), E^c_{J_k}(R, \delta, \varepsilon, a, \varepsilon_0, \rho, \eta, \alpha)\big) \\ \Big(u \mapsto \mathcal{T}^{s+k,0}_u f_0^{(s+k)}\Big)(t, Z_s), \end{aligned}$$

denoted as :

$$\mathcal{J}^{0,\delta}_{\substack{s,s+k-1\\ M_k,J_k}}(U^c, E^c) f_0^{(s+k)} = \mathcal{J}^{0,\delta}_{\substack{s,s+k-1\\ M_k,J_k}}(U^c, E^c)(R, \delta, \varepsilon, a, \varepsilon_0, \rho, \eta, \alpha) f_0^{(s+k)},$$

with :

$$U^c_{J_k}(\rho) = \big(U^c_{j_1}(\rho), \ldots, U^c_{j_k}(\rho)\big)$$

and

$$\begin{aligned} &E^c_{J_k}(R, \delta, \varepsilon, a, \varepsilon_0, \rho, \eta, \alpha) \\ &\qquad = \big(E^c_{j_1}(R, \delta, \varepsilon, a, \varepsilon_0, \rho, \eta, \alpha), \ldots, E^c_{j_k}(R, \delta, \varepsilon, a, \varepsilon_0, \rho, \eta, \alpha)\big), \end{aligned}$$

using, on the one hand, Definition 56 page 402 for the iterated, truncated in adjunction parameters, integrated in time collision-transport operator, and on the other hand, Definitions 58 page 432 and 59 page 433 for the truncated domains specified here.

The elementary term $\mathcal{J}^{0,\delta}_{\substack{s,s+k-1\\M_k,J_k}}(U^c,E^c)$ writes then explicitly :

$$\begin{aligned}
\mathcal{J}^{0,\delta}_{\substack{s,s+k-1\\M_k,J_k}}(U^c,E^c) = \mathbb{1}_{t\geq k\delta}\int_{(k-1)\delta}^{t-\delta}\mathbb{1}_{U^c_{j_1}(Z_s)}\int_{\mathbb{S}^{d-1}_{\omega_1}\times\mathbb{R}^d_{v_{s+1}}}(\pm_1)\mathbb{1}_{E^c_{j_1}(Z^0_{s,0}(M_k,J_k,t_1))}\\
\times\Big[\omega_1\cdot\big(v_{s+1}-v^{0,j_1}_{s,0}(M_k,J_k,t_1)\big)\Big]_{\pm_1}\\
\times\mathbb{1}_{t_1\geq(k-1)\delta}\int_{(k-2)\delta}^{t_1-\delta}\mathbb{1}_{U^c_{j_2}(Z^0_{s,1}(M_k,J_k,t_1))}\int_{\mathbb{S}^{d-1}_{\omega_2}\times\mathbb{R}^d_{v_{s+2}}}(\pm_2)\mathbb{1}_{E^c_{j_2}(Z^0_{s,1}(M_k,J_k,t_2))}\\
\times\Big[\omega_2\cdot\big(v_{s+2}-v^{0,j_2}_{s,1}(M_k,J_k,t_2)\big)\Big]_{\pm_2}\\
\cdots\\
\times\mathbb{1}_{t_{k-1}\geq\delta}\int_0^{t_{k-1}-\delta}\mathbb{1}_{U^c_{j_k}(Z^0_{s,k-1}(M_k,J_k,t_{k-1}))}\int_{\mathbb{S}^{d-1}_{\omega_k}\times\mathbb{R}^d_{v_{s+k}}}(\pm_k)\mathbb{1}_{E^c_{j_k}(Z^0_{s,k-1}(M_k,J_k,t_k))}\\
\times\Big[\omega_k\cdot\big(v_{s+k}-v^{0,j_k}_{s,k-1}(M_k,J_k,t_k)\big)\Big]_{\pm_k}\\
\times\mathbb{1}_{|V_{s+k}|\leq R}f_0^{(s+k)}\big(Z^0_{s,k}(M_k,J_k,0)\big)\\
\mathrm{d}\omega_k\,\mathrm{d}v_{s+k}\,\mathrm{d}t_k\ldots\mathrm{d}\omega_2\,\mathrm{d}v_{s+2}\,\mathrm{d}t_2\,\mathrm{d}\omega_1\,\mathrm{d}v_{s+1}\,\mathrm{d}t_1,
\end{aligned}$$

thanks to the notation for the pseudo-trajectories introduced in Definition 45 page 323.

Remark 44. *One sees that the excluded times defined in* (12.141)*, Definition 59, are in fact measurable subsets, depending also continuously on the configuration Z_k, so the elementary Boltzmann terms are well defined.*

One introduces similarly the elementary BBGKY terms.

Definition 61 (Elementary BBGKY terms)**.** *For any positive integer N and any strictly positive number ε, any positive integers s and k such that $s+k\leq N$, any elements $M_k\in\mathfrak{M}_k$ and $J_k\in\mathfrak{J}^s_k$, any strictly positive numbers $R,\delta,\varepsilon,a,\varepsilon_0,\rho,\eta$ and α, and any sequence of initial data $F_{N,0}=\big(f^{(s)}_{N,0}\big)_{1\leq s\leq N}$ belonging to $\boldsymbol{X}_{N,\varepsilon,\beta_0,\widetilde{\mu}^1}$, one defines the* elementary BBGKY term of type (M_k,J_k) of the s-th function of the hierarchy *as the function :*

$$(t,Z_s)\mapsto\mathcal{T}^{s,\varepsilon}_t\Big(\mathcal{I}^{N,\varepsilon,\delta}_{\substack{s,s+k-1\\M_k,J_k}}\big(U^c_{J_k}(\rho),E^c_{J_k}(R,\delta,\varepsilon,a,\varepsilon_0,\rho,\eta,\alpha)\big)f^{(s+k)}_{N,0}(t,Z_s)\Big),$$

denoted as :

$$\mathcal{J}^{N,\varepsilon,\delta}_{\substack{s,s+k-1\\M_k,J_k}}(U^c,E^c)f^{(s+k)}_{N,0}=\mathcal{J}^{N,\varepsilon,\delta}_{\substack{s,s+k-1\\M_k,J_k}}(U^c,E^c)(R,\delta,\varepsilon,a,\varepsilon_0,\rho,\eta,\alpha)f^{(s+k)}_{N,0},$$

with :

$$U^c_{J_k}(\rho)=\big(U^c_{j_1}(\rho),\ldots,U^c_{j_k}(\rho)\big)$$

and

$$\begin{aligned}&E^c_{J_k}(R,\delta,\varepsilon,a,\varepsilon_0,\rho,\eta,\alpha)\\&\qquad=\big(E^c_{j_1}(R,\delta,\varepsilon,a,\varepsilon_0,\rho,\eta,\alpha),\ldots,E^c_{j_k}(R,\delta,\varepsilon,a,\varepsilon_0,\rho,\eta,\alpha)\big),\end{aligned}$$

using Definition 57 page 429 and Definitions 58 page 432 and 59 page 433.

Remark 45. *Here, once again, the elementary BBGKY term cannot be written as a usual integral, at least without additional work. However, the explicit writing for the elementary Boltzmann term will hold also for the BBGKY hierarchy. This will be discussed below.*
One notices also that one is entering the final part of the comparison of the solutions. Therefore, one is not considering the solution of the conjugate BBGKY hierarchy anymore. This is why there is a hard sphere transport operator applied on the iterated, truncated in adjunction parameters, integrated in time transport-collision-transport operator.

Control of the error coming from the removal of the pathological adjunction parameters

One recalls that, after the preliminary cut-offs performed in Section 11 starting page 277, one has reduced the study of the respective solutions of the hierarchies to the functions :

$$\begin{aligned}F^{n,R,\delta}&=t\mapsto\Big(\mathcal{T}^{s,0}_t f^{(s)}_0(\cdot)\mathbb{1}_{|V_s|\le R}\\&\qquad+\sum_{k=1}^n\mathcal{I}^{0,\delta}_{s,s+k-1}\big(u\mapsto\mathcal{T}^{s+k,0}_u f^{(s+k)}_0\mathbb{1}_{|V_{s+k}|\le R}\big)(t,\cdot)\Big)_{s\ge1}\\&=t\mapsto\Big(\mathcal{T}^{s,0}_t f^{(s)}_0(\cdot)\mathbb{1}_{|V_s|\le R}\\&\qquad+\sum_{k=1}^n\sum_{M_k\in\mathfrak{M}_k}\sum_{J_k\in\mathfrak{J}^s_k}\mathcal{I}^{0,\delta}_{\substack{s,s+k-1\\M_k,J_k}}\big(u\mapsto\mathcal{T}^{s+k,0}_u f^{(s+k)}_0\mathbb{1}_{|V_{s+k}|\le R}\big)(t,\cdot)\Big)_{s\ge1}\end{aligned}$$

for the Boltzmann hierarchy, and :

$$F_N^{n,R,\delta} = t \mapsto \Big(\mathcal{T}_t^{s,\varepsilon}\Big(f_{N,0}^{(s)}(\cdot)\mathbb{1}_{|V_s|\leq R} + \sum_{k=1}^{n} \mathbb{1}_{s\leq N-k}\mathcal{I}_{s,s+k-1}^{N,\varepsilon,\delta}\big(f_{N,0}^{(s+k)}\mathbb{1}_{|V_{s+k}|\leq R}\big)(t,\cdot)\Big)\Big)_{1\leq s\leq N}$$

$$= t \mapsto \Big(\mathcal{T}_t^{s,\varepsilon} f_{N,0}^{(s)}(\cdot)\mathbb{1}_{|V_s|\leq R} + \sum_{k=1}^{n}\mathbb{1}_{s\leq N-k} \sum_{M_k\in\mathfrak{M}_k}\sum_{J_k\in\mathfrak{J}_k^s} \mathcal{T}_t^{s,\varepsilon}\mathcal{I}_{\substack{s,s+k-1\\M_k,J_k}}^{N,\varepsilon,\delta}\big(f_{N,0}^{(s+k)}\mathbb{1}_{|V_{s+k}|\leq R}\big)(t,\cdot)\Big)_{1\leq s\leq N}$$

for the BBGKY hierarchy.
The idea is of course to control the error obtained when the operators :

$$\mathcal{I}_{\substack{s,s+k-1\\M_k,J_k}}^{0,\delta}\mathcal{T}_t^{s+k,0} \text{ and } \mathcal{T}_t^{s,\varepsilon}\mathcal{I}_{\substack{s,s+k-1\\M_k,J_k}}^{N,\varepsilon,\delta}$$

are respectively replaced by :

$$\mathcal{J}_{\substack{s,s+k-1\\M_k,J_k}}^{0,\delta}(U^c,E^c) \text{ and } \mathcal{J}_{\substack{s,s+k-1\\M_k,J_k}}^{N,\varepsilon,\delta}(U^c,E^c),$$

so that the new cut-off leads to compare the solutions of the hierarchy with, respectively, the two new following expressions :

$$t \mapsto \Big(\mathcal{T}_t^{s,0} f_0^{(s)}(\cdot)\mathbb{1}_{|V_s|\leq R} + \sum_{k=1}^{n}\sum_{M_k\in\mathfrak{M}_k}\sum_{J_k\in\mathfrak{J}_k^s} \mathcal{J}_{\substack{s,s+k-1\\M_k,J_k}}^{0,\delta}(U^c,E^c)\big(f_0^{(s+k)}\mathbb{1}_{|V_{s+k}|\leq R}\big)(t,\cdot)\Big)_{s\geq 1}$$

and

$$t \mapsto \Big(\mathcal{T}_t^{s,\varepsilon} f_{N,0}^{(s)}(\cdot)\mathbb{1}_{|V_s|\leq R} + \sum_{k=1}^{n}\mathbb{1}_{s\leq N-k}\sum_{M_k\in\mathfrak{M}_k}\sum_{J_k\in\mathfrak{J}_k^s} \mathcal{J}_{\substack{s,s+k-1\\M_k,J_k}}^{N,\varepsilon,\delta}(U^c,E^c)\big(f_{N,0}^{(s+k)}\mathbb{1}_{|V_{s+k}|\leq R}\big)(t,\cdot)\Big)_{1\leq s\leq N}.$$

This is the purpose of the following proposition.

Definition 62 (Cut-off in pathological adjunction parameters)**.** *For any sequence of initial data $F_0 = \big(f_0^{(s)}\big)_{s\geq 1}$ belonging to $\boldsymbol{X}_{0,\beta_0,\mu_0^1}$, one defines the* truncated in high number of collisions, large energy, small time difference between collisions and in pathological adjunction parameters solution of the Boltzmann hierarchy associated to the initial datum F_0 *as the function :*

$$F^{n,R,\delta}(U^c,E^c) = t \mapsto \Big(\mathcal{T}_t^{s,0} f_0^{(s)}(\cdot)\mathbb{1}_{|V_s|\leq R} + \sum_{k=1}^{n}\sum_{M_k\in\mathfrak{M}_k}\sum_{J_k\in\mathfrak{J}_k^s} \mathcal{J}_{\substack{s,s+k-1\\M_k,J_k}}^{0,\delta}(U^c,E^c)\big(f_0^{(s+k)}\mathbb{1}_{|V_{s+k}|\leq R}\big)(t,\cdot)\Big)_{s\geq 1}.$$

For any sequence of initial data $F_{N,0} = \big(f_{N,0}^{(s)}\big)_{1\leq s\leq N}$ *belonging to* $\boldsymbol{X}_{N,\varepsilon,\beta_0,\mu_0^1}$, *one defines the* truncated in high number of collisions, large energy, small time difference between collisions and in pathological adjunction parameters solution of the BBGKY hierarchy associated to the initial datum $F_{N,0}$ *as the function :*

$$F_N^{n,R,\delta}(U^c,E^c) = t \mapsto \Big(\mathcal{T}_t^{s,\varepsilon} f_{N,0}^{(s)}(\cdot)\mathbb{1}_{|V_s|\leq R} + \sum_{k=1}^{n} \mathbb{1}_{s\leq N-k} \sum_{M_k\in\mathfrak{M}_k} \sum_{J_k\in\mathfrak{J}_k^s} \mathcal{J}_{s,s+k-1\,M_k,J_k}^{N,\varepsilon,\delta}(U^c,E^c)\big(f_{N,0}^{(s+k)}\mathbb{1}_{|V_{s+k}|\leq R}\big)(t,\cdot)\Big)_{1\leq s\leq N}.$$

Proposition 18 (Cut-off in pathological adjunction parameters)**.** *Let* β_0 *be a strictly positive number and* μ_0 *be a real number. Then there exist two strictly positive constants :*

$$c(d) \text{ and } C_4(d,\beta_0,\mu_0),$$

the first one depending only on the dimension, and the second one depending only on the dimension and on the numbers β_0 *and* μ_0*, such that the following holds :*
let $F_{N,0} = \big(f_{N,0}^{(s)}\big)_{1\leq s\leq N}$ *and* $F_0 = \big(f_0^{(s)}\big)_{s\geq 1}$ *be two sequences of initial data belonging respectively to* $\boldsymbol{X}_{N,\varepsilon,\beta_0,\mu_0^1}$ *and* $\boldsymbol{X}_{0,\beta_0,\mu_0^1}$.
For any positive integer n*, any strictly positive numbers* R *and* δ*, any positive integer* N *and any strictly positive number* ε *verifying the Boltzmann-Grad limit :*

$$N\varepsilon^{d-1} = 1,$$

and any strictly positive numbers $a, \varepsilon_0, \rho, \eta$ *and* α *such that*

$$R \geq 1,\ \eta \leq 1,\ 2\varepsilon \leq a,\ 4\sqrt{3}a \leq \varepsilon_0,\ \varepsilon_0 \leq \eta\delta,\ 3a \leq \rho \text{ and } \alpha \leq c(d),$$

one has for the BBGKY hierarchy :

$$\begin{aligned}
&\mathbb{1}_{\Delta_s(\varepsilon,R,\varepsilon_0,\alpha,\max(16Ra/\varepsilon_0,\varepsilon_0/\delta))}(Z_s)\\
&\qquad\times\Big|\mathcal{T}_t^{s,\varepsilon}\big(H_N^{n,R,\delta}\big)^{(s)}(t,Z_s) - \big(F_N^{n,R,\delta}(U^c,E^c)\big)^{(s)}(t,Z_s)\Big|\exp\Big(\frac{\widetilde{\beta}_\lambda(t)}{2}|V_s|^2\Big)\\
&\leq C_4(d,\beta_0,\mu_0)n(s+n)R\\
&\qquad\times\Big(\frac{\rho}{\alpha} + \eta^d + R^d\Big(\frac{a}{\rho}\Big)^{d-1} + nR^{2d-1}\Big(\frac{a}{\varepsilon_0}\Big)^{d-3/2}\\
&\qquad\qquad + nR^{d+1/2}\Big(\frac{\varepsilon_0}{\delta}\Big)^{d-3/2} + R^{d-1}\alpha + R^d\alpha^{1/8}\Big)\\
&\qquad\qquad\qquad\times\Big\|\big(f_{N,0}^{(s)}\big)_{1\leq s\leq N}\Big\|_{N,\varepsilon,\beta_0,\mu_0^1}
\end{aligned} \tag{12.142}$$

and for the Boltzmann hierarchy :

$$\begin{aligned}
&\mathbb{1}_{\Delta_s(\varepsilon,R,\varepsilon_0,\alpha,\max(16Ra/\varepsilon_0,\varepsilon_0/\delta))}(Z_s)\\
&\qquad\times\Big|\big(F^{n,R,\delta}\big)^{(s)}-\big(F^{n,R,\delta}(U^c,E^c)\big)^{(s)}\Big|(t,Z_s)\exp\Big(\frac{\widetilde{\beta}_\lambda(t)}{2}\big|V_s\big|^2\Big)\\
&\le C_4(d,\beta_0,\mu_0)n(s+n)R\\
&\quad\times\Big(\frac{\rho}{\alpha}+\eta^d+R^d\Big(\frac{a}{\rho}\Big)^{d-1}+nR^{2d-1}\Big(\frac{a}{\varepsilon_0}\Big)^{d-3/2}\\
&\qquad+nR^{d+1/2}\Big(\frac{\varepsilon_0}{\delta}\Big)^{d-3/2}+R^{d-1}\alpha+R^d\alpha^{1/8}\Big)\\
&\qquad\times\Big\|\big(f_0^{(s)}\big)_{s\ge1}\Big\|_{0,\beta_0,\mu_0^1},
\end{aligned}\tag{12.143}$$

where one used the notations of Definition 54 page 392 for Δ_s.

Proof. The proof will be written for the Boltzmann hierarchy, since it is exactly the same for the BBGKY hierarchy.
One considers, for any $t\in[0,T]$ and any positive integer s given, the difference :

$$\begin{aligned}
&\mathbb{1}_{\Delta_s\left(\varepsilon,R,\varepsilon_0,\alpha,\max(16Ra/\varepsilon_0,\varepsilon_0/\delta)\right)}(Z_s)\\
&\qquad\times\Big|\big(F^{n,R,\delta}\big)^{(s)}-\big(F^{n,R,\delta}(U^c,E^c)\big)^{(s)}\Big|(t,Z_s)\exp\Big(\frac{\widetilde{\beta}_\lambda(t)}{2}\big|V_s\big|^2\Big).
\end{aligned}$$

Starting by the term :

$$\Big|\big(F^{n,R,\delta}\big)^{(s)}-\big(F^{n,R,\delta}(U^c,E^c)\big)^{(s)}\Big|(t,Z_s),$$

that is :

$$\begin{aligned}
&\Bigg|\sum_{k=1}^n\sum_{M_k\in\mathfrak{M}_k}\sum_{J_k\in\mathfrak{J}_k^s}\prod_{l=1}^k(\pm_l)\mathcal{I}^{0,\delta}_{\substack{s,s+k-1\\M_k,J_k}}\big(u\mapsto\mathcal{T}_u^{s+k,0}f_0^{(s+k)}\mathbb{1}_{|V_{s+k}|\le R}\big)(t,Z_s)\\
&\qquad-\sum_{k=1}^n\sum_{M_k\in\mathfrak{M}_k}\sum_{J_k\in\mathfrak{J}_k^s}\prod_{l=1}^k(\pm_l)\mathcal{J}^{0,\delta}_{\substack{s,s+k-1\\M_k,J_k}}(U^c,E^c)\big(f_0^{(s+k)}\mathbb{1}_{|V_{s+k}|\le R}\big)(t,Z_s)\Bigg|,
\end{aligned}$$

which is of course bounded by :

$$\begin{aligned}
&\sum_{k=1}^n\Bigg|\sum_{M_k\in\mathfrak{M}_k}\sum_{J_k\in\mathfrak{J}_k^s}\prod_{l=1}^k(\pm_l)\Big(\mathcal{I}^{0,\delta}_{\substack{s,s+k-1\\M_k,J_k}}\big(u\mapsto\mathcal{T}_u^{s+k,0}f_0^{(s+k)}\mathbb{1}_{|V_{s+k}|\le R}\big)\\
&\qquad\qquad-\mathcal{J}^{0,\delta}_{\substack{s,s+k-1\\M_k,J_k}}(U^c,E^c)\big(f_0^{(s+k)}\mathbb{1}_{|V_{s+k}|\le R}\big)\Big)(t,Z_s)\Bigg|,
\end{aligned}$$

for all $1 \leq m \leq k$, one writes :

$$M_{1,m-1} = (\pm_1, \ldots, \pm_{m-1}), J_{1,m-1} = (j_1, \ldots, j_{m-1})$$

and

$$M_{m+1,k} = (\pm_{m+1}, \ldots, \pm_k), J_{m+1,k} = (j_{m+1}, \ldots, j_k)$$

so that :

$$M_k = (M_{1,m-1}, \pm_m, M_{m+1,k}) \text{ and } J_k = (J_{1,m-1}, j_m, J_{m+1,k}).$$

Each term

$$\mathcal{I}^{0,\delta}_{\substack{s,s+k-1\\ M_k,J_k}}$$

will be decomposed as :

$$\begin{aligned}\mathcal{I}^{0,\delta}_{\substack{s,s+k-1\\ M_k,J_k}} = \sum_{m=1}^{k} & \mathcal{I}^{0,\delta}_{\substack{s,s+m-2\\ M_{1,m-1},J_{1,m-1}}}\big(U^c_{J_{1,m-1}}, E^c_{J_{1,m-1}}\big) \\ & \circ \Big[\mathcal{I}^{0,\delta}_{\substack{s+m-1\\ \pm_m,j_m}}(U_{j_m}) + \mathcal{I}^{0,\delta}_{\substack{s+m-1\\ \pm_m,j_m}}\big(U^c_{j_m}, E_{j_m}\big)\Big] \circ \mathcal{I}^{0,\delta}_{\substack{s+m,s+k-1\\ M_{m+1,k},J_{m+1,k}}} \\ & + \mathcal{I}^{0,\delta}_{\substack{s,s+k-1\\ M_k,J_k}}\big(U^c_{J_k}, E^c_{J_k}\big).\end{aligned}$$

One has then :

$$\begin{aligned}&\mathcal{I}^{0,\delta}_{\substack{s,s+k-1\\ M_k,J_k}}\big(u \mapsto \mathcal{T}^{s+k,0}_u f_0^{(s+k)} \mathbb{1}_{|V_{s+k}|\leq R}\big) - \mathcal{J}^{0,\delta}_{\substack{s,s+k-1\\ M_k,J_k}}(U^c, E^c)\big(f_0^{(s+k)} \mathbb{1}_{|V_{s+k}|\leq R}\big) \\ &\quad = \sum_{m=1}^{k} \mathcal{I}^{0,\delta}_{\substack{s,s+m-2\\ M_{1,m-1},J_{1,m-1}}}\big(U^c_{J_{1,m-1}}, E^c_{J_{1,m-1}}\big) \\ &\qquad\qquad \circ \Big[\mathcal{I}^{0,\delta}_{\substack{s+m-1\\ \pm_m,j_m}}(U_{j_m}) + \mathcal{I}^{0,\delta}_{\substack{s+m-1\\ \pm_m,j_m}}\big(U^c_{j_m}, E_{j_m}\big)\Big] \circ \mathcal{I}^{0,\delta}_{\substack{s+m,s+k-1\\ M_{m+1,k},J_{m+1,k}}},\end{aligned}$$

and therefore :

$$
\begin{aligned}
&\Big|\big(F^{n,R,\delta}\big)^{(s)} - \big(F^{n,R,\delta}(U^c,E^c)\big)^{(s)}\Big| \\
&\quad\le \sum_{k=1}^{n}\sum_{m=1}^{k}\Bigg|\sum_{M_k\in\mathfrak{M}_k}\sum_{J_k\in\mathfrak{J}_k^s}\prod_{l=1}^{k}(\pm_l)\,\mathcal{I}^{0,\delta}_{\substack{s,s+m-2\\ M_{1,m-1},J_{1,m-1}}}\big(U^c_{J_{1,m-1}},E^c_{J_{1,m-1}}\big) \\
&\qquad\qquad\qquad\qquad \circ\Big[\mathcal{I}^{0,\delta}_{\substack{s+m-1\\ \pm_m,j_m}}(U_{j_m}) + \mathcal{I}^{0,\delta}_{\substack{s+m-1\\ \pm_m,j_m}}\big(U^c_{j_m},E_{j_m}\big)\Big] \\
&\qquad\qquad\qquad\qquad\qquad \circ\,\mathcal{I}^{0,\delta}_{\substack{s+m,s+k-1\\ M_{m+1,k},J_{m+1,k}}}\big(u\mapsto \mathcal{T}_u^{s+k,0}f_0^{(s+k)}\mathbb{1}_{|V_{s+k}|\le R}\big)\Bigg| \\
&\quad\le \sum_{k=1}^{n}\sum_{m=1}^{k}\Bigg|\sum_{M_k\in\mathfrak{M}_k}\sum_{J_k\in\mathfrak{J}_k^s}\prod_{l=1}^{k}(\pm_l)\,\mathcal{I}^{0,\delta}_{\substack{s,s+m-2\\ M_{1,m-1},J_{1,m-1}}}\big(U^c_{J_{1,m-1}},E^c_{J_{1,m-1}}\big) \\
&\qquad\qquad\qquad\qquad \circ\,\mathcal{I}^{0,\delta}_{\substack{s+m-1\\ \pm_m,j_m}}(U_{j_m})\circ\mathcal{I}^{0,\delta}_{\substack{s+m,s+k-1\\ M_{m+1,k},J_{m+1,k}}}\big(u\mapsto \mathcal{T}_u^{s+k,0}f_0^{(s+k)}\mathbb{1}_{|V_{s+k}|\le R}\big)\Bigg| \\
&\quad+\sum_{k=1}^{n}\sum_{m=1}^{k}\Bigg|\sum_{M_k\in\mathfrak{M}_k}\sum_{J_k\in\mathfrak{J}_k^s}\prod_{l=1}^{k}(\pm_l)\,\mathcal{I}^{0,\delta}_{\substack{s,s+m-2\\ M_{1,m-1},J_{1,m-1}}}\big(U^c_{J_{1,m-1}},E^c_{J_{1,m-1}}\big) \\
&\qquad\qquad\qquad\qquad \circ\,\mathcal{I}^{0,\delta}_{\substack{s+m-1\\ \pm_m,j_m}}\big(U^c_{j_m},E_{j_m}\big)\circ\mathcal{I}^{0,\delta}_{\substack{s+m,s+k-1\\ M_{m+1,k},J_{m+1,k}}}\big(u\mapsto \mathcal{T}_u^{s+k,0}f_0^{(s+k)}\mathbb{1}_{|V_{s+k}|\le R}\big)\Bigg|.
\end{aligned}
\tag{12.144}
$$

The first term of the right-hand side of the last inequality (12.144) contains all the terms such that the more demanding cut-off is the restriction to U_{j_m} (which is meant to be small) for the time domain.
Splitting the sums and products as follows :

$$
\begin{aligned}
\sum_{M_k\in\mathfrak{M}_k}\sum_{J_k\in\mathfrak{J}_k^s}\prod_{l=1}^{k} &= \sum_{(M_{1,m-1},\pm_m)\in\mathfrak{M}_m}\ \sum_{(J_{1,m-1},j_m)\in\mathfrak{J}_m^s}\prod_{l=1}^{m}(\pm_l) \\
&\qquad\times \sum_{M_{m+1,k}\in\mathfrak{M}_{k-m}}\ \sum_{J_{m+1,k}\in\mathfrak{J}_{k-m}^{s+m-1}}\prod_{l=m+1}^{k}(\pm_l),
\end{aligned}
$$

and writting :

$$
\sum_{M_{m+1,k}\in\mathfrak{M}_{k-m}}\ \sum_{J_{m+1,k}\in\mathfrak{J}_{k-m}^{s+m-1}}\prod_{l=m+1}^{k}(\pm_l)\mathcal{I}^{0,\delta}_{\substack{s+m,s+k-1\\ M_{m+1,k},J_{m+1,k}}} = \mathcal{I}^{0,\delta}_{s+m,s+k-1},
$$

the first term of (12.144) can be controlled thanks to the inequality (12.121) of Lemma 32 page 403, applied to :

$$
\Big(\mathcal{I}^{0,\delta}_{s+m,s+k-1}\big(u\mapsto \mathcal{T}_u^{s+k,0}\big(\mathbb{1}_{|V_{s+k}|\le R}f_0^{(s+k)}\big)\big)\Big)_{s\ge 0},
$$

which is possible since one has in fact :

$$\Big(\mathcal{I}^{0,\delta}_{s+m,s+k-1}\big(u \mapsto \mathcal{T}^{s+k,0}_u\big(\mathbb{1}_{|V_{s+k}|\leq R} f_0^{(s+k)}\big)\big)\Big)_{s\geq 0}$$
$$= \Big(\mathbb{1}_{|V_{s+m}|\leq R}\, \mathcal{I}^{0,\delta}_{s+m,s+k-1}\big(u \mapsto \mathcal{T}^{s+k,0}_u\big(\mathbb{1}_{|V_{s+k}|\leq R} f_0^{(s+k)}\big)\big)\Big)_{s\geq 0}$$

according to Proposition 13 page 285, about the stability of the cut-off in large energy by the integrated in time collision-transport operator of the Boltzmann hierarchy. Adding the indicator which prepares the initial configuration, this lemma provides :

$$\begin{aligned}
&\mathbb{1}_{\Delta_s(\varepsilon,R,\varepsilon_0,\alpha,\max(16Ra/\varepsilon_0,\varepsilon_0/\delta))}(Z_s)\\
&\times \sum_{k=1}^{n}\sum_{m=1}^{k}\Bigg|\sum_{M_k\in\mathfrak{M}_k}\sum_{J_k\in\mathfrak{J}_k^s} \mathcal{I}^{0,\delta}_{s,s+m-2\;M_{1,m-1},J_{1,m-1}}\big(U^c_{J_{1,m-1}},E^c_{J_{1,m-1}}\big)\\
&\qquad\circ \mathcal{I}^{0,\delta}_{s+m-1\;\pm_m,j_m}(U_{j_m})\circ \mathcal{I}^{0,\delta}_{s+m,s+k-1\;M_{m+1,k},J_{m+1,k}}\big(u\mapsto \mathcal{T}^{s+k,0}_u\big(\mathbb{1}_{|V_{s+k}|\leq R} f_0^{(s+k)}\big)\big)\Bigg|(t,Z_s)\\
&\qquad\qquad\times \exp\Big(\frac{\widetilde{\beta}_\lambda(t)}{2}\big|V_s\big|^2\Big)\\
&\leq C(d,\beta_0,\mu_0)\sum_{k=1}^{n}\sum_{m=1}^{k}\frac{(s+m-1)}{2^{m-1}}R\Big(\sup_{\substack{Z^0_{s,m-1}(t_{m-1})\\1\leq j_m\leq s+m-1}}\big|U_{j_m}(Z^0_{s,m-1}(t_{m-1}))\big|\Big)\\
&\qquad\times\Big|\!\Big|\!\Big|\Big(\mathcal{I}^{0,\delta}_{s+m,s+k-1}\big(u\mapsto \mathcal{T}^{s+k,0}_u\big(\mathbb{1}_{|V_{s+k}|\leq R} f_0^{(s+k)}\big)\big)\Big)_{s\geq 0}\Big|\!\Big|\!\Big|_{0,\widetilde{\beta}_\lambda,\widetilde{\mu}^1_\lambda},
\end{aligned}$$

and finally, thanks to the contracting property of the integrated in time collision-transport operator :

$$\begin{aligned}
&\sum_{k=1}^{n}\sum_{m=1}^{k}\Bigg|\sum_{M_k\in\mathfrak{M}_k}\sum_{J_k\in\mathfrak{J}_k^s} \mathcal{I}^{0,\delta}_{s,s+m-2\;M_{1,m-1},J_{1,m-1}}\big(U^c_{J_{1,m-1}},E^c_{J_{1,m-1}}\big)\\
&\qquad\circ \mathcal{I}^{0,\delta}_{s+m-1\;\pm_m,j_m}(U_{j_m})\circ \mathcal{I}^{0,\delta}_{s+m,s+k-1\;M_{m+1,k},J_{m+1,k}}\big(u\mapsto \mathcal{T}^{s+k,0}_u\big(\mathbb{1}_{|V_{s+k}|\leq R} f_0^{(s+k)}\big)\big)\Bigg|(t,Z_s)\\
&\qquad\qquad\times \exp\Big(\frac{\widetilde{\beta}_\lambda(t)}{2}\big|V_s\big|^2\Big)\\
&\leq C(d,\beta_0,\mu_0)\Big(\sup_{1\leq m\leq n}\sup_{\substack{Z^0_{s,m-1}(t_{m-1})\\1\leq j_m\leq s+m-1}}\big|U_{j_m}(Z^0_{s,m-1}(t_{m-1}))\big|\Big)\\
&\qquad\times R\sum_{k=1}^{n}\sum_{m=1}^{k}\frac{(s+m-1)}{2^{m-1}}\frac{1}{2^{k-m}}\Big|\!\Big|\!\Big|\Big(u\mapsto \mathcal{T}^{s,0}_u\big(\mathbb{1}_{|V_s|\leq R} f_0^{(s)}\big)_{s\geq 0}\Big|\!\Big|\!\Big|_{0,\widetilde{\beta}_\lambda,\widetilde{\mu}^1_\lambda}.
\end{aligned}$$

Then, using the conservation of the $|\cdot|_{\varepsilon,s,\widetilde{\beta}(t)}$ norm by the transport operator $\mathcal{T}^{s,0}_u$, and the embeddings of the spaces $X_{0,s,\widetilde{\beta}_\lambda(u)}\subset X_{0,s,\beta_0}$ on the one hand,

and $\mathbf{X}_{0,\widetilde{\beta}_\lambda(u),\widetilde{\mu}^1_\lambda(u)} \subset \mathbf{X}_{0,\beta_0,\mu_0^1}$ on the other hand (see Proposition 8 page 209) :

$$\sum_{k=1}^{n}\sum_{m=1}^{k}\Bigg|\sum_{M_k\in\mathfrak{M}_k}\sum_{J_k\in\mathfrak{J}_k^s}\mathcal{I}^{0,\delta}_{s,s+m-2\,M_{1,m-1},J_{1,m-1}}\big(U^c_{J_{1,m-1}},E^c_{J_{1,m-1}}\big)$$
$$\circ\,\mathcal{I}^{0,\delta}_{s+m-1\,\pm_m,j_m}(U_{j_m})\circ\mathcal{I}^{0,\delta}_{s+m,s+k-1\,M_{m+1,k},J_{m+1,k}}\big(u\mapsto\mathcal{T}^{s+k,0}_u\big(\mathbb{1}_{|V_{s+k}|\le R}f_0^{(s+k)}\big)\big)\Bigg|(t,Z_s)$$
$$\times\exp\Big(\frac{\widetilde{\beta}_\lambda(t)}{2}\big|V_s\big|^2\Big)$$
$$\le C(d,\beta_0,\mu_0)\Big(\sup_{1\le m\le n}\sup_{\substack{Z^0_{s,m-1}(t_{m-1})\\1\le j_m\le s+m-1}}\big|U_{j_m}(Z^0_{s,m-1}(t_{m-1}))\big|\Big)$$
$$\times R\sum_{k=1}^{n}\sum_{m=1}^{k}\frac{(s+m-1)}{2^{k-1}}\Big\|\big(f_0^{(s)}\big)_{s\ge0}\Big\|_{0,\beta_0,\mu_0^1}.$$

One notes also that :

$$\sum_{k=1}^{n}\sum_{m=1}^{k}\frac{(s+m-1)}{2^{k-1}}\le n(s+n)\sum_{k=1}^{+\infty}\frac{1}{2^{k-1}}=2n(s+n),$$

so the first term of (12.144) verifies the following control :

$$\sum_{k=1}^{n}\sum_{m=1}^{k}\Bigg|\sum_{M_k\in\mathfrak{M}_k}\sum_{J_k\in\mathfrak{J}_k^s}\mathcal{I}^{0,\delta}_{s,s+m-2\,M_{1,m-1},J_{1,m-1}}\big(U^c_{J_{1,m-1}},E^c_{J_{1,m-1}}\big)\circ\mathcal{I}^{0,\delta}_{s+m-1\,\pm_m,j_m}(U_{j_m})$$
$$\circ\,\mathcal{I}^{0,\delta}_{s+m,s+k-1\,M_{m+1,k},J_{m+1,k}}\big(u\mapsto\mathcal{T}^{s+k,0}_u\big(\mathbb{1}_{|V_{s+k}|\le R}f_0^{(s+k)}\big)\big)\Bigg|\exp\Big(\frac{\widetilde{\beta}_\lambda(t)}{2}\big|V_s\big|^2\Big)$$
$$\le C(d,\beta_0,\mu_0)n(s+n)R\Big(\sup_{1\le m\le n}\sup_{\substack{Z^0_{s,m-1}(t_{m-1})\\1\le j_m\le s+m-1}}\big|U_{j_m}(Z^0_{s,m-1}(t_{m-1}))\big|\Big)$$
$$\times\Big\|\big(f_0^{(s)}\big)_{s\ge0}\Big\|_{0,\beta_0,\mu_0^1},\tag{12.145}$$

replacing the constant $C(d,\beta_0,\mu_0)$ given by Lemma 32 page 403 by $2C(d,\beta_0,\mu_0)$.

For the second term of (12.144), the stronger cut-off is the restriction to E_{j_m}, and it is controlled thanks to the inequality (12.122) of Lemma 33 page 411, applied, as in the case of the first term, to

$$\Big(\mathcal{I}^{0,\delta}_{s+m,s+k-1}\big(u\mapsto\mathcal{T}^{s+k,0}_u\big(\mathbb{1}_{|V_{s+k}|\le R}f_0^{(s+k)}\big)\big)\Big)_{s\ge0},$$

again due to the stability of the cut-off in large energy, which is mandatory to apply Lemma 33. Again, adding the indicator preparing the initial configura-

tions, one obtains :

$$
\begin{aligned}
&\Bigg|\!\Bigg|\Bigg(\mathbb{1}_{\Delta_s(\varepsilon,R,\varepsilon_0,\alpha,\max(16Ra/\varepsilon_0,\varepsilon_0/\delta))}(Z_s)\\
&\quad\times\sum_{k=1}^{n}\sum_{m=1}^{k}\sum_{M_k\in\mathfrak{M}_k}\sum_{J_k\in\mathfrak{J}_k^s}\prod_{l=1}^{k}(\pm_l)\mathcal{I}^{0,\delta}_{\substack{s,s+m-2\\M_{1,m-1},J_{1,m-1}}}\big(U^c_{J_{1,m-1}},E^c_{J_{1,m-1}}\big)\\
&\circ\mathcal{I}^{0,\delta}_{\substack{s+m-1\\\pm_m,j_m}}\big(U^c_{j_m},E_{j_m}\big)\circ\mathcal{I}^{0,\delta}_{\substack{s+m,s+k-1\\M_{m+1,k},J_{m+1,k}}}\big(u\mapsto\mathcal{T}_u^{s+k,0}f_0^{(s+k)}\mathbb{1}_{|V_{s+k}|\le R}\big)\Bigg)_{s\ge1}\Bigg|\!\Bigg|_{0,\widetilde{\beta}_\lambda,\widetilde{\mu}^1_\lambda}\\
&\le\sum_{k=1}^{n}\sum_{m=1}^{k}\Bigg|\!\Bigg|\Bigg(\mathbb{1}_{\Delta_s(\varepsilon,R,\varepsilon_0,\alpha,\max(16Ra/\varepsilon_0,\varepsilon_0/\delta))}(Z_s)\\
&\qquad\times\sum_{M_k\in\mathfrak{M}_k}\sum_{J_k\in\mathfrak{J}_k^s}\prod_{l=1}^{k}(\pm_l)\mathcal{I}^{0,\delta}_{\substack{s,s+m-2\\M_{1,m-1},J_{1,m-1}}}\big(U^c_{J_{1,m-1}},E^c_{J_{1,m-1}}\big)\\
&\circ\mathcal{I}^{0,\delta}_{\substack{s+m-1\\\pm_m,j_m}}\big(U^c_{j_m},E_{j_m}\big)\circ\mathcal{I}^{0,\delta}_{\substack{s+m,s+k-1\\M_{m+1,k},J_{m+1,k}}}\big(u\mapsto\mathcal{T}_u^{s+k,0}f_0^{(s+k)}\mathbb{1}_{|V_{s+k}|\le R}\big)\Bigg)_{s\ge1}\Bigg|\!\Bigg|_{0,\widetilde{\beta}_\lambda,\widetilde{\mu}^1_\lambda}.
\end{aligned}
$$

Now thanks to the control of Lemma 33, one gets :

$$
\begin{aligned}
&\Bigg|\!\Bigg|\Bigg(\mathbb{1}_{\Delta_s(\varepsilon,R,\varepsilon_0,\alpha,\max(16Ra/\varepsilon_0,\varepsilon_0/\delta))}(Z_s)\\
&\quad\times\sum_{k=1}^{n}\sum_{m=1}^{k}\sum_{M_k\in\mathfrak{M}_k}\sum_{J_k\in\mathfrak{J}_k^s}\prod_{l=1}^{k}(\pm_l)\mathcal{I}^{0,\delta}_{\substack{s,s+m-2\\M_{1,m-1},J_{1,m-1}}}\big(U^c_{J_{1,m-1}},E^c_{J_{1,m-1}}\big)\\
&\circ\mathcal{I}^{0,\delta}_{\substack{s+m-1\\\pm_m,j_m}}\big(U^c_{j_m},E_{j_m}\big)\circ\mathcal{I}^{0,\delta}_{\substack{s+m,s+k-1\\M_{m+1,k},J_{m+1,k}}}\big(u\mapsto\mathcal{T}_u^{s+k,0}f_0^{(s+k)}\mathbb{1}_{|V_{s+k}|\le R}\big)\Bigg)_{s\ge1}\Bigg|\!\Bigg|_{0,\widetilde{\beta}_\lambda,\widetilde{\mu}^1_\lambda}\\
&\le C(\beta_0,\mu_0)R\sum_{k=1}^{n}\sum_{m=1}^{k}\frac{1}{2^{m-1}}\Big(\sup_{\substack{Z^0_{s,m-1}(t_m)\\1\le j_m\le s+m-1}}\big|E_{j_m}(Z^0_{s,m-1}(t_m))\big|\Big)\\
&\qquad\times\Bigg|\!\Bigg|\Big(\mathcal{I}^{0,\delta}_{s,s+k-m-1}\big(u\mapsto\mathcal{T}_u^{s+k-m,0}f_0^{(s+k-m)}\mathbb{1}_{|V_{s+k}|\le R}\big)\Big)_{s\ge1}\Bigg|\!\Bigg|_{0,\widetilde{\beta}_\lambda,\widetilde{\mu}^1_\lambda},
\end{aligned}
$$

and again using the contracting property of the iterated, integrated in time

collision-transport operator :

$$
\begin{aligned}
&\Bigg|\!\Bigg|\Bigg(\mathbb{1}_{\Delta_s(\varepsilon,R,\varepsilon_0,\alpha,\max(16Ra/\varepsilon_0,\varepsilon_0/\delta))}(Z_s)\\
&\qquad\times\sum_{k=1}^{n}\sum_{m=1}^{k}\sum_{M_k\in\mathfrak{M}_k}\sum_{J_k\in\mathfrak{J}_k^s}\prod_{l=1}^{k}(\pm_l)\mathcal{I}^{0,\delta}_{\substack{s,s+m-2\\M_{1,m-1},J_{1,m-1}}}\big(U^c_{J_{1,m-1}},E^c_{J_{1,m-1}}\big)\\
&\circ\mathcal{I}^{0,\delta}_{\substack{s+m-1\\\pm_m,j_m}}\big(U^c_{j_m},E_{j_m}\big)\circ\mathcal{I}^{0,\delta}_{\substack{s+m,s+k-1\\M_{m+1,k},J_{m+1,k}}}\big(u\mapsto\mathcal{T}_u^{s+k,0}f_0^{(s+k)}\mathbb{1}_{|V_{s+k}|\le R}\big)\Bigg)_{s\ge1}\Bigg|\!\Bigg|_{0,\widetilde{\beta}_\lambda,\widetilde{\mu}^1_\lambda}\\
&\le C(\beta_0,\mu_0)R\sum_{k=1}^{n}\sum_{m=1}^{k}\frac{1}{2^{k-1}}\Big(\sup_{\substack{Z^0_{s,m-1}(t_m)\\1\le j_m\le s+m-1}}\big|E_{j_m}(Z^0_{s,m-1}(t_m))\big|\Big)\\
&\qquad\qquad\times\Big|\!\Big|\Big|\big(u\mapsto\mathcal{T}_u^{s+k-m,0}f_0^{(s+k-m)}\mathbb{1}_{|V_{s+k}|\le R}\big)_{s\ge1}\Big|\!\Big|\Big|_{0,\widetilde{\beta}_\lambda,\widetilde{\mu}^1_\lambda}\\
&\le C(\beta_0,\mu_0)n(s+n)R\Big(\sup_{1\le m\le n}\sup_{\substack{Z^0_{s,m-1}(t_m)\\1\le j_m\le s+m-1}}\big|E_{j_m}(Z^0_{s,m-1}(t_m))\big|\Big)\\
&\qquad\qquad\times\Big|\!\Big|\big(f_0^{(s)}\big)_{s\ge1}\Big|\!\Big|_{0,\beta_0,\mu_0^1}.
\end{aligned}
$$

Translating the $|\!|\!|\cdot|\!|\!|_{0,\widetilde{\beta}_\lambda,\widetilde{\mu}_\lambda}$ norm of the last inequality, one obtains for the second term of (12.144) :

$$
\begin{aligned}
&\Bigg|\sum_{k=1}^{n}\sum_{m=1}^{k}\sum_{M_k\in\mathfrak{M}_k}\sum_{J_k\in\mathfrak{J}_k^s}\prod_{l=1}^{k}(\pm_l)\mathcal{I}^{0,\delta}_{\substack{s,s+m-2\\M_{1,m-1},J_{1,m-1}}}\big(U^c_{J_{1,m-1}},E^c_{J_{1,m-1}}\big)\\
&\quad\circ\mathcal{I}^{0,\delta}_{\substack{s+m-1\\\pm_m,j_m}}\big(U^c_{j_m},E_{j_m}\big)\circ\mathcal{I}^{0,\delta}_{\substack{s+m,s+k-1\\M_{m+1,k},J_{m+1,k}}}\big(u\mapsto\mathcal{T}_u^{s+k,0}f_0^{(s+k)}\mathbb{1}_{|V_{s+k}|\le R}\big)\Bigg|\\
&\qquad\qquad\times\mathbb{1}_{\Delta_s(\varepsilon,R,\varepsilon_0,\alpha,\max(16Ra/\varepsilon_0,\varepsilon_0/\delta))}(Z_s)\exp\Big(\frac{\widetilde{\beta}_\lambda(t)}{2}\big|V_s\big|^2\Big)\\
&\le C(\beta_0,\mu_0)n(s+n)\exp\big(-s\widetilde{\mu}_\lambda(t)\big)R\\
&\qquad\times\Big(\sup_{1\le m\le n}\sup_{\substack{Z^0_{s,m-1}(t_m)\\1\le j_m\le s+m-1}}\big|E_{j_m}(Z^0_{s,m-1}(t_m))\big|\Big)\Big|\!\Big|\big(f_0^{(s)}\big)_{s\ge1}\Big|\!\Big|_{0,\beta_0,\mu_0^1}\\
&\le C(\beta_0,\mu_0)n(s+n)\exp\big(-s\widetilde{\mu}_\lambda(T)\big)R\\
&\quad\times\Big(\sup_{1\le m\le n}\sup_{\substack{Z^0_{s,m-1}(t_m)\\1\le j_m\le s+m-1}}\big|E_{j_m}(Z^0_{s,m-1}(t_m))\big|\Big)\Big|\!\Big|\big(f_0^{(s)}\big)_{s\ge1}\Big|\!\Big|_{0,\beta_0,\mu_0^1}
\end{aligned}
\tag{12.146}
$$

since the function $\widetilde{\mu}_\lambda$ is decreasing.

Now combining the inequalities (12.145) page 442 and (12.146) page 444, one

finds :

$$
\begin{aligned}
&\mathbb{1}_{\Delta_s(\varepsilon,R,\varepsilon_0,\alpha,\max(16Ra/\varepsilon_0,\varepsilon_0/\delta))}(Z_s)\\
&\qquad\times\Big|\big(F^{n,R,\delta}\big)^{(s)}-\big(F^{n,R,\delta}(U^c,E^c)\big)^{(s)}\Big|(t,Z_s)\exp\Big(\frac{\widetilde{\beta}_\lambda(t)}{2}\big|V_s\big|^2\Big)\\
&\leq C(d,\beta_0,\mu_0)n(s+n)R\\
&\quad\times\Big(\sup_{1\leq m\leq n}\sup_{\substack{Z^0_{s,m-1}(t_{m-1})\\1\leq j_m\leq s+m-1}}\big|U_{j_m}(Z^0_{s,m-1}(t_{m-1}))\big|\\
&\qquad+\exp\big(-s\widetilde{\mu}_\lambda(T)\big)\sup_{1\leq m\leq n}\sup_{\substack{Z^0_{s,m-1}(t_m)\\1\leq j_m\leq s+m-1}}\big|E_{j_m}(Z^0_{s,m-1}(t_m))\big|\Big)\\
&\qquad\qquad\times\Big\|\big(f_0^{(s)}\big)_{s\geq 1}\Big\|_{0,\beta_0,\mu_0^1}.
\end{aligned}
$$

First, the fact that Z_s is chosen in $\Delta_s\big(\varepsilon,R,\varepsilon_0,\alpha,\max(16Ra/\varepsilon_0,\varepsilon_0/\delta)\big)$, and especially the fact that $\big|v_i\cdot e_1\big|\geq\alpha$ for all $1\leq i\leq s$, enables to use the inequality (12.116) of Proposition 17 page 394, that is, following Definition 59 page 433, one has :

$$\sup_{\substack{Z_s\in\Delta_s\\1\leq j_1\leq s}}\big|U_{j_1}(Z_s)\big|\leq 2\frac{\rho}{\alpha}.$$

Then one wants to show that if Z_s is chosen in Δ_s and $t-t_1$ outside $U_{j_1}(Z_k)$, it is possible to apply Proposition 16 page 347, so that the size of the pathological adjunction parameters will be controlled.
Using the notations for the pseudo-trajectories introduced in Definition 45 page 323, $t-t_1$ chosen outside $U_{j_1}(Z_s)$ means that :

$$x^{0,j_1}_{s,0}(t_1)\cdot e_1=\big(Z^0_{s,0}(t_1)\big)^{X,j_1}\cdot e_1=\big(T^{s,0}_{t_1-t}(Z_s)\big)^{X,j_1}\cdot e_1\geq\rho,$$

that is the particle j_1 is far enough from the obstacle at time t_1.
This is the bound (12.36) (of Proposition 16 page 347) from below on the distance to the obstacle of the particle undergoing the adjunction. Besides, another consequence of the fact that $Z_s\in\Delta_s$ is that thanks to Proposition 17 page 394, waiting long enough, that is, if $t-t_1\geq\delta$, the second point (12.114) of this proposition provides exactly that :

$$T^{s,0}_{t_1-t}(Z_s)=Z^0_{s,0}(t_1)$$

belongs to $\mathcal{G}^0_s(\varepsilon_0)$. As a consequence, if the parameters verify :

$$R\geq 1,\ 4\sqrt{3}a\leq\varepsilon_0,\ \varepsilon_0/\delta\leq 1,\ 2\varepsilon\leq a,\ \varepsilon_0\leq\eta\delta\text{ and }3a\leq\rho,$$

on the one hand, Proposition 16 can be applied and it provides the existence of the subset

$$\widetilde{\mathcal{B}_k}\big(T^{s,0}_{t_1-t}(Z_s)\big)(R,\delta,\varepsilon,a,\varepsilon_0,\rho,\eta)$$

of $\mathbb{S}^{d-1}\times\mathbb{R}^d$, so that outside this subset, the configuration $Z^0_{s,1}(t_1)$, after adjunction, and possibly after scattering according to the pre or post-collisional character of the configuration $\big(Z^0_{s,0}(t_1), x^{0,j_1}_{s,0}(t_1), v\big) = \big(T^{s,0}_{t_1-t}(Z_s), \big(T^{s,0}_{t_1-t}(Z_s)\big)^{X,j_1}, v\big)$, is a good configuration of $s+1$ particles for the free flow dynamics, separated by at least ε_0, after applying the free flow during δ. In other words :

$$T^{s+1,0}_{-\delta}\big(Z^0_{s,1}(t_1)\big) \in \mathcal{G}^0_{s+1}(\varepsilon_0).$$

Moreover, the size of this subset $\widetilde{\mathcal{B}_k}\big(T^{s,0}_{t_1-t}(Z_s)\big)$ is controlled by the inequality (12.37) page 348 of the same proposition.
On the other hand, the conditions

$$R \geq 1,\ \alpha \leq c(d)$$

enable to use Lemma 31 page 380.
Then, as in Definition 58 page 432, one considers the subset $E_{j_1}\big(Z^0_{s,0}(t_1)\big)$, which is composed of $\widetilde{\mathcal{B}_k}\big(T^{s,0}_{t_1-t}(Z_s)\big)$, and in addition of the angular and velocity adjunction parameters that could produce a configuration with a particle grazing the obstacle. The size of $E_{j_1}\big(Z^0_{s,0}(t_1)\big)$ is then controlled by Proposition 16 page 347 on the one hand, and on the other hand by the inequality (12.68) page 369 and Lemma 31 (the first of the two last results controls the size of parameters leading to grazing trajectory *without* scattering, while the second one controls the effect of the scattering *in addition*), so that :

$$\Big|E_{j_1}\big(Z^0_{s,0}(t_1)\big)\Big| \leq C(d)\Big(\ \eta^d + R^d\Big(\frac{a}{\rho}\Big)^{d-1} + kR^{2d-1}\Big(\frac{a}{\varepsilon_0}\Big)^{d-3/2} + kR^{d+1/2}\Big(\frac{\varepsilon_0}{\delta}\Big)^{d-3/2} + R^{d-1}\alpha + R^d\alpha^{1/8}\Big).$$

Here, since it will have no significant impact in the end of the proof to make a difference between the cases $d=2$ and $d\geq 3$, one has chosen the more general (but less accurate) bound

$$C(d)R^d\alpha^{1/8}$$

to control the size of the grazing velocities after the application of the scattering. It is indeed a more general control since for $R\geq 1$ and $\alpha\leq 1$, this quantity is larger than the two bounds provided by Lemma 31, namely

$$C(d)R^2\alpha^{1/8}$$

for the case $d=2$ and

$$C(d)R^{d-1}\alpha$$

for the case $d\geq 3$.

The cut-off in adjunction parameters implying a particle with a grazing trajectory enables to state that :

$$\sup_{\substack{Z_s\in\Delta_s\\ 1\leq j_1\leq s\\ t_1\in U^c_{j_1}(Z_s),\ (\omega_1,v_{s+1})\ \in\ E^c_{j_1}(Z^0_{s,0}(t_1))\\ 1\leq j_2\leq s+1}} \big|U_{j_2}\big(Z^0_{s,1}(t_1)\big)\big| \leq 2\frac{\rho}{\alpha}.$$

Moreover, since t_2 is chosen such that $t_1 - t_2$ does not belong to $U_{j_2}\big(Z^0_{s,1}(t_1)\big)$, one has :

$$\big(x^{0,j_2}_{s,1}(t_2)\big)\cdot e_1 = \Big(\big(Z^0_{s,1}(t_2)\big)^{X,j_2}\Big)\cdot e_1 \geq \rho,$$

and since $t_1 - t_2 \geq \delta$, one has

$$T^{s+1,0}_{t_2-t_1}\big(Z^0_{s,1}(t_1)\big) = Z^0_{s,1}(t_2) \in \mathcal{G}_{s+1}\big(\varepsilon_0\big),$$

one can again apply Proposition 16 page 347. Therefore, one can iterate the process, and one obtains a subset $E_{j_2}\big(Z^0_{s,1}(t_2)\big)$ of $\mathbb{S}^{d-1}\times\mathbb{R}^d$ such that for every t_3 with $t_2 - t_3 \geq \delta$, $Z^0_{s,2}(t_3) \in \mathcal{G}_{s+2}(\varepsilon_0)$ and $\big(x^{0,j_3}_{s,2}(t_3)\big)\cdot e_1 \geq \rho$, and so on.

At each step of the process, that is for any integer $1 \leq m \leq n$, one obtained two subsets $U_{j_m}\big(Z^0_{s,m-1}(t_{m-1})\big)$ and $E_{j_m}\big(Z^0_{s,m-1}(t_m)\big)$ belonging respectively to $[0,T]$ and $\mathbb{S}^{d-1}\times B_{\mathbb{R}^d}(0,R)$, and verifying :

$$\Big|U_{j_m}\big(Z^0_{s,m-1}(t_{m-1})\big)\Big| \leq 2\frac{\rho}{\alpha}$$

and

$$\begin{aligned}\Big|E_{j_m}\big(Z^0_{s,m-1}(t_m)\big)\Big| \leq C(d)\Big(\ \eta^d + R^d\Big(\frac{a}{\rho}\Big)^{d-1} + kR^{2d-1}\Big(\frac{a}{\varepsilon_0}\Big)^{d-3/2} \\ + kR^{d+1/2}\Big(\frac{\varepsilon_0}{\delta}\Big)^{d-3/2} + R^{d-1}\alpha + R^d\alpha^{1/8}\Big),\end{aligned}$$

the bound being uniform in the respective parameters $Z^0_{s,m-1}(t_{m-1})$ and $Z^0_{s,m-1}(t_m)$, that is, uniform in all the parameters defining the pseudo-trajectories :

$$\begin{gathered} Z_s \in \Delta_s,\ 1 \leq j_1 \leq s,\ t_1 \in U^c_{j_1}(Z_s), \\ (\omega_1, v_{s+1}) \in E^c_{j_1}\big(Z^0_{s,0}(t_1)\big) = E^c_{j_1}\big(T^{s,0}_{t_1-t}(Z_s)\big), \\ \dots \\ 1 \leq j_m \leq s+m-1,\ t_m \in U^c_{j_m}\big(Z^0_{s,m-1}(t_{m-1})\big). \end{gathered}$$

□

Chapter 13

Pointwise convergence of the pseudo -trajectories and dominated convergence argument

One starts here the very core of the proof of Lanford's theorem : the comparison of the pseudo-trajectories, and the consequence on the convergence on the elementary terms of the two dynamics.

13.1 Divergence between the pseudo-trajectories of the two dynamics

One studies here carefully the difference in position and velocity between the pseudo-trajectories of the BBGKY and the Boltzmann dynamics. This is the purpose of the following lemma.

One recalls that the notations used in the statement of the following lemma, such as $T_k \in \mathfrak{T}_k$, $J_k \in \mathfrak{J}_k^s$, $A_k \in \mathfrak{A}_k$, $Z^0(T_k, J_k, A_k)$ and $Z^\varepsilon(T_k, J_k, A_k)$ concerning the pseudo-trajectories are introduced in Section 12.1.1 starting page 319 (see in particular Definitions 44, 45 page 323 and 46 page 324), Δ_s concerning the subset of preparation of the initial configurations is introduced in Definition 54 page 392, and finally U_{j_i} and E_{j_i} concerning the excluded pathological subsets of adjunction parameters are introduced in Definitions 59 page 433 and 58 page 432.

Lemma 38 (Uniform comparison of the pseudo-trajectories)**.** *Let ε, R and δ be three strictly positive numbers, s and k be two positive integers, $T_k = (t_1, \ldots, t_k)$ be an element of $\mathfrak{T}_k$, $J_k = (j_1, \ldots, j_k)$ be an element of $\mathfrak{J}_k^s$ and*

$A_k = \big((\omega_1, v_{s+1}), \dots, (\omega_k, v_{s+k})\big)$ be an element of $\mathfrak{A}_k$. One considers the backwards pseudo-trajectory of the Boltzmann hierarchy $Z^0(T_k, J_k, A_k)$, and the backwards pseudo-trajectory of the BBGKY hierarchy $Z^\varepsilon(T_k, J_k, A_k)$, both starting from $Z_s \in \big(\Omega^c \times \mathbb{R}^d\big)^s$.
For all strictly positive numbers $a, \varepsilon_0, \rho, \eta$ and α, if one assumes that :

$$4\sqrt{3}a \le \varepsilon_0,\ 3a \le \rho,\ \varepsilon_0 \le \eta\delta, \tag{13.1}$$

$$R \ge 1,\ \eta \le 1 \text{ and } \alpha \le c(d), \tag{13.2}$$

where $c(d)$ is a strictly positive constant depending only on the dimension d and

$$2k\varepsilon \le a, \tag{13.3}$$

$$Z_s \in \Delta_s\big(\varepsilon, R, \varepsilon_0, \alpha, \max(16Ra/\varepsilon_0, \varepsilon_0/\delta)\big), \tag{13.4}$$

and for all $1 \le i \le k$:

$$t_{i-1} - t_i \ge \delta,$$

(with t_0 denoting T)

$$t_i \notin U_{j_i}\big(\rho, Z^0_{s,i-1}(t_{i-1})\big)$$

and

$$(\omega_i, v_{s+i}) \notin E_{j_i}\big(R, \delta, \varepsilon, a, \varepsilon_0, \rho, \eta, \alpha\big)\big(Z^0_{s,i-1}(t_i)\big),$$

then one has :

$$\forall\ 1 \le j \le s+k,\ \big|x^{\varepsilon,j}_{s,k}(0) - x^{0,j}_{s,k}(0)\big| \le 2k\varepsilon, \tag{13.5}$$

that is

$$\Big|\big(Z^\varepsilon_{s,k}(T_k, J_k, A_k)(0)\big)^{X,j} - \big(Z^0_{s,k}(T_k, J_k, A_k)(0)\big)^{X,j}\Big| \le 2k\varepsilon,$$

and

$$\forall\ 1 \le i \le k,\ v^{\varepsilon,j_i}_{s,i-1}(t_i) = v^{0,j_i}_{s,i-1}(t_i), \tag{13.6}$$

that is

$$\big(Z^\varepsilon_{s,i-1}(T_k, J_k, A_k)(t_i)\big)^{V,j_i} = \big(Z^0_{s,i-1}(T_k, J_k, A_k)(t_i)\big)^{V,j_i}.$$

Proof. The result is obtained by induction. To be more accurate, the second point (13.6) of the Lemma will be established this way, step by step starting from the case $i = 1$, while the first one (13.5) will be obtained recursively, by showing the three following intermediate results.
The first of those intermediate results is that, for any integer $1 \le i \le k$:

$$Z^\varepsilon_{s,i-1}(t_i) \in \mathcal{G}^\varepsilon_{s+i-1}(\varepsilon),\ Z^0_{s,i-1}(t_i) \in \mathcal{G}^0_{s+i-1}(\varepsilon_0), \tag{13.7}$$

which can be rewritten as follows : for any integers $1 \leq l < m \leq s+i-1$ and for any strictly positive number τ :

$$\Big| T^{1,\varepsilon}_{-\tau}\Big(\big(Z^{\varepsilon}_{s,i-1}(T_k, J_k, A_k)(t_i)\big)^{X,l}\Big) - T^{1,\varepsilon}_{-\tau}\Big(\big(Z^{\varepsilon}_{s,i-1}(T_k, J_k, A_k)(t_i)\big)^{X,m}\Big)\Big| > \varepsilon$$

and

$$\Big| T^{1,0}_{-\tau}\Big(\big(Z^{0}_{s,i-1}(T_k, J_k, A_k)(t_i)\big)^{X,l}\Big) - T^{1,0}_{-\tau}\Big(\big(Z^{0}_{s,i-1}(T_k, J_k, A_k)(t_i)\big)^{X,m}\Big)\Big| > \varepsilon_0.$$

The second intermediate result is that, for all integers $1 \leq j \leq s+i-1$ and for all nonnegative numbers τ :

$$\Big| \Big(T^{s+i-1,\varepsilon}_{-\tau}\big(Z^{\varepsilon}_{s,i-1}(T_k, J_k, A_k)(t_{i-1})\big)\Big)^{X,j} - \Big(T^{s+i-1,0}_{-\tau}\big(Z^{0}_{s,i-1}(T_k, J_k, A_k)(t_{i-1})\big)\Big)^{X,j}\Big| \leq (2i-1)\varepsilon, \quad (13.8)$$

$$v^{\varepsilon,j}_{s,i-1}(t_i) = v^{0,j}_{s,i-1}(t_i) \text{ or } v^{\varepsilon,j}_{s,i-1}(t_i) = \mathcal{S}_0\big(v^{0,j}_{s,i-1}(t_i)\big), \quad (13.9)$$

and finally the third intermediate result is that, for any nonnegative number τ, if $v^{\varepsilon,j}_{s,i-1}(\tau) \neq v^{0,j}_{s,i-1}(\tau)$, (that is if $v^{\varepsilon,j}_{s,i-1}(\tau) = \mathcal{S}_0\big(v^{0,j}_{s,i-1}(\tau)\big)$), then

$$\big|x^{0,j}_{s,i-1}(\tau) \cdot e_1\big| \leq (2i-3/2)\varepsilon. \quad (13.10)$$

The three last conditions are nothing more than a control on the positions, and on the velocities of the particles of the pseudo-trajectories of the two hierarchies, just before the adjunctions of the additional particles. The three first intermediate conditions (13.7), (13.8) and (13.9) will be useful to apply Proposition 16 page 347.

Case $i = 1$.

First, thanks to the hypothesis (13.4), since Z_s is chosen inside

$$\Delta_s\big(\varepsilon, R, \varepsilon_0, \alpha, \max(16Ra/\varepsilon_0, \varepsilon_0/\delta)\big),$$

Proposition 17 page 394 can be applied. First, one knows that $T^{s,0}_{-\delta}(Z_s) \in \mathcal{G}^0_s(\varepsilon_0)$. In particular, since (thanks to the cut-off in small time difference between adjunctions) $T - t_1 \geq \delta$, one has :

$$T^{s,0}_{-t_1}(Z_s) = Z^0_{s,0}(t_1) \in \mathcal{G}^0_s(\varepsilon_0). \quad (13.11)$$

Second, one has also (again thanks to Proposition 17 page 394) that $Z_s \in \mathcal{G}_s^\varepsilon(\varepsilon)$. Then on the first hand, one has in particular

$$T_{-t_1}^{s,\varepsilon}(Z_s) = Z_{s,0}^\varepsilon(t_1) \in \mathcal{G}_s^\varepsilon(\varepsilon). \tag{13.12}$$

The two conditions (13.11) and (13.12) mean that the first intermediate condition (13.7) holds for the first case $i = 1$.
On the other hand, one can assert that for all $t_1 \leq \tau \leq T$, $Z^\varepsilon(\tau) = Z_{s,0}^\varepsilon(\tau)$ is obtained from Z_s using only the free transport with boundary condition for particles of radius $\varepsilon/2$. So, for each particle $1 \leq j_1 \leq s$, one has

$$x_{s,0}^{\varepsilon,j_1}(\tau) = x_{s,0}^{0,j_1}(\tau)$$

as long as $x_{s,0}^{\varepsilon,j_1}(\tau) \cdot e_1 \geq \varepsilon/2$ and $x_{s,0}^{0,j_1}(\tau) \cdot e_1 \geq 0$.
If this condition is fulfilled for all $t_1 \leq \tau \leq T$, the intermediate condition (13.8) on the position of the particles of the system is obviously true, and so is the condition (13.6), since no velocity is modified during the time interval $[t_1, T]$, due to the absence of bouncing against the obstacle.
If there exists $t_1 \leq \tau_{s,0}^{0,j_1} \leq T$ such that

$$x_{s,0}^{0,j_1}(\tau_{s,0}^{0,j_1}) \cdot e_1 = 0,$$

then by continuity there exists $\tau_{s,0}^{\varepsilon,j_1}$ such that

$$\tau_{s,0}^{0,j_1} \leq \tau_{s,0}^{\varepsilon,j_1} \leq T$$

and

$$x_{s,0}^{\varepsilon,j_1}(\tau_{s,0}^{\varepsilon,j_1}) \cdot e_1 = \varepsilon/2$$

(that is the particle of the BBGKY dynamics bounces first against the obstacle here).
Then for all $\tau_{s,0}^{0,j_1} \leq \tau \leq \tau_{s,0}^{\varepsilon,j_1}$, one has

$$\begin{cases} x_{s,0}^{0,j_1}(\tau) & = x_{j_1} - (T-\tau)v_{j_1}, \\ x_{s,0}^{\varepsilon,j_1}(\tau) & = x_{j_1} - (T - \tau_{s,0}^{\varepsilon,j_1})v_{j_1} - (\tau_{s,0}^{\varepsilon,j_1} - \tau)\mathcal{S}_0\big(v_{j_1}\big). \end{cases}$$

This gives

$$\begin{aligned} \big|x_{s,0}^{\varepsilon,j_1}(\tau) - x_{s,0}^{0,j_1}(\tau)\big| &= \big| - 2(\tau_s^{\varepsilon,m_1} - \tau)v_{j_1} \cdot e_1 e_1\big| \\ &\leq 2\big|\tau_{s,0}^{\varepsilon,j_1} - \tau_{s,0}^{0,j_1}\big| \cdot \big|v_{j_1} \cdot e_1\big|, \end{aligned}$$

and one deduces that

$$\big|x_{s,0}^{\varepsilon,j_1}(\tau) - x_{s,0}^{0,j_1}(\tau)\big| \leq \varepsilon$$

since

$$\tau_{s,0}^{0,j_1} = (x_{j_1} \cdot e_1)/v_{j_1} \cdot e_1,$$

and

$$\tau_{s,0}^{\varepsilon,j_1} = \big(x_{j_1} \cdot e_1 - \varepsilon/2\big)/v_{j_1} \cdot e_1,$$

those two last times being well defined since $Z_s \in \Delta_s$ implies in particular that the velocities composing this initial configuration are not grazing, namely $v_{j_1} \cdot e_1 \geq \alpha > 0$.
Similarly, for all $t_1 \leq \tau \leq \tau_{s,0}^{0,j_1}$, one has :

$$\begin{cases} x_{s,0}^{0,j_1}(\tau) &= x_{j_1} - (T - \tau_{s,0}^{0,j_1})v_{j_1} - (\tau_{s,0}^{0,j_1} - \tau)\mathcal{S}_0\big(v_{j_1}\big), \\ x_{s,0}^{\varepsilon,j_1}(\tau) &= x_{j_1} - (T - \tau_{s,0}^{\varepsilon,j_1})v_{j_1} - (\tau_{s,0}^{\varepsilon,j_1} - \tau)\mathcal{S}_0\big(v_{j_1}\big). \end{cases}$$

This gives

$$\big|x_{s,0}^{\varepsilon,j_1}(\tau) - x_{s,0}^{0,j_1}(\tau)\big| = \big|2(\tau_{s,0}^{0,j_1} - \tau_{s,0}^{\varepsilon,j_1})v_{j_1} \cdot e_1 e_1\big| = \varepsilon.$$

Moreover, one has

$$v_{s,0}^{0,j_1}(t_1) = \mathcal{S}_0(v_{j_1})$$

and

$$v_{s,0}^{\varepsilon,j_1}(t_1) = \mathcal{S}_0(v_{j_1}),$$

so that the condition (13.6) on the velocity of the pseudo-trajectories, which is stated for the specific particle j_1 in the case $i = 1$, is fulfilled for this case $i = 1$ and when the particle j_1 of size zero (that is for the free flow dynamics) bounces against the obstacle before the time t_1.

Now one will show that either both of the particles $x_{s,0}^{0,j_1}(\tau)$ and $x_{s,0}^{\varepsilon,j_1}(\tau)$ of respective radius zero and $\varepsilon/2$ bounce against the obstacle during the same time interval $[t_1, T]$, or none of those particles bounces. Of course, this will be obtained thanks to the hypothesis of the cut-off in the time variable t_1, which prevents the particle j_1 to be too close to the obstacle during the adjunction. This is mandatory to obtain this result, otherwise, at adjunction time t_1, one can have :

$$v_{s,0}^{0,j_1}(t_1) \neq v_{s,0}^{\varepsilon,j_1}(t_1).$$

One has shown that if the particle of radius zero bounces, then so does the one of strictly positive radius, at an anterior time, for the specific case of $i = 1$. In other words, if $\tau_{s,0}^{0,j_1} \in [t_1, T]$, then one has also $\tau_{s,0}^{\varepsilon,j_1} \in [t_1, T]$.
Conversely, if the particle of radius $\varepsilon/2$ bounces against the obstacle, say at time $\tau_{s,0}^{\varepsilon,j_1}$ with $t_1 \leq \tau_{s,0}^{\varepsilon,j_1} \leq T$, one knows that the time interval delimited by t_1 and $\tau_{s,0}^{\varepsilon,j_1}$ is not too small. Indeed, by hypothesis $t_1 \notin U_{j_1}(\rho, Z_s)$, which means that :

$$x_{s,0}^{0,m_1}(t_1) \cdot e_1 \geq \rho.$$

Using the explicit expressions for the times $\tau_{s,0}^{\varepsilon,j_1}$ and $\tau_{s,0}^{\varepsilon,j_1}$, one obtains :

$$\tau_{s,0}^{\varepsilon,j_1} - \tau_{s,0}^{\varepsilon,j_1} = \frac{\varepsilon}{2v_{j_1} \cdot e_1} \leq \frac{\varepsilon}{2\alpha}.$$

But now thanks to the crucial hypothesis (13.3), and the relation between a and ρ asserted in (13.1), one has :

$$\tau_{s,0}^{0,j_1} - \tau_{s,0}^{\varepsilon,j_1} \leq \frac{a}{2k\alpha} \leq \frac{\rho}{6k\alpha} \leq \frac{\rho}{\alpha}.$$

In other words, the time $\tau_{s,0}^{\varepsilon,j_1}$ belongs to the time interval $U_{j_1}(\rho, Z_s)$, while by hypothesis t_1 does not belong to this time interval. As an obvious consequence

$$t_1 < \tau_{s,0}^{\varepsilon,j_1} < \tau_{s,0}^{0,j_1},$$

or in other words : if the particle j_1 bounces against the obstacle for the BBGKY dynamics in the time interval $[t_1, T]$, then so does the same particle for the Boltzmann dynamics.
Therefore, one has indeed that, in the time interval $[t_1, T]$, the particle j_1 of the Boltzmann dynamics bounces against the obstacle if and only if the particle j_1 of the BBGKY dynamics bounces against the obstacle too.
So, and in particular thanks to the condition (13.3), the intermediate control (13.8) stated at the beginning of the proof of the Lemma is fulfilled for the case $i = 1$.

For the difference between the velocities of all the particles of the two pseudo-trajectories, that is the intermediate result (13.9), one uses again the fact that the hard sphere flow from the initial configuration Z_s does not lead to any recollision between two particles of the system of s particles, that is

$$Z_s \in \mathcal{G}_s^{\varepsilon}(\varepsilon).$$

As a consequence, the only way to observe a modification in the velocities of the particles of the pseudo-trajectory of the BBGKY hierarchy starting from the configuration Z_s is through a bouncing against the obstacle. Therefore, one can have only, for all integer $1 \leq j \leq s$:

$$v_{s,0}^{\varepsilon,j}(t_1) = v_{s,0}^{\varepsilon,j}(T) = v_j$$

or

$$v_{s,0}^{\varepsilon,j}(t_1) = \mathcal{S}_0\big(v_{s,0}^{\varepsilon,j}(T)\big) = \mathcal{S}_0(v_j).$$

For the pseudo-trajectories of the Boltzmann hierarchy, this is also the only way to modify the velocities (for this dynamics, it is even true in general, regardless any preparation of the initial configuration), and since $v_{s,0}^{0,j}(T) = v_{s,0}^{\varepsilon,j}(T) = v_j$, the last intermediate result (13.9) holds for the case $i = 1$.
Finally, for the last intermediate result (13.10), one notices again that from the configuration Z_s (which is, at this step, the starting point for the pseudo-trajectories of the two dynamics), if a pair j of particles bounces with the obstacle (using one more time the fact that the trajectories are explicitly given for the two dynamics, because there is no recollision), then the particle of radius $\varepsilon/2$, that is, the one following the hard sphere flow, is the first to bounce. Before this bouncing, and after the bouncing of the particle following the free flow with boundary condition, the two particles of the pair have the same velocity, which is nothing else than

$$v_j = (Z_s)^{V,j}.$$

So the particles of the pair j have distinct velocities only on the time interval

$$[\tau_{s,0}^{0,j}, \tau_{s,0}^{\varepsilon,j}],$$

with

$$\big(x_j - \tau_{s,0}^{\varepsilon,j} v_j\big) \cdot e_1 = \varepsilon/2$$

and

$$\big(x_j - \tau_{s,0}^{0,j} v_j\big) \cdot e_1 = 0,$$

this time interval being therefore characterized by the condition

$$0 \leq (x_j - \tau v_j) \cdot e_1 \leq \varepsilon/2.$$

If $\tau = t_i$ belongs to $[\tau_{s,0}^{0,j}, \tau_{s,0}^{\varepsilon,j}]$, the intermediate result (13.10) page 451 is then obtained for the first case $i = 1$.

Case $i > 1$.

Now, let i be a positive integer smaller than k, strictly larger than 1. To perform the proof by recursion, one will assume that

$$\begin{aligned} Z_{s,i-2}^{\varepsilon}\big(T_k, J_k, A_k\big)(t_{i-1}) &\in \mathcal{G}_{s+i-2}^{\varepsilon}(\varepsilon), \\ Z_{s,i-2}^{0}\big(T_k, J_k, A_k\big)(t_{i-1}) &\in \mathcal{G}_{s+i-2}^{0}(\varepsilon_0). \end{aligned} \tag{13.13}$$

One assumes also that for all integers $1 \leq j \leq s+i-2$ and for all nonnegative numbers τ :

$$\Big| T_{-\tau}^{1,\varepsilon}\big(x_{s,i-2}^{\varepsilon,j}(t_{i-2})\big) - T_{-\tau}^{1,0}\big(x_{s,i-2}^{0,j}(t_{i-2})\big)\Big| \leq (2i-3)\varepsilon,$$

that is

$$\begin{aligned} \Big| \Big(T_{-\tau}^{s+i-2,\varepsilon}\big(Z_{s,i-2}^{\varepsilon}(T_k, J_k, A_k)(t_{i-2})\big)\Big)^{X,j} \\ - \Big(T_{-\tau}^{s+i-2,0}\big(Z_{s,i-2}^{0}(T_k, J_k, A_k)(t_{i-2})\big)\Big)^{X,j} \Big| \leq (2i-3)\varepsilon, \end{aligned} \tag{13.14}$$

the two assumptions (13.13) and (13.14) are the statements ot the two intermediate results (13.7) and (13.8) for the particular case $i-1$. One assumes finally that, for every integer $1 \leq j \leq s+i-2$,

$$v_{s,i-2}^{\varepsilon,j}(t_{i-1}) = v_{s,i-2}^{0,j}(t_{i-1}) \text{ or } v_{s,i-2}^{\varepsilon,j}(t_{i-1}) = \mathcal{S}_0\big(v_{s,i-2}^{0,j}(t_{i-1})\big), \tag{13.15}$$

and for every integer $1 \leq j \leq s+i-2$, and every nonnegative number τ, if $v_{s,i-2}^{\varepsilon,j}(\tau) \neq v_{s,i-2}^{0,j}(\tau)$, then

$$\big|\big(x_{s,i-2}^{0,j}(\tau)\big) \cdot e_1\big| \leq (2i-1/2)\varepsilon, \tag{13.16}$$

which are the last intermediate results (13.9) and (13.10) stated page 451 at the beginning of the Lemma, assumed to be true for the particular case $i-1$. One notes that those three hypotheses were obtained in the first part of the proof for the specific case $i = 1$.

By definition of the backwards pseudo-trajectories for the free-flow with boundary condition and for the hard sphere dynamics (see Definitions 45 page 323 and 46 page 324), a particle is added at time t_{i-1} next to the particle j_{i-1}, with an angular parameter ω_{i-1} and a velocity v_{s+i-1}. Explicitly, one recalls that the configuration obtained after the adjunction, first for the Boltzmann pseudo-trajectory, is

$$Z^0_{s,i-1}\big(T_k, J_k, A_k\big)(t_{i-1}) = \Big(Z^0_{s,i-2}\big(T_k, J_k, A_k\big)(t_{i-1}), x^{0,j_{i-1}}_{s,i-2}(t_{i-1}), v_{s+i-1}\Big)$$

if $m_{i-1} = -$ (pre-collisional case) and

$$\begin{aligned} &Z^0_{s,i-1}\big(T_k, J_k, A_k\big)(t_{i-1}) \\ &\qquad = \Big(Z^0_{s,i-2}\big(T_k, J_k, A_k\big)(t_{i-1}), x^{0,j_{i-1}}_{s,i-2}(t_{i-1}), v_{s+i-1}\Big)'_{j_{i-1},s+i-1} \end{aligned}$$

if $m_{i-1} = +$ (post-collisional case), the scattering being applied with the angular parameter ω_{i-1} (see Definition 1 page 51), and second for the BBGKY pseudo-trajectory, is

$$\begin{aligned} &Z^\varepsilon_{s,i-1}\big(T_k, J_k, A_k\big)(t_{i-1}) \\ &\qquad = \Big(Z^0_{s,i-2}\big(T_k, J_k, A_k\big)(t_{i-1}), x^{0,j_{i-1}}_{s,i-2}(t_{i-1}) + \varepsilon\omega_{i-1}, v_{s+i-1}\Big) \end{aligned}$$

if $m_{i-1} = -$ and

$$\begin{aligned} &Z^\varepsilon_{s,i-1}\big(T_k, J_k, A_k\big)(t_{i-1}) \\ &\qquad = \Big(Z^0_{s,i-2}\big(T_k, J_k, A_k\big)(t_{i-1}), x^{0,j_{i-1}}_{s,i-2}(t_{i-1}) + \varepsilon\omega_{i-1}, v_{s+i-1}\Big)'_{j_{i-1},s+i-1} \end{aligned}$$

if $m_{i-1} = +$, the scattering being also applied with the angular parameter ω_{i-1}.

Thanks to the cut-off in proximity with the obstacle (that is, thanks to the hypothesis stating that $t_{i-1} \notin U_{j_{i-1}}$, see Definition 59 page 433), one knows that

$$x^{0,j_{i-1}}_{s,i-2}(t_{i-1}) \cdot e_1 \geq \rho.$$

Moreover, one has assumed the conditions (13.15), (13.15) and (13.10). Therefore, this large enough distance between the particle j_{i-1} and the obstacle and the conditions on the velocities enable to use Proposition 16 page 347 : by hypothesis, the adjunction parameters ω_{i-1} and v_{s+i-1} are chosen outside $E_{j_{i-1}}\big(Z^0_{s,i-2}(t_{i-1})\big)$. As a consequence, by definition of $E_{j_{i-1}}$ (see Definition 58 page 432), with the hypotheses of good configuration (13.13), and if

$$(2i-3)\varepsilon \leq a$$

then, one has first :

$$T^{s+i-1,0}_{-\delta}\big(Z^0_{s,i-1}(t_{i-1})\big) \in \mathcal{G}^0_{s+i-1}(\varepsilon_0)$$

and

$$Z^{\varepsilon}_{s,i-1}(t_{i-1}) \in \mathcal{G}^{\varepsilon}_{s+i-1}(\varepsilon). \tag{13.17}$$

Therefore, one has of course :

$$T^{s+i-1,\varepsilon}_{-(t_{i-1}-t_i)}\big(Z^{\varepsilon}_{s,i-1}(t_{i-1})\big) = Z^{\varepsilon}_{s,i-1}(t_i) \in \mathcal{G}^{\varepsilon}_{s+i-1}(\varepsilon),$$

and since $t_{i-1} - t_i \geq \delta$:

$$T^{s+i-1,0}_{-(t_{i-1}-t_i)}\big(Z^{0}_{s,i-1}(t_{i-1})\big) = Z^{0}_{s,i-1}(t_i) \in \mathcal{G}^{0}_{s+i-1}(\varepsilon_0),$$

those two last condition being exactly the first intermediate result (13.7) page 450.

Now one studies the velocities of the particles of the two pseudo-trajectories at time t_{i-1}, *after* the adjunction of the particle $s+i-1$.
By definition the velocities of all the particles $1 \leq j \leq s+i-2$, with $j \neq j_{i-1}$, and also the velocity of the particles j_{i-1} in the case of a pre-collisional configuration, are not changed during the adjunction (for the two dynamics), that is :

$$v^{0,j}_{s,i-1}(t_{i-1}) = v^{0,j}_{s,i-2}(t_{i-1}) \text{ and } v^{\varepsilon,j}_{s,i-1}(t_{i-1}) = v^{\varepsilon,j}_{s,i-2}(t_{i-1}).$$

In the case of a pre-collisional configuration, by definition of the adjunction one has :

$$v^{0,s+i-1}_{s,i-1}(t_{i-1}) = v^{\varepsilon,s+i-1}_{s,i-1}(t_{i-1}) = v_{s+i-1}.$$

In the case of a post-collisional configuration, the velocities $v^{0,j_{i-1}}_{s,i-1}$ and $v^{\varepsilon,j_{i-1}}_{s,i-1}$ are computed from the velocities $v^{0,j_{i-1}}_{s,i-2}$ and $v^{\varepsilon,j_{i-1}}_{s,i-2}$ of the pair of particles j_{i-1} of the two pseudo-trajectories according to the scattering, computed with the same second velocity v_{s+i-1} for the second particles $s+i-1$ colliding, and with the same angular parameter ω_{i-1}, for both of the two dynamics. But the velocities before the adjunction are linked thanks to the hypothesis (13.15), hence one has, regardless whether the configuration after the adjunction is pre or post-collisional, for all $1 \leq j \leq s+i-1$:

$$v^{\varepsilon,j}_{s,i-1}(t_{i-1}) = v^{0,j}_{s,i-1}(t_{i-1}) \text{ or } v^{\varepsilon,j}_{s,i-1}(t_{i-1}) = \mathcal{S}_0\big(v^{0,j}_{s,i-1}(t_{i-1})\big). \tag{13.18}$$

One knows that the only way to observe a modification in the velocity of the particles along the trajectories of the free flow dynamics with boundary condition is during a bouncing against the obstacle. In other words, one has for every time $\tau \leq t_{i-1}$:

$$v^{0,j}_{s,i-1}(\tau) = v^{0,j}_{s,i-1}(t_{i-1}) \text{ or } v^{0,j}_{s,i-1}(\tau) = \mathcal{S}_0\big(v^{0,j}_{s,i-1}(t_{i-1})\big). \tag{13.19}$$

Besides, using the intermediate result (13.17) page 457, one knows that no recollision can occur in the past (since one considers the backwards dynamics) following the hard sphere flow dynamics starting from the configuration $Z^{\varepsilon}_{s,i-1}(t_{i-1})$.

As a consequence, and exactly as the free flow dynamics, only the bouncings can change the velocity of the particles following the hard sphere dynamics. Therefore, one can describe explicitly the velocities of the pseudo-trajectory of the BBGKY hierarchy starting from this configuration, and one has also :

$$v^{\varepsilon,j}_{s,i-1}(\tau) = v^{\varepsilon,j}_{s,i-1}(t_{i-1}) \text{ or } v^{\varepsilon,j}_{s,i-1}(\tau) = \mathcal{S}_0\big(v^{\varepsilon,j}_{s,i-1}(t_{i-1})\big). \tag{13.20}$$

Collecting the statements (13.18), (13.19) and (13.20) together with the obvious fact that the symmetry $\mathcal{S}_0$ is an involution, one obtains that for all integer $1 \leq j \leq s+i-1$:

$$v^{\varepsilon,j}_{s,i-1}(t_i) = v^{0,j}_{s,i-1}(t_i) \text{ or } v^{\varepsilon,j}_{s,i-1}(t_i) = \mathcal{S}_0\big(v^{0,j}_{s,i-1}(t_i)\big),$$

which is exactly the intermediate result (13.9) page 451, for the particular case i.

As for the difference between the positions of the same particle following the two pseudo-trajectories, first one considers the case of the particles which are not affected by the adjunctions. If the adjunction $i-1$ is post-collisional, the velocities of all particles j such that $j \neq j_{i-1}$ and $j \neq s+i-1$ are not modified by the adjunction, and if the adjunction $i-1$ is pre-collisional, the velocities of all particles j such that $j \neq s+i-1$ are not modified by the adjunction, as it was already recalled just above. Therefore, since such particles follow the free flow, one has for all $\tau \in [t_i, t_{i-1}]$:

$$\begin{aligned}
x^{\varepsilon,j}_{s,i-1}(\tau) &= T^{1,\varepsilon}_{-(t_{i-1}-\tau)}\big((x^{\varepsilon,j}_{s,i-1}(t_{i-1}), v^{\varepsilon,j}_{s,i-1}(t_{i-1}))\big) \\
&= \Big(T^{1,\varepsilon}_{-(t_{i-1}-\tau)}\big((x^{\varepsilon,j}_{s,i-2}(t_{i-1}), v^{\varepsilon,j}_{s,i-2}(t_{i-1}))\big)\Big)^X \\
&= \Big(T^{1,\varepsilon}_{-(t_{i-1}-\tau)}\big(T^{1,\varepsilon}_{-(t_{i-2}-t_{i-1})}\big((x^{\varepsilon,j}_{s,i-2}(t_{i-2}), v^{\varepsilon,j}_{s,i-2}(t_{i-2}))\big)\big)\Big)^X \\
&= \Big(T^{1,\varepsilon}_{-(t_{i-2}-\tau)}\big((x^{\varepsilon,j}_{s,i-2}(t_{i-2}), v^{\varepsilon,j}_{s,i-2}(t_{i-2}))\big)\Big)^X
\end{aligned}$$

for the BBGKY pseudo-trajectory, and similarly for the Boltzmann pseudo-trajectory :

$$x^{0,j}_{s,i-1}(\tau) = \Big(T^{1,0}_{-(t_{i-2}-\tau)}\big((x^{0,j}_{s,i-2}(t_{i-2}), v^{0,j}_{s,i-2}(t_{i-2}))\big)\Big)^X.$$

For such pairs of particles, the result concerning the difference of the positions is then a direct consequence of the intermediate hypothesis (13.14).
If the adjunction $i-1$ is post-collisional, at the time t_{i-1} of the adjunction, the velocities of the particles of the pair j_{i-1} are the same, equal to

$$v^{0,j_{i-1}}_{s,i-1}(t_{i-1}) = v^{0,j_{i-1}}_{s,i-1}(t_{i-1}) = \big(v^{0,j_{i-1}}_{s,i-2}(t_{i-1})\big)',$$

$\big(v^{0,j_{i-1}}_{s,i-2}(t_{i-1})\big)'$ denoting the first velocity obtained after applying the scattering operator to the pair of velocities $\big(v^{0,j_{i-1}}_{s,i-2}(t_{i-1}), v_{s+i-1}\big)$, with the angular parameter ω_{i-1}. According to Proposition 16, one knows that the two particles of

the pair j_i follow the free flow, so that one has for all $\tau \le t_{i-1}$:

$$x_{s,i-1}^{\varepsilon,j_{i-1}}(\tau) = T^{1,\varepsilon}_{-(t_{i-1}-\tau)}\Big(\big(x_{s,i-1}^{\varepsilon,j_{i-1}}(t_{i-1}), \big(v_{s,i-2}^{0,j_{i-1}}(t_{i-1})\big)'\big)\Big)$$

for the BBGKY pseudo-trajectory, and similarly for the Boltzmann pseudo-trajectory :

$$x_{s,i-1}^{0,j_{i-1}}(\tau) = \Big(T^{1,0}_{-(t_{i-2}-\tau)}\big(\big(x_{s,i-2}^{0,j}(t_{i-2}), \big(v_{s,i-2}^{0,j_{i-1}}(t_{i-1})\big)'\big)\big)\Big)^X.$$

Then, depending on the sign of the scalar product $\big(v_{s,i-2}^{0,j_{i-1}}(t_{i-1})\big)' \cdot e_1$, either both of the particles of the pair bounce against the obstacle, or none does.
If none does, then clearly

$$\begin{aligned}\big|x_{s,i-1}^{\varepsilon,j_{i-1}}(\tau) - x_{s,i-1}^{0,j_{i-1}}(\tau)\big| &= \Big|\Big(x_{s,i-1}^{\varepsilon,j_{i-1}}(t_{i-1}) - (t_{i-1}-\tau)\big(v_{s,i-2}^{0,j_{i-1}}(t_{i-1})\big)'\Big) \\ &\qquad - \Big(x_{s,i-1}^{0,j_{i-1}}(t_{i-1}) - (t_{i-1}-\tau)\big(v_{s,i-2}^{0,j_{i-1}}(t_{i-1})\big)'\Big)\Big| \\ &= \big|x_{s,i-1}^{\varepsilon,j_{i-1}}(t_{i-1}) - x_{s,i-1}^{0,j_{i-1}}(t_{i-1})\big|,\end{aligned}$$

the right-hand side being smaller than $(2i-3)\varepsilon$, since it is equal to

$$\Big|T^{1,\varepsilon}_{-(t_{i-2}-t_{i-1})}\big(x_{s,i-2}^{\varepsilon,j_{i-1}}(t_{i-2})\big) - T^{1,0}_{-(t_{i-2}-t_{i-1})}\big(x_{s,i-2}^{0,j_{i-1}}(t_{i-2})\big)\Big|,$$

this quantity being controlled thanks to the intermediate hypothesis (13.14) page 455.
If both particles of the pair j_{i-1} bounce against the obstacle, the trajectory of the particle j_{i-1} of the Boltzmann pseudo-trajectory is given by

$$x_{s,i-1}^{0,j_{i-1}}(\tau) = x_{s,i-1}^{0,j_{i-1}}(t_{i-1}) - (t_{i-1}-\tau)\big(v_{s,i-2}^{0,j_{i-1}}(t_{i-1})\big)'$$

if $\tau_{s,i-1}^{0,j_{i-1}} \le \tau \le t_{i-1}$, and

$$\begin{aligned}x_{s,i-1}^{0,j_{i-1}}(\tau) &= x_{s,i-1}^{0,j_{i-1}}(t_{i-1}) - (t_{i-1}-\tau_{s,i-1}^{0,j_{i-1}})\big(v_{s,i-2}^{0,j_{i-1}}(t_{i-1})\big)' \\ &\qquad - (\tau_{s,i-1}^{0,j_{i-1}}-\tau)\mathcal{S}_0\big(\big(v_{s,i-2}^{0,j_{i-1}}(t_{i-1})\big)'\big) \\ &= \mathcal{S}_0\big(x_{s,i-1}^{0,j_{i-1}}(t_{i-1}) - (t_{i-1}-\tau)\big(v_{s,i-2}^{0,j_{i-1}}(t_{i-1})\big)'\big),\end{aligned}$$

if $\tau \le \tau_{s,i-1}^{0,j_{i-1}}$, with $\tau_{s,i-1}^{0,j_{i-1}}$ verifying

$$\Big(x_{s,i-1}^{0,j_{i-1}}(t_{i-1}) - (t_{i-1}-\tau_{s,i-1}^{0,j_{i-1}})\big(v_{s,i-2}^{0,j_{i-1}}(t_{i-1})\big)'\Big) \cdot e_1 = 0.$$

The trajectory of the particle j_{i-1} of the BBGKY pseudo-trajectory is given similarly by

$$x_{s,i-1}^{\varepsilon,j_{i-1}}(\tau) = x_{s,i-1}^{\varepsilon,j_{i-1}}(t_{i-1}) - (t_{i-1}-\tau)\big(v_{s,i-2}^{0,j_{i-1}}(t_{i-1})\big)'$$

if $\tau_{s,i-1}^{\varepsilon,j_{i-1}} \leq \tau \leq t_{i-1}$, and

$$\begin{aligned} x_{s,i-1}^{\varepsilon,j_{i-1}}(\tau) &= x_{s,i-1}^{\varepsilon,j_{i-1}}(t_{i-1}) - (t_{i-1} - \tau_{s,i-1}^{\varepsilon,j_{i-1}})\big(v_{s,i-2}^{0,j_{i-1}}(t_{i-1})\big)' \\ &\qquad - (\tau_{s,i-1}^{\varepsilon,j_{i-1}} - \tau)\mathcal{S}_0\big(\big(v_{s,i-2}^{0,j_{i-1}}(t_{i-1})\big)'\big) \\ &= \mathcal{S}_\varepsilon\big(x_{s,i-1}^{\varepsilon,j_{i-1}}(t_{i-1}) - (t_{i-1} - \tau)\big(v_{s,i-2}^{0,j_{i-1}}(t_{i-1})\big)'\big) \end{aligned}$$

if $\tau \leq \tau_{s,i-1}^{\varepsilon,j_{i-1}}$, with $\tau_{s,i-1}^{\varepsilon,j_{i-1}}$ verifying

$$\Big(x_{s,i-1}^{\varepsilon,j_{i-1}}(t_{i-1}) - (t_{i-1} - \tau_{s,i-1}^{\varepsilon,j_{i-1}})\big(v_{s,i-2}^{0,j_{i-1}}(t_{i-1})\big)'\Big) \cdot e_1 = \varepsilon/2,$$

and where $\mathcal{S}_\varepsilon$ denotes the orthogonal symmetry with respect to the hyperplane $x \cdot e_1 = \varepsilon/2$.
Clearly, if $\tau \geq \max\big(\tau_{s,i-1}^{0,j_{i-1}}, \tau_{s,i-1}^{\varepsilon,j_{i-1}}\big)$, one has

$$\begin{aligned} \Big|x_{s,i-1}^{\varepsilon,j_{i-1}}(\tau) - x_{s,i-1}^{0,j_{i-1}}(\tau)\Big| &= \Big|\Big(x_{s,i-1}^{\varepsilon,j_{i-1}}(t_{i-1}) - (t_{i-1} - \tau)\big(v_{s,i-2}^{0,j_{i-1}}(t_{i-1})\big)'\Big) \\ &\qquad - \Big(x_{s,i-1}^{0,j_{i-1}}(t_{i-1}) - (t_{i-1} - \tau)\big(v_{s,i-2}^{0,j_{i-1}}(t_{i-1})\big)'\Big)\Big|, \end{aligned}$$

controlled as for the case of no bouncing, and then this quantity is smaller than $(2i-3)\varepsilon$.
If $\min\big(\tau_{s,i-1}^{0,j_{i-1}}, \tau_{s,i-1}^{\varepsilon,j_{i-1}}\big) \leq \tau \leq \max\big(\tau_{s,i-1}^{0,j_{i-1}}, \tau_{s,i-1}^{\varepsilon,j_{i-1}}\big)$, one has :

$$\begin{aligned} \Big|x_{s,i-1}^{\varepsilon,j_{i-1}}(\tau) - x_{s,i-1}^{0,j_{i-1}}(\tau)\Big| &= \Big|\Big(x_{s,i-1}^{\varepsilon,j_{i-1}}(t_{i-1}) - (t_{i-1} - \tau)\big(v_{s,i-2}^{0,j_{i-1}}(t_{i-1})\big)'\Big) \\ &\qquad - \mathcal{S}_0\big(x_{s,i-1}^{0,j_{i-1}}(t_{i-1}) - (t_{i-1} - \tau)\big(v_{s,i-2}^{0,j_{i-1}}(t_{i-1})\big)'\big)\Big| \end{aligned}$$

or

$$\begin{aligned} \big|x_{s,i-1}^{\varepsilon,j_{i-1}}(\tau) - x_{s,i-1}^{0,j_{i-1}}(\tau)\big| &= \Big|\mathcal{S}_\varepsilon\big(x_{s,i-1}^{\varepsilon,j_{i-1}}(t_{i-1}) - (t_{i-1} - \tau)\big(v_{s,i-2}^{0,j_{i-1}}(t_{i-1})\big)'\big) \\ &\qquad - \Big(x_{s,i-1}^{0,j_{i-1}}(t_{i-1}) - (t_{i-1} - \tau)\big(v_{s,i-2}^{0,j_{i-1}}(t_{i-1})\big)'\Big)\Big|. \end{aligned}$$

One uses here once again a result appearing in the proof of Lemma 27 page 335, namely, for all points x_1 and x_2 belonging to $x \cdot e_1 = \alpha/2$, one has

$$\big|x_1 - \mathcal{S}_\alpha(x_2)\big| \geq \big|x_1 - x_2\big|,$$

with $\mathcal{S}_\alpha$ denoting as usual the orthogonal symmetry with respect to the hyperplane $\big\{x \in \mathbb{R}^d \ / \ x \cdot e_1 = \alpha/2\big\}$.
As a consequence, if $\tau_{s,i-1}^{\varepsilon,j_{i-1}} \leq \tau_{s,i-1}^{0,j_{i-1}}$, and if $\tau_{s,i-1}^{\varepsilon,j_{i-1}} \leq \tau \leq \tau_{s,i-1}^{0,j_{i-1}}$, one has

$$\mathcal{S}_0\big(x_{s,i-1}^{0,j_{i-1}}(t_{i-1}) - (t_{i-1} - \tau)\big(v_{s,i-2}^{0,j_{i-1}}(t_{i-1})\big)'\big) \cdot e_1 \geq 0$$

and then :

$$\begin{aligned}
\big|x_{s,i-1}^{\varepsilon,j_{i-1}}(\tau)-x_{s,i-1}^{0,j_{i-1}}(\tau)\big| &= \Big|\Big(x_{s,i-1}^{\varepsilon,j_{i-1}}(t_{i-1})-(t_{i-1}-\tau)\big(v_{s,i-2}^{0,j_{i-1}}(t_{i-1})\big)'\Big) \\
&\qquad -\mathcal{S}_0\big(x_{s,i-1}^{0,j_{i-1}}(t_{i-1})-(t_{i-1}-\tau)\big(v_{s,i-2}^{0,j_{i-1}}(t_{i-1})\big)'\big)\Big| \\
&\le \Big|\Big(x_{s,i-1}^{\varepsilon,j_{i-1}}(t_{i-1})-(t_{i-1}-\tau)\big(v_{s,i-2}^{0,j_{i-1}}(t_{i-1})\big)'\Big) \\
&\qquad -\mathcal{S}_0\big(\mathcal{S}_0\big(x_{s,i-1}^{0,j_{i-1}}(t_{i-1})-(t_{i-1}-\tau)\big(v_{s,i-2}^{0,j_{i-1}}(t_{i-1})\big)'\big)\big)\Big| \\
&\le \big|x_{s,i-1}^{\varepsilon,j_{i-1}}(t_{i-1})-x_{s,i-1}^{0,j_{i-1}}(t_{i-1})\big|.
\end{aligned}$$

Again, the right-hand side is bounded by $(2i-3)\varepsilon$.
If $\tau_{s,i-1}^{0,j_{i-1}} \le \tau_{s,i-1}^{\varepsilon,j_{i-1}}$, and if $\tau_{s,i-1}^{0,j_{i-1}} \le \tau \le \tau_{s,i-1}^{\varepsilon,j_{i-1}}$, one has

$$\mathcal{S}_\varepsilon\big(x_{s,i-1}^{\varepsilon,j_{i-1}}(t_{i-1})-(t_{i-1}-\tau)\big(v_{s,i-2}^{0,j_{i-1}}(t_{i-1})\big)'\big)\cdot e_1 \ge \varepsilon/2$$

and since

$$\big(x_{s,i-1}^{0,j_{i-1}}(t_{i-1})-(t_{i-1}-\tau)\big(v_{s,i-2}^{0,j_{i-1}}(t_{i-1})\big)'+\frac{\varepsilon}{2}e_1\big)\cdot e_1 \ge \varepsilon/2,$$

then :

$$\begin{aligned}
\big|x_{s,i-1}^{\varepsilon,j_{i-1}}(\tau)-x_{s,i-1}^{0,j_{i-1}}(\tau)\big| &= \Big|\mathcal{S}_\varepsilon\big(x_{s,i-1}^{\varepsilon,j_{i-1}}(t_{i-1})-(t_{i-1}-\tau)\big(v_{s,i-2}^{0,j_{i-1}}(t_{i-1})\big)'\big) \\
&\qquad -\Big(x_{s,i-1}^{0,j_{i-1}}(t_{i-1})-(t_{i-1}-\tau)\big(v_{s,i-2}^{0,j_{i-1}}(t_{i-1})\big)'+\frac{\varepsilon}{2}e_1-\frac{\varepsilon}{2}e_1\Big)\Big| \\
&\le \Big|\mathcal{S}_\varepsilon\big(\mathcal{S}_\varepsilon\big(x_{s,i-1}^{\varepsilon,j_{i-1}}(t_{i-1})-(t_{i-1}-\tau)\big(v_{s,i-2}^{0,j_{i-1}}(t_{i-1})\big)'\big)\big) \\
&\qquad -\Big(x_{s,i-1}^{0,j_{i-1}}(t_{i-1})-(t_{i-1}-\tau)\big(v_{s,i-2}^{0,j_{i-1}}(t_{i-1})\big)'+\frac{\varepsilon}{2}e_1\Big)\Big| \\
&\qquad +\frac{\varepsilon}{2} \\
&\le \big|x_{s,i-1}^{\varepsilon,j_{i-1}}(t_{i-1})-\Big(x_{s,i-1}^{0,j_{i-1}}(t_{i-1})+\frac{\varepsilon}{2}e_1\Big)\big|+\frac{\varepsilon}{2}.
\end{aligned}$$

This time, the right-hand side is bounded by $(2i-2)\varepsilon$.
To finish the study of the case of the pair bouncing against the obstacle, if

$\tau \le \min\big(\tau_{s,i-1}^{0,j_{i-1}}, \tau_{s,i-1}^{\varepsilon,j_{i-1}}\big)$, one has

$$\begin{aligned}
\big|x_{s,i-1}^{\varepsilon,j_{i-1}}(\tau) - x_{s,i-1}^{0,j_{i-1}}(\tau)\big| &= \Big|\mathcal{S}_\varepsilon\big(x_{s,i-1}^{\varepsilon,j_{i-1}}(t_{i-1}) - (t_{i-1}-\tau)\big(v_{s,i-2}^{0,j_{i-1}}(t_{i-1})\big)'\big) \\
&\qquad - \mathcal{S}_0\big(x_{s,i-1}^{0,j_{i-1}}(t_{i-1}) - (t_{i-1}-\tau)\big(v_{s,i-2}^{0,j_{i-1}}(t_{i-1})\big)'\big)\Big| \\
&= \Big|\Big(x_{s,i-1}^{\varepsilon,j_{i-1}}(t_{i-1}) - (t_{i-1}-\tau_{s,i-1}^{\varepsilon,j_{i-1}})\big(v_{s,i-2}^{0,j_{i-1}}(t_{i-1})\big)' \\
&\qquad\qquad - (\tau_{s,i-1}^{\varepsilon,j_{i-1}}-\tau)\mathcal{S}_0\big(\big(v_{s,i-2}^{0,j_{i-1}}(t_{i-1})\big)'\big)\Big) \\
&\qquad - \Big(x_{s,i-1}^{0,j_{i-1}}(t_{i-1}) - (t_{i-1}-\tau_{s,i-1}^{0,j_{i-1}})\big(v_{s,i-2}^{0,j_{i-1}}(t_{i-1})\big)' \\
&\qquad\qquad - (\tau_{s,i-1}^{0,j_{i-1}}-\tau)\mathcal{S}_0\big(\big(v_{s,i-2}^{0,j_{i-1}}(t_{i-1})\big)'\big)\Big) \\
&\le \big|x_{s,i-1}^{\varepsilon,j_{i-1}}(t_{i-1}) - x_{s,i-1}^{0,j_{i-1}}(t_{i-1})\big| \\
&\quad + \Big|\big(\tau_{s,i-1}^{0,j_{i-1}} - \tau_{s,i-1}^{\varepsilon,j_{i-1}}\big)\Big(\mathcal{S}_0\big(\big(v_{s,i-2}^{0,j_{i-1}}(t_{i-1})\big)'\big) - \big(v_{s,i-2}^{0,j_{i-1}}(t_{i-1})\big)'\Big)\Big|.
\end{aligned}$$

Recalling that one has

$$t_{i-1} - \tau_{s,i-1}^{0,j_{i-1}} = \frac{x_{s,i-1}^{0,j_{i-1}}(t_{i-1})\cdot e_1}{\big(\big(v_{s,i-2}^{0,j_{i-1}}(t_{i-1})\big)'\big)\cdot e_1}$$

and

$$t_{i-1} - \tau_{s,i-1}^{\varepsilon,j_{i-1}} = \frac{\big(x_{s,i-1}^{\varepsilon,j_{i-1}}(t_{i-1})\cdot e_1 - \varepsilon/2\big)}{\big(\big(v_{s,i-2}^{0,j_{i-1}}(t_{i-1})\big)'\big)\cdot e_1},$$

one obtains therefore

$$\begin{aligned}
\big|x_{s,i-1}^{\varepsilon,j_{i-1}}(\tau) - x_{s,i-1}^{0,j_{i-1}}(\tau)\big| &= \Big|\mathcal{S}_\varepsilon\big(x_{s,i-1}^{\varepsilon,j_{i-1}}(t_{i-1}) - (t_{i-1}-\tau)\big(v_{s,i-2}^{0,j_{i-1}}(t_{i-1})\big)'\big) \\
&\qquad - \mathcal{S}_0\big(x_{s,i-1}^{0,j_{i-1}}(t_{i-1}) - (t_{i-1}-\tau)\big(v_{s,i-2}^{0,j_{i-1}}(t_{i-1})\big)'\big)\Big| \\
&= \Big|\Big(x_{s,i-1}^{\varepsilon,j_{i-1}}(t_{i-1}) - (t_{i-1}-\tau_{s,i-1}^{\varepsilon,j_{i-1}})\big(v_{s,i-2}^{0,j_{i-1}}(t_{i-1})\big)' \\
&\qquad\qquad - (\tau_{s,i-1}^{\varepsilon,j_{i-1}}-\tau)\mathcal{S}_0\big(\big(v_{s,i-2}^{0,j_{i-1}}(t_{i-1})\big)'\big)\Big) \\
&\qquad - \Big(x_{s,i-1}^{0,j_{i-1}}(t_{i-1}) - (t_{i-1}-\tau_{s,i-1}^{0,j_{i-1}})\big(v_{s,i-2}^{0,j_{i-1}}(t_{i-1})\big)' \\
&\qquad\qquad - (\tau_{s,i-1}^{0,j_{i-1}}-\tau)\mathcal{S}_0\big(\big(v_{s,i-2}^{0,j_{i-1}}(t_{i-1})\big)'\big)\Big) \\
&\le \big|x_{s,i-1}^{\varepsilon,j_{i-1}}(t_{i-1}) - x_{s,i-1}^{0,j_{i-1}}(t_{i-1})\big| \\
&\quad + \Big|\frac{-\varepsilon/2}{\big(\big(v_{s,i-2}^{0,j_{i-1}}(t_{i-1})\big)'\big)\cdot e_1}\big(-2\big(\big(v_{s,i-2}^{0,j_{i-1}}(t_{i-1})\big)'\big)\cdot e_1 e_1\big)\Big|,
\end{aligned}$$

using the explicit definition of the orthogonal symmetry $\mathcal{S}_0$, so that the right-hand side is bounded again by $(2i-2)\varepsilon$.
One has shown that in general

$$\Big|T_{-\tau}^{1,\varepsilon}\big(x_{s,i-2}^{\varepsilon,j_{i-1}}(t_{i-1}), v_{s,i-1}^{0,j_{i-1}}(t_{i-1})\big) - T_{-\tau}^{1,0}\big(x_{s,i-2}^{0,j_{i-1}}(t_{i-2}), v_{s,i-1}^{0,j_{i-1}}(t_{i-1})\big)\Big| \le (2i-2)\varepsilon.$$

To conclude the study of the difference of the positions of the particles of any pair, it remains to study the case of the pair $j = s+i-1$. Regardless the fact that the adjunction $i-1$ is pre or post-collisional (which has as only consequence that the two velocities $v^{\varepsilon,s+i-1}_{s,i-1}(t_{i-1}) = v^{\varepsilon,s+i-1}_{s,i-1}(t_{i-1})$ of the pair $s+i-1$ are both equal to v_{s+i-1} or v'_{s+i-1}, with v'_{s+i-1} denoting the second velocity obtained after applying the scattering operator to the pair $\big(v^{0,j_{i-1}}_{s,i-2}(t_{i-1}), v_{s+i-1}\big)$, with the angular parameter ω_{i-1}), one will study, for all $\tau \leq t_{i-1}$, the difference between the positions

$$\Big(T^{1,0}_{-(t_{i-1}-\tau)}\Big(\big(x^{0,s+i-1}_{s,i-1}(t_{i-1}), v^{0,s+i-1}_{s,i-1}(t_{i-1})\big)\Big)\Big)^X \\ = \Big(T^{1,0}_{-(t_{i-1}-\tau)}\Big(\big(x^{0,j_{i-1}}_{s,i-2}(t_{i-1}), v^{0,s+i-1}_{s,i-1}(t_{i-1})\big)\Big)\Big)^X$$

and

$$\Big(T^{1,\varepsilon}_{-(t_{i-1}-\tau)}\Big(\big(x^{\varepsilon,s+i-1}_{s,i-1}(t_{i-1}), v^{\varepsilon,s+i-1}_{s,i-1}(t_{i-1})\big)\Big)\Big)^X \\ = \Big(T^{1,\varepsilon}_{-(t_{i-1}-\tau)}\Big(\big(x^{0,j_{i-1}}_{s,i-2}(t_{i-1}) + \varepsilon\omega_{i-1}, v^{0,s+i-1}_{s,i-1}(t_{i-1})\big)\Big)\Big)^X.$$

One proceeds exactly as in the case $j = j_{i-1}$ in the post-collisional case. The only difference lies in the difference of position

$$x^{\varepsilon,j_{i-1}}_{s,i-2}(t_{i-1}) = x^{0,j_{i-1}}_{s,i-2}(t_{i-1}) + \varepsilon\omega_{i-1},$$

which simply induces an additional error smaller than $\big|\varepsilon\omega_{i-1}\big| = \varepsilon$. In this case, one finds

$$\Big|\Big(T^{1,0}_{-(t_{i-1}-\tau)}\Big(\big(x^{0,s+i-1}_{s,i-1}(t_{i-1}), v^{0,s+i-1}_{s,i-1}(t_{i-1})\big)\Big)\Big)^X \\ - \Big(T^{1,\varepsilon}_{-(t_{i-1}-\tau)}\Big(\big(x^{\varepsilon,s+i-1}_{s,i-1}(t_{i-1}), v^{\varepsilon,s+i-1}_{s,i-1}(t_{i-1})\big)\Big)\Big)^X\Big| \leq (2i-1)\varepsilon. \tag{13.21}$$

To summarize, one has obtained the intermediate result (13.8).

As for the link between the difference of velocities of the particles of the same pair, and the proximity with the obstacle, now that the trajectories of the particles can be explicitly written for the pseudo-trajectories associated to the two hierarchies, one will proceed as for the first case $i = 1$, and show that after t_{i-1}, for any pair $1 \leq j \leq s+i-1$ of particles of the two pseudo-trajectories, either they have the same velocity, or if they do not, then the particle of the Boltzmann pseudo-trajectory has to be close to the obstacle, that is one will obtain the intermediate result (13.10) page 451.
One proceeds as for the difference between the positions of the particles of the pairs.

First, for all times $\tau \leq t_{i-1}$ and for all pairs such that $j \neq j_{i-1}$ and $j \neq s+i-1$ if the adjunction $i-1$ is post-collisional, and just such that $j \neq s+i-1$ if the adjunction $i-1$ is pre-collisional, the free flow applied to the configurations of the particles of such pairs is

$$\begin{aligned} &T^{1,\varepsilon}_{-(t_{i-1}-\tau)}\big((x^{\varepsilon,j}_{s,i-1}(t_{i-1}), v^{\varepsilon,j}_{s,i-1}(t_{i-1}))\big) \\ &\qquad = T^{1,\varepsilon}_{-(t_{i-2}-\tau)}\big((x^{\varepsilon,j}_{s,i-2}(t_{i-2}), v^{\varepsilon,j}_{s,i-2}(t_{i-2}))\big) \end{aligned}$$

for the BBGKY pseudo-trajectory, and similarly for the Boltzmann pseudo-trajectory :

$$\begin{aligned} &T^{1,0}_{-(t_{i-1}-\tau)}\big((x^{0,j}_{s,i-1}(t_{i-1}), v^{0,j}_{s,i-1}(t_{i-1}))\big) \\ &\qquad = T^{1,0}_{-(t_{i-2}-\tau)}\big((x^{0,j}_{s,i-2}(t_{i-2}), v^{0,j}_{s,i-2}(t_{i-2}))\big). \end{aligned}$$

Therefore, the intermediate result (13.10) page 451 is obtained for the case i and for those particular pairs, since it is implied by the intermediate hypothesis (13.16) page 455, that is one has, if

$$\begin{aligned} &\Big(T^{1,\varepsilon}_{-(t_{i-1}-\tau)}\big((x^{\varepsilon,j}_{s,i-1}(t_{i-1}), v^{\varepsilon,j}_{s,i-1}(t_{i-1}))\Big)^V \\ &\qquad \neq \Big(T^{1,0}_{-(t_{i-1}-\tau)}\big((x^{0,j}_{s,i-1}(t_{i-1}), v^{0,j}_{s,i-1}(t_{i-1}))\Big)^V, \end{aligned}$$

then

$$\Big(T^{1,\varepsilon}_{-(t_{i-1}-\tau)}\big((x^{\varepsilon,j}_{s,i-1}(t_{i-1}), v^{\varepsilon,j}_{s,i-1}(t_{i-1}))\big)\Big)^X \cdot e_1 \leq (2i-5/2)\varepsilon. \tag{13.22}$$

Second, if the adjunction $i-1$ is post-collisional, one recalls that the configurations of the particles of the pair j_{i-1} at time t_{i-1} just after the adjunction are given by

$$\big(x^{\varepsilon,j_{i-1}}_{s,i-1}(t_{i-1}), v^{\varepsilon,j_{i-1}}_{s,i-1}(t_{i-1})\big) = \big(x^{\varepsilon,j_{i-1}}_{s,i-2}(t_{i-1}), \big(v^{0,j_{i-1}}_{s,i-2}(t_{i-1})\big)'\big)$$

and

$$\big(x^{0,j_{i-1}}_{s,i-1}(t_{i-1}), v^{0,j_{i-1}}_{s,i-1}(t_{i-1})\big) = \big(x^{0,j_{i-1}}_{s,i-2}(t_{i-1}), \big(v^{0,j_{i-1}}_{s,i-2}(t_{i-1})\big)'\big).$$

In particular, the two particles of this pair have the same velocity at time t_{i-1}. Therefore, the difference of velocities

$$\begin{aligned} &\Big(T^{1,\varepsilon}_{-(t_{i-1}-\tau)}\big((x^{\varepsilon,j_{i-1}}_{s,i-1}(t_{i-1}), v^{\varepsilon,j_{i-1}}_{s,i-1}(t_{i-1}))\Big)^V \\ &\qquad \neq \Big(T^{1,0}_{-(t_{i-1}-\tau)}\big((x^{0,j_{i-1}}_{s,i-1}(t_{i-1}), v^{0,j_{i-1}}_{s,i-1}(t_{i-1}))\Big)^V, \end{aligned}$$

can happen if and only if the two particles bounce against the obstacle, that is when

$$\big(\big(v^{0,j_{i-1}}_{s,i-2}(t_{i-1})\big)'\big) \cdot e_1 > 0,$$

(one recalls that this scalar product has to be *positive*, since one considers the *backwards* free flow), and then this difference of velocity happens exactly in the time interval determined by the respective times of bouncing $\tau_{s,i-1}^{0,j_{i-1}}$ and $\tau_{s,i-1}^{\varepsilon,j_{i-1}}$ of the two particles. If one has $\tau_{s,i-1}^{0,j_{i-1}} \leq \tau_{s,i-1}^{\varepsilon,j_{i-1}}$, that is if the particle of the BBGKY pseudo-trajectory bounces first against the obstacle, then during the time interval

$$\big[\tau_{s,i-1}^{0,j_{i-1}}, \tau_{s,i-1}^{\varepsilon,j_{i-1}}\big],$$

one has

$$\big(\big(v_{s,i-2}^{0,j_{i-1}}(t_{i-1})\big)'\big) \cdot e_1 > 0,$$

so that

$$\begin{aligned}
&\Big(T^{1,0}_{-(t_{i-1}-\tau)}\big((x_{s,i-1}^{0,j_{i-1}}(t_{i-1}), v_{s,i-1}^{0,j_{i-1}}(t_{i-1}))\big)^X \cdot e_1 \\
&\qquad = \big(x_{s,i-1}^{0,j_{i-1}} - (t_{i-1}-\tau) v_{s,i-1}^{0,j_{i-1}}(t_{i-1})\big) \cdot e_1 \\
&\qquad \leq \big(x_{s,i-1}^{0,j_{i-1}} - (t_{i-1}-\tau_{s,i-1}^{\varepsilon,j_{i-1}}) v_{s,i-1}^{0,j_{i-1}}(t_{i-1})\big) \cdot e_1 \\
&\qquad \leq \Big(T^{1,0}_{-(t_{i-1}-\tau_{s,i-1}^{\varepsilon,j_{i-1}})}\big((x_{s,i-1}^{0,j_{i-1}}(t_{i-1}), v_{s,i-1}^{0,j_{i-1}}(t_{i-1}))\big)^X \cdot e_1,
\end{aligned}$$

that is, in other words, the distance, during this time interval, between the obstacle and the particle of the Boltzmann pseudo-trajectory, is the largest when the BBGKY particle of the pair bounces against the obstacle. But one knows thanks to the intermediate result (13.21) page 463 obtained just above that in particular

$$\begin{aligned}
&\Big|\Big(T^{1,\varepsilon}_{-(t_{i-1}-\tau_{s,i-1}^{\varepsilon,j_{i-1}})}\big((x_{s,i-1}^{\varepsilon,j_{i-1}}(t_{i-1}), v_{s,i-1}^{\varepsilon,j_{i-1}}(t_{i-1}))\big)^X \\
&\qquad - \Big(T^{1,0}_{-(t_{i-1}-\tau_{s,i-1}^{\varepsilon,j_{i-1}})}\big((x_{s,i-1}^{0,j_{i-1}}(t_{i-1}), v_{s,i-1}^{0,j_{i-1}}(t_{i-1}))\big)^X\Big| \leq (2i-1)\varepsilon,
\end{aligned}$$

and since by definition

$$\Big(T^{1,\varepsilon}_{-(t_{i-1}-\tau_{s,i-1}^{\varepsilon,j_{i-1}})}\big((x_{s,i-1}^{\varepsilon,j_{i-1}}(t_{i-1}), v_{s,i-1}^{\varepsilon,j_{i-1}}(t_{i-1}))\big)^X \cdot e_1 = \varepsilon/2,$$

collecting the results together, one has therefore :

$$\begin{aligned}
&\Big(T^{1,0}_{-(t_{i-1}-\tau)}\big((x_{s,i-1}^{0,j_{i-1}}(t_{i-1}), v_{s,i-1}^{0,j_{i-1}}(t_{i-1}))\big)^X \cdot e_1 \\
&\qquad \leq \Big(T^{1,0}_{-(t_{i-1}-\tau_{s,i-1}^{\varepsilon,j_{i-1}})}\big((x_{s,i-1}^{0,j_{i-1}}(t_{i-1}), v_{s,i-1}^{0,j_{i-1}}(t_{i-1}))\big)^X \cdot e_1 \\
&\qquad \leq \Big|\Big(T^{1,\varepsilon}_{-(t_{i-1}-\tau_{s,i-1}^{\varepsilon,j_{i-1}})}\big((x_{s,i-1}^{\varepsilon,j_{i-1}}(t_{i-1}), v_{s,i-1}^{\varepsilon,j_{i-1}}(t_{i-1}))\big)^X \\
&\qquad\qquad - \Big(T^{1,0}_{-(t_{i-1}-\tau_{s,i-1}^{\varepsilon,j_{i-1}})}\big((x_{s,i-1}^{0,j_{i-1}}(t_{i-1}), v_{s,i-1}^{0,j_{i-1}}(t_{i-1}))\big)^X\Big| \\
&\qquad\quad + \Big(T^{1,\varepsilon}_{-(t_{i-1}-\tau^{\varepsilon,j_{i-1}})}\big((x_{s,i-1}^{\varepsilon,j_{i-1}}(t_{i-1}), v_{s,i-1}^{\varepsilon,j_{i-1}}(t_{i-1}))\big)^X \cdot e_1 \\
&\qquad \leq (2i-1/2)\varepsilon.
\end{aligned}$$

If $\tau_{s,i-1}^{\varepsilon,j_{i-1}} \leq \tau_{s,i-1}^{0,j_{i-1}}$, the reasoning is exactly the same and provides the same result.

Third and last, for the pair of particles $s+i-1$, again the reasoning is the same as in the previous case, using the intermediate result (13.21) page 463. One has in fact obtained the intermediate result (13.10) page 451 for all the pairs of particles, in the case i.

One is now able to conclude the proof. The intermediate result (13.8) page 451 provides immediately the first point (13.5), since one has for all integer $1 \leq j \leq s+k$:

$$\left(Z_{s,k}^{\varepsilon}(0)\right)^{X,k} - \left(Z_{s,k}^{\varepsilon}(0)\right)^{X,k} = \left(T_{-t_k}^{1,\varepsilon}\left(\left(x_{s,k}^{\varepsilon,j}(t_k), v_{s,k}^{\varepsilon,j}(t_k)\right)\right)\right)^{X} - \left(T_{-t_k}^{1,0}\left(\left(x_{s,k}^{0,j}(t_k), v_{s,k}^{0,j}(t_k)\right)\right)\right)^{X}.$$

The second point (13.6) is a direct consequence of the intermediate results (13.9) page 451 and (13.10) page 451, since thanks to the cut-off in proximity between the obstacle and the particle chosen for the adjunction at the time of this adjunction, one is able to exclude the fact that the respective velocities of the two particles of the pair chosen for the adjunction are different. The lemma is proved. □

13.2 Rewriting the elementary terms of the BBGKY hierarchy

13.2.1 Pseudo-trajectories reduced to free flow with boundary condition thanks to surgery

Along the proof of the previous lemma, one has in fact obtained that the pseudo-trajectories of the BBGKY hierarchy, for adjunction parameters chosen outside the subsets defined by the cut-offs, are given by the free flow between two adjunctions. In other words, the goal of removing recollisions has been reached. To be more accurate, one has obtained the intermediate result (13.7) page 450, which enables to state the following corollary of Lemma 38, which will be useful in the sequel.

Corollary 1 (Absence of recollision in the BBGKY pseudo-trajectories)**.** *Let ε be a strictly positive number, s and k be two positive integers. Let $T_k = (t_1, \ldots, t_k)$ be an element of $\mathfrak{T}_k$, $J_k = (j_1, \ldots, j_k)$ be an element of $\mathfrak{J}_k^s$ and $A_k = \big((\omega_1, v_{s+1}), \ldots, (\omega_k, v_{s+k})\big)$ be an element of $\mathfrak{A}_k$ (see Definition 44 page 323 for those notations).*
One considers then the backwards pseudo-trajectory of the Boltzmann hierarchy $Z^0(T_k, J_k, A_k)$ (see Definition 46 page 324), and the backwards pseudo-trajectory

of the BBGKY hierarchy $Z^\varepsilon(T_k, J_k, A_k)$ (see Definition 46 page 324), both starting from $Z_s \in \left(\Omega^c \times \mathbb{R}^d\right)^s$.
Under the assumptions of Lemma 38, for all $0 \leq i \leq k$ and for all $\tau \in [t_{i+1}, t_i]$, and for all $1 \leq j \leq s+k$ one has :

$$\left(Z^\varepsilon_{s,i}(T_k, J_k, A_k)(\tau)\right)^j = T^{1,\varepsilon}_{-(t_i-\tau)}\Big(\left(Z^\varepsilon_{s,i}(T_k, J_k, A_k)(t_i)\right)^j\Big), \tag{13.23}$$

where $T^{1,\varepsilon}$ is the hard sphere transport introduced in Definition 4 page 53.
In other words, between adjunctions, the particles following the BBGKY pseudo-trajectory starting from Z_s, with removed pathological adjunction parameters, evolve without interacting one with another.

13.2.2 Rewriting the elementary terms of the BBGKY hierarchy as a usual integral

Here one will provide a much simpler writing of the elementary terms of the BBGKY hierarchy, introduced in Definition 61 page 434. One recalls that those quantities were defined as iterations of operators, defined themselves as limits of Cauchy sequences of classical Lebesgue integrals. One will see that the cut-off performed along the initial configurations and the adjunction parameters defining the pseudo-trajectories will enable a very simple, though equivalent, definition of those elementary terms.

Proposition 19 (Rewriting the elementary terms of the BBGKY hierarchy as an usual integral)**.** *Let s, k and N be three positive integers, and ε be a strictly positive number.*
Let $J_k = (j_1, \ldots, j_k)$ be an element of $\mathfrak{J}^s_k$ and $M_k = (\pm_1, \ldots, \pm_k)$ be an element of $\mathfrak{M}_k$. Let $f^{(s+k)}_{N,0}$ be a function of $X_{\varepsilon,s+k,\beta_0}$ and let K be a compact set of the domain of local uniform convergence Ω_s (see Definition (53) page 392).
Then, there exist five strictly positive numbers $\overline{\varepsilon}, \overline{\varepsilon_0}, \overline{\alpha}, \overline{\gamma}$ and $\overline{R}$ (depending only on the compact set K) such that for every strictly positive numbers $R, \delta, \varepsilon, a, \varepsilon_0, \rho, \eta$ and α which satisfy

$$\left\{\begin{array}{c} \varepsilon \leq \overline{\varepsilon},\ \varepsilon_0 \leq \overline{\varepsilon_0},\ \alpha \leq \overline{\alpha},\ \max\left(16R\varepsilon/\varepsilon_0, \varepsilon_0/\delta\right) \leq \overline{\gamma} \\ \\ \text{and } R \geq \overline{R}, \end{array}\right. \tag{13.24}$$

and

$$\left\{\begin{array}{c} 4\sqrt{3}a \leq \varepsilon_0, \quad 3a \leq \rho, \quad \varepsilon_0 \leq \eta\delta, \quad R \geq 1, \quad \eta \leq 1 \text{ and } \alpha \leq c(d), \\ \text{with } c(d) \text{ a constant depending only on the dimension } d, \text{ and} \\ \\ 2k\varepsilon \leq a, \end{array}\right. \tag{13.25}$$

one has that the elementary term of the BBGKY hierarchy evaluated on the compact set K, that is :

$$\mathbb{1}_K \mathcal{J}^{N,\varepsilon,\delta}_{\substack{s,s+k-1\\ M_k,J_k}}(U^c, E^c)\big(f^{(s+k)}_{N,0}\mathbb{1}_{|V_{s+k}|\leq R}\big)$$

can be rewritten as the following Lebesgue integral :

$$
\begin{aligned}
&\mathbb{1}_K \mathbb{1}_{t\geq k\delta} \frac{(N-s)!}{(N-s-k)!} \varepsilon^{k(d-1)} \int_{(k-1)\delta}^{t-\delta} \mathbb{1}_{U^c_{j_1}(Z_s)}(t_1) \int_{\mathbb{S}^{d-1}_{\omega_1}\times\mathbb{R}^d_{v_{s+1}}} \mathbb{1}_{E^c_{j_1}(Z^0_{s,0(t_1)})}(\omega_1, v_{s+1}) \\
&\qquad\qquad \times (\pm_1)\Big[\omega_1\cdot\big(v_{s+1}-v^{\varepsilon,j_1}_{s,0}\big)\Big]_{\pm_1} \\
&\times \int_{(k-2)\delta}^{t_1-\delta} \mathbb{1}_{U^c_{j_2}(Z^0_{s,1}(t_1))}(t_2) \int_{\mathbb{S}^{d-1}_{\omega_2}\times\mathbb{R}^d_{v_{s+2}}} \mathbb{1}_{E^c_{j_2}(Z^0_{s,1}(t_2))}(\omega_2, v_{s+2}) \\
&\qquad\qquad \times (\pm_2)\Big[\omega_2\cdot\big(v_{s+2}-v^{\varepsilon,j_2}_{s,1}(t_2)\big)\Big]_{\pm_2} \\
&\qquad \dots \\
&\times \int_0^{t_{k-1}-\delta} \mathbb{1}_{U^c_{j_k}(Z^0_{s,k-1}(t_{k-1}))}(t_k) \int_{\mathbb{S}^{d-1}_{\omega_k}\times\mathbb{R}^d_{v_{s+k}}} \mathbb{1}_{E^c_{j_2}(Z^0_{s,k-1}(t_k))}(\omega_k, v_{s+k}) \\
&\qquad\qquad \times (\pm_k)\Big[\omega_k\cdot\big(v_{s+k}-v^{\varepsilon,j_k}_{s,k-1}(t_k)\big)\Big]_{\pm_k} f^{(s+k)}_{N,0}\big(Z^\varepsilon_{s,k}(0)\big) \\
&\qquad\qquad\qquad \mathrm{d}\omega_k\,\mathrm{d}v_{s+k}\,\mathrm{d}t_k\dots\mathrm{d}\omega_2\,\mathrm{d}v_{s+2}\,\mathrm{d}t_2\,\mathrm{d}\omega_1\,\mathrm{d}v_{s+1}\,\mathrm{d}t_1.
\end{aligned}
\tag{13.26}
$$

Proof. The proof is in two parts. First, one starts by showing that the expression (13.26) makes sense. Second, one shows that this expression is indeed another writing of the elementary term of the BBGKY hierarchy, provided that it is evaluated on the subset of preparation of the initial configurations, and provided also that the parameters of cut-off defining the excluded subsets of pathological adjunction parameters satisfy the conditions (13.25).

For the definition of the expression (13.26), one provides details for the case $k = 1$, which will enable to introduce naturally the relevant notations involved in the proof, and which will lead to a better understanding of the general case $k \in \mathbb{N}^*$.
Expliciting the pseudo-trajectories introduced in (46) page 324, the expression

(13.26) in the case $k=1$ is given by :

$$\begin{aligned}
&\mathbb{1}_K(Z_s)\mathbb{1}_{t\geq k\delta}\int_{(k-1)\delta}^{t-\delta}\mathbb{1}_{U_{j_1}^c(Z_s)}(t_1)\\
&\quad\times\mathcal{T}_{t-t_1}^{s,\varepsilon}\Bigg(\int_{\mathbb{S}_{\omega_1}^{d-1}\times\mathbb{R}_{v_{s+1}}^d}\mathbb{1}_{E_{j_1}^c(T_{t_1-t}^{s,0}(T_{t-t_1}^{s,\varepsilon}(Z_s)))}(\omega_1,v_{s+1})(\pm_1)\Big[\omega_1\cdot\big(v_{s+1}-v_{j_1}\big)\Big]_{\pm_1}\mathbb{1}_{|V_{s+1}|\leq R}\\
&\qquad\times\sum_{R_1\in\{0,1\}^{s+1}}\prod_{l_1=1}^{s+1}\mathbb{1}_{(-1)^{r_1^{l_1}}\left(S_s^{\pm_1}\left(Z_s,t_1,\omega_1,v_{s+1}\right)\right)^{X,l_1}\cdot e_1\geq\varepsilon}\\
&\qquad\qquad\times f_{N,0}^{(s+1)}\Big(\mathcal{S}^{R_1}\big(S_s^{\pm_1}\big(Z_s,t_1,\omega_1,v_{s+1}\big)\big)\Big)\,\mathrm{d}\omega_1\,\mathrm{d}v_{s+1}\Bigg)\,\mathrm{d}t_1,
\end{aligned}$$

that is, expliciting the effect of the transport $\mathcal{T}_{t-t_1}^{s,0}$:

$$\begin{aligned}
&\mathbb{1}_K(Z_s)\\
&\quad\times\mathbb{1}_{t\geq k\delta}\int_{(k-1)\delta}^{t-\delta}\mathbb{1}_{U_{j_1}^c(Z_s)}(t_1)\int_{\mathbb{S}_{\omega_1}^{d-1}\times\mathbb{R}_{v_{s+1}}^d}\mathbb{1}_{E_{j_1}^c(T_{-(t-t_1)}^{s,0}(Z_s))}(\omega_1,v_{s+1})\\
&\qquad\times(\pm_1)\Big[\omega_1\cdot\big(v_{s+1}-\big(T_{-(t-t_1)}^{s,\varepsilon}(Z_s)\big)^{V,j_1}\big)\Big]_{\pm_1}\mathbb{1}_{|V_{s+1}|\leq R}\\
&\qquad\quad\times\sum_{R_1\in\{0,1\}^{s+1}}\prod_{l_1=1}^{s+1}\mathbb{1}_{(-1)^{r_1^{l_1}}\left(S_s^{\pm_1}\left(T_{-(t-t_1)}^{s,\varepsilon}(Z_s),t_1,\omega_1,v_{s+1}\right)\right)^{X,l_1}\cdot e_1\geq\varepsilon/2}\\
&\qquad\qquad\times f_{N,0}^{(s+1)}\Big(\mathcal{S}^{R_1}\big(S_s^{\pm_1}\big(T_{-(t-t_1)}^{s,\varepsilon}(Z_s),t_1,\omega_1,v_{s+1}\big)\big)\Big)\,\mathrm{d}\omega_1\,\mathrm{d}v_{s+1}\,\mathrm{d}t_1,
\end{aligned}\tag{13.27}$$

where $R_1=(r_1^1,\ldots,r_1^{s+1})$ and

$$\mathcal{S}^{R_1}\big(\widetilde{Z}_{s+1}\big)=\mathcal{S}^{R_1}\big(\widetilde{x}_1,\widetilde{v}_1,\ldots,\widetilde{x}_{s+1},\widetilde{v}_{s+1}\big)$$

denotes

$$\Big(\mathcal{S}_\varepsilon^{r_1^1}\big(\widetilde{x}_1\big),\mathcal{S}_0^{r_1^1}\big(\widetilde{v}_1\big),\ldots,\mathcal{S}_\varepsilon^{r_1^{s+1}}\big(\widetilde{x}_{s+1}\big),\mathcal{S}_0^{r_1^{s+1}}\big(\widetilde{v}_{s+1}\big)\Big).$$

One notices that the argument of the indicator function $\mathbb{1}_{E_{j_1}^c}$ is $T_{-(t-t_1)}^{s,0}(Z_s)$, which is *not* the configuration of the BBGKY pseudo-trajectory at time t_1, but the configuration of the Boltzmann pseudo-trajectory. The decomposition

$$\sum_{R_1\in\{0,1\}^{s+1}}\prod_{l_1=1}^{s+1}\mathbb{1}_{(-1)^{r_1^{l_1}}\left(S_s^{\pm_1}\left(T_{-(t-t_1)}^{s,\varepsilon}(Z_s),t_1,\omega_1,v_{s+1}\right)\right)^{X,l_1}\cdot e_1\geq\varepsilon}$$

is here to consider the effects of the obstacle : if the position of the particle l_1, after the application of the free transport (that is, after the application of $S_s^{\pm_1}$), verifies :

$$S_s^{\pm_1}\big(T_{-(t-t_1)}^{s,\varepsilon}(Z_s),t_1,\omega_1,v_{s+1}\big)^{X,l_1}\cdot e_1<\varepsilon/2,$$

then according to the dynamics prescribed for the hard spheres, since there is no recollision between the particles (thanks to the cut-offs on the initial configuration Z_s and the adjunction parameters t_1, ω_1 and v_{s+1}), this particle should have bounced against the obstacle, that is its position is replaced by

$$\mathcal{S}_\varepsilon\Big(S_s^{\pm_1}\big(T^{s,\varepsilon}_{-(t-t_1)}(Z_s),t_1,\omega_1,v_{s+1}\big)^{X,l_1}\Big)$$

and its velocity is replaced by

$$\mathcal{S}_0\Big(S_s^{\pm_1}\big(T^{s,\varepsilon}_{-(t-t_1)}(Z_s),t_1,\omega_1,v_{s+1}\big)^{V,l_1}\Big).$$

Those modifications are taken into account by the function $\mathcal{S}^{R_1}$.
To give sense to the expression (13.27), one will follow the same idea as the one used to define the truncated transport-collision operator (see Section 5.1 page 88).
First, the mapping $f_{N,0}^{(s+1)}\circ\ \mathcal{S}^{R_1}$ is measurable and bounded almost everywhere (since it is the case for $f_{N,0}^{(s+1)}$ itself), and multiplied by the indicator function :

$$\mathbb{1}_{S_s^{\pm}(K\times[0,t-\delta]\times\mathbb{S}^{d-1}\times B_{\mathbb{R}^d}(0,R))},$$

which is the indicator of a compact set of $\mathcal{D}^\varepsilon_{s+1}$, one obtains an integrable function on $\mathcal{D}^\varepsilon_{s+1}$.
Using the fact that the Jacobian determinant of the mapping $S_s^{\pm}$ has been found in Propositions 3 page 92 and 4 page 98 to be equal to

$$\Big[\omega_1\cdot\big(v_{s+1}-v_{j_1}\big)\Big]_{\pm_1},$$

one knows thanks to the Fubini theorem that for almost every (Z_s,t_1), the function

$$\begin{aligned}(\omega_1,v_{s+1})\mapsto\ &\mathbb{1}_{(K\times[0,t-\delta]\times\mathbb{S}^{d-1}\times\mathbb{R}^d)}(\pm_1)\Big[\omega_1\cdot\big(v_{s+1}-v_{j_1}\big)\Big]_{\pm_1}\mathbb{1}_{|V_{s+1}|\le R}\\ &\qquad\times f_{N,0}^{(s+1)}\circ\ \mathcal{S}^{R_1}\circ\ S_s^{\pm_1}\\ =(\omega_1,v_{s+1})\mapsto\ &\mathbb{1}_{(K\times[0,t-\delta]\times\mathbb{S}^{d-1}\times\mathbb{R}^d)}(\pm_1)\Big[\omega_1\cdot\big(v_{s+1}-v_{j_1}\big)\Big]_{\pm_1}\mathbb{1}_{|V_{s+1}|\le R}\\ &\times\sum_{R_1\in\{0,1\}^{s+1}}\prod_{l_1=1}^{s+1}\mathbb{1}_{(-1)^{r_1^{l_1}}\left(S_s^{\pm_1}\left(Z_s,t_1,\omega_1,v_{s+1}\right)\right)^{X,l_1}\cdot e_1\ge\varepsilon}\\ &\qquad\times f_{N,0}^{(s+1)}\circ\ \mathcal{S}^{R_1}\circ\ S_s^{\pm_1}\end{aligned}$$

is well defined almost everywhere and integrable on $\mathbb{S}^{d-1}_{\omega_1}\times\mathbb{R}^d_{v_{s+1}}$, and so is

$$\begin{aligned}(\omega_1,v_{s+1})\mapsto\ &\mathbb{1}_{E^c_{j_1}(Z_s)}(\omega_1,v_{s+1})\mathbb{1}_{(K\times[0,t-\delta]\times\mathbb{S}^{d-1}\times\mathbb{R}^d)}(\pm_1)\Big[\omega_1\cdot\big(v_{s+1}-v_{j_1}\big)\Big]_{\pm_1}\\ &\times\mathbb{1}_{|V_{s+1}|\le R}\sum_{R_1\in\{0,1\}^{s+1}}\prod_{l_1=1}^{s+1}\mathbb{1}_{(-1)^{r_1^{l_1}}\left(S_s^{\pm_1}\left(Z_s,t_1,\omega_1,v_{s+1}\right)\right)^{X,l_1}\cdot e_1\ge\varepsilon}\\ &\qquad\times f_{N,0}^{(s+1)}\circ\ \mathcal{S}^{R_1}\circ\ S_s^{\pm_1},\end{aligned}$$

and its integral on $\mathbb{S}^{d-1}_{\omega_1} \times \mathbb{R}^d_{v_{s+1}}$ is a function well defined almost everywhere, integrable on $\mathcal{D}^\varepsilon_s \times [0, t-\delta]$.
Composing now the function obtained by the mapping

$$\big(t_1, Z_s, \omega_1, v_{s+1}\big) \mapsto \big(t_1, T^{s,\varepsilon}_{-(t-t_1)}(Z_s), \omega_1, v_{s+1}\big),$$

which is a measurable mapping defined and regular almost everywhere of Jacobian determinant 1, so that it does not change the results on the integrability and then the results obtained with the change of variable nor the Fubini theorem, one obtains that the function :

$$\begin{aligned}(t_1, Z_s) \mapsto \int_{\mathbb{S}^{d-1}_{\omega_1}\times\mathbb{R}^d_{v_{s+1}}} & \mathbb{1}_{E^c_{j_1}(T^{s,0}_{-(t-t_1)}(Z_s))}(\omega_1, v_{s+1}) \\ & \times (\pm_1)\Big[\omega_1 \cdot \big(v_{s+1} - \big(T^{s,\varepsilon}_{-(t-t_1)}(Z_s)\big)^{V,j_1}\big)\Big]_{\pm_1} \mathbb{1}_{|V_{s+1}|\le R} \\ & \times \sum_{R_1\in\{0,1\}^{s+1}} \prod_{l_1=1}^{s+1} \mathbb{1}_{(-1)^{r_1^{l_1}}\left(S_s^{\pm_1}\left(T^{s,\varepsilon}_{-(t-t_1)}(Z_s), t_1, \omega_1, v_{s+1}\right)\right)^{X,l_1}\cdot e_1 \ge \varepsilon/2} \\ & \times f^{(s+1)}_{N,0}\Big(\mathcal{S}^{R_1}\big(S_s^{\pm_1}\big(T^{s,\varepsilon}_{-(t-t_1)}(Z_s), t_1, \omega_1, v_{s+1}\big)\big)\Big)\,\mathrm{d}\omega_1\,\mathrm{d}v_{s+1}\end{aligned}$$

is well defined and integrable over $[0, t-\delta] \times \mathcal{D}^\varepsilon_s$.
Using again the Fubini theorem, for almost every $Z_s \in \mathcal{D}^\varepsilon_s$, the function

$$\begin{aligned}t_1 \mapsto \int_{\mathbb{S}^{d-1}_{\omega_1}\times\mathbb{R}^d_{v_{s+1}}} & \mathbb{1}_{E^c_{j_1}(T^{s,0}_{-(t-t_1)}(Z_s))}(\omega_1, v_{s+1}) \\ & \times (\pm_1)\Big[\omega_1 \cdot \big(v_{s+1} - \big(T^{s,\varepsilon}_{-(t-t_1)}(Z_s)\big)^{V,j_1}\big)\Big]_{\pm_1} \mathbb{1}_{|V_{s+1}|\le R} \\ & \times \sum_{R_1\in\{0,1\}^{s+1}} \prod_{l_1=1}^{s+1} \mathbb{1}_{(-1)^{r_1^{l_1}}\left(S_s^{\pm_1}\left(T^{s,\varepsilon}_{-(t-t_1)}(Z_s), t_1, \omega_1, v_{s+1}\right)\right)^{X,l_1}\cdot e_1 \ge \varepsilon/2} \\ & \times f^{(s+1)}_{N,0}\Big(\mathcal{S}^{R_1}\big(S_s^{\pm_1}\big(T^{s,\varepsilon}_{-(t-t_1)}(Z_s), t_1, \omega_1, v_{s+1}\big)\big)\Big)\,\mathrm{d}\omega_1\,\mathrm{d}v_{s+1}\end{aligned}$$

is well defined almost everywhere and integrable, this integral being a measurable function in Z_s on $\mathcal{D}^\varepsilon_s$, and so is the function

$$\begin{aligned}t_1 \mapsto \mathbb{1}_{U^c_{j_1}(Z_s)}(t_1) \int_{\mathbb{S}^{d-1}_{\omega_1}\times\mathbb{R}^d_{v_{s+1}}} & \mathbb{1}_{E^c_{j_1}(T^{s,0}_{-(t-t_1)}(Z_s))}(\omega_1, v_{s+1}) \\ & \times (\pm_1)\Big[\omega_1 \cdot \big(v_{s+1} - \big(T^{s,\varepsilon}_{-(t-t_1)}(Z_s)\big)^{V,j_1}\big)\Big]_{\pm_1} \mathbb{1}_{|V_{s+1}|\le R} \\ & \times \sum_{R_1\in\{0,1\}^{s+1}} \prod_{l_1=1}^{s+1} \mathbb{1}_{(-1)^{r_1^{l_1}}\left(S_s^{\pm_1}\left(T^{s,\varepsilon}_{-(t-t_1)}(Z_s), t_1, \omega_1, v_{s+1}\right)\right)^{X,l_1}\cdot e_1 \ge \varepsilon/2} \\ & \times f^{(s+1)}_{N,0}\Big(\mathcal{S}^{R_1}\big(S_s^{\pm_1}\big(T^{s,\varepsilon}_{-(t-t_1)}(Z_s), t_1, \omega_1, v_{s+1}\big)\big)\Big)\,\mathrm{d}\omega_1\,\mathrm{d}v_{s+1},\end{aligned}$$

which concludes the definition of the expression (13.27) page 469 in the case $k=1$.
One will now detail the definition of the expression (13.27) in the case $k=2$. This time, the expression writes :

$$
\begin{aligned}
&\mathbb{1}_K(Z_s)\\
&\quad\times \mathbb{1}_{t\geq 2\delta}\int_\delta^{t-\delta}\mathbb{1}_{U^c_{j_1}(Z_s)}(t_1)\int_{\mathbb{S}^{d-1}_{\omega_1}\times\mathbb{R}^d_{v_{s+1}}}\mathbb{1}_{E^c_{j_1}(T^{s,0}_{-(t-t_1)}(Z_s))}(\omega_1,v_{s+1})\\
&\qquad\qquad\times(\pm_1)\Big[\omega_1\cdot\big(v_{s+1}-\big(T^{s,\varepsilon}_{-(t-t_1)}(Z_s)\big)^{V,j_1}\big)\Big]_{\pm_1}\\
&\qquad\qquad\quad\times\sum_{R_1\in\{0,1\}^{s+1}}\prod_{l_1=1}^{s+1}\mathbb{1}_{(-1)^{r_1^{l_1}}\left(S^{\pm_1}_s\left(T^{s,\varepsilon}_{-(t-t_1)}(Z_s),t_1,\omega_1,v_{s+1}\right)\right)^{X,l_1}\cdot e_1\geq\varepsilon/2}\\
&\quad\times\int_0^{t_1-\delta}\mathbb{1}_{U^c_{j_2}((T^{s,0}_{-(t-t_1)}(Z_s),(T^{s,0}_{-(t-t_1)}(Z_s))^{X,j_1},v_{s+1}))}(t_2)\int_{\mathbb{S}^{d-1}_{\omega_2}\times\mathbb{R}^d_{v_{s+2}}}\mathbb{1}_{E^c_{j_2}(Z^0_{s,1}(t_2))}(\omega_2,v_{s+2})\\
&\qquad\qquad\times(\pm_2)\Big[\omega_2\cdot\big(v_{s+2}-\big(T^{s+1,\varepsilon}_{t_2}\big(S^{\pm_1}_s\big(T^{s,\varepsilon}_{-(t-t_1)}(Z_s),t_1,\omega_1,v_{s+1}\big)\big)\big)^{V,j_2}\big)\Big]_{\pm_2}\\
&\quad\times\sum_{R_2\in\{0,1\}^{s+2}}\\
&\quad\times\Big(\prod_{l_2=1}^{s+2}\mathbb{1}_{(-1)^{r_2^{l_2}}\left(S^{\pm_1}_{s+1}\left(T^{s,\varepsilon}_{t_2}\left(S^{\pm_1}_s\left(T^{s,\varepsilon}_{-(t-t_1)}(Z_s),t_1,\omega_1,v_{s+1}\right)\right),t_2,\omega_2,v_{s+2}\right)\right)^{X,l_2}\cdot e_1\geq\varepsilon/2}\Big)\\
&\qquad\times f^{(s+2)}_{N,0}\Big(\mathcal{S}^{R_2}\big(\mathcal{S}^{\pm_2}_{s+1}\big(\\
&\qquad\qquad\qquad T^{s+1,\varepsilon}_{t_2}\big(\mathcal{S}^{R_1}\big(S^{\pm_1}_s\big(T^{s,\varepsilon}_{-(t-t_1)}(Z_s),t_1,\omega_1,v_{s+1}\big)\big),t_2,\omega_2,v_{s+2}\big)\big)\big)\Big)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\mathrm{d}\omega_2\,\mathrm{d}v_{s+2}\,\mathrm{d}t_2\,\mathrm{d}\omega_1\,\mathrm{d}v_{s+1}\,\mathrm{d}t_1,
\end{aligned}
\tag{13.28}
$$

where $Z^0_{s,1}(t_2)$, the argument of the indicator function $\mathbb{1}_{E^c_{j_2}}$, means

$$T^{s+1,0}_{-(t_1-t_2)}\big(T^{s,0}_{-(t-t_1)}(Z_s),(T^{s,0}_{-(t-t_1)}(Z_s))^{X,j_1},v_{s+1}\big)$$

in the pre-collisional case $\pm_1=-$, and

$$T^{s+1,0}_{-(t_1-t_2)}\big(\big(T^{s,0}_{-(t-t_1)}(Z_s),(T^{s,0}_{-(t-t_1)}(Z_s))^{X,j_1},v_{s+1}\big)'_{j_1,s+1}\big)$$

in the post-collisional case $\pm_1=+$.
Once again, thanks to the cut-offs performed for the initial configuration and the adjunction parameters, since there is no recollision between the particles, rewriting the arguments of the second excluded subsets among the times t_2 and

the adjunction parameters (ω_2, v_{s+2}), namely :

$$\begin{aligned}\big(T^{s,\varepsilon}_{-(t-t_1)}(Z_s), \big(T^{s,\varepsilon}_{-(t-t_1)}(Z_s)\big)^{X,j_1} + \varepsilon\omega_1, v_{s+1}\big)\\ = T^{s+1,\varepsilon}_{t_1}\big(\mathcal{S}^{R_1}\big(S^{\pm_1}_s\big(T^{s,\varepsilon}_{-(t-t_1)}(Z_s), t_1, \omega_1, v_{s+1}\big)\big)\big)\end{aligned}$$

and

$$\begin{aligned}T^{s+1,\varepsilon}_{-(t_1-t_2)}\big(T^{s,\varepsilon}_{-(t-t_1)}(Z_s), \big(T^{s,\varepsilon}_{-(t-t_1)}(Z_s)\big)^{X,j_1} + \varepsilon\omega_1, v_{s+1}\big)\\ = T^{s+1,\varepsilon}_{t_2}\big(\mathcal{S}^{R_1}\big(S^{\pm_1}_s\big(T^{s,\varepsilon}_{-(t-t_1)}(Z_s), t_1, \omega_1, v_{s+1}\big)\big),\end{aligned}$$

one sees that the expression (13.28) for the case $k = 2$ will make sense using the work done for the expression (13.27), replacing $f^{(s+1)}_{N,0}$ by the function

$$\begin{aligned}(t_1, Z_{s+1}) \mapsto \int_0^{t_1-\delta} \mathbb{1}_{U^c_{j_2}\left(T^{s+1,0}_{t_1}(Z_{s+1})\right)}(t_2) \int_{\mathbb{S}^{d-1}_{\omega_2}\times\mathbb{R}^d_{v_{s+2}}} \mathbb{1}_{E^c_{j_1}\left(T^{s+1,0}_{t_2}(Z_{s+1})\right)}(\omega_2, v_{s+2})\\ \times (\pm_2)\Big[\omega_2 \cdot \big(v_{s+2} - \big(T^{s+1,\varepsilon}_{t_2}(Z_{s+1})\big)^{V,j_2}\big)\Big]_{\pm_2}\\ \times \sum_{R_2\in\{0,1\}^{s+2}} \prod_{l_2=1}^{s+2} \mathbb{1}_{(-1)^{r^{l_2}_2}\left(\mathcal{S}^{R_2}\left(S^{\pm_1}_{s+1}\left(T^{s,\varepsilon}_{t_2}(Z_{s+1}), t_2, \omega_2, v_{s+2}\right)\right)\right)^{X,l_2}\cdot e_1 \geq \varepsilon/2}\\ \times f^{(s+2)}_{N,0}\Big(\mathcal{S}^{R_2}\big(S^{\pm_2}_{s+1}\big(T^{s+1,\varepsilon}_{t_2}(Z_{s+1}), t_2, \omega_2, v_{s+2}\big)\big)\Big)\, \mathrm{d}\omega_2\, \mathrm{d}v_{s+2}\, \mathrm{d}t_2.\end{aligned} \tag{13.29}$$

Therefore, the last expression (13.29) has to be studied to conclude the definition of the expression (13.28), that is the case $k = 2$. But following the same steps as for the case $k = 1$, that is first showing that the function

$$\begin{aligned}(t_1, t_2, Z_{s+1}) \mapsto \int_{\mathbb{S}^{d-1}_{\omega_2}\times\mathbb{R}^d_{v_{s+2}}} \mathbb{1}_{E^c_{j_2}(Z_{s+1})}(\omega_2, v_{s+2})\\ \times (\pm_2)\Big[\omega_2 \cdot \big(v_{s+2} - v_{j_2}\big)\Big]_{\pm_2}\\ \times \sum_{R_2\in\{0,1\}^{s+2}} \prod_{l_2=1}^{s+2} \mathbb{1}_{(-1)^{r^{l_2}_2}\left(\mathcal{S}^{R_2}\left(S^{\pm_1}_{s+1}\left(Z_{s+1}, t_2, \omega_2, v_{s+2}\right)\right)\right)^{X,l_2}\cdot e_1 \geq \varepsilon/2}\\ \times f^{(s+2)}_{N,0}\Big(\mathcal{S}^{R_2}\big(S^{\pm_2}_{s+1}\big(Z_{s+1}, t_2, \omega_2, v_{s+2}\big)\big)\Big)\, \mathrm{d}\omega_2\, \mathrm{d}v_{s+2}\, \mathrm{d}t_2\end{aligned}$$

is well defined and integrable, composing it with the hard sphere transport $T^{s+1,\varepsilon}_{t_2}$, multiplying it by the function

$$\mathbb{1}_{U^c_{j_2}\left(T^{s+1,\varepsilon}_{t_1}(Z_{s+1})\right)},$$

and integrating the product of those two functions on $[0, t_1 - \delta]$ with respect to the time variable t_2, one obtains that the previous expression (13.29) makes

sense, and so the expression of the elementary term of the BBGKY hierarchy for the case $k = 2$.
For the general case, that is for a general integer k, one sees that the steps used for the case $k = 2$ enable in general to give sense to the expression (13.26) of the proposition, by a simple induction.

For the second part of the proof, the fact that K is a compact subset of the domain of local uniform convergence Ω_s (see Definition 53 page 392) implies that there exist five strictly positive numbers $\overline{\varepsilon}, \overline{\delta}, \overline{\varepsilon_0}, \overline{\alpha}$ and $\overline{\gamma}$ small enough, and a strictly positive number $\overline{R}$ large enough such that, for all strictly positive numbers $\varepsilon, R, \delta, \varepsilon_0$ and α such that

$$\varepsilon \leq \overline{\varepsilon},\ \varepsilon_0 \leq \overline{\varepsilon_0},\ \alpha \leq \overline{\alpha},\ \gamma \leq \overline{\gamma} \text{ and } R \geq \overline{R},$$

one has :

$$K \subset \Delta_s(\varepsilon, R, \varepsilon_0, \alpha, \gamma).$$

If in addition the parameters are chosen such that

$$2\sqrt{3}\varepsilon \leq \varepsilon_0, \quad \max(16R\varepsilon/\varepsilon_0, \varepsilon_0/\delta) \leq \gamma,$$

one can use the results (12.114) and (12.115) of Proposition 17 page 394.
Now the cut-offs performed respectively in the time domain (for the variable t_1) on the one hand, and in the angular parameter and velocity domain (for the variables ω_1 and v_{s+1}), namely :

$$t - t_1 \geq \delta,\ t_1 \notin U_{j_1}(Z_s)$$

and

$$(\omega_1, v_{s+1}) \notin E_{j_1}\big(T^{s,0}_{-(t-t_1)}(Z_s)\big)$$

together with the results enable to use Proposition 16 page 347, asserting that the new configuration after the adjunction to the particle j_1 (for the system of particles now in the configuration $T^{s,\varepsilon}_{-(t-t_1)}(Z_s)$, just before the adjunction, for the BBGKY pseudo-trajectory) is in a good configuration. Therefore, no recollision will happen for the new system of $s+1$ particles following the hard sphere flow.
In particular, the hard sphere flow starting from the configuration

$$\Big(T^{s,\varepsilon}_{-(t-t_1)}(Z_s), \big(T^{s,\varepsilon}_{-(t-t_1)}(Z_s)\big)^{X,j_1} + \varepsilon\omega_1, v_{s+1}\Big)$$

will coincide with the flow used to define the term (13.27) page 469, that is one has

$$\begin{aligned} &T^{s+1,\varepsilon}_{-\tau}\Big(T^{s,\varepsilon}_{-(t-t_1)}(Z_s), \big(T^{s,\varepsilon}_{-(t-t_1)}(Z_s)\big)^{X,j_1} + \varepsilon\omega_1, v_{s+1}\Big) \\ &\quad = \prod_{l_1=1}^{s+1} \mathbb{1}_{(-1)^{r_1^{l_1}}\left(S_s^{\pm 1}\left(T^{s,\varepsilon}_{-(t-t_1)}(Z_s),\tau,\omega_1,v_{s+1}\right)\right)^{X,l_1}} \\ &\qquad\qquad \times \mathcal{S}^{R_1}\big(S_s^{\pm 1}\big(T^{s,\varepsilon}_{-(t-t_1)}(Z_s), t_1, \omega_1, v_{s+1}\big)\big). \end{aligned}$$

By induction, one can show that the final configuration $Z^{\varepsilon}_{s,k}(0)$ can be expressed using only the mappings $S^{\pm}_{s}, \dots, S^{\pm}_{s+k-1}$ and the orthogonal symmetries $\mathcal{S}_0$ and $\mathcal{S}_\varepsilon$, applied to the configuration $T^{s,\varepsilon}_{-(t-t_1)}(Z_s)$.
Therefore, the elementary term of the BBGKY hierarchy, defined in Lemma 61 page 434, as the limit of a sequence of integrals, is in fact given by a classical integral. The expression (13.26) of the proposition is in fact a simple writing of the elementary term of the BBGKY hierarchy, which concludes the proof. □

13.3 The dominated convergence argument

Thanks to Proposition 19 page 467 of the previous section, one sees that the elementary term of the BBGKY hierarchy

$$\mathbb{1}_K \mathcal{J}^{N,\varepsilon,\delta}_{\substack{s,s+k-1\\ M_k,J_k}}(U^c,E^c)\big(f^{(s+k)}_{N,0}\mathbb{1}_{|V_{s+k}|\le R}\big),$$

evaluated on any compact subset K of the domain of uniform convergence (up to choosing small enough the parameters defining the pathological subsets for the adjunction parameters, removed by surgery, and up to choosing those parameters such that they satisfy some conditions), can be written as follows, as a usual Lebesgue integral :

$$\begin{aligned}
\mathbb{1}_K \mathbb{1}_{t\ge k\delta}\frac{(N-s)!}{(N-s-k)!}\varepsilon^{k(d-1)}\int_{(k-1)\delta}^{t-\delta}\mathbb{1}_{U^c_{j_1}(Z_s)}(t_1)\int_{\mathbb{S}^{d-1}_{\omega_1}\times\mathbb{R}^d_{v_{s+1}}}\mathbb{1}_{E^c_{j_1}(Z^0_{s,0(t_1)})}(\omega_1,v_{s+1})\\
\times(\pm_1)\Big[\omega_1\cdot\big(v_{s+1}-v^{\varepsilon,j_1}_{s,0}\big)\Big]_{\pm_1}\\
\times\int_{(k-2)\delta}^{t_1-\delta}\mathbb{1}_{U^c_{j_2}(Z^0_{s,1}(t_1))}(t_2)\int_{\mathbb{S}^{d-1}_{\omega_2}\times\mathbb{R}^d_{v_{s+2}}}\mathbb{1}_{E^c_{j_2}(Z^0_{s,1}(t_2))}(\omega_2,v_{s+2})\\
\times(\pm_2)\Big[\omega_2\cdot\big(v_{s+2}-v^{\varepsilon,j_2}_{s,1}(t_2)\big)\Big]_{\pm_2}\\
\dots\\
\times\int_0^{t_{k-1}-\delta}\mathbb{1}_{U^c_{j_k}(Z^0_{s,k-1}(t_{k-1}))}(t_k)\int_{\mathbb{S}^{d-1}_{\omega_k}\times\mathbb{R}^d_{v_{s+k}}}\mathbb{1}_{E^c_{j_k}(Z^0_{s,k-1}(t_k))}(\omega_k,v_{s+k})\\
\times(\pm_k)\Big[\omega_k\cdot\big(v_{s+k}-v^{\varepsilon,j_k}_{s,k-1}(t_k)\big)\Big]_{\pm_k}f^{(s+k)}_{N,0}\big(Z^{\varepsilon}_{s,k}(0)\big)\\
\mathrm{d}\omega_k\,\mathrm{d}v_{s+k}\,\mathrm{d}t_k\dots\mathrm{d}\omega_2\,\mathrm{d}v_{s+2}\,\mathrm{d}t_2\,\mathrm{d}\omega_1\,\mathrm{d}v_{s+1}\,\mathrm{d}t_1.
\end{aligned}\tag{13.30}$$

In Section 13.1 (in particular, in Lemma 38 page 449), one obtained a control on the difference of the pseudo-trajectories of the two hierachies, provided

that the cut-offs in pathological initial configurations and adjunction parameters have been performed. This control invites to consider a slightly different elementary term, which is an intermediary between the elementary terms of the two hierarchies, namely one will remove the prefactors

$$\frac{(N-s)!}{(N-s-k)!}\varepsilon^{k(d-1)},$$

and replace the $s+k$-th initial datum $f_{N,0}^{(s+k)}$ of the BBGKY hierarchy by $f_0^{(s+k)}$, the $s+k$-th initial datum of the Boltzmann hierarchy, but one will keep the BBGKY pseudo-trajectory. This gives :

$$\begin{aligned}
&\mathbb{1}_K \mathbb{1}_{t\geq k\delta} \int_{(k-1)\delta}^{t-\delta} \mathbb{1}_{U_{j_1}^c(Z_s)}(t_1) \int_{\mathbb{S}_{\omega_1}^{d-1}\times\mathbb{R}_{v_{s+1}}^d} \mathbb{1}_{E_{j_1}^c(Z_{s,0(t_1)}^0)}(\omega_1, v_{s+1}) \\
&\qquad\qquad \times (\pm_1)\Big[\omega_1\cdot\big(v_{s+1}-v_{s,0}^{\varepsilon,j_1}\big)\Big]_{\pm_1} \\
&\times \int_{(k-2)\delta}^{t_1-\delta} \mathbb{1}_{U_{j_2}^c(Z_{s,1}^0(t_1))}(t_2) \int_{\mathbb{S}_{\omega_2}^{d-1}\times\mathbb{R}_{v_{s+2}}^d} \mathbb{1}_{E_{j_2}^c(Z_{s,1}^0(t_2))}(\omega_2, v_{s+2}) \\
&\qquad\qquad \times (\pm_2)\Big[\omega_2\cdot\big(v_{s+2}-v_{s,1}^{\varepsilon,j_2}(t_2)\big)\Big]_{\pm_2} \\
&\qquad \dots \\
&\times \int_0^{t_{k-1}-\delta} \mathbb{1}_{U_{j_k}^c(Z_{s,k-1}^0(t_{k-1}))}(t_k) \int_{\mathbb{S}_{\omega_k}^{d-1}\times\mathbb{R}_{v_{s+k}}^d} \mathbb{1}_{E_{j_k}^c(Z_{s,k-1}^0(t_k))}(\omega_k, v_{s+k}) \\
&\qquad\qquad \times (\pm_k)\Big[\omega_k\cdot\big(v_{s+k}-v_{s,k-1}^{\varepsilon,j_k}(t_k)\big)\Big]_{\pm_k} f_0^{(s+k)}\big(Z_{s,k}^{\varepsilon}(0)\big) \\
&\qquad\qquad \mathrm{d}\omega_k\,\mathrm{d}v_{s+k}\,\mathrm{d}t_k\dots\mathrm{d}\omega_2\,\mathrm{d}v_{s+2}\,\mathrm{d}t_2\,\mathrm{d}\omega_1\,\mathrm{d}v_{s+1}\,\mathrm{d}t_1. \qquad (13.31)
\end{aligned}$$

One will denote this intermediate elementary term as

$$\mathbb{1}_K \overline{\mathcal{J}}_{\substack{s,s+k-1\\ M_k,J_k}}^{\varepsilon,\delta}(U^c,E^c)\big(f_0^{(s+k)}\mathbb{1}_{|V_{s+k}|\leq R}\big). \qquad (13.32)$$

The purpose of the present section is then to control the difference

$$\mathbb{1}_K\Big(\mathcal{J}_{\substack{s,s+k-1\\ M_k,J_k}}^{0,\delta}(U^c,E^c)\big(f_0^{(s+k)}\mathbb{1}_{|V_{s+k}|\leq R}\big) - \overline{\mathcal{J}}_{\substack{s,s+k-1\\ M_k,J_k}}^{\varepsilon,\delta}(U^c,E^c)\big(f_0^{(s+k)}\mathbb{1}_{|V_{s+k}|\leq R}\big)\Big).$$

Remark 46. *The intermediate elementary term defined above makes sense as an integral of a continuous function.*

Lemma 39 (Error coming from the divergence of the trajectories, qualitative version)**.** *Let s and n be two positive integers, and β_0 be a strictly positive number. For all integers $1\leq k\leq n$, let $f_0^{(s+k)}$ be a function of $X_{0,s+k,\beta_0}$, and*

let K be a compact set of the domain of local uniform convergence Ω_s.
Then, there exist five strictly positive numbers $\overline{\varepsilon}, \overline{\varepsilon_0}, \overline{\alpha}, \overline{\gamma}$ and $\overline{R}$ (depending only on the compact set K) such that for all strictly positive numbers $R, \delta, \varepsilon, a, \varepsilon_0, \rho, \eta$ and α which satisfy

$$\left\{ \begin{array}{c} \varepsilon \leq \overline{\varepsilon},\ \varepsilon_0 \leq \overline{\varepsilon_0},\ \alpha \leq \overline{\alpha},\ \max\big(16R\varepsilon/\varepsilon_0, \varepsilon_0/\delta\big) \leq \overline{\gamma} \\ \\ \text{and } R \geq \overline{R}, \end{array} \right. \tag{13.33}$$

and

$$\left\{ \begin{array}{c} 4\sqrt{3}a \leq \varepsilon_0, \quad 3a \leq \rho, \quad \varepsilon_0 \leq \eta\delta, \quad R \geq 1, \quad \eta \leq 1 \text{ and } \alpha \leq c(d), \\ \text{with } c(d) \text{ a constant depending only on the dimension } d, \text{ and} \\ \\ 2n\varepsilon \leq a, \end{array} \right. \tag{13.34}$$

one has that, in the Boltzmann-Grad limit $N \to +\infty$, $N\varepsilon^{d-1} = 1$, for s, n, R, δ, a, ε_0, η, ρ and α fixed, the following uniform convergence on the compact set K, and uniform on the time interval $[0, T]$, holds :

$$\begin{aligned} \Bigg\| \mathbb{1}_K \sum_{k=1}^{n} \sum_{M_k \in \mathfrak{M}_k} \sum_{J_k \in \mathfrak{J}_k^s} \Big(& \mathcal{J}^{0,\delta}_{\substack{s,s+k-1 \\ M_k, J_k}}(U^c, E^c)\big(f_0^{(s+k)} \mathbb{1}_{|V_{s+k}| \leq R}\big) \\ & - \overline{\mathcal{J}}^{\varepsilon,\delta}_{\substack{s,s+k-1 \\ M_k, J_k}}(U^c, E^c)\big(f_0^{(s+k)} \mathbb{1}_{|V_{s+k}| \leq R}\big)\Big)(Z_s) \Bigg\|_{L^\infty} \underset{\varepsilon \to 0}{\longrightarrow} 0. \end{aligned} \tag{13.35}$$

Proof. First, thanks to the second point (13.6) of Lemma 38 page 449, the velocities of the two pseudo-trajectories at the times of adjunctions are the

same, so that one has :

$$
\begin{aligned}
&\mathbb{1}_K\Big(\mathcal{J}^{0,\delta}_{\substack{s,s+k-1\\M_k,J_k}}(U^c,E^c)\big(f_0^{(s+k)}\mathbb{1}_{|V_{s+k}|\le R}\big)-\overline{\mathcal{J}}^{\varepsilon,\delta}_{\substack{s,s+k-1\\M_k,J_k}}(U^c,E^c)\big(f_0^{(s+k)}\mathbb{1}_{|V_{s+k}|\le R}\big)\Big)\\
&=\mathbb{1}_K\mathbb{1}_{t\ge k\delta}\int_{(k-1)\delta}^{t-\delta}\mathbb{1}_{U^c_{j_1}(Z_s)}(t_1)\int_{\mathbb{S}^{d-1}_{\omega_1}\times\mathbb{R}^d_{v_{s+1}}}\mathbb{1}_{E^c_{j_1}(Z^0_{s,0(t_1)})}(\omega_1,v_{s+1})\\
&\qquad\qquad\times(\pm_1)\Big[\omega_1\cdot\big(v_{s+1}-v^{0,j_1}_{s,0}\big)\Big]_{\pm_1}\\
&\quad\times\int_{(k-2)\delta}^{t_1-\delta}\mathbb{1}_{U^c_{j_2}(Z^0_{s,1}(t_1))}(t_2)\int_{\mathbb{S}^{d-1}_{\omega_2}\times\mathbb{R}^d_{v_{s+2}}}\mathbb{1}_{E^c_{j_2}(Z^0_{s,1}(t_2))}(\omega_2,v_{s+2})\\
&\qquad\qquad\times(\pm_2)\Big[\omega_2\cdot\big(v_{s+2}-v^{0,j_2}_{s,1}(t_2)\big)\Big]_{\pm_2}\\
&\qquad\cdots\\
&\quad\times\int_0^{t_{k-1}-\delta}\mathbb{1}_{U^c_{j_k}(Z^0_{s,k-1}(t_{k-1}))}(t_k)\int_{\mathbb{S}^{d-1}_{\omega_k}\times\mathbb{R}^d_{v_{s+k}}}\mathbb{1}_{E^c_{j_k}(Z^0_{s,k-1}(t_k))}(\omega_k,v_{s+k})\\
&\times(\pm_k)\Big[\omega_k\cdot\big(v_{s+k}-v^{0,j_k}_{s,k-1}(t_k)\big)\Big]_{\pm_k}\mathbb{1}_{|V_{s+k}|\le R}\\
&\qquad\qquad\times\Big(f_0^{(s+k)}\big(Z^0_{s,k}(0)\big)-f_0^{(s+k)}\big(Z^\varepsilon_{s,k}(0)\big)\Big)\\
&\qquad\qquad\qquad\mathrm{d}\omega_k\,\mathrm{d}v_{s+k}\,\mathrm{d}t_k\ldots\mathrm{d}\omega_2\,\mathrm{d}v_{s+2}\,\mathrm{d}t_2\,\mathrm{d}\omega_1\,\mathrm{d}v_{s+1}\,\mathrm{d}t_1.
\end{aligned}
$$

Using then the first point (13.5) of Lemma 38, one sees that, at time 0, for any particle $1\le j\le s+k$:

$$\Big|\big(Z^\varepsilon_{s,k}(0)\big)^{X,j}-\big(Z^0_{s,k}(0)\big)^{X,j}\Big|\underset{\varepsilon\to0}{\longrightarrow}0.$$

At time 0, the velocity of a given particle can be different for the BBGKY and the Boltzmann pseudo-trajectories, depending on a possible non empty interval of time between the respective bouncings of this particle for the two pseudo-trajectories. But since the positions of this particle for the two pseudo-trajectories are close, one can use the boundary condition verified by the initial datum of the Boltzmann hierarchy.

Namely, if the velocities of a particle j are different at time 0, only one of the two pseudo-trajectories induces a bouncing for the particle. But in all cases, the particle following the Boltzmann pseudo-trajectory will bounce as well, possibly after the time 0 though. One will denote t_0^j this time of bouncing. One has then by definition :

$$\big(Z^0_{s,k}(t_0^j)\big)^{X,j}\cdot e_1=0,$$

and thanks to the boundary condition verified by the initial datum

$$f_0^{(s+k)}\big(Z^0_{s,k}\big((t_0^j)^+\big)\big)=f_0^{(s+k)}\big(Z^0_{s,k}\big((t_0^j)^-\big)\big).$$

One denotes p the number of particles having different velocities at time 0 depending on the pseudo-trajectory followed. Those particles are denoted $j_1, \dots, j_p$. One will use p times the boundary condition in order to modify the diverging velocity, that is one writes :

$$\begin{aligned}
&f_0^{(s+k)}\big(Z^0_{s,k}(0)\big) - f_0^{(s+k)}\big(Z^\varepsilon_{s,k}(0)\big)\\
&= f_0^{(s+k)}\big(X^0_{s,k}(0), V^0_{s,k}(0)\big) - f_0^{(s+k)}\big(X^\varepsilon_{s,k}(0), V^\varepsilon_{s,k}(0)\big)\\
&= f_0^{(s+k)}\big(X^0_{s,k}(0), V^\varepsilon_{s,k}(0)\big)\\
&\qquad - \sum_{l=1}^{p}\Big(f_0^{(s+k)}\big(X^0_{s,k}(t_0^{j_l}), V_l\big) - f_0^{(s+k)}\big(X^0_{s,k}(t_0^{j_l}), V_l\big)\Big)\\
&\qquad - f_0^{(s+k)}\big(X^\varepsilon_{s,k}(0), V^\varepsilon_{s,k}(0)\big)\\
&= \Big(f_0^{(s+k)}\big(X^0_{s,k}(0), V^0_{s,k}(0)\big) - f_0^{(s+k)}\big(X^0_{s,k}(t_0^{j_1}), V^0_{s,k}(0)\big)\Big)\\
&\qquad + \sum_{l=1}^{p-1}\Big(f_0^{(s+k)}\big(X^0_{s,k}(t_0^{j_l}), V_l\big) - f_0^{(s+k)}\big(X^0_{s,k}(t_0^{j_{l+1}}), V_{l+1}\big)\Big)\\
&\qquad + \Big(f_0^{(s+k)}\big(X^0_{s,k}(t_0^{j_p}), V_p\big) - f_0^{(s+k)}\big(X^\varepsilon_{s,k}(0), V^\varepsilon_{s,k}(0)\big)\Big),
\end{aligned} \tag{13.36}$$

where V_l is defined as follows :

$$V_1 = V^0_{s,k}(0) = \big(v^{0,1}_{s,k}(0), \dots, v^{0,s+k}_{s,k}(0)\big),$$

$$\begin{aligned}
V_2 &= \big(v^{0,1}_{s,k}(0), \dots, v^{0,j_1-1}_{s,k}(0), \mathcal{S}_0\big(v^{0,s+k}_{s,k}(0)\big), v^{0,j_1+1}_{s,k}, \dots, v^{0,s+k}_{s,k}(0)\big)\\
&= \big(v^{0,1}_{s,k}(0), \dots, v^{0,j_1-1}_{s,k}(0), v^{\varepsilon,s+k}_{s,k}, v^{0,j_1+1}_{s,k}, \dots, v^{0,s+k}_{s,k}(0)\big),
\end{aligned}$$

and so on.

In other words, V_1 is defined as the collection of all the velocities of the Boltzmann pseudo-trajectory at time 0. Then the collection of velocities V_2 is defined from the collection V_1, in which the velocity of the particle j_1 of the Boltzmann pseudo-trajectory is replaced by the one of the same particle of the BBGKY pseudo-trajectory at the same time 0. Similarly, V_3 is defined from V_2 by replacing the velocity of the particle j_2 following the Boltzmann pseudo-trajectory at time 0 by the velocity of the same particle at the same time, but following the BBGKY pseudo-trajectory. All the V_l are defined by induction, so that in the end, one has $V_p = V^\varepsilon_{s,k}(0)$, and then the last term of the expression (13.36) above is

$$f_0^{(s+k)}\big(X^0_{s,k}(t_0^{j_p}), V^\varepsilon_{s,k}(0)\big) - f_0^{(s+k)}\big(X^\varepsilon_{s,k}(0), V^\varepsilon_{s,k}(0)\big).$$

One is now able to use the boundary condition satisfied by the $s+k$-th initial datum of the Boltzmann hierarchy, that is

$$\begin{aligned}
&f_0^{(s+k)}\big(X^0_{s,k}(t_0^{j_l}), V_l\big) - f_0^{(s+k)}\big(X^0_{s,k}(t_0^{j_{l+1}}), V_{l+1}\big)\\
&= f_0^{(s+k)}\big(X^0_{s,k}(t_0^{j_l}), V_{l+1}\big) - f_0^{(s+k)}\big(X^0_{s,k}(t_0^{j_{l+1}}), V_{l+1}\big)
\end{aligned}$$

(the velocities V_l in the first initial datum have been replaced by V_{l+1}, since at $t_0^{j_l}$, one has

$$x_{s,k}^{0,j_l}(t_0^{j_l}) \cdot e_1 = 0,$$

and then the boundary condition can be used).
For any compact subset K' of $\big(\{x \cdot e_1 > 0\} \times \mathbb{R}^d\big)^{s+k}$ and for any strictly positive number η, one will denote

$$\theta_{f_0^{(s+k)}}^{K'}(\eta) = \sup_{\substack{Z_{s+k}^1, Z_{s+k}^2 \in K' \\ |Z_{s+k}^1 - Z_{s+k}^2| \leq \eta}} \Big| f_0^{(s+k)}\big(Z_{s+k}^1\big) - f_0^{(s+k)}\big(Z_{s+k}^2\big) \Big|,$$

which is a well defined, finite quantity, and which goes to zero as η goes to zero thanks to the continuity of the $s+k$-th initial datum $f_0^{(s+k)}$.
For all the terms except the last of the expression (13.36), one uses again the following fact. As ε goes to zero, the difference between the positions of the particles of the two pseudo-trajectories is going to zero as well, so the times of bouncing t_0^j are converging to the final time $t = 0$, uniformly in the initial configuration Z_s and in the adjunction parameters (since, one recalls, those times are given by the pseudo-trajectory). Then the difference between the positions of any particle m between $t_0^{j_l}$ and $t_0^{j_{l+1}}$ verifies :

$$\big|x_{s,k}^{0,m}(t_0^{j_l}) - x_{s,k}^{0,m}(t_0^{j_{l+1}})\big| \leq |t_0^{j_l} - t_0^{j_{l+1}}| \big|v_{s,k}^{0,m}(t_0^{j_l})\big| \leq R|t_0^{j_l} - t_0^{j_{l+1}}|,$$

so that

$$\big|X_{s,k}^0(t_0^{j_l}) - X_{s,k}^0(t_0^{j_{l+1}})\big| \leq \sqrt{s+k} R |t_0^{j_l} - t_0^{j_{l+1}}|.$$

Since all the configurations Z_s and the adjunction parameters are lying in compact sets, so are the final configurations (that is, at time $t = 0$) of the pseudo-trajectories. One will denote (with a slight abuse, but the following notation emphasizes which quantities really matter here, recalling in particular that α denotes, following the notations introduced in Section 12.2.5 page 368, the parameter of cut-off in the grazing velocities)

$$Z_{s+k}(K, R, \alpha) \subset \big(\overline{\Omega^c} \times \mathbb{R}^d\big)^{s+k}. \tag{13.37}$$

the subset in which lie the final configurations $Z_{s,k}^0(0)$ and $Z_{s,k}^\varepsilon(0)$.
It is easy to provide a brief description of this subset, since the cut-off in large energy (performed along Section 11.2 page 280) enables to state that for any final configuration $Z_{s,k}^0(0) = (X_{s,k}^0(0), V_{s,k}^0(0))$ (see the points (12.14) and (12.15) of Definition 47 page 325 for those notations) belonging to $Z_{s+k}(K, R, \alpha)$, one has :

$$\big|V_{s,k}^0(0)\big| \leq R,$$

so that

$$X_{s,k}^0(0) \in \big(B_{\mathbb{R}^d}(0, r_K + R)\big)^{s+k},$$

with r_K denoting the radius of the smallest ball of the Euclidean space $\mathbb{R}^{2ds}$ containing the compact subset K.

Therefore, the $1+(p-1)$ first terms of the expression (13.36) are uniformly converging to zero as ε goes to zero, writing :

$$\begin{aligned}
&\Big|\Big(f_0^{(s+k)}\big(X_{s,k}^0(0),V_{s,k}^0(0)\big)-f_0^{(s+k)}\big(X_{s,k}^0(t_0^{j_1}),V_{s,k}^0(0)\big)\Big)\\
&\qquad+\sum_{l=1}^{p-1}\Big(f_0^{(s+k)}\big(X_{s,k}^0(t_0^{j_l}),V_{l+1}\big)-f_0^{(s+k)}\big(X_{s,k}^0(t_0^{j_{l+1}}),V_{l+1}\big)\Big)\Big|\\
&\le\Big|f_0^{(s+k)}\big(X_{s,k}^0(0),V_{s,k}^0(0)\big)-f_0^{(s+k)}\big(X_{s,k}^0(t_0^{j_1}),V_{s,k}^0(0)\big)\Big|\\
&\qquad+\sum_{l=1}^{p-1}\Big|f_0^{(s+k)}\big(X_{s,k}^0(t_0^{j_l}),V_{l+1}\big)-f_0^{(s+k)}\big(X_{s,k}^0(t_0^{j_{l+1}}),V_{l+1}\big)\Big|\\
&\le\sum_{l=0}^{p-1}\theta_{f_0^{(s+k)}}^{Z_{s+k}(K,R,\alpha)}\big(\sqrt{s+k}R\big|t_0^{j_l}-t_0^{j_{l+1}}\big|\big),
\end{aligned}$$

where $t_0^{j_0}$ denotes 0. Now, the last term

$$\begin{aligned}
&f_0^{(s+k)}\big(X_{s,k}^0(t_0^{j_p}),V_p\big)-f_0^{(s+k)}\big(X_{s,k}^\varepsilon(0),V_{s,k}^\varepsilon(0)\big)\\
&=f_0^{(s+k)}\big(X_{s,k}^0(t_0^{j_p}),V_{s,k}^\varepsilon(0)\big)-f_0^{(s+k)}\big(X_{s,k}^\varepsilon(0),V_{s,k}^\varepsilon(0)\big)
\end{aligned}$$

of the expression (13.36) converges uniformly to zero as ε goes to zero, thanks to the first point (13.5) of Lemma 38 page 449. Indeed, one writes :

$$\begin{aligned}
&\Big|f_0^{(s+k)}\big(X_{s,k}^0(t_0^{j_p}),V_{s,k}^\varepsilon(0)\big)-f_0^{(s+k)}\big(X_{s,k}^\varepsilon(0),V_{s,k}^\varepsilon(0)\big)\Big|\\
&\le\Big|f_0^{(s+k)}\big(X_{s,k}^0(t_0^{j_p}),V_{s,k}^\varepsilon(0)\big)-f_0^{(s+k)}\big(X_{s,k}^0(0),V_{s,k}^\varepsilon(0)\big)\Big|\\
&\qquad+\Big|f_0^{(s+k)}\big(X_{s,k}^0(0),V_{s,k}^\varepsilon(0)\big)-f_0^{(s+k)}\big(X_{s,k}^\varepsilon(0),V_{s,k}^\varepsilon(0)\big)\Big|. \qquad (13.38)
\end{aligned}$$

The first term of the right-hand side of (13.38) is controlled in the same way as for the p first terms :

$$\begin{aligned}
&\Big|f_0^{(s+k)}\big(X_{s,k}^0(t_0^{j_p}),V_{s,k}^\varepsilon(0)\big)-f_0^{(s+k)}\big(X_{s,k}^0(0),V_{s,k}^\varepsilon(0)\big)\Big|\\
&\qquad\le\theta_{f_0^{(s+k)}}^{Z_{s+k}(K,R,\alpha)}\big(\sqrt{s+k}R\big|t_0^{j_p}\big|\big),
\end{aligned}$$

and for the second term, since Lemma 38 provides

$$\begin{aligned}
\big|X_{s,k}^0(0)-X_{s,k}^\varepsilon(0)\big|&=\sqrt{\sum_{l=1}^{s+k}\big|x_{s,k}^{0,l}(0)-x_{s,k}^{\varepsilon,l}(0)\big|^2}\\
&\le\sqrt{\sum_{l=1}^{s+k}\big(2k\varepsilon\big)^2}=2(\sqrt{s+k})k\varepsilon,
\end{aligned}$$

one finds therefore

$$\Big|f_0^{(s+k)}\big(X^0_{s,k}(0),V^\varepsilon_{s,k}(0)\big)-f_0^{(s+k)}\big(X^\varepsilon_{s,k}(0),V^\varepsilon_{s,k}(0)\big)\Big| \\ \leq \theta^{Z_{s+k}(K,R,\alpha)}_{f_0^{(s+k)}}\big(2(\sqrt{s+k})k\varepsilon\big).$$

The decomposition presented in (13.36) enables then to obtain the following bound

$$\big|f_0^{(s+k)}\big(Z^0_{s,k}(0)\big)-f_0^{(s+k)}\big(Z^\varepsilon_{s,k}(0)\big)\big| \\ \leq (p+1)\theta^{Z_{s+k}(K,R,\alpha)}_{f_0^{(s+k)}}\Big(2(\sqrt{s+k})R\sup_{1\leq l\leq p}|t_0^{j_l}|\Big) \\ +\theta^{Z_{s+k}(K,R,\alpha)}_{f_0^{(s+k)}}\Big(2(\sqrt{s+k})k\varepsilon\Big).$$

Now one wants a bound on $\big|f_0^{(s+k)}\big(Z^0_{s,k}(0)\big)-f_0^{(s+k)}\big(Z^\varepsilon_{s,k}(0)\big)\big|$ which is uniform on the compact set $Z_{s+k}(K,R,\alpha)$.

On the one hand, one recalls that p is the number of particles having not the same velocity at time 0 following the Boltzmann or the BBGKY pseudo-trajectories. This number depends strongly on the pseudo-trajectories, that is on the initial data Z_s and the adjunction parameters, but this number is of course bounded by the number of particles of the system at time 0, that is $s+k$. On the other hand, the times $t_0^{j_l}$ are the times of bouncing against the obstacle following the Boltzmann pseudo-trajectory, of the particles having not the same velocity at time 0 following the Boltzmann or the BBGKY pseudo-trajectory. So those times depend also on the initial data and the adjunction parameters. However, it can be bounded uniformly, since on $Z_{s+k}(K,R,\alpha)$, several cut-offs have been performed so that at any time (and then in particular at time 0) the velocities of the particles of the system are not grazing. Using this fact together with the result of proximity of the two pseudo-trajectories obtained in Lemma 38, one deduces that the times $t_0^{j_l}$ all go to zero, as ε goes to zero, uniformly in l and the adjunction parameters, that is uniformly on $Z_{s+k}(K,R,\alpha)$. In other words, for any compact subset K of the phase space of s particles, for any positive integer k, there exists a function $T_0(K,k,R,\alpha)$ depending on ε such that

$$\forall 1\leq l\leq p,\ t_0^{j_l}\leq T_0(K,k,R,\alpha)(\varepsilon)$$

and

$$T_0(K,k,R,\alpha)(\varepsilon)\underset{\varepsilon\to 0}{\longrightarrow}0.$$

Taking the supremum of the quantity $\big|f_0^{(s+k)}\big(Z^0_{s,k}(0)\big)-f_0^{(s+k)}\big(Z^\varepsilon_{s,k}(0)\big)\big|$ over $Z_{s+k}(K,R,\alpha)$, one finds

$$\sup_{Z_{s+k}(K,R,\alpha)}\big|f_0^{(s+k)}\big(Z^0_{s,k}(0)\big)-f_0^{(s+k)}\big(Z^\varepsilon_{s,k}(0)\big)\big| \\ \leq (s+k+1)\theta^{Z_{s+k}(K,R,\alpha)}_{f_0^{(s+k)}}\Big(2(\sqrt{s+k})RT_0(K,k)(\varepsilon)|\Big) \\ +\theta^{Z_{s+k}(K,R,\alpha)}_{f_0^{(s+k)}}\Big(2(\sqrt{s+k})k\varepsilon\Big).$$

One can then bound each term

$$\Big|\mathbb{1}_K\Big(\mathcal{J}^{0,\delta}_{\substack{s,s+k-1\\M_k,J_k}}(U^c,E^c)\big(f_0^{(s+k)}\mathbb{1}_{|V_{s+k}|\le R}\big) - \overline{\mathcal{J}}^{\varepsilon,\delta}_{\substack{s,s+k-1\\M_k,J_k}}(U^c,E^c)\big(f_0^{(s+k)}\mathbb{1}_{|V_{s+k}|\le R}\big)\Big)\Big|,$$

of the decomposition (13.35) of the lemma, denoted here to simplify the notations as Q_{M_k,J_k}. Recalling that this quantity writes :

$$\begin{aligned}
Q_{M_k,J_k} = \mathbb{1}_K\mathbb{1}_{t\ge k\delta}\Bigg|&\int_{(k-1)\delta}^{t-\delta}\mathbb{1}_{U^c_{j_1}(Z_s)}(t_1)\int_{\mathbb{S}^{d-1}_{\omega_1}\times\left(B_{\mathbb{R}^d}(0,R)\right)_{v_{s+1}}}\mathbb{1}_{E^c_{j_1}(Z^0_{s,0(t_1)})}(\omega_1,v_{s+1})\\
&\qquad\times(\pm_1)\Big[\omega_1\cdot\big(v_{s+1}-v^{0,j_1}_{s,0}\big)\Big]_{\pm_1}\\
&\times\int_{(k-2)\delta}^{t_1-\delta}\mathbb{1}_{U^c_{j_2}(Z^0_{s,1}(t_1))}(t_2)\int_{\mathbb{S}^{d-1}_{\omega_2}\times\left(B_{\mathbb{R}^d}(0,R)\right)_{v_{s+2}}}\mathbb{1}_{E^c_{j_2}(Z^0_{s,1}(t_2))}(\omega_2,v_{s+2})\\
&\qquad\times(\pm_2)\Big[\omega_2\cdot\big(v_{s+2}-v^{0,j_2}_{s,1}(t_2)\big)\Big]_{\pm_2}\\
&\qquad\cdots\\
&\times\int_0^{t_{k-1}-\delta}\mathbb{1}_{U^c_{j_k}(Z^0_{s,k-1}(t_{k-1}))}(t_k)\int_{\mathbb{S}^{d-1}_{\omega_k}\times\left(B_{\mathbb{R}^d}(0,R)\right)_{v_{s+k}}}\mathbb{1}_{E^c_{j_k}(Z^0_{s,k-1}(t_k))}(\omega_k,v_{s+k})\\
&\times(\pm_k)\Big[\omega_k\cdot\big(v_{s+k}-v^{0,j_k}_{s,k-1}(t_k)\big)\Big]_{\pm_k}\mathbb{1}_{|V_{s+k}|\le R}\\
&\qquad\times\Big(f_0^{(s+k)}\big(Z^0_{s,k}(0)\big)-f_0^{(s+k)}\big(Z^\varepsilon_{s,k}(0)\big)\Big)\\
&\qquad\mathrm{d}\omega_k\,\mathrm{d}v_{s+k}\,\mathrm{d}t_k\ldots\mathrm{d}\omega_2\,\mathrm{d}v_{s+2}\,\mathrm{d}t_2\,\mathrm{d}\omega_1\,\mathrm{d}v_{s+1}\,\mathrm{d}t_1\Bigg|,
\end{aligned}$$

it can be bounded easily as

$$\begin{aligned}
Q_{M_k,J_k}\le\Big(2\big|\mathbb{S}^{d-1}\times B_{\mathbb{R}^d}(0,R)\big|R\Big)^k\Big(\int_0^t\int_0^{t_1}\ldots\int_0^{t_{k-1}}\mathrm{d}t_k\ldots\mathrm{d}t_2\,\mathrm{d}t_1\Big)\\
\times\Big((s+k+1)\theta^{Z_{s+k}(K,R,\alpha)}_{f_0^{(s+k)}}\Big(2(\sqrt{s+k})RT_0(K,k,\alpha)(\varepsilon)\Big)\\
+\theta^{Z_{s+k}(K,R,\alpha)}_{f_0^{(s+k)}}\Big(2(\sqrt{s+k})k\varepsilon\Big)\Big).
\end{aligned}$$

Since $k\le n$ and since the functions $\theta^{Z_{s+k}(K,R,\alpha)}_{f_0^{(s+k)}}$ are non decreasing, and recall-

ing that $R \geq 1$, one has

$$\begin{aligned}
&(s+k+1)\theta_{f_0^{(s+k)}}^{Z_{s+k}(K,R,\alpha)}\Big(2(\sqrt{s+k})RT_0(K,k,R,\alpha)(\varepsilon)\Big) \\
&\qquad\qquad + \theta_{f_0^{(s+k)}}^{Z_{s+k}(K,R,\alpha)}\Big(2(\sqrt{s+k})k\varepsilon\Big) \\
&\leq (s+n+1)\theta_{f_0^{(s+k)}}^{Z_{s+k}(K,R,\alpha)}\Big(2(\sqrt{s+n})RT_0(K,k,R,\alpha)(\varepsilon)\Big) \\
&\qquad\qquad + \theta_{f_0^{(s+k)}}^{Z_{s+k}(K,R,\alpha)}\Big(2n(\sqrt{s+n})R\varepsilon\Big).
\end{aligned}$$

Denoting $\psi(K,k,R,\alpha)(\varepsilon) = \max\big(\varepsilon, T_0(K,k,R,\alpha)(\varepsilon)\big)$, one finds

$$Q_{M_k,J_k} \leq \big(C(d)R^{d+1}\big)^k \frac{t^k}{k!}(s+n+2)\theta_{f_0^{(s+k)}}^{Z_{s+k}(K,R,\alpha)}\Big(2n(\sqrt{s+n})R\psi(K,k,R,\alpha)(\varepsilon)\Big).$$

Finally, one wants to sum over all the contributions $1 \leq k \leq n$, $M_k \in \mathfrak{M}_k$, $J_k \in \mathfrak{J}_k^s$, since the quantity controlled by the lemma is smaller than

$$\sum_{k=1}^{n} \sum_{M_k \in \mathfrak{M}_k} \sum_{J_k \in \mathfrak{J}_k^s} Q_{M_k,J_k}.$$

For K, s, R and α given, all the n functions of the family

$$\Big(\varepsilon \mapsto \theta_{f_0^{(s+k)}}^{Z_{s+k}(K,R,\alpha)}\Big(2n(\sqrt{s+n})R\psi(K,k,R,\alpha)(\varepsilon)\Big)\Big)_{1\leq k\leq n}$$

converge to zero as ε goes to zero. Therefore, one has also

$$\sup_{1\leq k\leq n}\Big(\theta_{f_0^{(s+k)}}^{Z_{s+k}(K,R,\alpha)}\Big(2n(\sqrt{s+n})R\psi(K,k,R,\alpha)(\varepsilon)\Big)\Big) \underset{\varepsilon\to 0}{\longrightarrow} 0,$$

and then

$$\begin{aligned}
&\sum_{k=1}^{n} \sum_{M_k \in \mathfrak{M}_k} \sum_{J_k \in \mathfrak{J}_k^s} Q_{M_k,J_k} \\
&\quad \leq \sum_{k=1}^{n} \sum_{M_k \in \mathfrak{M}_k} \sum_{J_k \in \mathfrak{J}_k^s} \Big(\big(C(d)R^{d+1}\big)^k \frac{t^k}{k!}(s+n+2) \\
&\qquad\qquad \times \theta_{f_0^{(s+k)}}^{Z_{s+k}(K,R,\alpha)}\Big(2n(\sqrt{s+n})R\psi(K,k,R,\alpha)(\varepsilon)\Big)\Big) \\
&\quad \leq (s+n+2)\Big(\sum_{M_n \in \mathfrak{M}_n} \sum_{J_n \in \mathfrak{J}_n^s} 1\Big)\Big(\sum_{k=1}^{n} \big(C(d)R^{d+1}\big)^k \frac{t^k}{k!}\Big) \\
&\qquad\qquad \times \sup_{1\leq k\leq n}\Big(\theta_{f_0^{(s+k)}}^{Z_{s+k}(K,R,\alpha)}\Big(2n(\sqrt{s+n})R\psi(K,k,R,\alpha)(\varepsilon)\Big)\Big) \\
&\quad \leq (s+n+2)\frac{(s+n-1)!}{(s-1)!}2^n \exp\big(C(d)Rt\big) \\
&\qquad\qquad \times \sup_{1\leq k\leq n}\Big(\theta_{f_0^{(s+k)}}^{Z_{s+k}(K,R,\alpha)}\Big(2n(\sqrt{s+n})R\psi(K,k,R,\alpha)(\varepsilon)\Big)\Big).
\end{aligned}$$

Denoting

$$\psi_1(s,n,R,\alpha)(\varepsilon) = \sup_{1\le k\le n}\Big(\theta_{f_0^{(s+k)}}^{Z_{s+k}(K,R,\alpha)}\Big(2n(\sqrt{s+n})R\psi(K,k,R,\alpha)(\varepsilon)\Big)\Big),$$

one obtains the result stated in the lemma. □

Chapter 14

Control of the last error terms

This part is now devoted to controlling the error produced by the changes made in the elementary term of the BBGKY hierarchy to define the intermediate elementary term, namely the removal of the prefactors (see Section 14.1 below) and the substitution of the initial datum of the BBGKY hierarchy by the one of the Boltzmann hierarchy (see Section 14.2 page 488 below).

14.1 Limit and rate of convergence of the prefactors of the collision operators

In the Boltzmann-Grad limit, one can control the error produced by the removal of the prefactor term thanks to the following lemma.

Lemma 40 (Limit and rate of convergence of the prefactor of the collision operators in the Boltzmann-Grad limit)**.** *Let s, n and N be three positive integers, and ε be a strictly positive number. For s and $k \leq n$ fixed, in the Boltzmann-Grad limit $N\varepsilon^{d-1} = 1$, one has :*

$$\frac{(N-s)!}{(N-s-k)!}\varepsilon^{k(d-1)} \underset{N\to+\infty}{\longrightarrow} 1. \tag{14.1}$$

Moreover, one has for all positive integers s and n, in the Boltzmann-Grad limit $N\varepsilon^{d-1} = 1$ and for N large enough :

$$1 - 2\frac{(s+n)^2}{N} \leq \frac{(N-s)!}{(N-s-k)!}\varepsilon^{k(d-1)} \leq 1. \tag{14.2}$$

Proof. Thanks to the Stirling formula, one has :

$$\begin{aligned}\frac{(N-s)!}{(N-s-k)!}\varepsilon^{k(d-1)} &\sim \frac{\sqrt{2\pi(N-s)}\left(\frac{N-s}{e}\right)^{(N-s)}}{\sqrt{2\pi(N-s-k)}\left(\frac{N-s-k}{e}\right)^{(N-s-k)}}N^{-k}\\ &\sim \left(\frac{N-s}{N-s-k}\right)^{N-s+1/2} e^{-k}(N-s-k)^k N^{-k}.\end{aligned}$$

Then one uses

$$\begin{aligned}\left(\frac{N-s}{N-s-k}\right)^{N-s+1/2} &= \exp\left((N-s+1/2)\ln\left(\frac{N-s}{N-s-k}\right)\right)\\ &\sim \exp\left((N-s+1/2)\frac{k}{N-s-k}\right)\\ &\sim e^k,\end{aligned}$$

which gives the first result (14.1).

For the second pair of inequalities (14.2), one has :

$$\begin{aligned}\frac{(N-s)!}{(N-s-k)!}\varepsilon^{k(d-1)} &= (N-s-k+1)(N-s-k+2)...(N-s)\varepsilon^{k(d-1)}\\ &\geq \left((N-s-k)\varepsilon^{d-1}\right)^k = e^{k\ln(\varepsilon^{d-1}(N-s-k))}.\end{aligned}$$

Then using the definition of the Boltzmann-Grad limit $N\varepsilon^{d-1}=1$, one has :

$$\ln\left(\varepsilon^{d-1}(N-s-k)\right) = \ln\left(\frac{N-s-k}{N}\right) \geq -2\frac{(s+k)}{N}$$

for N large enough ($2(s+k)\leq 2(s+n)\leq N$ is enough, since $-1<-\ln 2$), so that

$$\begin{aligned}\frac{(N-s)!}{(N-s-k)!}\varepsilon^{k(d-1)} &\geq 1+k\ln\left(\varepsilon^{d-1}(N-s-k)\right)\\ &\geq 1-2k\frac{(s+k)}{N} \geq 1-2\frac{(s+n)^2}{N}.\end{aligned}$$

□

14.2 Error coming from the substitution of the initial datum and the removal of the prefactor

Finally, one studies here the last error term, coming from, on the one hand the removal of the prefactors, and on the other hand the substitution of the initial

datum of the BBGKY hierarchy in the elementary term of the same hierarchy, by the initial datum of the Boltzmann hierarchy to define the intermediate elementary term $\overline{\mathcal{J}}^{\varepsilon,\delta}_{\substack{s,s+k-1\\M_k,J_k}}(U^c,E^c)$.

Lemma 41 (Error coming from the substitution of the initial data and the removal of the prefactors, qualitative version)**.** *Let s and n be two positive integers, β_0 be a strictly positive number, and μ_0 be a real number.*
Let $F_0=\big(f_0^{(\widetilde{s})}\big)_{\widetilde{s}\geq 1}$ be a sequence of functions belonging to $\boldsymbol{X}_{0,\beta_0,\mu_0^1}$, and for any positive integer N, let $F_{N,0}=\big(f_{N,0}^{(\widetilde{s})}\big)_{1\leq\widetilde{s}\leq N}$ be a sequence of functions belonging to $\boldsymbol{X}_{N,\varepsilon,\beta_0,\mu_0^1}$ such that for any $\widetilde{s}\in\mathbb{N}^*$,

$$f_{N,0}^{(\widetilde{s})}\underset{N\to+\infty}{\longrightarrow}f_0^{(\widetilde{s})}$$

locally uniformly on the phase space of $\widetilde{s}$ particles $\big(\overline{\Omega^c}\times\mathbb{R}^d\big)^{\widetilde{s}}$. Let in addition K be a compact set of Ω_s.
Then, there exist five strictly positive numbers $\overline{\varepsilon},\overline{\varepsilon_0},\overline{\alpha},\overline{\gamma}$ and $\overline{R}$ (depending only on the compact set K) such that for every strictly positive numbers R,δ,ε,a,ε_0,ρ,η and α which satisfy

$$\left\{\begin{array}{c}\varepsilon\leq\overline{\varepsilon},\ \varepsilon_0\leq\overline{\varepsilon_0},\ \alpha\leq\overline{\alpha},\ \max\big(16R\varepsilon/\varepsilon_0,\varepsilon_0/\delta\big)\leq\overline{\gamma}\\ \text{and } R\geq\overline{R},\end{array}\right. \tag{14.3}$$

and

$$\left\{\begin{array}{c}4\sqrt{3}a\leq\varepsilon_0,\quad 3a\leq\rho,\quad \varepsilon_0\leq\eta\delta,\quad R\geq 1,\quad \eta\leq 1 \text{ and } \alpha\leq c(d),\\ \text{with } c(d) \text{ a constant depending only on the dimension } d, \text{ and}\\ 2n\varepsilon\leq a,\end{array}\right. \tag{14.4}$$

one has that, in the Boltzmann-Grad limit $N\to+\infty$, $N\varepsilon^{d-1}=1$, for s, n, R, δ, a, ε_0, η, ρ and α fixed, the following uniform convergence on the compact set K, and uniform on the time interval $[0,T]$, holds :

$$\Bigg\|\mathbb{1}_K\sum_{k=1}^{n}\sum_{M_k\in\mathfrak{M}_k}\sum_{J_k\in\mathfrak{J}_k^s}\Big(\overline{\mathcal{J}}^{\varepsilon,\delta}_{\substack{s,s+k-1\\M_k,J_k}}(U^c,E^c)\big(f_0^{(s+k)}\mathbb{1}_{|V_{s+k}|\leq R}\big)$$
$$-\mathcal{J}^{N,\varepsilon,\delta}_{\substack{s,s+k-1\\M_k,J_k}}(U^c,E^c)\big(f_{N,0}^{(s+k)}\mathbb{1}_{|V_{s+k}|\leq R}\big)\Big)(Z_s)\Bigg\|_{L^\infty}\underset{\varepsilon\to 0}{\longrightarrow}0. \tag{14.5}$$

Proof. The difference

$$\mathbb{1}_K\Big(\overline{\mathcal{J}}^{\,\varepsilon,\delta}_{\substack{s,s+k-1\\M_k,J_k}}(U^c,E^c)\big(f_0^{(s+k)}\mathbb{1}_{|V_{s+k}|\le R}\big) \\ - \mathcal{J}^{N,\varepsilon,\delta}_{\substack{s,s+k-1\\M_k,J_k}}(U^c,E^c)\big(f_{N,0}^{(s+k)}\mathbb{1}_{|V_{s+k}|\le R}\big)\Big)$$

that one has to study here can be splitted in two terms Q_{1,M_k,J_k} and Q_{2,M_k,J_k} that write explicitly as follows.

The first one is defined as :

$$\begin{aligned}
Q_{1,M_k,J_k} = \mathbb{1}_K\mathbb{1}_{t\ge k\delta}\Big(1-\frac{(N-s)!}{(N-s-k)!}\varepsilon^{k(d-1)}\Big) \\
\times\int_{(k-1)\delta}^{t-\delta}\mathbb{1}_{U^c_{j_1}(Z_s)}(t_1)\int_{\mathbb{S}^{d-1}_{\omega_1}\times\mathbb{R}^d_{v_{s+1}}}\mathbb{1}_{E^c_{j_1}(Z^0_{s,0(t_1)})}(\omega_1,v_{s+1}) \\
\times(\pm_1)\Big[\omega_1\cdot\big(v_{s+1}-v^{0,j_1}_{s,0}(t_1)\big)\Big]_{\pm_1} \\
\times\int_{(k-2)\delta}^{t_1-\delta}\mathbb{1}_{U^c_{j_2}(Z^0_{s,1}(t_1))}(t_2)\int_{\mathbb{S}^{d-1}_{\omega_2}\times\mathbb{R}^d_{v_{s+2}}}\mathbb{1}_{E^c_{j_2}(Z^0_{s,1}(t_2))}(\omega_2,v_{s+2}) \\
\times(\pm_2)\Big[\omega_2\cdot\big(v_{s+2}-v^{0,j_2}_{s,1}(t_2)\big)\Big]_{\pm_2} \\
\dots \\
\times\int_0^{t_{k-1}-\delta}\mathbb{1}_{U^c_{j_k}(Z^0_{s,k-1}(t_{k-1}))}(t_k)\int_{\mathbb{S}^{d-1}_{\omega_k}\times\mathbb{R}^d_{v_{s+k}}}\mathbb{1}_{E^c_{j_k}(Z^0_{s,k-1}(t_k))}(\omega_k,v_{s+k}) \\
\times(\pm_k)\Big[\omega_k\cdot\big(v_{s+k}-v^{0,j_k}_{s,k-1}(t_k)\big)\Big]_{\pm_k}\mathbb{1}_{|V_{s+k}|\le R}f_0^{(s+k)}\big(Z^\varepsilon_{s,k}(0)\big) \\
\mathrm{d}\omega_k\,\mathrm{d}v_{s+k}\,\mathrm{d}t_k\dots\mathrm{d}\omega_2\,\mathrm{d}v_{s+2}\,\mathrm{d}t_2\,\mathrm{d}\omega_1\,\mathrm{d}v_{s+1}\,\mathrm{d}t_1,
\end{aligned}$$

(it takes into account the removal of the prefactors), and

$$
\begin{aligned}
Q_{2,M_k,J_k} = \mathbb{1}_K \mathbb{1}_{t\geq k\delta} \int_{(k-1)\delta}^{t-\delta} \mathbb{1}_{U^c_{j_1}(Z_s)}(t_1) \int_{\mathbb{S}^{d-1}_{\omega_1}\times\mathbb{R}^d_{v_{s+1}}} \mathbb{1}_{E^c_{j_1}(Z^0_{s,0(t_1)})}(\omega_1, v_{s+1}) \\
\times (\pm_1)\Big[\omega_1\cdot\big(v_{s+1}-v^{0,j_1}_{s,0}(t_1)\big)\Big]_{\pm_1} \\
\times \int_{(k-2)\delta}^{t_1-\delta} \mathbb{1}_{U^c_{j_2}(Z^0_{s,1}(t_1))}(t_2) \int_{\mathbb{S}^{d-1}_{\omega_2}\times\mathbb{R}^d_{v_{s+2}}} \mathbb{1}_{E^c_{j_2}(Z^0_{s,1}(t_2))}(\omega_2, v_{s+2}) \\
\times (\pm_2)\Big[\omega_2\cdot\big(v_{s+2}-v^{0,j_2}_{s,1}(t_2)\big)\Big]_{\pm_2} \\
\cdots \\
\times \int_0^{t_{k-1}-\delta} \mathbb{1}_{U^c_{j_2}(Z^0_{s,k-1}(t_{k-1}))}(t_k) \int_{\mathbb{S}^{d-1}_{\omega_k}\times\mathbb{R}^d_{v_{s+k}}} \mathbb{1}_{E^c_{j_k}(Z^0_{s,k-1}(t_k))}(\omega_k, v_{s+k}) \\
\times (\pm_k)\Big[\omega_k\cdot\big(v_{s+k}-v^{0,j_k}_{s,k-1}(t_k)\big)\Big]_{\pm_k} \mathbb{1}_{|V_{s+k}|\leq R} \\
\times \Big(f_0^{(s+k)}\big(Z^\varepsilon_{s,k}(0)\big) - f_{N,0}^{(s+k)}\big(Z^\varepsilon_{s,k}(0)\big)\Big) \\
\mathrm{d}\omega_k\,\mathrm{d}v_{s+k}\,\mathrm{d}t_k\ldots\mathrm{d}\omega_2\,\mathrm{d}v_{s+2}\,\mathrm{d}t_2\,\mathrm{d}\omega_1\,\mathrm{d}v_{s+1}\,\mathrm{d}t_1,
\end{aligned}
$$

for the second one (here one takes into account the replacement of the initial data of the BBGKY hierarchy by the initial data of the Boltzmann hierarchy). The first term is immediately controlled thanks to Lemma 40 page 487, and the contracting property of the integrated in time collision operator (each iterate of this operator provides a 1/2 term, thanks to Lemma 23 page 255).
The second one is controlled as in the proof of the previous Lemma 39 : one bounds crudely almost everywhere the integrand of the quantity Q_{2,M_k,J_k}. The argument of the difference

$$
\mathbb{1}_{Z_{s+k}(K,R)}\Big(f_0^{(s+k)}\big(Z^\varepsilon_{s,k}(0)\big) - f_{N,0}^{(s+k)}\big(Z^\varepsilon_{s,k}(0)\big)\Big)
$$

lies in a compact subset of the phase space of $s+k$ particles, since it is the case for the initial configuration Z_s and all the adjunction parameters. As in the proof of the previous Lemma 39, one denotes by $Z_{s+k}(K)$ this compact subset. One writes then

$$
\begin{aligned}
\Big|\mathbb{1}_{Z_{s+k}(K)}\Big(f_0^{(s+k)}\big(Z^\varepsilon_{s,k}(0)\big) - f_{N,0}^{(s+k)}\big(Z^\varepsilon_{s,k}(0)\big)\Big)\Big| \\
\leq \Big\|\mathbb{1}_{Z_{s+k}(K)}\Big(f_0^{(s+k)} - f_{N,0}^{(s+k)}\Big)\Big\|_{L^\infty},
\end{aligned}
$$

the right-hand side being by hypothesis a function which converges to zero as N goes to infinity. Considering then the supremum of this quantity for $1\leq k\leq n$,

one still has an upper bound which converges to zero as N goes to infinity, so that one has, performing the same control as in the end of the proof of Lemma 39 :

$$Q_{2,M_k,J_k} \leq \Big(2\big|\mathbb{S}^{d-1} \times B_{\mathbb{R}^d}(0,R)\big|R\Big)^k \Big(\int_0^t \int_0^{t_1} \dots \int_0^{t_{k-1}} \mathrm{d}t_k \dots \mathrm{d}t_2\, \mathrm{d}t_1\Big) \\ \times \sup_{1\leq k\leq n} \Big\| \mathbb{1}_{Z_{s+k}(K)} \Big(f_0^{(s+k)} - f_{N,0}^{(s+k)}\Big)\Big\|_{L^\infty},$$

so that, summing over all the contributions $1 \leq k \leq n$, $M_k \in \mathfrak{M}_k$, $J_k \in \mathfrak{J}_k^s$ for the quantities Q_{2,M_k,J_k}, one obtains

$$\sum_{k=1}^n \sum_{M_k\in\mathfrak{M}_k} \sum_{J_k\in\mathfrak{J}_k^s} Q_{2,M_k,J_k} \\ \leq \sum_{k=1}^n \sum_{M_k\in\mathfrak{M}_k} \sum_{J_k\in\mathfrak{J}_k^s} \big(C(d)R^{d+1}\big)^k \frac{t^k}{k!} \sup_{1\leq k\leq n} \Big\| \mathbb{1}_{Z_{s+k}(K)} \Big(f_0^{(s+k)} - f_{N,0}^{(s+k)}\Big)\Big\|_{L^\infty},$$

and then the conclusion of the lemma. □

Chapter 15

Gathering the estimates, and a first result : the convergence of the solutions of the BBGKY hierarchy towards the solutions of the Boltzmann hierarchy

In this section, one will finally put together all the cut-offs introduced previously, and one will be able to compare the solutions of the Boltzmann hierarchy with the solutions of the BBGKY hierarchy. This comparison is stated in the following theorem, which is one of the main results of this work.

One emphasizes on the fact that the convergence obtained in the following theorem is not quantitative. This is due, on the one hand, to the initial datum of the Boltzmann hierarchy : if the functions $f_0^{(s)}$ composing the sequence of initial data $\left(f_0^{(s)}\right)_{s\geq 1}$ of the Boltzmann hierarchy are assumed to be only continuous, then this lack of regularity prevents to use in a sharp way the result of the uniform convergence between the pseudo-trajectories obtained in Lemma 38 page 449.
On the other hand, another limitation comes from the convergence of the sequences of initial data of the BBGKY hierarchy towards the initial datum of the Boltzmann hierarchy : if this convergence is not quantified, then the final convergence of the two solutions starting from those respective initial data cannot be quantified either.

Theorem 7 (Convergence of the solutions of the BBGKY hierarchy towards the solutions of the Boltzmann hierarchy)**.** *Let β_0 be a strictly positive number and μ_0 be a real number. Then, there exists a time $T > 0$ such that the following holds :*
let $F_0 = \left(f_0^{(s)}\right)_{s\geq 1}$ be a sequence of initial data of the Boltzmann hierarchy belonging to $\boldsymbol{X}_{0,\beta_0,\mu_0^1}$ (see Definition 26 page 207), and for any positive integer N, let $F_{N,0} = \left(f_{N,0}^{(s)}\right)_{1\leq s\leq N}$ be a sequence of initial data of the BBGKY hierarchy, belonging to $\boldsymbol{X}_{N,\varepsilon,\beta_0,\mu_0^1}$ with $N\varepsilon^{d-1} = 1$ (see Definition 25 page 207). One assumes that for any $s \in \mathbb{N}^*$,

$$f_{N,0}^{(s)} \underset{N\to+\infty}{\longrightarrow} f_0^{(s)} \tag{15.1}$$

locally uniformly on the phase space of s particles $\left(\overline{\Omega^c} \times \mathbb{R}^d\right)^s$, with in addition

$$\sup_{N\geq 1} \|F_{N,0}\|_{N,\varepsilon,\beta_0,\mu_0^1} < +\infty. \tag{15.2}$$

Then, in the Boltzmann-Grad limit $N \to +\infty$, $N\varepsilon^{d-1} = 1$, if one denotes $F = \left(f^{(s)}\right)_{s\geq 1}$ the solution on $[0,T]$ of the Boltzmann hierarchy with initial data F_0, and $F_N = \mathcal{T}_t^{\varepsilon} H_N = \left(\mathcal{T}_t^{s,\varepsilon} h_N^{(s)}\right)_{1\leq s\leq N}$ where $H_N = \left(h_N^{(s)}\right)_{1\leq s\leq N}$ is the solution on $[0,T]$ of the conjugate BBGKY hierarchy with initial data $F_{N,0}$ (the time T and the solutions are provided by Theorem 6 page 259), one has that, for every positive integer s, the following locally uniform convergence on the domain of local uniform convergence Ω_s (see Definition 53 page 392), uniform on the time interval $[0,T]$, holds :

$$f_N^{(s)} \underset{N\to+\infty}{\longrightarrow} f^{(s)}.$$

Proof. For any positive integer N and any initial data $F_{N,0} = \left(f_{N,0}^{(s)}\right)_{1\leq s\leq N} \in \mathbf{X}_{N,\varepsilon,\beta_0,\mu_0^1}$ on the one hand, and for any initial datum $F_0 = \left(f^{(s)}\right)_{s\geq 1} \in \mathbf{X}_{0,\beta_0,\mu_0^1}$ on the other hand, one uses Theorem 6 page 259, which provides simultaneously, a unique solution $H_N = \left(h_N^{(s)}\right)_{1\leq s\leq N}$ to the conjugate BBGKY hierarchy with initial datum $F_{N,0}$, and a unique solution $F = \left(f^{(s)}\right)_{s\geq 1}$ to the Boltzmann hierarchy with initial datum F_0, on the same time interval $[0,T]$ (the upper bound T of this time interval depends only on the parameters β_0 and μ_0). Those solutions H_N and F belong respectively to the spaces $\widetilde{\mathbf{X}}_{N,\varepsilon,\widetilde{\beta}_\lambda,\widetilde{\mu}_\lambda^1}$ and $\widetilde{\mathbf{X}}_{0,\widetilde{\beta}_\lambda,\widetilde{\mu}_\lambda^1}$ (see Definitions 29 page 211 and 30 page 212 for the introduction of the spaces $\widetilde{\mathbf{X}}_{\cdot,\widetilde{\beta}_\lambda,\widetilde{\mu}_\lambda^1}$, and the statement of Theorem 6 for the introduction of the weights $\widetilde{\beta}_\lambda$ and $\widetilde{\mu}_\lambda$).
One recalls that Theorem 6 holds in the Boltzmann-Grad limit, which means that the diameter ε of the particles considered here satisfies

$$N\varepsilon^{d-1} = 1.$$

Using Proposition 11 page 278, one starts by cutting off the large number of adjunctions. Let n be the parameter of cut-off in large number of adjuntions (which means that one will consider only pseudo-trajectories with at most n adjunctions). For the solution of the BBGKY hierarchy, since the hard sphere transport preserves the measure, one finds, using the notations introduced in Definition 35 page 280 and thanks to the first point (11.1) of Proposition 11 :

$$\begin{aligned}\|\!|F_N - \mathcal{T}_t^\varepsilon H_N^n|\!\|_{N,\varepsilon,\widetilde{\beta}_\lambda,\widetilde{\mu}_\lambda^1} &= \big\|\!\big|\mathcal{T}_t^\varepsilon\big(H_N - H_N^n\big)\big|\!\big\|_{N,\varepsilon,\widetilde{\beta}_\lambda,\widetilde{\mu}_\lambda^1} \\ &= \|\!|H_N - H_N^n|\!\|_{N,\varepsilon,\widetilde{\beta}_\lambda,\widetilde{\mu}_\lambda^1} \leq \Big(\frac{1}{2}\Big)^n \|F_{N,0}\|_{N,\varepsilon,\beta_0,\mu_0^1} \\ &\leq \Big(\frac{1}{2}\Big)^n \Big(\sup_{N\geq 1} \|F_{N,0}\|_{N,\varepsilon,\beta_0,\mu_0^1}\Big). \end{aligned} \tag{15.3}$$

Similarly, for the solution of the Boltzmann hierarchy, thanks to the second point (11.2) of Proposition 11, one finds :

$$\|\!|F - F^n|\!\|_{0,\widetilde{\beta}_\lambda,\widetilde{\mu}_\lambda^1} \leq \Big(\frac{1}{2}\Big)^n \|F_0\|_{0,\beta_0,\mu_0^1}. \tag{15.4}$$

Now, cutting off in high velocities, let R be the parameter of cut-off in high velocities (which means that one will consider only pseudo-trajectories such that, evaluated at the final time $t = 0$, the norm of the vector composed of all the velocities of the system is smaller than R). One knows, thanks to Proposition 12 page 281, that there exist two positive constants $C_1(d, \beta_0, \mu_0)$ and $C(d, \beta_0, \mu_0)$, depending only on the dimension d and the parameters β_0 and μ_0, and a strictly positive and strictly decreasing affine weight $\widetilde{\beta}' < \widetilde{\beta}_\lambda$, such that, using the notations introduced in Definition 36 page 280 :

$$\begin{aligned}\Big\|\mathcal{T}_t^\varepsilon H_N^n - \mathcal{T}_t^\varepsilon H_N^{n,R}\Big\|_{N,\varepsilon,\widetilde{\beta}',\widetilde{\mu}_\lambda^1} &= \Big\|\mathcal{T}_t^\varepsilon\big(H_N^n - H_N^{n,R}\big)\Big\|_{N,\varepsilon,\widetilde{\beta}',\widetilde{\mu}_\lambda^1} \\ &\leq C_1 \exp\big(-C_2R^2\big)\|F_{N,0}\|_{N,\varepsilon,\widetilde{\beta}'(0),\widetilde{\mu}_0^1} \\ &\leq C_1 \exp\big(-C_2R^2\big)\|F_{N,0}\|_{N,\varepsilon,\beta_0,\mu_0^1} \\ &\leq C_1 \exp\big(-C_2R^2\big)\Big(\sup_{N\geq 1}\|F_{N,0}\|_{N,\varepsilon,\beta_0,\mu_0^1}\Big) \end{aligned} \tag{15.5}$$

for the truncated solution of the BBGKY hierarchy, and such that :

$$\big\|F^n - F^{n,R}\big\|_{0,\widetilde{\beta}',\widetilde{\mu}_\lambda^1} \leq C_1 \exp\big(-C_2R^2\big)\|F_0\|_{0,\beta_0,\mu_0^1} \tag{15.6}$$

for the truncated solution of the Boltzmann hierarchy.
Cutting off in small time difference between the adjunctions, let δ be the parameter of cut-off in small time difference between the adjunctions (which means that one considers only pseudo-trajectories such that the time interval between two adjunctions of particles is larger than δ). Proposition 15 page 308 states that there exists a positive constant $C_3(d, \beta_0, \mu_0)$ depending only on the dimension d and on the parameters β_0 and μ_0, such that one has (using the notations

introduced in Definition 40 page 297), for all positive integer s and uniformly on $[0,T]$:

$$\begin{aligned}\Big|\Big(\mathcal{T}_t^{s,\varepsilon}\big(H_N^{n,R}\big)^{(s)}\Big)(t,\cdot)-\Big(\mathcal{T}_t^{s,\varepsilon}\big(H_N^{n,R,\delta}\big)^{(s)}\Big)(t,\cdot)\Big|_{\varepsilon,s,\widetilde{\beta}_\lambda(t)}&\\ &=\Big|\big(H_N^{n,R}(t,\cdot)\big)^{(s)}-\big(H_N^{n,R,\delta}(t,\cdot)\big)^{(s)}\Big|_{\varepsilon,s,\widetilde{\beta}_\lambda(t)}\\ &\le C_3\sqrt{s}n^{3/2}\sqrt{\delta}\|F_{N,0}\|_{N,\varepsilon,\beta_0,\mu_0^1}\\ &\le C_3\sqrt{s}n^{3/2}\sqrt{\delta}\Big(\sup_{N\ge1}\|F_{N,0}\|_{N,\varepsilon,\beta_0,\mu_0^1}\Big)\end{aligned}\tag{15.7}$$

for the BBGKY hierarchy, and similarly for the Boltzmann hierarchy :

$$\Big|\big(F^{n,R}\big)^{(s)}(t,\cdot)-\big(F^{n,R,\delta}\big)^{(s)}(t,\cdot)\Big|_{0,s,\widetilde{\beta}_\lambda(t)}\le C_3\sqrt{s}n^{3/2}\sqrt{\delta}\|F_0\|_{0,\beta_0,\mu_0^1}.\tag{15.8}$$

Finally, proceeding to the last cut-off, in pathological adjunction parameters, one uses Proposition 18 page 437. There exists a positive constant $C_4(d,\beta_0,\mu_0)$ depending only on the dimension d and on the parameters β_0 and μ_0, and another positive constant $c(d)$ depending only on the dimension such that if the parameter R satisfies

$$R\ge1,\tag{15.9}$$

and for any strictly positive numbers a, ε_0, ρ, η and α such that :

$$2\varepsilon\le a,\ 4\sqrt{3}a\le\varepsilon_0,\ \varepsilon_0\le\eta\delta,\ \varepsilon_0/\delta\le1,\ 3a\le\rho\text{ and }\alpha\le c(d),\tag{15.10}$$

the following controls hold (using the notations introduced in Definition 62 page 436) :

$$\begin{aligned}&\mathbb{1}_{\Delta_s(\varepsilon,R,\varepsilon_0,\alpha,\max(16Ra/\varepsilon_0,\varepsilon_0/\delta))}(Z_s)\\ &\qquad\times\Big|\mathcal{T}_t^{s,\varepsilon}\big(H_N^{n,R,\delta}\big)^{(s)}(t,Z_s)-\big(F_N^{n,R,\delta}(U^c,E^c)\big)^{(s)}(t,Z_s)\Big|\\ &\le C_4n(s+n)R\Big(\frac{\rho}{\alpha}+\eta^d+R^d\Big(\frac{a}{\rho}\Big)^d+nR^{2d-1}\Big(\frac{a}{\varepsilon_0}\Big)^{d-3/2}\\ &\qquad+nR^{d+1/2}\Big(\frac{\varepsilon_0}{\delta}\Big)^{d-3/2}+R^{d-1}\alpha+R^d\alpha^{1/8}\Big)\\ &\qquad\times\exp\Big(-\frac{\widetilde{\beta}_\lambda(t)}{2}|V_s|^2\Big)\Big\|\big(f_{N,0}^{(s)}\big)_{1\le s\le N}\Big\|_{N,\varepsilon,\beta_0,\mu_0^1}\\ &\le C_4n(s+n)R\Big(\frac{\rho}{\alpha}+\eta^d+R^d\Big(\frac{a}{\rho}\Big)^d+nR^{2d-1}\Big(\frac{a}{\varepsilon_0}\Big)^{d-3/2}\\ &\qquad+nR^{d+1/2}\Big(\frac{\varepsilon_0}{\delta}\Big)^{d-3/2}+R^{d-1}\alpha+R^d\alpha^{1/8}\Big)\\ &\qquad\times\Big(\sup_{N\ge1}\Big\|\big(f_{N,0}^{(s)}\big)_{1\le s\le N}\Big\|_{N,\varepsilon,\beta_0,\mu_0^1}\Big)\end{aligned}\tag{15.11}$$

for the BBGKY hierarchy, and similarly for the Boltzmann hierarchy :

$$\begin{aligned}
&\mathbb{1}_{\Delta_s(\varepsilon,R,\varepsilon_0,\alpha,\max(16Ra/\varepsilon_0,\varepsilon_0/\delta))}(Z_s)\\
&\qquad\times\Big|\big(F^{n,R,\delta}\big)^{(s)}(t,Z_s)-\big(F^{n,R,\delta}(U^c,E^c)\big)^{(s)}(t,Z_s)\Big|\\
&\le C_4 n(s+n)R\Big(\frac{\rho}{\alpha}+\eta^d+R^d\Big(\frac{a}{\rho}\Big)^d+nR^{2d-1}\Big(\frac{a}{\varepsilon_0}\Big)^{d-3/2}+nR^{d+1/2}\Big(\frac{\varepsilon_0}{\delta}\Big)^{d-3/2}\\
&\qquad+R^{d-1}\alpha+R^d\alpha^{1/8}\Big)\Big\|\big(f_0^{(s)}\big)_{s\ge1}\Big\|_{0,\beta_0,\mu_0^1}.
\end{aligned}\tag{15.12}$$

For the solution of the BBGKY hierarchy, collecting the results (15.3), (15.5), (15.7) and (15.11) together, one obtains, for all $t\in[0,T]$ and almost every $Z_s\in\mathcal{D}_s^\varepsilon$:

$$\begin{aligned}
&\mathbb{1}_{\Delta_s}\Big|F_N^{(s)}(t,Z_s)-\Big(F_N^{n,R,\delta}(U^c,E^c)\Big)^{(s)}(t,Z_s)\Big|\\
&\le\Big(\sup_{N\ge1}\|F_{N,0}\|_{N,\varepsilon,\beta_0,\mu_0^1}\Big)\Big(\Big(\frac12\Big)^n\exp\Big(-\frac{\widetilde\beta_\lambda(t)}{2}|V_s|^2\Big)\exp\big(-s\widetilde\mu_\lambda(t)\big)\\
&\qquad+C_1\exp\big(-C_2R^2\big)\exp\Big(-\frac{\widetilde\beta'(t)}{2}|V_s|^2\Big)\exp\big(-s\widetilde\mu_\lambda(t)\big)\\
&\qquad+C_3\sqrt{s}n^{3/2}\sqrt{\delta}\exp\Big(-\frac{\widetilde\beta_\lambda(t)}{2}|V_s|^2\Big)\\
&\qquad+C_4n(s+n)R\Big(\frac{\rho}{\alpha}+\eta^d+R^d\Big(\frac{a}{\rho}\Big)^d+nR^{2d-1}\Big(\frac{a}{\varepsilon_0}\Big)^{d-3/2}\\
&\qquad+nR^{d+1/2}\Big(\frac{\varepsilon_0}{\delta}\Big)^{d-3/2}+R^{d-1}\alpha+R^d\alpha^{1/8}\Big)\Big)\\
&\le\Big(\sup_{N\ge1}\|F_{N,0}\|_{N,\varepsilon,\beta_0,\mu_0^1}\Big)\Big(\exp\big(-s\widetilde\mu_\lambda(T)\big)\Big(\Big(\frac12\Big)^n+C_1\exp\big(-C_2R^2\big)\Big)\\
&\qquad+C_3\sqrt{s}n^{3/2}\sqrt{\delta}\\
&\qquad+C_4n(s+n)R\Big(\frac{\rho}{\alpha}+\eta^d+R^d\Big(\frac{a}{\rho}\Big)^d+nR^{2d-1}\Big(\frac{a}{\varepsilon_0}\Big)^{d-3/2}\\
&\qquad+nR^{d+1/2}\Big(\frac{\varepsilon_0}{\delta}\Big)^{d-3/2}+R^{d-1}\alpha+R^d\alpha^{1/8}\Big)\Big).
\end{aligned}\tag{15.13}$$

One obtains the same control for the solution of the Boltzmann hierarchy.

$$\begin{aligned}&\mathbb{1}_{\Delta_s}\Big|F^{(s)}(t,Z_s)-\Big(F^{n,R,\delta}(U^c,E^c)\Big)^{(s)}(t,Z_s)\Big|\\&\leq \|F_0\|_{0,\beta_0,\mu_0^1}\Big(\exp\big(-s\widetilde{\mu}_\lambda(T)\big)\Big(\Big(\frac{1}{2}\Big)^n+C_1\exp\big(-C_2R^2\big)\Big)\\&\qquad\qquad+C_3\sqrt{s}n^{3/2}\sqrt{\delta}\\&\qquad+C_4n(s+n)R\Big(\frac{\rho}{\alpha}+\eta^d+R^d\Big(\frac{a}{\rho}\Big)^d+nR^{2d-1}\Big(\frac{a}{\varepsilon_0}\Big)^{d-3/2}\\&\qquad\qquad+nR^{d+1/2}\Big(\frac{\varepsilon_0}{\delta}\Big)^{d-3/2}+R^{d-1}\alpha+R^d\alpha^{1/8}\Big)\Big).\end{aligned}\tag{15.14}$$

Therefore, let r be any strictly positive number. One starts by choosing n large enough such that

$$\max\Big(\sup_{N\geq1}\|F_{N,0}\|_{N,\varepsilon,\beta_0,\mu_0^1},\|F_0\|_{0,\beta_0,\mu_0^1}\Big)\exp\big(-s\widetilde{\mu}_\lambda(T)\big)\Big(\frac{1}{2}\Big)^n\leq r/10.\tag{15.15}$$

Then one chooses R large enough such that

$$\begin{aligned}C_1\max&\Big(\sup_{N\geq1}\|F_{N,0}\|_{N,\varepsilon,\beta_0,\mu_0^1},\|F_0\|_{0,\beta_0,\mu_0^1}\Big)\\&\times\exp\big(-s\widetilde{\mu}_\lambda(T)\big)\exp\big(-C_2R^2\big)\leq r/10.\end{aligned}\tag{15.16}$$

Afterwards, since now n is fixed, one chooses δ small enough such that

$$C_3\max\Big(\sup_{N\geq1}\|F_{N,0}\|_{N,\varepsilon,\beta_0,\mu_0^1},\|F_0\|_{0,\beta_0,\mu_0^1}\Big)\sqrt{s}n^{3/2}\sqrt{\delta}\leq r/10.\tag{15.17}$$

Now considering any compact set K of the domain of local uniform convergence Ω_s, there exist five strictly positive numbers $\overline{\varepsilon}$, $\overline{\varepsilon_0}$, $\overline{\alpha}$, $\overline{\gamma}$ and $\overline{R}$ such that for any strictly positive numbers ε, ε_0, α, γ and R which satisfy :

$$\left\{\begin{array}{c}\varepsilon\leq\overline{\varepsilon},\ \varepsilon_0\leq\overline{\varepsilon_0},\ \alpha\leq\overline{\alpha},\ \gamma\leq\overline{\gamma},\\ R\geq\overline{R},\end{array}\right.\tag{15.18}$$

one has

$$K\subset\Delta_s\big(\varepsilon,R,\varepsilon_0,\alpha,\gamma\big).$$

One considers therefore the parameters n, R and δ fixed as soon as they are chosen such that they satisfy the conditions (15.15) for n, (15.9), (15.16) and (15.18) for R, and (15.17) for δ. The choice of those three parameters constitutes the first step of the final control.

The objective is now to fix all the other parameters but ε such that the quantity

$$\begin{aligned}n(s+n)R\Big(\frac{\rho}{\alpha}+\eta^d+R^d\Big(\frac{a}{\rho}\Big)^d+nR^{2d-1}\Big(\frac{a}{\varepsilon_0}\Big)^{d-3/2}\\+nR^{d+1/2}\Big(\frac{\varepsilon_0}{\delta}\Big)^{d-3/2}+R^{d-1}\alpha+R^d\alpha^{1/8}\Big)\end{aligned}$$

is arbitrarily small, and such that in addition one has (15.10), (15.18) and

$$\max(16Ra/\varepsilon_0, \varepsilon_0/\delta) \leq \overline{\gamma}, \tag{15.19}$$

the two last conditions implying then that

$$K \subset \Delta_s\big(\varepsilon, R, \varepsilon_0, \alpha, \max(16Ra/\varepsilon_0, \varepsilon_0/\delta)\big).$$

- <u>Condition on α</u> :
  One starts by choosing α such that

$$\alpha \leq c(d) \text{ and } \alpha \leq \overline{\alpha} \tag{15.20}$$

  (so that α fulfills the conditions (15.10) and (15.18)),

$$C_4 \max\Big(\sup_{N\geq 1} \|F_{N,0}\|_{N,\varepsilon,\beta_0,\mu_0^1}, \|F_0\|_{0,\beta_0,\mu_0^1}\Big) n(s+n)R^d\alpha \leq r/70, \tag{15.21}$$

  and

$$C_4 \max\Big(\sup_{N\geq 1} \|F_{N,0}\|_{N,\varepsilon,\beta_0,\mu_0^1}, \|F_0\|_{0,\beta_0,\mu_0^1}\Big) n(s+n)R^{d+1}\alpha^{1/8} \leq r/70. \tag{15.22}$$

  Actually one sees that the condition (15.22) implies the condition (15.21). The two conditions (15.20) and (15.22) provide an upper bound $\widetilde{\alpha}(n, R)$ on α, depending on the other truncation parameters n and R. No other truncation parameters are involved in the expression of the upper bound.

- <u>Condition on η</u> : In parallel, one chooses η such that

$$C_4 \max\Big(\sup_{N\geq 1} \|F_{N,0}\|_{N,\varepsilon,\beta_0,\mu_0^1}, \|F_0\|_{0,\beta_0,\mu_0^1}\Big) n(s+n)R\eta^d \leq r/70. \tag{15.23}$$

  The upper bound $\widetilde{\eta}(n, R)$ on η also depends only on the truncation parameters n and R.

The choice of the parameters α and η constitutes the second step of the final control. One can fix now the parameters of the third step.

- <u>Condition on ρ</u> :
  One chooses ρ such that

$$C_4 \max\Big(\sup_{N\geq 1} \|F_{N,0}\|_{N,\varepsilon,\beta_0,\mu_0^1}, \|F_0\|_{0,\beta_0,\mu_0^1}\Big) n(s+n)R\frac{\rho}{\alpha} \leq r/70. \tag{15.24}$$

  The upper bound $\widetilde{\rho}(n, R, \alpha)$ on the parameter ρ depends on n and R, but also on α, so that this parameter has to be chosen after α.

- Condition on ε_0 :
  In parallel, one chooses η such that

$$\varepsilon_0 \leq \eta\delta, \tag{15.25}$$

(providing from the condition (15.10))

$$\varepsilon_0 \leq \min(\overline{\varepsilon_0}, \delta\overline{\gamma}) \tag{15.26}$$

(providing from the condition (15.18) and (15.19)) and

$$C_4 \max\Big(\sup_{N\geq 1} \|F_{N,0}\|_{N,\varepsilon,\beta_0,\mu_0^1}, \|F_0\|_{0,\beta_0,\mu_0^1}\Big) n^2(s+n)R^{d+3/2}\Big(\frac{\varepsilon_0}{\delta}\Big)^{d-3/2} \leq r/70. \tag{15.27}$$

The upper bound $\widetilde{\varepsilon_0}(n, R, \delta, \eta)$ on the parameter ε_0 depends on n, R and δ, but also on η.

One can finally fix the parameter a. This is the fourth and last step of the final control.

Condition on a :
One chooses a such that

$$a \leq \min\Big(\frac{\varepsilon_0}{4\sqrt{3}}, \frac{\rho}{3}\Big), \tag{15.28}$$

(providing from the condition (15.10))

$$a \leq \frac{\overline{\gamma}\varepsilon_0}{16R}, \tag{15.29}$$

(providing from the condition (15.18))

$$C_4 \max\Big(\sup_{N\geq 1} \|F_{N,0}\|_{N,\varepsilon,\beta_0,\mu_0^1}, \|F_0\|_{0,\beta_0,\mu_0^1}\Big) n(s+n)R^{d+1}\Big(\frac{a}{\rho}\Big)^{d} \leq r/70, \tag{15.30}$$

and

$$C_4 \max\Big(\sup_{N\geq 1} \|F_{N,0}\|_{N,\varepsilon,\beta_0,\mu_0^1}, \|F_0\|_{0,\beta_0,\mu_0^1}\Big) n^2(s+n)R^{2d}\Big(\frac{a}{\varepsilon_0}\Big)^{d-3/2} \leq r/70. \tag{15.31}$$

The upper bound $\widetilde{a}(\rho, \varepsilon_0)$ depends on the truncation parameters ρ and ε_0, so that a was the last parameter that can be chosen.
Then finally, if one chooses ε smaller than $a/2$, all the hypotheses of Proposition 18 page 437 are fulfilled, and the controls (15.11) for the BBGKY hierarchy, and (15.12) for the Boltzmann hierarchy, together with the conditions (15.21), (15.22), (15.23), (15.24), (15.27), (15.30) and (15.31) provide first that

$$\mathbb{1}_{\Delta_s}(Z_s)\Big|\mathcal{T}_t^{s,\varepsilon}\big(H_N^{n,R,\delta}\big)^{(s)}(t, Z_s) - \big(F_N^{n,R,\delta}(U^c, E^c)\big)^{(s)}(t, Z_s)\Big| \leq r/10,$$

and

$$\mathbb{1}_{\Delta_s}(Z_s)\Big|\big(F^{n,R,\delta}\big)^{(s)}(t,Z_s)-\big(F^{n,R,\delta}(U^c,E^c)\big)^{(s)}(t,Z_s)\Big|\leq r/10.$$

In addition, the controls (15.3), (15.5) and (15.7) for the BBGKY hierarchy, and the controls (15.4), (15.6) and (15.8) for the Boltzmann hierarchy provide

$$\mathbb{1}_{\Delta_s}\Big|F_N^{(s)}(t,Z_s)-\Big(F_N^{n,R,\delta}(U^c,E^c)\Big)^{(s)}(t,Z_s)\Big|\leq 4r/10 \tag{15.32}$$

and

$$\mathbb{1}_{\Delta_s}\Big|F^{(s)}(t,Z_s)-\Big(F^{n,R,\delta}(U^c,E^c)\Big)^{(s)}(t,Z_s)\Big|\leq 4r/10. \tag{15.33}$$

In order to apply the two last Lemmas 39 page 476 and 41 page 489, one will require in addition that

$$2n\varepsilon\leq a. \tag{15.34}$$

One finds therefore that

$$\begin{aligned}\mathbb{1}_K\Big|F^{(s)}(t,Z_s)-F_N^{(s)}(t,Z_s)\Big|\leq{}&\mathbb{1}_K\Big|F^{(s)}(t,Z_s)-\Big(F^{n,R,\delta}(U^c,E^c)\Big)^{(s)}(t,Z_s)\Big|\\&+\mathbb{1}_K\Big|\Big(F^{n,R,\delta}(U^c,E^c)\Big)^{(s)}(t,Z_s)-\Big(F_N^{n,R,\delta}(U^c,E^c)\Big)^{(s)}(t,Z_s)\Big|\\&+\mathbb{1}_K\Big|\Big(F_N^{n,R,\delta}(U^c,E^c)\Big)^{(s)}(t,Z_s)-\Big(F_N^{(s)}(t,Z_s)\Big)\Big|.\end{aligned}$$

Since $K\subset\Delta_s$, the first and the third term of the right-hand side of the last inequality are controlled uniformly in time, in the phase space variable, and in ε by $4r/10$, as the controls (15.32) and (15.33) assert it. The second term is finally controlled thanks to Lemmas 39 and 41, which assert that this term, when all the truncation parameters except ε are fixed, converges uniformly to zero as N goes to infinity, that is as ε goes to zero, and the proof of the theorem is complete.

□

Chapter 16

Lanford's theorem

The previous part ended with a theorem describing a result of convergence of solutions of the BBGKY hierarchy towards solutions of the Boltzmann hierarchy. In this section, one goes back to the Boltzmann equation. To do so, one will consider solutions of the Boltzmann hierarchy with initial data built by a tensorization of a one particle density distribution. Besides, one will improve the convergence result stated in Theorem 7 above, by giving an explicit rate of convergence according to additionnal hypotheses on the initial datum. This is the purpose of Theorem 8 stated below, the final result of this work.

16.1 Adding hypotheses to the initial data

In this section one will construct a family of suitable initial data for the BBGKY hierarchy, following exactly [34][1]. Indeed, the work done in Section 10 page 267 shows clearly that the initial data play an important role in the explicit description of the solutions, besides the link between the initial data of the two hierarchies plays a crucial role in the comparison of the solutions of the two hierarchies, as it was shown in Section 13 starting page 449, and especially in the paragraph 13.3 page 475, dealing with the dominated convergence argument and the convergence of the pseudo-trajectories of the two hierarchies. First, one will discuss what kind of initial data one can choose for the Boltzmann hierarchy. Second, one will construct, for any initial datum of this kind, an associated initial datum of the BBGKY hierarchy.

[1] See Section 6.1 "Quasi-independence".

16.1.1 Admissible initial data for the Boltzmann hierarchy

For any positive number s, one recalls the definition of the domain of local uniform convergence Ω_s, introduced in Definition 53 page 392 :

$$\Omega_s = \Omega_s^1 \cap \Omega_s^2 \cap \Omega_s^3 \cap \Omega_s^4,$$

with :

$$\left\{ \begin{array}{ll} \Omega_s^1 & = \Big\{ Z_s \in \mathcal{D}_s^0 \ / \ \forall \ 1 \leq i \neq j \leq s, x_i \neq x_j \Big\}, \\ \Omega_s^2 & = \Big\{ Z_s \in \mathcal{D}_s^0 \ / \ \forall \ 1 \leq i \leq s, v_i \cdot e_1 \neq 0 \Big\}, \\ \Omega_s^3 & = \Big\{ Z_s \in \mathcal{D}_s^0 \ / \ \forall \ 1 \leq i < j \leq s, v_j \notin v_i + \mathrm{Vect}(x_i - x_j) \Big\}, \\ \Omega_s^4 & = \Big\{ Z_s \in \mathcal{D}_s^0 \ / \ \forall 1 \leq i < j \leq s, v_j \notin \mathcal{S}_0(v_i) + \mathrm{Vect}\big(\mathcal{S}_0(x_i) - x_j\big) \Big\}. \end{array} \right.$$

For any compact subset of this domain, one was able to perform cut-offs on the pathological adjunctions parameters so that all the pseudo-trajectories of the BBGKY hierarchy, starting from an initial configuration of this compact subset and obtained with non pathological adjunction parameters, converge uniformly as the number N of particles of the system goes to infinity to the associated pseudo-trajectory of the Boltzmann hierarchy. One was then naturally led to consider, for a sequence $\big(f_0^{(s)}\big)_{s \geq 1}$ of initial data of the Boltzmann hierarchy, a sequence (of sequences) $\Big(\big(f_{N,0}^{(s)}\big)_{s \geq 1}\Big)_{N \geq 1}$ of initial data of the BBGKY hierarchy, such that for any positive integer s :

$$f_{N,0}^{(s)} \underset{N \to +\infty}{\longrightarrow} f_0^{(s)}$$

uniformly on every compact set of the domain Ω_s. This convergence enabled to obtain Theorem 7 page 493.
Besides, recalling that the BBGKY and the Boltzmann hierarchies were obtained from the distribution function of a system of N hard spheres, verifying the Liouville equation, and the study of its sequence of marginals (see Section 2.1 page 65), it seems reasonable to work with initial data satisfying this "marginal relation".
One will therefore introduce a definition for sequences of initial data of the Boltzmann hierarchy that satisfy those two properties.

Definition 63 (Admissible Boltzmann data)**.** *For any positive integer s, let $f_0^{(s)}$ be a nonnegative function which is integrable and continuous over $\big(\overline{\Omega^c} \times \mathbb{R}^d\big)^s$. The family $\Big(f_0^{(s)}\Big)_{s \geq 1}$ is said to be an* admissible Boltzmann datum *if :*

- *(marginal property) for all $s \geq 1$:*

$$\int_{\Omega^c \times \mathbb{R}^d} f_0^{(s+1)}(Z_s, z_{s+1}) dz_{s+1} = f_0^{(s)}(Z_s),$$

- *(boundary condition) for all* $s \geq 1$,

$$f_0^{(s)}(Z_s) = f_0^{(s)}(\chi_s^0(Z_s))$$

for all Z_s *belonging to the boundary of* $\left(\overline{\Omega^c} \times \mathbb{R}^d\right)^s$, *that is such that there exists at least an integer* $1 \leq i \leq s$ *such that* $x_i \cdot e_1 = 0$ *and* $v_i \cdot e_1 > 0$, *with* χ_s^0 *introduced in Definition 11 page 79,*

- *there exist two numbers* $\beta_0 > 0$ *and* $\mu_0 \in \mathbb{R}$, *and for any positive integer* N, *a function* $f_{N,0}^{(N)}$ *such that if one defines the sequence* $\left(f_{N,0}^{(s)}\right)_{1 \leq s \leq N}$ *defined as, for any integer* $1 \leq s \leq N$ *:*

$$f_{N,0}^{(s)}(Z_s) = \int_{(\overline{\Omega^c} \times \mathbb{R}^d)^{N-s}} \mathbb{1}_{Z_N \in \mathcal{D}_N^\varepsilon} f_{N,0}^{(N)} \, dz_{s+1} ... dz_N,$$

all of the finite sequences of N *initial data* $F_{N,0} = (f_{N,0}^{(s)})_{1 \leq s \leq N}$ *belong to the space* $\boldsymbol{X}_{N,\varepsilon,\beta_0,\mu_0}$ *(see Definition 25 page 207) with* $N\varepsilon^{d-1} = 1$ *and*

$$\sup_{N \geq 1} \left\| \left(f_{N,0}^{(s)}\right)_{1 \leq s \leq N} \right\|_{N,\varepsilon,\beta_0,\mu_0} < +\infty,$$

and for all positive integer s, *the following convergence on* Ω_s *holds :*

$$f_{N,0}^{(s)} \underset{N \to +\infty}{\longrightarrow} f_0^{(s)},$$

locally uniformly.

The sequence of initial data $\left(F_{N,0}\right)_{N \geq 1}$ *is called the* associated sequence of BBGKY initial data.

In particular, any solution of the Boltzmann hierarchy with initial data that are admissible Boltzmann satisfy the convergence stated in Theorem 7 page 493.

16.1.2 Tensorized initial data

One gives here a crucial result stated in [34][2], which shows that the set of admissible Boltzmann data is not empty. Actually, it even shows that the functional spaces introduced in Section 7 page 205 provide a good setting to work on the problem of convergence of the solutions of the BBGKY hierarchy to the solutions of the Boltzmann hierarchy.

Proposition 20. *Let* $\beta_0 > 0$ *and* $\mu_0 \in \mathbb{R}$ *two real numbers. For any nonnegative, normalized function* f_0 *belonging to the space* $X_{0,1,\beta_0}$ *(see Definition 24 page 206) such that*

$$e^{\mu_0} |f_0|_{0,1,\beta_0} \leq 1,$$

[2]See Section 6.1.2 "Conditioning", and Section 6.1.3 "Characterization of admissible Boltzmann initial data".

the sequence of chaotic configurations $\left(f_0^{\otimes s}\right)_{s\geq 1}$ *defined, for all positive integer* s *as*

$$f_0^{\otimes s}(Z_s) = \prod_{i=1}^{s} f_0(z_i)$$

for all $1 \leq s \leq N$*, is a sequence of admissible Boltzmann data.*

One does not rewrite the proof, since the reader may find it in [34]. One just gives the main steps of the proof for the sake of completeness.
The result is based on the *conditioning on energy surfaces* (the reader may refer to the sources already given in [34], namely [36] and [48]). One defines the *conditioned datum built on* f_0 as the function

$$f_{N,0}^{(N)} : Z_N \mapsto \mathcal{Z}_N^{-1} \mathbb{1}_{Z_N \in \mathcal{D}_N^{\varepsilon}} f_0^{\otimes N}(Z_N),$$

with $\mathcal{Z}_N$ denoting the function

$$\mathcal{Z}_N : Z_N \mapsto \int_{(\overline{\Omega^c} \times \mathbb{R}^d)^N} \mathbb{1}_{Z_N \in \mathcal{D}_N^{\varepsilon}} f_0^{\otimes N}(Z_N)\, \mathrm{d}Z_N,$$

and, for all $1 \leq s \leq N-1$, the marginals $f_{N,0}^{(s)}$ of the function $f_{N,0}^{(N)}$. The sequence $\left(F_N\right)_{N\geq 1}$, with $F_N = \left(f_{N,0}^{(s)}\right)_{1\leq s\leq N}$ will then be an associated sequence of BBGKY initial data to the sequence of initial data $\left(f_0^{\otimes s}\right)_{s\geq 1}$. This result will be a consequence of the two controls (16.1) and (16.2) below.

Since the initial datum f_0 belongs to $X_{0,1,\beta_0}$, the L^1 norm in velocity of this function is uniformly bounded in position, that is

$$f_0 \in L^{\infty}\left(\overline{\Omega^c}_x, L^1\left(\mathbb{R}^d_v\right)\right).$$

One obtains first the following control :

$$1 \leq \mathcal{Z}_N^{-1} \mathcal{Z}_{N-s} \leq \left(1 - C(d)\varepsilon |f_0|_{L^{\infty}(\overline{\Omega^c}, L^1(\mathbb{R}^d))}\right)^{-s}, \tag{16.1}$$

with $C(d)$ a constant depending only on the dimension d, and decomposing :

$$f_{N,0}^{(s)} = \mathcal{Z}_N^{-1} \mathbb{1}_{Z_s \in \mathcal{D}_s^{\varepsilon}} f_0^{\otimes s}\left(\mathcal{Z}_{N-s} - \mathcal{Z}_{(s+1,N)}^{\flat}\right)$$

(where $\mathcal{Z}_{(s+1,N)}^{\flat}$ is entirely defined since the expressions of $f_{N,0}^{(s)}$, $f_0^{\otimes s}$, $\mathcal{Z}_N$ and $\mathcal{Z}_{N-s}$ are given), one finds in addition

$$0 \leq \mathcal{Z}_N^{-1} \mathcal{Z}_{(s+1,N)}^{\flat} \leq C(d) s\varepsilon |f_0|_{L^{\infty}(\overline{\Omega^c}, L^1(\mathbb{R}^d))} \left(1 - C(d)\varepsilon |f_0|_{L^{\infty}(\overline{\Omega^c}, L^1(\mathbb{R}^d))}\right), \tag{16.2}$$

with $C(d)$ a constant depending only on the dimension d (and which is the same as in (16.1)).

One finds therefore :

$$\begin{aligned}0 \leq \mathbb{1}_{Z_s \in \mathcal{D}_s^\varepsilon} f_0^{\otimes s} - f_{N,0}^{(s)} &\leq \Big(\big(1 - \mathcal{Z}_N^{-1} \mathcal{Z}_{N-s}\big) + \mathcal{Z}_N^{-1} \mathcal{Z}_{(s+1,N)}^b \Big) \mathbb{1}_{Z_s \in \mathcal{D}_s^\varepsilon} f_0^{\otimes s} \\ &\leq \Big(\big(1 - C(d)\varepsilon |f_0|_{L^\infty(\overline{\Omega^c}, L^1(\mathbb{R}^d))}\big)^{-s} - 1 \\ &+ C(d) s \varepsilon |f_0|_{L^\infty(\overline{\Omega^c}, L^1(\mathbb{R}^d))} \big(1 - C(d)\varepsilon |f_0|_{L^\infty(\overline{\Omega^c}, L^1(\mathbb{R}^d))}\big) \Big) \mathbb{1}_{Z_s \in \mathcal{D}_s^\varepsilon} f_0^{\otimes s}.\end{aligned}$$

Since

$$\big(1 - C(d)\varepsilon |f_0|_{L^\infty(\overline{\Omega^c}, L^1(\mathbb{R}^d))}\big)^{-s} - 1 \underset{\varepsilon \to 0}{\sim} C(d) s \varepsilon |f_0|_{L^\infty(\overline{\Omega^c}, L^1(\mathbb{R}^d))},$$

one can bound the quantity $\big(1 - C(d)\varepsilon |f_0|_{L^\infty(\overline{\Omega^c}, L^1(\mathbb{R}^d))}\big)^{-s} - 1$ by

$$2C(d) s \varepsilon |f_0|_{L^\infty(\overline{\Omega^c}, L^1(\mathbb{R}^d))}$$

for ε small enough, so that

$$0 \leq \mathbb{1}_{Z_s \in \mathcal{D}_s^\varepsilon} f_0^{\otimes s} - f_{N,0}^{(s)} \leq 3C(d) s \varepsilon |f_0|_{L^\infty(\overline{\Omega^c}, L^1(\mathbb{R}^d))}. \tag{16.3}$$

In other words, one has obtained a control on the difference between the tensorized sequence of initial data $\big(f_0^{\otimes s}\big)_{s \geq 1}$ and its associated sequence of BBGKY initial data, which is given by the tensorized initial data themselves multiplied by a constant converging to zero as $\varepsilon \to 0$, that is as $N \to +\infty$.
Here one provided only a sufficient condition for a sequence of initial data of the Boltzmann hierarchy to obtain, for every s-th term ($s \geq 1$) of the sequence, the locally uniform converge on Ω_s. In [34], the reader may in fact find a characterization[3].

16.1.3 Error coming from a tensorized sequence of initial data

In the case of a tensorized sequence of initial data for the Boltzmann hierarchy, it is possible to improve the results obtained in Lemmas 39 page 476 and 41 page 489, to obtain quantitative bounds.

Lemma 42 (Error coming from the divergence of the trajectories, quantitative version for sequences of initial data obtained with a tensorization of a square root Lipschitz function)**.** *Let s and n be two positive integers, β_0 be a strictly positive number and $\mu_0 \in \mathbb{R}$ be a real number. Then there exists a time T' satisfying $0 \leq T' < T$ (where T is given by Theorem 6 page 259) such that the following holds :*
let f_0 be a nonnegative, normalized function belonging to $X_{0,1,\beta_0}$ such that $\sqrt{f_0}$ is

[3]See Proposition 6.1.1 and its proof in Section 6.1.3 "Characterization of admissible Boltzmann initial data".

Lipschitz with respect to the position variable uniformly in the velocity variable. Let K be a compact set of the domain of local uniform convergence Ω_s.
Then, there exist five strictly positive numbers $\overline{\varepsilon}$, $\overline{\varepsilon_0}$, $\overline{\alpha}$, $\overline{\gamma}$ and $\overline{R}$ (depending only on the compact set K) such that for every strictly positive numbers R, δ, ε, a, ε_0, ρ, η and α which satisfy

$$\left\{\begin{array}{c} \varepsilon \leq \overline{\varepsilon},\ \varepsilon_0 \leq \overline{\varepsilon_0},\ \alpha \leq \overline{\alpha},\ \max\big(16R\varepsilon/\varepsilon_0, \varepsilon_0/\delta\big) \leq \overline{\gamma} \\ \\ \text{and } R \geq \overline{R}, \end{array}\right. \tag{16.4}$$

and

$$\left\{\begin{array}{c} 4\sqrt{3}a \leq \varepsilon_0, \quad 3a \leq \rho, \quad \varepsilon_0 \leq \eta\delta, \quad R \geq 1, \quad \eta \leq 1 \text{ and } \alpha \leq c(d), \\ \text{with } c(d) \text{ a constant depending only on the dimension } d, \text{ and} \\ \\ 2n\varepsilon \leq a, \end{array}\right. \tag{16.5}$$

one has that, in the Boltzmann-Grad limit $N \to +\infty$, $N\varepsilon^{d-1} = 1$, for s, n, R, δ, a, ε_0, η, ρ and α fixed, the following uniform convergence on the compact set K, uniform on the time interval $[0, T']$, of the sequence of the sum of the intermediate elementary terms for the sequence of tensorized initial data $\big(f_0^{\otimes s}\big)_{s\geq 1}$ towards the sum of the elementary Boltzmann terms for the same sequence of tensorized initial data $\big(f_0^{\otimes s}\big)_{s\geq 1}$ holds :

$$\begin{aligned} \Bigg\| \mathbb{1}_K \sum_{k=1}^{n} \sum_{M_k \in \mathfrak{M}_k} \sum_{J_k \in \mathfrak{J}_k^s} \Big(& \mathcal{J}^{0,\delta}_{\substack{s,s+k-1\\ M_k,J_k}}(U^c, E^c)\big(f_0^{\otimes(s+k)} \mathbb{1}_{|V_{s+k}|\leq R}\big) \\ & - \overline{\mathcal{J}}^{\varepsilon,\delta}_{\substack{s,s+k-1\\ M_k,J_k}}(U^c, E^c)\big(f_0^{\otimes(s+k)} \mathbb{1}_{|V_{s+k}|\leq R}\big)\Big)(Z_s) \Bigg\|_{L^\infty} \\ & \leq C(d)(s+n)^2 nR \frac{\varepsilon}{\alpha} \big|\nabla_x \sqrt{f_0}\big|_\infty \|F_0\|_{0,\beta_0/2,\mu_0}. \end{aligned} \tag{16.6}$$

It is important to notice here that the norm in the space $\mathbf{X}_{0,\beta_0/2,\mu_0}$ of the sequence of initial data appears in the right-hand side of the inequality of the Lemma, and not the norm in the smaller space $\mathbf{X}_{0,\beta_0,\mu_0}$.

Proof. The proof of the lemma is very similar to the proof of Lemma 39 page 476. The only difference lies in the uniform control of the difference

$$f_0^{(s+k)}\big(Z^0_{s,k}(0)\big) - f_0^{(s+k)}\big(Z^\varepsilon_{s,k}(0)\big).$$

As in the proof of Lemma 39, one makes sure that, possibly with a small change in position, one is comparing pseudo-trajectories with same velocities.
One denotes $j_1, \ldots, j_p$ the labels of the particles having different velocities at time 0 after following the BBGKY and the Boltzmann pseudo-trajectories, and $t_0^{j_l}$ (for $1 \leq l \leq p$) the times such that

$$x^{0,j_l}_{s,k}(t_0^{j_l}) \cdot e_1 = 0.$$

One has, following the notations of the proof of Lemma 39 :

$$\begin{aligned}
&\big|f_0^{(s+k)}\big(Z_{s,k}^0(0)\big) - f_0^{(s+k)}\big(Z_{s,k}^\varepsilon(0)\big)\big| \\
&\le \Big|f_0^{(s+k)}\big(X_{s,k}^0(0), V_{s,k}^0(0)\big) - f_0^{(s+k)}\big(X_{s,k}^0(t_0^{j_1}), V_{s,k}^0(0)\big)\Big| \\
&\quad + \sum_{l=1}^{p-1}\Big|\Big(f_0^{(s+k)}\big(X_{s,k}^0(t_0^{j_l}), V_{l+1}\big) - f_0^{(s+k)}\big(X_{s,k}^0(t_0^{j_{l+1}}), V_{l+1}\big)\Big)\Big| \\
&\quad + \Big|f_0^{(s+k)}\big(X_{s,k}^0(t_0^{j_p}), V_{s,k}^\varepsilon(0)\big) - f_0^{(s+k)}\big(X_{s,k}^\varepsilon(0), V_{s,k}^\varepsilon(0)\big)\Big|. \qquad (16.7)
\end{aligned}$$

Now one can use that the initial datum $f_0^{(s+k)}$ is tensorized. For example, for the first term of the right-hand side of the last inequality (16.7), one writes

$$\begin{aligned}
&\Big|f_0^{(s+k)}\big(X_{s,k}^0(0), V_{s,k}^0(0)\big) - f_0^{(s+k)}\big(X_{s,k}^0(t_0^{j_1}), V_{s,k}^0(0)\big)\Big| \\
&= \Big|f_0\big(x_{s,k}^{0,1}(0), v_{s,k}^{0,1}(0)\big)\dots f_0\big(x_{s,k}^{0,s+k}(0), v_{s,k}^{0,s+k}(0)\big) \\
&\qquad\qquad - f_0\big(x_{s,k}^{0,1}(t_0^{j_1}), v_{s,k}^{0,1}(0)\big)\dots f_0\big(x_{s,k}^{0,s+k}(t_0^{j_1}), v_{s,k}^{0,s+k}(0)\big)\Big|,
\end{aligned}$$

that is

$$\begin{aligned}
&\Big|f_0^{(s+k)}\big(X_{s,k}^0(0), V_{s,k}^0(0)\big) - f_0^{(s+k)}\big(X_{s,k}^0(t_0^{j_1}), V_{s,k}^0(0)\big)\Big| \\
&= \Big|\prod_{l=1}^{s+k} f_0\big(x_{s,k}^{0,l}(0), v_{s,k}^{0,l}(0)\big) - \prod_{l=1}^{s+k} f_0\big(x_{s,k}^{0,l}(t_0^{j_1}), v_{s,k}^{0,l}(0)\big)\Big|.
\end{aligned}$$

One now decomposes the first product as follows :

$$\begin{aligned}
&f_0\big(x_{s,k}^{0,1}(0), v_{s,k}^{0,1}(0)\big) f_0\big(x_{s,k}^{0,2}(0), v_{s,k}^{0,2}(0)\big)\dots f_0\big(x_{s,k}^{0,s+k}(0), v_{s,k}^{0,s+k}(0)\big) \\
&= f_0\big(x_{s,k}^{0,1}(0), v_{s,k}^{0,1}(0)\big) f_0\big(x_{s,k}^{0,2}(0), v_{s,k}^{0,2}(0)\big)\dots f_0\big(x_{s,k}^{0,s+k}(0), v_{s,k}^{0,s+k}(0)\big) \\
&\quad - f_0\big(x_{s,k}^{0,1}(t_0^{j_1}), v_{s,k}^{0,1}(0)\big) f_0\big(x_{s,k}^{0,2}(0), v_{s,k}^{0,2}(0)\big)\dots f_0\big(x_{s,k}^{0,s+k}(0), v_{s,k}^{0,s+k}(0)\big) \\
&\quad + f_0\big(x_{s,k}^{0,1}(t_0^{j_1}), v_{s,k}^{0,1}(0)\big) f_0\big(x_{s,k}^{0,2}(0), v_{s,k}^{0,2}(0)\big)\dots f_0\big(x_{s,k}^{0,s+k}(0), v_{s,k}^{0,s+k}(0)\big),
\end{aligned}$$

that is

$$\begin{aligned}
&\prod_{l=1}^{s+k} f_0\big(x_{s,k}^{0,l}(0), v_{s,k}^{0,l}(0)\big) \\
&= \Big(f_0\big(x_{s,k}^{0,1}(0), v_{s,k}^{0,1}(0)\big) - f_0\big(x_{s,k}^{0,1}(t_0^{j_1}), v_{s,k}^{0,1}(0)\big)\Big) \prod_{l=2}^{s+k} f_0\big(x_{s,k}^{0,l}(0), v_{s,k}^{0,l}(0)\big) \\
&\qquad + f_0\big(x_{s,k}^{0,1}(t_0^{j_1}), v_{s,k}^{0,1}(0)\big) \prod_{l=2}^{s+k} f_0\big(x_{s,k}^{0,l}(0), v_{s,k}^{0,l}(0)\big),
\end{aligned}$$

and now decomposing the second term, and so on, one obtains in the end :

$$\begin{aligned}
&\prod_{l=1}^{s+k} f_0\big(x_{s,k}^{0,l}(0), v_{s,k}^{0,l}(0)\big) - \prod_{l=1}^{s+k} f_0\big(x_{s,k}^{0,l}(t_0^{j_1}), v_{s,k}^{0,l}(0)\big) \\
&= \sum_{m=1}^{s+k} \Big[\prod_{l_m=1}^{m-1} f_0\big(x_{s,k}^{0,l_m}(t_0^{j_1}), v_{s,k}^{0,l_m}(0)\big) \\
&\qquad\qquad \times \Big(f_0\big(x_{s,k}^{0,m}(0), v_{s,k}^{0,m}(0)\big) - f_0\big(x_{s,k}^{0,m}(t_0^{j_1}), v_{s,k}^{0,m}(0)\big)\Big) \\
&\qquad\qquad\qquad\qquad \times \prod_{l_m=m+1}^{s+k} f_0\big(x_{s,k}^{0,l_m}(0), v_{s,k}^{0,l_m}(0)\big)\Big]. \qquad (16.8)
\end{aligned}$$

For the first term

$$\begin{aligned}
&f_0\big(x_{s,k}^{0,1}(0), v_{s,k}^{0,1}(0)\big) f_0\big(x_{s,k}^{0,2}(0), v_{s,k}^{0,2}(0)\big) \dots f_0\big(x_{s,k}^{0,s+k}(0), v_{s,k}^{0,s+k}(0)\big) \\
&\qquad - f_0\big(x_{s,k}^{0,1}(t_0^{j_1}), v_{s,k}^{0,1}(0)\big) f_0\big(x_{s,k}^{0,2}(0), v_{s,k}^{0,2}(0)\big) \dots f_0\big(x_{s,k}^{0,s+k}(0), v_{s,k}^{0,s+k}(0)\big)
\end{aligned}$$

obtained in the last decomposition (16.8), that is

$$\Big(f_0\big(x_{s,k}^{0,1}(0), v_{s,k}^{0,1}(0)\big) - f_0\big(x_{s,k}^{0,1}(t_0^{j_1}), v_{s,k}^{0,1}(0)\big)\Big) \prod_{l=2}^{s+k} f_0\big(x_{s,k}^{0,l}(0), v_{s,k}^{0,l}(0)\big),$$

one writes :

$$\begin{aligned}
&\Big| \Big(f_0\big(x_{s,k}^{0,1}(0), v_{s,k}^{0,1}(0)\big) - f_0\big(x_{s,k}^{0,1}(t_0^{j_1}), v_{s,k}^{0,1}(0)\big)\Big) \prod_{l=2}^{s+k} f_0\big(x_{s,k}^{0,l}(0), v_{s,k}^{0,l}(0)\big) \Big| \\
&\quad \le \Big| f_0\big(x_{s,k}^{0,1}(0), v_{s,k}^{0,1}(0)\big) - f_0\big(x_{s,k}^{0,1}(t_0^{j_1}), v_{s,k}^{0,1}(0)\big) \Big| \Big| \prod_{l=2}^{s+k} f_0\big(x_{s,k}^{0,l}(0), v_{s,k}^{0,l}(0)\big) \Big|,
\end{aligned}$$

and now, in order to use the Lipschitz control on the square root of the initial

data :

$$\begin{aligned}
&\Big|\Big(f_0\big(x^{0,1}_{s,k}(0),v^{0,1}_{s,k}(0)\big)-f_0\big(x^{0,1}_{s,k}(t_0^{j_1}),v^{0,1}_{s,k}(0)\big)\Big)\prod_{l=2}^{s+k}f_0\big(x^{0,l}_{s,k}(0),v^{0,l}_{s,k}(0)\big)\Big|\\
&\quad\le\Big|\sqrt{f_0\big(x^{0,1}_{s,k}(0),v^{0,1}_{s,k}(0)\big)}-\sqrt{f_0\big(x^{0,1}_{s,k}(t_0^{j_1}),v^{0,1}_{s,k}(0)\big)}\Big|\\
&\qquad\qquad\times\Big(\sqrt{f_0\big(x^{0,1}_{s,k}(0),v^{0,1}_{s,k}(0)\big)}+\sqrt{f_0\big(x^{0,1}_{s,k}(t_0^{j_1}),v^{0,1}_{s,k}(0)\big)}\Big)\\
&\qquad\qquad\qquad\qquad\times\Big|\prod_{l=2}^{s+k}f_0\big(x^{0,l}_{s,k}(0),v^{0,l}_{s,k}(0)\big)\Big|\\
&\quad\le\big|\nabla_x\sqrt{f_0}\big|_\infty\big|x^{0,1}_{s,k}(0)-x^{0,1}_{s,k}(t_0^{j_1})\big|\\
&\qquad\qquad\times\Big(\sqrt{f_0\big(x^{0,1}_{s,k}(0),v^{0,1}_{s,k}(0)\big)}+\sqrt{f_0\big(x^{0,1}_{s,k}(t_0^{j_1}),v^{0,1}_{s,k}(0)\big)}\Big)\\
&\qquad\qquad\qquad\qquad\times\Big|\prod_{l=2}^{s+k}f_0\big(x^{0,l}_{s,k}(0),v^{0,l}_{s,k}(0)\big)\Big|.
\end{aligned}$$

Then, one uses the fact that $f_0\in X_{0,1,\beta_0}$, so in particular $\sqrt{f_0}\in X_{0,1,\beta_0/2}$, and then

$$\begin{aligned}
&\Big|\Big(f_0\big(x^{0,1}_{s,k}(0),v^{0,1}_{s,k}(0)\big)-f_0\big(x^{0,1}_{s,k}(t_0^{j_1}),v^{0,1}_{s,k}(0)\big)\Big)\prod_{l=2}^{s+k}f_0\big(x^{0,l}_{s,k}(0),v^{0,l}_{s,k}(0)\big)\Big|\\
&\quad\le\big|\nabla_x\sqrt{f_0}\big|_\infty\big|x^{0,1}_{s,k}(0)-x^{0,1}_{s,k}(t_0^{j_1})\big|\\
&\qquad\qquad\times 2\exp\Big(-\frac{\beta_0}{4}\big|v^{0,1}_{s,k}(0)\big|^2\Big)\exp\Big(-\frac{\beta_0}{2}\sum_{l=2}^{s+k}\big|v^{0,l}_{s,k}(0)\big|^2\Big).
\end{aligned}$$

One now needs to obtain an explicit control on the time $t_0^{j_1}$.
One knows that $x^{0,1}_{s,k}(t_0^{j_1})$ and $x^{\varepsilon,1}_{s,k}(t_0^{j_1})$ are separated by a distance smaller than $2k\varepsilon$ (thanks to Lemma 38 page 449). But since the velocities of the pseudo-trajectories are not grazing, one has $\big|v^{0,1}_{s,k}(t_0^{j_1})\cdot e_1\big|=\big|v^{\varepsilon,1}_{s,k}(t_0^{j_1})\cdot e_1\big|\ge\alpha$ (thanks to the cut-off in pathological adjunction parameters), where $t_0^{j_1}$ is the time of bouncing of the particle j_1 following the Boltzmann pseudo-trajectory. If one denotes $t_\varepsilon^{j_1}$ the time of bouncing of the same particle following the BBGKY pseudo-trajectory, one has

$$x^{\varepsilon,j_1}_{s,k}(t_0^{j_1})\cdot e_1\le 2k\varepsilon,$$

and since by definition $x^{\varepsilon,j_1}_{s,k}(t_\varepsilon^{j_1})\cdot e_1=\varepsilon/2$, and using the fact that

$$x^{\varepsilon,j_1}_{s,k}(t_\varepsilon^{j_1})=x^{\varepsilon,j_1}_{s,k}(t_0^{j_1})+(t_\varepsilon^{j_1}-t_0^{j_1})v^{\varepsilon,j_1}_{s,k}(t_\varepsilon^{j_1}),$$

this implies that

$$\big|t_\varepsilon^{j_1}-t_0^{j_1}\big|=\frac{\big|(x^{\varepsilon,j_1}_{s,k}(t_\varepsilon^{j_1})-x^{\varepsilon,j_1}_{s,k}(t_0^{j_1}))\cdot e_1\big|}{\big|v^{\varepsilon,j_1}_{s,k}(t_\varepsilon^{j_1})\cdot e_1\big|}\le\frac{2k\varepsilon+\varepsilon/2}{\alpha}.$$

By definition, the particle j_1 does not have the same velocity following the Boltzmann or the BBGKY hierarchy, which means that the final time 0 lies in the time interval bounded by the two times of bouncing of this particle j_1 for each of the two pseudo-trajectories. This provides the control on the times $t_0^{j_l}$:

$$t_0^{j_l} \leq \frac{2k+1/2}{\alpha}\varepsilon,$$

and then, using the fact that the norms of the velocities of all the particles of the system are bounded by R :

$$\big|x_{s,k}^{0,1}(0) - x_{s,k}^{0,1}(t_0^{j_1})\big| \leq \frac{2k+1/2}{\alpha}\varepsilon R.$$

For the sum which is the second term of the right-hand side of the inequality (16.7), one obtains exactly the same bound on each term of the decomposition, but with a slight difference :

$$\big|x_{s,k}^{0,1}(0) - x_{s,k}^{0,1}(t_0^{j_1})\big|$$

has to be replaced by

$$\big|x_{s,k}^{0,1}(t_0^{j_l}) - x_{s,k}^{0,1}(t_0^{j_{l+1}})\big|,$$

this term being controlled by

$$\big|t_0^{j_{l+1}} - t_0^{j_l}\big| R \leq 2\frac{2k+1/2}{\alpha}\varepsilon R$$

(the only difference with the previous case is a factor 2).
The last term of the right-hand side of the inequality (16.7)

$$\begin{aligned}
&\Big|f_0^{(s+k)}\big(X_{s,k}^0(t_0^{j_p}), V_{s,k}^\varepsilon(0)\big) - f_0^{(s+k)}\big(X_{s,k}^\varepsilon(0), V_{s,k}^\varepsilon(0)\big)\Big| \\
&\quad = \Big|f_0\big(x_{s,k}^{0,1}(t_0^{j_p}), v_{s,k}^{\varepsilon,1}(0)\big) f_0\big(x_{s,k}^{0,2}(t_0^{j_p}), v_{s,k}^{\varepsilon,2}(0)\big) \dots f_0\big(x_{s,k}^{0,s+k}(t_0^{j_p}), v_{s,k}^{\varepsilon,s+k}(0)\big) \\
&\qquad - f_0\big(x_{s,k}^{\varepsilon,1}(0), v_{s,k}^{\varepsilon,1}(0)\big) f_0\big(x_{s,k}^{\varepsilon,2}(0), v_{s,k}^{\varepsilon,2}(0)\big) \dots f_0\big(x_{s,k}^{\varepsilon,s+k}(0), v_{s,k}^{\varepsilon,s+k}(0)\big)\Big|
\end{aligned}$$

that is :

$$\Big|\prod_{l=1}^{s+k} f_0\big(x_{s,k}^{0,l}(t_0^{j_p}), v_{s,k}^{\varepsilon,l}(0)\big) - \prod_{l=1}^{s+k} f_0\big(x_{s,k}^{\varepsilon,l}(0), v_{s,k}^{\varepsilon,l}(0)\big)\Big|$$

can be also decomposed as :

$$\begin{aligned}
\sum_{m=1}^{s+k} \Big[\prod_{l_m=1}^{m-1} & f_0\big(x_{s,k}^{\varepsilon,l_m}(0), v_{s,k}^{\varepsilon,l_m}(0)\big) \\
&\times \Big(f_0\big(x_{s,k}^{0,m}(t_0^{j_p}), v_{s,k}^{\varepsilon,m}(0)\big) - f_0\big(x_{s,k}^{\varepsilon,m}(0), v_{s,k}^{\varepsilon,m}(0)\big)\Big) \\
&\qquad \times \prod_{l_m=m+1}^{s+k} f_0\big(x_{s,k}^{0,l_m}(t_0^{j_p}), v_{s,k}^{\varepsilon,l_m}(0)\big)\Big].
\end{aligned}$$

Now decomposing

$$\Big|f_0\big(x_{s,k}^{0,m}(t_0^{j_p}),v_{s,k}^{\varepsilon,m}(0)\big)-f_0\big(x_{s,k}^{\varepsilon,m}(0),v_{s,k}^{\varepsilon,m}(0)\big)\Big|$$

as

$$\Big(\Big|f_0\big(x_{s,k}^{0,m}(t_0^{j_p}),v_{s,k}^{\varepsilon,m}(0)\big)-f_0\big(x_{s,k}^{0,m}(0),v_{s,k}^{\varepsilon,m}(0)\big)\Big| \\ +\Big|f_0\big(x_{s,k}^{0,m}(0),v_{s,k}^{\varepsilon,m}(0)\big)-f_0\big(x_{s,k}^{\varepsilon,m}(0),v_{s,k}^{\varepsilon,m}(0)\big)\Big|\Big),$$

one finds

$$\begin{aligned}
&\Big|\prod_{l=1}^{s+k}f_0\big(x_{s,k}^{0,l}(t_0^{j_p}),v_{s,k}^{\varepsilon,l}(0)\big)-\prod_{l=1}^{s+k}f_0\big(x_{s,k}^{\varepsilon,l}(0),v_{s,k}^{\varepsilon,l}(0)\big)\Big| \\
&\le\sum_{m=1}^{s+k}\Big|f_0\big(x_{s,k}^{0,m}(t_0^{j_p}),v_{s,k}^{\varepsilon,m}(0)\big)-f_0\big(x_{s,k}^{0,m}(0),v_{s,k}^{\varepsilon,m}(0)\big)\Big| \\
&\qquad\times\Big(\prod_{l_m=1}^{m-1}\big|f_0\big(x_{s,k}^{\varepsilon,l_m}(0),v_{s,k}^{\varepsilon,l_m}(0)\big)\big|\Big)\Big(\prod_{l_m=m+1}^{s+k}\big|f_0\big(x_{s,k}^{0,l_m}(t_0^{j_p}),v_{s,k}^{\varepsilon,l_m}(0)\big)\big|\Big) \\
&\quad+\sum_{m=1}^{s+k}\Big|f_0\big(x_{s,k}^{0,m}(0),v_{s,k}^{\varepsilon,m}(0)\big)-f_0\big(x_{s,k}^{\varepsilon,m}(0),v_{s,k}^{\varepsilon,m}(0)\big)\Big| \\
&\qquad\times\Big(\prod_{l_m=1}^{m-1}\big|f_0\big(x_{s,k}^{\varepsilon,l_m}(0),v_{s,k}^{\varepsilon,l_m}(0)\big)\big|\Big)\Big(\prod_{l_m=m+1}^{s+k}\big|f_0\big(x_{s,k}^{0,l_m}(t_0^{j_p}),v_{s,k}^{\varepsilon,l_m}(0)\big)\big|\Big).
\end{aligned}$$

The first term of the right-hand side of the last inequality is controlled as the first term of the inequality (16.7), while the second one is directly controlled thanks to Lemma 38 page 449.
Gathering all those results concerning the first decomposition of the proof :

$$\begin{aligned}
&\big|f_0^{(s+k)}\big(Z_{s,k}^0(0)\big)-f_0^{(s+k)}\big(Z_{s,k}^\varepsilon(0)\big)\big| \\
&\le\Big|f_0^{(s+k)}\big(X_{s,k}^0(0),V_{s,k}^0(0)\big)-f_0^{(s+k)}\big(X_{s,k}^0(t_0^{j_1}),V_{s,k}^0(0)\big)\Big| \\
&\quad+\sum_{l=1}^{p-1}\Big|\Big(f_0^{(s+k)}\big(X_{s,k}^0(t_0^{j_l}),V_{l+1}\big)-f_0^{(s+k)}\big(X_{s,k}^0(t_0^{j_{l+1}}),V_{l+1}\big)\Big)\Big| \\
&\quad+\Big|f_0^{(s+k)}\big(X_{s,k}^0(t_0^{j_p}),V_{s,k}^\varepsilon(0)\big)-f_0^{(s+k)}\big(X_{s,k}^\varepsilon(0),V_{s,k}^\varepsilon(0)\big)\Big|,
\end{aligned}$$

one obtains

$$
\begin{aligned}
&\big|f_0^{(s+k)}\big(Z^0_{s,k}(0)\big) - f_0^{(s+k)}\big(Z^\varepsilon_{s,k}(0)\big)\big| \\
&\qquad \le \Big(\big|\nabla_x\sqrt{f_0}\big|_\infty (s+k)\frac{2k+1/2}{\alpha}\varepsilon R \\
&\qquad\qquad\qquad \times 2\exp\Big(-\frac{\beta_0}{4}\big|v^{0,1}_{s,k}(0)\big|^2\Big)\exp\Big(-\frac{\beta_0}{2}\sum_{l=2}^{s+k}\big|v^{0,l}_{s,k}(0)\big|^2\Big)\Big) \\
&\qquad + (p-1)\Big(\big|\nabla_x\sqrt{f_0}\big|_\infty 2(s+k)\frac{2k+1/2}{\alpha}\varepsilon R \\
&\qquad\qquad\qquad \times 2\exp\Big(-\frac{\beta_0}{4}\big|v^{0,1}_{s,k}(0)\big|^2\Big)\exp\Big(-\frac{\beta_0}{2}\sum_{l=2}^{s+k}\big|v^{0,l}_{s,k}(0)\big|^2\Big)\Big) \\
&\qquad + \Big(\big|\nabla_x\sqrt{f_0}\big|_\infty (s+k)\frac{2k+1/2}{\alpha}\varepsilon R \\
&\qquad\qquad\qquad \times 2\exp\Big(-\frac{\beta_0}{4}\big|v^{0,1}_{s,k}(0)\big|^2\Big)\exp\Big(-\frac{\beta_0}{2}\sum_{l=2}^{s+k}\big|v^{0,l}_{s,k}(0)\big|^2\Big) \\
&\qquad + \big|\nabla_x\sqrt{f_0}\big|_\infty (s+k)k\varepsilon \\
&\qquad\qquad\qquad \times 2\exp\Big(-\frac{\beta_0}{4}\big|v^{0,1}_{s,k}(0)\big|^2\Big)\exp\Big(-\frac{\beta_0}{2}\sum_{l=2}^{s+k}\big|v^{0,l}_{s,k}(0)\big|^2\Big)\Big),
\end{aligned}
$$

or again

$$
\begin{aligned}
\big|f_0^{(s+k)}\big(Z^0_{s,k}(0)\big) - f_0^{(s+k)}\big(Z^\varepsilon_{s,k}(0)\big)\big| &\le \big|\nabla_x\sqrt{f_0}\big|_\infty (s+k)\Big(2p\frac{2k+1/2}{\alpha}R\varepsilon + k\varepsilon\Big) \\
&\quad \times 2\exp\Big(-\frac{\beta_0}{4}\big|v^{0,1}_{s,k}(0)\big|^2\Big)\exp\Big(-\frac{\beta_0}{2}\sum_{l=2}^{s+k}\big|v^{0,l}_{s,k}(0)\big|^2\Big).
\end{aligned}
$$

The number p depends on the pseudo-trajectory, but is bounded by definition by $s+k$, so finally one bounded uniformly the difference

$$
\big|f_0^{(s+k)}\big(Z^0_{s,k}(0)\big) - f_0^{(s+k)}\big(Z^\varepsilon_{s,k}(0)\big)\big|
$$

by the constant

$$
2\big|\nabla_x\sqrt{f_0}\big|_\infty (s+k)\Big(2(s+k)\frac{2k+1/2}{\alpha}R\varepsilon + k\varepsilon\Big),
$$

multiplied by the gaussian weight

$$
\begin{aligned}
&\exp\Big(-\frac{\beta_0}{4}\big|v^{0,1}_{s,k}(0)\big|^2\Big)\exp\Big(-\frac{\beta_0}{2}\sum_{l=2}^{s+k}\big|v^{0,l}_{s,k}(0)\big|^2\Big) \\
&\qquad\qquad \le \exp\Big(-\frac{\beta_0}{4}\big|v^{0,1}_{s,k}(0)\big|^2\Big)\exp\Big(-\frac{\beta_0}{4}\sum_{l=2}^{s+k}\big|v^{0,l}_{s,k}(0)\big|^2\Big).
\end{aligned}
$$

The conservation of the kinetic energy along the pseudo-trajectories enables to write then

$$\exp\Big(-\frac{\beta_0}{4}\big|v_{s,k}^{0,1}(0)\big|^2\Big)\exp\Big(-\frac{\beta_0}{4}\sum_{l=2}^{s+k}\big|v_{s,k}^{0,l}(0)\big|^2\Big)=\exp\Big(-\frac{\beta_0}{4}\big|V_{s+k}\big|^2\Big).$$

The inequality of the lemma is therefore obtained using the contracting property of the integrated in time transport-collision-operator. However, one has to be careful here : the time interval $[0,T]$ on which Theorem 6 page 259 provides the existence and uniqueness of respective solutions of the Boltzmann and BBGKY hierarchies, for initial data in $\mathbf{X}_{\cdot,\beta_0,\mu_0}$, depends strongly on β_0. Here, the gaussian weight obtained is not an element of this functional space $\mathbf{X}_{\cdot,\beta_0,\mu_0}$, but belongs to the larger space $\mathbf{X}_{\cdot,\beta_0/2,\mu_0}$. The integrated in time transport-collision-transport operator is still a contracting map for initial data taken in this space, but the time interval on which holds this contracting property is smaller (see the proof of Lemma 23 page 255 for the method to obtain this time interval). This explains the change in the time interval stated in the lemma. □

Remark 47. *One notices that one used in the previous lemma, in addition to a gaussian control in velocity on the initial datum f_0, an assumption on a Lipschitz control on the square root of this function f_0. Another way to obtain a similar result as the control of the Lemma 42 is to ask instead a gaussian control in velocity on the gradient in position of f_0. In fact, this other assumption is implied by the first one. Indeed, for a function f_0 with a Lipschitz control on its square root, one has :*

$$\nabla_x f_0=\nabla_x\big(\sqrt{f_0}\sqrt{f_0}\big)=2\sqrt{f_0}\nabla_x\big(\sqrt{f_0}\big)\leq 2\big|\nabla_x\big(\sqrt{f_0}\big)\big|_\infty\sqrt{f_0},$$

where of course $\sqrt{f_0}$ is controlled by a gaussian in velocity, since f_0 is also controlled in the same way by hypothesis.

Lemma 43 (Error coming from the substitution of the initial data and the removal of the prefactors, quantitative version for tensorized sequences of initial data)**.** *Let s and n be two positive integers, β_0 be a strictly positive number and $\mu_0\in\mathbb{R}$ be a real number.*
Let f_0 be a nonnegative, normalized function belonging to $\in X_{0,1,\beta_0}$ and let K be compact set of the domain of local uniform convergence Ω_s.
Then, there exist five strictly positive numbers $\overline{\varepsilon}$, $\overline{\varepsilon_0}$, $\overline{\alpha}$, $\overline{\gamma}$ and $\overline{R}$ (depending only on the compact set K) such that for every strictly positive numbers R, δ, ε, a, ε_0, ρ, η and α which satisfy

$$\begin{cases}\varepsilon\leq\overline{\varepsilon},\ \varepsilon_0\leq\overline{\varepsilon_0},\ \alpha\leq\overline{\alpha},\ \max\big(16R\varepsilon/\varepsilon_0,\varepsilon_0/\delta\big)\leq\overline{\gamma}\\ \qquad\qquad\qquad\qquad \text{and } R\geq\overline{R},\end{cases}\tag{16.9}$$

and

$$\left\{\begin{array}{c} 4\sqrt{3}a \leq \varepsilon_0, \quad 3a \leq \rho, \quad \varepsilon_0 \leq \eta\delta, \quad R \geq 1, \quad \eta \leq 1 \text{ and } \alpha \leq c(d), \\ \text{with } c(d) \text{ a constant depending only on the dimension } d, \text{ and} \\ \\ 2n\varepsilon \leq a, \end{array}\right. \tag{16.10}$$

one has that, for all positive integer N in the Boltzmann-Grad limit $N \to +\infty$, $N\varepsilon^{d-1} = 1$, for s, n, R, δ, a, ε_0, η, ρ and α fixed, the following uniform convergence on the compact set K, and uniform on the time interval $[0,T]$, of the sequence of the sum of the elementary BBGKY terms for the sequence of initial data $\left(F_{N,0}\right)_{N\geq 0}$ associated sequence of initial data to the sequence of tensorized initial data $\left(f_0^{\otimes s}\right)_{s\geq 1}$ (with $F_{N,0} = \left(f_{N,0}^{(s)}\right)_{1\leq s\leq N}$), towards the sum of the intermediate elementary terms for the sequence of tensorized initial data $\left(f_0^{\otimes s}\right)_{s\geq 1}$ holds :

$$\begin{aligned} \Bigg\| \mathbb{1}_K \sum_{k=1}^{n} \sum_{M_k\in\mathfrak{M}_k} \sum_{J_k\in\mathfrak{J}_k^s} \Big(\overline{\mathcal{J}}^{\varepsilon,\delta}_{\substack{s,s+k-1\\ M_k,J_k}}(U^c,E^c)\big(f_0^{\otimes(s+k)}\mathbb{1}_{|V_{s+k}|\leq R}\big) \\ - \mathcal{J}^{N,\varepsilon,\delta}_{\substack{s,s+k-1\\ M_k,J_k}}(U^c,E^c)\big(f_{N,0}^{(s+k)}\mathbb{1}_{|V_{s+k}|\leq R}\big)\Big)(Z_s)\Bigg\|_{L^\infty} \\ \leq C(d)(s+n)\varepsilon\big|f_0\big|_{L^\infty(\overline{\Omega^c},L^1(\mathbb{R}^d))}\|F_0\|_{0,\beta_0,\mu_0}. \end{aligned} \tag{16.11}$$

Proof. The lemma is an immediate consequence of the control (16.3) page 507, which bounds the integrand of each difference of elementary terms by a quantity going to zero as N goes to infinity multiplied by the tensorized initial datum, belonging to X_{0,β_0,μ_0}, space on which the integrated in time collision-transport operator is a contracting map. □

16.2 The final result

With the additional assumptions of Section 16.1, it is possible to obtain Lanford's theorem, which is a quantitative version of Theorem 7 page 493, for a smaller class of initial data.

This smaller class of initial data is composed of tensorized initial data. One knows in particular (see [25][4]), as it was already noticed in Section 3.2, that if f is a solution of the Boltzmann equation with initial datum f_0, then $\left(f^{\otimes s}\right)_{s\geq 1}$ is a solution of the Boltzmann hierarchy with initial data $\left(f_0^{\otimes s}\right)_{s\geq 1}$. Therefore, the following result can be applied in this particular setting, and for the case $s = 1$, it provides a rigorous derivation of the Boltzmann equation, with in addition an explicit rate of convergence of the one particle distribution of a system of hard spheres towards the solution of the Boltzmann equation.

[4]In particular, one can refer to the first section of Chapter IV : "Introduction : On the Boltzmann-Grad limit".

Theorem 8 (Lanford's theorem in the half-space)**.** *Let β_0 be a strictly positive number and μ_0 be a real number. Then there exist two times $0 \leq T' < T$ such that the following holds :*
let f_0 be a nonnegative normalized function belonging to $X_{0,1,\beta_0}$ which satisfies

$$|f_0|_{0,1,\beta_0} \leq \exp(-\mu_0)$$

and such that $\sqrt{f_0}$ is Lipschitz with respect to the position variable uniformly in the velocity variable.
Then if one considers the solution $F = \left(f^{(s)}\right)_{s\geq 1}$ on $[0,T]$ of the Boltzmann hierarchy with the tensorized initial datum $F_0 = \left(f_0^{\otimes s}\right)_{s\geq 1}$, and if one considers for every positive integer N the solution $F_N = \left(f_N^{(s)}\right)_{s\geq 1}$ on $[0,T]$ of the BBGKY hierarchy with the initial datum $F_{N,0} = \left(f_{N,0}^{(s)}\right)_{1\leq s\leq N}$, where $\left(F_{N,0}\right)_{N\geq 0}$ is the sequence of initial data associated to the sequence of tensorized initial data $\left(f_0^{\otimes s}\right)_{s\geq 1}$, (the time T and the solutions are provided by Theorem 6 page 259), one has that, in the Boltzmann-Grad limit $N \to +\infty$, $N\varepsilon^{d-1} = 1$, for every positive integer s, the following locally uniform convergence on the domain of local uniform convergence Ω_s, uniform on the time interval $[0,T']$, holds :

$$f_N^{(s)} \underset{N\to+\infty}{\longrightarrow} f^{(s)},$$

moreover, whatever the dimension d is, the rate of convergence is of order $O(\varepsilon^\gamma)$, for all $\gamma \in]0, 13/128[$.

The functional space $X_{0,1,\beta_0}$ quoted in the statement of the theorem is introduced in Definition 24 page 206, and the domain of local uniform convergence Ω_s is introduced in Definition 53 page 392.

Remark 48. *One can notice that the time interval on which holds the convergence of $f_N^{(s)}$ towards $f^{(s)}$ is smaller than the time interval on which the solutions of the Boltzmann and the BBGKY hierarchies exist simultaneously. This loss in the length of the time interval comes from Lemma 42 page 507 : in order to use in a quantitative way the control of the difference between the pseudo-trajectories of the two hierarchies, one has to consider the difference $\left|\sqrt{f_0}(z_1) - \sqrt{f_0}(z_2)\right|$ if one wants to take advantage of the Lipschitz control holding on $\sqrt{f_0}$.*
One cannot directly use a Lipschitz control on the difference $|f_0(z_1) - f_0(z_2)|$, since it is mandatory to keep a term $f_0^a(z_1)\dots f_0^a(z_{s+k})$ on which the integrated in time collision-transport operator acts, with $0 < a$, in order to use the contracting property of this operator. But therefore, taking $a < 1$ implies that one has to relax the weight β defining the space $X_{0,1,\beta}$ in which f_0^a lies (necessarily smaller than the weight β_0 such that $f_0 \in X_{0,1,\beta_0}$). As a consequence, this relaxation reduces the length of the time interval $[0,T']$ such that the integrated in time collision-transport operator is a contracting mapping on the space $\widetilde{\boldsymbol{X}}_{0,\widetilde{\beta},\widetilde{\mu}^1}$, with $\widetilde{\beta} : [0,T'] \to \mathbb{R}_+^*$ and $\widetilde{\beta}(0) = \beta$ (see Definition 30 page 212 for the introduction of the time-dependent weighted spaces $\widetilde{\boldsymbol{X}}_{0,\widetilde{\beta},\widetilde{\mu}^1}$, and Section 8.3 page 254

for the link between those functional spaces and the contracting property of the integrated in time collision-transport operator).

Proof of Theorem 8. The main difference with Theorem 7 is that, here, one is looking for an explicit rate of convergence. So back to the proof of this theorem, one had to find, for any compact set of the domain of local uniform convergence Ω_s, truncation parameters n, R, δ, a, ε_0, ρ η and α fulfilling on the one hand the conditions :

$$R \geq 1, \tag{16.12}$$

$$2\varepsilon \leq a,\ 4\sqrt{3}a \leq \varepsilon_0,\ \varepsilon_0 \leq \eta\delta,\ \eta \leq 1,\ 3a \leq \rho \text{ and } \alpha \leq c(d), \tag{16.13}$$

where $c(d)$ denotes a positive constant depending only on the dimension d, and

$$2n\varepsilon \leq a, \tag{16.14}$$

and on the other hand the conditions :

$$K \subset \Delta_s\big(\varepsilon, R, \varepsilon_0, \alpha, \max(16Ra/\varepsilon_0, \varepsilon_0/\delta)\big), \tag{16.15}$$

in order to apply the different lemmas used to compare properly the solutions of the two hierarchies. Besides, the truncation parameters were chosen in such a way that the errors obtained thanks to those lemmas were arbitrarily small. Namely, for example in the case of the BBGKY hierarchy, the errors were controlled by the inequalities (15.3) page 495, (15.5) page 495, (15.7) page 496 and (15.11) page 496.

If all the truncation parameters are written as functions of ε, and if in addition one has

$$\frac{n(\varepsilon)\varepsilon}{a(\varepsilon)} \underset{\varepsilon\to 0}{\longrightarrow} 0,\ \frac{a(\varepsilon)}{\varepsilon_0(\varepsilon)} \underset{\varepsilon\to 0}{\longrightarrow} 0,\ \frac{\varepsilon_0(\varepsilon)}{\eta(\varepsilon)\delta(\varepsilon)} \underset{\varepsilon\to 0}{\longrightarrow} 0,$$
$$\frac{\varepsilon_0(\varepsilon)}{\delta(\varepsilon)} \underset{\varepsilon\to 0}{\longrightarrow} 0,\ \frac{a(\varepsilon)}{\rho(\varepsilon)} \underset{\varepsilon\to 0}{\longrightarrow} 0,\ \alpha(\varepsilon) \underset{\varepsilon\to 0}{\longrightarrow} 0 \text{ and } R(\varepsilon) \underset{\varepsilon\to 0}{\longrightarrow} +\infty,$$

all the conditions (16.12), (16.13), (16.15) and (16.15) will hold for ε small enough. Following [34], one will therefore choose such functions of ε, such that the errors go to zero as ε goes to zero, with an explicit rate.
Choosing $n = C_5\big|\log(\varepsilon)\big|$ with C_5 a strictly positive constant, the right-hand side of the inequality (15.3) controlling the first error becomes

$$\begin{aligned}
&\Big(\frac{1}{2}\Big)^{-C_5\log(\varepsilon)}\Big(\sup_{N\geq 1}\|F_{N,0}\|_{N,\varepsilon,\beta_0,\mu_0^1}\Big)\\
&= \exp\big(-C_5\log(\varepsilon)\log(1/2)\big)\Big(\sup_{N\geq 1}\|F_{N,0}\|_{N,\varepsilon,\beta_0,\mu_0^1}\Big)\\
&= \exp\big(\log(2)C_5\log(\varepsilon)\big)\Big(\sup_{N\geq 1}\|F_{N,0}\|_{N,\varepsilon,\beta_0,\mu_0^1}\Big)\\
&= \Big(\sup_{N\geq 1}\|F_{N,0}\|_{N,\varepsilon,\beta_0,\mu_0^1}\Big)\varepsilon^{\log(2)C_5}.
\end{aligned}$$

Choosing $R = \sqrt{C_6 \big| \log(\varepsilon) \big|}$, the right-hand side of (15.5) becomes

$$\begin{aligned} & C_1 \exp\big(C_2 C_6 \log(\varepsilon)\big) \Big(\sup_{N \geq 1} \|F_{N,0}\|_{N,\varepsilon,\beta_0,\mu_0^1} \Big) \\ & = C_1 \Big(\sup_{N \geq 1} \|F_{N,0}\|_{N,\varepsilon,\beta_0,\mu_0^1} \Big) \varepsilon^{C_2 C_6}. \end{aligned}$$

It is now possible to choose all the other truncation parameters as powers of ε and $\big| \log(\varepsilon) \big|$ such that in the end the sum of the errors is controlled by a power of ε.

There are several restrictions on the rate of convergence coming from the intermediate lemmas. On the one hand, the right-hand side of (15.7) is given by a square root of δ, but δ has to be larger ε_0, itself larger than ε. Moreover ε_0 has to be smaller than $\eta\delta$. The rate of convergence cannot be better than $\varepsilon^{1/2}$. On the other hand, this rate is at most the rate of convergence of $\alpha^{1/8}$ (due to the right-hand side of (15.11)), but α has to be larger than ρ (otherwise the term ρ/α in (15.11) wouldn't be controlled), itself larger than a, and then larger than ε. Therefore, one cannot obtain a rate of convergence better than $\varepsilon^{1/8}$.

Choosing

$$\begin{aligned} \delta &= \varepsilon^{1/2} \big| \log(\varepsilon) \big|^3, \\ a &= \varepsilon \big| \log(\varepsilon) \big|^2, \\ \varepsilon_0 &= \varepsilon^{3/4} \big| \log(\varepsilon) \big|^2, \\ \eta &= \varepsilon^{1/4}, \\ \rho &= \varepsilon^{15/16} \big| \log(\varepsilon) \big|^2, \\ \alpha &= \varepsilon^{13/16} \big| \log(\varepsilon) \big|^2, \end{aligned}$$

one obtains the rate of convergence claimed in the theorem, since the last error terms, which were not quantitatively studied in Theorem 7, are controlled in the present case by Lemmas 42 page 507 and 43 page 515. The rate of convergence of Lemma 42 is $O(\varepsilon^{\gamma_1})$ for every $\gamma_1 < 3/16$ if α is chosen as $\varepsilon^{13/16} |\log(\varepsilon)|^2$, and the rate of convergence of Lemma 43 is $O(\varepsilon^{\gamma_2})$, for every $\gamma_2 < 1$. □

Remark 49. *The limitation in the rate of convergence comes from, essentially, the geometric estimates, and in particular from Lemmas 30 page 370 and 31 page 380. Those estimates are clearly not optimal, and therefore the limitation in the final rate of convergence obtained in the previous theorem is of technical nature. Nevertheless, it would be surprising to recover the rate of convergence* $O(\varepsilon)$.

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