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Thèse de doctorat



Modélisation mathématique des spring magnets

Thèse de doctorat de l'Institut Polytechnique de Paris
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Thèse présentée et soutenue à Palaiseau, le 3 novembre 2020, par

 **LÉA NICOLAS**

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Contents

1	Introduction	7
1	Modélisation micromagnétique	7
1.1	Modélisation	7
1.2	Aimants permanents	9
1.3	La courbe d’hysteresis	12
1.4	Maximisation du produit énergétique	13
1.5	Equation de Landau-Lifschitz-Gilbert	18
2	Homogénéisation	19
2.1	Convergence double-échelle	19
2.2	Micromagnétisme	21
2.3	Homogénéisation stochastique	21
3	Résumé par chapitre	22
2	Stochastic homogenization of Landau-Lifshitz-Gilbert equation	25
1	Introduction	25
2	Stochastic two-scale convergence	26
2.1	The stochastic framework	26
2.2	L^2 Two-scale convergence	32
2.3	Construction of $H^1(\Omega)$	36
2.4	Two-scale convergence compactness theorem in $H^1(\mathcal{D})$	40
3	Elliptic equations	42
4	The harmonic maps equation	46
5	The Landau-Lifshitz-Gilbert equation	50
5.1	The Beppo-Levi space and the variational formulation for the demagnetizing field	50
5.2	Two-scale convergence of the demagnetizing field	51
5.3	Homogenization Theorem	53

3	Nucleation of <i>spring magnets</i>	67
1	Introduction	67
1.1	Expression of the magnetic energy	69
1.2	Nucleation equation	70
2	Spring magnets	73
2.1	Homogenization of the nucleation equation	73
2.2	Convergence of the nucleation field	77
2.3	Multilayer structures	82
3	Analysis of nucleation	84
3.1	Ellipsoids	84
3.2	Infinite cylinders	96
4	Approximation of homogenized coefficients	115
1	Introduction of the approximation	115
1.1	Definitions of the approximations	117
1.2	Review of literature	118
2	Finite element method	119
2.1	The method	119
2.2	The one-dimensional case	121
2.3	The two-dimensional case	125
3	Multigrid methods	134
3.1	Basic principles	134
3.2	Algorithm	135
3.3	Results	140

Chapter 1

Introduction

1 Modélisation micromagnétique

La recherche d'efficacité énergétique est de plus en plus prédominante dans le secteur de l'énergie. Les matériaux magnétiques permettent d'améliorer considérablement l'efficacité et la performance de nombreux dispositifs électriques (moteurs, climatisation, conversion, transports, etc.), c'est pourquoi la fabrication et l'étude d'aimants de pointe est au coeur des préoccupations actuelles.

Les caractéristiques recherchées dans un aimant sont, d'une part, une grande aimantation, car c'est ce qui permet de générer une force magnétique, et d'autre part, une forte capacité à conserver son aimantation malgré les perturbations extérieures (dans des applications telles que les moteurs par exemple, de forts champs magnétiques extérieurs sont appliqués et sont susceptibles de désaimanter un aimant), ce qu'on appelle la *coercivité*. Ainsi, les matériaux magnétiques se divisent en deux catégories. D'une part, les matériaux *doux*, caractérisés par une *aimantation à saturation* (i.e. aimantation maximale) importante, mais une faible *aimantation rémanente* (i.e. l'aimantation du matériau en l'absence de champ magnétique extérieur), ainsi qu'un faible *champ coercitif* (i.e. champ magnétique minimal qu'il faut appliquer au matériau pour qu'il perde son aimantation), sont utilisés dans des applications liées à la génération d'électricité et à la transformation énergétique. D'autre part, les matériaux *durs*, à plus faible aimantation à saturation, mais forte aimantation rémanente et champ coercitif, sont appropriés pour la fabrication d'*aimants permanents*, utilisés dans les moteurs électriques et les supports d'enregistrement magnétiques (disques durs, disquettes, bandes magnétiques, etc.).

1.1 Modélisation

On modélise un continuum ferromagnétique par une distribution de spins, c'est-à-dire de mini-aimants permanents décrits par le champ de vecteurs, appelé l'*aimantation*,

$$\mathbf{M} : \mathcal{D} \mapsto \mathbb{R}^3, \quad (1.1)$$

où \mathcal{D} est un domaine borné de l'espace \mathbb{R}^3 correspondant à l'aimant, soumis à la contrainte suivante : pour presque tout $x \in \mathcal{D}$, $|\mathbf{M}(x)| = M_s(T)$, avec $M_s(T)$ l'*aimantation à*

saturation, dépendant uniquement du matériau et de la température, et dont l'unité est $A.m^{-1}$ dans le système S.I..

Les configurations physiquement admissibles sont sélectionnées par la minimisation de l'énergie libre (dite de Brown), où l'on note $\mathbf{M} = M_s \mathbf{m}$, avec $|\mathbf{m}| = 1$ presque-partout sur \mathcal{D} ,

$$\mathcal{E}(\mathbf{m}) = \int_{\mathcal{D}} A |\nabla \mathbf{m}|^2 + \int_{\mathcal{D}} J(\mathbf{m}) - \int_{\mathcal{D}} \mu_0 \mathbf{H}_{\text{ext}} \cdot M_s \mathbf{m} - \frac{1}{2} \int_{\mathcal{D}} \mu_0 \mathbf{H}_d(M_s \mathbf{m}) \cdot M_s \mathbf{m}, \quad (1.2)$$

dont les différents termes sont :

- l'énergie extérieure

$$\mathcal{E}_{\text{ext}}(\mathbf{m}) = - \int_{\mathcal{D}} \mathbf{H}_{\text{ext}} \cdot M_s \mathbf{m},$$

avec \mathbf{H}_{ext} le champ extérieur, en $A.m^{-1}$. Ce terme rend compte du fait qu'un échantillon magnétique placé dans un champ extérieur a tendance à voir son aimantation s'aligner dans la direction du champ,

- l'énergie d'échange

$$\mathcal{E}_{\text{éch}}(\mathbf{m}) = \int_{\mathcal{D}} A |\nabla \mathbf{m}|^2,$$

avec A la constante d'échange (en $J.m^{-1}$). Lorsque deux spins sont très proches, ils ont tendance à s'aligner. Cette énergie pénalise les variations de \mathbf{m} ,

- l'énergie démagnétisante

$$\mathcal{E}_{\text{dem}}(\mathbf{m}) = - \frac{1}{2} \int_{\mathcal{D}} \mu_0 \mathbf{H}_d(M_s \mathbf{m}) \cdot M_s \mathbf{m},$$

avec \mathbf{H}_d le champ créé par l'aimant. D'après les équations de Maxwell, le champ \mathbf{H}_d vérifie en tout point de l'espace

$$\begin{cases} \text{rot}(\mathbf{H}_d(\mathbf{M})) = 0, \\ \text{div}(\mathbf{H}_d(\mathbf{M}) + \mathbf{M}) = 0, \end{cases}$$

- l'énergie d'anisotropie

$$\mathcal{E}_{\text{an}}(\mathbf{m}) = \int_{\mathcal{D}} J(\mathbf{m}),$$

avec $J : \mathbb{S}^2 \rightarrow \mathbb{R}$ (\mathbb{S}^2 étant la sphère unité de \mathbb{R}^3) une fonction paire atteignant son minimum sur un ensemble de directions appelées *directions faciles d'aimantation*. Un exemple d'une telle fonction est l'anisotropie uniaxiale :

$$J(\mathbf{m}) = K (1 - (\mathbf{m} \cdot \mathbf{u})^2),$$

où $K > 0$ est la *constante d'anisotropie* et \mathbf{u} est la direction facile d'aimantation,

Un matériau dur se caractérise par une forte anisotropie, il va donc avoir tendance à rester stable dans l'une de ses directions faciles d'aimantation, même en présence d'un champ extérieur. Cependant, les meilleurs matériaux durs disponibles aujourd'hui ont une aimantation à saturation typiquement plus faible que beaucoup de matériaux doux.

Après cette brève introduction des grandeurs importantes, nous pouvons introduire les aimants permanents, et notamment ceux qui font l'objet de cette thèse, les *spring magnets*.

1.2 Aimants permanents

Dans les pays les plus développés, tels que les Etats-unis, près de 30% de l'électricité est consommée par le secteur industriel, dont 65% par les moteurs électriques, ce qui revient à près de 750 milliards de kWh, soit environ 45 milliards de dollars, pour les seuls Etats-Unis [31]. De plus, les transports électriques se développant, la consommation d'électricité est vouée à augmenter rapidement. C'est pourquoi une amélioration, même faible, de l'efficacité énergétique des moteurs électriques, a un impact économique et environnemental conséquent, d'où l'intérêt porté aux matériaux magnétiques doux et durs qui permettent de construire des moteurs plus petits et plus légers, tout en conservant la puissance et le couple nécessaire pour leurs applications.

Dans les machines électriques, les aimants permanents remplissent une fonction de base : fournir un flux magnétique, ce qui demande une aimantation à saturation M_s la plus grande possible, ainsi qu'une coercivité appropriée : en effet, des champs magnétiques extérieurs sont susceptibles de les désaimanter.

Durant le vingtième siècle, plusieurs matériaux magnétiques durs ont été découverts, ainsi que des techniques pour les fabriquer. Les terres rares, un ensemble de 17 éléments du tableau périodique, ont connu un regain d'intérêt, car elles permettent de construire de tels aimants permanents sous différentes températures. Malgré leur appellation, les terres rares sont relativement abondantes sur Terre, mais sont très coûteuses à produire car, d'une part, elles se trouvent ensemble dans la nature, et sont difficiles à séparer les unes des autres, et d'autre part, de par leurs propriétés géochimiques, elles sont très dispersées. La Chine domine ce marché, notamment grâce à la présence de grands gisements sur leur territoire [25].

La qualité d'un aimant permanent est mesurée par le *produit énergétique maximum*, noté $(BH)_{\max}$ et mesuré en *Mega-Gauss-Ersted* (MGOe) dans le système CGS, couramment utilisé dans le domaine du micromagnétisme. Celui-ci augmente avec le champ coercitif H_c et l'aimantation à saturation M_s [25]. Ce produit énergétique a été amélioré tout au long du vingtième siècle : d'environ 1 MGOe pour l'acier utilisé au début du vingtième siècle à environ 56 MGOe pour les aimants au néodyme récemment fabriqués. Ce sont les aimants permanents les plus performants disponibles sur le marché. Ceux-ci ont significativement contribué au développement de matériels informatiques périphériques, tels que les disques durs et les outils sans fil.

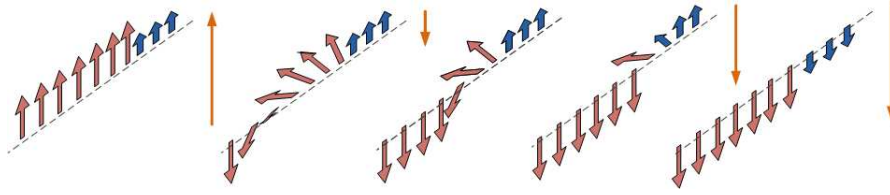


Figure 1.1: En bleu : les spins du matériau dur, et en rouge ceux du matériau doux. La flèche orange indique le champ magnétique extérieur. Illustration tirée de la page Wikipédia « spring magnets ».

Exchange spring magnets

L'approche la plus prometteuse et la plus active pour fabriquer les meilleurs aimants permanents est celle de nanocomposites qui comprendraient des phases de matériaux durs, couplées avec des phases de matériaux doux. Ce type d'aimant est appelé *exchange spring magnet*.

Le principe d'un spring magnet est de bénéficier du fait que les spins ont naturellement tendance à s'aligner (dû au terme d'*énergie d'échange*), et que, simultanément, les spins du matériau dur ont tendance à rester dans une direction favorable à l'anisotropie. Ceci est illustré dans la figure 1.1 : dans l'image de gauche tous les spins sont alignés sur le champ magnétique extérieur. Lorsqu'on inverse celui-ci et que l'on augmente sa puissance : les spins du matériau doux éloignés du matériau dur changent de direction, mais pour ceux qui sont proches du matériau dur, l'interaction d'échange est plus forte que la force extérieure, donc ils restent maintenus dans la même direction. Finalement, quand le champ magnétique dépasse le champ coercitif du matériau dur, tous les spins changent de direction.

Ce type de comportement a été observé pour la première fois par Coehoorn et al. [21] (1989) dans un échantillon de $\text{Nd}_{4.5}\text{Fe}_{77}\text{B}_{18.5}$ fondu, qui présentait des propriétés intéressantes pour en faire un aimant permanent, tout en étant constitué de 85% de matériau doux, ce qui réduit le coût du matériau.

Un certain nombre d'études théoriques physiques ont été faites sur la démagnétisation des spring magnets, qui ont montré que le paramètre le plus important pour caractériser leur démagnétisation était la dimension de la phase douce : pour une phase douce fine entre deux phases dures, il existe une épaisseur critique en-dessous de laquelle la phase douce est rigidement couplée avec la phase dure et les deux phases se renversent pour le même champ extérieur. Cette épaisseur a été calculée par E.E. Kneller et R. Hawig dans [40], et est donnée par

$$\delta_h = \pi \sqrt{\frac{A_h}{K_h}}$$

avec A_h et K_h les constantes d'échange et d'anisotropie (dans le cas d'une anisotropie uniaxiale) de la phase dure.

- Pour des couches douces d'épaisseur inférieures à δ_h , les deux phases sont inextricablement couplées et le système composite est caractérisé par la moyenne des propriétés magnétiques de chaque couche : le champ coercitif est donné par

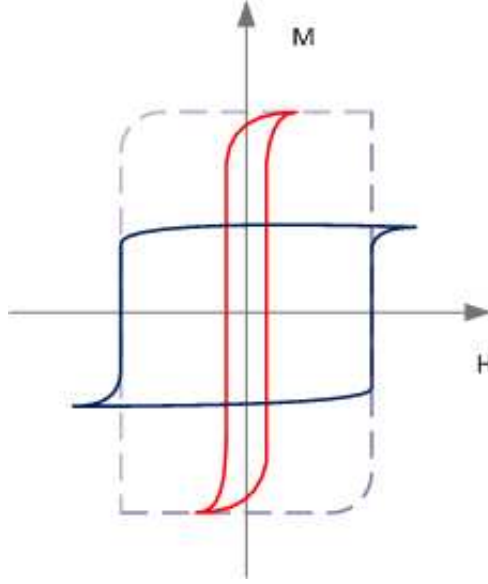


Figure 1.2: Courbe d'hysteresis idéale d'un spring magnet (en pointillés), ainsi que de ses composants isolés (dur en bleu, et doux en rouge). M est la densité magnétique, et H le champ magnétique externe.

$H_c = \frac{2(t_h K_h + t_s K_s)}{\mu_0(t_h M_h + t_s M_s)}$ (avec t_h et t_s les épaisseurs des couches, et K_h , K_s , M_h et M_s leurs constantes d'anisotropie et aimantations à saturation respectives). Skomski et Coey dans [61] ont estimé qu'une multi-couche de $\text{Sm}_2\text{Fe}_{17}\text{N}_3/\text{Fe}_{65}\text{Co}_{35}$ avec $t_h = 2.4\text{nm}$ et $t_s = 9.0\text{nm}$ (la phase dure étant $\text{Sm}_2\text{Fe}_{17}\text{N}_3$), pouvait avoir un $(BH)_{\max}$ de 120MGOe (soit trois fois celui des aimants permanents disponibles sur le marché), avec seulement 5% de Samarium.

- Pour des couches douces d'épaisseurs supérieures, la coercivité de la couche douce s'effondre brutalement, ce qui dégrade les propriétés du système composite. Goto et al. [30] et Thompson [62] ont étudié pour la première fois dans les années 1960 un film doux couplé ferromagnétiquement à une couche dure, auquel ils ont appliqué un champ extérieur opposé à l'aimantation du matériau dur. Ils ont déterminé que la couche douce reste parallèle à la couche dure pour des champs plus faible que $H_{\text{ex}} = \pi^2 \frac{A_s}{2M_s t_s^2}$, appelé champ d'échange. Pour des champs d'intensité supérieure, l'aimantation de la couche douce se retourne. Ce retournement de la couche douce est réversible : celle-ci se réaligne sur la couche dure si le champ extérieur est supprimé. Par analogie avec le mouvement élastique d'un ressort mécanique, on appelle ce phénomène *exchange-spring process*.

Ainsi, pour pouvoir améliorer le produit énergétique $(BH)_{\max}$ d'un aimant, il est nécessaire que les couches douces restent en-dessous de l'épaisseur δ_h , ce qui nécessite de fabriquer des films d'épaisseur nanométrique, et de pouvoir les incorporer dans des structures hétéromagnétiques adaptées. Des exchange-spring magnets ont été fabriqués principalement par trempe rapide et recuisson ou par des alliages de nanocomposites avec des grains de matériau dur orientés aléatoirement. On peut citer des films Sm-Co/FeCo [2], des films Pr-Co/Co [46] et des films et des multicouches $\alpha\text{-Fe}/\text{Nd-Fe-B}$ [59].

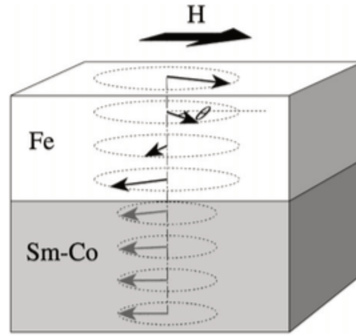


Figure 1.3: Illustration d'un couplage exchange-spring dans le cas d'une bi-couche (le matériau doux est Fe, le matériau dur est SmCo). L'angle de rotation augmente dans la couche douce à mesure que l'on s'éloigne de l'interface avec la couche dure.

Ces structures présentent des propriétés intéressantes, mais elles se montrent souvent bien différentes du modèle idéal précédent. En effet, une structure FePt/Fe admet de très fortes valeurs pour $(BH)_{\max}$ (supérieure à 40 MGOe [47]). Mais, les échantillons étant produits par recuisson thermique rapide de multicouches Fe/Pt, le film final est un matériau nanostructuré qui ne présente plus la structure multi-couches originelle.

Afin de comprendre le phénomène de désaimantation dans les spring magnets, il faut le comprendre dans le cas plus simple des matériaux homogènes. Nous rappelons ici l'étude bien connue de l'aimantation dans une particule homogène dans laquelle l'aimantation est supposée uniforme.

1.3 La courbe d'hysteresis

On introduit dans cette sous-section la courbe d'hysteresis, qui caractérise l'évolution de l'aimantation d'une particule en présence d'un champ extérieur. Pour une introduction plus complète, on peut se référer à [60].

Dans les petites particules, c'est l'énergie d'échange qui domine, on suppose ici la particule assez petite pour considérer que l'aimantation reste uniforme. On écrit l'énergie totale comme la somme d'une fonction d'anisotropie généralisée $g(\mathbf{m})$ qui, dans le cas d'une l'aimantation uniforme, inclut l'énergie démagnétisante, et de l'énergie du champ extérieur $E_{\text{ex}} = -\mathbf{H}_{\text{ex}} \cdot M_s \mathbf{m}$.

Dans le cas de couches fines, le problème est planaire. Un seul angle φ décrit l'aimantation, l'origine φ étant la direction facile d'aimantation, $\mathbf{m} = \begin{pmatrix} \cos \varphi \\ \sin \varphi \end{pmatrix}$, et deux composantes du champ, $H_{//}$, parallèle à la direction facile d'aimantation du matériau, et H_{\perp} , perpendiculaire à celle-ci, doivent être prises en compte. L'énergie totale d'une particule uniaxiale quelconque devient alors

$$e_{\text{tot}}(\varphi) = g(\varphi) - \mu_0 H_{//} M_s \cos \varphi - \mu_0 H_{\perp} M_s \sin \varphi. \quad (1.3)$$

A l'équilibre statique, la direction de l'aimantation doit satisfaire :

$$\frac{\partial e_{\text{tot}}}{\partial \varphi} = g'(\varphi) + \mu_0 H_{//} M_s \sin \varphi - \mu_0 H_{\perp} M_s \cos \varphi = 0. \quad (1.4)$$

La dérivée seconde doit être positive pour satisfaire un équilibre stable. C'est pourquoi la limite de stabilité nous intéresse particulièrement. Elle est définie par :

$$\frac{\partial^2 e_{\text{tot}}}{\partial \varphi^2} = g''(\varphi) + \mu_0 H_{//} M_s \cos \varphi + \mu_0 H_{\perp} M_s \sin \varphi = 0. \quad (1.5)$$

En atteignant la limite de stabilité, la particule précédemment dans un état stable devient instable et change d'état. La combinaison de (1.4) et (1.5) conduit à un système d'équations implicites pour la limite de stabilité ou la courbe de basculement :

$$\begin{aligned} H_{//}^* &= \frac{1}{\mu_0 M_s} (-g'(\varphi) \sin \varphi - g''(\varphi) \cos \varphi), \\ H_{\perp}^* &= \frac{1}{\mu_0 M_s} (g'(\varphi) \cos \varphi - g''(\varphi) \sin \varphi). \end{aligned} \quad (1.6)$$

Dans le cas d'une anisotropie uniaxiale, on a

$$\begin{aligned} g(\varphi) &= K (1 - (\mathbf{m} \cdot \mathbf{u})^2) = K \sin^2 \varphi, \\ g'(\varphi) &= 2K \sin \varphi \cos \varphi, \\ g''(\varphi) &= 2K (\cos^2 \varphi - \sin^2 \varphi). \end{aligned}$$

La courbe de basculement en termes de champ réduit $h = \frac{H_{\text{ex}} \mu_0 M_s}{2K}$:

$$\begin{aligned} h_{//}^* &= -\cos^3 \varphi, \\ h_{\perp}^* &= \sin^3 \varphi, \end{aligned} \quad (1.7)$$

est la célèbre astroïde de *Stoner-Wohlfarth*, représentée sur la figure 1.4. Pour chaque champ réduit \mathbf{h} à l'intérieur de l'astroïde, il existe quatre solutions, dont deux stables, et deux instables (en pointillés). Pour chaque champ réduit \mathbf{h} à l'extérieur de l'astroïde, les deux solutions stables et les deux solutions instables sont confondues. En combinant les solutions stables, on peut obtenir des courbes d'hystérésis d'une particule uniaxiale. Sur la figure 1.5, les composantes d'aimantation longitudinale $m_{//}$ et transversale m_{\perp} sont représentées en fonction du champ longitudinal $h_{//}$ (le long de l'axe facile de la particule \mathbf{u}) pour différentes valeurs d'un champ de polarisation transversal h_{\perp} . La généralisation de cette analyse à trois dimensions, dans laquelle deux angles de magnétisation ϑ et φ doivent être considérés, est directe.

Cette étude simple nous permet d'obtenir dans la section suivante le produit énergétique maximal des *aimants permanents*, c'est-à-dire des matériaux dont l'anisotropie est non nulle.

1.4 Maximisation du produit énergétique

Nous introduisons dans cette section le *produit énergétique maximal* $(BH)_{\text{max}}$, et expliquons en quoi il est lié à la qualité d'un aimant. Pour plus de détails, on pourra se référer à [22].

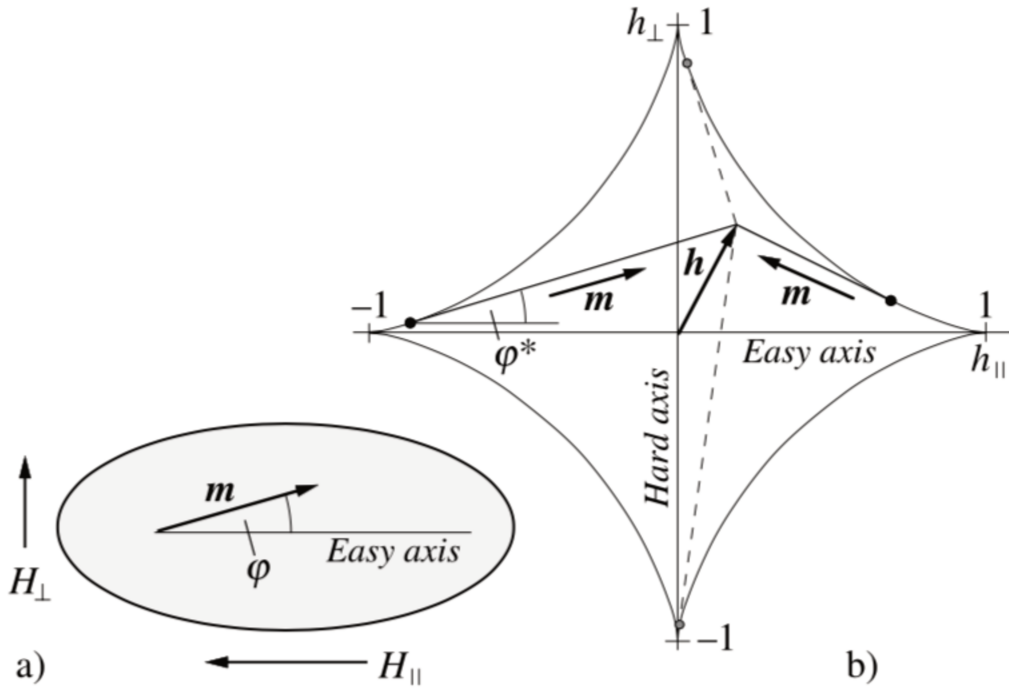


Figure 1.4: Astéroïde. Pour un champ (réduit) \mathbf{h} , les deux solutions stables pour \mathbf{m} à l'équilibre correspondant sont indiquées en gras, les solutions instables sont indiquées en pointillés. Illustration tirée de [37]

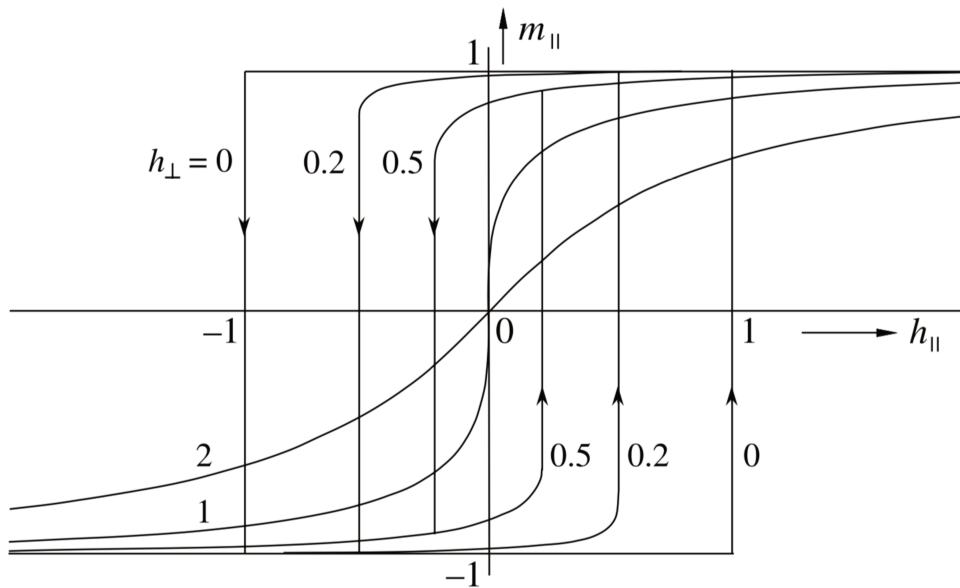


Figure 1.5: Courbe d'hystérésis, elle indique l'aimantation longitudinale $m_{//}$ en fonction du champ longitudinal extérieur $h_{//}$, pour différentes valeurs du champ transversal h_{\perp} fixées. Pour $h_{\perp} = 0$, on observe qu'une aimantation préexistante alignée selon la direction facile d'aimantation $m_{//} = 1$ se retourne pour le champ réduit $h_{//} = -1$, et inversement $m_{//} = -1$ se retourne pour le champ réduit $h_{//} = 1$ Illustration tirée de [37]

L'induction magnétique \mathbf{B} est définie dans tout l'espace par

$$\mathbf{B} = \mu_0 (\mathbf{M} + \mathbf{H}') ,$$

où l'on a noté $\mathbf{H}' = \mathbf{H}_d(\mathbf{M}) + \mathbf{H}_{\text{ext}}$, et avec $\mathbf{M} = 0$ dans l'espace libre autour de l'aimant. Les champs \mathbf{B} et \mathbf{H}' obéissent aux équations de Maxwell :

$$\begin{cases} \nabla \times \mathbf{H}' = 0 , \\ \nabla \cdot \mathbf{B} = 0 . \end{cases} \quad (1.8)$$

On définit l'énergie magnétostatique spécifique d'un aimant par

$$E_{ms} = \frac{1}{2\mu_0} \int_{\mathbb{R}^3} |\mathbf{B}|^2 .$$

Cette énergie représente l'interaction magnétostatique mutuelle des dipôles élémentaires dont l'aimant est composé. Elle contient des contributions de l'intérieur et de l'extérieur de l'aimant. Or, pour des raisons pratiques, seule l'énergie $E' = \frac{1}{2\mu_0} \int_{\mathbb{R}^3 \setminus \mathcal{D}} |\mathbf{B}|^2$ en-dehors de l'aimant est disponible. L'aimantation étant nulle en-dehors de l'aimant, on a $\mathbf{B} = \mu_0 \mathbf{H}'$, et donc

$$E' = \frac{1}{2} \int_{\mathbb{R}^3 \setminus \mathcal{D}} \mathbf{B} \cdot \mathbf{H}' .$$

Or, d'après les équations de Maxwell (1.8), on a $\int_{\mathbb{R}^3} \mathbf{B} \cdot \mathbf{H}' = 0$, donc

$$E' = -\frac{1}{2} \int_{\mathcal{D}} \mathbf{B} \cdot \mathbf{H}' .$$

C'est donc ce produit que nous cherchons à maximiser. Dans le cas d'une aimantation uniforme, nous nous intéressons à la densité $BH = -\mathbf{B} \cdot \mathbf{H}'$ à l'intérieur de l'aimant. La proposition suivante donne sa valeur dans le cas d'un aimant permanent idéal dont l'aimantation est représenté par une courbe d'hysteresis rectangulaire.

Proposition 1.1. *La densité d'énergie maximale des aimants permanents, est limitée par leur aimantation à saturation M_s et leur champ coercitif H_c de la façon suivante :*

- si $H_c \geq \frac{1}{2}M_s$, alors

$$(BH)_{max} = \mu_0 \frac{M_s^2}{4} ,$$

- si $H_c < \frac{1}{2}M_s$, alors

$$(BH)_{max} = \mu_0 (M_s - H_c) H_c .$$

Proof. Supposons l'aimantation uniforme. Le produit BH vaut à l'intérieur de l'aimant $BH = -\mu_0 (\mathbf{M} + \mathbf{H}') \cdot \mathbf{H}'$. Il est donc maximal quand \mathbf{M} et \mathbf{H}' sont antiparallèles. En notant $\mathbf{M} = M_s \mathbf{m}$, avec $\mathbf{m} = 1$ et $\mathbf{H}' = -H \mathbf{m}$, on a $BH = \mu_0 (M_s - H) H$.

La figure 1.6 représente la courbe d'hystéresis d'un aimant permanent, avec H' en abscisse, et M en ordonnée pour le graphe de gauche, et B en ordonnée pour le graphe de droite. Les cadrans qui nous intéressent sont ceux en haut à gauche ou en bas à droite

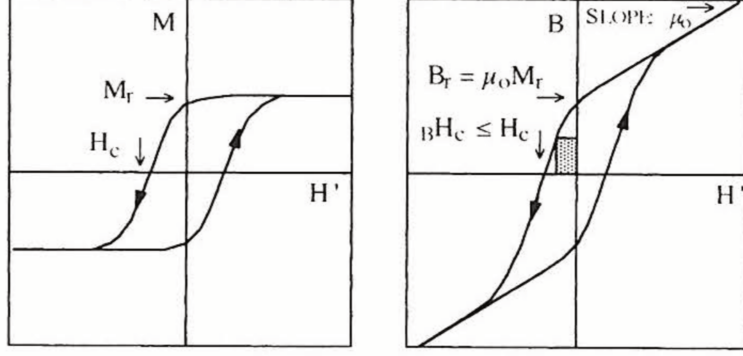


Figure 1.6: Courbe d'hystérésis, en termes de champ H (à gauche) et B (à droite). La surface du carré grisé sous la courbe de B correspond à $(BH)_{\max}$. Illustration tirée de [60]

car \mathbf{M} et \mathbf{H}' sont antiparallèles. Comme ils sont symétriques, on regarde le cadran en haut à gauche, i.e. $H' < 0$ et $B > 0$. Pour $H' > -H_c$, on a $M = M_s$, et pour $H' < -H_c$, on a $M = -M_s$. On regarde donc les valeurs de BH pour $H \in I = (0, \min(H_c, M_s))$. La fonction $H \mapsto \mu_0(M_s - H)H$ admet un maximum en $H = \frac{1}{2}M_s$. Deux cas se présentent :

- Le cas $\frac{1}{2}M_s < H_c$. Alors $\frac{1}{2}M_s \in I$ donc

$$(BH)_{\max} = \mu_0 \left(M_s - \frac{M_s}{2} \right) \frac{M_s}{2} = \mu_0 \frac{M_s^2}{4}.$$

- Le cas $\frac{1}{2}M_s > H_c$: alors, en particulier, $M_s > H_c$ donc $I = (0, H_c)$ et $\frac{1}{2}M_s \notin I$ donc le produit (BH) atteint son maximum en H_c :

$$(BH)_{\max} = \mu_0(M_s - H_c)H_c.$$

□

L'énergie spécifique d'un matériau est donc d'autant plus grande que l'aimantation à saturation M_s est grande. Or, plus M_s est grand, plus on tombe dans le second cas. On voit donc l'utilité de combiner des matériaux durs et des matériaux doux. En effet, pour les matériaux durs, on est dans le premier cas : comme l'anisotropie est grande, le champ coercitif est d'autant plus grand; tandis que pour les matériaux doux, on est dans le second cas : le champ coercitif est faible, mais l'aimantation à saturation est plus élevée.

On peut montrer de manière élémentaire le gain en produit énergétique que l'on obtient en combinant un matériau dur et un matériau doux. Skomski et Coey dans [61] calculent les produits énergétiques pour différents *spring magnets*. Prenons un matériau dur dont les paramètres sont M_1 et K_1 (respectivement, l'aimantation à saturation et la constante d'anisotropie), et un matériau doux dont les paramètres sont M_2 et K_2 . On suppose que les deux matériaux ont le même axe facile d'aimantation \mathbf{u} . En considérant une inclusion sphérique de matériau doux dans un matériaux dur, ils montrent que pour des inclusions de rayon D supérieure à $\delta_h = \pi \sqrt{\frac{A_h}{K_h}}$, le champ coercitif, dont la valeur

maximale est celui du matériau dur $\frac{2K_1}{M_1}$, chute proportionnellement à $\frac{1}{D^2}$. Ils considèrent donc un nombre plus important d'inclusions, pour que le gain en produit énergétique soit conséquent. Ils expliquent qu'ils assimilent ce matériau à un matériau uniforme et homogène avec des nouvelles constantes

$$e(\mathbf{m}) = \langle K \rangle (1 - (\mathbf{m} \cdot \mathbf{u})^2) - H_{\text{ext}} \cdot \langle M_s \rangle \mathbf{m},$$

où l'anisotropie et l'aimantation effectives sont respectivement

$$\begin{aligned} \langle M_s \rangle &= \theta M_1 + (1 - \theta) M_2, \\ \langle K \rangle &= \theta K_1 + (1 - \theta) K_2, \end{aligned}$$

avec θ la proportion de matériau dur. Il montrent, en supposant $K_2 = 0$, $M_2 > M_1$, et $K_1 \gg M_2^2 + M_1^2$, que le produit énergétique maximal est obtenu pour une fraction de matériau dur

$$\theta^* = \mu_0 \frac{M_2^2}{4K_1} \left(1 - \frac{\mu_0 M_2 (M_2 - M_1)}{2K_1} \right).$$

Dans ce cas, le produit énergétique maximal est

$$(BH)_{\text{max}} = \mu_0 \frac{M_2^2}{4} \left(1 - \frac{\mu_0 M_2 (M_2 - M_1)}{2K_1} + 5 \frac{\mu_0^2 M_2^2 (M_2 - M_1)^2}{16K_1^2} \right). \quad (1.9)$$

Ainsi, les aimantations à saturation des matériaux doux étant typiquement plus élevées que celles des matériaux durs, on obtient un gain de produit énergétique maximal par rapport au matériau dur seul. De plus, la fraction de matériau dur, plus coûteux que le matériaux doux, est faible.

Nous donnons ici le calcul, qui ne figure pas dans [61]. Ce calcul est élémentaire et peut être omis sans gêner la compréhension de notre chapitre introductif.

On a pour le champ coercitif $H_c = \frac{2\langle K \rangle}{\mu_0 \langle M_s \rangle}$. D'après la proposition 1.1, il s'agit de maximiser

$$\begin{aligned} (BH)_{\text{max}} &= \mu_0 \frac{\langle M_s \rangle^2}{4} \\ &= \mu_0 \frac{(M_2 - M_1)^2}{4} \left(\theta - \frac{M_2}{M_2 - M_1} \right)^2, \end{aligned}$$

sous la condition

$$\frac{2\langle K \rangle}{\mu_0 \langle M_s \rangle} \geq \frac{\langle M_s \rangle}{2} \Leftrightarrow \theta^2 \mu_0 (M_2 - M_1)^2 - 2\theta \mu_0 (M_2 (M_1 - M_2) + 2(K_1 - K_2)) + \mu_0 M_2^2 - 4K_2 \leq 0.$$

L'optimalité est donc réalisée pour $\theta \in [0, 1]$ qui réalise l'égalité

$$\theta^2 \mu_0 (M_2 - M_1)^2 - 2\theta \mu_0 (M_2 (M_2 - M_1) + 2(K_1 - K_2)) + \mu_0 M_2^2 - 4K_2 = 0.$$

L'unique $\theta \in [0, 1]$ qui vérifie l'égalité précédente est

$$\theta^* = \frac{\mu_0 M_2 (M_2 - M_1) + 2(K_1 - K_2) - \sqrt{\Delta}}{\mu_0 (M_2 - M_1)^2},$$

avec

$$\Delta = 4(K_1 - K_2)(\mu_0 M_2(M_2 - M_1) + (K_1 - K_2)) + 4\mu_0 K_2(M_2 - M_1)^2 \geq 0.$$

On vérifie que θ^* est dans $[0, 1]$ en utilisant $K_1 \gg \mu_0(M_2^2 + M_1^2)$. On peut calculer une estimation de θ^* et $(BH)_{\max}$ dans le cas $K_2 = 0$ et en utilisant $K_1 \gg \mu_0(M_2^2 + M_1^2)$:

$$\begin{aligned} \theta^* &= \frac{1}{\mu_0(M_2 - M_1)^2} \left(\mu_0 M_2(M_2 - M_1) + 2K_1 - 2K_1 \sqrt{1 + \frac{\mu_0 M_2(M_2 - M_1)}{K_1}} \right) \\ &\approx \frac{1}{\mu_0(M_2 - M_1)^2} \left(\mu_0 M_2(M_2 - M_1) + 2K_1 \right. \\ &\quad \left. - 2K_1 \left(1 + \frac{\mu_0 M_2(M_2 - M_1)}{2K_1} - \frac{\mu_0^2 M_2^2(M_2 - M_1)^2}{8K_1^2} + \frac{\mu_0^3 M_2^3(M_2 - M_1)^3}{16K_1^3} \right) \right) \\ &= \mu_0 \frac{M_2^2}{4K_1} \left(1 - \frac{\mu_0 M_2(M_2 - M_1)}{2K_1} \right), \end{aligned}$$

en faisant le développement limité à l'ordre 3 de la racine. Ceci donne la fraction de volume du matériau dur. Calculons maintenant le produit énergétique :

$$\begin{aligned} (BH)_{\max} &= \frac{K_1^2}{(M_2 - M_1)^2} \left(-1 + \sqrt{1 + \frac{\mu_0 M_2(M_2 - M_1)}{K_1}} \right)^2 \\ &\approx \mu_0 \frac{M_2^2}{4} \left(1 - \frac{\mu_0 M_2(M_2 - M_1)}{2K_1} + 5 \frac{\mu_0^2 M_2^2(M_2 - M_1)^2}{16K_1^2} \right), \end{aligned}$$

en faisant un développement limité à l'ordre 3 de la racine, et à l'ordre 2 du carré.

1.5 Equation de Landau-Lifschitz-Gilbert

Landau et Lifschitz [45] ont proposé un modèle phénoménologique pour décrire l'évolution temporelle de l'aimantation dans un continuum ferromagnétique. Dans cette thèse, nous étudions la variante de Gilbert de cette équation, qui est équivalente, mais mathématiquement plus pratique pour notre étude.

$$\frac{\partial \mathbf{m}}{\partial t} = -\gamma_G \mathbf{m} \times \mathbf{H}_{\text{eff}}(\mathbf{m}) + \alpha_G \mathbf{m} \times (\mathbf{m} \times \mathbf{H}_{\text{eff}}(\mathbf{m}))$$

avec le champ effectif H_{eff} donné par

$$H_{\text{eff}}(\mathbf{m}) = -\nabla_{\mathbf{m}} \mathcal{E}(\mathbf{m}) = A \Delta \mathbf{m} + \mu_0 M_s \mathbf{H}_{\text{ext}} + \mu_0 M_s^2 \mathbf{H}_d(\mathbf{m}) - \nabla_{\mathbf{m}} J(\mathbf{m}),$$

et la contrainte non-convexe

$$|\mathbf{m}| = 1 \text{ p.p. sur } \mathcal{D},$$

ainsi qu'une condition initiale $\mathbf{m}(\cdot, 0) = \mathbf{m}_0$.

Un couple $\mathbf{m} \times \mathbf{H}_{\text{eff}}(\mathbf{m})$ implique une réaction gyromagnétique, décrit par l'équation suivante :

$$\frac{\partial \mathbf{m}}{\partial t} = -\gamma \mathbf{m} \times \mathbf{H}_{\text{eff}}(\mathbf{m}).$$

Cette équation décrit le mouvement de précession de l'aimantation autour du champ effectif. Afin de prendre en compte les diverses pertes, Landau et Lifschitz ont introduit dans [45] un terme d'amortissement, en définissant un facteur empirique α_{LL} . L'équation devient

$$\frac{\partial \mathbf{m}}{\partial t} = -\gamma_{LL} \mathbf{m} \times \mathbf{H}_{\text{eff}}(\mathbf{m}) - \alpha_{LL} \mathbf{m} \times \frac{\partial \mathbf{m}}{\partial t}.$$

Cette équation a fait l'objet de nombreuses études, tant numériques que théoriques. On peut citer notamment une série d'articles par Baryakhtar, qui démontrent la validité générale de l'équation [10]. Pour une introduction plus détaillée, on pourra se référer à la section dédiée dans [37].

2 Homogénéisation

Mathématiquement parlant, l'étude des matériaux qui nous intéressent est difficile car les modèles habituels sont non linéaires et la dépendance matérielle des paramètres varient à une très petite échelle. Ainsi, résoudre numériquement l'équation de Landau-Lifschitz pour ces aimants est irréalisable, car les faibles dimensions des éléments constitutifs du matériau augmenteraient le nombre de cellules dans le maillage de manière prohibitive. Un moyen plus pratique est de dériver un modèle macroscopique en utilisant des techniques d'homogénéisation. Nous allons dans cette sous-section introduire la théorie de l'homogénéisation, c'est-à-dire un ensemble de méthodes de moyennisation dans les équations aux dérivées partielles. En d'autres termes, l'homogénéisation cherche des paramètres effectifs (aussi appelés *homogénéisés*, ou macroscopiques) pour décrire des milieux désordonnés ou très hétérogènes. Pour une présentation plus complète, nous nous référons à [4], [11], [20], [39].

Après avoir introduit le concept central de la convergence double-échelle, nous allons présenter les travaux d'homogénéisation dans le contexte du micromagnétisme, qui se limitent au cadre périodique. Ensuite, nous présenterons l'homogénéisation stochastique.

2.1 Convergence double-échelle

L'homogénéisation a d'abord été développée pour des structures périodiques. Un concept central qui a permis de développer la théorie de l'homogénéisation périodique est la convergence double-échelle, issue d'un raisonnement formel en termes de développement asymptotique.

Dans une structure périodique, nous notons ϵ le rapport de la période sur la taille caractéristique de la structure. L'homogénéisation consiste à effectuer une analyse asymptotique lorsque ce paramètre positif tend vers zéro. La limite ainsi obtenue est dite *homogénéisée*, *effective*, ou *macroscopique*. Dans le problème homogénéisé, la forte hétérogénéité de la structure périodique d'origine est moyennée et remplacée par l'utilisation

de coefficients homogénéisés. Moralement, on remplace ainsi le matériau microscopique et hétérogène par un matériau composite équivalent.

Historiquement, la première technique utilisée a été celle du développement asymptotique. Bien que purement formel, ce type de raisonnement permet d'obtenir le modèle macroscopique correspondant au modèle microscopique, tout en visualisant d'où proviennent les termes conservés à une échelle microscopique.

La solution du problème microscopique situé dans un domaine borné \mathcal{D} de \mathbb{R}^d dépend de ϵ . Soit u^ϵ une telle solution. L'hypothèse de départ est de supposer que la solution u^ϵ est donnée par un développement en série de ϵ , dit à *deux échelles*, du type

$$u^\epsilon(x) = u_0\left(x, \frac{x}{\epsilon}\right) + \epsilon u_1\left(x, \frac{x}{\epsilon}\right) + \epsilon^2 u_2\left(x, \frac{x}{\epsilon}\right) + \dots,$$

où chaque terme $u_k(x, y)$ est une fonction de deux variables $x \in \mathcal{D}$ et $y \in (0, 1)^d$, qui est la cellule standard reproduite périodiquement dans le domaine \mathcal{D} , conférant à la solution sa structure périodique. La variable x est dite lente ou macroscopique, tandis que y est dite rapide ou microscopique. Cette série est injectée dans l'équation du modèle microscopique, en considérant les variables x et $y = \frac{x}{\epsilon}$ comme indépendantes, et en utilisant la règle de dérivation composée suivante :

$$\nabla = \epsilon^{-1} \nabla_y + \nabla_x.$$

Puis, en identifiant chaque puissance de ϵ dans l'équation microscopique, on obtient une suite d'équations qui montrent, d'une part, que u_0 dépend uniquement de la variable lente x , et, d'autre part, donnent une équation macroscopique sur u_0 , dont les coefficients sont déterminés par une équation aux dérivées partielles en la variable rapide y sur le correcteur u_1 .

Le raisonnement que l'on vient de décrire est purement formel. Afin de prouver rigoureusement la convergence du modèle microscopique vers le modèle macroscopique, un outil essentiel est la convergence double-échelle. Définie pour la première fois par G. Nguetseng [51], la théorie a été développée par G. Allaire [3] et est couramment utilisée. Donnons-en la définition la plus simple, pour des fonctions ne dépendant pas du temps.

Definition 1.1. On dit qu'une suite de fonctions $(u^\epsilon)_{\epsilon>0}$ de $L^2(\mathcal{D})$ converge à double-échelle vers une limite $u_0(x, y)$ de $L^2(\mathcal{D}, L^2_{\text{per}}(\mathbb{R}^d))$, avec $L^2_{\text{per}}(\mathbb{R}^d)$ l'ensemble des fonctions L^2 périodiques de cellule de périodicité $(0, 1)^d$, si

$$\lim_{\epsilon \rightarrow 0} \int_{\mathcal{D}} u^\epsilon(x) \Psi\left(x, \frac{x}{\epsilon}\right) dx = \int_{\mathcal{D} \times (0, 1)^d} u_0(x, y) \Psi(x, y) dx dy,$$

pour tout fonction test $\Psi \in \mathcal{C}_c^\infty(\mathcal{D}, \mathcal{C}_{\text{per}}^\infty(\mathbb{R}^d))$, avec $\mathcal{C}_{\text{per}}^\infty(\mathbb{R}^d)$ l'ensemble des fonctions \mathcal{C}^∞ périodiques de cellule de périodicité $(0, 1)^d$.

La notion de convergence double-échelle prend tout son sens et son utilité dans la proposition suivante.

Proposition 1.2. De toute suite $(u^\epsilon)_\epsilon$ bornée dans $H^1(\mathcal{D})$, on peut extraire une sous-suite qui converge à double-échelle vers une fonction $u_0 \in H^1(\mathcal{D})$. De plus, il existe une fonction $u_1 \in L^2(\mathcal{D}, H^1_{\#}(\mathbb{R}^d))$ ($H^1_{\#}(\mathbb{R}^d)$ étant l'ensemble des fonctions $H^1(\mathbb{R}^d)$ $(0, 1)^d$ -périodiques) telle que la suite $(\nabla u^\epsilon)_\epsilon$ converge à double-échelle vers $\nabla u_0 + \nabla_y u_1$.

2.2 Micromagnétisme

Si on peut noter des approches précoces d'homogénéisation dans le contexte du micromagnétisme [15, 49] dès le début du vingtième siècle, ce n'est que récemment, grâce au potentiel que représentent certains matériaux composites, dont entre autres les *spring magnets*, que cette approche a été véritablement abordée. Santugini-Repiquet [58] a étudié l'homogénéisation de l'équation de Landau-Lifschitz-Gilbert dans des multicouches ferromagnétiques. Ces microstructures sont constituées d'une alternance de couches ferromagnétiques et de couches non magnétiques avec un couplage inter-couches aux interfaces, en tenant compte d'une énergie d'anisotropie de surface. Une équation homogénéisée est obtenue en utilisant la convergence double-échelle. Il a également étudié dans [57] l'homogénéisation du champ démagnétisant dans des structures périodiquement perforées avec la même méthode. Un modèle statique complet, comprenant tous les termes physiques classiques pertinents, a été dérivée dans [5], utilisant la notion de Γ -convergence de De Giorgi [26], dans des problèmes de minimisation liés à l'énergie de Brown de tels matériaux. Dans [18], l'homogénéisation de l'équation de Landau-Lifschitz-Gilbert est étudiée dans un milieu composé de deux matériaux ferromagnétiques différents présentant un fort contraste sur les propriétés magnétiques. Les auteurs utilisent la méthode de la convergence double-échelle avec un opérateur de dilatation approprié pour traiter les termes non-linéaires.

Toutes ces études ont été faites dans des contextes périodiques. Or, le caractère désordonné des *spring magnets* suggère l'étude dans un contexte stochastique.

2.3 Homogénéisation stochastique

Les premiers travaux portant sur l'homogénéisation stochastique remontent à 1981. Kozlov [41], Papanicolaou et Varadhan [54] ont étudié l'équation de la conduction thermique dans un milieu conducteur hétérogène aléatoire et ont obtenu un premier résultat d'homogénéisation stochastique pour des conductivités stationnaires et ergodiques. Kozlov [42] et Künnemann [44] ont étudié le problème analogue dans un cadre discret, c'est-à-dire la diffusion sur un réseau \mathbb{Z}^d avec des conductivités de liaison aléatoires. Ces derniers ont prouvé qu'à la limite d'homogénéisation, une conductivité effective émerge décrite par la formule

$$\xi \cdot a_{\text{hom}} \xi = \mathbb{E} ((\xi + \nabla \varphi_\xi) \cdot a(\xi + \nabla \varphi_\xi)) , \forall \xi \in \mathbb{R}^d , \quad (1.10)$$

où a_{hom} est la conductivité effective, symétrique, et déterministe, et $a(x)$ représente les conductivités de liaison : c'est un champ aléatoire sur le réseau \mathbb{Z}^d de matrices diagonales, stationnaire et ergodique. La formule d'homogénéisation implique un *correcteur*, qui dépend linéairement de ξ , $\varphi_\xi = \varphi_\xi(a, x)$, défini comme la solution au problème suivant

$$\begin{cases} -\operatorname{div}(a(x)(\xi + \nabla \varphi_\xi(a, x))) = 0 , \\ \nabla \varphi_\xi \text{ stationnaire} , \\ \mathbb{E}(\nabla \varphi_\xi) = 0 . \end{cases}$$

La première généralisation de la convergence à deux échelles au cadre stochastique, en termes de moyenne, a été proposée pour la première fois en 1994 dans [14] dans le cas des

structures non-singulières, mais les auteurs obtiennent un résultat de convergence applicable uniquement sur une moyenne de toutes les réalisations, et non sur une réalisation particulière. Au lieu de cela, nous utilisons dans le premier chapitre une théorie développée douze ans plus tard par Zhikov et Pyatniskii dans [65], qui donne l'existence d'une limite double-échelle presque-sûrement. Cette théorie est basée sur l'hypothèse que l'espace de probabilité est un espace métrique compact, avec des mesures aléatoires de Radon. Cela peut être considéré comme un handicap de la théorie, car les problèmes d'application ne donnent pas forcément lieu à une mesure aléatoire, mais plutôt à une géométrie aléatoire. Cette géométrie aléatoire peut être définie sur un espace de probabilité métrique compact, mais parfois, il se peut qu'aucun modèle concret de cet espace de probabilité ne soit donné. Dans notre cadre, nous allons procéder par compactification d'un espace abstrait.

Cette théorie commence à avoir un écho non négligeable. On peut citer par exemple le contexte des problèmes mécaniques continus [34, 55], et des processus stochastiques [48]. Notamment, Heida dans [32] propose un cadre qui part des géométries aléatoires, dans lequel [65] s'inscrit naturellement.

Actuellement, l'homogénéisation stochastique *quantitative* est un champ de recherche très actif. En effet, une question naturelle est d'étudier l'approximation de la conductivité effective. Par ergodicité, l'espérance dans (1.10) peut être remplacée par la moyenne spatiale, qui ne demande qu'une seule réalisation de a :

$$\lim_{p \rightarrow \infty} \frac{1}{p} \sum_{x \in ([0, p] \cap \mathbb{Z})^d} (\xi + \nabla \varphi_\xi(x)) \cdot a(x) (\xi + \nabla \varphi_\xi(x)). \quad (1.11)$$

Par ailleurs, pour les coefficients stationnaires et périodiques, le domaine infini \mathbb{Z}^d du problème du correcteur peut être remplacé par le domaine fini $(\mathbb{Z} \cap [0, N])^d$, en rajoutant une condition périodique aux bords, et (1.11) peut être calculé avec un tel correcteur. L'*approximation périodique* résulte de ces deux observations. La matrice aléatoire qui en résulte est en général non déterministe, puisque la périodisation détruit généralement l'ergodicité. Une convergence presque certaine peut être démontrée assez directement. Cependant, ces arguments ne donnent aucun taux de convergence.

Otto et Gloria ont développé dans une série d'articles [27–29] des méthodes quantitatives qui permettent d'estimer les erreurs d'approximation. Notamment, dans [29] ils montrent que l'erreur d'approximation globale pour les coefficients i.i.d. décroît avec le taux du théorème limite central. Leurs outils sont la dynamique de Glauber, l'étude des moments élevés du correcteur, des estimés de Green, et la régularisation des exposants spectraux. Une seconde approche, portée notamment par Armstrong, Kuusi et Mourrat [7], concerne la régularité des opérateurs élliptiques aléatoires.

3 Résumé par chapitre

La thèse consiste en trois chapitres qui sont en grande partie indépendants. Dans le chapitre 2, nous étudions l'homogénéisation stochastique de l'équation de Landau-Lifschitz afin d'obtenir un modèle homogène et déterministe pour les spring magnets, à partir d'un modèle hétérogène et aléatoire. Le troisième chapitre étudie le comportement d'un aimant permanent homogène, mais non uniforme. Enfin, le quatrième chapitre

s'intéresse au calcul effectif des coefficients homogénéisés, c'est-à-dire à la résolution des problèmes du correcteur issus du second chapitre.

Chapitre 2 : Homogénéisation stochastique de l'équation de Landau-Lifschitz-Gilbert L'objectif de ce chapitre est de caractériser l'évolution de l'aimantation à l'intérieur d'un matériau ferromagnétique composite, en considérant que les différents composés sont distribués de manière aléatoire à l'échelle microscopique à l'intérieur du composite macroscopique. Autrement dit, nous étudions l'homogénéisation stochastique de l'équation de Landau-Lifschitz-Gilbert. Ce problème présente plusieurs difficultés : seule l'existence de solutions faibles est garantie, et elles ne sont pas uniques [6, 64], la contrainte $|\mathbf{m}(x, t)| = 1$ est non convexe, et l'équation est fortement non-linéaire. Le modèle partage de nombreuses propriétés communes avec l'équation des applications harmoniques, et nous montrerons la méthodologie en traitant d'abord cet exemple (section 4).

Notre outil principal est une généralisation stochastique de la méthode de convergence à deux échelles, proposée dans [65] dans un contexte plus général. Afin que le présent document soit autonome, nous limitons l'approche de [65] à un cadre qui est suffisant pour nos besoins et nous présentons dans la section 2 une introduction détaillée à ce sujet. De plus, nous montrons comment faire une compactification de l'espace de probabilité afin de permettre à notre problème de remplir les conditions requises par ce cadre théorique.

Ensuite, nous traitons l'exemple classique des équations elliptiques (section 3), puis l'équation des applications harmoniques (section 4) pour illustrer la méthodologie qui sera utilisée dans l'équation de Landau-Lifschitz-Gilbert (section 5). Dans la sous-section 5.2, pour nous permettre d'étudier tous les termes compris dans l'énergie magnétique, et notamment le terme de champ démagnétisant, nous étendons la définition de cette convergence double-échelle stochastique afin d'obtenir un résultat sur la convergence double-échelle du champ démagnétisant (Proposition 2.30). Finalement, nous obtenons un théorème de convergence des solutions faibles de l'équation de Landau-Lifschitz vers une équation de Landau-Lifschitz dont les coefficients sont totalement identifiés (Théorème 2.34). Deux d'entre eux, à savoir le coefficient correspondant au terme d'échange, et un coefficient qui vient du champ démagnétisant, sont obtenus par la résolution d'un problème de cellule sur l'espace de probabilité, tandis que les autres consistent en des calculs d'espérance. Il est remarquable que l'énergie d'échange possède le même coefficient homogénéisé que celui qui apparaît dans l'homogénéisation des problèmes elliptiques, et ne tient donc pas compte de la contrainte.

La plupart du contenu de ce chapitre est à paraître dans une publication soumise, en collaboration avec François Alouges, Anne de Bouard et Benoît Merlet.

Chapitre 3. Nucléation dans les *spring magnets* Dans ce chapitre, nous étudions le comportement d'un aimant homogène, et notamment le processus de déstabilisation d'un état précédemment stable, appelé la *nucléation*. Suite à une étude de l'énergie magnétique (sous-section 1.1), ainsi qu'une justification via l'homogénéisation que l'étude des spring magnets peut se ramener à l'étude d'un aimant homogène (sous-section 2), nous identifions l'équation qui caractérise la nucléation (sous-section 1.2), qui consiste en

un problème aux valeurs propres. Nous étudions le cas particulier de domaines où les vecteurs constants sont à l'équilibre, et on étudie leur déstabilisation. Ces domaines sont les ellipsoïdes (sous-section 3.1) et les cylindres infinis (sous-section 3.2).

Notre étude de ce problème aux valeurs propres se fait dans ces deux cas selon une méthode de décomposition similaire, en étudiant des espaces stables pour le champ démagnétisant. Nous retrouvons des comportements connus par les physiciens, à savoir la déstabilisation à l'unisson dans le cas des ellipsoïdes, le *buckling* dans le cas des cylindres infinis pour les petites particules, et le *curling* pour les particules plus grandes.

Il est notable que, contrairement à la littérature sur le sujet, notre étude des cylindres infinis n'est pas limitée aux cylindres à section circulaire, bien que nous puissions dans le cas des sections circulaires, donner des résultats plus précis, mais inclut des cylindres à section quelconque.

Chapitre 4. Approximation des coefficients homogénéisés L'objectif de ce chapitre est de calculer numériquement les coefficients qui apparaissent dans l'équation de Landau-Lifschitz homogénéisée. Les deux coefficients que l'on cherche à homogénéiser sont calculés à partir de deux équations aux dérivées partielles similaires sur un espace infini. Après une introduction des quantités que nous cherchons à approcher et les problèmes posés par leur approximation en Section 1, nous proposons dans la Section 2 une méthode qui utilise des éléments finis. Nous avons mis en oeuvre numériquement cette méthode pour les problèmes unidimensionnels et bidimensionnels, avec des éléments finis d'ordre 1 et 2. En dimension 1, la méthode converge avec des éléments finis d'ordre 1 vers la valeur attendue, tandis qu'en dimension 2, la méthode converge, vers une valeur proche de la valeur attendue, mais pas exactement égale. En nous inspirant largement des outils développés par Armstrong, Kuusi, et Mourrat dans [8], nous donnons une justification théorique de cette convergence. Nous observons une nette amélioration de la valeur approchée pour le second ordre, et faisons l'hypothèse qu'augmenter l'ordre des éléments finis mène à la convergence vers la valeur attendue. Enfin, notre problème étant multi-échelles, nous explorons dans la Section 3 les méthodes multi-grilles.

Chapter 2

Stochastic homogenization of Landau-Lifshitz-Gilbert equation

1 Introduction

Mathematically speaking, the problem of studying *spring magnets* is challenging because usual models are highly non-linear and the material dependence of the parameters varies at a very small scale. It is therefore a problem of homogenization. The purpose of this chapter is to characterize the dynamic problem or, in other words, the evolution of the magnetization inside the composite ferromagnet, considering that the different compounds are randomly distributed at the microscopic scale inside the macroscopic composite (compare with [5, 19]).

Spring magnets being composed of several different materials, the coefficients contained in the effective field H_{eff} of the Landau-Lifschitz-Gilbert equation

$$\begin{cases} \frac{\partial u}{\partial t} = -u \times H_{\text{eff}}(u) + \lambda u \times (u \times H_{\text{eff}}(u)) & \text{in } \mathcal{D} \times (0, T), \\ |u| = 1 & \text{in } \mathcal{D} \times (0, T) \\ u(0, \cdot) = u_0 & \text{in } \mathcal{D}, \end{cases} \quad (2.1)$$

are likely to depend on the space variable, and we assume that the different materials are magnetically strongly coupled which amounts to say that the direction of the magnetization u does not jump at the interface between two materials. Assuming furthermore that the materials are randomly distributed on a small scale ϵ , we are led to consider the Landau-Lifschitz-Gilbert equation with random coefficients : the effective field H_{eff}^ϵ becomes

$$H_{\text{eff}}^\epsilon(u) = \text{div}(a^\epsilon \nabla u) + \mu_0 M^\epsilon H_d(M^\epsilon u) + \mu_0 M^\epsilon h_a - \nabla_u \varphi^\epsilon(u), \quad (2.2)$$

with $a^\epsilon(x) = a(\frac{x}{\epsilon}, \omega)$, $M^\epsilon(x) = M_s(\frac{x}{\epsilon}, \omega)$, $\varphi^\epsilon(x, u) = \varphi_{\text{an}}(\frac{x}{\epsilon}, \omega, u)$, and ω an element of a probability space. The problem we wish to solve is therefore the stochastic homogenization of the Landau-Lifschitz-Gilbert equation, or in other words, passing to the limit in (2.1) with the effective field defined by (2.2) as ϵ goes to 0. Notice that the problem possesses several difficulties that make it not obvious: global solutions of (2.1) are only

known to exist weakly and are not unique [6, 64], the constraint $|u(x, t)| = 1$ is not convex and the equation is highly non linear. It is well known that the model shares many common properties with the harmonic map equation, and we will show the methodology by treating this example first.

In this chapter, we use a theory developed in [65], which allows us to realize the proposed program: we prove that, almost-surely, and up to extraction, u^ϵ converges weakly in $H^1((0, T) \times \mathcal{D})$ to a weak solution of (2.1) where the effective coefficients are fully identified.

The chapter is organized as follows. In Section 2, we recall the probabilistic setting and introduce the stochastic two-scale convergence. We briefly recall the essential properties of this theory that will be needed for our purpose. Then, the applications are presented: after studying the classical example of stochastic homogenization of elliptic equations in Section 3, Section 4 is devoted to the harmonic map equation. Eventually, in Section 5, the Landau-Lifschitz-Gilbert equation is considered.

2 Stochastic two-scale convergence

A stochastic generalization of the two-scale convergence method, was proposed in [65] in a very general context. In order to have the present paper self-contained, we restrict the approach of [65] to a framework that is sufficient for our needs and we present a complete setting. We hope that the reader will find here a comprehensive introduction to this method.

2.1 The stochastic framework

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a standard probability space with \mathcal{F} a complete σ -algebra and let us consider a d -dimensional random field $a(x, \omega) \in \mathbb{R}^{d \times d}$, defined for x in \mathbb{R}^d and ω in Ω . We consider a group action of \mathbb{R}^d on the set Ω , for $x \in \mathbb{R}^d$, we note T_x the action on Ω and we call it the translation of vector x . We assume that a and T satisfy the following classical assumptions:

- (H1) Compatibility: the mapping $(x, \omega) \mapsto T_x \omega$ is measurable from $\mathbb{R}^d \times \Omega$ into Ω . Moreover, for every x in \mathbb{R}^d and ω in Ω , $a(\cdot, T_x \omega) = a(x + \cdot, \omega)$.
- (H2) Stationarity: for every x in \mathbb{R}^d , for every k in \mathbb{N} , for every Borelian \mathcal{B} of $(\mathbb{R}^{d \times d})^k$, for every y_1, \dots, y_k in \mathbb{R}^d ,

$$\mathbb{P} \{ (a(y_1, T_x \omega), \dots, a(y_k, T_x \omega)) \in \mathcal{B} \} = \mathbb{P} \{ (a(y_1, \omega), \dots, a(y_k, \omega)) \in \mathcal{B} \}.$$

- (H3) Ergodicity: the only measurable sets that are translation invariant (that is, $A \in \mathcal{F}$ such that (up to a null subset) $T_x A = A$ for every $x \in \mathbb{R}^d$) have null or full measure.

(H4) The random field a is symmetric, uniformly bounded and uniformly elliptic:

$$\exists c_1, c_2 > 0; \forall \omega \in \Omega, \forall \xi \in \mathbb{R}^d, \forall x \in \mathbb{R}^d, c_1 |\xi|^2 \leq \xi \cdot a(x, \omega) \xi \leq c_2 |\xi|^2.$$

(H5) The random field a is *stochastically continuous*: for every x in \mathbb{R}^d ,

$$\forall \epsilon > 0, \lim_{y \rightarrow x} \mathbb{P}(\|a(y, \cdot) - a(x, \cdot)\|_{\mathbb{R}^{d \times d}} \geq \epsilon) = 0.$$

As we shall see, the assumption (H5) allows us to restrict ourselves to the case where Ω is a compact metric space, which is the assumption used in [65]. In fact, all the results given in this paper are applicable if (H5) is replaced by assumptions (H5'-a)–(H5'-c) below.

(H5'-a) Ω is a compact metric space and \mathcal{F} is the completion of its Borel σ -algebra.

(H5'-b) The mapping $\omega \mapsto a(0, \omega)$ is continuous on Ω .

(H5'-c) The group action of \mathbb{R}^d on $(\Omega, \mathcal{F}, \mathbb{P})$ defined by $a(\cdot, T_x \omega) = a(x + \cdot, \omega)$ defines a continuous action of \mathbb{R}^d on $L^1(\Omega)$. Namely, for every $\Phi \in L^1(\Omega)$, $\Phi \circ T_x$ also belongs to $L^1(\Omega)$ and moreover the mapping $x \mapsto \Phi \circ T_x$ is continuous from \mathbb{R}^d into $L^1(\Omega)$.

Example 2.1. A class of examples satisfying assumptions (H1)–(H5), introduced in [8], can be constructed using a Poisson point process. Let us recall that a Poisson point process on a measurable space (E, \mathcal{E}) with intensity measure λ , is a random subset Π of E such that the following properties hold:

- for every measurable set A of \mathcal{E} , the number of points in $\Pi \cap A$, denoted by $N(A)$, follows a Poisson law with mean $\lambda(A)$,
- for every pairwise disjoint measurable sets A_1, \dots, A_k of \mathcal{E} , the random variables $N(A_1), \dots, N(A_k)$ are independent.

We now consider the case where Π is a Poisson point process on \mathbb{R}^d with intensity measure λ being the Lebesgue measure, defined on the Borel sets of \mathbb{R}^d , and a_0, a_1 are two distinct matrices in the set

$$\{\tilde{a} \in \mathbb{R}_{\text{sym}}^{d \times d} : \forall \xi \in \mathbb{R}^d, c_1 |\xi|^2 \leq \xi \cdot \tilde{a} \xi \leq c_2 |\xi|^2\},$$

with $c_1, c_2 > 0$. We define a random field a by setting, for every $x \in \mathbb{R}^d$, and every $\omega \in \Omega$,

$$a(x, \omega) = \begin{cases} a_0 & \text{if } \text{dist}(x, \Pi(\omega)) \leq \frac{1}{2}, \\ a_1 & \text{otherwise.} \end{cases}$$

Let us choose $0 < \epsilon < \|a_0 - a_1\|_{\mathbb{R}^{d \times d}}$. Then

$$\limsup_{y \rightarrow x} \mathbb{P}(\|a(y, \cdot) - a(x, \cdot)\|_{\mathbb{R}^{d \times d}} \geq \epsilon) = \mathbb{P}\left(\text{dist}(x, \Pi) = \frac{1}{2}\right) = 0,$$

and (H4) is satisfied. Defining $T_x \omega$ for $x, \omega \in \mathbb{R}^d, \Omega$ such that $\Pi(T_x \omega) = \Pi(\omega) - x$, hypotheses (H1), (H2) and (H3) are obviously satisfied.

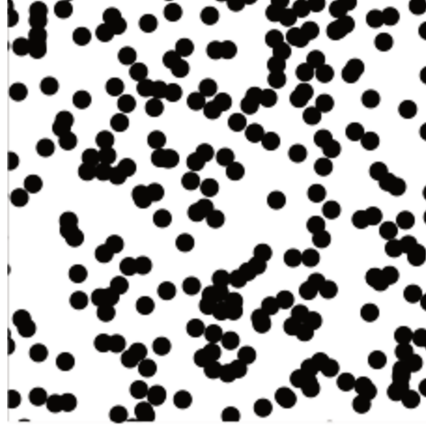


Figure 2.1: A sample of the coefficient field defined by the homogeneous Poisson point cloud. The matrix a is equal to a_0 in the black region and to a_1 in the white region. Illustration from [8].

In the next three propositions, we prove that (H1)–(H5) imply (H5') in the case where a is real valued for which the demonstration is simpler. The general case follows from direct modifications that we leave to the reader. (H5'-a) is established in Proposition 2.2, (H5'-b) in Proposition 2.3 and (H5'-c) in Proposition 2.5.

Hence we consider in what follows a real valued random field a satisfying assumptions (H1)–(H5). Let us start with a technical lemma.

Lemma 2.1. *Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a standard probability space with \mathcal{A} a complete σ -algebra and $b(x, \omega)$, x in \mathbb{R}^d , ω in Ω , be a real bounded random field. Let \mathcal{N} be a dense countable subset of \mathbb{R}^d and $\mathcal{F} \subset \mathcal{A}$ be the σ -algebra generated by $\{b(y, \cdot), y \in \mathcal{N}\}$. Assume that for every x in \mathbb{R}^d , and for every positive ϵ ,*

$$\lim_{y \rightarrow x} \mathbb{P}(|b(y, \cdot) - b(x, \cdot)| \geq \epsilon) = 0.$$

Then b is measurable with respect to \mathcal{F} .

Proof. For simplicity, we only show that the set $A = \{\omega \in \Omega : b(0, \omega) > 0\}$ is an element of \mathcal{F} . Let $(x_n)_{n \in \mathbb{N}}$ be a sequence of elements of \mathcal{N} that converges to 0. By assumption, for every $\epsilon > 0$,

$$\lim_{n \rightarrow \infty} \mathbb{P}(|b(x_n) - b(0)| > \epsilon) = 0.$$

By a diagonal argument and up to extraction, we can assume that for every n in \mathbb{N} ,

$$\mathbb{P}(|b(x_n) - b(0)| > 2^{-n}) \leq 2^{-n}.$$

Let us denote $E = \limsup_n \{\omega \in \Omega : |b(x_n, \omega) - b(0, \omega)| > 2^{-n}\}$. Borel-Cantelli lemma implies $\mathbb{P}(E) = 0$. Now, for $\omega \in \Omega \setminus E$, we have $b(0, \omega) = \liminf_n b(x_n, \omega)$, so that

$$\begin{aligned} A \setminus E &= \left\{ \omega \in \Omega \setminus E : \liminf_{n \rightarrow \infty} b(x_n, \omega) > 0 \right\} \\ &= \bigcap_{k > 0} \left\{ \omega \in \Omega : \liminf_{n \rightarrow \infty} b(x_n, \omega) \geq \frac{1}{k} \right\} \setminus E. \end{aligned}$$

Hence, A is an element of \mathcal{F} . □

We now build a compact metric space K and a random process $\tilde{a}(\cdot, f)$ indexed by $f \in K$ with the same law as $a(\cdot, \omega)$.

Let \mathcal{N} be a dense countable subset of \mathbb{R}^d containing 0. Let c_1, c_2 in \mathbb{R} be such that for every x in \mathbb{R}^d and ω in Ω , $c_1 \leq a(x, \omega) \leq c_2$ and let us set

$$K = \{\{f(x)\}_{x \in \mathcal{N}} : \forall x \in \mathcal{N}, f(x) \in [c_1, c_2]\} = [c_1, c_2]^{\mathcal{N}}.$$

Let $\{\alpha_x\}_{x \in \mathcal{N}}$ be a summable family of positive numbers, we define a distance on K by

$$d(f, g) = \sum_{x \in \mathcal{N}} \alpha_x |f(x) - g(x)|. \quad (2.3)$$

According to Tikhonov theorem, K equipped with the distance d is a compact metric space.

Now let us define a probability measure on the Borelians of K by

$$\mu(B) = \mathbb{P}\{\omega \in \Omega : (a(x, \omega))_{x \in \mathcal{N}} \in B\},$$

which is well defined by measurability of a on \mathcal{F} . Completing the Borelians with respect to μ , we obtain a σ -algebra \mathcal{E} . The space (K, \mathcal{E}, μ) is the canonical space for $\{a(x), x \in \mathcal{N}\}$.

Finally, let us define \tilde{a} , for every f in K , as follows: for every x in \mathcal{N} ,

$$\tilde{a}(x, f) = f(x).$$

Applying Lemma 2.1 to a , it follows that the family $(a(x), x \in \mathbb{R}^d)$ is a deterministic function of $(a(x), x \in \mathcal{N})$. Thus, we can define $\tilde{a}(x)$ for x in \mathbb{R}^d , such that the same goes. We easily check that \tilde{a} has the same law as a . We have established (H5'a), namely:

Proposition 2.2. *The real-valued random field $\tilde{a}(x)$, $x \in \mathbb{R}^d$ over the probability space (K, \mathcal{E}, μ) has the same law as a . Moreover (K, d) is a compact metric space and \mathcal{E} is the completion of its Borel sets with respect to μ .*

We check that (H5'-b) is satisfied.

Proposition 2.3. *Under the assumptions (H1)–(H5) and with the notation introduced in the previous proposition, $\tilde{a}(0, \cdot)$ is continuous on K for the distance d defined by (2.3).*

Proof. Let $f \in K$. Since $0 \in \mathcal{N}$, we have for every g in K ,

$$\alpha_0 |\tilde{a}(0, f) - \tilde{a}(0, g)| \leq \sum_{x \in \mathcal{N}} \alpha_x |\tilde{a}(x, f) - \tilde{a}(x, g)| = d(f, g).$$

As $\alpha_0 > 0$, it follows that $\tilde{a}(0, \cdot)$ is continuous in f . □

In order to prove (H5'-c) in Proposition 2.5 we first establish the following.

Lemma 2.4. *The space $C_f(K)$ of continuous functions on $K = [c_1, c_2]^{\mathcal{N}}$ that only depend on a finite number of variables is dense in $C(K)$. Similarly, $C_f(K)$ is dense in $L^p(K, \mu)$ for any $1 < p < \infty$.*

Proof. Let us enumerate $\mathcal{N} = \{x_1, x_2, \dots\}$. Let $\Phi \in C(K)$ and $N > 0$. For $f \in K$, we note $f_N = \Pi_N f$ the element of K defined as $f_N(x_j) = f(x_j)$ for $j = 1, \dots, N$ and $f_N(x_j) = c_1$ for $j > N$. The mapping Π_N is continuous from K into K and only depends on the first N variables. Next, we define $\Phi_N := \Phi \circ \Pi_N$ and by composition Φ_N belongs to $C_f(K)$. Let m be the modulus of continuity of Φ . We have

$$\begin{aligned} \|\Phi_N - \Phi\|_\infty &= \sup \{|\Phi(f) - \Phi(\Pi_N f)| : f \in K\} \\ &\leq \sup \{m(d(f, \Pi_N f)) : f \in K\} \\ &\leq m \left(|c_2 - c_1| \sum_{x \in \mathcal{N} \setminus \{x_1, \dots, x_N\}} \alpha_x \right) \xrightarrow{N \uparrow \infty} 0. \end{aligned}$$

We conclude that $C_f(K)$ is dense into $C(K)$. Eventually the density of $C_f(K)$ in $L^p(K)$ follows from that of $C(K)$ in $L^p(K)$ for $p > 1$. \square

Let us define the dynamical system $\{T_x : K \rightarrow K\}_{x \in \mathbb{R}^d}$ by

$$\tilde{a}(y, T_x f) = \tilde{a}(y + x, f),$$

for every $f \in K$ and $x, y \in \mathbb{R}^d$. Let us notice that for every x in \mathbb{R}^d , T_x is uniquely defined thanks to Lemma 2.1.

By definition, for every x, y in \mathbb{R}^d , $T_{x+y} = T_x \circ T_y$ and $T_0 = \text{Id}_K$. Moreover, the stationarity assumption (H2) implies for every event A of \mathcal{E} , for every x in \mathbb{R}^d ,

$$\mu(T_x^{-1}(A)) = \mu(A). \quad (2.4)$$

Eventually, (H3) gives that T is ergodic: if an event A of \mathcal{E} is T -invariant, then $\mu(A) = 0$ or $\mu(A) = 1$.

Let us now check that T complies to (H5'-c).

Proposition 2.5. *Let $\Phi \in L^1(K)$, then $\Phi \circ T_x$ belongs to $L^1(K)$ for $x \in \mathbb{R}^d$ and*

$$\lim_{y \rightarrow x} \int_K |\Phi(T_y f) - \Phi(T_x f)| d\mu(f) = 0, \text{ for every } x \in \mathbb{R}^d, \text{ and } \Phi \in L^1(K). \quad (2.5)$$

Remark 2.1. With the same arguments (see the proof below), we may also prove that for any $\Phi \in L^p(K)$, the mapping $x \mapsto \Phi \circ T_x$ is continuous from \mathbb{R}^d into $L^p(K)$.

Proof. Let $\Phi \in L^1(K)$. By measurability of $T_x : K \rightarrow K$, we see that $\Phi \circ T_x$ is measurable, moreover by the stationarity property (2.4), we have

$$\mu\{\Phi \circ T_x \in B\} = \mu\{\Phi \in B\}$$

for every $x \in \mathbb{R}^d$ and every Borel set $B \subset \mathbb{R}$. In particular, $\Phi \circ T_x \in L^1(K)$ with $\|\Phi \circ T_x\|_{L^1(K)} = \|\Phi\|_{L^1(K)}$.

Let us now check the continuity of $x \in \mathbb{R}^d \mapsto \Phi \circ T_x$. By density of $C(K)$ in $L^1(K)$, we can assume that Φ is continuous (hence bounded) in K and in fact by Lemma 2.4, we can assume that Φ only depends on a finite number of variables. Eventually, by stationarity

we only have to check the continuity at $x = 0$. So, let us assume that for every f in K , $\Phi(f) = \varphi(f(x_1), \dots, f(x_N))$ with $\varphi \in C([c_1, c_2]^N)$ and $x_1, \dots, x_N \in \mathcal{N}$. Let $\eta > 0$, for $y \in \mathbb{R}^d$ we denote

$$K_1(y) = \{f \in K : |(f(x_j + y) - f(x_j))_{j=1, \dots, N}| < \eta\}, \quad K_2(y) = K \setminus K_1(y).$$

Decomposing the integration on K over K_1 and K_2 , we have

$$\int_K |\Phi(T_y f) - \Phi(f)| d\mathbb{P}(f) \leq m(\eta) + 2\|\varphi\|_\infty \mu(K_2(y)),$$

where m is the modulus of continuity of φ . Now, by assumption (H5), $\mu(K_2(y))$ tends to 0 as y tends to 0 and since $\eta > 0$ is arbitray, we see that the integral in the left hand side also goes to 0. \square

From now on, we assume that $(\Omega, \mathcal{F}, \mathbb{P})$ is the canonical space associated with a , and that it satisfies (H1)-(H5'). As for every x in \mathbb{R}^d and ω in Ω , $a(x, \omega) = a(0, T_x \omega)$, for simplicity we will denote $a(x, \omega) = a(T_x \omega)$. By (H5'), it holds that a is in the space of continuous functions defined on Ω , which will be denoted $C(\Omega)$.

Remark 2.2. In Section 2.2 below, we introduce the notion of two-scale convergence and use test functions in $L^2(\mathcal{D} \times \Omega)$ where \mathcal{D} is a bounded domain of \mathbb{R}^d . In the theory developed thereafter, we use the continuous embedding of $C_c(\mathcal{D} \times \Omega)$ into $L^2(\mathcal{D} \times \Omega)$, which is a consequence of the finiteness of the measure $\lambda \otimes \mu$ in $\mathcal{D} \times \Omega$. The following fact is more crucial: since Ω is compact, the Banach space $(C_c(\mathcal{D} \times \Omega), \|\cdot\|_\infty)$ is separable, so there exists a countable subset $\Gamma \subset C_c(\mathcal{D} \times \Omega)$ which is dense in $(C_c(\mathcal{D} \times \Omega), \|\cdot\|_\infty)$ and in $L^2(\mathcal{D} \times \Omega)$.

We end this section by recalling the Birkhoff ergodic theorem, introduced in [12], and extended in [23] to the continuous setting. that plays a prominent role in the analysis. We first check that the quantities involved are well defined.

Lemma 2.6. *Let $p \geq 1$ and u be a function of $L^p(\Omega)$. Then, μ -almost surely, $x \mapsto u(T_x \omega)$ is in $L^p_{loc}(\mathbb{R}^d)$.*

Proof. For every bounded Borel set A of \mathbb{R}^d ,

$$\begin{aligned} \int_\Omega \int_A |u(T_x \omega)|^p dx d\mu(\omega) &= \int_A \int_\Omega |u(T_x \omega)|^p d\mu(\omega) dx \\ &= |A| \int_\Omega |u(\omega)|^p d\mu(\omega) < \infty, \end{aligned}$$

according to Fubini's theorem and thanks to the stationarity of a . It follows that $\int_A |u(T_x \omega)|^p dx$ is μ -almost surely finite. \square

Theorem 2.7. [Birkhoff ergodic theorem]. *Let f be a function of $L^1(\Omega)$. Then, for μ -almost every $\tilde{\omega}$ in Ω and for every bounded Borel set A of \mathbb{R}^d ,*

$$\frac{1}{t^d |A|} \int_{tA} f(T_x \tilde{\omega}) dx \xrightarrow{t \rightarrow \infty} \int_\Omega f(\omega) d\mu(\omega) = \mathbb{E}(f). \quad (2.6)$$

2.2 L^2 Two-scale convergence

Let us start by noticing that the Birkhoff ergodic theorem presented above is not sufficient to obtain results valid almost surely for all functions f , and thus, cannot be sufficient to obtain an homogenization theorem with almost sure convergence of the solution. One of the reasons is the fact that the set of $\tilde{\omega}$ for which the convergence hold depends on f . Therefore, we introduce the following definition.

Definition 2.1. Let $\tilde{\omega} \in \Omega$. We say that $\tilde{\omega}$ is a *typical trajectory*, if,

$$\lim_{t \rightarrow \infty} \frac{1}{t^d |A|} \int_{tA} g(T_x \tilde{\omega}) dx = \int_{\Omega} g(\omega) d\mu(\omega) = \mathbb{E}(g),$$

for every bounded Borelian $A \subset \mathbb{R}^d$ with $|A| > 0$ and every g in $C(\Omega)$.

Proposition 2.8. Let $\tilde{\Omega}$ be the set of typical trajectories. Then $\mu(\tilde{\Omega}) = 1$.

Proof. We first notice that the compactness of Ω entails that $C(\Omega)$ (endowed with the norm $\|g\|_{\infty} = \sup_{\Omega} |g|$) is separable. We thus consider

$$\Gamma = \{g_k, k \in \mathbb{N}\}$$

a dense countable subset of $C(\Omega)$. According to Birkhoff ergodic theorem, for every k in \mathbb{N} , there exists Ω_k in \mathcal{F} such that $\mu(\Omega_k) = 1$, and $\Omega_k \subset \Omega_{k-1} \subset \dots \subset \Omega_0$ and, for every $\omega' \in \Omega_k$,

$$\lim_{t \rightarrow \infty} \frac{1}{t^d |A|} \int_{tA} g_k(T_x \omega') dx = \int_{\Omega} g_k(\omega) d\mu(\omega) = \mathbb{E}(g_k),$$

for every bounded Borel set A with $|A| > 0$. Considering $\Omega' = \bigcap_{k \in \mathbb{N}} \Omega_k$, we have $\mu(\Omega') = 1$, and it only remains to show that $\Omega' \subset \tilde{\Omega}$.

Let $\omega' \in \Omega'$, $g \in C(\Omega)$, $\epsilon > 0$ and $g' \in \Gamma$ such that $\|g' - g\|_{\infty} < \epsilon$. For every bounded Borel set $A \subset \mathbb{R}^d$ and $t > 0$,

$$\begin{aligned} & \left| \frac{1}{t^d |A|} \int_{tA} g(T_x \omega') dx - \mathbb{E}(g) \right| \\ & \leq \left| \frac{1}{t^d |A|} \int_{tA} [g - g'](T_x \omega') dx \right| + \left| \frac{1}{t^d |A|} \int_{tA} g'(T_x \omega') dx - \mathbb{E}(g') \right| + |\mathbb{E}(g' - g)| \\ & \leq 2\|g' - g\|_{\infty} + \left| \frac{1}{t^d |A|} \int_{tA} g'(T_x \omega') dx - \mathbb{E}(g') \right| \\ & \leq 3\epsilon, \end{aligned}$$

for t large enough since $g' \in \Gamma$ and $\omega' \in \Omega'$. Therefore,

$$\lim_{t \rightarrow \infty} \frac{1}{t^d |A|} \int_{tA} g(T_x \omega') dx = \int_{\Omega} g(\omega) d\mu(\omega) = \mathbb{E}(g),$$

which gives $\Omega' \subset \tilde{\Omega}$. We deduce that $\mu(\tilde{\Omega}) \geq \mu(\Omega') = 1$. □

Proposition 2.9. [Mean-value property.] *Let g be a function in $C(\Omega)$. Then, for every φ in $C_c(\mathbb{R}^d)$, for every $\tilde{\omega}$ in $\tilde{\Omega}$,*

$$\lim_{\epsilon \downarrow 0} \int_{\mathbb{R}^d} \varphi(x) g(T_{x/\epsilon} \tilde{\omega}) dx = \int_{\mathbb{R}^d} \varphi(x) dx \int_{\Omega} g(\omega) d\mu(\omega) = \mathbb{E}(g) \int_{\mathbb{R}^d} \varphi(x) dx.$$

Proof. We use Proposition 2.8 to get the result for a simple function $\varphi = \sum_{i=1}^N a_i \mathbf{1}_{A_i}$ where $(A_i)_{1 \leq i \leq N}$ are bounded Borel sets of \mathbb{R}^d , and $\mathbf{1}_{A_i}$ denotes the characteristic function of A_i . The conclusion comes from the approximation of any function $\varphi \in C_c(\mathbb{R}^d)$ by simple functions in the $L^1(\mathbb{R}^d)$ norm. \square

We now state the two-scale convergence definition and prove the main compactness theorem.

Definition 2.2. Let $\tilde{\omega} \in \tilde{\Omega}$ be fixed, let $\{v^\epsilon\}_{\epsilon \in I}$ be a family of elements of $L^2(\mathcal{D})$ indexed by ϵ in a set $I \subset (0, +\infty)$ with $0 \in \bar{I}$ and let $v \in L^2(\mathcal{D} \times \Omega)$. We say that the family of elements $\{v^\epsilon\}_{\epsilon \in I}$ *weakly two-scale converges* to v as ϵ goes to zero if, for every φ in $C_c^\infty(\mathcal{D})$ and every b in $C(\Omega)$,

$$\lim_{\epsilon \rightarrow 0} \int_{\mathcal{D}} v^\epsilon(x) \varphi(x) b(T_{x/\epsilon} \tilde{\omega}) dx = \int_{\mathcal{D}} \int_{\Omega} v(x, \omega) \varphi(x) b(\omega) d\mu(\omega) dx.$$

We write

$$v^\epsilon \in L^2(\mathcal{D}) \xrightarrow{L^2} v \in L^2(\mathcal{D} \times \Omega).$$

It is worth noticing that this definition of two-scale convergence, and thus the limit, depends on the choice of $\tilde{\omega}$ in $\tilde{\Omega}$. From now on, we assume that $\tilde{\omega} \in \tilde{\Omega}$ is fixed.

The main result of this subsection is the following theorem. It is the stochastic equivalent of the two-scale compactness theorem provided in [3] for periodic homogenization.

Theorem 2.10. *Let $\{v^\epsilon\}_{\epsilon \in I}$ be a bounded family in $L^2(\mathcal{D})$, with I as in the above definition. Then, there exist a sequence $(\epsilon_k)_{k \geq 0}$ in $I^\mathbb{N}$ that tends to zero, and v^0 in $L^2(\mathcal{D} \times \Omega)$ such that $(v^{\epsilon_k})_k$ weakly two-scale converges to v^0 .*

Proof. Let $K = \{\varphi b, \varphi \in C_c^\infty(\mathcal{D}), b \in C(\Omega)\}$ be the set of test functions and $\langle K \rangle$ be its linear span, *i.e.* $\langle K \rangle$ is the tensor product $C_c^\infty(\mathcal{D}) \otimes C(\Omega)$. Using the Cauchy-Schwarz inequality and the boundedness of $\{v^\epsilon\}$ in $L^2(\mathcal{D})$, it holds that, for every $\epsilon > 0$ and every Φ in $\langle K \rangle$,

$$\left| \int_{\mathcal{D}} v^\epsilon(x) \Phi(x, T_{x/\epsilon} \tilde{\omega}) dx \right| \leq C \left(\int_{\mathcal{D}} \Phi^2(x, T_{x/\epsilon} \tilde{\omega}) dx \right)^{1/2}.$$

Decomposing Φ as $\Phi = \sum_{p=1}^N \Phi_p$ with $\Phi_1, \dots, \Phi_N \in K$, and using Proposition 2.9, it is easily checked that

$$\lim_{\epsilon \rightarrow 0} \int_{\mathcal{D}} \Phi^2(x, T_{x/\epsilon} \tilde{\omega}) dx = \int_{\mathcal{D} \times \Omega} \Phi^2(x, \omega) dx d\mu(\omega).$$

Therefore,

$$\limsup_{\epsilon \rightarrow 0} \left| \int_{\mathcal{D}} v^\epsilon(x) \Phi(x, T_{x/\epsilon} \tilde{\omega}) dx \right| \leq C \|\Phi\|_{L^2(\mathcal{D} \times \Omega)}. \quad (2.7)$$

In particular, for every $\Phi \in \langle K \rangle$, the family $\{\int_{\mathcal{D}} v^\epsilon(x)\Phi(x, T_{x/\epsilon}\tilde{\omega})dx\}_{\epsilon>0}$ is bounded in \mathbb{R} . Recalling Remark 2.2, we pick a countable subset $\Gamma \subset \langle K \rangle$ which is both dense in $C(\mathcal{D} \times \Omega)$ and in $L^2(\mathcal{D} \times \Omega)$. Using a diagonal process, there exists a sequence $(\epsilon_k)_k$ that tends to zero, such that for every Ψ in Γ ,

$$\lim_{k \rightarrow \infty} \int_{\mathcal{D}} v^{\epsilon_k}(x)\Psi(x, T_{x/\epsilon_k}\tilde{\omega})dx = \ell(\Psi), \quad (2.8)$$

for some real number $\ell(\Psi)$. By linearity, this relation extends to the linear span $\langle \Gamma \rangle$ of Γ , the function ℓ is a linear form on $\langle \Gamma \rangle$ and by (2.7), for every $\Psi \in \langle \Gamma \rangle$, there holds

$$\ell(\Psi) \leq C\|\Psi\|_{L^2(\mathcal{D} \times \Omega)}. \quad (2.9)$$

By density of Γ in $L^2(\mathcal{D} \times \Omega)$, we see that ℓ uniquely extends as a continuous linear map $\ell : L^2(\mathcal{D} \times \Omega) \rightarrow \mathbb{R}$ satisfying (2.9). Now, let $\Phi \in K$ and let $\eta > 0$, there exists $\Psi \in \Gamma$ such that $\|\Psi - \Phi\|_\infty < \eta$. Using (2.7), we compute

$$\begin{aligned} & \left| \int_{\mathcal{D}} v^{\epsilon_k}(x)\Phi(x, T_{x/\epsilon_k}\tilde{\omega})dx - \ell(\Phi) \right| \leq \left| \int_{\mathcal{D}} v^{\epsilon_k}(x)(\Phi - \Psi)(x, T_{x/\epsilon_k}\tilde{\omega})dx \right| \\ & \quad + \left| \int_{\mathcal{D}} v^{\epsilon_k}(x)\Psi(x, T_{x/\epsilon_k}\tilde{\omega})dx - \ell(\Psi) \right| + |\ell(\Psi - \Phi)| \\ & \leq C\sqrt{|\mathcal{D}|}\|\Phi - \Psi\|_\infty + \left| \int_{\mathcal{D}} v^{\epsilon_k}(x)\Psi(x, T_{x/\epsilon_k}\tilde{\omega})dx - \ell(\Psi) \right| + C\|\Psi - \Phi\|_{L^2} \\ & \leq 2C\sqrt{|\mathcal{D}|}\eta + \left| \int_{\mathcal{D}} v^{\epsilon_k}(x)\Psi(x, T_{x/\epsilon_k}\tilde{\omega})dx - \ell(\Psi) \right|. \end{aligned}$$

Since $\Psi \in \Gamma$, the last term tends to 0 as k tends to infinity and since $\eta > 0$ is arbitrary, we obtain that (2.8) holds for every $\Psi \in K$.

Eventually, since ℓ is a continuous linear form on the Hilbert space $L^2(\mathcal{D} \times \Omega)$, by Riesz representation theorem, there exists v^0 in $L^2(\mathcal{D} \times \Omega)$ such that for every Φ in $L^2(\mathcal{D} \times \Omega)$,

$$\ell(\Phi) = \int_{\mathcal{D} \times \Omega} v^0(x, \omega)\Phi(x, \omega)dx d\mu(\omega).$$

This entails, in particular that for every φ in $C_c^\infty(\mathcal{D})$ and b in $C(\Omega)$,

$$\lim_{k \rightarrow \infty} \int_{\mathcal{D}} v^{\epsilon_k}(x)\varphi(x)b(T_{x/\epsilon_k}\tilde{\omega})dx = \ell(\varphi b) = \int_{\mathcal{D} \times \Omega} v^0(x, \omega)\varphi(x)b(\omega)dx d\mu(\omega).$$

□

Two-scale convergence is mostly a weak notion. A corresponding strong two-scale convergence exists that allows us to use weak-strong convergence properties as stated in the following.

Definition 2.3. The sequence $(v^{\epsilon_k})_{k \geq 0}$ of $L^2(\mathcal{D})$ is said to *strongly two-scale converge* to a function v^0 in $L^2(\mathcal{D} \times \Omega)$ as ϵ goes to 0 if (v^{ϵ_k}) weakly two-scale converges to v^0 and if

$$\lim_{\epsilon_k \rightarrow 0} \|v^{\epsilon_k}\|_{L^2(\mathcal{D})} = \|v^0\|_{L^2(\mathcal{D} \times \Omega)}.$$

Remark 2.3. An important example of strong convergence is the following. If (v^{ϵ_k}) converges towards some function \bar{v} strongly in $L^2(\mathcal{D})$, then, by definition, (v^{ϵ_k}) weakly two-scale converges towards v^0 defined as $v^0(x, \omega) := \bar{v}(x)$. Moreover,

$$\|v^0\|_{L^2(\mathcal{D} \times \Omega)} = \|\bar{v}\|_{L^2(\mathcal{D})} = \lim_{\epsilon_k \downarrow 0} \|v^{\epsilon_k}\|_{L^2(\mathcal{D})},$$

and (v^{ϵ_k}) strongly two-scale converges towards v^0 .

Strong two-scale convergence allows us to have a two-scale version of the weak-strong convergence principle.

Proposition 2.11. *Let $\{v^\epsilon\}_{\epsilon>0}$ and $\{u^\epsilon\}_{\epsilon>0}$ be two families of functions of $L^2(\mathcal{D})$. If $\{v^\epsilon\}_{\epsilon>0}$ strongly two-scale converges to v^0 and $\{u^\epsilon\}_{\epsilon>0}$ weakly two-scale converges to u^0 , then, for every φ in $C_c^\infty(\mathcal{D})$ and b in $C(\Omega)$,*

$$\lim_{\epsilon \rightarrow 0} \int_{\mathcal{D}} u^{\epsilon_k}(x) v^{\epsilon_k}(x) \varphi(x) b(T_{x/\epsilon_k} \tilde{\omega}) dx = \int_{\mathcal{D}} \int_{\Omega} u^0(x, \omega) v^0(x, \omega) \varphi(x) b(\omega) d\mu(\omega) dx.$$

Proof. Let $\{u^\epsilon\}_{\epsilon>0}$, $\{v^\epsilon\}_{\epsilon>0}$, u^0 , and v^0 satisfying the assumptions of the proposition. Let $\delta > 0$, there exists $\Phi \in C_c^\infty(\mathcal{D}) \otimes C(\Omega)$ such that

$$\|\Phi - v^0\|_{L^2(\mathcal{D} \times \Omega)}^2 \leq \delta^2.$$

Since $v^\epsilon \xrightarrow{L^2} v^0$ with strong convergence, and using Proposition 2.9, there exists ϵ_δ such that for $0 < \epsilon < \epsilon_\delta$,

$$\begin{aligned} \left| \int_{\mathcal{D}} \Phi(x, T_{x/\epsilon} \tilde{\omega})^2 dx - \|\Phi\|_{L^2(\mathcal{D} \times \Omega)}^2 \right| &\leq \delta^2, \\ \left| \int_{\mathcal{D}} v^\epsilon(x) \Phi(x, T_{x/\epsilon} \tilde{\omega}) dx - \int_{\mathcal{D}} \int_{\Omega} v^0(x, \omega) \Phi(x, \omega) dx d\mu(\omega) \right| &\leq \delta^2, \\ \left| \int_{\mathcal{D}} (v^\epsilon(x))^2 dx - \int_{\mathcal{D} \times \Omega} (v^0(x, \omega))^2 dx d\mu(\omega) \right| &\leq \delta^2. \end{aligned}$$

Combining the preceding estimates leads to

$$\int_{\mathcal{D}} (v^\epsilon(x) - \Phi(x, T_{x/\epsilon} \tilde{\omega}))^2 dx \leq 5\delta^2. \quad (2.10)$$

Now, let $\varphi \in C_c^\infty(\mathcal{D})$ and $b \in C(\Omega)$, we write for $\epsilon > 0$,

$$\begin{aligned} \int_{\mathcal{D}} u^\epsilon(x) v^\epsilon(x) \varphi(x) b(T_{x/\epsilon} \tilde{\omega}) dx &= \int_{\mathcal{D}} u^\epsilon(x) \Phi(x, T_{x/\epsilon} \tilde{\omega}) \varphi(x) b(T_{x/\epsilon} \tilde{\omega}) dx \\ &\quad + \int_{\mathcal{D}} u^\epsilon(x) (v^\epsilon(x) - \Phi(x, T_{x/\epsilon} \tilde{\omega})) \varphi(x) b(T_{x/\epsilon} \tilde{\omega}) dx. \end{aligned}$$

Since $u^\epsilon \xrightarrow{L^2} u^0$, there exists $\epsilon'_\delta \in (0, \epsilon_\delta]$ such that for every $0 < \epsilon < \epsilon'_\delta$,

$$\left| \int_{\mathcal{D}} u^\epsilon(x) \Phi(x, T_{x/\epsilon} \tilde{\omega}) \varphi(x) b(T_{x/\epsilon} \tilde{\omega}) dx - \int_{\mathcal{D} \times \Omega} u^0 \Phi \varphi b dx d\mu \right| \leq \delta.$$

We obtain, for $0 < \epsilon < \epsilon'_\delta$,

$$\begin{aligned}
& \left| \int_{\mathcal{D}} u^\epsilon(x) v^\epsilon(x) \varphi(x) b(T_{x/\epsilon} \tilde{\omega}) dx - \int_{\mathcal{D} \times \Omega} u^0 v^0 \varphi b dx d\mu \right| \\
& \leq \delta + \left| \int_{\mathcal{D} \times \Omega} u^0 \varphi b (\Phi - v^0) dx d\mu \right| \\
& + \left| \int_{\mathcal{D}} u^\epsilon(x) (v^\epsilon(x) - \Phi(x, T_{x/\epsilon} \tilde{\omega})) \varphi(x) b(T_{x/\epsilon} \tilde{\omega}) dx \right| \\
& \leq \delta (1 + \|u^0 \varphi b\|_{L^2(\mathcal{D} \times \Omega)}) \\
& + \left| \int_{\mathcal{D}} u^\epsilon(x) (v^\epsilon(x) - \Phi(x, T_{x/\epsilon} \tilde{\omega})) \varphi(x) b(T_{x/\epsilon} \tilde{\omega}) dx \right|.
\end{aligned}$$

To estimate the last term, we use the Cauchy-Schwarz inequality to get,

$$\begin{aligned}
& \left| \int_{\mathcal{D}} u^\epsilon(x) (v^\epsilon(x) - \Phi(x, T_{x/\epsilon} \tilde{\omega})) \varphi(x) b(T_{x/\epsilon} \tilde{\omega}) dx \right| \\
& \leq C \|u^\epsilon\|_{L^2(\mathcal{D})} \left(\int_{\mathcal{D}} (v^\epsilon(x) - \Phi(x, T_{x/\epsilon} \tilde{\omega}))^2 dx \right)^{1/2} \leq C' \delta,
\end{aligned}$$

where we have used the boundedness of $(u_\epsilon)_\epsilon$ in $L^2(\mathcal{D})$ and (2.10) to get the last inequality. Consequently, for every $0 < \epsilon < \epsilon'_\delta$,

$$\left| \int_{\mathcal{D}} u^\epsilon(x) v^\epsilon(x) \varphi(x) b(T_{x/\epsilon} \tilde{\omega}) dx - \int_{\mathcal{D} \times \Omega} u^0 v^0 \varphi b dx d\mu \right| \leq C'' \delta.$$

This proves the proposition. \square

The key ingredient in the applications of two-scale convergence to homogenization problems, as introduced for the first time in [3], is the use of a two-scale compactness theorem on $H^1(\mathcal{D})$. This compactness is used to pass to the two-scale limit in the integral formulation of the equations for both the solution and its gradient. In order to extend such property to our setting, we first need to define a space $H^1(\Omega)$.

2.3 Construction of $H^1(\Omega)$

Definition 2.4. Let u be a function of $L^2(\Omega)$. We say that u is differentiable at ω in Ω if, for every i in $\{1, \dots, d\}$, the limit

$$\lim_{\delta \rightarrow 0} \frac{u(T_{\delta e_i} \omega) - u(\omega)}{\delta} =: (D_i u)(\omega) \tag{2.11}$$

exists, where (e_1, \dots, e_d) denotes the canonical basis of \mathbb{R}^d . In this case, we note $D_\omega u = (D_1 u, \dots, D_d u)$.

Lemma 2.12. Let u be a function of $L^2(\Omega)$, differentiable at every point of Ω . Then, for every x in \mathbb{R}^d and i in $\{1, \dots, d\}$,

$$\frac{\partial}{\partial x_i} [u(T_x \omega)] = (D_i u)(T_x \omega).$$

Proof. For every x in \mathbb{R}^d and i in $\{1, \dots, d\}$,

$$\begin{aligned} \frac{\partial}{\partial x_i} [u(T_x \omega)] &= \lim_{\delta \rightarrow 0} \frac{u(T_{x+\delta e_i} \omega) - u(T_x \omega)}{\delta} \\ &= \lim_{\delta \rightarrow 0} \frac{u(T_{\delta e_i}(T_x \omega)) - u(T_x \omega)}{\delta} = (D_i u)(T_x \omega). \end{aligned}$$

□

Definition 2.5. Let $C^1(\Omega)$ be the set of functions u of Ω that are continuous and differentiable at every ω in Ω and such that for every i in $\{1, \dots, d\}$, the function $D_i u$ is continuous on Ω .

Lemma 2.13. $C^1(\Omega)$ is dense in $L^2(\Omega)$.

Proof. Since $C(\Omega)$ is dense in $L^2(\Omega)$ (see [56, Theorem 3.14]), it suffices to show that $C^1(\Omega)$ is dense in $C(\Omega)$ for the L^2 -norm.

Let $\rho \in C_c^\infty(\mathbb{R}^d, \mathbb{R}_+)$ such that $\int_{\mathbb{R}^d} \rho = 1$. We define a standard family of approximations of unity $\{\rho^\delta\}_{\delta>0}$ as $\rho^\delta(x) = (1/\delta)^d \rho(x/\delta)$ for $x \in \mathbb{R}^d$, $\delta > 0$. Now, let φ be a function in $C(\Omega)$, we consider its mollifications $\{\varphi^\delta\}$ defined for $\delta > 0$ and $\omega \in \Omega$ as,

$$\varphi^\delta(\omega) := \int_{\mathbb{R}^d} \rho^\delta(y) \varphi(T_{(-y)} \omega) dy.$$

We easily check that $\varphi^\delta \in C^1(\Omega)$ with

$$D_i \varphi^\delta(\omega) = \int_{\mathbb{R}^d} \partial_{x_i} \rho^\delta(y) \varphi(T_{(-y)} \omega) dy.$$

Using assumption (H5'-c) (see also Remark 2.1) it is also standard to check that we have

$$\|\varphi^\delta - \varphi\|_{L^2(\Omega)} \xrightarrow{\delta \downarrow 0} 0.$$

This proves the lemma. □

Lemma 2.14. Let w in $C^1(\Omega)$, then $\mathbb{E}(D_i w) = 0$ for i in $\{1, \dots, d\}$. In particular, for u, v in $C^1(\Omega)$, there holds

$$\int_{\Omega} D_i u v d\mu + \int_{\Omega} u D_i v d\mu = 0. \quad (2.12)$$

Proof. Let w in $C^1(\Omega)$ and let χ in $C_c^\infty(\mathbb{R}^d)$ such that $\int \chi dx = 1$, we have by Proposition 2.9,

$$\mathbb{E}(D_i w) = \lim_{\epsilon \downarrow 0} \int_{\mathbb{R}^d} \chi(x) D_i w(T_{x/\epsilon} \tilde{\omega}) dx.$$

On the other hand, for $\epsilon > 0$, we have, using Lemma 2.12 and integrating by parts,

$$\begin{aligned} \int_{\mathbb{R}^d} \chi(x) D_i w(T_{x/\epsilon} \tilde{\omega}) dx &= \epsilon \int_{\mathbb{R}^d} \chi(x) \frac{\partial}{\partial x_i} [w(T_{x/\epsilon} \tilde{\omega})] dx \\ &= -\epsilon \int_{\mathbb{R}^d} \frac{\partial \chi}{\partial x_i}(x) w(T_{x/\epsilon} \tilde{\omega}) dx. \end{aligned}$$

By Proposition 2.9 again, the last expression tends to 0 as ϵ goes to 0 which proves the first part of the lemma.

Now if u, v are two functions of $C^1(\Omega)$, we easily see that $w := uv$ is also in $C^1(\Omega)$ with the product rule $D_i w = D_i u v + u D_i v$. The identity (2.12) then follows from $\mathbb{E}(D_i w) = 0$. \square

We now define the weak derivatives of a function $u \in L^2(\Omega)$ by duality.

Definition 2.6. Let u in $L^2(\Omega)$, i in $\{1, \dots, d\}$ and w_i in $L^2(\Omega)$; we say that u admits w_i as weak derivative in the i^{th} direction if

$$-\int_{\Omega} u D_i v d\mu = \int_{\Omega} w_i v d\mu \quad \text{for every } v \in C^1(\Omega).$$

We note $D_i u = w_i$ the weak derivative (which is uniquely defined) in $L^2(\Omega)$, noticing that both definitions of D_i coincide for C^1 random variable. If u admits weak derivatives in $L^2(\Omega)$ in all directions, we say that u belongs to $H^1(\Omega)$ and we note $D_\omega u = (D_1 u, \dots, D_d u)$. Obviously, $H^1(\Omega)$ is a vector space. We define an inner product on $H^1(\Omega)$ as

$$(u, v)_{H^1(\Omega)} := (u, v)_{L^2(\Omega)} + (D_\omega u, D_\omega v)_{L^2(\Omega)^d}.$$

We easily see that $H^1(\Omega)$ shares several properties with the usual Sobolev space $H^1(U)$ when U is a bounded open set of \mathbb{R}^N . First, using the isometric embedding $j : u \in H^1(\Omega) \mapsto (u, D_\omega u) \in L^2(\Omega) \times L^2(\Omega, \mathbb{R}^d)$, the closed graph theorem implies that $j(H^1(\Omega))$ is a closed subspace of $L^2(\Omega) \times L^2(\Omega, \mathbb{R}^d)$, hence $(H^1(\Omega), (\cdot, \cdot)_{H^1(\Omega)})$ is a Hilbert space. Moreover, using the mollifying procedure of Lemma 2.13, we may prove that $C^1(\Omega)$ is dense in $H^1(\Omega)$ and the embedding

$$C^1(\Omega) \hookrightarrow H^1(\Omega)$$

is continuous. Eventually, notice that for u in $L^2(\Omega)$, if there exists $C \geq 0$ such that

$$\|u(T_{\delta e_i} \cdot) - u\|_{L^2(\Omega)} \leq C\delta \quad \text{for } \delta > 0 \text{ and } i \text{ in } \{1, \dots, d\}, \quad (2.13)$$

then u is in $H^1(\Omega)$. This is a classical consequence of (2.11), of the Lebesgue dominated convergence theorem and of the Riesz representation theorem. Gathering these facts, we have:

Proposition 2.15. *$H^1(\Omega)$ is a separable Hilbert space; its subspace $C^1(\Omega)$ is dense. Moreover, a function u in $L^2(\Omega)$ belongs to $H^1(\Omega)$ if and only if (2.13) holds true.*

Lemma 2.16. *Let u be a function in $H^1(\Omega)$ and let us denote by $v_\omega : x \mapsto u(T_x \omega)$ and $z_\omega : x \mapsto (D_\omega u)(T_x \omega)$. Then, μ -almost surely, v_ω belongs to $H_{loc}^1(\mathbb{R}^d)$ and $z_\omega = \nabla_x v_\omega$ almost everywhere on \mathbb{R}^d .*

Proof. Let us proceed by density. Let $(u_k)_k$ be a sequence in $(C^1(\Omega))^{\mathbb{N}}$ such that u_k converges to u , and $D_\omega u_k$ converges to $D_\omega u$ in $L^2(\Omega)$, as k tends to infinity.

For every bounded Borel set A of \mathbb{R}^d ,

$$\int_{\Omega} \int_A (u_k(T_x \omega) - v_\omega(x))^2 dx d\mu(\omega) = |A| \int_{\Omega} (u_k(\omega) - u(\omega))^2 d\mu(\omega),$$

because of stationarity. The right-hand side term converges to 0 as k goes to infinity by definition of $(u_k)_k$. As a consequence, thanks to a diagonal argument, and up to extraction, $u_k(T_x \omega)$ converges to v_ω in $L^2_{\text{loc}}(\mathbb{R}^d)$, almost surely on Ω .

We have similarly for i in $\{1, \dots, d\}$,

$$\int_{\Omega} \int_A \left(\frac{\partial}{\partial x_i} u_k(T_x \omega) - z_{\omega, i}(x) \right)^2 dx d\mu(\omega) = |A| \int_{\Omega} (D_i u_k(\omega) - D_i u(\omega))^2 d\mu(\omega),$$

according to Lemma 2.12, and the stationarity assumption. The right-hand side converges to 0 as k tends to infinity. Then, up to extraction again, $\frac{\partial}{\partial x_i} u_k(T_x \omega)$ converges to $z_{\omega, i}$ in $L^2_{\text{loc}}(\mathbb{R}^d)$ almost surely in ω , and therefore, v_ω is in $H^1_{\text{loc}}(\mathbb{R}^d)$ and $\frac{\partial}{\partial x_i} v_\omega = z_{\omega, i}$ almost everywhere on \mathbb{R}^d . \square

Proposition 2.17. *For every u in $H^1(\Omega)$,*

$$D_\omega u \equiv 0 \implies u \text{ is constant } \mu\text{-a.s.}$$

The measure μ is said to be ergodic with respect to translations.

Proof. Let $u \in H^1(\Omega)$ be such that $D_\omega u \equiv 0$. Then, according to Lemma 2.16, μ -a.s., a.e. in $x \in \mathbb{R}^n$, $\nabla_x u(T_x \omega) = 0$ so as a function of $L^2_{\text{loc}}(\mathbb{R}^d)$, $x \mapsto u(T_x \omega)$ is constant. As a consequence, μ -a.s., for every t in \mathbb{R}^d ,

$$u(\omega) = \frac{1}{t^d} \int_{[0, t]^d} u(T_x \omega) dx,$$

and according to Birkhoff theorem, the right-hand side converges μ -a.s. to $\mathbb{E}(u)$ as t goes to infinity, therefore u is constant μ -a.s. \square

Now we define spaces that will be used in the definition of the effective matrix for the homogenized problems in the sequel.

Definition 2.7. We denote by $L^2_{\text{pot}}(\Omega)$ the closure of the set $\{D_\omega u : u \in C^1(\Omega)\}$ in $L^2(\Omega, \mathbb{R}^d)$, and $L^2_{\text{sol}}(\Omega) = L^2_{\text{pot}}(\Omega)^\perp$. We define the closure of a subspace M of $L^2_{\text{loc}}(\mathbb{R}^d)$ as the set of functions such that there exists a sequence of functions of M for which every sequence of restrictions to every bounded borelian of \mathbb{R}^d converges in L^2 to this function. Thus, we also denote by $L^2_{\text{pot,loc}}(\mathbb{R}^d)$ the closure of $\{\nabla u : u \in C_c^\infty(\mathbb{R}^d)\}$ in $L^2_{\text{loc}}(\mathbb{R}^d, \mathbb{R}^d)$, and $L^2_{\text{sol,loc}}(\mathbb{R}^d)$ as the closure of $\{\sigma \in C_c^\infty(\mathbb{R}^d, \mathbb{R}^d) : \text{div } \sigma \equiv 0\}$ in $L^2_{\text{loc}}(\mathbb{R}^d, \mathbb{R}^d)$.

Definition 2.8. Let σ be a random variable in $L^2(\Omega, \mathbb{R}^d)$. We say that σ is in $H_{\text{div}}(\Omega)$ if there exists g in $L^2(\Omega)$ such that

$$\forall u \in C^1(\Omega), \int_{\Omega} \sigma \cdot D_\omega u d\mu(\omega) = - \int_{\Omega} g u d\mu(\omega).$$

In that case, we denote by $\text{div}_\omega \sigma$ the function g , and call it the divergence of σ .

Remark 2.4. It is clear that $H^1(\Omega, \mathbb{R}^d) \subset H_{\text{div}}(\Omega)$ and that for σ in $H^1(\Omega, \mathbb{R}^d)$,

$$\text{div}_\omega \sigma = \sum_{i=1}^d D_i \sigma_i,$$

where σ_i , $i = 1, \dots, d$, denote the marginals of σ . Moreover, the definition of $L_{\text{sol}}^2(\Omega)$ is equivalent to

$$L_{\text{sol}}^2(\Omega) = \{\sigma \in H_{\text{div}}(\Omega) : \text{div}_\omega \sigma \equiv 0\}.$$

Theorem 2.18. *Let σ be a random variable in $L_{\text{sol}}^2(\Omega)$. Then, μ -almost surely, the function $x \mapsto \sigma(T_x \omega)$ is in $L_{\text{sol,loc}}^2(\mathbb{R}^d)$.*

Proof. Using a mollification as in the proof of Lemma 2.13, we see that $C^1(\Omega)$ is dense in $H_{\text{div}}(\Omega)$ for the natural norm $\|\sigma\|_{H_{\text{div}}}^2 := \|\sigma\|_{L^2(\Omega)}^2 + \|\text{div}_\omega \sigma\|_{L^2(\Omega)}^2$. The proof then proceeds as that of Lemma 2.16. \square

Proposition 2.19. *The intersection between $L_{\text{pot}}^2(\Omega)$ and the space of constant functions on Ω is null. The measure μ is said to be non-degenerate.*

Proof. Let $\xi \in L_{\text{pot}}^2(\Omega)$ be a constant function, and $(u_k)_k \in (C^1(\Omega))^{\mathbb{N}}$ be a sequence such that $D_\omega u_k$ converges to ξ in $L^2(\Omega)$. Lemma 2.14 implies $\mathbb{E}(D_\omega u_k) = 0$ and therefore,

$$|\xi| = |\mathbb{E}(\xi)| = |\mathbb{E}(D_\omega u_k - \xi)| \leq \mathbb{E}(|D_\omega u_k - \xi|) \leq \|D_\omega u_k - \xi\|_{L^2(\Omega)},$$

according to Cauchy-Schwarz inequality. Sending k to infinity, the right-hand side goes to 0 which leads to $\xi \equiv 0$. \square

2.4 Two-scale convergence compactness theorem in $H^1(\mathcal{D})$

Having defined $H^1(\Omega)$, the purpose of this subsection is to provide a two-scale compactness theorem in $H^1(\mathcal{D})$ in a manner similar to the periodic case (see [3]).

Lemma 2.20. *Let $\{v^\epsilon\}_{\epsilon \in I}$ be a family of elements of $H^1(\mathcal{D})$ (with I as in Definition 2.2) such that*

(i) $\{\|v^\epsilon\|_{L^2(\mathcal{D})}\}$ is bounded

(ii) $\|\epsilon \nabla v^\epsilon\|_{L^2(\mathcal{D})} \xrightarrow{\epsilon \downarrow 0} 0$.

Then, up to an extraction, there exists $v^0 \in L^2(\mathcal{D})$ such that

$$v^\epsilon(x) \xrightarrow{L^2} v^0(x) \text{ weakly in } L^2(\mathcal{D} \times \Omega).$$

Proof. According to Theorem 2.10, there exists $\hat{v}^0 \in L^2(\mathcal{D} \times \Omega)$ such that, up to the extraction of a subsequence, $(v^\epsilon)_\epsilon$ two-scale converges weakly to $\hat{v}^0(x, \omega)$.

Let $\sigma \in (C^1(\Omega))^d$ and $\varphi \in C_c^1(\mathcal{D})$, we have for $0 < \epsilon \leq 1$,

$$\int_{\mathcal{D}} v^\epsilon(x) \varphi(x) \operatorname{div}_\omega \sigma(T_{x/\epsilon} \tilde{\omega}) dx = - \int_{\mathcal{D}} \epsilon \nabla_x (v^\epsilon \varphi)(x) \sigma(T_{x/\epsilon} \tilde{\omega}) dx.$$

Passing to the limit as ϵ goes to 0, and using Cauchy-Schwarz inequality together with assumption (ii), and Proposition 2.9, we infer

$$\int_{\mathcal{D}} \varphi(x) \left[\int_{\Omega} \hat{v}^0(x, \omega) \operatorname{div}_\omega \sigma(\omega) d\mu(\omega) \right] dx = 0.$$

As a consequence, for almost every $x \in \mathcal{D}$ and every $\sigma \in C^1(\Omega)^d$,

$$\int_{\Omega} \hat{v}^0(x, \omega) \operatorname{div}_\omega \sigma(\omega) d\mu(\omega) = 0.$$

For such x , we have by definition that $\hat{v}^0(x, \cdot)$ is in $H^1(\Omega)$ with $D_\omega \hat{v}^0(x, \cdot) = 0$ and by Proposition 2.17, this means that $\hat{v}^0(x, \omega)$ does not depend on ω up to a μ -negligible set. Setting $v^0(x) := \mathbb{E}(\hat{v}^0(x, \cdot))$, we have $v^0 \in L^2(\mathcal{D})$ and $v^0 = \hat{v}^0$ in $L^2(\mathcal{D} \times \Omega)$ so that we still have the weak two-scale convergence $v^\epsilon(x) \xrightarrow{L^2} v^0$. \square

The following compactness theorem is the most important of this section.

Theorem 2.21. *Let $\{v^\epsilon\}_{\epsilon \in I}$ be a bounded sequence of $H^1(\mathcal{D})$. Then, up to an extraction, there exist v^0 in $H^1(\mathcal{D})$ and ξ in $L^2(\mathcal{D}, L^2_{\text{pot}}(\Omega))$ such that*

$$\begin{cases} v^\epsilon \xrightarrow{L^2} v^0(x) \text{ strongly,} \\ \nabla v^\epsilon \xrightarrow{L^2} \nabla v^0(x) + \xi(x, \omega) \text{ weakly, in } L^2(\mathcal{D} \times \Omega). \end{cases}$$

Moreover, if for every $\epsilon \in I$, v^ϵ is in $H_0^1(\mathcal{D})$, then v^0 is in $H_0^1(\mathcal{D})$.

Proof. We first apply Theorem 2.10 to the family $\{v^\epsilon\}$ so that for some subsequence (ϵ_k) , (v^{ϵ_k}) two-scale converges towards some $v_0 \in L^2(\Omega \times \mathcal{D})$. According to Lemma 2.20, v^0 does not depend on ω . On the other hand, since $\{v^\epsilon\}$ is bounded in $H^1(\mathcal{D})$, up to a further extraction, we may assume that v^ϵ converges weakly in $H^1(\mathcal{D})$ and strongly in $L^2(\mathcal{D})$ to a limit $\bar{v} \in H^1(\mathcal{D})$. By Remark 2.3, we have actually, $\bar{v} = v^0$ and the sequence strongly two-scale converges to v^0 .

Next, we apply Theorem 2.10 to the family $\{\nabla v^\epsilon\}$. Extracting again, the sequence (∇v^{ϵ_k}) two-scale converges towards some element w of $L^2(\Omega \times \mathcal{D}, \mathbb{R}^d)$. Let φ in $C_c^\infty(\mathcal{D})$. Integrating by parts, we compute,

$$\int_{\mathcal{D}} \nabla v^\epsilon(x) \varphi(x) dx = - \int_{\mathcal{D}} v^\epsilon(x) \nabla \varphi(x) dx,$$

Passing to the two-scale limit, we obtain

$$\int_{\mathcal{D}} \varphi(x) \left[\int_{\Omega} w(x, \omega) d\mu(\omega) \right] dx = - \int_{\mathcal{D}} v^0 \nabla \varphi dx.$$

Therefore, the distribution ∇v^0 is given by the function

$$\nabla v^0 = \int_{\Omega} w(\cdot, \omega) d\mu(\omega),$$

which belongs to $L^2(\mathcal{D})$.

Now, for σ in $L^2_{\text{sol}}(\Omega) \cap C^1(\Omega, \mathbb{R}^d)$ and φ in $C_c^\infty(\mathcal{D})$, we compute

$$\begin{aligned} \int_{\mathcal{D}} \nabla v^\epsilon(x) \cdot \sigma(T_{x/\epsilon} \tilde{\omega}) \varphi(x) dx &= \int_{\mathcal{D}} \nabla [v^\epsilon(x) \varphi(x)] \cdot \sigma(T_{x/\epsilon} \tilde{\omega}) dx \\ &\quad - \int_{\mathcal{D}} v^\epsilon(x) \nabla \varphi(x) \cdot \sigma(T_{x/\epsilon} \tilde{\omega}) dx \\ &= - \int_{\mathcal{D}} v^\epsilon(x) \nabla \varphi(x) \cdot \sigma(T_{x/\epsilon} \tilde{\omega}) dx. \end{aligned}$$

We have used the fact that by Theorem 2.18, the mapping $x \mapsto \sigma(T_{x/\epsilon} \tilde{\omega})$ is in $L^2_{\text{sol,loc}}(\mathbb{R}^d)$. Sending ϵ to 0 and integrating by parts, we get

$$\begin{aligned} \int_{\mathcal{D}} \int_{\Omega} w(x, \omega) \cdot \sigma(\omega) \varphi(x) d\mu dx &= - \int_{\mathcal{D}} \int_{\Omega} v^0(x) \nabla \varphi(x) \cdot \sigma(\omega) d\mu(\omega) dx \\ &= \int_{\mathcal{D}} \int_{\Omega} \nabla v^0(x) \cdot \sigma(\omega) \varphi(x) d\mu(\omega) dx. \end{aligned}$$

Let us note $\xi := w - \nabla v^0$. The last identity shows that for almost every x in \mathcal{D} , $\xi(x)$ lies in $((C(\Omega))^d \cap L^2_{\text{sol}}(\Omega))^\perp$. By density of $C(\Omega)$ in $L^2(\Omega)$ we conclude that

$$\xi \text{ is in } L^2(\mathcal{D}, (L^2_{\text{sol}}(\Omega))^\perp) = L^2(\mathcal{D}, L^2_{\text{pot}}(\Omega)).$$

This proves the main part of the theorem.

Finally, let us assume moreover, that for every ϵ , v^ϵ is in $H_0^1(\mathcal{D})$. Since the subspace $H_0^1(\mathcal{D})$ is closed in $H^1(\mathcal{D})$, we see that $\bar{v} = v^0$ is also in $H_0^1(\mathcal{D})$. \square

With the tools that we just defined, we are now able to provide homogenization theorems for some linear and non-linear problems with random coefficients. To introduce the method and set the notations, we start by considering the well-known problem of elliptic equations. We then turn to non-linear problems, namely harmonic maps and the Landau-Lifschitz-Gilbert equation which do not have in general a unique solution. From now on we assume \mathcal{D} to be a bounded domain of \mathbb{R}^3 .

3 Elliptic equations

We consider the problem

$$\begin{cases} -\operatorname{div}(a(T_{x/\epsilon} \tilde{\omega}) \nabla u^\epsilon(x)) = f(x) & \text{in } \mathcal{D} \\ u^\epsilon \in H_0^1(\mathcal{D}), \end{cases} \quad (2.14)$$

with f in $L^2(\mathcal{D})$ and a given $\tilde{\omega} \in \tilde{\Omega}$ (see Definition 2.2). The weak formulation of (2.14) reads

$$\int_{\mathcal{D}} a(T_{x/\epsilon}\tilde{\omega})\nabla u^\epsilon(x) \cdot \nabla \varphi(x) dx = \int_{\mathcal{D}} f(x)\varphi(x) dx, \quad \forall \varphi \in C_c^\infty(\mathcal{D}), \quad (2.15)$$

which, thanks to Lax-Milgram theorem, is well posed and admits a unique solution.

Definition 2.9. Let us define the homogenized problem as

$$\begin{cases} -\operatorname{div} \bar{a} \nabla u^0(x) = f(x), \\ u^0 \in H_0^1(\mathcal{D}), \end{cases} \quad (2.16)$$

where \bar{a} is the 3×3 matrix defined by

$$\bar{a}\nu = \int_{\Omega} a(\omega)(\nu + \eta_\nu(\omega)) d\mu(\omega) \quad \text{for all } \nu \text{ in } \mathbb{R}^3, \quad (2.17)$$

with η_ν defined as the unique solution of the problem

$$\begin{cases} \eta_\nu \in L_{\text{pot}}^2(\Omega), \\ a(\cdot)(\nu + \eta_\nu(\cdot)) \in L_{\text{sol}}^2(\Omega). \end{cases} \quad (2.18)$$

Proposition 2.22. *The matrix \bar{a} is well defined, symmetric and positive-definite, with the same bounds as $a(x, \omega)$, that is $c_1 \operatorname{Id} \leq \bar{a} \leq c_2 \operatorname{Id}$ (see assumption (H_4)). Moreover, it satisfies, for every ν in \mathbb{R}^3 ,*

$$\bar{a}\nu \cdot \nu = \inf_{v \in L_{\text{pot}}^2(\Omega)} \int_{\Omega} a(\omega)(\nu + v(\omega)) \cdot (\nu + v(\omega)) d\mu(\omega).$$

As a consequence, the homogenized problem is well posed and admits a unique solution.

Proof. The proof is very classical. For every ν in \mathbb{R}^3 , let us define

$$\begin{aligned} Q_\nu : L_{\text{pot}}^2(\Omega) &\longrightarrow \mathbb{R} \\ v &\longmapsto \int_{\Omega} a(\omega)(\nu + v(\omega)) \cdot (\nu + v(\omega)) d\mu(\omega). \end{aligned}$$

Due to the coercivity assumption on a , Q_ν is a strongly convex quadratic functional that possesses a unique minimizer ξ_ν in $L_{\text{pot}}^2(\Omega)$. Writing the Euler-Lagrange equation for ξ_ν leads to (2.18) which shows on the one hand that $\xi_\nu = \eta_\nu$. Using (2.17) and (2.18) shows on the other hand that

$$\bar{a}\nu_1 \cdot \nu_2 = \int_{\Omega} a(\omega)(\nu_1 + \eta_{\nu_1}(\omega)) \cdot (\nu_2 + \eta_{\nu_2}(\omega)) d\mu(\omega),$$

which gives that \bar{a} is symmetric. For the coercivity of \bar{a} , we compute

$$\begin{aligned} \bar{a}\nu \cdot \nu &= \int_{\Omega} a(\omega)(\nu + \eta_\nu(\omega)) \cdot (\nu + \eta_\nu(\omega)) d\mu(\omega) \geq c_1 \mathbb{E}(|\nu + \eta_\nu(\omega)|^2) \\ &\geq c_1 |\mathbb{E}(\nu + \eta_\nu)|^2 = c_1 |\nu|^2, \end{aligned}$$

where the identity $\mathbb{E}(\eta_\nu) = 0$ comes from the fact that η_ν is in $L^2_{pot}(\Omega)$, and Lemma 2.14. Eventually, we bound $\bar{a}\nu \cdot \nu$ by using the optimality of η_ν . We have

$$\bar{a}\nu \cdot \nu = Q_\nu(\eta_\nu) \leq Q_\nu(0) = \mathbb{E}(a)|\nu|^2 \leq c_2|\nu|^2.$$

This achieves the proof of the proposition. \square

Moreover, the following lemma holds

Lemma 2.23.

$$\mathbb{E}(a^{-1})^{-1}I_d \leq \bar{a} \leq \mathbb{E}(a)I_d, \quad (2.19)$$

in the sense of non-negative symmetric matrices.

Proof. By Proposition 2.22, for every $\nu \in \mathbb{R}^3$, it holds that

$$\bar{a}\nu \cdot \nu = \inf_{v \in L^2_{pot}(\Omega)} \int_{\Omega} a(\omega)(\nu + v(\omega)) \cdot (\nu + v(\omega)) d\mu(\omega).$$

It follows immediately that $\bar{a} \leq \mathbb{E}(a)$. For every $\nu \in \mathbb{R}^3$, we have

$$\mathbb{E}(a^{-1})\nu \cdot \nu = \int_{\Omega} a^{-1}(\omega)\nu \cdot \nu d\mu(\omega) \geq c_2^{-1}|\nu|^2 > 0.$$

Thus, $\xi = \mathbb{E}(a^{-1})^{-1}\nu$ is well defined. Denoting $p = a(\nu + \eta_\nu)$, it holds that

$$\begin{aligned} 0 &\leq \int_{\Omega} (p - \xi) \cdot a^{-1}(\omega)(p - \xi) d\mu(\omega) \\ &= \bar{a}\nu \cdot \nu - 2\xi \cdot \nu + \mathbb{E}(a^{-1})\xi \cdot \xi \\ &= \bar{a}\nu \cdot \nu - \mathbb{E}(a^{-1})^{-1}\nu \cdot \nu. \end{aligned}$$

The left inequality of (2.19) follows. \square

Remark 2.5. Let us note that the effective matrix \bar{a} may not be scalar, even if a is. The following proposition gives a sufficient condition on a so that \bar{a} is scalar (see [8] exercise 1.1). The proof is similar to the proof of Proposition 2.33 below.

Proposition 2.24. *Assume that a is scalar valued and isotropic in law in the following sense: for every linear isometry τ on \mathbb{R}^3 , which maps $U = \{\pm e_i : i \in \{1, 2, 3\}\}$ onto itself, and for every A in \mathcal{F} , there holds*

$$\mu\{a(\tau(\cdot), \omega) : \omega \in A\} = \mu(A).$$

Then \bar{a} is scalar.

We are now ready to state the homogenization theorem for Problem (2.14).

Theorem 2.25. *The solution u^ϵ of the problem (2.14) converges in $L^2(\mathcal{D})$, and weakly converges in $H^1(\mathcal{D})$ towards the solution u^0 of the homogenized problem (2.16).*

Proof. Let us first prove that $(u_\epsilon)_\epsilon$ is bounded in $H^1(\mathcal{D})$. For every $\epsilon > 0$, as a is bounded on Ω , and u^ϵ satisfies the variational formulation (2.15), it holds that

$$\begin{aligned} \|\nabla u^\epsilon\|_{L^2(\mathcal{D}, \mathbb{R}^3)}^2 &\leq \frac{1}{c_1} \int_{\mathcal{D}} a(T_{x/\epsilon} \tilde{\omega}) \nabla u^\epsilon(x) \cdot \nabla u^\epsilon(x) dx \\ &= \frac{1}{c_1} \int_{\mathcal{D}} f(x) u^\epsilon(x) dx \\ &\leq \frac{1}{c_1} \|f\|_{L^2(\mathcal{D})} \|u^\epsilon\|_{L^2(\mathcal{D})} \\ &\leq \frac{C}{c_1} \|f\|_{L^2(\mathcal{D})} \|\nabla u^\epsilon\|_{L^2(\mathcal{D}, \mathbb{R}^3)}, \end{aligned}$$

according to Cauchy-Schwarz and Poincaré inequalities. Thus, the sequence $(\|\nabla u^\epsilon\|_{(L^2(\mathcal{D}))^3}^2)_{\epsilon>0}$ is bounded, and the family of functions $(u^\epsilon)_{\epsilon>0}$ is bounded in $H^1(\mathcal{D})$ (again by Poincaré inequality), independently of ϵ . According to Theorem 2.21 and Rellich Theorem, up to the extraction of a subsequence, we can assume that

$$\begin{cases} u^\epsilon(x) \xrightarrow{L^2} u^0(x) \text{ strongly in } L^2(\mathcal{D}), \\ \nabla u^\epsilon(x) \xrightarrow{L^2} \nabla u^0(x) + \xi(x, \omega) \text{ in } L^2(\mathcal{D} \times \mathbb{R}^3), \\ u^\epsilon \rightharpoonup u^0 \text{ weakly in } H^1(\mathcal{D}), \end{cases}$$

for some u^0 in $H_0^1(\mathcal{D})$ and ξ in $L^2(\mathcal{D}, L_{\text{pot}}^2(\Omega))$. Rewriting equation (2.15) with the test function $\varphi(x) = \epsilon \psi(x) v(T_{x/\epsilon} \tilde{\omega})$, with $\psi \in C_c^\infty(\mathcal{D})$ and $v \in C^1(\Omega)$, it holds that

$$\begin{aligned} \int_{\mathcal{D}} a(T_{x/\epsilon} \tilde{\omega}) \nabla u^\epsilon(x) \cdot (\epsilon v(T_{x/\epsilon} \tilde{\omega}) \nabla \psi(x) + \psi(x) D_\omega v(T_{x/\epsilon} \tilde{\omega})) dx \\ = \epsilon \int_{\mathcal{D}} f(x) \psi(x) v(T_{x/\epsilon} \tilde{\omega}) dx. \end{aligned}$$

As a is in $C(\Omega, \mathbb{R}^{3 \times 3})$, we are allowed to pass to the two-scale limit:

$$\int_{\mathcal{D}} \int_{\Omega} a(\omega) (\nabla u^0(x) + \xi(x, \omega)) \cdot \psi(x) D_\omega v(\omega) d\mu(\omega) dx = 0;$$

hence, a.e. in \mathbb{R}^3 ,

$$\int_{\Omega} a(\omega) (\nabla u^0(x) + \xi(x, \omega)) \cdot D_\omega v(\omega) d\mu(\omega) = 0.$$

We thus deduce that $a(\cdot) (\nabla u^0(x) + \xi(x, \cdot))$ is in $L_{\text{sol}}^2(\Omega)$ for a.e. $x \in \mathcal{D}$. As $\xi(x, \cdot)$ is in $L_{\text{pot}}^2(\Omega)$, it follows that $\xi(x, \cdot)$ is the unique solution to the problem (2.18) with $\nu = \nabla u_0(x)$, thus,

$$\int_{\Omega} a(\omega) (\nabla u^0(x) + \xi(x, \omega)) d\mu(\omega) = \bar{a} \nabla u^0(x).$$

Moreover, passing to the limit in the two-scale sense in Equation (2.15), it holds that for every $\varphi \in C_c^\infty(\mathcal{D})$,

$$\int_{\mathcal{D}} \int_{\Omega} a(\omega) (\nabla u^0(x) + \xi(x, \omega)) \cdot \nabla \varphi(x) d\mu(\omega) dx = \int_{\mathcal{D}} f(x) \varphi(x) dx,$$

or, in other terms,

$$\int_{\mathcal{D}} \bar{a} \nabla u^0(x) \cdot \nabla \varphi(x) dx = \int_{\mathcal{D}} f(x) \varphi(x) dx .$$

Therefore, u^0 is the unique solution of the homogenized problem (2.16). \square

4 The harmonic maps equation

The subject of harmonic maps into manifold is very rich. Giving a complete bibliography on the topic is out of the scope of this manuscript and we refer the reader to the review book [38] for a rather complete overview of existing literature on the subject. In terms of physical applications, harmonic maps into the unit sphere of \mathbb{R}^3 have many common features with models that appear when studying liquid crystals or ferromagnetic materials which are the main incentives of this study.

Harmonic maps into the unit sphere are critical points of the Dirichlet energy

$$\mathcal{E}(u) = \frac{1}{2} \int_{\mathcal{D}} a(x) \nabla u(x) \cdot \nabla u(x) dx , \quad (2.20)$$

with $a \in L^\infty(\mathcal{D}, \mathbb{R}^{3 \times 3})$, under the constraint $u \in H^1(\mathcal{D}, \mathbb{S}^2)$ and a given Dirichlet boundary condition, where we recall that \mathbb{S}^2 is the unit sphere of \mathbb{R}^3 . They thus satisfy the harmonic maps equation

$$- \operatorname{div} (a \nabla u) = (a \nabla u \cdot \nabla u) u ,$$

in the weak sense, that is to say : for all $\varphi \in C_c^\infty(\mathcal{D}, \mathbb{R}^3)$,

$$\int_{\mathcal{D}} a(x) \nabla u(x) \cdot \nabla \varphi(x) dx = \int_{\mathcal{D}} (a(x) \nabla u(x) \cdot \nabla u(x)) u(x) \cdot \varphi(x) dx . \quad (2.21)$$

Existence of non-trivial solutions to (2.21) is guaranteed by the direct method of the calculus of variations, seeking minimizers of the energy (2.20) under the constraint $u \in H^1(\mathcal{D}, \mathbb{S}^2)$ and the boundary condition.

In view of applying the methodology described before and in preparation to the more complete model of ferromagnetic materials (see section 5 below), we now consider the problem of homogenizing the harmonic maps equation when one changes the energy (2.20) to

$$\mathcal{E}_\epsilon(u) = \frac{1}{2} \int_{\mathcal{D}} a(T_{x/\epsilon} \tilde{\omega}) \nabla u^\epsilon(x) \cdot \nabla u^\epsilon(x) dx ,$$

where again $\tilde{\omega} \in \tilde{\Omega}$ is fixed, a is a matrix valued random field satisfying assumptions (H1)–(H5'), and we denote $a(\omega) = a(0, \omega)$ for simplicity, u^ϵ belongs to $H^1(\mathcal{D}, \mathbb{S}^2)$, and the product \cdot is the scalar product of matrices. Critical points of this energy satisfy the following Euler-Lagrange equation

$$\begin{cases} - \operatorname{div} (a(T_{x/\epsilon} \tilde{\omega}) \nabla u^\epsilon(x)) = (a(T_{x/\epsilon} \tilde{\omega}) \nabla u^\epsilon(x) \cdot \nabla u^\epsilon(x)) u^\epsilon(x) , \\ |u^\epsilon(x)| = 1 \text{ a.e. on } \mathcal{D} . \end{cases} \quad (2.22)$$

The above equations must be understood componentwise on u^ϵ . An equivalent weak form of (2.21) was obtained in [17] with $a = Id$, which we can adapt in our case using the same method. Therefore, we have the following proposition whose proof is left to the reader.

Proposition 2.26. *The function $u^\epsilon \in H^1(\mathcal{D}, \mathbb{S}^2)$ is a weak solution to (2.22) if and only if u^ϵ satisfies*

$$\sum_{i,j=1}^3 \int_{\mathcal{D}} a_{i,j}(T_{x/\epsilon} \tilde{\omega}) (\partial_j u^\epsilon(x) \times u^\epsilon(x)) \cdot \partial_i \varphi(x) dx = 0, \quad \forall \varphi \in H_0^1(\mathcal{D}, \mathbb{R}^3). \quad (2.23)$$

Let us consider the following homogenized problem:

$$\begin{cases} -\operatorname{div}(\bar{a} \nabla u^0) = (\bar{a} \nabla u^0 \cdot \nabla u^0) u^0, \\ |u^0(x)| = 1 \text{ a.e. on } \mathcal{D}, \end{cases} \quad (2.24)$$

where \bar{a} is the matrix defined in last section by (2.17). Likewise, weak solutions $u^0 \in H^1(\mathcal{D}, \mathbb{S}^2)$ to (2.24) are characterized by

$$\sum_{i,j=1}^3 \bar{a}_{i,j} \int_{\mathcal{D}} (\partial_j u^0(x) \times u^0(x)) \cdot \partial_i \varphi(x) dx = 0, \quad \forall \varphi \in H_0^1(\mathcal{D}, \mathbb{R}^3). \quad (2.25)$$

Theorem 2.27. *Let $(u^\epsilon)_{\epsilon>0}$ be a sequence in $H^1(\mathcal{D}, \mathbb{S}^2)$ of weak solutions of (2.22), bounded in $H^1(\mathcal{D}, \mathbb{R}^3)$. Then, up to the extraction of a subsequence, $(u^\epsilon)_{\epsilon>0}$ weakly converges in $H^1(\mathcal{D}, \mathbb{R}^3)$ to $u^0 \in H^1(\mathcal{D}, \mathbb{S}^2)$ which is a solution of the homogenized problem (2.24).*

Remark 2.6. One may remark that the convergence only holds up to the extraction of subsequences. Indeed, the weak solutions of the homogenized problem are not unique, and therefore, a family of weak solutions of (2.22) may contain several subsequences that converge to different weak solutions of the same homogenized equation (2.24). In other words, the weak solutions of the homogenized problem are not unique. Only the homogenized equation can be uniquely defined.

Proof. By assumption, the sequence $(u^\epsilon)_{\epsilon>0}$ is bounded in $H^1(\mathcal{D}, \mathbb{R}^3)$. Therefore, according to Theorem 2.21, there exist

$$u^0 \in H^1(\mathcal{D}, \mathbb{R}^3), \quad \xi \in L^2(\mathcal{D}, L^2_{\text{pot}}(\Omega, \mathbb{R}^{3 \times 3}))$$

such that, up to the extraction of subsequences,

$$\begin{cases} u^\epsilon \xrightarrow{L^2} u^0 \text{ strongly in } L^2(\mathcal{D}, \mathbb{R}^3), \\ \forall i \in \{1, 2, 3\}, \partial_i u^\epsilon \xrightarrow{L^2} \partial_i u^0 + \xi_i \text{ in } L^2(\mathcal{D} \times \Omega, \mathbb{R}^3), \\ u^\epsilon \rightharpoonup u^0 \text{ weakly in } H^1(\mathcal{D}, \mathbb{R}^3). \end{cases}$$

By Rellich Theorem $(u^\epsilon)_\epsilon$ converges to u^0 in $L^2(\mathcal{D}, \mathbb{R}^3)$ and up to the extraction of a subsequence, we may assume that $(u^\epsilon(x))_\epsilon$ converges to $u^0(x)$ for a.e. $x \in \mathcal{D}$, and therefore $|u^0(x)| = 1$ for a.e. $x \in \mathcal{D}$.

Moreover, since for every $\epsilon > 0$ and a.e. on x , $|u^\epsilon(x)| = 1$, it holds that, for every i in $\{1, 2, 3\}$, $\partial_i u^\epsilon(x) \cdot u^\epsilon(x) = \frac{1}{2} \partial_i |u^\epsilon(x)|^2 = 0$ almost everywhere on \mathcal{D} . Thus, for every φ in $C_c^\infty(\mathcal{D})$, and ψ in $C(\Omega)$,

$$\int_{\mathcal{D}} u^\epsilon(x) \cdot \partial_i u^\epsilon(x) \varphi(x) \psi(T_{x/\epsilon} \tilde{\omega}) dx = 0.$$

According to Proposition 2.11, we are allowed to pass to the limit in the above equation to get

$$\int_{\mathcal{D}} \int_{\Omega} u^0(x) \cdot (\partial_i u^0(x) + \xi_i(x, \omega)) \varphi(x) \psi(\omega) d\mu(\omega) dx = 0.$$

Noticing that $u^0 \cdot \partial_i u^0 \equiv 0$ a.e. in \mathcal{D} , we deduce that, almost everywhere on \mathcal{D} and μ -almost surely,

$$u^0(x) \cdot \xi_i(x, \omega) = 0. \quad (2.26)$$

Rewriting the variational formulation (2.23) satisfied by u^ϵ with the test function $\varphi(x) = \epsilon \psi(x) v(T_{x/\epsilon} \tilde{\omega})$, where $\psi \in C_c^\infty(\mathcal{D})$ and $v \in C^1(\Omega, \mathbb{R}^3)$, it holds that

$$\begin{aligned} \epsilon \sum_{i,j=1}^3 \int_{\mathcal{D}} a_{i,j}(T_{x/\epsilon} \tilde{\omega}) (\partial_j u^\epsilon(x) \times u^\epsilon(x)) \cdot v(T_{x/\epsilon} \tilde{\omega}) \partial_i \psi(x) dx \\ + \sum_{i,j=1}^3 \int_{\mathcal{D}} a_{i,j}(T_{x/\epsilon} \tilde{\omega}) (\partial_j u^\epsilon(x) \times u^\epsilon(x)) \cdot D_i v(T_{x/\epsilon} \tilde{\omega}) \psi(x) dx = 0. \end{aligned}$$

The sequence $(u^\epsilon)_\epsilon$ strongly two-scale converges to u^0 , $(\nabla u^\epsilon)_\epsilon$ weakly two-scale converges to ∇u^0 , and a is in $C(\Omega, \mathbb{R}^{3 \times 3})$, therefore, applying again Proposition 2.11, we are allowed to pass to the limit and obtain

$$\sum_{i,j=1}^3 \int_{\mathcal{D}} \int_{\Omega} a_{i,j}(\omega) ((\partial_j u^0(x) + \xi_j(x, \omega)) \times u^0(x)) \cdot D_i v(\omega) \psi(x) d\mu(\omega) dx = 0.$$

As a consequence, almost everywhere in \mathcal{D} , and for all $v \in C^1(\Omega, \mathbb{R}^3)$,

$$\sum_{i,j=1}^3 \int_{\Omega} a_{i,j}(\omega) ((\partial_j u^0(x) + \xi_j(x, \omega)) \times u^0(x)) \cdot D_i v(\omega) d\mu(\omega) = 0,$$

or equivalently

$$\sum_{i,j=1}^3 \int_{\Omega} a_{i,j}(\omega) (\partial_j u^0(x) + \xi_j(x, \omega)) \cdot D_i (v(\omega) \times u^0(x)) d\mu(\omega) = 0. \quad (2.27)$$

For every v in $C^1(\Omega, \mathbb{R}^3)$ and x in \mathcal{D} , let us write v as $v = \lambda u^0(x) + v^\perp \times u^0(x)$, with

$\lambda = v \cdot u^0(x) \in C^1(\Omega)$ and $v^\perp = v \times u^0(x) \in C^1(\Omega, \mathbb{R}^3)$. Then,

$$\begin{aligned}
& \int_{\Omega} a(\omega)(\nabla u^0(x) + \xi(x, \omega)) \cdot D_{\omega} v(\omega) d\mu(\omega) \\
&= \sum_{i,j=1}^3 \int_{\Omega} a_{i,j}(\omega)(\partial_j u^0(x) + \xi_j(x, \omega)) \cdot D_i v(\omega) d\mu(\omega) \\
&= \sum_{i,j=1}^3 \int_{\Omega} a_{i,j}(\omega)(\partial_j u^0(x) + \xi_j(x, \omega)) \cdot D_i(v^\perp(\omega) \times u^0(x)) d\mu(\omega) \\
&\quad + \sum_{i,j=1}^3 \int_{\Omega} a_{i,j}(\omega)(\partial_j u^0(x) + \xi_j(x, \omega)) \cdot u^0(x) D_i \lambda(\omega) d\mu(\omega) \\
&= 0,
\end{aligned}$$

according to equations (2.26) and (2.27).

It follows that $a(\cdot)(\nabla u^0(x) + \xi(x, \cdot))$ is in $L^2_{\text{sol}}(\Omega, \mathbb{R}^{3 \times 3})$. As $\xi(x, \cdot)$ is in $L^2_{\text{pot}}(\Omega, \mathbb{R}^{3 \times 3})$, it is the unique solution to the problem (2.18), with $\nu = \nabla u^0(x)$, and as a consequence,

$$\int_{\Omega} a(\omega)(\nabla u^0(x) + \xi(x, \omega)) d\mu(\omega) = \bar{a} \nabla u^0(x).$$

Here, the equation must be understood as Equation (2.17) for each of the coordinate of the vector valued function u^0 (with the same matrix \bar{a}). It is worth noticing that, as in [5], in that case, the solution ξ to the corrector equation (2.18) automatically satisfies (2.26).

Let us now show that u^0 satisfies the variational formulation of the homogenized problem, i.e. (2.25). For every $\epsilon > 0$ and φ in $C_c^\infty(\mathcal{D}, \mathbb{R}^3)$, u^ϵ satisfies

$$\sum_{i,j=1}^3 \int_{\mathcal{D}} a_{i,j}(T_{x/\epsilon} \tilde{\omega})(\partial_j u^\epsilon(x) \times u^\epsilon(x)) \cdot \partial_i \varphi(x) dx = 0.$$

Applying Proposition 2.11 again, we are allowed to pass to the two limit in the above equation, and using the Fubini Theorem, we obtain:

$$\sum_{i,j=1}^3 \int_{\mathcal{D}} \left(\left(\int_{\Omega} a_{i,j}(\omega)(\partial_j u^0(x) + \xi_j(x, \omega)) d\mu(\omega) \right) \times u^0(x) \right) \cdot \partial_i \varphi(x) dx = 0.$$

Moreover,

$$\begin{aligned}
& \sum_{i,j=1}^3 \int_{\mathcal{D}} \left(\left(\int_{\Omega} a_{i,j}(\omega)(\partial_j u^0(x) + \xi_j(x, \omega)) d\mu(\omega) \right) \times u^0(x) \right) \cdot \partial_i \varphi(x) dx \\
&= \sum_{i=1}^3 \int_{\mathcal{D}} \left(((\bar{a} \nabla u^0(x)) \cdot e_i) \times u^0(x) \right) \cdot \partial_i \varphi(x) dx \\
&= \sum_{i,j=1}^3 \bar{a}_{i,j} \int_{\mathcal{D}} (\partial_j u^0(x) \times u^0(x)) \cdot \partial_i \varphi(x) dx,
\end{aligned}$$

and therefore

$$\sum_{i,j=1}^3 \bar{a}_{i,j} \int_{\mathcal{D}} (\partial_j u^0(x) \times u^0(x)) \cdot \partial_i \varphi(x) dx = 0,$$

for every φ in $C_c^\infty(\mathcal{D}, \mathbb{R}^d)$, and therefore, by density, for every φ in $H_0^1(\mathcal{D}, \mathbb{R}^d)$, which is nothing but (2.25). Therefore, u^0 is a weak solution to (2.24). \square

5 The Landau-Lifshitz-Gilbert equation

We can finally turn to the study of the Landau-Lifshitz-Gilbert equation. First, we need to recall properly the definition of the demagnetizing field and study its two-scale convergence. This is the stochastic counterpart of what has been done in [5] and [57].

5.1 The Beppo-Levi space and the variational formulation for the demagnetizing field

A given magnetization $m \in L^2(\mathcal{D}, \mathbb{R}^3)$ generates the *demagnetizing* (or *stray*) field $H_d(m) = \nabla u_m$, where the potential u_m solves in \mathbb{R}^3

$$\Delta u_m = -\operatorname{div}(m\chi_{\mathcal{D}}), \quad (2.28)$$

where $m\chi_{\mathcal{D}}$ is the extension of m that vanishes outside \mathcal{D} .

Let us introduce the weighted Lebesgue space $L_\nu^2(\mathbb{R}^3)$ of functions u such that $\nu u \in L^2(\mathbb{R}^3)$, with $\nu(x) = (1 + |x|^2)^{-1/2}$. Associated to the scalar product $(u, v)_{L_\nu^2(\mathbb{R}^3)} = \int_{\mathbb{R}^3} uv\nu^2$, $L_\nu^2(\mathbb{R}^3)$ is a Hilbert space. We now define the Beppo-Levi space :

$$BL^1(\mathbb{R}^3) = \{u \in L_\nu^2(\mathbb{R}^3) ; \nabla u \in L^2(\mathbb{R}^3, \mathbb{R}^3)\}.$$

It is a well-known fact that $BL^1(\mathbb{R}^3)$, equipped with the scalar product

$$(u, v)_{BL^1(\mathbb{R}^3)} = \int_{\mathcal{D}} \nabla u \cdot \nabla v,$$

is a Hilbert space, and there holds a Poincaré-Hardy-type inequality : for every u in $L_\nu^2(\mathbb{R}^3)$,

$$\|u\|_{L_\nu^2(\mathbb{R}^3)} \leq 2\|\nabla u\|_{L^2(\mathbb{R}^3, \mathbb{R}^3)} = 2\|u\|_{BL^1(\mathbb{R}^3)}. \quad (2.29)$$

It is straightforward to show, by the means of Lax-Milgram theorem, the existence and uniqueness of the solution of the variational formulation associated with equation (2.28): namely to prove the existence of a potential u_m in $BL^1(\mathbb{R}^3)$ such that for all φ in $BL^1(\mathbb{R}^3)$,

$$(u_m, \varphi)_{BL^1(\mathbb{R}^3)} = \int_{\mathbb{R}^3} \nabla u_m(x) \cdot \nabla \varphi(x) dx = - \int_{\mathcal{D}} m(x) \cdot \nabla \varphi(x) dx. \quad (2.30)$$

Moreover, writing (2.30) with $\varphi = u_m$, the following stability estimate holds :

$$\|H_d(m)\|_{L^2(\mathcal{D}, \mathbb{R}^3)} = \|u_m\|_{BL^1(\mathbb{R}^3)} \leq \|m\|_{L^2(\mathcal{D}, \mathbb{R}^3)}. \quad (2.31)$$

5.2 Two-scale convergence of the demagnetizing field

The purpose of the section is to characterize the behaviour of the demagnetizing field operator under the stochastic two-scale convergence defined above. More precisely, we suppose to have a bounded sequence $(m^\epsilon)_{\epsilon>0}$ in $L^2(\mathcal{D}, \mathbb{R}^3)$ which two-scale converges to some m in $L^2(\mathcal{D} \times \Omega, \mathbb{R}^3)$, and we want to understand if the two-scale limit of the sequence $H_d(m^\epsilon)$ exists, and in the affirmative case to characterize in some analytic sense such a limit.

We recall that $\tilde{\omega}$ is a fixed typical trajectory. The first proposition is a weighted variant of the compactness result for stochastic two-scale convergence.

Proposition 2.28. *Let $(u_\epsilon)_{\epsilon>0}$ be a bounded sequence of $L^2_\nu(\mathbb{R}^3)$. Then, there exists a function \bar{u} in $L^2_\nu(\mathbb{R}^3, L^2(\Omega))$ such that, up to the extraction of a subsequence,*

$$\lim_{\epsilon \rightarrow 0} \int_{\mathbb{R}^3} u_\epsilon(x) \varphi(x) b(T_{x/\epsilon} \tilde{\omega}) dx = \int_{\mathbb{R}^3 \times \Omega} \bar{u}(x, \omega) \varphi(x) b(\omega) d\mu(\omega) dx,$$

for every φ in $C_c^\infty(\mathbb{R}^3)$ and b in $C(\Omega)$. In this case, we say that the sequence $(u_\epsilon)_{\epsilon>0}$ L^2_ν -two-scale converges to u , and we denote it by

$$u_\epsilon \xrightarrow{L^2_\nu} \bar{u}.$$

Proof. As ν^{-1} is in $L^\infty(A)$ for every bounded domain A of \mathbb{R}^3 , $(u_\epsilon)_{\epsilon>0}$ is bounded in $L^2(A)$. Applying Theorem 2.10 on a non-decreasing sequence of bounded domains covering \mathbb{R}^3 , and using a diagonal argument, we find a function \bar{u} in $L^2_{\text{loc}}(\mathbb{R}^3 \times \Omega)$ and a sub-sequence extracted from $(u_{\epsilon_k})_k$ that L^2 two-scale converges to \bar{u} in every bounded domain of \mathbb{R}^3 . For every bounded $A \subset \mathbb{R}^3$, $\varphi \in C_c^\infty(\mathbb{R}^3)$, $b \in C(\Omega)$, $k \geq 0$,

$$\begin{aligned} \left| \int_{\mathbb{R}^3} \nu(x) u_{\epsilon_k}(x) \varphi(x) b(T_{x/\epsilon} \tilde{\omega}) dx \right| &\leq \|\varphi b(T_{\cdot/\epsilon})\|_{L^2(A)} \|\nu u_{\epsilon_k}\|_{L^2(A)} \\ &\leq \|\varphi b(T_{\cdot/\epsilon})\|_{L^2(\mathbb{R}^3)} \|u_{\epsilon_k}\|_{L^2_\nu(\mathbb{R}^3)}. \end{aligned}$$

Passing to the lower limit as k goes to infinity,

$$\left| \int_{A \times \Omega} \nu(x) \bar{u}(x, \omega) \varphi(x) b(\omega) d\mu(\omega) dx \right| \leq \|\varphi\|_{L^2(\mathbb{R}^3)} \|b\|_{L^2(\Omega)} \liminf_{k \rightarrow \infty} \|u_{\epsilon_k}\|_{L^2_\nu(\mathbb{R}^3)}.$$

By density of $C(\Omega)$ in $L^2(\Omega)$ and $C_c^\infty(\mathbb{R}^3)$ in $L^2(\mathbb{R}^3)$, it follows that for every bounded $A \subset \mathbb{R}^3$,

$$\int_{A \times \Omega} \nu^2(x) \bar{u}^2(x, \omega) d\mu(\omega) dx \leq \liminf_{k \rightarrow \infty} \|u_{\epsilon_k}\|_{L^2_\nu(\mathbb{R}^3)}^2 < +\infty.$$

Therefore, \bar{u} is in $L^2_\nu(\mathbb{R}^3, L^2(\Omega))$. □

Proposition 2.29. *Let $(u_\epsilon)_{\epsilon>0}$ be a sequence in $BL^1(\mathbb{R}^3)$ weakly convergent to $u_\infty \in BL^1(\mathbb{R}^3)$. Then, $(u_\epsilon)_{\epsilon>0}$ $L^2_\nu(\mathbb{R}^3)$ -two-scale converges to u_∞ , and there exists a function v in $L^2(\mathbb{R}^3, L^2_{\text{pot}}(\Omega))$ such that, up to the extraction of a subsequence,*

$$\nabla u_\epsilon \xrightarrow{L^2} \nabla u_\infty + v.$$

Proof. We observe that since $(u_\epsilon)_{\epsilon>0}$ is weakly convergent to u_∞ in $BL^1(\mathbb{R}^3)$, by the Poincaré-Hardy inequality (2.29), $(u_\epsilon)_{\epsilon>0}$ is weakly convergent to u_∞ in $L^2_\nu(\mathbb{R}^3)$, and is bounded in $L^2_\nu(\mathbb{R}^3)$. Therefore, according to the previous proposition, there exists \bar{u} in $L^2_\nu(\mathbb{R}^3, L^2(\Omega))$ such that

$$u_\epsilon \xrightarrow{L^2_\nu} \bar{u}.$$

Considering a sequence of bounded domains covering \mathbb{R}^3 , using the same diagonal argument as in the previous proof, noticing that $(\nabla u_\epsilon)_{\epsilon>0}$ is bounded in $L^2(\mathbb{R}^3, \mathbb{R}^3)$ and using Lemma 2.20, we have that $(u_\epsilon)_\epsilon$ two-scale converges to a function $w \in L^2_{\text{loc}}(\mathbb{R}^3)$. By uniqueness of the limit in $L^2_{\text{loc}}(\mathbb{R}^3)$, it follows that $\bar{u} = w \in L^2_{\text{loc}}(\mathbb{R}^3)$ while the uniqueness of the two-scale limit implies $w = u_\infty$. Thus, $u_\infty = \bar{u} \in L^2_\nu(\mathbb{R}^3)$.

Next, we observe that since $(u_\epsilon)_{\epsilon>0}$ weakly converges to u_∞ in $BL^1(\mathbb{R}^3)$, then $(\nabla u_\epsilon)_{\epsilon>0}$ weakly converges to ∇u_∞ in $L^2(\mathbb{R}^3, \mathbb{R}^3)$. Thus, using Theorem 2.10 and uniqueness of the L^2 limit for $(\nabla u_\epsilon)_\epsilon$, there exists a function v in $L^2(\mathbb{R}^3, L^2_{\text{pot}}(\Omega))$ such that, up to the extraction of a subsequence,

$$\nabla u_\epsilon \xrightarrow{L^2} \nabla u_\infty + v \text{ in } L^2(\mathbb{R}^3, \mathbb{R}^3).$$

□

We are now ready to prove the two-scale convergence of the demagnetizing field operator.

Proposition 2.30. *Let $(m^\epsilon)_{\epsilon>0}$ be a bounded family of $L^2(\mathcal{D}, \mathbb{R}^3)$ that two-scale converges to $m \in L^2(\mathcal{D} \times \Omega, \mathbb{R}^3)$. Then, the two-scale limit of the family $(H_d(m^\epsilon))_{\epsilon>0}$ of $(L^2(\mathbb{R}^3, \mathbb{R}^3))^{\mathbb{N}}$ exists and is given by*

$$H_\infty(m)(x, \omega) = H_d(\mathbb{E}(m))(x) + w_m(x, \omega),$$

where for every x in \mathbb{R}^3 , $w_m(x, \cdot)$ is the unique solution in $L^2_{\text{pot}}(\Omega)$ of the cell problem : $\forall v \in C^1(\Omega)$,

$$\chi_{\mathcal{D}}(x) \int_{\Omega} m(x, \omega) \cdot D_\omega v(\omega) d\mu(\omega) = - \int_{\Omega} w_m(x, \omega) \cdot D_\omega v(\omega) d\mu(\omega).$$

Proof. The sequence $(m^\epsilon)_{\epsilon>0}$ is bounded in $L^2(\mathcal{D}, \mathbb{R}^3)$, therefore, according to the equation of stability (2.31), the sequence of magnetostatic potentials $(u_{m^\epsilon})_{\epsilon>0}$ solutions of $\Delta u_{m^\epsilon} = -\text{div } m^\epsilon \chi_{\mathcal{D}}$ is bounded in the Hilbert space $BL^1(\mathbb{R}^3)$. Thus, there exists a function u_∞ such that $(u_{m^\epsilon})_{\epsilon>0}$ weakly converges to u_∞ in $BL^1(\mathbb{R}^3)$ and, according to Proposition 2.29, there exists a function w_m in $L^2(\mathbb{R}^3, L^2_{\text{pot}}(\Omega))$ such that

$$\begin{cases} u_{m^\epsilon} \xrightarrow{L^2_\nu} u_\infty, \\ \nabla u_{m^\epsilon} \xrightarrow{L^2} \nabla u_\infty + w_m. \end{cases}$$

For every $\epsilon > 0$, u_{m^ϵ} satisfies the variational formulation :

$$\int_{\mathbb{R}^3} \nabla u_{m^\epsilon}(x) \cdot \nabla \varphi(x) dx = - \int_{\mathcal{D}} m^\epsilon \cdot \nabla \varphi(x) dx, \quad (2.32)$$

for every φ in $C_c^\infty(\mathbb{R}^3)$. Passing to the limit as ϵ goes to zero, it holds that

$$\int_{\mathbb{R}^3} \nabla u_\infty(x) \cdot \nabla \varphi(x) dx = - \int_{\mathcal{D}} \mathbb{E}(m)(x) \cdot \nabla \varphi(x) dx.$$

Thus, we reach the conclusion that the weak limit u_∞ is a solution of the homogenized equation

$$\Delta u_\infty = - \operatorname{div}(\mathbb{E}(m)\chi_{\mathcal{D}}) \text{ in } \mathbb{R}^3,$$

or in other words, that $H_d(\mathbb{E}(m)) = \nabla u_\infty$.

Moreover, writing the variational formulation (2.32) with the test function $\psi(x) = \epsilon \varphi(x)b(T_{x/\epsilon}\tilde{\omega})$, with φ in $C_c^\infty(\mathbb{R}^3)$ and b in $C^1(\Omega)$, it holds that, for every $\epsilon > 0$,

$$\begin{aligned} \epsilon \int_{\mathbb{R}^3} \nabla u_{m^\epsilon}(x) \cdot \nabla \varphi(x)b(T_{x/\epsilon}\tilde{\omega}) dx + \int_{\mathbb{R}^3} \nabla u_{m^\epsilon}(x) \cdot D_\omega b(T_{x/\epsilon}\tilde{\omega})\varphi(x) dx \\ = -\epsilon \int_{\mathcal{D}} m^\epsilon(x) \cdot \nabla \varphi(x)b(T_{x/\epsilon}\tilde{\omega}) dx - \int_{\mathcal{D}} m^\epsilon(x) \cdot D_\omega b(T_{x/\epsilon}\tilde{\omega})\varphi(x) dx. \end{aligned}$$

Passing to the limit as ϵ goes to zero, we get

$$\begin{aligned} \int_{\mathbb{R}^3 \times \Omega} (\nabla u_\infty(x) + w_m(x, \omega)) \cdot D_\omega b(\omega)\varphi(x) d\mu(\omega) dx \\ = - \int_{\mathcal{D} \times \Omega} m(x, \omega) \cdot D_\omega b(\omega)\varphi(x) d\mu(\omega) dx; \end{aligned}$$

since $\int_{\Omega} D_\omega b(\omega) d\mu(\omega) = 0$, w_m satisfies the so-called cell problem: for a.e. x in \mathbb{R}^3 ,

$$-\chi_{\mathcal{D}}(x) \int_{\Omega} m(x, \omega) \cdot D_\omega b(\omega) d\mu(\omega) = \int_{\Omega} w_m(x, \omega) \cdot D_\omega b(\omega) d\mu(\omega).$$

The existence and uniqueness of the solution of the cell problem is a direct consequence of Lax-Milgram Theorem. \square

5.3 Homogenization Theorem

First, we need a proper notion of two-scale convergence for time-dependent problems that do not present fast oscillations in time, which can be found in [34] or [33].

Let us set $Q_T = (0, T) \times \mathcal{D}$, with $T > 0$, and let $\tilde{\omega}$ be a fixed typical trajectory.

Definition 2.10. We say that $v^\epsilon(t, x)$ weakly two-scale converges to $v(t, x, \omega)$ in $L^2(Q_T)$ if $(v^\epsilon)_{\epsilon > 0}$ is a bounded sequence of $L^2(Q_T)$ and for every ϕ in $C_c^\infty(Q_T)$, and b in $C(\Omega)$,

$$\lim_{\epsilon \rightarrow 0} \int_{Q_T} v^\epsilon(t, x)\phi(t, x)b(T_{x/\epsilon}\tilde{\omega}) dx dt = \int_{Q_T} \int_{\Omega} v(t, x, \omega)\phi(t, x)b(\omega) d\mu(\omega) dx dt.$$

Remark 2.7. 1. This is different from the weak two-scale convergence of $v^\epsilon(t, \cdot)$ to $v(t, \cdot)$ in $L^2(\mathcal{D})$, for every $t \in (0, T)$. However, if the sequence $(\|\partial_t v^\epsilon\|_{L^2(Q_T)})_\epsilon$ is bounded, then there exists u_t , a two-scale weak limit of $(\partial_t v^\epsilon)_\epsilon$, and by integration by parts, it can be proven that for every $\tau \in (0, T)$, $(v^\epsilon(\tau))_\epsilon$ weakly two-scale converges to $v(\tau) = v(0) + \int_0^\tau u_t$, as has been done in [33].

2. The definition of two-scale convergence in $L^2(Q_T)$ corresponds to Definition 2.2, with $T_{(t,x)} = T_x$. Defined as such, T is an ergodic action that does not depend on t . In other words, for every u in $L^2(\Omega)$, ω in Ω , and t in $(0, T)$, $u(T_{(t,0)}\omega) = u(\omega)$, and therefore $D_t u = 0$. It follows that every z in $L^2_{\text{pot}}(\Omega)$ has a vanishing first coordinate, and we thus consider $L^2_{\text{pot}}(\Omega)$ as naturally embedded into $L^2(\Omega, \mathbb{R}^3)$. The results of Section 2 thus still hold in this case, among which Theorem 2.21, stated in the following proposition with the needed modifications. A more extensive proof can be found in [33].

Proposition 2.31. *Let $(v^\epsilon)_{\epsilon>0}$ be a bounded sequence of $H^1(Q_T)$. Then, there exists v^0 in $H^1(Q_T)$ and ξ in $L^2(Q_T, L^2_{\text{pot}}(\Omega))$ such that, up to the extraction of a subsequence,*

$$\begin{cases} v^\epsilon \rightharpoonup v^0 \text{ weakly in } H^1(Q_T), \\ \forall i \in \{1, 2, 3\}, \partial_i v^\epsilon \xrightarrow{L^2} \partial_i v^0 + \xi_i, \\ \frac{\partial v^\epsilon}{\partial t} \xrightarrow{L^2} \frac{\partial v^0}{\partial t}, \\ \forall t \in (0, T), v^\epsilon(t) \xrightarrow{L^2} v^0(t). \end{cases}$$

Let us consider, for every $\epsilon > 0$, the problem

$$\begin{cases} \frac{\partial u_\epsilon}{\partial t} = u_\epsilon \times H_{\text{eff}}^\epsilon(u_\epsilon) - \lambda u_\epsilon \times (u_\epsilon \times H_{\text{eff}}^\epsilon(u_\epsilon)), \\ u_\epsilon(0, x) = u_0(x), \end{cases} \quad (2.33)$$

where u_0 is a function of $H^1(Q_T, \mathbb{S}^2)$, $\lambda > 0$ is the damping coefficient, and the effective field H_{eff}^ϵ is defined as

$$H_{\text{eff}}^\epsilon(u) = \text{div}(a^\epsilon \nabla u) + \mu_0 M^\epsilon H_d(M^\epsilon u) + \mu_0 M_\epsilon h_a - \nabla_u \varphi^\epsilon(u).$$

Let us define mathematically the different terms involved in the above expression, for a fixed $\omega \in \Omega$.

- The *exchange parameter* $a^\epsilon(x, \omega) = a(T_{x/\epsilon}\omega)$, with a in $C(\Omega, \mathbb{R}^{3 \times 3})$ and T a dynamical system that satisfy (H1)-(H5').
- the *saturation magnetization* $M^\epsilon(x, \omega) = M_s(T_{x/\epsilon}\omega)$, with M_s in $C(\Omega)$,
- the *external field* h_a , in \mathbb{R}^3 , which we assume to be constant for simplicity,
- the *anisotropy density energy* $\varphi^\epsilon(x, u, \omega) = \varphi_{\text{an}}(u, T_{x/\epsilon}\omega)$, with φ_{an} in $H^1(\mathbb{S}^2, C(\Omega, \mathbb{R}))$.

The parameters a , M_s and φ_{an} are continuous functions on Ω because they depend on the materials, and the distribution of the materials is represented by the dynamical system T .

Let us now define the notion of weak solution that we will consider. The formulation is taken from [6].

Definition 2.11. Let u_ϵ be a function defined on Q_T . We say that u_ϵ is a weak solution of (2.33) if it satisfies :

(D1) the function u_ϵ is in $H^1(Q_T, \mathbb{S}^2)$,

(D2) for every ϕ in $C_c^\infty(Q_T, \mathbb{R}^3)$,

$$\begin{aligned} & \int_{Q_T} \left(\frac{\partial u_\epsilon}{\partial t}(t, x) + \lambda u_\epsilon(t, x) \times \frac{\partial u_\epsilon}{\partial t}(t, x) \right) \cdot \phi(t, x) \, dx dt \\ &= (1 + \lambda^2) \left(\sum_{i,j=1}^3 \int_{Q_T} a_{i,j}(T_{x/\epsilon}\omega) (\partial_j u_\epsilon(t, x) \times u_\epsilon(t, x)) \cdot \partial_i \phi(t, x) \, dx dt \right. \\ & \quad \left. + \int_{Q_T} (u_\epsilon \times (M^\epsilon H_d(\mu_0 M^\epsilon u_\epsilon) + \mu_0 M^\epsilon h_a - \nabla_u \varphi^\epsilon(u_\epsilon))) \cdot \phi \, dx dt \right), \end{aligned}$$

(D3) in the trace sense, $u_\epsilon(0, \cdot) = u_0$,

(D4) the Dirichlet energy is uniformly bounded independently of t : there exists a constant K (which depends on ϵ and ω *a priori*) such that, for every t in $(0, T)$,

$$\begin{aligned} \mathcal{E}_\epsilon(u_\epsilon) &:= \frac{1}{2} \int_{\mathcal{D}} a^\epsilon \nabla u_\epsilon(t) \cdot \nabla u_\epsilon(t) - \frac{\mu_0}{2} \int_{\mathcal{D}} H_d(M^\epsilon u_\epsilon)(t) \cdot M^\epsilon u_\epsilon(t) \\ & \quad - \mu_0 \int_{\mathcal{D}} h_a \cdot M^\epsilon u_\epsilon(t) + \int_{\mathcal{D}} \varphi^\epsilon(u_\epsilon(t)) + \frac{\lambda}{1 + \lambda^2} \int_0^t \int_{\mathcal{D}} \left| \frac{\partial u_\epsilon}{\partial t} \right|^2 \\ & \leq K. \end{aligned}$$

Let us notice that the function u_ϵ implicitly depends on the fixed $\omega \in \Omega$ by (D2) and (D4).

Remark 2.8. The condition (D4) is not standard : we can usually be more precise and take $K = \mathcal{E}_\epsilon(u_0)$, the initial energy. However, in the present case, the initial energy $\mathcal{E}_\epsilon(u_0)$ does not converge to the homogenized energy as ϵ goes to zero. This will be further explained in Remark 2.12.

Remark 2.9. As noted in Remark 2.6 for Theorem 2.27, the convergence only holds up to the extraction of subsequences due to the non-uniqueness of solutions of the problem.

Existence of weak solutions of the Landau-Lifshitz-Gilbert equation has been established in [6, 64], taking into account only the exchange term, using a Galerkin approximation. However, it is easy to adapt the proof to the complete equation.

In order to introduce the homogenized problem, we need to define yet another homogenized matrix.

Definition 2.12. The homogenized demagnetizing matrix is defined by : for every ν in \mathbb{R}^3 ,

$$\overline{M}\nu = -\mathbb{E}(M_s \rho_\nu), \quad (2.34)$$

where ρ_ν is the unique solution to the problem

$$\begin{cases} \rho_\nu \in L_{\text{pot}}^2(\Omega), \\ M_s(\cdot)\nu + \rho_\nu(\cdot) \in L_{\text{sol}}^2(\Omega). \end{cases} \quad (2.35)$$

Proposition 2.32. *The matrix \overline{M} is well defined, symmetric and satisfies*

$$0 \leq \overline{M} \leq \text{Var}(M_s), \quad (2.36)$$

in the sense of non-negative symmetric matrices.

Proof. For every ν in \mathbb{R}^3 , ρ_ν is well defined and unique as it is the projection on $L^2_{\text{pot}}(\Omega)$ of $-M_s\nu$, in the sense of the orthogonal projection on $L^2(\Omega)$. Since ρ_ν depends linearly on ν , \overline{M} is a matrix. Moreover, for every ν, η in \mathbb{R}^3 ,

$$\begin{aligned} \overline{M}\nu \cdot \eta &= - \int_{\Omega} M_s(\omega) \rho_\nu(\omega) \cdot \eta d\mu(\omega) \\ &= \int_{\Omega} \rho_\eta(\omega) \cdot \rho_\nu(\omega) d\mu(\omega), \end{aligned}$$

by definition of ρ_η . It follows that \overline{M} is symmetric, and in particular,

$$\overline{M}\nu \cdot \nu = \|\rho_\nu\|_{L^2(\Omega)}^2 \geq 0; \quad (2.37)$$

In addition, by definition of ρ_ν and Cauchy-Schwarz inequality,

$$\begin{aligned} \|\rho_\nu\|_{L^2(\Omega)}^2 &= - \int_{\Omega} M_s(\omega) \nu \cdot \rho_\nu(\omega) d\mu(\omega) \\ &= - \int_{\Omega} (M_s(\omega) - \mathbb{E}(M_s)) \nu \cdot \rho_\nu(\omega) d\mu(\omega) \\ &\leq \|M_s - \mathbb{E}(M_s)\|_{L^2(\Omega)} \|\rho_\nu\|_{L^2(\Omega)} |\nu|, \end{aligned}$$

since ρ_ν is in $L^2_{\text{pot}}(\Omega)$ and therefore $\int_{\Omega} \rho_\nu d\mu = 0$. It follows that

$$\|\rho_\nu\|_{L^2(\Omega)} \leq \|M_s - \mathbb{E}(M_s)\|_{L^2(\Omega)} |\nu| = (\text{Var}(M_s))^{1/2} |\nu|.$$

This gives (2.37), hence (2.36). □

Remark 2.10. We may notice that, like \bar{a} , \overline{M} may not be scalar, even if $M_s(\omega)$ is scalar for every $\omega \in \Omega$. The following proposition gives a sufficient condition for \overline{M} to be scalar.

Proposition 2.33. *Assume that M_s is scalar valued and isotropic in law in the following sense: for every linear isometry τ on \mathbb{R}^3 , which maps $U = \{\pm e_i : i \in \{1, 2, 3\}\}$ onto itself, and for every A in \mathcal{F} , there holds*

$$\mu\{M_s(\tau(\cdot), \omega) : \omega \in A\} = \mu(A).$$

Then \overline{M} is scalar.

Proof. For the proof, Ω is the canonical space for M_s .

Let τ be a linear isometry which maps U to itself. Let us define the following function on $C(\Omega)$:

$$\begin{aligned} \varphi_\tau : C(\Omega) &\longrightarrow C(\Omega) \\ v &\longmapsto (\omega \mapsto v(\omega \circ \tau)). \end{aligned}$$

It is a bijection, with $\varphi_\tau^{-1} = \varphi_{\tau^{-1}}$. For every v in $C^1(\Omega)$, i in $\{1, 2, 3\}$, $\delta > 0$, and ω in Ω , let j in $\{1, 2, 3\}$ and $\epsilon = \pm 1$ be such that $\tau(e_i) = \epsilon e_j$, then

$$\begin{aligned} \frac{\varphi_\tau(v)(T_{\delta e_i}\omega) - \varphi_\tau(v)(\omega)}{\delta} &= \frac{v(\omega(\delta\tau(e_i) + \tau(\cdot))) - v(\omega \circ \tau)}{\delta} \\ &= \epsilon \frac{v(T_{\epsilon\delta e_j}(\omega \circ \tau)) - v(\omega \circ \tau)}{\epsilon\delta}. \end{aligned}$$

Passing to the limit as δ goes to zero, it follows that, for every ω in Ω ,

$$D_i[\varphi_\tau(v)](\omega) = \epsilon D_j v(\omega \circ \tau);$$

in other terms,

$$D_\omega[\varphi_\tau(v)](\omega) = \tau(D_\omega v(\omega \circ \tau)) = (\tau \circ \varphi_\tau(D_\omega v))(\omega).$$

In particular, $\varphi_\tau(v)$ is in $C^1(\Omega)$. Thus, the restriction of φ_τ to $C^1(\Omega)$ is a bijection. For every v in $C^1(\Omega)$ and k in $\{1, 2, 3\}$, as $M_s(\omega)$ is scalar for every ω and by isotropy in law of M_s ,

$$\begin{aligned} M_s(\omega)\tau(e_k) + D_\omega\varphi_\tau(v)(\omega) &= M_s(\omega)\tau(e_k) + \tau(D_\omega v(\omega \circ \tau)) \\ &= \tau(M_s(\omega \circ \tau)e_k + D_\omega v(\omega \circ \tau)) \end{aligned}$$

as $M_s(\omega \circ \tau) = M_s(\omega)$ by assumption. As φ_τ is a bijection of $C^1(\Omega)$, and by density of $\{D_\omega v, v \in C^1(\Omega)\}$ in $L^2_{\text{pot}}(\Omega)$ it follows that $\tau \circ \varphi_\tau(\rho_{e_k}) = \rho_{\tau(e_k)}$, where η_{e_k} is defined by (2.35) with $\nu = e_k$ (see Proposition 2.22). Finally, for every k, l in $\{1, 2, 3\}$ and for every τ that maps U to itself,

$$\begin{aligned} \overline{M}\tau(e_k) \cdot \tau(e_l) &= \int_{\Omega} \rho_{\tau(e_k)}(\omega) \cdot \rho_{\tau(e_l)}(\omega) d\mu(\omega) \\ &= \int_{\Omega} \tau \circ \varphi_\tau(\rho_{e_k})(\omega) \cdot \tau \circ \varphi_\tau(\rho_{e_l})(\omega) d\mu(\omega) \\ &= \int_{\Omega} \tau(\rho_{e_k})(\omega \circ \tau) \cdot \tau(\rho_{e_l})(\omega \circ \tau) d\mu(\omega) \\ &= \overline{M}e_k \cdot e_l. \end{aligned}$$

Now, we deduce that for every k, l in $\{1, 2, 3\}$, such that $k \neq l$, let σ be a linear isometry that maps U to itself and such that $\sigma(e_k) = e_l$. Then,

$$\overline{M}e_l \cdot e_l = \overline{M}\sigma(e_k) \cdot \sigma(e_k) = \overline{M}e_k \cdot e_k.$$

Moreover, let τ be a linear isometry that maps U to itself and such that $\tau(e_k) = e_l$ and $\tau(e_l) = -e_k$. Then

$$\overline{M}e_k \cdot e_l = -\overline{M}\tau(e_l) \cdot \tau(e_k) = -\overline{M}e_l \cdot e_k = -\overline{M}e_k \cdot e_l,$$

by symmetry of \overline{M} . Therefore, $\overline{M}e_k \cdot e_l = 0$. \square

We now introduce the homogenized problem,

$$\begin{cases} \frac{\partial \bar{u}}{\partial t} = \bar{u} \times \overline{H_{\text{eff}}}(\bar{u}) - \lambda \bar{u} \times (\bar{u} \times \overline{H_{\text{eff}}}(\bar{u})) , \\ \bar{u}(0, \cdot) = u_0 , \end{cases} \quad (2.38)$$

with the homogenized effective field $\overline{H_{\text{eff}}}$, defined by

$$\overline{H_{\text{eff}}}(u) = \text{div}(\bar{a}\nabla u) + \mu_0 \mathbb{E}(M_s)^2 H_d(u) - \mu_0 \overline{M}u + \mu_0 \mathbb{E}(M_s)h_a + \nabla_u \mathbb{E}(\varphi_{\text{an}}(u)) ,$$

with \bar{a} defined by (2.17), and \overline{M} defined by (2.34).

Let \bar{u} be a function defined on Q_T . We say that \bar{u} is a weak solution of (2.38) if it satisfies a new set of conditions (D'1)-(D'4). (D'1) and (D'3) are identical to (D1) and (D3), while the variational formulation (D'2) becomes : for every ϕ in $C_c^\infty(Q_T, \mathbb{R}^3)$,

$$\begin{aligned} & \int_{Q_T} \left(\frac{\partial \bar{u}}{\partial t}(t, x) + \lambda \bar{u}(t, x) \times \frac{\partial \bar{u}}{\partial t}(t, x) \right) \cdot \phi(t, x) \, dx dt \\ &= (1 + \lambda^2) \left(\sum_{i,j=1}^3 \int_{Q_T} \bar{a}_{i,j} (\partial_j \bar{u}(t, x) \times \bar{u}(t, x)) \cdot \partial_i \phi(t, x) \, dx dt \right. \\ & \quad \left. + \int_{Q_T} (\bar{u} \times (\mu_0 \mathbb{E}(M_s)^2 H_d(\bar{u}) + \mu_0 \overline{M}\bar{u} + \mu_0 \mathbb{E}(M_s)h_a - \nabla_u \mathbb{E}(\varphi_{\text{an}}(\bar{u}))) \cdot \phi \, dx dt \right) , \end{aligned}$$

and in (D'4), the Dirichlet energy bound condition becomes : for every t in $(0, T)$,

$$\begin{aligned} \bar{\mathcal{E}}(\bar{u}) &:= \frac{1}{2} \int_{\mathcal{D}} \bar{a} \nabla \bar{u}(t) \cdot \nabla \bar{u}(t) - \frac{\mu_0 \mathbb{E}(M_s)^2}{2} \int_{\mathcal{D}} H_d(\bar{u}(t)) \cdot \bar{u}(t) + \frac{\mu_0}{2} \int_{\mathcal{D}} \bar{u}(t) \cdot \overline{M}\bar{u}(t) \\ & \quad - \mu_0 \mathbb{E}(M_s) \int_{\mathcal{D}} h_a \cdot \bar{u}(t) + \int_{\mathcal{D}} \mathbb{E}(\varphi_{\text{an}})(\bar{u}(t)) + \frac{\lambda}{1 + \lambda^2} \int_0^t \int_{\mathcal{D}} \left| \frac{\partial \bar{u}}{\partial t} \right|^2 \\ & \leq K . \end{aligned}$$

Now let us state the homogenization Theorem. Let $\tilde{\omega}$ a typical trajectory be fixed.

Theorem 2.34. *Let $\tilde{\omega} \in \tilde{\Omega}$ be a typical trajectory, and $(u_\epsilon)_{\epsilon>0}$ be a family of weak solutions of (2.33), where $\omega = \tilde{\omega}$ is taken in Definition 2.11. If the energy $\mathcal{E}_\epsilon(u^\epsilon)$ is uniformly bounded independently of ϵ , then, up to the extraction of a subsequence, the family weakly converges in $H^1(\mathcal{D}, \mathbb{R}^3)$ to u^0 , a weak solution of the homogenized problem (2.38).*

Remark 2.11. As noted in Remark 2.6 for Theorem 2.27, the convergence only holds up to the extraction of subsequences due to the non-uniqueness of solutions of the problem.

Proof. The outline of the proof is as follows :

1. Showing the existence of a two-scale limit in $H^1(\mathcal{D})$ for $(u_\epsilon)_\epsilon$. This is the object of Step 1 below.
2. Showing that this limit satisfies the four conditions of a weak limit of (2.38). This is the object of the last three steps.

Step 1. Let us first show that the sequence $(u^\epsilon)_\epsilon$ is bounded in $H^1(Q_T, \mathbb{R}^3)$ in order to apply Proposition 2.10. As for every $\epsilon > 0$, u^ϵ is in $H^1(Q_T, \mathbb{S}^2)$, $(u^\epsilon)_{\epsilon>0}$ is obviously bounded in $L^2(Q_T, \mathbb{R}^3)$. Besides, the family $(u_\epsilon)_{\epsilon>0}$ is a family of weak solutions of (2.33) such that the Dirichlet energy $\mathcal{E}_\epsilon(u_\epsilon)$ is uniformly bounded independently of ϵ . Let K be such that, for every $\epsilon > 0$ and t in $(0, T)$,

$$\mathcal{E}_\epsilon(u_\epsilon)(t) \leq K. \quad (2.39)$$

Let us first notice that for every $\epsilon > 0$ and t in $(0, T)$,

$$-\frac{1}{2} \int_{\mathcal{D}} H_d(M^\epsilon u_\epsilon)(t, x) \cdot M^\epsilon u_\epsilon(t, x) dx \geq 0. \quad (2.40)$$

Indeed, for every m in $L^2(\mathcal{D}, \mathbb{R}^3)$, if $H_d(m) = \nabla u_m$, with $u_m \in BL1(\mathbb{R}^3)$, then

$$\begin{aligned} - \int_{\mathcal{D}} m(x) \cdot H_d(m)(x) dx &= - \int_{\mathcal{D}} m(x) \cdot \nabla u_m(x) dx \\ &= \int_{\mathbb{R}^3} \nabla u_m(x) \cdot \nabla u_m(x) dx \\ &\geq 0, \end{aligned}$$

by definition of u_m (see (2.30)).

Moreover, as M_s is continuous on a compact space and therefore bounded, and since almost everywhere on Q_T , for every ϵ , $|u_\epsilon(x, t)| = 1$, we have for every t in $(0, T)$,

$$- \int_{\mathcal{D}} h_a \cdot M^\epsilon u_\epsilon(t) = - \int_{\mathcal{D}} h_a \cdot M_s(T_{x/\epsilon} \tilde{\omega}) u_\epsilon(x, t) dx \leq |h_a| |\mathcal{D}| \sup_{\Omega} |M_s|.$$

It follows that, for every $\epsilon > 0$, using equations (2.39), (2.40) and the ellipticity of a ,

$$\begin{aligned} \|u_\epsilon\|_{\dot{H}^1(Q_T, \mathbb{R}^3 \times \mathbb{S}^2)}^2 &= \int_{Q_T} |\nabla u_\epsilon|^2 + \int_{Q_T} \left| \frac{\partial u_\epsilon}{\partial t} \right|^2 \\ &\leq \frac{1}{c_1} \int_0^T \int_{\mathcal{D}} a(T_{x/\epsilon} \tilde{\omega}) \nabla u_\epsilon \cdot \nabla u_\epsilon dx dt + \int_{Q_T} \left| \frac{\partial u_\epsilon}{\partial t} \right|^2 \\ &\leq \left(\frac{2T}{c_1} + 1 \right) (K + |h_a| |\mathcal{D}| \sup_{\Omega} |M_s| + |\mathcal{D}| \sup_{\mathbb{S}^2 \times \Omega} |\varphi_{\text{an}}|). \end{aligned}$$

As a consequence, according to Proposition 2.10 and Rellich Theorem, there exists \bar{u} in $H^1(Q_T, \mathbb{R}^3)$, ξ in $(L^2(Q_T, L^2_{\text{pot}}(\Omega)))^3$ such that, up to the extraction of a subsequence,

$$\left\{ \begin{array}{l} u_\epsilon \xrightarrow{L^2} \bar{u}, \\ \forall i \in \{1, 2, 3\}, \partial_i u_\epsilon \xrightarrow{L^2} \partial_i \bar{u} + \xi_i, \\ \frac{\partial u_\epsilon}{\partial t} \xrightarrow{L^2} \frac{\partial \bar{u}}{\partial t}, \\ \forall t \in (0, T), u^\epsilon(t) \xrightarrow{L^2} \bar{u}(t), \\ u_\epsilon \xrightarrow{L^2(Q_T, \mathbb{R}^3)} \bar{u} \text{ strongly}, \\ u_\epsilon \xrightarrow{H^1(Q_T, \mathbb{R}^3)} \bar{u} \text{ weakly}. \end{array} \right. \quad (2.41)$$

Step 2. Let us show that \bar{u} satisfies (D'3).

For every $\epsilon > 0$, $u_\epsilon(0, \cdot) = u_0$ in the trace sense, and since the trace is weakly continuous,

$$\bar{u}(0, \cdot) = \lim_{\epsilon \rightarrow 0} u_\epsilon(0, \cdot) = u_0,$$

in the trace sense.

Step 3. Let us show that \bar{u} satisfies (D'1) and (D'2).

Re-writing the variational formulation (D2) using the test function $\phi(t, x) = \epsilon \psi(t, x) v(T_{x/\epsilon} \tilde{\omega})$, with $\psi \in C_c^\infty(Q_T)$ and $v \in C^1(\Omega, \mathbb{R}^3)$, it holds that

$$\begin{aligned} & \epsilon \int_{Q_T} \left(\frac{\partial u_\epsilon}{\partial t} + \lambda u_\epsilon \times \frac{\partial u_\epsilon}{\partial t} \right) \cdot v(T_{x/\epsilon} \tilde{\omega}) \psi(t, x) dt dx \\ &= (1 + \lambda^2) \left(\epsilon \sum_{i,j=1}^3 \int_{Q_T} a_{i,j}^\epsilon (\partial_j u_\epsilon \times u_\epsilon) \cdot v(T_{x/\epsilon} \tilde{\omega}) \partial_i \psi(t, x) dt dx \right. \\ & \quad + \sum_{i,j=1}^3 \int_{Q_T} a_{i,j}^\epsilon (\partial_j u_\epsilon \times u_\epsilon) \cdot D_i v(T_{x/\epsilon} \tilde{\omega}) \psi(t, x) dt dx \\ & \quad + \epsilon \int_{Q_T} (u_\epsilon(t, x) \times (\mu_0 M^\epsilon H_d(M^\epsilon u_\epsilon) + \mu_0 M^\epsilon h_a \\ & \quad \left. - \nabla_u \varphi^\epsilon(u_\epsilon))) \cdot v(T_{x/\epsilon} \tilde{\omega}) \psi(t, x) dt dx \right). \end{aligned} \quad (2.42)$$

The sequence $(u_\epsilon)_{\epsilon > 0}$ is a bounded family of $L^2(Q_T, \mathbb{R}^3)$ that two-scale converges to $\bar{u} \in L^2(Q_T, \mathbb{R}^3)$, therefore, since M_s is in $C(\Omega)$, by definition of the two-scale convergence, we have

$$M^\epsilon(x) u_\epsilon(t, x) = M_s(T_{x/\epsilon} \tilde{\omega}) u_\epsilon(t, x) \xrightarrow{L^2} M_s(\omega) \bar{u}(t, x).$$

Then, according to Proposition 2.30, the two-scale limit of the family $(H_d(M^\epsilon u_\epsilon))_{\epsilon > 0}$ exists and is given by

$$H_\infty(M_s \bar{u}(t))(x, \omega) = \mathbb{E}(M_s) H_d(\bar{u}(t))(x) + w(t, x, \omega), \quad (2.43)$$

where for every (t, x) in Q_T , $w(t, x, \cdot)$ is the unique solution in $L^2_{\text{pot}}(\Omega)$ of the cell problem

$$\chi_{\mathcal{D}}(x) \int_{\Omega} M_s(\omega) \bar{u}(t, x) \cdot D_\omega v(\omega) d\mu(\omega) = - \int_{\Omega} w(t, x, \omega) \cdot D_\omega v(\omega) d\mu(\omega),$$

for all v in $C^1(\Omega)$. Put differently, for every (t, x) in Q_T , $w(t, x, \cdot)$ is the unique solution of the cell problem (2.35) for $\nu = \chi_{\mathcal{D}}(x) \bar{u}(t)$. It follows by Definition 2.12 that

$$\mathbb{E}(M_s w(t, x, \cdot)) = -\chi_{\mathcal{D}}(x) \overline{M} \bar{u}(t, x). \quad (2.44)$$

Moreover, the sequence $(u_\epsilon)_{\epsilon > 0}$ strongly converges to \bar{u} in $L^2(Q_T, \mathbb{R}^3)$. In particular, $\|u_\epsilon\|_{L^2(Q_T, \mathbb{R}^3)}$ converges to $\|\bar{u}\|_{L^2(Q_T, \mathbb{R}^3)}$ as ϵ goes to zero. As $(u_\epsilon)_{\epsilon > 0}$ weakly two-scale

converges to \bar{u} , it follows that $(u_\epsilon)_{\epsilon>0}$ strongly two-scale converges to \bar{u} . Thus, we are allowed to pass to the limit in equation (2.42), which yields

$$\sum_{i,j=1}^3 \int_{Q_T} \int_{\Omega} a_{i,j}(\cdot) ((\partial_j \bar{u}(t, x) + \xi_j(t, x, \cdot)) \times \bar{u}(t, x)) \cdot D_i v(\cdot) \psi(t, x) d\mu dx dt = 0. \quad (2.45)$$

Moreover, almost everywhere on Q_T , $|u_\epsilon(t, x)| = 1$, so that $|\bar{u}(t, x)| = 1$. On one hand, we have proven that \bar{u} satisfies (D'1), and on the other hand, it follows that, for every i in $\{1, 2, 3\}$, φ in $C_c^\infty(Q_T)$, and ψ in $C^1(\Omega)$,

$$\int_{Q_T} u_\epsilon(t, x) \cdot \partial_i u_\epsilon(t, x) \varphi(x) \psi(T_{x/\epsilon} \tilde{\omega}) dx dt = 0.$$

As the sequence $(u_\epsilon)_{\epsilon>0}$ strongly two-scale converges to \bar{u} and $(\partial_i u_\epsilon)_\epsilon$ weakly two-scale converges to $\partial_i \bar{u}$, we can pass to the limit in the above equation to get

$$\int_{Q_T} \int_{\Omega} \bar{u}(t, x) \cdot (\partial_i \bar{u}(t, x) + \xi_i(t, x, \omega)) \varphi(t, x) \psi(\omega) d\mu(\omega) dx dt = 0.$$

Consequently, almost everywhere on Q_T and μ -almost surely,

$$\bar{u}(t, x) \cdot \xi_i(t, x, \omega) = 0. \quad (2.46)$$

Decomposing the test function v in (2.45) along and orthogonally to \bar{u} and using the same arguments as in the proof of Theorem 2.27, we deduce from (2.45) and (2.46) that $a(\cdot)(\nabla \bar{u}(t, x) + \xi(t, x, \cdot))$ is in $(L_{\text{sol}}^2(\Omega))^3$. Thus, as $\xi(t, x, \cdot)$ is in $(L_{\text{pot}}^2(\Omega))^3$, $\xi(t, x, \cdot)$ is the unique solution to the problem (2.18), with $\nu = \nabla \bar{u}(t, x)$, and hence

$$\int_{\Omega} a(\omega)(\nabla \bar{u}(t, x) + \xi(t, x, \omega)) d\mu(\omega) = \bar{a} \nabla \bar{u}(t, x). \quad (2.47)$$

Let us now show that \bar{u} satisfies the variational formulation of the homogenized problem (2.38). For every $\epsilon > 0$ and ϕ in $C_c^\infty(Q_T, \mathbb{R}^3)$, u_ϵ is a weak solution of (2.33), therefore it satisfies the variational formulation :

$$\begin{aligned} & \int_{Q_T} \left(\frac{\partial u_\epsilon}{\partial t}(t, x) + \lambda u_\epsilon(t, x) \times \frac{\partial u_\epsilon}{\partial t}(t, x) \right) \cdot \phi(t, x) dx dt \\ &= (1 + \lambda^2) \left(\sum_{i,j=1}^3 \int_{Q_T} a_{i,j}(T_{x/\epsilon} \tilde{\omega}) (\partial_j u_\epsilon(t, x) \times u_\epsilon(t, x)) \cdot \partial_i \phi(t, x) dx dt \right. \\ & \left. + \int_{Q_T} (u_\epsilon(t, x) \times (\mu_0 M^\epsilon H_d(M^\epsilon u_\epsilon) + \mu_0 M^\epsilon h_a - \nabla_u \varphi^\epsilon(u_\epsilon))) \cdot \phi(t, x) dx dt \right). \end{aligned}$$

As seen above, we are allowed to pass to the limit in this equation, which yields

$$\begin{aligned} & \int_{Q_T} \left(\frac{\partial \bar{u}}{\partial t} + \lambda \bar{u} \times \frac{\partial \bar{u}}{\partial t} \right) \cdot \phi dx dt \\ &= (1 + \lambda^2) \left(\sum_{i,j=1}^3 \int_{Q_T} \left(\left(\int_{\Omega} a_{i,j}(\partial_j \bar{u} + \xi_j) d\mu \right) \times \bar{u} \right) \cdot \partial_i \phi dx dt \right. \\ & \left. + \int_{Q_T} (\bar{u} \times \left(\int_{\Omega} \mu_0 M_s H_\infty + \mu_0 M_s h_a - \nabla_u \varphi_{\text{an}}(\bar{u}) \right) d\mu(\omega) \cdot \phi) dx dt, \right. \end{aligned}$$

for every ϕ in $C_c^\infty(Q_T, \mathbb{R}^3)$. Using (2.44), in combination with (2.43) and (2.47) we get

$$\begin{aligned} \int_{Q_T} \left(\frac{\partial \bar{u}}{\partial t} + \lambda \bar{u} \times \frac{\partial \bar{u}}{\partial t} \right) \cdot \phi &= (1 + \lambda^2) \left(\sum_{i,j=1}^3 \bar{a}_{i,j} \int_{Q_T} (\partial_j \bar{u} \times \bar{u}) \cdot \partial_i \phi \, dx dt \right. \\ &\quad \left. + \int_{Q_T} (\bar{u} \times (\mu_0 \mathbb{E}(M_s)^2 H_d(\bar{u}) - \mu_0 \bar{M} \bar{u} + \mu_0 \mathbb{E}(M_s) h_a - \nabla_u \mathbb{E}(\varphi_{\text{an}}(\bar{u}))) \cdot \phi \right) dx dt, \end{aligned}$$

for every ϕ in $C_c^\infty(Q_T, \mathbb{R}^3)$. The function \bar{u} satisfies the variational formulation of the homogenized problem (D'2).

Step 4. Finally, let us show that \bar{u} satisfies the energy dissipation inequality (D'4). Let t in $(0, T)$ be fixed. As u^ϵ satisfies the inequality (D4), and since $|u^\epsilon(t, x)| = 1$ for a.e. $(t, x) \in Q_T$, it holds that, for every $\epsilon > 0$,

$$\begin{aligned} \|u^\epsilon(t, \cdot)\|_{H^1(\mathcal{D}, \mathbb{R}^3)}^2 &= \int_{\mathcal{D}} |u^\epsilon(t, x)|^2 dx + \int_{\mathcal{D}} |\nabla u^\epsilon(t, x)|^2 dx \\ &\leq |\mathcal{D}| + \frac{2c_2}{c_1} K, \end{aligned}$$

by the uniform ellipticity of a . Therefore, according to Theorem 2.21, there exists v in $H^1(\mathcal{D}, \mathbb{R}^3)$ and v_1 in $L^2(\mathcal{D}, L_{\text{pot}}^2(\Omega)^3)$ such that, up to the extraction of a subsequence,

$$\begin{cases} u_\epsilon(t) \xrightarrow{L^2} v, \\ \nabla u_\epsilon(t) \xrightarrow{L^2} \nabla v + v_1, \\ u_\epsilon(t) \xrightarrow{H^1(\mathcal{D}, \mathbb{R}^3)} v \text{ weakly}, \\ u_\epsilon(t) \xrightarrow{L^2(\mathcal{D}, \mathbb{R}^3)} v \text{ strongly}. \end{cases}$$

By (2.41) and uniqueness of the two-scale limit in $L^2(\mathcal{D})$, $v = \bar{u}(t)$. Let $(\tilde{v}_1^n)_{n \in \mathbb{N}}$ be a sequence of elements in the linear span of test functions in $C_c^\infty(\mathcal{D}) \times C(\Omega, \mathbb{R}^{3 \times 3})$ such that $\tilde{v}_1^n \rightarrow v_1$ in $L^2(\mathcal{D} \times \Omega, \mathbb{R}^{3 \times 3})$ as n tends to infinity. For every $\epsilon > 0$, t in $(0, T)$, by the positivity of a ,

$$\begin{aligned} \int_{\mathcal{D}} a(T_{x/\epsilon} \tilde{\omega}) (\nabla u^\epsilon(t, x) - \nabla \bar{u}(t, x) - \tilde{v}_1^n(x, T_{x/\epsilon} \tilde{\omega})) \\ \cdot (\nabla u^\epsilon(t, x) - \nabla \bar{u}(t, x) - \tilde{v}_1^n(x, T_{x/\epsilon} \tilde{\omega})) \, dx \geq 0. \end{aligned}$$

Expanding this inequality gives

$$\begin{aligned} \int_{\mathcal{D}} a(T_{x/\epsilon} \tilde{\omega}) \nabla u^\epsilon(t, x) \cdot \nabla u^\epsilon(t, x) \, dx \\ \geq 2 \int_{\mathcal{D}} a(T_{x/\epsilon} \tilde{\omega}) \nabla u^\epsilon(t, x) \cdot (\nabla \bar{u}(t, x) + \tilde{v}_1^n(x, T_{x/\epsilon} \tilde{\omega})) \, dx \\ - \int_{\mathcal{D}} a(T_{x/\epsilon} \tilde{\omega}) (\nabla \bar{u}(t, x) + \tilde{v}_1^n(x, T_{x/\epsilon} \tilde{\omega})) \cdot (\nabla \bar{u}(t, x) + \tilde{v}_1^n(x, T_{x/\epsilon} \tilde{\omega})) \, dx. \end{aligned}$$

For every n in \mathbb{N} , \tilde{v}_1^n is in the linear span of test functions, therefore we are allowed to pass to the two-scale limit in the right-hand side term. Thus, as we pass to the lower limit in the above inequality as ϵ goes to zero, it holds that,

$$\begin{aligned} & \liminf_{\epsilon \downarrow 0} \int_{\mathcal{D}} a(T_{x/\epsilon} \tilde{\omega}) \nabla u^\epsilon(t, x) \cdot \nabla u^\epsilon(t, x) dx \\ & \geq 2 \int_{\mathcal{D}} \int_{\Omega} a(\omega) (\nabla \bar{u}(t) + v_1(x, \omega)) \cdot (\nabla \bar{u}(t) + \tilde{v}_1^n(x, \omega)) d\mu(\omega) dx \\ & \quad - \int_{\mathcal{D}} \int_{\Omega} a(\omega) (\nabla \bar{u}(t) + \tilde{v}_1^n(x, \omega)) \cdot (\nabla \bar{u}(t) + \tilde{v}_1^n(x, \omega)) d\mu(\omega) dx. \end{aligned}$$

Passing now to the limit as n tends to infinity, we obtain

$$\begin{aligned} & \liminf_{\epsilon \downarrow 0} \int_{\mathcal{D}} a(T_{x/\epsilon} \tilde{\omega}) \nabla u^\epsilon(t, x) \cdot \nabla u^\epsilon(t, x) dx \\ & \geq \int_{\mathcal{D}} \int_{\Omega} a(\omega) (\nabla \bar{u}(t, x) + v_1(x, \omega)) \cdot (\nabla \bar{u}(t, x) + v_1(x, \omega)) d\mu(\omega) dx \\ & \geq \int_{\mathcal{D}} \bar{a} \nabla \bar{u}(t) \cdot \nabla \bar{u}(t) dx. \end{aligned} \tag{2.48}$$

By (2.41) and Proposition 2.11, it holds that

$$M_s(T_{x/\epsilon} \tilde{\omega}) u_\epsilon(t, x) \xrightarrow{L^2} M_s(\omega) \bar{u}(t),$$

strongly, and therefore

$$H_d(M^\epsilon u_\epsilon(t)) \xrightarrow{L^2} \mathbb{E}(M_s) H_d(\bar{u}(t)) + w(t, \cdot).$$

We are allowed to pass to the limit in a product of a strongly two-scale converging sequence and a weakly two-scale converging sequence:

$$\lim_{\epsilon \rightarrow 0} - \int_{\mathcal{D}} H_d(M^\epsilon u_\epsilon(t)) \cdot M^\epsilon u_\epsilon(t) = -\mathbb{E}(M_s)^2 \int_{\mathcal{D}} H_d(\bar{u}(t)) \cdot \bar{u}(t) + \int_{\mathcal{D}} \bar{u}(t) \cdot \bar{M} \bar{u}(t),$$

by (2.43).

Moreover, for every t in $(0, T)$, $(u_\epsilon)_{\epsilon > 0}$ weakly converges to \bar{u} in $H^1((0, t) \times \mathcal{D}, \mathbb{R}^3)$, $(u_\epsilon(t))_{\epsilon > 0}$ weakly converges to $\bar{u}(t)$ in $L^2(\mathcal{D}, \mathbb{R}^3)$ and $(\varphi^\epsilon(u_\epsilon(t)))_{\epsilon > 0}$ weakly converges to $\mathbb{E}(\varphi_{\text{an}}(\bar{u}(t)))$ in $L^2(\mathcal{D})$, therefore, passing to the lower limit in (2.39) as ϵ goes to zero, and using (5.3) and (2.48),

$$\begin{aligned} & \frac{1}{2} \int_{\mathcal{D}} \bar{a} \nabla \bar{u}(t) \cdot \nabla \bar{u}(t) - \frac{\mu_0 \mathbb{E}(M_s)^2}{2} \int_{\mathcal{D}} H_d(\bar{u}(t)) \cdot \bar{u}(t) + \frac{\mu_0}{2} \int_{\mathcal{D}} \bar{u}(t) \cdot \bar{M} \bar{u}(t) \\ & \quad - \mu_0 \mathbb{E}(M_s) \int_{\mathcal{D}} h_a \cdot \bar{u}(t) + \int_{\mathcal{D}} \mathbb{E}(\varphi_{\text{an}})(\bar{u}(t)) + \int_0^t \int_{\mathcal{D}} \left| \frac{\partial \bar{u}}{\partial t} \right|^2 \leq K. \end{aligned}$$

Finally, \bar{u} satisfies (D'1)-(D'4). Therefore, it is a weak solution of (2.38). □

Remark 2.12. 1. In [6], the existence of weak solutions of (2.33) has been proven with an even more precise bound on the dissipation of the energy given by the energy at time 0. This could be written here as

$$\begin{aligned} \mathcal{E}_\epsilon(u_0) &= \frac{1}{2} \int_{\mathcal{D}} a^\epsilon \nabla u_0(x) \cdot \nabla u_0(x) dx - \frac{\mu_0}{2} \int_{\mathcal{D}} H_d(M^\epsilon u_0) \cdot M^\epsilon u_0 \\ &\quad - \int_{\mathcal{D}} \mu_0 h_a \cdot M^\epsilon u_0 + \int_{\mathcal{D}} \varphi^\epsilon(u_0). \end{aligned}$$

However, we cannot be as precise in our definition of weak solutions : if $(\mathcal{E}_\epsilon(u_0))_{\epsilon>0}$ is indeed uniformly bounded, it does not converge to the homogenized energy at time 0, namely

$$\begin{aligned} \bar{\mathcal{E}}(u_0) &= \frac{1}{2} \int_{\mathcal{D}} \bar{a} \nabla u_0(x) \cdot \nabla u_0(x) dx - \frac{\mu_0 \mathbb{E}(M_s)^2}{2} \int_{\mathcal{D}} H_d(u_0) \cdot u_0 + \frac{\mu_0}{2} \int_{\mathcal{D}} u_0 \cdot \bar{M} u_0 \\ &\quad - \mu_0 \mathbb{E}(M_s) \int_{\mathcal{D}} h_a \cdot u_0 + \int_{\mathcal{D}} \mathbb{E}(\varphi_{\text{an}})(u_0). \end{aligned}$$

In fact, all of the terms behave as expected, except for the exchange energy term. By definition of a typical trajectory, we have

$$\lim_{\epsilon \downarrow 0} \frac{1}{2} \int_{\mathcal{D}} a(T_{x/\epsilon} \tilde{\omega}) \nabla u_0(x) \cdot \nabla u_0(x) dx = \frac{1}{2} \int_{\mathcal{D}} \mathbb{E}(a) \nabla u_0(x) \cdot \nabla u_0(x) dx$$

which is *not* inferior to the homogenized energy at time 0. Indeed,

$$\begin{aligned} &\frac{1}{2} \int_{\mathcal{D}} \bar{a} \nabla u_0(x) \cdot \nabla u_0(x) dx \\ &= \frac{1}{2} \int_{\mathcal{D}} \inf_{v \in (L^2_{\text{pot}}(\Omega))^3} \int_{\Omega} a(\omega) (\nabla u_0(x) + v(\omega)) \cdot (\nabla u_0(x) + v(\omega)) d\mu(\omega) dx \\ &\leq \frac{1}{2} \int_{\mathcal{D}} \mathbb{E}(a) \nabla u_0(x) \cdot \nabla u_0(x) dx, \end{aligned}$$

the last inequality being strict in general.

2. However, we can define, as in [35], weak solutions of (2.33) for well-prepared initial data u_0^ϵ , i.e.

$$u_0^\epsilon \xrightarrow{L^2} u_0 \text{ strongly, and } \nabla u_0^\epsilon \xrightarrow{L^2} \nabla u_0 + \chi,$$

where u_0 is in $H^1(\mathcal{D}, \mathbb{R}^3)$ and χ is in $L^2(\mathcal{D}, (L^2_{\text{pot}}(\Omega))^3)$ such that, for every $x \in \mathcal{D}$,

$$a(\cdot) (\nabla u_0(x) + \chi(x, \cdot)) \in (L^2_{\text{sol}}(\Omega))^3.$$

In this case, we can state the dissipation of energy (D4) of weak solutions u^ϵ of (2.33) as

$$\mathcal{E}_\epsilon(u^\epsilon) \leq \mathcal{E}_\epsilon(u_0^\epsilon),$$

and the Homogenization Theorem would hold, where in the definition of weak solutions of the homogenized problem, (D4) is replaced by

$$\bar{\mathcal{E}}(\bar{u}) \leq \bar{\mathcal{E}}(u_0).$$

Conclusion

In this chapter, we have presented a simple version of the stochastic two-scale convergence that was proposed in [65]. The technique provides tools that are very similar to the ones used in periodic homogenization, and mostly extends the techniques to the stochastic setting. We have proposed two application examples of the method in non-linear problems, one concerning the homogenization of the harmonic maps equation while the other deals with the homogenization of the Landau-Lifshitz-Gilbert equation, which is currently used for the modelization of ferromagnetic materials that are inhomogeneous at the small scale.

We emphasize the fact that both equations share common mathematical difficulties such as the non-linear behavior, a non convex constraint on the (vectorial) unknown and non uniqueness of the weak solutions. Nevertheless, we have shown that the stochastic two-scale convergence method may be applied to those equations, permitting to identify the limiting homogenized equation.

Chapter 3

Nucleation of *spring magnets*

1 Introduction

Let us recall that a ferromagnetic continuum is described by the vector field

$$\mathbf{M} : \mathcal{D} \mapsto \mathbb{R}^3, \quad (3.1)$$

where \mathcal{D} is a bounded domain of the space \mathbb{R}^3 representing the domain occupied by the ferromagnetic material. The *magnetization* \mathbf{M} is subject to the constraint: for almost every x in \mathcal{D} , $|\mathbf{M}(x)| = M_s$, with M_s the *saturation magnetization*. The physically admissible configurations are selected by minimization of the free energy

$$\mathcal{E}(\mathbf{m}) = \int_{\mathcal{D}} A |\nabla \mathbf{m}|^2 + \int_{\mathcal{D}} J(\mathbf{m}) - \int_{\mathcal{D}} \mu_0 \mathbf{H}_{\text{ext}} \cdot M_s \mathbf{m} - \frac{1}{2} \int_{\mathcal{D}} \mu_0 \mathbf{H}_d(M_s \mathbf{m}) \cdot M_s \mathbf{m}, \quad (3.2)$$

where the magnetization is denoted $\mathbf{M} = M_s \mathbf{m}$, with the constraint $|\mathbf{m}(x)| = 1$ on \mathcal{D} . The different terms are:

- The exterior energy

$$\mathcal{E}_{\text{ext}}(\mathbf{m}) = -\mu_0 \int_{\mathcal{D}} \mathbf{H}_{\text{ext}} \cdot M_s \mathbf{m},$$

where \mathbf{H}_{ext} is the exterior field. This term expresses the fact that a magnetic sample placed in an electric field tends to align its magnetization with the direction of the field.

- The exchange energy

$$\mathcal{E}_{\text{ex}}(\mathbf{m}) = \int_{\mathcal{D}} A |\nabla \mathbf{m}|^2,$$

where A is the energy constant. Two neighbouring spins tend to align with each other.

- The demagnetizing energy

$$\mathcal{E}_{\text{dem}}(\mathbf{m}) = -\frac{1}{2} \int_{\mathcal{D}} \mu_0 \mathbf{H}_d(M_s \mathbf{m}) \cdot M_s \mathbf{m},$$

where \mathbf{H}_d is an operator on \mathbf{M} called the *demagnetizing field*, and satisfies $-\int_{\mathcal{D}} \mathbf{H}_d(\mathbf{M}) \cdot \mathbf{M} \geq 0$.

- The anisotropy energy

$$\mathcal{E}_{\text{an}}(\mathbf{m}) = \int_{\mathcal{D}} J(\mathbf{m}) ,$$

where $J : \mathbb{S}^2 \rightarrow \mathbb{R}$ is an even function that vanishes on a set of directions called *easy axes of magnetization*. In this chapter, we focus on the case of uniaxial anisotropy, for which

$$J(\mathbf{m}) = K(1 - (\mathbf{m} \cdot \mathbf{u})^2),$$

where $K > 0$ is a constant, and the easy axis \mathbf{u} is a unit vector.

As explained in Chapter 1, the study of *coercivity*, i.e. the ability to maintain its magnetization despite external disturbances, of a permanent magnet is essential to evaluate its quality. For this reason, it is necessary to evaluate the exterior applied field at which the spins of a previously saturated ideal ferromagnetic particle cease to be aligned. It is defined as the *nucleation field*. Of interest are what we will call the *nucleation modes*, called *mechanisms of magnetization reversal* in the physics literature, i.e. the description of the magnetization after it has been destabilized from a constant equilibrium.

These have been studied by physicists, notably in [24], using calculus of variations, the nucleation field was calculated for an infinite cylinder and a sphere, *assuming* three mechanisms of magnetization reversal: spin rotation in unison (i.e. the magnetization reversal is uniform in the magnet), magnetization curling (i.e. the magnetization changes by spin rotation from the easy axis in a plane perpendicular to the radius), and magnetization buckling (i.e., in the case of cylinders only, the magnetization reversal changes along the easy axis). The critical size for single-domain behavior, defined as the largest size at which magnetization reversal proceeds by rotation in unison, is calculated for the prolate ellipsoid and is found to be practically independent of magnetocrystalline anisotropy and elongation and approximately equal to the *exchange length* $\ell_{\text{ex}} = \sqrt{\frac{A}{\mu_0 M_s^2}}$. They discuss available experimental data and find satisfactory agreements with this theory.

In this chapter, we prove that the three mechanisms (spin rotation in unison, magnetization curling, and magnetization buckling) are in fact the only mechanisms that need to be considered.

The outline of the chapter is the following. In the first section, by means of a study of magnetic energy, we identify the equation that characterizes nucleation, which is an eigenvalue problem. Then, we justify via homogenization how the study of spring magnets boils down to the study of a homogeneous magnet. Finally, we analyze the nucleation equation in the special case of domains where there exist a state of uniform magnetization in equilibrium, and we study their destabilization. These domains are ellipsoids and infinite cylinders. We are able to draw conclusions on the nucleation modes according to the size of the domain and the characteristic values of the magnet.

1.1 Expression of the magnetic energy

We choose to study the energy under the following form, for $\mathbf{m} \in \mathbb{R}^3$ such that $|\mathbf{m}| = 1$:

$$E(\mathbf{m}) = \int_{\mathcal{D}} A (|\nabla \times \mathbf{m}|^2 + (\nabla \cdot \mathbf{m})^2) + K(1 - (\mathbf{m} \cdot \mathbf{u})^2) - \mu_0 \mathbf{H} \cdot M_s \mathbf{m}, \quad (3.3)$$

with $\mathbf{H} = \mathbf{H}_{\text{ext}} + \frac{1}{2} \mathbf{H}_d(M_s \mathbf{m})$. This expression only differs from the classical one (3.2) by the exchange term. We justify hereafter this choice.

1.1.1 Physical modeling

In the mostly used expression of the energy (3.2), the energy term is $|\nabla \mathbf{m}|^2$, whereas in our expression (3.3), this term is replaced by the two terms $|\nabla \times \mathbf{m}|^2 + (\nabla \cdot \mathbf{m})^2$. However, both expressions express the fact that neighbouring spins tend to align with each other. The term $|\nabla \mathbf{m}|^2$ is mostly used in the micromagnetic literature because it derives naturally from the modeling of the ferromagnetic continuum by a set of spins. It is worth pointing out that in the context of liquid crystals, the increase in the free energy density caused by distortions from the uniformly aligned configuration of the liquid crystal, called the Distortion free energy density, has the form $|\nabla \times \mathbf{m}|^2 + (\nabla \cdot \mathbf{m})^2$.

Let us note that the expression (3.3) has a few occurrences in the micromagnetic literature, notably in the context of ideally soft ferromagnetic cylinders [9, 36]. For our purpose, as we will see in our following study of nucleation, it turns out that the study of expression (3.3) is more convenient.

1.1.2 Mathematical properties

Mathematically, we have the identity, for $\mathbf{m} \in H^1(\mathcal{D}, \mathbb{S}^2)$,

$$\begin{aligned} |\nabla \mathbf{m}|^2 &= -\Delta \mathbf{m} \cdot \mathbf{m} \\ &= -(\nabla(\nabla \cdot \mathbf{m})) \cdot \mathbf{m} + (\nabla \times (\nabla \times \mathbf{m})) \cdot \mathbf{m} \\ &= |\nabla \times \mathbf{m}|^2 + (\nabla \cdot \mathbf{m})^2 + \nabla \cdot ((\nabla \times \mathbf{m}) \times \mathbf{m} - (\nabla \cdot \mathbf{m}) \mathbf{m}). \end{aligned} \quad (3.4)$$

The replacement relies on the omission of a area integral of the expression $(\nabla \times \mathbf{m}) \times \mathbf{m} - (\nabla \cdot \mathbf{m}) \mathbf{m}$, which does not vanish in general.

Let us show that the two expressions are coherent with each other. Denoting $E_{\text{ex}} = \int_{\mathcal{D}} A (|\nabla \times \mathbf{m}|^2 + (\nabla \cdot \mathbf{m})^2)$ and $\mathcal{E}_{\text{ex}}(\mathbf{m}) = \int_{\mathcal{D}} \frac{A}{2} |\nabla \mathbf{m}|^2$, we have, for every $\mathbf{m} \in H^1(\mathcal{D}, \mathbb{S}^2)$,

$$\mathcal{E}_{\text{ex}}(\mathbf{m}) \leq E_{\text{ex}}(\mathbf{m}) \leq D \mathcal{E}_{\text{ex}}(\mathbf{m}), \quad (3.5)$$

with a constant D that depends on the domain \mathcal{D} .

Indeed, we have, for every $\mathbf{m} \in H^1(\mathcal{D}, \mathbb{R}^3)$, on one hand

$$(\nabla \cdot \mathbf{m})^2 = (\partial_x m_1)^2 + (\partial_2 m_2)^2 + (\partial_3 m_3)^2,$$

and on the other hand

$$\begin{aligned} |\nabla \times \mathbf{m}|^2 &= (\partial_2 m_3 - \partial_3 m_2)^2 + (\partial_1 m_3 - \partial_3 m_1)^2 + (\partial_2 m_1 - \partial_1 m_2)^2 \\ &= \sum_{\substack{i,j=1 \\ i \neq j}}^3 (\partial_i m_j)^2 - 2(\partial_2 m_3 \partial_3 m_2 + \partial_1 m_3 \partial_3 m_1 + \partial_2 m_1 \partial_1 m_2). \end{aligned}$$

It follows that

$$|\nabla \times \mathbf{m}|^2 + (\nabla \cdot \mathbf{m})^2 = |\nabla \mathbf{m}|^2 - 2(\partial_2 m_3 \partial_3 m_2 + \partial_1 m_3 \partial_3 m_1 + \partial_2 m_1 \partial_1 m_2).$$

We get the left inequality of (3.5) by noticing

$$2|\partial_2 m_3 \partial_3 m_2 + \partial_1 m_3 \partial_3 m_1 + \partial_2 m_1 \partial_1 m_2| \leq \sum_{\substack{i,j=1 \\ i \neq j}}^3 (\partial_i m_j)^2 \leq |\nabla \mathbf{m}|^2.$$

Moreover, for $\mathbf{m} \in H^1(\mathcal{D}, \mathbb{S}^2)$, using (3.4) and integrating by parts, we have that

$$\begin{aligned} \int_{\mathcal{D}} |\nabla \mathbf{m}|^2 &= \int_{\mathcal{D}} |\nabla \times \mathbf{m}|^2 + (\nabla \cdot \mathbf{m})^2 + \int_{\partial \mathcal{D}} ((\nabla \times \mathbf{m}) \times \mathbf{m}) \cdot \mathbf{n} - (\nabla \cdot \mathbf{m}) \mathbf{m} \cdot \mathbf{n} \\ &\leq (1 + D) \int_{\mathcal{D}} |\nabla \times \mathbf{m}|^2 + (\nabla \cdot \mathbf{m})^2, \end{aligned}$$

with a constant D that depends on \mathcal{D} , by continuity of the trace.

1.2 Nucleation equation

Magnetization reversal in most real materials begins with a process called *nucleation*. The term refers to a localized or delocalized instability of the initial magnetization state. Let us study in this section how a previously stable equilibrium is destabilized by an exterior field.

Definition 3.1. Let \mathcal{D} be a domain of \mathbb{R}^3 and let us assume that there exists a constant vector \mathbf{u} such that the demagnetizing field $\mathbf{H}_d(\mathbf{u})$ is a constant vector parallel to \mathbf{u} inside \mathcal{D} .

Let us consider a magnet with uniaxial anisotropy of easy axis \mathbf{u} . Then, the constant aligned state $\mathbf{m} = \mathbf{u}$ is an equilibrium of the energy (3.3).

We define the *nucleation field* of the magnet as the minimum of the following expression:

$$H_N = \inf_{\substack{\boldsymbol{\psi} \in H^1(\mathcal{D}, \mathbb{R}^3) \cap L^\infty(\mathcal{D}, \mathbb{R}^3) \\ \boldsymbol{\psi} \cdot \mathbf{m} = 0}} \frac{q(\boldsymbol{\psi})}{\|\boldsymbol{\psi}\|_{L^2(\mathcal{D})}^2}, \quad (3.6)$$

for the quadratic form q defined by

$$q(\boldsymbol{\psi}) = \frac{2}{\mu_0 M_s} \int_{\mathcal{D}} A (|\nabla \times \boldsymbol{\psi}|^2 + (\nabla \cdot \boldsymbol{\psi})^2) + K_{\text{eff}} |\boldsymbol{\psi}|^2 - \frac{\mu_0}{2} M_s \mathbf{H}_d(M_s \boldsymbol{\psi}) \cdot \boldsymbol{\psi},$$

where $K_{\text{eff}} = K + \frac{\mu_0}{2} M_s \mathbf{H}_d(M_s \mathbf{m}) \cdot \mathbf{m}$ is a constant.

We say that $\boldsymbol{\psi} \in H^1(\mathcal{D}, \mathbb{R}^3)$ such that $\mathbf{m} \cdot \boldsymbol{\psi} = 0$ is a *nucleation mode* for the magnet if it is a minimizer to the previous expression. In other terms, it is a non-trivial solution to the following variational problem: for every $\boldsymbol{\phi} \in H^1(\mathcal{D}, \mathbb{R}^3) \cap L^\infty(\mathcal{D}, \mathbb{R}^3)$,

$$\int_{\mathcal{D}} A(\nabla \times \boldsymbol{\psi}) \cdot (\nabla \times \boldsymbol{\phi}) + A(\nabla \cdot \boldsymbol{\psi})(\nabla \cdot \boldsymbol{\phi}) + \left(K_{\text{eff}} \boldsymbol{\psi} - \frac{\mu_0}{2} M_s \mathbf{H}_d(M_s \boldsymbol{\psi}) \right) \cdot \boldsymbol{\phi} - \frac{\mu_0}{2} M_s H_N \boldsymbol{\psi} \cdot \boldsymbol{\phi} = 0. \quad (3.7)$$

called the *nucleation equation*.

Let us justify this definition of the nucleation field.

Let \mathbf{m} be a magnetization field that is an equilibrium for the energy E , and let us study a small perturbation of this state:

$$\frac{\mathbf{m} + \boldsymbol{\psi}}{|\mathbf{m} + \boldsymbol{\psi}|}, \boldsymbol{\psi} \in H^1(\mathcal{D}, \mathbb{R}^3) \cap L^\infty(\mathcal{D}, \mathbb{R}^3); \boldsymbol{\psi} \cdot \mathbf{m} = 0. \quad (3.8)$$

Remark 3.1. We can indeed restrict ourselves to this setting : given a small perturbation $\boldsymbol{\psi} \in C_c^\infty(\mathcal{D}, \mathbb{R}^3)$ such that $\|\boldsymbol{\psi}\|_\infty < 1$, we have for every $\lambda \in C_c^\infty(\mathcal{D})$ with $\lambda < 1$,

$$E\left(\frac{\mathbf{m} + \boldsymbol{\psi} - \lambda(\mathbf{m} + \boldsymbol{\psi})}{|\mathbf{m} + \boldsymbol{\psi} - \lambda(\mathbf{m} + \boldsymbol{\psi})|}\right) = E\left(\frac{\mathbf{m} + \boldsymbol{\psi}}{|\mathbf{m} + \boldsymbol{\psi}|}\right).$$

Thus, taking $\lambda = \frac{\mathbf{m} \cdot \boldsymbol{\psi}}{1 + \mathbf{m} \cdot \boldsymbol{\psi}}$, we have $(\boldsymbol{\psi} - \lambda(\mathbf{m} + \boldsymbol{\psi})) \cdot \mathbf{m} = 0$, and $\boldsymbol{\psi} - \lambda(\mathbf{m} + \boldsymbol{\psi}) \in H^1(\mathcal{D}, \mathbb{R}^3) \cap L^\infty(\mathcal{D}, \mathbb{R}^3)$. It follows that we can study interchangeably perturbations of the form $\frac{\mathbf{m} + \boldsymbol{\psi}}{|\mathbf{m} + \boldsymbol{\psi}|}$, with $\boldsymbol{\psi} \in H^1(\mathcal{D}, \mathbb{R}^3) \cap L^\infty(\mathcal{D}, \mathbb{R}^3)$ and $\boldsymbol{\psi} \cdot \mathbf{m} = 0$, or $\boldsymbol{\psi} \in C_c^\infty(\mathcal{D}, \mathbb{R}^3)$.

Developing the expression (3.8) up to the second order, we get

$$\begin{aligned} \frac{\mathbf{m} + \boldsymbol{\psi}}{|\mathbf{m} + \boldsymbol{\psi}|} &= (\mathbf{m} + \boldsymbol{\psi}) (|\mathbf{m}|^2 + |\boldsymbol{\psi}|^2 + 2\mathbf{m} \cdot \boldsymbol{\psi})^{-1/2} \\ &= \mathbf{m} + \boldsymbol{\psi} - \frac{1}{2} |\boldsymbol{\psi}|^2 \mathbf{m} + o(|\boldsymbol{\psi}|^2). \end{aligned}$$

Therefore, in order to study the perturbations of an equilibrium state \mathbf{m} , we may expand the energy as

$$\begin{aligned} E\left(\frac{\mathbf{m} + \boldsymbol{\psi}}{|\mathbf{m} + \boldsymbol{\psi}|}\right) &= E\left(\mathbf{m} + \boldsymbol{\psi} - \frac{1}{2} |\boldsymbol{\psi}|^2 \mathbf{m} + o(|\boldsymbol{\psi}|^2)\right) \\ &= E(\mathbf{m}) + DE(\mathbf{m})(\boldsymbol{\psi}) + D^2E(\mathbf{m})(\boldsymbol{\psi}, \boldsymbol{\psi}) + o\left(\|\boldsymbol{\psi}\|_{H^1(\mathcal{D}, \mathbb{R}^3)}^2\right), \end{aligned}$$

where

$$\begin{aligned} DE(\mathbf{m})(\boldsymbol{\psi}) &= 2 \int_{\mathcal{D}} A(\nabla \times \mathbf{m}) \cdot (\nabla \times \boldsymbol{\psi}) + A(\nabla \cdot \mathbf{m})(\nabla \cdot \boldsymbol{\psi}) - K(\mathbf{m} \cdot \mathbf{u})(\boldsymbol{\psi} \cdot \mathbf{u}) \\ &\quad + \int_{\mathcal{D}} -\mu_0 M_s \mathbf{H}_d(M_s \mathbf{m}) \cdot \boldsymbol{\psi} - \mu_0 M_s \mathbf{H}_{\text{ext}} \cdot \boldsymbol{\psi}, \end{aligned}$$

and

$$\begin{aligned}
D^2E(\mathbf{m})(\boldsymbol{\psi}, \boldsymbol{\psi}) = & 2 \int_{\mathcal{D}} A(\nabla \times \mathbf{m}) \cdot \nabla \times \left(-\frac{1}{2} |\boldsymbol{\psi}|^2 \mathbf{m} \right) + A(\nabla \cdot \mathbf{m}) \left(\nabla \cdot \left(-\frac{1}{2} |\boldsymbol{\psi}|^2 \mathbf{m} \right) \right) \\
& + \int_{\mathcal{D}} \frac{1}{2} K |\boldsymbol{\psi}|^2 (\mathbf{m} \cdot \mathbf{u})^2 + \frac{\mu_0}{4} M_s |\boldsymbol{\psi}|^2 \mathbf{H}_d(M_s \mathbf{m}) \cdot \mathbf{m} \\
& + \int_{\mathcal{D}} A |\nabla \times \boldsymbol{\psi}|^2 + A(\nabla \cdot \boldsymbol{\psi})^2 - K(\boldsymbol{\psi} \cdot \mathbf{u})^2 - \frac{\mu_0}{2} M_s \mathbf{H}_d(M_s \boldsymbol{\psi}) \cdot \boldsymbol{\psi} \\
& + \frac{1}{2} \int_{\mathcal{D}} |\boldsymbol{\psi}|^2 M_s \mu_0 \mathbf{H}_{\text{ext}} \cdot \mathbf{m}.
\end{aligned}$$

The state \mathbf{m} is supposed to be an equilibrium of E , therefore, the first-order terms vanish, i.e., for every $\boldsymbol{\psi} \in H^1(\mathcal{D}, \mathbb{R}^3)$ such that $\boldsymbol{\psi} \cdot \mathbf{m} = 0$,

$$\int_{\mathcal{D}} A(\nabla \times \mathbf{m}) \cdot (\nabla \times \boldsymbol{\psi}) + A(\nabla \cdot \mathbf{m})(\nabla \cdot \boldsymbol{\psi}) - K(\mathbf{m} \cdot \mathbf{u})(\boldsymbol{\psi} \cdot \mathbf{u}) - \frac{\mu_0}{2} M_s \mathbf{H}_d(M_s \mathbf{m}) \cdot \boldsymbol{\psi} = 0. \quad (3.9)$$

This equation characterizes the equilibria of E . Let us notice that, when \mathbf{m} is constant and equal to \mathbf{u} , this condition reduces to:

$$\int_{\mathcal{D}} \mu_0 M_s \mathbf{H}_d(M_s \mathbf{m}) \cdot \boldsymbol{\psi} = 0, \forall \boldsymbol{\psi} \in H^1(\mathcal{D}, \mathbb{R}^3) \text{ s.t. } \boldsymbol{\psi} \cdot \mathbf{m} = 0. \quad (3.10)$$

It follows that, for the domain \mathcal{D} to admit a constant aligned equilibrium state $\mathbf{m} = \mathbf{u}$, the demagnetizing field $\mathbf{H}_d(\mathbf{u})$ must be a (possibly non-constant) vector parallel to \mathbf{u} inside \mathcal{D} . This condition is satisfied for ellipsoids [53], infinite cylinders (see subsection 3.1), but not for cubes [1].

Let us denote $H = -\mathbf{H}_{\text{ext}} \cdot \mathbf{m}$. We have

$$\begin{aligned}
E \left(\frac{\mathbf{m} + \boldsymbol{\psi}}{|\mathbf{m} + \boldsymbol{\psi}|} \right) = & E(\mathbf{m}) + \int_{\mathcal{D}} A |\nabla \times \boldsymbol{\psi}|^2 + A(\nabla \cdot \boldsymbol{\psi})^2 \\
& + |\boldsymbol{\psi}|^2 \left(K + \frac{\mu_0}{2} M_s \mathbf{H}_d(M_s \mathbf{m}) \cdot \mathbf{m} - \frac{\mu_0}{2} M_s H \right) - \int_{\mathcal{D}} \frac{\mu_0}{2} M_s \mathbf{H}_d(M_s \boldsymbol{\psi}) \cdot \boldsymbol{\psi} \\
& + o \left(\|\boldsymbol{\psi}\|_{H^1(\mathcal{D})}^2 \right), \quad (3.11)
\end{aligned}$$

Our study consists in finding the lowest value of H for which the equilibrium state \mathbf{m} becomes unstable, i.e. when the second order term vanishes and changes sign. Thus, the nucleation field is defined as

$$H_N = \inf_{\substack{\boldsymbol{\psi} \in H^1(\mathcal{D}, \mathbb{R}^3) \cap L^\infty(\mathcal{D}, \mathbb{R}^3); \\ \boldsymbol{\psi} \cdot \mathbf{m} = 0}} \frac{q(\boldsymbol{\psi})}{\|\boldsymbol{\psi}\|_{L^2(\mathcal{D}, \mathbb{R}^3)}^2}, \quad (3.12)$$

for the quadratic form q defined by

$$q(\boldsymbol{\psi}) = \frac{2}{\mu_0 M_s} \int_{\mathcal{D}} A (|\nabla \times \boldsymbol{\psi}|^2 + (\nabla \cdot \boldsymbol{\psi})^2) + K_{\text{eff}} |\boldsymbol{\psi}|^2 - \frac{\mu_0}{2} M_s \mathbf{H}_d(M_s \boldsymbol{\psi}) \cdot \boldsymbol{\psi},$$

denoting $K_{\text{eff}} = K + \frac{\mu_0}{2} M_s \mathbf{H}_d(M_s \mathbf{m}) \cdot \mathbf{m}$.

Then, a *nucleation mode* is a minimizer of (3.12) $\boldsymbol{\psi} \in H^1(\mathcal{D}, \mathbb{R}^3)$ such that $\boldsymbol{\psi} \cdot \mathbf{m} = 0$ and satisfies the following variational equation: for every $\boldsymbol{\phi} \in H^1(\mathcal{D}, \mathbb{R}^3) \cap L^\infty(\mathcal{D}, \mathbb{R}^3)$,

$$\int_{\mathcal{D}} A(\nabla \times \boldsymbol{\psi}) \cdot (\nabla \times \boldsymbol{\phi}) + A(\nabla \cdot \boldsymbol{\psi})(\nabla \cdot \boldsymbol{\phi}) + \left(K_{\text{eff}} \boldsymbol{\psi} - \frac{\mu_0}{2} M_s \mathbf{H}_d(M_s \boldsymbol{\psi}) \right) \cdot \boldsymbol{\phi} - \frac{\mu_0}{2} M_s H_N \boldsymbol{\psi} \cdot \boldsymbol{\phi} = 0.$$

2 Spring magnets

2.1 Homogenization of the nucleation equation

In order to apply our results to spring magnets, we must study how the expressions of energy (3.3), the nucleation equation (3.7) and the nucleation field (3.6) behave in the homogenization limit. We refer to chapter 2 for the stochastic homogenization of the Landau-Lifschitz-Gilbert equation. Likewise, we assume that the materials are randomly distributed on a small scale ϵ . Moreover, we limit ourselves to the case where both materials have a uniaxial anisotropy, with the same easy axis direction. Let us show that the study of the nucleation of a spring magnets boils down to the study of a permanent magnet which coefficients are introduced in Chapter 2.

Let us recall the framework of Chapter 2: $(\Omega, \mathcal{F}, \mu)$ is a probability setting, with Ω a compact metric space and T a dynamical system, $M_s, K \in C(\Omega, \mathbb{R}^+)$, continuous functions of Ω in \mathbb{R}^+ , $a \in C(\Omega, \mathbb{R}^{3 \times 3})$, a symmetric matrix, uniformly bounded and uniformly elliptic:

$$\exists c_1, c_2 > 0; \forall \omega \in \Omega, \forall \xi \in \mathbb{R}^d, \forall x \in \mathbb{R}^d, c_1 |\xi|^2 \leq \xi \cdot a(x, \omega) \xi \leq c_2 |\xi|^2.$$

We assume that the random field $a : (x, \omega) \mapsto a(T_x \omega)$ and T are stationary and ergodic.

Let $\tilde{\omega}$ be a typical trajectory (see Definition 2.1).

In order to be in the setting of the last section, we need to assume that the domain \mathcal{D} and the probability settings are such that there exists a constant vector \mathbf{m} that satisfies, for every $\epsilon > 0$,

$$\int_{\mathcal{D}} M_s(T_{x/\epsilon} \tilde{\omega}) \mathbf{H}_d(M_s(T_{x/\epsilon} \tilde{\omega}) \mathbf{m}) \cdot \boldsymbol{\psi} = 0, \quad \forall \boldsymbol{\psi} \in H^1(\mathcal{D}, \mathbb{R}^3); \boldsymbol{\psi} \cdot \mathbf{m} = 0 \quad (3.13)$$

and that this vector is the easy axis of magnetization for the materials involved.

Noticing that we have, for every $\boldsymbol{\phi} \in H^1(\mathcal{D}, \mathbb{R}^3)$,

$$\nabla \cdot \boldsymbol{\phi} = \text{Tr}(\nabla \boldsymbol{\phi}), \quad \nabla \times \boldsymbol{\phi} = \mathcal{L}(\nabla \boldsymbol{\phi}),$$

with $\mathcal{L} : \mathbb{R}^{3 \times 3} \rightarrow \mathbb{R}^3$ a linear application, we denote, for $\boldsymbol{\phi} \in H^1(\Omega, \mathbb{R}^3)$,

$$D_\omega \cdot \boldsymbol{\phi} = \text{Tr}(D_\omega \boldsymbol{\phi}), \quad D_\omega \times \boldsymbol{\phi} = \mathcal{L}(D_\omega \boldsymbol{\phi}).$$

The magnetic energy writes

$$\begin{aligned} E^\epsilon(\mathbf{m}) &= \int_{\mathcal{D}} \mathcal{L}(a(T_{x/\epsilon} \tilde{\omega}) \nabla \mathbf{m}) \cdot (\nabla \times \mathbf{m}) + \text{Tr}(a(T_{x/\epsilon} \tilde{\omega}) \nabla \mathbf{m})(\nabla \cdot \mathbf{m}) \\ &\quad + K(T_{x/\epsilon} \tilde{\omega}) (1 - (\mathbf{m} \cdot \mathbf{u})^2) - \mu_0 \left(\mathbf{H}_{\text{ext}} + \frac{1}{2} \mathbf{H}_d(M_s(T_{x/\epsilon} \tilde{\omega}) \mathbf{m}) \right) M_s(T_{x/\epsilon} \tilde{\omega}) \mathbf{m}. \end{aligned}$$

We notice that the exchange energy writes

$$\int_{\mathcal{D}} (\mathcal{L}^T \mathcal{L} + \text{Tr}^T \text{Tr}) a(T_{x/\epsilon} \tilde{\omega}) \nabla \mathbf{m} \cdot \nabla \mathbf{m} = \int_{\mathcal{D}} \mathcal{A}(T_{x/\epsilon} \tilde{\omega}) \nabla \mathbf{m} \cdot \nabla \mathbf{m},$$

with $\mathcal{A} \in C(\Omega, \mathbb{R}^{9 \times 9})$. We assume furthermore that for any $\omega \in \Omega$, $a(\omega)$ commutes with $\mathcal{L}^T \mathcal{L} + \text{Tr}^T \text{Tr}$. Then \mathcal{A} is a symmetric matrix, uniformly bounded and uniformly elliptic, since $\mathcal{L}^T \mathcal{L} + \text{Tr}^T \text{Tr}$ is positive-definite.

Thus, the nucleation equation (3.7) becomes, for a solution $\boldsymbol{\psi}^\epsilon \in H^1(\mathcal{D}, \mathbb{R}^3)$ such that $\boldsymbol{\psi}^\epsilon \cdot \mathbf{m} = 0$: for every $\boldsymbol{\phi} \in H^1(\mathcal{D}, \mathbb{R}^3)$,

$$\begin{aligned} \int_{\mathcal{D}} \mathcal{A}(T_{x/\epsilon} \tilde{\omega}) \nabla \boldsymbol{\psi}^\epsilon \cdot \nabla \boldsymbol{\phi} + K(T_{x/\epsilon} \tilde{\omega}) \boldsymbol{\psi}^\epsilon \cdot \boldsymbol{\phi} \\ + \frac{\mu_0}{2} M_s(T_{x/\epsilon} \tilde{\omega}) \left((\mathbf{H}_d(M_s(T_{x/\epsilon} \tilde{\omega}) \mathbf{m}) \cdot \mathbf{m}) \boldsymbol{\psi}^\epsilon - \mathbf{H}_d(M_s(T_{x/\epsilon} \tilde{\omega}) \boldsymbol{\psi}^\epsilon) \right) \cdot \boldsymbol{\phi} \\ - \frac{\mu_0}{2} M_s(T_{x/\epsilon} \tilde{\omega}) H_N \boldsymbol{\psi}^\epsilon \cdot \boldsymbol{\phi} = 0. \end{aligned} \quad (3.14)$$

Using the techniques explained in Chapter 2, we know that the homogenized problem reads: find a solution $\bar{\boldsymbol{\psi}} \in H^1(\mathcal{D}, \mathbb{R}^3)$ such that $\bar{\boldsymbol{\psi}} \cdot \mathbf{m} = 0$, and for every $\boldsymbol{\phi} \in H^1(\mathcal{D}, \mathbb{R}^3)$,

$$\begin{aligned} \int_{\mathcal{D}} \bar{\mathcal{A}} \nabla \bar{\boldsymbol{\psi}} \cdot \nabla \boldsymbol{\phi} + \left(\mathbb{E}(K) + \frac{\mu_0}{2} \mathbb{E}(M_s)^2 \mathbf{H}_d(\mathbf{m}) \cdot \mathbf{m} - \frac{\mu_0}{2} \mathbf{m} \cdot \bar{M} \mathbf{m} \right) \bar{\boldsymbol{\psi}} \cdot \boldsymbol{\phi} \\ - \frac{\mu_0}{2} \mathbb{E}(M_s)^2 \mathbf{H}_d(\bar{\boldsymbol{\psi}}) \cdot \boldsymbol{\phi} - \frac{\mu_0}{2} \bar{M} \bar{\boldsymbol{\psi}} \cdot \boldsymbol{\phi} - \frac{\mu_0}{2} \mathbb{E}(M_s) H_N \bar{\boldsymbol{\psi}} \cdot \boldsymbol{\phi} = 0, \end{aligned} \quad (3.15)$$

where the homogenized matrix $\bar{\mathcal{A}}$ is defined as the 9×9 matrix satisfying, for every $\boldsymbol{\nu} \in \mathbb{R}^9$

$$\bar{\mathcal{A}} \boldsymbol{\nu} = \int_{\Omega} \mathcal{A}(\omega) (\boldsymbol{\nu} + \boldsymbol{\eta}_\nu(\omega)) d\mu(\omega) \quad \text{for all } \boldsymbol{\nu} \text{ in } \mathbb{R}^9,$$

with $\boldsymbol{\eta}_\nu$ defined as the unique solution of the problem

$$\begin{cases} \boldsymbol{\eta}_\nu \in (L^2_{\text{pot}}(\Omega))^3, \\ \mathcal{A}(\cdot) (\boldsymbol{\nu} + \boldsymbol{\eta}_\nu(\cdot)) \in (L^2_{\text{sol}}(\Omega))^3. \end{cases} \quad (3.16)$$

Similarly, the homogenized demagnetizing matrix \bar{M} is defined by : for every $\boldsymbol{\nu}$ in \mathbb{R}^3 ,

$$\bar{M} \boldsymbol{\nu} = -\mathbb{E}(M_s \boldsymbol{\rho}_\nu),$$

where $\boldsymbol{\rho}_\nu$ is the unique solution to the problem

$$\begin{cases} \boldsymbol{\rho}_\nu \in L^2_{\text{pot}}(\Omega), \\ M_s(\cdot) \boldsymbol{\nu} + \boldsymbol{\rho}_\nu(\cdot) \in L^2_{\text{sol}}(\Omega). \end{cases} \quad (3.17)$$

By Proposition 2.32, we know that the matrix \bar{M} is well defined, symmetric and satisfies

$$0 \leq \bar{M} \leq \text{Var}(M_s) I_d,$$

in the sense of non-negative symmetric matrices.

By Proposition 2.22, the matrix $\bar{\mathcal{A}}$ is well defined, and satisfies by Lemma 2.19,

$$\mathbb{E}(a^{-1})^{-1} (\mathcal{L}^T \mathcal{L} + \text{Tr}^T \text{Tr}) \leq \bar{\mathcal{A}} \leq \mathbb{E}(a) (\mathcal{L}^T \mathcal{L} + \text{Tr}^T \text{Tr}) ,$$

in the sense of non-negative symmetric matrices.

Remark 3.2. In particular, by Proposition 2.30 of two-scale convergence of the demagnetizing field, if (3.13) is satisfied for every $\epsilon > 0$, then, the domain \mathcal{D} and the probability settings are such that

$$\int_{\mathcal{D}} \mathbb{E}(M_s)^2 \mathbf{H}_d(\mathbf{m}) \cdot \boldsymbol{\psi} - \bar{M} \mathbf{m} \cdot \boldsymbol{\psi} = 0 \quad \forall \boldsymbol{\psi} \in H^1(\mathcal{D}, \mathbb{R}^3) ; \boldsymbol{\psi} \cdot \mathbf{m} = 0 ,$$

which can be separated into two different assumptions on the domain \mathcal{D} and the probability setting $(\Omega, \mathcal{F}, \mu)$:

- The demagnetizing field $\mathbf{H}_d(\mathbf{m})$ must be a (possibly non-constant) vector parallel to \mathbf{m} inside \mathcal{D} . This condition is satisfied for ellipsoids [53], infinite cylinders (see subsection 3.1), but not for cubes [1].
- The vector \mathbf{m} is an eigenvector of the matrix \bar{M} . As we will see in Subsection 2.3, this condition is satisfied for multilayer structure of axis parallel or perpendicular to \mathbf{m} . Moreover, it is satisfied in the case where the probability setting is such that M_s is *isotropic in law*, see Proposition 2.33.

Proposition 3.1. *Let $\boldsymbol{\psi}^\epsilon$ be a solution of the problem (3.14) bounded in $H^1(\mathcal{D}, \mathbb{R}^3)$. The sequence $(\boldsymbol{\psi}^\epsilon)_\epsilon$ converges in $L^2(\mathcal{D}, \mathbb{R}^3)$, and weakly converges in $H^1(\mathcal{D}, \mathbb{R}^3)$ towards a solution $\bar{\boldsymbol{\psi}}$ of the homogenized problem (3.15).*

Proof. According to the Theorem of existence of a two-scale limit 2.21 and Rellich theorem, up to the extraction of a subsequence, it holds that

$$\left\{ \begin{array}{l} \boldsymbol{\psi}^\epsilon(x) \xrightarrow{L^2} \bar{\boldsymbol{\psi}}(x) \quad \text{strongly,} \\ \nabla \boldsymbol{\psi}^\epsilon(x) \xrightarrow{L^2} \nabla \bar{\boldsymbol{\psi}}(x) + \boldsymbol{\xi}(x, \omega), \\ \boldsymbol{\psi}^\epsilon \rightarrow \bar{\boldsymbol{\psi}} \quad \text{weakly in } H^1(\mathcal{D}, \mathbb{R}^2), \\ \mathbf{H}_d(M_s(T_{x/\epsilon} \tilde{\omega}) \boldsymbol{\psi}^\epsilon) \xrightarrow{L^2} \mathbb{E}(M_s) \mathbf{H}_d(\bar{\boldsymbol{\psi}}) + \mathbf{w}(x, \omega), \\ M_s(T_{x/\epsilon} \tilde{\omega}) \mathbf{m} \xrightarrow{L^2} M_s(\omega) \mathbf{m} \quad \text{strongly,} \\ \mathbf{H}_d(M_s(T_{x/\epsilon} \tilde{\omega}) \mathbf{m}) \xrightarrow{L^2} \mathbb{E}(M_s) \mathbf{H}_d(\mathbf{m}) + \mathbf{h}(\omega), \end{array} \right.$$

for some $\bar{\boldsymbol{\psi}}$ in $H^1(\mathcal{D}, \mathbb{R}^3)$, $\boldsymbol{\xi}$ in $L^2(\mathcal{D}, L^2_{\text{pot}}(\Omega, \mathbb{R}^3))$, $\mathbf{w} \in L^2(\mathbb{R}^3, L^2_{\text{pot}}(\Omega)^3)$, and $\mathbf{h} \in L^2_{\text{pot}}(\Omega)^3$. Moreover, for every $x \in \mathbb{R}^3$, $g \in C^1(\Omega)$,

$$\begin{aligned} \chi_{\mathcal{D}}(x) \int_{\Omega} M_s(\omega) \bar{\boldsymbol{\psi}}(x) \cdot D_\omega g(\omega) d\mu(\omega) &= - \int_{\Omega} \mathbf{w}(x, \omega) \cdot D_\omega g(\omega) d\mu(\omega), \\ \int_{\Omega} M_s(\omega) \mathbf{m} \cdot D_\omega g(\omega) d\mu(\omega) &= - \int_{\Omega} \mathbf{h}(\omega) \cdot D_\omega g(\omega) d\mu(\omega). \end{aligned}$$

Rewriting equation (3.14) with the test function $\phi(x) = \epsilon\psi(x)\mathbf{v}(T_{x/\epsilon}\tilde{\omega})$, with $\psi \in C_c^\infty(\mathcal{D})$ and $\mathbf{v} \in C^1(\Omega, \mathbb{R}^3)$, it holds that

$$\begin{aligned} & \int_{\mathcal{D}} \mathcal{A}(T_{x/\epsilon}\tilde{\omega}) \nabla \psi^\epsilon \cdot (\psi(x) D_\omega \mathbf{v}(T_{x/\epsilon}\tilde{\omega})) + \epsilon \int_{\mathcal{D}} \mathcal{A}(T_{x/\epsilon}\tilde{\omega}) \nabla \psi^\epsilon \cdot (\nabla \psi(x) : \mathbf{v}(T_{x/\epsilon}\tilde{\omega})) \\ & \quad + \epsilon \int_{\mathcal{D}} \left(K(T_{x/\epsilon}\tilde{\omega}) \psi^\epsilon + \frac{\mu_0}{2} M_s(T_{x/\epsilon}\tilde{\omega}) (\mathbf{H}_d(M_s(T_{x/\epsilon}\tilde{\omega}) \mathbf{m}) \cdot \mathbf{m}) \psi^\epsilon \right. \\ & \quad \left. - \frac{\mu_0}{2} M_s(T_{x/\epsilon}\tilde{\omega}) \mathbf{H}_d(M_s(T_{x/\epsilon}\tilde{\omega}) \psi^\epsilon) - \frac{\mu_0}{2} M_s(T_{x/\epsilon}\tilde{\omega}) H_N \right) \cdot \psi(x) \mathbf{v}(T_{x/\epsilon}\tilde{\omega}) = 0, \end{aligned}$$

As a , λ and M_s are continuous on Ω (and in particular, are bounded), we are allowed to pass to the two-scale limit and obtain

$$\int_{\mathcal{D}} \int_{\Omega} \mathcal{A}(\omega) (\nabla \bar{\psi}(x) + \boldsymbol{\xi}(x, \omega)) \cdot \psi(x) D_\omega \mathbf{v}(\omega) d\mu(\omega) = 0,$$

and, ψ being an arbitrary test function,

$$\int_{\Omega} \mathcal{A}(\omega) (\nabla \bar{\psi}(x) + \boldsymbol{\xi}(x, \omega)) \cdot D_\omega \mathbf{v}(\omega) d\mu(\omega) = 0, \quad \text{for } x \text{ a.e. in } \mathbb{R}^3.$$

We thus deduce that $\mathcal{A}(\cdot)(\nabla \bar{\psi}(x) + \boldsymbol{\xi}(x, \cdot))$ is in $(L_{\text{sol}}^2(\Omega))^3$ for a.e. $x \in \mathcal{D}$. As $\boldsymbol{\xi}(x, \cdot)$ is in $(L_{\text{pot}}^2(\Omega))^3$, it follows that $\boldsymbol{\xi}(x, \cdot)$ is the unique solution to the problem (3.16) with $\nu = \nabla \bar{\psi}(x)$, thus,

$$\int_{\Omega} \mathcal{A}(\omega) (\nabla \bar{\psi}(x) + \boldsymbol{\xi}(x, \omega)) d\mu(\omega) = \bar{\mathcal{A}} \nabla \bar{\psi}(x).$$

Moreover, passing to the limit in the two-scale sense in Equation (3.14), it holds that for every $\phi \in H^1(\mathcal{D}, \mathbb{R}^3)$,

$$\begin{aligned} & \int_{\mathcal{D}} \int_{\Omega} \mathcal{A}(\omega) (\nabla \bar{\psi} + \boldsymbol{\xi}(\cdot, \omega)) \cdot \nabla \phi \\ & \quad + \left(K + \frac{\mu_0}{2} \mathbb{E}(M_s) M_s \mathbf{H}_d(\mathbf{m}) \cdot \mathbf{m} + \frac{\mu_0}{2} \mathbf{m} \cdot M_s \mathbf{v} \right) \bar{\psi} \cdot \phi \\ & \quad - \frac{\mu_0}{2} \mathbb{E}(M_s) M_s \mathbf{H}_d(\bar{\psi}) \cdot \phi + \frac{\mu_0}{2} M_s \mathbf{w} \cdot \phi - \frac{\mu_0}{2} M_s H_N \bar{\psi} \cdot \phi d\mu(\omega) dx = 0, \end{aligned}$$

or, in other terms,

$$\begin{aligned} & \int_{\mathcal{D}} \bar{\mathcal{A}} \nabla \bar{\psi} \cdot \nabla \phi + \left(\mathbb{E}(K) + \frac{\mu_0}{2} \mathbb{E}(M_s)^2 \mathbf{H}_d(\mathbf{m}) \cdot \mathbf{m} - \frac{\mu_0}{2} \mathbf{m} \cdot \bar{M} \mathbf{m} \right) \bar{\psi} \cdot \phi \\ & \quad - \frac{\mu_0}{2} \mathbb{E}(M_s)^2 \mathbf{H}_d(\bar{\psi}) \cdot \phi - \frac{\mu_0}{2} \bar{M} \bar{\psi} \cdot \phi - \frac{\mu_0}{2} \mathbb{E}(M_s) H_N \bar{\psi} \cdot \phi = 0. \end{aligned}$$

Indeed, for every x in \mathcal{D} , $\mathbf{w}(x, \cdot)$ is in $L_{\text{pot}}^2(\Omega)^3$ and $M_s(\cdot) \bar{\psi}(x) + \mathbf{w}(x, \cdot)$ is in $L_{\text{sol}}^2(\Omega)^3$ by definition of \mathbf{w} , therefore, $\mathbf{w}(x, \cdot)$ is solution of the cell problem (3.17), and

$$\bar{M} \bar{\psi}(x) = - \int_{\Omega} M_s(\omega) \mathbf{w}(x, \omega) d\mu(\omega).$$

Likewise, we get

$$\bar{M} \mathbf{m} = - \int_{\Omega} M_s(\omega) \mathbf{v}(\omega) d\mu(\omega).$$

□

2.2 Convergence of the nucleation field

Proposition 3.2. *Let $(H_N^\epsilon)_{\epsilon>0}$ be the sequence of nucleation fields defined as the smallest value for which (3.14) admits a non-trivial solution.*

Then, $(H_N^\epsilon)_{\epsilon>0}$ converges to the nucleation field $\overline{H_N}$ defined as the smallest field for which (3.15) admits a non-trivial solution.

Proof. We will prove this proposition by the means of Γ -convergence, introduced by De Giorgi in [26]. We use the following definition of Γ -convergence.

Definition 3.2. Let (X, d) be a metric space. Consider for $\epsilon > 0$ a family of functionals

$$J_\epsilon : X \rightarrow \mathbb{R},$$

a limiting functional

$$J_0 : X \rightarrow \mathbb{R},$$

and the corresponding minimization problems

$$(\mathcal{P}_\epsilon) \quad \min_{u \in X} J_\epsilon, \quad (\mathcal{P}_0) \quad \min_{u \in X} J_0.$$

We say that the family of problems (\mathcal{P}_ϵ) $\Gamma(d)$ -converges to (\mathcal{P}_0) if for any $u_0 \in X$ one has

- Γ -lim inf. For every sequence $(u_\epsilon)_\epsilon$ that converges to u_0 in X , one has

$$\liminf_{\epsilon \rightarrow 0} J_\epsilon(u_\epsilon) \geq J_0(u_0). \quad (3.18)$$

- Γ -lim sup. There exists a sequence $(u_\epsilon)_\epsilon$ that converges to u_0 in X such that

$$\limsup_{\epsilon \rightarrow 0} J_\epsilon(u_\epsilon) \leq J_0(u_0). \quad (3.19)$$

The Γ -convergence is defined for a metric space, but can be extended for $H^1(\mathcal{D})$ endowed with the weak convergence topology.

For any $\epsilon > 0$, by a version of (3.6) adapted to the composite material, the nucleation field H_N^ϵ is defined by

$$H_N^\epsilon = \inf_{\substack{\psi \in H^1(\mathcal{D}, \mathbb{R}^3) \cap L^\infty(\mathcal{D}, \mathbb{R}^3) \\ \psi \cdot \mathbf{m} = 0}} \frac{2q^\epsilon(\psi)}{\int_{\mathcal{D}} \mu_0 M^\epsilon |\psi|^2},$$

for the quadratic form q^ϵ defined by

$$q^\epsilon(\psi) = \int_{\mathcal{D}} \mathcal{A}^\epsilon \nabla \psi \cdot \nabla \psi + \left(K^\epsilon + \frac{\mu_0 M^\epsilon}{2} \mathbf{H}_d(M^\epsilon \mathbf{m}) \cdot \mathbf{m} \right) |\psi|^2 - \frac{\mu_0 M^\epsilon}{2} \mathbf{H}_d(M^\epsilon \psi) \cdot \psi,$$

where we have denoted $\mathcal{A}^\epsilon = \mathcal{A}(T_{x/\epsilon} \tilde{\omega})$, $M^\epsilon = M(T_{x/\epsilon} \tilde{\omega})$ and $K^\epsilon = K(T_{x/\epsilon} \tilde{\omega})$, with $\tilde{\omega}$ is a typical trajectory. As $H^1(\mathcal{D}, \mathbb{R}^3) \cap L^\infty(\mathcal{D}, \mathbb{R}^3)$ is dense in $H^1(\mathcal{D}, \mathbb{R}^3)$, H_N^ϵ is the minimum of the minimization problem

$$(\mathcal{P}_\epsilon)_\epsilon \quad H_N^\epsilon = \min_{\substack{\psi \in H^1(\mathcal{D}, \mathbb{R}^3) \\ \psi \cdot \mathbf{m} = 0}} \frac{2q^\epsilon(\psi)}{\int_{\mathcal{D}} \mu_0 M^\epsilon |\psi|^2}.$$

Thus, we will prove the Γ -convergence in $H^1(\mathcal{D}, \mathbb{R}^3)$ of $(\mathcal{P}_\epsilon)_{\epsilon>0}$ to

$$(\mathcal{P}_0) \quad \inf_{\substack{\psi \in H^1(\mathcal{D}, \mathbb{R}^3) \cap L^\infty(\mathcal{D}, \mathbb{R}^3); \\ \psi \cdot \mathbf{m} = 0}} \frac{2\bar{q}(\psi)}{\mu_0 \mathbb{E}(M_s) \|\psi\|_{L^2(\mathcal{D}, \mathbb{R}^3)}^2},$$

for the quadratic form \bar{q} defined by

$$\begin{aligned} \bar{q}(\psi) = & \int_{\mathcal{D}} \bar{\mathcal{A}} \nabla \psi \cdot \nabla \psi + \left(\mathbb{E}(K) + \frac{\mu_0 \mathbb{E}(M_s)^2}{2} \mathbf{H}_d(\mathbf{m}) \cdot \mathbf{m} - \frac{\mu_0 \mathbf{m} \cdot \bar{M} \mathbf{m}}{2} \right) |\psi|^2 \\ & - \frac{\mu_0 \mathbb{E}(M_s)^2}{2} \mathbf{H}_d(\psi) \cdot \psi + \frac{\mu_0}{2} \psi \cdot \bar{M} \psi. \end{aligned}$$

This will prove the proposition by the following classical property of Γ -convergence.

Proposition 3.3. *Let X be a metric space, and let $(\mathcal{P}_\epsilon)_{\epsilon>0}$ be a minimization problem defined by*

$$(\mathcal{P}_\epsilon) \quad \min_{u \in X} J_\epsilon,$$

with J_ϵ a functional on X . Then, if the sequence $(\mathcal{P}_\epsilon)_{\epsilon>0}$ Γ -converges to a minimization problem

$$(\mathcal{P}_0) \quad \min_{u \in X} J_0,$$

and if we consider $(u^\epsilon)_{\epsilon>0}$ a sequence of minimizers of $(\mathcal{P}_\epsilon)_{\epsilon>0}$ that converges to u^0 , u^0 is a minimizer of (\mathcal{P}_0) and

$$\lim_{\epsilon \rightarrow 0} J_\epsilon(u^\epsilon) = J_0(u^0).$$

It is well-known that the Γ -lim sup definition (3.19) may be replaced by the following weaker statement for which this proposition still holds true: *For all $\eta > 0$, there exists a sequence $(u_\epsilon)_\epsilon$ that converges to u_0 such that*

$$\limsup J_\epsilon(u_\epsilon) \leq J_0(u_0) + \eta. \quad (3.20)$$

We will make use of this statement in our proof.

The proof of Γ -convergence typically divides into two steps: we show the Γ -lim inf by means of weak convergence methods, and then the Γ -lim sup using a recovery sequence.

Step 1. Let us show the Γ -lim inf: let $\psi^\epsilon \in H^1(\mathcal{D}, \mathbb{R}^3)$ such that $\psi^\epsilon \cdot \mathbf{m} = 0$ that weakly converges to $\bar{\psi}$ in $H^1(\mathcal{D}, \mathbb{R}^3)$. Let us show that

$$\liminf_{\epsilon \rightarrow 0} \frac{2q^\epsilon(\psi^\epsilon)}{\int_{\mathcal{D}} \mu_0 M^\epsilon |\psi^\epsilon|^2 dx} \geq \frac{2\bar{q}(\bar{\psi})}{\mu_0 \mathbb{E}(M_s) \|\bar{\psi}\|_{L^2(\mathcal{D}, \mathbb{R}^3)}^2}.$$

By boundedness of $(\psi^\epsilon)_{\epsilon>0}$ in $H^1(\mathcal{D}, \mathbb{R}^3)$, the Theorem of existence of a two-scale limit 2.21, Rellich theorem, and the uniqueness of the limit in $L^2(\mathcal{D}, \mathbb{R}^3)$, it holds that, up to

the extraction of a subsequence, it holds that

$$\left\{ \begin{array}{l} \boldsymbol{\psi}^\epsilon(x) \xrightarrow{L^2} \bar{\boldsymbol{\psi}}(x) \text{ strongly,} \\ \nabla \boldsymbol{\psi}^\epsilon(x) \xrightarrow{L^2} \nabla \bar{\boldsymbol{\psi}}(x) + \boldsymbol{\xi}(x, \omega), \\ \mathbf{H}_d(M_s(T_{x/\epsilon}\tilde{\omega})\boldsymbol{\psi}^\epsilon) \xrightarrow{L^2} \mathbb{E}(M_s)\mathbf{H}_d(\bar{\boldsymbol{\psi}}) + \mathbf{w}(x, \omega), \\ M_s(T_{x/\epsilon}\tilde{\omega})\mathbf{m} \xrightarrow{L^2} M_s(\omega)\mathbf{m} \text{ strongly,} \\ \mathbf{H}_d(M_s(T_{x/\epsilon}\tilde{\omega})\mathbf{m}) \xrightarrow{L^2} \mathbb{E}(M_s)\mathbf{H}_d(\mathbf{m}) + \mathbf{h}(\omega), \end{array} \right.$$

for some $\boldsymbol{\xi}$ in $L^2(\mathcal{D}, L^2_{\text{pot}}(\Omega, \mathbb{R}^3))$, $\mathbf{w} \in L^2(\mathbb{R}^3, L^2_{\text{pot}}(\Omega)^3)$, and $\mathbf{h} \in L^2_{\text{pot}}(\Omega)^3$ such that, for every $x \in \mathbb{R}^3$, $g \in C^1(\Omega)$,

$$\begin{aligned} \chi_{\mathcal{D}}(x) \int_{\Omega} M_s(\omega) \bar{\boldsymbol{\psi}}(x) \cdot D_{\omega} g(\omega) d\mu(\omega) &= - \int_{\Omega} \mathbf{w}(x, \omega) \cdot D_{\omega} g(\omega) d\mu(\omega), \\ \int_{\Omega} M_s(\omega) \mathbf{m} \cdot D_{\omega} g(\omega) d\mu(\omega) &= - \int_{\Omega} \mathbf{h}(\omega) \cdot D_{\omega} g(\omega) d\mu(\omega). \end{aligned}$$

Thus, we have immediately that

$$\lim_{\epsilon \rightarrow 0} \int_{\mathcal{D}} \mu_0 M^\epsilon |\boldsymbol{\psi}^\epsilon|^2 = \mu_0 \mathbb{E}(M_s) \|\bar{\boldsymbol{\psi}}\|_{L^2(\mathcal{D}, \mathbb{R}^3)}^2.$$

It remains to show that

$$\liminf_{\epsilon \rightarrow 0} q^\epsilon(\boldsymbol{\psi}^\epsilon) \geq \bar{q}(\bar{\boldsymbol{\psi}}).$$

First, by two-scale convergence of $\nabla \boldsymbol{\psi}^\epsilon$, the inferior limit of the exchange energy satisfies

$$\begin{aligned} \liminf_{\epsilon \rightarrow 0} \int_{\mathcal{D}} \mathcal{A}(T_{x/\epsilon}\omega) \nabla \boldsymbol{\psi}^\epsilon \cdot \nabla \boldsymbol{\psi}^\epsilon &\geq \int_{\mathcal{D}} \int_{\Omega} \mathcal{A}(\nabla \bar{\boldsymbol{\psi}} + \boldsymbol{\xi}) \cdot (\nabla \bar{\boldsymbol{\psi}} + \boldsymbol{\xi}) d\mu dx \\ &\geq \int_{\mathcal{D}} \inf_{\mathbf{v} \in L^2_{\text{pot}}(\Omega)} \int_{\Omega} \mathcal{A}(\nabla \bar{\boldsymbol{\psi}} + \mathbf{v}) \cdot (\nabla \bar{\boldsymbol{\psi}} + \mathbf{v}) d\mu dx \\ &= \int_{\mathcal{D}} \bar{\mathcal{A}} \nabla \bar{\boldsymbol{\psi}} \cdot \nabla \bar{\boldsymbol{\psi}} dx, \end{aligned}$$

by Proposition 2.22.

The other terms converge as they are products of a two-scale strongly converging sequence and a weakly two-scale convergence sequence, by the Proposition 2.10 of weak-strong convergence:

$$\begin{aligned} &\lim_{\epsilon \rightarrow 0} \int_{\mathcal{D}} \left(K^\epsilon + \frac{\mu_0 M^\epsilon}{2} \mathbf{H}_d(M^\epsilon \mathbf{m}) \cdot \mathbf{m} \right) |\boldsymbol{\psi}^\epsilon|^2 - \frac{\mu_0 M^\epsilon}{2} \mathbf{H}_d(M^\epsilon \boldsymbol{\psi}^\epsilon) \cdot \boldsymbol{\psi}^\epsilon \\ &= \int_{\mathcal{D}} \int_{\Omega} \left(K + \frac{\mu_0 \mathbb{E}(M_s) M_s}{2} \mathbf{H}_d(\mathbf{m}) \cdot \mathbf{m} + \frac{\mu_0 M_s}{2} \mathbf{m} \cdot \mathbf{h} \right) |\bar{\boldsymbol{\psi}}|^2 - \frac{\mu_0 \mathbb{E}(M_s) M_s}{2} \mathbf{H}_d(\bar{\boldsymbol{\psi}}) \cdot \bar{\boldsymbol{\psi}} \\ &\quad - \frac{\mu_0 M_s}{2} \bar{\boldsymbol{\psi}} \cdot \mathbf{w} d\mu dx \\ &= \int_{\mathcal{D}} \left(\mathbb{E}(K) + \frac{\mu_0 \mathbb{E}(M_s)^2}{2} \mathbf{H}_d(\mathbf{m}) \cdot \mathbf{m} - \frac{\mu_0 \mathbf{m} \cdot \bar{M} \mathbf{m}}{2} \right) |\bar{\boldsymbol{\psi}}|^2 - \frac{\mu_0 \mathbb{E}(M_s)^2}{2} \mathbf{H}_d(\bar{\boldsymbol{\psi}}) \cdot \bar{\boldsymbol{\psi}} \\ &\quad + \frac{\mu_0}{2} \bar{\boldsymbol{\psi}} \cdot \bar{M} \bar{\boldsymbol{\psi}} dx. \end{aligned}$$

The conclusion follows.

Step 2. Let us show the Γ -lim sup. Let $\bar{\psi} \in H^1(\mathcal{D}, \mathbb{R}^3)$. Then, by Proposition (2.22), we have, for every $x \in \mathcal{D}$,

$$\begin{aligned} \nabla \bar{\psi}(x) \cdot \bar{\mathcal{A}} \nabla \bar{\psi}(x) &= \inf_{\xi \in L^2_{\text{pot}}(\Omega)^3} \int_{\Omega} (\nabla \bar{\psi}(x) + \xi(\omega)) \cdot \mathcal{A}(\omega) (\nabla \bar{\psi}(x) + \xi(\omega)) d\mu(\omega) \\ &= \inf_{\mathbf{v} \in C^1(\Omega)^3} \int_{\Omega} (\nabla \bar{\psi}(x) + D_{\omega} \mathbf{v}(\omega)) \cdot \mathcal{A}(\omega) (\nabla \bar{\psi}(x) + D_{\omega} \mathbf{v}(\omega)) d\mu(\omega), \end{aligned} \quad (3.21)$$

by definition of $L^2_{\text{pot}}(\Omega)$. For every $x \in \mathcal{D}$, let us take $(\mathbf{v}^n(x))_n \in (C^1(\Omega))^{\mathbb{N}}$ a minimizing sequence of this quantity. By (3.21), for every $n \in \mathbb{N}$, \mathbf{v}^n is in $L^2(\mathcal{D}, C^1(\Omega)^3)$, as $\nabla \bar{\psi}$ is in $L^2(\mathcal{D})$. By Remark 2.2, the linear span of test functions $C_c^{\infty}(\mathcal{D}) \times C^1(\Omega)^3$ is dense in $L^2(\mathcal{D}, C^1(\Omega)^3)$. Thus, let us consider a sequence $(\mathbf{w}^n)_n$ in the linear span of test functions $C_c^{\infty}(\mathcal{D}) \times C^1(\Omega)^3$ such that,

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_{\mathcal{D}} \int_{\Omega} (\nabla \bar{\psi}(x) + D_{\omega} \mathbf{w}^n(x, \omega)) \cdot \mathcal{A}(\omega) (\nabla \bar{\psi}(x) + D_{\omega} \mathbf{w}^n(x, \omega)) d\mu(\omega) dx \\ = \int_{\mathcal{D}} \nabla \bar{\psi}(x) \cdot \bar{\mathcal{A}} \nabla \bar{\psi}(x) dx. \end{aligned} \quad (3.22)$$

For every $\epsilon > 0$ and $n \in \mathbb{N}$, let us define, for $x \in \mathcal{D}$,

$$\psi_n^{\epsilon}(x) = \bar{\psi}(x) + \epsilon \mathbf{w}^n(x, T_{x/\epsilon} \tilde{\omega}).$$

Then, for every $n \in \mathbb{N}$, $(\psi_n^{\epsilon})_{\epsilon}$ two-scale converges to $\bar{\psi}$, the sequence satisfies, for every $\varphi \in C_c^{\infty}(\mathcal{D})$, $b \in C(\Omega)^3$,

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \int_{\mathcal{D}} \psi_n^{\epsilon}(x) \varphi(x) b(T_{x/\epsilon} \tilde{\omega}) dx &= \lim_{\epsilon \rightarrow 0} \int_{\mathcal{D}} \bar{\psi}(x) \varphi(x) b(T_{x/\epsilon} \tilde{\omega}) dx + \epsilon \int_{\mathcal{D}} \mathbf{w}^n(x, T_{x/\epsilon} \tilde{\omega}) \varphi(x) b(T_{x/\epsilon} \tilde{\omega}) dx \\ &= \int_{\mathcal{D}} \bar{\psi}(x) \varphi(x) dx \int_{\Omega} b d\mu, \end{aligned}$$

by the Proposition 2.9 of mean-value property, and $(\nabla \psi_n^{\epsilon})_{\epsilon}$ two-scale converges to $\nabla \bar{\psi} + D_{\omega} \mathbf{w}^n$, for every $\varphi \in C_c^{\infty}(\mathcal{D})$, $b \in C(\Omega)^{3 \times 3}$,

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \int_{\mathcal{D}} \nabla \psi_n^{\epsilon}(x) \cdot \varphi(x) b(T_{x/\epsilon} \tilde{\omega}) dx &= \lim_{\epsilon \rightarrow 0} \int_{\mathcal{D}} (\nabla \bar{\psi}(x) + D_{\omega} \mathbf{w}^n(x, T_{x/\epsilon} \tilde{\omega})) \cdot \varphi(x) b(T_{x/\epsilon} \tilde{\omega}) dx \\ &\quad + \epsilon \int_{\mathcal{D}} \nabla_x \mathbf{w}^n(x, T_{x/\epsilon} \tilde{\omega}) \cdot \varphi(x) b(T_{x/\epsilon} \tilde{\omega}) dx \\ &= \int_{\mathcal{D}} \int_{\Omega} (\nabla \bar{\psi}(x) + D_{\omega} \mathbf{w}^n(x, \omega)) \cdot \varphi(x) b(\omega) d\mu(\omega) dx, \end{aligned}$$

by the same argument. It follows in particular that $(\psi_n^\epsilon)_\epsilon$ weakly converges to $\bar{\psi}$ in $H^1(\mathcal{D}, \mathbb{R}^3)$ and, moreover,

$$\left\{ \begin{array}{l} \psi_n^\epsilon(x) \xrightarrow{L^2} \bar{\psi}(x) \text{ strongly,} \\ \nabla \psi_n^\epsilon(x) \xrightarrow{L^2} \nabla \bar{\psi}(x) + D_\omega \mathbf{w}^n(x, \omega), \\ \mathbf{H}_d(M_s(T_{x/\epsilon} \tilde{\omega}) \psi_n^\epsilon) \xrightarrow{L^2} \mathbb{E}(M_s) \mathbf{H}_d(\bar{\psi}) + \mathbf{w}(x, \omega), \\ M_s(T_{x/\epsilon} \tilde{\omega}) \mathbf{m} \xrightarrow{L^2} M_s(\omega) \mathbf{m} \text{ strongly,} \\ \mathbf{H}_d(M_s(T_{x/\epsilon} \tilde{\omega}) \mathbf{m}) \xrightarrow{L^2} \mathbb{E}(M_s) \mathbf{H}_d(\mathbf{m}) + \mathbf{h}(\omega), \end{array} \right.$$

with $\mathbf{w} \in L^2(\mathbb{R}^3, L^2_{\text{pot}}(\Omega)^3)$ and $\mathbf{h} \in L^2_{\text{pot}}(\Omega)^3$ such that, for every $x \in \mathbb{R}^3$, $g \in C^1(\Omega)$,

$$\begin{aligned} \chi_{\mathcal{D}}(x) \int_{\Omega} M_s(\omega) \bar{\psi}(x) \cdot D_\omega g(\omega) d\mu(\omega) &= - \int_{\Omega} \mathbf{w}(x, \omega) \cdot D_\omega g(\omega) d\mu(\omega), \\ \int_{\Omega} M_s(\omega) \mathbf{m} \cdot D_\omega g(\omega) d\mu(\omega) &= - \int_{\Omega} \mathbf{h}(\omega) \cdot D_\omega g(\omega) d\mu(\omega). \end{aligned}$$

By two-scale convergence, we have, for the exchange energy,

$$\begin{aligned} \limsup_{\epsilon \rightarrow 0} \int_{\mathcal{D}} \mathcal{A}^\epsilon \nabla \psi_n^\epsilon \cdot \nabla \psi_n^\epsilon &= \limsup_{\epsilon \rightarrow 0} \int_{\mathcal{D}} (\nabla \bar{\psi}(x) + D_\omega \mathbf{w}^n(x, T_{x/\epsilon} \tilde{\omega}) + \epsilon \nabla_x \mathbf{w}^n(x, T_{x/\epsilon} \tilde{\omega})) \\ &\quad \cdot \mathcal{A}(T_{x/\epsilon} \tilde{\omega}) (\nabla \bar{\psi}(x) + D_\omega \mathbf{w}^n(x, T_{x/\epsilon} \tilde{\omega}) + \epsilon \nabla_x \mathbf{w}^n(x, T_{x/\epsilon} \tilde{\omega})) dx \\ &= \int_{\mathcal{D}} \int_{\Omega} (\nabla \bar{\psi}(x) + D_\omega \mathbf{w}^n(x, \omega)) \cdot \mathcal{A}(\omega) (\nabla \bar{\psi}(x) + D_\omega \mathbf{w}^n(x, \omega)) dx. \end{aligned}$$

Thus, for $\eta > 0$ fixed, taking by (3.22), $n \in \mathbb{N}$ such that

$$\left| \int_{\mathcal{D}} (\nabla \bar{\psi}(x) + D_\omega \mathbf{w}^n(x, \omega)) \cdot \mathcal{A}(\omega) (\nabla \bar{\psi}(x) + D_\omega \mathbf{w}^n(x, \omega)) d\mu(\omega) dx - \int_{\mathcal{D}} \nabla \bar{\psi}(x) \cdot \bar{\mathcal{A}} \nabla \bar{\psi}(x) dx \right| \leq \eta,$$

we have a sequence $(\psi_n^\epsilon)_\epsilon$ such that

$$\limsup_{\epsilon \rightarrow 0} \int_{\mathcal{D}} \mathcal{A}^\epsilon \nabla \psi_n^\epsilon \cdot \psi_n^\epsilon \leq \int_{\mathcal{D}} \nabla \bar{\psi}(x) \cdot \bar{\mathcal{A}} \nabla \bar{\psi}(x) dx + \eta. \quad (3.23)$$

The other terms converge as they are products of a two-scale strongly converging sequence and a weakly two-scale converging sequence, by the Proposition 2.10 of weak-strong

convergence:

$$\begin{aligned}
& \lim_{\epsilon \rightarrow 0} \int_{\mathcal{D}} \left(K^\epsilon + \frac{\mu_0 M^\epsilon}{2} \mathbf{H}_d(M^\epsilon \mathbf{m}) \cdot \mathbf{m} \right) |\psi_n^\epsilon|^2 - \frac{\mu_0 M^\epsilon}{2} \mathbf{H}_d(M^\epsilon \psi_n^\epsilon) \cdot \psi_n^\epsilon \\
&= \int_{\mathcal{D}} \int_{\Omega} \left(K + \frac{\mu_0 \mathbb{E}(M_s) M_s}{2} \mathbf{H}_d(\mathbf{m}) \cdot \mathbf{m} + \frac{\mu_0 M_s}{2} \mathbf{m} \cdot \mathbf{h} \right) |\bar{\psi}|^2 - \frac{\mu_0 \mathbb{E}(M_s) M_s}{2} \mathbf{H}_d(\bar{\psi}) \cdot \bar{\psi} \\
&\quad - \frac{\mu_0 M_s}{2} \bar{\psi} \cdot \mathbf{w} d\mu dx \\
&= \int_{\mathcal{D}} \left(\mathbb{E}(K) + \frac{\mu_0 \mathbb{E}(M_s)^2}{2} \mathbf{H}_d(\mathbf{m}) \cdot \mathbf{m} - \frac{\mu_0 \mathbf{m} \cdot \bar{M} \mathbf{m}}{2} \right) |\bar{\psi}|^2 - \frac{\mu_0 \mathbb{E}(M_s)^2}{2} \mathbf{H}_d(\bar{\psi}) \cdot \bar{\psi} \\
&\quad + \frac{\mu_0}{2} \bar{\psi} \cdot \bar{M} \bar{\psi} dx.
\end{aligned}$$

The conclusion follows. \square

2.3 Multilayer structures

Let us compute the homogenized quantities in the case of a multilayer structure where the layers are distributed along the axis of rotation of the cylinder \mathbf{e}_z . Let us assume that the exchange matrix $a(\omega)$ is a scalar for every $\omega \in \Omega$. In this case, the dynamical system, which represents the distribution of materials only depends on the variable z : for $(x, y, z) \in \mathbb{R}^3$, we have $T_{(x,y,z)} = T_z$. It follows that $L_{\text{pot}}^2(\Omega) \subset \{u \mathbf{e}_z; u \in L^2(\Omega), \mathbb{E}(u) = 0\}$. Moreover,

$$L_{\text{sol}}^2(\Omega) = \{\sigma \in H^1(\Omega)^3; D_\omega \cdot \sigma = 0\} = \{\sigma \in (L^2(\Omega))^2 \times H^1(\Omega); D_3 \sigma_3 = 0\}, \quad (3.24)$$

we deduce by the definition of $L_{\text{sol}}^2(\Omega)$ as the orthogonal of $L_{\text{pot}}^2(\Omega)$ in $L^2(\Omega, \mathbb{R}^3)$, that

$$L_{\text{pot}}^2(\Omega) = \{u \mathbf{e}_z; u \in L^2(\Omega), \mathbb{E}(u) = 0\}, \quad (3.25)$$

and therefore the solutions of (3.17) are given by

$$\begin{cases} \rho_{\mathbf{e}_x} = \rho_{\mathbf{e}_y} = 0, \\ \rho_{\mathbf{e}_z} = (\mathbb{E}(M_s) - M_s) \mathbf{e}_z. \end{cases}$$

It holds that

$$\mathbf{m} \cdot \bar{M} \mathbf{m} = -\mathbb{E}(\mathbb{E}(M_s) M_s - M_s^2) (\mathbf{m} \cdot \mathbf{e}_z)^2 = \text{Var}(M_s) (\mathbf{m} \cdot \mathbf{e}_z)^2. \quad (3.26)$$

Furthermore, after calculations, we get that,

$$\begin{aligned}
\bar{\mathbf{A}} \mathbf{m} &= \mathbb{E}(a) \mathcal{L}^T \mathcal{L}^T \mathbf{m} + \mathbb{E}(a^{-1})^{-1} \text{Tr}^T \text{Tr} \mathbf{m} \text{ for } \mathbf{m} \text{ such that } \mathbf{m} \cdot \mathbf{e}_z = 0, \\
\bar{\mathbf{A}} \mathbf{e}_z &= \mathbb{E}(a^{-1})^{-1} (\mathcal{L}^T \mathcal{L} + \text{Tr}^T \text{Tr}) \mathbf{e}_z.
\end{aligned}$$

Writing for $\mathbf{m} \in H^1(\mathcal{D}, \mathbb{R}^3)$, $\mathbf{m} = \mathbf{m}^\perp + m_z \mathbf{e}_z$, the homogenized exchange energy writes

$$\int_{\mathcal{D}} \mathbb{E}(a^{-1})^{-1} (|\nabla \times (m_z \mathbf{e}_z)|^2 + (\partial_z m_z)^2 + (\nabla \cdot \mathbf{m}^\perp)^2) + \mathbb{E}(a) |\nabla \times \mathbf{m}^\perp|^2.$$

As $\mathbb{E}(a^{-1})^{-1} \leq \mathbb{E}(a)$, this exchange energy favors energy spins in the direction of \mathbf{e}_z .

Thus, the most important factor in the evaluation of a magnet is the anisotropy, which writes, according to the last section,

$$\mathbb{E}(K) - \mu_0 \mathbf{m} \cdot \overline{M} \mathbf{m},$$

where $0 \leq \mathbf{m} \cdot \overline{M} \mathbf{m} \leq \text{Var}(M_s)$. It follows that the multilayer structures where the layers are perpendicular to the axis are the configurations with the best anisotropy, and the multilayer structure where the layers are parallel to the axis is the configuration with the worst anisotropy, for the distributions with equal moments. Considering that other terms are averages on the different values, and therefore do not depend on the structure of the spring magnets, and that for small particles, the exchange energy does not impact the value of the nucleation field, the multilayer structures where the layers are perpendicular to the axis are the best configurations for small particles, and the multilayer structure where the layers are parallel to the axis is the worst configuration. Nevertheless, for larger particles, we cannot conclude.

Knowing this fact, let us compute the maximum energy product in the case of multilayer structure where the layers are parallel to the axis, and deduce the optimal ratio of hard and soft material in this case. We use the same calculations as in the introduction.

The hard magnet is characterized by M_1 and K_1 , the soft magnet M_2 and K_2 . Let us denote θ the proportion of hard material. In the case of small particles, assuming that the nucleation occurs at unison, the coercive field writes

$$H_c = \frac{2\mathbb{E}(K) - \mu_0 \text{Var}(M_s)}{\mu_0 \mathbb{E}(M_s)} = \frac{2(\theta K_1 + (1 - \theta)K_2) - \mu_0(M_2 - M_1)^2 \theta(1 - \theta)}{\mu_0(\theta M_1 + (1 - \theta)M_2)},$$

It follows that the optimality is defined by

$$-\frac{\mathbb{E}(M_s)}{2} = -H_c \Leftrightarrow \mu_0 \mathbb{E}(M_s)^2 = 4\mathbb{E}(K) - 2\mu_0 \text{Var}(M_s).$$

Let us estimate these quantities for $K_2 = 0$ and $K_1 \gg M_2^2 + M_1^2$. Following the same elementary calculus as in our introductory chapter 1, we get

$$\theta^* \approx \mu_0 \frac{M_2^2}{4K_1} \left(1 - \mu_0 \frac{M_1(M_2 - M_1)}{2K_1} \right), \quad (3.27)$$

as well as

$$(BH)_{\max}^* \approx \mu_0 \frac{M_2^2}{4} \left(1 - \frac{\mu_0 M_2(M_2 - M_1)}{2K_1} + \frac{\mu_0 M_2(4M_1 + M_2)(M_2 - M_1)^2}{16K_1^2} \right),$$

computing an expansion to the second order in $\frac{\mu_0 M_2(M_2 - M_1)}{2K_1}$.

Comparing this result with equation (1.9), which does not take into account the additional term due to the homogenization of the demagnetizing field, knowing by Proposition 2.30 that this coefficient is comprised between 0 and its maximum value $\text{Var}(M_s)$, and that it is the only term that takes the stochastic geometry of the structure into account, we conclude that both values give us a minimum and a maximum value of the energy product of a spring magnet according to its stochastic geometry.

3 Analysis of nucleation

This section is dedicated to the analysis of the phenomenon of nucleation via the analysis of the *nucleation equation*, introduced in Definition 3.1. We recall that the nucleation field is defined as the lowest value of H such that there exists a non-trivial solution, called a nucleation mode, to the following variational problem: for every $\phi \in H^1(\mathcal{D}, \mathbb{R}^3)$,

$$\int_{\mathcal{D}} A(\nabla \times \psi) \cdot (\nabla \times \phi) + A(\nabla \cdot \psi)(\nabla \cdot \phi) + \left(K_{\text{eff}} \psi - \frac{\mu_0}{2} M_s \mathbf{H}_d(M_s \psi) \right) \cdot \phi - \frac{\mu_0}{2} M_s H \psi \cdot \phi = 0. \quad (3.28)$$

We make use of a method of decomposition of the solution into three well chosen functions, which gives rise to three distinct eigenvalue problems. However, the spaces of decomposition, which correspond to stable spaces for the demagnetizing field \mathbf{H}_d , are slightly different according to the domain of the magnet. Therefore, we will study the case of ellipsoids and infinite cylinders separately.

3.1 Ellipsoids

In this section only, \mathcal{D} is an ellipsoid. It is worth noting that most of this section applies to a bounded open domain \mathcal{D} of class \mathcal{C}^2 such that that exists an aligned equilibrium state. However, to the knowledge of the author, ellipsoids are the only known bounded domains for which this is true.

Proposition 3.4. *Let \mathcal{D} be an ellipsoid, and let us define $\alpha_1(\mathcal{D})$ as the lowest value of $\lambda \in \mathbb{R}$ such that there exists $\alpha \in H^1(\mathcal{D}, \mathbb{R}^3)$ with $\alpha \neq 0$ that satisfies*

$$\begin{cases} -\Delta \alpha = \lambda \alpha & \text{in } \mathcal{D}, \\ \nabla \cdot \alpha = 0 & \text{in } \mathcal{D}, \\ \alpha \times n = 0 & \text{on } \partial \mathcal{D}. \end{cases}$$

There exist two constants $D > 0$ and $\gamma \geq 0$, and a value V_0 defined as the value of the volume of \mathcal{D} such that the following equality holds

$$\frac{2A\alpha_1(\mathcal{D})}{\mu_0 M_s} = \frac{2M_s}{1 + \gamma}. \quad (3.29)$$

such that the value of the nucleation field for a magnet which easy axis is one of the ellipsoid axis, is given by the following expressions, depending on the volume V of the ellipsoid \mathcal{D} with

- if $V < V_0$, then

$$H_N = \frac{2K}{\mu_0 M_s} + 2M_s \left(\frac{1}{1 + \gamma} - D \right),$$

- and if $V \geq V_0$, then

$$H_N = \frac{2K}{\mu_0 M_s} - 2DM_s + \frac{2A\alpha_1(\mathcal{D})}{\mu_0 M_s}.$$

Remark 3.3. We may notice how the nucleation modes are characterized by the exchange length and hardness parameter of the magnet. Indeed, the equation that determines the size of the magnet for which the switching of nucleation mode occurs is equivalent to

$$\alpha_1(\mathcal{D}) = \frac{1}{1 + \gamma} \frac{\mu_0 M_s^2}{A} = \frac{\ell_{\text{ex}}^{-2}}{1 + \gamma}.$$

For a fixed \mathcal{D} and $\lambda > 0$, it holds that $\alpha_1(\lambda\mathcal{D}) = \frac{1}{\lambda^2} \alpha_1(\mathcal{D})$, thus the switching of nucleation mode occurs for a dilatation of \mathcal{D} by λ , with

$$\lambda = \sqrt{\frac{1 + \gamma}{\alpha_1(\mathcal{D})}} \ell_{\text{ex}}.$$

We find a theory coherent with the physics literature. Indeed, as we will see in the particular case of the ball, the mode in the case $V < V_0$ is typically a uniform mode, called *unison* in physicists' terms, and it is a well-known fact that non-uniform behavior appears for magnets of large size compared to ℓ_{ex} .

Let us first recall the definition of the demagnetizing field.

Definition 3.3. Denoting $\nu : x \mapsto (1 + |x|^2)^{-1/2}$, we define the Hilbert space, introduced in [50],

$$BL^1(\mathbb{R}^3) = \{u \in H_{\text{loc}}^1(\mathbb{R}^3); \nu u \in L^2(\mathbb{R}^3), \nabla u \in L^2(\mathbb{R}^3, \mathbb{R}^3)\},$$

equipped with the scalar product $(u, v)_{BL^1(\mathbb{R}^3)} = \int_{\mathbb{R}^3} \nabla u \cdot \nabla v$.

We define the demagnetizing field by $\mathbf{H}_d(\mathbf{m}) = \nabla u$, with $u \in BL^1(\mathbb{R}^3)$ the unique solution of the variational problem in $BL^1(\mathbb{R}^3)$: for every $\varphi \in BL^1(\mathbb{R}^3)$,

$$\int_{\mathbb{R}^3} \nabla u \cdot \nabla \varphi = - \int_{\mathcal{D}} \mathbf{m} \cdot \nabla \varphi. \quad (3.30)$$

Osborn shows in [53] that for every axis \mathbf{a} of an ellipsoid, there exists a constant $0 < D \leq 1$ such that $\mathbf{H}_d(\mathbf{a}) = -D\mathbf{a}$. Therefore, by (3.10), every axis of the ellipsoid can be an aligned equilibrium state. Let us work with the Cartesian base $(\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$ such that $\mathbf{e}_z = \mathbf{u}$ is one of the ellipsoid axis, and the easy axis of the magnet, and let $D \in (0, 1]$ be a constant such that $\mathbf{H}_d(\mathbf{e}_z) = -D\mathbf{e}_z$. In this case the effective anisotropy coefficient, that appears in the nucleation equation (3.28), writes

$$K_{\text{eff}} = K + \mu_0 M_s^2 \mathbf{H}_d(\mathbf{e}_z) \cdot \mathbf{e}_z = K - \mu_0 D M_s^2. \quad (3.31)$$

Let us show this fact in the case of the ball.

Lemma 3.5. *If \mathcal{D} is a ball: let us take a ball centered at the origin of the space $\mathcal{D} = B_R = \{\mathbf{x} \in \mathbb{R}^3, |\mathbf{x}| \leq R\}$, with $R \geq 0$, then for every constant vector \mathbf{m} , $\mathbf{H}_d(\mathbf{m}) = -\frac{1}{3}\mathbf{m}$ in \mathcal{D} .*

Proof. Let $\mathbf{m} \in \mathbb{R}^3$ be a constant vector such that $|\mathbf{m}| = 1$, and let us define the function

$$u(\mathbf{x}) = \begin{cases} -\frac{1}{3}\mathbf{x} \cdot \mathbf{m}, & |\mathbf{x}| \leq R, \\ -\frac{R^3}{3|\mathbf{x}|^3}\mathbf{x} \cdot \mathbf{m}, & |\mathbf{x}| > R. \end{cases}$$

A rapid calculation gives that u is in $BL^1(\mathbb{R}^3)$ and that it satisfies (3.30) for the constant vector \mathbf{m} , therefore $\mathbf{H}_d(\mathbf{m}) = \nabla u$. It follows that in B_R , $\mathbf{H}_d(\mathbf{m}) = -\frac{1}{3}\mathbf{m}$. The problem being stable by translation, the result is the same for balls which are not centered at the origin. \square

Let us introduce in the following section the decomposition of a function $\boldsymbol{\psi}$ in accordance with the spectral decomposition of \mathbf{H}_d , which is the key ingredient of the proof of Proposition 3.4.

3.1.1 Decomposition

Let us consider the following spaces

$$\begin{aligned} G^1(\mathcal{D}) &= \{ \psi \in H^2(\mathcal{D}) ; \psi = 0 \text{ on } \partial\mathcal{D} \} , \\ G^2(\mathcal{D}) &= \{ \psi \in H^2(\mathcal{D}) ; \Delta\psi = 0 \text{ in } \mathcal{D} \} , \\ G^3(\mathcal{D}) &= \{ \boldsymbol{\Theta} \in H^2(\mathcal{D}, \mathbb{R}^3) ; \nabla \cdot \boldsymbol{\Theta} = 0 \text{ in } \mathcal{D}, (\nabla \times \boldsymbol{\Theta}) \cdot \mathbf{n} = 0 \text{ on } \partial\mathcal{D} \} . \end{aligned}$$

The following proposition shows a decomposition that is appropriate for the study of equation (3.7).

Proposition 3.6. *Let \mathcal{D} be a bounded open set of class \mathcal{C}^2 . Then, it holds that*

$$H^1(\mathcal{D}, \mathbb{R}^3) = \nabla G^1(\mathcal{D}) \oplus \nabla G^2(\mathcal{D}) \oplus \nabla \times G^3(\mathcal{D}) ,$$

and the decomposition is orthogonal for the scalar product of $L^2(\mathcal{D}, \mathbb{R}^3)$. Moreover, the three spaces are stable for the demagnetizing operator \mathbf{H}_d restricted to \mathcal{D} , with in particular $\mathbf{H}_d(\nabla \times G^3(\mathcal{D})) = \{0\}$.

Proof. Let $\boldsymbol{\psi}$ be a function in $H^1(\mathcal{D}, \mathbb{R}^3)$. Let us define $\psi_1 \in H^1(\mathcal{D})$ as the solution of

$$\begin{cases} \Delta\psi_1 = \nabla \cdot \boldsymbol{\psi} \text{ in } \mathcal{D} , \\ \psi_1 = 0 \text{ on } \partial\mathcal{D} . \end{cases} \quad (3.32)$$

Then, by the regularity of \mathcal{D} , ψ_1 is in $H^2(\mathcal{D})$, and in $G^1(\mathcal{D})$. In particular, the trace of $\nabla\psi_1$ on $\partial\mathcal{D}$ is well defined and is in $L^2(\partial\mathcal{D})$. Thus, the following problem is well posed:

$$\begin{cases} \Delta\psi_2 = 0 \text{ in } \mathcal{D} , \\ \nabla\psi_2 \cdot \mathbf{n} = (\boldsymbol{\psi} - \nabla\psi_1) \cdot \mathbf{n} \text{ on } \partial\mathcal{D} . \end{cases} \quad (3.33)$$

Let $\psi_2 \in H^2(\mathcal{D})$ be a solution of the problem (3.33). Then, ψ_2 is in $G^2(\mathcal{D})$. Moreover, it holds that

$$\begin{cases} \nabla \cdot (\boldsymbol{\psi} - \nabla\psi_1 - \nabla\psi_2) = 0 \text{ in } \mathcal{D} , \\ (\boldsymbol{\psi} - \nabla\psi_1 - \nabla\psi_2) \cdot \mathbf{n} = 0 \text{ on } \partial\mathcal{D} . \end{cases} \quad (3.34)$$

Thus, we have by [52], Theorem 2, and the regularity of \mathcal{D} that there exists $\boldsymbol{\Theta} \in H^1(\mathcal{D}, \mathbb{R}^3)$ such that

$$\begin{cases} \boldsymbol{\psi} = \nabla\psi_1 + \nabla\psi_2 + \nabla \times \boldsymbol{\Theta} \text{ in } \mathcal{D} , \\ \nabla \cdot \boldsymbol{\Theta} = 0 \text{ in } \mathcal{D} , \\ (\nabla \times \boldsymbol{\Theta}) \cdot \mathbf{n} = 0 \text{ on } \partial\mathcal{D} . \end{cases} \quad (3.35)$$

Since in the sense of distributions, $\Delta \Theta = \nabla(\nabla \cdot \Theta) - \nabla \times (\nabla \times \Theta) = -\nabla \times \psi \in L^2(\mathcal{D}, \mathbb{R}^3)$, then Θ is in $H^2(\mathcal{D})$.

Thus, we have the embedding $H^1(\mathcal{D}, \mathbb{R}^3) \subset \nabla G^1(\mathcal{D}) + \nabla G^2(\mathcal{D}) + \nabla \times G^3(\mathcal{D})$. The inverse embedding is trivial by the definition of the three spaces. Let us now show that the decomposition is orthogonal for the canonical scalar product of $L^2(\mathcal{D}, \mathbb{R}^3)$.

Let $\varphi \in H^1(\mathcal{D})$, $\alpha \in G^3(\mathcal{D})$, by the definition of $G^3(\mathcal{D})$, it holds that

$$\int_{\mathcal{D}} \nabla \varphi \cdot (\nabla \times \alpha) = - \int_{\mathcal{D}} \nabla \cdot (\nabla \times \alpha) \varphi + \int_{\partial \mathcal{D}} \varphi (\nabla \times \alpha) \cdot n = 0. \quad (3.36)$$

This computation shows the orthogonality of $\nabla G^1(\mathcal{D})$ and $\nabla \times G^3(\mathcal{D})$, and of $\nabla G^2(\mathcal{D})$ and $\nabla \times G^3(\mathcal{D})$.

Let $\varphi_1 \in G^1(\mathcal{D})$ and $\varphi_2 \in G^2(\mathcal{D})$. It holds that

$$\int_{\mathcal{D}} \nabla \varphi_1 \cdot \nabla \varphi_2 = - \int_{\mathcal{D}} \Delta \varphi_2 \varphi_1 + \int_{\partial \mathcal{D}} \varphi_1 \nabla \varphi_2 \cdot n = 0.$$

Now let us show the stability of each of those spaces for the demagnetizing operator. Every function $\psi \in G^1(\mathcal{D})$ is extendable by 0 on \mathbb{R}^3 . Thus, this extension is in $BL^1(\mathbb{R}^3)$, and by definition of the demagnetizing field, $\mathbf{H}_d(\nabla \psi) = -\nabla \psi$ in \mathcal{D} .

For every $\alpha \in G^3(\mathcal{D})$, applying the computation (3.36) with a function $\varphi \in BL^1(\mathbb{R}^3)$, it follows that $\mathbf{H}_d(\nabla \times \alpha) = 0$.

Finally, for every $\varphi_2 \in G^2(\mathcal{D})$, let $u \in BL^1(\mathbb{R}^3)$ be such that $\mathbf{H}_d(\nabla \varphi_2) = \nabla u$. For every $\varphi_1 \in H_0^1(\mathcal{D})$,

$$\int_{\mathcal{D}} \varphi_1 \Delta u = - \int_{\mathcal{D}} \mathbf{H}_d(\nabla \varphi_2) \cdot \nabla \varphi_1 = \int_{\mathcal{D}} \nabla \varphi_1 \cdot \nabla \varphi_2 = 0.$$

It follows that the restriction of u to \mathcal{D} is in $G^2(\mathcal{D})$. □

3.1.2 Proof of Proposition 3.4

Let H be a scalar such that there exists a solution $\psi \in H^1(\mathcal{D}, \mathbb{R}^3)$ such that for every $\phi \in H^1(\mathcal{D}, \mathbb{R}^3)$,

$$\int_{\mathcal{D}} A(\nabla \times \psi) \cdot (\nabla \times \phi) + A(\nabla \cdot \psi)(\nabla \cdot \phi) + (K_{\text{eff}} \psi - \mu_0 M_s \mathbf{H}_d(M_s \psi)) \cdot \phi - \frac{\mu_0}{2} M_s H \psi \cdot \phi = 0. \quad (3.37)$$

Let $\psi_1 \in G^1(\mathcal{D})$, $\psi_2 \in G^2(\mathcal{D})$, $\Theta \in G^3(\mathcal{D})$ be such that $\psi = \nabla \psi_1 + \nabla \psi_2 + \nabla \times \Theta$, by Proposition (3.6).

The outline of the proof is the following : Proposition 3.6 allows us to obtain from (3.37) three independent problems for ψ_1 , ψ_2 and Θ , which correspond to three eigenvalue problems for H . Let us denote the three corresponding smallest eigenvalues of these problems H_1 , H_2 , and H_{Θ} . As the nucleation field is defined as the smallest value for which (3.37) admits a non-trivial solution, H_N is the smallest of the three solutions of these eigenvalue problems. As we will see, it holds that H_{Θ} is always less than H_1 , and therefore, if ψ is the nucleation solution, then $\psi_1 = 0$. To conclude, the comparison between H_2 and H_{Θ} will depend on the volume of the ellipsoid.

Step 1. Let us study the variational problem satisfied by Θ . For every $\phi \in H^1(\mathcal{D}, \mathbb{R}^3)$, decomposing $\phi = \nabla\varphi_1 + \nabla\varphi_2 + \nabla \times \alpha$, with $\varphi_1 \in G^1(\mathcal{D})$, $\varphi_2 \in G^2(\mathcal{D})$, $\alpha \in G^3(\mathcal{D})$, by Proposition 3.6, it holds that

$$\begin{aligned} & \int_{\mathcal{D}} A(\nabla \times (\nabla \times \Theta)) \cdot (\nabla \times \phi) + \left(K_{\text{eff}} - \frac{\mu_0}{2} M_s H\right) (\nabla \times \Theta) \cdot \phi \\ &= \int_{\mathcal{D}} A(\nabla \times (\nabla \times \Theta)) \cdot (\nabla \times (\nabla \times \alpha)) \\ & \quad + \left(K_{\text{eff}} - \frac{\mu_0}{2} M_s H\right) (\nabla \times \Theta) \cdot (\nabla \times \alpha) \end{aligned} \quad (3.38)$$

$$\begin{aligned} &= \int_{\mathcal{D}} A(\nabla \times \psi) \cdot (\nabla \times (\nabla \times \alpha)) + A(\nabla \cdot \psi) (\nabla \cdot (\nabla \times \alpha)) \\ & \quad + (K_{\text{eff}} \psi - \mu_0 M_s \mathbf{H}_d(M_s \psi)) \cdot (\nabla \times \alpha) - \frac{\mu_0}{2} M_s H \psi \cdot (\nabla \times \alpha) \end{aligned} \quad (3.39)$$

$$= 0. \quad (3.40)$$

The equalities (3.38) and (3.39) come from the fact that $\nabla \times (\nabla\varphi_1 + \nabla\varphi_2) = \nabla \times (\nabla\psi_1 + \nabla\psi_2) = 0$, by the orthogonality of $\nabla G^1(\mathcal{D})$ and $\nabla G^2(\mathcal{D})$ with $\nabla \times G^3(\mathcal{D})$ in $L^2(\mathcal{D})$, and as for every $\psi \in H^1(\mathcal{D}, \mathbb{R}^3)$, $\mathbf{H}_d(\psi)$ is in $\nabla G^1(\mathcal{D}) + \nabla G^2(\mathcal{D})$. Finally, the last equality (3.40) holds because ψ satisfies (3.37).

Let us assume that $\Theta \neq 0$ and show that Θ is a solution of an eigenvalue problem. Let us denote H_{Θ} the smallest value of H such that the variational problem (3.40) holds for $\Theta \neq 0$.

Integrating (3.40) by parts, we have, for every $\phi \in H^1(\mathcal{D}, \mathbb{R}^3)$,

$$\begin{aligned} & \int_{\mathcal{D}} \left(A(\nabla \times (\nabla \times \Theta)) + \left(K_{\text{eff}} - \frac{\mu_0}{2} M_s H\right) \Theta \right) \cdot (\nabla \times \phi) \\ & \quad + \left(K_{\text{eff}} - \frac{\mu_0}{2} M_s H\right) \int_{\partial\mathcal{D}} (\Theta \times \phi) \cdot n = 0. \end{aligned} \quad (3.41)$$

Considering this equation for all $\phi \in H_0^1(\mathcal{D}, \mathbb{R}^3)$, it follows by the path-connectedness of \mathcal{D} that there exists a function $g \in H^1(\mathcal{D})$ such that

$$A(\nabla \times (\nabla \times \Theta)) + \left(K_{\text{eff}} - \frac{\mu_0}{2} M_s H_{\Theta}\right) \Theta = \nabla g. \quad (3.42)$$

Considering (3.41) for all $\phi \in H^1(\partial\mathcal{D}, \mathbb{R}^3)$, it holds that

$$\Theta \times n = 0 \text{ on } \partial\mathcal{D}. \quad (3.43)$$

Thus, for every $\phi \in H^1(\mathcal{D}, \mathbb{R}^3)$,

$$\int_{\partial\mathcal{D}} \nabla g \cdot (\nabla \times \phi) = 0. \quad (3.44)$$

Moreover, by the classic vector calculus identities, and because Θ is in $G^3(\mathcal{D})$, it holds that $(\nabla \times (\nabla \times \Theta)) = -\Delta\Theta + \nabla(\nabla \cdot \Theta) = -\Delta\Theta$.

Let us show that g is a constant function, and therefore $\nabla g = 0$. Applying the divergence operator on both sides of (3.42), and using the fact that $\nabla \cdot \Theta = 0$ as Θ is in $G^3(\mathcal{D})$, we

get that $\Delta g = 0$ in $H^{-1}(\mathcal{D})$. Since g is in $H^1(\mathcal{D})$, it follows that g is in $H^2(\mathcal{D})$, and the trace of ∇g on $\partial\mathcal{D}$ is well defined and is in $L^2(\partial\mathcal{D})$. Thus, integrating (3.44) by parts, it holds that for every $\phi \in H^1(\mathcal{D}, \mathbb{R}^3)$,

$$0 = \int_{\mathcal{D}} \nabla g \cdot (\nabla \times \phi) = \int_{\mathcal{D}} (\nabla \times \nabla g) \cdot \phi + \int_{\partial\mathcal{D}} (\phi \times \nabla g) \cdot n = \int_{\partial\mathcal{D}} (\nabla g \times n) \cdot \phi.$$

Combining this with $\Delta g = 0$, it follows that $\nabla g = 0$ on \mathcal{D} .

Using this fact with (3.43) and the definition of $G^3(\mathcal{D})$, we have

$$H_{\Theta} = \frac{2K_{\text{eff}}}{\mu_0 M_s} + \frac{2A\alpha_1(\mathcal{D})}{\mu_0 M_s}, \quad (3.45)$$

where $\alpha_1(\mathcal{D})$ is the lowest value of $\lambda \in \mathbb{R}$ such that there exists $\alpha \in H^1(\mathcal{D}, \mathbb{R}^3)$ with $\alpha \neq 0$ that satisfies

$$\begin{cases} -\Delta \alpha = \lambda \alpha & \text{in } \mathcal{D}, \\ \nabla \cdot \alpha = 0 & \text{in } \mathcal{D}, \\ \alpha \times n = 0 & \text{on } \partial\mathcal{D}. \end{cases} \quad (3.46)$$

Step 2. Let us recall that we have $\psi = \nabla\psi_1 + \nabla\psi_2 + \nabla \times \Theta$, with $\psi_1 \in G^1(\mathcal{D})$, $\psi_2 \in G^2(\mathcal{D})$, and $\Theta \in G^3(\mathcal{D})$. Let us now study the variational problem satisfied by ψ_1 . For any $\phi \in H^1(\mathcal{D}, \mathbb{R}^3)$, writing $\phi = \nabla\varphi_1 + \nabla\varphi_2 + \nabla \times \alpha$, with $\varphi_1 \in G^1(\mathcal{D})$, $\varphi_2 \in G^2(\mathcal{D})$, $\alpha \in G^3(\mathcal{D})$,

$$\int_{\mathcal{D}} A\Delta\psi_1 (\nabla \cdot \phi) + \left(K_{\text{eff}} + \mu_0 M_s^2 - \frac{\mu_0}{2} M_s H \right) \nabla\psi_1 \cdot \phi \quad (3.47)$$

$$= \int_{\mathcal{D}} A\Delta\psi_1 \Delta\varphi_1 + \left(K_{\text{eff}} + \mu_0 M_s^2 - \frac{\mu_0}{2} M_s H \right) \nabla\psi_1 \cdot \nabla\varphi_1 \quad (3.48)$$

$$= \int_{\mathcal{D}} A (\nabla \times \psi) \cdot (\nabla \times (\nabla\varphi_1)) + A (\nabla \cdot \psi) \Delta\varphi_1 \\ + (K_{\text{eff}}\psi - \mu_0 M_s \mathbf{H}_d(M_s \psi)) \cdot \nabla\varphi_1 - \frac{\mu_0}{2} M_s H \psi \cdot \nabla\varphi_1 \quad (3.49)$$

$$= 0, \quad (3.50)$$

The equalities (3.48) and (3.49) come from the fact that $\nabla \cdot (\nabla\psi_2 + \nabla \times \Theta) = \nabla \cdot (\nabla\varphi_2 + \nabla \times \alpha) = 0$ in \mathcal{D} , by the orthogonality of $\nabla G^2(\mathcal{D})$ and of $\nabla \times G^3(\mathcal{D})$ with $\nabla G^1(\mathcal{D})$ in $L^2(\mathcal{D})$, and by the stability of each of those spaces by \mathbf{H}_d . Finally, the last equality (3.50) holds because ψ satisfies (3.7).

Let us assume that $\psi_1 \neq 0$ and show that ψ_1 is an eigenfunction of the Dirichlet Laplacian on \mathcal{D} . Let us denote H_1 the smallest value of H such that the variational problem (3.50) holds for $\psi_1 \neq 0$.

Integrating (3.50) by parts and using the fact that ψ_1 is in $H_0^1(\mathcal{D})$, we have, for every $\phi \in H^1(\mathcal{D}, \mathbb{R}^3)$,

$$\int_{\mathcal{D}} \left(A\Delta\psi_1 - \left(K_{\text{eff}} + \mu_0 M_s^2 - \frac{\mu_0}{2} M_s H \right) \psi_1 \right) (\nabla \cdot \phi) = 0. \quad (3.51)$$

Considering this equation for $\phi \in H_0^1(\mathcal{D}, \mathbb{R}^3)$, it follows that there exists a constant $C \in \mathbb{R}$ such that, in \mathcal{D} ,

$$-A\Delta\psi_1 + \left(K_{\text{eff}} + \mu_0 M_s^2 - \frac{\mu_0}{2} M_s H\right) \psi_1 = C. \quad (3.52)$$

Since, moreover, ψ_1 is in $H^2(\mathcal{D})$, then ψ_1 is in $H^4(\mathcal{D})$ and the trace of $\Delta\psi_1$ is well defined on $\partial\mathcal{D}$ and is in $L^2(\partial\mathcal{D})$. We can then integrate (3.51) by parts: for every $\phi \in H^1(\mathcal{D}, \mathbb{R}^3)$,

$$\begin{aligned} 0 &= \int_{\mathcal{D}} \left(A\Delta\psi_1 - \left(K_{\text{eff}} + \mu_0 M_s^2 - \frac{\mu_0}{2} M_s H \right) \psi_1 \right) (\nabla \cdot \phi) \\ &= - \int_{\mathcal{D}} (\nabla C) \cdot \phi + \int_{\partial\mathcal{D}} A\Delta\psi_1 (\phi \cdot n), \end{aligned}$$

as ψ_1 is in $G^1(\mathcal{D})$. It follows that $\Delta\psi_1 = 0$ on $\partial\mathcal{D}$, and then $C = 0$. Thus, ψ_1 is a solution of the Dirichlet Laplacian on \mathcal{D} , and we have

$$H_1 = \frac{2}{\mu_0 M_s} (K_{\text{eff}} + \mu_0 M_s^2 + A\lambda) > \frac{2}{\mu_0 M_s} (K_{\text{eff}} + A\alpha_1(\mathcal{D})) = H_{\Theta},$$

where λ is an eigenvalue of the Dirichlet Laplacian on \mathcal{D} . Indeed, let us take $f \in H^1(\mathcal{D})$ an eigenfunction of the Dirichlet Laplacian on \mathcal{D} of eigenvalue λ . Then, defining

$$\alpha = \begin{pmatrix} f \\ -f \\ 0 \end{pmatrix},$$

we have a solution of the problem (3.46). Thus, $\lambda \geq \alpha_1(\mathcal{D})$.

By definition of the nucleation field, it follows that if ψ is the nucleation solution, then $\psi_1 = 0$.

Step 3. Finally, let us study the variational problem satisfied by ψ_2 . Decomposing $\phi = \nabla\varphi_1 + \nabla\varphi_2 + \nabla \times \alpha$ and taking account of the fact that $\Delta\psi_2 = 0$, we obtain in the same way as above, for every $\phi \in H^1(\mathcal{D}, \mathbb{R}^3)$,

$$\int_{\mathcal{D}} \left(\left(K_{\text{eff}} - \mu_0 \frac{M_s}{2} H \right) \nabla\psi_2 - \mu_0 M_s^2 \mathbf{H}_d(\nabla\psi_2) \right) \cdot \phi = 0. \quad (3.53)$$

Let us denote H_2 the smallest value of H such that this equation holds for $\psi_2 \neq 0$. Then,

$$\mathbf{H}_d(\nabla\psi_2) = \frac{2K_{\text{eff}} - \mu_0 M_s H_2}{2\mu_0 M_s^2} \nabla\psi_2 \text{ in } \mathcal{D}. \quad (3.54)$$

Denoting $\mathbf{H}_d(\nabla\psi_2) = \nabla u$, with $u \in BL^1(\mathbb{R}^3)$, then by Proposition 3.6, u is harmonic on \mathcal{D} , and on $\mathbb{R}^3 \setminus \overline{\mathcal{D}}$, as, for every $\varphi \in C_c^\infty(\mathbb{R}^3 \setminus \overline{\mathcal{D}})$,

$$\int_{\mathbb{R}^3 \setminus \overline{\mathcal{D}}} \varphi \Delta u = - \int_{\mathbb{R}^3 \setminus \overline{\mathcal{D}}} \nabla\varphi \cdot \mathbf{H}_d(\nabla\psi_2) = \int_{\mathcal{D}} \nabla\varphi \cdot \nabla\psi_2 = 0.$$

Let us denote, for a function $f \in H^1(\mathcal{D} \cup (\mathbb{R}^3 \setminus \overline{\mathcal{D}}))$, by f^- and f^+ the trace of f on $\partial\mathcal{D}$ of the restriction of f on, respectively, \mathcal{D} and $\mathbb{R}^3 \setminus \overline{\mathcal{D}}$. By definition of the demagnetizing field, we have on $\partial\mathcal{D}$,

$$(\mathbf{H}_d(\nabla\psi_2) + \nabla\psi_2)^- \cdot n = (\mathbf{H}_d(\nabla\psi_2))^+ \cdot n,$$

with, inside \mathcal{D} , by (3.54),

$$(\mathbf{H}_d(\nabla\psi_2) + \nabla\psi_2)^- \cdot n = \left(1 + \frac{2\mu_0 M_s^2}{2K_{\text{eff}} - \mu_0 M_s H_2}\right) \frac{\partial u^-}{\partial n},$$

and outside \mathcal{D} ,

$$(\mathbf{H}_d(\nabla\psi_2))^+ \cdot n = \frac{\partial u^+}{\partial n}.$$

It follows that u is a solution of

$$\begin{cases} \Delta u = 0 & \text{in } \mathcal{D} \cup (\mathbb{R}^3 \setminus \overline{\mathcal{D}}), \\ u^- = u^+ & \text{on } \partial\mathcal{D}, \\ \left(1 + \frac{2\mu_0 M_s^2}{2K_{\text{eff}} - \mu_0 M_s H_2}\right) \frac{\partial u^-}{\partial n} = \frac{\partial u^+}{\partial n} & \text{on } \partial\mathcal{D}, \\ u(x) \rightarrow 0 & |x| \rightarrow \infty, \end{cases}$$

which boils down to the two-phase conductivity equation on the domain \mathcal{D} : search for u such that

$$\begin{cases} -\operatorname{div}(a\nabla u) = 0 & \text{in } \mathbb{R}^3, \\ |u(x)| \rightarrow 0 & |x| \rightarrow \infty, \end{cases} \quad (3.55)$$

where

$$a(x) = \begin{cases} 1 + \frac{2\mu_0 M_s^2}{2K_{\text{eff}} - \mu_0 M_s H_2} & \text{if } x \in \mathcal{D}, \\ 1 & \text{otherwise.} \end{cases}$$

It follows that we can use the method of layer potentials [50] to represent the solutions of (3.55).

Proposition 3.7. *Let us consider the two-phase conductivity equation (3.55), where*

$$a(x) = \begin{cases} -\gamma & \text{if } x \in \mathcal{D}, \\ 1 & \text{otherwise.} \end{cases}$$

Then, the set of real numbers γ for which there holds a non trivial solution is a discrete set, which accumulation point is 1 :

$$0 \leq \gamma_1^- \leq \gamma_2^- \leq \dots < 1 < \dots \leq \gamma_2^+ \leq \gamma_1^+ < \infty.$$

It follows that

$$H_2 = \frac{2K_{\text{eff}}}{\mu_0 M_s} + \frac{2M_s}{1 + \gamma_1^+}. \quad (3.56)$$

Finally, the nucleation field and nucleation modes are given by the following expressions, depending on whether H_2 is less than H_{Θ} . Let V_0 be a value of the volume of \mathcal{D} for which the following equality holds

$$\frac{2A\alpha_1(\mathcal{D})}{\mu_0 M_s} = \frac{2M_s}{1 + \gamma_1^+}.$$

Then, the nucleation modes are given by:

- if the volume of \mathcal{D} is such that $V < V_0$, $\boldsymbol{\psi} = \nabla\psi_2$, and by (3.56) and (3.31),

$$H_N = H_2 = \frac{2K}{\mu_0 M_s} + 2M_s \left(\frac{1}{1 + \gamma_1^+} - D \right),$$

- and if $V \geq V_0$, $\boldsymbol{\psi} = \nabla \times \boldsymbol{\Theta}$ and, by (3.45) and (3.31),

$$H_N = H_{\boldsymbol{\Theta}} = \frac{2K}{\mu_0 M_s} - 2DM_s + \frac{2A\alpha_1(\mathcal{D})}{\mu_0 M_s}.$$

where $\alpha_1(\mathcal{D})$ is the lowest value such that the problem (3.46) admits a non-trivial solution.

proof of Proposition 3.7. Let G be the Green function of the Laplace operator in \mathbb{R}^3 :

$$G : (x, y) \in \mathbb{R}^3 \times \mathbb{R}^3 \mapsto -\frac{1}{4\pi} \frac{1}{|x - y|}.$$

We define the single layer potential associated to a function $\phi \in \mathcal{C}(\partial\mathcal{D})$ by: for every $x \notin \partial\mathcal{D}$,

$$\mathcal{S}_{\mathcal{D}}\phi(x) = \int_{\partial\mathcal{D}} G(x, y)\phi(y)ds(y).$$

Then, the single layer potential $\mathcal{S}_{\mathcal{D}}\phi$:

- has a continuous trace across $\partial\mathcal{D}$: for every $x \in \partial\mathcal{D}$,

$$\mathcal{S}_{\mathcal{D}}\phi^-(x) = \mathcal{S}_{\mathcal{D}}\phi^+(x),$$

- has a derivative jump across $\partial\mathcal{D}$: for every $x \in \partial\mathcal{D}$,

$$\left(\frac{\partial}{\partial n} \mathcal{S}_{\mathcal{D}}\phi \right)^{\pm}(x) = \pm \frac{1}{2}\phi(x) + \mathcal{K}_{\mathcal{D}}^*\phi(x), \quad (3.57)$$

where $\mathcal{K}_{\mathcal{D}}^* : \mathcal{C}(\partial\mathcal{D}) \rightarrow \mathcal{C}(\partial\mathcal{D})$ is the *Neumann-Poincaré* operator, defined by: for every $x \in \partial\mathcal{D}$,

$$\mathcal{K}_{\mathcal{D}}^*\phi(x) = \int_{\partial\mathcal{D}} \frac{\partial G}{\partial n_x}(x, y)\phi(y)ds(y).$$

We consider the extension of $\mathcal{K}_{\mathcal{D}}^*$ on $L^2(\partial\mathcal{D})$, and the extension of $\mathcal{S}_{\mathcal{D}}$ on $H^{-1/2}(\partial\mathcal{D})$. Let $u \in H_{\text{loc}}^1(\mathbb{R}^3)$ be a solution of (3.55), with

$$a(x) = \begin{cases} -\gamma & \text{if } x \in \mathcal{D}, \\ 1 & \text{otherwise.} \end{cases}$$

Then, in particular, u satisfies

$$\begin{cases} \Delta u = 0 & \text{in } \mathcal{D} \cup (\mathbb{R}^3 \setminus \overline{\mathcal{D}}), \\ u^- = u^+ & \text{on } \partial\mathcal{D}, \\ u(x) \rightarrow 0 & |x| \rightarrow \infty. \end{cases}$$

By the Theorem of integral representation ([50], Thm 3.1.1), u can be written

$$u = \mathcal{S}_{\mathcal{D}}\phi,$$

with $\phi \in H^{-1/2}(\partial\mathcal{D})$.

Let μ be an eigenvalue of $\mathcal{K}_{\mathcal{D}}^*$, associated to the eigenfunction $\phi \in H^{-1/2}(\mathcal{D}) : \mathcal{K}_{\mathcal{D}}^*\phi = \mu\phi$. Then, by the jump relation (3.57), $\mathcal{S}_{\mathcal{D}}\phi$ is a solution of (3.55), with

$$\gamma = \frac{1 + 2\mu}{1 - 2\mu}. \quad (3.58)$$

It follows the set of γ such that (3.55) admits a non trivial solution is entirely determined by the spectrum of $\mathcal{K}_{\mathcal{D}}^*$. As \mathcal{D} is of class \mathcal{C}^2 , we have by [43], that the Neumann-Poincaré operator $\mathcal{K}_{\mathcal{D}}^* : L^2(\partial\mathcal{D}) \rightarrow L^2(\partial\mathcal{D})$ is compact, and that spectrum of $\mathcal{K}_{\mathcal{D}}^*$ is a discrete sequence of eigenvalues that satisfies:

$$-\frac{1}{2} \leq \mu_1^- \leq \mu_2^- \leq \dots < 0 < \dots \leq \mu_2^+ \leq \mu_1^+ < \frac{1}{2}.$$

The proposition follows by (3.58). □

3.1.3 The particular case of the ball

In the case of the ball, we can obtain more explicit solutions for the nucleation equation. Our problem being invariant by translation, let us take a ball centered at the origin $\mathcal{D} = B_R = \{\mathbf{x} \in \mathbb{R}^3 ; |\mathbf{x}| \leq R\}$.

Proposition 3.8. *In the case of a ball of radius R , we can decompose the nucleation according to the values of R : let us denote the coherence radius,*

$$R_{ch} = x_1^* \sqrt{\frac{A}{3\mu_0 M_s^2}} = \frac{x_1^*}{\sqrt{3}} \ell_{ex},$$

where x_1^* is a constant. We have

- for $R \leq R_{ch}$, the nucleation mode is uniform, with

$$H_N = \frac{2K}{\mu_0 M_s}.$$

- for $R > R_{ch}$, the nucleation mode is non-uniform, and called a rotation, with

$$H_N = \frac{2K}{\mu_0 M_s} + \frac{2}{3} M_s \left(\frac{R_{ch}^2}{R^2} - 1 \right).$$

We find a well-known result in the physics literature, where the nucleation is also called *switching*, the uniform mode is called *switching at unison*, and the rotation is also called *curling mode*.

Proof. First, by Lemma 3.5, we have $K_{\text{eff}} = K - \frac{1}{3}\mu_0 M_s^2$.
Let us work with the spherical coordinates $(r, \theta, \varphi) \in \mathbb{R}^+ \times (0, 2\pi) \times (0, \pi)$ such that

$$\mathbf{x} = \begin{cases} r \sin \theta \cos \varphi \\ r \sin \theta \sin \varphi \\ r \cos \theta \end{cases},$$

along with the canonical vector basis

$$\mathbf{u}_r = \begin{pmatrix} \sin \theta \cos \varphi \\ \sin \theta \sin \varphi \\ \cos \theta \end{pmatrix}, \mathbf{u}_\theta = \begin{pmatrix} \cos \theta \cos \varphi \\ \cos \theta \sin \varphi \\ -\sin \theta \end{pmatrix}, \mathbf{u}_\varphi = \begin{pmatrix} -\sin \theta \sin \varphi \\ \sin \theta \cos \varphi \\ 0 \end{pmatrix}.$$

Step 1. We first consider the eigenvalue problem solved by H_2 . Let H be such that there exist a nontrivial solution ψ_2 of the problem

$$\mathbf{H}_d(\nabla \psi_2) = \frac{2K_{\text{eff}} - \mu_0 M_s H}{2\mu_0 M_s^2} \nabla \psi_2 \text{ in } \mathcal{D}. \quad (3.59)$$

As seen in Step 3 of the proof of Proposition 3.4, the resolution of this problem amounts to finding γ , and u such that

$$\begin{cases} \Delta u = 0 & \text{in } B_R \cup (\mathbb{R}^3 \setminus \overline{B_R}), \\ u^- = u^+ & \text{on } \partial B_R, \\ -\gamma \frac{\partial u^-}{\partial n} = \frac{\partial u^+}{\partial n} & \text{on } \partial B_R, \\ u(x) \rightarrow 0 & |x| \rightarrow \infty. \end{cases} \quad (3.60)$$

As $\Delta u = 0$ in $B_R \cup (\mathbb{R}^3 \setminus \overline{B_R})$ with $u(x) \rightarrow 0$ at infinity, we can decompose u into spherical harmonics :

$$u(r, \theta, \varphi) = \begin{cases} \sum_{l,m} \alpha_{l,m} r^l Y_{l,m}(\theta, \varphi) & \text{for } r < R, \\ \sum_{l,m} \beta_{l,m} r^{-l-1} Y_{l,m}(\theta, \varphi) & \text{for } r > R, \end{cases}$$

where $\{Y_{l,m}\}_{l,m}$ are the Legendre polynomials. Writing the conditions of continuity and the jump of derivatives across ∂D_R , it holds that, for every l, m

$$\begin{aligned} \alpha_{l,m} R^l &= \beta_{l,m} R^{-l-1}, \\ -\gamma l R^{l-1} \alpha_{l,m} &= -(l+1) R^{-l-2} \beta_{l,m}. \end{aligned}$$

It follows that there exists $l \in \mathbb{N}^*$ such that

$$\gamma = \frac{l+1}{l}.$$

By (3.56),

$$H = \frac{2K_{\text{eff}}}{\mu_0 M_s} + \frac{2M_s}{1+\gamma},$$

and the definition of H_2 as the lowest value of H such that (3.53) holds, it holds that $\gamma = 2$, and

$$H_2 = \frac{2K_{\text{eff}}}{\mu_0 M_s} + \frac{2M_s}{3} = \frac{2K}{M_s},$$

Conversally, by Lemma 3.5, every constant vector \mathbf{x} satisfies $\mathbf{H}_d(\mathbf{x}) = -\frac{1}{3}\mathbf{x}$ in B_R , and thus every constant function orthogonal to \mathbf{e}_z satisfies equation (3.59) with $H = \frac{2K}{M_s}$.

Step 2. Now let us consider the eigenvalue problem solved by H_{Θ} and explain the *rotation mode*. We need to find Θ, λ such that

$$\begin{cases} -\Delta \Theta = \lambda \Theta & \text{in } D_R, \\ \nabla \cdot \Theta = 0 & \text{on } D_R, \\ \Theta \times \mathbf{n} = 0 & \text{on } \partial D_R, \end{cases}$$

(see Step 1 of the proof of Proposition 3.4). Then, $\alpha_1(\mathcal{D})$ is defined as the smallest value of λ such that there exists a nontrivial solution Θ , and we compute

$$H_{\Theta} = \frac{2K_{\text{eff}}}{\mu_0 M_s} + \frac{2A\alpha_1(\mathcal{D})}{\mu_0 M_s}.$$

We can write Θ in terms of the vector spherical harmonics :

$$\Theta = f(r)\Phi_{l,m} = f(r)\mathbf{r} \times \nabla Y_{l,m},$$

with $f(R) = 0$ for some integers $l \geq 1$ and $m \in \{-l, \dots, l\}$, as $\nabla Y_{0,0} = 0$. Using the fact that $\Delta \Phi_{l,m} = -\frac{1}{r^2}l(l+1)\Phi_{l,m}$ and $\Delta(f(r)\Phi_{l,m}) = \left(\frac{1}{r^2}\frac{d}{dr}r^2\frac{df}{dr}\right)\Phi_{l,m} + f(r)\Delta\Phi_{l,m}$, we have

$$-f''(r) - \frac{2}{r}f'(r) + \frac{l(l+1)}{r^2}f(r) - \lambda f(r) = 0.$$

Writing $f(r) = g(\sqrt{\lambda}r)$, we have

$$x^2 g''(x) + 2xg'(x) + (x^2 - l(l+1))g(x) = 0.$$

The solutions of this differential equations are the spherical Bessel functions of first kind j_l and of second kind y_l , which are not defined on 0. It follows that

$$f(r) = j_l(\sqrt{\lambda}r).$$

Writing $f(R) = 0$, we have

$$\sqrt{\lambda}R = x_l^*,$$

with x_l^* the first zero of j_l . Knowing that x_l^* is an increasing function of l , it follows that the smallest value of H occurs for $l = 1$, with $j_1(x) = \frac{\cos x - x \sin x}{x}$. Finally,

$$H_{\Theta} = \frac{2K}{\mu_0 M_s} - \frac{2}{3}M_s + \frac{2A(x_1^*)^2}{\mu_0 M_s R^2} = \frac{2K}{\mu_0 M_s} + \frac{2}{3}M_s \left(\frac{R_{ch}^2}{R^2} - 1 \right),$$

with the coherence radius $R_{ch} = x_1^* \sqrt{\frac{A}{3\mu_0 M_s^2}}$, and

$$\boldsymbol{\psi} = \nabla \times \left(j_1 \left(x_1^* \frac{r}{R} \right) \mathbf{r} \times \nabla Y_{1,m} \right),$$

with $m \in \{-1, 0, 1\}$. Conversely, defining

$$\begin{aligned} \boldsymbol{\psi}(\mathbf{x}) &= \nabla \times \left(j_1 \left(x_1^* \frac{r}{R} \right) \mathbf{r} \times \nabla Y_{1,0} \right) \\ &= \sqrt{\frac{3}{\pi}} \sin \theta \frac{-\sin \frac{x_1^* r}{R} + \cos \frac{x_1^* r}{R}}{r} \mathbf{u}_\varphi, \end{aligned} \quad (3.61)$$

then $\boldsymbol{\psi} \cdot \mathbf{e}_z = 0$ in \mathcal{D} and $\boldsymbol{\psi}$ satisfies (3.37) with $H = \frac{2K}{\mu_0 M_s} + \frac{2}{3} M_s \left(\frac{R_{ch}^2}{R^2} - 1 \right)$.

Finally, the nucleation modes are given by the following expressions, depending on the radius R , with

- if $R \leq R_{ch}$, then $\boldsymbol{\psi}$ is a constant vector orthogonal to \mathbf{e}_z , and

$$H_N = H_2 = \frac{2K}{M_s},$$

- and if $R \geq R_{ch}$, then $\boldsymbol{\psi}(\mathbf{x}) = \sqrt{\frac{3}{\pi}} \sin \theta \frac{-\sin \frac{x_1^* r}{R} + \cos \frac{x_1^* r}{R}}{r} \mathbf{u}_\varphi$, with

$$H_N = H_\Theta = \frac{2K}{\mu_0 M_s} + \frac{2}{3} M_s \left(\frac{R_{ch}^2}{R^2} - 1 \right).$$

□

Remark 3.4. We notice that the expression (3.61) of the nucleation mode corresponds to the definition of the *curling* in the physics literature as it is defined as *a rotation from the easy axis in a plane perpendicular to the radius* [24].

Now that we have justified and extended some well-known results of the physics literature, we can turn to the case of infinite cylinders.

3.2 Infinite cylinders

In this section, \mathcal{D} is an infinite cylinder, with a periodicity along its axis. The proof will follow the same outline as in the previous section, with a few differences due to the definition of the demagnetizing field. It is worth noting that we are able to provide results on the nucleation of infinite cylinders, without further information on the cross-section.

Let us work with a Cartesian base $(\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$ such that \mathbf{e}_z is the rotation axis of the cylinder, and let us identify $\text{Vect}(\mathbf{e}_x, \mathbf{e}_y)$ with \mathbb{R}^2 .

Let us assume that \mathcal{D} is an infinite cylinder with axis along \mathbf{e}_z , that we will consider as a periodic domain of period T in the z , and let us denote by S the area of the section \mathcal{D}^* . We will denote $\mathcal{D} = \mathcal{D}^* \times (0, T)$.

Proposition 3.9. *Let $\mathcal{D}^* \subset \mathbb{R}^2$ be a bounded open set of class \mathcal{C}^2 . Let $\alpha_1(\mathcal{D}^*)$ be the first eigenvalue of the Dirichlet Laplacian on \mathcal{D}^* . There exists a constant $\beta \in (0, 1]$ and a length λ defined by*

$$\lambda = \sqrt{\frac{\beta}{\alpha_1(\mathcal{D}^*)}} \ell_{ex},$$

such that the nucleation field of $\mathcal{D} = l\mathcal{D}^ \times (0, \frac{1}{l})$ for a magnet of easy axis \mathbf{e}_z , is given by*

- *if $l \leq \lambda$, then*

$$H_N = \frac{2K}{\mu_0 M_s} + 2\beta M_s, \quad (3.62)$$

- *and if $l \geq \lambda$, then*

$$H_N = \frac{2K}{\mu_0 M_s} + \frac{2A\alpha_1(\mathcal{D}^*)}{\mu_0 M_s}.$$

As we will see in the particular case of a circular section, the nucleation modes associated with the nucleation field values (3.62) and (3.9) are generalizations of respectively the *buckling mode* (defined as a mode where the perturbations of the magnetization are a periodic function in z) and *curling mode* (defined as a mode where the magnetization does not depend on n and is orthogonal to the radius).

We give a definition of the demagnetizing field in this context.

Definition 3.4. Let us denote

$$\begin{aligned} \nu : \mathbb{R}^2 &\rightarrow \mathbb{R} \\ x &\mapsto \frac{1}{\log(2 + |x|^2)(1 + |x|^2)^{1/2}}, \end{aligned}$$

and define the Hilbert space, introduced in [50],

$$\begin{aligned} BL_{\sharp}^1(\mathbb{R}^2 \times (0, T)) = \{ &u \in H_{loc}^1(\mathbb{R}^2 \times (0, T)); \nu(x, y)u(x, y, z) \in L^2(\mathbb{R}^2 \times (0, T)) \ , \\ &u(\cdot, 0) = u(\cdot, T), \nabla u \in L^2(\mathbb{R}^2 \times (0, T), \mathbb{R}^3) \} . \end{aligned}$$

As we notice that this space contains constant functions, we consider the class of equivalence of functions of $BL_{\sharp}^1(\mathbb{R}^2 \times (0, T))$ equal up to a constant. We equip this space with the scalar product $(u, v)_{BL_{\sharp}^1(\mathbb{R}^2 \times (0, T))} = \int_{\mathbb{R}^2 \times (0, T)} \nabla u \cdot \nabla v$.

We define the demagnetizing field by $\mathbf{H}_d(\mathbf{m}) = \nabla u$, with $u \in BL_{\sharp}^1(\mathbb{R}^2 \times (0, T))$ the unique solution of the variational problem in $BL_{\sharp}^1(\mathbb{R}^2 \times (0, T))$: for every $\varphi \in BL_{\sharp}^1(\mathbb{R}^2 \times (0, T))$,

$$\int_{\mathbb{R}^2 \times (0, T)} \nabla u \cdot \nabla \varphi = - \int_{\mathcal{D}} \mathbf{m} \cdot \nabla \varphi. \quad (3.63)$$

From now on, let us assume that the easy axis of anisotropy of our material is \mathbf{e}_z . Then, \mathbf{e}_z is an aligned equilibrium state: for every $\varphi \in BL_{\sharp}^1(\mathbb{R}^2 \times (0, T))$,

$$\int_{\mathbb{R}^2 \times (0, T)} \mathbf{H}_d(\mathbf{e}_z) \cdot \nabla \varphi = - \int_{\mathcal{D}} \nabla \varphi \cdot \mathbf{e}_z = - \int_{\mathcal{D}^*} \varphi(\cdot, T) - \varphi(\cdot, 0) = 0.$$

It follows that

$$\mathbf{H}_d(\mathbf{e}_z) = 0, \quad (3.64)$$

and the state $\mathbf{m} = \mathbf{e}_z$ satisfies the equilibrium condition (3.9). We have in this case

$$K_{\text{eff}} = K + \mu_0 M_s^2 \mathbf{H}_d(\mathbf{m}) \cdot \mathbf{m} = K.$$

3.2.1 Decomposition

In order to study the solutions $\boldsymbol{\psi}$ of equation (3.7), we will decompose the functions in accordance with the spectral decomposition of \mathbf{H}_d . Here, we consider periodic functions in the z variable. Thus, we define $H_{\sharp}^1(\mathcal{D})$ as the subset of $H^1(\mathcal{D})$ of functions f such that $f(\cdot, 0) = f(\cdot, T)$. By Remark 3.1, we can assume that $\boldsymbol{\psi}$ is in $H_{\sharp}^1(\mathcal{D}, \mathbb{R}^2)$.

Thus, the nucleation equation in this case writes: find $\boldsymbol{\psi} \in H_{\sharp}^1(\mathcal{D}, \mathbb{R}^2)$ such that, for every $\boldsymbol{\phi} \in H_{\sharp}^1(\mathcal{D}, \mathbb{R}^3)$

$$\int_{\mathcal{D}} A(\nabla \times \boldsymbol{\psi}) \cdot (\nabla \times \boldsymbol{\phi}) + A(\nabla \cdot \boldsymbol{\psi})(\nabla \cdot \boldsymbol{\phi}) + \left(K\boldsymbol{\psi} - \frac{\mu_0}{2} M_s \mathbf{H}_d(M_s \boldsymbol{\psi}) \right) \cdot \boldsymbol{\phi} - \frac{\mu_0}{2} M_s H \boldsymbol{\psi} \cdot \boldsymbol{\phi} = 0. \quad (3.65)$$

Let $\boldsymbol{\psi} \in H_{\sharp}^1(\mathcal{D}, \mathbb{R}^2)$ be given by the Fourier expansion

$$\boldsymbol{\psi}(x, y, z) = \sum_{k=0}^{+\infty} \boldsymbol{\psi}_k(x, y) \cos \frac{2\pi k z}{T}.$$

In this section, we will take

$$\boldsymbol{\psi}(x, y, z) = \begin{pmatrix} \psi_1 \cos \frac{2\pi k z}{T} \\ \psi_2 \cos \frac{2\pi k z}{T} \end{pmatrix} = \boldsymbol{\Phi}(x, y) \cos \frac{2\pi k z}{T}. \quad (3.66)$$

Indeed, in our variational problem (3.65), if $\boldsymbol{\phi}$ and $\boldsymbol{\psi}$ are modes of different orders, all the terms vanish. Moreover, we choose to work with a cosine, but working with a sine function will give exactly the same results.

Proposition 3.10. *Let \mathcal{D}^* be a bounded open domain of class \mathcal{C}^2 , and let $\boldsymbol{\psi} \in H_{\sharp}^1(\mathcal{D}, \mathbb{R}^3)$ be defined by (3.66), with $\psi_1, \psi_2 \in H^1(\mathcal{D}^*)$. Then, there exist $\phi_1, \phi_2 \in H_{\sharp}^2(\mathcal{D})$, $\boldsymbol{\Theta} \in H_{\sharp}^2(\mathcal{D}, \mathbb{R}^3)$, such that*

$$\boldsymbol{\psi} = \nabla \phi_1 + \nabla \phi_2 + \nabla \times \boldsymbol{\Theta},$$

where ϕ_1, ϕ_2 and $\boldsymbol{\Theta}$ are unique and defined by

$$\phi_1 = \varphi_1 \cos \frac{2\pi k z}{T}, \phi_2 = \varphi_2 \cos \frac{2\pi k z}{T}, \boldsymbol{\Theta} = \begin{pmatrix} a \sin \frac{2\pi k z}{T} \\ b \sin \frac{2\pi k z}{T} \\ \varphi_3 \cos \frac{2\pi k z}{T} \end{pmatrix},$$

and $\varphi_1, \varphi_3 \in H^2(\mathcal{D}^*) \cap H_0^1(\mathcal{D}^*)$, $\varphi_2 \in H^2(\mathcal{D}^*)$ such that $\Delta \varphi_2 - \frac{4\pi^2 k^2}{T^2} \varphi_2 = 0$ in \mathcal{D}^* , $a, b \in H^1(\mathcal{D}^*)$, $\nabla \cdot \boldsymbol{\Theta} = 0$ and $(\nabla \times \boldsymbol{\Theta}) \cdot \mathbf{n} = 0$ on $\partial \mathcal{D}$.

This decomposition is orthogonal for the scalar product of $L^2(\mathcal{D})$.

Proof. Let $\varphi_1 \in H^1(\mathcal{D}^*)$ be the unique solution in $H^1(\mathcal{D}^*)$ of

$$\begin{cases} \Delta\varphi_1 - \frac{4\pi^2 k^2}{T^2}\varphi_1 = \partial_x\psi_1 + \partial_y\psi_2 & \text{in } \mathcal{D}^*, \\ \varphi_1 = 0 & \text{on } \partial\mathcal{D}^*, \end{cases}$$

and let us define $\phi_1(x, y, z) = \varphi_1(x, y) \cos \frac{2\pi kz}{T}$. As \mathcal{D}^* is of class \mathcal{C}^2 , φ_1 is in $H^2(\mathcal{D}^*)$. In particular, the trace of $\nabla\varphi_1$ on $\partial\mathcal{D}$ is well defined, and so is the following problem:

$$\begin{cases} \Delta\varphi_2 - \frac{4\pi^2 k^2}{T^2}\varphi_2 = 0 & \text{in } \mathcal{D}^*, \\ \frac{\partial\varphi_2}{\partial n} = (\Phi - \nabla\varphi_1) \cdot n & \text{on } \partial\mathcal{D}^*, \end{cases} \quad (3.67)$$

of which the solution φ_2 is in $H^2(\mathcal{D}^*)$. Let us define $\phi_2(x, y, z) = \varphi_2(x, y) \cos \frac{2\pi kz}{T}$.

Let us define finally $\varphi_3 \in H^1(\mathcal{D}^*)$ as the unique solution in $H^1(\mathcal{D}^*)$ of

$$\begin{cases} \Delta\varphi_3 - \frac{4\pi^2 k^2}{T^2}\varphi_3 = \partial_y\psi_1 - \partial_x\psi_2 & \text{in } \mathcal{D}^*, \\ \varphi_3 = 0 & \text{on } \partial\mathcal{D}^*. \end{cases}$$

By the regularity of \mathcal{D}^* , φ_3 is in $H^2(\mathcal{D}^*)$. Let us define

$$\begin{aligned} a &= \frac{T}{2\pi k} (\psi_2 - \partial_y(\varphi_1 + \varphi_2) + \partial_x\varphi_3), \\ b &= \frac{T}{2\pi k} (-\psi_1 + \partial_x(\varphi_1 + \varphi_2) + \partial_y\varphi_3). \end{aligned}$$

A simple computation gives

$$\partial_x b - \partial_y a = \frac{2\pi k}{T} (\varphi_1 + \varphi_2), \quad (3.68)$$

as well as

$$\partial_x a + \partial_y b = \frac{2\pi k}{T} \varphi_3, \quad (3.69)$$

Let us denote

$$\Theta(x, y, z) = \begin{pmatrix} a(x, y) \sin \frac{2\pi kz}{T} \\ b(x, y) \sin \frac{2\pi kz}{T} \\ \varphi_3(x, y) \cos \frac{2\pi kz}{T} \end{pmatrix}. \quad (3.70)$$

By (3.69), Θ is a divergence-free operator, $\nabla \cdot \Theta = 0$. Moreover, it holds that

$$\nabla \times \Theta = \begin{pmatrix} (\psi_1 - \partial_x(\varphi_1 + \varphi_2)) \cos \frac{2\pi kz}{T} \\ (\psi_2 - \partial_y(\varphi_1 + \varphi_2)) \cos \frac{2\pi kz}{T} \\ -\frac{2\pi k}{T} (\varphi_1 + \varphi_2) \sin \frac{2\pi kz}{T} \end{pmatrix} = \psi - \nabla\phi_1 - \nabla\phi_2, \quad (3.71)$$

by the definitions of a , b and (3.68). It follows by (3.67) that $(\nabla \times \Theta) \cdot n = 0$ on $\partial\mathcal{D}$. Thus, we have decomposed ψ .

Now let us show that this decomposition is orthogonal in $L^2(\mathcal{D})$. For every $\phi_1 = f_1 \cos \frac{2k\pi z}{T}$, with $f_1 \in H_0^1(\mathcal{D}^*) \cap H^2(\mathcal{D}^*)$, $\phi_2 = f_2 \cos \frac{2k\pi z}{T}$, with $f_2 \in H^2(\mathcal{D}^*)$ such that $\Delta f_2 - \frac{4\pi^2 k^2}{T^2} f_2 = \Delta_{\mathbb{R}^3} \phi_2 = 0$, it holds that

$$\begin{aligned} \int_{\mathcal{D}} \nabla \phi_1 \cdot \nabla \phi_2 &= - \int_{\mathcal{D}} \Delta_{\mathbb{R}^3} \phi_2 \phi_1 + \int_{\partial \mathcal{D}} \phi_1 \nabla \phi_2 \cdot n \\ &= \int_0^T \int_{\partial \mathcal{D}^*} f_1 \cos \frac{2k\pi z}{T} \nabla \phi_2 \cdot n \\ &= 0, \end{aligned}$$

since f_1 is in $H_0^1(\mathcal{D}^*)$. Moreover, for every $\boldsymbol{\alpha} = \begin{pmatrix} a(x, y) \sin \frac{2\pi k z}{T} \\ b(x, y) \sin \frac{2\pi k z}{T} \\ \varphi_3(x, y) \cos \frac{2\pi k z}{T} \end{pmatrix}$, with $\varphi_3 \in H_0^1(\mathcal{D}^*) \cap H^2(\mathcal{D}^*)$, $a, b \in H^1(\mathcal{D}^*)$ such that $(\nabla \times \boldsymbol{\alpha}) \cdot n = 0$ on $\partial \mathcal{D}$, and $\phi \in H_{\#}^1(\mathcal{D})$, it holds that

$$\int_{\mathcal{D}} (\nabla \times \boldsymbol{\alpha}) \cdot \nabla \phi = - \int_{\mathcal{D}} \nabla \cdot (\nabla \times \boldsymbol{\alpha}) \phi + \int_{\partial \mathcal{D}} \phi (\nabla \times \boldsymbol{\alpha}) \cdot n = 0.$$

□

Proposition 3.11 (Properties of the demagnetizing operator). *The following properties hold :*

- For any function $\psi = \varphi \cos \frac{2\pi k z}{T}$, with $\varphi \in H^1(\mathcal{D}^*)$, we have

$$\mathbf{H}_d(\nabla \psi) = \nabla f,$$

with $f = u \cos \frac{2k\pi z}{T}$, $u \in BL^1(\mathbb{R}^2)$.

- If $\Delta_{\mathbb{R}^3} \psi = 0$ in \mathcal{D} , then $\Delta_{\mathbb{R}^3} f = 0$ on $\mathcal{D} \cup (\mathbb{R}^3 \setminus \overline{\mathcal{D}})$.
- The decomposition of Proposition 3.10 is stable for the demagnetizing operator \mathbf{H}_d .

Proof. Let a function $\psi = \varphi \cos \frac{2\pi k z}{T}$, with $\varphi \in H^1(\mathcal{D}^*)$, and let $f \in BL_{\#}^1(\mathbb{R}^2 \times (0, T))$ such that $\mathbf{H}_d(\nabla \psi) = \nabla f$. Decomposing the function in a Fourier expansion, we can write

$$f(x, y, z) = \sum_{n=0}^{+\infty} u_n(x, y) \cos \frac{2\pi n z}{T} + v_n(x, y) \sin \frac{2\pi n z}{T}, \quad (3.72)$$

with $u_n, v_n \in BL^1(\mathbb{R}^2)$, defined by

$$BL^1(\mathbb{R}^2) = \{u \in H_{\text{loc}}^1(\mathbb{R}^2); \nu u \in L^2(\mathbb{R}^2), \nabla u \in L^2(\mathbb{R}^2, \mathbb{R}^2)\}, .$$

Here, again, we consider the classes of equivalence of functions of $BL^1(\mathbb{R}^2)$ equal up to a constant. We have, for every $n \neq k$, for every $g \in BL^1(\mathbb{R}^2)$, by definition of the demagnetizing field,

$$\begin{aligned} \int_{\mathbb{R}^2 \times (0, T)} \mathbf{H}_d(\nabla \psi) \cdot \nabla \left(g \cos \frac{2\pi n z}{T} \right) &= - \int_{\mathcal{D}^* \times (0, T)} \nabla \left(\varphi \cos \frac{2\pi k z}{T} \right) \cdot \nabla \left(g \cos \frac{2\pi n z}{T} \right) \\ &= 0. \end{aligned}$$

Similarly, for every $n \in \mathbb{N}$,

$$\int_{\mathbb{R}^2 \times (0, T)} \mathbf{H}_d(\nabla \psi) \cdot \nabla \left(g \sin \frac{2\pi n z}{T} \right) = 0.$$

Therefore, for every $h \in BL_{\sharp}^1(\mathbb{R}^2 \times (0, T))$, decomposing h according to (3.72), we have

$$\int_{\mathbb{R}^2 \times (0, T)} \mathbf{H}_d(\nabla \psi) \cdot \nabla h = \int_{\mathbb{R}^2 \times (0, T)} \nabla \left(u_k \cos \frac{2\pi k z}{T} \right) \cdot \nabla h,$$

it follows by definition of the demagnetizing field (3.63) that $\mathbf{H}_d(\nabla \psi) = \nabla \left(u_k \cos \frac{2\pi k z}{T} \right)$. If we assume in addition that $\Delta \psi = 0$ in \mathcal{D} , then for every $g \in C_c^\infty(\mathcal{D})$,

$$\int_{\mathcal{D}} g \Delta f = - \int_{\mathcal{D}} \nabla f \cdot \nabla g = \int_{\mathcal{D}} \nabla \psi \cdot \nabla g = 0,$$

and for every $g \in C_c^\infty(\mathbb{R}^3 \setminus \overline{\mathcal{D}})$,

$$\int_{\mathbb{R}^3 \setminus \overline{\mathcal{D}}} g \Delta f = - \int_{\mathbb{R}^3 \setminus \overline{\mathcal{D}}} \nabla f \cdot \nabla g = \int_{\mathcal{D}} \nabla \psi \cdot \nabla g = 0.$$

It follows that, if $\psi = \varphi_2 \cos \frac{2\pi k z}{T}$, with $\Delta \varphi_2 - \frac{4\pi^2 k^2}{T^2} \varphi_2 = 0$ in \mathcal{D}^* , then $\Delta \psi = 0$ in \mathcal{D} , and thus $\mathbf{H}_d(\nabla \psi) = \nabla \left(u \cos \frac{2\pi k z}{T} \right)$, with $u \in BL^1(\mathbb{R}^2)$ such that $\Delta u - \frac{4\pi^2 k^2}{T^2} u = 0$ in $\mathcal{D}^* \cup (\mathbb{R}^2 \setminus \overline{\mathcal{D}^*})$.

Let $\psi_1 = \varphi_1 \cos \frac{2k\pi}{T}$, with $\varphi_1 \in H^2(\mathcal{D}^*) \cap H_0^1(\mathcal{D}^*)$. Then, ψ_1 is extendable by 0 on $\mathbb{R}^2 \times (0, T)$, with $\psi_1(\cdot, 0) = \psi_1(\cdot, T)$, therefore the extension of ψ_1 is in $BL_{\sharp}^1(\mathbb{R}^2 \times (0, T))$ and $\mathbf{H}_d(\nabla \psi_1) = -\nabla \psi_1$.

Finally, for any $\boldsymbol{\alpha} \in H_{\sharp}^1(\mathcal{D}, \mathbb{R}^3)$ such that $(\nabla \times \boldsymbol{\alpha}) \cdot \mathbf{n} = 0$ on $\partial \mathcal{D}$, and for every φ in $BL_{\sharp}^1(\mathbb{R}^2 \times (0, T))$, integrating by parts,

$$\int_{\mathcal{D}} (\nabla \times \boldsymbol{\alpha}) \cdot \nabla \varphi = - \int_{\mathcal{D}} \nabla \cdot (\nabla \times \boldsymbol{\alpha}) \varphi + \int_{\partial \mathcal{D}} \varphi (\nabla \times \boldsymbol{\alpha}) \cdot \mathbf{n} = 0.$$

Therefore by definition of the demagnetizing field (3.63), $\mathbf{H}_d(\nabla \times \boldsymbol{\alpha}) = 0$. \square

3.2.2 Proof of Proposition 3.9

Let H be a scalar and $\boldsymbol{\psi}$ defined by (3.66) such that for every $\boldsymbol{\phi} \in H_{\sharp}^1(\mathcal{D}, \mathbb{R}^3)$,

$$\begin{aligned} \int_{\mathcal{D}} A(\nabla \times \boldsymbol{\psi}) \cdot (\nabla \times \boldsymbol{\phi}) + A(\nabla \cdot \boldsymbol{\psi})(\nabla \cdot \boldsymbol{\phi}) + (K\boldsymbol{\psi} - \mu_0 M_s \mathbf{H}_d(M_s \boldsymbol{\psi})) \cdot \boldsymbol{\phi} \\ - \frac{\mu_0}{2} M_s H \boldsymbol{\psi} \cdot \boldsymbol{\phi} = 0. \end{aligned} \quad (3.73)$$

Let

$$\phi_1 = \varphi_1 \cos \frac{2\pi k z}{T}, \phi_2 = \varphi_2 \cos \frac{2\pi k z}{T}, \boldsymbol{\Theta} = \begin{pmatrix} a \sin \frac{2\pi k z}{T} \\ b \sin \frac{2\pi k z}{T} \\ \varphi_3 \cos \frac{2\pi k z}{T} \end{pmatrix},$$

be defined by Proposition 3.10 such that

$$\boldsymbol{\psi} = \nabla\phi_1 + \nabla\phi_2 + \nabla \times \boldsymbol{\Theta}.$$

The outline of the proof is the following: Proposition 3.10 allows us to obtain from (3.73) three independent variational problems for φ_1 , φ_2 and φ_3 , which correspond to three eigenvalue problems for H . Let us denote the three corresponding smallest eigenvalues of these problems H_1 , H_2 , and $H_{\boldsymbol{\Theta}}$. As the nucleation field is defined as the smallest value for which (3.73) admits a non-trivial solution, H_N is the smallest of the three solutions of these eigenvalue problems. As we will see, it holds that $H_3 < H_1$, therefore, if $\boldsymbol{\psi}$ is the nucleation solution, then $\varphi_1 = 0$, and the comparison between H_2 and $H_{\boldsymbol{\Theta}}$ will depend on the area of the cylinder section.

Step 1. Let us study the variational problem satisfied by φ_3 . Let $\boldsymbol{\phi} = \boldsymbol{\chi} \cos \frac{2\pi kz}{T}$, with $\boldsymbol{\chi} \in H^1(\mathcal{D}^*, \mathbb{R}^2)$, and $f_1, f_2, \boldsymbol{\alpha}$ defined by Proposition 3.10 such that $\boldsymbol{\phi} = \nabla f_1 + \nabla f_2 + \nabla \times \boldsymbol{\alpha}$. It holds that

$$\begin{aligned} & \int_{\mathcal{D}} A(\nabla \times (\nabla \times \boldsymbol{\Theta})) \cdot (\nabla \times \boldsymbol{\phi}) + \left(K - \frac{\mu_0}{2} M_s H\right) (\nabla \times \boldsymbol{\Theta}) \cdot \boldsymbol{\phi} \\ &= \int_{\mathcal{D}} A(\nabla \times (\nabla \times \boldsymbol{\Theta})) \cdot (\nabla \times (\nabla \times \boldsymbol{\alpha})) \\ & \quad + \left(K - \frac{\mu_0}{2} M_s H\right) (\nabla \times \boldsymbol{\Theta}) \cdot (\nabla \times \boldsymbol{\alpha}) \end{aligned} \quad (3.74)$$

$$\begin{aligned} &= \int_{\mathcal{D}} A(\nabla \times \boldsymbol{\psi}) \cdot (\nabla \times (\nabla \times \boldsymbol{\alpha})) + A(\nabla \cdot \boldsymbol{\psi}) (\nabla \cdot (\nabla \times \boldsymbol{\alpha})) \\ & \quad + (K\boldsymbol{\psi} - \mu_0 M_s \mathbf{H}_d(M_s \boldsymbol{\psi})) \cdot (\nabla \times \boldsymbol{\alpha}) - \frac{\mu_0}{2} M_s H \boldsymbol{\psi} \cdot (\nabla \times \boldsymbol{\alpha}) \end{aligned} \quad (3.75)$$

$$= 0. \quad (3.76)$$

Indeed, (3.74) holds by the orthogonality of the decomposition of Proposition 3.10, and because $\nabla \times (\nabla f_1 + \nabla f_2) = 0$, (3.75) holds by the orthogonality of the decomposition of Proposition 3.10, because $\nabla \times (\nabla \phi_1 + \nabla \phi_2) = 0$, and by the stability of the decomposition for the demagnetizing field, and (3.76) holds because $\boldsymbol{\psi}$ satisfies (3.73).

It follows that for every $\boldsymbol{\phi} \in H_{\sharp}^1(\mathcal{D}, \mathbb{R}^2)$,

$$\begin{aligned} 0 &= \int_{\mathcal{D}} A(\nabla \times (\nabla \times \boldsymbol{\Theta})) \cdot (\nabla \times \boldsymbol{\phi}) \\ & \quad + \left(K - \frac{\mu_0}{2} M_s H\right) \left(\int_{\mathcal{D}} \boldsymbol{\Theta} \cdot (\nabla \times \boldsymbol{\phi}) - \int_{\partial \mathcal{D}} (\boldsymbol{\Theta} \times \boldsymbol{\phi}) \cdot \boldsymbol{n} \right) \\ &= \int_{\mathcal{D}} \left(-A\Delta \boldsymbol{\Theta} + \left(K - \frac{\mu_0}{2} M_s H\right) \boldsymbol{\Theta} \right) \cdot (\nabla \times \boldsymbol{\phi}), \end{aligned}$$

as $\boldsymbol{\Theta} \cdot \boldsymbol{e}_z = 0$ on $\partial \mathcal{D}$. Writing this equality with $\boldsymbol{\phi} = \begin{pmatrix} f \\ 0 \\ 0 \end{pmatrix}$ and $\boldsymbol{\phi} = \begin{pmatrix} 0 \\ f \\ 0 \end{pmatrix}$ for any

$f \in H^1(\mathcal{D}^*)$, and using the fact that, by the definition of Θ (3.70) and equation (3.68),

$$\Delta \Theta = \begin{pmatrix} -\frac{2\pi k}{T} \psi_2 \sin \frac{2\pi k z}{T} \\ \frac{2\pi k}{T} \psi_1 \sin \frac{2\pi k z}{T} \\ \left(-\Delta \varphi_3 + \frac{4\pi^2 k^2}{T^2} \varphi_3\right) \cos \frac{2\pi k z}{T} \end{pmatrix},$$

We see that φ_3 is a solution of

$$\begin{cases} -A\Delta \varphi_3 + \left(K + \frac{4\pi^2 k^2 A}{T^2} - \frac{\mu_0}{2} M_s H\right) \varphi_3 = 0 & \text{in } \mathcal{D}^*, \\ \varphi_3 = 0 & \text{on } \partial \mathcal{D}^*, \end{cases}$$

and thus,

$$H = \frac{2}{\mu_0 M_s} \left(K + \frac{4\pi^2 k^2 A}{T^2} + A\alpha \right),$$

where α is an eigenvalue of the Dirichlet Laplacian on \mathcal{D}^* .

Conversely, taking $\psi = \nabla \times \begin{pmatrix} 0 \\ 0 \\ \varphi \end{pmatrix} = \begin{pmatrix} \partial_y \varphi \\ -\partial_x \varphi \\ 0 \end{pmatrix}$, where φ is the first eigenfunction of the

Dirichlet Laplacian on \mathcal{D}^* , ψ satisfies (3.73) with $H = \frac{2}{\mu_0 M_s} (K + A\alpha_1(\mathcal{D}^*))$. Thus, as the nucleation field is the smallest value for which (3.73) holds for $\psi \in H_{\sharp}^1(\mathcal{D}, \mathbb{R}^2)$, with $\psi \neq 0$, it follows that

$$H_{\Theta} = \frac{2}{\mu_0 M_s} (K + A\alpha_1(\mathcal{D}^*)),$$

where $\alpha_1(\mathcal{D}^*)$ is the first eigenvalue of the Dirichlet Laplacian on \mathcal{D}^* .

Step 2. Let us recall that we have $\psi = \nabla \phi_1 + \nabla \phi_2 + \nabla \times \Theta$, where $\phi_1 = \varphi_1 \cos \frac{2\pi k z}{T}$, ϕ_2 , and Θ are defined by Proposition 3.10, and let us study the variational problem satisfied by φ_1 . Let $\phi = \chi \cos \frac{2\pi k z}{T}$, with $\chi \in H^1(\mathcal{D}^*, \mathbb{R}^2)$, and f_1, f_2, α defined by Proposition 3.10 such that $\phi = \nabla f_1 + \nabla f_2 + \nabla \times \alpha$. Then, it holds that

$$\int_{\mathcal{D}} A\Delta \phi_1 (\nabla \cdot \phi) + \left(K + \mu_0 M_s^2 - \frac{\mu_0}{2} M_s H\right) \nabla \phi_1 \cdot \phi \quad (3.77)$$

$$= \int_{\mathcal{D}} A\Delta \phi_1 \Delta f_1 + \left(K + \mu_0 M_s^2 - \frac{1}{2} M_s H\right) \nabla \phi_1 \cdot \nabla f_1 \quad (3.78)$$

$$\begin{aligned} &= \int_{\mathcal{D}} A(\nabla \times \psi) \cdot (\nabla \times \nabla \phi_1) + A(\nabla \cdot \psi) \Delta f_1 \\ &\quad + (K\psi - \mu_0 M_s \mathbf{H}_d(M_s \psi)) \cdot \nabla f_1 - \frac{\mu_0}{2} M_s H \psi \cdot \nabla f_1 \end{aligned} \quad (3.79)$$

$$= 0. \quad (3.80)$$

Indeed, (3.78) holds by the orthogonality of the decomposition of Proposition 3.10, and because $\nabla \cdot (\nabla f_2 + \nabla \times \alpha) = 0$ in \mathcal{D} , (3.79) holds by the orthogonality of the decomposition of Proposition 3.10, because $\nabla \cdot (\nabla \phi_2 + \nabla \times \Theta) = 0$ in \mathcal{D} , and by the stability of the decomposition for the demagnetizing field, and (3.80) holds because ψ satisfies (3.73).

It follows that, for every $\phi = \chi \cos \frac{2k\pi z}{T}$, for $\chi \in H^1(\mathcal{D}^*)$, integrating by parts,

$$\begin{aligned}
0 &= \int_{\mathcal{D}} A \Delta \phi_1 (\nabla \cdot \phi) + \left(K + \mu_0 M_s^2 - \frac{\mu_0}{2} M_s H \right) \left(- \int_{\mathcal{D}} \phi_1 (\nabla \cdot \phi) + \int_{\partial \mathcal{D}} \phi_1 \phi \cdot n \right) \\
&= \int_{\mathcal{D}} \left(A \Delta \phi_1 + \left(K + \mu_0 M_s^2 - \frac{\mu_0}{2} M_s H \right) \phi_1 \right) (\nabla \cdot \chi) \sin \frac{2k\pi z}{T} \\
&= \int_{\mathcal{D}^*} \left(A \Delta \varphi_1 + \left(\frac{4\pi^2 k^2 A}{T^2} + K + \mu_0 M_s^2 - \frac{\mu_0}{2} M_s H \right) \varphi_1 \right) (\nabla \cdot \chi) \int_0^T \sin^2 \frac{2k\pi z}{T} dz \\
&= \frac{1}{2} \int_{\mathcal{D}^*} \left(A \Delta \varphi_1 + \left(\frac{4\pi^2 k^2 A}{T^2} + K + \mu_0 M_s^2 - \frac{\mu_0}{2} M_s H \right) \varphi_1 \right) (\nabla \cdot \chi), \tag{3.81}
\end{aligned}$$

as $\phi_1 = 0$ on $\partial \mathcal{D}^* \times (0, T)$ by definition, and $\phi \cdot e_z = 0$. It follows that there exists a constant $C \in \mathbb{R}$ such that, in \mathcal{D}^* ,

$$A \Delta \varphi_1 + \left(\frac{4\pi^2 k^2 A}{T^2} + K + \mu_0 M_s^2 - \frac{\mu_0}{2} M_s H \right) \varphi_1. \tag{3.82}$$

Since, moreover, φ is in $H^2(\mathcal{D}^*)$, then ϕ is in $H^4 \mathcal{D}^*$ and the trace of $\Delta \varphi_1$ is well defined on $\partial \mathcal{D}^*$ and is in $L^2(\partial \mathcal{D}^*)$. We can then integrate (3.81) by parts,

$$\begin{aligned}
0 &= \int_{\mathcal{D}^*} \left(A \Delta \varphi_1 + \left(\frac{4\pi^2 k^2 A}{T^2} + K + \mu_0 M_s^2 - \frac{\mu_0}{2} M_s H \right) \varphi_1 \right) (\nabla \cdot \chi) \\
&= \int_{\mathcal{D}^*} (\nabla C) \cdot \chi - \int_{\partial \mathcal{D}^*} A \Delta \varphi_1 (\chi \cdot n),
\end{aligned}$$

as φ_1 is in $H_0^1(\mathcal{D}^*)$. It follows that $\Delta \varphi_1 = 0$ on $\partial \mathcal{D}^*$, and then $C = 0$. Thus, it holds that

$$\begin{cases} A \Delta \varphi_1 + \left(\frac{4\pi^2 k^2 A}{T^2} + K + \mu_0 M_s^2 - \frac{1}{2} M_s H \right) \varphi_1 = 0 & \text{in } \mathcal{D}^*, \\ \varphi_1 = 0 & \text{on } \partial \mathcal{D}^*. \end{cases}$$

It follows that

$$H_1 \geq \frac{2}{\mu_0 M_s} (K + \mu_0 M_s^2 + A \alpha_1(\mathcal{D}^*)) > \frac{2}{\mu_0 M_s} (K + A \alpha_1(\mathcal{D}^*)) = H_{\Theta}.$$

By definition of the nucleation field, we conclude that if ψ is a nucleation mode, then $\varphi_1 = 0$.

Step 3. Let us recall that we have $\psi = \nabla \phi_1 + \nabla \phi_2 + \nabla \times \Theta$, where $\phi_1, \phi_2 = \varphi_2 \cos \frac{2\pi k z}{T}$, and Θ are defined by Proposition 3.10, and let us study the variational problem satisfied by φ_2 . Let $\phi = \chi \cos \frac{2\pi k z}{T}$, with $\chi \in H^1(\mathcal{D}^*, \mathbb{R}^2)$, and f_1, f_2, α defined by Proposition 3.10 such that $\phi = \nabla f_1 + \nabla f_2 + \nabla \times \alpha$. Using the same calculus as in the previous two steps, and the fact that $\Delta \phi_2 = 0$ by Proposition 3.10, we get,

$$\int_{\mathcal{D}} \left(\left(K - \frac{\mu_0}{2} M_s H \right) \nabla \phi_2 - \mu_0 M_s^2 \mathbf{H}_d(\nabla \phi_2) \right) \cdot \phi = 0. \tag{3.83}$$

By Proposition 3.11, we have $\mathbf{H}_d(\nabla\psi_2) = \nabla(u \cos \frac{2\pi kz}{T})$, with u in $BL^1(\mathbb{R}^2)$ and

$$\Delta u - \frac{4\pi^2 k^2}{T^2} u = 0 \text{ in } \mathcal{D}^* \cup (\mathbb{R}^2 \setminus \overline{\mathcal{D}^*}) . \quad (3.84)$$

Let us denote, for a function $f \in H^1(\mathcal{D}^* \cup (\mathbb{R}^2 \setminus \overline{\mathcal{D}^*}))$, by f^- and f^+ the trace of f on $\partial\mathcal{D}^*$ of the restriction of f on, respectively, \mathcal{D}^* and $\mathbb{R}^2 \setminus \overline{\mathcal{D}^*}$. By definition of the demagnetizing field, and the fact that $\Delta\phi_2 = 0$, we have on $\partial\mathcal{D}^* \times (0, T)$,

$$(\mathbf{H}_d(\nabla\phi_2) + \nabla\phi_2)^- \cdot n = (\mathbf{H}_d(\nabla\phi_2))^+ \cdot n ,$$

with, inside \mathcal{D}^* ,

$$(\mathbf{H}_d(\nabla\phi_2) + \nabla\phi_2)^- \cdot n = \left(1 + \frac{\mu_0 M_s^2}{2K - \mu_0 M_s H}\right) \frac{\partial u^-}{\partial n} \cos \frac{2k\pi z}{T} ,$$

as by (3.84), $\mathbf{H}_d(\nabla\phi_2) = \frac{2K - \mu_0 M_s H}{2\mu_0 M_s^2} \nabla\phi_2$, and n is orthogonal to \mathbf{e}_z . Outside \mathcal{D}^* ,

$$(\mathbf{H}_d(\nabla\phi_2))^+ \cdot n = \frac{\partial u^+}{\partial n} \cos \frac{2k\pi z}{T} .$$

It follows that u is a solution of

$$\begin{cases} \Delta u - \frac{4\pi^2 k^2}{T^2} u = 0 & \text{in } \mathcal{D}^* \cup (\mathbb{R}^2 \setminus \overline{\mathcal{D}^*}) , \\ u^- = u^+ & \text{on } \partial\mathcal{D}^* , \\ \left(1 + \frac{2\mu_0 M_s^2}{2K - \mu_0 M_s H}\right) \frac{\partial u^-}{\partial n} = \frac{\partial u^+}{\partial n} & \text{on } \partial\mathcal{D}^* , \\ u(x) \rightarrow 0 & \text{as } |x| \rightarrow \infty , \end{cases} \quad (3.85)$$

which boils down to the following equation: search for u such that

$$\begin{cases} -\operatorname{div}(a\nabla u) + \frac{4\pi^2 k^2}{T^2} u = 0 & \text{in } \mathbb{R}^2 , \\ |u(x)| \rightarrow 0 & |x| \rightarrow \infty , \end{cases} \quad (3.86)$$

where

$$a(x) = \begin{cases} 1 + \frac{2\mu_0 M_s^2}{2K - \mu_0 M_s H} & \text{if } x \in \mathcal{D}^* , \\ 1 & \text{otherwise.} \end{cases}$$

A necessary condition for equation (3.86) to admit a non-trivial solution is

$$1 + \frac{2\mu_0 M_s^2}{2K - \mu_0 M_s H} \leq 0 \Leftrightarrow \frac{2K}{\mu_0 M_s} < H \leq \frac{2K}{\mu_0 M_s} + 2M_s . \quad (3.87)$$

Let u be a solution of (3.86), $\lambda > 0$ and define

$$\begin{aligned} u_\lambda : \lambda\mathcal{D}^* &\rightarrow \mathbb{R} \\ x &\mapsto u\left(\frac{x}{\lambda}\right) . \end{aligned}$$

Then, u_λ satisfies

$$\begin{cases} -\operatorname{div}\left(a\left(\frac{\cdot}{\lambda}\right)\nabla u_\lambda\right)+\frac{4\pi^2k^2}{\lambda^2T^2}u_\lambda=0 & \text{in } \mathbb{R}^2, \\ |u_\lambda(x)|\rightarrow 0 & |x|\rightarrow\infty. \end{cases} \quad (3.88)$$

It follows that equation is stable by dilatation of \mathcal{D}^* by λ , as T is replaced by $\frac{T}{\lambda}$.

We consider a domain $\mathcal{D} = \mathcal{D}^* \times (0, T)$ and want to see how the nucleation behaves as \mathcal{D} is dilated. Let $T = 1$ and $k \in \mathbb{N}$ be fixed and H^k be the lowest value such that (3.86) admits a non-trivial solution. Let u be this solution and define for $(x, y, z) \in \mathcal{D}$,

$$\varphi(x, y, z) = \frac{2\mu_0M_s}{2K - \mu_0M_sH^k}u(x, y) \cos 2k\pi z.$$

Then $\boldsymbol{\psi} = \nabla\varphi$ satisfies (3.73). Let us take $k \in \mathbb{N}$ that gives the lowest value of H^k . Then

$$H_2 = \frac{2K}{\mu_0M_s} + 2\beta M_s,$$

where β is in $(0, 1]$ by (3.87).

This value must be compared to

$$H_{\Theta} = \frac{2K}{\mu_0M_s} + \frac{2A\alpha_1(\mathcal{D}^*)}{\mu_0M_s},$$

with $\alpha_1(\mathcal{D}^*)$ a decreasing function of the area of \mathcal{D}^* , which tends to zero at infinity, and towards infinity at zero. Thus, there exists a value of λ such that for $\mathcal{D} = l\mathcal{D}^* \times (0, \frac{1}{l})$, and $l \leq \lambda$ we have $H_N = H_2$ and for $l \geq \lambda$, $H_N = H_{\Theta}$. This value of λ is defined by

$$\frac{2A\alpha_1(\lambda\mathcal{D}^*)}{\mu_0M_s} = \frac{2A\alpha_1(\mathcal{D}^*)}{\lambda^2\mu_0M_s} = 2\beta M_s, .$$

It follows that λ is characterized by the exchange length ℓ_{ex} :

$$\lambda = \sqrt{\frac{A\alpha_1(\mathcal{D}^*)}{\beta\mu_0M_s^2}} = \sqrt{\frac{\beta}{\alpha_1(\mathcal{D}^*)}}\ell_{\text{ex}},$$

where β and $\alpha_1(\mathcal{D}^*)$ depend on the geometry of \mathcal{D}^* .

3.2.3 Cylinder with a circular section

In the particular case of infinite cylinders with a circular section, we can refine the results of Proposition 3.9. In this subsection, we set

$$\mathcal{D}^* = D_R = \{\mathbf{x} \in \mathbb{R}^2; |\mathbf{x}| < R\}.$$

We make use of the cylindrical coordinate system (ρ, θ, z) such that

$$\begin{cases} x = \rho \cos \theta, \\ y = \rho \sin \theta, \\ z = z, \end{cases}$$

along with the canonical vector basis $(\mathbf{e}_\rho, \mathbf{e}_\theta, \mathbf{e}_z)$.

Proposition 3.12. Let $\mathcal{D} = D_R \times (0, T)$ be an infinite cylinder with circular section, with $R, T > 0$. Let $R, \lambda = \frac{R}{T}$ be fixed. Let us define the coherence radius,

$$R_{ch} = x_1^* \sqrt{\frac{A}{\mu_0 M_s^2 (1 + \gamma(\lambda))}} = x_1^* \ell_{ex} \sqrt{\frac{1}{1 + \gamma(\lambda)}},$$

where $\gamma > 0$ depends on λ , and x_1^* is a constant. The nucleation mode depends on the radius of the disk R , with

- for $R \leq R_{ch}$, the nucleation mode writes

$$\boldsymbol{\psi} = \begin{pmatrix} \frac{2\pi k}{T} I_1 \left(2\pi k \frac{r}{T} \right) \cos \theta \cos \frac{2\pi k z}{T} \\ -\frac{1}{r} I_1 \left(2\pi k \frac{r}{T} \right) \sin \theta \cos \frac{2\pi k z}{T} \\ -\frac{2\pi k}{T} I_1 \left(2\pi k \frac{r}{T} \right) \cos \theta \sin \frac{2\pi k z}{T} \end{pmatrix}, \quad (3.89)$$

in cylindrical coordinates, where I_1 is the modified Bessel function of first order and of first kind, and the nucleation field is

$$H_N = \frac{2K}{\mu_0 M_s} + \frac{2M_s}{1 + \gamma(\lambda)}.$$

In particular, for $\lambda \rightarrow 0$, we have

$$H_N \rightarrow \frac{2K}{\mu_0 M_s} + M_s, \quad (3.90)$$

and the nucleation is equivalent to the uniform mode $\boldsymbol{\psi} = \mathbf{e}_x$.

- for $R > R_{ch}$, the nucleation mode writes

$$\boldsymbol{\psi} = \begin{pmatrix} -\frac{\sin \theta}{r} J_1 \left(x_1^* \frac{r}{R} \right) \\ -\frac{x_1^* \cos \theta}{R} J_1' \left(x_1^* \frac{r}{R} \right) \\ 0 \end{pmatrix}, \quad (3.91)$$

in cylindrical coordinates, where J_1 is the Bessel function of first order and of first kind, and the nucleation field is

$$H_N = \frac{2K}{\mu_0 M_s} + \frac{2A(x_1^*)^2}{\mu_0 M_s R^2}.$$

Comparing our results with the study of nucleation done by physicists in [24], we see that, despite our expression of the energy being different (see Subsection 1.1), the expressions of the nucleation modes (3.89) and (3.91) correspond to, respectively, the *buckling mode* and the *curling mode*. In Figure 3.1 is the graph of the (reduced) nucleation field obtained by the authors:

- for $R < \ell_{ex}$, the nucleation field is $H_N = H_c = \frac{2K}{M_s}$, and the nucleation mode is uniform (called *unison* by physicists). They obtained this result by showing, just like in (3.90), that the buckling mode is equivalent to the uniform mode.

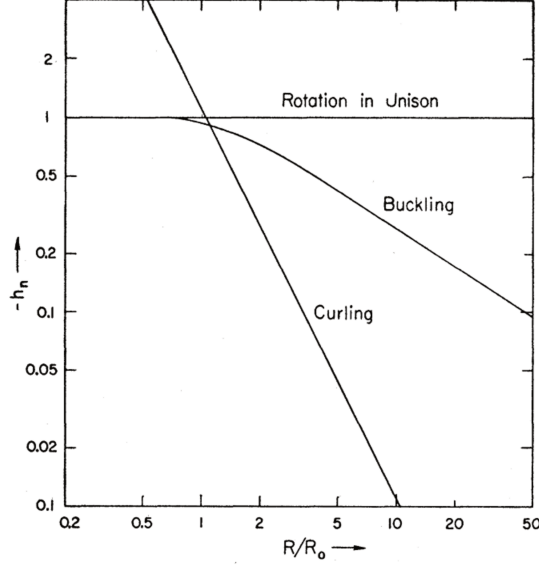


Figure 3.1: This graph from [24] gives the reduced nucleation field $h_n = \frac{H_N}{H_c}$, with $H_c = \frac{2K}{\mu_0 M_s}$, as a function of the radius of the cylinder $\frac{R}{\ell_{\text{ex}}}$.

- For $R \approx \ell_{\text{ex}}$, the nucleation mode is the *buckling*,
- for $R > \ell_{\text{ex}}$, the nucleation mode is the *curling*.

We see that our study is slightly more precise.

Proof. Let us resolve explicitly the eigenvalue problems that appear in the previous section.

Step 1. Let us first resolve the eigenvalue problem satisfied by ϕ_2 (3.83). By the proof of Proposition 3.2, this amounts to resolving completely the problem (3.85).

First, the case $k = 0$ must be treated separately. Following the same ideas as in the proof of Proposition 3.7, let us study the spectrum of the Neumann-Poincaré operator of D_R , defined by: for every $\mathbf{x} \in \partial D$,

$$\mathcal{K}_{D_R}^* \phi(\mathbf{x}) = \int_{\partial D_R} \frac{\partial G}{\partial n_x}(\mathbf{x}, \mathbf{y}) \phi(\mathbf{y}) ds(\mathbf{y}), \quad (3.92)$$

where G is the Green function of the Laplace operator in \mathbb{R}^2 , for functions $\phi \in \mathcal{C}(\partial D_R)$ such that $\int_{\partial D_R} \phi ds = 0$. We consider the extension of $\mathcal{K}_{D_R}^*$ on

$$H_0^{-1/2}(\partial D_R) = \left\{ \phi \in H^{-1/2}(\partial D_R) ; \int_{\partial D_R} \phi ds = 0 \right\}.$$

For every $\mathbf{x}, \mathbf{y} = (r_x, \theta_x), (r_y, \theta_y) \in \mathbb{R}^2$, the Green function G of the Laplace operator in \mathbb{R}^2 is defined by

$$\begin{aligned} G(\mathbf{x}, \mathbf{y}) &= \frac{1}{2\pi} \log |\mathbf{x} - \mathbf{y}| \\ &= \frac{1}{4\pi} \log \left((r_x \cos \theta_x - r_y \cos \theta_y)^2 + (r_x \sin \theta_x - r_y \sin \theta_y)^2 \right). \end{aligned}$$

On the boundary ∂D_R , the normal to the surface \mathbf{n} is equal to the unit vector \mathbf{e}_r , therefore, for every $\mathbf{x}, \mathbf{y} = (R, \theta_x), (R, \theta_y) \in \partial D_R$,

$$\begin{aligned} \frac{\partial G}{\partial n_x}(\mathbf{x}, \mathbf{y}) &= \frac{\partial G}{\partial r_x}(\mathbf{x}, \mathbf{y}) \\ &= \frac{1}{4\pi} \frac{2R(\cos^2 \theta_x - \cos \theta_y \cos \theta_x + \sin^2 \theta_x - \sin \theta_y \sin \theta_x)}{(R \cos \theta_x - R \cos \theta_y)^2 + (R \sin \theta_x - R \sin \theta_y)^2} \\ &= \frac{1}{4\pi R}. \end{aligned}$$

It follows by the definition of $\mathcal{K}_{D_R}^*$ (3.92) that, for every $\phi \in H_0^{-1/2}(\partial D_R)$, for every $\mathbf{x} \in \partial \mathcal{D}$,

$$\mathcal{K}_{D_R}^* \phi(\mathbf{x}) = \int_{\partial D_R} \frac{1}{4\pi R} \phi(\mathbf{y}) ds(\mathbf{y}) = 0, \quad (3.93)$$

by definition of $H_0^{-1/2}(\partial D_R)$. The equation

$$\begin{cases} \Delta u = 0 & \text{in } \mathcal{D}^* \cup (\mathbb{R}^2 \setminus \overline{\mathcal{D}^*}), \\ u^- = u^+ & \text{on } \partial \mathcal{D}^*, \\ \left(1 + \frac{2\mu_0 M_s^2}{2K - \mu_0 M_s H}\right) \frac{\partial u^-}{\partial n} = \frac{\partial u^+}{\partial n} & \text{on } \partial \mathcal{D}^*, \\ u(x) \rightarrow 0 & \text{as } |x| \rightarrow \infty, \end{cases}$$

admits a solution that writes $u = \mathcal{S}_{D_R} \phi$, with $\phi \in H_0^{-1/2}(\partial D_R)$. Using (3.93) and the jump of derivatives across ∂D_R , we have

$$\frac{\partial u^+}{\partial n} = \frac{1}{2} \phi = -\frac{\partial u^-}{\partial n},$$

and therefore

$$H = \frac{2K}{\mu_0 M_s} + M_s.$$

Let us explicit such a solution u . Let us define the function

$$u(r, \theta) = \begin{cases} r \cos \theta, & \text{for } r \leq R, \\ R^2 \frac{\cos \theta}{r}, & \text{for } r > R. \end{cases}$$

Then, we have

$$\nabla u(r, \theta) = \begin{cases} \cos \theta \mathbf{e}_r - \sin \theta \mathbf{e}_\theta = \mathbf{e}_x, & \text{for } r \leq R, \\ -\frac{R^2}{r^2} \cos \theta \mathbf{e}_r - \frac{R^2}{r^2} \sin \theta \mathbf{e}_\theta & \text{for } r > R. \end{cases}$$

It follows that the derivative of u across ∂D_R satisfy

$$\left(\frac{\partial u}{\partial n}\right)^+ = (\nabla u)^+ \cdot \mathbf{e}_r = -(\nabla u)^- \cdot \mathbf{e}_r = -\left(\frac{\partial u}{\partial n}\right)^-.$$

Moreover, we verify that $\Delta u = 0$ on $D_R \cup (\mathbb{R}^2 \setminus \overline{D_R})$. Thus, the constant function \mathbf{e}_x satisfies the nucleation equation (3.65), with

$$H = \frac{2K}{\mu_0 M_s} + M_s, \quad \boldsymbol{\psi} = \mathbf{e}_x. \quad (3.94)$$

Let us now resolve (3.85) for $k > 0$:

$$\left\{ \begin{array}{ll} \Delta u - \frac{4\pi^2 k^2}{T^2} u = 0 & \text{in } \mathcal{D}^* \cup (\mathbb{R}^2 \setminus \overline{\mathcal{D}^*}), \\ u^- = u^+ & \text{on } \partial \mathcal{D}^*, \\ -\gamma \frac{\partial u^-}{\partial n} = \frac{\partial u^+}{\partial n} & \text{on } \partial \mathcal{D}^*, \\ u(x) \rightarrow 0 & |x| \rightarrow \infty, \end{array} \right. \quad (3.95)$$

We resolve this equation by separation of variables. Taking u in the form of $u(r, \theta) = f(r) \cos n\theta$, $n \in \mathbb{N}$, it holds that

$$\Delta u - \frac{4\pi^2 k^2}{T^2} u = \left(\frac{f'(r) + r f''(r)}{r} - \frac{n^2}{r^2} f(r) - \frac{4\pi^2 k^2}{T^2} f(r) \right) \cos n\theta.$$

It follows that f is a solution in \mathbb{R}^+ of

$$r^2 f''(r) + r f'(r) - (n^2 + \frac{4\pi^2 k^2}{T^2} r^2) f(r) = 0.$$

Defining g such that $f(r) = g(2\pi k \frac{r}{T})$, we have

$$u^2 g''(u) = \frac{u^2 T^2}{4\pi^2 k^2} f'' \left(\frac{uT}{2\pi k} \right) = -\frac{uT}{2\pi k} f'(u) + (n^2 + u^2) f \left(\frac{uT}{2\pi k} \right) = -u g'(u) + (n^2 + u^2) g(u).$$

The solutions of this differential equation are the modified Bessel functions of first kind I_n and of second kind K_n . Knowing that the function K_n is not defined at 0 and the function I_n diverges at infinity, it follows that

$$f(r) = \begin{cases} a_1 I_n \left(2\pi k \frac{r}{T} \right), & \text{for } r < R \\ a_2 K_n \left(2\pi k \frac{r}{T} \right), & \text{for } r > R. \end{cases}$$

Writing the continuity condition of (3.95), it holds that $a_1 I_n(2\pi k \frac{R}{T}) = a_2 K_n(2\pi k \frac{R}{T})$. Moreover, with the derivative condition, we get

$$-\gamma_n k a_1 I_n' \left(2\pi k \frac{R}{T} \right) = k a_2 K_n' \left(2\pi k \frac{R}{T} \right),$$

therefore the value of γ depends on n and $k \frac{R}{T}$:

$$\gamma_n \left(k \frac{R}{T} \right) = -\frac{a_2 K_n'(2\pi k \frac{R}{T})}{a_1 I_n'(2\pi k \frac{R}{T})} = \begin{cases} \frac{I_0(2\pi k \frac{R}{T}) K_1(2\pi k \frac{R}{T})}{K_0(2\pi k \frac{R}{T}) I_1(2\pi k \frac{R}{T})}, & n = 0 \\ \frac{I_n(2\pi k \frac{R}{T})}{K_n(2\pi k \frac{R}{T})} \left(\frac{K_{n+1}(2\pi k \frac{R}{T}) + K_{n-1}(2\pi k \frac{R}{T})}{I_{n+1}(2\pi k \frac{R}{T}) + I_{n-1}(2\pi k \frac{R}{T})} \right), & n \in \mathbb{N}^*. \end{cases}$$

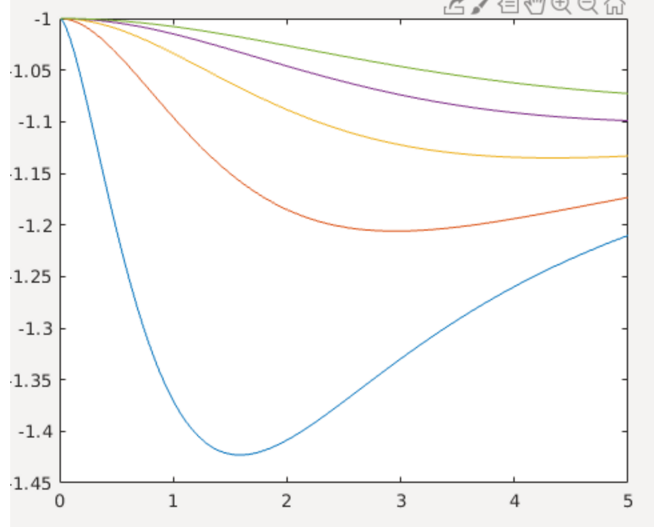


Figure 3.2: Graph of the function $\lambda \mapsto -\frac{I_n(\lambda)}{K_n(\lambda)} \frac{K_{n+1}(\lambda)+K_{n-1}(\lambda)}{I_{n+1}(\lambda)+I_{n-1}(\lambda)}$ for $n \in \{1, \dots, 5\}$, where each color corresponds to a value of n (respectively, blue for $n = 1$, orange for $n = 2$, yellow for $n = 3$, purple for $n = 4$ and green for $n = 5$).

It follows that

$$H = \frac{2K}{\mu_0 M_s} + \frac{2M_s}{1 + \gamma_n \left(k \frac{R}{T}\right)}.$$

Observing in Figure 3.2 that the maximum of $\gamma_n(\lambda)$ is obtained for $n = 1$ for every $\lambda > 0$. Thus, we define

$$\gamma \left(\frac{R}{T}\right) = \max_{k \in \mathbb{N}^*} \gamma_1 \left(k \frac{R}{T}\right).$$

By (3.94) and $\gamma \left(\frac{R}{T}\right) > 1$, it follows that the mode $k = 0$ always gives a higher value for H . Thus,

$$H_2 = \frac{2K}{\mu_0 M_s} + \frac{2M_s}{1 + \gamma \left(\frac{R}{T}\right)}, \quad (3.96)$$

associated to the nucleation mode

$$\boldsymbol{\psi} = \begin{pmatrix} \frac{2\pi k}{T} I_1' \left(2\pi k \frac{r}{T}\right) \cos \theta \cos \frac{2\pi k z}{T} \\ -\frac{1}{r} I_1 \left(2\pi k \frac{r}{T}\right) \sin \theta \cos \frac{2\pi k z}{T} \\ -\frac{2\pi k}{T} I_1 \left(2\pi k \frac{r}{T}\right) \cos \theta \sin \frac{2\pi k z}{T} \end{pmatrix},$$

in cylindrical coordinates.

Let us denote $\lambda = 2\pi \frac{R}{T}$. Then, for every $k \in \mathbb{N}^*$,

$$I_n(k\lambda) \sim_{\lambda \rightarrow 0} \frac{1}{n!} \left(\frac{k\lambda}{2}\right)^n,$$

$$K_n(k\lambda) \sim_{\lambda \rightarrow 0} \frac{(n-1)!}{2} \left(\frac{2}{k\lambda}\right)^n,$$

It follows that for every $n \in \mathbb{N}^*$,

$$\frac{I_n(k\lambda)}{K_n(k\lambda)} \left(\frac{K_{n+1}(k\lambda) + K_{n-1}(k\lambda)}{I_{n+1}(k\lambda) + I_{n-1}(k\lambda)}\right) \sim_{\lambda \rightarrow 0} \frac{1}{n!} \left(\frac{k\lambda}{2}\right)^n \frac{2}{(n-1)!} \left(\frac{k\lambda}{2}\right)^n \frac{n!}{2} \left(\frac{2}{k\lambda}\right)^{n+1} \frac{(n-1)!}{\left(\frac{k\lambda}{2}\right)^{n-1}} = 1,$$

which is also easily satisfied for $n = 0$.

Thus, for $\frac{R}{T} \rightarrow 0$, $\gamma\left(\frac{R}{T}\right) \rightarrow 1$, and

$$H_2 \rightarrow \frac{2K}{\mu_0 M_s} + M_s.$$

The nucleation mode is equivalent to the constant mode (3.94).

The value of H_2 (3.96) must be compared to $H_{\Theta} = \frac{2K}{\mu_0 M_s} + \frac{2A\alpha_1(D_R)}{\mu_0 M_s}$, where $\alpha_1(D_R)$ is the smallest eigenvalue of the Dirichlet Laplacian. We compute this latter value in the following step.

Step 2. Let us recall the computation of the first eigenvalue of the Dirichlet Laplacian on the disk D_R .

Let $\phi \in H^2(D_R)$ such that

$$\begin{cases} -\Delta\phi = \alpha\phi & D_R \\ \phi = 0 & \partial D_R. \end{cases}$$

We write $\phi(r, \theta) = f(r)g(\theta)$. Thus, by separation of variables, we get :

$$\begin{aligned} g(\theta) &= A \cos(n\theta) + B \sin(n\theta), \\ -f''(r) - \frac{1}{r}f'(r) &= \left(\alpha + \frac{n^2}{r^2}\right)f(r), \quad f(R) = 0, \end{aligned}$$

with $n \in \mathbb{N}$. It follows that

$$f(r) = J_n(\sqrt{\alpha}r),$$

with J_n the n -th Bessel function of first kind. Let x_n^* be the first zero of J_n . Then,

$$\alpha \geq \frac{(x_n^*)^2}{R^2}.$$

Knowing that x_n^* is an increasing function of n [16], the smallest eigenvalue holds for $n = 1$, and therefore

$$H_{\Theta} = \frac{2K}{\mu_0 M_s} + \frac{2A(x_1^*)^2}{\mu_0 M_s R^2}, \quad (3.97)$$

associated to the nucleation mode

$$\boldsymbol{\psi} = \nabla \times \begin{pmatrix} 0 \\ 0 \\ J_1\left(x_1^* \frac{r}{R}\right) \cos \theta \end{pmatrix} = \begin{pmatrix} -\frac{\sin \theta}{r} J_1\left(x_1^* \frac{r}{R}\right) \\ -\frac{x_1^* \cos \theta}{R} J_1'\left(x_1^* \frac{r}{R}\right) \\ 0 \end{pmatrix},$$

in cylindrical coordinates.

Finally, let $\mathcal{D} = D_R \times (0, T)$, for fixed $\lambda = \frac{R}{T} > 0$, the nucleation mode switches for $R = R_{\text{ch}}(\lambda)$ such that

$$H_2 = H_{\Theta} \Leftrightarrow \frac{2A(x_1^*)^2}{\mu_0 M_s (R_{\text{ch}})^2} = \frac{2M_s}{1 + \gamma(\lambda)} \Leftrightarrow R_{\text{ch}} = x_1^* \sqrt{\frac{A}{\mu_0 M_s^2 (1 + \gamma(\lambda))}} = \sqrt{\frac{x_1^*}{1 + \gamma(\lambda)}} \ell_{\text{ex}},$$

by the expressions of H_2 (3.96) and H_{Θ} (3.97).

Thus,

- if $R \leq R_{\text{ch}}(\lambda)$, the nucleation mode is $\boldsymbol{\psi} = \nabla\phi_2$, and the nucleation field

$$H_N = H_2 = \frac{2K}{\mu_0 M_s} + \frac{2M_s}{1 + \gamma(\lambda)},$$

- if $R > R_{\text{ch}}(\lambda)$, the nucleation mode is $\boldsymbol{\psi} = \nabla \times \boldsymbol{\Theta}$, and the nucleation field

$$H_N = H_{\boldsymbol{\Theta}} = \frac{2K}{\mu_0 M_s} + \frac{2A(x_1^*)^2}{\mu_0 M_s R^2}.$$

□

In this chapter, we have studied the stability of permanent magnets. In Section 3, we have exhibited the mechanisms of nucleation in the cases of ellipsoids and infinite cylinders. We were able to extend the well-known mechanisms of unison, curling, and buckling, and to give an exact expression for the nucleation field of a permanent magnet. By the tools of homogenization in Section 2, we have a framework that allows to extend this study to the stability of spring magnets.

Chapter 4

Approximation of homogenized coefficients

1 Introduction of the approximation

This chapter is dedicated to the computation of the quantities involved in the homogenized problems in micromagnetism (see Chapter 2 and Chapter 3, Section 2). We work with the framework of Chapter 2, which we recall here. Let $(\Omega, \mathcal{F}, \mu)$ be a probability space where Ω is a compact metric space and \mathcal{F} is a complete σ -algebra. We consider a d -dimensional random field $a(x, \omega) \in \mathbb{R}^{d \times d}$ (in our problem, d is in $\{1, 2, 3\}$) defined for $x \in \mathbb{R}^d$ and $\omega \in \Omega$, such that, for every $x \in \mathbb{R}^d$, $\omega \in \Omega$, $\xi \in \mathbb{R}^d$,

$$c_1 |\xi|^2 \leq \xi \cdot a(x, \omega) \xi \leq c_2 |\xi|^2, \quad (4.1)$$

where $c_1, c_2 > 0$ are constants, and a group action T of \mathbb{R}^d on the set Ω such that for every x in \mathbb{R}^d and ω in Ω , $a(\cdot, T_x \omega) = a(x + \cdot, \omega)$. For the sake of simplicity, we will denote $a(\omega) = a(0, \omega)$. We assume that T and a satisfy stationarity and ergodicity (see assumptions (H1)-(H5) in Chapter 2, Subsection 2.1 for details). Moreover, we define $C(\Omega)$ the set of continuous functions on Ω , and consider M_s in $C(\Omega)$. For simplicity, and without loss of generality, in this chapter we set $M_s = a$. We also consider $L_{\text{pot}}^2(\Omega)$ the set of gradients in the sense of $\omega \in \Omega$, where the gradient is denoted D_ω (see Chapter 2, subsection 2.3) and $L_{\text{sol}}^2(\Omega)$ its orthogonal in $L^2(\Omega)$.

As we have seen in Chapter 2, the behavior of solutions of elliptic equations, harmonic maps equations and the Landau-Lifschitz-Gilbert equation, at large scales, are described by constant-coefficient equations. Among the characteristic coefficients of these equations are the following matrices:

$$\bar{a}\nu = \int_{\Omega} a(\omega)(\nu + \eta_\nu(\omega)) d\mu(\omega) \quad \text{for all } \nu \text{ in } \mathbb{R}^d, \quad (4.2)$$

where η_ν is defined as the unique solution of the problem

$$\begin{cases} \eta_\nu \in L_{\text{pot}}^2(\Omega), \\ a(\cdot)(\nu + \eta_\nu(\cdot)) \in L_{\text{sol}}^2(\Omega), \end{cases} \quad (4.3)$$

and

$$\overline{M}\nu = -\mathbb{E}(M_s\rho_\nu), \quad (4.4)$$

where ρ_ν is the unique solution of the problem

$$\begin{cases} \rho_\nu \in L^2_{\text{pot}}(\Omega), \\ M_s(\cdot)\nu + \rho_\nu(\cdot) \in L^2_{\text{sol}}(\Omega). \end{cases} \quad (4.5)$$

We may notice that in Chapter 2, we defined \overline{M} only in a tri-dimensional setting, but the extension to a d -dimensional setting is direct, although it does not present a practical interest. However, we choose to work in dimension $d \in \{1, 2, 3\}$ in this chapter.

Since these formal mathematical definitions rely on cell problems stated in an abstract probability space, they do not give any practical recipe for constructing or approximating the effective characteristics. Thus, this chapter is dedicated to their approximation. It is a well-known fact in the homogenization literature that \bar{a} is also defined by

$$\bar{a} = \mathbb{E} \left(\int_{(0,1)^d} a(T_x\omega_0) \nabla\varphi(x) dx \right), \quad (4.6)$$

where φ is the *corrector*, unique (up to a constant) sublinear solution at infinity of the equation

$$-\operatorname{div}(a(T_x\omega_0)(\nabla\varphi + I)) = 0 \text{ in } \mathbb{R}^d. \quad (4.7)$$

Thus, the computation of \bar{a} has three sources of inaccuracy:

1. The equation (4.7) is posed on the infinite space \mathbb{R}^d .
2. The resolution of equation (4.7) requires a discretization.
3. The computation of the expectancy (4.6) requires an approximate calculation by averaging.

Likewise, \overline{M} is also defined by

$$\overline{M} = -\mathbb{E} \left(\int_{(0,1)^d} M(T_x\omega_0) \nabla\psi(x) dx \right),$$

where ψ is the *corrector*, unique (up to a constant) sublinear solution at infinity of the equation

$$-\operatorname{div}(\nabla\psi + M(T_x\omega_0)I) = 0 \text{ in } \mathbb{R}^d. \quad (4.8)$$

Thus the sources of inaccuracy are identical.

Thankfully, \bar{a} is the coefficient that appears in the stochastic homogenization of elliptic equations, and quantitative approximation of this coefficient has been an active field of research recently. We make use of the techniques developed for the approximation of both \bar{a} and \overline{M} , specifically by Armstrong, Kuusi and Mourrat in [8]. Indeed, as the definitions are very close, the extension of the results for the approximation of \overline{M} is straightforward.

After introducing the quantities of interest and the difficulties implied by their computation in Section 1, we propose an approximation via finite element method and study our results with this method in Section 2. Finally, in Section 3, we explore MultiGrid methods.

1.1 Definitions of the approximations

One widely used way is to approximate the effective tensors by averaging the operator on a truncated volume with various possible boundary conditions [13]. Let $\rho > 0$, $\square_\rho = (-\frac{1}{2}\rho, \frac{1}{2}\rho)^d$, $\omega_0 \in \Omega$ and let us define the three approximations:

(i) Periodic approximation:

For every z in \mathbb{R}^d , we denote the periodization of $a(T_z\omega_0)$: $a_{\text{per}}^\rho(z, \omega_0) = a(T_{z(\text{mod}\square_\rho)}\omega_0)$, where $z(\text{mod}\square_\rho)$ represents the unique vector $y \in \square_\rho$ such that $z - y$ is in $\rho\mathbb{Z}^d$. Let us define $\chi^\rho \in H_{\text{per},\rho}^1(\mathbb{R}^d, \mathbb{R}^d)$, where $H_{\text{per},\rho}^1(\mathbb{R}^d)$ is the subset of $H_{\text{loc}}^1(\mathbb{R}^d, \mathbb{R}^d)$ of \square_ρ -periodic function, the solution of the problem

$$-\text{div}(a_{\text{per}}^\rho(z)(\nabla\chi^\rho + I)) = 0 \text{ in } \mathbb{R}^d.$$

Then, the *periodic approximation* of the effective matrix is given by

$$\tilde{a}^\rho = \rho^{-d} \int_{\square_\rho} a(T_z\omega_0)(\nabla\chi^\rho(z) + I)dz. \quad (4.9)$$

(ii) Dirichlet approximation :

Let us define w^ρ as the solution of the problem

$$\begin{cases} -\text{div}(a(T_z\omega_0)(\nabla w^\rho + I)) = 0 \text{ in } \square_\rho, \\ w^\rho = 0 \text{ on } \partial\square_\rho. \end{cases} \quad (4.10)$$

Then, the *Dirichlet approximation* of the effective matrix is given by

$$\bar{a}^\rho = \rho^{-d} \int_{\square_\rho} a(T_z\omega_0)(\nabla w^\rho(z) + I)dz.$$

(iii) Neumann approximation :

Let us define ψ^ρ as the solution of the problem

$$\begin{cases} -\text{div}(a(T_z\omega_0)(\nabla\psi^\rho + I)) = 0 \text{ in } \square_\rho, \\ \frac{\partial}{\partial\nu}\psi^\rho = 0 \text{ on } \partial\square_\rho. \end{cases}$$

Then, the *Neumann approximation* of the effective matrix is given by

$$\hat{a}^\rho = \rho^{-d} \int_{\square_\rho} a(T_z\omega_0)(\nabla\psi^\rho(z) + I)dz.$$

We may notice that the field $a_{\text{per}}^\rho(z, \omega_0)$ is no longer ergodic, and thus the matrix $\tilde{a}^\rho = \tilde{a}^\rho(\omega_0)$ could possibly not be deterministic. The same goes for the Dirichlet and the Neumann approximation. However, a direct application of the homogenization Theorem for elliptic equations gives the following Theorem.

Theorem 4.1 ([13], Theorems 1,2,3). *Almost-surely, the following limit relations hold*

$$\begin{aligned}\tilde{a}^\rho &\xrightarrow{\rho \rightarrow +\infty} \bar{a}, \\ \bar{a}^\rho &\xrightarrow{\rho \rightarrow +\infty} \bar{a}, \\ \widehat{a}^\rho &\xrightarrow{\rho \rightarrow +\infty} \bar{a},\end{aligned}$$

where \bar{a} is the matrix defined by (4.2).

Similarly, we may define the periodic, Dirichlet, or Neumann approximations of the demagnetizing homogenized matrix. Let us define $\phi^\rho \in H^1(\square_\rho, \mathbb{R}^3)$, the solution of the problem

$$\begin{cases} -\operatorname{div}(\nabla \phi^\rho + M(T_z \omega_0)I) = 0 & \text{in } \square_\rho, \\ \phi^\rho = 0 & \text{on } \partial \square_\rho. \end{cases}$$

Then, the *Dirichlet approximation* of the effective matrix is given by

$$\overline{M}^\rho = -\rho^{-d} \int_{S_\rho} M_s(T_z \omega_0) \nabla \phi^\rho dz. \quad (4.11)$$

By a straightforward adaptation of the arguments used in Theorem 4.1, we get that almost-surely,

$$\overline{M}^\rho \xrightarrow{\rho \rightarrow +\infty} \overline{M}.$$

1.2 Review of literature

Although Theorem 4.1 is a direct consequence of the homogenization theorem for elliptic equation, convergence rates are more difficult to obtain, and require supplementary assumptions. The field dedicated to the study of convergence rates for \bar{a} and for the corrector defined by (4.7) is *quantitative homogenization*. Otto and Gloria have developed in a series of papers on quantitative methods for estimating approximation errors. Notably, in [29] they show a Central Limit Theorem-type scaling of the variance of the homogenized coefficient in terms of the system volume ρ^d , under an assumption of *Spectral Gap Estimate*. A Spectral Gap Estimate is the following: there exists a radius R only depending on d such that for any function $\zeta : \mathbb{R}^{3 \times 3} \rightarrow \mathbb{R}$, we have

$$\mathbb{E} \left(|\zeta(a) - \mathbb{E}\zeta(a)|^{1/2} \right)^2 \leq \mathbb{E} \left(\sum_{z \in \mathbb{Z}^d \cap [0, \rho]^d} (\operatorname{osc}_{B_R(z)} \zeta(a)) \right),$$

where, $B_R(z)$ is the ball of radius R centered in z , and, for a Lebesgue measurable subset D of $\mathbb{Z}^d \cap \square_\rho$, the *essential oscillation* $\operatorname{osc}_D \zeta$ of ζ with respect to D is a function defined through

$$\begin{aligned} \operatorname{osc}_D \zeta(\tilde{a}) &= \sup\{\zeta(a(\omega)) ; \omega \in \Omega \text{ s.t. } a(\omega) = \tilde{a} \text{ outside } D\} \\ &\quad - \inf\{\zeta(a(\omega)) ; \omega \in \Omega \text{ s.t. } a(\omega) = \tilde{a} \text{ outside } D\}. \end{aligned}$$

It measures how sensitively $\zeta(\tilde{a})$ depends on the values of \tilde{a} outside D . Let us notice that classical examples of stationary random fields satisfy this property, such as the example defined by a Poisson point process, in Chapter 2, Section 2.

Theorem 4.2 ([29], Theorem 1). *Suppose the field a is a stationary d -dimensional matrix field defined on a probability space Ω , with, for every $x, \xi \in \mathbb{R}^d$, $\omega \in \Omega$,*

$$\lambda|\xi|^2 \leq \xi \cdot a(x, \omega)\xi \leq |\xi|^2$$

with $\lambda > 0$, and satisfies the Spectral Gap Estimate. Then, we have the following estimate on the variance of the periodic approximation of the homogenized coefficient

$$\mathbb{E} (|\tilde{a}^\rho - \bar{a}|^2)^{1/2} \leq C(d, \lambda)\rho^{-d}.$$

Simultaneously, Armstrong, Kuusi and Mourrat in [8] obtained results for the Dirichlet approximation under an assumption of finite-range dependence. Indeed, for $\rho = 3^m$, with $m \in \mathbb{N}$, introducing the notation $\mathcal{O}_1(\cdot)$ to control the size of random variables: for every random variable X and $\theta > 0$,

$$X = \mathcal{O}_1(\theta) \Leftrightarrow \mathbb{E} (\exp(\theta^{-1}X_+)) \leq 2, \quad (4.12)$$

where $X_+ = \max(X, 0)$, they obtain the following Theorem.

Theorem 4.3 ([8], Theorem 2.4). *Suppose the field a is a stationary d -dimensional matrix field defined on a probability space Ω , with, for every $x, \xi \in \mathbb{R}^d$, $\omega \in \Omega$,*

$$\lambda|\xi|^2 \leq \xi \cdot a(x, \omega)\xi \leq |\xi|^2$$

with $\lambda > 0$, and has a unit range of dependence.

Fix $s \in (0, d)$. Then, there exists $\alpha(d, \lambda) \in [0, \frac{1}{2})$ and $C(s, d, \lambda) < \infty$ such that, for every $n \in \mathbb{N}$,

$$|\bar{a}^{3^n} - \bar{a}| \leq C3^{-n\alpha(d-s)} + \mathcal{O}_1(C3^{-ns}).$$

2 Finite element method

2.1 The method

Our approximation of the homogenized quantity is based on the Dirichlet approximation. A difficulty is that \bar{a}^ρ is not computable directly. However, the calculation of \bar{a}^ρ boils down to the resolution of a partial differential equation on a finite volume, which solution can be approximated via a finite element method. Here, we present and apply the method for $d = 2$ with quadrangles, which is appropriate for the random checkerboard problem, but it can be extended to $d = 3$.

Illustrated in Figure 4.1, the random checkerboard is a random structure where the space is paved by unit-sized squares, and each square corresponds to a value of a and M_s . Thus, the resolution via the finite element method with quadrangle fits perfectly



Figure 4.1: A segment of one realization of a random checkerboard. In the black region, the matrix a is I , and in the white region $2I$. Illustration from [8].

with this structure: each quadrangle corresponds to a square, and a is constant on this quadrangle.

Let us explain in details the method. We are trying to approximate the solution w^ρ of the problem

$$\begin{cases} -\operatorname{div}(a(T_z\omega_0)(\nabla w^\rho + I)) = 0 & \text{in } \square_\rho, \\ w^\rho = 0 & \text{on } \partial\square_\rho. \end{cases}$$

which is equivalent to the variational problem: for every $\varphi \in H_0^1(\square_\rho, \mathbb{R}^2)$,

$$\int_{\square_\rho} a(T_z\omega_0)\nabla w^\rho(z) \cdot \nabla\varphi(z) + a(T_z\omega_0) \cdot \nabla\varphi(z) dz = 0.$$

In the finite element method, we set $\rho = 2m + 1$, $m \in \mathbb{N}^*$, and we compute the solution of the variational problem: find $v_n^{2m+1} \in Q_{0,n}(\square_{2m+1}, \mathbb{R}^2)$, for every $\varphi \in Q_{0,n}(\square_{2m+1}, \mathbb{R}^2)$,

$$\int_{\square_{2m+1}} a(T_z\omega_0)\nabla v_n^{2m+1}(z) \cdot \nabla\varphi(z) + a(T_z\omega_0) \cdot \nabla\varphi(z) dz = 0,$$

where $Q_{0,n}(\square_{2m+1}, \mathbb{R}^2)$ is a set of piece-wise polynomial continuous functions on *quadrangles*.

Let us introduce the quadrangles, on which the coefficients a and M_s are constant. We partition \square_{2m+1} into unit size squares:

$$\{z + \square_1; z \in \square_{2m+1} \cap \mathbb{Z}^2\} = (\blacksquare^k)_{k \in \{1, \dots, 2^{2m+1}\}}. \quad (4.13)$$

We define

$$\mathbb{P}_n = \left\{ (x, y) \mapsto \sum_{0 \leq k, l \leq n} \lambda_{k,l} x^k y^l; \forall (k, l) \in \{0, \dots, n\}^2, \lambda_{k,l} \in \mathbb{R} \right\}.$$

Thus, we may define the set continuous piece-wise polynomial functions on quadrangles:

$$Q_n(\square_{2m+1}) = \{u \in \mathcal{C}(\square_{2m+1}); \forall k \in \{1, \dots, 2^m\}, u|_{\blacksquare^k} \in \mathbb{P}_n\}, \quad (4.14)$$

where $\mathcal{C}(\square_{2m+1})$ is the set of continuous functions on \square_{2m+1} , as well as

$$Q_{0,n}(\square_{2m+1}) = Q_n(\square_{2m+1}) \cap H_0^1(\square_{2m+1}).$$

Our *finite element Dirichlet approximation of \bar{a} of order n* is

$$\bar{a}_n^{2m+1} = (2m+1)^{-2} \int_{\square_{2m+1}} a(T_z \omega_0) (\nabla v_n^{2m+1}(z) + I) dz. \quad (4.15)$$

We also define the equivalent for the demagnetizing homogenized matrix \bar{M} . Let $g_n^{2m+1} \in Q_{0,n}(\square_{2m+1}, \mathbb{R}^2)$ be the solution of the variational problem: for every $\varphi \in Q_{0,n}(\square_{2m+1}, \mathbb{R}^2)$,

$$\int_{\square_{2m+1}} \nabla g_n^{2m+1}(z) \cdot \nabla \varphi(z) + M_s(T_z \omega_0) \nabla \cdot \varphi(z) dz;$$

and define the *finite element Dirichlet approximation of \bar{M} of order n* :

$$\bar{M}_n^{2m+1} = -(2m+1)^{-2} \int_{\square_{2m+1}} M_s(T_z \omega_0) \nabla g_n^{2m+1}(z) dz.$$

2.2 The one-dimensional case

Let us illustrate first this method in the simple one-dimensional case.

In the one-dimensional case, we can compute \bar{a} explicitly. In this case, we have

$$L_{\text{sol}}^2(\Omega) = \{\sigma \in H^1(\Omega); D_\omega \sigma = 0\},$$

and thus

$$L_{\text{pot}}^2(\Omega) = \{u \in L^2(\Omega); \mathbb{E}(u) = 0\}.$$

It follows that $\xi = a^{-1}(\mathbb{E}(a^{-1})^{-1} - a) \in L_{\text{pot}}^2(\Omega)$ is the solution of the problem (4.3), and

$$\bar{a} = \mathbb{E}(a(1 + \xi)) = \mathbb{E}(a + \mathbb{E}(a^{-1})^{-1} - a) = \mathbb{E}(a^{-1})^{-1}. \quad (4.16)$$

Let us study the one-dimensional checkerboard: assume that for every $n \in \mathbb{Z}$, a is constant on $(\frac{n}{2}, \frac{n}{2} + 1)$, and equal to 1 or 2 with probability $\frac{1}{2}$. Then, by (4.16),

$$\bar{a} = \frac{1}{\frac{1}{2} + \frac{1}{2} \frac{1}{2}} = \frac{4}{3}.$$

Moreover, $\eta = \mathbb{E}(M_s) - M_s \in L_{\text{pot}}^2(\Omega)$ is the solution of the problem (4.5), and therefore

$$\bar{M} = -\mathbb{E}(M_s \eta) = \mathbb{E}(M_s^2) - \mathbb{E}(M_s)^2 = \text{Var}(M_s).$$

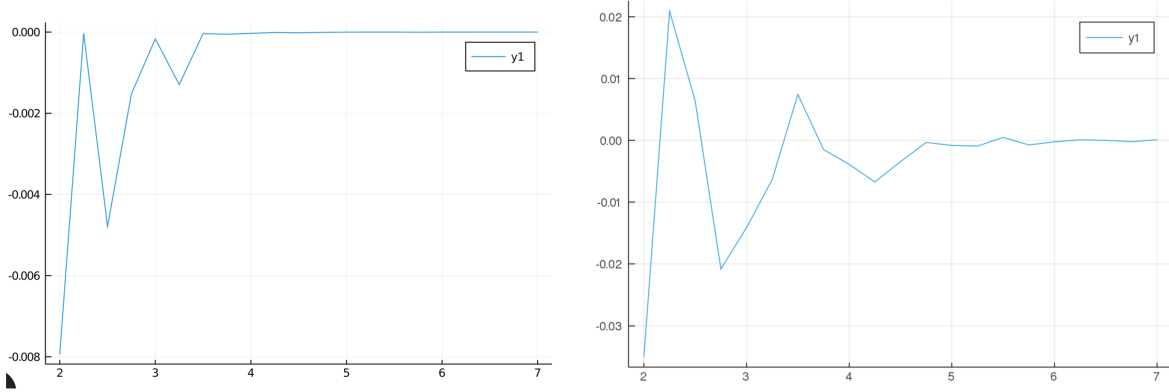


Figure 4.2: Graph of the relative errors $\frac{\overline{M}_1^\rho - \overline{M}}{\overline{M}}$ (left) and $\frac{\overline{a}_1^\rho - \overline{a}}{\overline{a}}$ (right) of the Dirichlet finite element approximation of first order for $\rho = 10^n$ with $n \in (2, 7)$, in the one-dimensional case.

It follows that in the random checkerboard problem,

$$\overline{M} = \frac{1}{2} + \frac{4}{2} - \left(\frac{3}{2}\right)^2 = \frac{1}{4}.$$

Figure 4.2 shows the relative errors $\frac{\overline{M}_1^\rho - \overline{M}}{\overline{M}}$ and $\frac{\overline{a}_1^\rho - \overline{a}}{\overline{a}}$ of the Dirichlet finite element approximations of first order for $\rho = 10^n$ with $n \in (2, 7)$. It shows a rapid convergence to the exact value.

Let us resolve the Dirichlet approximation of \overline{a} . For any $\rho \in \mathbb{N}$, let w^ρ be the solution of

$$\begin{cases} - (a(T_x \omega)((w^\rho)' + 1))' = 0 \text{ in } \left(-\frac{1}{2}\rho, \frac{1}{2}\rho\right), \\ w^\rho\left(-\frac{1}{2}\rho\right) = w^\rho\left(\frac{1}{2}\rho\right) = 0. \end{cases}$$

Then, there exists a constant C such that

$$(w^\rho)' = \frac{C}{a(T_x \omega)} - 1 = \text{constant on } \left(\frac{n}{2}, \frac{n}{2} + 1\right) \forall n \in \mathbb{Z} \cap \left(-\frac{\rho}{2}, \frac{\rho}{2}\right).$$

It follows that w^ρ is piecewise affine: w^ρ is in $Q_0^1(\square_\rho)$. Thus,

- the Dirichlet approximation of the corrector is exactly equal to the finite-element Dirichlet approximation of first order,
- and we have the equality

$$\overline{a}^\rho = \overline{a}_1^\rho.$$

It follows that the finite element approximation of first order converges to the expected value \overline{a} as ρ goes to infinity, with the convergence rate given by Theorem 4.3.

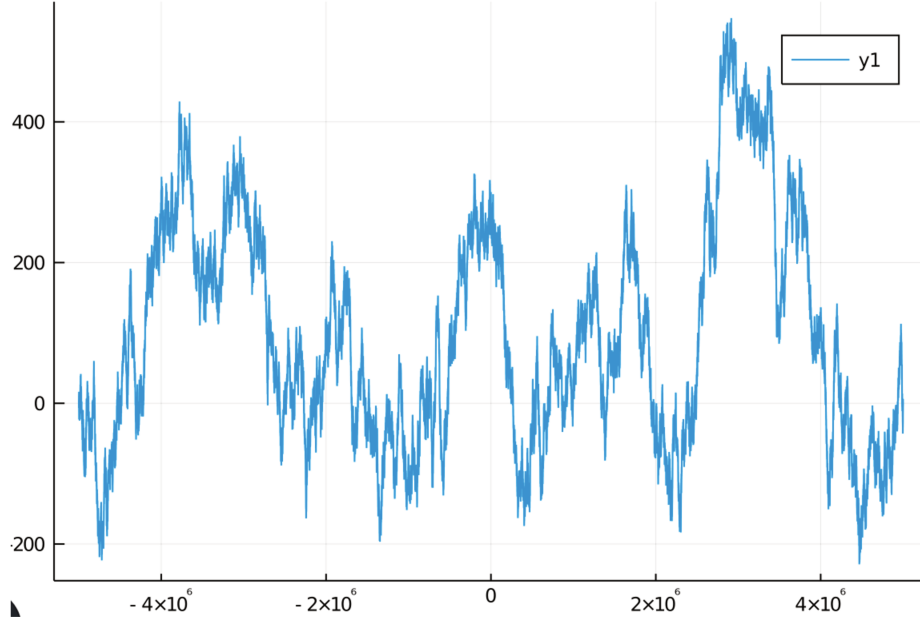


Figure 4.3: Graph of the Dirichlet corrector for \bar{a}, w^ρ for $\rho = 10^7$.

The plot of one realization of the Dirichlet approximation of the corrector w^ρ for $\rho = 10^7$ is given in Figure 4.3.

Let us resolve the Dirichlet approximation of \bar{M} . For any $\rho \in \mathbb{N}$, let ϕ^ρ be the solution of

$$\begin{cases} -((\phi^\rho)' + M_s(T_x \omega_0))' = 0 & \text{in } \left(-\frac{1}{2}\rho, \frac{1}{2}\rho\right), \\ \phi^\rho\left(-\frac{1}{2}\rho\right) = \phi^\rho\left(\frac{1}{2}\rho\right) = 0 \end{cases}. \quad (4.17)$$

Then, there exists a constant C such that

$$(\phi^\rho)' = C - M_s(T_x \omega_0) = \text{constant on } \left(\frac{n}{2}, \frac{n}{2} + 1\right) \forall n \in \mathbb{Z}.$$

It follows that ϕ^ρ is piecewise affine: ϕ^ρ is in $Q_0^1(\square_\rho)$ and the Dirichlet approximation of the corrector is exactly equal to the finite-element approximation of first order.

We have

$$\phi^\rho(x) = C \left(x + \frac{\rho}{2}\right) - \int_{-\frac{1}{2}\rho}^x M_s(T_u \omega_0) du.$$

Since $\phi^\rho\left(\frac{1}{2}\rho\right) = 0$ by (4.17), it follows that

$$C = \rho^{-1} \int_{-\frac{1}{2}\rho}^{\frac{1}{2}\rho} M_s(T_x \omega_0) dx = \rho^{-1} \sum_{k=0}^{\rho} X_k,$$

where $(X_k)_{k \in \{1, \dots, \rho\}}$ is a sequence of identically distributed independent random variables. Then, the finite element Dirichlet approximation of first order of \bar{M} is equal to the

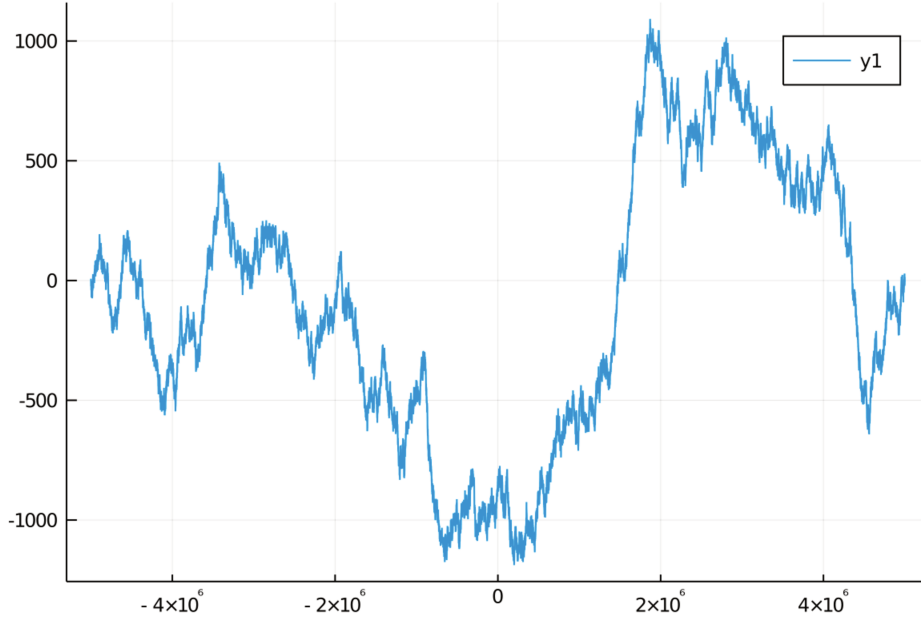


Figure 4.4: Graph of the Dirichlet corrector for \bar{M} , ϕ^ρ for $\rho = 10^7$.

Dirichlet approximation of \bar{M} , and writes, by (4.11),

$$\bar{M}^\rho = \bar{M}_1^\rho = -\rho^{-1} \int_{-\frac{1}{2}\rho}^{\frac{1}{2}\rho} M_s(T_x \omega_0) (\phi^\rho)'(x) dx = \rho^{-1} \sum_{k=0}^{\rho} X_k^2 - \left(\rho^{-1} \sum_{k=0}^{\rho} X_k \right)^2.$$

Thus, this quantity converges to

$$\mathbb{E}(M_s^2) - \mathbb{E}(M_s)^2 = \bar{M},$$

as ρ goes to infinity.

Moreover, we can compute the rate of convergence using the Central Limit Theorem. Indeed, on one hand, it holds that

$$\frac{\rho^{-1} \sum_{k=0}^{\rho} X_k^2 - \mathbb{E}(M_s^2)}{\frac{3}{2\sqrt{\rho}}},$$

converges in distribution to the standard normal distribution $\mathcal{N}(0, 1)$ as ρ goes to infinity. On the other hand,

$$\frac{\left| (\rho^{-1} \sum_{k=0}^{\rho} X_k)^2 - \mathbb{E}(M_s)^2 \right|}{\frac{1}{4\sqrt{\rho}}} \leq 2(\max M_s) \left| \frac{\rho^{-1} \sum_{k=0}^{\rho} X_k - \mathbb{E}(M_s)}{\frac{1}{2\sqrt{\rho}}} \right|,$$

where $\frac{\rho^{-1} \sum_{k=0}^{\rho} X_k - \mathbb{E}(M_s)}{\frac{1}{2\sqrt{\rho}}}$ converges in distribution to the standard normal distribution $\mathcal{N}(0, 1)$ as ρ goes to infinity.

The plot of one realization of the Dirichlet approximation of the corrector ϕ^ρ for $\rho = 10^7$ is given in Figure 4.4.

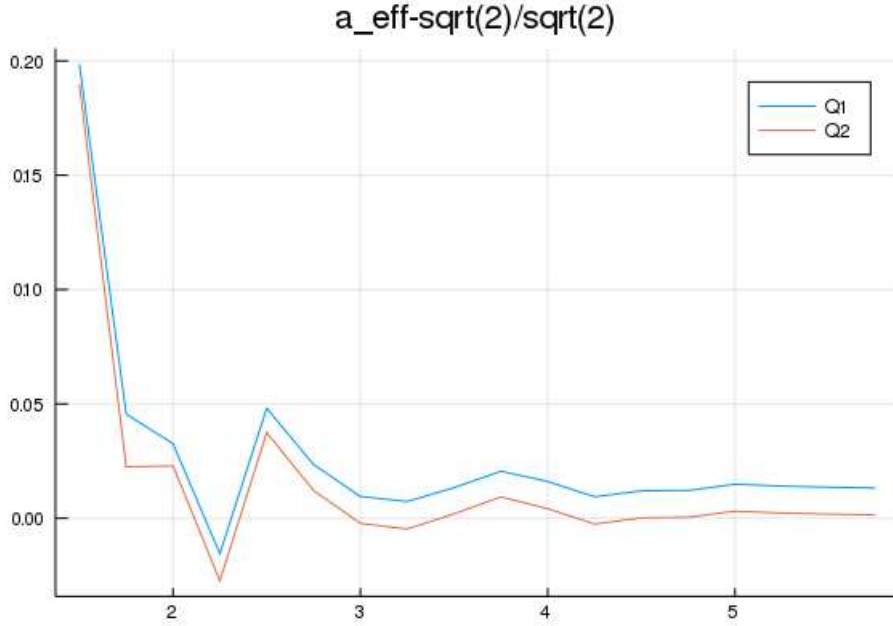


Figure 4.5: Graph of the relative errors $\frac{\bar{a}_1^\rho - \bar{a}}{\bar{a}}$ and $\frac{\bar{a}_2^\rho - \bar{a}}{\bar{a}}$ of finite element Dirichlet approximations of \bar{a} of order 1 (blue) and 2 (orange), for $\rho = 10^l + 1$, where l , in abscissa, is in $(0, 6)$. For each value of l , corresponds a realization of the checkerboard, for which both approximations are computed.

2.3 The two-dimensional case

2.3.1 Numerical results

In some two-dimensional problems, the homogenized matrix can be calculated [39]. Among them is the random checkerboard. Let us pave the space by unit-sized squares (see (4.13)) and color each square either white or black independently at probability p . Each color is then associated with a particular value of the matrix a . If $p = \frac{1}{2}$, and the associated values of a are I and $2I$, then, by [39], Section 7.2,

$$\bar{a} = \sqrt{2}I.$$

On Figure 4.5, we show the error of finite element Dirichlet approximations of \bar{a} of order 1 and 2 for realizations of checkerboard for increasing values of $\rho \in \{0, \dots, 10^6\}$ (each value of ρ corresponds to a different realization of the checkerboard, and for each ρ , the approximations of order 1 and 2 are computed for the same realization). We observe that the error rapidly converges to a small but nonzero error. For the approximation of order 2, the same convergence is observed, the limit is significantly smaller value. The next section is dedicated to the identification of these values. Our guess is that, increasing the degree of the finite element method, we can make significant improvement in the approximation of \bar{a} .

The demagnetizing homogenized matrix \overline{M} cannot be calculated explicitly. However, by Proposition 2.33, we know that \overline{M} is a scalar. In Figure 4.6, the finite element Dirichlet approximation seems to converge towards a particular value.

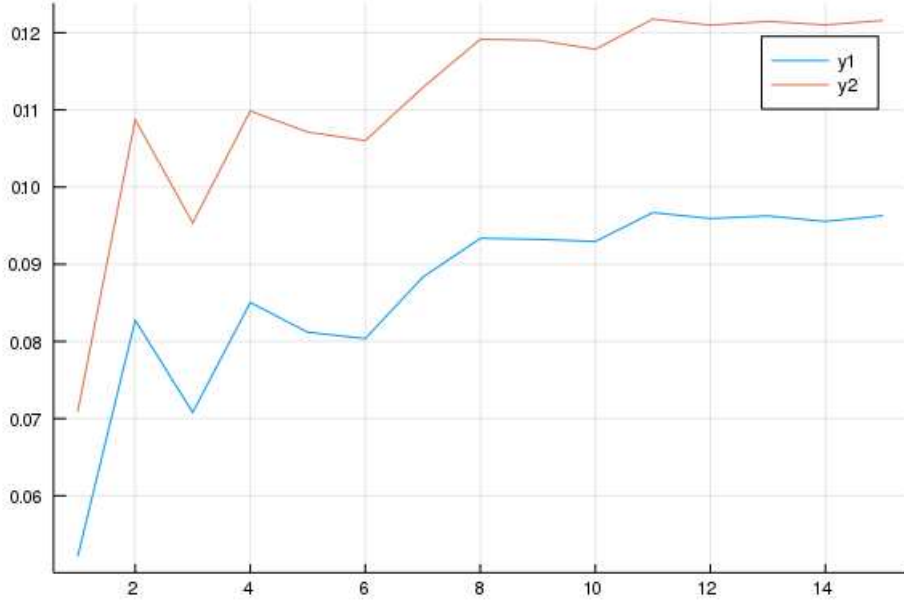


Figure 4.6: Graph of finite element Dirichlet approximations of \bar{M} of order 1 (blue) and 2 (orange), \bar{M}_1^ρ and \bar{M}_2^ρ , for $\rho = 10^l + 1$, where l , in abscissa, is in $(0, 6)$. For each value of l , corresponds a realization of the checkerboard, for which both approximations are computed.

2.3.2 Analysis of the results

We study the finite element Dirichlet approximation of \bar{a} via a straightforward adaptation of the arguments used in [8].

Let the order of approximation $n \in \mathbb{N}$ be fixed, i.e. the degree of the piece-wise polynomial functions used in our approximation (see (4.14)). We define, for $m \in \mathbb{N}$, $p \in \mathbb{R}^2$, the quadratic application

$$\mu_n(\square_{3^m}, p) = \frac{1}{2} p \cdot \bar{a}_n^{3^m} p, \quad (4.18)$$

where $\bar{a}_n^{3^m}$ is defined by (4.15). We notice that, as \bar{a}_n^m is not deterministic, $\mu_n(\square_{3^m})$ is not either. We will compare this quadratic application to the following, introduced in [8]:

$$\mu_H(\square_{3^m}, p) = \frac{1}{2} p \cdot \bar{a}^{3^m} p.$$

As shown by Theorem 4.3 in [8], for every $s \in (0, 2)$, there exists $\alpha \in [0, \frac{1}{2})$ and C such that

$$\left| \mu_H(\square_{3^m}, p) - \frac{1}{2} p \cdot \bar{a} p \right| \leq C 3^{-m\alpha(d-s)} + \mathcal{O}_1(3^{-ms}), \quad (4.19)$$

where \mathcal{O}_1 is defined by (4.12). As can be expected, μ_H and μ_n behave exactly the same.

Proposition 4.4. *Suppose the field a is a stationary two-dimensional matrix field defined on a probability space Ω , satisfies, for every $x, \xi \in \mathbb{R}^2$, $\omega \in \Omega$,*

$$c_1 |\xi|^2 \leq \xi \cdot a(x, \omega) \xi \leq c_2 |\xi|^2$$

with $c_1, c_2 > 0$, and has a unit range of dependence.

There exists a matrix \bar{a}_n , and $\alpha \in [0, \frac{1}{2})$ such that for every $s \in (0, 2)$, there exists $C < \infty$ such that, for every $m \in \mathbb{N}^*$,

$$\left| \mu_n(\square_{3^m}, p) - \frac{1}{2} p \cdot \bar{a}_n p \right| \leq C 3^{-m\alpha(d-s)} + \mathcal{O}_1(3^{-ms}). \quad (4.20)$$

In this section, we will show the basic properties that μ_H and μ_n share and that lead to Proposition 4.4, as in [8].

First, we begin by noticing that, by Lax-Milgram Theorem, we have

$$\mu_n(\square_{3^m}, p) = \inf_{u \in \ell_p + Q_{0,n}(\square_{3^m})} \int_{\square_{3^m}} \frac{1}{2} \nabla u \cdot a \nabla u, \quad (4.21)$$

where $\ell_p : x \mapsto p \cdot x$, and \int_D is the integral on D divided by the volume of D , as well as

$$\mu_H(\square_{3^m}, p) = \inf_{u \in \ell_p + H_0^1(\square_{3^m})} \int_{\square_{3^m}} \frac{1}{2} \nabla u \cdot a \nabla u. \quad (4.22)$$

As for every $m \in \mathbb{N}$, $Q_{0,n}(\square_{3^m}) \subset H_0^1(\square_{3^m})$, we have

$$\mu_n(\square_{3^m}, p) \geq \mu_H(\square_{3^m}, p), \quad (4.23)$$

and thus

$$\bar{a}_n \geq \bar{a}_n.$$

Furthermore, the following proposition holds.

Proposition 4.5. *The minimizer $v^n \in \ell_p + Q_{0,n}(\square_{3^m})$ of $\mu_n(\square_{3^m})$ is the projection of the Dirichlet approximation of the corrector $v \in \ell_p + H_0^1(\square_{3^m})$ into $Q_{0,n}(\square_{3^m})$, which is the minimizer of $\mu_H(\square_{3^m})$, for the following norm in $H_0^1(\square_{3^m})$:*

$$\|f\|_{H_0^1(\square_{3^m})} = \left(\int_{\square_{3^m}} \nabla f \cdot a \nabla f \right)^{1/2}.$$

Proof. By (4.21) and (4.22), we have, for every $w \in \ell_p + Q_{0,n}(\square_{3^m})$,

$$\begin{aligned} \|w - v\|_{H_0^1(\square_{3^m})}^2 &= \int_{\square_{3^m}} \nabla(w - v) \cdot a \nabla(w - v) \\ &= \int_{\square_{3^m}} \nabla w \cdot a \nabla w + \int_{\square_{3^m}} \nabla v \cdot a \nabla v - 2 \int_{\square_{3^m}} a \nabla w \cdot \nabla v \\ &= \int_{\square_{3^m}} \nabla w \cdot a \nabla w + \int_{\square_{3^m}} \nabla v \cdot a \nabla v \\ &\geq \int_{\square_{3^m}} \nabla v^n \cdot a \nabla v^n + \int_{\square_{3^m}} \nabla v \cdot a \nabla v \\ &= \|v^n - v\|_{H_0^1(\square_{3^m})}^2 \end{aligned}$$

because w is in $H_0^1(\square_{3^m})$ and by the variational problem satisfied by v , and by (4.21). \square

Moreover, since for every $n \in \mathbb{N}$, and every $m \in \mathbb{N}$, $Q_{0,n}(\square_m) \subset Q_{0,n+1}(\square_m)$, it holds that

$$\bar{a}^{3^m} \leq \bar{a}_{n+1}^{3^m} \leq \bar{a}_n^{3^m}.$$

Passing to the limit as m goes to infinity, we get

$$\bar{a} \leq \bar{a}_{n+1} \leq \bar{a}_n.$$

Thus, an improvement of the approximation can be obtained by applying the finite element method for a higher degree.

Now let us show the properties satisfied by μ_n which lead to the proof of Proposition 4.4. They are an adaptation from the properties satisfied by μ_H , presented in [8].

Lemma 4.6. Basic properties of μ_n . Fix $m \in \mathbb{N}$. Then $\mu_n(\square_{3^m}, p)$ and its minimizer $v_Q(\cdot, \square_{3^m}, p) = \ell_p + v^m \cdot p$ satisfy the following properties:

- Bounds: for every $p \in \mathbb{R}^2$,

$$\frac{c_1}{2}|p|^2 \leq \mu_n(\square_{3^m}, p) \leq \frac{c_2}{2}|p|^2,$$

where c_1 and c_2 are the bounds for a (see (4.1)).

- Subadditivity: let $l \leq m$, the set $\{z + \square_{3^l}; z \in 3^{2(m-l)}\mathbb{Z}^2 \cap \square_{3^m}\}$ is a partition of \square_{3^m} , and it holds that

$$\mu_n(\square_{3^m}, p) \leq \bigoplus_z \mu_n(z + \square_{3^l}, p),$$

where $\bigoplus_z = 3^{-2(m-l)} \sum_{z \in 3^{2(m-l)}\mathbb{Z}^2 \cap \square_{3^m}}$.

- Quadratic response For every $w \in \ell_p + Q_{0,n}(\square_{3^m})$,

$$\begin{aligned} \frac{1}{2} \int_{\square_{3^m}} |\nabla w - \nabla v^n(\cdot, \square_{3^m}, p)|^2 &\leq \frac{1}{2} \int_{\square_{3^m}} \nabla w \cdot a \nabla w - \mu_n(\square_{3^m}, p) \\ &\leq \frac{\Lambda}{2} \int_{\square_{3^m}} |\nabla w - \nabla v^n(\cdot, \square_{3^m}, p)|^2. \end{aligned} \quad (4.24)$$

Proof. Step 1. Let us show that

$$\frac{c_1}{2}|p|^2 \leq \mu_n(\square_{3^m}, p) \leq \frac{c_2}{2}|p|^2.$$

The right-hand side bound is obtained directly by testing equation (4.21) with ℓ_p . The left-hand side bound comes from Jensen inequality: for every $w \in Q_{0,n}(\square_{3^m})$,

$$\int_{\square_{3^m}} \frac{1}{2} (p + \nabla w) \cdot a (p + \nabla w) \geq \int_{\square_{3^m}} \frac{c_1}{2} |p + \nabla w|^2 \geq \frac{c_1}{2} \left| \int_{\square_{3^m}} p + \nabla w \right|^2 = \frac{c_1}{2} |p|^2.$$

taking the infimum on $Q_{0,n}(\square_{3^m})$, we get the inequality.

Step 2. Let us show the quadratic response (4.24). Let $w \in \ell_p + Q_{0,n}(\square_{3^m})$, then it holds that,

$$\begin{aligned} \int_{\square_{3^m}} \frac{1}{2} \nabla w \cdot a \nabla w - \mu_n(\square_{3^m}, p) &= \int_{\square_{3^m}} \frac{1}{2} (\nabla w - \nabla v) \cdot a (\nabla w - \nabla v^n) \\ &\quad + \int_{\square_{3^m}} (\nabla w - \nabla v^n) \cdot a \nabla v^n, \end{aligned}$$

as $w - v^n$ is in $Q_{0,n}(\square_{3^m})$, this last term vanishes.

Step 3. Proof of subadditivity: we assemble the minimizers of $\mu_n(z + \square_{3^l}, p)$, and compare the energy of the resulting function to the minimizer of $\mu_n(\square_{3^m}, p)$ v : let \tilde{v} be this resulting function, it holds, by definition of μ_n ,

$$\begin{aligned} \mu_n(\square_{3^m}, p) &\leq \int_{\square_{3^m}} \frac{1}{2} \nabla \tilde{v} \cdot a \nabla \tilde{v} \\ &= \bigoplus_z \mu_n(z + \square_{3^l}, p). \end{aligned}$$

□

An immediate consequence of the subadditivity of μ_n is the non-increasing behavior of $\mathbb{E}(\mu_n(\square_{3^m}, p))$: for every $m \in \mathbb{N}$,

$$\begin{aligned} \mathbb{E}(\mu_n(\square_{3^{m+1}}, p)) &\leq \mathbb{E} \left(\bigoplus_z \mu_n(z + \square_{3^m}, p) \right) \\ &\leq \bigoplus_z \mathbb{E}(\mu_n(z + \square_{3^m}, p)) \\ &= \mathbb{E}(\mu_n(\square_{3^m}, p)), \end{aligned}$$

by stationarity.

Moreover, this sequence is always non-negative, therefore, for every $p \in \mathbb{R}^2$, there exists a limit

$$\bar{\mu}_n(p) = \lim_{m \rightarrow +\infty} \mathbb{E}(\mu_n(\square_{3^m}, p)).$$

It follows that $\bar{\mu}_n$ is quadratic and satisfies

$$\frac{c_1}{2} |p|^2 \leq \bar{\mu}_n(p) \leq \frac{c_2}{2} |p|^2.$$

We can define the following matrix

Definition 4.1. Let us denote \bar{a}_n the unique matrix that satisfies

$$\forall p \in \mathbb{R}^3, \bar{\mu}_n(p) = \frac{1}{2} p \cdot \bar{a}_n p$$

Let us define the dual subadditive quantity μ_n^* by, for every $q \in \mathbb{R}^2$, $m \in \mathbb{N}$,

$$\mu_n^*(\square_{3^m}, q) = \sup_{u \in Q_n(\square_{3^m})} \int_{\square_{3^m}} -\frac{1}{2} \nabla u \cdot a \nabla u + q \nabla u.$$

Similarly to the dual subadditive quantity defined in [8],

$$\mu_H^*(\square_{3^m}, q) = \sup_{u \in H^1(\square_{3^m})} \int_{\square_{3^m}} -\frac{1}{2} \nabla u \cdot a \nabla u + q \nabla u = \frac{1}{2} q (\bar{a}_*^{3^m})^{-1} q,$$

where $\bar{a}_*^{3^m}$ converges to \bar{a} as m goes to infinity, with the same convergence rate as \bar{a}^{3^m} (4.19).

Since $Q_n(\square_{3^m}) \subset H^1(\square_{3^m})$, we get that

$$\mu_n^*(\square_{3^m}, q) \leq \mu_H^*(\square_{3^m}, q). \quad (4.25)$$

We also define the set of a -harmonic functions, in the sense of $Q_{0,n}$:

$$\mathcal{A}_n(\square_{3^m}) = \left\{ u \in Q_n(\square_{3^m}); \forall v \in Q_n(\square_{3^m}), \int_{\square_{3^m}} \nabla v \cdot a \nabla u = 0 \right\}.$$

Then, the minimizer of $\mu_n^*(\square_{3^m}, q)$ is in $\mathcal{A}_n(\square_{3^m})$. Indeed, let $v \in Q_n(\square_{3^m})$ be this minimizer, $w \in Q_{0,n}(\square_{3^m})$, $t \in \mathbb{R}$ and let $w_t = v + tw$. We get

$$\begin{aligned} \int_{\square_{3^m}} -\frac{1}{2} \nabla v \cdot a \nabla v + q \nabla v &\geq \int_{\square_{3^m}} -\frac{1}{2} \nabla w_t \cdot a \nabla w_t + q \nabla w_t \\ &= \int_{z+\square_{3^m}} -\frac{1}{2} \nabla v \cdot a \nabla v + q \nabla v + t^2 \int_{\square_{3^m}} -\frac{1}{2} \nabla w \cdot a \nabla w \\ &\quad - t \int_{\square_{3^m}} \nabla v \cdot a \nabla w, \end{aligned}$$

as w is in $Q_{0,n}(\square_{3^m})$. It follows that, for every $t \in \mathbb{R}$,

$$\int_{z+\square_{3^m}} \nabla w \cdot a \nabla v \geq -t \int_{z+\square_{3^m}} \frac{1}{2} \nabla w \cdot a \nabla w,$$

as t goes to 0 and applying the inequality for w and $-w$, we get

$$\int_{z+\square_{3^m}} \nabla w \cdot a \nabla v = 0.$$

Thus, we can write

$$\mu_n^*(\square_{3^m}) = \sup_{u \in \mathcal{A}_n(z+\square_{3^m})} \int_{z+\square_{3^m}} -\frac{1}{2} \nabla u \cdot a \nabla u + q \cdot \nabla u.$$

Testing the minimizer $v \in Q_n(\square_{3^m})$ of $\mu_n(\square_{3^m}, p)$ in the definition of $\mu_n^*(\square_{3^m}, q)$, we get

$$\begin{aligned} \mu_n^*(\square_{3^m}, q) &\geq \int_{\square_{3^m}} -\frac{1}{2} \nabla v \cdot a \nabla v + q \cdot \nabla v \\ &= -\mu_n(\square_{3^m}, p) + q \cdot p. \end{aligned} \quad (4.26)$$

We introduce naturally

$$J_n(\square_{3^m}, p, q) = \mu_n^*(\square_{3^m}, q) + \mu_n(\square_{3^m}, p) - p \cdot q.$$

Lemma 4.7. *for every $m \in \mathbb{N}$, $p \in \mathbb{R}^2$, $q \in \mathbb{R}^2$,*

$$J_n(\square_{3^m}, p, q) = \sup_{w \in \mathcal{A}_n(\square_{3^m})} \int_{\square_{3^m}} -\frac{1}{2} \nabla w \cdot a \nabla w - p \cdot a \nabla w + q \cdot \nabla w, \quad (4.27)$$

and the maximum is obtained in the difference between the minimizer of μ_n and the maximizer of μ_n^ .*

Proof. Let $v \in Q_n(\square_{3^m})$ the minimizer of $\mu_n(\square_{3^m}, p)$. For every $u \in \mathcal{A}_n(\square_{3^m})$,

$$\begin{aligned} \mu_n(\square_{3^m}, p) + \int_{\square_{3^m}} -\frac{1}{2} \nabla u \cdot a \nabla u + q \cdot \nabla u - p \cdot q \\ &= \int_{\square_{3^m}} \frac{1}{2} \nabla v \cdot a \nabla v - \frac{1}{2} \nabla u \cdot a \nabla u + q \cdot \nabla u - p \cdot q \\ &= \int_{\square_{3^m}} -\frac{1}{2} (\nabla u - \nabla v) \cdot a (\nabla u - \nabla v) - \nabla v \cdot a \nabla u + q \cdot (\nabla u - p) \\ &= \int_{\square_{3^m}} -\frac{1}{2} (\nabla u - \nabla v) \cdot a (\nabla u - \nabla v) - p \cdot a (\nabla u - \nabla v) + q \cdot (\nabla u - \nabla v). \end{aligned}$$

because as v is in $\ell_p + Q_{0,n}(\square_{3^m})$ and u is in $\mathcal{A}_n(\square_{3^m})$, we get $\int_{\square_{3^m}} \nabla v = p$ and $\int_{\square_{3^m}} \nabla u \cdot a \nabla v = \int_{\square_{3^m}} \nabla u \cdot ap$. As $u - v$ is in $\mathcal{A}(\square_{3^m})$, we obtain (4.27) by definition of $\mu_n^*(\square_{3^m})$, and by taking the infimum over $\mathcal{A}(\square_{3^m})$. \square

Lemma 4.8. *Basic properties of J_n . Fix $m \in \mathbb{N}$. The quantity $J_n(\square_{3^m}, p, q)$ and its maximizer $v(\cdot, \square_{3^m}, p, q)$ satisfy the following properties:*

- *Quadratic representation: there exists a matrix $\bar{a}_{n,*}^{3^m}$ such that*

$$c_1 Id \leq \bar{a}_{n,*}^{3^m} \leq \bar{a}_n^{3^m} \leq c_2 Id, \quad (4.28)$$

and

$$J_n(\square_{3^m}, p, q) = \frac{1}{2} p \cdot \bar{a}_{n,*}^{3^m} p + \frac{1}{2} q \cdot (\bar{a}_{n,*}^{3^m})^{-1} q - p \cdot q. \quad (4.29)$$

Moreover, it holds that

$$\begin{aligned} \bar{a}_n^{3^m} p &= - \int_{\square_{3^m}} a \nabla v(\cdot, \square_{3^m}, p, 0) \\ (\bar{a}_{n,*}^{3^m})^{-1} q &= \int_{\square_{3^m}} \nabla v(\cdot, \square_{3^m}, 0, q). \end{aligned} \quad (4.30)$$

- *Subadditivity: for every $l \leq m$, $p, q \in \mathbb{R}^2$*

$$J_n(\square_{3^m}, p, q) \leq \bigoplus_z J_n(z + \square_{3^l}, p, q).$$

- *First variation: for every $p, q \in \mathbb{R}^2$, the function $v(\cdot, \square_{3^m}, p, q)$ is characterized as the unique function $v \in \mathcal{A}_n(\square_{3^m})$ that satisfies for every $w \in \mathcal{A}_n(\square_{3^m})$,*

$$\int_{\square_{3^m}} \nabla w \cdot a \nabla v = \int_{\square_{3^m}} -p \cdot a \nabla w + q \cdot \nabla w. \quad (4.31)$$

- *Quadratic response:* for every $w \in \mathcal{A}_n(\square_{3^m})$, $p, q \in \mathbb{R}^2$,

$$\begin{aligned} & \int_{\square_{3^m}} (\nabla w - \nabla v(\cdot, \square_{3^m}, p, q)) \cdot a(\nabla w - \nabla v(\cdot, \square_{3^m}, p, q)) \\ &= J_n(\square_{3^m}, p, q) - \int_{\square_{3^m}} -\frac{1}{2} \nabla w \cdot a \nabla w - p \cdot a \nabla w + q \cdot \nabla w. \end{aligned} \quad (4.32)$$

Proof. Step 1. Proof of the first variation and of the quadratic response. Let $p, q \in \mathbb{R}^2$ be fixed, we denote $v = v(\cdot, \square_{3^m}, p, q)$. Let $w \in \mathcal{A}_n(\square_{3^m})$ and $t \in \mathbb{R}$. We denote $v_t = v + tw$, and we have

$$\begin{aligned} J_n(\square_{3^m}, p, q) &\geq \int_{\square_{3^m}} -\frac{1}{2} \nabla v_t \cdot a \nabla v_t - p \cdot a \nabla v_t + q \cdot \nabla v_t \\ &= J_n(\square_{3^m}, p, q) + t \int_{\square_{3^m}} -\nabla w \cdot a \nabla v - p \cdot a \nabla w + q \cdot \nabla w \end{aligned} \quad (4.33)$$

$$+ t^2 \int_{\square_{3^m}} -\frac{1}{2} \nabla w \cdot a \nabla w \quad (4.34)$$

Sending t to 0 and applying the inequality to w and $-w$, we get

$$\int_{\square_{3^m}} -\nabla w \cdot a \nabla v - p \cdot a \nabla w + q \cdot \nabla w = 0.$$

We have proven (4.31). Moreover, applying the equality (4.34) with $t = -1$, we get (4.32). We see that the maximizer associated with J_n is unique and that any function v satisfying (4.31) will be the maximizer associated with J_n , which proves that $(p, q) \mapsto v(\cdot, \square_{3^m}, p, q)$ is linear.

Step 2. Let us show that J_n is quadratic, and prove its quadratic representation. By (4.27), we have

$$\begin{aligned} J_n(\square_{3^m}, p, q) &= \int_{\square_{3^m}} -\frac{1}{2} \nabla v(\cdot, \square_{3^m}, p, q) \cdot a \nabla v(\cdot, \square_{3^m}, p, q) - p \cdot a \nabla v(\cdot, \square_{3^m}, p, q) \\ &\quad + q \cdot \nabla v(\cdot, \square_{3^m}, p, q) \\ &= \frac{1}{2} \int_{\square_{3^m}} \nabla v(\cdot, \square_{3^m}, p, q) \cdot a \nabla v(\cdot, \square_{3^m}, p, q), \end{aligned}$$

applying (4.31) for $w = v$. Therefore, by linearity of $(p, q) \mapsto v(\cdot, \square_{3^m}, p, q)$, J_n is quadratic and definite positive, and in particular, $q \mapsto \mu_n^*(\square_{3^m}, q) = J_n(\square_{3^m}, 0, q)$ is quadratic and definite positive, as well. Let $\bar{a}_{n,*}^{3^m}$ the symmetric matrix such that for every $q \in \mathbb{R}^2$,

$$\mu_n^*(\square_{3^m}, q) = \frac{1}{2} q \cdot (\bar{a}_{n,*}^{3^m})^{-1} q.$$

then, (4.29) is immediate. We deduce

$$q \cdot (\bar{a}_{n,*}^{3^m})^{-1} q = \int_{\square_{3^m}} \nabla v(\cdot, \square_{3^m}, 0, q) \cdot a \nabla v(\cdot, \square_{3^m}, 0, q),$$

thus, by linearity of $(p, q) \mapsto v(\cdot, \square_{3^m}, p, q)$, we have, for every $p, q \in \mathbb{R}^2$

$$\begin{aligned} p \cdot (\bar{a}_{n,*}^{3^m})^{-1} q &= (p+q) \cdot (\bar{a}_{n,*}^{3^m})^{-1} (p+q) - p \cdot (\bar{a}_{n,*}^{3^m})^{-1} p - q \cdot (\bar{a}_{n,*}^{3^m})^{-1} q \\ &= \int_{\square_{3^m}} \nabla v(\cdot, \square_{3^m}, 0, p) \cdot a \nabla v(\cdot, \square_{3^m}, 0, q) \\ &= p \cdot \int_{\square_{3^m}} \nabla v(\cdot, \square_{3^m}, 0, q), \end{aligned}$$

by (4.31). Likewise,

$$\begin{aligned} q \cdot \bar{a}_n^{3^m} p &= \int_{\square_{3^m}} \nabla v(\cdot, \square_{3^m}, q, 0) \cdot a \nabla v(\cdot, \square_{3^m}, p, 0) \\ &= -q \cdot \int_{\square_{3^m}} a \nabla v(\cdot, \square_{3^m}, p, 0). \end{aligned}$$

Step 3. We prove the bounds (4.28). The upper bound for $\bar{a}_n^{3^m}$ has already been shown in the lemma 4.6. The inequality $\bar{a}_{n,*}^{3^m} \leq \bar{a}_n^{3^m}$ derives directly from (4.26) by taking $q = \bar{a}_{n,*}^{3^m} p$. For the last inequality, we write using Young's inequality,

$$\begin{aligned} q \cdot (\bar{a}_{n,*}^{3^m})^{-1} q &= \int_{\square_{3^m}} q \cdot a \nabla v(\cdot, \square_{3^m}, 0, q) \\ &\leq \frac{1}{2} \int_{\square_{3^m}} (q \cdot a^{-1} q + \nabla v(\cdot, \square_{3^m}, 0, q) \cdot a \nabla v(\cdot, \square_{3^m}, 0, q)) \\ &= \frac{1}{2} \int_{\square_{3^m}} q \cdot a^{-1} q + \frac{1}{2} q \cdot (\bar{a}_{n,*}^{3^m})^{-1} q, \end{aligned}$$

therefore

$$\begin{aligned} q \cdot (\bar{a}_{*,n}^{3^m})^{-1} q &\leq \int_{\square_{3^m}} q \cdot a^{-1} q \\ &\leq c_1 |q|^2. \end{aligned}$$

Step 4. We verify the subadditivity J_n . Testing $J_n(\cdot, z_i + \square_{3^l}, p, q)$ with $v = v(\cdot, \square_{3^m}, p, q)$ according to the expression (4.27), we get

$$\int_{z_i + \square_{3^l}} -\frac{1}{2} \nabla v \cdot a \nabla v - p \cdot a \nabla v + q \cdot \nabla v \leq J_n(z_i + \square_{3^l}, p, q),$$

summing, we get the expected inequality. □

Lemma 4.9. *for every $m \in \mathbb{N}$, we have*

$$\begin{aligned} \bar{a}_{*,n}^{3^m} &\leq \bar{a}^{3^m} \leq \bar{a}_n^{3^m} \\ \bar{a}_{*,n}^{3^m} &\leq \bar{a}_{n,*}^{3^m} \leq \bar{a}_n^{3^m}. \end{aligned}$$

Proof. These inequalities are deduced directly from the quadratic representation (4.29) and the inequalities (4.23),(4.25) et (4.28). □

Thus, μ_n and μ_n^* satisfy the same basic properties as μ_H and μ_H^* which are necessary for the proof of Theorem 4.3 (Theorem 2.4 in [8]) on the rate of convergence. With these properties, the proof of Proposition 4.4 is a straightforward adaptation of the proof of Theorem 4.3.

We may conjecture that increasing the degree of the polynomials used in the finite element space would converge to the expected value \bar{a} . However, the inequalities we obtain are not sufficient to show this fact.

3 Multigrid methods

Our problem, as it exhibits multiscale behavior, is a typical case for the use of multigrid methods. Let us introduce in the first subsection the basic principles and why it is a subject of interest. Then, in the second subsection, we give details on the algorithm itself. These two subsections are largely borrowed from [63], to which we refer for more details. Finally, we exhibit our application of Multi-Grid to our finite element method.

3.1 Basic principles

Multigrid methods in numerical analysis are algorithms for solving differential equations using a hierarchy of discretizations. The methods are based on the observation that many basic relaxation methods exhibit different rates of convergence for short- and long-wavelength components, suggesting these different scales be treated differently.

We use the grid function oriented notation. We consider a space of finite dimension Ω_h , and write

$$L_h u_h = f_h \text{ in } \Omega_h. \quad (4.35)$$

Here, u_h and f_h are grid functions on Ω_h and L_h is a linear operator

$$L_h : \mathcal{G}(\Omega_h) \rightarrow \mathcal{G}(\Omega_h),$$

where $\mathcal{G}(\Omega_h)$ denotes the linear space of grid functions on Ω_h . We assume that L_h^{-1} exists.

Here, we will illustrate the principles of Multigrid with our finite element Dirichlet corrector approximation of first order: find $v_1^\rho \in \mathcal{G}(\Omega_h)$, where $\Omega_h = \square_\rho$, and $\mathcal{G}(\Omega_h)$ is the linear space of piece-wise polynomials of first order on \square_ρ , such that, in the sense of $\mathcal{G}(\Omega_h)$,

$$\begin{cases} -\operatorname{div}(a(T_z \omega_0)(\nabla v_1^\rho + I)) = 0 \text{ in } \square_\rho, \\ v_1^\rho = 0 \text{ on } \partial \square_\rho. \end{cases}$$

In dimension one, the problem is the following. Let $\rho \in \mathbb{N}$ and find a function v_1^ρ , piece-wise affine on intervals $(-\frac{\rho}{2}, \frac{\rho}{2})$, with $p \in \mathbb{Z}$ and such that $v^\rho = 0$ on $\partial \square_\rho$, for every function φ piece-wise affine on intervals $(-\frac{\rho}{2}, \frac{\rho}{2})$ and such that $\varphi = 0$ on $\partial \square_\rho$,

$$\int_{\square_\rho} a(T_z \omega_0)(v_1^\rho)'(z)\varphi'(z) + a(T_z \omega_0)\varphi'(z)dz = 0. \quad (4.36)$$

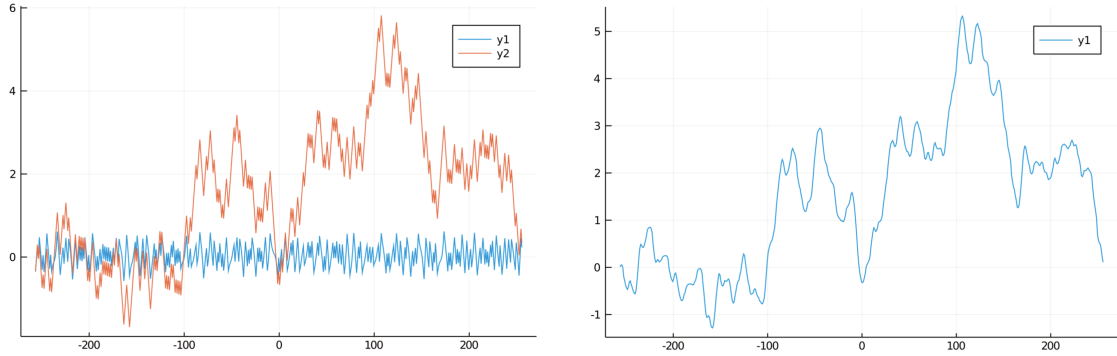


Figure 4.7: In the left figure, plot of the exact solution of the Dirichlet corrector in dimension one (blue) and the result of five iterations of the Gauss-Siedel algorithm (orange). In the right figure, plot of the difference.

In dimension two, the problem is the following. Let $\rho \in \mathbb{N}$, we are looking for the solution of the variational problem $v_1^\rho \in Q_{0,1}(\square_\rho, \mathbb{R}^2)$, for every $\varphi \in Q_{0,1}(\square_\rho, \mathbb{R}^2)$,

$$\int_{\square_\rho} a(T_z \omega_0) \nabla v_1^\rho(z) \cdot \nabla \varphi(z) + a(T_z \omega_0) \cdot \nabla \varphi(z) dz = 0. \quad (4.37)$$

Multigrid methods are based on two basic principles.

- Smoothing principle:
Many classical iterative methods of resolution of a linear system (Gauss-Seidel, gradient, etc.), if appropriately applied to discrete elliptic problems, have a strong smoothing effect on the error of any approximation. This is illustrated on figure 4.7. We resolve the one-dimensional problem (4.36), and show, on the left, the graphs of the exact solution in blue, and of the result of five iterations of the Gauss-Siedel algorithm. We see the low frequencies are not well approximated, but the high frequencies are. Indeed, showing the error on the right, we see that the error contains less high frequencies.
- Coarse grid principle:
A quantity that is smooth on a certain grid can, without any essential loss of information, also be approximated on a coarser grid, say a grid with double the mesh size. In other words: if the error of our approximation has become smooth after some iteration steps, we may approximate this error by a suitable procedure on a (much) coarser grid. Moreover, a coarse grid procedure is substantially less expensive (substantially fewer grid points) than a fine grid procedure.

A more extensive justification of these principles can be found in [63].

3.2 Algorithm

In this subsection, we explain the multi-grid algorithm. For a pedagogical purpose, we explain the case of a two-grid cycle, which has virtually no practical interest, but gives a

good understanding of the multigrid method. Then, we explain schematically the more general Multi-Grid cycle.

The multigrid method is a repetition of cycles. Thus, we will explain one cycle, which is repeated in an algorithm.

3.2.1 Two-grid cycle

The simplest multigrid algorithm is a two-grid algorithm. Let us explain the procedure in this simple case. We recall that the equation we wish to solve is the following

$$L_h u_h = f_h \text{ in } \Omega_h. \quad (4.38)$$

Let u_h^m be an approximation of the solution of (4.38) and let us describe a cycle of the two-grid algorithm.

One cycle of the algorithm consists in three steps. We denote one cycle as a function *TGCYC*:

$$u_h^{m+1} = \text{TGCYC}(u_h^m, L_h, f_h, \nu_1, \nu_2),$$

where $\nu_1, \nu_2 \in \mathbb{N}$ are two parameters of the algorithm.

1. Pre-smoothing $\bar{u}_h^m = \text{SMOOTH}^{\nu_1}(u_h^m, L_h, f_h)$: We compute ν_1 iterations of a so-called *smoothing procedure*, which is any iterative procedure aiming to solve (4.38). It has been proved that the Gauss-Siedel algorithm is the most effective in terms of smoothing.

Thus, we get a *defect* (or *residual*)

$$d_h^m = f_h - L_h \bar{u}_h^m.$$

Trivially, the *defect equation*

$$L_h v_h^m = d_h^m, \quad (4.39)$$

is equivalent to the original equation (4.38), as, by injectivity of L_h , $v_h^m = u_h - \bar{u}_h^m$.

2. Coarse grid correction:

The idea to approximately solve the defect equation is to use an appropriate approximation L_H of L_h on a coarser grid Ω_H , for instance the grid with mesh size $H = 2h$. This means that the defect equation (4.39) is replaced by

$$L_H \hat{v}_H^m = \bar{d}_H^m,$$

where $\bar{d}_H^m = I_h^H d_h^m$ is the restriction of d_h^m to the coarse grid. Here, we assume

$$L_H : \mathcal{G}(\Omega_H) \rightarrow \mathcal{G}(\Omega_H),$$

and L_H^{-1} to exist, as well as two linear transfer operators

$$I_h^H : \mathcal{G}(\Omega_h) \rightarrow \mathcal{G}(\Omega_H), \quad I_H^h : \mathcal{G}(\Omega_H) \rightarrow \mathcal{G}(\Omega_h),$$

to be given.

Then, we can interpolate the defect by

$$\widehat{v}_h^m = I_H^h \widehat{v}_H^m,$$

and compute the corrected approximation

$$\widetilde{u}_h^m = \bar{u}_h^m + \widehat{v}_h^m.$$

3. post-smoothing $u_h^{m+1} = \text{SMOOTH}^{\nu_2}(\widetilde{u}_h^m, L_h, f_h)$.

We may notice that in this cycle, the most costly operation is the resolution on the coarse grid (4.39). Thus, due to the still large complexity of the coarse grid problem, two-grid methods are of little practical significance. The idea of the multigrid is to reduce this cost, i.e. use several grids, until the coarsest grid is reduced to a few points and the exact resolution of the defect equation amounts to a negligible cost.

3.2.2 Multi-grid cycle

We give in this subsection a formal description of multigrid methods. We now use a sequence of coarser and coarser grids Ω_{h_k} , characterized by a sequence of mesh sizes h_k :

$$\Omega_{h_l}, \dots, \Omega_{h_0}.$$

The coarsest grid is characterized by the mesh size h_0 , whereas the index l corresponds to the finest grid. For each Ω_{h_k} , we assume the linear operators

$$\begin{aligned} L_k &: \mathcal{G}(\Omega_{h_k}) \rightarrow \mathcal{G}(\Omega_{h_k}), \\ I_k^{k-1} &: \mathcal{G}(\Omega_{h_k}) \rightarrow \mathcal{G}(\Omega_{h_{k-1}}), \\ I_{k-1}^k &: \mathcal{G}(\Omega_{h_{k-1}}) \rightarrow \mathcal{G}(\Omega_{h_k}), \end{aligned}$$

are given, where the L_k are discretizations of L on Ω_{h_k} , and where the original equation reads

$$L_l u_l = f_l \text{ in } \Omega_{h_l}. \quad (4.40)$$

Let u_k^m be an approximation of the solution of (4.40) on the grid Ω_{h_k} and let us describe a cycle of the multi-grid algorithm.

$$u_k^{m+1} = \text{MGCYC}(k, \gamma, u_k^m, L_k, f_k, \nu_1, \nu_2),$$

where $\nu_1, \nu_2, \gamma \in \mathbb{N}$ are parameters of the algorithm.

1. pre-smoothing $\bar{u}_k^m = \text{SMOOTH}^{\nu_1}(u_k^m, L_k, f_k)$,
2. coarse-grid correction
 - computation of the defect $d_k^m = f_k - L_k \bar{u}_k^m$,
 - restriction $\bar{d}_{k-1}^m = I_k^{k-1} d_k^m$,

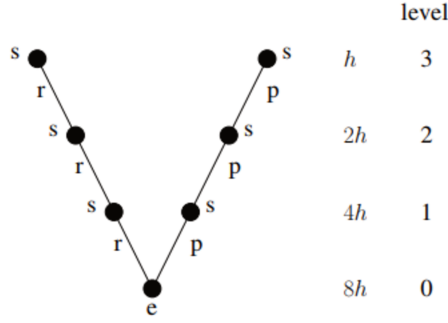


Figure 4.8: s – smoothing, r – restriction, p – prolongation, e – exact solution. Illustration from [63]

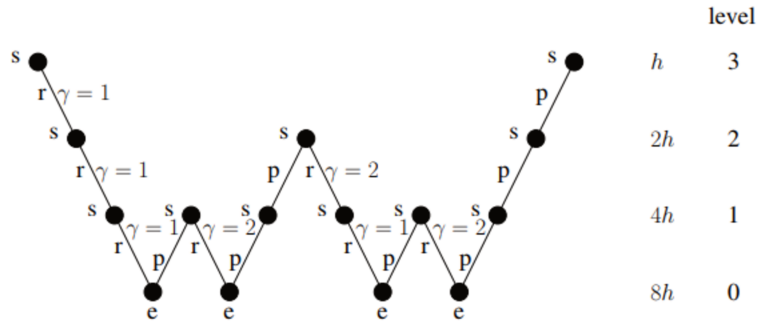


Figure 4.9: s – smoothing, r – restriction, p – prolongation, e – exact solution. Illustration from [63]

- resolution of $L_{k-1}\widehat{v}_{k-1}^m = \bar{d}_{k-1}^m$. We are presented with two cases:
 - if $k = 1$, then we compute the exact solution using a direct solver,
 - or else, we perform γ cycles on the Ω_{k-1} grid, using the zero grid function as a first approximation,

$$\widehat{v}_{k-1}^m = MGICYC^\gamma \left(k - 1, \gamma, 0, L_{k-1}, \bar{d}_{k-1}^m, \nu_1, \nu_2 \right).$$

- interpolation of the correction $\widehat{v}_k^m = I_{k-1}^k \widehat{v}_{k-1}^m$,
 - computation of the approximated correction $\widetilde{u}_k^m = \bar{u}_k^m + \widehat{v}_k^m$,
3. post-smoothing $u_k^{m+1} = SMOOTH^{\nu_2} (\widetilde{u}_k^m, L_k, f_k)$.

Different schemes of a multi-grid cycle are presented on figures 4.8 and 4.9. On figure 4.8 is the outline of a particular multi-grid cycle, called a *V-cycle*, such that $\gamma = 1$.

3.2.3 Convergence of the algorithm

We refer to [63] for a thorough analysis of the convergence of this analysis.

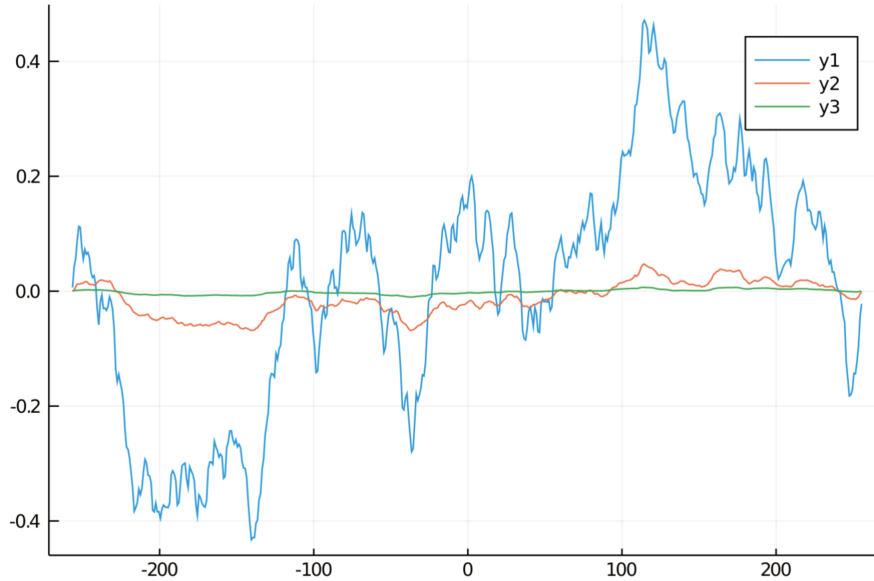


Figure 4.10: Visualizations of the error of the approximate solutions of the one-dimensional Dirichlet corrector for \bar{a} , w^ρ , for $\rho = 250$, obtained in 1 (blue), 2 (orange), 3 (green) iterations (cycles). $\nu_1 = 1$, $\nu_2 = 2$, $\gamma = 1$

One element to take away from this analysis is the following: though the convergence factors become better if the number of smoothing steps is increased, it is more efficient not to smooth the error too much but rather carry out a few more multigrid cycles. In practice, common choices are $\nu_1 + \nu_2 \leq 3$. In particular, $\nu_1 = 1$ and ν_2 is the most effective (see Table 3.3 in [63]).

Proposition 4.10 (Result 3.3.1 in [63]). *Let M_h be the iteration matrix of a Multi-Grid Cycle for which the finest mesh size is h , i.e. one iteration of a Multi-Grid cycle on u writes $M_h u$. We obtain the norm estimates*

$$\|M_h\|_S \leq \begin{cases} 0.17 & \text{for } \eta_1 = \eta_2 = 1, \\ 0.08 & \text{for } \eta_1 = 2, \eta_2 = 1, \end{cases}$$

where, for a matrix B ,

$$\|B\|_S = \sqrt{r(BB^*)}$$

and r is the spectral radius.

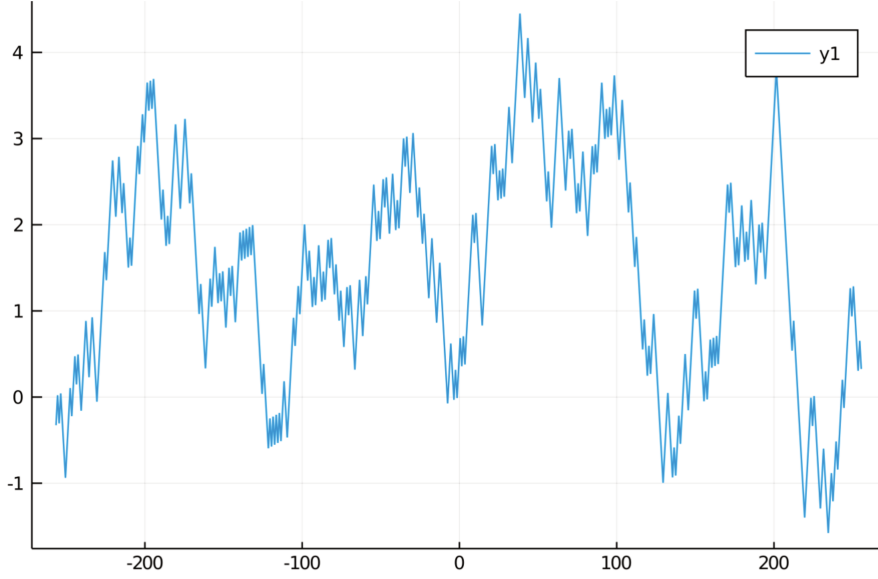


Figure 4.11: Plot of the exact Dirichlet approximation for \bar{a} , w^ρ , for $\rho = 250$.

3.3 Results

3.3.1 Results in the one-dimensional case

In Figure 4.10 is displayed the error in the Multi-grid approximation of the exact solution of the Dirichlet approximation, i.e. w^ρ such that

$$\begin{cases} - (a(T_x \omega)((w^\rho)' + 1))' = 0 \text{ on } \left(-\frac{1}{2}\rho, \frac{1}{2}\rho\right), \\ w^\rho \left(-\frac{1}{2}\rho\right) = w^\rho \left(\frac{1}{2}\rho\right) = 0, \end{cases}$$

with $\rho = 250$. By the results of the last section, we know that the associated quantity \bar{a}_1^ρ approximates well the expected value \bar{a} . To compare the smoothness of the error, in Figure 4.11 is displayed the exact solution.

In Figure 4.12 is displayed the error in the Multi-grid approximation of the exact solution of the Dirichlet approximation of the demagnetizing effective coefficient, i.e. ϕ^ρ such that

$$\begin{cases} - ((\phi^\rho)' + M_s(T_x \omega))' = 0 \text{ on } \left(-\frac{1}{2}\rho, \frac{1}{2}\rho\right), \\ \phi^\rho \left(-\frac{1}{2}\rho\right) = \phi^\rho \left(\frac{1}{2}\rho\right) = 0, \end{cases}$$

with $\rho = 250$. By the results of the last section, we know that the associated quantity \bar{M}_1^ρ approximates well the expected value \bar{M} . To compare the smoothness of the error, in Figure 4.13 is displayed the exact solution.

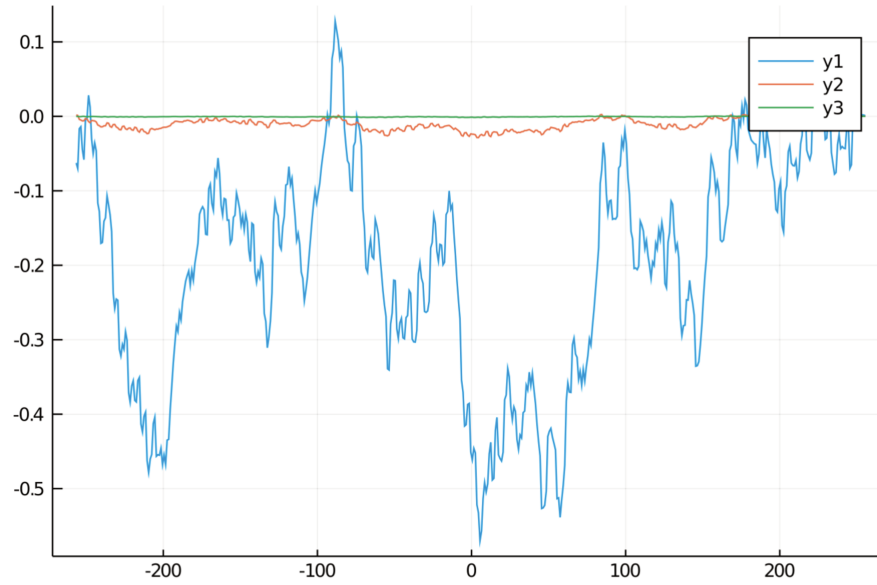


Figure 4.12: Visualizations of the error of the approximate solutions of the one-dimensional Dirichlet corrector for \overline{M} , ϕ^ρ , for $\rho = 250$, obtained in 1 (blue), 5 (orange), 20 (green) iterations (cycles) $\nu_1 = 1$, $\nu_2 = 2, \gamma = 2$

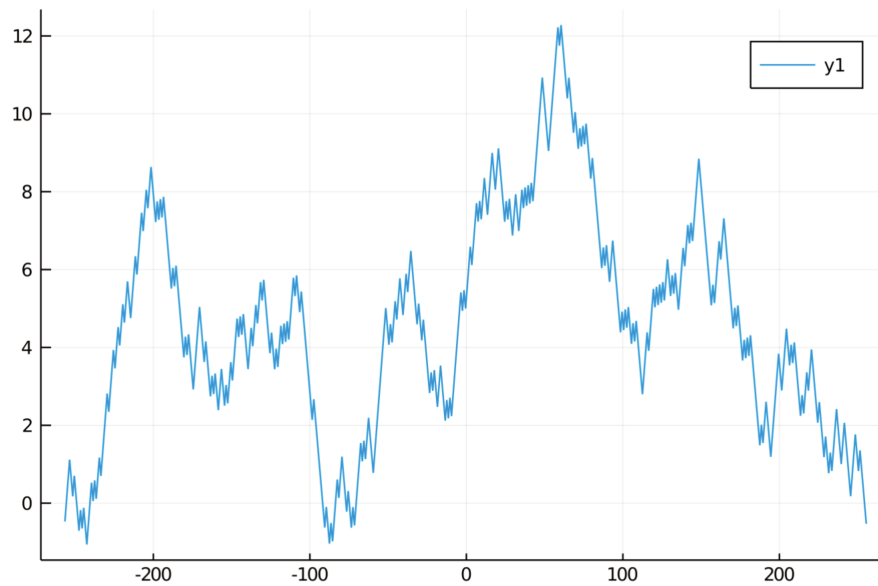


Figure 4.13: Plot of the exact Dirichlet approximation for \overline{M} , ϕ^ρ , for $\rho = 250$.

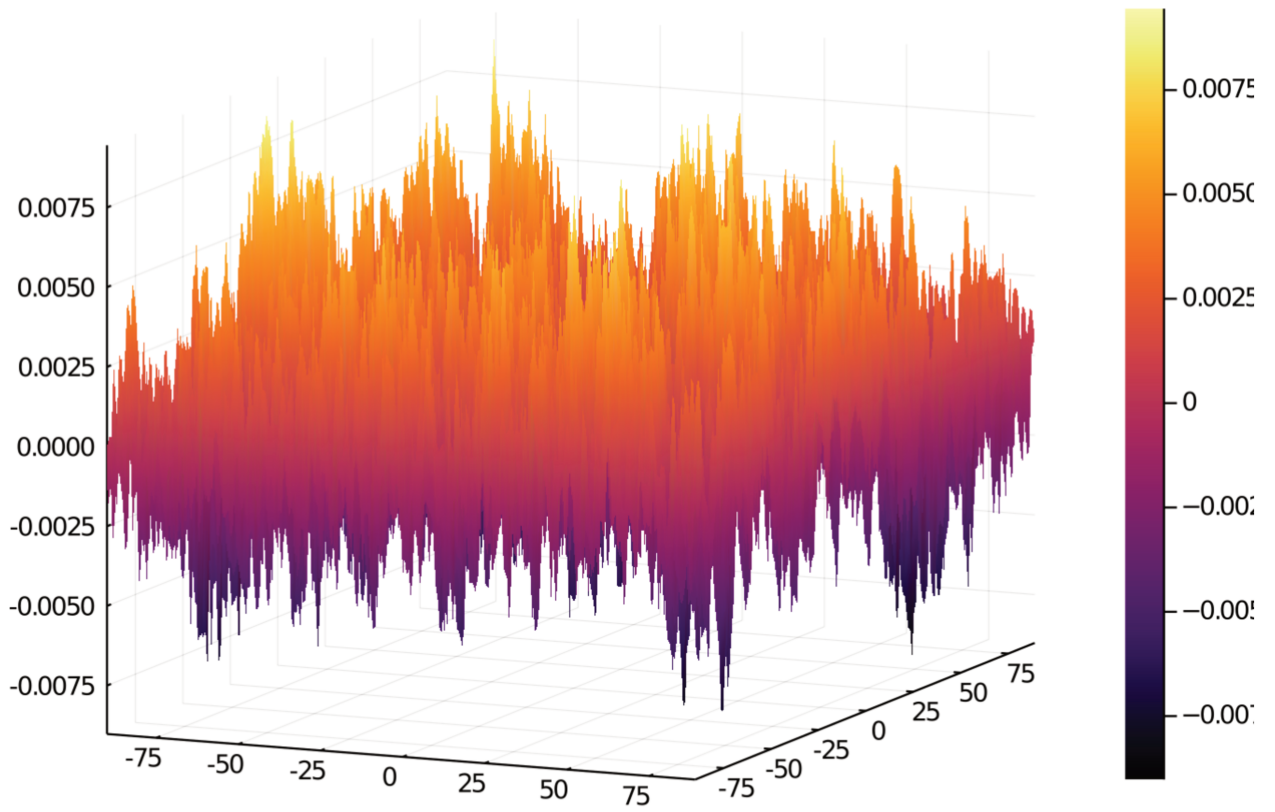


Figure 4.14: Exact solution of the Dirichlet finite element approximation of first order for \bar{a}, v_1^ρ for $\rho = 182$

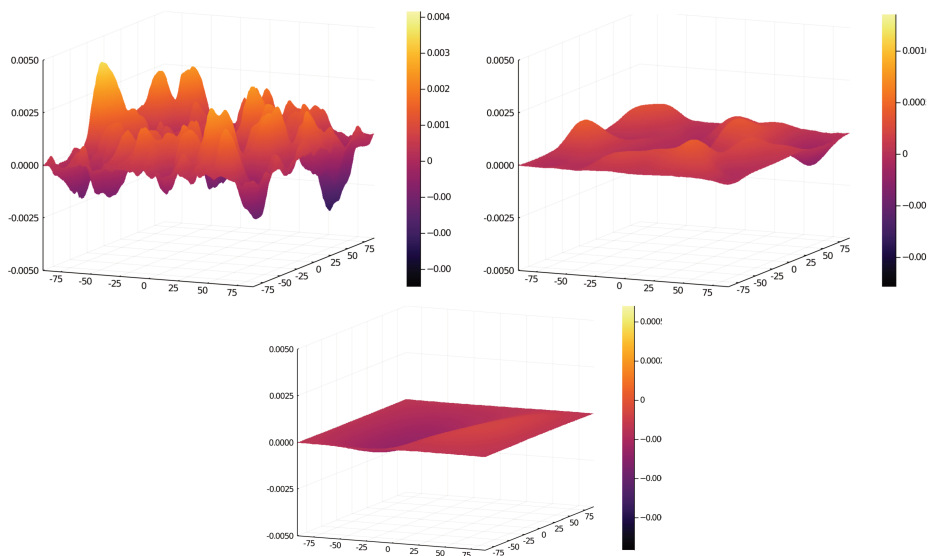


Figure 4.15: Visualizations of the error of the approximate solutions of the Dirichlet finite element approximation of first order for \bar{a}, v_1^ρ for $\rho = 250$ obtained in 1, 5, 20 iterations.

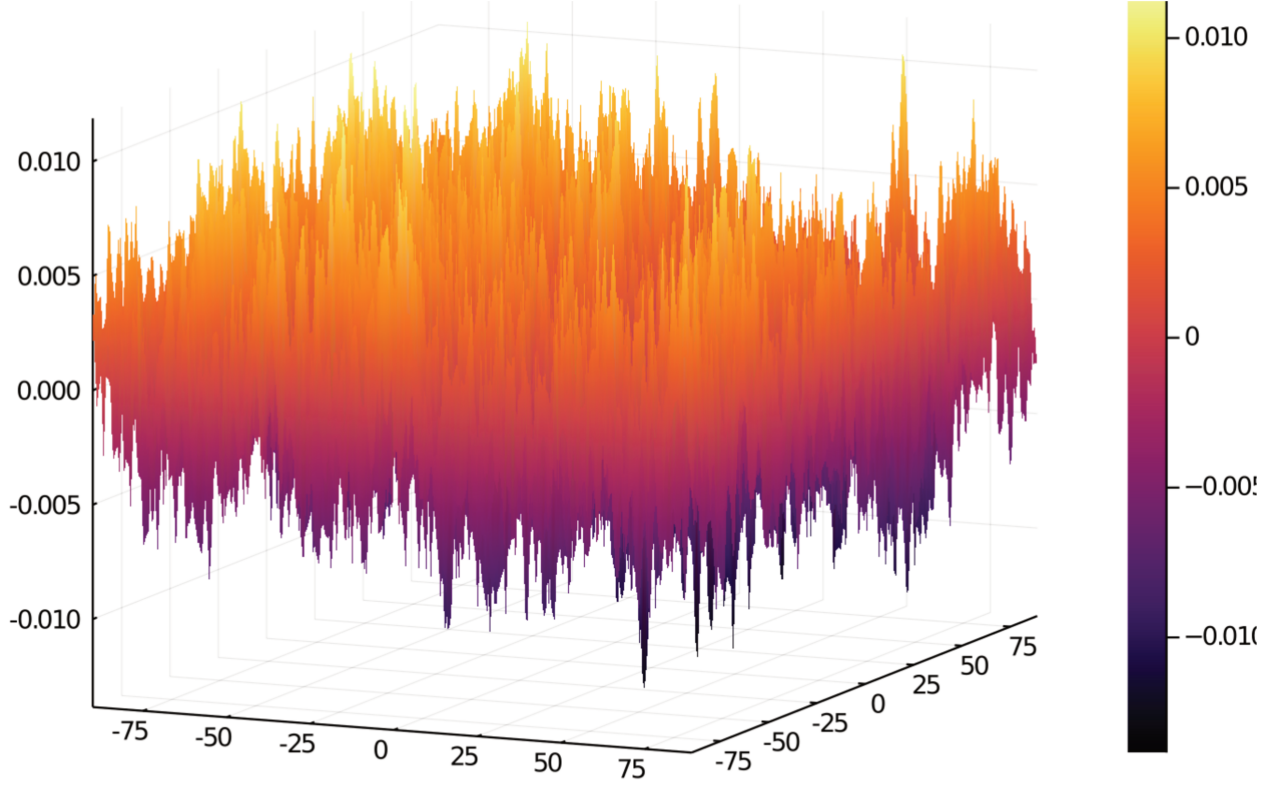


Figure 4.16: Exact solution of the Dirichlet finite element approximation of first order for \bar{M} , g_1^ρ for $\rho = 182$

3.3.2 Results in the two-dimensional case

In Figure 4.14 is displayed the exact solution of the Dirichlet approximation, i.e. $v_1^\rho \in Q_{0,1}(\square_\rho, \mathbb{R}^2)$, such that for every $\varphi \in Q_{0,1}(\square_\rho, \mathbb{R}^2)$,

$$\int_{\square_\rho} a(T_z \omega_0) \nabla v_1^\rho(z) \cdot \nabla \varphi(z) + a(T_z \omega_0) \cdot \nabla \varphi(z) dz = 0.$$

with $\square_\rho = (-91, 91)^2$. In Figure 4.15 are displayed the errors in the Multi-Grid approximations, for different counts of iterations.

In Figure 4.16 is displayed the exact solution of the Dirichlet approximation of the demagnetizing effective coefficient, i.e. $g_1^\rho \in Q_{0,1}(\square_\rho, \mathbb{R}^2)$ such that for every $\varphi \in Q_{0,1}(\square_\rho, \mathbb{R}^2)$,

$$\int_{\square_\rho} \nabla g_1^\rho(z) \cdot \nabla \varphi(z) + M_s(T_z \omega_0) \nabla \cdot \varphi(z) dz = 0,$$

with $\rho = 182$. In Figure 4.17 are displayed the errors in the Multi-Grid approximations, for different counts of iterations (cycles).

In this chapter, we have explored the numerical approximation of the homogenized coefficients which appear in the homogenized problem in micromagnetism. After introducing the quantities of interest and the difficulties implied by their computation in Section 1, we have proposed an approximation via finite element method and studied our

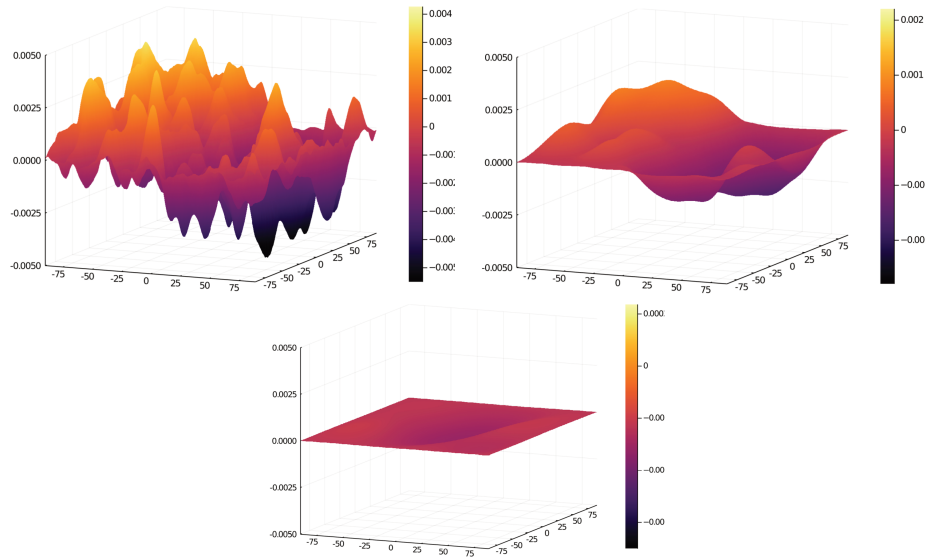


Figure 4.17: Visualizations of the error of the approximate solutions of the Dirichlet finite element approximation of first order for \overline{M} , g_1^ρ , for $\rho = 182$, obtained in 1, 5, 20 iterations.

results with this method in Section 2. Most of the results from this section are inspired by the work of Armstrong, Kuusi and Mourrat [8]. Finally, in Section 3, we have explored MultiGrid methods for this problem.

Conclusion

This thesis has consisted of four chapters, among which one introduction of our problem, and three largely independent chapters. Let us recall in this conclusion the work presented, and further perspectives.

The objective of the second chapter is to characterize the evolution of magnetization inside a composite ferromagnetic material, considering that the different compounds are randomly distributed at the microscopic scale within the macroscopic composite. In other words, we study the stochastic homogenization of the Landau-Lifschitz-Gilbert equation. Our main tool is a stochastic generalization of the two-scaled convergence method proposed in [65]. We extend the definition of this stochastic double-scale convergence in order to obtain a result on the double-scale convergence of the demagnetizing field. The final result of this chapter is a convergence theorem of the weak solutions of the Landau-Lifschitz-Gilbert equation to a Landau-Lifschitz-Gilbert equation whose coefficients are fully identified (Theorem 2.34). Two of them, namely the coefficient corresponding to the exchange term, and a coefficient that comes from the demagnetizing field, are obtained by solving a cell problem on the probability space, while the others consist of expectation calculations. It is remarkable that the exchange energy has the same homogenized coefficient as the one that appears in the homogenization of elliptic problems, and therefore does not take into account the nonlinear constraint which characterizes the Landau-Lifschitz-Gilbert equation.

In the third chapter, we study the behaviour of a spring magnet, by showing via homogenization techniques that the study boils down to a homogeneous magnet. Of interest is the process of destabilisation of a previously stable state, called *nucleation*. We identify the equation that characterizes nucleation, which consists of an eigenvalue problem. We study the particular case of domains where constant vectors are at equilibrium. These domains are ellipsoids and infinite cylinders. Our study of this eigenvalue problem is done in both cases using a similar decomposition method, exploiting stable spaces for the demagnetizing field. We were able to find behaviors known by physicists, namely the reversal in *unison* in the case of ellipsoids and *buckling* in the case of infinite cylinders for small particles, and the *curling* for larger particles, and to justify that these are the only occurring behaviors. It is notable that our study of infinite cylinders is not limited to cylinders with ellipsoidal cross-sections, although we can in the case of circular cross-sections give a complete theoretical study, but includes cylinders of any cross-section.

In the last chapter, we have explored the numerical approximation of the homogenized coefficients which appear in the homogenized problem in micromagnetism. The two

coefficients that we are trying to homogenize are calculated from two similar partial differential equations on an infinite space. We therefore use the same methodology in both cases. We propose a method in a two-dimensional setting, using finite elements on quadrangles, and compute it with finite elements of order 1 and 2. Both methods converge as the size of the truncated volume is increased towards a value close to the expected value, but not exactly equal. Largely using tools developed in [8], we justify theoretically this convergence. We see a clear improvement with the second degree, and therefore hint that the method should converge to the expected value as the degree is improved. Finally, as our problem exhibits multiple scales of behavior, we propose an algorithm using Multi-Grid cycles, and compute it numerically.

Numerical perspectives. Several numerical tracks remain to be explored in our problem. In chapter 4, we use a method based on finite elements on quadrangles. Firstly, the method must be further investigated, as increasing the degree of polynomials should improve the approximation results. A result on the convergence of the approximated quantity towards the expected value as the degree of polynomials is increased should be obtained. Moreover, we could adapt the method to the three-dimensional setting, which should be straightforward. Finally, as the method is based on finite elements on quadrangles, it is only to adapted to problems structured by a checkerboard. An alternative method must be developed for other less structured problems.

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Titre : Modélisation mathématique des spring magnets

Mots clés : micromagnétisme, homogénéisation stochastique, équation de Landau-Lifschitz

Résumé : Cette thèse est consacrée à l'étude de matériaux nanocomposites, qui sont l'approche la plus prometteuse et la plus active pour fabriquer les meilleurs aimants permanents actuellement. Ce type d'aimant est appelé spring magnet. Mathématiquement parlant, l'étude de ces matériaux est difficile car les modèles habituels sont non linéaires et la dépendance matérielle de les paramètres varient à une très petite échelle. Ainsi, résoudre directement les modèles magnétiques pour ces aimants est irréalisable, car les faibles dimensions des matériaux augmenteraient le nombre de cellules dans le maillage de manière prohibitive. Un moyen plus pratique est de dériver un modèle macroscopique en utilisant des techniques d'homogénéisation, ce qui désigne un ensemble de méthodes de moyennisation dans les équations aux dérivées partielles. En d'autres termes, l'homogénéisation cherche des paramètres effectifs (aussi appelés homogénéisés, ou macroscopiques) pour décrire des

milieux désordonnés ou très hétérogènes.

La thèse consiste en quatre chapitres, dont le premier introduit le cadre de notre étude, et les trois suivants sont en grande partie indépendants. Nous étudions dans le second chapitre l'homogénéisation stochastique de l'équation de Landau-Lifschitz-Gilbert, qui décrit l'évolution temporelle de l'aimantation dans un continuum ferromagnétique, afin d'obtenir un modèle homogène pour les spring magnets, à partir d'un modèle hétérogène. Une fois ce modèle homogène identifié, nous étudions dans le troisième chapitre le comportement d'un aimant permanent homogène, mais non uniforme, via la résolution d'un problème aux valeurs propres. Enfin, le quatrième chapitre s'intéresse au calcul numérique des coefficients homogénéisés, qui nécessite la résolution d'une équation aux dérivées partielles. Nous explorons dans ce chapitre une méthode utilisant des éléments finis, ainsi que des méthodes multi-grilles.

Title : Mathematical modelization of spring magnets

Keywords : micromagnetism, stochastic homogenization, Landau-Lifschitz equation

Abstract : This thesis is dedicated to the study of nanocomposite materials, which are the most promising and active approach to making the best permanent magnets today. This type of magnet is called a spring magnet. Mathematically speaking, the study of these materials is difficult, because the usual models are non-linear and the material dependence of the parameters vary on a very small scale. Thus, solving the magnetic models for these magnets directly is impractical, as the small dimensions of the materials would increase the number of cells in the mesh prohibitively. A more practical way is to derive a macroscopic model using homogenization techniques, which refers to a set of averaging methods in partial differential equations. In other words, homogenization searches for effective parameters (also called homogenized, or macroscopic) to describe disordered or highly heterogeneous media.

The thesis consists of four chapters, the first of which introduces the framework of our study, and the next three are largely independent. In the second chapter, we study the stochastic homogenization of the Landau-Lifschitz-Gilbert equation, which describes the temporal evolution of magnetization in a ferromagnetic continuum, in order to obtain a homogeneous model for spring magnets, from a heterogeneous model. Once this homogeneous model has been identified, we study in the third chapter the behaviour of a homogeneous, but not uniform permanent magnet, by solving an eigenvalue problem. Finally, the fourth chapter deals with the numerical calculation of homogenized coefficients, which requires the resolution of a partial differential equation. We explore in this chapter a method using finite elements, as well as multi-grid methods.