



Wave propagation and fractal boundary problems: mathematical analysis and applications

Anna Rozanova-Pierrat

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Wave propagation and fractal boundary problems:
mathematical analysis and applications

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- ARP-1** F. Magoulès, P.T.K. Ngyuen, P. Omnes, A. Rozanova-Pierrat, *Optimal absorbtion of acoustic waves by a boundary*. SIAM SICON to appear.
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- ARP-2** A. Dekkers, A. Rozanova-Pierrat, Models of nonlinear acoustics viewed as an approximation of the Navier-Stokes and Euler compressible isentropic systems, *to appear in Commun. Math. Sci.* (44 pages)
<https://hal.archives-ouvertes.fr/hal-01935515>
- ARP-3** A. Dekkers, V. Khodygo, A. Rozanova-Pierrat, Models of nonlinear acoustics viewed as approximations of the Kuznetsov equation. *DCDS-A*, Vol. 40, No. 7, 2020, (28 pages).
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- ARP-4** K. Arfi, A. Rozanova-Pierrat, Dirichlet-to-Neumann or Poincaré-Steklov operator on fractals described by d-sets. *Discrete & Continuous Dynamical Systems – S*, Vol. 12, No. 1, 2019, pp. 1–26
- ARP-5** A. Dekkers, A. Rozanova-Pierrat, Cauchy problem for the Kuznetsov equation. *Discrete & Continuous Dynamical Systems – A*, Vol. 39, No. 1, 2019, pp. 277–307
- ARP-6** C. Bardos, D. Grebenkov, A. Rozanova-Pierrat, Short-time heat diffusion in compact domains with discontinuous transmission boundary conditions. *Math. Mod. Meth. Appl. Sci.*, Vol. 26, No. 1, 2016, pp. 59–110
- ARPproc-7** A. Rozanova-Pierrat, Approximation of a compressible Navier-Stokes system by non-linear acoustical models, *Proceedings of the International Conference DAYS on DIFFRACTION*, 2015 May 25–29, 2015, St. Petersburg, Russia, pp. 270–276
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- ARP-11** A. Rozanova-Pierrat, On the Controllability for the Khokhlov-Zabolotskaya-Kuznetsov (KZK)-like Equation. *Applicable Analysis*, Vol. 89, No. 3, 2010, pp. 391–408
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- ARP-13** A. Rozanova-Pierrat, Qualitative Analysis of the Khokhlov-Zabolotskaya-Kuznetsov (KZK) Equation. *Math. Mod. Meth. Appl. Sci.*, Vol. 18, No. 5, 2008, pp. 781–812
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- ARP-17** A.V. Rozanova, Controllability in a Nonlinear Parabolic Problem with Integral Overdetermination. *Differential Equations*, Vol. 40, No. 6, 2004, pp. 853–872.

Book chapter

- BookChap** A. Rozanova-Pierrat, *Generalization of Rellich-Kondrachov theorem and trace compactness in the framework of d -set boundaries*. to appear in Volume “Fractals in engineering: Theoretical aspects and Numerical approximations”, ICIAM 2019 - SEMA SIMAI SPRINGER SERIES PUBLICATIONS. Preprint hal-02489325.

Preprints

- PrepShape2** F. Magoulès, P.T.K. Ngyuen, P. Omnes, A. Rozanova-Pierrat, *Optimal and efficient shapes in acoustic boundary absorption*. Preprint hal-02543993.
- PrepWestDir** A. Dekkers, A. Rozanova-Pierrat, *Dirichlet boundary valued problems for linear and nonlinear wave equations on arbitrary and fractal domains*. Preprint hal-02539158.
- PrepWestMixed** A. Dekkers, A. Rozanova-Pierrat, A. Teplyaev, *Mixed boundary valued problem for linear and nonlinear wave equations in domains with fractal boundaries*. Preprint hal-02514311. Submitted.
- NewPrep2** M. Hinz, F. Magoulès, A. Rozanova-Pierrat, M. Rynkovskaya, A. Teplyaev, *On the existence of optimal shapes in architecture*. Preprint hal-02956458. Submitted.

Remark

[ARP-2], [ARP-3], [ARP-5], [PrepWestDir], [PrepWestMixed] are publications and preprints of results obtained with my PhD student A. Dekkers:

A. Dekkers, “Analyse mathématique de l’équation de Kuznetsov : problème de Cauchy, questions d’approximations et problèmes aux bords fractals.”, Ph.D. thesis, 22 mars 2019. Mathématiques appliquées, CentraleSupélec, Université Paris Saclay. (PhD Adviser F. Abergel)

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Introduction générale

Il est bien connu qu'il n'est pas possible d'envisager la théorie des équations aux dérivées partielles sans l'analyse fonctionnelle et ses méthodes. En même temps, tout problème décrit par une équation aux dérivées partielles modélise plus ou moins étroitement un processus ou des phénomènes réels de la nature. Il est également bien connu que la même EDP peut être appliquée pour modéliser plusieurs problèmes de natures absolument différentes, car, par exemple, l'équation de la chaleur peut être utile pour modéliser la diffusion de la chaleur, pour modéliser des processus de mathématiques financières, pour l'évolution de la population de lapins et etc. Cependant, ces différentes applications utilisent les mêmes propriétés principales des solutions de cette équation, son irréversibilité en temps et la diffusion avec la décroissance exponentielle. Ainsi, il est important de présenter des propriétés mathématiques des solutions de différents modèles pour pouvoir les utiliser dans le cadre le plus adapté et aussi pouvoir choisir quand un modèle est meilleur que l'autre.

Cette philosophie de développer des résultats abstraits pour pouvoir comprendre les problèmes physiques concrets ou les préciser va à travers toutes mes recherches mathématiques et m'a permis de considérer, pour un regard rapide, des problèmes mathématiques très différents, mais tous sont fondamentalement liés entre eux. Donc, ma première tâche est d'expliquer leurs connexions.

Par exemple, pour la propagation des ultrasons, il existe différents modèles non linéaires comme les équations de Kuznetsov, Westervelt, Khokhlov-Zabolotskaya-Kuznetsov (KZK) et l'équation des ondes progressives non linéaires (NPE) (voir le chapitre 1). Mais dans le même temps pour les modèles de propagation des ondes non linéaires, il existe également des équations de Korteweg-de Vries (KdV), Kadomtsev-Petviashvili (KP), Benjamin-Ono et d'autres généralisations et d'autres équations d'ordre supérieur par rapport au temps. La principale propriété commune des équations KdV et KP est leur description des phénomènes de dispersion, mais, par exemple, l'équation de Kuznetsov, Westervelt et KZK sont des modèles dissipatifs pour un milieu visqueux. C'est en fait la principale raison pour laquelle les propriétés des solutions entre les modèles dispersifs et dissipatifs sont très différentes, ainsi que l'analyse mathématique permettant de montrer que les problèmes pour ces équations sont bien-posés. Il est intéressant de remarquer qu'en modifiant la signification physique des variables et des axes de l'équation KZK, il en résulte l'équation NPE qui est un modèle dispersif de propagation des ultrasons dans l'océan.

Par conséquent, je présente dans le chapitre 1 le contexte physique qui permet de systématiser la dérivation physique des modèles dissipatifs cités, et donc, de relier tous les modèles entre eux et surtout de considérer la question de l'estimation du temps pendant lequel deux solutions de modèles différents restent proches entre eux [ARP-2], [ARP-3], [ARP-12]. Nous obtenons ces résultats de stabilité une fois que nous connaissons l'existence et la

régularité des solutions des problèmes de Cauchy correspondants [ARP-5], [ARP-13], en prenant en compte dans le temps deux différences physiques principales: les cas de la propagation d'onde dans des milieux visqueux et non visqueux. L'absence ou la présence des termes de viscosité est connue pour être cruciale, à commencer par les résultats mathématiques sur les systèmes d'Euler et de Navier-Stokes.

Pour la validité de la dérivation des équations de Kuznetsov, Westervelt, KZK et NPE d'un système compressible de Navier-Stokes ou d'Euler, il est important de supposer l'absence de toute influence d'un bord sur le mouvement des ondes. Ainsi nous travaillons dans tout espace et considérons notamment les problèmes de Cauchy [ARP-2], [ARP-3], [ARP-5], [ARP-13], [ARP-14], [ARPproc-15].

Mais il y a une question sur la façon dont une onde interagit avec un bord ou une interface, surtout si elle est irrégulière ou fractale.

Pour considérer le cadre abstrait de l'analyse fonctionnelle permettant de considérer les équations différentielles partielles dans de tels domaines irréguliers, j'introduis dans le chapitre 2 un concept général des domaines Sobolev admissibles, contenant par exemple des domaines avec un bord donnée par un d -ensemble et (ϵ, ∞) -domaines [ARP-4], [BookChap], et je montre les résultats cruciaux de compacité pour l'opérateur de trace et pour les inclusions des espaces de Sobolev par une généralisation du théorème de Rellich-Kondrachov sur les domaines Sobolev admissibles. Une fois que je sais comment intégrer par parties et comment traiter les problèmes elliptiques sur les domaines Sobolev admissibles, je donne deux exemples d'application de la théorie abstraite développée:

1. la définition de l'opérateur de Dirichlet-à-Neumann sur un d -ensemble et la justification des articles numériques physiques de D. Grebenkov et ses co-auteurs en utilisant les propriétés de son spectre dans le cadre du transport du laplacien [ARP-4];
2. le caractère bien-posé (faiblement) d'un problème aux limites mixtes pour l'équation de Westervelt compte tenu des difficultés résultant de l'irrégularité de la frontière, sachant que dans de tels domaines il est impossible de s'assurer que la solution faible a la régularité de H^2 [PrepWestMixed], [PrepWestDir].

Dans le chapitre 3, je poursuis une question générale: comment l'irrégularité et la forme de la frontière modifient la propagation des ondes ou de la chaleur? En particulier, j'étudie l'influence sur la vitesse de propagation de la chaleur (dans le cadre de l'hypothèse de Gènes [ARP-6], [ARP-8]) ainsi que sur la dissipation des ondes (dans le cadre de l'optimisation de forme [ARP-1], [PrepShape2]). Je considère également la question d'approximation d'une onde ultrasonore décrite par l'équation de Westervelt dans un domaine avec un bord fractal (pour les conditions aux limites mixtes [PrepWestMixed]) ou même avec une frontière arbitraire (pour des conditions aux limites de Dirichlet homogènes [PrepWestDir]) par une autre onde ultrasonore dans un domaine avec un bord, disons, préfractal, qui approxime le bord fractal dans le cadre de la convergence de Mosco.

Dans chaque chapitre, je discute également des différentes manières possibles d'obtenir de nouveaux résultats, des améliorations possibles, des généralisations et des problèmes ouverts.

Je n'ai pas inclus dans ce manuscrit la description de la thématique des problèmes inverses [ARP-9], [ARP-10], [ARP-11], [ARP-16], [ARP-17], que je développe encore

en préparant plusieurs publications, et qui présente un autre exemple d'application de la théorie des EDPs.

La liste des publications est donnée au début du mémoire.

[ARP-2], [ARP-3], [ARP-5], [PrepWestDir], [PrepWestMixed] sont des publications de résultats obtenus avec mon doctorant A. Dekkers:

A. Dekkers, “Analyse mathématique de l'équation de Kuznetsov: problème de Cauchy, questions d'approximations et problèmes aux bords fractals. ”, Thèse de doctorat, 22 mars 2019. Mathématiques appliquées, CentraleSupélec, Université Paris Saclay. Directeur de thèse F. Abergel.

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General introduction

It is well-known that it is not possible to consider the theory of partial differential equations without the functional analysis and its methods. In the same time any problem described by a partial differential equation models more or less closely a real process or a phenomena of nature. It is also well known that the same PDE can be applied for modeling several problems of absolutely different natures, as for instance the heat equation can be useful for the heat diffusion, for the financial mathematics, for the evolution of the population of rabbits and ctr. However, these different applications use the same main properties of the solutions of this equation, its irrevertivity in time and the diffusion with the exponential decay. Thus, it is important to now the mathematical properties of solutions of different models to be able to use them in the most suitable framework and also to be able to chose when one model is better than other.

This philosophy to develop abstract results to be able to understand the concrete physical problems or to precise them goes through all my mathematical research and allowed me to consider, for a rapid look, very different mathematical problems, but all of them are fundamentally related together. So, my first task is to explain their connections.

For instance for the propagation of the ultrasounds there are different non-linear models as the Kuznetsov, Westervelt, Khokhlov-Zabolotskaya-Kuznetsov (KZK) equations and the Nonlinear Progressive wave Equation (NPE) (see Chapter 1). But in the same time for nonlinear wave propagation models there are also Korteweg-de Vries (KdV), Kadomtsev-Petviashvili (KP), Benjamin-Ono equations and other generalizations and other equations of higher order in time. The main common property of KdV and KP equations that they describe the dispersive phenomena, but, for instance, the Kuznetsov, the Westervelt and the KZK equation are dissipative models for a viscous medium. It is actually the main reason why the properties of solutions between dispersive and dissipative models are very different also as the mathematical analysis allowing to show their well-posedness. It is interesting to notice that by changing the physical meaning of variables and axis of the KZK equation it results in the NPE equation which is a dispersive model of ultrasound propagation in the ocean.

Therefore, I present in Chapter 1 the physical context which allows to systematize the physical derivation of the cited dissipative models and thus, relate all models together and especially to consider the question how long two solutions of different models stay closed to each other [ARP-2], [ARP-3], [ARP-12]. We obtain these stability results once we know the existence and the regularity of considered solutions of corresponding Cauchy problems [ARP-5], [ARP-13], taking all time into account two main physical differences: the viscous and inviscid media of the wave propagation. The absence or the presence of the viscosity terms are known to be crucial, starting by the mathematical results on the Euler

and Navier-Stokes systems.

For the validity of the derivation of the Kuznetsov, Westervelt, KZK and NPE equations from a compressible Navier-Stokes or Euler systems it is important to suppose the absence of any boundary influence on the wave motion. Thus we are working in all space and especially consider the Cauchy problems [ARP-2], [ARP-3], [ARP-5], [ARP-13], [ARP-14], [ARPproc-15].

But there is a question about how a wave interact with a boundary, especially if it is irregular or fractal.

To consider the abstract framework of the functional analysis allowing to consider the partial differential equations in such irregular domains, I introduce in Chapter 2 a general concept of Sobolev admissible domains, containing for instance domains with a d -set boundary and (ϵ, ∞) -domains [ARP-4], [BookChap], and show the crucial compactness results for the trace operator and for Sobolev embedding by a generalization of the Rellich-Kondrachov theorem on the introduced Sobolev admissible domains. Once I know how to integrate by parts and how to treat elliptic problems on the Sobolev admissible domains, I give two examples of the application of the developed abstract theory:

1. the definition of the Dirichlet-to-Neumann operator on a d -set and the justification of physical numerical articles of D. Grebenkov and his co-authors using the properties of its spectrum in the framework of the Laplacian transport [ARP-4];
2. the proof of the weak well-posedness for a mixed boundary valued problem for the Westervelt equation taking into account the difficulties coming from the irregularity of the boundary, as soon as in such domains it is impossible to ensure that the weak solution has the regularity of H^2 [PrepWestMixed], [PrepWestDir].

In Chapter 3 I follow a general question: how the irregularity and the shape of the boundary change the wave or heat propagation? In particular, I study the influence on the speed of the heat propagation (in the framework of de Gennes' hypothesis [ARP-6], [ARP-8]) and also on the wave dissipation (in the framework of the shape optimization [ARP-1], [PrepShape2]). I also consider the approximation question of an ultrasound wave describing by the Westervelt equation in a domain with a fractal (for mixed boundary conditions [PrepWestMixed]) or even arbitrary boundary (for homogeneous Dirichlet boundary conditions [PrepWestDir]) by an other ultrasound wave in, let us say, a prefractal domain approximating the fractal one in the framework of the Mosco convergence.

In each Chapter I also discuss the different possible ways for further results, possible improvements, generalizations and open problems.

I did not included in this thesis the description of the thematic of the inverse problems [ARP-9], [ARP-10], [ARP-11], [ARP-16], [ARP-17], which I still develop preparing several publications in the future, and which presents an other applicative example of the theory of PDEs.

Part I

Models of nonlinear acoustics

Chapter 1

Relations between different models of nonlinear acoustics

Introduction

My interest in the models of nonlinear acoustics comes from the beginning of my Ph.D. with Claude Bardos, whom the first day of my Ph.D. has taken me to a physical conference on ultrasound waves organized by one of the famous specialists of waves Mathias Fink. In this conference, we discovered the equation named the KZK (Khokhlov-Zabolotsaya-Kuznetsov) equation, well-known for physicists but unknown in mathematics. My other participation in physical conferences in nonlinear acoustics allowed me to discover that physicists use a lot of different, very complicated nonlinear models, almost all unknown (or few known) in the mathematical world. Hence I have also realized the difference between the mathematical and the physical studies of a model. As said me Zabolotskaya at the Congress of Acoustics 2008, by Khokhlov, his Ph.D. adviser, a physicist needs to find an exact solution to its new model be able to validate and publish it. However, it is not a priority from the mathematical point of view.

Hence, I am interested in how the models, as the KZK equation, the NPE, the Kuznetsov, and the Westervelt equations, are related to each other, how they can be obtained, and if there is a possibility to know in some kind the accuracy of the approximation fulfilled by the solutions of these models. Sure, the mentioned equations take part of “the simplest models” (see, for instance, for higher-order models as the nonlinear Jordan-Moore-Gibson-Thompson (JMGT) equation [103, 110, 108], containing the Kuznetsov equation as a particular or a limit case, or other variants [78]). Hence, my results on the chosen “simplest models” [ARP-2], [ARP-3], [ARP-12], using [ARP-5], [ARP-13], [ARP-14], [ARPproc-15], can be viewed as the first step to the understanding the relations between existing models of nonlinear acoustics, in complement to [112] and also to dispersive models of KP-type [121].

We give this introductive description of this chapter in French before proceeding to its content.

Introduction en français

Mon intérêt pour les modèles d'acoustique non linéaire vient du début de ma thèse que j'ai effectué avec Claude Bardos, qui le premier jour de ma thèse m'a emmené à une conférence physique sur les ondes ultrasonores organisée par l'un des célèbres spécialistes des ondes Mathias Fink. Dans cette conférence, nous avons appris l'existence d'une équation nommée l'équation de KZK (Khokhlov-Zabolotskaya-Kuznetsov), bien connue des physiciens mais inconnue en mathématiques. Mes autres participations à des conférences physiques en acoustique non linéaire m'ont permis de découvrir que les physiciens utilisent beaucoup de modèles non linéaires très compliqués presque tous inconnus (ou peu connus) dans le monde mathématique. Par conséquent, j'ai également réalisé la différence entre les études mathématiques et physiques d'un modèle. Comme me l'a dit Zabolotskaya au Congrès d'Acoustique 2008, d'après son directeur de thèse Khokhlov, un physicien doit trouver une solution exacte de son nouveau modèle pour pouvoir le valider et le publier. Mais ce n'est pas une priorité du point de vue mathématique.

Par conséquent, je me suis intéressée à comprendre comment les modèles comme l'équation de KZK, le NPE, les équations de Kuznetsov et de Westervelt sont liées les uns aux autres, comment ils peuvent être obtenus et s'il est possible de connaître en quelque sorte la précision de l'approximation remplies par les solutions de ces modèles. Bien sûr, les équations mentionnées font partie des “ modèles les plus simples ” (voir par exemple pour les modèles d'ordre supérieur comme l'équation non linéaire de Jordan-Moore-Gibson-Thompson (JMGT) [103, 110, 108], contenant l'équation de Kuznetsov comme cas particulier ou limite, ou d'autres variantes [78]). Mes résultats sur ces “ modèles les plus simples ” [ARP-2], [ARP-3], [ARP-12], en utilisant [ARP-5], [ARP-12], [ARP-14], [ARPproc-15], peuvent être considérés comme la première étape vers la compréhension des relations entre les modèles existants d'acoustique non linéaire, en complément de [112] et aussi aux modèles dispersives de type KP [121].

1.1 Introduction and derivation of the models

There is a renewed interest in the study of nonlinear wave propagation, in particular because of recent applications to ultrasound imaging (*e.g.* HIFU) or technical and medical applications such as lithotripsy or thermotherapy. Such new techniques rely heavily on the ability to model accurately the nonlinear propagation of a finite-amplitude sound pulse in thermo-viscous elastic media.

One of the most general model to describe an acoustic wave propagation in an homogeneous thermo-elastic medium is the compressible Navier-Stokes system in \mathbb{R}^n

$$\partial_t \rho + \operatorname{div}(\rho \mathbf{v}) = 0, \quad (1.1)$$

$$\rho[\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla) \mathbf{v}] = -\nabla p + \eta \Delta \mathbf{v} + \left(\zeta + \frac{\eta}{3} \right) \nabla \cdot \operatorname{div}(\mathbf{v}), \quad (1.2)$$

$$\begin{aligned} \rho T[\partial_t S + (\mathbf{v} \cdot \nabla) S] &= \kappa \Delta T + \zeta (\operatorname{div} \mathbf{v})^2 \\ &+ \frac{\eta}{2} \left(\partial_{x_k} v_i + \partial_{x_i} v_k - \frac{2}{3} \delta_{ik} \partial_{x_i} v_i \right)^2, \end{aligned} \quad (1.3)$$

where the pressure p is given by the state law $p = p(\rho, S)$. The density ρ , the velocity \mathbf{v} , the temperature T and the entropy S are unknown functions in system (1.1)–(1.3). The

coefficients ζ , κ and η are constant viscosity coefficients. For the acoustical framework the wave motion is supposed to be potential and the viscosity coefficients are supposed to be small in terms of a dimensionless small parameter $\epsilon > 0$, which also characterizes the size of the perturbations near the constant state $(\rho_0, 0, S_0, T_0)$. Here the velocity \mathbf{v}_0 is taken equal to 0 just using a Galilean transformation.

Actually, ϵ is the Mach number, which is supposed to be small [21] ($\epsilon = 10^{-5}$ for the propagation in water with an initial power of the order of 0.3 W/cm^2):

$$\frac{\rho - \rho_0}{\rho_0} \sim \frac{T - T_0}{T_0} \sim \frac{|\mathbf{v}|}{c} \sim \epsilon,$$

where $c = \sqrt{p'(\rho_0)}$ is the speed of sound in the unperturbed media. In addition,

$$S(x, t) = S_0 + \epsilon^2 \tilde{S}(x, t),$$

where the perturbation of the entropy is of order $O(\epsilon^2)$, since it is the smallest size on ϵ of right hand terms in Eq (1.3), due to the smallness of the viscosities. Hence as in [ARPproc-7], [ARP-2], system (1.1)–(1.3) becomes an isentropic Navier-Stokes system

$$\partial_t \rho_\epsilon + \text{div}(\rho_\epsilon \mathbf{v}_\epsilon) = 0, \quad (1.4)$$

$$\rho_\epsilon [\partial_t \mathbf{v}_\epsilon + (\mathbf{v}_\epsilon \cdot \nabla) \mathbf{v}_\epsilon] = -\nabla p(\rho_\epsilon) + \epsilon \nu \Delta \mathbf{v}_\epsilon, \quad (1.5)$$

with the approximate state equation $p(\rho, S) = p(\rho_\epsilon) + O(\epsilon^3)$:

$$p(\rho_\epsilon) = p_0 + c^2(\rho_\epsilon - \rho_0) + \frac{(\gamma - 1)c^2}{2\rho_0}(\rho_\epsilon - \rho_0)^2, \quad (1.6)$$

where $\gamma = C_p/C_V$ denotes the ratio of the heat capacities at constant pressure and at constant volume respectively and with a small enough and positive viscosity coefficient:

$$\epsilon \nu = \beta + \kappa \left(\frac{1}{C_V} - \frac{1}{C_p} \right).$$

If we go on physical assumptions of the wave motion [21, 82, 116, 173] for the perturbations of the density or of the velocity or of the pressure, the isentropic system (1.4)–(1.5) gives

1. the Westervelt equation for the potential of the velocity, derived initially by Westervelt [173] and later by other authors [1, 163]:

$$\partial_t^2 \Pi - c^2 \Delta \Pi = \epsilon \partial_t \left(\frac{\nu}{\rho_0} \Delta \Pi + \frac{\gamma + 1}{2c^2} (\partial_t \Pi)^2 \right) \quad (1.7)$$

with the same constants introduced for the Navier-Stokes system.

2. the Kuznetsov equation also for the potential of the velocity, firstly introduced by Kuznetsov [116] for the velocity potential, see also Refs. [82, 104, 112, 123] for other different methods of its derivation:

$$\partial_t^2 u - c^2 \Delta u = \epsilon \partial_t \left((\nabla u)^2 + \frac{\gamma - 1}{2c^2} (\partial_t u)^2 + \frac{\nu}{\rho_0} \Delta u \right). \quad (1.8)$$

3. the Khokhlov-Zabolotskaya-Kuznetsov (KZK) [21], [**ARProc-7**] for the density:

$$c\partial_{\tau z}^2 I - \frac{(\gamma+1)}{4\rho_0}\partial_\tau^2 I^2 - \frac{\nu}{2c^2\rho_0}\partial_\tau^3 I - \frac{c^2}{2}\Delta_y I = 0. \quad (1.9)$$

4. the Nonlinear Progressive wave Equation (NPE) derived in Ref. [131] also for the density:

$$\partial_{\tau z}^2 \xi + \frac{(\gamma+1)c}{4\rho_0}\partial_z^2[(\xi)^2] - \frac{\nu}{2\rho_0}\partial_z^3 \xi + \frac{c}{2}\Delta_y \xi = 0. \quad (1.10)$$

All these models were derived from a compressible nonlinear isentropic Navier-Stokes (for viscous media) and Euler (for the inviscid case) systems up to some small negligible terms. But all cited physical derivations of these models don't allow to say that their solutions approximate the solution of the Navier-Stokes or Euler system. The first work explaining it for the KZK equation is [**ARP-12**].

We consider system (1.4)–(1.6) as the exact model. The state law (1.6) is a Taylor expansion of the pressure up to the terms of the third order on ϵ . Therefore an approximation of system (1.4)–(1.6) for \mathbf{v}_ϵ and ρ_ϵ up to terms $O(\epsilon^3)$ would be optimal. We systematize in [**ARP-2**] the derivation of all these models using the ideas of [**ARP-12**], consisting to use correctors in the Hilbert type expansions of corresponding physical *ansatzs*.

More precisely, we show that all these models are approximations of the isentropic Navier-Stokes or Euler system up to third order terms of a small dimensionless parameter $\epsilon > 0$ measuring the size of the perturbations of the pressure, the density and the velocity to compare to their constant state $(p_0, \rho_0, 0)$ (see Fig. 1.1). In addition we show that

- the Kuznetsov equation can be obtained up to $O(\epsilon^3)$ terms from the Navier-Stokes/Euler systems just by small perturbations of the medium;
- the Westervelt equation is a nonlinear approximation of the Kuznetsov equation up to $O(\epsilon^2)$ -terms;
- the KZK and NPE equations can be obtained up to $O(\epsilon^3)$ terms by two steps from the Navier-Stokes/Euler systems:

1. by small perturbations firstly obtain the Kuznetsov equation

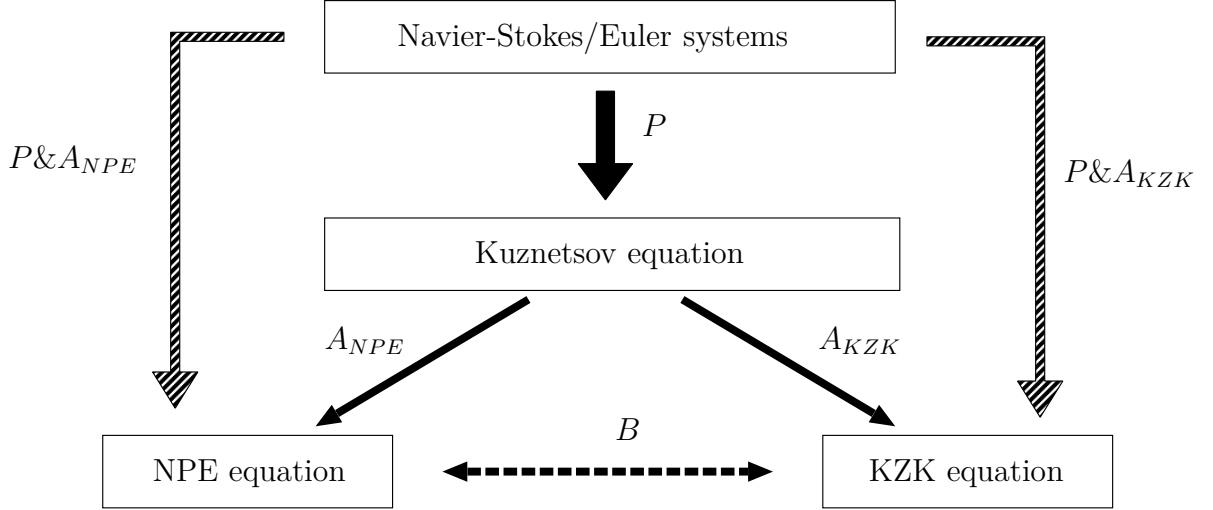
2. by performing a paraxial change of variables for the Kuznetsov equation,

and alternatively, by performing the small perturbations with a paraxial change of variables at the same time for Navier-Stokes/Euler systems.

The last point allows considering the approximation between the solutions of the KZK/NPE equations and the solutions of the Kuznetsov equation separately of the approximation between these models and the Navier-Stokes/Euler systems.

It is easy to understand working, for instance, only on the linear part of the KZK equation (1.45) (see Subsection 1.1.3 for a complete derivation). Let $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$. In the aim to describe the propagation of ultrasound beams, it is assumed that the variation of beam's propagation in the direction

$$x' = (x_2, x_3, \dots, x_n)$$



P : small perturbations (1.19)–(1.20)

A_{KZK} : KZK-paraxial approximation (Fig. 1.2)

B : bijection (1.56)

A_{NPE} : NPE-paraxial approximations (Fig. 1.3)

Figure 1.1 – Schema of derivation of the models of the nonlinear acoustics. All models, the Kuznetsov, the KZK and the NPE equations are approximations up to terms of the order of ϵ^3 of the isentropic Navier-Stokes or Euler system.

perpendicular to the x_1 -axis is much larger than its variation along the x_1 -axis, *i.e.* we suppose that the beam has the form $U(t - x_1/c, \epsilon x_1, \sqrt{\epsilon} x')$. The first argument $t - x_1/c$ describes the wave propagation in time along the x_1 -axis with the sound speed c , two last arguments ϵx_1 and $\sqrt{\epsilon} x'$ describe respectively the speed of the deformation of the wave along the x_1 -axis and along the x' -axis. We remark that $\epsilon \ll 1$ and consequently, $\epsilon \ll \sqrt{\epsilon}$.

For instance for the linear wave equation in \mathbb{R}^n ($n > 1$):

$$\frac{1}{c^2} \partial_t^2 u - \Delta u = 0, \quad (1.11)$$

the following *ansatz*

$$u_\epsilon = U\left(t - \frac{x_1}{c}, \epsilon x_1, \sqrt{\epsilon} x'\right) \quad (1.12)$$

containing a “profile” $U(\tau, z, y)$ (with small ϵ) leads to the formula:

$$\partial_{\tau,z}^2 U - \frac{c}{2} \Delta_y U = O(\epsilon), \quad (1.13)$$

or for functions $U(\tau, z, y) = A(z, y)e^{i\omega\tau}$, to the equation

$$i\omega \partial_z A - \frac{c}{2} \Delta_y A = O(\epsilon). \quad (1.14)$$

We observe that with $\epsilon = 0$ Eqs. (1.13) and (1.14) are two variants of the classical paraxial approximation and that Eq. (1.13) contains the linear non-diffusive terms of the KZK equation which usually has the following form for some positive constants β and γ :

$$\partial_{\tau,z}^2 U - \frac{1}{2} \partial_\tau^2 U^2 - \beta \partial_\tau^3 U - \gamma \Delta_y U = 0.$$

Conversely, the isentropic evolution of the thermo-elastic non-viscous media is given by the following Euler system:

$$\partial_t \rho + \operatorname{div}(\rho v) = 0, \quad \rho(\partial_t v + v \cdot \nabla v) = -\nabla p(\rho). \quad (1.15)$$

Any constant state (ρ_0, v_0) is a stationary solution of system (1.15). Linearization near this state introduces the variables

$$\rho = \rho_0 + \epsilon \tilde{\rho}, \quad v = v_0 + \epsilon \tilde{v}$$

and for $v_0 = 0$ the acoustic system:

$$\partial_t \tilde{\rho} + \rho_0 \nabla \cdot \tilde{v} = 0, \quad \rho_0 \partial_t \tilde{v} + p'(\rho_0) \nabla \tilde{\rho} = 0. \quad (1.16)$$

System (1.16) is equivalent to the wave equation:

$$\frac{1}{c^2} \partial_t^2 \tilde{\rho} - \Delta \tilde{\rho} = 0, \quad \partial_t \tilde{v} = -\frac{p'(\rho_0)}{\rho_0} \nabla \tilde{\rho}, \quad (1.17)$$

where $c = \sqrt{p'(\rho_0)}$ is the speed of sound in the unperturbed media.

We observe that Eq. (1.13), which is the linearized and inviscid part of the KZK equation, can be obtained in two steps. First, we consider small perturbations around a constant state of the isentropic Euler system, which are solutions to the acoustic equation and then we consider a paraxial approximation of such solutions.

The derivation of the complete KZK equation follows almost the same line. It considers the viscosity and the size of the nonlinear terms and is given in Subsection 1.1.3.

1.1.1 Kuznetsov equation

The Kuznetsov equation models a propagation of nonlinear acoustic waves in thermo-viscous elastic media and describes the evolution of the velocity potential. Initially the Kuznetsov equation was derived by Kuznetsov [116] from the isentropic Navier-Stokes system (1.4)–(1.6) for the small velocity potential $\mathbf{v}_\varepsilon(\mathbf{x}, t) = -\nabla \tilde{u}(\mathbf{x}, t)$, $\mathbf{x} \in \mathbb{R}^n$, $t \in \mathbb{R}^+$:

$$\partial_t^2 \tilde{u} - c^2 \Delta \tilde{u} = \partial_t \left((\nabla \tilde{u})^2 + \frac{\gamma - 1}{2c^2} (\partial_t \tilde{u})^2 + \frac{\varepsilon \nu}{\rho_0} \Delta \tilde{u} \right). \quad (1.18)$$

The derivation was latter discussed by a lot of authors [82, 104, 123].

Unlike in these physical derivations we introduce a Hilbert expansion type construction with a corrector $\varepsilon^2 \rho_2(\mathbf{x}, t)$ for the density perturbation, considering the following *ansatz*

$$\rho_\varepsilon(\mathbf{x}, t) = \rho_0 + \varepsilon \rho_1(\mathbf{x}, t) + \varepsilon^2 \rho_2(\mathbf{x}, t), \quad (1.19)$$

$$\mathbf{v}_\varepsilon(\mathbf{x}, t) = -\varepsilon \nabla u(\mathbf{x}, t). \quad (1.20)$$

The use of the second order corrector in (1.19) allows to ensure the approximation of (1.5) up to terms of order ε^3 and to open the question about the approximation between the exact solution of the isentropic Navier-Stokes system (1.4)–(1.6) and its approximation given by the solution of the Kuznetsov equation (see Section 1.3).

Putting expressions for the density and velocity (1.19)–(1.20) into the isentropic Navier-Stokes system (1.4)–(1.6), we obtain for the momentum conservation (1.5)

$$\begin{aligned} \rho_\varepsilon [\partial_t \mathbf{v}_\varepsilon + (\mathbf{v}_\varepsilon \cdot \nabla) \mathbf{v}_\varepsilon] + \nabla p(\rho_\varepsilon) - \varepsilon \nu \Delta \mathbf{v}_\varepsilon &= \varepsilon \nabla (-\rho_0 \partial_t u + c^2 \rho_1) \\ &+ \varepsilon^2 \left[-\rho_1 \nabla (\partial_t u) + \frac{\rho_0}{2} \nabla ((\nabla u)^2) + c^2 \nabla \rho_2 + \frac{(\gamma-1)c^2}{2\rho_0} \nabla (\rho_1^2) + \nu \nabla \Delta u \right] + O(\varepsilon^3). \end{aligned} \quad (1.21)$$

In order to have an approximation up to the terms $O(\varepsilon^3)$ we put the terms of order one and two in ε equal to 0, what allows us to find the expressions for the density correctors:

$$\rho_1(\mathbf{x}, t) = \frac{\rho_0}{c^2} \partial_t u(\mathbf{x}, t), \quad (1.22)$$

$$\rho_2(\mathbf{x}, t) = -\frac{\rho_0(\gamma-2)}{2c^4} (\partial_t u)^2 - \frac{\rho_0}{2c^2} (\nabla u)^2 - \frac{\nu}{c^2} \Delta u. \quad (1.23)$$

Indeed, we start by making $\varepsilon \nabla (-\rho_0 \partial_t u + c^2 \rho_1) = 0$ and find the first order perturbation of the density ρ_1 given by Eq. (1.22). Consequently, if ρ_1 satisfies (1.22), then Eq. (1.21) becomes

$$\begin{aligned} \rho_\varepsilon [\partial_t \mathbf{v}_\varepsilon + (\mathbf{v}_\varepsilon \cdot \nabla) \mathbf{v}_\varepsilon] + \nabla p(\rho_\varepsilon) - \varepsilon \nu \Delta \mathbf{v}_\varepsilon &= \varepsilon \nabla (-\rho_0 \partial_t u + c^2 \rho_1) \\ \varepsilon^2 \nabla \left[-\frac{\rho_0}{2c^2} (\partial_t u)^2 + \frac{\rho_0}{2} (\nabla u)^2 + c^2 \rho_2 + \frac{(\gamma-1)\rho_0}{2c^2} (\partial_t u)^2 + \nu \Delta u \right] &+ O(\varepsilon^3). \end{aligned} \quad (1.24)$$

Thus, taking the corrector ρ_2 by formula (1.23), we ensure that

$$\rho_\varepsilon [\partial_t \mathbf{v}_\varepsilon + (\mathbf{v}_\varepsilon \cdot \nabla) \mathbf{v}_\varepsilon] + \nabla p(\rho_\varepsilon) - \varepsilon \nu \Delta \mathbf{v}_\varepsilon = O(\varepsilon^3). \quad (1.25)$$

Now we put these expressions of ρ_1 from (1.22) and ρ_2 from (1.23) with *ansatz* (1.19)–(1.20) in Eq. (1.4) of the mass conservation to obtain

$$\begin{aligned} \partial_t \rho_\varepsilon + \operatorname{div}(\rho_\varepsilon \mathbf{v}_\varepsilon) &= \varepsilon \frac{\rho_0}{c^2} \left[\partial_t^2 u - c^2 \Delta u - \right. \\ &\left. \varepsilon \partial_t \left((\nabla u)^2 + \frac{\gamma-2}{2c^2} (\partial_t u)^2 + \frac{\nu}{\rho_0} \Delta u \right) - \varepsilon u_t \Delta u \right] + O(\varepsilon^3). \end{aligned} \quad (1.26)$$

Then we notice that the right hand term of the order ε in Eq. (1.26) is actually the linear wave equation up to smaller on ε terms:

$$\partial_t^2 u - c^2 \Delta u = O(\varepsilon).$$

Hence, we express

$$\varepsilon u_t \Delta u = \varepsilon \frac{1}{c^2} u_t u_{tt} + O(\varepsilon^2) = \varepsilon \frac{1}{2c^2} \partial_t ((u_t)^2) + O(\varepsilon^2),$$

and putting it in Eq. (1.26), we finally have

$$\begin{aligned} \partial_t \rho_\varepsilon + \operatorname{div}(\rho_\varepsilon \mathbf{v}_\varepsilon) &= \varepsilon \frac{\rho_0}{c^2} \left[\partial_t^2 u - c^2 \Delta u - \right. \\ &\left. \varepsilon \partial_t \left((\nabla u)^2 + \frac{\gamma-1}{2c^2} (\partial_t u)^2 + \frac{\nu}{\rho_0} \Delta u \right) \right] + O(\varepsilon^3). \end{aligned} \quad (1.27)$$

The right hand side of Eq. (1.27) gives us the Kuznetsov equation

$$\partial_t^2 u - c^2 \Delta u = \varepsilon \partial_t \left((\nabla u)^2 + \frac{\gamma - 1}{2c^2} (\partial_t u)^2 + \frac{\nu}{\rho_0} \Delta u \right), \quad (1.28)$$

which is the first order approximation of the isentropic Navier-Stokes system up to the terms $O(\varepsilon^3)$. Moreover, if u is a solution of the Kuznetsov equation, then with the relations for the density perturbations (1.22) and (1.23) and with *ansatz* (1.19)–(1.20) we have

$$\partial_t \rho_\varepsilon + \operatorname{div}(\rho_\varepsilon \mathbf{v}_\varepsilon) = O(\varepsilon^3), \quad (1.29)$$

$$\rho_\varepsilon [\partial_t \mathbf{v}_\varepsilon + (\mathbf{v}_\varepsilon \cdot \nabla) \mathbf{v}_\varepsilon] + \nabla p(\rho_\varepsilon) - \varepsilon \nu \Delta \mathbf{v}_\varepsilon = O(\varepsilon^3). \quad (1.30)$$

Hence, it is clear that the standard physical perturbative approach without the corrector ρ_2 (it is sufficient to take $\rho_2 = 0$ in our calculus) can't ensure (1.29)–(1.30).

Let us also notice, as it was originally mentioned by Kuznetsov, that the Kuznetsov equation (1.28) contains terms of different orders, and hence, it is a wave equation with small size nonlinear perturbations $\partial_t(\nabla u)^2$, $\partial_t(\partial_t u)^2$ and the viscosity term $\partial_t \Delta u$.

1.1.2 Westervelt equation

In the physical notations the Westervelt equation, historically derived [173] for the acoustic pressure fluctuation, has the following form

$$p_{tt} - c^2 \Delta p - \nu \varepsilon \Delta p_t = \frac{\gamma + 1}{c^2} \varepsilon p_t p_{tt}, \quad (1.31)$$

and can also be seen as an approximation of an isentropic Navier-Stokes system. Moreover, the Westervelt equation can be viewed as an approximation of the Kuznetsov equation by a nonlinear perturbation. Actually the only difference between these two models is that the Westervelt equation keeps only one of two nonlinear terms of the Kuznetsov equation, producing cumulative effects in a progressive wave propagation [1]. Let u be a solution of the Kuznetsov equation (1.18). Similarly as in Ref. [1] we set

$$\overline{\Pi} = u + \frac{1}{2c^2} \varepsilon \partial_t [u^2] \quad (1.32)$$

and obtain

$$\partial_t^2 \overline{\Pi} - c^2 \Delta \overline{\Pi} = \varepsilon \partial_t \left(\frac{\nu}{\rho_0} \Delta u + \frac{\gamma + 1}{2c^2} (\partial_t u)^2 + \frac{1}{c^2} u (\partial_t^2 - c^2 \Delta u) \right).$$

By definition (1.32) of $\overline{\Pi}$ we have

$$\partial_t^2 \overline{\Pi} - c^2 \Delta \overline{\Pi} = \varepsilon \partial_t \left(\frac{\nu}{\rho_0} \Delta \overline{\Pi} + \frac{\gamma + 1}{2c^2} (\partial_t \overline{\Pi})^2 \right) + O(\varepsilon^2). \quad (1.33)$$

We recognize the Westervelt equation (1.7) obtained up to remainder terms of order ε^2 .

1.1.3 KZK equation

The Khoklov-Zabolotskaya-Kuznetsov equation (KZK) equation typically models the ultrasonic propagation with strong diffraction phenomena, combining with finite amplitude effects (see Ref. [21], [ARP-12], [ARPproc-7] and the references therein). In [ARP-12] there is a review of different ways to obtain the KZK equation: it can be found in the framework of geometrical optic taking the $O(1)$ -oscillation of the order zero [47, 162] and in the framework of electromagnetic waves in a saturated ferromagnetic media [150].

Let us focus on the nonlinear acoustic framework. To be able to derive the KZK equation from the isentropic Navier-Stokes system or the Kuznetsov equation, we assume the following additional properties of beam's propagation:

1. the beams are concentrated near the x_1 -axis ;
2. the beams propagate along the x_1 -direction;
3. the beams are generated either by an initial condition or by a forcing term on the boundary $x_1 = 0$.

Derivation from the Navier-Stokes system

In [ARP-12] the considered *ansatz* allows to obtain the KZK equation only up $O(\epsilon^{\frac{5}{2}})$ -terms. I have improved it for the approximation up to $O(\epsilon^3)$ -terms in [ARPproc-7] which is optimal on the order of ϵ and was later used in [ARP-2]. Thus let us follow [ARP-2].

We perform the derivation in two steps:

1. Firstly we introduce small perturbations around a constant state of the compressible isentropic Navier-Stokes system according to the Kuznetsov *ansatz* (1.19)–(1.20):

$$\begin{aligned} \partial_t \rho_\epsilon + \nabla \cdot (\rho_\epsilon \mathbf{v}_\epsilon) = & \epsilon [\partial_t \rho_1 - \rho_0 \Delta u] \\ & + \epsilon^2 [\partial_t \rho_2 - \nabla \rho_1 \nabla u - \rho_1 \Delta u] + O(\epsilon^3), \end{aligned} \quad (1.34)$$

and we have again (1.21) for the conservation of momentum.

2. Secondly, we perform the paraxial change of variables (see Fig. 1.2):

$$\tau = t - \frac{x_1}{c}, \quad z = \epsilon x_1, \quad y = \sqrt{\epsilon} x'. \quad (1.35)$$

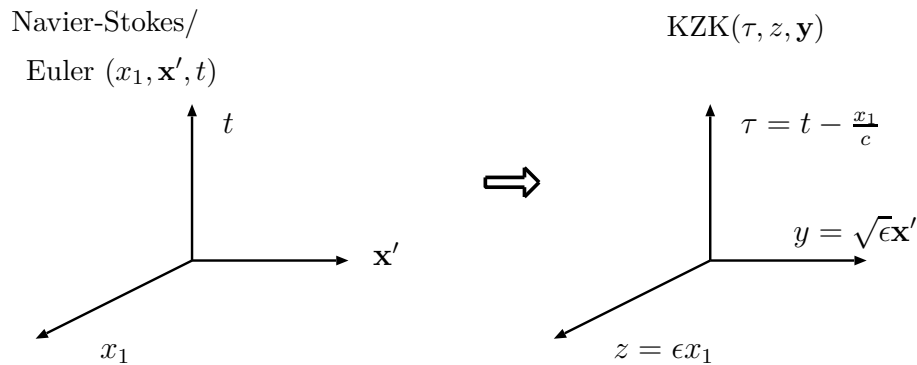


Figure 1.2 – Paraxial change of variables for the profiles $U(t - x_1/c, \epsilon x_1, \sqrt{\epsilon} x')$.

We notice that z becomes the propagation variable, τ is retarded time and there is rescaling transversal variables y . The paraxial change of variables (1.35) defines the axis of the propagation x_1 along which the wave changes its profile much slower than along the transversal axis x' . This is typical for the propagation of ultrasound waves. Since the gradient ∇ in the coordinates (τ, z, y) becomes depending on ϵ

$$\tilde{\nabla} = \left(\epsilon \partial_z - \frac{1}{c} \partial_\tau, \sqrt{\epsilon} \nabla_y \right)^t,$$

if we denote

$$u(x_1, \mathbf{x}', t) = \Phi(t - x_1/c, \epsilon x_1, \sqrt{\epsilon} \mathbf{x}') = \Phi(\tau, z, \mathbf{y}), \quad (1.36)$$

we need to take attention to have the paraxial correctors of the order $O(1)$:

$$\rho_1(x_1, \mathbf{x}', t) = I(\tau, z, \mathbf{y}), \quad \rho_2(x_1, \mathbf{x}', t) = H(\tau, z, \mathbf{y}) = J(\tau, z, \mathbf{y}) + O(\epsilon),$$

where actually $H(\tau, z, \mathbf{y})$ is the profile function obtained from ρ_2 (see Appendix A [ARP-2]) containing not only the terms of the order $O(1)$ but also terms up to ϵ^2 . Hence, we denote by J all terms of H of order 0 on ϵ , which are significant in order to have an approximation up to the terms $O(\epsilon^3)$.

The assumption to work directly with the velocity potential (1.36) immediately implies the following velocity expansion

$$\mathbf{v}_\epsilon(\mathbf{x}, t) = -\epsilon \left(-\frac{1}{c} \partial_\tau \Phi + \epsilon \partial_z \Phi; \sqrt{\epsilon} \nabla_y \Phi \right) (\tau, z, \mathbf{y}). \quad (1.37)$$

In new variables (τ, z, y) Eq. (1.21) becomes

$$\begin{aligned} \rho_\epsilon [\partial_t \mathbf{v}_\epsilon + (\mathbf{v}_\epsilon \cdot \nabla) \mathbf{v}_\epsilon] + \nabla p(\rho_\epsilon) - \epsilon \nu \Delta \mathbf{v}_\epsilon &= \epsilon \tilde{\nabla} [-\rho_0 \partial_\tau \Phi + c^2 I] \\ + \epsilon^2 \left[-I \tilde{\nabla} (\partial_\tau \Phi) + \frac{\rho_0}{2} \tilde{\nabla} \left(\frac{1}{c^2} (\partial_\tau \Phi)^2 \right) \right. \\ \left. + c^2 \tilde{\nabla} J + \frac{\gamma - 1}{2\rho_0} c^2 \tilde{\nabla} (I^2) + \nu \tilde{\nabla} \left(\frac{1}{c^2} \partial_\tau^2 \Phi \right) \right] &+ O(\epsilon^3). \end{aligned} \quad (1.38)$$

Consequently, we find the correctors of the density as functions of Φ :

$$I(\tau, z, y) = \frac{\rho_0}{c^2} \partial_\tau \Phi(\tau, z, \mathbf{y}), \quad (1.39)$$

$$J(\tau, z, y) = -\frac{\rho_0(\gamma - 1)}{2c^4} (\partial_\tau \Phi)^2 - \frac{\nu}{c^4} \partial_\tau^2 \Phi. \quad (1.40)$$

Indeed, we start by making

$$\epsilon \tilde{\nabla} [-\rho_0 \partial_\tau \Phi + c^2 I] = 0$$

and find the first order perturbation of the density I given by Eq. (1.39). Moreover, if ρ_1 satisfies (1.39), then Eq. (1.38) becomes

$$\begin{aligned} \rho_\epsilon [\partial_t \mathbf{v}_\epsilon + (\mathbf{v}_\epsilon \cdot \nabla) \mathbf{v}_\epsilon] + \nabla p(\rho_\epsilon) - \epsilon \nu \Delta \mathbf{v}_\epsilon &= \epsilon \tilde{\nabla} [-\rho_0 \partial_\tau \Phi + c^2 I] \\ \epsilon^2 \tilde{\nabla} \left[-\frac{\rho_0}{2c^2} (\partial_\tau \Phi)^2 + \frac{\rho_0}{2c^2} (\partial_\tau \Phi)^2 + c^2 J + \frac{(\gamma - 1)\rho_0}{2c^2} (\partial_\tau \Phi)^2 + \frac{\nu}{c^2} \partial_\tau^2 \Phi \right] &+ O(\epsilon^3). \end{aligned} \quad (1.41)$$

Thus, taking the corrector J in the expansion of ρ_ϵ

$$\rho_\epsilon(x_1, \mathbf{x}', t) = \rho_0 + \epsilon I(t - x_1/c, \epsilon x_1, \sqrt{\epsilon} \mathbf{x}') + \epsilon^2 J(t - x_1/c, \epsilon x_1, \sqrt{\epsilon} \mathbf{x}'), \quad (1.42)$$

by formula (1.40), we ensure that

$$\rho_\epsilon [\partial_t \mathbf{v}_\epsilon + (\mathbf{v}_\epsilon \cdot \nabla) \mathbf{v}_\epsilon] + \nabla p(\rho_\epsilon) - \epsilon \nu \Delta \mathbf{v}_\epsilon = O(\epsilon^3). \quad (1.43)$$

Now we put these expressions of I from (1.39) and J from (1.40) with the paraxial approximation in Eq. (1.34) of the mass conservation to obtain

$$\begin{aligned} \partial_t \rho_\epsilon + \nabla \cdot (\rho_\epsilon \mathbf{v}_\epsilon) = & \epsilon^2 \left[\frac{\rho_0}{c^2} (2c \partial_{\tau z}^2 \Phi - c^2 \Delta_y \Phi) - \frac{\rho_0}{2c^4} (\gamma + 1) \partial_\tau [(\partial_\tau \Phi)^2] - \frac{\nu}{c^4} \partial_\tau^3 \Phi \right] \\ & + O(\epsilon^3). \end{aligned} \quad (1.44)$$

All terms of the second order on ϵ in relation (1.44) give us the equation for Φ , which is the KZK equation. If we use relation (1.39), we obtain the usual form of the KZK equation often written [21] and [ARP-2], [ARP-3], [ARPproc-7], [ARP-12], [ARP-13] for the first perturbation I of the density ρ_ϵ :

$$c \partial_{\tau z}^2 I - \frac{(\gamma + 1)}{4\rho_0} \partial_\tau^2 I^2 - \frac{\nu}{2c^2 \rho_0} \partial_\tau^3 I - \frac{c^2}{2} \Delta_y I = 0. \quad (1.45)$$

We notice that, as the Kuznetsov equation, this model still contains terms describing the wave propagation $\partial_{\tau z}^2 I$, the nonlinearity $\partial_\tau^2 I^2$ and the viscosity effects $\partial_\tau^3 I$ of the medium but also adds a diffraction effects by the traversal Laplacian $\Delta_y I$. This corresponds to the description of the quasi-one-dimensional propagation of a signal in a homogeneous nonlinear isentropic medium. In addition, thanks to the paraxial approximation, this time all terms in the equation are of the same size in contrast with the Kuznetsov equation.

By our derivation we obtain that the KZK equation is the second-order approximation of the isentropic Navier-Stokes system up to terms of $O(\epsilon^3)$ by the introduced previously *ansatz* (1.36)–(1.42). In this sense, since the entropy and the pressure of the compressible Navier-Stokes system (1.1)–(1.3) are approximated up to terms of the order of ϵ^3 , the KZK equation *ansatz* (1.36)–(1.42) is optimal, as it also gives the approximation of the equations of the isentropic Navier-Stokes system up to $O(\epsilon^3)$ -terms.

Derivation from the Kuznetsov equation

If the velocity potential is given [116] by Eq. (1.36), then we directly obtain from the Kuznetsov equation (1.18) via the paraxial change of variables (1.35) that

$$\begin{aligned} & \partial_t^2 u - c^2 \Delta u - \epsilon \partial_t \left((\nabla u)^2 + \frac{\gamma - 1}{2c^2} (\partial_t u)^2 + \frac{\nu}{\rho_0} \Delta u \right) \\ & = \epsilon \left[2c \partial_{\tau z}^2 \Phi - \frac{\gamma + 1}{2c^2} \partial_\tau (\partial_\tau \Phi)^2 - \frac{\nu}{\rho_0 c^2} \partial_\tau^3 \Phi - c^2 \Delta_y \Phi \right] + O(\epsilon^2). \end{aligned} \quad (1.46)$$

Therefore, we find that the right-hand side ϵ -order terms in Eq. (1.46) is precisely the KZK equation (1.45).

1.1.4 NPE equation

In the difference to the KZK equation, the NPE (Nonlinear Progressive wave Equation) equation is usually used to describe short-time pulses and long-range propagation, for instance, in an ocean waveguide, where the refraction phenomena are important [26, 132].

The NPE equation, initially derived by McDonald and Kuperman [131], is an example of a paraxial approximation aiming to describe short-time pulses and long-range propagation, for instance, in an ocean waveguide, where the refraction phenomena are important. To compare to the KZK equation we use the following paraxial change of variables

$$u(t, x_1, x') = \Psi(\varepsilon t, x_1 - ct, \sqrt{\varepsilon}x') = \Psi(\tau, z, y), \quad (1.47)$$

with

$$\tau = \varepsilon t, \quad z = x_1 - ct, \quad y = \sqrt{\varepsilon}x'. \quad (1.48)$$

To compare to the KZK equation, the propagation follows the rescaled time variable τ and z take the role of τ from the KZK *ansatz*. For the velocity we have

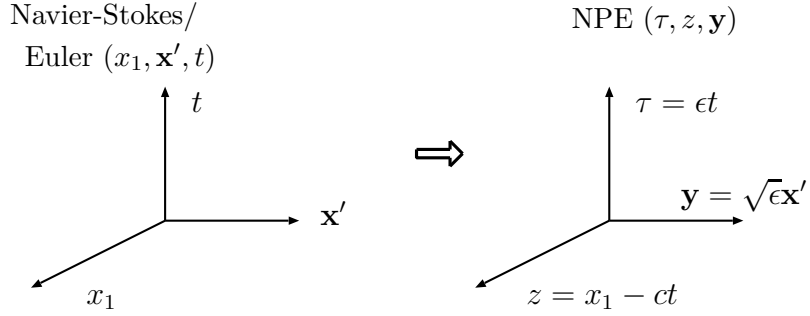


Figure 1.3 – Paraxial change of variables for the profiles $U(\varepsilon t, x_1 - ct, \sqrt{\varepsilon}x')$.

$$\mathbf{v}_\varepsilon(t, x_1, x') = -\varepsilon \nabla u(t, x_1, x') = -\varepsilon (\partial_z \Psi, \sqrt{\varepsilon} \nabla_y \Psi)(\tau, z, y). \quad (1.49)$$

Derivation from the Navier-Stokes system

If we compare the NPE equation to the isentropic Navier-Stokes system this method of approximation does not allow to keep the Kuznetsov *ansatz* of perturbations (1.19)–(1.20) imposing (1.22)–(1.23), just by introducing the new paraxial profiles Ψ for u , ξ for ρ_1 and χ for ρ_2 and taking the term of order 0 in ε as it was done in the case of the KZK-approximation. This time the paraxial change of variables (1.48) for ρ_1 and ρ_2 , defined in Eqs. (1.22)–(1.23), gives

$$\begin{aligned} \rho_1 &= -\frac{\rho_0}{c} \partial_z \Psi + \varepsilon \frac{\rho_0}{c^2} \partial_\tau \Psi, \\ \rho_2 &= -\frac{\rho_0(\gamma-2)}{2c^2} (\partial_z \Psi)^2 - \frac{\rho_0}{2c^2} (\partial_z \Psi)^2 - \frac{\nu}{\rho_0} \partial_z^2 \Psi \\ &\quad + \varepsilon \left[\frac{\rho_0(\gamma-2)}{2c^3} \partial_z \Psi \partial_\tau \Psi - \frac{\rho_0}{2c^2} (\nabla_y \Psi)^2 - \frac{\nu}{c^2} \Delta_y \Psi \right] \\ &\quad + \varepsilon^2 \left(-\frac{\rho_0(\gamma-2)}{2c^4} \right) (\partial_\tau \Psi)^2. \end{aligned}$$

Thus, one of the terms in the ρ_1 -extension takes part of the second order corrector of ρ_ε :

$$\rho_\varepsilon(t, x_1, x') = \rho_0 + \varepsilon \xi(\tau, z, y) + \varepsilon^2 \chi(\tau, z, y), \quad (1.50)$$

with

$$\xi(\tau, z, y) = -\frac{\rho_0}{c} \partial_z \Psi, \quad (1.51)$$

$$\chi(\tau, z, y) = \frac{\rho_0}{c^2} \partial_\tau \Psi - \frac{\rho_0(\gamma-1)}{2c^2} (\partial_z \Psi)^2 - \frac{\nu}{c^2} \partial_z^2 \Psi. \quad (1.52)$$

The obtained *ansatz* (1.49)–(1.50), applied to the Navier-Stokes system, gives

$$\begin{aligned} \partial_t \rho_\varepsilon + \operatorname{div}(\rho_\varepsilon \mathbf{v}_\varepsilon) &= \varepsilon^2 \left(-\frac{2\rho_0}{c} \right) \left(\partial_{\tau z}^2 \Psi - \frac{(\gamma+1)}{4} \partial_z (\partial_z \Psi)^2 - \frac{\nu}{2\rho_0} \partial_z^3 \Psi + \frac{c}{2} \Delta_y \Psi \right) \\ &\quad + O(\varepsilon^3), \end{aligned}$$

and

$$\begin{aligned} \rho_\varepsilon [\partial_t \mathbf{v}_\varepsilon + (\mathbf{v}_\varepsilon \cdot \nabla) \mathbf{v}_\varepsilon] + \nabla p(\rho_\varepsilon) - \varepsilon \nu \Delta \mathbf{v}_\varepsilon &= \varepsilon \nabla \left(\xi + \frac{\rho_0}{c} \partial_z \Psi \right) \\ &\quad + c^2 \varepsilon^2 \nabla \left[\chi - \frac{\rho_0}{c^2} \partial_\tau \Psi + \frac{\rho_0(\gamma-1)}{2c^2} (\partial_z \Psi)^2 + \frac{\nu}{c^2} \partial_z^2 \Psi \right] + O(\varepsilon^3). \end{aligned}$$

As all previous models, for this *ansatz*, the NPE equation

$$\partial_{\tau z}^2 \Psi - \frac{(\gamma+1)}{4} \partial_z (\partial_z \Psi)^2 - \frac{\nu}{2\rho_0} \partial_z^3 \Psi + \frac{c}{2} \Delta_y \Psi = 0 \quad (1.53)$$

appears as the second order approximation of the isentropic Navier-Stokes system up to the terms of the order of $\mathcal{O}(\varepsilon^3)$. In the sequel we work with

$$\xi(\tau, z, y) = -\frac{\rho_0}{c} \partial_z \Psi, \quad (1.54)$$

which satisfies the NPE equation

$$\partial_{\tau z}^2 \xi + \frac{(\gamma+1)c}{4\rho_0} \partial_z^2 [(\xi)^2] - \frac{\nu}{2\rho_0} \partial_z^3 \xi + \frac{c}{2} \Delta_y \xi = 0. \quad (1.55)$$

Looking at Figs 1.2 and 1.3 together with Eqs. (1.45) and (1.53), we see that there is a bijection between the variables of the KZK and NPE equations defined by the relations

$$z_{NPE} = -c \tau_{KZK} \text{ and } \tau_{NPE} = \varepsilon \tau_{KZK} + \frac{z_{KZK}}{c}, \quad (1.56)$$

which implies for the derivatives

$$\partial_{\tau_{NPE}} = c \partial_{z_{KZK}} \text{ and } \partial_{z_{NPE}} = -\frac{1}{c} \partial_{\tau_{KZK}}.$$

Thus, as it was mentioned in Introduction, the known mathematical results for the KZK equation can be directly applied for the NPE equation.

Derivation from the Kuznetsov equation

If we start with the Kuznetsov equation (1.18), putting inside the velocity potential (1.47) we directly obtain according the paraxial change of variable (1.48) that

$$\begin{aligned} & \partial_t^2 u - c^2 \Delta u - \varepsilon \partial_t \left((\nabla u)^2 + \frac{\gamma-1}{2c^2} (\partial_t u)^2 + \frac{\nu}{\rho_0} \Delta u \right) \\ &= \varepsilon \left(-2c \partial_{\tau z}^2 \Psi - c^2 \Delta_y \Psi + \frac{\nu}{\rho_0} c \partial_z^3 \Psi + \frac{\gamma+1}{2} c \partial_z (\partial_z \Psi)^2 \right) + O(\varepsilon^2). \end{aligned}$$

We obtain the NPE equation satisfying by $\partial_z \Psi$ modulo a multiplicative constant:

$$\partial_{\tau z}^2 \Psi - \frac{\gamma+1}{4} \partial_z (\partial_z \Psi)^2 - \frac{\nu}{2\rho_0} \partial_z^3 \Psi + \frac{c}{2} \Delta_y \Psi = 0.$$

1.2 Strong well posedness of the Cauchy problems

To explain the main techniques to study the well posedness of the nonlinear dissipative models, I start this section with the simplest case of the KZK or the NPE equation [ARP-13]. I finish by mention the main ideas which we use for establish the well posedness of the Cauchy problem for the Kuznetsov equation [ARP-5].

1.2.1 KZK and NPE equations

Although the physical context and the physical using of the KZK and the NPE equations are different, there is a bijection (see Eq. (1.56)) between the variables of these two models and they can be presented by the same type differential operator with constant positive coefficients:

$$Lu = 0, \quad L = \partial_{tx}^2 - c_1 \partial_x (\partial_x \cdot)^2 - c_2 \partial_x^3 \pm c_3 \Delta_y, \quad \text{for } t \in \mathbb{R}^+, x \in \mathbb{R}, y \in \mathbb{R}^{n-1}.$$

Therefore, the results on the solutions of the KZK equation from [ARP-13] are valid for the NPE equation. See also [92] for analogous results in \mathbb{R}^2 for these models in the viscous case.

We study the well posedness of the following Cauchy problem:

$$\begin{cases} c \partial_{\tau z}^2 I - \frac{(\gamma+1)}{4\rho_0} \partial_\tau^2 I^2 - \frac{\nu}{2c^2 \rho_0} \partial_\tau^3 I - \frac{c^2}{2} \Delta_y I = 0 \text{ on } \mathbb{T}_\tau \times \mathbb{R}_+ \times \mathbb{R}^{n-1}, \\ I(\tau, 0, y) = I_0(\tau, y) \text{ on } \mathbb{T}_\tau \times \mathbb{R}^{n-1} \end{cases} \quad (1.57)$$

in the class of L -periodic functions with respect to the variable τ and with mean value zero

$$\int_0^L I(\tau, z, y) d\tau = 0. \quad (1.58)$$

The introduction of the operator ∂_τ^{-1} , defined by formula

$$\partial_\tau^{-1} I(\tau, z, y) := \int_0^\tau I(s, z, y) ds + \int_0^L \frac{s}{L} I(s, z, y) ds, \quad (1.59)$$

allows us to consider instead of Eq. (1.45) the following equivalent equation

$$c \partial_z I - \frac{(\gamma+1)}{4\rho_0} \partial_\tau I^2 - \frac{\nu}{2c^2 \rho_0} \partial_\tau^2 I - \frac{c^2}{2} \partial_\tau^{-1} \Delta_y I = 0 \text{ on } \mathbb{T}_\tau \times \mathbb{R}_+ \times \mathbb{R}^{n-1}, \quad (1.60)$$

which for all initial data in H^s ($s > \left[\frac{n}{2}\right] + 1$) is at least locally well-posed [ARP-13] with $I \in C([0, T^*[, H^s(\mathbb{T}_\tau \times \mathbb{R}^{n-1}))$. Here by T^* is denoted the maximal value for the interval $[0, T[$ of z on which a such solution exists.

As it was mentioned in [120, 121, 137] for the KP type equations in \mathbb{R}^2 , the introduced operator ∂_τ^{-1} is singular in the sense that its Fourier transform gives a division [ARP-13] by a discrete variable m :

$$\mathcal{F}(\partial_\tau^{-1} \Delta_y I) = \frac{L \xi^2}{i 2 \pi m} \mathcal{F}(I)(m, \xi) \quad m \in \mathbb{Z}, \quad \xi \in \mathbb{R}.$$

If we suppose that I has the mean value zero in τ , it implies that $\mathcal{F}(I)(0, \xi) = 0$ for all ξ , which makes disappear the singularity for $m = 0$. For the same reason this requires Lemma 5.2 [ARP-13] the additional constraint for the initial data $\partial_\tau^{-1} \Delta_y I_0 = \phi_0 \in H^{s-2}$ to be able to ensure that the solution $I \in C([0, T[, H^s(\mathbb{T}_\tau \times \mathbb{R}^{n-1}))$ can be also considered in $C^1([0, T[, H^{s-2}(\mathbb{T}_\tau \times \mathbb{R}^{n-1}))$ (see also a similar situation for the KP type equations explained in [137]). In the same time as it is discussed in [120, 121, 137] in the non-periodic case this regularity constraint is not physical. However, if we work in the class of periodic functions with the mean value zero this condition can be omitted. Indeed, by definition (1.59) of the operator ∂_τ^{-1} , it preserves the property of a periodic function to have the mean value zero. Thus, if I_0 is a periodic function with the mean value zero on τ , the solution I also belongs in this class, where we find the equivalence between the Cauchy problem (1.57) and the analogous problem considered for Eq. (1.60). Formula (1.59), as it is noticed in [ARP-13] p. 796, allows to establish an analogue of the Poincaré inequality (which is false in the non periodic case of \mathbb{R}^n):

$$\|I\|_{H^s([0, L] \times \mathbb{R}_y^{n-1})} \leq C \|\partial_\tau I\|_{H^s([0, L] \times \mathbb{R}_y^{n-1})}, \quad (1.61)$$

coming from the following relation

$$I = \partial_\tau^{-1} \partial_\tau I = \int_0^\tau \partial_\tau I(s, y) ds + \int_0^L \frac{s}{L} \partial_\tau I(s, y) ds.$$

As, by (1.59), $\partial_\tau^{-1} I$ is L -periodic in τ and of mean value zero, this also gives us the following estimate

$$\|\partial_\tau^{-1} I\|_{H^s(\Omega_1)} \leq C \|\partial_\tau \partial_\tau^{-1} I\|_{H^s(\Omega_1)} = C \|I\|_{H^s(\Omega_1)}. \quad (1.62)$$

This means that in the class of periodic and of mean value zero functions as soon as $I_0 \in H^s(\Omega_1)$, it implies that $\partial_\tau^{-1} I_0$ is also in $H^s(\Omega_1)$ and in the same class. Hence the condition $\partial_\tau^{-1} \Delta_y I_0 \in H^{s-2}(\Omega_1)$ required in Theorem. 1.2, Point 4 [ARP-13] is automatically verified for I_0 from H^s which are periodic and of mean value zero in t ($\tau = t$ for $z = 0$).

To treat this kind of equations a priori estimates are still crucial as for the Kuznetsov and Westervelt equations. They are in particular a consequence of the following relation taking place in the class of periodic functions with the mean value zero:

$$\begin{aligned} \int_0^L \int_{\mathbb{R}_y^{n-1}} \partial_\tau^{-1}(\Delta_y I) I d\tau dy &= - \int_0^L \int_{\mathbb{R}_y^{n-1}} \partial_\tau^{-1}(\nabla_y I) \nabla_y I d\tau dy \\ &= \int_0^L \int_{\mathbb{R}_y^{n-1}} \partial_\tau^{-1}(\nabla_y I) \partial_\tau(\partial_\tau^{-1}(\nabla_y I)) d\tau dy = 0. \end{aligned} \quad (1.63)$$

To simplify the notations we rewrite Eq. (1.60) in the following form

$$\partial_z w - w \partial_\tau w - \beta \partial_\tau^2 w - \gamma \partial_\tau^{-1} \Delta_y w = 0, \quad (1.64)$$

where for instance β represents the viscosity ν multiplied by $(c^5 \rho_0 (\gamma + 1))^{-\frac{1}{2}}$. Thus it is sufficient to multiply (1.64) by w and integrate by parts to obtain

$$\frac{1}{2} \frac{d}{dz} \|w\|_{L_2}^2 + \beta \|\partial_\tau w\|_{L_2}^2 = 0.$$

The estimate with some strictly positive constants $C_1(L)$ and $C_2(s)$

$$\frac{1}{2} \frac{d}{dz} \|w\|_s^2 + \beta C_1(L) \|w\|_s^2 \leq C_2(s) \|w\|_s^3 \quad (1.65)$$

holds only in the periodic case and not on the whole space. In this latter case the H^s norm of $\partial_\tau w$ does not control the H^s norm of w . Nevertheless, it is crucial in the well-posedness result for the KZK equation. It can be observed that periodic solutions with mean value zero satisfy, for z small enough, the estimate:

$$\frac{1}{2} \frac{d}{dz} \|w\|_s^2 + \|w\|_s^2 (\beta C_1(L) - C_2(s) \|w\|_s) \leq 0. \quad (1.66)$$

Therefore, if $\beta > 0$ for $z = 0$

$$\beta C_1(L) - C_2(s) \|w(0, \cdot)\|_s \geq 0 \quad i.e., \quad \|u(0, \cdot)\|_s \leq \frac{\beta C_1(L)}{C_2(s)},$$

the quantity $\|w(z, \cdot)\|_s^2$ decays for $z > 0$, and, therefore, satisfies the same estimate on all the interval $[0, T^*]$, which can thus be extended after any finite value T^* . Hence, for $\beta > 0$ the maximum existence interval is \mathbb{R}^+ with $T^* = +\infty$. In addition, we are able to determinate the exponential decay if the function $y(z) = \|w\|_{H^s}$ is defined such that $y(0) = \|w_0\|_{H^s}$, thus, it satisfies the equation

$$\frac{d}{dz}(y^2) = C_2(s)y^3 - \beta C_1(L)y^2.$$

Solving it, we find that

$$y(z) = \left(\frac{C_2(s)}{\beta C_1(L)} - \left(\frac{C_2(s)}{\beta C_1(L)} - \frac{1}{\|w_0\|} \right) e^{\frac{\beta C_1(L)}{2} z} \right)^{-1},$$

from where, imposing $\|w_0\|_{H^s} \leq \frac{\beta C_1(L)}{C_2(s)}$, it follows as soon as $\beta > 0$ there exists a constant $C > 0$ depending only on β , $\|w_0\|$, $C_1(L)$ and $C_2(s)$ such that

$$\|w(z)\|_{H^s} \leq y(z) \leq C e^{-\frac{\beta C_1(L)}{2} z} \quad \forall z \in [0, +\infty[.$$

For $\beta = 0$, or equivalently $\nu = 0$ in the inviscid case, the power 3 in the right hand side of estimate (1.65) gives the influence on the possibility to have blow-up formations, since this time the problem is described by

$$\frac{d}{dz} y = C y^{\frac{3}{2}}, \quad y(0) = \|w_0\|_{H^s},$$

which solution is explicit and given by the formula:

$$y(z) = \frac{\|w_0\|_{H^s}}{(1 - \frac{1}{2}Cz\sqrt{\|w_0\|_{H^s}})^2}.$$

In particular we have a simple estimation of the maximal existence interval

$$T^* \geq \frac{2}{C\sqrt{\|w_0\|_{H^s}}}. \quad (1.67)$$

This kind of argument is useful to establish the local existence result for instance using the fractional step method or by the general operator theory of Kato (see [ARP-13]).

To prove the unicity, we also can apply the stability estimate between a regular solution of the KZK equation $I \in L_\infty([0, T[, H^s)$ with the initial data $I_0 \in H^s$ and a less regular (a weak) solution $J \in L^2([0, T[, L_2)$ with the initial data $J_0 \in L^2$:

$$|I(\cdot, z) - J(\cdot, z)|_{L^2} \leq e^{\int_0^z \sup_{\tau, y} |\partial_\tau I(\tau, y, s)| ds} |I_0 - J_0|_{L^2}. \quad (1.68)$$

This kind of estimates can be easily obtained by writing the resulting equation for the difference $I - J$ and taking the inner product of it with $I - J$ in L^2 and finally performing integration by parts. The second interest of this estimate, that it contains exactly the expression which can become infinite for a finite point (T^*, τ^*, y^*) , corresponding to the geometrical blow-up [7] (see Theorem 1.3 [ARP-13]):

$$\int_0^{T^*} \sup_{\tau, y} |\partial_\tau I(\tau, y, z)| dz = \infty.$$

To prove rigorously for the inviscid case for suitable initial data the existence of a point (τ^*, y^*, T^*) at which $\partial_\tau I$ becomes infinite

$$\lim_{z \rightarrow T^*} (T^* - z) \sup_{\tau, y} \partial_\tau I(\tau, y, z) > 0,$$

we use a generalization of the method of characteristics for Burgers' equation adapted to the multidimensional case with an application of the Nash-Moser theorem following Alinhac's method of working with so called “blow-up” system [8, 7]. From geometrical point of view, the first derivative of τ blow-up when the profile of the wave become vertical as it schematically presented in Fig. 1.4.

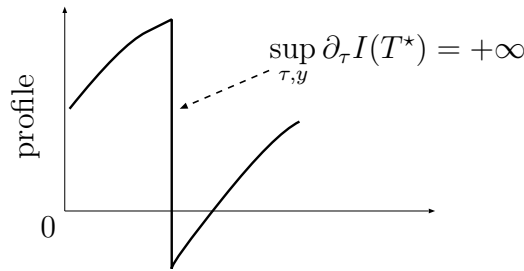


Figure 1.4 – The profile of the beam which provides the blow-up formation at final $T^* < +\infty$.

1.2.2 Kuznetsov and Westervelt equations

The Cauchy problem for the Kuznetsov equation reads for $\alpha = \frac{\gamma-1}{c^2}$, $\beta = 2$ and $\nu = \frac{\delta}{\rho_0}$ as

$$u_{tt} - c^2 \Delta u - \nu \varepsilon \Delta u_t = \alpha \varepsilon u_t u_{tt} + \beta \varepsilon \nabla u \cdot \nabla u_t, \quad x \in \mathbb{R}^n, \quad (1.69)$$

$$u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad x \in \mathbb{R}^n, \quad (1.70)$$

where c , ρ_0 , γ , δ are the velocity of the sound, the density, the ratio of the specific heats and the viscosity of the medium respectively. In what follows, we just suppose that α and β are some positive constants. Eq. (1.69) is a weakly quasi-linear damped wave equation that describes the propagation of a high amplitude wave in fluids. As we have seen, the Kuznetsov equation is one of the models derived from the Navier-Stokes system, and it is well suited for the plane, cylindrical and spherical waves in a fluid [82]. Most of the works on the Kuznetsov equation (1.69) are treated in the one space dimension [104] or a bounded spatial domain of \mathbb{R}^n [106, 107, 135]. For the viscous case, Kaltenbacher and Lasiecka [107] have considered the Dirichlet boundary valued problem and proved for sufficiently small initial data the global well-posedness for $n \leq 3$. Meyer and Wilke [135] have proved it for all n . In [106] it was proven a local well-posedness of the Neumann boundary valued problem for $n \leq 3$.

In the inviscid case for $\nu = 0$, the Cauchy problem for the Kuznetsov equation is a particular case of a general quasi-linear hyperbolic system of the second-order considered by Hughes, Kato, and Marsden [90]. The local well posedness result, proved in [90], does not use a priori estimate techniques and is based on the semi-group theory. Hence, thanks to [90], we have the well posedness of (1.69)–(1.70) in the Sobolev spaces H^s with a real $s > \frac{n}{2} + 1$:

Theorem 1.2.1 *Let $\nu = 0$, $n \in \mathbb{N}^*$ and $s > \frac{n}{2} + 1$. For all $u_0 \in H^{s+1}(\mathbb{R}^n)$ and $u_1 \in H^s(\mathbb{R}^n)$ such that $\|u_1\|_{L^\infty(\mathbb{R}^n)} < \frac{1}{2\alpha\varepsilon}$, $\|u_0\|_{L^\infty(\mathbb{R}^n)} < M_1$, $\|\nabla u_0\|_{L^\infty(\mathbb{R}^n)} < M_2$, with M_1 and M_2 in \mathbb{R}_+^* the following results hold:*

1. *For all $T > 0$, there exists $T' > 0$, $T' \leq T$, such that there exists a unique solution u of (1.69)–(1.70) with the following regularity*

$$u \in C^r([0, T']; H^{s+1-r}(\mathbb{R}^n)) \text{ for } 0 \leq r \leq s, \quad (1.71)$$

$$\forall t \in [0, T'], \quad \|u_t(t)\|_{L^\infty(\mathbb{R}^n)} < \frac{1}{2\alpha\varepsilon}, \quad \|u\|_{L^\infty(\mathbb{R}^n)} < M_1, \quad \|\nabla u\|_{L^\infty(\mathbb{R}^n)} < M_2. \quad (1.72)$$

2. *The map $(u_0, u_1) \mapsto (u(t, \cdot), \partial_t u(t, \cdot))$ is continuous in the topology of $H^{s+1} \times H^s$ uniformly in $t \in [0, T']$.*

The condition $\|u_t(t)\|_{L^\infty(\mathbb{R}^n)} < \frac{1}{2\alpha\varepsilon}$ in Eq. (1.72) ensures that the Kuznetsov equation stays hyperbolic and hence is crucial for its well posedness.

Therefore, actually, to extend the local well-posedness to a global one (for $n \geq 4$) and to estimate the maximal time interval on which there exists a regular solution, John [94] has developed a priori estimates for the Cauchy problem for a general quasi-linear wave equation. This time, due to the nonlinearities $u_t u_{tt}$ and $\nabla u \cdot \nabla u_t$ including the time derivatives, to have an a priori estimate for the Kuznetsov equation, we need to work with Sobolev spaces with a natural s , thus denoted in what follows by m .

Let us consider the structure of the Kuznetsov equation working just in L^2 and considering its simplified versions. This gives us the base for a priori estimates involving a high order

energies in the Sobolev spaces and also gives the ideas about the principal properties and difficulties in the mathematical analysis of this equation.

We notice that Eq. (1.69) is a wave equation containing a dissipative term Δu_t and two nonlinear terms: $\nabla u \nabla u_t$ describing local nonlinear effects and $u_t u_{tt}$ describing global or cumulative effects. Actually, the linear wave equation appears from Eq. (1.69) if we consider only the terms of the zero order on ε :

$$u_{tt} - c^2 \Delta u = 0. \quad (1.73)$$

The semi-group theory allows in the usual way to show that for $u_0 \in H^1(\mathbb{R}^n)$ and $u_1 \in L^2(\mathbb{R}^n)$ there exists a unique solution of the Cauchy problem (1.73), (1.70)

$$u \in C^0(\mathbb{R}^+; H^1(\mathbb{R}^n)) \cap C^1(\mathbb{R}^+; L^2(\mathbb{R}^n)).$$

So, the energy of the wave equation (1.73)

$$E(t) = \int_{\mathbb{R}^n} [(u_t)^2 + c^2 (\nabla u)^2](t, x) dx, \quad (1.74)$$

is well defined and conserved

$$\frac{d}{dt} E(t) = 0.$$

For $\nu > 0$ and without nonlinear terms, the Kuznetsov equation (1.69) becomes the known strongly damped wave equation:

$$u_{tt} - c^2 \Delta u - \nu \varepsilon \Delta u_t = 0, \quad (1.75)$$

which is well-posed [91]: for $m \in \mathbb{N}$, $u_0 \in H^{m+1}(\mathbb{R}^n)$ and $u_1 \in H^m(\mathbb{R}^n)$ there exists a unique solution of the Cauchy problem (1.75), (1.70)

$$u \in C^0(\mathbb{R}^+; H^{m+1}(\mathbb{R}^n)) \cap C^1(\mathbb{R}^+; H^m(\mathbb{R}^n)).$$

Multiplying Eq. (1.75) by u_t in $L^2(\mathbb{R}^n)$, we obtain for the energy of the wave equation (1.74)

$$\frac{d}{dt} E(t) = -2\nu \varepsilon \int_{\mathbb{R}^n} (\nabla u_t)^2(t, x) dx \leq 0,$$

what means that the energy $E(t)$ decreases in time, thanks to the viscosity term with $\nu > 0$. The decrease rate is found for more regular energies in [152] in accordance with the regularity of the initial conditions. Without the term $\nabla u \nabla u_t$ (local nonlinear effects), the Kuznetsov equation becomes similar to the Westervelt equation (1.31), initially derived by Westervelt [173] before Kuznetsov. Denoting conveniently p by u in (1.31), we multiply Eq. (1.31) by u_t and integrate over \mathbb{R}^n to obtain

$$\frac{1}{2} \frac{d}{dt} \left(\int_{\mathbb{R}^n} [(u_t)^2 + c^2 (\nabla u)^2] dx \right) + \nu \varepsilon \int_{\mathbb{R}^n} (\nabla u_t)^2 dx = \frac{1}{3} \frac{\gamma + 1}{c^2} \varepsilon \frac{d}{dt} \left(\int_{\mathbb{R}^n} (u_t)^3 dx \right).$$

Then we have

$$\frac{1}{2} \frac{d}{dt} \left(\int_{\mathbb{R}^n} \left[\left(1 - \frac{2}{3} \frac{\gamma + 1}{c^2} \varepsilon u_t \right) (u_t)^2 + c^2 (\nabla u)^2 \right] dx \right) + \nu \varepsilon \int_{\mathbb{R}^n} (\nabla u_t)^2 dx = 0.$$

For $\alpha = \frac{2}{3} \frac{\gamma+1}{c^2}$ we consider the energy

$$E_{nonl}(t) = \int_{\mathbb{R}^n} \left[(1 - \alpha \varepsilon u_t) (u_t)^2 + c^2 (\nabla u)^2 \right] dx, \quad (1.76)$$

which is monotonous decreasing for $\nu > 0$ and is conserved for $\nu = 0$. Let us also notice that, taking the same initial data for $\nu = 0$ and $\nu > 0$, we have:

$$\text{for all } \nu > 0 \text{ and } t > 0 \quad E_{nonl}(t, \nu = 0) > E_{nonl}(t, \nu) \geq 0,$$

in the assumption that $1 - \alpha \varepsilon u_t \geq 0$ almost everywhere.

While $\frac{1}{2} \leq 1 - \alpha \varepsilon u_t \leq \frac{3}{2}$, that is to say $\|u_t(t)\|_{L^\infty(\mathbb{R}^n)}$ remains small enough in time, then we can compare E_{nonl} to the energy of the wave equation

$$\frac{1}{2} E(t) \leq E_{nonl}(t) \leq \frac{3}{2} E(t).$$

Then a sufficiently regular solution of the Cauchy problem for the Westervelt equation has the energy E controlled by a decreasing in time function:

$$E(t) \leq 3E(0) - 4\nu\varepsilon \int_0^t \int_{\mathbb{R}^n} (\nabla u_t(\tau, x))^2 dx dl.$$

Now, let us consider the Kuznetsov equation (1.69). We multiply it by u_t and integrate on \mathbb{R}^n to obtain

$$\frac{1}{2} \frac{d}{dt} E_{nonl}(t) + \nu\varepsilon \int_{\mathbb{R}^n} (\nabla u_t)^2 dx = 2\varepsilon \int_{\mathbb{R}^n} \nabla u \nabla u_t u_t dx,$$

where $E_{nonl}(t)$ is given by Eq. (1.76) with $\alpha = \frac{2}{3} \frac{\gamma-1}{c^2}$. As

$$2\varepsilon \int_{\mathbb{R}^n} \nabla u \nabla u_t u_t dx = \varepsilon \frac{d}{dt} \int_{\mathbb{R}^n} u_t (\nabla u)^2 dx - \varepsilon \int_{\mathbb{R}^n} u_{tt} (\nabla u)^2 dx,$$

we find

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \left(\int_{\mathbb{R}^n} \left[\left(1 - \frac{2}{3} \frac{\gamma-1}{c^2} \varepsilon u_t \right) (u_t)^2 + (c^2 - 2\varepsilon u_t) (\nabla u)^2 \right] dx \right. \\ \left. + 2\varepsilon \int_0^t \int_{\mathbb{R}^n} u_{tt} |\nabla u|^2 dx dl \right) + \nu\varepsilon \int_{\mathbb{R}^n} (\nabla u_t)^2 dx = 0. \end{aligned} \quad (1.77)$$

Thus, for $\alpha = \frac{2}{3} \frac{\gamma-1}{c^2}$, the function

$$F_\nu(t) = \int_{\mathbb{R}^n} \left[(1 - \alpha \varepsilon u_t) (u_t)^2 + (c^2 - 2\varepsilon u_t) (\nabla u)^2 \right] dx + 2\varepsilon \int_0^t \int_{\mathbb{R}^n} u_{tt} |\nabla u|^2 dx dl$$

is constant if $\nu = 0$ and decreases if $\nu > 0$. Let us notice that while $\frac{1}{2} \leq 1 - \alpha \varepsilon u_t \leq \frac{3}{2}$, the coefficient $c^2 - 2\varepsilon u_t$ is always positive (since c is the sound speed in the chosen medium, $c^2 \gg 1$), hence the first integral in $F_\nu(t)$ is positive, but we a priori don't know the sign of the second integral, *i.e.* the sign of u_{tt} . However, for $\nu = 0$, $F_{\nu=0}(t)$ is positive, as soon as $0 \leq 1 - \alpha \varepsilon u_1$:

$$F_{\nu=0}(t) = F_{\nu=0}(0) = \int_{\mathbb{R}^n} \left[(1 - \alpha \varepsilon u_1) (u_1)^2 + (c^2 - 2\varepsilon u_1) (\nabla u_0)^2 \right] dx \geq 0,$$

and, if we take the same initial data for the Cauchy problems with $\nu = 0$ and $\nu > 0$, for all $t > 0$ (for all time where $F_{\nu=0}$ exists) it holds $F_{\nu=0}(t) = F_{\nu=0}(0) > F_{\nu>0}(t)$.

For $n \geq 3$, we can control the term $2\varepsilon \int_{\mathbb{R}^n} \nabla u \nabla u_t u_t \, dx$ using the Hölder inequality and the Sobolev embeddings (which fails in \mathbb{R}^2):

$$\left| \int_{\mathbb{R}^n} \nabla u \nabla u_t u_t \, dx \right| \leq \|\nabla u\|_{L^n} \|\nabla u_t\|_{L^2} \|u_t\|_{L^{\frac{2n}{n-2}}} \leq C \|\nabla u\|_{L^n} \|\nabla u_t\|_{L^2}^2.$$

Indeed, in \mathbb{R}^2 we don't have any estimates of the form

$$\|u\|_{L^p(\mathbb{R}^2)} \leq \|\nabla u\|_{L^2(\mathbb{R}^2)},$$

with $p > 2$. But such an estimate is essential to control the nonlinear term. Then, instead of Eq. (1.77) for F_ν , we have the relation for E_{nonl} :

$$\frac{1}{2} \frac{d}{dt} E_{nonl}(t) + \varepsilon(\nu - 2C \|\nabla u\|_{L^n}) \int_{\mathbb{R}^n} (\nabla u_t)^2 \, dx \leq 0.$$

So, if a solution of the Kuznetsov equation u is such that $\|\nabla u(t)\|_{L^n}$ and $\|u_t(t)\|_{L^\infty}$ stay small enough for all time, then E_{nonl} decreases in time and, as previously for the Westervelt equation, thanks to $\frac{1}{2}E(t) \leq E_{nonl}(t) \leq \frac{3}{2}E(t)$, the energy E has for upper bound a decreasing function. This fact leads especially to look for global well posedness results for the Cauchy problem for the Kuznetsov equation in the viscous case.

If we directly apply general results of Ref. [94] to our case of the Kuznetsov equation, we obtain a well posedness result with a high regularity of the initial data. We improve it in [ARP-5] and show John's results for the Kuznetsov equation with the minimal regularity on the initial data corresponding to the regularity obtained by Hughes, Kato and Marsden [90]. For instance, we prove the analogous energy estimates in H^m with $m \geq [\frac{n}{2} + 2]$ instead of John's $m \geq \frac{3}{2}n + 4$ and its slight modified version in H^m with $m \geq [\frac{n}{2} + 3]$ instead of $m \geq \frac{3}{2}n + 7$. The energy estimates, this time for the energy

$$E_m[u](t) = \|\nabla u(t)\|_{H^m(\mathbb{R}^n)}^2 + \sum_{i=1}^{m+1} \|\partial_t^i u(t)\|_{H^{m+1-i}(\mathbb{R}^n)}^2, \quad (1.78)$$

where however we can recognize the influence of the L^2 -wave energy, allow us to evaluate the maximal existence time interval $[0, T^*]$. In the form of one of the main a priori estimates we recognize the same structure of a priori estimate for the KZK equation coming from the common second order of the nonlinearity for the Kuznetsov and the KZK equations:

$$E_m[u](t) \leq B E_m[u](0) + \varepsilon C_m \int_0^t (E_m[u](\tau))^{\frac{3}{2}} \, d\tau, \quad (1.79)$$

with constants $B > 0$, depending only on c , and $C_m > 0$, depending only on α, β, m , on the dimension n and eventually on c if $c^2 < 0.5$. In \mathbb{R}^2 and \mathbb{R}^3 the optimality of obtained estimations for the maximal existence time is ensured by the results of Alinhac [8]. In Ref. [8] a geometric blow-up for small data is proved for $\partial_t^2 u$ and Δu at a finite time of the same order as predicted by our a priori estimates (our estimates of the minimum existence time correspond to Alinhac's maximum existence time results). The principle to obtain the estimations for the maximal existence time is the same as in the case of the KZK or NPE equations in Eq. (1.67). But this time we need to use as John in [94] the group of linear

transformations preserving the linear wave equation $u_{tt} - \Delta u = 0$. The generators of this group (the derivatives with respect to group parameters taken at the identity) are called generalized derivatives. Thus, using the Klainerman estimate [115], we develop an a priori estimate for a new energy involving the generalized derivatives (see Section 3.2 [ARP-5]), which allows us to obtain the estimations of T^* . More precisely, in the inviscid case with $s = m \geq n + 2$ and for sufficiently small initial data we have the following estimates of the maximum existence time T^* :

1. $\liminf_{\varepsilon \rightarrow 0} \varepsilon^2 T^* > 0$ for $n = 2$,
2. $\liminf_{\varepsilon \rightarrow 0} \varepsilon \log(T^*) > 0$ for $n = 3$,
3. $T^* = +\infty$ for $n \geq 4$.

From the other hand, the blow-up of $\partial_t^2 u$ and Δu is also confirmed by the stability estimate of the same nature as for the KZK equation case (1.68) which evaluates the difference between of a regular solution u and less regular solution v of the Kuznetsov equation for $\nu = 0$ defined on $[0, T^*[$: there exist constants $C_1 > 0$ and $C_2 > 0$, independent on time, such that

$$(\|(u - v)_t\|_{L^2}^2 + \|\nabla(u - v)\|_{L^2}^2)(t) \leq C_1 \exp \left(C_2 \varepsilon \int_0^t \sup(\|u_{tt}\|_{L^\infty(\mathbb{R}^n)}, \|\Delta u\|_{L^\infty(\mathbb{R}^n)}) d\tau \right) \cdot (\|u_1 - v_1\|_{L^2}^2 + \|\nabla(u_0 - v_0)\|_{L^2}^2). \quad (1.80)$$

Consequently, if the maximal existence time interval is finite and limited by T^* , by Eq. (1.80), we have the divergence

$$\int_0^{T^*} (\|u_{tt}\|_{L^\infty(\mathbb{R}^n)} + \|\Delta u\|_{L^\infty(\mathbb{R}^n)}) d\tau = +\infty. \quad (1.81)$$

For $n \geq 4$ and $\nu = 0$, we also improve the results of John [94] and show the global existence for sufficiently small initial data $u_0 \in H^{m+1}(\mathbb{R}^n)$ and $u_1 \in H^m(\mathbb{R}^n)$ with $m \geq n + 2$ instead of $m \geq \frac{3}{2}n + 7$. The smallness of the initial data here directly ensures the hyperbolicity of the Kuznetsov equation for all time, *i.e.* it ensures that $1 - \alpha \varepsilon u_t$ is strictly positive and bounded for all time. The proof uses the generalized derivatives for the wave type equations [94] and a priori estimate of Klainerman [114, 115].

In the presence of the term Δu_t for the viscous case $\nu > 0$, the regularity of the higher-order time derivatives of u is different (to compare to the inviscid case), and the way to control the nonlinearities in the a priori estimates becomes different. As it was shown in [152], this dissipative term changes a finite speed of propagation of the wave equation to the infinite one. Indeed, the linear part of Eq. (1.69) can be viewed as two compositions of the heat operator $\partial_t - \Delta$ in the following way:

$$u_{tt} - c^2 \Delta u - \nu \varepsilon \Delta u_t = \partial_t(\partial_t u - \varepsilon \nu \Delta u) - c^2 \Delta u.$$

For the viscous case we prove the global in time well posedness results in \mathbb{R}^n for small enough initial data, the size of which we specify according to the theorem of abstract nonlinear analysis due to M.F. Sukhinin [158, 1.5 Cor., p. 368]:

Theorem 1.2.2 *Let $\nu > 0$, $n \in \mathbb{N}^*$, $s > \frac{n}{2}$ and $\mathbb{R}^+ = [0, +\infty[$.*

Let

$$X := H^2(\mathbb{R}^+; H^s(\mathbb{R}^n)) \cap H^1(\mathbb{R}^+; H^{s+2}(\mathbb{R}^n)),$$

the initial data

$$u_0 \in H^{s+2}(\mathbb{R}^n) \quad \text{and} \quad u_1 \in H^{s+1}(\mathbb{R}^n)$$

and $C_1 = O(1)$ be the minimal constant such that the solution u^* of the corresponding linear Cauchy problem (i.e. with $\alpha = \beta = 0$) satisfies

$$\|u^*\|_X \leq \frac{C_1}{\sqrt{\nu\epsilon}} (\|u_0\|_{H^{s+2}(\mathbb{R}^n)} + \|u_1\|_{H^{s+1}(\mathbb{R}^n)}).$$

Then there exists a strictly positive constant $r_* = O(1)$ (for the definition see Eq. (38) [ARP-5]) such that for all $r \in [0, r_*[$ and all initial data satisfying

$$\|u_0\|_{H^{s+2}(\mathbb{R}^n)} + \|u_1\|_{H^{s+1}(\mathbb{R}^n)} \leq \frac{\sqrt{\nu\epsilon}}{C_1} r, \quad (1.82)$$

there exists the unique solution $u \in X$ of the Cauchy problem for the Kuznetsov equation (1.69)–(1.70) and $\|u\|_X \leq 2r$.

To be able to apply [158, 1.5 Cor., p. 368] we need to have an isomorphism between the space of source terms and the solutions of the linear problem, which holds thanks to L^2 -maximal regularity (see [37] Definition 2.1) on \mathbb{R}^+ of the strongly damped wave equation [63].

Knowing the existence of a solution u of the Kuznetsov equation in X , it follows that

$$u \in C(\mathbb{R}^+; H^{s+2}(\mathbb{R}^n)) \quad \text{and} \quad u_t \in H^1(\mathbb{R}^+; H^s(\mathbb{R}^n)) \cap L^2(\mathbb{R}^+; H^{s+2}(\mathbb{R}^n)).$$

By Theorem III.4.10.2 in [10], it implies that $u_t \in C(\mathbb{R}^+; H^{s+1}(\mathbb{R}^n))$, which gives that

$$u \in C^1(\mathbb{R}^+; H^{s+1}(\mathbb{R}^n)) \cap C(\mathbb{R}^+; H^{s+2}(\mathbb{R}^n))$$

and, this time with the help of the Kuznetsov equation, $u_{tt} \in C(\mathbb{R}^+; H^{s-1}(\mathbb{R}^n))$. Consequently, in the viscous case the regularity of the time derivatives of the order greater than two of the solutions differs from the regularity, obtained for the inviscid case. Thus we have to consider estimates with different energies: the energy $E_{\frac{m}{2}}[u](t)$, defined by

$$E_{\frac{m}{2}}[u](t) = \|\nabla u(t)\|_{H^m(\mathbb{R}^n)}^2 + \sum_{i=1}^{\frac{m}{2}+1} \|\partial_t^i u(t)\|_{H^{m-2(i-1)}(\mathbb{R}^n)}^2, \quad (1.83)$$

and the energy

$$S_{\frac{m}{2}}[u](t) = \sum_{i=1}^{\frac{m}{2}+1} \|\nabla \partial_t^i u(t)\|_{H^{m-2(i-1)}(\mathbb{R}^n)}^2,$$

defined, as $E_{\frac{m}{2}}[u](t)$, for $m \in \mathbb{N}$ and m even, which respect to the obtained regularity of u and its derivatives. For $n \geq 3$ we establish an a priori estimate which gives also a sufficient condition of the existence of a global solution for a sufficiently small initial energy of the same order on ϵ as in Theorem 1.2.2. More precisely, for $n \geq 3$, $s = m \in \mathbb{N}$ be even and $m \geq [\frac{n}{2} + 3]$ we show that there exists a constant $C = O(1) > 0$, independent on time, such that for all initial data $u_0 \in H^{m+1}(\mathbb{R}^n)$ and $u_1 \in H^m(\mathbb{R}^n)$ satisfying

$$E_{\frac{m}{2}}[u](0) < \epsilon C,$$

there exists a unique $u \in C^0(\mathbb{R}^+; H^{m+1}(\mathbb{R}^n)) \cap C^i(\mathbb{R}^+; H^{m+2-2i}(\mathbb{R}^n))$, for $i = 1, \dots, \frac{m}{2} + 1$ with the bounded energy

$$\forall t \in \mathbb{R}^+, \quad E_{\frac{m}{2}}[u](t) \leq O\left(\frac{1}{\epsilon}\right) E_{\frac{m}{2}}[u](0) = O(1).$$

Remark 1.2.1 *If we take such u , we can show the sharp character of Theorem 1.2.2 by the following direct energy estimation approach (see also Appendix [ARP-5]). It is important to take attention that this time all physical coefficients of the Cauchy problem for the Kuznetsov equation are expressed to compare to the powers of ϵ , which for instance, means that $c^2 = O(\frac{1}{\epsilon})$. If we take it into account we obtain the same types of smallness of the initial energy for the inviscid case: $\sqrt{E_{m_0}[u](0)} \leq O(\sqrt{\epsilon})$.*

So, for $n \geq 3$, $m \geq \left[\frac{n}{2} + 3\right]$ if

$$\sqrt{E_{\frac{m}{2}}[u](0)} = \sqrt{\|\nabla u(0)\|_{H^m(\mathbb{R}^n)}^2 + \sum_{i=1}^{\frac{m}{2}+1} \|\partial_t^i u(0)\|_{H^{m-2(i-1)}(\mathbb{R}^n)}^2} \leq O(\sqrt{\epsilon}),$$

then it follows in a sufficient way that for $u_0 \in H^{m+1}(\mathbb{R}^n)$ and for $u_1 \in H^m(\mathbb{R}^n)$ it holds

$$\|\nabla u_0\|_{H^m(\mathbb{R}^n)} + \|u_1\|_{H^m(\mathbb{R}^n)} \leq O(\sqrt{\epsilon^{m+1}}), \quad (1.84)$$

which implies the existence of a unique global solution $u \in C^0(\mathbb{R}^+; H^{m+1}(\mathbb{R}^n)) \cap C^1(\mathbb{R}^+; H^m(\mathbb{R}^n))$ of problem (1.69)–(1.70) such that for all $t \in \mathbb{R}^+$

$$E_{\frac{m}{2}}[u](t) \leq O\left(\frac{1}{\epsilon}\right) E_{\frac{m}{2}}[u](0) = O(1).$$

Thus we see that by this approach the sufficient condition to have for all $t \geq 0$ $E_{\frac{m}{2}}[u](t)$ bounded by a constant of order zero on ϵ is given by Eq. (1.84) and depends on the smooth properties of the initial data (more they are regular, more they should be small). Hence, it is much more restrictive to compare to (1.82).

Moreover, (see Theorem 4.4 [ARP-5]) we also have an analogue of Eq. (1.66) as for solutions of the KZK equation:

$$\frac{d}{dt}E(t) + \sqrt{2}\varepsilon S_{\frac{m}{2}}[u](t) \left(\sqrt{2}\nu - C_m \max(\alpha, \beta) \sqrt{E(t)} \right) \leq 0,$$

where by V is denoted the set of all multi-indexes $A = (A_0, A_1, \dots, A_n)$ with $|A| - A_0 \leq m - 2A_0$ and

$$E(t) = \sum_{A \in V} \int_{\mathbb{R}^n} (1 - \alpha \varepsilon u_t) (D^A u_t)^2 + c^2 (\nabla D^A u)^2(t, x) \, dx.$$

The same well posedness results hold in $(\mathbb{R}/L\mathbb{Z}) \times \mathbb{R}^{n-1}$ for $n \geq 2$ (with a periodicity and mean value zero on one variable) thanks to the Poincaré inequality (1.61).

We finish by noticing that the hyperbolicity condition (1.72) is automatically satisfied if we require conditions (1.82) and (1.2.2).

1.3 Approximation results

Once we know the well-posedness properties of all introduced models, we can validate the approximations of the compressible isentropic Navier-Stokes system [ARP-2] by the different models: by the Kuznetsov, the KZK, and the NPE equations. We also do the same for the Euler system in the inviscid case and justify the approximations between the Kuznetsov equation [ARP-3] and the Westervelt, the KZK and the NPE equations.

As we have seen previously, the main difference between the viscous and the inviscid cases is the time existence and regularity of the solutions. Typically in the inviscid case, the solutions of the models and also of the Euler system itself (actually strong solutions), due to the nonlinearity, can provide shock front formations at a finite time [7, 153, 176], [ARP-5], [ARP-13]. Thus, they are only locally well-posed, while in the viscous media, all approximative models are globally well-posed for small enough initial data [ARP-5], [ARP-13]. These existence properties of solutions for the viscous and the inviscid cases may also imply the difference in the definition of the domain where the approximations hold: for example [ARP-12], for the approximation between the KZK equation and the Navier-Stokes system, the approximation domain is a half-space, but for the analogous inviscid case of the KZK and the Euler system, it is a cone (see the summarizing Table 1.1).

To keep a physical sense of the approximation problems, we consider especially the two or three-dimensional cases, *i.e.* \mathbb{R}^n with $n = 2$ or 3 .

1.3.1 Approximations of the Navier-Stokes and Euler systems [ARP-2]

In what follows we denote by \mathbf{U}_ε a solution of the “exact” system - the Navier-Stokes/Euler system -

$$Exact(\mathbf{U}_\varepsilon) = 0$$

and by $\overline{\mathbf{U}}_\varepsilon$ an approximate solution, constructed by the derivation *ansatz* from a regular solution of one of the approximate models (typically of the Kuznetsov, the KZK or the NPE equations). In this case, the approximate solution $\overline{\mathbf{U}}_\varepsilon$ is a function which solves the Navier-Stokes/Euler system up to ε^3 terms, denoted by $\varepsilon^3 \mathbf{R}$:

$$Approx(\overline{\mathbf{U}}_\varepsilon) = Exact(\overline{\mathbf{U}}_\varepsilon) - \varepsilon^3 \mathbf{R} = 0.$$

To have the remainder term $\mathbf{R} \in C([0, T], L^2(\Omega))$ we ensure that

$$Exact(\overline{\mathbf{U}}_\varepsilon) \in C([0, T], L^2(\Omega)),$$

i.e. we need a sufficiently regular solution $\overline{\mathbf{U}}_\varepsilon$. The minimal regularity of the initial data to have a such $\overline{\mathbf{U}}_\varepsilon$ is given in the last line of Table 1.1.

Choosing for the exact system the same initial-boundary data found by the *ansatz* for $\overline{\mathbf{U}}_\varepsilon$ (the regular case) or the initial data taken in their small L^2 -neighborhood, *i.e.*

$$\|\mathbf{U}_\varepsilon(0) - \overline{\mathbf{U}}_\varepsilon(0)\|_{L^2(\Omega)} \leq \delta \leq \varepsilon, \quad (1.85)$$

with $\mathbf{U}_\varepsilon(0)$ not necessarily smooth, we prove the existence of constants $C > 0$ and $K > 0$ independent of ε , δ and the time t such that

$$\text{for all } 0 \leq t \leq \frac{C}{\varepsilon} \quad \|(\mathbf{U}_\varepsilon - \overline{\mathbf{U}}_\varepsilon)(t)\|_{L^2(\Omega)}^2 \leq K(\varepsilon^3 t + \delta^2) e^{K\varepsilon t} \leq 9\varepsilon^2 \quad (1.86)$$

with Ω a domain where the both solutions \mathbf{U}_ε and $\overline{\mathbf{U}}_\varepsilon$ exist.

To define the minimal regularity property of \mathbf{U}_ε for which stability estimate (1.86) holds, we introduce admissible weak solutions of a bounded energy using the entropy $\eta(\mathbf{U}_\varepsilon)$ of the Euler system which is known [43] to be convex with $\eta''(\mathbf{U}_\varepsilon)$ strictly positive defined. More precisely, we introduce

Definition 1.3.1 *The function $\mathbf{U}_\varepsilon = (\rho_\varepsilon, \rho_\varepsilon \mathbf{v}_\varepsilon)$ is called an admissible weak solution of a bounded energy of the Cauchy problem for the Navier-Stokes system (1.4)–(1.6) if it satisfies the following properties:*

1. *The pair $(\rho_\varepsilon, \mathbf{v}_\varepsilon)$ is a weak solution of the Cauchy problem for the Navier-Stokes system (1.4)–(1.6) (in the distributional sense).*
2. *The function \mathbf{U}_ε satisfies in the sense of distributions (see Ref. [43, p.52])*

$$\partial_t \eta(\mathbf{U}_\varepsilon) + \nabla \cdot \mathbf{q}(\mathbf{U}_\varepsilon) - \varepsilon \nu \mathbf{v}_\varepsilon \Delta \mathbf{v}_\varepsilon \leq 0, \text{ where } \mathbf{q}(\mathbf{U}_\varepsilon) = \mathbf{v}_\varepsilon (\eta(\mathbf{U}_\varepsilon) + p(\rho_\varepsilon)), \quad (1.87)$$

or equivalently, for any positive test function ψ in $\mathcal{D}(\mathbb{R}^n \times [0, \infty[)$ the function \mathbf{U}_ε satisfies

$$\begin{aligned} & \int_0^T \int_{\mathbb{R}^n} \left(\partial_t \psi \eta(\mathbf{U}_\varepsilon) + \nabla \psi \cdot \mathbf{q}(\mathbf{U}_\varepsilon) + \varepsilon \nu |\nabla \cdot \mathbf{v}_\varepsilon|^2 \psi + \varepsilon \nu \mathbf{v}_\varepsilon \cdot [\nabla \cdot \mathbf{v}_\varepsilon \nabla \psi] \right) dx dt \\ & + \int_{\mathbb{R}^n} \psi(x, 0) \eta(\mathbf{U}_\varepsilon(0)) dx \geq 0. \end{aligned}$$

3. *The function \mathbf{U}_ε satisfies the equality (with the notation $\mathbf{v}_\varepsilon = (v_1, \dots, v_n)$)*

$$\begin{aligned} & - \int_{\mathbb{R}^n} \frac{\mathbf{U}_\varepsilon^2(t)}{2} dx + \int_0^t \int_{\mathbb{R}^n} \left(\sum_{i=1}^n \mathbf{G}_i(\mathbf{U}_\varepsilon) \partial_{x_i} \mathbf{U}_\varepsilon - \varepsilon \nu \nabla(\rho_\varepsilon v_i) \cdot \nabla v_i \right) dx ds \\ & + \int_{\mathbb{R}^n} \frac{\mathbf{U}_\varepsilon^2(0)}{2} dx = 0, \end{aligned}$$

where with the notation \mathbf{e}_i for the vector number i of the canonical basis of \mathbb{R}^n

$$\mathbf{G}_i(\mathbf{U}_\varepsilon) = \begin{bmatrix} \rho_\varepsilon v_i \\ \rho_\varepsilon v_i \mathbf{v}_\varepsilon + p(\rho_\varepsilon) \mathbf{e}_i \end{bmatrix}.$$

Let us notice that any classical solution of (1.4)–(1.6) satisfies the entropy condition (1.87) by the equality and obviously it is sufficient regular to perform the integration by parts resulting in the relation of point 3. For existence results of global weak solutions of the Cauchy problem for the Navier-Stokes system (1.4)–(1.6) with sufficiently small initial data around the constant state $(\rho_0, 0)$ (actually, $\rho_0 - \rho(0)$ is small in L^∞ , $\mathbf{v}(0)$ is small in L^2 and bounded in L^{2^n}) and with the pressure $p(\rho) = K\rho^\gamma$ with $\gamma \geq 1$, we refer to results of D. Hoff [87, 88]. Therefore, from [87] it follows that a weak solution of the isentropic compressible Navier-Stokes system (1.4)–(1.6) is also an admissible weak solution of bounded energy in the sense of Definition 1.3.1. But in [ARP-2] we only consider the question of the validity of the stability estimate (1.86) for initial data closed to $\overline{\mathbf{U}}_\varepsilon(0)$ in L^2 norm (thus for initial data not necessarily satisfying Hoff's assumptions). We do not consider the existence question of an admissible weak solution of bounded energy of the Cauchy problem for the Navier-Stokes system. Following the ideas of [ARP-12] we prove (1.86)

for different approximations: the Navier-Stokes system by the Kuznetsov equation, by the KZK equation, and by the NPE equation.

As we have mentioned, in the viscous case all approximative models have a global unique classical solution for small enough initial data in their corresponding approximative domains (Ω varies for different models, see Table 1.1: it is equal to \mathbb{R}^n , $\mathbb{T}_{x_1} \times \mathbb{R}^{n-1}$ and $\mathbb{R}_+ \times \mathbb{R}^{n-1}$ for the Kuznetsov equation, the NPE equation and the KZK equation respectively). If we take regular initial data $\mathbf{U}_\varepsilon(0) = \overline{\mathbf{U}}_\varepsilon(0)$, the same thing is true for the Navier-Stokes system with the same regularity for the solutions [130]. But in the case of the half-space for the approximation between the Navier-Stokes system and the KZK equation, firstly considered in [ARP-12], when, due to the periodic in time boundary conditions, coming from the initial conditions for the KZK equation, we prove the well-posedness for all finite time. To obtain it we use [ARP-12] Theorem 5.5 and improve its proof for the new *ansatz*.

For the inviscid case, we verify that the existence time of (strong) solutions of all models is not less than $O(\frac{1}{\varepsilon})$ and estimate (1.86) still holds. We use here the known blow-up results for the Euler system [7, 153, 154, 155, 156, 176]. Once again, to obtain estimate (1.86) we don't need the regularity of the classical solution of the Euler system, it can be one of solutions in the sense of Luo and al. [127] for the Euler system satisfying the admissible conditions given in Definition 1.3.1 (see also Ref. [43] p.52 and [ARP-12] Definition 5.9). But this time there exist infinitely many weak solutions of the Euler system, so there is no any sense to consider an regular approximation of one of them.

1.3.2 Approximations of the Kuznetsov equation [ARP-3]

For the approximation framework for the solutions of the Kuznetsov and the KZK equations we study two cases. The first case considers the purely time periodic boundary problem in the *ansatz* variables (z, τ, y) moving with the wave. In this case the only viscous medium can be considered as the condition to be periodic in time is not compatible with shock formations providing the loss of the regularity which may occur in the inviscid medium (see Thm. 1.3 [ARP-13]). To be able to consider this approximation we prove the well posedness of the periodic in time Dirichlet boundary valued problem for the Kuznetsov equation in the half space $\mathbb{R}^+ \times \mathbb{R}^{n-1}$ for small enough boundary data. In this case the boundary condition is considered as the initial condition of the corresponding Cauchy problem in \mathbb{R}^n . The proof is based as previously in [ARP-5] on the maximal regularity result for the corresponding linear problem and on the application of a result of the nonlinear functional analysis from [158, 1.5. Cor., p. 368]. We also applied it to prove the well posedness needed for the second approximation case described by the initial boundary valued problem for the Kuznetsov equation in the half space, once again combining with the maximal regularity result for the linear problem. The second case approximation case studies the initial boundary-value problem for the Kuznetsov equation in the initial variables (t, x_1, x') with data coming from the solution of the KZK equation. This time we have the approximation results for the viscous and inviscid cases, as in the approximations by the solutions of the NPE and the Westervelt equations.

Denoting by u a solution of the “exact” problem for the Kuznetsov equation $Exact(u) = 0$ and by \bar{u} an approximate solution, constructed by the derivation *ansatz* from a regular solution of one of the approximate models (for instance of the KZK or of the NPE equations),

Table 1.1 – Approximation results for models derived from Navier-Stokes and Euler systems

	Kuznetsov		KZK		NPE	
	Navier-Stokes	Euler	Navier-Stokes	Euler	Navier-Stokes	Euler
<i>Ansatz</i>	$\rho_\varepsilon = \rho_0 + \varepsilon \rho_1 + \varepsilon^2 \rho_2,$ $\mathbf{v}_\varepsilon = -\varepsilon \nabla u,$ $\rho_1 = \frac{\rho_0}{c^2} \partial_t u,$ ρ_2 from (1.23)		paraxial approximation $u = \Phi(t - \frac{x_1}{c}, \varepsilon x_1, \sqrt{\varepsilon} \mathbf{x}')$ $\rho_\varepsilon = \rho_0 + \varepsilon I + \varepsilon^2 J,$ \mathbf{v}_ε from (1.37), $I = \frac{\rho_0}{c^2} \partial_\tau \Phi,$ J from (1.40)		paraxial approximation $u = \Psi(\varepsilon t, x_1 - ct, \sqrt{\varepsilon} \mathbf{x}')$ $\rho_\varepsilon = \rho_0 + \varepsilon \xi + \varepsilon^2 \chi,$ \mathbf{v}_ε from (1.49), $\xi = -\frac{\rho_0}{c} \partial_z \Psi,$ χ from (1.52)	
Models	$\partial_t^2 u - c^2 \Delta u =$ $\varepsilon \partial_t \left((\nabla u)^2 + \frac{\gamma-1}{2c^2} (\partial_t u)^2 \right.$ $\left. + \frac{\nu}{\rho_0} \Delta u \right)$		$c \partial_{\tau z}^2 I - \frac{(\gamma+1)}{4\rho_0} \partial_\tau^2 I^2$ $-\frac{\nu}{2c^2 \rho_0} \partial_\tau^3 I - \frac{c^2}{2} \Delta_y I = 0$		$\partial_{\tau z}^2 \xi + \frac{(\gamma+1)c}{4\rho_0} \partial_z^2 (\xi^2)$ $-\frac{\nu}{2\rho_0} \partial_z^3 \xi + \frac{c}{2} \Delta_y \xi = 0$	
Approximation Order	$O(\varepsilon^3)$					
Domain Ω	\mathbb{R}^3		the half space $\{x_1 > 0, x' \in \mathbb{R}^{n-1}\}$	the cone $\{ x_1 < \frac{R}{\varepsilon} - ct\}$ $\times \mathbb{R}_{x'}^{n-1}$	$\mathbb{T}_{x_1} \times \mathbb{R}^2$	
Approximation	$\ U_\varepsilon - \overline{U}_\varepsilon\ _{L^2} \leq \varepsilon$ for $t \leq \frac{T}{\varepsilon}$					
Initial data regularity	$u_0 \in H^5(\Omega)$ $u_1 \in H^4(\Omega)$	$u_0 \in H^4(\Omega)$ $u_1 \in H^3(\Omega)$	$I_0 \in H^{10}(\Omega)$	$I_0 \in H^{10}(\Omega)$	$\xi_0 \in H^5(\Omega)$	$\xi_0 \in H^5(\Omega)$
Data regularity for remainder boundness	$u_0 \in H^{s+2}(\Omega)$ $u_1 \in H^{s+1}(\Omega)$ $s > \frac{n}{2}$	$u_0 \in H^{s+2}(\Omega)$ $u_1 \in H^{s+1}(\Omega)$ $s > \frac{n}{2}$	$I_0 \in H^8(\Omega)$	$I_0 \in H^6(\Omega)$	$\xi_0 \in H^4(\Omega)$	$\xi_0 \in H^4(\Omega)$

i.e. \bar{u} is a function which solves the Kuznetsov equation up to ϵ terms, denoted by ϵR :

$$Approx(\bar{u}) = Exact(\bar{u}) - \epsilon R = 0.$$

In the approximation between the solutions of the Kuznetsov equation and of the Westervelt equation the remainder term appears with the size ϵ^2 (it is natural since both models contain terms of order ϵ^0 and ϵ).

We can summarize the obtained approximation results of the Kuznetsov equation in the following way: if, once again, u is a solution of the Kuznetsov equation and \bar{u} is a solution of the NPE or of the KZK (for the initial boundary value problem) or of the Westervelt equations found for rather closed initial data

$$\|\nabla_{t,\mathbf{x}}(u(0) - \bar{u}(0))\|_{L^2(\Omega)} \leq \delta \leq \epsilon,$$

then there exist constants $K, C_1, C_2, C > 0$ independent of ϵ, δ and on time, such that for all $t \leq \frac{C}{\epsilon}$ it holds

$$\|\nabla_{t,\mathbf{x}}(u - \bar{u})\|_{L^2(\Omega)} \leq C_1(\epsilon^2 t + \delta)e^{C_2 \epsilon t} \leq K\epsilon.$$

To obtain the last estimate we use stability estimate (1.80). For a more detailed comparison between different models we include the main points of our results to the comparative Table 1.2.

In Table 1.2 the line named “Initial data regularity” gives the information about the regularity of the initial data for the approximate model, which ensure the same regularity of the solutions of an approximate model and of the solution of the Kuznetsov equation, taken with the same initial data $u(0) = \bar{u}(0)$, coming from the corresponding *ansatz*.

As in the case of the approximations of the Navier-Stokes/Euler system, to have the remainder term $R \in C([0, T], L^2(\Omega))$ we ensure that $Exact(\bar{u}) \in C([0, T], L^2(\Omega))$, *i.e.* we need a sufficiently regular solution \bar{u} . The minimal regularity of the initial data to have a such \bar{u} is given in Table 1.2 in the last line named “Data regularity for remainder boundness”.

1.4 Further developments

It is obvious that the list of nonlinear models studied in this Chapter is not at all complete. For instance there are higher order in time equations and other more complicated models to describe the wave propagation in a fluid with bubbles [78]. So we can ask the same questions for them.

A first simple generalization of the well posedness results for the Kuznetsov equation is to consider it with non constant, for instance bounded, piecewise discontinuous coefficients. This kind of generalization could be very useful in order to develop the inverse problems for this equation, *i.e.* for the problem of reconstructing one of its coefficients, for example the nonlinearity coefficients α and β , since usually in the ultrasound imaging the properties of the media are not really known.

Going closer to the physics of the wave propagation there are a lot of possibilities to take into account different parameters. For instance, as it was mentioned, the models of the Kuznetsov, the KZK and the NPE equations are derived in the assumption that a contact with the boundary has no influence. So there is a question to study how models could be

Table 1.2 – Approximation results for models derived from the Kuznetsov equation

	KZK		NPE	Westervelt	
	periodic boundary condition problem	initial boundary value problem	viscous and inviscid case	viscous case	inviscid case
Derivation	paraxial approximation $u = \Phi(t - \frac{x_1}{c}, \varepsilon x_1, \sqrt{\varepsilon} \mathbf{x}')$		paraxial approximation $u = \Psi(\varepsilon t, x_1 - ct, \sqrt{\varepsilon} \mathbf{x}')$	$\Pi = u + \frac{1}{c^2} \varepsilon u \partial_t u$	
Approximation domain	the half space $\{x_1 > 0, x' \in \mathbb{R}^{n-1}\}$		$\mathbb{T}_{x_1} \times \mathbb{R}^2$	\mathbb{R}^n	
Approximation order	$O(\varepsilon)$		$O(\varepsilon)$	$O(\varepsilon^2)$	
Estimation	$\ I - I_{approx}\ _{L^2(\mathbb{T}_t \times \mathbb{R}^{n-1})} \leq \varepsilon$ $z \leq K$	$\ (u - \bar{u})_t(t)\ _{L^2}$ $+\ \nabla(u - \bar{u})(t)\ _{L^2}$ $\leq K\varepsilon.$ $t < \frac{T}{\varepsilon}$	$\ (u - \bar{u})_t(t)\ _{L^2}$ $+\ \nabla(u - \bar{u})(t)\ _{L^2}$ $\leq K\varepsilon$ $t < \frac{T}{\varepsilon}$	$\ (u - \bar{u})_t(t)\ _{L^2}$ $+\ \nabla(u - \bar{u})(t)\ _{L^2}$ $\leq K\varepsilon$ $t < \frac{T}{\varepsilon}$	
Initial data regularity	$I_0 \in H^{s+\frac{3}{2}}(\mathbb{T}_t \times \mathbb{R}_{x'}^{n-1})$ for $s > \max(\frac{n}{2}, 2)$	$I_0 \in H^s(\mathbb{T}_t \times \mathbb{R}_{x'}^{n-1})$ for $\lceil \frac{s}{2} \rceil > \frac{n}{2} + 2$	$\xi_0 \in H^{s+2}(\mathbb{T}_{x_1} \times \mathbb{R}_{x'}^{n-1})$ for $s > \frac{n}{2} + 1$	$u_0 \in H^{s+3}(\mathbb{R}^n)$ $u_1 \in H^{s+3}(\mathbb{R}^3)$ for $s > \frac{n}{2}$	$u_0 \in H^{s+3}(\mathbb{R}^n)$ $u_1 \in H^{s+2}(\mathbb{R}^3)$ for $s > \frac{n}{2}$
Data regularity for remainder boundness	$I_0 \in H^{s+\frac{3}{2}}(\mathbb{T}_t \times \mathbb{R}_{x'}^{n-1})$ for $s > \max(\frac{n}{2}, 2)$	$I_0 \in H^6(\mathbb{T}_t \times \mathbb{R}_{x'}^{n-1})$ for $n = 2, 3,$ $I_0 \in H^s(\mathbb{T}_t \times \mathbb{R}_{x'}^{n-1})$ for $\lceil \frac{s}{2} \rceil > \frac{n}{2} + 1, n \geq 4$	$\xi_0 \in H^4(\mathbb{T}_{x_1} \times \mathbb{R}_{x'}^{n-1})$ for $n = 2, 3.$ $\xi_0 \in H^s(\mathbb{T}_{x_1} \times \mathbb{R}_{x'}^{n-1})$ for $s > \frac{n}{2} + 2, n \geq 4.$	$u_0 \in H^{s+3}(\mathbb{R}^n)$ $u_1 \in H^{s+3}(\mathbb{R}^n)$ for $s > \frac{n}{2}$	$u_0 \in H^{s+3}(\mathbb{R}^n)$ $u_1 \in H^{s+2}(\mathbb{R}^n)$ for $s > \frac{n}{2}$

modified by the boundary effects (see for instance [42] for the Blackstock-Lesser-Seebass-Crighton equation).

Let us also mention that in general the viscosity coefficient in our models should depend on a frequency ω of a wave [1, 51]. Formally, when a wave propagates through a medium, its amplitude exponentially decreases while the distance grows, proportionally to the law $e^{-\omega^{\gamma_{att}} z}$ with $0 \leq \gamma_{att} \leq 2$ [34, 35]. In this Chapter we have considered only the cases $\gamma_{att} = 0$ when there is no dissipation and the limit case $\gamma_{att} = 2$. Indeed, the KZK equation with $\nu > 0$ corresponds to the quadratic case $\gamma_{att} = 2$ with the dissipative term equal to $\partial_\tau^{\gamma_{att}+1} I$. The Kuznetsov and Westervelt equations involve the dissipative term $\partial_t(-\Delta)^{\frac{\gamma_{att}}{2}} u$, which for $0 < \gamma_{att} < 2$ becomes a term with a fractional Laplacian. Experimentally, it is known that the dissipation with $\gamma_{att} = 2$ corresponds to the wave propagation in pure homogeneous liquids and gases, but [34] not for example to the propagation in the blood (as one of the liquids in the HIFU technique) which is a suspension in which it takes a fractional value $0 < \gamma_{att} < 2$. Thus it would be more realistic in the medical applications to consider the Kuznetsov, the Westervelt and the KZK equations for the fractional values of γ_{att} . For instance, by [159, 143] the fractional Westervelt equation with $1 \leq \gamma_{att} \leq 1.7$ is appropriate for the ultrasound propagation in human tissues. Therefore, this makes of interest the questions about the systematic derivation and the approximation of the fractional Navier-Stokes system by all other fractional models, as well as concerning their well posedness, by starting to analyze the existing preliminary results [89, 134]. However, this kind of improvement does not seem to be important in the case of the NPE equation which is considered for the ultrasound propagation in the ocean, knowing that the viscosity of the water has an exceptionally wide diapason of frequencies where it is almost constant [144].

Another possible generalization for the wave propagation in the human body where there are a lot of flow motion, for example of the blood, can come from works considering the moving media [19, 38]. In this case we could try the following type of *ansatz*

$$\begin{aligned} \rho(x, t) &= \rho_0 + \epsilon \rho_{1,a}(x, t) + \epsilon^2 \rho_{2,a}(x, t) + \mathcal{M} \rho_{1,m}(x) + \mathcal{M}^2 \rho_{2,m}(x), \\ \vec{u}(x, t) &= -\epsilon \nabla \phi_a(x_1, x', t) - \mathcal{M}^\ell \nabla \phi_m(x_1, x'). \end{aligned}$$

Here subscripts a and m denote acoustic and medium components respectively, ℓ is an unknown integer power to be defined.

Part II

Treatment of PDEs on domains with fractals boundaries

Chapter 2

Functional analysis for the weak well-posedness of PDEs on domains with irregular boundaries

Introduction

My interest in fractals has started thanks to my post-doctoral collaboration with Bernard Sapoval, Marcel Filoche, and Denis Grebenkov, the physicists working in the area of fractal interfaces for the wave or the heat propagation and the Laplacian transport. I have found a lot of interesting empirical, in my opinion, ideas on different phenomena due to the irregularity of the boundary or an interface between two media. For instance, the famous observation of the localization of the eigenfunctions of the $-\Delta$ in some zones near the Neumann boundary of an irregular or prefractal boundary. In particular, in Section 2.3 we define the operator Dirichlet-to-Neumann on d -set boundaries according to [ARP-4], the work which primary motivation was to justify the physical articles [60, 72, 73, 74]. I give more examples in Chapter 3. Therefore, up to now, my general goal is to justify and to study the physical problems from the mathematical point of view. But, as mentioned in the introduction, it is not possible to consider the theoretical questions for the PDEs on the domains with irregular and fractal boundaries without developing a suitable framework of the functional analysis. Hence, I develop it in this chapter in Sections 2.1 and 2.2 using [ARP-4], [ARP-6], [BookChap], [PrepWestMixed] and show its application for the weak well-posedness of the mixed boundary problem for the Westervelt equation in Section 2.4. The consideration of this example also allows discussing the main differences in the regularity properties of solutions found in domains with regular and irregular boundaries. I finish with a brief comment of the results for the Dirichlet boundary and the Robin boundary problems for the Westervelt equation considered in [PrepWestDir] and in the Ph.D. thesis of A. Dekkers.

We give this introductive description of this chapter in French before proceeding to its content.

Introduction en français

Mon intérêt pour les fractales a commencé grâce à ma collaboration post-doctorale avec Bernard Sapoval, Marcel Filoche et Denis Grebenkov, les physiciens travaillant dans le domaine des interfaces fractales pour la propagation des ondes ou de la chaleur et pour le transport du laplacien. J'ai trouvé beaucoup d'idées intéressantes, empiriques à mon avis, sur les différents phénomènes dus à l'irrégularité de la frontière ou d'une interface entre deux milieux. Par exemple, la fameuse observation de la localisation des fonctions propres du $-\Delta$ dans certaines zones proches de la frontière de Neumann d'une forme irrégulière ou préfractale. En particulier, dans la section 2.3 nous définissons l'opérateur Dirichlet-to-Neumann sur des bords qui sont d -ensembles selon [ARP-4], le travail dont la motivation principale était de justifier les articles physiques [60, 72, 73, 74]. Je donne plus d'exemples au chapitre 3. Par conséquent, jusqu'à présent, mon objectif général est de justifier et d'étudier les problèmes physiques d'un point de vue mathématique. Mais, comme cela a été mentionné dans l'introduction générale, il n'est pas possible de considérer les questions théoriques pour les EDPs sur les domaines à frontières irrégulières et fractales sans le développement d'un cadre adapté de l'analyse fonctionnelle. Par conséquent, je le développe dans ce chapitre dans les sections 2.1 et 2.2 en utilisant [ARP-4], [ARP-6], [BookChap], [PrepWestMixed] et montre son application pour l'étude du caractère bien-posé (faiblement) du problème aux limites mixtes pour l'équation de Westervelt dans la section 2.4. La prise en compte de cet exemple permet également de discuter des principales différences dans les propriétés de régularité des solutions trouvées dans les domaines à frontières régulières et irrégulières. Je termine par un bref commentaire des résultats pour les cas d'une condition de Dirichlet et également de Robin sur tout le bord toujours dans le cadre de l'équation de Westervelt considérés dans [PrepWestDir] issues de la thèse d'A. Dekkers.

2.1 Framework of Sobolev admissible domains

2.1.1 Introduction and Lipschitz boundary framework

From the theory of the partial differential equations, it is known that the irregularity of the boundary of the considered domain can be a serious obstacle even for the proof of the existence of a weak solution. In this chapter, we are interested in the question which is the worst boundary (the most irregular) or a class of boundaries for which we still have the weak well-posedness firstly of the elliptic problems and secondly of the Westervelt equation introduced in the previous chapter.

In the past, mathematics has been concerned largely with regular domains. Firstly domains with fractal boundaries like, for example, the Von Koch snowflake have mainly been considered as "pathological" and used only to produce counterexamples. Nevertheless, there has been a change of attitude as mathematicians and physicists have discovered that such Von Koch-like structures appear in nature as in the famous example [128] of the coast of Britain. There are many other appearances of fractal domains in mathematics and physics, including the following papers most relevant to my work: [151, 53, 85, 122, 166, 79, 31, 30, 86, 85]. To be able to solve mixed boundary valued problems of partial differential equations in domains with nonsmooth or fractal boundaries, it is important to describe a functional framework in which it is possible to consider the weak-well posedness of elliptic equations,

in particular of the simplest one, the Poisson equation:

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_{D,\Omega}, \\ \frac{\partial u}{\partial n} = 0 & \text{on } \Gamma_{N,\Omega}, \\ \frac{\partial u}{\partial n} + au = 0 & \text{on } \Gamma_{R,\Omega}, \end{cases} \quad (2.1)$$

with $\partial\Omega = \Gamma_{D,\Omega} \cup \Gamma_{N,\Omega} \cup \Gamma_{R,\Omega}$.

Thus the general approach is to start to find the weak formulation of this problem. Hence, it is important to be able to integrate by parts and to work with the trace operator on $\partial\Omega$. For at least Lipschitz $\partial\Omega$ it is classical and well-known (for sufficiently smooth boundary see Raviart-Thomas [149], for the Lipschitz case, see Marschall [129] and [75, 141]).

If $\partial\Omega$ is Lipschitz, then the normal unit vector ν to the boundary $\partial\Omega$ exists almost everywhere, the trace operator $\text{Tr} : H^1(\Omega) \rightarrow H^{\frac{1}{2}}(\partial\Omega)$ is linear continuous and surjective [125, 129, 75, 141] with a linear continuous right inverse, *i.e.* the extension operator

$$E : H^{\frac{1}{2}}(\partial\Omega) \rightarrow H^1(\Omega) \quad \text{is such that } \text{Tr}(E(u)) = u.$$

Moreover, for $u, v \in H^1(\Omega)$ with $\Delta u \in L^2(\Omega)$ it holds the usual Green formula in the following sense

$$\int_{\Omega} \nabla u \nabla v dx = \left\langle \frac{\partial u}{\partial \nu}, \text{Tr} v \right\rangle_{(H^{\frac{1}{2}}(\partial\Omega))', H^{\frac{1}{2}}(\partial\Omega)} - \int_{\Omega} \nabla v \nabla u dx. \quad (2.2)$$

This formula understands the existence of the normal derivative of u on $\partial\Omega$ as the existence of a linear continuous form on $H^{\frac{1}{2}}(\partial\Omega)$, where $H^{\frac{1}{2}}(\partial\Omega)$ is the image of $H^1(\Omega)$ for a Lipschitz domain Ω by the trace operator. The dual space $(H^{\frac{1}{2}}(\partial\Omega))'$ is usually denoted by $H^{-\frac{1}{2}}(\partial\Omega)$.

In this weak way for Lipschitz domains it is also possible to define the operator of divergence for vector valued functions (see for instance Theorem 2.5 § 2 [66]) or simply the usual integration by parts for all u and v from $H^1(\Omega)$ in the following weak sense

$$\langle uv_i, v \rangle_{(H^{-\frac{1}{2}}(\partial\Omega), H^{\frac{1}{2}}(\partial\Omega))} := \int_{\Omega} \frac{\partial u}{\partial x_i} v dx + \int_{\Omega} u \frac{\partial v}{\partial x_i} dx \quad i = 1, \dots, n, \quad (2.3)$$

where by uv_i is denoted the linear continuous functional on $H^{\frac{1}{2}}(\partial\Omega)$.

2.1.2 Sobolev extension domains

Thanks to the classical results of Calderon-Stein [27, 157] it is known that every Lipschitz domain Ω is an extension domain for the Sobolev space $W_p^k(\Omega)$ with $1 \leq p \leq \infty$, $k \in \mathbb{N}^*$, which means

Definition 2.1.1 (W_p^k -extension domains) A domain $\Omega \subset \mathbb{R}^n$ is called a W_p^k -extension domain ($k \in \mathbb{N}^*$) if there exists a bounded linear extension operator $E : W_p^k(\Omega) \rightarrow W_p^k(\mathbb{R}^n)$. This means that for all $u \in W_p^k(\Omega)$ there exists a $v = Eu \in W_p^k(\mathbb{R}^n)$ with $v|_{\Omega} = u$ and it holds

$$\|v\|_{W_p^k(\mathbb{R}^n)} \leq C \|u\|_{W_p^k(\Omega)} \quad \text{with a constant } C > 0.$$

It is known [96] that the results of Calderon and Stein [27, 157] about Sobolev extension domains for domains with Lipschitz boundaries can be improved by the class of (ϵ, δ) -domains, or locally uniform domains, which in the bounded case are simply called uniform domains [84].

Definition 2.1.2 ((ϵ, δ)-domain [96]) *An open connected subset Ω of \mathbb{R}^n is an (ϵ, δ) -domain, $\epsilon > 0$, $0 < \delta \leq \infty$, if whenever $(x, y) \in \Omega^2$ and $|x - y| < \delta$, there is a rectifiable arc $\gamma \subset \Omega$ with length $\ell(\gamma)$ joining x to y and satisfying*

1. $\ell(\gamma) \leq \frac{|x-y|}{\epsilon}$ and
2. $d(z, \partial\Omega) \geq \epsilon|x - z| \frac{|y-z|}{|x-y|}$ for $z \in \gamma$.

The (ϵ, δ) -domains give the optimal class of Sobolev extension domains in \mathbb{R}^2 (see [96] Theorem 3), but not in \mathbb{R}^3 , where there exist Sobolev extension domains which are not (ϵ, δ) -domains. Recently, this question was solved in terms of n -sets by [80] for $W^{k,p}$ -extension domains with $1 < p < \infty$ and $k \in \mathbb{N}$ for domains in \mathbb{R}^n . To be able to use it as in [ARP-4] we need to introduce the notion of d -sets:

Definition 2.1.3 (Ahlfors d -regular set or d -set [100, 101, 172, 164]) *Let F be a Borel non-empty subset of \mathbb{R}^n . The set F is called a d -set ($0 < d \leq n$) if there exists a d -measure μ on F , i.e. a positive Borel measure with support F ($\text{supp } \mu = F$) such that there exist constants $c_1, c_2 > 0$,*

$$c_1 r^d \leq \mu(F \cap \overline{B_r(x)}) \leq c_2 r^d, \quad \text{for } \forall x \in F, 0 < r \leq 1,$$

where $B_r(x) \subset \mathbb{R}^n$ denotes the Euclidean ball centered at x and of radius r .

As [100, Prop. 1, p. 30] all d -measures on a fixed d -set F are equivalent, it is also possible to define a d -set by the d -dimensional Hausdorff measure m_d , which in particular implies that F has Hausdorff dimension d in the neighborhood of each point of F [100, p.33]. The definition (2.1.3) includes the case $d = n$, i.e. n -sets. In \mathbb{R}^n Lipschitz domains and domains with more regular boundaries are n -sets and their boundaries are $(n - 1)$ -sets. Using [100, 172], the (ϵ, δ) domains in \mathbb{R}^n are n -sets:

$$\exists c > 0 \quad \forall x \in \overline{\Omega}, \forall r \in]0, \delta[\cap]0, 1[\quad \lambda(B_r(x) \cap \Omega) \geq C \lambda(B_r(x)) = cr^n,$$

where $\lambda(A)$ denotes the n -dimensional Lebesgue measure of a set A . This property is also called the measure density condition [80]. Let us notice that an n -set Ω cannot be “thin” close to its boundary $\partial\Omega$. At the same time [172], if Ω is an (ϵ, δ) -domain and $\partial\Omega$ is a d -set ($d < n$), then $\overline{\Omega} = \Omega \cup \partial\Omega$ is an n -set. A typical example of a d -set boundary it is the self-similar fractals as the Von Koch fractals.

In what follows we will use one of main results of [80]:

Theorem 2.1.1 (Sobolev extension [80]) *For $1 < p < \infty$, $k = 1, 2, \dots$ a domain $\Omega \subset \mathbb{R}^n$ is a W_p^k -extension domain if and only if Ω is an n -set and $W^{k,p}(\Omega) = C_p^k(\Omega)$ (in the sense of equivalent norms).*

In Theorem 2.1.1 the spaces $C_p^k(\Omega)$, $1 < p < +\infty$, $k = 1, 2, \dots$ are the spaces of fractional

sharp maximal functions,

$$C_p^k(\Omega) = \{f \in L^p(\Omega) \mid f_{k,\Omega}^\sharp(x) = \sup_{r>0} r^{-k} \inf_{P \in \mathcal{P}^{k-1}} \frac{1}{\lambda(B_r(x))} \int_{B_r(x) \cap \Omega} |f - P| dy \in L^p(\Omega)\}$$

with the norm $\|f\|_{C_p^k(\Omega)} = \|f\|_{L^p(\Omega)} + \|f_{k,\Omega}^\sharp\|_{L^p(\Omega)}$ and with the notation \mathcal{P}^{k-1} for the space of polynomials on \mathbb{R}^n of degree less or equal $k - 1$.

From [96] and [80] we directly have **[ARP-4]**

Corollary 2.1.1 *Let Ω be a bounded finitely connected domain in \mathbb{R}^2 and $1 < p < \infty$, $k \in \mathbb{N}^*$. The domain Ω is a 2-set with $W_p^k(\Omega) = C_p^k(\Omega)$ (with norms' equivalence) if and only if Ω is an (ϵ, δ) -domain and its boundary $\partial\Omega$ consists of a finite number of points and quasi-circles.*

Once we know the optimal class of the Sobolev extension domains, we need to define the trace operator on the boundaries of these domains.

2.1.3 Generalization of the trace on the boundary

Thanks to works [172, 102, 117] **[ARP-4]**, **[ARP-6]** it is possible to generalize the trace operator for more irregular boundaries, as for instance the d -sets or even on sets without a fixed dimension [98], **[BookChap]**. By the way, the d -sets are called “Ahlfors d -regular sets”, which finally gives an impression that in mathematics the fractals are regular sets.

Thus we define the trace for a regular distribution:

Definition 2.1.4 (Trace [100]) *For an arbitrary open set Ω of \mathbb{R}^n the trace operator Tr is defined for $u \in L_{loc}^1(\Omega)$ by*

$$Tru(x) = \lim_{r \rightarrow 0} \frac{1}{\lambda(\Omega \cap B_r(x))} \int_{\Omega \cap B_r(x)} u(y) d\lambda.$$

The trace operator Tr is considered for all $x \in \overline{\Omega}$ for which the limit exists.

By [172, 100] it is known that, if $\partial\Omega$ is a d -set with a positive Borel d -measure μ with $\text{supp } \mu = \partial\Omega$, the limit in Definition 2.1.4 exists μ -a.e. for $x \in \partial\Omega$. In addition it is possible to define the trace operator as a linear continuous operator from a Sobolev space on Ω to a Besov space on $\partial\Omega$ which is its image, i.e. there exists the right inverse extension $E_{\partial\Omega \rightarrow \Omega}$ operator and $\text{Tr}(E_{\partial\Omega \rightarrow \Omega} u) = u \in \text{Im}(\text{Tr})$. The image of $\text{Tr}(H^1(\Omega))$ in this case is the Besov space $B_\alpha^{2,2}(\partial\Omega)$ with $\alpha = 1 - \frac{n-d}{2} > 0$ [172, 100]. From where we obtain the restriction on the dimension of the boundary: $n - 2 < d < n$. By the way, for a connected boundary of a bounded domain the case $n - 2 < d < n - 1$ is impossible, so it is more realistic to impose $n - 1 \leq d < n$. Let us notice that if the image of the trace is a Besov space with $\alpha < 1$ then we don't need to have any additional geometrical restrictions on the boundary to have the continuity and the surjective property of the trace. But if $\alpha \geq 1$ we need to ensure [171, 2.1] that there exists a bounded linear extension operator \hat{E} of the Hölder space $C^{k-1, \alpha-k+1}(\partial\Omega)$ to the Hölder space $C^{k-1, \alpha-k+1}(\mathbb{R}^n)$, where for $k \in \mathbb{N}^*$ $k - 1 < \alpha \leq k$ (see also [100, p. 2]). This extension of Hölder spaces allows to show the existence of a linear continuous extension of the Besov space $B_\alpha^{p,p}(\partial\Omega)$ on $\partial\Omega$ to the Sobolev space $W_p^k(\mathbb{R}^n)$ with

$\alpha = k - \frac{(n-d)}{p} \geq 1$ and $k \geq 2$ [102]. To be able to ensure it, we need additionally to assume that the boundary $\partial\Omega$ preserves the Markov local inequality [100] p.39 (see [BookChap] for a detailed discussion). The geometrical characterization of sets preserving Markov's local inequality was initially given in [99] (see Theorem 1.3) and can be simply interpreted as sets which are not too flat anywhere. Smooth manifolds in \mathbb{R}^n of dimension less than n , as for instance a sphere, are examples of "flat" sets not preserving Markov's local inequality, but any d -set with $d > n - 1$ preserves it, as all \mathbb{R}^n . In the case $\alpha < 1$ (hence $k = 1$) the local Markov inequality is trivially satisfied on all closed sets of \mathbb{R}^n , and hence we do not need to impose it [102, p. 198]. Moreover, we are able to consider more general boundaries if we modify the definition of the image of the trace [BookChap] thanks to [97].

Let us apply the general results of Jonsson [97] to the trace of $W_p^1(\Omega)$, $1 < p < \infty$ as soon as only this Sobolev space is useful in the further considered applications. So, as detailed in [97, 98] (see also [BookChap]) we can consider Borel positive measures μ with a support $\text{supp } \mu = \partial\Omega$ which satisfies

1. the D_s -condition for an exponent $0 < s \leq n$ ensuring that there is a constant $c_s > 0$ such that

$$\mu(B_{kr}(x)) \leq c_s k^s \mu(B_r(x)), \quad x \in \partial\Omega, \quad r > 0, \quad k \geq 1, \quad 0 < kr \leq 1. \quad (2.4)$$

2. the L_d -condition for an exponent $0 \leq d \leq n$ ensuring that for some constant $c > 0$ it holds

$$\mu(B_{kr}(x)) \geq c_d k^d \mu(B_r(x)), \quad x \in \partial\Omega, \quad r > 0, \quad k \geq 1, \quad 0 < kr \leq 1. \quad (2.5)$$

3. the normalization condition

$$c_1 \leq \mu(B_1(x)) \leq c_2, \quad x \in \partial\Omega, \quad (2.6)$$

where $c_1 > 0$ and $c_2 > 0$ are constants independent of x .

Here $B_r(x) \subset \mathbb{R}^n$ denotes the Euclidean ball centered at x and of radius r . The D_s -condition (2.4) implies the doubling condition

$$\mu(B_{2r}(x)) \leq c \mu(B_r(x)), \quad x \in \partial\Omega, \quad 0 < r \leq 1/2,$$

where $c > 0$ is a suitable constant, [97, Section 1]. Moreover, combining (2.4) and (2.5) with (2.6) respectively, for some constants $c > 0$ and $c' > 0$ the measure μ also satisfies

$$c r^s \leq \mu(B_r(x)) \leq c' r^d, \quad x \in \partial\Omega, \quad 0 < r \leq 1.$$

We see that for $d = s$, the measure μ is a d -measure. For this general measure μ supported on a closed subset $\partial\Omega \subset \mathbb{R}^n$, which is actually a boundary of a domain Ω and hence at least $n - 1$ -dimensional, it is possible thanks to [97] to define the corresponding Lebesgue spaces $L^p(\partial\Omega, \mu)$ and Besov spaces $B_1^{p,p}(\partial\Omega)$ in a such way that we have the following theorem (to compare with Theorem 6 [BookChap]):

Theorem 2.1.2 *Let $0 < n-1 \leq d \leq s < n$, $1 \leq p \leq +\infty$, and let $\Omega \subset \mathbb{R}^n$ be a domain with a closed boundary $\partial\Omega \subset \mathbb{R}^n$ which is the support of a Borel measure μ satisfying (2.4), (2.5) and (2.6).*

Then, considering the Besov space $B_1^{p,p}(\partial\Omega)$ on $\partial\Omega$, defined as the space of μ -classes of real-valued functions f on $\partial\Omega$ such that the norm

$$\|f\|_{B_1^{p,p}(\partial\Omega,\mu)} := \|f\|_{L_p(\partial\Omega,\mu)} + \left(\sum_{\nu=0}^{\infty} 2^{\nu(1-\frac{n}{p})} \int \int_{|x-y|<2^{-\nu}} \frac{|f(x) - f(y)|^p}{\mu(B(x, 2^{-\nu}))\mu(B(y, 2^{-\nu}))} \mu(dy)\mu(dx) \right)^{1/p} \quad (2.7)$$

is finite, the following statements hold:

(i) $\text{Tr}_{\partial\Omega}$ is a continuous linear operator from $W^{1,p}(\mathbb{R}^n)$ onto $B_1^{p,p}(\partial\Omega)$, and

$$\|\text{Tr}_{\partial\Omega} f\|_{B_1^{p,p}(\partial\Omega)} \leq \hat{c} \|f\|_{W^{1,p}(\mathbb{R}^n)}, \quad f \in W^{1,p}(\mathbb{R}^n), \quad (2.8)$$

with a constant $\hat{c} > 0$ depending only on $s, d, n, c_s, c_d, c_1, c_2$.

(ii) There is a continuous linear extension operator $E_{\partial\Omega} : B_1^{p,p}(\partial\Omega) \rightarrow W^{1,p}(\mathbb{R}^n)$ such that $\text{Tr}_{\partial\Omega}(E_{\partial\Omega}f) = f$ for $f \in B_1^{p,p}(\partial\Omega)$.

Theorem 2.1.2 is a particular case of [97, Theorem 1].

The spaces $B_1^{p,p}(\partial\Omega)$ are Banach spaces, while $B_1^{2,2}(\partial\Omega)$ are Hilbert spaces, and their corresponding scalar product is denoted by $\langle \cdot, \cdot \rangle_{B_1^{2,2}(\partial\Omega)}$. In addition, the spaces $B_1^{p,p}(\partial\Omega)$ does not depend on μ (if there are two measures μ_1 and μ_2 with the support $\partial\Omega$ satisfying the conditions of the Theorem 2.1.2, then the norms (2.7) constructed on them are equivalent [97, Section 3.5]). It is important to notice that for a d -set boundary $\partial\Omega$ the space $\hat{B}_1^{p,p}(\partial\Omega)$ is equivalent to the Besov space $B_\alpha^{p,p}(\partial\Omega)$ with $0 < \alpha = 1 - \frac{n-d}{p} < 1$ (see Ref. [97] and for the spaces $B_\alpha^{p,p}(\partial\Omega)$ see Ref. [100]). In addition if $d = s = n - 1$, the trace space of $H^1(\Omega)$, as it also mentioned in [ARP-6], is given by the Besov space with $\alpha = \frac{1}{2}$ which coincides with $H^{\frac{1}{2}}(\partial\Omega)$:

$$B_1^{2,2}(\partial\Omega) = B_{\frac{1}{2}}^{2,2}(\partial\Omega) = H^{\frac{1}{2}}(\partial\Omega)$$

as usual in the case of the classical results [125, 129] for Lipschitz boundaries.

Therefore, thanks to Theorem 2.1.2, it is sufficient to replace $H^{\frac{1}{2}}(\partial\Omega)$ in the Green formula (2.2) and the formula of the integration by parts (2.3) by the Besov space $B_1^{2,2}(\partial\Omega)$ to obtain linear continuous functionals and hence to be able to apply these formulas for domains with boundaries defined by a measure μ as in Theorem 2.1.2.

2.1.4 Sobolev admissible domains [ARP-4], [BookChap]

Once the trace theorem is obtained, it gives us the class of domains

$$\mathcal{T} = \{\Omega \subset \mathbb{R}^n \mid \exists \mu \text{ on } \partial\Omega \text{ satisfying Theorem 2.1.2}\},$$

which we intersect with the optimal class of the Sobolev extension domains, defined by Theorem 2.1.1, to obtain the class of the Sobolev admissible domains [ARP-4], [BookChap]:

Definition 2.1.5 (Sobolev admissible domain) Let $1 < p < \infty$ and $k \in \mathbb{N}^*$ be fixed. A domain $\Omega \subset \mathbb{R}^n$ is called a Sobolev admissible domain if it is an n -set, such that $W_p^k(\Omega) = C_p^k(\Omega)$ as sets with equivalent norms (hence, Ω is a W_p^k -extension domain), with a closed boundary $\partial\Omega$ which is the support of a Borel measure μ satisfying the conditions of Theorem 2.1.2.

In other words, we introduce the class of all Sobolev extension domains with boundaries on which it is possible to define a surjective linear continuous trace operator with linear continuous right inverse. To insist on their extension nature, we thus called these domains Sobolev admissible domains (see Definition 2.1.5).

Example 2.1.1 *An example of a Sobolev admissible domain could be a bounded domain of \mathbb{R}^n with a boundary $\partial\Omega$ equal to a finite disjoint union of parts Γ_j which are d_j -sets respectively for $n - 1 \leq d_j < n$ ($j = 1, \dots, m$). For instance it is the case of a three-dimensional cylindrical domain constructed on a base of two-dimensional domain with a d -set boundary as considered for the Koch snowflake base in [118, 41].*

Example 2.1.2 *The fractal trees [3] and the domains with outgoing cusps are examples of domains which are not Sobolev admissible, as they are not Sobolev extension domains.*

Thus we summarize useful in what follows results (initially developed in the framework of d -set boundaries in [ARP-4]) on Sobolev admissible domains in the following trace theorem

Theorem 2.1.3 (Traces and extensions) *Let Ω be a Sobolev admissible domain in \mathbb{R}^n , $1 < p < +\infty$, $k \in \mathbb{N}^*$ be fixed. Then the following trace operators (see Definition 2.1.4)*

1. $Tr : W^{1,p}(\mathbb{R}^n) \rightarrow B_1^{p,p}(\partial\Omega),$
2. $Tr_\Omega : W^{k,p}(\mathbb{R}^n) \rightarrow W^{k,p}(\Omega),$
3. $Tr_{\partial\Omega} : W^{1,p}(\Omega) \rightarrow B_1^{p,p}(\partial\Omega)$

are linear continuous and surjective with linear bounded right inverse, i.e. extension, operators $E : B_1^{p,p}(\partial\Omega) \rightarrow W^{1,p}(\mathbb{R}^n)$, $E_\Omega : W^{k,p}(\Omega) \rightarrow W^{k,p}(\mathbb{R}^n)$, $E_{\partial\Omega} : B_1^{p,p}(\partial\Omega) \rightarrow W^{1,p}(\Omega)$.

2.2 Compactness of the trace operator

To be able to ensure the weak well-posedness of problem (2.1) and also for the associated spectral problem of $-\Delta$, we need to have in addition the compactness of the inclusion $H^1(\Omega)$ in $L^2(\Omega)$ and the compactness of the trace operator this time considered as an operator from $H^1(\Omega)$ to $L^2(\partial\Omega)$. Thanks to [50] Theorem V.4.17, it is known that if a domain Ω has a continuous boundary (in the sense of graphs, see [50] Definition V.4.1) then $H^1(\Omega)$ is compactly embedded in $L^2(\Omega)$. The general d -set boundaries with $d > n - 1$, as for instance, a von Koch curve, does not satisfy the assumption to have a continuous boundary. In our article [ARP-4] this fact was proven in the framework of Sobolev admissible domains with a d -set boundary. In [BookChap], [PrepWestMixed] we prove it also for more general boundaries described by Theorem 2.1.2 as in [97, 98].

The generalization of the Kondrachov-Rellich theorem in the framework of Sobolev admissible domains allows to extend the compactness studies of the trace from [15] and to update the results of [ARP-4]: for a Sobolev admissible domain with a compact boundary, the trace operator considered from $H^1(\Omega)$ to $L^2(\partial\Omega)$ is compact. To have a compact embedding, the domain Ω must be a Sobolev extension domain. Hence, a trace operator $H^1(\Omega) \rightarrow L^2(\partial\Omega)$ mapping the functions defined on a domain Ω to their values on the boundary $\partial\Omega$ (or on any part D of Ω , $H^1(\Omega) \rightarrow L^2(D)$) is compact if and only if the boundary $\partial\Omega$ (or the part \overline{D}) is compact.

Thus, as for the usual Lipschitz bounded case, the problem (2.1) is weakly well-posed. The

corresponding spectral problem has a countable number of eigenvalues going to $+\infty$ with the eigenfunctions forming an orthogonal basis in $H^1(\Omega)$, which becomes an orthonormal basis in $L^2(\Omega)$ by the classical Hilbert-Schmidt theorem for compact auto-adjoint operators on a Hilbert space.

More precisely we have the following generalization of the classical Rellich-Kondrachov theorem (see for instance Adams [5] p.144 Theorem 6.2):

Theorem 2.2.1 (Compact Sobolev embeddings for n -sets, [ARP-4]) *Let $\Omega \subset \mathbb{R}^n$ be a bounded n -set with $W_p^k(\Omega) = C_p^k(\Omega)$, $1 < p < \infty$, $k, \ell \in \mathbb{N}^*$. Then there hold the following compact embeddings:*

1. $W_p^{k+\ell}(\Omega) \subset\subset W_q^\ell(\Omega)$,
2. $W_p^k(\Omega) \subset\subset L_q(\Omega)$,

with $q \in [1, +\infty[$ if $kp = n$, $q \in [1, +\infty]$ if $kp > n$, and with $q \in [1, \frac{pn}{n-kp}]$ if $kp < n$.

We also prove [ARP-4], [BookChap] the compactness of embeddings for the Besov spaces on fractals. In particular we obtain that if $F \subset \mathbb{R}^n$ be a bounded closed set satisfying conditions of Theorem 2.1.2, then for $1 \leq q \leq p$ the embedding $B_1^{p,p}(F) \subset\subset L^q(F)$ is compact (in the case when F is not bounded, we only have the compactness of the embedding $B_1^{p,p}(F) \subset\subset L_{loc}^q(F)$). The compactness of this embedding actually follows from the compactness of the trace $\text{Tr} : W^{1,p}(\Omega) \rightarrow L^p(\partial\Omega)$ for a compact boundary $\partial\Omega$.

2.3 Application example: Dirichlet-to-Neumann operator on d -sets [ARP-4]

We introduce the Dirichlet-to-Neumann operator on d -sets in the framework of the Laplacian transports. Laplacian transports to and across irregular and fractal interfaces are ubiquitous in nature and industry: properties of rough electrodes in electrochemistry, heterogeneous catalysis, steady-state transfer across biological membranes (see [60, 72, 73, 74] and references therein).

To model it, there is a usual interest in considering truncated domains as an approximation of the exterior unbounded domain case.

Let Ω_0 and Ω_1 be two bounded domains in \mathbb{R}^n with disjoint boundaries $\partial\Omega_0 \cap \partial\Omega_1 = \emptyset$, denoted by Γ and S respectively, such that $\overline{\Omega_0} \subset \Omega_1$. Thus, we consider two types of domains constructed on Ω_0 :

1. the unbounded exterior domain to Ω_0 , denoted by $\Omega = \mathbb{R}^n \setminus \overline{\Omega_0}$;
2. a bounded, truncated by a boundary S , truncated domain $\Omega_S = (\mathbb{R}^n \setminus \overline{\Omega_0}) \cap \Omega_1$.

Let us notice that $\Gamma \cup S = \partial\Omega_S$ (for the unbounded case $S = \emptyset$ and $\partial\Omega = \Gamma$), see Fig. 2.1. As Ω_0 is bounded, its boundary Γ is supposed compact. Therefore, by the previous section, the trace operator $H^1(\Omega) \rightarrow L^2(\Gamma)$ is compact. The phenomenon of Laplacian transport to

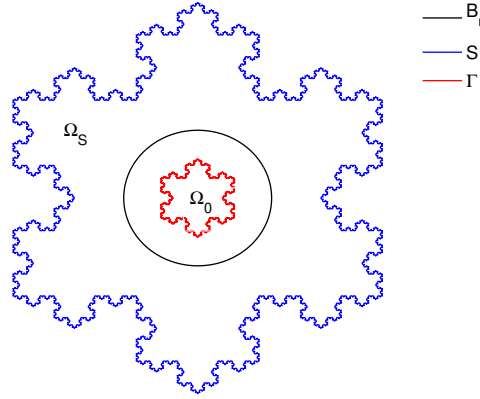


Figure 2.1 – Example of the considered domains: Ω_0 (the von Koch snowflake) is the bounded domain, bounded by a compact boundary Γ , which is a d -set (see Definition ??) with $d = \log 4 / \log 3 > n - 1 = 1$. The truncated domain Ω_S is between the boundary Γ and the boundary S (presented by the same von Koch fractal as Γ). The boundaries Γ and S have no an intersection and here are separated by the boundary of a ball B_r of a radius $r > 0$. The domain, bounded by S , is called $\Omega_1 = \overline{\Omega}_0 \cup \Omega_S$, and the exterior domain is $\Omega = \mathbb{R}^n \setminus \overline{\Omega}_0$.

Γ can be described by the following boundary value problem:

$$\begin{aligned} -\Delta u &= 0, \quad x \in \Omega_S \text{ or } \Omega, \\ \lambda u + \partial_\nu u &= \psi \quad \text{on } \Gamma, \\ u &= 0 \quad \text{on } S, \end{aligned} \tag{2.9}$$

where $\partial_\nu u$ denotes the normal derivative of u , in some appropriate sense, $\lambda \in [0, \infty[$ is the resistivity of the boundary and $\psi \in L_2(\Gamma)$. For $S = \emptyset$, we impose Dirichlet boundary conditions at infinity. The boundary Γ is supposed to be a d -set, and S can be another \hat{d} -set boundary. We also notice that, thanks to [BookChap], the boundaries Γ and S finally also can be described by Theorem 2.1.2. The generalization is trivial by replacing the d -Hausdorff measure on Γ (or S) by the measure μ on Γ (or on S) and taking into account that the image of the trace operator of $H^1(\Omega)$ on Γ is equal to $B_1^{2,2}(\Gamma)$.

The main difficulty when we work in the exterior domain Ω is the invalidity of the Poincaré inequality. For $n \geq 3$ it can be replaced by

$$\left\| u - \lim_{r \rightarrow \infty} \frac{1}{|B_r|} \int_{B_r} u dx \right\|_{L_{\frac{2n}{n-2}}(\mathbb{R}^n)} \leq c \|\nabla u\|_{L_2(\mathbb{R}^n)}.$$

Here B_r is a non trivial ball of \mathbb{R}^n . Thus, to work in the exterior domain Ω we need to introduce the functional space $W^D(\Omega)$ defined by the closure of the space

$$\{u|_\Omega : u \in \mathcal{D}(\mathbb{R}^n), \ n \geq 3\}$$

with respect to the norm $u \mapsto (\int_\Omega |\nabla u|^2 dx)^{1/2}$. Therefore, for the inner product $(u, v)_{W^D(\Omega)} = \int_\Omega \nabla u \cdot \nabla v dx$, the space $(W^D(\Omega), (\cdot, \cdot)_{W^D(\Omega)})$ is a Hilbert space (see a discussion about it on p. 8 of Ref. [126]). If in the same time

$$W(\Omega) := \{u \in H_{loc}^1(\Omega), \quad \int_\Omega |\nabla u|^2 dx < \infty\},$$

then for $n \geq 3$, $H^1(\Omega) \subset W(\Omega) \cap L_{\frac{2n}{n-2}}(\Omega) = W^D(\Omega)$, which is false for $n = 2$.

As $W^D(\Omega)$ is larger than $H^1(\Omega)$ we update the trace and extension results for this space staying analogous to the results for $H^1(\Omega)$ and give, following [17], the new sense to the Green formula, defining the normal derivative of $u \in W^D(\Omega)$ with $\Delta u \in L_2(\Omega)$ in the distributional sense in $L^2(\Gamma, \mu)$, denoting $\partial_\nu u = \psi$, if $\psi \in L_2(\Gamma)$ exists:

$$\forall v \in \mathcal{D}(\mathbb{R}^n) \quad \int_{\Omega} (\Delta u) v dx + \int_{\Omega} \nabla u \cdot \nabla v dx = \int_{\Gamma} \psi \text{Tr} v d\mu. \quad (2.10)$$

For all $\psi \in L^2(\Gamma)$ and $\lambda \geq 0$ we say that u is a weak solution of (2.9)

on Ω if $u \in W^D(\Omega)$ ($n \geq 3$) and for all $v \in W^D(\Omega)$

$$\int_{\Omega} \nabla u \nabla v dx + \lambda \int_{\Gamma} \text{Tr}_\Gamma u \text{Tr}_\Gamma v d\mu = \int_{\Gamma} \psi \text{Tr}_\Gamma v d\mu; \quad (2.11)$$

on Ω_S if $u \in \tilde{H}^1(\Omega_S) := \{u \in H^1(\Omega_S) : \text{Tr}_S u = 0\}$ and for all $v \in \tilde{H}^1(\Omega_S)$ it holds the variational formulation (2.11) with $\Omega = \Omega_S$;

on Ω_0 if $u \in H^1(\Omega_0)$ and for all $v \in H^1(\Omega_0)$ it holds (2.11) with $\Omega = \Omega_0$.

As Γ (and S) is compact, hence, by Section 2.1 and by [ARP-4], the trace $H^1(\Omega) \rightarrow L^2(\Gamma)$ is compact (the same is true for $W^D(\Omega) \rightarrow L^2(\Gamma)$), which each time implies the equivalence of the usual norm of $H^1(\Omega)$ (respectively of $W^D(\Omega)$) to the trace norm

$$\|u\|_{\text{Tr}}^2 = \|\nabla u\|_{L^2(\Omega)}^2 + \|\text{Tr} u\|_{L^2(\Gamma, \mu)}^2. \quad (2.12)$$

Consequently, if $(\cdot, \cdot)_{\text{Tr}}$ is the corresponding inner product, we can rewrite (2.11) in the form

$$(u, v)_{\text{Tr}} = (\psi, \text{Tr}_\Gamma v)_{L^2(\Gamma)} \quad (2.13)$$

and obtain the well-posedness by the Riesz representation theorem.

The case of a truncated domain Ω_S corresponds to an approximation of the exterior problem in the sense of Theorem 3.8 [ARP-4].

The Poincaré-Steklov operator, also named the Dirichlet-to-Neumann operator, was initially introduced by V.A. Steklov and usually defined by a map

$$A : u|_{\Gamma} \mapsto \frac{\partial u}{\partial \nu} \Big|_{\Gamma}$$

for a solution u of the elliptic Dirichlet problem: $-\Delta u = 0$ in a domain Ω and $u|_{\Gamma} = f$ (with $\partial\Omega = \Gamma$).

It is well-known that if Ω is a bounded domain with a C^∞ -regular boundary (a regular manifold with boundary), then the operator $A : C^\infty(\Gamma) \rightarrow C^\infty(\Gamma)$ is an elliptic self-adjoint pseudo-differential operator of the first order (see [161] §11 and 12 of Chapter 7) with a discrete spectrum

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots, \quad \text{with } \lambda_k \rightarrow +\infty \text{ as } k \rightarrow +\infty.$$

If A is considered as an operator $H^1(\Gamma) \rightarrow L_2(\Gamma)$, then its eigenfunctions form a basis in $L_2(\Gamma)$. For any Lipschitz boundary Γ of a bounded domain Ω , the Dirichlet-to-Neumann operator

$$A : H^{\frac{1}{2}}(\Gamma) \rightarrow H^{-\frac{1}{2}}(\Gamma)$$

is well-defined and it is a linear continuous self-adjoint operator. By the analogy, in [ARP-4] we define for a bounded domain Ω_0 with a d -set boundary Γ the Dirichlet-to-Neumann operator

$$A : B_{1-\frac{n-d}{2}}^{2,2}(\Gamma) \rightarrow B_{-(1-\frac{n-d}{2})}^{2,2}(\Gamma)$$

also as a linear continuous self-adjoint operator.

Thanks to [14], we also know that the Dirichlet-to-Neumann operator A has a compact resolvent. Hence, it has a discrete spectrum, as long as the trace operator $\text{Tr} : H^1(\Omega) \rightarrow L_2(\Gamma)$ is compact (see also [15] and [164] for an abstract definition of the elliptic operators on a d -set). Thus, thanks to the compactness of the trace operator, the property of the compact resolvent also holds for an Sobolev admissible domain Ω with a compact boundary Γ .

Since Γ (see Fig. 2.1) can be viewed not only as the boundary of Ω_0 , but also as the boundary of the exterior domain Ω and its truncated domain Ω_S , we also introduce the Poincaré-Steklov operator A on Γ for the exterior and truncated cases and relate their spectral properties. In all cases, the Poincaré-Steklov operator A can be defined as a positive self-adjoint operator on $L_2(\Gamma)$, and A has a discrete spectrum if and only if the boundary Γ is compact.

The main idea is to use Theorem 3.4 from [15], which is a kind of generalization of the Riesz representation theorem and the Lax-Milgram theorem [15, 14, 13] ensuring the existence of the linear continuous positive self-adjoint operator $A^{int} : L^2(\Gamma) \rightarrow L^2(\Gamma)$ associated to the bilinear form

$$a(u, v) = \int_{\Omega_0} \nabla u \nabla v dx : D(a) \times D(a) \rightarrow \mathbb{R}$$

for $D(a) = H^1(\Omega_0) \cap C(\overline{\Omega_0})$ dense in $H^1(\Omega_0)$ (see the discussion of Ref. [15]), and the compact trace operator $\text{Tr} : D(a) \rightarrow L_2(\Gamma)$ with the dense image $\text{Tr}(D(a))$ in $L_2(\Gamma)$. Thus, the operator A^{int} is defined for all $\phi \in L_2(\Gamma)$ in the following way

$$\begin{aligned} \phi \in D(A^{int}) \text{ and there exists an element } \psi = A^{int} \phi \text{ of } L_2(\Gamma) &\iff \\ \exists u \in H^1(\Omega_0) \text{ such that } \text{Tr} u = \phi \text{ and } \forall v \in H^1(\Omega_0) &\int_{\Omega_0} \nabla u \nabla v dx = \int_{\Gamma} \psi \text{Tr} v dm_d. \end{aligned}$$

From [24], we also have that $\text{Ker } A^{int} \neq \{0\}$, since 0 is the eigenvalue of the Neumann eigenvalue problem for the Laplacian. In the same way we define A^{ext} :

Definition 2.3.1 (*Dirichlet-to-Neumann operator for an exterior domain* $n \geq 3$) *Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$, be a Sobolev admissible exterior domain with a compact boundary Γ . The operator $A^{ext} : L_2(\Gamma) \rightarrow L_2(\Gamma)$, associated with the bilinear form $a^D : W^D(\Omega) \times W^D(\Omega) \rightarrow \mathbb{R}$ given by*

$$a^D(u, v) = \int_{\Omega} \nabla u \nabla v dx = \langle u, v \rangle_{W^D(\Omega)},$$

and the trace operator $\text{Tr} : W^D(\Omega) \rightarrow L_2(\Gamma)$, is called the Dirichlet-to-Neumann operator with the Dirichlet boundary condition at infinity.

Remark 2.3.1 Theorem 3.3 in Ref. [15] does not require to $D(a)$ the completeness, i.e. $a(\cdot, \cdot)$ can be equivalent to a semi-norm on $D(a)$, what is the case of $W^D(\Omega)$ with $a(u, u) = \int_{\Omega} |\nabla u|^2 dx$ for $n = 2$. Therefore, it allows us to define the Dirichlet-to-Neumann operator A^{ext} of the exterior problem in \mathbb{R}^2 , which can be understood as the limit case for $r \rightarrow +\infty$ of the problem for a truncated domain well-posed in $\tilde{H}^1(\Omega_{S_r})$ (we suppose that $\Omega_{S_r} \uparrow \Omega$ for $r \rightarrow +\infty$). In the case of $W^D(\Omega)$ in \mathbb{R}^n with $n \geq 3$, we have that $D(a) = W^D(\Omega)$ is the Hilbert space corresponding to the inner product $a(\cdot, \cdot)$.

The two dimensional case differs from the case of \mathbb{R}^n with $n \geq 3$ by the functional reason and gives different properties of the point spectrum of A^{ext} . Actually, for $n = 2$ the definition of A^{ext} is based on the bilinear form defined on $D(a) = H^1(\Omega)$ and for $n \geq 3$, as it was mentioned, on $D(a) = W^D(\Omega)$. In particular, in the exterior case for a compact d -set boundary Γ the Dirichlet-to-Neumann operator A^{ext} for $n = 2$ and $n \geq 3$ has different domains of definition:

- for A^{int} and for A^{ext} in the case $n = 2$

$$D(A^{int}) = D(A^{ext}) = B_{\frac{d}{2}}^{2,2}(\Gamma),$$

- for $A^{truncated}$ with $n \geq 2$ and for A^{ext} in the case $n \geq 3$

$$D(A^{truncated}) = D(A^{ext}) = L^2(\Gamma).$$

Let us denote the sets of all eigenvalues of A^{int} and A^{ext} , mapping $L_2(\Gamma)$ to $L_2(\Gamma)$, respectively by σ^{int} and σ^{ext} , which are subsets of \mathbb{R}^+ . For the Weil asymptotic formulas for the distribution of the eigenvalues of the Dirichlet-to-Neumann operator A^{int} , there are results for bounded smooth, compact Riemannian manifolds with C^∞ boundaries [68], for polygons [69] and more general class of plane domains [67] and also for a bounded domain with a fractal boundary [146]. In Theorem 4.1 in [ARP-4] we relate these spectral results, obtained for the Dirichlet-to-Neumann operator for a bounded domain, with the case of the exterior domain:

- For $n = 2$

$$\sigma^{int} = \sigma^{ext} \subset \mathbb{R}^+ \quad \text{and} \quad 0 \in \sigma^{ext}.$$

- For $n \geq 3$

$$\sigma^{int} = \{0\} \cup \sigma^{ext} \quad \text{with} \quad \sigma^{ext} \subset]0, +\infty[,$$

i.e. the Dirichlet-to-Neumann operator of the exterior problem, also as of the truncated problem, is an injective operator with the compact inverse.

We also show that the eigenvalues of the Dirichlet-to-Neumann operator for the truncated problem expanding with $r \rightarrow +\infty$ to the exterior domain converge to the eigenvalues of the exterior problem.

Specially, for the case of a d -set Γ or also more generally with a measure μ satisfying Theorem 2.1.2, we justify the method, developed in [73], true for smooth boundaries, to find the total flux Φ across the interface Γ using the spectral decomposition of 1_Γ (belonging to the domain of A) on the basis of eigenfunctions of the Dirichlet-to-Neumann operator $(V_k)_{k \in \mathbb{N}}$ in $L_2(\Gamma)$ and its eigenvalues $(\lambda_k)_{k \in \mathbb{N}}$:

$$\Phi \propto \sum_k \frac{\lambda_k (\mathbf{1}_\Gamma, V_k)_{L_2(\Gamma)}^2}{1 + \frac{\lambda_k}{\lambda}}. \quad (2.14)$$

2.4 Regularity problems and weak solutions of the Westervelt equation [PrepWestMixed], [PrepWestDir]

2.4.1 Regularity of the weak solution of the Poisson equation

To consider the weak solutions of the Poisson equation (2.1) let us start by discussing the Dirichlet homogeneous boundary condition, hence taking $\Gamma_N = \Gamma_R = \emptyset$. Let Ω be a bounded domain of \mathbb{R}^n . In the framework of weak solutions, the Dirichlet boundary valued problem for the Poisson equation is understood in the following variational form

$$\forall v \in H_0^1(\Omega) \quad \int_{\Omega} \nabla u \nabla v dx = \int_{\Omega} f v dx, \quad (2.15)$$

in which there is no more any boundary influence (to compare with (2.13)). Therefore, the unique weak solution $u \in H_0^1(\Omega)$ exists for an arbitrary bounded domain Ω by a simple application of the Riesz representation theorem. Moreover, thanks to Evans [52] Theorem 2 p. 304 and Theorem 3 p. 316, we have (even for solutions in $H^1(\Omega)$ and thus for different boundary conditions) the interior regularity of the weak solution, *i.e.*, the fact that for a subset V compactly included in Ω , $V \subset\subset \Omega$, the solution on Ω has on V the same regularity as for a domain with regular boundaries. For instance, if $f \in C^\infty(\Omega)$ then $u \in C^\infty(\Omega) \cap H_0^1(\Omega)$. So, for any boundary of Ω , even worse than a fractal or than a fractal tree and a domain with cusps, the weak solution of (2.15) is in $C^\infty(\Omega)$ for the same regularity of f . The key point here that Ω is open.

The property to be in $C(\overline{\Omega})$ is much more restrictive, since the continuity on a compact requires from u to be bounded and equicontinuous, and does not hold for arbitrary shapes of $\partial\Omega$ [50]. By very technical results of Nyström [142] the necessary condition for Ω is to be a non-tangentially accessible domain (a NTA domain):

Definition 2.4.1 (NTA domain) [93] *A bounded domain $\Omega \subset \mathbb{R}^n$ is called NTA when there exists constants M and r_0 such that:*

1. *Corkscrew condition: For any point $Q \in \partial\Omega$, $r < r_0$, there exists a point $A = A_r(Q) \in \Omega$ such that $M^{-1}r < |A - Q| < r$ and $d(A, \partial\Omega) > M^{-1}r$.*
2. *$\mathbb{R}^n \setminus \overline{\Omega}$ satisfies the Corkscrew condition.*
3. *Harnack chain condition: If $\epsilon > 0$ and points P_1 and P_2 belongs to Ω , $d(P_j; \partial\Omega) > \epsilon$ and $|P_1 - P_2| < C\epsilon$, then there exists a Harnack chain from P_1 to P_2 whose length depends on C and not on ϵ .*

For P_1, P_2 in Ω , a Harnack chain from P_1 to P_2 in Ω is a sequence of M non-tangential balls such that the first ball contains P_1 , the last contains P_2 , and such that consecutive balls have non empty intersections. Finally, a M non-tangential ball in a domain Ω is a ball $B(A, r)$ in Ω whose distance from $\partial\Omega$ is comparable to its radius:

$$Mr > d(B(A, r), \partial\Omega) > M^{-1}r.$$

Thanks to [142], if Ω is a bounded NTA domain characterized by M and r_0 , then Ω is an (ϵ, δ) -domain with ϵ and δ characterized by M and r_0 only. Thus, by [95] if $\Omega \subset \mathbb{R}^2$ is a bounded simply connected set, it is a NTA domain if and only if it is a quasidisc (*i.e.*

there exists a quasiconformal map mapping the domain to a disc). Moreover, by [96] and Definition 2.1.5, we continue this statement by noticing that if Ω is a Sobolev admissible domain then it is a NTA domain. Obviously, it is not anymore the case if $\Omega \subset \mathbb{R}^3$. A typical irregular example of a NTA domain in \mathbb{R}^2 is the von Koch snowflake.

For C^2 boundaries it is also known the H^2 -regularity [52, Thm. 4, p. 317] of the weak solutions: if $f \in L^2(\Omega)$ and $\partial\Omega \in C^2$, then the weak solution of (2.15) $u \in H^2(\Omega) \cap H_0^1(\Omega)$ (see also Theorem 5 p. 323 [52] for higher boundary regularity). But it is no more true in the general class of NTA domains. It is important to cite here the theorem of Nyström [142]:

Theorem 2.4.1 *Let $\Omega \subset \mathbb{R}^2$ be von Koch's snowflake. Let $f \in \mathcal{D}(\Omega)$ be non negative and non identically zero. Let $u \in H_0^1(\Omega)$ be the weak solution of the Poisson problem with homogeneous Dirichlet boundary condition. Then*

$$u \in H_0^1(\Omega) \cap C^\infty(\Omega) \text{ and } u \in C(\overline{\Omega}),$$

but

$$u \notin H^2(\Omega).$$

However, it holds for the convex polygonal domains [76]. The convexity condition does not allow the incoming angles, which create the singularities.

Another important question is whether the solutions of the Poisson problem belong to $C(\Omega) \cap L^\infty(\Omega)$ (a weaker condition than to be continuous up to the boundary) with an estimate of the form:

$$\|u\|_{L^\infty(\Omega)} \leq C\|f\|_{L^2(\Omega)}. \quad (2.16)$$

By Nyström [142] for $\partial\Omega = \Gamma_D$ the answer is positive in dimension $n = 2$ in the class of the NTA domains, and hence Sobolev admissible domains. By Xie [175] it is also positive for the three-dimensional case considering the solutions of (2.15) in arbitrary domains. If $\partial\Omega = \Gamma_R$, using Daners [44], we obtain that it is also possible to have for $n = 2$ or 3 if Ω is Sobolev admissible. Furthermore, we show the same result for the weak solutions of the mixed boundary valued problem.

Without the access to the H^2 -regularity for the general case of Sobolev admissible domains, we however able to improve the regularity of the weak solutions working in the domain of the Laplacian. In the framework of the mixed boundary condition for the Poisson problem (2.1), it is natural to consider an analogue of the space $\tilde{H}^1(\Omega_{S_r})$ introduced in Section 2.3

$$V(\Omega) = \{u \in H^1(\Omega) \mid \text{Tr } u|_{\Gamma_D} = 0\}, \quad (2.17)$$

endowed with the norm of the trace (2.12) equivalent to the usual norm of $H^1(\Omega)$.

Thanks to the compactness (see Section 2.2 and [ARP-4]) of the trace $\text{Tr} : V(\Omega) \rightarrow L^2(\partial\Omega)$ and of the inclusion $V(\Omega) \rightarrow L^2(\Omega)$ and by the assumption that $a > 0$ is real (thus $-\Delta$ is auto-adjoint positive operator), we have the usual properties of the spectral problem associated with (2.1). It means that if Ω is a bounded Sobolev admissible domain, then the point spectrum is discrete, all eigenvalues are strictly positive, form an unbounded sequence, and the corresponding eigenfunctions form an orthonormal basis of $L^2(\Omega)$. If $\Gamma_R = \emptyset$ it is still important to work in the class of Sobolev admissible domains to ensure the compactness of the embedding of $V(\Omega)$ into $L^2(\Omega)$. Nevertheless, in the case $\partial\Omega = \Gamma_D$,

it is possible to consider arbitrary domains since $H_0^1(\Omega) \rightarrow L^2(\Omega)$ is compact independently on the regularity of $\partial\Omega$.

As Ω is a bounded domain, we have $L^p(\Omega) \hookrightarrow L^2(\Omega)$ if $p \geq 2$, and consequently it is also possible to take $f \in L^p(\Omega)$ and consider the weak solutions in $V(\Omega)$ in the sense of (2.13). Therefore, there is the following generalization of the domain of the Laplacian in the L^p framework:

Definition 2.4.2 (Laplacian domain in L^p) *Let Ω be a Sobolev admissible domain and $p \geq 2$. We define*

$$\begin{aligned} -\Delta : \mathcal{D}(-\Delta) \subset V(\Omega) &\rightarrow L^p(\Omega) \\ u &\mapsto -\Delta u \end{aligned}$$

with the domain

$$\mathcal{D}(-\Delta) = \{u \in V(\Omega) \mid -\Delta u \in L^p(\Omega), \text{ i.e. } \exists f \in L^p(\Omega) \text{ such that it holds (2.13)}\}.$$

Then the operator $-\Delta$ is linear self-adjoint and coercive in the sense that

$$\forall u \in \mathcal{D}(-\Delta) \quad (-\Delta u, u)_{L^2(\Omega)} = (u, u)_{V(\Omega)},$$

and we use the notation $\|u\|_{\mathcal{D}(-\Delta)} = \|\Delta u\|_{L^p(\Omega)}$ for $u \in \mathcal{D}(-\Delta)$.

The L^p -framework for the Poisson problem (2.1) is in particular important for the study of the continuity of its solution [44]. But also it is useful to use the maximal regularity results for the linear part of the Westervelt equation and to control its nonlinear terms in the study of its weak well posedness in the Sobolev admissible domains (see for instance [PrepWest-Mixed]).

2.4.2 Weak solutions of the Westervelt equation on a bounded domain

To study the weak well-posedness of different boundary valued problems for the Westervelt equation in the most possible large class of domains, we modify a little bit our model: we derive Eq. (1.7) once on time and pose $u = \partial_t \Pi$. Thus we obtain for u a modified version of the Westervelt equation with the following nonlinear terms:

$$\begin{cases} \partial_t^2 u - c^2 \Delta u - \nu \Delta \partial_t u = \alpha u \partial_t^2 u + \alpha (\partial_t u)^2 + f & \text{on } [0, T] \times \Omega, \\ u = 0 & \text{on } \Gamma_D \times [0, T], \\ \frac{\partial}{\partial n} u = 0 & \text{on } \Gamma_N \times [0, T], \\ \frac{\partial}{\partial n} u + au = 0 & \text{on } \Gamma_R \times [0, T], \\ u(0) = u_0, \quad \partial_t u(0) = u_1. \end{cases} \quad (2.18)$$

The regularity of the solutions of the Westervelt equation on regular domains, typically with a C^2 boundary, is well known. Besides, the solutions become more regular up to the boundary if the initial data are more regular. We can cite Evans [52] for the linear wave equation and Refs. [105, 106, 107, 109, 135] and the references therein for the strongly damped wave equation and the Westervelt equation with the Dirichlet boundary conditions. However, it is possible [PrepWestDir] to have the same results as in [105] and [109], developed for C^2 regular boundaries, for the Westervelt problem for the Dirichlet nonhomogeneous boundary condition in the class of admissible domains in the sense of [ARP-4]: the Sobolev

extension domains with a d -set boundary preserving Markov's local inequality. This time we need to impose this geometrical restriction on $\partial\Omega$ to be able to work with the traces of $H^2(\Omega)$: the trace $\text{Tr} : H^2(\Omega) \rightarrow B_{2-\frac{n-d}{2}}^{2,2}(\partial\Omega)$ is linear continuous with the linear right inverse extension operator [172]. Thus it is possible to develop the analogous estimates to those used in Ref. [105] and [109] for a regular domain, with the Besov spaces replacing $H^{3/2}(\partial\Omega)$ and $H^{1/2}(\Omega)$. Nevertheless by [142] Main Theorem p. 337 we do not have in a general NTA domain or Lipschitz domain the estimate

$$\|\nabla w\|_{L^6(\Omega)} \leq C(\|\Delta w\|_{L^2(\Omega)} + \|Tr_{\partial\Omega} w\|_{B_{2-\frac{n-d}{2}}^{2,2}(\partial\Omega)}), \quad (2.19)$$

and hence we need to make a sly modification in the proof of Ref. [109]. In the dimension $n = 2$ this estimate stays true for convex polygonal domains by the work of Ref. [76] which allows to extend directly the results of well-posedness in Refs. [105, 106, 107, 109] found initially for a regular C^2 boundary. Instead of estimate (2.19), this time for admissible domains in \mathbb{R}^2 for fixed $p_1 > 2$ and $p'_1 > 2$ such that $2 < p_1 < q_0 + \epsilon$ (see by [142] Main Theorem p. 337) and $\frac{1}{p_1} + \frac{1}{p'_1} = \frac{1}{2}$ there exist $C_{p_1}, C_{p'_1} > 0$ such that $[PrepWestDir]$

$$\|\nabla w\|_{L^{p_1}(\Omega)} \leq C_{p_1}(\|\Delta w\|_{L^2(\Omega)} + \|Tr_{\partial\Omega} w\|_{B_{2-\frac{2-d}{2}}^{2,2}(\partial\Omega)}), \quad (2.20)$$

$$\|w\|_{L^{p'_1}(\Omega)} \leq C_{p'_1}(\|\nabla w\|_{L^2(\Omega)} + \|Tr_{\partial\Omega} w\|_{B_{1-\frac{2-d}{2}}^{2,2}(\partial\Omega)}). \quad (2.21)$$

Going back to the C^2 regularity of the boundary, it is a natural assumption for equations involving the spatial derivatives of the order less or equal to 2, as it is possible to define these derivatives in the classical way on the boundary. The same approach is obviously impossible for any less regular boundary case. Hence we work only with weak solutions taken the most possible regular in the sense that for the space variables they belong to the domain of the Laplacian $\mathcal{D}(-\Delta)$ (see Definition 2.4.2).

For different homogeneous boundary value problems, we prove the well-posed results in the following class of domains

Domain Ω	Linear equation	Nonlinear equation
$\partial\Omega = \Gamma_D$ in \mathbb{R}^2	arbitrary	NTA or limit of uniform NTA domains
$\partial\Omega = \Gamma_D$ in \mathbb{R}^3	arbitrary	arbitrary
$\Gamma_R \neq \emptyset$ in \mathbb{R}^2 or \mathbb{R}^3	Sobolev admissible	Sobolev admissible

As it was mentioned in Subsection 2.4.1 if we work in $H_0^1(\Omega)$ we do not have an influence of the shape of the boundary in the variational formulation, and thus, it is possible to work in the arbitrary domains studying the linear strong damping equation, since the results are based only on the properties of $-\Delta$. However, to control the nonlinearity of the Westervelt equation, we need to control the L^∞ -norm of the weak solution. In \mathbb{R}^3 , it is possible to do with the help of (2.16) holding thanks to [175] for the weak solutions of (2.15) in arbitrary domains. For two dimensional case, we use [142] and consequently estimate (2.16) holds only for the NTA domains, corresponding in \mathbb{R}^2 to the Sobolev admissible case. However, we improve this result, using Mosco convergence techniques, for all domains which can be obtained as a limit of a sequence of the NTA domains with uniform geometrical constants M and r_0 (see Definition 2.4.1). When the Robin boundary condition is posed on a part of

boundary or all boundary, the variational formulation contains the trace of a solution on this part of the boundary (see (2.12)). Thus the trace operator must be well-defined, which excludes for us the possibility to work in arbitrary domains.

We start by prove the L^p -maximal regularity result giving the weak well-posedness for the linear strong damped wave equation for $p \geq 2$ in the space

$$X^p := W^{1,p}([0, T]; \mathcal{D}(-\Delta)) \cap W^{2,p}([0, T]; L^p(\Omega)), \quad (2.22)$$

taking the initial data $(u_0, u_1) \in L^p(\Omega) \times L^p(\Omega)$ and the source term $f \in L^p([0, T]; L^p(\Omega))$. We notice that for $p = 2$ it is possible to consider directly the time spaces on $[0, +\infty[$, but for $p > 2$ the time interval can be arbitrary large but finite (see [63] for more details).

Remark 2.4.1 *As $-\Delta$ is a sectorial operator on $L^p(\Omega)$ [16, Thm. 5.6] admitting a bounded RH^∞ functional calculus of angle β with $0 < \beta < \frac{\pi}{2}$, then by Theorem 4.1 in Ref. [37] the boundary problems for the strong damped wave equation considered with the homogeneous initial data has L^p -maximal regularity.*

For the nonlinear problem (2.18) its weak solution $u \in X^p$ is understood in the following sense: for all $\phi \in L^2([0, T]; Y(\Omega))$

$$\begin{aligned} & \int_0^T (\partial_t^2 u, \phi)_{L^2(\Omega)} + c^2(u, \phi)_{Y(\Omega)} + \nu(\partial_t u, \phi)_{Y(\Omega)} ds \\ &= \int_0^T \alpha(u \partial_t^2 u + (\partial_t u)^2 + f, \phi)_{L^2(\Omega)} ds, \end{aligned} \quad (2.23)$$

with $u(0) = u_0$ and $\partial_t u(0) = u_1$. For $\partial\Omega = \Gamma_D$ $(u, \phi)_{Y(\Omega)} := (\nabla u, \nabla \phi)_{L^2(\Omega)}$, but for the mixed or pure Robin boundary cases

$$(u, \phi)_{Y(\Omega)} := (u, \phi)_{\text{Tr}} = (\nabla u, \nabla \phi)_{L^2(\Omega)} + a(u, \phi)_{L^2(\Gamma_R)}.$$

Thus, thanks to [PrepWestDir], [PrepWestMixed], our main result states that there exists $r^* > 0$ such that for all data (u_0, u_1, f) , taken in a ball $B_r(0)$ of $\mathcal{D}(-\Delta) \times \mathcal{D}(-\Delta) \times L^p(\mathbb{R}^+; L^p(\Omega))$ for $p > 2$ or of $\mathcal{D}(-\Delta) \times Y(\Omega) \times L^p(\mathbb{R}^+; L^p(\Omega))$ for $p = 2$ with $r \in [0, r^*[$, there exists the unique weak solution $u \in X^p$ of the nonlinear boundary problem (2.23) which stays in the ball $B_r(0)$ of X^p : i.e. $\|u\|_{X^p} \leq r$.

As in Chapter 1, the main idea for the proof is the application of the abstract theorem of Sukhinin [158], based on the maximal regularity properties of the corresponding linear model.

In the case of a plane domain for which there is a sequence of uniform NTA domains converging to it (for the definition of the convergence see Chapter 3 Section 3.2 Definition 3.2.2 and Definition 7.1 in [PrepWestDir]) instead of $u \in X^2$ we obtain a weaker solution $u \in H^1([0, +\infty[; H_0^1(\Omega)) \cap H^2([0, +\infty[; L^2(\Omega))$.

2.5 Comments and possible further developments

The interest of Eq. (2.14) is that all the relevant information on the geometry of the system is entirely represented via the eigenvalues $(\lambda_k)_{k \in \mathbb{N}}$ and the coefficients $((\mathbb{1}_\Gamma, V_k)_{L^2(\Gamma)})_{k \in \mathbb{N}}$. Similarly, the dependency of Φ concerning the physical characteristics is explicitly given in

terms of the constant λ and some multiplying factors (see [73] **section III.D** for details). Thus it allows posing the inverse problem of reconstruction of the geometry of Γ by Φ . It is of course an ill-posed problem. Inspiring of a famous analogous question of Mark Kac “Can one hear the shape of a drum?” in the framework of the spectrum of the Dirichlet-to-Neumann operator [68, 70, 71], we do not have the uniqueness of the shape Γ , since different shapes can have the same spectrum.

The next step in the area of the inverse problems in the framework of the imagery should be the generalization for the irregular boundaries the techniques of H. Ammari and al. [11]. The main question is to define the operators of the single and double layer potentials in the most possibly large class of boundaries. The first result in this area is given in [20], where the boundary is not supposed to be regular. Another possibility of an application of these operators is the theory of irregular obstacle problems, started in the fractal framework in [174].

Jonsson [98] gives the trace and extensions theorems by means of atomic decomposition and in particular write that for a set describing as the support of a measure satisfying only the upper bound condition

$$\mu(B_r(x)) \leq Cr^d, \quad 0 < r \leq 1, \quad x \in \mathbb{R}^n, \quad 0 \leq d \leq n.$$

In this case, the obtained extension operators are nonlinear (but the trace operator on the boundary stays linear). This opens many questions as is it possible to introduce these kinds of boundaries as admissible (see Section 2.1), and which kind of differences come from this fact for the solutions of the PDEs, starting by the Poisson equation.

In Subsection 2.4.2 we give the weak well-posedness of the Westervelt equation on the Sobolev admissible domains, but the analogous result for the Kuznetsov equation is an open problem. The main difficulty is the control of the nonlinear term $\nabla u \nabla u_t$ in the absence of the H^2 -regularity. The well-posedness of the KZK equation on the fractal domains is also an open question since the weak framework completely changes the developed techniques of the proof for strong solutions in H^s with $s > [\frac{n}{2}] + 1$ presented in Chapter 1.

Chapter 3

Approximations questions related with the boundary shape

Introduction

I apply the functional analysis obtained in Chapter 2 to the three types of problems related to the same question: how we can approximate a solution on a domain with a fractal or even arbitrary boundary, or how to characterize the class of boundaries on which it is possible to obtain the existence of an optimal shape in some sense.

I start in Section 3.1 with the asymptotic approximation of the speed of the propagation of the heat content for small times [ARP-6], [ARP-8], the approximation by the volume of the interior Minkowski sausage proposed by de Gennes [46] for any type of boundary on which we know the dimension. Then I introduce the Mosco convergence results for the Westervelt equation with different types of boundary conditions [PrepWestDir], [PrepWestMixed] in Section 3.2. These Mosco convergence results are not only helpful in \mathbb{R}^2 for a generalization of the well-posedness result for the Westervelt equation with homogeneous Dirichlet condition for any domain which is a limit of uniform NTA domains, but they also can be used in the framework of the shape optimization, considered for linear acoustical waves [ARP-1], [PrepShape2] in Section 3.3. Finally, in Section 3.4 I give some main ideas of my several other papers recently accepted and in preparation.

Introduction en français

J'applique l'analyse fonctionnelle obtenue au chapitre 2 aux trois types de problèmes liés à la même question: comment approcher une solution sur un domaine avec un bord fractal ou même arbitraire ou comment caractériser la classe des bords sur laquelle il se trouve possible d'obtenir une forme optimale dans un certain sens.

Je commence dans la section 3.1 par l'approximation asymptotique de la vitesse de propagation du contenu calorifique pour les temps petits [ARP-6], [ARP-8], l'approximation par le volume de la saucisse Minkowski intérieure proposée par de Gennes [46] pour tout type de frontière dont nous connaissons la dimension. J'introduis ensuite les résultats de convergence Mosco pour l'équation de Westervelt avec différents types de conditions aux limites [PrepWestDir], [PrepWestMixed] dans la section 3.2, qui ne sont pas seulement

utiles dans \mathbb{R}^2 pour une généralisation du résultat du caractère bien posé de l'équation de Westervelt avec une condition aux limites de Dirichlet homogène pour tout domaine qui est une limite de domaines NTA uniformes, mais peut également être utilisé dans le cadre de l'optimisation de forme, considérée pour les ondes acoustiques linéaires [ARP-1], [PrepShape2] dans la section 3.3. Enfin, dans la section 3.4, je donne quelques idées principales de mes autres articles en préparation.

3.1 Heat content and de Gennes' hypothesis [ARP-6], [ARP-8]

As mentioned in [ARP-8], the radiator's shape is significant for the speed of diffusive heat transfer. If we consider numerically (numerical calculus made with COMSOL Multiphysics for the model described by the linear heat equation) a cavity composed at the initial time with a hot and a cold medium (see Fig. 3.1 on the bottom) separated by an interface of different length, then we can notice that the speed of the heat propagation is an increasing function of the length of the interface for any fixed (rather small) time. We can compare the bottom pictures with the up pictures on Fig. 3.1, presenting the propagation of the heat by a boundary in a cold medium. In a cavity with two media, we can notice two propagations: heat propagation (from the hot to cold) and coldness propagation (from the cold to hot). We do not have this phenomenon in the up figures since the hot boundary has for all times the same constant temperature equal to 1. From a theoretical and practical point of view, the case of the two media is more interesting.

Once again, the beneficial interest is to make hot/cold the medium with the opposite temperature (cold/hot) the most rapidly possible. Hence, the aim is to study the behavior of the diffusion for short times. In addition, if we denote the mean heat content by $N(t)$, equal to the integral over the domain of the heat propagation of the solution of the heat equation in this domain with a thermal isolated exterior boundary, then we can observe on Fig. 3.2 that for the long times, $t \rightarrow +\infty$ there are no any influence of the geometry since the heat content converges to the constant state (the constant temperature of two media). However, the geometrical influence is very important in the regime of small times ($t \rightarrow +0$). Besides, we can see in Fig. 3.2 the existence of three times regions characterizing different speeds of the heat propagation following different asymptotes (the blue and red dotted lines). Thus these different speeds are asymptotically characterized by different powers of t . Exactly this dependence was pointed by de Gennes [46]. In the case when there are no resistivity of the boundary to the heat propagation, de Gennes [46] argued that as $t \rightarrow +0$, $N(t)$ is proportional to the volume $\mu(\partial\Omega, \sqrt{D_+t})$ of the interior Minkowski sausage of $\partial\Omega$ of the width equal to the diffusion length $\sqrt{D_+t}$:

$$\mu(\partial\Omega, \ell) = \text{Vol}(\{x \in \Omega \mid \text{dist}(x, \partial\Omega) < \ell\})$$

(see also Ref. [124]). In particular,

- for a regular boundary $\partial\Omega$, $N(t)$ is proportional to $\text{Vol}(\partial\Omega)\sqrt{D_+t}$;
- for a fractal boundary $\partial\Omega$ of the Hausdorff dimension d , $N(t)$ is proportional to $(D_+t)^{\frac{n-d}{2}}$.

The de Gennes scaling argument was further investigated in [ARP-8], both experimentally and numerically. It was shown that irregularly shaped passive coolers rapidly dissipate at

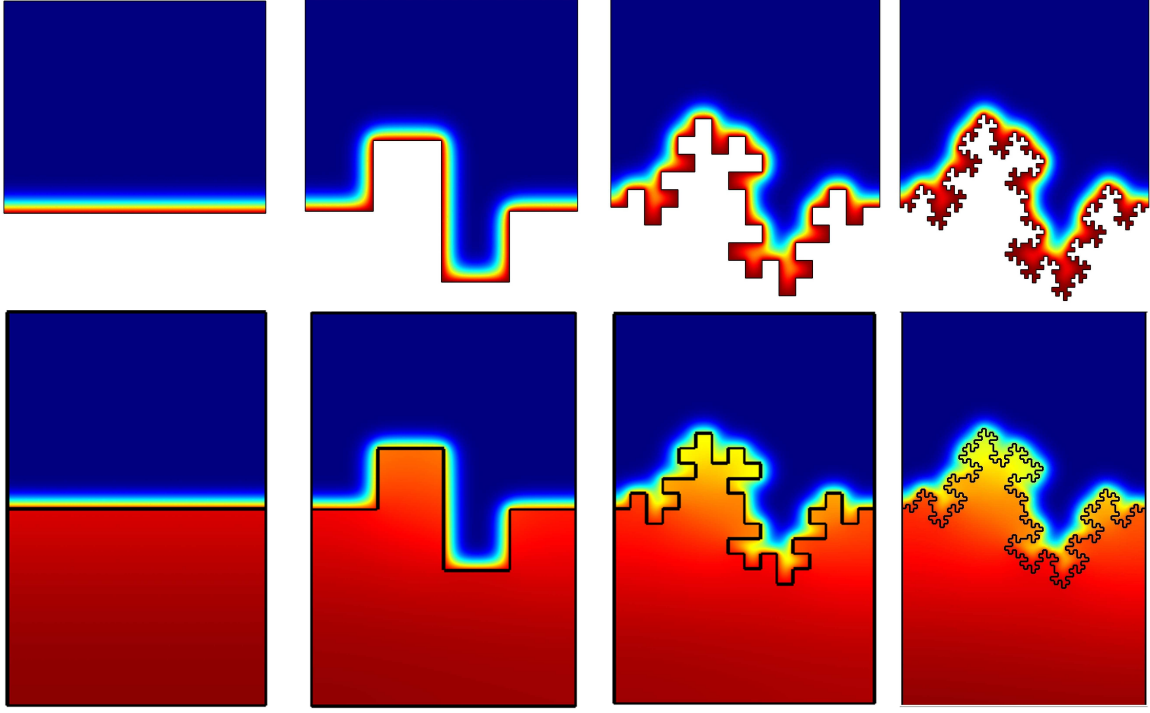


Figure 3.1 – Influence of geometry on heat propagation at the time moment $t = 0.1$. Red colors are hot, and blue colors are cold. On the top: the Dirichlet condition, equal to 1, is imposed on the bottom boundary. On the bottom: the propagation between a hot and a cold media in a thermo-isolated cavity.

short times, but their efficiency decreases with time. The de Gennes scaling argument was shown to be only a large scale approximation, which is not sufficient to describe the temperature distribution close to the irregular frontier adequately. See also Fig. 3.3 for the case of one cold medium with a hot boundary.

Thus the main goal of [ARP-6] is to develop the preliminary study gave in [ARP-8] and especially to formalize the seminal approach by de Gennes [46].

But the case of a bounded domain separated into two subdomains by an interface between two media is a too complicated case from the theoretical point of view as soon as there are two boundary points that belong at the same time to the interior and the exterior boundary. This means that in these points, we have formally everything discontinuous: the diffusion coefficients and the type of the boundary condition. To avoid this complicated situation, instead of a divided into two media cavity, we consider all space \mathbb{R}^n divided into two media by a bounded domain, homeomorphic to a ball, and by its exterior domain.

More precisely in [ARP-6], we consider a bounded domain $\Omega \subset \mathbb{R}^n$ with boundary $\partial\Omega$ that splits \mathbb{R}^n into “hot” and “cold” media, $\Omega_+ = \Omega$ and $\Omega_- = \mathbb{R}^n \setminus \overline{\Omega}$, characterized by (distinct) heat diffusion coefficients D_+ and D_- (Fig. 3.4). On the boundary $\partial\Omega$ is also defined a function $0 \leq \lambda(x) \leq \infty$ which describes the resistivity to heat exchange through the boundary.

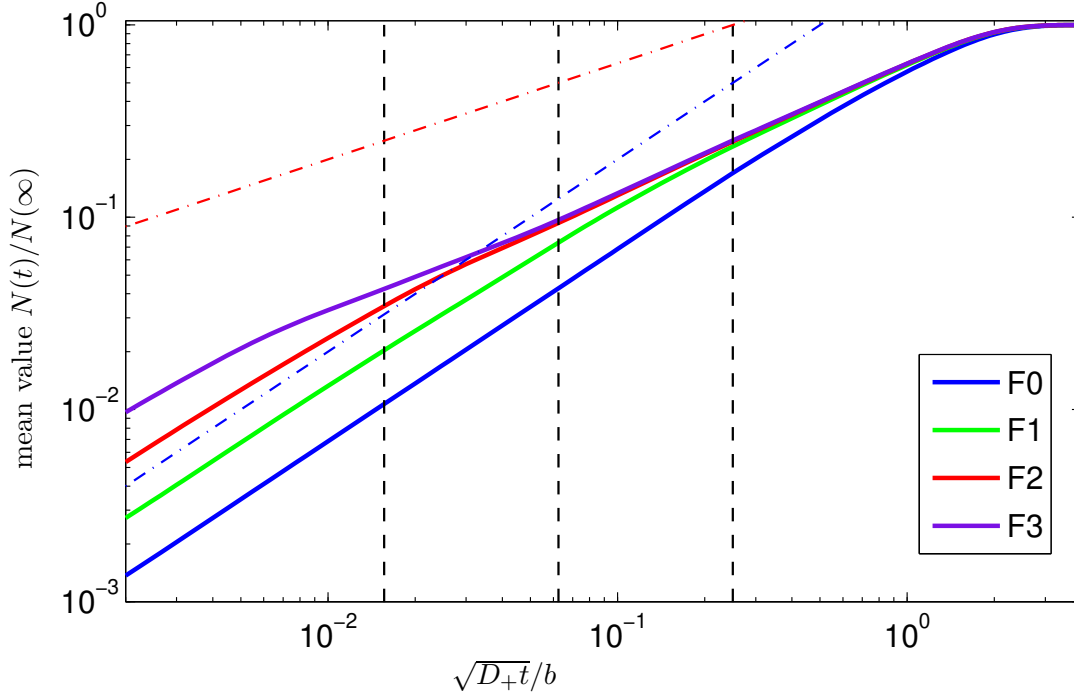


Figure 3.2 – Two dimensional propagation in time for two symmetric media of the volume equal to the volume of $[0, l] \times [0, b]$ with $l = 1$, $b = 3$ for four pre-fractal cavities given in the bottom on Fig. 3.1. At $t = 0$ one of the media is hot, and the other is cold (the diffusion coefficients of the cold and hot medium respectively are $D_- = 1$ and $D_+ = 1/100$). Asymptotes: $2^i \frac{\sqrt{D_+ t}}{b}$ with $i = 0, 1, 2, 3$ (fractal generation).

We are interested in propagation of the heat content associated with the following problem:

$$\partial_t u_{\pm} - D_{\pm} \Delta u_{\pm} = 0 \quad x \in \Omega_{\pm}, \quad t > 0, \quad (3.1)$$

$$u_+|_{t=0} = 1, \quad u_-|_{t=0} = 0, \quad (3.2)$$

$$D_- \frac{\partial u_-}{\partial n} \Big|_{\partial \Omega} = \lambda(x)(u_- - u_+) \Big|_{\partial \Omega}, \quad (3.3)$$

$$D_+ \frac{\partial u_+}{\partial n} \Big|_{\partial \Omega} = D_- \frac{\partial u_-}{\partial n} \Big|_{\partial \Omega}, \quad (3.4)$$

where $\partial/\partial n$ is the normal derivative directed outside the domain Ω .

We develop a rigorous analysis of problem (3.1)–(3.4) for irregular boundaries given by d -sets, which requires its variational formulation in appropriate functional spaces. The variational problem is shown to have a unique weak solution with the desired trace properties on the boundary $\partial \Omega$.

Once a unique solution u_{\pm} of the problem (3.1)–(3.4) is established, we study the asymptotic expansion of the heat content as $t \rightarrow 0$

$$N(t) = \int_{\mathbb{R}^n \setminus \Omega} u_-(x, t) dx = \text{Vol}(\Omega) - \int_{\Omega} u_+(x, t) dx. \quad (3.5)$$

Eqs. (3.1)–(3.4) describe heat exchange between two media prepared initially at different temperatures and separated by a partially isolating boundary [32, 40]. In fact, $u(x, t)$

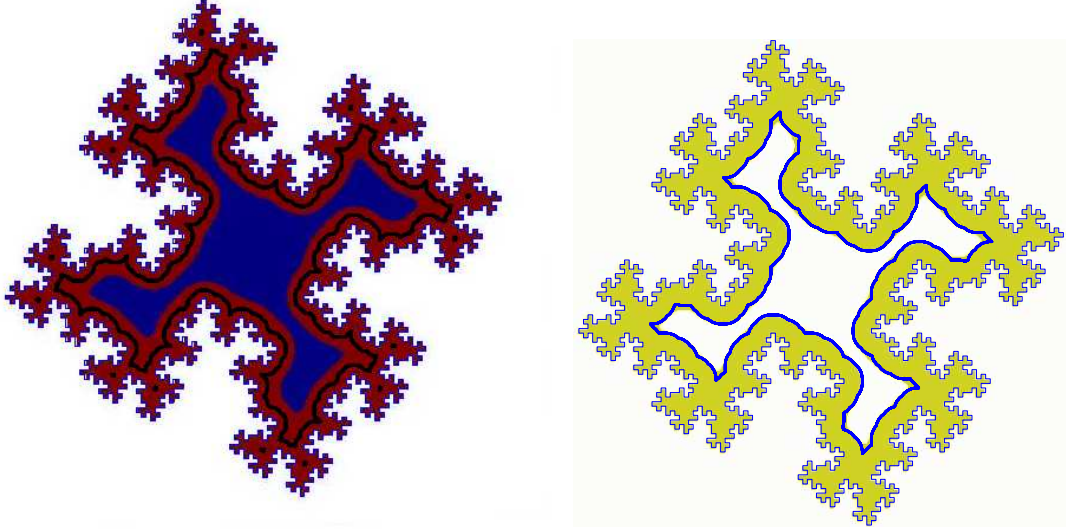


Figure 3.3 – The interior Minkowski sausages the width equal to $\ell_D = \sqrt{D_+ t_0}$ (on the left) and to $\ell_D = \sqrt{4D_+ t_0}$ (on the right) compared to the isolines $u(x, t_0) = 0.1$ of the temperature at $t_0 = 0.1$ (the solution of the heat equation with the constant Dirichlet condition equal to 1 and homogeneous initial data). On the left, the border of the interior Minkowski sausage is the black line, the limit between the red and blue media is the isoline. On the right, the border of the interior Minkowski sausage is the blue line, the limit between the yellow and white media is the isoline.

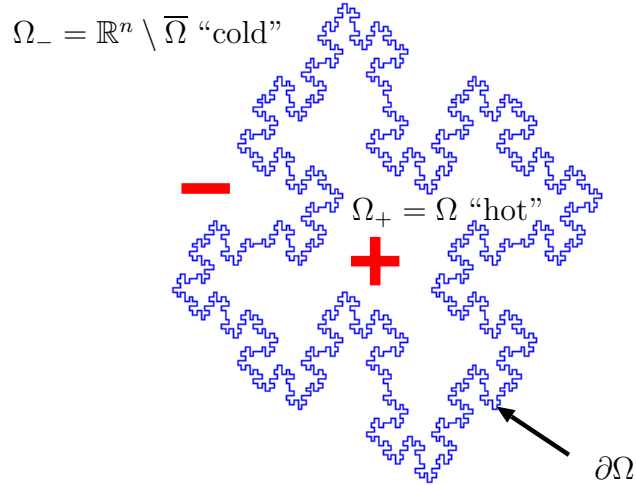


Figure 3.4 – Illustration of the heat content problem for a planar domain Ω with pre-fractal boundary $\partial\Omega$ presented by the third generation of the Minkowski fractal (of fractal dimension $3/2$). This boundary splits the plane into two complementary regions. At time $t = 0$, the inner region $\Omega_+ = \Omega$ is “hot” (functions on Ω_+ are denoted with subscript $+$), while the outer region $\Omega_- = \mathbb{R}^n \setminus \overline{\Omega}$ is “cold” (functions on Ω_- are denoted with subscript $-$).

can describe how the distribution of (normalized) temperature evolves with time. The transmission boundary conditions (3.3), (3.4) impose the continuity of the temperature flux across the boundary, and relate this flux to the temperature drop at the boundary due to thermal isolation. The growth rate of the heat content with time characterizes the efficiency of thermal isolation. Understanding this problem is relevant to improve heat exchanges,

e.g., cooling of metallic radiators or thermal isolation of pipes and buildings. Depending on application, cooling rate has to be either enhanced (e.g., in the case of microprocessors or nuclear reactors), or slowed down (e.g., in the case of pipes and buildings). For these purposes, one can either modify the thermal isolation (i.e., the resistivity λ), or the shape of the exchange boundary. It is therefore crucial to understand how the shape of the boundary influences heat exchange. In particular, would an irregular (e.g., fractal) boundary with a very large exchange area significantly speed up cooling?

Similar equations can describe molecular diffusion between two media across semi-permeable membranes [160, 148]. In that case, $u(x, t)$ represents the (normalized) concentration of molecules, while Eqs. (3.1)–(3.4) can model the leakage of molecules from a cell (Ω_+) to the extracellular space (Ω_-) or, more generally, the diffusive exchange between two compartments (e.g., oxygen or carbon dioxide exchange between air and blood across the alveolar membrane in the lungs). The resistance λ is related to the cellular membrane permeability. As for heat exchange, one may need to enhance or slow down the molecular leakage, and the shape of the boundary may play an important role.

The discontinuity of the initial condition, of the diffusion coefficient, and the solution $u(x, t)$ across the boundary between two domains constitutes one of the mathematical difficulties to be treated. From a physical point of view, such discontinuities might appear unrealistic. For instance, the diffusive flux at the boundary at time $t = 0$ is infinite. There would be an intermediate layer between two media in which the material properties would change rapidly but continuously for any physical setting of heat or molecular diffusion. When the thickness of this intermediate layer is much smaller than the size of the domain, the physical problem with continuously varying parameters can be approximated by the heat problem (3.1)–(3.4). Such an approximation is applicable starting from a small cut-off time while understanding the heat exchange at smaller time scales would need either restituting an intermediate layer, or introducing nonlinear terms into the heat equation. But let us, as in [ARP-6], focus on the mathematical problem (3.1)–(3.4).

The physical properties of the two media Ω_+ and Ω_- are supposed to be different: $D_+ \neq D_-$. This implies the discontinuity of the metric on $\partial\Omega$. The case of continuous metric ($g_-|_{\partial\Omega} = g_+|_{\partial\Omega}$) on smooth compact n -dimensional Riemannian manifolds with a smooth boundary $\partial\Omega$ was considered in Ref. [64]. The case of continuous transmission boundary conditions for the expansion of the heat kernel on the diagonal was treated in Ref. [147] (see also Ref. [170] for a survey of results on the asymptotic expansion of the heat kernel for different boundary conditions). The heat content asymptotic expansion with Dirichlet boundary condition was found

- up to the third-order term for a compact connected domain $\Omega \subset \mathbb{R}^n$ with a regular boundary $\partial\Omega \in C^3$ (Refs. [169, 167]);
- up to an exponentially small error for a compact connected domain $\Omega \subset \mathbb{R}^2$ with a polygonal $\partial\Omega$ (Ref. [168]) and for $\Omega \subset \mathbb{R}^2$ with $\partial\Omega$ given by the triadic Von Koch snowflake (Ref. [61]);
- up to the second-order term for the general case of self-similar fractal compact connected domains in \mathbb{R}^n (Ref. [124]).

In general, the boundary between two media can have some resistance to heat exchange, described by the function $\lambda(x) \geq 0$ ($x \in \partial\Omega$) that may account for partial thermal isolation.

We outline three cases of boundary conditions according to λ :

1. If $0 < \lambda(x) < \infty$ for all $x \in \partial\Omega$, u is discontinuous on $\partial\Omega$ and we have:

$$\left(\lambda(x)u_- - D_- \frac{\partial u_-}{\partial n} \right) |_{\partial\Omega} = \lambda(x)u_+ |_{\partial\Omega}, \quad D_+ \frac{\partial u_+}{\partial n} |_{\partial\Omega} = D_- \frac{\partial u_-}{\partial n} |_{\partial\Omega}.$$

2. If $\lambda = +\infty$ for all $x \in \partial\Omega$, u is continuous on $\partial\Omega$ due to the transmission condition and in this case

$$u_+ |_{\partial\Omega} = u_- |_{\partial\Omega}, \quad D_+ \frac{\partial u_+}{\partial n} |_{\partial\Omega} = D_- \frac{\partial u_-}{\partial n} |_{\partial\Omega}.$$

3. If $\lambda = 0$ for all $x \in \partial\Omega$, we have the Neumann boundary condition

$$\frac{\partial u_-}{\partial n} |_{\partial\Omega} = \frac{\partial u_+}{\partial n} |_{\partial\Omega} = 0$$

that models the complete thermal isolation of $\partial\Omega$ and implies the trivial solution given by $u_-(x, t) = 0$ and $u_+(x, t) = 1$ for all time $t \geq 0$.

We provide a mathematical foundation and further understanding for the de Gennes approach. In [ARP-6] we obtain three results valid for all bounded (ϵ, δ) -domains Ω in \mathbb{R}^n with connected boundary $\partial\Omega$, presented by a closed d -set (see Chapter 2 for the definitions of (ϵ, δ) -domains and d -sets):

1. the well-posedness of the problem (3.1)–(3.4),
2. the continuity of the solution on λ ,
3. the asymptotic expansion of the heat content (3.5).

For the well-posedness results, which finally, thanks to [BookChap] and [PrepWest-Mixed], it is possible for all bounded Sobolev admissible domains $\Omega = \Omega_+$ with connected boundary $\partial\Omega$ (see Definition 2.1.5), we need to introduce the space

$$V = \{f \in L^2(\mathbb{R}^n) \mid f_+ = f|_{\Omega_+} \in H^1(\Omega_+), \text{ and } f_- = f|_{\Omega_-} \in H^1(\Omega_-)\}$$

of functions $f = f_+ \mathbf{1}_{\Omega_+} + f_- \mathbf{1}_{\Omega_-}$ defined on $\Omega_+ \cup \Omega_-$ such that their restrictions $f_+ = f|_{\Omega_+}$ and $f_- = f|_{\Omega_-}$ belong to H^1 . We equip V with the norm:

$$\|u\|_V^2 = D_+ \int_{\Omega_+} |\nabla u_+|^2 dx + D_- \int_{\Omega_-} |\nabla u_-|^2 dx + \int_{\Omega_+ \cup \Omega_-} |u|^2 dx.$$

Therefore, V is a Hilbert space, $V \subset L^2(\Omega)$, and V is dense in $L^2(\Omega)$. In addition, $V \subset L^2(\mathbb{R}^n) \subset V'$, where V' is the dual space to V . Finally, since $\partial\Omega$ is not less irregular as in Theorem 2.1.2 defined by the support of a measure μ , the inclusion $V \subset L^2(\mathbb{R}^n)$ is compact. Thus, in the usual way using the continuous and coercive bilinear form on $V \times V$ with the notation μ for the measure on $\partial\Omega$

$$a(u, v) = D_+ \int_{\Omega_+} \nabla u_+ \nabla v_+ + D_- \int_{\Omega_-} \nabla u_- \nabla v_- + \int_{\partial\Omega} \lambda(x)(u_+ - u_-)(v_+ - v_-) d\mu, \quad (3.6)$$

we obtain the weak well-posedness of problem (3.1)–(3.4) for a positive continuous function $\lambda \in C(\partial\Omega)$ in the following sense: there exists a unique solution $u \in C(\mathbb{R}_t^+, L^2(\mathbb{R}^n)) \cap L^2(\mathbb{R}_t^+, V)$ of the variational problem

$$\forall v \in V \quad \frac{d}{dt} \langle u, v \rangle_{L^2(\mathbb{R}^n)} + a(u, v) = 0, \quad u(x, 0) = u_0 \in L^2(\mathbb{R}^n). \quad (3.7)$$

When $\lambda = +\infty$ on $\partial\Omega$ then the boundary term in Eq. (3.6) disappears and hence $V = H^1(\mathbb{R}^n)$. For the continuous dependence of u on λ see Theorem 2.2 [ARP-6].

Once we know the well-posedness results, for d -set boundaries with $n - 1 \leq d < n$ we show that the heat content $N(t)$ is approximated by the volume of the interior Minkowski sausage of $\partial\Omega$ of the radius $\sqrt{4D_+t}$:

$$N(t) = \tau_\lambda \left[C_\lambda(\partial\Omega) \mu \left(\partial\Omega, \sqrt{4D_+t} \right) + O \left(\mu^2 \left(\partial\Omega, \sqrt{4D_+t} \right) \right) \right], \quad (3.8)$$

where τ_λ is equal to 1 if $\lambda = \infty$ and \sqrt{t} if $\lambda > 0$ is finite. Here $C_\lambda(\partial\Omega)$ is a constant depending only on the shape of $\partial\Omega$ and finiteness of λ . Formula (3.8) is the first approximation, which allows to find $N(t)$ up to terms of the order $\tau_\lambda O \left(\sqrt{t} \mu \left(\partial\Omega, \sqrt{4D_+t} \right) \right)$.

Moreover, the asymptotic relation (3.8) remains valid even for mixed boundary conditions for three disjoint boundary parts, i.e. when $\lambda = \infty$ on one part of the boundary, $\lambda = 0$ on another part, and $0 < \lambda < \infty$ on the remaining boundary. However, changes of the type of the boundary condition should be continuous such that u remains a continuous function of λ . In this more general case, for $0 < \lambda < \infty$ the coefficient $C_\lambda(\partial\Omega)$ in Eq. (3.8) is given by

$$N(t) = \frac{2\sqrt{t} \mu(\partial\Omega, \sqrt{4D_+t})}{\sqrt{D_+} \text{Vol}(\partial\Omega)} \left[\int_{\partial\Omega} d\sigma \lambda(\sigma) \int_1^2 dz f(\sigma, z, t) - \int_1^2 dz (z-1)^{n-d} \int_{\partial\Omega} d\sigma \lambda(\sigma) f(\sigma, z, t) - \int_0^1 dz z^{n-d} \int_{\partial\Omega} d\sigma \lambda(\sigma) f(\sigma, z, t) \right] + O(\sqrt{t} \mu(\partial\Omega, \sqrt{t})^2), \quad (3.9)$$

where $d\sigma$ is understood in the sense of the d dimensional Hausdorff measure (see Ref. [113, 65]) on $\partial\Omega$, $\alpha = \frac{1}{\sqrt{D_-}} + \frac{1}{\sqrt{D_+}}$ and

$$f(\sigma, z, t) = \exp \left(2\lambda(\sigma)\alpha\sqrt{t}z + \lambda(\sigma)^2\alpha^2t \right) \text{Erfc}(z + \lambda(\sigma)\alpha\sqrt{t}). \quad (3.10)$$

In the case $\lambda = \infty$ it is defined by

$$C_\lambda(\partial\Omega) = \frac{2\sqrt{D_-} \beta_{n-d}}{\sqrt{D_-} + \sqrt{D_+}} \quad (3.11)$$

with the prefactor

$$\beta_x \equiv \int_0^2 \frac{z^x e^{-z^2}}{\sqrt{\pi}} dz = \frac{1}{2\sqrt{\pi}} \gamma \left(\frac{x+1}{2}, 4 \right) \quad (3.12)$$

expressed through the incomplete Gamma function. Or again, it is equal to 0 for $\lambda = 0$ (the boundary with $\lambda = 0$ does not contribute to the short-time asymptotics of the heat content). Finding the asymptotics for mixed boundary conditions with a discontinuous jump from a finite λ to $\lambda = \infty$ is still an open problem.

As expected, the resistivity of the boundary to heat transfer makes heat diffusion *slower* due to the presence of the coefficient $\tau_\lambda = \sqrt{t}$.

The comparison between the asymptotic formula (3.8) and a numerical solution of problem (3.1)–(3.4) for the unit square and a pre-fractal domain is shown in Fig. 3.5 for a finite λ and in Fig. 3.6 for $\lambda = +\infty$. The numerical solution was obtained in FreeFem++ by a

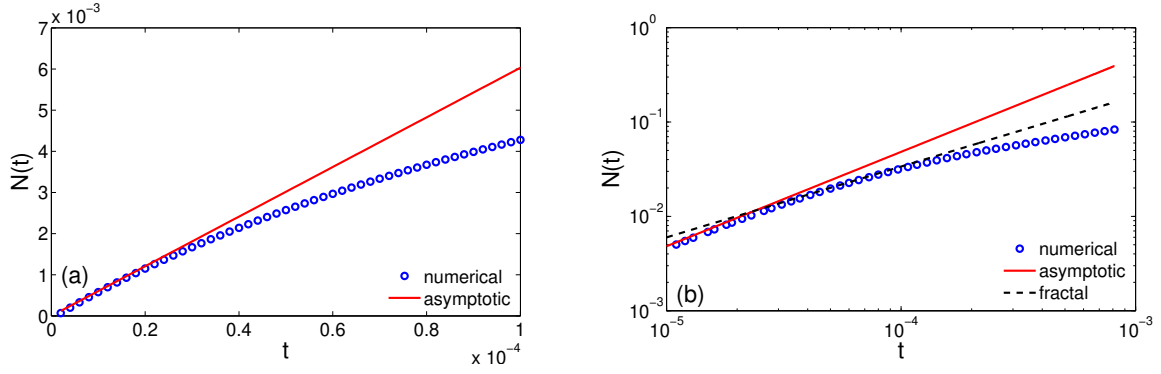


Figure 3.5 – Comparison between the asymptotic formula (3.8) (solid line) and a FreeFem++ numerical solution of the problem (3.1)–(3.4) (circles) for two domains: (a) the unit square ($\text{Vol}(\partial\Omega) = 4$) and (b) the third generation of the Minkowski fractal ($\text{Vol}(\partial\Omega) = 2^3 \cdot 4$), with $D_+ = 1/100$, $D_- = 1$, and $\lambda = 17$. Since the Hausdorff dimension of the boundaries of these domains is 1 (even for the *pre-fractal* case), Eq. (3.8) for a constant λ is reduced to $N(t) = 2\sqrt{t}C_0\lambda\mu(\partial\Omega, \sqrt{4D_+t}) + O(t^{\frac{3}{2}})$ with $\mu(\partial\Omega, \sqrt{4D_+t}) \simeq \sqrt{4D_+t} \text{Vol}(\partial\Omega)$ and $C_0 = 1 + \frac{3}{2}\text{erf}(1) - \frac{9}{4}\text{erf}(2) + \frac{1}{\sqrt{\pi}}\left(\frac{1}{e} - \frac{1}{e^4}\right) \approx 0.2218$. For plot (b), dashed line shows the fractal asymptotic (that would be exact for the infinite generation of the fractal) with de Gennes approximation of $\mu(\partial\Omega, \sqrt{4D_+t})$ in Eq. (3.8) by $(4D_+t)^{\frac{1}{4}}$. This approximation is valid for intermediate times.

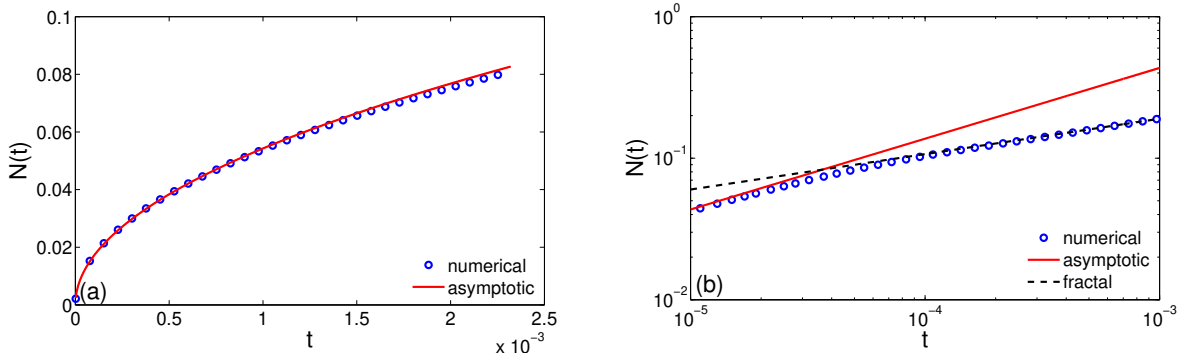


Figure 3.6 – Comparison between the asymptotic formula (3.8) (solid line) and a FreeFem++ numerical solution of the problem (3.1)–(3.4) (circles) for two domains: (a) the unit square ($\text{Vol}(\partial\Omega) = 4$), and (b) the third generation of the Minkowski fractal ($\text{Vol}(\partial\Omega) = 2^3 \cdot 4$), with $D_+ = 0.4$, $D_- = 1$, and $\lambda = \infty$. Since the *pre-fractal* boundary $\partial\Omega$ has the Hausdorff dimension 1, Eq. (3.8) is reduced to (3.13), i.e., $N(t) \propto \sqrt{t}$. In turn, dashed line shows the fractal asymptotic (that would be exact for the infinite generation of the fractal) with de Gennes approximation of $\mu(\partial\Omega, \sqrt{4D_+t})$ in Eq. (3.8) by $2.5(4D_+t)^{\frac{1}{4}}$. This approximation is valid for intermediate times.

finite element method with the implicit θ -schema, also known as Crank-Nicolson schema, for the time discretization with $\theta = \frac{1}{2}$ and $\Delta t = 10^{-6}$. The domain Ω was centered in a ball B of diameter (at least) twice bigger than the diameter of Ω . The Neumann boundary condition was imposed on the boundary of the ball. According to the principle “not feeling the boundary” [61], the heat content propagation in \mathbb{R}^2 with a prescribed boundary $\partial\Omega$ can be very accurately approximated at short times by the heat content propagation computed in B . The accuracy of this approximation can also be checked by changing the diameter of the ball. In the case of the square domain Ω , the ball was replaced by a square with

four times bigger edge. Each pre-fractal edge was discretized with 27 space points while 57 points were used in the external boundary of the ball. The mesh size was varied to check the accuracy of the presented numerical solutions. For the case of the discontinuous solution on the boundary (when $0 < \lambda < \infty$) we apply the domain decomposition method and match the boundary values of the respective solutions on $\partial\Omega$ by a Picard fixed point method. Therefore, we consider the numerical solution of heat propagation for small times as a reference, to which asymptotic formulas are compared. In particular, deviations between the numerical solution and the asymptotic formulas observed at longer times illustrate the range of validity of the short-time expansion.

For the regular case $\partial\Omega \in C^3$, we obtain the heat content approximation up to the third-order term. The formulas are given in Theorem 7.1 [ARP-6]. For the case $\lambda < \infty$, the coefficient in front of the second-order term ($t^{\frac{3}{2}}$) in the asymptotic expansion depends on the mean curvature. In turn, for $\lambda = \infty$, the second-order term (here, t) in the asymptotic expansion vanishes:

$$N(t) = 2 \frac{1 - e^{-4}}{\sqrt{\pi}} \frac{\sqrt{D_- D_+}}{\sqrt{D_+} + \sqrt{D_-}} \text{Vol}(\partial\Omega) \sqrt{t} + O(t^{\frac{3}{2}}). \quad (3.13)$$

The proof of these asymptotic expansion formulas is very technical. In several words, we need to calculate explicitly the Green function of the constant coefficient problem in the half space. We start by proving that the problem to find $N(t)$ can be replaced by a heat problem localized in $O(\sqrt{t})$ -interior Minkowski sausage of the boundary Ω_ϵ by a variant of the principle “not feeling the boundary” [61] in the general case in \mathbb{R}^n . This allows, due to the continuity of u on λ , to establish for a constant $\delta > 0$

$$N(t) = \int_{\Omega} (1 - u(x, t)) dx = \int_{\Omega_\epsilon} (1 - u(x, t)) dx + O(e^{-\frac{1}{t^\delta}}),$$

can be found as a sum of two heat contents according to the finiteness or not of λ (including the value zero) in the boundary conditions (*i.e.* for $\partial\Omega = \Gamma_\infty \cup \Gamma_\lambda \cup \Gamma_0$):

$$N(t) = \int_{\Omega_\epsilon^{\Gamma_\lambda}} (1 - u(x, t)) dx + \int_{\Omega_\epsilon^{\Gamma_\infty}} (1 - u(x, t)) dx + O(e^{-\frac{1}{t^\delta}}).$$

In order to prove Eq. (3.8) for a large class of (ϵ, δ) -bounded connected domains Ω in \mathbb{R}^n , we first prove it for the case of domains with regular boundary $\partial\Omega \in C^\infty$ or at least in C^3 . As Ω is bounded, for all types of connected $\partial\Omega$, the volume of Ω is finite and, therefore, the volume of the ϵ -neighborhood of $\partial\Omega$ in Ω is also finite and can be approximated by a sequence of volumes of Minkowski sausages with regular boundaries (the same argument was used in Ref. [61] p. 378). Considering a regular $\partial\Omega$ (at least in C^3) and using the localization properties of the heat propagation, we rewrite the formula for $N(t)$ in terms of the local coordinates. We firstly give the approximation of the heat problem solution through the solution of one-dimensional constant-coefficient problem. The key point is that, according to Ref. [133] p. 48–49, due to Varadhan’s bound property of Green functions, locally the difference between the Green function of the problem in the local coordinates with “frozen” coefficients in one boundary point and the analogous Green function of the constant coefficient problem in the half space in \mathbb{R}^n is exponentially small. Therefore, following the ideas of McKean and Singer [133] (p. 49), we approximate the Green function of the problem in the local coordinates by the Green function with the frozen coefficients

on one boundary point, whose replacement by the known explicitly the Green function of the constant coefficient problem in the half space yields only an exponentially small error. Thus we approximate locally the solution using Duhamel formula, constructing a parametrix [133]. To relate the heat content expansion with the interior Minkowski sausage, roughly speaking, we follow the strategy of the construction of the Riemann integral: we construct an elementary sum or a discretization and pass to the limit applying at the same time the mean value theorem. In addition, we also use the relation

$$\mu(\partial\Omega, \epsilon z) = z^{n-d} \mu(\partial\Omega, \epsilon) + O(\epsilon^{2(n-d)}),$$

which, for a fixed z and $\epsilon \rightarrow +0$, is evident for the regular case and can be proved by approximating the fractal volume by a converging sequence of the volumes for smooth boundaries.

3.2 Mosco-type convergence for the Westervelt initial-boundary value problems $[PrepWestMixed]$, $[PrepWestDir]$

As we could see in the previous section, for the numerics, it is not possible to have a von Koch or other type fractal boundary, but only its a finite pre-fractal generation. In the most common case, the third generation is already rather complex and needs much attention to a correct meshing [33]. Hence there is a question about the approximation of a solution on the fractal domain by a solution in a pre-fractal domain. A typical method to treat this kind of questions is to consider the Mosco convergence, or M -convergence, of the energies or weak formulations (thus functionals) of the considered PDE problem. There are at least three possible types of this convergence: for quadratic forms, functionals, and spaces. In addition [83, p.113] M -convergence is related with γ -convergence.

In the aim to approximate the solutions of the Westervelt equation we use the notion of M – convergence of functionals introduced in Ref. [138].

Definition 3.2.1 *A sequence of functionals $G^m : H \rightarrow (-\infty, +\infty]$ is said to M -converge to a functional $G : H \rightarrow (-\infty, +\infty]$ in a Hilbert space H , if*

1. (*lim sup condition*) *For every $u \in H$ there exists u_m converging strongly in H such that*

$$\overline{\lim} G^m[u_m] \leq G[u], \quad \text{as } m \rightarrow +\infty. \quad (3.14)$$

2. (*lim inf condition*) *For every v_m converging weakly to u in H*

$$\underline{\lim} G^m[v_m] \geq G[u], \quad \text{as } m \rightarrow +\infty. \quad (3.15)$$

M -convergence of functionals can be used to obtain a the well-posedness of the corresponding problem on the limit domain. Let us give the main ideas.

Let $(\Omega_m)_{m \in \mathbb{N}}$ converge to Ω in some sense (for example, a pre-fractal sequence of boundaries converging to the fractal one). The interest of the construction could be in the assumption that the boundaries of $(\Omega_m)_{m \in \mathbb{N}}$ are more regular than the boundary of their limit $\partial\Omega$ [57, Lemma 7.1]. Therefore, let us suppose that it is possible for a regular boundary

to have a weak well-posedness result for a boundary-valued problem. Taking then $G^m[u_m]$ as their weak formulations, the M -convergence implies that if $(u_m)_{m \in \mathbb{N}}$ the sequence of the weak-solutions on $(\Omega_m)_{m \in \mathbb{N}}$ uniformly bounded on m (i.e., independently on the shape of $\partial\Omega_m$) then there exists u , the weak limit of their extensions (or of a subsequence of their extensions) on a unique large domain, and $0 = G^m[u_m] \rightarrow G[u]$ for $m \rightarrow +\infty$ as a numerical sequence. Consequently, $G[u] = 0$ becomes the variational formulation of the limit domain, and hence u is its weak solution, unique by the uniqueness of the weak limit of extensions of u_m .

The main difficulty in applying this method is to ensure that the operators of the extension are uniform on the shape of Ω_m and the sequence of the weak solutions $(u_m)_{m \in \mathbb{N}}$ are uniformly bounded on m too. These two questions partially define the class of limit domains Ω and the approximate sequence $(\Omega_m)_{m \in \mathbb{N}}$ and very related with the framework of the shape optimization considered in the next section. For instance it is not the case of the solutions of the Poisson for the homogeneous Robin boundary condition

$$\frac{\partial u_m}{\partial n} + \frac{a}{\lambda(\partial\Omega_m)} u_m = 0$$

on a pre-fractal sequence of boundaries converging to a fractal one, i.e. the lengths $\lambda(\partial\Omega_m) \rightarrow +\infty$ for $m \rightarrow +\infty$, since the constant in Eq. (2.16) depends on $\frac{\text{Vol}(\partial\Omega_m)}{a}$ by [44]. Nevertheless, if there is a part of the boundary with the homogeneous Dirichlet condition, we have Eq. (2.16) with a constant independent on m by the Poincaré inequality (see Theorem 3.2 [PrepWestMixed]). In the homogeneous Dirichlet case the class of validity of Eq. (2.16) defines the class of the approximation domains Ω_m : arbitrary domains for $n = 3$ and NTA domains with the same geometrical constants M and r_0 for $n = 2$ [PrepWestDir] (see Definition 2.4.1 for the definition of these constants). In the case of arbitrary domains in \mathbb{R}^3 , we approximate Ω by a sequence of arbitrary domains $(\Omega_m)_{m \in \mathbb{N}^*}$ which converges to Ω in the following sense:

Definition 3.2.2 *We say that a sequence of domains $(\Omega^m)_{m \in \mathbb{N}^*}$ converges to Ω , $\Omega^m \rightarrow \Omega$ if the following two conditions are satisfied*

1. *for any compact $K \subset \Omega$ there is $m = m(K) \geq 0$ such that*

$$K \subset \Omega^m \text{ for all } m \geq m(K),$$

2. *the sets $\Omega^m \setminus \Omega$ are bounded and for $m \rightarrow +\infty$*

$$\lambda((\Omega \setminus \Omega^m) \cup (\Omega^m \setminus \Omega)) \rightarrow 0.$$

In \mathbb{R}^3 we can take arbitrary domains for $(\Omega_m)_{m \in \mathbb{N}^*}$ and Ω , since we have obtained the well-posedness of the Westervelt equation on them in \mathbb{R}^3 . This kind of arbitrary approximation or the approximation in the same class of domains is common to the shape optimization techniques [56], [ARP-1]. In the two dimensional case, we approximate arbitrary domains by sequences of NTA domains with the same geometrical constants M and r_0 . Moreover, thanks to Mosco convergence of the weak formulation functionals, we prove the well-posedness of the Westervelt equation with homogeneous Dirichlet boundary conditions on an arbitrary domain Ω , for which there exists a convergent (in the sense of Definition 3.2.2)

to Ω sequence of NTA domains with the same geometrical constants M and r_0 . The obtained solution is a little bit less regular because we cannot ensure that it belongs to the Laplacian domain, but only to $H_0^1(\Omega)$. However, it belongs to the domain of the Laplacian in the case of an NTA domain Ω (see also Chapter 2).

If we work with Robin boundary conditions or non-homogeneous Dirichlet boundary conditions, the trace and extension theorems discussed in Chapter 2 are essential. The first known results on the uniform bound of the extension operators are due to Chenais [36] in the class of uniformly Lipschitz domains satisfying the same property of ϵ -cone (see also next section). But the results of Jones [96] allow considering all (ϵ, ∞) -domains with the same ϵ . It is the reason why the pre-fractal approximation of the von Koch mixtures considered in \mathbb{R}^2 enjoys this uniform bound property [31, 30]. Thus, we extend the von Koch mixtures' two-dimensional case to the case of self-similar fractal boundaries in \mathbb{R}^n satisfying a “strong open set condition” **[PrepWestMixed]**.

More precisely, working in the class of (ϵ, ∞) or uniform domains in \mathbb{R}^n , we start in **[PrepWestMixed]** by defining the conditions on Ω and Ω_m so that they are all (ϵ, ∞) -domains with a fixed ϵ independent on m . In particular it is the case of Ω with a self-similar fractal boundary and a polyhedral approximation Ω_m satisfying a strong open set condition:

Assumption 3.2.1 (Fractal Self-Similar Face) *We assume that each K_m is a polygonal surface with $(n - 2)$ -dimensional hypersurface boundary that is the same as the $(n - 2)$ -dimensional hypersurface boundary of K_0 .*

Assumption 3.2.2 (A Strong Open Set Condition) *We assume the Open Set Condition for the sequence Φ_m is satisfied with two different convex open polygons $\mathcal{O} \subsetneq \mathcal{O}'$, not depending on m , such that*

$$\partial\mathcal{O} \cap K_0 = \partial\mathcal{O}' \cap K_0 = \partial\mathcal{O} \cap \partial\mathcal{O}' = \partial_{(n-2)}K_0.$$

This property to be (ϵ, ∞) -domain with the same ϵ is crucial to have the extension operators from Ω_m to \mathbb{R}^n with norms independent on m (see also [30, Thm 3.4]), what is important to be able to pass to the limit for $m \rightarrow +\infty$ in the Mosco convergence of the functionals corresponding to the weak formulations of the Westervelt mixed problem. In this way, we prove for a fixed self-similar boundary of a domain in \mathbb{R}^n the existence of a polyhedral boundary sequence of domains with the same ϵ as Ω itself. This result generalizes the known two-dimensional approximation results for von Koch mixtures of Refs. [31, 30]. Thus, we introduce the trace and extension properties for the fixed Ω and $(\Omega_m)_{m \in \mathbb{N}^*}$. The main result allowing to pass to the limit in the boundary integrals is the following: for any function $g \in H^1(\mathbb{R}^n)$

$$\frac{1}{\lambda(K_m)} \int_{K_m} Tr_{K_m} g ds \rightarrow \int_K Tr_K g d\mu \text{ for } m \rightarrow +\infty, \quad (3.16)$$

where by $\lambda(K_m)$ is denoted the length of the boundary K_m by the Lebesgue measure. It is a corollary of the weak* limit of measures having supports equal to K_m intersected with K to the measure μ on K (*i.e.* all measures are supposed to be on K).

Considering the mixed boundary valued problem for the Westervelt equation, we introduce the space

$$H(\Omega) := H^1([0, +\infty[; H^1(\Omega)) \cap H^2([0, +\infty[; L^2(\Omega)) \quad (3.17)$$

and let us consider Ω^* a Sobolev admissible domain such that $\Omega \subset \Omega^*$ for all $m \in \mathbb{N}^*$ $\Omega_m \subset \Omega^*$.

For $u \in H(\Omega^*)$ and $\phi \in L^2([0, +\infty[, H^1(\Omega^*))$ we define the following functional coming from the variational formulation for the Westervelt equation

$$\begin{aligned} F_m[u, \phi] := & \int_0^{+\infty} \int_{\Omega_m} \partial_t^2 u \phi + c^2 \nabla u \nabla \phi + \nu \nabla \partial_t u \nabla \phi \, d\lambda dt \\ & + \int_0^{+\infty} \int_{K_m} c^2 a_m \text{Tr}_{\partial\Omega_m} u \, \text{Tr}_{\partial\Omega_m} \phi + \nu a_m \text{Tr}_{\partial\Omega_m} \partial_t u \, \text{Tr}_{\partial\Omega_m} \phi \, d\lambda dt \\ & \int_0^{+\infty} \int_{\Omega_m} -\alpha(u \partial_t^2 u) \phi - \alpha(\partial_t u)^2 \phi + f \phi \, d\lambda dt \end{aligned} \quad (3.18)$$

and also

$$\begin{aligned} F[u, \phi] := & \int_0^{+\infty} \int_{\Omega} \partial_t^2 u \phi + c^2 \nabla u \nabla \phi + \nu \nabla \partial_t u \nabla \phi \, d\lambda dt \\ & + \int_0^{+\infty} \int_K c^2 a \text{Tr}_{\partial\Omega} u \, \text{Tr}_{\partial\Omega} \phi + \nu a \text{Tr}_{\partial\Omega} \partial_t u \, \text{Tr}_{\partial\Omega} \phi \, d\lambda dt \\ & \int_0^{+\infty} \int_{\Omega} -\alpha(u \partial_t^2 u) \phi - \alpha(\partial_t u)^2 \phi + f \phi \, d\lambda dt. \end{aligned} \quad (3.19)$$

Thus we introduce for $u \in L^2([0, +\infty[; L^2(\Omega^*))$

$$\overline{F}_m[u, \phi] = \begin{cases} F_m[u, \phi] & \text{if } u \in H(\Omega^*), \\ +\infty & \text{otherwise} \end{cases} \quad (3.20)$$

and

$$\overline{F}[u, \phi] = \begin{cases} F[u, \phi] & \text{if } u \in H(\Omega^*), \\ +\infty & \text{otherwise.} \end{cases} \quad (3.21)$$

We establish that the Mosco convergence for the functionals, coming from the variational formulation for the Westervelt equation, holds only in \mathbb{R}^2 or \mathbb{R}^3 , which once again is due to the impossibility to control nonlinear terms in a higher dimension. Thus, taking Ω as a fractal domain of \mathbb{R}^2 or \mathbb{R}^3 with its approximation $(\Omega_m)_{m \in \mathbb{N}^*}$ as the pre-fractal polyhedral sequence satisfying two Assumptions 3.2.1 and 3.2.2 (hence, by the fractal approximation, converging to Ω in the sense of the characteristic functions: $\|\mathbf{1}_{\Omega_m} - \mathbf{1}_{\Omega}\|_{L^1(\Omega^*)} \rightarrow 0$ for $m \rightarrow +\infty$), we ensure (see Theorem 6.1 [**PrepWestMixed**]) that they belong to the same class of (ϵ, ∞) -domains as Ω . For $\phi \in L^2([0, +\infty[; H^1(\Omega^*))$ and $a_m = \frac{a}{\text{Vol}(\partial\Omega_m)}$, we establish the Mosco convergence for the functionals $u \mapsto \overline{F}_m[u, \phi]$, defined in (3.20) in $L^2([0, +\infty[; L^2(\Omega^*))$ to the functional $u \mapsto \overline{F}[u, \phi]$ defined in (3.21) as $m \rightarrow +\infty$.

Moreover, we prove (see Theorem 6.5 [**PrepWestMixed**]) that for all $\phi \in L^2([0, +\infty[; H^1(\Omega^*))$ if $v_m \rightharpoonup u$ in $H(\Omega^*)$, then

$$F_m[v_m, \phi] \xrightarrow{m \rightarrow +\infty} F[u, \phi].$$

The result holds only in \mathbb{R}^2 and in \mathbb{R}^3 because of the control of the nonlinear terms. But the Mosco convergence of the linear part holds in \mathbb{R}^n for all $n \geq 2$.

Finally, we end up proving that the weak solutions u_m on the pre-fractal approximate domains Ω_m converge weakly to the weak solution u on the fractal domain (see Theorem 6.6 [**PrepWestMixed**]), a method often used in the case of shape optimization [**ARP-1**]. We notice that since our proof does not require any monotone assumption on Ω_m our approximation result works in particular for so-called Minkowski fractal domain [151, 53], [**ARP-8**] and their 3-dimensional analog.

3.3 Shape optimization for the wave absorption [ARP-1], [PrepShape2]

The diffraction and absorption of waves by a system with both absorbing properties and irregular geometry is an open physical problem. This problem must be solved to understand why anechoic chambers (electromagnetic or acoustic) do work better with irregular absorbing walls. The first studies relating to irregular geometry and absorption are performed in [58] numerically. The problem of the efficient boundary dissipation was considered numerically in [58], where the authors show the importance of the shape of the boundary, modeling very small absorption by the homogeneous Neumann boundary condition. It was also shown that the energy decay is related to the localization of eigenmodes on the boundary [58] and these localization phenomena are purely geometrical, *i.e.* created by the ("irregular" or multiscale) geometry of the boundary. In the same direction, the localized modes on the boundary between the air and a dissipative wall, giving the phenomena of astride localization [59], make bigger the wave interaction with the dissipative media and thus imply a more rapid decay of the acoustical energy. These ideas of creation of a bigger interaction of the wave with the absorbent media by a multiscale geometry favored the eigenmodes localization phenomena were successfully applied in the construction of the anti-noise barrier named Fractal Wall¹, the most efficient wall in the absorption of low-frequency traffic of auto-roads.

Therefore there is a question, treated numerically in [PrepShape2], about the existence of an optimal shape of an absorbent wall (for a fixed absorbing material), optimal in the sense that it is mostly dissipative for a large range of frequencies. At the same time, it is crucial to ensure that such a wall could effectively be constructed. In the framework of the propagation of acoustic waves, the wall's acoustic absorbent material is a porous medium. In [ARP-1], for a fixed frequency of the sound wave, we prove the existence of an optimal shape minimizing the acoustic energy for the Helmholtz frequency model with damping on the boundary.

To optimize acoustic performances of non-absorbing walls, Duhamel [48, 49] studies sound propagation in a two-dimensional vertical cut of a wall and uses genetic algorithms to obtain optimal shapes (some of them are, however, not connected and thus could not be easily manufactured). The author also uses a branch and bound (combinatorial optimization) type linear programming in order to optimize the sensors' positions that allow an active noise control, following former work introduced by Lueg [77] in 1934. Abe et al. [2] consider boundary elements based on the shape optimization of a non-absorbing two-dimensional wall in the framework of a two-dimensional sound scattering problem for a fixed frequency (for the Helmholtz equation), using a topological derivative with the principle that a new shape or topology is obtained by nucleating small scattering bodies. Also, for the Helmholtz equation for a fixed frequency, using the shape derivative of a functional representing the acoustical energy, Cao and Stanescu [28] consider a two-dimensional shape design problem for a non-absorbing part of the boundary to reduce the amount of noise radiated from aircraft turbofan engines. For the same problem, Farhadinia [55] developed a method based on measure theory, which does not require any information about gradients and the differentiability of the cost function.

¹product of Colas Inc., French patent N0-203404; U.S. patent 10" 508,119.

On the other hand, for shape optimization problems, there are theoretical results, reviewed in Refs. [9, 136], which rely on the topological derivatives of the cost functional to be minimized, with a numerical application of the gradient method in both two and three-dimensional cases (in the framework of solid mechanics). In particular, Achdou and Pironneau [4] considered the problem of optimization of a photocell, using a complex-valued Helmholtz problem with periodic boundary conditions to maximize the solar energy in a dissipative region. For acoustic waves in the two-dimensional case, optimization of the shape of an absorbing inclusion placed in a lossless acoustic medium was considered in Refs. [139, 140]. The considered model is the linear damped wave equation [39, 18]. Using the topology derivative approach, Munch and al. consider in [139, 140] the minimization of the acoustic energy of the solution of the damped wave equation at a given time $T > 0$ without any geometric restrictions and the purpose of the design of an absorbent wall. See also [12] for the shape optimization of shell structure acoustics.

In the absorbing acoustical wall framework, we first define our model to relate the energy damping with a fixed porous medium.

To describe the acoustic wave absorption by a porous medium, there are two possibilities. The first one is to consider wave propagation in two media, typically air and a wall, which corresponds to damping in the volume. The most common mathematical model for this is the damped wave equation [18]. The second one is to consider only one lossless medium, air, and to model energy dissipation by a damping condition on the boundary. In both cases, we need to ensure the same order of energy damping corresponding to the chosen porous medium's physical characteristics as its porosity ϕ , tortuosity α_h and resistivity to the passage of air σ [81].

Thanks to Ref. [81], we can define the coefficients in the damped wave equation (damping in volume) as functions of the characteristics mentioned above. More precisely, for a regular bounded domain $\Omega \subset \mathbb{R}^2$ (for instance $\partial\Omega \in C^1$) composed of two disjoint parts $\Omega = \Omega_0 \cup \Omega_1$ of two homogeneous media, the air in Ω_0 and a porous material in Ω_1 , separated by an internal boundary Γ , we consider the following boundary value problem (for the pressure of the wave)

$$\begin{cases} \xi(x)\partial_t^2 u + a(x)\partial_t u - \nabla \cdot (\eta(x)\nabla u) = 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial n}|_{\mathbb{R}_t \times \partial\Omega} \equiv 0, \quad [u]_\Gamma = [\eta\nabla u \cdot n]_\Gamma = 0, \\ u|_{t=0} = u_0 \mathbb{1}_{\Omega_0}, \quad \partial_t u|_{t=0} = u_1 \mathbb{1}_{\Omega_0}, \end{cases} \quad (3.22)$$

with $\xi(x) = \frac{1}{c_0^2}$, $a(x) = 0$, $\eta(x) = 1$ in air, *i.e.*, in Ω_0 , and

$$\xi(x) = \frac{\phi\gamma_p}{c_0^2}, \quad a(x) = \sigma \frac{\phi^2\gamma_p}{c_0^2\rho_0\alpha_h}, \quad \eta(x) = \frac{\phi}{\alpha_h}$$

in the porous medium, *i.e.*, in Ω_1 . The external boundary $\partial\Omega$ is supposed to be rigid, *i.e.*, Neumann boundary condition are applied, and on the internal boundary Γ we have no-jump conditions on u and $\eta\nabla u \cdot n$, where n denotes the normal unit vector to Γ . Here, c_0 and ρ_0 denote the sound velocity and the density of air, respectively, whereas $\gamma_p = 7/5$ denotes the ratio of specific heats. But instead of energy absorption in volume, we can also consider the following frequency model of damping by the boundary.

We suppose that the boundary $\partial\Omega$ is divided into three parts $\partial\Omega = \Gamma_D \cup \Gamma_N \cup \Gamma$ (see Fig. 3.8

for an example of Ω , chosen for the numerical calculations) and consider

$$\begin{cases} \Delta u + \omega^2 u = f(x), & x \in \Omega, \\ u = g(x) \quad \text{on } \Gamma_D, \quad \frac{\partial u}{\partial n} = 0 \quad \text{on } \Gamma_N, \quad \frac{\partial u}{\partial n} + \alpha(x)u = \text{Tr}h(x) \quad \text{on } \Gamma, \end{cases} \quad (3.23)$$

where $\alpha(x)$ is a complex-valued regular function with a strictly positive real part ($\text{Re}(\alpha) > 0$) and a strictly negative imaginary part ($\text{Im}(\alpha) < 0$). This particular choice of the signs of the real and the imaginary parts of α are needed for the well-posedness properties [62] and the energy decay of the corresponding time-dependent problem [23].

We find α from a minimization problem, minimizing the difference of H^1 norm between the solution of the damped wave equation in the frequency regime and the solution of the Helmholtz problem for the case of a straight line boundary Γ . The numerical values of α calculated for a porous material named ISOREL, frequently used in building insulation, are given in Fig. 3.7.

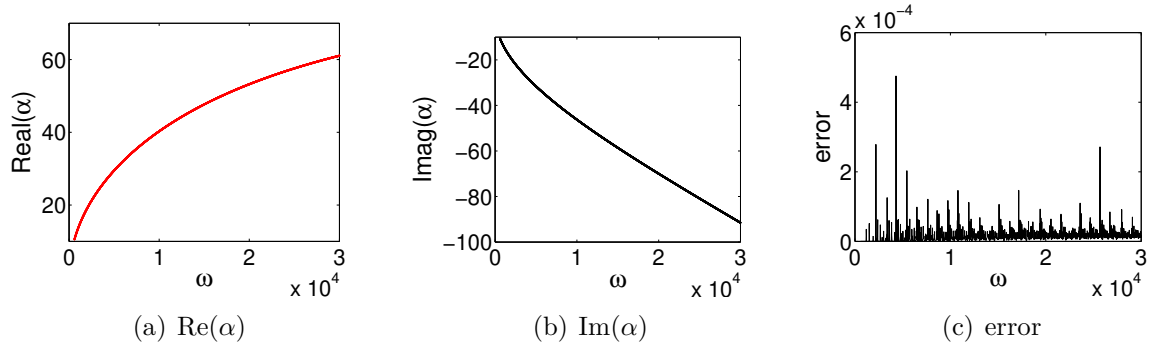


Figure 3.7 – The real (top left) and imaginary (top right) parts of α and the sum of the errors $e_{\Delta x}$ (in the bottom) as function of frequencies $\omega \in [600, 30000]$ calculated for the ISOREL porous material.

Thus we generalize the existing well-posedness results for the Helmholtz problem in domains with Lipschitz boundaries [62] to all admissible domains with d -set boundaries using [ARP-4]. Actually, it is also well-posed on Sobolev admissible domains (see Definition 2.1.5 and [BookChap]), since we work only on $H^1(\Omega)$.

Moreover, we stress that once a measure μ is fixed on the boundary $\partial\Omega$, it modifies the meaning of the Green formula in the following sense: for all u and v from $H^1(\Omega)$ with $\Delta u \in L_2(\Omega)$ the normal derivative of u is understood as the linear continuous functional on the Besov space $B_1^{2,2}(\partial\Omega)$ constructed by μ according to the definition

$$\left\langle \frac{\partial u}{\partial \nu}, \text{Tr}v \right\rangle_{(B_1^{2,2}(\partial\Omega))', B_1^{2,2}(\partial\Omega)} := \int_{\Omega} v \Delta u dx + \int_{\Omega} \nabla v \cdot \nabla u dx.$$

However, only the Lipschitz boundary case in [ARP-1] is considered for the shape optimization problem.

We consider the two-dimensional shape design problem, which consists of optimizing the shape of Γ with the Robin dissipative condition in order to minimize the acoustic energy of system 3.23. The boundaries with the Neumann and Dirichlet conditions Γ_D and Γ_N are supposed to be fixed.

We also define a fixed open set D with a Lipschitz boundary that contains all domains Ω . Actually, as only a part of the boundary (precisely Γ) changes its shape, we also impose that the changing part always lies inside the closure of a fixed open set G with a Lipschitz boundary: $\Gamma \subset \overline{G}$. The set G forbids Γ to be too close to Γ_D , making the idea of an

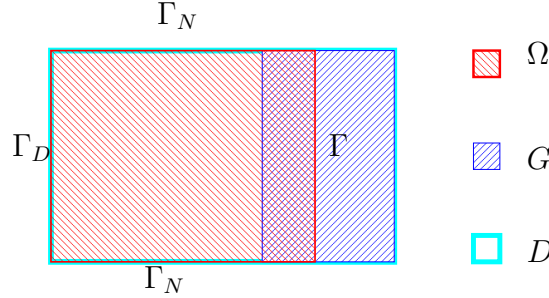


Figure 3.8 – Example of a domain Ω in \mathbb{R}^2 with three types of boundaries: Γ_D and Γ_N are fixed and Γ can be changed in the restricted area \overline{G} . Here $\Omega \cup G = D$ and obviously $\Omega \subset D$.

acoustical wall more realistic.

To introduce the class of admissible domains, on which we minimize the acoustical energy of system 3.23, we define \mathcal{Lip} as the class of all domains $\Omega \subset D$ for which

1. there exists a fixed $\epsilon > 0$ such that all domains $\Omega \in \mathcal{Lip}$ satisfy the ϵ -cone property [6, 36]: for all $x \in \partial\Omega$, there exists $\xi_x \in \mathbb{R}^2$ with $\|\xi_x\| = 1$ such that for all $y \in \overline{\Omega} \cap B_\epsilon(x)$

$$C(y, \xi_x, \epsilon) = \{z \in \mathbb{R}^2 | (z - y, \xi_x) \geq \cos(\epsilon)\|z - y\| \text{ and } 0 < \|z - y\| < \epsilon\} \subset \Omega.$$

2. there exists a fixed $\hat{c} > 0$ such that for any $\Omega \in \mathcal{Lip}$ and for all $x \in \Gamma$ we have

$$\int_{\Gamma \cap B_r(x)} d\lambda \leq \hat{c}r, \quad (3.24)$$

where $B_r(x)$ is the open Euclidean ball centered in x with radius r and λ is the usual one-dimensional Lebesgue measure on Γ .

The uniform ϵ -cone property implies, by Remark 2.4.8 [83, p. 55] and Theorem 2.4.7, that all boundaries of $\Omega \in \mathcal{Lip}$ are uniformly Lipschitz.

Let us notice that, by the boundness of D containing all Ω , condition (3.24) implies that all Γ for $\Omega \in \mathcal{Lip}$ have uniform length: there exists $M > 0$ depending on the chosen $\hat{c} > 0$ such that for all $\Omega \in \mathcal{Lip}$ it holds $\text{Vol}(\partial\Omega) = \int_{\partial\Omega} d\lambda \leq M$.

The constant M (and hence initially \hat{c}) can be chosen arbitrary large but finite. We denote by $\Omega_0 \in \mathcal{Lip}$ and $\Gamma_0 \subset \overline{G}$ the “reference” domain and the “reference” boundary respectively (actually $\partial\Omega_0 = \Gamma_D \cup \Gamma_N \cup \Gamma_0$) corresponding to the initial shape before optimization.

Thus, the admissible class of domains can be defined as

$$U_{ad}(\Omega_0, \epsilon, \hat{c}, G) = \left\{ \Omega \in \mathcal{Lip} \mid \Gamma_D \cup \Gamma_N \subset \partial\Omega, \Gamma \subset \overline{G}, M_0 \leq \int_{\Gamma} d\lambda \leq M(\hat{c}), \int_{\Omega} dx = \text{Vol}(\Omega_0) \right\}, \quad (3.25)$$

where \hat{c} is given sufficiently large in the aim to have a sufficiently large constant $M > 0$ in the sense that it is not less than $M_0 > 0$, which is the length of the straight line boundary.

Moreover the case when M is equal to the length of the plane boundary M_0 is the trivial case when $U_{ad}(\Omega_0, \epsilon, \hat{c}, G)$ contains only one unique domain with the plane boundary, which hence is trivially optimal. Therefore the problem becomes interesting for a sufficiently large M .

We show that the class of admissible domains $U_{ad}(\Omega_0, \epsilon, \hat{c}, G)$ is closed and compact by three types of convergence: in the sense of Hausdorff, of characteristic functions and compacts.

In what follows we denote by λ the 1-dimensional Lebesgue measure on the Lipschitz boundary Γ , by m_1 the 1-dimensional Hausdorff measure (which is equal to λ on Γ) and we denote by $u(\Omega, \mu)$ the weak solution of the Helmholtz problem on Ω satisfying for all $v \in V(\Omega)$ (see Eq. (2.17))

$$\int_{\Omega} \nabla u \cdot \nabla \bar{v} dx - \omega^2 \int_{\Omega} u \bar{v} dx + \int_{\Gamma} \alpha \operatorname{Tr} u \operatorname{Tr} \bar{v} d\mu = - \int_{\Omega} f \bar{v} dx + \int_{\Gamma} \operatorname{Tr} h \operatorname{Tr} \bar{v} d\mu$$

with 1-dimensional Radon measure μ .

We define

$$J(\Omega, u(\Omega, \mu), \lambda) = A \int_{\Omega} |u(\Omega, \mu)|^2 dx + B \int_{\Omega} |\nabla u(\Omega, \mu)|^2 dx + C \int_{\Gamma} |u(\Omega, \mu)|^2 d\lambda \quad (3.26)$$

for given μ and λ and with $A \geq 0$, $B \geq 0$, $C \geq 0$ positive constants for any fixed $\omega > 0$.

Ideally we would like to minimize $J(\Omega, u(\Omega, \lambda), \lambda)$ on $U_{ad}(\Omega_0, \epsilon, \hat{c}, G)$, however we are able to prove [ARP-1] the existence of Ω_{opt} in $U_{ad}(\Omega_0, \epsilon, \hat{c}, G)$ with a 1-measure μ^* , equivalent to λ , satisfying $\mu^*(\Gamma_{opt}) \geq \lambda(\Gamma_{opt})$ on its boundary Γ_{opt} , such that $J(\Omega, u(\Omega, \mu^*), \lambda)$ realizes the infimum of $J(\Omega, u(\Omega, \lambda), \lambda)$ on $U_{ad}(\Omega_0, \epsilon, \hat{c}, G)$. So, if $\mu^*(\Gamma_{opt}) = \lambda(\Gamma_{opt})$, *i.e.* $\mu^* = \lambda$ (this depends on the properties of the minimizing sequence which we don't know in advance), then Ω_{opt} realizes the minimum of $J(\Omega, u(\Omega, \lambda), \lambda)$.

The assumption to have a uniform upper bound for the boundary lengths inside non-trivial balls is crucial to prove the regularity of the measure μ^* , obtained as a weak* limit of Hausdorff measures of the boundaries of the minimizing sequence for J . The relation $\mu^* \geq \lambda$ follows from the lower semicontinuity of the perimeters. The measure μ^* is not necessarily equal to the Lebesgue measure of Γ , because $\mu^*(\Gamma)$ can be bigger than $\lambda(\Gamma)$. For instance, if Γ_n are oscillating boundaries with a constant length around a plane segment with a length two times smaller and such that $\Gamma_n \rightarrow \Gamma$ in the sense of Hausdorff, it easy to see that it is an example when $\lambda(\Gamma_n) \rightarrow \mu^*(\Gamma) = 2\lambda(\Gamma) > \lambda(\Gamma)$. Obviously, in the numerical framework, we don't have at all this kind of problem, we only have a finite number of possible boundaries Γ_n , and hence we can always choose the most efficient with all times $\mu^* = \lambda$.

As on the part of boundary Γ_D we impose the Dirichlet boundary condition, thanks also to the boundness of the global domain D , it is possible to control the solution of the Helmholtz problem with a constant independent of the shape of the domain, but only depending on its volume by the Poincaré inequality:

$$\|u\|_{H^1(\Omega)} \leq C(\alpha, \omega, \operatorname{Vol}(\Omega)) \left(\|f\|_{L_2(\Omega)} + \|g\|_{B_1^{2,2}(\Gamma_D)} + \|h\|_{V(\Omega, \lambda)} \right). \quad (3.27)$$

Space $V(\Omega, \lambda)$ is the same as defined in (2.17). It allows to take a sequence of solutions of the Helmholtz problem on the minimizing sequence of domains and deduce that it is bounded in $H^1(D)$. Using the uniform continuity of the extension operators on a closed class of domains satisfying the same property of ϵ -cone [36] and the same type of Sobolev embeddings

as in Mosco convergence framework of the previous section (see [PrepWestDir], [PrepWestMixed] and [29]), we show that the functional weak formulation defined with the Lebesgue boundary measure for the Helmholtz problem considered on the minimizing sequence of admissible domains converges to the analogous weak formulation defined with μ^* measure on the limit boundary Γ . From where follows our main optimal shape existence result. Moreover, we notice that in order to have the existence of an optimal shape in a higher dimensional case (for instance, in \mathbb{R}^3 or simply in \mathbb{R}^N), it is sufficient to replace everywhere the $N - 1$ -dimensional Lebesgue measure of the boundary by the $N - 1$ -dimensional Hausdorff measure since in that case the Lebesgue measure of the $N - 1$ -dimensional boundary is not equal to the Hausdorff measure as for one-dimensional curves, but proportional to it (see [54, Thm. 1.12, p. 13], for the optimization in \mathbb{R}^3 the Lebesgue measure of the boundary is equal to $\pi/4$ times the Hausdorff measure). See also Ref. [25] for a free discontinuity approach to a class of shape optimization problems involving a Robin condition on a free boundary.

In [PrepShape2] for the case of a regular boundary (at least C^3) we provide the shape derivative of the objective functional (3.26). Using the gradient descent method for the shape derivative, combined with the finite volume and level set methods introduced in [145], we find numerically optimal shapes for a fixed frequency in the two-dimensional case. We also show the stability of the numerical algorithm and the non-uniqueness of the optimal shape, which can be explained by the non-uniqueness of the geometry providing the same spectral properties (see [68, 71, 70]). Numerically, we show that for efficiency in the energy absorption, the wall's shape must be related to the half wavelength of the wave created by the source. Thus, it is not pertinent to add much smaller geometric variations, which finally confirms the possibility to create "not too complicated but most efficient" walls. Simultaneously, the multiscale nature of the wall geometry is necessary for an efficient absorption in a large band of frequencies. This shape is multiscale (see Ω^{opt} on Fig. 3.9). We

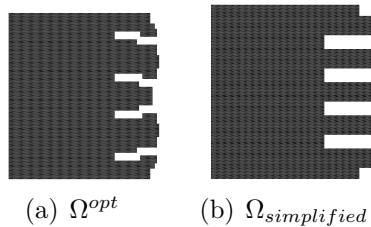


Figure 3.9 – The optimal shape Ω^{opt} is obtained by the shape optimization algorithm process, and the domain $\Omega_{simplified}$ is generated manually with the aim to simplify the shape of Ω^{opt} which is initially multiscale.

show that if we keep only the largest scale, the new shape (see $\Omega_{simplified}$ on Fig. 3.9) has the same good dissipation properties as the optimal one in the low frequencies corresponding to the chosen scale length. Nevertheless, the new shape is no more efficient in higher frequencies, for which the deleted geometry scales were important (see Fig. 3.10).

We can try to justify the efficiency of the half-wavelength scales in the following way.

Let us consider the penetration of the plane wave with the normal incidence to a porous medium with a plane shape. The Robin boundary problem for the Helmholtz equation can be viewed [ARP-1] as a reformulation of the frequency variant damped wave equation

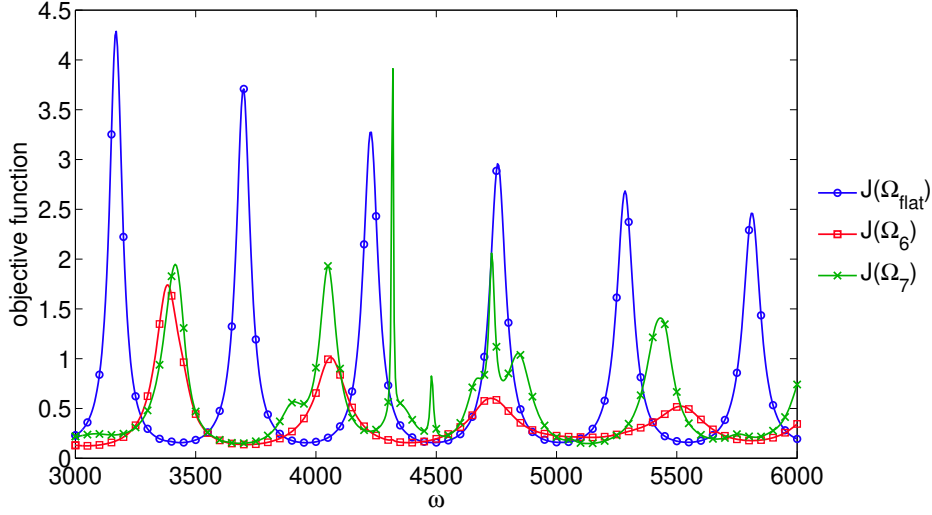


Figure 3.10 – Comparison of the dissipative properties of the flat shape Ω_{flat} , the optimal $\Omega^{\text{opt}} := \Omega_6$ and of its simplification $\Omega_{\text{simplified}} := \Omega_7$. The values of $J(\Omega_{\text{flat}})$, of $J(\Omega_6)$ and of $J(\Omega_7)$ ($A = 1, B = 0, C = 0$) as functions of $\omega \in [3000, 6000]$ are given by the lines with circles, squares and stars respectively.

providing the same solutions with the same energy damping. For more simplicity, let us consider the one dimensional case: the wave

$$p(|x|, t) = e^{i\omega t} e^{-ik(\omega)|x|} e^{-\frac{|x|}{\Lambda(\omega)}}$$

solves the damped wave equation with constant coefficients ($c > 0$, $a > 0$)

$$\frac{1}{c^2} \partial_t^2 p - \partial_x^2 p + a \partial_t p = 0, \quad (3.28)$$

if

$$k(\omega) = \frac{\omega}{\sqrt{2}c} \left(1 + \sqrt{1 + \frac{a^2 c^4}{4\omega^2}} \right)^{\frac{1}{2}}, \quad \text{and} \quad \Lambda(\omega) = \frac{2k(\omega)}{a\omega} = \frac{\sqrt{2}}{ac} \left(1 + \sqrt{1 + \frac{a^2 c^4}{4\omega^2}} \right)^{\frac{1}{2}}.$$

Here, $\Lambda(\omega)$ is the attenuation length of the wave, *i.e.* all longer lengths than $\Lambda(\omega)$ of the path traveled by the wave in the absorbing medium ensure its exponential damping.

Now, let consider the dissipative medium of a length ℓ : $a(x) = 1_{0 < x < \ell}$, and suppose that the end $x = \ell$ is reflective, *i.e.* if the wave reaches the end of the absorbent wall, it changes its propagation in the opposite direction. If, as previously, $\gamma(\omega)$ is the length of the path traveled by the wave in the wall, we can parametrize it with $r \in [0, \gamma(\omega)]$ and deduce that if $\gamma(\omega) \geq 2\ell > \Lambda(\omega)$ with $2\ell \geq \lambda$, the wave is exponentially damped by the wall *i.e.* to damp a wave with a wavelength λ we need a wall of minimal length $\frac{\lambda}{2}$.

If $\ell < \frac{\lambda}{2}$ and $\Lambda < 2\ell$ there is a part of the wave which is not exponentially absorbed, but when $\ell \ll \frac{\lambda}{2}$ known the properties of the porous material, the most common case is $\Lambda > 2\ell$ corresponding to a non-efficient wave damping.

3.4 Examples of further developments

One of possible open problems in the way to develop the results of [ARP-6] is to ask the same questions for the small time asymptotic developments of the heat content of the Venttsel problem [118, 119]. The main difficulty in the Venttsel problem is to find a Green function of an elementary half space problem with the same boundary condition:

$$\partial_t u - c_0 \Delta_{\partial\Omega} u = \left[\frac{\partial u}{\partial n} \right] \text{ on } \partial\Omega.$$

3.4.1 Optimal fractal shapes in acoustics

Knowing from [ARP-1] the existence of an optimal shape for a fixed frequency of a two-dimensional shape optimization problem for a Helmholtz equation with damping on the boundary, the further step is to consider the same question for a frequency range.

The idea is to relate the question about the existence of the most simple and the most efficient in energy absorption shapes for a band of frequencies with their multiscale geometries which must correspond to the wavelengths of the noise to be dissipated. The key point is to use the following physical principle: a wave with a wavelength λ_0 does not fit into a shape of characteristic scale much smaller than λ_0 . It is actually a direct corollary of the continuity of the acoustical energy 3.26 corresponding to the domain. Naturally, bigger the length of the path traveled by the wave in the absorbing wall, more the wave is absorbed, and its energy is dissipated. In addition, here we are in the framework of the case when the wavelength of the source λ_0 is supposed to be much bigger than the added geometrical fragments. Therefore, this situation is opposite to the case when it is possible to capture the wave by the geometry of wall fragments following [22] and the wave propagation can be treated as just propagation of raises following the rules of the geometric optic.

Besides, more significant wave interaction with the dissipative media decreases the acoustical energy. This statement on the decreasing energy property for a fixed frequency follows from the fact that the adding of smaller scales to Γ_{λ_0} (a boundary with geometrical parts of the size $\lambda_0/2$) means to make its length (perimeter) bigger. A bigger perimeter can increase the interaction of the wave with the absorbing medium, giving the relation $J(\Omega_\lambda)(\omega_0) \leq J(\Omega_{\lambda_0})(\omega_0)$ for a domain Ω_λ obtained from Ω_{λ_0} by adding a geometrical scale of the order of $\lambda/2$ ($\lambda \ll \lambda_0$). Following [58], it can also be viewed as a creation of localized eigenmodes taking their maximal values on the dissipative boundary. The presence of such eigenmodes increases energy dissipation. To finish, we notice that on U_{ad} anyway the acoustic energy takes its minimum or infimum on a $\Omega_{\lambda_0}^o \in U_{ad}$ and hence $J(\Omega_{\lambda_0}^o)(\omega_0) \leq J(\Omega_\lambda)(\omega_0)$. We can add here the hypothesis discussed at the end of Section 3.3 for the one-dimensional case that any optimal shape on U_{ad} for a fixed frequency ω_0 have the geometrical scale $\lambda_0/2$.

Knowing empirically that for the efficient energy dissipation of an acoustic wave, its wavelength λ must be related with a geometric scale of the wall, we can confirm this fact by one-dimensional examples and also numerically, by calculating the impact of different geometric scales on the energy dissipation in time. These partial results would contribute to a partial proof of the guess that the wall length scale ℓ must be of the order of $\lambda/2$. For instance, we can give the following illustration².

²F. Magoulès, P.T.K. Ngyuen, P. Omnes, A. Rozanova-Pierrat, *Fractal boundaries in acoustic energy wave absorption*, in preparation.

We consider the three cavities $\Omega = \Omega_0 \sqcup \Omega_1 =]0, 1[\times]-2, 2[$, partially shown on Fig. 3.11 with two homogeneous media, air (lower part) and a porous material (upper part), separated by an internal boundary Γ_i , $i = 0, 1, 2$. To preserve the volume of each medium and to model the increasing irregularity of the interface, as compared to the plane Γ_0 (at $y = 0$), we choose Γ_1 and Γ_2 as the first two fractal generations of a symmetric element. The external boundary $\partial\Omega$ is supposed to be perfectly rigid (Neumann boundary condition). Air is considered as a loss-less medium, and the porous medium (ISOREL) is considered as a dissipative homogeneous medium. Thus we solve numerically by finite volume method the damped wave equation (3.22) with an initial data chosen as a Gaussian, centered in a fixed point $x_0 = (0.75, -1.5)$ of Ω_0 .

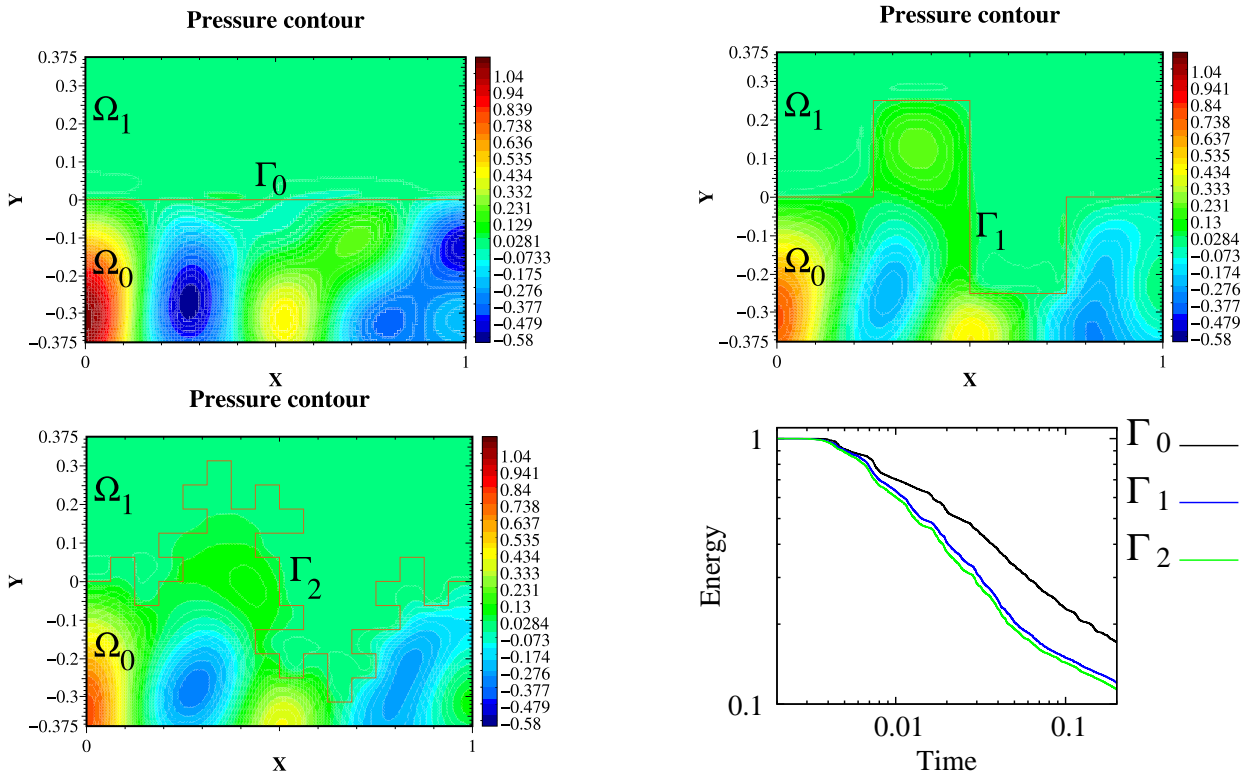


Figure 3.11 – Pressure contours at $t = 0.01$ in cavities with an internal boundary of different Minkowski fractal generations (from left to right and top to bottom: Γ_0 (flat), Γ_1 and Γ_2) and the corresponding energy damping. The size of the mesh is 128×512 .

Fig. 3.11 shows that an irregular shape of the internal boundary can significantly increase the dissipation properties of the porous medium ($\Gamma_{1,2}$ as compared to Γ_0). The energy damping by Γ_1 , compared to the damping performances of Γ_0 , is much better, and we notice that the wavelength λ of the wave, created by the initial data, is comparable (twice bigger) to the characteristic length scale size of the geometry Γ_1 . At the same time, the small difference in the energy decays corresponding to the internal boundaries Γ_1 , and Γ_2 confirms the physical hypothesis: the wave does not penetrate in the smallest geometry parts of size $\lambda/8$, but the wave still keeps a good penetration for the scales of the order $\lambda/2$ as for Γ_1 . This finally implies that the internal boundary's shape does not need to be "too complicated" for being an efficient acoustic absorbent for a fixed frequency.

The optimization algorithm developed in [PrepShape2] also confirms that the optimal shape has its largest scale length of the order of $\lambda/2$ (see Fig. 3.12 where the optimal shape

$\hat{\Omega}_{10}$ keeps the largest characteristic geometrical size of order $\frac{\lambda}{2}$ and deletes the smaller scales of the initial shape).

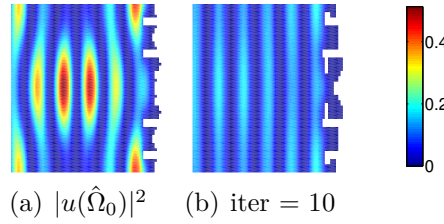


Figure 3.12 – The values of $|u|^2$ on the domains (from the left to right) $\hat{\Omega}_0$, which is the initial shape and $\hat{\Omega}_{10}$, which is the optimal shape for $\omega = 3170$, respectively with the same scale of colors.

Let us make attention that the general proof of this fact is an open "cross over" problem. At the same time, the cases of $\lambda \ll \ell$, corresponding to the wave propagation following laws of geometrical optic [22], and of $\lambda \gg \ell$, corresponding to the phenomena of homogenization, are much more understood.

The decreasing property of the energy of system (3.23) for $\omega \rightarrow +\infty$ is crucial. Fortunately, it is possible to find conditions on the source terms when the decay holds:

Proposition 3.4.1 *Let f , h and α be smooth functions of ω (of the class C^1), such that in the high frequency limit verify for $\omega \rightarrow +\infty$*

$$\frac{f}{\omega^2} \rightarrow 0 \text{ in } L_2(\Omega), \quad \frac{h}{\omega^2} \rightarrow 0 \text{ in } V(\Omega), \quad \text{Re} \alpha \rightarrow c_r \geq 0, \quad |\text{Im} \alpha| \rightarrow c_i, \quad (3.29)$$

where c_i is either a strictly positive constant or $+\infty$.

Then the acoustical energy $\|u\|_{L^2(\Omega)} \rightarrow 0$ for $\omega \rightarrow +\infty$.

Therefore, the main result which we prove³ in this framework is that for a finite frequency interval $[\omega_0, \omega_1]$ with $\omega_1 \geq 2^k \omega_0$, $k \geq 1$, the "most simple" and efficient geometry must be pre-fractal with k characteristic scales equal to $\frac{\lambda}{2^k}$ with a wavelength $\lambda \in [\frac{2\pi}{\omega_1}, \frac{2\pi}{\omega_0}]$. Consequently, in the infinite case $[\omega_0, +\infty[$ the "most simple" and efficient geometry must be fractal.

As a fractal shape could be "almost optimal" to dissipate the acoustical energy, there is a sense to consider the shape optimization framework for the Helmholtz system (3.23) in the class of admissible shapes allowing to have fractal boundaries. More precisely, it is possible to prove [ARP-New] that the class of the admissible domains defined by all (ε, ∞) -domains, included all in a bounded domain $D \subset \mathbb{R}^n$, with a boundary $\partial\Omega$ defined by the support of a Borel probability measure μ satisfying Theorem 2.1.2 and (2.4), (2.5) and (2.6) with uniform geometrical constants as ϵ , s , d , n , c_s , c_d , c_1 , c_2 , is closed and compact corresponding the usual three types of convergence: the Hausdorff, in the sense of compacts and characteristic functions. We also provide for this class of shape admissible domains (which is a subset of Sobolev admissible domains) the Mosco convergence of the

³F. Magoulès, P.T.K. Ngyuen, P. Omnes, A. Rozanova-Pierrat, *Fractal boundaries in acoustic energy wave absorption*, in preparation.

weak formulations for the Helmholtz problem (3.23). Moreover, since the minimization problem is understood in the sense of each time changing boundary measure, defining the absorbing part of the boundary denoted by Γ , the weak* limit measure μ^* , defining the optimal shape, provides all time the minimum of the considered shape minimization problem. Hence, this result could be viewed as a kind of relaxation of the optimization on (3.25).

Another problem of optimal perforated dissipative acoustic walls⁴, which can be formulated in the framework of the parametric or topological optimization. The holes in the non-absorbent material allow the wave to interact with the absorbent one. Therefore, let $\chi : \Gamma \rightarrow \{0; 1\}$ be the density of absorbent materials (actually, $\chi(x)$ is the characteristic function, which is equal to 1, if there is an absorption on $x \in \Gamma$, or is equal to 0, otherwise). In this case our previous model (3.23) is considered with the following boundary condition on Γ :

$$\frac{\partial u}{\partial n} + \alpha \chi u = 0 \text{ on } \Gamma; \quad (3.30)$$

The parameter to be optimized is χ . For solutions $u(\chi)$ of (3.23) with the boundary condition (3.30), we still want to minimize the energy functional (3.26), which this time is a function of χ with a constant Ω and Γ satisfying Theorem 2.1.2 on

$$\mathcal{U}_{ad} = \{\chi \in L^\infty(\Gamma) : \forall x \in \Gamma, \chi(x) \in \{0; 1\}, \int_\Gamma \chi d\mu = \beta\},$$

where β corresponds to the volume of absorbent materials. We only consider cases where $0 < \beta \leq \mu(\Gamma)$. Indeed, the opposite case gives a solution where the wall is entirely made of absorbent. Knowing that the set of characteristic functions is not closed for the weak* convergence in L^∞ , we use the relaxation method [9] consisting to take [83] the convex closure of \mathcal{U}_{ad} for the weak*-topology on L^∞

$$\mathcal{U}_{ad}^* = \{\chi \in L^\infty(\Gamma) : \forall x \in \Gamma, \chi(x) \in [0, 1], \int_\Gamma \chi d\mu = \beta\},$$

which ensures $\mathcal{U}_{ad} \subset \mathcal{U}_{ad}^*$ and

$$\inf_{\chi \in \mathcal{U}_{ad}} \mathcal{J}(\chi) = \min_{\chi \in \mathcal{U}_{ad}^*} \mathcal{J}^*(\chi)$$

with \mathcal{J}^* an extension of \mathcal{J} on \mathcal{U}_{ad}^* . Thus the relaxed optimization problem is

$$\text{to find } \chi^* \in \mathcal{U}_{ad}^* : \mathcal{J}^*(\chi^*) = \min_{\chi \in \mathcal{U}_{ad}^*} \mathcal{J}^*(\chi).$$

We prove the existence of an optimal χ^* and use the numerical algorithm given in [9] to simulate it numerically. Going back to our discussion about the geometrical size of wall's fragments to compare to the half wavelength of the wave, the found numerical size of absorbing holes is also equal to $\lambda/2$ as we can see from Fig. 3.13. To finalize the theoretical part of subject for the topological approach, we are developing the results of [165] to the Helmholtz equation.

⁴M. Boschhat, F. Magoulès, E. Savin, A. Rozanova-Pierrat, *Parametric and topological shape optimization in linear acoustics*, in preparation.

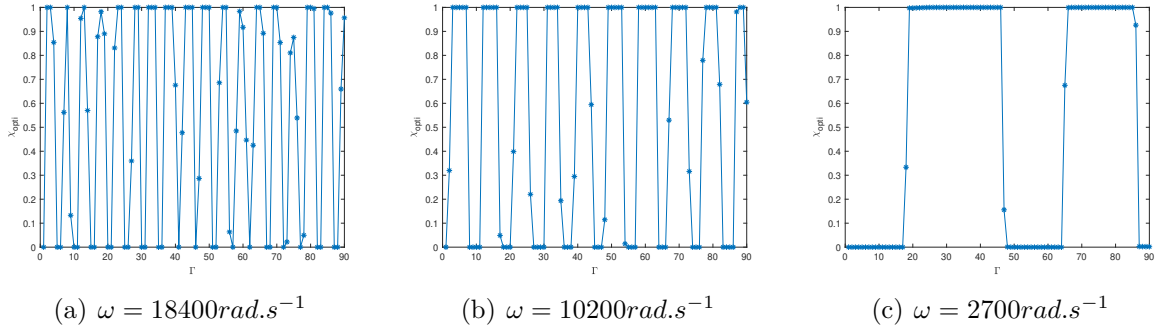


Figure 3.13 – The volume of absorbent is of 50%. The width of the peaks is close to $\lambda/2$.

3.4.2 Existence results for other optimal shape problems: in the non-linear acoustics for the Westervelt equation and the architecture for the linear elasticity system

Thanks to the results of [ARP-1] we can apply this method of the proof for other situations as the shape optimization for the Westervelt equation posed with homogeneous Robin boundary condition⁵ with the help of results on the Mosco convergence from [PrepWest-Mixed] or to give the existence of the optimal shape for the architecture problem [New-Prep2] posed and then solved numerically in [45]. In two situations, we use the close and compact properties of U_{ad} given in (3.25) and the existence of an equivalent measure μ^* on the changing part of the boundary.

For instance, it is impossible to obtain the Mosco convergence of the weak formulations of the Westervelt equation on a pre-fractal domain sequence to the same kind of weak formulation on a domain with a fractal boundary if on all boundary (fractal and prefractal) are only imposed the homogeneous Robin type boundary conditions without any Dirichlet homogeneous part. It is due to the facts that $a_m = \frac{a}{\lambda(\partial\Omega_m)}$ in the Robin boundary conditions $\frac{\partial u_m}{\partial n} + a_m u_m|_{\partial\Omega_m} = 0$ with $\lambda(\partial\Omega_m) \rightarrow +\infty$ while estimate (2.16) thanks to [44] holds with a constant proportional to $\max\left(1, \frac{1}{a_m}\right)$. Hence, in this case it does not possible to prove the a uniform boundness of the sequence of pre-fractal solutions. But if we stay in the closed class of shape admissible domains (3.25), taking only $\frac{\partial u_m}{\partial n} + a u_m|_{\partial\Omega_m} = 0$ as the boundary condition with a positive constant $a > 0$ for all $m \in \mathbb{N}^*$, we obtain the existence of an optimal shape which minimizes or gives the infimum of $\|u\|_{H^1([0,+\infty[, H^1(\Omega))}$ on U_{ad} , defined in (3.25), with $\Gamma = \partial\Omega$ and $\Gamma_D = \Gamma_N = \emptyset$ in the sense of an equivalent measure $\mu^* \leq \lambda$ of $\partial\Omega$. To obtain the result we just need to add that $\|\text{Tr } u_m\|_{L^2(\partial\Omega_m, \lambda)}^2 \rightarrow \|\text{Tr } u\|_{L^2(\partial\Omega, \mu^*)}^2$ and $\|\text{Tr } \partial_t u_m\|_{L^2(\partial\Omega_m, \lambda)}^2 \rightarrow \|\text{Tr } \partial_t u\|_{L^2(\partial\Omega, \mu^*)}^2$ for $m \rightarrow +\infty$ following the same proof as in [30].

Therefore, it is also possible to show the optimal shape existence for the Westervelt equation and of the linear elasticity system [45] in the uniform class of (ϵ, ∞) -domains.

The shape derivative for the Westervelt shape optimization problem was found in [111] for regular domains.

⁵A. Dekkers, A. Rozanova-Pierrat *Optimal absorption of ultrasound waves by a partial reflective boundary*, in preparation.

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