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# High-gain observer design for systems of PDEs

Constantinos Kitsos

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Constantinos Kitsos. High-gain observer design for systems of PDEs. Automatic Control Engineering. Université Grenoble Alpes [2020-..], 2020. English. NNT: 2020GRALT031 . tel-03097631

**HAL Id: tel-03097631**

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## THÈSE

pour obtenir le grade de

**DOCTEUR DE L' UNIVERSITÉ GRENOBLE ALPES**

Spécialité : **Automatique-Productique**

Arrêté ministériel : 25 mai 2016

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préparée au sein du **Laboratoire Grenoble Images Parole Signal  
Automatique**  
dans l' **École Doctorale Electronique, Electrotechnique,  
Automatique, Traitement du Signal (EEATS)**

**Synthèse d'observateurs grand gain pour  
des systèmes d'EDP**

**High-gain observer design for systems of  
PDEs**

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# Remerciements

Depuis mes premières études sur la Théorie du Contrôle, j'étais impressionné par les travaux notamment de chercheurs en France. Gildas Besançon et Christophe Prieur m'ont donné l'opportunité de collaborer avec cette *équipe du contrôle* française. Par conséquent, je voudrais adresser ma plus vive reconnaissance à tous les deux pour leur encadrement unique qui m'a formé à devenir chercheur. Grâce à la combinaison de maîtrises de chacun d'eux, notamment aux observateurs nonlinéaires et aux systèmes d'EDP, je suis arrivé à bien profiter scientifiquement. J'ai apprécié leur vastes connaissances, leur aptitude à me communiquer leur enthousiasme aux époques d'incertitude, leur soutien constant, et leur appréciation remarquable envers moi. Merci à tous les deux pour votre temps infini, pour votre patience de lire mes notes chaotiques pendant nos rencontres hebdomadaires, pour me donner l'opportunité de voyager (nombreuses conférences et collaborations) jusqu'au bout du monde (Chili) et à plusieurs autres pays pour des grandes périodes pendant ces trois années de thèse, et d'élargir mon réseau des collègues international. Nos échanges m'ont beaucoup apporté autant sur le plan scientifique que sur le plan humaine.

Mes remerciements profondes vont à Professeur Jean-Michel Coron qui m'a fait l'honneur de présider du jury et je lui en suis très reconnaissant. J'ai eu une chance inestimable de pouvoir le rencontrer pour la première fois, car je le connaissais justement par ses ouvrages monumentals sur la Théorie du Contrôle moderne. Merci au jury de m'avoir fait l'honneur d'assister à ma présentation et de me poser des questions très intéressantes. Je remercie Vincent Andrieu pour avoir passé beaucoup de temps sur son rapport très appréciable et pour ses remarques très importantes. Fabian Wirth pour avoir lu ma thèse avec telle rigueur et qui a préparé un rapport vraiment très exhaustif et précis. Je le remercie également pour m'avoir accueilli en Allemagne avec telle chaleur et m'avoir donné la chance d'apprendre beaucoup de choses par sa culture mathématique, et m'avoir montré des nouvelles pistes de recherches. Iasson Karafyllis pour avoir été examinateur et pour ses questions précises. Je le remercie également pour ses enseignements et nos échanges à l'Ecole Polytechnique Nationale d'Athènes (NTUA) qui me sont scientifiquement extrêmement profitables et aussi pour sa contribution à ma selection comme thésard en France. Merci à Federico Bribiesca Argomedeo d'être venu à ma soutenance comme examinateur et pour ses questions importantes.

J'adresse aussi ma vive reconnaissance à John Tsinias, Professeur à NTUA. C'est lui en fait qui m'a initié à la Théorie du Contrôle avant ma thèse et qui m'a enseigné la rigueur mathématique. Au cours de mes années à NTUA, les discussions scientifiques avec lui et la collaboration qu'on a effectué ont été d'une importance capitale dans mon travail.

J'ai eu beaucoup de plaisir de travailler avec Eduardo Cerpa pendant mon séjour à Santiago. Il a été un collaborateur et enseignant exceptionnel, qui a dirigé mes recherches avec beaucoup de rigueur mathématique et patience. Je le remercie pour avoir passé plusieurs heures avec moi et pour son accueil inoubliable au Chili.

Je suis très heureux d'avoir fait la connaissance avec des jeunes chercheurs, pas justement

pour leur vaste connaissance sur la Théorie du Contrôle et les Mathématiques et pour leurs conseils professionnels inestimables, mais aussi pour leur amitié dont j'ai profité. Notamment, je suis énormément reconnaissant à Sebin Gracy avec qui j'ai eu nombreuses échanges. Il est un très bon ami et qui a toujours des solutions à tous les problèmes. À Nicolas Espitia pour nos discussions très enrichissantes et ses conseils professionnelles. À Andrii Mirochenko pour sa gentillesse. Sa rigueur aux mathématiques est rare. Aux mathématiciens admirables, mes amis du Chili, Patricio Guzmán et Esteban Hernández. À Swann Marx que j'ai rencontré virtuellement à la fin de ma thèse pour sa chaleur, qui surpasse ses compétences mathématiques, et aussi pour ses conseils scientifiques.

Je suis aussi heureux d'avoir fait partie au GIPSA-lab où la grande gentillesse de tous, de permanents et de doctorants, qui ont contribué à une ambiance très conviviale et stimulante, va être inoubliable. J'ai eu la chance de construire des relations très amicales avec beaucoup d'eux pendant nos nombreux échanges et j'ai passé des bons moments avec Phong, Hùng, et Karthik. Merci pour des nombreuses expériences dans les laboratoire ces trois années.

Enfin, je remercie mes très proches amis Owen et Alex et surtout ma famille en Grèce pour son constant soutien pendant ces trois années.

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# Notation and Acronyms

- For given positive integers  $n, m$ , we adopt the notations  $\mathbb{R}^n, \mathbb{R}^{n \times m}$  for the set of the  $n$ -dimensional real space and the set of  $n \times m$  real matrices, respectively. Also,  $\mathbb{R}_+ := [0, +\infty)$ .
- For a given  $w$  in  $\mathbb{R}^n$ ,  $|w|$  denotes its usual Euclidean norm. For a given matrix  $A \in \mathbb{R}^{n \times n}$ ,  $|A| := \sup\{|Aw|, |w| = 1\}$  is its induced norm.
- For a given matrix  $A$  in  $\mathbb{R}^{n \times n}$ ,  $A^\top$  denotes its transpose, and  $\text{Sym}(A) = \frac{A+A^\top}{2}$  stands for its symmetric part. By  $\text{eig}(A)$  (or  $\lambda_{\min}(A)$  in Chapter 4) we denote the minimum eigenvalue of a symmetric matrix  $A$ . By  $I_n$  we denote the identity matrix of dimension  $n$ . By  $\text{diag}(a_1, \dots, a_n)$  we denote the  $n \times n$  diagonal matrix with elements  $a_1, \dots, a_n$  on its diagonal.
- For given constants  $l, L$ , with  $l < L$ , function  $u : [0, +\infty) \times [l, L] \rightarrow \mathbb{R}^n$  and time  $t \geq 0$ , we use the notation  $u(t)(x) := u(t, x)$ , for all  $x$  in  $[l, L]$  to refer to the profile at certain time.
- Given constants  $l, L$ , with  $l < L$ , and mapping  $[0, +\infty) \times [l, L] \ni (t, x) \mapsto u(t, x) \in \mathbb{R}^n$ ,  $u_t$  (or  $\partial_t u$ ) and  $u_x$  (or  $\partial_x u$ ) stand for the partial derivative of  $u$  with respect to  $t$  and  $x$ , respectively. Also,  $u_{xx}, u_{xxx}$ , and  $\partial_x^q u$  stand for the second, third and  $q$ -th partial derivative with respect to  $x$ , respectively. When mapping  $u$  is only time-dependent,  $\dot{u}$  or  $\frac{d}{dt}u$  denotes its first time-derivative. When  $u$  is scalar and depends only on  $x$ , then  $u', u'', u'''$ , and  $\frac{d^q}{dx^q}u$ , stand for the first, second, third, and  $q$ -th derivative, respectively.
- By  $D_u f$  (or simply  $Df$ ) we denote the Jacobian of a differentiable mapping  $\mathbb{R}^n \ni u \mapsto f(u) \in \mathbb{R}^n$ . For a differentiable in its second argument mapping  $f$ , by  $\langle D_u f(\cdot, \mathcal{F}[u]), h \rangle$  we denote its Fréchet derivative w.r.t.  $u$  acting on  $h$ , where  $\mathcal{F}$  is any Fréchet differentiable operator acting on functions  $u$ . Note that by the chain rule,  $D_u f(\cdot, \mathcal{F}[u]) = D_w f(\cdot, w)|_{w=\mathcal{F}[u]} D_u \mathcal{F}[u]$ , where  $D_u \mathcal{F}[u]$  denotes the Fréchet derivative of  $\mathcal{F}$ .
- Given constants  $T, l, L$ , with  $T > 0, L > l$ , we denote by  $C^q(\mathcal{D}; \mathbb{R}^n)$  the space of  $q$ -times continuously differentiable functions on  $\mathcal{D}$ , taking values in  $\mathbb{R}^n$ , with  $D := [0, T] \times [l, L]$ , or  $[0, +\infty) \times [l, L]$ , or  $[0, +\infty)$ , or  $[l, L]$ . For  $q = 0$ , this space corresponds to the space of continuous functions on  $\mathcal{D}$ . For a continuous ( $C^0$ ) mapping  $[l, L] \ni x \mapsto u(x) \in \mathbb{R}^n$  (or  $[0, L] \ni x \mapsto A(x) \in \mathbb{R}^{n \times n}$ ) we adopt the notation  $\|u\|_\infty := \max\{|u(x)|, x \in [l, L]\}$  ( $\|A\|_\infty := \max\{|A(x)|, x \in [0, L]\}$  respectively) for the sup-norm, with  $\|u\|_0 := \|u\|_\infty$ . If this mapping is  $q$ -times continuously differentiable ( $C^q$ ), we adopt the notation  $\|u\|_q := \sum_{i=0}^q \|\frac{d^i}{dx^i}u\|_\infty$  for the  $q$ -norm.
- For given constants  $l, L$ , with  $l < L$ , and  $p$  a positive integer,  $L^p((l, L); \mathbb{R}^n)$  (or  $L^p(l, L)^n$ ) denotes the space of equivalence classes of measurable functions  $u : [l, L] \rightarrow \mathbb{R}^n$ , for which  $\|u\|_{L^p((l, L); \mathbb{R}^n)} := \left( \int_l^L |u(x)|^p dx \right)^{1/p} < +\infty$ . For given  $T > 0$ ,  $L^\infty(0, T)$  denotes

the space of essentially bounded measurable functions on  $(0, T)$ . By  $L_{loc}^\infty(0, +\infty)$  we denote the space of  $L^\infty$  functions on every compact subset of  $(0, +\infty)$ .

- By  $H^2((l, L); \mathbb{R}^n)$  we denote the Sobolev space of functions in  $L^2((l, L); \mathbb{R}^n)$  with all their weak derivatives up to order 2 in  $L^2((l, L); \mathbb{R}^n)$ , equipped with the norm  $\|u\|_{H^2((l, L); \mathbb{R}^n)} := \left( \int_l^L (|u(x)|^2 + |u_x(x)|^2 + |u_{xx}(x)|^2) dx \right)^{1/2}$ . The fractional Sobolev space  $H^s(0, L)$ , for  $s \in (0, 1)$ , is simply mentioned in Chapter 4, referring to  $H^s(0, L) := \left\{ u \in L^2(0, L) : \frac{|u(x)-u(y)|}{|x-y|^{s+\frac{1}{2}}} \in L^2((0, L) \times (0, L)) \right\}$ .
- In Chapter 2, for a mapping  $f(\cdot, \cdot)$  we use the difference operator  $\Delta_{\hat{\xi}}[f(\cdot, \mathcal{F})](\xi)(x) := f(x, \mathcal{F}[\hat{\xi}](x)) - f(x, \mathcal{F}[\xi](x))$ , parametrized by  $\hat{\xi}$ , where  $\mathcal{F}$  denotes any chosen operator acting on  $\xi$ . In Chapter 3, for a mapping  $f(\cdot, \cdot)$ , we use the difference operator given by  $\Delta_{\hat{\xi}_2}[f(\xi_1, \xi_2)] := f(\xi_1, \hat{\xi}_2) - f(\xi_1, \xi_2)$ , parametrized by  $\hat{\xi}_2$ .
- In Chapter 2, we define  $\mathcal{B}_\delta := \{\xi \in C^1([0, L]; \mathbb{R}^n) : \|\xi\|_1 \leq \delta\}$ . In Chapter 3, we define  $\mathcal{B}(\delta) := \{\xi \in H^2((0, 1); \mathbb{R}^2) : \|\xi\|_{H^2((0, 1); \mathbb{R}^2)} \leq \delta\}$ .
- By  $\text{sgn}(x)$  we denote the sign function  $\text{sgn}(x) := \frac{d}{dx}|x|, x \neq 0$ .
- By  $\mathcal{L}(X, Y)$  we denote the space of bounded linear operators from  $X$  to  $Y$ , where  $X, Y$  are Banach spaces.
- By  $\bar{\Pi}$  and  $\partial\Pi$  we denote the closure and the boundary of a set  $\Pi$ , respectively.

<b>H-GODP</b>	High-Gain Observer Design Problem
<b>ISS</b>	Input-to-State Stability
<b>KdV</b>	Korteweg-de Vries
<b>ODE</b>	Ordinary Differential Equation
<b>PDE</b>	Partial Differential Equation
<b>PIDE</b>	Partial Integro-Differential Equation
<b>SIR</b>	Susceptible, Infectious, Recovered

<b>inf, sup</b>	infimum, supremum
<b>min, max</b>	minimum, maximum

# Résumé de la thèse

Au cours des dernières décennies, de nombreuses études ont été consacrées au problème de la synthèse d'observateurs grand gain pour des systèmes nonlinéaires de dimension finie (voir par exemple [Khalil (2017)] et ses références). Cette thèse s'intéresse principalement à l'extension de cette conception générale à certaines classes de systèmes de dimension infinie, à savoir, des systèmes d'équations aux dérivées partielles (EDP) couplées à structure triangulaire, et considérant la mesure d'une partie de l'état sur tout le domaine. Elle propose ainsi des solutions à un problème de synthèse d'observateur grand gain (H-GODP), qui se révèle être étendu à dimension infinie de manière non triviale. La thèse considère, enfin, une application de l'approche de synthèse d'observateur proposée à un problème de commande de retour de sortie. Les problèmes principaux abordés par cette thèse peuvent être résumés comme suit.

**Problème 1.** Construire un observateur pour un système d'EDP triangulaire, lorsque moins d'observations internes que le nombre d'états sont disponibles.

Un problème qui montre une certaine dualité, ceci de contrôlabilité pour les systèmes d'EDP en cascade avec un nombre de contrôles internes réduit, a déjà été envisagé [Alabau-Boussouira et al (2017)]. Dans [Lissy et Zuazua (2019)], l'observabilité des systèmes linéaires d'EDP couplés, avec un nombre d'observations réduit est étudiée. Un problème de stabilisation frontière est résolu en plaçant un contrôle sur un bord de l'un des deux états d'un système hyperbolique dans [Coron et al (2013)]. Cette approche nécessite des conditions spécifiques aux limites, qui permettent une transformation dite backstepping en un système stable. Ce n'est pas le cas ici, car nous considérons des conditions aux limites plus générales. En général, le problème de commande/observateur avec un nombre de contrôles/observations réduit, inférieur au nombre d'états, est un problème difficile. À notre connaissance, la synthèse d'observateurs pour des systèmes qui ont un nombre d'observations réduit n'a pas encore apparue dans la littérature. Les approches proposées ici comptent sur une structure triangulaire nonlinéaire appropriée pour la dynamique du système comme dans les systèmes de dimension finie, afin de proposer une solution au Problème 1. Cependant, le type d'opérateur différentiel, associé à chaque système d'EDP, augmente la difficulté et ça rend le problème non trivial.

**Problème 2.** Construire un observateur grand gain pour toute classe de systèmes d'EDP, qui satisfont une structure triangulaire comme dans le cas de dimension finie.

Ce problème n'a pas été pris en compte dans la littérature au meilleur de nos connaissances. Les observateurs grand gain pour des systèmes non linéaires de dimension finie présentent un taux de convergence arbitraire, qui peut être réglé via un seul paramètre dans la dynamique de l'observateur. Pour certaines classes de systèmes d'EDP, en particulier hyperboliques, une convergence arbitraire serait souhaitable, car les observateurs de frontière proposés dans la littérature, éprouvent une limitation en ce qui concerne la vitesse de convergence. Pour trouver

une solution, le système doit satisfaire une structure triangulaire spécifique comme dans la dimension finie. En dimension infinie, plusieurs difficultés se posent, la plus importante d'entre elles provenant des propriétés de l'opérateur différentiel associé au système, qui peut avoir des coefficients non triviaux. Ces propriétés ajoutent des difficultés à l'analyse, ce qui peut ne pas permettre des synthèses d'observateurs pour un grand nombre d'états. De plus, la présence de termes non locaux dans la dynamique, la généralité des conditions aux limites pour les systèmes hyperboliques et les types de non-linéarités dans les termes sources augmentent la complexité de celle synthèse. Par conséquent, pour résoudre partiellement le Problème 2, nous considérons des classes spécifiques de systèmes triangulaires, tandis que la mesure d'une partie de l'état (uniquement le premier) dans la sortie est considérée être distribuée dans tout le domaine. Tout au long de la thèse, il est montré que, de manière contre-intuitive, une extension naturelle de la conception d'observateurs grand gain des systèmes de dimension finie à des systèmes de dimension infinie s'avère être non triviale. Nous notons que les solutions aux problèmes 1 et 2 sont données simultanément dans cette thèse, car la solvabilité du problème H-GODP considéré (Problème 2) implique la solvabilité du Problème 1.

**Problème 3.** Fournir des lois de retour de sortie pour des systèmes d'EDP en cascade, en mesurant une partie de l'état et en plaçant le contrôle sur une autre partie de l'état.

Une solution à ce problème pour un système d'équations de Korteweg-de Vries (KdV) en cascade est présentée ici, en utilisant l'observateur grand gain proposé.

La résumé des chapitres précise suit.

## Outline

- Le Chapitre 1 présente la classe de systèmes considérée, la définition du problème H-GODP et deux approches de ses solutions: une synthèse d'observateur directe et une indirecte dépendant d'une propriété de l'opérateur différentiel associé au système considéré. Les difficultés de ces approches sont analytiquement décrites.
- Le Chapitre [ref chap:H-GODP1](#) est dédié à une solution au problème H-GODP pour une classe de systèmes d'équations integrodifférentielles hyperboliques quasilineaires avec  $n$  états et une seule vitesse, écrite sous forme triangulaire et considérant la mesure interne du premier état. La preuve de stabilité de l'observateur repose sur une fonctionnelle de Lyapunov introduite. La conception est ensuite appliquée à un modèle d'épidémie SIR (il semble être d'un intérêt particulier pour la pandémie COVID-19, qui s'est produite au moment de la rédaction de cette thèse).
- Le Chapitre 3 aborde le problème de la synthèse d'observateurs grand gain pour des systèmes hyperboliques avec des vitesses distinctes et des systèmes paraboliques avec des diffusivités distinctes en suivant une approche indirecte. Les systèmes considérés sont d'abord mappés dans ceux qui conviennent à la synthèse d'observateurs via une transformation d'état à dimension infinie. Des conditions suffisantes appropriées conduisent à la solvabilité du problème H-GODP pour des systèmes hyperboliques quasi-linéaires à deux

états, des systèmes paraboliques semilinéaires à deux ou trois états, et enfin des systèmes hyperboliques hétérodirectionnels linéaires avec un nombre quelconque d'états. La dynamique de l'observateur contient des dérivées spatiales (d'ordre élevé) de la mesure distribuée. La convergence de l'observateur est prouvée par l'utilisation des fonctionnelles de Lyapunov alors qu'un phénomène de perte de dérivées apparaît. Enfin, la synthèse d'observateur proposée est appliquée aux réacteurs chimiques à flux continu et aux systèmes diffusionnels de type Lotka-Volterra.

- Le Chapitre 4 traite de la stabilisation d'un système d' $n$  équations de Korteweg-de Vries linéarisées en cascade dans un intervalle borné. Il considère un contrôle de retour de sortie placée sur le bord gauche de la dernière équation, tandis que la sortie n'implique que la solution de la première équation. Les problèmes de contrôle frontière étudiés comprennent deux cas: le contrôle classique sur la condition aux limites de Dirichlet et un contrôle moins courant sur sa deuxième dérivée. La loi de contrôle de retour utilise les solutions estimées par un système d'observateur grand gain et le contrôle de retour de sortie nous amène à la stabilisation pour tout  $n$  pour le premier cas de conditions aux limites et à la stabilisation quand  $n = 2$  pour le second cas.

## Contributions

Les contributions de la thèse peuvent être résumées comme suit

- extension des synthèses d'observateurs grand gain des systèmes à dimension finie à certaines classes de systèmes à dimension infinie;
- des solutions aux problèmes de synthèses d'observateurs pour les systèmes en cascade/triangulaires, où seule une partie de l'état est disponible comme mesure;
- introduction de certaines transformations d'état à dimension infinie, qui conduisent à des systèmes cibles d'EDP, qui sont appropriés pour la synthèse d'observateurs;
- introduction de fonctionnelles de Lyapunov appropriées conduisant à une analyse de stabilité des équations d'erreur d'observateur dans certains espaces fonctionnels;
- des solutions au problème H-GODP pour des systèmes en cascade/triangulaires, lorsque l'opérateur différentiel du système a des coefficients triangulaires inférieurs ou diagonaux avec des éléments distincts;
- des solutions au problème H-GODP pour les systèmes en cascade/triangulaires en présence de termes non linéaires non locaux dans la dynamique, de nonlinéarités générales (localement Lipschitz) et de quasilinearités dans l'opérateur différentiel du système;
- des lois de retour de sortie pour les systèmes en cascade de KdV avec un nombre de contrôles et d'observations réduit;

- application de la méthodologie théorique introduite aux systèmes de la physique et de la biologie, y compris les modèles épidémiques, les réacteurs chimiques et la dynamique des populations.

## Liste des publications relatives à ce manuscrit

### Journaux

- C. Kitsos, G. Besançon, and C. Prieur, High-gain observer design for a class of quasi-linear integro-differential hyperbolic systems - application to an epidemic model, cond. accepted in *IEEE Transactions on Automatic Control*, 2019.
- C. Kitsos, G. Besançon, and C. Prieur, High-gain observer for  $3 \times 3$  linear heterodirectional hyperbolic systems, accepted in *Automatica*, 2020.
- C. Kitsos, G. Besançon, and C. Prieur, High-gain observer design for some semilinear reaction-diffusion systems: a transformation-based approach, to appear in *IEEE Control Systems Letters (L-CSS)*, 2020.
- C. Kitsos, E. Cerpa, G. Besançon, and C. Prieur, Output feedback control of a cascade system of linear Korteweg-de Vries equations, submitted to *SIAM J. on Control and Optimization (SICON)*, 2020.

### Conférences Internationales

- C. Kitsos, G. Besançon, and C. Prieur, High-gain observer design for a class of hyperbolic systems of balance laws, *IEEE Conference on Decision and Control*, Miami, USA, pp. 1361–1366, 2018.
- C. Kitsos, G. Besançon, and C. Prieur, A high-gain observer for a class of 2x2 hyperbolic systems with  $C^1$  exponential convergence, *IFAC Workshop on Control of Systems Governed by Partial Differential Equations*, Oaxaca, Mexico, pp. 174–179, 2019.
- C. Kitsos, G. Besançon, and C. Prieur, High-gain observers for a class of 2x2 quasilinear hyperbolic systems with 2 different velocities, *Joint 8th IFAC Symposium on MECHATRONICS and 11th IFAC Symposium on NOLCOS*, Vienne, Austria, pp. 210–215, 2019.

En plus des publications présentées dans ce manuscrit, citons

- J. Tsinias and C. Kitsos, Observability and state estimation for a class of nonlinear systems, *IEEE Transactions on Automatic Control*, vol. 64, no. 6, pp. 2621–2628, 2019

qui a été finalisée au début de cette thèse (en dehors de la supervision de C. Prieur and G. Besançon).

# Introduction

In the last decades, many studies have been dedicated to the problem of the so-called high-gain observer design for nonlinear finite-dimensional systems (see for instance [Khalil (2017)] and references therein). The present thesis is mainly concerned with the extension of this general design to some classes of infinite-dimensional systems, namely, systems of coupled Partial Differential Equations (PDEs) with triangular structure, and considering measurement of part of the state, distributed throughout the whole domain. It, thereby, proposes some solutions to a High-Gain Observer Design Problem (H-GODP), which turns out to be non-trivially extended to infinite dimensions. The thesis considers, finally, an application of the proposed observer design approach to some problem of output feedback control. The main problems this thesis addresses can be summarized as follows.

**Problem 1.** Design an observer for a triangular system of PDEs, when fewer internal observations than the number of states are available.

Somehow dual problems of controllability for cascade systems of PDEs with reduced number of internal controls have already been considered [Alabau-Boussouira et al (2017)]. In [Lissy and Zuazua (2019)], observability for coupled systems of linear PDEs with reduced number of observations is studied. In [Coron et al (2013)] a stabilization problem is solved by placing a boundary control at one endpoint of one of the two states of an hyperbolic system. This approach requires specific boundary conditions, which allow a so-called backstepping transformation into a stable system. This is not the case here, since we consider more general boundary conditions. In general, the problem of control/observer design with reduced number of controls/observations, less than the number of the states, is a difficult problem. To the best of our knowledge, observer design for systems with reduced number of observations has not been considered in the literature. The proposed approaches here rely on the consideration of an appropriate nonlinear triangular structure for the system dynamics as in the finite-dimensional systems, in order to propose a solution to Problem 1. The type of differential operator, associated to each system of PDEs, however, adds to the difficulty rendering the problem nontrivial.

**Problem 2.** Design a high-gain observer for any class of systems of PDEs, which satisfy a triangular structure as in the finite-dimensional case.

This problem has not been considered in the literature to the best of our knowledge. High-gain observers for nonlinear finite-dimensional systems exhibit a property of arbitrary convergence rate, which can be tuned via a single parameter in the observer dynamics. For some classes of systems of PDEs, particularly hyperbolic ones, arbitrary convergence would be desirable, since the boundary observers proposed in the literature, for instance, experience a limitation with respect to convergence speed. To find a solution, system must satisfy a specific triangular structure as in the finite dimensions. In infinite dimensions, several difficulties



arise, with the most important of them coming from the properties of the differential operator associated to the system, which might have nontrivial coefficients. These properties add difficulties to the analysis, which might not allow designs for any large number of states. Also, the presence of nonlocal terms in the dynamics, the generality of the boundary conditions for hyperbolic systems, and types of nonlinearities in the source terms increase the complexity of the design. Therefore, to partially solve Problem 2 we consider specific classes of triangular systems, while the measurement of a part of the state (only the first one) in the output is considered distributed within the whole domain. Along the thesis, it is shown that, counter-intuitively, a natural extension of high-gain observer design from finite-dimensional systems to infinite-dimensional ones turns out to be nontrivial. We note here that solutions to both Problems 1 and 2 are given simultaneously in this thesis, since the solvability of the considered H-GODP (Problem 2) implies solvability of Problem 1.

**Problem 3.** Provide output feedback laws for cascade systems of PDEs, by measuring a part of the state and placing the control on another part of the state.

A solution to this problem for a cascade system of Korteweg-de Vries (KdV) equations is presented here, by use of the proposed high-gain observer.

The precise outline of the manuscript follows.

## Outline

- Chapter 1 presents the considered class of systems, the definition of the H-GODP, and two approaches towards its solutions: a direct and an indirect observer design depending on some property of the differential operator associated to the considered system. The difficulties of these approaches are also mentioned.
- Chapter 2 is dedicated to a solution to the H-GODP for a class of quasilinear hyperbolic systems of partial integro-differential equations with  $n$  states and a single velocity written in triangular form and considering internal measurement of the first state. The observer stability proof relies on an introduced Lyapunov functional. The design is then applied to an SIR epidemic model (it appears to be of special interest for the COVID-19 pandemic, which occurred at the time this thesis was written).
- Chapter 3 tackles the problem of high-gain observer design for hyperbolic systems with distinct velocities and parabolic systems with distinct diffusivities by following an indirect approach. The considered systems are first mapped into ones suitable for observer design via an infinite-dimensional state transformation. Appropriate sufficient conditions lead to the solvability of the H-GODP for quasilinear hyperbolic systems with two states, semilinear parabolic systems with two or three states, and finally linear heterodirectional hyperbolic systems with any number of states. The observer dynamics contain (high-order) spatial derivatives of the distributed measurement. The observer convergence is proven by use of Lyapunov functionals while a loss of derivatives phenomenon

appears. Finally, the proposed observer design is applied to plug and flow chemical reactors and diffusional Lotka-Volterra systems.

- Chapter 4 is about the stabilization of a cascade system of  $n$  linearized Korteweg-de Vries equations in a bounded interval. It considers an output feedback control placed at the left endpoint of the last equation, while the output involves only the solution to the first equation. The boundary control problems investigated include two cases: the classical control on the Dirichlet boundary condition and a less common one on its second-order derivative. The feedback control law uses the estimated solutions of a high-gain observer system and the output feedback control leads to stabilization for any  $n$  for the first boundary conditions case and for  $n = 2$  for the second one.

## Contributions

The main contributions of the thesis can be summarized as follows

- extension of high-gain observer designs from finite-dimensional systems to some classes of infinite-dimensional ones;
- solutions to observer design problems for cascade/triangular systems, where only a part of the state is available as measurement;
- introduction of some infinite-dimensional state transformations, which lead to target systems of PDEs, suitable for observer design;
- introduction of appropriate Lyapunov functionals leading to stability analysis of the observer error equations in some function spaces;
- solutions to the H-GODP for cascade/triangular systems, when system's differential operator has coefficients being lower triangular and diagonal with distinct elements;
- solutions to the H-GODP for cascade/triangular systems in the presence of nonlinear nonlocal terms in the dynamics, general (locally Lipschitz) nonlinearities, and quasilinearities in system's differential operator;
- output feedback laws for cascade systems of KdVs with reduced number of controls and observations;
- application of the introduced theoretical methodology to physical and biological systems, including epidemic models, chemical reactors, and population dynamics.

## List of publications

### Journals

- C. Kitsos, G. Besançon, and C. Prieur, High-gain observer design for a class of quasi-linear integro-differential hyperbolic systems - application to an epidemic model, cond. accepted in *IEEE Transactions on Automatic Control*, 2019.
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- C. Kitsos, G. Besançon, and C. Prieur, High-gain observers for a class of 2x2 quasilinear hyperbolic systems with 2 different velocities, *Joint 8th IFAC Symposium on MECHATRONICS and 11th IFAC Symposium on NOLCOS*, Vienne, Austria, pp. 210–215, 2019.

In addition to the publications presented in this manuscript, let us cite

- J. Tsiniias and C. Kitsos, Observability and state estimation for a class of nonlinear systems, *IEEE Transactions on Automatic Control*, vol. 64, no. 6, pp. 2621–2628, 2019

that has been finalized in the beginning of this thesis (outside the supervision of C. Prieur and G. Besançon).

# The High-Gain Observer Design Problem (H-GODP)

## Contents

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This chapter is devoted to the introduction of the main topics, this thesis deals with, whose extensive analysis follows in the next chapters. A brief overview of observer designs for infinite-dimensional systems is first presented. Then, the general class of the considered cascade systems (which satisfy a particular triangular form) is introduced, with emphasis to some application examples including epidemiology and chemical reactions. The main problem of high-gain observer design this thesis deals with is next described underlining also the difficulties that are encountered, whence a description of proposed solutions follows.

## 1.1 A short overview of observer designs for infinite-dimensional systems

The problem of state estimation is important and is mainly used in model-based feedback control when the full state of the system is not available. To solve this problem, several observer design techniques have appeared in the literature for finite-dimensional systems governed by linear and nonlinear ordinary differential equations (ODEs). In the area of nonlinear systems, many of these designs lean heavily on a triangular structure (see [Bernard (2019)]) of a given nonlinear system and the most famous include the Luenberger-type observer design (see [Besancon (2007)]). As part of the latter, the high-gain observer design has gained significant consideration during the last decades, see [Khalil (2017)]. This triangular structure has been shown to be suitable for state estimation, while weakening of the conditions has been a goal of several approaches, see for instance [Theodosis et al (2016)],

[Tsinias and Kitsos (2019)]. This thesis is concerned with the introduction of a similar triangular structure to the case of infinite-dimensional systems governed by partial differential equations (PDEs), which allows a high-gain observer design to be feasible for some classes of these systems under suitable sufficient conditions.

Characterizations and conditions for observability for infinite-dimensional systems have arisen in the literature during the last decades. In [Tucsnak and Weiss (2000)], observability is studied for linear systems via semigroups and operator-theoretic techniques. Observability for coupled systems of hyperbolic equations has been introduced in [Lions (1988a)], [Lions (1988b)], where three notions are considered: complete, partial and simultaneous observability. In [Li (2008)], exact boundary observability for  $n \times n$  quasilinear hyperbolic systems via observation on the boundaries of the whole state vector is studied. In [Alabau-Boussouira (2003)], the problem of boundary observability for linear weakly coupled hyperbolic systems via observation of only one of the two components of the state is investigated. In [Lissy and Zuazua (2019)], internal observability for a coupled system of linear PDEs with reduced number of observations is studied.

Following the fundamental results on observability, observer design for infinite-dimensional systems has been developed with fewer results than for finite-dimensional systems so far, see [Wouwer and Zeitz (2001)] for a survey. In fact, two main techniques have been considered: early-lumping and late-lumping ones. Early-lumping approach relies on suitable finite-dimensional approximations and reduces the systems of PDEs into a system of ODEs, see for instance [Besancon et al (2006)]. This, however, experiences the problem of loss of the PDE properties, including the physical system information and additionally the state estimation of the true state might not be ensured, see [Balas (1998)]. In the late-lumping techniques, approaches are based on semigroup and spectral theory [Curtain (1982)], [Curtain and Zwart (1995)], [Demetriou (2004)], Lyapunov-based analysis, backstepping [Di Meglio et al (2013)], [Hasan et al (2016)], [Anfinson et al (2016)], and [Vazquez and Krstic (2010)], or optimization methods [Nguyen et al (2016)]. The case of state estimation for nonlinear infinite-dimensional systems, which is significantly more complicated, has been addressed in [Xu et al (1995)], [Bounit and Hammouri (1998)], [Christophides and Daoutidis (1996)], [Meurer (2013)], [Besancon et al (2013)], [Schaum et al (2015)], [Karafyllis et al (2019)], [Castillo et al (2013)] amongst others, but considering the full state vector on the boundaries as measurement.

Unlike all the above approaches, the present thesis considers observer problems where a part of the state is fully unknown (including at the boundaries). The known part is however distributed and the explored observers strongly rely on high gain, extending techniques and performances of finite-dimensional cases.

## 1.2 The class of systems

In this section, let us introduce the considered class of systems and present some examples which satisfy this form.

In short, the systems can be written as

$$\dot{u} = \mathfrak{A}(u)u + A(u_1)u + \mathfrak{F}(u) \quad (1.1)$$

in some Banach space of functions on an interval with values in  $\mathbb{R}^n$ , with an output given by

$$\begin{aligned} y &= Cu; \\ C &= (1 \quad 0 \quad \cdots \quad 0). \end{aligned} \quad (1.2)$$

To complete the definition of the class of systems, let us consider some initial conditions (in general unknown)  $u^0$ .

We assume that  $\mathfrak{A}(u)$  is, for each fixed argument, the infinitesimal generator of a  $C^0$ -semigroup and is written in the form

$$\mathfrak{A}(u)u = \sum_{l=1}^k \Pi_l(u) \partial_x^l u, \quad (1.3)$$

being a differential operator, acting on vector-valued functions  $u$ , with its domain  $dom(\mathfrak{A}(\cdot))$  containing appropriate functions satisfying suitable boundary conditions for each class of system of PDEs (with  $u^0 \in dom(\mathfrak{A}(\cdot))$ ). The mappings  $\Pi_l(\cdot)$  belong to appropriate function spaces, depending on the considered problem, take values in the space of lower triangular matrices and their elements satisfy the following triangularity

$$(\Pi_l)_{ij}(u) := (\Pi_l)_{ij}(u_1, \dots, u_j), i = 1, \dots, n, j = 1, \dots, i, l = 1, \dots, k. \quad (1.4)$$

Also,  $A(\cdot)$  is a mapping belonging to appropriate function spaces and taking values in  $\mathbb{R}^{n \times n}$  for each fixed argument. The nonlinear source term  $\mathfrak{F}(\cdot)$  acts on vector-valued functions  $u$ , which might be seen as a general nonlinear operator. Mappings  $A(u_1)$  and  $\mathfrak{F}(u)$  in general satisfy the following forms

$$A(u_1) = \begin{pmatrix} 0 & a_2(u_1) & 0 & \cdots & 0 \\ & \ddots & \ddots & & \vdots \\ \vdots & & & & a_n(u_1) \\ 0 & \cdots & & & 0 \end{pmatrix}, \quad (1.5)$$

$$\mathfrak{F}(u) = \begin{pmatrix} \mathfrak{F}_1(u_1) \\ \mathfrak{F}_2(u_1, u_2) \\ \vdots \\ \mathfrak{F}_n(u_1, \dots, u_n) \end{pmatrix}.$$

Note that more about function spaces and system's regularity will be said in each chapter depending on the considered class of systems. The above form corresponds to a cascade system of partial differential equations (PDEs) looking like a natural extension of the triangular forms that are appropriate for high-gain observer design in finite dimensions [Khalil (2017)], [Gauthier and Bornard (1981)], [Gauthier et al (1992)].

**Remark 1.1.** *Although in this chapter we introduce a system written in a very general triangular form which encompasses all classes of systems considered in the following chapters, observer design is performed for some subclasses only of this form. Contrary to the results we would expect to obtain coming from a natural extension of triangular forms for finite-dimensional systems to infinite-dimensional ones, observer design is not achieved for all classes of (1.1)-(1.2). For instance, throughout the thesis mappings  $\Pi_l$  are mostly considered diagonal and only  $u_1$ -dependent, except for Chapter 3.1 concerning  $2 \times 2$  hyperbolic systems, where these mappings exhibit some triangularity. Also, mapping  $A(u_1)$  is mostly considered constant, except for Chapter 3.2 concerning a class of  $2 \times 2$  diffusional Lotka-Volterra system. Observer design for systems written in the here-considered triangular form, in its full generality, remains open.*

Cascade systems of hyperbolic or parabolic type have been widely considered in the literature. For instance, internal controllability with reduced numbers of controls for first order hyperbolic systems in cascade form has been studied in [Alabau-Boussouira et al (2017)] (see also [Alabau-Boussouira (2013)] for observability of second order hyperbolic systems with reduced observations). In [Coron et al (2010)] the null controllability is studied for a parabolic system in a cascade form with a nonlinear coupling term. In [Alabau-Boussouira (2003)] boundary observability is studied for systems in cascade form with possibly different differential operators on the diagonal, i.e., couplings of different types of PDEs, for instance coupled wave-Petrowsky equations.

In what follows, we present some models that can be written in the triangular form (1.1) with output (1.2), for which, high-gain observers will be designed in the following chapters.

- **SIR epidemic models:** In the field of epidemiology, mathematical models are used to explain epidemic phenomena and to assess vaccination strategies through control mechanisms, as for instance in the recent COVID-19 pandemic. For infectious diseases, a fundamental model was formulated by Kermack and McKendrick [Kermack and McKendrick (1927)] (see [Bastin and Coron (2016)], Chapter 1 and also [Ianeli (1995)], Chapter III for detailed presentation of such systems). This model classifies population into three groups: (i) the individuals who are uninfected and susceptible ( $S$ ) of catching the disease, (ii) the individuals who are infected ( $I$ ) by the concerned pathogen, (iii) the recovered ( $R$ ) individuals who have acquired a permanent immunity to the disease. In the case when the age of patients is taken into account,  $S(t, x), I(t, x), R(t, x)$  represent the age distribution of the population of each group at time  $t$ . As a result, the integral from  $x_1$  to  $x_2$  of  $S, I$  and  $R$  is the number of individuals of each group with ages between  $x_1$  and  $x_2$ .

The dynamics of the disease propagation in the population are then described by the following set of hyperbolic partial integro-differential equations on  $[0, +\infty) \times [0, L]$

$$\begin{aligned} \partial_t S(t, x) + \partial_x S(t, x) + \mu(x)S(t, x) + \mathcal{G}[S(t), I(t)](x) &= 0, \\ \partial_t I(t, x) + \partial_x I(t, x) + (\gamma(x) + \mu(x))I(t, x) - \mathcal{G}[S(t), I(t)](x) &= 0, \\ \partial_t R(t, x) + \partial_x R(t, x) + \mu(x)R(t, x) - \gamma(x)I(t, x) &= 0, \end{aligned} \quad (1.6)$$

where  $\mathcal{G}[S(t), I(t)](x) := \beta(x)S(t, x) \int_0^L I(t, s)ds$  represents the disease transmission

rate by contact between susceptible and infected individuals, which is assumed to be proportional to the sizes of both groups, with  $\beta(x) > 0$  being the age-dependent transmission coefficient between all infected individuals and susceptibles having age  $x$ . The maximal life duration in the considered population is denoted by  $L$  and, thus,  $S(t, L) = I(t, L) = R(t, L) = 0$ . Parameter  $\mu(x) > 0$  denotes the natural age-dependent per capita death rate in the population and  $\gamma(x) > 0$  is the age-dependent rate at which infected individuals recover from the disease.

The boundary conditions are written in the form

$$S(t, 0) = B(t), \quad I(t, 0) = 0, \quad R(t, 0) = 0,$$

where  $B(t)$  stands for the inflow of newborn individuals in the susceptible part of the population at time  $t$ . Assume that we are able to measure the number of people in the group  $R$  of recovered patients between ages 0 and  $x$ , for every age  $x \in [0, L]$  and time  $t \geq 0$ , i.e., system's output is given by

$$y(t, x) = \int_0^x R(t, s) ds.$$

The above system is written in the form (1.1)-(1.2) by applying a nonlocal transformation, as it is shown in Chapter 2.2. Note that the proposed measurement of the infected patients upon recovery is following the remarks in the recent paper [Park and Bolker (2020)] about the COVID-19 pandemic. An alternative formulation would be plausible, where measurement of just the number of infected patients  $\int_0^x I(t, s) ds$  is considered.

- **Tubular chemical reactors**

Control and observer designs for chemical reactors in the sense of distributed parameter systems have been widely investigated, see for instance [Boscovic and Krstic (2002)] and [Christophides and Daoutidis (1996)]. We present here a model of a plug flow chemical reactor (see [Bastin and Coron (2016)], Chap. 1.7). A plug flow chemical reactor is a tubular reactor where a liquid reaction mixture circulates. The reaction proceeds as the reactants travel through the reactor. Here, we consider the case of a horizontal reactor, where a simple mono-molecular reaction takes place between  $A$  and  $B$ , where  $A$  is the reactant species and  $B$  is the desired product. The reaction is supposed to be exothermic and a jacket is used to cool the reactor. The cooling fluid flows around the wall of the tubular reactor. The dynamics are described by the following hyperbolic equations on  $[0, +\infty) \times [0, L]$

$$\begin{aligned} \partial_t T_c - V_c \partial_x T_c - k_0(T_c - T_r) &= 0, \\ \partial_t T_r + V_r \partial_x T_r + k_0(T_c - T_r) - k_1 r(T_r, C_A, C_B) &= 0, \\ \partial_t C_A + V_r \partial_x C_A + r(T_r, C_A, C_B) &= 0, \\ \partial_t C_B + V_r \partial_x C_B - r(T_r, C_A, C_B) &= 0, \end{aligned} \tag{1.7}$$

where  $V_c$  is the coolant velocity in the jacket,  $V_r$  is the reactive fluid velocity in the reactor,  $k_0$  and  $k_1$  are some positive constants,  $T_c(t, x)$  is the coolant temperature,



$T_r(t, x)$  is the reactor temperature,  $C_A(t, x)$  and  $C_B(t, x)$  denote the concentrations of the chemicals in the reaction medium. The function  $r(T_r, C_A, C_B)$  represents the reaction rate and is given by

$$r(T_r, C_A, C_B) = (aC_A - bC_B) \exp\left(-\frac{E}{RT_r}\right),$$

where  $a, b$  are rate constants,  $E$  is the activation energy and  $R$  is the Boltzmann constant. We consider boundary conditions as follows

$$T_r(t, 0) = T_r^{\text{in}}, \quad T_c(t, 0) = T_c^{\text{in}}, \quad C_A(t, 0) = C_A^{\text{in}}, \quad C_B(t, 0) = 0.$$

Noting that the sum of concentrations  $C_A + C_B$  is simply described by a delay equation, we can assume constant sum  $C_A(t, x) + C_B(t, x) = C_A^{\text{in}}$  and thus the last hyperbolic equation may be ignored. Assuming that the measured quantity is the coolant temperature  $T_c$ , i.e.,

$$y(t, x) = T_c(t, x),$$

we can transform system into a form as (1.1)-(1.2) by applying the invertible transformation

$$u_1 = T_c, \quad u_2 = T_r, \quad u_3 = ((a + b)C_A - bC_A^{\text{in}}) \exp\left(-\frac{E}{RT_r}\right).$$

- **Hyperbolic Lotka-Volterra systems** We consider a class of hyperbolic cooperative Lotka-Volterra model, see for instance [Zhang et al (2018)],

$$\begin{aligned} \partial_t w_i + \lambda_i \partial_x w_i &= w_i \left( c_i + \sum_{j=1}^{i+1} m_{ij} w_j \right), \text{ in } [0, +\infty) \times [0, L], i = 1, \dots, n-1, \\ \partial_t w_n + \lambda_n \partial_x w_n &= c_n w_n + w_n \sum_{j=1}^n m_{nj} w_j, \text{ in } [0, +\infty) \times [0, L], \end{aligned}$$

where  $w_i(t, x)$  is the population size of the  $i$ -th species,  $c_i$  depicts the intrinsic growth rate of each species,  $m_{ij}$  is interspecific ( $m_{ij} \geq 0$ , as  $i \neq j$ ) or intraspecific ( $m_{ij} \leq 0$ , as  $i = j$ ) interaction, and  $\lambda_i$  are the transport speeds of each species. For observer design purposes, we assume  $m_{i-1,i} \neq 0, \forall i = 2, \dots, n$ . Assuming that  $\lambda_i < 0, i = 1, \dots, m$  and  $\lambda_i > 0, i = m + 1, \dots, n$ , boundary conditions can be written as

$$\begin{pmatrix} w^-(L, t) \\ w^+(0, t) \end{pmatrix} = K \begin{pmatrix} w^-(0, t) \\ w^+(L, t) \end{pmatrix}, \forall t \geq 0,$$

where  $w = (w^- \ w^+)^T$ ;  $w^- \in \mathbb{R}^m, w^+ \in \mathbb{R}^{n-m}$  and  $K \in \mathbb{R}^{n \times n}$ . We assume also that initial conditions are positive and, by standard results, there exist unique global classical positive solutions for this class of systems. Then, linearizing the above class of systems

around their steady state, the following hyperbolic dynamics are satisfied

$$\begin{aligned}\partial_t \bar{w}_i + \lambda_i \partial_x \bar{w}_i &= \sum_{j=1}^{i+1} \bar{m}_{ij}(x) \bar{w}_j, \quad i = 1, \dots, n-1, \\ \partial_t \bar{w}_n + \lambda_n \partial_x \bar{w}_n &= \sum_{j=1}^n \bar{m}_{nj}(x) \bar{w}_j,\end{aligned}\tag{1.8}$$

with  $\bar{m}_{i-1,i}(x) > 0, \forall x \in [0, L], i = 2, \dots, n$  and boundary conditions as in the nonlinear model, but perturbed by a function of  $x$ . We make the hypothesis that the population of the first species is measured within the domain, i.e.,

$$y(t, x) = \bar{w}_1(t, x).$$

Hence, this linearized system takes the form of (1.1)-(1.2) (in its linear version). Such class of hyperbolic systems will be considered in Section 3.3 and their diffusional version in Section 3.2 of Chapter 3.

## 1.3 Description of the H-GODP

In this section, we present the main problem this thesis deals with along with some directions towards the proposed solutions, on which the next chapters elaborate. It concerns the high-gain observer design problem for infinite-dimensional systems. This thesis is devoted to the extension of the high-gain observer design for finite-dimensional systems to infinite-dimensional ones, which turns out to be a nontrivial problem. The classical high-gain observer design for finite-dimensional nonlinear systems has been extensively studied in the literature and remains widely considered, see [Khalil (2017)] and references therein. In short, it relies on a single tuning coefficient, to be chosen large enough so as to ensure exponential - and possibly arbitrarily fast - convergence. High-gain observers apply to a large class of cases corresponding to uniformly observable systems [Gauthier and Bornard (1981)], [Gauthier et al (1992)].

### 1.3.1 Problem statement

In this subsection, we present a general form of the proposed high-gain observer. More explicit forms of such observers will be specified in the forthcoming chapters, depending on the considered classes of systems. For each of the cases, the considered differential operator  $\mathfrak{A}(\cdot)$  of system (1.1), its domain, along with the type of nonlinearities will lead to different observer designs and specifications. The main design difficulty encountered in the thesis comes from the presence of distinct velocities in the hyperbolic operators or distinct diffusivities in the parabolic ones. Additionally, the type of nonlinearities, which are either locally or globally Lipschitz, possible existence of nonlocal terms, and quasilinearity/semilinearity of system of PDEs require various adjustments and treatments. Also, particularly hyperbolic systems experience some difficulties with respect to observer design, coming from the relationship on

the boundaries coupling the incoming information with the outgoing one (they might be general and nonlinear), which exhibits the same behavior and imposes the same obstacles as the nonlinearities in the source terms of the hyperbolic dynamics. Note here, that although we consider (1.1) as the general form, some slight modifications on the dynamics will be made in some cases of the following chapters.

Let us first consider a symmetric positive definite  $n \times n$  matrix  $P$  satisfying a Lyapunov inequality of the following form

$$\text{Sym}(PA(y)) - C^\top C \leq -\frac{\eta}{2}I_n, \quad (1.9)$$

for some constant  $\eta > 0$ . Such an inequality is always feasible for  $A(y)$  and  $C$  satisfying algebraic forms as the ones we already assumed in (1.2), (1.5) (borrowed from the observability canonical forms for finite-dimensional systems, see for instance [Khalil (2017)]) and, furthermore, if  $A(\cdot)$  is uniformly bounded and  $\inf_{u \in \text{dom}(\mathfrak{A}(\cdot))} a_i(Cu) > 0, i = 2, \dots, n$ . The existence of such a  $P$  can be easily proven, refer for instance to [Hammouri et al (2002)]. Define also a constant matrix  $\Theta$  by

$$\Theta := \text{diag}(\theta, \theta^2, \dots, \theta^n), \quad (1.10)$$

where  $\theta > 0$  is a large enough constant and is called the *high gain* constant of the observer.

Consider now an observer system for (1.1)-(1.2) described by the following equation

$$\dot{\hat{u}} = \hat{\mathfrak{A}}(y)\hat{u} + A(y)\hat{u} + \Theta P^{-1}C^\top(y - C\hat{u}) + \hat{\mathfrak{F}}(y, \hat{u}), \quad (1.11)$$

where  $\hat{\mathfrak{A}}(y)$  is the infinitesimal generator of a  $C^0$ -semigroup for fixed  $y$ . The domain of  $\hat{\mathfrak{A}}(y)$  contains functions satisfying same type of boundary conditions as  $\mathfrak{A}$ , but possibly perturbed by nonlocal mappings of the output  $y$  and  $\hat{\mathfrak{A}}$  might perform higher-order differentiations than the ones of  $\mathfrak{A}$ . Also,  $\hat{\mathfrak{F}}(y, \hat{u})$  is an appropriately defined mapping for each of the considered design problems, which might include higher-order differentiations of  $y$  in its domain and has the following dependency on  $\hat{u}$

$$\hat{\mathfrak{F}}(y, \hat{u}) = \begin{pmatrix} \hat{\mathfrak{F}}_1(y, \hat{u}_1) \\ \hat{\mathfrak{F}}_2(y, \hat{u}_1, \hat{u}_2) \\ \vdots \\ \hat{\mathfrak{F}}_n(y, \hat{u}_1, \dots, \hat{u}_n) \end{pmatrix}.$$

**Remark 1.2.** *In this thesis, most of the times we assume mapping  $A(\cdot)$  constant and containing 1s on its sup-diagonal. In that case, we can substitute the term  $\Theta P^{-1}C^\top(y - C\hat{u})$  by  $-\Theta KC(y - C\hat{u})$  in (1.11), where  $K$  is a vector, which renders  $A + KC$  Hurwitz. The latter is always possible due to the observability of the pair  $(A, C)$ .*

The problem this thesis deals with is described by the following definition.

**Definition 1.1.** *(H-GODP) The High-Gain Observer Design Problem is solvable for a system given by (1.1) with output (1.2), while output's spatial derivatives of order at most  $n - 1$  might*

also be available, if there exists a well-posed observer system, which estimates the state of (1.1) with a convergence speed that can be arbitrarily tuned via a single parameter (high-gain constant)  $\theta$ . More precisely, for every  $\kappa > 0$ , there exists  $\theta_0 > 1$ , such that for every  $\theta \geq \theta_0$ , solutions to (1.1), (1.11), with initial conditions  $u^0, \hat{u}^0$  satisfy

$$\|\hat{u} - u\|_{\mathcal{X}_1} \leq \ell e^{-\kappa t} \|\hat{u}^0 - u^0\|_{\mathcal{X}_2} \quad (1.12)$$

for some  $\ell > 0$  polynomial in  $\theta$ , where by  $\|\cdot\|_{\mathcal{X}_1}, \|\cdot\|_{\mathcal{X}_2}$  we denote norms of appropriately chosen function spaces  $\mathcal{X}_1, \mathcal{X}_2$  for each of the considered problems.

The proposed observer designs rely on a measurement of part of the state. The main feature of this described high-gain observer design is the arbitrarily fast convergence rate, similarly as in the finite dimensions and to achieve this property, distributed measurement on the whole domain is assumed. Furthermore, as indicated in the H-GODP definition, stronger regularity of the solutions to the initial systems is required for some classes of systems, since the observer dynamics may include higher-order spatial derivatives of the output (in the weak or classical sense). This requirement of stronger regularity reveals some links to previous studies on internal controllability for cascade systems with reduced number of controls, see for instance the work of [Alabau-Boussouira et al (2017)]. We note here that, although boundary observers with the full-state measurement are preferred for practical reasons, see for instance [Castillo et al (2013)], in the present formulation distributed measurement of part of the state might be available in many cases of distributed parameter systems. For instance, some setups include thermal cameras for chemical reactors or alternative methods, see for instance [Zogg et al (2004)], [Pradere et al (2009)], providing the desired distributed measurements. Also, approximations with distributed measurements within the domain would provide an approximated measurement on the whole domain. In Lotka-Volterra systems, measurements of one of the  $n$  species might be possible through monitoring it in time. Additionally, the required higher-order spatial derivatives of the output can be available in real-time, since these constitute causal measurements, contrary to the time-derivatives of the output, which are strictly not included in observer designs, as the knowledge of them is noncausal. Although this requirement of the availability of space derivatives of the output might seem restrictive, approximations via kernel convolutions might be an alternative realization. In the following remark, we discuss the problem of the solvability of the H-GODP for hyperbolic systems, if instead of the distributed measurement (1.2), we had a boundary one.

**Remark 1.3.** *If we consider hyperbolic systems of the form (1.1), we see that the problem of minimum-time control of such systems arises (see [Coron and Nguyen(2019)] and [Auriol and di Meglio (2016)]), suggesting that a faster observer than a boundary one would be desirable. Solvability of the H-GODP suggests that a high-gain observer would be arbitrarily fast, without any limitation in the convergence speed. H-GODP is not solvable in case of boundary measurement, instead of internal measurement as in (1.2).*

*First, arbitrary convergence condition would not be fulfilled, since a boundary observer for hyperbolic systems would experience a limitation with respect to convergence speed. The rate of convergence is limited by a minimal observation time which depends on the size of the domain and the characteristic velocities in that case (see [Li (2008)] for minimum time of*

observability due to transport phenomena, and [Deutschmann et al (2016)] for comments on the convergence of boundary observers).

Second, following a boundary observer design methodology as in [Castillo et al (2013)], in the presence of a general form of boundary conditions, where a general (nonlinear) law couples the incoming with the outgoing information on the boundaries, boundary measurement of the whole state vector would be required, instead for just the first state, for the boundary observer to be feasible. In [Coron et al (2013)], control design is achieved for a  $2 \times 2$  hyperbolic system with some particular boundary conditions, via boundary control on one end of only one state. Here, however, where we consider the dual problem of observer design with one observation, such an approach would not be feasible, because of the generality of the boundary conditions that we might consider. For general boundary conditions, by just one observation we cannot achieve a dissipativity of the boundary conditions as in this work, which would lead to stability of the observation error system (see [Coron and Bastin (2015)] about linking dissipativity of boundary conditions with stability).

### 1.3.2 Solutions to the H-GODP

In this subsection, we introduce the main approaches, which lead to solutions to the H-GODP for some classes of systems of PDEs. These are more explicitly developed in the next chapters. We distinguish two approaches, the direct observer design and the indirect observer design, depending on a specific property of the differential operator associated to the system.

**Definition 1.2** (Property (S)). *We say that system (1.1) satisfies Property (S) for its associated differential operator  $\mathfrak{A}(\cdot)$ , given by (1.3), if mappings  $\Pi_l(\cdot)$  satisfy*

$$\Pi_l(u) = \pi_l(Cu)I_n, \forall u \in \mathbb{R}^n, l = 1, \dots, k, \quad (1.13)$$

for some scalar functions  $\pi_l(\cdot)$ .

**Remark 1.4.** *When Property (S) in the above definition is satisfied, we deduce a commutative property between  $\Pi_l(\cdot)$  and  $P$ , where  $P$  is a Lyapunov matrix satisfying (1.9) for some  $\eta > 0$ . This Lyapunov matrix is used in the Lyapunov stability analysis of the observer error equations and such a Property (S) implying the aforementioned commutativity, is needed to perform integrations by parts. In this thesis, we are mostly concerned with systems associated with diagonal differential operators  $\mathfrak{A}$ , although a system (1.1) might be suitable for observer design if it has a more general form, with its differential operator being lower triangular, i.e., matrices  $\Pi_l(\cdot)$  are lower triangular. It would be plausible, thereby, to state a "generalized" version of Property (S) corresponding to these cases, where again integrations by parts would be feasible. "We say that system (1.1) satisfies Property (S) for its differential operator  $\mathfrak{A}$ , if there exists a constant matrix  $P$  symmetric and positive definite satisfying (1.9) for some constant  $\eta > 0$ , with*

$$P\Pi_l(u) \text{ symmetric, for all } u \in \text{dom}(\mathfrak{A}(\cdot)), l = 1, \dots, k."$$

Observer design has been achieved for a class of hyperbolic systems with  $\Pi_{l=1}(\cdot)$  only  $u_1$ -dependent, which, under a finite-dimensional state transformation, can be written in a system,

whose hyperbolic operator satisfies the above-mentioned "generalized" version of Property (S), see [Kitsos et al (2019b)]. However, this work is not included in the present manuscript, as some specifications of the high-gain observer designs were not met via this approach.

Following the above definition, we distinguish two cases for high-gain observer design. The first corresponds to solvability of the H-GODP when Property (S) holds and the second to solvability when Property (S) does not hold.

#### A. Direct observer design (Property (S) holds)

Direct observer design is feasible when Property (S) holds and this is shown in Chapter 2 for a class of  $n \times n$  quasilinear diagonal hyperbolic systems of partial integro-differential equations with one characteristic velocity, satisfying structure (1.1). Property (S) allows the design of a high-gain observer of the form (1.11) with  $\tilde{\mathfrak{A}}(\cdot)$  the same as  $\mathfrak{A}(\cdot)$  (while the argument  $u_1$  of the first operator  $\tilde{\mathfrak{A}}(\cdot)$  is substituted by  $y$  in the second operator  $\mathfrak{A}(\cdot)$ ) and all the mappings of this equation not containing any spatial derivatives of the output. Thus, for this observer design, in the definition of H-GODP, spatial derivatives of the output need not be available. In the recent work of [Alabau-Boussouira et al (2017)] and previous works of these authors, it is shown that for a case of diagonal hyperbolic systems with a single characteristic velocity (diagonal hyperbolic operator with one velocity, obviously, satisfies Property (S) defined here), written in a cascade form, the controllability problem is less complicated compared to the case of distinct velocities. This problem, for instance, does not encounter the phenomenon of loss of derivatives, which arises in the presence of distinct characteristic velocities. In this case, no stronger regularity of system's solutions is required. In the context of the present thesis, observer design with reduced number of observation and under Property (S), it is simpler compared to the case where Property (S) does not hold and then direct observer design is the solution to the former case. In Chapter 2, it will be shown that, under Property (S) integration by parts is possible, needed in the Lyapunov stability analysis of the observer error. Although Property (S) seems restrictive for the class of hyperbolic systems, since it implies the presence of a single velocity, there are examples as the previously mentioned SIR epidemic models, which indeed satisfy this property, see Section 1.2. Property (S) also holds for cascade systems of linearized Korteweg-de Vries equations, considered in Chapter 4.

#### B. Indirect observer design (Property (S) does not hold)

In this thesis, we consider some systems, where Property (S) does not hold. Chemical reactors and hyperbolic Lotka-Volterra systems, considered in Section 1.2, do not in general satisfy Property (S). Without property (S) holding, a different strategy than in the case A is employed, where indirect observer design is applied. This indirect method, first, requires system's differential operator  $\mathfrak{A}$  to be decomposed into a sum of a) a new differential operator, satisfying this fundamental Property (S), b) a differential operator (and possibly nonlinear) including only the first measured state in its domain, and c) a bilinear mapping between a function of the unmeasured state and a differential operator, including only the first state in its domain. To achieve this, we apply an invertible infinite-dimensional state transformation, where, simultaneously, the systems of PDEs preserve their triangular structure. In this way, the differential operator, associated to the new system, satisfies Property (S). This proposed

decomposition of the differential operator leads us to inject the spatial derivatives of the output of higher orders in the observer dynamics. Existence of such infinite-dimensional transformations is shown in Chapter 3, where quasilinear  $2 \times 2$  hyperbolic systems with different characteristic velocities,  $2 \times 2$  and  $3 \times 3$  semilinear parabolic systems with distinct diffusivities, and finally  $n \times n$  linear heterodirectional hyperbolic systems are considered.

The proposed solutions to the H-GODP via this indirect design is connected to the problem of controllability/observability and exhibits similar difficulties that arise in these problems. Although these links are not investigated in this thesis, since its main goal is to provide a solution to the H-GODP, we refer to some significant works on controllability, which might present some duality with the considered problem here of observer design with reduced number of observations, and could be part of a future research. The difficulties arising in the controllability of hyperbolic systems with internal control and reduced number of controls and in the presence of distinct characteristic velocities is described in the notable work [Alabau-Boussouira et al (2017)], as well as in previous works of Coron and Alabau-Boussouira, see for instance [Coron and Lissy (2014)] for controllability of Navier-Stokes systems. The difficulties with respect to the presence of distinct velocities come from the algebraic solvability, a notion which appears in the fixed point theorem of M. Gromov [Gromov (1986)] for partial differential operators and its use of the framework of control theory was introduced in [Coron (2007)]. In these works, it is shown that the complexity to prove controllability augments with the number of equations and the strength of the nonlinearities. The same difficulties reasonably arise in the problem of H-GODP, solved in Chapter 3, where the larger the number of states of the systems, the number of distinct velocities or diffusivities and the number of the nonlinearities are, the solvability analysis is more complicated. In this context, the proposed indirect observer design is only applicable to some subclasses of the general system (1.1).

Stronger regularity is required for the controllability problem in [Alabau-Boussouira et al (2017)], see also [Alabau-Boussouira (2013)] (Hypothesis H1 in Section 4). In these works, the phenomenon of loss of derivatives arises, as the regularity of the dynamics is stronger than the regularity of the control laws, whenever the velocities are distinct (see Theorem 3.1 in [Alabau-Boussouira et al (2017)]). In the present framework of the solutions to the H-GODP, the regularity of system's dynamics needs to be stronger than the regularity of the space in the norm of which the asymptotic convergence of the observer is exhibited (refer to the norm  $\|\cdot\|_{\mathcal{X}_1}$  in Definition 1.1). Furthermore, when dealing with  $n \times n$  hyperbolic systems with  $n \geq 4$  velocities, a condition of space periodicity for the boundary conditions is required for the solutions of the initial system, in order to solve the H-GODP in Chapter 3.3 here. A similar condition of space periodicity for the boundary conditions appears in controllability problems in [Alabau-Boussouira et al (2017)]. Hence, revealing the nature of these mentioned works, it is plausible to understand that this indirect observer design, which is proposed in this thesis, requires some stronger sufficient conditions than the ones in the case of direct observer design.

## 1.4 Conclusion

After a short review of existing literature on observer design for infinite-dimensional systems, the new problem that is considered in the present manuscript has been introduced (and called H-GODP). It extends in short high gain techniques already available for finite-dimensional systems to some infinite-dimensional ones, highlighting some specific issues in such cases and giving rise to two types of solutions to be presented in next chapters: direct or indirect design, depending on some property of the associated differential operator.





# Solving H-GODP for a system of quasilinear hyperbolic PIDEs with a single velocity

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This chapter is devoted to presenting a solution to the H-GODP for a class of quasi-linear hyperbolic systems, possibly including nonlocal source terms, making them systems of Partial Integro-Differential Equations (PIDEs). The results of the present chapter are presented in [Kitsos et al (2018)] and [Kitsos et al (2020d)].

The systems considered here are written in a triangular form, as in the general form (1.1) introduced in Chapter 1, while the output is a distributed measurement of part of the state, see (1.2). This class of systems can be found in various cases, like chemical reactors [Boscovic and Krstic (2002)], where by measuring the temperature of the reactor we would expect to estimate the concentrations of masses of chemicals, or some age-dependent epidemic models, where the objective would be to estimate the remaining population groups arbitrarily fast with the only measurement of the population that has recovered from a disease. Stability and controllability of age-structured population models have been already studied (see [Bastin and Coron (2016)] [Chapter 1] and references therein, [Schmidt et al (2018)]) and in general PIDEs or PDEs with nonlocal terms have been considered for instance in [Coron et al (2016)], [Karafyllis and Krstic (2017)], [Deutschmann et al (2016)].

Concerning the quasilinear hyperbolic operator domain, here it is considered quite general with respect to boundary conditions, while it is diagonal with only one characteristic velocity,

in order to satisfy Property (S), introduced in Definition 1.2 of Chapter 1. We recall from Subsection 1.3.2 that Property (S) leads to direct observer design (see A in Subsection 1.3.2), since in that case, a Lyapunov matrix commutes with the matrix of the characteristic velocities. Then, under Property (S), integration by parts, needed in the Lyapunov stability analysis of the observer error equations, can directly be used. Note that constraints on the source term can be found in some studies of stability problems as in [Bastin and Coron (2016)], [Coron and Bastin (2015)] and in [Prieur et al (2014)] (Proposition 2.1), which allow a similar commutativity as the previously described one, while this is not the case here.

The main contribution of this chapter is a solution to the H-GODP for the considered class of systems: a high-gain observer is designed, while system's solutions remain bounded in an appropriate space, an assumption which is made in order to deal with the quasilinearity of the hyperbolic operator. The nonlocal source terms in the dynamics are considered locally Lipschitz. The high-gain observer then is a copy of the system up to some output injection terms, while the nonlinearities are substituted by their composition with sufficiently smooth saturation functions. The exponential stability of the observer error is proven for the  $C^1$ -norm, following analysis inspired by [Bastin and Coron (2016)] and [Coron and Bastin (2015)] for instance; and finally, the applicability of this approach is illustrated to an epidemic model.

The chapter is organized as follows. The solution to the H-GODP, sufficient conditions and detailed proofs are provided in Section 2.1, where Theorem 2.1 constitutes our main result. In Section 2.2 an application to an age-dependent SIR-type epidemic model is proposed. Conclusions and perspectives are discussed in Section 2.3.

## 2.1 Main observer result

In this section, we present the considered class of systems, the proposed high-gain observer, and its convergence proof.

### 2.1.1 Problem statement and requirements

We are concerned with one-dimensional, first-order quasi-linear hyperbolic systems of balance laws, described by the following equations on a strip  $\Pi := [0, +\infty) \times [0, L]; L > 0$

$$\xi_t(t, x) + \lambda(\xi_1(t, x))\xi_x(t, x) = A\xi(t, x) + f(x, \mathcal{F}[\xi(t)](x)), \quad (2.1a)$$

where  $\xi = (\xi_1 \ \cdots \ \xi_n)^\top$ .

Consider also a distributed measurement, available in the output, of the form

$$y(t, x) = C\xi(t, x), \quad (2.1b)$$

where  $y : [0, +\infty) \times [0, L] \rightarrow \mathbb{R}$ .

We assume that the involved constant matrices satisfy the following particular structures

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ & \ddots & \ddots & & \vdots \\ \vdots & & & & 1 \\ 0 & \cdots & & & 0 \end{pmatrix}, \quad C = (1 \ 0 \ \cdots \ 0),$$

and the nonlinear balance terms are of the form

$$f(\cdot, \mathcal{F}[\xi]) = \begin{pmatrix} f_1(\cdot, \mathcal{F}_1[\xi_1]) \\ f_2(\cdot, \mathcal{F}_2[\xi_1, \xi_2]) \\ \vdots \\ f_n(\cdot, \mathcal{F}_n[\xi_1, \dots, \xi_n]) \end{pmatrix}.$$

From the previous equations, we observe that the system satisfies some triangular structure, which presents an analogy to the finite-dimensional case (see [Khalil (2017)]), as the one introduced in Section 1.2 of Chapter 1.

In the sequel, for mapping  $f$  we use the definition of the following difference operator

$$\Delta_{\hat{\xi}}[f(\cdot, \mathcal{F})](\xi)(x) := f(x, \mathcal{F}[\hat{\xi}](x)) - f(x, \mathcal{F}[\xi](x)),$$

parametrized by  $\hat{\xi}$ .

We assume the following regularity for the dynamics:

- the characteristic velocity  $\lambda$  is continuously differentiable, i.e.,  $\lambda$  in  $C^1(\mathbb{R}; \mathbb{R})$  and, without loss of generality,  $\lambda(\xi_1) > 0$ , for all  $\xi_1$  in  $\mathbb{R}$  (hyperbolicity condition).
- the nonlinear balance term  $f$  is continuously differentiable, i.e.,  $f$  in  $C^1([0, L] \times \mathbb{R}^n; \mathbb{R}^n)$ . As a result, for every  $R > 0$ , there exists  $L_R > 0$ , such that for every  $w, \hat{w}$  in  $\mathbb{R}^n$ , with  $|w|, |\hat{w}| \leq R$ ,  $\max_{x \in [0, L]} |f(x, w) - f(x, \hat{w})| \leq L_R |w - \hat{w}|$ . In addition, suppose that for every  $R > 0$ , there exists  $L'_R > 0$ , such that for every  $w, \hat{w}$  in  $\mathbb{R}^n$ , with  $|w|, |\hat{w}| \leq R$ ,  $\max_{x \in [0, L]} |D_w f(x, w) - D_{\hat{w}} f(x, \hat{w})| \leq L'_R |w - \hat{w}|$ . Furthermore,  $\mathcal{F} : C^0([0, L]; \mathbb{R}^n) \rightarrow C^1([0, L]; \mathbb{R}^n)$  is a Fréchet differentiable mapping that can include terms  $g_1(\xi)$ , where  $g_1$  in  $C^1(\mathbb{R}^n; \mathbb{R}^n)$  and also nonlocal terms (nonlinear Volterra integrals) of the form  $\int_0^x g_2(\xi(s)) ds$ , where  $g_2$  in  $C^0(\mathbb{R}^n; \mathbb{R}^n)$ . More explicitly, suppose that for every  $R > 0$ , there exist  $L_{1,R}, L_{2,R}, L_{3,R}, L'_{1,R}, L'_{2,R}, L'_{3,R} > 0$ , such that for every  $\xi, \hat{\xi}$  in  $C^0([0, L]; \mathbb{R}^n)$ , with  $\|\xi\|_\infty, \|\hat{\xi}\|_\infty \leq R$ ,  $|\Delta_{\hat{\xi}}[\mathcal{F}](\xi)(x)| \leq L_{1,R} |\xi(x) - \hat{\xi}(x)| + L_{2,R} |\xi(L) - \hat{\xi}(L)| + L_{3,R} \int_0^L |\xi(s) - \hat{\xi}(s)| ds$ , and  $|\Delta_{\hat{\xi}} D_{\xi}[\mathcal{F}](\xi)(x)| \leq L'_{1,R} |\xi(x) - \hat{\xi}(x)| + L'_{2,R} |\xi(L) - \hat{\xi}(L)| + L'_{3,R} \int_0^L |\xi(s) - \hat{\xi}(s)| ds$ , for all  $x$  in  $[0, L]$ .

We consider initial and boundary conditions of the following general form

$$\xi(0, x) = \xi^0(x), \quad x \in [0, L], \quad (2.2a)$$

$$\xi(t, 0) = \mathcal{H}(\xi(t, L)), \quad t \in [0, +\infty), \quad (2.2b)$$

where we assume the following regularity:

- Initial condition  $\xi^0$  is continuously differentiable and satisfies zero-order and one-order compatibility conditions (see [Bastin and Coron (2016)][App. B] for precise definition of compatibility conditions). The mapping  $\mathcal{H}$  is continuously differentiable, i.e.,  $\mathcal{H}$  in  $C^1(\mathbb{R}^n; \mathbb{R}^n)$ . More explicitly, suppose that for every  $R > 0$ , there exist  $L_{4,R}, L'_{4,R} > 0$ , such that for every  $\xi, \hat{\xi}$  in  $\mathbb{R}^n$ , with  $|\xi|, |\hat{\xi}| \leq R$ ,  $|\Delta_{\hat{\xi}}[\mathcal{H}](\xi)(L)| \leq L_{4,R}|\xi(L) - \hat{\xi}(L)|$ , and  $|\Delta_{\hat{\xi}}[D_{\xi}\mathcal{H}](\xi)(L)| \leq L'_{4,R}|\xi(L) - \hat{\xi}(L)|$ .

**Remark 2.1.** *The specific type of boundary conditions input/output relation (2.2b) stands as a sufficient condition for the well-posedness of the observer. This will be shown in the sequel. Also, the main result of the present section would remain unchanged if we considered the case where boundary conditions satisfy the relation  $\xi(t, 0) = h(t, \xi(t, L))$ , with  $h(\cdot, \cdot)$  a  $C^1$  map. This case would only require slight modifications.*

The assumption that follows is essential to assert the well-posedness of our system, along with an observer design requirement of *forward completeness*. Furthermore, it imposes global boundedness of classical solutions in the  $C^1$ -norm, which is essential in the design of our nonlinear observer. The latter assumption arises from the quasi-linearity of the system (the dependence of  $\lambda$  on  $\xi_1$ ) and can be dropped for the case of semilinear systems, but then a stronger assumption on the nonlinear source terms would be imposed in its place. For more detailed presentation of the nature of the following assumption, the reader can refer to [Bastin and Coron (2016)], [Li (1985)] and references therein, where sufficient conditions for the well-posedness and existence of classical solutions for quasi-linear hyperbolic systems of balance laws are given. If we had nonlocal conservation laws, i.e., if velocity was of the form  $\lambda[\xi_1(t)](x) := \lambda(\int_0^L \xi_1(t, s) ds)$ , this assumption would be met more easily, see for instance [Coron et al (2020)], [Keimer et al (2018)].

**Assumption 2.1.** *Consider a set  $\mathcal{M} \subset C^1([0, L]; \mathbb{R})$  nonempty and bounded, consisting of functions satisfying zero-order and one-order compatibility conditions for problem (2.1a)-(2.2). Then for any initial condition  $\xi^0$  in  $\mathcal{M}$ , problem (2.1a)-(2.2) admits a unique classical solution in  $C^1([0, +\infty) \times [0, L]; \mathbb{R}^n)$ . Moreover, there exists  $\delta > 0$ , such that for all  $\xi^0$  in  $\mathcal{M}$ ,  $\|\xi(t, \cdot)\|_1 \leq \delta$ , for all  $t$  in  $[0, +\infty)$ .*

Along the chapter, in order to refer to solutions satisfying Assumption 2.1, we use the definition of the following set

$$\mathcal{B}_{\delta} := \{\xi \in C^1([0, L]; \mathbb{R}^n) : \|\xi\|_1 \leq \delta\},$$

in which they belong.

Define now a  $C^1$  vector-valued function  $\mathbb{R}^n \ni \zeta \mapsto s_{\delta}(\zeta) = (s_{\delta}^1(\zeta_1), \dots, s_{\delta}^n(\zeta_n))$ , parametrized by  $\delta$ , and satisfying the following properties.

For every  $\delta > 0$  and  $v, w$  in  $\mathbb{R}^n$ , such that  $|w| \leq \delta$ , there exists  $\omega_{\delta} > 0$ , such that the following inequality is satisfied

$$|s_{\delta}(v) - w| \leq \omega_{\delta}|v - w|. \tag{2.3a}$$

Moreover, there exists  $m_\delta > 0$ , such that for every  $v$  in  $\mathbb{R}^n$ ,

$$|s_\delta(v)|, |D_v s_\delta(v)| \leq m_\delta. \quad (2.3b)$$

Note that function

$$s_\delta^i(\zeta_i) = \begin{cases} \zeta_i, & |\zeta_i| \leq \delta \\ \text{sgn}(\zeta_i) ((|\zeta_i| - \delta) e^{-|\zeta_i| + \delta} + \delta), & |\zeta_i| > \delta \end{cases} \quad (2.4)$$

satisfies (2.3) with  $\omega_\delta = \sqrt{n}$ ,  $m_\delta = \sqrt{n} (e^{-1} + \delta)$ .

We are now in a position to introduce our candidate observer dynamics and its boundary conditions for system (2.1a), (2.1b), (2.2). Define, first, a diagonal matrix  $\Theta$  by

$$\Theta := \text{diag}(\theta, \theta^2, \dots, \theta^n), \quad (2.5)$$

where  $\theta > 1$  is the candidate high-gain constant of the observer, which will be selected precisely later. Let also  $K$  in  $\mathbb{R}^n$ , chosen in a way that  $A + KC$  is Hurwitz. Note that for such a  $K$ , one can find a symmetric and positive definite  $n \times n$  matrix  $P$  satisfying a quadratic Lyapunov equation of the following form

$$2\text{Sym}(P(A + KC)) = -I_n. \quad (2.6)$$

The previous equation is always feasible, due to the observability of the pair  $(A, C)$ . Let us remark that  $P$  satisfying (2.6) cannot be diagonal, since matrix  $A$  fails by its definition to be a diagonally stabilizable matrix.

### High-Gain Observer

With the previous assumptions, the observer that will solve the H-GODP can be given by the following equations on  $\Pi$

$$\hat{\xi}_t(t, x) + \lambda(y(t, x)) \hat{\xi}_x(t, x) = A\hat{\xi}(t, x) - \Theta K \left( y(t, x) - C\hat{\xi}(t, x) \right) + f \left( x, \mathcal{F} \left[ s_\delta \left( \hat{\xi}(t) \right) \right] (x) \right), \quad (2.7a)$$

$$\hat{\xi}(t, 0) = \mathcal{H} \left( s_\delta \left( \hat{\xi}(t, L) \right) \right). \quad (2.7b)$$

We can easily deduce some difference inequalities for the dynamics, to be invoked later, as a direct consequence of the regularity assumptions and properties (2.3) of  $s_\delta(\cdot)$ . Precisely, for any  $\delta > 0$ , there exist constants  $L_{1,\delta}, L_{2,\delta}, L_{3,\delta}, L_{4,\delta}, L'_{1,\delta}, L'_{2,\delta}, L'_{3,\delta}, L'_{4,\delta} > 0$  depending on constants  $\omega_\delta$  and  $m_\delta$ , such that for every  $\xi, \hat{\xi}$  in  $C^0([0, L]; \mathbb{R}^n)$ , with  $\|\xi\|_\infty \leq \delta$ , the following

inequalities are satisfied for all  $x$  in  $[0, L]$

$$|\Delta_{s_\delta(\hat{\xi})} [f(\cdot, \mathcal{F})](\xi)(x)| \leq L_{1,\delta} |\hat{\xi}(x) - \xi(x)| + L_{2,\delta} |\hat{\xi}(L) - \xi(L)| + L_{3,\delta} \int_0^L |\hat{\xi}(s) - \xi(s)| ds, \quad (2.8a)$$

$$|\Delta_{s_\delta(\hat{\xi})} [\mathcal{H}](\xi)(L)| \leq L_{4,\delta} |\hat{\xi}(L) - \xi(L)|, \quad (2.8b)$$

$$|\Delta_{s_\delta(\hat{\xi})} [D_\xi f(\cdot, \mathcal{F})](\xi)(x)| \leq L'_{1,\delta} |\hat{\xi}(x) - \xi(x)| + L'_{2,\delta} |\hat{\xi}(L) - \xi(L)| + L'_{3,\delta} \int_0^L |\hat{\xi}(s) - \xi(s)| ds, \quad (2.8c)$$

$$|\Delta_{s_\delta(\hat{\xi})} [D_\xi \mathcal{H}](\xi)(L)| \leq L'_{4,\delta} |\hat{\xi}(L) - \xi(L)|. \quad (2.8d)$$

**Remark 2.2.** *Assumption 2.1 guarantees the existence of a preassigned ball in  $C^1$  in which system's solutions take values. This, in conjunction with the regularity assumptions, would allow us to avoid assuming that the nonlinear dynamics of the system are globally Lipschitz, which in general constitutes a natural assumption in classical high-gain observer designs for finite dimensions. To avoid such a restricting assumption, we plug function  $s_\delta$  in observer dynamics (2.7). Exploiting this function guarantees that observer system's dynamics are globally Lipschitz. We note also that, although it is not described explicitly by the observer's equations, we avoid injecting  $s_\delta$  in potential linear terms included in  $f$  and  $\mathcal{H}$ . In that case, we set  $s_\delta(\hat{\xi}) = \hat{\xi}$ , since linear terms are globally Lipschitz.*

The following lemma guarantees the existence of unique global classical solutions for the candidate observer. The proof of the lemma follows from classical arguments and the fact that observer's nonlinearities are globally Lipschitz.

**Lemma 2.1.** *(Existence/uniqueness of global classical solutions to the observer system) Under the regularity assumptions for the dynamics and for any  $y$  in  $C^1([0, +\infty) \times [0, L]; \mathbb{R})$ , the problem described by (2.7) on domain  $\Pi$  with initial condition  $\hat{\xi}^0(x) := \hat{\xi}(0, x)$ , for all  $x$  in  $[0, L]$ , satisfying zero-order and one-order compatibility conditions, admits a unique classical solution in  $\Pi$ , i.e., there exists a unique solution  $\hat{\xi}$  to (2.7) in  $C^1([0, +\infty) \times [0, L]; \mathbb{R}^n)$ .*

*Proof.* See Appendix 2.4 at the end of the chapter for a detailed proof. □

We are now in a position to present our main result on the solvability of the H-GODP.

**Theorem 2.1.** *(Observer convergence) Consider system (2.1a), (2.2), defined on  $\Pi$  with output (2.1b) and suppose that Assumption 2.1 holds for initial condition  $\xi^0 \in \mathcal{M}$ . Let also  $K$  in  $\mathbb{R}^n$ , chosen in such a way that  $A + KC$  is Hurwitz. Then, the H-GODP for system (2.1a) - (2.2) is solvable by system (2.7) for  $\theta > 1$  as a high gain and initial condition  $\hat{\xi}^0$  in  $C^1([0, L]; \mathbb{R}^n)$ , with  $\hat{\xi}^0(x) = \hat{\xi}(0, x)$ , satisfying zero-order and one-order compatibility conditions. This is a high-gain observer for  $\xi$ , in the sense that for  $\theta$  large enough it admits a unique classical solution in  $\Pi$  on the one hand, providing an estimate for the state of system (2.1a) - (2.2) on the other hand. More precisely, for every  $\kappa > 0$ , there exists  $\theta_0 \geq 1$ , such*

that for every  $\theta > \theta_0$ , the following inequality holds

$$\|\hat{\xi}(t, \cdot) - \xi(t, \cdot)\|_1 \leq \ell e^{-\kappa t} \|\hat{\xi}^0(\cdot) - \xi^0(\cdot)\|_1, \forall t \geq 0, \quad (2.9)$$

for some  $\ell > 0$ , polynomial in  $\theta$ .

This theorem states that for system (2.1a) - (2.2) with output (2.1b) we have a high-gain observer design providing an estimate of its full state, with a convergence rate adjustable via  $\theta$ . The well-posedness results from Lemma 2.1. The convergence result is established in next subsection.

### 2.1.2 Observer convergence proof

This subsection is dedicated to the proof of Theorem 2.1.

Consider the observer error  $e := \hat{\xi} - \xi$  which satisfies

$$e_t(t, x) + \lambda(y(t, x))e_x(t, x) = (A + \Theta KC) e(t, x) + \Delta_{s_\delta(\hat{\xi}(t))} [f(\cdot, \mathcal{F})](\xi(t))(x), \quad (2.10)$$

$$e(t, 0) = \Delta_{s_\delta(\hat{\xi}(t))} [\mathcal{H}](\xi(t))(L). \quad (2.11)$$

We now define the linearly transformed error by  $\varepsilon := \Theta^{-1}e$  and we derive the following hyperbolic equations for  $\varepsilon$  on  $\Pi$

$$\varepsilon_t(t, x) + \lambda(y(t, x))\varepsilon_x(t, x) = \theta(A + KC)\varepsilon(t, x) + \Theta^{-1}\Delta_{s_\delta(\hat{\xi}(t))} [f(\cdot, \mathcal{F})](\xi(t))(x), \quad (2.12)$$

$$\varepsilon(t, 0) = \Theta^{-1}\Delta_{s_\delta(\hat{\xi}(t))} [\mathcal{H}](\xi(t))(L). \quad (2.13)$$

At this point, let us introduce an operator  $\mathcal{K} : C^1([0, L]; \mathbb{R}^n) \rightarrow C^0([0, L]; \mathbb{R}^{n \times n})$  defined by

$$\mathcal{K}[\xi] := \lambda'(C\xi) (\lambda(C\xi))^{-1} I_n C \left( -\lambda(C\xi) \frac{\partial}{\partial x} \xi + A\xi + f(\cdot, \mathcal{F}[\xi]) \right). \quad (2.14)$$

Define also  $\mathcal{K}_1^{\hat{\xi}} : C^1([0, L]; \mathbb{R}^n) \rightarrow C^0([0, L]; \mathbb{R}^n)$ , parametrized by  $\hat{\xi} \in C^0([0, L]; \mathbb{R}^n)$ , and given by

$$\mathcal{K}_1^{\hat{\xi}}[\xi] := -\mathcal{K}[\xi]\Theta^{-1}\Delta_{s_\delta(\hat{\xi})} [f(\cdot, \mathcal{F})](\xi) + \Theta^{-1} \left\langle \Delta_{s_\delta(\hat{\xi})} [D_\xi f(\cdot, \mathcal{F})](\xi), \xi_t \right\rangle. \quad (2.15)$$

Next, by temporarily assuming that  $\varepsilon$  is of class  $C^2$ , we derive the following hyperbolic equations for  $\varepsilon_t$

$$\begin{aligned} & \varepsilon_{tt}(t, x) + \lambda(y(t, x))\varepsilon_{tx}(t, x) \\ &= \mathcal{K}[\xi(t)](x)\varepsilon_t(t, x)\Theta^{-1} \left\langle D_u f(\cdot, \mathcal{F}[u])|_{u=s_\delta(\hat{\xi}(t))}, D_{\hat{\xi}} s_\delta(\hat{\xi}(t)) \Theta \varepsilon_t(t) \right\rangle (x) \\ &+ \theta(A + KC)\varepsilon_t(t, x) - \theta \mathcal{K}[\xi(t)](x)(A + KC)\varepsilon(t, x) + \mathcal{K}_1^{\hat{\xi}(t)}[\xi(t)](x), \end{aligned} \quad (2.16)$$

$$\begin{aligned} \varepsilon_t(t, 0) &= \Theta^{-1} \left( \Delta_{s_\delta(\hat{\xi}(t))} [D_\xi \mathcal{H}](\xi(t))(L)\xi_t(t, L) \right. \\ &\left. + D_u \mathcal{H}(u)|_{u=s_\delta(\hat{\xi}(t))}(L) D_{\hat{\xi}} s_\delta(\hat{\xi}(t, L)) \Theta \varepsilon_t(t, L) \right). \end{aligned} \quad (2.17)$$



**Remark 2.3.** *Due to cumbersome notation in the previous relations, we provide an example to make the calculations of the Fréchet derivative more clear. Let, for instance,*

$$f(x, \mathcal{F}[u](x)) = g(x)g_1(u(x)) + g_2(u(L)) + \int_0^x g_3(u(s)) ds.$$

Then,

$$\begin{aligned} \langle D_u f(\cdot, \mathcal{F}[u]), u_t \rangle(x) &= g(x)D_u g_1(u)(x)u_t(x) \\ &+ D_u g_2(u)(L)u_t(L) + \int_0^x D_u g_3(u)(s)u_t(s)ds. \end{aligned}$$

Now, notice that whenever  $\xi$  in  $\mathcal{B}_\delta$ , due to continuity of  $\lambda(\cdot)$ , the fact that  $\lambda(\xi_1) > 0$ , for all  $\xi_1$  in  $\mathbb{R}$  and the fact that  $\|\xi\|_\infty \leq \delta$ , the quantities  $\sup_{\xi \in \mathcal{B}_\delta}(\lambda(C\xi))$ ,  $\inf_{\xi \in \mathcal{B}_\delta}(\lambda(C\xi))$  are positive and finite. In addition, whenever  $\xi$  in  $\mathcal{B}_\delta$  and invoking the hyperbolic dynamics (2.1a), we can easily calculate constant  $\delta_1 > 0$ , dependent on  $\delta$ , such that

$$\|\xi_t\|_\infty = \| -\lambda(\xi_1)\xi_x + A\xi + f(\cdot, \mathcal{F}[\xi]) \|_\infty \leq \delta_1. \quad (2.18)$$

By combining the above arguments, the fact that  $\theta > 1$ , (2.18), continuity of the involved mappings and inequalities (2.8), we can easily calculate positive constants  $\gamma_{i,\delta}$ ,  $i = 1, \dots, 6$ , such that whenever  $\xi$  in  $\mathcal{B}_\delta$ ,  $\hat{\xi}$  in  $C^0([0, L]; \mathbb{R}^n)$ , the following inequalities are satisfied for all  $x$  in  $[0, L]$ :

$$\begin{aligned} |\Theta^{-1} \langle D_u f(\cdot, \mathcal{F}[u])|_{u=s_\delta(\hat{\xi})}, D_{\hat{\xi}} s_\delta(\hat{\xi}) \Theta \varepsilon_t \rangle(x)| \\ \leq \gamma_{1,\delta} \left( |\varepsilon_t(x)| + |\varepsilon_t(L)| + \int_0^L |\varepsilon_t(s)| ds \right), \end{aligned} \quad (2.19a)$$

$$|\Theta^{-1} \Delta_{s_\delta(\hat{\xi})} [f(\cdot, \mathcal{F})](\xi)(x)| \leq \gamma_{2,\delta} \left( |\varepsilon(x)| + |\varepsilon(L)| + \int_0^L |\varepsilon(s)| ds \right), \quad (2.19b)$$

$$|\Theta^{-1} \Delta_{s_\delta(\hat{\xi})} [\mathcal{H}](\xi)(L)| \leq \gamma_{3,\delta} \theta^{n-1} |\varepsilon(L)|, \quad (2.19c)$$

$$|\Theta^{-1} \Delta_{s_\delta(\hat{\xi})} [D_\xi f(\cdot, \mathcal{F})](\xi)(x)| \leq \gamma_{4,\delta} \left( |\varepsilon(x)| + |\varepsilon(L)| + \int_0^L |\varepsilon(s)| ds \right), \quad (2.19d)$$

$$|\Theta^{-1} D_u \mathcal{H}[u]|_{u=s_\delta(\hat{\xi})}(L) D_{\hat{\xi}} s_\delta(\hat{\xi})(L) \Theta| \leq \gamma_{5,\delta} \theta^{n-1}, \quad (2.19e)$$

$$|\Theta^{-1} \Delta_{s_\delta(\hat{\xi})} [D_\xi \mathcal{H}](\xi)(L)| \leq \gamma_{6,\delta} \theta^{n-1} |\varepsilon(L)|. \quad (2.19f)$$

Define also constants

$$\gamma_{7,\delta} := \sup_{\xi \in \mathcal{B}_\delta} |\lambda'(C\xi)|, \gamma_{8,\delta} := \inf_{\xi \in \mathcal{B}_\delta} \lambda(C\xi), \gamma_{9,\delta} := \sup_{\xi \in \mathcal{B}_\delta} \lambda(C\xi), \quad (2.20a)$$

and derive the following bounds for all  $x$  in  $[0, L]$ :

$$|\mathcal{K}[\xi](x)| \leq \gamma_{10,\delta} := \frac{\delta_1 \gamma_{7,\delta}}{\gamma_{8,\delta}}, \quad (2.20b)$$

$$|\mathcal{K}_1^{\hat{\xi}}[\xi](x)| \leq \gamma_{11,\delta} \left( |\varepsilon(x)| + |\varepsilon(L)| + \int_0^L |\varepsilon(s)| ds \right); \quad (2.20c)$$

$$\gamma_{11,\delta} := \gamma_{10,\delta} \gamma_{2,\delta} + \delta_1 \gamma_{4,\delta}.$$

**Remark 2.4.** Note here that all the above constants  $\gamma_{1,\delta}, \dots, \gamma_{11,\delta}$  are calculated by taking bounds of the involved mappings on  $\mathcal{B}_\delta$  and depend on the global bound  $\delta$  of the system's trajectories (and subsequently on  $\delta_1$  in (2.18)), constants in (2.8), the order of the system  $n$ , and are independent of the observer gain  $\theta$ . More explicitly,  $\gamma_{1,\delta}, \gamma_{2,\delta}, \gamma_{4,\delta}$ , and  $\gamma_{11,\delta}$  being independent of  $\theta$  is a direct consequence of the assumed triangularity of the involved mappings, similarly as in the classical high-gain observer designs [Gauthier et al (1992)]. This property turns out to be sufficient for the solvability of the H-GODP. More precisely, in the sequel, while bounding the Lyapunov derivative from above, the independence of these parameters on  $\theta$  shall not add positive terms with linear (or higher-order) dependency on  $\theta$ . On the other hand, negative terms will appear depending linearly on  $\theta$  as a direct consequence of the assumed observability of the pair  $(A, C)$ . This will render the negativity of the Lyapunov derivative feasible, as this will be more clear in the forthcoming Lyapunov analysis.

To prove exponential stability of the error system at the origin, we first need to define a Lyapunov functional  $\mathcal{W}_p : C^1([0, L]; \mathbb{R}^n) \rightarrow \mathbb{R}$  by

$$\mathcal{W}_p[\varepsilon] := \left( \int_0^L \pi(x) \exp(p\mu_{\theta,\delta}x) \mathcal{G}_p[\varepsilon](x) dx \right)^{1/p}, \quad (2.21a)$$

where  $\mathcal{G}_p : C^1([0, L]; \mathbb{R}^n) \rightarrow \mathbb{R}$  is given by

$$\mathcal{G}_p[\varepsilon] := \left( \varepsilon^\top P \varepsilon + \rho_0 \varepsilon_t^\top P \varepsilon_t \right)^p \quad (2.21b)$$

and  $\rho_0$  in  $(0, 1)$  is a constant (to be chosen appropriately),  $p$  in  $\mathbb{N}$ ,  $P$  in  $\mathbb{R}^{n \times n}$  is a symmetric positive definite matrix satisfying (2.6),  $\pi : [0, L] \rightarrow \mathbb{R}$  is given by

$$\pi(x) := (\bar{\pi} - 1) \frac{x}{L} + 1; \quad \bar{\pi} := \frac{\sup_{\xi \in \mathcal{B}_\delta} (\lambda(C\xi))}{\inf_{\xi \in \mathcal{B}_\delta} (\lambda(C\xi))} \quad (2.22)$$

and  $\mu_{\theta,\delta}$  is given by

$$\mu_{\theta,\delta} := \frac{1}{L} \ln(\mu_\delta \theta^{2n-2}), \quad (2.23a)$$

where

$$\mu_\delta := \frac{|P|}{\underline{\text{eig}}(P)} \max(\gamma_{3,\delta}^2, \gamma_{5,\delta}^2, \gamma_{6,\delta}^2 \delta_1^2, \gamma_{5,\delta} \gamma_{3,\delta} \delta_1). \quad (2.23b)$$

Note here that, by its definition,  $\pi$  is bounded as follows

$$1 \leq \pi(x) \leq \bar{\pi}, \forall x \in [0, L]. \quad (2.24)$$

Let us also define functional  $\mathcal{V} : C^1([0, L]; \mathbb{R}^n) \rightarrow \mathbb{R}$  by

$$\mathcal{V}[\varepsilon] := \|\exp(\mu_{\theta,\delta} \cdot) \varepsilon^\top P \varepsilon\|_\infty + \|\rho_0 \exp(\mu_{\theta,\delta} \cdot) \varepsilon_t^\top P \varepsilon_t\|_\infty. \quad (2.25)$$

The idea is to let  $p \rightarrow \infty$  in the end, to approximate  $V$  by  $W_p$ . This convergence property is presented in details in Appendix 2.4.2. By invoking Lemma 2.1 and Assumption 2.1, which

establish global unique classical solutions for both observer system (2.7) and system (2.1a)-(2.2) respectively, we are now in a position to define  $G_p, W_p, V : [0, +\infty) \rightarrow \mathbb{R}$  by

$$G_p(t) := \mathcal{G}_p[\varepsilon](t), W_p(t) := \mathcal{W}_p[\varepsilon](t), V(t) := \mathcal{V}[\varepsilon](t), t \geq 0, \forall t \geq 0. \quad (2.26)$$

Calculating the time-derivative  $\dot{W}_p$  along the classical solutions to (2.12) - (2.13), (2.16) - (2.17), we get

$$\begin{aligned} \dot{W}_p &= \frac{1}{p} W_p^{1-p} \int_0^L p \pi(x) \exp(p\mu_{\theta, \delta} x) G_{p-1}(x) \\ &\times \left( \varepsilon_t(x)^\top P \varepsilon(x) + \varepsilon^\top(x) P \varepsilon_t(x) + \rho_0 \varepsilon_{tt}^\top(x) P \varepsilon_t(x) + \rho_0 \varepsilon_t(x)^\top P \varepsilon_{tt}(x) \right) dx. \end{aligned} \quad (2.27)$$

After substituting the dynamical equations (2.12) and (2.16) into the above equation and performing integration by parts (which is possible, since error system satisfies Property (S) for its hyperbolic operator, see Definition 1.2 of Chapter 1),  $\dot{W}_p$  can be written in the following form

$$\dot{W}_p = W_p^{1-p} \left( \frac{1}{p} T_{1,p} + \frac{1}{p} T_{2,p} + T_{3,p} + T_{4,p} + T_{5,p} \right), \quad (2.28)$$

where

$$T_{1,p} := -\pi(L) \lambda(y(L)) \exp(p\mu_{\theta, \delta} L) G_p(L) + \pi(0) \lambda(y(0)) G_p(0), \quad (2.29a)$$

$$T_{2,p} := \int_0^L d_x [\pi(x) \exp(p\mu_{\theta, \delta} x) \lambda(y(x))] G_p(x) dx, \quad (2.29b)$$

$$\begin{aligned} T_{3,p} := & 2 \int_0^L \pi(x) \exp(p\mu_{\theta, \delta} x) G_{p-1}(x) \left[ \varepsilon^\top(x) P \Theta^{-1} \Delta_{s_\delta(\hat{\xi})} [f(\cdot, \mathcal{F})](\hat{\xi})(x) + \rho_0 \varepsilon_t^\top(x) P \mathcal{K}_1^{\hat{\xi}}[\xi](x) \right. \\ & \left. + \rho_0 \varepsilon_t^\top \text{Sym}(P \mathcal{K}[\xi]) \varepsilon_t \right] dx, \end{aligned} \quad (2.29c)$$

$$\begin{aligned} T_{4,p} := & 2 \int_0^L \pi(x) \exp(p\mu_{\theta, \delta} x) G_{p-1}(x) \rho_0 \varepsilon_t(x)^\top P \Theta^{-1} \\ & \times \left\langle D_u f(\cdot, \mathcal{F}[u])|_{u=s_\delta(\hat{\xi})}, D_{\hat{\xi}} s_\delta(\hat{\xi}) \left( \hat{\xi} \right) \Theta \varepsilon_t \right\rangle (x) dx, \end{aligned} \quad (2.29d)$$

$$\begin{aligned} T_{5,p} := & \theta \int_0^L \pi(x) \exp(p\mu_{\theta, \delta} x) G_{p-1}(x) \left[ 2 \varepsilon^\top \text{Sym}(P(A + KC)) \varepsilon + 2 \rho_0 \varepsilon_t^\top \text{Sym}(P(A + KC)) \varepsilon_t \right. \\ & \left. - \rho_0 \varepsilon_t^\top P \mathcal{K}[\xi](A + KC) \varepsilon - \rho_0 \varepsilon^\top (A + KC)^\top \mathcal{K}^\top[\xi] P \varepsilon_t \right] dx. \end{aligned} \quad (2.29e)$$

After substituting boundary equations (2.13) and (2.17) in  $T_{1,p}$  and by virtue of (2.8) and (2.22), using the previously calculated bounds and trivial inequalities, we can easily obtain the following inequality

$$T_{1,p} \leq \sup_{\xi \in \mathcal{B}_\delta} (\lambda(C\xi)) G_p(L) \left( -\exp(p\mu_{\theta, \delta} L) + (\theta^{2n-2} \mu_\delta)^p \right) \quad (2.30)$$

and, subsequently, by (2.23a)

$$T_{1,p} \leq 0. \quad (2.31)$$

By the fact that  $\pi(x) \geq 1$ , for all  $x \in [0, L]$ , we can derive the following bound for  $T_{2,p}$

$$T_{2,p} \leq (\omega_{1,\delta} + p|\mu_{\theta, \delta}| \omega_{2,\delta}) W_p^p, \quad (2.32)$$

where

$$\omega_{1,\delta} := \frac{|P|\delta\gamma_{7,\delta}}{\underline{\text{eig}}(P)}, \omega_{2,\delta} := \frac{|P|\gamma_{9,\delta}}{\underline{\text{eig}}(P)}.$$

By exploiting (2.19), (2.20),  $T_{3,p}$  can be bounded as follows

$$\begin{aligned} T_{3,p} &\leq 2 \int_0^L \pi(x) \exp(p\mu_{\theta,\delta}x) G_{p-1}(x) |P| \left[ \gamma_{2,\delta} |\varepsilon(x)| \left( |\varepsilon(x)| + |\varepsilon(L)| + \int_0^x |\varepsilon(s)| ds \right) \right. \\ &\quad \left. + \rho_0 \gamma_{11,\delta} |\varepsilon_t(x)| \left( |\varepsilon(x)| + |\varepsilon(L)| + \int_0^x |\varepsilon(s)| ds \right) + \rho_0 \gamma_{10,\delta} |\varepsilon_t(x)|^2 \right] dx \\ &\leq \int_0^L \pi(x) \exp(p\mu_{\theta,\delta}x) G_{p-1}(x) \frac{|P|}{\underline{\text{eig}}(P)} \max \{4\gamma_{2,\delta} + \rho_0 \gamma_{11,\delta}, 3\gamma_{11,\delta} + \gamma_{10,\delta}\} G_1(x) dx \\ &\quad + \int_0^L \pi(x) \exp(p\mu_{\theta,\delta}x) G_{p-1}(x) \frac{|P|}{\underline{\text{eig}}(P)} (\gamma_{2,\delta} + \rho_0 \gamma_{11,\delta}) \left( G_1(L) + \int_0^x G_1(s) ds \right) dx \end{aligned} \quad (2.33)$$

$$\begin{aligned} &\leq \omega_{3,\delta} W_p^p + \omega_{4,\delta} \|\exp(p\mu_{\theta,\delta}\cdot) G_1(\cdot)\|_\infty W_{p-1}^{p-1} \\ &\leq \omega_{3,\delta} W_p^p + \omega_{4,\delta} V W_{p-1}^{p-1}, \end{aligned} \quad (2.34)$$

where

$$\omega_{3,\delta} := \frac{|P|}{\underline{\text{eig}}(P)} \max \{4\gamma_{2,\delta} + \rho_0 \gamma_{11,\delta}, 3\gamma_{11,\delta} + \gamma_{10,\delta}\}, \omega_{4,\delta} := \frac{|P|}{\underline{\text{eig}}(P)} (\gamma_{2,\delta} + \rho_0 \gamma_{11,\delta})(1+L).$$

Similarly, we can bound  $T_{4,p}$  as follows

$$T_{4,p} \leq \omega_{5,\delta} W_p^p + \omega_{6,\delta} V W_{p-1}^{p-1}, \quad (2.35)$$

where

$$\omega_{5,\delta} := 2 \frac{|P|}{\underline{\text{eig}}(P)} \gamma_{1,\delta}, \omega_{6,\delta} := 2 \frac{|P|}{\underline{\text{eig}}(P)} \gamma_{1,\delta} (1+L).$$

The term  $T_{5,p}$  of  $\dot{W}_p$  can be rewritten in the following way

$$T_{5,p} := -\theta \int_0^L \pi(x) \exp(p\mu_{\theta,\delta}x) G_{p-1}(x) \begin{pmatrix} \varepsilon^\top(x) & \varepsilon_t^\top(x) \end{pmatrix} \mathfrak{S}[\xi](x) \begin{pmatrix} \varepsilon(x) \\ \varepsilon_t(x) \end{pmatrix} dx, \quad (2.36)$$

where, after utilizing (2.6),  $\mathfrak{S} : \mathcal{B}_\delta \rightarrow C^0([0, L]; \mathbb{R}^{2n \times 2n})$  is given by

$$\mathfrak{S}[\xi] := \begin{pmatrix} I_{n \times n} & -\rho_0(A + KC)^\top \mathcal{K}^\top[\xi] P \\ -\rho_0 P \mathcal{K}[\xi](A + KC) & \rho_0 I_{n \times n} \end{pmatrix}. \quad (2.37)$$

Now, we can easily verify (Schur complement) that for all  $w \in \mathbb{R}^{2n} \setminus \{0\}$ , we have  $\inf_{\xi \in \mathcal{B}_\delta} \frac{w^\top \mathfrak{S}[\xi] w}{|w|^2} \in (0, +\infty)$ , if

$$0 < \rho_0 < \min \left( \frac{1}{|P|^2 |A + KC|^2 \gamma_{10,\delta}^2}, 1 \right). \quad (2.38)$$

It turns out that for every choice of matrices  $P$  and  $K$  satisfying (2.6), there always exists a  $\rho_0$ , such that (2.38) is satisfied and this fact renders  $\mathfrak{S}$  positive. Consequently, for appropriate choice of  $\rho_0$ , there exists  $\sigma_\delta > 0$ , such that

$$T_{5,p} \leq -\theta \frac{\sigma_\delta}{|P|} W_p^p. \quad (2.39)$$

Combining (2.31), (2.32), (2.34), (2.35) and (2.39) with (2.28), we obtain

$$\dot{W}_p \leq (-\theta\omega_{7,\delta} + \omega_{8,\delta} \ln(\theta) + \omega_{9,\delta})W_p + (\omega_{4,\delta} + \omega_{6,\delta}) W_p^{1-p} W_{p-1}^{p-1} V, \quad (2.40)$$

where

$$\omega_{7,\delta} := \frac{\sigma_\delta}{|P|}, \omega_{8,\delta} := \frac{\omega_{2,\delta}(2n-2)}{L}, \omega_{9,\delta} := \omega_{1,\delta} + \omega_{3,\delta} + \frac{\omega_{2,\delta}}{L} |\ln \mu_\delta|.$$

Now, using Hölder's inequality, one can obtain

$$W_{p-1}^{p-1} \leq W_p^{p-1} \|\pi(\cdot)\|_\infty^{1/p}.$$

Utilizing the above inequality, (2.40) gives

$$\dot{W}_p \leq (-\theta\omega_{7,\delta} + \omega_{8,\delta} \ln(\theta) + \omega_{9,\delta})W_p + (\omega_{4,\delta} + \omega_{6,\delta}) \bar{\pi}^{1/p} V. \quad (2.41)$$

We obtained the estimate (2.41) of  $\dot{W}_p$  for  $\varepsilon$  of class  $C^2$ , but, by invoking density arguments, the results remain valid with  $\varepsilon$  only of class  $C^1$  (see [Coron and Bastin (2015)] for further details).

From (2.41), taking the limit as  $p \rightarrow +\infty$  and using the fact that  $\lim_{p \rightarrow +\infty} \bar{\pi}^{1/p} = 1$ , we get in the distribution sense in  $(0, +\infty)$ ,

$$\dot{V}(t) \leq (-\theta\omega_{7,\delta} + \omega_{8,\delta} \ln(\theta) + \omega_{10,\delta}) V(t) \quad (2.42)$$

where

$$\omega_{10,\delta} := \omega_{4,\delta} + \omega_{6,\delta} + \omega_{9,\delta}.$$

For more details on how to obtain the above estimate, see Properties 1 and 2 in Appendix 2.4.2.

Now, one can select the high gain  $\theta$ , such that

$$\theta > \theta_0, \quad (2.43)$$

where  $\theta_0 > 1$  is such that

$$-\theta\omega_{7,\delta} + \omega_{8,\delta} \ln(\theta) + \omega_{10,\delta} \leq -2\kappa_\delta, \forall \theta > \theta_0 \quad (2.44)$$

for some  $\kappa_\delta > 0$ . One can easily check that for any  $\kappa_\delta > 0$ , there always exists a  $\theta_0 > 1$ , dependent on the involved constants, such that the previous inequality is satisfied.

Subsequently, (2.42) yields to the following differential inequality in the distribution sense in  $(0, +\infty)$

$$\dot{V}(t) \leq -2\kappa_\delta V(t) \quad (2.45)$$

and by the comparison lemma, we get

$$V(t) \leq e^{-2\kappa_\delta t} V(0), \forall t \geq 0. \quad (2.46)$$

Now, by the dynamics (2.12), in conjunction with (2.19), (2.20), we can obtain the following inequalities

$$\begin{aligned} \|\varepsilon_t\|_\infty &\geq \gamma_{8,\delta} \|\varepsilon_x\|_\infty - s_{\delta,\theta} \|\varepsilon\|_\infty, \\ \|\varepsilon_t\|_\infty &\leq \gamma_{9,\delta} \|\varepsilon_x\|_\infty + s_{\delta,\theta} \|\varepsilon\|_\infty, \end{aligned}$$

where  $s_{\delta,\theta} := \theta|A + KC| + \gamma_{2,\delta}(2 + L)$ . Invoking these inequalities, (2.23a), estimate (2.46) and the following inequality,

$$\frac{\rho_0}{2} e^{\frac{\mu_{\theta,\delta} - |\mu_{\theta,\delta}|}{2} L} \underline{\text{eig}}(P) (\|\varepsilon\|_\infty + \|\varepsilon_t\|_\infty)^2 \leq V \leq e^{\frac{\mu_{\theta,\delta} + |\mu_{\theta,\delta}|}{2} L} |P| (\|\varepsilon\|_\infty + \|\varepsilon_t\|_\infty)^2$$

we obtain

$$\|\varepsilon\|_1 \leq l_{\delta,\theta} e^{-\kappa_\delta t} \|\varepsilon^0\|_1, \forall t \geq 0, \quad (2.47)$$

where  $\varepsilon^0(x) := \varepsilon(0, x)$  and

$$l_{\delta,\theta} := \sqrt{\frac{|P|}{\rho_0 \underline{\text{eig}}(P)}} (\mu_\delta)^{\frac{1}{2L}} \theta^{\frac{n-1}{L}} \max\left(s_{\delta,\theta} + 1, \frac{1}{\gamma_{8,\delta}}\right) \max(1 + 2s_{\delta,\theta}, 2\gamma_{9,\delta}).$$

By (2.47), we derive the following estimate, which holds for every  $t \geq 0$

$$\|\hat{\xi}(t, \cdot) - \xi(t, \cdot)\|_1 \leq \bar{l}_{\delta,\theta} e^{-\kappa_\delta t} \|\hat{\xi}^0 - \xi^0\|_1, \quad (2.48)$$

where  $\bar{l}_{\delta,\theta} := \theta^{n-1} l_{\delta,\theta}$ .

Concluding, we solved the H-GODP for (2.1a), (2.1b), (2.2) by designing an exponential in the  $C^1$ -norm high-gain observer of adjustable convergence rate  $\kappa_\delta$ , dependent on the selection of  $\theta$ . The proof of Theorem 2.1 is complete. ■

**Remark 2.5.** *It is worthwhile to remark here that  $l_{\delta,\theta}$  depends polynomially (and not exponentially) on  $\theta$ , which is also a known property for high-gain observer designs in finite dimensions. The particular type of asymptotic convergence (2.48) of the error equation indicates that the convergence of the observer can be arbitrarily fast.*

## 2.2 Application to SIR epidemic models

As an illustrative example, let us consider the case of an epidemic model as in (1.2) of Chapter 1. Let us recall its form here (see [Bastin and Coron (2016)], Chapter 1 and also [Ianneli (1995)], Chapter III for detailed presentation of such systems)

$$\begin{aligned} \partial_t S(t, x) + \partial_x S(t, x) + \mu(x)S(t, x) + \mathcal{G}[S(t), I(t)](x) &= 0, \\ \partial_t I(t, x) + \partial_x I(t, x) + (\gamma(x) + \mu(x))I(t, x) - \mathcal{G}[S(t), I(t)](x) &= 0, \\ \partial_t R(t, x) + \partial_x R(t, x) + \mu(x)R(t, x) - \gamma(x)I(t, x) &= 0, \end{aligned} \quad (2.49)$$

where  $S(t, x), I(t, x), R(t, x)$  represent the age distribution of Susceptible, Infected and Recovered populations at time  $t$  and with age  $x$ ,  $\mathcal{G}[S(t), I(t)](x) := \beta(x)S(t, x) \int_0^L I(t, s)ds$  represents the disease transmission rate by contact between susceptible and infected individuals and  $\beta(x), \gamma(x)$ , and  $\mu(x)$  are positive coefficients of class  $C^2$ . In addition, if  $L$  is the maximal life duration in the considered population, we have  $S(t, L) = I(t, L) = R(t, L) = 0$ , with the following boundary conditions

$$S(t, 0) = B(t), \quad I(t, 0) = 0, \quad R(t, 0) = 0. \quad (2.50)$$

Here,  $B(t)$  stands for the inflow of newborn individuals in the susceptible part of the population at time  $t$ . Let us assume that the number of people in the group  $R$  of recovered patients between ages 0 and  $x$ , for every age  $x \in [0, L]$  and time  $t \geq 0$ , is the system's output

$$y(t, x) = \int_0^x R(t, s)ds. \quad (2.51)$$

Remember that this is consistent with the remarks in the recent paper [Park and Bolker (2020)] about the COVID-19 pandemic and an alternative measurement can be the number of infected patients  $\int_0^x I(t, s)ds$ .

We perform the following coordinates invertible transformation, in order to write (2.49) in the appropriate form for observer design

$$\xi_1(t, x) = \int_0^x R(t, s)ds, \quad (2.52a)$$

$$\xi_2(t, x) = \int_0^x \gamma(s)I(t, s)ds, \quad (2.52b)$$

$$\xi_3(t, x) = \int_0^x \beta(s)\gamma(s)S(t, s)ds \int_0^L I(t, s)ds. \quad (2.52c)$$

Using these coordinates, (2.49) is written as follows

$$\partial_t \xi_1(t, x) + \partial_x \xi_1(t, x) = \xi_2(t, x) + \int_0^x \mu'(s)\xi_1(t, s)ds - \mu(x)\xi_1(t, x), \quad (2.53a)$$

$$\partial_t \xi_2(t, x) + \partial_x \xi_2(t, x) = \xi_3(t, x) + \int_0^x (\kappa_1'(s) - \kappa_2'(s)) \xi_2(t, s)ds + (\kappa_2(x) - \kappa_1(x)) \xi_2(t, x), \quad (2.53b)$$

$$\begin{aligned} \partial_t \xi_3(t, x) + \partial_x \xi_3(t, x) &= \xi_3(t, x)g_1[\xi_2(t), \xi_3(t)](x) \\ &- \int_0^x (k_3'(s) - \mu'(s) - g_2[\xi_2(t)](L)\beta'(s)) \xi_3(t, s)ds + g_2[\xi_2(t)](L)B(t)\beta(0)\gamma(0), \end{aligned} \quad (2.53c)$$

$$y(t, x) = C\xi(t, x), \quad (2.53d)$$

$$\xi(t, 0) = 0, \quad (2.53e)$$

where

$$\begin{aligned}
\kappa_1(x) &:= \mu(x) + \gamma(x), \kappa_2(x) := \frac{\gamma'(x)}{\gamma(x)}, \kappa_3(x) := \frac{(\beta(x)\gamma(x))'}{\beta(x)\gamma(x)}, \\
g_1[\xi_2(t), \xi_3(t)](x) &:= \kappa_3(x) - \mu(x) - \beta(x)g_2[\xi_2(t)](L) \\
&+ \frac{1}{g_2[\xi_2(t)](L)} \left( \int_0^L \left( \frac{\kappa_1(s)}{\gamma(s)} \right)' \xi_2(t, s) ds \right. \\
&\left. - \frac{\kappa_1(L)}{\gamma(L)} \xi_2(t, L) + \int_0^L \frac{\gamma'(s)}{\gamma^2(s)} \xi_3(t, s) ds + \frac{\xi_3(t, L)}{\gamma(L)} \right), \\
g_2[\xi_2(t)](L) &:= \frac{\xi_2(t, L)}{\gamma(L)} + \int_0^L \frac{\gamma'(s)}{\gamma^2(s)} \xi_2(t, s) ds, \\
C &= (1 \quad 0 \quad 0).
\end{aligned}$$

Notice that hyperbolic dynamics (2.53) of the transformed system satisfy a simpler version of structure (2.1a) (semilinear, with  $\lambda = 1$ ), with  $\mathcal{H} = 0$  in (2.2b). Note also, that  $g_2[\xi_2(t)](L)$ , which is equal to  $\int_0^L I(t, b) db$ , is positive, because of the positiveness of system (2.53) for positive initial conditions and, therefore, mapping  $g_1$  is well defined.

Choose now some initial conditions, such that Assumption 2.1 of previous section is satisfied for system (2.53) with  $\delta = 20$  so we meet sufficient conditions for the solvability of the H-GODP for this particular choice of initial conditions (existence and uniqueness of classical solutions for (2.49) is established in [Inaba (2006)]).

The high-gain observer dynamics, as in (2.7), are given by

$$\partial_t \hat{\xi}_1(t, x) + \partial_x \hat{\xi}_1(t, x) = \hat{\xi}_2(t, x) + \int_0^x \mu'(s) \hat{\xi}_1(t, s) ds - \mu(x) \hat{\xi}_1(t, x) - \theta k_1(y(t, x) - \hat{\xi}_1(t, x)), \quad (2.54a)$$

$$\begin{aligned}
\partial_t \hat{\xi}_2(t, x) + \partial_x \hat{\xi}_2(t, x) &= \hat{\xi}_3(t, x) \\
&+ \int_0^x (\kappa_1'(s) - \kappa_2'(s)) \hat{\xi}_2(t, s) ds + (\kappa_2(x) - \kappa_1(x)) \hat{\xi}_2(t, x) - \theta^2 k_2(y(t, x) - \hat{\xi}_1(t, x)), \quad (2.54b)
\end{aligned}$$

$$\begin{aligned}
\partial_t \hat{\xi}_3(t, x) + \partial_x \hat{\xi}_3(t, x) &= s_\delta^3(\hat{\xi}_3(t, x)) g_1 \left[ s_\delta^2(\hat{\xi}_2(t)), s_\delta^3(\hat{\xi}_3(t)) \right] (x) \\
&- \int_0^x \left( k_3'(s) - \mu'(s) - g_2 \left[ s_\delta^2(\hat{\xi}_2(t)) \right] (L) \beta'(s) \right) s_\delta^3(\hat{\xi}_3(t, s)) ds + g_2 \left[ \hat{\xi}_2(t) \right] (L) B(t) \beta(0) \gamma(0) \\
&- \theta^3 k_3(y(t, x) - \hat{\xi}_1(t, x)), \quad (2.54c)
\end{aligned}$$

$$\hat{\xi}(t, 0) = 0, \quad (2.54d)$$

where  $s_\delta^i(\hat{\xi}_i)$ ,  $i = 2, 3$  are given by (2.4).

We assume for the sake of simplicity that the inflow of the newborns  $B(t)$  is not time-varying, but zero. This time invariance simplification is made in order to meet system's (2.1a) time invariance. The case where  $B$  is time-varying is not included in this framework, but it is not hard to adjust the present analysis to this case.



In Figure 2.1 we represent the considered functions  $\mu, \gamma$ , and  $\beta$  (academic choices) and we choose  $L = 100$ .

We now apply Theorem 2.1, with  $\theta = 20$  and  $K = (k_1 \ k_2 \ k_3)^\top = (-2 \ -1 \ -1)^\top$ , which establishes the solvability of the H-GODP for system (2.53) with observer (2.54). After inverting the coordinates transformation, the convergence of the observation errors between states  $S, I, R$  and  $\hat{S}, \hat{I}, \hat{R}$  (the observer states in the original coordinates) is guaranteed by Theorem 2.1.

In Figure 2.2 we represent the output  $\xi_1$ . In Figures 2.3 - 2.5 we see the observation errors for each of the original states  $S, I$  and  $R$ , after choosing arbitrary observer's initial conditions satisfying compatibility conditions.

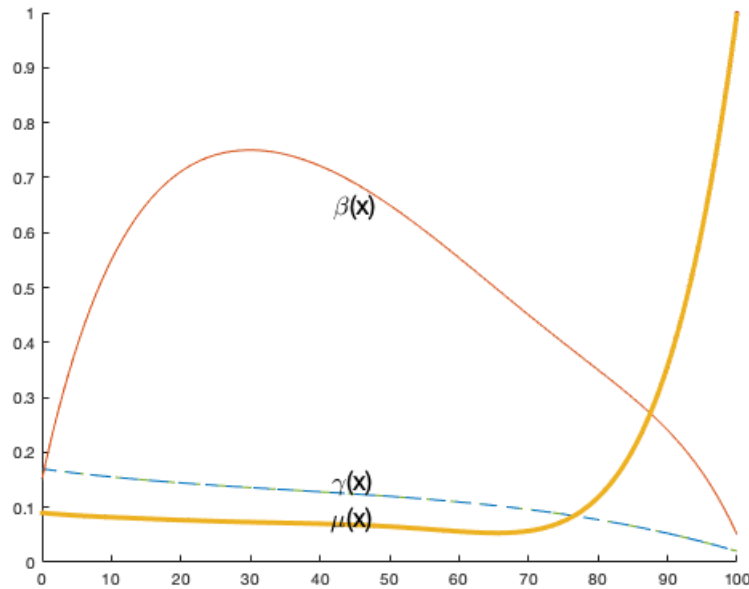


Figure 2.1: Transmission, recovery, and death rates as functions of age

**Remark 2.6.** *In Theorem 2.1 we proved a type of stability in the  $C^1$ -norm. It turns out that this is essential for the particular case of systems that we study in the current section and it is due to the nonlocal transformation we performed in (2.52). More precisely, the exponential stability of the spatial derivative of the error between (2.53) and (2.54) induces the exponential stability in the sup-norm for the estimation error of the initial system (2.49).*

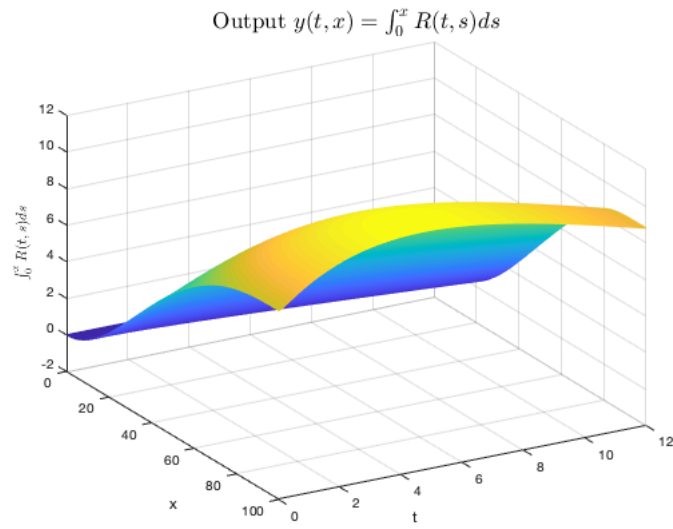


Figure 2.2: Time and age evolution of simulated system's output

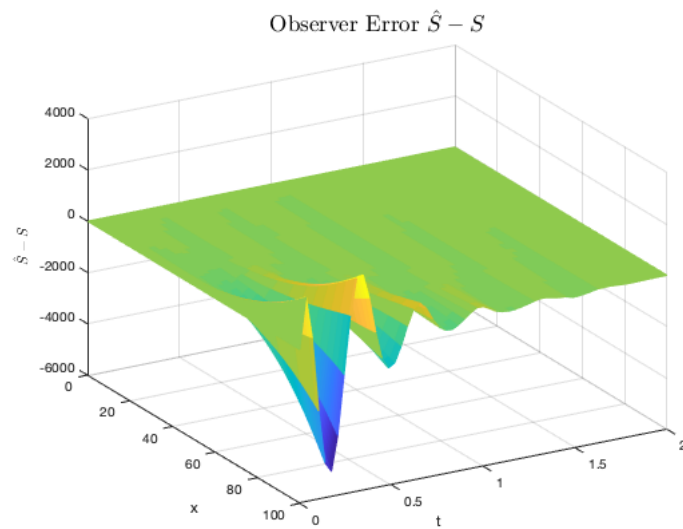


Figure 2.3: Time and age evolution of the first estimation error

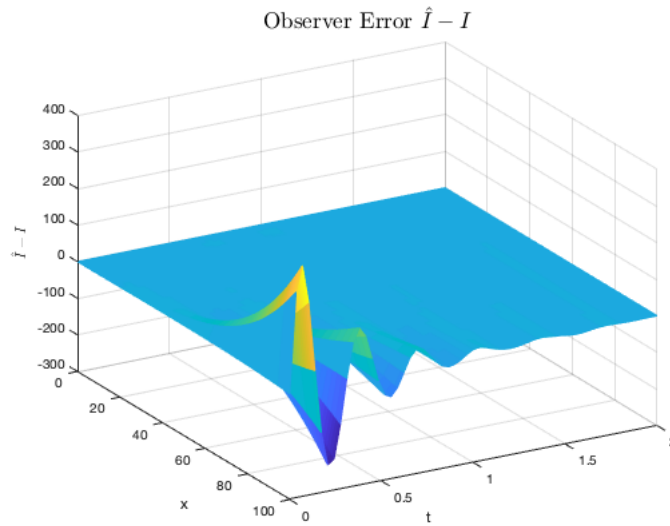


Figure 2.4: Time and age evolution of the second estimation error

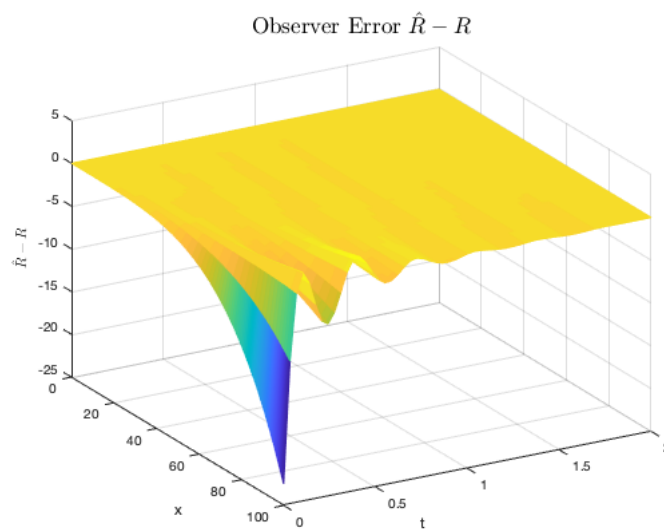


Figure 2.5: Time and age evolution of the third estimation error

## 2.3 Conclusion

In this chapter, we designed a high-gain observer for a class of cascade systems of hyperbolic PIDEs with distributed measurement, which also satisfy Property (S) (Definition 1.2 in Subsection 1.3.2) for their hyperbolic operator. We proved the exponential decay of the observer error in the  $C^1$ - norm step by step by first choosing an appropriate Lyapunov functional and we illustrated our methodology with its application to an SIR epidemic model.

As it was shown in the present chapter, the requirement of Property (S) to be satisfied is crucial in the Lyapunov stability analysis. Also, the proposed triangular structure allows the choice of a sufficiently large high-gain constant, which compensates for both system's nonlinear source terms (as in the finite-dimensional case) and also for appearing terms, due to the boundary conditions. Although intuition suggests that high-gain observer design would be directly extended from finite-dimensional systems to infinite-dimensional ones, if the latter satisfies some triangular structure, the properties of system's differential operator do not allow this extension to be direct. These facts indicate that the solution to the H-GODP is nontrivial and the main difficulty comes from the expression and the domain of system's differential operator. This ignites the idea of different manipulations that we need to apply with respect to system's differential operator, when system does not satisfy Property (S). These ideas are explored in the next chapter.

Finally, some extensions of the present analysis might include the consideration of observer design for more general epidemic models, as in [Inaba (2006)] for vertical transmission, or epidemic models which capture the characteristics of the recent COVID-19 pandemic.

## 2.4 Appendix of Chapter 2

### 2.4.1 Proof of Lemma 2.1

To prove the existence and uniqueness of global classical solutions to the observer system, which is a semilinear system of hyperbolic PIDEs, we follow a fixed-point methodology inspired by [Kmit (2008)] (Theorem 2.1), where a similar result is proven for a system of semilinear hyperbolic PDEs.

We first represent (2.7) in its integral form, as follows

$$\begin{aligned} \hat{\xi}(t, x) &= \mathcal{G}[\hat{\xi}](t, x); \\ \mathcal{G}[\hat{\xi}](t, x) &:= \Phi(t, t_0(t, x)) \mathcal{R}[\hat{\xi}](t, x) + \int_{t_0(t, x)}^T \Phi(t, s) \left( f(\omega(s; t, x), \mathcal{F}[s_\delta(\hat{\xi}(s))]) (\omega(s; t, x)) \right. \\ &\quad \left. - \Theta K y(s, \omega(s; t, x)) \right) ds; \\ \mathcal{R}[\hat{\xi}](t, x) &= \begin{cases} \hat{\xi}^0(\omega(0; t, x)), & \text{when } t_0(t, x) = 0 \\ \mathcal{H}(s_\delta(\hat{\xi}(t_0(t, x), L))), & \text{when } t_0(t, x) \neq 0 \end{cases} \end{aligned} \tag{2.55}$$

where  $\omega(s; t, x)$  denotes the characteristic (for characteristic velocity  $\lambda \circ y$ ) passing through  $(t, x) \in \Pi$ ,  $t_0(t, x)$  is the smallest value of  $s$  at which  $x_0 = \omega(s; t, x)$  reaches  $\partial\Pi$  and by  $\Phi(t, t_0)$  we denote the fundamental matrix of  $A + \Theta KC$ . More precisely,  $\omega(s; t, x)$  is the unique solution of

$$\begin{cases} \partial_s \omega(s; t, x) = (\lambda \circ y)(s, \omega(s; t, x)) \\ \omega(t; t, x) = x \end{cases} \quad (2.56)$$

It is sufficient to prove this lemma in  $\Pi^\top := (0, T) \times (0, L)$  for some fixed  $T > 0$ .

In conjunction with our regularity assumptions, it is easy to see that for every  $\delta > 0$ , there exist  $L_f, L_{\mathcal{H}} \geq 0$ , such that

$$\begin{aligned} \sup_{z \in C^0(\Pi; \mathbb{R}^n)} \left\{ \| D_u f(\cdot, \mathcal{F}[u])|_{u=s_\delta(z)} \|_{\text{op}} |D_z s_\delta(z)| \right\} &\leq L_f, \\ \sup_{z \in C^0(\Pi; \mathbb{R}^n)} |D_u \mathcal{H}(u)|_{u=s_\delta(z)} |D_z s_\delta(z)| &\leq L_{\mathcal{H}}, \end{aligned}$$

where  $\|\cdot\|_{\text{op}} := \|\cdot\|_{\mathcal{L}(C^0(\Pi; \mathbb{R}^n); C^0(\Pi; \mathbb{R}^n))}$ .

First, we prove that there exists a unique continuous solution in  $\Pi^{\tau_0}$ , for some  $\tau_0 > 0$ , such that

$$\omega(t; \tau, 0) < L; \tau > 0, t \in [\tau, \tau + \tau_0]. \quad (2.57)$$

From the integral representation (2.55) we obtain the following equation for  $t$  in  $[0, \tau_0]$ :

$$\begin{aligned} \hat{\xi}(t, L) = &\Phi(t, 0) \hat{\xi}^0(\omega(0; t, L)) + \int_0^t \Phi(t, s) \left( f(\omega(s; t, L), \mathcal{F}[s_\delta(\hat{\xi}(s))]) (\omega(s; t, L)) \right. \\ &\left. - \Theta K y(s, \omega(s; t, L)) \right) ds. \end{aligned} \quad (2.58)$$

Now, choosing  $\hat{\xi}^1, \hat{\xi}^2$  in  $C^0(\overline{\Pi^{\tau_0}}; \mathbb{R}^n)$  with identical initial conditions, from the mean value theorem, we obtain the following

$$\begin{aligned} &\sup_{(t,x) \in \overline{\Pi^{\tau_0}}} |f(x, \mathcal{F}[s_\delta(\hat{\xi}^1(t))](x)) - f(x, \mathcal{F}[s_\delta(\hat{\xi}^2(t))](x))| \\ &\leq \sup_{s \in [0,1]} \left\{ \| D_u f(\cdot, \mathcal{F}[s_\delta(u)])|_{u=\hat{\xi}^1+s(\hat{\xi}^2-\hat{\xi}^1)} \|_{\text{op}} \right. \\ &\quad \left. \times |D_u s_\delta(u)|_{u=\hat{\xi}^1+s(\hat{\xi}^2-\hat{\xi}^1)} \right\} \sup_{(t,x) \in \overline{\Pi^{\tau_0}}} |\hat{\xi}^1(t, x) - \hat{\xi}^2(t, x)| \\ &\leq L_f \sup_{(t,x) \in \overline{\Pi^{\tau_0}}} |\hat{\xi}^1(t, x) - \hat{\xi}^2(t, x)|. \end{aligned}$$

This, in conjunction with (2.55), directly leads to the following

$$\begin{aligned} \sup_{(t,x) \in \overline{\Pi^{\tau_0}}} |\mathcal{G}[\hat{\xi}^1] - \mathcal{G}[\hat{\xi}^2]| &\leq \tau_0 q_0 \sup_{(t,x) \in \overline{\Pi^{\tau_0}}} |\hat{\xi}^1 - \hat{\xi}^2|; \\ q_0 &:= \max_{(t,x) \in \overline{\Pi^{\tau_0}}} |\Phi(t, t_0(t, x))| L_f (1 + L_{\mathcal{H}}). \end{aligned}$$

Now, choosing  $\tau_0 = (2q_0)^{-1}$  and applying the Banach fixed-point theorem, we prove the existence and uniqueness of a  $C^0(\overline{\Pi^{\tau_0}}; \mathbb{R}^n)$  solution  $\hat{\xi}$ . In order to prove the existence and uniqueness of a continuous solution in  $\Pi^\top$ , for arbitrary  $T$ , we follow the previous procedure in  $\lceil T/\tau_0 \rceil$  steps, iterating the local existence-uniqueness result in domains  $(\Pi^{j\tau_0} \cap \Pi^\top) \setminus \overline{\Pi^{(j-1)\tau_0}}$ , where  $j \leq \lceil T/\tau_0 \rceil$ .

Next, we follow the same procedure to prove the existence-uniqueness of  $C^1$  solutions in  $\Pi^\top$ , by considering the problem for  $\hat{\xi}_x$ .

By (2.55), we obtain

$$\begin{aligned} \hat{\xi}_x(t, x) &= \frac{d}{dx} \Phi(t, t_0(t, x)) \mathcal{R}[\hat{\xi}](t, x) + \Phi(t, t_0(t, x)) \mathcal{R}_1[\hat{\xi}](t, x) \\ &+ \int_{t_0(t, x)}^T \Phi(t, s) \left[ \left( \left\langle D_u f(z, \mathcal{F}[u])|_{u=s_\delta(\hat{\xi}(s))(z)}, D_{\hat{\xi}s_\delta}(\hat{\xi}(s))(z) \hat{\xi}_z(s, z) \right\rangle \right. \right. \\ &\quad \left. \left. - \partial_z(\lambda \circ y)(s, z) \hat{\xi}_z(s, z) - \Theta K y_z(s, z) \right) \right] \Big|_{z=\omega(s; t, x)} ds, \\ \mathcal{R}_1[\hat{\xi}](t, x) &= \frac{d}{du} \hat{\xi}^0(u) \Big|_{u=\omega(0; t, x)}, \text{ when } t_0(t, x) = 0 \\ \text{and } \mathcal{R}_1[\hat{\xi}](t, x) &= (\lambda \circ y)^{-1}(\tau, 0) \left( f\left(0, \mathcal{F}\left[s_\delta(\hat{\xi}(\tau))\right](0)\right) - D_u \mathcal{H}(u) \Big|_{u=s_\delta(\hat{\xi})}(\tau, L) \right. \\ &\quad \left. \times D_{\hat{\xi}s_\delta}(\hat{\xi}(\tau))(L) \hat{\xi}_\tau(\tau, L) \right) \Big|_{\tau=t_0(t, x)}, \text{ when } t_0(t, x) \neq 0. \end{aligned} \quad (2.59)$$

We prove the existence-uniqueness of continuous solutions  $\hat{\xi}_x$  in  $\overline{\Pi^{\tau_1}}$ , for some  $\tau_1$  satisfying condition (2.57) with  $\tau_0$  replaced by  $\tau_1$ . Combining observer system's dynamics (2.7a) and (2.59), we get the following expression for the right boundary

$$\begin{aligned} \hat{\xi}_t(t, L) &= -(\lambda \circ y)(t, L) \hat{\xi}_x(t, L) + A \hat{\xi}(t, L) \\ &- \Theta K \left( y(t, L) - C \hat{\xi}(t, L) \right) + f\left(L, \mathcal{F}\left[s_\delta(\hat{\xi}(t))\right](L)\right) \\ &= -(\lambda \circ y)(t, L) \left( \Phi(t, 0) \frac{d}{du} \hat{\xi}^0(u) \Big|_{u=\omega(0; t, L)} \right. \\ &+ \int_0^T \Phi(t, s) \left[ \left( \left\langle D_u f(z, \mathcal{F}[u])|_{u=s_\delta(\hat{\xi}(s))(z)}, D_{\hat{\xi}s_\delta}(\hat{\xi}(s))(z) \hat{\xi}_z(s, z) \right\rangle \right. \right. \\ &\quad \left. \left. - \partial_z(\lambda \circ y)(s, z) \hat{\xi}_z(s, z) - \Theta K y_z(s, z) \right) \right] \Big|_{z=\omega(s; t, L)} ds \Big) + (A + \Theta K C) \hat{\xi}(t, L) - \Theta K y(t, L) + f\left(L, \mathcal{F}\left[s_\delta(\hat{\xi}(t))\right](L)\right). \end{aligned} \quad (2.60)$$

Taking into account that  $\hat{\xi}$  is a known continuous function, we apply the operator defined by the right hand side of (2.59) to continuous functions  $\partial_x \hat{\xi}^1$  and  $\partial_x \hat{\xi}^2$ , starting from the same

initial conditions. We can, therefore, obtain the estimate

$$\sup_{(t,x) \in \overline{\Pi^{\tau_1}}} |\partial_x \hat{\xi}^1 - \partial_x \hat{\xi}^2| \leq \tau_1 q_1 \sup_{(t,x) \in \overline{\Pi^{\tau_1}}} |\partial_x \hat{\xi}^1 - \partial_x \hat{\xi}^2|;$$

$$q_1 := \max_{(t,x) \in \overline{\Pi^{\tau_1}}} |\Phi(t, t_0(t, x))| \left( L_f + \max_{(t,x) \in \overline{\Pi^{\tau_1}}} \partial_x |(\lambda \circ y)(t, x)| \right) \left( 1 + L_{\mathcal{H}} \max_{(t,x) \in \overline{\Pi^{\tau_1}}} ((\lambda \circ y)(t, x)) \right. \\ \left. \times \max_{(t,x) \in \overline{\Pi^{\tau_1}}} ((\lambda \circ y)(t, x))^{-1} \right).$$

Choosing  $\tau_1$ , such that  $q_1 \tau_1 < 1$  and in conjunction with the fact that  $\hat{\xi}$  is  $C^1$  function in both arguments (as a result of (2.7a)), we obtain the existence-uniqueness of classical solutions in  $\overline{\Pi^{\tau_1}}$ . Iterating the previous local existence-uniqueness result in domains  $(\Pi^{j\tau_1} \cap \Pi^{\tau_1}) \setminus \overline{\Pi^{(j-1)\tau_1}}$ , where  $j \leq \lceil T/\tau_1 \rceil$ , we obtain classical solutions in  $\Pi^{\tau_1}$  and since  $T$  is arbitrary, we obtain unique classical solutions in  $\Pi$ . The proof is complete. ■

## 2.4.2 Properties of the Lyapunov Functional

We prove here two properties, which we have invoked in the stability proof of Theorem 2.1.

**Property 1:** For every continuous functions (or more generally,  $L^\infty$  functions, where  $\|\cdot\|_\infty$  denotes the ess.sup-norm)  $\varepsilon, \varepsilon_t : [0, L] \rightarrow \mathbb{R}^n; L > 0$ , matrix  $P$ ,  $\pi : [0, L] \rightarrow \mathbb{R}$  as in (3.93),  $\rho_0 > 0$ , and  $\mu$  in  $\mathbb{R}$ , we have

$$\lim_{p \rightarrow +\infty} W_p = \|e^{\mu \cdot} (\varepsilon^\top P \varepsilon + \rho_0 \varepsilon_t^\top P \varepsilon_t)\|_\infty \quad (2.61)$$

where  $W_p$ , as in (3.92a), is given by

$$W_p := \left( \int_0^L \pi(x) e^{p\mu x} G_p(x) dx \right)^{1/p}; \quad (2.62a)$$

$$G_p := (\varepsilon^\top P \varepsilon + \rho_0 \varepsilon_t^\top P \varepsilon_t)^p. \quad (2.62b)$$

To prove this, define, first for  $\bar{\varepsilon} > 0$  arbitrarily small, the set

$$B_{\bar{\varepsilon}} := \left\{ x \in [0, L] : |e^{\mu x} (\varepsilon^\top(x) P \varepsilon(x) + \rho_0 \varepsilon_t^\top(x) P \varepsilon_t(x))| \right. \\ \geq \|e^{\mu \cdot} (\varepsilon^\top P \varepsilon + \rho_0 \varepsilon_t^\top P \varepsilon_t)\|_\infty - \bar{\varepsilon}; \\ \left. \bar{\varepsilon} < \|e^{\mu \cdot} (\varepsilon_t P \varepsilon + \varepsilon_t P \varepsilon_t)\|_\infty \right\}. \quad (2.63)$$

Then, for all  $p$  in  $[1, +\infty)$ ,

$$W_p \geq \left( \int_{B_{\bar{\varepsilon}}} \left( \|e^{\mu \cdot} (\varepsilon^\top P \varepsilon + \rho_0 \varepsilon_t^\top P \varepsilon_t)\|_\infty - \bar{\varepsilon} \right)^p dx \right)^{1/p} \\ = \left( \|e^{\mu \cdot} (\varepsilon^\top P \varepsilon + \rho_0 \varepsilon_t^\top P \varepsilon_t)\|_\infty - \bar{\varepsilon} \right) \mu(B_{\bar{\varepsilon}})^{1/p} \quad (2.64)$$

where  $\mu(B_{\bar{\varepsilon}})$  denotes the Lebesgue measure of  $B_{\bar{\varepsilon}}$ .

The above gives

$$\liminf_{p \rightarrow +\infty} W_p \geq \|e^{\mu \cdot} (\varepsilon^\top P \varepsilon + \rho_0 \varepsilon_t^\top P \varepsilon_t)\|_\infty. \quad (2.65)$$

Furthermore,

$$W_p \leq \|e^{\mu \cdot} (\varepsilon^\top P \varepsilon + \rho_0 \varepsilon_t^\top P \varepsilon_t)\|_\infty (L\bar{\pi})^{1/p}. \quad (2.66)$$

This implies

$$\limsup_{p \rightarrow +\infty} W_p \leq \|e^{\mu \cdot} (\varepsilon^\top P \varepsilon + \rho_0 \varepsilon_t^\top P \varepsilon_t)\|_\infty \quad (2.67)$$

since  $L\bar{\pi}$  is finite.

By virtue of (2.65) and (2.67), we obtain (2.61).

**Property 2:**

$$\dot{W}_p \xrightarrow{*} \frac{d}{dt} \left( \|e^{\mu \cdot} \varepsilon^\top P \varepsilon\|_\infty + \|\rho_0 e^{\mu \cdot} \varepsilon_t^\top P \varepsilon_t\|_\infty \right) \text{ in the weak* topology } \sigma(L^\infty(0, +\infty), L^1(0, +\infty)),$$

as  $p \rightarrow +\infty$  (2.68)

where by  $\sigma(X, X^*)$  we denote the weak\* topology on  $X$ , with  $X^*$  the topological dual of  $X$ .





# Solving H-GODP for hyperbolic and parabolic systems with distinct velocities and diffusivities

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This chapter is devoted to introducing a main approach leading to solvability of the H-GODP, when Property (S) (see Definition 1.2 in Section 1.3.2) does not hold. We recall that Property (S) is satisfied, when system's differential operator (see  $\mathfrak{A}(\cdot)$  in (1.1) in Chapter 1) has scalar coefficients. Property (S) allows direct observer design, as it was shown for classes of systems considered in Chapter 2 (in particular, since it allows the integration needed in the Lyapunov stability analysis of the observer error). Without this property holding, a different strategy should be employed, leading to indirect observer design. As it was introduced in Subsection 1.3.2, this indirect method first requires system's differential operator  $\mathfrak{A}(\cdot)$  to be decomposed into a sum of a) a new differential operator, satisfying this fundamental Property (S), b) a differential operator (and possibly nonlinear) including only the first measured state in its domain, and c) a bilinear mapping between a function of the unmeasured state and a differential operator, including only the first state in its domain. This is done via an

infinite-dimensional state transformation, which preserves the initial triangular structure. As a consequence, the new differential operator satisfies Property (S). This proposed decomposition of the differential operator oblige us to inject the spatial derivatives of the output of higher orders in the observer dynamics, provided that these spatial derivatives are available, since the measurement is distributed.

To deal with this operator decomposition problem in its full generality, we consider both nonlinear and linear systems, the former of which experience limitations with respect to the maximum number of states that they can have, while for the latter ones, a solution is given for arbitrary number of states. The proposed approaches are first applied to a  $2 \times 2$  quasilinear hyperbolic system, where  $k = 1$  and  $\Pi_1(\cdot)$  in (1.3) is a lower triangular map, making the differential operator  $\mathfrak{A}(\cdot)$  a quasilinear hyperbolic one. Then, the approach is applied to  $2 \times 2$  and  $3 \times 3$  Lotka-Volterra-like parabolic systems, with  $k = 2$ ,  $\Pi_1(\cdot) = 0$  and  $\Pi_2(\cdot)$  a constant diagonal linear map, while choosing appropriate domain of the corresponding parabolic operator  $\mathfrak{A}(\cdot)$  in (1.3). However, for nonlinear systems with more than three states, accompanied with more than three different elements on the diagonal differential operator  $\mathfrak{A}(\cdot)$ , such a decomposition is hard to be implemented, even if the candidate state transformation was nonlinear. To overcome this limitation with respect to the distinct elements on the differential operator, we pick general  $n \times n$  nonuniform inhomogeneous linear hyperbolic systems, instead of nonlinear ones, with any number of distinct characteristic velocities of any sign, in order to show that for these systems such decomposition is feasible. The complexity of the methodology augments with the number of distinct velocities, and, thus, even for linear systems, the required infinite-dimensional state transformation requires tedious calculations. The present approach is possible under appropriate sufficient conditions, including strong regularity of system solutions and a space periodicity on the boundaries for hyperbolic systems with more than three states. As it was commented in Subsection 1.3.2 (see case B. Indirect observer design, therein), these conditions are linked to the conditions for internal controllability for cascade hyperbolic systems with reduced numbers of controls, which are assumed in the notable work of [Alabau-Boussouira et al (2017)]. Similarly to that work, the present analysis experiences the loss of derivatives phenomenon, since we require stronger regularity for system's solutions, in order to obtain an observer exponential convergence in a norm of a space of lower regularity.

In the first section, we consider a  $2 \times 2$  quasilinear hyperbolic system of balance laws, written in an appropriate triangular form, and considering a non-diagonal hyperbolic operator, whose diagonalization would equip system with distinct and positive characteristic velocities. The high-gain observer is designed, utilizing the first spatial derivative of the output, apart from the classical output injection terms, and its convergence is proven for the  $C^1$ -spatial norm. Section 3.2 is devoted to a class of  $2 \times 2$  and  $3 \times 3$  semilinear reaction-diffusion systems with distinct diffusivities. First, a Lotka-Volterra system of two species is considered, where a similar technique to the one for the hyperbolic systems is adopted, dealing in addition with the particular type of nonlinearities of those systems. Then, the methodology is extended to  $3 \times 3$  semilinear parabolic systems, where an infinite-dimensional state transformation leads to a new system of PDEs, for which the observer utilizes output's spatial derivatives of order depending on the number of different diffusivities in the parabolic operator. Additionally, to tackle the

fact that nonlinearities are not globally Lipschitz, appropriate saturation functions are injected in the observer's nonlinearities. Finally, seeing that extension of the aforementioned approach to nonlinear systems with more than three states is difficult, the third section goes back to hyperbolic systems with linear dynamics this time, but in the general  $n \times n$  case, with up to  $n$  distinct velocities of any sign (positive or negative), showing how the transformation-based approach is applied. We present, in this way, the applicability of this method in its maximal generality, which turns out to be achieved only for linear systems. The calculation of such a state transformation is performed via an introduced algorithm. We also underline that the features of high-gain approach are instrumental for these linear cases, contrary to the linear finite-dimensional case, where simpler Luenberger designs are sufficient. In each of the sections, the convergence of the observer is proven in appropriate spatial norms. The stability proof relies on the choice of appropriate Lyapunov functionals, and, finally, suitable simulations illustrate those theoretical results.

### 3.1 A class of $2 \times 2$ quasilinear hyperbolic systems

The results of this section have been presented in [Kitsos et al (2019a)]. They constitute a first extension of the results presented in Chapter 2 to systems which do not satisfy Property (S) (Definition 1.2) for the hyperbolic operator, meaning, in that case, that the considered system might have distinct characteristic velocities. We consider, here, a triangular form of the hyperbolic operator  $\mathfrak{A}(\cdot)$ , as it was introduced in (1.3), with matrix  $\Pi_1(\cdot)$  full-state dependent and lower triangular. In the present approach, the differential operator is decomposed into a diagonal hyperbolic one with only one characteristic velocity, plus a first-order nonlinear spatial differentiator acting on the measured first state. The high-gain observer dynamics, then, includes the latter spatial differential operator in its dynamics, acting on the known output, and, as a consequence, the observer error hyperbolic operator satisfies the desired Property (S). The observer convergence, then, is proven for the  $C^1$ -norm, noting also a loss of derivatives, since system's solutions are assumed to be of class  $H^2$ . This methodology is a preliminary step towards the generalization of this approach to systems with more than 2 states. As an alternative approach to this, in [Kitsos et al (2019b)], another transformation-based approach, which is finite-dimensional this time, and avoiding the use of spatial differentiators, was introduced for  $2 \times 2$  systems, succeeding to satisfy Property (S) for the new system's differential operator. However, this approach imposes limitations on the considered system and on the performance of the observer and, thus, we omit to present it in this manuscript.

The first subsection is devoted to the class of system which is considered and the proposed observer, along with the main result on the solvability of the H-GODP, described in Theorem 3.1. The second subsection deals with the proof of Theorem 3.1, while in the last subsection we illustrate this methodology via an academic example.

### 3.1.1 Class of systems and solvability of the H-GODP

Let us consider the  $2 \times 2$  first-order quasilinear hyperbolic system satisfying the general form introduced in Section 1.2 of Chapter 1. described by the following equations on the strip  $\Pi := [0, +\infty) \times [0, 1]$

$$\partial_t \xi_1(t, x) + \lambda_{11}(\xi_1(t, x)) \partial_x \xi_1(t, x) = \xi_2(t, x) + f_1(\xi_1(t, x)), \quad (3.1a)$$

$$\begin{aligned} \partial_t \xi_2(t, x) + \lambda_{21}(\xi_1(t, x)) \partial_x \xi_1(t, x) + \lambda_{22}(\xi_1(t, x)) \partial_x \xi_2(t, x) &= f_2(\xi(t, x)), \\ y(t, x) &= \xi_1(t, x), \end{aligned} \quad (3.1b)$$

where  $\xi := (\xi_1 \quad \xi_2)^\top$  is the state and  $y : [0, +\infty) \times [0, 1] \rightarrow \mathbb{R}$  is the distributed output (measurement).

As we noted in the introduction of the present chapter, we seek to solve here the H-GODP for more general systems than the ones considered in the previous chapter with respect to their hyperbolic operator. In fact, the present  $2 \times 2$  system has a hyperbolic operator with lower triangular coefficient. Furthermore, its elements on the diagonal are distinct. In Chapter 2, this coefficient was diagonal with a single element on the diagonal (one characteristic velocity). It will be shown in the sequel that the consideration of such systems is far more complex.

Assume that in addition to  $\xi_1$ , space derivative (in the classical sense)  $\partial_x \xi_1$  is also available as a measurement, which is not very restrictive since  $\xi_1$  is available on the full  $x$ -domain. This assumption on the derivative is an extra assumption, compared to the ones of the previous chapter and it comes from the requirements of the indirect observer design, as explained in Subsection 1.3.2. Assume also that  $\lambda_{11}(\xi_1), \lambda_{22}(\xi_1) > 0, \forall \xi_1 \in \mathbb{R}$ .

We consider initial and boundary conditions as follows

$$\xi(0, x) =: \xi^0(x), x \in [0, 1], \quad (3.2a)$$

$$\xi(t, 0) = H(\xi(t, 1)), t \in [0, +\infty), \quad (3.2b)$$

where  $H = (H_1 \quad H_2)^\top$ .

We make the following regularity assumption.

**Assumption 3.1.** *Functions  $\lambda_{11}(\cdot), \lambda_{22}(\cdot), \lambda_{21}(\cdot), f_i(\cdot), H_i(\cdot), i = 1, 2$  are of class  $C^1$  in their arguments.*

Prior to the rest of the assumptions, we provide the definition of compatibility conditions:

**Definition 3.1** (Compatibility conditions). *We say that the initial condition  $\xi^0$  of system (3.1a) satisfies zero-order and one-order compatibility conditions (or  $C^1$  conditions), if  $\xi^0$*

satisfies the following two equations

$$\xi^0(0) = H(\xi^0(1)), \quad (3.3a)$$

$$\begin{aligned} & \begin{pmatrix} -\lambda_{11}(\xi_1^0(0))\partial_x \xi_1^0(0) + \xi_2^0(0) + f_1(\xi_1^0(0)) \\ -\lambda_{21}(\xi^0(0))\partial_x \xi_1^0(0) - \lambda_{22}(\xi_1^0(0))\partial_x \xi_2^0(0) + f_2(\xi^0(0)) \end{pmatrix} = \frac{\partial H}{\partial \xi}(\xi^0(1)) \\ & \times \begin{pmatrix} -\lambda_{11}(\xi_1^0(1))\partial_x \xi_1^0(1) + \xi_2^0(1) + f_1(\xi_1^0(1)) \\ -\lambda_{21}(\xi^0(1))\partial_x \xi_1^0(1) - \lambda_{22}(\xi_1^0(1))\partial_x \xi_2^0(1) + f_2(\xi^0(1)) \end{pmatrix} \end{aligned} \quad (3.3b)$$

The following assumption is essential for the well-posedness of our system, along with a minimal observer design requirement of "forward completeness" and, furthermore, it imposes uniform boundedness of the solutions in the  $H^2$ -norm, which is essential in the design of the nonlinear observer, when dealing with quasilinear system operator. For further details, the reader can refer to [Bastin and Coron (2016)], [Li (1985)] and references therein, where sufficient conditions for the well-posedness of quasilinear hyperbolic systems of balance laws are presented. Similar assumption for the space  $C^1$  was made in the previous Chapter and it turned out to be fundamental in the proofs.

**Assumption 3.2.** Consider a set  $\mathcal{M} \subset H^2((0, 1); \mathbb{R}^2)$  nonempty and bounded, consisting of functions satisfying zero-order and one-order compatibility conditions for problem (3.1a)-(3.2). Then for any initial condition  $\xi^0$  in  $\mathcal{M}$ , problem (3.1a)-(3.2) admits a unique solution  $C^0([0, +\infty); H^2((0, 1); \mathbb{R}^2))$ . Moreover, there exists  $\delta > 0$ , such that for any  $\xi^0$  in  $\mathcal{M}$ ,  $\|\xi(t, \cdot)\|_{H^2((0,1), \mathbb{R}^2)} \leq \delta, \forall t \in [0, +\infty)$ .

In this section, we use the definition of the set

$$\mathcal{B}(\delta) := \{\xi \in H^2((0, 1); \mathbb{R}^2) : \|\xi\|_{H^2((0,1), \mathbb{R}^2)} \leq \delta\},$$

corresponding the subspace, where solutions satisfying Assumption 3.2 belong.

Note, that by simple inclusion arguments, Assumption 3.2 implies that solutions  $\xi$  belong to  $C^1([0, +\infty) \times [0, 1]; \mathbb{R}^2)$  (classical solutions).

In the sequel, for a mapping  $g(\cdot, \cdot)$  we use the definition of the following difference operator

$$\Delta_{\hat{\xi}_2} [g(\xi_1, \xi_2)] := g(\xi_1, \hat{\xi}_2) - g(\xi_1, \xi_2),$$

parametrized by  $\hat{\xi}_2$ .

In this section, for simplification reasons, the considered balance terms  $f_1, f_2$  are assumed simpler than the balance terms of Chapter 2, as they do not anymore include nonlocal terms. Also, they are not considered anymore locally Lipschitz (as in Chapter 2) but they have stronger properties. These simplifications on the balance terms are made as there is no need to repeat tedious calculations of the previous chapter, since the objective of this section is not to generalize the type of the balance terms of Chapter 2 but only the type of the hyperbolic operator. We make, therefore, the following assumption which will be needed in the stability analysis of the observer error equation.

**Assumption 3.3.** *There exist continuous functions  $L_{f_2}, L_{\lambda_{21}}, L_H, L_{f'_2}, L_{\lambda'_{21}}, L_{H'} : \mathbb{R} \rightarrow \mathbb{R}_+$ , such that for all  $\xi \in \mathbb{R}^2, \hat{\xi}_2 \in \mathbb{R}$ , we have  $|\frac{\partial f_2}{\partial \hat{\xi}_2}| \leq L_{f_2}(\xi_1), |\frac{\partial \lambda_{21}}{\partial \hat{\xi}_2}| \leq L_{\lambda_{21}}(\xi_1), |\frac{\partial H}{\partial \hat{\xi}_2}| \leq L_H(\xi_1), |\Delta_{\hat{\xi}_2} [Df_2(\xi)]| \leq L_{f'_2}(\xi_1)|\hat{\xi}_2 - \xi_2|, |\Delta_{\hat{\xi}_2} [D\lambda_{21}(\xi)]| \leq L_{\lambda'_{21}}(\xi_1)|\hat{\xi}_2 - \xi_2|, |\Delta_{\hat{\xi}_2} [DH(\xi)]| \leq L_{H'}(\xi_1)|\hat{\xi}_2 - \xi_2|$ .*

Prior to our main result, we must emphasize the fact that according to a well-known Sobolev inequality (coming from compact injections, see for instance [Brezis (1983)]), for all  $\xi$  in  $H^2((0, 1); \mathbb{R}^2)$ , there exists  $c_0 > 0$ , such that

$$\|\xi\|_\infty + \|\xi_x\|_\infty \leq c_0 \|\xi\|_{H^2((0,1); \mathbb{R}^2)}. \quad (3.4)$$

At this point, let us introduce some operators, parametrized by  $\hat{\xi}_2$

$$\begin{aligned} \eta_0, \eta_1^{\hat{\xi}_2}, \eta_2^{\hat{\xi}_2} &: H^2((0, 1); \mathbb{R}^2) \rightarrow C^0([0, 1]; \mathbb{R}^2), \\ \eta_3^{\hat{\xi}_2} &: H^2((0, 1); \mathbb{R}^2) \rightarrow L^2((0, 1); \mathbb{R}^2), \\ \Lambda_1, \Lambda_2^{\hat{\xi}_2} &: H^2((0, 1); \mathbb{R}^2) \rightarrow C^0([0, 1]; \mathbb{R}^{2 \times 2}) \end{aligned}$$

acting on  $\xi$  and defined by

$$\eta_0[\xi] := \lambda'_{22}(\xi_1) \lambda_{22}^{-1}(\xi_1) \partial_t \xi_1, \quad (3.5a)$$

$$\eta_1^{\hat{\xi}_2}[\xi] := \frac{\partial f_2(\xi_1, \hat{\xi}_2)}{\partial \hat{\xi}_2} - \partial_x \xi_1 \frac{\partial \lambda_{21}(\xi_1, \hat{\xi}_2)}{\partial \hat{\xi}_2} + \eta_0[\xi], \quad (3.5b)$$

$$\eta_2^{\hat{\xi}_2}[\xi] := \Delta_{\hat{\xi}_2} [f_2(\xi_1, \xi_2)] - \partial_x \xi_1 \Delta_{\hat{\xi}_2} [\lambda_{21}(\xi_1, \xi_2)], \quad (3.5c)$$

$$\begin{aligned} \eta_3^{\hat{\xi}_2}[\xi] &:= \Delta_{\hat{\xi}_2} [Df_2(\xi_1, \xi_2)] \partial_t \xi - \Delta_{\hat{\xi}_2} [\lambda_{21}(\xi_1, \xi_2)] \partial_{xt} \xi \\ &\quad - \partial_x \xi_1 \Delta_{\hat{\xi}_2} [D\lambda_{21}(\xi_1, \xi_2)] \xi_t - \eta_0[\xi] \eta_2^{\hat{\xi}_2}[\xi], \end{aligned} \quad (3.5d)$$

$$\Lambda_1(\xi) := \begin{pmatrix} \lambda_{11}(\xi_1) & 0 \\ \lambda_{21}(\xi) & \lambda_{22}(\xi_1) \end{pmatrix}, \quad (3.5e)$$

$$\Lambda_2^{\hat{\xi}_2}[\xi] := \text{diag}(\eta_0[\xi], \eta_1^{\hat{\xi}_2}[\xi]), \quad (3.5f)$$

noting also time-derivatives can be substituted by space-derivatives, by use of the hyperbolic dynamics. Now, notice that whenever  $\xi \in \mathcal{B}(\delta)$ , due to continuity and positiveness of  $\lambda_{ii}, i = 1, 2$  and further the fact that  $\|\xi(t, \cdot)\|_\infty \leq c_0 \delta, \forall t \geq 0$ , as a result of (3.4), the quantities  $\max_{\xi \in \mathcal{B}(\delta)}(\lambda_{ii}(\xi_1)), \min_{\xi \in \mathcal{B}(\delta)}(\lambda_{ii}(\xi_1))$  are well-defined and positive. In addition, note that whenever  $\xi \in \mathcal{B}(\delta)$ , as a result of (3.4) and the use of hyperbolic dynamics (3.1a), we easily calculate constants  $\delta_1, \delta_2 > 0$ , such that

$$\|\xi_t(t, \cdot)\|_\infty \leq \delta_1, \|y_{tx}(t, \cdot)\|_{L^2((0,1); \mathbb{R}^2)} \leq \delta_2, \forall t \geq 0. \quad (3.6)$$

Note that to obtain  $\delta_2$ , we use the fact that

$$\begin{aligned} &\|y_{tx}(t, \cdot)\|_{L^2((0,1); \mathbb{R}^2)} \\ &= \left\| -\lambda'_{11}(y(t, \cdot)) y_x^2(t, \cdot) - \lambda_{11}(y(t, \cdot)) y_{xx}(t, \cdot) + \partial_x \xi_2(t, \cdot) + f'_1(y(t, \cdot)) y_x(t, \cdot) \right\|_{L^2((0,1); \mathbb{R}^2)}, \end{aligned} \quad (3.7)$$

and we subsequently apply trivial inequalities. By virtue of (3.4), (3.6), continuity and Assumption 3.3 concerning the nonlinearities of the involved mappings, we can easily calculate positive constants  $\gamma_i, i = 1, \dots, 5$ , such that for  $\xi \in \mathcal{B}(\delta)$ , the following inequalities are satisfied for all  $t \geq 0, x \in [0, 1]$ :

$$\begin{aligned} |\eta_1^{\hat{\xi}_2(t,x)}[\xi](t,x)| &\leq \gamma_1, |\Lambda_2[\xi](t,x)| \leq \gamma_2, |\eta_0[\xi](t,x)| \leq \gamma_3, \\ |\eta_2^{\hat{\xi}_2(t,x)}[\xi](t,x)| &\leq \gamma_4 |\hat{\xi}_2(t,x) - \xi_2(t,x)|, \\ |\eta_3^{\hat{\xi}_2(t,x)}[\xi](t,x)| &\leq \gamma_5 |\hat{\xi}_2(t,x) - \xi_2(t,x)|, \forall \hat{\xi}_2(t,x) \in \mathbb{R}. \end{aligned} \quad (3.8)$$

Let us now introduce our candidate observer dynamics on  $\Pi$ .

### High-Gain Observer

$$\begin{aligned} \partial_t \hat{\xi}_1(t,x) + \lambda_{22}(y(t,x)) \partial_x \hat{\xi}_1(t,x) &= \hat{\xi}_2(t,x) + (\lambda_{22}(y(t,x)) - \lambda_{11}(y(t,x))) y_x(t,x) + f_1(y(t,x)) \\ &\quad + \theta k_1 (\hat{\xi}_1(t,x) - y(t,x)), \end{aligned} \quad (3.9a)$$

$$\begin{aligned} \partial_t \hat{\xi}_2(t,x) + \lambda_{22}(y(t,x)) \partial_x \hat{\xi}_2(t,x) &= f_2(y(t,x), \hat{\xi}_2(t,x)) - \lambda_{21}(y(t,x), \hat{\xi}_2(t,x)) y_x(t,x) \\ &\quad + \theta^2 k_2 (\hat{\xi}_1(t,x) - y(t,x)), \end{aligned} \quad (3.9b)$$

with high-gain constant  $\theta > 1$ , and boundary conditions satisfying

$$\hat{\xi}(t,0) = H \left( y(t,1), \hat{\xi}_2(t,1) \right). \quad (3.10)$$

The following lemma guarantees the existence of a unique global classical solution to the candidate observer (3.9), (3.10) for any initial condition of class  $C^1$ . We invoke paper [Kmit (2008)], where an analogous result is proven under Lipschitz properties of the dynamics. It is easy to check that our candidate observer satisfies semilinear hyperbolic laws and is written in a well-posed characteristic form. Assumptions 3.1 - 3.3, in conjunction with the previously mentioned comments (details are left to the reader) are compatible with the sufficient conditions of Theorem 2.1 in [Kmit (2008)] and, thereby similar global existence result is established for our observer system. This yields to the following result.

**Lemma 3.1** (Existence/uniqueness for the observer). *Under Assumptions 3.1 - 3.3, for any initial conditions  $\hat{\xi}^0(x) := \hat{\xi}(0,x), \forall x \in [0,1]$  satisfying zero-order and one-order compatibility conditions, and for any  $y \in C^1([0,+\infty) \times [0,1])$ , the problem described by equations (3.9), (3.10) in  $\Pi$  admits a unique classical solution in  $\Pi$ , namely, there exists a unique solution  $\hat{\xi} \in C^1([0,+\infty) \times [0,1]; \mathbb{R}^2)$ .*

We now present our main result on the observer design.

**Theorem 3.1** (Solvability of the H-GODP). *Consider system (3.1a)-(3.2), defined on  $\Pi$  with output (3.1b) and suppose that initial condition  $\xi^0 \in \mathcal{M}$  and Assumptions 3.1 - 3.3 hold. Let also  $P \in \mathbb{R}^{2 \times 2}$  be a symmetric and positive definite matrix satisfying<sup>1</sup>*

$$2\text{Sym}(P(A + KC)) = -qI_2, \quad (3.11)$$

<sup>1</sup>This is always possible since  $(A, C)$  is observable.



where

$$A := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, C = (1 \ 0), \mathbb{R}^2 \ni K = (k_1 \ k_2)^\top, q > 0 \quad (3.12)$$

Then, for  $\theta > 1$ , system (3.9), (3.10), with initial condition  $\hat{\xi}^0 \in C^1([0, 1]; \mathbb{R}^2)$ ;  $\hat{\xi}(0, x) =: \hat{\xi}^0(x)$  satisfying zero-order and one-order compatibility conditions, is a well-posed high-gain observer in the sense that it admits a unique classical solution in  $\Pi$  on the one hand, providing an estimate for the state of the system for  $\theta$  large enough on the other hand. More precisely, for every  $\kappa > 0$ , there exists a constant  $\theta_0 \geq 1$ , such that for every  $\theta > \theta_0$ , the following inequality holds for the solutions to (3.1a)- (3.2) and (3.9)-(3.10)

$$\|\xi(t, \cdot) - \hat{\xi}(t, \cdot)\|_1 \leq \ell e^{-\kappa t} \|\xi^0(\cdot) - \hat{\xi}^0(\cdot)\|_1, t \geq 0,$$

with  $\ell$  a polynomial in  $\theta$ .

This theorem states that the H-GODP is solvable for system (3.1a)-(3.2), with output (3.1b) via observer system (3.9), (3.10), utilizing also the first spatial derivative of the output. The proof of the observer convergence in the  $C^1$  spatial norm follows the Lyapunov-based methodology employed in Chapter 2, noting also that a loss of derivatives appears, since system's solutions are of class  $H^2$ , while the stability can be proven for a less regular space, namely, at most  $C^1$ .

### 3.1.2 Observer convergence proof

In this subsection, we prove Theorem 3.1 by following Lyapunov-based techniques.

We first define the linearly transformed observer error  $\varepsilon = (\varepsilon_1, \varepsilon_2)^\top$  by

$$\varepsilon_1 = \theta^{-1}(\hat{\xi}_1 - \xi_1); \varepsilon_2 = \theta^{-2}(\hat{\xi}_2 - \xi_2). \quad (3.13)$$

By use of (3.1a) and (3.9), the error components  $\varepsilon_1, \varepsilon_2$  satisfy the following equations in  $\Pi$ :

$$\partial_t \varepsilon_1 + \lambda_{22}(y) \partial_x \varepsilon_1 = \theta(k_1 \varepsilon_1 + \varepsilon_2), \quad (3.14a)$$

$$\partial_t \varepsilon_2 + \lambda_{22}(y) \partial_x \varepsilon_2 = \theta k_2 \varepsilon_1 + \theta^{-2} \eta_2^{\hat{\xi}_2}[\xi]. \quad (3.14b)$$

Furthermore, the following equation is satisfied on the boundaries for  $t \geq 0$ , as a consequence of (3.2b) and (3.10):

$$\varepsilon_i(t, 0) = \theta^{-i} \Delta_{\hat{\xi}_2} [H_i(y, \xi_2)](t, 1), i = 1, 2. \quad (3.15)$$

Next, by temporarily assuming that  $\varepsilon$  is of class  $C^2$ , we perform time differentiation of equations (3.14), (3.15) and we can easily obtain the following hyperbolic equations for  $\varepsilon_t$  in  $\Pi$ :

$$\partial_{tt} \varepsilon_1 + \lambda_{22}(y) \partial_{tx} \varepsilon_1 = \theta(k_1 \partial_t \varepsilon_1 + \partial_t \varepsilon_2) + \eta_0[\xi] \partial_t \varepsilon_1 - \theta \eta_0[\xi] (k_1 \varepsilon_1 + \varepsilon_2), \quad (3.16a)$$

$$\partial_{tt} \varepsilon_2 + \lambda_{22}(y) \partial_{tx} \varepsilon_2 = \theta k_2 \partial_t \varepsilon_1 + \eta_1^{\hat{\xi}_2}[\xi] \partial_t \varepsilon_2 - \theta \eta_0[\xi] k_2 \varepsilon_1 + \theta^{-2} \eta_3^{\hat{\xi}_2}[\xi], \quad (3.16b)$$

and boundary conditions for  $t \geq 0$  given by

$$\partial_t \varepsilon_i(t, 0) = \theta^{-i} \Delta_{\hat{\xi}_2} \left[ \frac{\partial H_i(y, \xi_2)}{\partial (y, \xi_2)} \right] (t, 1) \xi_t(t, 1) + \theta^{2-i} \frac{\partial H_i(y, \hat{\xi}_2)}{\partial \hat{\xi}_2} (t, 1) \partial_t \varepsilon_2(t, 1), i = 1, 2. \quad (3.17)$$

To proceed to the stability analysis, we first define functionals  $\mathcal{G}_p$  and  $\mathcal{W}_p : C^1([0, 1]; \mathbb{R}^2) \rightarrow \mathbb{R}$  by

$$\mathcal{G}_p[\varepsilon] := \left( \varepsilon^\top P \varepsilon + \rho_0 \varepsilon_t^\top P \varepsilon_t \right)^p, \quad (3.18a)$$

$$\mathcal{W}_p[\varepsilon] := \left( \int_0^1 \pi(x) e^{p\mu x} \mathcal{G}_p[\varepsilon] dx \right)^{1/p}, \quad (3.18b)$$

where  $\rho_0 \in (0, 1]$  is a constant (to be chosen appropriately),  $p \in \mathbb{N}$ ,  $P \in \mathbb{R}^{2 \times 2}$  is a symmetric positive definite matrix satisfying (3.11),  $\pi : [0, 1] \rightarrow \mathbb{R}$  is given by

$$\pi(x) := \left( \frac{\max_{\xi \in \mathcal{B}(\delta)} (\lambda_{22}(\xi_1))}{\min_{\xi \in \mathcal{B}(\delta)} (\lambda_{22}(\xi_1))} - 1 \right) x + 1, x \in [0, 1], \quad (3.19)$$

and  $\mu \in \mathbb{R}$  is given by

$$\mu := \ln(\mu_0 \theta^2); \quad (3.20a)$$

$$\mu_0 := \frac{|P|}{\underline{\text{eig}}(P)} \max \left( (L_{H'} + L_H \delta_1)^2, L_H (L_H + L_{H'} \delta_1) \right), \quad (3.20b)$$

where the above-used constants are defined in (3.6) and Assumption 3.3. Notice that

$$\pi(x) \in \left[ 1, \frac{\max_{\xi \in \mathcal{B}(\delta)} (\lambda_{22}(\xi_1))}{\min_{\xi \in \mathcal{B}(\delta)} (\lambda_{22}(\xi_1))} \right], \forall x \in [0, 1]. \quad (3.21)$$

By invoking Lemma 3.1 and Assumption 3.2, which establish global unique classical solutions for observer system (3.9), (3.10) and system (3.1a), (3.2) respectively, we are now in a position to define  $G_p, W_p : [0, +\infty) \rightarrow \mathbb{R}$  by

$$G_p(t) := \mathcal{G}_p[\varepsilon](t), W_p(t) := \mathcal{W}_p[\varepsilon](t), t \geq 0. \quad (3.22)$$

(we use the notation  $\varepsilon(t)(x) := \varepsilon(t, x)$ ).

These chosen  $p$ -functionals have similarities with the ones appeared in Chapter 2 and the main idea is to let  $p \rightarrow \infty$ , in order to establish the asymptotic convergence in the sup-norms. The calculations done for the time-derivative of  $\dot{W}_p$  follow the steps of the previous chapter, with some technical differences appearing, as for instance the existence of the second derivative of the output.

Calculating the time-derivative  $\dot{W}_p$  along the classical solutions  $\varepsilon, \varepsilon_t$  of (3.14) - (3.15), (3.16) - (3.17), we get

$$\dot{W}_p = \frac{1}{p} W_p^{1-p} \int_0^1 p \pi(x) e^{p\mu x} G_{p-1} \left( \varepsilon_t^\top P \varepsilon + \varepsilon^\top P \varepsilon_t + \rho_0 \varepsilon_{tt}^\top P \varepsilon_t + \rho_0 \varepsilon_t^\top P \varepsilon_{tt} \right) dx. \quad (3.23)$$

After substituting the dynamical equations (3.14) and (3.16) into the above equation and applying integration by parts (which is possible, since observer error satisfies Property (S) for its hyperbolic operator),  $\dot{W}_p$  can be written in the following form

$$\dot{W}_p = W_p^{1-p} \left( \frac{1}{p} T_{1,p} + \frac{1}{p} T_{2,p} + T_{3,p} + T_{4,p} \right), \quad (3.24)$$

where

$$T_{1,p} := -\pi(1)\lambda_{22}(y(1))e^{p\mu}G_p(1) + \pi(0)\lambda_{22}(y(0))G_p(0), \quad (3.25a)$$

$$T_{2,p} := \int_0^1 d_x(\pi(x)e^{p\mu x}\lambda_{22}(y))G_p dx, \quad (3.25b)$$

$$\begin{aligned} T_{3,p} := & 2 \int_0^1 \pi(x)e^{p\mu x}G_{p-1} \left( \varepsilon^\top P\theta^{-2} \begin{pmatrix} 0 \\ \eta_2^{\xi_2}[\xi] \end{pmatrix} \right. \\ & \left. + \rho_0\varepsilon_t^\top P\theta^{-2} \begin{pmatrix} 0 \\ \eta_3^{\xi_2}[\xi] \end{pmatrix} + \rho_0\varepsilon_t^\top \text{Sym}(P\Lambda_2^{\xi_2}[\xi])\varepsilon_t \right) dx, \end{aligned} \quad (3.25c)$$

$$\begin{aligned} T_{4,p} := & \theta \int_0^1 \pi(x)e^{p\mu x}G_{p-1} \left( 2\varepsilon^\top \text{Sym}(P(A+KC))\varepsilon + 2\rho_0\varepsilon_t^\top \text{Sym}(P(A+KC))\varepsilon_t \right. \\ & \left. - \rho_0\eta_0[\xi]\varepsilon_t^\top P(A+KC)\varepsilon - \rho_0\eta_0[\xi]\varepsilon^\top (A+KC)^\top P\varepsilon_t \right) dx. \end{aligned} \quad (3.25d)$$

After substituting boundary equations (3.15) and (3.17) in  $T_{1,p}$  and by virtue of Lipschitz properties (Assumption 3.3), equations (3.6), (3.19) and trivial inequalities, we can easily obtain the following bound

$$T_{1,p} \leq \max_{\xi \in \mathcal{B}(\delta)} (\lambda_{22}(\xi_1))G_p(1) (-e^{p\mu} + (\theta^2\mu_0)^p). \quad (3.26)$$

As a result of (3.20), we get

$$T_{1,p} \leq 0. \quad (3.27)$$

Next, observe that term  $T_{2,p}$  is written as follows

$$T_{2,p} = \int_0^1 ((\pi(1) - 1)\lambda_{22}(y) + p\mu\pi(x)\lambda_{22}(y) + \pi(x)\lambda'_{22}(y)y_x) e^{p\mu x}G_p dx. \quad (3.28)$$

By the fact that  $\pi(x) \geq 1, \forall x \in [0, 1]$ , we obtain the following inequality

$$T_{2,p} \leq \left( \alpha + p|\mu| \max_{\xi \in \mathcal{B}(\delta)} (\lambda_{22}(\xi_1)) \right) W_p^p, \quad (3.29)$$

where  $\alpha := (\pi(1) - 1) \max_{\xi \in \mathcal{B}(\delta)} (\lambda_{22}(\xi_1)) + c_0\delta \max_{\xi \in \mathcal{B}(\delta)} (\lambda'_{22}(\xi_1))$ .

By exploiting bounds given in (3.8),  $T_{3,p}$  can be bounded as follows

$$\begin{aligned} T_{3,p} & \leq \int_0^1 \pi(x)e^{p\mu x}G_{p-1} (2|P|(\gamma_4|\varepsilon|^2 + \rho_0\gamma_5|\varepsilon||\varepsilon_t| + \rho_0\gamma_2|\varepsilon_t|^2)) dx \\ & \leq \int_0^1 \pi(x)e^{p\mu x}G_{p-1} \frac{|P|}{\underline{\text{eig}}(P)} \left( 2\gamma_4\varepsilon^\top P\varepsilon + 2\rho_0\gamma_2\varepsilon_t^\top P\varepsilon_t + \gamma_5w_1 \right) dx \\ & \leq (\gamma_5 + 2\max(\gamma_2, \gamma_4)) \frac{|P|}{\underline{\text{eig}}(P)} W_p^p. \end{aligned} \quad (3.30)$$

Term  $T_{4,p}$  can be written in the following form

$$T_{4,p} := -\theta \int_0^1 \pi(x) e^{p\mu x} G_{p-1} \left( \begin{pmatrix} \varepsilon^\top & \varepsilon_t^\top \\ \varepsilon & \varepsilon_t \end{pmatrix} \Sigma[\xi] \begin{pmatrix} \varepsilon \\ \varepsilon_t \end{pmatrix} \right) dx, \quad (3.31)$$

where, after utilizing (3.11), mapping  $\Sigma : \mathcal{B}(\delta) \rightarrow C^0([0, 1]; \mathbb{R}^{4 \times 4})$  is given by

$$\Sigma[\xi] := \begin{pmatrix} qI_2 & -\rho_0 \eta_0[\xi](A + KC)^\top P \\ -\rho_0 \eta_0[\xi]P(A + KC) & \rho_0 qI_2 \end{pmatrix}. \quad (3.32)$$

Now, we can easily verify that, by Schur complement, we get  $0 < \inf_{\xi \in \mathcal{B}(\delta)} \frac{w^\top \Sigma[\xi] w}{|w|^2} < +\infty, \forall w \in \mathbb{R}^4 \setminus \{0\}$ , if we choose

$$0 < \rho < \min \left( \frac{q^2}{\gamma_3^2 |P|^2 |A + KC|^2}, 1 \right), \quad (3.33)$$

where  $\gamma_3$  is defined in (3.8). It turns out that for every choice of matrices  $P$  and  $K$  and constant  $q$  satisfying equation (3.11), there always exists a  $\rho_0$  (sufficiently small), such that (3.33) is satisfied and this fact renders  $\Sigma$  positive definite. Consequently, for appropriate choice of  $\rho_0$ , there exists  $\sigma > 0$ , such that

$$\begin{aligned} T_{4,p} &\leq -\sigma \theta \int_0^1 \pi(x) e^{p\mu x} G_{p-1} (|\varepsilon|^2 + |\varepsilon_t|^2) dx \\ &\leq -\sigma \theta \int_0^1 \pi(x) e^{p\mu x} G_{p-1} (|\varepsilon|^2 + \rho_0 |\varepsilon_t|^2) dx \leq -\theta \frac{\sigma}{|P|} W_p^p. \end{aligned} \quad (3.34)$$

Combining equations (3.27), (3.29), (3.30), (3.34) with (3.24) and taking into account that  $p \geq 1$ , we obtain

$$\dot{W}_p \leq (-\theta \omega_1 + \omega_2 \ln \theta + \omega_3) W_p, \quad (3.35)$$

where  $\omega_1 := \frac{\sigma}{|P|}$ ,  $\omega_2 := 2 \max_{\xi \in \mathcal{B}(\delta)} (\lambda_{22}(\xi_1))$ ,  $\omega_3 := a + \max_{\xi \in \mathcal{B}(\delta)} (\lambda_{22}(\xi_1)) \ln \mu_0 + (\gamma_5 + 2 \max(\gamma_2, \gamma_4)) \frac{|P|}{\text{eig}(P)}$ .

We obtained the estimate (3.35) of  $\dot{W}_p$  for  $\varepsilon$  of class  $C^2$ , but the proof so far implies that the result does not depend on the  $C^2$ -norms. Therefore, as in the proof of Theorem [Othm] of the previous chapter, by invoking density arguments, the results remain valid with  $\varepsilon$  only of class  $C^1$  (see [Coron and Bastin (2015)] for further details).

Applying the comparison lemma to (3.35), we obtain an estimate for  $W_p$  of the following form:

$$W_p(t) \leq e^{-(\omega_1 \theta - \omega_2 \ln \theta - \omega_3)t} W_p(0), \forall t \geq 0. \quad (3.36)$$

Now, one can select the high gain  $\theta$ , such that

$$\theta > \theta_0, \quad (3.37)$$

where  $\theta_0 > 1$  is such that

$$-\omega_1 \theta + \omega_2 \ln \theta + \omega_3 \leq -2\kappa, \forall \theta > \theta_0,$$

for  $\kappa > 0$ . It can be checked that for any  $\kappa > 0$ , there exists  $\theta_0 > 1$ , such that the previous inequality is satisfied for any  $\theta > \theta_0$ .

Next, by taking into account (3.21) we obtain the following property

$$\lim_{p \rightarrow \infty} W_p = \lim_{p \rightarrow \infty} \|\pi(\cdot)^{\frac{1}{p}} e^{\mu x} (\varepsilon^\top P \varepsilon + \rho_0 \varepsilon_t^\top P \varepsilon_t)\|_{L^p((0,1); \mathbb{R}^2)} = \|e^{\mu x} \varepsilon^\top P \varepsilon\|_0 + \|\rho_0 e^{\mu x} \varepsilon_t^\top P \varepsilon_t\|_0, \quad (3.38)$$

which holds for continuous  $\varepsilon$  and  $\varepsilon_t$ . We are now in a position to define functional  $V : C^1([0, 1]; \mathbb{R}^2) \rightarrow \mathbb{R}$  by

$$\mathcal{V}[\varepsilon] := \|e^{\mu \cdot} \varepsilon^\top P \varepsilon\|_0 + \|e^{\mu \cdot} \varepsilon_x^\top P \varepsilon_x\|_0, \quad (3.39)$$

and as in (3.22),

$$V(t) := \mathcal{V}[\varepsilon](t), t \geq 0. \quad (3.40)$$

In conjunction with property (3.38), (3.36), we obtain the following

$$V(t) \leq c \rho_0^{-1} e^{-2\kappa t} V(0), \forall t \geq 0, \quad (3.41)$$

where  $0 < c \sim \theta^2$  ( $c$  is obtained after exploiting (3.14) and bounds derived from the Assumptions and (3.8), in order to calculate the bound in the sup-norm for  $\varepsilon_x$  with respect to the bounds of  $\varepsilon_t$  and  $\varepsilon$ . Details are left to the reader). By use of the inequality

$$e^{-\frac{\mu+|\mu|}{2}} \underline{\text{eig}}(P) \|\varepsilon(t, \cdot)\|_1^2 \leq V(t) \leq e^{-\frac{\mu-|\mu|}{2}} |P| \|\varepsilon(t, \cdot)\|_1^2,$$

we easily obtain

$$\|\varepsilon(t, \cdot)\|_1 \leq c^{1/2} \rho_0^{-1/2} e^{|\mu|/2} \sqrt{|P| \underline{\text{eig}}^{-1}(P) e^{-\kappa t}} \|\varepsilon^0\|_1, t \geq 0, \quad (3.42)$$

where  $\varepsilon^0(x) := \varepsilon(0, x)$ . By virtue of (3.13), we derive the following estimate, which holds for every  $t \geq 0$

$$\|\xi(t, \cdot) - \hat{\xi}(t, \cdot)\|_1 \leq \ell e^{-\kappa t} \|\xi^0 - \hat{\xi}^0\|_1, \quad (3.43)$$

where  $\ell := \theta c^{1/2} \rho_0^{-1/2} e^{|\mu|/2} \sqrt{|P| \underline{\text{eig}}^{-1}(P)}$ . Concluding, we designed an exponential in the  $C^1$ -norm high-gain observer of adjustable convergence rate  $\kappa$ , dependent on the selection of  $\theta$ , namely, for every  $\theta > \theta_0$ , there exist  $\ell, \kappa > 0$ , such that (3.43) is satisfied. The higher the values  $\theta$  attains, the faster the observation error converges to zero. This concludes the proof of Theorem 3.1.

### 3.1.3 Simulation

In this subsection, the obtained theoretical results of the previous subsection are illustrated via an example.

**Example 3.1.** Consider the following coupled hyperbolic equations on  $\Pi$

$$\begin{aligned}\partial_t \xi_1 + 0.1(2 + \cos(\xi_1))\partial_x \xi_1 &= \xi_2 + \sin(\xi_1), \\ \partial_t \xi_2 - 0.1\xi_1 \sin(\xi_1 \xi_2)\partial_x \xi_1 + 0.1(2 + \sin(\xi_1))\partial_x \xi_2 &= \sin(\xi_2 - \xi_1), \\ y &= \xi_1\end{aligned}$$

and boundary conditions satisfying

$$\begin{aligned}\xi_1(t, 0) &= -\frac{3}{7}\xi_1(t, 1) - \xi_2(t, 1), \\ \xi_2(t, 0) &= -\frac{2}{7}\xi_1(t, 1).\end{aligned}$$

Consider the initial condition  $\xi_1^0(x) = \pi(1 - x)$ ,  $\xi_2^0(x) = -\pi x$ . All Assumptions 3.1 - 3.3 are satisfied for the above system and, thus, we can design the high-gain observer (3.9) on  $\Pi$  for a choice of high gain  $\theta = 5$  and  $k_1 = -1, k_2 = -2$

$$\begin{aligned}\partial_t \hat{\xi}_1 + 0.1(2 + \sin(y))\partial_x \hat{\xi}_1 &= \hat{\xi}_2 + 0.1(\sin(y) - \cos(y))y_x + \sin(y) + \theta k_1(\hat{\xi}_1 - y), \\ \partial_t \hat{\xi}_2 + 0.1(2 + \sin(y))\partial_x \hat{\xi}_2 &= \sin(\hat{\xi}_2 - y) + 0.1y \sin(y \hat{\xi}_2)y_x + \theta^2 k_2(\hat{\xi}_1 - y), \\ \hat{\xi}_1(t, 0) &= -\frac{3}{7}y(t, 1) - \hat{\xi}_2(t, 1), \hat{\xi}_2(t, 0) = -\frac{2}{7}y(t, 1).\end{aligned}$$

We choose observer initial conditions in accordance with the compatibility conditions. Figure 3.1 shows the measured state, while figures 3.2, 3.3 illustrate the estimation error functions for system's states, which exhibit exponential convergence to zero, as predicted by Theorem 3.1.

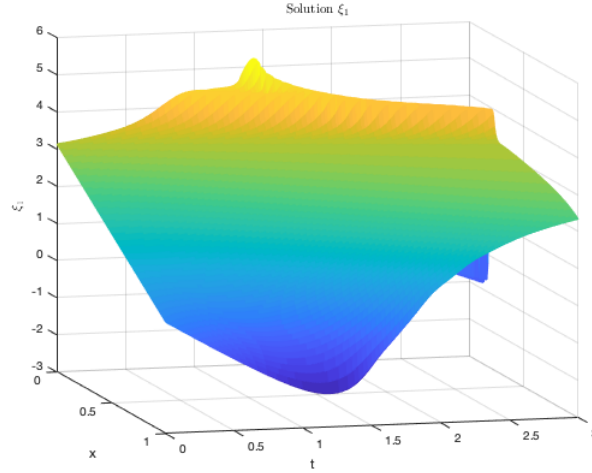


Figure 3.1: Time and space evolution of measured state  $\xi_1$

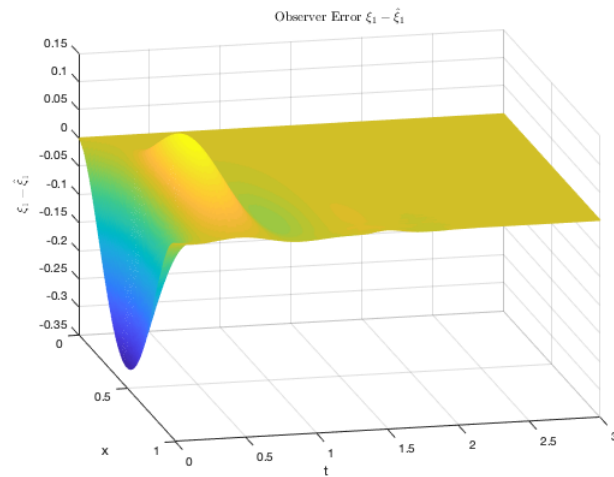


Figure 3.2: Time and space evolution of the observer error of the first state

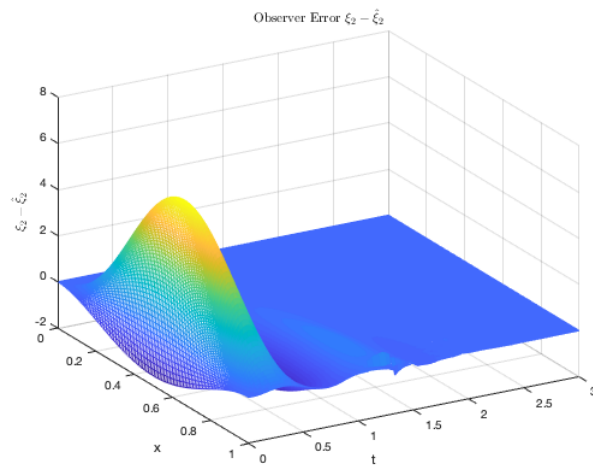


Figure 3.3: Time and space evolution of the observer error of the second state

## 3.2 A class of $2 \times 2$ and $3 \times 3$ semilinear parabolic systems

In this section, the results of [Kitsos et al (2020b)] are presented concerning a solution to the H-GODP for some Lotka-Volterra-like semilinear parabolic cascade systems of order  $n = 2$  and  $n = 3$  and considering measurement of the first state. The class of systems that we study might describe biological predator-prey models and other population and social dynamics phenomena [Britton (1986)], [Berryman (1992)]. Such systems have gained significant interest with respect to controllability and one can refer to [Sakthivel et al (2010)] (see also [Crepeau and Prieur (2008)]). For Lyapunov techniques on parabolic systems, one can refer to [Mazenc and Prieur (2011)]. Moreover, observer design for finite-dimensional Lotka-Volterra systems has been addressed in [Bourat and Saif (2013)].

The problem is technically nontrivial, as the distinct diffusivities of the parabolic operator do not allow the observer design to be directly feasible (Property (S) would not be fulfilled for the parabolic operator, see Definition 1.2 in Chapter 1). Following the philosophy of the present chapter, an indirect transformation-based approach is thus adopted. The main contribution here is a solution to this H-GODP, in the presence of distinct diffusivities of the parabolic operator and considering also semilinear dynamics. An appropriate infinite-dimensional and lower triangular state transformation is first performed, in order to map the considered systems into new sets of PDEs, where the parabolic operator is decomposed into a new one with only one diffusivity and a mapping including spatial derivations of the measured state in its domain. As a consequence, this methodology results in requiring, additionally to the output correction terms, the injection of output's spatial derivatives in the high-gain observer dynamics, up to an order depending on the number of distinct diffusivities. The convergence of the observer is then proven for the sup spatial norm, noting a loss of derivatives, since we require system's dynamics to have stronger regularity. Moreover, in the presence of the nonlinearities, we inject sufficiently smooth saturation functions in the observer dynamics, to tackle the absence of globally Lipschitz continuity of the considered class of systems (the nonlinearities are just locally Lipschitz). The present approach extends on the one hand the approach of the previous section to  $2 \times 2$  systems with different type of nonlinearity and on the other hand the one for  $2 \times 2$  nonlinear systems to up to  $3 \times 3$  systems. Notice, however, that the application of the present methodology to systems with more than 3 states is not solved, due to the nonlinearities, which impose extra difficulties.

The sufficient conditions and a solution to the H-GODP are presented in Subsection 3.2.1, where Theorem 3.2 constitutes the main result. The infinite-dimensional transformation that we use to obtain a target system for observer design, along with the proof of Theorem 3.2, are presented in Section 3.2.2. In Section 3.2.3 we apply our methodology to a  $2 \times 2$  Lotka-Volterra system.

### 3.2.1 Problem statement and solvability of the H-GODP

Motivated by applications to information diffusion by multiple sources in social media [Wang et al (2013)], we consider a system of two or three one-dimensional semilinear



parabolic equations with Neumann boundary conditions, written in the following form

$$u_t = Du_{xx} + A(u_1)u + f(u) \text{ in } (0, +\infty) \times (l, L), \quad (3.46a)$$

$$u(0, x) = u^0(x), x \in (l, L), \quad (3.46b)$$

$$u_x(t, l) = 0, u_x(t, L) = 0, t \in (0, +\infty). \quad (3.46c)$$

where  $L > l \geq 0$  define the space domain, and  $u = (u_1 \ \cdots \ u_n)^\top$  is the 2 or 3-dimensional state vector (namely  $n = 2$  or  $n = 3$ ),

$$D = \text{diag}(d_1, \dots, d_n); d_i > 0, i = 1, \dots, n$$

is the constant diffusion matrix, and

$$A(u_1) := \begin{cases} \begin{pmatrix} 0 & a_{12}u_1 \\ 0 & 0 \end{pmatrix}, & \text{if } n = 2, \\ \begin{pmatrix} 0 & a_{12} & 0 \\ 0 & 0 & a_{23} \\ 0 & 0 & 0 \end{pmatrix} (=: A), & \text{if } n = 3 \end{cases}$$

$$f(u) := \begin{cases} \begin{pmatrix} r_1u_1 \left(1 - \frac{u_1}{K_1}\right) \\ r_2u_2 \left(1 - \frac{u_2}{K_2}\right) + a_{21}u_2u_1 \end{pmatrix}, & \text{if } n = 2, \\ \begin{pmatrix} r_1u_1 \left(1 - \frac{u_1}{K_1}\right) \\ r_2u_2 \left(1 - \frac{u_2}{K_2}\right) + a_{21}u_2u_1 \\ r_3u_3 \left(1 - \frac{u_3}{K_3}\right) + a_{31}u_3u_1 + a_{32}u_3u_2 \end{pmatrix}, & \text{if } n = 3 \end{cases}$$

for some constants  $a_{21}, a_{31}, a_{32}$  and positive constants  $a_{12}, a_{23}, r_i, K_i$ . The above system satisfies the general form introduced in Section 1.2 of Chapter 1. Note here that these two types of systems are henceforth parametrized by  $n$ , thus, whenever we say  $n = 2$  or  $n = 3$ , we refer to the above  $2 \times 2$  or  $3 \times 3$  system, respectively. Assume, also, that initial conditions  $u^0(\cdot)$ , with  $u^0(\cdot) \geq 0$ , belong to the Banach space  $\mathcal{X}$ , where

$$\mathcal{X} := C^{q_0}([l, L]; \mathbb{R}) \times C^2([l, L]; \mathbb{R}^{n-1}); \quad (3.47)$$

$$q_0 := \max(2, 2q - 2);$$

$$q := \min \{i : d_i = d_j, \forall j = i, i + 1, \dots, n\},$$

and  $\mathcal{X}$  is equipped with the norm

$$\|u\|_{\mathcal{X}} := \|u_1\|_{q_0} + \|(u_2, u_3)\|_2, \text{ when } n = 3.$$

From the above definition of  $q$ , note that  $q \in \{1, 2, 3\}$  (thus  $q_0 \in \{2, 4\}$ ) and when diffusivities  $d_i$  are distinct, we get  $q = n$ , while  $q = 1$ , when all diffusivities are equal.

We consider a distributed measurement of the first state, written as follows

$$y(t, x) = Cu(t, x), \quad (3.48)$$

where  $C = (1 \ 0)$ , if  $n = 2$ , or  $C = (1 \ 0 \ 0)$ , if  $n = 3$ .

**Remark 3.1.** *Compatibility conditions for the space  $X$  might be understood in the following sense. Consider the more general case for  $n = q = 3$ . Then, for initial condition  $u^0$ , the following conditions are satisfied*

$$u_x^0(b) = 0,$$

$$d_1 \partial_x^3 u_1^0(b) + r_1 \partial_x u_1^0(b) - 2 \frac{r_1}{K_1} u_1^0(b) \partial_x u_1^0(b) + a_{12} u_2(b) = 0, \text{ for } b = l \text{ and } b = L.$$

*For the case  $n = q = 2$ , only the first of the above equations should be satisfied.*

**Remark 3.2.** *The above-considered system (3.46) for  $n = 2$  is a diffusional Lotka-Volterra system. We underline that for the  $3 \times 3$  system, we consider some simpler dynamics with respect to  $A(\cdot)$  constant in this case, and, thus, not satisfying the exact form of classical  $3 \times 3$  Lotka-Volterra systems in cascade form. An extension of the method we adopt here to general diffusional Lotka-Volterra cascade systems of  $n$  species is difficult due to the nonlinearities, and the difficulty augments with the number of distinct diffusivities. Notice that similar difficulties, which increase with the number of characteristic velocities and the types of nonlinearities, appear in the study of controllability for cascade hyperbolic systems with reduced number of controls [Alabau-Boussouira et al (2017)] and come from the notion of algebraic solvability. For such problems, stronger regularity of the solutions is imposed; in a similar way stronger regularity conditions are assumed here as well, in order to solve the problem of observer design with reduced number of observations.*

In [Pierre (2010)] it is shown that systems of the form (3.46), with the present regularity conditions, admit unique local solutions in the classical sense defined therein (see Lemma 1.1) and their regularity can be understood in the usual sense, since the nonlinear source term  $A(u_1)u + f(u)$  is regular enough. Particularly, there exist  $T > 0$  and a unique solution  $u$  to (3.46) on  $[0, T)$ , with  $u \in C^1([0, T) \times [l, L]; \mathbb{R}^n)$  and  $u(t, \cdot) \in C^2([l, L]; \mathbb{R}^n), \forall t \in [0, T)$ . Furthermore, "quasipositivity" (see [Pierre (2010)]) of the nonlinearities implies that these local solutions are non-negative for non-negative initial conditions  $u^0$ . Here, for observer design issues, we introduce an assumption on global existence of nonnegative solutions which are also uniformly bounded, assuming additionally some extra regularity. In addition, we assume that, for the case  $n = 2$ , there exists a lower positive bound for system's first state, so for this case  $n = 2$  only, we assume that the solutions are not just nonnegative, but strictly positive with  $u_1$  lower bounded, contrary to the case  $n = 3$ . The latter presents an analogy to the observer design conditions for finite-dimensional systems, see [Hammouri et al (2002)].

**Assumption 3.4.** *System (3.46) with initial condition  $u^0$  satisfying compatibility conditions for the space  $X$ , admits a unique and uniformly bounded solution  $u$  in  $C^{\max(1, q_0 - 2)}([0, +\infty) \times [l, L]; \mathbb{R}) \times C^1([0, +\infty) \times [l, L]; \mathbb{R}^{n-1})$ , with  $u(t, \cdot) \in X$ , for all  $t \in [0, +\infty)$ . For  $n = 3$ , considering non-negative initial condition  $u^0$ , this solution remains non-negative. In the case where  $n = 2$ , we have  $u^0(\cdot) > 0$  and its corresponding solution  $u_1$  satisfies*

$$\inf_{(t,x) \in [0, +\infty) \times [l, L]} u_1(t, x) > 0.$$

**Remark 3.3.** *It is not unusual to consider global solutions for such systems, by showing that solutions might not blow up on some maximal time interval of existence (see assertions in*

[Pierre (2010)]). Following Theorem 3.5 in [Pierre (2010)] we can deduce that a sufficient condition for global existence when  $n = 3$  is  $a_{21}, a_{31}, a_{32} \leq 0$ . The stronger regularity that we impose on the solutions can be seen as a consequence of the regularity of the initial conditions and the nonlinear source. The method to prove this by finding a priori estimates is standard, see for instance Chap. V [Ladyzenskaja et al (1968)] and Chap. 8.3.2 [Evans (1998)].

We are in a position to propose an indirect observer design, as explained below, which deals with the problem of the presence of distinct diffusivities. Let us first consider a symmetric and positive definite matrix  $P$ , satisfying a Lyapunov equation of the following form for  $(t, x) \in [0, +\infty) \times [l, L]$

$$\text{Sym}(PA(y(t, x))) - C^\top C \leq -\frac{\eta}{2}I_n, \quad (3.49)$$

for some constant  $\eta > 0$ . Such an inequality is always feasible for  $A(y)$  and  $C$  satisfying particular structures as the ones we already assumed, forming an observability canonical form as in the finite dimensions. More explicitly, for systems with two states ( $n = 2$ ), feasibility of such an inequality requires additionally to the algebraic structure,  $y = u_1$  to satisfy Assumption 3.4 (having upper and lower bounds). To prove the existence of such a  $P$ , one can refer to [Hammouri et al (2002)] for the case of  $A(\cdot)$  satisfying similar conditions as in the present case. When  $n = 3$  such an inequality is feasible, because of the observability of the pair  $A, C$ . Furthermore, let us note that such a  $P$  is never diagonal. In a possible direct observer design, observer asymptotic convergence would be proven by choosing  $P$  as a Lyapunov matrix, simultaneously commuting with the diffusion matrix  $D$ . This commutative property would allow the integration by parts in the Lyapunov analysis and can be only satisfied when  $D$  is a scalar matrix as in Property (S), see Chapter 1, which is not the general case here. For this reason, we propose a transformation into a new system where the parabolic operator is decomposed into a sum of a) a new differential operator, satisfying the previously mentioned commutativity, b) a differential operator (and possibly nonlinear) including only the first measured state in its domain and c) a bilinear mapping between a function of the unmeasured state and a differential operator, including only the first state in its domain. Moreover, this transformation is assumed to preserve the triangular structure of  $A$  and  $f$ . That kind of transformation is, therefore, infinite-dimensional and lower triangular. More precisely, we show the existence of a linear bounded injective transformation  $\mathcal{T} : (X, \|\cdot\|_X) \rightarrow (X, \|\cdot\|_X)$ , with bounded inverse, which maps initial system into a target system  $v$ , as follows:

$$\begin{aligned} v &= \mathcal{T}u; \\ v_1 &= u_1. \end{aligned} \quad (3.50)$$

Such an infinite-dimensional transformation always exists for the considered systems with  $n = 2$  or  $3$ , as shown in the sequel, although existence of a transformation (possibly nonlinear) for more general  $n \times n$  Lotka-Volterra systems with distinct diffusivities remains open (see Remark 3.2). The nature of this transformation indicates the need for stronger regularity of system's solutions as in Assumption 3.4 and this requirement is linked to the sufficient conditions for controllability problems for cascade systems with reduced number of controls [Alabau-Boussouira et al (2017)] (see also Remark 3.2).

The target system (T) of PDEs, which is suitable for observer design, satisfies the following equations in  $(0, +\infty) \times (l, L)$ :

$$(T) \begin{cases} v_t(t, x) = d_n v_{xx}(t, x) + A(v_1(t, x))v(t, x) \\ \quad + f(v(t, x)) + \mathcal{M}_1[v_1(t)](x) + \mathcal{M}_2[v_1(t)](x)v(t, x) \\ v_x(l) = \mathcal{K}v_1(l), v_x(L) = \mathcal{K}v_1(L), \\ y_v(t, x) = y(t, x) = Cv(t, x), \end{cases}$$

with initial condition  $v(0, x) = v^0(x) = \mathcal{T}u^0(x)$ , where  $\mathcal{M}_1 : C^{q_0}([l, L]; \mathbb{R}) \rightarrow C^0([l, L]; \mathbb{R}^n)$ ,  $\mathcal{M}_2 : C^{q_0}([l, L]; \mathbb{R}) \rightarrow C^{\max(0, 2q-4)}([l, L]; \mathbb{R}^{n \times n})$ ,  $\mathcal{K} : C^{q_0}([l, L]; \mathbb{R}) \rightarrow \mathbb{R}^n$  are nonlinear differential operators acting on  $v_1$ , to be determined in the sequel, depending on the choice of  $\mathcal{T}$ , and  $y_v$  is target system's output, which remains equal to original system's output  $y$ . The existence of such a transformation  $\mathcal{T}$  is shown in the following subsection.

We are now in a position to propose a high-gain observer for target system (T).

### High-Gain Observer

The observer is given by the following equations in  $(0, +\infty) \times (l, L)$

$$\begin{aligned} \hat{v}_t(t, x) = & d_n \hat{v}_{xx}(t, x) + A(y(t, x))\hat{v}(t, x) + \Theta P^{-1} C^\top (y(t, x) - C\hat{v}(t, x)) + (f \circ s_\delta)(\hat{v}(t, x)) \\ & + \mathcal{M}_1[y(t)](x) + \mathcal{M}_2[y(t)](x)\hat{v}(t, x), \end{aligned} \quad (3.52a)$$

$$\hat{v}_x(l) = \mathcal{K}y(l), \hat{v}_x(L) = \mathcal{K}y(L), \quad (3.52b)$$

with initial condition  $\hat{v}(0, x) = \hat{v}^0(x)$  (for some function  $\hat{v}^0$  in  $\mathcal{X}$ , satisfying compatibility conditions for this space). Also,

$$\Theta := \text{diag}(\theta, \theta^2, \dots, \theta^n), \quad (3.53)$$

where  $\theta > 1$  is the candidate high-gain constant of the observer, to be selected sufficiently large and precisely determined in the sequel, and  $P$  is symmetric and positive definite, satisfying (3.49) for some  $\eta > 0$ . We also injected in the nonlinear dynamics a function  $\mathbb{R}^n \ni \hat{v} \mapsto s_\delta(\hat{v}) = (s_{\delta,1}(\hat{v}_1), \dots, s_{\delta,n}(\hat{v}_n))$ , parametrized by  $\delta > 0$ , which is a global bound of the solution  $v$ , i.e.,

$$\delta \in \{\delta_0 > 0 : \|v(t, \cdot)\|_X \leq \delta_0, \forall t \geq 0\}.$$

The previous set, in which  $\delta$  belongs, is nonempty by Assumption 3.4, in conjunction with (3.50) and boundedness of  $\mathcal{T}$ . There is no need, however, to inject this function in the appearing linear terms, since they are already globally Lipschitz, although this is not explicitly written here. Alternatively, one can avoid injecting  $s_{\delta,1}(\hat{v}_1)$  in observer's nonlinear dynamics, but use  $y$  instead and we adopt this simplification in the example of the last subsection. We assume that  $s_\delta$  is of class  $C^2$  and satisfies the following properties:

1) For every  $\delta > 0$  and  $w, \hat{w}$  in  $\mathbb{R}$ , such that  $|w| \leq \delta$ , there exists  $\omega_\delta > 0$ , such that the following inequality is satisfied:

$$|s_{\delta,i}(\hat{w}) - w| \leq \omega_\delta |\hat{w} - w|, i = 1, \dots, n. \quad (3.54a)$$

2) There exists  $m_\delta > 0$ , such that for every  $\hat{w}$  in  $\mathbb{R}$ ,

$$\left| \frac{d^j}{d\hat{w}^j} s_{\delta,i}(\hat{w}) \right| \leq m_\delta, j = 0, 1, 2, i = 1, \dots, n. \quad (3.54b)$$

We are now in a position to present our main result on the convergence of the proposed high-gain observer.

**Theorem 3.2** (Solvability of the H-GODP). *Consider system (3.46) with output (3.48) and suppose that Assumption 3.4 holds. Then, there exists a linear bounded injective operator  $\mathcal{T}$  with bounded inverse, transforming system into system (T). Let also  $P$  be a symmetric and positive definite matrix, satisfying (3.49) for some  $\eta > 0$ . Then, for  $\theta$  large enough,  $\mathcal{T}^{-1}\hat{v}$  provides an estimate for the solution  $u$  to (3.46), where  $\hat{v}$  is the unique solution to observer system (3.52). More precisely, for every  $\kappa > 0$ , there exists  $\theta_0 \geq 1$ , such that for every  $\theta \geq \theta_0$ , the following holds for all  $t \geq 0$ :*

$$\|\mathcal{T}^{-1}\hat{v}(t, \cdot) - u(t, \cdot)\|_\infty \leq ce^{-\kappa t} \|\mathcal{T}^{-1}\hat{v}^0(\cdot) - u^0(\cdot)\|_X, \quad (3.55)$$

for some  $c > 0$  polynomial in  $\theta$ .

### 3.2.2 Observer Convergence Proof

In this subsection, we prove Theorem 3.2.

First, we show the existence of  $\mathcal{T}$  of the form (3.50) mapping (3.46)-(3.48) into target system (T) of the previous subsection. Let us choose

$$\mathcal{T} := \begin{cases} I_2, & n = 2, \\ \begin{pmatrix} 1 & 0 & 0 \\ b\partial_x^2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & n = 3 \end{cases} ;$$

where  $b := \frac{d_3 - d_2}{a_{12}}$ ,

with  $\mathcal{T}$  obviously bounded, invertible with bounded inverse from  $X$  to  $X$ .

Observe in the above mapping that differentiation is only required when  $d_2 \neq d_3$ . This implies that indicator  $q$  is equal to 3. As we mentioned in the introduction of the present chapter, the order of the essential regularity depends on the indicator  $q$ . For instance, for  $n = q = 3$ , we would need a regularity of order 4 for  $u_1$  and of order 2 for  $u_2$  and  $u_3$ .

Then, applying this transformation to the initial system, we obtain system (T) with

$$\mathcal{M}_1[v_1] := \begin{cases} \begin{pmatrix} (d_1 - d_3)\partial_x^2 v_1 \\ 0 \end{pmatrix}, & n = 2, \\ \begin{pmatrix} (d_1 - d_3 - a_{12}b)\partial_x^2 v_1 \\ [b(d_1 - d_3)\partial_x^4 + b(r_1 - r_2)\partial_x^2] v_1 - \\ -b\left(a_{21} + 2\frac{r_1}{K_1}\right)u_1\partial_x^2 v_1 - \\ -2b\frac{r_1}{K_1}(\partial_x v_1)^2 - b^2\frac{r_2}{K_2}(\partial_x^2 v_1)^2 \\ 0 \end{pmatrix}, & n = 3, \end{cases}$$

$$\mathcal{M}_2[v_1] := \begin{cases} 0, & n = 2, \\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2b\frac{r_2}{K_2}\partial_x^2 v_1 & 0 \\ 0 & 0 & -ba_{32}\partial_x^2 v_1 \end{pmatrix}, & n = 3, \end{cases}$$

$$\mathcal{K} = 0, \text{ if } n = 2,$$

$$\mathcal{K} = \begin{pmatrix} 0 \\ b\partial_x^2 \\ 0 \end{pmatrix}, \text{ if } n = 3.$$

We note here some properties, that will be invoked in the well-posedness of the observer and its convergence proof. First, notice that by virtue of Assumption 3.4 on boundedness of the system's solutions in  $\|\cdot\|_x$ , boundedness of mapping  $\mathcal{T}$ , and the dynamics of target system (T),  $v_t$ ,  $v_{xx}$ , and  $v_{txx}$  are uniformly bounded in the sup-norm. Now, due to continuity of the nonlinear operators  $\mathcal{M}_1, \mathcal{M}_2$ , we get

$$\sup_{\|y\|_{q_0} \leq \delta} |\mathcal{M}_1[y]| < +\infty,$$

$$\sigma_\delta := \sup_{\|y\|_{q_0} \leq \delta} |\mathcal{M}_2[y]|, \sigma'_\delta := \sup_{\|v\|_x \leq \delta} |\mathcal{M}_2[Cv_t]| < +\infty. \quad (3.56)$$

Furthermore, in view of boundedness of solutions (Assumption 3.4), dynamics of  $v$ , and properties (3.54), for any  $\delta > 0$ , there exist constants  $L_\delta, L'_\delta > 0$ , such that for every  $v, \hat{v}$  in  $X$ , with  $\|v\|_x \leq \delta$ , the following inequalities are satisfied for all  $x$  in  $[l, L]$

$$|(f \circ s_\delta)(\hat{v}(x)) - f(v(x))| \leq L_\delta |\hat{v}(x) - v(x)|, \quad (3.57a)$$

$$|D(f \circ s_\delta)(\hat{v}(x)) \cdot \hat{v}_t(x) - Df(v(x)) \cdot v_t(x)| \leq L'_\delta (|\hat{v}(x) - v(x)| + |\hat{v}_t(x) - v_t(x)|). \quad (3.57b)$$

Now, to show well-posedness of the observer system, we recall general existence results for systems written in the form  $\dot{z}(t) = \mathcal{A}z(t) + F(y(t), z(t))$ , see for instance [Brezis (1983)] (see also Example 3.6, p. 75, [Henry (1981)]). Here  $\mathcal{A}$  is the parabolic operator with domain  $\mathcal{D}(\mathcal{A}) = \{z \in C^2([l, L]; \mathbb{R}^n), z'(l) = \mathcal{K}y(l), z'(L) = \mathcal{K}y(L)\}$  and  $F(y(t), z(t)) := A(y(t))z(t) + \mathcal{M}_2[y(t)]z(t) + (f \circ s_\delta)(z(t)) + \Theta P^{-1}C^\top(y(t) - Cz(t)) + \mathcal{M}_1[y(t)]$  is the nonlinear source term of the observer, which is uniformly Lipschitz continuous with respect to  $z$  when system's output  $y$  satisfies Assumption 3.4. Then, the existence of a unique local solution  $z$  in  $C^1([0, T_1] \times [l, L]; \mathbb{R}^n)$  with  $z(t) \in C^2([l, L]; \mathbb{R}^n), \forall t \in [0, T_1]$  for some  $T_1 > 0$  is

guaranteed. Following the continuous differentiability of the nonlinear source, existence in space  $X$  for  $n = q = 3$  can be deduced by considering the initial value problem for  $\dot{z}_1$ , which can be written as  $\frac{d}{dt}\dot{z}_1(t) = \mathcal{A}_1\dot{z}_1(t) + G(t, y(t), \dot{y}(t), z(t), \dot{z}(t))$ ;  $\mathcal{A}_1 w := d_n w''$ , with  $\mathcal{D}(\mathcal{A}_1) = \{\dot{z}_1 \in C^2((l, L); \mathbb{R}), \dot{z}'_1(l) = CK\dot{y}(l), \dot{z}'_1(L) = CK\dot{y}(L)\}$  and initial conditions  $\dot{z}_1(0)$  satisfy compatibility conditions of order 1.  $G$  is uniformly Lipschitz with respect to  $\dot{z}$ , when  $y$  satisfies Assumption 3.4 and, thus, system admits a unique solution  $\dot{z}_1$  on  $[0, T_2)$  for some  $T_2 > 0$ , belonging to  $C^1([0, T_2) \times [l, L]; \mathbb{R})$  with  $\dot{z}_1(t) \in C^2([l, L]; \mathbb{R}), \forall t \in [0, T_2)$ . Concluding, observer system admits a unique solution on  $[0, T^*)$ , with  $T^* := \min(T_1, T_2)$ , belonging to  $C^{\max(1, q_0 - 2)}([0, T^*) \times [l, L]; \mathbb{R}) \times C^1([0, T^*) \times [l, L]; \mathbb{R}^{n-1})$ , with  $\hat{u}(t, \cdot) \in X$ , for all  $t \in [0, T^*)$ .

Let us now proceed to the stability proof.

We define a scaled observer error by

$$\varepsilon := \Theta^{-1}(\hat{v} - v),$$

for which we derive the following parabolic equations in  $(0, +\infty) \times (l, L)$ :

$$\begin{aligned} \varepsilon_t(t, x) = & d_n \varepsilon_{xx}(t, x) + \theta \left( A(y(t, x)) - P^{-1}C^\top C \right) \varepsilon(t, x) + \mathcal{M}_2[y(t)](x) \varepsilon(t, x) \\ & + \Theta^{-1}((f \circ s_\delta)(\hat{v}(x)) - f(v(x))), \end{aligned} \quad (3.58a)$$

$$\varepsilon_x(l) = \varepsilon_x(L) = 0. \quad (3.58b)$$

Furthermore, for systems with  $n = q = 3$ , by temporarily assuming some extra regularity for  $v_2, \hat{v}_2, v_3, \hat{v}_3$ , which will be dropped by density arguments, we get the following parabolic equations for  $\varepsilon_t$ :

$$\begin{aligned} \varepsilon_{tt}(t, x) = & d_n \varepsilon_{ttx}(t, x) + \theta \left( A - P^{-1}C^\top C \right) \varepsilon_t(t, x) + \mathcal{M}_2[y_t(t)](x) \varepsilon(t, x) + \mathcal{M}_2[y(t)](x) \varepsilon_t(t, x) \\ & + \Theta^{-1}(D(f \circ s_\delta)(\hat{v}(x)) \cdot \hat{v}_t(x) - Df(v(x)) \cdot v_t(x)), \end{aligned} \quad (3.59a)$$

$$\varepsilon_{tx}(l) = \varepsilon_{tx}(L) = 0. \quad (3.59b)$$

To prove the error's exponential stability with respect to its origin, we adopt a Lyapunov-based approach. Let us define a Lyapunov functional  $\mathcal{W}_p : X \rightarrow \mathbb{R}$  by

$$\begin{aligned} \mathcal{W}_p[\varepsilon] := & \left( \int_l^L \mathcal{G}_p[\varepsilon](x) dx \right)^{1/p}; \\ \mathcal{G}_p[\varepsilon](x) := & \left( \sum_{i=0}^{\bar{q}} \partial_t^i \varepsilon^\top(x) P \partial_t^i \varepsilon(x) \right)^p, \end{aligned} \quad (3.60)$$

with  $p$  in  $\mathbb{N}$  and  $\bar{q} := \frac{(q-1)(q-2)}{2}$ . Denoting  $W_p(t) := \mathcal{W}_p[\varepsilon(t)], G_p(t) := \mathcal{G}_p[\varepsilon(t)], t \in [0, T^*)$ , we calculate the time-derivative  $\dot{W}_p$  along the solutions  $\varepsilon, \varepsilon_t$  to the error equations (3.91) as follows:

$$\begin{aligned} \dot{W}_p = & \frac{1}{p} W_p^{1-p} \int_l^L p G_{p-1}(x) \left( \varepsilon_t(x)^\top P \varepsilon(x) + \varepsilon(x)^\top P \varepsilon_t(x) \right. \\ & \left. + \bar{q} \varepsilon_{tt}(x)^\top P \varepsilon_t(x) + \bar{q} \varepsilon_t(x)^\top P \varepsilon_{tt}(x) \right) dx. \end{aligned}$$

After substituting (3.58), (3.59) and performing an integration by parts (which can be possible, since the observer error in the target coordinates satisfies Property (S) of Chapter 1),  $\dot{W}_p$  can be written as follows:

$$\dot{W}_p = W_p^{1-p} \left( \frac{1}{p} T_{1,p} + T_{2,p} + T_{3,p} \right), \quad (3.61)$$

where

$$\begin{aligned} T_{1,p} &:= d_n^p [\partial_x G_p]_l^L, \\ T_{2,p} &:= \int_l^L G_{p-1} \left[ 2\varepsilon^\top P \Theta^{-1} ((f \circ s_\delta)(\hat{v}) - f(u)) + 2\bar{q} \varepsilon_t^\top P \Theta^{-1} (D(f \circ s_\delta)(\hat{v}) \hat{v}_t - Df(v) v_t) \right. \\ &\quad + \varepsilon^\top (P \mathcal{M}_2[y] + \mathcal{M}_2[y] P) \varepsilon + \bar{q} \varepsilon_t^\top (P \mathcal{M}_2[y] + \mathcal{M}_2[y] P) \varepsilon_t \\ &\quad \left. + \frac{\bar{q}}{2} \varepsilon_t^\top P \mathcal{M}_2[y_t] \varepsilon + \frac{\bar{q}}{2} \varepsilon^\top \mathcal{M}_2[y_t] P \varepsilon_t - 2\varepsilon_x^\top P \varepsilon_x - 2\bar{q} \varepsilon_{tx}^\top P \varepsilon_{tx} \right] dx, \\ T_{3,p} &:= 2\theta \int_l^L G_{p-1} \left( \varepsilon^\top [\text{Sym}(PA(y)) - C^\top C] \varepsilon + \bar{q} \varepsilon_t^\top [\text{Sym}(PA) - C^\top C] \varepsilon_t \right) dx. \end{aligned}$$

By use of boundary conditions (3.59b) and also (3.49), (3.56), and (3.57a), we get

$$\begin{aligned} T_{1,p} &= 0, \\ T_{2,p} &\leq \sigma W_p^p, \\ T_{3,p} &\leq -\theta \frac{\eta}{|P|} W_p^p, \end{aligned}$$

with

$$\sigma := 2 \frac{|P|}{\underline{\text{eig}}(P)} (\sqrt{n} (L_\delta + \bar{q} L'_\delta) + \sigma_\delta (\bar{q} + 1) + \bar{q} \sigma'_\delta).$$

Now, selecting high-gain  $\theta > \max\left(1, \frac{|P|}{\eta} \sigma\right)$ , we get by (3.101) that there exists a  $\kappa > 0$ , such that

$$\dot{W}_p(t) \leq -2\kappa W_p(t), t \in [0, T^*]. \quad (3.63)$$

By comparison lemma, we get the following estimate

$$W_p(t) \leq e^{-2\kappa t} W_p(0), t \in [0, T^*].$$

By this estimate, we deduce that solutions in  $X$  to the observer equations (3.58) exist globally in time and, therefore, the previous inequality holds for all  $t \geq 0$ . We invoke, next, the following property:

$$\lim_{p \rightarrow \infty} W_p = \sum_{i=0}^{\bar{q}} \|\partial_t^i \varepsilon^\top(\cdot) P \partial_t^i \varepsilon(\cdot)\|_\infty,$$

holding for continuous  $\varepsilon, \varepsilon_t$  and also the following inequality which is derived by error equations (3.58):

$$a_1 (\|\varepsilon_{xx}\|_\infty - \|\varepsilon\|_\infty) \leq \|\varepsilon_t\|_\infty \leq a_2 (\|\varepsilon_{xx}\|_\infty + \|\varepsilon\|_\infty),$$



for some positive constants  $a_1$  and  $a_2$  that we can easily calculate. Following the above arguments, we can calculate a positive  $\bar{c}$  (polynomial in  $\theta$ ), for which the following inequality holds:

$$\sum_{i=0}^{\bar{q}} \|\partial_x^{2i} \hat{v}(t, \cdot) - \partial_x^{2i} v(t, \cdot)\|_\infty \leq \bar{c} e^{-\kappa t} \sum_{i=0}^{\bar{q}} \|\partial_x^{2i} \hat{v}^0(\cdot) - \partial_x^{2i} v^0(\cdot)\|_\infty, t \geq 0.$$

Now, from continuous embedding of  $\mathcal{X}$  in  $C^2([l, L]; \mathbb{R}^n)$  and boundedness of  $\mathcal{T}$ , from the continuous embedding of  $C^2([l, L]; \mathbb{R}^n)$  in  $C^0([l, L]; \mathbb{R}^n)$  and from boundedness of the continuous extension of  $\mathcal{T}^{-1}$  on  $C^0([l, L]; \mathbb{R}^n)$ , we deduce stability inequality (3.55).

The proof of Theorem 3.2 is complete.

### 3.2.3 Simulation for a Lotka-Volterra system

In this subsection, we apply the proposed high-gain observer design to a  $2 \times 2$  Lotka-Volterra system (3.46) as an illustration, with  $l = 0, L = 10$ , diffusivities  $d_1 = 2, d_2 = 1$  and  $a_{12} = 0.2, a_{21} = -0.2, K_1 = 15, K_2 = 0.1, r_1 = 0.5, r_2 = 0.01$ . We choose initial conditions  $u_1^0(x) = \cos(\pi x/10) + 6, u_2^0(x) = -3 \cos(\pi x/10) + 9$ , such that Assumption 3.4 is satisfied with a global bound for the solution, which can be known a priori, to be  $\delta = 20$ . The corresponding output is represented in Figure 3.4. System's unknown second state is represented in Figure 3.5.

The proposed high-gain observer has the form (3.70). Exploiting the a priori known bounds of system's output, we choose  $\eta = 0.5, P = \begin{pmatrix} 13.4 & 0.667 \\ 0.667 & 0.14 \end{pmatrix}$ , for which (4.25) holds. We next apply Theorem 3.3, with observer given by (3.70) and  $\theta = 4$ . As expected, the convergence of observer state to the unknown state  $u$  is guaranteed.

In figures 3.6, 3.7 we see the observation errors for each of the states  $u_1, u_2$ , after choosing arbitrary observer's initial conditions, satisfying also observer's boundary conditions.

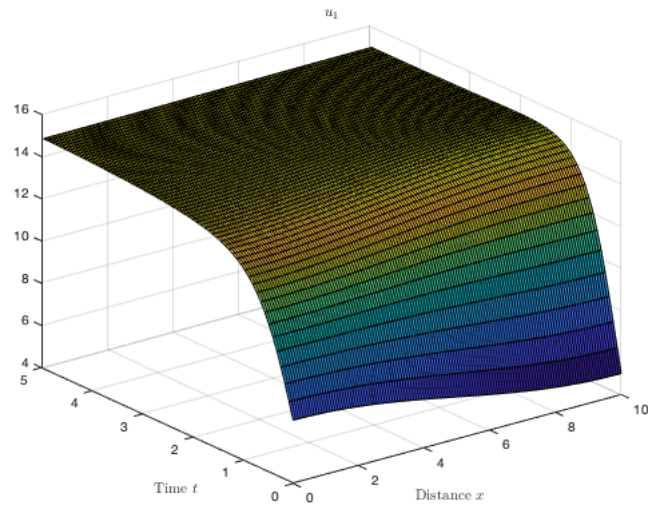
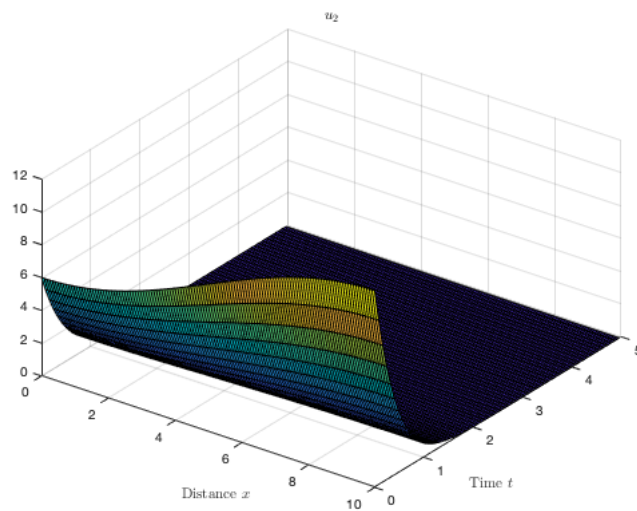
Figure 3.4: Time and space evolution of measured state  $u_1$ 

Figure 3.5: Time and space evolution of system's second state

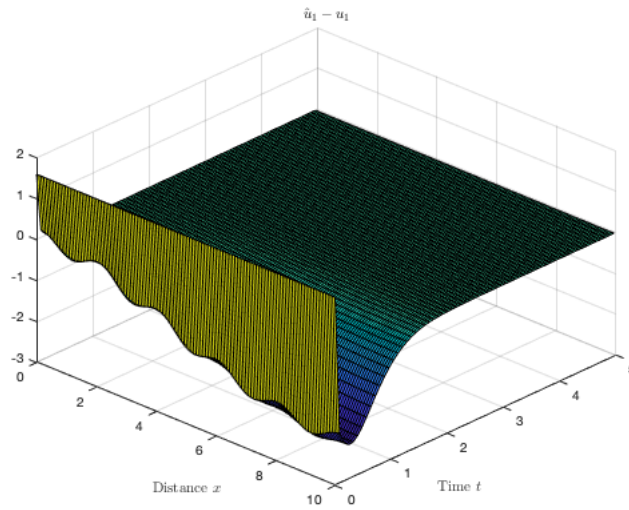


Figure 3.6: Time and space evolution of the observer error of the first state

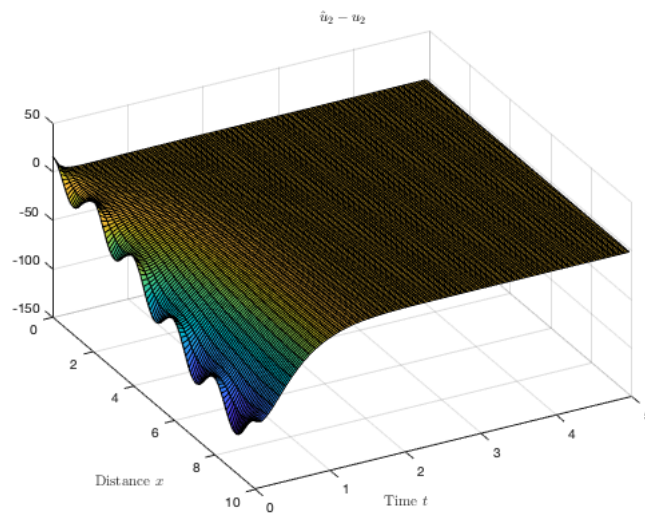


Figure 3.7: Time and space evolution of the observer error of the second state

### 3.3 A class of $n \times n$ linear heterodirectional hyperbolic systems

This final section suggests a solution to the H-GODP for a case of a nonuniform inhomogeneous linear hyperbolic system, which does not satisfy Property (S) for its hyperbolic operator (see Definition 1.2 in Chapter 1), following some of the results in [Kitsos et al (2020a)]. In view of the previous sections, one would expect that the proposed transformation-based approach would solve the problem for general systems of any number of states and for any number of distinct characteristic velocities or diffusivities, depending on the type of PDEs. It turns out that the complexity of the solvability of the H-GODP increases with the number of distinct velocities/diffusivities for hyperbolic/parabolic systems. Furthermore, possible nonlinearities, both in the coefficients of the differential operators and the source terms do not allow the direct extension of this approach to systems with large number of characteristic velocities/diffusivities. In this context,  $n \times n$  hyperbolic systems, with the largest possible number of distinct velocities are considered (equal to  $n$ ) nonuniform inhomogeneous, but also linear, in view of the fact that nonlinear systems with more than 3 states seem to not achieve to be directly handled by this transformation-based approach.

The class of heterodirectional hyperbolic systems that we study, written in an appropriate canonical form, can be found in various situations, like plug flow chemical reactors, where, by measuring the jacket temperature, we would expect to estimate the concentrations of masses of chemicals, see Chapter 1.2. Stability and controllability for linear hyperbolic systems have been widely studied, see for instance [Bastin and Coron (2016)], [Prieur and Winkin (2018)], [Espitia et al (2017)] employing Lyapunov and operator theoretic strategies.

The main contribution here is the proof of solvability of the H-GODP for the considered class of systems. Following the philosophy of this chapter, an important idea of this section is to perform an appropriate infinite - dimensional and lower triangular transformation to overcome the absence of Property (S) yet needed in the Lyapunov stability analysis and due to distinct velocities. The original hyperbolic system is mapped into a target system of PDEs. This infinite-dimensional transformation solves a generalized Sylvester operator equation and its computation is provided via a proposed algorithm. The proposed methodology results in requiring, additionally to the output correction terms, the injection of output's spatial derivatives up to order  $q - 1$  in the high-gain observer dynamics, where  $q$  is an indicator of the number of different velocities. Sufficient conditions, both on strong regularity of the dynamics and space periodicity on the boundary conditions, allow the extension to system with any number of states. These conditions are related to [Alabau-Boussouira et al (2017)], which studies internal controllability of hyperbolic systems with reduced number of controls. Similarly to this work, the analysis experiences the loss of derivatives phenomenon, since we require stronger regularity for system's solutions, in order to obtain an asymptotic observer convergence in a norm of a space with lower regularity. Last but not least, it can be noticed that, even though the system is linear, the use of high gain is instrumental, as a result of the considered domain of the hyperbolic operator, namely the generality of the boundary conditions. The convergence of the observer is then proven for an appropriate spatial norm.

The Section is organized as follows. The sufficient conditions for solvability and a solution to the H-GODP are presented in Subsection 3.3.1, where Theorem 3.3 constitutes the main result. The infinite-dimensional transformation and the derivation of the target system, for which we design the observer, are presented in Subsection 3.3.2. In Subsection 3.3.3 Theorem 3.3 is proven and, finally, in Subsection 3.3.4, we illustrate the methodology via a  $3 \times 3$  system, with application to plug flow chemical reactors.

### 3.3.1 Problem statement and main result

We are concerned with one-dimensional, first-order linear hyperbolic systems of balance laws, described by the following equations on a strip  $\Sigma := [0, +\infty) \times [0, L]$

$$\xi_t(t, x) + \Lambda(x)\xi_x(t, x) = M(x)\xi(t, x), \quad (3.64a)$$

where  $\xi = (\xi_1 \ \cdots \ \xi_n)^\top$  is the state.

Consider also a distributed measurement of the form

$$y(t, x) = C\xi(t, x), \quad (3.64b)$$

where  $y$  is a mapping from  $\Sigma$  to  $\mathbb{R}$  and

$$\Lambda(x) = \text{diag}(\lambda_1(x), \dots, \lambda_n(x))$$

contains  $m$  characteristic velocities  $\lambda_i(x)$ , which are positive and  $n - m$  negative for all  $x$  in  $[0, L]$ . We have the following algebraic structures for the involved matrices

$$M(x) = \begin{pmatrix} m_{1,1}(x) & m_{1,2}(x) & 0 & \cdots & 0 \\ & \ddots & \ddots & & \vdots \\ \vdots & & & & m_{n-1,n}(x) \\ m_{n,1}(x) & \cdots & & & m_{n,n}(x) \end{pmatrix},$$

$$C = (1 \ 0 \ \cdots \ 0).$$

Note that the above system satisfies the general form introduced in Section 1.2 of Chapter 1.

To give appropriate boundary conditions, let us first define a permutation matrix denoted by  $I_\pi$ , which re-orders components of  $\xi$  according to the signs of the related characteristic velocities, namely, putting the  $m$  elements corresponding to positive velocities in  $\xi^+$  and the  $n - m$  to negative velocities in  $\xi^-$ . In addition, if  $\lambda_1(\cdot) > 0$ ,  $\xi^+$  is put ahead, and if  $\lambda_1(\cdot) < 0$ ,  $\xi^-$  is put in the first place, keeping in both cases  $\xi_1$  in the upper first place. More explicitly,

$$I_\pi \xi := \begin{pmatrix} \xi^+ \\ \xi^- \end{pmatrix}; \xi_1^+ = \xi_1, \quad \text{when } \lambda_1(\cdot) > 0,$$

$$I_\pi \xi := \begin{pmatrix} \xi^- \\ \xi^+ \end{pmatrix}; \xi_1^- = \xi_1, \quad \text{when } \lambda_1(\cdot) < 0,$$

where  $\xi^+ \in \mathbb{R}^m, \xi^- \in \mathbb{R}^{n-m}$ .

We consider initial conditions

$$\xi(0, x) =: \xi^0(x), x \in [0, L] \quad (3.65a)$$

and boundary conditions (distinguishing cases  $\lambda_1(\cdot) > 0$  and  $\lambda_1(\cdot) < 0$ ), written in the form

$$\begin{pmatrix} \mathcal{B}_0^{1,m} & \mathcal{B}_L^{m+1,n} \end{pmatrix} I_\pi \xi = K \begin{pmatrix} \mathcal{B}_L^{1,m} & \mathcal{B}_0^{m+1,n} \end{pmatrix} I_\pi \xi, \text{ when } \lambda_1(\cdot) > 0, \quad (3.65b)$$

$$\begin{pmatrix} \mathcal{B}_L^{1,n-m} & \mathcal{B}_0^{n-m+1,n} \end{pmatrix} I_\pi \xi = K \begin{pmatrix} \mathcal{B}_0^{1,n-m} & \mathcal{B}_L^{n-m+1,n} \end{pmatrix} I_\pi \xi, \text{ when } \lambda_1(\cdot) < 0, \quad (3.65c)$$

where  $\mathcal{B}_l^{i,j}$ , for  $l = 0$  or  $l = L$ , performs the following trace operation

$$\mathcal{B}_l^{i,j}(\xi_i, \dots, \xi_j)^\top := (\xi_i(l), \dots, \xi_j(l))^\top, 1 \leq i \leq j,$$

and  $K$  is a matrix of the form

$$K = \frac{1 + \operatorname{sgn}(\lambda_1(x))}{2} \begin{pmatrix} K_{00} & K_{01} \\ K_{10} & K_{11} \end{pmatrix} + \frac{1 - \operatorname{sgn}(\lambda_1(x))}{2} \begin{pmatrix} K_{11} & K_{10} \\ K_{01} & K_{00} \end{pmatrix}; \quad (3.66)$$

$$K_{00} \in \mathbb{R}^{m \times m}, K_{01} \in \mathbb{R}^{m \times (n-m)}, K_{10} \in \mathbb{R}^{(n-m) \times m}, K_{11} \in \mathbb{R}^{(n-m) \times (n-m)}.$$

At this point, let us define a, roughly speaking, “measure of ordered strict hyperbolicity” as follows:

$$q := \min \{i : \lambda_i \equiv \lambda_j, \forall j = i, i+1, \dots, n\},$$

where we used the equivalence relation  $\lambda_i \equiv \lambda_j \Leftrightarrow \lambda_i(x) = \lambda_j(x), \forall x \in [0, L]$ . By this definition, we have  $q \in [1, n]$ , and in case of a strictly hyperbolic system, we have  $q = n$ . The case where  $q = 1$  (one identical characteristic velocity) as a particular case of the general formulation here, has been already addressed in Chapter 2. We define also

$$q_i := \max(1, 2q - 3 - i), i = 1, \dots, n-1, q_n := q_{n-1}.$$

We are now in position to present the main assumptions.

**Assumption 3.5.** *Functions  $\lambda_i$  and  $m_{i,j}$  are in  $C^{q_1}[0, L]$ . Initial conditions  $\xi^0$  in  $C^{q_1}([0, L]; \mathbb{R}^n)$  satisfy compatibility conditions of order  $q_1$  (see [Bastin and Coron (2016)], Chapter 4.5.2 for definition of compatibility conditions of order  $k$ ).*

The previous assumption leads to higher-order regularity of system’s solutions and, thus, of system’s output  $y$ . The following fact results from classical arguments borrowed from the theory of linear hyperbolic systems (method of characteristics) and fixed-point theory, combined with the manipulation of the extra regularity that is required (see [Bastin and Coron (2016)], App. A, Chap. 4.5 and references therein).

**Fact 3.1.** *Under Assumption 3.5, there exists a unique global solution to (3.64a), (3.65) in  $C^{q_1}([0, +\infty) \times [0, L]; \mathbb{R}^n)$ .*

The nature of the following assumption is revealed in [Kitsos et al (2018)], where a triangular form is introduced for the case of quasi-linear hyperbolic systems and an adaptation to the present case suggests that this assumption meets the requirements of the aforementioned work. This assumption allows us to obtain a target system, that we introduce later, and is also a sufficient condition for the observer design.

**Assumption 3.6.** *The following condition is satisfied for all  $x$  in  $[0, L]$*

$$m_{1,2}(x), m_{2,3}(x), \dots, m_{n-1,n}(x) \neq 0.$$

Given the previous assumption, system (3.64a), (3.64b) satisfies some triangular structure, which presents an analogy to the finite-dimensional case (see [Khalil (2017)]).

Last, we make an assumption having two alternative versions, which in general do not need to be satisfied simultaneously. Both of them are sufficient conditions to lead to an appropriate target system. The first version (A) imposes a space  $L$ -periodicity to system (3.64a), (3.65) with  $n > 3$ . The second version (B) of this assumption imposes limitations on its characteristic velocities.

**Assumption 3.7.** *One of the two following versions is true for systems larger than  $3 \times 3$ .*

(A) *Whenever  $n > 3$ , the following space  $L$ -periodicity holds*

$$\begin{aligned} \xi(t, 0) &= \xi(t, L), \forall t \in [0, +\infty) \text{ (in other words, } K = I_n \text{ in (3.65b))} \\ \text{and } \partial_x^j \lambda_i(0) &= \partial_x^j \lambda_i(L), \partial_x^j M(0) = \partial_x^j M(L), i = 1, \dots, n, j = 0, \dots, q_1. \end{aligned}$$

(B) *The characteristic velocities satisfy the following relations on the boundaries for  $n > 3$*

$$\begin{aligned} \lambda_3(0) &= \dots = \lambda_n(0), \lambda_3(L) = \dots = \lambda_n(L), \\ \lambda'_4(0) &= \dots = \lambda'_n(0), \lambda'_4(L) = \dots = \lambda'_n(L). \end{aligned}$$

Also,  $K_{11}$  is invertible when  $\lambda_n(\cdot) > 0$  and  $K_{00}$  is invertible, when  $\lambda_n(\cdot) < 0$ , where  $K_{00}, K_{11}$  are defined in (3.66).

In case Assumption 3.7 (A) is fulfilled, we simultaneously derive the following periodicity property for all spatial derivatives up to the maximal order of regularity, as a consequence of system (3.64a) hyperbolic dynamics

$$\partial_x^j \xi(t, 0) = \partial_x^j \xi(t, L), \forall t \in [0, +\infty), j = 0, \dots, q_1. \quad (3.67)$$

When this Assumption 3.7 (A) holds, then the unique solution of the initial system lies in the Banach space of  $x$   $L$ -periodic functions, denoted by  $C_L^{q_1}([0, +\infty) \times [0, L]; \mathbb{R}^n)$ .

**Remark 3.4.** *It should be emphasized that version (A) of Assumption 3.7 is natural in many contexts and a similar assumption can be also found in [Alabau-Boussouira et al (2017)], which deals with a dual problem of controllability. A case where it is automatically satisfied is when  $K = I_n$  and  $\Lambda(x)$  constant. Version (B) restricts system's characteristic velocities. Assuming that  $\lambda_i$ 's are uniform, this version would hold if system possessed at most three distinct velocities in the first three places, with all the subsequent ones identical to  $\lambda_3$  ( $q = 3$  in that case).*

The methodology we need to follow hereafter, in order to solve the H-GODP, relies on indirect observer design, as it was introduced in Chapter 1.3.2. Solvability of H-GODP by use of a distributed observer has been shown for specific classes of quasi-linear hyperbolic systems in the first section of the present chapter, with only one velocity. Although system (3.64a), (3.64b), (3.65) is written in a triangular form, as it was introduced in [Kitsos et al (2018)], it seems that with Property (S) not holding, direct observer design is not applicable. To address this problem, we perform a transformation including spatial differentiations of the state up to order  $q - 2$ , in order to write the system in an appropriate form for which the desired observer design is possible. Then, for the obtained target system, we design the high-gain observer and, finally, returning to the initial coordinates, solvability of H-GODP is guaranteed. We present these appropriate transformations in the next subsections, starting from examples of  $2 \times 2$  and  $3 \times 3$  systems.

Define a Banach space  $\mathcal{X}$  by

$$\mathcal{X} := C^{q_1}[0, L] \times C^{q_2}[0, L] \times \cdots \times C^{q_n}[0, L],$$

equipped with a norm

$$\|(\xi_1, \dots, \xi_n)\|_{\mathcal{X}} := \|\xi_1\|_{q_1} + \|\xi_2\|_{q_2} + \cdots + \|\xi_n\|_{q_n}.$$

Assume that there exists a bounded injective linear transformation  $\mathcal{T} : (\mathcal{X}, \|\cdot\|_{\mathcal{X}}) \rightarrow (\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ , with bounded inverse, which maps system (3.64a), (3.65) into a target system  $\zeta$ , as follows

$$\begin{aligned} \zeta &= \mathcal{T}\xi; \\ \text{with } \zeta_1 &= \xi_1. \end{aligned} \tag{3.68}$$

The target system of PDEs that we consider in this chapter satisfies the following equations on  $\Sigma$ , distinguishing two boundary cases, depending on the sign of  $\lambda_1(\cdot)$

$$(T) \left\{ \begin{array}{l} \zeta_t(t, x) + \lambda_n(x)\zeta_x(t, x) = \bar{M}(x)\zeta(t, x) + \mathcal{M}\zeta_1(t)(x), \\ \left( \mathcal{B}_0^{1,m} \quad \mathcal{B}_L^{m+1,n} \right) I_{\pi}\zeta = K \left( \mathcal{B}_L^{1,m} \quad \mathcal{B}_0^{m+1,n} \right) I_{\pi}\zeta \\ + \mathcal{K}_1\zeta_1(0) + \mathcal{K}_2\zeta_1(L), \text{ when } \lambda_1(\cdot) > 0, \\ \left( \mathcal{B}_L^{1,n-m} \quad \mathcal{B}_0^{n-m+1,n} \right) I_{\pi}\zeta = K \left( \mathcal{B}_0^{1,n-k} \quad \mathcal{B}_L^{n-m+1,n} \right) I_{\pi}\zeta \\ + \mathcal{K}_1\zeta_1(0) + \mathcal{K}_2\zeta_1(L), \text{ when } \lambda_1(\cdot) < 0, \\ y_{\zeta}(t, x) = y(t, x) = C\zeta(t, x), \end{array} \right. \tag{3.69}$$

with initial condition  $\zeta(0, x) := \zeta^0(x) = \mathcal{T}\xi^0(x)$ , where  $\mathcal{M} : C^{q-1}[0, L] \rightarrow C^0([0, L]; \mathbb{R}^n)$ ,  $\mathcal{K}_1, \mathcal{K}_2 : C^{q-1}([0, L]; \mathbb{R}) \rightarrow \mathbb{R}^n$  are linear differential operators acting on  $\zeta_1$ , to be determined in the sequel, depending on the choice of  $\mathcal{T}$ ,  $\bar{M}(x)$  is matrix to be precised, having the same algebraic structure as  $M(x)$ , and  $y_{\zeta}$  is target system's output, which remains equal to original system's output  $y$ . The existence of such transformation  $\mathcal{T}$  is shown in the following subsection.

### High-Gain Observer



The proposed high-gain observer for (3.69) satisfies the following equations on  $\Sigma$

$$\begin{aligned} \hat{\zeta}_t(t, x) + \lambda_n(x)\hat{\zeta}_x(t, x) = & \bar{M}(x)\hat{\zeta}(t, x) \\ & - \Theta L(x) \left( y(t, x) - C\hat{\zeta}(t, x) \right) + \mathcal{M}y(t)(x), \end{aligned} \quad (3.70a)$$

$$\begin{aligned} \begin{pmatrix} \mathcal{B}_0^{1,m} & \mathcal{B}_L^{m+1,n} \end{pmatrix} I_\pi \hat{\zeta} = & K \begin{pmatrix} \mathcal{B}_L^{1,m} & \mathcal{B}_0^{m+1,n} \end{pmatrix} I_\pi \hat{\zeta} \\ & + \mathcal{K}_1 y(0) + \mathcal{K}_2 y(L), \text{ when } \lambda_1(\cdot) > 0, \end{aligned} \quad (3.70b)$$

$$\begin{aligned} \begin{pmatrix} \mathcal{B}_L^{1,n-m} & \mathcal{B}_0^{n-m+1,n} \end{pmatrix} I_\pi \hat{\zeta} = & K \begin{pmatrix} \mathcal{B}_0^{1,n-m} & \mathcal{B}_L^{n-m+1,n} \end{pmatrix} I_\pi \hat{\zeta} \\ & + \mathcal{K}_1 y(0) + \mathcal{K}_2 y(L), \text{ when } \lambda_1(\cdot) < 0, \end{aligned} \quad (3.70c)$$

with initial condition  $\hat{\zeta}^0(x) := \hat{\zeta}(0, x)$  (for a function  $\hat{\zeta}^0$  in  $\mathcal{X}$ ), where

$$\Theta := \text{diag}(\theta, \theta^2, \dots, \theta^n), \quad (3.71)$$

with  $\theta > 1$  the candidate high-gain constant of the observer, to be selected precisely later. In the above equations, we considered also a vector gain  $L(\cdot)$  in  $C^{q_1}([0, L]; \mathbb{R}^n)$ , selected in a way such that for  $P(\cdot)$  in  $C^{q_1}([0, L]; \mathbb{R}^{n \times n})$  symmetric and positive definite, a Lyapunov equation of the following form is satisfied

$$2\text{Sym}(P(x)(M_1(x) + L(x)C)) = -Q(x) \quad (3.72)$$

for some positive definite symmetric  $Q(x)$  of class  $C^{q_1}$ , where  $M_1(x)$  is derived by  $M(x)$  keeping only its sup-diagonal, namely,

$$M_1(x) = \begin{pmatrix} 0 & m_{1,2}(x) & 0 & \cdots & 0 \\ & \ddots & \ddots & & \vdots \\ \vdots & & & & m_{n-1,n}(x) \\ 0 & \cdots & & & 0 \end{pmatrix}.$$

The Lyapunov equation (4.25) is solvable by a positive definite  $P(x)$  for choice of  $L(x)$ , such that  $M_1(x) + L(x)C$  is Hurwitz for all  $x$  in  $[0, L]$ . The latter is feasible due to the observability of the pair  $(M_1(x), C)$  (Assumption 3.6). In addition, we note that the solution  $P(\cdot)$  of (4.25) is never diagonal, meaning that it would not commute in general with  $\Lambda(\cdot)$  for distinct characteristic velocities. This is a problem for Lyapunov-based convergence analysis, which motivates for a proposed indirect observer design relying on an infinite-dimensional transformation, in order to obtain a target system, where such a commutativity holds.

We are now in a position to present our main result on the solvability of the H-GODP.

**Theorem 3.3.** (*Observer convergence*) Consider system (3.64a), (3.65), defined on  $\Sigma$  with output (3.64b) and suppose that Assumptions 3.5 - 3.7 hold. Let also  $P$  in  $C^{q_1}([0, L]; \mathbb{R}^{n \times n})$  be symmetric and positive definite and let  $L$  in  $C^{q_1}([0, L]; \mathbb{R}^n)$ , both satisfying (4.25) for some  $Q$  in  $C^{q_1}([0, L]; \mathbb{R}^{n \times n})$ . Then, the H-GODP is solvable by  $\mathcal{T}^{-1}\hat{\zeta}$  (where  $\hat{\zeta}$  is the unique solution to (3.70)), for  $\theta > 1$  as a high gain and initial condition  $\mathcal{T}^{-1}\hat{\zeta}^0(x)$ , with  $\hat{\zeta}^0$  satisfying compatibility

conditions of order  $q_n$ . More precisely, for every  $\kappa > 0$ , there exists  $\theta_0 \geq 1$ , such that for every  $\theta > \theta_0$ , the following holds for all  $t \geq 0$

$$\|\mathcal{T}^{-1}\hat{\zeta}(t, \cdot) - \xi(t, \cdot)\|_\infty \leq \ell e^{-\kappa t} \|\mathcal{T}^{-1}\hat{\zeta}^0(\cdot) - \xi^0(\cdot)\|_x, \quad (3.73)$$

with  $\ell > 0$  a polynomial in  $\theta$ .

This observer convergence result is based on the existence of a transformation  $\mathcal{T}$ , as introduced in (3.68). In the next subsection, we prove that this transformation always exists and we give an algorithm to determine it, in order to design the observer (3.70) considered in Theorem 3.3.

### 3.3.2 A target system for observer design

In this subsection we present a constructive methodology to obtain target system (3.69) via a lower triangular transformation (3.68). We start from the case of a  $2 \times 2$  and  $3 \times 3$  system and, then, we proceed to the  $n \times n$  case, which is more tedious and needs more requirements, see Assumption 3.7.

Let us consider the cases  $n = 2$  and  $n = 3$ , and, for simplification, assume that  $m = 2$  and  $m = 3$  for each of the two cases respectively, namely, systems have only positive characteristic velocities. In this case, permutation matrix  $I_\pi$  is the identity, i.e.,  $I_\pi = I_3$ . All other cases, for possible existence of negative characteristic velocities, can be considered following similar procedure and are omitted here, since the calculations would easily follow from different choices of the permutation matrix  $I_\pi$ .

We aim at obtaining target system (3.69), but with boundary conditions reducing to the following, for these particular cases of positive velocities

$$\zeta(0) = K_{00}\zeta(L) + \mathcal{K}_1\zeta_1(0) + \mathcal{K}_2\zeta_1(L), \quad (3.74)$$

To achieve this, we need to show the existence of an invertible infinite-dimensional coordinates transformation. Let us, therefore, perform a lower-triangular transformation of the following type

$$\zeta = \mathcal{T}\xi; \quad \mathcal{T} := \begin{cases} I_2, & \text{if } n = 2, \\ \begin{pmatrix} 1 & 0 & 0 \\ \tau(x)\partial_x & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & \text{if } n = 3, \end{cases} \quad (3.75)$$

with  $\tau(\cdot)$  given by

$$\tau(x) := \frac{\lambda_2(x) - \lambda_3(x)}{m_{1,2}(x)}.$$

Obviously, this transformation is bounded, invertible with bounded inverse from  $\mathcal{X}$  to  $\mathcal{X}$ , independently of boundary conditions. Applying this transformation to system (3.64a), (3.65),

we obtain the following equations for each of the cases  $n = 2$  (we just rewrite the hyperbolic equations in a different way, since the transformation  $\mathcal{T}$  is the identity)

$$\partial_t \zeta_1 + \lambda_2 \partial_x \zeta_1 = m_{1,1} \zeta_1 + m_{1,2} \zeta_2 + (\lambda_2 - \lambda_1) \partial_x \zeta_1, \quad (3.76a)$$

$$\partial_t \zeta_2 + \lambda_2 \partial_x \zeta_2 = m_{2,1} \zeta_1 + m_{2,2} \zeta_2, \quad (3.76b)$$

$$\zeta(0) = K_{00} \zeta(L), \quad (3.76c)$$

and for  $n = 3$

$$\partial_t \zeta_1 + \lambda_3 \partial_x \zeta_1 = m_{1,1} \zeta_1 + m_{1,2} \zeta_2 + a_1 \partial_x \zeta_1, \quad (3.77a)$$

$$\partial_t \zeta_2 + \lambda_3 \partial_x \zeta_2 = (m_{2,1} + \tau m'_{1,1}) \zeta_1 + (m_{2,2} + \tau m'_{1,2}) \zeta_2 + m_{2,3} \zeta_3 + a_2 \partial_x \zeta_1 + a_3 \partial_x^2 \zeta_1, \quad (3.77b)$$

$$\partial_t \zeta_3 + \lambda_3 \partial_x \zeta_3 = \sum_{i=1}^3 m_{3,i} \zeta_i + a_4 \partial_x \zeta_1, \quad (3.77c)$$

$$\zeta(0) = K_{00} \zeta(L) + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \tau(0) \partial_x \zeta_1(0) - K_{00} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \tau(L) \partial_x \zeta_1(L), \quad (3.77d)$$

where

$$a_1(x) := \lambda_3(x) - \lambda_1(x) - m_{1,2}(x) \tau(x),$$

$$a_2(x) := \tau(x) m_{1,1}(x) - \tau(x) \lambda'_1(x) - \tau(x) (m_{1,2}(x) \tau(x))' - \tau(x) m_{2,2}(x) + \tau'(x) \lambda_2(x),$$

$$a_3(x) := (\lambda_2(x) - \lambda_1(x) - \tau(x) m_{1,2}(x)) \tau(x),$$

$$a_4(x) := -\tau(x) m_{3,2}(x).$$

Thus, we have obtained the hyperbolic dynamics of target system (3.69) with boundary conditions (3.74), with

$$\bar{M}(x) := \begin{cases} M(x), & \text{if } n = 2 \\ M(x) + \tau(x) \begin{pmatrix} 0 & 0 & 0 \\ 0 & m'_{1,2}(x) & 0 \\ 0 & 0 & 0 \end{pmatrix}, & \text{if } n = 3 \end{cases}, \quad (3.78)$$

$$\mathcal{M} := \begin{cases} \begin{pmatrix} (\lambda_2(x) - \lambda_1(x)) \partial_x \\ 0 \end{pmatrix}, & \text{if } n = 2 \\ \begin{pmatrix} a_1(x) \partial_x \\ \tau(x) m'_{1,1}(x) + a_2(x) \partial_x + a_3(x) \partial_x^2 \\ a_4(x) \partial_x \end{pmatrix}, & \text{if } n = 3 \end{cases},$$

$$\mathcal{K}_1 = \mathcal{K}_2 = 0, \text{ if } n = 2,$$

$$\mathcal{K}_1 := \begin{pmatrix} 0 \\ \tau(0) \partial_x \\ 0 \end{pmatrix}, \mathcal{K}_2 := -K_{00} \begin{pmatrix} 0 \\ \tau(L) \partial_x \\ 0 \end{pmatrix}, \text{ if } n = 3.$$

Observe that the right-hand side of the target hyperbolic dynamics (3.76) contains a spatial derivative of the measured first state  $\zeta_1$  of order 1, whereas, parabolic dynamics (3.77) contain spatial derivatives of the first state up to order 2.

Next, we aim at generalizing this method to systems with more than three states.

Consider system (3.64a), (3.64b), (3.65), assuming  $n > 3$ . We show here the existence of a lower triangular infinite-dimensional transformation, which maps system into a target system (T).

Let us consider a transformation (3.68), with

$$\mathcal{T} := I_n + \tilde{\mathcal{T}}, \quad (3.79)$$

where  $\tilde{\mathcal{T}} : \mathcal{X} \rightarrow \mathcal{X}$  is a matrix operator given by

$$\tilde{\mathcal{T}} := \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \tau_{2,1}^1(x)\partial_x & 0 & 0 & \cdots & 0 & 0 & 0 \\ \sum_{i=1}^2 \tau_{3,1}^i(x)\partial_x^i & \tau_{3,2}^1(x)\partial_x & 0 & \cdots & 0 & 0 & 0 \\ \vdots & & & & \vdots & & \\ \sum_{i=1}^{n-2} \tau_{n-1,1}^i(x)\partial_x^i & \sum_{i=1}^{n-3} \tau_{n-1,2}^i(x)\partial_x^i & \sum_{i=1}^{n-4} \tau_{n-1,3}^i(x)\partial_x^i & \cdots & \tau_{n-1,n-2}^1(x)\partial_x & 0 & 0 \\ \sum_{i=1}^{n-3} \tau_{n,1}^i\partial_x^i & \sum_{i=1}^{n-4} \tau_{n,2}^i\partial_x^i & \sum_{i=1}^{n-5} \tau_{n,3}^i\partial_x^i & \cdots & 0 & 0 & 0 \end{pmatrix} \quad (3.80)$$

and  $\tau_{i,j}^k \in C^{q_j}[0, L]$  are appropriate functions to be chosen, obeying, in addition, to

$$\tau_{i,j}^k \equiv 0, \forall k = q - 1, \dots, n - 2 \quad (3.81)$$

(observe that the last row of  $\tilde{\mathcal{T}}$ , counter-intuitively, recurs to the form of two rows before it, i.e., row  $n - 2$ ). One can verify that differential operator (3.79) is bounded, invertible with bounded inverse from  $\mathcal{X}$  to  $\mathcal{X}$  for any choice of  $\tau_{i,j}^k$  and its inverse is given by

$$\mathcal{T}^{-1} = I_n + \sum_{i=1}^{n-1} (-1)^i \tilde{\mathcal{T}}^i. \quad (3.82)$$

Next, let us represent mapping  $I_\pi \tilde{\mathcal{T}} I_\pi^\top$  in a block form as

$$I_\pi \tilde{\mathcal{T}} I_\pi^\top = \frac{1 + \operatorname{sgn}(\lambda_1(x))}{2} \begin{pmatrix} \tilde{\mathcal{T}}_{00} & \tilde{\mathcal{T}}_{01} \\ \tilde{\mathcal{T}}_{10} & \tilde{\mathcal{T}}_{11} \end{pmatrix} + \frac{1 - \operatorname{sgn}(\lambda_1(x))}{2} \begin{pmatrix} \tilde{\mathcal{T}}_{11} & \tilde{\mathcal{T}}_{10} \\ \tilde{\mathcal{T}}_{01} & \tilde{\mathcal{T}}_{00} \end{pmatrix}, \quad (3.83)$$

with each block  $\tilde{\mathcal{T}}_{00}, \tilde{\mathcal{T}}_{01}, \tilde{\mathcal{T}}_{10}, \tilde{\mathcal{T}}_{11}$  having the same algebraic dimensions as the blocks of (3.66).

We aim at obtaining target system (3.69) via  $\mathcal{T}$ .

To obtain target system (3.69), we use transformation (3.68), (3.79), (3.80), and by using (3.64a), (3.65b) we derive the following equations

$$\begin{aligned} & \zeta_i(t, x) + \lambda_n(x)\zeta_x(t, x) = M(x)\zeta(t, x) \\ & + \left[ (\lambda_n(x)I_n\partial_x - M(x))\tilde{\mathcal{T}} + \tilde{\mathcal{T}}((\lambda_n(x)I_n - \Lambda(x))\partial_x + M(x)) + (\lambda_n(x)I_n - \Lambda(x))\partial_x \right] \xi(t)(x), \end{aligned}$$

$$\begin{aligned}
\begin{pmatrix} \mathcal{B}_0^{1,m} & \mathcal{B}_L^{m+1,n} \end{pmatrix} I_\pi \zeta &= K \begin{pmatrix} \mathcal{B}_L^{1,m} & \mathcal{B}_0^{m+1,n} \end{pmatrix} I_\pi \zeta \\
&+ \begin{pmatrix} \tilde{\mathcal{T}}_{00}(0) & \tilde{\mathcal{T}}_{01}(L) \\ \tilde{\mathcal{T}}_{10}(0) & \tilde{\mathcal{T}}_{11}(L) \end{pmatrix} \begin{pmatrix} \mathcal{B}_0^{1,m} & \mathcal{B}_L^{m+1,n} \end{pmatrix} I_\pi \xi \\
&- K \begin{pmatrix} \tilde{\mathcal{T}}_{00}(L) & \tilde{\mathcal{T}}_{01}(0) \\ \tilde{\mathcal{T}}_{10}(L) & \tilde{\mathcal{T}}_{11}(0) \end{pmatrix} \begin{pmatrix} \mathcal{B}_L^{1,k} & \mathcal{B}_0^{m+1,n} \end{pmatrix} I_\pi \xi, \text{ when } \lambda_1(\cdot) > 0, \\
\begin{pmatrix} \mathcal{B}_L^{1,n-m} & \mathcal{B}_0^{n-m+1,n} \end{pmatrix} I_\pi \zeta &= K \begin{pmatrix} \mathcal{B}_0^{1,n-m} & \mathcal{B}_L^{n-m+1,n} \end{pmatrix} I_\pi \zeta \\
&+ \begin{pmatrix} \tilde{\mathcal{T}}_{11}(0) & \tilde{\mathcal{T}}_{10}(L) \\ \tilde{\mathcal{T}}_{01}(0) & \tilde{\mathcal{T}}_{00}(L) \end{pmatrix} \begin{pmatrix} \mathcal{B}_L^{1,n-k} & \mathcal{B}_0^{n-m+1,n} \end{pmatrix} I_\pi \xi \\
&- K \begin{pmatrix} \tilde{\mathcal{T}}_{11}(L) & \tilde{\mathcal{T}}_{10}(0) \\ \tilde{\mathcal{T}}_{01}(L) & \tilde{\mathcal{T}}_{00}(0) \end{pmatrix} \begin{pmatrix} \mathcal{B}_0^{1,n-k} & \mathcal{B}_L^{n-k+1,n} \end{pmatrix} I_\pi \xi, \text{ when } \lambda_1(\cdot) < 0.
\end{aligned}$$

Comparing the previous equations with target system (T), the following equations must be satisfied for all  $\xi \in \mathcal{X}$

$$\begin{aligned}
&\left[ (\lambda_n(x) I_n \partial_x - M(x)) \tilde{\mathcal{T}} + \tilde{\mathcal{T}} ((\lambda_n(x) I_n - \Lambda(x)) \partial_x + M(x)) \right. \\
&\left. + (\lambda_n(x) I_n - \Lambda(x)) \partial_x \right] \xi(x) = \mathcal{M} C \xi(x) + (\bar{M}(x) - M(x)) \xi(x), \tag{3.85a}
\end{aligned}$$

$$\begin{aligned}
&\begin{pmatrix} \tilde{\mathcal{T}}_{00}(0) & \tilde{\mathcal{T}}_{01}(L) \\ \tilde{\mathcal{T}}_{10}(0) & \tilde{\mathcal{T}}_{11}(L) \end{pmatrix} \begin{pmatrix} \mathcal{B}_0^{1,m} & \mathcal{B}_L^{m+1,n} \end{pmatrix} I_\pi \xi \\
&- K \begin{pmatrix} \tilde{\mathcal{T}}_{00}(L) & \tilde{\mathcal{T}}_{01}(0) \\ \tilde{\mathcal{T}}_{10}(L) & \tilde{\mathcal{T}}_{11}(0) \end{pmatrix} \begin{pmatrix} \mathcal{B}_L^{1,m} & \mathcal{B}_0^{m+1,n} \end{pmatrix} I_\pi \xi \\
&= \mathcal{K}_1 C \xi(0) + \mathcal{K}_2 C \xi(L), \text{ when } \lambda_1(\cdot) > 0, \tag{3.85b}
\end{aligned}$$

$$\begin{aligned}
&\begin{pmatrix} \tilde{\mathcal{T}}_{11}(0) & \tilde{\mathcal{T}}_{10}(L) \\ \tilde{\mathcal{T}}_{01}(0) & \tilde{\mathcal{T}}_{00}(L) \end{pmatrix} \begin{pmatrix} \mathcal{B}_L^{1,n-m} & \mathcal{B}_0^{n-m+1,n} \end{pmatrix} I_\pi \xi \\
&- K \begin{pmatrix} \tilde{\mathcal{T}}_{11}(L) & \tilde{\mathcal{T}}_{10}(0) \\ \tilde{\mathcal{T}}_{01}(L) & \tilde{\mathcal{T}}_{00}(0) \end{pmatrix} \begin{pmatrix} \mathcal{B}_0^{1,n-m} & \mathcal{B}_L^{n-m+1,n} \end{pmatrix} I_\pi \xi \\
&= \mathcal{K}_1 C \xi(0) + \mathcal{K}_2 C \xi(L), \text{ when } \lambda_1(\cdot) < 0. \tag{3.85c}
\end{aligned}$$

Now, we choose the involved operators  $\bar{M}(x)$ ,  $\mathcal{M}$ ,  $\mathcal{K}_1$ , and  $\mathcal{K}_2$  to be as follows

$$\begin{aligned}
&\bar{M}(x) := M(x) \\
&+ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \tilde{\mathcal{T}}_{2,*} M_{*,2}(x) & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \tilde{\mathcal{T}}_{3,*} M_{*,2}(x) & \tilde{\mathcal{T}}_{3,*} M_{*,3} & \tilde{\mathcal{T}}_{3,*} M_{*,4}(x) & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & & & & & & \vdots & \\ 0 & \tilde{\mathcal{T}}_{n-1,*} M_{*,2}(x) & \tilde{\mathcal{T}}_{n-1,*} M_{*,3}(x) & \tilde{\mathcal{T}}_{n-1,*} M_{*,4}(x) & \tilde{\mathcal{T}}_{n-1,*} M_{*,5}(x) & \cdots & \tilde{\mathcal{T}}_{n-1,*} M_{*,n-2}(x) & \tilde{\mathcal{T}}_{n-1,*} M_{*,n-1}(x) & 0 \\ 0 & \tilde{\mathcal{T}}_{n,*} M_{*,2}(x) & \tilde{\mathcal{T}}_{n,*} M_{*,3}(x) & \tilde{\mathcal{T}}_{n,*} M_{*,4}(x) & \tilde{\mathcal{T}}_{n,*} M_{*,5}(x) & \cdots & \tilde{\mathcal{T}}_{n,*} M_{*,n-2}(x) & 0 & 0 \end{pmatrix}, \tag{3.86}
\end{aligned}$$

where by  $\tilde{\mathcal{T}}_{i,*}$ ,  $M_{*,j}(x)$ ,  $i, j = 1, \dots, n-1$  we denote the  $i$ -th row of  $\tilde{\mathcal{T}}$  and  $j$ -th column of

$M(x)$ , respectively. Also, we choose

$$\begin{aligned}\mathcal{M} &:= \left[ (\lambda_n(x)I_n \partial_x - M(x)) \tilde{\mathcal{T}} + \tilde{\mathcal{T}} ((\lambda_n(x)I_n - \Lambda(x)) \partial_x + M(x)) \right. \\ &\quad \left. + (\lambda_n(x) - \lambda_1(x)) I_n \partial_x \right] C^\top, \\ \mathcal{K}_1 &:= \frac{1 + \operatorname{sgn}(\lambda_1(x))}{2} I_\pi \tilde{\mathcal{T}} C^\top - \frac{1 - \operatorname{sgn}(\lambda_1(x))}{2} K I_\pi \tilde{\mathcal{T}} C^\top, \\ \mathcal{K}_2 &:= -\frac{1 + \operatorname{sgn}(\lambda_1(x))}{2} K I_\pi \tilde{\mathcal{T}} C^\top + \frac{1 - \operatorname{sgn}(\lambda_1(x))}{2} I_\pi \tilde{\mathcal{T}} C^\top.\end{aligned}\tag{3.87}$$

The next step is to determine  $\tilde{\mathcal{T}}$ . Let us, first, examine equation (3.85a), which should be satisfied. This, by utilizing (3.87), can be alternatively written as the following generalized Sylvester operator equation, with unknown  $\tilde{\mathcal{T}}$

$$\begin{aligned}\mathcal{R} &:= \left[ (\lambda_n I_n \partial_x - M) \tilde{\mathcal{T}} + \tilde{\mathcal{T}} ((\lambda_n I_n - \Lambda) \partial_x + M) \right] (I_n - C^\top C) \\ &\quad + (\lambda_n I_n - \Lambda) \partial_x + (\lambda_1 - \lambda_n) C^\top C \partial_x + M - \bar{M} = 0.\end{aligned}\tag{3.88}$$

We do not focus here on the proof of solvability of the previous generalized Sylvester operator equation (see for instance [Bhatia and Rosenthal (1997)] for operator equations of such a type), we provide, however, its solution  $\tilde{\mathcal{T}}$ , which turns out to be computed algorithmically. Finding the solution reduces to an exhaustive calculation of functions  $\tau_{i,j}^k(\cdot)$ . The unique existence of such a solution  $\tilde{\mathcal{T}}$  is a consequence of the particular algebraic structure of the operators and matrices, in conjunction with Assumption 3.6. To calculate  $\tau_{i,j}^k$  and, therefore, find  $\tilde{\mathcal{T}}$ , we follow the steps of the algorithm presented below, by which we obtain each new  $\tau_{i,j}^k$  as a linear function of the previously calculated  $\tau_{i,j}^k$  and their derivatives and also system's dynamics, in a cascade manner resulting from the structure of the involved mappings. This cascade manner of calculation can be inductively proven to provide the solution  $\mathcal{T}$ , for every system dimension  $n$ . Let us first consider all  $n$  equations resulting from equation  $\mathcal{R}\xi = 0$ , attributing to each of them an equation index  $i$ . The steps of the calculations are described by the following algorithm.

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**Algorithm 1** Computation of transformation  $\mathcal{T}$

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- 1: **procedure** COMPUTE  $\tau_{i,j}^k, i \in [2, \dots, n], j, k \in [1, \dots, n-2]$ .
  - 2:      $i = n - 1$ .
  - 3:     **while** not all  $\tau_{i,j}^k$  are computed, **do**
  - 4:         **while**  $i \geq 2$  **do**
  - 5:             In equation  $i$ , find  $j \in [2, n-1]$  as the largest subscript of  $\partial_x \xi$  and set to zero all the coefficients of  $\partial_x^{\bar{k}} \xi_j, \dots, \partial_x^2 \xi_j, \partial_x \xi_j$  sequentially, with  $\bar{k}$  the maximal order of spatial derivative of  $\xi_j$  existing in the current equation  $i$ . For each elimination, calculate function  $\tau_{i,j-1}^k, k = 1 \dots, \bar{k}$ , where each of the latter corresponds to elimination of the coefficient of  $\partial_x^k \xi_j$  in current equation  $i$ .
  - 6:              $i \leftarrow i - 1$ .
  - 7:             Pick  $i \in [2, n]$  as the largest equation index for which there exists a  $k$ -spatial derivative of  $\xi_j$  in equation  $i$ , for some  $k \in [1, n-2]$  and  $j \in [1, n-2]$ .
-

After acquiring the above-mentioned solutions  $\tau_{i,j}^k$  and for the chosen  $\mathcal{K}_1, \mathcal{K}_2$ , we need to validate that, additionally, (3.85b) or (3.85c) are satisfied on the boundaries. To proceed, we invoke Assumption 3.7. Under the case of version (A) of Assumption 3.7, it holds

$$\tilde{\mathcal{T}}\xi(0) = \tilde{\mathcal{T}}\xi(L),$$

thus (3.85b), (3.85c) are satisfied. Then, target system boundary conditions reduce to

$$\zeta(0) = \zeta(L), \quad (3.89)$$

since  $K = I_n$  in that case. Under the case of version (B) of Assumption 3.7, (3.85b) or (3.85c) is automatically satisfied. In fact, for (3.85b) or (3.85c) to be true, we need  $\tau_{i,j}^k(\cdot)$  to vanish at  $x = 0$  and  $x = L$ , for all  $j > 1$ . Indeed, following the steps of the algorithm, any new calculated  $\tau_{i,j}^k(\cdot), j > 1$  is a linear combination of all calculated  $\tau_{i,j}^k(\cdot), j > 1$  in preceding steps and, also, of differences  $\lambda_i(\cdot) - \lambda_j(\cdot)$  and  $\lambda'_{i'}(\cdot) - \lambda'_{j'}(\cdot)$ , for  $i, j \geq 3, i', j' \geq 4$ . Since all the latter differences vanish at  $x = 0$  and  $x = L$  (by this version (B) of Assumption 3.7) and all  $\tau_{i,j}^k(\cdot), j > 1$  calculated in preceding steps, starting from the first calculation  $\tau_{n-1, n-2}^1(\cdot)$ , successively vanish at 0 and  $L$ , the desired properties are satisfied.

As an example of this algorithmic calculation, we provide the solution  $\mathcal{T}$  for  $5 \times 5$  systems. It is computationally tedious, however, to provide general formulas for  $n \times n$  systems.

**Example 3.2.** *Considering a  $5 \times 5$  system, we calculate each of the functions  $\tau_{i,j}^k(\cdot), i = 2, \dots, 5, j = 1, 2, 3, k = 1, 2, 3$ , and we present them in the cascade order that the above-mentioned algorithm suggests (recall that all are functions of  $x$ , but we omit their arguments below).*

*While loop 1*

$$\tau_{4,3}^1 = \frac{\lambda_4 - \lambda_5}{m_{3,4}}, \tau_{3,2}^1 = \frac{\lambda_3 - \lambda_5 + m_{3,4}\tau_{4,3}^1}{m_{2,3}}, \tau_{2,1}^1 = \frac{\lambda_2 - \lambda_5 + m_{2,3}\tau_{3,2}^1}{m_{1,2}}.$$

*While loop 2*

$$\begin{aligned} \tau_{5,2}^1 &= \frac{m_{5,4}\tau_{4,3}^1}{m_{2,3}}, \tau_{4,2}^1 = \frac{(\lambda_3 - 2\lambda_5)\tau_{4,3}^1}{m_{2,3}}, \\ \tau_{4,2}^1 &= \frac{-\lambda_5 \frac{d\tau_{4,3}^1}{dx} + (m_{4,4} - m_{3,3} + (\lambda_3 - \lambda_5)')\tau_{4,3}^1}{m_{2,3}}, \\ \tau_{3,1}^2 &= \frac{(\lambda_2 - 2\lambda_5)\tau_{3,2}^1 + m_{3,4}\tau_{4,2}^2}{m_{1,2}}, \\ \tau_{3,1}^1 &= \frac{1}{m_{1,2}} \left( -\lambda_5 \frac{d\tau_{3,2}^1}{dx} + (m_{3,3} - m_{2,2} + (\lambda_2 - \lambda_5)')\tau_{3,2}^1 + m_{3,4}\tau_{4,2}^1 - 2m'_{1,2}\tau_{3,1}^2 \right). \end{aligned}$$

While loop 3

$$\begin{aligned}\tau_{5,1}^2 &= \frac{(\lambda_2 - 2\lambda_5)\tau_{5,2}^1 + m_{5,4}\tau_{4,2}^2}{m_{1,2}}, \\ \tau_{5,1}^1 &= \frac{1}{m_{1,2}} \left( -\lambda_5 \frac{d\tau_{5,2}^1}{dx} + (m_{5,5} - m_{2,2} + (\lambda_2 - \lambda_5)') \tau_{5,2}^1 + m_{5,4}\tau_{4,2}^1 + m_{5,3}\tau_{3,2}^1 - 2m'_{1,2}\tau_{5,1}^2 \right), \\ \tau_{4,1}^3 &= \frac{(\lambda_2 - 2\lambda_5)\tau_{4,2}^2}{m_{1,2}}, \\ \tau_{4,1}^2 &= \frac{1}{m_{1,2}} \left( (\lambda_2 - 2\lambda_5)\tau_{4,2}^1 - \lambda_5 \frac{d\tau_{4,2}^2}{dx} + (m_{4,4} - m_{2,2} + 2(\lambda_2 - \lambda_5)') \tau_{4,2}^2 - 3m'_{1,2}\tau_{4,1}^3 \right), \\ \tau_{4,1}^1 &= \frac{1}{m_{1,2}} \left( -\lambda_5 \frac{d\tau_{4,2}^1}{dx} + (m_{4,4} - m_{2,2} + (\lambda_2 - \lambda_5)') \tau_{4,2}^1 + m_{4,3}\tau_{3,2}^1 + m_{4,5}\tau_{5,2}^1 \right. \\ &\quad \left. - m_{3,2}\tau_{4,3}^1 - 2m'_{1,2}\tau_{4,1}^2 - 3m''_{1,2}\tau_{4,1}^3 + ((\lambda_2 - \lambda_5)'' - 2m'_{2,3} - 2m'_{2,2}) \tau_{4,2}^2 \right).\end{aligned}$$

### 3.3.3 Observer convergence proof

In this subsection, we prove that the proposed observer is a high-gain observer for the target system, which is mapped from the original system via  $\mathcal{T}$ . Observer's exponential convergence is proven for appropriate spatial norm. Injectivity of  $\mathcal{T}$  and boundedness of its inverse, then, guarantees that  $\mathcal{T}^{-1}\hat{\zeta}$  approaches exponentially the state  $\xi$  of the original system and, thus, solves the H-GODP.

We start from a prerequisite lemma for the well-posedness of the observer, which is a direct consequence of Assumption 3.5 and linearity of the system. The proof relies on the method of characteristics and a combination of classical arguments from proofs found in [Bastin and Coron (2016)], Chap. 4.5, App. A, therefore, it is omitted here.

**Lemma 3.2.** (*Existence/uniqueness of global solutions for the observer system*) Under the regularity assumptions for the dynamics and for any  $y$  in  $C^{q_1}([0, +\infty) \times [0, L]; \mathbb{R})$ , the problem described by (3.70) on domain  $\Sigma$  with initial condition  $\hat{\zeta}^0(x)$  satisfying compatibility conditions of order  $q_n$  admits a unique solution  $\hat{\zeta}$  in  $C^{q_n}([0, +\infty) \times [0, L]; \mathbb{R}^n)$

Consider, now, observer (3.70) for target system (T), with  $\bar{M}(x)$  given by (3.78), when  $n = 2, 3$  and  $\bar{M}(x) = M$  (constant), for  $n > 3$ . We define a scaled observer error by

$$\varepsilon := \Theta^{-1}(\hat{\zeta} - \zeta), \quad (3.90)$$

for which we derive the following hyperbolic equations on  $\Sigma$

$$\varepsilon_t(t, x) + \lambda_n(x)\varepsilon_x(t, x) = \theta(M_1(x) + L(x)C)\varepsilon(t, x) + \Theta^{-1}(\bar{M}(x) - M_1(x))\Theta\varepsilon(t, x), \quad (3.91a)$$

$$\varepsilon(t, 0) = \Theta^{-1}\bar{K}\Theta\varepsilon(t, L), \text{ when } \lambda_n(\cdot) > 0, \quad (3.91b)$$

$$\varepsilon(t, L) = \Theta^{-1}\bar{\bar{K}}\Theta\varepsilon(t, 0), \text{ when } \lambda_n(\cdot) < 0, \quad (3.91c)$$



where

$$\bar{K} := K_{00}, \text{ when } n = 2, 3$$

and in case  $n > 3$ , for each of versions (A) or (B) of Assumption 3.7,  $\bar{K}$  and  $\bar{\bar{K}}$  (with  $\bar{\bar{K}}^{-1} = \bar{K}$ ) are given by

$$\bar{K} := \begin{cases} I_n, & \text{for case (A)} \\ I_\pi^\top \tilde{\bar{K}} I_\pi, & \text{for case (B)} \end{cases}$$

$$\bar{\bar{K}} := \begin{cases} I_n, & \text{for case (A)} \\ I_\pi^\top \tilde{\bar{\bar{K}}} I_\pi, & \text{for case (B)} \end{cases},$$

where

$$\tilde{\bar{K}} := \frac{1 + \operatorname{sgn}(\lambda_1(x))}{2} \begin{pmatrix} K_{00} - K_{01}K_{11}^{-1}K_{10} & K_{01}K_{11}^{-1} \\ -K_{11}^{-1}K_{10} & K_{11}^{-1} \end{pmatrix}$$

$$+ \frac{1 - \operatorname{sgn}(\lambda_1(x))}{2} \begin{pmatrix} K_{11}^{-1} & -K_{11}^{-1}K_{10} \\ K_{01}K_{11}^{-1} & K_{00} - K_{01}K_{11}^{-1}K_{10} \end{pmatrix},$$

$$\tilde{\bar{\bar{K}}} := \frac{1 + \operatorname{sgn}(\lambda_1(x))}{2} \begin{pmatrix} K_{00}^{-1} & -K_{00}^{-1}K_{01} \\ K_{10}K_{00}^{-1} & K_{11} - K_{10}K_{00}^{-1}K_{01} \end{pmatrix}$$

$$+ \frac{1 - \operatorname{sgn}(\lambda_1(x))}{2} \begin{pmatrix} K_{11} - K_{10}K_{00}^{-1}K_{01} & K_{10}K_{00}^{-1} \\ -K_{00}^{-1}K_{01} & K_{00}^{-1} \end{pmatrix}.$$

To prove the error's exponential stability with respect to its origin, we adopt a Lyapunov-based approach inspired by methodologies presented in [Bastin and Coron (2016)]. Similar  $p$ -functionals have appeared in [Kitsos et al (2018)], [Kitsos et al (2020d)]. The stability is proven for the  $q_n$ -norm. We define a Lyapunov functional  $\mathcal{W}_p : C^{q_n}([0, L]; \mathbb{R}^n) \rightarrow \mathbb{R}$  by

$$\mathcal{W}_p[\varepsilon] := \left( \int_0^L \pi(x) e^{p\mu_\theta x} \mathcal{G}_p[\varepsilon](x) dx \right)^{1/p}, \quad (3.92a)$$

where  $\mathcal{G}_p : C^{q_n}([0, L]; \mathbb{R}^n) \rightarrow \mathbb{R}$  is given by

$$\mathcal{G}_p[\varepsilon](x) := \left( \sum_{i=0}^{q_n} \partial_i^i \varepsilon^\top(x) P(x) \partial_i^i \varepsilon(x) \right)^p \quad (3.92b)$$

and  $p$  in  $\mathbb{N}$ ,  $P(\cdot)$  is of class  $C^{q_1}$ , symmetric and positive definite, satisfying (4.25),  $\pi : [0, L] \rightarrow \mathbb{R}$  is a function given by

$$\pi(x) := (\bar{\pi} - 1) \frac{x}{L} + 1; \bar{\pi} := \frac{\max_{x \in [0, L]} (\lambda_n(x))}{\min_{x \in [0, L]} (\lambda_n(x))} \quad (3.93)$$

and  $\mu_\theta$  in  $\mathbb{R}$  is given by

$$\mu_\theta := \frac{1 + \operatorname{sgn}(\lambda_n(x))}{2L} \ln \left( \frac{|\bar{K}|^2 \|P(\cdot)\|_\infty}{\min_{x \in [0, L]} \operatorname{eig}(P(x))} \theta^{2n-2} \right)$$

$$+ \frac{1 - \operatorname{sgn}(\lambda_n(x))}{2L} \ln \left( \frac{\min_{x \in [0, L]} \operatorname{eig}(P(x))}{|\bar{\bar{K}}|^2 \|P(\cdot)\|_\infty} \theta^{2n-2} \right). \quad (3.94)$$

Note here that, by its definition,  $\pi$  is bounded as follows

$$1 \leq \pi(x) \leq \bar{\pi}, \forall x \in [0, L]. \quad (3.95)$$

By invoking Lemma 3.2 and Fact 3.1, we are in a position to define  $G_p, W_p : [0, +\infty) \rightarrow \mathbb{R}$  by

$$G_p(t) := \mathcal{G}_p[\varepsilon](t), W_p(t) := \mathcal{W}_p[\varepsilon](t), t \geq 0. \quad (3.96)$$

By temporarily assuming that  $\varepsilon$  has some extra regularity, i.e., it is  $C^{q_n+1}$ , we obtain the following equations for each  $i = 1, \dots, q_n$

$$\begin{aligned} & \partial_t^{i+1} \varepsilon(t, x) + \lambda_n(x) \partial_t^i \partial_x \varepsilon(t, x) \\ &= \theta (M_1(x) + L(x)C) \partial_t^i \varepsilon(t, x) + \Theta^{-1}(\bar{M}(x) - M_1(x)) \Theta \partial_t^i \varepsilon(t, x), \end{aligned} \quad (3.97a)$$

$$\partial_t^i \varepsilon(t, 0) = \Theta^{-1} \bar{K} \Theta \partial_t^i \varepsilon(t, L), \lambda_n(\cdot) > 0, \quad (3.97b)$$

$$\partial_t^i \varepsilon(t, L) = \Theta^{-1} \bar{K} \Theta \partial_t^i \varepsilon(t, 0), \lambda_n(\cdot) < 0. \quad (3.97c)$$

We calculate the time-derivative  $\dot{W}_p$  along the solutions of (3.97) as follows

$$\begin{aligned} \dot{W}_p &= \frac{1}{p} W_p^{1-p} \int_0^L p \pi(x) e^{p\mu_\theta x} G_{p-1}(x) \\ &\quad \times \sum_{i=0}^{q_n} \left( \partial_t^{i+1} \varepsilon^\top(x) P(x) \partial_t^i \varepsilon(x) + \partial_t^i \varepsilon^\top(x) P(x) \partial_t^{i+1} \varepsilon(x) \right) dx \\ &= W_p^{1-p} \left( \frac{1}{p} T_{1,p} + T_{2,p} \right), \end{aligned} \quad (3.98)$$

where

$$\begin{aligned} T_{1,p} &:= - \int_0^L \lambda_n(x) \pi(x) e^{p\mu_\theta x} \left[ \partial_x G_p(x) - p G_{p-1}(x) \sum_{i=0}^{q_n} \partial_t^i \varepsilon^\top(x) P_x(x) \partial_t^i \varepsilon(x) \right] dx, \\ T_{2,p} &:= \int_0^L \left( 2\theta \pi(x) e^{p\mu_\theta x} G_{p-1}(x) \sum_{i=0}^{q_n} \partial_t^i \varepsilon^\top(x) \text{Sym} \left( (P(x)(M_1(x) + L(x)C) \partial_t^i \varepsilon(x) \right. \right. \right. \\ &\quad \left. \left. \left. + 2\pi(x) e^{p\mu_\theta x} G_{p-1}(x) \sum_{i=0}^{q_n} \partial_t^i \varepsilon^\top(x) \text{Sym} \left( P(x) \Theta^{-1} (\bar{M}(x) - M_1(x)) \Theta \right) \partial_t^i \varepsilon(x) \right) \right) dx. \end{aligned}$$

Using an integration by parts in term  $T_{1,p}$  (possible, since observer error in the new coordinates satisfies Property (S) introduced in Definition 1.2 of Chapter 1) and utilizing (3.95) and other trivial bounds, we get

$$\begin{aligned} T_{1,p} &\leq - \lambda_n(L) \pi(L) e^{p\mu_\theta L} G_p(L) + \lambda_n(0) \pi(0) G_p(0) \\ &\quad + (\omega_1 + p\omega_2 + p|\mu_\theta| \|\lambda_n(\cdot)\|_\infty) W_p^p, \end{aligned}$$

where

$$\omega_1 := \frac{\bar{\pi} - 1}{L} \|\lambda_n(\cdot)\|_\infty + \|\lambda_n'(\cdot)\|_\infty, \omega_2 := \frac{\|\lambda_n(\cdot)\|_\infty \|P_x(\cdot)\|_\infty}{\min_{x \in [0, L]} \underline{\text{eig}}(P(x))}.$$

Substituting (3.97b), for each of the cases  $\lambda_n(\cdot) > 0$  and  $\lambda_n(\cdot) < 0$ , the above yields

$$\begin{aligned}
 T_{1,p} &\leq \left[ -\min_{x \in [0,L]} \lambda_n(x) \pi(L) \left( \min_{x \in [0,L]} \underline{\text{eig}}(P(x)) \right)^p e^{p\mu_\theta L} \right. \\
 &\quad \left. + \pi(0) \|\lambda_n(\cdot)\|_\infty (\theta^{n-1} |\bar{K}|)^{2p} \|P(\cdot)\|_\infty^p \right] \left( \sum_{i=0}^{q_n} |\partial_t^i \varepsilon(L)|^2 \right)^p \\
 &\quad + \left( \omega_1 + p\omega_2 + p|\mu_\theta| \max_{x \in [0,L]} \lambda_n(x) \right) W_p^p, \text{ when } \lambda_n(x) > 0, \\
 T_{1,p} &\leq \left[ -\min_{x \in [0,L]} \lambda_n(x) \pi(L) \|P(\cdot)\|_\infty^p (\theta^{n-1} |\bar{K}|)^{2p} e^{p\mu_\theta L} \right. \\
 &\quad \left. + \pi(0) \|\lambda_n(\cdot)\|_\infty \left( \min_{x \in [0,L]} \underline{\text{eig}}(P(x)) \right)^p \right] \left( \sum_{i=0}^{q_n} |\partial_t^i \varepsilon(0)|^2 \right)^p \\
 &\quad + (\omega_1 + p\omega_2 + p|\mu_\theta| \|\lambda_n(\cdot)\|_\infty) W_p^p, \text{ when } \lambda_n(x) < 0,
 \end{aligned}$$

which, by virtue of (3.93), (3.94), is bounded as follows

$$T_{1,p} \leq (\omega_1 + p\omega_2 + p|\mu_\theta| \|\lambda_n(\cdot)\|_\infty) W_p^p. \quad (3.99)$$

Next, for  $\theta > 1$  and invoking (4.25), we obtain for  $T_{2,p}$

$$T_{2,p} \leq (-\theta\omega_3 + \omega_4) W_p^p, \quad (3.100)$$

where

$$\begin{aligned}
 \omega_3 &:= \frac{\min_{x \in [0,L]} \underline{\text{eig}}(P(x)) \min_{x \in [0,L]} \underline{\text{eig}}(Q(x))}{\|P(\cdot)\|_\infty}, \\
 \omega_4 &:= 2 \frac{\|P(\cdot)\|_\infty \|\bar{M}(\cdot) - M_1(\cdot)\|_\infty}{\min_{x \in [0,L]} \underline{\text{eig}}(P(x))}.
 \end{aligned}$$

By (3.98), in conjunction with (3.99), (3.100) we get

$$\dot{W}_p \leq (-\theta\omega_3 + |\mu_\theta| \|\lambda_n(\cdot)\|_\infty + \omega_1 + \omega_2 + \omega_4) W_p. \quad (3.101)$$

We obtained the above estimate of  $\dot{W}_p$  for  $\varepsilon$  of class  $C^{q_n+1}$ . Similarly as in the proof of Theorem 2.1 of Chapter 2 we invoke density arguments, so the results remain valid with  $\varepsilon$  only of class  $C^{q_n}$  (see also [Coron and Bastin (2015)]). Now, one can select the high gain  $\theta$ , such that

$$\theta > \theta_0, \quad (3.102)$$

where  $\theta_0 > 1$  is such that

$$-\theta\omega_3 + |\mu_\theta| \|\lambda_n(\cdot)\|_\infty + \omega_1 + \omega_2 + \omega_4 \leq -2\kappa_\theta, \forall \theta > \theta_0,$$

for some  $\kappa_\theta > 0$ . One can easily check that for any  $\kappa_\theta > 0$ , there always exists a  $\theta_0 > 1$ , dependent on the involved constants, such that the previous inequality is satisfied. Subsequently, (3.101) yields

$$W_p(t) \leq e^{-2\kappa_\theta t} W_p(0). \quad (3.103)$$

Taking also into account (3.95), we get the following property holding for continuous  $\varepsilon, \varepsilon_t$

$$\lim_{p \rightarrow \infty} W_p = \lim_{p \rightarrow \infty} \|e^{\mu\theta} \pi(\cdot)^{\frac{1}{p}} G_p^{1/p}(\cdot)\|_{L^p(0,L)} = \sum_{i=0}^{q_n} \|e^{\mu\theta} \partial_t^i \varepsilon^\top(\cdot) P(\cdot) \partial_t^i \varepsilon(\cdot)\|_\infty. \quad (3.104)$$

By (3.103), in conjunction with (3.104), we derive

$$\sum_{i=0}^{q_n} \|e^{\mu\theta} \partial_t^i \varepsilon^\top(\cdot) P(\cdot) \partial_t^i \varepsilon(\cdot)\|_\infty \leq e^{-2\kappa\theta t} \sum_{i=0}^{q_n} \|e^{\mu\theta} \partial_t^i \varepsilon_0^\top(\cdot) P(\cdot) \partial_t^i \varepsilon_0(\cdot)\|_\infty, \quad (3.105)$$

where  $\varepsilon_0(x) := \varepsilon(0, x)$ . Now, from error dynamics (3.97) in conjunction with continuous differentiability of the dynamics, we can perform simple differentiations, so as to calculate constants  $\rho_{\theta,i}, \sigma_{\theta,i}$ , depending polynomially on  $\theta$ , such that

$$\sum_{i=0}^{q_n} \rho_{\theta,i} \|\partial_x^i \varepsilon(\cdot)\|_\infty \leq \sum_{i=0}^{q_n} \|\partial_t^i \varepsilon(\cdot)\|_\infty \leq \sum_{i=0}^{q_n} \sigma_{\theta,i} \|\partial_x^i \varepsilon(\cdot)\|_\infty.$$

Combining the above estimates with the following inequality

$$\begin{aligned} e^{\frac{\mu\theta - |\mu\theta|}{2}L} \min_{x \in [0,L]} \text{eig}(P(x)) \left( \sum_{i=0}^{q_n} \|\partial_t^i \varepsilon(\cdot)\|_\infty \right)^2 &\leq \sum_{i=0}^{q_n} \|e^{\mu\theta} \partial_t^i \varepsilon^\top(\cdot) P(\cdot) \partial_t^i \varepsilon(\cdot)\|_\infty \\ &\leq e^{\frac{\mu\theta + |\mu\theta|}{2}L} \|P(\cdot)\|_\infty \left( \sum_{i=0}^{q_n} \|\partial_t^i \varepsilon(\cdot)\|_\infty \right)^2, \end{aligned}$$

and also (3.90), we obtain the desired stability result of the following type

$$\|\hat{\zeta}(t, \cdot) - \zeta(t, \cdot)\|_{q_n} \leq \bar{\ell}_\theta e^{-\kappa\theta t} \|\hat{\zeta}^0 - \zeta^0\|_{q_n}, \quad (3.106)$$

where  $\bar{\ell}_\theta$  is a polynomial in  $\theta$  (as in high-gain observers in finite dimensions). Finally, we have that  $\mathcal{T}$  is bounded,  $\mathcal{X}$  is continuously embedded in  $C^{q_n}([0, L]; \mathbb{R}^n)$ , and also the extension of  $\mathcal{T}^{-1}$  on  $C^0[0, L]; \mathbb{R}^n$  is bounded in  $C^0[0, L]; \mathbb{R}^n$  and  $C^{q_n}([0, L]; \mathbb{R}^n)$  is continuously embedded in  $C^0([0, L]; \mathbb{R}^n)$ . Thereby, by (3.106), we can calculate constant  $\ell_\theta$ , polynomial again in  $\theta$  and depending on the values of the functions  $\tau_{i,j}^k$ , such that

$$\|\mathcal{T}^{-1} \hat{\zeta}(t, \cdot) - \xi(t, \cdot)\|_\infty \leq \ell_\theta e^{-\kappa\theta t} \|\mathcal{T}^{-1} \hat{\zeta}^0 - \xi^0\|_{\mathcal{X}}.$$

The proof of Theorem 3.3 is complete.

**Remark 3.5.** Notice that although the system here is linear, the high-gain technique is of special interest to dominate "extra terms" in the Lyapunov derivative, similarly to the nonlinear terms in finite-dimensional high-gain observers. In the present case, indeed, there appears a term  $\mu\theta$  in the Lyapunov derivative (see (3.99)), coming from the boundary conditions, and having an effect similar to nonlinearities in finite-dimensional systems. In a more abstract sense, passing from the finite dimensions to the infinite dimensions, the domain of the hyperbolic (differential) operator imposes extra difficulties, even for linear source terms.

### 3.3.4 Simulation for a chemical reactor

In this subsection, we apply the high-gain observer design to a  $3 \times 3$  system of an exothermic plug flow chemical reactor. Control and observer designs for chemical reactors have been widely investigated (see for instance [Boskovic and Krstic (2002)] and [Christophides and Daoutidis (1996)]).

Here we consider a linearized model, where system's states  $\xi_1, \xi_2, \xi_3$  represent the deviation with respect to their steady values, i.e.,  $\xi_1 = T_c - T_c^*, \xi_2 = T_r - T_r^*, \xi_3 = C_A - C_A^*$ , where  $T_c$  is the coolant temperature,  $T_r$  is the reactor temperature and  $C_A$  is the concentration of the chemical components (see [Bastin and Coron (2016)], Section 5.1.1 for more details). The hyperbolic dynamics satisfy (3.64a) with

$$\Lambda = \begin{pmatrix} V_c & 0 & 0 \\ 0 & V_r & 0 \\ 0 & 0 & V_r + \epsilon \end{pmatrix},$$

$$M(x) = \begin{pmatrix} k_0 & -k_0 & 0 \\ -k_0 & k_0 + k_1\phi_1(x) & k_1\phi_0(x) \\ 0 & -\phi_1(x) & -\phi_0(x) \end{pmatrix}$$

for positive  $V_c, V_r, \epsilon, k_0, k_1$ , and boundary conditions

$$\xi_1(t, 0) = 2\xi(t, L), \quad \xi_2(t, 0) = \xi_2(t, L), \quad \xi_3(t, 0) = \xi_3(t, L),$$

(being fictitious in order to get unstable trajectories for the sake of illustration), while

$$\phi_0(x) = (a + b) \exp\left(-\frac{E}{RT_r^*(x)}\right),$$

$$\phi_1(x) = \left(C_A^*(x) - \frac{b}{a + b} C_A^{\text{in}}\right) \frac{E}{R(T_r^*(x))^2} \phi_0(x),$$

for constants  $a, b, E, R, C_A^{\text{in}}$ . The steady states satisfy the following differential equations over  $[0, L]$

$$V_r \frac{d}{dx} T_r^* = -k_0(T_c^* - T_r^*) + k_1 r(T_r^*, C_A^*),$$

$$V_r \frac{d}{dx} C_A^* = -r(T_r^*, C_A^*),$$

$$V_c \frac{d}{dx} T_c^* = k_0(T_c^* - T_r^*),$$

with  $r(T_r, C_A) = ((a + b)C_A - bC_A^{\text{in}}) \exp\left(-\frac{E}{RT_r}\right)$  the reaction rate and boundary conditions

$$T_r^*(0) = T_r^{\text{in}}, C_A^*(0) = C_A^{\text{in}}, T_c^*(0) = T_c^{\text{in}}$$

For simulation, numerical values are as follows:  $T_r^{\text{in}} = 340\text{K}, C_A^{\text{in}} = 0.02\text{mol} \cdot \text{L}^{-1}, T_c^{\text{in}} = 293\text{K}$ . The length of the reactor is  $L = 1\text{m}$ , the reactive fluid velocity in the reactor is  $V_r = 0.025\text{m} \cdot \text{s}^{-1}$ , the coolant velocity in the jacket is  $V_c = 1.13\text{m} \cdot \text{s}^{-1}$ , the activation energy

is  $E = 11250\text{cal} \cdot \text{mol}^{-1}$ ,  $a = 0.56\text{s}^{-1}$  and  $b = 0.12\text{s}^{-1}$  are rate constants, and  $R = 1.986\text{cal} \cdot \text{mol}^{-1} \cdot \text{K}^{-1}$  is the Boltzmann constant. We also add an artificial constant  $\epsilon = 0.005\text{m} \cdot \text{s}^{-1}$  in the third characteristic velocity to make system strictly hyperbolic and deal with the problem in its full generality (with  $\epsilon \neq 0$ , we have  $q = 3$  instead of  $q = 2$ , for  $\epsilon = 0$ ). Assume that measured output is the cooling temperature, i.e.,

$$y(t, x) = \xi_1(t, x).$$

We now follow the steps of the H-GODP described in the previous subsections. In particular, we apply the transformation (3.75) for  $3 \times 3$  systems, introduced in Subsection 3.3.2, with  $\tau = \epsilon/k_0$ . Choose some initial conditions, such that Assumption 3.5 is satisfied, and now system meets all sufficient conditions for solvability of the H-GODP. We apply Theorem 3.3, with  $\theta = 4$  and  $L(x) = (-1 \ 5 \ 10)^\top$ . As expected, the convergence of the inversely transformed observer state to the unknown state  $\xi$  is guaranteed by Theorem 3.3.

In Figure 3.8 we represent the output  $\xi_1$ . In Figures 3.9-3.11 we see the observation errors for each of the original states  $\xi_1, \xi_2$  and  $\xi_3$ , after choosing arbitrary observer's initial conditions satisfying compatibility conditions of order 2. As expected, the convergence of the inversely transformed observer state to the unknown state  $\xi$  is guaranteed by Theorem 3.3.

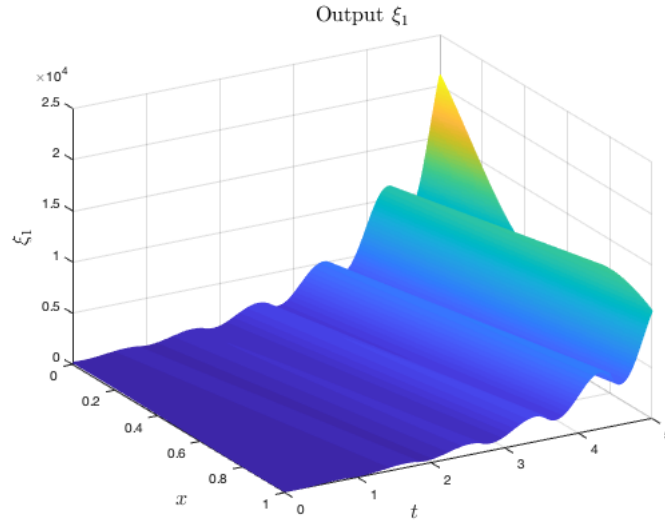


Figure 3.8: Time and space evolution of system's output

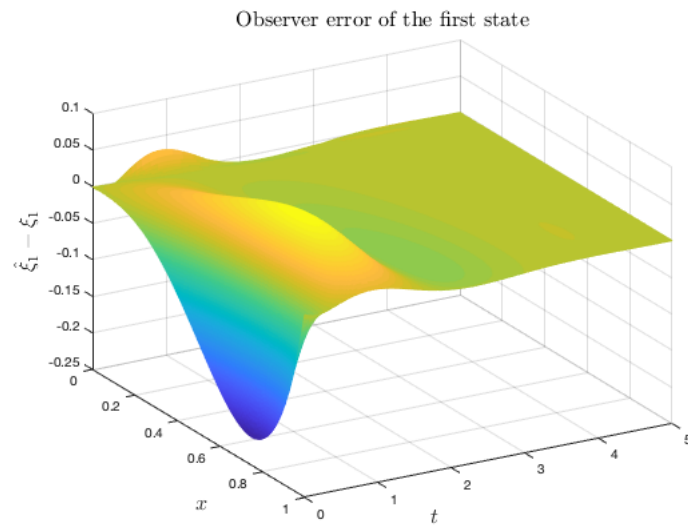


Figure 3.9: Time and space evolution of the first estimation error

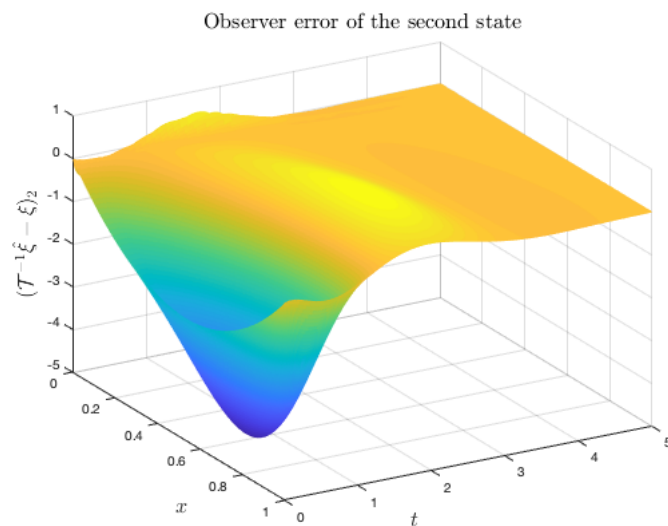


Figure 3.10: Time and space evolution of the second estimation error

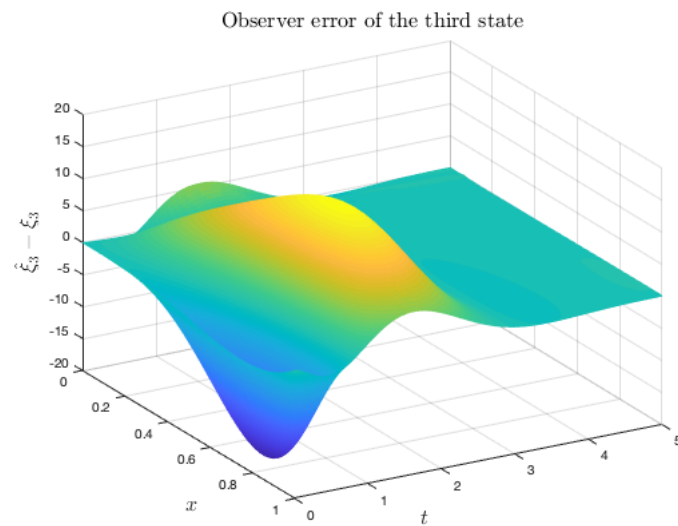


Figure 3.11: Time and space evolution of the third estimation error



### 3.4 Conclusion

In this chapter, an approach to solve the H-GODP for some systems of PDEs was introduced, when fundamental for observer design Property (S) defined in Definition 1.2, is not fulfilled. This problem seems unsolvable by performing direct observer design in the sense of the analysis in Chapter 2 and therefore an indirect approach was adopted, which requires the transformation of the system into target systems of PDEs. These new systems of PDEs retain the triangularity of the initial systems but their differential operators are decomposed in a part which satisfies Property (S) and a part containing the measured state in its domain. This last part of the differential operators can be handled suitably by simply injecting it in the observer dynamics, while they are acting on the measured function.

This general transformation-based methodology, that we introduced, was proven to be applicable to classes of triangular  $2 \times 2$  nondiagonal quasilinear systems with distinct velocities, to  $2 \times 2$  and  $3 \times 3$  Lotka-Volterra-like parabolic systems with distinct diffusivities, and finally to  $n \times n$  linear hyperbolic systems with  $n$  distinct velocities of any sign. The complexity of the methodologies was related to the number of distinct velocities/diffusivities. The method was proven to apply to nonlinear systems of PDEs with up to three states, but it could not be applied for systems with more than three states. However, a solution was given for a class of heterodirectional linear hyperbolic systems of arbitrary number of states and any number of velocities, where the high-gain features turn to be essential to deal with the domain of the hyperbolic operator. Technical difficulties included the types of nonlinearities, which were handled appropriately for the considered systems, the algorithmic calculation of the infinite-dimensional state transformations, and finally the observer convergence proof in appropriate norms, related to the required regularity of the output, which in turns depends on the number of distinct velocities/diffusivities. Also, the theoretical methodology was accompanied with simulations.

Although for finite-dimensional systems a specific observability structure (the so-called triangular structure) leads to solvability of the H-GODP, for the infinite-dimensional spaces, the aforementioned limitations, coming from system's differential operator, do not allow the H-GODP to be solvable for any type of systems, even in the case when their source terms satisfy a triangular structure. Concerning this matter, future developments might include solutions to this H-GODP for more general infinite-dimensional systems and possibly avoiding design limitations, such as compatibility conditions of higher order or the use of spatial derivatives of the output. Also, robustness and Input-to-State Stability (ISS) properties of the observers proposed herein, would be of special interest for future research.

# Output feedback control of a cascade system of Korteweg-de Vries equations

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In this chapter, we present an application of the high-gain observer design to the stabilization of a cascade system of so-called Korteweg-de Vries (KdV for short) equations via output feedback control. The results of this chapter are found in [Kitsos et al (2020c)]. It will be shown that this high-gain design is needed in achieving system stabilization. The considered system includes  $n$  KdV equations in a bounded interval  $(0, L)$  via an output feedback law acting on the left endpoint of the last equation, while the measurement involves only the solution to the first equation. Two boundary control problems are investigated: the classical control on the Dirichlet boundary condition and a less common one on its second-order derivative. The feedback control law utilizes the estimated state provided by a high-gain observer system and the output feedback control leads to two stabilization results: 1) for any  $n$  in the case of Dirichlet boundary conditions and 2) for  $n = 2$  for the second case of boundary conditions.

The nonlinear version of a single KdV equation can describe propagation of waters with small amplitude in closed channels. It was introduced in 1895 and since then its properties have gained much consideration, see for instance [Cerpa et al (2013)]. Surveys on recent progresses and open problems on control and stabilization of such models can be found in [Rosier and Zhang (2009)] and [Cerpa (2014)].

More precisely, some results are as follows. In [Cerpa and Coron (2013)], backstepping method is used for feedback controls placed at the left boundary. Output feedback laws for single linearized and nonlinear KdV equations have been already established via boundary observers in [Marx and Cerpa (2014)] and [Marx and Cerpa (2018)] (see also

[Tang and Krstic (2013)]), by means of backstepping and Lyapunov techniques. In these two works, the measurement injected in the observer involves the right endpoint of the domain, more precisely, the second derivative of the boundary or the Dirichlet condition, depending on the boundary conditions. Output feedback control laws for systems written in the cascade form considered here have not yet appeared in the literature, while controllability of coupled KdV equations but with couplings different from the ones here studied have been investigated for instance in [Cerpa and Pazoto (2011)]. Furthermore, placing the control on the second derivative of the left boundary, as in the second boundary control problem is even more original and its investigation exhibits some technical difficulties.

Here, we aim at observing the full state of a system of KdV equations written in a cascade form and finally controlling it, by considering a one only observation. The cascade form considered here allows a high-gain observer design, which relies on a choice of a sufficiently large parameter in its equations, while appropriate choice of the latter leads simultaneously to the closed-loop output feedback stabilization. This high-gain observer design is instrumental in the technical analysis, while a simpler Luenberger observer design, which is used for linear systems, would not be sufficient. In summary, the contribution of the present chapter first lies in stabilizing the trajectory of the last equation by means of a high-gain observer relying on the measurement of the first state. The control placed at the left boundary, combined with the observer gain brings this trajectory asymptotically to zero in an arbitrarily fast manner. Subsequently, it is proven that 1) the whole cascade system becomes asymptotically stable for any  $L > 0$ , when the first boundary conditions are considered, and 2) this result holds for the second boundary conditions, only when  $n = 2$  (number of equations), noting that for  $n > 2$  stabilization is achieved for quite small  $L$ . The methodology relies on backstepping techniques and appropriate Lyapunov analysis, which are more challenging for the second boundary problem. Exponential stabilization for the second boundary problem is proven here to be linked to the solvability of an ordinary differential equations problem, similar to the differential equation satisfied by the eigenvectors of the associated operator to these KdV equations and subject to some constraints.

In Section 4.1, we prove a preliminary result on the stability of a single damped KdV equation and then we prove the full state stabilization of the cascade system for both boundary condition problems. In Section 4.2, we first present the high-gain observer design for the coupled system and finally the main output feedback stabilization result.

## 4.1 Full State Feedback Stabilization

In this chapter, we study the following cascade system of  $n$  linear KdV equations posed in a bounded interval of length  $L$ :

$$v_t + v_x + v_{xxx} = (A_n - B)v, \text{ in } (0, +\infty) \times (0, L), \quad (4.1)$$

where  $v = (v_1 \ \cdots \ v_n)^\top$  is the state and

$$A_n = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ & \ddots & \ddots & & \vdots \\ \vdots & & & & 1 \\ 0 & \cdots & & & 0 \end{pmatrix}, \quad B = \text{diag}(1, 1, \dots, 1, -1).$$

Note that this system is written in the general form introduced in Section 1.2 of Chapter 1 and used throughout the thesis.

Let us consider two different types of boundary conditions, where the input control  $u$  in both of them is placed at the left side ( $x = 0$ ) and only acts on the  $n$ -th coordinate of the state ( $v_n$ ).

*Boundary conditions A (BC-A):*

$$\begin{aligned} v_i(t, 0) &= 0, \quad i = 1, \dots, n-1, \quad \text{for all } t > 0, \\ v_n(t, 0) &= u(t), \quad \text{for all } t > 0, \\ v(t, L) &= 0, \quad v_x(t, L) = 0, \quad \text{for all } t > 0. \end{aligned} \tag{4.2a}$$

*Boundary conditions B (BC-B):*

$$\begin{aligned} v_{i,xx}(t, 0) &= 0, \quad i = 1, \dots, n-1, \quad \text{for all } t > 0, \\ v_{n,xx}(t, 0) &= u(t), \quad \text{for all } t > 0, \\ v(t, L) &= 0, \quad v_x(t, L) = 0, \quad \text{for all } t > 0. \end{aligned} \tag{4.2b}$$

In order to complete our control system we add an initial condition given by

$$v(0, x) = v^0(x), \quad x \in (0, L) \tag{4.3}$$

and a distributed measurement given by the following output

$$\begin{aligned} y(t, x) &= Cv(t, x); \\ C &= (1 \ 0 \ \cdots \ 0). \end{aligned} \tag{4.4}$$

This system is unstable due to the instability of the trajectory corresponding to its last equation, as it can be seen by following classical energy arguments.

#### 4.1.1 Stability of a single KdV equation

Prior to the stabilization of the cascade system, we present a preliminary result about the stability of a single damped linear KdV equation, which will be invoked in the sequel. Consider a single damped KdV equation in the domain  $(0, L)$ ,

$$w_t + w_x + w_{xxx} + \lambda w = 0, \quad \text{in } (0, +\infty) \times (0, L), \tag{4.5}$$

satisfying one of the following distinct cases of boundary conditions

$$w(t, 0) = w(t, L) = w_x(t, L) = 0, t \geq 0, \quad (4.6a)$$

$$w_{xx}(t, 0) = w(t, L) = w_x(t, L) = 0, t \geq 0, \quad (4.6b)$$

and initial condition of the form

$$w(0, x) = w^0(x), x \in (0, L). \quad (4.7)$$

The stability result for solutions  $w$  to the above problem is presented in the following propositions. Although asymptotic stability for boundary conditions (4.6a) is ensured for every  $\lambda > 0$ , for (4.6b) there is a minimal  $\lambda_0$  for which asymptotic stability is guaranteed only when  $\lambda \geq \lambda_0$ . These results are stated precisely in the next two propositions and will be used throughout this work.

**Proposition 4.1.** *Consider system (4.5) with boundary conditions (4.6a) and initial condition  $w^0 \in L^2(0, L)$ . Then for all  $\lambda > 0$ , we have*

$$\|w(t, \cdot)\|_{L^2(0, L)} \leq e^{-\lambda t} \|w^0(\cdot)\|_{L^2(0, L)}, t \geq 0, \quad (4.8)$$

for every  $L > 0$ .

Proposition 4.1 concerning boundary conditions (4.6a) is a standard result and can be derived from energy estimates. Let us note here, that asymptotic stability for this case can be proven even when the damping is not constant in the domain but localized to a part of it, see for instance [Perla Menzala et al (2002)], and even when the damping is saturated, see [Marx et al (2017)].

To proceed to the stability result for boundary conditions (4.6b), we utilize the following lemma.

**Lemma 4.1.** *There exists  $\lambda_0 > 0$ , such that the following assertions hold true.*

*Assertion 1: For every  $\lambda \geq \lambda_0$ , there exist  $\pi(\cdot)$  in  $C^3(0, +\infty)$  and  $b > 0$ , such that the following holds for all  $x \geq 0$*

$$\begin{cases} \pi'''(x) + \pi'(x) - 2\lambda\pi(x) = -2b\pi(x), \\ \pi''(0)\pi(0) + (\pi'(0))^2 + \pi^2(0) \leq 0, \\ \pi(x) > 0, \\ \pi'(x) \geq 0. \end{cases} \quad (4.9)$$

*Assertion 2: For every  $\lambda \in (0, \lambda_0)$ , there exist  $\bar{L}, b > 0$  and  $\pi(\cdot)$  in  $C^3(0, +\infty)$  satisfying (4.9) for all  $x \in [0, \bar{L}]$ .*

*Proof of Lemma 4.1.* To prove Assertions 1 and 2, it is more convenient to write the characteristic equation of the differential equation in (4.9) as

$$r^3 + r - s^3 - s = 0 \quad (4.10)$$

(as in [Rosier (2004)]), where

$$s^3 + s = 2\lambda - 2b \quad (4.11)$$

and considering  $s$  as the unique real root of the latter equation. Then, solutions to (4.10) are given by

$$r_1 = s, r_2 = -\frac{s}{2} + i\frac{\sqrt{3s^2 + 4}}{2}, r_3 = -\frac{s}{2} - i\frac{\sqrt{3s^2 + 4}}{2}$$

and, therefore, a complete parametrization of solutions to the differential equation in (4.9) is given by

$$\pi(x) = \alpha e^{sx} + \beta e^{-\frac{s}{2}x} \cos \frac{\sqrt{3s^2 + 4}}{2}x + \gamma e^{-\frac{s}{2}x} \sin \frac{\sqrt{3s^2 + 4}}{2}x \quad (4.12)$$

with  $\alpha, \beta, \gamma \in \mathbb{R}$  chosen such that restriction on initial conditions in (4.9) is satisfied. We can check numerically that there exists a number  $\epsilon > 0$  near zero, such that for  $s \geq 1 - \epsilon$ ,  $\pi(\cdot)$  given by (4.12) with initial conditions  $\pi(0) = 4, \pi'(0) = 2, \pi''(0) = -5$  (corresponding to  $\alpha = 56/25, \beta = 44/25, \gamma = 8/25$ ) is positive and increasing and, therefore, satisfies (4.9). Defining a small constant  $\delta > 0$  by

$$\delta := \frac{\epsilon^3}{5} - \frac{3\epsilon^2}{5} + \frac{7\epsilon}{5}, \quad (4.13)$$

we see from (4.11) that for  $s \geq 1 - \epsilon$  we have  $\lambda \geq \lambda_0 := 1 - \delta$  for choice  $b = \frac{11}{16}\lambda$ . Thus, for all  $\lambda \geq \lambda_0$ , there exists  $b > 0$ , such that conditions (4.9) are satisfied. Hence, Assertion 1 is proven to hold for  $\lambda_0 = 1 - \delta$ , where  $\delta$  is defined above. Now, notice that for  $s < 1 - \epsilon$ , corresponding to  $\lambda < \lambda_0$ , and for any initial condition of  $\pi$ , satisfying second equation of (4.9), there is a  $\bar{L} > 0$ , such that for  $x > \bar{L}$ ,  $\pi$  becomes decreasing and, thus, fails to satisfy all conditions (4.9). This implies that for  $0 < \lambda < \lambda_0$ , Assertion 2 is satisfied for some small  $\bar{L} > 0$ . Letting  $s \rightarrow 0^+$ , and choosing initial conditions  $\pi(0) = 4, \pi'(0) = 2, \pi''(0) = -5$  as before,  $\pi$  approaches the trajectory of  $\pi(x) = -1 + 5 \cos(x) + 2 \sin(x)$ , for which  $\pi'(x) < 0$  for  $L > \arctan(2/5)$ . By this, for  $\lambda \rightarrow 0^+, b \rightarrow 0^+$ , Assertion 2 is satisfied with  $\bar{L} = \arctan(2/5)$ .

In Figure 4.1 we see the evolution of  $\pi(x)$  for choice of initial condition  $\pi(0) = 4, \pi'(0) = 2, \pi''(0) = -5$  and various values of  $s$ , corresponding to various values of  $\lambda$ . For small values of  $s$ , corresponding to small values  $\lambda$ ,  $\pi(\cdot)$  is increasing until some point  $x = \bar{L}$  quite small, but for  $x > \bar{L}$ , it is decreasing and, thus, fails to satisfy fourth equation of (4.9) after this point, in accordance with Assertion 2. We also see that for all  $s \geq 1 - \epsilon$ , for  $\epsilon > 0$  small as before,  $\pi(\cdot)$  is everywhere increasing, verifying Assertion 1. The proof is complete.  $\square$

The following proposition concerns the second case of boundary conditions.

**Proposition 4.2.** *Consider system (4.5) with boundary conditions (4.6b) and initial condition  $w^0 \in L^2(0, L)$ . Then, there exists  $\lambda_0 > 0$ , such that:*

1) *For all  $\lambda \geq \lambda_0$ , there exist  $a, b > 0$ , such that the solution to (4.5)-(4.7)-(4.6b) satisfies the following:*

$$\|w(t, \cdot)\|_{L^2(0, L)} \leq ae^{-bt} \|w^0(\cdot)\|_{L^2(0, L)}, t \geq 0, \quad (4.14)$$

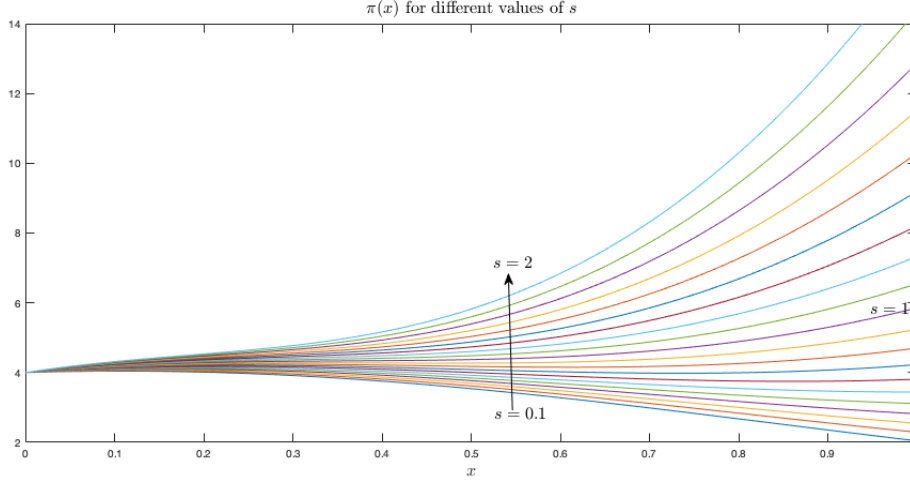


Figure 4.1: Solutions to (4.9) for different  $s$

for every  $L > 0$ .

2) For all  $\lambda \in (0, \lambda_0)$ , there exist  $\bar{L}, a, b > 0$  such that (4.14) is satisfied for all  $L \in (0, \bar{L})$ .

*Proof of Proposition 4.2.* In this context, we are interested by unique solutions in  $L^2(0, L)$ . Well-posedness of the initial boundary value problem (4.5)-(4.7)-(4.6b) can be easily proven by invoking classical arguments, although these boundary conditions are less common in the literature. We consider solutions  $w \in C([0, T]; L^2(0, L))$ . The reader can refer for instance to [Caicedo and Zhang (2017)] for details about the well-posedness in spaces  $H^s(0, L)$ ,  $s \in [0, 3]$ , by noticing hidden regularities and in conjunction with the Kato smoothing property.

Let us consider the following weighted  $L^2$ -norm

$$E(t) := \int_0^L \pi(x) w^2(x) dx$$

along the  $L^2$  solutions to (4.5)-(4.7)-(4.6b), for some choice of positive  $\pi(\cdot) \in C^3[0, L]$ . Calculating its time-derivative along the solutions to (4.5) and applying integrations by parts, we obtain

$$\begin{aligned} \dot{E}(t) = & \int_0^L (\pi'''(x) + \pi'(x) - 2\lambda) w^2(x) dx - 3 \int_0^L \pi'(x) w_x^2(x) dx \\ & + [(-\pi''(x) - \pi(x)) w^2(x) - 2\pi(x) w_{xx}(x) w(x) + \pi(x) w_x^2(x) \\ & + 2\pi'(x) w_x(x) w(x)]_0^L. \end{aligned}$$

Substituting boundary conditions (4.6b) we get

$$\begin{aligned} \dot{E}(t) = & \int_0^L (\pi'''(x) + \pi'(x) - 2\lambda) w^2(x) dx - 3 \int_0^L \pi'(x) w_x^2(x) dx \\ & - \begin{pmatrix} w_x(0) & w(0) \end{pmatrix} \begin{pmatrix} \pi(0) & -\pi'(0) \\ -\pi'(0) & -\pi(0) - \pi''(0) \end{pmatrix} \begin{pmatrix} w_x(0) \\ w(0) \end{pmatrix}. \end{aligned}$$

To ensure the exponential decay of  $E(t)$ , we invoke assertions of Lemma 4.1 for  $\pi(\cdot)$ . By Assertions 1 and 2, second and third term of the above equation become nonpositive and we obtain the existence of a constant  $b > 0$ , such that

$$\dot{E}(t) \leq -2bE(t)$$

and, therefore, (4.14) holds with  $a = \sqrt{\frac{\pi(L)}{\pi(0)}}$ . This completes the proof of Proposition 4.2.  $\square$

#### 4.1.2 Full state stabilization

Following the previous results, we are in a position to study the closed-loop stabilization. Here, the considered state feedback controls, which are placed in the last equation, will be proven to be of the following form for each of the problems (BC-A) and (BC-B)

$$(BC-A): u(t) = \int_0^L p(0, y)v_n(t, y)dy, \quad (4.15a)$$

$$(BC-B): u(t) = -\frac{\omega + 1}{3}Lv_n(t, 0) + \int_0^L p_{xx}(0, y)v_n(t, y)dy, \quad (4.15b)$$

with  $\omega > 0$  to be chosen appropriately and kernel function  $p : \Pi \rightarrow \mathbb{R}$  depending on  $\omega$ , where  $\Pi := \{(x, y); x \in [0, L], y \in [x, L]\}$ .

We now present the exponential decay result of the solution  $v$  to the cascade system (4.1) via the control (4.15), which utilizes the full state. The proof uses backstepping tools appearing in [Cerpa and Coron (2013)], [Cerpa (2014)] for single KdV equations.

**Theorem 4.1.** *Consider system (4.1) with boundary conditions (BC-A) or (BC-B), feedback control laws of the form (4.15) and initial condition  $v^0 \in L^2(0, L)^n$ .*

a) *If (BC-A) holds and  $n \geq 2$ , then for every  $L > 0$ , there exist constants  $c, d > 0$ , such that the solution  $v$  to (4.1) satisfies the following*

$$\|v\|_{L^2(0, L)^n} \leq ce^{-dt}\|v^0\|_{L^2(0, L)^n}, \forall t \geq 0. \quad (4.16)$$

b) *If (BC-B) holds and  $n = 2$ , then for every  $L > 0$ , there exist constants  $c, d > 0$ , such that solution  $v$  to (4.1) satisfies (4.16).*

c) *If (BC-B) holds and  $n > 2$ , then there exists  $\bar{L} > 0$ , such that (4.16) is guaranteed for all  $L \in (0, \bar{L}]$ .*

*Proof of Theorem 4.1.* We first prove a preliminary result concerning the exponential stability of  $v_n$ . We apply the following Volterra transformation

$$z(x) = \mathcal{T}[v_n](x) := v_n(x) - \int_x^L p(x, y)v_n(y)dy \quad (4.17)$$



to the solution to the last equation of the cascade system, with  $p$  defined on  $\Pi$ . Under appropriate choice of  $p(\cdot, \cdot)$ , we prove that this transformation maps solution  $v_n$  to the trajectory  $z$  satisfying the following target equation in  $[0, +\infty) \times [0, L]$

$$\begin{aligned} z_t + z_x + z_{xxx} + \omega z &= 0, \\ (\text{BC-A}, z) : z(t, 0) = z(t, L) = z_x(t, L) &= 0, \\ (\text{BC-B}, z) : z_{xx}(t, 0) = z(t, L) = z_x(t, L) &= 0. \end{aligned} \tag{4.18}$$

with control given by (4.15). Indeed, performing standard differentiations and integrations by parts (for more intuition the reader can refer to [Cerpa and Coron (2013)]), we derive the following equations

$$\begin{aligned} z_t(t, x) + z_x(t, x) + z_{xxx}(t, x) + \omega z(t, x) &= \\ - \int_x^L (p_{xxx}(x, y) + p_{yyy}(x, y) + p_y(x, y) + (\omega + 1)p(x, y)) v_n(t, y) dy & \\ + p(x, L)v_{n,xx}(t, L) + p(x, L)v_n(t, L) + p_{yy}(x, L)v_n(t, L) - p_y(x, L)v_{n,x}(t, L) & \\ + \left( \omega + 1 + \frac{d^2}{dx^2}p(x, x) + \frac{d}{dx}p_x(x, x) + p_{xx}(x, x) - p_{yy}(x, x) \right) v_n(t, x) & \\ + \left( p_x(x, x) + p_y(x, x) + 2\frac{d}{dx}p(x, x) \right) v_{n,x}(t, x). & \end{aligned}$$

By choosing  $p(\cdot, \cdot)$  satisfying the following equations

$$\begin{cases} p_{xxx} + p_{yyy} + p_x + p_y + (\omega + 1)p = 0, & (x, y) \in \Pi \\ p(x, x) = p(x, L) = 0, & x \in [0, L] \\ p_x(x, x) = \frac{\omega+1}{3}(L-x), & x \in [0, L] \end{cases} \tag{4.19}$$

we achieve to obtain target system (4.18) for both boundary problems (BC-A,  $z$ ) and (BC-B,  $z$ ). Solutions to (4.19) are proven to be unique in  $C^3(\Pi)$  in [Cerpa and Coron (2013)], by following successive approximation methods. The feedback control  $u$  is easily checked to satisfy (4.15), if we use (4.17) and also calculate the value of the second derivative

$$\begin{aligned} z_{xx}(x) &= v_{n,xx}(x) + \frac{d}{dx}p(x, x)v_n(x) + p(x, x)v_{n,x}(x) + p_x(x, x)v_n(x) \\ &\quad - \int_0^L p_{xx}(x, y)v_n(y) dy \end{aligned}$$

for  $x = 0$ .

Now, as we saw in the proof of Proposition 4.1 and Proposition 4.2 of the previous subsection, solution  $z$  to target system (4.18) is asymptotically stable for every  $L > 0$ , if  $\omega > 0$  when (BC A,  $z$ ) holds and if  $\omega \geq 1$  when (BC B,  $z$ ) holds. This implies the asymptotic stability of  $v_n$ , solution to (4.1), with control given by (4.15) for each of the boundary problems (BC-A) and (BC-B). The latter follows from the fact that, as proven in [Cerpa and Coron (2013)], transformation (4.17), mapping solution  $v_n$  to  $z$ , is bounded and invertible with bounded inverse. So, for every  $\bar{d} > 0$ , there exist  $\omega_0, \bar{c} > 0$ , such that for all  $\omega \geq \omega_0$ ,

$$\|v_n\|_{L^2(0,L)} \leq \bar{c}e^{-\bar{d}t} \|v_n^0\|_{L^2(0,L)}, \forall t \geq 0. \tag{4.20}$$

To prove the asymptotic stability of the full state, consider vector  $v_{[n-1]} := (v_1 \ \cdots \ v_{n-1})^\top$ . Then,  $v_{[n-1]}$  satisfies the following equations

$$\begin{cases} v_{[n-1],t} + v_{[n-1],x} + v_{[n-1],xxx} = (A_{n-1} - I_{n-1})v_{[n-1]} + \ell v_n, \\ \text{(BC-A, } v_{[n-1]}) : v_{[n-1]}(t, 0) = v_{[n-1]}(t, L) = v_{[n-1],x}(t, L) = 0, \\ \text{(BC-B, } v_{[n-1]}) : v_{[n-1],xx}(t, 0) = v_{[n-1]}(t, L) = v_{[n-1],x}(t, L) = 0, \end{cases}$$

where

$$\ell := (0 \ \cdots \ 0 \ 1)^\top.$$

To prove stability of this system, consider a Lyapunov functional of the following form

$$W(t) = \int_0^L \pi(x) |v_{[n-1]}(x)|^2 dx$$

along the  $L^2(0, L)^{n-1}$  solutions  $v_{[n-1]}$  to the last equations, where  $\pi(\cdot)$  is a positive increasing  $C^3$  function to be chosen. After substituting the above equations satisfied by  $v_{[n-1]}$  and applying integrations by parts, we obtain for the time-derivative of  $W$

$$\begin{aligned} \dot{W}(t) &= \int_0^L (\pi'''(x) + \pi'(x)) |v_{[n-1]}(x)|^2 dx - 3 \int_0^L \pi'(x) |v_{[n-1],x}(x)|^2 dx \\ &\quad - \int_0^L \pi(x) v_{[n-1]}^\top(x) \left( 2I_{n-1} - A_{n-1}^\top - A_{n-1} \right) v_{[n-1]}(x) dx \\ &\quad + 2 \int_0^L \pi(x) v_{n-1}(x) v_n(x) dx + W_0, \end{aligned}$$

with

$$\begin{aligned} W_0 &:= \left[ -(\pi''(x) + \pi(x)) |v_{[n-1]}(x)|^2 + \pi(x) \left( |v_{[n-1],x}(x)|^2 - 2v_{[n-1],xx}^\top(x) v_{[n-1]}(x) \right) \right. \\ &\quad \left. + 2\pi'(x) v_{[n-1],x}^\top(x) v_{[n-1]}(x) \right]_0^L. \end{aligned} \quad (4.21)$$

Matrix  $2I_{n-1} - A_{n-1}^\top - A_{n-1}$  is positive definite and its eigenvalues are

$$\rho := 2 - 2 \cos \frac{\pi j}{n}, j = 1, \dots, n-1.$$

Hence, its minimal eigenvalue is given by

$$\rho_n := \lambda_{\min}(2I_{n-1} - A_{n-1}^\top - A_{n-1}) = 2 - 2 \cos \frac{\pi}{n}, \mathbb{N} \ni n \geq 2. \quad (4.22)$$

Since  $\pi'(x) \geq 0$ , by use of Young's inequality we obtain

$$\begin{aligned} \dot{W}(t) &\leq \int_0^L (\pi'''(x) + \pi'(x) - \rho_n \pi(x)) |v_{[n-1]}(x)|^2 dx \\ &\quad + 2\delta \int_0^L \pi(x) |v_{[n-1]}(x)|^2 dx + \frac{1}{2\delta} \int_0^L \pi(x) v_n^2 dx + W_0, \end{aligned}$$

and  $\delta > 0$  is chosen sufficiently small as in (4.13) in the proof of Lemma 4.1 of the previous subsection.

Now, we choose  $\pi(\cdot)$  for each of the two boundary problems as follows.

For (BC-A,  $v_{[n-1]}$ ) we choose

$$\pi(\cdot) = 1.$$

From this, taking also into account the exponential stability of  $v_n$  (4.20), we get for the case (BC-A,  $v_{[n-1]}$ ) the following estimate

$$\dot{W}(t) \leq -2dW(t) + \frac{1}{2\delta}\pi(L)\bar{c}^2 e^{-2\bar{d}t} \|v_n^0\|_{L^2(0,L)}^2 \quad (4.23)$$

with  $d = \rho_n/2 - \delta$ .

For (BC-B,  $v_{[n-1]}$ ) we choose a positive and increasing  $\pi(\cdot)$  satisfying (4.9) (see Assertion 1 in Lemma 4.1) with  $\lambda = \frac{\rho_n}{2} - \delta$  and  $b > 0$ . It turns out by Assertion 1 that there is  $\pi(\cdot)$  and  $b > 0$  satisfying this equation for any  $L > 0$ , when  $\lambda = 1 - \delta$ , corresponding to  $\rho_2 = 2$  for  $n = 2$ . Then, the exponential decay of the Lyapunov functional is ensured similarly as in Proposition 4.2. More precisely, there exists  $d > 0$ , such that for all  $L > 0$ , (4.23) is satisfied for (BC-B,  $v_{[n-1]}$ ) as well. Also, as shown in Proposition 4.2, for  $n > 2$ , which renders  $\rho_n < 2$ , (4.23) is satisfied for some  $\pi(\cdot)$ ,  $b > 0$ , only when  $0 < L \leq \bar{L}$ , with  $\bar{L}$  depending on  $n$ .

Combining the above results, from (4.23), which holds for both (BC-A,  $v_{[n-1]}$ ) and (BC-B,  $v_{[n-1]}$ ), we derive by Gronwall's inequality

$$W(t) \leq e^{-2dt}W(0) + \frac{\pi(L)\bar{c}^2}{4\delta(d-\bar{d})} \left( e^{-2\bar{d}t} - e^{-2dt} \right) \|v_n^0\|_{L^2(0,L)}^2, \quad (4.24)$$

recalling also, that  $\bar{d}$  depending on the parameter  $\omega$  of the control laws, can be chosen, such that  $\bar{d} > d$ . Combining (4.24) and (4.20), we get

$$\begin{aligned} \|v\|_{L^2(0,L)^n} &\leq \|v_{[n-1]}\|_{L^2(0,L)^{n-1}} + \|v_n\|_{L^2(0,L)} \leq \sqrt{\frac{\pi(L)}{\pi(0)}} e^{-dt} \|v_{[n-1]}(0, \cdot)\|_{L^2(0,L)^{n-1}} \\ &\quad + \frac{\bar{c}\sqrt{\pi(L)}}{2\sqrt{\pi(0)\delta(d-\bar{d})}} \sqrt{e^{-2dt} - e^{-2\bar{d}t}} \|v_n^0\|_{L^2(0,L)} + \bar{c}e^{-\bar{d}t} \|v_n^0\|_{L^2(0,L)}. \end{aligned}$$

The last inequality leads to (4.16) for a suitable choice of  $c$ .

This concludes the proof and shows, also, that although the exponential convergence to zero of  $v_n$  can become arbitrarily fast by the choice of parameter  $\omega$  inside the controls, solution  $v$  to the whole cascade system has a fixed convergence rate.  $\square$

## 4.2 Observer Design and Output Feedback Stabilization

In this section, we first present the proposed observer, along with its convergence proof for each of the boundary control problems (BC-A) and (BC-B). Then, we study the output feedback stabilization of system (4.1) with controls placed at the left boundaries as described in each

of problems (BC-A) and (BC-B). We note here that, even though the considered system is linear, the use of the high-gain observer design is instrumental in the output feedback control in the two following manners. 1) For (BC-B), the choice of the high-gain parameter is needed to establish convergence of the observer, contrary to a simpler Luenberger observer design, which would be sufficient for (BC-A); 2) The high-gain parameter is used in the stabilization of the closed-loop system for both boundary control problems (BC-A) and (BC-B).

In the following subsection we present the observer for the cascade system, whose exponential stability relies on the result presented in Proposition 4.2 of Section 4.1.

### 4.2.1 Observer

Define, first, diagonal matrix  $\Theta_n$  by

$$\Theta_n := \text{diag}(\theta, \theta^2, \dots, \theta^n),$$

where  $\theta > 1$  represents a gain, which will be selected later. Consider a vector gain  $K_n = (k_1 \ \dots \ k_n)^\top$  and let  $P \in \mathbb{R}^{n \times n}$  be a symmetric and positive definite matrix satisfying a quadratic Lyapunov equation of the form

$$2\text{Sym}(P(A_n + K_n C)) = -I_n. \quad (4.25)$$

The previous equation is always feasible, due to the observability of the pair  $(A_n, C)$ .

Then, our observer is defined to satisfy the following equations in  $(0, +\infty) \times (0, L)$

$$\hat{v}_t(t, x) + \hat{v}_x(t, x) + \hat{v}_{xxx}(t, x) = (A_n - B)\hat{v}(t, x) - \Theta_n K_n (y(t, x) - C\hat{v}(t, x)) \quad (4.26)$$

with boundary conditions for each of (BC-A) and (BC-B) as follows

$$\begin{aligned} & \hat{v}_i(t, 0) = 0, i = 1, \dots, n-1, \text{ for all } t > 0, \\ \text{(BC-A): } & \hat{v}_n(t, 0) = u(t), \text{ for all } t > 0, \\ & \hat{v}(t, L) = \hat{v}_x(t, L) = 0, \text{ for all } t > 0, \end{aligned} \quad (4.27a)$$

$$\begin{aligned} & \hat{v}_{i,xx}(t, 0) = 0, i = 1, \dots, n-1, \text{ for all } t > 0, \\ \text{(BC-B): } & \hat{v}_{n,xx}(t, 0) = u(t), \text{ for all } t > 0, \\ & \hat{v}(t, L) = \hat{v}_x(t, L) = 0, \text{ for all } t > 0. \end{aligned} \quad (4.27b)$$

and initial condition

$$\hat{v}(0, x) = \hat{v}^0(x), x \in (0, L).$$

The main observer result is stated in the following theorem.

**Theorem 4.2.** *Consider system (4.1) with output (4.4) and boundary conditions satisfying (4.2) ((BC-A) or (BC-B)) and  $v^0 \in L^2(0, L)^n, u \in L^2_{loc}(0, +\infty)$ . Consider, also,  $P$  and  $K_n$  satisfying a Lyapunov equation as in (4.25). Then, (4.26), with boundary conditions (4.27) and initial condition  $\hat{v}^0 \in L^2(0, L)^n$  is an observer for solution of (4.1), in the sense that for*

$\theta$  large it estimates the state  $v$  arbitrarily fast. More precisely, for every  $\kappa > 0$ , there exist  $\theta_0, \nu > 0$ , such that for every  $\theta > \theta_0$ , the following holds for all  $v^0, \hat{v}^0 \in L^2(0, L)^n$ ,  $t \geq 0$ :

$$\|\hat{v}(t, \cdot) - v(t, \cdot)\|_{L^2(0, L)^n} \leq \nu e^{-\kappa t} \|\hat{v}^0(\cdot) - v^0(\cdot)\|_{L^2(0, L)^n}, \quad (4.28)$$

with  $\nu$  increasing with  $\theta$ ,  $n$  and  $L$ .

*Proof.* Let us define a scaled observer error  $\varepsilon$  by

$$\varepsilon = \Theta_n^{-1}(\hat{v} - v). \quad (4.29)$$

Then,  $\varepsilon$  satisfies the following equations

$$\varepsilon_t + \varepsilon_x + \varepsilon_{xxx} = \theta(A_n + K_n C)\varepsilon - B\varepsilon \quad (4.30)$$

and boundary conditions for each of the cases (BC-A) and (BC-B) as follows

$$\varepsilon(t, 0) = \varepsilon(t, L) = \varepsilon_x(t, L) = 0, \quad (4.31a)$$

$$\varepsilon_{xx}(t, 0) = \varepsilon(t, L) = \varepsilon_x(t, L) = 0. \quad (4.31b)$$

We expect that solutions to the previous coupled equations can approach zero exponentially fast, since  $A_n + K_n C$  being Hurwitz will exhibit a damping effect (as in the single KdV equation), with its magnitude being controlled by  $\theta$ . Indeed, to prove exponential stability, we choose the following Lyapunov functional defined on the  $L^2(0, L)^n$  solutions to the observer error equations

$$V(t) := \int_0^L \mu(x) \varepsilon^\top(x) P \varepsilon(x) dx, \quad (4.32)$$

with positive  $\mu(\cdot) \in C^3[0, L]$  to be chosen suitably for each of the boundary conditions cases. Taking its time-derivative and substituting (4.30) and Lyapunov equation (4.25), yields

$$\begin{aligned} \dot{V}(t) = & \int_0^L \mu(x) \left[ -\partial_x^3 \left( \varepsilon^\top(x) P \varepsilon(x) \right) - \partial_x \left( \varepsilon^\top(x) P \varepsilon(x) \right) + 3\partial_x \left( \varepsilon_x^\top(x) P \varepsilon_x(x) \right) \right. \\ & \left. - \theta \varepsilon^\top(x) \varepsilon(x) - 2\varepsilon^\top(x) P B \varepsilon(x) \right] dx. \end{aligned}$$

Performing successive integrations by parts, we obtain

$$\begin{aligned} \dot{V}(t) \leq & \int_0^L \left( \mu'''(x) + \mu'(x) + \left( -\theta \frac{1}{|P|} + 2 \frac{|P|}{\lambda_{\min}(P)} \right) \mu(x) \right) \varepsilon^\top(x) P \varepsilon(x) dx \\ & - 3 \int_0^L \mu'(x) \varepsilon_x^\top(x) P \varepsilon_x(x) dx + V_0, \end{aligned}$$

where

$$\begin{aligned} V_0 := & \left[ \left( -\mu''(x) - \mu(x) \right) \varepsilon^\top(x) P \varepsilon(x) - \mu(x) \left( \varepsilon_{xx}^\top(x) P \varepsilon(x) + \varepsilon^\top(x) P \varepsilon_{xx}(x) \right) \right. \\ & \left. + \mu(x) \varepsilon_x^\top(x) P \varepsilon_x(x) + \mu'(x) \left( \varepsilon_x^\top(x) P \varepsilon(x) + \varepsilon^\top(x) P \varepsilon_x(x) \right) \right]_0^L \quad (4.33) \end{aligned}$$

and  $\lambda_{\min}(P)$  is the minimal eigenvalue of  $P$ .

Let us now choose for boundary conditions case (4.31a)

$$\mu(\cdot) := 1, \quad (4.34)$$

for which we obtain

$$V_0 = -\varepsilon_x^\top(0)P\varepsilon_x(0) \leq 0.$$

Note that given (4.34) for the boundary conditions case (4.31a), for every  $\theta > \theta_{0,A}$ , with

$$\theta_{0,A} := 2 \frac{|P|^2}{\lambda_{\min}(P)},$$

we get

$$\dot{V}(t) \leq -2\kappa_A V(t), t \geq 0, \quad (4.35)$$

for some  $\kappa_A > 0$ .

Considering boundary conditions of case (4.31b), (4.33) is written as

$$V_0 := - \begin{pmatrix} \varepsilon_x^\top(0) & \varepsilon^\top(0) \end{pmatrix} \begin{pmatrix} P\mu(0) & P\mu'(0) \\ P\mu'(0) & -P(\mu''(0) + \mu(0)) \end{pmatrix} \begin{pmatrix} \varepsilon_x(0) \\ \varepsilon(0) \end{pmatrix}$$

For this case (4.31b), we see here that for all

$$\theta \geq \theta_{0,B} := 2 \frac{|P|^2}{\lambda_{\min}(P)} + 2|P|,$$

Assertion 1 ((4.9) in Lemma 4.1) is satisfied with  $\mu(\cdot)$  in the place of  $\pi(\cdot)$ ,  $\lambda = \theta \frac{1}{2|P|} - \frac{|P|}{\lambda_{\min}(P)}$ ,  $\lambda_0 = 1$  and  $b = \kappa_B$  for some  $\kappa_B > 0$ . For all  $\theta \geq \theta_0$ , we choose, therefore,  $\pi(\cdot) = \mu(\cdot)$  satisfying (4.9) and we derive again (4.35) satisfied for every  $\theta \geq \theta_{0,B}$ , with  $\kappa_A$  substituted by  $\kappa_B$ .

Combining the previous estimates, we directly obtain (4.28) with

$$\nu := \theta^{n-1} \sqrt{\frac{\mu(L)}{\mu(0)}} \sqrt{\frac{|P|}{\lambda_{\min}(P)}}$$

and this concludes the proof of Theorem 4.2.  $\square$

## 4.2.2 Output feedback stabilization

Next, it is proven that plugging the observer's state considered in Theorem 4.2 in the feedback laws (4.15) of the previous section, the closed-loop system is stabilized. This is done in two steps. First, it is proven that the considered output feedback law stabilizes arbitrarily fast the solution of the last KdV equation and second, the stabilization of the whole cascade system of KdV equations follows. However, for system with boundary conditions (BC-B), stabilization for any  $L$  is only achieved when  $n = 2$ , corresponding to a cascade system of two equations, while for  $n > 2$ , stabilization is achieved for small  $L$ , following the result of Proposition 4.2 of the previous section. These statements are presented in the following theorem.

**Theorem 4.3.** Consider the closed-loop system (4.1)-(4.26), output (4.4) and boundary conditions being of the form (BC-A) or (BC-B). Then, for any  $\bar{d} > 0$ , there exist an output feedback law  $u(t)$  of the form (4.15), where  $v$  is substituted by the observer state  $\hat{v}$ , and constants  $\theta_0, \omega_0, \bar{c} > 0$ , such that for any design parameters  $\theta > \theta_0, \omega > \omega_0$  (with  $\theta$  involved in the observer and  $\omega$  involved in the control laws), the closed-loop system solution with  $v^0, \hat{v}^0 \in L^2(0, L)^n$  satisfies the following stability inequality (on the estimation error and last observer state)

$$\|\hat{v} - v\|_{L^2(0,L)^n} + \|\hat{v}_n\|_{L^2(0,L)} \leq \bar{c}e^{-\bar{d}t} (\|\hat{v}^0 - v^0\|_{L^2(0,L)^n} + \|\hat{v}_n^0\|_{L^2(0,L)}), \forall t \geq 0. \quad (4.36)$$

Moreover, whenever the previous assertion holds, we get the following (full state convergence)

a) When boundary conditions (BC-A) hold with  $n \geq 2$ , for every  $L > 0$ , there exist constants  $c, d > 0$ , such that solutions  $v, \hat{v}$  satisfy the following

$$\|\hat{v} - v\|_{L^2(0,L)^n} + \|\hat{v}\|_{L^2(0,L)^n} \leq ce^{-dt} (\|\hat{v}^0 - v^0\|_{L^2(0,L)^n} + \|\hat{v}^0\|_{L^2(0,L)^n}), \forall t \geq 0, \quad (4.37)$$

with  $d$  depending on  $n$ .

b) When boundary conditions (BC-B) hold with  $n = 2$ , for every  $L > 0$ , there exist constants  $c, d > 0$ , such that solutions  $v, \hat{v}$  satisfy (4.37).

c) When (BC-B) holds, with  $n > 2$  there exists  $\bar{L} > 0$  small, such that asymptotic stability (4.37) is guaranteed for all  $L \in (0, \bar{L}]$ .

*Proof.* To address the closed-loop control problem, let us rewrite observer error and observer equations coupling (see (4.30), (4.26)):

$$\begin{cases} \varepsilon_t + \varepsilon_x + \varepsilon_{xxx} &= \theta(A_n + K_n C)\varepsilon - B\varepsilon, \\ \hat{v}_t + \hat{v}_x + \hat{v}_{xxx} &= (A_n - B)\hat{v} + \theta\Theta_n K_n \varepsilon_1, \end{cases} \quad (4.38a)$$

with boundary conditions (4.31), (4.27).

Let us perform a Volterra transformation to the solution of the  $n$ -th equation of the observer, which by (4.26) is written as

$$\hat{v}_{n,t} + \hat{v}_{n,x} + \hat{v}_{n,xxx} = \hat{v}_n + k_n \theta^{n+1} \varepsilon_1. \quad (4.39)$$

The Volterra transformation

$$q(x) = \mathcal{T}[\hat{v}_n](x) := (k_n \theta^{n+1})^{-1} \hat{v}_n(x) - (k_n \theta^{n+1})^{-1} \int_x^L p(x, y) \hat{v}_n(y) dy, \quad (4.40)$$

under appropriate choice of  $p(\cdot, \cdot)$  maps (4.39) into target system

$$q_t + q_x + q_{xxx} = -\omega q + \varepsilon_1 - \int_x^L p(x, y) \varepsilon_1(t, y) dy, \quad (4.41)$$

with  $\omega$  a constant involved in the controller, and boundary conditions for each of the two considered cases as follows

$$q(t, 0) = q(t, L) = q_x(t, L) = 0, \quad (4.42a)$$

$$q_{xx}(t, 0) = q(t, L) = q_x(t, L) = 0. \quad (4.42b)$$

Then, the kernel functions  $p(\cdot, \cdot)$  satisfy (4.19) for both problems (4.42a), (4.42b). It is easy to check this if we apply successive differentiations of (4.40) as in Theorem 4.1, we obtain the above target system, by choosing  $p(\cdot, \cdot)$  satisfying (4.19). Subsequently, the output feedback control  $u(\cdot)$  for (BC-A) is given by

$$u(t) = \int_0^L p(0, y) \hat{v}_n(t, y) dy, \quad (4.43a)$$

and for (BC-B),

$$u(t) = -\frac{\omega + 1}{3} L \hat{v}_n(t, 0) + \int_0^L p_{xx}(0, y) \hat{v}_n(t, y) dy. \quad (4.43b)$$

As noticed in proof of Theorem 4.1, it has been proven that the kernel equations (4.19) are solvable in  $\Pi$  and the corresponding Volterra transformation is bounded and injective with bounded inverse.

Consider now the Lyapunov function

$$U_1(t) = U_{1,1}(t) + U_{1,2}(t);$$

$$U_{1,1}(t) := \int_0^L \mu(x) \varepsilon^\top(x) P \varepsilon(x) dx, U_{1,2}(t) := \int_0^L \sigma(x) q^2(x) dx,$$

along the solutions to (4.30)-(4.31) and (4.41)-(4.42), where  $U_{1,1}$  is the same as (4.32) of Theorem 4.2 and  $\sigma(\cdot)$  is a positive  $C^3$  increasing function in  $[0, L]$  to be chosen later.

Taking the time-derivative of  $U_{1,2}$  and substituting (4.41), we infer

$$\begin{aligned} \dot{U}_{1,2}(t) &= \int_0^L (\sigma'''(x) + \sigma'(x) - 2\omega) q^2(x) dx - 3 \int_0^L \sigma'(x) q_x^2(x) dx \\ &\quad + 2 \int_0^L \sigma(x) q(x) \varepsilon_1(x) dx - 2 \int_0^L \sigma(x) q(x) \int_x^L p(x, y) \varepsilon_1(y) dy dx \\ &\quad + [(-\sigma''(x) - \sigma(x)) q^2(x) - 2\sigma(x) q_{xx}(x) q(x) + \sigma(x) q_x^2(x) \\ &\quad + 2\sigma'(x) q_x(x) q(x)]_0^L. \end{aligned}$$

By using

$$\begin{aligned} 2 \int_0^L \sigma(x) q(x) \int_x^L p(x, y) \varepsilon_1(y) dy dx &\leq U_{1,2}(t) + \sigma(L) \int_0^L \left( \int_x^L p(x, y) \varepsilon_1(y) dy \right)^2 dx \\ &\leq U_{1,2}(t) + L^2 \sigma(L) \max_{x, y \in [0, L]} p^2(x, y) \int_0^L \varepsilon_1^2(y) dy \\ &\leq U_{1,2}(t) + L^2 \frac{\sigma(L)}{\mu(0) \lambda_{\min}(P)} \max_{x, y \in [0, L]} p^2(x, y) U_{1,1}(t), \end{aligned}$$



we get

$$\begin{aligned} \dot{U}_{1,2}(t) \leq & \int_0^L (\sigma'''(x) + \sigma'(x) - 2(\omega - 1)) q^2(x) dx - 3 \int_0^L \sigma'(x) q_x^2(x) dx + hU_{1,1}(t) \\ & + [(-\sigma''(x) - \sigma(x)) q^2(x) - 2\sigma(x) q_{xx}(x) q(x) + \sigma(x) q_x^2(x) \\ & + 2\sigma'(x) q_x(x) q(x)]_0^L, \end{aligned} \quad (4.44)$$

where  $h := (L^2 \max_{x,y \in [0,L]} p^2(x,y) + 1) \frac{\sigma(L)}{\mu(0)\lambda_{\min}(P)}$ .

We can prove that for each of the two cases of boundary conditions we get

$$\dot{U}_1(t) \leq -2\bar{d}U_1(t). \quad (4.45)$$

**Case (BC-A):**

We choose  $\mu(\cdot) = \sigma(\cdot) = 1$  and we obtain:

$$\dot{U}_{1,2}(t) \leq -2(\omega - 1)U_{1,2}(t) + hU_{1,1}(t).$$

As seen in Theorem 4.2, for  $\mu(\cdot) = 1$ , we have:

$$\dot{U}_{1,1}(t) \leq \left( -\theta \frac{1}{|P|} + 2 \frac{|P|}{\lambda_{\min}(P)} \right) U_{1,1}(t).$$

Combining the last two equations, if we choose  $\theta, \omega$  as follows:

$$\theta > h|P| + 2 \frac{|P|^2}{\lambda_{\min}(P)}, \omega > 1,$$

we get a  $\bar{d} > 0$ , such that (4.45) holds.

**Case (BC-B):**

We see here that for all

$$\theta \geq \theta_0 := 2 \frac{|P|^2}{\lambda_{\min}(P)} + h|P| + 2|P|,$$

Assertion 1 ((4.9) in Lemma 4.1) is satisfied with  $\mu(\cdot)$  in the place of  $\pi(\cdot)$ ,  $\lambda = \theta \frac{1}{2|P|} - \frac{|P|}{\lambda_{\min}(P)} - \frac{h}{2}$ ,  $\lambda_0 = 1$ . For all  $\theta \geq \theta_0$ , we choose, therefore,  $\mu(\cdot)$  satisfying (4.9) and we get that the first term of the right hand side of

$$\begin{aligned} \dot{U}_1(t) \leq & \int_0^L \left( \mu'''(x) + \mu'(x) - 2 \left( \theta \frac{1}{2|P|} - \frac{|P|}{\lambda_{\min}(P)} - \frac{h}{2} \right) \mu(x) \right) \varepsilon^\top(x) P \varepsilon(x) dx \\ & + \dot{U}_{1,2}(t), t \geq 0. \end{aligned}$$

becomes negative.

Similarly, for every

$$\omega \geq 2,$$

we can find  $\sigma(\cdot) = \pi(\cdot)$  satisfying (4.9), with  $\lambda = \omega - 1$  and  $\lambda_0 = 1$  and by virtue of Proposition 4.2, right hand side of (4.44) becomes negative. Hence, going back to  $\dot{U}_1$  and choosing  $\theta \geq \theta_0$  and  $\omega \geq 2$ , we can always find  $\mu(\cdot), \sigma(\cdot)$  as in Assertion 1 of Lemma 4.1, in a such way that we always get a  $\bar{d} > 0$ , satisfying again (4.45).

Consequently, for each of the two problems (BC-A) and (BC-B), for each  $\bar{d} > 0$  we can find  $\theta, \omega$ , chosen as before in a way that there exists constant  $\gamma > 0$  depending polynomially on  $\theta$ , such that

$$\|\hat{v} - v\|_{L^2(0,L)^n} + \|q\|_{L^2(0,L)} \leq \gamma e^{-\bar{d}t} (\|\hat{v}^0 - v^0\|_{L^2(0,L)^n} + \|q(0, \cdot)\|_{L^2(0,L)}), \forall t \geq 0.$$

Transformation  $\mathcal{T}$  is bounded with bounded inverse (see the comments in Theorem 4.1) and, therefore, we obtain an inequality as (4.36).

We are now in a position to prove the closed-loop stability for the whole system following the methodology of Theorem 4.1. Let  $\hat{v}_{[n-1]} := (\hat{v}_1 \cdots \hat{v}_{n-1})^\top$ . Then,  $\hat{v}_{[n-1]}$  satisfies the following equations

$$\begin{aligned} \hat{v}_{[n-1],t} + \hat{v}_{[n-1],x} + \hat{v}_{[n-1],xxx} &= (A_{n-1} - I_{n-1})\hat{v}_{[n-1]} + \ell\hat{v}_n + \Theta_{n-1}K_{n-1}(\hat{v}_1 - v_1), \\ (\text{BC-A}, \hat{v}_{[n-1]}) : \hat{v}_{[n-1]}(t, 0) &= \hat{v}_{[n-1]}(t, L) = \hat{v}_{[n-1],x}(t, L) = 0 \\ (\text{BC-B}, \hat{v}_{[n-1]}) : \hat{v}_{[n-1],xx}(t, 0) &= \hat{v}_{[n-1]}(t, L) = \hat{v}_{[n-1],x}(t, L) = 0 \end{aligned}$$

where  $\ell := [0 \ \cdots \ 0 \ 1]^\top$  and  $\Theta_{n-1}, K_{n-1}$  are involved in observer (4.26).

By choosing

$$U_2(t) = \int_0^L \pi(x) |\hat{v}_{[n-1]}(x)|^2 dx$$

as a Lyapunov functional along the  $L^2(0, L)^{n-1}$  solutions to the last equations, with  $\pi(\cdot)$  a positive increasing  $C^3$  function, we obtain

$$\begin{aligned} \dot{U}_2(t) &= \int_0^L (\pi'''(x) + \pi'(x)) |\hat{v}_{[n-1]}(x)|^2 dx - 3 \int_0^L \pi'(x) |\hat{v}_{[n-1],x}(x)|^2 dx \\ &\quad - 2 \int_0^L \pi(x) \hat{v}_{[n-1]}^\top(x) \text{Sym}(I_{n-1} - A_{n-1}) \hat{v}_{[n-1]}(x) dx \\ &\quad + 2 \int_0^L \pi(x) \hat{v}_{n-1}(x) \hat{v}_n(x) dx + 2 \int_0^L \pi(x) \hat{v}_{[n-1]}^\top \Theta_{n-1} K_{n-1} (\hat{v}_1 - v_1) dx + U_{2,0}, \end{aligned}$$

where  $U_{2,0}$  is as  $W_0$  in (4.21) (see the proof of Theorem 4.1), while  $v_{[n-1]}$  is substituted by  $\hat{v}_{[n-1]}$ . Applying Young's inequality, we get

$$\begin{aligned} \dot{U}_2(t) &\leq \int_0^L (\pi'''(x) + \pi'(x) - (\rho_n - 2\delta)\pi(x)) |\hat{v}_{[n-1]}(x)|^2 dx \\ &\quad + \frac{1}{\delta} \int_0^L \pi(x) \hat{v}_n^2(x) dx + \frac{1}{\delta} \theta^{2n-2} |K_{n-1}|^2 \int_0^L \pi(x) |\hat{v}_1(x) - v_1(x)|^2 dx + U_{2,0}, \quad (4.46) \end{aligned}$$

with  $\delta > 0$  chosen sufficiently small, as in (4.13), determined in the proof of Lemma 4.1 of previous section, and  $\rho_n$  defined in (4.22).

Now, to ensure negativity of the Lyapunov derivative, we choose  $\pi(\cdot)$  for each of the two boundary problems as follows.

**Case (BC A,  $\hat{v}_{[n-1]}$ ):**

$$\pi(\cdot) = 1.$$

Then, in conjunction with the previously proven equation (4.36), we get from equation (4.46)

$$\dot{U}_2(t) \leq -2dU_2(t) + me^{-2\bar{d}t} (\|\hat{v}^0 - v^0\|_{L^2(0,L)^n} + \|\hat{v}_n^0\|_{L^2(0,L)})^2 \quad (4.47)$$

where  $d := \rho_n - 2\delta > 0$  and

$$m := \frac{1}{\delta}\pi(L)\bar{c}^2 \max(1, \theta^{2n-2}|K_{n-1}|^2). \quad (4.48)$$

**Case (BC B,  $\hat{v}_{[n-1]}$ ):**

For boundary conditions (BC B,  $\hat{v}_{[n-1]}$ ), to obtain an asymptotic stability result, we first check that for  $n = 2$ , we have  $\rho_n = 2$ . For this  $\rho_2$ , proof of Lemma (4.1) suggests that there exists  $\pi(\cdot)$  satisfying (4.9) for some  $b > 0$ , with  $\pi(\cdot) = \pi(\cdot)$ ,  $\lambda = \frac{\rho_2}{2} - \delta$ . Then, a similar inequality as (4.47) is satisfied for all  $L > 0$ ,  $d = b$  and  $m$  as in (4.48). Additionally, following Assertion 2 in the proof of Lemma 4.1, we see that for any  $n > 2$ , implying  $\rho_n < 2$ , there exist again  $\pi(\cdot)$ ,  $d = b > 0$ , such that (4.9) holds for  $L \in (0, \bar{L}]$ , with  $\bar{L}$  depending on  $n$ .

Now, we see that for both boundary problems (BC-A) and (BC-B), (4.47) gives

$$U_2(t) \leq e^{-2dt}W(0) + \frac{m}{2d - 2\bar{d}}(e^{-2\bar{d}t} - e^{-2dt}) (\|\hat{v}^0 - v^0\|_{L^2(0,L)^n} + \|\hat{v}_n^0\|_{L^2(0,L)})^2.$$

The latter implies

$$U_2(t) \leq \left( e^{-2dt} + \frac{m}{2d - 2\bar{d}}(e^{-2\bar{d}t} - e^{-2dt}) \right) (\|\hat{v}^0 - v^0\|_{L^2(0,L)^n} + \|\hat{v}_n^0\|_{L^2(0,L)})^2. \quad (4.49)$$

Recalling that  $\bar{d}$  depends on the observer adjustable parameter  $\theta$ , we suppose, without loss of generality, that can be chosen such that  $\bar{d} > d$ , so that the previous inequality has meaning.

Now, using trivial inequalities and by virtue of (4.49) and (4.36), we easily get

$$\begin{aligned} \|\hat{v} - v\|_{L^2(0,L)^n} + \|\hat{v}\|_{L^2(0,L)^n} &\leq \|\hat{v} - v\|_{L^2(0,L)^n} + \|\hat{v}_n\|_{L^2(0,L)} + \|\hat{v}_{[n-1]}\|_{L^2(0,L)^{n-1}} \\ &\leq \left[ \bar{c}e^{-\bar{d}t} + \frac{1}{\sqrt{\pi(0)}} \sqrt{e^{-2dt} + \frac{m}{2d - 2\bar{d}}(e^{-2\bar{d}t} - e^{-2dt})} \right] \\ &\quad \times (\|\hat{v}^0 - v^0\|_{L^2(0,L)^n} + \|\hat{v}_n^0\|_{L^2(0,L)}). \end{aligned}$$

The latter completes the proof of Theorem 4.3, suggesting also that the asymptotic rate of the whole closed-loop cascade system is no larger than  $d$ , which is decreasing with  $n$ , contrary to the asymptotic rate for the last state  $v_n$ , which is adjusted by the observer and control parameters.  $\square$

### 4.3 Conclusion

The purpose of this chapter was to show a way the high-gain observers, considered in this thesis, can be applied to the output feedback stabilization of cascade systems of linearized KdV equations. The considered system is written in the cascade forms that we consider throughout the whole thesis. The distributed measurement involves the first state, while the control is placed at the left boundary of the last state. Two types of boundary conditions were considered, the second of which required special treatment, in order to prove the stability result. Via the proposed observer, the output feedback control law stabilizes the solution to the last equation of the cascade system, making the convergence to zero arbitrarily fast. At the same time, the cascade system is stabilized for both boundary control problems, while for the second one, stabilization experiences a limitation with respect to the number of equations.

Future developments of the present framework might consider the inclusion of nonlinearities, as in the original KdV equation.



# Conclusion and perspectives

This thesis proposed solutions to a high-gain observer design problem for some classes of infinite-dimensional systems written as systems of  $n$  PDEs and having a triangular form. The output of the system was assumed to be given by a distributed measurement of a part of the state on the whole domain, namely, its first component only. Solutions were proposed for systems including nonlinearities and nonlocal components in their source terms and also associated with differential operators of various forms. Finally, these designs were applied to the output feedback control of a cascade system of linearized KdV equations. More explicitly, some solutions have been proposed for each of Problems 1, 2, and 3, listed in the Introduction of the thesis.

For **Problem 1**, it was shown that under sufficient conditions for the nonlinearities and system solutions, observers can be designed for some systems including quasilinear systems of PIDEs in triangular form with one velocity and considering a reduced number of observations. Also, under sufficient conditions, including strong regularity of system solutions and, additionally, space periodicity of the boundary conditions for large systems, observers with reduced number of observations can be designed for hyperbolic systems. However, limitations appeared with respect to the generality of the considered hyperbolic operators, the nonlinearities, and the number of the states. Furthermore, observer design with reduced number of observations was achieved for semilinear coupled systems of reaction-diffusion systems with distinct diffusivities, assuming stronger regularity, and also cascade systems of linearized KdV equations. These solutions were provided by using tools from Lyapunov theory, namely, the choice of appropriate Lyapunov functionals, leading to stability in norms of appropriate spaces (with strong regularity). Also, a fundamental tool used in this analysis, was the investigation of existence of infinite-dimensional state transformations, which solve generalized Sylvester operator equations.

Solutions to **Problem 2** were achieved, since the proposed observers along this thesis achieve to exhibit performances similar to the ones of the high-gain observers in finite dimensions. More explicitly, besides the limitations cited in the previous paragraph, the triangular structure of the dynamics allowed the design of a high-gain observer. It relies on a choice of a sufficiently large high-gain constant, which compensates for both nonlinearities and terms coming from the boundary conditions. In this way, arbitrary convergence of the observer was achieved, noting also that for the class of hyperbolic systems it overcomes the minimum time of convergence that characterizes boundary observers. The tools used to solve this problem included extended methods inspired by finite-dimensional high-gain observer designs and appropriate treatment of the nonlinearities in the observer dynamics (by use of smooth saturations) and also of the terms coming from boundary conditions.

**Problem 3** was shown to have a solution for a class of linearized KdV equations. A high-gain observer, relying on an internal measurement of the first stated, provided an estimation of the full state, which was used in the output feedback controls placed in the last equation

on the left boundary. Two boundary control problems were considered: a rather common Dirichlet condition and a special one on the second derivative. The high-gain constant, along with the boundary controls achieved to stabilize the coupled system, although this was possible when system consists of only two equations for the second boundary control problem. This framework introduced new tools of stability analysis and choice of Lyapunov functionals.

The main contributions of the thesis were summarized in the Introduction. The general tools used along the thesis included techniques inspired by finite-dimensional high-gain observer design, nonlinear partial differential equation theory, Lyapunov stability theory for infinite-dimensional systems, stabilization theory, operator equations, and backstepping (in Chapter 4), amongst others.

## Perspectives

Some challenges, related to this work, might be:

- investigation of observability properties for cascade/triangular forms as the ones proposed here, see for instance [Alabau-Boussouira et al (2017)]. Many problems remain open;
- links between observability and observer design for coupled infinite-dimensional systems, similarly as in the finite dimensions;
- observer design for cascade/triangular systems of PDEs with different (more general) coupling operators. Also, systems with differential operators of different types on the diagonal, namely coupled PDEs of different type. In [Alabau-Boussouira (2003)], such open problems with respect to controllability, having various types of couplings, are presented. The observation might be internal, as in the present thesis, but localized in a part of the domain;
- solutions to problems of observer design for nonlinear triangular infinite-dimensional systems, with associated diagonal differential operators of nontrivial coefficients and any number of states. This problem has been partially answered in Chapter 3 here;
- introduction of new observer design techniques for the class of nonlinear triangular infinite-dimensional systems, as the one considered here, and extension of approaches for finite dimensions in the case of infinite-dimensional observer design, inspired by [Tsinias and Kitsos (2019)]. These observer designs would lead, for instance, to finite-time state estimation;
- ISS properties in appropriate norms of the proposed observers for triangular infinite-dimensional systems, with respect to boundary or internal disturbances, related to the considered ones here, see [Bastin et al (2020)];
- ISS properties of finite-dimensional approximated observer systems versus the real PDE observers, as the ones proposed here;

- 
- approximation methods for the use of the here-proposed output spatial derivatives (of high order) in the high-gain observer designs as in Chapter 3, via kernel convolutions for instance, and analysis of the performance;
  - various performance improvements to these high-gain observer designs for infinite-dimensional systems, similarly to the ones adopted for finite-dimensional ones, see for instance [Astolfi et al (2017)]. Some extensions to the infinite dimensions are not straightforward;
  - investigation of the application and realization of the proposed theoretical designs to wider classes of real systems of PDEs;
  - extension of the proposed output feedback controls for cascade systems of linearized KdV equations in Chapter 4 to the case of nonlinear systems of KdV equations. Consideration, also, of different couplings between KdV equations.





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**Résumé** — Cette thèse introduit quelques extensions non-triviales de la synthèse classique des observateurs grand gain pour des systèmes nonlinéaires de dimension finie à quelques classes de systèmes de dimension infinie, ayant la forme de systèmes triangulaires décrites par des équations différentielles aux dérivées partielles (EDP) couplées, où une seule coordonnée de l'état dans tout le domaine spatial est considérée comme la sortie du système. Pour aborder ce problème, des synthèses directes et indirectes d'observateurs sont proposées, en fonction d'une propriété de l'opérateur différentiel, associé à chaque système d'EDP. D'abord, en suivant la synthèse directe, la solvabilité de ce problème de synthèse des observateurs grand gain est prouvée pour une classe de systèmes d'équations integrodifférentielles hyperboliques quasilineaires avec termes sources et une seule vitesse de propagation. Ensuite, pour le cas de vitesses distinctes, une synthèse indirecte est proposée pour une classe de systèmes quasilineaires hyperboliques  $2 \times 2$  et une classe de systèmes linéaires inhomogènes hyperboliques  $n \times n$ . Ce type de synthèse est aussi appliqué à une classe de systèmes semilineaires de reaction-diffusion de 2 ou de 3 équations. La synthèse indirecte introduit des transformations d'état de dimension infinie des systèmes considérés vers des systèmes cibles d'EDP, qui permettent l'injection de dérivées spatiales de la sortie dans la dynamique de l'observateur. La convergence des observateurs proposés dans des normes d'espaces de régularité appropriés est basée sur des outils de type Lyapunov. La thèse contient aussi des applications des résultats théoriques obtenus à des exemples de modèles épidémiques, de réacteurs chimiques et de systèmes Lotka-Volterra avec diffusion. Enfin, les synthèses d'observateurs proposées sont appliquées à la stabilisation par retour de sortie d'un système d'équations linéaires de Korteweg-de Vries en cascade, où deux problèmes différents de contrôle frontière sont considérés.

**Mots clés :** observateur grand gain, systèmes nonlinéaires, systèmes hyperboliques, systèmes paraboliques, équation Korteweg-de Vries, commande de systèmes de dimension infinie.

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**Abstract** — This thesis introduces some non-trivial extensions of the classical high-gain observer design for finite-dimensional nonlinear systems to some classes of infinite-dimensional systems, written as triangular systems of coupled partial differential equations (PDEs), where an observation of one coordinate of the state along the spatial domain is considered as system's output. To deal with this problem, depending on a property of the differential operator associated to each system of PDEs, direct and indirect observer design is proposed. First, via direct observer design, solvability of this high-gain observer design problem is proven for a class of systems of quasilinear hyperbolic partial integro-differential equations of balance laws with a single characteristic velocity. Then, for the case of distinct velocities, indirect observer

design is proposed for a class of  $2 \times 2$  quasilinear and a class of  $n \times n$  linear inhomogeneous hyperbolic systems. This design is also applied to semilinear reaction-diffusion systems of 2 and 3 equations. The indirect design introduces infinite-dimensional state transformations of the considered systems to target systems of PDEs and this leads to the injection of spatial derivatives of the output in the observer dynamics. The convergence of the proposed observers in norms of appropriate regularity spaces is based upon various introduced Lyapunov tools. The thesis also addresses the application of the proposed theoretical results to epidemic models, chemical reactors, and diffusional Lotka-Volterra systems. Finally, the proposed observer designs are applied to the output feedback stabilization of a cascade system of linear Korteweg-de Vries equations, where two different boundary control problems are considered.

**Keywords:** high-gain observer, nonlinear systems, hyperbolic systems, parabolic systems, Korteweg-de Vries equation, control of infinite-dimensional systems.

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