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Yassine Tahraoui

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Yassine Tahraoui. Parabolic problems with constraints, deterministic and stochastic.. Analysis of PDEs [math.AP]. Université de Pau et des Pays de l'Adour; Ecole normale supérieure de Kouba (Alger), 2020. English. NNT: 2020PAUU3026 . tel-03126849

HAL Id: tel-03126849

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THÈSE

Pour l'obtention du grade de

Docteur de l'université de Pau et des Pays de l'Adour
&
de l'École Normale Supérieure de Kouba

ED 211

Domaine de recherche: **Mathématiques Appliquées.**

Présentée par

Yassine TAHRAOUI

Problèmes paraboliques à contraintes, déterministes et stochastiques.

Directeurs de thèse : **Guy VALLET & Abdelhafid MOKRANE**

Co-directeur de thèse : **Bénédicte PUIG**

Soutenue le 04 décembre 2020 à 10h en Salle 02 de l'IPRA

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Acknowledgements

First and foremost, words are not enough to express my deepest gratitude to Guy Vallet. Thank you very sincerely for your quality of supervision, from my Master thesis until the end of my PhD thesis. Thank you for your availability, patience and valuable comments which helped me to learn new things every day. Thank you for your support and helps every time I needed. Without Guy, this work would not have succeeded. I'm a lucky person to work with such an outstanding supervisor! Now, I would like to thank Bénédicte Puig for her support, remarques, comments and valuable discussions to enrich this work. I would like to thank as well Abdelhafid Mokrane for his advices, supports and continuous efforts to give me the best opportunities.

I would thank Caroline Bauzet for her detailed relecture of this manuscript and comments which helped to improve this manuscript. Thank you as well to Gilles Carbou for having accepted preside the jury of my defense. My sincere thanks go to Imran Biswas, Anis Matoussi and José Francisco Rodrigues for their time and effort in reviewing this work. I would thank Bénédicte Alziary-Chassat for having accepted be part of the jury in my defense. My thanks go to my collaborator Olivier Guibé for the valuable discussion that we have, it was a pleasure to work with him.

I would to thank my parents, my brother and sisters, all my friends and my colleagues for their encouragement and support.

Yassine T.

ملخص

عنوان الأطروحة: المسائل المكافئة ذات حواجز في الإطارين الحتمي و التصادفي.

كاتب الأطروحة: ياسين طهراوي.

تحت إشراف الأستاذين:

◦ في فاليه (Guy Vallet) ، أستاذ محاضر HDR ، مخبر الرياضيات و تطبيقاتها،
جامعة بو بفرنسا.

◦ عبد الحفيظ مقران (Abdelhafid Mokrane) ، أستاذ تعليم عالي، مخبر المعادلات التفاضلية
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لغة الأطروحة: الإنجليزية.

تعتبر المتباينات التغيرية من المواضيع المهمة في الرياضيات و لها عدة تطبيقات. في هذه الأطروحة سنهتم بدراسة بعض المسائل الناقصية و المكافئة في الإطارين الحتمي و التصادفي. بعبارة أدق، سندرس وجود الحلول و متراجحات لوي - ستامباكيا (Lewy - Stampacchia) المرفقة بها. في الفصل الأول نقوم بدراسة مسألة ناقصية ذات حازرين في إطار فضاءات سوبولاف بأس متغير حيث المؤثر الرئيسي من نوع لوراي - ليونس (Leray - Lions) و يتم إثبات وجود حل يحقق متراجحة لوي - ستامباكيا بإستعمال تقنية إرباك المؤثر. هذه النتيجة تعمم النتائج و تقلص الفرضيات الموجودة في الأعمال السابقة. في الفصل الثاني ندرس مسألة مكافئة مع حازر في إطار فضاءات سوبولاف حيث المؤثر الرئيسي من نوع لوراي - ليونس ثمّ نثبت وجود حل يحقق متراجحة لوي - ستامباكيا بإستعمال طريقة الجزاء و طريقة إرباك المؤثر المستعملة في الفصل الأول. في الفصل الأخير نثبت الوجود، الوحدانية و متراجحة لوي - ستامباكيا المرفقة ببعض المسائل المكافئة التصادفية مع حازر و مؤثرات رتبية، للوصول إلى النتيجة المذكورة نقوم بإستعمال تقنية إرباك رد الفعل التصادفي.

في الأخير عرضنا في ملاحق بعض النتائج العددية بإستعمال برنامج سيلاب (Scilab) إضافة إلى إثبات بعض النتائج المستعملة في دراسة المسائل محل الدراسة.

كلمات مفتاحية: متباينات تغيرية، مؤثرات شبه - رتبية، متراجحة لوي - ستامباكيا، طريقة الجزاء، معادلات تفاضلية جزئية تصادفية، عملية وينر، مسائل ذات حواجز.

Abstract

Title : Parabolic problems with constraints, deterministic and stochastic.

Author : Yassine TAHRAOUI

Supervisors of the doctoral thesis :

- Guy Vallet, LMAP-UMR CNRS 5142, Université de Pau et des Pays de l'Adour, France.
- Abdelhafid Mokrane, Laboratoire d'EDPNL et HM, École Normale Supérieure de Kouba, Algeria.

In this thesis, our aim is to study elliptic and parabolic problems with constraints in the frame of deterministic and stochastic settings. More precisely, we are interested in the existence of solutions and the associated Lewy-Stampacchia (L-S) inequalities.

In the first chapter, we are interested in the proof of L-S inequalities associated with a bilateral elliptic problem governed by a pseudomonotone operator in the frame of Sobolev spaces with variable exponents, we prove a result of existence of solutions satisfying L-S inequalities by using a technique of perturbation of the operator. In the second chapter, we study a parabolic variational inequality with constraint where we prove a result of existence of a solution satisfying L-S inequalities; by a method of penalization of the constraint and a technique of perturbation of the operator. In the last chapter, we are interested in a stochastic parabolic obstacle problem governed by a T -monotone operator in the presence of a stochastic reaction where we prove a result of existence and uniqueness of the solution satisfying L-S inequalities; by using a method of penalization of the constraint and perturbation of the stochastic reaction.

Finally, we present some numerical illustrations of the previous problems in the one- dimensional space setting.

Keywords: Variational inequalities, Pseudomonotone operator, Lewy-Stampacchia inequality, Penalization, Stochastic PDE, Wiener process, Obstacle problem.

Résumé succinct

Titre : Problèmes paraboliques à contraintes, déterministes et stochastiques.

L'auteur : Yassine TAHRAOUI

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Dans cette thèse, notre but est d'étudier des problèmes elliptiques et paraboliques avec des contraintes dans les cadres déterministes et stochastiques. Plus précisément, nous nous intéressons à l'existence de solutions et aux inégalités de Lewy-Stampacchia (L-S) associées.

Dans le premier chapitre, nous nous intéressons à la preuve des inégalités de L-S associées à un problème elliptique bilatéral gouverné par un opérateur pseudomonotone dans le cadre des espaces de Sobolev avec des exposants variables, nous prouvons un résultat d'existence de solutions satisfaisant les inégalités de L-S en utilisant une technique de perturbation de l'opérateur. Dans le deuxième chapitre, nous étudions une inégalité variationnelle parabolique avec contrainte où nous prouvons un résultat d'existence d'une solution satisfaisant les inégalités de L-S ; par une méthode de pénalisation de la contrainte et une technique de perturbation de l'opérateur. Dans le dernier chapitre, nous nous intéressons à un problème d'obstacle parabolique stochastique régi par un opérateur T -monotone en présence d'une réaction stochastique où nous prouvons un résultat d'existence et unicité de la solution satisfaisant les inégalités de L-S ; en utilisant une méthode de pénalisation de la contrainte et une perturbation de la réaction stochastique.

Enfin, nous présentons quelques illustrations numériques des problèmes précédents.

Mots clés: Inéquations variationnelles, Opérateur pseudo-monotone, Inégalité de Lewy-Stampacchia, Pénalisation, EDP stochastique, Problème d'obstacle, Processus de Wiener.

General notations

- $d \geq 1$ Dimension of the space domain.
- D A bounded domain of \mathbb{R}^d with a Lipschitz boundary if $d \geq 2$.
- ∂D The boundary of D for $d \geq 2$.
- $T > 0$ Maximal time of the study.
- $]0, T[$ Time interval of the study.
- Q The product space $]0, T[\times D$.
- \xhookrightarrow{d} The dense embedding.
- \tilde{q} The minimum between p and 2 .
- H The Lebesgue space $L^2(D)$.
- V A separable reflexive Banach space.
- V' The dual space of V .
- $\langle \cdot, \cdot \rangle$ The duality product between the space and its dual.
- f^+ The positive part of the function f i.e. $f^+ = \max(f, 0)$.
- f^- The negative part of the function f i.e. $f^- = \max(-f, 0)$.
- ψ, ψ_1, ψ_2 The obstacle functions.
- (Ω, \mathcal{F}, P) A complete probability space.
- Ω_T The product space $]0, T[\times \Omega$.
- $(\mathcal{F}_t)_{t \geq 0}$ A right continuous filtration.
- $W(t)$ A Wiener process.
- E The expectation, i.e. the integral on Ω with respect to the measure P .

Functional spaces

$\mathcal{D}(D), \mathcal{C}_c^\infty(D)$ The space of $\mathcal{C}^\infty(D)$ – functions with compact support in D .

$W_0^{m,p}(D)$ The closure of $\mathcal{D}(D)$ in the Sobolev space $W^{m,p}(D)$.

$W^{-m,p'}(D)$ The dual space of $W_0^{m,p}(D)$, $(\frac{1}{p} + \frac{1}{p'} = 1)$.

$\mathcal{C}(I, V)$ The space of continuous function in I with values in V .

$\mathcal{D}'(I, V)$ The space of distributions on I with values in V .

$L^p(D) := \{f : D \rightarrow \mathbb{R} \text{ measurable and } \int_D |f(x)|^p dx < \infty\}$, $(1 \leq p < \infty)$.

$L^\infty(D) := \{f : D \rightarrow \mathbb{R} \text{ measurable and } \exists C > 0 \text{ such that } |f(x)| \leq C \text{ a.e. in } D\}$.

$W^{m,p}(D) := \{f \in L^p(D) | D^\alpha f \in L^p(D) \alpha \in \mathbb{N}^d \text{ with } |\alpha| := \sum_{i=1}^d \alpha_i \leq m\}$, $(1 \leq p < \infty)$.

$L^p(I, V) := \{f : I \rightarrow V \text{ measurable and } \int_D \|f(x)\|_V^p dt < \infty\}$, $(1 \leq p < \infty)$.

$L^\infty(I, V) := \{f : I \rightarrow V \text{ measurable and } \exists C > 0 \text{ such that } \|f(x)\|_V \leq C \text{ a.e. in } I\}$.

$W(0, T) = \{u \in L^p(0, T, V), \partial_t u \in L^{p'}(0, T, V')\}$.

$W^{1,\infty}(0, T)$ The element $u \in L^\infty(0, T)$ such that $\partial_t u \in L^\infty(0, T)$.

$(L^{p'}(0, T, V'))^+$ The non-negative cone of $L^{p'}(0, T, V')$, i.e. the elements $u \in L^{p'}(0, T, V')$ such that $\langle u, \varphi \rangle \geq 0$ for any $\varphi \in L^p(0, T, V)$ such that $\varphi \geq 0$.

$L^p(0, T, V)^*$ The order dual equals $(L^{p'}(0, T, V'))^+ - (L^{p'}(0, T, V'))^+$.

$W^{1,p,q}(0, T, V, E)$ The space of functions $u \in L^p(0, T; V)$ such that $\partial_t u \in L^q(0, T; E)$.

$\mathcal{N}_W^2(0, T, H)$ The space of all predictable processes of $L^2(\Omega_T, H)$.

Résumé élargi

Dans cette thèse, notre but est d'étudier des problèmes elliptiques et paraboliques avec contraintes dans les deux cas déterministes (sans bruit) et stochastiques (avec bruit). En d'autres termes, on s'intéresse à des Équations aux Dérivées Partielles (EDP) elliptiques et paraboliques sous forme d'inéquations variationnelles ou d'équations, où la solution est recherchée dans un convexe fermé lié aux contraintes.

Plus précisément, on s'intéresse à des problèmes avec obstacle : ψ étant donné, trouver u tel que

$$\partial_t u + Au \geq f \quad \text{ou} \quad \partial_t(u - \int_0^\cdot h(u)dW) + Au \geq f \quad \text{sous la contrainte} \quad u \geq \psi.$$

Les inéquations variationnelles sont bien connues dans la littérature des mathématiques appliquées et conduisent à de nombreuses applications (cf. [26, 62]). Du point de vue mathématique, il est important d'avoir des informations sur le défaut dans l'équation $\partial_t u + Au = f$ (resp. $\partial_t(u - \int h(u)dW) + Au = f$), qui n'est satisfaite *a priori* que dans l'ensemble libre $\{u > \psi\}$ alors que l'inégalité $\partial_t u + Au - f \geq 0$ (resp. $\partial_t(u - \int h(u)dW) + Au - f \geq 0$) doit toujours être vérifiée. C'est ce que l'on appelle " **l'inégalité de Lewy-Stampacchia**" où la valeur par défaut $\partial_t u + Au - f$ (resp. $\partial_t(u - \int h(u)dW) + Au - f$) est contrôlée, sur le complémentaire de l'ensemble libre, par le terme similaire $\partial_t \psi + A\psi - f$ (resp. $\partial_t(\psi - \int h(\psi)dW) + A\psi - f$) agissant sur la contrainte ψ .

On trouve quelques travaux dans la littérature sur ce sujet, en déterministe et lorsque la partie principale A est un opérateur T -monotone dans le cas parabolique (cf. [22]) et plusieurs travaux dans le cas elliptique [41, 48, 49, 51]. Notre objectif est de généraliser de tels résultats à une classe plus générale d'opérateurs dans le cas déterministe. Ensuite, nous étudions des problèmes bruités en introduisant le caractère aléatoire (traduit les effets aléatoires dans le problème) en considérant un second membre de l'équation de type "intégrale d'Itô". La présence de termes stochastiques d'Itô ne nous permet pas d'adapter facilement les techniques de compacité-monotonie développées pour des opérateurs pseudo-monotones dans nos travaux en déterministe à la recherche de solutions martingales. L'enjeu dans cette partie est donc d'adapter principalement les techniques de monotonie du cas déterministe au cas stochastique.

Faisant suite à une première étude de A. Mokrane et G. Vallet [51] sur l'inégalité de Lewy-Stampacchia dans les espaces de Sobolev à exposants variables pour un opérateur pseudo-monotone, le **Chapitre I** propose, en collaboration avec ces deux auteurs, une généralisation de ce travail. Plus précisément, il s'agit du papier [50] où on considère un problème elliptique de même nature avec une contrainte bilatérale associée à deux obstacles ψ_1 et ψ_2 et en simplifiant les hypothèses de façon significative. Les inégalités de Lewy-Stampacchia associées aux contraintes $\psi_1 \leq u$ et/ou $u \leq \psi_2$ s'écrivent alors :

$$\begin{cases} LS_1 & A(u) + a_0(u) - f \leq (A(\psi_1) + a_0(\psi_1) - f)^+, \\ LS_2 & -(A(\psi_2) + a_0(\psi_2) - f)^- \leq A(u) + a_0(u) - f, \end{cases}$$

avec un opérateur pseudo-monotone non linéaire de type Leray-Lions A , un opérateur monotone de Nemitsky a_0 et une solution u de l'inégalité variationnelle

$$u \in K, \quad \langle A(u) + a_0(u), v - u \rangle \geq \langle f, v - u \rangle, \quad \forall v \in K,$$

où K est un convexe fermé de $W_0^{1,p(\cdot)}(D)$ lié aux contraintes. Nous utilisons une technique de perturbation *ad hoc* de l'opérateur et une pénalisation des contraintes pour prouver les inégalités de Lewy-Stampacchia. Nous discutons également sous quelles hypothèses les deux parties des inégalités de Lewy-Stampacchia ci-dessus sont simultanément vérifiées.

Dans le **Chapitre II**, nous nous intéressons à des inéquations variationnelles paraboliques sous une contrainte d'obstacle. Plus précisément, nous étudions l'existence d'une solution pour le problème:

$$\int_0^T \langle \partial_t u, v - u \rangle dt + \int_Q a(t, x, u, \nabla u) \nabla(v - u) dx dt \geq \int_0^T \langle f, v - u \rangle dt$$

qui satisfait l'inégalité de Lewy-Stampacchia suivante :

$$0 \leq \partial_t u - \operatorname{div}[a(\cdot, \cdot, u, \nabla u)] - f \leq g^- = (f - \partial_t \psi + \operatorname{div}[a(\cdot, \cdot, \psi, \nabla \psi)])^-,$$

où $u \mapsto -\operatorname{div}[a(t, x, u, \nabla u)]$ est un opérateur pseudo-monotone sous la contrainte $u \geq \psi$, $f \in L^{p'}(0, T; V')$ et $\psi \in W^{1,p,p'}(0, T, W^{1,p}(D) \cap L^2(D), V')$. Cette étude fait partie d'un travail avec O. Guibé, A. Mokrane et G. Vallet [33]. Nous exhiberons un résultat d'existence d'une solution qui satisfait l'inégalité de Lewy-Stampacchia par une méthode de pénalisation de la contrainte et la transformation du problème en un

problème monotone dans l'ensemble libre $\{u \leq \psi\}$ par une technique de perturbation de l'opérateur.

Dans le **Chapitre III**, nous étudions un problème d'obstacle parabolique stochastique associé à un opérateur T -monotone et une force stochastique par la présence d'une réaction stochastique. Plus précisément, on a prouvé un résultat d'existence et d'unicité d'une solution u pour une famille de problèmes d'obstacles qui peuvent être écrits sous la forme

$$f - \partial_t \left(u - \int_0^\cdot G(u, \cdot) dW \right) - A(u, \cdot) \in \partial I_K(u),$$

où K est un convexe fermé de $L^p(\Omega_T, V)$ lié à la contrainte stochastique ψ , A est un opérateur T -monotone non linéaire défini sur V , $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$ est un espace probabilisé filtré et $W(t)$ est un processus de Wiener dans un espace de Hilbert séparable H . Ensuite, nous donnons les inégalités de Lewy-Stampacchia associées, à savoir

$$0 \leq \partial_t \left(u - \int_0^\cdot G(u, \cdot) dW \right) + A(u, \cdot) - f \leq \left(f - \partial_t \left(\psi - \int_0^\cdot G(\psi, \cdot) dW \right) - A(\psi, \cdot) \right)^-.$$

Nous utilisons pour cela une perturbation *ad hoc* de la réaction stochastique et une pénalisation de la contrainte pour prouver l'existence et l'unicité de la solution variationnelle, c'est-à-dire, une solution forte au sens probabiliste de l'inéquation variationnelle stochastique, puis les inégalités de Lewy-Stampacchia associées au problème.

Finalement, nous présentons quelques exemples et illustrations numériques avec le logiciel libre *Scilab* suivi par un rappel de quelques résultats d'analyse fonctionnelle, utilisés dans la thèse, tels qu'un résultat de continuité pour des fonctions à valeurs vectorielles non classique puisque u et $\partial_t u$ ne sont pas dans des espaces en relation de dualité, quelques explications sur le lemme d'Aubin-Lions-Simon quand $1 < p < 2$, une formule d'intégration par parties de Mignot-Bamberger [4] / Alt-Luckhaus [2], un résultat de densité pour les espaces de Sobolev à valeurs vectorielles et aussi une démonstration de l'existence d'une solution, pour un problème parabolique pseudo-monotone, via la méthode de Galerkin. Ensuite, nous proposons certaines extensions aux situations où l'obstacle et la solution ne sont pas dans le même espace, ou aux problèmes d'obstacles bilatéraux.

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General Introduction

In this thesis, we are interested in the study of elliptic and parabolic problems with constraints in the deterministic (without noise) and stochastic (with noise) frameworks. More specifically, we are interested in elliptic and parabolic PDE in the form of variational inequalities or equations, where we look for the solution in a close convex set related to the constraint. In other words, we are interested in problems like: ψ being given, find u such that

$$\partial_t u + Au \geq f \quad \text{or} \quad \partial_t(u - \int_0^\cdot h(u)dW) + Au \geq f \quad \text{under the constraint} \quad u \geq \psi.$$

From a mathematical point of view, it is important to have information about the equation $\partial_t u + Au = f$ (resp. $\partial_t(u - \int h(u)dW) + Au = f$), which is satisfied *a priori* only in the free set $\{u > \psi\}$ while the inequality $\partial_t u + Au - f \geq 0$ (resp. $\partial_t(u - \int h(u)dW) + Au - f \geq 0$) must always be satisfied. This is what we call "**Lewy-Stampacchia inequality**" where the default $\partial_t u + Au - f$ (resp. $\partial_t(u - \int h(u)dW) + Au = f$) is controlled, on the complement of the free set, by the similar term $\partial_t \psi + A\psi - f$ (resp. $\partial_t(\psi - \int h(\psi)dW) + A\psi - f$) acting on the constraint ψ .

One of the main parts of this thesis is devoted to prove this type of inequalities associated with some problems in different frames; after proving the existence of a solution to such problems.

1.1 Variational inequalities and obstacle problems

Variational inequalities are well known in the literature of applied mathematics and lead to many applications. Historically, the theory of variational inequalities bourned in Italy in the sixties as mentioned in the book of J.F. Rodrigues [62] with the work of G. Fichera in 1963 on elasticity problem and the work of G. Stampacchia in 1964 in the frame of potential theory in connection with capacity. Then, the study of variational inequalities evolved over time with several contributions in pure mathematics, PDE's and variational calculus as well as in applied mathematics.

To give an illustration about the physical origins of variational inequalities, let us present the following examples, the first one concerning an elliptic problem, which is presented in [62] and is related to the following problem: find the equilibrium position $u = u(x), x \in D \subset \mathbb{R}^2$ of an elastic membrane constrained to lie above a given obstacle $\psi = \psi(x)$. It is solved by the unique solution of the minimization problem

$$\min_{v \in K} \int_D |\text{grad}_x v|^2 dx,$$

where K is a convex set of functions in an appropriate space greater or equal to ψ . This problem is equivalent to a variational inequality

$$u \in K \quad \text{and} \quad \int_D \text{grad}_x u \cdot \text{grad}_x (v - u) dx \geq 0 \quad \forall v \in K.$$

Another form of the obstacle problem, assuming u regular, is the complementary problem

$$u \geq \psi, \quad -\left(\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2}\right) \geq 0 \quad (u - \psi)\left(\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2}\right) = 0 \quad \text{in } D,$$

D is divided into two regions, the *coincidence set* $I = \{x, u(x) = \psi(x)\}$ and its complement, the *free set* $S = \{x, u(x) > \psi(x)\}$. The common boundary $\partial I = \partial S$ in D is called the *free boundary*, and formally one has

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} = 0 \quad \text{in } S, \quad u = \psi \quad \text{and} \quad \text{grad}_x u = \text{grad}_x \psi \quad \text{on } \partial I.$$

The second example concerns a parabolic problem from [26], consider $u(t, X)$ which represents the pressure of a fluid at point $X = (x, y, z)$ at the time t , occupying a region D of \mathbb{R}^3 limited by a membrane Γ of negligible thickness but semi-permeable (allowing

the free passage of the fluid entering but forbidding on the contrary any fluid outlet). By [26], u satisfies the following equation

$$\frac{\partial u}{\partial t} - \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) = g \quad \text{in } D, t > 0, \quad (1.1)$$

g is a given function, with boundary conditions in the form of inequalities:

$$\begin{cases} u(t, X) > 0 \implies \partial u(t, X)/\partial n = 0, & X \in \Gamma, \\ u(t, X) = 0 \implies \partial u(t, X)/\partial n \geq 0, & X \in \Gamma, \end{cases} \quad (1.2)$$

and the initial condition $u(0, X) = u_0(X)$.

The free boundary conditions imply for any moment t fixed, that we have two regions Γ_0^t and Γ_1^t where $u(t, X) = 0$ and $\partial u(t, X)/\partial n = 0$ respectively. These regions are not given *a priori*, so this is a "free boundary problem". Let us introduce the set K which allows us to rephrase (1.1) and (1.2) in the following equivalent form:

$$\begin{cases} K = \{v \in H^1(D), \quad v \geq 0 \quad \text{on } \Gamma\}, \\ u(t, \cdot) \in K \quad \forall t \geq 0, \\ \int_D \left(\frac{\partial u}{\partial t}(v - u) + \text{grad}_X u \cdot \text{grad}_X(v - u) - g(v - u) \right) dX \geq 0 \quad \forall v \in K. \end{cases} \quad (1.3)$$

This problem with the initial condition u_0 called "Evolution inequality of parabolic type".

Let us refer to some applications of stochastic obstacle problems. First, in the pricing of American contingent claims. We recall that the American option is a contract which gives the right to the holder to exercise the option at any time before the maturity time. It is necessary to introduce superstrategies with a value greater than the payoff of the option to hedge the additional risk of early exercise which leads, in terms of stochastic differential equations, to a reflected backward stochastic differential equation. In other words, the American option is forced to stay above a given stochastic process and corresponds to the solution of reflected backward SDEs of the following form (formally),

$$\begin{cases} Y_t = X_T + \int_t^T f(s, Y_s, Z_s) ds + K_T - K_t - \int_t^T Z_s^* dW_s, \\ Y_t \geq \psi_t \quad 0 \leq t \leq T, \\ \int_0^T (Y_t - \psi_t) dK_t = 0 \end{cases}$$

where Y is the American price process, K is an increasing process which pushes the solution upward to remain above the obstacle and active only when the constraint is saturated, Z is the portfolio process and X_T is the terminal condition. One can consult [10, 36] for more details. In this direction, it's worth noting that obstacle problems for SPDE are a natural generalization of PDEs with obstacle and also related to reflected SDEs where the authors in [27] proved that the solution of a reflected BSDE provides a probabilistic formula for the unique viscosity solution of a parabolic deterministic obstacle problem.

As another application, let us refer to [7] where the authors presented a stochastic obstacles parabolic problem, which can be used to describe several physical phenomena. They were interested, in particular, by the evolution of damage in a continuous medium taking into account the microscopic description which leads to the presence of stochastic dynamics. If one denotes by u the damage parameter, that is, the local proportion of active cohesive bonds in the micro-structure of the material. The damage parameter is forced to take values in the interval $[0, 1]$ where $u = 1$ means that the material is completely undamaged, $u = 0$ that it is completely damaged, while $u \in]0, 1[$ describes an intermediate situation. Then, the evolution of the damage parameter can be described by an Allen-Cahn equation of the following form

$$\begin{cases} g(u) + f - \partial_t(u - \int_0^\cdot h(u)dW) + \Delta u \in \partial I_{[0,1]}(u), \\ \text{Neumann boundary condition} + \text{An initial condition,} \end{cases} \quad (1.4)$$

where h is a function which depends on the damage parameter and reflect the fact that the phenomenon of damage is related to microscopic changes in the structure and configuration of the material lattice, f represents the external source of damage (mechanical or chemical), g is a positive Lipschitz function related to the internal cohesion of the material and it may depend on the damage parameter, vanishing in the case of complete damage. The sub-differential $\partial I_{[0,1]}$ represents the physical constraint on u and W is a standard adapted continuous Brownian motion.

Generally, different approaches lead to the so called variational formulation or weak formulation, the question of in what sense this type of approach solves the original problems is crucial and therefore the analysis and the study of the smoothness of the solutions is very important from a mathematical point of view where Lewy-

Stampacchia inequalities help to answer such question.

To conclude this section, without exhaustiveness, let us mention some other motivations to study variational inequalities and obstacle problems, we start with a famous application: the dam problem in porous media in dimension two which is an appropriate model for the flow in the channels and interesting in hydrodynamics [17] (for dimension three see [72]). Some phase transition questions based on Stefan's problem studied in [62]. In petroleum engineering, a mass conservation given by a Buckley-Leverett equation under a pressure constraint to maintain the gas dissolution in oil presented in [42]; but this can also be related to metastable systems where a constraint is imposed to avoid phase changes as in the stochastic model presented in [74]. As an application in population dynamics, a mathematical model of glass eels migration in an river is exposed in [55] where the light and salinity are constraints on the dynamics of glass eals. We recommend the interested reader to consult the following references [26, 62] and their references.

1.2 Elliptic Lewy-Stampacchia inequalities

In this section, we propose to present a quick overview of some results concerning Lewy-Stampacchia inequalities in the case of elliptic problems.

In 1969, H. Lewy and G. Stampacchia studied in [40] the smoothness of the solution to some variational inequalities, by using the Green function G associated with a linear operator L ($Lu = - \sum_{i,j=1}^d \frac{\partial}{\partial x_j} (a_{ij}(x) \frac{\partial u}{\partial x_i})$) and its associated measure μ ; which satisfies $Lu = \mu$ in the sense of distributions. They proved that the first derivative of the solution is continuous. Then they proposed in [41] a strictly local method to study the regularity of the solution of a superharmonic problem by presenting the solution as following:

$$u(x) = \int_D G(x, y) d\mu(y), \quad x \in D$$

where D is a smooth bounded domain and μ is a non-negative Radon measure supported in the coincidence (contact) set. Then, they proved the following result:

Theorem 1.1. Assume ψ to be continuous in $D \cup \partial D$; $\psi < 0$ on ∂D and

$$-c_d \Delta \psi = \nu (= \text{signed measure}) \quad \text{in the sense of distributions,} \quad (\star)$$

where $c_d > 0$. Let $\bar{\nu}$ be the positive part of ν , μ be the measure associated with u i.e:

$$u(x) = \int_D G(x, y) d\mu(y)$$

with $G(x, y)$ is the Green's function (for $-c_d \Delta$) of D . Then, for every Borel set $F \subset D$ the following inequality holds

$$\mu(F) \leq \bar{\nu}(F). \quad (\text{The first L-S inequality})$$

The assumption (\star) can be written in the form:

$$\psi(x) = \int_D G(x, y) d\nu(y) + h(x), \quad \psi(x) = h(x) \quad \text{on} \quad \partial D;$$

$h(x)$ is harmonic in D and continuous in $D \cup \partial D$.

The authors used Theorem 1.1 to prove that the first derivatives of the solution are Hölder-continuous with an exponent $\alpha > 0$ in the case $x \in D, d > 2$.

If $d = 2$, the solution is Hölder-continuous with exponent $\alpha', 0 < \alpha' < \alpha$ and they conclude that the second derivatives are in $L^p(D)$ if $\Delta \psi \in L^p(D), p > 1$ if one uses an inequality of Calderon-Zygmund type.

In 1973, U. Mosco and G.M. Troianiello in [53] proved Lewy-Stampacchia inequality i.e: $\mu(F) \leq \nu^+(F)$, for all Borel set $F \subset D$, associated with the following problem:

$$\begin{cases} \psi \in H^1(D) \cap C^0(\bar{D}), \psi \leq 0 \quad \text{on} \quad \partial D, \\ L\psi = \nu = \nu^+ - \nu^- \quad (\text{signed measure}), \\ K = \{v \in H_0^1(D) : v \geq \psi \text{ in } H^1(D)\} \\ u \in K, \langle Lu, v - u \rangle \geq 0 \quad \forall v \in K, \end{cases}$$

where L is a uniformly elliptic operator in the bounded domain D of the form

$$Lu = - \sum_{i,j=1}^d \frac{\partial}{\partial x_j} (a_{ij}(x) \frac{\partial u}{\partial x_i}) = \mu, \quad a_{ij}(x) \quad \text{are bounded measurable real coefficients.}$$

The proof used a capacity potential instead of the explicit representation of the solution by Green function which was used where $L = -c_d \Delta$.

After the result of H. Lewy and G. Stampacchia [41], many authors have been interested in the so called Lewy-Stampacchia inequality and used to prove existence and regularity results of nonlinear elliptic operators. In 1997, M.C. Palmeri in [56] studied a variational inequality involving a pseudomonotone operators of Leray-Lions type. The author used an approximation method via "a bounded penalization" called homographic approximation to prove that the sequence of solutions of approximated problems converges strongly to the solution of the variational inequality which satisfies the constraint and the associated Lewy-Stampacchia inequality.

It is worth noting that the penalization is a method used to prove the existence of solutions of variational inequalities, it consists in approximating the variational inequality by a family of nonlinear Dirichlet problems depending on a small positive parameter ε . There exists in the literature several penalization methods. The classical one, as mentioned in [56], was introduced by J.L. Lions [43] and is applied to study nonlinear elliptic unilateral problems, this kind of penalization which involve unbounded mappings in the penalization term is called "unbounded penalization". We will use this method in the study of different problems in this thesis.

In 1998, A. Mokrane and F. Murat in [48] proved the following Lewy-Stampacchia inequality

$$\operatorname{div}(a(x, u, \operatorname{grad}_x u)) + f \geq (\operatorname{div}(a(x, \psi, \operatorname{grad}_x \psi)) + f)^-$$

associated to the following problem:

$$\begin{cases} K(\psi) = \{v \in W_0^{1,p}(D), v \geq \psi \text{ in } D\}, & \psi \leq 0 \text{ on } \partial D \\ u \in K(\psi), & \int_D a(x, u, \operatorname{grad}_x u) \cdot (\operatorname{grad}_x v - \operatorname{grad}_x u) dx \geq \langle f, v - u \rangle, \forall v \in K(\psi), \end{cases}$$

under appropriate assumptions on a, f and ψ .

It's worth noting that the authors used a classical penalization method to prove the above result. It seems to be the first one using this penalization method to prove Lewy-Stampacchia inequality associated with a pseudomonotone operator. In 2004, the authors generalized in [49] the previous result to the case of bilateral obstacle problem in the presence of second order Leray-Lions operator, by using again the classical penalization. Note that the difference between the works of the authors in [56] and [48] is in the method of penalization used to study the problems.

In 2005, A. Azevedo, J.F. Rodrigues and L. Santos in [3] derived a regularity result and the stability of the coincidence set by using Lewy-Stampacchia inequality for N -membrane problem associated with degenerate elliptic operators.

In 2008, J.F. Rodrigues, M. Sanchón and J. M. Urbano [64] proved the existence and uniqueness of an entropy solution to an obstacle problem with L^1 -data and nonlinear elliptic operator where the principal part is a $p(\cdot)$ -Laplacian with variable exponents. Then Lewy-Stampacchia inequalities are extended in the L^1 -framework. The authors used Lewy-Stampacchia inequalities to show some convergence and stability properties of the corresponding coincidence set.

In 2011, J.F. Rodrigues and R. Teymurazyan in [65] studied a bilateral obstacle problem for some monotone operators in the context of Orlicz-Sobolev spaces. Then the authors proved Lewy-Stampacchia inequalities in abstract form associated with a bilateral problem for T -monotone operators, and derived some results of regularity for the solution. As another application of Lewy-Stampacchia inequalities, the authors studied quasi-variational inequalities which can be related to a stochastic switching game.

In 2014, A. Mokrane and G. Vallet in [51] proved Lewy-stampacchia inequality in the context of variable exponents Sobolev spaces by using a penalization method. It's worth noting that the authors generalized the result presented in [48] by adapting the same assumptions to the framework of variable exponents Sobolev spaces. We wish to draw the reader's attention onto the fact that there is some additional assumptions -now useless- used to establish Lewy-Stampacchia inequality.

The subject of Lewy-Stampacchia inequality associated with unilateral and bilateral obstacle elliptic problems in the context of variable exponents Sobolev spaces with assuming only the usual assumptions on the data will be presented in the first chapter.

1.3 Parabolic Lewy-Stampacchia inequalities

In this section, we would refer to some results about Lewy-Stampacchia inequalities associated with some parabolic problems.

In 1982, F. Donati in [22] used an approach based on a penalization method to prove a result of existence and uniqueness of the solution to a parabolic variational inequality governed by a hemicontinuous, bounded, coercive and T -monotone operator from \mathcal{V} into its dual \mathcal{V}' , namely

$$u \geq \psi \text{ in } Q, \quad u(0) = u_0 \geq \psi(0), \quad \langle u' + Au - f, v - u \rangle \leq 0 \quad \forall v \in \mathcal{V}, \quad v \geq \psi \text{ in } Q,$$

where $Q = D \times]0, T[$, $D \subset \mathbb{R}^d$ is a smooth domain and $\mathcal{V} = L^p(0, T; W_0^{1,p}(D) \cap L^2(D))$, $u_0 \in L^2(D)$ and appropriate assumptions on the obstacle ψ and f . Then, the author proved the following Lewy-Stampacchia inequality associated with the problem:

$$u' + Au - f \leq P^+,$$

where

$$\psi' + A\psi - f = P^+ - P^- \quad \text{with} \quad P^\pm \in \mathcal{V}', P^\pm \geq 0.$$

In 1983, M. Biroli in [11] used Lewy-Stampacchia inequality and some regularity results for quasilinear parabolic equations to establish some regularity results about some parabolic obstacle problems.

In 1995, L. Mastroeni and M. Matzeu in [45] used some estimates of Lewy-Stampacchia type and a fixed point argument to prove the existence and regularity for a linear integro-differential parabolic variational inequality connected with the problem of the American option pricing.

In 2013, T. Klimsiak in [37] generalized Lewy-Stampacchia inequality for non-Radon measures. The author proved the existence, uniqueness and stochastic representation of solutions of the Cauchy problem for semilinear parabolic equations in divergence form with two time-dependent obstacles, by using methods of the theory of backward stochastic differential equations.

The subject of Lewy-Stampacchia inequality associated with unilateral obstacle parabolic problems governed by a pseudo-monotone operator of Leray-Lions type in the deterministic setting with assuming the usual assumptions on the data will be presented in the second chapter.

1.4 Stochastic obstacle problems

In this section, we would refer to some results about stochastic partial differential equations with obstacles.

A. Bensoussan and J.L. Lions studied in [8] some second order partial differential equations with obstacles, some stochastic control and optimal stopping-time problems. They were interested, among other things, in the strong relation between these kind of problems where they proved that certain variational inequalities possess a probabilistic interpretation. It's worth noting that the authors used the analytic and probabilistic methods in the proofs, the probabilistic methods are more intuitive and the analytic methods are more powerful when the variational formulation and energy techniques can be applied. They showed also the correspondence between Stefan problem and some optimal stopping-time problems, they proved that some problems of differential games can be formulated as free boundary problems and variational inequalities.

In 1989, U.G. Haussmann and E. Pardoux in [34] proved the existence and uniqueness of strong solution of stochastic partial differential equations of parabolic type with reflection in one-dimensional setting. The authors formulated the problem in the form of a stochastic variational inequality where the principal operator is a linear one, then penalization method and compactness arguments have been used to prove the result where the requirement of the space dimension to be one is necessary. It's worth noting that they discussed the absolute continuity of the reflected measure with respect to Lebesgue measure.

In 1993, C. Donati-Martin and E. Pardoux in [23] studied a reflected solutions of a nonlinear heat equation, driven by a space time white noise, on the spatial interval

$[0, 1]$ with Dirichlet boundary conditions. The authors proved the existence of a non negative solution by using a penalization method, a construction of a monotone approximated sequence and a comparison theorem for solutions of white-noise driven SPDEs. It's worth noting that they didn't prove the uniqueness result.

Over time, many authors have been interested in the study of stochastic obstacle problem via different approaches. The interested reader can consult, for example, the following references [59, 76, 77] and their references. We will refer now to some contributions about stochastic obstacle problem in the last decade.

In 2010, A. Matoussi and L. Stoica in [47] proved an existence and uniqueness result for the obstacle problem of quasilinear stochastic PDE on the whole space \mathbb{R}^d and driven by a finite dimensional Brownian motion with a given terminal condition. The authors used the doubly stochastic calculus, the probabilistic representation of the divergence term in the treatment of the quasilinear part and the penalization method in the study of the problem.

In 2014, L. Denis, A. Matoussi and J. Zhang in [18] proved an existence and uniqueness result for a quasilinear stochastic PDE with obstacle in an open bounded domain of \mathbb{R}^d driven by an infinite dimensional Brownian motion, with Dirichlet boundary condition. The analytical approach based on the parabolic potential theory, developed by M. Pierre in the stochastic framework, have been used to study the problem. It's worth noting that the authors proved a quasi-continuity result of the solution by mixing pathwise argument and some existence result for some deterministic PDEs. They proved also that the reflected measure is a random regular measure satisfying a minimal condition.

In 2017, C. Bauzet et. al. in [7] studied a time noise-driven Allen-Cahn equation which represents the evolution of damage in continuous media in the presence of stochastic dynamics (see (1.4)). The authors used a time discretization method and Yosida approximation to prove the global-in-time existence and uniqueness of solution to the problem.

In 2017, A. Rascanu and E. Rotenstein studied in [61] an obstacle problem for parabolic SPDEs in the frame of Gelfand-Lions triple space setup. More precisely, the authors were interested on the qualitative analysis of a stochastic variational inequality with a regular deterministic obstacle in the context of stochastic differential inclusions driven by a Hölder continuous multiplicative noise, with the Hölder exponent $\alpha \in [1/2, 1]$. A result of existence and uniqueness of strong variational solution has been proved, by using a regularization by Yosida approximation of the sub-differential operator, where the assumption that the barriers cancel the diffusion coefficients played a crucial role to estimate this term, and therefore to prove the existence of a strong variational solution. When $\alpha \in [1/2, 1[$, they considered a linear operator, independent of time, in a Hilbert space setup where the initial condition was a deterministic one. An existence result of weak variational solution satisfying some energy inequalities on a new stochastic base and a new Wiener process has been established when the assumption that the barriers have to cancel the diffusion coefficients was given up. The proof in the latter case has been based on Prokhorov's and Skorokhod's theorems and it's worth noting that no uniqueness result was established in this case. Our aim in the third chapter is to revisit similar variational inequalities by adding random dependences for the operator, the source and the stochastic reaction terms, and the obstacle to establish the existence and uniqueness of strong variational solution with Lipschitz multiplicative noise.

The third chapter is devoted to prove the existence, uniqueness and Lewy-Stampacchia inequality for some stochastic parabolic obstacle problems.

1.5 Organisation of the manuscript

We propose in the organization of this manuscript to resume the following works: the first work in collaboration with O. Guibé, A. Mokrane and G. Vallet [33], the second work with A. Mokrane and G. Vallet [50] and finally, the work with G. Vallet [73]. We chose to keep the same form as the articles. Each chapter begins with a brief summary and a quick review about the literature on some former results; then the content of the study. Finally, we add Appendices to provide some detailed proofs of some results used in Chapter 2 and some numerical illustrations.

In **Chapter I**, after the presentation of the results of A. Mokrane and G. Vallet [51] on Lewy-Stampacchia inequality in the framework of Sobolev spaces with variable exponents, a bilateral elliptic problem will be exposed, resulting from a work in collaboration with A. Mokrane and G. Vallet [50]. More precisely, we generalize the results and simplify the proofs known in the literature to the cases of bilateral problems, that is, Lewy-Stampacchia inequalities associated with constraints $\psi_1 \leq u$ and/or $u \leq \psi_2$ and a bilateral elliptic problem in Sobolev spaces with variable exponents, *i.e.*

$$\begin{cases} LS_1 & A(u) + a_0(u) - f \leq (A(\psi_1) + a_0(\psi_1) - f)^+, \\ LS_2 & -(A(\psi_2) + a_0(\psi_2) - f)^- \leq A(u) + a_0(u) - f, \end{cases}$$

with a nonlinear pseudomonotone operator of the Leray-Lions type A , a Nemitsky monotone operator a_0 and the solution u to the variational inequality

$$u \in K, \quad \langle A(u) + a_0(u), v - u \rangle \geq \langle f, v - u \rangle, \quad \forall v \in K,$$

where K is a closed convex set of $W_0^{1,p(\cdot)}(D)$ related to the constraints. By using an *ad hoc* perturbation of the operator and a penalization of the constraints, we are able to prove Lewy-Stampacchia inequalities. We also discuss under what assumptions the two parts of Lewy-Stampacchia inequalities hold simultaneously.

In **Chapter II**, we are interested in parabolic variational inequalities under a constraint, which generalize the result of F. Donati [22]. Namely, we prove the existence of a solution to the following problem:

$$\int_0^T \langle \partial_t u, v - u \rangle dt + \int_Q a(t, x, u, \nabla u) \nabla(v - u) dx dt \geq \int_0^T \langle f, v - u \rangle dt$$

which satisfies the following Lewy-Stampacchia inequality:

$$0 \leq \partial_t u - \operatorname{div}[a(\cdot, \cdot, u, \nabla u)] - f \leq g^- = (f - \partial_t \psi + \operatorname{div}[a(\cdot, \cdot, \psi, \nabla \psi)])^-,$$

where $u \mapsto -\operatorname{div}[a(t, x, u, \nabla u)]$ is a pseudomonotone operator under the constraint $u \geq \psi$, $f \in L^{p'}(0, T; V')$ and $\psi \in W^{1,p,p'}(0, T, W^{1,p}(D) \cap L^2(D), V')$. This study is a joint work with O. Guibé, A. Mokrane and G. Vallet [33]. The result is proved by using a method of penalization of the constraint and transform the problem to a monotone one in the set $\{u \leq \psi\}$ by using a technique of operator perturbation.

In **Chapter III**, we study a stochastic parabolic problem with a constraint governed by a T -monotone operator, a stochastic force in the presence of a stochastic reaction. More specifically, we prove the existence and uniqueness of the solution u to some obstacle problems that can be written as

$$f - \partial_t \left(u - \int_0^\cdot G(u, \cdot) dW \right) - A(u, \cdot) \in \partial I_K(u),$$

where K is a closed convex set of $L^p(\Omega_T, V)$ related to the stochastic constraint ψ , A is a nonlinear T -monotone operator defined on V , $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$ is a filtered probability space and $W(t)$ is a Wiener process with values in a separable Hilbert space H . Then we prove Lewy-Stampacchia inequalities, namely

$$0 \leq \partial_t \left(u - \int_0^\cdot G(u, \cdot) dW \right) + A(u, \cdot) - f \leq \left(f - \partial_t \left(\psi - \int_0^\cdot G(\psi, \cdot) dW \right) - A(\psi, \cdot) \right)^-.$$

We use an *ad hoc* perturbation of the stochastic reaction and a penalization of the constraint to prove the existence, uniqueness of the variational solution and Lewy-Stampacchia inequalities associated with the problem.

Finally, this document ends with Appendices. In Appendix A, we present some numerical examples of the studied problems with and without constraints in deterministic and stochastic cases. In Appendix B, we present: a continuity result for vector valued functions that is not usual since u and $\partial_t u$ are not in spaces being in a duality, some explanations about the lemma of Aubin-Lions-Simon when $1 < p < 2$, an integration by part formula of Mignot-Bamberger [4]/Alt-Luckhaus [2] and a density result for vector valued functions in Sobolev spaces. A proof of the existence of a solution for a pseudomonotone parabolic problem via the Galerkin method is presented in Appendix C. Appendix D is devoted to a presentation of some extensions, in the stochastic context, to situations where the obstacle and the solution are not in the same space, or to bilateral obstacle problems.

On Lewy-Stampacchia inequalities for elliptic bilateral problems

In this chapter, we are interested in non linear elliptic problems with constraints and homogeneous Dirichlet boundary conditions in the framework of variable exponents Sobolev spaces. This study is a part of a joint work with A. Mokrane and G.Vallet [50].

More precisely, we are interested in proving Lewy-Stampacchia (LS) inequalities associated with constraints $\psi_1 \leq u$ and/or $u \leq \psi_2$, namely,

$$\left\{ \begin{array}{l} \text{The right constraint LS inequality} \quad - (A(\psi_2) + a_0(\psi_2) - f)^- \leq A(u) + a_0(u) - f, \\ \text{The left constraint LS inequality} \quad A(u) + a_0(u) - f \leq (A(\psi_1) + a_0(\psi_1) - f)^+, \end{array} \right.$$

in the general framework of a nonlinear Leray-Lions pseudomonotone operator A , a monotone Nemitsky operator a_0 and a solution u to the variational inequality

$$u \in K, \quad \langle A(u) + a_0(u), v - u \rangle \geq \langle f, v - u \rangle, \quad \forall v \in K,$$

where K is a closed convex subset from $W_0^{1,p(\cdot)}(D)$ related to the constraints. We discuss also under which assumptions the two parts of the above Lewy-Stampacchia inequalities simultaneously hold. We use an *ad hoc* perturbation of the operator and a penalization of the constraint to prove the Lewy-Stampacchia inequalities. The aim of this chapter is given in detail sketches of this proof.

Keywords: Variational inequalities, pseudomonotone operator, Lewy-Stampacchia inequality, variable exponents.

1.1 Introduction

1.1.1 Former results / Our result

H. Lewy and G. Stampacchia [41] proved the first inequality in the frame of superharmonic problems, then, many authors have been interested in the so-called Lewy-Stampacchia inequality associated with obstacle problems. Without trying to be exhaustive, let us cite the monograph of J.F. Rodrigues [62] and the papers of A. Mokrane and F. Murat [48] for pseudo-monotone elliptic problems, A. Mokrane and G. Vallet [51] in the context of Sobolev spaces with variable exponents, J.F. Rodrigues, M. Sanchón and J. M. Urbano [64] proved the existence and uniqueness of an entropy solution to an obstacle problem for nonlinear elliptic equations with variable growth and L^1 -data, A. Pinamonti and E. Valdinoci [58] in the framework of Heisenberg group, R. Servadei and E. Valdinoci [69] for nonlocal operators or N. Gigli and S. Mosconi [32] concerning an abstract presentation. Concerning the bilateral problem, let us cite A. Mokrane and F. Murat [49] where the authors proved the existence of a solution satisfying LS inequality for a rather general Leray-Lions operator of second order by assuming the existence of a perturbed problem satisfying a uniqueness property. Let us also cite J. F. Rodrigues and R. Teymurazyan [65] where the authors proved LS inequality for the two obstacles problem in abstract form for a T-monotone operator in the frame of (generalized) Orlicz-Sobolev spaces.

To the best of our knowledge, there don't exist in the literature such general LS inequalities for pseudomonotone operators, nor generalizations of A. Mokrane and F. Murat work [49] to the case of variable exponents Sobolev spaces $W_0^{1,p(\cdot)}(D)$. Consequently, the aim of this chapter is to consider Lewy-Stampacchia inequalities for pseudomonotone elliptic operators in very general situations. This generalizes the results, and simplifies the proofs, proposed in the unilateral obstacle, as well as the one in the bilateral case.

1.1.2 Content of the study

In this chapter, we are interested in proving Lewy-Stampacchia (LS) inequalities associated with constraints $\psi_1 \leq u$ and/or $u \leq \psi_2$, namely,

$$\begin{cases} \text{The right constraint LS inequality} & -(A(\psi_2) + a_0(\psi_2) - f)^- \leq A(u) + a_0(u) - f, \\ \text{The left constraint LS inequality} & A(u) + a_0(u) - f \leq (A(\psi_1) + a_0(\psi_1) - f)^+, \end{cases}$$

in the general framework of a nonlinear Leray-Lions pseudomonotone operator A , a monotone Nemitsky operator a_0 and a solution u to the variational inequality

$$u \in K, \quad \langle A(u) + a_0(u), v - u \rangle \geq \langle f, v - u \rangle, \quad \forall v \in K,$$

where K is a closed convex subset from $W_0^{1,p(\cdot)}(D)$ related to the constraints. We discuss also under which assumptions the two parts of the above Lewy-Stampacchia inequalities simultaneously hold. In this chapter, we propose such results by using a method of penalization, associated with a suitable perturbation of the operator as proposed *e.g.* by [35, p.102] and [12] for sub/super solutions to obstacle quasilinear elliptic problems. This perturbation is one of the main point of the proof: to make it possible and to reduce to the minimum the list of assumptions. We discuss also about additional conditions proposed *e.g.* in [49] to derive a result in the general case.

The chapter is organized in the following way: after giving the hypotheses and the main result (Th. 1.3) in Section 1.2, Section 1.3 is devoted to the proof of this result. A first step is devoted to the existence of a solution to the penalized/perturbed problem associated with a parameter ε ; then, some *a priori* estimates and passage to the limit with respect to η (subsequence of ε) are considered with regular non-negative elements g_1^+ and g_2^- , associated with decompositions of certain elements assumed to be in the order dual. We prove first the two parts of Lewy-Stampacchia inequality when g_1^+ and g_2^- are still regular; finally, the proof of Lewy-Stampacchia inequality is extended to the general case in the context of unilateral problems. In the Section 1.4, we discuss some additional assumptions to get the two parts of Lewy-Stampacchia inequality for bilateral problems simultaneously.

1.2 Notation, hypotheses and main result

In this chapter, the exponent $p : D \rightarrow [1, +\infty[$ is a measurable function, and we set $p_- = \operatorname{ess\,inf}_D p$ and $p^+ = \operatorname{ess\,sup}_D p$. We assume also that p is a log-Holder continuous function (see e.g. [21, p. 98]).

Consider the following variable exponents Sobolev spaces $L^{p(x)}(D)$ and $W_0^{1,p(x)}(D)$, one can consult [31] for the basic properties and some results concerning this type of spaces.

Denote, for given measurable functions $(\psi_i)_{i=1,2} : D \rightarrow \bar{\mathbb{R}}$, by

$$K_{\psi_1} = \{u \in W_0^{1,p(\cdot)}(D) : \psi_1 \leq u \text{ a.e in } D\},$$

$$K^{\psi_2} = \{u \in W_0^{1,p(\cdot)}(D) : u \leq \psi_2 \text{ a.e in } D\},$$

$$\text{and } K(\psi_1, \psi_2) = K_{\psi_1} \cap K^{\psi_2} = \{u \in W_0^{1,p(\cdot)}(D) : \psi_1 \leq u \leq \psi_2 \text{ a.e in } D\}.$$

Assume that:

H₁ : A is a Leray-Lions pseudomonotone operator of the form

$$v \mapsto A(v) = - \operatorname{div} [a(x, v, \nabla v)],$$

which acts from $W^{1,p(\cdot)}(D)$ into $W^{-1,p'(\cdot)}(D)$ where

H_{1,1} $a : (x, u, \vec{\xi}) \in D \times \mathbb{R} \times \mathbb{R}^d \mapsto a(x, u, \vec{\xi}) \in \mathbb{R}^d$ is a Carathéodory function on $D \times \mathbb{R}^{d+1}$,

H_{1,2} a is strictly monotone with respect to its last argument:

$$\forall x \in D \text{ a.e.}, u \in \mathbb{R}, \forall \vec{\xi}, \vec{\eta} \in \mathbb{R}^d, \vec{\xi} \neq \vec{\eta} \Rightarrow [a(x, u, \vec{\xi}) - a(x, u, \vec{\eta})] \cdot (\vec{\xi} - \vec{\eta}) > 0.$$

H_{1,3} there exist constants $\bar{\alpha} > 0, \bar{\beta} > 0$ and $\bar{\gamma} \geq 0$, a function \bar{h} in $L^1(D)$ and a function \bar{k} in $L^{p(\cdot)}(D)$ and $1 \leq q(x), r(x) \leq q^+ < p^-$ two exponents such that, for a.e. $x \in D, \forall u \in \mathbb{R}, \forall \vec{\xi} \in \mathbb{R}^d$,

$$a(x, u, \vec{\xi}) \cdot \vec{\xi} \geq \bar{\alpha} |\vec{\xi}|^{p(x)} - [\bar{\gamma} |u|^{q(x)} + |\bar{h}(x)|], \quad (1.1)$$

$$|a(x, u, \vec{\xi})| \leq \bar{\beta} \left[|\bar{k}(x)| + |u|^{\frac{r(x)}{p(x)}} + |\vec{\xi}| \right]^{p(x)-1}. \quad (1.2)$$

H₂: a_0 is a nonlinear superposition operator acting from $L^{p(\cdot)}(D)$ into its dual $L^{p'(\cdot)}(D)$, which is defined by

$$a_0(u) = a_0(x, u), \quad (1.3)$$

where the function $a_0 : D \times \mathbb{R} \rightarrow \mathbb{R}$ is a monotone (non decreasing) Carathéodory function, i.e.,

$\forall s \in \mathbb{R}, x \mapsto a_0(x, s)$ is measurable, a.e. $x \in D$, $s \mapsto a_0(x, s)$ is continuous, and a.e. $x \in D, \forall s \in \mathbb{R}, \forall t \in \mathbb{R}, (a_0(x, s) - a_0(x, t))(s - t) \geq 0$.

We also assume that there exist a constant $\bar{\beta}_0 > 0$ and a function \bar{k}_0 in $L^{p(\cdot)}(D)$, $q_1(x) \leq p(x)$ and a function $v \geq 0$ such that for a.e. $x \in D$ and for all $s \in \mathbb{R}$ one has

$$\begin{cases} |a_0(x, s)| \leq \bar{\beta}_0(|\bar{k}_0(x)| + |s|)^{q_1(x)-1}, \\ \nabla q_1 \cdot \nabla v \geq 0, \quad |\nabla v| \neq 0 \quad \text{in } \bar{D}. \end{cases} \quad (1.4)$$

Remark 1.1. *The second assumption on q_1 in (1.4) allows us to use Poincaré's inequality in modular form i.e.*

$$\exists C_p > 0, \quad \forall v \in W_0^{1, q_1(x)}(D), \quad \int_D |v(x)|^{q_1(x)} dx \leq C_p \int_D |\nabla v(x)|^{q_1(x)} dx,$$

(see [1, Th. 1]). Instead, we can also assume that $q_1(x) \leq p_-$ without assuming the second condition in (1.4) used to prove Lemma 1.5.

$H_3 : f \in W^{-1, p'(\cdot)}(D), (\psi_i)_{i=1,2} : D \rightarrow \bar{\mathbb{R}}$ are measurable functions such that there exists v^* in $W_0^{1, p(\cdot)}(D)$ such that $\psi_1 \leq v^* \leq \psi_2$.

$H_4 : Define, for all $v \in W^{1, p(\cdot)}(D)$, the operator B by $B(v) = A(v) + a_0(v) - f$ and denote the order dual space by$

$$V_{p(\cdot)}^* = (W^{-1, p'(\cdot)}(D))^+ - (W^{-1, p'(\cdot)}(D))^+.$$

$H_{4,1} : \psi_1 \in W^{1, p(\cdot)}(D), B(\psi_1) \in V_{p(\cdot)}^*$
i.e. $B(\psi_1) = g_1^+ - g_1^-, g_1^+, g_1^- \in W^{-1, p'(\cdot)}(D), g_1^+, g_1^- \geq 0$.

$H_{4,2} : \psi_2 \in W^{1, p(\cdot)}(D), B(\psi_2) \in V_{p(\cdot)}^*$
i.e. $B(\psi_2) = g_2^+ - g_2^-, g_2^+, g_2^- \in W^{-1, p'(\cdot)}(D), g_2^+, g_2^- \geq 0$.

Remark 1.2. *Of course, H_3 and $H_{4,i}$ ($i = 1, 2$) yield $\psi_1 \leq 0$ and $\psi_2 \geq 0$ on ∂D . Reciprocally, by [21, Prop. 7.1.8 p. 244], assuming $H_{4,1}$ with $\psi_1 \leq 0$ on ∂D yields ψ_1^+ belongs to K_{ψ_1} . Likewise, $H_{4,2}$ with $\psi_2 \geq 0$ on ∂D yields $-\psi_2^-$ belongs to K^{ψ_2} . Then, assuming ψ_1 and ψ_2 in $W^{1, p(\cdot)}(D)$ with $\psi_1 \leq \psi_2$ in D and $\psi_1 \leq 0 \leq \psi_2$ on ∂D ensures that $v^* = \psi_1^+ - \psi_2^-$ belongs to $K(\psi_1, \psi_2)$.*

Our aim is to prove the following result and discuss under which assumptions the full Lewy-Stampacchia inequality (1.8) holds in general framework.

Theorem 1.3. *Under the above assumptions (H_1) - (H_3) , there exists at least one solution $u \in K(\psi_1, \psi_2)$ which is a solution of the variational inequality*

$$\int_D \left[a(x, u, \nabla u) \nabla(v - u) + a_0(x, u)(v - u) \right] dx \geq \langle f, v - u \rangle, \forall v \in K(\psi_1, \psi_2). \quad (1.5)$$

- Assuming $H_{4,1}$, there exists a solution of (1.5) such that

$$B(u) \in V_{p(\cdot)}^* \quad \text{and} \quad (B(u))^+ \leq (B(\psi_1))^+. \quad (1.6)$$

- Assuming $H_{4,2}$, there exists a solution of (1.5) such that

$$B(u) \in V_{p(\cdot)}^* \quad \text{and} \quad (B(u))^- \leq (B(\psi_2))^- . \quad (1.7)$$

- If H_4 holds true with g_1^+ and g_2^- in $W_0^{1,p(\cdot)}(D) \cap L^\infty(D)$, then there exists a solution of (1.5) such that $B(u) \in V_{p(\cdot)}^*$ and

$$-(A(\psi_2) + a_0(\psi_2) - f)^- \leq A(u) + a_0(u) - f \leq (A(\psi_1) + a_0(\psi_1) - f)^+. \quad (1.8)$$

- If H_4 holds true and if there exists a Nemitsky operator j on $D \times \mathbb{R}$, satisfying H_2 like a_0 , such that the solution $u \in K(\psi_1, \psi_2)$ to

$$\int_D a(x, u, \nabla u) \nabla(v - u) dx + \int_D [a_0(x, u) + j(x, u)](v - u) dx \geq \langle f, v - u \rangle$$

for all $v \in K(\psi_1, \psi_2)$ is unique, then (1.8) is satisfied for any solution to (1.5).

1.3 Proof of Theorem 1.3

We will prove Theorem 1.3 in three steps. First, we introduce the perturbed operator and some preliminary results, then, one concludes the existence of a solution to the variational inequality. Secondly, we prove the two parts of Lewy-Stampacchia inequality independently when g_1^+ and g_2^- are regular, by passing the limit in the same subsequence satisfying the penalized problem. Finally, the proof of Lewy Stampacchia inequality in the general case for unilateral problems will be presented, by using a density lemma.

1.3.1 The perturbed operator and some preliminary results

Denote by $\tilde{a}(x, u, \xi) = a(x, \max(\psi_1, \min(u, \psi_2)), \xi)$ and \tilde{A} is the operator associated with \tilde{a} .

Remark 1.4. *We wish to draw the reader's attention to the fact that with the proposed perturbation: $\tilde{a}(x, u, \xi) = a(x, \max(\psi_1, \min(u, \psi_2)), \xi)$, the operator is formally monotone and not pseudomonotone any more on the free set where the constraints are violated.*

One will perform the proof in the bilateral case, but the unilateral cases correspond to $\psi_1 = -\infty$ or $\psi_2 = +\infty$.

Note that, the above assumption H_1 still holds for \tilde{a} . Indeed,

$$\tilde{a}(x, u, \vec{\xi}) \cdot \vec{\xi} \geq \bar{\alpha} |\vec{\xi}|^{p(x)} - \left[\bar{\gamma} |\max(\psi_1, \min(u, \psi_2))|^{q(x)} + |\bar{h}(x)| \right], \quad (1.9)$$

$$|\tilde{a}(x, u, \xi)| \leq \bar{\beta} \left[|\bar{k}(x)| + |\max(\psi_1, \min(u, \psi_2))|^{\frac{r(x)}{p(x)}} + |\vec{\xi}| \right]^{p(x)-1}. \quad (1.10)$$

Since by assumption H_3 , $|\max(\psi_1, \min(u, \psi_2))|^{q(x)} \leq |u|^{q(x)} + |v^*|^{q(x)}$, one gets that

$$|\max(\psi_1, \min(u, \psi_2))|^{\frac{r(x)}{p(x)}} \leq |u|^{\frac{r(x)}{p(x)}} + |v^*|^{\frac{r(x)}{p(x)}},$$

(1.1) and (1.2) are satisfied by replacing \bar{h} by $\bar{h} + \bar{\gamma}|v^*|^{q(x)}$ and \bar{k} by $\bar{k} + |v^*|^{\frac{r(x)}{p(x)}}$.

Lemma 1.5. *Assume $H_1 - H_2$ and H_3 . There exists a constant $M > 0$ such that, for any $u, v \in W_0^{1,p(\cdot)}(D)$,*

$$\begin{aligned} & \int_D \tilde{a}(x, u, \nabla u) \nabla(u - v) dx + \int_D a_0(x, u)(u - v) dx + M \\ & \geq \frac{\bar{\alpha}}{2} \min(\|u\|_{W_0^{1,p(\cdot)}}^{p^+}, \|u\|_{W_0^{1,p(\cdot)}}^{p^-}) - [M\delta \int_D |\nabla v(x)|^{p(x)} dx + M \int_D |v(x)|^{q_1(x)} dx]. \end{aligned}$$

Proof. From [51, Lemma 4], there exist positive constants C, δ and C_1 such that, for any $u, v \in W_0^{1,p(\cdot)}(D)$,

$$\begin{aligned} & \int_D \tilde{a}(x, u, \nabla u) \nabla(u - v) dx + C \\ & \geq (\bar{\alpha} - \delta) \int_D |\nabla u|^{p(x)} dx - \delta [\|u\|_{W_0^{1,p(\cdot)}}^{q^+} + \|u\|_{W_0^{1,p(\cdot)}}^{r^+}] - C_1 \delta \int_D |\nabla v|^{p(x)} dx. \end{aligned}$$

On other hand, for any $u, v \in W_0^{1,p(\cdot)}(D)$, one has

$$a_0(x, u)(u - v) = (a_0(x, u) - a_0(x, v))(u - v) + a_0(x, v)(u - v) \geq a_0(x, v)(u - v),$$

Since $|\int_D a_0(x, v)v dx| \leq C \int_D |\bar{k}(x)|^{q_1(x)} dx + C \int_D |v(x)|^{q_1(x)} dx$ where C is a positive constant. Thanks Young inequality and Remark 1.1, one has

$$\begin{aligned} |\int_D a_0(x, v)u dx| &\leq C_\delta \int_D (|\bar{k}(x)|^{q_1(x)} + |v(x)|^{q_1(x)}) dx + \delta \int_D |u(x)|^{q_1(x)} dx, \\ &\leq C_\delta \int_D (|\bar{k}(x)|^{q_1(x)} + |v(x)|^{q_1(x)}) dx + C_p \delta \int_D |\nabla u(x)|^{q_1(x)} dx, \end{aligned}$$

where C_p is the positive constant of modular Poincaré's inequality. Using again Young inequality, one has

$$|\int_D a_0(x, v)u dx| \leq C_\delta \int_D (|\bar{k}(x)|^{q_1(x)} + |v(x)|^{q_1(x)}) dx + \delta \int_D |\nabla u(x)|^{p(x)} dx + C'$$

where C_δ and C' are positive constants. Therefore

$$\begin{aligned} &\int_D \tilde{a}(x, u, \nabla u) \nabla(u - v) dx + \int_D a_0(x, u)(u - v) dx + C \\ &\geq (\bar{\alpha} - \delta) \int_D |\nabla u(x)|^{p(x)} dx - \delta [\|u\|_{W_0^{1,p(\cdot)}}^{q^+} + \|u\|_{W_0^{1,p(\cdot)}}^{r^+}] - C_1 \delta \int_D |\nabla v(x)|^{p(x)} dx \\ &\quad - [C \int_D |\bar{k}(x)|^{q_1(x)} dx + C \int_D |v(x)|^{q_1(x)} dx] \\ &\quad - [C_\delta \int_D (|\bar{k}(x)|^{q_1(x)} + |v(x)|^{q_1(x)}) dx + \delta \int_D |\nabla u(x)|^{p(x)} dx + C']. \end{aligned}$$

Since $\bar{k} \in L^{p(\cdot)}(D)$, by using Young inequality and with a suitable choice of δ , there exists a constant $M > 0$ such that

$$\begin{aligned} &\int_D \tilde{a}(x, u, \nabla u) \nabla(u - v) dx + \int_D a_0(x, u)(u - v) dx + M \\ &\geq \frac{\bar{\alpha}}{2} \min(\|u\|_{W_0^{1,p(\cdot)}}^{p^+}, \|u\|_{W_0^{1,p(\cdot)}}^{p^-}) - M [\int_D [\delta |\nabla v(x)|^{p(x)} + |v(x)|^{q_1(x)}] dx]. \end{aligned}$$

□

Now, we consider the penalized problem.

Theorem 1.6. *Assume H_1 - H_2 and H_3 . Then for each $\varepsilon > 0$ there exists at least one u_ε such that*

$$\left\{ \begin{array}{l} u_\varepsilon \in W_0^{1,p(\cdot)}(D), \quad (u_\varepsilon - \psi_1)^- \in L^2(D), \quad (u_\varepsilon - \psi_2)^+ \in L^2(D) \\ \int_D \tilde{a}(x, u_\varepsilon, \nabla u_\varepsilon) \nabla v dx + \int_D a_0(x, u_\varepsilon) v dx - \frac{1}{\varepsilon} \int_D (u_\varepsilon - \psi_1)^- v dx \\ \quad + \frac{1}{\varepsilon} \int_D (u_\varepsilon - \psi_2)^+ v dx = \langle f, v \rangle, \quad \forall v \in W_0^{1,p(\cdot)}(D) \cap L^2(D). \end{array} \right. \quad (1.11)$$

Moreover, for all $v \in K(\psi_1, \psi_2)$, $(u_\varepsilon - \psi_1)^-(u_\varepsilon - v)$ and $(u_\varepsilon - \psi_2)^+(u_\varepsilon - v)$ are in $L^1(D)$, and

$$\begin{aligned} & \int_D \tilde{a}(x, u_\varepsilon, \nabla u_\varepsilon) \nabla(u_\varepsilon - v) dx + \int_D a_0(x, u_\varepsilon)(u_\varepsilon - v) dx \\ & - \frac{1}{\varepsilon} \int_D (u_\varepsilon - \psi_1)^-(u_\varepsilon - v) dx + \frac{1}{\varepsilon} \int_D (u_\varepsilon - \psi_2)^+(u_\varepsilon - v) dx = \langle f, u_\varepsilon - v \rangle. \end{aligned} \quad (1.12)$$

Proof. Note that \tilde{A} is a coercive pseudomonotone operator [51, Rem. 1]. Then, denoting by T_n the truncation at height n , the operator

$$L : w \mapsto -\operatorname{div}[\tilde{a}(x, w, \nabla w)] + a_0(x, w) - \frac{1}{\varepsilon} T_n(w - \psi_1)^- + \frac{1}{\varepsilon} T_n(w - \psi_2)^+,$$

L is well defined from $W_0^{1,p(\cdot)}(D)$ in $W^{-1,p'(\cdot)}(D)$, and is a strongly continuous perturbation of \tilde{A} . For every $\varepsilon > 0$, consider the problem

$$u_\varepsilon^n \in W_0^{1,p(\cdot)}(D), \quad Lu_\varepsilon^n = f. \quad (1.13)$$

The existence of solution for (1.13) follows from [66, Th. 2.6]. With cosmetic updating of the proof [48, Th. 6.1], we get (1.11). The proof of (1.12) is similar to the one of [48, Prop. 6.1], one just needs to verify that

$$(u_\varepsilon - \psi_1)^-(u_\varepsilon - v) \in L^1(D) \quad \text{and} \quad (u_\varepsilon - \psi_2)^+(u_\varepsilon - v) \in L^1(D).$$

We know that: $\forall v \in W_0^{1,p(\cdot)}(D)$, $T_n(v)^+$ (resp. $T_n(v)^-$) belongs to $W_0^{1,p(\cdot)}(D) \cap L^2(D)$, and $T_n(v)^+$ (resp. $T_n(v)^-$) tends strongly to v^+ (resp. v^-) in $W_0^{1,p(\cdot)}(D)$.

Let $v \in K(\psi_1, \psi_2)$, consider $T_n(u_\varepsilon - v)^-$ as test function in (1.11) and remark that:

$$\begin{cases} (u_\varepsilon - \psi_2)^+ T_n(u_\varepsilon - v)^- = 0 & \text{a.e. in } D, \\ -\frac{1}{\varepsilon} \int_D (u_\varepsilon - \psi_1)^- T_n(u_\varepsilon - v)^- dx & \text{is bounded independently of } n. \end{cases}$$

The monotone convergence theorem then implies that $(u_\varepsilon - \psi_1)^-(u_\varepsilon - v) \in L^1(D)$ and

$$\begin{aligned} -\frac{1}{\varepsilon} \int_D (u_\varepsilon - \psi_1)^- T_n(u_\varepsilon - v)^- dx & \longrightarrow -\frac{1}{\varepsilon} \int_D (u_\varepsilon - \psi_1)^-(u_\varepsilon - v)^- dx \\ & = \frac{1}{\varepsilon} \int_D (u_\varepsilon - \psi_1)^-(u_\varepsilon - v) dx. \end{aligned}$$

Using $T_n(u_\varepsilon - v)^+$ as test function in (1.11) we remark similarly that:

$$\begin{cases} (u_\varepsilon - \psi_1)^- T_n(u_\varepsilon - v)^+ = 0 & \text{a.e. in } D, \\ \frac{1}{\varepsilon} \int_D (u_\varepsilon - \psi_2)^+ T_n(u_\varepsilon - v)^+ dx & \text{is bounded independently of } n. \end{cases}$$

Again, by monotone convergence theorem, $(u_\varepsilon - \psi_2)^+(u_\varepsilon - v) \in L^1(D)$ and

$$\begin{aligned} \frac{1}{\varepsilon} \int_D (u_\varepsilon - \psi_2)^+ T_n(u_\varepsilon - v)^+ dx &\longrightarrow \frac{1}{\varepsilon} \int_D (u_\varepsilon - \psi_2)^+(u_\varepsilon - v)^+ dx \\ &= \frac{1}{\varepsilon} \int_D (u_\varepsilon - \psi_2)^+(u_\varepsilon - v) dx. \end{aligned}$$

□

For all $v \in K(\psi_1, \psi_2)$, we have

$$\left\{ \begin{aligned} -\frac{1}{\varepsilon} \int_D (u_\varepsilon - \psi_1)^-(u_\varepsilon - v) dx &= \frac{1}{\varepsilon} \int_D |(u_\varepsilon - \psi_1)^-|^2 + (u_\varepsilon - \psi_1)^-(v - \psi_1) dx, \\ &\geq \frac{1}{\varepsilon} \int_D |(u_\varepsilon - \psi_1)^-|^2 dx \geq 0, \\ \frac{1}{\varepsilon} \int_D (u_\varepsilon - \psi_2)^+(u_\varepsilon - v) dx &= \frac{1}{\varepsilon} \int_D |(u_\varepsilon - \psi_2)^+|^2 + (u_\varepsilon - \psi_2)^+(\psi_2 - v) dx, \\ &\geq \frac{1}{\varepsilon} \int_D |(u_\varepsilon - \psi_2)^+|^2 dx \geq 0. \end{aligned} \right. \quad (1.14)$$

Thanks to Lemma 1.5 and previous calculations, there exists a constant $C > 0$ independent of ε such that

$$\left\{ \begin{aligned} \|u_\varepsilon\|_{W_0^{1,p(\cdot)}(D)} + \|\tilde{a}(x, u_\varepsilon, \nabla u_\varepsilon)\|_{(L^{p'(\cdot)}(D))^d} &\leq C, \\ \|(u_\varepsilon - \psi_1)^-\|_{L^2(D)}^2 + \|(u_\varepsilon - \psi_2)^+\|_{L^2(D)}^2 &\leq C\varepsilon. \end{aligned} \right. \quad (1.15)$$

Thus, we can extract a subsequence, denoted by η , such that

$$u_\eta \rightharpoonup u \quad \text{in } W_0^{1,p(\cdot)}(D) \quad \text{and a.e. in } D, \quad (1.16)$$

$$\tilde{a}(x, u_\eta, \nabla u_\eta) \rightharpoonup \chi \quad \text{in } (L^{p'(\cdot)}(D))^d. \quad (1.17)$$

In view of (1.15), we get

$$u \in K(\psi_1, \psi_2). \quad (1.18)$$

By (1.16), the Sobolev embedding theorem and the growth condition (1.4), we have

$$a_0(x, u_\eta) \rightarrow a_0(x, u) \quad \text{in } L^{p'(\cdot)}(D). \quad (1.19)$$

Using (1.14) and passing to the limit in (1.12), we obtain, for all $v \in K(\psi_1, \psi_2)$,

$$\limsup_\eta \int_D \tilde{a}(x, u_\eta, \nabla u_\eta) \nabla u_\eta dx - \int_D \chi \nabla v dx + \int_D a_0(x, u)(u - v) dx \leq \langle f, u - v \rangle. \quad (1.20)$$

Using (1.18) and taking $v = u$, we get

$$\limsup_\eta \int_D \tilde{a}(x, u_\eta, \nabla u_\eta) \nabla u_\eta dx \leq \int_D \chi \nabla u dx. \quad (1.21)$$

Since $v \mapsto \tilde{A}(v) = -\operatorname{div}[\tilde{a}(x, v, \nabla v)]$ is a pseudomonotone operator, one has $\operatorname{div}\chi = \operatorname{div}[\tilde{a}(x, u, \nabla u)]$ and

$$\int_D \tilde{a}(x, u_\eta, \nabla u_\eta) \nabla u_\eta dx \rightarrow \int_D \tilde{a}(x, u, \nabla u) \nabla u dx. \quad (1.22)$$

Moreover, $\chi = \tilde{a}(x, u, \nabla u)$.

Indeed, similarly to the proof of [13, Lemma 1], we get that $\nabla u_\eta \rightarrow \nabla u$ in measure. Therefore, there exists a subsequence denoted by the same way such that $\nabla u_\eta \rightarrow \nabla u$ a.e. Using the continuity of \tilde{a} with respect to its second and third arguments, we get

$$\tilde{a}(x, u_\eta, \nabla u_\eta) \rightarrow \tilde{a}(x, u, \nabla u) \quad \text{a.e. in } D$$

therefore $\chi = \tilde{a}(x, u, \nabla u)$. Thanks to the previous calculations, we deduce

Theorem 1.7. *Assume H_1 - H_3 hold true. Then there exists at least a solution $u \in K(\psi_1, \psi_2)$ to the variational inequality*

$$\int_D a(x, u, \nabla u) \nabla(v - u) dx + \int_D a_0(x, u)(v - u) dx \geq \langle f, v - u \rangle, \quad \forall v \in K(\psi_1, \psi_2).$$

Note that the cases corresponding to K_{ψ_1} and K^{ψ_2} are similar by assuming formally $\psi_2 = +\infty$ or $\psi_1 = -\infty$.

1.3.2 Proof of Lewy-Stampacchia inequality in the regular case

Define $\mu_\eta^1 = \frac{1}{\eta}(u_\eta - \psi_1)^- \geq 0$, $\mu_\eta^2 = \frac{1}{\eta}(u_\eta - \psi_2)^+ \geq 0$. We have (see Th. 1.6)

$$\mu_\eta^1 \in L^2(D), \quad \mu_\eta^2 \in L^2(D).$$

Take $v \in \mathcal{C}_c^\infty(D)$ as test function in (1.11) and $\varepsilon = \eta$, we get

$$\mu_\eta^1 - \mu_\eta^2 \rightharpoonup -\operatorname{div}(\tilde{a}(x, u, \nabla u) + a_0(x, u) - f) \quad \text{in } W^{-1, p'(\cdot)}(D). \quad (1.23)$$

In this subsection, we consider the subsequence $(u_\eta)_\eta$ which satisfies the penalized problem (1.11). Thanks to the selected test function, we prove the two parts of Lewy-Stampacchia inequality independently and we get at the limit the two parts of Lewy-Stampacchia inequality since $(u_\eta)_\eta$ converges to the same limit u .

First Lewy-Stampacchia inequality.

In this subsection, we assume that $0 \leq g_1^+ = (B(\psi_1))^+ \in W_0^{1,p(\cdot)}(D) \cap L^\infty(D)$. Denote by

$$z_\eta = g_1^+ - \frac{1}{\eta}(u_\eta - \psi_1)^-.$$

Note that $(u_\eta - \psi_1)^- \in W_0^{1,p(\cdot)}(D)$ since $\psi_1 \leq 0$ on ∂D , and from Theorem 1.6, $(u_\eta - \psi_1)^- \in L^2(D)$. Therefore $z_\eta \in W_0^{1,p(\cdot)}(D) \cap L^2(D)$.

Lemma 1.8. *There exists a constant C , such that for any η ,*

$$\begin{aligned} \int_D \left| [\tilde{a}(x, u_\eta, \nabla u_\eta) - \tilde{a}(x, \psi_1, \nabla \psi_1)] \cdot \nabla (u_\eta - \psi_1)^- \right| dx &\leq C\eta \|g_1^+\|_{L^2(D)}^2, \\ \frac{1}{\eta} \|(u_\eta - \psi_1)^-\|_{L^2(D)}^2 &\leq C\eta \|g_1^+\|_{L^2(D)}^2. \end{aligned}$$

Proof. With the admissible test-function $v = -(u_\eta - \psi_1)^-$ in (1.11), one has

$$\left\{ \begin{aligned} & - \int_D [\tilde{a}(x, u_\eta, \nabla u_\eta) - \tilde{a}(x, \psi_1, \nabla \psi_1)] \nabla (u_\eta - \psi_1)^- dx \\ & - \int_D [a_0(x, u_\eta) - a_0(x, \psi_1)] (u_\eta - \psi_1)^- dx + \frac{1}{\eta} \int_D |(u_\eta - \psi_1)^-|^2 dx \\ & - \frac{1}{\eta} \int_D (u_\eta - \psi_2)^+ (u_\eta - \psi_1)^- dx \\ & = - \langle f + \operatorname{div}(\tilde{a}(x, \psi_1, \nabla \psi_1)) - a_0(x, \psi_1), (u_\eta - \psi_1)^- \rangle \\ & = \langle g_1^+ - g_1^-, (u_\eta - \psi_1)^- \rangle \leq 2\eta \|g_1^+\|_{L^2(D)}^2 + \frac{1}{2\eta} \int_D |(u_\eta - \psi_1)^-|^2 dx. \end{aligned} \right.$$

Since $\psi_1 \leq \psi_2$ in D , then $(u_\eta - \psi_2)^+ (u_\eta - \psi_1)^- = 0$ a.e. in D . Therefore

$$-\frac{1}{\eta} \int_D (u_\eta - \psi_2)^+ (u_\eta - \psi_1)^- dx = 0.$$

Since $-(u_\eta - \psi_1)^- = (u_\eta - \psi_1)1_{\{u_\eta < \psi_1\}}$ and a_0 is non decreasing with respect to its second argument, one has

$$-\int_D [a_0(x, u_\eta) - a_0(x, \psi_1)] (u_\eta - \psi_1)^- dx \geq 0.$$

Then

$$\begin{aligned} \int_{\{u_\eta - \psi_1 < 0\}} [\tilde{a}(x, u_\eta, \nabla u_\eta) - \tilde{a}(x, \psi_1, \nabla \psi_1)] \cdot \nabla (u_\eta - \psi_1) dx + \frac{1}{2\eta} \int_D |(u_\eta - \psi_1)^-|^2 dx \\ \leq 2\eta \|g_1^+\|_{L^2(D)}^2. \end{aligned}$$

Since $\tilde{a}(x, u_\eta, \nabla u_\eta) = a(x, \psi_1, \nabla u_\eta)$ in $\{u_\eta < \psi_1\}$, one gets that

$$\int_D \left| \left[\tilde{a}(x, u_\eta, \nabla u_\eta) - \tilde{a}(x, \psi_1, \nabla \psi_1) \right] \cdot \nabla (u_\eta - \psi_1)^- \right| dx + \frac{1}{2\eta} \int_D |(u_\eta - \psi_1)^-|^2 dx \leq 2\eta \|g_1^+\|_{L^2(D)}^2.$$

□

We have $\tilde{A}(u_\eta) - A(\psi_1) + a_0(u_\eta) - a_0(\psi_1) + z_\eta + \frac{1}{\eta}(u_\eta - \psi_2)^+ = g_1^-$, where

$$z_\eta = g_1^+ - \frac{1}{\eta}(u_\eta - \psi_1)^-.$$

Our aim is to prove the strong convergence of z_η^- to 0 in $L^2(D)$. Using $-z_\eta^-$ as test function in (1.11), one has

$$\begin{aligned} & - \int_D \left[\tilde{a}(x, u_\eta, \nabla u_\eta) - \tilde{a}(x, \psi_1, \nabla \psi_1) \right] \nabla z_\eta^- dx - \int_D \left[a_0(x, u_\eta) - a_0(x, \psi_1) \right] z_\eta^- dx \\ & + \int_D |z_\eta^-|^2 dx - \frac{1}{\eta} \int_D (u_\eta - \psi_2)^+ z_\eta^- dx = \langle g_1^-, -z_\eta^- \rangle \leq 0. \end{aligned}$$

On the one hand, since $u_\eta < \psi_1$ on $\{z_\eta < 0\}$, one has

$$-\frac{1}{\eta} \int_D (u_\eta - \psi_2)^+ z_\eta^- dx = 0, \quad - \int_D \left[a_0(x, u_\eta) - a_0(x, \psi_1) \right] z_\eta^- dx \geq 0.$$

On the other hand, note that, \mathcal{B} being the set $\{g_1^+ - \frac{1}{\eta}[(u_\eta - \psi_1)^-] < 0\}$,

$$\begin{aligned} & - \int_D \left[\tilde{a}(x, u_\eta, \nabla u_\eta) - \tilde{a}(x, \psi_1, \nabla \psi_1) \right] \cdot \nabla z_\eta^- dx \\ & = \int_D 1_{\mathcal{B}} \left[\tilde{a}(x, u_\eta, \nabla u_\eta) - \tilde{a}(x, \psi_1, \nabla \psi_1) \right] \nabla \left[g_1^+ - \frac{1}{\eta}[(u_\eta - \psi_1)^-] \right] dx \\ & = \int_D 1_{\mathcal{B}} \left[\tilde{a}(x, \psi_1, \nabla u_\eta) - \tilde{a}(x, \psi_1, \nabla \psi_1) \right] \nabla \left[g_1^+ - \frac{1}{\eta}[(u_\eta - \psi_1)^-] \right] dx. \end{aligned}$$

Since in this situation, the integration holds in the set $\{u_\eta < \psi_1\}$. Thus,

$$\begin{cases} \left[\tilde{a}(x, \psi_1, \nabla u_\eta) - \tilde{a}(x, \psi_1, \nabla \psi_1) \right] \nabla \left[g_1^+ - \frac{1}{\eta}[(u_\eta - \psi_1)^-] \right] \\ \geq \frac{1}{\eta} \left[\tilde{a}(x, \psi_1, \nabla u_\eta) - \tilde{a}(x, \psi_1, \nabla \psi_1) \right] \nabla (u_\eta - \psi_1) \\ \quad - \left| \tilde{a}(x, \psi_1, \nabla u_\eta) - \tilde{a}(x, \psi_1, \nabla \psi_1) \right| |\nabla g_1^+| \\ \geq - \left| \tilde{a}(x, \psi_1, \nabla u_\eta) - \tilde{a}(x, \psi_1, \nabla \psi_1) \right| |\nabla g_1^+|. \end{cases}$$

Thanks to the first estimate of Lemma 1.8,

$$\left[\tilde{a}(x, \psi_1, \nabla u_\eta) - \tilde{a}(x, \psi_1, \nabla \psi_1) \right] \nabla (u_\eta - \psi_1)^- \rightarrow 0 \text{ in } L^1(D).$$

Then, by assumptions $H_{1,1}$ to $H_{1,3}$, up to a subsequence denoted in the same way, one gets that

$$\nabla(u_\eta - \psi_1)^-(x) \rightarrow 0 \quad \text{a.e. in } D. \quad (1.24)$$

Indeed, up to a subsequence denoted in the same way, u_η converges to u a.e. in D with $u \geq \psi_1$ a.e. and

$$\left| (\tilde{a}(x, \psi_1, \nabla u_\eta) - \tilde{a}(x, \psi_1, \nabla \psi_1)) \cdot \nabla(u_\eta - \psi_1)^- \right| \rightarrow 0 \quad \text{a.e. in } D. \quad (1.25)$$

Consider x such that the above limit (1.25) holds.

Thanks to Young inequality, there exist $\delta > 0$ and $C > 0$ such that

$$|\tilde{a}(x, \psi_1, \nabla u_\eta) \cdot \nabla \psi_1| \leq C(p^+, p_-, \delta, \bar{\beta}) |\nabla \psi_1|^{p(x)} + \delta 3^{p^+-1} (|\bar{k}| + |\psi_1|^{r(x)} + |\nabla u_\eta|),$$

with suitable choice of δ , one gets

$$|\tilde{a}(x, \psi_1, \nabla u_\eta) \cdot \nabla \psi_1| \leq C(p^+, p_-, r, \psi_1, \nabla \psi_1, \bar{k}, \bar{\beta}) + \frac{\bar{\alpha}}{2} |\nabla u_\eta|.$$

$$\begin{aligned} \text{Since} \quad & -\tilde{a}(x, \psi_1, \nabla u_\eta) \cdot \nabla(u_\eta - \psi_1)^- \\ & \geq \left[\bar{\alpha} |\nabla u_\eta|^{p(x)} - \bar{\gamma} |\psi_1|^{q(x)} - |\bar{h}| - \tilde{a}(x, \psi_1, \nabla u_\eta) \cdot \nabla \psi_1 \right] 1_{\{u_\eta < \psi_1\}}, \end{aligned}$$

$$\begin{aligned} \text{then} \quad & -\tilde{a}(x, \psi_1, \nabla u_\eta) \cdot \nabla(u_\eta - \psi_1)^- \\ & \geq \left[\frac{\bar{\alpha}}{2} |\nabla u_\eta|^{p(x)} - C(p^+, p_-, r, \psi_1, \nabla \psi_1, \bar{k}, \bar{h}, \bar{\beta}) \right] 1_{\{u_\eta < \psi_1\}}. \end{aligned}$$

$$\begin{aligned} \text{We have} \quad & |\tilde{a}(x, \psi_1, \nabla \psi_1) \cdot \nabla(u_\eta - \psi_1)^-| \\ & \leq \bar{\beta} \left[|\bar{k}| + |\psi_1|^{\frac{r(x)}{p(x)}} + |\nabla \psi_1| \right]^{p(x)-1} \left[|\nabla u_\eta| + |\nabla \psi_1| \right] 1_{\{u_\eta < \psi_1\}}. \end{aligned}$$

Thanks to Young inequality, there exist $\delta > 0$ and $C > 0$ such that

$$\begin{aligned} & |\tilde{a}(x, \psi_1, \nabla \psi_1) \cdot \nabla(u_\eta - \psi_1)^-| \\ & \leq \left[C(p^+, p_-, \delta, \bar{\beta}) \left[|\bar{k}| + |\psi_1|^{\frac{r(x)}{p(x)}} + |\nabla \psi_1| \right]^{p(x)} + \delta \left[|\nabla u_\eta| + |\nabla \psi_1| \right]^{p(x)} \right] 1_{\{u_\eta < \psi_1\}}. \end{aligned}$$

With suitable choice of δ , one gets

$$|\tilde{a}(x, \psi_1, \nabla \psi_1) \cdot \nabla(u_\eta - \psi_1)^-| \leq \left[C(p^+, p_-, r, \psi_1, \nabla \psi_1, \bar{k}, \bar{\beta}) + \frac{\bar{\alpha}}{4} |\nabla u_\eta|^{p(x)} \right] 1_{\{u_\eta < \psi_1\}}.$$

$$\begin{aligned} \text{Therefore} \quad & \left(-(\tilde{a}(x, \psi_1, \nabla u_\eta) - \tilde{a}(x, \psi_1, \nabla \psi_1)) \right) \cdot \nabla(u_\eta - \psi_1)^- \\ & \geq \left[\frac{\bar{\alpha}}{4} |\nabla u_\eta|^{p(x)} - C(p^+, p_-, r, \psi_1, \nabla \psi_1, \bar{k}, \bar{h}, \bar{\beta}) \right] 1_{\{u_\eta < \psi_1\}}. \end{aligned}$$

Using (1.25), one gets that $(\nabla u_\eta 1_{\{u_\eta < \psi_1\}}(x))_\eta$ is a bounded sequence in \mathbb{R}^d . Thus $(\nabla(u_\eta - \psi_1)^-(x))_\eta$ is a bounded sequence in \mathbb{R}^d .

Since $\nabla(u_\eta - \psi_1)^-(x) = -\nabla(u_\eta - \psi_1)(x)1_{\{u_\eta < \psi_1\}}(x)$, it converges to 0 if $u(x) > \psi_1(x)$. Else, at the limit, one has that $u(x) = \psi_1(x)$.

If one assumes that $\nabla(u_\eta - \psi_1)^-(x)$ is not converging to 0, then there exists a subsequence η' (depending on x) such that $\|\nabla(u_{\eta'} - \psi_1)^-(x)\| \geq \delta > 0$ for a positive δ . Then, necessarily $-\nabla(u_{\eta'} - \psi_1)^-(x) = \nabla(u_{\eta'} - \psi_1)(x)$ and, since it is a bounded sequence in \mathbb{R}^d , there exists $\vec{\xi} \in \mathbb{R}^d$ and a new subsequence still labeled η' such that $\nabla u_{\eta'}(x)$ converges to $\vec{\xi}$, with the additional information: $\|\vec{\xi} - \nabla\psi_1(x)\| \geq \delta > 0$. Therefore, since $\vec{\xi} \neq \nabla\psi_1(x)$

$$\begin{aligned} & \left[\tilde{a}(x, \psi_1(x), \nabla u_{\eta'}(x)) - \tilde{a}(x, \psi_1(x), \nabla\psi_1(x)) \right] \nabla(u_{\eta'} - \psi_1)^-(x) \\ &= - \left[\tilde{a}(x, \psi_1(x), \nabla u_{\eta'}(x)) - \tilde{a}(x, \psi_1(x), \nabla\psi_1(x)) \right] \nabla(u_{\eta'} - \psi_1)(x), \end{aligned}$$

the last term converges to $-\left[\tilde{a}(x, \psi_1(x), \vec{\xi}) - \tilde{a}(x, \psi_1(x), \nabla\psi_1(x)) \right] [\vec{\xi} - \nabla\psi_1(x)] < 0$. But, this is in contradiction with the convergence of the same sequence to 0 and (1.24) holds. Note that for x a.e. in D ,

$$\begin{cases} \left[\tilde{a}(x, \psi_1(x), \nabla u_\eta(x)) - \tilde{a}(x, \psi_1(x), \nabla\psi_1(x)) \right] 1_{\{u_\eta < \psi_1\}} \\ = \left[\tilde{a}(x, \psi_1(x), \nabla u_\eta 1_{\{u_\eta < \psi_1\}}(x)) - \tilde{a}(x, \psi_1(x), \nabla\psi_1 1_{\{u_\eta < \psi_1\}}(x)) \right] \end{cases}$$

and $\nabla u_\eta 1_{\{u_\eta < \psi_1\}}(x) - \nabla\psi_1 1_{\{u_\eta < \psi_1\}}(x)$ converges to 0. Then

$$\left[\tilde{a}(x, \psi_1, \nabla u_\eta) - \tilde{a}(x, \psi_1, \nabla\psi_1) \right] 1_{\{u_\eta < \psi_1\}} \quad \text{converges a.e. to 0.}$$

Lemma 1.9. *Let $(u_n)_n$ be a bounded sequence in $L^{p(x)}(D)$ and assume moreover that u_n converges a.e. to u . Then, $u_n \rightharpoonup u$ in $L^{p(x)}(D)$.*

Indeed, the result is true when p is constant. Then, $(u_n)_n$ is a bounded sequence in $L^{p^-}(D)$ and

$$u_n \rightharpoonup u \quad \text{in } L^{p^-}(D) \quad \text{and in } \mathcal{D}'(D).$$

Since $L^{p(x)}(D)$ is a reflexive Banach space, there exists a subsequence denoted by the same way and $v \in L^{p(x)}(D)$ such that

$$u_n \rightharpoonup v \quad \text{in } L^{p(x)}(D) \quad \text{and in } \mathcal{D}'(D).$$

The uniqueness of the limit, in $\mathcal{D}'(D)$, ensures the proof of the lemma.

Since $\left[\tilde{a}(x, \psi_1, \nabla u_\eta) - \tilde{a}(x, \psi_1, \nabla \psi_1) \right] 1_{\{u_\eta < \psi_1\}})_\eta$ is bounded in $L^{p'(x)}(D)$, by Lemma 1.9 it converges weakly to 0 in $L^{p'(x)}(D)$ and

$$\int_D 1_{\{g_1^+ - \frac{1}{\eta}[(u_\eta - \psi_1)^-] < 0\}} \left| \tilde{a}(x, \psi_1, \nabla u_\eta) - \tilde{a}(x, \psi_1, \nabla \psi_1) \right| |\nabla g_1^+| dx \rightarrow 0.$$

As a conclusion, z_η^- converges to 0 in $L^2(D)$. Since $z_\eta = g_1^+ - \mu_\eta^1$ and $\mu_\eta^1 \geq 0$, this implies that

$$0 \leq \mu_\eta^1 \leq g_1^+ + z_\eta^-.$$

Since $z_\eta^- \rightarrow 0$ in $L^2(D)$, $(\mu_\eta^1)_\eta$ is bounded in $L^2(D)$, by extracting a subsequence, there exists a non negative function μ_1 such that

$$\mu_\eta^1 \rightharpoonup \mu^1 \quad \text{in } L^2(D) \quad \text{resp. in } \mathcal{D}'(D) \quad \text{and} \quad 0 \leq \mu^1 \leq g_1^+. \quad (1.26)$$

But (1.23) implies that there exists a measure μ^2 such that

$$\mu_\eta^2 \rightharpoonup \mu^2 \quad \text{in } \mathcal{D}'(D) \quad \text{and} \quad \mu^2 \geq 0, \quad \mu^1 - \mu^2 = -\text{div}[a(x, u, \nabla u)] + a_0(x, u) - f.$$

Since $g_1^+ \in L^\infty(D)$ then $\mu^1 \in L^\infty(D)$ and therefore μ^1 belongs to $W^{-1, p'(\cdot)}(D)$ and μ^2 belongs to $W^{-1, p'(\cdot)}(D)$. We have proved $B(u) \in V_{p(\cdot)}^*$ and

$$(B(u))^+ \leq g_1^+ = (B(\psi_1))^+$$

which implies that $B(u) \leq (B(\psi_1))^+$.

Remark 1.10. We can prove the above Lewy-Stampacchia inequality without proving $B(u) \in V_{p(\cdot)}^*$ as following, $z_\eta = g_1^+ - \frac{1}{\eta}[(u_\eta - \psi_1)^-]$

$$\begin{aligned} &\Rightarrow z_\eta^+ - \text{div}[\tilde{a}(\cdot, u_\eta, \nabla u_\eta)] + a_0(\cdot, u_\eta) + \frac{1}{\eta}(u_\eta - \psi_2)^+ - f = g_1^+ + z_\eta^- \\ &\Rightarrow -\text{div}[\tilde{a}(\cdot, u, \nabla u)] + a_0(\cdot, u) - f \leq g_1^+. \end{aligned}$$

Since $u \in K(\psi_1, \psi_2)$, then $\tilde{a}(\cdot, u, \nabla u) = a(\cdot, u, \nabla u)$. Therefore

$$-\text{div}[a(\cdot, u, \nabla u)] + a_0(\cdot, u) - f \leq g_1^+.$$

Second Lewy-Stampacchia inequality.

In this subsection, we assume that $0 \leq g_2^- = (B(\psi_2))^- \in W_0^{1,p(\cdot)}(D) \cap L^\infty(D)$.

We just give a sketch of the proof of this second Lewy-Stampacchia inequality since this will be done similarly to the one proposed in Subsection 1.3.2. By considering the same subsequence $(u_\eta)_\eta$ used in the Subsection 1.3.2, which satisfies the penalized problem (1.11), denote by

$$z_\eta = \frac{1}{\eta}(u_\eta - \psi_2)^+ - g_2^-.$$

Note that $(u_\eta - \psi_2)^+ \in W_0^{1,p(\cdot)}(D)$ since $\psi_2 \geq 0$ on ∂D , and from Theorem 1.6, $(u_\eta - \psi_2)^+ \in L^2(D)$. Therefore $z_\eta \in W_0^{1,p(\cdot)}(D) \cap L^2(D)$.

Lemma 1.11. *There exists a constant C , such that for any η ,*

$$\begin{aligned} \int_D \left| \tilde{a}(x, u_\eta, \nabla u_\eta) - \tilde{a}(x, \psi_2, \nabla \psi_2) \cdot \nabla (u_\eta - \psi_2)^+ \right| dx &\leq C\eta \|g_2^-\|_{L^2(D)}^2 \quad (1.27) \\ \frac{1}{\eta} \|(u_\eta - \psi_2)^+\|_{L^2(D)}^2 &\leq C\eta \|g_2^-\|_{L^2(D)}^2. \end{aligned}$$

Proof. The proof is similar to the one of Lemma 1.8 by using $v = (u_\eta - \psi_2)^+$ in (1.11).

□

$$\text{We have } \tilde{A}(u_\eta) - A(\psi_2) + a_0(u_\eta) - a_0(\psi_2) + z_\eta - \frac{1}{\eta}(u_\eta - \psi_1)^- = -g_2^+$$

where $z_\eta = \frac{1}{\eta}(u_\eta - \psi_2)^+ - g_2^-$.

Our aim is to prove the strong convergence of z_η^+ to 0 in $L^2(D)$.

Using z_η^+ as test function in (1.11), one has

$$\begin{aligned} \int_D [\tilde{a}(x, u_\eta, \nabla u_\eta) - \tilde{a}(x, \psi_2, \nabla \psi_2)] \nabla z_\eta^+ dx + \int_D [a_0(x, u_\eta) - a_0(x, \psi_2)] z_\eta^+ dx \\ + \int_D |z_\eta^+|^2 dx - \frac{1}{\eta} \int_D (u_\eta - \psi_1)^- z_\eta^+ dx = \langle -g_2^+, z_\eta^+ \rangle \leq 0. \end{aligned}$$

On the one hand, since $\psi_1 \leq \psi_2$ and $u_\eta > \psi_2$ on $\{z_\eta > 0\}$, one has

$(u_\eta - \psi_1)^- z_\eta^+ = 0$ a.e. in D . Therefore

$$-\frac{1}{\eta} \int_D (u_\eta - \psi_1)^- z_\eta^+ dx = 0.$$

Since $u_\eta > \psi_2$ on $\{z_\eta > 0\}$ and a_0 is non decreasing with respect to the last argument, one has $[a_0(x, u_\eta) - a_0(x, \psi_1)]z_\eta^+ \geq 0$ a.e. in D . Therefore

$$\int_D [a_0(x, u_\eta) - a_0(x, \psi_2)]z_\eta^+ dx \geq 0.$$

On the other hand, note that, \mathcal{B} being the set $\{\frac{1}{\eta}[(u_\eta - \psi_2)^+ - g_2^-] > 0\}$,

$$\begin{aligned} & \int_D [\tilde{a}(x, u_\eta, \nabla u_\eta) - \tilde{a}(x, \psi_2, \nabla \psi_2)] \cdot \nabla z_\eta^+ dx \\ &= \int_D 1_{\mathcal{B}} [\tilde{a}(x, u_\eta, \nabla u_\eta) - \tilde{a}(x, \psi_2, \nabla \psi_2)] \nabla \left[\frac{1}{\eta} [(u_\eta - \psi_2)^+ - g_2^-] \right] dx \\ &= \int_D 1_{\mathcal{B}} [\tilde{a}(x, \psi_2, \nabla u_\eta) - \tilde{a}(x, \psi_2, \nabla \psi_2)] \nabla \left[\frac{1}{\eta} [(u_\eta - \psi_2)^+ - g_2^-] \right] dx, \end{aligned}$$

since in this situation, the integration holds in the set $\{u_\eta > \psi_2\}$. Thus,

$$\begin{aligned} & [\tilde{a}(x, \psi_2, \nabla u_\eta) - \tilde{a}(x, \psi_2, \nabla \psi_2)] \nabla \left[\frac{1}{\eta} [(u_\eta - \psi_2)^+ - g_2^-] \right] \\ & \geq \frac{1}{\eta} [\tilde{a}(x, \psi_2, \nabla u_\eta) - \tilde{a}(x, \psi_2, \nabla \psi_2)] \nabla (u_\eta - \psi_2) \\ & \quad - \left| \tilde{a}(x, \psi_2, \nabla u_\eta) - \tilde{a}(x, \psi_2, \nabla \psi_2) \right| |\nabla g_2^-| \\ & \geq - \left| \tilde{a}(x, \psi_2, \nabla u_\eta) - \tilde{a}(x, \psi_2, \nabla \psi_2) \right| |\nabla g_2^-|. \end{aligned}$$

Thanks to (1.27), one gets

$$\left[\tilde{a}(x, \psi_2, \nabla u_\eta) - \tilde{a}(x, \psi_2, \nabla \psi_2) \right] \nabla (u_\eta - \psi_2)^+ \rightarrow 0 \text{ in } L^1(D).$$

Similar arguments detailed previously yield,

$$\int_D 1_{\{\frac{1}{\eta}[(u_\eta - \psi_2)^+ - g_2^-] > 0\}} \left| \tilde{a}(x, \psi_2, \nabla u_\eta) - \tilde{a}(x, \psi_2, \nabla \psi_2) \right| |\nabla g_2^-| dx \rightarrow 0.$$

As a conclusion, z_η^+ converges strongly to 0 in $L^2(D)$.

Since $z_\eta = \mu_\eta^2 - g_2^-$ and $\mu_\eta^2 \geq 0$, this implies that

$$0 \leq \mu_\eta^2 \leq g_2^- + z_\eta^+,$$

since $z_\eta^+ \rightarrow 0$ in $L^2(D)$, $(\mu_\eta^2)_\eta$ is bounded in $L^2(D)$, by extracting a subsequence, there exists a non negative function μ^2 such that

$$\mu_\eta^2 \rightharpoonup \mu^2 \text{ in } L^2(D) \quad \text{resp. in } \mathcal{D}'(D) \text{ and } 0 \geq -\mu^2 \geq -g_2^-.$$

By (1.26), one deduces

$$\mu^1 - \mu^2 = -\operatorname{div}[a(x, u, \nabla u)] + a_0(x, u) - f.$$

We know already that $B(u) \in V_{p(\cdot)}^*$ and we can add that

$$-(B(u))^- \geq -g_2^- = -(B(\psi_2))^-$$

which implies that $B(u) \geq -(B(\psi_2))^-$. This completes the proof of full Lewy-Stampacchia inequality (1.8) in regular case.

Remark 1.12. *As in Remark 1.10, one can prove Lewy-Stampacchia inequality without proving $B(u) \in V_{p(\cdot)}^*$, following the same type of arguments.*

1.3.3 Proof of Lewy Stampacchia inequalities in the general case

Let us consider now general data as assumed in H_4 . Thanks to [51, section 3.3], there exist g_n^1 and g_n^2 such that:

$$\begin{cases} g_n^1 \in W_0^{1,p(\cdot)}(D) \cap L^\infty(D), & g_n^1 \geq 0 & g_n^1 \rightarrow g_1^+ & \text{strongly in } W^{-1,p'(\cdot)}(D), \\ g_n^2 \in W_0^{1,p(\cdot)}(D) \cap L^\infty(D), & g_n^2 \geq 0 & g_n^2 \rightarrow g_2^- & \text{strongly in } W^{-1,p'(\cdot)}(D). \end{cases} \quad (1.28)$$

The first Lewy-Stampacchia inequality

Associated with g_n^1 , denote the following f_n^1 by,

$$f_n^1 = A(\psi_1) + a_0(\psi_1) - g_n^1 + g_1^-, \quad g_1^- \in W^{-1,p'(\cdot)}(D), \quad g_1^- \geq 0. \quad (1.29)$$

Note that $f_n^1 \in W^{-1,p'(\cdot)}(D)$ and f_n^1 converges strongly to f in $W^{-1,p'(\cdot)}(D)$. We also define B^n by

$$\forall v \in W^{1,p(\cdot)}(D), \quad B^n(v) = A(v) + a_0(v) - f_n^1.$$

Then $B^n(\psi_1) = g_n^1 - g_1^-$. By Theorem 1.7, there exists $u_n \in K(\psi_1, \psi_2)$ such that, $\forall v \in K(\psi_1, \psi_2)$

$$\int_D \left[a(x, u_n, \nabla u_n) \nabla(v - u_n) + a_0(x, u_n)(v - u_n) \right] dx \geq \langle f_n^1, v - u_n \rangle. \quad (1.30)$$

And satisfying (see subsection 1.3.2),

$$B(u_n) \in V_{p(\cdot)}^*, \quad B(u_n) \leq (B(u_n))^+ \leq g_n^1. \quad (1.31)$$

Since this solution comes from the above penalization method, and C in (1.15) can be chosen independent of n , one gets that

$$\|u_n\|_{W_0^{1,p(\cdot)}(D)} + \|a(x, u_n, \nabla u_n)\|_{(L^{p'(\cdot)}(D))^d} \leq C.$$

Up to a subsequence denoted similarly,

$$\begin{cases} u_n \rightharpoonup u & \text{in } W_0^{1,p(\cdot)}(D), \quad \text{strongly in } L^{p(\cdot)}(D) \quad \text{and a.e. in } D, \\ a_0(x, u_n) \rightarrow a_0(x, u) & \text{strongly in } L^{p'(\cdot)}(D), \\ a(x, u_n, \nabla u_n) \rightharpoonup \chi & \text{weakly in } (L^{p'(\cdot)}(D))^d. \end{cases}$$

Since $K(\psi_1, \psi_2)$ is a closed convex subset of $W_0^{1,p(\cdot)}(D)$, one gets $u \in K(\psi_1, \psi_2)$.

Taking $v = u$ in (1.30), one has

$$\int_D \left[a(x, u_n, \nabla u_n) \nabla(u - u_n) + a_0(x, u_n)(u - u_n) \right] dx \geq \langle f_n^1, u - u_n \rangle, \quad (1.32)$$

and passing to the limit, we get

$$\limsup_n \int_D a(x, u_n, \nabla u_n) \nabla u_n dx \leq \int_D \chi \nabla u dx.$$

The pseudomonotonicity of the operator $A(v) = -\operatorname{div}[a(x, v, \nabla v)]$ yields

$\operatorname{div} \chi = \operatorname{div}[a(x, u, \nabla u)]$ and

$$\lim_n \int_D a(x, u_n, \nabla u_n) \nabla u_n dx = \int_D a(x, u, \nabla u) \nabla u dx.$$

Arguments already detailed previously yield $\chi = a(x, u, \nabla u)$.

Passing to the limit in (1.30), there exists $u \in K(\psi_1, \psi_2)$ such that

$$\int_D \left[a(x, u, \nabla u) \nabla(v - u) + a_0(x, u)(v - u) \right] dx \geq \langle f, v - u \rangle, \quad \forall v \in K(\psi_1, \psi_2).$$

Passing to the limit in $B(u_n) \leq g_n^1$, one gets $B(u) \leq g_1^+$ in $W^{-1,p'(\cdot)}(D)$. Therefore,

$$\exists \kappa \in W^{-1,p'(\cdot)}(D), \quad \kappa = g_1^+ - B(u) \geq 0 \quad \text{such that } B(u) = g_1^+ - \kappa,$$

which implies $B(u) \in V_{p(\cdot)}^*$. Since $(B(u_n))^+ \leq g_n^1$, one has at the limit $B(u)^+ \leq g_1^+$.

Therefore $B(u) \in V_{p(\cdot)}^*$ and $B(u) \leq (B(u))^+ \leq g_1^+$.

This completes the proof of the first Lewy-Stampacchia inequality (1.6) of the main theorem.

The second Lewy Stampacchia inequality

Associated with g_n^2 , denote the following f_n^2 by,

$$f_n^2 = A(\psi_2) + a_0(\psi_2) - g_2^+ + g_n^2, \quad g_2^+ \in W^{-1,p'(\cdot)}(D), \quad g_2^+ \geq 0. \quad (1.33)$$

Note that $f_n^2 \in W^{-1,p'(\cdot)}(D)$ and f_n^2 converges strongly to f in $W^{-1,p'(\cdot)}(D)$. We also define B^n by

$$\forall v \in W^{1,p(\cdot)}(D), \quad B^n(v) = A(v) + a_0(v) - f_n^2.$$

Then $B^n(\psi_2) = g_2^+ - g_n^2$. By Theorem 1.7, there exists $u_n \in K(\psi_1, \psi_2)$ such that, $\forall v \in K(\psi_1, \psi_2)$

$$\int_D a(x, u_n, \nabla u_n) \nabla(v - u_n) dx + \int_D a_0(x, u_n)(v - u_n) dx \geq \langle f_n^1, v - u_n \rangle.$$

And satisfying (see subsection 1.3.2),

$$B(u_n) \in V_{p(\cdot)}^*, \quad B(u_n) \geq -(B(u_n))^- \geq -g_n^2 \quad (1.34)$$

A similar proof to the one of subsection 1.3.3 (The first Lewy Stampacchia inequality), one gets there exists $u \in K(\psi_1, \psi_2)$ such that

$$\int_D a(x, u, \nabla u) \nabla(v - u) dx + \int_D a_0(x, u)(v - u) dx \geq \langle f, v - u \rangle, \forall v \in K(\psi_1, \psi_2).$$

We know already that $B(u) \in V_{p(\cdot)}^*$. Passing to the limit in $B(u_n) \geq -g_n^2$, one gets $B(u) \geq -g_2^-$ in $W^{-1,p'(\cdot)}(D)$ and we can add that

$$B(u) \in V_{p(\cdot)}^*, \quad -g_2^- \leq -(B(u))^- \leq B(u).$$

Remark 1.13.

- By avoiding assumptions (2.7)-(2.9) of [51], this result is a significant generalization of Lewy-Stampacchia inequality for pseudomonotone operators.
- Note that in general the solution to the variational inequality is not a priori unique. So that, satisfying both Lewy-Stampacchia inequalities simultaneously is still an issue.

1.4 An exemple of problem satisfying both Lewy Stampacchia inequalities

Following [49], we propose in this section a situation where both Lewy Stampacchia inequalities are satisfied. Let j be a nonlinear superposition operator associated with

a Carathéodory function denoted with the same name on $D \times \mathbb{R}$ satisfying H_2 like a_0 . One assumes moreover that it is strictly monotone ($\lambda \mapsto j(\cdot, \lambda)$ is increasing). Let $U \in K(\psi_1, \psi_2)$ and note that

$$A(\psi_i) + a_0(\psi_i) + j(\psi_i) - f - j(U) = g_i^+ - g_i^- + j(\psi_i) - j(U) \in V_{p(\cdot)}^* \quad (i = 1, 2).$$

Then, from Section 1.3, there exist u_1 and u_2 in $K(\psi_1, \psi_2)$ satisfying, for any $v \in K(\psi_1, \psi_2)$,

$$\left\{ \begin{array}{l} \int_D a(\cdot, u_1, \nabla u_1) \nabla(v - u_1) + [a_0(u_1) + j(u_1)](v - u_1) dx \\ \qquad \qquad \qquad \geq \langle f, v - u_1 \rangle + \int_D j(U)(v - u_1) dx, \\ \int_D a(\cdot, u_2, \nabla u_2) \nabla(v - u_2) + [a_0(u_2) + j(u_2)](v - u_2) dx \\ \qquad \qquad \qquad \geq \langle f, v - u_2 \rangle + \int_D j(U)(v - u_2) dx, \end{array} \right. \quad (1.35)$$

with the additional information that $B_j(u_i) - j(U) \in V_{p(\cdot)}^*$ ($i = 1, 2$) where one denotes $B_j(u) = B(u) + j(u)$ and

$$\left\{ \begin{array}{l} B_j(u_1) - j(U) \leq (B_j(u_1) - j(U))^+ \leq (B_j(\psi_1) - j(U))^+, \\ -(B_j(\psi_2) - j(U))^- \leq -(B_j(u_2) - j(U))^- \leq B_j(u_2) - j(U). \end{array} \right. \quad (1.36)$$

Assuming furthermore that the solution to (1.35) is unique (this can be obtained by adapting *e.g.* the proof of [49, Prop. 2.2] in the framework of variable exponents Sobolev spaces), one gets that $u_1 = u_2$. If moreover $U = u$ is chosen from the solutions given by Prop. 1.7, it is also a solution to (1.35) and $u = u_1 = u_2$. Consequently, $B(u) \in V_{p(\cdot)}^*$ and

$$-(B(\psi_2) + j(\psi_2) - j(u))^- \leq B(u) \leq (B(\psi_1) + j(\psi_1) - j(U))^+.$$

Since $\lambda \mapsto j(\cdot, \lambda)$ is an increasing function, $\psi_1 \leq u \leq \psi_2$ yields

$$B(\psi_1) + j(\psi_1) - j(u) \leq g_1^+, \quad B(\psi_2) + j(\psi_2) - j(u) \geq -g_2^-$$

and $B(u) \in V_{p(\cdot)}^*$ with

$$-(A(\psi_2) + a_0(\psi_2) - f)^- \leq A(u) + a_0(u) - f \leq (A(\psi_1) + a_0(\psi_1) - f)^+. \quad (1.37)$$

To finish, let us give examples of situations leading to the uniqueness of the solution. Consider u_1 and u_2 two given solutions to (1.35) and $p_\delta : \mathbb{R} \rightarrow \mathbb{R}$ a Lipschitz-continuous, non decreasing function, such that $p_\delta(0) = 0$.

Denote by $w_1 = u_1 - \frac{1}{c}p_\delta(u_1 - u_2)$ where $c = \|p'_\delta\|_\infty$.

If $u_1 \geq u_2$, then $0 \leq \frac{1}{c}p_\delta(u_1 - u_2) \leq u_1 - u_2$ and $u_2 \leq w_1 \leq u_1$; similarly, if $u_1 \leq u_2$, then $u_1 \leq w_1 \leq u_2$.

Thanks to the chain-rule, $w_1 \in K(\psi_1, \psi_2)$, as well as $w_2 = u_2 + \frac{1}{c}p_\delta(u_1 - u_2)$. Then, using w_1 in the first part of (1.35), w_2 in the second part of (1.35), and adding the corresponding inequalities, one gets that

$$\begin{aligned} & \int_D \left[a(\cdot, u_1, \nabla u_1) - a(\cdot, u_2, \nabla u_2) \right] \nabla p_\delta(u_1 - u_2) \\ & + [a_0(u_1) - a(u_2) + j(u_1) - j(u_2)] p_\delta(u_1 - u_2) dx \leq 0 \end{aligned}$$

Set $p_\delta(r) = \min \left[1, \ln \left(\frac{re}{\delta} \right)^+ \right]$ (see e.g. [68]). p_δ is compatible with the assumptions, $p'_\delta(r) = \frac{1}{r} \mathbf{1}_{\{\frac{\delta}{e} < r < \delta\}}$ and it converges pointwise to the sign^+ -function. So, Fatou's lemma yields

$$\liminf_\delta \int_D [a_0(u_1) - a(u_2) + j(u_1) - j(u_2)] p_\delta(u_1 - u_2) dx \geq \int_D [j(u_1) - j(u_2)]^+ dx.$$

Concerning the main operator, assume in a first case that a is Lipschitz-continuous in the following sense: if $u \in W^{1,p(\cdot)}(D)$,

$$|a(\cdot, t, \nabla u) - a(\cdot, s, \nabla u)| \leq g_1(\nabla u) |t - s| \quad \text{where } g_1(\nabla u) \in L^{p'(\cdot)}(D).$$

$$\begin{aligned} \text{Thus, } & \int_D \left[a(\cdot, u_1, \nabla u_1) - a(\cdot, u_2, \nabla u_2) \right] \nabla p_\delta(u_1 - u_2) dx \\ & = \int_D p'_\delta(u_1 - u_2) \left[a(\cdot, u_1, \nabla u_1) - a(\cdot, u_1, \nabla u_2) \right] \nabla(u_1 - u_2) dx \\ & \quad + \int_D p'_\delta(u_1 - u_2) \left[a(\cdot, u_1, \nabla u_2) - a(\cdot, u_2, \nabla u_2) \right] \nabla(u_1 - u_2) dx \\ & \geq \int_D p'_\delta(u_1 - u_2) \left[a(\cdot, u_1, \nabla u_2) - a(\cdot, u_2, \nabla u_2) \right] \nabla(u_1 - u_2) dx \\ & \geq - \int_D p'_\delta(u_1 - u_2) |u_1 - u_2| g_1(\nabla u_2) |\nabla(u_1 - u_2)| dx \\ & \geq - \int_{\frac{\delta}{e} < u_1 - u_2 < \delta} g_1(\nabla u_2) |\nabla(u_1 - u_2)| dx. \end{aligned}$$

Assume in a second case a Hölder-continuous property with a stronger monotony in the following sense*:

$$\begin{aligned} [a(\cdot, \lambda, \vec{\xi}_1) - a(\cdot, \lambda, \vec{\xi}_2)](\vec{\xi}_1 - \vec{\xi}_2) &\geq c_0 |\vec{\xi}_1 - \vec{\xi}_2|^{\alpha(\cdot)}, \\ |a(\cdot, t, \nabla u) - a(\cdot, s, \nabla u)| &\leq g_2(\nabla u) |t - s|^{\theta(\cdot)} \\ \text{where } c_0 > 0, \quad \alpha' &\geq \frac{1}{\theta} > 1 \text{ and } g_2(\nabla u) \in L^{\alpha'(\cdot)}(D) \text{ if } u \in W^{1,p(\cdot)}(D). \end{aligned}$$

$$\begin{aligned} \text{Thus, } \int_D [a(\cdot, u_1, \nabla u_1) - a(\cdot, u_2, \nabla u_2)] \nabla p_\delta(u_1 - u_2) dx \\ &\geq c_0 \int_D p'_\delta(u_1 - u_2) |\nabla(u_1 - u_2)|^{\alpha(\cdot)} dx - c_0 \int_D p'_\delta(u_1 - u_2) |\nabla(u_1 - u_2)|^{\alpha(\cdot)} dx \\ &\quad - C \int_D p'_\delta(u_1 - u_2) |u_1 - u_2|^{\theta(\cdot)\alpha'(\cdot)} |g_2(\nabla u_2)|^{\alpha'(\cdot)} dx \\ &\geq -C \int_{\frac{\delta}{\epsilon} < u_1 - u_2 < \delta} |u_1 - u_2|^{\theta(\cdot)\alpha'(\cdot)-1} |g_2(\nabla u_2)|^{\alpha'(\cdot)} dx \\ &\geq -C \int_{\frac{\delta}{\epsilon} < u_1 - u_2 < \delta} |g_2(\nabla u_2)|^{\alpha'(\cdot)} dx. \end{aligned}$$

where C is a positive constant independent of δ .

In both situations, Lebesgue theorem yields

$$\liminf_{\delta} \int_D [a(\cdot, u_1, \nabla u_1) - a(\cdot, u_2, \nabla u_2)] \nabla p_\delta(u_1 - u_2) dx \geq 0$$

and $\int_D [j(u_1) - j(u_2)]^+ dx = 0$.

As j is increasing with respect to its second argument, one gets that $u_1 \leq u_2$. Interchanging u_1 and u_2 , the result of uniqueness holds.

Remark 1.14. We invite the reader interested in more general situations, like local continuity assumptions, to consult [49] concerning the bilateral problem and [29, 54] and their references for uniqueness methods.

*additional assumptions are made on the exponents to make sense to the integrals.

On Lewy-Stampacchia inequality for parabolic problem

In this chapter, we are interested in non linear parabolic problems with constraint and homogeneous Dirichlet boundary conditions. This study is a part of a joint work with O. Guibé, A. Mokrane and G.Vallet [33].

In this work, we prove the existence of a solution satisfying the following Lewy-Stampacchia inequality

$$0 \leq \partial_t u - \operatorname{div}[a(\cdot, \cdot, u, \nabla u)] - f \leq g^- = (f - \partial_t \psi + \operatorname{div}[a(\cdot, \cdot, \psi, \nabla \psi)])^-,$$

associated with the following problem

$$\int_0^T \langle \partial_t u, v - u \rangle dt + \int_Q a(t, x, u, \nabla u) \nabla(v - u) dx dt \geq \int_0^T \langle f, v - u \rangle dt,$$

where $u \mapsto -\operatorname{div}[a(t, x, u, \nabla u)]$ is a pseudomonotone operator under the constraint $u \geq \psi$. We use a method of penalization of the constraint associated with a suitable perturbation of the operator. The aim of this chapter is to give in detail sketches of this proof; we start by proving an existence theorem then the proof of Lewy-Stampacchia inequality associated with the problem will be given.

Keywords: Lewy-Stampacchia inequality, Variational inequalities, Pseudomonotone, Penalization.

2.1 Introduction

2.1.1 Former results / Our result

After the first results of H. Lewy and G. Stampacchia [41] concerning inequalities in the context of superharmonic problems, many authors have been interested in the so-called Lewy-Stampacchia inequality associated with obstacle problems. We refer to chapter 1 to have an idea about the literatures of elliptic Lewy-Stampacchia inequalities.

The literature on Lewy-Stampacchia inequality is mainly aimed at elliptic problems, or close to elliptic problems and fewer papers are concerned with other type of problems. Let us cite J. F. Rodrigues [63] for hyperbolic problems, F. Donati [22] for parabolic problems with a monotone operator or L. Mastroeni and M. Matzeu [46] in the case of a double obstacle.

To the best of our knowledge, F. Donati's work [22] has not been extended to pseudomonotone parabolic problems with a Leray Lions operator. This chapter concerns the study of Lewy-Stampacchia inequality associated with pseudomonotone parabolic problems.

2.1.2 Content of the study

Our aim is to propose a result of existence of solution for

$$\int_0^T \langle \partial_t u, v - u \rangle dt + \int_Q a(t, x, u, \nabla u) \nabla(v - u) dx dt \geq \int_0^T \langle f, v - u \rangle dt,$$

by using a method of penalization associated with an *ad hoc* perturbation of the operator, where $u \mapsto -\operatorname{div}[a(t, x, u, \nabla u)]$ is a pseudomonotone operator under the constraint $u \geq \psi$. Then we prove the following Lewy-Stampacchia inequality

$$0 \leq \partial_t u - \operatorname{div}[a(\cdot, \cdot, u, \nabla u)] - f \leq g^- = (f - \partial_t \psi + \operatorname{div}[a(\cdot, \cdot, \psi, \nabla \psi)])^-.$$

This chapter is organized in the following way: after giving the hypotheses and the main result, a second section is devoted to an existence result for the penalized problem associated with a new perturbation of the operator. After some *a priori* estimates and passing to the limit, one gets the existence of solution to our problem, Lewy-Stampacchia inequality is proved for regular data in a new section. Finally, the general case is considered by using some adapted density arguments.

2.2 Assumptions and main results

Let us denote by $p \in (1, +\infty)$. As usual, p' denotes the conjugate exponent of p , $V = W_0^{1,p}(D)$ if $p \geq 2$ and $V = W_0^{1,p}(D) \cap L^2(D)$ with the graph-norm else. Then, the corresponding dual spaces are $V' = W^{-1,p'}(D)$ if $p \geq 2$ and $V' = W^{-1,p'}(D) + L^2(D)$ else (cf. e.g. [28, p.24]).

In this situation, the Lions-Guelfand triple $V \xhookrightarrow{d} L^2(D) \xhookrightarrow{d} V'$ holds.

Assume in the sequel the following:

H_1 : A is a Leray-Lions pseudomonotone operator of the form

$$v \mapsto A(v) = -\operatorname{div} \left[a(t, x, v, \nabla v) \right],$$

which acts from $W^{1,p}(D)$ into $W^{-1,p'}(D)$ where

$H_{1,1}$ $a : (t, x, u, \vec{\xi}) \in Q \times \mathbb{R} \times \mathbb{R}^d \mapsto a(t, x, u, \vec{\xi}) \in \mathbb{R}^d$ is a Carathéodory function on $Q \times \mathbb{R}^{d+1}$,

$H_{1,2}$ a is strictly monotone with respect to its last argument:

$$\begin{aligned} & \forall (t, x) \in Q \text{ a.e.}, u \in \mathbb{R}, \forall \vec{\xi}, \vec{\eta} \in \mathbb{R}^d, \\ & \vec{\xi} \neq \vec{\eta} \Rightarrow [a(t, x, u, \vec{\xi}) - a(t, x, u, \vec{\eta})] \cdot (\vec{\xi} - \vec{\eta}) > 0. \end{aligned}$$

$H_{1,3}$ a is coercive and bounded: there exist constants $\bar{\alpha} > 0$, $\bar{\beta} > 0$ and $\bar{\gamma} \geq 0$, a function \bar{h} in $L^1(Q)$ and a function \bar{k} in $L^p(Q)$ and two exponents $q, r < p$ such that, for a.e. $(t, x) \in Q$, for all $u \in \mathbb{R}$ and for all $\vec{\xi} \in \mathbb{R}^d$,

$$a(t, x, u, \vec{\xi}) \cdot \vec{\xi} \geq \bar{\alpha} |\vec{\xi}|^p - \left[\bar{\gamma} |u|^q + |\bar{h}(t, x)| \right], \quad (2.1)$$

$$|a(t, x, u, \vec{\xi})| \leq \bar{\beta} \left[|\bar{k}(t, x)| + |u|^{r/p} + |\vec{\xi}| \right]^{p-1}. \quad (2.2)$$

H_2 : assume that the obstacle ψ belongs to $L^p(0, T, W^{1,p}(D)) \cap L^p(0, T, L^2(D))$; that $\partial_t \psi$ belongs to $L^{p'}(0, T, V')$ and $\psi \leq 0$ on ∂D (see Lemma 2.1 for the time regularity of such elements).

Lemma 2.1. *We have the following result:*

$$u \in L^p(0, T; W^{1,p}(D) \cap L^2(D)), \partial_t u \in L^{p'}(0, T; W^{-1,p'}(D) + L^2(D)) \Rightarrow u \in \mathcal{C}([0, T], L^2(D)).$$

Remark 2.2. *A detailed proof of Lemma 2.1 is given in Appendix B.1.*

Denote by $\mathcal{K}(\psi) := \{u \in W(0, T), u \geq \psi\}$.

Remark 2.3. $\mathcal{K}(\psi)$ is a not empty convex set.

Proof. Indeed, if one denotes by v^* , the solution in $W(0, T)$ to

$$\partial_t v^* - \Delta_p v^* = \partial_t \psi - \Delta_p \psi \in L^{p'}(0, T, V'), \quad v^*(t=0) = \psi(0).$$

$-(v^* - \psi)^- \in L^p(0, T, V)$ is an admissible test-function and one has that

$$-\langle \partial_t(v^* - \psi), (v^* - \psi)^- \rangle + \int_D 1_{\{v^* - \psi < 0\}} [|\nabla v^*|^{p-2} \nabla v^* - |\nabla \psi|^{p-2} \nabla \psi] \cdot \nabla (v^* - \psi) dx = 0.$$

Then, Corollary 3.28 with $\beta = 1$ and $\alpha = 1$ yields for any t

$$\begin{aligned} 0 &\geq - \int_0^t \langle \partial_t(v^* - \psi), (v^* - \psi)^- \rangle ds = - \int_D \int_0^{(v^* - \psi)(t)} s^- ds dx + \int_D \int_0^{(v^* - \psi)(0)} s^- ds dx \\ &= \frac{1}{2} \|(v^* - \psi)^-(t)\|_{L^2}^2 \end{aligned}$$

since $(v^* - \psi)^-(0) = 0$. As a consequence, $v^* \geq \psi$ and $v^* \in \mathcal{K}(\psi)$. \square

H_3 : the right hand side f , which is assumed to be such that

$$g = f - \partial_t \psi - A(\psi) = g^+ - g^-$$

belongs to the order dual $L^p(0, T, V)^* = (L^{p'}(0, T, V'))^+ - (L^{p'}(0, T, V'))^+$, i.e. $g^+, g^- \in (L^{p'}(0, T, V'))^+$, the non-negative elements of $L^{p'}(0, T, V')$.

$$(f \in (L^{p'}(0, T; V'))^+ \Leftrightarrow \{\forall \varphi \in L^p(0, T; V), \varphi \geq 0 \Rightarrow \langle f, \varphi \rangle \geq 0\}).$$

H_4 : $u_0 \in L^2(D)$ satisfies the constraint, i.e. $u_0 \geq \psi(0)$.

The main result in the sequel is the following.

Theorem 2.4. *Under the above assumptions (H_1) - (H_4) , there exists at least $u \in \mathcal{K}(\psi)$ with $u(t=0) = u_0$ and such that, for any $v \in L^p(0, T, V)$, $v \geq \psi$ implies that*

$$\int_0^T \langle \partial_t u, v - u \rangle dt + \int_Q a(t, x, u, \nabla u) \nabla (v - u) dx dt \geq \int_0^T \langle f, v - u \rangle dt.$$

Moreover, the following Lewy-Stampacchia inequality holds

$$0 \leq \partial_t u - \operatorname{div}[a(\cdot, \cdot, u, \nabla u)] - f \leq g^- = (f - \partial_t \psi + \operatorname{div}[a(\cdot, \cdot, \psi, \nabla \psi)])^-.$$

2.3 Proof of the main result

Theorem 2.4 will be proved in four steps. In a first part, we establish the existence of a solution to a problem where the constraint $u \geq \psi$ is penalized. Moreover, the crucial point in the method developed in the present chapter is to replace $a(\cdot, \cdot, u, \vec{\xi})$ by $a(\cdot, \cdot, \max(u, \psi), \vec{\xi})$. The aim of this additional perturbation is to ensure, formally, a monotone behavior of the operator when u violates the constraint. This is the aim of Theorem 2.6.

For technical reasons, some *a priori* estimates and the passage to the limit will be obtained firstly by assuming that g^- is regular. This is the object of Lemmas 2.10, 2.11, 2.12 and Theorem 2.8. Then a proof of Lewy-Stampacchia inequality, still with a regular g^- , will be presented in Lemma 2.14.

Finally, one will be able to prove Lewy-Stampacchia inequality in the general case.

2.3.1 Penalization

Denote by $\tilde{q} = \min(p, 2)$ and let us define the function Θ

$$\Theta : \mathbb{R} \rightarrow \mathbb{R}, \quad x \mapsto -[x^-]^{\tilde{q}-1},$$

and the perturbed operator

$$\tilde{a}(t, x, u, \vec{\xi}) : Q \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}^d \quad (x, t, u, \vec{\xi}) \mapsto \tilde{a}(t, x, u, \vec{\xi}) = a(t, x, \max(u, \psi(t, x)), \vec{\xi}). \quad (2.3)$$

Remark 2.5. *We wish to draw the reader's attention to the fact that with the proposed perturbation: $\tilde{a}(t, x, u, \vec{\xi}) = a(t, x, \max(u, \psi), \vec{\xi})$, the idea is to make formally the operator monotone and not pseudomonotone any more on the free-set where the constraint is violated.*

We define $\mathcal{A} : L^p(0, T; V) \rightarrow L^{p'}(0, T; V')$ such that

$$[\mathcal{A}(u)](t) := \tilde{A}(u(t)) = -\operatorname{div}[\tilde{a}(t, x, u, \nabla u)].$$

Note that, the above assumption H_1 still holds. Indeed,

$$\tilde{a}(t, x, u, \vec{\xi}) \cdot \vec{\xi} \geq \bar{\alpha}|\vec{\xi}|^p - \left[\bar{\gamma} |\max(u, \psi)|^q + |\bar{h}(t, x)| \right], \quad (2.4)$$

$$|\tilde{a}(t, x, u, \vec{\xi})| \leq \bar{\beta} \left[|\bar{k}(t, x)| + |\max(u, \psi)|^{r/p} + |\vec{\xi}| \right]^{p-1}. \quad (2.5)$$

Since $|\max(u, \psi)|^q \leq |u|^q + |\psi|^q$, $|\max(u, \psi)|^{r/p} \leq |u|^{r/p} + |\psi|^{r/p}$, (2.1) and (2.2) are satisfied by replacing \bar{h} by $\bar{h} + \bar{\gamma}|\psi|^q$ and \bar{k} by $\bar{k} + \psi^{r/p}$. For any positive ε , we have the following result.

Theorem 2.6. *There exists $u_\varepsilon \in W(0, T)$ such that $u_\varepsilon(t = 0) = u_0$ and*

$$\partial_t u_\varepsilon - \operatorname{div} \left[\tilde{a}(t, x, u_\varepsilon, \nabla u_\varepsilon) \right] + \frac{1}{\varepsilon} \Theta(u_\varepsilon - \psi) = f, \quad (2.6)$$

$$\text{i.e. } \partial_t u_\varepsilon - \operatorname{div} \left[a(t, x, \max[u_\varepsilon, \psi], \nabla u_\varepsilon) \right] + \frac{1}{\varepsilon} \Theta(u_\varepsilon - \psi) = f.$$

Remark 2.7. *A detailed proof of Theorem 2.6 is given in Appendix C.*

2.3.2 Existence result for a regular g^-

Following Assumption H_3 let us recall that $f - \partial_t \psi - A\psi = g = g^+ - g^-$ belongs to the order dual $L^p(0, T; V)^*$. In this subsection we impose an additional regularity on g^- , namely $0 \leq g^- \in L^{\tilde{q}'}(Q) \hookrightarrow L^{p'}(Q)$. This subsection is devoted to prove the following result.

Theorem 2.8. *Assume $H_1 - H_4$, $f - \partial_t \psi - A\psi = g = g^+ - g^- \in L^p(0, T; V)^*$ where $g^- \in L^{p'}(Q) \cap L^2(Q)$. There exists at least $u \in \mathcal{K}(\psi)$ with $u(t = 0) = u_0$ such that, for any $v \in L^p(0, T; V)$ with $v \geq \psi$,*

$$\int_0^T \langle \partial_t u, v - u \rangle dt + \int_Q a(t, x, u, \nabla u) \cdot \nabla(v - u) dx dt \geq \int_0^T \langle f, v - u \rangle dt.$$

Remark 2.9. *Note that the pseudomonotone assumption of the operator doesn't ensure the uniqueness of the solution. Observe that under additional assumptions on the operator a , namely a local Lipschitz continuity with respect to the third variable, standard arguments allow one to prove the uniqueness of the solution obtained in Theorem 2.8.*

First, we establish some *a priori* estimates with respect to ε then we pass to the limit when $\varepsilon \rightarrow 0$ to get Theorem 2.8.

A priori estimates with respect to ε

Let us test the penalized problem (2.6) with $u_\varepsilon - v^*$,

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|u_\varepsilon - v^*\|_{L^2(D)}^2 + \int_D \tilde{a}(t, x, u_\varepsilon, \nabla u_\varepsilon) \cdot \nabla u_\varepsilon dx + \frac{1}{\varepsilon} \int_D \Theta(u_\varepsilon - \psi)(u_\varepsilon - v^*) dx \\ & = \langle f - \partial_t v^*, u_\varepsilon - v^* \rangle + \int_D \tilde{a}(t, x, u_\varepsilon, \nabla u_\varepsilon) \cdot \nabla v^* dx. \end{aligned}$$

Thus, by using (2.1), for any positive δ_1 , there exists C_{δ_1} depending on δ_1 and D such that

$$\begin{aligned} \int_D \tilde{a}(t, x, u_\varepsilon, \nabla u_\varepsilon) \cdot \nabla u_\varepsilon dx &\geq \int_D (\bar{\alpha} |\nabla u_\varepsilon|^p - \bar{\gamma} |\max(u_\varepsilon, \psi)|^q - |\bar{h}|) dx \\ &\geq \bar{\alpha} \|u_\varepsilon\|_{W_0^{1,p}(D)}^p - \bar{\gamma} \|u_\varepsilon\|_{L^q(D)}^q - \bar{\gamma} \|\psi\|_{L^q(D)}^q - \|\bar{h}\|_{L^1(D)} \\ &\geq \bar{\alpha} \|u_\varepsilon\|_{W_0^{1,p}(D)}^p - \delta_1 \|u_\varepsilon\|_{L^p(D)}^p - \bar{\gamma} \|\psi\|_{L^q(D)}^q - \|\bar{h}\|_{L^1(D)} - C_{\delta_1}. \end{aligned}$$

For the third term, $\Theta \leq 0$ and $v^* \geq \psi$ yield

$$\frac{1}{\varepsilon} \int_D \Theta(u_\varepsilon - \psi)(u_\varepsilon - v^*) dx \geq \frac{1}{\varepsilon} \int_D \Theta(u_\varepsilon - \psi)(u_\varepsilon - \psi) dx.$$

By using (2.2), for any positive δ_2 , there exists C_{δ_2} depending on δ_2 and D such that

$$\begin{aligned} \int_D \tilde{a}(t, x, u_\varepsilon, \nabla u_\varepsilon) \cdot \nabla v^* dx &\leq \int_D \bar{\beta} \left[|\bar{k}| + |\max(u_\varepsilon, \psi)|^{r/p} + |\nabla u_\varepsilon| \right]^{p-1} |\nabla v^*| dx \\ &\leq C_{\delta_2} \|v^*\|_{W_0^{1,p}(D)}^p + \delta_2 \left[\|\bar{k}\|_{L^p(D)}^p + \|\psi\|_{L^r(D)}^r + \|u_\varepsilon\|_{L^r(D)}^r + \|u_\varepsilon\|_{W_0^{1,p}(D)}^p \right] \\ &\leq \delta_2 \|u_\varepsilon\|_{W_0^{1,p}(D)}^p + \delta_2 \frac{r}{p} \|u_\varepsilon\|_{L^p(D)}^p + C_{\delta_2} \|v^*\|_{W_0^{1,p}(D)}^p + \delta_2 \|\bar{k}\|_{L^p(D)}^p + C_{\delta_2}. \end{aligned}$$

Finally, for any positive δ_3 , there exists C_{δ_3} depending on δ_3 and D such that

$$\langle f - \partial_t v^*, u_\varepsilon - v^* \rangle \leq \delta_3 [\|u_\varepsilon\|_V^p + \|v^*\|_V^p] + C_{\delta_3} \|f - \partial_t v^*\|_{V'}^{p'}.$$

In conclusion we have

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|u_\varepsilon - v^*\|_{L^2(D)}^2 + \bar{\alpha} \|u_\varepsilon\|_{W_0^{1,p}(D)}^p + \frac{1}{\varepsilon} \int_D \Theta(u_\varepsilon - \psi)(u_\varepsilon - \psi) dx \\ \leq \delta_1 \|u_\varepsilon\|_{L^p(D)}^p + \delta_2 \|u_\varepsilon\|_{W_0^{1,p}(D)}^p + \delta_2 \frac{r}{p} \|u_\varepsilon\|_{L^p(D)}^p + \delta_3 \|u_\varepsilon\|_V^p \\ + \bar{\gamma} \|\psi\|_{L^q(D)}^q + C_{\delta_2} \|v^*\|_{W_0^{1,p}(D)}^p + \delta_3 \|v^*\|_V^p \\ + C_{\delta_3} \|f - \partial_t v^*\|_{V'}^{p'} + \|\bar{h}\|_{L^1(D)} + \delta_2 \|\bar{k}\|_{L^p(D)}^p + C_{\delta_1} + C_{\delta_2}. \end{aligned}$$

Then, using Young's inequality and a convenient choice of parameters $\delta_1, \delta_2, \delta_3$ yield that for any positive δ there exists C depending on the listed parameters such that

$$\begin{aligned} \sup_t \|u_\varepsilon\|_{L^2(D)}^2(t) + \|u_\varepsilon\|_{L^p(0,T;W_0^{1,p}(D))}^p + \frac{1}{\varepsilon} \|\Theta(u_\varepsilon - \psi)(u_\varepsilon - \psi)^-\|_{L^1(Q)} \\ \leq C(\delta, \|v^*\|_{W(0,T)}, \|\psi\|_{L^p(0,T;V)}, \|\bar{k}\|_{L^p(D)}, \|\bar{h}\|_{L^1(Q)}, \|f\|_{L^{p'}(0,T;V')}) + \delta \|u_\varepsilon\|_{L^p(0,T;V)}^p. \end{aligned} \tag{2.7}$$

Lemma 2.10. *There exists a constant C_1 depending on $\|v^*\|_{W(0,T)}$, $\|\psi\|_{L^p(0,T;V)}$, $\|\bar{k}\|_{L^p(D)}$, $\|\bar{h}\|_{L^1(Q)}$ and $\|f\|_{L^{p'}(0,T;V')}$ such that, for any $\varepsilon > 0$,*

$$\sup_t \|u_\varepsilon\|_{L^2(D)}^2(t) + \|u_\varepsilon\|_{L^p(0,T;V)}^p + \frac{1}{\varepsilon} \|(u_\varepsilon - \psi)^-\|_{L^{\bar{q}}(Q)}^{\bar{q}} \leq C_1. \quad (2.8)$$

Proof.

If $p \geq 2$, $W_0^{1,p}(D) = V$ so that Lemma 2.10 is a straightforward consequence of (2.7).

If $p < 2$, it is enough to remark that

$$\begin{aligned} \sup_t \|u_\varepsilon\|_{L^2(D)}^2(t) + \|u_\varepsilon\|_{L^p(0,T;V)}^p &= \sup_t \|u_\varepsilon\|_{L^2(D)}^2(t) + \int_0^T [\|u_\varepsilon(t)\|_{L^2(D)} + \|u_\varepsilon(t)\|_{W_0^{1,p}(D)}]^p dt \\ &\leq \sup_t \|u_\varepsilon\|_{L^2(D)}^2(t) + 2^{p-1} \int_0^T [\|u_\varepsilon(t)\|_{L^2(D)}^p + \|u_\varepsilon(t)\|_{W_0^{1,p}(D)}^p] dt \\ &\leq \sup_t \|u_\varepsilon\|_{L^2(D)}^2(t) + 2^{p-1} \int_0^T \left[\frac{p}{2} \|u_\varepsilon(t)\|_{L^2(D)}^2 + \frac{2-p}{2} + \|u_\varepsilon(t)\|_{W_0^{1,p}(D)}^p \right] dt \\ &\leq (1 + 2^{p-2}pT) \sup_t \|u_\varepsilon\|_{L^2(D)}^2(t) + 2^{p-1} \int_0^T \|u_\varepsilon(t)\|_{W_0^{1,p}(D)}^p dt + 2^{p-2}T(2-p). \end{aligned}$$

□

It is worth noting that Lemma 2.10 gives that $\frac{1}{\varepsilon} \int_Q ((u_\varepsilon - \psi)^-)^{\bar{q}} dx dt$ is bounded (with respect to ε) so that we cannot expect to have a bound of the penalized term $\frac{1}{\varepsilon} \Theta(u_\varepsilon - \psi)$ in $L^{p'}(Q)$ nor in $L^{p'}(0, T; V')$.

Using the additional regularity $g^- \in L^{\bar{q}'}(Q)$ we prove in the following lemma more precise estimates on $(u_\varepsilon - \psi)^-$.

Lemma 2.11. *There exists a constant C_2 depending on C_1 of Lemma 2.10, such that for any $\varepsilon > 0$,*

$$\sup_{t \in (0,T)} \|(u_\varepsilon - \psi)^-(t)\|_{L^2(D)}^2 \leq C_2 \|g^-\|_{L^{\bar{q}'}(Q)} \varepsilon^{1/\bar{q}}, \quad (2.9)$$

$$\int_Q \left| \left[\tilde{a}(t, x, u_\varepsilon, \nabla u_\varepsilon) - \tilde{a}(t, x, \psi, \nabla \psi) \right] \cdot \nabla (u_\varepsilon - \psi)^- \right| dx ds \leq C_2 \|g^-\|_{L^{\bar{q}'}(Q)} \varepsilon^{1/\bar{q}}, \quad (2.10)$$

$$\frac{1}{\varepsilon} \|(u_\varepsilon - \psi)^-\|_{L^{\bar{q}}(Q)}^{\bar{q}-1} \leq C_2 \|g^-\|_{L^{\bar{q}'}(Q)}. \quad (2.11)$$

Proof. With the admissible test-function $(u_\varepsilon - \psi)^-$, one gets that

$$\begin{aligned} & - \left\langle \frac{d}{dt}(u_\varepsilon - \psi), (u_\varepsilon - \psi)^- \right\rangle - \int_{D \cap \{u_\varepsilon - \psi < 0\}} \left[\tilde{a}(t, x, u_\varepsilon, \nabla u_\varepsilon) - \tilde{a}(t, x, \psi, \nabla \psi) \right] \cdot \nabla (u_\varepsilon - \psi)^- dx \\ & - \frac{1}{\varepsilon} \int_D \Theta(u_\varepsilon - \psi)(u_\varepsilon - \psi)^- dx = - \langle f - \partial_t \psi + \operatorname{div} [\tilde{a}(t, x, \psi, \nabla \psi)], (u_\varepsilon - \psi)^- \rangle. \end{aligned}$$

Then, since $(u_\varepsilon - \psi)^- \in L^p(0, T; V)$ with $(u_\varepsilon - \psi)^-(0) = 0$, Corollary 3.28 yields: for any $t \in (0, T)$,

$$\begin{aligned} & \frac{1}{2} \|(u_\varepsilon - \psi)^-(t)\|_{L^2(D)}^2 + \int_0^t \int_{\{u_\varepsilon - \psi < 0\}} \left[\tilde{a}(s, x, u_\varepsilon, \nabla u_\varepsilon) - \tilde{a}(s, x, \psi, \nabla \psi) \right] \cdot \nabla (u_\varepsilon - \psi)^- dx ds \\ & - \frac{1}{\varepsilon} \int_0^t \int_D \Theta(u_\varepsilon - \psi)(u_\varepsilon - \psi)^- dx ds = - \int_0^t \langle f - \partial_s \psi + \operatorname{div} [\tilde{a}(s, x, \psi, \nabla \psi)], (u_\varepsilon - \psi)^- \rangle ds. \end{aligned}$$

In view of the definition of \tilde{a} we have $\tilde{a}(t, x, u_\varepsilon, \nabla u_\varepsilon) = a(t, x, \psi, \nabla u_\varepsilon)$ in the set $\{u_\varepsilon < \psi\}$. Therefore using assumption $H_{1,2}$ we obtain

$$\begin{aligned} & \frac{1}{2} \|(u_\varepsilon - \psi)^-(t)\|_{L^2(D)}^2 + \int_0^t \int_D \left| \left[\tilde{a}(s, x, u_\varepsilon, \nabla u_\varepsilon) - \tilde{a}(s, x, \psi, \nabla \psi) \right] \cdot \nabla (u_\varepsilon - \psi)^- \right| dx ds \\ & \quad + \frac{1}{\varepsilon} \int_0^t \|\Theta(u_\varepsilon - \psi)(u_\varepsilon - \psi)^-\|_{L^1(D)} ds \\ & \leq - \int_0^t \langle g, (u_\varepsilon - \psi)^- \rangle ds = - \int_0^t \langle g^+, (u_\varepsilon - \psi)^- \rangle ds + \int_0^t \int_D g^- (u_\varepsilon - \psi)^- dx ds. \end{aligned}$$

We recall that Lemma 2.10 yielded $\|(u_\varepsilon - \psi)^-\|_{L^{\tilde{q}}(Q)} \leq C_1 \varepsilon$, so that

$$\begin{aligned} & \frac{1}{2} \|(u_\varepsilon - \psi)^-(t)\|_{L^2(D)}^2 + \int_0^t \int_D \left| \left[\tilde{a}(s, x, u_\varepsilon, \nabla u_\varepsilon) - \tilde{a}(s, x, \psi, \nabla \psi) \right] \cdot \nabla (u_\varepsilon - \psi)^- \right| dx ds \\ & \quad + \frac{1}{\varepsilon} \int_0^t \|(u_\varepsilon - \psi)^-\|_{L^{\tilde{q}}(D)} ds \\ & \leq - \int_0^t \langle g, (u_\varepsilon - \psi)^- \rangle ds = - \int_0^t \langle g^+, (u_\varepsilon - \psi)^- \rangle ds + \int_0^t \int_D g^- (u_\varepsilon - \psi)^- dx ds \\ & \leq \|g^-\|_{L^{\tilde{q}'(Q)}} \|(u_\varepsilon - \psi)^-\|_{L^{\tilde{q}}(Q)} \leq \sqrt[\tilde{q}]{C_1 \varepsilon} \|g^-\|_{L^{\tilde{q}'(Q)}} \end{aligned}$$

and Lemma 2.11 holds. \square

Gathering Lemmas 2.10 and 2.11 we prove the following estimates.

Lemma 2.12. *There exists a constant C_3 depending on C_1 , C_2 and $\|g^-\|_{L^{\tilde{q}'(Q)}}$ such that for any $\varepsilon > 0$*

$$\|\partial_t u_\varepsilon\|_{L^{p'}(0, T; V')} + \|\tilde{a}(t, x, u_\varepsilon, \nabla u_\varepsilon)\|_{L^{p'}(Q)} + \|\tilde{A}(u_\varepsilon)\|_{L^{p'}(0, T; V')} \leq C_3.$$

Proof.

The growth condition (2.5) on \tilde{a} and Lemma 2.10 imply that

$$\begin{aligned} |\tilde{a}(t, x, u_\varepsilon, \nabla u_\varepsilon)|^{p'} &= |a(t, x, \max(u_\varepsilon, \psi), \nabla u_\varepsilon)|^{p'} \\ &\leq \tilde{\beta}^{p'} \left[|\bar{k}| + |u_\varepsilon|^{r/p} + |\psi|^{r/p} + |\nabla u_\varepsilon| \right]^p \\ &\leq C \left[|\bar{k}|^p + |u_\varepsilon|^p + |\psi|^p + |\nabla u_\varepsilon|^p + 1 \right] \end{aligned}$$

and then $\tilde{a}(t, x, u_\varepsilon, \nabla u_\varepsilon)$ is bounded in $L^{p'}(Q)^d$. The boundedness of $\|\tilde{A}(u_\varepsilon)\|_{L^{p'}(0,T;V')}$ is a direct consequence of the above inequality.

Recalling that $\partial_t u_\varepsilon = f - \tilde{A}(u_\varepsilon) - \frac{1}{\varepsilon} \Theta(u_\varepsilon - \psi)$ it remains to estimate $\frac{1}{\varepsilon} \Theta(u_\varepsilon - \psi)$ in $L^{p'}(0, T; V')$. We distinguish the two cases $p \geq 2$ and $1 < p < 2$.

If $p \geq 2$ then $\tilde{q} = 2$. From Lemma 2.11 we have $\frac{1}{\varepsilon} \|(u_\varepsilon - \psi)^-\|_{L^2(Q)} \leq C$ and since

$$\left\| \frac{1}{\varepsilon} \Theta(u_\varepsilon - \psi) \right\|_{L^{p'}(0,T;V')} = \sup_{\|v\|_{L^p(0,T;V)} \leq 1} \left\langle \frac{1}{\varepsilon} \Theta(u_\varepsilon - \psi), v \right\rangle \leq \frac{1}{\varepsilon} \|(u_\varepsilon - \psi)^-\|_{L^2(Q)} \|v\|_{L^2(Q)} \leq C,$$

it follows that $(\frac{1}{\varepsilon} \Theta(u_\varepsilon - \psi))_\varepsilon$ is bounded in $L^{p'}(0, T; V')$.

If $1 < p < 2$ then $\tilde{q} = p$. From Lemma 2.11 we have $\frac{1}{\varepsilon} \|(u_\varepsilon - \psi)^-\|_{L^p(Q)}^{p-1} \leq C$ and we have

$$\left\| \frac{1}{\varepsilon} \Theta(u_\varepsilon - \psi) \right\|_{L^{p'}(0,T;V')} = \sup_{\|v\|_{L^p(0,T;V)} \leq 1} \left\langle \frac{1}{\varepsilon} \Theta(u_\varepsilon - \psi), v \right\rangle \leq \frac{1}{\varepsilon} \|(u_\varepsilon - \psi)^-\|_{L^p(Q)}^{p-1} \leq C,$$

which concludes the proof of Lemma 2.12. \square

At the limit when $\varepsilon \rightarrow 0$.

The sequence $(u_\varepsilon)_\varepsilon$ is bounded in $W(0, T)$, therefore, up to a subsequence denoted the same, there exists $u \in W(0, T)$ such that u_ε converges weakly to u in $W(0, T)$. In particular, one gets that $u(t=0) = u_0$.

Then, by classical compactness arguments of type Aubin-Lions-Simon [71], the convergence is strong in $L^p(Q)$, and a.e. in Q^\dagger . Therefore, $(u_\varepsilon - \psi)^- \rightarrow (u - \psi)^-$ in $L^p(Q)$ and thanks to Lemma 2.11, one gets that $(u - \psi)^- = 0$ i.e. $u \in \mathcal{K}(\psi)$.

Moreover from Lemma 2.12 there exists $\vec{\xi} \in L^{p'}(Q)^d$ such that

$$\tilde{a}(\cdot, \cdot, u_\varepsilon, \nabla u_\varepsilon) \text{ converges weakly to } \vec{\xi} \text{ in } L^{p'}(Q)^d. \quad (2.12)$$

[†]Some arguments are given in Appendix A.2 when $p < 2$.

By (2.2), the following estimate holds for any $v \in L^p(0, T; V)$,

$$|\tilde{a}(t, x, u, \nabla v)|^{p'} \leq C \left[1 + |\bar{k}|^p + |u|^p + |\psi|^p + |\nabla v|^p \right],$$

so that, since $u \in \mathbb{R} \mapsto a(t, x, u, \nabla v)$ is a continuous function, the theory of Nemytskii operators (see e.g. [66, Theorem 1.27 p. 19]) gives that

$$\tilde{a}(t, x, u_\varepsilon, \nabla u) \rightarrow \tilde{a}(t, x, u, \nabla u) \text{ in } L^{p'}(Q)^d \quad (2.13)$$

and

$$\int_Q \tilde{a}(t, x, u_\varepsilon, \nabla u) \cdot \nabla(u_\varepsilon - u) dx dt \rightarrow 0. \quad (2.14)$$

Testing the penalized equation (2.6) introduced in Theorem 2.6 by $u_\varepsilon - u$ yields

$$\begin{aligned} \int_0^t \langle \partial_t u_\varepsilon, u_\varepsilon - u \rangle ds + \int_0^t \int_D \tilde{a}(s, x, u_\varepsilon, \nabla u_\varepsilon) \cdot \nabla(u_\varepsilon - u) dx ds \\ = \int_0^t \langle f, u_\varepsilon - u \rangle ds - \frac{1}{\varepsilon} \int_0^t \int_Q \Theta(u_\varepsilon - \psi)(u_\varepsilon - u) dx ds. \end{aligned}$$

Since $\int_0^t \langle f, u_\varepsilon - u \rangle ds \rightarrow 0$, the following decomposition

$$\begin{aligned} -\frac{1}{\varepsilon} \int_0^t \int_D \Theta(u_\varepsilon - \psi)(u_\varepsilon - u) dx ds \\ = \underbrace{-\frac{1}{\varepsilon} \int_0^t \int_D \Theta(u_\varepsilon - \psi)(u_\varepsilon - \psi) dx ds}_{\leq 0} \underbrace{-\frac{1}{\varepsilon} \int_0^t \int_D \Theta(u_\varepsilon - \psi)(\psi - u) dx ds}_{\leq 0} \end{aligned}$$

leads to

$$\limsup_\varepsilon \left[\int_0^t \langle \partial_t u_\varepsilon, u_\varepsilon - u \rangle ds + \int_0^t \int_D \tilde{a}(s, x, u_\varepsilon, \nabla u_\varepsilon) \cdot \nabla(u_\varepsilon - u) dx ds \right] \leq 0.$$

Using (2.14) we obtain

$$\limsup_\varepsilon \left[\int_0^t \langle \partial_t(u_\varepsilon - u), u_\varepsilon - u \rangle ds + \int_0^t \int_D [\tilde{a}(s, x, u_\varepsilon, \nabla u_\varepsilon) - \tilde{a}(s, x, u_\varepsilon, \nabla u)] \cdot \nabla(u_\varepsilon - u) dx ds \right] \leq 0.$$

The monotone character of the operator $\tilde{a}(t, x, u, \vec{\xi})$ with respect to $\vec{\xi}$ (see Assumption H_{1,2} and (2.3)) implies

$$\frac{1}{2} \limsup_\varepsilon \|(u_\varepsilon - u)(t)\|_{L^2(D)}^2 = \limsup_\varepsilon \int_0^t \langle \partial_t(u_\varepsilon - u), u_\varepsilon - u \rangle ds \leq 0$$

and

$$\lim_\varepsilon \int_0^t \int_D [\tilde{a}(s, x, u_\varepsilon, \nabla u_\varepsilon) - \tilde{a}(s, x, u_\varepsilon, \nabla u)] \cdot \nabla(u_\varepsilon - u) dx ds = 0. \quad (2.15)$$

It follows that

$$u_\varepsilon(t) \rightarrow u(t) \text{ in } L^2(D) \text{ for any } t \quad (2.16)$$

and in view of (2.14)

$$\lim_\varepsilon \int_0^T \int_D \tilde{a}(s, x, u_\varepsilon, \nabla u_\varepsilon) \cdot \nabla(u_\varepsilon - u) dx ds = 0. \quad (2.17)$$

Set $\vec{v} \in L^p(Q)^d$. Since

$$\begin{aligned} 0 &\leq \int_Q [\tilde{a}(t, x, u_\varepsilon, \nabla u_\varepsilon) - \tilde{a}(t, x, u_\varepsilon, \vec{v})] \cdot [\nabla u_\varepsilon - \vec{v}] dx dt \\ &\leq \int_Q [\tilde{a}(t, x, u_\varepsilon, \nabla u_\varepsilon) - \tilde{a}(t, x, u_\varepsilon, \vec{v})] \cdot \nabla(u_\varepsilon - u) + [\tilde{a}(t, x, u_\varepsilon, \nabla u_\varepsilon) - \tilde{a}(t, x, u_\varepsilon, \vec{v})] \cdot [\nabla u - \vec{v}] dx dt \\ &\leq \int_Q [\tilde{a}(t, x, u_\varepsilon, \nabla u_\varepsilon) - \tilde{a}(t, x, u_\varepsilon, \nabla u)] \cdot \nabla(u_\varepsilon - u) + [\tilde{a}(t, x, u_\varepsilon, \nabla u) - \tilde{a}(t, x, u_\varepsilon, \vec{v})] \cdot \nabla(u_\varepsilon - u) dx dt \\ &\quad + \int_Q [\tilde{a}(t, x, u_\varepsilon, \nabla u_\varepsilon) - \tilde{a}(t, x, u_\varepsilon, \vec{v})] \cdot [\nabla u - \vec{v}] dx dt, \end{aligned}$$

using (2.15) and information similar to (2.14) allow one to pass to the limit and to conclude that

$$0 \leq \int_Q [\vec{\xi} - \tilde{a}(t, x, u, \vec{v})] \cdot [\nabla u - \vec{v}] dx dt.$$

By the classical Minty's trick, considering $\vec{v} = \nabla u + \lambda \vec{w}$, $\vec{w} \in L^p(Q)^d$ and $\lambda \in \mathbb{R}$, we have necessarily

$$0 = \lim_{\lambda \rightarrow 0} \int_Q [\vec{\xi} - \tilde{a}(t, x, u, \nabla u + \lambda \vec{w})] \cdot \vec{w} dx dt.$$

Thus, a classical property of radial continuity coming from the assumptions on a yields, for any $\vec{w} \in L^p(Q)^d$,

$$\int_Q \vec{\xi} \cdot \vec{w} dx dt = \int_Q \tilde{a}(t, x, u, \nabla u) \cdot \vec{w} dx dt = \int_Q a(t, x, u, \nabla u) \cdot \vec{w} dx dt,$$

i.e. $\vec{\xi} = \tilde{a}(t, x, u, \nabla u) = a(t, x, u, \nabla u)$, since $u \geq \psi$.

Remark 2.13. Note that, following [13, Proof of Lemma 1], (2.15) yields the convergence in measure, then the a.e. convergence of ∇u_ε to ∇u (up to a subsequence if needed), so that this is also a way to identify $\vec{\xi}$ as being $a(t, x, u, \nabla u)$.

We are now in a position to pass to the limit in the penalized problem and to conclude the existence of a solution to the obstacle problem under the additional regularity on g^- .

Let us consider $v \in L^p(0, T; V)$, $v \geq \psi$ as a test function in the penalized problem (2.6),

$$\begin{aligned} \int_0^T \langle \partial_t u_\varepsilon, v - u_\varepsilon \rangle dt + \int_Q \tilde{a}(t, x, u_\varepsilon, \nabla u_\varepsilon) \cdot \nabla(v - u_\varepsilon) dx dt \\ + \frac{1}{\varepsilon} \int_Q \Theta(u_\varepsilon - \psi)(v - u_\varepsilon) dx dt = \int_0^T \langle f, v - u_\varepsilon \rangle dt. \end{aligned} \quad (2.18)$$

In view of (2.16) we have

$$\begin{aligned} \int_0^T \langle \partial_t u_\varepsilon, v - u_\varepsilon \rangle dt &= \int_0^T \langle \partial_t u_\varepsilon, v \rangle dt - \frac{1}{2} \|u_\varepsilon(T)\|_{L^2(D)}^2 + \frac{1}{2} \|u_0\|_{L^2(D)}^2 \\ &\rightarrow \int_0^T \langle \partial_t u, v \rangle dt - \frac{1}{2} \|u(T)\|_{L^2(D)}^2 + \frac{1}{2} \|u_0\|_{L^2(D)}^2 = \int_0^T \langle \partial_t u, v - u \rangle dt. \end{aligned}$$

From (2.17) and the identification $\tilde{\xi} = \tilde{a}(t, x, u, \nabla u) = a(t, x, u, \nabla u)$ it follows that

$$\begin{aligned} \int_Q \tilde{a}(t, x, u_\varepsilon, \nabla u_\varepsilon) \cdot \nabla(v - u_\varepsilon) dx dt &= \int_Q \tilde{a}(t, x, u_\varepsilon, \nabla u_\varepsilon) \cdot \nabla(v - u) dx dt \\ &\quad + \int_Q \tilde{a}(t, x, u_\varepsilon, \nabla u_\varepsilon) \cdot \nabla(u - u_\varepsilon) dx dt \\ &\rightarrow \int_Q \tilde{a}(t, x, u, \nabla u) \cdot \nabla(v - u) dx dt = \int_Q a(t, x, u, \nabla u) \cdot \nabla(v - u) dx dt. \end{aligned}$$

The weak convergence of u_ε to u in $L^p(0, T; V)$ yields that

$$\int_0^T \langle f, v - u_\varepsilon \rangle \rightarrow \int_0^T \langle f, v - u \rangle.$$

At last splitting the penalized term in the following way

$$\frac{1}{\varepsilon} \int_Q \Theta(u_\varepsilon - \psi)(v - u_\varepsilon) dx dt = \underbrace{\frac{-1}{\varepsilon} \int_Q [(u_\varepsilon - \psi)^-]^{\tilde{q}-1} (v - \psi) dx dt}_{\leq 0} - \underbrace{\frac{1}{\varepsilon} \|(u_\varepsilon - \psi)^-\|_{L^{\tilde{q}}(Q)}^{\tilde{q}}}_{\rightarrow 0 \text{ thanks to (2.11)}}$$

allows one to pass to the limit in (2.18). One concludes that Theorem 2.8 holds.

2.3.3 Lewy-Stampacchia inequality for a regular g^- .

Note that $\mu_\varepsilon := \partial_t u_\varepsilon - \operatorname{div}[\tilde{a}(\cdot, \cdot, u_\varepsilon, \nabla u_\varepsilon)] - f = \frac{1}{\varepsilon} [(u_\varepsilon - \psi)^-]^{\tilde{q}-1} \geq 0$, so that the limit $\mu := \partial_t u - \operatorname{div}[\tilde{a}(\cdot, \cdot, u, \nabla u)] - f$ is a non-negative Radon measure which is by Lemma 2.12 an element of $L^{p'}(0, T; V')$.

Using an idea from A. Mokrane and F. Murat [48], denote by

$$z_\varepsilon := g^- - \frac{1}{\varepsilon} [(u_\varepsilon - \psi)^-]^{\tilde{q}-1},$$

we have

$$\partial_t u_\varepsilon + A(u_\varepsilon) + z_\varepsilon = g^+ + \partial_t \psi + A(\psi) \quad \text{i.e.} \quad \partial_t (u_\varepsilon - \psi) + A(u_\varepsilon) - A(\psi) + z_\varepsilon = g^+.$$

Observing that

$$\partial_t u_\varepsilon + A(u_\varepsilon) - f = -z_\varepsilon + g^-.$$

As in [48] in the elliptic case and under more restrictive assumptions on the operator a , proving that z_ε^- converges to 0 in an appropriate space leads to the Lewy-Stampacchia inequality. Due to the time variable and the weak assumption on a we have to face additional difficulties. For technical reasons, we will assume in this section only that, on top of $g^- \in L^{p'}(Q) \cap L^p(0, T; V)$, $g^- \geq 0$, that $\partial_t g^- \in L^{\tilde{q}'}(Q)$. Roughly speaking it allows one to use a test function depending on g^- and together with Lemma 2.15 to perform an integration by part formula and then the convergence analysis of z_ε^- . The general case will be obtained in the next section by a regularization argument based on Lemma 2.18.

Our aim is now to show the convergence of z_ε^- to 0 in $L^2(Q)$ to prove the following lemma.

Lemma 2.14. *Under the assumptions of Theorem 2.8 and assuming moreover that $g^- \in L^{p'}(Q) \cap L^p(0, T; V)$, $g^- \geq 0$ with $\partial_t g^- \in L^{\tilde{q}'}(Q)$, the solution u satisfies*

$$0 \leq \partial_t u - \operatorname{div}[a(\cdot, \cdot, u, \nabla u)] - f \leq g^- \text{ in } L^{p'}(0, T; V').$$

For the proof of Lemma 2.14 we need the following Lemma of time integration by part formula adapted to our situation.

Lemma 2.15. *Consider $u \in L^p(0, T, W^{1,p}(D)) \cap L^p(0, T, L^2(D))$ such that $\partial_t u \in L^{p'}(0, T, V')$.*

Let $\Psi : Q \times \mathbb{R} \rightarrow \mathbb{R}$ be a function such that $(t, x) \mapsto \Psi(t, x, \lambda)$ is measurable, $\lambda \mapsto \Psi(t, x, \lambda)$ is non-decreasing (càdlàg[‡], or càglàd[§]) and denote by $\Lambda : Q \times \mathbb{R} \rightarrow \mathbb{R}$, $(t, x, \lambda) \mapsto \int_a^\lambda \Psi(t, x, \tau) d\tau$ where a is any arbitrary real number. Assume moreover that $|\Psi(t = 0)| \leq$

[‡]right continuous with left limit

[§]left continuous with right limit

$h + |\lambda|^\alpha$ and that $\partial_t \Psi$ exists with $|\Psi(\lambda = 0)| + |\partial_t \Psi| \leq h$ where $h \in L^2(Q)$ and $\alpha \in [0, 1]$. If $\Psi(t, x, u) \in L^p(0, T, V)$, then, for any $\beta \in W^{1, \infty}(0, T)$ and any $0 \leq s < t \leq T$,

$$\begin{aligned} \int_s^t \langle \partial_t u, \Psi(\sigma, x, u) \rangle \beta d\sigma &= \int_D \Lambda(t, x, u(t)) \beta(t) dx - \int_D \Lambda(s, x, u(s)) \beta(s) dx \\ &\quad - \int_s^t \int_D \Lambda(\sigma, x, u) \beta' dx d\sigma - \int_s^t \int_D \partial_t \Lambda(\sigma, x, u) \beta dx d\sigma. \end{aligned}$$

Remark 2.16. A detailed proof of Lemma 2.15 inspired by [25] is given in Appendix B.3.

A priori, following Lemma's 2.15 notations, one should denote by $\Psi(t, x, \lambda) = -(g^- - \frac{1}{\varepsilon} [\lambda^-]^{\tilde{q}-1})^-$ and $\Lambda(t, x, \lambda) = \int_0^\lambda \Psi(t, x, \sigma) d\sigma$. For that, we need $\Psi(t, x, u)$ to be a test-function. Since $x \mapsto [x^-]^{\tilde{q}-1}$ is not a priori a Lipschitz-continuous function (e.g. if $p < 2^{\mathbb{1}}$), therefore, for any positive k , we will denote by

$$\eta_k(x) = (\tilde{q} - 1) \int_0^{x^+} \min(k, s^{\tilde{q}-2}) ds, \quad \Psi_k(t, x, \lambda) = -(g^- - \frac{1}{\varepsilon} \eta_k(\lambda^-))^-$$

and $\Lambda_k(t, x, \lambda) = \int_0^\lambda \Psi_k(t, x, \sigma) d\sigma$. Note that $\Psi_k(t, x, 0) = 0$ and $\partial_t \Psi_k(t, x, \lambda) = \partial_t g^- \mathbf{1}_{\{g^- - \frac{1}{\varepsilon} \eta_k(\lambda^-) < 0\}}$ so that, since $\Psi_k(t, x, u)$ is a test-function, by Lemma 2.15, for any t ,

$$\begin{aligned} & - \int_0^t \int_D \partial_t \Lambda_k(s, x, u_\varepsilon - \psi) dx ds \\ & \quad + \int_D \Lambda_k(t, x, u_\varepsilon(t) - \psi(t)) dx - \int_D \Lambda_k(0, x, u_\varepsilon(0) - \psi(0)) dx \\ & - \int_0^t \langle A(u_\varepsilon) - A(\psi), (g^- - \frac{1}{\varepsilon} \eta_k[(u_\varepsilon - \psi)^-])^- \rangle ds - \int_Q z_\varepsilon (g^- - \frac{1}{\varepsilon} \eta_k[(u_\varepsilon - \psi)^-])^- dx ds \\ & \quad = - \int_0^t \langle g^+, (g^- - \frac{1}{\varepsilon} \eta_k[(u_\varepsilon - \psi)^-])^- \rangle ds \leq 0. \quad (2.19) \end{aligned}$$

We now pass to the limit first as $k \rightarrow \infty$ and then as $\varepsilon \rightarrow 0$. Since $g^- \geq 0$, one has that $\Psi_k(t, x, \lambda) = 0$ if $\lambda \geq 0$ and as $u_\varepsilon(0) = u_0 \geq \psi(0)$, one gets that

$$\int_D \Lambda_k(t, x, u_\varepsilon(t) - \psi(t)) dx - \int_D \Lambda_k(0, x, u_\varepsilon(0) - \psi(0)) dx = \int_D \Lambda_k(t, x, u_\varepsilon(t) - \psi(t)) dx.$$

Note that $(\Psi_k(t, x, \lambda))_k$ is a non-increasing sequence of functions with non-positive values so that by monotone convergence theorem

$$\int_D \Lambda_k(t, x, u_\varepsilon(t) - \psi(t)) dx \xrightarrow[k]{} - \int_D \int_0^{(u_\varepsilon - \psi)(t)} (g^- - \frac{1}{\varepsilon} [\sigma^-]^{\tilde{q}-1})^- d\sigma dx \geq 0$$

$\mathbb{1}\tilde{q} = \min(2, p)$

since the integration holds on the set of negative values of $u_\varepsilon(t) - \psi(t)$.

Due to the definition of z_ε we have

$$\begin{aligned} - \int_Q z_\varepsilon (g^- - \frac{1}{\varepsilon} \eta_k [(u_\varepsilon - \psi)^-])^- dxdt &= - \int_Q (g^- - \frac{1}{\varepsilon} [(u_\varepsilon - \psi)^-]^{\tilde{q}-1}) (g^- - \frac{1}{\varepsilon} \eta_k [(u_\varepsilon - \psi)^-])^- dxdt \\ &= \int_Q (g^- - \frac{1}{\varepsilon} [(u_\varepsilon - \psi)^-]^{\tilde{q}-1})^- (g^- - \frac{1}{\varepsilon} \eta_k [(u_\varepsilon - \psi)^-])^- dxdt \\ &\quad - \int_Q (g^- - \frac{1}{\varepsilon} [(u_\varepsilon - \psi)^-]^{\tilde{q}-1})^+ (g^- - \frac{1}{\varepsilon} \eta_k [(u_\varepsilon - \psi)^-])^- dxdt, \end{aligned}$$

from which it follows using again the monotone convergence theorem

$$- \int_Q z_\varepsilon (g^- - \frac{1}{\varepsilon} \eta_k [(u_\varepsilon - \psi)^-])^- dxdt \xrightarrow[k]{} \int_0^T \|z_\varepsilon^-\|_{L^2(D)}^2 dt.$$

As far as the first term of (2.19) is concerned we obtain

$$\begin{aligned} - \int_Q \partial_t \Lambda_k(t, x, u_\varepsilon - \psi) dxds &= - \int_Q \partial_t g^- \int_0^{u_\varepsilon - \psi} 1_{\{g^- - \frac{1}{\varepsilon} \eta_k(\tau^-) < 0\}} d\tau dxds \\ &= - \int_Q \partial_t g^- \int_0^{-(u_\varepsilon - \psi)^-} 1_{\{g^- - \frac{1}{\varepsilon} \eta_k(\tau^-) < 0\}} d\tau dxds \geq - \int_Q |\partial_t g^-| |(u_\varepsilon - \psi)^-| dxds \xrightarrow[\varepsilon]{} 0. \end{aligned}$$

For the fourth term of (2.19) we have the following equality

$$\begin{aligned} &- \int_0^T \langle A(u_\varepsilon) - A(\psi), (g^- - \frac{1}{\varepsilon} \eta_k [(u_\varepsilon - \psi)^-])^- \rangle dt \\ &= \int_Q 1_{\{g^- - \frac{1}{\varepsilon} \eta_k [(u_\varepsilon - \psi)^-] < 0\}} [\tilde{a}(t, x, u_\varepsilon, \nabla u_\varepsilon) - \tilde{a}(t, x, \psi, \nabla \psi)] \nabla [g^- - \frac{1}{\varepsilon} \eta_k [(u_\varepsilon - \psi)^-]] dxdt \\ &= \int_Q 1_{\{g^- - \frac{1}{\varepsilon} \eta_k [(u_\varepsilon - \psi)^-] < 0\}} [\tilde{a}(t, x, \psi, \nabla u_\varepsilon) - \tilde{a}(t, x, \psi, \nabla \psi)] \nabla [g^- - \frac{1}{\varepsilon} \eta_k [(u_\varepsilon - \psi)^-]] dxdt, \end{aligned}$$

since in this situation, the integration holds in the set where $u_\varepsilon \leq \psi$. Thus,

$$\begin{aligned} &[\tilde{a}(t, x, \psi, \nabla u_\varepsilon) - \tilde{a}(t, x, \psi, \nabla \psi)] \nabla [g^- - \frac{1}{\varepsilon} \eta_k [(u_\varepsilon - \psi)^-]] \\ &\quad \geq \overbrace{\frac{1}{\varepsilon} \eta'_k [(u_\varepsilon - \psi)^-] [\tilde{a}(t, x, \psi, \nabla u_\varepsilon) - \tilde{a}(t, x, \psi, \nabla \psi)] \nabla (u_\varepsilon - \psi)}^{\geq 0} \\ &\quad \quad - |\tilde{a}(t, x, \psi, \nabla u_\varepsilon) - \tilde{a}(t, x, \psi, \nabla \psi)| |\nabla g^-| \\ &\quad \geq - |\tilde{a}(t, x, \psi, \nabla u_\varepsilon) - \tilde{a}(t, x, \psi, \nabla \psi)| |\nabla g^-|. \end{aligned}$$

We now claim that estimate (2.10) of Lemma 2.11 which gives

$$[\tilde{a}(t, x, \psi, \nabla u_\varepsilon) - \tilde{a}(t, x, \psi, \nabla \psi)] \nabla (u_\varepsilon - \psi)^- \xrightarrow[\varepsilon]{} 0 \text{ in } L^1(Q)$$

and Assumptions $H_{1,1}$ to $H_{1,3}$ imply that, up to a subsequence (still denoted by ε), $\nabla(u_\varepsilon - \psi)^-$ converges to 0 a.e. in Q .

Indeed, up to a subsequence (still denoted by ε), u_ε converges to u a.e. in Q with $u \geq \psi$ a.e. and $\left| \tilde{a}(t, x, \psi, \nabla u_\varepsilon) - \tilde{a}(t, x, \psi, \nabla \psi) \right| |\nabla(u_\varepsilon - \psi)^-| \rightarrow 0$ a.e. in Q .

Consider (t, x) such that the above limits hold. Since,

$$\begin{aligned} -\tilde{a}(t, x, \psi, \nabla u_\varepsilon) \cdot \nabla(u_\varepsilon - \psi)^- &\geq \left[\bar{\alpha} |\nabla u_\varepsilon|^p - \bar{\gamma} |\psi|^q - |\bar{h}| - \tilde{a}(t, x, \psi, \nabla u_\varepsilon) \cdot \nabla \psi \right] 1_{\{u_\varepsilon < \psi\}} \\ &\geq \left[\bar{\alpha} |\nabla u_\varepsilon|^p - \bar{\gamma} |\psi|^q - |\bar{h}| - \bar{\beta} \left[|\bar{k}| + |\psi|^{r/p} + |\nabla u_\varepsilon| \right]^{p-1} |\nabla \psi| \right] 1_{\{u_\varepsilon < \psi\}} \\ &\geq \left[\bar{\alpha}/2 |\nabla u_\varepsilon|^p - C(\psi, \bar{h}, \bar{k}, \nabla \psi) \right] 1_{\{u_\varepsilon < \psi\}}, \end{aligned}$$

and

$$|\tilde{a}(t, x, \psi, \nabla \psi) \cdot \nabla(u_\varepsilon - \psi)^-| \leq \bar{\beta} \left[|\bar{k}| + |\psi|^{r/p} + |\nabla \psi| \right]^{p-1} \left[|\nabla u_\varepsilon| + |\nabla \psi| \right] 1_{\{u_\varepsilon < \psi\}},$$

one gets that $(\nabla(u_\varepsilon - \psi)^-(t, x))_\varepsilon$ is a bounded sequence.

Since $\nabla(u_\varepsilon - \psi)^-(t, x) = -\nabla(u_\varepsilon - \psi)(t, x) 1_{\{u_\varepsilon < \psi\}}(t, x)$, it converges to 0 if $u(t, x) > \psi(t, x)$.

Else, at the limit, one has that $u(t, x) = \psi(t, x)$. If one assumes that $\nabla(u_\varepsilon - \psi)^-(t, x)$ is not converging to 0, then there exists a subsequence ε' (depending on (t, x)) and a positive δ such that $\|\nabla(u_{\varepsilon'} - \psi)^-(t, x)\| \geq \delta > 0$. Then, necessarily $-\nabla(u_{\varepsilon'} - \psi)^-(t, x) = \nabla(u_{\varepsilon'} - \psi)(t, x)$ and, since it is a bounded sequence in \mathbb{R}^d , there exists $\vec{\xi} \in \mathbb{R}^d$ and a new subsequence still labeled ε' such that $\nabla u_{\varepsilon'}(t, x)$ converges to $\vec{\xi}$, with the additional information: $\|\vec{\xi} - \nabla \psi(t, x)\| \geq \delta > 0$. Therefore, since $\vec{\xi} \neq \nabla \psi(t, x)$

$$\begin{aligned} &\left[\tilde{a}(t, x, \psi, \nabla u_{\varepsilon'}(t, x)) - \tilde{a}(t, x, \psi, \nabla \psi(t, x)) \right] \cdot \nabla(u_{\varepsilon'} - \psi)^-(t, x) \\ &= - \left[\tilde{a}(t, x, \psi, \nabla u_{\varepsilon'}(t, x)) - \tilde{a}(t, x, \psi, \nabla \psi(t, x)) \right] \cdot \nabla(u_{\varepsilon'} - \psi)(t, x) \\ &\xrightarrow{\varepsilon'} - \left[\tilde{a}(t, x, \psi, \vec{\xi}) - \tilde{a}(t, x, \psi, \nabla \psi(t, x)) \right] \cdot [\vec{\xi} - \nabla \psi(t, x)] < 0. \end{aligned}$$

But, this is in contradiction with the convergence of the same sequence to 0 and the result holds.

Note that $(t, x) \in Q$ a.e.,

$$\left[\tilde{a}(t, x, \psi, \nabla u_\varepsilon) - \tilde{a}(t, x, \psi, \nabla \psi) \right] 1_{\{u_\varepsilon < \psi\}} = \left[\tilde{a}(t, x, \psi, \nabla u_\varepsilon 1_{\{u_\varepsilon < \psi\}}) - \tilde{a}(t, x, \psi, \nabla \psi 1_{\{u_\varepsilon < \psi\}}) \right]$$

and $\nabla u_\varepsilon 1_{\{u_\varepsilon < \psi\}} - \nabla \psi 1_{\{u_\varepsilon < \psi\}}$ converges to 0 with $\nabla \psi 1_{\{u_\varepsilon < \psi\}}$ bounded. Then, the continuity of \tilde{a} with respect to its fourth argument ensures that $\left[\tilde{a}(t, x, \psi, \nabla u_\varepsilon) - \tilde{a}(t, x, \psi, \nabla \psi) \right] 1_{\{u_\varepsilon < \psi\}}$ converges a.e. to 0. Since it is bounded in $L^{p'}(Q)$, it converges weakly to 0 in $L^{p'}(Q)$ and

$$\int_Q \left| \tilde{a}(t, x, \psi, \nabla u_\varepsilon) - \tilde{a}(t, x, \psi, \nabla \psi) \right| |\nabla g^-| 1_{\{u_\varepsilon < \psi\}} dx dt \rightarrow 0.$$

As a conclusion, z_ε^- converges to 0 in $L^2(Q)$. On the one hand we have

$$\begin{aligned} 0 \leq \mu_\varepsilon &= \frac{1}{\varepsilon} [(u_\varepsilon - \psi)^-]^{\tilde{q}-1} = \partial_t u_\varepsilon - \operatorname{div}[\tilde{a}(\cdot, \cdot, u_\varepsilon, \nabla u_\varepsilon)] - f \\ \Rightarrow 0 &\leq \partial_t u - \operatorname{div}[\tilde{a}(\cdot, \cdot, u, \nabla u)] - f; \end{aligned}$$

On the other hand

$$\begin{aligned} z_\varepsilon &= g^- - \frac{1}{\varepsilon} [(u_\varepsilon - \psi)^-]^{\tilde{q}-1} \Rightarrow z_\varepsilon^+ + \partial_t u_\varepsilon - \operatorname{div}[\tilde{a}(\cdot, \cdot, u_\varepsilon, \nabla u_\varepsilon)] - f = g^- + z_\varepsilon^- \\ &\Rightarrow 0 \leq \partial_t u - \operatorname{div}[\tilde{a}(\cdot, \cdot, u, \nabla u)] - f \leq g^-. \end{aligned}$$

Since $\tilde{a}(\cdot, \cdot, u, \nabla u) = a(\cdot, \cdot, u, \nabla u)$, Lemma 2.14 is proved.

Remark 2.17. Note that, for any $\varphi \in L^p(0, T; V)$,

$$\begin{aligned} &\int_0^T \langle \partial_t u_\varepsilon - \operatorname{div}[\tilde{a}(\cdot, \cdot, u_\varepsilon, \nabla u_\varepsilon)] - f, \varphi \rangle dt \\ &= \int_0^T \langle \partial_t u_\varepsilon - \operatorname{div}[\tilde{a}(\cdot, \cdot, u_\varepsilon, \nabla u_\varepsilon)] - f, \varphi^+ \rangle dt - \int_0^T \langle \partial_t u_\varepsilon - \operatorname{div}[\tilde{a}(\cdot, \cdot, u_\varepsilon, \nabla u_\varepsilon)] - f, \varphi^- \rangle dt \\ &\leq \int_0^T \langle \partial_t u_\varepsilon - \operatorname{div}[\tilde{a}(\cdot, \cdot, u_\varepsilon, \nabla u_\varepsilon)] - f, \varphi^+ \rangle dt \leq \int_0^T \langle g^-, \varphi^+ \rangle dt. \end{aligned}$$

In such a way, $\|\partial_t u_\varepsilon - \operatorname{div}[\tilde{a}(\cdot, \cdot, u_\varepsilon, \nabla u_\varepsilon)] - f\|_{L^{p'}(0, T; V')} \leq \|g^-\|_{L^{p'}(0, T; V')}$.

2.3.4 Proof of Theorem 2.4

Let us first recall the following result.

Lemma 2.18. *The positive cone of $L^p(0, T; V) \cap L^2(Q)$ is dense in the positive cone of \mathcal{V}' , the dual set of $\mathcal{V} = L^p(0, T, V)$.*

Remark 2.19. *By truncation argument, the same result holds for the positive cone of $L^p(0, T; V) \cap L^{p'}(Q)$ when $p < 2$. We refer to Appendix A.4 for a detailed proof of Lemma 2.18.*

In this section, H_3 is assumed and $g = f - \partial_t \psi - A(\psi) = g^+ - g^-$ where $g^+, g^- \in (L^{p'}(0, T; V'))^+$ are non-negative elements of $L^{p'}(0, T; V')$.

Thanks to Lemma 2.18, there exists positives $(g_n^-) \subset L^{p'}(Q)$ such that $g_n^- \rightarrow g^-$ in $L^{p'}(0, T; V')$. Then, by a regularization procedure, one can assume that $g_n^- \in L^{p'}(Q) \cap L^p(0, T; V)$, $g_n^- \geq 0$ with $\partial_t g_n^- \in L^{q'}(Q)$. Then, the corresponding sequence f_n converges to f in $L^{p'}(0, T; V')$.

Remark 2.20. *In fact, since $D(Q)^+$ is dense in $L^{p'}(Q)^+$, one can consider g_n^- as regular as needed.*

Associated with g_n^- , Theorem 2.8 and Lemma 2.14 provide the existence of $u_n \in \mathcal{K}(\psi)$ with $u_n(t=0) = u_0$ and such that, for any $v \in L^p(0, T; V)$, $v \geq \psi$ implies that

$$\int_0^T \langle \partial_t u_n, v - u_n \rangle dt + \int_Q a(t, x, u_n, \nabla u_n) \cdot \nabla (v - u_n) dx dt \geq \int_0^T \langle f_n, v - u_n \rangle dt$$

and satisfying the Lewy-Stampacchia inequality

$$0 \leq \partial_t u_n - \operatorname{div}[a(\cdot, \cdot, u_n, \nabla u_n)] - f_n \leq g_n^-.$$

Since this solution comes from the above penalization method, and as C_1 of Lemma 2.10 can be chosen independent of n , one gets that

$$\sup_t \|u_n\|_{L^2(D)}^2(t) + \|u_n\|_{L^p(0, T; V)}^p \leq C_1.$$

Thus, $(a(\cdot, u_n, \nabla u_n))_n$ is bounded in $L^{p'}(Q)^d$ and, thanks to the above Lewy-Stampacchia inequality, $(\partial_t u_n)_n$ is bounded in $L^{p'}(0, T; V')$.

Up to a subsequence denoted similarly, u_n converge weakly to an element $u \in \mathcal{K}(\psi)$ in $W(0, T)$ and strongly in $L^p(Q)$; and $a(\cdot, u_n, \nabla u_n)$ converge weakly to an element $\vec{\xi}$ in $L^{p'}(Q)^d$.

Finally, the embedding of $W(0, T)$ in $C([0, T], L^2(D))$ yields the weak convergence of $u_n(t)$ to $u(t)$ in $L^2(D)$, for any t .

Since $u \in \mathcal{K}(\psi)$,

$$\int_0^T \langle \partial_t u_n, u - u_n \rangle dt + \int_Q a(t, x, u_n, \nabla u_n) \cdot \nabla (u - u_n) dx dt \geq \int_0^T \langle f_n, u - u_n \rangle dt.$$

Therefore, passing to the limit with respect to n in

$$\begin{aligned} & \int_0^T \langle \partial_t u_n, u \rangle dt + \int_Q a(t, x, u_n, \nabla u_n) \cdot \nabla u dx dt + \frac{1}{2} \|u_0\|_{L^2(D)}^2 \\ & \geq \int_0^T \langle f_n, u - u_n \rangle dt + \frac{1}{2} \|u_n(T)\|_{L^2(D)}^2 + \int_Q a(t, x, u_n, \nabla u_n) \cdot \nabla u_n dx dt \end{aligned}$$

yields

$$\int_0^T \langle \partial_t u, u \rangle dt + \int_Q \vec{\xi} \cdot \nabla u dx dt + \frac{1}{2} \|u_0\|_{L^2(D)}^2 \geq \frac{1}{2} \|u(T)\|_{L^2(D)}^2 + \limsup_n \int_Q a(t, x, u_n, \nabla u_n) \cdot \nabla u_n dx dt.$$

Since $\int_0^T \langle \partial_t u, u \rangle dt = \frac{1}{2} \|u(T)\|_{L^2(D)}^2 - \frac{1}{2} \|u_0\|_{L^2(D)}^2$, one gets that

$$\limsup_n \int_Q a(t, x, u_n, \nabla u_n) \cdot \nabla u_n dx dt \leq \int_Q \vec{\xi} \cdot \nabla u dx dt.$$

Thus, (2.2) and the continuity property of Nemytskii operator ensures the following limit argument:

$$\begin{aligned} 0 &\leq \int_Q [a(t, x, u_n, \nabla u_n) - a(t, x, u_n, \nabla u)] \cdot \nabla (u_n - u) dx dt \\ &\leq \int_Q a(t, x, u_n, \nabla u_n) \cdot \nabla u_n dx dt - \int_Q a(t, x, u_n, \nabla u_n) \cdot \nabla u dx dt - \int_Q a(t, x, u_n, \nabla u) \cdot \nabla (u_n - u) dx dt, \end{aligned}$$

thus

$$0 \leq \liminf_n \int_Q a(t, x, u_n, \nabla u_n) \cdot \nabla u_n dx dt - \int_Q \vec{\xi} \cdot \nabla u dx dt.$$

Then, $\lim_n \int_Q a(t, x, u_n, \nabla u_n) \cdot \nabla u_n dx dt = \int_Q \vec{\xi} \cdot \nabla u dx dt$ and arguments already developed previously based on Minty's trick for the pseudomonotone operator A yield the identification $\vec{\xi} = a(t, x, u, \nabla u)$ and one has

$$\lim_n \int_Q a(t, x, u_n, \nabla u_n) \cdot \nabla (u_n - v) dx dt = \int_Q a(t, x, u, \nabla u) \cdot \nabla (u - v) dx dt.$$

From the weak lower semicontinuity of $|\cdot|_{L^2(D)}$, one has

$$\limsup_n \int_0^T \langle \partial_t u_n, v - u_n \rangle dt \leq \int_0^T \langle \partial_t u, v - u \rangle dt.$$

Since $\int_0^T \langle f_n, v - u_n \rangle dt \rightarrow \int_0^T \langle f, v - u \rangle dt$, we deduce the existence result of Theorem 2.4 for general f . At last the Lewy-Stampacchia inequality is a consequence of passing to the limit in the one satisfied by u_n .

Lewy-Stampacchia inequality for a stochastic T-monotone obstacle problem

The aim of this chapter is to study a stochastic obstacle problem governed by a T -monotone operator, a random force and a multiplicative stochastic reaction in the context of Sobolev spaces. This study is a part of a joint work with G.Vallet [73].

In other words, we are interested in proving the existence and uniqueness of the solution u to some obstacle problems that can be written (formally)

$$f - \partial_t \left(u - \int_0^\cdot G(u, \cdot) dW \right) - A(u, \cdot) \in \partial I_K(u),$$

where K is a closed convex subset of $L^p(\Omega_T, V)$ related to the stochastic constraint ψ , A is a nonlinear T -monotone operator defined on a space V , $(\Omega, (\mathcal{F}_t)_{t \geq 0}, P)$ is a filtered probability space with the usual assumptions and $W(t)$ is a Wiener process in some separable Hilbert space H . Then, we give the corresponding Lewy-Stampacchia inequalities, namely

$$0 \leq \partial_t \left(u - \int_0^\cdot G(u, \cdot) dW \right) + A(u, \cdot) - f \leq \left(f - \partial_t \left(\psi - \int_0^\cdot G(\psi, \cdot) dW \right) - A(\psi, \cdot) \right)^-.$$

We use an *ad hoc* perturbation of the stochastic reaction and a penalization of the constraint to prove the existence of the variational solution, then Lewy-Stampacchia inequalities associated with the problem.

Keywords: Stochastic PDEs, Obstacle problem, Wiener process, Lewy-Stampacchia inequality.

3.1 Introduction

3.1.1 Former results / Our result

Concerning stochastic obstacle problems, without seeking to be exhaustive, let us mention the papers of U. G. Haussmann & E. Pardoux [34] where the authors proved the well-posedness of a reflected parabolic problem governed by a bounded linear operator. The question of the semi-linear case was studied by A. Rascanu [59], C. Donati-Martin and E. Pardoux [23], and T. Xua and T. Zhang [76]. A penalty method approach is used as in the deterministic case. We also cite the recent book of L. Zambotti [77] where a study of the nonlinear heat equation with an additive noise is considered.

Several studies on the quasilinear case have been proposed by L. Denis, A. Matoussi and J. Zhang. In [18], a homogeneous SPDE with obstacle, under Lipschitz hypotheses and L^2 -integrability conditions on the coefficients, have been studied by using technics of Parabolic potential theory. After the introduction of the notion of parabolic capacity, the authors constructed a solution which admits a quasi-continuous version *via* the penalization method by mixing pathwise arguments and some existence result of the deterministic obstacle problem. The result has been extended in [19] by considering a weaker $L^{p,q}$ -integrability conditions on the coefficients. Then, they used the same approach to study the case of non-homogeneous operator as they derived also a local maximum principle in [20].

In a differential inclusion approach, we mention the works of A. Rascanu [60] and A. Bensoussan & A. Rascanu [9] where a maximal monotone operator is considered on a Hilbert space; V. Barbu [6] for nonlinear heat problems and C. Bauzet *et al.* [7] for an Allen-Cahn type equation.

Concerning monotone operators in a non-Hilbertian case, A. Rascanu & E. Rotenstein [61] were interested, among other things, in strong solutions to some stochastic variational inequalities when the barriers cancel the diffusion coefficients. Our aim in this paper is to revisit similar variational inequalities by adding random dependences for the operator, the source and the stochastic reaction terms, and the obstacle. We will in particular assume that $f - \partial_t \left(\psi - \int_0^\cdot G(\psi, \cdot) dW \right) - A(\psi, \cdot)$ can be written as the difference of two non-negative elements of a dual-space to derive Lewy-Stampacchia's inequalities. Then, we propose in Appendix D some extensions to situations where the obstacle and the solution are not in the same space, or to bilateral obstacle problems.

The deterministic Lewy-Stampacchia (L-S) inequalities have been largely studied and we advice the reader to consult Chapter 1 and Chapter 2 to get an overview of the literature about the subject.

To the best of the author's knowledge, there doesn't exist in the literature a result of existence and uniqueness associated with corresponding L-S inequalities, of the solution to a stochastic obstacle problem with a nonlinear operator associated with a random obstacle that doesn't cancel the diffusion coefficients. Our aim, in this chapter, is to propose such a result with general assumptions on the T -monotone operator and a general multiplicative noise.

3.1.2 Content of the study

We are interested in proving the existence and uniqueness of the solution u to some obstacle problems which can be written (formally)

$$f - \partial_t \left(u - \int_0^\cdot G(u, \cdot) dW \right) - A(u, \cdot) \in \partial I_K(u),$$

where K is a closed convex subset of $L^p(\Omega_T, V)$ related to the stochastic constraint ψ , A is a nonlinear T -monotone operator defined on a space V , $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$ is a filtered probability space with the usual assumptions and $W(t)$ is a Wiener process in some separable Hilbert space H . Then, we give the corresponding Lewy-Stampacchia inequalities, namely

$$0 \leq \partial_t \left(u - \int_0^\cdot G(u, \cdot) dW \right) + A(u, \cdot) - f \leq \left(f - \partial_t \left(\psi - \int_0^\cdot G(\psi, \cdot) dW \right) - A(\psi, \cdot) \right)^-.$$

By using a penalization method of the constraint, associated with a suitable perturbation of the stochastic reaction to formally lead to an additive stochastic source on the free-set where the constraint is violated, we are able to prove on one hand the existence of a solution to the stochastic obstacle problem, and on the other hand, to prove the corresponding stochastic Lewy-Stampacchia inequalities.

The chapter is organized in the following way: after giving the hypotheses, a result of uniqueness (Lemma 3.12) and the main result (Theorem 3.10) in Section 3.2, Section 3.3 is devoted to the proof of the results. A first step concerns the existence of a solution

to the approximating problem associated with a parameter ε . Additionally, some *a priori* estimates and passage to the limit with respect to ε are considered when h^- is a regular non-negative element. A first proof of Lewy-Stampacchia inequality is given when h^- is still regular. Finally, the proof of Lewy-Stampacchia inequality is extended to the general case.

Remark 3.1. *It's worth noting that the presentation of our results is in an abstract way so that one can easily extend them to more general Riesz separable reflexive Banach spaces V . We will not develop this point of view because V' will not be a Banach lattice.*

3.2 Notation and hypotheses

By maintaining the same notations used in Chapter II, let us denote by $p \in (1, +\infty)$, p' its conjugate, $V = W_0^{1,p}(D)$ if $p \geq 2$ and $V = W_0^{1,p}(D) \cap L^2(D)$ with the graph-norm else. Then, the corresponding dual spaces are $V' = W^{-1,p'}(D)$ if $p \geq 2$ and $V' = W^{-1,p'}(D) + L^2(D)$ else.

Let (Ω, \mathcal{F}, P) be a complete probability space (e.g. the classical Wiener space) endowed with a right-continuous filtration $\{\mathcal{F}_t\}_{t \geq 0}$ completed with respect to the measure P . $W(t)$ is a Wiener process in H with nuclear covariance operator Q with $\text{tr}Q < \infty$. Denote by $\Omega_T = (0, T) \times \Omega$ and \mathcal{P}_T the predictable σ -algebra on Ω_T^* .

Let $L_Q(H)$ denotes the spaces of linear operators Φ defined on $Q^{\frac{1}{2}}H$ with values in H such that $\Phi Q^{\frac{1}{2}} \in \mathcal{HS}(H)$ (the space of Hilbert-Schmidt operators from H to H). $L_Q(H)$ is a separable Hilbert space relatively to the scalar product $(\Phi, \Psi)_Q = \text{tr} \Phi Q^{\frac{1}{2}} (\Psi Q^{\frac{1}{2}})^*$. The norm in this space is denoted by $|\cdot|_Q$. We recall that the stochastic integrals over a Wiener process are defined for predictable operators B such that $E[\int_0^t |B(s)|_Q^2 ds] < \infty$ for any $t \geq 0$ [38, Sec. I-2].

We will consider in the sequel the following assumptions:

H_1 : Let $A : V \times \Omega_T \rightarrow V'$, $G : H \times \Omega_T \rightarrow L_Q(H)$, $\psi : \Omega_T \rightarrow V$, $f : \Omega_T \rightarrow V'$ and $u_0 : \Omega \rightarrow H$ such that:

$H_{1,1}$: For any $v \in V$ and $u \in H$, $A(v, \cdot)$, $G(u, \cdot)$, ψ and f are predictable.

$H_{1,2}$: u_0 is \mathcal{F}_0 -measurable.

* $\mathcal{P}_T := \sigma(\{[s, t] \times F_s \mid 0 \leq s < t \leq T, F_s \in \mathcal{F}_s\} \cup \{\{0\} \times F_0 \mid F_0 \in \mathcal{F}_0\})$ (see [44, p. 33]). Then, a process defined on Ω_T with values in a given space E is predictable if it is \mathcal{P}_T -measurable.

$H_2 : \exists \alpha, K > 0, \lambda_T, \lambda \in \mathbb{R}, l_1 \in L^1(\Omega_T)$ and $g \in L^{p'}(\Omega_T)$, both predictable, such that:

$$H_{2,1} : (t, \omega) \in \Omega_T \text{ a.e., } \forall v \in V, \quad \langle A(v, t, \omega), v \rangle + \lambda \|v\|_H^2 + l_1(t, \omega) \geq \alpha \|v\|_V^p.$$

$$H_{2,2} : (T\text{-monotonicity [52, p. 120]}) \quad (t, \omega) \in \Omega_T \text{ a.e., } \forall v_1, v_2 \in V,$$

$$\lambda_T(v_1 - v_2, (v_1 - v_2)^+)_H + \langle A(v_1, t, \omega) - A(v_2, t, \omega), (v_1 - v_2)^+ \rangle \geq 0.$$

Note that since $v_1 - v_2 = (v_1 - v_2)^+ - (v_2 - v_1)^+$, $\lambda_T Id + A$ is also monotone.

Remark 3.2. If $H = L^2(D)$, the result presented in this chapter holds true even if $\lambda_T \leq 0$.

$$H_{2,3} : (t, \omega) \in \Omega_T \text{ a.e., } \forall v \in V, \quad \|A(v, t, \omega)\|_{V'} \leq K \|v\|_V^{p-1} + g(t, \omega).$$

$$H_{2,4} : (\text{Hemi-continuity}) \quad (t, \omega) \in \Omega_T \text{ a.e.,}$$

$$\forall v, v_1, v_2 \in V, \quad \eta \in \mathbb{R} \mapsto \langle A(v_1 + \eta v_2, t, \omega), v \rangle \text{ is continuous.}$$

Remark 3.3. Assumptions $H_{2,2}$ and $H_{2,4}$ yield (e.g [66, Lemma 2.16 p.38]) the continuity of $\lambda_T Id + A$, thus of A with respect to v from V -strong to V' -weak. Thus, for any $v_1 \in V$, the application $A_{v_1} : V \times \Omega_T \rightarrow \mathbb{R}, (v, t, \omega) \mapsto \langle A(v, t, \omega), v_1 \rangle$ is a Carathéodory function. Therefore, it is $\mathcal{B}(V) \otimes \mathcal{P}_T$ measurable and, $\langle A(v(t, \omega), t, \omega), v_1 \rangle$ is predictable too for any V -valued predictable process v [16, p.9]. If V is separable, $A(v, \cdot)$ is predictable with values in $(V', \mathcal{B}(V'))$ since the weak and the strong measurabilities coincide thanks to Pettis's Theorem.

$H_3 : \exists M > 0$ and $l \in L^1(\Omega_T)$, predictable, such that

$$H_{3,1} : (t, \omega) \in \Omega_T \text{ a.e., } \forall \theta, \sigma \in H, \quad |G(\theta, t, \omega) - G(\sigma, t, \omega)|_Q^2 \leq M \|\theta - \sigma\|_H^2.$$

$$H_{3,2} : (t, \omega) \in \Omega_T \text{ a.e., } \forall u \in H, \quad |G(u, t, \omega)|_Q^2 \leq l(t, \omega) + M \|u\|_H^2.$$

Remark 3.4. Thanks to Assumption H_3 , $G : L^2(D) \times \Omega_T \rightarrow L_Q(L^2(D))$ is a Carathéodory function. It is $\mathcal{B}(H) \otimes \mathcal{P}_T$ measurable and, $G(u(t, \omega), t, \omega)$ is predictable too for any H -valued predictable process u .

$H_4 : \psi \in L^p(\Omega, L^p(0, T, V)), \partial_t \left(\psi - \int_0^\cdot G(\psi, \cdot) dW \right) \in L^{p'}(\Omega_T, V')$ predictables.

$H_5 : f \in L^{p'}(\Omega_T, V')$ is predictable and one assumes moreover that

$$h = f - \partial_t \left(\psi - \int_0^\cdot G(\psi, \cdot) dW \right) - A(\psi, \cdot) \in L^p(\Omega_T, V)^*,$$

where $L^p(\Omega_T, V)^* = (L^{p'}(\Omega_T, V'))^+ - (L^{p'}(\Omega_T, V'))^+ \subset L^{p'}(\Omega_T, V')$

Definition 3.5. Denote by K the convex set of admissible functions

$$K = \{v \in L^p(\Omega_T, V), \quad v(x, t, \omega) \geq \psi(x, t, \omega) \quad \text{a.e. in } D \times \Omega_T\}.$$

Note that K is a not empty convex set, e.g. $\psi^+ \in K$.

H_5 : $f \in L^{p'}(\Omega_T, V')$ is predictable and one assumes moreover that

$$h = f - \partial_t \left(\psi - \int_0^\cdot G(\psi, \cdot) dW \right) - A(\psi, \cdot) \in L^p(\Omega_T, V)^*.$$

$L^p(\Omega_T, V)^* = (L^{p'}(\Omega_T, V'))^+ - (L^{p'}(\Omega_T, V'))^+$ denotes the order dual: the difference of two non-negative elements of $L^{p'}(\Omega_T, V')$, i.e. $h = h^+ - h^-$ where $h^+, h^- \in (L^{p'}(\Omega_T, V'))^+$ are non-negative elements of $L^{p'}(\Omega_T, V')$. f, h^+, h^- are also assumed to be predictable.

We recall that $h^\pm \in (L^{p'}(\Omega_T, V'))^+$ in the sense:

$$\forall \varphi \in L^p(\Omega_T, V), \quad \varphi \geq 0 \Rightarrow E \int_0^T \langle h^\pm, \varphi \rangle ds \geq 0.$$

H_6 : $u_0 \in L^2(\Omega, H)$ satisfies the constraint, i.e. $u_0 \geq \psi(0)$.

3.2.1 Formulation of the problem and the main result

Our aim is to look for (u, k) , in a space defined straight after, solution to

$$\left\{ \begin{array}{ll} du + A(u, \cdot) ds + k ds = f ds + G(u, \cdot) dW & \text{in } D \times \Omega_T, \\ u(t=0) = u_0 & \text{in } H, \text{ a.s.}, \\ u \geq \psi & \text{in } D \times \Omega_T, \\ u = 0 & \text{on } \partial D \times \Omega_T, \\ \langle k, u - \psi \rangle = 0 \text{ and } k \leq 0 & \text{in } \Omega_T. \end{array} \right. \quad (3.1)$$

Remark 3.6. Taking into account Assumptions H_4 and H_5 , it's worth noticing that ψ solves the following stochastic problem

$$d\psi + A(\psi, \cdot) dt = G(\psi, \cdot) dW + (f - h) dt,$$

and the obstacle can be understood as a constraint in the coupling of stochastic PDEs. For exemple, $\psi(t, x) = \sin(\pi x) \exp(\beta(t) - \pi^2 t)$ in $(0, T) \times (0, 1) \times \Omega$ where β is the standard

Brownian motion. ψ is the solution to $\partial_t(\psi - \int_0^\cdot \psi dW) - \partial_x^2 \psi - \frac{1}{2}\psi = 0$ and satisfies H_4 and H_5 with $p = 2$, $A(v, \cdot) = \partial_x^2 v - \frac{1}{2}v$ and $G(v, \cdot) = v$.

When the obstacle is with values in V , one can observe that the problem can reduce to the question of a positivity obstacle problem with a stochastic reaction term vanishing at 0.

Indeed, by setting $\hat{u} = u - \psi$, $\hat{u}_0 = u_0 - \psi(0)$, $\hat{A}(\hat{u}) = A(\hat{u} + \psi) - A(\psi)$, with $\hat{G}(\hat{u}) = G(\hat{u} + \psi) - G(\psi)$ and $\hat{f} = f - \partial_t(\psi - \int_0^\cdot G(\psi)dW) - A(\psi)$, the equation becomes $d\hat{u} + \hat{A}(\hat{u}, \cdot)ds + \hat{k}ds = \hat{f}ds + \hat{G}(\hat{u}, \cdot)dW$ in $D \times \Omega_T$ with $\hat{G}(0, \cdot) = 0$ and under the constraint $\hat{u} \geq 0$.

In case the obstacle ψ is not with values in V , if for example ψ has non-positive values on the boundary of D , or in case of a bilateral obstacle problem, this change of problem may not be helpful and we present some extensions in this direction in Appendix D.

Let us introduce the concept of a solution for Problem (3.1).

Definition 3.7. The pair (u, k) is a solution to Problem (3.1) if:

- $u \in L^p(\Omega_T, V)$ and $k \in L^{p'}(\Omega_T, V')$ are predictable and $u \in L^2(\Omega, \mathcal{C}([0, T], H))$.
- $u(t=0) = u_0$ and $u \in K$.
- P-a.s, for all $t \in [0, T]$,

$$u(t) + \int_0^t k ds + \int_0^t A(u, \cdot) ds = u_0 + \int_0^t G(u, \cdot) dW(s) + \int_0^t f ds.$$

- $-k \in (L^{p'}(\Omega_T, V'))^+$ and $\forall v \in K, \langle k, u - v \rangle \geq 0$ a.e. in Ω_T .

Remark 3.8. Since the embedding $V \hookrightarrow H$ is continuous, u is equally a predictable process with values in H or in V (thanks to Kuratowski's theorem [75, Th. 1.1 p. 5]).

Remark 3.9. We remind that $(L^p(\Omega_T, V))^+ = \{u \in L^p(\Omega_T, V), u(t, \omega) \in V^+ \text{ a.e. in } \Omega_T\}$, therefore, $-k \in (L^{p'}(\Omega_T, V'))^+$ if and only if $-k(t, \omega) \in (V')^+, \text{ a.e. in } \Omega_T$.

Indeed, If one assumes first that $-k \in (L^{p'}(\Omega_T, V'))^+$. Then, for any given $\varphi \in V^+$, any $A \in \mathcal{F}$ and any $B \in \mathcal{B}(0, T)$, $1_{A \times B} \varphi \in (L^p(\Omega_T, V))^+$.

Thus, $\int_{A \times B} \langle k(t, \omega), \varphi \rangle dt dP \leq 0$ for any such A and B and $\langle k(t, \omega), \varphi \rangle \leq 0$ on a subset of Ω_T of full measure, depending a priori on φ .

Since V is separable, for a given dense family $\{\varphi_n, n \in \mathbb{N}\} \subset V$, there exists $\tilde{\Omega}_T \subset \Omega_T$ a subset of full measure such that $\langle k(t, \omega), \varphi_n^+ \rangle \leq 0$ for any n and all $(t, \omega) \in \tilde{\Omega}_T$.

Let $\varphi \in V^+$ and $(\varphi_l) \subset \{\varphi_n, n \in \mathbb{N}\}$ satisfying $\varphi_l \rightarrow \varphi$ in V . Thus, $\varphi_l^+ \rightarrow \varphi^+ = \varphi$ in V and since $\langle k(t, \omega), \varphi_l^+ \rangle \leq 0$, the same inequality holds for φ . Thus, $-k(t, \omega) \in (V')^+$, $(t, \omega) \in \Omega_T$ a.e.

The converse is immediate since if $\varphi(t, \omega) \in V^+$ a.e. in Ω_T , one gets $\langle k(t, \omega), \varphi(t, \omega) \rangle \leq 0$ a.e. in Ω_T and $\int_{\Omega_T} \langle k(t, \omega), \varphi(t, \omega) \rangle dt dP \leq 0$.

As a consequence, knowing that $-k \in (L^{p'}(\Omega_T, V'))^+$ and $u \in K$, imply that the condition: $\langle k, u - \psi \rangle = 0$ a.e. in Ω_T is equivalent to the condition: $\forall v \in K, \langle k, u - v \rangle \geq 0$ a.e. in Ω_T .

Let us state our main result.

Theorem 3.10. *Under Assumptions (H_1) - (H_6) , there exists a unique solution (u, k) to Problem (3.1) in the sense of Definition 3.7. Moreover, the following Lewy-Stampacchia inequality holds*

$$0 \leq \partial_t \left(u - \int_0^\cdot G(u, \cdot) dW \right) + A(u, \cdot) - f \leq h^- = \left(f - \partial_t \left(\psi - \int_0^\cdot G(\psi, \cdot) dW \right) - A(\psi, \cdot) \right)^-.$$

Remark 3.11. *Note that Problem (3.1) can be written in the equivalent form:*

$$f - \partial_t \left(u - \int_0^\cdot G(u, \cdot) dW \right) - A(u, \cdot) \in \partial I_K(u)$$

where $\partial I_K(u)$ represents the sub-differential of $I_K : L^p(\Omega_T, V) \rightarrow \bar{\mathbb{R}}$ defined as

$$I_K(u) = \begin{cases} 0, & u \in K, \\ +\infty, & u \notin K, \end{cases}$$

and $\partial I_K(u) = N_K(u) = \{y \in L^{p'}(\Omega_T, V'); E \int_0^T \langle y, u - v \rangle ds \geq 0, \forall v \in K\}$ (see [5, p. 7 – 8]).

Before entering in the proof of our main theorem, we start with the following result.

Lemma 3.12. *If (u_1, k_1) and (u_2, k_2) are two solutions to (3.1) in the sense of Definition 3.7 associated with two different forces f_1 and f_2 then: there exists a positive constant $C > 0$ such that*

$$E \sup_{t \in [0, T]} \|(u_1 - u_2)(t)\|_H^2 \leq C \|f_1 - f_2\|_{L^{p'}(\Omega_T, V')} \|u_1 - u_2\|_{L^p(\Omega_T, V)}.$$

Remark 3.13. *Note that Lemma 3.12 ensures the uniqueness of the solution to (3.1) in the general framework.*

Proof. For any $t \in [0, T]$ and P-a.s we have

$$\begin{aligned} & u_1(t) - u_2(t) + \int_0^t [k_1 - k_2] ds + \int_0^t [A(u_1, \cdot) - A(u_2, \cdot)] ds \\ &= \int_0^t [G(u_1, \cdot) - G(u_2, \cdot)] dW(s) + \int_0^t [f_1 - f_2] ds. \end{aligned}$$

Applying Ito's formula with $F(t, v) = \frac{1}{2} \|v\|_H^2$, one gets for any $t \in [0, T]$

$$\begin{aligned} \frac{1}{2} \|(u_1 - u_2)(t)\|_H^2 + \int_0^t \langle A(u_1, \cdot) - A(u_2, \cdot), u_1 - u_2 \rangle ds + \int_0^t \langle k_1 - k_2, u_1 - u_2 \rangle ds \\ = \int_0^t \langle f_1 - f_2, u_1 - u_2 \rangle ds + \int_0^t \langle [G(u_1, \cdot) - G(u_2, \cdot)] dW(s), u_1 - u_2 \rangle \\ + \frac{1}{2} \int_0^t |G(u_1, \cdot) - G(u_2, \cdot)|_Q^2 ds. \end{aligned}$$

- Since $u_1, u_2 \in K$, Remark 3.9 yields a.e. in Ω_T ,

$$\langle k_1 - k_2, u_1 - u_2 \rangle = \langle k_1, u_1 - u_2 \rangle + \langle k_2, u_2 - u_1 \rangle \geq 0.$$

Therefore, for any t

$$\int_0^t \langle k_1 - k_2, u_1 - u_2 \rangle ds = \int_0^t (\langle k_1, u_1 - u_2 \rangle + \langle k_2, u_2 - u_1 \rangle) ds \geq 0 \text{ a.s.}$$

- $\forall t \in [0, T]$, $\frac{1}{2} \int_0^t |G(u_1, \cdot) - G(u_2, \cdot)|_Q^2 ds \leq M \int_0^t \|u_1 - u_2\|_H^2 ds$.
- Since $\lambda_T Id + A$ is T-monotone, $\forall t \in [0, T]$,

$$\int_0^t \langle A(u_1, \cdot) - A(u_2, \cdot), u_1 - u_2 \rangle ds \geq -\lambda_T \int_0^t \|u_1 - u_2\|_H^2 ds.$$

- By Burkholder-Davis-Gundy's inequality [38, Theorem 2.5 p.1240] (see also [61, p.652]) and Young's inequality, there exists a positive δ such that

$$\begin{aligned} & E \left[\sup_{t \in [0, T]} \left| \int_0^t \langle [G(u_1, \cdot) - G(u_2, \cdot)] dW(s), u_1 - u_2 \rangle \right| \right] \\ & \leq \frac{3\delta}{2} E \sup_{t \in [0, T]} \|(u_1 - u_2)(t)\|_H^2 + \frac{3M}{2\delta} E \int_0^T \|(u_1 - u_2)(s)\|_H^2 ds. \end{aligned}$$

- $E \sup_{t \in [0, T]} \left| \int_0^t \langle f_1 - f_2, u_1 - u_2 \rangle ds \right| \leq \|f_1 - f_2\|_{L^{p'}(\Omega_T, V')} \|u_1 - u_2\|_{L^p(\Omega_T, V)}.$

With a convenient choice of δ (e.g. $\delta = \frac{1}{4}$), we deduce the existence of a positive constant c such that

$$E \sup_{t \in [0, T]} \|(u_1 - u_2)(t)\|_H^2 \leq c \|f_1 - f_2\|_{L^{p'}(\Omega_T, V')} \|u_1 - u_2\|_{L^p(\Omega_T, V)} + c \int_0^T E \sup_{\tau \in [0, s]} \|(u_1 - u_2)(\tau)\|_H^2 ds. \quad (3.2)$$

Then, Grönwall's lemma ensures that

$$E \sup_{t \in [0, T]} \|(u_1 - u_2)(t)\|_H^2 \leq ce^{cT} \|f_1 - f_2\|_{L^{p'}(\Omega_T, V')} \|u_1 - u_2\|_{L^p(\Omega_T, V)}. \quad (3.3)$$

□

3.3 Proof of Theorem 3.10

We will prove Theorem 3.10 as follows: a first step concerning the proof of existence of the solution via penalization method and a perturbation of the stochastic reaction; assuming an additional regularity on h^- . Then; still with regular h^- , we prove the second part of Lewy-Stampacchia inequality by an idea inspired by [22]. The last step is devoted to the proof of the main result in the general case by using a density argument similar to the one used in the second Chapter (see Appendix B.4).

3.3.1 Existence of the solution and a first LS inequality in the regular case.

Penalization

Let $\varepsilon > 0$ and consider the following approximating problem:

$$\begin{cases} u_\varepsilon(t) + \int_0^t (A(u_\varepsilon, \cdot) - \frac{1}{\varepsilon} [(u_\varepsilon - \psi)^-]^{\tilde{q}-1} - f) ds = u_0 + \int_0^t \tilde{G}(u_\varepsilon, \cdot) dW(s) \\ u_\varepsilon(0) = u_0, \end{cases} \quad (3.4)$$

where $\tilde{q} = \min(p, 2)$ and $\tilde{G}(u_\varepsilon, \cdot) = G(\max(u_\varepsilon, \psi), \cdot)$. The idea of the perturbation of G is to have formally an additive stochastic source on the free-set where the constraint is violated.

Note that \tilde{G} satisfies also Assumptions H₁ and H₃, as well as Assumption H₅ since $\tilde{G}(\psi, \cdot) = G(\psi, \cdot)$. Indeed, since ψ is predictable in H , $\max(u, \psi)$ is also predictable for

any $u \in H$ and $\tilde{G}(u, \cdot)$ is predictable thanks to Remark 3.4.

For any $u, v \in V$, $|\tilde{G}(u, \cdot) - \tilde{G}(v, \cdot)|_Q^2 \leq M \|\max(u, \psi) - \max(v, \psi)\|_H^2 \leq M \|u - v\|_H^2$.

The only difference in the assumptions lies in $H_{3,2}$ where one gets now that

$$|\tilde{G}(u, \cdot)|_Q^2 \leq l + 2M \|\psi\|_H^2 + 2M \|u\|_H^2 = \tilde{l} + \tilde{M} \|u\|_H^2$$

where \tilde{l} is a $L^1(\Omega_T)$ -predictable element by composition of functions, depending only on the data.

Consider $\bar{A}(u_\varepsilon, \cdot) = A(u_\varepsilon, \cdot) - \frac{1}{\varepsilon} [(u_\varepsilon - \psi)^-]^{\tilde{q}-1} - f$ and note that:

- By construction, \bar{A} is an operator defined on $V \times \Omega_T$ with values in V' .
- Since ψ and f are predictable with values in V and V' respectively, $u \mapsto u^-$ is a Lipschitz-continuous mapping then, for any $v \in V$, $-\frac{1}{\varepsilon} [(v - \psi)^-]^{\tilde{q}-1} - f$ is predictable with values in V' and therefore \bar{A} satisfies Assumption $H_{1,1}$.
- Since $x \mapsto -x^-$ is non-decreasing, $\lambda_T Id + \bar{A}$ is T-monotone.
- The structure of the penalization operator yields the hemi-continuity of \bar{A} in the sense of $H_{2,4}$.
- (Coercivity): Note that for any $\delta > 0$, there exists $C_{\delta, \varepsilon} > 0$ such that: $\forall v \in V$,

$$\begin{aligned} \langle f, v \rangle &\leq C_\delta \|f\|_{V'}^{p'} + \delta \|v\|_V^p, \\ \langle -\frac{1}{\varepsilon} [(v - \psi)^-]^{\tilde{q}-1}, v \rangle &\geq \langle -\frac{1}{\varepsilon} [(v - \psi)^-]^{\tilde{q}-1}, \psi \rangle \\ &\geq -\delta \|v\|_{L^{\tilde{q}}(D)}^{\tilde{q}} - C_{\delta, \varepsilon} \|\psi\|_{L^{\tilde{q}}(D)}^{\tilde{q}} \geq -\delta C \|v\|_V^p - C_{\delta, \varepsilon} \|\psi\|_{L^{\tilde{q}}(D)}^{\tilde{q}} \end{aligned}$$

where C is related to the continuous embedding of V in $L^{\tilde{q}}(D)$.

Denote by $\tilde{l}_1 = l_1 + C_\delta \|f\|_{V'}^{p'} + C_{\delta, \varepsilon} \|\psi\|_{L^{\tilde{q}}(D)}^{\tilde{q}}$. It is a $L^1(\Omega_T)$ predictable element thanks to the assumptions on f and ψ , depending only on the data. Therefore, by a convenient choice of δ , \bar{A} satisfies $H_{2,1}$ by considering \tilde{l}_1 instead of l_1 .

- (Boundedness): $\forall v \in V$,

$$\begin{aligned} \left\| -\frac{1}{\varepsilon} [(v - \psi)^-]^{\tilde{q}-1} \right\|_{L^{\tilde{q}'}(D)} &= \frac{1}{\varepsilon} \|(v - \psi)^-\|_{L^{\tilde{q}}(D)}^{\tilde{q}-1} \leq C_\varepsilon \left(\|v\|_{L^{\tilde{q}}(D)}^{\tilde{q}-1} + \|\psi\|_{L^{\tilde{q}}(D)}^{\tilde{q}-1} \right) \\ &\leq C_\varepsilon \left(\|v\|_{L^{\tilde{q}}(D)}^{p-1} + \|\psi\|_{L^{\tilde{q}}(D)}^{p-1} \right) + C_p \end{aligned}$$

since $\tilde{q} < p$ may be possible. Now, since the embeddings of $L^{\tilde{q}}(D)$ in V' and of V in $L^{\tilde{q}}(D)$ are continuous,

$$\left\| -\frac{1}{\varepsilon}[(v - \psi)^-]^{\tilde{q}-1} \right\|_{V'} \leq C_\varepsilon \left(\|v\|_V^{p-1} + \|\psi\|_V^{p-1} \right) + C_p$$

and Assumption $H_{2,3}$ is satisfied with K replaced by $K + C_\varepsilon$ and g by $\tilde{g} = g + C_\varepsilon \|\psi\|_V^{p-1} + f + C_p$ which is a predictable element of $L^{p'}(\Omega_T)$.

- (The noise): Let us denote by $U = Q^{\frac{1}{2}}(H)$, we recall that U is a separable Hilbert space endowed with the scalar product $(u, v)_U = (Q^{-\frac{1}{2}}u, Q^{-\frac{1}{2}}v)_H$ (see [44, Prop. C.0.3 p. 221]) and note that $\tilde{G} \in HS(Q^{\frac{1}{2}}(H), H)$. Since $(W(t))_{t \in [0, T]}$ is a Wiener process in H with a nuclear covariance operator Q then $(W(t))_{t \in [0, T]}$ is a cylindrical Wiener process with a covariance operator I in U .

By [44, Th. 4.2.4 p.91] and Remark 3.8, for all $\varepsilon > 0$, there exists a unique solution $u_\varepsilon \in L^p(\Omega_T, V)$ predictable such that $u_\varepsilon \in L^2(\Omega, \mathcal{C}([0, T], H))$ and satisfying (3.4) for all $t \in [0, T]$ and P-a.s. in Ω .

Moreover, thanks to [44, Th. 4.2.5 p.91], $(u_\varepsilon)_{\varepsilon > 0}$ is bounded in $L^p(\Omega_T, V) \cap L^2(\Omega_T, H)$. Thanks to Assumptions $H_{2,3}$, we get the following lemma.

Lemma 3.14. • $(u_\varepsilon)_{\varepsilon > 0}$ is bounded in $L^p(\Omega_T, V) \cap C([0, T], L^2(\Omega, H))$.

- $(A(u_\varepsilon, \cdot))_{\varepsilon > 0}$ is bounded in $L^{p'}(\Omega_T, V')$.

Proof. Let $\varepsilon > 0$ and $v^* \in K$ such that $\partial_t(v^* - \int_0^\cdot \tilde{G}(v^*, \cdot) dW) \in L^{p'}(\Omega_T, V')$ with predictable assumptions. Note that $v^* = \psi$ holds in this situation and

$$\begin{aligned} & u_\varepsilon(t) - v^*(t) + \int_0^t \left(A(u_\varepsilon, \cdot) - \frac{1}{\varepsilon}[(u_\varepsilon - \psi)^-]^{\tilde{q}-1} \right) ds \\ &= u_0 - v^*(0) + \int_0^t [f - \partial_t(v^* - \int_0^\cdot \tilde{G}(v^*, \cdot) dW)] ds + \int_0^t [\tilde{G}(u_\varepsilon, \cdot) - \tilde{G}(v^*, \cdot)] dW(s). \end{aligned}$$

Itô's stochastic energy yields

$$\begin{aligned} & \|u_\varepsilon - v^*\|_H^2(t) + 2 \int_0^t \langle A(u_\varepsilon, \cdot), u_\varepsilon - v^* \rangle ds \\ & - 2 \int_0^t \int_D \frac{1}{\varepsilon} [(u_\varepsilon - \psi)^-]^{\tilde{q}-1} (u_\varepsilon - v^*) dx ds - \|u_0 - v^*(0)\|_H^2 \\ &= 2 \int_0^t \langle f - \partial_s(v^* - \int_0^\cdot \tilde{G}(v^*, \cdot) dW), u_\varepsilon - v^* \rangle ds \\ & + 2 \int_0^t \left(u_\varepsilon - v^*, [\tilde{G}(u_\varepsilon, \cdot) - \tilde{G}(v^*, \cdot)] dW(s) \right)_H + \int_0^t |\tilde{G}(u_\varepsilon, \cdot) - \tilde{G}(v^*, \cdot)|_Q^2 ds. \end{aligned}$$

Note that

$$\begin{aligned} & -2 \int_0^t \int_D \frac{1}{\varepsilon} [(u_\varepsilon - \psi)^-]^{\tilde{q}-1} (u_\varepsilon - v^*) dx ds \\ &= -\frac{2}{\varepsilon} \int_0^t \int_D [(u_\varepsilon - \psi)^-]^{\tilde{q}-1} (u_\varepsilon - \psi) dx ds - \frac{2}{\varepsilon} \int_0^t \int_D [(u_\varepsilon - \psi)^-]^{\tilde{q}-1} (\psi - v^*) dx ds \geq 0, \end{aligned}$$

$$\begin{aligned} \langle A(u_\varepsilon, \cdot), u_\varepsilon - v^* \rangle &\geq \alpha \|u_\varepsilon\|_V^p - \lambda \|u_\varepsilon\|_H^2 - l_1(t, \omega) - \langle A(u_\varepsilon, \cdot), v^* \rangle \\ &\geq \alpha \|u_\varepsilon\|_V^p - \lambda \|u_\varepsilon\|_H^2 - l_1(t, \omega) - \bar{K} \|u_\varepsilon\|_V^{p-1} \|v^*\|_V - g(t, \omega) \|v^*\|_V \\ &\geq \frac{\alpha}{2} \|u_\varepsilon\|_V^p - \lambda \|u_\varepsilon\|_H^2 - l_1(t, \omega) - C(v^*)(t, \omega) \end{aligned}$$

where $C(v^*) \in L^1(\Omega_T)$. Thus, for any positive γ , Young's inequality yields the existence of a positive constant C_γ that may change from line to line, such that

$$\begin{aligned} & E \|u_\varepsilon - v^*\|_H^2(t) + 2E \int_0^t \frac{\alpha}{2} \|u_\varepsilon\|_V^p(s) ds \\ &\leq \lambda E \int_0^t \|u_\varepsilon\|_H^2(s) ds + \|l_1 + C(v^*)\|_{L^1(\Omega_T)} + C_\gamma(f, \partial_s(v^* - \int_0^\cdot \tilde{G}(v^*, \cdot) dW)) \\ &\quad + \gamma E \int_0^t \|u_\varepsilon - v^*\|_V^p(s) ds + E \int_0^t |\tilde{G}(u_\varepsilon, \cdot) - \tilde{G}(v^*, \cdot)|_Q^2 ds. \\ &\leq CE \int_0^t \|u_\varepsilon - v^*\|_H^2(s) ds + \frac{\alpha}{2} E \int_0^t \|u_\varepsilon\|_V^p(s) ds + C, \end{aligned}$$

for a suitable choice of γ and thanks to $H_{3,2}$.

Then, the first part of the lemma is proved by Gronwall's lemma, and the second one by adding $H_{2,3}$ to the first estimate. \square

A priori estimates with a regular h^- .

H_7 : We will assume in this subsection that h^- is a predictable non negative element of $L^{\tilde{q}'}(\Omega_T, L^{\tilde{q}'}(D))$.

Lemma 3.15. *Under H_7 , $(\frac{1}{\varepsilon} [(u_\varepsilon - \psi)^-]^{\tilde{q}-1})_{\varepsilon>0}$ is bounded in $L^{\tilde{q}'}(\Omega_T \times D)$.*

Proof. Let $\delta > 0$ and consider the following approximation from [57, p. 152].

$$F_\delta(r) = \begin{cases} r^2 - \frac{\delta^2}{6} & \text{if } r \leq -\delta, \\ -\frac{r^4}{2\delta^2} - \frac{4r^3}{3\delta} & \text{if } -\delta \leq r \leq 0, \\ 0 & \text{if } r \geq 0. \end{cases} \quad (3.5)$$

Note that $(-\frac{1}{2}F'_\delta)_\delta$ approximates the negative part. Moreover, $F_\delta(\cdot) \in \mathcal{C}^2(\mathbb{R})$, and satisfies:

$$\begin{cases} |F_\delta(r)| \leq r^2, \\ |F'_\delta(r)| \leq 2r \quad \text{and} \quad \forall r \in \mathbb{R}, F'_\delta(r) \leq 0, \\ |F''_\delta(r)| \leq \frac{8}{3} \quad \text{and} \quad \forall r \in \mathbb{R}, F''_\delta(r) \geq 0. \end{cases}$$

Set $\varphi_\delta(v) = \int_D F_\delta(v(x))dx$, $v \in L^2(D)$ and denote by S the set $\{u_\varepsilon \leq \psi\}$. Applying Ito's formula [57, Th. 4.2 p. 65] (see also [61, Lemma 4]) to the process $u_\varepsilon - \psi$, one gets for any $t \in [0, T]$

$$\begin{aligned} \varphi_\delta(u_\varepsilon(t) - \psi(t)) + \int_0^t \langle A(u_\varepsilon, \cdot) - A(\psi, \cdot), F'_\delta(u_\varepsilon - \psi) \rangle ds - \frac{1}{\varepsilon} \int_0^t \langle [(u_\varepsilon - \psi)^-]^{\bar{q}-1}, F'_\delta(u_\varepsilon - \psi) \rangle ds \\ = \underbrace{\varphi_\delta(u_\varepsilon(0) - \psi(0))}_{=0} + \underbrace{\int_0^t \langle h^+, F'_\delta(u_\varepsilon - \psi) \rangle ds}_{\leq 0} - \int_0^t \langle h^-, F'_\delta(u_\varepsilon - \psi) \rangle ds \\ + \int_0^t (\{\tilde{G}(u_\varepsilon, \cdot) - \tilde{G}(\psi, \cdot)\} dW(s), F'_\delta(u_\varepsilon - \psi)) \\ + \frac{1}{2} \int_0^t \text{Tr}(F''_\delta(u_\varepsilon - \psi) \{\tilde{G}(u_\varepsilon, \cdot) - \tilde{G}(\psi, \cdot)\} Q \{\tilde{G}(u_\varepsilon, \cdot) - \tilde{G}(\psi, \cdot)\}^*) ds. \end{aligned}$$

Since $\tilde{G}(u_\varepsilon, \cdot) = \tilde{G}(\psi, \cdot)$ on the set S , we deduce

$$\frac{1}{2} \int_0^t \text{Tr}(F''_\delta(u_\varepsilon - \psi) \{\tilde{G}(u_\varepsilon, \cdot) - \tilde{G}(\psi, \cdot)\} Q \{\tilde{G}(u_\varepsilon, \cdot) - \tilde{G}(\psi, \cdot)\}^*) ds = 0.$$

Taking the expectation, one has

$$\begin{aligned} E\varphi_\delta(u_\varepsilon(t) - \psi(t)) + E \int_0^t \langle A(u_\varepsilon, \cdot) - A(\psi, \cdot), F'_\delta(u_\varepsilon - \psi) \rangle ds - \frac{1}{\varepsilon} E \int_0^t \langle [(u_\varepsilon - \psi)^-]^{\bar{q}-1}, F'_\delta(u_\varepsilon - \psi) \rangle ds \\ \leq E \int_0^t \langle -h^-, F'_\delta(u_\varepsilon - \psi) \rangle ds. \end{aligned}$$

Claim: a.e. $t \in [0, T]$ and P-a.s, $F'_\delta(u_\varepsilon - \psi)$ converges to $-2(u_\varepsilon - \psi)^-$ in V .

Indeed, we have

$$F'_\delta(r) = \begin{cases} 2r & \text{if } r \leq -\delta, \\ -2\frac{r^3}{\delta^2} - 4\frac{r^2}{\delta} & \text{if } -\delta \leq r \leq 0, \\ 0 & \text{if } r \geq 0. \end{cases} \quad (3.6)$$

Let us estimate $\|F'_\delta(u_\varepsilon - \psi) + 2(u_\varepsilon - \psi)^-\|_V$,

$$\begin{aligned} \|F'_\delta(u_\varepsilon - \psi) + 2(u_\varepsilon - \psi)^-\|_V &= \left(\int_D |F'_\delta(u_\varepsilon(x) - \psi(x)) + 2(u_\varepsilon(x) - \psi(x))^-|^2 dx \right)^{\frac{1}{2}} \\ &\quad + \left(\int_D |\nabla F'_\delta(u_\varepsilon(x) - \psi(x)) + 2\nabla(u_\varepsilon(x) - \psi(x))^-|^p dx \right)^{\frac{1}{p}}. \end{aligned}$$

We denote by B the set $\{-\delta \leq u_\varepsilon - \psi \leq 0\}$. On one hand, one has

$$\begin{aligned} & \int_D |F'_\delta(u_\varepsilon(x) - \psi(x)) + 2(u_\varepsilon(x) - \psi(x))^-|^2 dx \\ &= \int_B \left| -\frac{2}{\delta^2}(u_\varepsilon(x) - \psi(x))^3 - \frac{4}{\delta}(u_\varepsilon(x) - \psi(x))^2 - 2(u_\varepsilon(x) - \psi(x)) \right|^2 dx \\ &\leq C_2 \int_D 8\delta^2 dx = 8C_2\delta^2 \text{mes}(D) \rightarrow 0 \text{ as } \delta \rightarrow 0. \end{aligned}$$

On the other hand, setting $F = \{-\delta < u_\varepsilon - \psi < 0\}$ one has by the chain rule in the Sobolev spaces

$$\begin{aligned} & \int_D |\nabla F'_\delta(u_\varepsilon(x) - \psi(x)) + 2\nabla(u_\varepsilon(x) - \psi(x))^-|^p dx \\ &= \int_F \left| \frac{2}{\delta^2}\nabla(u_\varepsilon(x) - \psi(x))^3 + \frac{4}{\delta}\nabla(u_\varepsilon(x) - \psi(x))^2 + 2\nabla(u_\varepsilon(x) - \psi(x)) \right|^p dx \\ &\leq \int_F \left| \left(\frac{6}{\delta^2}(u_\varepsilon(x) - \psi(x))^2 + \frac{8}{\delta}(u_\varepsilon(x) - \psi(x)) + 2 \right) \nabla(u_\varepsilon(x) - \psi(x)) \right|^p dx. \end{aligned}$$

We have $\left| \frac{6}{\delta^2}(u_\varepsilon(x) - \psi(x))^2 + \frac{8}{\delta}(u_\varepsilon(x) - \psi(x)) + 2 \right| \mathbb{I}_F \leq 2\mathbb{I}_{\{-\delta < u_\varepsilon - \psi < 0\}} \rightarrow 0$ a.e. $x \in D$ as $\delta \rightarrow 0$ and

$$\left| \left(\frac{6}{\delta^2}(u_\varepsilon(x) - \psi(x))^2 + \frac{8}{\delta}(u_\varepsilon(x) - \psi(x)) + 2 \right) \mathbb{I}_F \nabla(u_\varepsilon(x) - \psi(x)) \right|^p \leq 2|\nabla(u_\varepsilon(x) - \psi(x))|^p \in L^1(D).$$

Dominated convergence theorem ensures that

$$\int_D |\nabla F'_\delta(u_\varepsilon(x) - \psi(x)) + 2\nabla(u_\varepsilon(x) - \psi(x))^-|^p dx \rightarrow 0.$$

Therefore a.e. $t \in [0, T]$ and P-a.s., one gets when $\delta \rightarrow 0$

- $\forall t \in [0, T], \quad \varphi_\delta(u_\varepsilon(t) - \psi(t)) \rightarrow \|(u_\varepsilon - \psi)^-(t)\|_{L^2(D)}^2,$
- $\langle A(u_\varepsilon, \cdot) - A(\psi, \cdot), F'_\delta(u_\varepsilon - \psi) \rangle \rightarrow \langle A(u_\varepsilon, \cdot) - A(\psi, \cdot), -2(u_\varepsilon - \psi)^- \rangle \geq -2\lambda_T \|(u_\varepsilon - \psi)^-\|_{H'}^2,$
since this last term is equal to $2\langle A(\psi, \cdot) - A(u_\varepsilon, \cdot), (\psi - u_\varepsilon)^+ \rangle \geq -2\lambda_T \|(\psi - u_\varepsilon)^+\|_{H'}^2,$
thanks to $H_{2,2}$.
- $\langle -[(u_\varepsilon - \psi)^-]^{\tilde{q}-1}, F'_\delta(u_\varepsilon - \psi) \rangle \rightarrow \langle -[(u_\varepsilon - \psi)^-]^{\tilde{q}-1}, -2(u_\varepsilon - \psi)^- \rangle$
 $= 2\|(u_\varepsilon - \psi)^-\|_{L^{\tilde{q}}(D)}^{\tilde{q}},$
- $\langle -h^-, F'_\delta(u_\varepsilon - \psi) \rangle \rightarrow \langle -h^-, 2(u_\varepsilon - \psi) \mathbb{I}_{\{u_\varepsilon < \psi\}} \rangle = 2\langle h^-, (u_\varepsilon - \psi)^- \rangle.$

Again, dominated convergence theorem ensures that for any t

$$\begin{aligned} & E\|(u_\varepsilon - \psi)^-(t)\|_{L^2(D)}^2 + \frac{2}{\varepsilon} E \int_0^t \|(u_\varepsilon - \psi)^-(s)\|_{L^{\tilde{q}}(D)}^{\tilde{q}} ds \\ &\leq 2E \int_0^t \langle h^-(s), (u_\varepsilon - \psi)^-(s) \rangle ds + 2\lambda_T E \int_0^t \|(u_\varepsilon - \psi)^-(s)\|_{H'}^2 ds. \end{aligned} \quad (3.7)$$

To continue our proof, we will consider two cases.

- If $p \geq 2$ then $\tilde{q} = 2$. By multiplying (3.7) by $\frac{1}{\varepsilon}$, one gets

$$\begin{aligned} & \frac{1}{2\varepsilon} E \|(u_\varepsilon - \psi)^-(T)\|_{L^2(D)}^2 + \frac{1}{\varepsilon^2} E \int_0^T \|(u_\varepsilon - \psi)^-(s)\|_{L^2(D)}^2 ds \\ & \leq E \int_0^T \langle h^-(s), \frac{1}{\varepsilon} (u_\varepsilon - \psi)^-(s) \rangle ds + \frac{\varepsilon \lambda_T^+}{\varepsilon^2} E \int_0^t \|(u_\varepsilon - \psi)^-(s)\|_H^2 ds. \end{aligned}$$

Since $E \int_0^T \langle h^-(s), \frac{1}{\varepsilon} (u_\varepsilon - \psi)^-(s) \rangle ds \leq \frac{1}{2\varepsilon^2} E \int_0^T \|(u_\varepsilon - \psi)^-(s)\|_{L^2(D)}^2 ds + \frac{1}{2} E \int_0^T \|h^-(s)\|_{L^2(D)}^2 ds$, one has, for $\varepsilon \leq \frac{1}{4\lambda_T^+ + 1}$,

$$\frac{1}{2\varepsilon} E \|(u_\varepsilon - \psi)^-(T)\|_{L^2(D)}^2 + \frac{1}{4\varepsilon^2} E \int_0^T \|(u_\varepsilon - \psi)^-(s)\|_{L^2(D)}^2 ds \leq E \int_0^T \|h^-(s)\|_{L^2(D)}^2 ds.$$

Therefore $(\frac{1}{\varepsilon} (u_\varepsilon - \psi)^-)_{\varepsilon > 0}$ is bounded in $L^2(\Omega_T \times D)$.

- If $2 > p > 1$ then $\tilde{q} = p$. From Grönwall's lemma applied to (3.7), one gets

$$\begin{aligned} \frac{1}{\varepsilon} \|(u_\varepsilon - \psi)^-\|_{L^p(\Omega_T \times D)}^p &= \frac{1}{\varepsilon} E \int_0^T \|(u_\varepsilon - \psi)^-(s)\|_{L^p(D)}^p ds \\ &\leq C(T) E \int_0^T \langle h^-(s), (u_\varepsilon - \psi)^-(s) \rangle ds \\ &\leq C \|h^-\|_{L^{p'}(\Omega_T \times D)} \|(u_\varepsilon - \psi)^-\|_{L^p(\Omega_T \times D)}, \end{aligned}$$

hence $\frac{1}{\varepsilon} \|(u_\varepsilon - \psi)^-\|_{L^p(\Omega_T \times D)}^{p-1} \leq \|h^-\|_{L^{p'}(\Omega_T \times D)}$. On the other hand, we have

$$\frac{1}{\varepsilon} [(u_\varepsilon - \psi)^-]^{p-1} \|_{L^{p'}(\Omega_T \times D)} = \frac{1}{\varepsilon} (E \int_0^T \int_D [(u_\varepsilon - \psi)^-]^{(p-1)p'} dx ds)^{\frac{1}{p'}} = \frac{1}{\varepsilon} \|(u_\varepsilon - \psi)^-\|_{L^p(\Omega_T \times D)}^{p-1}.$$

Consequently, $(\frac{1}{\varepsilon} [(u_\varepsilon - \psi)^-]^{p-1})_{\varepsilon > 0}$ is bounded in $L^{p'}(\Omega_T \times D)$.

□

As a consequence the following lemma holds.

Lemma 3.16. *Under H_7 , $(u_\varepsilon)_{\varepsilon > 0}$ is a Cauchy sequence in the space $L^2(\Omega, \mathcal{C}([0, T], H))$.*

Proof. Let $1 > \varepsilon \geq \delta > 0$ and consider the process $u_\varepsilon - u_\delta$, which satisfies the following equation

$$\begin{aligned} u_\varepsilon(t) - u_\delta(t) &+ \int_0^t (A(u_\varepsilon, \cdot) - A(u_\delta, \cdot)) + \left(-\frac{1}{\varepsilon} [(u_\varepsilon - \psi)^-]^{\tilde{q}-1} + \frac{1}{\delta} [(u_\delta - \psi)^-]^{\tilde{q}-1} \right) ds \\ &= \int_0^t (\tilde{G}(u_\varepsilon, \cdot) - \tilde{G}(u_\delta, \cdot)) dW(s). \end{aligned}$$

Applying Ito's formula with $F(t, v) = \frac{1}{2} \|v\|_H^2$, one gets for any $t \in [0, T]$

$$\begin{aligned} & \frac{1}{2} \|(u_\varepsilon - u_\delta)(t)\|_H^2 + \int_0^t \langle A(u_\varepsilon, \cdot) - A(u_\delta, \cdot), u_\varepsilon - u_\delta \rangle ds \\ & \quad + \int_0^t \left\langle -\frac{1}{\varepsilon} [(u_\varepsilon - \psi)^-]^{\tilde{q}-1} + \frac{1}{\delta} [(u_\delta - \psi)^-]^{\tilde{q}-1}, u_\varepsilon - u_\delta \right\rangle ds \\ & = \int_0^t \langle (\tilde{G}(u_\varepsilon, \cdot) - \tilde{G}(u_\delta, \cdot)) dW(s), u_\varepsilon - u_\delta \rangle + \frac{1}{2} \int_0^t |\tilde{G}(u_\varepsilon, \cdot) - \tilde{G}(u_\delta, \cdot)|_Q^2 ds. \end{aligned}$$

We argue as in the proof of (3.2) with $f_1 = f_2$ and note that we need only to discuss the penalization term.

On one hand, using the monotonicity of the penalization operator, one deduces

$$\int_0^t \left\langle -\frac{1}{\varepsilon} [(u_\varepsilon - \psi)^-]^{\tilde{q}-1} + \frac{1}{\delta} [(u_\delta - \psi)^-]^{\tilde{q}-1}, u_\varepsilon - u_\delta \right\rangle ds \geq \frac{\varepsilon - \delta}{\varepsilon \delta} \int_0^t \langle [(u_\delta - \psi)^-]^{\tilde{q}-1}, u_\varepsilon - u_\delta \rangle ds.$$

On the other hand, we have

$$\begin{aligned} & \frac{\varepsilon - \delta}{\varepsilon \delta} \int_0^t \langle [(u_\delta - \psi)^-]^{\tilde{q}-1}, u_\varepsilon - u_\delta \rangle ds \\ & = \frac{\varepsilon - \delta}{\varepsilon \delta} \left(\int_0^t \langle [(u_\delta - \psi)^-]^{\tilde{q}-1}, u_\varepsilon - \psi \rangle ds + \int_0^t \langle [(u_\delta - \psi)^-]^{\tilde{q}-1}, -(u_\delta - \psi) \rangle ds \right). \end{aligned}$$

Since $\frac{\varepsilon - \delta}{\varepsilon \delta} \int_0^t \langle [(u_\delta - \psi)^-]^{\tilde{q}-1}, -(u_\delta - \psi) \rangle ds \geq 0$, it holds that

$$\begin{aligned} \frac{\varepsilon - \delta}{\varepsilon \delta} \int_0^t \langle [(u_\delta - \psi)^-]^{\tilde{q}-1}, u_\varepsilon - u_\delta \rangle ds & \geq \frac{\varepsilon - \delta}{\varepsilon \delta} \int_0^t \langle [(u_\delta - \psi)^-]^{\tilde{q}-1}, u_\varepsilon - \psi \rangle ds \\ & \geq -\frac{\varepsilon - \delta}{\varepsilon \delta} \int_0^t \langle [(u_\delta - \psi)^-]^{\tilde{q}-1}, (u_\varepsilon - \psi)^- \rangle ds. \end{aligned}$$

We distinguish two cases:

- If $p \geq 2$, then $\tilde{q} = 2$. Since $(\frac{1}{\varepsilon} (u_\varepsilon - \psi)^-)_{\varepsilon > 0}$ is bounded in $L^2(\Omega_T \times D)$, we get

$$0 \leq \frac{\varepsilon - \delta}{\varepsilon \delta} E \int_0^T \langle [(u_\delta - \psi)^-]^{\tilde{q}-1}, (u_\varepsilon - \psi)^- \rangle ds = (\varepsilon - \delta) E \int_0^T \left\langle \frac{1}{\delta} (u_\delta - \psi)^-, \frac{1}{\varepsilon} (u_\varepsilon - \psi)^- \right\rangle ds \leq C\varepsilon.$$

- If $1 < p < 2$, then $\tilde{q} = p$. Since $(\frac{1}{\varepsilon} [(u_\varepsilon - \psi)^-]^{p-1})_{\varepsilon > 0}$ is bounded in $L^{p'}(\Omega_T \times D)$, we get

$$\begin{aligned} 0 \leq \frac{\varepsilon - \delta}{\varepsilon \delta} E \int_0^T \langle [(u_\delta - \psi)^-]^{p-1}, (u_\varepsilon - \psi)^- \rangle ds & = \frac{\varepsilon - \delta}{\varepsilon} E \int_0^T \left\langle \frac{1}{\delta} [(u_\delta - \psi)^-]^{p-1}, (u_\varepsilon - \psi)^- \right\rangle ds \\ & \leq \frac{\varepsilon - \delta}{\varepsilon} C \| (u_\varepsilon - \psi)^- \|_{L^p(\Omega_T \times D)} \leq C\varepsilon^{\frac{1}{p-1}}. \end{aligned}$$

By arguments similar to the ones used to obtain (3.2), we deduce

$$E \sup_{t \in [0, T]} \|(u_\varepsilon - u_\delta)(t)\|_H^2 \leq C(\varepsilon + \varepsilon^{\frac{1}{p-1}}) + C \int_0^T E \sup_{\tau \in [0, s]} \|(u_\varepsilon - u_\delta)(\tau)\|_H^2 ds$$

and Grönwall's lemma ensures that $(u_\varepsilon)_{\varepsilon > 0}$ is a Cauchy sequence in the space $L^2(\Omega, \mathcal{C}([0, T], H))$.

□

At the limit as $\varepsilon \rightarrow 0$.

From Lemma 3.14, Lemma 3.15 and Lemma 3.16, we deduce the following result.

Lemma 3.17. *There exist $u \in L^p(\Omega_T, V) \cap L^2(\Omega, \mathcal{C}([0, T], H)) \cap \mathcal{N}_W^2(0, T, H)^\dagger$ and $(\rho, \chi) \in L^{\tilde{q}'}(\Omega_T, L^{\tilde{q}'}(D)) \times L^{p'}(\Omega_T, V')$, each one predictable, such that the following convergences hold, up to sub-sequences denoted by the same way,*

$$u_\varepsilon \rightharpoonup u \quad \text{in } L^p(\Omega_T, V), \quad (3.8)$$

$$u_\varepsilon \rightarrow u \quad \text{in } L^2(\Omega, \mathcal{C}([0, T], H)), \quad (3.9)$$

$$A(u_\varepsilon, \cdot) \rightharpoonup \chi \quad \text{in } L^{p'}(\Omega_T, V'), \quad (3.10)$$

$$-\frac{1}{\varepsilon} [(u_\varepsilon - \psi)^-]^{\tilde{q}-1} \rightharpoonup \rho, \quad \rho \leq 0 \quad \text{in } L^{\tilde{q}'}(\Omega_T \times D). \quad (3.11)$$

Proof. By compactness with respect to the weak topology in the spaces $L^p(\Omega_T, V)$, $L^{p'}(\Omega_T, V')$ and $L^{\tilde{q}'}(\Omega_T \times D)$, there exist $u \in L^p(\Omega_T, V)$, $\chi \in L^{p'}(\Omega_T, V')$ and $\rho \in L^{\tilde{q}'}(\Omega_T \times D)$ such that (3.8), (3.10) and (3.11) hold (for sub-sequences).

Thanks to Lemma 3.16, we get the strong convergence of u_ε to u in $L^2(\Omega, \mathcal{C}([0, T], H)) \hookrightarrow L^2(\Omega_T \times D)$. Moreover,

- Since $u_\varepsilon \in \mathcal{N}_W^2(0, T, H)$, a Hilbert space, $u \in \mathcal{N}_W^2(0, T, H)$ too.
- Since $(A(u_\varepsilon, \cdot))_\varepsilon$ is predictable with values in V' (cf. Rmk. 3.3), the same applies to χ .
- Since $u_\varepsilon, \psi \in \mathcal{N}_W^2(0, T, H)$, $-\frac{1}{\varepsilon} [(u_\varepsilon - \psi)^-]^{\tilde{q}-1}$ is a predictable process with values in $L^{\tilde{q}'}(D)$. Hence ρ is a predictable process with values in $L^{\tilde{q}'}(D)$ and $\rho \leq 0$ since the set of non positive functions of $L^{\tilde{q}'}(\Omega_T \times D)$ is a closed convex subset of $L^{\tilde{q}'}(\Omega_T \times D)$.

[†] $\mathcal{N}_W^2(0, T, H)$ denotes the space of all predictable process of $L^2(\Omega_T, H)$ (see [44, p. 36]).

□

Remark 3.18. (initial condition and constraint).

- Since u_ε converges to u in $L^2(\Omega, \mathcal{C}([0, T], H))$ then $u_\varepsilon(0) = u_0$ converges to $u(0)$ in $L^2(\Omega, H)$ and $u(0) = u_0$ in $L^2(\Omega, H)$.
- Thanks to Lemma 3.15, we deduce that $(u_\varepsilon - \psi)^- \rightarrow (u - \psi)^- = 0$ in $L^{\tilde{q}}(\Omega_T \times D)$ and $u \in K$.

Lemma 3.19. Under H_7 , $\int_0^\cdot \tilde{G}(u_\varepsilon, \cdot) dW(s) \rightarrow \int_0^\cdot G(u, \cdot) dW(s)$ in $L^2(\Omega, \mathcal{C}([0, T], H))$ when $\varepsilon \rightarrow 0$.

Proof. By Burkholder-Davis-Gundy's inequality [61, p.652], one gets

$$E \sup_{t \in [0, T]} \left| \int_0^t (\tilde{G}(u_\varepsilon, \cdot) - \tilde{G}(u, \cdot)) dW(s) \right|_H^2 \leq 3E \int_0^T |\tilde{G}(u_\varepsilon, \cdot) - \tilde{G}(u, \cdot)|_Q^2 ds$$

$$\text{(by using } H_3) \leq 3ME \int_0^T \|u_\varepsilon - u\|_H^2 ds.$$

Since $u_\varepsilon \rightarrow u$ in $L^2(\Omega, \mathcal{C}([0, T], H))$ with $u \in K$, one deduces

$$\int_0^\cdot \tilde{G}(u_\varepsilon, \cdot) dW(s) \rightarrow \int_0^\cdot \tilde{G}(u, \cdot) dW(s) = \int_0^\cdot G(u, \cdot) dW(s) \quad \text{in } L^2(\Omega, \mathcal{C}([0, T], H)).$$

□

Lemma 3.20. We have $\rho(u - \psi) = 0$ a.e. in Ω_T and, $\forall v \in K$, $\rho(u - v) \geq 0$ a.e. in Ω_T .

Proof. On one hand, by Lemma 3.15, we have

$$0 \leq -\frac{1}{\varepsilon} E \int_0^t \langle [(u_\varepsilon - \psi)^-]^{\tilde{q}-1}, u_\varepsilon - \psi \rangle ds = \frac{1}{\varepsilon} E \int_0^t \|[(u_\varepsilon - \psi)^-](s)\|_{L^{\tilde{q}}}^{\tilde{q}} ds \leq C\varepsilon^{\tilde{q}'-1} \rightarrow 0.$$

On the other hand, by Lemma 3.17, we distinguish two cases:

- If $p \geq 2$ then $-\frac{1}{\varepsilon}(u_\varepsilon - \psi)^- \rightharpoonup \rho$ in $L^2(\Omega_T \times D)$ and $u_\varepsilon - \psi \rightarrow u - \psi$ in $L^2(\Omega_T \times D)$ by Lemma 3.16. Hence $E \int_0^T \int_D \rho(u - \psi) dx dt = 0$ and $\rho(u - \psi) = 0$ since the integrand is always non-positive.
- If $2 > p > 1$ then $-\frac{1}{\varepsilon}[(u_\varepsilon - \psi)^-]^{p-1} \rightharpoonup \rho$ in $L^{p'}(\Omega_T \times D)$ and $u_\varepsilon - \psi \rightarrow u - \psi$ in $L^p(\Omega_T \times D)$ by Lemma 3.16 and the same conclusion holds.

One finishes the proof by noticing that if $v \in K$, one has a.e. in Ω_T that,

$$\langle \rho, u - v \rangle = \overbrace{\langle \rho, u - \psi \rangle}^{=0} + \overbrace{\langle \rho, \psi - v \rangle}^{\geq 0} \geq 0.$$

□

Our aim now is to prove that $A(u, \cdot) = \chi$. We have for any $t \in [0, T]$

$$u_\varepsilon(t) + \int_0^t (A(u_\varepsilon, \cdot) - \frac{1}{\varepsilon} [(u_\varepsilon - \psi)^-]^{\bar{q}-1} - f) ds = u_0 + \int_0^t \tilde{G}(u_\varepsilon, \cdot) dW(s),$$

and

$$u(t) + \int_0^t (\chi + \rho - f) ds = u_0 + \int_0^t \tilde{G}(u, \cdot) dW(s).$$

Hence

$$u_\varepsilon(t) - u(t) + \int_0^t ((A(u_\varepsilon, \cdot) - \chi) + (-\frac{1}{\varepsilon} [(u_\varepsilon - \psi)^-]^{\bar{q}-1} - \rho)) ds = \int_0^t (\tilde{G}(u_\varepsilon, \cdot) - \tilde{G}(u, \cdot)) dW(s).$$

Note that $(A(u_\varepsilon, \cdot) - \chi) + (-\frac{1}{\varepsilon} [(u_\varepsilon - \psi)^-]^{\bar{q}-1} - \rho) \in L^{p'}(\Omega_T, V')$ is predictable and that $\int_0^t (\tilde{G}(u_\varepsilon, \cdot) - \tilde{G}(u, \cdot)) dW(s)$ is a square integrable \mathcal{F}_t -martingale. Thus, we can apply Ito's formula [57, Theorem 4.2 p. 65] to the process $u_\varepsilon - u$ with $F(v) = \frac{1}{2} \|v\|_H^2$ to get

$$\begin{aligned} \frac{1}{2} \|(u_\varepsilon - u)(t)\|_H^2 &+ \overbrace{\int_0^t \langle A(u_\varepsilon, \cdot) - \chi, u_\varepsilon - u \rangle ds}^{I_1} + \overbrace{\int_0^t \langle -\frac{1}{\varepsilon} [(u_\varepsilon - \psi)^-]^{\bar{q}-1} - \rho, u_\varepsilon - u \rangle ds}^{I_2} \\ &= \overbrace{\int_0^t \langle (\tilde{G}(u_\varepsilon, \cdot) - \tilde{G}(u, \cdot)) dW(s), u_\varepsilon - u \rangle}^{I_3} + \overbrace{\frac{1}{2} \int_0^t |\tilde{G}(u_\varepsilon, \cdot) - \tilde{G}(u, \cdot)|_Q^2 ds}^{I_4}. \end{aligned}$$

Let us consider in the sequel a given $v \in L^p(\Omega_T, V) \cap L^2(\Omega, \mathcal{C}([0, T], H))$ and $t \in]0, T]$.

- Note that $I_1 = \int_0^t \langle A(u_\varepsilon, \cdot), u_\varepsilon \rangle ds - \int_0^t \langle A(u_\varepsilon, \cdot), u \rangle ds - \int_0^t \langle \chi, u_\varepsilon - u \rangle ds$ and

$$\begin{aligned} \int_0^t \langle A(u_\varepsilon, \cdot), u_\varepsilon \rangle ds &= \int_0^t \langle A(u_\varepsilon, \cdot) - A(v, \cdot), u_\varepsilon - v \rangle ds \\ &\quad + \int_0^t \langle A(v, \cdot), u_\varepsilon - v \rangle ds + \int_0^t \langle A(u_\varepsilon, \cdot), v \rangle ds \\ (\lambda_T Id + A \text{ is T-monotone}) &\geq \int_0^t \langle A(v, \cdot), u_\varepsilon - v \rangle ds + \int_0^t \langle A(u_\varepsilon, \cdot), v \rangle ds - \lambda_T \int_0^t \|u_\varepsilon - v\|_H^2 ds. \end{aligned}$$

- $E(I_2) = E \int_0^t \langle -\frac{1}{\varepsilon} [(u_\varepsilon - \psi)^-]^{\bar{q}-1}, u_\varepsilon - u \rangle ds - E \int_0^t \langle \rho, u_\varepsilon - u \rangle ds$
 $\geq E \int_0^t \langle -\frac{1}{\varepsilon} [(u_\varepsilon - \psi)^-]^{\bar{q}-1}, \psi - u \rangle ds - E \int_0^t \langle \rho, u_\varepsilon - u \rangle ds.$

- Since I_3 is a \mathcal{F}_t -martingale then $E(I_3) = 0$.
- Thanks to H_3 we have $E(I_4) \leq ME \int_0^t \|u_\varepsilon(s) - u(s)\|_H^2 ds$.

By gathering the previous computation and taking the expectation, one has for any $t \in]0, T]$

$$\begin{aligned} & \frac{1}{2}E\|(u_\varepsilon - u)(t)\|_H^2 + E \int_0^t \langle A(v, \cdot), u_\varepsilon - v \rangle ds + E \int_0^t \langle A(u_\varepsilon, \cdot), v - u \rangle ds - E \int_0^t \langle \chi, u_\varepsilon - u \rangle ds \\ & + E \int_0^t \langle -\frac{1}{\varepsilon}[(u_\varepsilon - \psi)^-]^{\tilde{q}-1}, \psi - u \rangle ds - E \int_0^t \langle \rho, u_\varepsilon - u \rangle ds \leq (M + \lambda_T)E \int_0^t \|u_\varepsilon(s) - u(s)\|_H^2 ds. \end{aligned}$$

By passing to the limit as $\varepsilon \rightarrow 0$, thanks to Lemmas 3.17 and 3.20 and by setting $t = T$, we get

$$E \int_0^T \langle A(v, \cdot) - \chi, u - v \rangle ds \leq E \int_0^T \langle \rho, u - \psi \rangle ds = 0.$$

We are now in a position to use "Minty's trick" [66, Lemma 2.13 p.35] and deduce that $A(u, \cdot) = \chi$.

So, the conclusion of this section is: under assumption H_7 , there exists a unique $(u, \rho) \in L^p(\Omega_T, V) \times L^{\tilde{q}'}(\Omega_T, L^{\tilde{q}'}(D))$, both predictable, satisfying:

- $u \in L^2(\Omega, \mathcal{C}([0, T], H)) \cap K$ and $\rho \leq 0$.
- For any $t \in [0, T]$: $u(t) + \int_0^t (A(u, \cdot) + \rho - f) ds = u_0 + \int_0^t G(u, \cdot) dW(s)$.
- The first part of Lewy-Stampacchia inequality holds:

$$\partial_t(u - \int_0^\cdot G(u, \cdot) dW) + A(u, \cdot) - f = -\rho \geq 0 \quad \text{in } L^{\tilde{q}'}(\Omega_T \times D).$$

- $\langle \rho, u - \psi \rangle = 0$ a.e. in Ω_T and, for any $v \in K$, $\langle \rho, u - v \rangle \geq 0$ a.e. in Ω_T .

3.3.2 The second Lewy-Stampacchia inequality in the regular case

The aim of this subsection is to prove the second part of Lewy-Stampacchia inequality. For this, we used an idea inspired by [22]. Let u be the unique solution of Subsection 3.3.1 and denote by K_1 the closed convex set

$$K_1 = \{v \in L^p(\Omega_T, V), \quad v \leq u \quad \text{a.e. in } D \times \Omega_T\}.$$

We recall that u satisfies

$$(f + h^-) - \partial_t(u - \int_0^\cdot G(u, \cdot) dW) - A(u, \cdot) = h^- + \rho, \quad \rho \leq 0, \quad \rho \in L^{\tilde{q}'}(\Omega_T \times D).$$

Consider the following auxiliary problem: $(z, \nu) \in L^p(\Omega_T, V) \times L^{\tilde{q}'}(\Omega_T, L^{\tilde{q}'}(D))$, predictable, such that

$$\left\{ \begin{array}{l} \text{i.) } z \in L^2(\Omega, \mathcal{C}([0, T], H)), \quad z(0) = u_0 \quad \text{and} \quad z \in K_1, \\ \text{ii.) } \nu \geq 0, \quad \langle \nu, z - u \rangle = 0 \text{ a.e. in } \Omega_T \text{ and } \forall v \in K_1, \langle \nu, z - v \rangle \geq 0 \text{ a.e. in } \Omega_T. \\ \text{iii.) } \text{P-a.s. and for any } t \in [0, T] : \\ z(t) + \int_0^t \nu ds + \int_0^t A(z, \cdot) ds = u_0 + \int_0^t G(z, \cdot) dW(s) + \int_0^t (f + h^-) ds. \end{array} \right. \quad (3.12)$$

Note that the result of existence and uniqueness of the solution (z, ν) can be proved either by noting that $(-z, -\nu)$ is the solution to the above problem with data: $\tilde{f} = -f - h^-$, $\tilde{G}(v, \cdot) = -G(-v, \cdot)$, $\tilde{A}(v, \cdot) = -A(-v, \cdot)$, $\tilde{\psi} = -u$, $\tilde{h}^+ = -\rho$ and $\tilde{h}^- = h^-$. This can also be obtained by cosmetic changes of what has been done in Subsection 3.3.1, by passing to the limit in the following penalized problem:

$$\left\{ \begin{array}{l} z_\varepsilon(t) + \int_0^t (A(z_\varepsilon, \cdot) + \frac{1}{\varepsilon} [(z_\varepsilon - u)^+]^{\tilde{q}-1} - (f + h^-)) ds = u_0 + \int_0^t \tilde{G}(z_\varepsilon, \cdot) dW(s) \\ z_\varepsilon(0) = u_0, \end{array} \right.$$

where $\tilde{G}(z_\varepsilon, \cdot) = G(\min(z_\varepsilon, u), \cdot)$.

Moreover, $\partial_t(z - \int_0^\cdot G(z, \cdot) dW) + A(z, \cdot) - (f + h^-) = -\nu \leq 0$ in $L^{\tilde{q}'}(\Omega_T \times D)$ and z satisfies the following Lewy-Stampacchia inequality:

$$\partial_t(z - \int_0^\cdot G(z, \cdot) dW) + A(z, \cdot) - f \leq h^- \quad \text{in } L^{\tilde{q}'}(\Omega_T \times D).$$

We know already that $z \leq u$ and our aim is now to prove that $z = u$. For that, it is sufficient to prove that $z \geq \psi$. Indeed, let us assume for a moment that $z \geq \psi$, then

$$\langle \rho, u - z \rangle = \langle \rho, u - \psi \rangle + \langle \rho, \psi - z \rangle = \langle \rho, \psi - z \rangle \geq 0.$$

Thus we have $\langle \nu, z - u \rangle = 0$ and $\langle \rho - \nu, u - z \rangle = \langle \rho, u - z \rangle + \langle \nu, z - u \rangle \geq 0$.

Therefore, applying Ito's energy to

$$u(t) - z(t) + \int_0^t (\rho - \nu) ds + \int_0^t (A(u, \cdot) - A(z, \cdot)) ds = \int_0^t (G(u, \cdot) - G(z, \cdot)) dW(s) - \int_0^t h^- ds.$$

yields for any $t \in [0, T]$

$$\begin{aligned} & \frac{1}{2} \|(u - z)(t)\|_H^2 + \underbrace{\int_0^t \langle A(u, \cdot) - A(z, \cdot), u - z \rangle ds}_{(\lambda_T Id + A \text{ is T-monotone}) \geq -\lambda_T \|u - z\|_H^2} + \underbrace{\int_0^t \langle \rho - \nu, u - z \rangle ds}_{\geq 0} + \underbrace{\int_0^t \langle h^-, u - z \rangle ds}_{(u \geq z) \geq 0} \\ & = \int_0^t \langle (G(u, \cdot) - G(z, \cdot)) dW(s), u - z \rangle + \frac{1}{2} \int_0^t |G(u, \cdot) - G(z, \cdot)|_Q^2 ds. \end{aligned}$$

By similar arguments leading to (3.3), we conclude that $u = z$.

To conclude this subsection, we need to prove that $z \geq \psi$.

We know that $u \geq \psi$ so that $u - z = (u - z)^+ \geq (\psi - z)^+$ and

$$u \geq z + (\psi - z)^+ = z + (z - \psi)^-.$$

Using $v = z + (z - \psi)^- \in K_1$ in (3.12)[ii.] yields $\langle v, (z - \psi)^- \rangle \leq 0$.

We have

$$z(t) - \psi(t) + \int_0^t (v - h^+) ds + \int_0^t (A(z, \cdot) - A(\psi, \cdot)) ds = u_0 - \psi(0) + \int_0^t (G(z, \cdot) - G(\psi, \cdot)) dW(s).$$

As in the proof of Lemma 3.15, consider $\varphi_\delta(v) = \int_D F_\delta(v(x)) dx$ and $S = \{z \leq \psi\}$.

Applying Ito's formula [57, Th. 5.3 p. 78] to the process $z - \psi$, one gets for any $t \in [0, T]$

$$\begin{aligned} & \varphi_\delta(z(t) - \psi(t)) + \int_0^t \langle A(z, \cdot) - A(\psi, \cdot), F'_\delta(z - \psi) \rangle ds + \int_0^t \langle v - h^+, F'_\delta(z - \psi) \rangle ds - \overbrace{\varphi_\delta(u_0 - \psi(0))}^{=0} \\ &= \int_0^t \langle (G(z, \cdot) - G(\psi, \cdot)) dW(s), F'_\delta(z - \psi) \rangle + \frac{1}{2} \int_0^t |\sqrt{F''_\delta(z - \psi)} [G(z, \cdot) - G(\psi, \cdot)]|_Q^2 ds. \end{aligned}$$

Note that

$$\int_0^t |\sqrt{F''_\delta(z - \psi)} [G(z, \cdot) - G(\psi, \cdot)]|_Q^2 ds \leq \frac{8M}{3} \int_0^t \|z(s) - \psi(s)\|_H^2 \mathbb{I}_S ds = \frac{8M}{3} \int_0^t \|(z - \psi)^-(s)\|_H^2 ds.$$

Taking the expectation and passing to the limit when $\delta \rightarrow 0$,

- $\forall t \in [0, T], \quad E\varphi_\delta(z(t) - \psi(t)) \longrightarrow E\|(z - \psi)^-(t)\|_{L^2(D)}^2,$
- $E \int_0^t \langle A(z, \cdot) - A(\psi, \cdot), F'_\delta(z - \psi) \rangle ds \longrightarrow E \int_0^t \langle A(z, \cdot) - A(\psi, \cdot), -2(z - \psi)^- \rangle ds$
 $= 2E \int_0^t \langle A(\psi, \cdot) - A(z, \cdot), (\psi - z)^+ \rangle ds$
 $\geq -2\lambda_T E \int_0^t \|(\psi - z)^+\|_H^2 ds,$
- $E \int_0^t \langle v - h^+, F'_\delta(z - \psi) \rangle ds \rightarrow -2E \int_0^t \langle v - h^+, (z - \psi)^- \rangle ds$
 $= 2(E \int_0^t \langle h^+, (z - \psi)^- \rangle + \underbrace{\langle v, -(z - \psi)^- \rangle ds}_{\text{(thanks (3.12)[ii.])} \geq 0} \geq 0.$

Those limits may be obtained by Lebesgue's theorem and, for any $t \in [0, T]$:

$$E\|(z - \psi)^-(t)\|_H^2 \leq C \int_0^t E\|(z - \psi)^-(s)\|_H^2 ds.$$

Finally, Grönwall's lemma ensures that $\psi \leq z$, and, as conclusion of this subsection, we get $z = u$. Hence, u satisfies the second part of Lewy-Stampacchia inequality:

$$\partial_t(u - \int_0^\cdot G(u, \cdot) dW) + A(u, \cdot) - f \leq h^- \quad \text{in } L^{\tilde{q}'}(\Omega_T \times D).$$

From Subsection 3.3.1 and Subsection 3.3.2, we deduce the following theorem.

Theorem 3.21. *Under Assumptions (H₁)-(H₆) and assuming moreover that $h^- \in L^{\tilde{q}'}(\Omega_T, L^{\tilde{q}'}(D))$ is predictable, there exists a unique predictable stochastic process $(u, k) \in L^p(\Omega_T, V) \times L^{\tilde{q}'}(\Omega_T, L^{\tilde{q}'}(D))$ such that:*

i. $u \in L^2(\Omega, \mathcal{C}([0, T], H)) \cap K$, $u(0) = u_0$.

ii. $k \leq 0$ and $\forall v \in K$, $\langle k, u - v \rangle \geq 0$ a.e. in Ω_T .

iii. *P*-a.s, for all $t \in [0, T]$,

$$u(t) + \int_0^t k ds + \int_0^t A(u, \cdot) ds = u_0 + \int_0^t G(u, \cdot) dW(s) + \int_0^t f ds.$$

iv. The following Lewy-Stampacchia inequality holds:

$$0 \leq \partial_t(u - \int_0^\cdot G(u, \cdot) dW) + A(u, \cdot) - f \leq h^- = \left(f - \partial_t(\psi - \int_0^\cdot G(\psi, \cdot) dW) - A(\psi, \cdot) \right)^-.$$

3.3.3 Proof of the main theorem in the general case

First, we prove the following lemma which allows us to pass from the regular to the general case.

Density result in the positive cone of the dual

Lemma 3.22. *The positive cone of $L^p(\Omega_T, V) \cap L^2(\Omega_T, L^2(D))$ is dense in the positive cone of $L^{p'}(\Omega_T, V')$. Moreover, the positive cone of predictable elements of $L^p(\Omega_T, V) \cap L^2(\Omega_T, L^2(D))$ is dense in the positive cone of predictable elements of $L^{p'}(\Omega_T, V')$.*

By a truncation argument, the same result holds for the positive cone of $L^p(\Omega_T, V) \cap L^{p'}(\Omega_T, L^{p'}(D))$ (resp. predictable).

Proof. Since the proof of the lemma is mainly based on monotone arguments, it is similar to the one proposed in [33, Lemma 4.1] (see Appendix B.4) where one has just to add the predictable assumption to the spaces of type $L^r(0, T, X)$ in [33, Lemma 4.1] if needed.

□

Proof of Theorem 3.10

Let $h^- \in (L^{p'}(\Omega_T, V'))^+$ predictable. Thanks to Lemma 3.22, there exists $h_n \in L^{\tilde{q}'}(\Omega_T, L^{\tilde{q}'}(D))$ predictable and non negative such that

$$h_n \longrightarrow h^- \quad \text{in } L^{p'}(\Omega_T, V').$$

Associated with h_n , denote the following f_n by,

$$f_n = \partial_t(\psi - \int_0^\cdot G(\psi, \cdot) dW) + A(\psi, \cdot) + h^+ - h_n, \quad h^+ \in (L^{p'}(\Omega_T, V'))^+ \text{ predictable too.}$$

Note that $f_n \in L^{p'}(\Omega_T, V')$ is predictable and f_n converges strongly to f in $L^{p'}(\Omega_T, V')$.

Denote by (u_n, k_n) the sequence of solutions given by Theorem 3.21 where h^- is replaced by h_n .

By Lewy-Stampacchia inequality, one has $0 \leq -k_n \leq h_n$.

For any $\varphi \in L^p(\Omega_T, V)$, it holds that

$$\begin{aligned} E \int_0^T |\langle k_n, \varphi \rangle| ds &\leq E \int_0^T \langle -k_n, \varphi^+ \rangle ds + E \int_0^T \langle -k_n, \varphi^- \rangle ds \\ &\leq E \int_0^T \langle h_n, \varphi^+ \rangle ds + E \int_0^T \langle h_n, \varphi^- \rangle ds \leq 2 \|h_n\|_{L^{p'}(\Omega_T, V')} \|\varphi\|_{L^p(\Omega_T, V)}. \end{aligned}$$

Since $(h_n)_n$ converges to h in $L^{p'}(\Omega_T, V')$, one gets that $(h_n)_n$ is bounded independently of n in $L^{p'}(\Omega_T, V')$ and therefore $(k_n)_n$ is bounded independently of n in $L^{p'}(\Omega_T, V')$.

Let $n \in \mathbb{N}^*$ and applying Ito's energy formula to the process u_n , one gets for any $t \in [0, T]$

$$\begin{aligned} \frac{1}{2} \|u_n(t)\|_H^2 + \int_0^t \langle A(u_n, \cdot), u_n \rangle ds &= \frac{1}{2} \|u_0\|_H^2 + \int_0^t \langle -k_n, u_n \rangle ds + \int_0^t \langle f_n, u_n \rangle ds \\ &\quad + \int_0^t \langle G(u_n, \cdot) dW(s), u_n \rangle + \frac{1}{2} \int_0^t |G(u_n, \cdot)|_Q^2 ds. \end{aligned}$$

Since f_n converges to f in $L^{p'}(\Omega_T, V')$, it holds that $(f_n)_n$ is bounded independently of n in $L^{p'}(\Omega_T, V')$. Therefore, by Young's inequality, we get

$$E \int_0^T |\langle f_n - k_n, u_n \rangle| ds \leq \frac{\alpha}{2} E \int_0^T \|u_n(s)\|_V^p ds + C \|f_n - k_n\|_{L^{p'}(\Omega_T, V')}^{p'}.$$

By Burkholder-Davis-Gundy's inequality and Young's inequality, there exists $\delta > 0$ such that

$$E \left[\sup_{t \in [0, T]} \left| \int_0^t \langle G(u_n, \cdot) dW(s), u_n \rangle \right| \right] \leq \frac{3\delta}{2} E \sup_{t \in [0, T]} \|u_n(t)\|_H^2 + \frac{3M}{2\delta} E \int_0^T \|u_n(s)\|_H^2 ds + \frac{3}{2\delta} \|l\|_{L^1(\Omega_T)}.$$

With a convenient choice of δ (e.g. $\delta = \frac{1}{4}$) and using $H_{2,1}, H_{3,2}$, one deduces

$$E \sup_{t \in [0, T]} \|u_n(t)\|_H^2 + E \int_0^T \|u_n(s)\|_V^p ds \leq C(1 + E \int_0^T \sup_{\tau \in [0, s]} \|u_n(\tau)\|_H^2 ds).$$

By using Grönwall's lemma, one concludes that $(u_n)_n$ is bounded in $L^p(\Omega_T, V) \cap L^2(\Omega, L^\infty(0, T, H))$.

Now, we present the following lemma about the strong convergence of $(u_n)_n$.

Lemma 3.23. *$(u_n)_n$ is a Cauchy sequence in the space $L^2(\Omega, \mathcal{C}([0, T], H))$.*

Proof. Let $m, n \in \mathbb{N}^*$ and $\varepsilon > 0$. For any $t \in [0, T]$ and P-a.s, we have

$$\begin{aligned} & u_n(t) - u_m(t) + \int_0^t (k_n - k_m) ds + \int_0^t (A(u_n, \cdot) - A(u_m, \cdot)) ds \\ &= \int_0^t (G(u_n, \cdot) - G(u_m, \cdot)) dW(s) + \int_0^t (f_n - f_m) ds. \end{aligned}$$

Applying Ito's energy formula, one gets for any $t \in [0, T]$,

$$\begin{aligned} & \frac{1}{2} \|(u_n - u_m)(t)\|_H^2 + \int_0^t \langle A(u_n, \cdot) - A(u_m, \cdot), u_n - u_m \rangle ds \\ &= - \int_0^t \langle k_n - k_m, u_n - u_m \rangle ds + \int_0^t \langle f_n - f_m, u_n - u_m \rangle ds \\ & \quad + \int_0^t \langle (G(u_n, \cdot) - G(u_m, \cdot)) dW(s), u_n - u_m \rangle + \frac{1}{2} \int_0^t |G(u_n, \cdot) - G(u_m, \cdot)|_Q^2 ds. \end{aligned}$$

Similarly to the proof of Lemma 3.12, one deduces

$$E \sup_{t \in [0, T]} \|(u_n - u_m)(t)\|_H^2 \leq C \|f_n - f_m\|_{L^{p'}(\Omega_T, V')} \|u_n - u_m\|_{L^p(\Omega_T, V)}.$$

Since f_n converges strongly to f in $L^{p'}(\Omega_T, V')$ and $(u_n)_n$ is bounded in $L^p(\Omega_T, V)$, it holds that

$$E \int_0^T \langle f_n - f_m, u_n - u_m \rangle ds \leq \|f_n - f_m\|_{L^{p'}(\Omega_T, V')} \|u_n - u_m\|_{L^p(\Omega_T, V)} \leq C\varepsilon,$$

for big values of n and m . Therefore $(u_n)_n$ is a Cauchy sequence in the space $L^2(\Omega, \mathcal{C}([0, T], H))$.

□

Since $(u_n)_n$ is bounded sequence in $L^p(\Omega_T, V)$ of predictable processes, Rmk. 3.3 and $H_{2,3}$ yield that $(A(u_n, \cdot))_n$ is a bounded sequence in $L^{p'}(\Omega_T, V')$ of predictable processes.

By compactness with respect to the weak topology in the spaces $L^p(\Omega_T, V)$ and $L^{p'}(\Omega_T, V')$, there exist $u \in L^p(\Omega_T, V)$, $\chi \in L^{p'}(\Omega_T, V')$ and $k \in L^{p'}(\Omega_T, V')$, each one being predictable, such that (up to sub-sequences denoted by the same way)

$$u_n \rightharpoonup u \quad \text{in } L^p(\Omega_T, V), \quad (3.13)$$

$$A(u_n, \cdot) \rightharpoonup \chi \quad \text{in } L^{p'}(\Omega_T, V'), \quad (3.14)$$

$$k_n \rightharpoonup k \quad \text{in } L^{p'}(\Omega_T, V'). \quad (3.15)$$

Thanks to Lemma 3.23, we have the strong convergence of u_n to u in $L^2(\Omega, \mathcal{C}([0, T], H))$ thus in $L^2(\Omega_T, L^2(D))$ and $u \in \mathcal{N}_W^2(0, T, H)$.

Since $(-k_n) \in (L^{p'}(\Omega_T, V'))^+$ and $-k_n \rightharpoonup -k$ in $L^{p'}(\Omega_T, V')$, we deduce that $-k \in (L^{p'}(\Omega_T, V'))^+$. Indeed, let $\varphi \in L^p(\Omega_T, V)$, $\varphi \geq 0$ then

$$E \int_0^T \langle -k, \varphi \rangle ds = \lim_{n \rightarrow \infty} E \int_0^T \langle -k_n, \varphi \rangle ds \geq 0.$$

Remark 3.24. (initial condition and constraint).

- Since u_n converges to u in $L^2(\Omega, \mathcal{C}([0, T], H))$ with $u_n(0) = u_0$, one has that $u(0) = u_0$.
- Since K is a closed convex subset of $L^p(\Omega_T, V)$, it holds that $u \in K$.

Similarly to the proof of Lemma 3.19, one gets

$$\int_0^\cdot G(u_n, \cdot) dW(s) \rightarrow \int_0^\cdot G(u, \cdot) dW(s) \quad \text{in } L^2(\Omega, \mathcal{C}([0, T], H)) \quad \text{when } n \rightarrow \infty.$$

So, at the limit, we have a.s. and for any $t \in [0, T]$

$$u(t) + \int_0^t k ds + \int_0^t \chi ds = u_0 + \int_0^t G(u, \cdot) dW(s) + \int_0^t f ds.$$

For any $n \in \mathbb{N}^*$, we have a.s. and for any $t \in [0, T]$

$$u_n(t) + \int_0^t k_n ds + \int_0^t A(u_n, \cdot) ds = u_0 + \int_0^t G(u_n, \cdot) dW(s) + \int_0^t f_n ds.$$

Note that $(A(u_n, \cdot) - \chi) + (k_n - k) + (f_n - f) \in L^{p'}(\Omega_T, V')$ is predictable and $\int_0^t (G(u_n, \cdot) - G(u, \cdot)) dW(s)$ is a square integrable \mathcal{F}_t -martingale. We can apply Ito's formula [57, Theorem 4.2 p. 65] to the process $u_n - u$ with $F(v) = \frac{1}{2} \|v\|_H^2$ to get

$$\begin{aligned} & \frac{1}{2} \|(u_n - u)(t)\|_H^2 + \int_0^t \langle A(u_n, \cdot) - \chi, u_n - u \rangle ds + \int_0^t \langle k_n - k, u_n - u \rangle ds \\ &= \underbrace{\int_0^t \langle (G(u_n, \cdot) - G(u, \cdot)) dW(s), u_n - u \rangle}_{I_1(t)} + \underbrace{\frac{1}{2} \int_0^t |G(u_n, \cdot) - G(u, \cdot)|_Q^2 ds}_{I_2(t)} + \underbrace{\int_0^t \langle f_n - f, u_n - u \rangle ds}_{I_3(t)}. \end{aligned}$$

Thanks to Lemma 3.20, $\langle k_n, u_n - \psi \rangle = 0$ and one has

$$\begin{aligned} \langle k_n - k, u_n - u \rangle &= \langle k_n, u_n - u \rangle - \langle k, u_n - u \rangle \\ &= \langle k_n, u_n - \psi \rangle + \langle k_n, \psi - u \rangle - \langle k, u_n - u \rangle = \langle k_n, \psi - u \rangle - \langle k, u_n - u \rangle. \end{aligned}$$

Therefore

$$\begin{aligned} E \int_0^T \langle k_n - k, u_n - u \rangle ds &= E \int_0^T \langle k_n, \psi - u \rangle ds - \overbrace{E \int_0^T \langle k, u_n - u \rangle ds}^{\rightarrow 0} \\ &\longrightarrow E \int_0^T \langle k, \psi - u \rangle ds \geq 0. \end{aligned}$$

Since f_n converges strongly to f in $L^{p'}(\Omega_T, V')$, (3.13) ensures that $E(I_3(T)) \rightarrow 0$.

Similarly to the last part of Subsection 3.3.1, one has: $E(I_1(t)) = 0$, $E(I_2(T)) \rightarrow 0$ and, for any $v \in L^p(\Omega_T, V) \cap L^2(\Omega, \mathcal{C}([0, T], H))$,

$$E \int_0^T \langle k, \psi - u \rangle ds + E \int_0^T \langle A(v, \cdot) - \chi, u - v \rangle ds \leq 0. \quad (3.16)$$

By setting $v = u$ in (3.16), one has $E \int_0^T \langle k, \psi - u \rangle ds \leq 0$. Therefore

$$E \int_0^T \langle k, \psi - u \rangle ds = 0.$$

Since $-k \in (L^{p'}(\Omega_T, V'))^+$, $-k(t, \omega) \in (V')^+$ a.e. in Ω_T . Hence, $\langle k(s, \omega), \psi - u \rangle \geq 0$ and $\langle k(s, \omega), \psi - u \rangle = 0$ a.e. in Ω_T .

By (3.16), we get $E \int_0^T \langle A(v, \cdot) - \chi, u - v \rangle ds \leq 0$, then, using "Minty trick" one concludes that $\chi = A(u, \cdot)$.

Let $v \in K$, then a.e. Ω_T , we have $\langle k, u - v \rangle = \langle k, u - \psi \rangle + \langle k, \psi - v \rangle \geq 0$.

We deduce the existence result of Theorem 3.10 for general f . At last, Lewy-Stampacchia inequality is a consequence of the passage to the limit in the one satisfied by u_n .

Work in progress & Outlooks

I-A Lewy Stampacchia inequality for a pseudomonotone parabolic problem with measure data.

As an extension of the result presented in the second chapter, an interesting question is to study the same problem with non regular data (measure and L^1 -data). In a joint work with O. Guibé and G. Vallet, our aim is to investigate the Lewy-Stampacchia inequality associated with the following problem

$$\begin{cases} \frac{\partial u}{\partial t} - \operatorname{div}[a(t, x, u, \nabla u)] - f = \mu & \text{in } Q, \\ u(t=0) = u_0 & \text{in } D, \\ u \geq \psi & \text{in } Q, \\ u(t, x) = 0 & \text{on } \partial D \times [0, T], \\ \mu(u - \psi) = 0 \text{ and } \mu \geq 0 & \text{in } Q, \end{cases}$$

where D is a Lipschitz bounded domain of $\mathbb{R}^d, d \geq 2$ with Lipschitz boundary ∂D , $1 < p < d, T$ a positive real number and denote $Q =]0, T[\times D$. $u \mapsto -\operatorname{div}[a(t, x, u, \nabla u)]$ is a pseudomonotone operator.

Note that the non regularity of the data leads us to look for an appropriate formulation to define the solution of the problem. Following the same arguments as in [64], an entropy formulation seems to be an adequate one. The idea is to regularize the data and use the result of the second chapter, then get some *a priori* estimates and passing to the limit. It's worth noting that a density argument is needed to pass to the limit.

II-A Lewy Stampacchia inequality for a stochastic conservation law.

In a joint work with I. Biswas and G. Vallet, we would like to study a stochastic non-linear first order conservation law under an obstacle condition associated with a Lewy-Stampacchia inequality. *i.e.*, given an obstacle ψ , one looks for a $L^2(\mathbb{R})$ -valued predictable process u such that $0 \leq u \leq \psi$ and satisfying formally the problem.

$$\begin{cases} du(t, x) - \operatorname{div} f(t, x, u(t, x)) dt = g(t, x, u(t, x)) dt + G(t, u(t, x)) dW(t), & x \in Q_T, \\ u(0, x) = u_0(x), & x \in \mathbb{R}^d, \end{cases}$$

where $Q_T = \mathbb{R}^d \times (0, T)$ with $T > 0$ fixed, $u_0(x)$ is the given initial function satisfying the constraint.

The idea is to use the result of the third chapter to show the existence and uniqueness of a viscous solution and study the regularity and boundedness results of such a solution. Then, using the compactness argument of Young measure theory to establish the existence of Young measure valued process. The last step is to compare between the viscous solution and the Young measure valued process to obtain the uniqueness by using Kato's inequality.

Appendices

Appendix A: Examples of numerical illustrations

In this part, we propose some numerical illustrations of the solution of the obstacle problem (A.1) and, at the same time, we compare them to the numerical solution of the free problem *i.e* the stochastic heat equation when the constraint $u \geq 0$ is ignored. To the best of the author's knowledge, there doesn't exist in the literature numerical studies of stochastic obstacle problems.

Consider the following problem:

$$\left\{ \begin{array}{ll} du - \alpha u_{xx} ds + k ds = f ds + \sigma u dW & \text{in }]0, 1[\times \Omega \times]0, 1[, \\ u(t=0) = u_0 \geq 0 & \text{in } L^2(0, 1), \text{ a.s.}, \\ u \geq 0 & \text{in } [0, 1] \times \Omega \times [0, 1], \\ u(0, t) = u(1, t) = 0 & \text{on } \Omega \times [0, 1], \\ \langle k, u \rangle = 0 \text{ and } k \leq 0 & \text{in } \Omega \times [0, 1] \end{array} \right. \quad (\text{A.1})$$

where $\alpha > 0, \sigma \in \mathbb{R}$ and f is a smooth function. By [61, Thm 5] or the above Theorem 3.10, there exists a unique solution (u, k) to Problem (A.1) in the sense of Definition 3.7 with $p = 2$ and $D =]0, 1[$. Moreover, the following Lewy-Stampacchia inequality holds

$$0 \leq \partial_t \left(u - \sigma \int_0^\cdot u dW \right) - \alpha u_{xx} - f \leq h^- = f^-.$$

Note that thanks to Remark 3.6, this basic situation of a constraint of positivity with a vanishing stochastic reaction term at 0 can be an illustration of a more general situation.

Inspired by Chapter 3, our aim is to present some numerical illustrations of the stochastic obstacle problem (A.1) *via* a penalty method, i.e. an approximation by the family $(P_\varepsilon)_{\varepsilon>0}$ of penalized problems:

$$P_\varepsilon : \begin{cases} u^\varepsilon(t) - \int_0^t (\alpha u_{xx}^\varepsilon + \frac{1}{\varepsilon} [(u^\varepsilon)^-] + f) ds = u_0 + \sigma \int_0^t u^\varepsilon(s) dW(s) \\ u^\varepsilon(0) = u_0, \\ u^\varepsilon(0, t) = u^\varepsilon(1, t) = 0 \quad \text{on } \Omega \times [0, 1]. \end{cases} \quad (\text{A.2})$$

For that, one needs a suitable choice of the small parameter ε compatible with the space and time discretization steps.

Let us denote by $\Delta t = \frac{1}{N}$ the time step of the uniform discretization of the time-interval $[0, 1]$, $\{t_0, \dots, t_N\}$ are the points of this discretization. Similarly, $\Delta x = \frac{1}{M}$ is the uniform space step discretization of the space-interval $[0, 1]$ and $\{x_0, \dots, x_M\}$ are the points of the space discretization. Then, following what is usually done in the deterministic case (see e.g. [67]), one sets $\varepsilon = \Delta t = (\Delta x)^2$ to ensure the convergence of the scheme (A.3) below to the solution of (A.1).

Denote by U_j^i the approximate solutions at time t_i , computed at x_j when U_0 is given by the initial condition, *via* $U_0 = \{u_0(x_1), \dots, u_0(x_{M-1})\}$.

We consider the following approximate discretized problem obtained *via* a penalty method, a stochastic "Saul'yev scheme" (see [30]) *i.e*

$$\begin{cases} U_i^j = \frac{1}{1+\beta} (\beta U_{i-1}^{j+1} + (1-\beta) U_{i-1}^j + \beta U_i^{j-1}) + \frac{\sigma U_{i-1}^j}{1+\beta} (W(t_i) - W(t_{i-1})) + \frac{\Delta t}{1+\beta} f(t_{i-1}, x_j) \\ \quad + \frac{\Delta t}{\varepsilon(1+\beta)} (U_{i-1}^j)^-, \quad 1 \leq j \leq M-1, \quad 1 \leq i \leq N, \\ U_0^j = u_0(x_j), \quad 1 \leq j \leq M-1 \\ U_i^0 = U_i^{M+1} = 0, \quad 0 \leq i \leq N, \end{cases} \quad (\text{A.3})$$

where $\beta = \alpha \frac{\Delta t}{(\Delta x)^2}$.

We wish to draw the reader's attention that all the numerical simulations are implemented with the free software *Scilab*.

Example 1 (see Figure 3.1 and 3.2)

- This example is implemented with the following data:

$$u_0(x) = \sin(\pi x), \alpha = 1, f(x, t, \omega) = 0, \sigma = 2, N = 900 \text{ and } M = 30.$$

Example 2 (see Figure 3.3, 3.4 and 3.5)

- This example is implemented with the following data:

$$u_0(x) = \sin(\pi x), \alpha = 1, f(x, t, \omega) = 3 \cos(4\pi t), \sigma = 2, N = 900 \text{ and } M = 30.$$

Comments on the numerical examples

- Figure 3.1 is devoted to the numerical illustration of the first example, we present the deterministic problem; free (without the penalization-term) and obstacle in the Figures 3.1a and 3.1b resp., a sample path for the associated stochastic problems is plotting in the Figures 3.1c and 3.1d. Then we plot in the Figures 3.1e and 3.1f the mean of the stochastic problems; free and obstacle, by using Monte Carlo method with 2000 sample paths. Finally, we chose two points, one close to the boundary $x = 0.1$ (Figure 3.1h) and the second in the middle $x = 0.5$ (Figure 3.1g) to present the trajectories corresponding to the previous figures. Then, to be more precise, we plot the trajectories of the free problem in full line and the obstacle problem in red dotted-line at the points $x = 0.1$ and $x = 0.5$, separately as shown in Figure 3.2

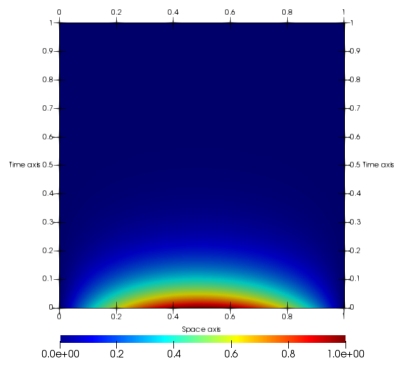
One can see that, as expected, the solution of the free and obstacle problems coincide and both of the solutions are positive (satisfy the constraint), this fits perfectly with the theoretical results, since the maximum principle is fulfilled in the first example.

- Figure 3.3 is devoted to the numerical illustration of the second example, we present the deterministic problem; free and obstacle (without the penalization-term) in the Figures 3.3a and 3.3b resp., a sample path for the associated stochastic problems is plotted in the Figures 3.3c and 3.3d. Then we plot in the Figures 3.3e and 3.3f the mean of the stochastic problems with 5000 sample paths; free and obstacle. Finally, we chose two points, one close to the boundary $x = 0.1$ (Figure 3.3h) and the second in the middle $x = 0.5$ (Figure 3.3g) to present the trajectories corresponding to the previous figures. Then, to be more precise, we plot the trajectories of the free problem in black-line and the obstacle problem in red-line at the points $x = 0.1$ and $x = 0.5$, separately as shown in Figure 3.4

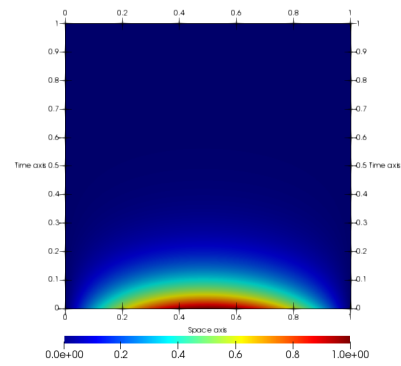
One can see that, as expected, the trajectories of the free and obstacle problems are the same before the first time-contact with the obstacle. When the constraint is active for Problem (A.3), the solution is equal to the constraint 0, else it is positive.

- In the Figure 3.5, we present the simulation of the deterministic problem (*i.e.* when $\sigma = 0$) in black-line and the mean of 5000 trajectories of the stochastic problem in red-dotted-line. Figures 3.5a and 3.5b represent the simulations of the problem without constraint and Figures 3.5c and 3.5d are concerned by Problem (A.3).

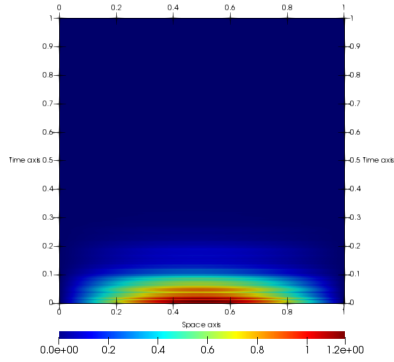
As expected for the linear heat equation (case of Figure 3.5a and 3.5b), the mean of the stochastic paths coincides with the solution to the deterministic problem. The situation is slithy different for the problem with constraint. Indeed, even if the constraint is deterministic, the penalization, and the Lagrange multiplier at the limit, induces a non linear term. Thus, the mean and the deterministic solution may differ.



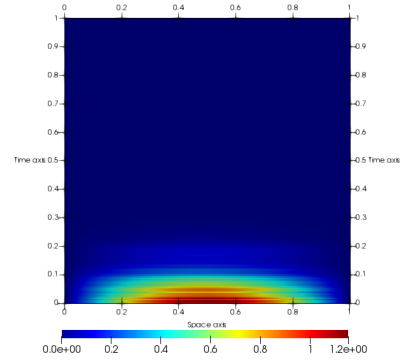
(a) The deterministic free problem.



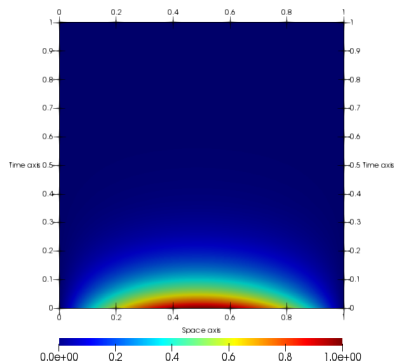
(b) The deterministic obstacle problem.



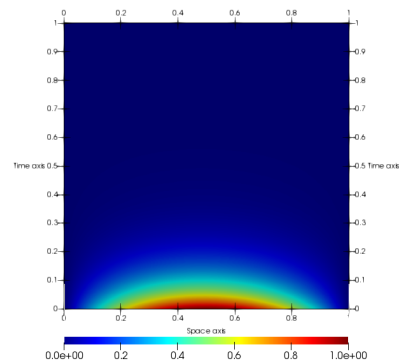
(c) One sample path of the stochastic free problem.



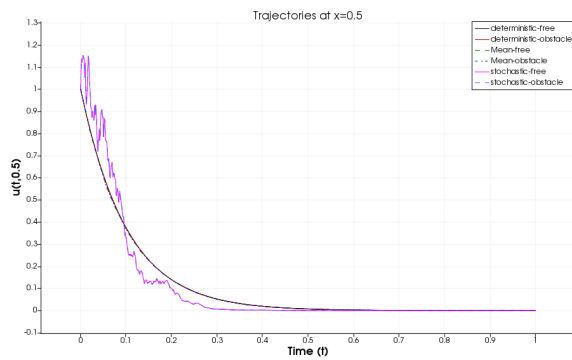
(d) One sample path of the stochastic obstacle problem.



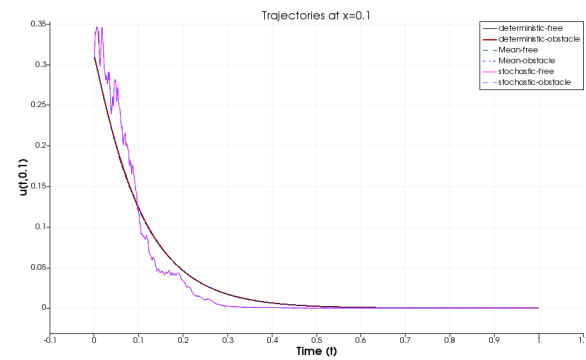
(e) The mean of the stochastic free problem with 2000 trajectories.



(f) The mean of the stochastic obstacle problem with 2000 trajectories.

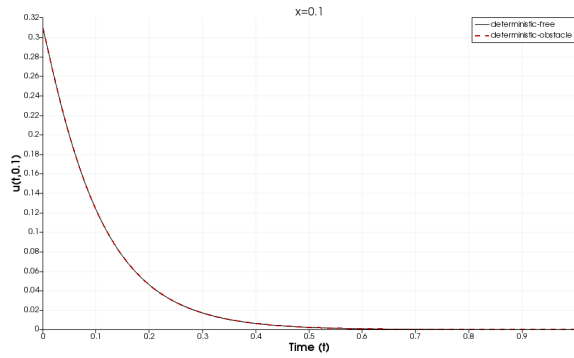


(g) Pathwise trajectory at $x=0.5$.

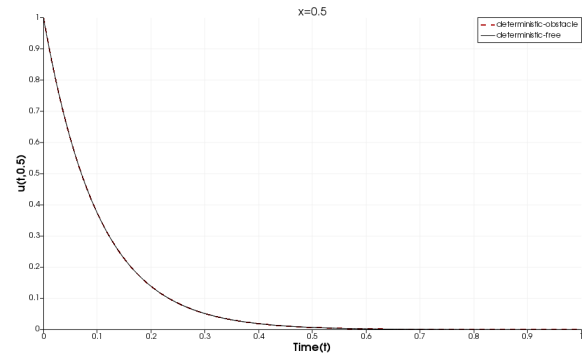


(h) Pathwise trajectory at $x=0.1$.

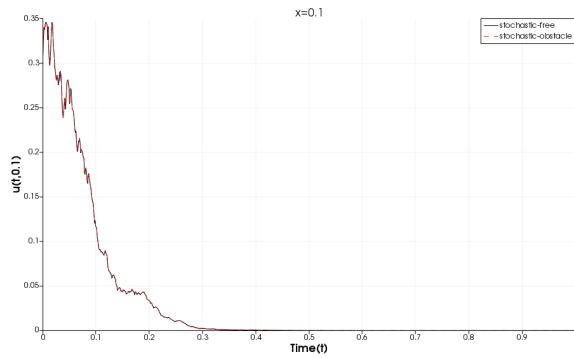
Figure 3.1: A numerical illustration of the 1st example



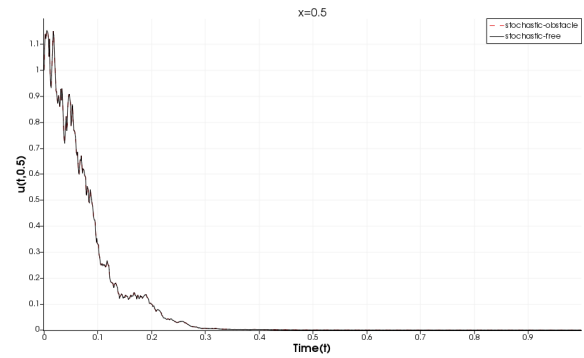
(a) The deterministic free and obstacle problem at $x=0.1$



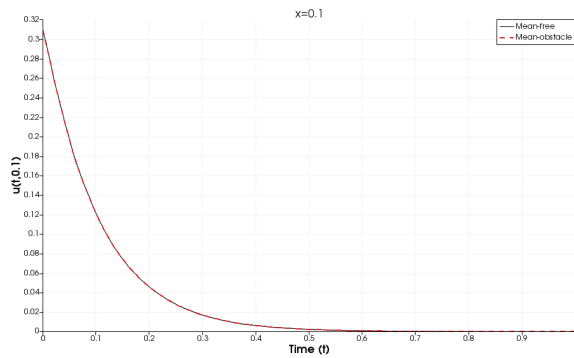
(b) The deterministic free and obstacle problem at $x=0.5$



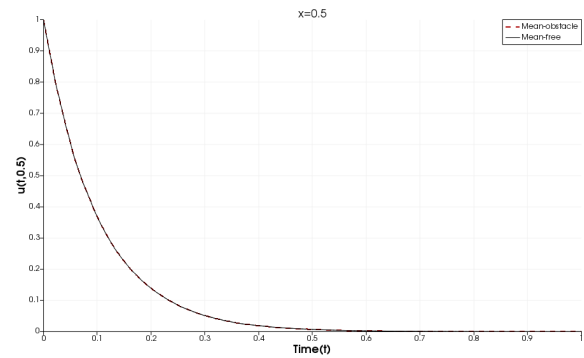
(c) Pathwise of stochastic free and obstacle problem at $x=0.1$



(d) Pathwise of stochastic free and obstacle problem at $x=0.5$

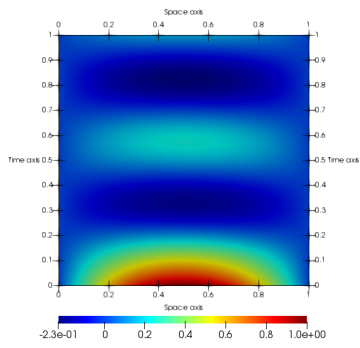


(e) The mean of free and obstacle problem at $x=0.1$

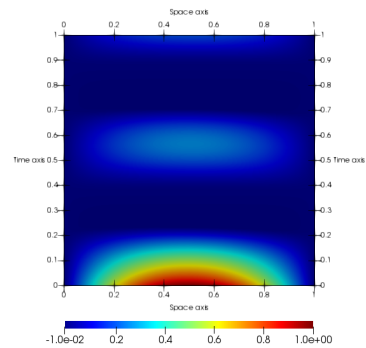


(f) The mean of free and obstacle problem at $x=0.5$

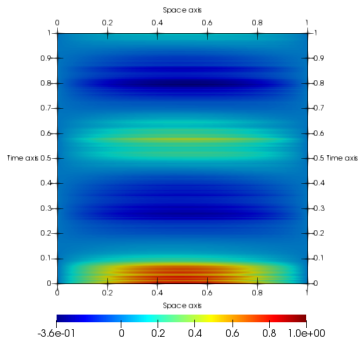
Figure 3.2: Pathwise of deterministic and stochastic problems at $x=0.1$, $x=0.5$ of the 1st example.



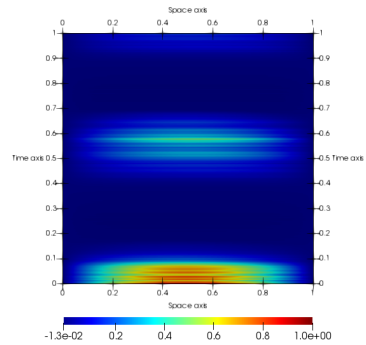
(a) The deterministic free problem.



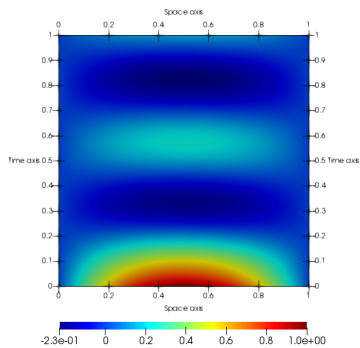
(b) The deterministic obstacle problem.



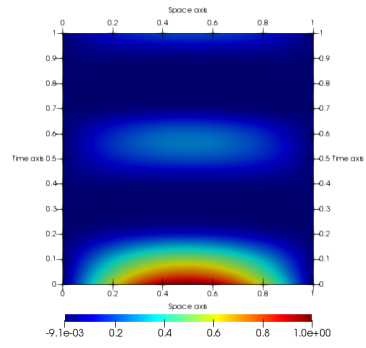
(c) One sample path of the stochastic free problem.



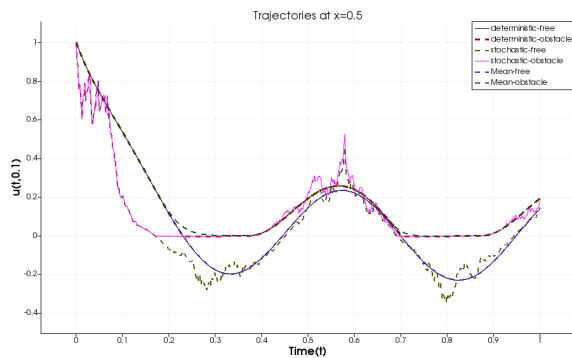
(d) One sample path of the stochastic obstacle problem.



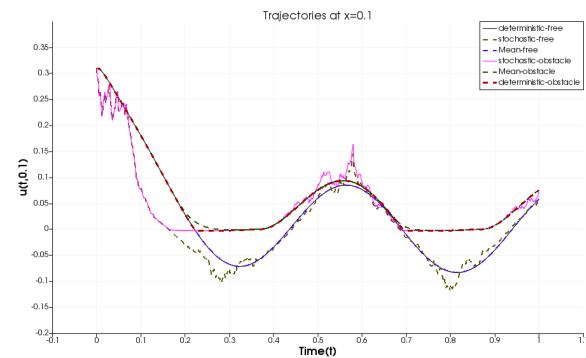
(e) The mean of the stochastic free problems with 5000 trajectories.



(f) The mean of the stochastic obstacle problems with 5000 trajectories.

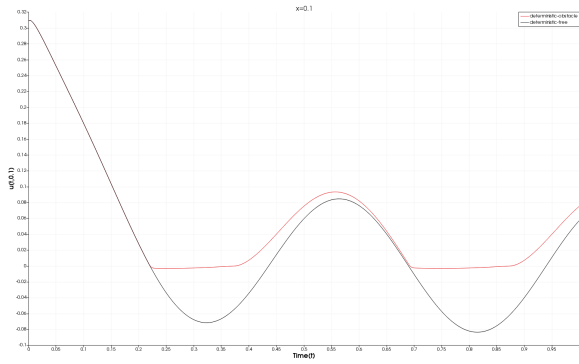


(g) Pathwise trajectory at $x=0.5$.

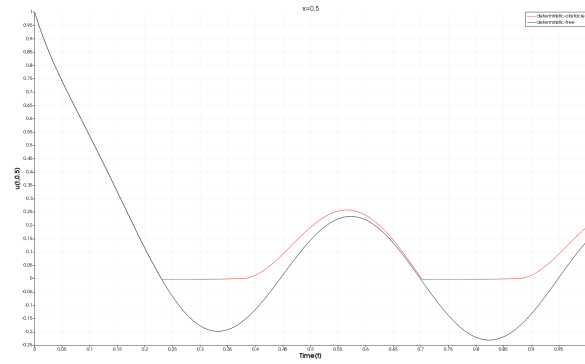


(h) Pathwise trajectory at $x=0.1$.

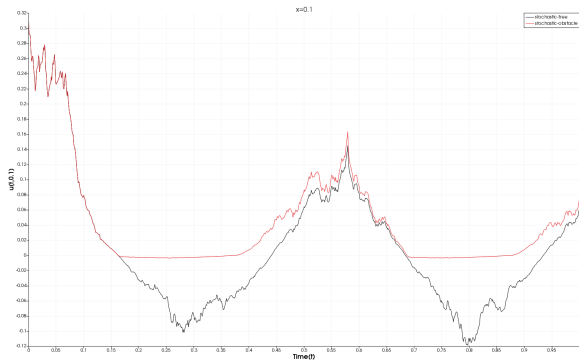
Figure 3.3: A numerical illustration of the 2nd example.



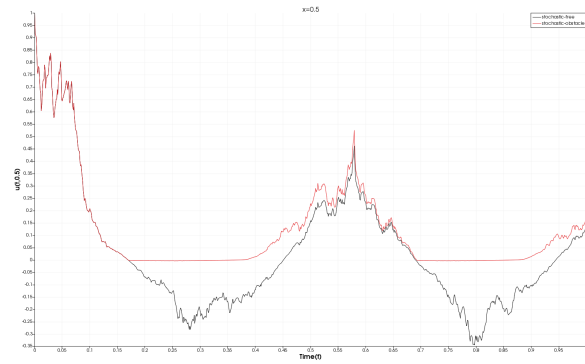
(a) The deterministic free and obstacle problem at $x=0.1$



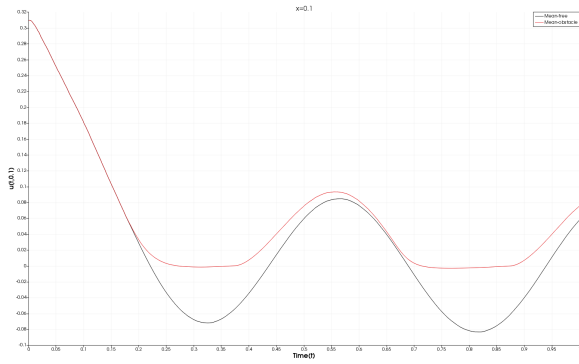
(b) The deterministic free and obstacle problem at $x=0.5$



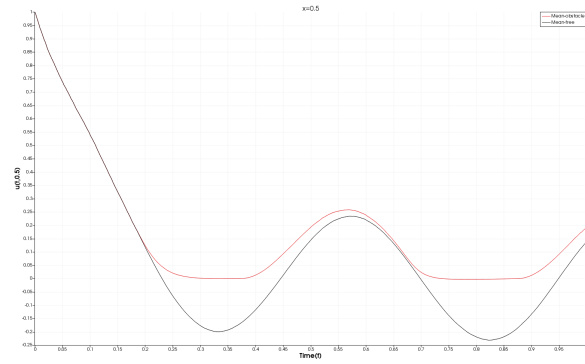
(c) Pathwise of stochastic free and obstacle problem at $x=0.1$



(d) Pathwise of stochastic free and obstacle problem at $x=0.5$

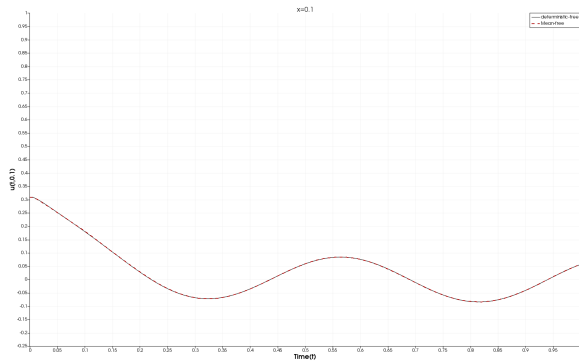


(e) The mean of free and obstacle problem at $x=0.1$

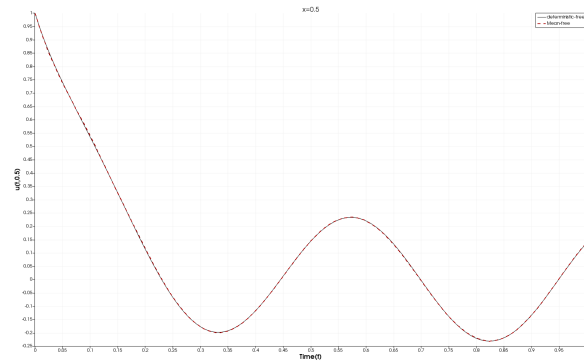


(f) The mean of free and obstacle problem at $x=0.5$

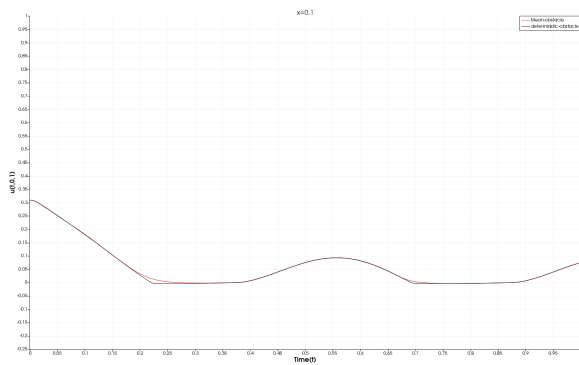
Figure 3.4: Pathwise of deterministic and stochastic problems at $x=0.1$, $x=0.5$ of the 2^{nd} example.



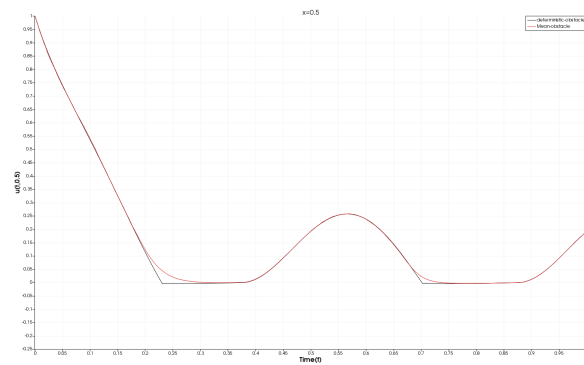
(a) The mean and deterministic solution the free problem at $x=0.1$



(b) The mean and deterministic solution of the free problem at $x=0.5$



(c) The mean and deterministic solution of the obstacle problem at $x=0.1$



(d) The mean and deterministic solution of the obstacle problem at $x=0.5$

Figure 3.5: The deterministic and the mean stochastic problems with 5000 paths at $x=0.1$, $x=0.5$ of the 2^{nd} example.

Appendix B: Some results on functional analysis

1. Strong continuity in $L^2(D)$

Proof of Lemma 2.1

We consider the following notations in the sequel: $V(D) = W^{1,p}(D) \cap L^2(D)$, $V_0(D) = W_0^{1,p}(D) \cap L^2(D)$ and $V'(D) = W^{-1,p'}(D) + L^2(D)$.

Remark 3.25. *This result is not the usual one since u and $\partial_t u$ are not in spaces being in duality relation and few words are needed concerning the time-derivative. Note that both $V(D)$ and $V_0(D)$ are dense subspaces of the chosen pivot space $L_2(D)$ so that it can be identify to a subspace of $V'(D)$ or $(V(D))'$. Therefore, u , as an element of $L^p(0, T; V(D)) \hookrightarrow L^p(0, T; L^2(D))$, has a time derivative in the sense of $\mathcal{D}'(0, T; L^2(D)) \hookrightarrow \mathcal{D}'(0, T; V'(D))$ and it is assumed to belong to $L^p(0, T; V'(D))$.*

Proof.

1. For $D = \mathbb{R}^N$, we have $W_0^{1,p}(\mathbb{R}^N) = W^{1,p}(\mathbb{R}^N)$, therefore we can identify $V'(\mathbb{R}^N)$ with the dual of $V(\mathbb{R}^N)$. By considering the triple $V(\mathbb{R}^N) \xrightarrow{d} L^2(\mathbb{R}^N) \xrightarrow{d} V'(\mathbb{R}^N)$, thanks to [70] (Prop. 1.2 p. 106), one has $u \in \mathcal{C}([0, T], L^2(\mathbb{R}^N))$.
2. $D = \mathbb{R}_+^N = \{(x', x_N) \in \mathbb{R}^N; x_N > 0\}$ and $\mathbb{R}_-^N = \{(x', x_N) \in \mathbb{R}^N; x_N < 0\}$.
Claim: $u \in L^p(0, T; V(\mathbb{R}_+^N)), \partial_t u \in L^{p'}(0, T; V'(\mathbb{R}_+^N)) \Rightarrow u \in \mathcal{C}([0, T], L^2(\mathbb{R}_+^N))$.

Indeed, consider the following extension

$$\tilde{u}(t, x', x_N) = \begin{cases} u(t, x', x_N); & x_N > 0 \\ -3u(t, x', -x_N) + 4u(t, x', -2x_N); & x_N < 0. \end{cases}$$

Note that $\tilde{u} \in L^p(0, T; V(\mathbb{R}^N))$.

Let's estimate $\partial_t \tilde{u}$, for $\varphi \in C_c^\infty(]0, T[\times \mathbb{R}^N)$ one gets

$$\begin{aligned} \int_0^T \int_{\mathbb{R}^N} \tilde{u}(t, x) \partial_t \varphi(t, x) dx dt &= \int_0^T \int_{\mathbb{R}_-^N} (-3u(t, x', -x_N) + 4u(t, x', -2x_N)) \partial_t \varphi(t, x', x_N) dx dt \\ &\quad + \int_0^T \int_{\mathbb{R}_+^N} u(t, x) \partial_t \varphi(t, x) dx dt. \end{aligned}$$

$$\begin{aligned} \text{Since } \int_0^T \int_{\mathbb{R}_-^N} -3u(t, x', -x_N) \partial_t \varphi(t, x', x_N) dx dt &= \int_0^T \int_{\mathbb{R}_+^N} -3u(t, x', x_N) \partial_t \varphi(t, x', -x_N) dx dt \\ \int_0^T \int_{\mathbb{R}_-^N} 4u(t, x', -2x_N) \partial_t \varphi(t, x', x_N) dx dt &= \int_0^T \int_{\mathbb{R}_+^N} 2u(t, x', x_N) \partial_t \varphi(t, x', -\frac{x_N}{2}) dx dt. \end{aligned}$$

Then

$$\begin{aligned} \int_0^T \int_{\mathbb{R}^N} \tilde{u}(t, x) \partial_t \varphi(t, x) dx dt \\ = \int_0^T \int_{\mathbb{R}_+^N} (\partial_t (\varphi(t, x', x_N) - 3\varphi(t, x', -x_N) + 2\varphi(t, x', -\frac{x_N}{2})) u(t, x, x_N) dx dt. \end{aligned}$$

Remark that

$$\psi(t, x) = \varphi(t, x', x_N) - 3\varphi(t, x', -x_N) + 2\varphi(t, x', -\frac{x_N}{2}) = 0 \quad \text{if } x_N = 0$$

and $\partial_t \psi(t, x) = 0$ if $x_N = 0$, which implies $\psi \in W^{1, \infty}(0, T; V_0(\mathbb{R}_+^N))$.

Since

$$\begin{aligned} \|\psi\|_{L^p(0, T; V_0(\mathbb{R}_+^N))} &= \|\varphi(t, x', x_N) - 3\varphi(t, x', -x_N) + 2\varphi(t, x', -\frac{x_N}{2})\|_{L^p(0, T; V_0(\mathbb{R}_+^N))} \\ &\leq \|\varphi(t, x', x_N)\|_{L^p(0, T; V_0(\mathbb{R}_+^N))} + 3\|\varphi(t, x', x_N)\|_{L^p(0, T; V_0(\mathbb{R}_-^N))} \\ &\quad + 3\sqrt{2}\|\varphi(t, x', x_N)\|_{L^p(0, T; V_0(\mathbb{R}_-^N))} \\ &\leq 6\sqrt{2}\|\varphi\|_{L^p(0, T; V(\mathbb{R}^N))}. \end{aligned}$$

Then

$$\begin{aligned} \left| \int_0^T \langle \partial_t \tilde{u}, \varphi \rangle dt \right| &= \left| \int_0^T \int_{\mathbb{R}_+^N} u \partial_t \psi dx dt \right| \leq \|\partial_t u\|_{L^{p'}(0, T; V'(\mathbb{R}_+^N))} \|\psi\|_{L^p(0, T; V_0(\mathbb{R}_+^N))} \\ &\leq 6\sqrt{2} \|\partial_t u\|_{L^{p'}(0, T; V'(\mathbb{R}_+^N))} \|\varphi\|_{L^p(0, T; V(\mathbb{R}^N))} = C \|\varphi\|_{L^p(0, T; V(\mathbb{R}^N))}. \end{aligned}$$

Thus $\partial_t \tilde{u} \in L^{p'}(0, T; V'(\mathbb{R}^N))$. By (1.) we conclude that $\tilde{u} \in \mathcal{C}([0, T], L^2(\mathbb{R}^N))$ i.e $u \in \mathcal{C}([0, T], L^2(\mathbb{R}_+^N))$.

3. Let us recall the following result from [24](p.3).

For all $a \in \partial D$, there exist an open set $O \subset \mathbb{R}^N$ such that $a \in O$, an open set $W \subset \mathbb{R}^{N-1}$, $r > 0$ and $T : W \times [0, r[\rightarrow \mathbb{R}^N$ satisfying:

- T is a bilipschitzian homeomorphism between $W \times [0, r[$ and $\bar{D} \cap O$, $T(W \times \{0\}) = \partial D \cap O$ and there exist $\alpha > 0$ such that $JT \geq \alpha$ on $W \times]0, r[$ where JT denote the Jacobien of T .

4. Thank's to (3.) we cover the boundary ∂D by the transport $(O_i, T_i)_{i=1, \dots, n}$ and we add an open set O_0 relatively compact in D such that $\bar{D} \subset \bigcup_{i=0}^n O_i$. Then, we define a partition of unity $(\gamma_i)_{i=0, \dots, n}$ associated with the previous covering. First, we prove that

$$u \in L^p(0, T; V(D)), \partial_t u \in L^{p'}(0, T; V'(D))$$

↓

$$\forall i = 0, \dots, n; \gamma_i u \in L^p(0, T; V(D \cap O_i)), \partial_t(\gamma_i u) \in L^{p'}(0, T; V'(D \cap O_i)).$$

Note that $\frac{\partial}{\partial x_k}(\gamma_i u) = \frac{\partial \gamma_i}{\partial x_k} u + \frac{\partial u}{\partial x_k} \gamma_i$, $k = 1, \dots, N$ and $\frac{\partial}{\partial t}(\gamma_i u) = \gamma_i \frac{\partial u}{\partial t}$ where the derivative is taken in the sense of distributions in $]0, T[\times (D \cap O_i)$.

Then we can deduce that $\gamma_i u \in L^p(0, T; V(D \cap O_i))$.

Let's estimate $\partial_t(\gamma_i u)$, for $\varphi \in \mathcal{C}_c^\infty(]0, T[\times (D \cap O_i))$, one has

$$\begin{aligned} \left| \int_0^T \langle \partial_t(\gamma_i u), \varphi \rangle dt \right| &= \left| \int_0^T \langle \partial_t u, \gamma_i \varphi \rangle dt \right| = \left| \int_0^T \int_{D \cap O_i} u(t, x) \partial_t(\gamma_i \varphi)(t, x) dx dt \right| \\ &\text{(since } \gamma_i \varphi = 0 \text{ outside } D \cap O_i) = \left| \int_0^T \int_D u(t, x) \partial_t(\gamma_i \varphi)(t, x) dx dt \right| \\ &\leq \|\partial_t u\|_{L^{p'}(0, T; V'(D))} \|\gamma_i \varphi\|_{L^p(0, T; V_0(D))}. \end{aligned}$$

Note that $\|\gamma_i \varphi\|_{L^p(0, T; V_0(D))} = \|\gamma_i \varphi\|_{L^p(0, T; V_0(D \cap O_i))} \leq \|\gamma_i\|_{L^\infty(O_i)} \|\varphi\|_{L^p(0, T; V_0(D \cap O_i))}$.

Then $\left| \int_0^T \langle \partial_t(\gamma_i u), \varphi \rangle dt \right| \leq \|\partial_t u\|_{L^{p'}(0, T; V'(D))} \|\gamma_i\|_{L^\infty(O_i)} \|\varphi\|_{L^p(0, T; V_0(D \cap O_i))} \leq C \|\varphi\|_{L^p(0, T; V_0(D \cap O_i))}$.

Therefore $\partial_t(\gamma_i u) \in L^{p'}(0, T; V'(D \cap O_i))$.

Secondly, we have $[\forall i = 0, \dots, n; \gamma_i u \in \mathcal{C}([0, T], L^2(D \cap O_i))] \Rightarrow u \in \mathcal{C}([0, T], L^2(D))$.

Indeed, Since $\sum_{i=0}^n \gamma_i(x) u(t, x) \in \mathcal{C}([0, T], L^2(\bigcup_{i=0}^n (D \cap O_i))) = \mathcal{C}([0, T], L^2(D))$.

Then $u(t, x) = \sum_{i=0}^n \gamma_i(x) u(t, x) \in \mathcal{C}([0, T], L^2(D))$.

Hence, it sufficient to prove, for any $i = 1, \dots, n$, the following

$$[\gamma_i u \in L^p(0, T; V(D \cap O_i)), \gamma_i \partial_t u \in L^{p'}(0, T; V'(D \cap O_i))] \Rightarrow \gamma_i u \in \mathcal{C}([0, T], L^2(D \cap O_i)).$$

5. For any $i = 1, \dots, n$; the property is transportable.

Note that, if $(\gamma_i u) \circ T_i \in \mathcal{C}([0, T], L^2(W_i \times]0, r_i[))$, then $\gamma_i u \in \mathcal{C}([0, T], L^2(D \cap O_i))$.

We have the following result

$$\gamma_i u \in L^p(0, T; V(D \cap O_i)), \gamma_i \partial_t u \in L^{p'}(0, T; V'(D \cap O_i))$$

\Downarrow

$$(\gamma_i u) \circ T_i \in L^p(0, T; V(W_i \times]0, r_i[)), (\gamma_i \partial_t u) \circ T_i \in L^{p'}(0, T; V'(W_i \times]0, r_i[)).$$

Indeed, we have $\int_0^T \int_{D \cap O_i} |\gamma_i(x) u(t, x)|^p dx dt = \int_0^T \int_{W_i \times]0, r_i[} |\gamma_i(T_i(x)) u(t, T_i(x))|^p |JT_i|^p dx dt$.

Hence $\int_0^T \int_{D \cap O_i} |\gamma_i(x) u(t, x)|^p dx dt \geq \alpha^p \int_0^T \int_{W_i \times]0, r_i[} |\gamma_i(T_i(x)) u(t, T_i(x))|^p dx dt$.

Since $\nabla[(\gamma_i(x) u(t, x)) \circ T_i(x)] = (DT_i)^T \nabla[(\gamma_i(T_i(x)) u(t, T_i(x)))]$. Then

$$[(DT_i)^T]^{-1} \nabla[(\gamma_i(x) u(t, x)) \circ T_i(x)] = \nabla[(\gamma_i(T_i(x)) u(t, T_i(x)))]$$

Note that the inverse of $(DT_i)^T$ exists because $JT_i \geq \alpha_i > 0$ and $((DT_i)^T)^{-1}$ is uniformly bounded and we can deduce that

$$\begin{aligned} |\nabla[(\gamma_i(T_i(x)) u(t, T_i(x)))]| &\leq \|((DT_i)^T)^{-1}\|_\infty |\nabla[(\gamma_i(x) u(t, x)) \circ T_i(x)]| \\ &\leq M_i |\nabla[(\gamma_i(x) u(t, x)) \circ T_i(x)]|. \end{aligned}$$

So, one has

$$M_i^p \int_0^T \int_{D \cap O_i} |\nabla[\gamma_i(x) u(t, x)]|^p dx dt \geq \int_0^T \int_{W_i \times]0, r_i[} |\nabla[(\gamma_i(T_i(x)) u(t, T_i(x)))]|^p |JT_i|^p dx dt.$$

Hence

$$\frac{M_i^p}{\alpha_i^p} \int_0^T \int_{D \cap O_i} |\nabla[\gamma_i(x) u(t, x)]|^p dx dt \geq \int_0^T \int_{W_i \times]0, r_i[} |\nabla[(\gamma_i(T_i(x)) u(t, T_i(x)))]|^p dx dt.$$

Therefore $\gamma_i u \in L^p(0, T; V(D \cap O_i)) \Rightarrow (\gamma_i u) \circ T_i \in L^p(0, T; V(W_i \times]0, r_i[))$.

Now, for $\varphi \in \mathcal{C}_c^\infty([0, T] \times W_i \times]0, r_i[)$, we can estimate

$$\begin{aligned} \left| \int_0^T \langle \partial_t(\gamma_i u) \circ T_i, \varphi \rangle dt \right| &= \left| \int_0^T \int_{W_i \times]0, r_i[} \gamma_i(T_i(x)) u(t, T_i(x)) \partial_t \varphi(t, x) dx dt \right| \\ &= \left| \int_0^T \int_{D \times O_i} u(t, y) \gamma_i(y) \partial_t \varphi(t, T_i^{-1}(y)) |JT_i^{-1}| dy dt \right| \\ &\leq M_i \|\partial_t u\|_{L^{p'}(0, T; V'(D \times O_i))} \|\varphi \circ T_i^{-1}\|_{L^p(0, T; V_0(D \times O_i))} \\ &\leq M_i \|\partial_t u\|_{L^{p'}(0, T; V'(D \times O_i))} C(T_i) \|\varphi\|_{L^p(0, T; V_0(W_i \times]0, r_i[))} \\ &\leq C_i \|\varphi\|_{L^p(0, T; V_0(W_i \times]0, r_i[))} \end{aligned}$$

and $\partial_t(\gamma_i u) \circ T_i \in L^{p'}(0, T; V'(W_i \times]0, r_i[))$.

6. Extension an elements of $L^{p'}(0, T; V')$:

If O and U are two open sets of \mathbb{R}^N , $K \subset O$ a compact. Then there exists C depends on K and O such that: for all $L \in L^{p'}(0, T; V'(U \cap O))$ with a support on K , an element L can be extended by 0 for an element \tilde{L} of $L^{p'}(0, T; V'(U))$ and

$$\|\tilde{L}\|_{L^{p'}(0, T; V'(U))} \leq C \|L\|_{L^{p'}(0, T; V'(U \cap O))}.$$

Let $\xi \in C_c^\infty(O)$ such that $\xi \equiv 1$ on neighborhood of K .

For $\varphi \in C_c^\infty(]0, T[\times U)$, define

$$\int_0^T \langle \tilde{L}, \varphi \rangle_{V'(U), V_0(U)} dt = \int_0^T \langle L, \xi \varphi \rangle_{V'(U \cap O), V_0(U \cap O)} dt.$$

This map is well defined because $\xi \varphi \in C_c^\infty(]0, T[\times (O \cap U))$, since φ ensures the annulment on the neighborhood of ∂U and ξ ensures the annulment on the neighborhood of ∂O .

By definition, L with a support K means $\int_0^T \langle L, \varphi \rangle dt = 0$ for all φ satisfies $\text{supp}(\varphi) \cap K = \emptyset$. Hence

$$\begin{aligned} \int_0^T \langle \tilde{L}, \varphi \rangle_{V'(U), V_0(U)} dt &\leq \|L\|_{L^{p'}(0, T; V'(U \cap O))} \|\xi \varphi\|_{L^p(0, T; V_0(O \cap U))} \\ &\leq C(\xi) \|L\|_{L^{p'}(0, T; V'(U \cap O))} \|\varphi\|_{L^p(0, T; V_0(U))}. \end{aligned}$$

Since ξ depends only on K and O , then $C(\xi)$ also.

7. We prove the property on D as following:

- For $i = 0$, since $\text{supp}(\gamma_0)$ is compact on D we can extend $\gamma_0 u$ on \mathbb{R}^N by 0. Then (1.) and (6.) ensures that $\gamma_0 u \in C([0, T], L^2(D \cap O_0))$.

- For $i \geq 1$, we start by using (5.) which ensures us that it is sufficient to prove the property for $\gamma_i u \circ T_i$ on $D_i = W_i \times]0, r_i[$ but the support of this function contained in $\text{supp}(\gamma_i \circ T)$ which is compact in $W_i \times]0, r_i[$.

- Denote by $\widehat{\gamma_i u \circ T_i}$ The extension by 0 of the function $\gamma_i u \circ T_i$ on \mathbb{R}_+^N .

- Thank's (6.) and (2.), one gets that this extension $\widehat{\gamma_i u \circ T_i}$ is an element of $C([0, T], L^2(\mathbb{R}_+^N))$, therefore $\gamma_i u \circ T_i \in C([0, T], L^2(W_i \times]0, r_i[))$.

□

2. Compactness when $p < 2$.

Concerning the compactness argument in $L^p(Q)$ when $p < 2$: note that there exists an integer $k \geq 1$ such that $W_0^{k,p}(D) \xhookrightarrow{d} L^{p'}(D)$ so that

$$W_0^{k,p}(D) \xhookrightarrow{d} V \xhookrightarrow{d} L^p(D) \equiv [L^{p'}(D)]' \xhookrightarrow{d} W^{-k,p'}(D) \text{ and } V' \xhookrightarrow{d} W^{-k,p'}(D).$$

Remark 3.26. *Let us justify that the identification $L^p(D) \equiv [L^{p'}(D)]'$ is possible if $L^2(D)$ is already chosen as the pivot-space.*

Indeed, one has: $L^{p'}(D) \xhookrightarrow{d} L^2(D) \xhookrightarrow{d} L^p(D)$ with reflexive B-spaces, so that $L^p(D)' \xhookrightarrow{d} L^2(D)' \xhookrightarrow{d} L^{p'}(D)'$.

Consider $T \in L^{p'}(D)'$ and $T_n \in L^2(D)'$ such that $T_n \rightarrow T$ in $L^{p'}(D)'$. Then, by the pivot-space identification, there exists $u_n \in L^2(D)$ such that $T_n = u_n$ in the sense of Riesz-identification.

Then, for any $v \in L^{p'}(D)$ with norm 1,

$$\left| \int_D u_n v dx \right| \leq \|T\|_{L^{p'}(D)'} + \|T_n - T\|_{L^{p'}(D)'}$$

By considering $v = \text{Sgn}(u_n) \frac{|u_n|^{p-1}}{\|u_n\|_{L^p}^{p-1}}$, one has that the sequence (u_n) is bounded in $L^p(D)$ and that, up to a subsequence if needed, it converges weakly to a given u in $L^p(D)$.

Thus, for any $v \in L^{p'}(D)$,

$$\langle T, v \rangle = \lim_n \langle T_n, v \rangle = \lim_n \int_D u_n v dx = \int_D u v dx.$$

Since this element u is unique in its way, the identification holds.

Then, since the embedding of V is compact in $L^p(D)$, by Aubin-Lions-Simon compactness theorems, if a sequence is bounded in $W(0, T)$, it is also bounded in $W^{1,p,p'}(0, T, V, W^{-k,p'}(D))^\ddagger$ and relatively compact in $L^p(Q)$.

[‡]the space of functions $u \in L^p(0, T; V)$ such that $\partial_t u \in L^{p'}(0, T; W^{-k,p'}(D))$.

3. Mignot-Bamberger / Alt -Luckhaus integration by part formula

Proof of Lemma 2.15

Thanks to the assumptions, Ψ is a measurable function on $Q \times \mathbb{R}$ and Λ is a Carathéodory function on $Q \times \mathbb{R}$. Moreover,

$$|\Psi(t, x, \lambda)| \leq |\Psi(t=0)| + \int_0^t |\partial_t \Psi(s, x, \lambda)| ds \leq (T+1) \cdot h(t, x) + |\lambda|^\alpha,$$

$$|\Lambda(t, x, \lambda)| \leq |\lambda - a| \left[(T+1) \cdot h(t, x) + |\lambda|^\alpha \right] \leq C(T, a) \left[|\lambda|^2 + h^2(t, x) + h(t, x) + 1 \right]$$

so that $\Lambda, \Psi \in L^2_{loc}(\mathbb{R}, L^2(Q))$ and the Nemitskii operator associated with Λ is continuous from $L^2(Q)$ to $L^1(Q)$. Concerning the time-derivation of Λ , for any $\varphi \in D(Q \times \mathbb{R})$, Fubini's theorem yields

$$\begin{aligned} - \int_{Q \times \mathbb{R}} \Lambda(t, x, \lambda) \partial_t \varphi(t, x, \lambda) dt dx d\lambda &= - \int_{Q \times \mathbb{R}} \int_a^\lambda \Psi(t, x, \tau) d\tau \partial_t \varphi(t, x, \lambda) dt dx d\lambda \\ &= - \int_a^\lambda \int_{Q \times \mathbb{R}} \Psi(t, x, \tau) \partial_t \varphi(t, x, \lambda) dt dx d\lambda d\tau \\ &= \int_a^\lambda \int_{Q \times \mathbb{R}} \partial_t \Psi(t, x, \tau) \varphi(t, x, \lambda) dt dx d\lambda d\tau \\ &= \int_{Q \times \mathbb{R}} \int_a^\lambda \partial_t \Psi(t, x, \tau) d\tau \varphi(t, x, \lambda) dt dx d\lambda. \end{aligned}$$

As a consequence,

$$\partial_t \Lambda(t, x, \lambda) = \int_a^\lambda \partial_t \Psi(t, x, \tau) d\tau, \quad \left| \partial_t \Lambda(t, x, \lambda) \right| \leq |\lambda - a| h(t, x) \leq |\lambda|^2 + h^2(t, x)/4 + |a| h(t, x)$$

so that the Nemitskii operator associated with $\partial_t \Lambda$ is continuous from $L^2(Q)$ to $L^1(Q)$.

Thanks to the assumptions, $u \in C([0, T], L^2(D))$ and one extends u to \bar{u} in \mathbb{R} by $\bar{u}(t) = u_0$ if $t < 0$ and $\bar{u}(t) = u(T)$ si $t > T$.

Therefore, if $I_1 := (-1, T+1)$, $\bar{u} \in L^p(I_1, W^{1,p}(D)) \cap L^\infty(I_1, L^2(D)) \cap C(\bar{I}_1, L^2(D))$ such that $\partial_t \bar{u} \in L^{p'}(I_1, V')$ with $\partial_t \bar{u} = 0$ when $t < 0$ or $t > T$.

Similarly to u , denote by $\bar{\Psi}$ the extension to I_1 of Ψ in the same way and by $\bar{\Lambda}$ the corresponding integral as introduced in the Lemma.

For any fixed $0 < h \ll 1$, let us denote by

$$v_h : t \mapsto \frac{\bar{u}(t+h) - \bar{u}(t)}{h}, \quad w_h : t \mapsto \frac{\bar{u}(t) - \bar{u}(t-h)}{h}.$$

Consider $\beta \in \mathcal{D}(I_1)$ and h , small enough so that $\text{supp}\beta + [-h, h] \subset I_1$. Then,

$$\begin{aligned} \int_{I_1} v_h(t)\beta(t)dt &= \frac{1}{h} \int_{I_1} [\bar{u}(t+h) - \bar{u}(t)]\beta(t)dt = \frac{1}{h} \int_{I_1} \bar{u}(t)\beta(t-h)dt - \frac{1}{h} \int_{I_1} \bar{u}(t)\beta(t)dt \\ &= \frac{1}{h} \int_{I_1} \bar{u}(t)[\beta(t-h) - \beta(t)]dt \rightarrow - \int_{-1}^{T+1} \bar{u}(t)\beta'(t)dt \quad \text{in } L^2(D) \\ &= - \int_0^T u(t)\beta'(t)dt + u(T)\beta(T) - u_0\beta(0); \end{aligned}$$

similarly,

$$\begin{aligned} \int_{I_1} w_h(t)\beta(t)dt &= \frac{1}{h} \int_{I_1} [\bar{u}(t) - \bar{u}(t-h)]\beta(t)dt = \frac{1}{h} \int_{I_1} \bar{u}(t)\beta(t)dt - \frac{1}{h} \int_{I_1} \bar{u}(t)\beta(t+h)dt \\ &= \frac{1}{h} \int_{I_1} \bar{u}(t)[\beta(t) - \beta(t+h)]dt \rightarrow - \int_{-1}^{T+1} \bar{u}(t)\beta'(t)dt \quad \text{in } L^2(D) \\ &= - \int_0^T u(t)\beta'(t)dt + u(T)\beta(T) - u_0\beta(0), \end{aligned}$$

so that v_h and w_h converge to $\partial_t \bar{u}$ in $\mathcal{D}'[I_1, L^2(D)]$, thus in $\mathcal{D}'[I_1, V']$; and to $\partial_t u$ in $\mathcal{D}'[0, T, L^2(D)]$ and $\mathcal{D}'[0, T, V']$.

Moreover, by [14, Corollary A.2 p.145], the properties of Bochner integral and since $\partial_t \bar{u} = 0$ outside $(0, T)$,

$$\begin{aligned} \int_{I_1} \|v_h(t)\|_{V'}^{p'} dt &= \int_{I_1} \frac{1}{h^{p'}} \left\| \int_t^{t+h} \partial_t \bar{u}(s) ds \right\|_{V'}^{p'} dt \leq \int_{I_1} \frac{1}{h} \int_t^{t+h} \|\partial_t \bar{u}(s)\|_{V'}^{p'} ds dt \\ &\leq \frac{1}{h} \int_{I_1} \int_{-1}^{t+h} \|\partial_t \bar{u}(s)\|_{V'}^{p'} ds dt - \frac{1}{h} \int_{I_1} \int_{-1}^t \|\partial_t \bar{u}(s)\|_{V'}^{p'} ds dt \\ &= \frac{1}{h} \int_{-1+h}^{T+1+h} \int_{-1}^t \|\partial_t \bar{u}(s)\|_{V'}^{p'} ds dt - \frac{1}{h} \int_{I_1} \int_{-1}^t \|\partial_t \bar{u}(s)\|_{V'}^{p'} ds dt \\ &= \frac{1}{h} \int_{T+1}^{T+1+h} \int_{-1}^t \|\partial_t \bar{u}(s)\|_{V'}^{p'} ds dt - \frac{1}{h} \int_{-1}^{-1+h} \int_{-1}^t \|\partial_t \bar{u}(s)\|_{V'}^{p'} ds dt = \int_0^T \|\partial_t u(s)\|_{V'}^{p'} ds. \end{aligned}$$

Since v_h already converges in the sense of distributions, as a consequence of the above estimate, one may conclude that v_h converges weakly to $\partial_t \bar{u}$ in $L^{p'}[I_1, V']$ and to $\partial_t u$ in $L^{p'}[0, T, V']$.

Similarly, w_h converges weakly to $\partial_t \bar{u}$ in $L^{p'}[I_1, V']$ and to $\partial_t u$ in $L^{p'}[0, T, V']$.

For any $\beta \in \mathcal{D}(I_1)$, one has that $\Psi(\cdot, \bar{u})\beta \in L^p(I_1, V)$, since $L^2(D)$ is identified with its dual, one gets that

$$\begin{aligned} \int_{I_1 \times D} v_h \bar{\Psi}(\cdot, u(t))\beta dx dt &= \int_{I_1} \langle v_h, \bar{\Psi}(\cdot, \bar{u}(t)) \rangle \beta dt \rightarrow \int_{I_1} \langle \partial_t \bar{u}, \bar{\Psi}(\cdot, \bar{u}) \rangle \beta dt, \\ \int_{I_1 \times D} w_h \bar{\Psi}(\cdot, \bar{u}(t))\beta dx dt &= \int_{I_1} \langle w_h, \bar{\Psi}(\cdot, \bar{u}(t)) \rangle \beta dt \rightarrow \int_{I_1} \langle \partial_t \bar{u}, \bar{\Psi}(\cdot, \bar{u}) \rangle \beta dt. \end{aligned}$$

Let us recall that a is a given real and $\bar{\Lambda}(t, x, \lambda) = \int_a^\lambda \bar{\Psi}(t, x, \tau) d\tau$. Since $\bar{\Psi}$ is a non-decreasing function of its third variable, for any real numbers u and v , one has

$$(v - u)\bar{\Psi}(t, x, u) \leq \bar{\Lambda}(t, x, v) - \bar{\Lambda}(t, x, u) = \int_u^v \bar{\Psi}(t, x, \tau) d\tau \leq (v - u)\bar{\Psi}(t, x, v).$$

Thus, assuming moreover that β is non-negative,

$$\begin{aligned} [\bar{u}(t+h, x) - \bar{u}(t, x)]\bar{\Psi}(t, x, \bar{u}(t))\beta &\leq [\bar{\Lambda}(t, x, \bar{u}(t+h)) - \bar{\Lambda}(t, x, \bar{u}(t))]\beta \\ &\leq [\bar{u}(t+h, x) - \bar{u}(t, x)]\bar{\Psi}(t, x, \bar{u}(t+h))\beta, \\ [\bar{u}(t, x) - \bar{u}(t-h, x)]\bar{\Psi}(t, x, \bar{u}(t-h))\beta &\leq [\bar{\Lambda}(t, x, \bar{u}(t)) - \bar{\Lambda}(t, x, \bar{u}(t-h))]\beta \\ &\leq [\bar{u}(t, x) - \bar{u}(t-h, x)]\bar{\Psi}(t, x, \bar{u}(t))\beta. \end{aligned}$$

and, for h small enough to have $\text{supp}\beta + [-h, h] \subset I_1$,

$$\begin{aligned} \int_{I_1 \times D} v_h \beta \bar{\Psi}(\cdot, u(t)) dx dt &\leq \int_{I_1 \times D} \frac{\bar{\Lambda}(\cdot, \bar{u}(t+h)) - \bar{\Lambda}(\cdot, \bar{u}(t))}{h} \beta dx dt \leq \int_{I_1 \times D} v_h \beta \bar{\Psi}(\cdot, \bar{u}(t+h)) dx dt, \\ \int_{I_1 \times D} w_h \beta \bar{\Psi}(\cdot, \bar{u}(t-h)) dx dt &\leq \int_{I_1 \times D} \frac{\bar{\Lambda}(\cdot, \bar{u}(t)) - \bar{\Lambda}(\cdot, \bar{u}(t-h))}{h} \beta dx dt \leq \int_{I_1 \times D} w_h \beta \bar{\Psi}(\cdot, \bar{u}(t)) dx dt, \end{aligned}$$

so that

$$\begin{aligned} \liminf \int_{I_1 \times D} \frac{\bar{\Lambda}(\cdot, \bar{u}(t+h)) - \bar{\Lambda}(\cdot, \bar{u}(t))}{h} \beta dx dt &\geq \int_{I_1} \langle \partial_t \bar{u}, \bar{\Psi}(\cdot, \bar{u}) \rangle \beta dt = \int_0^T \langle \partial_t u, \Psi(\cdot, u) \rangle \beta dt, \\ \limsup \int_{I_1 \times D} \frac{\bar{\Lambda}(\cdot, \bar{u}(t)) - \bar{\Lambda}(\cdot, \bar{u}(t-h))}{h} \beta dx dt &\leq \int_{I_1} \langle \partial_t \bar{u}, \bar{\Psi}(\cdot, \bar{u}) \rangle \beta dt = \int_0^T \langle \partial_t u, \Psi(\cdot, u) \rangle \beta dt. \end{aligned}$$

Moreover,

$$\begin{aligned} &\int_{I_1 \times D} \frac{\bar{\Lambda}(t, x, \bar{u}(t+h)) - \bar{\Lambda}(t, x, \bar{u}(t))}{h} \beta(t) dx dt \\ &= \frac{1}{h} \int_{I_1 \times D} \bar{\Lambda}(t-h, x, \bar{u}(t)) \beta(t-h) dx dt - \frac{1}{h} \int_{I_1 \times D} \bar{\Lambda}(t, x, \bar{u}(t)) \beta(t) dx dt \\ &= \int_{I_1 \times D} \frac{\bar{\Lambda}(t-h, x, \bar{u}(t)) - \bar{\Lambda}(t, x, \bar{u}(t))}{h} \beta(t-h) dx dt + \int_{I_1 \times D} \frac{\beta(t-h) - \beta(t)}{h} \bar{\Lambda}(t, x, \bar{u}(t)) dx dt \end{aligned}$$

and

$$\begin{aligned} &\int_{I_1 \times D} \frac{\bar{\Lambda}(t, x, \bar{u}(t)) - \bar{\Lambda}(t, x, \bar{u}(t-h))}{h} \beta(t) dx dt \\ &= \int_{I_1 \times D} \frac{\bar{\Lambda}(t, x, \bar{u}(t)) - \bar{\Lambda}(t+h, x, \bar{u}(t))}{h} \beta(t+h) dx dt + \int_{I_1 \times D} \frac{\beta(t) - \beta(t+h)}{h} \bar{\Lambda}(t, x, \bar{u}(t)) dx dt. \end{aligned}$$

One gets, by passing to the limit, and thanks to the time-extension procedure,

$$\begin{aligned}
& \liminf \int_{I_1 \times D} \frac{\bar{\Lambda}(t-h, x, \bar{u}(t)) - \bar{\Lambda}(t, x, \bar{u}(t))}{h} \beta(t-h) dx dt \\
& \geq \int_0^T \langle \partial_t u, \Psi(\cdot, u) \rangle \beta dt + \int_{I_1 \times D} \bar{\Lambda}(\cdot, \bar{u}) \beta' dt \\
& = \int_0^T \langle \partial_t u, \Psi(\cdot, u) \rangle \beta dt + \int_Q \Lambda(\cdot, u) \beta' dt + \int_D \Lambda(0, x, u_0) \beta(0) dx - \int_D \Lambda(T, x, u(T)) \beta(T) dx \\
& \geq \limsup \int_{I_1 \times D} \frac{\bar{\Lambda}(t, x, \bar{u}(t)) - \bar{\Lambda}(t+h, x, \bar{u}(t))}{h} \beta(t+h) dx dt.
\end{aligned}$$

Note that

$$\int_{I_1 \times D} \frac{\bar{\Lambda}(t-h, x, \bar{u}(t)) - \bar{\Lambda}(t, x, \bar{u}(t))}{h} \beta(t-h) dx dt = - \int_{I_1 \times D} \frac{1}{h} \int_{t-h}^t \partial_t \bar{\Lambda}(s, x, \bar{u}(t)) \beta(t-h) ds dx dt.$$

Since, $|\partial_t \bar{\Lambda}(s, x, \bar{u}(t)) \beta(t-h)| \leq \|\beta\|_\infty |\bar{u}(t, x) - a| h(s, x)$ is an integrable function, the properties of the point of Lebesgue (steklov average) yields

$$\begin{aligned}
\int_{I_1 \times D} \frac{\bar{\Lambda}(t-h, x, \bar{u}(t)) - \bar{\Lambda}(t, x, \bar{u}(t))}{h} \beta(t-h) dx dt & \rightarrow - \int_{I_1 \times D} \partial_t \bar{\Lambda}(t, x, \bar{u}(t)) \beta(t) dx dt \\
& = - \int_Q \partial_t \Lambda(t, x, u(t)) \beta(t) dx dt.
\end{aligned}$$

Since the same holds for $\limsup \int_{I_1 \times D} \frac{\bar{\Lambda}(t, x, \bar{u}(t)) - \bar{\Lambda}(t+h, x, \bar{u}(t))}{h} \beta(t+h) dx dt$, and if β is regular and non negative, one gets that:

$$\begin{aligned}
& \forall \beta \in D^+([0, T]), \quad \int_0^T \langle \partial_t u, \Psi(\cdot, u) \rangle \beta dt \\
& = \int_D \Lambda(T, x, u(T)) \beta(T) dx - \int_D \Lambda(0, x, u_0) \beta(0) dx - \int_Q \Lambda(\cdot, u) \beta' dt - \int_Q \partial_t \Lambda(t, x, u(t)) \beta(t) dx dt.
\end{aligned}$$

Since β is involved in linear integral terms, a classical argument of regularisation yields the result for any non-negative elements of $W^{1,\infty}(0, T)$, then for any elements of $W^{1,\infty}(0, T)$.

Since T is arbitrary, the result holds for any t and $s = 0$, then for any t and s by subtracting the integral from 0 to s to the one from 0 to t . \square

Remark 3.27. As a consequence,

$$\frac{d}{dt} \left[\int_D \Lambda(t, x, u) dx \right] = \langle \partial_t u, \Psi(t, x, u) \rangle + \int_D \partial_t \Lambda(t, x, u) dx \text{ in } \mathcal{D}'(0, T)$$

and has it an integrable function, one concludes that $t \mapsto \int_D \Lambda(t, x, u) dx$ is an absolutely-continuous in $[0, T]$.

Corollary 3.28. Consider $u \in L^p(0, T, W^{1,p}(D)) \cap L^\infty(0, T, L^2(D))$ such that $\partial_t u \in L^{p'}(0, T, V')$, $\alpha \in L^2(D)$, $\alpha \geq 0$ and $\Psi : \mathbb{R} \rightarrow \mathbb{R}$ a given non-decreasing function. Assume that $\Psi(u)\alpha \in L^p(0, T, V)$, then, for any $\beta \in W^{1,\infty}(0, T)$ and any $0 \leq s < t \leq T$,

$$\begin{aligned} \int_s^t \langle \partial_t u, \Psi(u)\alpha \rangle \beta d\sigma &= - \int_s^t \int_D \int_a^u \Psi(\tau) d\tau \alpha \beta' dx d\sigma + \int_D \int_a^{u(t)} \Psi(\tau) d\tau \alpha \beta(t) dx \\ &\quad - \int_D \int_a^{u(s)} \Psi(\tau) d\tau \alpha \beta(s) dx, \end{aligned}$$

where a is any arbitrary real number.

Remark 3.29. Note that, by linearity, the same result holds if $\alpha = \alpha_1 - \alpha_2$ with $\Psi(u)\alpha_i \in L^p(0, T, V)$ ($i = 1, 2$) or if $\Psi = \Psi_1 - \Psi_2$ and Ψ_i ($i = 1, 2$) satisfies the assumptions.

4. Positive cones in the dual

Proof of Lemma 2.18

Note that by truncation argument, the same result holds for the positive cone of $L^p(0, T; V) \cap L^{p'}(Q)$ when $p < 2$. This result is given in [22, Lemma p.593]. We propose here a scetch of a proof following the idea of [48]. In other terms, let f be such that:

$$f \in L^{p'}(0, T; V'), \quad f \geq 0: \text{i.e. } \forall \varphi \in L^p(0, T; V), \varphi \geq 0 \Rightarrow \langle f, \varphi \rangle \geq 0.$$

We will construct $f_\varepsilon \in L^p(0, T; V)$, $f_\varepsilon \geq 0$ for each $\varepsilon > 0$ and

$$f_\varepsilon \rightarrow f \text{ in } L^{p'}(0, T; V') \text{ strongly.}$$

Proof. Consider the following operator J defined as following: $J : L^p(0, T; V) \rightarrow L^{p'}(0, T; V')$ where

$$(Jv)(x, t) = - \sum_{i=1}^d \frac{\partial}{\partial x_i} \left[\left| \frac{\partial v(x, t)}{\partial x_i} \right|^{p-2} \frac{\partial v(x, t)}{\partial x_i} \right] + \|v(t)\|_{L^2(D)}^{p-2} v(x, t). \quad (\text{A.4})$$

J is monotone, bounded, continuous and coercive operator from $L^p(0, T; V)$ into $L^{p'}(0, T; V')$.

1. Consider the following problem:

$$\begin{cases} Jv = f & \text{in } L^{p'}(0, T; V') \hookrightarrow \mathcal{D}'(Q), \\ v \in L^p(0, T; V). \end{cases} \quad (\text{A.5})$$

The properties of J and [66, section 2.1] ensure the existence of v .

2. For any $2 \leq p < \infty$, consider the following problem:

$$\begin{cases} Jv_\varepsilon + \frac{1}{\varepsilon}(v_\varepsilon - v) = 0 & \text{in } L^{p'}(0, T; V') \hookrightarrow \mathcal{D}'(Q), \\ v_\varepsilon \in L^p(0, T; V). \end{cases} \quad (\text{A.6})$$

If $p \geq 2$, the properties of J ensure the existence of v_ε .

When $1 < p < 2$ is close to 1, it is unclear whether the problem

$$\begin{cases} Jv_\varepsilon + \frac{1}{\varepsilon}(v_\varepsilon - v) = 0 & \text{in } \mathcal{D}'(Q), \\ v_\varepsilon \in L^p(0, T; V). \end{cases} \quad (\text{A.7})$$

has a solution or not. For this we replace (A.7) by the following problem:

$$\begin{cases} \int_0^T \langle Jv_\varepsilon, w \rangle dt + \frac{1}{\varepsilon} \int_Q (v_\varepsilon - v) w dx dt = 0 & \forall w \in L^p(0, T; V) \cap L^2(Q), \\ v_\varepsilon \in L^p(0, T; V), \quad v_\varepsilon - v \in L^2(Q). \end{cases} \quad (\text{A.8})$$

- Approximation and a *a priori* estimates: Consider

$$\begin{cases} Jv_\varepsilon^n + \frac{1}{\varepsilon}T_n(v_\varepsilon^n - v) = 0 & \text{in } \mathcal{D}'(Q), \\ v_\varepsilon^n \in L^p(0, T; V). \end{cases} \quad (\text{A.9})$$

where T_n is the truncation at the height n . Note that $w \rightarrow Jw + \frac{1}{\varepsilon}T_n(w - v)$ is monotone, bounded, continuous and coercive from $L^p(0, T; V)$ into $L^{p'}(0, T; V')$.

Thanks to [66, section 2.1] there exists $v_\varepsilon^n \in L^p(0, T; V)$.

Using $v_\varepsilon - v$ as test function in (A.9), one has

$$\int_0^T \langle Jv_\varepsilon^n, v_\varepsilon^n - v \rangle dt + \frac{1}{\varepsilon} \int_Q T_n(v_\varepsilon^n - v)(v_\varepsilon^n - v) dx dt = 0.$$

Then

$$\int_0^T \langle Jv_\varepsilon^n, v_\varepsilon^n \rangle dt + \frac{1}{\varepsilon} \int_Q T_n(v_\varepsilon^n - v)(v_\varepsilon^n - v) dx dt = \int_0^T \langle Jv_\varepsilon^n, v \rangle dt.$$

Thanks to Young inequality, there exist $\delta, C > 0$ such that

$$\begin{aligned} \int_0^T \langle Jv_\varepsilon^n, v \rangle dt &\leq \delta \left[\int_0^T (\|\nabla v_\varepsilon^n(t)\|_{L^p(D)}^p + \|v_\varepsilon^n(t)\|_{L^2(D)}^p) dt \right] \\ &\quad + C \left[\int_0^T \|v(t)\|_{L^p(D)}^p + \|v(t)\|_{L^2(D)}^p dt \right]. \end{aligned}$$

Since $|T_n(v_\varepsilon^n - v)|^2 \leq T_n(v_\varepsilon^n - v)(v_\varepsilon^n - v)$. Then, a convenient choice of δ yields

$$\int_0^T (\|\nabla v_\varepsilon^n(t)\|_{L^p(D)}^p + \|v_\varepsilon^n(t)\|_{L^2(D)}^p) dt + \frac{1}{\varepsilon} \int_Q |T_n(v_\varepsilon^n - v)|^2 dx dt \leq C_1$$

where C_1 is a constant independent of n and ε . Since J is bounded, we have

$$\|v_\varepsilon^n\|_{L^p(0, T; V)} \leq C_1, \quad (\text{A.10})$$

$$\|Jv_\varepsilon^n\|_{L^{p'}(0, T; V')} \leq C_1, \quad (\text{A.11})$$

$$\frac{1}{\varepsilon} \int_Q |T_n(v_\varepsilon^n - v)|^2 dx dt \leq C_1. \quad (\text{A.12})$$

- At the limit: Extracting a subsequence denoted by the same way, there exist v_ε, ξ, h such that:

$$v_\varepsilon^n \rightharpoonup v_\varepsilon \quad \text{in } L^p(0, T; V), \quad (\text{A.13})$$

$$Jv_\varepsilon^n \rightharpoonup \xi \quad \text{in } L^{p'}(0, T; V'), \quad (\text{A.14})$$

$$T_n(v_\varepsilon^n - v) \rightharpoonup h \quad \text{in } L^2(Q). \quad (\text{A.15})$$

By classical arguments, one can deduce that $h = v_\varepsilon - v \in L^2(Q)$.

Indeed, for any $\varphi \in \mathcal{D}(Q)$,

$$(Id - T_n)(v_\varepsilon^n - v) = v_\varepsilon^n - v - T_n(v_\varepsilon^n - v) \rightarrow v_\varepsilon - v - h \text{ in } \mathcal{D}'(Q)$$

$$\text{and } \left| \int_Q (Id - T_n)(v_\varepsilon^n - v) \varphi dt dx \right| \leq C(\varphi) \int_{|v_\varepsilon^n - v| \geq n} |v_\varepsilon^n - v| dt dx \leq C \frac{\|v_\varepsilon^n - v\|_{L^{\min(2,p)}(Q)}^{\min(2,p)}}{n^{\min(2,p)-1}} \rightarrow 0.$$

Passing to the limit in

$$\int_0^T \langle Jv_\varepsilon^n, w \rangle dt + \frac{1}{\varepsilon} \int_Q T_n(v_\varepsilon^n - v) w dx dt = 0, \quad \forall w \in L^p(0, T; V) \cap L^2(Q), \quad (\text{A.16})$$

one has

$$\int_0^T \langle \xi, w \rangle dt + \frac{1}{\varepsilon} \int_Q (v_\varepsilon - v) w dx dt = 0, \quad \forall w \in L^p(0, T; V) \cap L^2(Q). \quad (\text{A.17})$$

We will prove that: $\limsup_n \left[-\frac{1}{\varepsilon} \int_Q T_n(v_\varepsilon^n - v) \cdot (v_\varepsilon^n - v_\varepsilon) dx dt \right] \leq 0$.

Indeed,

$$\int_Q T_n(v_\varepsilon^n - v) \cdot (v_\varepsilon^n - v_\varepsilon) dx dt = \int_Q T_n(v_\varepsilon^n - v) \cdot (v_\varepsilon^n - v) dx dt + \int_Q T_n(v_\varepsilon^n - v) \cdot (v - v_\varepsilon) dx dt.$$

We have $\int_Q T_n(v_\varepsilon^n - v) \cdot (v - v_\varepsilon) dx dt \rightarrow -\int_Q |v - v_\varepsilon|^2 dx dt$.

Since $\int_Q T_n(v_\varepsilon^n - v) \cdot (v_\varepsilon^n - v) dx dt \geq \int_Q |T_n(v_\varepsilon^n - v)|^2 dx dt$ and thanks to the weak convergence in $L^2(Q)$ of $T_n(v_\varepsilon^n - v)$ to $v_\varepsilon - v$, one has

$$\liminf_n \int_Q T_n(v_\varepsilon^n - v) \cdot (v_\varepsilon^n - v) dx dt \geq \int_Q |v_\varepsilon - v|^2 dx dt.$$

Thus

$$\liminf_n \int_Q T_n(v_\varepsilon^n - v) \cdot (v_\varepsilon^n - v_\varepsilon) dx dt \geq 0.$$

Therefore

$$\limsup_n \left[-\frac{1}{\varepsilon} \int_Q T_n(v_\varepsilon^n - v) \cdot (v_\varepsilon^n - v_\varepsilon) dx dt \right] \leq 0. \quad (\text{A.18})$$

Using $v_\varepsilon^n - v_\varepsilon$ as test function in (A.9), one has

$$\int_0^T \langle Jv_\varepsilon^n, v_\varepsilon^n - v_\varepsilon \rangle dt + \frac{1}{\varepsilon} \int_Q T_n(v_\varepsilon^n - v) (v_\varepsilon^n - v_\varepsilon) dx dt = 0. \quad (\text{A.19})$$

Using (A.18) and (A), we deduce that

$$\limsup_n \int_0^T \langle Jv_\varepsilon^n, v_\varepsilon^n \rangle dt \leq \int_0^T \langle \xi, v_\varepsilon \rangle dt. \quad (\text{A.20})$$

Since J is monotone, continuous and bounded, we deduce $\xi = J(v_\varepsilon)$. Thus (A.8) holds.

3. Using $(v_\varepsilon - v)^+$ as test function in the difference of (A.8)-(A.5), one has

$$\int_0^T \langle Jv_\varepsilon - Jv, (v_\varepsilon - v)^+ \rangle dt + \frac{1}{\varepsilon} \int_Q |(v_\varepsilon - v)^+|^2 dx dt = - \int_0^T \langle f, (v_\varepsilon - v)^+ \rangle dt \leq 0 \quad (\text{A.21})$$

Since J is monotone, one has $\int_Q |(v_\varepsilon - v)^+|^2 dx dt \leq 0$, which implies in particular that: $v_\varepsilon - v \leq 0$ a.e. in Q .

We thus have constructed $f_\varepsilon = -\frac{1}{\varepsilon}(v_\varepsilon - v) \in L^p(0, T; V)$, $f_\varepsilon \geq 0$ (also $f_\varepsilon \in L^2(Q)$).

4. The strong convergence of f_ε to f in $L^{p'}(0, T; V')$:

Using $v_\varepsilon - v$ as test function in (A.8), one has

$$\int_0^T \langle Jv_\varepsilon, v_\varepsilon \rangle dt + \frac{1}{\varepsilon} \int_Q |v_\varepsilon - v|^2 dx dt = \int_0^T \langle Jv_\varepsilon, v \rangle dt. \quad (\text{A.22})$$

Similarly to what has been done above, thanks to Young inequality, there exist $\delta, C > 0$ such that

$$\int_0^T \langle Jv_\varepsilon, v \rangle dt \leq \delta \left[\int_0^T (\|\nabla v_\varepsilon(t)\|_{L^p(D)}^p + \|v_\varepsilon(t)\|_{L^2(D)}^p) dt \right] + C \left[\int_0^T \|v(t)\|_{L^p(D)}^p + \|v(t)\|_{L^2(D)}^p dt \right].$$

With a convenient choice of δ one has

$$\|v_\varepsilon\|_{L^p(0, T; V)} + \frac{1}{\varepsilon} \int_Q |v_\varepsilon - v|^2 dx dt \leq C_1. \quad (\text{A.23})$$

Therefore

$$\begin{aligned} v_\varepsilon - v &\rightarrow 0 \quad \text{in } L^2(Q), \\ v_\varepsilon &\rightharpoonup v \quad \text{in } L^p(0, T; V). \end{aligned}$$

Rewriting (A.22) as

$$\int_0^T \langle Jv_\varepsilon - Jv, v_\varepsilon - v \rangle dt + \frac{1}{\varepsilon} \int_Q |v_\varepsilon - v|^2 dx dt = - \int_0^T \langle Jv, v_\varepsilon - v \rangle dt \rightarrow 0, \quad \text{as } \varepsilon \rightarrow 0.$$

Since

$$0 \leq \int_Q [|\nabla v_\varepsilon|^{p-2} \nabla v_\varepsilon - |\nabla v|^{p-2} \nabla v] \cdot [\nabla v_\varepsilon - \nabla v] dx dt \leq \int_0^T \langle Jv_\varepsilon - Jv, v_\varepsilon - v \rangle dt, \quad (\text{A.24})$$

one concludes

$$\int_Q [|\nabla v_\varepsilon|^{p-2} \nabla v_\varepsilon - |\nabla v|^{p-2} \nabla v] \cdot [\nabla v_\varepsilon - \nabla v] dx dt \rightarrow 0 \quad \varepsilon \rightarrow 0. \quad (\text{A.25})$$

When $p \geq 2$, we have for some $\alpha_p > 0$

$$\alpha_p \int_Q |\nabla v_\varepsilon - \nabla v|^p dx dt \leq \int_Q [|\nabla v_\varepsilon|^{p-2} \nabla v_\varepsilon - |\nabla v|^{p-2} \nabla v] \cdot [\nabla v_\varepsilon - \nabla v] dx dt \rightarrow 0 \quad \varepsilon \rightarrow 0,$$

which implies $v_\varepsilon \rightarrow v$ strongly in $L^p(0, T; W_0^{1,p}(D)) = L^p(0, T; V)$.

When $1 < p < 2$, we have for some $\alpha_p > 0$

$$\alpha_p \int_Q \frac{|\nabla v_\varepsilon - \nabla v|^2}{(|\nabla v_\varepsilon| + |\nabla v|)^{2-p}} dx dt \leq \int_Q [|\nabla v_\varepsilon|^{p-2} \nabla v_\varepsilon - |\nabla v|^{p-2} \nabla v] \cdot [\nabla v_\varepsilon - \nabla v] dx dt \rightarrow 0. \quad (\text{A.26})$$

Using Holder's inequality with $q = 2/p$ and $q' = 2/(2-p)$, one has

$$\begin{aligned} \int_Q |\nabla v_\varepsilon - \nabla v|^p dx dt &= \int_Q \frac{|\nabla v_\varepsilon - \nabla v|^p}{(|\nabla v_\varepsilon| + |\nabla v|)^{(2-p)p/2}} (|\nabla v_\varepsilon| + |\nabla v|)^{(2-p)p/2} dx dt \\ &\leq \left(\int_Q \frac{|\nabla v_\varepsilon - \nabla v|^2}{(|\nabla v_\varepsilon| + |\nabla v|)^{2-p}} dx dt \right)^{p/2} \left(\int_Q (|\nabla v_\varepsilon| + |\nabla v|)^p dx dt \right)^{\frac{1}{q'}. \end{aligned}$$

Thanks to (A.23) and (A.26), one has $v_\varepsilon \rightarrow v$ strongly in $L^p(0, T; W_0^{1,p}(D))$.

Since $v_\varepsilon - v \rightarrow 0$ strongly in $L^2(Q) \hookrightarrow L^p(0, T; L^2(D))$, then $v_\varepsilon \rightarrow v$ strongly in $L^p(0, T; V)$.

Since J is continuous from $L^p(0, T; V)$ into $L^{p'}(0, T; V')$, we can deduce

$$-\frac{1}{\varepsilon}(v_\varepsilon - v) = Jv_\varepsilon \rightarrow Jv = f \quad L^{p'}(0, T; V').$$

Remark 3.30. Using the convexity of $x \in \mathbb{R}^d \mapsto \|x\|^p$, one has

$$\begin{aligned} |\nabla v|^p &\geq |\nabla v_\varepsilon|^p + p|\nabla v_\varepsilon|^{p-2} \nabla v_\varepsilon \cdot \nabla[v - v_\varepsilon] \\ &= |\nabla v_\varepsilon|^p + p[|\nabla v_\varepsilon|^{p-2} \nabla v_\varepsilon - |\nabla v|^{p-2} \nabla v] \cdot \nabla[v - v_\varepsilon] + p|\nabla v|^{p-2} \nabla v \cdot \nabla[v - v_\varepsilon]. \end{aligned}$$

Since $v_\varepsilon \rightharpoonup v$ in $L^p(0, T; V)$ and by (A.25) above,

$$\int_Q |\nabla v|^p dxdt \geq \limsup \int_Q |\nabla v_\varepsilon|^p dxdt$$

and the strong convergence of $L^p(0, T; W_0^{1,p}(D))$ comes from the uniform convexity.

This complete the proof of the lemma. □

Appendix C: On the existence for pseudomontone parabolic equation.

Remark 3.31. For the convenience of the reader, we consider u instead of u_ε in the following proof.

1. Proof of Theorem 2.6

Proof. Denote in the following $\mathcal{A}(u) + \frac{1}{\varepsilon}\Theta(u - \psi)$ by $B(u)$.

We use Faedo-Galerkin method, let $w_1, w_2, \dots, w_m, \dots$ be a countable basis of V , we introduce the approximate solution $u_m(t) = \sum_{j=1}^m g_{mj}(t)w_j$ and we consider the subspace $V_m \subset V$ of finite dimension generated by w_1, \dots, w_m .

Then our problem is rewritten

$$\begin{cases} (u'_m(t), w_j) + \langle B(u_m(t)), w_j \rangle = \langle f(t), w_j \rangle, 1 \leq j \leq m, \\ u_m(0) = u_{0m} \in V_m, u_{0m} \longrightarrow u_0 \text{ in } H. \end{cases} \quad (\mathcal{A})$$

Therefore

$$\begin{cases} \sum_{i=1}^m g'_{mi}(t)(w_i, w_j) + \langle B(u_m(t)), w_j \rangle = \langle f(t), w_j \rangle, 1 \leq j \leq m, \\ u_m(0) = u_{0m} \in V_m, u_{0m} \longrightarrow u_0 \text{ in } H. \end{cases} \quad (\text{A.26})$$

Lemma 3.32. There is a solution u_m for

$$\begin{cases} \sum_{i=1}^m g'_{mi}(t)(w_i, w_j) + \langle B(u_m(t)), w_j \rangle = \langle f(t), w_j \rangle, 1 \leq j \leq m, \\ u_m(0) = u_{0m} \in V_m, u_{0m} \longrightarrow u_0 \text{ in } H. \end{cases}$$

Moreover, there exist $C > 0$ independent of m and u_m satisfying

$$\sup_t \|u_m(t)\|_{L^2(D)} + \|u_m\|_{L^p(0,T;V)} \leq C. \quad (\text{A.27})$$

Since $V \subset H$ with continuous injection and the family w_1, \dots, w_m is linearly independent, then the matrix $M = (w_i, w_j)_{1 \leq i, j \leq m}$ is invertible because M is a positive-definite matrix. Then, If we denote $G_m := (g_{mj})_{1 \leq j \leq m}$ we can write (A.26) as

$$G'_m(t) = M^{-1} \left[\langle f(t) - B(u_m(t)), w_j \rangle_{1 \leq j \leq m} \right] = F(t, G_m(t))$$

For all $1 \leq j \leq m$, by the assumptions $\langle B(u_m(t)), w_j \rangle$ is continuous with respect to $u_m(t)$ (therefore is continuous with respect to $g_{mj}(t)$) and since $f \in L^{p'}(0, T; V')$ then f is measurable from $[0, T]$ with values in V' then $\langle f(t), w_j \rangle$ is also measurable with respect to t . So F is a Carathéodory mapping.

By "Local in time existence theorem" [66, Theorem 1.44 page 25] for Cauchy problem, we deduce the existence of a solution in $[0, t_m]$, $t_m > 0$. But we note that

$$\int_D a(t, x, u, \nabla u) \cdot \nabla u dx \geq \bar{\alpha} \int_D |\nabla u|^p dx - \left[\bar{\gamma} \int_D |u|^q dx + \int_D |\bar{h}(t, x)| dx \right].$$

Using Young's and Poincaré's inequalities, there exists $\delta > 0$ such that

$$\int_D a(t, x, u, \nabla u) \cdot \nabla u dx \geq (\bar{\alpha} - \bar{\gamma} \delta C_{Poin}) \int_D |\nabla u|^p dx - \left(\int_D (C + |\bar{h}(t)|) dx \right).$$

We have also

$$\int_D |(u - \psi)^{-|\bar{q}-1}| |u| dx \leq \delta C_{Poin} \int_D |\nabla u|^p dx + |W_\psi(t)|, \quad W_\psi \in L^1([0, T]).$$

By choosing an appropriate δ , we get

$$\int_D a(t, x, u, \nabla u) \cdot \nabla u dx + \frac{1}{\varepsilon} \int_D \Theta(u - \psi) u dx \geq \frac{\bar{\alpha}}{2} \int_D |\nabla u|^p dx - |W_{\varepsilon, \psi, \bar{h}}(t)|, \quad W_{\varepsilon, \psi, \bar{h}} \in L^1([0, T]).$$

Then

$$\begin{aligned} \int_0^t (u'_m(s), u_m(s)) ds + \frac{\bar{\alpha}}{2} \int_0^t \int_D |\nabla u_m(s)|^p dx ds - \int_0^t |W_{\varepsilon, \psi, \bar{h}}(s)| ds &\leq \int_0^t \|f(s)\|_{V'} \|u_m(s)\| ds \\ &\leq \int_0^t \|f(s)\|_{V'} |\nabla u_m(s)|_{L^p(D)} ds \\ &\quad + \int_0^t \|f(s)\|_{V'} |u_m(s)|_{L^2(D)} ds. \end{aligned}$$

Therefore

$$\begin{aligned} \int_0^t (u'_m(s), u_m(s)) ds + \frac{\bar{\alpha}}{2} \int_0^t \int_D |\nabla u_m(s)|^p dx ds &\leq C \int_0^t \|f(s)\|_{V'}^{p'} ds + \frac{\bar{\alpha}}{4} \int_0^t \int_D |\nabla u_m(s)|_{L^p(D)}^p dx ds \\ &+ \int_0^t \|f(s)\|_{V'} \left[\frac{1}{2} + \frac{1}{2} |u_m(s)|_{L^2(D)}^2 \right] ds \\ &+ \int_0^T |W_{\varepsilon, \psi, \bar{h}}(s)| ds. \end{aligned}$$

Hence

$$\frac{1}{2} |u_m(t)|_{L^2(D)}^2 + \frac{\bar{\alpha}}{4} \int_0^t \int_D |\nabla u_m(s)|^p dx ds \leq C_{u_0, W, f} + \frac{1}{2} \int_0^t \|f(s)\|_{V'} |u_m(s)|_{L^2(D)}^2 ds \quad (\text{A.28})$$

Using Grönwall's inequality to deduce that $(u_m)_m$ is bounded in $C([0, T]; L^2(D)) \cap L^p(0, T; V)$.

Thanks to [66, Theorem 1.45-(i) page 25], one gets $t_m = T$.

Lemma 3.33. $\mathcal{A} : L^p(0, T; V) \longrightarrow L^{p'}(0, T; V')$ and $\Theta : L^p(0, T; V) \rightarrow L^{p'}(0, T; V')$ are bounded.

Proof. For all $u \in L^p(0, T; V)$, one has

$$\begin{aligned} \|\mathcal{A}(u)\|_{L^{p'}(0, T; V')} &= \sup_{\|v\|_{L^p(0, T; V)} \leq 1} \langle \mathcal{A}(u), v \rangle = \sup_{\|v\|_{L^p(0, T; V)} \leq 1} \int_Q \tilde{a}(t, x, u, \nabla u) \nabla v dx dt \\ &\leq \left(\int_Q |\tilde{a}(t, x, u, \nabla u)|^{p'} dx dt \right)^{1/p'} \\ &\leq \tilde{\beta} \left(\int_Q \left[|\bar{k}(t, x)| + |u|^{r/p} + |\psi|^{r/p} + |\vec{\nabla} u| \right]^{(p-1)p'} dx dt \right)^{1-\frac{1}{p}} \\ &\leq C \tilde{\beta} \left(\int_Q \left[|\bar{k}(t, x)|^p + |u|^r + |\psi|^r + |\vec{\nabla} u|^p \right] dx dt \right)^{1-\frac{1}{p}}. \end{aligned}$$

Since $\bar{k} \in L^p(Q)$, $r < p$, we deduce that \tilde{A} is bounded from $L^p(0, T; V)$ into $L^{p'}(0, T; V')$.

$$\begin{aligned} \|\Theta(u)\|_{L^{p'}(0, T; V')} &= \sup_{\|v\|_{L^p(0, T; V)} \leq 1} \langle \Theta(u), v \rangle = \sup_{\|v\|_{L^p(0, T; V)} \leq 1} \frac{1}{\varepsilon} \int_Q [(u - \psi)^-]^{\tilde{q}-1} v dx dt \\ &\leq \frac{1}{\varepsilon} \left(\int_Q [(u - \psi)^-]^{(\tilde{q}-1)p'} dx dt \right)^{1/p'}. \end{aligned}$$

We have two cases:

- If $p \geq 2$, then $\tilde{q} = 2$ and $p \geq p'$ which implies $L^p(Q) \hookrightarrow L^{p'}(Q)$ and the boundedness of Θ holds.
- If $2 > p > 1$, then $\tilde{q} = p$ which implies $(\tilde{q} - 1)p' = p$ and the boundedness of Θ holds.

□

Since $(u_m)_m$ is bounded in $L^p(0, T; V)$, then $(B(u_m))_m$ is bounded in $L^{p'}(0, T; V')$ thank's to Lemma 3.33.

Remark 3.34. From the proof of Lemma 3.33, one remarks that $\Theta : L^p(Q) \rightarrow L^{p'}(Q)$ is bounded. So if $u_m \rightarrow u$ in $L^p(Q)$, by the theory of Nemytskii operators, one gets that

$$\lim_{m \rightarrow \infty} \Theta(u_m - \psi) = \Theta(u - \psi) \quad \text{in } L^{p'}(Q).$$

Remark 3.35. [66, Remark 8.41]

Let $H_0^s(D)$ satisfying $H_0^s(D) \hookrightarrow W^{1,p}(D)$ (which requires $s \geq 1 + d \frac{p-2}{2p}$) and $H_0^s(D) \hookrightarrow_c L^2(D)$. Consider the operator Δ^s defined on $E = \{\Delta^s u \in L^2(D), u \in L^2(D)\}$ endowed with $\|\cdot\|_E = \|\cdot\|_{H_0^s(D)}$. One can consult ([15]p.115, [39] p.28) and spectral theorem for compact and self-adjoint operator to conclude the existence of orthonormal basis $(v_i)_{i=1}^\infty$ of $L^2(D)$ and orthonormal basis $(v_i/\sqrt{\lambda_i})_{i=1}^\infty$ of $H_0^s(D)$ solves the eigenvalue problem

$$(-\Delta)^s v_i = \lambda_i v_i.$$

Consider the following projectors $P_m(u) := \sum_{i=1}^m \left(\int_D u v_i dx \right) v_i$.

Then the projector $P_m : L^2(D) \rightarrow L^2(D)$ is selfadjoint satisfying

$$P_m L^2(D) = P_m [W_0^{1,p}(D) \cap L^2(D)] = V_m = \text{span}\langle v_1, \dots, v_m \rangle$$

where V_m the spaces used in Faedo-Galerkin method.

After estimating the norm of projectors, one gets

$$\|P_m\|_{\mathcal{L}(L^2(D), L^2(D))} \leq 1 \quad \& \quad \|P_m\|_{\mathcal{L}(H_0^s(D), H_0^s(D))} \leq 1.$$

Lemma 3.36. There exists $s \geq 1$ such that $(\frac{du_m}{dt})_m$ is bounded in $L^{p'}(0, T; H^{-s}(D))$.

Proof. Let $v \in L^p(0, T; H_0^s(D))$, since $P_m u_m = u_m$ and $P_m^* = P_m$ we can estimate

$$\begin{aligned} \left\langle \frac{du_m}{dt}, v \right\rangle &= \left\langle P_m \frac{du_m}{dt}, v \right\rangle = \left\langle \frac{du_m}{dt}, P_m v \right\rangle = \langle f(t) - B(u_m), P_m v \rangle \\ &\leq (\|B(u_m)\|_{L^{p'}(0, T; H^{-s}(D))} + \|f\|_{L^{p'}(0, T; H^{-s}(D))}) \|P_m v\|_{L^p(0, T; H_0^s(D))} \\ &\leq (\|B(u_m)\|_{L^{p'}(0, T; H^{-s}(D))} + \|f\|_{L^{p'}(0, T; H^{-s}(D))}) \|P_m\|_{\mathcal{L}((H_0^s(D), H_0^s(D)))} \|v\|_{L^p(0, T; H_0^s(D))}. \end{aligned}$$

Since $(B(u_m))_m$ is bounded in $L^{p'}(0, T; V') \hookrightarrow L^{p'}(0, T; H^{-s}(D))$ and $f \in L^{p'}(0, T; H^{-s}(D))$, one has

$$\left\langle \frac{du_m}{dt}, v \right\rangle \leq C \|P_m\|_{\mathcal{L}((H_0^s(D), H_0^s(D)))} \|v\|_{L^p(0, T; H_0^s(D))} \leq C \|v\|_{L^p(0, T; H_0^s(D))}.$$

Therefore $(\frac{du_m}{dt})_m$ is bounded in $L^{p'}(0, T; H^{-s}(D))$. \square

By the reflexivity of $L^p(0, T; V)$, $L^{p'}(0, T; H^{-s}(D))$, there exists a subsequences denoted by the same way and $u \in L^p(0, T; V)$, $\frac{du}{dt} \in L^{p'}(0, T; H^{-s}(D))$ such that

$$u_m \rightharpoonup u \text{ in } L^p(0, T; V) \ \& \ \frac{du_m}{dt} \rightharpoonup \frac{du}{dt} \text{ in } L^{p'}(0, T; H^{-s}(D)).$$

By the weak continuity of the mapping $u \mapsto u(t) : W^{1,p,p'}(0, T; V, H^{-s}(D)) \rightarrow H^{-s}(D)$ one has, for any t ,

$$u_m(t) \rightharpoonup u(t) \text{ in } H^{-s}(D).$$

Then, thanks to a classical compactness argument, (A.28) yields $u_m(t) \rightharpoonup u(t)$ in $L^2(D)$ for any t . In particular, $u_m(T) \rightharpoonup u(T)$ and $u_m(0) \rightarrow u_0$ in $L^2(D)$.

Let us fix a positive integer M , $m \geq M$ and $v_M(t) = \theta(t)w_M$ where $\theta \in \mathcal{D}(0, T)$ and $w_M \in V_M$. Then,

$$\int_0^T \left\langle \frac{du_m}{dt}(s), v_M(s) - u_m(s) \right\rangle ds + \int_0^T \langle B(u_m(s)), v_M(s) - u_m(s) \rangle ds = \int_0^T \langle f, v_M(s) - u_m(s) \rangle ds.$$

As $u_m \rightharpoonup u$ in $L^p(0, T; V)$, one has $\int_0^T \langle f, v_M(s) - u_m(s) \rangle ds \rightarrow \int_0^T \langle f, v_M(s) - u(s) \rangle ds$.

We have

$$\int_0^T \left\langle \frac{du_m}{dt}(s), v_M \right\rangle ds + \int_0^T \langle B(u_m(s)), v_M \rangle ds = \int_0^T \langle f, v_M \rangle ds.$$

Since $(B(u_m(s)))_m$ is bounded in $L^{p'}(0, T; V')$, then there exist a subsequence denoted by the same way and $\kappa \in L^{p'}(0, T; V')$ such that $B(u_m(s)) \rightharpoonup \kappa$ in $L^{p'}(0, T; V')$.

At the limit, one has

$$\int_0^T \left\langle \frac{du}{dt} + \kappa - f, v_M \right\rangle ds = 0, \quad \forall v_M, v_M \in W.$$

W , the linear span of $\mathcal{D}(0, T) \otimes \cup_m V_m$, is dense in $L^p(0, T; V)$, for any $v \in L^p(0, T; V)$ and any positive ε , there exists $v_\varepsilon \in W$ such that $v_\varepsilon \rightarrow v$ in $L^p(0, T; V)$, then

$$\int_0^T \left\langle \frac{du}{dt} + \kappa - f, v \right\rangle ds = \int_0^T \left\langle \frac{du}{dt} + \kappa - f, v - v_\varepsilon \right\rangle ds \rightarrow 0 \text{ as } \varepsilon \rightarrow 0.$$

Then $\frac{du}{dt} = f - \kappa \in L^{p'}(0, T; V')$. Therefore $u \in \mathcal{C}(0, T; H)$ and

$$\int_0^T \left\langle \frac{du}{dt}(s), u(s) \right\rangle ds = \frac{1}{2} |u(T)|_{L^2(D)}^2 - \frac{1}{2} |u(0)|_{L^2(D)}^2.$$

By the weak lower semicontinuity of $|\cdot|_{L^2(D)}$, $|u_m(0)|_{L^2(D)}$ converges to $|u_0|_{L^2(D)}$ and since $v_M \in H_0^s(D)$, we can estimate

$$\begin{aligned} \limsup_m \int_0^T \left\langle \frac{du_m}{dt}(s), v_M(s) - u_m(s) \right\rangle ds \\ &= \lim_m \int_0^T \left\langle \frac{du_m}{dt}(s), v_M(s) \right\rangle ds - \frac{1}{2} \liminf_m |u_m(T)|_{L^2(D)}^2 + \frac{1}{2} \lim_m |u_m(0)|_{L^2(D)}^2 \\ &= \int_0^T \left\langle \frac{du}{dt}(s), v_M(s) \right\rangle ds - \frac{1}{2} \liminf_m |u_m(T)|_{L^2(D)}^2 + \frac{1}{2} \lim_m |u_m(0)|_{L^2(D)}^2 \\ &\leq \int_0^T \left\langle \frac{du}{dt}(s), v_M(s) \right\rangle ds - \frac{1}{2} |u(T)|_{L^2(D)}^2 + \frac{1}{2} |u(0)|_{L^2(D)}^2 \\ &\leq \int_0^T \left\langle \frac{du}{dt}(s), v_M(s) - u(s) \right\rangle ds. \end{aligned}$$

Since $u_m \rightharpoonup u$ in $W^{1,p,p'}(0, T; V, H^{-s}(D))$ then $u_m \rightarrow u$ in $L^p(Q)$ by using a classical compactness arguments of type Aubin-Lions-Simon. By remark 3.34 one has

$$\lim_m -\frac{1}{\varepsilon} \int_0^T \int_D [(u_m - \psi)^-]^{\tilde{q}-1} (v_M - u_m) dx dt = -\frac{1}{\varepsilon} \int_0^T \int_D [(u - \psi)^-]^{\tilde{q}-1} (v_M(s) - u(s)) dx ds.$$

Since

$$\int_0^T \left\langle \mathcal{A}(u_m(s)), v_M(s) - u_m(s) \right\rangle ds = \int_0^T \left\langle f - \frac{du_m}{dt} - \frac{1}{\varepsilon} \Theta(u_m - \psi), v_M(s) - u_m(s) \right\rangle ds.$$

Then

$$\begin{aligned}
& \liminf_m \int_0^T \langle \mathcal{A}(u_m(s)), v_M(s) - u_m(s) \rangle ds \\
&= \liminf_m \int_0^T \langle f - \frac{du_m}{dt} - \frac{1}{\varepsilon} \Theta(u_m - \psi), v_M(s) - u_m(s) \rangle ds \\
&= \int_0^T \langle f - \frac{1}{\varepsilon} \Theta(u - \psi), v_M(s) - u(s) \rangle ds + \liminf_m \int_0^T \langle -\frac{du_m}{dt}, v_M(s) - u_m(s) \rangle ds \\
&= \int_0^T \langle f - \frac{1}{\varepsilon} \Theta(u - \psi), v_M(s) - u(s) \rangle ds - \limsup_m \int_0^T \langle \frac{du_m}{dt}, v_M(s) - u_m(s) \rangle ds \\
&\geq \int_0^T \langle f - \frac{1}{\varepsilon} \Theta(u - \psi), v_M(s) - u(s) \rangle ds - \int_0^T \langle \frac{du}{dt}, v_M(s) - u(s) \rangle ds.
\end{aligned}$$

Then

$$\liminf_m \int_0^T \langle \mathcal{A}(u_m(s)), v_M(s) - u_m(s) \rangle ds \geq \int_0^T \langle f - \frac{du}{dt}(s) - \frac{1}{\varepsilon} \Theta(u - \psi), v_M(s) - u(s) \rangle ds.$$

Since M is some integer, one can conclude that for any $w \in \cup_m V_m$ and any $\theta \in \mathcal{D}(0, T)$, if $v = \theta w$,

$$\limsup_m \int_0^T \langle \mathcal{A}(u_m(s)), u_m - v \rangle ds \leq \int_0^T \langle \frac{du}{dt}(s) + \frac{1}{\varepsilon} \Theta(u - \psi) - f, v - u(s) \rangle ds \quad (\text{A.29})$$

Then, the sub-linearity of the \limsup yields the same inequality of any v in W : the linear span of $\mathcal{D}(0, T) \otimes \cup_m V_m$. This latter space being dense in $L^p(0, T; V)$, for any $v \in L^p(0, T; V)$ and any positive ε , there exists $v_\varepsilon \in W$ such that, thanks to the boundedness of $(\mathcal{A}(u_m))_m$ in $L^{p'}(0, T; V')$,

$$\begin{aligned}
\limsup_m \int_0^T \langle \mathcal{A}(u_m(s)), u_m - v \rangle ds &\leq \limsup_m \int_0^T \langle \mathcal{A}(u_m(s)), u_m - v_\varepsilon \rangle ds + C\varepsilon \\
&\leq \int_0^T \langle \frac{du}{dt}(s) + \frac{1}{\varepsilon} \Theta(u - \psi) - f, v_\varepsilon(s) - u(s) \rangle ds + C\varepsilon \\
&(\text{ as } \varepsilon \rightarrow 0) \leq \int_0^T \langle \frac{du}{dt}(s) + \frac{1}{\varepsilon} \Theta(u - \psi) - f, v(s) - u(s) \rangle ds.
\end{aligned}$$

Thus, (A.29) holds for any $v \in L^p(0, T, V)$, in particular, for $v = u \in W^{1,p,p'}(0, T; V, V')$ we have got $\limsup_m \int_0^T \langle \mathcal{A}(u_m(s)), u_m - u \rangle ds \leq 0$.

Lemma 3.37. *If $u_m \rightharpoonup u$ in $W^{1,p,p'}(0, T; V, H^{-r}(D))$ and $\limsup_m \int_0^T \langle \mathcal{A}(u_m), u_m - u \rangle ds \leq 0$. Then*

$$\liminf_m \int_0^T \langle \mathcal{A}(u_m), u_m - v \rangle ds \geq \int_0^T \langle \mathcal{A}(u), u - v \rangle ds.$$

Proof. Since $u_m \rightharpoonup u$ in $W^{1,p,p'}(0, T; V, H^{-s}(D))$ then $u_m \rightarrow u$ in $L^p(Q)$ by using a classical compactness arguments of type Aubin-Lions-Simon. Note that

$$\limsup_m \int_0^T \langle \mathcal{A}(u_m), u_m - u \rangle ds \leq 0 \iff \limsup_m \int_Q \tilde{a}(t, x, u_m, \nabla u_m) \nabla(u_m - u) dx ds \leq 0.$$

By (1.2), one has that $(\tilde{a}(t, x, u_m, \nabla u_m))_m$ is bounded in $(L^{p'}(Q))^d$. Therefore, there exists $\vec{\xi} \in (L^{p'}(Q))^d$ such that

$$\tilde{a}(t, x, u_m, \nabla u_m) \rightharpoonup \vec{\xi} \text{ in } (L^{p'}(Q))^d.$$

Since

$$0 \leq \int_Q [\tilde{a}(t, x, u_m, \nabla u_m) - \tilde{a}(t, x, u_m, \nabla u)] \nabla(u_m - u) dx ds,$$

one has

$$\int_Q \tilde{a}(t, x, u_m, \nabla u_m) \nabla(u_m - u) dx ds \geq \int_Q \tilde{a}(t, x, u_m, \nabla u) \nabla(u_m - u) dx ds$$

By (1.2), one has that for any $v \in L^p(0, T, V)$,

$$|\tilde{a}(t, x, u, \nabla v)|^{p'} \leq C [1 + |\bar{k}|^p + |u|^p + |\psi|^p + |\nabla v|^p],$$

so that, since $u \in \mathbb{R} \mapsto a(t, x, u, \nabla v)$ is a continuous function, by the theory of Nemytskii operators, one gets that

$$\tilde{a}(t, x, u_m, \nabla u) \rightarrow \tilde{a}(t, x, u, \nabla u) \text{ in } L^{p'}(Q),$$

and

$$\int_Q \tilde{a}(t, x, u_m, \nabla u) \nabla(u_m - u) dx dt \rightarrow 0.$$

Therefore $\liminf_m \int_Q \tilde{a}(t, x, u_m, \nabla u_m) \nabla(u_m - u) dx ds \geq 0$. By the hypothesis on the upper limit one has

$$\lim_m \int_Q \tilde{a}(t, x, u_m, \nabla u_m) \nabla(u_m - u) dx ds = 0.$$

Using the assumptions, for $v \in L^p(0, T, W_0^{1,p}(D))$, we have

$$\begin{aligned} 0 &\leq \int_Q [\tilde{a}(t, x, u_m, \nabla u_m) - \tilde{a}(t, x, u_m, \nabla v)] \nabla(u_m - v) dx ds \\ &\leq \int_Q \tilde{a}(t, x, u_m, \nabla u_m) \nabla(u_m - u) dx ds + \int_Q \tilde{a}(t, x, u_m, \nabla u_m) \nabla(u - v) dx ds \\ &\quad - \int_Q \tilde{a}(t, x, u_m, \nabla v) \nabla(u_m - v) dx ds. \end{aligned}$$

Then

$$0 \leq \int_Q [\vec{\xi} - \tilde{a}(t, x, u, \nabla v)] \nabla(u - v) dx ds.$$

By classical "Minty trick", $v = u + \lambda w$, $\lambda \in \mathbb{R}^*$ and $w \in L^p(0, T; V)$.

$$\lim_{\lambda \rightarrow 0} \int_Q [\vec{\xi} - \tilde{a}(t, x, u, \nabla(u + \lambda w))] \nabla w dx dt = 0 \Rightarrow \vec{\xi} = \tilde{a}(t, x, u, \nabla u).$$

As conclusion, $\liminf_m \int_Q \tilde{a}(t, x, u_m, \nabla u_m) \nabla(u_m - v) dx ds \geq \int_Q \tilde{a}(t, x, u, \nabla u) \nabla(u - v) dx dt$. □

Thank's to Lemma 3.37, one can conclude that

$$\liminf_m \int_0^T \langle \mathcal{A}(u_m(s)), u_m - v \rangle ds \geq \int_0^T \langle \mathcal{A}(u(s)), u(s) - v(s) \rangle ds, \quad \forall v \in L^p(0, T; V).$$

Then

$$\begin{aligned} \int_0^T \langle \mathcal{A}(u(s)), u(s) - v(s) \rangle ds &\leq \int_0^T \langle f, u(s) - v(s) \rangle ds - \int_0^T \langle \frac{du}{dt}(s), u(s) - v(s) \rangle ds \\ &\quad - \int_0^T \langle \frac{1}{\varepsilon} \Theta(u - \psi), u(s) - v(s) \rangle ds. \end{aligned}$$

Since v is arbitrary in $L^p(0, T; V)$, we can conclude

$$\int_0^T \langle \mathcal{A}(u(s)), v(s) \rangle ds = \int_0^T \langle f, v(s) \rangle ds - \int_0^T \langle \frac{du}{dt}(s), v(s) \rangle ds - \int_0^T \langle \frac{1}{\varepsilon} \Theta(u - \psi), v(s) \rangle ds.$$

Therefore $\mathcal{A}(u) - f + \frac{du}{dt} = -\frac{1}{\varepsilon} \Theta(u - \psi)$ holds a.e. on $[0, T]$. □

Appendix D: Some extensions on the stochastic obstacle problems.

1. Itô's formula with non-nul trace on the boundary.

We are interested in this subsection in replacing the assumption $\psi \in L^p(\Omega_T, V)$ of H_4 by $\psi \in L^p(\Omega_T, W^{1,p}(D) \cap L^2(D)) \cap L^{\max(p,p')}(\Omega, C([0, T], L^2(D)))$ with a non-positive trace on the boundary[§]. This situation appears for exemple if A is a Leray-Lions type differential operator of the form $A(u, t, \omega) = -\text{div}(a(t, \omega, x, \nabla u)) + b(t, \omega, x, u)$ that can be defined on $\Omega_T \times W^{1,p}(D) \cap L^2(D)$ with values in V' by: $(t, \omega) \in \Omega_T$ a.e.,

$$\langle A(u, t, \omega), v \rangle = \int_D a(t, \omega, x, \nabla u) \nabla v dx + \int_D b(t, \omega, x, u) v dx, \quad \forall v \in V.$$

In order to be able to follow the same steps of our demonstration, only two major points need to be adapted: the first one is in the proof of Lemma 3.14 where choosing $v^* = \psi$ is not possible anymore; the second one is in the proof of Lemma 3.15 since, in this new situation, $u - \psi$ is not with values in V anymore and the classical Itô formula no longer applies. The other modifications are minor ones based on embeddings of V into some Lebesgue's spaces that still hold when replacing $W_0^{1,p}(D)$ by $W^{1,p}(D)$.

Concerning the question of v^* , one can chose for it the solution to the problem

$$\partial_t \left[v^* - \int_0^\cdot \tilde{G}(v^*, \cdot) dW \right] - \Delta_p v^* = \partial_t \left[\psi - \int_0^\cdot \tilde{G}(\psi, \cdot) dW \right] - \Delta_p \psi = \tilde{f}$$

[§]Note that the pathwise continuity assumption can be implicit thanks to arguments similar to [33, Lemma 4.7].

associated with Dirichlet boundary conditions, $v^*(0) = \psi(0)$ and where, by assumption \bar{f} is a predictable process in $L^{p'}(\Omega_T, V')$. v^* exists with the convenient regularity and one still need to prove that $v^* \geq \psi$ to have it in K and use it in the proof. This is achieved by applying formally Itô's formula to the process $(v^* - \psi)^-$ where

$$d(v^* - \psi) - [\Delta_p v^* - \Delta_p \psi]dt = [\tilde{G}(v^*, \cdot) - \tilde{G}(\psi, \cdot)]dW.$$

The question related to Lemma 3.15 is similar since the proof is based on the possibility to apply Itô's formula to the process $(u_\varepsilon - \psi)^-$ where

$$\begin{aligned} d(u_\varepsilon - \psi) + [A(u_\varepsilon, \cdot) - A(\psi, \cdot)]dt - \frac{1}{\varepsilon}[(u_\varepsilon - \psi)^-]^{\tilde{q}-1}dt \\ = hdt + [\tilde{G}(u_\varepsilon, \cdot) - \tilde{G}(\psi, \cdot)]dW. \end{aligned} \quad (\text{A.30})$$

In both situation, one has a predictable process X , being $v^* - \psi$ in the first case and $u_\varepsilon - \psi$ is the second one, with values in $W^{1,p}(D) \cap L^2(D)$ and not *a priori* V , such that $dX + Adt = GdW$ where A is with values in V' and G in $L_Q(L^2(D))$. This is not a classical situation and Itô's formula associated with the negative-part should apply since X has a positive trace on the boundary of D and thus X^- is with values in V .

For any positive integer n , denote by Φ_n the function $x \mapsto \min(1, nd(x, \partial D))$. This is a sequence of bounded 1-Lipschitz continuous functions that converges a.e. to 1 in D . Thus, for any $u \in W^{1,p}(D) \cap L^2(D)$, the product $u\Phi_n$ is in V and if moreover u belongs to V , then $u\Phi_n$ converges to u in V .

Indeed, convergences of $u\Phi_n$ to u in $L^p(D) \cap L^2(D)$ and $\Phi_n \nabla u$ to ∇u in $L^p(D)$ are just applications of Lebesgue Theorem, and

$$\begin{aligned} u \nabla \Phi_n &= nu \nabla d(\cdot, \partial D) 1_{\{0 < d(\cdot, \partial D) < \frac{1}{n}\}}, \\ |u \nabla \Phi_n| &\leq n |u| 1_{\{0 < d(\cdot, \partial D) < \frac{1}{n}\}} \leq \frac{|u|}{d(\cdot, \partial D)} 1_{\{0 < d(\cdot, \partial D) < \frac{1}{n}\}}, \end{aligned}$$

and $u \nabla \Phi_n$ tends to 0 in $L^p(D)$ since, by Hardy's inequality, $\frac{u}{d(\cdot, \partial D)}$ is in $L^p(D)$.

Since the product $\Phi_n A$ for A in V' is $\varphi \in V \mapsto \langle A, \Phi_n \varphi \rangle$, one gets that

$$dX\Phi_n + \Phi_n Adt = \Phi_n GdW$$

with now $X\Phi_n$ with values in V so that Itô's formula is applicable, in particular with the function F_δ introduced in (3.5). Thus, with the notations of the proof of Lemma 3.15

$$\begin{aligned} \varphi_\delta(X\Phi_n) + \int_0^t \langle A, \Phi_n F'_\delta(X\Phi_n) \rangle ds \\ = \int_0^t \Phi_n G F'_\delta(X\Phi_n) dW + \frac{1}{2} \int_0^t \text{Tr}(F''_\delta(X\Phi_n) \{ \Phi_n G \} Q \{ \Phi_n G \}^*) ds. \end{aligned}$$

Note that $F'_\delta(X\Phi_n) = F'_\delta(-X^-\Phi_n)$ and since X^- is in V , passing to the limit in n is possible. Thus, the desired Itô formula is proved for X and Theorem 3.10 holds when one assumes that the obstacle may have a non-positive value on the boundary of D .

2. On bilateral problems.

We are interested in this subsection in saying few words about the situation of double obstacles problems. First, let us precise assumptions on obstacles.

H_4^* : ψ_1, ψ_2 satisfy H_4 with $\psi_2 \geq \psi_1$ a.e. in $D \times \Omega_T$.

H_5^* : Assumption H_5 is satisfied by both obstacles ψ_i $i = 1, 2$:

$$h_i = h_i^+ - h_i^- = f - \partial_t \left(\psi_i - \int_0^\cdot G(\psi_i, \cdot) dW \right) - A(\psi_i, \cdot)$$

with the associated regularity information.

H_6^* : u_0 satisfies the constraints, i.e. $\psi_2(0) \geq u_0 \geq \psi_1(0)$.

H_7^* : h_1^-, h_2^+ are predictable non negative elements of $L^{\tilde{q}'}(\Omega_T, L^{\tilde{q}'}(D))$.

The convex set of admissible functions becomes

$$K_{\psi_1}^{\psi_2} = \{v \in L^p(\Omega_T, V), \quad \psi_1(x, t, \omega) \leq v(x, t, \omega) \leq \psi_2(x, t, \omega) \quad \text{a.e. in } D \times \Omega_T\},$$

and note that $K_{\psi_1}^{\psi_2}$ is not empty since $\psi_i \in K_{\psi_1}^{\psi_2}$, $i = 1, 2$.

The idea is to follow the same strategy than the one used in the one obstacle case. In other words, we consider the same assumptions on the operator A , the multiplicative noise G and update the other assumptions. The corresponding penalized problem is

$$\begin{cases} u_\varepsilon(t) + \int_0^t (A(u_\varepsilon, \cdot) - \frac{1}{\varepsilon} [(u_\varepsilon - \psi_1)^-]^{\tilde{q}-1} + \frac{1}{\varepsilon} [(u_\varepsilon - \psi_2)^+]^{\tilde{q}-1} - f) ds \\ u_\varepsilon(0) = u_0, \end{cases} = u_0 + \int_0^t \tilde{G}(u_\varepsilon, \cdot) dW(s) \quad (\text{A.31})$$

where $\tilde{G}(u_\varepsilon, \cdot) = G(\max(\min(u_\varepsilon, \psi_2), \psi_1), \cdot)$, which satisfies properties similar to G and behaves formally as an additive stochastic source on the free-set where the constraints are violated.

By cosmetic changes of what has been done in Subsection 3.3 and by noticing that the penalized term is the sum of two parts with disjoint supports, one can prove the boundedness of the two parts of penalized terms independently. Then, passing to the limit in (A.31) to prove the existence of a solution. Finally, we can prove the two parts of Lewy-Stampacchia inequalities independently by adapting the arguments used in subsection 3.3.2; and the one of the proof of Lemma 3.12 to get the uniqueness result. Thus, one gets

Theorem 3.38. *Under Assumptions (H_1) - (H_3) and $(H_i^*, i=4,5,6,7)$, there exists a unique predictable stochastic process $(u, \rho_1, \rho_2) \in L^p(\Omega_T, V) \times L^{\bar{q}'}(\Omega_T, L^{\bar{q}'}(D)) \times L^{\bar{q}'}(\Omega_T, L^{\bar{q}'}(D))$ such that:*

i. $u \in L^2(\Omega, \mathcal{C}([0, T], H)) \cap K_{\psi_1}^{\psi_2}, u(0) = u_0.$

ii. $-\rho_1, \rho_2 \geq 0$ and $\forall v \in K_{\psi_1}^{\psi_2}, \langle \rho_i, u - v \rangle \geq 0, \quad i = 1, 2$ a.e. in $\Omega_T.$

iii. *P-a.s, for all $t \in [0, T],$*

$$u(t) + \int_0^t (\rho_1 + \rho_2) ds + \int_0^t A(u, \cdot) ds = u_0 + \int_0^t G(u, \cdot) dW(s) + \int_0^t f ds.$$

iv. *The following Lewy-Stampacchia's inequality holds:*

$$\begin{aligned} -h_2^+ &= - \left(f - \partial_t(\psi_2 - \int_0^\cdot G(\psi_2, \cdot) dW) - A(\psi_2, \cdot) \right)^+ \\ &\leq \partial_t(u - \int_0^\cdot G(u, \cdot) dW) + A(u, \cdot) - f \\ &\leq h_1^- = \left(f - \partial_t(\psi_1 - \int_0^\cdot G(\psi_1, \cdot) dW) - A(\psi_1, \cdot) \right)^-. \end{aligned}$$

The reader interested in relaxing Assumption H_7^* could be inspired by the strategy of [51], for example.

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