



Fluid dynamic instabilities in complex flow systems

Raffaele Longobardi

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THÈSE

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**FLUID DYNAMIC INSTABILITIES IN COMPLEX
FLOW SYSTEMS**

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Dedication

To my cousin Irene, who will live forever in my heart...

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List of publications

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- Studying the sound production in the hole–tone configuration using compressible and incompressible global stability analyses; R. Longobardi, D. Fabre, P. Bonnefis, V. Citro, F. Giannetti, P. Luchini. In proceeding of IUTAM Symposium on “Critical flow dynamics involving moving/deformable structures with design applications”; in press, Springer.
- Stability and sensitivity analysis of a T–shaped micro–mixer with superhydrophobic surfaces; R. Longobardi, V. Citro, L. Siconolfi, F. Giannetti, P. Luchini. In proceeding of XXIII conference of the Italian Association of Theoretical and Applied Mechanics, 2017; ISBN: 978-889-42484-7-0.

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Abstract

Stability analysis is a very powerful tool in order to investigate the properties of a complex fluid system. For example, it turns out to be very useful for understanding the laminar–turbulent transition scenario or to investigate the dynamic evolution of a fluid in very complex situations such as wakes, jets, recirculation bubbles *etc.* In this work, linear stability theory has been applied to very different situations. In the first part, we investigate the stability characteristics of a 2D T-shaped micro–mixer, a very common device in micro– and nano–fluidics, fitted with an anisotropic superhydrophobic texture on the walls of the outlet channel, using a global stability approach. A parametric analysis has been carried out by varying the surface properties, i.e. the equivalent slip length of the grooves and their orientation angle with respect to the direction of the main pressure gradient. We characterize both the primary and the secondary instability of such kind of flow. We show that in some conditions, the presence of the SHS generates an unsteady instability apt to improve the mixing in the channel. The second and third parts concern the linearized study of an incompressible laminar viscous jet passing through a circular aperture. In particular, in the second part we considered the flow passing through a hole of zero thickness. We compute the response of such kind of flow to harmonic perturbations. We characterize both the spatial amplification of perturbations and the impedance, defined as the ratio between the pressure jump and the flow rate across the hole, which is a key quantity to investigate the response of the jet to an acoustic forcing. Owing to the strong spatial amplification of the perturbation the computation requires a special treatment of the downstream boundary conditions, and quickly becomes impossible when the Reynolds number is increased. We introduce a method based

on the analytical continuation in the complex plane of the axial coordinate, thus extending the range of Reynolds number investigated up to $Re = 3000$. The third part concerns the stability of a jet through a circular aperture in a thick plate. Experiments and simulations show that if the plate is thick enough, strong periodic oscillations can occur and lead to characteristic whistling tones, suggesting the existence of a feedback mechanism that supports self-sustained oscillations. We show that, contrary to previous expectations, the feedback mechanism is not related to acoustics and an instability can exist even in a purely incompressible description. We investigate the stability properties of such kind of flow using both the Nyquist criterion, based on the impedance analysis, and the classical global stability approach. Finally, we perform a structural-sensitivity analysis showing that the instability of such kind of flows is connected to the presence of a recirculation region in the hole. In the last part of the thesis we apply the stability analysis to the production of sound in a more traditional configuration, namely the birdcall, where the flow is constrained to pass through two successive holes in curved rigid plates. Although the production of sound in this classical whistle is a compressible phenomenon, an incompressible approach can provide some useful information at least in the region near the hole. We thus initially perform a purely incompressible stability analysis. We identify the critical conditions, the global frequencies, and discuss the structure of the resulting global eigenmodes. In order to reintroduce and evaluate compressible effects, which can be relevant in the cavity between the two holes, we model the cavity as a Helmholtz resonator and couple it to the incompressible simulation. Finally, a fully compressible stability analysis is performed in order to check the accuracy of these simplified approaches in term of critical conditions, global frequencies and structure of the modes.

Abstract in French

L'analyse de stabilité est un outil très puissant pour étudier les propriétés d'un système fluide complexe, telles que des sillages, des jets, des bulles de recirculation, etc. Dans ce travail, la théorie de la stabilité linéaire a été appliquée à des situations très différentes. Dans la première partie, nous étudions les caractéristiques de stabilité d'un micro-mélangeur bidimensionnel en forme de T avec une texture super hydrophobe anisotrope à la surface du canal de sortie, utilisant une approche de stabilité globale. Une analyse paramétrique a été réalisée en faisant varier les propriétés de la surface, c'est-à-dire la longueur équivalente de glissement des rainures et leur angle d'orientation par rapport à la direction principale du gradient de pression. Nous avons caractérisé à la fois l'instabilité primaire et secondaire de ce type de flux. Nous avons montré que dans certaines conditions, la présence de la SHS génère une instabilité instable, capable d'améliorer le mélange dans le canal. Les deuxième et troisième parties concernent l'étude d'un jet laminaire visqueux à travers une ouverture circulaire utilisant une approche linéarisée incompressible. En particulier, dans la deuxième partie, nous avons considéré un écoulement passant par un trou d'épaisseur nulle. Nous avons calculé la réponse de ce type de flux à des perturbations harmoniques. Nous avons caractérisé à la fois l'amplification spatiale des perturbations et l'impédance, qui est une quantité essentielle pour étudier la réponse du jet avec un forçage acoustique, défini comme le rapport entre le saut de pression et le flux traversant le trou. La nature convective de l'instabilité conduit à des très grandes amplifications spatiales, Surtout à grand nombre de Reynolds, qui rendent impossible le calcul direct en coordonnées physiques de la perturbation linéaire et de l'impédance associée. Nous avons introduit une méthode basée sur la

continuation analytique de la coordonnée axiale dans le plan complexe qui nous permet d'étendre la gamme du nombre de Reynolds étudié jusqu'à $Re = 3000$. La troisième partie concerne la stabilité d'un jet à travers une ouverture circulaire dans une plaque épaisse. Les expériences et les simulations montrent que si la plaque est suffisamment épaisse, de fortes oscillations périodiques peuvent se produire et conduire à des sifflements caractéristiques, suggérant l'existence d'un mécanisme de rétroaction conduisant à des oscillations auto-entretenues. Nous avons montré que, contrairement aux attentes précédentes, le mécanisme de rétroaction n'est pas lié à l'acoustique, mais qu'une instabilité peut exister dans un cadre purement incompressible. Nous étudions les propriétés de stabilité de ce type d'écoulement en utilisant à la fois le critère de Nyquist, basé sur l'analyse d'impédance, et l'approche classique de stabilité globale. Enfin, l'analyse de sensibilité structurelle a montré que l'instabilité est associée à l'existence d'une région de recirculation dans le trou. Dans la dernière partie, nous avons appliqué l'analyse de stabilité pour étudier la production sonore d'une configuration plus réaliste, à savoir l'appau des chasseurs, où le flux est contraint de passer par deux trous successifs dans des plaques incurvées. Bien que le sifflet soit lié à des phénomènes compressibles, l'approche incompressible peut fournir des informations utiles, du moins dans la région proche du trou, où, dans certaines conditions, l'écoulement peut être considéré incompressible. Nous avons utilisé initialement une approche de stabilité purement incompressible pour identifier les conditions critiques, les fréquences globales et la structure des modes propres globaux résultants. Afin d'évaluer les effets compressibles, qui peuvent être pertinents dans la cavité entre les deux trous, nous avons modélisé la cavité comme un résonateur de Helmholtz. Enfin, une analyse de la stabilité compressible complète est effectuée afin de vérifier la validité de ces approches simplifiées.

Introduction

0.1 A "romantic" introduction to fluid–dynamic instability

Every-day life experiences contain some simple insights for the understanding of flow stability. If you observe the smoke of a cigarette, for example, you can see that just above the cigarette it ascends right upward in a regular manner, but at a certain distance above the cigarette it appears irregular and chaotic. In the first case the flow motion is defined "*laminar*", while in the second one it is defined "*turbulent*" (see figure 1). The transition from laminar to turbulent isn't a sudden and well defined phenomenon and understanding how a fluid develops from a laminar regime to turbulence has been the attention of very much researcher during the last century and still it isn't fully understood.

Laminar-turbulent transition can be related to the concept of *flow stability*: as for a generic dynamical system, a laminar flow is said to be *stable* if it returns to its initial state after the application of any small perturbation; on the contrary a laminar flow is defined *unstable* if the disturbances generated by small perturbations don't disappear but they grow (in space and time) causing the transition to turbulence.

From a stoical point of view, the first experiments on transition were carried out by Osborne Reynolds in 1883 (Reynolds, 1884), which used a glass pipe injecting ink at its central line in order to observe the fluid motion when varying quantities such as the fluid velocity U , the pipe's radius r and the cinematic viscosity of the flow ν . Finally he defined a non-dimensional parameter $Re = \frac{Ur}{\nu}$, after known as Reynolds number, that was found to govern the laminar-turbulent transition phenomenon.

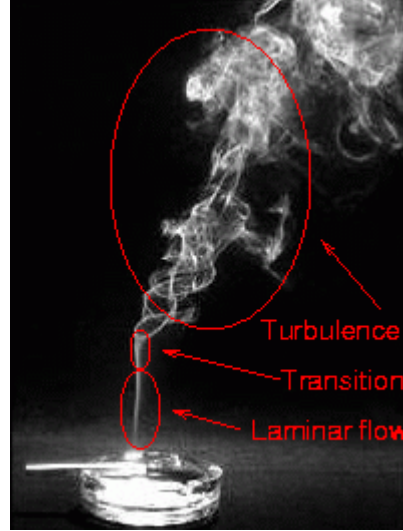


Figure 1: The smoke of a cigarette representing the laminar–turbulent transition phenomenon.

In particular, he observed that at "low" Reynolds numbers the flow was laminar while, increasing the Reynolds number over a certain threshold, known as critical Reynolds number (Re_{cr}), the flow became turbulent, as sketched in figure 2. It is important to remark that the Re_{cr} represents only the beginning of the transition phenomenon: it doesn't mean that the flow will become turbulent, since damping effects could avoid it and the critical Reynolds number can be interpreted as the start of the transition process. It is possible to define also a transitional Reynolds number, Re_{tr} , as the value at which the fluid flow shows a fully turbulent behaviour. However, defining exactly when the transition starts and when the flow can be considered fully turbulent is not so simple and the study of laminar–turbulent transition phenomenon remains still an open problem, involving the interest of both academic research and industry, above all in the aeronautical field. Air transport, in fact, contributes about 3% to the global gas emissions, and the air traffic expected to triple by 2050. Meeting the EU's climate and energy objectives will require a drastic reduction of the sector's environmental impact by re-

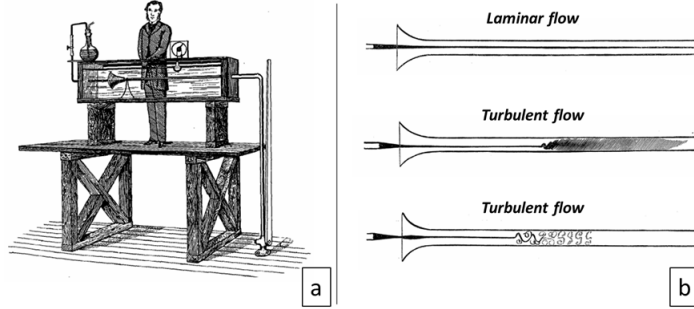


Figure 2: (a) Picture of Osborne Reynolds doing its experiments; (b) General schematization of Reynolds experimental results.

ducing its emissions. In this context, a great contribution is given by the program CleanSky2, a joint technology initiative (JTI) between public and private institutions, that is aimed to reduce the civil aircraft noise and gas emission; in particular, the principal objectives is to reduce the CO_2 and NO_X emissions between the 20% and 30%¹. In order to reach these objectives, the principal aeronautical industries focused their attention both on the design of new efficient engines but also on the aerodynamical design of more performing wings. In particular, it is important to underline that to maintain a laminar and attached flow over the wing generates a drag reduction respect to a turbulent flow and, in this matter, the aeronautical industry started to develop "Laminar Flow Control" techniques (see (Jahanmiri, 2011) for an exhaustive review), able to reduce drag also of about the 15%, as reported in figure 3, leading to a fuel saving and to a reduction of gas emissions. In this context (but also in other industrial context where a drag reduction is required), it is clear the importance of a full comprehension of the transition phenomena, a problem still open and not fully solved, in order to develop efficient and robust control techniques.

¹<http://www.cleansky.eu/>

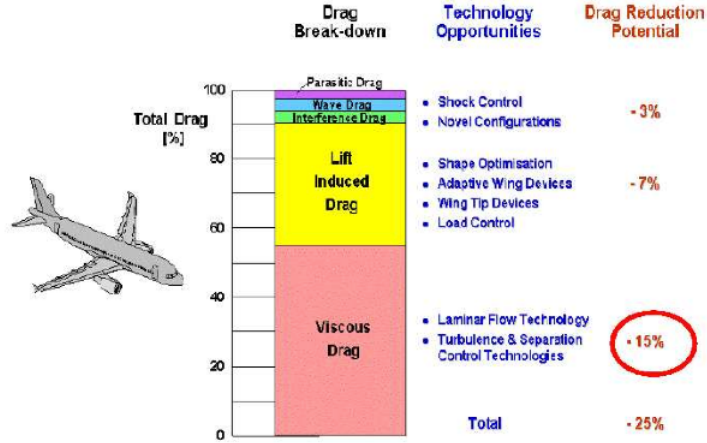


Figure 3: Drag reduction of an aircraft due to the development of new technologies.

0.2 Classification of fluid dynamic instabilities

Before starting to talk about the mathematical theory of the flow instabilities, it seems important to give a general (but not exhaustive) classification of the fluid dynamic instabilities. A first great classification can be made between *asymptotically stable systems*, if the solution tends to zero when the time tends to infinity and *marginally stable systems* if the systems are stable, but not asymptotically: it seems that in this second case the solution is bounded while in the first one it tends to zero.

Another classification can be done about the spatio-temporal evolution of an instability, In particular, it is possible to make a difference between *convective* and *absolute* instability (Drazin and Reid, 2004).

- *Convective instability*

This kind of instability is characterized by the fact that disturbances are convected away and amplified by the mean flow. In this particular case, the instability grows in time while travelling away from the region in which the initial disturbance is applied and

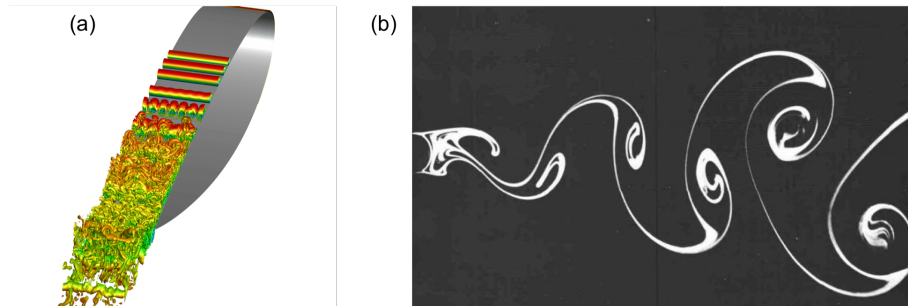


Figure 4: (a) Boundary layer transition over a wing. (b) Von–Karman wake around a circular cylinder at $Re = 50$.

the flow can remain laminar until the disturbance has travelled a certain distance from the source over which it grows following a specific amplification rate and eventually becoming turbulent. On the other hand, in the absence of continuous forcing, the flow eventually returns to its initial state. This is the case of a boundary layer forced with small amplitude perturbations, as depicted in figure 4(a).

- *Absolute instability*

On the contrary of the previous situation, the absolute instability is characterized by the fact that disturbances grow exponentially in time in a fixed point of the space, leading to turbulence. This kind of instability can be found in different flow regimes, as for example in the wake of bluff body, as reported in figure 4(b).

A comparison between the evolution in space and time of a stable, convectively unstable and absolutely unstable solution is given in figure 5.

Another important classification of fluid dynamics instabilities can be given on the basis of the characteristic forces that trigger them.

- *Inviscid Instability*

This kind of instability is relative to large Reynolds number flow, in which the viscous effects are neglected (inviscid). According to

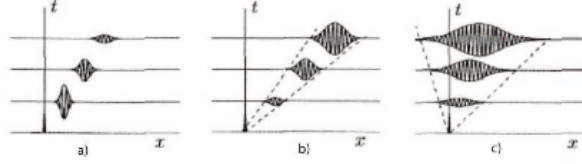


Figure 5: Classification of the stability of a flow in terms of impulse response: (a) stable, (b) convectively unstable and (c) absolutely unstable (from Huerre and Monkewitz (1990))

this theory, the flow with an inflection point of the velocity field are unstable, while the flow without any inflection point are stable at high Reynolds numbers.

- *Viscous Instability*

During the past, it was believed that also viscous profiles without any inflection point were stable: indeed it was found that viscosity can have a destabilizing effect as for example in Poiseuille flows. In this case the stability equation is known as *Orr-Sommerfeld equation*, derived at first for parallel flows but applied, during the years, also to weakly non parallel flows such as boundary layers (the mathematical justification can be found in the multiple-scale analysis, explained in §1.5).

- *Algebraic instability*

There instabilities are not related to the modal growth of the perturbation and they can be inviscid or viscous. In such kind of instabilities the linear modal growth is by-passed and they are relevant to study the by-pass transition of a boundary layer subjected to high level of disturbances (Zuccher et al., 2004). In this case the transition is promoted by low frequency three-dimensional disturbances, as known as streaks.

0.3 Scopes of the thesis

This thesis is about the use of the linear stability theory in complex fluid systems in order to explain some arising peculiar phenomena. In fact, the stability theory is able not only to study the transition to turbulence but also to explain some particular features of a fluid system, such as symmetry breaking, self-sustained processes, the occurrence of bifurcations *etc...* Moreover, using the properties of the direct and adjoint eigenmodes, it is possible to investigate the sensitivity of the flow to the instability, in order to localize the instability core and clarify the instability mechanism.

To these purposes, numerical tools have been developed using the open source code FreeFem++², considering both cartesian and axial-symmetric geometry. A brief description of the various problems considered in this thesis is given in the following subsections; however, all the details within the main results can be found in dedicated chapters.

0.3.1 Superhydrophobic surfaces

Natural surfaces are never smooth, but on the contrary they are always rough, porous, irregular, anisotropic *etc...* Although these "irregularities" have dimensions of some micron, interacting with the flow passing over them, they are able to significantly change its properties and characteristics.

A classical example of relevant natural surface is the shark skin, shown in figure 6(a) as it appears if seen by a microscope. One can see that it is not smooth but it is constituted a series of so called *denticles* that are able to reduce the skin friction and leading to the shark to swim faster (Oeffner and Lauder, 2012).

Another interesting example of natural surfaces are the superhydrophobic surfaces (SHS), consisting in micro grooves containing trapped gas bubbles. Thanks to this properties, SHS reduce the solid-fluid interaction leading, in the general case, to a drag reduction. A famous example of SHS is the lotus leaf, depicted in figure 6(b).

As function of the topology of the surface, it is possible to classify the superhydrophobic surfaces in two macro families:

²<http://www.freefem.org/>

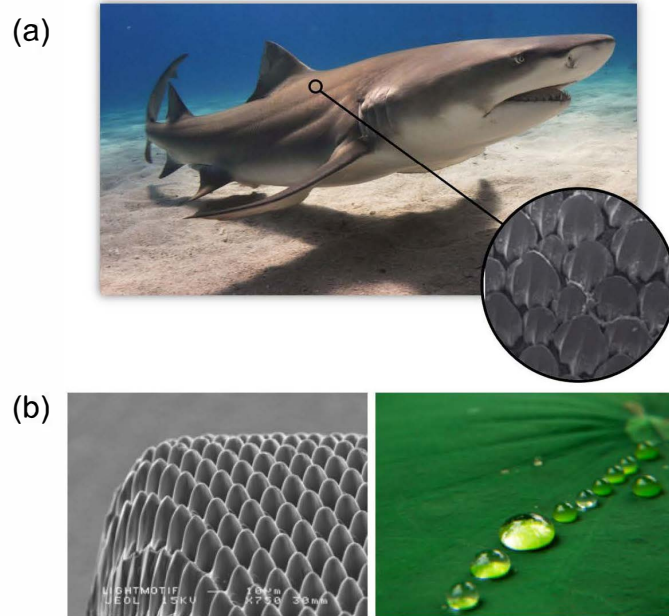


Figure 6: Examples of natural surfaces: (a) the shark skin with the *denticles*; (b) the lotus leaf as example of superhydrophobic surfaces.

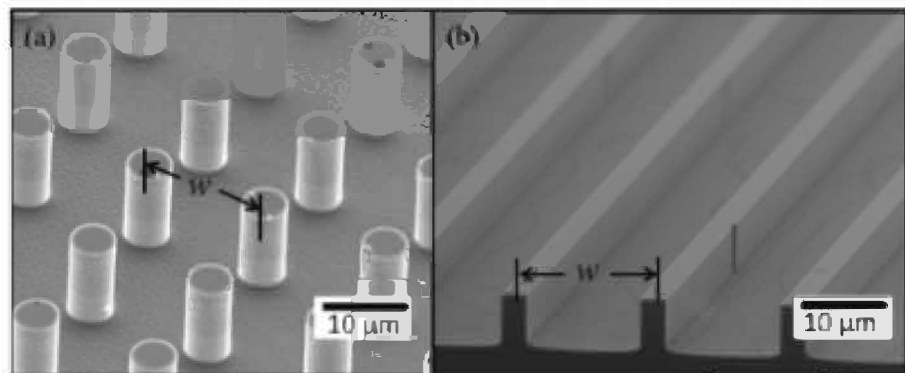


Figure 7: (a) Isotropic vs (b) anisotropic superhydrophobic surface (from Clavijo et al. (2015)).

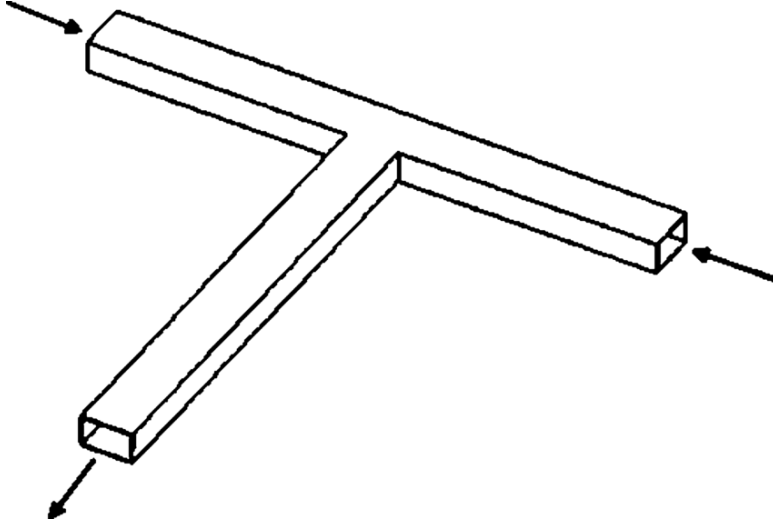


Figure 8: Representation of a T-mixer.

- Isotropic surfaces, characterized by the fact that grooves has not a preferential direction of alignment (figure 7 (a));
- Anisotropic surfaces, where grooves are all aligned with a precise direction (figure 7 (b)).

0.3.2 Micro T-Mixer with superhydrophobic surfaces

The first problem addressed in this thesis is the stability of the flow in a T-shaped micro mixer with superhydrophobic surfaces (SHS) on the outlet channel, as shown in figure 8. As reported in dedicated literature (see for example Daniello et al. (2009)), SHS are able to reduce the turbulent skin friction also in macro channels since, in this case, what is important is that they act on the viscous sub-layer. Instead, as far as the stability and transition are concerned, SHS result to be effective only in micro-channels with characteristic dimensions of few millimetres (Rothstein, 2010). We choose to investigate the flow in a T-mixer since they are very common in microfluidics and often they are also used simply as junction elements in more complex micro systems (Fani, 2013). We

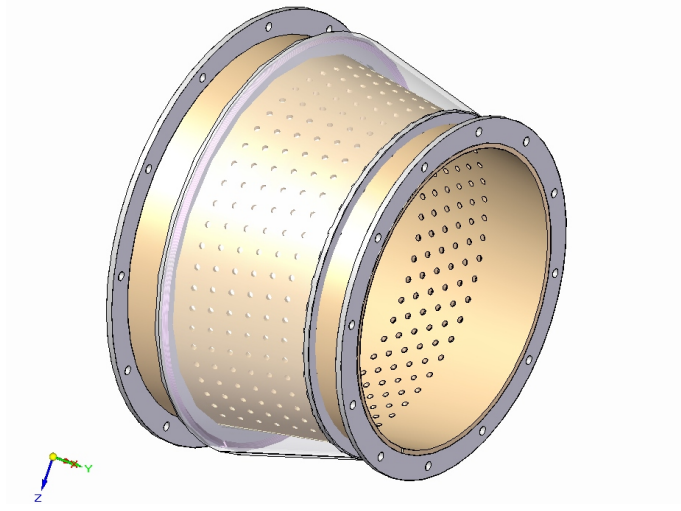


Figure 9: The case of a gas turbine with a perforated acoustic liner, useful both to cool the mechanical parts and to absorb the sound.

show that the presence of such kind of surfaces can largely influence the stability and so the mixing properties of the flow into the T-mixer. From a technical point of view, the presence of the SHS has been taken into account using the Navier slip boundary condition (Navier, 1823), namely considering a partial slip of the flow on the SHS surface: this argument will be further explore in dedicated paragraphs.

0.3.3 Acoustic properties of the flow passing through one or two circular apertures

It is known that the flow passing through one or two circular holes can absorb or generate sound, typically in the form of a strong whistle, and it is relevant in many practical situations such as the flow passing through a cooling system of a gas turbine (see figure 9), the human or bird whistle, the wind instruments, tea kettles *etc...* However, the causes and mechanism of the whistle generation are still not fully understood. In this thesis we try to clarify some mechanics in whistle generation: starting from the crucial hypothesis that whistle is generated



Figure 10: An example of bird-call.

by self-sustained oscillations, we use the stability approach in order to clarify some mechanism leading to the sound emissions of various flow configuration. In particular, we start from the most simple configuration, namely the viscous flow passing through a circular aperture in a thin plate. This is a classical problem first introduced by Rayleigh (1896) in a purely acoustic regime, namely without considering the presence of a mean flow. Moreover, Rayleigh introduced a key quantities, the so called Rayleigh conductivity, playing a crucial role in the study of the stability characteristic of such kind of flow configurations. The solution of this classical problem allows us to introduce the linearized theory and to substantiate its validity comparing the linear results with the ones obtained using non linear simulations, namely the solution in time of the full Navier–Stokes equations .

After this first step, we consider the effect of the thickness of the plate on the flow and whistle properties of such kind of flow configuration. In particular, we introduce a Nyquist stability criterion based on the analysis of the acoustic impedance, a key quantities strictly linked with the Rayleigh conductivity; then we solve a classical eigenvalue problem in order to validate the stability criterion previously introduced, varying the thickness of the plate. Moreover, using the concept of structural sensitivity, we identify the instability core.

Finally, we investigate a more complicate and realistic configuration, namely the flow passing through two circular apertures in curved plates:

this configuration is also known as bird–call and it is shown in figure 10. We use a global stability analysis in order to clarify the mechanism of whistle emission. We find that the mechanism of whistle generation is quite different for this configuration respect to the previous since the cavity between the two holes plays a crucial role both in whistle generation and in the frequency selection process. We introduce also the effect of the flow compressibility considering both a simplified model based on the Helmholtz equations and the full compressible Navier–Stokes equations, discussing about the range of validity of the simplified model.

0.4 Thesis outline

The present thesis is organized in 5 chapters as follow:

- In Chapter 1 the general equations, the theory and the methodology are introduced. In particular, we first introduce the equation governing the motion of an incompressible flows, namely the incompressible Navier–Stokes equations. Then, we introduce the linear stability theory, that is largely used in literature to characterize the dynamical response of numerous systems to little disturbances (see for example Reed et al. (1996), Chomaz (2005)). We first start introducing the linear stability theory applied to parallel (such as pipe flows) and weakly non parallel flows (such as boundary layers), also known as *local stability* theory. Then we relax some hypothesis introducing the so called *global stability* theory, that is more relevant in the context of this thesis. In fact, global stability theory, is a very powerful tools to study the dynamical behaviour of strongly non parallel flows, as for example cavity flows, bluff body wakes *etc...* We introduce also another very powerful tool in the stability context, namely the adjoint Navier–Stokes equations, largely used to identify optimal perturbations (Luchini, 2000), namely the initial conditions that maximize the energy growth of a disturbance, study the receptivity of a flow to external disturbances (Zuccher and Luchini, 2014), in fluid control (Marquet and Sipp, 2010; Carnarius et al., 2010; Carini et al., 2015) and to identify the most sensitive regions of a flow to an instability (Giannetti and Luchini, 2007; Marquet et al., 2008). Here, we use the adjoint equations to

calculate the *structural sensitivity* of the instability for both the T-mixer and the flow passing through a circular hole, in order to identify the *instability core*. Finally, we introduce the global stability theory applied to the compressible Navier–Stokes equations, that is applied to study the bird–call configuration.

- In Chapter 2 we report the main numerical tools used in this thesis. In particular, in the first part we introduce the Finite Element Method (FEM) and the FreeFem++ open source library. The second part, instead, is dedicated to the numerical algorithms.
 - Chapter 3 is about the stability and sensitivity analysis of a T-shaped micro-mixer with superhydrophobic surfaces. We perform a global stability analysis of such kind of flow configuration varying the properties of the SHS, namely its height and orientation respect to the main pressure gradient direction, and we compare results with a classical T-mixer without SHS. We investigate the transition scenario, first characterized by a pitchfork supercritical bifurcation that breaks the symmetry of the flow. The presence of the SHS leads to a reduction of the critical Reynolds number of the first bifurcation. Then, we investigate the existence of a 3D instability of the 2D asymmetric flow. We find that the presence of SHS reduce the critical Reynolds number also for the second bifurcation. However, the more interesting results is that, for particular parameters of the SHS, the secondary instability of the flow is characterized by an Hopf supercritical bifurcation, meaning that the flow shows self-sustained oscillations on 3D plane, leading to an improvement of the mixing efficiency between the two fluids coming from the two inlet channels. The contents of this chapter have been presented to the AIMETA conference (Salerno, September 2017) and published in the conference proceeding.
 - Chapter 4 is about the acoustic impedance of the laminar flow passing through a circular aperture in a thin plate. Actually, this is a classical problem already introduced and solved by Rayleigh (1896) without considering the mean flow and by Howe (1979) in the inviscid case. In this context, we introduce the effect of viscosity and we calculate the impedance for the thin hole using the
-

solution of the linearized Navier–Stokes equations. The solution of such kind of problem is difficult from a computational point of view since the flow results to be highly convective unstable increasing the Reynolds number, so that the perturbation levels can become so great to be compared to the machine precision, leading to numerical round–off problems and so to an inaccurate evaluation of the acoustic impedance. In order to solve this problem, an original method based on the complex mapping of the axial coordinate has been rigorously introduced and tested. This trick lead us to compute the acoustic impedance up to $Re = 3000$. Finally, we verify the validity of the linearized theory by solving in time the non linear Navier–Stokes equations. We find that, although results show a strongly non linear behaviour, the acoustic properties of such kind of flow configuration are well predict by using the linearized approach.

The contents of this chapter have been presented to the Euromech colloquium 591 on "Three–dimensional instability mechanisms in transitional and turbulent flows" (Bari, September 2017) and a full paper is under review for the publication in Journal of Fluid Mechanics.

- Chapter 5 discusses the same topic introduced in chapter 4, but for a circular hole in a thick plate. Using the same numerical tools previously introduced in chapter 4, we calculate the acoustic impedance varying the Reynolds number and the thickness of the hole. Using a Nyquist stability criterion, we demonstrate, for some parameters, the possibility of the purely hydrodynamical system to show self–sustained oscillation that are directly linked to the whistle emission. Moreover, we describe also the possibility to have an instability if the purely hydrodynamical system is coupled with an outer system, ad for example an acoustic resonator. Finally, using the global stability approach, we confirm the results found using the Nyquist stability criterion and, thanks to the direct–adjoint properties, we calculate the structural sensitivity of the instability. We demonstrate the crucial role that the thickness of the plate plays into the instability generation: in fact we find that the instability core is localized on the edge of the recirculation bubble

forming below the hole.

Results of this chapter have been presented to the Euromech colloquium 591 on "Three-dimensional instability mechanisms in transitional and turbulent flows" (Bari, September 2017) and a full paper has been submitted to Journal of Fluid Mechanics.

- In Chapter 6 we study the whistling properties of a bird-call, that is a more realistic geometry, using a global stability approach. In particular, an "augmented model" based on the Helmholtz equation has been introduced in order to take into account the effects of the local compressibility of the flow in the region near the holes. Results have been compared with the ones obtained considering the full compressible Navier–Stokes equations, but with very low Mach numbers. In particular we found that the model is able to give suitable results when the hypothesis of acoustically compact geometry is respected, namely the acoustic wavelength must be greater than the main geometrical dimensions. We discuss in detail the limit of validity of the model and finally we give an example of a real experiment³.

The contents of this chapter have been presented to the IUTAM Symposium on "Critical flow dynamics involving moving/deformable structures with design applications" (Santorini, June 2018) and the full paper has been accepted for the publication as conference proceeding.

Finally, in the last chapter, we try to give some general conclusions of this thesis.

³Actually some preliminary experiments have been carried out at IMFT and it is planned to deepen this topic in next year.

Chapter 1

Theoretical background

1.1 The Navier–Stokes equations: incompressible formulation

The motion of fluids is described by the Navier–Stokes equations, a system of partial differential equations (PDEs) first proposed by Claude–Louis Navier and Geroge Gabriel Stokes. Assuming the flow as a continuous medium, with a constant density ρ (hypothesis of incompressibility) and a direct proportionality between the viscous stress and the strain rate, namely a Newtonian fluid, the Navier–Stokes equations can be written in their non–dimensional form as follow:

$$\left. \begin{aligned} \nabla \cdot \mathbf{u} &= 0 \\ \partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} + \nabla p - \frac{1}{Re} \Delta_2 \mathbf{u} &= 0 \end{aligned} \right\}, \quad (1.1)$$

where $\mathbf{u}(\mathbf{x}, t)$ is the velocity vector, namely (u, v, w) for the cartesian case and (u_x, u_r, u_θ) in the axis–symmetric one, $\mathbf{x}$ is the vector containing the spatial coordinate, namely (x, y, z) for the cartesian case and (x, r, θ) in the axis–symmetric one and $p(\mathbf{x}, t)$ is the reduced pressure. Re is the Reynolds number defined as:

$$Re = \frac{U_{ref} L_{ref}}{\nu}, \quad (1.2)$$

with U_{ref} and L_{ref} respectively the velocity and length characteristic scale of a certain problem and ν the cinematic viscosity of the flow. In

this thesis, we use the cartesian formulation of the Navier–Stokes equations in order to study the stability of the flow in the T–mixer whereas, for geometrical consideration, the acoustic of circular hole has been studied using the axial–symmetric version of the Navier–Stokes equations. The solution in time of the Navier–stokes equations (1.1) fully describes the motion of a fluid. Analytical solutions of the system (1.1) are known only in very simple cases whereas, in order to study more complicated configuration, a numerical resolution is necessary. However, the numerical resolution in time of the full three–dimensional problem can be difficult (or also impossible) from a computational point of view both in term of memory required for the computation but also for the computational time. In this thesis, since we are interested to the stability and transition, we use a linearized approach, assuming that the flow can be decomposed in a steady base flow and a perturbation of small amplitude so that a linearization is justified. We show that the use of the linearized dynamic of the equations (1.1) can give very important information about the stability of the flow and it can be applied to study very different phenomena, even if at the beginning they can seem so different.

1.1.1 Boundary conditions on the SHS: the Navier slip length

In order to be solved, the Navier–Stokes equations 1.1 must be completed with suitable boundary and initial conditions: these conditions are different for each problem and so they will be specified in dedicated chapters. However, classical boundary conditions are no–slip at wall ($\mathbf{u} = \mathbf{0}$), an imposed velocity profile at inlet, ad–hoc symmetry boundary conditions on the symmetry axis in the cylindrical case, and traction–free at an open outlet ($-p\mathbf{n} + Re^{-1}\nabla\mathbf{u} \cdot \mathbf{n} = 0$). One of the arguments of this thesis is the stability of the flow over a superhydrophobic surface and a crucial point is the use of a correct boundary condition on such kind of surfaces. The SHS, in fact, can be seen as an alternation between no slip (wall) and no shear (gas bubbles) zones, as depicted in figure 1.1, and in fact the *hamletic* question arising in many dedicated articles and textbooks treating SHS is *to slip or not to slip?*, or, in other words, how is it possible to take into account this alternation between no–slip and free–shear regions? The most accepted answer is to use the partial slip

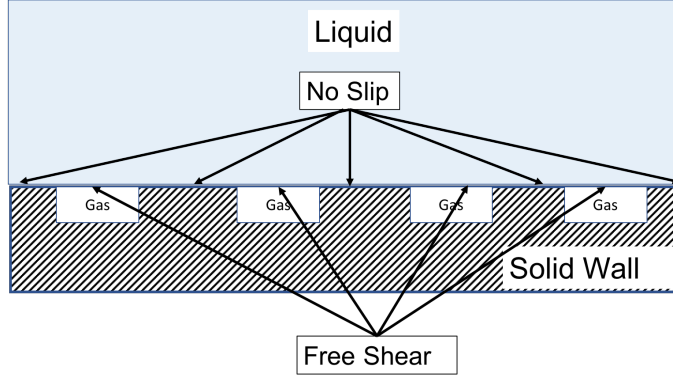


Figure 1.1: Schematic representation of a superhydrophobic surface highlighting the no slip and free shear regions.

Navier boundary condition (Navier, 1823) that can be written as:

$$\mathbf{u}_{\parallel}|_{wall} = \lambda \frac{\partial \mathbf{u}_{\parallel}}{\partial \mathbf{n}}|_{wall} \quad (1.3)$$

with $\mathbf{u}_{\parallel}$ is the vector containing the velocity components tangential to the surface and $\mathbf{n}$ is the outer normal vector. According to the equation (1.3), the velocity at the wall is proportional to the shear strain rate $\partial \mathbf{u}_{\parallel} / \partial \mathbf{n}$ via the Navier slip length λ , representing the fictitious distance below the surface where the velocity vanishes if the flow field is extended linearly through the solid wall Lauga and Stone (2003). If $\lambda = 0$ a classical no slip boundary conditions is retrieved whereas, on the contrary, for $\lambda \rightarrow \infty$ a perfect slip boundary condition is obtained: the geometrical interpretation of the Navier slip length is depicted in figure 1.2. The boundary condition for the velocity component normal to the surface, instead, derives from the continuity equation (that for incompressible fluids can be seen as a constrain equation) and it is a non penetration boundary condition, namely $\mathbf{u} \cdot \mathbf{n} = 0$. However, for superhydrophobic walls, it is correct to impose a null vertical velocity on the groove but a question could arise in case of wetted state, namely when the liquid penetrates into the gas bubble (also known as Wenzel state): is it possible to define a slip length also for the vertical velocity component? Luchini (2013) demonstrated that even for turbulent flows the near wall region

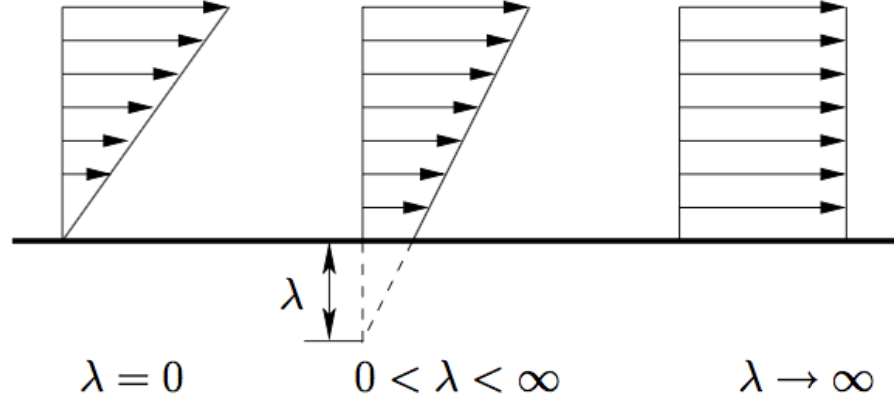


Figure 1.2: Geometrical interpretation of the Navier slip length λ .

can be described using the Stokes equations and the near wall Stokes problem has two degrees of freedom, so that only two slip length can be defined: imposing a null vertical velocity at a certain y rather than at another one has only a secondary effects on results.

The effectiveness of the Navier slip boundary condition has been experimentally observed and some classical results for the Poiseuille flow over a superhydrophobic surface are reported in figure 1.3. In equation (1.3) the same slip length λ has been used for all the velocity components, meaning that there are no preferential direction of the surface: the condition (1.3) is able to describe an isotropic texture. In chapter 3 we will show how to extend the condition (1.3) to an anisotropic texture.

1.2 The linearized Navier–Stokes Equations (LNSE)

In order to study the linearized dynamic of the Navier–Stokes equations, the following flow decomposition has to be introduced:

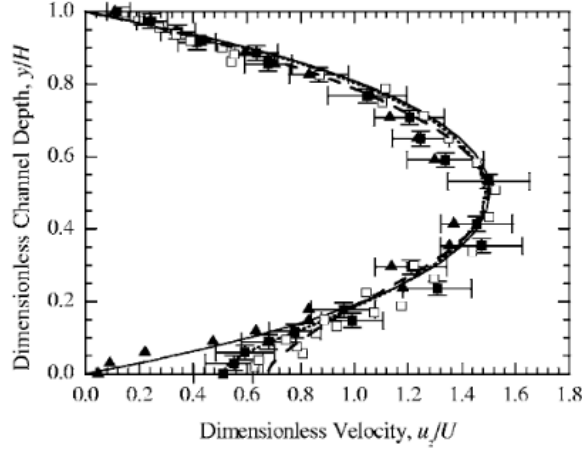


Figure 1.3: Comparison between the numerical solution (lines) and the PIV measurement (symbols) of a channel flow over a superhydrophobic surface: the filled symbols show measurements above the ribs, whereas the empty symbols are taken above the gas-water interface (from Ou and Rothstein (2005)).

$$\left. \begin{aligned} \mathbf{u}(\mathbf{x}, t) &= \mathbf{U}(\mathbf{x}) + \varepsilon \hat{\mathbf{u}}(\mathbf{x}, t) + \mathcal{O}(\varepsilon^2) \\ p(\mathbf{x}, t) &= P(\mathbf{x}) + \varepsilon \hat{p}(\mathbf{x}, t) + \mathcal{O}(\varepsilon^2) \end{aligned} \right\}, \quad (1.4)$$

where the quantities in capital letters are the steady base flow whereas the quantities with the hat are the perturbation of *small* amplitude ε . Inserting the flow decomposition (1.4) into the Navier–Stokes equations (1.1) and linearizing, two sets of PDE’s are obtained: one for the base flow (order $\mathcal{O}(\varepsilon^0)$) and one for the perturbation $\mathcal{O}(\varepsilon^1)$. Here we suppose that the base flow is steady and so it is a solution of the steady version of the non linear Navier–Stokes equations (1.1): a more complicated theory (the Floquet stability analysis) can be used for time periodical base flow but this is out from our purposes (see (Barkley and Henderson, 1996) for details).

The temporal evolution of the perturbation, instead, is governed by the so called Linearized Navier–Stokes equations (LNSE) that can be written as follow:

$$\left. \begin{aligned} \nabla \cdot \hat{\mathbf{u}} &= 0 \\ \partial_t \hat{\mathbf{u}} + \mathcal{L}\{\mathbf{U}, Re\} \hat{\mathbf{u}} + \nabla \hat{p} &= 0 \end{aligned} \right\}, \quad (1.5)$$

where $\mathcal{L}$ is the linearized Navier–Stokes operator defined as:

$$\mathcal{L}\{\mathbf{U}, Re\} \hat{\mathbf{u}} = (\mathbf{U} \cdot \nabla) \hat{\mathbf{u}} + (\hat{\mathbf{u}} \cdot \nabla) \mathbf{U} - \frac{1}{Re} \Delta_2 \hat{\mathbf{u}}. \quad (1.6)$$

In order to solve the differential problems (1.5), we have to impose the appropriate conditions at the boundaries of the domain under investigation and a suitable initial condition as well. These conditions will be specified in dedicated chapters for each different problem studied in this thesis.

1.3 Linear Stability Theory (LST)

If we are interested to study the stability of the flow system for $t \rightarrow \infty$, we can represent the perturbation, namely $\hat{\mathbf{q}}(\mathbf{x}, t) = [\hat{\mathbf{u}}, \hat{p}]^T(\mathbf{x}, t)$ with a Fourier expansion¹

$$\hat{\mathbf{q}}(\mathbf{x}, t) = \tilde{\mathbf{q}}(\mathbf{x}) e^{\sigma t} + c.c. \quad (1.7)$$

where $\tilde{\mathbf{q}}(\mathbf{x}) = [\tilde{\mathbf{u}}, \tilde{p}]^T(\mathbf{x})$, $\sigma = \gamma + i\omega$ is, in general, a complex quantity (particular considerations will be done in dedicated part of this manuscript when necessary) and *c.c.* stands for the complex conjugate. Inserting this ansatz into the equations (1.5), we can rewrite the LNSE as follow:

$$\left. \begin{aligned} \nabla \cdot \tilde{\mathbf{u}} &= 0 \\ \sigma \tilde{\mathbf{u}} + \mathcal{L}\{\mathbf{U}, Re\} \tilde{\mathbf{u}} + \nabla \tilde{p} &= 0 \end{aligned} \right\}. \quad (1.8)$$

In the more general case the system (1.8) is a generalized eigenvalue problem and $\sigma = \gamma + i\omega$ is the complex eigenvalue. The real part γ of the eigenvalue σ is the temporal growth rate of the perturbation whereas

¹ Actually the name of Fourier transform is not so appropriate since this is more similar to a Laplace transform because the exponent σ is, in the more general case, a complex quantity.

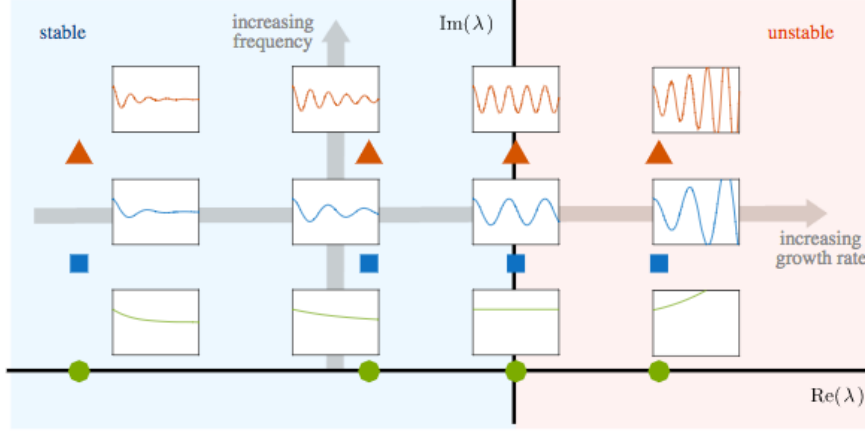


Figure 1.4: Real vs. imaginary part of the complex eigenvalue depicting the possible stability scenarios within the relative time evolution of the perturbation (from Taira et al. (2017)).

the imaginary part ω represents the frequency. For $\gamma < 0$, the flow is stable whereas for $\gamma > 0$ it is unstable and self-sustained oscillation of frequency ω can be observed. The neutral stability condition, instead, is observed for $\gamma = 0$: in this case the perturbation has a purely harmonic time evolution and it is neither dumped nor amplified. The scenarios described above are summarized in figure 1.4. Up until now, no considerations have been made about the spatial coordinate. Actually, as function of the specific problem, it is possible to find some homogeneous directions of the flow: in this way a Fourier representation of the homogeneous direction is licit, leading to a simplification of the problem and a reduction of the computational cost. In a general case, following the notation adopted by Juniper M. *et al.* (Juniper et al., 2014), we can apply the following transformation to the amplitude of the perturbation:

$$\tilde{\mathbf{q}} = \mathbf{q}' e^{i\Theta} \quad (1.9)$$

where Θ changes as function of the considerations about the directional flow homogeneities.

1.4 Local stability theory

From an historical point of view, the linear stability analysis was applied to parallel channel flows, as for examples plane Couette or Poiseuille flows, characterized by two homogeneous spatial directions, namely the streamwise and the spanwise one. This implies that the wall normal and the streamwise gradient of the base flow are negligible, leading to a monodimensional base flow ($U(y)$) and the following representation of the perturbation:

$$\tilde{\mathbf{q}}(\mathbf{x}) = \mathbf{q}'(y)e^{i(\alpha x + kz)} + c.c \quad (1.10)$$

where α and k are respectively the streamwise and the spanwise wavenumbers. Inserting the expansion (1.10) into the LNSE (1.8), we obtain a set of ODEs (Ordinary Differential Equations), known also as Orr–Sommerfeld equations:

$$\left. \begin{aligned} i\alpha u' + \partial_y v' + ikw' &= 0 \\ \sigma u' + i\alpha U u' + \partial_y U v' + i\alpha p' - \frac{1}{Re}(\partial_y^2 u' - \alpha^2 u' - k^2 u') &= 0 \\ \sigma v' + i\alpha U v' + \partial_y p' - \frac{1}{Re}(\partial_y^2 v' - \alpha^2 v' - k^2 v') &= 0 \\ \sigma w' + i\alpha U w' + ikp' - \frac{1}{Re}(\partial_y^2 w' - \alpha^2 w' - k^2 w') &= 0 \end{aligned} \right\}. \quad (1.11)$$

The equations (1.11), completed by homogeneous boundary conditions, represent a generalized eigenvalue problem. The eigenvalue $\mathbf{q}'$ exists only for values of α , k and σ satisfying the following dispersion relation:

$$\mathcal{D}(\alpha, k, \sigma, Re) = 0. \quad (1.12)$$

At this point, two different scenario are possible. We can be interested to study the temporal growth of the disturbance: in this case α and k are real and $\sigma = \gamma + i\omega$ is the complex eigenvalue, that is provided by the dispersion relation (1.12) and gives informations about the stability of the system. The eigenvalue problem (1.11) has to be solved varying the Reynolds number: we can define a critical Reynolds number Re_{cr} as the Re at which the flow pass from a stable state to an unstable one, namely when the growth rate is equal to zero: this condition is also

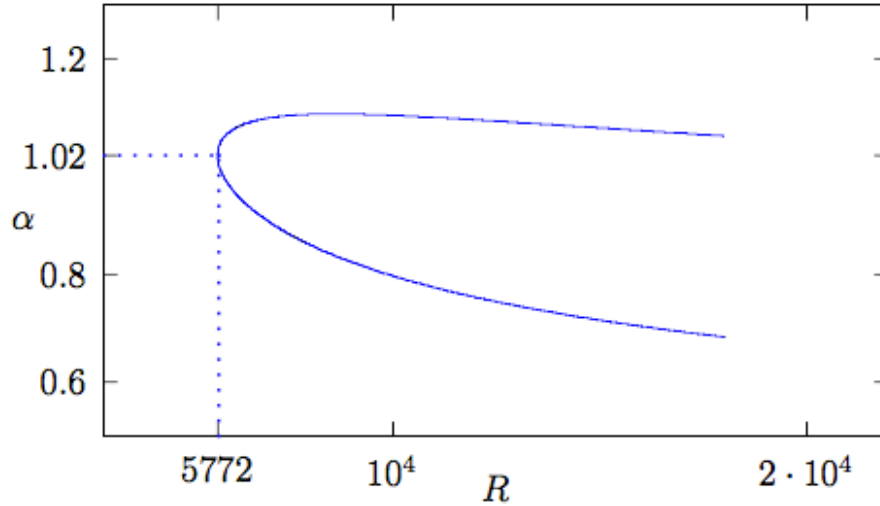


Figure 1.5: Neutral stability curve for plane Poiseuille flow for k (taken from HarrarII and Osborne (2003)).

known as marginally stability. A very important result of the stability analysis is the neutral curve, namely the locus of point on the plane $\alpha - Re$ at which $\gamma = 0$: in figure 1.5 we report as example the neutral curve for the Poiseuille flow, showing that the critical Reynolds number is $Re_{cr} \approx 5772$ at $\alpha \approx 1.02$. The neutral curve has been obtained for $k = 0$: in fact, for plane and parallel flows the Squire theorem states that the first instability is for bidimensional disturbances.

On the other hand, we could also be interested in the spatial growth of the disturbance in space, for example in the streamwise direction: in this case k is real, $\sigma = -i\omega$ is purely imaginary and $\alpha = \alpha_r + i\alpha_i$ is the complex eigenvalue provided by the dispersion relation (1.12). In this case, if $\alpha_i < 0$, the system results to be unstable, whereas it is stable for $\alpha_i > 0$; finally, α_r represents the frequency of oscillation of the disturbances.

The computational solution of such kind of problems is very fast: usually for such kind of flows the base flow is expressed with an analytical function and the stability problem consists in a one dimensional eigenvalue

problem that can be solved in a very efficient way using, for example, finite difference or spectral collocation methods. Note that the spatial stability analysis is a little more complicated as the eigenvalue problem is non linear respect to α .

The theory reported in this paragraph has been written for plane flows. however, without any difficulties, it can be written also for axisymmetric monodimensional flows, such as a Poiseuille flow in a pipe. In this case, the expansion (1.10) becomes:

$$\tilde{\mathbf{q}}(\mathbf{x}) = \mathbf{q}'(r)e^{i(\alpha x + m\theta)} + c.c \quad (1.13)$$

where the azimuthal wavenumber must be integer, namely $m \in \mathbb{N}$. The stability analysis shown about the parallel flow is also known as *local*, because the study of the stability properties of the fluid is reduced to the solution of a one dimensional problem independent from the streamwise location.

1.5 Stability of weakly non parallel flows

The spatial stability analysis can be applied, for example, to study the receptivity and the transition to the turbulence of a boundary layer. Actually the boundary layer is a weakly non parallel flow, because it shows a slow growth in the streamwise direction, almost in the region far from the leading edge. However, it is possible to generalize the stability theory written for parallel flows to the weakly non parallel ones, using a multiple scale or WKBJ approach (Bender and Orszag, 2013). Without going into the details, this method consist of introduce a *slowness parameter* ϵ through the changing of scale $X = \epsilon x$. The fundamental hypothesis of this method is that it is possible to write the solution in a power series, namely:

$$\tilde{\mathbf{q}}(X, y, z) = e^{i(\phi(X)/\epsilon) + kz} \sum_{n=0}^{\infty} \mathbf{q}'_n(X, y) \epsilon^n, \quad (1.14)$$

where the streamwise wavenumber $\alpha(X) = \partial_X \phi(X)$. Substituting the expansion (1.14) into the equations (1.8), separating the various order ϵ , a hierarchy of equation is obtained (Zuccher and Luchini, 2014). What is

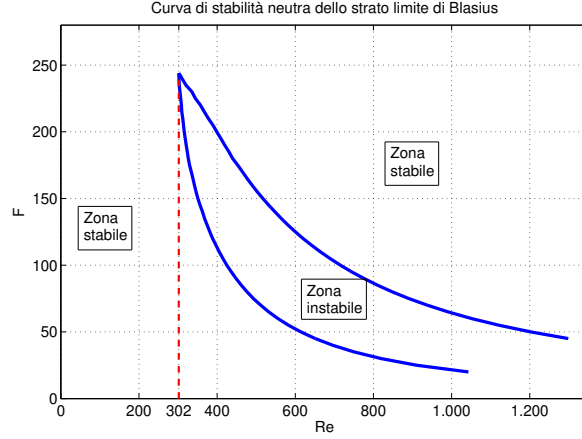


Figure 1.6: Neutral stability curve for a Blasius boundary layer for $k = 0$ (taken from Longobardi (2015)).

possible to demonstrate is that at the order ϵ^0 the same equations for the parallel flow stability are obtained (i.e. equations (1.11)): fixing ω and k real, the eigenvalue problem has to be solved at each streamwise location and the eigenvalue α is no more constant, but it results to be function of the slowing variable X . The location at which the eigenvalue changes its sign is called neutral point. The higher order equations, instead, give the correction to the eigenvalue due to the spatial growth and the non-parallelism of the boundary layer. An example of neutral stability curve for a Blasius boundary layer is reported in figure 1.6 for a bidimensional disturbance ($k = 0$): the critical Reynolds number, based on the boundary layer thickness, is $Re_{cr} \approx 302$ at a non dimensional frequency $F = \omega\nu/U_\infty \times 10^6 \approx 240$. The unstable waves growing into the boundary layer are also known as *Tollmien–Schlichting waves*, from the names of the scientists that for first observed this phenomenon (Tollmien et al., 1961). The instability of a boundary layer is *convective* and boundary layer flows are also known as *noise amplifier* (Sipp and Marquet, 2013): in fact they can be seen as black boxes in which a disturbance enters and it is convectively amplified (or attenuated if the system is stable) at a certain streamwise distance. The study of the spatial stability of a

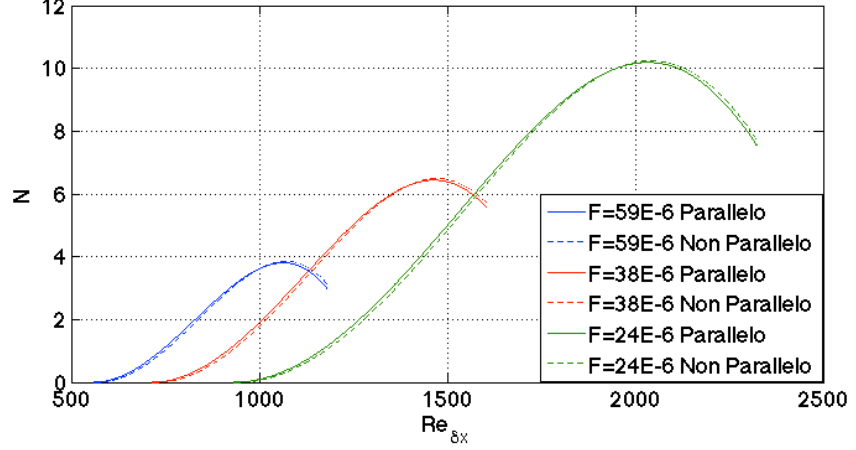


Figure 1.7: N-factor for a Blasius boundary layer at various frequencies: — parallel flow assumption; - - - non parallel flow correction (taken from Longobardi (2015)).

boundary layer is fundamental for understanding its transition to turbulence. There are various semi-empirical methods in literature that use the results of the stability analysis to predict the boundary layer transition. In fact, the transition location and the corresponding is different from the one at which there is the first neutral point x_N , i.e. the abscissa at which the imaginary part of the eigenvalue changes in sign, because of the convective nature of the instability. One of the most used method is the so called e^N . According to this method, it is possible to define the N-factor as:

$$N = \int_{x_N}^x -\alpha_i(\epsilon x) dx'. \quad (1.15)$$

From experiments, it has been observed that the transition location is achieved when $N = 9$, corresponding to an amplification of the initial disturbance of $\sim 80dB$. The equation (1.15) can be improved solving the higher order equations of the multiple scale expansion and adding to α_i the correction due to the growth of the boundary layer. For a flat plate Blasius boundary layer the non parallel correction is not so



Figure 1.8: Flow visualization of the wake of a circular cylinder at $Re = 2000$: it is possible to observe a large recirculation region characterized by strongly non parallel effects.

important, as depicted in figure 1.7: however, it can become crucial for boundary layers over more complicated geometries, as for example the wing of an air plane. The stability analysis of boundary layer is also called *non-local* because it consists in the solution of a one dimensional problem at each streamwise coordinate.

1.6 Stability of non parallel flows

The stability theory described in the previous paragraphs is applicable only to parallel or weakly non parallel flows. In many real applications, however, the fluid shows strongly non parallel effects, as for example recirculation regions in bluff body wakes (see figure 1.8): in these situations the parallel assumption is no more valid and the local stability analysis cannot be applied. In this context, no assumptions can be made about the spatial structure of the modes and all the spatial directions, in the more general case, must be treated as eigendirections, namely they are considered inhomogeneous. Since no considerations are

made about the spatial directions, the stability analysis of such kind of flows is usually called *global*, a term first introduced by Joseph (1966). In particular, when no assumptions can be made on all the three spatial directions (namely in the more general case), the stability analysis is called TriGlobal (Theofilis, 2011). However, there are situations in which the fluid can be considered homogeneous in one spatial direction: the base flow is usually bidimensional whereas it is possible to consider a three dimensional perturbation expanding its solution in a Fourier series along the homogeneous direction. In this case, the arising stability analysis is also known as BiGlobal (Theofilis, 2011). In this thesis we use only the BiGlobal stability approach and, for simplicity, we will call it simply global, as already done by numerous authors in literature.

1.6.1 Global stability analysis

If the fluid shows only one homogeneous direction, it is possible to simplify the full three-dimensional problem in a two dimensional one. Let us consider the cartesian case: if the flow configuration shows a spatial direction, as for example the spanwise one (z), greater than the others, the derivative of the base flow respect to z are equal to zero and so the base flow is bidimensional, namely $\mathbf{U}(\mathbf{x}) = [U(x, y), V(x, y)]$ (save for the case discussed in chapter 3 where the flow is three dimensional but dependent only by two spatial directions). The Fourier expansion (1.9) for the perturbation, instead, has to be written as:

$$\tilde{\mathbf{q}}(\mathbf{x}) = \mathbf{q}'(x, y)e^{ikz} + c.c., \quad (1.16)$$

where $k \in \mathbb{R}$ is the real spanwise wavenumber.

Putting the ansatz (1.16) into the LNSE (1.8) written in cartesian coor-

ordinates, the following set of PDEs is obtained:

$$\left. \begin{aligned} \partial_x u' + \partial_y v' + i k w' &= 0 \\ \sigma u' + U \partial_x u' + u' \partial_x U + V \partial_y u' + v' \partial_y U + i k U w' + \partial_x p' &= 0 \\ \frac{1}{Re} (\partial_{x^2}^2 u' + \partial_{y^2}^2 u' - k^2 u') & \\ \sigma v' + U \partial_x v' + u' \partial_x V + V \partial_y v' + v' \partial_y V + k V w' + \partial_y p' &= \\ = \frac{1}{Re} (\partial_{x^2}^2 v' + \partial_{y^2}^2 v' - k^2 v') & \\ \sigma w' + U \partial_x w' + V \partial_y w' + i k w' - \frac{1}{Re} (\partial_{x^2}^2 w' + \partial_{y^2}^2 w' - k^2 w') &= 0 \end{aligned} \right\}. \quad (1.17)$$

In axialsymmetric flows, under such geometrical symmetries, it is possible to consider as homogeneous direction the azimuthal one (θ). All the derivatives of the baseflow respect to the azimuthal coordinate θ are equal to zero so that the resulting baseflow is bidimensional ($\mathbf{U}(\mathbf{x}) = [U_x(x, r), U_r(x, r)]$)². For such kind of flows, the perturbation can be expanded in Fourier series as:

$$\tilde{\mathbf{q}}(\mathbf{x}) = \mathbf{q}'(x, r) e^{im\theta} + c.c., \quad (1.18)$$

where the azimuthal wavenumber $m \in \mathbb{N}$ is a natural number for periodicity reasons. Putting the ansatz (1.18) into the LNSE (1.8) written

²Note that without difficulties it is possible to consider also swirling flows in which the baseflow is three dimensional but dependent only by the radial and axial coordinates (see for example Montagnani (2018))

in axialsymmetric coordinate, the following set of PDEs is obtained:

$$\left. \begin{aligned} \partial_x u'_x + \frac{1}{r} \partial_r (r u'_r) + \frac{im}{r} u'_\theta &= 0 \\ \sigma u'_x + 2\partial_x (U_x u'_x) + \frac{1}{r} \partial_r (r U_x u'_r + r U_r u'_x) + \frac{im}{r} U_x u'_\theta + \partial_x p' &= \\ &= \frac{1}{Re} \left(\partial_{x^2}^2 u'_x \frac{1}{r} \partial_r (r \partial_r u'_x) - \frac{m^2}{r^2} u'_\theta \right) \\ \sigma u'_r + \partial_x (U_x u'_r + u'_x U_r) + \frac{1}{r} (2\partial_r (r U_r u'_r) + im U_r u'_\theta) + \partial_r p' &= \\ &= \frac{1}{Re} \left(\partial_{x^2}^2 u'_x + \partial_r \left(\frac{1}{r} \partial_r (r u'_r) \right) + \frac{m^2}{r^2} (u'_r - 2u'_\theta) \right) \\ \sigma u'_\theta + \partial_x (U_x u'_\theta) + \frac{1}{r} \partial_r (r U_r u'_\theta) + \frac{U_r u'_\theta}{m} + \frac{1}{r} \partial_\theta p' &= \\ &= \frac{1}{Re} \left(\partial_{x^2}^2 u'_\theta + \frac{1}{r} \partial_{r^2}^2 (r u'_\theta) + \frac{m^2}{r^2} (u'_\theta - 2u'_r) \right) \end{aligned} \right\}. \quad (1.19)$$

However, in this thesis, we are interested only to axialsymmetric disturbances, thus we consider only the case for $m = 0$. Boundary conditions for problems (1.17) and (1.19) will be specified for the specific problems in dedicated chapters.

Both the problems (1.17) and (1.19) can be recast in the form of a generalized linear twodimensional eigenvalue problem

$$[\mathbf{A}(U, Re) - \sigma \mathbf{B}] \mathbf{q}' = \mathbf{0} \quad (1.20)$$

where $\mathbf{A}$ can be seen as a stiffness matrix and $\mathbf{B}$ is the mass matrix (Cano-Lozano et al., 2016); the exact expression of these matrices is given in dedicated chapters. As per the local stability, the flow results to be unstable if the growth rate $\gamma > 0$ whereas ω is the oscillation frequency of the disturbances; the Reynolds number at which γ changes in sign is called, also in this case, *critical Reynolds number* (Re_{cr}).

The eigenvector $\mathbf{q}'$ corresponding to an unstable eigenvalue σ is also called *global mode*: its shape is very important because it gives a picture of what happens into the flow and how the oscillations are driven.

The nature of such kind of instability is very different from the local and non local ones. In fact, in this case, the disturbances don't grow while they are convected but rather they growth in time manifesting *self-sustained* spatial oscillations: a typical example is the Von-Karman

wake behind a circular cylinder, that occurs at a critical Reynolds number equal to $Re_{cr} \approx 47$. For this reasons, such kind of flow instability is called *global* and it is an absolute instability; moreover, the flows which show this features are called *global oscillators* (Sipp et al., 2010).

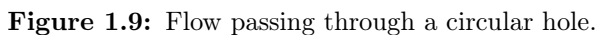
1.6.2 A little hint to the TriGlobal stability analysis

In many real applications it can happen that the flow has no homogeneous spatial directions. Typical examples are, for example, a jet in crossflow (see for example Bagheri et al. (2009)), the flow past a sphere (see for example Fabre et al. (2008)) or past an emispherical roughness element (see for example Citro et al. (2015b)). In this cases, the generalized three-dimensional eigenvalue problem (1.8) has to be solved and the computation is very hard. In fact, the discretized matrix of a one dimensional eigenvalue problem is in the order of few *MB*, the one of a bidimensional eigenvalue problem is in the order of some *GB* whereas the discretization of a full tridimensional eigenvalue problem leads to a huge matrix in the order of *TB* (Theofilis, 2003). It is evident that the direct inversion (but also the storage) of the discretization matrix for a three dimensional problem is impossible and some special tricks are required, as time marching combined with stabilization methods for computing a stable base flow also post bifurcation (for more details see Åkervik et al. (2006); Citro et al. (2017)).

1.7 The acoustic conductivity/impedance and the Nyquist stability criterion

1.7.1 Definitions

In chapters 4 and 5, we study the acoustic and stability properties of the flow passing through a circular hole; the general flow configuration is sketched in figure 1.9. The hole connects two semi-infinite open spaces; the pressure difference between the inlet and outlet generates a net flow rate through the hole: the result is the generation of a jet at the edge of the hole characterized by the vena contracta phenomenon. Since the main objective of this study is to characterize the interaction between the jet and an acoustic perturbation, it is fair to decompose the pressure


$$\left. \begin{aligned} p_{in}(t) &= P_{in} + \varepsilon p'_{in} e^{-i\omega t} \\ p_{out}(t) &= P_{out} + \varepsilon p'_{out} e^{-i\omega t} \\ q(t) &= Q + \varepsilon q' e^{-i\omega t} \end{aligned} \right\}, \quad (1.21)$$

Let's focus, now, the attention on the unsteady harmonic part of the flow and in particular on the relation between the pressure jump and the flow across the hole. Rayleigh (1896) introduced the concept of conductivity (K_R), a key quantity defined as the proportionality coefficient between the acceleration of the fluid particles located within the hole and the pressure jump across the hole. More specifically,

The conductivity is, in the general case, a complex quantity, and has the dimension of a length. The conductivity is classically decomposed

as (Howe, 1979):

$$K_R = 2R_h(\gamma - i\delta), \quad (1.23)$$

where R_h is the radius of the hole, the real part γ of the conductivity represents the inertia of the system whereas its imaginary part δ is linked to the average value of the power absorbed by the hole. In fact, for harmonic perturbations, it is possible to write the power exchanged between the perturbation and the mean flow as:

$$\langle \Pi \rangle = \langle ([p'_{in} - p'_{out}]e^{-i\omega t} + c.c.)(q'e^{-i\omega t} + c.c.) \rangle = 2\Re([p'_{in} - p'_{out}]\bar{q}'), \quad (1.24)$$

where the brackets $\langle \cdot \rangle$ represent the averaging over a complete oscillation period $2\pi/\omega$, $\Re$ stands for the real part and the overbar denotes the complex conjugate. Using the definition of the conductivity (1.22), the formula (1.24) directly leads to:

$$\langle \Pi \rangle = \frac{4R_h\delta}{\rho\omega} |p'_{in} - p'_{out}|^2. \quad (1.25)$$

So, when $\delta > 0$, this term represents a resistance (or the ability to absorb acoustic energy), meaning that exciting the jet at a given frequency necessitates the provision of energy by an outer system. Conversely, a negative δ means that oscillations of the jet can supply to an outer system, which can be for instance an acoustic resonator.

As an alternative to the Rayleigh conductivity, we can also define another key quantity, namely the *impedance* of the aperture (Z_h). Following the electronic analogy, it is defined as the ratio between the driven force and the effect, and so, in this specific case, as the ratio between the pressure jump and the flow rate across the hole:

$$Z_h = \frac{(p'_{in} - p'_{out})}{q'} \quad \left(= -\frac{i\omega\rho}{K_R} \right) \quad (1.26)$$

From (1.26), it is easy to verify the direct link existing between the conductivity and the impedance meaning that, using some due adjustments, the two concepts are completely interchangeable, at least from a theoretical and conceptual point of view. As a consequence, the impedance is also a complex quantity as well, with physical dimension $Mass \cdot Length^{-2}$.

$Time^{-1}$. In the following, we decompose the impedance as

$$Z_h = \frac{\rho U_M}{R_h^2} (Z_R + iZ_I), \quad (1.27)$$

where U_M is the mean velocity of the flow through the hole, Z_R is the dimensionless *resistance* and Z_I is the dimensionless *reactance*. It is easy to verify that the equation (1.24) for the power absorbed by the hole can be written as function of Z_R as follows:

$$\langle \Pi \rangle = 2 \frac{\rho U_M}{R_h^2} Z_R |q'|^2, \quad (1.28)$$

So, when $Z_R > 0$, this term represents a resistance whereas a negative Z_R means that there is power generation and oscillations of the jet can supply to an outer system, which can be for instance an acoustic resonator.

1.7.2 Nyquist stability criterion

The situation in which $Z_R < 0$ could lead to think that the system is unstable since it is generating energy rather than dissipate it. However, following Concinauro and Puglisi (1981) who studied the stability of electrical circuits, a negative value of the resistance does not directly involve that the system is unstable but it means that the concerned dynamical system is *active*: in fact, the condition of $Z_R < 0$ is only necessary but not sufficient to have an instability of the system. In conclusion, the situation with $Z_R < 0$ is sometimes referred as *conditional stability* since the coupling with an outer system, for example in this context an outer acoustic resonator, *could* bring to an instability and so to the presence of self-sustained processes.

Until now, we focus our attention on only the real part of the impedance that is linked to the energy dissipation/production of the system. However, the inertial effects play a crucial role on the stability of the system and in fact, in order to understand the stability properties of the jet, a combined analysis of the sign of both the real and imaginary part of the impedance is necessary. In fact, we stated that the condition of $Z_R < 0$ is necessary in order to have an instability. However, if the imaginary part of the impedance, namely the reactance or equivalently the inertia of the

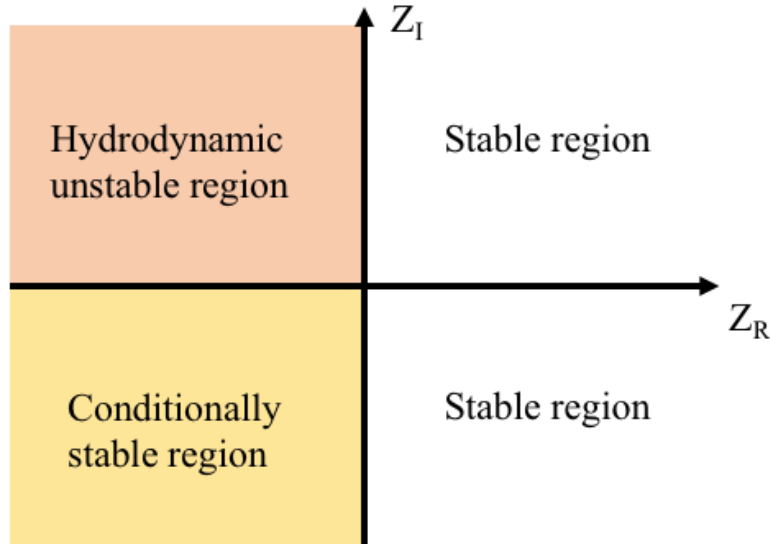


Figure 1.10: Example of Nyquist diagram. In white the stable region, in yellow the area of conditional stability and in orange the area of hydrodynamic instability.

perturbation, is positive ($Z_I > 0$), the purely hydrodynamical system becomes unstable: under this conditions the jet shows self-sustained oscillations leading to whistle generation (Karlsson and Åbom, 2011). Finally, the condition at which the jet results to be marginally stable is when the whole impedance shows a complex zero, namely $Z_h = 0$. The marginally stable condition has also an easy physical interpretation: since the impedance is defined as the ratio between the pressure jump and the flow rate across the hole, $Z_h = 0$ means that there is the possibility of a flow rate without a pressure jump and it is related to the instability of the system.

The stability criteria described above can be easily summarized in an unique *Nyquist stability criterion*. If we plot the real part of the impedance versus its imaginary part (the Nyquist diagram), three different region can be identified: the right side of the diagram ($Z_R > 0$) is the region of stability, the third quadrant ($Z_R < 0$ and $Z_I < 0$) is the region of conditional stability whereas the second quadrant ($Z_R < 0$

and $Z_I > 0$) is the region of hydrodynamical instability; an example of Nyquist diagram is depicted in figure 1.10. Following Kierkegaard et al. (2012), the number of critical zeros can be identified as the number of times the contour of the impedance $Z_h(\omega)$ encircles the origin. However, the disadvantage of the Nyquist diagram is that we lose information about the frequencies so that the associated frequencies of the critical zeros should be evaluated plotting the real and imaginary part of the impedance versus the frequency in a separate graphic.

Finally, the curious reader could ask why we introduce both the concept of conductivity and impedance but we use only the latter one to write the Nyquist stability criterion. The answer is that the same conclusions can be reached using the concept of conductivity rather than the impedance one. However, when the system is marginally stable the impedance goes to zero and the conductivity goes to infinity ($Z_h = 0 \Rightarrow K_R \rightarrow \infty$). For such kind of system, an analytical solution for the impedance and/or conductivity does not exist but it must be reconstructed from numerical or experimental data in which it results easier to identify something that is equal to zero rather than something that tends to infinity. So, in the case of thin holes (chapter 4) acting as sound attenuators ($\delta > 0$ and equivalently $Z_R > 0$), most authors have used the conductivity as initially introduced by Rayleigh and also in this thesis we used both the concept when studying thin holes. On the other hand, in case of thick hole (chapter 5) we use only the concept of impedance, for the reasons explained above.

1.8 The Navier–Stokes equations: compressible formulation

The final part of this thesis is about the stability of the flow passing through a bird–call, considering the effects of the compressibility. In particular, these effects have been taken into account using both the full compressible Navier–Stokes equations and a simplified augmented model coupling a Helmholtz acoustic resonator to the incompressible equations. We first report the compressible formulation of the Navier–Stokes equations and the linearized compressible Navier–Stokes equations; finally we describe the augmented model. We consider an ideal gas with a Prandtl

number $Pr = \mu c_p / \kappa$ equal to 0.7, where c_p is the constant specific heat, κ is the thermal conductivity and μ is the dynamic viscosity. Moreover, we hypothize that the viscosity and the thermal conductivity don't change with the temperature Yamouni et al. (2013). Under these assumptions, the non dimensionalized compressible Navier–Stokes equations can be written as:

$$\left. \begin{aligned} \partial_t \rho + \mathbf{u} \cdot \nabla \rho + \rho \nabla \cdot \mathbf{u} &= 0 \\ \rho \partial_t \mathbf{u} + \rho \mathbf{u} \cdot \nabla \mathbf{u} + \frac{1}{\gamma Ma^2} \nabla p - \frac{1}{Re} \nabla \cdot \tau(\mathbf{u}) &= \mathbf{0} \\ \rho \partial_t T + \rho \mathbf{u} \cdot \nabla T + (\gamma - 1) p \nabla \cdot \mathbf{u} &= \\ = \gamma(\gamma - 1) \frac{Ma^2}{Re} \tau(\mathbf{u}) : \mathbf{d}(\mathbf{u}) + \frac{\gamma}{Pr Re} \Delta_2 T & \\ p - \rho T &= 0 \end{aligned} \right\}, \quad (1.29)$$

where γ is the ratio of specific heats (equal to 1.4 for the air), ρ and T are respectively the density and the temperature, $\mathbf{d}(\mathbf{u})$ and $\tau(\mathbf{u})$ are respectively the strain and stress tensor, whose explicit expression is:

$$\mathbf{d}(\mathbf{u}) = \frac{1}{2} (\nabla \mathbf{u} + \nabla \mathbf{u}^T), \quad \tau(\mathbf{u}) = [2\mathbf{d}(\mathbf{u}) - \frac{2}{3}(\nabla \cdot \mathbf{u})\mathbf{I}] \quad (1.30)$$

The first equation is the mass conservation, the second the momentum, the third the internal energy and the forth the ideal gas law, for a total of six scalar equations for the six unknown variables (density, three velocities components, temperature and pressure) in the full three dimensional case (or four scalar equations for four unknown terms in the bidimensional case)³. The equations (1.29) clearly result to be valid for both cartesian and axialsymmetric flows. However, in this thesis, we consider only the axialsymmetric formulation for the compressible case; thus the velocity vector is defined as $\mathbf{u}(x, r, t) = (u_x, u_r)$ where x and r represent the axial and radial coordinates whereas u_x and u_r are respectively the axial and radial velocity components. Equations (1.29) have been non dimensionalized using some reference quantities U_{ref} , P_{ref} , T_{ref} , ρ_{ref} and L_{ref} respectively for the velocity, pressure, temperature, density and length. However, it is possible to observe into the momentum

³This isn't the only possible set of equations for compressible flows: for example if we replace the internal energy equation with the total entropy one, a different set of PDEs describing the motion of a compressible flow arises

equation that the pressure gradient is proportional to Ma^{-2} , leading to a singularity when $Ma \rightarrow 0$ (the incompressible limit): this generates numerical issues when solving numerically the equations (1.29) at very low Mach number, as done in this thesis. In order to avoid these problems, following Fani et al. (2018), we define the dimensionless pressure as $\frac{P - P_{ref}}{\rho_{ref} U_{ref}^2}$; thus, the compressible Navier Stokes equations can be written as:

$$\left. \begin{aligned} \partial_t \rho + \mathbf{u} \cdot \nabla \rho + \rho \nabla \cdot \mathbf{u} &= 0 \\ \rho \partial_t \mathbf{u} + \rho \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p - \frac{1}{Re} \nabla \cdot \boldsymbol{\tau}(\mathbf{u}) &= \mathbf{0} \\ \rho \partial_t T + \rho \mathbf{u} \cdot \nabla T + (\gamma - 1) \rho T \nabla \cdot \mathbf{u} &= \\ = \gamma(\gamma - 1) \frac{Ma^2}{Re} \boldsymbol{\tau}(\mathbf{u}) : \mathbf{d}(\mathbf{u}) - \frac{\gamma}{Pr Re} \Delta_2 T \\ \rho T - 1 - \gamma Ma^2 p &= 0 \end{aligned} \right\}, \quad (1.31)$$

where no terms proportional to Ma^{-2} are present into the equations. Note that with this last formulation the incompressible limit is retrieved simply putting $Ma = 0$ into the equations (and also in the numerical code) without a limit operation. Finally, the Reynolds and Mach numbers are respectively defined as:

$$Re = \frac{\rho_{ref} U_{ref} L_{ref}}{\mu}, \quad Ma = \frac{U_{ref}}{\sqrt{\gamma R T_{ref}}}, \quad (1.32)$$

where R is the ideal gas constant ($R \approx 8.314472 J/molK$). The references scales, within the boundary conditions, are specified in the dedicated chapter.

1.9 Global stability theory applied to compressible Navier–Stokes equations

The linear stability theory reported in previous paragraph can be clearly applied also to investigate the stability properties of a compressible flow. In order to study the linearized dynamic of the compressible Navier–Stokes equations, the following flow decomposition has to be

introduced:

$$\left. \begin{aligned} \rho(\mathbf{x}, t) &= \rho_B(\mathbf{x}) + \varepsilon \hat{\rho}(\mathbf{x}, t) + \mathcal{O}(\varepsilon^2) \\ \mathbf{u}(\mathbf{x}, t) &= \mathbf{U}(\mathbf{x}) + \varepsilon \hat{\mathbf{u}}(\mathbf{x}, t) + \mathcal{O}(\varepsilon^2) \\ p(\mathbf{x}, t) &= P(\mathbf{x}) + \varepsilon \hat{p}(\mathbf{x}, t) + \mathcal{O}(\varepsilon^2) \\ T(\mathbf{x}, t) &= T_B(\mathbf{x}) + \varepsilon \hat{T}(\mathbf{x}, t) + \mathcal{O}(\varepsilon^2) \end{aligned} \right\}, \quad (1.33)$$

Introducing the flow decomposition (1.33) into the compressible Navier Stokes equations (1.31) and linearizing, we obtain a set linear PDEs describing the linear dynamic of the perturbation, namely the linearized compressible Navier Stokes equations (LCNSE). Then, as function of the spatial and temporal behaviour of the perturbation, it is possible to do the same expansions done for the incompressible case, leading to local or global stability equations. In this thesis, we are interested to the global modes for axialsymmetric perturbation; thus we use the Fourier expansion (1.18) particularized for $m = 0$ but with the state vector $q' = [\rho', \mathbf{u}', p', T']$, leading to the following set of equations for the perturbation:

$$\left. \begin{aligned} \sigma \rho' + \mathbf{U} \cdot \nabla \rho' + \mathbf{u}' \cdot \nabla \rho_B + \rho_B \nabla \cdot \mathbf{u}' + \rho' \nabla \cdot \mathbf{U} &= 0 \\ \sigma \rho_B \mathbf{u}' + \rho' \mathbf{U} \cdot \nabla \mathbf{U} + \rho_B \mathbf{u}' \cdot \nabla \mathbf{U} + \rho_B \mathbf{U} \cdot \nabla \mathbf{u}' + \nabla p' - \frac{1}{Re} \nabla \cdot \tau(\mathbf{u}') &= \mathbf{0} \\ \sigma \rho_B T' + \rho' \mathbf{U} \cdot \nabla T_B + \rho_B \mathbf{u}' \cdot \nabla T_B + \rho_B \mathbf{U} \cdot \nabla T' + \\ + (\gamma - 1) (\rho' T_B \nabla \cdot \mathbf{U} + \rho_B T' \nabla \cdot \mathbf{U} + \rho_B T_B \nabla \cdot \mathbf{u}') + \\ - \gamma (\gamma - 1) \frac{Ma^2}{Re} [\tau(\mathbf{u}') : \mathbf{d}(\mathbf{U}) + \tau(\mathbf{U}) : \mathbf{d}(\mathbf{u}')] - \frac{\gamma}{Pr Re} \Delta_2 T' &= 0 \\ \rho' T_B + \rho_B T' - 1 - \gamma Ma^2 p' &= 0 \end{aligned} \right\}. \quad (1.34)$$

Finally, equations (1.34) can be recast in the form of a generalized eigenvalue problem as in equation (1.20) and solved respect to the complex eigenvalue σ .

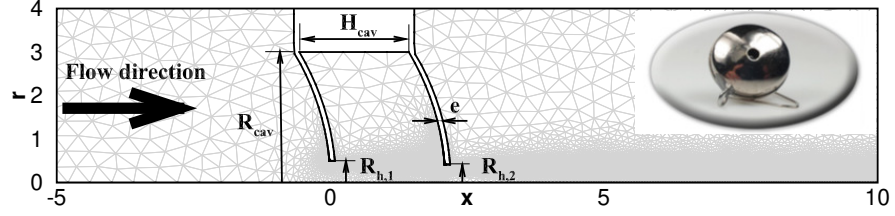


Figure 1.11: Sketch of the birdcall configuration. An example of the real configuration used in this paper is depicted in the upper right corner of the figure.

1.10 Modelling the local effect of the compressibility using the Helmholtz resonator

In chapter 6 we study the stability and acoustic properties of a birdcall configuration, depicted in figure 1.11. The birdcall is constituted by two successive holes forming a cavity. If we want to consider the effect of the compressibility of the flow, we can solve the compressible Navier–Stokes equations. However, the numerical solution of the compressible equations is notoriously more expensive than the solution of the incompressible ones. Motivated by this fact, we try to model the effect of the compressibility using a simplified model coupled to the incompressible Navier–Stokes equations. The main hypothesis of this model is that the flow can be assumed locally incompressible, namely the acoustic wavelength is greater than the main geometrical parameters. In particular, for the geometry investigated here:

$$\lambda_{ac} = \frac{2\pi}{Ma\omega} \gg \{D_h, D_{cav}, H_{cav}\}, \quad (1.35)$$

with ω the angular frequency of the acoustic wave. The condition (1.35) is also known as acoustic compactness of the geometry.

Under this hypothesis, in fact, it is possible to consider, in first approximation, that the flow is locally incompressible so that pressure can be considered constant inside the cavity between the two holes (and also density since we are in the incompressible regime). Thus, it is fair to

model the cavity as an Helmholtz resonator (Bonnefis, 2014) and the compressibility effects are taken into account imposing a spring-like impedance boundary condition on the upper wall of the cavity rather than a no slip one. The variation of the mass into the cavity can be written, in dimensional form, as (Fry, 2016):

$$\partial_t m_{cav}^d = -\rho^d Q_{cav}^d \quad (1.36)$$

where $m_{cav}^d = \rho^d V_{cav}^d$ and Q_{cav}^d are respectively the mass of the fluid inside the cavity and the flow rate outgoing from the cavity, whereas V_{cav}^d is the volume of the cavity. Note that the superscript " d " refers to dimensional quantities. Under the hypothesis of adiabatic and isoentropic system, the link between pressure and density inside the cavity is:

$$p_{cav}^d = c_0^{d^2} \rho_{cav}^d, \quad (1.37)$$

where $c_0^{d^2}$ is the speed of sound. Using the isoentropic condition (1.37) in equation (1.36) and applying the non dimensionalization of the variables, the following equation is obtained:

$$\partial_t p_{cav} + \frac{1}{\chi_c} Q_{cav} = 0 \quad \text{with} \quad \chi_c = V_{cav} Ma^2. \quad (1.38)$$

The coefficient χ_c can be defined as a compressibility parameter: it is interesting to observe that the compressibility effects are influenced both by the Mach number and the volume of the cavity. The unknown terms p_{cav} and Q_{cav} are called *augmented* variables and they are linked with the incompressible unknown terms through their definition:

$$\left. \begin{aligned} p_{cav} &= \frac{1}{S_{cav}} \int_{S_{cav}} p dS \\ Q_{cav} &= \int_{S_{cav}} \mathbf{u} \cdot \mathbf{n} dS \end{aligned} \right\}, \quad (1.39)$$

where S_{cav} is the surface of the upper wall of the cavity. Since we are interested to the global modes, we apply the Fourier decomposition to the equation (1.38) that becomes:

$$\sigma p'_{cav} + \frac{1}{\chi_c} Q'_{cav} = 0 \quad (1.40)$$

Finally, coupling the equations (1.40) and (1.39) with the incompressible LNSE (1.19), we are able to write the *augmented model* as:

$$\left. \begin{aligned} \nabla \cdot \mathbf{u}' &= 0 \\ \sigma \mathbf{u}' + (\mathbf{u}' \cdot \nabla) \mathbf{U} + (\mathbf{U} \cdot \nabla) \mathbf{u}' + \nabla p' - \frac{1}{Re} \Delta_2 \mathbf{u}' &= 0 \\ \sigma p'_{cav} + \frac{1}{\chi_c} Q'_{cav} &= 0 \\ \frac{1}{S_{cav}} \int_{S_{cav}} p' dS &= p'_{cav} \\ \int_{S_{cav}} \mathbf{u}' \cdot \mathbf{n} dS &= Q'_{cav} \end{aligned} \right\}, \quad (1.41)$$

where the operators gradient, divergence and laplacian are written in axialsymmetric coordinates.

Equations (1.41) can be recast in a generalized eigenvalue problem and its discretization matrix is the same of the incompressible problem with the adjoint of two rows and two columns for the two new variables p'_{cav} and Q'_{cav} which are called *augmented* unknown variables. In chapter 6 we discuss the range of validity and applicability of this augmented model.

1.11 Adjoint equations and structural sensitivity

1.11.1 Adjoint Navier–Stokes equations

A very powerful tool in functional analysis is the adjoint of a linear operator. The use of adjoint equations in the context of fluid dynamic stability has been recently reviewed by Luchini and Bottaro (2014). In fluid mechanics, the adjoint equations are fundamental to understand the receptivity process of a boundary layer (Hill, 1995), the nature of the global instabilities (Giannetti and Luchini, 2007) or in problem involving optimization and flow control (Luchini, 2000).

The adjoint linearized Navier–Stokes operator can be defined using the generalized lagrangian identity (Ince, 1926). Given a pair of suitably differentiable fields $\hat{\mathbf{u}} = [\hat{u}, p]$ and $\hat{\mathbf{u}}^\dagger = [\hat{u}^\dagger, p^\dagger]$, using the differentiation

by parts, the following lagrangian identity can be defined:

$$\begin{aligned}
& [(\partial_t \hat{\mathbf{u}} + \mathcal{L}\{\mathbf{U}, Re\} \hat{\mathbf{u}} + \nabla \hat{p}) \cdot \mathbf{u}^\dagger + \nabla \cdot \mathbf{u}^\dagger \hat{p}] + \\
& + [\hat{\mathbf{u}} \cdot (\partial_t \hat{\mathbf{u}}^\dagger + \mathcal{L}^\dagger\{\mathbf{U}, Re\} \hat{\mathbf{u}}^\dagger + \nabla \hat{p}^\dagger) + \hat{p} \nabla \cdot \hat{\mathbf{u}}^\dagger] = \\
& = \partial_t (\hat{\mathbf{u}} \cdot \hat{\mathbf{u}}^\dagger) + \nabla \cdot \mathbf{J}(\hat{\mathbf{u}}, \hat{\mathbf{u}}^\dagger, \hat{p}, \hat{p}^\dagger),
\end{aligned} \tag{1.42}$$

where $\mathbf{J}$ is known as *bilinear contaminator* (Giannetti and Luchini, 2007)

$$\mathbf{J}(\hat{\mathbf{u}}, \hat{\mathbf{u}}^\dagger, \hat{p}, \hat{p}^\dagger) = (\mathbf{U} \cdot \nabla) \hat{\mathbf{u}}^\dagger - \nabla \mathbf{U} \cdot \hat{\mathbf{u}}^\dagger + \frac{1}{Re} \Delta_2 \hat{\mathbf{u}}^\dagger, \tag{1.43}$$

$\mathcal{L}\{\mathbf{U}, Re\}$ is the linearized Navier–Stokes operator defined by equation (1.6) and $\mathcal{L}^\dagger\{\mathbf{U}, Re\}$ is the adjoint linearized Navier–Stokes operator defined as:

$$\mathcal{L}^\dagger\{\mathbf{U}, Re\} \hat{\mathbf{u}}^\dagger = (\mathbf{U} \cdot \nabla) \hat{\mathbf{u}}^\dagger - \nabla \mathbf{U} \cdot \hat{\mathbf{u}}^\dagger + \frac{1}{Re} \Delta_2 \hat{\mathbf{u}}^\dagger. \tag{1.44}$$

Finally, analysing the second term between square brackets in equation (1.42), it is possible to define the adjoint linearized Navier–Stokes equations as:

$$\left. \begin{aligned} \nabla \cdot \hat{\mathbf{u}}^\dagger &= 0 \\ \partial_t \hat{\mathbf{u}}^\dagger + \mathcal{L}^\dagger\{\mathbf{U}, Re\} \hat{\mathbf{u}}^\dagger + \nabla \hat{p}^\dagger &= 0 \end{aligned} \right\}, \tag{1.45}$$

It is interesting to observe that the convective part in (1.44) has an opposite sign with respect to the direct operator (1.6): in fact the adjoint solution is convected in the opposite direction of the direct one.

In order to obtain a generalized adjoint eigenvalue problem, we can follow the normal mode ansaltz:

$$\hat{\mathbf{q}}^\dagger(\mathbf{x}, t) = \tilde{\mathbf{q}}^\dagger(\mathbf{x}) e^{-\sigma t} + c.c., \tag{1.46}$$

leading to the following adjoint eigenvalue problem:

$$\left. \begin{aligned} \nabla \cdot \tilde{\mathbf{u}}^\dagger &= 0 \\ -\sigma \tilde{\mathbf{u}}^\dagger + \mathcal{L}^\dagger\{\mathbf{U}, Re\} \tilde{\mathbf{u}}^\dagger + \nabla \tilde{p}^\dagger &= 0 \end{aligned} \right\}. \tag{1.47}$$

The corresponding set of boundary conditions for the adjoint system is obtained by imposing the elimination of the boundary terms after the

application of the generalized lagrangian identity. The system (1.47) is a generalized three-dimensional adjoint eigenvalue problem: then the same considerations about the spatial properties of the flow, discussed previously for the direct global modes, can be done also for the adjoint, leading both to one, two or three dimensional adjoint eigenproblems.

From a physical point of view, the adjoint eigenmode can be interpreted as the initial condition which has maximum projection along the direction of the corresponding eigenmode. This implies that the adjoint of the most amplified direct mode corresponds to the optimal perturbation which maximizes the growth of energy as $t \rightarrow \infty$. Moreover, the adjoint eigenmodes give us important informations about the receptivity of the instability, namely the capability of the flow to accept and amplify external disturbances. In particular, it is possible to demonstrate that $|\tilde{\mathbf{u}}^\dagger|$ represents the receptivity to a momentum forcing whereas $|\tilde{p}^\dagger|$ is the receptivity to mass injections (Giannetti and Luchini, 2007).

1.11.2 Structural Sensitivity

One interesting property is that the direct and adjoint eigenvalues are complex conjugate whereas the direct and adjoint eigenvectors have very different structures. The reason is that the Navier–Stokes equations are not self-adjoint, namely $\mathcal{L} \neq \mathcal{L}^\dagger$: this property is also known as non normality of the Navier–Stokes operator (Chomaz, 2005). This means that a separate analysis of the direct and adjoint modes is not satisfactory for identifying the instability mechanism. Giannetti and Luchini (2007) performed a structural sensitivity analysis of the Navier–Stokes equations. Thus, they studied the effect of any structural modification of the Navier–Stokes operator on the modification of the eigenvalue. They modelled the feedback mechanism triggering the instability using a local force proportional to the velocity disturbance acting as a momentum source in the LNSE. This procedure leads to the definition of the so called *Structural Sensitivity* tensor:

$$\mathbf{S}(\mathbf{x}) = \frac{\tilde{\mathbf{u}}^\dagger \otimes \tilde{\mathbf{u}}}{\int_V (\tilde{\mathbf{u}}^\dagger \tilde{\mathbf{u}}) dV} \quad (1.48)$$

A spatial map of this tensor can be build considering a norm: in particular it is possible to demonstrate that the spectral norm leads to the maximum coupling between the component of the structural sensitivity tensor (Luchini et al., 2008). The maximum values of the spatial map indicate the regions where the feedback mechanism is stronger, or, in other words, the regions where the instability mechanism acts: the region where the structural sensitivity map reaches its maximum is sometimes called also *wavemaker* in order to remark that it depicts the zones where the instability, and so the unstable waves, rises up. This technique results to be very efficient to study the instability mechanism of complex fluid systems and in this thesis has been used to characterize the instability mechanism of both the T-mixer and the flow passing through a thick circular hole.

Chapter 2

Numerical methods

The following chapter can be divided in three main parts. In the first part, it is reported a description of the Finite Element Method (FEM) applied to discretize the Navier–Stokes equations (base flow, LNSE, direct and adjoint eigenvalue problems...): in particular, we focus the attention on FreeFem++ (<http://www.freefem.org/>), the open source library used in this thesis, giving in parallel the theoretical notions about the finite element discretization.

The second part, instead, describes the main algorithms used to solve the various problems of this thesis.

Finally, in the third part, we report two numerical tricks to treat the boundaries. In particular, we detail the complex mapping method used to compute the impedances of the flow passing through one hole; then, we explain the the sponge zone technique used to solve the compressible equations.

2.1 FreeFem++

FreeFem++ is an open source finite element library developed at INRIA (Institut national de recherche en informatique et en automatique) by F. Hecht (Hecht, 2012) and it has an its own language based on the C++ syntax. It is a compilative object oriented language, meaning that

in order to solve the problem it is necessary to import the appropriate libraries. FreeFem++ implements all the necessary structures to solve a 2D or 3D differential problem: in particular with FreeFem++ it is possible to solve both linear and non-linear elliptical, hyperbolic and parabolic problems.

A typical FreeFem++ code can be divided in the following three steps:

- Definition of the numerical domain and mesh generation;
- Choose of the appropriate finite element space;
- Definition of the problem, within its boundary conditions, in its weak formulation;
- Choose of the appropriate solver and solution of the discretized problem.

Actually, from a practical point of view, the finite element space is defined after the definition of the variational formulation of the problem. In fact, choosing the FEM space independently from the equations that must be solved could bring to numerical instability problems.

2.2 Mesh generation and automatic mesh adaptation

The first step to solve a numerical problem is the definition of a numerical domain and a computational grid, usually known as mesh. In FreeFem++, it is possible both to import geometry and mesh as input files (as for example geometry generated by a CAD software and then meshed with dedicated external software) or to build the geometry and the mesh by using internal commands and libraries, as done in this thesis. In particular, the geometry can be simply defined using parametric curves. Then, at each line, it is attributed a label: usually the same label is assigned to boundaries with the same boundary conditions since, as we will see in the dedicated paragraph, it results easier to impose the boundary conditions.

The triangular mesh, for 2D problems, can be generated using the **bamg** (Bidimensional Anisotropic Mesh Generator) library (Hecht, 1998b), based

on the the Delaunay-Voronoi triangulation (Delaunay, 1934) of the domain. In other words, the density of the mesh in the domain is function of the number of vertices on its boundaries. The distribution of the vertices on the boundaries, instead, is regulated by the definition of a metric M : one possibility is to take as metric of the mesh the parametric function used to define the geometry. In other words, if the geometry has been defined using linear parametric functions, the vertices of the mesh are equispaced whereas for polynomial distributions, for examples, the density of the mesh is higher in some regions rather than in other ones. The FreeFem++ syntax to generate a border is given below:

```
border border_name (t=min,max){x=f1(t); y=f2(t);
                        label=label_name;};
```

where **border_name** is the name of the border, **label_name** is the label, **t** is the parameter, **f1(t)** and **f2(t)** are the functions defining the metrics M . Moreover, it is possible to define interior boundaries so that the domain results to be splitted in different regions where different grid densities can be imposed. Thus, it is possible to refine the mesh where the gradient of the solution are expected to be higher, as for example in the wake of a bluff body. However, the inconvenient of this methodology is that the design of the mesh is made on the base of a priori consideration of the solution.

A valid alternative to this method is to use an automatic mesh adaptation algorithm. A very powerful tool implemented in this FreeFem++, in fact, is the **adaptmesh** command (Hecht, 1998a). This command uses as metric the Hessian matrix of an objective function u_{obj} , namely $M = \nabla \nabla u_{obj}$. To build an optimal mesh, it is possible to choose as objective function the solution of the equation, as for example the base flow solution or the direct and/or adjoint global modes¹. So, the step to obtain an adapted mesh are the following ones:

- Define a coarse mesh;
- Solve the (linear or non linear) problem, with the method that are explained in next paragraphs;

¹Note that the optimal mesh adapted on the structure of the baseflow can be very different from the one adapted on the global modes and so the choice of the objective function is clearly linked to the scope of the dedicated analysis.

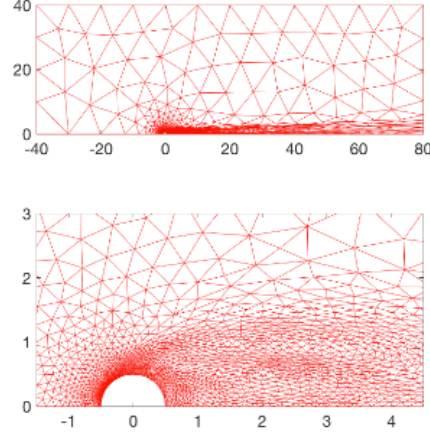


Figure 2.1: Example of mesh obtained using the automatic mesh adaptation procedure for the wake of a circular cylinder (from Fabre et al. (2018a)).

- Adapt the mesh on the solution of the problem;
- Solve the problem on the new mesh.

Note that the procedure described above can be iterated until a suitable mesh is reached. The precision of the adaptation procedure can be controlled by specifying an objective value for the interpolation error of the function on the new mesh. The automatic mesh adaptation method brings to two main advantages: no a priori consideration must be done in order to build the mesh; the mesh is adapted only where it is necessary and it can results also very coarse in regions where gradient of the solution are absent. This methodology has been recently reviewed by Fabre et al. (2018a)² using as benchmark case the linear and non linear global modes in the wake of a circular cylinder: they found that this procedure allows to obtain suitable results by using coarser meshes with respect to the ones used by the current literature; an example of adapted mesh for the wake of a cylinder is reported in figure 2.1.

The methodology described above has been used in this thesis to adapt the mesh when studying the flow passing through a circular aperture:

²<https://github.com/erbafeidavid/StabFem>

in fact the crucial point of that analysis is to capture the deep gradient that are present in the forming jet; an accurate description is given in the dedicated chapters.

2.3 Weak formulation of the Navier–Stokes equations: cartesian incompressible case

To solve Navier–Stokes equations using finite element method, we need of their variational formulation. Here only the discretization of the non linear equations for the base flow (steady state Navier–Stokes equations) is reported. However, the discretization of the LNSE can be achieved in the same way without difficulties. We report the strong (or differential) formulation of the steady state Navier–Stokes equations for the baseflow:

$$\left. \begin{aligned} \nabla \cdot \mathbf{U} &= 0 \\ (\mathbf{U} \cdot \nabla) \mathbf{U} + \nabla P - \frac{1}{Re} \Delta_2 \mathbf{U} &= 0 \end{aligned} \right\}. \quad (2.1)$$

We classically multiply equations (2.1) by test functions $[\mathbf{U}^+, P^+]$ and then we integrate over the domain V :

$$\forall [\mathbf{U}^+, P^+], \int_V ((\mathbf{U} \cdot \nabla) \mathbf{U} + \nabla P - \frac{1}{Re} \Delta_2 \mathbf{U}) \cdot \mathbf{U}^+ dV + \int_V P^+ (\nabla \cdot \mathbf{U}) dV = 0. \quad (2.2)$$

The following identity is used to rewrite the pressure gradient term:

$$\int_V \nabla \cdot (P \mathbf{U}^+) dV = \int_V \nabla P \cdot \mathbf{U}^+ dV + \int_V P (\nabla \cdot \mathbf{U}^+) dV. \quad (2.3)$$

Using the theorem of the divergence, the l.h.s of the equation (2.3) can be recast in this way:

$$\int_V \nabla \cdot (P \mathbf{U}^+) dV = \int_S P (\mathbf{U}^+ \cdot \mathbf{n}) dS, \quad (2.4)$$

where $\mathbf{n}$ is the normal unit vector and where S is the boundary of the integration domain V .

As regard the laplacian term, instead, it is written using the Gauss–Green

lemma, namely the generalization of the integration by part to a multi dimensional domain:

$$\begin{aligned} - \int_V \Delta_2 \mathbf{U} \cdot \mathbf{U}^+ dV &= - \int_V (\nabla \cdot \nabla \mathbf{U}) \cdot \mathbf{U}^+ dV = \\ &= \int_V \nabla \mathbf{U} : \nabla \mathbf{U}^+ dV - \int_S (\nabla \mathbf{U} \cdot \mathbf{n}) dS, \end{aligned} \quad (2.5)$$

where $:$ is the tensorial product between two vectors, namely $\sum_{k=1}^d \sum_{j=1}^d \partial_j u_k \partial_j u_k^+$,

with d the dimension of the velocity vector ($d = 2$ for bidimensional flows and $d = 3$ in three dimensional cases). Joining the border integral present in the equations (2.4) and (2.5), and remembering the definition of directional derivative, we obtain the following integral:

$$\int_S \left(\frac{1}{Re} \partial_n \mathbf{U} - P \mathbf{n} \right) \cdot \mathbf{U}^+ dS. \quad (2.6)$$

A typical outflow boundary conditions for open boundaries is the expression into the integral (2.6) equal to zero. In order to impose a Dirichelet boundary condition, as for example the no slip at wall or a velocity profile at inlet, there are two main possibilities. The former is to choose a test function $\mathbf{U}^+$ that is null on the "Dirichelet" border. The latter is to use the penalization method, namely putting a very great number on the diagonal of the discrete matrix: this is the method used in FreeFem++ to impose the Dirichelet boundary condition. In conclusion, it is possible to write the weak formulation of the incompressible Navier–Stokes equations as follow:

$$\begin{aligned} \forall [\mathbf{U}^+, P^+], NS(\mathbf{U}, P) &= \\ &= \int_V [((\mathbf{U} \cdot \nabla) \mathbf{U}) \mathbf{U}^+ - P(\nabla \cdot \mathbf{U}^+) + \frac{1}{Re} \nabla \mathbf{U} : \nabla \mathbf{U}^+ - P^+(\nabla \cdot \mathbf{U})] dV = 0. \end{aligned} \quad (2.7)$$

Finally, the last problem is about the imposition of a Robin (mixed) boundary condition, as for example the partial slip Navier boundary condition (1.3) used to simulate the superhydrophobic surface. In the general case, the normal derivative of the velocity present into the integral (2.6) is not equal to zero but it is equal to a prescribed value,

for example $\partial_n \mathbf{U} = \mathbf{g}$ on S_R , where S_R is the portion of the domain border on which the Robin boundary condition is applied. In order to impose such kind of boundary condition, it is sufficient to add to the weak formulation of the Navier–Stokes equations (2.7) the border integral $\int_{S_R} \mathbf{g} dS$.

2.4 Choose of the Finite Element space

A finite element space is a space of polynomial functions on elements, with certain matching properties at edges, vertices etc. In Navier–Stokes equations, the unknown variables are, in the incompressible formulation, the velocity $\mathbf{u}$ and pressure p : the choice of the finite element space on which to project these two variables is not independent but they must respect the Ladyzhenskaya-Babuska-Brezzi compatibility condition (Boffi et al., 2013). This condition establishes that in a 2D domain with triangular finite elements, the discretized unknown variables $\mathbf{u}_h$ and p_h (where the subscript h stands for the discrete counterpart of the continuous unknown variables) must belong to finite element spaces defined by polynomial elements which have the following ratio between their degrees of freedom (respectively for velocity and pressure):

$$\mathcal{P}_{k+2}/\mathcal{P}_k. \quad (2.8)$$

The exception to the condition (2.8) is given by the Taylor–Hood elements, generalized as:

$$\mathcal{P}_k/\mathcal{P}_{k-1} \text{ with } k > 1. \quad (2.9)$$

It is possible to demonstrate that the truncation error of such kind of elements is $\mathcal{O}(h^k)$. In this thesis, classical Taylor–Hood $\mathcal{P}_2 - \mathcal{P}_1$ have been used to discretize respectively velocity and pressure, resulting in a second order accurate discretization. The degrees of freedom used on each triangle for velocity and pressure are reported in figure 2.2.

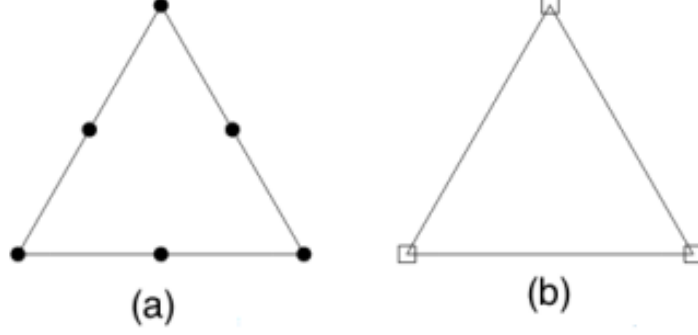


Figure 2.2: (a) $\mathcal{P}_2$ and (b) $\mathcal{P}_1$ finite element.

2.5 Discretization of the variational problem

Once the variational formulation of the problem has been defined and the FE spaces have been chosen, it is possible to assemble the discrete matrices. Let's define n_m the total number of nodes of the mesh. Since we use quadratic element for the velocity field and linear ones for the pressure, the total number of degrees of freedom is given by $n_{d.o.f} = n_m(2n_m - 1)^2$. Using the Galerkin approximation method (for more details see Ern and Guermond (2013)), the discrete unknown variables can be projected on the finite element space through these relation:

$$\left. \begin{aligned} \mathbf{U}(\mathbf{x}) &\approx \sum_{i=1}^{2n_m-1} \mathbf{U}_i \Phi_i^u(\mathbf{x}) \\ P(\mathbf{x}) &\approx \sum_{k=1}^{n_m} P_k \Phi_k^p(\mathbf{x}), \end{aligned} \right\}. \quad (2.10)$$

where the unknown terms are $\mathbf{U}_i$ and P_k with $i \in [1, 2n_m - 1]$ and $k \in [1, n_m]$. The Φ_i^u and Φ_k^p , instead, are the basis functions of the finite element space and it is possible to choose as basis functions the test functions $\mathbf{U}^+$ and P^+ . Putting the approximation (2.10) into the continuous variational formulation (2.7), it is possible to obtain the discrete system (for more details see for example Canton (2013)). This operation is made automatically by FreeFem++ libraries through the command `int2d()`.

```

1 // Th is the mesh
2 fespace XXMh(Th,[P2,P2,P1]); //FEspace (Taylor Hood)
3 XXMh [u1,u2,p]; //unknown functions (u,v,p)
4 XXMh [v1,v2,q]; //test functions (u^+,v^+,p^+)
5
6 //Definition of some useful macro function
7 macro ugrad(u1,u2,v) (u1*dx(v)+u2*dy(v)) //
8 macro Ugrad(u1,u2,v1,v2) [ugrad(u1,u2,v1),ugrad(u1,u2,v2)]//
9
10 ///Definition of the Navier-Stokes problem
11 varf vNS ([u1,u2,p],[v1,v2,v3,q]) =
12     int2d(Th)(
13         + nu * ( dx(u1)*dx(v1) + dy(u1)*dy(v1)
14         + dx(u2)*dx(v2) + dy(u2)*dy(v2)
15         + dx(u3)*dx(v3) + dy(u3)*dy(v3)
16         )
17         + p*q*(1e-10) ///Sabilization term
18         - p*dx(v1) - p*dy(v2) //
19         + dx(u1)*q+ dy(u2)*q
20         + Ugrad(u1,u2,u1,u2)'*[v1,v2]
21     )
22 //Add here the boundary conditions
23 + boundary conditions;

```

Figure 2.3: Sketch of the FreeFem++ script with the variational formulation of the Navier–Stokes equations.

An example of Navier–Stokes definition in FreeFem++, for cartesian coordinates, is given in figure 2.3.

2.6 Weak formulation of the Navier–Stokes equations in cylindrical coordinates

The variational formulation for the Navier–Stokes equations in cylindrical coordinate is a little different from the one in cartesian coordinates. In fact, integrating by parts the laplacian terms leads to different formulations due to the difference in the expression of the gradient of a vector

field in the two reference systems. In particular, in cartesian coordinate the weak formulation of the laplacian terms can be written as:

$$\begin{aligned} \nabla \mathbf{U} : \nabla \mathbf{U}^+|_{cart} &= \partial_x(U) \partial_x(U^+) + \partial_y(U) \partial_y(U^+) + \\ &+ \partial_x(V) \partial_x(V^+) + \partial_y(V) \partial_y(V^+). \end{aligned} \quad (2.11)$$

In cylindrical coordinates, instead, it results:

$$\begin{aligned} \nabla \mathbf{U} : \nabla \mathbf{U}^+|_{cyl} &= \partial_x(U_x) \partial_x(U_x^+) + \partial_r(U_x) \partial_r(U_x^+) + \\ &+ \partial_x(U_r) \partial_x(U_r^+) + \partial_r(U_r) \partial_r(U_r^+) \\ &+ \frac{1}{r^2} U_r U_r^+ \end{aligned} \quad (2.12)$$

Moreover, the elementary volume, in cylindrical coordinate, is expressed as $dV = 2\pi r dx dr$, whereas in cartesian coordinate it is classically $dV = dx dy$. Since FreeFem++ computes only cartesian integral, and in order to have the same definition for the differential operators in both cartesian and cylindrical coordinates (so that it is easy to adapt a cartesian code to a cylindrical one), the Navier–Stokes variational formulation in axialsymmetric coordinate can be arranged as:

$$\begin{aligned} \forall [\mathbf{U}^+, P^+], NS(\mathbf{U}, P) &= \\ &= \int_V r((\mathbf{U} \cdot \nabla) \mathbf{U}) \mathbf{U}^+ dV \\ &- \int_V r P (\nabla \cdot \mathbf{U}^+) dV - \int_V P U_r^+ dV \\ &+ \frac{1}{Re} \int_V r \nabla \mathbf{U} : \nabla \mathbf{U}^+ dV + \frac{1}{Re} \int_V \frac{1}{r} U_r U_r^+ dV \\ &- \int_V r P^+ (\nabla \cdot \mathbf{U}) dV - \int_V U_r P^+ dV = 0, \end{aligned} \quad (2.13)$$

where the volume $dS = dx dr$, the divergence is $\nabla \cdot (\cdot) = \partial_x(\cdot) + \partial_r(\cdot)$, the gradient is $\nabla(\cdot) = [\partial_x(\cdot); \partial_r(\cdot)]$ and the tensorial product is the one defined by equation (2.11).

Finally, the choose of the finite element space is the same already discussed in the previous paragraph. The FreeFem++ code for the Navier–Stokes equations in cylindrical coordinates is the same sketched in figure 2.3 with the already discussed adjoint of the terms reported in equation (2.13).

2.7 Finite element discretization of the augmented model

In this section, we report the approach used to discretize and implement in FreeFem++ the augmented model (1.41). Let's suppose to have discretized the incompressible eigenvalue problem (1.19); the arising discrete eigenvalue problem can be written as:

$$[\mathbf{A}_{inc} - \sigma \mathbf{B}_{inc}] \mathbf{q}'_{inc} = \mathbf{0}, \quad (2.14)$$

where the vector $\mathbf{q}'_{inc}$ is:

$$\mathbf{q}'_{inc} = [u'_{x,1}, \dots, u'_{x,2n(m)-1}, u'_{r,1}, \dots, u'_{r,2n(m)-1}, p'_1, \dots, p'_{n(m)}]^T. \quad (2.15)$$

In a finite element fashion, the augmented model, instead, can be written as:

$$[\mathbf{A}_{aug} - \sigma \mathbf{B}_{aug}] \mathbf{q}'_{aug} = \mathbf{0}, \quad (2.16)$$

with

$$\begin{aligned} \mathbf{q}'_{aug} &= [\mathbf{q}'_{inc}, p'_{cav}, Q'_{cav}]^T = \\ &= [u'_{x,1}, \dots, u'_{x,2n(m)-1}, u'_{r,1}, \dots, u'_{r,2n(m)-1}, p'_1, \dots, p'_{n(m)}, p'_{cav}, Q'_{cav}]^T. \end{aligned} \quad (2.17)$$

The matrix $\mathbf{A}_{aug}$ and $\mathbf{B}_{aug}$ has two more rows and columns respect to $\mathbf{A}_{inc}$ and $\mathbf{B}_{inc}$ and their explicit expression is given below:

$$\mathbf{A}_{aug} = \begin{bmatrix} \mathbf{A}_{inc} + TGV \int_{S_{cav}} (\mathbf{u}' \cdot \mathbf{n})(\mathbf{u}'^+ \cdot \mathbf{n}) dS & 0 & -\frac{TGV}{S_{cav}} \int_{S_{cav}} \mathbf{u}'^+ \cdot \mathbf{n} dS \\ 0 & 0 & -\frac{1}{\chi_c} \\ \frac{1}{H_{cav}} \int_{H_{cav}} p' dx & -1 & 0 \end{bmatrix}, \quad (2.18)$$

$$\mathbf{B}_{aug} = \begin{bmatrix} \mathbf{B}_{inc} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad (2.19)$$

where S_{cav} is the lateral surface of the cavity due to the revolution of H_{cav} around the symmetry axis ($S_{cav} = \int_{H_{cav}} 2\pi r dr dx$, H_{cav} is sketched

in figure 1.11) and $TGV = 10^{30}$ is a very great value (in french *Tres Grand Valeur*) and it is necessary to correctly impose Dirichelet boundary conditions since FreeFem++ uses the penalization method.

2.8 Weak formulation of the compressible Navier–Stokes equations

The variational formulation of the compressible Navier–Stokes equations (1.31) can be retrieved using the same approach seen in previous paragraph. Since for compressible flows there are two more unknown variables, namely the density ρ and temperature T , it is necessary to define two additional test functions, respectively ρ^+ and T^+ . Multiplying the equations (1.31) for the complete set of test functions, integrating over the whole domain and applying the same theorems already seen for the incompressible case, the arising variational formulation for compressible Navier–Stokes equations in cylindrical coordinates (1.31) is eventually written as:

$$\begin{aligned}
& \forall [\rho^+, \mathbf{U}^+, P^+, T^+], NS(\rho_B, \mathbf{U}, P, T_B) = \\
& = - \int_V r \rho^+ (\rho_B \nabla \cdot \mathbf{U} + \mathbf{U} \cdot \nabla \rho_B) dV - \int_V \rho^+ \rho_B U_r dV + \int_V r P (\nabla \cdot \mathbf{U}^+) dV \\
& - \frac{2}{Re} \int_V r \mathbf{d}(\mathbf{U}) : \nabla \mathbf{U}^+ dV + \frac{2}{3Re} \int_V r (\nabla \cdot \mathbf{U}) (\nabla \cdot \mathbf{U}^+) dV \\
& - \frac{4}{3Re} \int_V \frac{1}{r} U_r U_r^+ dV - \int_V r \rho_b (((\mathbf{U} \cdot \nabla) \mathbf{U}) \mathbf{U}^+) dV - \int_V r T^+ \mathbf{U} \cdot \nabla T_B dV \\
& - (\gamma - 1) \int_V r T^+ \rho_B T_B \nabla \cdot \mathbf{U} dV - (\gamma - 1) \int_V T^+ \rho_B T_B U_r dV \\
& + \gamma(\gamma - 1) \frac{2Ma^2}{Re} \int_V r T^+ \mathbf{d}(\mathbf{U}) : \mathbf{d}(\mathbf{U}) dV \\
& - \gamma(\gamma - 1) \frac{2Ma^2}{3Re} \int_V r T^+ (\nabla \cdot \mathbf{U})^2 dV
\end{aligned}$$

$$\begin{aligned}
& + \gamma(\gamma - 1) \frac{4Ma^2}{3Re} \int_V T^+ \frac{1}{r} U_r^2 dV - \frac{\gamma}{PrRe} \int_V r \nabla T \nabla T^+ dV \\
& - \int_V r P^+ (1 + \gamma Ma^2 P - \rho_B T_B) dV.
\end{aligned}$$

Here, dV and all the operators are the one already specified for the axialsymmetric case.

Finally, the variational formulation written above has been discretized using $\mathcal{P}_2$ elements for the velocity field and $\mathcal{P}_1$ elements for the pressure, density and temperature.

2.9 Resolution algorithms

2.9.1 Base flow computation

The steady base flow is solution of the non linear Navier–Stokes equations. The solution has been achieved using a classical Newton–Raphson method. Introducing the weak formulation of the Linearized Navier–Stokes equation as³:

$$\begin{aligned}
& \forall [\mathbf{U}^+, P^+], LNSE(\mathbf{U}, P)(\delta \mathbf{U}, \delta P) = \\
& \int_V [((\delta \mathbf{U} \cdot \nabla) \mathbf{U} + (\mathbf{U} \cdot \nabla) \delta \mathbf{u}) \mathbf{U}^+ - \delta P (\nabla \cdot \mathbf{U}^+) \\
& + \frac{1}{Re} \nabla \delta \mathbf{U} : \nabla \mathbf{U}^+ - p^+ (\nabla \cdot \delta \mathbf{U})] dV = 0,
\end{aligned} \tag{2.20}$$

the Newton algorithm can be written as follow:

In practice, at point 1 the guess field is generally taken as the base flow computed at a lower value of the Reynolds number. The whole algorithm is thus repeated for increasing values of the Reynolds number to generate a family of base flow. The initial guess for the lower initial Reynolds number, instead, can be generated or using few steps of a time integration of the equation (1.1) or solving the Stokes equations. At each step, the matrix inversion at the point 3 has been achieved using

³For sake of brevity only the weak formulation of the incompressible LNSE in cartesian coordinates has been introduced; however, following the theory and the considerations of previous paragraphs it is easy to write the LNSE also in axialsymmetric coordinates or for compressible flows.

Algorithm 1 Newton–Raphson Algorithm

```

1: Choose a guess  $\mathbf{W}^g = [\mathbf{U}^g, P^g]$  and a tolerance  $\epsilon$  (usually  $\epsilon = 10^{-12}$ );
2: procedure NEWTON–RAPHSON METHOD( $[\mathbf{U}^g, P^g]$ )
3:   Solve the linear system  $LNSE(\mathbf{W}^g)(\delta\mathbf{W}) = -NS(\mathbf{W}^g)$  where
      $NS(\mathbf{W}^g)$  is defined by equation (2.7);
4:   Update the initial guess as  $\mathbf{W}^g \leftarrow \mathbf{W}^g + \delta\mathbf{W}$ ;
5:   if  $\|\delta\mathbf{W}\|_2 > \epsilon$  then
6:     Go to the step 3;
7:   else
8:     return

```

the library UMFPACK⁴ (Davis, 2003) for incompressible flows and the parallel library MUMPS⁵ (Amestoy et al., 2000) in the compressible case; these libraries are both already implemented in FreeFem++.

2.9.2 Eigenvalue computation

Once obtained the base flow, the generalized eigenvalues problem for the direct and the adjoint eigenvalues problems have to be solved.

Direct problem After the spatial discretization the problem (1.20), together with the appropriate boundary conditions, can be written as follows:

$$[A(U, Re) - \sigma B] \cdot z = 0 \quad (2.21)$$

where z is the discrete direct or right eigenvector. If we are interested to the whole eigenvalue spectrum, we can use the ARPACK library⁶ (Lehoucq et al., 1998) implemented in FreeFem++: this library uses the Arnoldi iterative method to obtain the solution of the eigenvalue problem. In particular, the Arnoldi iterations are performed on a Krylov subspace: the dimension of the subspace’s basis is set equal to $2N_{ev} + 1$, where N_{ev} is the number of required eigenvalues. At each iteration, the solution of the linear problem is performed with the same solver used for

⁴<http://faculty.cse.tamu.edu/davis/suitesparse.html>

⁵<http://mumps.enseeiht.fr/index.php?page=home>

⁶<https://www.caam.rice.edu/software/ARPACK/>

the base flow computation and the tolerance of the iteration process has been set equal to 10^{-9} .

Instead of searching the whole eigenvalue spectrum, we could also be interested to find only one eigenvalue, namely the most unstable one. For this purpose, we used the so called inverse iteration algorithm which can be written as follow:

Algorithm 2 Shift and Invert algorithn

- 1: Choose a shift σ_{sh} , and an initial guess for the eigenvalue $\sigma^{(0)}$, the eigenvector $\mathbf{z}^0$ and a tolerance ϵ ;
 - 2: **procedure** SHIFT AND INVERT($\mathbf{z}, \sigma$)
 - 3: Define $\mathbf{C} = [\mathbf{A} - \sigma_{sh}\mathbf{B}]$;
 - 4: Solve $\mathbf{C}\mathbf{z}^{(n+1)} = \mathbf{B}\mathbf{z}^{(n)}$
 - 5: Choose a projection vector $\mathbf{z}_{proj}$, as for example $\mathbf{z}_{proj} = \mathbf{z}^{(n+1)}$;
 - 6: Calculate $\Xi = \frac{\langle \mathbf{z}^{(n+1)}, \mathbf{z}^{(n+1)} \rangle}{\langle \mathbf{z}^{(n+1)}, \mathbf{z}^{(n)} \rangle}$
 - 7:
 - 8: Update $\sigma^{(n+1)} = \sigma_{sh} + \frac{1}{\Xi}$
 - 9: Calculate $err = \|\sigma^{(n+1)} - \sigma^{(n)}\|$;
 - 10: **if** $err > \epsilon$ **then**
 - 11: $\sigma^{(n)} \leftarrow \sigma^{(n+1)}$, $\mathbf{z}^{(n)} \leftarrow \frac{\mathbf{z}^{(n+1)}}{\sqrt{|\langle \mathbf{z}^{(n+1)}, \mathbf{z}^{(n+1)} \rangle|}}$ and go to the
step 4;
 - 12: **else**
 - 13: return
-

Note that at step 11 we choose to normalize the eigenvector at each iteration only to avoid possible floating–points. The exit tolerance has been setted equal to 10^{-9} . If the initial guess on the eigenvalue and the eigenvector is close to the solution, the convergence of this algorithm is very fast (3 or 4 iteration). So, usually, the standard procedure is to search the whole spectrum using ARPACK at a certain Reynolds number, in order to have an idea of the location of the most unstable eigenvalues; then, the most unstable eigenvalues are followed using the inverse iteration algorithm (that is more rapid with respect to ARPACK) increasing the Reynolds number and choosing as initial guess of the al-

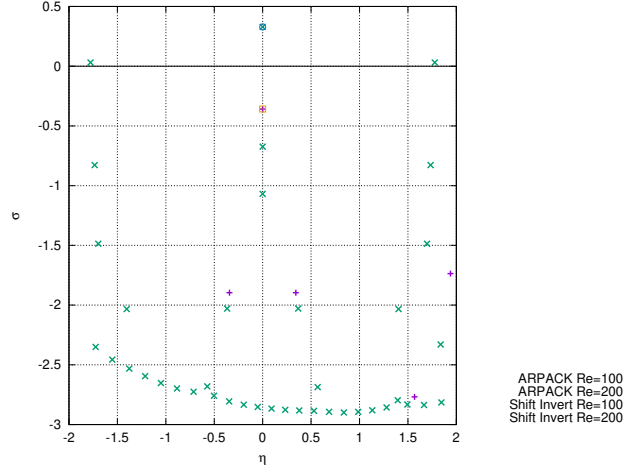


Figure 2.4: Comparison between ARPACK library and the inverse iteration algorithm 2 for a disk in a confined pipe.

gorithm the solution computed for the previous Reynolds number.

The inverse iteration algorithm 2 has been implemented in FreeFem++ and it has been tested for a more simple problem, namely a disk in a confined pipe. In particular, we calculate the whole spectrum using ARPACK and then we compute the most unstable eigenvalues using the inverse iteration algorithm 2. Results are depicted in figure 2.4, showing a perfect agreement between the two methods.

Adjoint problem The solution of the adjoint problem can be achieved using two different approaches, as sketched in figure 2.5. The first option is to solve the adjoint equations and it is indicated with continuous arrows in figure 2.5: the adjoint equations are here defined applying the generalized lagrangian identity to the continuous linear direct problem, as seen in previous chapter. As results, we obtain the continuous adjoint equations (1.47) that can be discretized and then solved with the methods seen in previous paragraphs. This approach requires two different discretizations, one for the direct and one for the adjoint equations. Actually, in a finite element fashion, this is not properly true since the integration by parts used to define the adjoint equations is the same

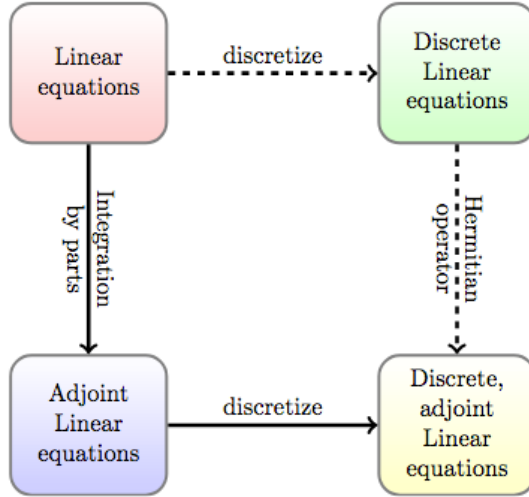


Figure 2.5: Possible routes for the solution of the adjoint problem (from Meneghello (2013)).

used to obtain the weak formulation from the differential one. Thus, the adjoint equations can be obtained simply inverting unknown functions and test functions. However, in this case, it is necessary to define the boundary conditions for the adjoint problem, operation that can results to be complicated and articulated in some circumstances.

The alternative way to compute the adjoint solution is represented by the dashed arrows in figure 2.5. In this case, the adjoint equations can be defined starting from the discretized linear direct problem rather than the continuous one. Thus, in this case, it is possible to demonstrate that the adjoint discrete matrix is equal to the hermitian (transpose and complex conjugate) of the discrete direct one (Luchini and Bottaro, 2014), namely:

$$\mathcal{L}_h^\dagger = \mathcal{L}_h^H, \quad (2.22)$$

where the subscript $(\cdot)_h$ indicates the discretized version of the linear operators. This is valid for any kind of discretization method. As a consequence, the adjoint eigenvector can be seen as the left eigenvector

of the direct linear problem (2.14), namely:

$$\mathbf{z}^\dagger \cdot [\mathbf{A}(\mathbf{U}, Re) - \sigma \mathbf{B}] = 0 \quad (2.23)$$

The eigenvalue problem (2.21) can thus be solved using the methods seen in previous paragraph, or better direct and adjoint solutions can be achieved in the same moment choosing the adjoint eigenvector as projection in step 5 of the inverse iteration algorithm (2). Finally, the discrete adjoint approach has the advantage that the direct and adjoint eigenvalues are the same at machine precision (Luchini and Bottaro, 2014).

Normalization of the eigenvectors Eigenvectors are defined up to a constant and so they must be normalized. The choice of the normalization is arbitrary; however, we prefer to spend some words about our choices, above all on the adjoint. The normalization of the direct vector can be made, for example, imposing that the maximum of a velocity component is equal to one. A convenient (but arbitrary) choice for the adjoint, instead, is to impose that $\int_V (\mathbf{u}'^\dagger \cdot \mathbf{u}') dV = 1$: in fact, it is possible to recognize in this term the denominator of the structural sensitivity (1.48) so that we have to compute only the numerator of the equation (1.48) to compute the structural sensitivity. Finally, it is interesting to observe that the structural sensitivity tensor is independent from the normalization of the adjoint eigenvector.

2.10 The complex coordinate mapping

In chapter 4 and 5, we treat the problem of the viscous flow passing through a circular aperture subject to an harmonic perturbation, as for example an acoustic gust; the flow configuration is depicted in figure 1.9. The total flow field can thus be separated in a steady base flow, governed by the steady non-linear Navier–Stokes equations, and a purely harmonic perturbation with a little amplitude, whose dynamic is ruled by the LNSE. The main characteristics of such kind of flows are the follows:

- The steady base flow is driven by the pressure drop between the inlet and the outlet part of the domain, generating a flow rate through the hole. Thus, the arising jet forming at the lip of the hole becomes parallel after several diameters but with a radius smaller than the one of the hole (vena contracta phenomenon). Moreover, for thick holes, another peculiar characteristic of the flow is the recirculation bubble forming below the hole.
- On the other hand, the harmonic perturbation is characterized by a vortex sheet generated at the edge of the hole and surrounding the steady jet. A crucial point is that, due to the strongly spatially unstable nature of the jet, all perturbations are strongly amplified along the axial direction.

The latter point leads to numerical difficulties when computing the solution of the LNSE subjected to an harmonic perturbation. In fact, the pressure field $p'(x, r)$ can reach huge levels (reaching 10^{15} or even more for $Re \approx 3000$) for large x , and this conflicts with the necessity of imposing the boundary condition $p'_{out} = 0$ at a finite distance x_{max} corresponding to the boundary of the computational domain. Moreover, even using very large values for x_{max} , the ratio between the pressure level in the vicinity of the hole and downstream can become of the order of the numerical round-off error (10^{-15} using double-precision), leading to the impossibility to accurately compute the structure of the perturbation in the vicinity of the hole and so to a wrong pressure level at inlet. We demonstrate in chapter 4 that without any numerical tricks is impossible to achieve suitable results in term of impedance for $Re \gtrsim 1500$.

In order to substantiate this fact, we briefly review the classical Kelvin–Helmholtz instability for a planar shear layer; in particular, since this is only a simplified and a qualitative analysis, we use the inviscid model whose solution is available in close form (see for an exhaustive treatment the textbooks by Drazin and Reid (2004) or Charru (2011)). Let us consider as base flow a shear layer separating two regions of constant axial velocity, namely $u'_x = U$ for $r < 0$ and $u'_x = 0$ for $r > 0$. Now assume that the perturbation consists of a displacement of the shear layer (see figure 1.9 in the form:

$$\eta(x, r, t) \propto e^{i\alpha x - i\omega t}, \quad (2.24)$$

and assume a similar modal expansion for the velocity potential in the upper and lower regions. Imposing the matching conditions at the interface of the two regions leads to the following expression of the phase velocity:

$$c \equiv \frac{\omega}{\alpha} = \frac{1 \pm i}{2} U. \quad (2.25)$$

The relation (2.25) implies that in a spatial stability framework a perturbation with real frequency ω is spatially amplified downstream with a complex wavenumber α and the solution diverges at $x \rightarrow +\infty$. So, when the axial coordinate x is real, it is not possible to write a finite solution for the displacement of the shear layer $\eta(x, r, t)$. However, the problem can be bypassed considering an analytical continuation of the function $\eta(x, t)$ with a *complex variable* x . In the specific case of the inviscid jet, as $\arg(\alpha) = -\pi/4$, the function $\eta(x, r, t)$ converges as soon as $|x| \rightarrow \infty$ in a direction of the complex plane verifying $\pi/4 < \arg(x) < 5\pi/4$. These considerations, made on a simplified model, suggest to extend this methodology to the viscous case in order to try to overcome the problem, namely using a *complex coordinate change* $x = \mathcal{G}_x(X)$ which maps a (real) numerical coordinate X defined over a finite-size computational downstream domain $X \in [-L_{in}; L_{out}]$, onto the physical coordinate x in a way that it enters the complex plane and follows a direction where the perturbation is spatially damped. Note that the idea is conceptually similar to the Perfectly Matching Layer (PML) method, which is a numerical approach largely used in electromagnetics, acoustics or compressible fluid dynamics to impose non-reflection boundary condition in wave-propagation problems (see Colonius (2004) for a complete review). Moreover, another peculiarities of this problem is to correctly impose the outflow boundary condition very far from the hole. In order to overcome to this requirement, we combine the complex mapping with a coordinate stretching in order to have short numerical domains and large physical ones.

Finally, in order to have a physical and a numerical domain with the same proportions, we also used a stretching $r = \mathcal{G}_r(R)$ for the radial coordinate from $R \in [0, R_{out}]$ to $r \in [0, r_{out}]$. However, for the radial coordinate no complex deformation has been used, since it is not crucial in terms of perturbation amplification. Details about the design of the stretching and mapping functions are given in chapter 4 and 5, since

they are a little bit different between the cases of thin and thick holes. Although the complex mapping has been implemented to tackle problems linked to perturbation issues, it has been applied also to the base flow computation, in order to be consistent with the change of coordinates. Moreover, this allows to use shorter numerical domain.

Finally, it remains to specify the numerical boundary conditions effectively used at the boundaries of the numerical domain $R = R_{out}$ (corresponding to $r = r_{out}$) and $X = L_{out}$ (corresponding to $x = x_{max}$). In the framework of finite elements, it is usual to impose outlet boundary conditions in order to take advantage of the integration by parts leading to the weak formulation. Thus, the most natural condition for open domains emerging in this way is the zero-traction condition, namely $-\mathbf{p}\mathbf{n} + Re^{-1}\nabla\mathbf{u} \cdot \mathbf{n} = 0$. In the present case, we used the zero-traction condition as an approximation of the physical condition $p = 0$ for both the base flow and perturbation computations. However, this is fair since the viscous stresses are negligible in the vicinity of the boundaries of an open domain.

A peculiarity of this method is that outflow boundary conditions are effectively applied at a location x_{max} located in the complex plane. The validity of this method is not justified by rigorous mathematical argument, but only by the fact that it effectively works, as demonstrated in chapter 4.

Finally, we want to underline that the use of complex coordinate mapping for linear problems involving a single spatial coordinate is customary in stability studies, and mathematical theorems are available to justify how to choose the integration contour as function of the singularities of the problem (see for example Bender and Orszag (2013)). On the other hand, its use for solving a non linear problem (i.e. computation of the base flow) involving two spatial coordinates is totally new to our knowledge.

2.11 Boundary treatment for compressible flow simulation

Boundary treatment is a crucial point in unsteady compressible flow simulations. In fact, the unsteadiness of the equations implies that the

sound waves (that are not contemplated in the incompressible model) are free to travel into the domain. Once one wave arrives on a numerical boundary, it is naturally reflected by it and, if the intent is to simulate an open boundary, this reflection has not any physical sense. This implies that in order to achieve suitable results, a special treatment of the open boundaries is necessary in the simulation of unsteady compressible flows. In order to solve this problem, several strategies are available in literature and an exhaustive review can be found in Colonius (2004). In chapter 6, we use the stability analysis applied to the compressible Navier–Stokes equations in order to characterize the acoustic properties of a birdcall (see figures 10 and 1.11). The computation of the base flow has been achieved without any special boundary treatment, since we are interested to a steady base flow whereas a special boundary treatment has been used in order to solve the stability problem for the harmonic perturbation. In particular, we choose to use the sponge zone technique. According to this technique, the whole domain is splitted in two regions: a physical domain, namely the domain on which we are interested to the physical solution and a sponge region which is able to absorb without reflect the sound waves generated into the physical domain. In order to implement the sponge region, following Rowley et al. (2002), we introduce the term $-\beta(\hat{r})\mathbf{q}'$ on the r.h.s of the linearized compressible Navier–Stokes equations (1.34), where $\hat{r} = \sqrt{x^2 + r^2}$, whereas the damping function $\beta > 0$, according to Yamouni et al. (2013), has been designed as follow:

$$\left. \begin{aligned} \hat{\beta}(\hat{r}) &= \left| \frac{1}{Ma} \right| \frac{2\alpha_{in}}{R_{s,in}^2} (\hat{r} - R_{in}), & \text{if } x < 0 \text{ and } \hat{r} > R_{in} \\ \hat{\beta}(\hat{r}) &= \left| \frac{1}{Ma} \right| \frac{2\alpha_{out}}{R_{s,out}^2} (\hat{r} - R_{out}), & \text{if } x > 0 \text{ and } \hat{r} > R_{out} \\ \hat{\beta}(\hat{r}) &= 0, & \text{otherwise} \end{aligned} \right\}, \quad (2.26)$$

where R_{in} and R_{out} are the inlet and outlet ray of the physical domain; $R_{s,in}$ and $R_{s,out}$, instead, are the length of the sponge zones (see figure 2.6). The length of the sponge region is chosen so that it is ten times greater than the acoustic wavelength $\lambda_{ac} = 2\pi\omega/Ma$: this means that the dimension of the sponge region is greater as the frequency increase or the Mach decrease. The coefficient α_{in} and α_{out} , instead, can be seen as

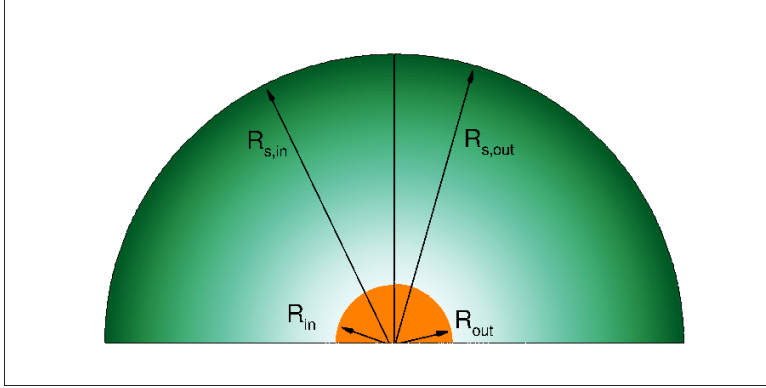


Figure 2.6: Example of computational domain (in scale) used for compressible stability analysis: in orange the physical domain; the sponge is coloured according to its intensity from the light green (minimum value) to the dark green (maximum value).

the coefficient of the exponential attenuation of the acoustic waves into the sponge: they have been fixed so that $2\alpha_{in}/R_{s,in}^2 = 2\alpha_{out}/R_{s,out}^2 \approx O(10^{-4})$. Moreover, a coarser grid has been used in the sponge region since this helps the dissipation and so the absorption of the sound waves.

An example of computational domain used for the compressible stability analysis (chapter 6) is reported in figure 2.6: the orange region is the physical domain whereas the exterior one is the sponge that is coloured as function of its intensity. Note that, according to equation (2.26), the sponge reaches its minimum value on the boundary of the physical domain (light green in figure 2.6) and then it grows until the external boundary where it reaches its maximum value (dark green in figure 2.6).

Chapter 3

Stability and sensitivity analysis of a T-shaped micro-mixer with superhydrophobic surfaces

Superhydrophobic surfaces (SHSs) are characterized by a stable layer of gas trapped in micro- or nano- grooves, with a consequently reduction of the solid-fluid contact area and of the corresponding skin-friction drag. SHSs have been largely studied and applied for drag-reduction of turbulent flows in macro-channels, while as concern transition to turbulence, they result to be effective only in micro-channels with dimensions of few millimetres.

In this context, this work aims at investigating the stability properties of the flow in a 2D T-mixer employing anisotropic SHSs in the outlet channel. Global stability analyses of the resulting flow fields have been carried out varying the characteristics of the SHSs, i.e. the slip equivalent length λ and the orientation angle θ of the grooves to the main pressure gradient direction. Starting from a steady and symmetric flow solution at low Reynolds numbers, the transition scenario is first characterized by a pitchfork supercritical bifurcation that drives the system towards a new asymmetric steady state. The use of SHSs always leads to a reduction of the critical Reynolds number for the onset of this first

bifurcation, compared to the T-mixer with solid walls. The neutral stability curves of this bifurcation together with the destabilising global modes will be discussed. In addition, the existence of a 3D bifurcation on the nominal 2D flow configuration has been investigated. Finally, using the properties of the adjoint operator, a structural sensitivity analysis is performed in order to localise the core of the instability and to better understand its nature.

3.1 Introduction

For a long time, people tried to obtain drag reduction employing surfaces as smooth as possible, but nature seems not to be in agreement. Actually, natural surfaces are never smooth and they present some small surface irregularities that, interacting with the fluid flowing over them, can change significantly its properties. An example is the shark skin that is covered by the so called *denticles* (Bottaro, 2014), i.e. flexible surface protrusions of characteristic length of $100\mu m$, that are able to reduce the hydrodynamic resistance (Oeffner and Lauder, 2012). Another very interesting example of natural surfaces are the superhydrophobic surfaces (SHSs), consisting of grooves containing trapped gas, as for example the lotus leaves. The grooves can be considered as micro and nano structure on the solid wall and, thanks to the trapped gas, they are able to reduce the fluid–solid interaction, reducing the skin friction (Rothstein, 2010). Recent studies have demonstrated that they are able to reduce the skin friction in turbulent macro-channels (Ou and Rothstein, 2005), while, as concern transition to turbulence, they result to be effective only in micro-channels with characteristic dimensions of few millimetres (Ou et al., 2004).

As function of the topology of the surface, it is possible to classify the superhydrophobic surfaces in two macro families: in particular we talk about isotropic superhydrophobic surface when the grooves have not a preferential direction of alignment, while we refer to an anisotropic superhydrophobic surface when the grooves are all aligned with some direction, forming an array (Clavijo et al., 2015).

In the past, the problem of small dimensions has limited the use of superhydrophobic surfaces for industrial applications because of the lack

of technology to produce them but, during the last years, their use is catching on thanks to new technologies.

The stability properties of flow configurations involving superhydrophobic walls have been largely investigated in literature for parallel flows using modal (Lauga and Cossu, 2005) and non modal (Min and Kim, 2005) stability theory. As concern the study of more complex fluid systems, instead, we can find some examples about stability properties of a superhydrophobic cylinder, investigated using the global stability approach (Auteri et al., 2016) and the direct numerical simulation (DNS) (Legendre et al., 2009). In all these previous works, the superhydrophobicity of the walls have been taken into account using a simple slip boundary conditions for the velocity component parallel to the mean flow. Muralidhar et al. (2011) carried out experiments over a superhydrophobic cylinder, showing that the flow properties as the vortex shedding frequency and the separation point are strongly influenced by the orientation of the grooves respect to the mean flow. Pralits et al. (2017) investigated the stability of the channel flow with superhydrophobic surfaces using a tensorial form of the slip boundary condition.

In this paper, we investigate the stability properties of the flow into a micro T-mixer employing anisotropic superhydrophobic surfaces on the outlet central channel: the existence of some kind of instability leads to an improvement of the mixing efficiency. T-shaped micromixers are very used in microfluidics to promote mixing between two fluids and their stability and mixing properties have been largely studied in literature (Fani et al., 2013, 2014; Siconolfi et al., 2015).

A global stability analysis has been carried out varying the slip length and the orientation of the grooves with respect to the main pressure gradient direction. We compared our results against the ones obtained considering a classical T-mixer with smooth walls.

3.2 Flow configuration and problem formulation

We consider the incompressible flow of a newtonian fluid inside a T-shaped micro-mixer made by two inlet channels and one outlet channel. The flow configuration is shown in Figure 3.1, together with the adopted frame of reference. The same height h has been considered for

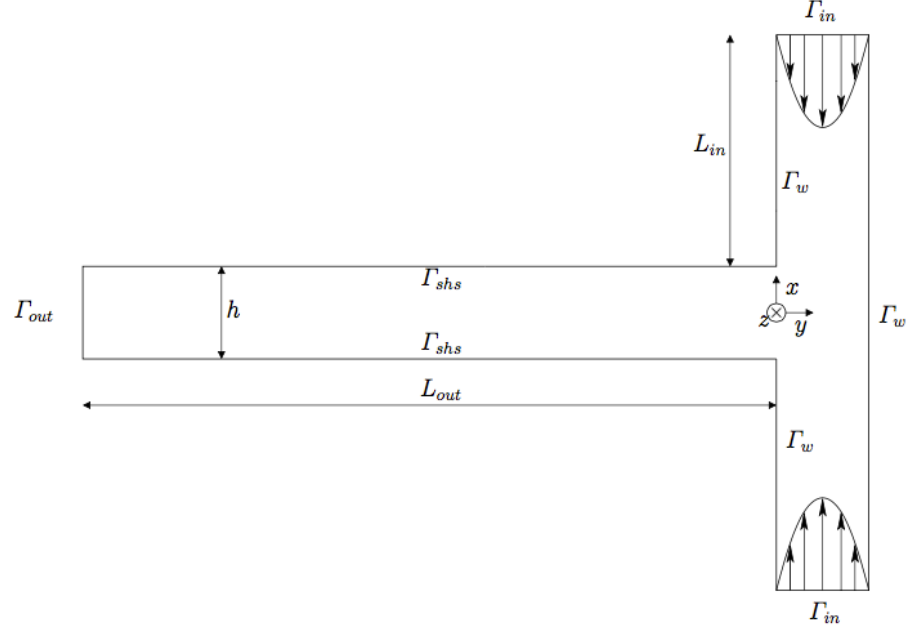


Figure 3.1: Flow configuration (not in scale) and frame of reference.

both the inflow and the outflow channels, and their lengths, L_{in} and L_{out} respectively, have been chosen in order to guarantee that the proposed results are independent of the domain size (Lashgari et al., 2014). In particular, the selected dimensions are $L_{in} = 10$, $L_{out} = 30$ and $h = 1$ (see Figure 3.1).

The fluid motion can be described by the unsteady incompressible Navier–Stokes (NS) equations that, in their non–dimensional form, can be written as follows:

$$\nabla \cdot \mathbf{U} = 0 \quad (3.1a)$$

$$\frac{\partial \mathbf{U}}{\partial t} + (\mathbf{U} \cdot \nabla) \mathbf{U} + \nabla P - \frac{1}{Re} \Delta \mathbf{U} = 0 \quad (3.1b)$$

where $\mathbf{U}(\mathbf{x}, t) = (U, V, W)^T$ is the velocity vector and $P(\mathbf{x}, t)$ is the reduced pressure. The equations (3.1) are made dimensionless considering the height of the channel h as characteristic length scale and the maximum velocity at the center of the two inlet boundaries, i.e.

$U_{max} = U(0, h/2, 0)$, as characteristic velocity. Thus, the Reynolds number Re is defined as $Re = U_{max}h/\nu$, where ν is the cinematic viscosity. The system (3.1) has to be supplemented by appropriate boundary conditions. In particular, a Poiseuille velocity profile, i.e. $U(y) = -4U_{max}y/h(1 - y/h)\text{sgn}(x)$, is imposed at the two inlet boundaries (Γ_{in}), where $\text{sgn}(x) = x/|x|$. At the outflow (Γ_{out}), the standard traction-free boundary condition has been considered, i.e. $\mathbf{n} \cdot [(Re)^{-1}\nabla \mathbf{U} - P\mathbf{I}] = \mathbf{0}$, where $\mathbf{n}$ is the wall-normal direction and $\mathbf{I}$ is the identity matrix. Concerning the lateral walls, different boundary conditions are imposed in order to highlight their effect on the dynamics of the flow. In particular, on Γ_w homogeneous boundary conditions are imposed for all the components of the velocity vector ($\mathbf{U} = \mathbf{0}$), while Γ_{shs} are the superhydrophobic surfaces: in the section 3.3 we will discuss about the boundary conditions to impose on these walls.

3.3 Boundary conditions over superhydrophobic surfaces

The classical no-slip hypothesis, that can be observed from a macroscopic point of view, can not be applied when micro and nano scales are considered (Stone et al., 2004). This phenomenon has been largely investigated in experimental (Cottin-Bizonne et al., 2005; Joly et al., 2006), theoretical (Bocquet and Barrat, 1994; Belyaev and Vinogradova, 2010) and numerical (Venkatesan and Ganesan, 2015; Jang et al., 2016) studies, in which the use of the Navier boundary conditions (Navier, 1823) is suggested instead of the classical no-slip ones. If we considered isotropic SHSs, the Navier boundary condition can be written as:

$$\mathbf{U}_{\parallel}|_{wall} = \lambda \frac{\partial \mathbf{U}_{\parallel}}{\partial \mathbf{n}} \Big|_{wall} \quad (3.2)$$

where $\mathbf{U}_{\parallel}$ is the vector containing the velocity components tangential to the surface and $\mathbf{n}$ is the outer normal vector.

According to the equation (3.2), the velocity at the wall is proportional to the shear strain rate $\partial \mathbf{U}_{\parallel} / \partial \mathbf{n}$ via the λ parameter, that can be interpreted as a slip length, i.e. the fictitious distance below the surface where the velocity vanishes if the flow field is extended linearly through

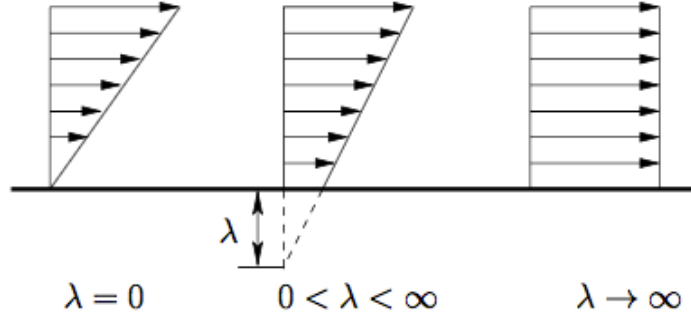


Figure 3.2: Interpretation of the Navier slip length Rothstein (2010)

the solid wall (Lauga and Stone, 2003). Thus, the Navier condition models the presence of alternate no-slip and free-slip regions, under the hypothesis that the gas inside the grooves does not exert any shear stress on the liquid that flows above (Luchini and Bottaro, 2014).

If $\lambda = 0$, the classical no-slip boundary condition is obtained while, for $\lambda \rightarrow \infty$, a slip condition is recovered (Figure 3.2).

In the case of isotropic SHSs, only one slip length is defined for all the flow directions. On the other hand, if we consider an anisotropic SHS, in which periodic arrays of micro or nano grooves are aligned along the flow direction, it is necessary to extend the Navier slip condition (3.2) introducing a slip tensor (Bazant and Vinogradova, 2008), because the slip length is different for the two tangential components of the velocity vector.

With reference to Figure 3.3, the tensorial Navier boundary condition on Γ_{shs} can be written as:

$$U_{\parallel}|_{\Gamma_{shs}} = \mathbf{\Lambda} \frac{\partial U_{\parallel}}{\partial \mathbf{n}}|_{\Gamma_{shs}} \quad (3.3)$$

where, in our frame of reference, $U_{\parallel} = (V, W)^T$ and $\mathbf{\Lambda}$ is the slip tensor defined as:

$$\mathbf{\Lambda} = \mathbf{R}_{\theta} \begin{bmatrix} \lambda^{\parallel} & 0 \\ 0 & \lambda^{\perp} \end{bmatrix} \mathbf{R}_{\theta}^T, \quad \mathbf{R}_{\theta} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \quad (3.4)$$

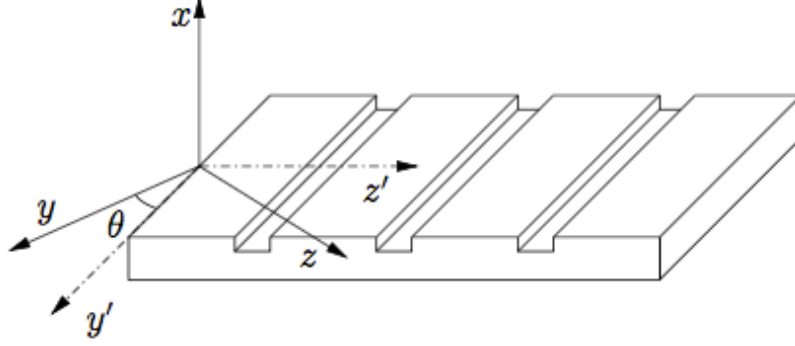


Figure 3.3: Three-dimensional view of Γ_{shs} and definition of the θ angle.

In equation (3.4), $\lambda^{\parallel}$ and $\lambda^{\perp}$ represent respectively the transverse and longitudinal slip length, i.e. the eigenvalues of the slip tensor $\mathbf{\Lambda}$ respectively for $\theta = 0^\circ$, i.e. with the grooves aligned with y , and $\theta = 90^\circ$, i.e. with the grooves aligned with z : thanks to the rotation matrix, equation (3.4) allows us to calculate the effective slip lengths in any direction defined by the angle θ .

For anisotropic SHS with grooves aligned to the mean pressure gradient and under the hypothesis of small separation between slip regions (Lauga and Stone, 2003), we have that $\lambda^{\parallel} = 2\lambda^{\perp}$ (Belyaev and Vinogradova, 2010; Teo and Khoo, 2009): in this paper we will present the results as function of only $\lambda^{\parallel}$ and in order to simplify the notation we will use just λ . Finally, as concern the boundary conditions on the wall-normal component of the velocity vector, the boundary condition on Γ_{shs} is $\mathbf{U} \cdot \mathbf{n} = 0$. This condition derives from the continuity equation and it means that there is no-penetration of the fluid at wall.

3.4 Global stability analysis

The stability characteristics have been here studied using a modal analysis. The total flow field $\mathbf{Q}(x, y, z, t) = [\mathbf{U}, P]^T = [U, V, W, P]^T$ is considered as the sum of a steady base flow $\mathbf{Q}_b(x, y) = [\mathbf{U}_b, P_b]^T = [U_b, V_b, W_b, P_b]^T$ and an unsteady perturbation $\mathbf{q}'(x, y, z, t) = [\mathbf{u}', p']^T =$

$[u', v', w', p']^T$ of small amplitude ϵ , i.e.:

$$\mathbf{U}(x, y, z, t) = \mathbf{U}_b(x, y) + \epsilon \mathbf{u}'(x, y, z, t) + \dots \quad (3.5a)$$

$$P(x, y, z, t) = P_b(x, y) + \epsilon p'(x, y, z, t) + \dots \quad (3.5b)$$

Inserting the equations (3.5) in the NS equations (3.1) and linearizing around the base state $\mathbf{Q}_b(x, y)$, two systems of PDEs are obtained, describing the structure of the base flow and the linear evolution of the perturbation. The base flow is governed by the steady version of (3.1). In this case, although the problem is bidimensional, also a spanwise component $W_b(x, y)$ is present (Vinogradova and Belyaev, 2011; Quéré, 2008) because of the Navier boundary condition (3.3) and it is equal to zero only for the two special cases of $\theta = 0^\circ$ and $\theta = 90^\circ$. The dynamics of the perturbation is then described by the linearized Navier–Stokes equations (LNSE), that can be written as follows:

$$\nabla \cdot \mathbf{u}' = 0 \quad (3.6a)$$

$$\frac{\partial \mathbf{u}'}{\partial t} + \mathcal{L}\{\mathbf{U}_b, Re\} \mathbf{u}' + \nabla p' = 0 \quad (3.6b)$$

where $\mathcal{L}\{\mathbf{U}_b, Re\}$ is the linearized Navier–Stokes operator:

$$\mathcal{L}\{\mathbf{U}_b, Re\} \mathbf{u}' = (\mathbf{U}_b \cdot \nabla) \mathbf{u}' + (\mathbf{u}' \cdot \nabla) \mathbf{U}_b - \frac{1}{Re} \Delta \mathbf{u}' \quad (3.7)$$

In the present study, the perturbation is searched in the form of normal modes. Since the flow is homogeneous in the spanwise direction, we can decompose the perturbation into Fourier modes in time and spanwise direction:

$$\mathbf{q}'(x, y, z, t) = \mathbf{q}(x, y) \exp(\sigma t + ikz) + c.c. \quad (3.8)$$

where $\mathbf{q}(x, y) = [\mathbf{u}, p]^T = [u, v, w, p]^T$, $k \in \mathbb{R}$ is the Fourier spanwise wavenumber, $\sigma \in \mathbb{C}$ is the complex angular frequency and c.c. stands for the complex conjugate term. Introducing the ansatz (3.8) into the LNSE (3.6), the following set of PDEs is obtained:

$$\nabla_k \cdot \mathbf{u} = 0 \quad (3.9a)$$

$$\sigma \mathbf{u} + (\mathbf{U}_b \cdot \nabla_k) \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{U}_b + \nabla_k p - \frac{1}{Re} \Delta_k \mathbf{u} = 0 \quad (3.9b)$$

where we define $\nabla_k = [\partial_x(\cdot), \partial_y(\cdot), ik]^T$ and $\Delta_k = (\partial_{x^2}(\cdot) + \partial_{y^2}(\cdot) - k^2)$.

The set of equations (3.9) can be rewritten also as

$$[\mathcal{A}(\mathbf{U}_b, Re, k) - \sigma \mathcal{B}] \mathbf{q} = \mathbf{0} \quad (3.10)$$

where linear operators $\mathcal{A}$ and $\mathcal{B}$ have the following expressions:

$$\begin{aligned} \mathcal{A} &= \begin{bmatrix} \mathcal{C} + \mathcal{D} + \partial_x U_b & \partial_y U_b & 0 & \partial_x(\cdot) \\ \partial_x V_b & \mathcal{C} + \mathcal{D} + \partial_y V_b & 0 & \partial_y(\cdot) \\ \partial_x W_b & \partial_y W_b & \mathcal{C} + \mathcal{D} & ik \\ \partial_x(\cdot) & \partial_y(\cdot) & ik & 0 \end{bmatrix}, \\ \mathcal{B} &= \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{aligned} \quad (3.11)$$

in which $\mathcal{C} = (\mathbf{U}_b \cdot \nabla_k)$ and $\mathcal{D} = -(Re)^{-1} \Delta_k$ represent respectively the advection of the perturbation by the base flow and its viscous diffusion (Citro et al., 2015c).

The global stability problem (3.9) has to be completed with the boundary conditions, that can be directly derived from the flow the decomposition (3.5). Thus, homogeneous Dirichlet boundary conditions, i.e. $(\mathbf{u} = \mathbf{0})$, is considered on Γ_{in} and Γ_w , while Neumann traction-free boundary conditions on Γ_{out} ($\mathbf{n} \cdot [(Re)^{-1} \nabla \mathbf{u} - p \mathbf{I}] = 0$).

As regard the SHS walls, i.e. on Γ_{shs} , Navier slip boundary conditions of the same type of (3.3) are imposed on the tangential components of the perturbation (Lauga and Cossu, 2005), while no-penetration is imposed on the wall-normal direction, i.e.:

$$\mathbf{u}_{||}|_{\Gamma_{shs}} = \Lambda \frac{\partial \mathbf{u}_{||}}{\partial \mathbf{n}} \Big|_{\Gamma_{shs}}, \quad \mathbf{u}|_{\Gamma_{shs}} \cdot \mathbf{n} = 0 \quad (3.12)$$

The linear system (3.10), together with the boundary conditions, is a generalized eigenvalue problem for the complex angular frequency $\sigma = \gamma + i\omega$, where γ and ω are the growth rate and the angular frequency of

the perturbations, respectively. In particular, if $\gamma > 0$, the disturbances grow exponentially in time and, thus, the flow is unstable.

The adjoint equations can be derived using the generalized Lagrangian identity as reviewed by Luchini and Bottato (Luchini and Bottaro, 2014), that leads to the following set of PDEs:

$$\nabla_k \cdot \mathbf{u}^\dagger = 0 \quad (3.13a)$$

$$-\sigma \mathbf{u}^\dagger + (\mathbf{U}_b \cdot \nabla_k) \mathbf{u}^\dagger - \nabla \mathbf{U}_b \cdot \mathbf{u}^\dagger + \nabla_k p^\dagger + \frac{1}{Re} \Delta_k \mathbf{u}^\dagger = 0 \quad (3.13b)$$

The corresponding set of boundary conditions for system (3.13) is here obtained by imposing the elimination of the boundary terms after the application of the generalized Lagrangian identity and the Green theorem (Marquet et al., 2009).

From the combination of the direct and adjoint global modes, the structural sensitivity tensor can be evaluated, identifying the core of the instability (Giannetti and Luchini, 2007). This tensor can be expressed as follow:

$$\mathbf{S}(x_0, y_0) = \frac{\mathbf{u}^\dagger(x_0, y_0) \otimes \mathbf{u}(x_0, y_0)}{\iint_{\Omega} (\mathbf{u}^\dagger \cdot \mathbf{u}) d\Omega} \quad (3.14)$$

where Ω is the whole domain and $\mathbf{u}^\dagger \otimes \mathbf{u}$ is the dyadic product between the direct and the adjoint mode¹.

3.5 Numerical implementation

The equations for the base flow, for the direct and adjoint linear problems have been discretized in space using a finite-element formulation with Taylor-Hood elements. The implementation has been carried out using the open-source solver FreeFem++ (<http://www.freefem.org/>). The domain has been discretized using the mesh generator BAMG implemented in FreeFem++ (Hecht, 2012), that employs a Delaunay-Voronoi

¹Different norm of the $\mathbf{S}$ tensor can be used in order to build a spatial map of the structural sensitivity. In the present study, the spectral norm have been chosen, that gives the maximum coupling among the velocity components (Pralits et al., 2010).

triangulation of the domain (Weatherill, 1992). Finer grid regions have been generated near the corners at the confluence between the two inlets and the outlet channel, in order to correctly describe the high velocity gradients.

First, the base flow has been here solved using the classical Newton–Raphson procedure for all the different angles θ and the slip lengths λ considered. Then, the spatial discretization of the direct linear problem (3.9) leads to a definition of following eigenvalue problem (Tammiola et al., 2014):

$$[\mathbf{A}(\mathbf{U}_b, Re, k) - \sigma \mathbf{B}] \cdot \mathbf{z} = \mathbf{0} \quad (3.15)$$

where matrices $\mathbf{A}$ and $\mathbf{B}$ represent the discretization of the inherent operators, and $\mathbf{z}$ is the discrete direct eigenvector. The discrete adjoint is used for the sensitivity analysis: the eigenvalues of the direct and adjoint problems coincide to machine precision (see Luchini and Bottaro (2014) for more detail). Considering the same mesh grid used for the evaluation of the base flow, the eigenvalue problems have been solved exploiting an Arnoldi method already implemented in the ARPACK library (Lehoucq et al., 1997) in FreeFem++.

3.6 Results and discussions

The stability analysis of both the first and the secondary instability has been carried out, varying the angle θ between 0° and 90° and $\lambda = [0.001; 0.006; 0.01; 0.02; 0.03; 0.04; 0.05]$. Pralits et al. (2017) demonstrated that, for parallel channel flows, the Navier–slip boundary conditions is a good approximation for $\lambda < 0.05$ and, already for $\lambda = 0.1$, the relative error committed in the evaluation of the eigenvalue is about 10%. In the present analysis, due to the complexity of the flow configuration, the maximum value of λ have been fixed at $\lambda^* = \max(\lambda) = 0.05$.

In Figure 3.4, two examples of the velocity distribution at $Re = 500$ and different parameters λ and θ are presented. In particular, figure 3.4 (a) shows the case of the T-mixer with classical no-slip boundary conditions on the wall: the flow is symmetric and it is characterized by two recirculation regions in the first part of the outlet channel, between the sections at $y = 0$ and $y = -2$. Figure 3.5 (a) shows the velocity profiles for the two components U_b and V_b extracted at $y = -1$. In Figure 3.4

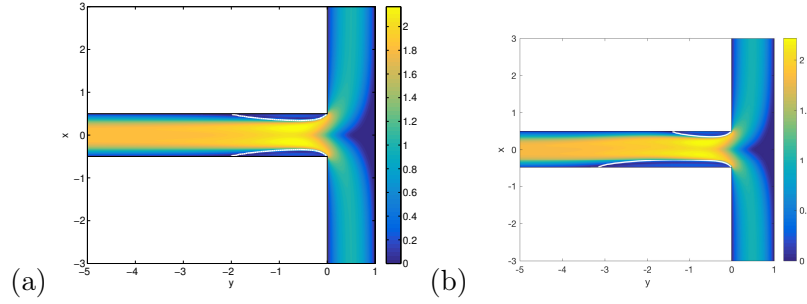


Figure 3.4: Total velocity distribution for $Re=500$: (a) symmetric steady flow for $\lambda = 0$ and $\theta = 0^\circ$; (b) asymmetric steady supercritical flow for $\lambda = 0.02$ and $\theta = 45^\circ$. In both figures, the white lines represent the edges of the recirculation bubbles.

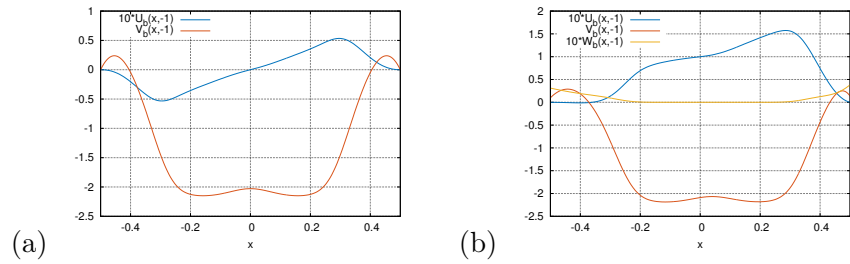


Figure 3.5: Velocity components at $y = -1$ for $Re=500$: (a) symmetric steady flow for $\lambda = 0$ and $\theta = 0^\circ$; (b) asymmetric steady supercritical flow for $\lambda = 0.02$ and $\theta = 45^\circ$.

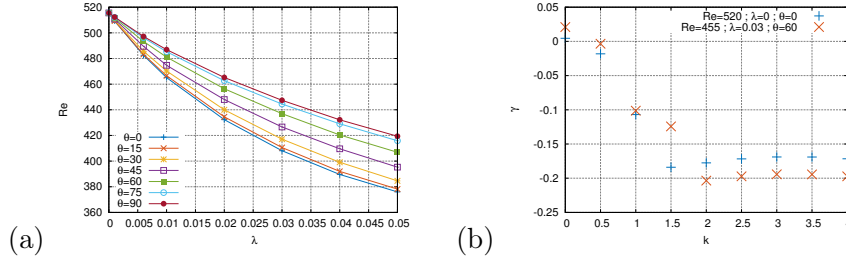


Figure 3.6: (a) Critical Reynolds number of the first bifurcation as function of λ and θ . (b) Growth rate of the leading eigenvalue as function of the spanwise wavenumber k .

(b), the velocity distribution obtained in presence of superhydrophobic surfaces is presented at $Re = 500$. In this case, the outflow channel walls are characterized by $\lambda = 0.02$ and $\theta = 45^\circ$. The symmetric configuration becomes unstable and, thus, the resulting stable flow field is asymmetric and characterized by two recirculation regions of different length. The symmetry breakdown is also observed in Figure 3.5 (b), where velocity profiles extracted at $y = -1$ are reported. Here, it is possible to observe that the spanwise velocity component of the base flow W_b , induced by the presence of the grooves on the surface, becomes relevant only approaching the walls, while it is almost null in the center of the channel, i.e. at $x = 0$.

In order to investigate the properties of the first instability, a global stability analysis has been carried out, varying the properties of the surfaces. As first test, stability analyses of a T-mixer with classical no-slip surfaces have been performed in order to check the convergence of our code. The critical Reynolds numbers for the first bifurcation as a function of the superhydrophobic surface parameters λ and θ are reported in Figure 3.6 (a). As general results, it is clearly visible that the critical Reynolds number obtained considering the superhydrophobic surfaces is always smaller with respect to that obtained for the configuration with no-slip wall conditions. Moreover, fixing the θ angle, it decreases monotonically with the slip length λ and it results to be smaller for grooves aligned to the main pressure gradient ($\theta = 0^\circ$) rather than perpendicular ones ($\theta = 90^\circ$). Finally, from Figure 3.6 (b), we can conclude that

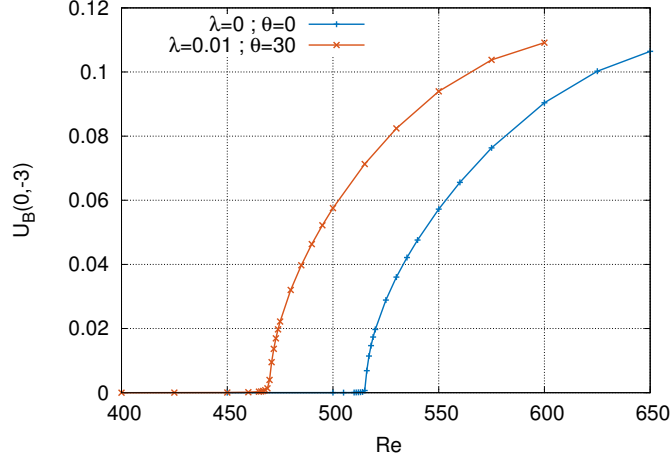


Figure 3.7: Examples of bifurcation diagrams.

the first bifurcation is a steady, because the growth rate of the leading eigenvalue is maximum for $k = 0$.

In order to further investigate the characteristics of the first instability, we introduce a measure of the asymmetry of the base flow: the velocity U_b component along the x-axis extracted at the point $(x^*, y^*) = (0, -3)$ (see Lashgari et al. (2014)). This quantity is here reported in Figure 3.7, confirming that the first instability is a supercritical pitchfork bifurcation.

Figures 3.8 (a–c) show the real part of the velocity components of the leading global mode for the unstable symmetric base flow at $Re = 455$, $\lambda = 0.03$, $\theta = 60^\circ$ and $k = 0$. It is clear that the most significant component of the global mode is the v -component, that reaches the maximum value in the regions of the recirculation bubbles. In particular, in one recirculation region the velocity perturbation is in the same direction of the base flow, while it has opposite direction in the other one. The same behaviour can be observed on the pressure field shown in Figure 3.8 (d). This leads to the symmetry breakdown and to the formation of an asymmetric base flow, as already shown in Figure 3.4. Finally, from Figure 3.8 (a,b) we can note that the v -component of the perturbation is one order of magnitude bigger than the vertical one, in-

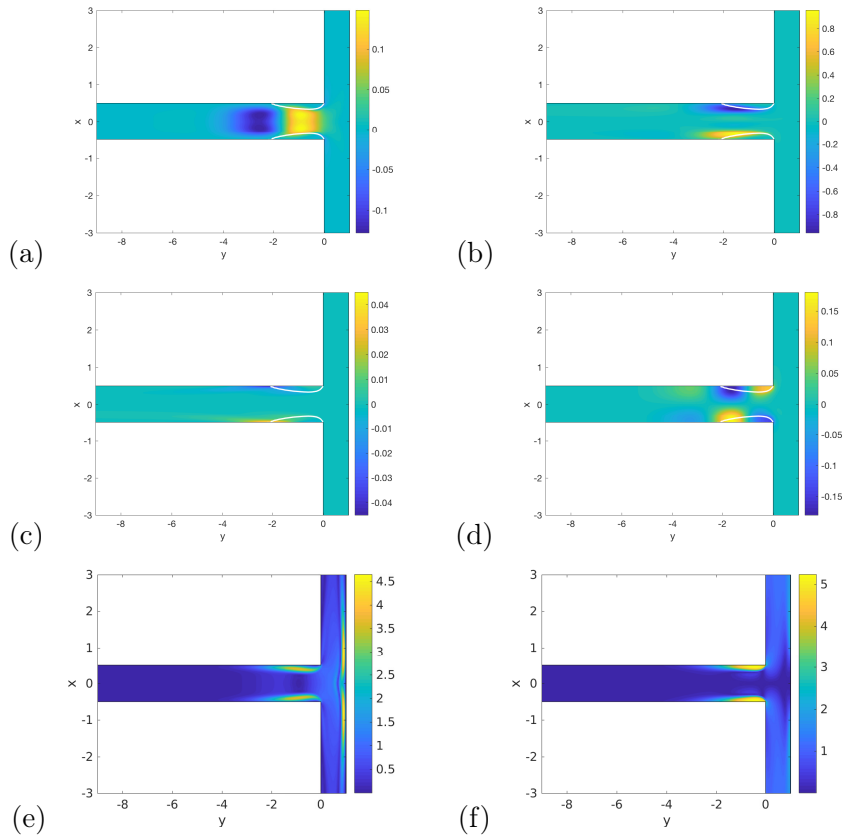


Figure 3.8: Real part of the direct global modes of the symmetric unstable base flow for $\text{Re}=455$, $\lambda = 0.03$, $\theta = 60^\circ$ and $k = 0$: (a) u -component, (b) v -component, (c) w -component, (d) pressure. (e) Modulus of the adjoint velocity field $|\mathbf{u}^\dagger|$; (f) modulus of the adjoint pressure $|p^\dagger|$ of the symmetric unstable base flow for $\text{Re}=455$, $\lambda = 0.03$, $\theta = 60^\circ$ and $k = 0$. The white lines represent the edges of the recirculation bubbles of the base flow.

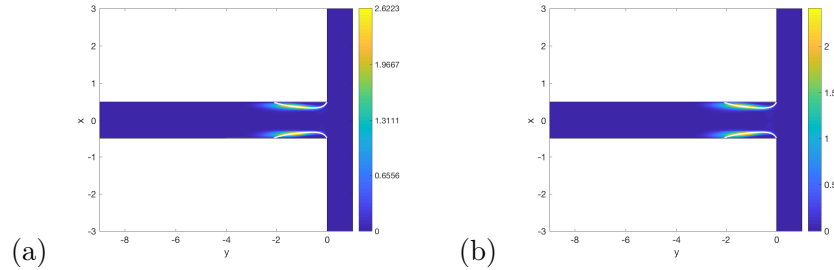


Figure 3.9: Structural sensitivity map for (a) $Re=520$ without superhydrophobic surfaces and (b) $Re=455$, $\lambda = 0.03$, $\theta = 60^\circ$ and $k = 0$.

dicating a lift-up mechanism of the instability (Brandt, 2014).

In order to investigate the receptivity properties of this flow configuration, the adjoint field has been also computed varying the parameters θ and λ . In Figure 3.8 (e) it is shown the absolute value of the adjoint velocity field for $Re = 455$, $\lambda = 0.03$, $\theta = 60^\circ$ and $k = 0$. This quantity represents the receptivity of the instability to a momentum forcing (Giannetti and Luchini, 2003) and it reaches its maximum value in the region of the recirculation bubble and near the vertical wall of the inlet channels. In addition, the most receptive region to mass injection are reported in Figure 3.8 (f), where the modulus of the adjoint pressure $|p^\dagger|$ is depicted.

Finally, starting from the direct and the adjoint modes, the structural sensitivity tensor has been evaluated for different values of λ and θ and the general shape of the corresponding sensitivity maps can be seen in Figure 3.9: in particular Figure 3.9 (a) represents the structural sensitivity for a T-mixer with smooth surfaces at $Re = 520$ whereas Figure 3.9 (b) reports the sensitivity map at $Re = 455$ with a surface characterized by $\lambda = 0.03$, $\theta = 60^\circ$ and $k = 0$. The instability core consists in a couple of symmetric lobes located on the edges of the recirculation bubbles, that is very common for this kind of flows (see, for examples, Lashgari et al. (2014) or Marquet et al. (2009)). Moreover, the shape of the structural sensitivity map is not affected by the presence of the superhydrophobic surfaces, meaning that they are not able to modify the instability mechanism with respect to the smooth case.

The asymmetric base flow that grows up after the first bifurcation re-

mains stable until the second critical Reynolds number is reached: in this case the flow bifurcates again but, this time, the leading global modes result to be three-dimensional, i.e. characterized by a Fourier number k into the spanwise direction. The critical Reynolds numbers of the sec-

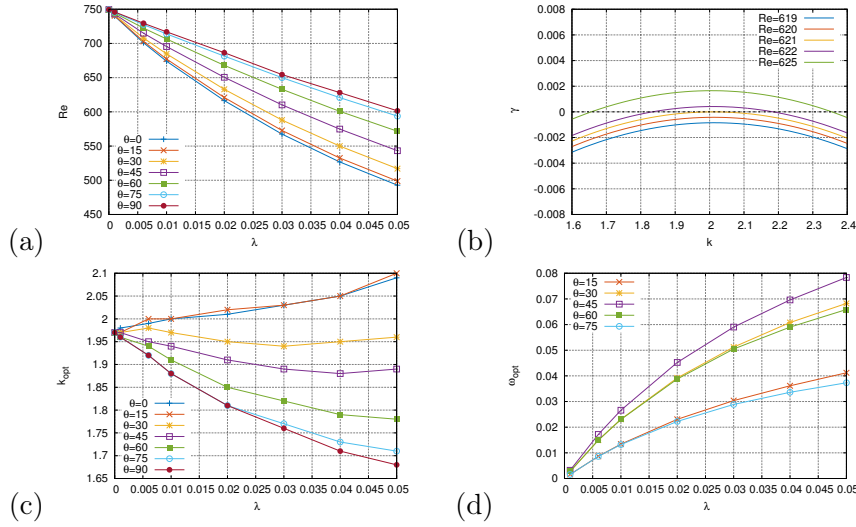


Figure 3.10: (a) Critical Reynolds number of the secondary bifurcation as function of λ and θ . (b) Growth Rate of the leading unstable eigenvalue as function of k and Re for $\lambda = 0.02$ and $\theta = 15^\circ$. (c) Optimal wavenumber k and (d) frequency ω corresponding to the critical conditions as function of λ and θ . Note that for $\theta = 0^\circ$ and 90° the modes are steady and the optimal frequency is equal to zero.

ondary instability are reported in Figure 3.10 (a) as function of λ and θ and the general trend is the same of the one discussed for the first instability. In Figure 3.10 (b), instead, we reported the growth rate of the leading eigenvalue as function of the spanwise wavenumber k at various Reynolds number, for $\lambda = 0.02$ and $\theta = 15^\circ$. It is possible to highlight that, for this condition, the critical Reynolds number is $Re_{cr,II} \approx 621$ for $k_{opt} \approx 2.0$. Figure 3.10 (c) shows the optimal spanwise wavenumber corresponding to all the critical conditions explored here: we can observe that for $\theta = 15^\circ$ and $\theta = 75^\circ$, the curves are similar to the ones obtained respectively for $\theta = 0^\circ$ and $\theta = 90^\circ$. Finally, a very important feature emerges analyzing the Figure 3.10 (d): the bifurcation, in fact, results to

be unsteady for the angle $\theta \neq 0^\circ, 90^\circ$. This characteristic is very important since the oscillation of the flow can improve the mixing efficiency. The optimal frequency ω_{opt} , reported in Figure 3.10 (d), reaches its maximum for $\theta = 45^\circ$ and, moreover, for little value of the slip length, it is equal for angle complementary to 90° while the various curves are no more overlapped as λ is increased. This transition from steady to unsteady modes has been found also in other system depending by multiple parameters, as for example for the stability analysis of non-Newtonian flows in closed (Haque et al., 2012) and open (Citro et al., 2014) cavities.

In Figure 3.11 (a–d), the real parts of the leading unstable global mode at $Re = 639$, $\lambda = 0.03$, $\theta = 60^\circ$ and $k = 1.7$ are shown. We can note that the asymmetry of the base flow is reflected also on the structure of the mode. Also in this case, the vertical component of the perturbation (u -component) is very small with respect to the other ones. The receptivity of the instability to a momentum forcing is here investigated by using the adjoint field, and reported in Figure 3.11 (e) for the case of $Re = 639$, $\lambda = 0.03$, $\theta = 60^\circ$, $k = 1.7$. It results to be not symmetric as for the first instability and its maximum value is near the lateral wall of the inlet channel opposite to the longer recirculation bubble. However, a moderate value is reached into the longer recirculation bubble. Similar considerations can be done for the receptivity to mass injection, represented in Figure 3.11 (f).

In order to localize the instability core, Figure 3.12 shows the structural sensitivity of the secondary instability for the unstable base flow. In particular, as for the first instability, we report in Figure 3.12 (a) the case with smooth walls at $Re = 754$ and $k = 2.0$ and in Figure 3.12 (b) the case with a superhydrophobic surface at $Re = 639$ and characterized by $\lambda = 0.03$, $\theta = 60^\circ$ and $k = 1.7$. This field reaches its maximum into the longer recirculation bubble for both cases meaning, as per the first instability, that the presence of the superhydrophobic surface does not change the instability mechanism.

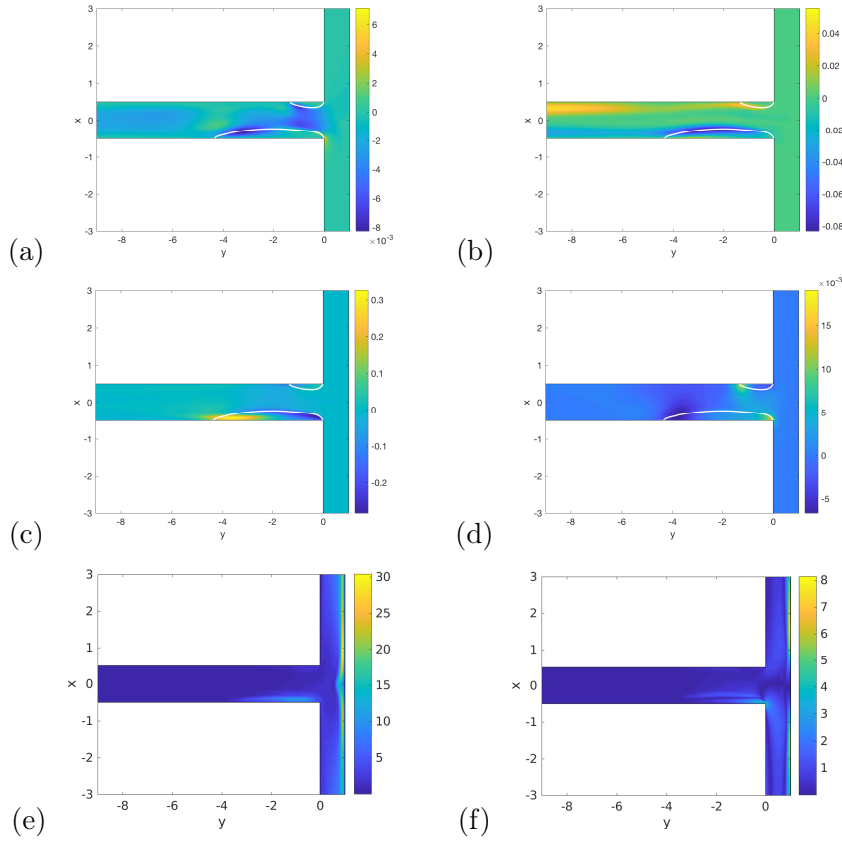


Figure 3.11: Real part of the direct global modes for the unstable asymmetric base flow for $Re = 639$, $\lambda = 0.03$, $\theta = 60^\circ$, $k = 1.7$ and $\omega \approx 0.474$: (a) u -component, (b) v -component, (c) w -component, (d) pressure. (e) Modulus of the adjoint velocity field $|\mathbf{u}^\dagger|$; (f) modulus of the adjoint pressure $|p^\dagger|$ of the unstable asymmetric base flow for $Re=639$, $\lambda = 0.03$, $\theta = 60^\circ$, $k = 1.7$ and $\omega \approx 0.474$. The white lines represent the edges of the recirculation bubbles of the base flow.

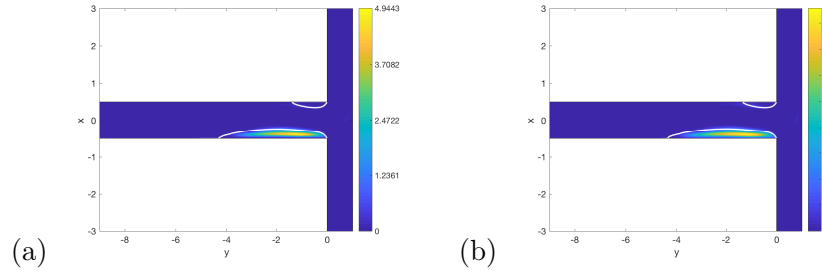


Figure 3.12: Structural sensitivity map of the secondary instability for (a) $Re=754$, $k = 2.0$ without superhydrophobic surfaces and (b) $Re=639$, $\lambda = 0.03$, $\theta = 60$ and $k = 1.7$.

3.7 Conclusions

In this paper we investigate the first and secondary instability characteristic of a 2D T-shaped micro-mixer with an anisotropic superhydrophobic texture on the surfaces of the outlet channel. In particular, parametric analyses have been carried out varying the surface's properties, i.e. the equivalent length λ of the grooves and their orientation angle θ with respect to the main pressure gradient direction.

The superhydrophobic surface has been treated using a tensorial version of the Navier slip boundary conditions, according to which the velocity components parallel to the surface are proportional via λ to their normal derivative and, following the dedicated literature (see for example Lauga and Cossu (2005), Pralits et al. (2017) or Min and Kim (2005)). This condition has been imposed both to the base flow and to the perturbation.

A large set of global stability analyses has been carried out and, using the direct-adjoint properties, we have identified the core of the instability.

We found that, as for the benchmark case of a classical T-mixer without superhydrophobic surfaces, the flow undergoes a first pitchfork supercritical bifurcation, that drives the flow towards a new asymmetric steady stable state. After the first bifurcation, as the Reynolds number increases, a three dimensional instability of the asymmetric base flow occurs. In particular, we found that the second bifurcation is unsteady,

except for $\theta = 0^\circ, 90^\circ$, i.e. when the grooves are respectively parallel and orthogonal to the main pressure gradient direction. For all the investigated cases, the critical Reynolds number of both first and secondary instability have been found to be smaller in comparison with the configuration without superhydrophobic surfaces. This means that the onset of the instability is in general anticipated by the presence of the superhydrophobic surfaces.

Finally, the core of the first instability is localized on the edges of both the recirculation bubbles, while the core of the secondary instability is just concentrated into the longer one.

Chapter 4

The acoustic impedance of a laminar viscous jet passing through a circular aperture in a thin plate

The unsteady axisymmetric flow through a circular aperture in a thin plate subjected to harmonic forcing (for instance under the effect of an incident acoustic wave) is a classical problem first considered by Howe (*Proc. R. Soc. London. A*, vol. 366, 1979, pp. 205 – 223), using an inviscid model. The purpose of this work is to reconsider this problem through a numerical resolution of the incompressible linearized Navier–Stokes equations (LNSE) in the laminar régime, corresponding to $Re = [500, 5000]$. We first compute a steady base flow which allows us to describe the *vena contracta* phenomenon in agreement with experiments. We then solve a linear problem allowing to characterize both the spatial amplification of perturbations and the impedance (or equivalently the Rayleigh conductivity), which is a key quantity to investigate the response of the jet to acoustic forcing. Since the linear perturbation is characterized by a strong spatial amplification, the numerical resolution requires the use of a complex mapping of the axial coordinate in order to enlarge the range of Reynolds number investigated. The results show that the impedances computed with $Re \gtrsim 1500$ collapse onto a sin-

gle curve, indicating that a large-Reynolds number asymptotic regime is effectively reached. However, expressing the results in terms of conductivity leads to substantial deviation with respect to Howe’s model. Finally, we investigate the case of finite amplitude perturbations through direct numerical simulations (DNS). We show that the impedance predicted by the linear approach remains valid for amplitudes up to order 10^{-1} , despite the fact that the spatial evolution of perturbations in the jet is strongly nonlinear.

4.1 Introduction

The problem of the flow passing through a circular aperture in a plate is encountered in many practical applications, as for example fuel injectors, cooling system for gas turbines or wind instruments. When subjected to harmonic forcing, for instance under the effect of an incident acoustic wave, the vortex sheet formed at the rim of the aperture becomes periodically modulated and acts as a spatial amplifier of Kelvin-Helmholtz instability, reorganizing the jet into an array of vortex rings. This feature is an essential part of the sound production mechanism in situations where the jet subsequently passes through a second aperture, a configuration known as “hole-tone” and encountered for instance in tea kettles (Henrywood and Agarwal, 2013) and birdcalls (Fabre et al., 2014a). The generation of vorticity is also an efficient mechanism to dissipate the acoustic energy. As a consequence, the use of multiply perforated plates traversed by a mean flow (or bias flow) is widely used as a sound attenuator device in many industrial applications, such as combustion system (Hughes and Dowling, 1990; Rupp et al., 2012).

The unsteady, periodic flow through a circular hole in a zero-thickness plate was initially solved by Rayleigh (1945) using inviscid, potential theory. The key result of his solution is the proportionality between the net pressure force felt from both sides of the hole and the acceleration of the fluid, so that the whole situation can be modeled by assuming that there is a rigid plug of fluid, with area $A_h = \pi R_h^2$ and equivalent length ℓ_{eff} , oscillating across the aperture, where R_h is the radius of the hole.

The case where the flow has a mean component (or bias flow) in addition to the oscillating component was considered by Howe (1979).

He introduced a key quantity, the Rayleigh conductivity K_R , defined as the ratio of the acceleration of the fluid particles located within the aperture to the net force exerted on it. The real part of the conductivity generalizes the concept of equivalent length ℓ_{eff} previously introduced by Rayleigh, while its imaginary part is directly proportional to the flux of energy transferred from the imposed oscillatory flow to the jet. Under the hypothesis of high Reynolds number, low Mach number, and assuming that the oscillating flow is of small amplitude with respect to the mean (or bias) flow, Howe derived a theoretical model describing the vorticity shed at the rim of the aperture and predicting the real and imaginary parts of the conductivity by analytical formulas. The main features and caveats of Howe's model will be reviewed in section 4.2.5.

In recent years, a number of studies have considered the interaction between acoustics and perforated plates in more complex situations including multiple holes (Hughes and Dowling, 1990), turbulent flows either parallel or tangential to the plates (Eldredge et al., 2007; ?; Mann et al., 2013), different geometries including honeycombs (Zhang and Bodony, 2016) and slit resonators (Tam et al., 2005), or additional physical effects such as thermoacoustic instabilities (Rupp et al., 2012). In the case where the thickness of the hole is not small compared to its radius, results substantially deviate from Howe's predictions, and a number of studies have proposed improvements of the original model to enlarge its range of validity (Bellucci et al., 2004; Jing and Sun, 2000; Yang and Morgans, 2017). In the case where the amplitude of the oscillating flow becomes comparable to that of the mean flow, nonlinearities also lead to substantial deviations (Jing and Sun, 2002; Scarpato, 2014). However, in the case of small-amplitude oscillations and short holes, Howe's model still constitutes the cornerstone for theoretical modelling of such flows (Scarpato et al., 2012).

In view of the above discussed literature, we can note that all available theoretical models are of inviscid nature and describe the vorticity production in terms of vortex sheets, thus these models are expected to be relevant only in the large-Reynolds limit. An alternative way, which allows to incorporate viscous effects in a rigorous way and to consider arbitrary values of the Reynolds number, is to use Linearized Navier-Stokes equations (LNSE). A number of studies have considered jet flows under this framework. Garnaud et al. (2013) considered the spatial am-

plification properties of an incompressible jet using a laminar base-flow solution for $Re \leq 1000$. Even more recent works have considered the case of compressible jets for $Ma \approx 0.9$ in the turbulent range ($Re \approx 10^6$) using a mean-flow obtained from experimental results (?), RANS simulations (Jeun et al., 2016), or LES simulations (Schmidt et al., 2017; ?). However, the focus of these studies was to characterize the spatial amplification properties of the jet and the sound radiation in the downstream domain due to vortex-shedding effects, which are different questions to the one we are considering here. Moreover, all the cited works considered a jet with imposed outlet velocity profile and did not consider the whole process of formation of a jet through a plate from an upstream domain to a downstream one, which is a necessity to correctly treat our problem.

It should also be noted that the application of LNSE to jet flows is much more difficult for the high-Reynolds, laminar range $Re \approx [500 - 5000]$ which is considered in this paper than for to the turbulent range. In effect, in the laminar range, the shear layers bounding the jet remain very sharp far downstream, leading to strong amplifications of convective instabilities extending very far away. It is thus difficult to design a method capturing both the spatial growth of perturbations in the axial direction, which can reach huge levels when the axial distance and the Reynolds number are large, and the coupling between the flow rate and the pressure jump, which is relevant when considering the possible coupling with an acoustical system. On the other hand, in the turbulent range, the shear layers of the jet spread rapidly in the downstream direction, leading to stabilization of the convective instabilities within a distance of about ten diameters of the jet.

The objectives of the present paper can thus be summarized in three main points.

- (i) First, we wish to design a numerical approach based on Linearized Navier-Stokes Equations, to compute the Rayleigh conductivity of the flow through a hole in the laminar but high-Reynolds range. We will introduce a convenient method based on a change of variable of the axial coordinate x in the complex plane (inspired by the PML method used in linear acoustics) which allows to perform accurate computations up to $Re \approx 10^4$.
 - (ii) Secondly, we wish to reconsider the case of a hole of zero thickness
-

initially investigated by Howe. We document the structure of base flow, with particular focus on the *vena contracta* phenomenon. We then describe the spatial structures corresponding to the linear response of the jet to harmonic forcing. The velocity and vorticity components of these structures allow to describe the spatial amplification by the jet, while the pressure components give access to the Rayleigh conductivity. We will compute and display the Rayleigh conductivities (as well as the equivalent concept of impedance) as function of forcing frequency and Reynolds number in the range $10^2 - 10^4$ and compare with the inviscid predictions of Howe.

- (iii) Finally, the third objective is to assess the validity of the linearized Navier-Stokes Equations with respect to perturbations of finite amplitude ε . For this purpose, we will conduct a Direct Numerical Simulation (DNS) of the forced axial-symmetric Navier-Stokes equations in the range $\varepsilon = [10^{-4} - 10^{-1}]$. Results show that the impedances are effectively well predicted by linearized Navier-Stokes equations (LNSE) up to $\varepsilon = 10^{-1}$, despite the fact that the evolution of vorticity perturbations in the jets are strongly nonlinear.

As briefly discussed in the bibliographical review, in the case where the plate is not thin and the holes are sufficiently long, different mechanisms take place and the jet can cease to act as a sound damper to become a sound generator (Jing and Sun, 2000; Yang and Morgans, 2017). The conductivity/impedance concepts are useful tools to characterize the mechanisms in this case. A full characterization of the impedance of finite-thickness holes using the method introduced here as well as a discussion of impedance-based instability criteria will be presented in a forthcoming paper.

4.2 Problem definition and review of inviscid models

4.2.1 Problem definition

The situation considered here is the flow of a viscous fluid of density ρ and viscosity ν through a circular hole of radius R_h and area $A_h = \pi R_h^2$

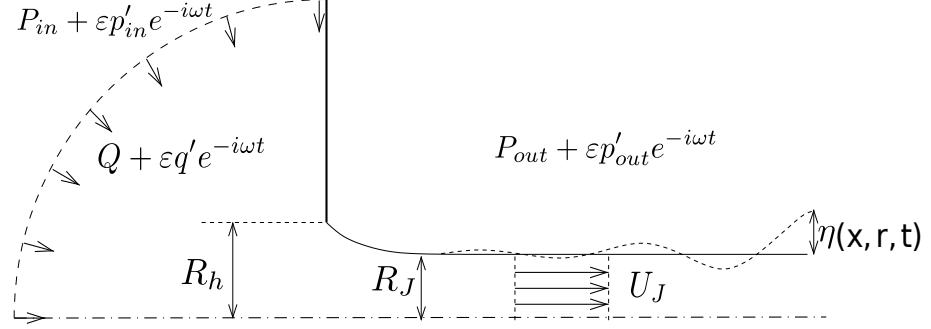


Figure 4.1: Sketch of the oscillating flow through a circular aperture in a thin plate.

inside a planar thin plate of thickness negligible respect to the radius, connecting an inner and an outer open domain, as shown in figure 4.1. We note Q the mean volumetric flow rate across the aperture, and from that later quantity we classically define the mean velocity as $U_M = Q/A_h$. Thus the Reynolds number of the flow is defined as:

$$Re = \frac{2R_h U_M}{\nu} \equiv \frac{2Q}{\pi R_h \nu}. \quad (4.1)$$

When subjected to harmonic forcing with frequency ω , a second dimensionless parameter naturally emerges: the dimensionless frequency Ω (or Strouhal number) defined as:

$$\Omega = \frac{\omega R_h}{U_M} \quad (4.2)$$

The final goal of our study is to characterize the interaction of the jet with acoustic waves, and in the general case this calls for a description using compressible equations. However, in many situations, it is justified to consider that the flow is locally incompressible, and hence to assume a uniform density ρ of the fluid. This simplification is justified under two hypotheses. First, the Mach number $Ma = U_M/c_0$ based on the mean velocity of the jet must be small enough. Secondly, all lengthscales

characterizing the aperture (here, the only relevant one is the radius R_h since the thickness is assumed to be zero) have to be small compared to the acoustic wavelength $\lambda_{ac} = 2\pi c_0/\omega$. This latter hypothesis is often referred to in acoustic textbooks as the *acoustic compactness* hypothesis. In dimensionless terms, the hypothesis can be formulated as $\lambda_{ac}/R_h = 2\pi/(Ma\Omega) \gtrsim 10$, so for the largest frequencies considered here ($\Omega \approx 6$) it is valid up to $Ma \approx 0.1$. For a flow of air through a typical hole of radius $R_h = 1\text{mm}$, the condition $Ma \approx 0.1$ corresponds to $Re \approx 4500$, confirming the relevance of the range of parameters investigated in the present paper.

Under these two hypotheses, it is justified to assume that far away from the hole, the pressure levels in the upstream and downstream regions tend to uniform values noted respectively $p_{in}(t)$ and $p_{out}(t)$. In the harmonic regime, the upstream and downstream pressure levels as well as the flow rate will be expanded as :

$$\begin{pmatrix} p_{in}(t) \\ p_{out}(t) \\ q(t) \end{pmatrix} = \begin{pmatrix} P_{in} \\ P_{out} \\ Q \end{pmatrix} + \varepsilon \begin{pmatrix} p'_{in} \\ p'_{out} \\ q' \end{pmatrix} e^{-i\omega t} + c.c., \quad (4.3)$$

where ε is a given amplitude of the harmonic perturbation and $\omega \in \mathbb{R}$ is the oscillation rate. The interaction of the jet with external systems can thus be characterized by the sole relationship between the pressure drop $[p'_{in} - p'_{out}]$ and the flow rate q' of the harmonic part.

Ultimately, if one wants to introduce the jet in the description of an acoustic system of much larger dimensions, the description (4.3) can be matched with an external solution derived from the equations of acoustics. Such a matching is not conducted here but examples will be given in a forthcoming paper.

4.2.2 Steady flow

The steady flow corresponding to the present situation is globally characterized by the mean pressure drop $[P_{in} - P_{out}]$ and the mean flow rate Q . In the inviscid case, a classical model to relate these quantities was proposed by Levi–Civita and Prandtl. The model consists of a vortex sheet formed at the hole and surrounding the jet (see figure 1). After several diameters, the jet becomes parallel, but with a radius R_J

smaller than that of the hole. We classically call the ratio of surfaces $\alpha = (\pi R_J^2)/(\pi R_h^2)$ the *vena contracta* coefficient. This coefficient is classically associated to the pressure loss across the aperture. Assuming a constant velocity U_J inside the jet (see figure 4.1), the conservation of flux through the hole leads to $Q = \pi R_J^2 U_J = \pi R_h^2 U_M^2$. Applying the Bernoulli theorem along streamlines passing through the hole thus leads to

$$[P_{in} - P_{out}] = \frac{\rho U_J^2}{2} = \frac{\rho U_M^2}{2\alpha^2}, \quad (4.4)$$

that links the pressure jump across the hole and the mean velocity (or flow rate) inside it. Theoretical inviscid calculations by Prandtl and Levi-Civita provided the value $\alpha = 0.5$, that represents also the lower limit for this coefficient. Smith and Walker (1923), instead, estimated the *vena contracta* coefficient $\alpha = \pi/(2 + \pi) \approx 0.611$ for round inviscid jets discharging in open spaces. This value has been found to agree very well with experiments (Cummings and Eversman, 1983) and numerical calculations (Scarpato et al., 2011) at very high Reynolds number.

4.2.3 Unsteady flow : Conductivity and Impedance concepts

We now consider the relationship between the pressure jump and the flow rate in the unsteady case, under the hypothesis of harmonic perturbations 4.3. As explained in the introduction, the Rayleigh conductivity (K_R) is defined as the proportionality coefficient between the acceleration of the fluid particules located within the hole and the pressure jump across the hole. More specifically,

$$K_R = \frac{-i\omega\rho q'}{(p'_{in} - p'_{out})}. \quad (4.5)$$

The conductivity is, in the general case, a complex quantity, and has the dimension of a length. Following Howe, it is classically noted $K_R = 2R_h(\gamma - i\delta)$. The real part γ represents the inertia of the system, while the imaginary part δ is directly related to the average value of the power absorbed by the hole. In effect, for harmonic perturbations described

with the convention (2.2), the power is given by

$$\langle \Pi \rangle = \langle ([p'_{in} - p'_{out}]e^{-i\omega t} + c.c.)(q'e^{-i\omega t} + c.c.) \rangle = 2\Re([p'_{in} - p'_{out}]\bar{q}') \quad (4.6)$$

Where the brackets $\langle \cdot \rangle$ represent the averaging over a complete period of oscillation $2\pi/\omega$, $\Re$ means the real part and the overbar denotes the complex conjugate. Using the definition of the conductivity, this formula directly leads to

$$\langle \Pi \rangle = \frac{4R_h\delta}{\rho\omega} |p'_{in} - p'_{out}|^2. \quad (4.7)$$

So, when $\delta > 0$, this term represents a resistance (or the ability to absorb acoustic energy), meaning that exciting the jet at a given frequency necessitates the provision of energy by an outer system.

As an alternative to the Rayleigh conductivity, we can also define the *impedance* of the aperture (Z_h) as the ratio between the pressure jump and the flow rate:

$$Z_h = \frac{(p'_{in} - p'_{out})}{q'} \left(\frac{-i\omega\rho}{K_R} \right) \quad (4.8)$$

The impedance is also a complex quantity, with physical dimension $Mass \cdot Length^{-2} \cdot Time^{-1}$. In the following we decompose it as

$$Z_h = \frac{\rho U_M}{R_h^2} (Z_R + iZ_I), \quad (4.9)$$

where Z_R is the dimensionless *resistance* and Z_I is the dimensionless *reactance*. It is easy to verify that the equation (4.6) for the power absorbed by the hole can be written as function of Z_R as follows:

$$\langle \Pi \rangle = 2 \frac{\rho U_M}{R_h^2} Z_R |q'|^2, \quad (4.10)$$

The Rayleigh conductivity and the impedance are conceptually and practically interchangeable quantities, and both have been used in the literature to characterize the interaction of a jet flow with acoustic fields. In the case of thin holes acting as a sound attenuators, most authors have used the conductivity as initially introduced by Howe. On the other hand, in cases where the jet can act as an energy source for external

acoustic systems and lead to instabilities, it proves to be more convenient to employ the impedance (Fabre et al., 2014a; Yang and Morgans, 2016). In the present paper, we will use both concepts. A more detailed discussion of impedance-based instability criteria and a parametric study of the impedance of long holes will be given in a forthcoming paper.

4.2.4 The classical Rayleigh solution in the absence of mean flow

The problem initially solved by (Rayleigh, 1945) is the simplest situation corresponding to the absence of mean flow. In this case, the problem admits an analytical solution under the framework of potential flow theory. This solution yields a direct proportionality between the flow acceleration and the pressure jump, namely

$$(p'_{in} - p'_{out}) = -\frac{i\rho\omega}{2R_h}q'. \quad (4.11)$$

The classical interpretation of this result is that the fluid in the vicinity of the hole behaves as a simple solid plug with mass $m = \rho\pi R_h^2 \ell_{eff}$ oscillating across the hole, where ℓ_{eff} is the equivalent length of the plug given by $\ell_{eff} = \pi R_h/2$.

When reformulated in terms of conductivity (resp. impedance) and using the nondimensionalization choices introduced in the previous section, the Rayleigh solution thus corresponds to $\gamma = 1; \delta = 0$ (resp. $Z_R = 0; Z_I = -i\Omega/2$). An obvious consequence is that under this model, the power absorbed by the hole predicted by (4.10) is exactly zero.

4.2.5 Review and criticism of Howe's inviscid model

We now review and discuss in more detail the classical model of Howe already mentioned in the introduction. Howe models the jet as a cylindrical vortex sheet of constant radius R_h formed at the rim of the aperture. He subsequently assumes a vorticity perturbation of this vortex sheet with the form

$$\gamma' = \sigma H(x)\delta(r - R_h)\exp[-i\omega(t - x/U_c)], \quad (4.12)$$

where δ and H are respectively the Dirac and Heaviside functions, U_c the assumed convection velocity of vorticity structures, and σ the amplitude

of the vorticity perturbation. This later parameter is determined by imposing a Kutta condition (Crighton, 1985), requiring finite velocity and pressure fluctuations on the rim of the hole. Starting from this point, and going through a series of very technical mathematical transformations, Howe was eventually able to predict the Rayleigh conductivity under the following analytical form:

$$K_R = 2R_h(\gamma - i\delta) = 2R_h \left\{ 1 + \frac{(\pi/2)I_1(\Omega^H)e^{-\Omega^H} - iK_1(\Omega^H)\sinh(\Omega^H)}{\Omega^H[(\pi/2)I_1(\Omega^H)e^{-\Omega^H} + iK_1(\Omega^H)\cosh(\Omega^H)]} \right\}, \quad (4.13)$$

where I_1 and K_1 are the order one modified Bessel functions of respectively first and second kind and $\Omega^H = \omega R_h / U_c$ is the Strouhal number.

Despite its mathematical rigor, a number of starting hypotheses of Howe's model are questionable. The main caveats of the model can be summarized in four points:

- First, the study models the mean flow as a cylindrical vortex sheet with radius R_h , and constant velocity U_M , hence completely overlooks the vena contracta phenomenon discussed above. In a subsequent step of his analysis (page 215 of his paper), Howe intended to incorporate partially this effect in his model, but this a posteriori modification remains imperfect.
- Secondly, Howe's model assumes that the perturbation affects only the strength of the vortex sheet but not its location, so that the perturbed vortex sheet is assumed to remain perfectly cylindrical. A better starting point would be to assume a vortex sheet with location given by (see figure 4.1):

$$r_J(r) = R_J + \varepsilon\eta(x, r, t) = R_J + \varepsilon\eta'(r)\exp[ik(\omega)x - i\omega t], \quad (4.14)$$

where $k(\omega) = k_r + ik_i$ is complex wavenumber which has to be determined as function of the frequency ω . The inviscid stability analysis of this model is a classical problem whose solution can be found, for instance, in Batchelor and Gill (1962) or in Abid et al. (1993). For completeness, this problem is reviewed in the appendix A.1.

- Thirdly, the starting point of Howe's analysis (4.12) assumes that the perturbations are convected at a constant velocity U_c which is assumed to be half the velocity of the jet. This choice is justified by analogy with the classical result for the Kelvin-Helmholtz instability of a planar vortex sheet. This choice is a questionable simplification and it would seem more rigorous to predict U_c using spatial stability analysis of the cylindrical vortex sheet model, namely $U_c = k_r/\omega$. This analysis shows that for small frequencies the convection velocity is actually closer to the velocity of the jet than half its value (see appendix A.1).
- Finally, Howe completely ignores the fact that perturbations of the vortex sheet are spatially amplified in addition to being convected.

According to the two last criticisms, it would thus seem more appropriate to replace the starting point (4.12) by the following ansatz:

$$\zeta' = \sigma H(x) \delta(r_J(r) - R_J - \varepsilon \eta'(r) \exp[ik(\omega)x - i\omega t]) \exp[ik(\omega)x - i\omega t]. \quad (4.15)$$

We have not intended to reconstruct the whole analysis from this modified starting point, an option which would anyway not address the first criticism discussed above (*vena contracta* effect) and would remain limited to the high-Reynolds numbers range. Instead, our chosen approach to address the problem is to compute the impedance (or alternatively the conductivity) through a global resolution of the linearized Navier-Stokes equations (LNSE) for given values of the Reynolds number.

4.3 The viscous problem: analysis and numerical method for the linear approach

4.3.1 General equations

Taking the diameter of the hole $D_h = 2R_h$ as a length scale and the mean velocity U_M as a velocity scale, the problem is governed by the axial-symmetric incompressible dimensionless Navier-Stokes equations:

$$\left. \begin{aligned} \nabla \cdot \mathbf{u} &= 0 \\ \partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} + \nabla p - \frac{1}{Re} \nabla^2 \mathbf{u} &= 0 \end{aligned} \right\}, \quad (4.16)$$

where $\mathbf{u}(x, r, t) = (u_x, u_r)$ and p is the reduced pressure. The variable x and r are respectively the axial and radial coordinate while u_x and u_r represent the axial and radial velocity components.

The flow is further decomposed into a base flow $(\mathbf{U}, P)$ associated with the mean flux Q and a harmonic perturbation $\varepsilon(\mathbf{u}', p')e^{-i\omega t}$ associated with the oscillating flow rate $q'e^{-i\omega t}$. A crucial hypothesis in this treatment is that the amplitude of the harmonic perturbation is small, namely $\varepsilon \ll 1$. Inserting this decomposition into the Navier–Stokes equations (4.16) and linearizing, two different sets of PDE's are obtained:

- First, the leading order yields the base flow equations:

$$\left. \begin{aligned} \nabla \cdot \mathbf{U} &= 0 \\ (\mathbf{U} \cdot \nabla)\mathbf{U} + \nabla P - \frac{1}{Re}\nabla^2\mathbf{U} &= 0 \end{aligned} \right\}. \quad (4.17)$$

The link between the base flow $(\mathbf{U}, P)$ and the quantities P_{in} , P_{out} , Q introduced in §4.2 is given by the asymptotic matching conditions and flow rate definition as follows:

$$P(x, r) \rightarrow P_{in} \quad \text{for } \sqrt{x^2 + r^2} \rightarrow \infty \text{ and } x < 0, \quad (4.18)$$

$$P(x, r) \rightarrow P_{out} \quad \text{for } \sqrt{x^2 + r^2} \rightarrow \infty \text{ and } x > 0, \quad (4.19)$$

$$\int_S \mathbf{U} \cdot \mathbf{n} dS = Q, \quad (4.20)$$

where S is any surface traversed by the flow and $\mathbf{n}$ a normal unitary vector oriented in the direction of the flow.

- Secondly, the ε -order yields the linearized Navier–Stokes equations (LNSE) governing the perturbation:

$$\left. \begin{aligned} \nabla \cdot \mathbf{u}' &= 0 \\ -i\omega\mathbf{u}' + (\mathbf{U} \cdot \nabla)\mathbf{u}' + (\mathbf{u}' \cdot \nabla)\mathbf{U} + \nabla p' - \frac{1}{Re}\nabla^2\mathbf{u}' &= 0 \end{aligned} \right\}. \quad (4.21)$$

The link with the quantities p'_{in} , p'_{out} , q' introduced in section 2

and allowing to define the impedance/conductivity is:

$$p'(x, r) \rightarrow p'_{in} \quad \text{for } \sqrt{x^2 + r^2} \rightarrow \infty \text{ and } x < 0, \quad (4.22)$$

$$p'(x, r) \rightarrow p'_{out} \quad \text{for } \sqrt{x^2 + r^2} \rightarrow \infty \text{ and } x > 0, \quad (4.23)$$

$$\int_S \mathbf{u}' \cdot \mathbf{n} dS = q'. \quad (4.24)$$

Note that, as is customary when dealing with incompressible flows, the pressure is defined up to an arbitrary constant. We can choose this constant by setting $P_{out} = 0$ and $p'_{out} = 0$ in equations (4.19) and (4.23), so that the mean pressure and fluctuating pressure drops is actually given by $[P_{in} - P_{out}] = P_{in}$, $[p'_{in} - p'_{out}] = p'_{in}$.

With the addition of no-slip conditions $\mathbf{U} = \mathbf{u}' = \mathbf{0}$ on the upstream and downstream surfaces of the plate (noted Γ_w) and symmetry conditions $U_r = u'_r = 0$; $\partial U_x / \partial r = \partial u'_x / \partial r = 0$ at the axis (noted Γ_{axis}), the set of equations (4.17–4.24) completely defines the nonlinear problem allowing to compute the *vena contracta* coefficient α and the linear problem allowing to compute the impedance/conductivity.

In practice, the boundary conditions at $\sqrt{x^2 + r^2}$ have to be imposed at the boundaries of a finite computational domain, both upstream and downstream. Treatment of these boundary conditions requires special attention and is detailed in the next sections.

4.3.2 Upstream domain

As sketched in figure 4.1, the upstream domain is expected to originate from an upstream container of large dimension, and sufficiently far away from the hole. Moreover, the flow is assumed to be radially convergent. However, in the numerical implementation, it is required to specify a given geometry for this upstream domain. Here, we chose to assume that the upstream region is a closed cavity of cylindrical section, with radius R_{in} and length L_{in} . The volumetric flux conditions (4.20) and (4.24) are imposed by assuming that both the base flow and the perturbation velocities are constant along the bottom of the cavity, noted Γ_{in} (see figure 4.2), i.e.

$$\left. \begin{aligned} \mathbf{U} &= Q/S_{in} \mathbf{n} \\ \mathbf{u}' &= q'/S_{in} \mathbf{n} \end{aligned} \right\} \text{ on } \Gamma_{in}, \quad (4.25)$$

where $S_{in} = \pi R_{in}^2$ is the area of the bottom wall. The values of Q and q' have been selected in order to have a mean velocity equal to one into the hole, for both the base flow and the perturbation. The pressure levels P_{in} and p'_{in} , which are required for the calculation of the mean pressure loss (and the *vena contracta* coefficient) and the impedance (or conductivity), are extracted by averaging along the inlet boundary :

$$\left. \begin{aligned} P_{in} &= 2\pi/S_{in} \int_0^{R_{in}} P(r)rdr \\ p'_{in} &= 2\pi/S_{in} \int_0^{R_{in}} p'(r)rdr \end{aligned} \right\} \text{ on } \Gamma_{in}. \quad (4.26)$$

Since the upstream cavity used in our mesh definition is expected to represent an upper domain of infinite extend, its precise geometry has no real importance, but its dimension has to be large enough so that the results are independent of this geometry. In practice we verified that the choice $L_{in} = R_{in} = 10R_h$ fulfills this conditions. Finally, at the lateral wall of the cavity for $r = R_{in}$ (noted Γ_{lat}), we simply choose non-penetration ($u_r = 0$) and no-stress ($\partial_r u_x = 0$) conditions for both base flow and perturbation. This condition ensures that the volumic flux imposed at the bottom of the cavity effectively corresponds with the one traversing the hole preventing the occurrence of an unphysical boundary layer that would be obtained using a no-slip condition.

4.3.3 Downstream domain : boundary conditions and change of coordinates

The treatment of the outlet boundary conditions is a delicate point here, as the structure of the perturbation leads to some difficulties, especially when the Reynolds number becomes large. In effect, due to the strongly spatially unstable nature of the jet, all perturbations are strongly amplified along the axial direction. In particular, the pressure field $p'(x, r)$ can reach huge levels (reaching 10^{15} or even more for $Re \approx 3000$) along the axis ($r = 0$) for large x , and this conflicts with the necessity of imposing the boundary condition $p'_{out} = 0$ at a finite distance x_{max} corresponding to the boundary of the computational domain.

To detail the origin of the problem and introduce the idea used to overcome it, let us review the classical modeling of the Kelvin-Helmholtz

instability for a planar shear layer of zero thickness in the inviscid case. The formal derivation can be found in any classical textbook on hydrodynamical stability (see for example Drazin and Reid (2004) or Charru (2011)). Consider as base flow a shear layer separating two regions of constant axial velocity, namely $u_x = U$ for $r < 0$ and $u_x = 0$ for $r > 0$. Now assume that the perturbation consists of a displacement of the shear layer with the form

$$\eta(x, r, t) \propto e^{ikx - i\omega t}, \quad (4.27)$$

and assume a similar modal expansion for the velocity potential in the upper and lower regions. Matching the two regions at the interface leads to the classical dispersion relation:

$$c \equiv \frac{\omega}{k} = \frac{1 \pm i}{2} U, \quad (4.28)$$

In a temporal stability framework, this means that a perturbation with a real wavenumber k is convected downstream with a phase velocity $U/2$ and temporally amplified with a growth rate $Uk/2$. On the other hand, in a spatial stability framework which is more relevant here, a perturbation with real frequency ω will be spatially amplified downstream with a complex wavenumber k and will diverge at $x \rightarrow +\infty$. This divergence forbids a global resolution of the function $\eta(x, t)$ when the variable x is real. However, the problem disappears if we consider an analytical continuation of the function $\eta(x, t)$ with a *complex variable* x . More specifically, as $\arg(k) = -\pi/4$, the function $\eta(x, t)$ becomes convergent as soon as $|x| \rightarrow \infty$ in a direction of the complex plane verifying $\pi/4 < \arg(x) < 5\pi/4$. These considerations suggest a possible way to overcome the problem, namely using a *complex coordinate change* $x = \mathcal{G}_x(X)$ which maps a (real) numerical coordinate X defined over a finite-size computational downstream domain $X \in [-L_{in}; L_{out}]$, onto the physical coordinate x in a way that it enters the complex plane and follows a direction where the perturbation is spatially damped.

The coordinate mapping effectively transforms the outlet location $X = L_{out}$ into a location $x = x_{max} = \mathcal{G}_x(L_{out})$ located into the complex plane. In order for the boundary conditions at the outlet $X = L_{out}$ of the computational domain to best represent the physical boundary condition at $|x| \rightarrow \infty$, it is desirable for $x_{max} = \mathcal{G}_x(L_{out})$ to be as large as possible.

This can be achieved using coordinate stretching in order to have short numerical domains and large physical ones.

Combining both ideas, namely stretching and complex mapping, we designed the following mapping function from numerical coordinate X to physical coordinate x :

$$\begin{aligned} x = \mathcal{G}_x(X) &= \frac{X}{\left[1 - \left(\frac{X}{L_A}\right)^2\right]^2} \left[1 + i\gamma_c \tanh\left(\left(\frac{X}{2L_C}\right)^2\right)\right] & \text{for } X > 0, \\ &= X & \text{for } X < 0. \end{aligned} \quad (4.29)$$

This function is characterized by three parameters which have the following interpretation. First, the parameter L_C controls the transition range from real coordinate to complex coordinate. For $X \ll L_C$ the mapping is almost identity ($\mathcal{G}_x(X) \approx X$) so that the transition with the upstream, unmapped domain is as smooth as possible. For $X \approx L_c$ the imaginary part of the corresponding physical coordinate x gradually increases. For $X \gg L_c$ the argument of x asymptotes to a constant value, namely $\arg(x) \approx \tan^{-1}(\gamma_c)$. The third parameter L_A controls the stretching effect associated to the coordinate mapping. This parameter has to be chosen so that $L_A > L_{out}$. $L_A \rightarrow \infty$ means no coordinate stretching, so that the real part of x_{max} is the same as the dimension L_{out} of the computational domain, while if $L_A - L_{out}$ is small the corresponding x_{max} is rejected very far away in the complex plane.

Finally, although the issue is less crucial respect to the axial coordinate, we also used a mapping $r = \mathcal{G}_r(R)$ to stretch the radial coordinate from $R \in [0, R_{out}]$ to $r \in [0, r_{out}]$ in order to enlarge the effective radial dimension of the physical domain. Here there is no point in using a complex deformation, so we used the following mapping function :

$$\begin{aligned} r = \mathcal{G}_r(R) &= R_M + \frac{R - R_M}{\left[1 - \left(\frac{R - R_M}{R_A - R_M}\right)^2\right]^2} & \text{for } X > 0 \text{ and } R > R_A, \\ &= R & \text{otherwise} \end{aligned} \quad (4.30)$$

This function leaves the radial coordinate unchanged in the region

$r < R_M$ where the jet develops, but it stretches the limit of the domain from R_{out} to $r_{out} = \mathcal{G}_r(R_{out})$ which is very large as soon as R_A is close to R_{out} (with the constraint $R_A > R_{out}$).

Having explained this change of coordinates, it remains to specify the numerical boundary conditions effectively used at the boundaries of the numerical domain $R = R_{out}$ (corresponding to $r = r_{out}$) and $X = L_{out}$ (corresponding to $x = x_{max}$). In the framework of finite elements, the usual way to impose outlet conditions is to take advantage of the integration by parts leading to the weak formulation. The most natural condition emerging in this way is the zero-traction condition, namely $-p\mathbf{n} + Re^{-1}\nabla\mathbf{u} \cdot \mathbf{n} = 0$. In the present case, we used the zero-traction condition as an approximation of the physical condition $p = 0$ for both the base flow and perturbation computations. This choice is justified if the viscous stresses are negligible in the vicinity of the boundaries of the domain, which turns to be the case here.

We stress that using the present method, outflow boundary conditions are effectively applied at a location x_{max} located in the complex plane. The validity of the method is not justified by rigorous mathematical argument, but only by the fact that it effectively works. Detailed validations are given in appendix A.2.1 of this paper. In particular, we show that at low Reynolds numbers results obtained with and without complex mapping are identical, and are independent upon the precise choice of the parameters (γ_c, L_C, L_A) of the mapping function.

Note that the idea of using a complex coordinate mapping is not completely new. Indeed the method is conceptually similar to the Perfectly Matching Layer (PML) method, which is a numerical approach largely used in electromagnetics and acoustics to impose non-reflection boundary conditions in wave-propagation problems (see Colonius (2004) for a complete review). In stability studies of incompressible flows, complex coordinate mappings have also been used in linear problems involving a single spatial coordinate and characterized by a critical layer singularity (see for instance ?) and mathematical theorems are available to justify how to choose the mapping as function of the singularities of the problem (see for example Bender and Orszag (2013)). However, we are not aware of any usage of such methods to get rid of convective amplifications (which is a different issue compared to reflexion of waves along boundaries and critical-layer singularities). The usage of complex map-

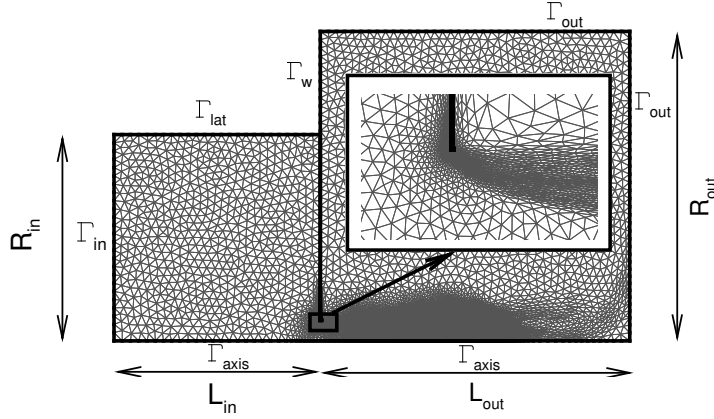


Figure 4.2: Structure of the mesh M_1 obtained at the end of the mesh adaptation process, and nomenclature of boundaries. The mesh is adapted to both the base flow for $Re = 1000$ and the harmonic perturbation for $\omega = 2$. The insert shows a zoom of the mesh structure in the range $X \in [-0.5; 0.8]R_h$ and $R \in [0.5; 1.3]R_h$. Note that owing to the coordinate mapping, the actual dimension of the outlet domain is $[x_{max}, r_{max}] = [1022 + 306i, 337]$.

pings for solving a nonlinear problem (i.e. computation of the base flow) involving two spatial coordinates is also totally new to our knowledge.

4.3.4 Numerical implementation

The numerical resolution of the problem was performed with a finite element method, using the FreeFem++ (<http://www.freefem.org>) open source code (Hecht (2012)). The procedure follows the classical approach initially introduced by Sipp and Lebedev (2007). The only notable originalities are the introduction of the complex mapping into the weak formulation, and the use of mesh adaptation using the `adaptmesh` command provided by the Freefem++ (see Fabre et al. (2018c) for a demonstration of the efficiency of this method for solving linear and nonlinear problems arising from stability analysis).

In order to solve the problem, the base flow and perturbation equa-

tions have first to be expressed in terms of the mapped coordinates. The treatment of both sets of equations is very similar so we document only the base flow equations. First, the spatial derivatives have to be modified as follows :

$$\left. \begin{aligned} \partial_x &\equiv \mathcal{H}_x(X)\partial_X & \text{with } \mathcal{H}_x &= \frac{1}{\partial_X \mathcal{G}_x(X)} \\ \partial_r &\equiv \mathcal{H}_r(R)\partial_R & \text{with } \mathcal{H}_r &= \frac{1}{\partial_R \mathcal{G}_r(R)} \end{aligned} \right\}, \quad (4.31)$$

The steady incompressible Navier-Stokes equations (4.17) thus take the following form :

$$\left. \begin{aligned} \mathcal{H}_x \partial_X U_x + \mathcal{H}_r / r \partial_R (r U_r) &= 0 \\ \mathcal{C}\{U_x\} + \mathcal{H}_x \partial_X P - \frac{1}{Re} [\mathcal{H}_x \partial_X (\mathcal{H}_x \partial_X U_x) + \mathcal{H}_r / r \partial_R (r \mathcal{H}_r \partial_R U_x)] &= 0 \\ \mathcal{C}\{U_r\} + \mathcal{H}_r \partial_R P - \frac{1}{Re} [\mathcal{H}_x \partial_X (\mathcal{H}_x \partial_X U_r) - U_r / r^2 + \mathcal{H}_r / r \partial_R (r \mathcal{H}_r \partial_R U_r)] &= 0 \end{aligned} \right\} \quad (4.32)$$

where, from (4.30), $r = \mathcal{G}_r(R)$ and $\mathcal{C}\{\cdot\} = U_x \mathcal{H}_x \partial_X (\cdot) + U_r \mathcal{H}_r \partial_R (\cdot)$. The weak formulation is classically obtained by multiplying by test functions $[U_x^+, U_r^+, P^+]$ and integrating over the domain. Note that this integration has to be done over the physical domain, so in terms of the numerical variables the elementary volume of integration is $dV = 2\pi r dr dx = 2\pi (\mathcal{H}_x \mathcal{H}_r)^{-1} r dR dX \equiv 2\pi (\mathcal{H}_x \mathcal{H}_r)^{-1} \mathcal{G}_r(R) dR dX$. After integration by parts of the pressure gradient and Laplacian terms of the equation (4.32), we are thus lead to the following weak formulation of the mapped Navier–Stokes equations:

$$\begin{aligned} & - \int [U_x^+ (U_x \mathcal{H}_x \partial_X U_x + U_r \mathcal{H}_r \partial_R U_x) + U_r^+ (U_x \mathcal{H}_x \partial_X U_r + U_r \mathcal{H}_r \partial_R U_r)] dV \\ & + \int [P (\mathcal{H}_x \partial_X U_x^+ + \mathcal{H}_r \partial_R U_r^+ + U_r^+ / r) - P^+ (\mathcal{H}_x \partial_X U_x + \mathcal{H}_r \partial_R U_r + U_r / r)] dV \\ & - \frac{1}{Re} \int [\mathcal{H}_x^2 \partial_X U_x \partial_X U_x^+ + \mathcal{H}_r^2 \partial_R U_x \partial_R U_x^+] dV \\ & - \frac{1}{Re} \int [\mathcal{H}_x^2 \partial_X U_r \partial_X U_r^+ + \mathcal{H}_r^2 \partial_R U_r \partial_R U_r^+ + U_r U_r^+ / r^2] dV \\ & = 0. \end{aligned} \quad (4.33)$$

Note that with this formulation, the no-traction boundary conditions at the outlet boundary, as well as the symmetry condition at the axis and the zero tangential stress condition at the lateral wall of the cavity are automatically satisfied thanks to the integration by parts. The other boundary conditions are imposed by penalization.

The LNSE (4.21) is treated in a similar way, leading to the following weak formulation:

$$\begin{aligned}
 & i\omega \int [U_x^+ u'_x + U_r u'_r] dV \\
 & + \int [U_x^+ (U_x \mathcal{H}_x \partial_X u'_x + u'_x \mathcal{H}_x \partial_X U_x + u'_r \mathcal{H}_r \partial_R U_x + U_r \mathcal{H}_r \partial_R u'_x)] dV \\
 & + \int [+U_r^+ (U_x \mathcal{H}_x \partial_X u_r + u'_x \mathcal{H}_x \partial_X U_r + u'_r \mathcal{H}_r \partial_R U_r + U_r \mathcal{H}_r \partial_R u'_r)] dV \\
 & - \int [p' (\mathcal{H}_x \partial_X U_x^+ + \mathcal{H}_r \partial_R U_r^+ + U_r^+/r) + P^+ (\mathcal{H}_x \partial_X u'_x + \mathcal{H}_r \partial_R u'_r + u'_r/r)] dV \\
 & + \frac{1}{Re} \int [\mathcal{H}_x^2 \partial_X u'_x \partial_X U_x^+ + \mathcal{H}_r^2 \partial_R u'_r \partial_R U_r^+] dV \\
 & + \frac{1}{Re} \int [\mathcal{H}_x^2 \partial_X u'_r \partial_X U_r^+ + \mathcal{H}_r^2 \partial_R u'_r \partial_R U_r^+ + u'_r U_r^+/r^2] dV \\
 & = 0.
 \end{aligned} \tag{4.34}$$

Once the weak formulation is written, the discrete matrix are assembled using classical Taylor–Hood (P_2, P_2, P_1) finite elements for the spatial discretization.

The use of mesh adaptation to generate a efficient mesh is done in a way very similar as explained in Fabre et al. (2018c). The procedure is as follows :

(i) we generate an initial coarse mesh using the Delaunay-Voronoi triangulation of the domain.

(ii) we use Newton iteration to compute a base flow at a moderate value of the Reynolds (for instance $Re = 10$).

(iii) we adapt the mesh to the base flow solution of the previous step and recompute the base flow on the resulting mesh.

(iv) we repeat points (ii) and (iii) for gradually increasing values the Reynolds number up $Re = 1000$.

After this stage, we are guaranteed to have a mesh yielding converged results as for base flow characteristics.

(*v*) we solve the linear problem for a value of ω in the range of interest, adapt the mesh to fit with the corresponding structure, and recompute the base flow on the resulting mesh.

After this stage, we are ensured to have a mesh yielding converged results for both the base flow and the perturbation for a given ω . For even better efficiency, it is also possible to do the last mesh adaptation (*v*) for two values of ω spanning the range of parameters in which converged results are expected.

To obtain the results presented in the next sections, two different meshes were designed in this way. The first mesh, noted $\mathbb{M}_0$ is generated without the use of complex mapping, with a large domain corresponding to $L_{out} = x_{max} = 80$. This mesh was used to compute impedances at low Reynolds (up to 1000) and to plot the base flow characteristics. The second, noted $\mathbb{M}_1$, uses complex mapping and was used for most results at larger Reynolds values. The structure of this mesh $\mathbb{M}_1$ is illustrated in figure 4.2

Additional meshes were designed for convergence tests and for demonstrations of the robustness of the complex mapping technique. Details are given in appendix A.2. The full characteristics of all meshes designed in this study (including $\mathbb{M}_0$ and $\mathbb{M}_1$) are given in table A.4, in appendix A.2.2. We mention that the number of grid points n_t in $\mathbb{M}_1$ is about half the value compared to mesh $\mathbb{M}_0$.

4.4 Results for the steady base flow

The first step in the approach is the computation of the base flow. We report two examples of base flows calculated at $Re = 500$ (figure 4.3) and $Re = 3000$ (figure 4.4). Here computations are done in physical coordinates using mesh $\mathbb{M}_0$. In both cases the streamlines show the transition across the hole from a radially converging flow to a quasi-parallel flow. They also indicate an entrainment effect of the outer flow which is also a well-known feature of such flows. Moreover, observing the axial velocity profiles in the upper part of figures 4.3(b) and 4.4(b), we can note that the jet becomes more parallel as far as the Reynolds number increases. In these figures, we also reported the velocity profile into the hole, consisting in an almost constant profile with $U_M = 1$ and dimensions equal

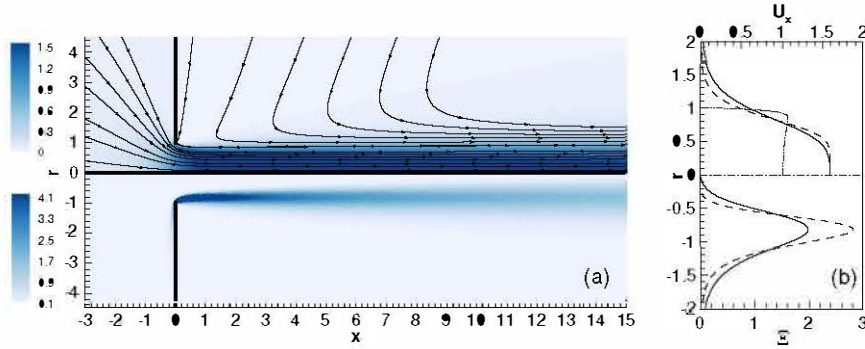


Figure 4.3: (a) Base flow for $Re = 500$ (in physical coordinates (x, r) , without mapping). Upper part : axial velocity U_x and streamlines. Lower part : vorticity Ξ . (b) Profiles of the axial velocity (upper) and vorticity (lower) at $x = 0$ (— · — · —), $x = 5R_h$ (---) and $x = 10R_h$ (—).

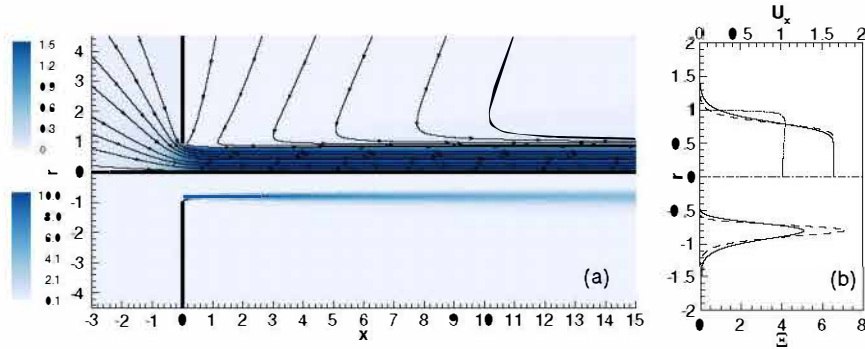


Figure 4.4: Same of figure 4.3 but for $Re = 3000$.

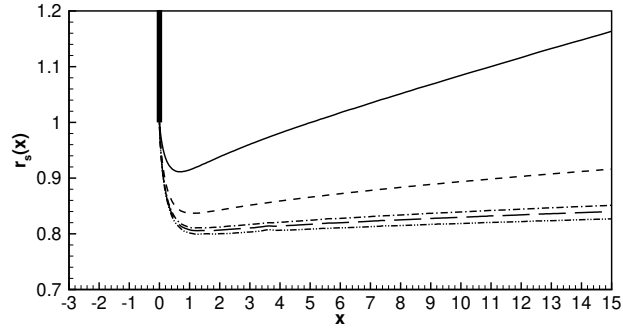


Figure 4.5: Radius of the shear layer $r_s(x)$ for $Re = 100$ (—), $Re = 500$ (---), $Re = 1500$ (- · -), $Re = 2000$ (— —) and $Re = 3000$ (- · · -). The vertical thick line represents the edge of the hole.

to the radius of the hole R_h .

We calculated also the vorticity of the base flow as $\Xi = \partial_x U_r - \partial_r U_x$, reported in the lower part of figures 4.3 and 4.4. As can be observed the jet is bounded by a very thin shear layer with high levels of vorticity, especially at high Reynolds numbers, agreeing with the inviscid theory used by many authors (Howe, 1979; Yang and Morgans, 2016, 2017). Moreover, from the lower part of figures 4.3(b) and 4.4(b), we noted that the vorticity reaches highest levels near the hole and then it is attenuated while it is convected downstream.

The radius of the shear layer $r_s(x)$ can be extracted from the base flow fields by localizing the streamline growing up from the edge of the hole. The actual shape of the jet has a great influence on the calculation of the impedance and many analytical model are based on its reconstruction from experimental or numerical datas (Jing and Sun, 2000; Mendez and Eldredge, 2009; Yang and Morgans, 2016). We reported our results in figure 4.5 for various Reynolds numbers. For $Re = 100$ the jet is not parallel: the inertia of the flow is low and the jet is accelerated only for a short distance; then, the effect of viscosity leads to a diffusion of the jet. For higher Reynolds numbers, instead, the inertial effects dominate the system: the fluid is accelerated for a long distance and the radius of

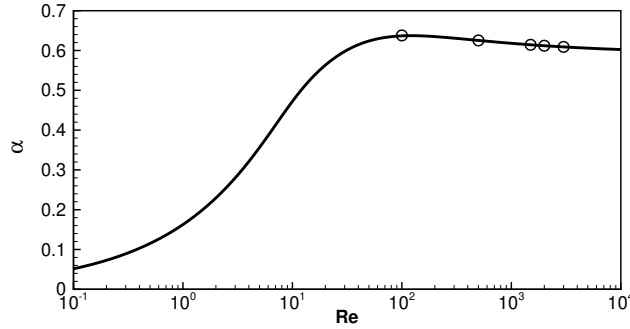


Figure 4.6: Vena contracta coefficient as function of Re . The circles ($\circ$) indicate the values of the vena contracta coefficient corresponding to the Reynolds number reported in figure 4.5.

the jet diverges very far (not reported here). Moreover, it is very clear that the jet is almost parallel and for high Reynolds numbers and the radius of the jet results to be approximatively $R_J \approx 0.8$.

As recalled in §4.2.2, this effect is classically measured through the introduction of the *vena contracta* coefficient α which is directly related to the pressure drop $[P_{in} - P_{out}]$. We calculated the *vena contracta* coefficient inverting the equation (4.4) as function of the Reynolds number. The results are shown in figure 4.6: we noted the curve grows with the Reynolds number, then it reach a maximum at $Re \approx 120$ and then it assumes an asymptotic behavior as $Re \rightarrow \infty$, leading to $\alpha \approx 0.61$, which is in agreement with classical results (Smith and Walker, 1923). Finally, we estimated the radius of the ideal jet at large Reynolds number using the relation $R_J \approx R_h \sqrt{\alpha} \approx 0.78$, where α is classically assumed 0.61: this value is in very good agreement with the value $R_J \approx 0.8$ extracted from the figure 4.5.

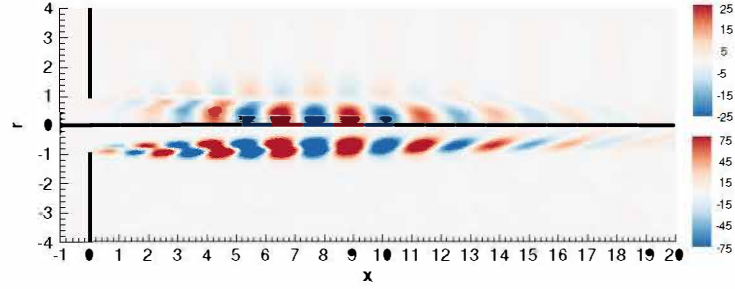


Figure 4.7: Harmonic perturbation for $Re = 500, \Omega = 3$ computed in physical coordinates (x, r) (mesh M_0). Real part of the axial velocity component u'_x (upper) and vorticity ξ' (lower).

4.5 Results for the unsteady flow

4.5.1 Structure of the unsteady flow for $Re = 500$

Let us now investigate the structure of the flow perturbation due to harmonic forcing. Figure 4.7 displays this structure for a moderate value of the Reynolds number, namely $Re = 500$, and for $\Omega = 3$, computed in physical coordinates without mapping (mesh M_0). As can be observed, the effect of a periodic forcing is to generate vortical structures in the jet which are amplified and convected in the downstream direction. In the case plotted, the maximum level is reached at about $x \approx 8$. Progressing further downstream, the perturbations are no longer amplified but begin to slowly decrease, until vanishing for $x \gtrsim 20$. This is consistent with the fact that for $x \gtrsim 8$ the shear layer bounding the jet has diffused (as documented in figure 4.3(b)) and is not steep enough to sustain a spatial instability.

In figure 4.8 we display the values taken along the axis of the jet by the axial velocity $u'_x(x, 0)$ and the pressure $p'(x, 0)$ associated to the harmonic perturbation previously described. One can clearly observe that the pressure perturbation asymptotes to different limit values in the upstream and downstream domains, allowing to deduce the pressure jump $[p'_{in} - p'_{out}]$ which is the key parameter allowing to define the impedance

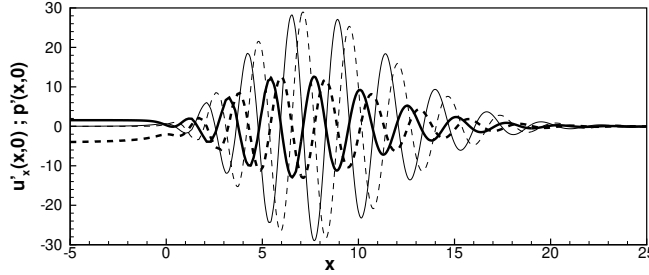


Figure 4.8: Harmonic perturbation for $Re = 500, \Omega = 3$ (in physical coordinates (x, r) ; mesh $\mathbb{M}_0$) on the axis of symmetry. Real (—) and imaginary (---) part of the axial velocity component u'_x (thin lines) and pressure p' (thick lines).

and/or the conductivity. In the upstream region the asymptote is reached rapidly and the pressure is almost constant with $p'(x, 0) \approx p'_{in} = 1.8 - 3i$ for $x \lesssim -3$. On the other hand, in the downstream region, the asymptotic limit (which amounts to $p'_{out} = 0$ owing to the way the boundary conditions are introduced in the problem, see section 3) is only reached for $x \gtrsim 25$, after the spatial growth and subsequent decay.

4.5.2 Efficiency of the complex mapping technique

Figure 4.9 displays the structure of the perturbation for the same parameters as in figure 4.7, but using the complex mapping technique (with mesh $\mathbb{M}_1$). Accordingly, the structure is plotted as function of the (real) numerical coordinates $[X, R]$. As one can observe, the complex mapping technique completely fulfills the goal of getting rid of the strong spatial amplification in the downstream direction.

Of course, the spatial structure displayed in figure 4.9 has no physical meaning as soon as $X > 0$, because a point (X, R) in the numerical domain correspond to a point (x, r) in the physical domain *for some complex* x defined by $x = G(X)$, and there is no easy way to access the structure of the perturbation for some real x . However, since our focus is on the impedance of the jets, we are not interested in a full characterization of the perturbation field but only by a determination of

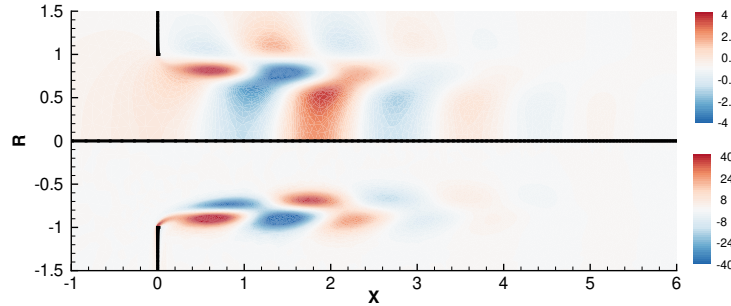


Figure 4.9: Harmonic perturbation for $Re = 500, \Omega = 3$ (in numerical coordinates (X, R) with complex mapping ; mesh $\mathbb{M}_1$). Real part of the axial velocity component u'_x (upper) and vorticity (lower).

the associated pressure jump.

Figure 4.10 compares the results obtained with and without complex mapping (again for $Re = 500$ and $\Omega = 3$), focussing on the pressure component p' (real part) of the harmonic disturbance. In figure 4.10(a) the same iso-levels are used for both results without complex mapping (upper half) and with complex mapping (lower half). The comparison shows again that the structure computed without complex mapping quickly grows to reach large values saturating the iso-levels, while the structure computed with complex mapping nicely decays to rapidly reach zero. Figure 4.10(b) complements the comparison with plots of the pressure field along the axis. The comparison confirms that in the inlet region ($X < 0$) the results exactly coincide. For $0 < X \lesssim 1.25$ the results with complex mapping remain qualitatively similar to the ones without mapping while for $X \gtrsim 1.25$ they become completely different and rapidly decay to zero. This not surprising, as our definition the mapping function defining our mesh contains a parameter $L_C = 1.25$ such that for $X < L_C$ the corresponding physical variable $x = G_x(X)$ is almost real while for $X > L_C$ it is fully complex.

Let us now consider the case $Re = 1000, \Omega = 3$. The pressure p' of the harmonic disturbance is plotted in figures 4.11(a) and (b), with the same conventions as in figure 4.10. Inspection shows that the results without complex mapping lead to the same difficulties as for $Re = 500$ but the

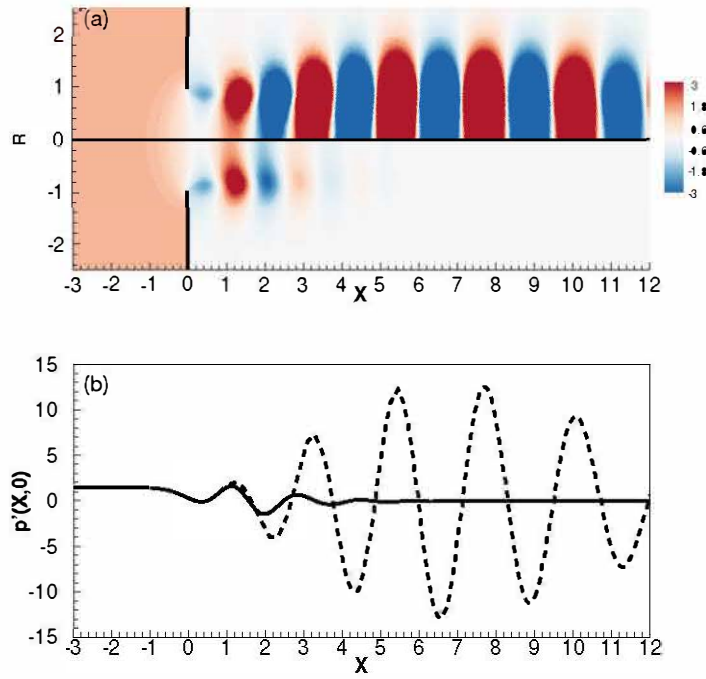


Figure 4.10: (a) Pressure contours of the perturbation p' computed in physical coordinates (mesh M_0 ; upper part) and with the complex mapping (mesh M_1 ; lower part). (b) Pressure of the perturbation on the symmetry axis $p'(X, 0)$ with (—) and without (---) the complex mapping ($Re = 500$; $\Omega = 3$).

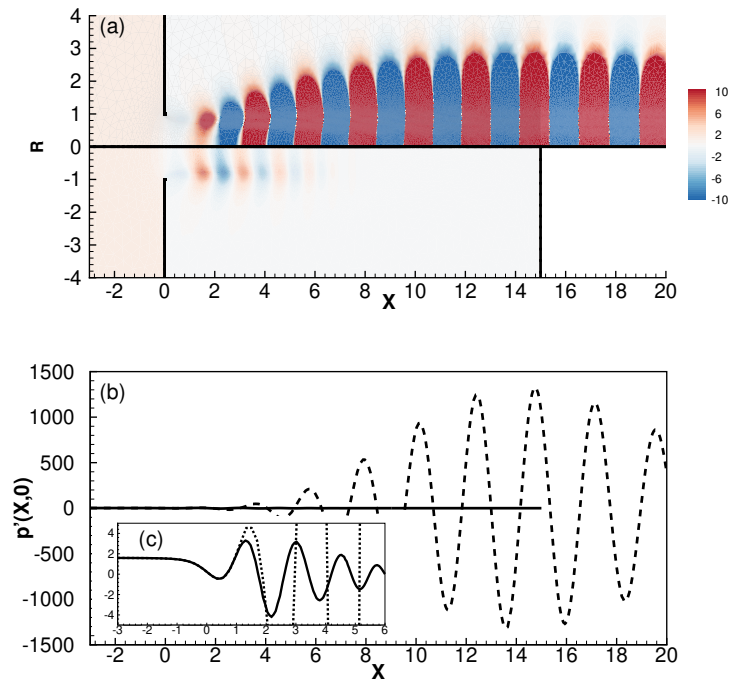


Figure 4.11: Same of figures 4.10(a) and (b) but for $Re = 1000$. The inserted plot (c) displays a zoom in the near hole region in order to catch the pressure jump across the hole.

spatial growth is much more pronounced. In this case, the pressure levels reach an amplitude of order 10^3 for $x \approx 15$ and the asymptotic value $p'_{out} = 0$ is only reached for $x \gtrsim 70$. This justifies that to correctly resolve this mode, we had to design the dimension of mesh $\mathbb{M}_0$ as large as $L_{out} = 80$. On the other hand, results obtained with complex mapping behave very similarly as for $Re = 500$, and the asymptotic limit $p'_{out} = 0$ is reached quite rapidly, for $X \gtrsim 8$.

In figures 4.11(b), the pressure levels of the results with and without complex mapping are so different that it is impossible to check that the pressure p' effectively asymptotes to the same limits p'_{in} and p'_{out} in both cases. To remedy this, figure 4.11(c) shows a zoom of the results in figure 4.11(b), in the region close to the hole. This representation confirms that the two computations lead to identical results in the upstream region where no mapping is used (and in particular that the upstream limit p'_{in} is the same), and that for the case using complex mapping the downstream limit is reached after only a few oscillations with amplitudes of order one.

Figures 4.10 and 4.11 thus confirms that the use of complex mapping is a convenient and efficient way to access to the pressure jump $p'_{in} - p'_{out}$ associated to the harmonic perturbation without having to deal with the strong spatial amplifications. It is worth pointing out that in this method is also computationally economical, as the number of point in mesh $\mathbb{M}_1$ is about half that of the unmapped case $\mathbb{M}_0$. The figures also indicate that the difficulties encountered when solving the problem in physical coordinates without mapping become worse as the Reynolds number becomes large. In practice, as soon as $Re \gtrsim 1500$, the huge levels reached by the perturbations in the far-field region lead to round-off errors and it becomes impossible to resolve accurately the near-field region. We verified that enlarging the domain to dimensions L_{out} to even larger than 80 does not improve the results. Using "sponge" regions with artificially large viscosity was also tried as an alternative idea to get rid of the problems associated to huge spatial amplifications, but this idea proved to be unsuccessful. At the end the only efficient way we found to obtain reliable results for $Re \gtrsim 1500$ was to use the complex mapping technique. An illustration of the failure of the resolution in physical variables for large Reynolds is given in appendix B.1 (figures A.3 and A.4).

Note that in order to be consistent, the base flow also needs to be

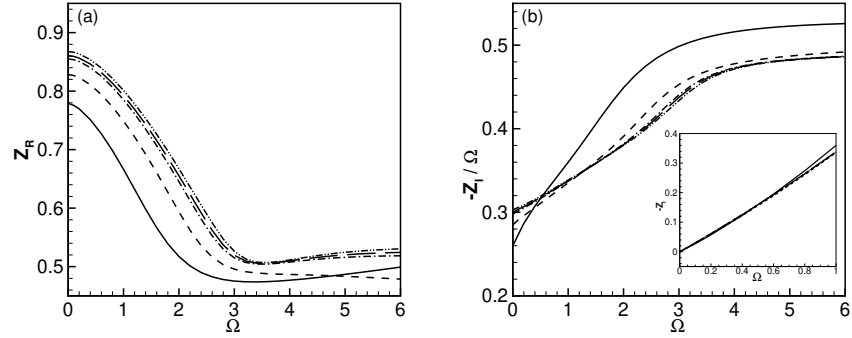


Figure 4.12: (a) Resistance Z_R and (b) reactance $-Z_I/\Omega$ for $Re = 100$ (—), $Re = 500$ (---), $Re = 1500$ (- · -), $Re = 2000$ (— —) and $Re = 3000$ (- · · -).

computed with the same numerical coordinates. The structure of the base flow in mapped coordinates has no physical meaning for the same reasons as the harmonic perturbation, but we verified that the pressure jump and the associated *vena contracta* coefficient are identical to results in physical coordinates. We detail this in appendix A.3, and display an example of base flow obtained in such a way in figure A.8.

4.5.3 Impedance and conductivity

Having illustrated the structure of the perturbation due to a harmonic forcing and justified the validity of the numerical method, we now come to the most important result of this work, namely prediction of the impedance as function of Re and Ω .

Figure 4.12 displays the real and imaginary parts of the impedance, calculated according to the equation (4.8), as a function of the forcing frequency Ω at various Reynolds numbers

As for the resistance (plot *a*), only the case $Re = 100$ is notably different from the other ones. On the other hand, the results obtained for $Re > 1500$ seem to collapse in a unique curve, indicating that a large-Reynolds number asymptotic regime is attained after this value. The resistance is maximum in the small-frequency limit $\Omega \approx 0$, with a

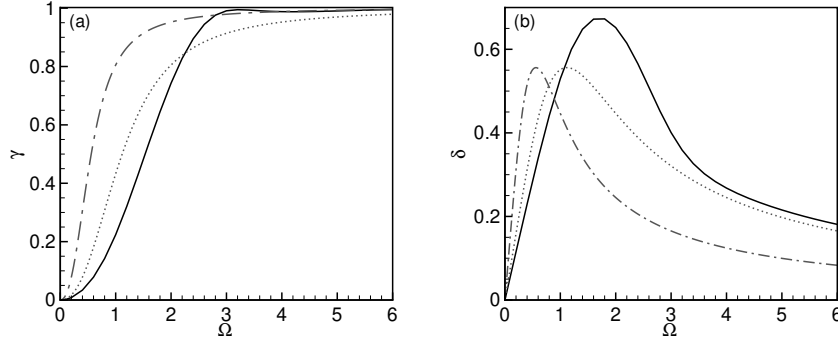


Figure 4.13: (a) Real part γ and (b) imaginary part δ of the Rayleigh conductivity. Plain lines: LNSE results for $Re = 3000$. Dash-dotted lines: Howe's original model. Dotted lines: Howe's modified model.

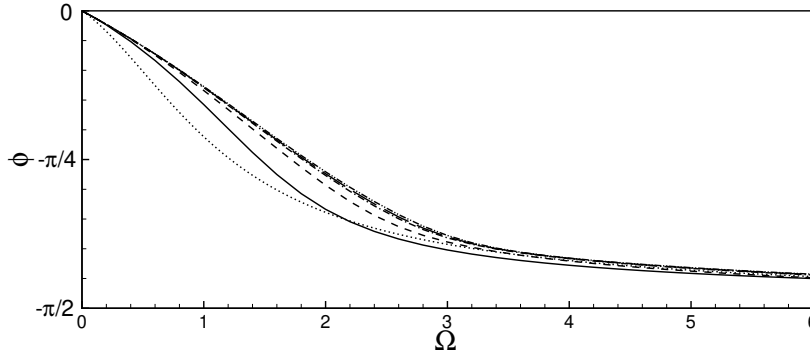


Figure 4.14: Argument ϕ of the impedance, for $Re = 100$ (—), $Re = 500$ (---), $Re = 1500$ (- · -), $Re = 2000$ (- · · -) and $Re = 3000$ (- · · · -), and from Howe's modified model (· · ·).

value $Z_R \approx 0.85$ which will be explained in the next section. As Ω is increased, Z_R first decreases to reach a minimum for $\Omega \approx 3.5$ and then it reaches a quasi-constant value equal to 0.53. Moreover, one can observe that the resistance increases with the Reynolds number for all the value of the frequency Ω . One can note that the resistance is always positive, meaning that, according to the equation (4.10), the jet is an energy sink and so, in order to excite the jet at a given frequency, we need to provide energy from an outer system, as just observed by Howe (1979) for an inviscid flow.

The reactance Z_I is documented in plot *b*, but as this quantity turns out to be a negative and approximately linear function of Ω , it is more practical to plot $-Z_I/\Omega$ as function of Ω . Under this representation, we can make the same observation as for the resistance, regarding the existence of a high-Reynolds asymptotic regime for $Re \gtrsim 1500$, and the notable difference of the case $Re = 100$. Note that in the large-frequency range, the curves indicate an asymptotic trend given by $Z_I/\Omega \approx 0.5$. This value matches with that predicted by the simple Rayleigh model (section 2.3), indicating that for large frequencies, the impedance of the hole is at leading order the same as in the absence of a mean flow.

We now document the results using the equivalent concept of conductivity (see section 2.2), and compare with the predictions of Howe's model. Just as for the impedance, the results for $Re \gtrsim 1500$ collapse onto a single curve characterizing a High-Reynolds number asymptotic regime. In figure 4.13 we plot with a thick line the results obtained for $Re = 3000$ which are representative of this regime.

As explained in section 2.5, Howe's expression for the conductivity (4.13) is expressed in terms of $\Omega^H = \omega R_h/U_c$ where U_c is the convection velocity of the structures along the vortex sheet, whose choice is one of the most questionable points of the analysis. In the initial model, Howe disregarded the vena contracta phenomenon and assumed a jet with radius R_h and velocity $U_J = U_M$. Then, assuming $U_c = U_M/2$ leads to $\Omega^H = 2\Omega$. The values for γ and δ obtained with this choice are plotted in the figure with dash-dotted lines. As can be seen, in this initial version Howe's model rather badly agrees with our LNSE results. In a subsequent step of his analysis, Howe argued that the effect of the vena contracta can be partly accounted for by using the more appropriate choice $U_J = 2U_M$. This choice leads to $U_c = U_M$ and hence $\Omega^H = \Omega$.

The predictions of this modified model are plotted in dashed lines on the figure. As can be seen, the agreement with LNSE results is improved but the curves still differ notably, especially as for the imaginary part δ (plot *b*) in the range $\Omega \approx 2$ where Howe's modified model underestimates the numerically computed one by approximately 30%. On the other hand, the model overestimates the real part γ for $\Omega \lesssim 2$ by about 10% and underestimates it for $\Omega \gtrsim 2$ with the same amount.

As discussed in section 2.5, the result of Howe is expressed in terms of a nondimensional frequency $\Omega^H = \Omega R_h / U_c$ based on the convection velocity of vorticity structures along the vortex sheet U_c , whose precise value is questionable. In figure 4.13 we followed the original choice of Howe $U_c = U_M$ which leads to $\Omega^H = \Omega$. We also tried to compare the results using improved modelings of U_c , leading only to mild ameliorations of the agreement

Finally, a useful quantity which can be extracted from the impedance is the delay angle of the pressure with respect to the velocity:

$$\phi = \arg(Z_h) = \tan^{-1} \left(\frac{Z_I}{Z_R} \right) \quad (4.35)$$

This quantity has been used in a number of experiments, as it allows to discriminate the cases where the impedance is mainly resistive ($\phi \approx 0$) from the ones where it is mainly reactive ($\phi \approx -\pi/2$). This quantity is plotted in figure 4.14, confirming that the behavior switches from purely resistive to purely reactive as the frequency is increased. We also observe in this plot a collapse of the curves obtained in the high-Reynolds asymptotic regime $Re \gtrsim 1500$. The angle ϕ extracted from Howe's modified model is also plotted in the figure (note that in terms of conductivity, the definition of ϕ translates into $\phi = \pi/2 - \arg(K_R) = -\tan^{-1}(\gamma/\delta)$). Again, a substantial deviation is observed, especially in the range of intermediate frequencies $\Omega \approx 2$ where the deviation can be as large as $\pi/12 \equiv 15^\circ$. Oddly, the inviscid Howe model turns out to give better predictions for the case $Re = 100$ than for the high-Reynolds number regime.

4.5.4 The quasi-static limit for $\Omega \rightarrow 0$.

We have observed that in the limit of small frequencies ($\Omega \rightarrow 0$), the impedance becomes purely real and tends to a constant value. This limit value can be predicted using a quasi-static approximation, and this property will be used to verify the consistency of our impedance calculations. As explained in section 2.3 for a steady flow, the pressure jump and the mean velocity across the hole are related through the Bernoulli equation which can be written under the form (4.4)

$$\Delta p = \frac{\rho u_M^2}{2\alpha^2(Re_M)}, \quad (4.36)$$

Assuming $\Delta p = \Delta P + \Delta p'$ and $u_M = U_M + u'_M$, inserting into (4.36) with $Re_M = (U_M + u'_M)R_h/\nu = Re(1 + u'_M/U_M)$ and linearizing lead to

$$\Delta P + \Delta p' \approx \frac{\rho U_M^2}{2\alpha^2} + \frac{\rho u'_M U_M}{\alpha^2} \left(1 - \frac{1}{\alpha} \frac{\partial \alpha}{\partial Re} \right). \quad (4.37)$$

Remembering now that $\Delta P = (\rho U_M^2)/(2\alpha^2)$, this equation allows to obtain a prediction for the impedance which is assumed to be valid in the quasi-static limit ($\Omega \rightarrow 0$):

$$Z_{QS} = \frac{\Delta p'}{\pi R_h^2 u'_M} = \frac{\rho U_M}{\alpha^2 \pi R_h^2} \left(1 - \frac{1}{\alpha} \frac{\partial \alpha}{\partial Re} Re \right). \quad (4.38)$$

Table 1 compares the impedance computed using the method of the previous section for a small value of the frequency, namely $\Omega = 10^{-6}$, to the quasi-static prediction (4.38) obtained using the base-flow characteristics computed in section 4. One can note that the results agree with less than 1% of error. Finally, we can note that the term $(1/\alpha)(\partial\alpha/\partial Re)Re$ in equation (4.38) is small because α is a slowly varying function of Re . The fourth column of table 4.1 gives the prediction of the quasi-static impedance obtained when neglecting this term. The comparison shows that this simplified prediction is still an excellent approximation, and slightly overestimates the actual value except for the case $Re = 100$, where it underestimates it. This is consistent with the fact that the $\alpha - Re$ curve reaches a maximum for $Re \approx 120$ (see figure 4.6).

The low-frequency limit was also addressed by Howe in the framework of his model. A Taylor series of the expression (4.13) leads to $\delta \approx \pi\Omega^H/4$

Re	$Z_R(\Omega = 10^{-6})$	Z_{QS}	$\frac{\rho U_M}{\alpha^2 \pi R_h^2}$
100	0.778957	0.778985	0.784964
500	0.828280	0.828228	0.813912
1500	0.854970	0.854510	0.843178
2000	0.860562	0.868020	0.849696
3000	0.867572	0.866437	0.857986

Table 4.1: Values of the impedance in the low-frequency range. Comparison of values obtained numerically with a very small Ω , quasi-static approximation (4.38), and simplified approximation obtained assuming $\partial\alpha/\partial Re = 0$.

(equation 3.15(b) of Howe’s paper), which, when expressed in terms of impedance, translates into $Z_R \approx (2/\pi)(U_c/U_M) \approx 0.637(U_c/U_M)$. Thus, the choice $U_c/U_M = 1$ made by Howe actually yields a prediction for Z_R which underestimates the High-Reynolds value by approximately 37%. Note that this mismatch can also be observed in figure 4.13(b) regarding the initial slope of the curve $\delta(\Omega)$. This error in the quasi-static limit may be cancelled using an ad-hoc choice of U_c/U_M , but as previously explained, such a modification does not improve substantially the agreement in other ranges of Ω .

4.6 Direct Numerical Simulations of a harmonically forced jet

In order both to validate the linearized approach for small amplitudes and to investigate the influence of nonlinearities for larger amplitudes, we performed Direct Numerical Simulations by integrating in time the Navier–Stokes equations (4.16) for a harmonically forced jet. The DNS are performed using FreeFem++ on the same mesh $\mathbb{M}_0$ as used in the previous section for resolution in physical coordinates (note that the complex mapping technique is fitted to the resolution of the linearized problem but is not relevant for nonlinear simulations). The numerical code used for time-integration is very similar to the one used by in Marquet et al. (2008). The equations are advanced in time using a partly implicit

second-order accurate scheme. The time derivative are approximated using a three-step backward finite difference scheme. The pressure, the laplacian term and the continuity equation are implicit while the convective terms are explicit and treated using a characteristics methods (Boukir et al., 1997).

As initial conditions, we used the steady solution of the Navier–Stokes equations $[\mathbf{U}; P]$ obtained as as described in §4.3.4. We used the same boundary conditions as for the LNSE, namely no-stress on Γ_w , symmetry on Γ_{axis} , stress-free conditions on Γ_{lat} and traction-free on Γ_{out} . At the inflow, we forced the problem on the axial velocity component as follows :

$$\left. \begin{aligned} u_x(t) &= U_{in}[1 + \varepsilon \sin(\omega t)] \\ u_r &= 0 \end{aligned} \right\} \text{ on } \Gamma_{in}, \quad (4.39)$$

where $U_{in} = Q/S_{in}$ and Q has been selected as just discussed in section 4.3.2. The pressure drop $\Delta p(t)$ necessary to define the impedance is is then extracted using

$$\Delta p(t) = p_{in}(t) - p_{out}(t) = 2\pi/S_{in} \int_0^{R_{in}} p(r, t) r dr - 2\pi/S_{out} \int_0^{R_{out}} p(r, t) r dr. \quad (4.40)$$

For the simulations, we fixed the Reynolds number to $Re = 1000$ and investigated the effect of both the frequency in the range $\Omega \in [0.5 - 4]$ and the amplitude in the range $\varepsilon \in [10^{-4}, 10^{-1}]$. Simulations were run for sufficiently long time so that a periodical behaviour of the solution is achieved. Actually, the transient time is very short and a we observed a limit cycle after only a few periods of forcing. In order to calculate the impedance, about 10 periods were simulated.

Figure 4.15 displays a snapshot of the vorticity for $\varepsilon = 10^{-2}$ and $\Omega = 0.5, 2$ and 4 . We also display the streamline originating from the edge of the hole, which can be identified with the surface of the jet $\eta(x, t)$ of figure 4.1. We can observe that under the effect of forcing, the shear layer bounding the jet reorganises into an array of vortices which are convected downstream. Note that the distance at which the vortex array appears is much larger in the lower frequency case, because the spatial instability of the jet is less active at low frequencies. The overall structure of the vorticity is consistent with studies which have used DNS

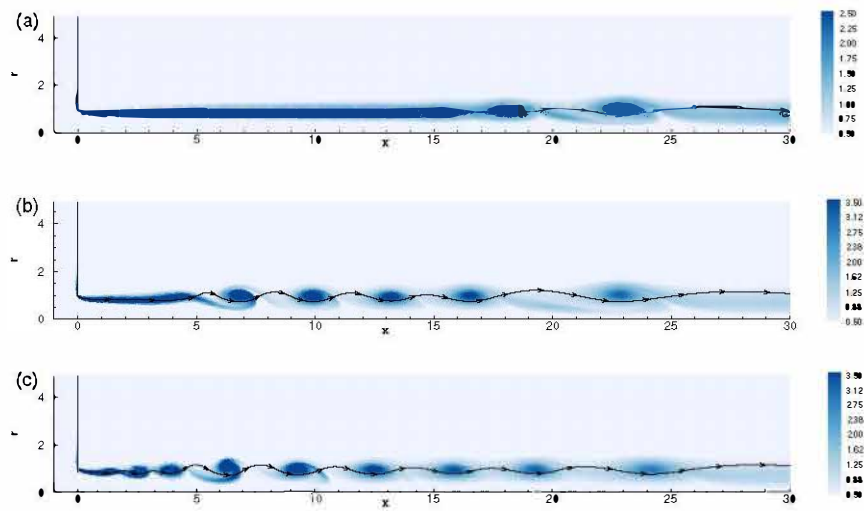


Figure 4.15: Vorticity snapshot at (a) $\Omega = 0.5$, (b) $\Omega = 2$, (c) $\Omega = 4$ for $\varepsilon = 10^{-2}$, $Re = 1000$ and $t = 25$. The line with arrows is the edge of the jet, i.e. the streamline originating from the edge of the hole.

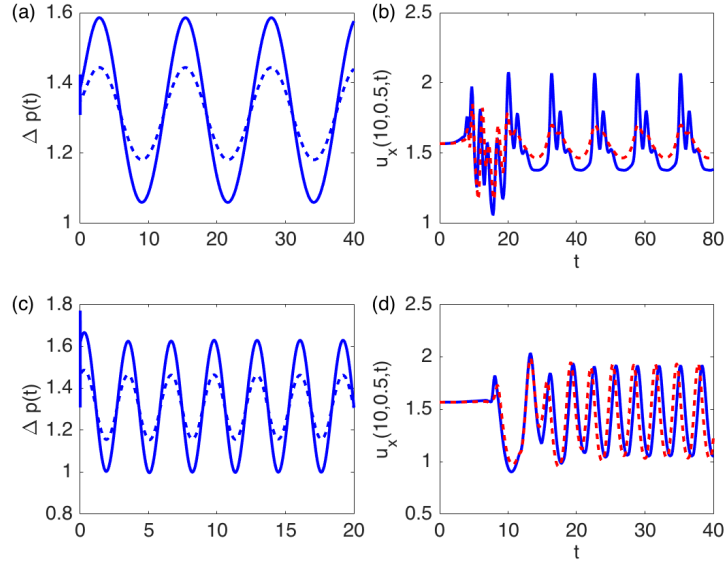


Figure 4.16: Time series of the pressure drop $\Delta p(t)$ and the velocity at $(x, r) = (10, 0.5)$ for $\Omega = 0.5$ (a and b) and $\Omega = 2$ (c and d). Full line is for $\varepsilon = 0.1$ and dashed line is for $\varepsilon = 0.05$.

of a harmonically forced to characterize the spatial amplification process (Kiya et al., 1996; Shaabani-Ardali et al., 2017).

Figure 4.16 displays the time series of the pressure drop $\Delta p(t)$ (plots a and b) and the axial velocity at position $(x, r) = (10, 0.5)$, chosen to be in the region where roll-up occurs. Two values of the frequency and amplitude are chosen, namely $\Omega = 0.5; 2$ and $\varepsilon = 0.1; 0.05$. The plots of the pressure drop show that this quantify is remarkably sinusoidal, even for the largest values of the amplitude. The oscillation levels for $\varepsilon = 0.1$ are about twice those for $\varepsilon = 0.05$, confirming that the pressure drop is indeed proportional to the forcing. Remarkably, in all cases displayed, the pressure almost instantaneously stabilises to a sinusoidal limit cycle, and a transient regime is almost unnoticeable, except for the very first time steps. The velocity signals show a rather different behaviour.

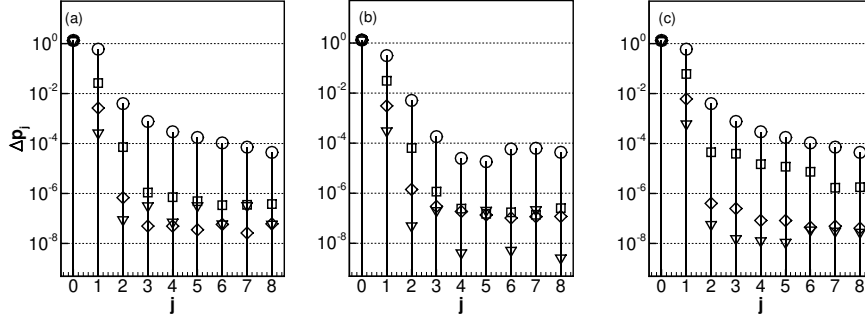


Figure 4.17: Discrete spectra of $\Delta p(t)$ for $\varepsilon = 10^{-1}$ ($\circ$), $\varepsilon = 10^{-2}$ ($\square$), $\varepsilon = 10^{-3}$ ($\diamond$), $\varepsilon = 10^{-4}$ (∇) and (a) $\Omega = 0.5$, (b) $\Omega = 2$ and (c) $\Omega = 4$.

For $\Omega = 0.5$, the signal is very far from sinusoidal and displays a rich harmonical content. This confirms that the roll-up process occurring in this region is strongly nonlinear. The limit cycle at $x = 10$ also takes time to establish, a fact associated to the time of convection of vortex structures. It is finally remarkable that the amplitude of oscillation for $\varepsilon = 0.1$ is not double compared to that for $\varepsilon = 0.05$. This is also observed for $\Omega = 2$, where the cases with $\varepsilon = 0.1$ and $\varepsilon = 0.05$ saturate to a limit cycle (closer to sinusoidal in this case) of same amplitude.

The pressure signal can be analysed in a finer way by decomposing into as a Fourier series in the form:

$$\Delta p(t) = \Delta p_0 + \sum_{j=1}^{\infty} \Delta p_j \sin(j\Omega t - \phi_j). \quad (4.41)$$

In practise, to discard the transient effects, the Fourier transformation was applied to the time for $t > 10$ (even if in most cases there is actually almost no transient effects, as documented above). Figure 4.17 displays the discrete spectra, namely the absolute value of the amplitudes $|\Delta p_j|$ as function of j , extracted from all performed DNS. The $j = 0$ component Δp_0 corresponds to the pressure drop associated to the mean flow, and is almost independent of ε . The $j = 1$ component Δp_1 corresponds to the amplitude at the fundamental forcing frequency. This quantity is

observed to be proportional to ε , confirming that the response to forcing is essentially linear. The components Δp_j with $j > 0$ corresponds to the higher harmonics. These components are generally smaller than 10^{-4} hence negligible. The case $\varepsilon = 10^{-1}$ leads to the largest values for the higher harmonics, but they still remain one order of magnitude smaller than the response at the driving frequency.

Truncating the Fourier series to the two first terms, i.e.

$$\Delta p(t) \approx \Delta p_0 + \Delta p_1 \sin(\omega t - \phi_1) \quad (4.42)$$

and remembering that the flow rate can be written as

$$q(t) = \underbrace{U_{in}\pi R_{in}^2}_{q_0} + \underbrace{U_{in}\pi R_{in}^2 \varepsilon \sin(\omega t)}_{q_1}, \quad (4.43)$$

we can calculate the impedance using only the first Fourier component of the pressure:

$$\left. \begin{aligned} Z_R &= \Delta p_1/q_1 \cos(\phi_1), \\ Z_I &= \Delta p_1/q_1 \sin(\phi_1). \end{aligned} \right\} \quad (4.44)$$

Table 4.2 displays the mean pressure drop Δp_0 and the impedance deduced from the DNS results for all cases simulated, and compares them to the LNSE results of sections 4 and 5.

As for Δp_0 , the LNSE results displayed in the table actually corresponds to the pressure drop associated to the *base flow*, namely steady solution of the NSE, while the DNS results correspond to the *mean flow* obtained by time-averaging. There is a subtle difference between these concepts (Barkley, 2006), and the difference is expected to be of order ε^2 . This is in agreement with the fact that deviations are only notable for the largest amplitude $\varepsilon = 10^{-1}$.

As for the impedances, it is remarkable that the LNSE results provide an excellent approximation to the DNS results, with a relative error less than 1% except for high frequency and large where it increases a little (we found the maximum relative error about 4% at $\omega = 4$ and $\varepsilon = 0.1$).

4.7 Summary and discussion

The main goal of this study was to reconsider the classical problem of the response of an axisymmetric jet through a circular aperture through

Method, Frequency	ε	Δp_0	Z_R	Z_I
DNS, $\Omega = 0.5$	10^{-1}	1.315197	0.823836	0.306126
	10^{-2}	1.308922	0.823949	0.304707
	10^{-3}	1.308850	0.823755	0.305271
	10^{-4}	1.308849	0.822438	0.305119
LNSE, $\Omega = 0.5$	—	1.308657	0.823451	0.307814
DNS, $\Omega = 2$	10^{-1}	1.318342	0.642373	0.384001
	10^{-2}	1.308899	0.631512	0.379576
	10^{-3}	1.308842	0.628868	0.376664
	10^{-4}	1.308849	0.628365	0.375493
LNSE, $\Omega = 2$	—	1.308657	0.627929	0.377853
DNS, $\Omega = 4$	10^{-1}	1.332050	0.524558	0.458023
	10^{-2}	1.309361	0.506036	0.467145
	10^{-3}	1.308858	0.502835	0.465519
	10^{-4}	1.308850	0.503785	0.465163
LNSE, $\Omega = 4$	—	1.308657	0.502442	0.467931

Table 4.2: Comparison between the DNS and the linear approximation in term of pressure drop of the mean (base) flow and impedances.

a plate of small thickness to harmonic forcing. This problem was initially considered by Howe who proposed an inviscid model which is still the basis of most studies of this problem. However a number of starting hypotheses of Howe's model are questionable. In order to reconsider the problem on more rigorous grounds, our chosen approach has been to numerically solve the problem using Linearized Navier–Stokes Equations (LNSE).

The first step of the LNSE approach consists of computing a base flow corresponding to the steady laminar flow through the aperture. Section 4 was devoted to the description of this base flow. Upstream of the aperture, it essentially consists of a radially convergent flow, while downstream of the aperture, the flow forms a quasi parallel jet bounded by a thin vorticity layer originating from the rim. As classically observed in experiments, the radius of the jet is smaller than the radius of the aperture. We documented this effect in terms of the *vena contracta* coefficient α . Our numerical results indicate an almost constant value $\alpha \approx 0.61$ in the range $10^3 < Re < 10^4$, in agreement with classical experiments.

The second step of the LNSE approach consists of solving a linear problem for small-amplitude disturbances with harmonic temporal dependence. A standard implementation of this method, starting from a formulation in terms of physical coordinates (x, r) on a numerical domain "large enough" to resolve correctly the structure of the linear perturbation (typically $[r_{max}, x_{max}] = [20, 80]$), was first tried. This first implementation was found to lead to difficulties in the high-Reynolds number range, leading to the impossibility to obtain reliable results as soon as $Re \gtrsim 1000$. These difficulties were analyzed, and the problem was found to be linked to the strong spatial amplification properties of the jet. To overcome these difficulties, a convenient method was designed, which consists of reformulating the problems in terms of a mapped complex coordinate $X(x)$. The idea of using complex coordinates is not new in linear acoustics, and is at the basis of the Perfectly Matched Layer (PLM) method to prevent reflections of the acoustic waves on the boundaries of the domain. However, to our knowledge, the use of such a method in strictly incompressible problems is new. We show that an appropriate choice of the mapping function allows to get rid of the spatial amplification of the perturbation in the axial, mapped direction.

Although the spatial structure of the perturbation has no longer a

physical interpretation when computed using complex coordinates, we demonstrated that the global quantities depending only from the pressure jump across the hole, such as the *vena contracta* coefficient and the impedances are well resolved. This method thus allows to obtain meaningful results using a much smaller numerical domain (typically $[R_{max}, L_{out}] = [15, 15]$) and incidentally a much lighter numerical grid.

Using this method, we then characterized the response of the jet to harmonic forcing by computing its *impedance*, namely the ratio between the fluctuating pressure jump and fluctuating flow rate across the aperture, which is a key quantity used by acousticians to characterize the interaction of jet flows with acoustic fields. In all cases the real part of the impedance was found to be positive, meaning that exciting the jet at a given frequency necessitates the provision of energy from an outer system. Moreover, the impedance was found to become independent of the precise value of Re as soon as $Re \gtrsim 1500$, indicating the existence of a high-Reynolds number asymptotic regime.

Results in this high-Reynolds number regime were compared to predictions of Howe's model. The comparison was done in terms of the *Rayleigh conductivity*, which is a concept directly related to the impedance and used by a fraction of the acoustic community as an alternative. Comparison shows substantial deviations, especially in the range of intermediate nondimensional frequencies, indicating that some of the hypotheses underlying Howe's model are too restrictive.

Finally, to confirm the validity of our linearized approach, we also performed direct numerical simulations considering harmonic perturbation with small but finite amplitude ε . The spatial structure of the perturbations computed in this way showed a rapid saturation of the spatial instability towards an array of vortex rings, very different from the structure computed using LNSE. Despite this, the values of the impedance extracted from these DNS, as well as the properties of the mean flow, were found to be in excellent agreement with LNSE results, with a maximum relative error of only a few percents for $\varepsilon = 0.1$. This result confirms that the LNSE is an efficient method to predict the impedance, even in cases where the spatial evolution of the perturbations is rapidly dominated by nonlinear effects.

We end this discussion with a few closing remarks. First, coming back on the complex mapping technique used in the LNSE approach,

we stress that this method was designed to overcome a mathematical difficulty linked to the *linear problem*, namely strong spatial amplification extending very far away in the axial direction. As so, this method is not suited to a direct numerical solution in the nonlinear regime, and the DNS presented in section 6 were thus performed in physical coordinates. On the other hand, the method is potentially usable for studying the linear stability of large class of flows characterized by nearly-parallel spatially unstable regions, such as the wakes of blunt or profiled bodies. We are currently investigating the applicability of complex mapping for such problems.

Secondly, since our whole approach relies on an assumed laminar base flow, one may question the applicability of our results when considering turbulent jets. Although the precise threshold is difficult to predict, transition to turbulence in such jets is typically thought to take place in the range $Re \in [10^3 - 10^4]$. However, when transition takes place, turbulence is only observed in the downstream region located after the near-field *vena contracta* region which remains essentially laminar. Having observed in our DNS that the nonlinear evolution of vortex structures in the far-field do not affect the value of the impedance, we can postulate that the same is true regarding nonlinear effects due to turbulence, as thus that our results, obtained under the hypothesis of a laminar flow, are actually applicable to turbulent jets in a large range of parameters.

Third, in the whole study, we have only considered axisymmetric disturbances to the flow. Non-axisymmetric disturbances also exist in such flows, and in high-Reynolds jets their signature has been detected among turbulent structures ?. However, such non-axisymmetric disturbances are not associated to a net flow rate through the hole. So, they are expected to be much less coupled to acoustic disturbances, and indeed it is not possible to describe them in terms of an impedance.

Finally, we have mentioned in the introduction that in the case where the thickness of the plate is not small compared to the radius of the hole, the jet can cease to act as a sound damper to become a sound generator, leading to the possibility of self-sustained oscillations of the jet. In such a case, the impedance concept is a useful tool to characterize the instability mechanism, and the numerical method designed in the present paper is directly applicable to investigation of such instabilities. A parametric study of the response of jets through plates of finite thickness to harmonic

forcing is underway and will be presented in a forthcoming paper.

Chapter 5

Acoustic impedance and hydrodynamic instability of the flow through a circular aperture in a thick plate

We study the unsteady flow of a viscous fluid passing through a circular aperture in a plate characterized by a non-zero thickness. We investigate this problem by solving the incompressible Linearized Navier-Stokes Equations (LNSE) around a laminar base flow, in both the forced regime (allowing to characterize the coupling of the flow with acoustic resonators) and the autonomous regime (allowing to identify the possibility of purely hydrodynamical instabilities). In the forced regime, we calculate the impedances and discuss the stability properties in terms of a Nyquist diagram. We show that such diagrams allow to predict two kinds of instabilities: (i) a conditional instability linked to the overreflexion of an acoustic wave but requiring the existence of an conveniently tuned external acoustic resonator, and (ii) a purely hydrodynamic instability existing even in a strictly incompressible framework. A parametric study is conducted to predict the range of existence of both instabilities in terms of the Reynolds number and the aspect ratio of the aperture. Analysing the structure of the linearly forced flow allows to show that the instability mechanism is closely linked to the existence of a recirculation

region within the thickness of the plate. We then investigate the autonomous regime using a standard normal-mode analysis. The analysis confirms the existence of the purely hydrodynamical instability in accordance with the impedance-based criterion. The spatial structure of the unstable eigenmodes are found to be similar to the structure of the corresponding unsteady flows computed using the forced problem. Analysis of the adjoint eigenmodes and of the adjoint-based structural sensitivity confirms that the origin of the instability lies in the recirculation region existing within the thickness of the plate.

5.1 Introduction

The unsteady flow through an aperture separating two fluid domains, either closed (ducts, chambers, resonators) or open, is encountered in a large number of applications. This situation is also of fundamental importance in the design of musical instruments. A fundamental milestone in the study of such problems is the classical Rayleigh (1945) solution of the inviscid, potential flow through a circular hole, in the absence of mean flow. This solution shows that the situation is globally equivalent to the simple assumption of a rigid plug of fluid with an "effective length" l_{eff} oscillating across the aperture. This Rayleigh solution is often invoked in simple models of acoustic devices and is, for instance, a key ingredient in the modelling of the so-called Helmholtz resonator.

In the case where the aperture is traversed by a mean flow, the fluid no longer behaves as an ideal, rigid plug but generally acts as an energy dissipator. This property is used in many industrial applications where one wants to suppress acoustical waves (see for instance the bibliography cited in Fabre et al. (2018b)). This energy dissipation is generally associated to a transfer of energy to the flow through the excitation of vortical structures along the shear layer bounding the jet. The situation was investigated theoretically by Howe (1979), who introduced a complex quantity called *conductivity* K_R which generalizes Rayleigh's "effective length". The knowledge of $K_R(\omega)$ as function of the forcing frequency ω , or of the closely related quantity $Z(\omega) = -i\omega/K_R(\omega)$ called the *impedance*, allows to fully characterize the possible interaction of the flow with acoustic waves. In particular, the real part of the impedance

(which is positive for a zero-thickness hole), is directly linked to the energy flux transferred from the waves to the flow. Howe subsequently derived a potential model predicting the conductivity (and impedance) in the case of a hole of zero thickness. Despite its mathematical rigour, Howe's model starts from very simplified hypotheses regarding the shape and the location of the vortex sheet and its convective velocity. Recently, Fabre et al. (2018b) reviewed Howe's problem using Linearized Navier–Stokes equations in order to take into account the effect of the viscosity and the exact shape of the vortex sheet. They showed that for $Re \gtrsim 1500$, results are quite independent from the Reynolds number but significantly deviates from Howe's ones, above all for intermediate frequencies. Nevertheless, in both Howe's model and Fabre et al. (2018b)'s improved solution, the behaviour of the hole remains dissipative (associated to a positive real part of the impedance), in accordance with experimental and numerical investigations.

The case where the thickness of the plate in which the hole is drilled is not small compared to its diameter leads to a completely different situation, as the jet flow can now act as a *sound generator* instead of a sound attenuator. The first observation of this property seems to have been made by Bouasse (1929), who reported that jets through thick plates could produce a well-reproducible whistling, with a frequency roughly proportional to the hole thickness. This observation remained unnoticed (as many other findings of the surprisingly rich experimental work of Bouasse), but was rediscovered in the 21st century by Jing and Sun (2000) and Su et al. (2015) who, in an effort to improve the design of perforated plates used as sound dampers, reported that in some circumstances, these devices could lose their ability to damp acoustic waves and lead to self-sustained whistlings. Numerical simulations by Kierkegaard et al. (2012) showed that in the range of parameters where such whistlings occur, the mean flow through the hole is characterized by a recirculation bubble, either trapped within the thickness of the plate, or fully detached. However, the precise role of this recirculation bubble in the sound-production phenomenon remains to be clarified.

The ability of the jet flow to provide acoustical energy is associated to a positive real part of the impedance, so computation or measurement of this quantity offers a convenient way to characterise these phenomena. A number of analytical and semi-empirical models (Jing and Sun (2000),

Bellucci et al. (2004)) have been proposed to predict the impedance of such finite-length holes. Confrontation with experiments Su et al. (2015) and numerical simulations Eldredge et al. (2007) have revealed the lack of robustness of such models which all contain *ad-hoc* parameters. Yang and Morgans (2016) and Yang and Morgans (2017) developed a more elaborate semi-analytical model based on the actual shape of the vortex sheet, and furthermore including the effect of compressibility within the thickness of the hole. However, their approach remains potential and cannot account for the effect of viscosity within the thickness of the shear layer, nor for the dependence of the impedance with respect to the Reynolds number.

Linearised Navier-Stokes Equations (LNSE), offers a more satisfying framework to access the impedance of such holes, with a full incorporation of viscous effects. As already pointed out, this approach has been carried out in Fabre et al. (2018b) for a zero-thickness hole, leading to notable improvements of Howe's classical inviscid model. This approach has also been applied to the flow through a finite-thickness hole by Kierkegaard et al. (2012) in a range of parameters characterized by self-sustained whistlings. These authors also showed that computation of the impedance for a forced problem (with real ω) can be used, thanks to Nyquist diagrams, to predict an instability criterion for eigenmodes (with complex ω) of the unforced, self-sustained problem. A similar approach will be carried out in the present paper. However, our work differs from that of Kierkegaard et al. (2012) by a number of points. First, Kierkegaard et al. (2012) considered a constriction within a long pipe, while we consider a small aperture connecting an upstream and a downstream domain considered of large dimension. Secondly, Kierkegaard et al. (2012) considered a case where the Mach number based on the maximum velocity is not small and used compressible LNSE. On the other hand, our study starts from the incompressible equations under an acoustical compactness hypothesis. Third, Kierkegaard et al. (2012) considered a High-Reynolds turbulent situation and built their linearized approach around a mean flow obtained by RANS simulations. On our side, we consider a lower range of Reynolds number and build our approach around a laminar base flow consistent with the LNSE framework. Finally Kierkegaard et al. (2012) explicitly considered the coupling with an acoustic resonator by introducing a wave reflection condition at the

inlet of the pipe, and hence their Nyquist-based stability criterion is only relevant to this situation. In our case, we wish to characterize the potential of the jet to lead to self-sustained oscillations regardless of the nature of the acoustic environment, and even in the case where there are no acoustic resonators at all. The situation we investigate is thus more simple and "academic", but by ruling out the geometry of the upstream and downstream domains and the Mach number parameter, we are able to conduct a full parametric study of the problem, an objective which was not achievable considering the choices of Kierkegaard et al. (2012).

The remainder of the paper is organized as follows :

- In section 2, after defining the geometry and the parameters of the study, we define the concept of impedance, and explain how, thanks to the use of Nyquist diagrams, this quantity can be used to predict the stability properties of the jet flow. We show that two kinds of instabilities are possible in this context : (i) a *conditional instability* corresponding to an overreflexion of acoustic waves in some range of frequencies, leading to an effective instability only if the jet is coupled to a conveniently tuned acoustic resonator, and (ii) a *purely hydrodynamical instability* which manifests regardless of the existence of an acoustic resonator, and exists even in the case of a strictly incompressible flow.
- In section 3, we recall the Linearized Navier Stokes Equations, and the numerical resolution method, as already presented in Fabre et al. (2018b). We show how this formalism can be used to solve both a harmonically forced problem for real frequencies ω allowing to compute the impedances, and a homogeneous eigenvalue problem allowing to compute the complex frequencies $\omega_r + i\omega_i$ allowing to characterize the purely hydrodynamical instabilities.
- In section 4, we detail the structure of the base flow corresponding to the steady jet as function of the Reynolds number Re and aspect ratio β of the hole. We detail in particular the *vena contracta* coefficient characterizing the relationship between the mean pressure drop and mean flux through the hole, and the range of existence and spatial structure of the recirculation region which forms within the thickness of the hole.

- In section 5, we present results of the LNSE approach in the harmonically forced regime. We present the computed impedances for selected values of Re and β . We document the structure of the linearly forced flows, in particular within the recirculation region. We eventually provide a parametric map allowing to predict the ranges of existence of both conditional and hydrodynamical instabilities in the $Re - \beta$ parameter plane.
- In section 6, we present results of the LNSE approach in the homogeneous regime. We confirm the existence of the purely hydrodynamical instability, in accordance with the impedance-based predictions. We further detail the structure of the eigenmodes, the adjoint eigenmodes and the adjoint-based structural sensitivity, allowing to highlight once again the role of the recirculation region on the instability mechanism.
- Finally, section 7 summarizes the findings and discusses a few perspectives opened by our work.

5.2 Problem definition

5.2.1 Geometry, parameters, and modelling hypotheses

The situation investigated in the present paper is sketched in figure 5.1. We consider a fluid of viscosity ν and density ρ discharging through a circular aperture of radius R_h in a planar thick plate with thickness L_h . The domains located upstream and downstream of the hole are supposed of large dimensions compared to the dimensions of the hole, so that the geometry is characterized by a single dimensionless parameter, the aspect ratio β defined as

$$\beta = \frac{L_h}{2R_h}. \quad (5.1)$$

The zero-thickness limit case ($\beta = 0$) is investigated in detail in Fabre et al. (2018b); in the present paper we consider holes with finite thickness in the range $\beta \in [0.1 - 2]$.

The pressure difference between the inlet and the outlet domain, namely $\Delta P = [P_{in} - P_{out}]$, generates a net flow $Q = U_M A_h$ through

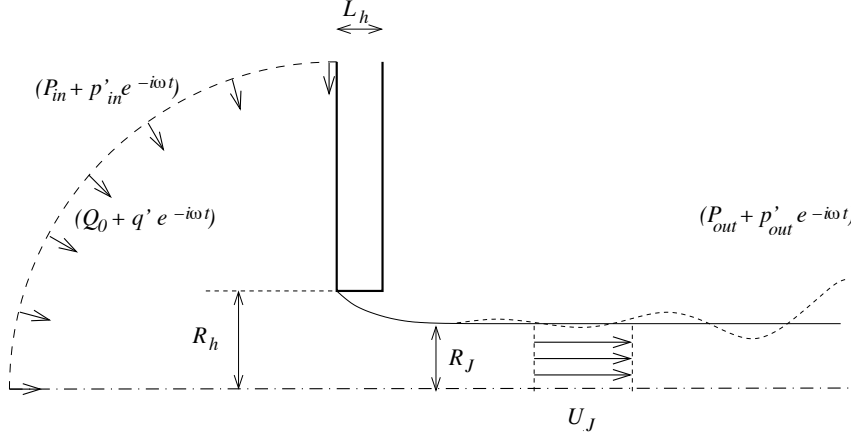


Figure 5.1: Sketch of the flow configuration (not in scale) representing the oscillating flow through a circular hole in a thick plate and definition of the geometrical parameters. We report also the decomposition of the flow quantities in the upstream and downstream boundaries.

the hole, where $A_h = \pi R_h^2$ is the area of the hole and U_M is the mean velocity. This mean flow is characterized by a Reynolds number defined as :

$$Re = \frac{2R_h U_M}{\nu} \equiv \frac{2Q}{\pi R_h \nu}. \quad (5.2)$$

Following Fabre et al. (2018b), we will suppose that the mean Mach number is small, and that the dimensions of the hole are small compared to the acoustical wavelengths (acoustical compactness hypothesis). These hypotheses allow to assume that the flow is locally incompressible in the region of the hole. An example of matching with an outer acoustic field is presented in appendix A.

5.2.2 Characterization of the unsteady regime and Impedance definition

To characterize the behaviour of the jet in the unsteady regime, we assume that far away from the hole the pressure levels in the upstream and downstream regions tend to uniform values denoted as $p_{in}(t)$ and

$p_{out}(t)$. We will further assume that both the pressure drop $\Delta p(t)$ and the flow rate $q(t)$ are perturbed by a small-amplitude deviations from the mean state characterized by a frequency ω (possibly complex):

$$\begin{pmatrix} \Delta p(t) \\ q(t) \end{pmatrix} = \begin{pmatrix} [P_{in} - P_{out}] \\ Q \end{pmatrix} + \varepsilon \begin{pmatrix} [p'_{in} - p'_{out}] \\ q' \end{pmatrix} e^{-i\omega t} + c.c., \quad (5.3)$$

$$Z_h(\omega) = \frac{[p'_{in} - p'_{out}]}{q'} \quad (5.4)$$

Note that with the present definition the impedance has physical dimensions $M \cdot T^{-1} L^{-4}$. We will also introduce a nondimensional impedance defined as

$$Z = \frac{R_h^2}{\rho U_M} Z_h \equiv Z_R + iZ_I, \quad (5.5)$$

where the real part of the impedance Z_R is the dimensionless resistance while its imaginary part Z_I is the reactance. In presentation of the results, the frequency will be represented in a nondimensional way by introducing the Strouhal number Ω as follows:

$$\Omega = \frac{\omega R_h}{U_M}. \quad (5.6)$$

As already stated in the introduction, the sign of the real part of the impedance Z_R (or *resistance*) is directly proportional to the energy flux $\langle \Pi \rangle$ transferred to the flow through $\langle \Pi \rangle = 2\rho U_M / (R_h^2) Z_R |q'|^2$. The demonstration of this property can be found in Howe (1979), and is also reproduced in Fabre et al. (2018b).

5.2.3 Impedance-based instability criteria

We now explain the links between impedance and instabilities, and show how simple instability criteria can be formulated using Nyquist diagrams (namely representations of Z_r versus Z_i).

- First, as already discussed, the sign of the real part of the impedance $Z_R(\omega)$ (or *resistance*) as function of the *real* frequency ω is a direct indicator of a possible instability. However, one should insist

that the condition $Z_R < 0$ is a necessary but not sufficient condition for instability. In the context of electrical circuits (Conciauro and Puglisi (1981)), a system with negative resistance is said to be *active* in the sense that it effectively leads to an instability if connected to a reactive circuit allowing oscillations in the right range of frequencies. In the present context, this situation is referred as *conditional instability* and requires the presence of a correctly tuned acoustic oscillator (a cavity and/or a pipe) connected upstream (or downstream) of the aperture.

The demonstration that $Z_R < 0$ is a necessary condition for *conditional instability* can be explicated in two ways. First, as already stated, Z_R is directly linked to the energy flux transferred from acoustic waves to the jet. Thus, if $Z_R > 0$ the jet behaves as an energy sink, while if $Z_R < 0$ it acts as an energy source. Secondly, one can also establish this link by studying the reflection of acoustic waves onto the hole. This argument is carried out in appendix A, where we conduct an asymptotic matching between the locally incompressible solution in the vicinity of the hole and an outer solution of the acoustic problem. The conclusion of this analysis is that in the limit of small Mach number, an incident acoustic wave coming from the upstream domain is overreflected if and only if $Z_R < 0$.

A situation leading to *conditional instability* is illustrated in figure 5.2a – b. Plot (a) shows the real and imaginary parts of the impedance in a situation where Z_R is negative in an interval $[\omega_1, \omega_2]$, and Z_i does not change sign. When represented in a Nyquist diagram, the criterion can be formulated as follows: *the system is conditionally unstable if the Nyquist curves enter the half-plane $Z_R < 0$.*

- Secondly, when considered as an analytical function of the *complex* frequency $\omega = \omega_r + i\omega_i$, the impedance can be used to formulate a second instability criterion, namely: the system is unstable, regardless of the properties of its environment if there exists a complex zero of the impedance function such that $\omega_i > 0$. Indeed, for complex values of ω the modal dependence reads $e^{-i\omega t} = e^{-i\omega_r t} e^{\omega_i t}$, thus solutions with the form 5.3 are exponentially growing if $\omega_i > 0$.

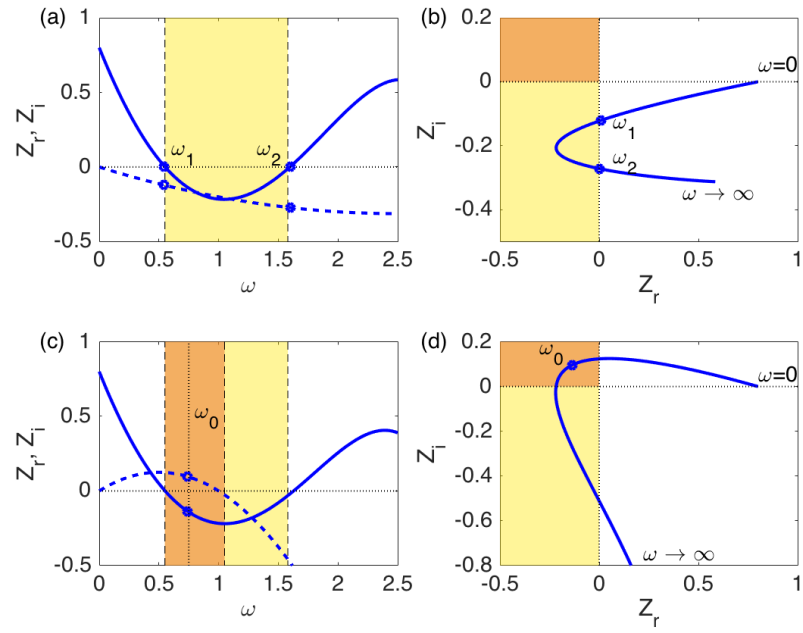


Figure 5.2: (a) and (b) example of situation leading to conditional instability. (c) and (d) example of situation leading to hydrodynamical instability. The regions of conditional and hydrodynamic instabilities are represented by yellow and orange areas, respectively.

In the context of electrical circuits, this situation is referred to as *absolute* instability in opposition to the conditional instability discussed above. Since the term "absolute" has a different meaning in the hydrodynamic stability community (as opposed to convective instabilities, see e.g. Huerre and Monkewitz (1990)), we prefer to adopt the term *purely hydrodynamical instabilities* to describe this case, emphasizing the fact that they can occur in a strictly incompressible framework.

Physically, the condition $Z_h(\omega) = 0$ implies that there exist modal solutions of the linearized problem in which pressure jump $[p'_{in} - p'_{out}]$ is exactly zero. In other terms, the total pressure jump across the hole is imposed as a constant (i.e. $[p_{in}(t) - p_{out}(t)] = [P_{in} - P_{out}]$) but the flow rate $q(t)$ is allowed to vary. This kind of boundary condition is a bit uncommon for incompressible flow problems. However, one must keep in mind that the incompressible solution is only valid locally in the vicinity of the hole. In appendix A, we conduct an asymptotic matching with an outer acoustic solution and show that in the limit of small Mach number, the condition $Z_h(\omega) = 0$ with complex ω and $\omega_i > 0$ corresponds to a spontaneous self-oscillation of the flow across the hole associated to the radiation of acoustical waves in both the upstream and downstream domains.

In practise, the number of complex zeros of the analytically continued impedance $Z_h(\omega)$ and their location in the complex plane can be deduced from the representation of $Z_h(\omega)$ for real values ω using classical Nyquist criterion, which states that *there exists an unstable zero of the impedance if and only if the Nyquist curve encircles the origin in the anticlockwise direction*. A weaker but practically equivalent version of this criterion can be formulated as follows: *the system is unstable in a purely hydrodynamical way if the Nyquist curve enters the quarter-plane defined by $Z_R < 0$; $Z_I > 0$* . A situation leading to *purely hydrodynamical instability* is illustrated in figure 5.2c – d.

In addition to providing an instability criterion, the knowledge of the impedance for real ω can also be used to predict an approximation of the complex zeros in the case where ω_i is small. for

this sake, let us suppose that the Nyquist curve passes close to the origin, and let us note ω_0 the value for which the norm of the complex impedance $|Z(\omega)|$ is smallest. The location of this point is illustrated in figures 5.2c – d. Searching for the complex zero as $\omega = \omega_0 + \delta\omega$ and working with a Taylor series around ω_0 leads to $Z(\omega_0) + (\partial Z/\partial\omega)_{\omega_0}\delta\omega = 0$, hence providing an estimation as follows :

$$\omega \approx \omega_0 - \frac{Z(\omega_0)\overline{(\partial Z/\partial\omega)_{\omega_0}}}{|(\partial Z/\partial\omega)_{\omega_0}|^2} \quad (5.7)$$

It can be shown that $\Re(Z(\omega_0)\overline{(\partial Z/\partial\omega)_{\omega_0}}) = 0$ (a simple geometrical interpretation being that the line joining the point $Z(\omega_0)$ to the origin and the line tangent to the Nyquist curve at ω_0 are orthogonal to each other). Hence, the correction appearing in 5.7 directly provides an estimation of the amplification rate ω_i .

5.3 Linearized Navier Stokes Equations and numerical methods

In the previous section, the linearly perturbed flow across a hole was considered from a general point of view, focussing on the impedance and its link with possible instabilities. In the present section, we introduce the LNSE framework, and show how this framework can be used both to compute the impedance through resolution of a *forced problem* and to directly address the instability problem through resolution of an *autonomous problem*.

5.3.1 Starting equations

The fluid motion is governed by the Navier-Stokes equations:

$$\frac{\partial}{\partial t} \begin{bmatrix} \mathbf{u} \\ 0 \end{bmatrix} = \mathcal{NS} \begin{bmatrix} \mathbf{u} \\ p \end{bmatrix} = \begin{bmatrix} -\mathbf{u} \cdot \nabla \mathbf{u} - \nabla p + Re^{-1} \nabla^2 \mathbf{u} \\ \nabla \cdot \mathbf{u} \end{bmatrix} \quad (5.8)$$

where p is the reduced pressure field. Since we are in axial-symmetric flow configuration, we decompose the velocity vector $\mathbf{u}$ in an axial component u_x and in a radial component u_r .

The Linearized Navier-Stokes framework consists of expanding the flow as a steady base-flow plus a small-amplitude modal perturbation as follows:

$$\begin{bmatrix} \mathbf{u} \\ p \end{bmatrix} = \begin{bmatrix} \mathbf{u}_0 \\ p_0 \end{bmatrix} + \epsilon \begin{bmatrix} \mathbf{u}' \\ p' \end{bmatrix} e^{-i\omega t}. \quad (5.9)$$

5.3.2 Base-flow equations

The base flow is the solution of the steady version of the Navier-Stokes equations:

$$\mathcal{NS}[\mathbf{u}_0; p_0] = 0$$

with the following set of boundary conditions:

$$\begin{aligned} P(x, r) &\rightarrow P_{in} & \text{as } \sqrt{x^2 + r^2} \rightarrow \infty & \text{ and } x < 0 \\ P(x, r) &\rightarrow P_{out} & \text{as } \sqrt{x^2 + r^2} \rightarrow \infty & \text{ and } x > 0 \end{aligned} \quad (5.10)$$

$$\int_S \mathbf{u}_0 \cdot \mathbf{n} dS = Q.$$

where $\mathcal{S}$ is any surface traversed by the flow. It is convenient to choose $\mathcal{S}$ as the inlet of the domain so that the latter equation can be imposed as a Dirichlet boundary condition.

5.3.3 Linear equations

The linear perturbation obeys the following equations :

$$-i\omega \mathcal{B}[\mathbf{u}'; p'] = \mathcal{LNS}_0[\mathbf{u}'; p'], \quad (5.11)$$

where $\mathcal{LNS}_0$ is the linearized Navier-Stokes operator around the base flow and $\mathcal{B}$ is a weight operator defined as follows:

$$\mathcal{LNS}_0 \begin{bmatrix} \mathbf{u}' \\ p' \end{bmatrix} = \begin{bmatrix} -(\mathbf{u}_0 \cdot \nabla \mathbf{u}' + \mathbf{u}' \cdot \nabla \mathbf{u}_0) - \nabla p' + Re^{-1} \nabla^2 \mathbf{u}' \\ \nabla \cdot \mathbf{u}' \end{bmatrix}; \quad \mathcal{B} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad (5.12)$$

This set of equations is complemented by the following boundary conditions :

$$p'(x, r) \rightarrow p'_{in} \quad \text{for } \sqrt{x^2 + r^2} \rightarrow \infty \text{ and } x < 0, \quad (5.13)$$

$$p'(x, r) \rightarrow p'_{out} \quad \text{for } \sqrt{x^2 + r^2} \rightarrow \infty \text{ and } x > 0, \quad (5.14)$$

$$\int_S \mathbf{u}' \cdot \mathbf{n} dS = q'. \quad (5.15)$$

This set of equations for the perturbations is relevant to both the forced problem and the autonomous problem. The difference is in the handling of the boundary conditions:

- For the *forced problem*, a non-zero q' is imposed (fixed arbitrarily to $q' = \pi R_h^2$; such that the associated mean velocity u'_M is 1). Eq. (5.15) thus constitutes a non-homogeneous boundary condition at the inlet plane. On the other hand, since only $p'_{in} - p'_{out}$ is relevant, one can set $p'_{out} = 0$ without loss of generality. Eq. (5.14) thus leads to a homogeneous boundary condition at the outlet plane. The problem can be symbolically written as

$$[\mathcal{LNS}_0 - i\omega\mathcal{B}][\mathbf{u}'; p'] = \mathcal{F},$$

where the definition of $\mathcal{LNS}_0$ implicitly contains the homogeneous boundary condition at the outlet, and $\mathcal{F}$ represents symbolically the non-homogeneous boundary condition at the inlet. This problem is nonsingular and readily solved. The pressure jump p'_{in} allowing to define the impedance is subsequently deduced from Eq. (5.13).

- For the *homogeneous problem*, as discussed in sec. 2, the relevant boundary conditions are $p'_{in} = p'_{out}$ and we can take $p'_{in} = p'_{out} = 0$ without loss of generality. Thus, Eqs. (5.13) and (5.14) both lead to homogeneous boundary conditions. Using the eigenmode notation $[\mathbf{u}', p'] = [\hat{\mathbf{u}}, \hat{p}]$ for the perturbation, the problem can be symbolically written in the form

$$[\mathcal{LNS}_0^* - i\omega\mathcal{B}][\hat{\mathbf{u}}; \hat{p}] = 0$$

where the operator $\mathcal{LNS}_0^*$ implicitly contains the homogeneous conditions at both upstream and downstream boundaries.

This problem is a generalized eigenvalue one, and thus admits solutions for a discrete set of complex eigenvalues $\omega = \omega_r + i\omega_i$. The flow rate $\hat{q}$ associated to the eigenmodes through 5.15 is generally nonzero, but the eigenmodes can be arbitrarily rescaled such that $\hat{q} = \pi R_h^2$.

Aside from the determinations of the (direct) eigenmodes $[\hat{\mathbf{u}}, \hat{p}]$, it is also useful to study the structure of the adjoint eigenmodes $[\hat{\mathbf{u}}^\dagger, \hat{p}^\dagger]$, namely the eigenmodes of the adjoint operator $\mathcal{LN}\mathcal{S}_0^{*\dagger}$. The importance of adjoint eigenmodes in fluid mechanics has been reviewed by ?. Here, since this concept has been largely discussed in a lot of previous papers like Citro et al. (2016b) and Citro et al. (2015a), we prefer to avoid to derive the adjoint problem. We refer to Luchini and Bottaro (2014) for a detailed discussion of the topic.

The structural sensitivity of a hydrodynamic oscillator is also used in the present manuscript to identify the flow region where the mechanism of instability acts. The so-called wavemaker can be spatially localized by inspecting the spatial map obtained from the sensitivity tensor:

$$\mathbf{S}(x, r) = \frac{\hat{\mathbf{u}}^\dagger(x, r) \otimes \hat{\mathbf{u}}(x, r)}{\int_D \hat{\mathbf{u}}^\dagger(x, r) \hat{\mathbf{u}}(x, r) dD} \quad (5.16)$$

where D is the computational domain. This region is responsible for the instability mechanism, i.e. one can think that the direct mode is emanated from the wavemaker region.

5.3.4 Numerical method

The results presented here are obtained with the same numerical code adopted in Fabre et al. (2018b). In particular, we used the open source code FreeFem++ that implements the finite-element method to solve the several problems described in the present paper. The main originalities of the present implementation are the use of complex mapping in the axial direction to overcome problems associated to the large convective amplification of structures in the downstream direction (see Fabre et al. (2018b)), and the systematic use of mesh adaptation to substantially reduce the required number of d.o.f. (following a methodology described in Fabre et al. (2018c) ; Fabre et al. (2018b)). An example of unstructured grid obtained in this way is displayed in figure 5.3. Note that the downstream dimension L_{out} in numerical coordinates seems rather short; however, as the coordinate mapping used in this case involves a stretching the actual dimension in physical coordinates is much larger (see details in appendix B).

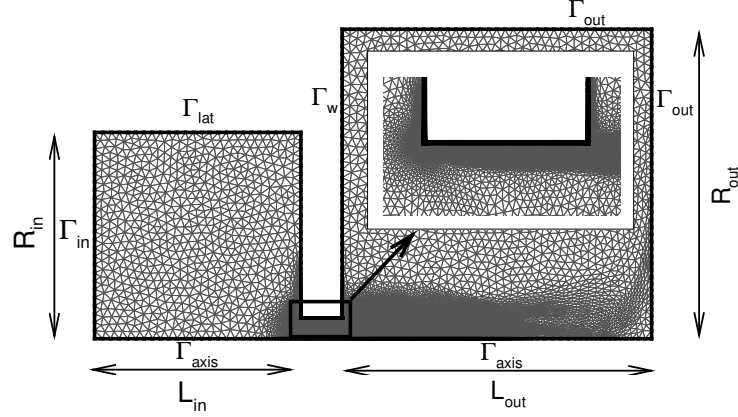


Figure 5.3: Structure of the mesh $\mathbb{M}_1$ obtained using complex mapping and mesh adaptation for $\beta = 1$, and nomenclature of the boundaries (see appendix B for details on mesh generation and validation). A zoom of the mesh is reported in the range $X \in [-2.5; 0.5]R_h$ and $R \in [0.1; 1.8]R_h$.

All the codes and the scripts used in the present manuscript are available online on the github page dedicated to the **StabFem** project (github.com/erbafeDavid/StabFem). As a consequence, most of the result presented herein can be easily obtained by running the inherent script in the StabFem project. On a standard laptop, all the computations discussed below can be obtained in only a few hours. The validation of the solver and of the proposed numerical approach is presented in the review dedicated to StabFem (Fabre et al., 2018c). The complex mapping adopted in the downstream domain is discussed in detail by Fabre et al. (2018b). A brief survey about this topic is also reported in Appendix B.

5.4 Base flows : study of the recirculation region

A typical base flow is depicted in figure 5.4 for a Reynolds number $Re = 1500$ and $\beta = 1$. The flow is characterized by an upstream radially converging flow turning into an almost parallel jet. However, an important feature is the occurrence of a recirculation region within the

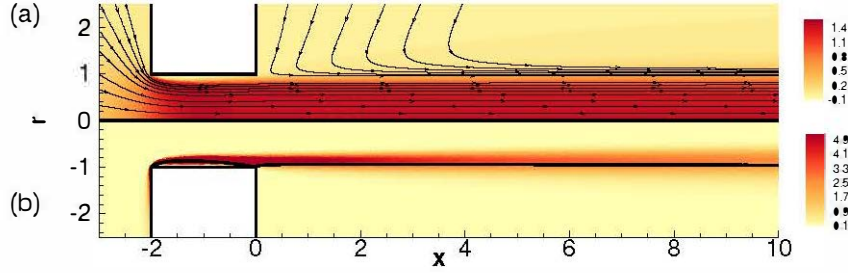


Figure 5.4: Contour plot of (a) axial velocity of the base flow and (b) vorticity field computed at $Re = 1500$ and $\beta = 1$.

thickness of the hole. The vorticity field reaches its maximum near the leading edge, namely the left edge of the hole, and is highly concentrated in the region of maximum shear stress. Figure 5.5 illustrates the structure of the flow in the close vicinity of the aperture, for $\beta = 1$. The recirculation region at $Re = 800$ takes the form of a narrow bubble trapped close to the upstream corner. As the Reynolds is increased, this bubble expands towards the downstream corner, until it opens up and involves an entrainment of the outer fluid which enters inside the thickness of the plate. Note that for $Re = 800$, the recirculation region still contains a bubble of closed streamlines, but detached from the wall. Further on, this bubble disappears and for $Re = 1600$ the recirculation region is fully open.

The intensity of the recirculation region can be characterized by the maximum level of negative velocity within the thickness of the hole, namely $U_{max} = \max(-u_{x0})$. This quantity is plotted in figure 5.6(a) as function of the Reynolds number for $\beta = 0.3, 0.6$ and 1 . It is observed that in all cases, the recirculation region shows up for $Re \approx 400$. The intensity of the recirculation region first grows as the trapped bubble extends to reach the downstream corner, and then decreases as it turns into a fully open one. Not surprisingly, the intensity is larger in the case of a thicker hole, as the bubble is able to extend over a longer region.

The steady flow is characterized by the classical phenomenon of the *vena contracta*, i.e. a reduction in the area/diameter of the jet after it emerges from the circular hole. This process produces a pressure loss

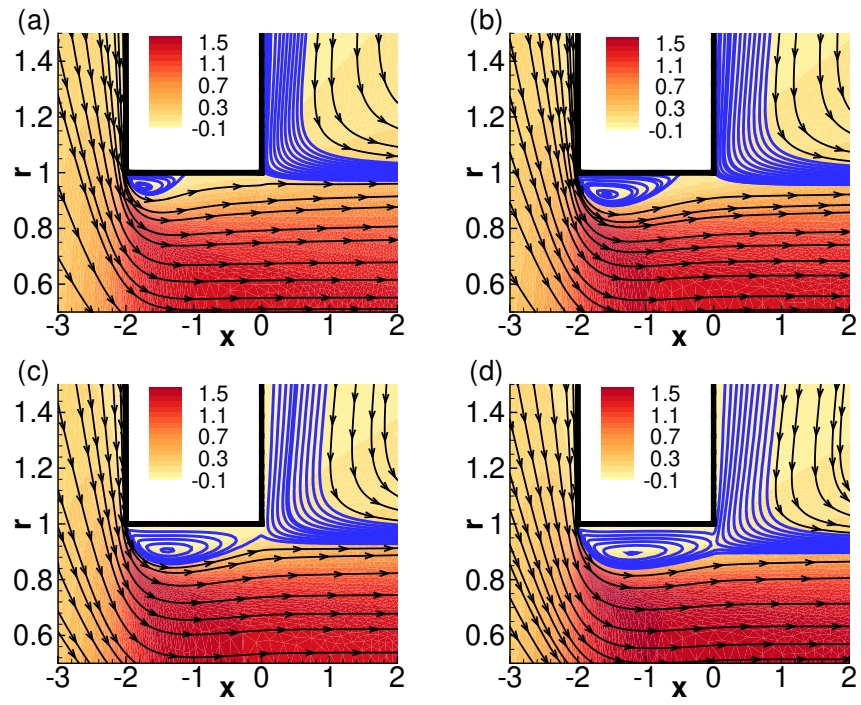


Figure 5.5: Contour plot of the axial component of the base flow at: (a) $Re = 800$, (b) $Re = 1200$, (c) $Re = 1600$, (d) $Re = 2000$. The structure of the recirculation region is highlighted using streamlines. The aspect ratio β is equal to 1.

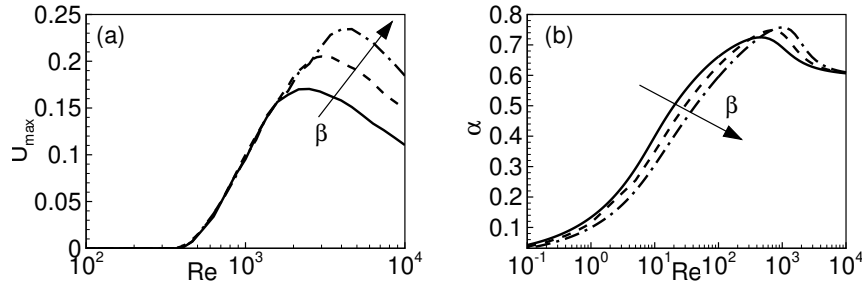


Figure 5.6: (a) Intensity of the recirculation flow inside the hole and (b) vena contracta coefficient as function of Re . Here we use a solid line for $\beta = 0.3$, dashes for $\beta = 0.6$, long dashes for $\beta = 1$.

across the aperture. To estimate this loss, we can apply the Bernoulli theorem along a streamline passing through the hole:

$$P_{in} - P_{out} = \frac{\rho U_J^2}{2} = \frac{\alpha^2 \rho U_M^2}{2}; \quad (5.17)$$

here, the classical *vena contracta* coefficient α is introduced as follows:

$$\alpha = (\pi R_J^2) / (\pi R_h^2), \quad (5.18)$$

where R_J is the jet radius where the flow becomes parallel. There exist several estimations of this coefficient: theoretical prediction by Borda reads $\alpha = 1/2$ while the hodograph method (Gilbarg, 1960) provides $\alpha \approx 0.61$.

We document on figure 5.6(b) the vena contracta coefficient α deduced from the pressure drop computed from the base flows. It is found that for $Re \approx 10^4$ the vena contracta coefficient reaches a value close to 0.61 in all cases, again in accordance with literature results. Note that for the thicker case ($\beta = 1$) α is lower than in the other cases for $Re \lesssim 100$, meaning that the pressure drop is weaker, but it is maximal for $Re \approx 2000$, a value corresponding approximately to the transition from a closed to an open recirculation region.

Note that ? reports that for $\beta = 0.3$ the vena contracta coefficient decreases from 0.70 to 0.61 as Re raises from 10^3 to 10^4 . This is consistent

with our findings. The literature generally attributes this decrease of α to the laminar-turbulent transition. Since our base-flow solution is strictly laminar, we can rule out this argument. It seems more relevant to attribute the decrease of α to the transition between attached and fully detached recirculation region.

5.5 Linear results for the forced problem

We turn now to analyse the results obtained by the numerical resolution of the forced problem. We chose two different cases characterized by $\beta = 0.3$ and $\beta = 1$.

5.5.1 Case $\beta = 0.3$

As previously introduced, the most important quantity associated to the unsteady flow is the impedance $Z = Z_R + iZ_I$. This quantity is plotted as function of the frequency in figure 5.7 for Reynolds ranging from 800 to 2000. The plots in the left column display Z_R and Z_I as function of Ω (note that as Z_I is generally negative and increasing with Ω , it is convenient to plot $-Z_I/\Omega$). The right column display the corresponding Nyquist diagrams.

For $Re = 800$ (plots (a) and (b)), the system presents a small frequency interval near $\Omega \approx 2.2$ with negative values of the real part of the impedance Z_R . As explained in section 5.2.3, this property is directly related to a possible instability. On the other hand, the imaginary part Z_I is always negative in the range of frequencies considered.

As the Reynolds number is increased further, one observes that the region of negative Z_R gets larger and reaches larger values. Note also that the negative, minimum value of Z_R is associated to a maximum of $-Z_I/\Omega$. Increasing the Reynolds number enlarges the range of ω where the system has negative values of Z_R . The cases (e), (g) associated to $Re = 1600, 2000$ show a second region of conditional instability for higher frequencies in the range near $\Omega \approx 8.5$. This is again associated with a maximum of $-Z_I/\Omega$. Note that for Reynolds numbers up to 2000 we do not find a hydrodynamic instability. We recall that the number of unstable modes (absolute instability) is associated to the number of

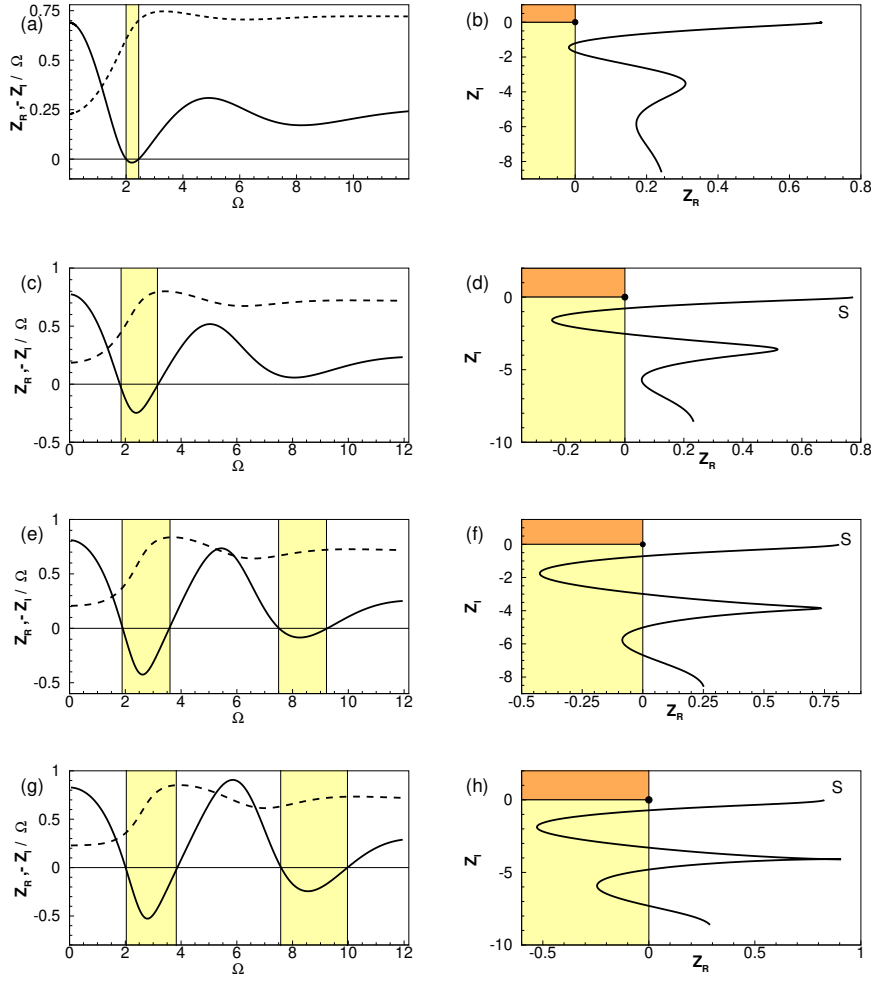


Figure 5.7: Impedance of the flow through a circular aperture with aspect ratio $\beta = 0.3$. Left: Plot of Z_R (solid line) and Z_I (dashed line) as a function of the perturbation frequency Ω ; Right: Nyquist diagrams for (a, b), $Re = 800$, (c, d), $Re = 1200$, (e, f), $Re = 1600$, (g, h), $Re = 2000$.

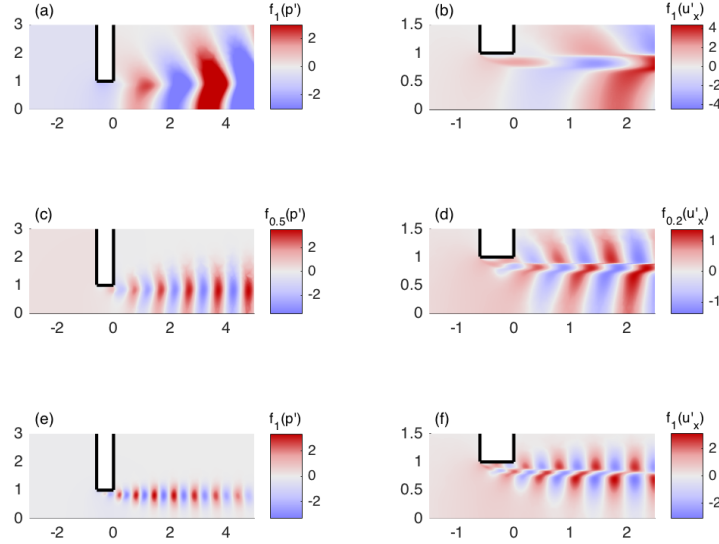


Figure 5.8: Structure of the unsteady flow for $\beta = 0.3$ and $Re = 1600$. Left row: real part of the pressure; right row: real part of the axial velocity. First row (a,b): $\Omega = 2.6$ and $Z_h = -0.425 - 1.719i$, Second row (c,d): $\Omega = 5.45$ and $Z_h = 0.736 - 3.861i$, third row (e,f): $\Omega = 8.25$ and $Z_h = -0.085 - 5.762i$. The color range is rescaled using the distortion function f_S defined in (5.19).

times the contour of the complex impedance Z_h encircles the origin. This condition is never satisfied in figure 5.7.

To explain these trends, and in particular the possibility for negative Z_R , we now depict in figure 5.8 the structure of the flow perturbation for three values of the frequency, corresponding to the two first minima (5.8a, b and 5.8e, f) and a positive maximum (5.8c, d) of $Z_R(\Omega)$. The plots on in the left column display the pressure component p' , and show that the conditionally unstable cases (5.8a and 5.8e) are associated to negative pressure levels in the upstream region, while they are positive

in the stable case (5.8c). Recalling that the pressure level far away in the downstream region is set to zero, this means that in the unstable cases the fluctuating flow goes against the pressure gradient.

The plots in the right column (5.8b, d, f) display the axial velocity component u'_x . This quantity displays strong gradients in the region of the shear layer, confirming that the perturbed flow actually corresponds to oscillations of this shear layer, as assumed in Howe's model and represented schematically in figure 5.1.

An important feature visible on both the pressure plots and the axial velocity plots is that the number of structures of opposite sign (red/blue patches in the figures) within the thickness of the hole is quantified. In effect, we observe respectively 1, 2 and three structures within the hole. This observation for the three first extrema of Z_R can be generalized as follows : *Minima of Z_R (potentially unstable situations) are associated to odd number of structures within the thickness while maxima of Z_R (most stable situations) are associated to even number of structures within the thickness of the hole.* This point was observed for all cases investigated, and will be further demonstrated in next paragraph for $\beta = 1$.

Note that, due to the strongly convective nature of the instability, it is impossible to use a linearly scaled color range to visualize the structures both in the region of the hole (where p' and u'_x are of order one) and in the jet region (where these quantities may reach levels 100 times larger or higher). This feature was already identified for the zero-thickness case by Fabre et al. (2018b). To overcome this difficulty, in figure 5.8 and subsequent figures, we rescale the color range thanks to a distortion function f_S defined as

$$f_s(\zeta) = S \text{sign}(\zeta) \log(1 + |\zeta|/S). \quad (5.19)$$

This function is chosen so that the colorange is linear when the plotted quantity ζ verifies $|\zeta| \ll S$ and turns to logarithmic when $|\zeta| \gg S$. In the plots the value of S is adjusted on order to allow the best visualization.

5.5.2 Case $\beta = 1$

We now consider the case of a thicker hole with aspect ratio $\beta = 1$. Figure 5.9 plots the impedance for Re from 800 to 2000. As in the

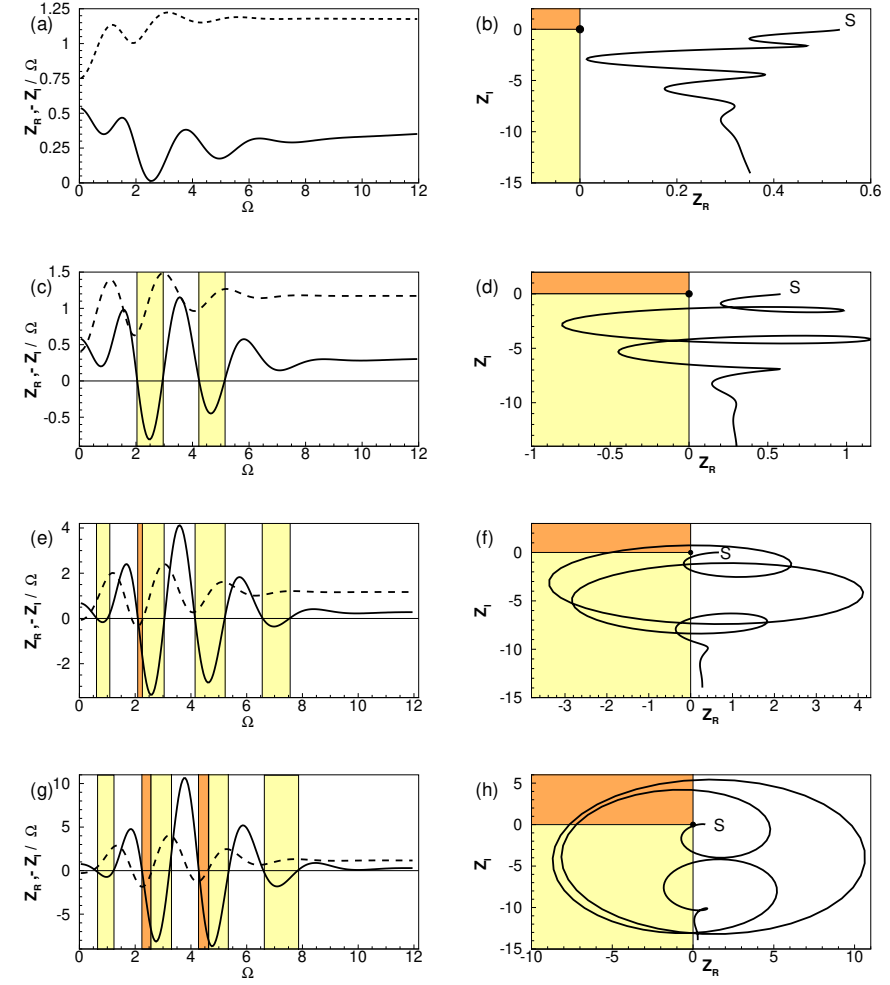


Figure 5.9: Impedance results for $\beta = 1$. Left: Plot of Z_R (solid line) and Z_I (dashed line) as a function of the perturbation frequency Ω ; Right: Nyquist diagrams for (a,b), $Re = 800$, (c,d), $Re = 1200$, (e,f), $Re = 1600$, (g,h), $Re = 2000$.

previous case detailed in sec. 5.5.1, one can see the existence of several frequency intervals where Z_R becomes negative.

The real and imaginary part of the impedance Z_h are always positive for $Re = 800$ (see fig. 5.9a). As a consequence, the associated Nyquist curve plotted in fig. 5.9b does not cross the $Z_R = 0$ axis. The system displays two intervals of conditional instability at $Re = 1200$, around $\Omega \approx 2.5$ and $\Omega \approx 4.7$, respectively. Note that the real part Z_R presents larger oscillations than in the corresponding case at $\beta = 0.3$.

When the Reynolds number is increased, both real and imaginary parts of the impedance reach very large values. Figure 5.9e plots Z_R and $-Z_I/\Omega$ for $Re = 1600$ and reveals four intervals of conditional instability and one interval of hydrodynamical instability. Another important result which can be seen in this figure is the existence of true zeros of the impedance. This happens in particular at $\Omega \approx 2.07$. This property reveals the existence of a purely hydrodynamical instability as discussed in sec. 3. This point will be further confirmed in sec. 5.6. Further increasing the Reynolds number to $Re = 2000$ produces a second interval of hydrodynamical instability around $\Omega = 4.4$.

Figure 5.10 depicts the structure of the oscillating flows for five values of ω corresponding to three minima and two maxima of Z_R . Inspection of these plots allows to confirm the observations made in the previous paragraph for $\beta = 0.3$. First, the pressure level in the upstream region is positive (resp. negative) for the conditionally unstable cases where Z_R is minimum (resp. for the most stable cases where Z_R is maximum). Secondly, as can be seen especially in the axial velocity plots on the right column, the number of structures (patches of alternating colors) within the thickness of the hole is respectively 1,2,3,4,5 for the cases plotted here. This fully confirms the rule enunciated in the previous paragraph, namely that the conditionally unstable cases correspond to an odd number of structures within the hole.

5.5.3 Parametric study

In the previous sections, we documented the impedance results for $\beta = 0.3$ and $\beta = 1$. In both cases, when increasing the Reynolds number, we observed the emergence of an increasing number of intervals of conditional instability, associated with the crossing of the real axis in the

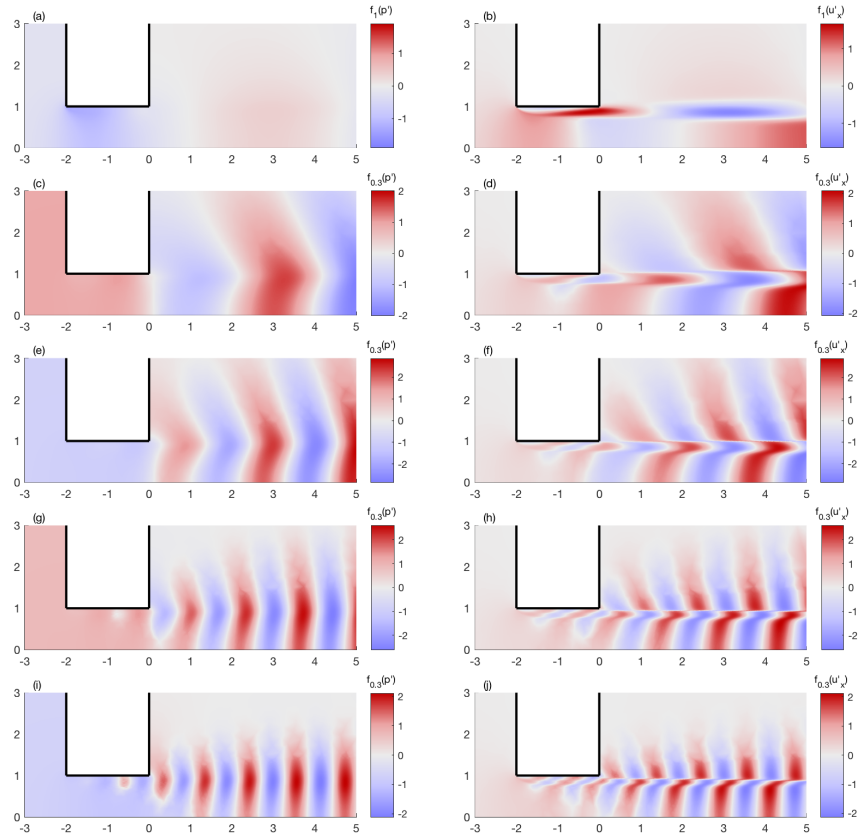


Figure 5.10: Structure of the unsteady flows for $\beta = 1$ and $Re = 1600$. Left: real part of the pressure; right: real part of the axial velocity. First row (a, b) $\Omega = 0.8$; second row (c, d) $\Omega = 1.6$; third row (e, f) $\omega = 2.5$; fourth row (g, h) $\omega = 3.6$; fifth row (i, j) $\omega = 4.5$. The color range is rescaled using the distortion function f_S defined in (5.19).

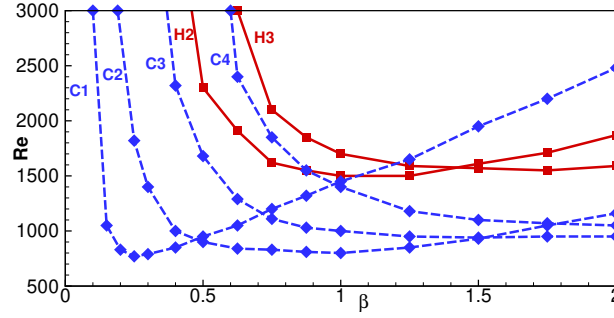


Figure 5.11: Thresholds for the onset of conditional instability ($C1$ to $C4$) and of hydrodynamical instability ($H2$ and $H3$).

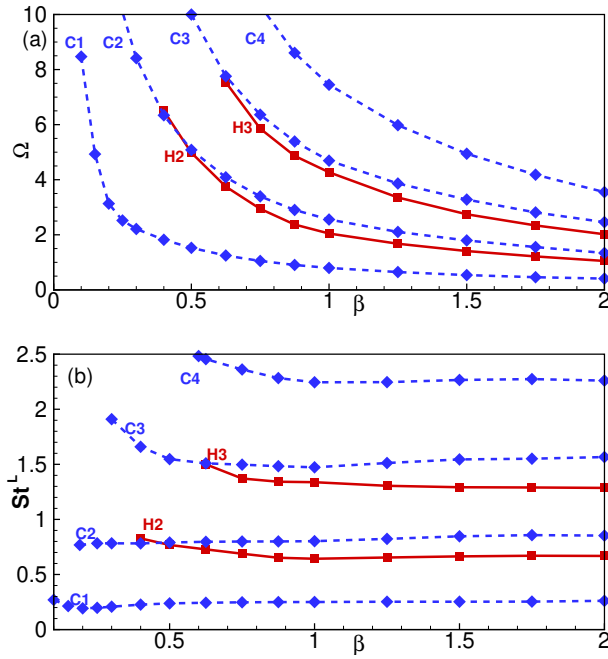


Figure 5.12: Frequencies corresponding to conditional instability ($C1$ to $C4$) and of hydrodynamical instability ($H2$ and $H3$).

Nyquist diagram by successive "loops" of the Nyquist curve. In addition, but only for $\beta = 1$, we observed the emergence of an increasing number of purely hydrodynamical instabilities associated to the encircling of the origin by successive loops of the same curve. In this section, we present the results of a parametric study which allowed to identify the regions of conditional and hydrodynamical instabilities in the range $\beta = [0.1 - 2]$; $Re = [500 - 2000]$.

Figure 5.11 shows the critical Reynolds number associated to each instability branch as a function of the aspect ratio β . In this figure, curves labelled C1 to C4 correspond to the first four branches of conditional instabilities, while branches H2 and H3 correspond to the first two branches of hydrodynamical instabilities. We adopted this labelling because these instabilities are associated to the same "loops" in the Nyquist curve as modes C2 and C3. Note that no crossing of the origin was ever observed along the first loop; this is why the figure does not display any H1 branch.

For short holes, branch C1 is the first to become unstable and branches C2, C3 etc... are only encountered at substantially larger Re . This is compatible with the results of figure 5.7 for $\beta = 0.3$, which indicates that branch C1 becomes unstable slightly below $Re = 800$ and branch C2 between 1200 and 1600. The situation is different for longer holes as branches C2, C3 successively become the most unstable ones. For instance, for $\beta = 1$, conditional instability first happens along branch C2 just above $Re = 800$, and as Re is further increased branch C3, C4 and C1 are then encountered in this order. This is again fully compatible with the Nyquist diagrams of figure 5.9.

Hydrodynamical instabilities generally occur at larger Reynolds than conditional instabilities, and are encountered only for sufficiently thick holes ($\beta > 0.5$). For $\beta = 1$, branch H2 becomes unstable for $Re \approx 1500$ and branch H3 for $Re \approx 1700$. This is again fully compatible with the Nyquist representations in figure 5.9.

We finally notice that for $\beta < 0.1$ no instability is found in the range investigated. This suggests that the limit case of zero thickness is unconditionally stable, in accordance with the classical model of Howe and our previous investigation of this case (Fabre et al., 2018b).

The frequencies associated to the each of the instability branches are plotted in figure 5.12. We start by plotting the Strouhal number based

on the hole radius R_h as a function of the aspect ratio β . Note that the frequencies associated to hydrodynamical instabilities $H2$ and $H3$ closely follows those associated to conditional instabilities $C2$ and $C3$, thus confirming our nomenclature choice.

It is interesting to note that all branches indicate that the frequency is inversely proportional to the aspect ratio of the hole. This suggest that instead of the definition Ω used up to here, it may be better to define a Strouhal number based on the thickness of the hole as follows:

$$St_L = \frac{fL}{U_M} \equiv \frac{\Omega\beta}{\pi}. \quad (5.20)$$

Plotting results using this definition leads to figure 5.12*b*, which confirms that the Strouhal number is almost independant of the aspect ratio for all branches.

The Strouhal number associated to branch C1 corresponds to $St_L \approx 0.25$. This value is in good accordance with observations of the leading whistling frequency of jets in a number of experimental and numerical studies. For instance, Kierkegaard et al. (2012) indicate $St \approx 0.26$, Testud et al. (2009), reports values in the range $[0.2-0.3]$, while ? recorded values in the range $[0.26-0.29]$. This accordance suggests that in all of these works, the instability is of conditional type.

The branch H2 indicates the existence of a purely hydrodynamical instability associated to an almost constant value of the Strouhal number $St_L \approx 0.65$ in the whole range $\beta \in [0.4 - 1.5]$. This implies that a jet through a hole joining two open domains would spontaneously whistle at such frequencies, even in the absence of an acoustic resonator. We are not aware in the recent literature of such an observation, as in all the cited works the hole was fitted at the outlet of a long pipe which played the role of the acoustic resonator needed for conditional instability.

To our knowledge, the only observations of whistling of the flow through a large plate is the work of Bouasse (1929). This author indeed reported that the whistling frequency is proportional to the thole thickness, but unfortunately did not express this result in terms of a Strouhal number.

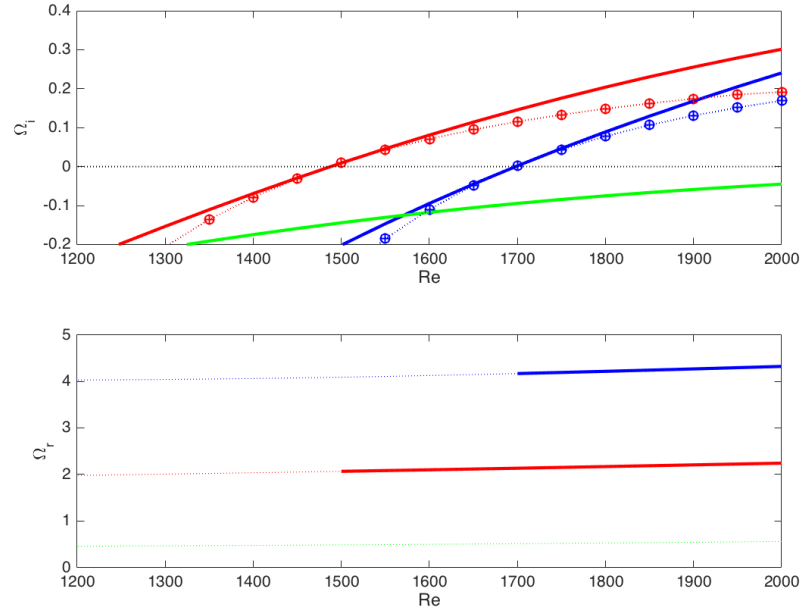


Figure 5.13: (a) Nondimensional growth rates $\Omega_i = (R_h/U_M)\omega_i$ and (b) nondimensional oscillation rates $\Omega_r = (R_h/U_M)\omega_r$ as function of Re , computed through the linear stability approach (lines) and the order-one expansion based on impedance predictions (symbols).

5.6 Linear stability results

The possible existence of a purely hydrodynamical instability, indicated by the impedance results of the previous sections, will now be confirmed through a global stability approach, which consists of solving an autonomous eigenvalue problem arising from the LNSE, as explained in section 3.

5.6.1 Eigenvalues

The stability characteristics of the base flow are assessed monitoring the evolution of the leading global modes. Figure 5.13(a) shows the growth rate ω_i for three least stable modes for $\beta = 1$. Two of them

become unstable in the plotted range of Re . The first branch becomes unstable at $Re \approx 1500$ while the second one presents a critical Reynolds number equal to $Re \approx 1700$. This is fully compatible with the impedance predictions corresponding to branches H2 and H3 discussed in the previous section.

Figure 5.13(b) displays the oscillation rate ω_r for the same three modes. The three branches display an almost constant value of the radius-based Strouhal number Ω . The values for the unstable modes are $\Omega \approx 2.1$ and $\Omega \approx 4.2$, in perfect accordance with the expected values for modes H2 and H3.

Note that figure 5.13(a – b) displays the existence of a third branch of eigenvalue which is always stable. The corresponding frequency is observed for $\Omega \approx 0.5$, which corresponds to a value for which the first "loop" of the Nyquist curve comes close to zero, but does not encircle it. This allows to identify this mode with the "H1" mode which was missing in fig. (5.11). This mode actually exists as a global mode but remains stable for all values of Re and β in the investigated range.

As discussed in section 2, in addition to providing an instability criterion, knowledge of the impedance for real ω also provides an estimation of the eigenvalues associated to the purely hydrodynamical instability valid in the case where ω_i is small. To demonstrate this, we have plotted with symbols in figure 5.13(a) the prediction of the asymptotic formula (5.7). As can be seen, this formula coincides very well with the numerically computed eigenvalues, but deviations are observed as soon as the dimensionless growth rate exceeds a value of about 0.1.

5.6.2 Eigenmodes and adjoint-based sensitivity

We now depict in the upper part of figure 5.14 the structure of the unstable modes computed for $Re = 1500$ and $Re = 1700$, respectively. We display the pressure component (a, e) and the axial velocity component (b, f) using the same representation as for the forced structures in figure 5.10.

The structure of the modes are dominated by axially extended stream-wise velocity disturbances located downstream of the aperture and is indeed very similar to the structures obtained in the linearly forced problem. Note that the levels of the pressure components are now tending to

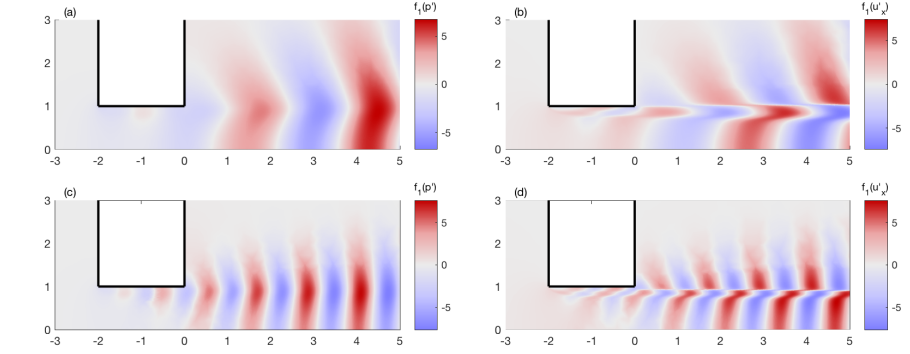


Figure 5.14: Structure of the unstable eigenmodes H2 for $\beta = 1$; $Re = 1500$ (a, b) and H3 for $\beta = 1$; $Re = 1570$ (c, d). Same representation as in figure 5.10.

zero both upstream and downstream, in accordance with the boundary conditions expected for the purely hydrodynamical instabilities. Apart from this, the eigenmode H2 has strong similarities with the structure of the forced mode C2 for $Re = 1600$ (figure 5.10e – f) and the eigenmode H3 with the forced mode C3 (figure 5.10i – j). The spatial wavelengths are slightly larger than the corresponding forced modes, in accordance with the fact that the frequencies are slightly smaller.

Finally, figure 5.15 completes the description of the eigenmodes by a plot of their associated adjoint fields and structural sensitivity. The adjoint modes (plots a, b) show that the region of maximum receptivity to momentum forcing is localized near the leading edge of the hole. The spatial oscillations develop in the upstream region. In striking contrast with the direct mode structure, the receptivity decays rapidly both upstream and downstream of the aperture. The distribution of the adjoint fields are also preserved over the range of Reynolds numbers investigated here.

The sensitivity is displayed by plotting the quantity S_w corresponding to the norm of the structural sensitivity tensor defined by eq. (5.16). The sensitivity for both eigenmodes is essentially localized along the shear layer detaching from the upstream corner of the hole. This confirms that the region responsible for the instability mechanism (the wavemaker region) is the boundary of the recirculation bubble formed within the thickness of the plate.

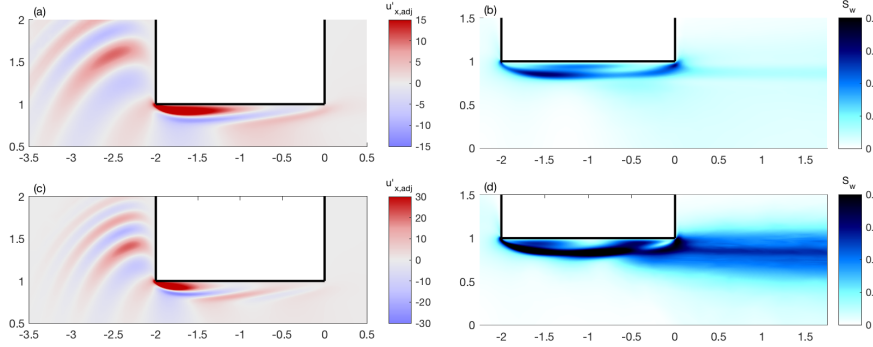


Figure 5.15: Structure of the adjoint eigenmodes (*a, c*) and structural sensitivity fields (*b, d*) associated to the eigenmodes plotted in figure 5.14.

Interestingly, the structural sensitivity also reaches significant levels in a second region located downstream of the aperture, especially for the mode H3. Note that a similar feature was also observed for instabilities of co-flowing jets (Canton et al., 2017). This result indicates that a positive instability feedback enhancing the instability mechanism may also come from the downstream region. This finding may be linked to the role of wavepackets propagating along the shear layer bounding the jet on the emergence of self-sustained oscillations (Schmidt et al., 2017).

5.7 Conclusions and perspectives

In this paper, we investigated the unsteady behaviour of a laminar viscous jet through a circular aperture in a thick plate, using Linearised Navier-Stokes Equations (LNSE). This method allows us to compute the impedance of the flow, which provides useful information on the coupling between the flow and the acoustic waves, and on the prediction of the stability or instability of the system. Impedance calculations allowed us to map the regime of existence of two kind of possible instabilities: (*i*) a conditional instability associated to an overreflection of acoustic waves, and (*ii*) a purely hydrodynamical instability associated to the spontaneous self-oscillation related to sound radiation both upstream and downstream. Both these instabilities can be predicted in a simple

way by plotting the impedance in a Nyquist diagram.

The main outcome of our study is the parametric study of sec. 5.5.3 providing a cartography of the regions of instability as function of the Reynolds number and the aspect ratio β of the hole. The zero-thickness case ($\beta = 0$) is stable in accordance with previous studies. For $\beta \gtrsim 0.1$ we observe conditional instabilities in several frequency intervals, the preferred mode of conditional instability (C1) for short holes corresponds to a Strouhal number $St \approx 0.25$, a value for which experimental observations confirm the existence of an instability mechanism coupling the jet to its acoustic environment. The purely hydrodynamical instability, on the other hand, is observed for longer holes ($\beta \gtrsim 0.5$) and higher Reynolds numbers ($Re \gtrsim 1500$). The preferred mode for $\beta \approx 1$ is associated to a higher value of the Strouhal number, namely $St \approx 0.65$.

In addition to the characterization of both type of instabilities through impedance calculations, we conducted a standard linear stability analysis (based on the computation of eigenvalues) which confirmed the range of existence of the purely hydrodynamical instability and allowed to characterize the spatial structure of the eigenmodes. Downstream of the aperture, the eigenmodes are characterized by a strong spatial amplification due to the convectively unstable nature of the jet. The instability mechanism is better revealed by inspecting the adjoint eigenmodes and the adjoint-based structural sensitivity, which reveal that the core of the instability mechanism lies in the shear layer detaching from the upstream edge of the hole. This observation suggests that the recirculation region existing within the thickness of the hole plays a key role.

By considering a locally incompressible flow and an idealized geometry corresponding to a circular hole with sharp corners connecting two domains of large extension, we have been able to focus on the hydrodynamical aspects of the whistling jet phenomenon, and characterize them without any precise reference of the acoustic environment. However, the study shows that the most powerful instability is the conditional one which requires the presence of an acoustic resonator. We thus plan to continue this study considering more realistic situations involving a resonator. Three configurations are particularly interesting. The first is the case where the upstream domain is a closed cavity acting as a Helmholtz resonator. The second is the case where the hole is fitted at the outlet of a long pipe. This configuration is called the Pfeifenton and has made

the object of investigations in the 1950's (see ?) which have to be reconsidered in view of the present model. The last one is the hole-tone configuration corresponding to a jet passing through two successive holes. NLSE has been recently applied to this case (Longobardi et al., 2018) considering both a fully compressible approach and an "augmented incompressible approach" in which resonators are modelled by equivalent impedances. Such an approach is a promising one for the whole class of problems considered here, and more generally for the study of musical instruments (Fabre et al., 2014b).

Aside from the characterization of the conditional instability in more realistic geometries, future works should be conducted to confirm the existence of the purely hydrodynamical instability in absence of acoustic resonators. To our knowledge, the only reporting of a whistling jet in the case of a hole connecting two open domains of large dimensions is the work by Bouasse in the 1920's. Experiments and numerical simulations should be conducted in this range to confirm our predictions.

Finally, since our study points out the important role of the shear layer formed at the upstream corner of the hole, future experimental and numerical studies should pay special attention to the sharpness of this corner. A preliminary study using LNSE and considering rounded corners indeed reveals that even a very small radius of curvature notably delays the onset of instabilities.

Chapter 6

Studying the sound production in the hole–tone configuration using compressible and incompressible global stability analyses

We study the jet passing through two successive circular holes, also known as hole-tone configuration. Such flow is relevant to many applications like human whistling, wind instruments and tea kettles.

Recently, Fabre et al. (2014c) investigated this flow configuration adopting a global stability approach, showing that the whistling is linked to a purely incompressible instability of the jet between the two holes. In this work, we focus our attention on a little different and more realistic geometry, known as birdcall configuration, consisting into two successive holes in curved thick plates.

Although the whistle is related to compressible phenomena, the incompressible approach can provide some useful information, at least in the region near the hole, where, in some conditions, the flow can be con-

sidered incompressible. We thus initially perform a purely incompressible stability approach. We identify the critical conditions, the global frequencies and discuss the structure of the resulting global eigenmodes. In order to reintroduce and evaluate compressible effects, which can be relevant into the cavity between the two holes, we model the cavity as a Helmholtz resonator and couple it to the incompressible simulation. Finally, a full compressible stability analysis is performed in order to check the accuracy of these simplified approaches in term of critical conditions, global frequencies and structure of the modes.

6.1 Introduction

It is known that the flow passing through two circular holes in thick plates, also known as hole–tone configuration, gives rise to a whistle tone and this situation is encountered in various practical situations, such as human whistling, wind instruments or tea kettles. Such kind of problems attracted the interest of numerous acoustic researches such as Helmholtz (1878), Rayleigh (1896) and Bouasse (1929), which investigated the problem majorly from an acoustic point of view, namely without considering the existence of a mean flow and its dynamics. More recently Henrywood and Agarwal (2013) investigated the hole–tone configuration from an experimental point of view, identifying two regimes: at low velocities the whistle frequency is selected by the cavity between the two holes whereas at high speed regimes the jet dynamics is more relevant in the frequency selection process. Recently, Fabre et al. (2014c) studied this problem by using an incompressible analysis thanks to the assumption of acoustically compact holes: they assumed the wavelength of the sound wave greater than the characteristic length scales of the cavity and holes. In particular, they used the global approach to compute the stability characteristics of the flow system. They found that the frequency selection is triggered by the hydrodynamic regime, although the whistle is related to compressible phenomena.

In this paper we study a more realistic geometry, namely the *birdcall configuration* (more details about the geometry are given in section 6.2). We investigate the whistling properties of this flow configuration using a global stability analysis. In particular, the main objectives of the paper

can be summarized as follows:

- (i) Characterization of the incompressible (hydrodynamic) mechanism; we apply the classical global stability approach to the Navier–Stokes system, showing the existence of various unstable branches.
- (ii) Modeling the effect of compressibility by assuming the cavity as an Helmholtz resonator; we impose a complex spring–like impedance boundary condition on the upper wall of the cavity.
- (iii) Validation of the model by using a full compressible stability analysis.

6.2 Geometry configuration and governing equations

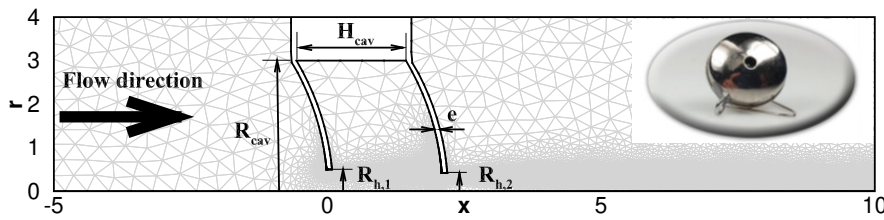


Figure 6.1: Sketch of the birdcall configuration, frame of reference and definition of the main geometrical parameters. An example of computational mesh is also reported in light gray. An example of the real configuration used in this paper is depicted in the upper right corner of the figure.

The birdcall configuration is a more realistic evolution of the classical hole–tone one. It consists in two successive holes in thick curved plates with the two diameters of similar dimensions. Figure 6.1 shows the geometry considered in this paper and an example of the mesh used for the computations. This geometry models a real whistle shown in the upper right corner of figure 6.1. In the actual case, the first hole is greater than the second one, whereas the thickness is considered the same for both the plates. More details about the geometrical parameters of the birdcall

R_{cav}	H_{cav}	$R_{h,1}$	$R_{h,2}$	e
3	2	0.5	0.42	0.1

Table 6.1: Geometrical parameters of the birdcall. The labels are referred to the one reported in figure 6.1. All the quantities are non-dimensionalized using the diameter of the first hole.

are reported in table 6.1. The birdcall connects two open spaces, whose dimensions are taken sufficiently large in order to guarantee domain size independent results. The mean flow moves from left to right driven by a pressure difference and it is constrained to pass through the holes, forming a recirculation region into the cavity and free shear layers into the cavity and past the second hole.

Here, we present the theoretical framework for the compressible Navier–Stokes equations: the incompressible formulation can be retrieved just taking the limit $Ma \rightarrow 0$. In particular, we assumed an ideal gas with a Prandtl number $Pr = \mu c_p / \kappa$ equal to 0.7, where c_p is the constant specific heat, κ is the thermal conductivity and μ is the dynamic viscosity. Moreover, we suppose that the viscosity and the thermal conductivity don't change with the temperature (Yamouni et al., 2013). Under these assumptions, the compressible Navier–Stokes equations can be written as:

$$\left. \begin{aligned} \partial_t \rho + \mathbf{u} \cdot \nabla \rho + \rho \nabla \cdot \mathbf{u} &= 0 \\ \rho \partial_t \mathbf{u} + \rho \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p - \frac{1}{Re} \nabla \cdot \boldsymbol{\tau}(\mathbf{u}) &= \mathbf{0} \\ \rho \partial_t T + \rho \mathbf{u} \cdot \nabla T + (\gamma - 1) \rho T \nabla \cdot \mathbf{u} &= \\ = \gamma(\gamma - 1) \frac{Ma^2}{Re} \boldsymbol{\tau}(\mathbf{u}) : \mathbf{d}(\mathbf{u}) - \frac{\gamma}{Pr Re} \Delta_2 T & \\ \rho T - 1 - \gamma Ma^2 p &= 0 \end{aligned} \right\}, \quad (6.1)$$

where γ is the ratio of specific heats (here equal to 1.4), ρ and T are respectively the density and the temperature, $\mathbf{d}(\mathbf{u}) = \frac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^T)$ is the strain tensor and $\boldsymbol{\tau}(\mathbf{u}) = [2\mathbf{d}(\mathbf{u}) - \frac{2}{3}(\nabla \cdot \mathbf{u})\mathbf{I}]$ is the stress tensor per unit viscosity. The velocity vector is defined as $\mathbf{u}(x, r, t) = (u_x, u_r)$ where x and r represent the axial and radial coordinates whereas u_x and u_r are respectively the axial and radial velocity components. The equations are non dimensionalized using the diameter of the first hole as length scale,

the mean velocity into the first hole U_m as velocity scale and the internal density ρ_{in} and temperature T_{ext} as density and temperature references; the dimensionless pressure, following Fani et al. (2018), is defined as $\frac{p - p_{ext}}{\rho_{in} U_m^2}$. As direct consequence of these choices, the Reynolds number Re and Mach number Ma are defined as:

$$Re = \frac{2R_{h,1}\rho_{in}U_m}{\mu} = \frac{2\dot{m}}{\mu\pi R_{h,1}}, \quad Ma = \frac{U_m}{\sqrt{\gamma RT_{ext}}}$$

where R is the ideal gas constant and $\dot{m}$ the mass flow rate across the first hole. System (6.1) has to be completed by suitable boundary conditions. In particular, we assume no-slip and adiabatic conditions on the solid walls and appropriate conditions on the axis (see Fabre et al. (Under review) for more details). The flow is forced to move through the holes by a pressure jump; thus, we should impose a given pressure on both inlet and outlet. Since the pressure jump is not known a priori, we prefer to impose velocity at the inlet as in Fabre et al. (Under review). In particular, we impose the asymptotical Stokes solution provided by Harrison (1919) with the density equal to its reference value ρ_{in} . The inlet mass flow rate is chosen in order to have a unitary mean velocity into the first hole in the incompressible case; then, we use the same mass flow rate also for the compressible simulation. On the other side, we impose the reference value of the temperature T_{ext} and no stress boundary conditions at the outlet. In this way, the pressure jump across the two holes is automatically provided by the solution of the system (6.1).

6.3 Global Stability Approach

The main hypothesis of this work is that sound emissions are related to self-sustained oscillations caused by an instability of the flow. Here we use the global stability approach to shed light on this mechanism since it is largely applied in literature to explain self-sustained instabilities (Citro et al., 2014) of various flow configurations, such as jets and wakes (Citro et al., 2016a). In order to tackle the problem, we decompose the total flow field into a steady base flow and a time harmonic

perturbation, namely:

$$q(\mathbf{x}, t) = Q_B(\mathbf{x}) + \varepsilon q'(\mathbf{x}) \exp(\sigma t), \quad (6.2)$$

with $\varepsilon \ll 1$. Inserting the ansatz (6.2) into the Navier–Stokes equations and linearizing, we obtain two sets of PDEs; in particular, we find that the base flow is described by the steady state Navier–Stokes equations whereas the perturbation is governed by the Linearized Navier–Stokes Equations (LNSE). Imposing suitable boundary conditions to the LNSE, we are left with a generalized eigenvalue problem. The arising leading complex eigenvalue σ provides important information about the dynamic evolution of the system: if $\Re(\sigma) < 0$ the system is asymptotically stable whereas $\Re(\sigma) > 0$ indicates a system asymptotically unstable. The imaginary part of the eigenvalue, namely $\Im(\sigma) = \omega$, is the frequency of the global mode.

6.3.1 Incompressible analysis

In the limit of $Ma = 0$, the dynamic evolution of the flow is well described by the incompressible Navier–Stokes equations. As mentioned in the introduction, one of our aim is to use the incompressible limit ($Ma = 0$) to characterize the dynamics of the birdcall. In this case, system (6.1) is reduced to the standard incompressible Navier–Stokes equations. As described above, introducing the flow decomposition (6.2) into the governing equations and linearizing, we obtain two problems. The resulting eigenvalue problem can be written as follow:

$$\left. \begin{aligned} \nabla \cdot \mathbf{u}' &= 0 \\ \sigma \mathbf{u}' + (\mathbf{U}_B \cdot \nabla) \mathbf{u}' + (\mathbf{u}' \cdot \nabla) \mathbf{U}_B + \nabla \mathbf{p}' - \frac{1}{Re} \nabla^2 \mathbf{u}' &= 0 \end{aligned} \right\}. \quad (6.3)$$

6.3.2 Modeling the effect of compressibility of the cavity in an "augmented incompressible approach"

The aim of this section is to include the effect of the compressibility by using a simple model coupled to the incompressible equations. In particular, the main hypothesis of this model is that the geometry is acoustically compact, namely the main geometrical parameters of the

birdcall (diameter and distance between the two holes) result to be much smaller than the acoustic wavelength: under this hypothesis, in fact, we can retain that the flow is locally incompressible, leading to a constant value of the pressure inside the cavity (and also of the density since we are in the incompressible regime). If the pressure is constant, we can model the cavity as an Helmholtz resonator (Bonnefis, 2014): in this case we take into account of the compressibility effect imposing a spring-like impedance boundary condition on the upper wall of the cavity rather than a no slip one. The variation of the mass into the cavity can be written, in dimensional form, as (Fry, 2016):

$$\partial_t m_{cav}^d = -\rho^d Q_{cav}^d \quad (6.4)$$

where $m_{cav}^d = \rho^d V_{cav}^d$ and Q_{cav}^d are respectively the mass of the fluid inside the cavity and the flow rate outgoing from the cavity, whereas V_{cav}^d is the volume of the cavity. Note that the superscript d refers to dimensional quantities. For an adiabatic and isoentropic thermodynamical system, pressure and density are linked through the following relation:

$$p_{cav}^d = c_0^{d^2} \rho_{cav}^d, \quad (6.5)$$

where $c_0^{d^2}$ is the speed of sound. Using the isoentropic condition (6.5) in equation (6.4), applying the non dimensionalization of the variables, and using the Fourier transform for the time derivative, the following equation is obtained:

$$\sigma p_{cav} + \frac{1}{\chi_c} Q_{cav} = 0 \quad \text{with} \quad \chi_c = V_{cav} Ma^2. \quad (6.6)$$

The coefficient χ_c can be defined as a compressibility parameter and it is clear that the compressibility effects are influenced both by the Mach number and the volume of the cavity. The unknown terms p_{cav} and Q_{cav} are the augmented variables and they are linked with the incompressible unknown terms through:

$$\left. \begin{aligned} p_{cav} &= \frac{1}{S_{cav}} \int_{S_{cav}} p' dS \\ Q_{cav} &= \int_{S_{cav}} \mathbf{u}' \cdot \mathbf{n} dS \end{aligned} \right\}, \quad (6.7)$$

where S_{cav} is the surface of the upper wall of the cavity. Coupling the equations (6.6) and (6.7) with the incompressible linearized Navier–Stokes system (6.3), a generalized eigenvalue problem is obtained: its solution provides information about the stability of the augmented system with the modeled compressibility.

6.3.3 Full compressible approach

We also use the full compressible stability analysis in order to check the accuracy of the results obtained with the two previously described approaches. The system of equations governing the stability of the compressible flow reads as:

$$\left. \begin{aligned} \sigma \rho' + \mathbf{U} \cdot \nabla \rho' + \mathbf{u}' \cdot \nabla \rho_B + \rho_B \nabla \cdot \mathbf{u}' + \rho' \nabla \cdot \mathbf{U} &= 0 \\ \sigma \rho_B \mathbf{u}' + \rho' \mathbf{U} \cdot \nabla \mathbf{U} + \rho_B \mathbf{u}' \cdot \nabla \mathbf{U} + \rho_B \mathbf{U} \cdot \nabla \mathbf{u}' + \nabla p' - \frac{1}{Re} \nabla \cdot \tau(\mathbf{u}') &= \mathbf{0} \\ \sigma \rho_B T' + \rho' \mathbf{U} \cdot \nabla T_B + \rho_B \mathbf{u}' \cdot \nabla T_B + \rho_B \mathbf{U} \cdot \nabla T' + \\ + (\gamma - 1) (\rho' T_B \nabla \cdot \mathbf{U} + \rho_B T' \nabla \cdot \mathbf{U} + \rho_B T_B \nabla \cdot \mathbf{u}') + \\ - \gamma(\gamma - 1) \frac{Ma^2}{Re} [\tau(\mathbf{u}') : \mathbf{d}(\mathbf{U}) + \tau(\mathbf{U}) : \mathbf{d}(\mathbf{u}')] - \frac{\gamma}{Pr Re} \Delta_2 T' &= 0 \\ \rho' T_B + \rho_B T' - 1 - \gamma Ma^2 p' &= 0 \end{aligned} \right\}. \quad (6.8)$$

6.4 Numerical methods

We use the finite element method implemented in the open source code FreeFem++ (Hecht, 2012) (<http://www.freefem.org/>) in order to solve the various problems of this paper. The unknown terms have been discretized using a triangular unstructured mesh, generated by the built-in Bamg routine. We use classical Taylor–Hood elements ($P2 - P2 - P1$) for the incompressible equations. On the other hand, in the compressible case, we adopt $P2$ elements for the velocity and $P1$ for the other variables, namely pressure, density and temperature. After having obtained the variational formulations of the various problems, matrices of the arising discrete systems have been assembled by FreeFem++ libraries. The nonlinear equations for the base flow have

been solved using a classical Newton method: at each iterative step the matrix inversion has been performed using the parallel MUMPS library. As far as the stability problems are concerned, we first use ARPACK library in order to localize the eigenvalues in the complex plain; then, the leading ones have been followed using the inverse iteration algorithm in order to have cheapest computations. In the compressible computation, in order to avoid the unphysical reflections of the acoustic waves from the inflow and outflow boundaries, we use a sponge zone technique combined with a grid stretching in order to assorbe and dissipate the acoustic waves (for more details see Rowley et al. (Rowley et al., 2002)).

6.5 Results

6.5.1 Incompressible results

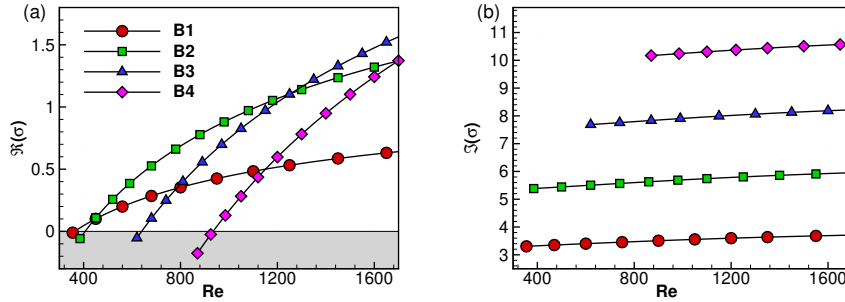


Figure 6.2: (a) Growth Rate and (b) frequency of the most unstable modes as function of Re . The stable region, namely $\Re(\sigma) < 0$ is filled in gray.

In this section, we report results obtained in the incompressible regime. In particular, figure 6.2 shows the growth rates and the frequencies of the most unstable eigenvalues as function of the Reynolds number. We find the existence of four unstable branches quantized in frequency, which is almost constant with the Reynolds number. At low Reynolds numbers, the dynamic is driven by the first branch (B1), with a frequency of $\omega \approx 3.3$; as the Reynolds number increases, the growth rates of the second (B2) and then with the third (B3) branch become dominant

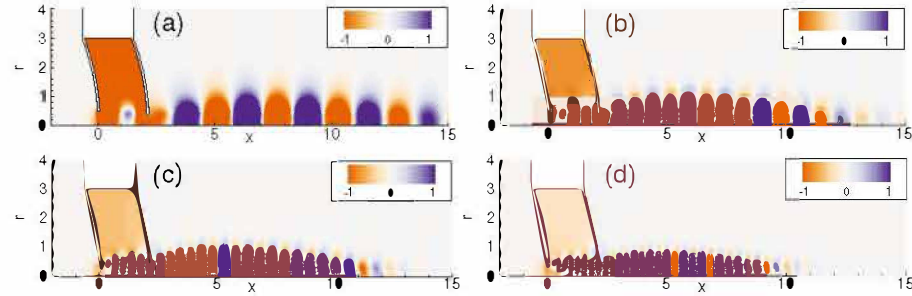


Figure 6.3: Real part of the pressure ($\Re(p')$) for the four unstable branches at the critical Reynolds number: (a) $Re = 363$, $\omega \approx 3.3$; (b) $Re = 406$, $\omega \approx 5.4$; (c) $Re = 639$, $\omega \approx 7.7$; (d) $Re = 934$, $\omega \approx 10.2$.

with frequencies respectively of $\omega \approx 5.5$ and $\omega \approx 7.7$. The fourth unstable branch (B4), on the other hand, is characterized by a frequency of $\omega \approx 10.2$, and never becomes dominant in term of the growth rate respect to the other ones, almost in the range of Reynolds numbers investigated. Figure 6.3 depicts the structure of the pressure for the four unstable branches at their critical Reynolds numbers. First, it is possible to observe that at low frequencies the spatial structure of the global modes extends for a longer distance from the birdcall. Secondly, the four different unstable branches are characterized by very different structures between the two holes. The first branch is characterized by one pressure node between the two holes, the second one by two pressure nodes and so on: this means that there is a direct link between the frequency quantization and the pressure oscillations between the two holes (Matsuura and Nakano, 2012).

6.5.2 Effect of compressibility

Once characterized the incompressible dynamics, we investigate the effect of the compressibility of the flow. In particular, we compare the full compressible results with the one obtained using the Helmholtz resonator augmented model described in section 6.3.2, in order to validate it and discuss its range of validity. Figure 6.4 depicts the growth rates and the frequencies obtained using the augmented model and the full

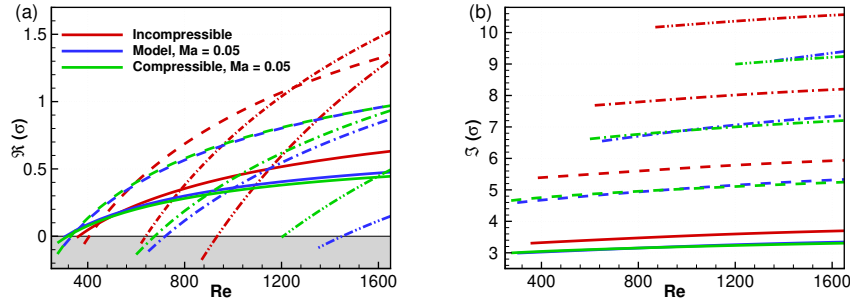


Figure 6.4: (a) Growth Rate and (b) frequency of the most unstable modes as function of Re . The red lines are the incompressible results, the blue lines are the model's results whereas the green lines are the full compressible results, both computed at $Ma = 0.05$. The first branch $B1$ is reported with full lines, the second branch $B2$ with dashed lines, the third branch $B3$ with dash-dot lines and the fourth branch $B4$ with dash-double dot lines. The stable region, namely $\mathfrak{R}(\sigma) < 0$ is filled in gray.

compressible approach at $Ma = 0.05$, within the incompressible results: even if the Mach number is very low, the compressibility seems to have a considerable effect on such kind of flow configuration, as already observed by Yamouni et al. (Yamouni et al., 2013). In particular, we can see that compressibility has a destabilizing effect on the first two unstable branches, in opposition to what happens in the compressible wakes (Meliga et al., 2010). Moreover, compressible effects tends to reduce the frequency of the unstable modes and this effect is as strong as the frequency gets larger. As can be noted from numerical results, the model well predicts both the growth rates and the frequencies of the unstable modes for the lower branches, namely the branch $B1$ and $B2$, whereas it gives less accurate results for the higher ones. In figure 6.5 we report the comparison between the model and compressible results at $Ma = 0.1$. We can observe that we don't find any unstable eigenvalues belonging to the branch $B1$, almost in the range of Reynolds number investigated. Moreover, as far as the other branches goes, it easy to note that the model is not able to predict the results of the full compressible simulations.

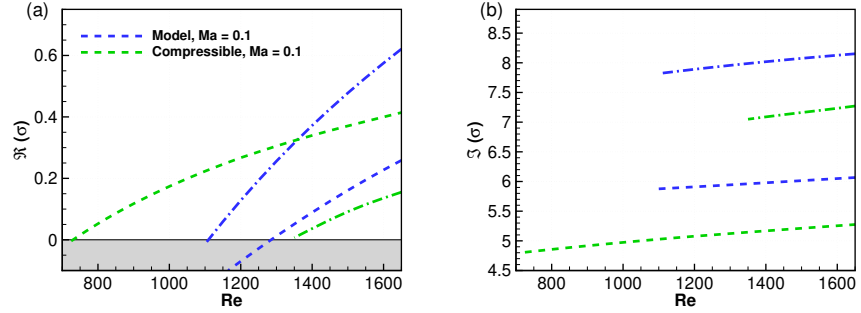


Figure 6.5: Same of figure 6.4 but for $Ma = 0.1$. In these figures we omit the incompressible results.

The prediction capability of the model is strictly related to the acoustic wavelength:

$$\lambda_{ac} = \frac{2\pi}{\omega} \frac{1}{Ma}. \quad (6.9)$$

In fact, the main hypothesis of the model is the local incompressibility of the flow, meaning that the acoustic wavelength must be greater than the characteristic lengthscale of the considered geometry. However, from equation (6.9) it is easy to verify that the acoustic wavelength decreases when the Mach number and the frequency increase and this explains why the model fails at high frequencies and larger Mach numbers. For the geometry considered in this paper, the greater characteristic lengthscale is the diameter of the cavity, namely $D_{cav} = 2R_{cav}$ (see figure 6.1 and table 6.1). Numerical simulations confirm that the model is able to provide accurate results until $\lambda_{ac} > 2D_{cav} = 12$ for the geometry considered here: if such relation does not hold, the acoustic waves are able to penetrate into the cavity and the pressure cannot be considered constant anymore contradicting the hypothesis of the model.

Finally, in figure 6.6, we depict the real part of the pressure of the global modes computed using the compressible equations. In particular, in figure 6.6(a) we report $\Re(p')$ for $Re = 800$, $Ma = 0.05$ and $\omega \approx 4.95$, a case where the model returns good results, as it is possible to verify from figure 6.4. In particular, it is possible to observe that the acoustic waves, propagating into the far field as spherical waves, have a wavelength equal

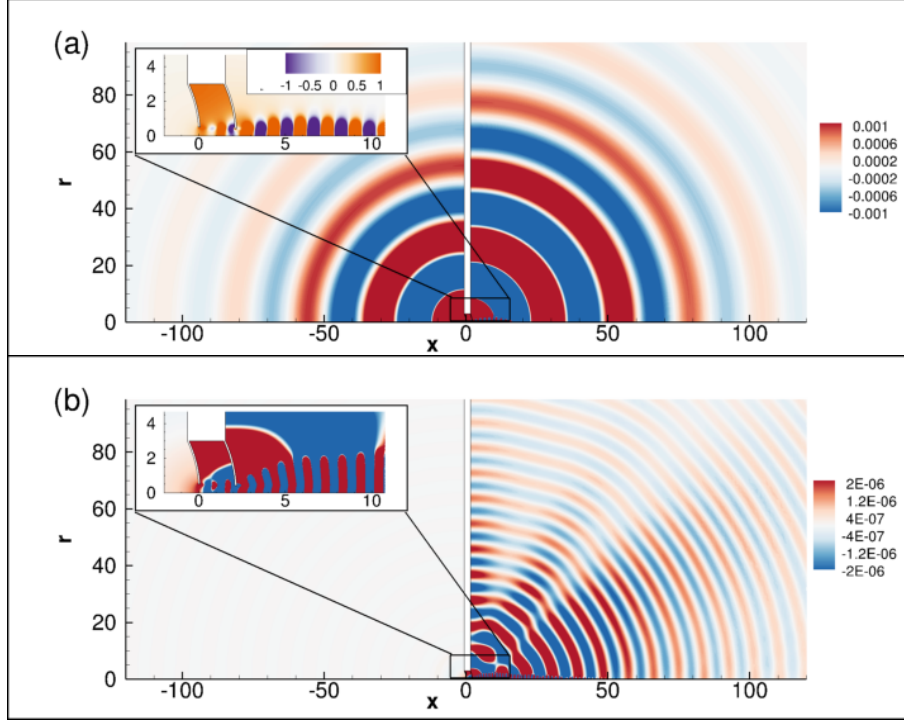


Figure 6.6: Real part of the pressure global modes $\Re(p')$ for: (a) $Re = 800$, $Ma = 0.05$ and $\omega \approx 4.95$; (b) $Re = 1400$, $Ma = 0.1$ and $\omega \approx 7.1$

to $\lambda_{ac} \approx 25$, so that the relation of validity of the model is respected. In figure 6.6(a) it is also shown the zoom of the near field, with a different color scale, showing the same pressure patterns already observed for the incompressible mode and a constant pressure into the cavity. In figure 6.6(b), instead, we report $\Re(p')$ for $Re = 1400$, $Ma = 0.1$ and $\omega \approx 7.1$. In this case, the acoustic wavelength is equal to $\lambda_{ac} \approx 8.85$ and the model is outside its range of validity, as it is possible to verify from figure 6.5. Here, we can observe that pressure is not constant anymore into the cavity. Moreover, also the acoustic directivity seems to change when the acoustic wave is able to penetrate into the cavity, as largely reviewed by Yamouni et al. (2013).

6.5.3 An example of practical application

In this last paragraph we try to set up a real experiment, that is the next step of our research. In the previous paragraph, we varied both the Reynolds and the Mach numbers independently, in order to test the augmented model and define its range of validity. Actually, if we want to simulate a real situation, this is not true, since the Reynolds and Mach numbers result to be linked, as shown by Fabre et al. (Ercoftac Sig. 33, La certosa di Pontignano (Siena). In particular, the Mach number results to be proportional to the Reynolds number:

$$Ma = \frac{U_m^d}{c_0^d} = \frac{\nu^d}{\underbrace{D_{h,1}^d c_0^d}_{K_p}} Re. \quad (6.10)$$

In order to simulate the experiment, we use the measures of a real bird-call depicted in the upper right corner of figure 6.1. In particular, the dimensional diameter of the first hole results to be equal to $D_{h,1}^d = 3mm$ whereas the air temperature is hypothesized to be $T^d = 300K$: the other properties of the air have been obtained from a standard table, leading to a value of $K_p \approx 1.5 \cdot 10^{-5}$. The range of Reynolds number investigated here is $Re \in [300 - 1650]$ leading to a range of Mach number $Ma \in [0.004 - 0.025]$. The use of the model rather than the full compressible simulation is here justified, at least for this specific geometry, by three main statements: first, the higher Mach number is small enough to have a good accuracy of the model, as demonstrated in the previous section; secondly, the full compressible simulation at very low Mach numbers can result very expansive since the acoustic wavelength grows, requiring very big domains with very long sponge zones; finally, the model has a very fast computation respect to the full compressible case, since we don't need sponge regions.

In figure 6.7 we report the comparison between the incompressible results and the one obtained with the augmented model. One can note that for the first two lower branches results are very slowly affected by compressibility, both in term of growth rate and frequency and the incompressible approximation is able to give good results. On the other hand, the effects of the compressibility are larger for the branches three

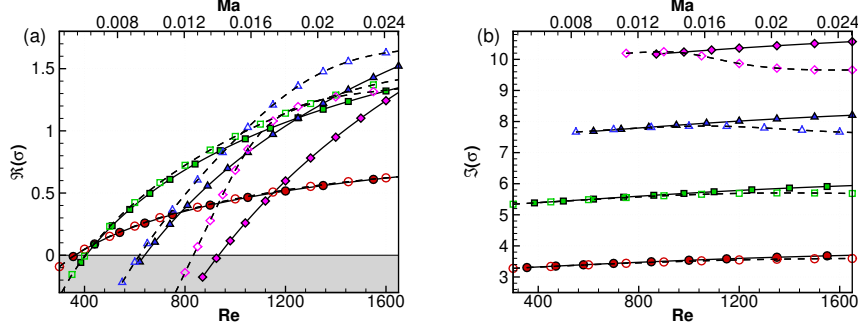


Figure 6.7: (a) Growth Rate and (b) frequency of the most unstable modes as function of Re and Ma . The legend is the same of figure 6.2: the full symbols with solid lines are the incompressible results whereas the empty symbols connected by dashed lines are the compressible augmented ones. The stable region, namely $\mathfrak{R}(\sigma) < 0$ is filled in gray.

and four. However, for all the unstable branches, we can observe two common features: (1) the compressibility has a destabilizing effect for this flow configuration; (2) the frequency is usually smaller than it is in the incompressible case. This effect is more enhanced when the Reynolds (and consequently the Mach number) increases.

6.6 Conclusions

In this paper we investigate the whistling properties of a birdcall using the global stability approach. In particular, we first use a full incompressible approach in order to characterize the dynamic of such kind of flow configuration. We find four unstable branches which are quantized in frequency. The associate pressure field of the leading global modes shows pressure oscillations between the two holes and such pattern is conserved along each branch: in fact the first branch has only one pressure node, the second one is characterized by two pressure nodes and so on.

Once the incompressible dynamic has been characterized, we have considered the effect of the flow compressibility. In particular, we have first modeled the cavity between the two holes as an Helmholtz resonator.

Then, we have performed a full compressible stability analysis in order to test the model and figure out the range of validity of both the incompressible approximation and the augmented model. We have found that the compressibility has an important effect on the stability of such kind of system, in term of both growth rate and frequency: in general, the compressibility reduces the frequency of the global modes, for almost all low Mach numbers investigated.

Numerical results have shown that the model is able to give accurate results only when the acoustic wavelength is greater than two cavity diameters, namely $\lambda_{ac} > 2D_{cav}$, almost for the configuration investigated in this paper.

The last part of the paper, finally, is about a practical application: in fact we have simulated a real experiment, in which the Reynolds and Mach number are proportional through a parameter that is function of the real geometry and air conditions. We have considered the effect of the compressibility using the model since the relation of validity is respected. We have found that, in a real experiment, the incompressible approximation gives good results for low Reynolds (and Mach) numbers and low frequencies, whereas the effect of compressibility is relevant at higher frequencies and Reynolds (and Mach) numbers.

Chapter 7

Concluding remarks

In this final section we summarize the main conclusions of this thesis. However, more detailed conclusions can be found at the end of chapters 3, 4, 5 and 6.

The main topic of this thesis is the application of the linear stability analysis in order to investigate the dynamical properties of complex flow systems. Linear stability theory consists to separate the flow field in a steady part, solution of the steady state Navier–Stokes equations, and an unsteady perturbation that can be considered harmonic. Thus, the perturbation is supposed to be of small amplitude so that the equations governing the perturbation can be linearized, leading to the Linearized Navier–Stokes equations. The study of such kind of equations allowed us to investigate the stability characteristics of different flow systems. Moreover, we introduce also the adjoint equations, an other very powerful tool in stability analysis, that allowed us to investigate the nature of the instabilities.

In order to tackle the various problems encountered in this thesis, we use the open source code FreeFem++ which turned out to be very versatile and adaptable. Moreover, some of the results obtained in chapter 5 have been obtained using StabFem (Fabre et al., 2018c), a powerful and practical interface between FreeFem++ and Matlab (that is more user friendly), developed by David Fabre in collaboration with many other colleagues; moreover, the source codes are available at the following github link: <https://github.com/erbafeidavid/StabFem>.

The first problem investigated in this thesis is the stability of the flow in a micro T-mixer with superhydrophobic surfaces on the outlet channel. The presence of such kind of surface has been taken into account using the Navier slip boundary condition. We demonstrate that the presence of such kind of surfaces has a destabilizing effect on the flow. In particular, the flow encounter a first bidimensional pitchfork bifurcation that drives the base flow from a symmetric steady condition to a new stable asymmetric steady one. Then, increasing the Reynolds number, the flow encounter a three dimensional bifurcation that is steady in case of smooth wall and unsteady when a superhydrophobic surface is considered, leading to an improvement of the mixing properties of such kind of devices.

The Linearized Navier–Stokes equations have been used to study the acoustic and hydrodynamical properties of the flow passing through a circular aperture as well. This situation is encountered in many practical and industrial applications as for example the injectors, the cooling holes of the gas turbine or wind instruments. The arising main flow is characterized by a jet with a diameter shorted than the one of the hole, due to the vena contracta phenomenon, and a recirculation bubble below the hole, when the thickness is enough large. When subject to acoustic forcing, the jet can act both as acoustic dissipator and acoustic amplifier, generating, in the latter case, a strong whistle with a precise frequency. We first consider the simplest situation of the hole in a thin plate subject to harmonic forcing, generated for example by an acoustic field. This is a classical problem already considered by many authors as for example Rayleigh (1896) and Howe (1979) in the inviscid regime. In particular, these authors found that in case of zero thickness hole, the jet acts as an energy sink. the use of the Linearized Navier–Stokes equations, however, allowed us to rigorously introduce the effect of the viscosity. In order to study the acoustic and stability characteristic of such kind of flow, we introduce the concept of acoustic impedance Z , a key quantity defined as the ratio between the pressure jump and the flow rate across the hole. In particular, its real part is a resistance Z_R and it is related to dissipative effects whereas its imaginary part Z_I is linked to inertial effects and so it is a reactance. We calculate the impedance

varying the Reynolds number and the frequency of the forcing term and we found that this flow system is always stable also in the viscous case, although results are quite different from the inviscid ones. However, the numerical resolution of the Linearized Navier–Stokes for such kind of flow is notoriously difficult since they are strongly convectively unstable, above all when the Reynolds number is high. In order to overcome to this problem, we introduce a technique based on the analytical continuation of the axial coordinate in the complex plane, where the strong oscillations of the perturbation are absorbed; this technique allowed us to obtain suitable results up to $Re = 10^5$.

Using the same numerical technique, we investigate the situation of the flow passing through a hole with a non zero thickness. In particular, we identify three main regimes as function of the signs of the real and imaginary part of the impedance. First, if Z_R and Z_I are both positive, the flow is stable. Secondly, the situation in which both Z_R and Z_I are negative is called conditional stability: in fact, although the jet acts as an energy source ($Z_R < 0$), the negative inertial effects don't allow the existence of an instability of the purely hydrodynamical system. However, the system can become unstable if coupled with an outer system able to change the sign of the reactance, as for example an acoustic resonator. Finally, if $Z_R < 0$ and $Z_I > 0$ the purely hydrodynamical system results to be unstable. We conduct a full parametric analysis varying the Reynolds number, the forcing frequency and the ratio between the length and the diameter of the hole β , identifying three conditional stable and two purely hydrodynamical unstable branches. In order to verify this last results, we use the classical global stability approach, finding a very good agreement between the impedance and classical stability results. Finally, we calculate the adjoint fields and the structural sensitivity for the unstable branches: we found that the instability core is located along the shear layer detaching from the upstream corner of the hole.

Finally, in the last part of this work, we investigate the whistling properties of a birdcall, a more realistic geometry constituted by two successive holes in thick plates, using the global stability approach. In particular, we first characterize the incompressible regime in term of unstable branches and spatial structure of the global modes. Then, under the hypothesis of acoustic compactness geometry, we introduce the effect

of compressibility using a simplified augmented model considering the cavity between the two holes a Helmholtz resonator (constant pressure level). Finally, we perform a full compressible stability analysis using the compressible Navier–Stokes equations in order to check the validity of both the incompressible and augmented model. Numerical results have shown that the model is able to give accurate results only when the acoustic wavelength is greater than two cavity diameters, namely $\lambda_{ac} > 2D_{cav}$, almost for the configuration investigated in this paper. However, in the last part we try to simulate a real experiment in which Reynolds and Mach are not independent parameters but they are linked by a precise relation once the real geometry is fixed. We have found that, in a real experiment, the incompressible approximation gives good results for low Reynolds (and Mach) numbers and low frequencies, whereas the effect of compressibility is relevant at higher frequencies and Reynolds (and Mach) numbers.

Chapter 8

Estratto in lingua italiana

8.1 Introduzione

Il presente lavoro di tesi è basato sull'applicazione della teoria della stabilità lineare alle equazioni di Navier–Stokes per spiegare fenomeni atipici in sistemi fluidodinamici complessi. La teoria della stabilità lineare è largamente utilizzata nel campo della fluidodinamica per investigare fenomeni quali la transizione laminare–turbolento dello strato limite, rottura di simmetrie, oscillazioni autosostenute, l'occorrenza di biforcazioni *etc.* Inoltre, usando le proprietà degli autovettori diretti e aggiunti (Luchini and Bottaro, 2014), è stato possibile studiare la sensibilità dei flussi alle instabilità, al fine di localizzare il nucleo dell'instabilità e chiarirne i meccanismi (Giannetti and Luchini, 2007).

I temi affrontati in questa tesi possono essere riassunti come segue:

- Stabilità del flusso in un micro canale a T con superfici superidrofobiche;
- Calcolo dell'impedenza acustica per un flusso oscillante passante per un foro circolare a spessore nullo;
- Impedenza acustica e stabilità del flusso passante per un foro circolare con spessore non nullo;
- Studio del suono emesso da un fischietto per il richiamo di uccelli: confronto tra approccio comprimibile e incomprimibile.

In questo estratto verranno presentati i principali risultati per ognuna delle tematiche affrontate. La teoria generale e le equazioni di governo sono state omesse; tuttavia, quando necessario, saranno fatti alcuni richiami teorici.

8.2 Stabilità e sensitività del flusso in un canale a T con superfici superidrofobiche

La prima problematica affrontata in questa tesi riguarda la stabilità di flussi su superfici superidrofobiche; un famoso esempio di superficie superidrofobica è la foglia di loto. Tali superfici sono costituite da scalanature in cui è imprigionata aria: grazie a questa proprietà l'interazione tra la parete solida e il fluido che vi scorre sopra è ridotta, portando, per esempio, alla riduzione dell'attrito viscoso. Le superfici superidrofobiche sono largamente utilizzate in macro canali turbolenti al fine di ridurre l'attrito mentre, nei confronti della stabilità, esse risultano essere effettive solo in micro-canali (Rothstein, 2010). L'obiettivo principale di questo lavoro è lo studio della stabilità di un flusso in un canale a T dotato di superfici superidrofobiche. In particolare, uno studio parametrico è stato fatto al variare delle proprietà della superficie, ovvero altezza delle scalanature e angolo di orientazione rispetto al gradiente principale di pressione; infine, i risultati sono stati confrontati con quelli ottenuti considerando le superfici del canale lisce.

La superficie superidrofobica può essere rappresentata, dal punto di vista matematico, tramite la condizione di scorrimento parziale teorizzata da Navier (1823); nella sua forma più generale essa è scritta come:

$$U_{\parallel}|_{\Gamma_{shs}} = \Lambda \frac{\partial U_{\parallel}}{\partial \mathbf{n}} \Big|_{\Gamma_{shs}} \quad (8.1)$$

dove $U_{\parallel}$ è il vettore velocità costituito dalle componenti parallele alla parete e Λ è il tensore di slip definito come:

$$\Lambda = \mathbf{R}_{\theta} \begin{bmatrix} \lambda^{\parallel} & 0 \\ 0 & \lambda^{\perp} \end{bmatrix} \mathbf{R}_{\theta}^T, \quad \mathbf{R}_{\theta} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \quad (8.2)$$

dove θ è l'angolo formato tra le scalanature e la direzione del gradiente di pressione nel canale, $\lambda^{\parallel}$ and $\lambda^{\perp}$ sono rispettivamente le lunghezze di slip in direzione parallela e ortogonale al flusso e $\mathbf{R}_{\theta}$ è la matrice di rotazione. Tale condizione è stata applicata sia al calcolo del flusso base che alla perturbazione. In particolare, sono state considerate superfici superidrofobiche anisotropiche, per cui vale la relazione $\lambda^{\parallel} = 2\lambda^{\perp}$.

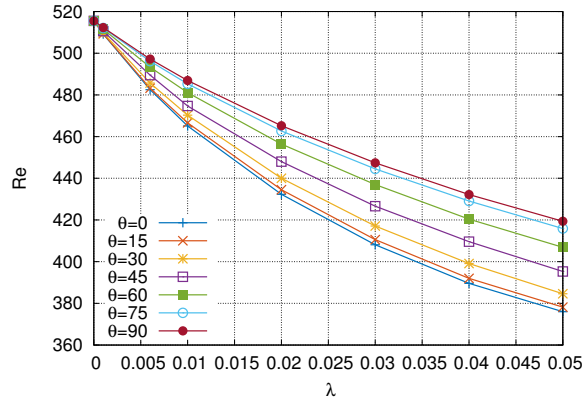


Figure 8.1: Curva di stabilità neutra al variare di λ e θ .

In figura 8.1 è riportata la curva di stabilità neutra del flusso al variare della lunghezza di slip e dell'angolo tra le scanalature ed il gradiente di pressione; in particolare, la condizione di parete liscia si verifica per $\lambda = 0$. La presenza della superficie superidrofobica ha un effetto destabilizzante sul flusso rispetto al caso di parete liscia: infatti il numero di Reynolds critico risulta essere più basso all'aumentare della lunghezza di scorrimento; inoltre, a parità di lunghezza di scorrimento, il flusso risulta essere più instabile per scanalature parallele al gradiente di pressione ($\theta = 0$) piuttosto che per scanalature ortogonali ($\theta = 90$). Infine, la frequenza degli autovalori instabili risulta essere nulla, ovvero vi è una rottura della simmetria del flusso che però trova un'altra condizione di stabilità stazionaria asimmetrica (vedi figure 8.2), senza l'insorgere di oscillazioni, ovvero il flusso attraversa una biforcazione di tipo *pitchfork* supercritica.

Una volta superata la prima biforcazione, il flusso asimmetrico rappresentato in figure 8.2(b) rimane stabile fino a quando non si raggiunge il numero di Reynolds critico della seconda instabilità. Le curve neutre della seconda instabilità sono riportate in figure 8.3. In particolare, il trend generale rimane lo stesso di quello discusso per la prima instabilità. Tuttavia, due differenze sostanziali possono essere osservate in questo caso. La prima è che l'instabilità è tridimensionale, ovvero caratterizzata da un certo numero d'onda trasversale k diverso da zero. Inoltre,

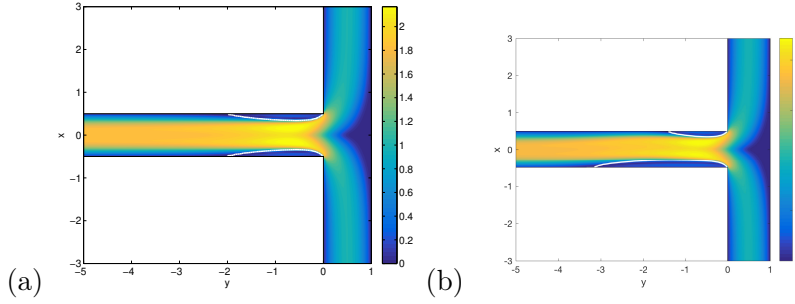


Figure 8.2: Campo di velocità a $Re=500$: (a) flusso stazionario simmetrico calcolato per $\lambda = 0$ e $\theta = 0^\circ$; (b) flusso asimmetrico supercritico ottenuto per $\lambda = 0.02$ e $\theta = 45^\circ$. In entrambe le figure, la linea bianca rappresenta il contorno della bolla di ricircolo.

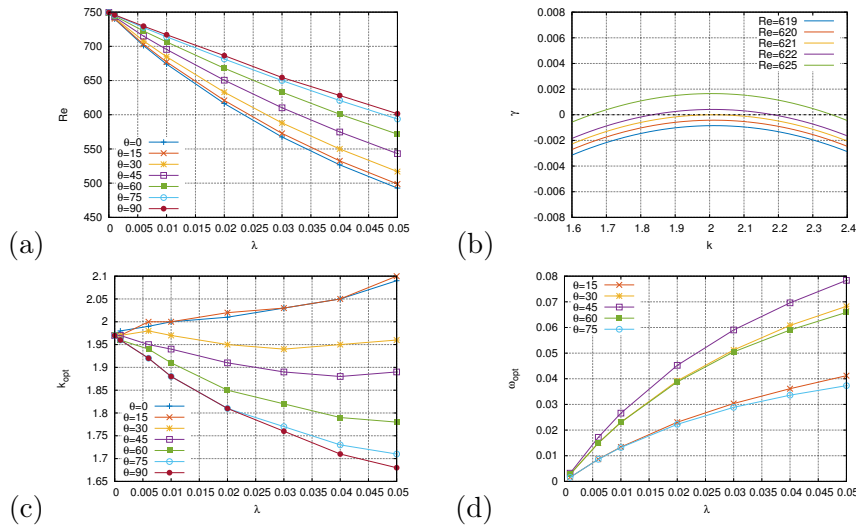


Figure 8.3: (a) Numero di Reynolds critico della seconda instabilità funzione di λ e θ . (b) Tasso di crescita dei disturbi funzione di k and Re per $\lambda = 0.02$ e $\theta = 15^\circ$. (c) Numero d'onda trasversale k_y e (d) frequenza ω_y nelle condizioni critiche funzione di λ and θ . Per $\theta = 0^\circ$ e 90° i modi risultano essere stazionari.

tranne per i casi $\theta = 0, 90$, l'instabilità risulta essere instazionaria: la frequenza aumenta all'aumentare della lunghezza di slip ed è massima per $\theta = 45$. Questo fenomeno è molto importante poichè l'instazionarietà indotta dalla presenza della superficie superidrofobica favorisce il miscelamento del fluido all'interno del canale d'uscita. Infine, sfruttando le proprietà degli autovettori diretti $\mathbf{u}$ e aggiunti $\mathbf{u}^\dagger$, è stata calcolata la sensitività strutturale sia della prima che della seconda instabilità come la norma del seguente tensore:

$$\mathbf{S}(x_0, y_0) = \frac{\mathbf{u}^\dagger(x_0, y_0) \otimes \mathbf{u}(x_0, y_0)}{\iint_{\Omega} (\mathbf{u}^\dagger \cdot \mathbf{u}) d\Omega}, \quad (8.3)$$

dove Ω è l'intero dominio di integrazione.

La mappa spaziale della sensitività strutturale è riportata in figure 8.4 per la prima instabilità (figure (a) e (b)) e per la seconda instabilità (figure (c) e (d)), sia nel caso di pareti lisce (colonna di sinistra) che con pareti dotate di superfici superidrofobiche (colonna di destra). Per quanto riguarda la prima instabilità, è possibile notare che il nucleo dell'instabilità (il massimo) è localizzato sul bordo della bolla ricircolo, mentre per quanto riguarda la seconda instabilità esso è localizzato all'interno della bolla di ricircolo più lunga. Infine, è possibile notare che la presenza della superficie superidrofobica non cambia il meccanismo dell'instabilità.

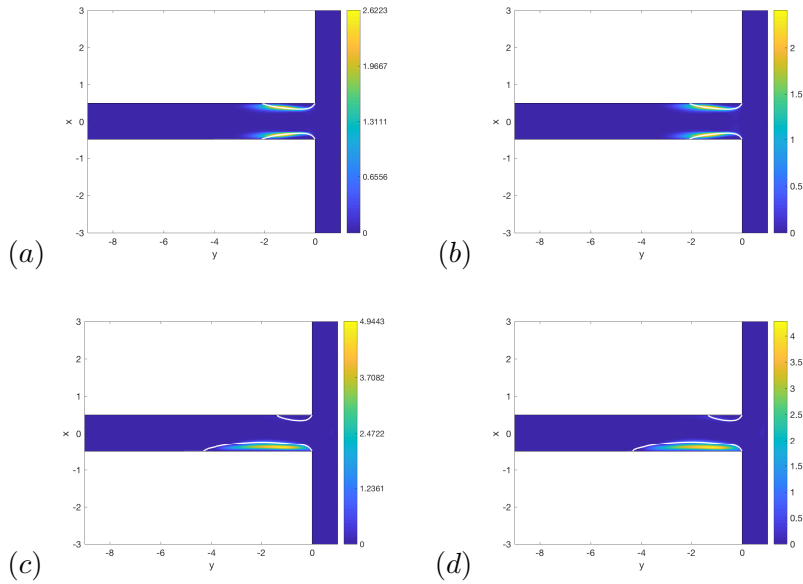


Figure 8.4: Sensitività strutturale per (a) $Re=520$ senza superfici superidrofobiche, (b) $Re=455$, $\lambda = 0.03$, $\theta = 60^\circ$ and $k = 0$, (c) $Re=754$ senza superfici superidrofobiche e (d) $Re=639$, $\lambda = 0.03$, $\theta = 60^\circ$ e $k = 1.7$.

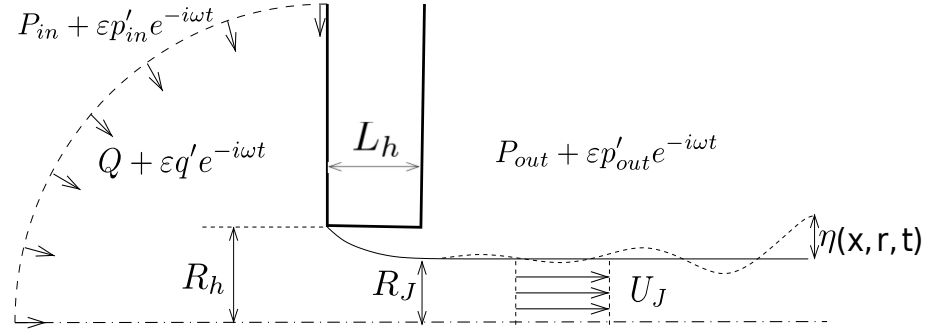


Figure 8.5: Flusso oscillante passante attraverso un foro circolare.

8.3 Calcolo dell' impedenza acustica e stabilità del flusso passante per un foro circolare

Il flusso passante attraverso un foro è una situazione che si incontra in numerosi applicazioni reali, come per esempio i sistemi di raffreddamento delle turbine, gli iniettori o gli strumenti musicali. Esso può talvolta generare un forte e caratteristico fischio ma, tuttavia, le cause e i meccanismi che sono alla base della generazione di tale fischio non sono ancora ben noti.

In questo paragrafo ci proponiamo di caratterizzare le proprietà acustiche e fluidodinamiche di un flusso viscoso passante attraverso un foro circolare soggetto ad un forzamento armonico, per esempio un campo acustico: la configurazione utilizzata è riportata in figura 8.5. E' possibile definire un parametro caratteristico del foro come il rapporto tra la sua lunghezza e il suo diametro, ovvero $\beta = L_h/2R_h$. Il foro mette in comunicazione due spazi semi-infiniti; l'inlet si trova a una certa pressione p_{in} mentre l'outlet si trova ad un livello di pressione differente p_{out} , eventualmente uguale a zero. Considerando il regime armonico, è possibile decomporre il flusso totale in una parte stazionaria ed una parte

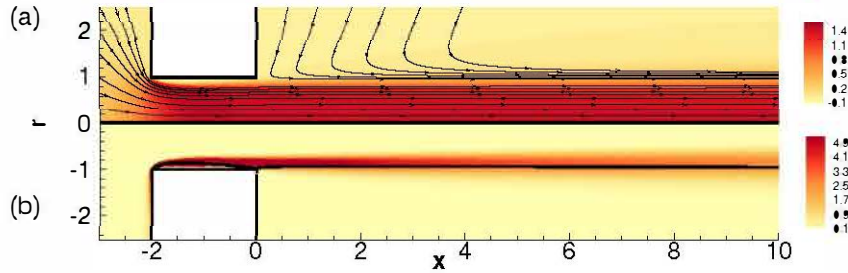


Figure 8.6: (a) Velocità assiale e (b) campo di vorticità del flusso base a $Re = 1500$ e $\beta = 1$.

oscillante, ovvero:

$$\begin{pmatrix} p_{in}(t) \\ p_{out}(t) \\ q(t) \end{pmatrix} = \begin{pmatrix} P_{in} \\ P_{out} \\ Q \end{pmatrix} + \varepsilon \begin{pmatrix} p'_{in} \\ p'_{out} \\ q' \end{pmatrix} e^{-i\omega t} + c.c., \quad (8.4)$$

dove ε è l'ampiezza della perturbazione e $\omega \in \mathbb{R}$ è la frequenza. Il flusso base è caratterizzato da un getto che si stacca dal bordo del foro. Il getto, a sua volta, ha un raggio minore di quello del foro stesso per via del fenomeno della vena contracta. Inoltre, se il parametro β non è nullo, vi è la formazione di una bolla di ricircolo all'interno del foro. Un esempio di soluzione per il flusso base per $Re = 1500$ e $\beta = 1$ è riportata in figura 8.6 in termini di velocità assiale (parte superiore) e vorticità (parte inferiore). E' possibile notare come il flusso, a monte del foro, converga radialmente, per poi formare, alcuni diametri a valle del foro, un getto parallelo all'asse. La vorticità, invece, è tutta concentrata sul bordo della bolla di ricircolo e raggiunge il suo massimo vicino al bordo del foro. La figura 8.7 mostra, invece, la struttura della zona di ricircolo all'aumentare del numero di Reynolds. A $Re = 800$, la zona di ricircolo è piccola e concentrata vicino al bordo sinistro del foro. All'aumentare del numero di Reynolds, invece, essa si allunga fino a raggiungere una lunghezza maggiore del foro stesso, aprendosi verso l'esterno.

Per quanto riguarda, invece, la parte instazionaria, è possibile definire una grandezza chiave, ovvero l'impedenza acustica, come il rapporto tra

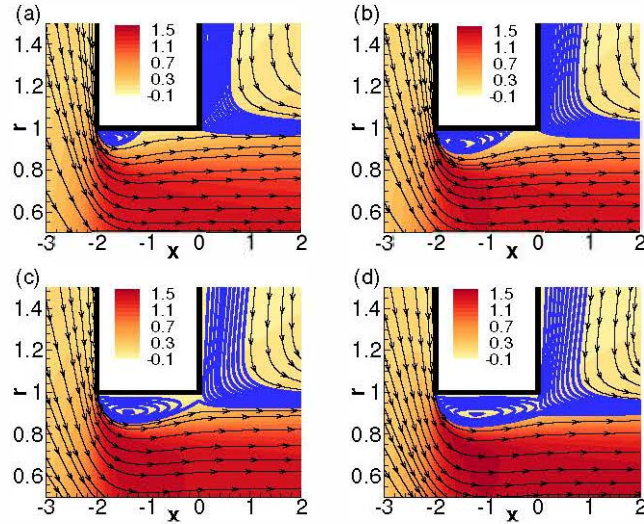


Figure 8.7: Velocità assiale del flusso base a $\beta = 1$ e: (a) $Re = 500$, (b) $Re = 1200$, (c) $Re = 1600$, (d) $Re = 2000$. La struttura della zona di ricircolo è evidenziata dalle linee di corrente.

il salto di pressione e il flusso netto attraverso il foro:

$$Z_h = \frac{p'_{in} - p'_{out}}{q'} \quad (8.5)$$

L'impedenza definita nell'equazione (8.5) è, nel caso generale, una quantità complessa, ovvero $Z_h = Z_R + iZ_I$. La sua parte real Z_R è rappresentata la resistenza mentre la sua parte immaginaria Z_I tiene conto degli effetti di inerzia del sistema ed è quindi una reattanza. E' possibile dimostrare che $Z_R > 0$ è direttamente collegato ad un assorbimento di potenza, ovvero il sistema dinamico si comporta come una resistenza; al contrario $Z_R < 0$ è associato a generazione di potenza. Tuttavia, quest'ultima condizione non è direttamente collegata all'instabilità del sistema stesso (Conciauro and Puglisi, 1981), ma piuttosto significa che il sistema è *attivo*. Un valore negativo della resistenza è condizione necessaria ma non sufficiente per avere un'instabilità del sistema. Questa situazione è anche nota come *stabilità condizionale* poiché l'accoppiamento con un sistema esterno (per esempio un risonatore acustico) potrebbe generare

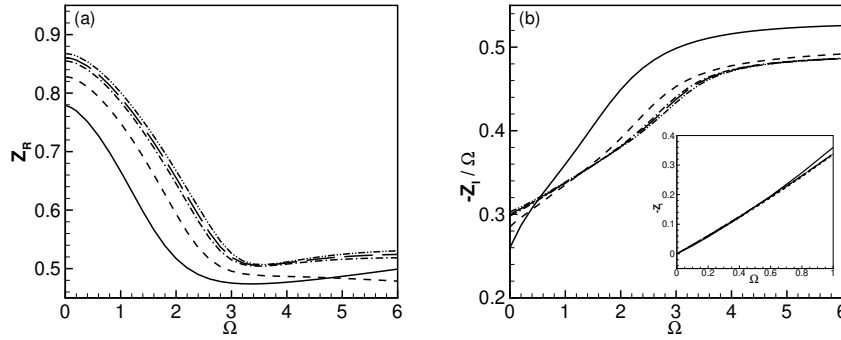


Figure 8.8: (a) Resistenza Z_R e (b) reattanza $-Z_I/\Omega$ calcolate $\beta = 0$ e: $Re = 100$ (—), $Re = 500$ (---), $Re = 1500$ (- · -), $Re = 2000$ (— —) and $Re = 3000$ (- · · -).

instabilità e presenza di oscillazioni autosostenute. Ciò che determina, invece, l'instabilità di un sistema attivo è il segno della reattanza, poichè è collegata agli effetti inerziali. In particolare, per un sistema attivo (ovvero con $Z_R < 0$), se $Z_I > 0$ il sistema risulta instabile, implicando l'esistenza di oscillazioni autosostenute e generazione sonora. Infine, la condizione di stabilità marginale, si ha se $Z_h = 0$.

Il calcolo dell'impedenza si basa sulla risoluzione numerica delle equazioni linearizzate di Navier–Stokes con un forzaggio armonico all'ingresso del dominio. Tuttavia, la soluzione di tali equazioni può creare problemi numerici poichè la perturbazione risulta essere convettivamente molto instabile e il suo valore può facilmente diventare maggiore di 10^{15} (precisione macchina), portando a problemi di arrotondamento e quindi ad una stima sbagliata del salto di pressione. Per ovviare questo problema è stata sviluppata una tecnica basata sulla mappatura nel piano complesso della coordinata assiale: in questo modo le oscillazioni vengono assorbite nel piano complesso, portando quindi ad una corretta valutazione del salto di pressione fino a $Re \approx 10^5$. Per maggiori dettagli e per l'espressione matematica della funzione di mapping si rimanda a Fabre et al. (2018b). Si riportano, in figura 8.8, le impedenze calcolate per il

foro a spessore nullo ($\beta = 0$) al variare del numero di Reynolds. Come è possibile vedere, tale configurazione risulta essere sempre stabile e i risultati tendono a collassare in una sola curva per $Re \gtrsim 1500$.

All'aumentare dello spessore del foro (e quindi di β), invece, emergono situazioni più interessanti. Si riportano, in figura 8.9, le impedenze in funzione della frequenza di oscillazione della perturbazione (colonna di sinistra) ed i relativi diagrammi di Nyquist (colonna di destra). La resistenza e la reattanza per $Re = 800$ risultano essere sempre positive, per cui il sistema è stabile. A $Re = 1200$, invece, vi sono due zone di stabilità condizionata, attorno a $\Omega \approx 2.5$ e $\Omega \approx 4.7$. A $Re = 1600$, invece, il sistema incontra la prima regione di instabilità idrodinamica, attorno ad $\Omega \approx 2.07$. E' possibile osservare, per questo caso, che il rispettivo diagramma di Nyquist ha fatto un giro attorno all'origine degli assi. Infine, per $Re = 2000$, vi sono 4 zone di stabilità condizionata e due di instabilità idrodinamica, per $\Omega \approx 2.2$ e $\Omega \approx 4.5$.

Infine, nelle figure 8.10 e 8.11, si riporta l'analisi parametrica completa al variare di β e Re . In particolare, è possibile individuare 4 rami condizionalmente stabili e 2 rami instabili. Tutte le curve in figura 8.10 presentano un minimo per un certo β che cresce all'aumentare del numero del ramo considerato. Inoltre, è possibile notare che per tutti i β la condizione di stabilità condizionale insorge sempre prima dell'instabilità idrodinamica. In figura 8.11 si riportano le frequenze corrispondenti alle curve riportate in figura 8.10. E' interessante notare che se si varia la lunghezza del foro (figura (c)) piuttosto che il suo raggio (figura (b)) per adimensionalizzare la frequenza, i vari rami sono caratterizzati da frequenze quasi costanti. Ciò significa che le frequenze di oscillazione del sistema sono inversamente proporzionali allo spessore del foro, come già osservato da Bouasse (1929).

Al fine di verificare i risultati ottenuti tramite l'analisi delle impedenze in termini di instabilità idrodinamica, è stata condotta un'analisi di stabilità globale. Si riporta, in figura 8.12, il confronto tra i risultati ottenuti dall'analisi di stabilità globale (linee piene) e dalle impedenze (linee tratteggiate). E' possibile osservare che, per quanto riguarda il tasso di crescita dei disturbi, i due metodi restituiscono esattamente gli stessi risultati al punto neutro ($\sigma = 0$) e divergono invece all'allontanarsi dal punto neutro stesso. Per quanto riguarda le frequenze, invece, i due metodi restituiscono esattamente gli stessi valori.

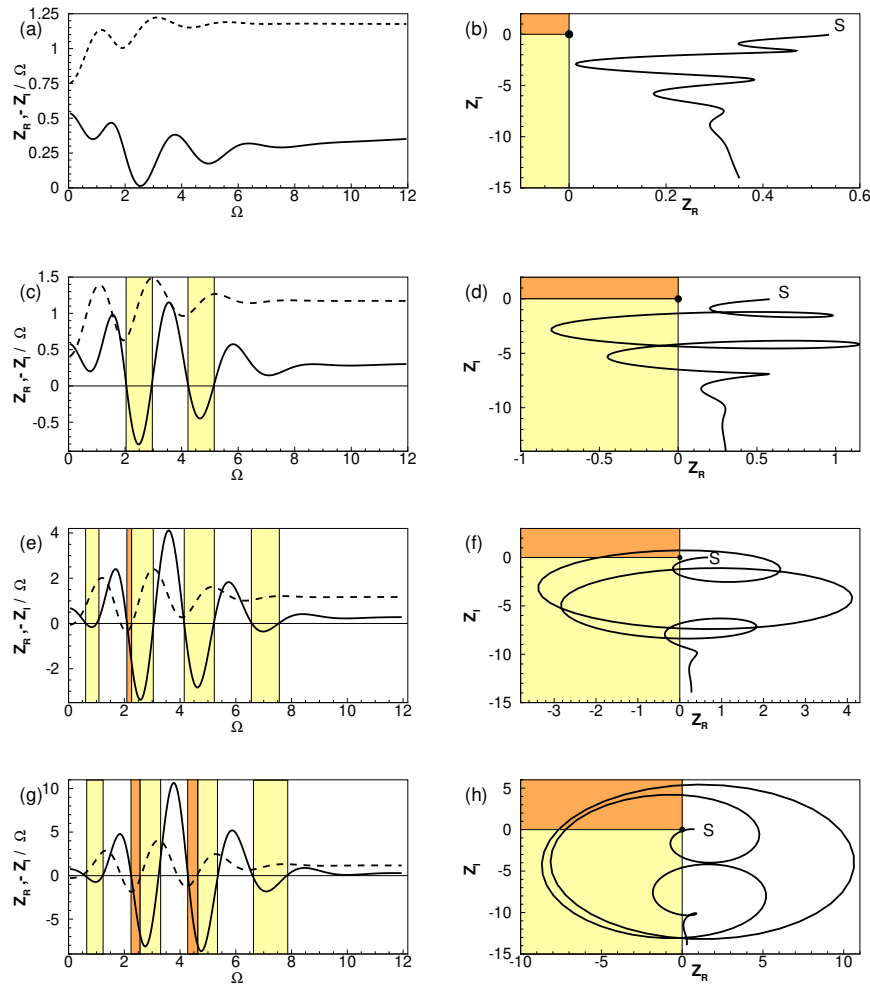


Figure 8.9: Impedenze calcolate a $\beta = 1$. sinistra) resistenza Z_R (linea piena) e reattanza Z_I (linea tratteggiata) funzione della frequenza Ω ; destra) rispettivi diagrammi di Nyquist per (a, b), $Re = 800$, (c, d), $Re = 1200$, (e, f), $Re = 1600$, (g, h), $Re = 2000$.

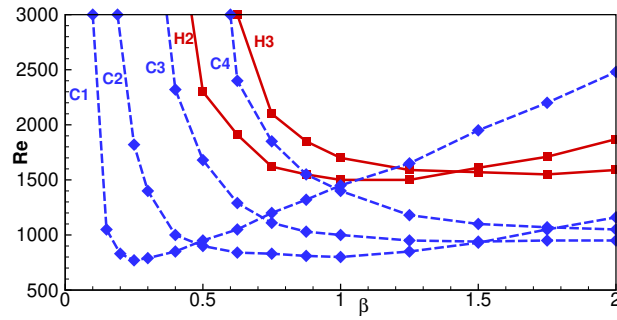


Figure 8.10: Valori di soglia per la stabilità condizionale ($C1$ to $C4$) e l'instabilità idrodinamica ($H2$ and $H3$).

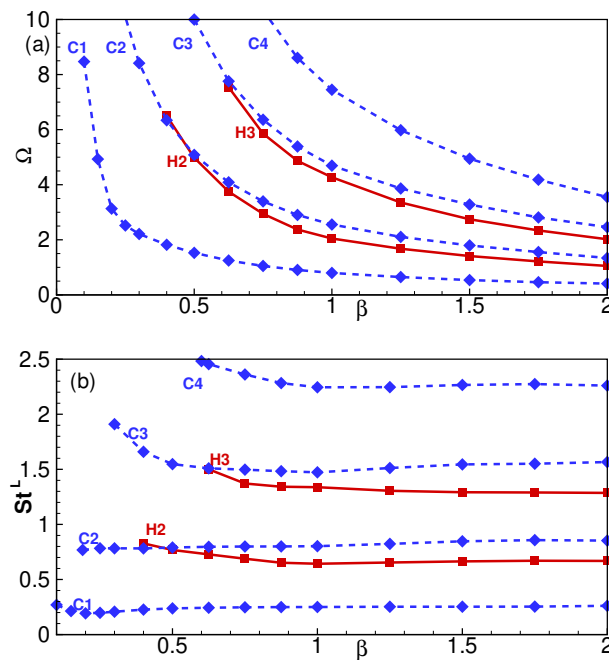


Figure 8.11: Frequenze corrispondenti alla stabilità condizionale ($C1$ to $C4$) ed all'instabilità idrodinamica ($H2$ and $H3$).

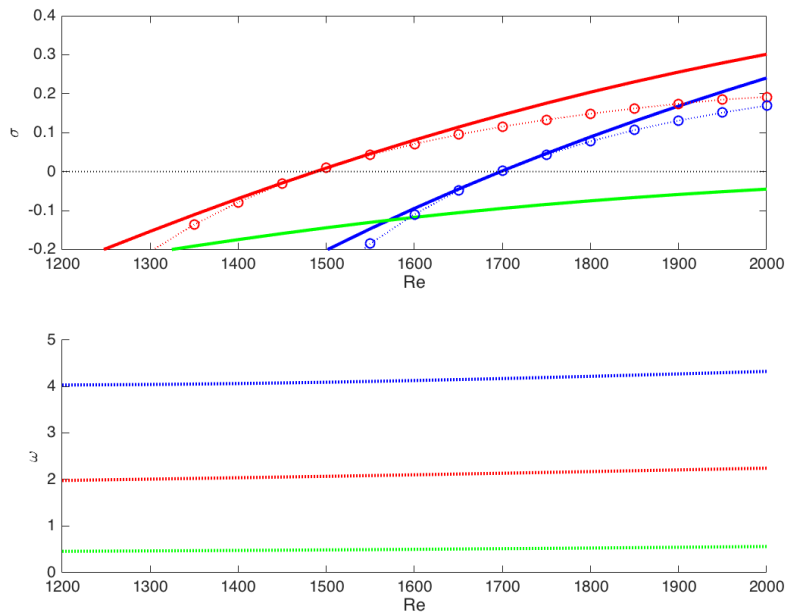


Figure 8.12: (a) Tasso di crescita dei disturbi σ e (b) frequenza ω funzione di Re per il foro a $\beta = 1$. ottenuti dall'analisi di stabilità globale (linee piene) e dall'analisi delle impedenze (linee tratteggiate).

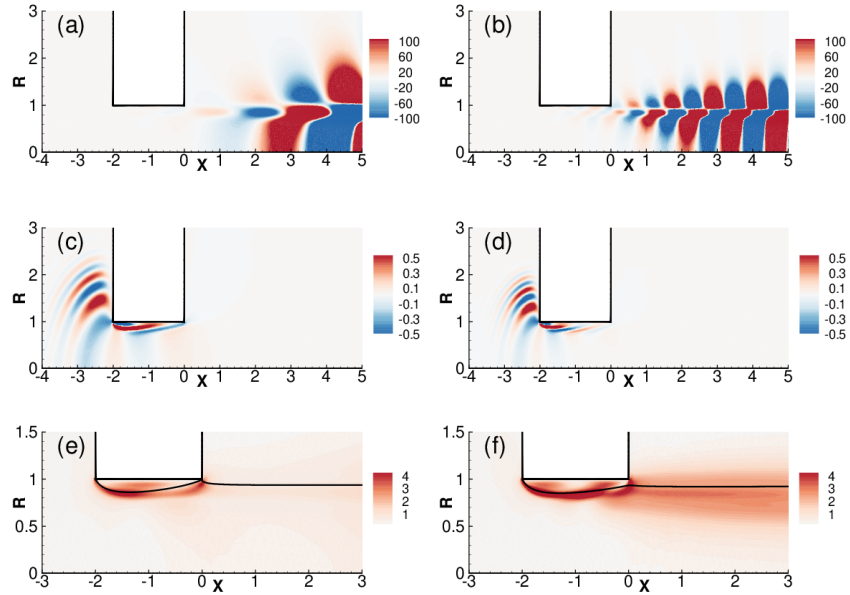


Figure 8.13: Modi diretti (a & b), modi aggiunti (c & d) e sensitività strutturale (e & f) per gli autovalori instabili a $Re = 1500$ (colonna di sinistra) e $Re = 1700$ (colonna di destra). Le linee nere nelle figure (e) ed (f) rappresentano il contorno della bolla di ricircolo.

Infine, si riportano in figura 8.13 i modi diretti (figure (a) e (b)) e i modi aggiunti (figure (c) e (d)) ottenuti per i modi instabili rispettivamente a $Re = 1500$ e $Re = 1700$. Il modo diretto propaga a valle del foro mentre quello aggiunto propaga a monte. In particolare, quest'ultimo ha un massimo vicino al bordo sinistro del foro, che è quindi la zona del flusso più ricettiva ad un forzamento esterno.

Infine, applicando l'equazione (8.3), è stata calcolata la sensitività strutturale per i due casi considerati (figure (e) ed (f)). Per entrambi i casi la sensitività strutturale risulta essere massima sul bordo della bolla di ricircolo, confermando che l'instabilità è strettamente collegata all'esistenza della zona di ricircolo stesso.

8.4 Studio del suono prodotto da un fischietto per il richiamo di uccelli: confronto tra approccio comprimibile ed incompressibile

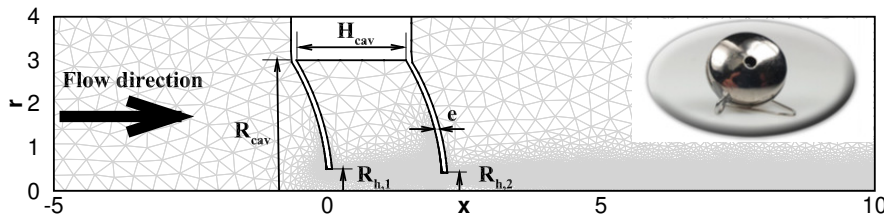


Figure 8.14: Configurazione generale studiata in questo paragrafo e definizione dei parametri geometrici.

L'ultima parte di questa tesi riguarda lo studio di una configurazione più realistica, ovvero un fischietto utilizzato per il richiamo di uccelli, rappresentato in figura 8.14. Il flusso passa attraverso due fori circolari consecutivi situati in pareti curve; tra i due fori vi è una cavità. L'obiettivo principale di questo paragrafo è la comprensione, tramite l'analisi di stabilità globale, del suono prodotto dal fischietto. In particolare, in primo luogo è stato utilizzato un approccio incompressibile per identificare e caratterizzare i vari rami instabili. In secondo luogo, invece, l'effetto della comprimibilità è stato modellato considerando la cavità tra i due fori come un risonatore di Helmholtz, nella condizione di basso numeri di Mach. L'ipotesi principale di questo modello è quella della compattezza acustica della geometria, ovvero che il numero d'onda acustico $\lambda_{ac} = 2\pi/\omega Ma$ deve essere molto maggiore dei parametri geometrici che caratterizzano la geometria stessa, quali diametri o dimensioni della cavità. Sotto quest'ipotesi è possibile considerare la cavità a pressione (e densità costante) e quindi modellarla come un risonatore di Helmholtz: l'effetto della comprimibilità del fluido è incluso sostituendo una condizione al contorno sull'impedenza sulla parete superiore della cavità piuttosto che una classica condizione di non penetrazione (Bonafis, 2014). Considerando, all'interno della cavità, il flusso isoentropico, dopo vari passaggi (vedi Fry (2016)), è possibile scrivere la relazione tra

pressione della cavità e portata volumetrica come segue:

$$\partial_t p_{cav} + \frac{1}{\chi_c} Q_{cav} = 0 \quad \text{with} \quad \chi_c = V_{cav} Ma^2, \quad (8.6)$$

dove il coefficiente χ_c è il parametro di comprimibilità, funzione del numeri di Mach e del volume effettivo della cavità.

La pressione e la portata volumetrica, invece, sono definite come:

$$\left. \begin{aligned} p_{cav} &= \frac{1}{S_{cav}} \int_{S_{cav}} p' dS \\ Q_{cav} &= \int_{S_{cav}} \mathbf{u}' \cdot \mathbf{n} dS \end{aligned} \right\}, \quad (8.7)$$

dove S_{cav} è la superficie laterale della cavità. Le equazioni (8.6) e (8.7) vengono poi accoppiate alle equazioni di Navier–Stokes incomprimibili, a cui poi è applicata la teoria della stabilità globale per il calcolo della stabilità.

Infine, sono state considerate le equazioni di Navier–Stokes comprimibili per discutere la validità del suddetto modello. In figura (8.15) si ripor-

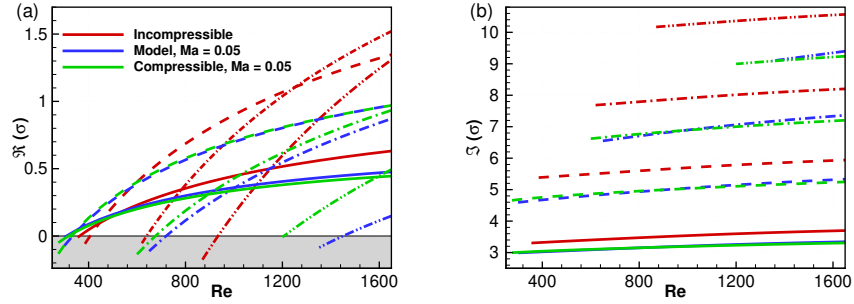


Figure 8.15: (a) Tasso di crescita dei disturbi e (b) frequenza dei modi instabili al variare del numero di Reynolds. Le linee rosse sono i risultati ottenuti usando le equazioni incomprimibili, le linee blu sono i risultati ottenuti utilizzando il modello mentre le linee verdi sono i risultati ottenuti utilizzando le equazioni comprimibili, entrambe per $Ma = 0.05$.

tano i risultati ottenuti per $Ma = 0.05$ insieme ai risultati incomprimibili. Sebbene il numero di mach è relativamente basso, la comprimibilità

sembra avere un effetto notevole su questa configurazione, come già osservato da Yamouni et al. (2013). In particolare, la comprimibilità ha un effetto destabilizzante per i primi due rami instabili, cioè quelli a frequenza $\omega \approx 3$ ed $\omega \approx 4.5$. Inoltre, la comprimibilità tende ad abbassare la frequenza dei modi globali. I risultati ottenuti utilizzando il modello sono, per questo numero di Mach e questa particolare geometria, molto buoni rispetto a quelli ottenuti con le equazioni comprimibili, soprattutto per basse frequenze. In figura 8.16, invece, si riportano i risultati ottenuti

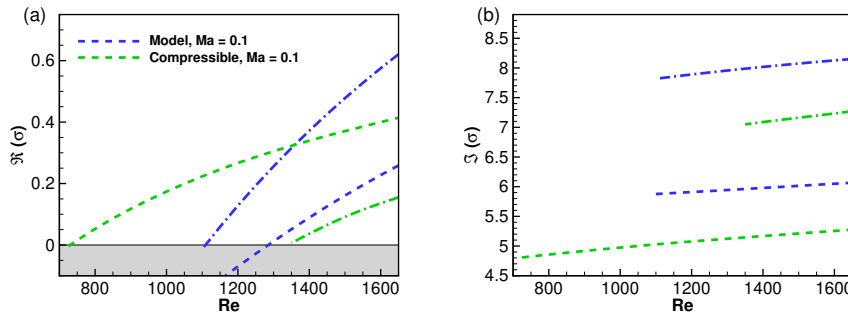


Figure 8.16: Lo stesso della figura 8.15 ma per $Ma = 0.1$. In questo caso i risultati incomprimibili sono stati omessi.

per $Ma = 0.1$. Non si osservano modi instabili appartenenti al primo ramo, perlomeno per i numeri di Reynolds considerati. Inoltre, è facile notare che il modello non è in grado di predire i risultati ottenuti tramite le equazioni comprimibili. L'ipotesi fondamentale del modello è che il numero d'onda acustico λ_{ac} deve essere molto maggiore delle caratteristiche fondamentali della geometria o, più precisamente, della più grande di esse. In questo caso, la dimensione geometria più grande è il diametro della cavità $D_{cav} = 2R_{cav}$. Le simulazioni numeriche mostrano che il modello funziona fin quando $\lambda_{ac} > 2D_{cav} = 12$ per la geometria considerata qui. Se tale relazione non è rispettata, l'onda acustica è in grado di penetrare nella cavità e la pressione non può più essere considerata costante, contraddicendo una delle ipotesi del modello.

Infine, si riportano in figura 6.6 la parte reale della pressione del modo globale ottenuto simulando le equazioni comprimibili; in particolare, in figura 6.6(a) si riporta il modo globale ottenuto per $Re = 800$, $Ma = 0.05$

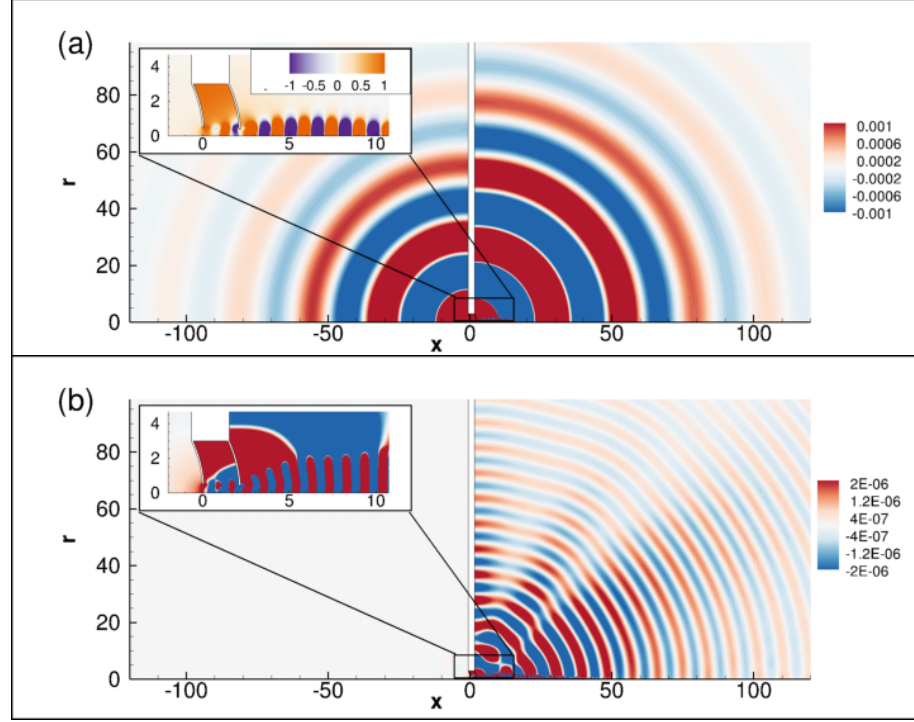


Figure 8.17: RParte reale della pressione del modo globale $\Re(p')$ per: (a) $Re = 800$, $Ma = 0.05$ e $\omega \approx 4.95$; (b) $Re = 1400$, $Ma = 0.1$ e $\omega \approx 7.1$

e $\omega = 4.95$, un caso in cui il modello è in grado di predire i risultati comprimibili: infatti la lunghezza dell'onda acustica, in questo caso, è pari a $\lambda_{ac} \approx 25$: l'ingrandimento riportato nella medesima figura della zona vicino al foro mostra che la pressione risulta essere costante nella cavità. In figura 6.6(b) si riporta, invece, il modo globale ottenuto per $Re = 1400$, $Ma = 0.1$ e $\omega = 7.1$, un caso in cui il modello non è in grado di predire i risultati comprimibili. In questo caso la lunghezza dell'onda acustica è pari a $\lambda_{ac} \approx 8.85$, ovvero fuori dal range di validità del modello. Da un ingrandimento fatto nella zona della cavità si nota che la pressione, in questo caso, non è costante ma si nota una sorta di discontinuità causata dalla penetrazione dell'onda acustica all'interno della cavità.

Chapter 9

Résumé en français

9.1 Introduction

Le travail de thèse que je présente est basé sur l'application de la théorie de la stabilité linéaire à les équations de Navier–Stokes pour expliquer des phénomènes atypiques dans systèmes hydrodynamiques complexes. La théorie de la stabilité linéaire est largement utilisée en matière de dynamique des fluides pour examiner phénomènes comme la transition laminaire–turbulent de la couche limite, rupture de symétrie, oscillations auto-entretenues, l'apparition de bifurcations, *etc...* Par ailleurs, en utilisant les propriétés des vecteurs propres directs et adjoints (Luchini and Bottaro, 2014), la sensibilité des de flux à les instabilités a pu être étudiée. Cela a été fait avec la finalité de localiser le centre de l'instabilité et en clarifier ses mécanismes (Giannetti and Luchini, 2007). Les thèmes traités dans cette thèse peuvent être résumés comme suivant:

- Stabilité du flux dans un microcanal en forme de T avec des surfaces super hydrophobes;
 - Calcul de l'impédance acoustique pour un flux oscillant passant par un trou circulaire avec épaisseur nulle;
 - Impédance acoustique et stabilité du flux passant par un trou circulaire avec épaisseur pas nulle;
 - Étude du son émis par un sifflement pour l'appareil des chasseurs:
-

comparaison entre approche compressible et approche incompressible.

Les principaux résultats par chacune des thèmes abordés vont être présentés dans cet extrait. La théorie générale et les équations gouvernantes ont été omises; toutefois, quand il sera nécessaire, des références théoriques seront fournies.

9.2 Stabilité du flux dans un microcanal T avec des surfaces super hydrophobes

La première problématique abordée dans cette thèse porte sur la stabilité des flux sur surfaces superhydrophobes; un exemple de surface superhydrophobe bien connu c'est la feuille de lotus. Ces surfaces sont constituées par des rainures qui retiennent l'air: grâce à cette composition, l'interaction entre la paroi solide et le fluide qui coule dessus est réduite, en produisant une réduction du frottement visqueux. Les surfaces superhydrophobes sont largement utilisées dans macro canaux afin de réduire le frottement, alors que pour la stabilité elles sont effectives seulement dans les microcanaux (Rothstein, 2010). L'objectif principal de ce travail c'est l'étude de la stabilité d'un flux dans un canal en forme de T avec surfaces superhydrophobes. En particulier, une étude paramétrique a été faite à mesure que les propriétés de la surface changent, c'est-à-dire la hauteur des rainures et l'angle d'orientation par rapport au gradient principal de la pression; finalement, les résultats ont été comparés avec ceux obtenus en considérant des surfaces du canal lisses. La surface superhydrophobe peut être représentée, sur le plan mathématique, avec la condition de glissement partiel théorisé par Navier (1823); dans sa forme la plus générale la condition est écrite comme:

$$U_{\parallel}|_{\Gamma_{shs}} = \Lambda \frac{\partial U_{\parallel}}{\partial \mathbf{n}} \Big|_{\Gamma_{shs}} \quad (9.1)$$

ou $U_{\parallel}$ c'est le vecteur de la vitesse constitué par les composantes parallèles à la paroi et Λ c'est le tenseur de slip, défini comme:

$$\Lambda = \mathbf{R}_{\theta} \begin{bmatrix} \lambda^{\parallel} & 0 \\ 0 & \lambda^{\perp} \end{bmatrix} \mathbf{R}_{\theta}^T, \quad \mathbf{R}_{\theta} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \quad (9.2)$$

ou θ c'est l'angle formé par les rainures et la direction du scalanature e la direction du gradient de pression dans le canal, $\lambda^{\parallel}$ et $\lambda^{\perp}$ sont respectivement les longueurs de slip en direction parallèle et orthogonale au flux et $\mathbf{R}_{\theta}$ c'est la matrice de rotation. Cette condition a été appliquée tant pour la calcul du flux base que pour la perturbation. En particulier, des surfaces superhydrophobes anisotropes ont été considérés,

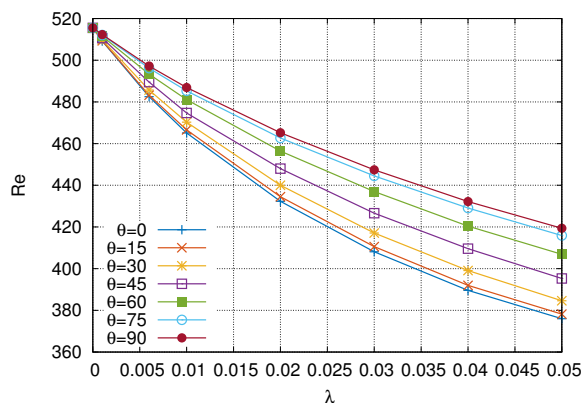


Figure 9.1: Courbe de stabilité neutre en fonction de λ et θ .

pour lesquelles nous appliquons la relation $\lambda^{\parallel} = 2\lambda^{\perp}$.

À la figure 9.1 la courbe de stabilité neutre du flux est reportée en fonction au changement de la longueur de slip et du angle entre les rainures et le gradient de pression; en particulier, la condition de paroi lisse se produit pour $\lambda = 0$. La présence de la surface superhydrophobe déstabilise le flux par rapport au cas de paroi lisse: en fait le nombre de Reynolds critique s'abaisse avec l'augmentation de la longueur de glissement; en outre, à égalité de longueur d'écoulement, le flux s'avère être plus instable pour rainures parallèles au gradient de pression ($\theta = 0$) plutôt que pour rainures orthogonales ($\theta = 90$). Finalement, la fréquence des résultats instables est nulle, c'est à dire que il y a une rupture de la symétrie du flux que, toutefois, trouve une autre condition de stabilité stationnaire asymétrique. (figure 9.2), sans oscillations, à savoir que le flux traverse une bifurcation du type supercritique *pitchfork*.

Une fois dépassée la première bifurcation, le flux asymétrique représenté dans la figure 9.2(b) reste stable jusqu'à quand le nombre de Reynolds critique de la deuxième instabilité est atteint. Les courbes neutres de la deuxième instabilité sont présentées dans les figures 9.3. En particulier, la tendance générale reste la même de cela discutée pour la première instabilité. Toutefois, deux différences substantielles peuvent être observées dans ce cas. La première c'est que l'instabilité est tridimensionnelle, c'est-à-dire caractérisée par un certain nombre de nombre d'onde

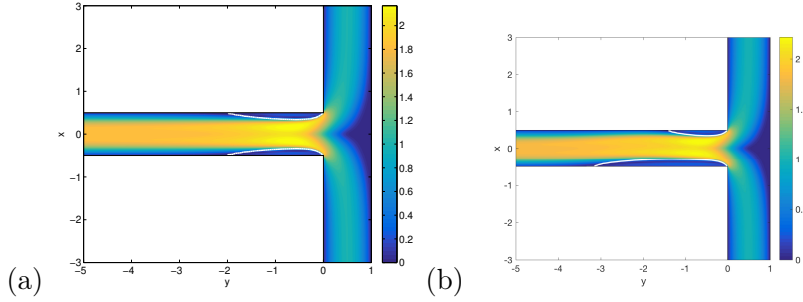


Figure 9.2: Gamme de vitesse $Re=500$: (a) flux stationnaire symétrique calculé pour $\lambda = 0$ et $\theta = 0^\circ$; (b) flux asymétrique supercritique obtenu pour $\lambda = 0.02$ et $\theta = 45^\circ$. Dans les deux figures, la ligne blanche représente le contour de la bulle de recirculation.

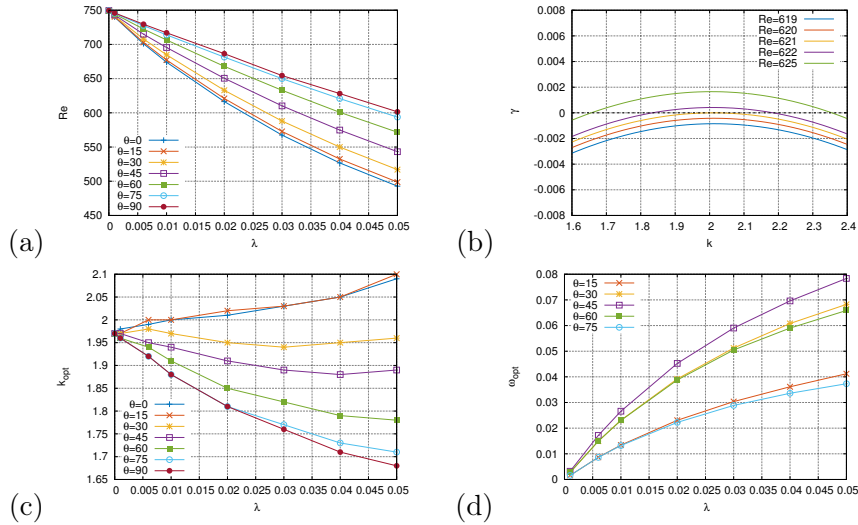


Figure 9.3: (a) Nombre de Reynolds critique de la deuxième instabilité en fonction de λ et θ . (b) Taux de croissance des troubles fonction de k et Re pour $\lambda = 0.02$ et $\theta = 15^\circ$. (c) Nombre d'onde transversale k et (d) fréquence ω dans les conditions critiques fonction de λ et θ . Pour $\theta = 0^\circ$ et 90° les modes sont stabilisés.

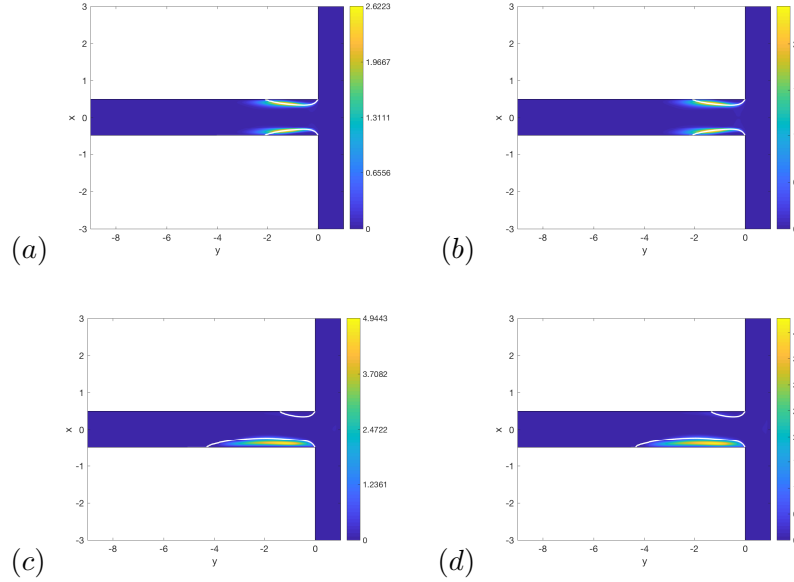


Figure 9.4: Sensitivité structurelle pour (a) $Re=520$ sans surfaces superhydrophobes, (b) $Re=455$, $\lambda = 0.03$, $\theta = 60^\circ$ and $k = 0$, (c) $Re=754$ sans surfaces superhydrophobes e (b) $Re=639$, $\lambda = 0.03$, $\theta = 60^\circ$ e $k = 1.7$.

transversale k différent de zéro. En outre, sauf pour les cas $\theta = 0, 90$, l'instabilité n'est pas stationnaire: la fréquence augmente avec l'augmentation de la longueur de slip et est maximale pour $\theta = 45$. Ce phénomène est très important puisque l'instationarité induite par la présence de la surface superhydrophobe favorise le mélange du fluide dans le canal de sortie. Enfin, en utilisant les propriétés des vecteurs propres directs $\mathbf{u}$ et en adjoint $\mathbf{u}^\dagger$, nous avons calculé la sensibilité structurelle tant de la première que de la deuxième instabilité, comme la norme du tenseur suivant:

$$\mathcal{S}(x_0, y_0) = \frac{\mathbf{u}^\dagger(x_0, y_0) \otimes \mathbf{u}(x_0, y_0)}{\iint_{\Omega} (\mathbf{u}^\dagger \cdot \mathbf{u}) d\Omega}, \quad (9.3)$$

où Ω est l'ensemble du domaine d'intégration.

Le plan spatial de la sensibilité structurelle est indiqué dans les fig-

ures 9.4 pour la première instabilité (figure (a) et (b)) et pour la deuxième instabilité (figure (c) et (d)), aussi bien dans le cas de parois lisses (colonne de gauche) et pour le cas avec parois avec surface superhydrophobe (colonne de droite). Pour ce qui concerne la première instabilité, c'est possible de remarquer que le noyau d'instabilité (le maximum) est localisé au bord de la bulle de recirculation, alors que pour ce qui concerne la deuxième instabilité le noyau est localisé à l'intérieur de la bulle de recirculation la plus longue. Enfin, il est possible de remarquer que la présence de la surface superhydrophobe ne change pas le mécanisme d'instabilité.

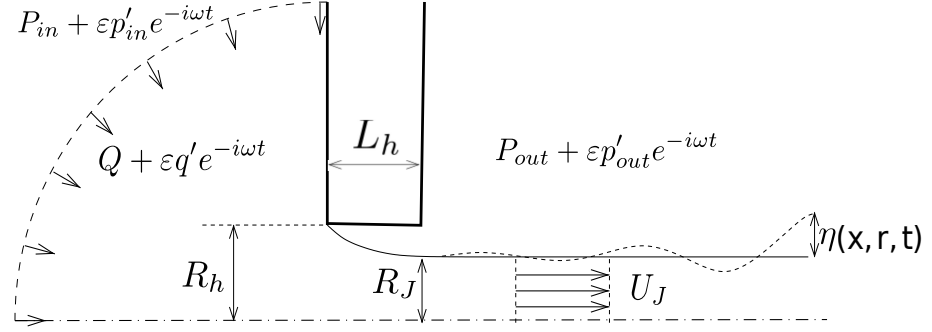


Figure 9.5: Flux oscillant passant par un trou circulier.

9.3 Calcul de l'impédance acustique pour un flux oscillant passant par un trou circulier

Le flux passant à travais d'un trou c'est une situation que peut être rencontrée dans plusieurs applications réels, par exemple les systèmes de refroidissement des turbines, les injecteurs ou les instruments de musique a vent. Le flux peut parfois générer un sifflement fort et caractéristique mais toutefois les causes et les mécanismes qui sont à la base de la génération de ce sifflement ne sont pas encore bien connus.

L'objectif de ce paragraphe est caractériser les propriétés acustiques et hydrodynamiques d'un écoulement visqueux passant pour un trou circulier soumis à un forçement harmonique, par exemple un champ acoustique: la configuration utilisée est montrée dans la figure 9.5. Il est possible de définir un paramètre caractéristique du trou comme le rapport entre sa longueur et son diamètre, c'est-à-dire $\beta = L_h/2R_h$. Le trou crée un lien entre deux espaces semi-infinis; l'inlet se trouve à une certaine pression p_{in} alors que l'outlet se trouve à un niveau de pression différent p_{out} , éventuellement égale à zero. Considérant le régime harmonique, il est possible de décomposer le flux total dans une partie stationnaire et

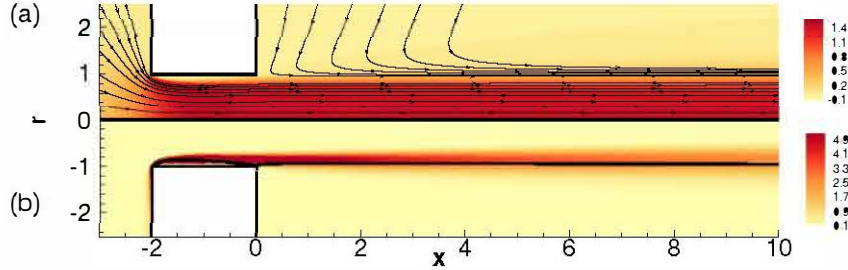


Figure 9.6: (a) Vitesse axiale et (b) champ de vorticit  du flux base   $Re = 1500$ et $\beta = 1$.

une partie oscillante, c'est-  dire:

$$\begin{pmatrix} p_{in}(t) \\ p_{out}(t) \\ q(t) \end{pmatrix} = \begin{pmatrix} P_{in} \\ P_{out} \\ Q \end{pmatrix} + \varepsilon \begin{pmatrix} p'_{in} \\ p'_{out} \\ q' \end{pmatrix} e^{-i\omega t} + c.c., \quad (9.4)$$

ou ε c'est l'ampleur de la perturbation et $\omega \in \mathbb{R}$ est la fr quence. Le champ de base est caract ris  par un jet que se d tache du bord du trou. Le jet,   son tour, a un rayon inf rieur de celui du trou   cause du ph nom ne de la *vena contracta*. En outre, si le param tre β n'est pas nulle, il y a la formation d'une bulle de recirculation   l'int rieur du trou. Un exemple de solution pour le champ de base pour $Re = 1500$ et $\beta = 1$ est montr  dans la figure 9.6 en termes de vitesse axiale (partie sup rieure) et vorticit  (partie inf rieure). Il est possible de r marquer que le flux, en amont du trou, converge radialement, pour former,   une distance de quelques diam tres en aval du trou, un jet parall le   l'axe. La vorticit , par contre, est totalement concentr  sur le bord de la bulle de recirculation et atteint sa valeur maximale pr s du bord du trou. La figure 9.7 montre, par contre, la structure de la zone de r circulation lorsque le nombre de Reynolds augmente. Avec $Re = 800$, la zone de r circulation est petite et concentr e pr s du al bord gauche du trou. Par contre, avec l'augmentation du nombre de Reynolds, la zone s'allonge jusqu'  atteindre une longueur plus grande du trou-m me, en s'ouvrant vers l'ext rieur.

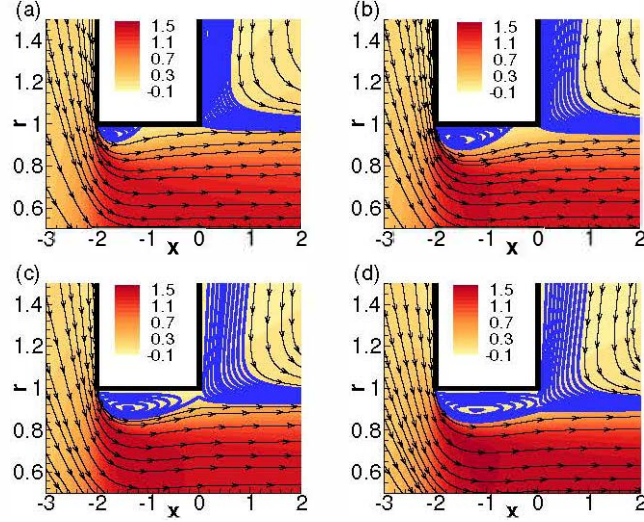


Figure 9.7: Vitesse axiale du flux base à $\beta = 1$ e: (a) $Re = 800$, (b) $Re = 1200$, (c) $Re = 1600$, (d) $Re = 2000$. La structure de la zone de recirculation est surlignée par les lignes de courant.

Pour ce qui concerne la partie non-stationnaire, il est possible de définir une grandeur clé, c'est-à-dire l'impédance acoustique, comme le rapport entre la différence de pression et le flux net par le trou:

$$Z_h = \frac{p'_{in} - p'_{out}}{q'} \quad (9.5)$$

L'impédance définie dans l'équation (9.5) est, dans le cas générale, une quantité complexe, c'est-à-dire $Z_h = Z_R + iZ_I$. Sa partie réelle Z_R est représentée par la résistance alors que la partie imaginaire Z_I tient compte des effets d'inertie du système et donc c'est une réactance. Il est possible de démontrer que $Z_R > 0$ est directement lié à une absorption de puissance, à savoir le système dynamique agit comme une résistance; par contre, $Z_R < 0$ est associé avec une génération de puissance. Toutefois, cette dernière condition n'est pas directement liée avec l'instabilité du système (Conciauro and Puglisi, 1981), mais plutôt ça signifie que le système est *actif*. Une valeur négative de la résistance est une condition nécessaire mais pas suffisant pour avoir une instabilité du système.

Cette situation est connue aussi comme *stabilité conditionnelle*, puis que le couplage avec un système extérieur (par exemple un système acoustique) pourrait générer de l'instabilité avec la présence des oscillations autosoutenues. Ce que détermine, par contre, l'instabilité d'un système actif c'est le signe de la réactance, puis que elle est liée aux effets inertiels. En particulier, pour un système actif (donc avec $Z_R < 0$), si $Z_I > 0$ le système se montre instable, impliquant l'existence des oscillations autosoutenues et génération sonore. Finalement, on a une condition de stabilité marginale si $Z_h = 0$.

Le calcul de l'impédance est basé sur la résolution numérique des équations linéarisées de Navier–Stokes avec un forçage harmonique à l'entrée du domaine. Toutefois, la solution de ces équations peut créer problèmes numériques parce que la perturbation semble être très instable numériquement et sa valeur peut aisément devenir plus de 10^{15} (précision machine), emmenant problèmes de arrondissement et donc à une estimation erronée du saut de pression. Pour remédier à ce problème, une technique basée sur la cartographie dans le plan complexe de la coordonnée axiale a été développée: de cette façon les oscillations sont absorbées dans le plan complexe, en obtenant une évaluation correcte du saut de pression jusqu'à $Re \approx 10^5$. Pour plus de détails sur l'expression mathématique de la fonction de mapping merci de consulter Fabre et al. (2018b). Dans la figure 9.8, ils sont reportées les impédances calculée pour le trou avec épaisseur nulle ($\beta = 0$) proportionnellement au changement du nombre de Reynolds. Comme nous pouvons remarquer, cette configuration s'avère être toujours stable et les résultats tendent à s'effondrer dans une seule courbe pour $Re \gtrsim 1500$.

Avec l'augmentation d'épaisseur du trou (et donc de β), par contre, des résultats plus intéressants émergent. Dans la figure 9.9, il y a reportés les impédances en fonction de la fréquence d'oscillation de la perturbation (colonne de gauche) et les relatifs diagrammes de Nyquist (colonne de droite). La résistance et la réactance pour $Re = 800$ apparaissent comme toujours positifs, donc le système est stable. Avec $Re = 1200$, par contre, il y a deux zones de stabilité conditionnelle, autour de $\Omega \approx 2.5$ et $\Omega \approx 4.7$. Avec $Re = 1600$, en revanche, le système face la première région d'instabilité hydrodynamique pour $\Omega \approx 2.07$. Il est possible de remarquer, pour ce cas, que respectif diagramme de Nyquist a fait un tour autour l'origine des axes. Enfin, pour $Re = 2000$, il y a 4 zones de sta-

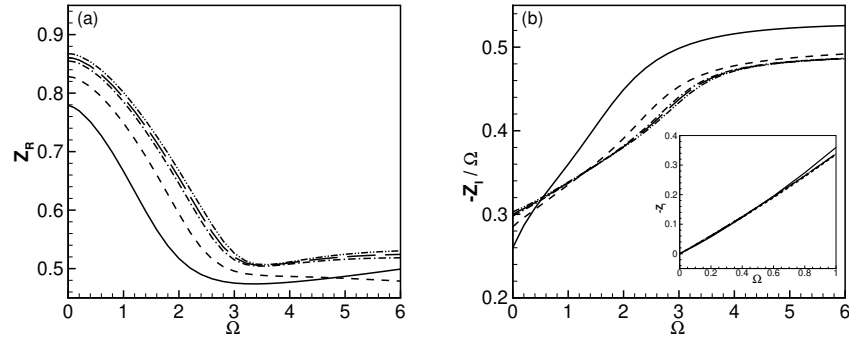


Figure 9.8: (a) Resistance Z_R e (b) reactance $-Z_I/\Omega$ calculées $\beta = 0$ et: $Re = 100$ (—), $Re = 500$ (---), $Re = 1500$ (- · -), $Re = 2000$ (— —) and $Re = 3000$ (- · · -).

bilité conditionnelle e deux de instabilité hydrodynamique, pour $\Omega \approx 2.2$ et $\Omega \approx 4.5$.

Finalement, dans les figures 9.10 et 9.11, on réporte l'analyse paramétrique totale en fonction de la variation de β et Re . En particulier, il est possible d'identifier 4 branches conditionnellement stables et 2 branches instables. Toutes les courbes dans la figure 9.10 présentent un minimum pour un certain β qui grandit avec l'augmentation du nombre du branche considéré. En outre, il est possible de remarquer que pour tous les β la condition de stabilité conditionnelle se lève toujours avant de l'instabilité hydrodynamique. Dans la figure 9.11 on indique les frequences corrispondenti alle curve riportate in figura 9.10. Il est interessant de noter que si on utilise la longueur du trou (figure (c)) plutôt que son rayon (figura (b)) pour dedimensionaliser la fréquence, les différents branches sont caracterisés par fréquences presque constants. Cela signifie que les fréquences d'oscillation du système sont inversement proportionnelles à l'épaisseur du trou, comme déjà observe par Bouasse (1929).

À fin de verifier les résultats obtenus à travers l'analyse des impédances en termes de instabilité hydrodynamique, une analyse de stabilité globale a été faite. Dans la figure 9.12, on présente la comparaison en-

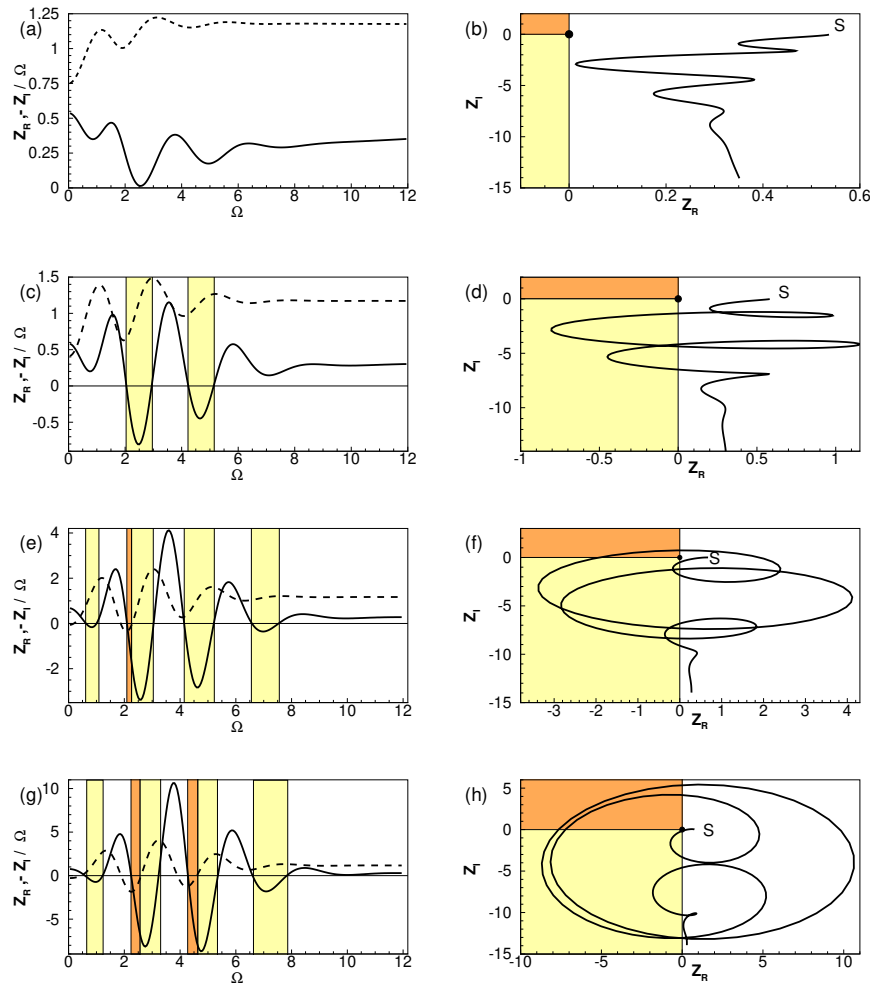


Figure 9.9: Impédances calculés à $\beta = 1$. gauche) résistance Z_R (ligne pleine) et réactance Z_I (ligne pointillée) fonction de la fréquence Ω ; droite) respectifs diagrammes de Nyquist pour (a, b), $Re = 800$, (c, d), $Re = 1200$, (e, f), $Re = 1600$, (g, h), $Re = 2000$.

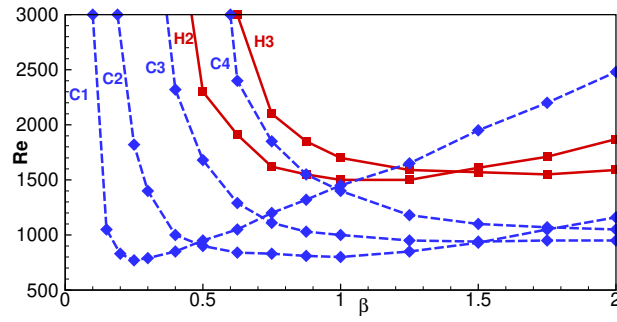


Figure 9.10: Valeurs de seuil pour la stabilité conditionnelle ($C1$ to $C4$) et l'instabilité hydrodynamique ($H2$ and $H3$).

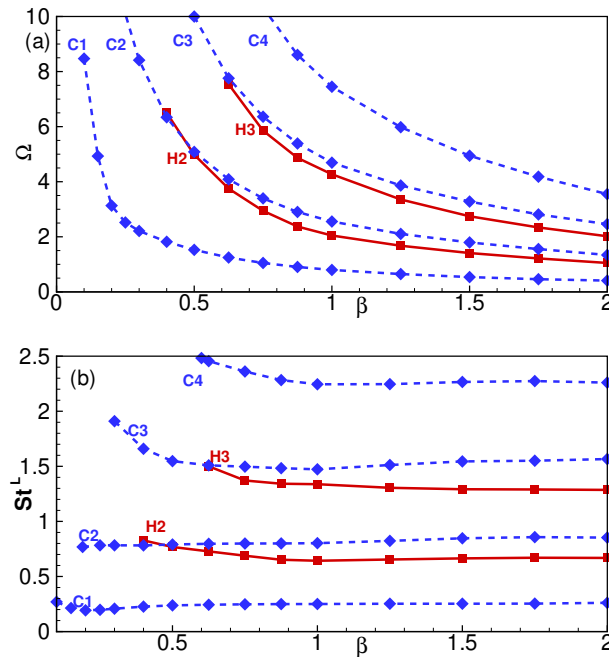


Figure 9.11: Fréquences correspondants à la stabilité conditionnelle ($C1$ to $C4$) et à l'instabilité hydrodynamique ($H2$ and $H3$).

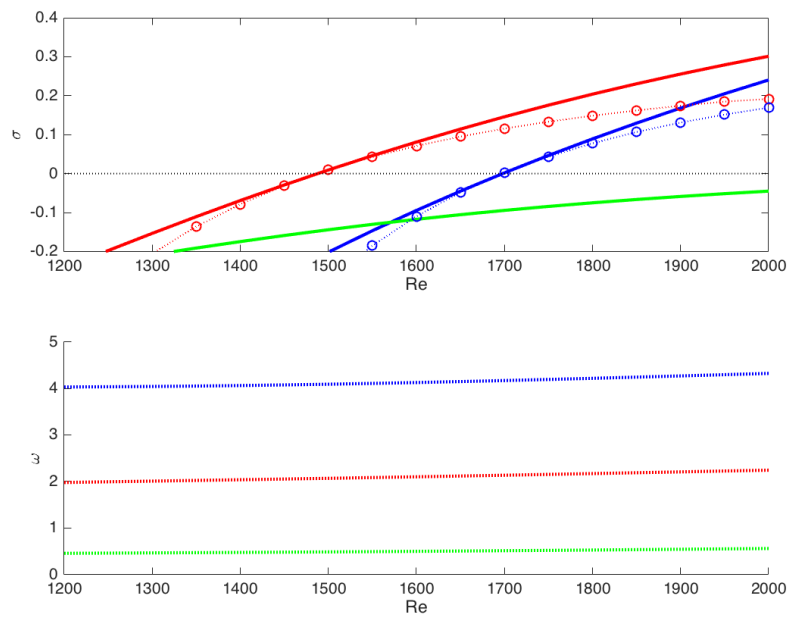


Figure 9.12: (a) Taux de croissance des troubles σ et (b) fréquence ω fonction de Re pour le trou à $\beta = 1$. Obtenus de l'analyse de stabilité globale (lignes pleines) et de l'analyse des impédances (lignes pointillées).

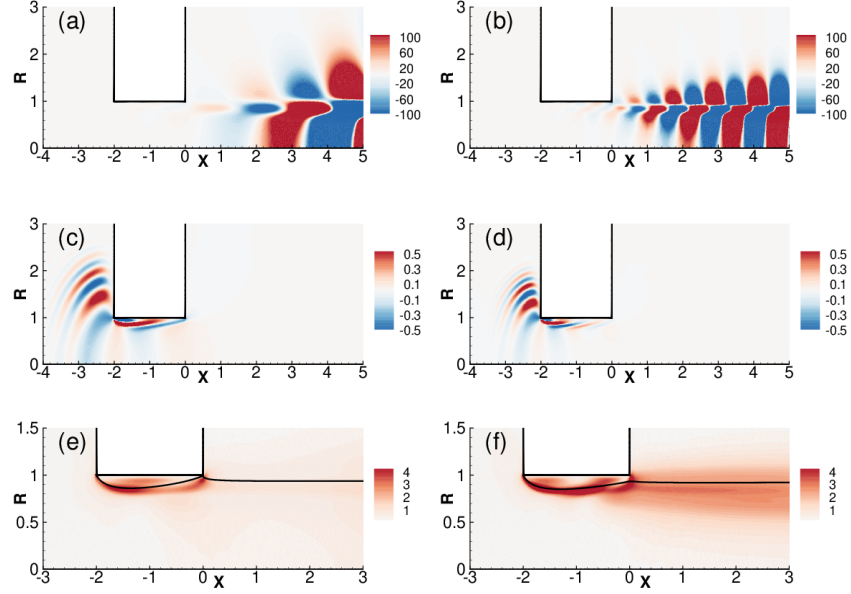


Figure 9.13: Mode directs (a & b), mode adjoint (c & d) et sensibilité structurale (e & f) pour les valeurs propres instables à $Re = 1500$ (colonne de gauche) et $Re = 1700$ (colonne de droite). Les lignes noires dans les figures (e) et (f) représentent le contour de la bulle de recirculation.

tre les résultats obtenus de l'analyse de stabilité globale (lignes pleines) et des impédances (lignes pointillées). Pour ce qui concerne le taux de croissance des troubles, il est possible de remarquer que les deux méthodes restituent les mêmes résultats au point neutre ($\sigma = 0$) et divergent, par contre, avec l'éloignement du point neutre même. En concernant les fréquences, par contre, les deux méthodes restituent exactement les mêmes valeurs.

Finalement, on réporte dans la figure 9.13 les modes directs (figure (a) et (b)) et les modes adjoint (figure (c) et (d)) obtenus pour les modes instables respectivement à $Re = 1500$ et $Re = 1700$. Le mode direct diffuse en aval du trou, alors que le mode adjoint diffuse en amont. En particulier, celui-ci a un maximum proche au bords gauche du trou, que donc est la zone du flux la plus réceptive à un forçement extérieur.

Enfin, en utilisant l'équation (9.3), on a calculé la sensibilité structurelle pour les deux cas considérés (figure (e) et (f)). Pour les deux cas la sensibilité structurelle est maximale au bord de la boucle de recirculation, et cela confirme que l'instabilité est étroitement liée à l'existence de la zone de recirculation.

9.4 Étude du son produit par un sifflement l'appeau des chasseurs: comparaison entre l'approche compressible et l'approche incompressible

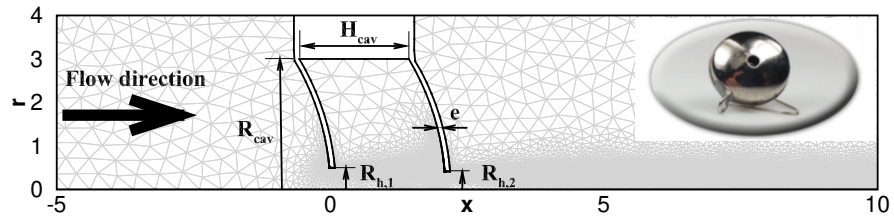


Figure 9.14: Configuration générale étudiée dans ce paragraphe et définition des paramètres géométriques.

La dernière partie de cette thèse couvre l'étude d'une configuration plus réaliste, c'est-à-dire un sifflet réalisé pour l'appeau des chasseurs, représenté dans la figure 9.14. Le flux passe au travers de deux trous circuliers consécutives situés dans parois incurvées; entre les deux trous il y a une cavité. L'objectif principale de ce paragraphe c'est la compréhension, par une analyse de stabilité globale, du son produit par le sifflet. En particulier, premièrement on a utilisé un incompressible pour identifier et caractériser les branches instables. Ensuite, l'effet de la compressibilité a été modélisé en considérant la cavité entre les deux trous comme résonateur de Helmholtz, dans la condition de bas nombres de Mach. L'hypothèse principale de ce modèle est cela de la compacité acoustique de la géométrie, c'est-à-dire que le nombre d'onde acoustique $\lambda_{ac} = 2\pi/\omega Ma$ doit être beaucoup plus grand des paramètres géométriques qui caractérisent la géométrie, comme les diamètres ou les dimensions de la cavité. Sous cet hypothèse, il est possible de considérer la cavité avec une pression et une densité constantes et donc on peut la modéliser comme un résonateur de Helmholtz: l'effet de la compressibilité du fluid est inclus en remplaçant une condition au contour sur l'impédance sur la paroi supérieure de la cavité, plutôt que une classique condition de non pénétration (Bonnefis, 2014). En considérant, à l'intérieur de la cavité, le flux isentropique, après différentes étapes(vedi Fry (2016)), il

est possible d'écrire la relation entre la pression de la cavité et le débit volumétrique comme suivant:

$$\partial_t p_{cav} + \frac{1}{\chi_c} Q_{cav} = 0 \quad \text{with} \quad \chi_c = V_{cav} Ma^2, \quad (9.6)$$

ou le coefficient χ_c est le paramètre de compressibilité, la fonction du nombre de Mach et du volume effectif de la cavité.

La pression et le débit volumétrique sont définis comme suivant:

$$\left. \begin{aligned} p_{cav} &= \frac{1}{S_{cav}} \int_{S_{cav}} p' dS \\ Q_{cav} &= \int_{S_{cav}} \mathbf{u}' \cdot \mathbf{n} dS \end{aligned} \right\}, \quad (9.7)$$

ou S_{cav} c'est la surface latérale de la cavité. Les équations (9.6) et (9.7) sont ensuite couplées avec les équations de Navier–Stokes incompressibles, auxquels ensuite est appliqué la théorie de la stabilité globale pour le calcul de la stabilité.

Finalement, on a considéré les équations de Navier–Stokes compressibles pour discuter la validité de ce model. Dans la figure (9.15) on réporte

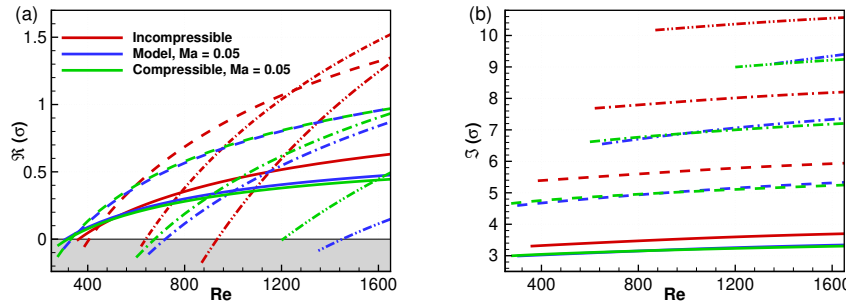


Figure 9.15: (a) Taux de croissance des troubles et (b) fréquence des mods instables en variation du nombre de Reynolds. Les lignes rouges représentent les résultats obtenus avec l'utilise des équations incompressibles, les lignes bleues représentent les résultats obtenus en utilisant le model et les lignes vertes sont les résultats obtenus avec les équations compressibles, les deux pour $Ma = 0.05$.

les résultats obtenus pour $Ma = 0.05$ avec les résultats incompressi-

bles. Bien que le nombre de Mach soit relativement bas, la compressibilité semble avoir un effet sur cette configuration, comme déjà observé par Yamouni et al. (2013). En particulier, la compressibilité a un effet déstabilisateur pour les premiers deux branches instables, donc ceux à fréquence $\omega \approx 3$ et $\omega \approx 4.5$. En outre, la compressibilité tend à baisser la fréquence des modes globaux. Les résultats obtenus en utilisant le modèle sont, pour ce nombre de Mach et cette géométrie particulière, très bons si comparés avec ceux obtenus avec les équations compressibles, surtout pour basses fréquences. Dans la figure 9.16, par contre, ils se trouvent

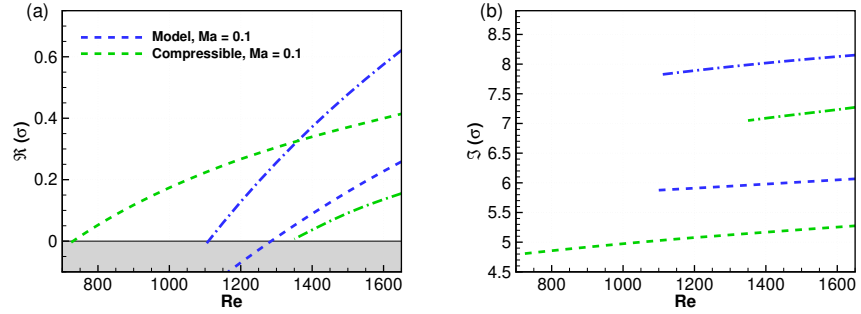


Figure 9.16: Le même que la figure 9.15 mais pour $Ma = 0.1$. Dans ce cas les résultats incompressibles ont été omis.

les résultats obtenus pour $Ma = 0.1$. Il n'y a pas de modes instables au premier branch, au moins pour les nombres de Reynolds considérés. En outre, c'est facile de remarquer que le modèle ne peut pas prédire les résultats obtenus avec les équations compressibles. L'hypothèse fondamentale du modèle c'est que le nombre d'onde acoustique λ_{ac} doit être beaucoup plus grand que les caractéristiques fondamentales de la géométrie ou, plus exactement, de la plus grande entre elles. Dans ce cas, la dimension géométrique la plus grande est le diamètre de la cavité $D_{cav} = 2R_{cav}$. Les simulations numériques montrent que le modèle marche jusqu'à $\lambda_{ac} > 2D_{cav} = 12$ pour la géométrie ici considérée. Si cette relation n'est pas considérée, l'onde acoustique peut pénétrer dans la cavité et la pression ne peut plus être considérée constante, en contredisant une des hypothèses du modèle.

Finalement, dans la figure 6.6 se trouve la partie réelle de la pression

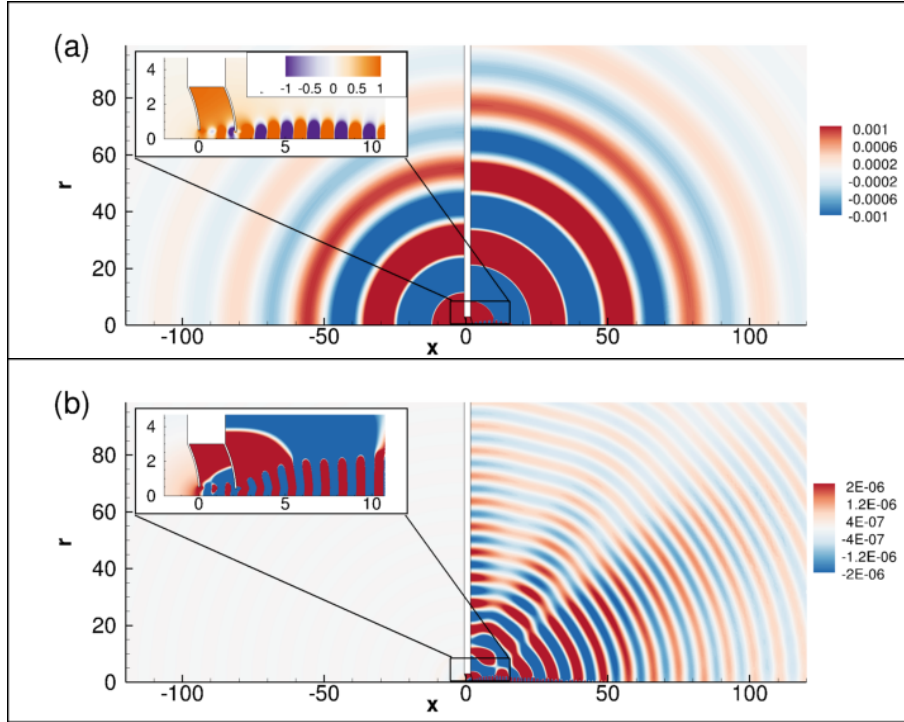


Figure 9.17: RPartie réelle de la pression du mode globale $\Re(p')$ pour: (a) $Re = 800$, $Ma = 0.05$ e $\omega \approx 4.95$; (b) $Re = 1400$, $Ma = 0.1$ e $\omega \approx 7.1$

du mode global obtenu en simulant les équations comprimibles ; dans la figure 6.6(a) nous réportons le mode globale obtenu pour $Re = 800$, $Ma = 0.05$ e $\omega = 4.95$, un cas dans le quel le model peut prédire les resultats comprimibles: en fait la longueur de l'onde acoustique, dans ce cas, est de $\lambda_{ac} \approx 25$: l'ingrandissement reporté dans la même figure de la zone proche du trou montre que la pression est constante dans la cavité.

Dans la figure 6.6(b) nous avons, par contre, le mode globale obtenu pour $Re = 1400$, $Ma = 0.1$ e $\omega = 7.1$, un cas dans le quel le model ne peut pas prédire les résultats comprimibles. Dans ce cas, la longueur de l'onde acoustique est s'élève à $\lambda_{ac} \approx 8.85$, donc hors du range de validité du model. Grâce à un agrandissement fait dans la zone de la cavité nous

puovons remarquer que la pr  ssion, dans ce cas, n'est pas constante mais nous pouvons remarquer une discontinuit   caus  e par la p  n  tration de l'onde acoustique    l'int  rieur de la cavit  .

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Appendix A

Appendix

A.1 Inviscid stability analysis of a cylindrical vortex sheet

In this appendix we review the stability analysis of a cylindrical vortex sheet, a classical problem first addressed by Batchelor and Gill (1962).

A.1.1 Equations

We consider as a base flow a cylindrical jet with a top-hat profile, with radius R_J and velocity U_J :

$$U_x(r) = \begin{cases} U_J & \text{if } r < R_J; \\ 0 & \text{if } r > R_J. \end{cases} \quad (\text{A.1})$$

This corresponds to a cylindrical shear layer. The stability analysis of this flow can be studied by adding small perturbations in potential form, both inside (ϕ_o) and outside (ϕ_i) the jet. These perturbations are searched in eigenmode form as follows :

$$\phi_i = AI_m(kr)e^{i(kx-\omega t)}; \quad \phi_o = BK_m(kr)e^{i(kx-\omega t)}; \quad \eta = Ce^{i(kx-\omega t)} \quad (\text{A.2})$$

Where $r = R_J + \eta$ is the location of the jet edge.

The matching conditions at $r = R$ are continuity of the pressure ($p_i = p_o$), and kinematical conditions connecting the temporal derivative of η to the radial velocity $\partial\phi/\partial r$. Hence :

$$i(\omega - kU_J)\phi_i(R_J) = i\omega\phi_o(R_J),$$

$$-i\omega\eta = (\partial\phi_o/\partial r)_{r=R_J},$$

$$i(kU_J - \omega)\eta = (\partial\phi_i/\partial r)_{r=R_J}.$$

Eliminating constants A, B, C , we get the following dispersion relation :

$$D(\omega, k) = (\omega - kU_J)^2 + L_0(kR_J)\omega^2 = 0 \quad (\text{A.3})$$

Where

$$L_0(k) = -\frac{I'_0(kR_J)K_0(kR_J)}{I_0(kR_J)K'_0(kR_J)}$$

Note that this dispersion relation generalizes the classical one for Kelvin-Helmholtz instability of a infinitely thin shear layer (obtained by replacing $L_0(k)$ by one). In the short-wavelength range ($kR_J \gg 1$), L_0 is close to one and the problem is effectively equivalent to the planar shear layer. On the other hand, in the long-wavelength range ($kR_J \ll 1$), L_0 tends to zero leading to different trends.

A.1.2 Temporal stability analysis

In a temporal stability framework ($k \in \mathbb{R}$), the dispersion relation leads to :

$$c = \frac{\omega}{k} = U_J \frac{1 \pm i\sqrt{L_0}}{1 + L_0}, \quad (\text{A.4})$$

where $c = c_r + ic_i$ is the phase velocity. The real part of the phase velocity represents the convection velocity of the disturbance, which corresponds to the term noted U_c in Howe's model. In the short-wave range, $L_0 \approx 1$ leading to $c_r = U_J/2$, which is the classical result for a planar shear layer. On the other hand, in the long-wave range, the asymptotic trends becomes $c_r \approx U_J$.

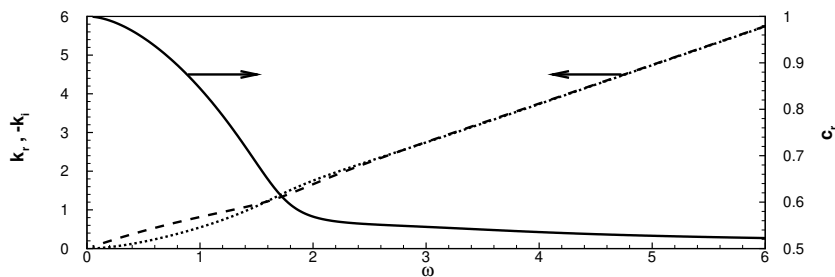


Figure A.1: Spatial stability analysis of a top-hat jet: $-$, k_r ; $\cdots$, $-k_i$; $—$, $c_r = \text{Re}(\omega/k)$.

A.1.3 Spatial stability analysis

In a spatial stability framework, which is more relevant here, $\omega \in \mathbb{R}$ and the problem has to be solved for the complex eigenvalue k . The dispersion relation has no analytical solution but is easily solved numerically. Results are reported in figure A.1. The spatial amplification rate $-k_i$ and the real part of the wavenumber k_r (related to the wavelength) are both increasing functions of ω . In the spatial framework one can still define the convection velocity of perturbations as $c_r = \text{Re}(\omega/k)$. This quantity is plotted in the figure with a thick line. We observe that the spatial analysis essentially leads to the same conclusion, namely the convection velocity of perturbations is close to $U_J/2$ in the high-frequency (i.e. short-wavelength) regime and close to U_J in the low-frequency (i.e. long-wavelength) regime.

A.2 Numerical Validations

In this appendix, we provide additional results obtained by varying the defining the mesh dimensions, grid density density as well as the parameters defining the complex mapping function. All meshes used in the study are described in table A.4. Meshes $\mathbb{M}_0$ and $\mathbb{M}_1$ are the reference meshes used in the paper, respectively without and with the use of complex mapping. Meshes $\mathbb{M}_{2-7}$ are additional meshes using

Mesh	R_{in}	L_{in}	R_{out}	L_{out}	L_C	L_A	γ_c	R_M	R_A	x_{max}	r_{max}	n_t
$\mathbb{M}_0$	10	10	20	80	—	—	—	—	—	80	20	101403
$\mathbb{M}_1$	10	10	15	15	1.25	16	0.3	5	16	1022+306i	337	53352
$\mathbb{M}_2$	10	10	15	15	1.25	16	0.5	5	16	1022 + 511i	337	26649
$\mathbb{M}_3$	10	10	15	15	1.25	20	0.3	5	20	78 + 23i	38	52587
$\mathbb{M}_4$	10	10	15	15	2.50	16	0.3	5	16	1022 + 306i	337	37972
$\mathbb{M}_5$	15	15	15	15	1.25	16	0.3	5	16	1022 + 306i	337	54782
$\mathbb{M}_6$	10	10	15	15	1.25	16	0.3	5	16	1022 + 306i	337	28579
$\mathbb{M}_7$	10	10	15	15	1.25	16	0.3	5	16	1022 + 306i	337	254093
$\mathbb{M}_8$	10	10	20	40	—	—	—	—	—	40	20	66759
$\mathbb{M}_9$	10	10	20	160	—	—	—	—	—	160	20	182659

Table A.1: Descriptions of numerical meshes $\mathbb{M}_{0-9}$ in term of dimensions, mapping parameters, and number of triangles n_t .

complex mapping, with different choices for the dimensions, parameters, and/or density. Meshes $\mathbb{M}_{8-9}$ are additional meshes without complex mapping, with different dimensions in the axial direction.

A.2.1 Complex mapping validation

We first provide a few additional results to illustrate the failure of the resolution in physical coordinates to compute the impedance for large Reynolds numbers, and the efficiency of the complex mapping technique to resolve it.

Figure A.2 displays a comparison between the impedances for $Re = 500$ calculated using the reference meshes as well as an additional mesh $\mathbb{M}_8$ designed without complex mapping and a with a shorter axial dimension. As one can observe, in this range of Reynolds number, results obtained with and without complex mapping are almost identical : the curves are perfectly overlapped for Z_R whereas for Z_I a little difference exists but with relative errors less than 1%.

Figure A.3 present a similar comparison for $Re = 2000$, using this time an additional mesh without complex mapping of longer axial dimension (mesh $\mathbb{M}_9$ with $L_{out} = 160R_h$). From the figure, it is clear that the computation of the impedance without the mapping is impossible. The reference mesh $\mathbb{M}_0$ leads to non-physical oscillations in the range $\Omega \gtrsim 2$. Doubling the size of the domain does not lead to notable im-

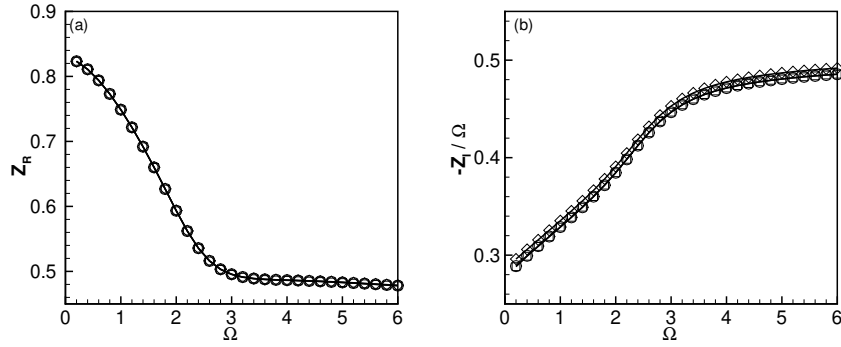


Figure A.2: Comparison between the results of impedances obtained using the complex coordinate mapping ($\diamond$, mesh $\mathbb{M}_1$) and without mapping with $L_{out} = 40R_h$ ($\circ$, mesh $\mathbb{M}_8$) and $L_{out} = 80R_h$ ($\square$, mesh $\mathbb{M}_0$) at $Re = 500$.

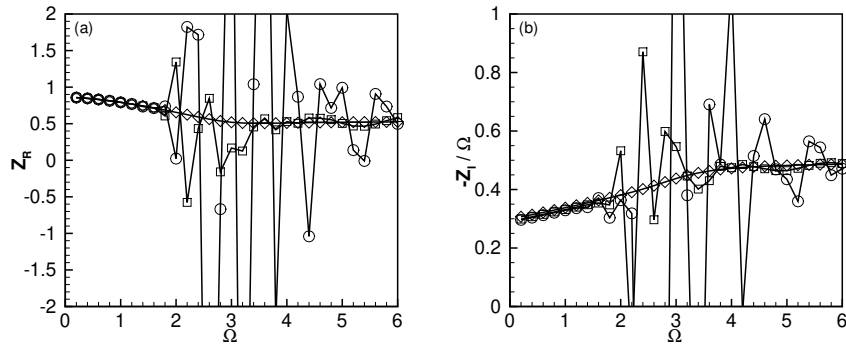


Figure A.3: Impedances computed using the complex coordinate mapping ($\diamond$, mesh $\mathbb{M}_1$) and without mapping with $L_{out} = 80R_h$ ($\circ$, mesh $\mathbb{M}_0$) and $L_{out} = 160R_h$ ($\square$, mesh $\mathbb{M}_9$) at $Re = 2000$.

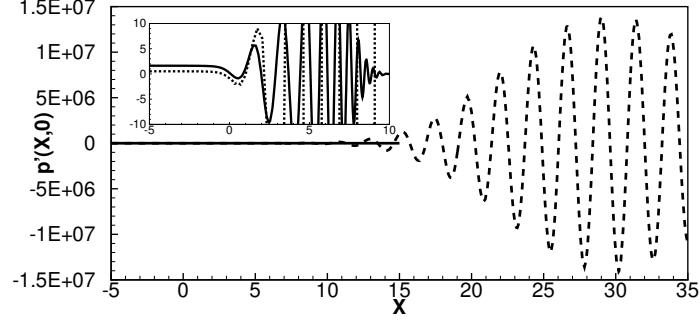


Figure A.4: Pressure perturbation on the symmetry axis for $Re = 2000$ and $\Omega = 3$ computed with complex mapping (mesh $\mathbb{M}_1$; —) and without complex mapping (mesh $\mathbb{M}_9$ with $L_{out} = 160R_h$; - -).

provements, and the use of complex mapping proves to be the only way to obtain reliable results.

To illustrate further the failure of the numerical resolution without complex mapping, we report in figure A.4 the pressure of the perturbation on the symmetry axis for $Re = 2000$ and $\Omega = 3$. As can be seen, the amplitudes in physical coordinates reach levels of order 10^7 . As a consequence, round off errors can occur, leading to an error propagation in all the domain and so a wrong pressure level at inlet. This is visible in the insert plot displaying a zoom in the inlet region, showing a mismatch between the results with and without complex mapping.

A.2.2 Robustness of the complex mapping

In order to validate the mapping and to verify its robustness, we performed a sensitivity analysis of the impedances to geometrical and mapping parameters variation. In figure A.5 we compare the impedances for at $Re = 3000$, computed with meshes $\mathbb{M}_{1-5}$. One can observe that the curves are all overlaid to the reference curves, namely the mesh $\mathbb{M}_1$ used to calculate the impedances in section 4.5.3, showing the robustness and the efficiency of the mapping formula used.

Finally, the last numerical issue is about the thickness of the hole. In the whole paper we assume a zero thickness hole, but to generate the

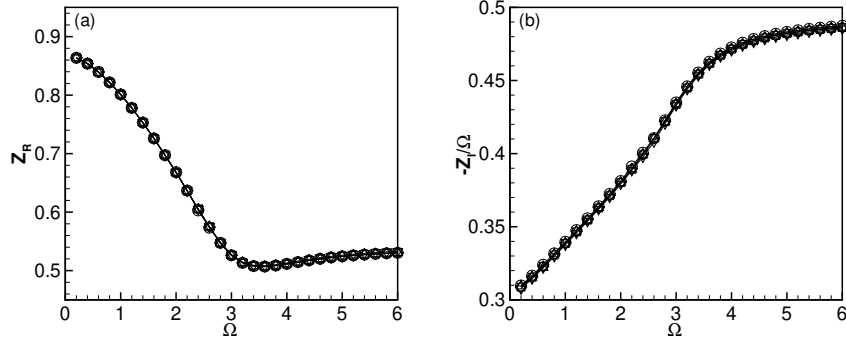


Figure A.5: (a) Real part and (b) imaginary part of the impedance at $Re = 3000$ computed with the mesh $\mathbb{M}_1$ ($-\square-$), $\mathbb{M}_2$ ($-\triangle-$), $\mathbb{M}_3$ ($-\nabla-$), $\mathbb{M}_4$ ($-\diamond-$) and $\mathbb{M}_5$ ($-\circ-$).

Mesh	n_t	$n_{d.o.f.}$	$\alpha(Re = 500)$	$\alpha(Re = 2000)$
$\mathbb{M}_1$	53352	241797	0.625367	0.612052
$\mathbb{M}_6$	28579	129911	0.625370	0.612059
$\mathbb{M}_7$	254093	1146799	0.625366	0.612052

Table A.2: Characteristic of the meshes used for the convergence analysis using the mapping parameters reported in table A.4 and convergence of the base flow. Note that $n_{d.o.f.}$ is the number of degrees of freedom of the mesh.

mesh we had to specify a small but finite value. We set it to 10^{-4} and we verified that results were insensitive to this length.

A.2.3 Mesh convergence

The last issue to consider for numerical validation is the sensitivity of results to grid density. As explained in section 3, the mesh generation process involves mesh adaptation thanks to the `adaptmesh` command of the `FreeFem++` software. Although the procedure is automatic, its efficiency can be tuned by specifying an interpolation error parameter.

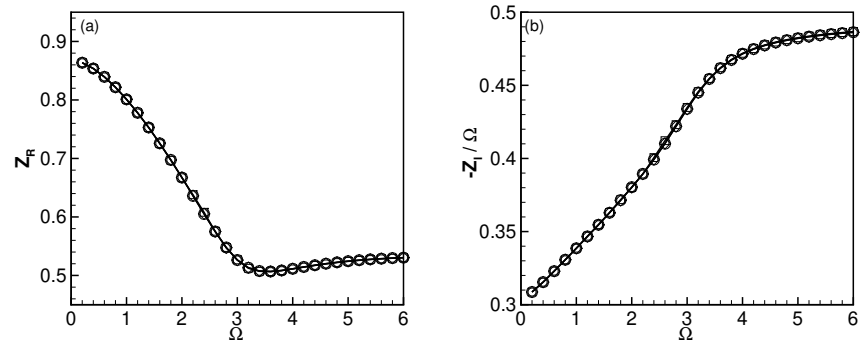


Figure A.6: (a) Real part and (b) imaginary part of the impedance at $Re = 3000$ computed with the mesh $\mathbb{M}_1$ ($-\square-$), $\mathbb{M}_6$ ($-\circ-$) and $\mathbb{M}_7$ ($-\diamond-$).

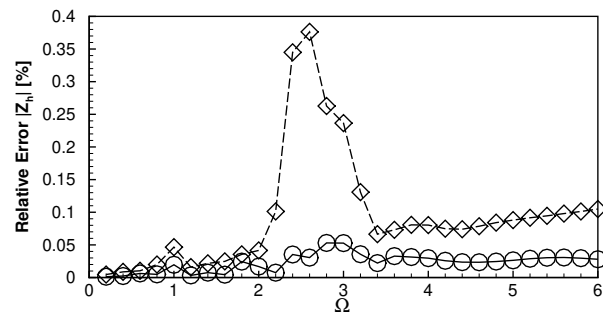


Figure A.7: Relative error calculated on the absolute value of the impedance between the meshes $\mathbb{M}_6 - \mathbb{M}_7$ ($--\diamond--$) and the meshes $\mathbb{M}_1 - \mathbb{M}_7$ ($-\circ-$) at $Re = 3000$.

Case	Base flow	Single linear problem	full parametric study
Mesh $\mathbb{M}_0$, $Re = 1500$	11 min 15 s	1 min 55 s	144 min 25 s
Mesh $\mathbb{M}_1$, $Re = 3000$	4 min 48 s	1 min 15 s	78 min 55 s

Table A.3: CPU time on a standard computer (MacBook Pro 2012, 2.5 GHz Intel Core i5, 4 Gb RAM) required for (a) computation of a base-flow and generation of an adapted mesh following the procedure explained in section 4, (b) resolution of a linear problem for a single value of Ω , and (c) full parametric study of the impedance, including generation of base flow and mesh, and resolution of 100 linear problems in the range $\Omega \in [0 - 6]$.

Mesh $\mathbb{M}_1$ was obtained using the default value $5 \cdot 10^{-3}$. Two additional meshes were designed, respectively with interpolation error 10^{-2} (mesh $\mathbb{M}_6$) and 10^{-3} (mesh $\mathbb{M}_7$). Table A.2 gives the number of triangles n_t of each of these meshes, as well as the corresponding number of degrees of freedom ($n_{d.o.f}$) of the finite-element discretization. In the two least columns of the table, we reported the values obtained for the *vena contracta* coefficient for $Re = 500$ and $Re = 2000$ with each of these meshes. One can observe that results are accurate up to the fourth digit. In figure A.6 we compare the impedances computed with each of these meshes for $Re = 3000$. The figure shows that the curves are completely overlapped. We also quantified the relative error of the meshes $\mathbb{M}_1$ and $\mathbb{M}_6$ with respect to $\mathbb{M}_7$ for all the frequencies. From figure A.7 we can observe that the maximum relative error committed using the mesh $\mathbb{M}_6$ is around 0.38%, whereas the maximum error is reduced to 0.005% using the mesh $\mathbb{M}_1$.

A.2.4 Numerical efficiency

One of the advantages of using LNSE compared to full DNS is that the first approach only requires resolution of a small amount of linear problems, while the second one requires a time-integration of the equations over a long time covering several periods of oscillation to ensure convergence. In this section we demonstrate the numerical efficiency of the method by indicating in table 3 the CPU time required for the various steps of the analysis. As can be seen, the amount of time to obtain an impedance curve $Z(\Omega)$ in the range $\Omega \in [0 - 6]$ (including generation

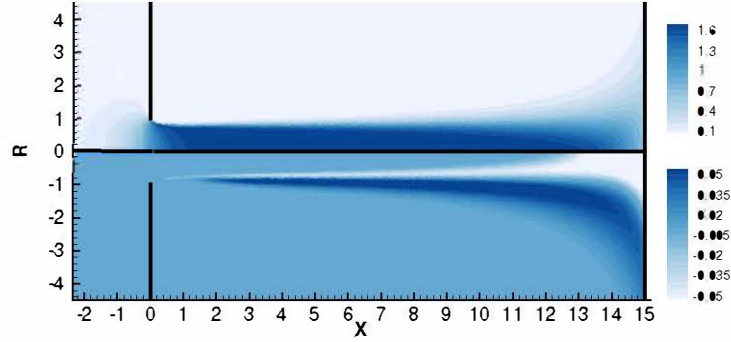


Figure A.8: Real (upper part) and imaginary (lower part) of the axial velocity U_x of the base flow at $Re = 3000$.

of the base flow and adapted mesh, and resolution of elementary problems for 100 values of Ω spanning the range) on a standard computer is of the order of 144 minutes using resolution in real coordinates, and can be reduced to as small as 78 minutes using the complex mapping method.

As a comparison, using the DNS time-stepped we have used for generating the results displayed in section 6, using M_0 , a time-step $dt = 2 \cdot 10^{-3}$ and performing 50000 time steps to cover several oscillation periods required a total CPU time of about 90 hours, for a single value of the parameters Ω and ϵ .

A.3 The complex base flow

As mentioned in section 3 and detailed in appendix B, as soon as $Re > 1500$, converged results for the impedance can only be obtained using the complex mapping technique. For consistency of the whole approach, in such cases the base flow also has to be computed in the same coordinates.

In this appendix we briefly document the structure of the base flow when computed in terms of complex mapped coordinates. Figure A.8 displays the axial velocity component of the base flow U_x obtained in

this way. This quantity is actually the analytical continuation of the axial velocity (displayed in figure 4.4) in the complex x -plane. The real part (upper plot) has a structure similar to the one in real coordinates plotted in figure 4.4, but one can observe that the thin shear layer rapidly enlarges as one progresses along the X direction. This is mostly an effect of the stretching involved in the coordinate mapping: the computation is made with $L_{out} = 15$ and $L_A = 16$, so that the position in the X direction corresponding to $|x| \rightarrow \infty$ is actually just a little outside of the computational domain.

The lower part displays the imaginary part of U_x . Here the complex coordinate mapping is done with $L_C = 1.25$, so the imaginary part becomes significant above this value in the X direction.

A.4 Link between impedance and reflection coefficient

The objective of this appendix is to establish the link between the impedance of the aperture and the reflection coefficient of an acoustic wave. For this purpose, we will perform an asymptotic matching between the incompressible "inner" solution investigated in the main part of the paper and a compressible "outer solution" expressed in terms of spherical acoustic waves.

We thus consider an outer solution composed in the upstream domain of an incident convergent spherical wave of amplitude A and a reflected divergent spherical wave of amplitude B , and in the downstream region of a transmitted spherical diverging wave of amplitude C . We use spherical coordinates and assume a pressure field $p'(r_s, t)$ and a velocity field $\mathbf{u}' = u'_{rs}(r_s, t)\mathbf{e}_{rs}$ where $r_s = \sqrt{r^2 + x^2}$ is the spherical radial coordinate and $\mathbf{e}_{rs}$ is the unit vector in the radial direction. The pressure and axial velocity fields have the classical expressions:

$$p'(r_s, t) = \begin{cases} \frac{A}{r_s} e^{-i(kr_s + \omega t)} + \frac{B}{r_s} e^{i(kr_s - \omega t)} & x < 0; \\ \frac{C}{r_s} e^{i(kr_s - \omega t)} & x > 0. \end{cases} \quad (\text{A.5})$$

$$u'_{rs}(r_s, t) = \begin{cases} \frac{A}{i\rho\omega} \left(\frac{1}{r_s^2} - \frac{ik}{r_s} \right) e^{-i(kr_s + \omega t)} + \frac{B}{i\rho\omega} \left(\frac{1}{r_s^2} + \frac{ik}{r_s} \right) e^{i(kr_s - \omega t)} & x < 0; \\ \frac{C}{i\rho\omega} \left(\frac{1}{r_s^2} + \frac{ik}{r_s} \right) e^{i(kr_s - \omega t)} & x > 0. \end{cases} \quad (\text{A.6})$$

where $k = \omega c_0$ is the acoustical wavenumber and c_0 is the speed of sound. The inner limit ($r_s \rightarrow 0$) of this outer solution can be expressed as follows:

$$p'(r_s, t) \approx \begin{cases} \left(\frac{(A+B)}{r_s} + ik(B-A) \right) e^{-i\omega t} & x < 0; \\ \left(\frac{C}{r_s} + ikC \right) e^{-i\omega t} & x > 0. \end{cases} \quad (\text{A.7})$$

$$u'_{rs}(r_s, t) \approx \begin{cases} \frac{(A+B)}{\rho i \omega r_s^2} e^{-i\omega t} & x < 0; \\ \frac{C}{\rho i \omega r_s^2} e^{-i\omega t} & x > 0. \end{cases} \quad (\text{A.8})$$

The outer limit of the inner solution (i.e the incompressible solution considered in the main part of the paper) is a spherical source (resp. sink) of flow rate q' in the downstream (resp. upstream) domain and reads:

$$u'_{rs}(r_s, t) \approx \begin{cases} \frac{-q'}{2\pi r_s^2} e^{-i\omega t} & x < 0; \\ \frac{q'}{2\pi r_s^2} e^{-i\omega t} & x > 0. \end{cases} \quad (\text{A.9})$$

$$p'(r_s, t) \approx \begin{cases} \left(p'_{in} + \frac{\rho i \omega q'}{2\pi r_s} \right) e^{-i\omega t} & x < 0; \\ \left(p'_{out} - \frac{\rho i \omega q'}{2\pi r_s} \right) e^{-i\omega t} & x > 0. \end{cases} \quad (\text{A.10})$$

Note that the latter expressions comprise both the constant levels p'_{out} , p'_{in} and a subdominant term proportional to $1/r_s$ which was not mentioned in the main part of the paper. The latter corresponds to the pressure field associated to an unsteady incompressible source/sink.

The matching is done by identifying the coefficients of similar terms in Eqs. (A.7), (A.8), (A.9), (A.10). This leads to:

$$(A + B) = \frac{-\rho i \omega q'}{2\pi}, \quad (\text{A.11})$$

$$ik(B - A) = p'_{in}, \quad (\text{A.12})$$

$$C = \frac{-\rho i \omega q'}{2\pi}, \quad (\text{A.13})$$

$$ikC = p'_{out} \quad (\text{A.14})$$

The two latter relations can be combined with the introduction of the *radiation impedance* Z_{rad} :

$$Z_{rad} = \frac{p'_{out}}{q'} = \frac{\rho \omega^2}{2\pi c_0} \quad (\text{A.15})$$

The expressions can be eventually combined to express the amplitude reflection coefficient B/A in terms of the hole impedance Z_h and the radiation impedance just introduced:

$$\frac{B}{A} = \frac{-Z_h}{Z_h + 2Z_{rad}} \quad (\text{A.16})$$

The energy reflection coefficient R is eventually deduced as:

$$R = \frac{|B|^2}{|A|^2} = \frac{|Z_h|^2}{|Z_h|^2 + 4Z_{rad}(Re(Z_h) + Z_{rad})} \quad (\text{A.17})$$

These expressions yield the following conclusions :

- the energy reflection R is larger than 1 (overreflexion condition) if and only if $Re(Z_h) + Z_{rad} < 0$. In dimensionless terms, this leads to

$$Z_R + \frac{M\Omega^2}{2\pi} < 0$$

(where M is the Mach number), which reduces to the simpler condition $Z_R < 0$ given in section 2 in the limit $M \ll 1$.

- B/A is infinite if and only if $Re(Z_h) + 2Z_{rad} = 0$. The situation $B/A = \infty$ corresponds to a situation where a wave is emitted

upstream ($B \neq 0$) in the absence of an incident wave ($A = 0$), hence to a spontaneous self-oscillation associated to emission of sound both upstream and downstream. We recognize the definition of the purely hydrodynamical instability described in sec. 2. In dimensionless terms, the condition leads to

$$Z + \frac{M\Omega^2}{\pi} = 0$$

which reduces to the simpler condition $Z_R = Z_I = 0$ given in section 2 in the limit $M \ll 1$.

Note that the assumption of an incident converging spherical wave coming from a semi-infinite space adopted here is questionable; clearly other choices are possible for modelling the upper domain. For instance, the case where the upper domain is a long pipe of radius $R_p \gg R_h$ and the incident wave is a plane wave can also be considered, and the analysis leads to practically identical conclusions.

A.5 Details on the complex mapping technique and mesh validations

As identified in Fabre et al. (2018b), a severe numerical difficulty arises in the resolution of the LNSE equations (for both forced and autonomous problems) due to the strong spatial amplification of linear perturbations. In this previous paper, usage of a complex coordinate mapping was proposed as an efficient way to overcome this difficulty. Fabre et al. (2018b) demonstrated that in conjunction with mesh adaptation, this method allows both to significantly reduce the required number of mesh points and to extend the range of application of the LNSE up to $Re \approx 3000$.

In this appendix we give some detail about the implementation and efficiency of this technique for present study. The technique has been used for both forced (impedance) and autonomous (eigenvalues) computations, but we only document its performances for the autonomous problem, restricting to the case $\beta = 1$.

	L_{out}	R_{out}	L_M	L_C	L_A	γ_c	R_M	R_A	x_{max}	r_{max}	δ_M	Adapt.	N_v
$\mathbb{M}_1$	15	15	0	2.5	17	0.3	5	17	503+149i	337	1	BF+F	19320
$\mathbb{M}_2$	20	15	5	1	—	0.5	—	—	20+8i	15	0.5	BF+M	19075
$\mathbb{M}_3$	30	20	—	—	—	—	—	—	30	20	1	BF+M+A	30695
$\mathbb{M}_4$	60	20	—	—	—	—	—	—	60	20	0.25	BF+F	49999

Table A.4: Description of meshes $\mathbb{M}_1$ - $\mathbb{M}_4$ built for $\beta = 1$ following four different strategies. $[L_M, L_C, L_A, \gamma_c, R_M, R_A]$: parameters defining the coordinate mapping. $[x_{max}, r_{max}]$: effective dimensions in physical coordinates. δ_M : prescribed value of the maximum grid step. Adapt.: mesh adaptation strategy (see text). N_v : number of vertices of the mesh obtained at the outcome of the adaptation process.

In the present paper, the mappings from numerical coordinates (X, R) to physical coordinates (x, r) are slightly different from the ones used in Fabre et al. (2018b), and defined as follows:

$$\begin{aligned}
 x = \mathcal{G}_x(X) &= L_M + \frac{X - L_M}{\left[1 - \left(\frac{X - L_M}{L_A - L_M}\right)^2\right]^2} \left[1 + i\gamma_c \tanh\left(\frac{X - L_M}{2L_C}\right)^2\right] \quad \text{for } X < L_M, \\
 &= X \quad \text{for } L_M < X < L_{out}.
 \end{aligned} \tag{A.18}$$

$$\begin{aligned}
 r = \mathcal{G}_r(R) &= R_M + \frac{R - R_M}{\left[1 - \left(\frac{R - R_M}{R_A - R_M}\right)^2\right]^2} \quad \text{for } R > 0 \text{ and } R_A < R < R_{out}, \\
 &= R \quad \text{otherwise}
 \end{aligned} \tag{A.19}$$

Note that the mapping of the x -mapping involves both an imaginary part (controlled by the parameter γ_c) and a stretching (controlled by the parameter L_A .) The difference with Fabre et al. (2018b) is the presence of an additional parameter L_m such that the complex mapping only applies for $x > L_m$.

The set of parameters used and the corresponding dimension of the domain in complex coordinates are reported in table A.4.

For validation of the method it is essential to demonstrate that the

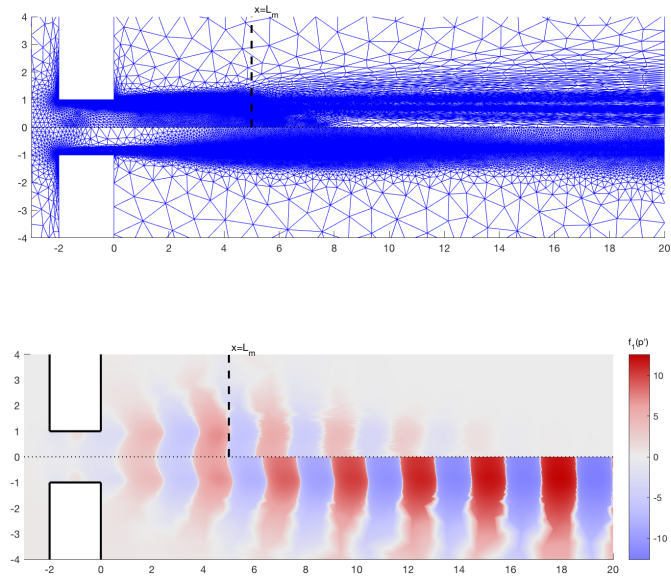


Figure A.9: (a) Structure of meshes M_2 (upper) and M_4 (lower) ; (b) pressure component of the eigenmode H2 as computed using mesh M_2 (upper) and M_4 (lower).

results are effectively independent of the values of the parameters. In the present study we have mainly used two kind of meshes involving complex mapping, with properties detailed in table A.4. The first one, named $\mathbb{M}_1$, and already plotted in fig. 3, is very similar to the one used in Fabre et al. (2018b) for the case of the zero-thickness hole. This kind of mesh has been used for the impedance-based parametric study of section 5.3. On the other hand, since the coordinate mappings applies for $x > L_M = 0$, it is not suited to represent the linear forced flow and eigenmode structures. The second one kind of mesh, named $\mathbb{M}_2$, has no stretching (thus parameters L_A and R_A are not relevant) but only complex mapping. This kind of mesh has been used to plot the structures (figures 8,10 and 14) since complex mapping only applies for $x > L_M = 5$, outside of the chosen range of these figures. The two meshes also differ by the mesh adaptation strategy : mesh $\mathbb{M}_1$ is adapted to the base flow for $Re = 2000$ and two forced flow structures computed for two values of Ω spanning the range of the parametric study, namely $\Omega = 0.5$ and $\Omega = 4.5$, following the same strategy as in Fabre et al. (2018b). On the other hand, mesh $\mathbb{M}_1$ is adapted to the base flow and the two leading eigenmodes H1 and H2, following the same strategy as in Fabre et al. (2018c).

For validation purposes, we have also designed two meshes $\mathbb{M}_3$ and $\mathbb{M}_4$ which do not involve coordinate mapping. These meshes are designed with a longer axial dimension L_{out} , and are characterized and are significantly heavier in terms of number of vertices.

Figure A.9(a) displays the structure of meshes $\mathbb{M}_2$ $\mathbb{M}_4$. It is found that the mesh adaptation strategy used for mesh $\mathbb{M}_2$ is most efficient to concentrate the grid points in the most significant regions of the flow (inside the hole) while $\mathbb{M}_4$ concentrates a much larger number of points in the far downstream regions.

Figure A.10 superposes the numerically computed spectra using meshes $\mathbb{M}_1$, $\mathbb{M}_3$ and $\mathbb{M}_4$ for $Re = 1700$ and 2000 . As usual, along with the eigenvalues of the physically relevant modes H1, H2, H3, the spectra display a large number of spurious eigenvalues. As can be seen, both meshes $\mathbb{M}_3$ and $\mathbb{M}_4$ lead to the presence of spurious modes in the unstable part ($\omega_i > 0$) of the complex plane, and as the Reynolds number is increase they come dangerously close to the physical eigenvalues. On the other hand, the complex mapping used for mesh $\mathbb{M}_1$ results in a good separa-

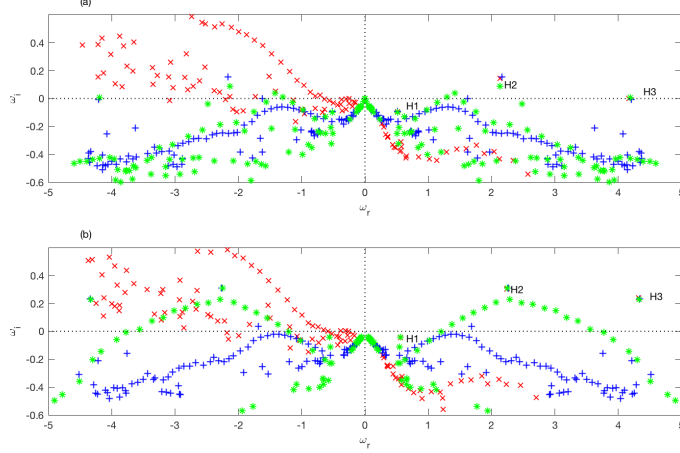


Figure A.10: Spectra computed with three different meshes : $\times$ (red online): mesh $\mathbb{M}_1$ (complex mapping); $*$ (green online): mesh $\mathbb{M}_3$ (no mapping, $X_{max} = 30$); $+$ (blue online): mesh $\mathbb{M}_4$ (no mapping, $X_{max} = 60$). (a): $\beta = 1$; $Re = 1700$; (b): $\beta = 1$; $Re = 2000$

tion between the physical eigenvalues and the spurious ones, which are substantially shifted in the stable part ($\omega_i < 0$) of the complex plane. Note, however, that use of the complex mapping does not allow to compute the complex conjugates of modes H1, H2, H3 located in the $\omega_r < 0$ half-plane. For reasons discussed in Fabre et al. (2018b), using a complex mapping with $\gamma_c > 0$ only allows to suppress the spatial amplification of linear forced structures (or eigenmodes) with $\omega_r > 0$. Instead, choosing $\gamma_c < 0$ would give access to the other half of the spectrum.

Table A.5 displays the eigenvalues H1, H2, H3 computed for $Re = 1600$ and $Re = 2000$ using all meshes considered here. The table confirms that the results obtained using complex mapping are independent upon the values of the parameters (value for $\mathbb{M}_1$ and $\mathbb{M}_2$ are very close to each other despite the fact that the parameters are very different). They also show that the meshes $\mathbb{M}_3$ and $\mathbb{M}_4$ are less reliable despite the fact that they contain a larger number of vertices.

Finally, figure A.9(b) depicts the structure of the eigenmode H2 computed using meshes $\mathbb{M}_2$ and $\mathbb{M}_4$ for $Re = 1600$. As the complex mapping

	$Re = 1600$		
	H1	H2	H3
$\mathbb{M}_1$	$-0.1176i + 0.5000$	$0.0809i + 2.1007$	$-0.0942i + 4.1245$
$\mathbb{M}_2$	$-0.1164i + 0.5014$	$0.0813i + 2.1108$	$-0.0925i + 4.1207$
$\mathbb{M}_3$	$-0.1259i + 0.5017$	$0.13916i + 2.1051$	$-0.1051i + 4.1359$
$\mathbb{M}_4$	$-0.1189i + 0.5017$	$0.0840i + 2.0953$	$-0.1955i + 4.0984$

	$Re = 2000$		
	H1	H2	H3
$\mathbb{M}_1$	$-0.0450i + 0.5610$	$0.3010i + 2.2434$	$0.2408i + 4.3205$
$\mathbb{M}_2$	$-0.0438i + 0.5619$	$0.3042i + 2.2476$	$0.2427i + 4.3170$
$\mathbb{M}_3$	$-0.0421i + 0.5645$	$0.3114i + 2.2467$	$0.2287i + 4.3268$
$\mathbb{M}_4$	$-0.0420i + 0.5628$	$0.2965i + 2.2399$	$0.1232 + 4.2807$

Table A.5: Eigenvalues computed with four different meshes for $Re = 1600$ and $Re = 2000$ ($\beta = 1$)

for mesh $\mathbb{M}_2$ only applies for $x > L_m = 5$, the structure for $x < L_m$ is expected to be identical as when computed without this method. The figure confirms that this is effectively the case. On the other hand, for $x > L_m$ the eigenmode computed in physical coordinates still displays a spatial amplification up to a very large downstream distance. On the other hand, the complex mapping results in a suppression of this spatial amplification.

Note that figure A.9(b) makes use of a nonuniform colormap by plotting $f_S(p')$ as explained in sec. 5.1. Without this trick it would be impossible to give a good representation of the structure, as the maximum values p' are of order $1.8 \cdot 10^3$ and $3 \cdot 10^5$ for $\mathbb{M}_2$ and $\mathbb{M}_4$, respectively. Hence use of the complex mapping limits the round-off errors due to the very large maximal levels reached far downstream. Note that on the other hand, this visualization method enhances the numerical imprecision in the external parts of the flow ($|R| > 2$) where the mesh is less refined (but where mesh refinement is not necessary for accurate computation of the eigenvalues).